QUASIFREE MARTINGALES

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Abstract. A noncommutative Kunita-Watanabe-type representation theorem is established for the martingales of quasifree states of CCR algebras. To this end the basic theory of quasifree stochastic integrals is developed using the abstract Itô integral in symmetric Fock space, whose interaction with the operators of Tomita-Takesaki theory we describe. Our results extend earlier quasifree martingale representation theorems in two ways: the states are no longer assumed to be gauge-invariant, and the multiplicity space may now be infinite-dimensional. The former involves systematic exploitation of Araki’s Duality Theorem. The latter requires the development of a transpose on matrices of unbounded operators, defying the lack of complete boundedness of the transpose operation.

Introduction

In this paper we consider martingales adapted to a filtration of von Neumann algebras determined by a quasifree state of the CCR algebra over an $L^2$-space of vector valued functions on the half-line. The main tools of our analysis are the abstract Itô integral in Fock space whose interaction with the operators of Tomita-Takesaki theory enables us to develop the basic theory of quasifree quantum stochastic integrals, and Araki’s Duality Theorem for generating Type III factors with a cyclic and separating vector from the Fock representation of a CCR algebra. A transpose operation on the relevant class of integrands also plays a crucial role. The main result is a noncommutative Kunita-Watanabe-type representation theorem for quasifree martingales.

Our results extend previous work in two ways. First the multiplicity space of the noise may now be infinite dimensional, and secondly, the class of quasifree states is much wider than hitherto considered; it is subject only to natural constraints, in particular we go beyond gauge-invariant states. The importance of the former generalisation is underlined by the fact that the stochastic flows arising in the dilation of Markov semigroups on operator algebras typically require infinite-dimensional multiplicity spaces. A consequence of the latter is that (without gauge invariance) creation and annihilation integrals need no longer be mutually orthogonal at the Hilbert space level. As with [HuL], and its fermionic counterpart [L1], the full filtration of the quasifree noise is used here, rather than that generated by a fixed linear combination of quasifree quantum stochastic integrators, as in [BSW] (the connection between these is elucidated in [LiW]).

Recent developments in the use of quantum probabilistic models (e.g. [AlJ], [Bel]) demonstrate the need for quasifree stochastic analysis. In a sister paper ([LM]) we develop a stochastic calculus for the quasifree integrals defined here.

Noncommutative martingale representation theorems have been established in a variety of other contexts. The original one was for the Clifford filtration, which is the fermionic analogue of the Wiener filtration of canonical Brownian motion.

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Its free analogue was obtained in [BSW]. A representation theorem for martingales with respect to the operator filtration of (minimal variance) quantum Brownian motion as Hudson-Parthasarathy quantum stochastic integrals, was obtained in [HLP] for the classes of essentially Hilbert-Schmidt and unitary martingales, in [PS1] for so-called regular martingales, and in [PS2] for regular martingales with respect to infinite dimensional quantum noise (see also [Mey] and, for a recent coordinate-free treatment not reliant on extra set-theoretic axioms, [L3]). These results lie at a deeper level of noncommutativity than the Clifford and free cases, which make essential use of the finite trace available in those contexts. So far they cover only a class of bounded (as opposed to $L^2$) martingales, however they do extend very satisfactorily to an algebra of semimartingales whose martingales precisely comprise the Parthasarathy-Sinha class ([AII]). White noise extensions of the latter form of martingale representation have been obtained in which explicit expression is found for the ‘stochastic derivatives’ (see [JIO] and references therein).

The plan of the paper is as follows. In Section 1 an extension of the well-known vector-operator correspondence for operators affiliated to a von Neumann algebra with cyclic and separating vector is established. The transpose operation that we need for defining quasifree stochastic integrals is identified, and its properties described, in Section 2. Commutation relations between the abstract Itô integral in Fock space and operators which respect the Fock space filtration are proved in Section 3. In Sections 4 and 5 the general context for our stochastic calculus is set, through a detailed discussion of relevant sufficient conditions for Araki’s Duality Theorem to apply. Natural assumptions for the stochastic setting then emerge and these are shown to imply the sufficient conditions. We also describe classes of examples of quasifree states for stochastic calculus which are covered by our general assumptions. Section 6 establishes the underlying vector process theory by means of a modified Itô integral and its commutation relations with the relevant Tomita-Takesaki $S$ operator, using results of Section 3. In the last section, quasifree stochastic integrals are defined and are shown to yield all the martingales of the theory, moreover adjointability of a martingale is shown to correspond precisely to the adjointability of the quasifree integrand process. Various facts that we need about the behaviour of unbounded operators under composition, orthogonal sum and tensor product are gathered in an appendix.

Notational conventions. For any vector-valued function $f: \mathbb{R}_+ \to V$ and subinterval $I$ of $\mathbb{R}_+$, $f_I$ denotes the function agreeing with $f$ on $I$ and taking the value 0 outside $I$. All Hilbert spaces are complex, with inner products linear in the second argument, in sinc with the following natural and very convenient (Dirac-inspired) notations: for a vector $u$ in the Hilbert space $h$, we write $|u\rangle \in |h\rangle := B(C; h)$ and $\langle u| \in (h) := B(h; \mathbb{C})$ for the respective operators $\lambda \mapsto \lambda u$ and $v \mapsto \langle u, v\rangle$. We abbreviate $h \oplus h$ to $h^{\oplus 2}$. The linear span of a set of vectors $S$ is denoted $\text{Lin} S$. For subspaces $U_1$ and $U_2$ of Hilbert spaces $h_1$ and $h_2$ we write $U_1 \oplus U_2$ for $\text{Lin}\{u_1 \otimes u_2 : u_1 \in U_1, u_2 \in U_2\}$, the linear tensor product of $(U_1, U_2)$ realised in the Hilbert space tensor product $h_1 \otimes h_2$. Blanks replace zero entries in matrices.

The following notation is used for the symmetric Fock space over a Hilbert space $h$: $\Gamma(h) = \bigoplus_{n \geq 0} h^{\otimes n}$, where $h^{\otimes 0} = \mathbb{C}$ and, for $n \geq 1$, $h^{\otimes n}$ denotes the $n$-fold symmetric tensor power of $h$. The (normalised) exponential vectors are given by

$$\varpi(u) := \exp(-\|u\|^2/2) \varepsilon(u) \quad \text{where} \quad \varepsilon(u) := ((n!)^{-1/2}u^{\otimes n})_{n \geq 0} \quad (u \in h),$$

and the Fock vacuum vector $\Omega_h$, by $\varpi(0) = \varepsilon(0) \in \Gamma(h)$. For $S \subset h$, we set $\mathcal{E}(S) := \text{Lin}\{\varepsilon(v) : v \in S\}$. For $u \in h$, the Fock-Weyl operator $W_0(u)$ is the unitary obtained by continuous linear extension of the inner-product preserving prescription

$$\varpi(v) \mapsto e^{-i \text{Im}(u,v)} \varpi(u + v) \quad (v \in h).$$
We also use the gradient operator $\nabla$ on Fock space (which will be freely amplified without change of notation). This is the unique closed operator from $\Gamma(h)$ to $h \otimes \Gamma(h)$ with core $E := E(h)$ satisfying

$$\nabla \varepsilon(v) = v \otimes \varepsilon(v) \quad (v \in h).$$

1. **Affiliated operators and matrix-operator correspondence**

The following notations will be used for classes of unbounded operators. For a subspace $\mathcal{D}_1$ of the Hilbert space $H_1$, write $\mathcal{O}(\mathcal{D}_1; H_2)$ for the linear space of operators from $H_1$ to $H_2$ with domain $\mathcal{D}_1$ and, for dense subspaces $\mathcal{D}_1$ of $H_1$ and $\mathcal{D}_2$ of $H_2$, set

$$\mathcal{O}^\dagger(\mathcal{D}_1, \mathcal{D}_2) := \{ T \in \mathcal{O}(\mathcal{D}_1; H_2) : \text{Dom} T^* \supset \mathcal{D}_2 \} \quad \text{and} \quad T^\dagger := (T^*)_{|\mathcal{D}_2}.$$ 

Clearly the dagger operation is a conjugate-linear isomorphism

$$\dagger : \mathcal{O}^\dagger(\mathcal{D}_1, \mathcal{D}_2) \to \mathcal{O}^\dagger(\mathcal{D}_2, \mathcal{D}_1)$$

(1.1)
satisfying $T^{\dagger\dagger} = T$. In case the Hilbert spaces are the same, we abbreviate $\mathcal{O}^\dagger(\mathcal{D}, \mathcal{D})$ to $\mathcal{O}^\dagger(\mathcal{D})$.

**Remark.** By the Closed Graph Theorem, $\mathcal{O}^\dagger(H_1, \mathcal{D}_2) = B(H_1; H_2)$, for any dense subspace $\mathcal{D}_2$ of $H_2$.

For this section we fix a von Neumann algebra $(M, H)$. There will be supplementary Hilbert spaces $h$, $h_1$ and $h_2$ appearing. The following definition extends standard terminology (for the case where $h_1 = h_2 = C$).

**Definition.** A possibly unbounded operator $T$, from $h_1 \otimes H$ to $h_2 \otimes H$, is **affiliated to $M$**, written $T \eta B(h_1; h_2) \otimes M$, if for all unitaries $u$ in $M'$, $(I_2 \otimes u^*) T(I_1 \otimes u) = T$; in particular $(I_1 \otimes u) \text{Dom} T = \text{Dom} T$.

**Remark.** If $T$ is closed and densely defined then $T \eta B(h_1; h_2) \otimes M$ if and only if $P_G \in B(h_1 \otimes h_2) \otimes M$, where $G = \text{Graph} T$.

For a subspace $\mathcal{D}$ of $H$, set

$$\mathcal{O}_M(h_1 \otimes \mathcal{D}; h_2 \otimes H) := \{ T \in \mathcal{O}(h_1 \otimes \mathcal{D}; h_2 \otimes H) : T \eta B(h_1; h_2) \otimes M \},$$

and if $\mathcal{D}$ is dense, also set

$$\mathcal{O}^\dagger_M(h_1 \otimes \mathcal{D}, h_2 \otimes \mathcal{D}) := \{ T \in \mathcal{O}^\dagger(h_1 \otimes \mathcal{D}, h_2 \otimes \mathcal{D}) : T \eta B(h_1; h_2) \otimes M \},$$

and abbreviate $\mathcal{O}^\dagger_M(h \otimes \mathcal{D}, h \otimes \mathcal{D})$ to $\mathcal{O}^\dagger_M(h \otimes \mathcal{D})$. It is easily seen that, when $\mathcal{D}_1$ is $h_1 \otimes \mathcal{D}$ and $\mathcal{D}_2$ is $h_2 \otimes \mathcal{D}$, the conjugate-linear isomorphism \(1.1\) restricts to an isomorphism

$$\mathcal{O}^\dagger_M(h_1 \otimes \mathcal{D}, h_2 \otimes \mathcal{D}) \to \mathcal{O}^\dagger_M(h_2 \otimes \mathcal{D}, h_1 \otimes \mathcal{D}).$$

For the rest of the section suppose that $M$ has a cyclic and separating vector $\xi$, set $\Xi = M'\xi$, and let $S_T$ be the associated Tomita-Takesaki operator (\cite[Chapter VI; \cite[Chapter 10]{STZ}]). Define operators $E_\xi := I_\mathcal{H} \otimes \langle \xi \rangle$ and $E^\xi := (E_\xi)^* = I_\mathcal{H} \otimes \langle \xi \rangle$ where the Hilbert space $H$ is determined by context. Note that

$$\mathcal{O}^\dagger_M(h_1 \otimes \Xi, h_2 \otimes \Xi) = \{ T \in \mathcal{O}_M(h_1 \otimes \Xi, h_2 \otimes H) : h_2 \subset \text{Dom} T^* E_\xi \}.$$ 

The following class of operators helps us manage adjoints of affiliated operators through bounded operators:

$$B^\dagger_M(h_1; h_2 \otimes H) := \{ B \in B(h_1; h_2 \otimes H) :$$

$$\exists B_{\alpha} \in B(h_2; h \otimes H) \forall x' \in M' \quad B^*(I_2 \otimes x') E_\xi = E^\xi(I_1 \otimes x') B_{\alpha} \}.$$ 

When such an operator exists it is unique. The map

$$B^\dagger_M(h_1; h_2 \otimes H) \to B^\dagger_M(h_2; h_1 \otimes H), \quad B \mapsto B_{\alpha}$$
is manifestly a conjugate-linear isomorphism satisfying $B_{\dagger\dagger} = B$. Clearly, for $B \in B(h_1; h_2 \otimes H)$, to be in $B_{M,\xi}^1(h_1; h_2 \otimes H)$ is for there to be a $B_1 \in B(h_2; h_1 \otimes H)$ satisfying
\[
(c_1 \otimes x, B_1(c_2)) = \langle Bc_1, c_2 \otimes x' \xi \rangle \quad (c_1 \in h_1, c_2 \in h_2, x' \in M').
\]
(1.2)
Moreover, for $A \in B(h_1; h_2)$ and $\eta \in \text{Dom} S_{\xi}$,
\[
A \otimes |\eta\rangle \in B_{M,\xi}^1(h_1; h_2 \otimes H) \quad \text{and} \quad (A \otimes |\eta\rangle)_\dagger = A^* \otimes |S_{\xi}\eta\rangle.
\]
Note also that, when $T \in \mathcal{O}_M^2(h_1 \boxtimes \Xi, h_2 \boxtimes \Xi)$, the operator $TE_\xi$ is everywhere defined and closed, and thus bounded. The ‘matrix-operator’ correspondences contained in the straightforward proposition below play a significant role in the sequel.

**Proposition 1.1.** The map
\[
\mathcal{O}_M(h_1 \boxtimes \Xi, h_2 \otimes H) \to \mathcal{O}(h_1; h_2 \otimes H), \quad T \mapsto TE_\xi := TE_\xi
\]
is a linear isomorphism with inverse given by $B \mapsto B^\xi$, where $B^\xi$ is the linearisation of the bilinear map
\[
(c_1, x' \xi) \mapsto (I_2 \otimes x')Bc_1,
\]
which restricts to an isomorphism
\[
\mathcal{O}_M^1(h_1 \boxtimes \Xi, h_2 \boxtimes \Xi) \to B_{M,\xi}^1(h_1; h_2 \otimes H),
\]
intertwining the operations $\dagger$ and $\check{}$:
\[
(TE_\xi)_\dagger = T^\dagger E_\xi \quad \text{and} \quad (B^\xi)^\dagger = (B_1)^\xi.
\]
**Remarks.** To illustrate on simple tensors, let
\[
A \in \mathcal{O}(h_1; h_2), \quad B \in B(h_1; h_2), \quad R \in \mathcal{O}_M(\Xi; H), \quad X \in \mathcal{O}_M(\Xi) \quad \text{and} \quad Z \in \mathcal{O}_M^1(\Xi).
\]
Then, setting $\xi = Z\xi$,
\[
(A \otimes R)E_\xi = A \otimes |R\xi\rangle \quad \text{and} \quad (A \otimes |X\xi\rangle)^\xi = A \otimes X, \quad \text{so}
\]
\[
((B \otimes R)E_\xi)_\dagger = (B \otimes |R\xi\rangle)_\dagger = B^* \otimes |S_{\xi}R\xi\rangle = B^* \otimes |R^\dagger \xi\rangle = (B \otimes R)^\dagger E_\xi, \quad \text{and}
\]
\[
((B \otimes |\zeta\rangle)^\xi)_\dagger = (B^* \otimes |S_{\xi}\zeta\rangle)^\xi = B^* \otimes |Z^\dagger \subset (B \otimes Z)^* = ((B \otimes |\zeta\rangle)^\xi)^\dagger,
\]
thus $((B \otimes |\zeta\rangle)^\xi)_{\dagger\dagger} = ((B \otimes |\zeta\rangle)^\xi)^{\dagger\dagger}$.

When $h_1 = h_2 = \mathbb{C}$ the above correspondences reduce to the well-known linear isomorphism
\[
\mathcal{O}_M(\Xi; H) \to H, \quad X \mapsto X\xi,
\]
and its restriction, the isomorphism
\[
\mathcal{O}_M^1(\Xi) \to \text{Dom} S_{\xi},
\]
under which $S_{\xi}(X\xi) = X^\dagger\xi$ (see, for example, [BrR], Proposition 2.5.9). Specifically, $\mathcal{O}_M(\mathbb{C}; \mathbb{C} \otimes H) = \{H\}$ and
\[
B_{M,\xi}^1(\mathbb{C}; \mathbb{C} \boxtimes H) = \{|\zeta\rangle : \zeta \in \text{Dom} S_{\xi}\} \quad \text{with} \quad |\zeta\rangle_{\dagger} = |S_{\xi}\zeta\rangle.
\]
In the next section we shall see how this connection can be raised to the matrix level.

We end this section with another very useful elementary fact.

**Lemma 1.2.** Let $\mathcal{V}$ be a subspace of $M'$.

(a) If $\mathcal{V}$ is dense in $M'$ in the strong operator topology then $\mathcal{V}\xi$ is a common core for all operators in $\mathcal{O}_M^1(\Xi)$.

(b) If $\mathcal{V}$ is dense in $M'$ in the ultraweak topology then $h_1 \boxtimes \mathcal{V}\xi$ is a common core for all operators in $\mathcal{O}_M^1(h_1 \boxtimes \Xi, h_2 \boxtimes \Xi)$. 
2. Transpose and conjugate for matrices of unbounded operators

For this section we fix a von Neumann algebra $(M, H)$ with cyclic and separating vector $\xi$, let $S_t$ and $F_t$ denote the corresponding Tomita-Takesaki operators, and set $\Xi = M'\xi$. Also Hilbert spaces $k$, and $k_i$ ($i = 0, 1, \ldots$), will appear which are complexifications of real Hilbert spaces; we denote the action of their associated conjugations $k_i$ respectively $k_i'$ by $c \mapsto \tau_i$. We consider a transpose operation on a class of abstract matrix spaces over a space of unbounded operators affiliated to $M$. We then detail its relation to the dagger (adjoint) operation and to $S_t$. This is needed to handle quasifree stochastic integrals for infinite dimensional noise; it also enables multiple quasifree integrals to be defined in [LM], where they are used for solving quasifree stochastic differential equations.

For $B \in B(k_1; k_2)$, its conjugate operator is defined by

$$\mathcal{B} := k_2Bk_1 \in B(k_1; k_2), \ c \mapsto \mathcal{B}^c,$$

and its transpose by $B^t := \mathcal{B}^c = \mathcal{B}^\tau$. The transpose maps $B(k_1; k_2)$ linearly and isometrically onto $B(k_2; k_1)$. Due to the lack of complete boundedness of the transpose, the map

$$B(k_1; k_2) \otimes B(H_1; H_2) \rightarrow B(k_2; k_1) \otimes B(H_1; H_2),$$

given by linearisation of the bilinear map $(B, X) \mapsto B^t \otimes X$, is unbounded unless $B(k_1; k_2)$ or $B(h_1; h_2)$ is finite-dimensional (see, for example [EFR]). We need to overcome this obstruction whilst transposing a class of unbounded operators.

We exploit the fact that the transpose restricts to a unitary operator between the Hilbert-Schmidt classes, say $U : HS(k_1; k_2) \rightarrow HS(k_1; k_2)$ and so, for any Hilbert spaces $h_1$ and $h_2$, there is a partial transpose

$$U \otimes I : HS(k_1; k_2) \otimes HS(h_1; h_2) = HS(k_1 \otimes h_1; k_2 \otimes h_2) \rightarrow HS(k_2 \otimes h_1; k_1 \otimes h_2)$$

which we denote by $H \mapsto H_T$. This is characterised by

$$\langle c_1 \otimes v_2, H_T(c_2 \otimes v_1) \rangle = \langle c_1 \otimes v_2, H(c_2 \otimes v_1) \rangle \quad (c_i \in k_i, v_i \in h_i, i = 1, 2). \ (2.1)$$

The class of unbounded operators that we need to transpose is defined next. Recall the linear isomorphisms described in Proposition [LiJ].

**Definition.** The $(k_1, k_2)$-matrix space associated to $(M, \xi)$ is the following class of operators:

$$\mathcal{M}_{k_1, k_2}(M, \xi) := \{ T \in \mathcal{O}_M(k_1 \otimes \Xi; k_2 \otimes H) : TE_\xi \in HS(k_1; k_2 \otimes H) \},$$

and for $T \in \mathcal{M}_{k_1, k_2}(M, \xi)$, its (matrix) transpose is given by

$$T^T := ((TE_\xi)\tau)^\xi,$$

thus, for $B \in HS(k_1; k_2 \otimes H)$, $(B^c)^T = (B_T)^c$. The corresponding column and row spaces are given by

$$C_k(M, \xi) := \mathcal{M}_{k, k}(M, \xi) \text{ and } R_k(M, \xi) := \mathcal{M}_{k, c}(M, \xi).$$

**Remarks.** This construction evidently enjoys the following properties:

- $\mathcal{M}_{k_1, k_2}(M, \xi) = \{ B^c : B \in HS(k_1; k_2 \otimes H) \};$
- $HS(k_1; k_2) \otimes \mathcal{O}_M(\Xi) \subset \mathcal{M}_{k_1, k_2}(M, \xi)$, with equality if $M = \mathbb{C}$, or if $k_1 = k_2 = \mathbb{C}$;
- $(H \otimes X)^T = H^* \otimes X \quad (H \in HS(k_1; k_2), X \in \mathcal{O}_M(\Xi));$
- $C_k(M, \xi) = \mathcal{O}_M(\Xi; k \otimes H)$, whereas
- $R_k(M, \xi) = \{ R \in \mathcal{O}_M(k_1 \otimes \Xi; H) : RE_\xi \in HS(k_1; H) \};$
- $(B(k_2; k_1) \otimes M)\mathcal{M}_{k_1, k_2}(M, \xi)(B(k_0; k_1) \otimes I_H) \subset \mathcal{M}_{k_0, k_2}(M, \xi).$
Moreover, $\mathcal{M}_{k_1,k_2}(M,\xi)$ is a left $B(k_2)\boxtimes M$-module and a right $B(k_1)$-module, and the matrix transpose is characterised by

$$\langle c_1 \otimes x', T^T(c_2 \otimes \xi) \rangle = \langle \tau T(c_2 \otimes \xi), c_1 \otimes x' \rangle \quad (c_1 \in k_1, c_2 \in k_2, x' \in M').$$

We now need to relate the transpose operation

$$\mathcal{M}_{k_1,k_2}(M,\xi) \to \mathcal{M}_{k_2,k_1}(M,\xi), \quad T \mapsto T^T$$

with the adjoint operation

$$\mathcal{O}_M^+(k_1 \otimes \Xi, k_2 \otimes \Xi) \to \mathcal{O}_M^+(k_2 \otimes \Xi, k_1 \otimes \Xi) \quad T \mapsto T^\dagger.$$

Specifically, we seek the appropriate space of operators/matrices compatible with both operations. To this end we define

$$HS_{M,\xi}^I(k_1; k_2 \otimes H) := \{ B \in HS(k_1; k_2 \otimes H) \cap B_{k_1,\xi}^I(k_1; k_2 \otimes H) : B_1 \in HS(k_2; k_1 \otimes H) \}.$$

The proposition below justifies our choice. Its corollary, Theorem 2.2 below, is key for the construction of quasifree stochastic integrals in Section 7. For $i = 1, 2$, let $k_i$ denote the conjugations on $k_i$.

**Proposition 2.1.** Let $B \in HS(k_1; k_2 \otimes H)$. Then the following are equivalent:

(i) $B \in HS_{M,\xi}^I(k_1; k_2 \otimes H)$.

(ii) $B_T \in HS_{M,\xi}^I(k_2; k_1 \otimes H)$.

(iii) $\text{Ran } B \subset \text{Dom } k_2 \otimes S_\xi$ and $\mathcal{B} := (k_2 \otimes S_\xi)Bk_1 \in HS(k_1; k_2 \otimes H)$.

In this case,

$$B_{1T} = B_{T1} = \mathcal{B}.$$

**Proof.** For all $c_1 \in k_1$, $c_2 \in k_2$ and $x' \in M'$, we have

$$(B_{T}c_2, c_1 \otimes x'^* \xi) = (Bk_1c_1, (k_2 \otimes F_\xi)(c_2 \otimes x' \xi)).$$

Since $k_2 \otimes \Xi$ is a core for $k_2 \otimes F_\xi$ and $(k_2 \otimes F_\xi)^* = k_2 \otimes S_\xi$, it follows (using the characterisation (1.2)) that (ii) and (iii) are equivalent, and also that when they hold, $B_{1T} = B_{T1}$.

If (i) holds then, for all $c_1 \in k_1$, $c_2 \in k_2$ and $x \in M'$,

$$(B_{T}c_2, c_1 \otimes x'^* \xi) = (B_{T}c_2, c_1 \otimes x'^* \xi) = (\tau c_1 \otimes x', B_{T}c_2) = (c_2 \otimes x', B_{T1}c_1)$$

so, by the characterisation (1.2),

$$B_T \in B_{T1}^I(k_2; k_1 \otimes H) \quad \text{and} \quad B_{T1} = B_{T1} \in HS(k_2; k_1 \otimes H).$$

Thus (i) implies (ii), and since $B_{1T} = B$, also (ii) implies (i). This completes the proof. □

**Definition.** The $(k_1,k_2)$-**adjointable matrix space associated to** $(M,\xi)$ is the class of operators defined by

$$\mathcal{M}_{k_1,k_2}^I(M,\xi) := \{ T \in \mathcal{M}_{k_1,k_2}(M,\xi) \cap \mathcal{O}_M^+(k_1 \otimes \Xi, k_2 \otimes \Xi) : T^\dagger \in \mathcal{M}_{k_2,k_1}(M,\xi) \}.$$

The corresponding column and row spaces are given by

$$\mathcal{C}_k^I(M,\xi) := \mathcal{M}_{K,k}^I(M,\xi) \quad \text{and} \quad \mathcal{R}_k^I(M,\xi) := \mathcal{M}_{k,K}^I(M,\xi).$$

We now have a matrix space of affiliated operators having adjoints and transposes; the key properties are summarised next.
Theorem 2.2. The following hold
\[ \mathcal{M}_{k_1,k_2}(M, \xi) = \begin{cases} \mathcal{B}^\xi : B \in HS_{M,\xi}(k_1; k_2 \otimes H) \\ \mathcal{C}^\xi(\Xi; k_2 \otimes \Xi) \end{cases} \]
where \( \mathcal{B}^\xi = \{ T \in O_M^\dagger(k_1 \otimes \Xi, k_2 \otimes \Xi) : TE_\xi \in HS(k_1; k_2 \otimes H), T^\dagger E_\xi \in HS(k_2; k_1 \otimes H) \} \); \( HS(k_1; k_2 \otimes O_M^\dagger(\Xi)) \subset \mathcal{M}_{k_1,k_2}(M, \xi) \), with equality if \( M = C, \) or if \( k_1 = k_2 = C \); \( H \otimes X)^\dagger = H^\dagger \otimes X^\dagger = \overline{H} \otimes X^\dagger \) (H \in HS(k_1; k_2), X \in O_M^\dagger(\Xi)); 
\( \mathcal{C}^\xi(\mathcal{M}, \xi) = \{ C \in O_M^\dagger(\Xi, k \otimes \Xi) : C^T E_\xi \in HS(k; H) \} \) (restoring symmetry with) \( \mathcal{R}^\xi(k, \xi) = \{ R \in O_M^\dagger(k \otimes \Xi, \Xi) : RE_\xi \in HS(k; H) \} \) (but also) \( \mathcal{C}^\xi(\mathcal{M}, \xi) = \{ |\xi\rangle : \xi \in \text{Dom } k \otimes S_\xi \} \).
Moreover, for all \( T \in \mathcal{M}_{k_1,k_2}(M, \xi) \) and \( C \in \mathcal{C}^\xi(\mathcal{M}, \xi) \),
\[ T^\dagger, T^\dagger \in \mathcal{M}_{k_2,k_1}(M, \xi), \quad T^T \in \mathcal{M}_{k_2,k_1}(M, \xi), \quad T^T = T^\dagger = T, \quad T^\dagger = T^T, \]
\[ T^T E_\xi = (k_2 \otimes S_\xi)TE_\xi k_1, \quad \text{and} \quad C^T \xi = (k \otimes S_\xi)C_\xi. \] (2.2)
Remarks. Note further that \( (B(k_2; k_3) \otimes I_H)\mathcal{M}_{k_1,k_2}(M, \xi)(B(k_0; k_1) \otimes I_H) \subset \mathcal{M}_{k_0,k_3}(M, \xi) \), \( \mathcal{M}_{k_1,k_2}(M, \xi) \) is a left \( B(k_2) \)-module and a right \( B(k_1) \)-module.

The relationship between the various spaces is seen in the following commutative diagram, in which the horizontal arrows represent linear isomorphisms and all other arrows represent inclusions.

![Diagram](image)

We end this section by introducing a transform between matrices and columns which is one of the ingredients of the construction of quasi-free integrals in Section 7.

Denote by \( \pi \) the sum-flips on both \( k^{\otimes 2} \) and \( k^{\otimes 2} \otimes H = (k \otimes H)^{\otimes 2} \), set \( k := C \oplus k \),
\[ \mathcal{M}_k(M, \xi)_0 := \{ \begin{bmatrix} C \end{bmatrix} : C \in \mathcal{C}_k(M, \xi) \} \] and \( \mathcal{M}_k(M, \xi)_0 := \mathcal{M}_k(M, \xi)_0 \cap \mathcal{M}_k^\dagger(M, \xi) \),
and set \( k^{\pi} := (k \oplus k) \circ \pi \).

Corollary 2.3. The map
\[ \mathcal{M}_k(M, \xi)_0 \to \mathcal{C}_k^{\otimes 2}(M, \xi) = \mathcal{O}_M(\Xi; k^{\otimes 2} \otimes H), \quad T = \begin{bmatrix} C & R \end{bmatrix} \mapsto T^{\dagger} := \begin{bmatrix} C^T \\ R \end{bmatrix} \]
is a linear isomorphism which restricts to an isomorphism \( \mathcal{M}_k^\dagger(M, \xi)_0 \to \mathcal{C}_k^{\otimes 2}(M, \xi) \) satisfying \( T^{\dagger} = \pi \circ T^{\dagger} \circ T \), and thus
\[ T^{\dagger} \xi = (k^{\pi} \oplus S_\xi)T^{\dagger} \xi. \]
3. Itô Integral and Commutation Relations

In this section we prove a commutation relation between second quantisation and the abstract Itô integral. First we set up notation for stochastic analysis in Fock space. Fix a Hilbert space $\mathfrak{h}$ and a separable Hilbert space $k$. For a subinterval $I$ of $\mathbb{R}_+$, set

$$K_I = L^2(I; k), \quad \mathcal{F}_{k,I} = \mathfrak{h} \otimes \mathcal{F}_{k,I}, \quad \Omega_{k,I} = (1, 0, 0, \cdots) \in \mathcal{F}_{k,I},$$

dropping the $I$ when it is all of $\mathbb{R}_+$. The tensor decompositions

$$\mathcal{F}_k = \mathcal{F}_{k,[0,s]} \otimes \mathcal{F}_{k,[s,t]} \otimes \mathcal{F}_{k,[t,\infty]} \quad (0 \leq s \leq t \leq \infty)$$

are witnessed by exponential vectors. Write

$$p_t \text{ for } M_{1,[0,t]} \text{ on } \mathcal{K} \text{ and } P_t \text{ for } I_h \otimes \Gamma(p_t) \text{ on } h \otimes \mathcal{F}_k \quad (t \geq 0),$$

where $M$ denotes multiplication operator and $h$ can be $\mathbb{C}$, $\mathfrak{h}$ (or $k \otimes \mathfrak{h}$), depending on context, and let $K_I$, $\mathcal{F}_{k,I}$ and $\Omega_{k,I}$ be the images of the respective orthogonal projections. Then $\mathcal{K} \otimes \mathcal{S}_k = L^2(\mathbb{R}_+; k \otimes \mathcal{S}_k)$ and, by Fubini’s Theorem,

$$\{ y \in \mathcal{K} \otimes \mathcal{S}_k : \text{ for a.a. } t \in \mathbb{R}_+, y_t = y_{(t)} \otimes \Omega_{k,[t,\infty]} \text{ for some } y_{(t)} \in \mathcal{S}_{k,[0,t]} \} \quad \text{and}$$

$$\{ y \in \mathcal{K} \otimes \mathcal{S}_k : \forall t \geq 0 \ (p_t \otimes \mathcal{I}_{k})y \in \mathcal{K} \otimes \mathcal{S}_{k,t} \},$$

the divergence operator $\mathcal{S} := \nabla^*$ being an abstract Hitsuda-Skorohod integral [3]. We further define

$$L^2_{\Omega,\text{loc}}(\mathbb{R}_+; k \otimes \mathcal{S}_k) := \{ y \in L^2_{\text{loc}}(\mathbb{R}_+; k \otimes \mathcal{S}_k) : \forall t \geq 0 \ y_{[0,t]} \in \mathcal{K} \otimes \mathcal{S}_{k,t} \}$$

and for $t \in \mathbb{R}_+$ and $z \in L^2_{\Omega,\text{loc}}(\mathbb{R}_+; k \otimes \mathcal{S}_k)$, $T_t y := \mathcal{T}_y_{[0,t]}$. The following characterisation of operators affiliated to the von Neumann algebra $L^\infty(\mathbb{R}_+) \overline{\otimes} B(k)$ is useful.

**Lemma 3.1.** Let $T$ be a closed and densely defined operator on $\mathcal{K}$. Then the following are equivalent.

(i) $T$ is affiliated to $L^\infty(\mathbb{R}_+) \overline{\otimes} B(k)$.

(ii) $T$ satisfies the invariance condition

$$T p_t \supset p_t T \quad (t \geq 0).$$

(iii) $T$ is ‘pointwise adjointable’, that is for all $f \in \text{Dom} T^*$ and $g \in \text{Dom} T$,

$$(f(t), (T g)(t)) = ((T^* f)(t), g(t)) \quad \text{for a.a. } t \geq 0.$$ 

**Proof.** Since, for all $t \geq 0$, $p_t \in L^\infty(\mathbb{R}_+) \otimes I_k$, the commutant of $L^\infty(\mathbb{R}_+) \overline{\otimes} B(k)$, (i) implies (ii). On the other hand, viewing $L^\infty(\mathbb{R}_+)$ as the dual of $L^1(\mathbb{R}_+)$, for $f \in \text{Dom} T^*$ and $g \in \text{Dom} T$ the set

$$\{ \varphi \in L^\infty(\mathbb{R}_+) : ||\varphi||_\infty \leq 1 \text{ and } (T^* f, \varphi \cdot g) = (f, \varphi \cdot T g) \}$$

is compact and metrizable in the relative weak topology, and step functions with $L^\infty$-bound at most one are dense in the unit ball of $L^\infty(\mathbb{R}_+)$. It follows that (ii) implies (i).
The equivalence of (i) and (iii) is evident from the identities

$$
\int dt \varphi(t) \langle f(t), (Tg)(t) \rangle = \langle f, \varphi \cdot Tg \rangle, \quad \text{and}
$$

$$
(T^* f, \varphi \cdot g) = \int dt \varphi(t) \langle (T^* f)(t), (g(t)) \rangle,
$$

for \( f \in \text{Dom } T^* \) and \( g \in \text{Dom } T \) and \( \varphi \in L^\infty(\mathbb{R}_+) \).

**Remark.** A good reference for the identification of \( L^\infty(\mathbb{R}_+) \otimes \mathcal{M} \) and \( L^\infty(\mathbb{R}_+; \mathcal{M}) \), for a von Neumann algebra \( \mathcal{M} \) with separable predual, is Theorem 1.22.13 of [Sak].

**Lemma 3.2.** Let \( R = T \otimes X \), where \( T \) and \( X \) are closed densely defined operators on \( \mathcal{K} \) and \( \mathfrak{H}_k \) respectively, satisfying

\[
T \eta L^\infty(\mathbb{R}_+) \otimes B(k), \quad \text{equivalently } T p_t \supset p_t T, \quad \text{and}
\]

\[
X (\mathfrak{H}_{k,t} \cap \text{Dom } X) \subset \mathfrak{H}_{k,t}, \quad \text{equivalently } X P = P X P \quad (t \geq 0).
\]

Then

\[
(T \otimes X) \left( L^2(\mathbb{R}_+; k \otimes \mathfrak{H}_k) \cap \text{Dom } T \otimes X \right) \subset L^2(\mathbb{R}_+; k \otimes \mathfrak{H}_k), \quad \text{equivalently}
\]

\[
(T \otimes X) P^\Omega = P^\Omega (T \otimes X) P^\Omega.
\]

**Proof.** Set \( I := I_{\mathfrak{H}_k} \). By Part (f) of Proposition [A.3] and Corollary [A.4] we have

(a) \( R(p_t \otimes I) \supset (p_t \otimes I) R \), and

(b) \( R(K \otimes \mathfrak{H}_{k,t} \cap \text{Dom } R) \subset K \otimes \mathfrak{H}_{k,t} \), for all \( t \geq 0 \).

Let \( z \in L^2(\mathbb{R}_+; k \otimes \mathfrak{H}_k) \cap \text{Dom } R \) and \( t \geq 0 \). By adaptedness and (a), \( (p_t \otimes I) z \in (K \otimes \mathfrak{H}_{k,t}) \cap \text{Dom } R \) and

\[
R(p_t \otimes I) z = (p_t \otimes I) R z
\]

so, by (b), \( R(p_t \otimes I) z \in K \otimes \mathfrak{H}_{k,t} \) and thus \( (p_t \otimes I) R z \in K \otimes \mathfrak{H}_{k,t} \). Therefore, by \( \mathcal{F}_2 \), \( R z \in L^2(\mathbb{R}_+; k \otimes \mathfrak{H}_k) \), as required.

**Notation.** For operators \( T \) and \( X \) of the above form we set

\[
T \otimes X := V^*_\Omega (T \otimes X) V_\Omega \tag{3.4}
\]

where \( V_\Omega \) is the inclusion map \( L^2(\mathbb{R}_+; k \otimes \mathfrak{H}_k) \rightarrow K \otimes \mathfrak{H}_k \).

**Remark.** Operators of the form \( T \otimes X \) are closed, as is easily verified.

The next two results involve the (ampliated) gradient operator on Fock space (which is defined in the introduction), and the second quantised operators of Proposition [A.5]

**Lemma 3.3.** Let \( A \) and \( T \) be closed densely defined operators on \( \mathfrak{h} \) and \( \mathcal{K} \) respectively. Then

\[
\nabla (A \otimes \Gamma(T)) \subset (T \otimes A \otimes \Gamma(T)) \nabla.
\]

**Proof.** For \( v \in \text{Dom } A \) and \( g \in \text{Dom } T \),

\[
\nabla \epsilon(g) \in \text{Dom } \nabla, \quad Av \otimes \epsilon(Tg) \in \text{Dom } \nabla,
\]

\[
\nabla v \epsilon(g) = g \otimes v \otimes \epsilon(g) \in \text{Dom } (T \otimes A \otimes \Gamma(T)), \quad \text{and}
\]

\[
(T \otimes A \otimes \Gamma(T)) \nabla v \epsilon(g) = Tg \otimes Av \otimes \epsilon(Tg) = \nabla (Av \otimes \epsilon(Tg)).
\]

The result follows.

With these we are able to establish a key commutation relation between the operations of second quantisation and Itô integration.
**Theorem 3.4.** Let \( X = A \otimes \Gamma(T) \) where \( A \) and \( T \) are closed densely defined operators on \( \mathcal{H} \) and \( K \) respectively, with \( T \) affiliated to \( L^\infty(\mathbb{R}_+) \otimes B(k) \). Then
\[
X \mathcal{I} = \mathcal{I} \circ (T \otimes \mathcal{O} X)
\]
and, for any core \( \mathcal{C} \) for \( X \), \( D(\mathcal{C}) \) is a core for \( X \mathcal{I} \).

**Proof.** The strategy of proof is as follows. We prove successively:

(a) For all \( t \geq 0 \), \( X(\mathcal{O}_{k,t} \cap \text{Dom } X) \subset \mathcal{O}_{k,t} \).

(b) \( X \mathcal{I} \supset \mathcal{I} \circ (T \otimes \mathcal{O} X) \).

(c) The operators \( X \mathcal{I} \) and \( \mathcal{I} \circ (T \otimes \mathcal{O} X) \) are both closed.

(d) If \( \mathcal{C} \) is a core for \( X \) then \( D(\mathcal{C}) \) is a core for \( X \mathcal{I} \).

(e) Setting \( D := \text{Dom } T \otimes \text{Dom } A \otimes \mathcal{E}(\text{Dom } T) \), we have
\[
P^\Omega(D) \subset \text{Dom } T \otimes X.
\]

Then, setting \( \mathcal{C} = \text{Dom } A \otimes \mathcal{E}(\text{Dom } T) \), we have
\[
V_\Omega D(\mathcal{C}) \subset P^\Omega D \subset \text{Dom } T \otimes X.
\]

Thus, by (d), \( D(\mathcal{C}) \) is a core for \( X \mathcal{I} \) contained in \( \text{Dom } (T \otimes \mathcal{O} X) \), which equals \( \text{Dom } (\mathcal{I} \circ (T \otimes \mathcal{O} X)) \). Since \( \mathcal{I} \circ (T \otimes \mathcal{O} X) \) is closed, it follows that the inclusion in (b) is an equality and the proof will then be complete.

(a) Let \( t \geq 0 \). First note that \( T_{p_1} p_1 T_{p_1} \in \text{Dom } T_{p_1} \) and \( T_{p_1} p_1 f = T_{p_1} p_1 f = p_1 T_{p_1} f \). Now let \( \zeta \in \mathcal{O}_{k,t} \cap \text{Dom } \Gamma(T) \). By Proposition A.3, we have
\[
\Gamma(T) \zeta = \Gamma(T) \Gamma(p_1) \zeta = \Gamma(T_{p_1}) \zeta \in \text{Ran } \Gamma(T_{p_1}) \subset \text{Ran } \Gamma(p_1) = \mathcal{F}_t.
\]

Thus \( \Gamma(T) (\mathcal{O}_{k,t} \cap \text{Dom } \Gamma(T)) \) \( \subset \mathcal{O}_{k,t} \), and Corollary A.4 implies that \( \text{Dom } (\mathcal{O}_{k,t} \cap \text{Dom } \Gamma(T)) \) \( \subset \mathcal{O}_{k,t} \), as required.

(b) By (a), Lemma 3.3 applies, thus
\[
(T \otimes X)V_\Omega = P^\Omega (T \otimes X)V_\Omega \tag{3.5}
\]
and we may form the operator \( T \otimes \mathcal{O} X \). Let \( z \in \text{Dom } T \otimes \mathcal{O} X \) and \( \zeta \in \text{Dom } A^* \otimes \mathcal{E}(\text{Dom } T^*) \).

Then, by Lemma 3.3 and Proposition A.3
\[
(\zeta, \mathcal{I} (T \otimes \mathcal{O} X)z) = (\nabla \zeta, V_\Omega (T \otimes X)V_\Omega z)
= (\nabla \zeta, (T \otimes A \otimes \Gamma(T))V_\Omega z)
= (\nabla (A^* \otimes \Gamma(T)^*) \zeta, V_\Omega z) = (A^* \otimes \Gamma(T)^*) \zeta, \mathcal{I} z).
\]

Since \( \text{Dom } A^* \otimes \mathcal{E}(\text{Dom } T^*) \) is a core for \( A^* \otimes \Gamma(T)^* = X^* \), this implies that \( \mathcal{I} z \in \text{Dom } X \) and \( X \mathcal{I} z = \mathcal{I} (T \otimes \mathcal{O} X)z \). This proves (b).

(c) Being a closed operator composed with a bounded operator, \( X \mathcal{I} \) is closed (Lemma A.4). To see that \( R := \mathcal{I} \circ (T \otimes \mathcal{O} X) \) is closed too, let \( (z_n) \) be a sequence in \( \text{Dom } R = \text{Dom } (T \otimes X)V_\Omega \) such that \( z_n \rightarrow z \) and \( Rz_n \rightarrow w \). Then \( V_\Omega z_n \rightarrow V_\Omega z \) and, by (3.5),
\[
(T \otimes X)V_\Omega z_n = P^\Omega (T \otimes X)V_\Omega z_n = V_\Omega D \mathcal{I} (T \otimes \mathcal{O} X)z_n = V_\Omega DRz_n \rightarrow V_\Omega Dw.
\]

Therefore, since \( T \otimes X \) is closed, \( V_\Omega z \in \text{Dom } T \otimes X \) and \( (T \otimes X)V_\Omega z = V_\Omega Dw \).

Thus, since \( w \in \text{Ran } \mathcal{I} \), \( z \in \text{Dom } (T \otimes X)V_\Omega = \text{Dom } R \) and
\[
Rz = \mathcal{I} V_\Omega Dw = \mathcal{I} Dw = w.
\]

Thus \( R \) is closed too.
(d) This follows from Part (c) of Lemma 4.1 since X is closed, \( \mathcal{I} \) is isometric \(\mathcal{I}D = \mathcal{I}T^* = I_0 \otimes \Gamma(0)\), and the evident inclusion

\[
(I_0 \otimes \Gamma(0)) A^* \otimes \Gamma(T^*) \subset A^* \otimes \Gamma(T^*) \{I_0 \otimes \Gamma(0)\}
\]

implies that \( \mathcal{X}T^* \supset \mathcal{I}T^* X \), by the adjoint-product-inclusion relation and Proposition 4.5.

(e) Let \( \zeta = f^1 \otimes u \otimes \varepsilon(f^2) \) and \( \eta = g^1 \otimes v \otimes \varepsilon(g^2) \), where \( f^1, f^2 \in \text{Dom} T^* \), \( u \in \text{Dom} A^* \), \( g^1, g^2 \in \text{Dom} T \) and \( v \in \text{Dom} A \). Then, by Lemma 4.1,

\[
\langle (T \otimes X)^* \zeta, P^\Omega \eta \rangle
\]

\[
= \int dt ((T^* f^1)(t) \otimes A^* u \otimes \varepsilon(T^* f^2), g^1(t) \otimes v \otimes \varepsilon(p_t g^2))
\]

\[
= \int dt (f^1(t) \otimes u \otimes \varepsilon(f^2), (T g^1)(t) \otimes A v \otimes \varepsilon(p_t g^2)) = \langle \zeta, P^\Omega (T \otimes X) \eta \rangle.
\]

Thus \((T \otimes X)P^\Omega \supset P^\Omega (T \otimes X)\), in particular \(P^\Omega(D) \subset \text{Dom} T \otimes X\). \(\square\)

Remarks. For comparison, note that if \( X \) is bounded (equivalently, if \( A \) is bounded and \( T \) is a contraction) then

\[
\overline{X \mathcal{S}} = \mathcal{S} \circ (T \otimes X),
\]

but \( X \mathcal{S} \) is typically not closed (e.g. \( T = 0 \)).

We shall use this result with \( A \) and \( T \) being conjugate-linear operators.

**Corollary 3.5.** For all \( t \geq 0 \),

\[
DP_t = M^\Omega_t D \text{ where } M^\Omega_t := p_t \otimes \Omega I \text{ and } I = I_{k \otimes \mathcal{H}_1}.
\]

**Proof.** Let \( t \geq 0 \). In view of the identity \((p_t \otimes P_t)\mathcal{V}_0 = (p_t \otimes I)\mathcal{V}_0\), the theorem implies that \( P_t \mathcal{I} = \mathcal{I}(p_t \otimes P_t) = \mathcal{I}(p_t \otimes \Omega I) \), and (3.6) follows on taking adjoints. \(\square\)

4. CCR algebras and quasifree states

For any nondegenerate symplectic space \( (V, \sigma) \) there is an associated simple \( C^* \)-algebra, denoted \( CCR(V, \sigma) \); it is generated by elements \( \{w_v : v \in V\} \) satisfying the canonical commutation relations in Weyl form:

\[
w_u w_v = e^{-i\sigma(u,v)} w_{u+v} \text{ and } w_u^* = w_{-u} \quad (u, v \in V).
\]

Every *-algebra morphism from \( CCR_0(V, \sigma) := \text{Lin}\{w_v : v \in V\} \) to a \( C^* \)-algebra \( A \), extends uniquely to a \( C^* \)-morphism from \( CCR(V, \sigma) \) to \( A \), and every symplectic map \( R \) from \( V \) into another nondegenerate symplectic space \( V' \) induces a \( C^* \)-monomorphism \( \phi_R : CCR(V, \sigma) \to CCR(V', \sigma') \) satisfying \( \phi_R(w_v) = w_{Rv} \) \( (v \in V) \) ([Siu], [Man]; see Theorem 5.2.8 of [BrR], and Chapter 2 of [Ped]). When \( (V', \sigma') = (V, \sigma) \) and \( R \) is a symplectic automorphism, \( \phi_R \) is known as a Bogoliubov transformation. Typically \( V \) is a real subspace of a complex Hilbert space and \( \sigma = \text{Im}(\cdot, \cdot) \) (in this case we write \( CCR(V) \)); when \( V \) is a complex subspace, the gauge transformations of \( CCR(V) \) are the Bogoliubov transformations \( \phi_z \) induced by the symplectic automorphisms \( v \mapsto zv \) \( (z \in \mathbb{T}) \). The characteristic function of a state \( \varphi \) on \( CCR(V, \sigma) \) is the complex-valued function \( \hat{\varphi} := \varphi \circ w \) on \( V \). Given any nonnegative quadratic form \( a \) on \( V \) satisfying

\[
\sigma(u, v)^2 \leq a[u]a[v] \quad (u, v \in V),
\]

there is a unique state \( \varphi \) on \( CCR(V, \sigma) \) whose characteristic function is given by

\[
\hat{\varphi} : v \mapsto e^{-\frac{1}{2}a[v]} \quad (4.1)
\]
Such states are called (mean zero) quasi-free states. When $V$ is a complex subspace of a Hilbert space, a state $\varphi$ on $CCR(V)$ is called gauge-invariant if it is invariant under the group of gauge transformations. Thus the above quasi-free state is gauge invariant if its covariance satisfies $a(zv) = a(v)$ ($v \in V$, $z \in T$). Quasi-free states are obviously regular, that is $t \in \mathbb{R} \mapsto \hat{\varphi}(tv) \in \mathbb{C}$ is continuous for all $v \in V$. As a consequence their GNS representations yield field operators $R_\varphi(v)$ as Stone-generators of the unitary group $(\pi_\varphi(wv))_{v \in \mathbb{R}}$ and thus, when $(V, \sigma)$ is a complex subspace of $(H, \text{Im}(\cdot, \cdot))$ for a complex Hilbert space $H$, also annihilation and creation operators $a_\varphi(v) := \frac{i}{2}(R_\varphi(v) + iR_\varphi(iv))$, respectively $a^*_\varphi(v) := \frac{i}{2}(R_\varphi(v) - iR_\varphi(iv))$ ($v \in V$). The latter are fully formed closed mutually adjoint operators satisfying the canonical commutation relations in the form

\[ \|a^*_\varphi(v)\zeta\|^2 - \|a_\varphi(v)\zeta\|^2 = \|v\|^2\zeta, \quad (\zeta \in \text{Dom } a^*_\varphi(v) = \text{Dom } a_\varphi(v)) \]

([BrR], Lemma 5.1.12). Warning: We use the probabilists’ normalisation rather than that of the mathematical physicists. The case where $(V, \sigma) = (H, \text{Im}(\cdot, \cdot))$ and $a = \|\cdot\|^2$, for a complex Hilbert space $H$, is the Fock state. Its GNS representation is given by the Fock-Weyl operators defined in the introduction and Fock vacuum vector. For any nondegenerate symplectic space $(V, \sigma)$ and symplectic map $R : V \to H$ satisfying $|\sigma(u, v)| \leq \|Rv\|\|Rv\|$ ($u, v \in V$), there is a representation $\pi_R$ of $CCR(V, \sigma)$ on $\Gamma(H)$ satisfying $\pi_R(wv) = W_\varphi(Rv)$ and a quasi-free state with characteristic function (4.1) in which $a[v] = \|Rv\|^2$ ($v \in V$). There is an extensive literature on quasi-free states; the notes [Ped] are useful, and [BrR] provides their context in quantum statistical mechanics.

Remark. The analogue of quasi-free states in free probability is investigated in [Shi].

A pair $(H_1, H_2)$, consisting of closed subspaces of a real Hilbert space, is said to be in generic position if $H_1 \cap H_2$, $H_1^\perp \cap H_2$, $H_1 \cap H_2^\perp$ and $H_1^\perp \cap H_2^\perp$ are all trivial ([Hal]). Araki’s Duality Theorem, which we quote next, is central to the understanding of von Neumann algebras associated with quasi-free states of CCR algebras.

**Theorem 4.1** ([Ar1,2]). Let $H_1$ and $H_2$ be closed real subspaces of a complex Hilbert space $H$. Suppose that $(H_1, H_2)$ is in generic position and let $\pi$ be the Fock representation of $CCR(H)$. For $i = 1, 2$, let $\pi_i = \pi \circ \phi_i$ where $\phi_i$ is the natural $C^*$-monomorphism $CCR(H_i) \to CCR(H)$, then $\pi_i$ is a faithful, irreducible representation which generates a Type III factor $\mathcal{N}_i$ for which the Fock vacuum $\Omega_{\mathcal{N}_i}$ is cyclic and separating and $\mathcal{N}_2 = (\mathcal{N}_1)'$.

In this section $H = K^{\otimes 2}$ where $K$ is the complexification of a real Hilbert space. Viewing $K$ and $K^{\otimes 2} := K \oplus K$ as real vector spaces, they carry the symplectic forms $\text{Im}(\cdot, \cdot)_K$ and $\text{Im}(\cdot, \cdot)_{K^{\otimes 2}}$ respectively, and the real inner products $\text{Re}(\cdot, \cdot)_K$ and $\text{Re}(\cdot, \cdot)_{K^{\otimes 2}}$. The symbol $^\perp$ denotes symplectic complement with respect to the symplectic form $\text{Im}(\cdot, \cdot)$, and $\text{Re}^\perp$ means orthogonality with respect to the real inner product $\text{Re}(\cdot, \cdot)$. The conjugation on both $K$ and $K^{\otimes 2}$ is denoted by $\overline{\cdot}$, and we employ the conjugate-linear operator $K^\pi := K \circ \pi = \pi \circ K$, where $\pi$ is the sum-flip on $K^{\otimes 2}$, and the real-linear operator

\[ \iota := \begin{bmatrix} I \\ -K \end{bmatrix} : K \to K^{\otimes 2}, \quad f \mapsto \begin{pmatrix} f \\ -f \end{pmatrix}. \]

Let $(\Sigma^o, \mathcal{X})$ consist of a real subspace $\mathcal{X}$ of $K$ and an operator $\Sigma^o$ on $K^{\otimes 2}$ with domain $\text{Lin}_C \iota(\mathcal{X})$, and assume that the following hold:

\[ \mathcal{X} \text{ is dense in } K, \quad \text{(4.3a)} \]

\[ \Sigma^o \text{ is closable, and} \quad \text{(4.3b)} \]
\( \Sigma^o \circ \iota \) is symplectic. \hspace{1cm} (4.3c)

Set \( \Sigma := \Sigma^o \). Note the following, in which \( R := \text{Ran} \iota \):

\[ \mathcal{X} \cap i\mathcal{X} \text{ is dense in } \mathcal{K}; \]
\[ R \cap iR = \{ 0 \} \text{ and } R + iR = \mathcal{K}^{\oplus 2}; \]
\[ \text{Dom } \Sigma^o \text{ is dense in } \mathcal{K}^{\oplus 2}. \]

Recalling the Fock-Weyl operator notation described in the introduction, we define

\[ N_{(\Sigma, \mathcal{X})} := (W_{\Sigma^o})' \text{ where } W_{\Sigma^o} := \text{Lin}\{ W(f) : f \in \mathcal{X} \} \text{ and } W := W_0 \circ \Sigma^o \circ \iota; \]
\[ \Omega := \Omega_{\mathcal{K}^{\oplus 2}}, \mathcal{F} := \mathcal{F}_{\mathcal{K}^{\oplus 2}}, \text{ and write } V^{(1)} \text{ for the natural isometry } \mathcal{K}^{\oplus 2} \to \mathcal{F}; \]
\[ H_1 := \Sigma^o(\mathcal{X}) \text{ and } H_2 := H_1^{\perp}{\Re} = (iH_1)^{\Re \perp}. \hspace{1cm} (4.4) \]

Thus \( H_1 \) and \( H_2 \) are closed real subspaces of \( \mathcal{K}^{\oplus 2} \) and \( V^{(1)}V^{(1)*} = P_{\mathcal{F}^{(1)}}, \) where \( \bigoplus_{n \geq 0} F^{(n)} \) is the eigendecomposition for the number operator on \( \mathcal{F} \).

The map \( w_f \mapsto W(f) \) defines a representation of \( CCR(\mathcal{X}) \), and the vacuum vector induces the quasifree state on \( CCR(\mathcal{X}) \) with characteristic function \( \hat{\vfi}(f) = e^{-\frac{1}{2}||\Sigma^o(f)||^2}. \)

To the above assumptions on \( (\Sigma^o, \mathcal{X}) \) we add the following:

\[ \text{Ran } \Sigma^o \text{ is dense in } \mathcal{K}^{\oplus 2}. \hspace{1cm} (4.5a) \]

the pair \((H_1, H_2)\) is in generic position. \hspace{1cm} (4.5b)

Thus \( \Omega \) is cyclic and separating for \( N_{(\Sigma, \mathcal{X})} \).

**Theorem 4.2.** Let \( (\Sigma^o, \mathcal{X}) \) be as above, satisfying (4.3c) and (4.5). Set \( s_\Omega := V^{(1)*}S_\Omega V^{(1)} \) and \( f_\Omega := V^{(1)*}F_\Omega V^{(1)} \). Then the following hold.

(a) \( S_\Omega P_{\mathcal{F}^{(1)}} \supset P_{\mathcal{F}^{(1)}} S_\Omega, \quad V^{(1)*}S_\Omega \subset s_\Omega V^{(1)*}, \) and \( s_\Omega \Sigma^o = \Sigma^o K^\pi. \)

(b) \( s_\Omega \) is closed and densely defined with core \( \text{Ran } \Sigma^o \). Moreover,

\[ \text{Dom } s_\Omega^2 \ni H_1 + iH_1 \text{ and } s_\Omega^2 \eta = \eta \quad (\eta \in H_1 + iH_1), \]

with \( s_\Omega \zeta = -\zeta \) for \( \zeta \in H_1 \) and \( s_\Omega(\zeta) = \zeta \) for \( \zeta \in iH_1 \).

Let \( j_\Omega \Omega^{1/2} \) be the polar decomposition of \( s_\Omega \).

(c) \( j_\Omega H_1 = H_2, \quad j_\Omega = V^{(1)*}j_\Omega V^{(1)}, \quad \delta_\Omega^{1/2} = V^{(1)*}\Delta_\Omega^{1/2}V^{(1)}, \) and

\[ j_\Omega W_{\Omega}(G)j_\Omega = W_{\Omega}(j_\Omega G) \quad (G \in H_1). \]

Let \( \Sigma^o' := j_\Omega \Sigma^o(K \oplus K) \) and note that \( \Sigma^o' \) is closable and

\[ \text{Dom } \Sigma^o' = (K \oplus K) \text{ Dom } \Sigma^o = \text{Lin}_\mathcal{C} \iota(K\mathcal{X}). \]

Define \( N_{(\Sigma', \mathcal{K} \mathcal{X})} := (W_{\Sigma'})' \) where \( W_{\Sigma'} := \text{Lin}\{ W'(g) : g \in K\mathcal{X} \} \) and \( W' := W_0 \circ \Sigma^o' \circ \iota \).

(d) \( \Sigma^o' \circ \iota \) is a symplectic map from \( K\mathcal{X} \) to \( \mathcal{K}^{\oplus 2}. \)

(e) \( N_{(\Sigma', \mathcal{K} \mathcal{X})} = (N_{(\Sigma, \mathcal{X})})'. \)

(f) \( F_\Omega P_{\mathcal{F}^{(1)}} \supset P_{\mathcal{F}^{(1)}} F_\Omega, \quad V^{(1)*}F_\Omega \subset f_\Omega V^{(1)*}, \) and \( f_\Omega \Sigma^o' = \Sigma^o' K^\pi. \)

(g) \( f_\Omega \) is closed and densely defined with core \( \text{Ran } \Sigma^o'. \)

(h) \( f_\Omega = s_\Omega^\ast, \quad \Gamma(s_\Omega) = S_\Omega \) and \( \Gamma(f_\Omega) = F_\Omega. \)

**Proof.** (a) For \( f \in \mathcal{X}, \) since \( S_\Omega \vfi(s_\Omega^o(f)) = \vfi(-it\Sigma^o(f)), \)

\[ t^{-1} \vfi(-it\Sigma^o(f)) \to V^{(1)} \Sigma^o_t(f) = P_{\mathcal{F}^{(1)}} \vfi(t\Sigma^o(f)), \]

\[ S_\Omega t^{-1} \vfi(s_\Omega^o(f)) \to -V^{(1)} \Sigma^o_t(f) = -P_{\mathcal{F}^{(1)}} \vfi(t\Sigma^o(f)). \]
Thus \( P_{\mathcal{F}^{(0)}} W(f) \Omega \in \text{Dom} \ S_\Omega \) and \( S_\Omega P_{\mathcal{F}^{(0)}} W(f) \Omega = P_{\mathcal{F}^{(0)}} S_\Omega W(f) \Omega \). Since \( \mathcal{W}_\Xi^\eta \Omega \) is a core for \( S_\Omega \), this implies the first inclusion. The second inclusion follows, as does the identity
\[
s_\Omega \Sigma^\eta \circ \iota = -\Sigma^\eta \circ \iota. \tag{4.6}
\]
Since \( K^\pi \circ \iota = -\iota \), the conjugate-linear operators \( s_\Omega \Sigma^\eta \) and \( \Sigma^\eta K^\pi \) agree on \( \iota(\mathcal{F}) \), and therefore coincide.

(b) Since \( S_\Omega \) is closed with core \( \mathcal{W}_\Xi^\eta \Omega \), (a) and the adjoint-product-inclusion relation \( \text{[A.3]} \) imply that \( s_\Omega \) is closed with core \( \mathcal{V}^{(1)} \mathcal{W}_\Xi^\eta \Omega \), which is dense by assumption. Now let \( \zeta \in H_1 \). Then \( \zeta = \lim \zeta_n \) for a sequence \( \{\zeta_n\} \) in \( \Sigma^\eta \iota(\mathcal{F}) \). By \( \text{(1.6)} \), \( s_\Omega \zeta_n = -\zeta_n \to -\zeta \). Since \( s_\Omega \) is closed, this implies that \( \zeta \in \text{Dom} \ s_\Omega \) and \( s_\Omega \zeta = -\zeta \). Also, by conjugate linearity, \( s_\Omega \zeta = i\zeta \). It follows that \( H_1 + iH_1 \subset \text{Dom} \ s_\Omega^2 \) and \( s_\Omega^2 \eta = \eta \) for \( \eta \in H_1 + iH_1 \). This proves (b).

(c) This is proved in \( \text{[ExO]} \) using Halmos’ two subspaces paper \( \text{[Hal]} \); see also Chapter 7 of \( \text{[Dor]} \).

(d) \( \Sigma^\eta \circ \iota \) is symplectic since, for \( f, g \in \mathcal{F} \),
\[
\text{Im}(\Sigma^\eta \iota(\mathcal{F}), \Sigma^\eta \iota(\mathcal{F})) = \text{Im}(j_\Omega \Sigma^\eta \iota(f), j_\Omega \Sigma^\eta \iota(g))
\]
\[
= -\text{Im}(\Sigma^\eta \iota(f), \Sigma^\eta \iota(g)) = -\text{Im}(f, g) = \text{Im}(\overline{f}, \overline{g}).
\]
Since \( (K \oplus K) \circ \iota = \iota \circ K \) and \( j_\Omega \) is isometric, the density of \( \text{Ran} \Sigma^\eta \) follows from (c):
\[
\text{Ran} \Sigma^\eta = j_\Omega \text{Ran} \Sigma^\eta = j_\Omega H_1 = H_2.
\]

(e) By (c),
\[
W^*(\overline{\mathcal{F}}) = W_0(\Sigma^\eta \iota(\mathcal{F})) = W_0(j_\Omega \Sigma^\eta \iota(f))
\]
\[
= J_\Omega W_0(\Sigma^\eta \iota(f)) J_\Omega = J_\Omega W(f) J_\Omega
\]
so, by Tomita’s Theorem,
\[
N_{\Sigma^\eta \iota, \mathcal{F}} = (W_{\mathcal{F}^{(0)}})^\eta = (J_\Omega W_{\mathcal{F}^{(0)}}, J_\Omega) = J_\Omega (W_{\mathcal{F}^{(0)}})^\eta J_\Omega = J_\Omega N_{\Sigma^\eta \iota, \mathcal{F}} J_\Omega = (N_{\Sigma^\eta \iota, \mathcal{F}})^\eta.
\]

(f)\&(g) By the assumptions on \( (\mathcal{F}, \Sigma^\eta, H_1, H_2) \), and what has been already proved, the pair \( (K \mathcal{F}, \Sigma^\eta) \) consists of a dense real subspace of \( K \) and a closable operator satisfying \( \text{[A.4]} \), with \( (H_2, H_1) \) in place of \( (H_1, H_2) \). Since, by (e), the \( S \)-operator for \( (N_{\Sigma^\eta \iota, \mathcal{F}}) \) is \( F_{\mathcal{F}_1} \), (f) and (g) are precisely what results from applying (a) and (b) to the pair \( (K \mathcal{F}, \Sigma^\eta) \).

(h) The identity \( s_\Omega^2 = f_\Omega \) follows from (a) and Part (c) of Lemma \( \text{[A.4]} \). For \( f \in \mathcal{F} \),
\[
\Gamma(s_\Omega \varepsilon(\Sigma^\eta \iota(f))) = \varepsilon(s_\Omega \Sigma^\eta \iota(f)) = \varepsilon(-\Sigma^\eta \iota(f)) = S_\Omega \varepsilon(\Sigma^\eta \iota(f)).
\]
The closed operators \( S - \Omega \) and \( \Gamma(s_\Omega) \) therefore agree on \( \mathcal{E}(\Sigma^\eta \iota(\mathcal{F})) = \mathcal{W}_\Xi^\eta \Omega \), which is a core for \( S_\Omega \), so \( S_\Omega \subset \Gamma(s_\Omega) \). Applying this with \( (\Sigma^\eta, S_\Omega) \) replaced by \( (\Sigma^\eta, F_{\mathcal{F}_1}) \) gives \( F_{\mathcal{F}_1} \subset \Gamma(f_\Omega) \), so we also have
\[
S_\Omega = F_{\mathcal{F}_1} \supset \Gamma(f_\Omega)^* = \Gamma(f_\Omega) = \Gamma(s_\Omega).
\]
Therefore the required equality holds, and the proof is complete.

We make two simple observations, as motivation for the following result.

Remarks. If \( \Sigma^\eta \) is closed (so that \( \Sigma = \Sigma^\eta \)), then
\[
\text{Ran} \Sigma \subset H_1 + iH_1.
\]
Thus, if \( \Sigma^\eta \) is surjective (and thus also closed) then
\[
H_1 + iH_1 = K^{\oplus 2}.
\tag{4.7}
\]

**Proposition 4.3.** Let \( (\Sigma^\eta, \mathcal{F}) \) be as in Theorem \text{[4.3]} and assume \text{[4.7]}. Then the following hold:
(a) $s^2_\Omega = I_{K^\otimes 2}$, in particular $s_\Omega$ is bounded; it is given by
\[ s_\Omega(\zeta + i\eta) = -\zeta + i\eta \quad (\zeta, \eta \in H_1). \]

(b) If also $\Sigma$ is surjective then
(i) $s_\Omega \Sigma = \Sigma K^\pi$, so $s_\Omega = \Sigma K^\pi \Sigma^{-1}$.
(ii) $(s_\Omega \otimes S_\Omega)(\Sigma \otimes I_F) \subset s_\Omega \Sigma \otimes S_\Omega \subset (\Sigma \otimes I_F)(K^\pi \otimes S_\Omega)$, moreover, the second operator is the closure of the first.
(c) If $\Sigma$ is surjective and we assume further that there is a real subspace $D$ of $\mathcal{X}$ such that
\[ \iota(KD) \subset \text{Dom} \Sigma^* \Sigma' \quad \text{and} \quad \text{Lin}_\Sigma \Sigma' \iota(KD) \text{ is dense in } K^\otimes 2, \] then the conclusion in (b)(ii) has the following refinement:
\[ \text{Dom}(s_\Omega \otimes S_\Omega)(\Sigma \otimes I_F) = \text{Dom} s_\Omega \Sigma \otimes S_\Omega \cap \text{Dom} \Sigma \otimes I_F. \]

Proof. (a) This follows immediately from Part (b) of Theorem 4.2.
(b) (i) We have $s_\Omega \Sigma^0 = \Sigma^0 K^\pi$ and so, since $s^2_\Omega = I_{K^\otimes 2}$, $\Sigma^0 = s_\Omega \Sigma^0 K^\pi$. Since also $(K^\pi)^2 = I_{K^\otimes 2}$, it follows that $\Sigma = s_\Omega \Sigma^0 K^\pi = s_\Omega \Sigma K^\pi$ and (i) follows.
(b) (ii) Since $s_\Omega \Sigma$ is closed and $\Sigma^{-1}$ is bounded we have
\[ s_\Omega \otimes S_\Omega = s_\Omega \Sigma \Sigma^{-1} \otimes S_\Omega = (s_\Omega \Sigma \otimes S_\Omega)(\Sigma^{-1} \otimes I_F) = (\Sigma K^\pi \otimes S_\Omega)(\Sigma^{-1} \otimes I_F) \]
(by Part (d) of Proposition A.3), therefore
\[ (s_\Omega \otimes S_\Omega)(\Sigma \otimes I_F) \subset s_\Omega \Sigma \otimes S_\Omega = \Sigma K^\pi \otimes S_\Omega \]
\[ \subset (\Sigma \otimes I_F)(K^\pi \otimes S_\Omega), \]
by Part (c) of Proposition A.3. Since $s_\Omega \Sigma \otimes S_\Omega$ is closed and the domain of the LHS of this inclusion contains $\text{Dom} \Sigma \otimes \text{Dom} S_\Omega$ which is a core for the middle term, (ii) follows.
(c) Let $x \in \text{Dom} s_\Omega \Sigma \otimes S_\Omega \cap \text{Dom} \Sigma \otimes I_F$. Since $(s_\Omega \Sigma \otimes S_\Omega)^* = \Sigma^* f_\Omega \otimes F_\Omega$, to see that $x \in \text{Dom}(s_\Omega \otimes S_\Omega)(\Sigma \otimes I_F)$ if suffices to verify that
\[ \langle (f_\Omega \otimes F_\Omega)\alpha, (\Sigma \otimes I_F)x \rangle = \langle (\Sigma^* f_\Omega \otimes F_\Omega)\alpha, x \rangle \quad (4.9) \]
for all vectors $\alpha$ from a subset of $\text{Dom} \Sigma^* f_\Omega \otimes F_\Omega$ which is a core for $f_\Omega \otimes F_\Omega$. Since $f_\Omega$ is bounded, it suffices to verify (4.9) for vectors $\alpha$ of the form $u \otimes \Omega$ where $T \in N'_{\Sigma, X}$ and $u$ is from a total subset of $K^\otimes 2$. By assumption we may take $u$ from $\Sigma^* \iota(KD)$. Now
\[ (f_\Omega \otimes F_\Omega)\Sigma^* \iota(g) \otimes T \Omega = f_\Omega \Sigma^* \iota(g) \otimes T^* \Omega = -\Sigma^* \iota(g) \otimes T^* \Omega \]
for all $g \in D$ and $T \in N'_{\Sigma, X}$ and so, for such $\alpha$,
\[ \text{LHS of (4.9)} = \langle -\Sigma^* \Sigma^* \iota(g) \otimes T^* \Omega, x \rangle = \text{RHS of (4.9)}, \]
as required. \hfill \Box

The elementary observation contained in the following lemma is relevant to the examples below.

**Lemma 4.4.** For any real subspace $V$ of $K$,
\[ V \oplus \{0\} = \{i(f) - i(i)f : f \in V\} \quad \text{and} \quad \{0\} \oplus KV = \{i(f) + i(i)f : f \in V\}. \]
In particular, if $V$ is a complex subspace of $K$ then
\[ \text{Lin}_\Sigma \iota(V) = V \oplus KV. \]
Proof. Let $J$ be the real-linear map $f \mapsto if$ on $K$. Then

$$\iota = \begin{bmatrix} 1 & 0 \\ -K & K \end{bmatrix} \quad \text{and} \quad (J \oplus J) \iota = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix},$$

so

$$\iota - (J \oplus J) \iota J = 2 \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \quad \iota + (J \oplus J) \iota J = -2 \begin{bmatrix} 0 \\ K \end{bmatrix}.$$

The result follows. \qed

**Example** (Guage-invariant quasifree states). Let $X$ be the complex subspace $\text{Dom} T^{1/2}$ of $K$, where $T$ is a nonnegative selfadjoint operator on $K$, and let $\Sigma^o$ be the nonnegative selfadjoint operator

$$\Sigma_T := \begin{bmatrix} \sqrt{I + T} & K \sqrt{T} \\ K \sqrt{T} & K \end{bmatrix}.$$  

It follows from the functional calculus for $T$ that $\Sigma^o \circ \iota$ is symplectic and $\|\Sigma^o(v)\| = \| \sqrt{I + 2T} v \| \geq \|v\|$ ($v \in X$), so there is a unique quasifree state on $\text{CCR}(X)$ with characteristic function

$$\hat{\varphi}_T : v \mapsto e^{-\frac{1}{2} \|\sqrt{I + 2T} v\|^2}.$$  

Moreover, since $\hat{\varphi}_T(zv) = \hat{\varphi}_T(v)$ ($z \in \mathbb{T}$), the state is gaug invariant. Note also that $H_1$ and $H_2$ are the closures of the ranges of the respective operators

$$\sqrt{I + T} - K \sqrt{T} \quad \text{and} \quad \begin{bmatrix} \sqrt{I + T} \\ -K \sqrt{T} \end{bmatrix}.$$  

The degenerate case where $T = 0$ is the Fock state. On the other hand if $T$ is injective then $\Sigma_T$ has dense range and it is straightforward to verify that $(H_1, H_2)$ is in generic position, so Theorem 4.2 applies. The associated operators are then

$$J = \begin{bmatrix} K \\ -I \end{bmatrix}, \quad \delta_{1/2} = \begin{bmatrix} \sqrt{I + T}^{-1} \\ K \sqrt{T} \end{bmatrix}$$

and $\Sigma' = \Sigma_T$, where

$$\Sigma'_T := \begin{bmatrix} K \sqrt{I + T} & \sqrt{T} \\ \sqrt{T} & K \end{bmatrix}.$$  

Thus $J = K^\dagger$. Note that $\Sigma_T$ and $\Sigma'_T$ are both closed, and (4.13) holds with $D$ equal to $\text{Dom} T$ since

$$\Sigma_T \Sigma'_T = \frac{1}{2} \begin{bmatrix} K \sqrt{T(I + T)} \\ K \sqrt{T(I + T)} \end{bmatrix}.$$  

Thus, if $T$ is bijective then so is $\Sigma_T$ and Proposition 4.3 applies. Note that in this case $T^{-1}$ is bounded so the boundedness of $\delta_{1/2}$, and thus also of $s_{1/2}$, is manifest. Moreover, setting $A = \log(I + T^{-1})$, we have $I + 2T = \coth A$. The case $A = \frac{\beta}{2} I$ then corresponds to the temperature state of $\text{CCR}(K)$ with inverse temperature $\beta$ (BrR).

**Example** (Squeezed states). The above guage-invariant quasifree states may be `squeezed' by composing with the Bogoliubov automorphism $\phi_Q$ of $\text{CCR}(X)$ induced by a symplectic automorphism $Q$ of $X$. We use the following structure theorem from HoR. If either $K$ is separable, or $Q$ is bounded (as a densely defined operator on $K$, viewed as a real Hilbert space), then $Q$ is the restriction of an operator of the form

$$U(\cosh P - K' \sinh P)$$

to $X$, where $U$, $K'$ and $P$ are operators on $K$, $U$ being unitary, $K'$ another conjugation, and $P$ a second nonnegative selfadjoint operator, and the following consistency conditions hold:
(a) For $R \in \{U \cosh P, UK' \sinh P, \cosh P U^*, \sinh P K' U\}$, $X \subset \text{Dom } R$ and $R(X) \subset X$.

(b) $K'$ commutes with the spectral projectors of $P$.

(c) $X$ is a core for $\sinh^2 P$.

Moreover if $(\tilde{U}, \tilde{K}'', \tilde{P})$ is another such parameterisation of $Q$ then $(\tilde{U}, \tilde{P}) = (U, P)$, and $\tilde{K}'$ and $K'$ agree on $\text{Ran } P$. In terms of these, $\Sigma_T \circ \iota \circ Q = \Sigma_{T,Q} \circ \iota$ where

$$\Sigma_{T,Q} = \Sigma_T(U \oplus KU'K') \Gamma(I \oplus K'K)$$

for $\Gamma = \begin{bmatrix} \cosh P & \sinh P \\ \sinh P & \cosh P \end{bmatrix}$, the corresponding quasifree state on $C\!\!\!\!R(X)$ has characteristic function

$$\hat{\varphi}_{T,Q}(v) = e^{-\frac{1}{2} \sqrt{t^{\gamma(\pi)}} Q |v|^2}.$$}

If $\Sigma^o := \Sigma_{T,Q}$ is closable with dense range (for example if $P$ is bounded) then Theorem 4.3 applies, and $H_1, H_2, J_\Omega$ and $\delta_\Omega^{1/2}$ are as in the gauge-invariant case above, and

$$\Sigma_{T,Q}' := \Sigma_{T}(K \oplus K')(U \oplus KU'K') \Gamma(K \oplus K').$$

5. QUASIFREE STATES FOR STOCHASTIC ANALYSIS

We now specialise our quasifree states for stochastic analysis, and we identify natural conditions on a pair $(\Sigma^o, X)$ — consisting of a dense real subspace $X$ of $K$ and closable operator $\Sigma^o$ on $K^{\otimes 2}$ with domain $\text{Linc}(\iota(X))$ — for Assumptions (4.3) and (4.4) to hold, so that Theorem 4.3 applies. We then show that this entails a key commutation relation between Itô integration and the Tomita-Takesaki operators.

The notation is as for the previous section, but now $K = L^2(\mathbb{R}_+; k)$ as in Section 3 except that now $k$ is the complexification of a separable real Hilbert space $k$. Thus $K^{\otimes 2} = L^2(\mathbb{R}_+; k^{\otimes 2})$ and $K$ is the complexification of $L^2(\mathbb{R}_+; k)$; the conjugation on $K$ being that induced by the conjugation on $k$ pointwise:

$$f(t) := \overline{f(t)} \quad (t \in \mathbb{R}_+).$$

Assumptions. Setting $\Sigma := \Sigma^o$ and $\Sigma_t := V_t^* \Sigma V_t$, where $V_t$ is the inclusion map $K^{\otimes 2}_t \to K^{\otimes 2}$, we now make the following assumptions on the pair $(\Sigma^o, X)$:

(a) $\Sigma^o \circ \iota$ is symplectic and, for all $t \in \mathbb{R}_+$,

(b) $X_t := p_t(X) \subset X$,

(c) $p_t \Sigma^o \subset \Sigma^o p_t$,

(d) $\Sigma_t$ is bijective with bounded inverse,

and consider the further alternative assumptions:

(e) $V_t^*(H_1 + iH_1) = K^{\otimes 2}_t$ and there is a real subspace $D_t$ of $W_t^* X_t$, where $W_t$ is the inclusion $K_t \to K$, such that $\iota(K_t D_t) \subset \text{Dom } \Sigma_t' \Sigma_t'$ and $\text{Linc} \iota(\iota(\Sigma_t' \iota(\iota(D_t)))$ is dense in $K^{\otimes 2}_t$.

(f) $\Sigma_t$ is bounded for all $t \in \mathbb{R}_+$.

Remarks. (i) Here are some consequences of Assumptions (a)–(d).

(\alpha) $\Sigma \in L^\infty(\mathbb{R}_+) \supset B(k^{\otimes 2})$; this follows from Lemma 3.1.

(\beta) For all $t \in \mathbb{R}_+$, $\Sigma_t$ is closed with core $\text{Linc} \iota(\iota(X_t))$; this follows from Part (c) of Lemma 3.

(\gamma) $\Sigma$ is injective.

(\delta) $\bigcup_{t \geq 0} \text{Linc} \iota(X_t)$ is a core for $\Sigma$.

(e) For all $t \in \mathbb{R}_+$, $\text{Ran } \Sigma_t$ is dense in $K^{\otimes 2}_t$, where $\Sigma^o := V_t^* \Sigma^o V_t$.

(\zeta) For all $t \in \mathbb{R}_+$, $p_t \Sigma \subset \Sigma p_t$; this follows from Lemma 3.4 (a).

(ii) Notice that (e) is a localised version of the hypotheses in Proposition 4.3.

Indeed (e) implies the local boundedness property $V_t^* s_\Omega V_t \in B(K^{\otimes 2}_t)$ for every
t \in \mathbb{R}_+\) as follows. Setting \(X' := (I - p_I)X\), the assumptions (a)-(d) give us a decomposition \(\Sigma = \Sigma_t \oplus \Sigma'\) on \(X = X_t \oplus X'\), and thus a pair of von Neumann algebras \(N(\Sigma_x, x_t)\) and \(N(\Sigma_{x'}, x_{t'})\) for each \(t \geq 0\). Using Weyl operators one sees that \(N(\Sigma_x, x_t) \cong N(\Sigma_{x'}, x_{t'})\) for each \(t \geq 0\). Using Weyl operators one sees that \(N(\Sigma_x, x_t) \cong N(\Sigma_{x'}, x_{t'})\) for each \(t \geq 0\).

Recall that \(H_0 := \Sigma_0 \cap \perp\) is defined on all of \(\iota\) and \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\).

Recall that \(H_0 := \Sigma_0 \cap \perp\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\).

Recall that \(H_0 := \Sigma_0 \cap \perp\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\).

Recall that \(H_0 := \Sigma_0 \cap \perp\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\).

Recall that \(H_0 := \Sigma_0 \cap \perp\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\). Thus \(\iota\) is defined on all of \(\iota\).
(iv) In view of the identity $H^+_T \cap H^+_S = iH^+_2 \cap iH^+_1 = i(H^+_1 \cap H^+_2)$, (i) implies that this subspace is trivial too. Therefore (a) holds.

(b) Since $f_\Omega = s_\Omega^t$, it suffices to show that $s_\Omega$ is so affiliated. Let $t \geq 0$. Then, for $f \in \mathcal{F}$,

$$p_t \Sigma^\alpha t(f) = \Sigma^\alpha t(f_0, t) \in V^{(1)} W^\Omega_\Sigma, \quad \text{and}$$
$$s_\Omega p_t \Sigma^\alpha t(f) = -\Sigma^\alpha t(f_0, t) = p_t s_\Omega \Sigma^\alpha t(f).$$

Thus $(p_t s_\Omega)_{V^{(1)} W^\Omega_\Sigma} \subset s_\Omega p_t$. Since $s_\Omega p_t$ is closed and, by Part (b) of Theorem 4.2, $V^{(1)} W^\Omega_\Sigma$ is a core for $s_\Omega$, this implies that $p_t s_\Omega \subset s_\Omega p_t$. (b) therefore follows from Lemma 3.1.

(c) This now follows from Theorem 3.4. \qed

Remark. In [HH], an abstract noncommutative stochastic calculus is related to squeezed states, additive cocycles with respect to the natural shift are considered, and an Itô table derived. In [LM] we derive the Itô table for the general quasifree setting considered here.

Examples. For the squeezed quasifree states discussed in Section 4 Assumptions (a)-(d) are satisfied if $T$ and $P$ are affiliated to $L^\infty(\mathbb{R}_+)^{\otimes 2}(k)$, $P$ is locally bounded, $K^\prime$ is a pointwise conjugation on $K$: $(K^\prime f)(t) = k f(t)$ for some conjugation $k^\prime$ on $k$, and there is $\alpha \in L^\infty(\mathbb{R}_+)$ such that $\alpha > 0$ almost everywhere and $T \geq M_{\alpha^{-1}} \otimes I_k$. Assumption (e) is satisfied too if $\alpha$ may be chosen so that also $T \leq M_{\alpha} \otimes I_k$.

On the other hand, if $P$ is bounded and $T = I_{L^2(\mathbb{R}_+)} \otimes Q$ where $Q$ is a closed, densely defined, unbounded and bijective operator then the resulting pairs $(\Sigma^\alpha, \mathcal{X})$ satisfy (a)-(e), but not (e$_+$).

6. Modified Itô integral

In this section we establish the appropriate analogue of the abstract Kunita-Watanabe Theorem at the vector process level.

Let $(\Sigma^\alpha, \mathcal{X})$ be as in Section 5, take the notations $\Sigma$, $\Omega$, $N(\Sigma, \mathcal{X})$ and $V_t$ from Sections 5 and 4 and fix a von Neumann algebra $A$ acting on a separable Hilbert space $h$, which we refer to as the initial algebra, with cyclic and separating vector $\nu$. Assumptions (a)-(e) are in operation. and we set $N = N(\Sigma, \mathcal{X})$.

$$M = A \otimes N, \quad \xi = \nu \otimes \Omega, \quad \Xi = M^\prime \xi, \quad S = S_\xi, \quad \text{and} \quad \tilde{\mathcal{H}} = h \otimes \mathcal{F}_{k^{\otimes 2}}. \quad (6.1)$$

Thus the vector $\xi$ is cyclic and separating for the von Neumann algebra $M$, $S = S_\nu \otimes S_\Omega$ (S7Z, 10.7), and the Hilbert space $\Xi$ is separable. Also write $P^\Omega_t$ for $V_t^\alpha P_\Sigma V_\Omega$, the restriction of $P_t$ on $K^{\otimes 2} \otimes \tilde{\mathcal{H}}$ to the subspace $L^2_\Omega(\mathbb{R}_+; k^{\otimes 2} \otimes \tilde{\mathcal{H}}) = (K^{\otimes 2} \otimes \tilde{\mathcal{H}}) \rightarrow L^2_\Omega(\mathbb{R}_+; k^{\otimes 2} \otimes \tilde{\mathcal{H}})$ in a natural way. Finally, set

$$k^\pi := k \circ \pi = \pi \circ k, \quad K^\pi := K \circ \pi \quad \text{and} \quad K^\pi_t := K_t \circ \pi,$$

where $k$, $K$ and $K_t$ are the conjugations on $k^{\otimes 2}$, $K^{\otimes 2}$ and $K^{\otimes 2}_t$ respectively, and $\pi$ is the sum-flip on each of these orthogonal sums.

Lemma 6.1. The following holds:

$$(s_\Omega \otimes S)(\Sigma V_t \otimes I_\Omega) \subset (\Sigma V_t \otimes I_\Omega)(K^\pi_t \otimes S).$$

Under Assumption $(e_+)$ this can be strengthened to an equality.
Proof. Recall that \( s_\Omega = s_\Omega \uparrow \ominus s_\Omega \) and, by Proposition A.3 (a), \( s_\Omega \) is bounded, hence \( s_\Omega \Sigma = \Sigma K_\Sigma \). Now, since \( V_\Sigma \) is an isometry, applying Lemma A.1 (d) together with Proposition A.3 (b) we get

\[
(s_\Omega \otimes S)(\Sigma V_\Sigma \otimes I_B) = (s_\Omega \otimes S)(P_\Sigma \otimes I)(\Sigma V_\Sigma \otimes I_B) = (V_\Sigma \otimes I)(s_\Omega \otimes S)(\Sigma \otimes I_B) \subset (V_\Sigma \otimes I)(\Sigma V_\Sigma \otimes I_B) = (\Sigma V_\Sigma \otimes I_B)(K_\Sigma \otimes S) \quad (6.2)
\]

If Assumption \((e+)\) holds then \( (s_\Omega \otimes S)(\Sigma V_\Sigma \otimes I_B) \) is closed and defined on \( \text{Dom} \Sigma V_\Sigma \), which is a core for \( s_\Omega V_\Sigma \otimes S \). Thus \( (s_\Omega \otimes S)(\Sigma V_\Sigma \otimes I_B) \subset s_\Omega \Sigma V_\Sigma \otimes S \) and \((6.2)\) can be strengthened to an equality. \( \square \)

Lemma 6.2. The following holds

\[
(s_\Omega \otimes S)(\Sigma \otimes I_B)P_t^\Omega \subset (\Sigma \otimes I_B)P_t^\Omega \quad (t \in \mathbb{R}_+). \quad (6.3)
\]

Moreover, under Assumption \((e+)\), this is an equality.

Proof. Let \( t \in \mathbb{R}_+ \). In view of the identity

\[
(V_\Sigma \otimes I_B)(V_\Sigma^* K_\Sigma V_\Sigma \otimes S)(V_\Sigma^* \otimes I_B) = (K_\Sigma \otimes S)(V_\Sigma^* \otimes I_B)
\]

(which follows from Part (d) of Proposition A.3), applying Lemma A.2 first with \( T = \Sigma \) and \( X = I_B \) and last with \( T = K \) and \( X = S \), and Part (b) of Proposition A.3 again, we have

\[
\text{LHS of (6.3)} = V_\Omega^*(s_\Omega \otimes S)V_\Omega V_\Omega^*(\Sigma \otimes I_B)V_\Omega P_t^\Omega = V_\Omega^*(s_\Omega \otimes S)V_\Omega V_\Omega^*(\Sigma \otimes I_B)P_t^\Omega \subset V_\Omega^*(s_\Omega \otimes S)V_\Omega V_\Omega^*(\Sigma \otimes I_B)(P_\Sigma \otimes I_B)V_\Omega = V_\Omega^*(s_\Omega \otimes S)V_\Omega \subset \Sigma \otimes I_B)(V_\Sigma^* \otimes I_B)V_\Omega = V_\Omega^*(\Sigma \otimes I_B)(K_\Sigma \otimes S)(V_\Sigma^* \otimes I_B)V_\Omega = V_\Omega^*(\Sigma \otimes I_B)(K^\Sigma \otimes S)(V_\Sigma^* \otimes I_B)V_\Omega \subset V_\Omega^*(\Sigma \otimes I_B)(K_\Sigma \otimes S)V_\Omega P_t^\Omega = \text{RHS of (6.3)}
\]

with equality if assumption \((e+)\) holds. \( \square \)

Lemma 6.3. The operator \( K^\Sigma \otimes S \) on \( L^2(\mathbb{R}_+; k^{\otimes 2} \otimes \mathcal{H}) \) is \( K^\Sigma \otimes \mathcal{H} \) may be characterised as follows:

\[
\text{Dom} K^\Sigma \otimes S = \{ f \in L^2(\mathbb{R}_+; k^{\otimes 2} \otimes \mathcal{H}) : f(t) \in \text{Dom} k^\Sigma \otimes S \text{ for } a.a. \ t, \text{ and } (k^\Sigma \otimes S)f(\cdot) \in L^2(\mathbb{R}_+; k^{\otimes 2} \otimes \mathcal{H}) \}
\]

\[
(K^\Sigma \otimes S)f = (k^\Sigma \otimes S)f(\cdot).
\]

Proof. Call the operator defined above \( R \). The inclusions \( K^\Sigma \otimes S \subset R \subset K^\Sigma \otimes S \) are easily verified, it therefore suffices to show that \( R \) is closed. Letting \( (f_n) \) be a sequence in \( K^{\otimes 2} \otimes \text{Dom} S \) satisfying \( f_n \to f \) and \( Rf_n \to g \), we may pass to a subsequence and assume that the convergence is almost everywhere. Then, for almost all \( t \in \mathbb{R}_+ \),

\[
f(t) = \lim f_n(t) \text{ and } g(t) = \lim(Rf_n)(t) = \lim(k^\Sigma \otimes S)f_n(t),
\]

and so, since \( k^\Sigma \otimes S \) is closed, \( f(t) \in \text{Dom} k^\Sigma \otimes S \) and \( (k^\Sigma \otimes S)f(t) = g(t) \). Since \( g \) is square-integrable, it follows that \( f \in \text{Dom} \) and \( Rf = g \). Thus \( R \) is closed, as required. \( \square \)
Define the following modified Itô integral:
\[ I_t^z := I \circ (\Sigma \otimes \Omega) P^\Omega_t \] (6.4)
and set \( I_t^z := I \circ (\Sigma \otimes \Omega) P^\Omega_t \) \((t \in \mathbb{R}_+)\).

**Remark.** Under Assumption \((e+)\), the integral \( I_t^z \) is bounded and has full domain \( L^2_{\Omega}(\mathbb{R}_+; k^{\mathbb{H}_2} \otimes \mathcal{H}) \), for all \( t \in \mathbb{R}_+ \). Without Assumption \((e+)\) the domains may be smaller. Accordingly, let \( \text{Dom} \Sigma \otimes \Omega I_\beta \) denote the set of (measure equivalence classes of) functions \( z : \mathbb{R}_+ \to k^{\mathbb{H}_2} \otimes \mathcal{H} \) such that, for all \( t \in \mathbb{R}_+ \),
\[ z_t \in k^{\mathbb{H}_2} \otimes \mathcal{H}_t \text{ and } z|_{[0,t]} \in \text{Dom} \Sigma \otimes \Omega I_\beta. \] (6.5)

**Proposition 6.4.** Let \( z \in \text{Dom}_{\text{loc}} \Sigma \otimes \Omega I_\beta \). Then, for all \( t \in \mathbb{R}_+ \),
\[ \|I_t^z\| = \|\Sigma \otimes \Omega z|_{[0,t]}\| \]
and \( I_t^z z = 0 \) if and only if \( z|_{[0,t]} = 0 \).

**Proof.** The first part follows immediately from Itô isometry. For the second part, note that we also have \( \|I_t^z z\| = \|\Sigma_t V_t^* z|_{[0,t]}\| \), and so the result follows from the injectivity of \( \Sigma \) and the fact that \( V_t^* \) is isometric on \( \mathcal{H} \).

By a vector martingale in \( \mathcal{H} \) we mean a family \((x_t)_{t \geq 0} \) in \( \mathcal{H} \) satisfying \( P_s x_t = x_s \) for all \( 0 \leq s \leq t \).

**Theorem 6.5.** Let \( x \) be a vector martingale in \( \mathcal{H} \). Then the following hold.

(a) There is a unique \( z \in \text{Dom}_{\text{loc}} \Sigma \otimes \Omega I_\beta \) such that
\[ x_t - x_0 = I_t^z z \quad (t \in \mathbb{R}_+) \]

(b) The following are equivalent:
(i) \( x \) is Dom-\( S \)-valued.
(ii) \( x_0 \in \text{Dom} S \) and \( z|_{[0,t]} \in \text{Dom} \Sigma K^\pi \otimes \Omega S \) for all \( t \in \mathbb{R}_+ \).

When these hold,
\[ ((K^\pi \otimes \Omega S)z|_{[0,t]}) (s) = (k^\pi \otimes S)z_s \quad \text{for a.a. } s \in [0,t], \]
and, for all \( t \in \mathbb{R}_+ \),
\[ Sx_t - Sx_0 = I_t^z ((k^\pi \otimes S)z) = I_t^z ((K^\pi \otimes \Omega S)z|_{[0,t]}). \] (6.6)

(c) If also \((e+)\) holds then we have the following further equivalences:
(iii) \( x_0 \in \text{Dom} S \) and \( z|_{[0,t]} \in \text{Dom} \Sigma K^\pi \otimes \Omega S \) for all \( t \in \mathbb{R}_+ \).
(iv) \( x_0 \in \text{Dom} S, z \) is almost everywhere \( \text{Dom}(k^\pi \otimes S) \)-valued, and the function \( s \mapsto (k^\pi \otimes S)z_s \) is locally square-integrable.

**Proof.** (a) Uniqueness follows from Proposition 6.4. By the abstract Kunita-Watanabe Theorem (see [6,3]), there is \( y \in L^2_{\text{loc}, \text{loc}}(\mathbb{R}_+; k^{\mathbb{H}_2} \otimes \mathcal{H}) \) such that \( x_t - x_0 = I_t^y (t \in \mathbb{R}_+) \).
Letting \( z \in \text{Dom}_{\text{loc}} \Sigma \otimes \Omega I_\beta \) be the process defined by
\[ z|_{[0,t]} = (\Sigma^{-1} \otimes \Omega I_\beta) y|_{[0,t]}, \]
we have \( x_t - x_0 = I_t^z z \) \((t \in \mathbb{R}_+)\).

(b) By Lemma 6.2 we have
\[ (s_1 \otimes \Omega S) (\Sigma \otimes \Omega I_\beta) P^\Omega_t \subset (\Sigma \otimes \Omega I_\beta) (K^\pi \otimes \Omega S) P^\Omega_t \]
\((t \in \mathbb{R}_+)\). Therefore, by Part (c) of Theorem 6.1 (which happily ampliates to the current setting),
\[ S \mathcal{I} \circ (\Sigma \otimes \Omega I_\beta) P^\Omega_t = \mathcal{I} \circ (s_1 \otimes \Omega S) (\Sigma \otimes \Omega I_\beta) P^\Omega_t \]
\[ \subset \mathcal{I} \circ (\Sigma \otimes \Omega I_\beta) (K^\pi \otimes \Omega S) P^\Omega_t \]
so \( S \mathcal{I} \circ P^\Omega_t \subset \mathcal{I} \circ (K^\pi \otimes S) P^\Omega_t \) \((t \in \mathbb{R}_+)\). This gives (i) \(\Rightarrow\) (ii) and, when (i) holds, identity 6.6. Conversely, if (ii) holds then \((V_t^* \otimes I_\beta)|_{[0,t]} \) is in \( \text{Dom} \Sigma_t K^\pi \otimes
S ∩ Dom Σ_t ⊗ I so, by Proposition 1.3 (c), \((V^*_t ⊗ I_Ω)z_{[0,t]} \in \text{Dom}(s_{Ω_t} ⊗ S)(Σ_t ⊗ I_Ω)\) and it follows from Theorem 3.3 that \(z_{[0,t]} \in \text{Dom} S^*\), so (i) holds.

(c) Now assume that \((e_+)\) holds. Lemma 5.2 yields equality in (6.7), so (i) is equivalent to (iii). The equivalence of (iii) and (iv) follows from Lemma 6.3. □

7. Quasifree processes, martingales and integrals

For this section the setup is the same as in Section 4, and we write \(Ξ\) for the domain \(M'ξ\), as in Sections 1 and 2. Quasifree martingales and stochastic integrals are defined and the martingale representation theorem is established.

We rely heavily on the vector-operator linear isomorphisms (1.4) and (1.3), and on the transpose operation on unbounded operators treated in Section 2. Filtrations of \(O_M(Ξ; S)\) and \(O^0_M(Ξ)\), and conditional expectations, are defined by

\[
\begin{align*}
O_M(Ξ; S)_t := \{ T ∈ O_M(Ξ; S) : Tξ ∈ S_t \}, \\
O^0_M(Ξ)_t := O_M(Ξ; S)_t ∩ O^0_M(Ξ), \\
E^Σ_t : O_M(Ξ; S)_t → O_M(Ξ; S), \quad E^Σ_t[T]ξ = PtTξ \quad (t ∈ \mathbb{R}^+).
\end{align*}
\]

Thus \(O_M(Ξ; S)_t = \text{Ran} E^Σ_t\) and \(E^Σ_t[O^0_M(Ξ)] = O^0_M(Ξ)_t\) \((t ∈ \mathbb{R}^+)\). A quasifree process is a family \(X = (X_t)_{t≥0}\) in \(O_M(Ξ; S)\) adapted to the above filtration; it is a quasifree martingale if it satisfies

\[E^Σ_t[X_s] = X_t \quad (s ≤ t),\]

equivalently, \((X_t)_{t≥0}\) is a vector martingale with respect to the filtration \((S_t)_{t≥0}\) (cf. 4.4). Thus, for example, if \(T ∈ O_M(Ξ; S)\) then \((E^Σ_t[T])_{t≥0}\) is a martingale; these are called closed martingales.

Remark. The maps \(E^Σ_t\) induce conditional expectations in the standard sense of Umegaki (norm-one projections) from \(M\) to \(M_t := A ⊗ N_t\) which leave the vector state \(ω_ξ\) invariant. Here \(N_t := W_0(X_t)''\) In general, due to Takesaki’s No Go Theorem, the existence of such conditional expectations is not guaranteed; it rests on the subalgebras being left invariant by the modular automorphism group associated with \((M, ξ)\) (4.2; see Theorem IX.4.2 of [4]).

Write \(P_Σ(k, A, v)\) and \(M_Σ(k, A, v)\) for the collection of quasifree processes, respectively martingales, and set

\[
\begin{align*}
P^1_Σ(k, A, v) := \{ X ∈ P_Σ(k, A, v) : X_t ∈ O^1_M(Ξ) \text{ for all } t ∈ \mathbb{R}^+ \}, \quad \text{and} \\
M^1_Σ(k, A, v) := M_Σ(k, A, v) ∩ P^1_Σ(k, A, v),
\end{align*}
\]

referring to such processes and martingales as adjointable. We are ready to define quasifree stochastic integrals. Recall Corollary 2.3.

Definition. A quasifree integrand is a family \(F = (F_t)_{t≥0}\) in \(M_Σ(M, ξ)_0\) such that

\[F^↓_t ξ ∈ \text{Dom}_{loc} Σ ⊗ Ω I_Ω. \quad (7.1)\]

Write \(I_Σ(k, A, v)\) for the collection of these, and \(I^↓_Σ(k, A, v)\) for the subcollection of adjointable integrands, that is those for which

\[F_t ∈ M^↓_Σ(k, ξ)_0 \text{ for all } t ∈ \mathbb{R}^+ \text{ and } F^↓ := (F^↓_t)_{t≥0} ∈ I_Σ(k, A, v).\]

For \(F ∈ I_Σ(k, A, v)\) define \(Λ^Σ_↓(F) ∈ O_M(Ξ; S)\) by

\[Λ^Σ_↓(F)ξ = T^Σ_↓(F^↓_t)ξ \quad (t ∈ \mathbb{R}^+).\]
Remarks. (i) By Lemma 1.2, the operators of an adjointable quasifree process have common core $\Xi_0 := \mathcal{A}' \otimes \mathcal{W}_2 \cdot \Omega$; those of an adjointable quasifree integrand have common core $k \otimes \Xi_0$.

(ii) The explicit action of quasifree integrals on vectors from the dense subspace $\Xi_0$ is given by a Hitsuda-Skorohod integral ([LM]); it is obtained from commutation relations between Weyl operators and such integrals ([LM]).

(iii) For $F \in \mathcal{L}_c(k, A, v)$ with block matrix form $[L_M^L M^0_0]$, $F^\dagger$ is the family $[L_M^L]$ in $C_{k^{\otimes 2}}(M, \xi) = \mathcal{O}_M(\Xi; k^{\otimes 2} \otimes \mathcal{F})$, and if $F \in \mathcal{I}_c^L(k, A, v)$ then $F^\dagger \in C^*_L(k^{\otimes 2}(M, \xi) \subset \mathcal{O}_M^L(\Xi; k^{\otimes 2} \otimes \mathcal{F})$. The top left zero in the block matrix form of $F$ is available for a time-integral.

(iv) The bottom right zero is related to the fact that there is no number/exchange/gauge process affiliated to the quasifree filtration.

(v) From Proposition 6.3 we have a form of Itô isometry (cf. [BSW2]):

$$\|A^\Sigma_1(F)(\cdot t)\|^2 = \|\Sigma_t V^*_t z_{[0, t]}\|^2, \quad \text{for all } t \in \mathbb{R}_+, \quad \text{where } z := F^\dagger\xi.$$

(vii) Quasifree creation and annihilation integrals are defined by

$$A^*_t(L) + A_t(M) = \Lambda^\Sigma_1(F) \quad \text{where } F = [L_M^L].$$

The proposition below confirms that, for adjointable $L$, $A^*_t(L) \dagger = A_t(L^\dagger)$.

(viii) When $X$ is a complex subspace of $K$, as in the case of squeezed states, quasifree creation and annihilation operators may be formed, and may be viewed as quasifree Wiener integrals:

$$\langle a^*(f) + a(g) \rangle = \Lambda^\Sigma_1(H), \quad \text{so } \langle a^*(f) + a(g) \rangle \xi = \mathcal{I}^\Sigma(h \otimes \xi) \quad (f, g \in K),$$

where

$$H = \left[(f) \otimes I_\Sigma \langle g \rangle \otimes I_\Sigma \right] \quad \text{and } h = \left(\frac{f}{g}\right).$$

(ix) In the gauge-invariant case we have orthogonality of creation and annihilation integrals on the cyclic and separating vector, entailing some simplification in the analysis for that case:

$$A^*_t(L)\xi \perp A_t(M)\xi \quad (t \in \mathbb{R}_+).$$

(x) Under $(e_+)$, the condition of adjointability for $F \in \mathcal{I}_c(k, A, v)$ is equivalent to

$$(F^\dagger\xi)_{[0, t]} \in \text{Dom } K \otimes \mathcal{S} \quad \text{for all } t \in \mathbb{R}_+, \quad \text{which is in turn equivalent to}$$

$$F_t \in \mathcal{M}^L_k(M, \xi_0) \quad \text{for a.a. } t \in \mathbb{R}_+, \quad \text{and}$$

$$(k \otimes \mathcal{S})F^\dagger\xi \quad \text{is locally square integrable.}$$

Example (Exponential martingales). Elementary examples of bounded quasifree martingales are given by

$$E^f_t = e^{t \langle (f_0, \xi) \rangle} \mathcal{W}(f_{[0, t]} \quad (t \in \mathbb{R}_+),$$

where $f \in L^2_{\mathcal{F}^T}(\mathbb{R}_+; k)$ is such that $\iota(f) \in \text{Dom}_{\mathcal{F}^T} \Sigma$ (so $A = \mathcal{C}$ here). These martingales are adjointable, with $(E^f)^\dagger = E^{-\dagger}$, and have the following stochastic integral representation:

$$E^f_t = I_F + \Lambda^\Sigma_1(F) \quad \text{where } F_t = \iota \left[\frac{1}{f(t)} \langle f(t) \rangle\right] \otimes E^f_t \quad (t \in \mathbb{R}_+).$$

In other words, they satisfy the basic quasifree stochastic differential equation

$$dE^f_t = E^f_t dX^f_t \quad E^f_0 = I_F.$$
where $X^f$ is the martingale formed from the field operators $(iR(f_{[0,t]}))_{t \in \mathbb{R}_+}; E^f$ is said to be the \textit{stochastic exponential} of $X^f$.

**Proposition 7.1.** Let $F \in \mathbb{I}_\Sigma(k, A, v)$. Then $\Lambda \Sigma^i(F) \in M_\Sigma(k, A, v)$.

**Proof.** This follows from the fact that, for any $z \in \text{Dom}_{\text{loc}}(\Sigma \otimes \Omega \ I_\beta)$, $\mathcal{I}_\Sigma^z(z)$ is (an Itô-integral process and thus) a vector martingale. □

We conclude with the converse, which may be viewed as confirmation that the general definition of quasifree integrals given here is the correct one.

**Theorem 7.2.** Let $X \in M_\Sigma(k, A, v)$. Then the following hold.

(a) There is a unique $F \in \mathbb{I}_\Sigma(k, A, v)$ such that

$$X_t - X_0 = \Lambda \Sigma^i(F) \quad (t \geq 0). \quad (7.2)$$

(b) The martingale $X$ is adjointable if and only if the operator $X_0$ is adjointable and the integrand process $F$ is adjointable. In this case

$$X_t^\dagger - X_0^\dagger = \Lambda \Sigma^i(F^\dagger) \quad (t \geq 0).$$

**Proof.** (a) Uniqueness follows from uniqueness in Theorem 6.5. Let $x = (X_t \xi)_{t \geq 0}$ be the corresponding vector process in $\mathfrak{H}$. Then, by Theorem 6.5, there is a unique $z \in \text{Dom}_{\text{loc}} \Sigma \otimes \Omega \ I_\beta$ such that $x_t - x_0 = \mathcal{I}_\Sigma^z(z)$ for all $t \in \mathbb{R}_+$. Now define

$$Q_t := |z_t\rangle \in \mathcal{O}_{\mathfrak{H}(\Xi; k_{\Xi} \otimes \mathfrak{H})} = C_{k_{\Xi}}(M, \xi)$$

and, recalling Corollary 2.3, define $F_t \in M_{\Sigma}(M, \xi) \text{ by } F_t^\dagger = Q_t \ (t \in \mathbb{R}_+)$. Then $Q \xi = z \in \text{Dom}_{\text{loc}} \Sigma \otimes \Omega \ I_\beta$ and so $F \in \mathbb{I}_\Sigma(k, A, v)$ and (7.2) holds since

$$\Lambda \Sigma^i(F) \xi = \mathcal{I}_\Sigma^z(Q \xi) = x_t - x_0 = (X_t - X_0) \xi.$$

(b) Now suppose that the operator $X_0$ is adjointable. By Theorem 2.2 the adjointability of the integrand process $F$ is equivalent to

$$Q \xi \text{ is a.e. Dom}(k \otimes S)\text{-valued, and}$$

$$(k \otimes S)Q \xi \in \text{Dom}_{\text{loc}} \Sigma \otimes \Omega \ I_\beta.$$

Since $\pi$ is unitary, $k$ may be replaced by $k^\pi = \pi \circ k$ and so, by Theorem 6.5, this is equivalent to

$$(x_t - x_0) \in \text{Dom} S \text{ for all } t \in \mathbb{R}_+,$$

in which case,

$$Sx_t - Sx_0 = \mathcal{I}_\Sigma^z((k^\pi \otimes S)z) \quad \text{ for all } t \in \mathbb{R}_+.$$  

Thus $F$ is adjointable if and only if $X$ is adjointable, in which case, by Corollary 2.3

$$X_t^\dagger \xi - X_0^\dagger \xi = \mathcal{I}_\Sigma^z((k^\pi \otimes S)Q \xi) = \mathcal{I}_\Sigma^z(F^\dagger \xi) = \Lambda \Sigma^i(F^\dagger) \xi.$$

(b) follows and so the proof is complete. □

**Remark.** If Assumption (e+) also holds then, by Remark (x) following the definition of quasifree integrands, the conditions for $F$ to be adjointable simplify.
Appendix: Unbounded Operators and Tensor Products

In this appendix we collect some basic facts about the behaviour of unbounded linear and conjugate-linear operators under composition, adjoint, orthogonal sum and tensor operations, for ease of reference in the paper.

For compatible densely defined Hilbert space operators we have the following inclusions

\[(S_1 + \lambda S_2)^* \supset S_1^* + \lambda S_2^*, \quad \text{with equality if } S_1 \text{ is bounded,}\]
\[(S_3S_4)^* \supset S_3^*S_4^*, \quad \text{with equality if } S_4 \text{ is bounded,}\] (A.3)

whenever \(S_1 + \lambda S_2\) and \(S_3S_4\) are also densely defined and \(\lambda \in \mathbb{C} \setminus \{0\}\). We refer to (A.3) as the adjoint-product-inclusion relation. We call a Hilbert space operator \(T\), with target \(H\), injective/surjective/bijective if it has that property as a map from \(\text{Dom } T\) to \(H\). Thus if \(T\) is injective then \(T^{-1}\) is the operator given by \(\text{Dom } T^{-1} = \text{Ran } T, Tu \mapsto u\); if \(T\) is closed and bijective then \(T^{-1}\) is everywhere defined and, by the Closed Graph Theorem, bounded — as is usual, we refer to such operators as invertible. Here are some more detailed relations. They each follow, in turn, from the definitions; proofs of (a) and (b) may be found, for example, in [Wei]. Recall that a core for an operator \(T\) is a subspace of its domain which is dense in the graph norm of \(T\).

Lemma A.1. Compatible Hilbert space operators satisfy the following.

(a) Let \(S, B, R, E\) and \(F\) be operators, with \(S\) closable, \(B\) bounded, \(R\) closed and injective with bounded inverse, \(E\) bounded, everywhere defined and bijective, and \(F\) bounded and injective with bounded inverse. Then (when defined)
   (i) \(\overline{SB}\) and \(\overline{RS}\) are closed;
   (ii) if \(BS\) is closable and \(\text{Dom } B \supset \text{Ran } S\) then \(\overline{BS} = \overline{BS}\);
   (iii) \(FSE\) is closable and
       \[\overline{FSE} = F\overline{SE},\]
       in particular, \(F\overline{SE}\) is closed with core \(E^{-1}\text{Dom } S\).

(b) Let \(T\) be a closed and densely defined operator, and let \(D\) be a closed, densely defined and bijective operator. Then (when defined)
   \[(TD)^* = D^*T^*.\]

(c) Let \(S\) be a closable operator and \(V\) an (everywhere defined) isometric operator satisfying \(\overline{SVV^*} \supset \overline{VV^*S}\). Then \(V^*\overline{SV}\) is closed and \(V^*(\text{Dom } S)\) is a core for both \(\overline{SV}\) and \(V^*\overline{SV}\). Moreover, if \(S\) is also densely defined then
   \[(V^*\overline{SV})^* = V^*S^*V.\]

We need to consider tensor products of unbounded operators. The following commonly used notation is convenient. For operators \(T_1\) and \(T_2\), \(T_1 \otimes T_2\) denotes the unique operator \(T\) satisfying
\[\text{Dom } T := \text{Dom } T_1 \otimes \text{Dom } T_2\]
\[T(u_1 \otimes u_2) = T_1u_1 \otimes T_2u_2 \quad (u_1 \in \text{Dom } T_1, u_2 \in \text{Dom } T_2).\]

The elegant proof of part (c) below is from [Wei], it perhaps deserves to be better known; for other proofs, see Section VII.10 of [RS] and Chapter 9 of [SLZ]. Recall that, for an operator \(T\) on \(H\), a vector \(x \in H\) is analytic for \(T\) if \(x \in \bigcap_{n \in \mathbb{N}} \text{Dom } T^n\) and \(\sum_{n \geq 0} (n!)^{-1} \|(T)^nx\| < \infty\), for some \(t > 0\).

Lemma A.2. Let \(T = T_1 \otimes T_2\) for Hilbert space operators \(T_1\) and \(T_2\).
(a) If $T_1$ and $T_2$ are closable then $T$ is too.
(b) If $T_1$ and $T_2$ are closable and densely defined then
   (i) $T^* = \overline{T_1 \otimes T_2}$,
   (ii) $T = (T_1 \otimes T_2)^*$.
(c) If $T_1$ and $T_2$ are essentially selfadjoint then $T$ is too.

Proof. (c) First note that, being densely defined and symmetric, $T$ is closable, $\overline{T}$
is symmetric and $\overline{T} \supset \overline{T_1 \otimes T_2}$. Let $A_1$, $A_2$ and $A$ denote respectively
the space of analytic vectors for the operators $\overline{T_1}$, $\overline{T_2}$ and $\overline{T}$. It is easily verified that
$A \supset A_1 \otimes A_2$. Since a closed symmetric operator is selfadjoint if and only if its
space of analytic vectors is dense ([Ne]: see Theorem X.39 of [RS]), (c) follows.

(b) (ii) follows from (i) by taking adjoints. We prove (i). It is easily seen
that $T^* \supset \overline{T_1 \otimes T_2}$, so $T$ is closable, and that $\overline{T_1 \otimes T_2} \subset \overline{T}$. We must show that
Dom $T_1^* \otimes T_2$ is a core for $T^*$. Suppose therefore that $z \in$ Dom $T^*$ is orthogonal to
Dom $T_1^* \otimes$ Dom $T_2^*$ with respect to the graph inner product of $T^*$; we must show that
$z = 0$. Setting $A := \overline{T_1 \otimes T_2}$, we have $A \subset \overline{T}T^*$ and, for all $u \in$ Dom $A$,

$$0 = \langle z, u \rangle + \langle T^* z, T^* u \rangle = \langle z, (I + A)u \rangle.$$

By (c) $A$ is essentially selfadjoint and so $\overline{T}T^* = A$. Now $I + \overline{T}T^*$ is invertible, so
$I + A$ has dense range and thus $z = 0$, as required.

(a) This follows by applying (b) to the operators obtained by viewing $T_1$, $T_2$ and $T$ as densely defined operators from the Hilbert spaces $\overline{\text{Dom}T_1}$, $\overline{\text{Dom}T_2}$ and $\overline{\text{Dom}T}$
respectively. □

Notation. For closed operators $R_1$ and $R_2$ (following common practice) we set

$$R_1 \otimes R_2 := \overline{R_1 \otimes R_2}.$$

Thus, for closable densely defined operators $T_1$ and $T_2$, we have

$$(T_1 \otimes T_2)^* = T_1^* \otimes T_2^* = \overline{T_1 \otimes T_2}^*.$$  \hfill (A.4)

The useful facts collected together next may all be proved by systematic application of the above two lemmas.

Proposition A.3. For $i = 1, 2$, let $R_i$, $\tilde{R}_i$, $T_i$, $B_i$, $\tilde{B}_i$, $E_i$ and $F_i$ be Hilbert
space operators, with $R_i$ and $\tilde{R}_i$ closed, $T_i$ closed and densely defined, $B_i$ and $\tilde{B}_i$
bounded and everywhere defined, $E_i$ bounded, everywhere defined and bijective, and $F_i$
bounded and injective with bounded inverse, and set

$$R = R_1 \otimes R_2, \quad T = T_1 \otimes T_2, \quad B = B_1 \otimes B_2, \quad E = E_1 \otimes E_2, \quad F = F_1 \otimes F_2,$$

and $\tilde{R} = \tilde{R}_1 \otimes \tilde{R}_2$. Then the following hold (when the compositions are defined):

(a) $RB \supset R_1 B_1 \otimes R_2 B_2$.
(b) $TB = T_1 B_1 \otimes T_2 B_2$ if $T_1 B_1$ and $T_2 B_2$ are densely defined.
(c) $RE = R_1 E_1 \otimes R_2 E_2$.
(d) If $BR$, $B_1 R_1$ and $B_2 R_2$ are closable then $\overline{BR} = B_1 R_1 \otimes B_2 R_2$, in particular,
   $FR = F_1 R_1 \otimes B_2 R_2$.
(e) $T = (T_1 \otimes I_2)(I_1 \otimes T_2)$, and if either $T_1$ is injective with bounded inverse,
   or $T_2$ is bounded, then $(T_1 \otimes I_2)(I_1 \otimes T_2)$ is closed, so $T = (T_1 \otimes I_2)(I_1 \otimes T_2)$.
(f) If $R_1 B_1 \supset B_1 \tilde{R}_1$ then $R(B_1 \otimes I_2) \supset (B_1 \otimes I_2) \tilde{R}$.

The following corollary is also useful.

Corollary A.4. Let $T = T_1 \otimes T_2$ and $U = U_1 \otimes U_2$ where, for $i = 1, 2$, $T_i$ is a
closed and densely defined operator from $H_i$ to $H_i'$, $U_i$ is a closed subspace of $H_i$, and $T_i(U_i \cap \text{Dom} T_i) \subset U_i$. Then $T(U \cap \text{Dom} T) \subset U$. 

Proof. Letting $V_1$, $V_2$ and $V$ be the inclusion maps of $U_1$, $U_2$ and $U$ in $H_1$, $H_2$ and $H$ respectively, Part (b) of Proposition \[\ref{prop:conjugate-linear-operator}\] implies that

$$T|_U = TV = T_1V_1 \otimes T_2V_2 = T_1|_{U_1} \otimes T_2|_{U_2},$$

from which the result is evident. \[\square\]

For a sequence of operators $(T_n)$ from $H_n$ to $H'_n$, an operator $T = \bigoplus T_n$ from $H = \bigoplus H_n$ to $H' = \bigoplus H'_n$ is defined in the obvious way:

$$\text{Dom } T = \left\{ \xi \in H : \forall n \geq 0 \sum_n \|T_n\xi_n\|^2 < \infty \right\}, \quad T\xi = (T_n\xi_n).$$

Elementary properties of this construction include the following:

* If each $T_n$ is closed then so is $T$.
* If each $T_n$ is densely defined then so is $T$, and $T^* = \bigoplus T_n^*$.
* If each $T_n$ has core $C_n$ then $T$ has core $\bigoplus C_n$ (algebraic sum).
* $T$ is bounded if and only if each $T_n$ is bounded and $\sup_n \|T_n\| < \infty$.

Recall the notation $E(S) := \text{Lin}\{\varepsilon(v) : v \in S\}$. For a closed operator $R$ from $h_1$ to $h_2$, operators from $\Gamma(h_1)$ to $\Gamma(h_2)$ are defined by

$$\Gamma(R) := \bigoplus R^{(n)}, \quad \text{where, for } n \geq 0, \quad R^{(n)} := V_n^* R^{\otimes n} V_n, \quad \text{and} \quad \Gamma(R|_1) := \Gamma(R) \varepsilon(\text{Dom } R),$$

where $h_{1,2}$ and $V_{1,2}$ are the inclusions $h_1^{\otimes n} \to h_1^{\otimes n}$ and $h_2^{\otimes n} \to h_2^{\otimes n}$.

**Proposition \[\ref{prop:second-quantisation}\].** Let $R$, $S$ and $T$ be operators from $h_1$ to $h_2$ such that $S$ is densely defined, $R$ is closed and $T$ is closed and densely defined, and let $C$ be an everywhere defined contraction operator from $h_0$ to $h_1$. Then the following hold.

(i) $\Gamma(R)$ is closed.
(ii) If $C$ is a core for $R$ then $E(C)$ is a core for $\Gamma(R)$.
(iii) $\Gamma(S^*) = \Gamma(S^*)$.
(iv) $\Gamma(C)$ is an everywhere defined contraction operator.
(v) $\Gamma(RC) \subset \Gamma(R)\Gamma(C)$.
(vi) $\Gamma(TC) = \Gamma(T)\Gamma(C)$, when $TC$ is densely defined.

**Proof.** (i), (iii) and (iv) follow easily from the elementary properties of orthogonal sums of operators listed above. (ii) follows from the fact that $E(C)$ is dense in $\Gamma(h^*)$, where $h^*$ denotes $\text{Dom } R$ in the graph norm of $R$, and this in turn implies (v), in view of the obvious inclusion

$$\Gamma(RC) \subset \Gamma(R)\Gamma(C),$$

and the closedness of the RHS (by Part (a) of Lemma \[\ref{lem:closedness}\]).

(vi) follows from (v) and the fact that $T^{\otimes n}C^{\otimes n} = (TC)^{\otimes n}$ $(n \in \mathbb{N})$, cf. Part (b) of Proposition \[\ref{prop:conjugate-linear-operator}\]. \[\square\]

**Remark.** For an everywhere-defined contraction operator $C$, $\Gamma(C)$ is known as the *second quantisation* of $C$ (see, for example, \[\cite{RS1}\]).

We also need to consider conjugate-linear operators, including the Tomita-Takesaki operators associated with a von Neumann algebra with cyclic and separating vector. Thus, for a conjugate linear operator $T$ from $H_1$ to $H_2$ with domain $D$, its *adjoint* is the conjugate-linear operator from $H_2$ to $H_1$ defined as follows:

$$\text{Dom } T^* := \{ x \in h' : \text{ the linear functional } u \in D \to \langle Tu, x \rangle \text{ is bounded} \}$$

$$\langle T^* x, u \rangle = \langle Tu, x \rangle \quad (u \in D, x \in \text{Dom } T^*).$$
In terms of any antiunitary operator $J : H_2 \rightarrow H_1$, $T J$ being a linear operator with domain $J^{-1}D$. Compositions, orthogonal sums and tensor products of conjugate-linear operators enjoy corresponding properties to those of their linear sisters listed above. Thus, for closable conjugate-linear operators $T_1$ and $T_2$, $T_1 \otimes T_2$ is closable and its closure is denoted $T_1 \otimes T_2$, and $\Gamma(T)$ enjoys the properties listed in Proposition A.5.

Caution. If $T_1$ is a linear operator and $T_2$ a conjugate-linear operator then (except in the trivial case where one is a zero operator) $T_1 \otimes T_2$ makes no sense, let alone $T_1 \otimes T_2$.

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