The quantum affine origin of the AdS/CFT secret symmetry

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Abstract

We find a new quantum affine symmetry of the S-matrix of the one-dimensional Hubbard chain. We show that this symmetry originates from the quantum affine superalgebra \( \hat{U}_q(\mathfrak{gl}(2\mid 2)) \), and in the rational limit exactly reproduces the secret symmetry of the AdS/CFT worldsheet S-matrix.

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1. Introduction

Recent progress in exploring the quantum-deformed one-dimensional Hubbard model [1] was inspired by the construction of the deformed quantum affine algebra \( \hat{Q} \) [2]. The algebra \( \hat{Q} \) may be viewed as the quantum affine lift of the centrally extended superalgebra \( \mathfrak{sl}(2\mid 2) \) that governs the worldsheet S-matrix of the AdS/CFT correspondence\textsuperscript{5} [4, 5]. The central extension makes the situation significantly different from the conventional case. This algebra is described by two parameters, the deformation parameter \( q \) and the coupling parameter \( g \), and it is currently unknown how \( \hat{Q} \) fits into the classification of quantum algebras. For this reason, extra work is required to establish the symmetries of the S-matrix.

The S-matrix is at the core of the integrability of the system effectively describing the spectrum of anomalous dimensions in AdS/CFT. Many of the properties of the worldsheet S-matrix can therefore be reinterpreted in the light of their quantum affine lift. For example, the Yangian symmetry of the worldsheet S-matrix [6] is equivalent to the rational \( q \rightarrow 1 \) limit of \( \hat{Q} \). In a similar way, the S-matrices for bound states (i.e. symmetric short representations) [7, 8] can be obtained from the bound state S-matrices of the deformed Hubbard model [9].

\textsuperscript{5} We will not attempt a self-contained introduction to the topic. For a recent review, we refer to [3]. We will also not specify a real form of the algebras we will treat in this paper, although it will always be possible to make such a choice at any stage.
One of the most peculiar features of the worldsheet $S$-matrix is the so-called secret symmetry, which would normally extend the superalgebra $su(2|2)$ to $gl(2|2)$ [10]. This symmetry was shown to be present only starting from the first Yangian level, since the corresponding Lie algebra charge is not a symmetry of the worldsheet $S$-matrix. Symmetries of a similar origin were found in other studies of AdS/CFT. For instance, they reveal themselves as symmetries of the boundary scattering matrices [11] and also appear as a so-called bonus Yangian symmetry in [12, 13]. Thus, the secret symmetry should perhaps be regarded as an integral part of the symmetries of the model. The need for such an extension seems to respond to a consistency issue of the underlying quantum group description of the integrable structure, following a general prescription by Khoroshkin and Tolstoy [14]. According to this argument, in the case of superalgebras with a degenerate Cartan matrix (as the present one is), one may adopt the $R$-matrix of the smallest non-degenerate algebra containing the original one. The $R$-matrix found in such a way intertwines a fortiori the co-products of the original algebra. This leads to the natural question of whether a similar symmetry is also hidden in the $S$-matrices of the deformed Hubbard model.

A first hint that this is the case is found in the so-called classical limit. For the rational case, the secret symmetry plays a crucial role, where it is needed to achieve factorization of the classical $r$-matrix in the form of a quantum double [15–18]. Similarly, the secret symmetry generator also appears in the factorized expression of the $q$-deformed classical $r$-matrix [19].

Another natural limit to investigate is the ‘conventional’ affine limit of the algebra $\widehat{Q}$. This limit is obtained by sending one of the (complex) parameters of the relevant representation (namely, the coupling constant $g$) to zero, followed by a suitable transformation that removes the twist factors of [20, 21]. In this limit, two of the three central charges of $\widehat{Q}$ vanish; thus, the algebra becomes isomorphic to the conventional quantum affine superalgebra $U_q(\hat{gl}(2|2))$. By adjoining the non-super-traceless Cartan generators $h_{4,0}$ and $h_{4,\pm1}$, one may extend $U_q(\hat{gl}(2|2))$ to $U_q(\hat{gl}(2|2))$. The representations of $U_q(\hat{gl}(2|2))$ can be obtained from [22]. We derive the corresponding co-products, whose structure forms our intuition on what to look for in the case of $\widehat{Q}$.

The full $\widehat{Q}$ $r$-matrix in this conventional limit is naturally found to have $U_q(\hat{gl}(2|2))$ symmetry. In other words, we automatically find an extended symmetry in this limit, corresponding to the operators $h_{4,i}$. However, at non-zero $g$, we see the appearance of the same phenomenon as in the rational case: the level one non-super-traceless generator is once again a symmetry, while the level zero is not. We find two secret symmetries which we call $B_E$ and $B_F$, and which extend to all the bound state $\widehat{Q}$ $S$-matrices. More precisely, while these symmetries are an analogue of the Cartan generators $h_{4,\pm1}$ of $U_q(\hat{gl}(2|2))$, they get promoted to full $\widehat{Q}$ symmetries only in specific linear combinations. In the rational $q \to 1$ limit, they exactly reproduce the secret symmetry of the worldsheet $S$-matrix [10]. We have checked these facts numerically for the total bound state number $M_1 + M_2 \leq 5$.

The paper is organized as follows. In section 2, we review the Chevalley–Serre and Drinfeld’s second realization of the quantum affine superalgebra $U_q(\hat{gl}(1|1))$. We consider its fundamental representation and give the explicit realization of the corresponding $R$-matrix and the non-super-traceless charges $h_{2,0}$ and $h_{2,\pm1}$. This can be considered both as a warm-up exercise and as a treatment relevant to a wealth of subsectors of the full algebra and corresponding $R$-matrix, later discussed in section 4. In section 3, we review the superalgebra $U_q(\hat{gl}(2|2))$ and its fundamental representation, and give the necessary background for building the secret symmetry of $\widehat{Q}$. In section 4, bearing on the construction presented in section 3, we

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6 With ‘being a symmetry’ we mean that the $R$-matrix intertwines the co-product of the corresponding generator, as will be explained in the main text.
build the secret symmetry of the bound state $S$-matrices of $\hat{Q}$ in both the conventional limit ($g \to 0$) and the full case of $\hat{Q}$. In conclusions, we present comments on our findings and possible future directions of investigation.

Throughout this paper, we assume the reader to be familiar with the terminology of superalgebras (details can be found in [23, 24]) and their $q$-deformed enveloping algebras (see for instance [25, 26]). Specific details concerning the representations of the quantum affine superalgebra $U_q(\hat{gl}(n|n))$ can be found in [22]. However, we will try and provide the necessary formulas relevant to our purposes.

2. The quantum affine superalgebra $U_q(\hat{gl}(1|1))$

In this section, we provide both the Chevalley–Serre realization and the so-called Drinfeld second realization [27] of the quantum affine superalgebra $U_q(\hat{gl}(1|1))$, in the conventions of [22] (see also [28–33]). We choose a complex number $q \neq 0$ and not a root of unity, and define

$$[y]_q = \frac{q^y - q^{-y}}{q - q^{-1}}. \quad (2.1)$$

We will also set the central charge $c$ of the quantum affine algebra to zero for the rest of this section, and generically indicate with $[,]$ the graded (or super-)commutator. We instead reserve the symbol $\{,\}$ for the anti-commutator.

2.1. Chevalley–Serre realization

In the Chevalley–Serre realization, the Lie superalgebra $U_q(\hat{gl}(1|1))$ is generated by fermionic Chevalley generators $\xi^\pm_1$, Cartan generators $h_1, h_2$, with $h_2$ the non-super-traceless element completing the superalgebra $sl(1|1)$ to $gl(1|1)$ and the affine fermionic Chevalley generators $\xi^\pm_0$ and corresponding Cartan generator $h_0$.

The generalized symmetric Cartan matrix is given by

$$(a_{ij})_{0 \leq i,j \leq 2} = \begin{pmatrix}
0 & 0 & -2 \\
0 & 2 & 0 \\
-2 & 0 & 0
\end{pmatrix}. \quad (2.2)$$

Note that this matrix is degenerate, but the Lie superalgebra block $1 \leq i, j \leq 2$ is not.

The defining relations are as follows, for $0 \leq i, j \leq 2$ (Chevalley generators corresponding to the Cartan generator $h_2$ are absent):

$$[h_i, h_j] = 0, \quad [h_i, \xi^\pm_j] = \pm a_{ij} \xi^\pm_j, \quad \{\xi^+_i, \xi^-_j\} = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \quad (2.3)$$

supplemented by a suitable set of Serre relations. We refer to [22] for the explicit form of the Serre relations, as we will instead spell out the complete set of relations in Drinfeld’s second realization, see (2.6).

One can define a Hopf algebra structure with the following co-product, antipode and counit:

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad S(h_i) = -h_i,$$

$$\Delta(\xi^+_i) = \xi^+_i \otimes 1 + q^{h_i} \otimes \xi^+_i, \quad S(\xi^+_i) = -q^{x h_i} \xi^+_i,$$

$$\Delta(\xi^-_i) = \xi^-_i \otimes q^{-h_i} + 1 \otimes \xi^-_i, \quad \epsilon(h_i) = \epsilon(\xi^*_i) = 0. \quad (2.4)$$
2.2. Drinfeld’s second realization

The same algebra is also generated by an infinite set of Drinfeld’s generators, which in some sense make explicit the infinite set of ‘levels’ of the quantum affine algebra obtained, in the Chevalley–Serre realization, by subsequent commutations with the affine generators $\xi_0^\pm$. Drinfeld’s generators are

$$\xi^-_{1,m}, h_i, n, \quad \text{with} \quad i = 1, 2, \quad m, n \in \mathbb{Z}. \quad (2.5)$$

The defining relations are as follows:

$$[h_i, n, h_j, m] = 0, \quad [\xi^+_{1, n}, \xi^-_{1, m}] = \frac{1}{q - q^{-1}} (\psi^+_1 n + m - \psi^-_1 n + m),$$

$$[h_i, 0, \xi^\pm_{1, m}] = \pm a_i \xi^\pm_{1, m}, \quad [\xi^\pm_{1, n}, \xi^\pm_{1, m}] = 0,$$  

$$[h_i, n, \xi^\pm_{1, m}] = \pm \frac{a_{i1}}{n} \xi^\pm_{1, n} + m, \quad \text{for} \quad n \neq 0. \quad (2.6)$$

We have used the definition

$$\psi^\pm_1 (z) = q^{\pm h_1} \exp \left( \pm (q - q^{-1}) \sum_{m=0}^\infty h_{1, k} z^m \right) = \sum_{n \in \mathbb{Z}} \psi^\pm_{1, n} z^n. \quad (2.7)$$

The above expression (2.7) should be understood as defining a generating function for the individual $\psi^\pm_{1, n}$s, which in turn can be obtained by Laurent expanding both sides of the equation and matching the powers of the parameter $z$.

We call ‘level’ the index $n$ of Drinfeld’s generators. One typically introduces a ‘derivation’ operator $d$ that counts the level in the following way:

$$[d, \tau_n] = d \tau_n, \quad (2.8)$$

for any generator $\tau_n$ at level $n$.

The map between the Chevalley–Serre and Drinfeld’s second realization, which constitutes a Hopf algebra isomorphism, is given by the following assignment, for $i = 1, 2$:

$$h_i = h_i, 0, \quad \xi^\pm_i = \xi^\pm_{1, 0},$$

$$h_0 = -h_i, 0, \quad \xi^\pm_0 = \pm \xi^\pm_{1, 0} q^{\pm h_0, 0}. \quad (2.9)$$

where we have used the fact that $a_{11} = 0$. As one can see, the Chevalley generator associated with the positive (respectively, negative) affine root generates the positive (respectively, negative) tower of levels in Drinfeld’s second realization.

The co-algebra structure in Drinfeld’s second realization satisfies the following triangular decomposition, for $n \in \mathbb{Z}$, $n \neq 0$ (for $n = 0$ the co-product can be obtained directly from (2.9) and (2.4)):

$$\Delta(h_{1,n}) = h_{1,n} \otimes 1 + 1 \otimes h_{1,n} \text{ mod } N_- \otimes N_+,$$

$$\Delta(\xi^\pm_1, n) = \xi^\pm_1, n \otimes 1 + q^{\pm \text{sign}(n) h_0} \otimes \xi^\pm_1, n$$

$$+ \sum_{k=\frac{1}{2} \text{sign}(n)}^{[n]-1} \psi^\pm_1, \text{sign}(-k) \otimes \xi^\pm_1, \text{sign}(-k) \text{ mod } N_- \otimes N^2_+,$$

$$\Delta(\xi^-_1, n) = \xi^-_1, n \otimes q^{\pm \text{sign}(n) h_0} + 1 \otimes \xi^-_1, n$$

$$+ \sum_{k=\frac{1}{2} \text{sign}(n)}^{[n]-1} \xi^-_1, \text{sign}(n) \otimes q^{\pm \text{sign}(n) \text{sign}(-k)} \text{ mod } N^2_- \otimes N_+, \quad (2.10)$$

with $N_+$ (respectively $N^2_+$) the left ideals generated by $\xi^\pm_1, n$ (respectively $\xi^\pm_1, n \otimes 1$), with $m, n' \in \mathbb{Z}$. 

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The co-product for the generators $h_{2,n}$ is obtained by imposing that $\Delta$ is an algebra homomorphism, namely that it respects the defining relations (2.6). Making use of (2.10), we obtain for instance
\[
\Delta h_{2,+1} = h_{2,+1} \otimes 1 + 1 \otimes h_{2,+1} + (q^{-2} - q^2)\xi_{1,+1}^- \otimes \xi_{1,0}^+,
\]
\[
\Delta h_{2,-1} = h_{2,-1} \otimes 1 + 1 \otimes h_{2,-1} - (q^{-2} - q^2)\xi_{1,0}^- \otimes \xi_{1,-1}^+.
\]  
(2.11)

### 2.3. Fundamental representation

We provide here what we will call the ‘fundamental evaluation’ representation in Drinfeld’s second realization, as obtained from [22] by specializing to a particular case. By the terminology ‘fundamental evaluation’ representation we mean a representation which coincides with the fundamental representation at level zero, while the level one generators of the quantum affine algebra are obtained by multiplying the entries of the level zero generators by appropriate linear polynomials in a certain (sometimes called ‘evaluation’ or ‘spectral’) parameter $\zeta$. To obtain the corresponding representation in the Chevalley–Serre realization, one can make use of Drinfeld’s map (2.9). For $v_1$ and $v_2$, a bosonic state and a fermionic state, respectively, $\eta_{ij}$ the matrix with one in position $(i,j)$ and zero elsewhere, and $\zeta$ a spectral parameter counting the level, we have for instance
\[
\xi_{1,0}^+ = \eta_{12}, \quad \xi_{1,0}^- = \eta_{21}, \quad h_{1,0} = \eta_{11} + \eta_{22},
\]
\[
h_{2,0} = \eta_{11} - \eta_{22}, \quad h_{2,\pm 1} = \frac{1}{2}(zq)^{\pm 1}[2\eta]\eta_{11} - \eta_{22}),
\]
\[
\xi_{1,\pm 1}^+ = (zq)^{\pm 1}\eta_{12}, \quad \xi_{1,\pm 1}^- = (zq)^{\pm 1}\eta_{21}.
\]  
(2.12)

Derivation (2.8) in this representation is given by $d = \zeta \frac{d}{\zeta}$. The $R$-matrix ($R$) is defined by the requirement that it satisfies the intertwining property
\[
\Delta^{op}(\tau) R = R \Delta(\tau),
\]  
(2.13)

with $\Delta^{op}(\tau)$ being defined as $\Delta(\tau)$ followed by a graded permutation and $\tau$ is any generator of the algebra. In this specific instance, such an $R$-matrix is given (up to an overall factor) by (see also [34])
\[
R = \eta_{11} \otimes \eta_{11} + \frac{\zeta - 1}{q^2 \zeta - q^{-1}}(\eta_{11} \otimes \eta_{22} + \eta_{22} \otimes \eta_{11})
\]
\[
+ \frac{\zeta(q - q^{-1})}{q^2 \zeta - q^{-1}}[(\eta_{21} \otimes \eta_{12} - \frac{w}{z} \eta_{12} \otimes \eta_{21}) + \frac{q^{-1} \zeta - q}{q^2 \zeta - q^{-1}} \eta_{22} \otimes \eta_{22}],
\]  
(2.14)

where $\zeta$ and $w$ are the spectral parameters corresponding to the first and second copies of the algebra, respectively.

Finally, we want to translate expressions (2.11) into the Chevalley–Serre realization, as this shall be important to us later on. This can be done with the help of (2.9). However, the charges $h_{2,\pm 1}$ have no canonical image under Drinfeld’s map. For this reason, let us introduce new charges
\[
B_{\pm} = \frac{(zq)^{\pm 1}}{q^{-1} - q} h_{2,0}.
\]  
(2.15)

In the Chevalley–Serre realization, (2.11) then reads as
\[
\Delta B_+ = B_+ \otimes 1 + 1 \otimes B_+ + 2\xi_{1,+1}^- \otimes \xi_{1,0}^+;
\]
\[
\Delta B_- = B_- \otimes 1 + 1 \otimes B_- + 2\xi_{1,-1}^- \otimes \xi_{1,0}^+.
\]  
(2.16)
3. The quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$

We will now specialize the presentation of [22] to the case of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$. While the previous section is strictly related to certain subsectors of the $q$-deformed AdS/CFT algebra (which we will treat in the second part of the paper), this section is related to the full algebra and corresponding $R$-matrix. We will directly focus on Drinfeld’s second realization for simplicity, referring to [22] for further details (see also [35]).

3.1. Drinfeld’s second realization

The algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ (for an all-fermionic Dynkin diagram) is generated by an infinite set of Drinfeld’s generators

$$\xi^{\pm}_{i,m}, h_{j,n}, \quad \text{with} \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4, \quad m, n \in \mathbb{Z}. \quad (3.1)$$

The defining relations are as follows:

$$[h_{j,n}, h_{r,s}] = 0,$$

$$[h_{j,n}, \xi^{\pm}_{i,m}] = \pm a_{ji} \xi^{\pm}_{i,m},$$

$$[h_{j,n}, \xi^{\pm}_{i,m}] = \pm \frac{[a_{im} n]}{n} \xi^{\pm}_{i,n+m}, \quad n \neq 0,$$

$$[\xi^{+}_{i,n}, \xi^{-}_{j,m}] = \frac{\delta_{ij}}{q - q^{-1}} (\psi^+_{i,n+m} - \psi^-_{i,n+m}), \quad (3.2)$$

combined with a suitable set of Serre relations [22] which read

$$[\xi^{\pm}_{i,m}, \xi^{\pm}_{i,n}] = 0, \quad \text{if} \quad a_{ii} = 0,$$

$$[\xi^{\pm}_{i,m+1}, \xi^{\mp}_{i,n}] q^{a_{ii}} = [\xi^{\mp}_{i,m+1}, \xi^{\pm}_{i,n}] q^{-a_{ii}},$$

$$(\xi^{\pm}_{2,m}, \xi^{\pm}_{3,n}) q, [\xi^{\pm}_{2,m}, \xi^{\mp}_{3,n}] q^{-1} = [[\xi^{\pm}_{2,m}, \xi^{\pm}_{3,n}] q, [\xi^{\pm}_{2,m}, \xi^{\pm}_{3,n}] q^{-1}]. \quad (3.3)$$

The symmetric Cartan matrix reads

$$(a_{ij})_{1 \leq i,j \leq 4} = \begin{pmatrix}
0 & 1 & 0 & 2 \\
1 & 0 & -1 & -2 \\
0 & -1 & 0 & 2 \\
2 & -2 & 2 & 0
\end{pmatrix}. \quad (3.4)$$

We have once again used the definition

$$\psi^\pm_i (z) = q^{\pm h_{i,0}} \exp \left( \pm (q - q^{-1}) \sum_{m > 0} h_{i,\pm m} z^{\pm m} \right) = \sum_{n \in \mathbb{Z}} \psi^\pm_{i,n} z^{-n}. \quad (3.5)$$

The ‘derivation’ operator $d$ counting the level is once again introduced in the following way:

$$[d, \tau_n] = n \tau_n. \quad (3.6)$$

for any generator $\tau_n$ at level $n$.

Let us comment on the Serre relations (3.3). The first line expresses the fermionic nature of the generators associated with the simple roots, while the second one ensures that a good filtration is preserved. This means that one is free to combine levels in different ways to obtain one and the same ‘sum’ level as a result. The third line, taken at level 0 (namely, for $m = n = p = r = 0$), tells us that there are only three generators associated with the non-simple roots, two obtained as $[\xi^{\pm}_{2,0}, \xi^{\mp}_{1,0}] q$ and $[\xi^{\pm}_{2,0}, \xi^{\mp}_{1,0}] q^{-1}$, and one obtained by commuting, for instance, the very first of these generators with $\xi^{\pm}_{3,0}$. In fact, the third Serre relation implies that commuting the two generators associated with the non-simple roots with each other returns to zero, which truncates any further growth in the number of generators.
The co-product has the natural structure (we define \(\text{sign}(0) \equiv +1\))

\[
\Delta(h_{i,n}) = h_{i,n} \otimes 1 + 1 \otimes h_{i,n} \mod N_- \otimes N_+,
\]

\[
\Delta(\xi^+_{i,n}) = \xi^+_{i,n} \otimes 1 + q^{\text{sign}(n)h_{i,n}} \otimes \xi^+_{i,n} + \sum_{k=1}^{[n]-1} \psi_{i,\text{sign}(n)([n]-k)} \otimes \xi^+_{i,\text{sign}(n)k} \mod N_- \otimes N_+^2,
\]

\[
\Delta(\xi^-_{i,n}) = \xi^-_{i,n} \otimes q^{\text{sign}(n)h_{i,n}} + 1 \otimes \xi^-_{i,n} + \sum_{k=1}^{[n]-1} \xi^-_{i,\text{sign}(n)k} \otimes \psi_{i,\text{sign}(n)([n]-k)} \mod N_-^2 \otimes N_+,
\] (3.7)

with \(N_h\) (respectively \(N^2_h\)) the left ideals generated by \(\xi^\pm_{i,m}\) (respectively \(\xi^\pm_{i,m} \xi^\pm_{i,m'}\)), with \(m, m' \in \mathbb{Z}\) and \(i = 1, 2, 3\).

The co-product for the generators \(h_{4,n}\) is obtained by imposing that \(\Delta\) respects the defining relations (3.2). With respect to the case of \(U_q(\hat{\mathfrak{g}}(1|1))\), the ‘tail’ of the co-product (i.e. the quadratic part that comes after the trivial comultiplication rule for the generator itself) now contains generators associated with non-simple roots (which before were simply absent). By carefully taking into account (3.7), we find

\[
\Delta(h_{4,+1}) = h_{4,+1} \otimes 1 + 1 \otimes h_{4,+1} + (q^{-1} - q) \sum_{i=1}^{3} (a_{41})^i \xi^+_{i,+1} \otimes \xi^+_{i,0} + \text{non-simple roots},
\]

\[
\Delta(h_{4,-1}) = h_{4,-1} \otimes 1 + 1 \otimes h_{4,-1} - (q^{-1} - q) \sum_{i=1}^{3} (a_{41})^i \xi^-_{i,-1} \otimes \xi^+_{i,0} + \text{non-simple roots}.
\] (3.8)

We will specify the non-simple part of the tail of the co-product in the fundamental representation in the following section.

### 3.2. Fundamental representation

The fundamental evaluation representation in Drinfeld’s second realization can be obtained from [22] in a particular case. For \(v_1, v_2\) and \(v_3, v_4\), two bosonic and two fermionic states, respectively, with \(h_{ij}\) being the matrix with one in position \((i, j)\) and zero elsewhere, and \(z\) being a spectral parameter counting the level, we have

\[
\begin{align*}
\xi^+_{1,0} &= \eta_{13}, & \xi^+_{2,0} &= \eta_{32}, & \xi^+_{3,0} &= \eta_{24}, \\
\xi^+_{1,1} &= \eta_{31}, & \xi^+_{2,1} &= -\eta_{23}, & \xi^+_{3,1} &= \eta_{42}, \\
\xi^+_{1,2} &= (q^{\pm 1}) \eta_{11} + \eta_{33}, & h_{1,0} &= (\eta_{33} + \eta_{22}), & h_{2,0} &= -(\eta_{33} + \eta_{22}), & h_{3,0} &= (\eta_{22} + \eta_{44}).
\end{align*}
\]

\[
\begin{align*}
h_{4,0} &= \sum_{k=1}^{4} (-1)^{[k]} \eta_{kk}, \\
h_{4,1} &= (zq)^{\pm 1} \eta_{13}, & \xi^+_{2,\pm 1} &= \pm 1 \eta_{32}, & \xi^+_{3,\pm 1} &= (zq)^{\pm 1} \eta_{24}, \\
h_{4,2} &= (zq)^{\pm 1} \eta_{31}, & \xi^+_{2,\pm 1} &= -\pm 1 \eta_{23}, & \xi^+_{3,\pm 1} &= (zq)^{\pm 1} \eta_{42}, \\
h_{4,3} &= z^{\pm 1} (\eta_{11} + \eta_{33}), & h_{2,\pm 1} &= -(zq \pm 1) (\eta_{22} + \eta_{33}), & h_{3,\pm 1} &= z^{\pm 1} (\eta_{22} + \eta_{44}), \\
h_{4,4} &= z^{\pm 1} (2q \eta_{11} + (y^\pm + 1 - q^\mp 1) \eta_{22} + (y^\pm - q^\mp 1) \eta_{33} + (y^\pm + 1 - 2q^\mp 1) \eta_{44}).
\end{align*}
\] (3.9)
with \(|k|\) being the grading of the state \(\eta_k\). Derivation (3.6) in the fundamental evaluation representation (3.9) is given by \(d = z \frac{\delta}{\delta u}\). The algebra \(gl(n|m)\) is non-semisimple (with \(sl(n|m)\) being a non-trivial ideal strictly contained in it). Hence, one can always add at constant times the identity to the non-super-traceless generator which lives outside the ideal (and, therefore, never appears on the right-hand side of any commutation relations). The generator \(h_{\pm,1}\) of the quantum-affine version also does not appear on the rhs of any commutation relations, and one can use the freedom we just mentioned to redefine this generator by adding a multiple of the identity. This is reflected in the choice of \(y^\pm\) (which we tacitly fixed to a convenient value in the previous section). The term multiplying \(y^\pm\) is a multiple of the identity matrix, and its co-product is trivial; hence, it drops out of the defining relation for the \(R\)-matrix (2.13).

Let us spell out co-product (3.8) in this representation \((z \text{ and } w \text{ once again refer to the first and second factors, respectively, in the tensor product}):

\[
\Delta h_{4,1} = h_{4,1} \otimes 1 + 1 \otimes h_{4,1} + (q^{-2} - q^2)z(q\eta_{31} \otimes \eta_{13} + (q - 1)\eta_{21} \otimes \eta_{12} \\
+ (2q - 1)\eta_{41} \otimes \eta_{14} + \eta_{32} \otimes (q - 1)\eta_{43} \otimes \eta_{34} + q\eta_{42} \otimes \eta_{24}).
\]

\[
\Delta h_{4,-1} = h_{4,-1} \otimes 1 + 1 \otimes h_{4,-1} - (q^{-2} - q^2)w^{-1}(q^{-1}\eta_{31} \otimes \eta_{13} + (q - 1)\eta_{21} \otimes \eta_{12} \\
+ (2q^{-1} - 1)\eta_{41} \otimes \eta_{14} + \eta_{32} \otimes (1 - q^{-1})\eta_{43} \otimes \eta_{34} + q^{-1}\eta_{42} \otimes \eta_{24}).
\]

Note that the bosonic part of the tail is higher order in the \(q \rightarrow 1\) limit, and therefore it disappears in the Yangian limit. The parameter \(y\) does not appear in the coefficients of the tail, according to the above discussion. We can once again fix the constant \(y\) to a convenient value, for instance,

\[
y^\pm = q^{\pm 1} - \frac{1}{z},
\]

which produces the following representation:

\[
h_{4,\pm 1} = z^{\pm 1}[2]q^1((q^{-1} - \frac{1}{z})\eta_{11} + \frac{1}{z}\eta_{22} - \frac{1}{z}\eta_{33} - (q^{\pm 1} - \frac{1}{z})\eta_{44}).
\]

The \(R\)-matrix satisfying the interwining property (2.13) is given (up to an overall factor) by (see also [36])

\[
R = \eta_{11} \otimes \eta_{11} + \eta_{22} \otimes \eta_{22} + \frac{q^2 - z}{1 - q^2z}(\eta_{33} \otimes \eta_{33} + \eta_{44} \otimes \eta_{44})
+ \frac{q(1 - \frac{z}{w})}{1 - q^2z} \sum_{i,j} \eta_{ij} \otimes \eta_{ij} - \frac{q^2 - 1}{q^2z - 1} \left( \sum_{(i,j) \in A} \eta_{ij} \otimes \eta_{ij} - \eta_{12} \otimes \eta_{21} - \eta_{32} \otimes \eta_{23} \right) \\
+ \frac{q^2 - 1}{q^2 - \frac{z}{w}} \sum_{(i,j) \in B} \eta_{ij} \otimes \eta_{ji} - \eta_{23} \otimes \eta_{32} - \eta_{43} \otimes \eta_{34}).
\]

As a consistency check, one can note that in the scaling limit \(q = e^h\) and \(z/w = e^{2\Delta h}\) with \(h \rightarrow 0\), the above \(R\)-matrix reduces to the Yangian \(R\)-matrix

\[
R_F = -\frac{\delta u}{\delta u} + 1 \left( 1 + \frac{P}{\delta u} \right),
\]

with \(P\) being the graded permutation operator \(P = \sum_{i,j=1}^{4} (-)^i \eta_{ij} \otimes \eta_{ji} \).

One can show that the combination

\[
B_{\pm} = \frac{q^{\pm 1}}{q^1 - q} \left( \frac{2}{q^{\pm 1}[2]q} h_{4,\pm 1} + (q^{\pm 1} - 1)(h_{1,\pm 1} - h_{3,\pm 1}) \right)
\]
is such that, in the representation \( (3.9) \), one obtains an analogue of \( (2.15) \),
\[
B_\pm = \frac{(qz)_{\pm 1}}{q^{-1} - q} \sum_{j=1}^{4} (-1)^{j}\eta_j.
\]
Then, using \( (3.10) \) and
\[
\Delta h_{1,1} = h_{1,1} \otimes 1 + 1 \otimes h_{1,1} + (q^{-1} - q)z(\eta \otimes \eta)_{b},
\]
\[
\Delta h_{1,-1} = h_{1,-1} \otimes 1 + 1 \otimes h_{1,-1} - (q^{-1} - q)w^{-1}(\eta \otimes \eta)_{b},
\]
\[
\Delta h_{3,1} = h_{3,1} \otimes 1 + 1 \otimes h_{3,1} - (q^{-1} - q)z(\eta \otimes \eta)_{b},
\]
\[
\Delta h_{3,-1} = h_{3,-1} \otimes 1 + 1 \otimes h_{3,-1} + (q^{-1} - q)w^{-1}(\eta \otimes \eta)_{b},
\]
we find
\[
\Delta B_+ = B_+ \otimes 1 + 1 \otimes B_+ + 2qz(\eta_{31} \otimes \eta_{13} + \eta_{23} \otimes \eta_{32} + \eta_{41} \otimes \eta_{14} + \eta_{42} \otimes \eta_{24}),
\]
\[
\Delta B_- = B_- \otimes 1 + 1 \otimes B_- + 2(wq)^{-1}(\eta_{31} \otimes \eta_{13} + \eta_{23} \otimes \eta_{32} + \eta_{41} \otimes \eta_{14} + \eta_{42} \otimes \eta_{24}).
\]
\[
(3.19)
\]
As in the previous section, we translate these expressions into the Chevalley–Serre realization. The map between the Chevalley–Serre and Drinfeld second realization, in the fundamental representation which is relevant to the present discussion, is given by the following assignment:
\[
h_{i} = h_{i,0}, \quad \xi_{i}^{\pm} = \xi_{i,0}^{\pm},
\]
\[
h_{0} = -h_{1,0} - h_{2,0} - h_{3,0}, \quad \xi_{0}^{\pm} = \pm(zq)^{\pm 1}[[\xi_{1,0}^{+}, \xi_{2,0}^{+}], \xi_{3,0}^{+}]q_{\pm}^{\pm(h_{1,0} + h_{2,0} + h_{3,0})}.
\]
Thus, with the help of \( (3.9) \), we find
\[
\Delta B_+ = B_+ \otimes 1 + 1 \otimes B_+ + 2(\xi_{0}^{+} k_{123} \otimes \xi_{123}^{+} + \xi_{012}^{+} k_{3} \otimes \xi_{3}^{+}
\]
\[ - q^{2} \xi_{013}^{+} k_{2} \otimes \xi_{2}^{+} + \xi_{230}^{+} k_{1} \otimes \xi_{1}^{+})
\]
\[
\Delta B_- = B_- \otimes 1 + 1 \otimes B_- + 2(\xi_{123}^{+} k_{12} \otimes \xi_{3}^{+} + \xi_{5}^{+} k_{3} \otimes \xi_{123}^{-})
\]
\[ - q^{-2} \xi_{2}^{-} \otimes \xi_{230}^{+} k_{123} + \xi_{1}^{-} \otimes k_{1}^{-1} \xi_{230}^{+})
\]
\[
(3.21)
\]
where we have used the short-hand notations \( k_{jk} = k_{j}k_{j}k_{k} \) and \( \xi_{ijk} = [[\xi_{i}, \xi_{j}], \xi_{k}] \). One can observe that these expressions can formally be written as
\[
\Delta B_+ = B_+ \otimes 1 + 1 \otimes B_+ + 2 \sum_{a \in \Phi_{0}} c_{a} \xi_{-a} k_{a} \otimes \xi_{a},
\]
\[
\Delta B_- = B_- \otimes 1 + 1 \otimes B_- + 2 \sum_{a \in \Phi_{0}} c_{a} \xi_{-a} \otimes k_{a}^{-1} \xi_{-a},
\]
\[
(3.22)
\]
where \( \Phi_{0} \) is the set of all positive non-affine roots, \( \delta \) is the affine root and \( c_{a} \)'s are complex parameters.

Let us make a final remark concerning the symmetry we have just obtained. We derived co-product \( (3.19) \) starting from an all-fermionic Dynkin diagram, and the pattern of simple and non-simple roots which appear in the tail of the co-product respects the original choice of the Dynkin diagram. For later purposes, it will turn out to be convenient to work with a so-called distinguished Dynkin diagram. This is associated with a basis with only one fermionic root. The assignment of simple roots will be different and this will reflect on the appearance of the generators associated with non-simple roots in the tail. In order to be able to match with the
expressions we will later find, it is useful to perform a twist of the co-algebra structure (and of the corresponding \( R \)-matrix) in the spirit of \([26]\) (see also \([30]\)), where it is explained that such twists may involve factors of the universal \( R \)-matrix itself. One can check that the following transformation
\[
\Psi = \text{Id} - (q - q^{-1})(\eta_{23} \otimes \eta_{32} + \eta_{32} \otimes \eta_{23}) - \frac{w}{z} \eta_{22} \otimes \eta_{33} - \frac{z}{w} \eta_{33} \otimes \eta_{22} - \eta_{34} \otimes \eta_{43} - \eta_{43} \otimes \eta_{34} - \eta_{33} \otimes \eta_{44} - \eta_{44} \otimes \eta_{33}
\]
(3.23)
is such that
\[
\Delta' = \Psi \Delta \Psi^{-1} \quad \text{and} \quad R' = \Psi \Psi R \Psi^{-1}
\]
(3.24)
give
\[
\Delta'(B_+) = B_+ \otimes 1 + 1 \otimes B_+ + 2zq(\eta_{31} \otimes \eta_{13} + \eta_{32} \otimes \eta_{23} + \eta_{41} \otimes \eta_{14} + \eta_{42} \otimes \eta_{24}),
\]
\[
\Delta'(B_-) = B_- \otimes 1 + 1 \otimes B_- + \frac{2}{wq} (\eta_{31} \otimes \eta_{13} + \eta_{32} \otimes \eta_{23} + \eta_{41} \otimes \eta_{14} + \eta_{42} \otimes \eta_{24}),
\]
(3.25)
which is an analogue of (3.19) for the case of the distinguished Dynkin diagram. The inverse of (3.23) can be explicitly calculated, and it reads
\[
\Psi^{-1} = \text{Id} + \tau_1 \eta_{22} \otimes \eta_{33} + \tau_2 \eta_{33} \otimes \eta_{22} + \tau_3 \eta_{23} \otimes \eta_{32} + \tau_4 \eta_{32} \otimes \eta_{23},
\]
(3.26)
with
\[
\tau_1 = -((1 - w/z) + (q^{-1} - q)^2) \omega^{-1}, \quad \tau_3 = \tau_4 = (q - q^{-1}) \omega^{-1},
\]
\[
\tau_2 = -((1 - z/w) + (q^{-1} - q)^2) \omega^{-1},
\]
(3.27)
and
\[
\omega = (1 - z/w)(1 - w/z) + (q^{-1} - q)^2.
\]
(3.28)

The non-super-traceless generator we have been focusing our attention on is what will be promoted to the secret symmetry of the full \( q \)-deformed AdS/CFT model in the next section. While in the conventional case we have just been treating, this generator literally extends the superalgebra \( su(2|2) \) to \( gl(2|2) \), it will instead only appear at the first quantum-affine level in the subsequent treatment, in parallel to the rational case. The need for such an extension is however the same as in the conventional situation. Its presence corresponds to a consistency issue of the underlying quantum group description of the integrable structure, according to the prescription of Khoroshkin and Tolstoy \([14]\). In their analysis, an additional Cartan generator is needed to invert the otherwise degenerate Cartan matrix. In turn, the invertibility of the Cartan matrix allows one to write down the universal \( R \)-matrix, which appears to be in an exponential form with precisely the inverse Cartan matrix appearing at the exponent (see also \([37]\)).

4. Deformed quantum affine algebra \( \hat{Q} \)

Having explored the fundamental evaluation representations of the algebras \( \mathcal{U}_q(\hat{gl}(1|1)) \) and \( \mathcal{U}_q(\hat{gl}(2|2)) \), we are now ready to turn to the quantum affine algebra \( \hat{Q} \) constructed in \([2]\). We start by reviewing its bound state representations, put forward in \([9]\). Then, bearing on the construction presented in the previous sections, we build the secret symmetry of the representations of \( \hat{Q} \) considered in \([9]\). Finally, we show that this new symmetry is a quantum analogue of the secret symmetry discovered in \([10]\).
4.1. Chevalley–Serre realization

The algebra $\tilde{Q}$ of the quantum-deformed one-dimensional Hubbard chain is a double deformation of the centrally extended affine superalgebra $\tilde{\mathfrak{sl}}(2|2)$ whose Dynkin diagram has two bosonic (1, 3) and two fermionic (2, 4) nodes [2]. It is generated by four sets of Chevalley generators $K_i \equiv q^{\alpha_i}E_i, F_i$ ($i = 1, 2, 3, 4$) and two sets of central elements $U_k$ and $V_k$ ($k = 2, 4$), with $U_k$ being responsible for the so-called braiding of the co-product.

Let us start by recalling the symmetric matrix $DA$ and the normalization matrix $D$ associated with the Cartan matrix $A$ for $\tilde{\mathfrak{sl}}(2|2)$:

$$DA = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \quad D = \text{diag}(1, -1, -1, -1). \quad (4.1)$$

The algebra is then defined accordingly by the following commutation relations:

$$K_i E_j = q^{DA_{ij}} F_j K_i, \quad K_i F_j = q^{-DA_{ij}} F_j K_i,$n

$$\{E_2, F_4\} = -\tilde{g}\tilde{a}^{-1}(K_4 - U_2 U_4^{-1} K_2^{-1}), \quad \{E_4, F_2\} = \tilde{g}\tilde{a}^{-1}(K_2 - U_4 U_2^{-1} K_4^{-1}),$$

$$\{E_j, F_j\} = D_{jj} q^{-1} \delta_j - K_j^{-1}, \quad \{E_i, F_j\} = 0, \quad i \neq j, \ i + j \neq 6. \quad (4.2)$$

These are supplemented by a set of Serre relations ($j = 1, 3$):

$$[E_j, [E_j, E_k]] - (q - 2 + q^{-1}) E_j E_k E_j = 0, \quad [E_1, E_3] = E_3 E_2 = E_4 E_4 = [E_2, E_4] = 0,$$

$$[F_j, [F_j, F_k]] - (q - 2 + q^{-1}) F_j F_k F_j = 0, \quad [F_1, F_3] = F_3 F_2 = F_4 F_4 = [F_2, F_4] = 0. \quad (4.3)$$

The central elements are linked to the symmetric matrix $D$ and the central element $g_{\tilde{a}}^{-1}$ (4.4) as follows:

$$[[E_1, E_3], [E_1, E_3]] - (q - 2 + q^{-1}) E_1 E_3 E_1 E_3 = g_{\tilde{a}}^{-1} (1 - V_k^2 U_k^2),$$

$$[[F_1, F_3], [F_3, F_3]] - (q - 2 + q^{-1}) F_1 F_3 F_1 F_3 = g_{\tilde{a}}^{-1} (V_k^{-2} - U_k^{-2}). \quad (4.4)$$

This algebra has three central charges:

$$C_1 = K_1 K_2 K_3,$$

$$C_2 = [[E_2, E_1], [E_2, E_3]] - (q - 2 + q^{-1}) E_2 E_1 E_3 E_2,$$

$$C_3 = [[F_2, F_1], [F_2, F_3]] - (q - 2 + q^{-1}) F_2 F_1 F_3 F_2. \quad (4.5)$$

The central elements $V_k$ are constrained by the relation $K_1^{-1} K_2^{-2} K_3^{-1} = V_k^2.$

**Hopf algebra.** The elements $X \in \{1, K_j, U_k, V_k\}$ ($j = 1, 2, 3, 4$ and $k = 2, 4$) satisfy a standard group-like comultiplication rule defined by $\Delta(X) = X \otimes X$, while for the remaining Chevalley–Serre generators the co-product is deformed by the central elements $U_k$. Similar considerations work for the antipode $S$ and co-unit $\epsilon$. Summarizing, we have [7]

$$\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U_2^{\pm k;2} U_4^{\pm k;4} \otimes E_j, \quad \Delta(F_j) = F_j \otimes K_j + U_2^{\pm k;2} U_4^{\pm k;4} \otimes F_j. \quad (4.6)$$

[7] Note that these co-products differ from the ones in (2.4) not only by the $U$-deformation, but also by the fact that the elements $K_i$ in this section appear now with the inverse power. However, this difference does not play any significant role and merely represents different choices of twist of the algebra. We hope that this shall not bring much confusion to the reader. We have kept this different choice of the twist in order to be consistent and facilitate the comparison with the references we are relying on in the various sections.
Representation. We shall be using the \( q \)-oscillator representation constructed in [9]. The bound state representation is defined by vectors

\[
|m, n, k, l \rangle = (a^+_1)^m(a^+_2)^n(a^+_3)^k(a^+_4)^l |0 \rangle ,
\]

where the indices 1 and 2 denote the bosonic oscillators and 3 and 4 denote the fermionic ones. The total number of excitations \( k + l + m + n = M \) is the bound state number and the dimension of the representation is \( \text{dim} = 4M \). This representation constrains the central elements as \( U := U_2 = U^{-1}_2 \) and \( V := V_2 = V^{-1}_2 \), and describes a spin-chain excitation with quasi-momentum \( p \) related to the deformation parameter as \( U = e^{ip} \).

The triples corresponding to the bosonic and fermionic \( U_q(a(2)) \) in this representation are given by

\[
H_{1}\{m, n, k, l \} = (l - k)\{m, n, k, l \}, \quad H_{2}\{m, n, k, l \} = (n - m)\{m, n, k, l \},
\]

\[
E_{1}\{m, n, k, l \} = [k]_q\{m, n, k - 1, l + 1 \}, \quad E_{2}\{m, n, k, l \} = [m + 1, n - 1, k, l],
\]

\[
F_{1}\{m, n, k, l \} = [l]_q\{m, n, k + 1, l - 1 \}, \quad F_{2}\{m, n, k, l \} = \{m - 1, n + 1, k, l \}.
\]

The supercharges act on basis states as

\[
H_{2}\{m, n, k, l \} = -\left\{ C - \frac{k - l + m - n}{2} \right\} \{m, n, k, l \},
\]

\[
E_{2}\{m, n, k, l \} = a(-1)^m[l]_q\{m, n + 1, k - 1, l \} + b\{m - 1, n, k + 1, l \},
\]

\[
F_{2}\{m, n, k, l \} = c[k]_q\{m + 1, n - 1, k - 1, l \} + d(-1)^m[m - 1, n, k, l + 1].
\]

Here, \([n]_q = (q^n - q^{-n})/(q - q^{-1})\) and \( C \) is related to the central element \( V = q^C \) and represents the energy of the state. The representation labels \( a, b, c, d \) satisfy constraints

\[
\frac{ab}{[M]_q} = \frac{g}{[M]_q}(1 - U^2V^2), \quad \frac{cd}{[M]_q} = \frac{g}{[M]_q}(V^{-2} - U^{-2}),
\]

which altogether give the multiplet-shortening (mass-shell) condition

\[
\frac{g^2}{[M]_q^2}(V^{-2} - U^{-2})(1 - U^2V^2) = \frac{(V - q^M V^{-1})(V - q^{-M} V^{-1})}{(q^M - q^{-M})^2}.
\]

The explicit \( x^\pm \) parametrization of the representation labels is

\[
\begin{align*}
  a &= \sqrt{\frac{g}{[M]_q}} \gamma, & b &= \sqrt{\frac{g}{[M]_q}} \frac{a x^+ - x^-}{\gamma}, \\
  c &= \sqrt{\frac{g}{[M]_q}} \frac{\gamma}{a^V} \frac{\text{ig}q^V}{g(x^+ + \xi)}, & d &= \sqrt{\frac{g}{[M]_q}} \frac{\text{ig}^V}{\gamma} \frac{x^+ - x^-}{\xi x^+ + 1},
\end{align*}
\]

where \( \xi = -\text{ig}(q - q^{-1}) \), \( g^2 = g^2/(1 - g^2(q - q^{-1})^2) \) and the parameters \( x^\pm \) satisfy

\[
q^{-M} \left( x^+ + \frac{1}{x^+} \right) - q^M \left( x^- + \frac{1}{x^-} \right) = \left( q^M - \frac{1}{q^M} \right) \left( \xi + \frac{1}{\xi} \right).
\]

The central elements in this parametrization read as

\[
U^2 = \frac{1}{q^M} \frac{x^+ + \xi}{x^- + \xi} = q^M x^+ + \frac{1}{\xi x^- + 1}, \quad V^2 = \frac{1}{q^M} \frac{\xi x^+ + 1}{\gamma x^- + 1} = q^M x^+ + \frac{1}{\xi x^- + 1}.
\]

The action of the affine charges \( H_4, E_4 \) and \( F_4 \) is defined in exactly the same way as for the regular supercharges subject to the following substitutions \( C \to -C \) and
by replacing
\[ V \rightarrow V^{-1}, \quad \chi^\pm \rightarrow \frac{1}{\chi^\pm}, \quad \gamma \rightarrow i\tilde{\alpha} \gamma \quad \alpha \rightarrow \alpha \tilde{\alpha}^2, \quad \tilde{\alpha} \rightarrow -\frac{1}{\tilde{\alpha}}. \] (4.15)

Finally, we introduce the multiplicative spectral parameter of the algebra
\[ z = \frac{1 - U^2 V^2}{V^2 - U^2}, \] (4.16)
which will play an important role in constructing the secret symmetry.

### 4.2. Conventional affine limit

Before moving to the analysis of the secret symmetry of \( \hat{Q} \), we would like to first consider the conventional affine limit obtained by setting \( g \rightarrow 0 \) [2]. It is going to be a warm-up exercise and also shall serve as a bridge between the secret symmetry of \( \hat{Q} \) and the symmetries of \( \mathcal{U}_q(\hat{\mathfrak{gl}}(2|2)) \) considered in the previous section. In fact, we will prepare all formulas in such a way that it will be easy for the reader to appreciate the cross-over to the full \( q \)-deformed case. Note that the ‘braiding’ by the element \( U \) is preserved in the \( g \rightarrow 0 \) limit, while the Serre relations (4.4) are restored to their usual form. A suitable twist could remove the \( U \)-deformation; however, we choose to keep it to facilitate once again the transition to the AdS/CFT case later on. Thus, we obtain what we will call a ‘\( U \)-deformed’ \( \mathcal{U}_q(\hat{\mathfrak{sl}}(2|2)) \).

**Parametrization.** To find the explicit relation with \( \mathcal{U}_q(\hat{\mathfrak{gl}}(2|2)) \), we need to parametrize the conventional affine limit of \( \hat{Q} \) in terms of the spectral parameter \( z \). This may be achieved by expanding parameters \( \chi^\pm \) in a series of \( g \),
\[ \chi^\pm = \frac{i}{2} q^{M \pm 1} \frac{1}{\sqrt{g}} + \mathcal{O}(g). \] (4.17)

Upon rescaling \( \gamma \rightarrow \tilde{\gamma} \), we find the representation labels to be
\[ a = \gamma, \quad b = 0, \quad c = 0, \quad d = \frac{1}{\tilde{\gamma}}, \]
\[ \tilde{a} = 0, \quad \tilde{b} = \frac{\tilde{\alpha} \tilde{z}}{\tilde{\gamma}}, \quad \tilde{c} = -\frac{\gamma}{\tilde{\alpha} \tilde{z}}, \quad \tilde{d} = 0. \] (4.18)

The central elements of the algebra become
\[ U^2 = U_2^2 = U_4^{-2} = \frac{1 - q^{4M}}{q^M - 1}, \quad V^2 = V_2^2 = V_4^{-2} = q^M. \] (4.19)

**Fundamental representation.** The algebra \( \mathcal{U}_q(\hat{\mathfrak{gl}}(2|2)) \) is larger than the one obtained from \( \hat{Q} \) in the conventional limit due to the presence of the non-super-traceless operators. Let us denote these additional generators originating from \( \mathcal{U}_q(\hat{\mathfrak{gl}}(2|2)) \) as
\[ B_F = \frac{\gamma^{-1}}{\gamma^{1 - q}} B_0, \quad B_E = \frac{\gamma^{-1} q}{\gamma^{-1} - q} B_0 \quad \text{and} \quad B_0 = \text{diag}(1, 1, -1, -1). \] (4.20)
They are equivalent to (3.16) up to the redefinition \( z \mapsto z^{-1} \). The charge \( B_0 \) has a trivial co-product, while the co-products of the charges \( B_{E/F} \) are defined to have the following form:
\[ \Delta B_F = B_F \otimes 1 + 1 \otimes B_F - 2\alpha \tilde{\alpha} \left( U^{-1} F_2 \otimes K_4 F_{123} + U^{-1} F_{143} \otimes K_{43} F_{21} + U^{-1} F_{14} \otimes K_{143} F_{32} + U^{-1} F_{34} \otimes K_{43} F_2, \right), \]
\[ \Delta B_E = B_E \otimes 1 + 1 \otimes B_E - \frac{2}{\alpha \tilde{\alpha}} \left( U^{-1} E_3 K_{34}^{-1} \otimes E_{143} + U^{-1} E_{23} K_{41}^{-1} \otimes E_{41} + U^{-1} E_{12} K_{14}^{-1} \otimes E_{34} + U^{-1} E_{123} K_{41}^{-1} \otimes E_4. \right) \] (4.21)
Here, $K_{ij} = K_i K_j$, $K_{ijk} = K_i K_j K_k$, $E_{ij} = [E_i, E_j]$, $E_{ijk} = [[E_i, E_j], E_k]$ and similar expressions hold for the $F$. The explicit matrix representation is

\[ E_1 = \eta_{11}, \quad E_2 = \tilde{\gamma} \eta_{42}, \quad E_3 = \eta_{34}, \quad E_4 = a \tilde{\alpha} z \eta_{13}, \]
\[ F_1 = \eta_{12}, \quad F_2 = \tilde{\gamma}^{-1} \eta_{24}, \quad F_3 = \eta_{43}, \quad F_4 = -(a \tilde{\alpha} z)^{-1} \eta_{31}, \quad (4.22) \]

and

\[ K_1 = \text{diag}(q^{-1}, q, 1, 1), \quad K_2 = \text{diag}(1, q^{-1}, 1, q^{-1}), \]
\[ K_3 = \text{diag}(1, 1, q^{-1}, q), \quad K_4 = \text{diag}(q, 1, q, 1). \quad (4.23) \]

All three charges $B_0, B_\pm/F$ are symmetries of the $(g \to 0)$ fundamental $S$-matrix of $\hat{Q}$. This is because in this limit, the central charges $C_2$ and $C_3$ vanish and the $S$-matrix becomes equivalent to (3.13) up to the $U$-deformation and similarity transformation (3.23).

The co-products in (4.21) are of the generic form (3.22) and are equivalent to (3.21). Let us be more precise on this equivalence. By removing the $U$-deformation, setting the representation parameters to $\alpha = \tilde{\alpha} = 1$ and mapping the spectral parameter as $z \mapsto z^{-1}$, the above expressions (4.21) exactly coincide with (3.25).

The algebra $\hat{Q}$ has an outer automorphism which flips the nodes 2 and 4 of its Dynkin diagram [2]. This automorphism leads to the ‘doubling’ of the charges (4.20),

\[ B_F \to B_F^\pm = \frac{z q^{\pm 1}}{q^{-1} - q} B_0 \quad \text{and} \quad B_E \to B_E^\pm = \frac{z q^{\pm 1}}{q^{-1} - q} B_0, \quad (4.24) \]

The co-products of $B_E^-$ and $B_F^+$ are given by (4.21), while the co-products of $B_E^+$ and $B_F^-$ are obtained by interchanging indices $2 \leftrightarrow 4$ and inverting the $U$-deformation $U^{-1} \to U$. These new charges shall be important in obtaining the correct Yangian limit. In the following sections, we shall concentrate on the charges $B_F^\pm$ and $B_E^\pm$, or in a shorthand notation $B_E/F$.

**Bound state representation.** Let us lift the definitions presented in the previous paragraph to the case of generic bound state representations. For this purpose, we redefine the charges in (4.24) as

\[ B_F^\pm = \frac{z q^{\pm M}}{q^{-1} - q} B_0, \quad B_E^\pm = \frac{z q^{\pm M}}{q^{-1} - q} B_0 \quad \text{and} \quad B_0 = N_1 + N_2 - N_3 - N_4, \quad (4.25) \]

where $M$ is the bound state number and $N_i$ are the number operators (see [9] for their realization in terms of quantum oscillators). The charge $B_0$ has a trivial co-product. In order to define the explicit realization of the co-products of $B_E/F$ for arbitrary bound states, we need to introduce the notion of (twisted) right adjoint action,

\[ (\text{ad}_r E_i) A = K_i A E_i - (-1)^{[i][A]} K_i E_i A, \]
\[ (\text{ad}_r F_i) A = A F_i - (-1)^{[i][A]} F_i K_i^{-1} A K_i, \]
\[ (\text{ad}_r K_i) A = K_i A K_i^{-1}, \quad (4.26) \]

for any $A \in \hat{Q}$. Here, $(-1)^{[i][A]}$ represents the grading factor of the supercharges. We shall also be using the shorthand notations $\text{ad}_r A_0 \ldots A_i = \text{ad}_r A_0 \ldots \text{ad}_r A_i$ and $E' = K_i E_i$. The right adjoint action is used to define the bound state representation of generators corresponding to non-simple roots in the co-products of charges (4.25). In such a way, we obtain expressions

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8 In terms of (3.22), this automorphism corresponds to the shifting of the affine root $\delta$ from the left to the right factor of the tensor product, and vice versa.
Alternatively, we would also like to point the reader to the asymmetry between the indices possibly be included, and the different representations would only see a part of them survive.

\[ \Delta B_E^\pm = B_E^\pm \otimes 1 + 1 \otimes B_E^\pm \]
\[ - 2 \frac{a \tilde{a}}{} (U^{-1}F_3 \otimes (\text{ad}_{F_3}F_3)K_3 + U^{-1}(\text{ad}_{F_3}F_3)F_3 \otimes F_3K_3^{-1} + U^{-1}(\text{ad}_{F_3}F_3)F_3 \otimes K_{14} + U^{-1}(\text{ad}_{F_3}F_3)K_{14} \otimes (\text{ad}_{F_2}F_2)F_1 \]
\[ - F_3 \otimes ((\text{ad}_{F_2}F_1)F_3)K_3 - (\text{ad}_{F_2}F_1)F_3 \otimes F_3K_3^{-1}) \]
\[ \Delta B_E^\pm = B_E^\pm \otimes 1 + 1 \otimes B_E^\pm \]
\[ - \frac{2}{\alpha \tilde{a}} (UE'_1 \otimes K_4(\text{ad}_{E_3}E_2)E'_1 + U(\text{ad}_{E_1}E_4)E'_1 \otimes E_2 + U(\text{ad}_{E_1}E_3)E'_1 \otimes K_{14} + U(\text{ad}_{E_1}E_3)K_{14} \otimes (\text{ad}_{E_2}E_4)E'_1 \]
\[ - E'_1 \otimes K_3(\text{ad}_{E_2}E_1)E'_1 - (\text{ad}_{E_2}E_1)E'_1 \otimes E_3) \].

(4.27)

The co-products of \( B_{E/F}^\pm \) are obtained from the ones of \( B_{E/F}^\pm \) above in the same fashion as for the fundamental representation, i.e. by interchanging indices \( 2 \leftrightarrow 4 \) and \( U \leftrightarrow U^{-1} \). Note the extra two ‘bosonic’ terms in (4.27) in contrast to (4.21). These terms ensure that \( \Delta B_{E/F}^\pm \) are symmetries of the bound state \( S \)-matrix\(^9\).

We would like to point out that the extra terms in the tail display a quite surprising discrepancy between the two \( \mathcal{U}_q(\mathfrak{sl}(2)) \) subalgebras generated by \( E_1 \), \( F_1 \) and \( E_3 \), \( F_3 \). We do not fully understand the algebraic reason for this fact. The natural explanation would be that the bound state representations manifestly break the symmetry between bosons and fermions and hence between the two \( \mathcal{U}_q(\mathfrak{sl}(2)) \). This means that in the case of the \( S \)-matrix of the anti-bound states (for anti-supersymmetric representations), we might expect the tail to be modified by interchanging indices \( 1 \leftrightarrow 3 \) for the last two terms. For the case of a generic \( R \)-matrix, all four extra terms (the ones in (4.27) plus the ones with indices \( 1 \leftrightarrow 3 \) interchanged) would then possibly be included, and the different representations would only see a part of them survive.

Alternatively, we would also like to point the reader to the asymmetry between the indices \( 1, 2 \) (corresponding to bosons) and \( 3, 4 \) (corresponding to fermions) in (4.23), meaning that these bosonic terms could also be an artifact of the choice of the Dynkin diagram. It would be interesting to gain a better understanding of the origin of this discrepancy.

Finally, we note that \( \Delta B_E^\pm \) is related to \( \Delta B_E^\pm \) by renaming \( E'_i \rightarrow F_i \) and transposing the ordering \( K_iA \rightarrow AK_i \), where \( A \) represents any \( \text{ad}_A \)-type operator; thus, \( E_i \rightarrow F_iK_i^{-1} \).

**Restriction to the \( \mathcal{U}_q(\widehat{\mathfrak{g}}(1|1)) \) subsectors.** The bound state representations of \( \widehat{\mathcal{Q}} \) provided by the vectors (4.7) have four \( \mathcal{U}_q(\widehat{\mathfrak{g}}(1|1)) \)-invariant subsectors. These subsectors are spanned by the vectors
\[ |m, 0, k, 0 \rangle_1, \quad |0, n, 0, l \rangle_II, \quad |0, n, k, 0 \rangle_III, \quad |m, 0, 0, l \rangle_IV, \]
\[ |m, 0, k, 0 \rangle_1, \quad |0, n, 0, l \rangle_II, \quad |0, n, k, 0 \rangle_III, \quad |m, 0, 0, l \rangle_IV, \]
(4.28)

where Roman subscripts enumerate the different subsectors. Each of these subsectors is isomorphic to the bound state representations of the superalgebra \( \mathcal{U}_q(\widehat{\mathfrak{g}}(1|1)) \) considered in section 2. They lead to four independent copies of the corresponding bound state \( S \)-matrix embedded into the (complete) bound state \( S \)-matrix. Thus, one can introduce a formal

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\(^9\) Note that, in the case of the fundamental representation, these symmetries differ from (3.19) for the addition of precisely the above-mentioned bosonic terms. However, these terms are by themselves a symmetry of the \( R \)-matrix in the fundamental representation, and can therefore always be added to the co-product.
the total bound state number up to intertwining property for these new symmetries for the bound states representations with $\Delta^1$.

In this fashion, for each subsector we obtain charges equivalent to (2.16). The last two terms in the tails of (4.27) do not play any role in this case, as they vanish on these subsectors.

4.3. q-deformed AdS/CFT: the secret symmetry

Having prepared all the suitable formulas, we can now turn to the full q-deformed AdS/CFT case. In the previous section, we have explored the symmetries of the conventional affine limit of $\hat{Q}$ whose S-matrix is effectively isomorphic to the one of $\mathcal{U}_q(\mathfrak{gl}(2|2))$, thus the charges $B_0$ and $B^{\pm}_{E/F}$ are proper symmetries. The question we want to answer is whether any of these charges are symmetries of the bound state representations of $\hat{Q}$. Naturally, $B_0$ is not a symmetry. However, we find that the charges $B^{\pm}_{E/F}$ are symmetries of $\hat{Q}$, upon a redefinition

$$ B^+_F = \frac{g^{-1}[M]_q}{U^2 - V^{-2}} B_0, \quad B^+_E = \frac{g^{-1}[M]_q}{U^{-2} - V^{-2}} B_0, $$

$$ B^-_F = \frac{g^{-1}[M]_q}{V^2 - U^{-2}} B_0, \quad B^-_E = \frac{g^{-1}[M]_q}{V^{-2} - U^{-2}} B_0, \quad (4.30) $$

while keeping the form of co-products as in (4.27). We have checked numerically the intertwining property for these new symmetries for the bound states representations with the total bound state number up to $M_1 + M_2 \leq 5$. It is important to note that in the conventional limit these charges exactly reduce to (4.25), and so they correspond to the natural lift of the conventional affine limit case to the generic representations of $\hat{Q}$.

This striking similarity between $B^{\pm}_{E/F}$ is not accidental. The charges $B^+_F$ and $B^-_F$ (and equivalently $B^-_F$ and $B^{+}_F$) are related to each other by the map $U \mapsto U^{-1}$ and $E_i \mapsto F_i$ (as described above) as this is the automorphism of the co-algebra which interchanges lowering and raising Chevalley generators. The relation between $B^+_E$ and $B^-_E$ (and equivalently $B^-_E$ and $B^{+}_E$) corresponds to the algebra automorphism of flipping the nodes 2 and 4 of the Dynkin diagram and represents the symmetry between states (particles) and anti-states (anti-particles), i.e. the corresponding representations are self-adjoint. Thus, $B^{\pm}_{E/F}$ and $B^{\pm}_{E/F}$ are not independent, but rather two isomorphic representations of charges $B_{E/F}$.

An important difference between $\hat{Q}$ and its conventional affine limit is that the previously mentioned extra two ‘bosonic’ terms in (4.27) are no longer a symmetry of the fundamental S-matrix by themselves and thus (4.27) is unique for all bound state representations. Another important difference is that the $\mathcal{U}_q(\mathfrak{gl}(1|1))$-invariant subsectors I–IV become entangled from the algebra point of view. This is because the generators $E_{2/4}$ and $F_{2/4}$ act non-trivially on two subsectors simultaneously, while in the conventional affine limit this was not the case (as can
This is because the rational limit of the linear combinations needs to be modified by identifying the delta functions with indices I and II, and with indices III and IV.

Finally, we can consider the rational limit of the symmetry we have just found. Accordingly, we write \( q \sim 1 + h \) with \( h \to 0 \). In this limit, the secret charges we have constructed become

\[
\begin{align*}
B_F^\pm &= -B_E^\pm - Mx^\pm x^\mp B_0 + \mathcal{O}(h) \quad \text{and} \quad B_E^\pm &= -B_F^\pm = Mx^\pm x^\mp B_0 + \mathcal{O}(h). \quad (4.31)
\end{align*}
\]

Thus,

\[
\lim_{q \to 1} \frac{1}{q} (B_F^+ - B_F^-) = \lim_{q \to 1} \frac{1}{q} (B_E^+ - B_E^-) = i \mu_s B_0, \quad (4.32)
\]

where \( \mu_s = \frac{1}{4} (x^+ - \frac{1}{x^+} + x^- - \frac{1}{x^-}) \) is the rapidity found for the secret symmetry [10]. Subsequently, at the co-algebra level, we find

\[
\lim_{q \to 1} \frac{1}{q} (\Delta B_E^+ - \Delta B_E^-) = \lim_{q \to 1} \frac{1}{q} (\Delta B_F^+ - \Delta B_F^-) = \Delta \hat{\mathcal{B}}, \quad (4.33)
\]

where

\[
\Delta \hat{\mathcal{B}} = \hat{\mathcal{B}} \otimes 1 + 1 \otimes \hat{\mathcal{B}} - \frac{1}{2} (U \hat{\Sigma}_a \otimes \hat{\Sigma}_a + U^{-1} \hat{\Sigma}_a \otimes \hat{\Sigma}_a) \quad \text{and} \quad \hat{\mathcal{B}} = i \mu_s \mathcal{B}, \quad (4.34)
\]

precisely coincides with the secret symmetry of the AdS/CFT S-matrix [10] (we have kept the notation used in [10] here for comparison).

We remark that the outer-automorphism flipping roots 2 and 4, which lead to the doubling of the charges \( B_{E/F} \to B_{E/F}^\pm \), turn out to be crucial in obtaining the secret Yangian charge \( \mathcal{B} \). This is because the rational limit of the linear combinations \( B_E^+ - B_F^+ \) corresponds instead to a bilinear combination of Lie algebra charges plus a central element.

5. Discussion

In this work, we have constructed the so-called secret symmetry of the bound state S-matrices of the deformed Hubbard chain [9]. This new symmetry is represented by the charges \( B_{E/F}^{\pm} \) (4.30) having co-products (4.27), and it is the quantum affine analogue of the secret symmetry of the AdS/CFT S-matrix found in [10]. The nature of this generator can be traced back to the non-super-traceless charges \( h_{\pm 1} \) of the quantum affine superalgebra \( U_q(\hat{\mathfrak{gl}}(2|2)) \). We have checked numerically the intertwining property for these new symmetries for the bound states representations with the total bound state number up to \( M_1 + M_2 \leq 5 \). As a consistency check, we have verified that the symmetry we have found reduces in the Yangian limit to the known secret symmetry of the rational case [10]. We have also checked analytically the intertwining property for all \( \mathfrak{g}_s(1|1) \) subsectors of \( \hat{\mathcal{Q}} \) for generic bound state numbers \( M_1 \) and \( M_2 \).

Finding a realization of this symmetry enhancement in the context of deformations of AdS/CFT gives us a solid base for stating the universality of the secret symmetry, as we can construct the origin of this symmetry of the S-matrix and of its strictly related cousins found in [11–13] as coming from a much more general framework. In this respect, it would be interesting to see how, for example, the twisted secret symmetries of the boundary problem reported in [11] could be lifted to the quantum affine level in the spirit of [39]. Furthermore, it is intriguing to note how the deformation we have been studying has a strong connection with the so-called Pohlmeyer reduction of the string sigma model [40–42], as motivated in [43, 44]. It would be very interesting to investigate whether one can find the presence of the

\[10\]The rational factor \( (q^{-1} - q)^{-1} \) is already included in the definition of the charges, as one can easily trace back using (4.25) and (4.30).
secret symmetry in the Pohlmeyer-reduced model in terms of a non-local classically conserved charge, and what its implications are for the consistency of the model at the quantum level. We do not expect this generator to be a fundamental symmetry of the universal \( R \)-matrix of \( \hat{Q} \) in its present form. Rather, it is likely to be a projection of a more general symmetry of \( \hat{Q} \), with the projection operator being a function of the multiplet-shortening condition (see also [38]). As the universal \( R \)-matrix of \( \hat{Q} \) is not known, it is not possible at the moment to perform such a check for some of its blocks [8]. To gain more data one might need to study a wealth of short representations, starting from the anti-symmetric bound state one, then move to long representations and, possibly, infinite-dimensional representations.

A plausible path to a universal form could also involve considering Drinfeld’s second realization \( \hat{Q} \) based on an all-fermionic Dynkin diagram, and deriving the commutation relations of the secret generator with the supercharges of the algebra. The growth of the algebra is determined by how many independent elements are obtained in this process, and it is very interesting to ask whether any restriction can be put on this growth. It is known that in the rational case new supercharges are generated, which bear a different dependence on the spectral parameters with respect to the original ones [10]. The challenge one already faces in the rational case is precisely how to accommodate this type of relations in the framework of Drinfeld’s. Analogously, one needs to find a way to quantize the classical co-brackets of [17, 19], to which the full quantum relations must tend in the classical limit. We believe that identifying the presence of the secret symmetry into a much wider environment of parametric deformations, as we did in this work, may help resolving certain degeneracies of the strict rational limit, by exploiting the fact that several inequivalent limits can now be taken on the deformation parameter.

Another interesting question is whether such symmetries exist for the higher order quantum affine superalgebras \( \mathcal{U}_q(\mathfrak{psl}(n|n)) \), when \( n > 2 \). This has become even more pressing after the findings of [12], and the potential application to the determination of scattering amplitudes from integrability arguments. There is a powerful interplay between the degeneracy of the Cartan matrix of the relevant Lie superalgebras (\( \mathfrak{psl}(4|4) \) in the case of the amplitudes studied in [12]), the vanishing of the Killing form and the presence of the secret symmetry. In relation to this issue, an intriguing question concerns the role of the exceptional Lie superalgebras \( \mathfrak{D}(2, 1; \alpha) \) and of other superalgebras, like \( \mathfrak{osp}(2n + 2|2n) \), which share with \( \mathfrak{psl}(n|n) \) the feature of a vanishing Killing form, and furnish conformal string theory sigma models [47] (see also [48]). In the case of \( \mathfrak{D}(2, 1; \alpha) \), one may try to establish a deformation of the representation constructed for the rational case in [45], and understand whether it is possible to obtain the secret symmetry in a similar fashion as is done in that paper. In this respect, a first step has been undertaken in [46], where Drinfeld’s second realization of quantum affine \( \mathfrak{D}(2, 1; \alpha) \) has been obtained. The vanishing of the Killing form is a very important element of consistency for the integrable string sigma model. It is fascinating to think that the secret symmetry precisely arises in such a setting, although apparently from a quite different need: the need of consistency of an underlying quantum group with a universal \( R \)-matrix. In fact, if the universal \( R \)-matrix has to be of the Khoroshkin–Tolstoy form [14], an extension of the Cartan subalgebra which allows for an invertible Cartan matrix is in order.

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