Hyperbolic monopoles from hyperbolic vortices

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Abstract

Yang–Mills–Higgs monopoles and vortices in hyperbolic space can be constructed from \( \text{SO}(2) \) and \( \text{SO}(3) \) invariant Yang–Mills instantons, respectively. We use this fact to describe a large class of hyperbolic monopoles directly in terms of hyperbolic vortices embedded into 3 dimensions, yielding a remarkably simple relation between their Higgs fields. The class of monopoles we obtain are fixed relative to a plane in hyperbolic space, in a way which will be made clear by a study of the monopole spectral curve. We will use the correspondence between vortices and monopoles to give new insight into the moduli space of hyperbolic monopoles. Finally, our technique allows an explicit construction of the fields of a hyperbolic monopole invariant under a \( \mathbb{Z} \) action, which we compare to periodic monopoles in Euclidean space.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Finite action solutions of the self-dual Yang–Mills equations in 4-dimensional Riemannian spacetimes have long been of interest due to their non-perturbative contribution to the path integral and the rich geometrical techniques used to describe them. Such solutions are invariant under conformal rescalings of the background metric and are characterised by an integer topological charge \( N \) known as the ‘instanton number’. The total energy of a solution is
quantised in units of $N$, leading to the interpretation of a solution as containing this quantity of finite energy particles [38]. Another attractive feature of the self-duality equations is their relation to many other integrable systems by dimensional or algebraic reductions. The most well-known example of such a reduction is the set of self-duality equations invariant under translations in one Cartesian direction, which leads to the Bogomolny equations describing non-Abelian magnetic monopoles on Euclidean $\mathbb{R}^3$. An explicit description of this system is given in [20].

In this paper we consider the relation between the 2 and 3-dimensional systems obtained from reducing the self-duality equations by an $\text{SO}(3)$ or an $\text{SO}(2)$ action, respectively. There is a conformal equivalence between Euclidean $\mathbb{R}^4$ and hyperbolic space,

$$\mathbb{R}^4 \setminus \mathbb{R}^{3-n} \sim H^{4-n} \times S^n$$  \hspace{1cm} (1)

with $n = 1, 2$, where indicates the removal of a plane or a line, respectively. By applying this conformal equivalence we will see that the self-duality equations for gauge group $\text{SU}(2)$ reduce to:

- for $n = 1$, the $\text{SU}(2)$ Bogomolny (monopole) equations on hyperbolic 3-space $H^3$ with a prescribed boundary condition which will be discussed in section 1.2, [1];
- for $n = 2$, the Taubes equation on $H^2$, describing Abelian–Higgs vortices at critical coupling [38].

By construction, these two systems are related to one another by a lift to an $\text{SO}(3)$-invariant instanton. Our goal is to make use of the instanton to provide a direct relation between hyperbolic monopoles and hyperbolic vortices. In particular, we will find that any hyperbolic vortex can be lifted to a monopole, and that there is a large class of hyperbolic monopoles, accounting for half of the moduli space of solutions (the parameter space of gauge-inequivalent solutions within the same topological sector), which arise as hyperbolic vortices embedded into $H^3$. This point of view picks out a preferred slice of $H^3$, with reference to which we obtain insight into hyperbolic monopoles.

The paper is arranged as follows. In the remainder of this section we will describe the Taubes and Bogomolny equations in hyperbolic space, together with outlined methods of solution, including the JNR construction and the description in terms of spectral data. Section 2 discusses how the hyperbolic monopole and vortex equations are obtained from instanton reductions and shows how hyperbolic vortices can be used to construct hyperbolic monopoles. A description of this procedure in terms of JNR data is given in section 3. In section 4 we look at the spectral curve of the resulting hyperbolic monopoles and compare to the spectral data of Euclidean monopoles. The metric on the 2-hyperbolic-monopole moduli space (defined via the connection at infinity) is compared to the physical metric on the underlying hyperbolic vortex moduli space in section 5. Finally, in section 6 we use the procedure of section 2 to construct a new hyperbolic monopole periodic with respect to a chosen element of $\text{SL}_2(\mathbb{C})$, for which a direct construction in terms of JNR or ADHM data is not currently known.

1.1. Yang–Mills instantons

The Yang–Mills equations are defined by the action

$$E = -\frac{1}{8} \int \text{tr}(F \wedge *F)$$  \hspace{1cm} (2)
where the connection $A$ is an $\mathfrak{su}(2)$-valued one-form with curvature $F = dA + A \wedge A$, or, in component notation, $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$, and $\text{tr}$ denotes the $\mathfrak{su}(2)$ trace. The action has local minima, which are found by rewriting it as follows:

$$E = -\frac{1}{16} \int \text{tr} (\langle F - *F \rangle \wedge * (F - *F) + 2F \wedge F) \geq \pi^2 N$$  \hfill (3)

The action is minimised if the term in square brackets vanishes, i.e. if $F$ satisfies the self-duality equations with respect to the Hodge operator:

$$F = *F \Rightarrow F_{ij} = \frac{1}{2} \epsilon_{ijkl} F_{kl}$$  \hfill (4)

and solutions to these equations are known as Yang–Mills instantons. The topological charge $N$ of an instanton configuration is given by the remaining term in (3), which is proportional to the Chern number. Solutions to (4) contain a number of additional parameters, known as ‘moduli’, which alter the internal structure of the solution but retain the topologically invariant charge $N$.

A method to construct all solutions to the self-duality equations was provided by Atiyah, Drinfeld, Hitchin and Manin [3]. However, it will be sufficient for our purposes to concentrate on the restricted solution set provided by Jackiw, Nohl and Rebbi [18]. Known as the JNR Ansatz, this solution set allows the construction of all instantons of charge $N = 1$ and $N = 2$. Within the Ansatz, an $N$-instanton is generated by a harmonic function with $N + 1$ poles $\gamma_j$ located at arbitrary positions in Euclidean $\mathbb{R}^4$, each of which has an associated positive weight $\lambda_j$:

$$\psi = \sum_{j=0}^{N} \frac{\lambda_j^2}{|x - \gamma_j|^2}.$$  \hfill (5)

here with $x$ and $\gamma_j$ considered as vectors in $\mathbb{R}^4$, which gives the instanton gauge potentials

$$A_i = \frac{i}{2} \left( \epsilon_{ijk} \partial_j \log(\psi) \tau_k + \partial_4 \log(\psi) \tau_i \right), \quad A_4 = -\frac{i}{2} \partial_4 \log(\psi) \tau_i.$$  \hfill (6)

where $\tau_i$ are the Pauli matrices. JNR configurations with collinear or coplanar poles can be dimensionally reduced to monopoles or vortices in 3 and 2-dimensional hyperbolic space, respectively, by using the conformal equivalence (1).

A key property of the self-dual Yang–Mills equations is their conformal invariance: the instanton solution is unchanged by conformal rescalings of the background metric. This property will be amply used in section 2 when we dimensionally reduce the self-duality equations using the conformal equivalence (1).

1.2. Hyperbolic vortices

Abelian Higgs vortices consist of a complex Higgs field $\phi$ and a one-form gauge potential $a$. On a background space with metric $ds^2 = \Omega(x, y) \left( dx^2 + dy^2 \right)$ the action is

$$E = \frac{1}{2} \int \left( \frac{B^2}{2\Omega} + |D_x \phi|^2 + |D_y \phi|^2 + \frac{\lambda \Omega}{2} (1 - |\phi|^2)^2 \right) dx dy,$$  \hfill (7)

where $B = \partial_y a_x - \partial_x a_y$ is the curvature of $a$ and $D = d - ia$ is the covariant derivative. At critical coupling, $\lambda = 1$, the action can be minimised by completing the square:
\[ E = \frac{1}{2} \int \left( \frac{1}{2\Omega} [B - \Omega(1 - |\phi|^2)]^2 + \frac{1}{4} |D_\phi|^2 + B \right) \ dx \, dy \geq \pi N \]  
(8)

for \( N \in \mathbb{Z} \), with equality if the terms in square brackets vanish:

\[ B = \Omega(1 - |\phi|^2), \quad D_\phi = 0, \]  
(9)

the ‘real’ and ‘complex’ vortex equations, respectively.

The topological energy bound \( E \geq N \) fixes the number of zeros of \( \phi \), which are interpreted as the locations of the vortices. The above equations (9) can be combined to eliminate \( a \), such that away from its zeros \( |\phi|^2 \) obeys the Taubes equation [36]:

\[ \Delta \log |\phi|^2 + 2(1 - |\phi|^2) = 0, \]  
(10)

where \( \Delta \) is the Laplace–Beltrami operator, which for a conformally flat background is given by \( \Omega^{-1} \nabla^2 \), where \( \Omega \) is the conformal factor and \( \nabla^2 \) is the Euclidean Laplacian\(^1\). On the hyperbolic plane of Gauss curvature \(-1\) it was shown by Witten [38] that the Taubes equation can be reduced to the Liouville equation, which is integrable. Working in the Poincaré disk model of \( H^2 \), with metric

\[ ds^2 = \frac{4|dw|^2}{(1 - |w|^2)^2}, \]  
(11)

solutions to the Taubes equation are given in terms of a holomorphic function \( f(w) \) satisfying \( |f(w)| \leq 1 \), with equality on the boundary of the disk \( |w| = 1 \). Explicitly,

\[ \phi = \frac{1 - |w|^2}{1 - |f(w)|^2} \frac{df}{dw}, \quad a_\phi = -i \partial_\phi \log(|\phi|), \]  
(12)

which is defined up to a \( U(1) \) phase. The gauge potential is thus completely defined, via the complex vortex equation, in terms of a solution of the Taubes equations and a choice of phase on \( \phi \). Vortex configurations are completely described by the positions of the zeros of \( \phi \), and for a finite number of vortices it is always possible to construct the required function \( f(w) \) as a Blaschke product\(^2\).

13. Hyperbolic monopoles

\( \text{SU}(2) \) monopoles consist of an \( \mathfrak{su}(2) \)-valued scalar \( \Phi \) and one-form gauge potential \( A = A_i dx^i \) with two-form curvature \( F \). On a background with metric \( g = \Omega g_{E3} \) conformal to Euclidean \( \mathbb{R}^3 \), the Bogomolny monopole equations in component form are

\[ F = *D\Phi \quad \Rightarrow \quad F_{ij} = \sqrt{\Omega} \epsilon_{ijk} D_k \Phi, \]  
(13)

where \( \{x^i\} \) are coordinates on \( E^3 \) and the covariant derivative \( D \) acts on scalars as \( D_k \Phi = \partial_k \Phi + [A_k, \Phi] \). The Bogomolny equations (13) are minimisers of the action

\[ E = -\frac{1}{4} \int \text{tr} (F \wedge *F + D\Phi \wedge *D\Phi), \]  
(14)

\(^1\)We remark that the equation (10) was originally studied in the context of Ginzburg–Landau theory at critical coupling. In this regime, there are smooth solutions to (10) which describe dissipative vortices which have no net force between them and are hence static. Further specialising to a background hyperbolic space, the equation is integrable, and its solutions are the ones we consider in this paper.

\(^2\)For infinite lattices of vortices the Blaschke product may diverge and the function \( f(w) \) used in (12) must be obtained by other means.
and as for the instanton and vortex actions, solutions have quantised energies \( E \geq 2\pi N \). As we will see, \( N \) is typically the number of distinct maxima of the energy density, although in contrast to the vortex case, one can no longer identify \( N \) with the number of zeros of the scalar field \( \Phi \) in general.

Solutions to (13) in hyperbolic space are simplest when the boundary condition \( \| \Phi \|^2 := -\frac{1}{2} \text{tr}(\Phi^2) \rightarrow v^2 \) has half-integer \( v \). Although arbitrary values of \( v \) are possible, only those solutions with half-integer \( v \) are equivalent to circle-invariant instantons. The simplest case has \( v = \frac{1}{2} \), for which a large family of monopoles are obtained from JNR instantons by placing the poles of the harmonic function \( \psi \), (5), on a plane in \( \mathbb{R}^4 \) corresponding to the boundary of \( H^3 \). We will review the details of this procedure in section 3. Examples of \( v = \frac{1}{2} \) hyperbolic monopoles have been obtained from the JNR Ansatz include those with axial [10] and tetrahedral [28] symmetry. Hyperbolic monopoles of large charge have been modelled as magnetic bags in this way [6]. More generally, to obtain the full moduli space of \( v = \frac{1}{2} \) monopoles, one should use circle-invariant ADHM data, while for half-integer \( v \geq \frac{1}{2} \) one obtains a discrete version of the Nahm equations, known as the Braam–Austin equations [7]. Explicit monopole fields with \( v \neq \frac{1}{2} \) are only known for \( N = 1 \), [8, 30]. In general, these solutions do not have a description in terms of JNR data and the relation to the underlying instanton is more complicated. For instance, the instanton charge is \( 2v \) times the monopole charge [1].

Yang–Mills–Higgs monopoles are equipped with a spectral curve, which is an algebraic curve on \( \mathbb{C} \times \mathbb{C}^* \) which encodes the symmetry of the monopole. The spectral curve is the set of spectral lines: geodesics on the base space, the distance along which is parametrised by \( s \), on which there are \( L^2 \) normalisable 2-vectors \( w(s) \) satisfying \((\partial_s + A_i - i\Phi)w = 0\), where \( A_i \) indicates the component of the gauge potential along the chosen geodesic. In [5] explicit formulae were presented for the spectral curve of a hyperbolic monopole in terms of its JNR data, and the case of general \( v \) has also been tackled [32]. In section 4 we will describe the spectral curves corresponding to our class of monopoles, and we will use our results in section 5 to describe the metric on the moduli space.

2. Symmetric instantons

In order to relate hyperbolic vortices to hyperbolic monopoles we will make use of the underlying \( \text{SO}(3) \)-invariant instanton. First of all we lift the general vortex solution to an instanton using Witten’s approach [38], which is suited to the upper half space model of hyperbolic space. The instanton is then reduced to a monopole by imposing invariance under a circle action. This leads to a simple expression relating the monopole and vortex Higgs fields. We then confirm that for this class of monopoles, the Bogomolny equations (13) imply the Taubes equation (10) on the vortex fields.

2.1. Conformal rescalings

Before we proceed, let us fix our conventions. The Euclidean metric on \( \mathbb{R}^4 \) is

\[
\text{d}s_{E}^2 = (\text{d}x^4)^2 + (\text{d}x^1)^2 + (\text{d}x^2)^2 + (\text{d}x^3)^2 = (\text{d}\rho^2 + \rho^2 \text{d}\xi^2, (15)
\]

with \( x^2 = \rho \cos(\xi) \) and \( x^3 = \rho \sin(\xi) \), where \( \rho \) and \( \xi \) are cylindrical coordinates \( \rho \geq 0 \) and \( \xi \in [0, 2\pi] \). We now follow Manton & Sutcliffe in implementing the conformal equivalence (1) explicitly, by expressing the metric (15) as
Removing the overall conformal factor of \( \rho^2 \) and reducing by the angular \( \xi \) dependence we obtain hyperbolic 3-space with the upper half space metric

\[
dx_{H^3}^2 = \frac{1}{\rho^2} \left( (dx^4)^2 + (dx^1)^2 + d\rho^2 \right). \tag{16}
\]

Notice that the coordinate \( \rho \) has become the hyperbolic coordinate and that in performing the conformal equivalence (1) we are to remove the plane \( \rho = 0 \), the boundary of \( H^3 \).

Now introduce in (15) the coordinates \( r \geq 0 \) and \( \theta \in [0, \pi] \) via \( x^1 = r \cos(\theta) \), \( \rho = r \sin(\theta) \). Then \( r \), \( \theta \) and \( \xi \) are standard spherical polar coordinates with respect to which

\[
dx_{H^3}^2 = (dx^4)^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\xi^2 \right), \tag{17}
\]

where \( r^2 = (x^1)^2 + \rho^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \). Following the same steps as in the previous reduction, we quotient by the angular dependence and remove the line \( r = 0 \) to give a metric on the hyperbolic plane \( H^2 \),

\[
dx_{H^2}^2 = \frac{1}{\xi^2} \left( (dx^4)^2 + dr^2 \right). \tag{18}
\]

The relation between the metrics (19) and (17) is interesting. Note that in obtaining (19) one is free to set \( \theta \) to take any constant value. Restricting to \( \theta = \pi/2 \), \( r = \rho \) and (19) reads

\[
\left. dx_{H^2}^2 \right|_{\theta = \pi/2} = \frac{1}{\xi^2} \left( (dx^4)^2 + d\rho^2 \right), \tag{20}
\]

which by comparison with (17) is a slice of \( H^3 \) (an equatorial slice of the unit ball model of \( H^3 \) is a unit disc carrying a hyperbolic metric). There is a more subtle reduction if we restrict to \( \theta = 0 \). Then (19) becomes

\[
\left. dx_{H^2}^2 \right|_{\theta = 0} = \frac{1}{(x^1)^2} \left( (dx^4)^2 + (dx^1)^2 \right). \tag{21}
\]

This is the boundary of \( H^3 \) equipped with a hyperbolic metric. Since \( x^1 \) can take either sign, this is two copies of the hyperbolic plane glued along the \( x^4 \)-axis. By extension there is a family of hyperbolic metrics on slices of the upper half space containing the \( x^4 \) axis, according to a choice of the angle \( \theta \) (the choice of fixed axis within the \( x^1 \sim x^4 \) plane is arbitrary).

We will frequently use the ball model of \( H^3 \), where cyclic symmetry is more apparent than in the upper half space model. The ball model coordinates are given in terms of the upper half space coordinates by

\[
X^1 + iX^2 = \frac{2(x^4 + ix^1)}{(x^1)^2 + (x^4)^2 + (\rho + 1)^2}, \quad X^3 = \frac{(x^1)^2 + (x^4)^2 + (\rho^2 - 1)}{(x^1)^2 + (x^4)^2 + (\rho + 1)^2},
\]

\[
R^2 = (X^1)^2 + (X^2)^2 + (X^3)^2 = \frac{(x^1)^2 + (x^4)^2 + (\rho - 1)^2}{(x^1)^2 + (x^4)^2 + (\rho + 1)^2}, \tag{22}
\]

and the ball metric is

\[
dx^2 = \frac{4}{(1 - R^2)^2} \left( (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right). \tag{23}
\]
For completeness, we invert these expressions to give the upper half space coordinates in terms of ball model coordinates:

\[
x^4 + ix^1 = \frac{2(X^1 + iX^2)}{1 + R^2 - 2X^2}, \quad \rho = \frac{1 - R^2}{1 + R^2 - 2X^2}.
\] (24)

2.2. Dimensional reductions

Monopoles and vortices in hyperbolic space are constructed by dimensional reductions of instantons on \(E^4\). The self-duality (instanton) equations are conformally invariant, so solutions are unchanged under the conformal rescalings of the background metric described above. Instantons invariant under a circle symmetry (i.e. those instanton solutions for which all gauge invariant quantities are independent of the angle \(\xi\) in (16)) can then be dimensionally reduced to monopoles on \(H^3\), while \(SO(3)\)-invariant instantons, independent of \(\xi\) and \(\theta\), give rise to hyperbolic vortices.

The reduction of circle-invariant instantons to hyperbolic monopoles was first considered by Atiyah [1] and carried out by Chakrabarti [8] and Nash [30]. Given an instanton gauge potential \(A_i(x^2, x^4, \tau_+, \tau_-)\), one must search for a gauge transformation \(G\) for which the transformed gauge potential \(A_i^G\) is explicitly independent of \(\xi = \tan^{-1}(x^2/x^3)\). If such a gauge exists, the instanton is said to be independent of \(\xi\) and the instanton can be interpreted as a hyperbolic monopole whose Higgs field \(\Phi\) is identified with the \(\xi\) component of \(A_i^G\) and the monopole gauge potential has components \(A_i^G, A_i^{1G}, A_i^{2G}\).

The relation between \(SU(2)\) instantons and \(U(1)\) hyperbolic vortices first arose in Witten’s search for cylindrically symmetric instantons [38]. In the upper half plane model of \(H^3\), (19), a vortex consists of a Higgs field \(\phi = \phi_1(x^2, r) + i\phi_2(x^4, r)\) and a gauge potential \(a = a_0(x^2, r)dx^2 + a_r(x^4, r)dr\). The fields satisfy the vortex equations (9) and we assume they are in Coulomb gauge, \(\partial_\tau a = 0\). An \(SO(3)\)-invariant instanton defined in \(E^4\) (with metric (15)) is then constructed by embedding the vortex fields into an instanton gauge potential

\[
A_j = \frac{i}{2} \left( \frac{\phi_2 + 1}{r^2} \epsilon_{k\ell} x^k \tau_\ell + \frac{\phi_1 r^2}{r^2} \tau_+ - x^l x^j \tau_\ell \right) + \frac{a_4 x^j}{r^2} \tau_+ \tau_-, \quad A_4 = \frac{i a_4}{2r} x^j \tau_+ \tau_-
\] (25)

where \(i\) runs from 1 to 3, \(\{\tau_\ell\}\) are the Pauli matrices, \(r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2\) and all the \(x^j\) dependence is encoded in the vortex fields (\(\phi, a\)).

We would like to explore the class of hyperbolic monopoles obtained by lifting hyperbolic vortices to instantons and then reducing by a circle action. To do this, we first of all combine the \(A_2\) and \(A_4\) components of the instanton gauge potential (25) into radial and angular parts \(A_\rho = (x^2 A_2 + x^4 A_4)/\rho\) and \(A_\xi = -x^2 A_2 + x^4 A_4\),

\[
A_\rho = \frac{i}{2} \left( \frac{\phi_2 + 1}{r^2} x^1 [-s T_2 + c T_3] + \frac{\phi_1 r^2}{r^2} [-\rho T_1 + x^1 c T_2 + x^1 s T_3] + \frac{a_4 \rho}{r^2} x^j \tau_+ \tau_- \right)
\]

\[
A_\xi = \frac{i}{2} \left( \frac{\phi_2 + 1}{r^2} \rho [\rho T_1 - x^1 c T_2 - x^1 s T_3] + \frac{\phi_1 r^2}{r^2} [-s T_2 + c T_3] \right).
\] (26)

where \(c = \cos(\xi)\), \(s = \sin(\xi)\) and we recall that \(\rho^2 = (x_1)^2 + (x_3)^2\). Now the \(A_i\) are rendered explicitly independent of \(\xi\) by application of the gauge transformation

\[
A_i \mapsto A_i^G = G^{-1} A_i G + G^{-1} \partial G.
\] (27)
with $G = \exp \left( -i \xi \tau / 2 \right)$. The instanton, originally defined in $E^4$, may now be interpreted as the fields of a monopole on the hyperbolic space (17):

$$A_i^G = \frac{i}{2} \left( -\frac{\phi_2 + 1}{r^2} \rho \tau_3 + \frac{\phi_1 r}{r^2} [\rho \tau_1 - x^1 \tau_2] + \frac{a_i x^1}{r^2} [x^1 \tau_1 + \rho \tau_2] \right)$$  \hspace{1cm} (28)

$$A_\rho^G = \frac{i}{2} \left( \frac{\phi_2 + 1}{r^2} x^1 \tau_3 - \frac{\phi_1 x^1}{r^2} [\rho \tau_1 - x^1 \tau_2] + \frac{a_\rho r}{r^2} [x^1 \tau_1 + \rho \tau_2] \right)$$  \hspace{1cm} (29)

$$\Phi = A_i^G \cdot \frac{4}{2r} [x^1 \tau_1 + \rho \tau_2].$$  \hspace{1cm} (31)

The instanton fields (25) will match the standard gauge of the JNR construction, (6), if

$$a_4 = \frac{\phi_2 + 1}{r} \quad \text{and} \quad a_r = \frac{\phi_1}{r}. \hspace{1cm} (32)$$

By using the ‘complex’ vortex equation (9), we see that this condition is equivalent to imposing the Coulomb gauge $\partial_4 a_4 + \partial_r a_r = 0$ on the vortex fields, and with this choice the monopole gauge potential is automatically also in Coulomb gauge, $\partial \Phi^G_i = 0$.

From (30) we obtain the key formula relating the norms of the vortex and monopole Higgs fields:

$$||\Phi||^2 = \frac{\rho^2 |\phi|^2 + (x^1)^2}{4 r^2}, \hspace{1cm} (33)$$

where $r^2 = (x^1)^2 + \rho^2$ and $\phi$ is a function of $x^4$ and $r$. Let us analyse this formula in more detail. Recall that we are working on the upper half space whose boundary is the $(x^4, x^1)$ plane. $||\Phi||^2$ has the correct boundary behaviour for a monopole with $\nu = \frac{1}{2} (||\Phi||^2 \rightarrow \frac{1}{4} \text{ as we approach the boundary } \rho \rightarrow 0)$, and its zeros occur where $x^1 = 0$ and $\phi = 0$. In the equatorial plane $x^1 = 0$, the monopole Higgs field $||\Phi||^2$ is proportional to the vortex Higgs field $|\phi|^2$, providing an obvious interpretation of the monopole as an embedded vortex.

Now take $(x_0^4, r_0)$ to be the position of a vortex zero. Setting $r = r_0$ defines a geodesic in the upper half space: a semicircle which meets the boundary at $(x^4, x^1) = (x_0^4, \pm r_0)$, as shown in figure 1. As a function of the hyperbolic distance $d_4$ from the monopole zero, measured along this geodesic, the Higgs field is

$$||\Phi||^2_{\phi=0} = \frac{(x^1)^2}{4 r^2} = \frac{1}{4} \tanh^2(d_4). \hspace{1cm} (34)$$

This is precisely the radial profile function of a single hyperbolic monopole, and the result (34) is independent of the multiplicity of the associated monopole zero and of its position relative to any other monopoles in the configuration. In section 4 we will see that these distinguished geodesics are always spectral lines.

A similar analysis to that given in this section allowed Cockburn [10] to relate axially symmetric monopoles to charge one monopoles of half-integer mass $\nu > \frac{1}{2}$.

$^3$The energy density, which depends on derivatives of $||\Phi||^2$, is proportional to the radial energy density profile of a single hyperbolic monopole. The constant of proportionality depends on the leading behaviour of $||\Phi||^2$ near its zeros.
2.3. Field equations

Now let us check that the Bogomolny equations imply the vortex equations. Note that \( \partial_1 \tilde{\phi} = x^1 (\partial_\rho \tilde{\phi})/r \), \( \partial_\rho \tilde{\phi} = \rho (\partial_\rho \tilde{\phi})/r \), where \( \tilde{\phi} \) represents any of the vortex fields, which are independent of \( \theta \). Then, using the fields (28)–(31) but suppressing the superscript \( G \) for clarity, we check component-wise:

\[
F_{41} = \partial_4 A_1 - \partial_1 A_4 + [A_4, A_1] = \frac{i}{2\rho^2} \left( \frac{\rho}{r} (\partial_4 \phi_1 + a_4 \phi_2) \tau_a + x^1 (\partial_4 a_r - \partial_r a_4) \tau_b - \rho (\partial_4 \phi_2 - a_4 \phi_1) \tau_3 \right) \tag{35}
\]

\[
D_\rho \Phi = \partial_\rho \Phi + [A_\rho, \Phi] = \frac{i \rho}{2\rho^2} \left( \frac{\rho}{r} (\partial_\rho \phi_2 - a_r \phi_1) \tau_a + x^1 \left( 1 - \phi_1^2 - \phi_2^2 \right) \tau_b + \rho (\partial_\rho \phi_1 + a_r \phi_2) \tau_3 \right). \tag{36}
\]

where \( \tau_a = [\rho \tau_1 - x^1 \tau_2] \) and \( \tau_b = [x^1 \tau_1 + \rho \tau_2] \). The Bogomolny equations

\[
F_{ij} = -\frac{1}{\rho} \epsilon_{ijk} D_k \Phi \tag{37}
\]

with \( \epsilon_{41\rho} = 1 \) imply the ‘complex’ vortex equation (9). By considering the other components of the Bogomolny equations one similarly obtains the ‘real’ vortex equation. Finally, by eliminating the gauge potential between the real and complex vortex equations one arrives at the Taubes equation (10). It would be interesting to study whether a similar embedding of the vortex equations into \( \mathfrak{su}(2) \) is possible for \( v \neq \frac{1}{2} \).

2.4. Monopole number

We should check that the fields (28)–(31) have the correct topology to be monopoles. The fields are reflection-symmetric. In other words the replacement \( x^1 \to -x^1 \) reverses the orientation and changes the fields by a gauge:

\[
A'_1 = -\tau_2 A_1 \tau_2, \quad A'_\rho = \tau_2 A_\rho \tau_2, \quad \Phi' = -\tau_2 \Phi \tau_2, \quad A'_4 = \tau_2 A_4 \tau_2. \tag{38}
\]
Now we compute the Chern number by performing the integral

\[ c_1 = -\int_{\partial H^3} \frac{\text{tr}(F \Phi)}{4\pi \| \Phi \|} \]  

(39)

over the boundary of $H^3$. In upper half space coordinates this is the $x^4 - x^1$ plane. Setting $\rho = 0$, we have from (30) and (35) that

\[ \Phi^0 = -\frac{i}{2} \tau_1, \quad F_{41}^0 = \frac{i}{2} (\partial_4 a_r - \partial_r a_4) \tau_1 = \frac{i}{2} B \tau_1. \]  

(40)

Evaluating (39), we get

\[ c_1 = -\frac{1}{2\pi} \int_{\partial H^3} \frac{1}{2} B \text{d}x^4 \wedge \text{d}x^1 \]  

(41)

On the boundary $\partial H^3$ we have $\rho = 0$ and hence $r^2 = (x^1)^2$. We are thus working with the metric (21) with $(x^1, x^4) \in \mathbb{R}^2$, and the integral (41) is performed over two copies of the upper half plane. This gives

\[ c_1 = -\frac{1}{2\pi} \int_{\mathbb{H}^2} \frac{1}{2} B \Omega (2 \Omega \text{d}x^4 \wedge \text{d}r) = -N, \]  

(42)

where $\Omega = r^{-2}$ is the conformal factor appearing in (21). A vortex of multiplicity $N$ in $H^2$ therefore lifts to a charge $N$ monopole when embedded into $H^3$.

2.5. Energy density

The energy density of a monopole is obtained by applying the Laplace–Beltrami operator to $\| \Phi \|^2$. In the upper-half-space model we are using, this is

\[ E = \rho^2 \left( \partial_1^2 + \partial_4^2 + \partial_\rho^2 - \frac{1}{\rho} \partial_\rho \right) \| \Phi \|^2. \]  

(43)

Written in terms of derivatives of the vortex Higgs field $|\phi|^2$ gives

\[ E = \frac{\rho^4}{r^4} \left( \frac{1}{4} \Delta |\phi|^2 + \frac{1}{2} (1 - |\phi|^2) \right), \quad \text{where} \quad \Delta = r^2 (\partial_1^2 + \partial_4^2) \]  

(44)

is the Laplace–Beltrami operator acting on the vortex Higgs field in the upper half plane model of $H^2$. We recognise the bracketed term in (44) as the energy density of the vortex [25]. Integrating over the upper half space we find

\[ E = \int E \frac{1}{\rho^4} d\rho \text{d}x^1 \text{d}x^4 = 2\pi N. \]  

(45)

2.6. Example: a charge 1 monopole

Let us illustrate our discussion with a simple example. A vortex with one zero at the origin of the Poincaré disk has

\[ |\phi| = \frac{2|w|}{1 + |w|^2}. \]  

(46)

Now convert to upper half plane coordinates $z = x^4 + i r$ using
\[ w = \frac{i - z}{iz - 1}, \tag{47} \]

This gives
\[ |\phi|^2 = \frac{((x^4)^2 + (r - 1)^2)((x^4)^2 + (r + 1)^2)}{((x^4)^2 + r^2 + 1)^2}, \tag{48} \]

then from (33) and (22) we find, after some manipulation,
\[ \|\Phi\|^2 = \frac{1}{4} \frac{(x^4)^2 + (x^4)^2 + \rho^2 + 1}{(1 + R^2)^2} \tag{49} \]

Applying the energy density formula (44) to the vortex (46) gives
\[ E_1 = \frac{3}{2} \left( 1 - R^2 \right)^4 \tag{50} \]

as expected for the charge one monopole (49). In section 6 we will use this method of embedding a vortex in \( H^2 \) into \( H^3 \) using equation (33) to obtain a new explicit hyperbolic monopole solution.

### 3. JNR construction

Recall from section 1.1 the JNR Ansatz for the construction of Yang–Mills instantons. The procedure takes a harmonic function \( \psi \) with \( N + 1 \) poles at arbitrary locations \( \gamma_i \in \mathbb{R}^4 \) in \( E^4 \) and arbitrary real weights \( \lambda_i \)
\[ \psi = \sum_{j=0}^{N} \frac{\lambda_j^2}{|x - \gamma_j|^2} \tag{51} \]

and generates a self-dual gauge potential
\[ A_i = \frac{i}{2} (\epsilon_{ijk} \partial_j \log(\psi) \tau_k + \partial_k \log(\psi) \tau_j), \quad A_4 = -\frac{i}{2} \partial_4 \log(\psi) \tau_1 \tag{52} \]

where \( \tau_i \) are the Pauli matrices.

The dimensional reductions of the preceding section can be made at the level of JNR data. Circle-invariant JNR data gives a subset of hyperbolic monopoles. Circle invariance restricts the pole of \( \psi \) to lie on a plane (the fixed set of a circle action) in \( E^4 \), which becomes the boundary of \( H^2 \) in the coordinates (17)). Counting parameters suggests that all hyperbolic monopoles for \( N \leq 3 \) can be generated in this way [5]. To reduce the monopoles to vortices, we have the additional constraint that the poles must be on the fixed set of an \( SO(3) \) action, i.e. on the line \( r = 0 \) in \( E^4 \) corresponding to the boundary of \( H^2 \) in the coordinates (19).

It was shown by Manton [23] that the JNR Ansatz generates all hyperbolic vortices, i.e. that it is gauge-equivalent to the formulation of section 1.2.

A suitable definition of the centre of mass of a hyperbolic monopoles is given in [29].\(^4\) The centered hyperbolic monopole is described by \( 4(N - 1) \) parameters, while centered hyperbolic vortices require \( 2(N - 1) \) parameters. There is an \( S^2 \) worth of freedom in our choice of

\(^4\) Due to the lack of translations in hyperbolic space, the definition of a centre of mass of an arbitrary hyperbolic monopole is a subtle question, which was analysed by Murray et al [29]. Our interest here is in the charge 2 monopole, and in this case the centre of mass is simply located at the midpoint between the two monopole zeros.
embedding of the hyperbolic vortices into $H^3$, so the construction presented here gives a $2N$ dimensional family of centered hyperbolic monopoles. In particular, we obtain all centered 2-monopoles, whose moduli space is explored in section 5.

Using the same upper half space coordinates as before, a monopole Higgs field is constructed using the JNR function

$$
\psi = \lambda \sum_{j=0}^{N} \frac{\lambda^2_j}{|x^4 + 1x^1 - \gamma_j|^2 + \rho^2}
$$

in

$$
||\Phi||^2 = \frac{\rho^2}{4\psi^2} \left( \left( \frac{\partial \psi}{\partial x^4} \right)^2 + \left( \frac{\psi}{\rho} + \frac{\partial \psi}{\partial \rho} \right)^2 \right)
$$

(54)

(here the restriction of $\gamma_j$ to a plane allows us to take $\gamma_j \in \mathbb{C}$). Placing all the poles of (53) on the real $x^4$-axis gives

$$
\psi = \lambda \sum_{j=0}^{N} \frac{\lambda^2_j}{(x^4 - \gamma_j)^2 + \rho^2},
$$

(55)

and the vortex Higgs field is given by [23]

$$
|\phi|^2 = \frac{\rho^2}{\psi^2} \left( \left( \frac{\partial \psi}{\partial x^4} \right)^2 + \left( \frac{\psi}{\rho} + \frac{\partial \psi}{\partial \rho} \right)^2 \right) = -r^2 \left( \partial_{x^4}^2 + \partial_{\rho}^2 \right) \log(r\psi).
$$

(56)

Fixing the phase of $\phi$ by specialising to Coulomb gauge and using the relations (32) gives the components of the gauge potential as

$$
a_4 = -\partial_\rho \log \psi, \quad a_r = \partial_4 \log \psi.
$$

(57)

Using (55) in (54) and changing variables again gives the relation (33). In fact, there are certain vortex configurations, in particular those with an infinite number of vortex zeros, [21], which have only been constructed by Witten’s method (12) and for which no JNR function $\psi$ is known. The more general argument of section 2 ensures that (33) is still valid, and it is for these configurations that the construction of monopoles as an embedding of vortices provides truly novel monopole solutions.

The remarkable similarity between (54) and (56) invites us to consider a further dimensional reduction. The resulting one-dimensional field theory describes the SO(4)-invariant instanton. Using the radial coordinate $\varrho^2 = r^2 + (x^4)^2$ we define

$$
\varphi^2 = \frac{\rho^2}{\psi^2} \left( \frac{\psi}{\rho} + \frac{d\psi}{d\varrho} \right)^2,
$$

(58)

where $\psi$ is a function of $\varrho$ only. Combining (58) with (56), the corresponding vortex Higgs field is

$$
|\phi|^2 = \frac{\varrho^2\varphi^2 + (x^4)^2}{\varrho^2 + (x^4)^2}.
$$

(59)

Mimicking the procedure of section 2.3, we substitute (59) into the Taubes equation (10), to yield

$${}^5$$There is, of course, only one SO(4) symmetric ‘t Hooft function, namely $\psi = 1 + \lambda^2 / \varrho^2$, but we will stick to using $\psi$ in order to highlight the analogy with the previous reduction from 3 to 2 dimensions.
\[
\frac{d\varphi}{d \log(\varrho)} = 1 - \varphi^2,
\]
(60)

which is the Bogomolny equation for a \(\varphi^4\) kink, a 1-dimensional topological soliton. In other words, we can obtain the charge 1 hyperbolic vortex (48) by embedding the (unique up to translations) \(\varphi^4\) kink into \(H^2\). Lifting to \(H^3\), the hyperbolic tangent function describing the \(\varphi^4\) kink shows up when the Higgs field of a single monopole is expressed as a function of hyperbolic distance from the Higgs zero, (34). By a change of coordinates we regain the BPST instanton [4]. The position coordinate \(\varrho\) of the kink is precisely the scale size \(\lambda\) of the instanton.

To conclude this section we summarise in the following table the possible configurations which can be generated by the JNR Ansatz:

| Instanton | Centered monopole | Centered vortex |
|-----------|-------------------|-----------------|
| Number of parameters | 8N | 4N − 4 | 2N − 2 |
| JNR data | 5N + 7 | 3N − 1 | 2N − 1 |
| \(N_{\text{max}}\) | 2 | 3 | All |

Here \(N\) is the soliton or instanton number, such that the JNR function \(\psi\) has \(N + 1\) poles. \(N_{\text{max}}\) is the highest value of \(N\) for which the JNR Ansatz provides a complete description of the system. Note that all possible vortex configurations can in principle be generated by the JNR Ansatz. By embedding a centered hyperbolic vortex into \(H^3\) one obtains a centered hyperbolic monopole, as described previously. By rotating the plane in which the vortices are embedded, one augments the dimension of the parameter space by 2. The embedded centered hyperbolic vortex thus provides a \(2N\)-parameter family of the \(4(N−1)\)-dimensional centered monopole parameter space. In particular, then, all centered hyperbolic monopoles of charges \(N = 1\) and \(N = 2\) can be generated by hyperbolic vortices. We will make ample use of this result when considering the moduli space of hyperbolic monopoles in section 5.

4. Spectral data

Geodesics in \(H^3\) are parametrised by their endpoints \(\zeta\) and \(-\bar{\eta}^{-1}\) on the boundary \(S^2 \cong \mathbb{C}^*\). To any monopole configuration there is an associated complex curve \(S\) in the space of geodesics \(\mathbb{C} \times \mathbb{C}^*\) on the background \(H^3\). This curve was first studied by Hitchin [16] and Atiyah [1] and is defined by means of the monopole scattering data. One is to solve the differential equation

\[
(\partial_s + A_s - i\Phi)w = 0
\]
(61)

along geodesics in the background space, the arc length along each of which is parametrised by \(s\), for 2-vectors \(w(s)\). We will refer to geodesics along which \(w(s)\) is \(L^2\)-normalisable as spectral lines. The set of spectral lines makes up the spectral curve. As a complex curve, the spectral curve is given explicitly in terms of the positions and weights of JNR poles [5]:

\[
S : \sum_{j=0}^{N} \lambda_j^2 \prod_{k \neq j} (\zeta - \gamma_k)(1 + \eta \bar{\gamma}_k) = 0,
\]
(62)

where, as explained in section 3, we have \(\lambda_j \in \mathbb{R}, \gamma_i \in \mathbb{C}\).

We are interested in the monopoles obtained from embedding vortices into \(H^3\), such that the JNR poles are collinear. We will choose them to lie on the real axis, so \(\gamma_k = \gamma_k\). Any
charge 2 monopole can be cast into this form by an appropriate choice of centre of mass and orientation of the two zeros of the monopole Higgs field $\Phi$.

In the following sections we will use the well-known formulae above to identify and study three distinguished classes of spectral lines. Our analysis of the spectral curve will show that many of the results established for the spectral curve of monopoles in Euclidean space [2] carry through to the hyperbolic case.

4.1. Spectral lines through the monopole zeros

Consider a vortex configuration embedded in $H^3$ as described in section 2.2, where it was observed that geodesics through monopole zeros orthogonal to the plane $x^1 = 0$ have $\phi = 0$. The monopole Higgs field $\Phi$ along this line is the radial field of a unit charge hyperbolic monopole. It then follows from the definition (61) that such geodesics are spectral lines, by virtue of the fact that all spectral lines of a charge 1 monopole pass through the zero. We see this by expressing $\phi$ in terms of JNR data, such that

$$\phi(z_0) = 0 \Rightarrow ((\bar{z} - z) \partial_z \log(\psi))|_{z=z_0} = 1 \Rightarrow \sum_{j=0}^{N} \lambda_j^2 \prod_{k \neq j} (z_0 - \gamma_k)^2 = 0,$$

where $z = x^4 + i r$. Solutions for $z = z_0$ define geodesics in $H^3$ which meet the boundary of the upper half space at $\zeta = z_0$ and $\zeta = \bar{z}_0$. By comparison with (62), we see that these geodesics are in fact the unique spectral lines with $\eta = -\zeta^{-1}$, i.e. which intersect the plane $x^1 = 0$ at right angles. This observation should be contrasted with the case of Euclidean monopoles, when the spectral lines of a generic charge 2 monopole only approximately pinpoint the zero.

4.2. Spectral lines in the plane of the vortices

We now analyse some of the spectral lines described by (62). Firstly, as was noted in [5], geodesics between any pair of JNR poles are spectral lines. It is also clear that there are precisely $N$ spectral lines for each choice of $\zeta$ on the boundary, and that any geodesic with $\zeta \in \mathbb{R}$ also has $\eta \in \mathbb{R}$. Specialising to $N = 2$ with $\zeta \in \mathbb{R}$ leads to an interesting geometric picture in terms of Poncelet’s theorem, which has already given insight into the geometry of instantons [15] and indeed hyperbolic monopoles [17]. We will work through the details explicitly in our case, making use of various theorems of Daepp–Gorkin–Mortini [11] and Singer [33].

We work with the ball model of $H^3$, where the equatorial slice defined by $\zeta \in \mathbb{R}$ is a Poincaré disk with complex coordinate $w = X^1 + i X^3$. The boundary $w = i e^{-\theta}$ is related to the coordinate $\zeta$ by stereographic projection: $\zeta = \cot(\theta/2)$. For notational convenience we will consider a centered 2-monopole aligned with the $X^3$-axis, although the discussion follows through for any value of the (vortex) moduli. The spectral curve can be parametrised as

$$\gamma^2(\zeta^2 - \gamma^2)(1 - \eta^2 \gamma^2) - (1 - \gamma^4)\eta(\zeta - \eta \gamma^2) = 0,$$

with $\frac{1}{3} \leq \gamma^2 < 1$, and the relation between $\gamma$ and the monopole separation will be clarified in section 5.

Recall from section 1.2 that a centered charge 2 hyperbolic vortex can be constructed from the $C_2$ symmetric Blaschke product

$$f(w) = w \frac{w^2 + a^2}{1 + a^2 w^2},$$

(64)
where vortex zeros are located at the critical points of \( f(w) \) and \( a^2 \) is related to \( \gamma^2 \) by \((\gamma^2 + 1)(a^2 + 3) = 4\). Restricting to the action of \( f \) on the boundary, it is established in [11] that \( f \) is a surjection and that a point \( w = w_0 \) has exactly 3 preimages \( \{w_1, w_2, w_3\} = f^{-1}(\{w_0\}) \), defining an ideal triangle. The edges of this triangle are spectral lines, a fact that is readily checked by direct computation in simple cases, or numerically for more generic values of the parameters. The prescribed Blaschke product (64) then generates all of the spectral lines (with \( \zeta \in \mathbb{R} \)) and hence a family of ideal triangles corresponding to the gauge freedom in the JNR data. It was shown in [33] that the envelope of this family of triangles is a hyperbolic ellipse (the locus of points for which the sum of the geodesic distances from the foci is constant) whose foci are at the critical points of \( f \), i.e. at the vortex zeros. Figure 2 shows the hyperbolic ellipse for the monopole with \( \gamma^2 = \frac{1}{4} \).

### 4.3. Principal axes and spectral radii

Atiyah and Hitchin [2] observed that there are two spectral lines through the centre of a charge 2 monopole. This fact is used to define the principal axes of the monopole, which in turn define the Euler angles, as natural coordinates on the moduli space. A similar definition is possible in the hyperbolic case. Spectral lines through the origin of the hyperbolic ball have \( \eta = \zeta \). Taking a configuration of the form (63) with \( \gamma^2 \leq \frac{1}{4} \), these spectral lines are always contained in the plane \( X^2 = 0 \), and coalesce along the \( X^2 \) axis when \( \gamma^2 = \frac{1}{4} \). The axis \( e_1 \) is defined as the bisector of the angle between these spectral lines. The second bisection defines the axis \( e_2 \), which lies in the plane of the JNR poles. The third principal axis, \( e_3 \), is parallel to the line of separation of the monopole zeros.

The three spectral radii of a Euclidean 2-monopole are defined as half the separation between the unique two spectral lines parallel to each of the three principal axes, [2]. In the

---

On the other hand, joining the triples of points \( w_i \) by Euclidean triangles would yield a Euclidean ellipse with foci at \( w = \pm i a \), the 'non-zero zeros' of \( f \), [11].
hyperbolic setting we will define the spectral radii as the minimal geodesic separation between each pair of spectral lines orthogonal to one of the principal axes. This gives two of the spectral radii as the semi-major and semi-minor axes of the hyperbolic ellipse discussed above:

\[ d_\pm = \cosh^{-1} \left( \frac{2}{\sqrt{3 + 2a^2 - a^4}} \right) . \]  

(65)

In section 4.1 we showed that the only pair of spectral lines which meet the equatorial plane at right angles are those through the monopole zeros. This gives the third spectral radius as half the hyperbolic distance between the zeros,

\[ d_3 = \cosh^{-1} \left( \frac{3 + 2a^2 - a^4}{3 - 2a^2 - a^4} \right) . \]  

(66)

Atiyah & Hitchin’s observation [2] that the three spectral radii define a right-angled triangle also holds in the hyperbolic case, i.e. \( \cosh(d_-) \cosh(d_3) = \cosh(d_+) \). From our description, we see that this fact follows immediately from the definition of an ellipse. The area of this triangle is minimal when \( a = 2^{2/3} - 1 \). Curiously, this corresponds precisely to the critical radius at which there is a closed geodesic in Hitchin’s metric [17].

5. Moduli space

Low energy scattering of solitons has successfully been modelled by geodesic motion on the moduli space [27]. The metric on the moduli space is defined by the \( L^2 \) norm of small perturbations to the fields (\( \phi \to \phi + \delta \phi, a \to a + \delta a \)), which must satisfy the linearised field equations and are subject to the gauge-fixing constraint that perturbed fields should be orthogonal to gauge transformations.

It is well known that the requisite integral diverges for hyperbolic monopoles, and various alternative metrics have been proposed. Examples are Hitchin’s metric on the space of spectral curves [17], the \( L^2 \) metric on the space of circle-invariant instantons [12] and the metric defined through the physical force between monopoles [13]. We will focus on the metric defined via the connection on the boundary of \( H^3 \), [5, 7, 29], and compare this metric to the \( L^2 \) metric on the moduli space of the underlying hyperbolic vortices. In the charge 1 case these metrics are both proportional to the underlying hyperbolic metric. We thus focus on vortices and monopoles of charge 2 and fixed centre of mass.

The centered 2-vortex metric was computed by Strachan [34]. For vortices located at \( z = \pm \alpha e^{i\theta} \) in the Poincaré disk, the gauge orthogonality condition is

\[ 2 \partial_i(\delta a_i) + i \left( \bar{\phi} \delta \phi - \phi \delta \bar{\phi} \right) = 0 \]  

(67)

and the metric takes the form

\[ ds^2 = \int \left( \frac{1}{2} \delta \phi \delta \bar{\phi} + \frac{1}{4} g^{ij} \delta a_i \delta a_j \right) \sqrt{g} \, dx \, dr \]

\[ = \frac{2\pi \alpha^2}{(1 + \alpha^2)^2} \left( 1 + \frac{4(1 + \alpha^4)}{\sqrt{1 + 14 \alpha^4 + \alpha^8}} \right) \frac{4 (\alpha^2 + \alpha^2 d\theta^2)}{(1 - \alpha^2)^2} . \]  

(68)

Note that the gauge condition (67) does not allow the variations in the fields to be computed by varying the JNR function \( \psi \) in the gauge defined through (32), (57).
5.1. Boundary fields

In order to define a metric on the hyperbolic monopole moduli space, we consider the fields on the boundary of $H^3$. In this section, we use coordinates $z = x + i r = x^4 + i x^1$ with metric (21),

$$d s^2 = \frac{1}{r^2} (d x^2 + d r^2).$$  \hfill (69)

The boundary fields are obtained by taking the limit $\rho = 0$ and $r = x^1$ in (28)–(31):

$$A^0_4 = i \frac{2}{\lambda^2} a_4 \tau_1, \quad A^0_1 = i \frac{2}{\lambda^2} a_1 \tau_1,$$

$$A^0_\rho = \frac{i}{2} \left( \frac{\phi_1}{r^2} \tau_2 + \frac{\phi_2 + 1}{r} \tau_3 \right), \quad \Phi^0 = -\frac{i}{2} \tau_1.$$  \hfill (70)

As the Higgs field tends to a constant, the relevant gauge fixing condition is simply the Coulomb gauge $\partial_i (a_i) = 0$, which holds identically for fields of the form (57),

$$a_\tau = -\partial_\tau \log \psi, \quad a_r = \partial_r \log \psi,$$  \hfill (71)

allowing us to obtain the metric by varying $\psi$. The metric is then defined by

$$d s^2 = \int g_{ij} \delta a_i \delta a_j \sqrt{g} \, d x \, d r = \int \delta^{ij} \delta a_i \delta a_j \, d x \, d r,$$  \hfill (72)

where $g_{ij}$ is the hyperbolic metric on the boundary. The gauge potentials (71) are simply those of a vortex in the hemisphere model of $H^2$. However, the lack of a Higgs field contribution and the different gauge condition will give a metric different from (68).

The moduli space metric (72) is invariant both under gauge transformations and conformal rescalings of the boundary metric (69). The Coulomb gauge condition leaves a residual gauge freedom to multiply $\psi$ by the modulus-squared of a holomorphic function, and we use this to remove the poles in $\psi$. The resulting JNR function can equivalently be obtained from the spectral curve polynomial by setting $(\zeta, \eta) = (z, -\bar{z}^{-1})$ and multiplying by $\bar{z}^N$. We denote the resulting function $h$.

5.2. Monopole metric: radial component

We wish to compare (68) to the radial component of the metric of two hyperbolic monopoles obtained from lifting a charge 2 hyperbolic vortex to $H^3$. To compute the metric for two hyperbolic monopoles whose zeros are in the plane $x^1 = 0$, we take the ‘t Hooft function

$$\psi = 1 + \frac{\lambda^2}{(x^4 - \gamma)^2 + r^2} + \frac{\lambda^2}{(x^4 + \gamma)^2 + r^2},$$  \hfill (73)

where $r^2 = \rho^2 + (x^1)^2$ and the poles are fixed to lie on the $x^4$ axis. A geodesic one-parameter family is obtained by imposing dihedral symmetry $D_2$, which requires that $2 \lambda^2 = \gamma^2 - \gamma^2$, and this is centered by the definition of [29].

To relate $\gamma$ to the positions of the Higgs zeros we must locate, from (56), the zeros of $\nabla^2 \log (r \psi)$. There are two regimes: for $\gamma^2 \in [0, \frac{1}{4}]$, the zeros are found at $x^4 = x^1 = 0$ and

$$\rho_0^{\pm 2} = \frac{1}{2 \gamma^2} \left( (1 - 3 \gamma^4) + \sqrt{(1 - 3 \gamma^4)^2 - 4 \gamma^4} \right),$$  \hfill (74)

while for $\gamma^2 \in [\frac{1}{4}, 1]$ they are at $x^1 = 0$ and
Converting back to the ball model of $H^3$, the monopoles are located at

$$(X^1, X^2, X^3) = \left(0, \pm \frac{x_0}{\rho_0 + 1}, 0\right), \quad \frac{1}{3} \leq \gamma^2 \leq 1$$

or

$$(X^1, X^2, X^3) = \left(0, 0, \pm \frac{\rho_0 - 1}{\rho_0 + 1}\right), \quad 0 \leq \gamma^2 \leq \frac{1}{3},$$

defined from which we define

$$\alpha = \frac{\rho_0 - 1}{\rho_0 + 1}.$$  

For ease of numerical computation we recast the JNR function (73) into the form

$$h = |z|^4 - A(\gamma) (z^2 + \bar{z}^2) + B(\gamma) |z|^2 + 1,$$

with $A = \gamma^2$ and $B = \gamma^{-2} - \gamma^2$. We now obtain the radial component of the moduli space metric from (72), using the relations (74) and (77) to change to the coordinate $\alpha$:

$$g_{\alpha \alpha} d\alpha^2 = \frac{4 d\alpha^2}{(1 - \alpha^2)^2} \rho_0^4 \left(\frac{d\gamma}{d\rho_0}\right)^2 \int \frac{\partial \alpha_i}{\partial (\gamma^2)} \frac{\partial \alpha_i}{\partial (\gamma^2)} dx dr \equiv f^2(\alpha) d\alpha^2.$$  

The integral in (79) is evaluated numerically and the profile function is compared with the metric of the corresponding vortex, (68), in figure 3. In both cases the asymptotic metric approaches that of the underlying $H^3$.

5.3. Monopole metric: angular components

SO(3) and dihedral symmetry imply that the moduli space metric of two hyperbolic monopoles is diagonal when expressed in terms of the SO(3)-invariant one-forms $\sigma_i$, [2]:

![Figure 3. Radial component of the metric as a function of $\alpha$, the distance of each Higgs zero from the origin. Solid line: analytic result (68) for the vortex metric (rescaled by 32/9). Dashed line: monopole metric (79). In both cases we have divided by the factor $4(1 - \alpha^2)^{-2}$.](image-url)
\[ g = f^2(\alpha) \alpha^2 + a^2(\alpha) \sigma_1^2 + b^2(\alpha) \sigma_2^2 + c^2(\alpha) \sigma_3^2. \]  

(80)

The function \( f(\alpha) \) was defined in the previous section. To compute \( a, b, c \) we rotate the poles of the standard JNR function (73) so as to align each of the principal axes \( e_1, e_2, e_3 \) (identified in section 4.3) with the \( X_3 \) coordinate axis in turn, as shown in figure 4.

The gauge potential is still determined by functions of the form (78), with

\[ a^2 = \frac{1 - 3\gamma^2}{1 + \gamma^2}, \quad b^2 = \frac{1 - \gamma^2}{1 + 3\gamma^2}, \quad c^2 = \frac{\gamma^2}{\gamma^2}. \]

(81)

Deformations are now parametrised by a rotation by an angle \( \omega \) in the \( z \) plane (which is a stereographic projection of the boundary of the unit ball). This choice of parametrisation fixes the gauge freedom in the JNR data, and \( \omega \) represents a rotation about one of the principal axes. For each choice of \( A \) and \( B \) the relevant component of the metric is given by the integral

\[ \int \left[ (\partial_\omega a_i)^2 + (\partial_\omega a_i)^2 \right] \, dx \, dr = 64A^2 \int \left[ \left( \partial_x \left( \frac{px}{h} \right) \right)^2 + \left( \partial_r \left( \frac{rx}{h} \right) \right)^2 \right] \, dx \, dr. \]

(82)

Plots of these functions are given in figure 5. Expanding near \( \alpha = 0 \) gives the metric coefficients

\[ f^2 = c_1 \alpha^2 + \mathcal{O}(\alpha^4), \quad b^2 = c_2 + c_3 \alpha^2 + \mathcal{O}(\alpha^4), \quad a^2 = \alpha^2 f^2 + \mathcal{O}(\alpha^6), \quad c^2 = c_2 - c_3 \alpha^2 + \mathcal{O}(\alpha^4). \]

(83)

where

\[ c_1 = \frac{32\pi}{9} \left( 81 - 10\sqrt{3}\pi \right), \quad c_2 = \frac{8\pi}{27} \left( 2\sqrt{3}\pi - 9 \right), \quad c_3 = \frac{4\pi}{9} \left( 8\sqrt{3}\pi - 27 \right). \]

(84)

A numerical computation of the coefficients for \( \alpha \rightarrow 1 \) is consistent with the expansions

\[ f^2 = \frac{16\pi}{3} \frac{4}{(1 - \alpha^2)^2} \left( 1 - 4(1 - \alpha)^4 + \ldots \right), \]
\[ a^2 = b^2 = \frac{16\pi}{3} \frac{4\alpha^2}{(1 - \alpha^2)^2} \left( 1 - (1 - \alpha)^2 + \ldots \right), \quad c^2 \propto (1 - \alpha)^8, \]

(85)
where the leading power of $(1 - \alpha)$ can be determined by suitable approximations of the integrand. When $\alpha$ is converted to a hyperbolic distance, these expressions are exponentially close to the background hyperbolic metric. In this regime ($\alpha \to 1$) the coefficient of proportionality between the moduli space metric and the background hyperbolic metric can be interpreted as a mass, arising from the construction of the moduli space metric as a kinetic energy. In the vortex case [34] the mass of a single vortex is given by $3\pi/2$, while in our monopole construction we find the monopole can be assigned a mass $16\pi/3$.

It would be interesting to relate our asymptotic metric with that obtained from consideration of the forces between well-separated monopoles. The extension of Manton’s calculation [24] to the hyperbolic case was performed by Gibbons & Warnick [13]. For large monopole separations the resulting metric was found to approach LeBrun’s hyperbolic analogue of the Gibbons-Hawking metric, [19].

6. Periodic monopoles

The original motivation for this work was to obtain new examples of hyperbolic monopoles. The method presented in section 2 is particularly useful for periodic arrays of monopoles, for which the JNR and ADHM constructions are not currently known. However, periodic and large charge vortex configurations have been studied [21, 26, 35] and they are easily lifted to $H^3$.

Periodic monopoles in Euclidean space have previously been studied in some depth via the Nahm transform and spectral curve [9]. These tools demonstrated the splitting of the monopole into constituents [37] and allowed a study of the moduli space dynamics [22].

The periodic monopole we will construct in this section is the one lifted from a vortex on the hyperbolic cylinder [26], in which the Higgs zeros are periodic along a geodesic in $H^2$. The JNR data for such a periodic vortex is not known, so the formula (33) provides a novel example of a hyperbolic monopole. The vortex Higgs field is given in terms of Jacobi elliptic functions, where the elliptic modulus $k$ determines the periodicity. Explicitly, we use the formula (12) with

$$f(w) = \frac{\text{cn}_k(2\kappa \tan^{-1}(w)) - \text{dn}_k(2\kappa \tan^{-1}(w))}{\text{cn}_k(2\kappa \tan^{-1}(w)) + \text{dn}_k(2\kappa \tan^{-1}(w))},$$

(86)
where $\pi \kappa = 2K_k$. Using the coordinate $iu = \log(x_4 + ir)$ in the upper half space model of $H^2$ gives the Higgs field
\[
|\phi|^2 = \kappa^2 |\text{cn}_k(\kappa u)|^2 \left( \frac{\text{sin}(\text{Re}(u))}{\text{Re}(\text{sn}_k(\kappa u))} \right)^2.
\]
(87)
The monopole constructed from this vortex has zeros at $x^4_0 = x^3_0 = 0$ and $\rho_0 = e^{n\lambda/2}$, with $n \in \mathbb{Z}$ and
\[
\lambda = \frac{\pi K'_k}{K_k},
\]
(88)
where $\frac{1}{2} \lambda$ is the hyperbolic distance between neighbouring zeros of the Higgs field. The energy density is computed using (44) and plotted in figure 6 for various values of $k$. For small $k$ the monopoles are well separated and the energy density is peaked at the Higgs zeros. As $k$ is increased the monopoles get closer together and widen in the $X^1$ direction. Then at some critical value the energy peaks break apart and move away from the $X^3$ axis, leaving the positions of the Higgs zeros as saddle points of energy density.

Expansions of the Higgs field (87) at the half period points of the periodic vortex (in the Poincaré disk model) were given in [21]. Applying the formula (44) to these expansions yields explicit expressions for the maximal and minimal values taken by the Higgs field along the $X^3$ axis:
\[
E_{\text{min}} = \frac{1}{2} \left( 1 - \left( \frac{2K_k}{\pi} (1 - k) \right)^2 \right)^2, \quad E_{\text{saddle}} = \frac{1}{2} + \left( \frac{2K_k}{\pi} \sqrt{1 - k^2} \right)^4.
\]
(89)
The critical value of $k$ at which the maximum energy on the $X^3$ axis becomes a saddle point is found by expanding $|\phi|^2$ to higher order. Performing this expansion and converting back to the upper half-plane model, the energy density at $x^4 + ir = i + |\delta|e^{i\theta}$ restricted to $x^1 = 0$ is
\[
E|_{x^1=0} = \left( \frac{1}{2} + a^2 \right) + 3a|\delta|^2(b\cos(2\theta) - a) + \mathcal{O}(|\delta|^3),
\]
(90)
where $a$ and $b$ are the coefficients at order $w^2$ and $w^4$ in an expansion of (86), and are given by
\[ a = -\frac{4K^2}{\pi^2}(1 - k^2), \quad b = -\frac{8K^2}{3\pi^2}(1 - k^2) \left( \frac{4K^2}{\pi^2}(1 + k^2) - 1 \right). \] (91)

When \( \theta = 0 \), the order \( |\delta|^2 \) term in the expansion changes sign when \( a = b \), i.e.
\[ 8K^2_0(1 + k_0^2) = 5\pi^2 \Rightarrow k_0 \approx 0.780. \] (92)

Considering the next term in the expansion (90) shows that the hyperbolic distance of the energy peaks from the \( X^3 \) axis grows like \( d \propto (k - k_0)^{1/2} \). Differentiating (44) with respect to \( x^1 \) shows that the plane \( x^1 = 0 \) is everywhere a stationary point of the energy density, with a local maximum at the vortex zero. A similar splitting can be observed for chains of finite length. However, such a splitting has not been observed in the periodic monopole obtained from the axially symmetric Harrington–Shepard periodic instanton [14], which gives Higgs zeros equally spaced on a horocycle. Causing these monopoles on a horocycle to split into constituents is likely to require the introduction of further parameters which are not available in the vortex starting point considered in this paper.

7. Conclusions and outlook

In this paper we have studied the direct correspondence between vortices and monopoles in hyperbolic space. We have seen that all hyperbolic vortices can be lifted to \( v = \frac{1}{2} \) hyperbolic monopoles. The spectral curve of this class was analysed, and it was used to shed light on the 2-monopole moduli space. Finally, we used the construction to obtain a novel hyperbolic monopole with periodic symmetry. Open questions include whether a similar technique can be used to generate hyperbolic monopoles of higher rank gauge group, with point singularities or with \( v > \frac{1}{2} \). Work in these directions is currently underway. A similar embedding to the one described in this article has been observed for Skyrmions in 2 and 3 spatial dimensions, [31], and it would be interesting to determine whether these findings can be extended to hyperbolic space.

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