The Lyapunov dimension, convergency and entropy for a dynamical model of Chua memristor circuit

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INTRODUCTION

For the study of chaotic dynamics and dimension of attractors the concepts of the Lyapunov exponents [1] was found useful and became widely spread [2–5]. Such characteristics of chaotic behavior, as the Lyapunov dimension [6] and the entropy rate [7–9], can be estimated via the Lyapunov exponents [6, 10–12]. In this work an analytical approach to the study of the Lyapunov dimension, convergency and entropy for a dynamical model of Chua memristor circuit is demonstrated.

I. A DYNAMICAL MODEL OF THE CHUA MEMRISTOR CIRCUIT

Consider one of the Chua memristor models [13, eq.25]

\[ \begin{align*}
\dot{x} &= \alpha(m_0 - 1)x + ax^2 + \gamma \dot{z}, \\
\dot{y} &= x - y + z, \\
\dot{z} &= \beta y - \gamma z
\end{align*} \]

(1)

with real parameters \( \alpha, \beta, m_0, m_1, \gamma, x_0 \), and suppose that \( \alpha m_1 > 0 \). For a survey on memristor circuits see, e.g. [14]. System (1) with \( x_0 = 0 \) describes the dynamics of the Chua oscillator with cubic nonlinearity [15, 16]. If \( \gamma = 0 \) and \( x_0 = 0 \), then system (2) has equilibria \( u_{eq}^0 = (0, 0, 0) \) and \( u_{eq}^\pm = \left( \pm \sqrt{\frac{m_0 - 1}{m_1}}, 0, \mp \sqrt{\frac{m_0 - 1}{m_1}} \right) \) for \( m_0 > 1 \). Represent system (1) as an autonomous differential equation of general form:

\[ \dot{u} = f(u), \]

(2)

where \( u = (x, y, z) \in U = \mathbb{R}^3 \) and the continuously differentiable vector-function \( f: \mathbb{R}^3 \to \mathbb{R}^3 \) is the right-hand side of system (1). Define by \( u(t, u_0) \) a solution of (2) such that \( u(0, u_0) = u_0 \), and consider the evolutionary operator \( \varphi^t(u_0) = u(t, u_0) \). We assume the uniqueness and existence of solutions of (2) for \( t \in [0, +\infty) \). Then system (2) generates a dynamical system \( \{\varphi^t\}_{t \geq 0} \). Let a nonempty set \( K \subset U \) be invariant with respect to \( \{\varphi^t\}_{t \geq 0} \), i.e. \( \varphi^t(K) = K \) for all \( t \geq 0 \). For example, as the set \( K \) one can consider various types of attractors of system (2) (see, e.g. examples from [16, 17]). Recently the classification of local attractors as being hidden or self-excited was introduced in connection with the discovery of the first hidden attractor in the classical Chua model with the saturation nonlinearity [18–24]: an attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise, it is called a hidden attractor [19, 25–27]. For example, hidden attractors can be found in various memristive circuits, see, e.g. [28–43]. Remark that hidden attractors are not connected with equilibrium and, thus, do not related with the Shilnikov scenario of chaos [44]. In the works [45–48] it is demonstrated the difficulties of reliable simulation of the phase-locked loops circuits in SPICE and MATLAB Simulink, caused by hidden attractors with narrow basins of attraction.

Consider linearization of system (2) along the solution \( u(t, u_0) = \varphi^t(u) \):

\[ \dot{v} = J(\varphi^t(u))v, \quad J(u) = Df(u), \]

(3)

where \( J(u) \) is the \( 3 \times 3 \) Jacobian matrix

\[ J(u) = \begin{pmatrix}
\alpha(m_0 - 1) - 3\alpha m_1 x^2 & \alpha & 0 \\
1 & -1 & 1 \\
0 & \beta & -\gamma
\end{pmatrix} \]

and it can be represented as \( J = J(0) - 3\alpha m_1 x^2 I_1 \) with

\[ J_0 = J(0) = \begin{pmatrix}
\alpha(m_0 - 1) & \alpha & 0 \\
1 & -1 & 1 \\
0 & \beta & -\gamma
\end{pmatrix}, \quad I_1 = \begin{pmatrix}1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0\end{pmatrix}. \]

Let for any \( t > 0 \) and any \( u \in U \) the ordered sequence \( \lambda_1(u) \geq \cdots \geq \lambda_n(u) \), where \( \lambda_i(u) = \lambda_i(J(u) + J(u)^*), \)

\( i = 1, \ldots, n \) be the eigenvalues of the symmetrized Jacobi matrix \( \frac{1}{2}(J(u) + J(u)^*) \).

Lemma 1 \( \lambda_j(0) \geq \lambda_j(u), \quad j = 1, 2, 3 \)

Then from Corollary 2 (see in the Appendix) we get

Theorem 1 \( d_{Ly}^K(\{\lambda_i(u)\}_{i=1}^3) \leq d_{Ly}^K(\{\lambda_i(0)\}_{i=1}^3) \)

If \( J(0) \) have simple real eigenvalues \( \lambda_i(J(0)) \), then \( \lambda_i(J(0)) = \lambda_i(J(0) + J^*(0)) \) and we get the following result

Corollary 1 Let \( u_{eq}^0 = (0, 0, 0) \) be one of the equilibria of system (1) and the matrix \( J(0) \) have simple real eigenvalues. Then the exact Lyapunov dimension of any compact invariant set \( K \ni u_{eq}^0 \) is defined as

\[ \dim_L K = d_{Ly}^K(\{\lambda_i(0)\}_{i=1}^3). \]

By Theorem 4 (see in the Appendix) we get

Theorem 2 If \( \lambda_1(0) + \lambda_2(0) < 0 \), then any bounded solution of system (2) tends to the stationary set.
APPENDIX. EXACT AND FINITE-TIME
LYAPUNOV DIMENSION

Suppose that det $J(u) \neq 0 \quad \forall u \in U$. Consider a fundamental matrix of linearized system (3) $D\varphi^t(u)$ such that $D\varphi^0(u) = I$, where $I$ is a unit 3 × 3 matrix. Let $\sigma_i(t, u) = \sigma_i(D\varphi^t(u)), \; i = 1, 2, 3$, be the singular values of $D\varphi^t(u)$ with respect to their algebraic multiplicity ordered so that $\sigma_1(t, u) \geq \sigma_2(t, u) \geq \sigma_3(t, u) > 0$ for any $u \in U$ and $t \geq 0$. Consider the set of the finite-time Lyapunov exponents at the point $u_0$ \( LE_{\varphi}(t, u_0) = \frac{1}{t} \ln \sigma_j(t, u_0) \) ordered by decreasing for $t > 0$.

Introduce the Kaplan-Yorke formula [6] with respect to the ordered set $\lambda_1 \geq \ldots \geq \lambda_n$:

$$ d_{KY} (\{\lambda_i\}_{i=1}^3) = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_j + 1|} \quad j = \max \{m : \sum_{i=1}^m \lambda_i \geq 0\}, $$

(4)

where $d_{KY} (\{\lambda_i\}_{i=1}^3) = 0$ for $j = 0$ and $d_{KY} (\{\lambda_i\}_{i=1}^3) = 3$ for $j = 3$. Then the finite-time local Lyapunov dimension \([49, 50]\) at a certain point $u_0$ can be defined as

$$ \dim_{l}(t, u_0) = d_{KY} (\{LE_{\varphi}(t, u_0)\}_{i=1}^3). $$

and the finite-time Lyapunov dimension of invariant closed bounded set $K$ is as follows

$$ \dim_{l}(t, K) = \sup_{u \in K} \dim_{l}(t, u_0). $$

(5)

In this approach the use of Kaplan-Yorke formula (4) with the finite-time Lyapunov exponents is justified by the Dowady-Oesterlé theorem [51], which implies that for any fixed $t > 0$ the Lyapunov dimension of the map $\varphi^t$ with respect to a closed bounded invariant set $K$, defined by (5), is an upper estimate of the Hausdorff dimension of the set $K$: $\dim_{H} K \leq \dim_{l}(t, K)$. For the estimation of the Hausdorff dimension of invariant closed bounded set $K$ one can use the map $\varphi^t$ with any time $t$ (e.g. $t = 0$ leads to the trivial estimate $\dim_{H} K \leq 3$) and, thus, the best estimation is $\dim_{H} K \leq \inf_{t \geq 0} \dim_{l}(t, K)$. The following property

$$ \inf_{t \geq 0} \sup_{u \in K} \dim_{l}(t, u) = \liminf_{t \to +\infty} \sup_{u \in K} \dim_{l}(t, u) $$

(6)

allows one to introduce the Lyapunov dimension [49]

$$ \dim_{l} K = \lim_{t \to +\infty} \inf_{u \in K} \dim_{l}(t, u). $$

(7)

If the maximum of local Lyapunov dimensions on the global attractors, which involves all equilibria, is achieved at an equilibrium point $u^{eq}$, i.e. \( \dim_{l}(u^{eq}) = \max_{u \in K} \dim_{l}(u_0) \), then this allows one to get the exact Lyapunov dimension. (this term was suggested by Doering et al. in [52]). In general, a conjecture on the Lyapunov dimension of self-excited attractor [50] is that for a typical system the Lyapunov dimension of a self-excited attractor does not exceed the Lyapunov dimension of one of unstable equilibria, the unstable manifold of which intersects with the basin of attraction and visualize the attractor.

In contrast to the finite-time Lyapunov dimension (5), the Lyapunov dimension (7) is invariant under smooth change of coordinates [49, 53]. This property and a proper choice of smooth change of coordinates may significantly simplify the estimation of the Lyapunov dimension of dynamical system.

Consider an effective analytical approach, proposed by Leonov [49, 54, 56]. Let for any $t > 0$ and any $u_0 \in U$ the ordered sequence $\lambda_1(u_0, S) \geq \ldots \geq \lambda_n(u_0, S)$, where $\lambda_i(u_0, S) = \lambda_i(S\varphi^t(u_0)), \; i = 1, \ldots, n$ be the eigenvalues of the symmetrized Jacobian matrix

$$ \frac{1}{2} \left( SJ(\varphi^t(u_0))S^{-1} + (S J(\varphi^t(u_0))S^{-1})^* \right). $$

(8)

**Theorem 3** If there exist an integer $j \in \{1, \ldots, n - 1\}$, a real $s \in [0, 1]$, a differentiable scalar function $V : U \subseteq \mathbb{R}^n \to \mathbb{R}^1$, and a nonsingular $n \times n$ matrix $S$ such that

$$ \sup_{u \in U} (\lambda_1(u, S) + \cdots + \lambda_j(u, S) + s\lambda_{j+1}(u, S) + V(u)) < 0, $$

(9)

where $V(u) = (grad(V))^T f(u)$, then for a compact invariant set $K \subseteq U$ we have

$$ \dim_{H} K \leq \dim_{l}(\{\varphi^t\}_{t \geq 0}, K) < j + s. $$

In the work [49] it is shown how the method can be justified by the invariance of the Lyapunov dimension of compact invariant set with respect to the special smooth change of variables $h$ with $Dh(u) = e^{V(u)(j+1)}S$, where $V$ is a differentiable scalar function and $S$ is a nonsingular $n \times n$ matrix. For $S = 0$ and $V(u) \equiv 0$ we have

**Corollary 2** [49, 51, 54, 56]

$$ \dim_{H} K \leq \dim_{l} K \leq \sup_{u \in K} d_{KY} (\{\lambda_j(u)\}_{i=1}^3). $$

The following result [57] is useful for the study of global convergence.

**Theorem 4** If there exist a continuously differentiable scalar function $V : U \subseteq \mathbb{R}^n \to \mathbb{R}^1$ and a non-degenerate $n \times n$ matrix $S$ exist such that

$$ \sup_{u \in U} (\lambda_1(u, S) + \lambda_2(u, S) + V(u)) < 0, $$

(10)

then any bounded solution of system (2) with any initial data $u_0 \in U$ tends to the stationary set of dynamical system $\{\varphi^t\}_{t \geq 0}$ as $t \to +\infty$.

Remark that the stationary set can have any structure, e.g. be a line of equilibria.

In [58, 59] it is demonstrated how the above technique can be effectively used to a derive constructive upper bound of the sum of positive Lyapunov exponents and the topological entropy [9] (the topological entropy is an analogue of the entropy defined earlier by Kolmogorov and Sinai [7, 8]).
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