Lower bounds for warping functions on warped-product AHE manifolds

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Abstract
Let $\gamma$ be the conformal boundary of a warped product $C^{3,\alpha}$ AHE metric $g = g_M + u^2 h$ on $N = M \times F$, where $(F, h)$ is compact with unit volume and nonpositive curvature. We show that if $\gamma$ has positive Yamabe constant, then $u$ has a positive lower bound that depends only on $[\gamma]$.

1 Introduction
Let $\tilde{N} = N \cup \partial M$ be a smooth manifold with nonempty boundary $\partial N$. Let $\rho$ be a smooth defining function for $N$, i.e. $\rho$ is a nonnegative smooth function on $N$ such that $\rho^{-1}(0) = \partial N$ and $d\rho \neq 0$ on $\partial N$. A smooth metric $g$ on $N$ is said to be conformally compact if for some smooth defining function $\rho$, the tensor $\tilde{g} = \rho^2 g$ has a continuous extension to $\tilde{N}$, and the restriction of $\rho^2 g$ to $\partial N$ is a positive definite 2-tensor. We define

$$\Pi(g) = [\rho^2 g|_{\partial N}],$$

where $[\gamma]$ is the conformal class of $\gamma$. The metric $g$ is said to be $C^{m,\alpha}$ conformally compact if $\tilde{g} \in C^{m,\alpha}(\tilde{N})$. In particular, if $g$ is $C^{m,\alpha}$ conformally compact

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compact, then the induced metric on the boundary $\hat{g} = \bar{g}|_{\partial N}$ is a metric of class $C^{m,\alpha}$.

We say $g$ is asymptotically hyperbolic (AH) if the sectional curvatures of $g$ approach -1 on approach to the boundary. If $g$ is $C^{m,\alpha}$ conformally compact and $|d\rho|_\bar{g} = 1$, then $g$ is AH; see [3]. In general, we say $g$ is AH of order $C^{m,\alpha}$, if $g$ is conformally compact of order $C^{m,\alpha}$ and $|d\rho|_\bar{g} = 1$.

Many of the results and topics in the area of conformally compact metrics, in one way or another, describe the relationship between the interior metric $g$ with the conformal class of the induced boundary metric. A typical result makes geometric conclusions based on geometric conditions on the conformal class of the induced boundary (for example, its Yamabe invariant).

In this paper, we only consider the class of warped product conformally compact Einstein metrics. Let $\bar{N} = \bar{M} \times F$ be a smooth $(n + 1)$-dimensional manifold with boundary $\partial \bar{N} = \partial \bar{M} \times F$. Here and throughout, $F$ is a smooth compact $p$-dimensional manifold without boundary. We assume that $g$ is a warped-product metric:

$$g = g_M + u^2 h ,$$

where $g_M$ is a smooth complete metric on $M$, the warping function $u$ is smooth and positive on $M$, and $h$ is a smooth metric on $F$. To avoid confusion in the choice of the warping function, we will always assume $(F, h)$ has unit volume. Moreover, we assume that $g$ is Einstein, i.e. (after rescaling the metric) we have

$$\text{Ric}_g = -ng .$$

Let $E^{m,\alpha}_W(N)$ denote the set of warped product AHE metrics of class $C^{m,\alpha}$ on $N$ such that the metric on the fiber $F$ has nonpositive scalar curvature. Also let $\Pi_W$ denote the restriction of the boundary map to $E^{m,\alpha}_W$, and define

$$C^{m,\alpha}_W = \Pi_W(E^{m,\alpha}_W) .$$

We will prove the following theorem in section 3.

**Theorem 1.** Suppose $[\gamma] \in \text{Im}(\Pi_W)$ has positive Yamabe constant. If $g \in \Pi_W^{-1}(\gamma)$, then the warping function of $g$ has a positive lower bound that depends only on $[\gamma]$.

The main motivation for obtaining results of this sort is to study the properness of the map $\Pi_W$. Given a sequence in $C^{m,\alpha}_W$, one would like to show that there is a subsequence $\gamma_i$ such that $\Pi_W^{-1}(\gamma_i)$ is convergent in a suitable topology to a metric in $E^{m,\alpha}_W$. Such results have been obtained in the case of static circle actions on 4-manifolds [1].
2 Warped Product AHE metrics

In this section, we assume \( g \) is an AH Einstein metric on \( N \), and derive the induced equations on \( g_M \) and \( u \). Recall that a metric \( g \) on \( N \) is called Einstein, if its Ricci curvature tensor is a constant multiple of the metric, i.e.

\[
\text{Ric}_g = \Lambda g ,
\]

for some \( \Lambda \in \mathbb{R} \). In our case, \( N \) is AH, and so \( \Lambda < 0 \). In fact, by scaling the metric \( g \) if necessary, we shall assume \( g \) satisfies the equation (2). In this note, geometric notations such as \( g, \nabla, \) or \( \Delta \) refer to the metric \( g \). If we are dealing with corresponding notations with respect to \( g_M \) or \( h \), we will clarify our intentions by accompanying \( M \) or \( h \) as a subscript or a superscript. For example \( \Delta_M \) is the Laplacian on \((M, g_M)\), defined by

\[
\Delta_M u = \sum_{i=1}^{q} g_M \left( \nabla_{X_i}^M u, X_i \right),
\]

where \( X_i \)'s form an orthonormal basis.

**Lemma 2.** Let \((M^q, g_M)\) and \((F^p, h)\) be Riemannian manifolds such that \((N, g) = (M \times K, g_M + u^2 h)\), for some positive function \( u \in C^\infty(M) \). Suppose \((N, g)\) satisfies the Einstein equations (2). Then \((F, h)\) has constant Ricci curvature, and \( u \) satisfies:

\[
\frac{\Delta_M u}{u} = \frac{s_F}{p} - (p - 1) \frac{|\nabla u|^2}{u^2} + n ,
\]

where \( s_F \) is the scalar curvature of \((F, h)\).

**Proof.** Let \( T(x, y)N = T_xM \oplus T_yF \) be the natural splitting of the tangent space of \( N \) to horizontal and vertical vectors. We fix an orthonormal basis \( \{E_1, \ldots, E_q\} \) for \( T_xM \) and an orthonormal basis \( \{E_{q+1}, \ldots, E_{q+p}\} \) for \( T_yF \) (in the \( h \)-metric). Then, one calculates:

\[
\begin{align*}
\nabla_{E_i} E_j &= \nabla_{E_i}^M E_j, \ i, j \leq q , \quad (3) \\
\nabla_{E_\alpha} E_\beta &= \nabla_{E_\alpha}^h E_\beta - g(E_\alpha, E_\beta) \frac{\nabla u}{u} , \ q + 1 \leq \alpha, \beta , \quad (4) \\
\nabla_{E_\alpha} E_i &= \nabla_{E_i} E_\alpha = \frac{E_i \cdot u}{u} E_\alpha , \ i \leq q < \alpha . \quad (5)
\end{align*}
\]
It follows that
\[ \nabla_{E_i} \nabla_{E_\alpha} E_\alpha = \nabla_{E_i} \left( \nabla^h_{E_\alpha} E_\alpha - u \nabla u \right) = \frac{E_i \cdot u}{u} \nabla^h_{E_\alpha} E_\alpha - \nabla_{E_i} (u \nabla u), \quad (6) \]
\[ \nabla_{E_\alpha} \nabla_{E_i} E_\alpha = \nabla_{E_\alpha} \left( \frac{E_i \cdot u}{u} E_\alpha \right) = \frac{E_i \cdot u}{u} \left( \nabla^h_{E_\alpha} E_\alpha - u \nabla u \right). \quad (7) \]

Hence,
\[ \sum_{i=1}^{q} R(E_i, E_\alpha, E_\alpha, E_i) = \sum_{i=1}^{q} g(-u \nabla_{E_i} \nabla u, E_i) = -u \Delta_M u. \]

Similarly, we compute:
\[ \sum_{\beta=q+1}^{q+p} R(E_\beta, E_\alpha, E_\alpha, E_\beta) = u^2 \sum_{\beta=q+1}^{q+p} R^h(E_\beta, E_\alpha, E_\alpha, E_\beta) - (p-1)|\nabla u|^2. \]

Since \( \text{Ric}_g(E_\alpha, E_\alpha) = -(p+q-1)u^2 \), we get (by adding the above two equations):
\[ -nu^2 = -u \Delta_M u + u^2 \text{Ric}_h(E_\alpha, E_\alpha) - (p-1)|\nabla u|^2, \]
and the lemma follows. \[ \square \]

Next, we derive the equation for \( \Delta \ln u \). First, we have
\[ \Delta \ln u = \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} = \frac{1}{u} \left( \Delta_M u + \sum_{\alpha=q+1}^{q+p} \frac{1}{u^2} g(\nabla_{E_\alpha} \nabla u, E_\alpha) \right) - \frac{|\nabla u|^2}{u^2}. \]

On the other hand,
\[ g(\nabla_{E_\alpha} \nabla u, E_\alpha) = \Delta_M u + \sum_{\alpha=q+1}^{q+p} g(\frac{|\nabla u|^2}{u} E_\alpha, E_\alpha) = \frac{|\nabla u|^2}{u}. \]

And so, we get:
\[ \Delta \ln u = \frac{\Delta_M u}{u} + (p-1) \frac{|\nabla u|^2}{u^2} = \frac{s_F}{p} + n. \quad (8) \]
3 Proof of Theorem 1

If $g$ is $C^{m,\alpha}$ conformally compact, then so is $g_M$. Moreover, if $\rho$ is a smooth defining function for $g$, then $\rho u$ extends to a $C^{m,\alpha}$ positive function on $\partial N$ which is in fact the warping function of the induced boundary metric.

**Lemma 3.** If $g$ is AHE of order $C^{3,\alpha}$, $0 < \alpha < 1$, then for any smooth defining function $\rho$ on $N$, there exists a smooth, strictly positive function $w$ on $N$ such that

1. $\Delta w = (p + q)w$ ,
2. $w - \rho^{-1}$ is bounded,
3. $\Delta(|dw|^2 - w^2) \geq 0$ ,
4. $|dw|^2 - w^2$ has a continuous extension to $\partial N$, and is equal to

$$-\frac{\hat{s}}{n(n-1)}$$

on the boundary, where $\hat{s}$ is the scalar curvature of the induced boundary metric $\hat{g} = \rho^2 g|_{\partial N}$.

**Proof.** These results are Propositions 4.1, 4.2, and 5.3 of [2].

Now, we are ready to present the proof of Theorem 1.

**Proof of Theorem 1** Let $\rho_1$ be a defining function for $(N, g)$ such that $\rho_1^2 g \in C^{3,\alpha}(\bar{N})$, and $\gamma = \rho_1^2 g|_{\partial N}$ be the induced boundary metric. Since $\mathcal{Y}[\gamma] > 0$, there exists a positive function $f$ such that $\eta = f^2 \gamma$ is the Yamabe representative of the conformal class $[\gamma]$, and

$$s_\eta = \mathcal{Y}[\gamma] > 0 ,$$

where $s_\eta$ is the scalar curvature of $\eta$. Then $\rho = f \rho_1$ is a defining function for $(N, g)$. By Lemma 3, there exists a function $w$ on $N$ such that $\Delta w = (n+1)w$. It follows that

$$\Delta \ln w = (n+1) - \frac{|dw|^2}{w^2} .$$

(9)
Equations 8 and 9 imply that
\[ \Delta \ln \frac{u}{w} = \frac{s_F}{p} - 1 + \frac{|dw|^2}{w^2} \leq \frac{|dw|^2 - w^2}{w^2}, \] (10)
since \( F \) has nonpositive Ricci curvature. By part (4) of the Lemma 3 we have:
\[ \max_{\partial N}(|dw|^2 - w^2) = -\frac{s_\eta}{n(n-1)}. \]
On the other hand, by part (3) of Lemma 3 we know that \(|dw|^2 - w^2\) is subharmonic, and so it attains its maximum on the boundary. It follows that
\[ |dw|^2 - w^2 < -\frac{s_\eta}{n(n-1)}, \] (11)
throughout \( N \). In particular, we have:
\[ w^2 > \frac{s_\eta}{n(n-1)}, \] (12)
throughout \( N \). Equations (10) and (11) imply that
\[ \Delta \ln \frac{u}{w} < 0, \]
i.e. \( \ln(u/w) \) is superharmonic, and so its minimum occurs on approach to the boundary. By part (2) of Lemma 3 we have \( \rho w = 1 \) on \( \partial N \). Hence
\[ \min_{\partial N} \ln \left( \frac{u}{w} \right) = \min_{\partial N} \ln \left( \frac{\rho u}{\rho w} \right) = \min_{\partial N} \ln(\rho u) = C(\eta). \]
Note that \( C(\eta) \) depends only on \( \eta \), since the extension of \( \rho u \) to the boundary is the warping function of \( \eta \).
\[ \ln \left( \frac{u}{w} \right) \geq \min_{\partial N} \ln(\rho u), \] (13)
throughout \( N \). Inequalities (12) and (13) imply that
\[ u \geq wC(\eta) \geq \left( \frac{\gamma [\gamma]}{n(n-1)} \right)^{1/2} C(\eta). \]
This completes the proof of the theorem.
References

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