On Weyl-type Solutions of Differential Systems with a Singularity. The Case of Discontinuous Potential*

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Abstract—In this paper, we study the Weyl-type solutions of the differential system with a singularity \( y' - x^{-1} Ay - q(x)y = \rho By \) in the case of an integrable potential \( q(\cdot) \).

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1. INTRODUCTION

We consider the differential system

\[
y' - x^{-1} Ay - q(x)y = \rho By, \quad x > 0,
\]

with \( n \times n \) matrices \( A, B \), \( q(x) \), \( x \in (0, \infty) \), where \( A, B \) are constant and the matrix function \( q(\cdot) \) will be referred to as the potential.

Differential equations with coefficients having nonintegrable singularities at the end of or inside the interval often appear in various areas of natural sciences and engineering. As to the case of \( n = 2 \), there exists an extensive literature devoted to different aspects of the spectral theory of radial Dirac operators; see, for instance, [1]–[5]. Although this case is very important and has many applications, its investigation does not form a comprehensive picture. Systems of the form (1.1) with \( n \geq 2 \) and noncollinear complex eigenvalues of the matrix \( B \) show much more complicated behavior and are considerably more difficult for investigation even in the "regular" case \( A = 0 \) [6]. Some difficulties of principal character also appear due to the presence of a singularity at \( x = 0 \). Whereas the "regular" case \( A = 0 \) has been studied fairly completely to date [6]–[8], for system (1.1) with \( A \neq 0 \), there are no similar general results.

In this paper, we concentrate on the construction and investigation of a distinguished basis of generalized eigenfunctions for (1.1). We call them Weyl-type solutions. The Weyl-type solutions play a central role in studying both the direct and inverse spectral problems (see, for instance, [9], [10]). In the presence of a singularity at \( x = 0 \), this approach encounters certain difficulties that do not appear in the "regular" case \( A = 0 \). The approach presented in [11] for the case of the scalar differential operators

\[
\ell y = y^{(n)} + \sum_{j=0}^{n-2} \left( \frac{\nu_j}{x^{n-j}} + q_j(x) \right) y^{(j)}
\]

(closely connected with (1.1)) is based on some special solutions of the equation \( \ell y = \lambda y \) that also satisfy certain Volterra integral equations. In this way, various results concerning both the direct and inverse spectral problems for (1.2) were obtained (see, for instance, [11]–[13] and also [14] with references therein). But the approach has an important restriction; namely, it assumes some additional decay condition for the coefficients \( q_j(x) \) as \( x \to 0 \). In this paper, we use another approach that allows us not to impose any additional restrictions of such type.

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In short, the approach can be described as follows. We consider some auxiliary systems with respect to the functions with values in the exterior algebra $\wedge \mathbb{C}^n$. Our study of these auxiliary systems centers on two families of their solutions that also satisfy some asymptotical conditions as $x \to 0$ and $x \to \infty$, respectively, and can be constructed as solutions of certain Volterra integral equations. We call these distinguished tensor solutions fundamental tensors. The main difference from the above-mentioned method used in [11] is that we use integral equations to construct the fundamental tensors rather than the solutions for the original system. Since each of the fundamental tensors has minimal growth (as $x \to 0$ or $x \to \infty$) among the solutions of the same auxiliary system, this approach does not require any decay of $q(x)$ as $x \to 0$. As the next step, we show that the fundamental tensors are decomposable. Moreover, they can be represented as wedge products of some solutions of the original system (1.1), and these solutions can be shown to be Weyl-type solutions of (1.1). This strategy allows us to construct Weyl-type solutions via a purely algebraic procedure and investigate their properties.

This approach was first presented in [9] for higher-order differential operators with regular (from the Schwarz space) coefficients on the whole line and was mostly used for evaluating the behavior of Weyl-type solutions and of the scattering data for $\rho \to 0$. In [15] the approach was modified and adapted to the case of system (1.1). The developed method was used for the investigation of the asymptotical behavior of Weyl-type solutions as $\rho \to \infty$ and $\rho \to 0$; moreover, a certain uniqueness result for the inverse scattering problem for (1.1) was also obtained.

In [15], the potential $q(\cdot)$ was assumed to be integrable and absolutely continuous on the semi-axis. In this paper, we consider the case in which the function $q(\cdot)$ is allowed to be nondifferential and even discontinuous. Differential systems with discontinuous coefficients have been investigated intensively during the last two decades (see, for instance, [16], [17]); one should also mention the important role played by such systems in the theory of ordinary differential operators with distribution coefficients (see, for instance, [18]–[20]). Investigation of such systems required further nontrivial development of spectral theory methods (see [20]–[22]).

The results obtained in the present paper include, in particular, the asymptotics of Weyl-type solutions as $\rho \to \infty$ with $o(1)$ remainder’s estimates. Results of such type play an auxiliary role; more important results are formulated in terms of belonging of the Weyl-type solutions of certain classes of integrable functions of the spectral parameter $\rho$. We also establish the continuity of Weyl-type solutions with respect to the potential $q(\cdot)$.

2. SOLUTIONS OF THE UNPERTURBED SYSTEM

In this section, we briefly discuss the unperturbed system:

$$y' - x^{-1}Ay = \rho By \quad (2.1)$$

and introduce some fundamental systems of its solutions.

Here and throughout the paper, we assume the following.

**Condition A.** $A$ is off-diagonal. The eigenvalues $\{\mu_j\}_{j=1}^n$ of the matrix $A$ are distinct and such that $\mu_j - \mu_k \not\in \mathbb{Z}$ for $j \neq k$, moreover, $\text{Re}\mu_1 < \text{Re}\mu_2 < \cdots < \text{Re}\mu_n$ and $\text{Re}\mu_k \neq 0$, $k = 1, \ldots, n$.

**Condition B.** $B = \text{diag}(b_1, \ldots, b_n)$, where the entries $b_1, \ldots, b_n$ are nonzero distinct complex numbers such that any three of them are noncolinear and, moreover, $\sum_{j=1}^n b_j = 0$.

We start with considering (2.1) for $\rho = 1$:

$$y' - x^{-1}Ay = By \quad (2.2)$$

but for complex values of $x$.

Let $\Sigma$ be the following union of lines through the origin in $\mathbb{C}$:

$$\Sigma = \bigcup_{(k,j): j \neq k} \{ x : \text{Re}(xb_j) = \text{Re}(xb_k) \}.$$

Consider some (arbitrary) open sector $S \subset \mathbb{C} \setminus \Sigma$ with the vertex at $x = 0$. It is well-known that there exists the ordering $R_1, \ldots, R_n$ of the numbers $b_1, \ldots, b_n$ such that

$$\text{Re}(R_1x) < \text{Re}(R_2x) < \cdots < \text{Re}(R_nx)$$
for any $x \in S$. For $x \in S \setminus \{0\}$ there exist the following fundamental matrices for system (2.2) (see [23]–[25]):

- $c(x) = (c_1(x), \ldots, c_n(x))$, where $c_k(x) = x^{\mu_k} \hat{c}_k(x)$, $\det c(x) \equiv 1$, all the $\hat{c}_k(\cdot)$ are entire functions, $\hat{c}_k(0) = \xi_k$, and $\xi_k$ is the eigenvector of the matrix $A$ corresponding to the eigenvalue $\mu_k$;

- $e(x) = (e_1(x), \ldots, e_n(x))$, where

$$e_k(x) = \exp(xR_k) \left( f_k + O(x^{-1}) \right), \quad |x| \geq 1, \quad x \in S,$$

$(f_1, \ldots, f_n) = f$ is a permutation matrix such that $(R_1, \ldots, R_n) = (b_1, \ldots, b_n)f$.

**Condition I(S).** For all $k = 2, \ldots, n$, the Wronskian determinants

$$\Delta_k^0 := \det(e_1(x), \ldots, e_{k-1}(x), e_k(x), \ldots, e_n(x))$$

are not equal to 0.

Under Condition I(S) for $x \in S \setminus \{0\}$, the fundamental matrix $\psi_0(x) = (\psi_{01}(x), \ldots, \psi_{0n}(x))$ exists (and is unique) and satisfies

$$\psi_0(x)t = \exp(xR_k)(f_k + o(1)), \quad t \to +\infty, \quad x \in S, \quad \psi_0(x) = O(x^{\mu_k}), \quad x \to 0. \tag{2.3}$$

Since $\psi_0(x)$ admits the representation

$$\psi_0(x) = \sum_{j=k}^n l_{jk} \hat{c}_j(x) = \sum_{j=k}^n l_{jk} x^{\mu_j} \hat{c}_j(x)$$

with constant coefficients $\{l_{jk}\}$, we see that the function $x^{-\mu_k} \psi_0(x)$ admits a continuous extension onto $S$.

Now we return to the unperturbed system (2.1) with an arbitrary $\rho \in S \setminus \{0\}$ and a real positive $x$. Notice that if some function $y = y(x)$ satisfies (2.2), then $Y(x, \rho) := y(px)$ satisfies (2.1). Taking this fact into account, we define the matrix solutions $C(x, \rho), E(x, \rho), \Psi_0(x, \rho)$ of (2.1) as follows:

$$C(x, \rho) := c(px), \quad E(x, \rho) := e(px), \quad \Psi_0(x, \rho) := \psi_0(px).$$

The $k$th column $\Psi_0(x, \rho)$ will be referred to as the $k$th Weyl-type solution of system (2.1). For $\rho \in S$, it satisfies the asymptotic conditions:

$$\Psi_0(x, \rho) = \exp(pxR_k)(f_k + o(1)), \quad x \to +\infty, \quad \Psi_0(x, \rho) = O(x^{\mu_k}), \quad x \to 0.$$

In the sequel, we shall also use the following estimate that can be obtained directly from the definition of the matrix $\Psi_0(x, \rho)$ and equalities (2.3):

$$||Psi_0(x, \rho)|| \leq M|W_k(px)|. \tag{2.4}$$

Here and below, $W_k(\cdot), k = 1, \ldots, n$ denotes the weight function defined as follows:

$$W_k(\xi) := \begin{cases} W_0(\xi^{\mu_k} \exp(R_k \xi)), & |\xi| \leq 1, \\
\exp(R_k \xi), & |\xi| > 1, \end{cases} \quad W_0(\xi) := \begin{cases} (1 - |\xi|)\xi + |\xi|^2, & |\xi| \leq 1, \\
(W_0(\xi^{-1}))^{-1}, & |\xi| > 1, \end{cases}$$

where $M$ is an absolute constant (we agree to use the same symbol $M$ to denote various, possibly different, absolute constants).

From the construction and the properties of the matrices $\psi_0(x)$ and $\Psi_0(x, \rho)$ one can deduce, moreover, that the functions

$$\tilde{\Psi}_0(x, \rho) := (W_k(px))^{-1}\Psi_0(x, \rho)$$

admit continuous extensions onto the set $\{(x, \rho) : x \in [0, \infty), \rho \in S\}$. 
3. FUNDAMENTAL TENSORS

Proceeding in the way described above, for \( m = 1, \ldots, n \), we consider the following auxiliary systems:

\[
Y' = (U(x, \rho))^m Y,
\]

where \( Y \) is a function with values in the exterior product \( \wedge^m \mathbb{C}^n \). Here and below

\[
U(x, \rho) := x^{-1} A + q(x) + \rho B
\]

and for \( n \times n \) matrix \( V \) the symbol \( V^{(m)} \) denotes the operator acting in \( \wedge^m \mathbb{C}^n \) so that, for any vectors \( v_1, \ldots, v_m \) the following identity holds:

\[
V^{(m)}(v_1 \wedge v_2 \wedge \cdots \wedge v_m) = \sum_{j=1}^m v_1 \wedge v_2 \wedge \cdots \wedge v_{j-1} \wedge V_j \wedge v_{j+1} \wedge \cdots \wedge v_m.
\]

The system \( Y' = U^{(m)} Y \) is just the system which is satisfied by \( m \)-vector \( Y = y_1 \wedge \cdots \wedge y_m \) constructed by solutions \( y_1, \ldots, y_m \) of the system \( y' = U y \). Systems (3.1) can be written, of course, in the similar form as the original system. The idea is to restrict our analysis with only some special particular solutions of (3.1), namely the solutions which have a minimal growth (among the solutions of the same auxiliary system) as \( x \to 0 \) or as \( x \to \infty \). The key technical observation consists with the fact that such solutions of (3.1) can be constructed as solutions of certain Volterra integral equations.

In order to proceed in the way described above, we must introduce some notation. Namely,

- \( \mathcal{A}_m \) denotes the set of all ordered multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_m), \) \( \alpha_1 < \alpha_2 < \cdots < \alpha_m, \) \( \alpha_j \in \{1, 2, \ldots, n\}. \)
- For a set of vectors \( u_1, \ldots, u_n \) from \( \mathbb{C}^n \) and a multi-index \( \alpha \in \mathcal{A}_m \), define \( u_\alpha := u_{\alpha_1} \wedge \cdots \wedge u_{\alpha_m}. \)
- Let \( \{a_1, \ldots, a_n\} \) be a numerical sequence. For \( \alpha \in \mathcal{A}_m \), define \( a_\alpha := \sum_{j \in \alpha} a_j; \) for \( k \in \{1, \ldots, n\} \), denote \( a_k := \sum_{j=1}^n a_j, \) \( a^k := \sum_{j=k}^n a_j. \)
- For a multi-index \( \alpha \), the symbol \( \alpha' \) denotes the ordered multi-index that complements \( \alpha \) to \( (1, 2, \ldots, n). \)
- For \( h \in \wedge^n \mathbb{C}^n \), we define \( |h| \) as a constant in the representation \( h = |h| e_1 \wedge e_2 \wedge \cdots \wedge e_n, \) where (and everywhere below) \( e_1, e_2, \ldots, e_n \) is the standard basis in \( \mathbb{C}^n. \)
- The space \( \wedge^m \mathbb{C}^n \) will be considered with \( l_1 \) norm:

\[
\left\| \sum_{\alpha \in \mathcal{A}_m} h_\alpha e_\alpha \right\| := \sum_{\alpha \in \mathcal{A}_m} |h_\alpha|.
\]

As above, we fix some arbitrary open sector \( \mathcal{S} \subset \mathbb{C} \setminus \Sigma. \)

Consider the following Volterra integral equations

\[
Y(x) = T^0_k(x, \rho) + \int_0^x G_{n-k+1}(x, t, \rho) \left(q^{(n-k+1)}(t)Y(t)\right) dt,
\]

(3.2)

\[
Y(x) = F^0_k(x, \rho) - \int_x^\infty G_k(x, t, \rho) \left(q^{(k)}(t)Y(t)\right) dt,
\]

(3.3)

where

\[
T^0_k(x, \rho) := C_k(x, \rho) \wedge \cdots \wedge C_n(x, \rho),
\]

\[
F^0_k(x, \rho) := E_1(x, \rho) \wedge \cdots \wedge E_k(x, \rho) = \Psi_{01}(x, \rho) \wedge \cdots \wedge \Psi_{0k}(x, \rho)
\]
and $G_m(x, t, \rho)$ is the operator acting in $\wedge^m C^n$ as follows:

$$G_m(x, t, \rho) f = \sum_{\alpha \in A_m} \sigma_\alpha \left| f \wedge C_{\alpha'}(t, \rho) \right| C_\alpha(x, \rho)$$

$$= \sum_{\alpha \in A_m} \chi_\alpha \left| f \wedge \Psi_{\alpha'}(t, \rho) \right| \Psi_{\alpha}(x, \rho) = \sum_{\alpha \in A_m} \chi_\alpha \left| f \wedge E_{\alpha'}(t, \rho) \right| E_\alpha(x, \rho).$$

Here $\sigma_\alpha = |b_\alpha \wedge b_{\alpha'}|$, $\chi_\alpha = |f_\alpha \wedge f_{\alpha'}|$.

Repeating the arguments used in proof of [15, Theorem 3.1], we obtain the following result.

**Proposition 1.** Suppose that $q \in L_1(0, \infty)$. Then, for any $\rho \in \mathfrak{S} \setminus \{0\}$, equations (3.2) and (3.3) have unique solutions $T_k(\cdot, \rho)$ and $F_k(\cdot, \rho)$, respectively, such that

$$\|T_k(x, \rho)\| \leq M \left\{ \begin{array}{ll} \left| (\rho x) \mathcal{E}_k \right|, & |\rho x| \leq 1, \\ \exp (\rho x \mathcal{R}_k), & |\rho x| > 1, \end{array} \right.$$  

$$\|F_k(x, \rho)\| \leq M \left\{ \begin{array}{ll} \left| (\rho x) \mathcal{E}_k \right|, & |\rho x| \leq 1, \\ \exp (\rho x \mathcal{R}_k), & |\rho x| > 1. \end{array} \right.$$  

The following asymptotics hold:

$$T_k(x, \rho) = \exp (\rho x \mathcal{R}_k) (f_1 \wedge \cdots \wedge f_k + o(1)), \quad x \to \infty,$$

$$T_k(x, \rho) = (\rho x) \mathcal{E}_k (h_1 \wedge \cdots \wedge h_n + o(1)), \quad x \to 0.$$

In what follows we consider the fundamental tensors as functions of *three* arguments $q, x, \rho$, where the potential $q$ will be assumed to belong to $X_p$, where $X_p$ denotes the Banach space consisting of all off-diagonal matrix functions with entries from $X_p := L_1(0, \infty) \cap L_p(0, \infty)$.

For a closed unbounded subset $L$ of the closed sector $\mathfrak{S}$ we denote by $C_0(L)$ the Banach space of continuous on $L$ and vanishing at infinity functions with the standard sup norm.

**Theorem 1.** Suppose that $p > 2$. Then:

1. The function $T_k(q, x, \rho)$, $q \in X_p, x \in (0, \infty), \rho \in \mathfrak{S} \setminus \{0\}$ admits the representation:

$$T_k(q, x, \rho) = T_k(q, x, \rho) + \mathcal{H}(q) T_k(q, x, \rho),$$

where $T_k \in C(X_p, BC([0, \infty), C_0(\mathfrak{S}))$. Moreover, for any ray of the form $\{\rho = zt, t \in [0, \infty)\}$ with $z \in \mathfrak{S} \setminus \{0\}$ the restriction $T_k|_l$ belongs to the space $C(X_p, BC([0, \infty), \mathcal{H}(l)))$, where $\mathcal{H}(l) := C_0(l) \cap L_2(l)$.

II. The function $F_k(q, x, \rho)$, $q \in X_p, x \in (0, \infty), \rho \in \mathfrak{S} \setminus \{0\}$ admits the representation:

$$F_k(q, x, \rho) = F_k(q, x, \rho) + \mathcal{H}(q) F_k(q, x, \rho),$$

where $F_k \in C(X_p, BC([0, \infty), C_0(\mathfrak{S}))$. Moreover, for any ray $\{\rho = zt, t \in [0, \infty)\}$ with $z \in \mathfrak{S} \setminus \{0\}$ the restriction $F_k|_l$ belongs to the space $C(X_p, BC([0, \infty), \mathcal{H}(l)))$.

**Proof.** 1. The substitution

$$T_k(q, x, \rho) = T_k(q, x, \rho) + \mathcal{H}(q) T_k(q, x, \rho)$$

converts (3.2) to the form

$$\hat{T}_k(q, x, \rho) = \mathcal{K}(q) \hat{T}_k(q, x, \rho) + \hat{T}_k(q, x, \rho),$$

where

$$\hat{T}_k(q, x, \rho) = \left( W_k(q) \right)^{-1} \int_0^x G_{n-k+1}(x, t, \rho)(q^{(n-k+1)}(t) T_k(t, \rho)) dt,$$
\( \mathcal{K}(q) \) — linear operator mapping a function \( f = f(x, \rho) \) into the function:

\[
(\mathcal{K}(q)f)(x, \rho) = \int_0^x G_{n-k+1}(x, t, \rho) \left( q^{(n-k+1)}(t)f(t, \rho) \right) dt,
\]

(3.4)

\[
G_{n-k+1}(x, t, \rho) := \frac{\overline{W}_k(\rho t)}{W_k(\rho x)} G_{n-k+1}(x, t, \rho).
\]

From the results of [26] it follows that \( \hat{T}_k^1 \in C(\mathcal{X}_p, BC([0, \infty), C_0(\mathcal{S})) \) and, moreover, for any ray \( \{\rho = zt, t \in [0, \infty)\} \) with \( z \in \mathcal{S} \setminus \{0\} \), the restriction \( \hat{T}_k^1 \big|_l \) belongs to the space \( C(\mathcal{X}_p, BC([0, \infty), \mathcal{H}(l))) \).

I. 1) Let us show that the operator \( \mathcal{K}(q) \) defined by (3.4) acts and is continuous in \( BC([0, \infty), C_0(\mathcal{S})) \).

We notice first that the function \( G_{n-k+1}(x, x\tau, \rho) \) is continuous in \( (x, \tau, \rho) \in [0, \infty) \times [0, 1] \times \mathcal{S} \) and therefore for any \( f(\cdot, \cdot) \in BC([0, \infty), C_0(\mathcal{S})) \) the function

\[
(\mathcal{K}(q)f)(x, \rho) = \int_0^1 G_{n-k+1}(x, x\tau, \rho) \left( q^{(n-k+1)}(x\tau)f(x\tau, \rho) \right) x \, d\tau
\]

is continuous in \( (x, \rho) \in [0, \infty) \times \mathcal{S} \). Furthermore, (2.4) implies that the function \( G_{n-k+1}(x, x\tau, \rho) \) is bounded. This yields the estimate:

\[
\|(\mathcal{K}(q)f)(x, \rho)\| \leq M \int_0^x \|q(t)\| \|f(t, \rho)\| \, dt.
\]

(3.5)

From (3.5), one can deduce, in particular:

\[
\|(\mathcal{K}(q)f)(x, \rho)\| \leq M \sup_{t \in [0, x]} \|f(t, \rho)\| \int_0^x \|q(t)\| \, dt.
\]

(3.6)

For any \( f(\cdot, \cdot) \in BC([0, \infty), C_0(\mathcal{S})) \), we have

\[
\lim_{\rho \to \infty, \rho \in \mathcal{S}} \sup_{t \in [0, x]} \|f(t, \rho)\| = 0
\]

and, from (3.6), it follows that \( (\mathcal{K}(q)f)(x, \cdot) \in C_0(\mathcal{S}) \) for any fixed \( x \); moreover,

\[
\sup_{x \in [0, T]} \|(\mathcal{K}(q)f)(x, \rho)\| \to 0 \quad \text{as} \quad \rho \to \infty,
\]

and \( \rho \in \mathcal{S} \) for any finite \( T > 0 \). Since \( (\mathcal{K}(q)f)(\cdot, \cdot) \in C([0, \infty) \times \mathcal{S}) \), we have

\[
(\mathcal{K}(q)f)(\cdot, \cdot) \in C([0, \infty), C_0(\mathcal{S})).
\]

Finally, estimate (3.6) implies the following estimate:

\[
\|(\mathcal{K}(q)f)(x, \rho)\| \leq M \|f\|_{BC([0, \infty), C_0(\mathcal{S})} \cdot \|q\|_{L_1([0, \infty))}.
\]

(3.7)

From (3.7), we conclude that \( (\mathcal{K}(q)f)(\cdot, \cdot) \in BC([0, \infty), C_0(\mathcal{S})) \) and \( \mathcal{K}(q) \in \mathcal{L}(BC([0, \infty), C_0(\mathcal{S}))) \). Moreover, since the mapping \( q \to \mathcal{K}(q) \) is linear, it follows from (3.7) that the operator

\[
\mathcal{K}(q) \in \mathcal{L}(BC([0, \infty), C_0(\mathcal{S})))
\]

depends continuously on \( q \in \mathcal{X}_p \).
I. 2) Inequality (3.5) implies the following estimate for the iterated operators $K^r(q)$:

$$\|(K^r(q)f)(x, \rho)\| \leq \frac{M^r}{r!} \left( \int_0^x \|q(t)\| \, dt \right)^r \sup_{\tau \in [0,x]} \|f(\tau, \rho)\|. \quad (3.8)$$

This yields

$$\|K^r(q)\| \leq \frac{M^r}{r!} \|q\|_{L_1(0,\infty)}, \quad (3.9)$$

where the norm on the left-hand side assumes the $\mathcal{L}(BC([0, \infty), C_0(\mathcal{S})))$ norm. Therefore, the operator $Id - K(q)$ is invertible in $BC([0, \infty), C_0(\mathcal{S}))$ for any $q \in \mathcal{X}_p$, moreover, the operator $(Id - K(q))^{-1}$ depends continuously on $q$. Since

$$\hat{T}_k(q) = (Id - K(q))^{-1} \hat{T}_k^1(q),$$

this means that $\hat{T}_k \in C(\mathcal{X}_p, BC([0, \infty), C_0(\mathcal{S}))).$

I. 3) Consider an arbitrary ray $l = \{\rho = zt, t \in [0, \infty)\}$ with $z \in \mathcal{S} \setminus \{0\}$. For an arbitrary function $f \in BC([0, \infty), \mathcal{H}(l))$, the arguments from I. 1) are still valid and yield, in particular, that $(K(q)f)(x, \rho)$ is continuous in $(x, \rho) \in [0, \infty) \times l$. Moreover, for $\rho \in l$ inequalities (3.5), (3.6) are true and therefore $(K(q)f)(\cdot, \cdot) \in BC([0, \infty), C_0(l))$.

In what follows, we denote by $l^+(R)$ the open ray $l \cap \{\rho : |\rho| > R\}$. Using the estimates

$$\|(K(q)f)(x, \cdot)\|_{L_2(l)} \leq M \int_0^x \|q(t)\| \|f(t, \cdot)\|_{L_2(l)} \, dt,$$

$$\|(K(q)f)(x, \cdot)\|_{L_2(l^+(R))} \leq M \int_0^x \|q(t)\| \|f(t, \cdot)\|_{L_2(l^+(R))} \, dt,$$

with an arbitrary $R > 0$ and [26, Lemma 3.1], we deduce that $(K(q)f)(x, \cdot) \in L_2(l)$ for any $x \in [0, \infty)$ and, moreover, $(K(q)f)(\cdot, \cdot) \in BC([0, \infty), L_2(l))$. It also follows from (3.6), (3.10) that

$$\|(K(q)f)(x, \rho)\| \leq M \|f\|_{BC([0, \infty), C_0(l))} \|q\|_{L_1(0,\infty)},$$

$$\|(K(q)f)(x, \cdot)\|_{L_2(l)} \leq M \|f\|_{BC([0, \infty), L_2(l))} \|q\|_{L_1(0,\infty)},$$

which allows us to conclude that $K(q) \in \mathcal{L}(BC([0, \infty), \mathcal{H}(l)))$ and depends continuously on $q \in \mathcal{X}_p$.

I. 4) From (3.10), we deduce the estimate:

$$\|(K^r(q)f)(x, \cdot)\|_{L_2(l)} \leq \frac{M^r}{r!} \left( \int_0^x \|q(t)\| \, dt \right)^r \sup_{\tau \in [0,x]} \|f(\tau, \cdot)\|_{L_2(l)},$$

which, together with (3.8), imply an estimate similar to (3.9), but with the $\mathcal{L}(BC([0, \infty), \mathcal{H}(l)))$ norm on the left-hand side. Proceeding as in I. 2), we conclude that $\hat{T}_k \in C(\mathcal{X}_p, BC([0, \infty), \mathcal{H}(l)))$.

II. 1) Proceeding as in I. 1), we consider $\hat{F}_k(q, \cdot, \cdot)$ as a solution of the equation

$$\hat{F}_k(q) = K^+(q)\hat{F}_k(q) + \hat{F}_k^1(q), \quad \text{where} \quad (K^+(q)f)(x, \rho) = -\int_x^\infty G_k(x, t, \rho) \left( q^{(k)}(t)f(t, \rho) \right) \, dt,$$

while the free term $\hat{F}_k^1(q)$ can be evaluated by using the results of [26]. This yields

$$\hat{F}_k^1 \in C(\mathcal{X}_p, BC([0, \infty), C_0(\mathcal{S})))$$

and, for any ray $\{\rho = zt, t \in [0, \infty)\}$ with $z \in \mathcal{S} \setminus \{0\}$, we see that the restriction $\hat{F}_k^1|_l \in C(\mathcal{X}_p, BC([0, \infty), \mathcal{H}(l)))$. 

MATHEMATICAL NOTES Vol. 108 No. 6 2020
Since the function $G_k(x,t,\rho)$ is continuous in $(x,t,\rho)$, for any $f(\cdot,\cdot) \in \mathcal{BC}([0,\infty),C_0(\overline{S}))$ the function $(\mathcal{K}^+ (q)f)(x,\rho)$ is continuous in $(x,\rho) \in [0,\infty) \times \overline{S}$. From the boundedness of the function $G_k(x,t,\rho)$, we obtain the estimate
\[
\|(\mathcal{K}^+ (q)f)(x,\rho)\| \leq M \int_x^\infty \|q(t)\| \|f(t,\rho)\| \, dt. \tag{3.11}
\]

Let us show that
\[
\lim_{\rho \to \infty} \sup_{x \in [0,\infty)} \|(\mathcal{K}^+ (q)f)(x,\rho)\| = 0. \tag{3.12}
\]

Fix an arbitrary $\varepsilon > 0$ and choose $T_\ast = T_\ast (\varepsilon)$ such that
\[
M\|f\| \int_{T_\ast}^\infty \|q(t)\| \, dt < \frac{\varepsilon}{2},
\]
where $\|f\| = \|f\|_{\mathcal{BC}([0,\infty),C_0(\overline{S}))}$ and $M$ is the same as in (3.11). For the chosen (finite) $T_\ast$, one can find $R$ such that
\[
M\|f(x,\rho)\| \int_0^{T_\ast} \|q(t)\| \, dt < \frac{\varepsilon}{2}
\]
for all $x \in [0,T_\ast]$, $\rho \in \overline{S}$: $|\rho| > R$. It follows that, for $x \in [0,T_\ast]$ and $\rho \in \overline{S}$: $|\rho| > R$, we can write $\|(\mathcal{K}^+ (q)f)(x,\rho)\| < \varepsilon$, while, for $x \in [T_\ast,\infty)$, $\rho \in \overline{S}$ we have $\|(\mathcal{K}^+ (q)f)(x,\rho)\| < \varepsilon/2$. Thus, (3.12) is established. Since the function $(\mathcal{K}^+ (q)f)(x,\rho)$ is continuous in $x \in [0,\infty)$, it follows that $\rho \in \overline{S}$ and (3.12) implies $(\mathcal{K}^+ (q)f)(\cdot,\cdot) \in \mathcal{BC}([0,\infty),C_0(\overline{S}))$.

Finally, (3.11) implies the estimate
\[
\|(\mathcal{K}^+ (q)f)(x,\rho)\| \leq M\|q\|_{L^1([0,\infty))} \|f\|_{\mathcal{BC}([0,\infty),C_0(\overline{S}))},
\]
from which we deduce that the operator $\mathcal{K}^+ (q) \in \mathcal{L}(\mathcal{BC}([0,\infty),C_0(\overline{S})))$ and continuous with respect to $q \in \mathcal{X}_p$.

II. 2) From (3.11), we obtain the estimate
\[
\|((\mathcal{K}^+ (q))^r f)(x,\rho)\| \leq \frac{M^r}{r!} \left( \int_x^\infty \|q(t)\| \, dt \right)^r \sup_{\tau \in [x,\infty)} \|f(\tau,\rho)\|, \tag{3.13}
\]
which yields
\[
\|((\mathcal{K}^+ (q))^r) \| \leq \frac{M^r}{r!} \|q\|_{L^1([0,\infty))}^r, \tag{3.14}
\]
where the norm on the left-hand side assumes the $\mathcal{L}(\mathcal{BC}([0,\infty),C_0(\overline{S})))$ norm. Thus, one can conclude that the operator $Id - \mathcal{K}^+ (q)$ is invertible in $\mathcal{BC}([0,\infty),C_0(\overline{S}))$ for any $q \in \mathcal{X}_p$, moreover, the operator $(Id - \mathcal{K}^+ (q))^{-1}$ is continuous with respect to $q$. Therefore, we have proved that $\tilde{F}_k \in C(\mathcal{X}_p,\mathcal{BC}([0,\infty),C_0(\overline{S})))$.

II. 3) Consider an arbitrary ray $l = \{\rho = zt, t \in [0,\infty)\}$ with $z \in \overline{S} \setminus \{0\}$. For an arbitrary function $f \in \mathcal{BC}([0,\infty),\mathcal{H}(l))$ the arguments from II. 1) are still valid. In particular, one has $((\mathcal{K}(q)f)(\cdot,\cdot) \in \mathcal{BC}([0,\infty),C_0(l))$ and for $\rho \in l$ inequality (3.11) is true, which yields the estimate:
\[
\|(\mathcal{K}^+ (q)f)(x,\rho)\| \leq M\|q\|_{L^1([0,\infty))} \|f\|_{\mathcal{BC}([0,\infty),C_0(l))}. \tag{3.15}
\]
Furthermore, using the estimates

$$
\| (K^+(q) f)(x, \cdot ) \|_{L_2(l)} \leq M \int_x^{\infty} \| q(t) \| \| f(t, \cdot ) \|_{L_2(l)} \, dt,
$$

(3.16)

$$
\| (K^+(q) f)(x, \cdot ) \|_{L_2(l^+(R))} \leq M \int_x^{\infty} \| q(t) \| \| f(t, \cdot ) \|_{L_2(l^+(R))} \, dt,
$$

with an arbitrary $R > 0$ and [26, Lemma 3.1], we see that $(K^+(q) f)(x, \cdot ) \in L_2(l)$ for all $x \in [0, \infty)$, $(K^+(q) f)(\cdot , \cdot ) \in BC([0, \infty), L_2(l))$. Moreover, from (3.15) and the estimate

$$
\| (K^+(q) f)(x, \cdot ) \|_{L_2(l)} \leq M \| f \|_{BC([0, \infty), L_2(l))} \cdot \| q \|_{L_1([0, \infty)}
$$

it follows that $K^+(q)$ belongs to $\mathcal{L}(BC([0, \infty), \mathcal{H}(l)))$ and is continuous with respect to $q \in \mathcal{X}_p$.

II. 4) From (3.16), we obtain the estimate

$$
\| ((K^+(q))^r f)(x, \cdot ) \|_{L_2(l)} \leq M^r \left( \int_x^{\infty} \| q(t) \| \, dt \right)^r \sup_{\tau \in [x, \infty)} \| f(\tau, \cdot ) \|_{L_2(l)},
$$

which yields (together with (3.13)) estimates similar to (3.14), but with $\mathcal{L}(BC([0, \infty), \mathcal{H}(l)))$ norm on the left-hand side. Proceeding as in II. 2), we conclude that

$$
\hat{F}_k \in C(\mathcal{X}_p, BC([0, \infty), \mathcal{H}(l))).
$$

4. WEYL-TYPE SOLUTIONS

Suppose that $q(\cdot ) \in L_1([0, \infty), \rho \in \mathbb{C} \setminus \Sigma, k \in \{1, \ldots, n\}$.

Definition 1. A function $y(x), x \in (0, \infty)$ is called a $k$th Weyl-type solution if it satisfies (1.1) and the following asymptotics hold:

$$
y(x) = O\left(x^{\mu_k}\right), \quad x \to 0, \quad y(x) = \exp(\rho R_k x)(f_k + o(1)), \quad x \to \infty.
$$

Let $\{T_k(\cdot , \rho)\}_{k=1}^n$, and let $\{F_k(\cdot , \rho)\}_{k=1}^n$ be the fundamental tensors constructed in the previous section. We define the characteristic functions as follows:

$$
\Delta_k(\rho) := |F_{k-1}(x, \rho) \wedge T_k(x, \rho)|, \quad k = 2, \ldots, n, \quad \Delta_1(\rho) := 1.
$$

We can repeat the arguments of [15] and obtain the following results similar to [15, Theorem 4.1] and [15, Lemma 4.2].

Proposition 2. If $\Delta_k(\rho) \neq 0$, then the $k$th Weyl-type solution exists and is unique. Moreover, it coincides with the (unique for each fixed $x \in (0, \infty)$) solution $\Psi_k(x, \rho)$ of the following system of linear algebraic equations:

$$
F_{k-1}(x, \rho) \wedge \Psi_k(x, \rho) = F_k(x, \rho), \quad \Psi_k(x, \rho) \wedge T_k(x, \rho) = 0.
$$

Our further considerations will be concentrated mostly on the properties of Weyl-type solutions as functions of the spectral parameter $\rho$ and the potential $q(\cdot )$. In what follows, we shall use the notation $\Psi_k(q, x, \rho)$ for the $k$th Weyl-type solution and the notation $\Psi(q, x, \rho)$ for the matrix $\langle \Psi_1(q, x, \rho), \ldots, \Psi_n(q, x, \rho) \rangle$. We also use the notation: $\hat{\Psi}(q, x, \rho) := \Psi(q, x, \rho)(W(\rho x))^{-1}$, where $W(\xi) := \text{diag}(W_1(\xi), \ldots, W_n(\xi))$.

As above we fix an arbitrary open sector $S \subset \mathbb{C} \setminus \Sigma$ with a vertex at $\rho = 0$. 
Theorem 2. Consider the matrix
\[ \beta(q, x, \rho) := (\tilde{\Psi}_0(x, \rho))^{-1} \tilde{\Psi}(q, x, \rho). \]
Suppose that \( p > 2 \). Then the following representations hold:
\[ \beta_{jk}(q, x, \rho) = \frac{\delta_{i,k} + d_{jk}(q, x, \rho)}{1 + d_k(q, x, \rho)}, \]
where \( \delta_{j,k} \) is the Kronecker delta and \( d_{jk}, d_k \in C(\mathcal{X}_p, BC([0, \infty), C_0(\mathcal{S}))) \). Moreover, for any ray \( l = \{ \rho = z t, t \in [0, \infty) \}, z \in \mathcal{S} \setminus \{0\} \) the restrictions \( d_{jk} \big|_l, d_k \big|_l \) belong to \( C(\mathcal{X}_p, BC([0, \infty), \mathcal{H}(l))) \).

**Proof.** Our definition of the matrix \( \beta(q, x, \rho) \) is equivalent to the following relations:
\[ \tilde{\Psi}_k(q, x, \rho) = \sum_{j=1}^n \beta_{jk}(q, x, \rho) \tilde{\Psi}_{0j}(x, \rho), \quad k = 1, \ldots, n. \]
Fix an arbitrary \( k \in \{1, \ldots, n\} \) and some \( (q, x, \rho) \). Substituting the above relations into the system
\[ \tilde{F}_{k-1} \land \tilde{\Psi}_k = \tilde{F}_k, \quad \tilde{\Psi}_k \land \tilde{T}_k = 0, \]
where
\[ \tilde{F}_k(q, x, \rho) := (\tilde{W}^k(\rho x))^{-1} F_k(q, x, \rho), \quad \tilde{T}_k(q, x, \rho) := (\tilde{W}^k(\rho x))^{-1} T_k(q, x, \rho), \]
we obtain
\[ \begin{cases} \sum_{j=1}^n \beta_{jk} \tilde{F}_{k-1} \land \tilde{\Psi}_{0j} = \tilde{F}_k, \\ \sum_{j=1}^n \beta_{jk} \tilde{\Psi}_{0j} \land \tilde{T}_k = 0. \end{cases} \]
Thus, we arrive at the following system of linear algebraic equations with respect to the values \( \{\beta_{jk}\}_{j=1}^k \):
\[ \sum_{j=1}^n m_{ij} \beta_{jk} = u_i, \quad (4.1) \]
\[ m_{ij} = |\tilde{F}_{k-1} \land \tilde{\Psi}_{0j} \land \tilde{\Psi}_{0k} \land \cdots \land \tilde{\Psi}_{0,i-1} \land \tilde{\Psi}_{0,i+1} \land \cdots \land \tilde{\Psi}_{0n}|, \quad i = k, \ldots, n, \]
\[ m_{ij} = |\tilde{\Psi}_{01} \land \cdots \land \tilde{\Psi}_{0,i-1} \land \tilde{\Psi}_{0,i+1} \land \cdots \land \tilde{\Psi}_{0,k-1} \land \tilde{\Psi}_{0j} \land \tilde{T}_k|, \quad i = 1, \ldots, k-1, \]
\[ u_i = |\tilde{F}_k \land \tilde{\Psi}_{0k} \land \cdots \land \tilde{\Psi}_{0,i-1} \land \tilde{\Psi}_{0,i+1} \land \cdots \land \tilde{\Psi}_{0n}|, \quad u_i = 0, \quad i = 1, \ldots, k-1. \]

Using Theorem 1, from the relations given above, we deduce the following representations:
\[ m_{ij}(q, x, \rho) = \left( m_{ij}^0 \frac{W_i(q, \rho x)}{W_j(q, \rho x)} + m_{ij}(q, x, \rho) \right) \cdot \frac{1}{W_n(q, \rho x)}, \]
\[ m_{ij}^0 = |\tilde{F}_{k-1}^0 \land \tilde{\Psi}_{0j} \land \tilde{\Psi}_{0k} \land \cdots \land \tilde{\Psi}_{0,i-1} \land \tilde{\Psi}_{0,i+1} \land \cdots \land \tilde{\Psi}_{0n}|, \quad i = k, \ldots, n, \]
\[ m_{ij}^0 = |\tilde{\Psi}_{01} \land \cdots \land \tilde{\Psi}_{0,i-1} \land \tilde{\Psi}_{0,i+1} \land \cdots \land \tilde{\Psi}_{0,k-1} \land \tilde{\Psi}_{0j} \land \tilde{T}_k^0|, \quad i = 1, \ldots, k-1, \]
\[ u_i(q, x, \rho) = \left( u_i^0 \frac{W_i(q, \rho x)}{W_j(q, \rho x)} + u_i(q, x, \rho) \right) \cdot \frac{1}{W_n(q, \rho x)}, \]
\[ u_i^0 = |\tilde{F}_k^0 \land \tilde{\Psi}_{0k} \land \cdots \land \tilde{\Psi}_{0,i-1} \land \tilde{\Psi}_{0,i+1} \land \cdots \land \tilde{\Psi}_{0n}|, \quad i = k, \ldots, n. \]

Here the scalar functions \( \hat{m}_{ij} = \hat{m}_{ij}(q, x, \rho), \hat{u}_i = \hat{u}_i(q, x, \rho) \) are such that
\[ \hat{m}_{ij}, \hat{u}_i \in C(\mathcal{X}_p, BC([0, \infty), C_0(\mathcal{S}))) \]
and, moreover, for any ray \( l = \{ \rho = z t, t \in [0, \infty) \} \) and \( z \in \mathcal{S} \setminus \{0\} \), the restrictions \( \hat{m}_{ij} \big|_l, \hat{u}_i \big|_l \) belong to \( C(\mathcal{X}_p, BC([0, \infty), \mathcal{H}(l))) \).
Using Kramer’s rule to solve (4.1) and taking into account the relations

\[
\begin{align*}
    m_{ij}^0 &= \delta_{i,j}(-1)^{i-k+1}\Delta_k, & i &= 1, \ldots, k - 1, \\
    m_{ij}^0 &= \delta_{i,j}(-1)^{i-k+1}|f|, & i &= k, \ldots, n, \\
    u_i^0 &= \delta_{i,k}|f|
\end{align*}
\]

we obtain the desired assertion. \(\square\)

More detailed results may be obtained if the characteristic function is a priori known to possess only nonzero values on some subset of \(\mathcal{F}\).

**Definition 2.** Let \(L\) be some closed (possibly unbounded) subset of \(\mathcal{F}\). We say that a matrix function \(q \in \mathcal{X}_p\) belongs to \(G_0^p(L)\) if \(\prod_{k=1}^n \Delta_k(\rho) \neq 0\) for all \(\rho \in L\).

**Theorem 3.** Suppose that \(\Psi_k(q, x, \rho)\) is written in the following form:

\[
\Psi_k(q, x, \rho) = \Psi_{0k}(x, \rho) + W_k(\rho)\hat{\Psi}_k(q, x, \rho).
\]

Then, for any finite segment \([0, T]\) and any closed subset \(L \subset \mathcal{F}\), the following insertion holds: \(\hat{\Psi}_k \in C(G_0^p(L), C([0, T], C_0(L)))\).

**Proof.**

1) Using the same arguments as in the proof of [15, Theorem 4.4], we can deduce that, for any \(q \in G_0^p(L)\), the function \(\hat{\Psi}_k(q, x, \rho)\) admits a continuous extension onto the set \([0, \infty) \times L\). Moreover, for any fixed \((x, \rho) \in [0, \infty) \times L, k = 1, \ldots, n\), the tensors \(\hat{T}_k(q, x, \rho), \hat{F}_k(q, x, \rho)\) are decomposable. By virtue of [9, Lemma 7.1], the condition \(\Delta_k(\rho) \neq 0, \rho \in L\) guarantees the unique solvability (for any fixed \((x, \rho) \in [0, \infty) \times L)\) of the system of linear algebraic equations

\[
\hat{F}_{k-1} \land \hat{\Psi}_k = \hat{F}_k, \quad \hat{\Psi}_k \land \hat{T}_k = 0.
\]

Thus, we can conclude that \(\hat{\Psi}_k \in C(G_0^p(L), C([0, T] \times K))\) for any finite segment \([0, T]\) and any compact set \(K \subset L\). This is equivalent to \(\hat{\Psi}_k \in C(G_0^p(L), C([0, T], C(K)))\). Since the compact set \(K\) is arbitrary, we have also obtained \(\hat{\Psi}_k(q, \cdot, \cdot) \in C([0, T] \times L)\).

2) It follows from Theorem 2 that, for any fixed \(q \in \mathcal{X}_p\) one has \(\hat{\Psi}_k(q, x, \rho) \to 0\) as \(\rho \to \infty\) uniformly with respect to \(x \in [0, T]\) for any finite \(T\). Therefore, we have \(\hat{\Psi}_k(q, \cdot, \cdot) \in C([0, T], C_0(L))\) for any fixed \(q \in G_0^p(L)\).

3) Let a finite \(T > 0\) be arbitrary fixed. Fix an arbitrary \(q_0 \in G_0^p(L)\), and consider a sequence \(q_m \in G_0^p(L)\) converging to \(q_0\) in \(\mathcal{X}_p\) norm.

By virtue of Theorem 2, we have (using the same notation) \(d_k(q_0, \cdot, \cdot) \in BC([0, \infty), C_0(\mathcal{F}))\). Therefore, there exists a \(R > 0\) such that \(|d_k(q_0, x, \rho)| \geq 1/2\) for all \(x \in [0, T], \rho \in L : |\rho| > R\).

Further, it follows from Theorem 2 that:

\[
\sup_{x \in [0, T], \rho \in \mathcal{F}, |\rho| > R} |d_k(q_m, x, \rho) - d_k(q_0, x, \rho)| \to 0,
\]

\[
\sup_{x \in [0, T], \rho \in \mathcal{F}, |\rho| > R} |d_{jk}(q_m, x, \rho) - d_{jk}(q_0, x, \rho)| \to 0.
\]

Using this, we obtain

\[
\sup_{x \in [0, T], \rho \in \mathcal{F}, |\rho| > R} |\beta_{jk}(q_m, x, \rho) - \beta_{jk}(q_0, x, \rho)| \to 0
\]

as \(m \to \infty\). This yields

\[
\sup_{x \in [0, T], \rho \in \mathcal{F}, |\rho| > R} \|\hat{\Psi}_k(q_m, x, \rho) - \hat{\Psi}_k(q_0, x, \rho)\| \to 0, \quad m \to \infty,
\]

MATHEMATICAL NOTES Vol. 108 No. 6 2020
Therefore, the functions \( f \) which yield

\[
  \Phi_k(q_m, x, \rho) = \Phi_k(q_0, x, \rho)
\]

where

\[
  f = \Phi_k(q, x, \rho).
\]

Substituting (4.3) into (4.2), we arrive, after some algebra, at the following representations:

\[
  \lim_{m \to \infty} \| \hat{\Phi}_k(q_m, \cdot, \cdot) - \hat{\Phi}_k(q_0, \cdot, \cdot) \|_{C([0, T], C_0(L))} = 0.
\]

\[ \square \]

**Theorem 4.** Consider the representation:

\[
  \Psi_k(q, x, \rho) = \Psi_{0k}(x, \rho) + W_k(\rho x) \hat{\Psi}_k(q, x, \rho),
\]

Suppose \( p > 2 \). Then \( \hat{\Psi}_k \in C(G^p_0(l), C([0, T], C_0(L))) \) for any ray \( l = \{ \rho = \rho_0, t \in [0, \infty) \}, z \in \mathcal{S} \setminus \{0\} \) and any finite segment \([0, T]\).

**Proof.** Fix arbitrary \( T \in (0, \infty) \) and ray \( l = \{ \rho = \rho_0, t \in [0, \infty) \}, z \in \mathcal{S} \setminus \{0\} \). For arbitrary fixed \( q \in G^p_0(l), x \in [0, T], \rho \in L \), we consider the system of linear algebraic equations

\[
  F_{k-1} \wedge \Psi_k = F_k, \quad \Psi_k \wedge T_k = 0.
\]

After the substitutions (where \( W_j \) assumes \( W_j(\rho x) \))

\[
  F_{k-1} = F^0_{k-1} + \overline{W}^{k-1} \hat{F}_{k-1}, \quad F_k = F^0_k + \overline{W}^k \hat{F}_k, \quad T_k = T^0_k + \overline{W}^k \hat{T}_k, \quad \Psi_k = \Psi_{0k} + W_k \hat{\Psi}_k
\]

the system of linear algebraic equations becomes

\[
  \hat{F}^0_{k-1} \wedge \hat{\Psi}_k = f_k, \quad \hat{\Psi}_k \wedge \hat{T}^0_k = g_k,
\]

where

\[
  f_k = \hat{F}_k - \hat{F}_{k-1} \wedge \hat{\Psi}_{0k}, \quad g_k = -\hat{\Psi}_{0k} \wedge \hat{T}_k - \hat{\Psi}_k \wedge \hat{T}_k.
\]

It follows from Theorem 1 that the functions \( \hat{F}_{k-1}, \hat{F}_k, \hat{T}_k \) belong to the space \( C(X_{0p}, BC([0, \infty), C_0(L))) \). From Theorem 3, it follows that \( \hat{\Psi}_k, \hat{T}_k \in C(G^p_0(l), C([0, T], C_0(L))) \). Therefore, the functions \( f_k, g_k \) belong to \( C(G^p_0(l), C([0, T], C_0(L))) \).

We write the function \( \hat{\Psi}_k(q, x, \rho) \) in the form

\[
  \hat{\Psi}_k(q, x, \rho) = \sum_{j=1}^n \hat{\beta}_{jk}(q, x, \rho) \hat{\Psi}_{0j}(x, \rho).
\]

Substituting (4.3) into (4.2), we arrive, after some algebra, at the following representations:

\[
  \hat{\beta}_{jk} = (-1)^{j-k} (\Delta_{0k} \overline{W}^n)^{-1} \cdot |\hat{\Psi}_{01} \wedge \cdots \wedge \hat{\Psi}_{0j-1} \wedge \hat{\Psi}_{0j+1} \wedge \cdots \wedge \hat{\Psi}_{0k-1} \wedge g_k|,
\]

\[
  \hat{\beta}_{jk} = (-1)^{j-k} |f| (\overline{W}^n)^{-1} \cdot |f_k \wedge \hat{\Psi}_{0k} \wedge \cdots \wedge \hat{\Psi}_{0j-1} \wedge \hat{\Psi}_{0j+1} \wedge \cdots \wedge \hat{\Psi}_{0n}|,
\]

which yield \( \hat{\beta}_{jk}(\cdot, \cdot, \cdot) \in C(G^p_0(l), C([0, T], C_0(L))) \).

\[ \square \]

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