Explicit description of all deflators for markets under random horizon

Tahir Choulli and Sina Yansori

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada

October 31, 2018

Abstract

This paper considers an initial market model, specified by its underlying assets $S$ and its flow of information $F$, and an arbitrary random time $\tau$ which might not be an $F$-stopping time. In this setting, our principal goal resides in describing as explicit as possible the set of all deflators, which constitutes the dual set of all “admissible” wealth processes, for the stopped model $S^\tau$. Since the death time and the default time (that $\tau$ might represent) can be seen when they occur only, the progressive enlargement of $F$ with $\tau$ sounds tailor-fit for modelling the new flow of information that incorporates both $F$ and $\tau$. Thanks to the deep results of Choulli et al. [8], on martingales classification and representation for progressive enlarged filtration, our aim is fully achieved for both cases of local martingale deflators and general supermartingale deflators. The results are illustrated on several particular models for $(\tau, S, F)$ such as the discrete-time and the jump-diffusion settings for $(S, F)$, and the case when $\tau$ avoids $F$-stopping times.

1 Introduction

This paper considers an initial market model represented by the pair $(S, F)$, where $S$ represents the discounted stock prices for $d$-stocks, and $F$ is the flow of “public” information which is available to all agents. To this initial market model, we add a random time $\tau$ that might not be seen through $F$ when it occurs. Mathematically speaking, this means that $\tau$ might not be an $F$-stopping time. Thus, for modelling the new flow of information, we adopt the progressive enlargement of $F$ with $\tau$, that we denote throughout the paper by $G$. Hence, our resulting informational market model is the pair $(S^\tau, G)$. This information modelling allows us to apply our obtained results to credit risk theory and life insurance (mortality and/or longevity risk), where the progressive enlargement of filtration sounds tailor-fit, and the initial enlargement of filtration –as in the insider trading framework– is totally inadequate. In fact the death time of an agent can not be seen with certainty before its occurrence, and there is no single financial literature that models the information in the default of a firm $\tau$ as fully seen from the beginning as in the case of insider trading.

For this new market model $(S^\tau, G)$, which includes the two important settings of default and mortality, many challenging questions arise in finance (both theoretical and empirical) and mathematical finance. Most of these questions are still open problems nowadays and are essentially concerned with measuring the impact of $\tau$ on the financial and economical concepts, theories, rules, models, methodologies, ...., etcetera. Among these we cite the (consumption-based) capital asset pricing model (s), equilibrium, arbitrage theory, market’s viability, the fundamental theorem of asset pricing, the optimal portfolios (e.g. the log-optimal portfolio, the numéraire portfolio and other types of portfolios to cite few),

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utility maximization, the various pricing rules, ..., etcetera. The first fundamental question to all these aforementioned problems, lies in the impact of the random time on the market’s viability and the corresponding no-arbitrage concept(s). In virtue of [11], see also [22, 23] for related discussions, the market’s viability in its various weakest form, the existence of the numéraire portfolio, and the no-unbounded-profit-with-bounded-risk (NUPBR) concept are equivalent or intimately related. In this spirit, there were an upsurge interest in studying first the effect of the random time on NUPBR in a series of papers, see [1, 2, 3, 4, 9, 10, 24] for details. This very recent literature answers fully the question when NUPBR is altered for \((S^*, G)\). Some of these papers, especially [2, 3, 4, 24], construct examples of deflators for special and very particular cases (such as when \((S, \mathcal{F})\) is local martingale under the physical probability), while the following question remains open and beyond reach up to now.

How can we describe the set of all deflators for the model \((S^*, G)\)? \hspace{1cm} (1)

The importance of this set and its numerous roles in optimization problems intrinsic to financial problems sound clear and without reproach. Indeed, the set of deflators represents somehow the dual set of all “admissible” wealth processes. No matter what is the optimization criterion, any optimal portfolio corresponds uniquely to an optimal deflator, and they are linked to each other via “some duality form”. Furthermore, in many (probably all) cases even when the utility is nice enough such as log utility, it is more convenient, more efficient, and more easier to solve a dual problem and describe the optimal deflator than getting the optimal portfolio directly. For this latter fact, we refer the reader to [12], where the authors prove that dealing with the dual problem gives sharp and precise results. When considering the impact of \(\tau\) on optimal portfolio, we refer the reader to [13] for direct application of the current paper.

This paper contains four sections including the current one. Section 2 presents the mathematical model and its preliminaries. Section 3 states the explicit parametrization of deflators (that are local martingales) for \((S^*, G)\) in terms of deflators of \((S, \mathcal{F})\) and the “survival” processes associated with the random time. Section 4 addresses the case of general supermartingale deflators, while Section 5 illustrates the results on particular models for the triplet \((\tau, S, \mathcal{F})\). Among these cases, we consider the jump-diffusion and the discrete-times settings for \((S, \mathcal{F})\). The paper contains an appendix where some proofs are relegated and some useful technical (new and existing) results are detailed.

2 Preliminaries

This section defines the notations, the financial and the mathematical concepts that the paper addresses or uses, the mathematical model that we focus on, and some useful existing results. Throughout the paper, we consider the complete probability space \((\Omega, \mathcal{F}, P)\). By \(\mathbb{H}\) we denote an arbitrary filtration that satisfies the usual conditions of completeness and right continuity. For any process \(X\), the \(\mathbb{H}\)-optional projection and dual \(\mathbb{H}\)-optional projection of \(X\), when they exist, will be denote by \(\text{p}^{-}\mathbb{H}X\) and \(\text{p}^{+}\mathbb{H}X\) respectively. Similarly, we denote by \(\text{p}^{-}\mathbb{H}X\) and \(\text{p}^{+}\mathbb{H}X\) the \(\mathbb{H}\)-predictable projection and dual predictable projection of \(X\) when they exist. The set \(\mathcal{M}(\mathbb{H}, Q)\) denotes the set of all \(\mathbb{H}\)-martingales under \(Q\), while \(\mathcal{A}(\mathbb{H}, Q)\) denotes the set of all optional processes with integrable variation under \(Q\). When \(Q = P\), we simply omit the probability for the sake of simple notations. For an \(\mathbb{H}\)-semimartingale \(X\), by \(L(X, \mathbb{H})\) we denote the set of \(\mathbb{H}\)-predictable processes that are \(X\)-integrable in the semimartingale sense. For \(\varphi \in L(X, \mathbb{H})\), the resulting integral of \(\varphi\) with respect to \(X\) is denoted by \(\varphi \cdot X\). For \(\mathbb{H}\)-local martingale \(M\), we denote by \(L_{\text{loc}}^{1}(M, \mathbb{H})\) the set \(\mathbb{H}\)-predictable processes \(\varphi\) that are \(X\)-integrable and the resulting integral \(\varphi \cdot M\) is an \(\mathbb{H}\)-local martingale. If \(\mathcal{C}(\mathbb{H})\) is the set of processes that are adapted to \(\mathbb{H} \in \{\mathcal{F}, G\}\), then \(\mathcal{C}_{\text{loc}}(\mathbb{H})\) is the set of processes, \(X\), for which there exists
a sequence of $\mathbb{H}$-stopping times, $(T_n)_{n \geq 1}$, that increases to infinity and $X^{T_n}$ belongs to $\mathcal{C}(\mathbb{H})$, for each $n \geq 1$. For any $\mathbb{H}$-semimartingale, $L$, we denote by $\mathcal{E}(L)$ the Doeleans-Dade (stochastic) exponential, it is the unique solution to the stochastic differential equation $dX = X \cdot dL$, $X_0 = 1$, given by

$$\mathcal{E}_t(L) = \exp(L_t - \frac{1}{2}(L^2)_t) \prod_{0<s \leq t} (1 + \Delta L_s) e^{-\Delta L_s}.$$ 

Below, we recall the mathematical definition of deflator, where we distinguish the cases of local martingale deflators and general deflators.

**Definition 2.1.** Consider the triplet $(X, \mathbb{H}, Q)$ such that $\mathbb{H}$ is a filtration, $X$ is an $\mathbb{H}$-semimartingale, and $Q$ be a probability. Let $Z$ be a process.

(a) We call $Z$ an $\mathbb{H}$-local martingale deflator for $X$ under $Q$ (or a local martingale deflator for $(X, Q, \mathbb{H})$) if $Z > 0$ and there exists a real-valued and $\mathbb{H}$-predictable process $\varphi$ such that $0 < \varphi \leq 1$ and both processes $Z$ and $Z(\varphi \cdot X)$ are $\mathbb{H}$-local martingales under $Q$. Throughout the paper, the set of all local martingale deflators for $(X, Q, \mathbb{H})$ will be denoted by $Z_{\text{loc}}(X, Q, \mathbb{H})$.

(b) We call $Z$ an $\mathbb{H}$-deflator for $X$ under $Q$ (or a deflator for $(X, Q, \mathbb{H})$) if $Z > 0$ and $Z\mathcal{E}(\varphi \cdot X)$ is an $\mathbb{H}$-supermartingale under $Q$, for any $\varphi \in L(X, \mathbb{H})$ such that $\varphi \Delta X \geq -1$. The set of all deflators for $(X, Q, \mathbb{H})$ will be denoted by $D(X, Q, \mathbb{H})$. When $Q = P$, for the sake of simplicity, we simple omit the probability in notations and terminology.

The rest of this section is divided into two subsections. The first subsection introduces the mathematical model that we are interested in studying, while the second subsection recalls an important martingales representation results.

### 2.1 The mathematical model

Throughout the paper, we consider $(\Omega, \mathcal{F}, \mathbb{P} := (\mathcal{F}_t)_{t \geq 0}, P)$ a filtered probability space satisfying the usual conditions of right continuity and completeness. Here $\mathbb{F}$ is the public flow of information. On this stochastic basis, we suppose given an $\mathbb{F}$-semimartingale, $S$, that represents the discounted price process of risky assets. In addition to this initial market model, we consider a random time $\tau$, that might represent the death time of an agent or the default time of a firm, and hence it might not be an $\mathbb{F}$-stopping time. Throughout the paper, we will be using the following associated non-decreasing process $D$ and the filtration $\mathcal{G} := (\mathcal{G}_t)_{t \geq 0}$ given by

$$D := I_{[\tau, +\infty]}, \quad \mathcal{G}_t := \mathcal{G}_t^0, \quad \text{where} \quad \mathcal{G}_t^0 := \mathcal{F}_t \lor \sigma (D_s, s \leq t). \quad (2)$$

The agent who has access to $\mathbb{F}$, can only get information about $\tau$ through the survival probabilities, called Azéma supermartingales in the literature,

$$G_t := \sigma^\mathcal{G} (I_{[0, \tau]} \mathcal{F}_t) \quad \text{and} \quad \tilde{G}_t := \sigma^\mathcal{G} (I_{[0, \tau]} \mathcal{F}_t). \quad (3)$$

The process

$$m := G + D^\mathbb{F}, \quad (4)$$

is a BMO $\mathbb{F}$-martingale. Then thanks to [8, Theorem 3], the process

$$\mathcal{T}(M) := M^\tau - \tilde{G}^{-1} I_{[0, \tau]} \cdot [M, m] + I_{[0, \tau]} \cdot (\sum \Delta MI_{\{G = 0 < \mathcal{G} \rightarrow \} \mathcal{G}})_{p, \mathbb{P}}, \quad (5)$$

is a $\mathcal{G}$-local martingale for any $M \in \mathcal{M}_{\text{loc}}(\mathbb{F})$. In [8, Theorem 2.3], the authors introduced

$$N^\mathcal{G} := D - \tilde{G}^{-1} I_{[0, \tau]} \cdot D^\mathbb{F}, \quad (6)$$

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which is a $G$-martingale with integrable variation such that $H \cdot N^G$ is a $G$-local martingale with locally integrable variation for any $H$ belonging to

$$T^G_{loc}(N^G, G) := \left\{ K \in \mathcal{O}(F) \mid |K| G^{-1} I_{\{G > 0\}} \cdot D \in \mathcal{A}^+_{loc}(G) \right\}. \quad (7)$$

For $p \in [1, +\infty)$ and a $\sigma$-algebra $\mathcal{H}$ on $\Omega \times [0, +\infty)$, we define $L^p_{loc}(\mathcal{H}, P \otimes D)$ as the set of all processes $X$ for which there exists a sequence of $\mathcal{F}$-stopping times $(T_n)_{n \geq 1}$ that increases to infinity almost surely and $X^{T_n}$ belongs to $L^p(\mathcal{H}, P \otimes D)$ given by

$$L^p(\mathcal{H}, P \otimes D) := \left\{ X \mathcal{H}\text{-measurable} \mid \mathbb{E}[|X_t|^p I_{\{T < +\infty\}}] < +\infty \right\}. \quad (8)$$

### 2.2 A martingale representation under $G$: Choulli et al. [8]

Below we recall (and slightly extend) a martingale representation result of Choulli et al. [8], which plays vital role in our analysis.

**Theorem 2.2.** ([8, Theorems 2.19, 2.22, 2.23]): Suppose $G > 0$. Let $h$ be an element of $L^1(\mathcal{O}(F), P \otimes D)$, and $M^{(h)}$ and $J$ be two processes given by

$$M^{(h)} := o_F\left(\int_0^\infty h_a dD_o^F\right), \quad J := \left(M^{(h)} - h \cdot D^o_F\right)^{-1}. \quad (9)$$

Then the following assertions hold.

(a) The $G$-martingale $H^{(h)} := o_G(h_\tau)_t = \mathbb{E}[h_\tau|G_t]$ satisfies

$$H^{(h)} = H_0^{(h)} + G^{-1}_- \cdot T(M^{(h)}) - J_- G^{-1}_- \cdot T(m) + (h - J) \cdot N^G. \quad (10)$$

(b) For any $G$-martingale $M^G$, there exists a unique triplet $(M^F, \varphi^{(o)}, \varphi^{(pr)})$ belonging to $\mathcal{M}_{0,loc}(F) \times T^G_{loc}(N^G, G) \times L^1_{loc}\left(\hat{\Omega}, \text{Prog}(F), P \otimes D\right)$ such that $\mathbb{E}\left[\varphi^{(pr)} \mid \mathcal{F}_\tau\right] I_{\{\tau < +\infty\}} = 0, \ P\text{-a.s. and}

$$\left(M^G\right)^\tau = M_0^G + G^{-2}_- \cdot T(M^F) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot D. \quad (11)$$

For the sake of a self-contained paper, we outline the key ideas with some details for the proof of this theorem in Appendix [3]. This version, whose proof is less technical than that of [8, Theorems 2.19, 2.22, 2.23] due to $G > 0$, will be used throughout the paper. The following extends slightly Theorem 2.2 (b) to the local martingales setting.

**Theorem 2.3.** Suppose that $G > 0$. Then for any $G$-local martingale $M^G$, there exists a unique triplet $(M^F, \varphi^{(o)}, \varphi^{(pr)})$ satisfying the following properties: $M^F \in \mathcal{M}_{0,loc}(F)$, $\varphi^{(o)} \in T^G_{loc}(N^G, G)$, $\varphi^{(pr)} \in L^1_{loc}(\text{Prog}(F), P \otimes D)$,

$$\mathbb{E}\left[\varphi^{(pr)} \mid \mathcal{F}_\tau\right] I_{\{\tau < +\infty\}} = 0, \ P\text{-a.s.}, \quad (12)$$

and

$$\left(M^G\right)^\tau = M_0^G + G^{-2}_- I_{\lfloor 0, \tau \rfloor} \cdot T(M^F) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot D. \quad (13)$$

**Proof.** Let $M^G \in \mathcal{M}_{0,loc}(G)$, then there exists a sequence of $G$-stopping times that increases to infinity such that $(M^G)^{T_n}$ is a $G$-martingale. On the one hand, due to $G > 0$ and [2] Proposition B.2-(b)], we deduce the existence of $\mathcal{F}$-stopping times $(\sigma_n)_n$ that increases to infinity and $T_n \land \tau = \sigma_n \land \tau$ for any $n \geq 1$. On the other hand, by applying Theorem 2.2 to each $(M^G)^{T_n} - (M^G)^{T_{n-1}}$, we deduce the existence
of a unique triplet \((M^F, n, \varphi^{(o, n)}, \varphi^{(pr, n)})\) belonging to \(\mathcal{M}_{0, \text{loc}}(\mathbb{F}) \times \mathcal{T}_{\text{loc}}^0(N^G, \mathcal{G}) \times L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D)\) and satisfying
\[
\mathbb{E}\left[\varphi^{(pr, n)}_\tau \mid \mathcal{F}_\tau\right] I_{\{\tau < +\infty\}} = 0, \quad P\text{-a.s.,}
\]
and
\[
(M^G)_{\tau \wedge T_n} - (M^G)_{\tau \wedge T_{n-1}} = G^{-2}I_{[0, \tau]} \cdot \mathcal{T}(M^F, n) + \varphi^{(o, n)} \cdot N^G + \varphi^{(pr, n)} \cdot D.
\]
Then notice that \((M^G)_{\tau} - M^G_0 = \sum_{n \geq 1} ((M^G)_{\tau \wedge T_n} - (M^G)_{\tau \wedge T_{n-1}}), \) and put
\[
\sigma_0 := 0, \quad \varphi^{(o)} := \sum_{n \geq 1} I_{[\sigma_{n-1}, \sigma_n]} \varphi^{(o, n)}, \quad \varphi^{(pr)} := \sum_{n \geq 1} I_{[\sigma_{n-1}, \sigma_n]} \varphi^{(pr, n)},
\]
and
\[
M^F := \sum_{n \geq 1} I_{[\sigma_{n-1}, \sigma_n]} \cdot M^F, n.
\]
This ends the proof of the theorem. \(\square\)

We end this section by the following

**Lemma 2.4.** Let \(\sigma\) be an \(\mathbb{H}\)-stopping time. \(Z\) is a deflator for \((X^\sigma, \mathbb{H})\) if and only if there exists unique pair of processes \((K_1, K_2)\) such that \(K_1 = (K_1)^\sigma, \mathcal{E}(K_1)\) is also a deflator for \((X^\sigma, \mathbb{H})\), \(K_2\) is any \(\mathbb{H}\)-local supermartingale satisfying \((K_2)^\sigma \equiv 0, \Delta K_2 > -1, \) and \(Z = \mathcal{E}(K_1 + K_2) = \mathcal{E}(K_1)\mathcal{E}(K_2).\)

The proof of this lemma is straightforward and will be omitted. This lemma shows, in a way or another, that when dealing with the stopped model \((X^\sigma, \mathbb{H})\), there is no loss of generality in focusing on the part up-to-\(\sigma\) of deflectors, and assume that the deflator is flat after \(\sigma\).

### 3 Local martingale deflectors for \((S^T, \mathcal{G})\)

This subsection focuses on describing completely the set of all local martingale deflectors, defined in Definition 2.1 (a), for \((S^T, \mathcal{G})\) in terms of those of \((S, \mathbb{F})\).

**Theorem 3.1.** Suppose \(G > 0\), and let \(K^G\) be a \(\mathcal{G}\)-local martingale. Then the following assertions are equivalent.

(a) \(Z^G := \mathcal{E}(K^G)\) is a \(\mathcal{G}\)-local martingale deflator for \((S^T, \mathcal{G})\) (i.e. \(Z^G \in \mathcal{Z}_{\text{loc}}(S^T, \mathcal{G})\)).

(b) There exists \((K^F, \varphi^{(o)}, \varphi^{(pr)})\) such that \((K^F, \varphi^{(o)}) \in \mathcal{M}_{0, \text{loc}}(\mathbb{F}) \times \mathcal{T}_{\text{loc}}^0(N^G, \mathcal{G}), \varphi^{(pr)} \in L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D)\) and \(\mathbb{E}[\varphi^{(pr)}_\tau \mid \mathcal{F}_\tau] I_{\{\tau < +\infty\}} = 0, P\text{-a.s., and the following three conditions hold:}

(b.1) \(Z^F := \mathcal{E}(K^F)\) is a local martingale deflator for \((S, \mathbb{F})\) (i.e. \(Z^F \in \mathcal{Z}_{\text{loc}}(S, \mathbb{F})\)).

(b.2) The following inequalities hold.
\[
\varphi^{(pr)} > -[G_{-} (1 + \Delta K^F) + \varphi^{(o)} G_{-}] / G_{-}, \quad P \otimes D\text{-a.e.,} \tag{14}
\]
\[
-G_{-} (1 + \Delta K^F) < \varphi^{(o)} < (1 + \Delta K^F) G_{-} / \Delta D_{o, F}, \quad P \otimes D_{o, F}\text{-a.e.} \tag{15}
\]

(b.3) The following decomposition holds
\[
K^G = \mathcal{T}(K^F) - G_{-1} \cdot \mathcal{T}(m) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot D. \tag{16}
\]

**Proof.** The proof will be achieved in two steps. The first step proves that \(\mathcal{E}(K^G)\) is a local martingale for which there exists a \(\mathcal{G}\)-predictable process \(\varphi^G\) satisfying \(0 < \varphi^G \leq 1\) and \(\mathcal{E}(K^G)(\varphi^G \cdot S^T)\) is a \(\mathcal{G}\)-local martingale if and only if there exist \(K^F \in \mathcal{M}_{0, \text{loc}}(\mathbb{F})\) and a triplet
\[
\left(\varphi^{(o)}, \varphi^{(pr)}, \varphi^F\right) \in \mathcal{T}_{\text{loc}}^0(N^G, \mathcal{G}) \times L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D) \times L(S, \mathbb{F})
\]
such that (16) holds, \( \mathbb{E}[\varphi_T^{(pr)} \mid \mathcal{F}_t]I_{\tau < +\infty} = 0 \) P-a.s., \( 0 < \varphi \leq 1 \) and \( \mathcal{E}(K^G)(\varphi \cdot S) \) is an \( \mathbb{F} \)-local martingale. The second step proves that \( \mathcal{E}(K^G) > 0 \) if and only if the triplet \((K^G, \varphi_0, \varphi^{(pr)})\), found in the first step satisfying (16), should fulfill (14)-(15) and \( 1 + \Delta K^G > 0 \).

**Step 1.** Suppose that \( Z^G \) is a local martingale deflator for \((S^r, \mathbb{G})\). Then, on the one hand, thanks to Theorem proof, there exists \((N^G, \varphi_0, \varphi^{(pr)})\) that belongs to \( \mathcal{M}_{0, loc}(\mathbb{F}) \times T_{loc}^0(N^G, \mathbb{G}) \times I_{loc}^1(\text{Prog}(\mathbb{F}), \mathbb{P} \otimes D) \), \( \mathbb{E}[\varphi_T^{(pr)} \mid \mathcal{F}_t]I_{\tau < +\infty} = 0 \) P-a.s.

\[
K^G = K^G_0 + G_{-2}I_{[0, \tau]} \cdot \mathcal{T}(N^G) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot D.
\]

(17)

On the other hand, thanks to a combination of Definition 2.1 and Lemma A.1-(a), we deduce the existence of an \( \mathbb{F} \)-predictable process \( \varphi \) such that \( 0 < \varphi \leq 1 \) and \( Z^G(\varphi \cdot S^r) \) is \( \mathbb{G} \)-local martingale or equivalently

\[
\varphi \cdot S^r + [\varphi \cdot S^r, K^G] \quad \text{is a } \mathbb{G} \text{-local martingale.}
\]

Thus, using the decomposition \( S = S_0 + M + A + \sum \Delta SI_{\{\Delta S > 1\}} \), where \( M \) is an \( \mathbb{F} \)-locally bounded local martingale and \( A \) an \( \mathbb{F} \)-predictable process with finite variation. (9), and the fact that the processes \( \varphi \cdot [A^r, K^G], \varphi \Delta M \varphi^{(o)} \cdot N^G \) and \( \varphi \Delta M \varphi^{(pr)} \cdot D \) are \( \mathbb{G} \)-local martingales, we derive

\[
\varphi \cdot S^r + \varphi \cdot [K^G, S^r] = \varphi \cdot M^r + \varphi \cdot A^r + \sum \varphi \Delta S^r I_{\{\Delta S > 1\}} + \varphi \cdot [K^G, S^r],
\]

\[
=G \text{-local martingale + } \frac{\varphi}{G} I_{[0, \tau]} \cdot [M, m] + \varphi \cdot A^r + \left( \frac{\varphi}{G} I_{[0, \tau]} \right) [N^F, M]
\]

\[
+ \sum \varphi \Delta SI_{\{\Delta S > 1\}} \left( (1 + \frac{\Delta N^G}{G})I_{[0, \tau]} + \varphi^{(o)} G N^G + \varphi^{(pr)} D \right).
\]

Then \( \varphi \cdot S^r + \varphi \cdot [K^G, S^r] \) is a \( \mathbb{G} \)-local martingale if and only if

\[
W := \sum \varphi \Delta SI_{\{\Delta S > 1\}} I_{[0, \tau]} \left( (1 + \frac{\Delta N^G}{G})I_{[0, \tau]} + \varphi^{(o)} G N^G + \varphi^{(pr)} D \right),
\]

(19)

has \( \mathbb{G} \)-locally integrable variation (i.e. \( W \in \mathcal{A}_{loc}(\mathbb{G}) \)) and (due to Lemma A.2)

\[
0 \equiv \frac{\varphi}{G} I_{[0, \tau]} \cdot (M, m)^F + \varphi \cdot A^r + \left( \frac{\varphi}{G} I_{[0, \tau]} \right) \cdot (N^F, M)^F + W^{\mathbb{P}, G}.
\]

(20)

In virtue of Lemma A.3, we conclude that \( W \in \mathcal{A}_{loc}(\mathbb{G}) \) iff both processes

\[
W^{(1)} := \sum \varphi \Delta SI_{\{\Delta S > 1\}} \left( (1 + \frac{\Delta N^G}{G})I_{[0, \tau]} \right)
\]

and \( W^{(2)} := \sum \varphi \Delta SI_{\{\Delta S > 1\}} \left( \varphi^{(o)} G N^G + \varphi^{(pr)} D \right) \) belong to \( \mathcal{A}_{loc}(\mathbb{G}) \).

It is clear that \( W^{(2)} \) belongs to \( \mathcal{A}_{loc}(\mathbb{G}) \) if and only if it is a \( \mathbb{G} \)-local martingale, and hence in this case we get

\[
W^{\mathbb{P}, G} = (W^{(1)})^{\mathbb{P}, G} = G^{-1}I_{[0, \tau]} \cdot \left( \sum \varphi \Delta SI_{\{\Delta S > 1\}} \left( \frac{\Delta N^G}{G} + \Delta N^F \right) \right)^{\mathbb{P}, G}.
\]

As a result, by inserting these remarks in (20), we obtain

\[
0 \equiv \varphi \cdot (M, m)^F + \varphi G \cdot A + \left( \frac{\varphi}{G} \right) \cdot (N^F, M)^F +
\]

\[
+ G \cdot \left( \sum \varphi \Delta SI_{\{\Delta S > 1\}} \left( 1 + \frac{\Delta m}{G} + \frac{\Delta N^F}{G^2} \right) \right)^{\mathbb{P}, G},
\]

(21)
This ends the proof of the theorem.

or equivalently

\[ 0 = \varphi \cdot \langle M, \frac{1}{G_-} \cdot m + \frac{1}{G_-^2} \cdot N \rangle_Y + \varphi \cdot A + \left( \sum \varphi \Delta S I_{\{\Delta S > 1\}} \left[ 1 + \frac{\Delta m}{G_-} + \frac{\Delta N}{G_-^2} \right] \right) \]

Thanks to Itô’s formula, this is equivalent to \( \mathcal{E}(K) (\varphi \cdot S) \) is an \( \mathbb{F} \)-local martingale with \( K := G^{-1} \cdot m + G^{-2} \cdot N \), and the first step is completed.

**Step 2.** Herein, we assume that \((16)\) holds, and prove that \( \mathcal{E}(K^G) > 0 \) if and only if \((14)-(15)\) and \( 1 + \Delta K^F > 0 \) hold. To this end, we put

\[ \Gamma := \frac{G}{G_-} (1 + \Delta K^F) - 1, \]

and we derive

\[
\Delta K^G = \Delta T(K^F) - G^{-1} \Delta T(m) + \varphi^{(o)} \Delta N^G + \varphi^{(pr)} \Delta D
\]

\[
= \left[ \Gamma - \frac{\varphi^{(o)} \Delta D^{o,F}}{G} \right] I_{[0,\tau]} + \left[ \Gamma + \varphi^{(pr)} + \varphi^{(o)} \frac{G}{G_-} \right] I_{[\tau]}.
\]

Therefore, \( \mathcal{E}(K^G) > 0 \) if and only if \( 1 + \Delta K^G > 0 \) which is equivalent to

\[ \left\llbracket 0, \tau \right\rrbracket \subset \left\{ \frac{G}{G_-} (1 + \Delta K^F) - \varphi^{(o)} \frac{\Delta D^{o,F}}{G} > 0 \right\}, \text{ and} \quad (21) \]

\[ \left\llbracket \tau \right\rrbracket \subset \left\{ \frac{G}{G_-} (1 + \Delta K^F) + \varphi^{(o)} \frac{G}{G_-} + \varphi^{(pr)} > 0 \right\}. \quad (22) \]

Thus, by putting \( \Sigma := \left\{ G_- G^{-1} (1 + \Delta K^F) - \varphi^{(o)} G^{-1} \Delta D^{o,F} > 0 \right\} \cap \llbracket 0, +\infty \rrbracket \), \((21)\) is equivalent to \( I_{[0,\tau]} \leq I_{\Sigma_1} \). Hence, by taking the \( \mathbb{F} \)-optional projection on both sides of this inequality, we get \( 0 < G \leq I_{\Sigma_1} \) on \( \llbracket 0, +\infty \rrbracket \). This proves the right inequality in \((15)\). Notice that \((22)\) is equivalent to

\[ G_- G^{-1} (1 + \Delta K^F) + \varphi^{(o)} G \tilde{G}^{-1} + \varphi^{(pr)} > 0, \quad P \otimes D - a.e., \]

and \((14)\) is proved. Now, we focus on proving that \( 1 + \Delta K^F > 0 \) and the left inequality in \((15)\). Thanks to \( \mathbb{E}_{\mathbb{F} \otimes D}[\varphi^{(pr)} | \mathcal{O}(\mathbb{F})] = 0, P \otimes D - a.e., \) by taking conditional expectation under \( P \otimes D \) with respect to \( \mathcal{O}(\mathbb{F}) \) on the both sides of the above inequality, we get

\[ (23) \]

or equivalently \( I_{[\tau]} \leq I_{\llbracket \Sigma > 0 \rrbracket} \). Remark that \((23)\) is equivalent to the left inequality in \((15)\) since \( \Sigma \) is \( \mathbb{F} \)-optional, and hence the proof of \((15)\) is completed. By taking the \( \mathbb{F} \)-optional projection in both sides of \( I_{[\tau]} \leq I_{\llbracket \Sigma > 0 \rrbracket} \), we get \( \Delta D^{o,F} \leq I_{\llbracket \Sigma > 0 \rrbracket} \). Therefore, we derive

\[ \left\{ \Delta D^{o,F} > 0 \right\} \subset \left\{ G_- (1 + \Delta K^F) > -\varphi^{(o)} G \right\}. \quad (24) \]

On the one hand, due to the right inequality in \((15)\), we deduce that on \( \{ \Delta D^{o,F} = 0 \} \), we have \( 1 + \Delta K^F > 0 \). On the other hand, using \((24)\) and the right inequality in \((15)\) afterwards again, we get

\[ \{ 1 + \Delta K^F \leq 0 \} \cap \{ \Delta D^{o,F} > 0 \} \subset \{ \varphi^{(o)} > 0, 1 + \Delta K^F \leq 0, \Delta D^{o,F} > 0 \} = \emptyset, \]

or equivalently \( \{ \Delta D^{o,F} > 0 \} \subset \{ 1 + \Delta K^F > 0 \} \). Thus, \( 1 + \Delta K^F > 0 \), and the second step is completed.

This ends the proof of the theorem. \[ \square \]
In the following, we derive a multiplicative version of Theorem 3.1 that seems adequate for logarithm utilities, see [13] for related discussions.

**Theorem 3.2.** Suppose $G > 0$. Then $Z^G \in Z_{loc}(S^r, G)$ iff there exists unique $(Z^G, \varphi^{(o)}, \varphi^{(pr)}) \in Z_{loc}(S, \mathbb{F}) \times T^G_{loc}(N^G, \mathbb{G}) \times L^1_{loc}(\text{Prog}^{\mathbb{F}}, P \otimes D)$ such that

$$
\varphi^{(pr)} > -1, \quad \frac{-\hat{G}}{G} < \varphi^{(o)} < \frac{\hat{G}}{G - \hat{G}}, \quad P \otimes D \text{-a.e.}
$$

(25)

and

$$
\mathbb{E}[\varphi^{(pr)}_T | \mathcal{F}_T]I_{\{T < +\infty\}} = 0 \quad P \text{-a.s.}
$$

(26)

Furthermore, put

$$
Z^G = \frac{(Z^G)^\tau}{\mathcal{E}(G^{-1} \cdot m)^\tau} \mathcal{E}(\varphi^{(o)} \cdot N^G) \mathcal{E}(\varphi^{(pr)} \cdot D).
$$

(27)

**Proof.** Thanks to Theorem 3.1, we conclude that $Z^G$ is a local martingale deflator for $(S^r, \mathbb{G})$ if and only if there exists a triplet $(Z^G, \varphi^{(o)}, \varphi^{(pr)})$ such that $Z^G = \mathcal{E}(K^F)$ belongs to $Z_{loc}(S, \mathbb{F})$, $\varphi^{(o)}$ belongs to $T^G_{loc}(N^G, \mathbb{G})$, $\varphi^{(pr)}$ belongs to $L^1_{loc}(\text{Prog}^{\mathbb{F}}, P \otimes D)$ and $\mathbb{E}[\varphi^{(pr)}_T | \mathcal{F}_T]I_{\{T < +\infty\}} = 0$ $P$-a.s., and (14), (15) and (16) hold. Thus, we put

$$
Y := T(K^F) - G^{-1} \cdot T(m), \quad X := Y + \varphi^{(o)} \cdot N^G
$$

and calculate

$$
\varphi^{(o)} := \frac{\tilde{G} \varphi^{(o)}}{G^{-1}(1 + \Delta K^F)} \quad \varphi^{(pr)} := \frac{\tilde{G} \varphi^{(pr)}}{G^{-1}(1 + \Delta K^F) + \varphi^{(o)} G}.
$$

(28)

Since the pair $(\varphi^{(o)}, \varphi^{(pr)})$ satisfies (14)-(15), we conclude that the pair $(\varphi^{(o)}, \varphi^{(pr)})$ satisfies (25)-(26). Furthermore, put

$$
\Gamma := G^{-1}(1 + \Delta K^F) - 1, \quad \tilde{\Omega} := \Omega \times [0, +\infty),
$$

and calculate

$$
1 + \Delta X = \left[ \Gamma + 1 - \Delta D^{\varphi^{(o)} / G} \right] I_{[0, \tau]} + \left[ \Gamma + 1 + \varphi^{(o)} G \right] I_{[\tau]} + I_{[\tau, +\infty]} > 0.
$$

$$
1 + \Delta Y = \frac{G^{-1}(1 + \Delta K^F) I_{[0, \tau]} + I_{[\tau, +\infty]} > 0}.
$$

Thanks to Yor’s formula (i.e. $\mathcal{E}(X_1)\mathcal{E}(X_2) = \mathcal{E}(X_1 + X_2 + [X_1, X_2])$) we derive

$$
\mathcal{E}(X_1 + X_2) = \mathcal{E}(X_1)\mathcal{E}(X_2) - \frac{1}{1 + \Delta X} \cdot [X_1, X_2],
$$

for any semimartingales $X_1, X_2$ with $1 + \Delta X > 0$. By applying this formula repeatedly, and using $\varphi^{(o)} = \varphi^{(o)} / (1 + \Delta Y)$ and $\varphi^{(pr)} = \varphi^{(pr)} / (1 + \Delta X)$ $P \otimes D$-a.e. which follow directly from (28), we obtain

$$
\mathcal{E}(K^G) = \mathcal{E}(X + \varphi^{(pr)} \cdot D) = \mathcal{E}(X)\mathcal{E}\left( \frac{\varphi^{(pr)}}{1 + \Delta X} \cdot D \right) = \mathcal{E}(X)\mathcal{E}(\varphi^{(pr)} \cdot D)
$$

$$
= \mathcal{E}(Y)\mathcal{E}\left( \frac{\varphi^{(o)}}{1 + \Delta Y} \cdot N^G \cdot \varphi^{(pr)} \cdot D \right) = \mathcal{E}(Y)\mathcal{E}(\varphi^{(o)} \cdot N^G)\mathcal{E}(\varphi^{(pr)} \cdot D).
$$

Therefore, the equality (27) follows immediately from combining this equality with $\mathcal{E}(Y) = \mathcal{E}(K^F)^\tau / \mathcal{E}(G^{-1} \cdot m)^\tau$. This latter equality is a direct consequence of $1 / \mathcal{E}(G^{-1} \cdot m)^\tau = \mathcal{E}(G^{-1} \cdot T(m))$ and

$$
\mathcal{E}(K)^\tau \mathcal{E}(-G^{-1} \cdot T(m)) = \mathcal{E}(T(K) - G^{-1} \cdot T(m)).
$$

This ends the proof of the theorem. \[ \square \]
As a direct consequence of Theorem 3.1 (or equivalently Theorem 3.2), we describe below a family of $\mathcal{G}$-local martingales.

**Corollary 3.3.** For any $\mathcal{F}$-local martingale (respectively an element of $\mathcal{Z}_{\text{loc}}(S, \mathcal{F})$) $Z^\mathcal{F} := \mathcal{E}(K^\mathcal{F})$, the process $Z^\mathcal{G}$ is given by

$$Z^\mathcal{G} := \mathcal{E}\left(T(K^\mathcal{F}) - G^{-1} \cdot T(m)\right) = \frac{(Z^\mathcal{F})^\tau}{\mathcal{E}(G^{-1} \cdot m)^\tau},$$

is a $\mathcal{G}$-local martingale (respectively an element of $\mathcal{Z}_{\text{loc}}(S^\tau, \mathcal{G})$).

**4 General deflators for $(S^\tau, \mathcal{G})$**

This section focuses on explicitly parametrizing the set of all deflators for $(S^\tau, \mathcal{G})$ in terms of deflators for $(S, \mathcal{F})$. Throughout the rest of the paper, processes will be compared to each other in the following sense.

**Definition 4.1.** Let $X$ and $Y$ be two processes with $X_0 = Y_0$. Then

$$X \succeq Y \quad \text{if} \quad X - Y \quad \text{is an increasing process.}$$

We start this section by parametrizing deflators as follows.

**Lemma 4.2.** Let $X$ be an $\mathbb{H}$-semimartingale, and $Z$ be a positive $\mathbb{H}$-supermartingale. Then the following assertions are equivalent.

(a) $Z$ is a deflator for $(X, \mathbb{H})$.

(b) There exists unique $(N, V)$ such that $N \in \mathcal{M}_{0,\text{loc}}(\mathbb{H})$, $V$ is nondecreasing and $\mathbb{H}$-predictable,

$$Z := Z_0 \mathcal{E}(N) \mathcal{E}(-V), \quad N_0 = V_0 = 0, \quad \Delta N > -1, \quad \Delta V < 1$$

$$\sup_{0<s\leq t} \left| \Delta Y(\varphi) \right| \in \mathcal{A}_{\text{loc}}^+(\mathbb{H}) \quad \text{and} \quad \frac{1}{1 - \Delta V} \cdot V \succeq (Y(\varphi))^{p,\mathbb{H}},$$

where $(Y(\varphi))^{p,\mathbb{H}}$ is predictable with finite variation such that

$$Y(\varphi) - (Y(\varphi))^{p,\mathbb{H}} \in \mathcal{M}_{\text{loc}}(\mathbb{H}) \quad \text{and} \quad Y(\varphi) := \varphi \cdot X + [\varphi \cdot X, N],$$

for any bounded $\varphi$ that belongs to $\mathcal{L}(X, \mathbb{H})$ given by

$$\mathcal{L}(X, \mathbb{H}) := \left\{ \varphi \text{ is } \mathbb{H}\text{-predictable} \mid \varphi \Delta X > -1 \right\}.$$

For the sake of easy exposition, we postpone the proof of this lemma to Appendix C. In the lemma, by “abuse of notations” for the sake of simplicity, we use $Y^{p,\mathbb{H}}$ to denote the predictable with finite variation part in the Doob-Meyer decomposition of $Y$ whenever this process is a special $\mathbb{H}$-semimartingale.

**Theorem 4.3.** Suppose $G > 0$, and let $Z^G$ be a $\mathcal{G}$-semimartingale. Then the following assertions are equivalent.

(a) $Z^G$ is a deflator for $(S^\tau, \mathcal{G})$ (i.e. $Z^G \in \mathcal{D}(S^\tau, \mathcal{G})$).

(b) There exists a unique $(K^\mathcal{F}, V^\mathcal{F}, \varphi^{(o)}, \varphi^{(pr)})$ such that $K^\mathcal{F} \in \mathcal{M}_{0,\text{loc}}(\mathcal{F})$, $V^\mathcal{F}$ is an $\mathcal{F}$-predictable and nondecreasing process, $\varphi^{(o)} \in \mathcal{T}_{\text{loc}}^0(N^G, \mathcal{G})$, $\varphi^{(pr)}$ belongs to $L^1_{\text{loc}}(\text{Prog}(\mathcal{F}), \mathcal{P} \otimes D)$ such that $\mathcal{E}(K^\mathcal{F}) \mathcal{E}(-V^\mathcal{F}) \in \mathcal{D}(S, \mathcal{F})$,

$$\varphi^{(pr)} > -\left[G_\cdot (1 + \Delta K^\mathcal{F}) + \varphi^{(o)} G\right] / \tilde{G}, \quad P \otimes D - a.e.,$$

$$\tilde{G} = G \left(1 + \Delta K^\mathcal{F}\right) + K^\mathcal{F} / \mathbb{E} \left(1 + \Delta K^\mathcal{F}\right)^\tau, \quad \tilde{F} = F \left(1 + \Delta \left(G^\mathcal{F}\right)^\tau\right) + F^\mathcal{F} / \mathbb{E} \left(1 + \Delta \left(G^\mathcal{F}\right)^\tau\right)^\tau,$$
Thus, due to Lemma 4.2, we deduce the existence of a nondecreasing process such that
\[ \phi < \frac{1}{G} (1 + \Delta K) < \frac{1 + \Delta K}{\Delta D^{\phi,F}} \cdot P \otimes D^{\phi,F} - \text{a.e.}, \]
(34)
\[ Z^G = \mathcal{E}(K^G) \mathcal{E}(-V^F), \quad K^G = T(K^F - \frac{1}{G} \cdot m) + \phi^{(o)} \cdot N^G + \phi^{(pr)} \cdot D. \]
(35)

(c) There exists unique \((Z^F, \phi^{(o)}, \phi^{(pr)})\) such that \(Z^F \in \mathcal{D}(S, \mathbb{F}), (\phi^{(o)}, \phi^{(pr)})\) belongs to \(\mathcal{T}_{\text{loc}}^o(N^G, \mathbb{G}) \times L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D)\),
\[ \phi^{(pr)} > -1, \quad P \otimes D - \text{a.e.}, \quad -\frac{\bar{G}}{G} < \phi^{(o)} < \frac{\bar{G}}{G - \bar{G}}, \quad P \otimes D^{\phi,F} - \text{a.e.}, \]
(36)
and
\[ Z^G = \frac{(Z^F)^\tau}{\mathcal{E}(K^G) \mathcal{E}(-V^F)} \mathcal{E}(\phi^{(o)} \cdot N^G) \mathcal{E}(\phi^{(pr)} \cdot D). \]
(37)

**Proof.** The proof will be achieved in three steps, where we prove the implications (a)⇒(b), (b)⇒(c), and (c)⇒(a) respectively.

**Step 1.** Herein, we prove (a)⇒(b). To this end, we suppose that \(Z^G\) is a deflator for \((S^T, \mathbb{G})\). Thus, due to Lemma \[4,2\] we deduce the existence of \(K^G \in \mathcal{M}_{0, \text{loc}}(\mathbb{G})\) and \(V^G\) a \(\mathbb{G}\)-predictable and nondecreasing process such that
\[ K^G = K^G_0 + \frac{1}{G} I_{[0,T]} \cdot T(N^F) + \phi^{(o)} \cdot N^G + \phi^{(pr)} \cdot D, \quad V^G = V^\tau. \]

Consider a bounded \(\phi \in \mathcal{L}(S^T, \mathbb{G})\), and remark that \(\phi \cdot S^T + [\phi \cdot S^T, K^G] \in \mathcal{A}_{\text{loc}}(\mathbb{G})\) is equivalent to \(W := \sum (\phi \Delta S^T + [\phi \Delta S^T, K^G]) I_{[|\Delta S^T| > 1]} \in \mathcal{A}_{\text{loc}}(\mathbb{G})\). As a result,
\[ W_{\phi,G} = \lim_{n \to +\infty} (I_{\{n > |\Delta S^T| > 1\}} \cdot W)^{G,F}. \]

By combining
\[ \phi \cdot S^T + [\phi \cdot S^T, K^G] = \phi \cdot S^T + \frac{1}{G} I_{[0,T]} \cdot T(N^F) + \phi^{(o)} \cdot N^G + \phi^{(pr)} \cdot D, \]
the fact that \(\phi \Delta S, \phi^{(o)} I_{[|\Delta S| \leq n]} \cdot N^G\) and \(\phi \Delta S, \phi^{(pr)} I_{[|\Delta S| \leq n]} \cdot D\) are \(\mathbb{G}\)-local martingales for any \(n \geq 1\), and \([T(N^F), \phi \cdot S^T] = G^{-1} \cdot G^{(o)} \cdot I_{[0,T]} \cdot [N^F, \phi \cdot S^T] \), we deduce that
\[ (\phi \cdot S^T + [\phi \cdot S^T, K^G])^{G,F} = (\phi \cdot S^T + \frac{1}{G} I_{[0,T]} \cdot [T(N^F), \phi \cdot S])^{G,F} \]
\[ = (\phi \cdot S^T + \frac{1}{G} I_{[0,T]} \cdot [N^F, \phi \cdot S])^{G,F}. \]
By inserting in this equation the following decomposition of $S$,

$$S = S_0 + M + A + \sum \Delta SI_{[\Delta S > 1]},$$

where $M$ is a locally bounded $\mathbb{F}$-local martingale and $A$ is an $\mathbb{F}$-predictable process with finite variation, we obtain

$$(\varphi \cdot S^\tau + [\varphi \cdot S^\tau, K^G])^{p,G} = \varphi \cdot A^\tau + \frac{I^0_{\frac{\tau}{\tau}}}{{G}_-} \cdot \langle m, \varphi \cdot M \rangle^\mathbb{F} + \frac{I^0_r}{G^-} \cdot \langle N^F, \varphi \cdot M \rangle^\mathbb{F} + + \left( \sum \varphi \Delta S(1 + \frac{\Delta N^F}{G^-})I_{[0,\tau]}I_{[\Delta S > 1]} \right)^{p,G}

= \varphi \cdot A^\tau + \frac{I^0_{\frac{\tau}{\tau}}}{{G}_-} \cdot \langle m, \varphi \cdot M \rangle^\mathbb{F} + \frac{I^0_r}{G^-} \cdot \langle N^F, \varphi \cdot M \rangle^\mathbb{F} + + G^{-1}I_{[0,\tau]} \left( \sum \varphi \Delta S(\bar{G} + \frac{\Delta N^F}{G^-})I_{[\Delta S > 1]} \right)^{p,F}. $$

As a result, for any bounded $\varphi \in \mathcal{L}(S, \mathbb{F})$, $V(\varphi) := \varphi \cdot S + [\varphi \cdot S, \frac{1}{G_-} \cdot m + \frac{1}{G_-} \cdot N^F]$ has an $\mathbb{F}$-compensator, and

$$\frac{1}{1 - \Delta V^F} \cdot V^F \geq \varphi \cdot A + \left( \frac{1}{G_-} \cdot m + \frac{1}{G^-} \cdot N^F, \varphi \cdot M \right)^{p,F} + \left( \sum \Delta V(\varphi)I_{[\Delta S > 1]} \right)^{p,F}

= \left( \varphi \cdot S + [\varphi \cdot S, \frac{1}{G_-} \cdot m + \frac{1}{G^-} \cdot N^F] \right)^{p,F}. $$

On the one hand, this is equivalent to $\mathcal{E}(K^F)\mathcal{E}(-V)\mathcal{E}(\varphi \cdot S)$ is an $\mathbb{F}$-supermartingale for any bounded $\varphi \in \mathcal{L}(S, \mathbb{F})$, where $K^F := G_-^{-1} \cdot m + G^{-2} \cdot N^F$. On the other hand, as in the proof of Theorem 3.1 (see step 2 of that proof), it is clear that $\mathcal{E}(K^G) > 0$ if and only if $1 + \Delta K^G > 0$ if and only if the triplet $(K^F, \varphi^{(o)}, \varphi^{(pr)})$ satisfies $1 + \Delta K^F > 0$ and (33)-(34). This proves assertion (b), and the first step is completed.

**Step 2.** This step proving (b)$\implies$(c). Hence, we suppose that assertion (b) holds. Then there exists a unique $(K^F, V^F, \varphi^{(o)}, \varphi^{(pr)})$ such that

$$Z^G = \mathcal{E}\left(T(K^F) - G^{-1} \cdot T(m) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot D\right)\mathcal{E}(-V^F)^\tau.$$

Then by mimicking the analysis (calculations) that starts from (28), we derive

$$Z^G = \frac{\mathcal{E}(K^F)^\tau \mathcal{E}(-V^F)^\tau \mathcal{E}(\varphi^{(o)} \cdot N^G)\mathcal{E}(\varphi^{(pr)} \cdot D)}{\mathcal{E}(G^{-1} \cdot m)^\tau}$$

$$= \frac{(Z^F)^\tau}{\mathcal{E}(G^{-1} \cdot m)^\tau} \mathcal{E}(\varphi^{(o)} \cdot N^G)\mathcal{E}(\varphi^{(pr)} \cdot D).$$

Here $\varphi^{(o)} := \bar{G} \varphi^{(o)}/G_-(1 + \Delta K^F)$ and $\varphi^{(pr)} := \bar{G} \varphi^{(pr)}/G_-(1 + \Delta K^F) + G \varphi^{(o)}$ are the $\mathbb{F}$-optional and $\mathbb{F}$-progressive processes respectively that satisfy (36) as a direct consequence of the conditions (33)-(34) fulfilled by the pair $(\varphi^{(o)}, \varphi^{(pr)})$. This ends the proof of (b)$\implies$(c).

**Step 3.** Herein, we deal with (c)$\implies$(a). Thus, we suppose that assertion (c) holds, and deduce the existence of a triplet $(Z^F, \varphi^{(o)}, \varphi^{(pr)})$ that belongs to $\mathcal{D}(S, \mathbb{F}) \times \mathcal{I}_{loc}(N^G, \mathbb{G}) \times L^1_{loc}(\text{Prog}(\mathbb{F}), P \otimes D)$ satisfying (36) and

$$Z^G = \frac{(Z^F)^\tau}{\mathcal{E}(G^{-1} \cdot m)^\tau} \mathcal{E}(\varphi^{(o)} \cdot N^G)\mathcal{E}(\varphi^{(pr)} \cdot D). \quad (38)$$
Then for any bounded \( \varphi \in \mathcal{L}(S, F) \), \( Z^E\mathcal{E}(\varphi \cdot S) \) is an \( F \)-supermartingale, and hence there exist \( N \in \mathcal{M}_{0,loc}(F) \) and \( V \) is an \( F \)-predictable and non decreasing process such that

\[
 Z^E\mathcal{E}(\varphi \cdot S) = \mathcal{E}(N)\mathcal{E}(-V).
\]

Therefore, by combining this with (38), we deduce that, for any bounded \( \varphi \in \mathcal{L}(S, F) \),

\[
 Z^G\mathcal{E}(\varphi \cdot S)^\tau = \frac{\mathcal{E}(N)^\tau}{\mathcal{E}(G^{-1}_- \cdot m)^\tau} \mathcal{E}(\varphi^{(o)} \cdot N^G)\mathcal{E}(\varphi^{(pr)} \cdot D)\mathcal{E}(-V)^\tau.
\]

Thus, thanks to Corollary 3.3 which allows us to conclude that the process \( (\mathcal{E}(N)^\tau / \mathcal{E}(G^{-1}_- \cdot m)^\tau)\mathcal{E}(\varphi^{(o)} \cdot N^G)\mathcal{E}(\varphi^{(pr)} \cdot D) \) is in fact a \( G \)-local martingale, we deduce that \( Z^G\mathcal{E}(\varphi \cdot S)^\tau \) is a \( G \)-supermartingale, for any bounded \( \varphi \in \mathcal{L}(S, F) \). Then assertion (a) follows immediately from combining this with the fact that, for any bounded \( \varphi^G \) that belongs to \( \mathcal{L}(S^\tau, G) \), there exists a bounded \( \varphi^F \in \mathcal{L}(S, F) \) such that \( \varphi^G = \varphi^F \) on \( [0, \tau] \) (see Lemma A.1(a)). This ends the proof of the theorem. \( \Box \)

5 Particular cases and examples

In this section, we illustrate the obtained results on particular cases and/or examples. Precisely, in three subsections, we discuss the case when \( \tau \) avoids \( F \)-stopping times or all \( F \)-martingales are continuous, the case of jump-diﬀusion for \( (S, F) \), and the case of discrete time model for \( (S, F) \).

5.1 When \( \tau \) avoids \( F \)-stopping times or all \( F \)-martingales are continuous

As explained in [8], when either \( \tau \) avoids all \( F \)-stopping times or all \( F \)-martingales are continuous, the \( G \)-local martingales \( \mathcal{T}(M) \) and \( N^G \) –given by (5) and (6)– coincide with \( M \) and \( N^G \) respectively where

\[
 M := M^\tau - G^{-1}_- I_{[0, \tau]} \cdot (M, m)^F, \quad N^G := D - G^{-1}_- I_{[0, \tau]} \cdot D^pF.
\]

(39)

It is clear that both \( M \) and \( N^G \) are the \( G \)-local martingale parts in the Doob-Meyer decomposition, under \( G \), of \( M^\tau \) and \( D \) respectively.

**Theorem 5.1.** Suppose that \( G > 0 \), and either \( \tau \) avoids \( F \)-stopping times or all \( F \)-martingales are continuous. Let \( K^G \) be a \( G \)-local martingale and \( V^G \) be a nondecreasing and \( G \)-predictable. Then the following assertions are equivalent.

(a) \( Z^G := \mathcal{E}(K^G)\mathcal{E}(-V^G) \) is a deflator for \( (S^\tau, G) \) if and only if there exists a unique \( (K^F, V^F, \varphi^{(p)}) \) such that \( (K^F, \varphi^{(p)}) \in M_{loc}(F) \times L^1_{loc}(N^G, G) \), \( V^F \) is nondecreasing with finite values and \( F \)-predictable, \( \mathcal{E}(K^F)\mathcal{E}(-V^F) \in \mathcal{D}(S, F) \),

\[
 -\frac{G_-}{pF} < \frac{\varphi^{(p)}}{G_- - pF}, \quad P \otimes D^{pF} - a.e.,
\]

\[
 V^G = (V^F)^\tau, \quad \text{and} \quad K^G = K^F - G^{-1}_- \cdot m + \varphi^{(p)} \cdot N^G.
\]

(40)

(41)

(b) \( Z^G := \mathcal{E}(K^G)\mathcal{E}(-V^G) \) is a deflator for \( (S^\tau, G) \) if and only if there exists a unique pair \( (Z^F, \varphi^{(p)}) \) in \( \mathcal{D}(S, F) \times L^1_{loc}(N^G, G) \) such that \( P \otimes D^{pF} - a.e. \)

\[
 -\frac{G_-}{pF} < \frac{\varphi^{(p)}}{G_- - pF}, \quad \text{and} \quad Z^G = (Z^F)^\tau \frac{\mathcal{E}(\varphi^{(p)} \cdot N^G)}{\mathcal{E}(G^{-1}_- \cdot m)^\tau}.
\]

(42)
Proof. (a) The proof of assertion (a) follows from combining Theorem 5.1-(a) with the following three facts. Under the assumption that either $\tau$ avoids $\mathbb{F}$-stopping times or all $\mathbb{F}$-martingales are continuous, the following facts hold:

1) For $\varphi^{(pr)}\in L_{loc}^1(\text{Prog}(\mathbb{F}), P \otimes D)$, there exists $\varphi \in L^1_{loc}(\mathcal{O}(\mathbb{F}), P \otimes D)$ such that $\varphi^{(pr)} = \varphi \cdot P$-a.s. on $\{\tau < +\infty\}$. Therefore, we deduce that $P$-a.s. on $\{\tau < +\infty\}$ we have $\varphi^{(pr)} = E(\varphi^{(pr)}|\mathcal{F}_\tau) = 0$, or equivalently $\varphi^{(pr)} = 0$. This proves assertion (a).

(b) The proof of assertion (b) follows immediately from combining assertion (a) and the fact that, under the assumption that either $\tau$ avoids $\mathbb{F}$-stopping times or all $\mathbb{F}$-martingales are continuous, we have $\mathbb{E}(\varphi^{(p)} \cdot \mathbb{N}^G) = \mathbb{E}(\mathbb{M}) \mathbb{E}(\varphi^{(p)} \cdot \mathbb{N}^G)$, for $\mathbb{M} \in \mathcal{M}_{0,loc}(\mathbb{F})$ with $1 + \Delta \mathbb{M} > 0$. This ends the proof of the theorem.

Theorem 5.1 states universal results that work for both cases of $\tau$ avoids $\mathbb{F}$-stopping times and when all $\mathbb{F}$-martingales are continuous. The main difference between the two cases lies in the condition on the parameter $\varphi^{(p)}$. Indeed, for the case when $\tau$ avoids $\mathbb{F}$-stopping times, the condition (D) (or equivalently the inequalities in (12)) becomes $-1 < \varphi^{(p)} \cdot P \otimes D^{\mathbb{P}}-a.e.$ instead. This is due to $\hat{G} = \hat{G}$ which follows from the avoidance property of $\tau$. However, when all $\mathbb{F}$-martingales are continuous, (D) takes the form of $-\hat{G} < \varphi^{(p)} < \hat{G} - G \cdot P \otimes D^{\mathbb{P}}-a.e.$, since in this case $\hat{G} = \hat{G}$ and $P^{\mathbb{F}}(G) = G$.

5.2 The case of jump-diffusion for $(S, \mathbb{F})$

This subsection focuses on the important case of a jump-diffusion framework for the market model $(S, \mathbb{F})$. For this model, we consider two situations depending whether $\tau$ is left to be an arbitrary random time or a particular example. Herein, we suppose that a standard Brownian motion $W$ and a Poisson process $\lambda > 0$ are defined on the probability space $(\Omega, \mathcal{F}, P)$, and the filtration $\mathbb{F}$ is the completed and right continuous filtration generated by $W$ and $N$. The stock’s price process is supposed to have the following dynamics

$$S_t = S_0 \mathcal{E}(X)_t, \quad X_t = \int_0^t \sigma_s dW_s + \int_0^t \zeta_s dN^\mathbb{F}_s + \int_0^t \mu_s ds, \quad N^\mathbb{F}_t := N_t - \lambda t, \quad (43)$$

where the processes $\mu$, $\zeta$, and $\sigma$ are bounded, $\mathbb{F}$-adapted and there exists a constant $\delta \in (0, +\infty)$ such that

$$\sigma > 0, \quad \zeta > -1, \quad \sigma + |\zeta| \geq \delta, \quad P \otimes dt-a.e.. \quad (44)$$

Since $m$ is an $\mathbb{F}$-martingale, then there exists two $\mathbb{F}$-predictable processes $\varphi^{(m)}$ and $\psi^{(m)}$ such that

$$\int_0^t(\varphi^{(m)})^2 + |\psi^{(m)}|)ds < +\infty \quad P$-a.s. \ for any $t \geq 0$$

$$\frac{1}{G} \cdot m = \varphi^{(m)} \cdot W + (\psi^{(m)} - 1) \cdot N^\mathbb{F} \quad (45)$$

Theorem 5.2. Suppose $S$ given by (43), $G > 0$, and let $Z^G := \mathcal{E}(K^G)$ be a positive $\mathbb{G}$-local martingale. Then the following assertions are equivalent.

(a) $Z^G$ is a local martingale deflator for $(S^+, \mathbb{G})$,
(b) There exist \((\psi_1, \psi_2) \in L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^F, \mathbb{F})\), \(\varphi^{(o)} \in T^o_{loc}(N^G, \mathbb{G})\) and \(\varphi^{(pr)} \in L^1_{loc}(\text{Prog}^o(\mathbb{F}), P \otimes D)\) satisfying the following:

\[
K^G = \psi_1 \cdot T(W) + (\psi_2 - 1) \cdot T(N^F) - G^{-1} \cdot T(m) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot D.
\]

\[
\varphi^{(pr)} > -[G_\psi - \varphi^{(o)} G] / \tilde{G}, \quad \text{and} \quad -\frac{\psi_2 G_\psi}{G} < \varphi^{(o)} < \frac{\psi_2 G_\psi}{G} \quad P \otimes D\text{-a.e.,}
\]

and \(\mu + \psi_1 \sigma + (\psi_2 - 1) \lambda \equiv 0, \quad \psi_2 > 0 \quad P \otimes dt - \text{a.e.}\)

(c) There exists unique quadruplet \((\psi_1, \psi_2, \varphi^{(o)}, \varphi^{(pr)})\) that belongs to the set \(L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^F, \mathbb{F}) \times T^o_{loc}(N^G, \mathbb{G}) \times L^1_{loc}(\text{Prog}^o(\mathbb{F}), P \otimes D)\), and satisfies

\[
Z^G = \frac{\mathcal{E}(\psi_1 \cdot W + (\psi_2 - 1) \cdot N^F)^r}{\mathcal{E}(G^{-1} \cdot m)^r} \mathcal{E}(\varphi^{(o)} \cdot N^G) \mathcal{E}(\varphi^{(pr)} \cdot D),
\]

\[
\varphi^{(pr)} > -1, \quad P \otimes D\text{-a.e.,} \quad -\frac{G}{\psi_2} < \varphi^{(o)} < \frac{G}{\psi_2} \quad P \otimes D^o\text{-a.e.,}
\]

and \(\mu + \psi_1 \sigma + (\psi_2 - 1) \lambda \equiv 0, \quad \psi_2 > 0 \quad P \otimes dt - \text{a.e.}\)

Proof. The proof follows immediately from Theorems 3.1 and 3.2 and the fact that for any \(\text{calculations for the three processes}\)

There exist \((\psi_1, \psi_2) \in L^1_{loc}(W, \mathbb{F}) \times L^1_{loc}(N^F, \mathbb{F})\) such that \(M = M_0 + \psi_1 \cdot W + (\psi_2 - 1) \cdot N^F\). \(\square\)

In the following, we discuss a particular model for \(\tau\) that was considered in [2] Example 2.12 and [3] Subsection 5.2.2, page 108. 

**Example 5.3.** Consider the same model for \((S, \mathbb{F})\) as in Theorem 5.2 and let \(\tau := (aT_2) \wedge T_1\), where \(a \in (0, 1)\) and \(T_1\) and \(T_2\) are the first and the second jump times of the Poisson process \(N\) (i.e. \(N := \sum_{n=1}^{+\infty} I_{[T_n, +\infty[}\)). Since \(\mathbb{F}\) is generated by \((W, N)\) and \(W\) is independent of \(\tau\), the same calculations for the three processes \((G, G_\tau, \tilde{G})\) as in [2, 5] remain valid. Thus, we get

\[
\tilde{G}_t = e^{-\beta \tau}(\beta t + 1)I_{[0, T_1]}(t) + e^{-\beta \tau}I_{[T_1, T_2]}(t), \quad G_t = e^{-\beta \tau}(\beta t + 1)I_{[0, T_1]}(t)
\]

\[
G_{t-} = \tilde{G}_{t-} = e^{-\beta \tau}(\beta t + 1)I_{[0, T_1]}(t).
\]

However, the arguments for the calculations of \(m\) and \(D^{o, F}\) differ slightly from that of [2, 5]. Let \(m^c\) be the continuous local martingale part of \(m\), and hence \(m - m^c\) is a pure jump local martingale with jumps equal to

\[
\Delta m = \tilde{G} - G_{t-} = \phi \Delta N^F, \quad \text{where} \quad \phi_t := -\beta te^{-\beta t}, \quad \beta := \lambda(a^{-1} - 1).
\]

Hence \(m = m^c + \phi \cdot N^F\) on the one hand. On the other hand, by writing

\[
G_t = e^{-\beta \tau}(\beta t + 1)(1 - H^{(1)}_t), \quad H^{(1)}_t := I_{[T_1, +\infty[}, \quad M^{(1)} := H^{(1)} - \lambda(t \wedge T_1) = (N^F)^T_t,
\]

and by applying Itô’s formula to the process \(G\) and using \(G = m + D^{o, F}\) (see [4]), we deduce that \(m^c \equiv 0\),

\[
m = m_0 + \phi \cdot N^F \quad \text{and} \quad D^{o, F}_t = \int_0^t e^{-\beta s}dH^{(1)}_s + (\beta + \lambda)\beta \int_0^{t \wedge T_1} se^{-\beta s}ds.
\]

Since in the current case we have \(\tilde{G} = 0 < G_{t-}\), we derive

\[
T(W) = W^r, \quad T(N^F) = (N^F)^r + \frac{\beta t}{1 + \beta t} \cdot (N)^r, \quad \frac{1}{G_{t-}} \cdot T(m) = -\frac{\beta t + 1}{\beta} \cdot T(N^F),
\]

\[
N^G = I_{(aT_2 < T_1)}I_{[aT_2, +\infty[} - (\lambda + \beta) \int_0^{t \wedge T_1} \frac{\beta s}{1 + \beta s}ds.
\]
Furthermore, we have \( [0, \tau] \subset \{ G_\tau > 0 \} = [0, T_1] \), and hence the condition \( G > 0 \) is redundant in the current case, as one can work with \( S^{T_1} \) instead. This confirms our claim that the condition \( G > 0 \) is technical and can be relaxed at the expenses of technicalities (in both the statements of the results and the proofs) that we tried to avoid in this paper. A combination of this analysis with Theorem 5.2 leads to the following

**Corollary 5.4.** \( Z^G \in \mathcal{D}(S^T, \mathbb{G}) \) is equivalent to each of the following:
(a) There exist \( (\psi_1, \psi_2) \in L^1_{\text{loc}}(W; \mathbb{F}) \times L^1_{\text{loc}}(N^\mathbb{F}, \mathbb{F}) \), \( \varphi^{(o)} = \varphi^{(o)}I_{[0,T_1]} \) belongs to \( T^o_{\text{loc}}(N^G, \mathbb{G}) \) and \( \varphi^{(pr)} \in L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), \mathbb{P} \otimes \mathcal{D}) \) satisfying the following

\[
K^G = \psi_1 \cdot W^\tau + (\psi_2 - \frac{1}{1 + \beta t}) \cdot T(N^\mathbb{F}) + \varphi^{(o)} \cdot N^G + \varphi^{(pr)} \cdot \mathcal{D},
\]

\[
\mu + \psi_1 \sigma + (\psi_2 - 1)\zeta \lambda \equiv 0, \quad \psi_2 > 0, \quad -\psi_2 < \varphi^{(o)}I_{[0,T_1]} \mathcal{D}, \quad \mathcal{P} \otimes \mathcal{D} - \text{a.e.}
\]

and the following inequalities hold \( \mathcal{P} \)-a.s.

\[
\varphi^{(o)}(T_1) < \psi_2(T_1)(1 + \beta T_1),
\]

\[
\varphi^{(pr)}(aT_2 \land T_1) > -[\psi_2(aT_2) + \varphi^{(o)}(aT_2)I_{(aT_2 < T_1)} - \psi_2(T_1)[1 + \beta T_1]I_{(aT_2 \geq T_1)},
\]

(b) There exists unique quadruplet \( (\psi_1, \psi_2, \varphi^{(o)}, \varphi^{(pr)}) \) belonging to the set \( L^1_{\text{loc}}(W; \mathbb{F}) \times L^1_{\text{loc}}(N^\mathbb{F}, \mathbb{F}) \times T^o_{\text{loc}}(N^G, \mathbb{G}) \times L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), \mathbb{P} \otimes \mathcal{D}) \), and satisfies

\[
Z^G = \mathcal{E}(L)^\tau \mathcal{E}(\varphi^{(o)} \cdot N^G)^\tau \mathcal{E}(\varphi^{(pr)} \cdot \mathcal{D}),
\]

\[
L := \psi_1 \cdot W + ((1 + \beta t)\psi_2 - 1) \cdot N^\mathbb{F} + \int_0^\tau \frac{\lambda t}{1 + \beta t}[(1 + \beta t)\psi_2(t) - 1] \cdot dt,
\]

\[
\varphi^{(pr)}(aT_2 \land T_1) > -1, \quad \mathcal{P} \text{-a.s.}, \quad \varphi^{(o)}(T_1) < \psi_2(T_1)(1 + \beta T_1) \quad \mathcal{P} \text{-a.s.},
\]

and \( \mu + \psi_1 \sigma + (\psi_2 - 1)\zeta \lambda \equiv 0, \quad \psi_2 > 0, \quad -1 < \varphi^{(o)}I_{[0,T_1]} \mathcal{D}, \quad \mathcal{P} \otimes \mathcal{D} - \text{a.e.} \)

### 5.3 The discrete-time market models

In this subsection, we suppose that the trading times are \( t = 0, 1, \ldots, T \), and hence on \( (\Omega, \mathcal{F}, \mathcal{P}) \), we have

\[
\mathbb{F} := (\mathcal{F}_n)_{n=0,1,\ldots,T}, \quad \mathcal{G}_n = \mathcal{F}_n \lor \sigma(\tau \land 1, \ldots, \tau \land n), \quad S = (S_n)_{n=0,1,\ldots,T}, \quad \tau \in (0, T), \quad \mathbb{G} := \mathcal{G}_n \land \sigma(\tau \land 1, \ldots, \tau \land n), \quad \mathbb{G}_n = \mathcal{F}_n \lor \sigma(\tau \land 1, \ldots, \tau \land n)
\]

Then the discrete-time version of the operator \( \mathcal{T} \) and the \( \mathbb{G} \)-martingale \( N^\mathbb{G} \) defined in (5) and (6) respectively are given by

\[
\mathcal{T}_n(M) = \sum_{k=1}^{n\land \tau} \frac{P(\tau \geq k|\mathcal{F}_{k-1})}{P(\tau \geq k|\mathcal{F}_k)} \Delta M_k + \sum_{k=1}^{n\land \tau} E(\Delta M_k I_{\{P(\tau \geq k|\mathcal{F}_k) = 0\}}|\mathcal{F}_{k-1}),
\]

and

\[
N^\mathbb{G}_n := I_{\{n \leq \tau\}} - \sum_{k=1}^{n\land \tau} \frac{P(\tau = k|\mathcal{F}_{k-1})}{P(\tau \geq k|\mathcal{F}_k)}
\]

for all \( n = 1, \ldots, T, \Delta M_n := M_n - M_{n-1} \) and \( M \) is an \( \mathbb{F} \)-martingale.

Then \( \tau \) is a \( \mathbb{G} \)-stopping time and \( \mathcal{G}_\tau \) is defined as usual, while \( \mathcal{F}_\tau \) is given by \( \mathcal{F}_\tau := \sigma(\{X_t, X \in \mathbb{F} \}-\text{adapted and bounded}) \).

Below, we discuss the relationship between \( \mathcal{G}_\tau \) and \( \mathcal{F}_\tau \), as this role is very important in our analysis.
**Lemma 5.5.** Consider the discrete-time setting of (46). Then $\sigma$-fields $F_\tau$ and $G_\tau$ coincide, and hence for any $G_\tau$-measurable random variable $X$, there exists an $F$-adapted process $\xi$ such that $X = \xi_\tau$ $P$-a.s.

**Proof.** Since $\tau$ is a $G$-stopping time and due to [16] Theorem 64, Chapter IV, we conclude that for any $G_\tau$-measurable random variable $X$, there exists a $G$-adapted process, $\xi^G = (\xi^G_n)_{n=0,1,\ldots,T}$ such that $X = \xi^G_\tau$. Thus, the lemma follows immediately if we prove that for any $n \in \{0,\ldots,T\}$, and any $G_n$-measurable random variable $X_n^G$, there exists an $F_n$-measurable random variable $X_n^F$ such that

$$X_n^G = X_n^F \quad \text{on} \quad (\tau = n).$$

(49)

Thus, on the one hand, it is clear that (49) holds for random variables having the form of $X_n^G = \xi_n^F f(\tau \wedge 1, \ldots, \tau \wedge n)$, where $\xi_n^F$ is a bounded and $F_n$-measurable random variable and $f$ is a bounded and $G_n$-measurable real-valued function on $\mathbb{R}^n$. On the other hand, these random variables generate the vector space of bounded and $G_n$-measurable random variables. Hence the fulfillment of (49) for general random variables, follows from this remark and the class monotone theorem (see [16] Theorem 21, Chapter I). This proves the lemma.

The impact of Lemma 5.5 can be noticed immediately in the discrete-time version of Theorem 5.6 that we state below.

**Theorem 5.6.** Let $M^G$ be a $G$-martingale. Then there exists a unique pair $(M^F, \varphi)$ of $F$-adapted processes such that $M^F$ is an $F$-martingale and

$$M_{n\wedge \tau}^G = M^G_0 + \sum_{k=1}^n \frac{\Delta k(M^F)}{P(\tau \geq k \mid F_{k-1})^2} + \sum_{k=1}^n \varphi_k \Delta N^G_k.$$  

(50)

**Proof.** The proof follows from combining Theorem 2.3 and Lemma 5.5.

Below, we state our main result in this subsection.

**Theorem 5.7.** Let $Z^G$ be a $G$-adapted process and $\tilde{Q}$ be a probability given by

$$\tilde{Q} := Z_T \cdot P \quad \text{and} \quad Z_n := \prod_{k=1}^n \left( \frac{\tilde{G}_k}{G_{k-1}} I_{\{G_{k-1} > 0\}} + I_{\{G_{k-1} = 0\}} \right).$$

(51)

Then the following assertions are equivalent.

(a) $Z^G$ is a deflator for $(S^*, G)$ (i.e. $Z^G \in \mathcal{D}(S^*, G)$).

(b) There exists a unique pair $\left( Z^{(\tilde{Q}, F)}, \varphi \right)$ such that $Z^{(\tilde{Q}, F)} \in \mathcal{D}(S, \tilde{Q}, F)$, $\varphi$ is an $F$-adapted process satisfying for all $n = 0, \ldots, T$ $P$-a.s.

$$- \frac{P(\tau \geq n \mid F_n)}{P(\tau > n \mid F_n)} < \varphi_n < \frac{P(\tau \geq n \mid F_n)}{P(\tau = n \mid F_n)}, \quad \text{and} \quad Z^G = (Z^{(\tilde{Q}, F)})^\tau Z^{(\varphi)}.$$  

(52)

Here $Z^{(\varphi)}$ is given by

$$Z_t^{(\varphi)} := \prod_{n=1}^t \left( 1 + \varphi_n \frac{P(\tau > n \mid F_n)}{P(\tau \geq n \mid F_n)} I_{\{\tau = n\}} - \frac{P(\tau = n \mid F_n)}{P(\tau > n \mid F_n)} I_{\{\tau > n\}} \right).$$

(53)

**Proof.** We start this proof by making the following three remarks:

1) It is easy to check that (see also [19] for details and related results) the process $\bar{Z}$ is a martingale and hence $\tilde{Q}$ is a well defined probability. Furthermore, the process $\bar{Z}$ is the discrete-time version of
the process $\mathcal{E}(G^{-1}I_{\{G>_0\}} \cdot m)$ (which is well defined even in the case where $G$ might vanish) see \[21\] Subsection 2.3.

2) It is clear that $X$ is a supermartingale under $\tilde{Q}$ if and only if $Y := \hat{Z}X$ is a supermartingale.

3) Thanks to \[18\], the discrete-time version of $\mathcal{E}(\varphi \cdot \eta^G)$ coincides with $Z(\varphi)$ given in \[53\], for any $\mathbb{F}$-optional process $\varphi$.

Then by combining these remarks and Theorem 4.3, the proof of the theorem follows immediately. \[\square\]

**APPENDIX**

A Some $\mathbb{G}$-properties versus those in $\mathbb{F}$

**Lemma A.1.** The following assertions hold.

(a) For any $\mathbb{G}$-predictable process $\varphi^G$, there exists an $\mathbb{F}$-predictable process $\varphi^F$ such that $\varphi^G = \varphi^F I_{[0,\tau]}$. Furthermore, if $\varphi^G$ is bounded, then $\varphi^F$ is bounded with the same constants.

(b) Suppose that $G > 0$. Then for any bounded $\theta \in L(S^\tau, \mathbb{G})$, there exists a bounded $\varphi \in L(S, \mathbb{F})$ that coincides with $\theta$ on $[0, \tau]$.

(c) Suppose $G > 0$, and let $V^G$ be a $\mathbb{G}$-predictable and nondecreasing process with finite values and $(V^G)^\tau = V^G$. Then there exists a unique nondecreasing with finite values and $\mathbb{F}$-predictable process $V$, such that $V^G = V^\tau$.

If furthermore $\Delta V^G < 1$, then $\Delta V < 1$ holds also.

**Proof.** Remark that the boundedness condition for $\varphi^G$ can be reduced to the condition $0 \leq \varphi^G \leq 1$. Thus, assertion (a) is a particular case of the general case treated in [2, Lemma B.1] (see also [20, Lemma 4.4 (b)]), hence its proof will be omitted and we refer the reader to this paper. Thus the remaining part of this proof focuses on proving assertions (b) and (c) in two parts.

**Part 1.** Here we prove assertion (b). Consider a bounded $\theta \in L(S^\tau, \mathbb{G})$. Then $\theta$ is a bounded and $\mathbb{G}$-predictable process satisfying $\theta^\tau \Delta S^\tau > -1$. Thus, in virtue of assertion (a), there exists a bounded and $\mathbb{F}$-predictable process $\varphi$ such that

$$\theta I_{[0,\tau]} = \varphi I_{[0,\tau]}.$$  \(54\)

Then by inserting this equality in $\theta^\tau \Delta S^\tau > -1$, we deduce that

$$\varphi^\tau \Delta S I_{[0,\tau]} > -1,$$

which is equivalent to $I_{[0,\tau]} \leq I_{(\varphi^\tau \Delta S)^\tau > 1}$. By taking the $\mathbb{F}$-optional projection on both sides of this inequality, we get $0 < G \leq I_{(\varphi^\tau \Delta S)^\tau > 1}$ on $[0, +\infty[$, or equivalently $\varphi^{\tau \Delta S} > -1$. Hence $\varphi$ belongs to $L(S, \mathbb{F})$, and the proof of assertion (b) is complete.

**Part 2.** This part proves assertion (c). Consider a $\mathbb{G}$-predictable and nondecreasing process with finite values $V^G$ such that $(V^G)^\tau = V^G$. It is clear that there is no loss of generality in assuming that $V^G$ is bounded. Then due to [20, Lemma 4.4 (b)] (see also [2, Lemma B.1]), there exists an $\mathbb{F}$-predictable process $V$ such that

$$V^G I_{[0,\tau]} = V I_{[0,\tau]}.$$  \(54\)

By writing $V^G I_{[0,\tau]} = V^G - V^G I_{[\tau, +\infty[}$—which is obviously a RCLL bounded $\mathbb{G}$-semimartingale—and by taking the $\mathbb{F}$-optional projection on both sides of (54), we get $V = o^G (V^G I_{[0,\tau]})/G$. Hence $V$ is
a RCCLL $F$-semimartingale that is predictable. As a result, there exists a continuous $F$-martingale $L$ with $L_0 = 0$ and an $F$-predictable process with finite variation $B$ such that $V = L + B$. Since $V^G$ is predictable with finite variation and

$$V^G = V^\tau = L^\tau + B^\tau = \left( L^\tau - G^{-1}_0 I_{[0,\tau]} \cdot (L,m)^F \right) + G^{-1}_0 I_{[0,\tau]} \cdot (L,m)^F + B^\tau,$$

then we conclude that the $G$-local martingale $L^\tau - G^{-1}_0 I_{[0,\tau]} \cdot (L,m)^F$ is null. This implies that $[L,L]^\tau$ is also a null process since $L$ is continuous, or equivalently $L \equiv 0$ due to the assumption $G > 0$. This proves that $V = B$ has a finite variation. To prove that $V$ is nondecreasing it is enough to remark that $(V^G)^{p,F}$ is nondecreasing and $V = G^{-1}_0 \cdot (V^G)^{p,F}$. This proves the first statement of assertion (c), while the proof of the last statement of assertion (c) follows the same steps of part 1). Indeed $\Delta V^G = \Delta V I_{[0,\tau]} \leq 1$ holds if and only if $I_{[0,\tau]} \leq I_{[\Delta V < 1]}$ holds, and this implies that –after taking the $F$-predictable projection on both sides of this inequality– 0 < $G_\tau \leq I_{[\Delta V < 1]}$ on $\|0, +\infty\|$. This is equivalent in fact to $\Delta V < 1$. The fact $G_\tau > 0$ follows from the assumption $G > 0$ and the fact that both sets $\{G_\tau > 0\}$ and $\{G > 0\}$ have the same début (see [20 Lemme (4.3)]). This ends the proof of the lemma.

The following lemma recalls the $G$-compensator of any $F$-optional process stopped at $\tau$.

**Lemma A.2.** Let $V \in A_{loc}(F)$, then we have

$$(V^\tau)^{p,G} = I_{[0,\tau]} G^{-1}_0 \cdot (\tilde{G} \cdot V)^{p,F}.$$ 

For the proof of this lemma and other related results, we refer to [2, 3, 4].

**Lemma A.3.** Let $\varphi$ is a real-valued and $F$-predictable process, $N^G \in M_{0,loc}(F)$, $\varphi^{(o)} \in T_{loc}(N^G,G)$, and $\varphi^{(pr)} \in L_{loc}(\text{Prog}(F), P \otimes D)$ such that

$$\left( 1 + \frac{\Delta N^G}{G_G} \right) I_{[0,\tau]} + \varphi^{(o)} \Delta N^G + \varphi^{(pr)} \Delta D > 0, \quad 0 < \varphi \leq 1. \quad (55)$$

Then the process

$$W := \sum \varphi \Delta S I_{[|\Delta S| > 1]} \left[ \left( 1 + \frac{\Delta N^G}{G_G} \right) I_{[0,\tau]} + \varphi^{(o)} \Delta N^G + \varphi^{(pr)} \Delta D \right]$$

has a $G$-locally integrable variation if and only if both processes

$$W^{(1)} := \sum \varphi \Delta S I_{[|\Delta S| > 1]} \left( 1 + \frac{\Delta N^G}{G_G} \right) I_{[0,\tau]} \quad \text{and}$$

$$W^{(2)} := \sum \varphi \Delta S I_{[|\Delta S| > 1]} \varphi^{(o)} \Delta N^G + \varphi^{(pr)} \Delta D$$

belong to $A_{loc}(G)$.

**Proof.** Due to the first condition in (55), it is clear that $W \in A_{loc}(G)$ iff

$$W^{+} := \sum |\varphi \Delta S| I_{[|\Delta S| > 1]} \left[ \left( 1 + \frac{\Delta N^G}{G_G} \right) I_{[0,\tau]} + \varphi^{(o)} \Delta N^G + \varphi^{(pr)} \Delta D \right] \in A_{loc}^{+}(G).$$
By stopping, there is no loss of generality to assume that $E[W^+_\infty] < +\infty$. Thus, since both $\sum |\varphi \Delta S| I_{\{k \geq |\Delta S| > 1\}} \varphi^{(o)} \cdot N^G$ and $\sum |\varphi \Delta S| I_{\{k \geq |\Delta S| > 1\}} \varphi^{(pr)} \Delta D = |\varphi \Delta S| I_{\{k \geq |\Delta S| > 1\}} \varphi^{(pr)} \cdot D$ are $\mathbb{G}$-local martingale, we derive

$$E \left[ \sum |\varphi \Delta S| I_{\{|\Delta S| > 1\}} (1 + \frac{\Delta N^F}{G - G}) I_{[0,\tau]} \right]$$

$$= \lim_{k \to +\infty} E \left[ \sum |\varphi \Delta S| I_{\{1 < |\Delta S| \leq k\}} (1 + \frac{\Delta N^F}{G - G}) I_{[0,\tau]} \right]$$

$$= \lim_{k \to +\infty} E \left[ (I_{\{|\Delta S| \leq k\}} \cdot W^+) \right] \leq E[W^+_\infty] < +\infty.$$ 

This proves that $W^{(1)} \in A^+_{\text{loc}}(\mathbb{G})$, and hence $W^{(2)} = W - W^{(1)} \in A^+_{\text{loc}}(\mathbb{G})$. Thus, the proof of the lemma is complete. \qed

**B Proof of Theorem 2.2**

*Proof*. We start this proof by highlighting some useful implications of the assumption $G > 0$. Indeed, thanks to [20, Lemma (4.3)], which states that the three sets $\{G = 0\}$, $\{G_- = 0\}$ and $\{\tilde{G} = 0\}$ have the same début, we deduce that the assumption $G > 0$ implies that the following three properties hold:

1) The three processes $G$, $G_-$ and $\tilde{G}$, are positive (strictly) processes,

2) $G_-$ is locally bounded, and $G^{-1}$ is a well defined RCLL semimartingale.

3) The operator $\mathcal{T}$ defined in (55) takes the following form

$$\mathcal{T}(M) = M^\tau - \tilde{G}^{-1} I_{[0,\tau]} \cdot [M, m], \text{ for all } M \in \mathcal{M}_{0,\text{loc}}(\mathbb{F}).$$

(56)

The rest of the proof is divided into three steps. The first step discusses some integrability properties useful for both assertions (a) and (b). The second step proves assertion (a) of the theorem, while the third step proves assertion (b).

**Step 1.** Consider $k \in L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D)$ and let $h$ be an $\mathbb{F}$-optional process such that $k_\tau = h_\tau$ P-a.s. on $(\tau < +\infty)$. In this step, we prove that $(\varphi^{(0)}, \varphi^{(pr)})$ belongs to $\mathcal{I}^D_{\text{loc}}(N^G, \mathbb{G}) \times L^1_{\text{loc}}(\text{Prog}(\mathbb{F}), P \otimes D)$, where

$$\varphi^{(pr)} := k - h, \quad \varphi^{(o)} := h - J^{(h)}.$$ 

and

$$J := \frac{Y^{(h)}}{G}, \quad Y^{(h)} := o^P(h_\tau I_{[0,\tau]}).$$

(57)

Since $k \in L^1(\text{Prog}(\mathbb{F}), P \otimes D)$, then $h \in L^1(\mathcal{O}(\mathbb{F}), P \otimes D)$ and hence $\varphi^{(pr)}$ belongs to $L^1(\text{Prog}(\mathbb{F}), P \otimes D)$ on the one hand. On the other hand, we derive

$$E \left[ |h| |G \tilde{G}^{-1} \cdot D| \right] \leq E \left[ |h_\tau| I_{\{\tau < +\infty\}} \right] < +\infty,$$

which is equivalent to $h \in \mathcal{I}^D(\mathbb{N}^G, \mathbb{G}) \subset \mathcal{I}^D_{\text{loc}}(\mathbb{N}^G, \mathbb{G})$. Now consider the sequence of $\mathbb{F}$-stopping times $T_n := \inf\{t \geq 0 \mid |J_t^{(h)}| > n\}$ with the convention $\inf(\emptyset) = +\infty$. Then $(T_n)$ increases to infinity and
Then we combine these remarks (see full details in the proof of [8, Theorem 2.22]) and write deduce the existence of a unique equality we refer the reader to [8]).

Step 3. Hence, by inserting this latter equality in (58) and using (6), the representation (10) follows immediately.

|J^{(h)}|I_{[0,T_{\tau}]}| \leq n, and hence

\[
E \left[ (|J^{(h)}|GG^{-1} \cdot D)_{T_{\tau}} \right] \leq n + E \left[ \left| Y^{(h)}_{T_{\tau}} G^{-1}_{T_{\tau}} I_{\{\tau = T_{\tau} < +\infty\}} \right| \right] = n + E \left[ \left| Y^{(h)}_{T_{\tau}} \tilde{G}^{-1}_{T_{\tau}} \Delta D^{o,F}_{T_{\tau}} I_{\{T_{\tau} < +\infty\}} \right| \right] \leq n + E \left[ \left| h_{\tau} I_{\{T_{\tau} < \tau\}} \tilde{G}^{-1}_{T_{\tau}} \Delta D^{o,F}_{T_{\tau}} \right| \right] \leq n + E \left[ \left| h_{\tau} I_{\{T_{\tau} < \tau\}} \right| < +\infty. \right]
\]

This proves that \( J^{(h)} \) belongs to \( T^{o,F}_{loc}(N^G, G) \) and hence \( \varphi^{(o)} \) does also.

**Step 2.** Here, we prove assertion (a). To this end, we remark that the process \( H := o,F \) \( (h_{\tau}) \) can be decomposed as follows

\[
H = h \cdot D + J^{(h)}_{1_{[0,\tau]} = (h - J^{(h)}) \cdot D + (J^{(h)})^\tau, (58)}
\]

\[
Y^{(h)} := M^{(h)} - h \cdot D^{o,F}, \quad M^{(h)} := o,F \left( \int_0^{+\infty} h_s dD_s \right) (59)
\]

For full details about these facts, we refer the reader to [8]. Thus, thanks to Itô (applied to 1/G) and \( \Delta G = \Delta m - \Delta D^{o,F} \), we get

\[
d \left( \frac{1}{G} \right) = -\frac{1}{(G^-)^2} dm + \frac{1}{G(G^-)^2} d[m,m] + \frac{G_- - \Delta m}{G(G^-)^2} dD^{o,F}.
\]

By combining this equation, (59) and \( \tilde{G}(G_- - \Delta m) + (\Delta m)^2 = G^2 \), we obtain

\[
d \left( \frac{1}{G} \right) = -\frac{1}{(G^-)^2} dT(m) + \frac{1}{GG} dD^{o,F}. (60)
\]

Then again Itô and (59) combined with (59) and \( Y = GJ \), we derive (for full details about the following equalities we refer the reader to [8])

\[
d(J^{(h)})^\tau = d \left( \frac{Y^\tau}{G} \right) = \frac{1}{G^\tau} dY^\tau + Y^\tau d \left( \frac{1}{G^\tau} \right) + d \left[ \frac{1}{G^\tau}, Y^\tau \right] = -\frac{J^{(h)}}{G^-} dT(m) + \frac{1}{G^-} dT(M^{(h)}) + \frac{G_- J^{(h)} + \Delta M^{(h)} - h\tilde{G}}{GG} I_{[0,\tau]} dD^{o,F} = -\frac{J^{(h)}}{G^-} dT(m) + \frac{1}{G^-} dT(M^{(h)}) + \frac{J^{(h)} - h}{G} I_{[0,\tau]} dD^{o,F}.
\]

Hence, by inserting this latter equality in (60) and using (61), the representation (10) follows immediately, and the proof of assertion (a) is complete.

**Step 3.** This step proves assertion (b). Consider a \( G \)-martingale \( M^G \). Therefore, see [8] for full details, there exist a unique \( k \in L^1(\text{Prog}(\mathbb{F}), P \otimes D) \) such that \( M^G = k_{\tau} \). Hence, to this process \( k \), we deduce the existence of a unique \( h \in L^1(\mathcal{O}(\mathbb{F}), P \otimes D) \) satisfying

\[
E(k_{\tau} | \mathcal{F}_\tau) = h_{\tau} \quad \text{in fact we have} \quad h := E_\mu(k | \mathcal{O}(\mathbb{F})), \quad \mu := P \otimes D.
\]

Then we combine these remarks (see full details in the proof of [8, Theorem 2.22]) and write

\[
(M^G)^\tau = o,G \left( M^G \right) = o,G \left( k_{\tau} \right) = (k - h) \cdot D + o,G \left( h_{\tau} \right).
\]
By applying assertion (a) to the $G$-martingale $o^G(h_F)$ and putting

$$\varphi^{(pr)} := k - h, \quad \varphi^{(o)} := h - J^{(h)} := h - \frac{M^{(h)} - h \cdot D^{\alpha_F}}{G},$$

$$M^F := G_\cdot M^{(h)} - G_\cdot J^{(h)} \cdot m,$$

where $M^{(h)}$ is given by \([59]\), \([13]\) follows immediately, and hence assertion (b) is proved. This ends the proof of the theorem.

\[\Box\]

\section{Proof of the lemma 4.2}

\textbf{Proof.} The proof of this lemma will be achieved in two steps. The first step proves that there exists a unique pair $(N, V)$ satisfying \((30)\) as soon as $Z$ is a deflator for $(X, \mathbb{H})$. The second step shows that for a process $Z$, for which there exists a pair $(N, V)$ satisfying \((30)\), there is equivalence between $Z \mathcal{E}(\varphi \cdot X)$ is supermartingale and \((31)\), for any bounded $\varphi \in \mathcal{L}(X, \mathbb{H})$.

\textbf{Step 1.} Suppose that $Z$ is a deflator. This implies that $Z$ is a positive supermartingale (since $\varphi = 0 \in \mathcal{L}(X, \mathbb{H})$), and hence $X := Z^{-1} \cdot Z$ is a local supermartingale having the unique Doob-Meyer decomposition $X := K - V$, where $K \in \mathcal{M}_{0,loc}(\mathbb{H})$ and $V$ is nondecreasing and predictable with $\Delta V < 1$ (since $Z > 0$). It is clear that the predictable process $(1 - \Delta V)^{-1}$ is well defined and is locally bounded. Hence

$$N := \frac{1}{1 - \Delta V} \cdot K \in \mathcal{M}_{0,loc}(\mathbb{H}), \quad \Delta N > -1 \quad \text{and} \quad Z = Z_0 \mathcal{E}(N) \mathcal{E}(-V).$$

This ends the first step.

\textbf{Step 2.} Suppose that there exists a pair $(N, V)$ such that $Z = Z_0 \mathcal{E}(N) \mathcal{E}(-V)$ and \((30)\) holds. Let $\varphi$ be a bounded element of $\mathcal{L}(X, \mathbb{H})$. Then by applying Itô to $Z \mathcal{E}(\varphi \cdot X) = Z_0 \mathcal{E}(N) \mathcal{E}(-V) \mathcal{E}(\varphi \cdot X)$, one get

$$Z \mathcal{E}(\varphi \cdot X) = Z_0 \mathcal{E}(N) \mathcal{E}(\varphi \cdot X) \mathcal{E}(-V) = Z_0 \mathcal{E}(N + \varphi \cdot X + \varphi \cdot [X,N]) \mathcal{E}(-V) = Z_0 \mathcal{E}(Y^{(\varphi)}) \mathcal{E}(-V) = Z_0 \mathcal{E}(1 - \Delta V \cdot Y^{(\varphi)} - V),$$

where $Y^{(\varphi)} := N + \varphi \cdot X + \varphi \cdot [X,N]$. Since $Z$ is positive and $\varphi \in \mathcal{L}(X, \mathbb{H})$, then the process $Z \mathcal{E}(\varphi \cdot X)$ is an $\mathbb{H}$-supermartingale if and only if $(1 - \Delta V) \cdot Y^{(\varphi)} - V$ is a local $\mathbb{H}$-supermartingale, or equivalently $Y^{(\varphi)}$ is a special semimartingale (which is equivalent to the first condition of \((31)\)) and its predictable with finite variation part, $(Y^{(\varphi)})^{p,H}$, satisfies $(Y^{(\varphi)})^{p,H} \leq (1 - \Delta V)^{-1} \cdot V$. This finishes the second step, and the proof of the lemma as well.

\textbf{Acknowledgements:} This research is fully supported financially by the Natural Sciences and Engineering Research Council of Canada, through Grant G121210818. The authors would like to thank Ferooz Alharbi, Safa Alsheyab, Jun Deng, Monique Jeanblanc, Youri Kabanov, Martin Schweizer and Michele Vanmaele for several comments and/or suggestions, fruitful discussions on the topic, and/or for providing important and useful related references.

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