A kinetic theory for a dilute gas-solid suspension under a simple shear is developed. With the aid of the corresponding Boltzmann equation, it is found that the flow curve (stress-strain rate relation) has a S-shape as a crossover from the Newtonian to the Bagnoldian for a granular suspension or from the Newtonian to a fluid having a viscosity proportional to the square of the shear rate for a suspension consisting of elastic particles. The existence of the S-shape in the flow curve directly leads to a discontinuous shear thickening (DST). This DST corresponds to the discontinuous transition of the kinetic temperature between a quenched state and an ignited state. The results of the event-driven Langevin simulation of hard spheres perfectly agree with the theoretical results without any fitting parameter. The simulation confirms that the DST takes place in the linearly unstable region of the uniformly sheared state.

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I. INTRODUCTION

Shear thickening is a drastic rheological process in which the viscosity increases as the shear rate increases. The shear thickening fluid typically behaves as a liquid at rest or weakly stirred situation, but its resistance becomes large as if it is a solid above a critical shear rate. The shear thickening can be a continuous shear thickening (CST) or a discontinuous shear thickening (DST) depending on its situation. The DST can be easily observed in densely packed suspensions of cornstarch in water. There are many industrial applications of the DST such as a body armor and a traction control.

The DST attracts much attention even from physicists [1–7]. Typical suspensions exhibiting DSTs have some common features. The first one is that the discontinuous jump can be only observed below the jamming point [8, 9]. The second characteristic feature is that the normal stress difference becomes large when the DST takes place [4, 5]. The third characteristic feature is that the mutual frictions between grains play important roles in the DSTs for dense suspensions and dry granular materials [4, 9–15]. There are some phenomenologies to understand the mechanism of the DST for dense suspensions and dry granular materials [16–20]. The most successful phenomenology, so far, is presented by Wyart and Cates [20] in which the rescaled pressure exhibits a saddle-node bifurcation at a critical density. Then, their flow curve exhibits a S-shape above the critical density.

So far, the most of studies on the shear-thickening focus on the rheological behavior of dense suspensions. However, we can look for the possibility of the occurrence of shear-thickening like processes even in dilute gas-solid suspensions. Such gas-solid suspensions are usually discussed in the context of fluidized beds [21, 22] which might be categorized as one of typical inertial suspensions [23], though the uniform flows are often unstable [24–28]. Nevertheless, there is a homogeneous phase if we control the injected gas flow rate from the bottom of the container. Indeed, Tsao and Koch [29] demonstrated the existence of a discontinuous phase transition for the kinetic temperature between a quenched state (a low temperature state) and an ignited state (a high temperature state) in a simple shear flow of such an inertial suspension consisting of elastic particles [23] in terms of the analysis of the Boltzmann equation. They also illustrated the existence of a strong shear thickening, i. e. a rapid increment of the shear viscosity as the shear rate increases, though it is not clear whether it is the DST or the CST [30]. Sangani et al. extended the analysis of Ref. [29] to the case of finite density and found that the discontinuous transition of the kinetic temperature for dilute suspensions becomes continuous at relatively low density [31]. Santos et al. also demonstrated the existence of a CST in moderately dense hard core gases by using the revised Enskog theory [32].

One recent paper suggests the existence of a DST for relatively dilute suspensions in terms of the analysis of a phenomenological BGK equation [33] as a result of a saddle-node bifurcation of two branches, the Newtonian and the branch of the viscosity proportional to the square of the shear rate. Note that the asymptotic form in high shear limit agrees with those in Ref. [29, 35]. It should be noted that the BGK model is only valid for elastic particles and the relaxation time scale cannot be determined within the theory. If we are interested in the behavior for granular suspensions, we need to add some extra terms. Therefore, the previous theory is only a qualitative theory but not a quantitative one. We also indicate that there is no quantitative test of the validity of the BGK model by the comparison between the simulation and the theoretical prediction in the previous paper [33]. Nevertheless, their paper also predicted that the DST becomes the CST as the density increases [33] as Sangani et al. indicated [31]. Such a behavior in gas-solid suspensions is contrast to that observed typical dense suspensions.

The purpose of this paper is to extract the mechanism of the DST in terms of the Boltzmann equation for dilute gas-solid suspensions in terms of a proper treatment of the statistical mechanics by identifying the discontinuous transition between the quenched state and the ignited state as the DST. We also examine the quantitative validity of our simple theory from the comparison of results between our theory and the event-driven Langevin simulation of hard spheres (EDLSHS) [34]. Therefore, we demonstrate that (i) the mutual friction between grains is not always necessary for the DST and (ii) the DST can take place even in the dilute inertia suspensions in this paper.

The organization of this paper is as follows. In the next section, we give a microscopic basic equation, the Boltzmann equation under the influence of the background fluid which is equivalent to the Langevin equation associated with collisions of particles. We also derive a set of equations for the kinetic temperature $T$, the anisotropy of the temperature $\Delta T$ which is proportional to the normal stress difference and the shear stress $P_{xy}$ when we adopt Grad’s approximation. In Sec. III, we obtain the flow curve which exhibits the S-shape curve as a result of the saddle-node bifurcation as well as $T$, $\Delta T$ and the stress ratio $\mu = -P_{xy}/P$, where $P$ is the pressure. In Sec. IV, we perform the simulation (EDLSHS) to verify the quantitative validity of our theoretical predictions. In the final section, we discuss and summarize our obtained results. We have four appendices to support the detailed calculation in our paper. In Appendix A, we discuss the connection between Grad’s approximation and Green-Kubo formula within the BGK approximation. Appendix B is devoted to some detailed calculations of integrals used in this paper. In Appendix C, we discuss the linear stability analysis of the stationary solution of our model. In Appendix D, we give a brief explanation of the method of EDLSHS.
II. KINETIC THEORY

Let us consider a collection of smooth mono-disperse spherical grains (the diameter $\sigma$, the mass $m$ and the restitution coefficient $\epsilon$ which is ranged $0 < \epsilon \leq 1$) distributed in a $d$-dimensional space influenced by the background fluid. We assume that the macroscopic velocity field $\mathbf{u} = (u_x, u_\perp)$ satisfies the simple shear flow

$$ u_x = \dot{\gamma} y, \quad u_\perp = 0, \quad (1) $$

where $\dot{\gamma}$ is the shear rate. Because we are interested in the homogeneous phase in fluidized beds, we assume that the effects of the gravity for the motion of particles are negligible. Introducing the peculiar momentum of $i$-th particle as $\mathbf{p}_i \equiv m(\mathbf{v}_i - \dot{\gamma} y \mathbf{e}_x)$ with the unit vector $\mathbf{e}_x$ parallel to $x$-direction with the velocity of $i$--th particle $\mathbf{v}_i$, a reasonable starting point for the motion of grains at low Reynolds number flows is the Langevin equation

$$ \frac{dp_i}{dt} = -\zeta p_i + \mathbf{F}_i^{(\text{imp})} + m\xi_i, \quad (2) $$

where we have introduced the impulsive force $\mathbf{F}_i^{(\text{imp})}$ to express collisions and the noise $\xi_i(t) = \xi_{i,\alpha}(t)e_\alpha$ satisfying

$$ \langle \xi_i(t) \rangle = 0, \quad \langle \xi_{i,\alpha}(t)\xi_{j,\beta}(t') \rangle = 2\zeta T_{\text{ex}}\delta_{ij}\delta_{\alpha\beta}\delta(t - t'). \quad (3) $$

Here, we have introduced $\zeta$ and $T_{\text{ex}}$ to characterize the drag from the background fluid and the environmental temperature under the unit of the Boltzmann constant $k_B = 1$, respectively. This model is essentially a dilute version of the model used by Kawasaki et al. [35]. We, here, assume that the inertia of the particles is important but the inertia of the fluid (or the gas) is negligible. Typically these conditions hold for the particles with the diameters in the range 1-70 μm [23]. Even when we consider such gas-solid suspensions, the realistic drag coefficient $\zeta$ must be a resistance matrix which strongly depend on the configuration of particles [28]. For simplicity, however, we regard $\zeta$ as a scalar constant which is independent of the configuration of particles as in Refs. [29, 31]. This treatment might be justified if we consider cases in the dilute limit or the dense limit.

In this paper, we also assume that $\zeta$ is proportional to $\sqrt{T_{\text{ex}}}$, because the drag coefficient is proportional to the viscosity of the solvent which is proportional to $\sqrt{T_{\text{ex}}}$ if the solvent consists of hard core molecules. (If we regard the gas as a dilute hard core gas for $d = 3$, the drag coefficient is given by $\zeta = 3\pi\eta_0\sigma/m$ where $\eta_0$ and $\sigma_0$ are $\eta_0 = (5/16\sigma_0^2)\sqrt{m_0 T_{\text{ex}}/\pi}$ with the mass of the molecule $m_0$ and the diameter of the molecule, respectively.)

We emphasize that Eq. (2) contains both the collision term and the thermal noise, where the noise represents random and uncorrelated collisions between gas molecules and the suspended grains. Although some previous papers [29, 31] ignored the thermal noise, the existence of the thermal noise is crucially important because (i) thanks to this term, the system can reach a thermal equilibrium state characterized by $T_{\text{ex}}$, (ii) the background viscosity $\eta_0$ and the drag $\zeta$ become zero if we take the limit $T_{\text{ex}} \to 0$, (iii) relatively small suspensions which are the target of our study are affected by the thermal noise if there is no external shear and (iv) the Newtonian rheology cannot be recovered at $T_{\text{ex}} = 0$ in zero shear limit, as will be shown. Of course, the thermal noise is not important if collisions between grains plays important roles in large shear cases. We also note that the inertial effects are believed to be important for inertial suspensions for large shear rate. Indeed, the large shear regime can be characterized by the large Stokes number $St = \rho_p\sigma^2\dot{\gamma}/\eta_0$ where $\rho_p$ is the mass density of a suspended particle. The Stokes number can be interpreted as the ratio of the kinetic energy $mC_0^2/2$ to the work $\eta_0\sigma^2 C_0$ due to the drag force proportional to $\eta_0\sigma$, where $C_0 = \sigma^2\dot{\gamma}$ is the characteristic speed for collisions of particles. (In other words, the Stokes number is expressed as $St = \tau_r/\tau_p$ where $\tau_r = m/3\pi\eta_0\sigma = \zeta^{-1}$ and $\tau_p = \sigma/C_0 = \dot{\gamma}^{-1}$ are, respectively, the relaxation time due to the drag force and the passing time of the particle scale $\sigma$ with the speed $C_0$ [28]. Therefore, small (large) $St$ corresponds to collisionless (collisional) regime. We also note that $St$ is proportional to the Péclet number $Pe = 3\pi\eta_0\sigma^3/(4T_{\text{ex}}) \propto \dot{\gamma}/\sqrt{T_{\text{ex}}} \sim St$. Thus, the low shear regime is dominated by the thermal motion.) Therefore, both contributions can coexist only for the transient regime between two limiting (low shear and high shear) cases.

It is well known that the Langevin equation (2) can be converted into the equation for the $N$–body distribution function $f^{(N)}(\{\mathbf{r}_i, \mathbf{V}_i\}, t)$ with $\mathbf{V}_i = \mathbf{p}_i/m = \mathbf{v}_i - \mathbf{u}$, which is represented by a sum of the Fokker–Planck type equation and the collision term. If we are interested in a dilute suspension or a moderately dense suspension, the equation of $N$–body distribution function is reduced to that of the one-body distribution function $f(\mathbf{V}, t)$ under the simple shear as [29, 31, 36–38]

$$ \left( \frac{\partial}{\partial t} - \dot{\gamma} \mathbf{V}_y \frac{\partial}{\partial \mathbf{V}_x} \right) f(\mathbf{V}, t) = \zeta \frac{\partial}{\partial \mathbf{V}} \cdot \left( \mathbf{V} + \frac{T_{\text{ex}}}{m} \frac{\partial}{\partial \mathbf{V}} \right) f(\mathbf{V}, t) + J(\mathbf{V}|f), \quad (4) $$

where we have ignored the spatial fluctuating term in Eq. (4) and introduced the peculiar velocity $\mathbf{V} \equiv \mathbf{v} - \mathbf{u}$, because the uniform shear flow is stable as long as we have checked. The collisional integral $J(\mathbf{V}|f)$ for a dilute suspension is
assumed to be given by

\[ J(V_1|f) = \sigma^{d-1} \int d\sigma \Theta(\sigma \cdot \hat{\sigma}) |v_{12} \cdot \hat{\sigma}| \left\{ \frac{f(V_1^{**})f(V_2^{**})}{e^2} - f(V_1)f(V_2) \right\}, \]

(5)

where \( \hat{\sigma} \) is the normal unit vector at contact, \( \Theta(x) = 1 \) for \( x \geq 0 \) and \( \Theta(x) = 0 \) otherwise and \( V_i^{**} = v_i^{**} - u \) for \( i = 1, 2 \) is the pre-collisional velocity of \( V_i \) defined through the pre-collisional velocities \( v_i^{**} \):

\[ v_1^{**} = v_1 - \frac{1 + e}{2e} (v_{12} \cdot \hat{\sigma}) \hat{\sigma}, \quad v_2^{**} = v_2 + \frac{1 + e}{2e} (v_{12} \cdot \hat{\sigma}) \hat{\sigma} \]

(6)

with \( v_{12} = v_1 - v_2 \). Once we adopt Eq. (4) associated with Eqs. (5) and (6) instead of Eq. (2), we can construct a theory describing the shear thickening.

One of the most important quantities to characterize the rheology of the dilute suspension is the pressure tensor

\[ P_{\alpha\beta} = m \int dV \alpha V_{\beta} f(V, t). \]

(7)

This is related to the pressure as \( P \equiv P_{\alpha\alpha}/d \), where we adopt Einstein’s notation for the sum rule \( i.e. \quad P_{\alpha\alpha} = \sum_{\alpha=1}^{d} P_{\alpha\alpha}. \)

Multiplying \( mV_{\alpha} V_{\beta} \) by Eq. (4) and integrate it over \( \nu \), we obtain

\[ \frac{d}{dt} P_{\alpha\beta} + 5(\delta_{\alpha x} P_{y\beta} + \delta_{\beta x} P_{y\alpha}) = -\Lambda_{\alpha\beta} + 2\zeta(nT \delta_{\alpha\beta} - P_{\alpha\beta}), \]

(8)

where we have introduced the number density \( n \equiv \int dV f(V, t) \) and

\[ \Lambda_{\alpha\beta} \equiv -m \int dV \alpha V_{\beta} J(V|f). \]

(9)

Because Eqs. (8) and (9) are not closed equations, we adopt Grad’s approximation [33, 38–44]

\[ f(V) = f_{eq}(V) \left[ 1 + \frac{m}{2T} \left( \frac{P_{\alpha\beta}}{nT} - \delta_{\alpha\beta} \right) V_{\alpha} V_{\beta} \right] \]

(10)

with

\[ f_{eq}(V) = n \left( \frac{m}{2nT} \right)^{d/2} \exp \left( -\frac{mV^2}{2T} \right), \]

(11)

where we have introduced the kinetic temperature \( T \) defined by \( T \equiv \int d\nu (\nu - u)^2 f(V)/(dn) \). Note that the pressure satisfies the equation of state for an ideal gas \( P = nT \) in our model. Grad’s approximation or Grad’s 13 moments method for \( d = 3 \) is the well established method to describe the slow motion of nonequilibrium gases [33, 38–44]. In fact, Grad’s 13 moment method is a natural extension of the Chapman-Enskog expansion [45] which can be regarded as Grad’s 5 moments method in terms of \( d + 2 \) collisional invariance (the number of particles, the components of momentum and the energy). Grad’s moment method consists of \( d + 2 \) collisional invariants plus the heat flux and the stress tensor. Note that the number of independent components of the stress tensor for Grad’s expansion is \( (d - 1)(d + 2)/2 \to 5 \) for \( d = 3 \), because the stress tensor is symmetric and the trace of the stress tensor is proportional to the kinetic energy. We also note that Grad’s approximation satisfies the Green-Kubo formula within the BGK approximation (see Appendix A). It should be noted that the contribution from the Chapman-Enskog expansion is irrelevant in the present analysis, because its contribution disappears if the system is spatially uniform. We also note that the heat flux is irrelevant for our problem. Therefore, Eq. (10) is a natural assumption to describe the nonequilibrium fluid under the shear. The quantitative justification of Eq. (10) will be examined through the comparison between the theoretical results in terms of Eq. (10) and the results of simulation of Eqs. (2) and (3) in Sec. IV.

When we adopt Eq. (10), it is straightforward to show the relation

\[ \Lambda_{\alpha\beta} = \nu(P_{\alpha\beta} - nT \delta_{\alpha\beta}) + \lambda nT \delta_{\alpha\beta}, \]

(12)

where \( \nu \) and \( \gamma \) are, respectively, given by [43] (see Appendix B for details)

\[ \nu = n\sqrt{T} v_0; \quad \nu_0 = \frac{2\pi^{(d-1)/2} \sigma^{d-1}(1 + e)(2d + 3 - 3e)}{d(d + 2)\Gamma(d/2)\sqrt{m}}, \]

(13)

\[ \lambda = (1 - e^2) n\sqrt{T} \lambda_0; \quad \lambda_0 = \frac{2\pi^{(d-1)/2} \sigma^{d-1}}{d\Gamma(d/2)\sqrt{m}}, \]

(14)
where $\Gamma(x) \equiv \int_0^\infty dt t^{x-1} e^{-t}$. Here, we have introduced constants $\nu_0$ and $\lambda_0$ whose details are unimportant for later discussion. Note that the nonlinear corrections from $f_{eq}(V)$ are ignored to obtain $\nu$ and $\lambda$ in Eqs. (13) and (14).

From Eq. (12) with Eqs. (13) and (14), Eq. (8) can be rewritten as three coupled equations:

\[
\frac{dT}{dt} = -\frac{2\gamma_1}{\nu_0} P_{xy} - \lambda T + 2\zeta (T_{ex} - T),
\]

\[
\frac{d\Delta T}{dt} = -\frac{2}{\nu} \gamma P_{xy} - (\nu + 2\zeta) \Delta T,
\]

\[
\frac{dP_{xy}}{dt} = \gamma_1 \left( \frac{\Delta T}{\nu} - T \right) - (\nu + 2\zeta) P_{xy},
\]

where we have introduced $\Delta T \equiv (P_{xx} - P_{yy})/n$ and used $P_{yy} = P_{\perp \perp}$ with $P_{yy} = (P_{yy} - P_{xx})/d + P_{\alpha \alpha}/d$ with the notation of $P_{\perp \perp}$ for any perpendicular component to $x$, i.e., $\perp = y, z, \cdots$. It should be noted that $P_{yy}$ is not always equal to $P_{zz}$ or $P_{\perp \perp}$ in general [31, 38], but the equality $P_{yy} = P_{\perp \perp}$ is held when we ignore nonlinear contributions from $f_{eq}(V)$ as in Eqs. (13) and (14) for dilute suspensions.

We stress that Eqs. (15)–(17) are coupled equations for the pressure, the shear stress and the normal stress difference. Thanks to this set of coupled equations once the normal stress difference becomes large, the shear stress and the pressure can be large. Therefore, we expect that the DST can take place if the discontinuous transition of the kinetic temperature between a quenched state and an ignited state exists or the sudden increment of the normal stress difference exists.

## III. RHEOLOGY

Now, let us derive a relation between the shear rate $\dot{\gamma}$ and the viscosity $\eta \equiv -P_{xy}/\dot{\gamma}$ from Eqs. (15)-(17) as well as the relations of $T$ and $\Delta T$ against $\dot{\gamma}$. Here, we introduce the following dimensionless quantities:

\[
\nu^* = \frac{\nu}{\sqrt{\theta^*}}, \quad \lambda^* = \frac{\lambda}{\sqrt{\theta^*}}, \quad \dot{\gamma}^* = \frac{\dot{\gamma}}{\zeta}
\]

where we have introduced $\theta \equiv T/T_{ex}$. Note that $\dot{\gamma}^*$ corresponds to the Stokes number $St = \rho_0 \sigma^2 \dot{\gamma}/\eta_0 = 18\dot{\gamma}^*$ in Refs. [29, 31]. Because of Eqs. (13), (14) and $\zeta \propto \sqrt{T_{ex}}$, $\nu^*$ and $\lambda^*$ are independent of both $T$ and $T_{ex}$.

In a steady state, Eqs. (15) and (16) are reduced to

\[
\frac{\Delta \theta}{\theta} = \frac{d[\lambda^* \sqrt{\theta} + 2(1 - \theta^{-1})]}{\nu^* \sqrt{\theta} + 2},
\]

where $\Delta \theta \equiv \Delta T/T_{ex}$. Substituting this into Eq. (16) we obtain the equation for $P_{xy}^* = P_{xy}/(nT_{ex})$:

\[
P_{xy}^* = \frac{d\theta}{2\dot{\gamma}^*} \left\{ \lambda^* \sqrt{\theta} + 2(1 - \theta^{-1}) \right\}.
\]

Then, substituting Eqs. (19) and (20) into the steady equation of Eq. (17) we obtain

\[
\dot{\gamma}^* = (\nu^* \sqrt{\theta} + 2) \sqrt{\frac{d[\lambda^* \sqrt{\theta} + 2(1 - \theta^{-1})]}{2[\nu^* - \lambda^*] \sqrt{\theta} + 2}},
\]

Therefore, the dimensionless viscosity $\eta^* = -P_{xy}^*/\dot{\gamma}^*$ is given by

\[
\eta^* = \frac{\theta \{(\nu^* - \lambda^*) \sqrt{\theta} + 2 \theta^{-1})\}}{(\nu^* \sqrt{\theta} + 2)^2}.
\]

Unfortunately, we cannot express $\eta^*$ in Eq. (22) as a function of $\dot{\gamma}^*$ in Eq. (21) explicitly, but $\eta^*$ and $\dot{\gamma}^*$ can be parametrically expressed as functions of $\theta$. We can also give the explicit asymptotic forms in the limits $\dot{\gamma}^* \to 0$ and $\dot{\gamma}^* \to \infty$. This model exhibits the crossover from the Newtonian regime for $\dot{\gamma} \to 0$:

\[
\eta^* \to \frac{\theta_{\min}}{\nu^* \sqrt{\theta_{\min}} + 2}
\]
at $\theta = \theta_{\text{min}}$, where $\theta_{\text{min}}$ is a real solution of $\theta = (1 + \lambda^* \sqrt{\theta}/2)^{-1}$ ($\theta_{\text{min}} = 1$ for $e = 1$), to the Bagnold viscosity

$$
\eta^* \to \sqrt{2 \left( \frac{\nu^* - \lambda^*}{\sqrt{\lambda^* \nu^*}} \right)^{3/2}} \gamma^* \, \theta \to \frac{2(\nu^* - \lambda^*) \gamma^* \nu^*}{d \nu^* \lambda^*},
$$

in the high shear rate limit for $e < 1$. Note that the asymptotic forms

$$
\eta^* \to \frac{\gamma^* \nu^*}{d \nu^*}, \quad \theta \to \frac{\gamma^* \nu^*}{d \nu^*},
$$

in the high shear limit for the elastic case $e = 1$ are quite different from those for $e < 1$. The asymptotic form of $\eta^*$ for $e = 1$ in Eq. (25) is observed in Ref. [35] and theoretically predicted in Ref. [33]. We also note that $\nu^*$ and $\lambda^*$ depend on the restitution coefficient $e$ as shown in Eqs. (13) and (14), where $\lambda^*$ becomes zero at $e = 1$.

Equations (23)–(25) can be rewritten as

$$
\eta \to \frac{n T_{\text{min}}}{2 \zeta},
$$

$$
\eta \to \sqrt{\frac{2(\nu_0 - (1 - e^2) \lambda_0)}{n}} \gamma^* \, \frac{1}{\sqrt{(1 - e^2) d \lambda_0 n^2}} \gamma^* \, \frac{2(\nu_0 - (1 - e^2) \lambda_0)}{(1 - e^2) d \lambda_0 n^2} \gamma^* \, \frac{1}{\sqrt{(1 - e^2) d \lambda_0 n^2}} \gamma^*,
$$

respectively, in dimensional forms, where $T_{\text{min}} = \theta_{\text{min}} T_{\text{ex}}$ reduces to $T_{\text{ex}}$ in the elastic limit $e \to 1$. To derive Eq. (26) we have used $\nu^* \sqrt{\theta_{\text{min}}} \ll 1$ in the dilute limit because of $\nu^* \approx \varphi$ and $\theta_{\text{min}} \approx 1$ for $e \lesssim 1$. The expression in Eq. (26) is proportional to $\eta_0$ for $e = 1$ because of $T_{\text{min}} \to T_{\text{ex}}$ and $\zeta \approx \eta_0 \approx \sqrt{\nu_{\text{ex}}}$. Equations (26)-(28) have interesting characteristics. In the low shear regime in Eq. (26), the viscosity is Newtonian which approaches zero if $T_{\text{ex}} \to 0$. Thus, our theory rescues the drawback of the previous analysis in Ref. [29] without adding the Newtonian viscosity by hand [46]. Note that both $\eta$ and $T$ diverge in the dilute limit $n \to 0$ and the high shear limit $\gamma \to \infty$ (see Eqs. (27) and (28)). Therefore, there exist the large contrasts in both the viscosity and the kinetic temperature between the low shear (quenched) regime and the high shear (ignited) regime for the dilute gas. We also note that Eqs. (27) and (28) satisfy $\eta = \{1 - (1 - e^2) \lambda_0/\nu_0\} \eta_0^{-1} \sqrt{T}$ and $\eta = \sqrt{T}/(\sqrt{(1 - e^2) d \lambda_0})$, respectively, which are known results [43].

Figure 1 (a) plots the behavior of the dimensionless kinetic temperature $\theta$ against $\gamma^*$, which exhibits a discontinuous transition between the quenched state for low shear regime and the ignited state for high shear regime. This result corresponds to that obtained by Tsao and Koch [29], but the behavior for $\gamma^* \to 0$ is different corresponding to the different asymptotic viscosities in the low shear regime. Indeed, the theoretical prediction on $T$ in Ref. [29] approaches zero satisfying $T \sim \text{St}^3 \sim \gamma^3$ but our result becomes $T \to T_{\text{min}}$ for $\gamma^* \to 0$ [46]. Needless to say, it is natural that the temperature of suspension is identical to the environmental temperature $\theta_{\text{env}}$. Thus, $T = T_{\text{ex}}$ in the absence of the shear in the elastic limit $e \to 1$. Note that there is a linearly unstable region for the steady solution which are indicated by the cross points in Fig. 1 (a). See Appendix C for the details of the linear stability analysis.

As shown in Fig. 1 (b) (the solid line for $e = 0.9$ and the dashed line for $e = 1$), we have also confirmed the existence of DSTs, in which the flow curves have S-shapes. Then, if we gradually increases/decreases $\gamma^*$, the viscosity $\eta^*$ discontinuously increases/decreases at a certain value of the shear rate. Therefore, this S-shape flow curve directly leads to the existence of the DST. Therefore, the saddle-node bifurcation takes place as the result of the connection between shear branch and the Newtonian branch at finite $T_{\text{ex}}$. This picture agrees with that in Ref. [29]. Note that the viscosity also has the linearly unstable region of the steady solution indicated by the cross points in Fig. 1 (b).

Let us provide an additional explanation on the mechanism of the DST for granular suspensions in a different manner. The viscosity for $e < 1$ in the limit $\gamma \to \infty$ is equivalent to that for a dilute dry granular gas, where the viscosity diverges in this limit as in Eq. (27). On the other hand, the Newtonian dimensionless viscosity $\eta^*$ is finite, which is determined by $\zeta$. Thus, the ratio of the viscosity for large $\gamma$ to that for small $\gamma$ is quite large. This is another background to have a large discontinuity at the DST when two branches are connected.

We also plot the theoretical prediction of $\Delta \theta$ against $\gamma^*$ (see Fig. 2). It is remarkable that $\Delta \theta$ is insensitive to $e$. This is because the asymptotic forms, $\Delta \theta \to \theta_{\text{min}}^2 \gamma^* \gamma^* \left\{ (\nu^* - \lambda^*) \sqrt{T_{\text{min}} + 2 \theta_{\text{min}}} \right\}/4$ in the limit $\theta \to \theta_{\text{min}}$, and $\Delta \theta \to 2(\nu^* - \lambda^*) \gamma^* \gamma^* \left\{ (\nu^* - \lambda^*) \right\}/4$ in the limit $\gamma^* \to \infty$, do not have any singularity at $e = 1$.

The stress ratio $\mu = -P_{xy}/P$ is also an important quantity to characterize the rheology [47]. Because we do not control $P$, we plot $\mu$ against $\gamma^*$ (see Fig. 3). The stress ratio satisfies $\mu \to \gamma^* \sqrt{T_{\text{min}}}/2 \gamma^* \gamma^*$ in the limit $\gamma^* \to 0$ and has a peak around $\gamma^* \approx 3$. Then $\mu$ takes multiple values as the result of the S-shape in the flow curve. The asymptotic form of $\mu$ in the limit $\gamma^* \to \infty$ strongly depends on $e$ as $\mu \to \sqrt{d \lambda}/(\nu^* \sqrt{\nu^* - \lambda^*})$ for $e < 1$ and $\mu \to d/\gamma^*$ for $e = 1$. It is interesting that inelastic collisions ($e < 1$) create a finite stress ratio, which might be one of important characteristics of macroscopic collections of granular particles.
IV. SIMULATION

Let us check the quantitative validity of our theoretical results by using the three dimensional simulation under Lees-Edwards boundary condition [34, 48, 49]. Because the Boltzmann equation (4) with Eq. (5) is equivalent to the Langevin equation (2) with Eq. (3) for dilute suspensions, we simulate Eq. (2) instead of numerically solving Eq. (4). It is difficult to implement the standard event-driven code for hard spheres because of the existence of the drag term in Eq. (2). On the other hand, it is almost impossible to adopt soft-core models for the simulation of Eq. (2) to reproduce the DST in this setup, because spheres are largely overlapped if the shear rate \( \dot{\gamma} \) is large. Moreover, there is numerical difficulty to simulate the situation if \( \eta^* \) discontinuously changes with the order of \( 10^3 \) at a fixed \( \dot{\gamma}^* \). To overcome such difficulties, the event-driven Langevin simulation of hard spheres (EDLSHS) [34] is a powerful simulator for hard spheres with the aid of Trotter decomposition. See Appendix D for the outline of the method of the EDLSHS.

In our simulation, we fix the number of particles \( N = 1000 \) and the density \( n_{\sigma^3} = 0.01 \) which corresponds to the volume fraction \( \varphi = (\pi/6)n_{\sigma^3} \approx 0.0052 \) for \( d = 3 \). In the vicinity of the DST for \( \dot{\gamma}^* \in [0.400, 0.798] \), we gradually change the shear rate from \( \dot{\gamma}^*_{0} = 0.400(0.798) \) to sequentially increasing (decreasing) values as \( \dot{\gamma}^* = \dot{\gamma}^*_{0}, a\dot{\gamma}^*_{0}, a^2\dot{\gamma}^*_{0}, \ldots, a^{15}\dot{\gamma}^*_{0} = 0.798(0.400) \) with the rate \( a = 10^{\pm0.02} \).
The main results of our simulation are presented in Figs. 1–3. All of the numerical results perfectly agree with the theoretical results without introduction of any fitting parameters. We also find that the discontinuous jumps of $\eta^*$ take place at different $\dot{\gamma}^*$ depending on the protocol (if $\dot{\gamma}^*$ increases/decreases). This protocol dependence means that there is a hysteresis in the DST.

Thus, we have confirmed the quantitative validity of our simple kinetic theory in terms of the Boltzmann equation to describe the DST. We also confirm that dilute suspension described by Eq. (4) or Eq. (2) can exhibit the DST.

Note that the spatial inhomogeneities for $\theta$, $V$, and $n$ have not been observed for all $\dot{\gamma}^*$, at least, within our simulation (see Figs. 4 and 5). These results are partially because the thermal noise in Eq. (2) stabilizes the homogeneous structure for small $\dot{\gamma}^*$, though such homogeneity might be violated for large and highly dissipative systems. Because we need to obtain an approximate inhomogeneous solution of the Boltzmann equation at Navier-Stokes order around the uniform solution in Eq. (10) for the discussion of the stability of the spatially uniform flow, we will discuss such stability in a separated paper.

V. DISCUSSION AND CONCLUSION

Let us discuss our results. Although we have demonstrated the existence of the DST in a dilute gas-solid suspension, there are many problems to be solved to understand the DST.

One taking home message here is that the DST can take place in dilute gas-solid suspensions without mutual frictions between particles, though we have ignored the hydrodynamic interaction. Absence of the hydrodynamic interactions can be justified because we are only interested in suspensions in the dilute limit. Indeed, some previous papers used the density dependent $\zeta$ which reduces to $\zeta \propto 1 + 3\sqrt{\phi}/2 \to 1$ in the dilute limit [29, 31]. This encourages experimentalists to try to find the DST for solid-gas suspensions such as aerosols in which particles are only influenced by the Stokesian drag and the collisional force. The easiest experimental setup might be a sheared granular gas under vibration, because we often use Eq. (4) for a simplified model of such a system.

One can indicate that the coupling among the pressure, the normal stress difference and the shear stress leads to a cubic equation to exhibit the S-shape in the flow curve. Therefore, the normal stress difference becomes suddenly large in the vicinity of the DST as observed in dense suspensions as in Ref. [5]. One can stress that this scenario might be universal, though we have analyzed dilute gas-solid suspensions. Therefore, it is not surprising that the DST can take place without the mutual friction between particles.

The Langevin equation (2) employed in our study assumes that the gravity force is perfectly balanced with the drag force immersed by the air flow. This assumption is only true if the homogeneous state is stable. On the other hand, the homogeneous phase becomes unstable if the injection rate of the gas flow exceeds a critical value. If the homogeneous state is unstable, one would need to consider the time evolution of local structure as well as the consideration of the inhomogeneous drag. The research in the unstable region would be an interesting subject in near future.

Let us compare our results with those of the pioneer paper [29] which found the discontinuous transition of the kinetic temperature. Our theoretical results $T \sim \dot{\gamma}^4/n^2$ and $\eta \sim \dot{\gamma}^2/n$ agree with theirs in the high shear regime of elastic suspensions ($e = 1$). Nevertheless, there are some differences between theirs and ours as explained below.
FIG. 4. Plots of the displacement vectors (black arrows) during the interval $\Delta t_{\text{interval}} = 1.0/\zeta$ for (a) $\dot{\gamma}^* = 0.527$ (linearly unstable region), (b) 0.552 (lower branch), (c) 0.552 (upper branch), and (d) 1.0 in Fig. 1 for $\epsilon = 0.90$ in the case of $d = 3$ and $n\sigma^3 = 0.01$. It is noted that the uniform shear term is subtracted in the displacement vector. In addition, we display the one third of the displacement vectors in the case of (iv) for visibility. We also show the temperature for the $i$-th particle $T_i \equiv \frac{1}{N} \sum_{i=1}^{N} m(v_i - u)^2/d$. The color indicates the magnitude of $T_i/T - 1$.

First, Ref. [29] only focuses on the rheology for elastic suspensions ($\epsilon = 1$), but we include the results for granular suspensions ($\epsilon < 1$) which have distinct behavior in the high shear regime as shown Figs. 1 and 3. (Note that Sangani et al. derived Banoldian expressions in Eq. (27) for $\dot{\gamma}^* \gg 1$, though they have not written their explicit forms [31].) Second, their kinetic calculation is only used for the ignited state, whereas physical quantities in the quenched state are calculated separately. Our analysis, however, can use a unified calculation for both the ignited and the quenched state. This is because any observable is not uniquely determined as a function of St or $\dot{\gamma}^*$ but can be uniquely determined as a function of $T$ or $\theta$. Third, we believe that their calculation is not applicable to the behavior in the low shear limit, because there is neither the (Newtonian) viscosity nor the kinetic temperature in this limit. Indeed, their calculation in this regime suggests $T \sim \varphi \text{St}^3$ and $\eta \sim \varphi^2 \text{St}^2$ which approach zero in the limit $\text{St} \to 0$. Moreover, their viscosity becomes zero in the dilute suspensions because of $\eta \propto \varphi^2$, which is the reason why non-existence of quenched viscosity in Fig. 1 of Ref. [29]. We believe that they recognized the drawback of their analysis, because they added the Newtonian viscosity to their calculated viscosity in low shear regime (see the argument to derive Eq. (4.13) in Ref. [29]) by hand. Our model, however, can describe the Newtonian rheology in the low shear regime without any artificial trick. Therefore, we believe that it is crucially important to introduce the temperature dependence of the drag and the thermal noise in Eq. (2). In other words, the previous model without the noise term is structurally unstable, because once we introduce very small $T_{\text{ex}}$ the rheology for the low shear regime is changed completely. It is obvious that the suspensions must be equilibrated in the absence of the external shear, which can be achieved only if we take into account the thermal noise. Therefore, we believe that our model is more appropriate than the previous model in the low shear regime.

It is straightforward to extend the present analysis to a moderately dense gases by using Enskog equation [31, 37]. Therefore, the extension of the analysis presented here is important. After the completion of this work, we have already performed both the simulation of a moderately dense suspension and the Enskog equation instead of the Boltzmann equation in Eqs. (4) and (5) [50]. As a result, we have verified that the DST disappears at a quite low
density around $\varphi \approx 0.0176$, though the CST still survives above the critical density. This result is qualitatively consistent with the previous theory for the temperature [31]. Therefore, the mechanism of the DST presented here is not directly related to typical DSTs observed in dense suspensions. Therefore, our finding of the DST in the dilute suspension is not related to recent arguments for dense suspensions in which the mutual friction between particles is necessary.

We can indicate that Eq. (10) is consistent with the Green-Kubo formula [51–54] as shown in Appendix A. Nevertheless, the model only based on the Green-Kubo formula has isotropic stress as shown in Appendix A. This is because the model based on the Green-Kubo formula is only valid for linearly nonequilibrium situations, while the model based on Eq. (10) recovers the correct linearly nonequilibrium state and is a natural extension to nonlinear nonequilibrium situations. Therefore, it is essential for the DST to adopt Grad's approximation (10). An extended Grad's approximation is applicable to dense non-Brownian suspensions near the jamming point [55], in which we can successfully reproduce the divergent pressure viscosity and shear viscosity observed in experiments [56, 57].

Our model does not include any mutual friction between grains, though many papers stressed important roles of the friction in DST for dense suspensions and dense dry granular flows [9–15]. In this sense, our analysis does not answer the mechanism of DST observed in typical experiments and simulations for dense suspensions. The difficulty including the mutual friction in terms of statistical mechanics is to have the contributions from the coupled stress and the spin temperature in addition to the shear stress, the normal stress difference and the pressure. A recent paper has
already partially reproduced a hysteresis in the flow curve, which is one of the characteristics of the DST by choosing the shear stress, the temperature (the pressure) and the spin temperature [58].

In conclusion, we have demonstrated the existence of a S-shape flow curve which means the existence of the DST in fluid coupled with the drag between particles and the background fluid at $T_{ex}$ within framework of (inelastic) Boltzmann equation. This model exhibits the crossovers from the Newtonian viscosity to the Bagnoldian viscosity for $e < 1$ and from the Newtonian to the viscosity proportional to $\gamma^2$ for $e = 1$. The even-driven Langevin simulation for hard spheres reproduces the DST and perfectly agrees with the theoretical results without any fitting parameters. Therefore, we confirm the existence of the DST for dilute gas-solid suspensions.

Note added in proof: After our submission of this paper, we realize that a new similar paper to ours has been submitted several months later [59]. This paper can be regarded as the modernized revision of Ref. [29] where they do not include any thermal noise. Although they are only interested in dilute gases, they have also illustrated the existence of the critical density between the discontinuous and the continuous quenched-ignited transitions similar to Refs. [31, 50] thanks to the fact of the low critical density as stated in the discussion.

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Appendix

This appendix contains some detailed information. In Appendix A, we explain the relationship between Grad’s approximation and Green-Kubo formula within the BGK approximation. In Appendix B we give some detailed calculation for $\Lambda_{\alpha\beta}$ in Eq. (9) as well as some formulas of Gaussian integrals and angle integrals. This calculation is basically written in Refs. [42, 43] but we write their details as a self-contained form. Appendix C is devoted to the detailed calculation of the linear stability analysis. In Appendix D, we explain the outline of the algorithm of EDLSHS.

Appendix A: Relationship between Grad’s approximation and Green-Kubo formula

The result in this paper strongly depends on the choice Eq.(10). In this section, we discuss the validity of the Grad expansion in Eq. (10).

For simplicity, let us consider

$$\left(\frac{\partial}{\partial t} - \dot{\gamma}_x \frac{\partial}{\partial V_x}\right)f(V, t) = \zeta \frac{\partial}{\partial V} \cdot \left(\left\{V + \frac{T_{ex}}{m} \frac{\partial}{\partial V}\right\} f(V, t)\right) + \frac{f_{eq}(V) - f(V, t)}{\tau},$$

(A1)

where the relaxation time $\tau$ satisfies $\tau = (16/5)\pi a^2 \sqrt{T/m}$ for $d = 3$ and $e = 1$ [32]. BGK equation (A1) is a well known simplified model to reduce to the linearized Boltzmann equation, which is compatible with Chapman-Enskog approximation with the proper choice of $\tau$ as in Ref. [32]. We restrict our interest to the case of $d = 3$ and $e = 1$ in this Appendix.

Within the framework of BGK equation (A1) is reduced to

$$\frac{\delta f}{f_{eq}} = -\dot{\gamma}^* \frac{m V_x V_y}{T} + \tau^* \left(1 - \frac{T_{ex}}{T}\right) \left(3 - \frac{p^2}{m T}\right),$$

(A2)

where $p = mV$, $\delta f \equiv f - f_{eq}$ and $\tau^* = \zeta \tau$. To derive Eq. (A2) we have used the relations:

$$\left(\frac{y \frac{\partial}{\partial x} - p_y \frac{\partial}{\partial p_x}}{p + m T_{ex} \frac{\partial}{\partial p}}\right) f_{eq} = \frac{m V_x V_y}{T},$$

(A3)

$$\left(p + m T_{ex} \frac{\partial}{\partial p}\right) f_{eq} = \left(1 - \frac{T_{ex}}{T}\right) p f_{eq}.$$  

(A4)

It should be noted that there is no contribution of the second term on the right hand side of Eq. (A1) for the macroscopic stress obtained by $P_{xy} = m \int dV f(V) V_x V_y$ because of its parity. Therefore, the first term on the right hand side of Eq. (A2) is only important for the consistency between Eqs. (10) and (A2).
The macroscopic shear stress $\sigma_{xy}$ and the viscosity $\eta = -P_{xy}/\dot{\gamma}$ are determined by

$$P_{xy} = m \int dV \delta f(V) V_x V_y = -\frac{\gamma^* \tau^* m^2}{T} (V_x^2 V_y^2)_{eq},$$  \hspace{1cm} (A5)$$

where $\langle \cdot \rangle_{eq}$ is the average in terms of the equilibrium distribution $f_{eq}(V)$. It should be noted that Eq. (A5) is identical to the Green-Kubo formula under an exponential relaxation [54].

On the other hand, if we adopt Eq. (10), the normal stress difference disappears within the framework of the linearized BGK equation. Indeed, substituting Eq. (A2) into the expression of normal stress difference $\Delta \sigma_{xx}$ and the Green-Kubo formula. Nevertheless, Grad’s expansion (10) should include the nonlinear term which causes the response theory can be used, though there still exists a little disagreement between asymptotic forms in this appendix.

Newtonian situation is reasonable, because the system remains in an early equilibrium situation in which the linear response to the applied forces is still valid.

First, let us prove the following identity

$$\int d\tilde{\sigma} \Theta(c \cdot \tilde{\sigma})(c \cdot \tilde{\sigma})^n = \beta_n c^n,$$

where

$$\beta_n = \pi^{(d-1)/2} \frac{\Gamma(n+1)}{\Gamma(n+d)},$$

It is straightforward to rewrite the left hand side (LHS) of Eq. (B1) as

$$\int d\tilde{\sigma} \Theta(c \cdot \tilde{\sigma})(c \cdot \tilde{\sigma})^n = S_d c^n \int_0^{\pi/2} d\theta (\sin \theta)^{d-2} \cos^n \theta = \frac{S_{d-1} c^n}{2} B \left( \frac{d-1}{2}, \frac{n+1}{2} \right)$$

$$= \pi^{(d-1)/2} \frac{\Gamma(n+1)}{\Gamma(n+d)} c^n,$$

where we have introduced the Beta function $B(x, y)$, the Gamma function $\Gamma(x)$ and the area of unit hypersurface $S_d$:

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)} = 2 \int_0^{\pi/2} d\theta (\sin \theta)^{2x-1} (\cos \theta)^{2y-1}, \quad \Gamma(x) = \int_0^\infty dt t^{x-1} e^{-t}, \quad S_d = \pi^{d/2} \frac{\pi^d}{\Gamma(d/2)}.$$  \hspace{1cm} (B4)$$

This is the end of proof of Eqs. (B1) and (B2).
2. \[ \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma} = \beta_{n+1} c^{n-1} c. \] (B5)

In this subsection, let us prove the identity:

\[ \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma} = \beta_{n+1} c^{n-1} c. \] (B5)

Let us introduce the angle \( \theta_i \) with \( 1 \leq i \leq d - 1 \) to characterize the unit hypersurface. In this case \( \hat{\sigma} \) can be expressed as

\[ \hat{\sigma} = \left( \begin{array}{c} \cos \theta_1 \\ \sin \theta_1 \cos \theta_2 \\ \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ \vdots \\ \sin \theta_1 \cdots \sin \theta_{d-2} \cos \theta_{d-1} \\ \sin \theta_1 \cdots \sin \theta_{d-2} \sin \theta_{d-1} \end{array} \right), \] (B6)

where \( \theta_{d-1} \) ranges over \([0, \pi]\) and the other angles \( \theta_i \) \( (1 \leq i \leq d - 2) \) ranges over \([0, \pi]\). The angle \( \theta_1 \) is regarded as the angle between \( c \) and the normal unit vector \( \hat{\sigma} \). The integral (B5) is

\[ \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma} = S_{d-1} c^n \int_0^{\pi/2} d\theta_1 \int_0^{\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_{d-1}(\sin \theta_1)^{d-2}(\sin \theta_2)^{d-3} \cdots \sin \theta_{d-2} \hat{\sigma}, \] (B7)

where the first component of \( \hat{\sigma} \) gives finite contribution but all the other components cancel because \( i \)-th component satisfies \( \int_0^\pi d\theta_i \sin \theta_i \sin^{d-i-1} \theta_i = [\sin^{d-i} \theta_i]_{\theta_i=0}^{\pi} / (d-i) = 0 \). Therefore, only the first component in LHS of Eq. (B5) which is parallel to \( c \) survives. Therefore, the LHS of Eq. (B5) can be evaluated as

\[ \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma} = S_{d-1} c^n \int_0^{\pi/2} d\theta_1 \int_0^{\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_{d-1}(\sin \theta_1)^{d-2} \cos^n \theta(c \cdot \hat{\sigma}) \hat{c} \\ = S_{d-1} c^n \int_0^{\pi/2} d\theta_1 \int_0^{\pi} d\theta_2 \cdots \int_0^{2\pi} d\theta_{d-1}(\sin \theta_1)^{d-2} \cos^{n+1} \theta \hat{c} = \beta_{n+1} c^n \hat{c} = \beta_{n+1} c^{n-1} c. \] (B8)

This is the end of proof of Eq. (B5).

3. \[ \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma}_\alpha \hat{\sigma}_\beta \]

Let us prove the following identity

\[ \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma}_\alpha \hat{\sigma}_\beta = \frac{\beta_n}{n + d} c^{n-2} (\alpha c_\alpha c_\beta + c^2 \delta_{\alpha\beta}). \] (B9)

Let us assume the form

\[ \hat{\sigma}_\alpha \hat{\sigma}_\beta = a(\delta_{\alpha\beta} + bc_\alpha c_\beta) \] (B10)

Because \( \hat{\sigma} \) is the unit vector and has the relation \( \hat{\sigma}_\alpha \hat{\sigma}_\alpha = 1 \), the trace of the LHS of Eq. (B9) is given by

\[ \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma}_\alpha \hat{\sigma}_\alpha = a(d + b) \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n = a(d + b) \beta_n c^n = \beta_n c^n. \] (B11)

Therefore we obtain

\[ a = \frac{1}{d + b} \] (B12)

On the other hand, \( \hat{c} \cdot \text{LHS} \cdot \hat{c} \)

\[ \hat{\sigma} \cdot \int d\hat{\sigma} \Theta(c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^n \hat{\sigma} \hat{\sigma} \hat{\sigma} = a \int d\hat{\sigma} \Theta(\hat{\sigma} \cdot c)(\hat{\sigma} \cdot c)^n (\hat{c}^2 + b c^2) = a(1 + b) \beta_n c^n \]

\[ = \int d\hat{\sigma} \Theta(\hat{\sigma} \cdot c)(\hat{\sigma} \cdot c)^n (\hat{\sigma} \cdot \hat{\sigma})^2 = \beta_{n+2} c^n. \] (B13)

Therefore, we obtain the relation

\[ \beta_{n+2} = a(1 + b) \beta_n = \beta_n \frac{n + 1}{n + d} \] (B14)

From Eqs. (B12) and (B14) we reach Eq. (B9).
4. Evaluation of $\Lambda_{\alpha\beta}$

In this subsection, we evaluate $\Lambda_{\alpha\beta}$ introduced in Eq. (9). Substituting Eq. (3) into Eq. (9) we obtain

$$
\Lambda_{\alpha\beta} = -m\sigma^{d-1} \int dv_1 \int dv_2 \int d\hat{\sigma} \Theta(-v_{12} \cdot \hat{\sigma}) |v_{12} \cdot \hat{\sigma}| V_{1,\alpha} V_{1,\beta} \left\{ \frac{f(V_{1}^{**})f(V_{2}^{**})}{e^2} - f(V_1)f(V_2) \right\} = -m\sigma^{d-1} \int dv_1 \int dv_2 \int d\hat{\sigma} \Theta(-v_{12} \cdot \hat{\sigma}) |v_{12} \cdot \hat{\sigma}| f(V_1)f(V_2)(V_{1,\alpha}V_{2,\beta}^* - V_{1,\alpha}V_{2,\beta}),
$$

(B15)

where we have introduced the post-collisional velocities $v_1^*$ and $v_2^*$ defined by

$$
v_1^* = v_1 - \frac{1+e}{2}(v_{12} \cdot \hat{\sigma})\hat{\sigma}, \quad v_2^* = v_2 + \frac{1+e}{2}(v_{12} \cdot \hat{\sigma})\hat{\sigma}.
$$

(B16)

To obtain the final expression of Eq. (B15), we have converted $(V_{1}^{**}, V_{2})$ into $(V_{1}, V_{2}^{*})$ and used the Jacobian $dv_1^* dv_2^* = dv_1 dv_2/e$ and the collision rule $(v_{12} \cdot \hat{\sigma})\hat{\sigma} = -e(v_{12} \cdot \hat{\sigma})\hat{\sigma}$. Equation (B15) can be symmetrized as

$$
\Lambda_{\alpha\beta} = -\frac{m\sigma^{d-1}}{2} \int dv_1 \int dv_2 \int d\hat{\sigma} \Theta(-v_{12} \cdot \hat{\sigma}) f(V_1)f(V_2)(V_{1,\alpha}V_{1,\beta} + V_{2,\alpha}V_{2,\beta}^* - V_{1,\alpha}V_{1,\beta} - V_{2,\alpha}V_{2,\beta}).
$$

(B17)

With the aid of Eq. (B16) we have the relation

$$
V_{1,\alpha}^* V_{1,\beta} + V_{2,\alpha} V_{2,\beta}^* - V_{1,\alpha} V_{1,\beta} - V_{2,\alpha} V_{2,\beta} = -\frac{1+e}{2}(v_{12} \cdot \hat{\sigma})(v_{12,\alpha}\hat{\sigma}_\beta + \hat{\sigma}_\alpha v_{12,\beta}) + \frac{(1+e)^2}{2}(v_{12} \cdot \hat{\sigma})^2\hat{\sigma}_\alpha \hat{\sigma}_\beta.
$$

(B18)

Substituting Eqs. (10) and (B18) into Eq. (B17) with the linearization around $f_{eq}(V)$ we obtain

$$
\Lambda_{\alpha\beta} = -\frac{m\sigma^{d-1}}{2} n^2 \left( \frac{m}{2\pi T} \right)^d \int dG \int dv_{12} \int d\hat{\sigma} \Theta(-v_{12} \cdot \hat{\sigma}) |v_{12} \cdot \hat{\sigma}| \exp\left(-\frac{mg^2}{T}\right) \exp\left(-\frac{mv_{12}^2}{4T}\right)
\times \left[ 1 + \frac{m}{2T} \left( \frac{P_{\gamma\delta}}{nT} - \delta_{\gamma\delta} \right)\left( 2G_{\gamma \delta} + \frac{1}{2}v_{12,\gamma}v_{12,\delta} \right) \right]
\times \left[ -A(v_{12} \cdot \hat{\sigma})(v_{12,\alpha}\hat{\sigma}_\beta + \hat{\sigma}_\alpha v_{12,\beta}) + 2A^2(v_{12} \cdot \hat{\sigma})^2\hat{\sigma}_\alpha \hat{\sigma}_\beta \right]
$$

(B19)

where we have introduced $G = (v_1 + v_2)/2$ and $A = (1+e)/2$. Equation (B19) is further rewritten as

$$
\Lambda_{\alpha\beta} = -\frac{n^2v_{12}^2m\sigma^{d-1}}{2\pi^d} \int dC \int dc \int d\hat{\sigma} \Theta(-c \cdot \hat{\sigma}) |c \cdot \hat{\sigma}| \exp\left(-2C^2\right) \exp\left(-\frac{1}{2}c^2\right)
\times \left[ 1 + \left( P_{\gamma\delta} - \delta_{\gamma\delta} \right) \left( 2C_{\gamma}C_{\delta} + \frac{1}{2}c_{\gamma}c_{\delta} \right) \right] [2A^2(c \cdot \hat{\sigma})^2\hat{\sigma}_\alpha \hat{\sigma}_\beta - A(c \cdot \hat{\sigma})(c_{\alpha}\hat{\sigma}_\beta + \hat{\sigma}_\alpha c_{\beta})] = \frac{n^2v_{12}^2m\sigma^{d-1}}{2\pi^d} A\tilde{\Lambda}_{\alpha\beta}
$$

(B20)

with

$$
\tilde{\Lambda}_{\alpha\beta} \equiv \Lambda_{\alpha\beta}^{(1)} - 2AA_{\alpha\beta}^{(2)} + (P_{\gamma\delta} - \delta_{\gamma\delta})(\Lambda_{\alpha\beta\gamma\delta}^{(3)} - 2AA_{\alpha\beta\gamma\delta}^{(4)}),
$$

(B21)

where we have introduced $P_{\gamma\delta} = P_{\alpha\beta}/(nT)$, $v_T = \sqrt{2T/m}$, $C = G/v_T$ and $c = v_{12}/v_T$ in the first expression. We have also introduced $\Lambda_{\alpha\beta}^{(i)}$ ($i = 1, 2$) and $A_{\alpha\beta\gamma\delta}$ ($j = 3, 4$) in Eq. (B20).
The expression of $\Lambda^{(1)}_{\alpha\beta}$ is given by

$$
\Lambda^{(1)}_{\alpha\beta} = \int dC \int dc \int d\hat{\sigma} \Theta(-c \cdot \hat{\sigma}) e^{-2C^2} e^{-c^2/2} (c \cdot \hat{\sigma})^2 (c_\alpha \hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta)
$$

$$
= (\frac{\pi}{2})^{d/2} \beta_3 \int dce^{-c^2/2}cc_\alpha c_\beta
$$

$$
= 2\delta_{\alpha\beta} \left( \frac{\pi}{2} \right)^{d/2} \frac{\pi(d-1)/2}{\Gamma(\frac{d+3}{2})} S_d \int_0^\infty dce^{d+2}e^{-c^2/2} \Gamma(\frac{d+3}{2})
$$

$$
= 2\delta_{\alpha\beta} \left( \frac{\pi}{2} \right)^{d/2} \frac{\pi(d-1)/2}{\Gamma(\frac{d+3}{2})} S_d \int_0^\infty dce^{d+2}e^{-c^2/2} \Gamma(\frac{d+3}{2})
$$

$$
= 4\sqrt{2} \pi^{(3d-1)/2} \delta_{\alpha\beta},
$$

(B22)

where we have used Eq. (B5) in the expression in the third line, Eqs. (B2) and (B39) in the fourth line and Eq. (B4) for the last expression. We also note the relation $\int_0^\infty dce^{d+a}e^{-c^2/2} = 2(d+a-1)/\Gamma((d+a+1)/2)$.

Similarly, $\Lambda^{(2)}_{\alpha\beta}$ is given by

$$
\Lambda^{(2)}_{\alpha\beta} = \int dC \int dc \int d\hat{\sigma} \Theta(-c \cdot \hat{\sigma}) e^{-2C^2} e^{-c^2/2} (c \cdot \hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
$$

$$
= (\frac{\pi}{2})^{d/2} \beta_3 \int dce^{-c^2/2}c(3c_\alpha c_\beta + c^2 \delta_{\alpha\beta})
$$

$$
= \delta_{\alpha\beta} \left( \frac{\pi}{2} \right)^{d/2} \frac{\pi(d-1)/2}{\Gamma(\frac{d+3}{2})} S_d \int_0^\infty dce^{d+2}e^{-c^2/2} \Gamma(\frac{d+3}{2})
$$

$$
= \delta_{\alpha\beta} \left( \frac{\pi}{2} \right)^{d/2} \frac{\pi(d-1)/2}{\Gamma(\frac{d+3}{2})} S_d \int_0^\infty dce^{d+2}e^{-c^2/2} \Gamma(\frac{d+3}{2})
$$

$$
= 2\sqrt{2} \pi^{(3d-1)/2} \delta_{\alpha\beta},
$$

(B23)

where we have used Eqs. (B9) and (B39) for the third line. Therefore, we obtain the relation

$$
\Lambda^{(1)}_{\alpha\beta} - 2\Lambda^{(2)}_{\alpha\beta} = \frac{4\sqrt{2} \pi^{(3d-1)/2}}{\Gamma(d/2)} (1 - A) \delta_{\alpha\beta}.
$$

(B24)

The expression of $\Lambda^{(3)}_{\alpha\beta\gamma\delta}$ consists of two parts

$$
\Lambda^{(3)}_{\alpha\beta\gamma\delta} = \int dC \int dc \int d\hat{\sigma} \Theta(-c \cdot \hat{\sigma}) e^{-2C^2} e^{-c^2/2} \left( 2C_\gamma C_\delta + \frac{1}{2} c_\gamma c_\delta \right) (c \cdot \hat{\sigma})^2 (c_\alpha \hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta)
$$

$$
= 2\Lambda^{(3,1)}_{\alpha\beta\gamma\delta} + \frac{1}{2} \Lambda^{(3,2)}_{\alpha\beta\gamma\delta}
$$

(B25)

where $\Lambda^{(3,1)}_{\alpha\beta\gamma\delta}$ is given by

$$
\Lambda^{(3,1)}_{\alpha\beta\gamma\delta} = \int dC \int dc \int d\hat{\sigma} \Theta(-c \cdot \hat{\sigma}) e^{-2C^2} e^{-c^2/2} \int d\hat{\sigma} \Theta(-c \cdot \hat{\sigma})(c \cdot \hat{\sigma})^2 (c_\alpha \hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta)
$$

$$
= \delta_{\alpha\beta} \delta_{\gamma\delta} \frac{1}{2} \Gamma(\frac{d+1}{2})^2 \Gamma(\frac{d+3}{2}) \Gamma(\frac{d}{2}) (d+1) \times 2(d+1)/\Gamma(\frac{d}{2})
$$

$$
= \delta_{\alpha\beta} \delta_{\gamma\delta} \frac{\pi^{(3d-1)/2}}{d\sqrt{2} \Gamma(\frac{d}{2})}
$$

(B26)
with the relation \( \int_0^\infty dCC^{d+1}e^{-2C^2} = 2^{-d/2-2}(\Gamma(d/2 + 1)) \), and \( \Lambda^{(3,2)}_{\alpha\beta\gamma\delta} \) is given by

\[
\Lambda^{(3,2)}_{\alpha\beta\gamma\delta} = \int dC e^{-2C^2} \int dce^{-c^2/2}c_\gamma c_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^2(c_\alpha\hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta) = 2\beta_3 \left( \frac{\pi}{2} \right)^{d/2} \int dce^{-c^2/2}cc_\gamma c_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^2(c_\alpha\hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta)
\]

\[
= 2\pi(d-1)/2 \frac{\Gamma(3d-2)}{\Gamma(\frac{d}{2})^2} \int \int dce^{-c^2/2}cc_\gamma c_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^2(c_\alpha\hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta)
\]

\[
= 2\pi(d-1)/2 \frac{\Gamma(3d-2)}{\Gamma(\frac{d}{2})^2} \int \int dce^{-c^2/2}cc_\gamma c_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^2(c_\alpha\hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta)
\]

\[
= 4\cdot 2^{3/2} \frac{\pi^{(3d-1)/2}}{\Gamma(d+2)} \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^2(c_\alpha\hat{\sigma}_\beta + \hat{\sigma}_\alpha c_\beta)
\]

\[
(\ref{B28})
\]

where we have used Eq. (B40) in the third line. Substituting Eqs. (B26) and (B27) into Eq. (B25) we obtain

\[
\Lambda^{(3)}_{\alpha\beta\gamma\delta} = \sqrt{2}\pi^{(3d-1)/2} d(d+2) \Gamma(\frac{d}{2}) \int (2d + 7)\delta_{\alpha\beta}\delta_{\gamma\delta} + 2(d+3)(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \}
\]

\[
(\ref{B28})
\]

Similarly, \( \Lambda^{(4)}_{\alpha\beta\gamma\delta} \) also consists of two parts

\[
\Lambda^{(4)}_{\alpha\beta\gamma\delta} = \int dC \int dc \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})e^{-2C^2} e^{-c^2/2} \left( 2C_\gamma C_\delta + 1 \right) (c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
= 2\Lambda^{(4,1)}_{\alpha\beta\gamma\delta} + \frac{\Lambda^{(4,2)}_{\alpha\beta\gamma\delta}}{2}
\]

(\ref{B29})

where \( \Lambda^{(4,1)}_{\alpha\beta\gamma\delta} \) is given by

\[
\Lambda^{(4,1)}_{\alpha\beta\gamma\delta} = \int dC e^{-2C^2} C_\gamma C_\delta \int dce^{-c^2/2} \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
= \delta_{\alpha\beta} \delta_{\gamma\delta} \int \int \int dce^{-c^2/2} \left( \frac{\beta_3}{3 + d} \right) c_\alpha^2 c_\beta^2 \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
= \delta_{\alpha\beta} \delta_{\gamma\delta} \int \int \int dce^{-c^2/2} \left( \frac{\beta_3}{3 + d} \right) c_\alpha^2 c_\beta^2 \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
= \delta_{\alpha\beta} \delta_{\gamma\delta} \int \int \int dce^{-c^2/2} \left( \frac{\beta_3}{3 + d} \right) c_\alpha^2 c_\beta^2 \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
(\ref{B30})
\]

and \( \Lambda^{(4,2)}_{\alpha\beta\gamma\delta} \) is given by

\[
\Lambda^{(4,2)}_{\alpha\beta\gamma\delta} = \int dC e^{-2C^2} \int dce^{-c^2/2} C_\gamma C_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
= \left( \frac{\pi}{2} \right)^{d/2} \int dce^{-c^2/2} C_\gamma C_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
= \left( \frac{\pi}{2} \right)^{d/2} \int dce^{-c^2/2} C_\gamma C_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
= \left( \frac{\pi}{2} \right)^{d/2} \int dce^{-c^2/2} C_\gamma C_\delta \int d\hat{\sigma}\Theta(-c\cdot\hat{\sigma})(c\cdot\hat{\sigma})^3 \hat{\sigma}_\alpha \hat{\sigma}_\beta
\]

\[
(\ref{B31})
\]
Substituting Eq. (B24) and (B33) into Eq. (B21) we obtain
\[
\Lambda^{(4)}_{\alpha\beta\gamma\delta} = \frac{\sqrt{2\pi}(3d-1)^{1/2}}{d(d+2)\Gamma(d/2)} \{ (2d + 7)\delta_{\alpha\beta}\delta_{\gamma\delta} + 3(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \}.
\] (B32)

With the aid of Eqs. (B28) and (B32) and taking into account the relation \((P_{\alpha\beta} - \delta_{\alpha\beta})\delta_{\alpha\beta} = P_{\alpha\beta}^* - \delta_{\alpha\alpha} = 0\), we obtain
\[
(P_{\gamma\delta}^* - \delta_{\gamma\delta})(\Lambda_{\alpha\beta\gamma\delta}^{(3)} - 2A\Lambda_{\alpha\beta\gamma\delta}^{(4)}) = \frac{4\sqrt{2\pi}(3d-1)^{1/2}}{d(d+2)\Gamma(d/2)} (d + 3 - 3A)(P_{\alpha\beta}^* - \delta_{\alpha\beta}).
\] (B33)

Substituting Eqs. (B24) and (B33) into Eq. (B21) we obtain
\[
\tilde{\Lambda}_{\alpha\beta} = \frac{4\sqrt{2\pi}(3d-1)^{1/2}}{d\Gamma(d/2)} \left\{ (1 - A)\delta_{\alpha\beta} + \frac{1}{d+2} (d + 3 - 3A)(P_{\alpha\beta}^* - \delta_{\alpha\beta}) \right\}.
\] (B34)

Substituting Eq. (B34) into Eq. (B20) we finally obtain
\[
\Lambda_{\alpha\beta} = \frac{2\sqrt{2\pi}(d-1/2)n^2(\pi^2)^{d-1}}{d\Gamma(d/2)} \left\{ A(1 - A)\delta_{\alpha\beta} + \frac{1}{d+2} (d + 3 - 3A)(P_{\alpha\beta}^* - \delta_{\alpha\beta}) \right\}
\] (B35)

\[
= \frac{\sqrt{2\pi}(d-1/2)n\pi^2\sigma^{d-1}}{d\Gamma(d/2)} \left\{ (1 - e^2)nT\delta_{\alpha\beta} + \frac{1 + e}{d+2} (2d + 3 - 3e)(P_{\alpha\beta}^* - nT\delta_{\alpha\beta}) \right\}.
\]

5. Gaussian integrals

Let us summarize Gaussian integrals used in the previous subsections.

\[
\int dC e^{-aC^2} = \left(\frac{\pi}{a}\right)^{d/2},
\] (B36)

\[
\int dce^{-c^2/2c_\alpha c_\beta} = \frac{\delta_{\alpha\beta}}{d} \int dce^{-c^2/2c_\alpha c_\beta c_\gamma c_\delta} = \int dce^{-c^2/2c_\alpha c_\beta c_\gamma c_\delta} = \frac{S_d}{d(d+2/\Gamma(d/2))} \left( \frac{d + a + 4}{2} \right) (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}),
\] (B37)

where we have used the following calculations: First let us show
\[
\int d\hat{\sigma} \hat{c}_\alpha \hat{c}_\beta = S_d \delta_{\alpha\beta}.
\] (B39)

Indeed, the trace of the left hand side of this equation gives \(\int d\hat{\sigma} \hat{c}_\alpha \hat{c}_\alpha = \int d\hat{\sigma} = S_d\), while we can write \(\int d\hat{\sigma} \hat{c}_\alpha \hat{c}_\alpha = S_d K_0 \delta_{\alpha\beta}\) if we set \(\int d\hat{\sigma} \hat{c}_\alpha \hat{c}_\beta = S_d K_0 \delta_{\alpha\beta}\). Therefore, we must have \(K_0 = 1/d\). Second, we show
\[
\int d\hat{\sigma} \hat{c}_\alpha \hat{c}_\beta \hat{c}_\gamma \hat{c}_\delta = S_d \left( \frac{1}{d+2} \right) (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}).
\] (B40)

Indeed, let us assume
\[
\int d\hat{\sigma} \hat{c}_\alpha \hat{c}_\beta \hat{c}_\gamma \hat{c}_\delta = S_d K_1 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}).
\] (B41)

If we set \(\alpha = \beta\), then using \(\hat{c}_\alpha \hat{c}_\alpha = 1\) we obtain
\[
\int d\hat{\sigma} \hat{c}_\gamma \hat{c}_\delta = S_d K_1 (d+2)\delta_{\gamma\delta} = S_d K_0 \delta_{\gamma\delta}.
\] (B42)

Therefore, we obtain \(K_1 = 1/(d(d+2))\).
Appendix C: Linear stability analysis

In this section, we analyze the linear stability of the steady state \((19)\)–\((22)\) obtained in the main text. Let us rewrite the set of equations \((15)\)–\((17)\) as

\[
\frac{d\theta}{dt^*} = -\frac{2}{d} \gamma^s P^*_s \theta^s - \lambda^s \theta^{3/2} + 2(1 - \theta),
\]

\[
\frac{d\Delta \theta}{dt^*} = -2 \gamma^s P^*_s \theta^s - (\nu^s \sqrt{\theta^s} + 2) \Delta \theta,
\]

\[
\frac{dP^*_s}{dt^*} = -\frac{1}{d} \gamma^s (d\theta - \Delta \theta) - (\nu^s \sqrt{\theta^s} + 2) P^*_s,
\]

where we have introduced \(t^* \equiv t\zeta\). We can linearize Eqs. \((C1)\)–\((C3)\) around the steady solution \((\theta_s, \Delta \theta_s, P^*_s, - P^*_s, xy, s, \nu_s)^T\) in the presence of the shear rate \(\dot{\gamma}^s\) as

\[
\frac{d}{dt^*} \Psi = A \Psi,
\]

where \(\Psi = (\delta \theta, \delta \Delta \theta, \delta P^*_s)^T = (\theta - \theta_s, \Delta \theta - \Delta \theta_s, P^*_s - P^*_s, xy, s, \nu_s)^T\). The explicit form of the \(3 \times 3\) matrix \(A \equiv (A_{ij})\) is given by

\[
A = \begin{pmatrix}
-\left(\frac{2}{d} \gamma^s P^*_s \nu_s + \frac{3}{2} \lambda^s \theta^{3/2}\right) & 0 & -\frac{2}{d} \gamma^s \\
-\left(2 \gamma^s P^*_s \theta^s + \frac{1}{2} \nu^s \theta^s - \frac{1}{2} \lambda^s \gamma^s \theta^s - \frac{1}{2} \nu^s \theta^s - \frac{1}{2} \lambda^s \gamma^s \theta^s\right) & -(\nu^s \sqrt{\theta^s} + 2) & -2 \gamma^s \\
\frac{1}{d} \gamma^s \Delta \theta^s - (\gamma^s + \gamma^s \theta^s) - \frac{1}{2} \nu^s \theta^s - \frac{1}{2} \lambda^s \gamma^s \theta^s & -2 \gamma^s & -(\nu^s \sqrt{\theta^s} + 2)
\end{pmatrix},
\]

where we have introduced \(\gamma^s \theta^s \equiv (\partial \gamma^s / \partial \theta^s)\). Introducing \(\tilde{\Psi}(s)\) as the Laplace transform of \(\Psi(t)\), the Laplace transform of Eq. \((C4)\) is written as

\[
\tilde{\Psi}(s) = (s1 - A)^{-1} \Psi(0)
\]

under the initial value \(\Psi(0)\). When the real part of the eigenvalue in the eigenequation \((C4)\) is positive, the steady solution is unstable under a perturbation \([60]\). The eigenvalues are given by

\[
\det(s1 - A) = s^3 - (A_{11} + A_{22} + A_{33}) s^2 + (A_{11} A_{22} + A_{22} A_{33} + A_{33} A_{11} - A_{12} A_{21} - A_{23} A_{32} - A_{31} A_{13}) s
\]

\[
- A_{11} A_{22} A_{33} - A_{12} A_{23} A_{31} - A_{21} A_{32} A_{13} + A_{11} A_{23} A_{31} + A_{12} A_{21} A_{33} + A_{23} A_{32} A_{11} + A_{31} A_{13} A_{22} - A_{11} A_{22} A_{33} - A_{12} A_{23} A_{31} - A_{21} A_{32} A_{13} + A_{11} A_{23} A_{31} + A_{12} A_{21} A_{33} + A_{23} A_{32} A_{11} + A_{31} A_{13} A_{22}
\]

\[
= s^3 + \left[\frac{2}{d} \gamma^s \theta^s P^*_s \nu_s + \frac{3}{2} \lambda^s \theta^{3/2}\right] s^2
\]

\[
+ \left[(\nu^s \sqrt{\theta^s} + 2) + 2(\nu^s \sqrt{\theta^s} + 2) \left(\frac{2}{d} \gamma^s \theta^s P^*_s \nu_s + \frac{3}{2} \lambda^s \theta^{3/2}\right) - \frac{1}{d^2} \gamma^s \theta^s (d\theta - \Delta \theta) - \frac{1}{d} \nu^s \gamma^s \theta^s - \frac{1}{2} \nu^s \theta^s - \frac{1}{2} \lambda^s \gamma^s \theta^s\right] s
\]

\[
+ (\nu^s \sqrt{\theta^s} + 2) \left(\frac{2}{d} \gamma^s \theta^s P^*_s \nu_s + \frac{3}{2} \lambda^s \theta^{3/2}\right) + \frac{1}{d} \gamma^s (\nu^s \sqrt{\theta^s} + 2) \left(1 \gamma^s \theta^s - (\gamma^s + \gamma^s \theta^s) - \frac{1}{2} \nu^s \theta^s - \frac{1}{2} \lambda^s \gamma^s \theta^s\right)
\]

\[
- \frac{1}{d^2} \nu^s \gamma^2 \theta^s - \frac{1}{3} \lambda^s \gamma^2 \theta^s + \frac{4}{d^2} \gamma^2 = 0.
\]

Figure 6 plots the real parts of the eigenvalues (lines) as the solutions of Eq. \((C7)\) for \(d = 3\). The linear stability is determined by the largest eigenvalue, which is approximately given by the linearized solution of Eq. \((C7)\)

\[
s = \frac{N}{D},
\]

with

\[
N = (\nu^s \sqrt{\theta^s} + 2) \left(\frac{2}{d} \gamma^s \theta^s P^*_s \nu_s + \frac{3}{2} \lambda^s \theta^{3/2}\right) + \frac{2}{d} \gamma^s (\nu^s \sqrt{\theta^s} + 2) \left(1 \gamma^s \theta^s - (\gamma^s + \gamma^s \theta^s) - \frac{1}{2} \nu^s \theta^s - \frac{1}{2} \lambda^s \gamma^s \theta^s\right)
\]

\[
- \frac{1}{d^2} \nu^s \gamma^2 \theta^s - \frac{1}{3} \lambda^s \gamma^2 \theta^s + \frac{4}{d^2} \gamma^2,
\]

\[
D = -(\nu^s \sqrt{\theta^s} + 2) \left(\frac{2}{d} \gamma^s \theta^s P^*_s \nu_s + \frac{3}{2} \lambda^s \theta^{3/2}\right) + \frac{1}{d} \gamma^s \theta^s (d\theta - \Delta \theta) + \frac{1}{d} \nu^s \gamma^s \theta^s - \frac{1}{2} \nu^s \theta^s - \frac{1}{2} \lambda^s \gamma^s \theta^s.
\]
FIG. 6. Plot of the real part of three eigenvalues $\Re s_i$ ($i = 1, 2, 3$) against $\theta$ for $\epsilon = 0.9$, where the lines and the open circles are, respectively, obtained from Eq. (C7) and (C8). Here, the largest eigenvalue becomes positive in the intermediate $\theta$, while the other eigenvalues (see the inset) are always negative and are degenerated above a threshold $\theta$.

This solution (open circles in Fig. 6) well reproduces the largest eigenvalue (the solid line) in Fig. 6. In the intermediate regime $47 \leq \theta \leq 2850$, the largest eigenvalue becomes the positive corresponding to the linearly unstable regime.

**Appendix D: Outline of EDLSHS**

In this section, we explain the outline of the event-driven Langevin simulation of hard spheres (EDLSHS) [34] under a plane shear [48, 51] with the aid of the Lees-Edwards boundary condition [49]. The time evolution of $i$-th particle at the position $r_i$ and the peculiar momentum of $i$-th particle are given by Eqs. (2) and (3). With the aid of Eq. (D1), the velocity increment from the time $t$ to $t + \Delta t$ can be expressed as

$$v_{i, \alpha}(t + \Delta t) = e^{-\zeta \Delta t}v_{i, \alpha}(t) + \sqrt{\frac{T_{ex}}{m}(1 - e^{-2\zeta \Delta t})}\Gamma,$$

(D1)

where $\Gamma$ represents a zero mean random number whose variance is 1. In this paper, we use $\Delta t = 0.1/\zeta$ [34]. To consider the effect of particle collisions, we calculate the minimum time interval $\Delta \tau$ without the random force among the binary collisions of $i$-th and $j$-th particles $\Delta \tau_{ij}$ and the time for the $i$-th particle to reach the Lee-Edwards boundary $\Delta \tau_{i, \text{wall}}$ [48]. For $t < n\Delta t < t + \Delta \tau$ ($n$ is an integer) the positions of particles are updated without any collisions satisfying Eq. (D1). At $\Delta \tau = \Delta \tau_{ij}$, $i$-th and $j$-th particles collide and therefore their velocities change according to Eq. (4), while only the position of $i$-th particle is updated as $r_i \mp \gamma L \Delta t \rightarrow r_i$ at $\Delta t = \Delta \tau_{i, \text{wall}}$, where $L$ is the system size and the minus (plus) sign is selected if the velocity is positive (negative).

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