THE ALTERNATING MARKED POINT PROCESS OF $h$–SLOPES OF THE DRIFTED BROWNIAN MOTION

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Abstract. We show that the slopes between $h$–extrema of the drifted 1D Brownian motion form a stationary alternating marked point process, extending the result of J. Neveu and J. Pitman for the non drifted case. Our analysis covers the results on the statistics of $h$–extrema obtained by P. Le Doussal, C. Monthus and D. Fisher via a Renormalization Group analysis and gives a complete description of the slope between $h$–extrema covering the origin by means of the Palm–Khinchin theory. Moreover, we analyze the behavior of the Brownian motion near its $h$–extrema.

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1. Introduction

Let $B$ be a two–sided standard Brownian motion with drift $-\mu$. Given $h > 0$ we say that $B$ admits an $h$–minimum at $x \in \mathbb{R}$, and that $x$ is a point of $h$–minimum, if there exist $u < x < v$ such that $B_t \geq B_x$ for all $t \in [u,v]$, $B_u \geq B_x + h$ and $B_v \geq B_x + h$. Similarly, we say that $B$ admits an $h$–maximum at $x \in \mathbb{R}$, and that $x$ is a point of $h$–maximum, if there exist $u < x < v$ such that $B_t \leq B_x$ for all $t \in [u,v]$, $B_u \leq B_x - h$ and $B_v \leq B_x - h$. We say that $B$ admits an $h$–extremum at $x \in \mathbb{R}$, and that $x$ is a point of $h$–extremum, if $x$ is a point of $h$–minimum or a point of $h$–maximum. Finally, the truncated trajectory $B$ going from an $h$–minimum to an $h$–maximum will be called upward $h$–slope, while the truncated trajectory $B$ going from an $h$–maximum to an $h$–minimum will be called downward $h$–slope.

Our first object of investigation is the statistics of $h$–slopes. The non drifted case $\mu = 0$ has been studied in [NP]. Here we assume $\mu \neq 0$ and show (see Theorem 1) that the statistics of $h$–slopes is well described by a stationary alternating marked simple point process on $\mathbb{R}$ whose points are the points of $h$–extrema of the Brownian motion, and each point $x$ is marked by the $h$–slope going from $x$ to the subsequent point of $h$–extremum. We will show that the $h$–slopes are independent and specify the laws $P^+_{\mu\pm}, P^-_{\mu\pm}$ of upward $h$–slopes and downward $h$–slopes non covering the origin, respectively. The $h$–slope covering the origin shows a different distribution that can be derived by means of the Palm–Khinchin theory [DVJ], [FKAS].

Our proof is based both on fluctuation theory for Lévy processes, and on the theory of marked simple point processes. The part of fluctuation theory follows strictly the scheme of [NP] and can be generalized to spectrally one–sided Lévy processes, i.e. real valued random processes with stationary independent increments and with no positive jumps or with no negative jumps [B][Chapter VII]. In fact, some of the identities of Lemma 1 and Proposition 1 below have already been obtained with more sophisticated methods for general spectrally one–sided Lévy processes (see [PS], [AKP], [C] and references therein).
On the other hand, the description of the $h$–slopes as a stationary alternating marked simple point process allows to use the very powerful Palm–Khintchin theory, which extends renewal theory and leads to a complete description of the $h$–slope covering the origin. This analysis can be easily extended to more general Lévy processes, as the ones treated in [C].

As discussed in Section 3 our results concerning the statistics of $h$–extrema of drifted Brownian motion correspond to the ones obtained in [DFM] via a non rigorous Real Space Renormalization Group method applied to Sinai random walk with a vanishing bias. In addition of a rigorous derivation, we are able here to describe also the statistics of the $h$–slopes, lacking in [DFM].

In section 5 (see Theorem 2), we analyze the behavior of the drifted Brownian motion around its $h$–extrema. While in the non–drifted case a generic $h$–slope non covering the origin behaves in proximity of its extremes as a 3–dimensional Bessel process, in the drifted case it behaves as a process with a cothangent drift, satisfying the SDE

$$\begin{cases}
    dX_t = d\beta_t \pm \mu \coth (\mu X_t) \, dt, & t \geq 0, \\
    X_0 = 0,
\end{cases}$$

(1.1)

where $\beta_t$ is an independent standard Brownian motion and the sign in the r.h.s. depends on the kind of $h$–slope (downward or upward) and on the kind of $h$–extrema ($h$–minimum or $h$–maximum). In addition, we show that the process (1.1) is simply the Brownian motion on $[0, \infty)$, starting at the origin, with drift $\pm \mu$, Doob–conditioned to hit $+\infty$ before 0.

The interest in the statistics of $h$–slopes and their behavior near to the extremes comes also from the fact that, considering the diffusion in a drifted Brownian potential, the piecewise linear path obtained by connecting the $h$–extrema of the Brownian potential is the effective potential for the diffusion at large times [BCGD].

## 2. Statistics of $h$–slopes of drifted Brownian motion

Given $\mu, x \in \mathbb{R}$ we denote by $\mathbf{P}_x^{\mu}$ the law on $C(\mathbb{R}, \mathbb{R})$ of the standard two–sided Brownian motion $B$ with drift $-\mu$ having value $x$ at time zero, i.e. $B_t = x + B^*_t - \mu t$ where $B^*: \mathbb{R} \rightarrow \mathbb{R}$ denotes the two–sided Brownian motion s.t. $B_t$ has expectation zero and variance $t$. We denote the expectation w.r.t. $\mathbf{P}_x^{\mu}$ by $E_x^{\mu}$. If $\mu = 0$ we simply write $\mathbf{P}_x$, $E_x$.

Recall the definitions of $h$–maximum, $h$–minimum and $h$–extremum given in the Introduction. It is simple to verify that $\mathbf{P}_x^{\mu}$–a.s. the set of points of $h$–extrema is locally finite, unbounded from below and from above, and that points of $h$–minima alternate with points of $h$–maxima. The $h$–slope between two consecutive points of $h$–extrema $\alpha$ and $\beta$ is defined as the truncated trajectory $\gamma := (B_t : t \in [\alpha, \beta])$. We call it an upward slope if $\alpha$ is a point of $h$–minimum (and consequently $\beta$ is a point of $h$–maximum), otherwise we call it downward slope. The length $\ell(\gamma)$ and the height $h(\gamma)$ of the slope $\gamma$ are defined as $\ell(\gamma) = \beta - \alpha$ and $h(\gamma) = |\gamma(\beta) - \gamma(\alpha)|$, respectively. Moreover, to the slope $\gamma$ we associate the translated path $\theta(\gamma) := (B_{t+\alpha} - B_{\alpha} : t \in [0, \beta - \alpha])$. With some abuse of notation (as in the Introduction) we call also $\theta(\gamma)$ the $h$–slope between the points of $h$–extrema $\alpha$ and $\beta$. When the context can cause some ambiguity, we will explicitly distinguish between the $h$–slope $\gamma$ and the translated path $\theta(\gamma)$.

Finally, we introduce the following notation: given $\alpha \in \mathbb{R}$, the constant $\hat{\alpha}$ is defined as

$$\hat{\alpha} = \alpha + \mu^2/2.$$  

(2.1)
2.1. **The building blocks of the $h$–slopes**. Given a two–sided Brownian $B$ with law $\mathbb{P}_0^\mu$ we define the random variables $b_t, \tau, \beta, \sigma$ as follows (see figure 1):

$$
\begin{align*}
& b_t = \min \{ B_s : 0 \leq s \leq t \}, \\
& \tau = \min \{ t \geq 0 : B_t = b_t + h \}, \\
& \beta = b_\tau = \min \{ B_s : 0 \leq s \leq \tau \}, \\
& \sigma = \max \{ s : s \leq \tau, B_s = \beta \}.
\end{align*}
$$

(2.2)

Note that $\mathbb{P}_0^\mu$–a.s. there exists a unique time $s \in [0, \tau]$ with $B_s = \beta$, which by definition coincides with $\sigma$.

![Figure 1. The random variables $\beta, \sigma, \tau$.](image)

Our analysis of the statistics of $h$–slopes for the drifted Brownian motion is based on the following lemma which extends to the drifted case the lemma in Section 1 of [NP]:

**Lemma 1.** Let $\mu \neq 0$, $\hat{\alpha} > 0$ and $x > 0$. Under $\mathbb{P}_0^\mu$, the two trajectories

$$(B_t, 0 \leq t \leq \sigma), \quad (B_{\sigma+t} - \beta, 0 \leq t \leq \tau - \sigma)$$

are independent; in particular $(\beta, \sigma)$ and $\tau - \sigma$ are independent.

Furthermore $-\beta$ is exponentially distributed with mean

$$
E_0^\mu(-\beta) = \frac{\sinh(\mu h)}{\mu e^{-\mu h}},
$$

(2.3)

and

$$
E_0^\mu[\exp(-\alpha \sigma) | \beta = -x] = \exp \left\{ -x \left[ \sqrt{2\hat{\alpha}} \coth \left( \sqrt{2\hat{\alpha}} h \right) - \mu \coth(\mu h) \right] \right\}.
$$

(2.4)

In particular, $E_0^\mu(\exp(-\alpha \sigma))$ is finite if and only if

$$
\sqrt{2\hat{\alpha}} \coth \left( \sqrt{2\hat{\alpha}} h \right) > \mu.
$$

(2.5)

If (2.5) is fulfilled, then

$$
E_0^\mu(\exp(-\alpha \sigma)) = \frac{\mu e^{-\mu h}}{\sinh(\mu h) \left( \sqrt{2\hat{\alpha}} \coth \left( \sqrt{2\hat{\alpha}} h \right) - \mu \right)}.
$$

(2.6)

Finally, it holds

$$
E_0^\mu(\exp(-\alpha(\tau - \sigma))) = \frac{\sqrt{2\hat{\alpha}} \sinh(\mu h)}{\mu \sinh \left( \sqrt{2\hat{\alpha}} h \right)}.
$$

(2.7)
The proof of the above lemma is based on excursion theory and identities concerning hitting times of the drifted Brownian motion. It will be given in Section 4.

2.2. The probability measures $P^\mu_+$ and $P^\mu_-$ on $h$–slopes. We define the path space $W$ as the set

$$W = \cup_{T \geq 0} C([0, T]).$$

Given $\gamma \in W$, we define $\ell(\gamma)$ as the nonnegative number such that $\gamma \in C[0, \ell(\gamma)]$ and we define the path $\gamma^*: [0, \infty) \to \mathbb{R}$ as

$$\gamma^*_t = \begin{cases} \gamma_t & \text{if } 0 \leq t \leq \ell(\gamma), \\ \gamma_{\ell(\gamma)} & \text{if } t \geq \ell(\gamma). \end{cases}$$

Then the space $W$ is a Polish space endowed of the metric $d_W$ defined as

$$d_W(\gamma_1, \gamma_2) = |\ell(\gamma_1) - \ell(\gamma_2)| + \|\gamma^*_1 - \gamma^*_2\|_{\infty}.$$ 

On $W$ we define the Borel probability measures $P^\mu_+$, $P^\mu_-$ as follows. Let $B, B'$ be independent Brownian motions with law $P^\mu_0$. Recall the definition (2.2) of $\tau, \beta, \sigma$ and define $b'_t, \tau', \beta', \sigma'$ as (see figure 2):

$$\begin{cases} b'_t = \max \{ B'_s : 0 \leq s \leq t \}, \\ \tau' = \min \{ t \geq 0 : B'_t = b'_t - h \}, \\ \beta' = b'_{\tau'} = \max \{ B'_s : 0 \leq s \leq \tau' \}, \\ \sigma' = \max \{ s : s \leq \tau', B'_s = \beta' \}. \end{cases}$$  \hspace{1cm} (2.8)

Then $P^\mu_+$ is the law of the path $\gamma$, with $\ell(\gamma) = \tau - \sigma + \sigma'$, defined as

$$\gamma_t = \begin{cases} B_{\sigma+t} - \beta, & \text{if } t \in [0, \tau - \sigma], \\ B_{t-(\tau-\sigma)} + h, & \text{if } t \in [\tau - \sigma, \tau - \sigma + \sigma'], \end{cases}$$  \hspace{1cm} (2.9)

while $P^\mu_-$ is the law of the path $\gamma$, with $\ell(\gamma) = \tau' - \sigma' + \sigma$, defined as

$$\gamma_t = \begin{cases} B'_{\sigma'+t} - \beta', & \text{if } t \in [0, \tau' - \sigma'], \\ B_{t-(\tau'-\sigma')} - h, & \text{if } t \in [\tau' - \sigma', \tau' - \sigma' + \sigma]. \end{cases}$$  \hspace{1cm} (2.10)

Note that $P^\mu_-$ equals the law of the path $-\gamma$ if $\gamma$ is chosen with law $P^\mu_+$. 

![Figure 2. The random variables $\beta', \sigma', \tau'$.](image-url)
We introduce two disjoint subsets $W_+$ and $W_-$ of $W$:

$W_+ = \{ \gamma \in W : \gamma_0 = \min\{\gamma_t : t \in [0, \ell(\gamma)]\} = 0, \; \gamma_{\ell(\gamma)} = \max\{\gamma_t : t \in [0, \ell(\gamma)]\} \geq h \}$,

$W_- = \{ \gamma \in W : \gamma_0 = \max\{\gamma_t : t \in [0, \ell(\gamma)]\} = 0, \; \gamma_{\ell(\gamma)} = \min\{\gamma_t : t \in [0, \ell(\gamma)]\} \leq -h \}$.

Then the probability measure $P^\mu_+$ is concentrated on $W_\pm$. Below we will prove that, given the two–sided BM with law $P^\mu_+$, $P^\mu_-$ is the law of the generic upward $h$–slope not covering the origin, while $P^\mu_-$ is the law of the generic downward $h$–slope not covering the origin.

We collect in what follows some results derived from Lemma 1 and straightforward computations, which will be useful in what follows:

**Proposition 1.** Fix $\mu \neq 0$. Let $(\ell_+, \zeta_+)$ be the random vector distributed as $(\ell(\gamma), \gamma_{\ell(\gamma)} - h)$ where $\gamma$ is chosen with law $P^\mu_+$ and let $(\ell_-, \zeta_-)$ be the random vector distributed as $(\ell(\gamma), -(\gamma_{\ell(\gamma)} + h))$ where $\gamma$ is chosen with law $P^\mu_-$. Then $\zeta_+$, $\zeta_-$ are exponential variables of mean

$$E(\zeta_+) = \frac{\sinh(\mu h)}{\mu e^{\mu h}}, \quad E(\zeta_-) = \frac{\sinh(\mu h)}{\mu e^{-\mu h}}. \quad (2.11)$$

Fix $\alpha$ such that $\hat{\alpha} := \alpha + \mu^2/2 > 0$. Then for all $x > 0$

$$E\left( e^{-\alpha \ell_+} | \zeta_+ = x \right) = \frac{\sqrt{2\hat{\alpha}}}{\mu} \frac{\sinh(\mu h)}{\sinh(\sqrt{2\hat{\alpha} h})} \exp\left\{ -x \left( \sqrt{2\hat{\alpha}} \coth\left( \frac{\sqrt{2\hat{\alpha} h}}{\mu} \right) - \mu \coth(\mu h) \right) \right\}. \quad (2.12)$$

In particular, the expectation $E\left( e^{-\alpha \ell_+ - \lambda \zeta_+} \right)$ is finite if and only if

$$\sqrt{2\hat{\alpha}} \coth\left( \frac{\sqrt{2\hat{\alpha} h}}{\mu} \right) + (\lambda \pm \mu) > 0. \quad (2.13)$$

If $(2.13)$ is fulfilled, then

$$E\left( e^{-\alpha \ell_+ - \lambda \zeta_+} \right) = \frac{\sqrt{2\hat{\alpha}} e^{\pm \mu h}}{\sqrt{2\hat{\alpha}} \cosh\left( \frac{\sqrt{2\hat{\alpha} h}}{\mu} \right) + (\lambda \pm \mu) \sinh\left( \frac{\sqrt{2\hat{\alpha} h}}{\mu} \right)}. \quad (2.14)$$

Hence

$$E(\ell_+) = \mu^2 \left( \mu h - \sinh(\mu h)e^{-\mu h} \right), \quad (2.15)$$

$$E(\ell_-) = \mu^2 \left( e^{\mu h} \sinh(\mu h) - \mu h \right). \quad (2.16)$$

Consider $\ell := \ell_- + \ell_+$, where $\ell_-$ and $\ell_+$ are chosen independently. Then,

$$E(\ell) = \frac{2}{\mu^2} \sinh^2(\mu h). \quad (2.17)$$

Given $\alpha \in \mathbb{R}$, $E(e^{-\alpha \ell})$ is finite if and only if

$$\begin{cases} \hat{\alpha} := \alpha + \mu^2/2 > 0, \\
2\hat{\alpha} \cosh^2(\sqrt{2\hat{\alpha} h}) + \mu^2 > 0. \end{cases} \quad (2.18)$$

If $(2.18)$ is fulfilled, then

$$E(e^{-\alpha \ell}) = \frac{2\hat{\alpha}}{2\hat{\alpha} \cosh^2(\sqrt{2\hat{\alpha} h}) + \mu^2}. \quad (2.19)$$
Remark 1. Due to the identity
\[ 2\alpha \cosh^2(\sqrt{2\alpha h}) + \mu^2 = \frac{\cosh(\sqrt{2\alpha h})}{h^2} \left( 2\alpha h^2 - \mu^2 h^2 \tanh(\sqrt{2\alpha h}) \right), \]
by straightforward computations one can check that for \( \mu h > 1 \) condition (2.18) is fulfilled if and only if \( \alpha > -\mu^2/2 + y_*/(2h^2) \), where \( y_* \) is the only positive solution of the equation \( y = \mu h \tanh(y) \). If \( \mu h \leq 1 \) then condition (2.18) is fulfilled if and only if \( \alpha > -\mu^2/2 \).

2.3. The stationary alternating marked simple point process \( \mathcal{P}^\mu \). We denote by \( \mathcal{N} \) the space of sequences \( \xi = \{(x_i, \gamma_i) : i \in \mathbb{Z}\} \) such that 1) \( (x_i, \gamma_i) \in \mathbb{R} \times \mathcal{W} \), 2) \( x_i < x_{i+1} \), 3) \( x_0 \leq 0 < x_1 \) and 4) \( \lim_{i \to \pm\infty} x_i = \pm \infty \). In what follows, \( \xi \) will be often identified with the counting measure \( \sum_{i \in \mathbb{Z}} \delta_{(x_i, \gamma_i)} \) on \( \mathbb{R} \times \mathcal{W} \).

\( \mathcal{N} \) is a measurable space with \( \sigma \)-algebra of measurable sets generated by
\[ \{\xi \in \mathcal{N} : \xi(A \times B) = j\} \quad A \subset \mathbb{R} \text{ Borel}, B \subset \mathcal{W} \text{ Borel}, j \in \mathbb{N}. \]
One can characterize the above \( \sigma \)-algebra as follows. Consider the space \( \mathcal{S} := (0, \infty) \times (0, \infty)^2 \times \mathcal{W}^2 \) as a measurable space with \( \sigma \)-algebra of measurable sets given by the Borel subsets associated to the product topology. Call \( \mathcal{S}' \) the subset of \( \mathcal{S} \) given by the elements where the first entry is not larger than the entry with index 0 of the factor space \( (0, \infty)^2 \).

Then by the same arguments leading to \cite{FKAS} Proposition 7.1.X one can prove that the map
\[ \mathcal{N} \ni \{(x_i, \gamma_i)\}_{i \in \mathbb{Z}} \rightarrow \{x_1\} \times \{\tau_1\}_{i \in \mathbb{Z}} \times \{\gamma_i\}_{i \in \mathbb{Z}} \in \mathcal{S}', \quad \tau_i := x_{i+1} - x_i, \quad (2.20) \]
is bijective and both ways measurable. We note that the introduction of \( \mathcal{S}' \) is due to the constrain \( x_1 \leq x_0 \).

Let us define \( \mathcal{P}_{0,\pm}^\mu \) as the law of the sequences \( \{(x_i, \gamma_i)\}_{i \in \mathbb{Z}} \in \mathcal{N} \) such that
- \( \{\gamma_i\}_{i \in \mathbb{Z}} \) are independent random paths,
- \( \{\gamma_{2i}\}_{i \in \mathbb{Z}} \) are i.i.d. random paths with law \( P_{\pm}^\mu \),
- \( \{\gamma_{2i+1}\}_{i \in \mathbb{Z}} \) are i.i.d. random paths with law \( P_{\mp}^\mu \),
- \( x_0 = 0 \),
- \( x_{i+1} - x_i = \ell(\gamma_i) \).

Note that \( \mathcal{P}_{0,\pm}^\mu \) is concentrated on the measurable subset \( \mathcal{N}_0 \) defined as
\[ \mathcal{N}_0 = \{\{(x_i, \gamma_i)\}_{i \in \mathbb{Z}} : x_0 = 0\} \]
Finally we consider the convex combination
\[ \mathcal{P}_0^\mu = \frac{1}{2} \mathcal{P}_{0,+}^\mu + \frac{1}{2} \mathcal{P}_{0,-}^\mu. \]
Let \( \theta : \mathcal{N} \rightarrow \mathcal{N} \) and, for all \( t \in \mathbb{R} \), let \( T_t : \mathcal{N} \rightarrow \mathcal{N} \) be the maps defined as
\[ \theta \xi = \sum_{i \in \mathbb{Z}} \delta_{(x_i-x_{i-1}, \gamma_i)}, \quad T_t \xi = \sum_{i \in \mathbb{Z}} \delta_{(x_i-\epsilon_t \gamma_i)} \quad \text{if} \quad \xi = \sum_{i \in \mathbb{Z}} \delta_{(x_i, \gamma_i)}. \]
We stress that the above translation map \( T_t \) coincides with the map \( T_{-t} \) of \cite{FKAS} and with the map \( S_{-t} \) of \cite{DVJ}. A probability measure \( \mathcal{Q} \) on \( \mathcal{N} \) is called stationary if \( T_t \mathcal{Q}(A) := \mathcal{Q}(T_t A) = \mathcal{Q}(A) \) for all \( t \in \mathbb{R} \) and all \( A \subset \mathcal{N} \) measurable, while it is called \( \theta \)-invariant if \( \theta \mathcal{Q}(A) := \mathcal{Q}(\theta A) = \mathcal{Q}(A) \) for all \( A \subset \mathcal{N} \) measurable.

Note that \( \mathcal{P}_0^\mu \) is \( \theta \)-invariant. Moreover, due to (2.17) in Lemma 1
\[ \mathbf{E}_{\mathcal{P}_0^\mu}(x_1) = \mathbf{E}_{\mathcal{P}_0^\mu}(\ell(\gamma_0)) = \mathbf{E}(\ell)/2 = \sinh^2(\mu h)/\mu^2. \quad (2.21) \]
Hence, due to the Palm–Khinchin theory (see Theorem 1.3.1 and formula (1.2.15) in [PKAS], and Theorem 12.3.Π in [DVJ]) there exists a unique stationary measure $\mathcal{P}^\mu$ on $\mathcal{N}$ such that

$$
\mathcal{P}^\mu(A) = \frac{2}{\mathcal{E}(\ell)} \mathcal{E}_{\mathcal{P}_0^\mu} \left[ \int_0^{x_1} \chi(T_{-t} \{ \{x_i, \gamma_i\}_{i \in \mathbb{Z}}\} \in A) \, dt \right],
$$

(2.22)

where $\chi(\cdot)$ denotes the characteristic function. We simply say that $\mathcal{P}^\mu$ is the law of the stationary alternating marked point process on $\mathbb{R}$ with alternating mark laws given by $\mathcal{P}_+^\mu, \mathcal{P}_-^\mu$. The probability measure $\mathcal{P}_0^\mu$ is the so called Palm distribution associated to $\mathcal{P}^\mu$.

One can write

$$
\mathcal{P}^\mu(\cdot) = \mathcal{P}^\mu(\cdot \mid \gamma_0 \in \mathcal{W}_+)\mathcal{P}^\mu(\gamma_0 \in \mathcal{W}_+) + \mathcal{P}^\mu(\cdot \mid \gamma_0 \in \mathcal{W}_-)\mathcal{P}^\mu(\gamma_0 \in \mathcal{W}_-).
$$

(2.23)

From (2.22) we obtain that

$$
\mathcal{P}^\mu(\gamma_0 \in \mathcal{W}_\pm) = \frac{2\mathcal{E}_{\mathcal{P}_0^\mu}[x_1 \chi(\gamma_0 \in \mathcal{W}_\pm)]}{\mathcal{E}(\ell)} = \frac{\mathcal{E}_{\mathcal{P}_\pm^\mu}(\ell(\gamma))}{\mathcal{E}(\ell)} = \frac{\mathcal{E}(\ell_\pm)}{\mathcal{E}(\ell)}.
$$

(2.24)

Hence, by (2.15) and (2.16) of Lemma 11 we can conclude that

$$
\mathcal{P}^\mu(\gamma_0 \in \mathcal{W}_\pm) = \frac{\pm \mu h \mp \sinh(\mu h)e^{\mp \mu h}}{2\sinh^2(\mu h)}.
$$

(2.25)

In order to describe the conditional probability measure $\mathcal{P}^\mu(\cdot \mid \gamma_0 \in \mathcal{W}_\pm)$ we first observe that $x_1$ and $\{\gamma_i\}_{i \in \mathbb{Z}}$ univocally determined the set $\{(x_i, \gamma_i)\}_{i \in \mathbb{Z}}$. Moreover from (2.22) we derive that, given Borel subsets $A \subset \mathbb{R}$, $B_j \subset \mathcal{W}$ for $-m \leq j \leq n$, it holds

$$
\mathcal{P}^\mu(x_1 \in A, \gamma_j \in B_j \forall j : -m \leq j \leq n \mid \gamma_0 \in \mathcal{W}_\pm) = \mathcal{E}_{\mathcal{P}_0^\mu,\pm} \left( \int_0^{x_1} \chi(x_1 - t \in A)dt, \gamma_j \in B_j \forall j : -m \leq j \leq n \right) / \mathcal{E}(\ell_\pm) = \mathcal{E}_{\mathcal{P}_\pm^\mu} \left( \int_0^{\ell(\gamma)} \chi(t \in A)dt, \gamma \in B_0 \right) \prod_{-m \leq j \leq n} \mathcal{P}_\pm^\mu(B_j) \prod_{-m \leq j \leq n} \mathcal{P}_\pm^\mu(B_j) / \mathcal{E}(\ell_\pm).
$$

This identity implies that under $\mathcal{P}^\mu(\cdot \mid \gamma_0 \in \mathcal{W}_\pm)$ the random paths $\gamma_i, i \in \mathbb{Z}$, are independent, the paths $\{\gamma_{2i}\}_{i \in \mathbb{Z}} \setminus \{0\}$ have common law $\mathcal{P}_0^\mu$ while the paths $\{\gamma_{2i+1}\}_{i \in \mathbb{Z}}$ have common law $\mathcal{P}_\pm^\mu$ and that the path $\gamma_0$ has law

$$
\mathcal{P}^\mu(\gamma_0 \in A \mid \gamma_0 \in \mathcal{W}_\pm) = \frac{\mathcal{E}_{\mathcal{P}_\pm^\mu}(\ell(\gamma)\chi(\gamma \in A))}{\mathcal{E}_{\mathcal{P}_\pm^\mu}(\ell(\gamma))} = \frac{\mathcal{E}_{\mathcal{P}_\pm^\mu}(\ell(\gamma)\chi(\gamma \in A))}{\mathcal{E}(\ell_\pm)}.
$$

(2.26)

Finally, we claim that under $\mathcal{P}^\mu(\cdot \mid \gamma_0 \in \mathcal{W}_\pm)$ $x_1$ has probability density function on $[0, \infty)$ given by $(1 - F_\pm(x)) / \mathcal{E}(\ell_\pm)$ where

$$
F_\pm(x) = \mathcal{P}(\ell_\pm \leq x), \quad x \geq 0.
$$
Indeed, given \( x \geq 0 \), due to (2.22) and (2.24) we can write
\[
\mathcal{P}^{\mu}(x_1 \geq x \mid \gamma_0 \in \mathcal{W}_\pm) = \frac{\mathcal{P}^{\mu}(x_1 \geq x, \gamma_0 \in \mathcal{W}_\pm)}{\mathcal{P}^{\mu}(\gamma_0 \in \mathcal{W}_\pm)} = \mathbb{E}_{\mathcal{P}^{\mu}_{\gamma_0, \pm}} \left( \int_0^{x_1} \chi(x_1 - t \geq x) dt \right) = \mathbb{E}(\ell_\pm) = \mathbb{E}(\ell_\pm - x) = \frac{\mathbb{E}(\ell_\pm - x)}{\mathbb{E}(\ell_\pm)}.
\]

In particular, we point out that due to (2.23) and (2.24) under \( \mathcal{P}^{\mu} \) the random variable \( x_1 \) has probability density on \([0, \infty)\) given by \((2 - F_+(x) - F_-(x))/\mathbb{E}(\ell)\).

### 2.4. The statistics of \( h \)-slopes.

We can finally prove the main theorem of this section:

**Theorem 1.** Let \( B \) be the Brownian motion with law \( \mathbb{P}^{\mu}_0 \) and let \((m_i)_{i \in \mathbb{Z}}\) be the sequence of points of \( h \)-extrema of \( B \), increasingly ordered, with \( m_0 \leq 0 < m_1 \). For each \( i \in \mathbb{Z} \) define the \( h \)-slope \( \gamma_i \) as
\[
\gamma_i = (B_t - B_{m_i} : 0 \leq t \leq m_{i+1} - m_i).
\]

Then the marked point process
\[
\{(m_i, \gamma_i) : i \in \mathbb{Z}\}
\]
has law \( \mathcal{P}^{\mu}_0 \).

In order to prove the above result we need to further elucidate the relation between \( \mathcal{P}^{\mu}_0 \) and its Palm distribution \( \mathcal{P}^{\mu}_0 \), benefiting of the Palm–Khinchin theory. To this aim we need the ergodicity of \( \mathcal{P}^{\mu}_0 \):

**Lemma 2.** The probability measure \( \mathcal{P}^{\mu}_0 \) is \( \theta \)-ergodic, i.e. if \( A \subset \mathcal{N} \) is a measurable set such that \( \mathcal{P}^{\mu}_0(A \Delta \theta A) = 0 \) then \( \mathcal{P}^{\mu}_0(A) \in \{0, 1\} \).

**Proof.** Let us suppose that \( A \subset \mathcal{N} \) is a Borel set with \( \mathcal{P}^{\mu}_0(A \Delta \theta A) = 0 \). Due to the characterization of the \( \sigma \)-algebra of measurable sets in \( \mathcal{N} \) given by the bijective and both ways measurable map (2.20) and since \( \mathcal{P}^{\mu}_0 \) is concentrated on \( \mathcal{N}_0 \), for each \( \varepsilon > 0 \) we can find a measurable set \( A_\varepsilon \subset \mathcal{N}_0 \) and an integer \( k = k(\varepsilon) \) such that \( A_\varepsilon \) depends only on the random variables \( \gamma_i \) with \( -k \leq i \leq k \) and \( \mathcal{P}^{\mu}_0(A \Delta A_\varepsilon) \leq \varepsilon \). Since
\[
\mathcal{P}^{\mu}_0(A \Delta A_\varepsilon) = \frac{1}{2} \mathcal{P}^{\mu}_0(A \Delta A_\varepsilon) + \frac{1}{2} \mathcal{P}^{\mu}_0(A \Delta A_\varepsilon)
\]
we can conclude that
\[
\mathcal{P}^{\mu}_0(A \Delta A_\varepsilon) \leq 2\varepsilon, \quad \mathcal{P}^{\mu}_0(A \Delta A_\varepsilon) \leq 2\varepsilon.
\]

Since \( \mathcal{P}^{\mu}_0 \) is \( \theta \)-invariant and \( \mathcal{P}^{\mu}_0(A \Delta \theta A) = 0 \), we get for each positive integer \( n \) that \( \mathcal{P}^{\mu}_0(A \Delta \theta^n A) = 0 \) and therefore
\[
\mathcal{P}^{\mu}_0(A \Delta \theta^n A) = \mathcal{P}^{\mu}_0(\theta^n A \Delta \theta^n A) = \mathcal{P}^{\mu}_0(\theta^n(A \Delta A_\varepsilon)) = \mathcal{P}^{\mu}_0(A \Delta A_\varepsilon) \leq \varepsilon.
\]

This implies that
\[
\mathcal{P}^{\mu}_0(A \Delta \theta^n A_\varepsilon) \leq \mathcal{P}^{\mu}_0(A \Delta A) + \mathcal{P}^{\mu}_0(A \Delta \theta^n A) \leq 2\varepsilon.
\]

Let us now write \( o(1) \) for any quantity which goes to \( 0 \) as \( \varepsilon \downarrow 0 \). We note that for \( n \) large enough and even it holds
\[
\mathcal{P}^{\mu}_{0, +}(A_\varepsilon \cap \theta^n A_\varepsilon) = \mathcal{P}^{\mu}_{0, +}(A_\varepsilon)^2 = \mathcal{P}^{\mu}_{0, +}(A)^2 + o(1),
\]
\[
\mathcal{P}^{\mu}_{0, -}(A_\varepsilon \cap \theta^n A_\varepsilon) = \mathcal{P}^{\mu}_{0, -}(A_\varepsilon)^2 = \mathcal{P}^{\mu}_{0, -}(A)^2 + o(1);
\]
while for \( n \) large enough and odd it holds
\[
\mathcal{P}^\mu_{0,+}(A \cap \theta^n A) = \mathcal{P}^\mu_{0,-}(A \cap \theta^n A) = \mathcal{P}^\mu_{0,+}(A \cap \theta^n A) = \mathcal{P}^\mu_{0,-}(A) = \mathcal{P}^\mu_{0,+}(A)\mathcal{P}^\mu_{0,-}(A) + o(1). \tag{2.32}
\]

Let \( a := \mathcal{P}^\mu_{0,+}(A) \) and \( b := \mathcal{P}^\mu_{0,-}(A) \). Due to \( [2.28],...,[2.32] \) we can conclude that
\[
o(1) = \mathcal{P}^\mu_{0}(A \Delta \theta^n A) = \mathcal{P}^\mu_{0}(A \cap \theta^n A) - 2\mathcal{P}^\mu_{0}(A \cap \theta^n A) = 2\mathcal{P}^\mu_{0}(A) - 2\mathcal{P}^\mu_{0}(A \cap \theta^n A) = \]
\[
\mathcal{P}^\mu_{0,+}(A) + \mathcal{P}^\mu_{0,-}(A) - \left[ \mathcal{P}^\mu_{0,+}(A \cap \theta^n A) + \mathcal{P}^\mu_{0,-}(A \cap \theta^n A) \right] = \tag{2.33}
\]
\[
\begin{cases}
  a + b - (a^2 + b^2) + o(1) & \text{if } n \text{ is even and large,} \\
  a + b - 2ab + o(1) & \text{if } n \text{ is odd and large.}
\end{cases}
\]

We conclude that
\[
a + b - (a^2 + b^2) = o(1), \quad a + b - 2ab = o(1).
\]

hence, by subtraction, \((a - b)^2 = o(1)\), i.e. \( a = b + o(1) \). It is simple to check that there are only two possible cases: (i) \( a = o(1) \) and \( b = o(1) \), (ii) \( a = 1 + o(1) \) and \( b = 1 + o(1) \). Since \( \mathcal{P}^\mu_{0}(A) = (a + b)/2 \), in the first case we get \( \mathcal{P}^\mu_{0}(A) = o(1) \) while in the latter \( \mathcal{P}^\mu_{0}(A) = 1 + o(1) \). Due to the arbitrariness of \( \varepsilon \) we conclude that \( \mathcal{P}^\mu_{0}(A) \in (0,1) \).

\[
\square
\]

Let us now introduce the space \( \mathcal{N}_s \) given by the counting measures \( \xi = \sum_{j \in J} n_j \delta_{(x(j), \gamma(j))} \) on \( \mathbb{R} \times \mathcal{W} \), where \( n_j \in \mathbb{N} \) and the set \( \{ (x(j), \gamma(j)) \}_{j \in J} \) has finite intersection with sets of the form \([a, b] \times \mathcal{W} \). Note that if \( n_j = 1 \) for all \( j \in J \) we can identify \( \xi \) with its support.

Hence we can think of \( \mathcal{N} \) as a subset of \( \mathcal{N}_s \).

As discussed in [FKAS][Section 1.1.5] one defines on \( \mathcal{N}_s \) a suitable metric \( d_{\mathcal{N}_s} \) such that i) \( \mathcal{N}_s \) is a Polish space, ii) \( \mathcal{N} \) is a Borel subset of \( \mathcal{N}_s \) and iii) the \( \sigma \)-algebra of Borel subsets of \( \mathcal{N}_s \) is generated by the sets \( \{ \xi \in \mathcal{N}_s : \xi(A \times B) = j \} \), \( A \subset \mathbb{R} \) Borel, \( B \subset \mathcal{W} \) Borel, \( j \in \mathbb{N} \).

In particular, the \( \sigma \)-algebra of measurable subsets of \( \mathcal{N} \) introduced above coincides with the \( \sigma \)-algebra of Borel subsets of \( \mathcal{N} \) and we can think of \( \mathcal{P}^\mu, \mathcal{P}^\mu_{0}, \mathcal{P}^\mu_{0,+} \) as Borel probability measures on \( \mathcal{N}_s \) concentrated on \( \mathcal{N} \).

Due to Lemma [2] and [FKAS][Theorem 1.3.13] we get

**Corollary 1.** Given \( \mu \neq 0 \), the probability measure \( T_{t_0} \mathcal{P}^\mu_{0} \) weakly converges to \( \mathcal{P}^\mu \) as \( t_0 \downarrow -\infty \), i.e. for any continuous bounded function \( f : \mathcal{N}_s \rightarrow \mathbb{R} \) it holds
\[
\lim_{t_0 \downarrow -\infty} \mathbf{E}_{T_{t_0} \mathcal{P}^\mu_{0}}(f) = \mathbf{E}_{\mathcal{P}^\mu}(f). \tag{2.34}
\]

Let \( \nu \) be the intensity measure associated to \( \mathcal{P}^\mu \), i.e. \( \nu \) is the probability measure on \( \mathbb{R} \times \mathcal{W} \) such that
\[
\nu(A \times B) = \mathbf{E}_{\mathcal{P}^\mu}(\xi(A \times B)), \quad A \subset \mathbb{R} \text{ Borel, } B \subset \mathcal{W} \text{ Borel}.
\]

Then due to [FKAS][Theorem 1.1.16], the weak convergence of \( T_{t_0} \mathcal{P}^\mu_{0} \) to \( \mathcal{P}^\mu \) stated in Corollary [1] is equivalent to the following fact: given a finite family \( X_1, X_2, \ldots, X_k \) of disjoint sets
\[
X_j = [a_j, b_j] \times L_j, \quad a_j, b_j \in \mathbb{R}, \quad L_j \subset \mathcal{W} \text{ Borel},
\]
satisfying
\[
\nu(\partial X_j) = 0, \quad j = 1, \ldots, k,
\]
where \( \partial X \) denotes the boundary of \( X \), it holds
\[
\lim_{t_0 \to -\infty} T_{t_0} \mathcal{P}^\mu (\xi(X_1) = j_1, \xi(X_2) = j_2, \ldots, \xi(X_k) = j_k) = \mathcal{P}^\mu (\xi(X_1) = j_1, \xi(X_2) = j_2, \ldots, \xi(X_k) = j_k)
\]
for all \( j_1, j_2, \ldots, j_k \in \mathbb{N} \).

We have now the main tools in order to prove Theorem 1.

\[\begin{figure}
\begin{center}
\includegraphics[width=0.5\textwidth]{figure3.png}
\end{center}
\caption{The sequence \( \sigma_n, \tau_n \).}
\end{figure}\]

Proof of Theorem 1 Let \( B \) be a two-sided Brownian motion with law \( \mathcal{P}^\mu_0 \). Set \( \tau_{-1} = t_0 \) and define the random variables \( \tau_n, \beta_n, \sigma_n \) inductively on \( n \in \mathbb{N} \) as follows (see figure 3):

For \( n \) even set
\[
\tau_n = \min \left\{ t \geq \tau_{n-1} : B_t = \min_{\tau_{n-1} \leq s \leq t} (B_s) + h \right\}, \tag{2.35}
\]
\[
\beta_n = \min \left\{ B_s : \tau_{n-1} \leq s \leq \tau_n \right\}, \tag{2.36}
\]
\[
\sigma_n = \max \left\{ s : \tau_{n-1} \leq s \leq \tau_n, B_s = \beta_n \right\}; \tag{2.37}
\]
for \( n \) odd set
\[
\tau_n = \min \left\{ t \geq \tau_{n-1} : B_t = \max_{\tau_{n-1} \leq s \leq t} (B_s) - h \right\}, \tag{2.38}
\]
\[
\beta_n = \max \left\{ B_s : \tau_{n-1} \leq s \leq \tau_n \right\}, \tag{2.39}
\]
\[
\sigma_n = \max \left\{ s : \tau_{n-1} \leq s \leq \tau_n, B_s = \beta_n \right\}. \tag{2.40}
\]

Note that by construction \( \sigma_n \) is a point of \( h \)-maximum for \( n \) odd, while \( \sigma_n \) is a point of \( h \)-minimum for \( n \neq 0 \) even. Moreover, due to Lemma 1 and the strong Markov property of Brownian motion at the Markov times \( \tau_n \), the slopes
\[
(B_{\sigma_n + s} - B_{\sigma_n} : 0 \leq s \leq \sigma_{n+1} - \sigma_n) \quad n \geq 1
\]
are independent, having law \( \mathcal{P}^- \) if \( n \) is odd and law \( \mathcal{P}^+ \) if \( n \) is even.

In what follows we will use the independent random variables \( X, V \) with the following laws: \( X \) is distributed as \( \sigma_1 - t_0 \), i.e. \( X \) is distributed as \( \tau + \sigma' \) where \( \tau, \sigma' \) are independent copies of \( \tau, \sigma' \) defined in (2.2), (2.8) respectively. \( V \) is distributed as \( \ell_+ \), i.e. as the length \( \ell(\gamma) \) of the random path \( \gamma \) chosen with law \( \mathcal{P}^+_+ \).
Given a realization of the two–sided Brownian motion $B$ with law $\mathbf{P}_0^\mu$, let $\xi(B)$ be the associated marked simple point process defined in (2.27), while let $\xi$ denote a generic element of $\mathcal{N}$. Fix a finite family $X_1, X_2, \ldots, X_k$ of disjoint sets $X_j = [a_j, b_j) \times L_j$, with $a_j, b_j \in \mathbb{R}$ and $L_j \subset W$ Borel, and consider the event

$$A := \{\xi : \xi(X_1) = j_1, \xi(X_2) = j_2, \ldots, \xi(X_k) = j_k\}$$

for given $j_1, j_2, \ldots, j_k \in \mathbb{N}$. Finally, set

$$a := \min\{a_1, a_2, \ldots, a_k\}.$$

Due to the discussion after Corollary 1, we only need to show that

$$\mathbf{P}_0^\mu(\xi(B) \in A) = \mathcal{P}_0^\mu(A).$$

(2.41)

To this aim, we set

$$g(u) = T_u \mathcal{P}_0^\mu(A)$$

and restrict in what follows to the case $t_0 < a$. Then our initial considerations imply that

$$\mathbf{P}_0^\mu(\xi(B) \in A, \sigma_1 < a) = \mathcal{E}(g(t_0 + X), t_0 + X < a)$$

(2.42)

and therefore

$$\left| \mathbf{P}_0^\mu(\xi(B) \in A) - \mathcal{E}g(t_0 + X) \right| =$$

$$\left| \mathbf{P}_0^\mu(\xi(B) \in A, \sigma_1 \geq a) - \mathcal{E}(g(t_0 + X), t_0 + X \geq a) \right| \leq 2P(t_0 + X \geq a).$$

(2.43)

In what follows, we will frequently apply the above argument in order to get estimates from above without explicit mention.

Let us consider the probability measure $T_{t_0} \mathcal{P}_0^\mu$. By definition

$$T_{t_0} \mathcal{P}_0^\mu(A) = \frac{1}{2} T_{t_0} \mathcal{P}_{0,+}^\mu(A) + \frac{1}{2} T_{t_0} \mathcal{P}_{0,-}^\mu(A),$$

while

$$\left| T_{t_0} \mathcal{P}_{0,+}^\mu(A) - \mathcal{E}g(t_0 + V) \right| \leq 2P(t_0 + V \geq a).$$

(2.44)

Hence we can estimate

$$\left| T_{t_0} \mathcal{P}_0^\mu(A) - \mathcal{E}g(t_0 + V) / 2 - g(t_0) / 2 \right| \leq 2P(t_0 + V \geq a).$$

(2.45)

Due to (2.43), (2.45) and Corollary 1 in order to prove the theorem it is enough to show that

$$\lim_{t_0 \downarrow -\infty} \left| \mathcal{E}g(t_0 + X) - \mathcal{E}g(t_0 + V) / 2 - g(t_0) / 2 \right| = 0.$$  

(2.46)

We will derive from the local central limit theorem that, given a generic positive random variable $W$ having a (bounded) probability density and bounded third moment, it holds

$$\lim_{t_0 \downarrow -\infty} \left| \mathcal{E}g(t_0 + W) - g(t_0) \right| = 0.$$  

(2.47)

Due to Lemma 3 below, this result allows to derive (2.46). In order to prove (2.47) define $S_n$ as the sum of $n$ independent copies of the random variable $\ell$ introduced in Proposition 1. Moreover, let $S_n$ and $W$ be independent. Then

$$\left| \mathcal{E}g(t_0 + W + S_n) - \mathcal{E}g(t_0 + W) \right| \leq 2P(t_0 + W + S_n \geq a),$$

$$\left| \mathcal{E}g(t_0 + S_n) - \mathcal{E}g(t_0) \right| \leq 2P(t_0 + S_n \geq a).$$

Hence in order to prove (2.47) it is enough to prove that

$$\lim_{n \to \infty} \limsup_{t_0 \downarrow -\infty} \left| \mathcal{E}g(t_0 + W + S_n) - \mathbf{E}g(t_0 + S_n) \right| = 0.$$
In general, given a r.w. $Z$ we write $p_Z$ for its probability density (if it exists). Moreover, we denote by $\| \cdot \|_1$ the norm in $L^1(\mathbb{R}, du)$. Setting $\bar{S}_n = S_n - nE(\ell)$, we can bound

$$|Eg(t_0 + W + S_n) - Eg(t_0 + S_n)| \leq \int_\mathbb{R} g(t_0 + u)|p_{W + S_n}(u) - p_{S_n}(u)| du \leq \|p_{W + S_n} - p_{S_n}\|_1.$$  

(2.48)

Since

$$p_{W + S_n}(u) du = p_{(W + \bar{S}_n)/\sqrt{n}}(u/\sqrt{n}) du,$$

$$p_{S_n}(u) du = p_{\bar{S}_n/\sqrt{n}}(u/\sqrt{n}) du,$$

by a change of variables we can conclude that

$$|Eg(t_0 + W + S_n) - Eg(t_0 + S_n)| \leq \|p_{(W + \bar{S}_n)/\sqrt{n}} - p_{\bar{S}_n/\sqrt{n}}\|_1.$$  

(2.49)

Let $N(u) = \exp(-u^2/(2\lambda))/\sqrt{2\pi\lambda}$ be the gaussian distribution with variance $\lambda = Var(\ell)$. Due to (2.48) and (2.49) in order to conclude we only need to show that

$$\lim_{n \to \infty} \|p_{(W + \bar{S}_n)/\sqrt{n}} - p_{W/\sqrt{n}} * N\|_1 = 0,$$  

(2.50)

$$\lim_{n \to \infty} \|p_{W/\sqrt{n}} * N - N\|_1 = 0,$$  

(2.51)

$$\lim_{n \to \infty} \|N - p_{\bar{S}_n/\sqrt{n}}\|_1 = 0,$$  

(2.52)

where $f * g$ denotes the convolution of $f$ and $g$.

We note that (2.50) follows from (2.52) since $p_{(W + \bar{S}_n)/\sqrt{n}} = p_{W/\sqrt{n}} * p_{\bar{S}_n/\sqrt{n}}$ and for generic functions $h, h', w$ in $L^1(\mathbb{R}, du)$ it holds $\|h * w - h' * w\|_1 \leq \|h - h'\|_1 \|w\|_1$. The limit (2.51) follows from straightforward computations while (2.52) corresponds to the $L^1$–local central limit theorem for densities since $W$ has bounded probability density and bounded third moment (see [PR][page 193] or Theorem 18 in [Pe][Chapter VII] where the boundedness of $p_W$ is required).

\[\square\]

**Lemma 3.** The random variables $X$ and $V$ in the proof of Theorem 4 have bounded continuous probability densities. Moreover, they have finite $n$–th moment for all $n \in \mathbb{N}$.

**Proof.** Due to Theorem 3 in [P][Section XV.3] in order to prove that $X$ and $V$ have bounded continuous probability densities it is enough that the associated Fourier transforms are in $L^1(\mathbb{R}, dx)$. Since $X = (\tau - \sigma) + \sigma + \sigma'$ and $\ell_+ = (\tau - \sigma) + \sigma'$ where the random variables $\tau - \sigma, \sigma$ and $\sigma'$ are independent, it is enough to prove that the Fourier transform of $\tau - \sigma$ is integrable. To this aim we observe that due to Lemma 1 the expectation $E_0^\mu(e^{-\alpha(\tau - \sigma)})$ is finite for $\alpha > -\mu^2/2$. This implies that the complex Laplace transform

$$\mathcal{C} \ni \alpha \to E_0^\mu(e^{-\alpha(\tau - \sigma)}) \in \mathbb{C}$$

is well-defined (i.e. the integrand is integrable) and analytic on the complex halfplane $\Re(\alpha) > -\mu^2/2$. Indeed, integrability is stated in Section 2.2 of [D1] and analyticity is stated in Satz 1 (Proposition 1) in Section 3.2 of [D1]. We point out that in [D1] the author considers the complex Laplace transform of functions, but all arguments and results can be easily extended to the complex Laplace transform of probability measures. In particular, we get that the Fourier transform $\tau - \sigma$ is given by

$$E_0^\mu(e^{-ia(\tau - \sigma)}) = \frac{\sqrt{2ai + \mu^2}}{\mu} \frac{\sinh(\mu h)}{\sinh \left(\sqrt{2ai + \mu^2}h\right)}, \quad a \in \mathbb{R},$$

\[Q.E.D.\]
where the square–root is defined by analytic extension as $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$ on the simply connected set $\{re^{i\theta} : r \geq 0, \theta \in (-\pi, \pi)\}$. From the above expression, one easily derives the integrability of the above Fourier transform.

Since the Laplace transforms $\mathbb{E}[z^{\Gamma}]$ and $\mathbb{E}[e^{t\delta}]$ are analytic in the origin, it follows from [DFM] that $\sigma$, $\tau - \sigma$ and $\sigma'$ have finite $n$–th moments for all $n \in \mathbb{N}$. Hence, the same holds for $X$ and $V$.

\[ \square \]

3. Comparison with the RG–approach

In this section we give some comments on the results concerning the $h$–extrema of drifted Brownian motion obtained in [DFM] via the non–rigorous Real Space Renormalization Group (RSRG) method and applied for the analysis of 1d random walks in random environments. We present the results obtained in [DFM] in the formalism of Sinai’s random walk, keeping the discussion at a non–rigorous level.

Start with a sequence of i.i.d. random variables $\{\omega_x\}_{x \in \mathbb{Z}}$ such that $\omega_x \in (0, 1)$ and

\[ A := \mathbb{E} \left[ \log \frac{1 - \omega_0}{\omega_0} \right] \in \mathbb{R} \setminus \{0\}, \quad \text{Var} \left[ \log \frac{1 - \omega_0}{\omega_0} \right] = 2\sigma \in (0, \infty). \]

Defining $\delta := A/(2\sigma)$, the random variable $\log \omega_x/(1-\omega_x)$ corresponds then to the random variable $f$ in (27) of [DFM]. Without loss of generality we set $\sigma = 1$ as in [DFM], thus implying that $A = 2\delta$.

The associated Sinai’s random walk is the nearest neighbor random walk on $\mathbb{Z}$ where $\omega_x$ is the probability to jump from $x$ to $x+1$ and $1-\omega_{x-1}$ is the probability to jump from $x$ to $x-1$. Consider the function $V : \mathbb{R} \rightarrow \mathbb{R}$ defined on $\mathbb{Z}$ as

\[ V(x) = \begin{cases} \sum_{i=0}^{x-1} \log \frac{1-\omega_x}{\omega_x}, & \text{if } x \geq 1, x \in \mathbb{Z}, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x}^{-1} \log \frac{1-\omega_x}{\omega_x} & \text{if } x < 0, x \in \mathbb{Z}, \end{cases} \]

and extended to all $\mathbb{R}$ by linear interpolation. Morally, the above Sinai’s random walk is well described by a diffusion in the potential $V$. In [DFM] the authors obtain results on the statistics of the $\Gamma$–extrema of $V$ taking the limits $\Gamma \uparrow \infty$, $\delta \downarrow 0$ with $\Gamma\delta$ fixed. In what follows, we show the link between their results and our analysis of the statistics of $h$–extrema of drifted Brownian motion.

By the Central Limit Theorem applied to $V(x) - 2\delta x$, one concludes that for $\Gamma$ large

\[ \frac{V(x\Gamma^2)}{\sqrt{2\Gamma}} \sim B^*_x + \sqrt{2\delta} \Gamma x, \quad x \in \mathbb{R}, \]

where $B^*$ is the standard two–sided Brownian motion (i.e. $B^*$ has law $P_0$). If we set

\[ \mu = -\sqrt{2\delta} \Gamma \]  \tag{3.1} \]

and consider the limits $\Gamma \uparrow \infty$ and $\delta \downarrow 0$ with $\mu$ fixed we get that the rescaled potential $V$ is well approximated by the Brownian motion $B$ with law $P_0^\mu$. In particular, for $\Gamma$ large one expects that the ordered family $\{(x_k, V(x_k))\}_{k \in \mathbb{Z}}$ of $\Gamma$–extrema of $V$ is well approximated by family $\{ (\Gamma^2 m_k, \sqrt{2\Gamma} B_{m_k}) \}_{k \in \mathbb{Z}}$ where $\{m_k\}_{k \in \mathbb{Z}}$ is the ordered sequence of points of $h$–extrema of $B$ with $h = 1/\sqrt{2}$ (we follow the convention that $x_0 \leq 0 < x_1$). Setting as in [DFM]

\[ \zeta := \left| V(x_{k+1}) - V(x_k) \right| - \Gamma, \quad l := x_{k+1} - x_k, \]
morally we get
\[ \zeta/\Gamma \sim \sqrt{2}|B_{m+1} - B_m| - 1, \quad l/\Gamma^2 \sim m_{k+1} - m_k. \] (3.2)

In [DFM] the authors write \( P^+(\zeta, l)dl d\zeta \) for the joint probability density of the random variables \( \zeta, l \) if \( x_k \) is a point of \( \Gamma \)-minimum or a point \( \Gamma \)-maximum respectively, they set \( P^\pm(\zeta, p) := \int_0^\infty e^{-lp} P_t(\zeta, l) dl \) and derive via the RSRG method the limiting form of \( P_t(\zeta, p) \) (note that in [DFM] the authors erroneously do not distinguish the \( \Gamma \)-slope covering the origin from the other \( \Gamma \)-slopes, but in order to have a correct result one must take \( k \neq 0 \)).

It is simple to check that all the computations obtained in [DFM] equal the results obtained in Proposition 1 by approximating the random variables \( \zeta, l \) with distribution \( P^\pm(\zeta, l) dl d\zeta \) by means of the random variables \( \sqrt{2\Gamma} \zeta^\pm, \Gamma^2 \ell^\pm \) of Proposition 1, where \( \mu = -\sqrt{2}\delta, h = 1/\sqrt{2} \). This confirms (3.2).

4. Proof of Lemma 1 via fluctuation theory

Given a path \( f \in C([0, \infty), \mathbb{R}) \), define the hitting time of \( f \) at \( x \) as
\[ T_x(f) = \inf \{ t > 0 : f_t = x \}, \quad x \in \mathbb{R}. \] (4.1)

Consider the process \( \{B_t, t \geq 0\} \) carrying the law \( P^\mu_0 \) and define (see figure 4)
\[ \begin{align*}
    b_t &= \min \{ B_s : 0 \leq s \leq t \} \\
    L_t &= -b_t, \\
    Y_t &= B_t - b_t.
\end{align*} \] (4.2)

The process \( Y_t \) is the so called one-sided drifted BM reflected at its last infimum. It has the following properties:

![Figure 4. The process \( Y_t \).](image)

**Lemma 4.** [RW2][Lemma VI.55.1]

The process \( Y = \{Y_t, t \geq 0\} \) is a diffusion and \( L = \{L_t, t \geq 0\} \) is a local time of \( Y \) at 0. The transition density function of the process \( Y \) stopped at 0, i.e. \( \{Y_{t \wedge T_0}, t \geq 0\} \), is
\[ \bar{p}_t(x, y) = (2\pi t)^{-1/2} e^{\mu(x-y)-\frac{1}{2} \mu^2 t} \left[ e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t} \right], \quad x, y > 0. \] (4.3)

The entrance law \( n_t, t > 0 \), associated with the excursions of \( Y \) from 0 w.r.t the time \( L \) is given by \( n_t(dy) = n_t(y) dy, \) where:
\[ n_t(y) = 2y(2\pi t)^{-1/2} \exp \left[ -(y + \mu t)^2/2t \right], \quad y > 0. \] (4.4)
Let us comment the above result and fix some notation (for a general treatment of excursion theory see for example [RW2][Chapter VI] and [B][Chapter IV]).

We denote by $U$ the space of excursions from 0, i.e. continuous functions $f : [0, \infty) \to \mathbb{R}$ satisfying the coffin condition

$$f(t) = f(H) = 0, \quad \forall t \geq H,$$

where $H$ is the life time of $f$, namely

$$H = H(f) = T_0(f) \in [0, \infty].$$

(5.4)

The excursion space $U$ is endowed of the smallest $\sigma$-algebra which makes each evaluation map $f \to f(t)$ measurable. One can prove that this $\sigma$-algebra coincides with the Borel $\sigma$-algebra of the space $U$ endowed of the Shohorod metric.

Write $\gamma_t$ for the right continuous inverse of $L_t$, namely

$$\gamma_t = \inf \{ u > 0 : L_u > t \} = \inf \{ u > 0 : \min_{0 \leq s < u} B_s < -t \},$$

and define the excursion $e_t \in U$, $t > 0$, as

$$e_t(s) = \begin{cases} Y(\gamma_t-s) & \text{if } 0 \leq s \leq \gamma_t - \gamma_{t-}, \\ 0 & \text{if } s \geq \gamma_t - \gamma_{t-}. \end{cases}$$

Then the random point process $\nu$ of excursions of $Y$ from 0 is defined as

$$\nu = \{ (s,e_t) : t > 0, \gamma_t \neq \gamma_{t-} \} .$$

In what follows, we will often identify the random discrete set $\nu$ with the random measure $\sum_{t>0; \gamma_t \neq \gamma_{t-}} \delta_{(t,e_t)}$ on $(0, \infty) \times U$.

Decomposing $U$ as $U = U_\infty \cup U_0$, where

$$U_\infty = \{ f \in U : H(f) = \infty \}, \quad U_0 = \{ f \in U : H(f) < \infty \},$$

Itô Theorem [RW2][Theorem VI.47.6] states that there exists a $\sigma$-finite measure $n$ on $U$ (called Itô measure) with $n(U_\infty) < \infty$ such that, if $\nu'$ is a Poisson point process on $(0, \infty) \times U$ with intensity measure $dt \times n$ and if

$$\zeta = \inf \{ t > 0 : \nu'((0,t] \times U_\infty) > 0 \},$$

then the point process $\nu$ under $\mathbf{P}^0_\infty$ has the same law of $\nu'|_{(0,\zeta] \times U}$:

$$\nu \sim \nu'|_{(0,\zeta] \times U}.$$ 

(5.7)

Here and in what follows, given a measurable space $X$ with measure $m$ and a measurable subset $A \subset X$ we denote $m|_A$ the measure on $X$ such that

$$m|_A(B) = m(A \cap B), \quad \forall B \subset X \text{ measurable}.$$ 

Given $t > 0$ the entrance law $n_t(dy)$, with support in $(0, \infty)$, is defined as

$$n_t(dy) = n( \{ f : H(f) > t, f_t \in dy \}).$$

(5.8)

Since the process $Y$ (starting at 0) defined via (5.2) is Markov and visits each $y \geq 0$ a.s., the definition of the process $Y$ starting at $y$ is obvious. Due to Lemma [4] the process $Y$ starting at $y$ and stopped at 0 is a strong Markov process with transition probability $\tilde{p}_t(\cdot, \cdot)$. In what follows we denote its law by $Q_y$.

Then, given $t > 0$, measurable subsets $\mathcal{A}, \mathcal{C} \subset U$ with $\mathcal{A} \in \sigma(f_s, 0 \leq s \leq t)$, it holds

$$n( f : f \in \mathcal{A}, H(f) > t, \theta_t f \in \mathcal{C}) = \int_0^\infty n( f \in \mathcal{A}, H(f) > t, f_t \in dy) Q_y(\mathcal{C}),$$

(5.9)
where \((\theta_tf)_s = f_{t+s}\). In particular, due to (4.9) the transition density functions (4.3) and the entrance laws (4.4) determine univocally the Itô measure \(n\).

In order to get more information on the Itô measure \(n\) of the point process of excursions of \(Y\) from 0, we give an alternative probabilistic interpretation of the transition density \(\bar{p}(x,y)\). To this aim recall that Girsanov formula implies that

\[
E^\mu_x(g) = E^x(gZ_t), \quad Z_t = \exp\left\{ -\mu(B_t - x) - \mu^2t/2 \right\},
\]

for each \(\mathcal{F}_t\)-measurable function \(g\), where \(\mathcal{F}_t = \sigma(B_s: 0 \leq s \leq t)\). Due to (4.10), we get for all \(x,y,z,s,t > 0\),

\[
P^\mu_z(B_{s+t} \in dy, T_0(\theta_sB) > t | B_s = x) = P^\mu_x(B_t \in dy, T_0 > t) = \bar{p}_t^\mu(x, y). \tag{4.11}
\]

In fact, the second identity follows from (4.10) while the last identity follows from (4.3) by computing \(P^\mu_x(B_t = y, T_0 > t)\) via a reflection argument. Hence, given \(z > 0\), \(\bar{p}_t^\mu(\cdot, \cdot)\) is the transition density function of the process \((B_t \wedge T_0, t \geq 0)\) under \(P^\mu_z\), whose law equals \(Q_z\). In particular, (4.9) can be reformulated as

\[
n(f : f \in A, H(f) > t, \theta_tf \in C) = \int_0^\infty n(f \in A, H(f) > t, f_t \in dy) P^\mu_y(B_t \wedge T_0 \in C). \tag{4.12}
\]

The above identity will be frequently used in what follows.

We point out that, as stated in Theorem 1 of [B][Section VII.1], the content of Lemma 4 is valid in more generality for spectrally positive Lévy processes s.t. the origin is a regular point, i.e. real valued processes starting at the origin with stationary independent increments, with no negative jumps and returning to the origin at arbitrarily small times. Moreover, defining

\[
\bar{b}_t := 0 \wedge \inf\{B_s : 0 \leq s \leq t\},
\]

it is simple to check that the process \(Y\) starting at \(x > 0\) has the same law of the process \((\tilde{Y}_t := B_t - \bar{b}_t, t \geq 0)\), where \(B\) is chosen with law \(P^\mu_x\). This implies that \(\tilde{Y}_t = B_t\) if \(t < T_0(B)\), hence once gets again that \(\tilde{p}_t^\mu(x, y) = P^\mu_x(B_t \in dy, T_0 > t)\) as in (4.11).

In order to state our results it is useful to fix some further notation. Given \(h > 0\) we denote by \(U^{h,+}\) the family of excursions with height at least \(h\) and by \(U^{h,-}\) the family of excursions with height less than \(h\), namely

\[
U^{h,+} = \left\{ f \in U : \sup_{s \geq 0} f_s \geq h \right\}, \tag{4.13}
\]

\[
U^{h,-} = \left\{ f \in U : \sup_{s \geq 0} f_s < h \right\} = U \setminus U^{h,+}. \tag{4.14}
\]

One of the main technical tools in order to extend the proof in [NP] to the drifted case is the following lemma, whose proof is postpone to Section 5.
Lemma 5. If \( \alpha > 0, \mu \neq 0 \), then
\[
n(U^h) = \frac{\mu e^{-\mu h}}{\sinh(\mu h)},
\]
where \( n(U^h) \) is the number of excursions of \( U \).

\[
n(U^h) = \frac{\sqrt{2\alpha}}{\mu} \sinh(\mu h).
\]

Finally, we can prove Lemma 1.

Proof of Lemma 1. One can recover the Brownian motion from the point process \( \nu \) of excursions of \( Y \) from 0 by the formula
\[
B_t = -a + f(t - S), \quad \text{for} \ t \in [S, S + H(f)],
\]
which is valid for each couple \((a, f) \in \nu\) by setting
\[
S = \int_{(0,a) \times U} H(f') \nu(da', df').
\]

It is convenient to associate to \( U^h, U^h \) the measures \( \nu^* = \nu|_{[0,\infty) \times U^h}, \nu_* = \nu|_{[0,\infty) \times U^h} \), \( n^* = n|_{U^h} \) and \( n_* = n|_{U^h} \). Moreover, we set
\[
a^* = \inf \left\{ a > 0 : \exists f \in U^h, (a, f) \in \nu \right\}.
\]

If \( a^* \) is finite, let \( f^* \) be the only excursion such that \((a^*, f^*) \in \nu\). Due to (4.16), (4.17) and (4.18)
\[
P_0^n(a^* > a) = P_0^n(\nu^* ((0, a) \times U) = 0) = \exp \{-a n^*(U)\},
\]
therefore \( a^* \) is an exponential variable with parameter \( n^*(U) \) (in particular, \( a^* \) is finite a.s.). Due to the representation (4.19), \( \beta = -a^* \). Together with (4.15) this implies that \( -\beta \) is an exponential variable with mean \( \beta \). Moreover, (4.19) implies that
\[
\sigma = \int_{(0,a^*) \times U} H(f') \nu(da', df') = \int_{(0,a^*) \times U} H(f') \nu_*(da', df').
\]

Due to the above expression and the representation (4.19), the trajectory \((B_t, 0 \leq t \leq \sigma)\) depends only on \( \nu_* \) and \( a^* \), while the trajectory \((B_{\beta^*} - \beta, 0 \leq t \leq \tau - \sigma)\) coincides with the excursion \( f^* \) stopped when it reaches level \( h \). Since \( \nu_* \) and \( a^* \) are independent from \( f^* \) we get the independence of the trajectories.

In order to prove (2.4) we observe that \( \beta = x \) means that \( a^* = x \). Therefore, conditioning on \( \beta = x \), it holds
\[
\sigma = \int_{(0,x) \times U} H(f') \nu_*(da', df'),
\]
thus implying that
\[
E_0^n[\exp(-\alpha \sigma) | \beta = -x] = E_0^n \left( \exp \left\{ -\alpha \int_{(0,x) \times U} H(f') \nu_*(da', df') \right\} \right).
\]

Note that, in order to derive the above identity, we have used that \( \nu \) is the superposition of the independent point processes \( \nu^* \) and \( \nu_* \).
We claim that

\[
E_0^\mu \left( \exp \left\{ -\alpha \int_{(0,x) \times U} H(f') \nu_\ast (da', df') \right\} \right) = \exp \left\{ -x \int_U \left( 1 - e^{-\alpha H(f)} \right) n_\ast (df) \right\}.
\]

(4.22)

In order to prove this claim we note that, due to Itô Theorem and (4.16), the point process \( \nu_\ast \) has the same distribution of the drifted BM with law \( P \). Theorem 2, but one can derive Lemma 1 from the cited references as follows. The Laplace and direct proof based on simple computations, which will be useful also for the proof of (4.11) by means of more sophisticated arguments always based on fluctuation theory, excursion proved for more general spectrally one–sided Lévy processes \([AKP]\) [Section O.5]. Suppose now that \( \alpha < 0 \) and \( \alpha > 0 \). Given \( m > 0 \) and \( f \in U \) we define \( H_m(f) \) as \( -\infty \) if \( H(f) \leq m \) and as \( H(f) \) if \( H(f) > m \). Due to (4.8) and (4.12), we get the bound

\[
\int e^{-\alpha H_m(f)} n_\ast (df) \leq e^{-\alpha m} \int_0^h n_m(dy) E_0^\mu \left( e^{-\alpha T_0 \mathbb{I}_{T_0 < T_h}} \right)
\]

where the r.h.s. is finite due to the form of \( n_m \) and identity (6.3). This allows to conclude that the integral \( \int e^{-\alpha H_m(f)} n_\ast (df) \) is finite, and therefore the same holds for the smaller integral \( \int |1 - e^{-\alpha H_m(f)}| n_\ast (df) \). This last property allows to apply again the exponential formula for Poisson point process and to deduce that

\[
E_0^\mu \left( \exp \left\{ -\alpha \int_{(0,x) \times U} H_m(f') \nu_\ast (da', df') \right\} \right) = \exp \left\{ -x \int_U \left( 1 - e^{-\alpha H_m(f)} \right) n_\ast (df) \right\}.
\]

(4.22)

Taking the limit \( m \downarrow 0 \) and applying the Monotone Convergence Theorem we derive (4.22) from the above identity. Hence, (2.4) follows from (4.17), (4.21) and (4.22), while trivially (2.6) follows from (2.4).

Finally, in order to prove (2.7), we observe that \( \tau - \sigma = T_h(f^\ast) \). Since the path \( f^\ast \) has law \( n^\ast / n^\ast(\bar{U}) \),

\[
E_0^\mu \left( \exp \left\{ -\alpha (\tau - \sigma) \right\} \right) = n^\ast(\bar{U})^{-1} \int_U n^\ast (df) e^{-\alpha T_h(f)}
\]

and the thesis follows from (4.18). \( \square \)

**Remark 2.** As already remarked, the analogous of Lemma 1 (restricted to \( \alpha > 0 \)) has been proved for more general spectrally one–sided Lévy processes \([AKP], [P]\) and \([C]\) [Proposition 1], by means of more sophisticated arguments always based on fluctuation theory, excursion theory and the analysis of the hitting times of the process. We have given a self–contained and direct proof based on simple computations, which will be useful also for the proof of Theorem 2, but one can derive Lemma 1 from the cited references as follows. The Laplace exponent of the drifted BM with law \( P_0^\mu \) is given by \( \psi(\lambda) = \frac{1}{2} \lambda^2 - \lambda \mu \), i.e. \( E_0^\mu (e^{\lambda B_t}) = e^{\psi(\lambda)} \) for \( \lambda \in \mathbb{R} \). Given \( \alpha > 0 \) we define the function \( W^{(\alpha)} \) as

\[
W^{(\alpha)} = e^{\mu x} \left( e^{x \sqrt{2\alpha}} - e^{-x \sqrt{2\alpha}} \right) / \sqrt{2\alpha}.
\]

Then it is simple to check that

\[
\int_0^\infty e^{-\lambda x} W^{(\alpha)}(x) dx = \frac{1}{\psi(\lambda) - \alpha}, \quad \forall \lambda \geq \Phi(\alpha),
\]

where the value \( \Phi(\alpha) \) is defined as the largest root of \( \psi(\lambda) = \alpha \), i.e. \( \Phi(\alpha) = \mu + \sqrt{2\alpha} \). The function \( W^{(\alpha)} \) is related to the exit of the BM from a given interval. More precisely, due to (6.4), it holds

\[
E_0^\mu (e^{-\alpha T_y} T_y < T_x) = W^{(\alpha)}(x) / W^{(\alpha)}(x + y), \quad \forall x, y > 0.
\]

(4.23)
To the function $W^{(\alpha)}$ one associates the function $Z^{(\alpha)}$ given by

$$Z^{(\alpha)}(x) = 1 + \alpha \int_0^x W^{(\alpha)}(z)dz = \frac{\alpha e^{\mu x}}{\sqrt{2\alpha}} \left( \frac{e^{\sqrt{2\alpha x}}}{\mu + \sqrt{2\alpha}} - \frac{e^{-\sqrt{2\alpha x}}}{\mu - \sqrt{2\alpha}} \right)$$

Knowing the values of $W^{(\alpha)}$ and $Z^{(\alpha)}$ one can compute the expressions in Lemma 2 for $\alpha > 0$ by applying for example Proposition 1 in [C].

5. The behavior of the drifted Brownian motion near to an $h$–extremum

In this section we characterize the behavior of an $h$–slope not covering the origin, near to its extremes. To this aim we recall the definition of the drifted Brownian motion Doob–conditioned to hit $+\infty$ before $0$, referring to [B][Section VII.2] and references therein for a more detailed discussion. First, we write $W(x)$ for the function

$$W(x) := W^{(0)}(x) = \frac{e^{2x\mu} - 1}{\mu}$$

($W^{(\alpha)}$ has been defined in Remark [2]. Defining $\Phi(0)$ has the largest zero of $\psi(\lambda) := \lambda^2/2 - \lambda\mu$, i.e. $\Phi(0) := 0 \lor (2\mu)$, the function $W$ is a positive increasing function with Laplace transform

$$\int_0^\infty e^{-\lambda x}W(x)dx = \frac{1}{\psi(\lambda)}, \quad \forall \lambda > \Phi(0),$$

satisfying the identity (see (4.23))

$$P^\mu_0(T_y < T_{-x}) = W(x)/W(x + y), \quad \forall x, y > 0. \quad (5.1)$$

Due to the above considerations, the function $W$ is the so called scale function of the drifted Brownian motion with law $P^\mu_0$.

For each $x > 0$ consider the new probability measure $P^\mu_{x\uparrow}$ on the path space $C([0, \infty), \mathbb{R})$ characterized by the identity

$$P^\mu_{x\uparrow}(\Lambda) = \frac{1}{W(x)} E^\mu_x(W(X_t), \Lambda, t < T_0), \quad \Lambda \in \mathcal{F}_t, \quad (5.2)$$

where $(X_t, t \geq 0)$ denotes a generic element of the path space $C([0, \infty), \mathbb{R})$ and $\mathcal{F}_t := \sigma\{X_s : 0 \leq s < t\}$. As discussed in [B][Section VII.3], the above probability measure is well defined, the weak limit $P^\mu_{0\uparrow} := \lim_{x \to 0} P^\mu_{x\uparrow}$ exists and the process $(P^\mu_{x\uparrow}, x \geq 0)$ is a Feller process, hence strong Markov (we point out that in [B][Section VII.3] the above results are proven in the Skorohod path space $D([0, \infty), \mathbb{R})$, but one can adapt the proofs to $C([0, \infty), \mathbb{R})$). As explained in [B][Section VII.3], this process can be thought of as the Brownian motion with drift $-\mu$ Doob–conditioned to hit $+\infty$ before $0$. In the case of positive drift, i.e. $\mu < 0$, this can realized very easily by observing that due to (5.1)

$$P^\mu_{x}(T_0 = \infty) = P^\mu_{0}(T_{-x} = \infty) = \lim_{y \to \infty} P^\mu_{0}(T_y < T_{-x}) = \lim_{y \to \infty} \frac{W(x)}{W(x + y)} = 1 - e^{2x\mu} = -\mu W(x), \quad \forall x > 0,$$

and that this identity together with the Markov property implies that

$$P^\mu_{x}(\Lambda|T_0 = \infty) = P^\mu_{x\uparrow}(\Lambda), \quad x > 0, \Lambda \in \mathcal{F}_t.$$
For negative drift the event \( \{ T_0 = \infty \} \) has zero probability, and a more subtle discussion is necessary.

**Lemma 6.** The process \( P_{0}^{\mu, \uparrow} \) is a diffusion characterized by the SDE

\[
    dX_t = dB_t + \mu \coth(\mu X_t) \, dt, \quad X_0 = 0,
\]

where \( B_t \) is the standard Brownian motion.

*Proof.* As already discussed, the above process has continuous paths and it is strong Markov, i.e. it is a diffusion.

Due to (5.2) and (4.11), given \( x, y > 0 \),

\[
    q_t(x, y) := P_{x}^{\mu, \uparrow}(X_t \in dy) = \frac{W(y)}{W(x)} P_{x}^{\mu}(X_t \in dy, t < T_0) = \frac{W(y) \tilde{p}_{t}^{\mu}(x, y)}{W(x)} = \frac{\sinh(\mu y) e^{-\frac{1}{2} \mu y^2}}{\sinh(\mu x) \sqrt{2\pi t}} \left[ e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t} \right].
\]

From the above expression, by direct computations one derives that

\[
    \frac{\partial}{\partial t} q_t(x, y) = -\frac{\partial}{\partial y} (\mu \coth(\mu y) q_t(x, y)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} q_t(x, y).
\]

Hence, the generator of the process is given by

\[
    Lf(y) = \mu \coth(\mu y) \frac{d}{dy} f(y) + \frac{1}{2} \frac{d^2}{dy^2} f(y).
\]

and this implies the SDE (5.3).

\[\square\]

Let us consider now the Brownian motion with drift \(-\mu\) Doob–conditioned to hit \( h \) before 0 and killed when it reaches \( h \). In order to precise its meaning when the Brownian motion starts at the origin, given \( 0 < x < h \), we define \( P_{x, h}^{\mu, \uparrow} \) as the conditioned law on \( C([0, \infty), \mathbb{R}) \)

\[
    P_{x, h}^{\mu, \uparrow}(\Lambda) = P_{x}^{\mu}(\Lambda \mid T_h < T_0), \quad \Lambda \in \bigcup_{s \geq 0} \mathcal{F}_s.
\]

Note that the above definition is well posed since by (5.1) \( P_{x}^{\mu}(T_h < T_0) = W(x)/W(h) > 0 \).

**Lemma 7.** Given \( 0 < x < h \), let \( Q_{x, h}^{\mu, \uparrow} \) and \( R_{x, h}^{\mu, \uparrow} \) be the law of the path \( (X_t : 0 \leq t \leq T_h) \) killed when level \( h \) is reached, where \( X \) is chosen with law \( P_{x}^{\mu, \uparrow} \) and \( P_{x}^{\mu, \uparrow} \) respectively. Then \( Q_{x, h}^{\mu, \uparrow} = R_{x, h}^{\mu, \uparrow} \) and the weak limit \( Q_{0, h}^{\mu, \uparrow} := \lim_{x \downarrow 0} Q_{x, h}^{\mu, \uparrow} \) exists and equals \( R_{0, h}^{\mu, \uparrow} \).

*Proof.* Given \( 0 < x_1 < x_2 < \cdots < x_n < h \) and times \( t_1 < t_2 < \cdots < t_n \), we denote by \( A \) the event

\[
    A := \{ X_{t_1} \in dx_1, X_{t_2} \in dx_2, \ldots, X_{t_n} \in dx_n, t_n < T_h \}.
\]

Then, by definition of \( P_{x, h}^{\mu, \uparrow} \) and the Markov property of the Brownian motion, for each \( 0 < x < h \) we get that

\[
    P_{x, h}^{\mu, \uparrow}(A) = P_{x}^{\mu}(A, t_n < T_0) P_{x}^{\mu}(T_h < T_0)/P_{x}^{\mu}(T_h < T_0) = P_{x}^{\mu}(A, t_n < T_0) W(x_n)/W(x).
\]

Due to (5.2), the last expression in the r.h.s. equals the probability \( P_{x}^{\mu, \uparrow}(A) \), hence we can conclude that \( P_{x, h}^{\mu, \uparrow}(A) = P_{x}^{\mu, \uparrow}(A) \). Hence \( Q_{x, h}^{\mu, \uparrow} = R_{x, h}^{\mu, \uparrow} \) for \( 0 < x < h \). The last statement concerning \( Q_{0, h}^{\mu, \uparrow} \) follows from the fact that the weak limit \( \lim_{x \downarrow 0} P_{x}^{\mu, \uparrow} \) exists and equals \( P_{0}^{\mu, \uparrow} \). \[\square\]
Due to the first part of the above lemma, we can think of \((P^{\mu,1}_{x,h}, 0 \leq x \leq h)\) as the Brownian motion with drift \(-\mu\) Doob–conditioned to hit \(h\) before 0 and killed when it hits \(h\).

We have now all the tools in order to describe the behavior of the \(h\)–slopes not covering the origin, near to their extremes. In order to simplify the notation, in what follows we denote by \(B^{(\mu)}\) the two–sided Brownian motion with drift \(-\mu\), starting at the origin. Moreover, given \(r \in \mathbb{R}\), we define

\[
T^{(h,+)}_r = \inf \left\{ s > 0 : \left| B^{(\mu)}_{r+s} - B^{(\mu)}_r \right| = h \right\},
\]

\[
T^{(h,-)}_r = \inf \left\{ s > 0 : \left| B^{(\mu)}_{r-s} - B^{(\mu)}_r \right| = h \right\}.
\]

**Theorem 2.** Let \(\mu \neq 0\) and let \(m < m'\) be consecutive points of \(h\)–extrema for the drifted Brownian motion \(B^{(\mu)}\), both non negative or both non positive.

If \(m\) is a point of \(h\)–minimum and \(m'\) is a point of \(h\)–maximum, then the processes

\[
\left\{ B^{(\mu)}_{m+t} - B^{(\mu)}_m, 0 \leq t \leq T^{(h,+)}_m \right\},
\]

\[
\left\{ B^{(\mu)}_{m'-t} - B^{(\mu)}_{m'}, 0 \leq t \leq T^{(h,-)}_{m'} \right\},
\]

have the same law of the Brownian motion starting at the origin, with drift \(-\mu\), Doob–conditioned to reach \(+\infty\) before 0 and killed when it hits \(h\). Moreover, they have the same law of the Brownian motion starting at the origin, with drift \(-\mu\), Doob–conditioned to reach \(h\) before 0 and killed when it hits \(h\). In particular, they satisfy the SDE (5.3) up to the killing time.

If \(m\) is a point of \(h\)–maximum and \(m'\) is a point of \(h\)–minimum, then the processes

\[
\left\{ B^{(\mu)}_m - B^{(\mu)}_{m+t}, 0 \leq t \leq T^{(h,+)}_m \right\},
\]

\[
\left\{ B^{(\mu)}_{m'-t} - B^{(\mu)}_{m'}, 0 \leq t \leq T^{(h,-)}_{m'} \right\},
\]

have the same law of the Brownian motion starting at the origin, with drift \(\mu\), Doob–conditioned to reach \(+\infty\) before 0 and killed when it hits \(h\). Moreover, they have the same law of the Brownian motion starting at the origin, with drift \(\mu\), Doob–conditioned to reach \(h\) before 0 and killed when it hits \(h\). In particular, they satisfy the SDE (5.3) with \(\mu\) replaced by \(-\mu\), up to the killing time.

**Proof.** The second part of the theorem follows from the first part by taking the reflection w.r.t. the coordinate axis.

As follows from the proof of Lemma 1 in Section 3, the law of the process (5.7) coincides with the law of the excursion \(f\) killed when it reaches \(h\), where \(f\) is chosen with probability measure

\[
n(\cdot|T_h < T_0) = n(\cdot, T_h < T_0)/n(T_h < T_0).
\]

Due to Proposition 15 in [3][Section VII.3], there exists a positive constant \(c\) such that

\[
n(\Lambda, t < T_0) = cE^{\mu,1}_0(W(X_t)^{-1}, \Lambda), \quad \forall \Lambda \in \mathcal{F}_t,
\]

\[
n(T_h < T_0) = c/W(h)
\]

(5.11) (5.12)
(note that (5.12) corresponds to (4.15) with $c = 2$). Hence, given numbers $x_1, x_2, \ldots, x_n$ in $(0, h)$ and increasing times $0 < t_1 < t_2 < \cdots < t_n$, we have

\[
n(f_1 \in dx_1, f_2 \in dx_2, \ldots, f_n \in dx_n, t_n < T_h < T_0) = \\
n(f_1 \in dx_1, f_2 \in dx_2, \ldots, f_n \in dx_n, t_n < T_h, t_n < T_0) \mathbb{P}_{x_n}^\mu(T_h < T_0) = \\
c \mathbb{P}_0^{\mu_1}(f_1 \in dx_1, f_2 \in dx_2, \ldots, f_n \in dx_n, t_n < T_h) / W(h)
\]  

(5.13)

(the first identity follows from the Markov property (4.12), while the latter follows from (5.1) and (5.11)). From (5.12) and (5.13) one derives that

\[
n(f_1 \in dx_1, f_2 \in dx_2, \ldots, f_n \in dx_n, t_n < T_h | T_h < T_0) = \\
\mathbb{P}^{\mu_1}_0(f_1 \in dx_1, f_2 \in dx_2, \ldots, f_n \in dx_n, t_n < T_h)
\]  

(5.14)

This concludes the proof that the process (5.7) has the same law of the Brownian motion starting at the origin, with drift $-\mu$, Doob–conditioned to reach $+\infty$ before 0 and killed when it hits $h$. All the other statements follow from this property, Lemmas 6 and 7 and by reflection arguments.

\[
\Box
\]

6. Proof of Lemma 5

Knowing the Laplace transform of the hitting times of the non–drifted Brownian motion [KS][Chapter 2], the following lemma follows by applying Girsanov formula (1.10):

Lemma 8. Let $x < 0 < y$ and $\alpha > 0$, then

\[
\mathbb{E}_0^\mu(e^{\alpha T_x} \mathbb{I}_{T_x < T_y}) = e^{-\mu x} \frac{\sinh(y \sqrt{2\alpha})}{\sinh((y - x) \sqrt{2\alpha})},
\]  

(6.1)

\[
\mathbb{E}_0^\mu(e^{\alpha T_y} \mathbb{I}_{T_y < T_x}) = e^{-\mu y} \frac{\sinh(-x \sqrt{2\alpha})}{\sinh((y - x) \sqrt{2\alpha})}.
\]  

(6.2)

Proof. Set $Z_t = \exp\{-\mu B_t - \mu^2 t/2\}$. We claim that

\[
\mathbb{E}^\mu_0(e^{-\alpha T_x} \mathbb{I}_{T_x < T_y}) = \lim_{t \to \infty} \mathbb{E}^\mu_0(e^{-\alpha T_x} \mathbb{I}_{T_x < T_y} \mathbb{I}_{T_x < t}) = \\
\lim_{t \to \infty} \mathbb{E}_0(e^{-\alpha T_x} \mathbb{I}_{T_x < T_y} \mathbb{I}_{T_x < t} Z_t) = \lim_{t \to \infty} \mathbb{E}_0(e^{-\alpha T_x} \mathbb{I}_{T_x < T_y} \mathbb{I}_{T_x < t} \mathbb{E}_0(Z_t | \mathcal{F}_{t \wedge T_x})).
\]

Indeed, the first identity follows from the Monotone Convergence Theorem, the second one from (1.10), and the last one by conditioning on $\mathcal{F}_{t \wedge T_x}$ and observing that $e^{-\alpha T_x} \mathbb{I}_{T_x < T_y} \mathbb{I}_{T_x < t}$ is $\mathcal{F}_{t \wedge T_x}$ measurable.

Since, under $\mathbb{P}_0$, $Z_t$ is a martingale and $t \wedge T_x$ is a bounded stopping time, the optional sampling theorem implies that $\mathbb{E}(Z_t | \mathcal{F}_{t \wedge T_x}) = Z_{t \wedge T_x}$. For $T_x < t$, $Z_{t \wedge T_x}$ equals $\exp(-\mu x - \alpha T_x + \alpha T_x)$, hence

\[
\mathbb{E}^\mu_0(e^{-\alpha T_x} \mathbb{I}_{T_x < T_y}) = \lim_{t \to \infty} e^{-\mu x} \mathbb{E}_0(e^{-\alpha T_x} \mathbb{I}_{T_x < T_y} \mathbb{I}_{T_x < t}) = e^{-\mu x} \mathbb{E}_0(e^{-\alpha T_x} \mathbb{I}_{T_x < T_y}).
\]

Since $\alpha > 0$, the above identity together with formula (8.27) in [KS][Chapter 2] implies (6.1).
The proof of (6.2) can be obtained by similar arguments and by formula (8.28) in [KS] [Chapter 2] or simply by observing that
\[ E_0^\mu \left( e^{-\alpha T_y} \mathbb{I}_{T_y < T_x} \right) = E_0^{-\mu} \left( e^{-\alpha T_y} \mathbb{I}_{T_y < T_x} \right) \]
and then applying (6.1). □

Due to the above lemma, given \( 0 < y < h \) and \( \hat{\alpha} > 0 \),
\[ E_y^\mu \left( e^{-\alpha T_0} \mathbb{I}_{T_0 < T_h} \right) = e^{\mu y} \frac{\sinh \left( (h - y) \sqrt{2\hat{\alpha}} \right)}{\sinh (h \sqrt{2\hat{\alpha}})}, \quad (6.3) \]
\[ E_y^\mu \left( e^{-\alpha T_h} \mathbb{I}_{T_h < T_0} \right) = e^{\mu (y - h)} \frac{\sinh (y \sqrt{2\hat{\alpha}})}{\sinh (h \sqrt{2\hat{\alpha}})}, \quad (6.4) \]
\[ P_y^\mu (T_0 < T_h) = e^{\mu y} \frac{\sinh (\mu (h - y))}{\sinh (\mu h)}, \quad (6.5) \]
\[ P_y^\mu (T_h < T_0) = e^{\mu (y - h)} \frac{\sinh (\mu y)}{\sinh (\mu h)}. \quad (6.6) \]

By taking the limit \( h \to \infty \) in (6.3) we get for all \( y > 0 \) and \( \hat{\alpha} > 0 \)
\[ E_y^\mu \left( e^{-\alpha T_0} \mathbb{I}_{T_0 < \infty} \right) = e^{\mu y - y \sqrt{2\hat{\alpha}}} = e^{\mu y - |\mu| y \sqrt{1 + \frac{2\hat{\alpha}}{\mu^2}}}. \quad (6.7) \]

By considering the Taylor expansion around \( \alpha = 0 \) in above identity, one can compute the expectation of \( T_0^k \mathbb{I}_{T_0 < \infty} \). In particular, for all \( y > 0 \) it holds
\[ E_y^\mu (T_0 \mathbb{I}_{T_0 < \infty}) = e^{\mu y - |\mu| y / |\mu|}. \quad (6.8) \]

We collect some identities (obtained by straightforward computations) which will be very useful below. First we observe that given \( a, b, w \in \mathbb{R} \) and \( t > 0 \) it holds
\[ \int_a^b 2y e^{-\frac{(y+w)^2}{2t}} dy = \frac{2}{\sqrt{2\pi t}} e^{-\frac{(a+w)^2}{2t}} - \frac{2}{\sqrt{2\pi t}} e^{-\frac{(b+w)^2}{2t}} - \frac{2w}{\sqrt{2\pi t}} \int_{a/\sqrt{t}}^{b/\sqrt{t+w}} e^{-\frac{z^2}{2}} dz. \quad (6.9) \]

In particular, fixed \( a > 0 \) and \( c, w \in \mathbb{R} \), it holds as \( t \downarrow 0 \)
\[ \int_0^a \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+w)^2}{2t}} dy = \frac{2}{\sqrt{2\pi t}} - w + o(1) \quad (6.10) \]
\[ \int_a^\infty \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+w)^2}{2t}} dy = o(1) \quad (6.11) \]
\[ \int_0^a \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+w)^2}{2t}} dy - cy \frac{2}{\sqrt{2\pi t}} = (w + c) + o(1) \quad \text{as } t \downarrow 0. \quad (6.12) \]

Note that the last identity can be derived from (6.9) by observing that
\[ \int_0^a \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+w)^2}{2t}} - cy dy = e^{\frac{1}{2} (c^2 + 2cw)} e^{-\frac{(a+w)^2}{2t}}. \]

As first application of the above observations and (4.12), we prove the following result:

**Lemma 9.** Let \( \hat{\alpha} > 0 \). Then
\[ \lim_{t \downarrow 0} \int_U \left| 1 - e^{-\alpha H(f)} \mathbb{I}_{H(f) \leq t} \right| n(df) = 0. \quad (6.13) \]
Proof. Since for a suitable positive constant $c > 0$ it holds $|1 - e^y| \leq c|y|$ if $|y| \leq 1$, the limit (6.13) is implied by
\[\lim_{t \downarrow 0} \int_U H(f)1_{H(f) \leq t} n(df) = 0.\]
Since excursions are continuous paths and $\lim_{t \downarrow 0} 1_{H(f) \leq t} = 0$ pointwise, by the Dominated Convergence Theorem in order to prove the above limit it is enough to show that
\[\int_U H(f)1_{H(f) \leq t} n(df) < \infty.\] (6.14)
By the Monotone Convergence Theorem
\[\int_U H(f)1_{H(f) \leq t} n(df) = \lim_{\varepsilon \downarrow 0} \int_U H(f)1_{\varepsilon < H(f) \leq t} n(df),\] (6.15)
and due to (4.12)
\[\int_U H(f)1_{\varepsilon < H(f) \leq t} n(df) = \int n_{\varepsilon}(dy) \mathcal{E}_y^\mu (\{T_0 = \varepsilon \}1_{T_0 \leq 1-\varepsilon}) \leq \int n_{\varepsilon}(dy) \mathcal{E}_y^\mu (\{T_0 = \varepsilon \}) + \varepsilon \int n_{\varepsilon}(dy).\] (6.16)
Due to (6.10) and (6.11), the last term $\varepsilon \int n_{\varepsilon}(dy)$ is negligible as $\varepsilon \downarrow 0$ while, due to (6.8),
\[\int n_{\varepsilon}(dy) \mathcal{E}_y^\mu (\{T_0 < \infty \}) = \frac{1}{\varepsilon |\mu|} \int_0^\infty \frac{2y^2}{\sqrt{2\pi \varepsilon}} e^{-\frac{(y+|\mu|)^2}{2\varepsilon}} dy.\] (6.17)
In order to conclude it is enough to observe that
\[\text{R.h.s. of (6.17)} = \frac{1}{\varepsilon |\mu|} \int_0^\infty \frac{2y^2}{\sqrt{2\pi \varepsilon}} e^{-\frac{(y+|\mu|)^2}{2\varepsilon}} dy \leq \frac{1}{\varepsilon |\mu|} \int_0^\infty \frac{2(y+|\mu|)^2}{\sqrt{2\pi \varepsilon}} e^{-\frac{(y+|\mu|)^2}{2\varepsilon}} dy = \frac{1}{|\mu|} < \infty.\]

Now we have all the technical tools in order to prove Lemma 5.

6.1. **Proof of (4.15).** Consider the measurable subsets $U_{t}^{h,+} = \{ f \in U : \sup_{s \geq t} f_s \geq h, H(f) > t \}$. Then $U_{t_2}^{h,+} \subset U_{t_1}^{h,+}$ for $t_1 \leq t_2$ and, by the continuity of excursions, $U^{h,+} = \cup_{t>0} U_{t}^{h,+}$. This implies that $n(U^{h,+}) = \lim_{t \downarrow 0} n(U_{t}^{h,+})$. Due to (4.12)
\[n(U_{t}^{h,+}) = \int_0^\infty n_t(dy) \mathcal{P}_y^\mu (\sup_{s \geq 0} B_{s \wedge T_0} \geq h) = \int_0^\infty n_t(dy) \mathcal{P}_y^\mu (T_h < T_0).\]
Hence (see also (6.9)) we get that
\[n(U^{h,+}) = \lim_{t \downarrow 0} (I_1(t) + I_2(t)),\]
where
\[I_1(t) = \int_0^\infty \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+|\mu|)^2}{2t}} dy, \quad I_2(t) = \int_0^h \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+|\mu|)^2}{2t}} e^{\mu(y-h)} \frac{\sinh(\mu y)}{\sinh(\mu h)} dy.\]
Due to (6.11) $\lim_{t \downarrow 0} I_1(t) = 0$. In order to treat the term $I_2$ we write
\[I_2(t) = (I_3(t) - I_4(t)) / (1 - e^{2\mu h}),\]
where
\[ I_3(t) = \int_0^h \frac{2y}{\sqrt{2\pi t}} e^{-\frac{(y+\mu)^2}{2t}} dy, \]
\[ I_4(t) = \int_0^h \frac{2y}{\sqrt{2\pi t}} e^{-\frac{(y+\mu)^2}{2t}+2\mu y} dy = \int_0^h \frac{2y}{\sqrt{2\pi t}} e^{-\frac{(y-\mu)^2}{2t}} dy. \]
Due to (6.10), \( I_3(t) = \frac{1}{\sqrt{2\pi t}} - \mu + o(1) \) and \( I_4(t) = \frac{2}{\sqrt{2\pi t}} + \mu + o(1) \). In particular,
\[ n(U^{h,+}) = \lim_{t \downarrow 0} I_2(t) = \frac{2\mu}{e^{2\mu h} - 1} = \frac{\mu e^{-\mu h}}{\sinh(\mu h)}, \]
thus concluding the proof of (4.15).

\[ \square \]

6.2. Proof of (4.16). Consider the subsets \( U_t \subset U \) defined as \( U_t = \{ f \in U : H(f) = \infty, \sup_{s \geq t} f_s < h \} \). Then \( U^{h,-} \cap U^{\infty} \subset U_t \) and, in order to prove (4.16), it is enough to show that \( \lim_{t \downarrow 0} n(U_t) = 0 \). Due to (4.12) we can write
\[ n(U_t) = \int_0^\infty n_t(dy) P_y^\mu(T_h = T_0 = \infty). \]
Note that if \( \mu > 0 \) then \( P_y^\mu(T_0 = \infty) = 0 \) for all \( y > 0 \), while if \( \mu < 0 \) then \( P_y^\mu(T_h = \infty) = 0 \) for all \( y < h \). Hence
\[ n(U_t) = \int_h^\infty n_t(dy) P_y^\mu(T_h = T_0 = \infty) \leq \int_h^\infty n_t(dy). \]
By (6.11) the last member above goes to 0 as \( t \downarrow 0 \). This implies that \( n(U_t) = o(1) \), thus concluding the proof of (4.16).

\[ \square \]

6.3. Proof of (4.17). We claim that
\[ \int_{U^{h,-}} \left( 1 - e^{-\alpha H(f)} \right) n(df) = \lim_{t \downarrow 0} \int_U \left( 1 - e^{-\alpha H(f)} \right) I_{\{\sup_{[t,\infty)} f < h\}} I_{H(f) > t} n(df). \quad (6.18) \]
Indeed, the above identity follows from the Dominated Convergence Theorem if we prove that
\[ \int_{U^{h,-}} |1 - e^{-\alpha H(f)}| n(df) < \infty. \quad (6.19) \]
To this aim, we observe that, for all \( t > 0 \),
\[ \int_{U^{h,-}} |1 - e^{-\alpha H(f)}| n(df) \leq \int_U |1 - e^{-\alpha H(f)}| I_{\{\sup_{[t,\infty)} f < h\}} I_{H(f) > t} n(df), \quad (6.20) \]
and by (6.13)
\[ \limsup_{t \downarrow 0} \text{r.h.s. of (6.20)} = \limsup_{t \downarrow 0} \int_U |1 - e^{-\alpha H(f)}| I_{\{\sup_{[t,\infty)} f < h\}} I_{H(f) > t} n(df) = \]
\[ \limsup_{t \downarrow 0} \text{sgn} (\alpha) \int_U (1 - e^{-\alpha H(f)}) I_{\{\sup_{[t,\infty)} f < h\}} I_{H(f) > t} n(df). \quad (6.21) \]
We claim that
\[ \lim_{t \downarrow 0} \int_U (1 - e^{-\alpha H(f)}) I_{\{\sup_{[t,\infty)} f < h\}} I_{H(f) > t} n(df) = \sqrt{2\alpha} \coth \left( h\sqrt{2\alpha} \right) - \mu \coth(\mu h). \quad (6.22) \]
Note that (6.22) implies that the r.h.s. of (6.20) is bounded and this implies (6.19), which implies (6.18), which together with (6.22) implies (4.17).

Let us prove (6.22). By (4.12)
\[
\text{l.h.s. of (6.22)} = \lim_{t \downarrow 0} \int_0^\infty n_t(y) E_y^\mu \left( I_{T_0 < T_h} \left( 1 - e^{-\alpha T_0 + \alpha t} \right) \right) \, dy = \lim_{t \downarrow 0} \left( J_1(t) - e^{\alpha t} J_2(t) \right),
\]
where
\[
J_1(t) = \int_0^\infty n_t(y) P_y^\mu \left( T_0 < T_h \right) \, dy,
\]
\[
J_2(t) = \int_0^\infty n_t(y) E_y^\mu \left( I_{T_0 < T_h} e^{-\alpha T_0} \right) \, dy = \int_0^h n_t(y) E_y^\mu \left( I_{T_0 < T_h} e^{-\alpha T_0} \right) \, dy.
\]
Due to the identities derived at the beginning of the proof of (4.15) we can write
\[
\lim_{t \downarrow 0} \int_0^\infty n_t(y) P_y^\mu \left( T_0 < T_h \right) = n(U^{h,+}) = \frac{\mu e^{-\mu h}}{\sinh(\mu h)},
\]
while due to (6.10) and (6.11)
\[
\int_0^\infty n_t(y) \, dy = 2 \sqrt{\frac{2}{\pi t}} - \mu + o(1).
\]
The above identities give
\[
J_1(t) = \frac{2}{\sqrt{2\pi t}} - \mu - \frac{\mu e^{-\mu h}}{\sinh(\mu h)} + o(1) = \frac{2}{\sqrt{2\pi t}} - \mu \coth(\mu h).
\]
Due to (6.3)
\[
J_2(t) = \int_0^h \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+\mu)^2}{2t} + \mu y} \sinh \left( \frac{(h - y) \sqrt{2\alpha}}{2} \right) \, dy =
\]
\[
\frac{e^{h\sqrt{2\alpha}}}{2 \sinh \left( h \sqrt{2\alpha} \right)} \int_0^h \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+\mu)^2}{2t} - y(\sqrt{2\alpha} - \mu)} \, dy - \frac{e^{-h\sqrt{2\alpha}}}{2 \sinh \left( h \sqrt{2\alpha} \right)} \int_0^h \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+\mu)^2}{2t} + y(\sqrt{2\alpha} + \mu)} \, dy.
\]
Hence, due to (6.12),
\[
J_2(t) = \frac{e^{h\sqrt{2\alpha}}}{2 \sinh \left( h \sqrt{2\alpha} \right)} \left( \frac{2}{\sqrt{2\pi t}} - \sqrt{2\alpha} + o(1) \right) - \frac{e^{-h\sqrt{2\alpha}}}{2 \sinh \left( h \sqrt{2\alpha} \right)} \left( \frac{2}{\sqrt{2\pi t}} - \sqrt{2\alpha} + o(1) \right) = \frac{2}{\sqrt{2\pi t}} - \sqrt{2\alpha} \coth \left( h \sqrt{2\alpha} \right).
\]
Then (6.22) follows from (6.23), (6.24) and (6.25), thus concluding the proof of (4.17). □
6.4. Proof of (4.18). Due to the Monotone Convergence Theorem

\[
\int_U e^{-\alpha T_h} \mathbb{I}_{T_h < T_0} n(df) = \lim_{t \downarrow 0} \int_U e^{-\alpha T_h} \mathbb{I}_{t < T_0} n(df).
\]

(6.26)

It is convenient to write the last integral as \( A(t) - B(t) \), where

\[
A(t) = e^{-\alpha t} \int_U e^{-\alpha T_h(\theta_t f)} \mathbb{I}_{T_h(\theta_t f) < T_0(\theta_t f)} \mathbb{I}_{H(f) > t} n(df),
\]

\[
B(t) = e^{-\alpha t} \int_U e^{-\alpha T_h(\theta_t f)} \mathbb{I}_{T_h(\theta_t f) < T_0(\theta_t f)} \mathbb{I}_{H(f) > t} \mathbb{I}_{T_h \leq t} n(df).
\]

Hence

\[
\int_U e^{-\alpha T_h} \mathbb{I}_{T_h < T_0} n(df) = \lim_{t \downarrow 0} A(t) - \lim_{t \downarrow 0} B(t).
\]

(6.27)

By (4.12)

\[
\lim_{t \downarrow 0} A(t) = \lim_{t \downarrow 0} e^{-\alpha t} \int_0^\infty n_t(dy) \mathbb{E}_y^\mu \left( e^{-\alpha T_h(\theta_t f)} \mathbb{I}_{T_h < T_0} \right) = \lim_{t \downarrow 0} (K_1(t) + K_2(t)),
\]

(6.28)

where

\[
K_1(t) = \int_0^h n_t(dy) \mathbb{E}_y^\mu \left( e^{-\alpha T_h(\theta_t f)} \mathbb{I}_{T_h < T_0} \right),
\]

\[
K_2(t) = \int_h^\infty n_t(dy) \mathbb{E}_y^\mu \left( e^{-\alpha T_h(\theta_t f)} \mathbb{I}_{T_h < T_0} \right) = \int_h^\infty n_t(dy) \mathbb{E}_y^\mu \left( e^{-\alpha T_h(\theta_t f)} \mathbb{I}_{T_h < \infty} \right)
\]

\[
= \int_h^\infty n_t(dy) \mathbb{E}_y^\mu \left( e^{-\alpha T_0(\theta_t f)} \mathbb{I}_{T_0 < \infty} \right).
\]

Due to (6.4)

\[
K_1(t) = \int_0^h \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+\mu)^2}{4t} + \mu(y-h)} \frac{\sinh \left( y\sqrt{2\alpha} \right)}{\sinh \left( h\sqrt{2\alpha} \right)} dy = \\
\frac{e^{-\mu h}}{2 \sinh \left( h\sqrt{2\alpha} \right)} \left[ \int_0^h \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+\mu)^2}{2t} + \mu y} \left( e^{y\sqrt{2\alpha}} - e^{-y\sqrt{2\alpha}} \right) dy \right].
\]

(6.29)

By applying (6.12) to the r.h.s. we get that

\[
K_1(t) = \frac{e^{-\mu h \sqrt{2\alpha}}}{\sinh \left( h\sqrt{2\alpha} \right)} + o(1).
\]

(6.30)

By (6.7)

\[
K_2(t) = \int_h^\infty \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+\mu)^2}{2t} + (\mu - \sqrt{2\alpha})(y-h)} dy = e^{\alpha t + h(\sqrt{2\alpha} - \mu)} \int_h^\infty \frac{2y}{\sqrt{2\pi t^3}} e^{-\frac{(y+\sqrt{2\alpha} \mu)^2}{2t}} dy
\]

and due to (6.11) \( \lim_{t \downarrow 0} K_2(t) = 0 \). This limit together with (6.28) and (6.30) implies that

\[
\lim_{t \downarrow 0} A(t) = \frac{e^{-\mu h \sqrt{2\alpha}}}{\sinh \left( h\sqrt{2\alpha} \right)}.
\]

(6.31)

(6.32)

We claim that \( \lim_{t \downarrow 0} B(t) = 0 \). Note that this together with (6.27), (6.32) and (4.15) implies (4.18). Hence, in order to conclude we only need to prove our claim. To this
aim we apply Hölder inequality with exponents $p, q > 1$ with $1/p + 1/q = 1$ and $\hat{\alpha}p > 0$, deriving that

$$B(t) \leq e^{-\alpha t} \left[ \int_U e^{-\alpha\hat{p} T_h(\theta f)} \mathbb{I}_{T_h(\theta f) \leq T_0(\theta f) + \hat{H}(f) > t} n(df) \right]^{1/p} n(T_h \leq t)^{1/q}.$$  

As $t \downarrow 0$ the first factor in the r.h.s. goes to 1, the second factor has finite limit due to (6.32) where $\alpha$ has to be replaced by $\alpha p$ (here we use that $\hat{\alpha}p > 0$). Hence we only need to prove that $\lim_{t \downarrow 0} n(T_h \leq t) = 0$. By the Monotone Convergence Theorem it is enough to show that $n(T_h \leq 1) < \infty$. But $n(T_h \leq 1) \leq n(U^{\hat{h},+})$ which is bounded by (4.15).

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