THE DISCRIMINANTS ASSOCIATED TO ISOTROPY REPRESENTATIONS OF SYMMETRIC SPACES

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ABSTRACT. We consider a generalized discriminant associated to a symmetric space which generalizes the discriminant of real symmetric matrices, and note that it can be written as a sum of squares of real polynomials. A method to estimate the minimum number of squares required to represent the discriminant is developed and applied in examples.

1. INTRODUCTION

In his doctoral dissertation held in Königsberg in 1885, Minkowski proposed the conjecture that, unlike the quadratic case, nonnegative homogeneous polynomials of higher degree and more than two variables in general cannot be written as a sum of squares of real polynomials. The problem attracted the attention of Hilbert who in 1888 proved nonconstructively the existence of such polynomials. However, the first concrete example of a nonnegative polynomial which is not a sum of squares seems to have been given only in 1967 by Motzkin [Mot67]. The question of which nonnegative polynomials admit such representations is of interest in real algebraic geometry and practical importance in applied mathematics, but it is left open. On the other hand, Blekherman [Ble06] has shown that there are significantly more nonnegative polynomials than sums of squares of polynomials by computing asymptotic bounds on the sizes of these sets.

The discriminant of a real symmetric $n \times n$ matrix $Y$ is

$$\delta(Y) = \Pi_{i<j}(\theta_i(Y) - \theta_j(Y))^2,$$

where the $\theta_i(Y)$ are the eigenvalues of $Y$. It is well known that the discriminant $\delta(Y)$ is a nonnegative homogeneous polynomial of degree $n(n - 1)$ in the entries of $Y$, and it vanishes if and only if $Y$ has an eigenvalue of multiplicity bigger than one. In the nineteenth century, Kummer exhibited an explicit representation of $\delta$ as a sum of squares for $n = 3$ (the case $n = 2$ is immediate), and Borchardt generalized it for arbitrary $n$. More recently, several authors have rediscovered and refined these results in one or another form (see [Wat56, New73, Ily92, Lax98, Par02, Dom10] and the references therein). In particular, the approaches of Lax [Lax98] and Domokos [Dom10] (see also [Wat56]) make important use of the action of the orthogonal group on the space of symmetric matrices by conjugation: since conjugate matrices have the same set of eigenvalues, this action leaves the discriminant unchanged.

In this note, we remark that some of these results can be viewed in the realm of symmetric spaces (or, slightly more generally, polar representations). Indeed the isotropy representations of Riemannian symmetric spaces constitute a remarkable class of representations of compact Lie groups (see e.g. [BCO03, ch. 3] for a discussion). Herein we are interested in the functional which computes the volume of the orbits of those representations; it turns out that its square is an invariant polynomial that can be considered as a generalized discriminant associated to the symmetric space (the case of symmetric matrices alluded to above corresponds to the symmetric space $GL(n, \mathbb{R})/SO(n)$, see example below).

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More generally, the following result is probably well known, but perhaps has not been related to the present context.

**Theorem 1.** The functional computing the squared volume of the principal orbits of an orthogonal representation of a compact connected Lie group uniquely extends to an invariant homogeneous polynomial function on the representation space. Moreover, this polynomial can be expressed as a sum of squares of polynomials.

The number of squares involved in this representation is big (for instance, for isotropy representations of symmetric spaces of maximal rank this number is the binomial coefficient \( \binom{d}{r} \), where \( d \) and \( r \) are respectively the dimension and rank of the symmetric space). In the case of polar representations, it is possible to show the existence of expressions with much smaller number of squares, based on Theorem 4 below. In particular, for isotropy representations of symmetric spaces of maximal rank, we prove Theorem 5 below, which supplies an effective method to construct those expressions. Theorems 4 and 5 generalize some ideas from [Lax98, Dom10]. By applying this method, we recover

**Theorem 2 ([Dom10]).** The discriminant of \( n \times n \) real symmetric matrices (with respect to the orthogonal group) can be written as the sum of \( \binom{2n-1}{n-1} - \binom{2n-3}{n-1} \) squares.

We also apply our method to complex symmetric matrices, or the symmetric space \( Sp(n, \mathbb{R})/U(n) \). We prove

**Theorem 3.** The discriminant of \( n \times n \) complex symmetric matrices (with respect to the unitary group) can be written as the sum of \( 2 \binom{2n-1}{n} \) squares.

**Conventions.** Let a compact connected Lie group \( G \) act linearly on a real vector space \( V \). We consider the space \( \mathbb{R}[V] \) of real polynomials on \( V \), and note that \( G \) acts on a polynomial \( f \) by the rule \( (gf)(v) = f(g^{-1}v) \), where \( v \in V \). The space of invariants is denoted \( \mathbb{R}[V]^G \). The homogeneous component of degree \( m \) of \( \mathbb{R}[V] \) is denoted \( \mathbb{R}[V]_m \). An element \( v \in V \) and its orbit \( Gv \) are called regular if \( \dim Gv \) is maximal amongst \( G \)-orbits, and singular otherwise. Suppose \( v \) is a regular element; then the dimension of its isotropy group \( G_v \) is as small as possible. In this case, \( Gv \) is called a principal orbit if in addition the number of connected components of \( G_v \) is as small as possible; otherwise, \( Gv \) is called an exceptional orbit. We also consider the complexified representation on \( V^c \) (of \( G \) or \( G^c \)) and the space of complex polynomials \( \mathbb{C}[V^c] \). On the other hand, if \( W \) is a already a complex representation, \( W^r \) denotes its realification.

## 2. The volume functional as a sum of squares

Let \( V \) be a real orthogonal finite-dimensional representation space of a compact connected Lie group \( G \). The principal orbit type theorem asserts that the union of all principal orbits is an open, dense, invariant set with connected orbit space. Indeed every principal orbit is a homogeneous Riemannian manifold of \( G/H \)-type with \( G \)-invariant metric induced from \( V \), where \( H \) is a fixed closed subgroup of \( G \) (more generally, if \( v \) is a regular point, the orbit \( Gv \) is finitely covered by \( G/H \)), and every orbit meets the fixed point subspace \( V^H \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \) equipped with an \( \text{Ad}_H \)-invariant inner product, and let \( \mathfrak{h} \) be the Lie algebra of \( H \). Fix orthonormal bases \( x_1, \ldots, x_m \) of \( \mathfrak{h}^c \) and \( v_1, \ldots, v_d \) of \( V \). Denote by \( \tilde{x}_1, \ldots, \tilde{x}_m \in \mathfrak{so}(V) \) the induced Killing fields on \( V \), namely \( \tilde{x}(v) = x_1 \cdot v \) for \( v \in V \). For a point \( v \in V^H \), it is easy to compute the Jacobian determinant of the \( G \)-equivariant diffeomorphism \( G/G_v \to Gv \) to deduce that the \( m \)-dimensional volume of the orbit \( Gv \) is

\[
\text{vol}(Gv) = \begin{cases} \\
\frac{1}{k} ||\tilde{x}_1(v) \wedge \cdots \wedge \tilde{x}_m(v)|| \ \text{vol}(G/H) & \text{if } Gv \text{ is a principal orbit,} \\
||\tilde{x}_1(v) \wedge \cdots \wedge \tilde{x}_m(v)|| \ \text{vol}(G/H) & \text{if } Gv \text{ is an exceptional orbit,} \\
0 & \text{if } Gv \text{ is a singular orbit,}
\end{cases}
\]
where \( k \) is the index of the covering \( G/H \rightarrow Gv \). The value of the constant \( \text{vol}(G/H) \) is unimportant, and in the sequel we shall normalize the metric in \( g \) so that this constant becomes 1.

**Definition 1.** The **discriminant** of the representation \((G, V)\) is the \( G \)-invariant function on \( V \) defined by

\[
\delta(v) = \begin{cases} 
k^2 \text{vol}^2(Gv) & \text{if } v \text{ is a regular point}, \\
0 & \text{if } v \text{ is a singular point},
\end{cases}
\]

where \( v \in V \) and \( k \) is the index of the covering \( G/H \rightarrow Gv \) \((k = 1 \text{ if } Gv \text{ is a principal orbit; in particular, if there are no exceptional orbits, then } \delta(v) = \text{vol}^2(Gv) \text{ for all } v \in V)\).

If \( v \in V^H \), the formula

\[
\delta(v) = ||\tilde{x}_1(v) \wedge \cdots \wedge \tilde{x}_m(v)||^2 = \det (\langle \tilde{x}_i(v), \tilde{x}_j(v) \rangle)
\]

shows that \( \delta|_{V^H} \) is a homogeneous polynomial of degree \( 2m \) on \( V^H \). The restriction \( \delta|_{V^H} \) must also be invariant under the subgroup of \( G \) stabilizing \( V^H \), which is the normalizer \( N(H) \) of \( H \) in \( G \). The Luna-Richardson theorem \([LK79]\), asserting that the restriction map from \( V \) to \( V^H \) induces a graded algebra isomorphism \( R[V]^G \rightarrow R[V^H]^N(H) \), can now be used to conclude that \( \delta \) is a homogeneous polynomial of degree \( 2m \) on \( V \). Recall that by a result of Hilbert and Hurwitz, the algebra \( R[V]^G \) is finitely generated.

On the other hand, it is very easy to make explicit the polynomial structure of \( \delta \). We shall recall that while introducing some new objects. Take an orthonormal basis \( \{v_{i_1} \wedge \cdots \wedge v_{i_m}\} \) of \( \Lambda^m V \), where \( \{i_1 < \cdots < i_m\} \) is an increasing multi-index, and write, for \( v \in V^H \),

\[
\delta(v) = ||\tilde{x}_1(v) \cdots \wedge \tilde{x}_m(v)||^2 \\
= \sum_{1 \leq i_1 < \cdots < i_m \leq d} \langle \tilde{x}_1(v) \wedge \cdots \wedge \tilde{x}_m(v), v_{i_1} \wedge \cdots \wedge v_{i_m} \rangle^2 \\
= \sum_{1 \leq i_1 < \cdots < i_m \leq d} [\det ||(x_b \cdot v, v_{i_a})||_{a,b=1}^m]^2.
\]

Next, we consider the morphism \( \rho : V \rightarrow \text{Hom}(g, V) \) encoding the representation, namely \( \rho(v)(x) = x \cdot v \) for \( v \in V \), \( x \in g \). Plainly, \( \rho \) is \( G \)-equivariant: \( \rho(gv) = g \rho(v) \text{Ad}_g^{-1} \) for \( g \in G \). Extend \( \{x_j\}_{j=1}^p \) to a basis \( \{x_j\}_{j=1}^p \) of \( g \). Then \( \rho(v) \) is represented by the matrix \((\rho(v)_{ij})\), where \( \rho(v)(x_j) = \sum_{i=1}^d \rho(v)_{ij} v_i \) for \( j = 1, \ldots, p \). Since \( (x_b \cdot v, v_{i_a}) = \rho(v)_{i_a,b} \), we have

\[
\delta(v) = \sum_{1 \leq i_1 < \cdots < i_m \leq d} [\det ||\rho(v)_{i_a,b}||_{a,b=1}^m]^2
\]

for \( v \in V^H \) and, in fact, we can write

\[
\delta(v) = \sum_{1 \leq i_1 < \cdots < i_m \leq d} [\det ||\rho(v)_{i_a,j_0}||_{a,b=1}^m]^2
\]

as the last \( p - m \) columns of \((\rho(v)_{ij})\) contain only zeros.

Define a morphism \( \Phi : \Lambda^m g \otimes \Lambda^m V \rightarrow R[V]^m \) by setting

\[
\Phi(x_{j_1} \wedge \cdots \wedge x_{j_m} \otimes v_{i_1} \wedge \cdots \wedge v_{i_m})(v) = (x_{j_1}(v) \wedge \cdots \wedge x_{j_m}(v), v_{i_1} \wedge \cdots \wedge v_{i_m}).
\]

Then \( \Phi \) is equivariant, so by Schur’s lemma, it maps a \( G \)-irreducible component of \( \Lambda^m g \otimes \Lambda^m V \) either to zero or onto an isomorphic \( G \)-irreducible component of \( R[V]^m \). Note that \( \Phi(x_{j_1} \wedge \cdots \wedge x_{j_m} \otimes v_{i_1} \wedge \cdots \wedge v_{i_m})(v) = \det ||\rho(v)_{i_a,j_0}||_{a,b} \) for all \( v \in V \), so the image of \( \Phi \) lies in the ideal of polynomials vanishing along the variety of singular points.

The following lemma has been used in different contexts (e.g. [Dom10] Lemma 2.1 or [BCO03] p. 53)].
Lemma 1. Let $V, W$ be real orthogonal representations of a compact Lie group $G$, and let $\Psi : W \to \mathbb{R}[V]$ be a $G$-equivariant map. Then, for an orthonormal basis $\{w_i\}$ of $W$, the polynomial $f = \sum_i \Psi(w_i)^2$ is $G$-invariant and independent of the basis used to construct it.

Proof. Let $v \in V$, $g \in G$. Since the $G$-action on $W$ is orthogonal, there is an orthogonal matrix $(a_{ij})$ such that $gw_j = \sum_i a_{ij}w_i$. Now

$$f(g^{-1}v) = \sum_j \Psi(w_j)(g^{-1}v)^2$$
$$= \sum_j \Psi(gw_j)(v)^2$$
$$= \sum_j \left[ \sum_i a_{ij} \Psi(w_i)(v) \right]^2$$
$$= \sum_{i,k} \left( \sum_j a_{ij}a_{kj} \right) \Psi(w_i)(v)\Psi(w_k)(v)$$
$$= \sum_i \Psi(w_i)(v)^2$$
$$= f(v),$$
as desired. The last assertion in the statement is proven similarly. $\square$

It follows from the lemma that the right hand-side of equation (1) represents a $G$-invariant polynomial on $V$. Hence we have found a polynomial expression for $\delta$, namely

$$\delta = \sum_{I,J} [\Phi(x_J \otimes v_I)]^2,$$

where $I, J$ are increasing multi-indices; indeed, this is a sum of squares whose number is the dimension of the domain $\Lambda^m g \otimes \Lambda^m V$ of $\Phi$. This proves Theorem 1 stated in the introduction. The lemma also says that, for every $G$-irreducible subspace $W$ of $\Lambda^m g \otimes \Lambda^m V$ with $\Phi(W) \neq 0$, and $\{w_i\}$ an orthonormal basis of $W$, the polynomial $f_W := \sum_i \Phi(w_i)^2$ is $G$-invariant.

2.1. The case of polar representations. We can get better results if we assume that $(G, V)$ is polar, as we henceforth do. This means there exists a subspace $c \subset V$, called a Cartan subspace, that meets every $G$-orbit, and meets always orthogonally. We can choose the basis of $V$ so that $v_1, \ldots, v_r \in c$, where $r = \dim c$. Then the tangent spaces $T_v(Gv)$ for regular $v \in c$ are all parallel and indeed spanned by $v_{r+1}, \ldots, v_d$. Now the matrix $\rho(v)$ for regular $v \in c$ has the block form

$$\begin{pmatrix} 0 & 0 \\ \ast & 0 \end{pmatrix}.$$  

We recall that polar representations do not admit exceptional orbits [BCO03, Cor. 5.4.3].

Consider the special element

$$\vartheta = x_1 \wedge \cdots \wedge x_m \otimes v_{r+1} \wedge \cdots \wedge v_d \in \Lambda^m g \otimes \Lambda^m V.$$  

It follows from the block form of the above matrix that the restriction map $V \to c$ takes all determinants of minors $\Phi(x_J \otimes v_I)$ to zero but $\Phi(\vartheta)$. Therefore

$$\delta|_c = \Phi(\vartheta)^2|_c.$$  

and
\[ f_W|_c = \sum_i [\Phi(w_i) |_c]^2 \]
\[ = \sum_i [(\langle w_i, \partial \rangle \Phi(\partial) |_c]^2 \]
\[ = \left( \sum_i \langle w_i, \partial \rangle^2 \right) [\Phi(\partial) |_c]^2 \]
\[ = c \cdot \delta |_c, \]

where \( c \) denotes a positive constant. Since \( f_W \) and \( \delta \) are both \( G \)-invariant polynomials, this implies that \( f_W = c \cdot \delta \) on \( V \). Hence:

**Theorem 4.** If \( (G, V) \) is polar, then each \( G \)-irreducible component of the image of \( \Phi \) gives rise to a decompositions of \( \delta \) as a sum of squares of polynomials.

**Remark 1.** Polarity is necessary in the statement of Theorem 4. In fact, the diagonal action of \( SO(2) \) on \( \mathbb{R}^2 \oplus \mathbb{R}^2 \) is not polar. Let \( \mathfrak{so}(2) = \langle x \rangle \). Then
\[ x \cdot \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -a_2 \\ a_1 \\ -b_2 \\ b_1 \end{pmatrix} \]
and
\[ \rho \left( \begin{array}{c} a_1 \\ a_2 \\ b_1 \\ b_2 \end{array} \right) = \begin{pmatrix} -a_2 \\ a_1 \\ -b_2 \\ b_1 \end{pmatrix}. \]

Of course, \( \delta = a_1^2 + a_2^2 + b_1^2 + b_2^2 \) is \( SO(2) \)-invariant. On the other hand, \( m = 1 \) and \( \Lambda^n \mathfrak{g} \otimes \Lambda^n V = \mathfrak{so}(2) \otimes (\mathbb{R}^2 \oplus \mathbb{R}^2) \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \). By taking \( W = \mathbb{R}^2 \oplus 0 \) or \( 0 \oplus \mathbb{R}^2 \), we get the \( SO(2) \)-invariant \( f_W = a_1^2 + a_2^2 \) or \( b_1^2 + b_2^2 \), each of which is different from \( \delta \).

**Baby example.** The direct product action \( (SO(2) \times SO(2), \mathbb{R}^2 \oplus \mathbb{R}^2) \) is clearly polar with rank \( r = 2 \). Let \( x_1, x_2 \) be generators of the summands in \( \mathfrak{so}(2) + \mathfrak{so}(2) \), let \( \{v_1, v_2; v_3, v_4\} \) be an orthonormal basis of \( \mathbb{R}^2 \oplus \mathbb{R}^2 \), and let \( \{a_1, a_2; b_1, b_2\} \) be the dual basis. Then
\[ x_1 \cdot \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -a_2 \\ a_1 \\ 0 \\ 0 \end{pmatrix} \]
and \[ x_2 \cdot \begin{pmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -b_2 \\ b_1 \end{pmatrix}, \]
and
\[ \rho \left( \begin{array}{c} a_1 \\ a_2 \\ b_1 \\ b_2 \end{array} \right) = \begin{pmatrix} -a_2 \\ a_1 \\ 0 \\ -b_2 \\ 0 \\ b_1 \end{pmatrix}. \]

Here \( \delta = a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + a_4^2 b_4^2 = (a_1^2 + a_2^2)(b_1^2 + b_2^2) = f_1 f_2 \), where \( f_1 = a_1^2 + a_2^2, f_2 = b_1^2 + b_2^2 \) is a complete system of invariants. Moreover, \( m = 2 \) and
\[ \Lambda^2 \mathfrak{g} \otimes \Lambda^2 V \cong \Lambda^2 (\mathbb{R}^2 \oplus \mathbb{R}^2) \cong \Lambda^2 \mathbb{R}^2 \oplus \Lambda^2 \mathbb{R}^2 \oplus \mathbb{R}^2 \otimes \mathbb{R}^2. \]
The first two summands on the left hand-side are spanned by \( \{v_1 \wedge v_2, v_3 \wedge v_4\} \) and mapped to zero under \( \Phi \). On the other hand, \( \mathbb{R}^2 \otimes \mathbb{R}^2 \) decomposes into \( \mathbb{R}^2 \oplus \mathbb{R}^2 \) yielding
\[ \delta = (a_1 b_1 + a_2 b_2)^2 + (a_1 b_2 - a_2 b_1)^2 = (a_1 b_1 - a_2 b_2)^2 + (a_1 b_2 + a_2 b_1)^2. \]
3. Systems of restricted roots for symmetric spaces

Dadok \cite{Dad85} has shown that a polar representation of a compact Lie group has the same orbits as the isotropy representation of a symmetric space, so we shall assume, with no loss of generality, that \((G, V)\) is already the isotropy representation of a symmetric space. This means that \(I := g + V\) has the structure of a Lie algebra of which \(g + V\) is the decomposition into \(±1\)-eigenspaces of an involutive automorphism of \(I\), and then the associated symmetric space is \(L/G\) where \(L\) is a Lie group with Lie algebra \(I\). However, for the sake of tradition, henceforth we change the notation and write \((K, p)\) (\(K\) connected) for the basic representation, \(g = \mathfrak{t} + p\) for the corresponding Lie algebra with involution, and \(G/K\) for the associated symmetric space. Now \((K, p)\) is equivalent to the isotropy representation of \(K\) on \(T_{1K}(G/K)\). Since a symmetric space and its dual have equivalent isotropy representations, we may restrict our attention to symmetric spaces of noncompact type. In this case the inner product on \(p\) is the restriction of the Cartan-Killing form of \(g\) to \(p\), and the inner product on \(\mathfrak{t}\) is the negative of the restriction of the Cartan-Killing form of \(g\) to \(\mathfrak{t}\).

It is instructive to express the discriminant associated to a symmetric space in terms of its system of restricted roots. A Cartan subspace of \(p\) is the same as a maximal Abelian subspace \(a\) of \(p\). The system of restricted roots of \(g\) with respect to \(a\) is the set of linear functionals \(\Lambda \subset a^* \setminus \{0\}\) such that \(\lambda \in \Lambda\) if and only if \(g_\lambda \neq 0\), where \(g_\lambda = \{x \in g | ad_a x = \lambda(a)x\} for all a \in a\) \cite[ch.6, \S1]{Loo69}. We have the real orthogonal restricted root decomposition \(g = t_0 + a + \sum_{\lambda \in \Lambda} g_\lambda\) where \(t_0\) denotes the centralizer of \(a\) in \(g\). Introduce a lexicographic order in the dual \(a^*\) (with respect to some basis) and let \(\Lambda^+\) denote the set of positive restricted roots. Since the involution of \(g\) interchanges \(g_\lambda\) and \(g_{-\lambda}\), we have decompositions

\[
\mathfrak{t} = t_0 + \sum_{\lambda \in \Lambda^+} g_\lambda \quad \text{and} \quad p = a + \sum_{\lambda \in \Lambda^+} p_\lambda,
\]

where \(t_\lambda = \mathfrak{t} \cap (g_\lambda + g_{-\lambda})\), \(p_\lambda = p \cap (g_\lambda + g_{-\lambda})\), and \(\dim g_\lambda = \dim V_\lambda = m_\lambda\) is defined to be the multiplicity of \(\lambda\). For each \(\lambda \in \Lambda^+\), there exist bases \(\{x_{\lambda,j}\}_{j=1}^{m_\lambda}\) and \(\{y_{\lambda,j}\}_{j=1}^{m_\lambda}\) of \(t_\lambda\) and \(p_\lambda\) resp., such that \(ad_a(x_{\lambda,j}) = \lambda(a)y_{\lambda,j}\) and \(ad_a(y_{\lambda,j}) = \lambda(a)x_{\lambda,j}\) for \(a \in a\) \cite[p.61]{Loo69}. It easily follows that

\[
(2) \quad \delta(a) = \Pi_{\lambda \in \Lambda^+} \lambda(a)^{2m_\lambda}
\]

for \(a \in a\).

Example 1. The symmetric space \(GL(n, \mathbb{R})/SO(n)\) has associated Cartan decomposition \(\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{so}(n) + \mathfrak{sym}(n, \mathbb{R})\) where \(\mathfrak{sym}(n, \mathbb{R})\) is the space of real symmetric \(n \times n\) matrices. A maximal Abelian subspace \(a\) is given by the subspace of diagonal matrices and then the restricted roots are \(\theta_i - \theta_j, i \neq j\), where \(\theta_i \in a^*\) is the \(i\)th diagonal coordinate, and one can take the positive ones to correspond to \(i < j\). All the multiplicities are one. The diagonal elements of a symmetric matrix are its eigenvalues. Hence in this case the discriminant of \((SO(n), \mathfrak{sym}(n, \mathbb{R}))\)

\[
\delta = \Pi_{i < j} (\theta_i - \theta_j)^2
\]

coincides with the usual discriminant of symmetric matrices, where \(\theta_1(Y), \ldots, \theta_n(Y)\) denote the eigenvalues of \(Y \in \mathfrak{sym}(n, \mathbb{R})\).

Some other examples of volume functionals of isotropy representations of symmetric spaces given in terms of formula (2) are listed in the final tables of \cite{HLZ1} (the spaces listed there are in fact the compact dual symmetric spaces, but the corresponding isotropy representations are equivalent).

The Weyl group of \(g\) with respect to \(a\), denoted \(W(a)\), is a finite group generated by reflections on the singular hyperplanes in \(a\) (which are the kernels of the restricted roots) and can be realized as the quotient of the normalizer of \(a\) in \(K\) by the centralizer of \(a\) in \(K\); it acts on \(a^*\) by permuting
the restricted roots. The Chevalley restriction theorem says that the restriction map from \( p \) to \( a \) induces an algebra isomorphism \( R[p]^K \to R[a]^{W(a)} \). We use this result in the proof of the following proposition.

**Proposition 1.** The discriminant of \((K, p)\) is an irreducible polynomial if and only if the diagram of restricted roots is connected and does not contain a multiple link, namely it is of type \( A_n \) \((n \geq 1)\), \( D_n \) \((n \geq 4)\), \( E_6 \), \( E_7 \) or \( E_8 \), and all the multiplicities are 1.

**Proof.** It is clear that \( \delta \) can be irreducible only if the representation \((K, p)\) is irreducible, which is to say that the diagram is connected; so we may as well make this assumption throughout.

We first claim that \( \delta \) is reducible if and only if there exists a nontrivial factorization of \( \delta|a \) into \( W(a) \)-invariant polynomials. In fact, suppose \( \delta = f \cdot g \) is a nontrivial factorization. Then \( \delta = \text{Ad}_k f \cdot \text{Ad}_k g \) for \( k \in K \) is a continuous family of nontrivial factorizations. Since a real polynomial has a unique decomposition into irreducible factors, up to permutation of the factors and multiplication by units, it is clear that \( f \) and \( g \) must be \( K \)-invariant polynomials (recall that \( K \) is assumed connected); thus, \( \delta|a = (f|a)(g|a) \) where \( f|a \) and \( g|a \) are \( W(a) \)-invariant. The converse follows from Chevalley’s theorem.

On the other hand, it is known that the action of \( W(a) \) preserves the length and the multiplicities of the roots, and is transitive on the sets of roots of the same length. Moreover, for each \( \lambda \in \Lambda^+ \), \( W(a) \) contains an element that maps \( \lambda \) to \( -\lambda \) and induces a permutation on \( \Lambda^+ \setminus \{\lambda\} \). Now it is clear that a nontrivial \( W(a) \)-invariant factor of \( \delta|a = \Pi_{\lambda \in \Lambda^+} \lambda^{2m_\lambda} \) exists if and only if \( W(a) \) is transitive on \( \Lambda \) or some multiplicity is bigger than 1. We finish the proof by noting that \( W(a) \) is transitive on \( \Lambda \) precisely if the diagram is one of those listed in the statement. \( \square \)

The zero set \( Z(\delta) \) of the discriminant is the singular set \( S \), which is the union of the singular orbits, and we now have a good description of its structure. The intersection \( S \cap a \), being the union of singular hyperplanes, is stratified by the intersections of the various subfamilies of singular hyperplanes, and \( S = \bigcup_{\lambda \in \Lambda^+} K(\ker \lambda) \) has a natural, induced invariant stratification, which coincides with the stratification by orbit types. Denote by \( (\ker \lambda)^0 \) the open dense subset of the singular hyperplane \( \ker \lambda \subset a \) consisting of points not lying in any other singular hyperplane. Then \( K((\ker \lambda)^0) \) is a stratum of \( S \) of dimension \( \dim p - 1 - m_\lambda \), as is easy to see. Therefore \( S \) has codimension \( 1 + \min_\lambda m_\lambda \geq 2 \). It is interesting to note that despite being defined as the zero set of a single polynomial, \( S \) has codimension bigger than one. Of course, this is related to the fact that \( \delta \) is a sum of squares of real polynomials.

4. Symmetric Spaces of Maximal Rank

Although being a sum of squares is ingrained in the definition of \( \delta \), in this section we propose to use Theorem 4 to find representations of it as sums of squares with as few terms as possible. In the case of the discriminant of real symmetric matrices, this problem has its roots in classical papers of Kummer and Borchardt, and more recently has been studied by Lax [Lax98] and Domokos [Dom10]. In practice, we need to decompose \( \Phi : \Lambda^m t \otimes \Lambda^m p \to R[p]_m \) into \( K \)-irreducible components and identify components which are not mapped to zero under \( \Phi \). In the sequel, we shall discuss this problem in the special case of locally free actions, namely, when the principal isotropy is discrete. For the symmetric space \( G/K \), this means that \( \text{rank}(G/K) = \text{rank}(G) \), i.e. it is of maximal rank; equivalently, \( G/K \) has uniform multiplicities 1. Then \( t_0 = 0, \dim t = m \text{ and } \Phi : \Lambda^m p \to R[p]_m \).

Note that \( m + r = \dim p \); in the sequel, it will be convenient to identify the representations \( \Lambda^m p \cong \Lambda^r p \) and view \( \Phi : \Lambda^r p \to R[p]_m \) (this can be done because \( p \) is a real orthogonal representation of \( K \)). Fix a basis \( \{y_1, \ldots, y_d\} \) of \( p \). Inspired by [Grü10], we define a linear map \( A : \Lambda^r p \to t \otimes \Lambda^{r-2} p \).
Theorem 5. Summarize this discussion in the following theorem.

Proof. Fix a Cartan subspace \( a \subset \mathfrak{p} \) and an orthonormal basis \( \{ y_i \}_{i=1}^d \) of \( \mathfrak{p} \) such that \( y_1, \ldots, y_r \in a \). Since the \( K \)-orbits meet \( a \) orthogonally, for \( a \in a \) we have

\[
\Phi(y_i_1 \wedge \cdots \wedge y_i_r)(a) \neq 0
\]

if and only if \( y_1 \wedge \cdots \wedge y_a \wedge \cdots \wedge y_b \wedge \cdots \wedge y_r = y_1 \wedge \cdots \wedge y_r \). However, in this case, \( y_a, y_b \in a \) implies that \( (ad_x y_a, y_b) = 0 \) for every \( x \in \mathfrak{t} \). This shows that \( \Phi \circ A^\ast|_a = 0 \). Let \( \{ \alpha_i \} \) be an orthonormal basis of \( \mathfrak{t} \otimes \Lambda^r \mathfrak{p} \). Since \( \Phi \circ A^\ast \) is \( K \)-equivariant, Lemma 1 says that \( f = \sum_i |(\Phi \circ A^\ast)(\alpha_i)|^2 \) is \( K \)-invariant, and we already know that \( f|_a = 0 \). It follows that \( f = 0 \) on \( \mathfrak{p} \). Hence \( \Phi \circ A^\ast(\alpha_i) = 0 \) for all \( i \) and thus \( \Phi \circ A^\ast = 0 \). \( \square \)

Since \( (\text{im}A^\ast)^\perp = \ker A \), in view of Proposition 2 we need only to consider the restriction of \( \Phi \) to the kernel of \( A \). It is often easier to deal with complex representations, so we complexify everything. Now our problem is equivalent to identifying \( \mathfrak{t}^c \)-irreducible components of \( \ker A^c \subset \Lambda^r(\mathfrak{p}^c) \) which are not mapped to zero under \( \Phi^c \). Let \( t \) be the Lie algebra of a maximal torus of \( K \). Then we have the (complex) root space decomposition

\[
\mathfrak{t} = \mathfrak{t}^c + \sum_{\alpha \in \Delta^+_{t^c}} (\mathfrak{t}^c)_{\alpha}
\]

and the weight space decomposition

\[
\mathfrak{p} = (\mathfrak{p}^c)_0 + \sum_{\alpha \in \Delta^-_{\mathfrak{p}^c}} (\mathfrak{p}^c)_{\alpha},
\]

where \( (\mathfrak{p}^c)_0 \) is the centralizer of \( \mathfrak{t}^c \) in \( \mathfrak{p}^c \), \( \mathfrak{t}^c + (\mathfrak{p}^c)_0 \) is a Cartan subalgebra of \( \mathfrak{g}^c \) and \( \dim(\mathfrak{t}^c)_{\alpha} = \dim(\mathfrak{p}^c)_{\alpha} = 1 \) (see e.g. [Pan01, p.15]). Choose a system of positive roots \( \Delta^+_{t^c} \subset \Delta^+_{t^c} \), and let \( \mathfrak{b} = \mathfrak{t}^c + \sum_{\alpha \in \Delta^+_{t^c}} (\mathfrak{t}^c)_{\alpha} \) denote the corresponding Borel subalgebra of \( \mathfrak{t}^c \). It is clear that the map \( \mathfrak{v} \mapsto [\Lambda^r \mathfrak{v}] \) sets up a bijective correspondence between the \( r \)-dimensional \( \mathfrak{b} \)-stable subspaces of \( \mathfrak{p}^c \) and the highest weight vectors of \( \Lambda^r(\mathfrak{p}^c) \).

One often combines the aforementioned Chevalley restriction theorem with the Chevalley theorem for finite reflection groups, which says that the algebra of invariants \( R[\mathfrak{g}]_{\mathfrak{w}(a)} \) is a free polynomial algebra, namely it has \( r \) algebraically independent homogeneous generators \( f_1, \ldots, f_r \). Now \( f_1, \ldots, f_r \) are also algebraically independent homogeneous generators for \( C[\mathfrak{p}^c]^{K^c} \). By a result of Panyushev [Pan84], \( y \in \mathfrak{p}^c \) is regular if and only if the set of linear forms on \( \mathfrak{p}^c \) \( \{ (df_1)_y, \ldots, (df_r)_y \} \) is linearly independent. One can find a \( r \)-dimensional complex subspace \( \mathfrak{v} \subset \mathfrak{p}^c \) (necessarily transversal to the orbit \( K^c \cdot y \)) such that the set of restricted linear forms \( \{ (df_1)_y|_\mathfrak{v}, \ldots, (df_r)_y|_\mathfrak{v} \} \) is linearly independent. Since \( \mathfrak{v} \) is transversal to \( K^c \cdot y \), \( \Phi^c([\Lambda^r \mathfrak{v}]) \neq 0 \). This remark is effective in the case \( \mathfrak{v} \) can be taken to be a \( \mathfrak{b} \)-stable subspace of \( \mathfrak{p}^c \), for in that case \([\Lambda^r \mathfrak{v}] \) is a highest weight vector of \( \Lambda^r(\mathfrak{p}^c) \), whence determines an irreducible component not mapped to zero under \( \Phi^c \). We summarize this discussion in the following theorem.

Theorem 5. Let \( \mathfrak{v} \) be a \( \mathfrak{b} \)-stable subspace of \( \mathfrak{p}^c \) such that for some \( y \in \mathfrak{p}^c \) the set of linear forms on \( \mathfrak{p}^c \) \( \{ (df_1)_y|_\mathfrak{v}, \ldots, (df_r)_y|_\mathfrak{v} \} \) is linearly independent. Then \([\Lambda^r \mathfrak{v}] \) is the highest weight vector of an irreducible component of \( \Lambda^r(\mathfrak{p}^c) \) which is not mapped to zero under \( \Phi^c \).
Remark 2. In this remark, assume for simplicity that \((K, p)\) is irreducible. Then the Casimir element \(\omega\) of \(\mathfrak{t}^c\) with respect to the Cartan-Killing form of \(g^c\) can be normalized so as to act as the identity on \(p^c\) (even if \((K, p)\) is not absolutely irreducible). We quote a result of Kostant [Kos65] and Panyushev [Pan01] stating that: The maximal eigenvalue of \(\omega\) on \(\Lambda^r(p^c)\) is \(r\). The corresponding eigenspace \(\mathcal{M}\) is spanned by decomposable elements. Indeed, \(\mathcal{M}\) is spanned by \([\lambda^r]\), where \(\lambda\) runs over all \(r\)-dimensional Abelian subalgebras of \(p^c\). Hence \(\mathcal{M} \subset \ker A\), but in general the inclusion is strict, as there are simple examples with \(r = 3\) in which \(\ker A\) is not spanned by decomposable elements. In any case, \(\mathcal{M}\) is a first approximation to \(\ker A\).

5. Applications

In this section we apply Theorem [5] to effectively compute upper bounds for the minimum number of squares required to represent \(\delta\) in some concrete examples. In particular, we recover Domokos estimate in the case of the discriminant of \(n \times n\) real symmetric matrices [Dom10 Thm. 6.2].

We start with the symmetric space \(G/K = Sp(n, \mathbb{R})/U(n)\). Its isotropy representation is the realification of the representation of \(U(n)\) on \(V = \text{Sym}(n, \mathbb{C})\) given by \(\rho(g)X = gXg^*\), so the associated discriminant can also be interpreted as the discriminant of complex \(n\)-ary quadratic forms. As Cartan subspace, one can take the set of real diagonal matrices.

The complexified symmetric space \(G^c/K^c\) has [GW09 p.594]

\[ G^c = Sp(2n, \mathbb{C}) = \{g \in M(n, \mathbb{C}) : g^tJg = J\}, \]

the complex symplectic group with respect to \(J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\) where \(I_n\) denotes an \(n \times n\) identity block. As involution of \(G^c\), we have conjugation by \(\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}\), which yields that \(K^c \cong GL(n, \mathbb{C})\) consisting of the matrices

\[ \begin{pmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{pmatrix} \quad \text{with} \quad g \in GL(n, \mathbb{C}), \]

whereas \(p^c\) consists of the matrices \((n \times n\) blocks)

\[ y = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \quad \text{with} \quad X^t = X, \ Y^t = Y. \tag{3} \]

Further, \((K^c, p^c)\) is equivalent to \(\rho \oplus \rho^*\); here we can identify \(V^*\) with \(V\) as a vector space and then \(\rho^*(g)X = (g^t)^{-1}Xg^{-1}\). Note that \(\rho\) is irreducible with highest weight \(2\theta_1\).

The polynomials \(f_j(y) = \text{tr}((XY)^j)\) for \(j = 1, \ldots, n\) and \(\nu\) as in (5) form a complete set of invariants. One easily computes

\[ (df_j)_y(\tilde{X}, \tilde{Y}) = j \left( \langle (XY)^{j-1}X, \tilde{Y} \rangle + \langle Y(XY)^{j-1}, \tilde{X} \rangle \right), \]

where \((\tilde{X}, \tilde{Y}) \in V \oplus V^*\) and \(\langle \cdot, \cdot \rangle\) denotes a constant multiple of the Cartan-Killing form of \(g^c\). In particular, by taking \(Y\) to be the identity and \(\tilde{Y} = 0\), we get \((df_j)_y(\tilde{X}, 0) = j \langle X^{j-1}, \tilde{X} \rangle\), whence \(y\) is regular if and only if \(\{I, X, \ldots, X^{n-1}\}\) is a linearly independent set of matrices. In the following, we construct \(X\) such that already the first columns of the matrices \(I, X, \ldots, X^{n-1}\) form a linearly independent set. It follows that the corresponding \(y\) is regular and moreover we can take the subspace \(v\) of \(p^c\) as in Theorem [5] as being the subspace of \(V\) formed by symmetric matrices \(\tilde{X}\) whose only nonzero entries lie in the first column or first line.

Let \(Z\) be a real symmetric matrix with distinct eigenvalues, say diagonal. It is known that such a matrix admits a cyclic vector, that is a vector \(w_1 \in \mathbb{R}^n\) such that \(\{w_1, Zw_1, \ldots, Z^{n-1}w_1\}\) is a basis of \(\mathbb{R}^n\). We can assume \(w_1\) is a unit vector and then complete it to form an orthonormal basis
\{w_1, \ldots, w_n\} of \mathbb{R}^n. Set X = M^{-1}ZM, where M is the orthogonal matrix whose column vectors are the w_j. Then X^{j-1} is a symmetric matrix whose first column is
\[
\begin{pmatrix}
    w_1^t Z^{j-1}w_1 \\
    \vdots \\
    w_n^t Z^{j-1}w_1
\end{pmatrix},
\]

namely, contains the coordinates of the vector \(Z^{j-1}w_1\) with respect to the basis \(\{w_1, \ldots, w_n\}\); hence the set of such columns for \(j = 1, \ldots, n\) form a linearly independent set, as we wished.

We have that \(E_{ij} + E_{ji} \in V\) is a weight vector of \(\rho\) of weight \(\theta_i + \theta_j\). It follows that \([\Lambda^2 v]\) is a highest weight vector of \(\mathfrak{gl}(n, \mathbb{C})\) with highest weight (compare [GW09, p.274])
\[
2\theta_1 + (\theta_1 + \theta_2) + \cdots + (\theta_1 + \theta_n) = n\theta_1 + (\theta_1 + \cdots + \theta_n),
\]
so on \(\mathfrak{sl}(n, \mathbb{C})\).

The polynomials \(X^j\) computes that the symmetric square \(\text{Sym}^2(C^n)\) representation on \(ZM\) is the \(C\) representation on \(\{w_1, \ldots, w_n\}\); hence the corresponding representation of \(U(n)\) is the \(n\)-th symmetric power \(S^n(C^n)\) of the standard representation on \(C^n\). Since the real dimension of this representation is \(2^{(n-1)}\), this proves Theorem 3 stated in the introduction.

The lowest nontrivial case \(n = 2\) is special, for the theorem then says \(\delta\) is the sum of six squares; however, it is easy to see the decomposition into \(U(2)\)-irreducible representations \(\Lambda^2(S^2(C^2)^r) \cong (C^3)^r \oplus \mathbb{R}^5 \oplus \mathbb{R}^3 \oplus \mathbb{R}\). We have that \((C^3)^r \oplus \mathbb{R}^5 \oplus \mathbb{R}^3 \oplus \mathbb{R}, \Phi\) yields \(\delta\) as a sum of six squares as in Theorem 3, but \(\mathbb{R}^5\), a real form of \(S^4(C^2)\) on which the center of \(U(2)\) does not act, yields the better result that \(\delta\) is a sum of five squares. Moreover, one computes directly for a complex symmetric matrix \(\begin{pmatrix} z_1 & z_3 \\ z_3 & z_2 \end{pmatrix}\) that
\[
\delta = |z_1 z_2 - z_3 z_3|^2 \left( |z_1|^2 - |z_2|^2 \right)^2 + 4 |z_1 z_3 + \bar{z}_2 z_3|^2,
\]
which shows in fact the best result that \(\delta\) is a sum of two squares. For \(n \geq 3\), Theorem 3 is probably neither optimal, though it should get closer to that.

Finally, we quickly revisit the case of \(SO(n)\)-conjugation of traceless real symmetric matrices, or, the isotropy representation of \(SL(n, \mathbb{R})/SO(n)\). It is usual and convenient to model the complexified representation \((SO(n, \mathbb{C}), \text{Sym}_0(n, \mathbb{C}))\) using the nondegenerate symmetric bilinear form given by
\[
Q = \begin{pmatrix} 0 & I_{\ell} \\ I_{\ell} & 0 \end{pmatrix} \quad \text{if } n = 2\ell \quad \text{and} \quad Q = \begin{pmatrix} 0 & I_{\ell} & 0 \\ I_{\ell} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{if } n = 2\ell + 1,
\]
so that \(SO(n, \mathbb{C}) \cong SO(Q) := \{g| g^t Q g = Q\} \) with Lie algebra \(\mathfrak{so}(n, \mathbb{C}) \cong \mathfrak{so}(Q) := \{A| A^t Q + QA = 0\}\) and \(\text{Sym}_0(n, \mathbb{C}) \cong \mathbb{V} := \{X| X^t = QXQ^{-1}, \text{tr} X = 0\}\). Now \((K^c, p^c)\) is the representation \(\rho\) of \(SO(Q)\) on \(\mathbb{V}\) given by \(\rho(g)X = gXg^{-1}\). We have that \(F : S^2_0(C^n) \to \mathbb{V}\) given by \(F(v \cdot w) = \frac{1}{2}(vw^t + wv^t)\) is an equivariant isomorphism; here \(S^2_0(C^n)\) denotes the nontrivial component of the symmetric square \(S^2(C^n)\). It follows that \(\rho\) is irreducible with highest weight \(2\theta_1\).

The polynomials \(f_j(X) = \text{tr}(X^j)\) for \(j = 2, \ldots, n\) form a complete set of invariants. One easily computes that
\[
(df_j)(\tilde{X}) = j(X^{j-1}, \tilde{X}),
\]
where \(\tilde{X} \in \mathbb{V}\) and \(\langle \cdot, \cdot \rangle\) denotes a constant multiple of the Cartan-Killing form of \(\mathfrak{g}^c\). Therefore \(X\) is regular if and only if \(\{X, X^2, \ldots, X^{n-1}\}\) is a linearly independent set of matrices.

The matrix \(X\) chosen in [Dom10, p.13] (denoted \(A\) there) acting on the canonical basis \(\{e_1, \ldots, e_n\}\) of \(C^n\) as
\[
e_1 \mapsto e_{\ell+1} \mapsto e_{\ell+2} \mapsto \cdots \mapsto e_n \mapsto e_\ell \mapsto e_{\ell-1} \mapsto \cdots \mapsto e_2 \mapsto e_1
\]
lies in $V$ and has $e_1$ as cyclic vector. Thus the first columns of the matrices $I, X, \ldots, X^{n-1}$ form a linearly independent set. Since $X^{n-1} = X^{-1} = X'$, the set $\{I, X, \ldots, X^{n-1}\}$ is invariant under transposition and hence also the first lines of $I, X, \ldots, X^{n-1}$ form a linearly independent set. In particular, taking out the identity $I$ from this set, it suffices to consider first lines minus the (1, 1)-entry of the matrices $X, \ldots, X^{n-1}$ to see that they form a linearly independent set. Thus $X$ is regular and we can take the subspace $v$ of $\mathfrak{p}^*$ as in Theorem 5 as being the subspace of $V$ formed by matrices $X$ whose only nonzero entries are in positions (1, 2) through (1, $n$). Note that $v = F(e_1^1, e_1 e_2, \ldots, e_1 e_{\ell+1}, \ldots, e_1 e_n))$, and the weight of $e_i$ in $C^n$ is

$$\begin{align*}
\theta_i & \quad \text{if } i = 1, \ldots, \ell, \\
-\theta_i & \quad \text{if } i = \ell + 1, \ldots, 2\ell, \\
0 & \quad \text{if } i = 2\ell + 1.
\end{align*}$$

It follows that $[\Lambda^{n-1}v]$ is a highest weight vector of weight

$$\begin{align*}
2\theta_1 + (\theta_1 + \theta_2) + \ldots + (\theta_1 + \theta_\ell) + (\theta_1 - \theta_2) + \ldots + (\theta_1 - \theta_\ell) & \quad \text{if } n = 2\ell, \\
2\theta_1 + (\theta_1 + \theta_2) + \ldots + (\theta_1 + \theta_\ell) + (\theta_1 - \theta_2) + \ldots + (\theta_1 - \theta_\ell) + \theta_1 & \quad \text{if } n = 2\ell + 1,
\end{align*}$$

which in both cases equals $n\theta_1$, which corresponds to the representation on the space of $n$-variable spherical harmonics of degree $n$, of dimension $\dim S^n(\mathbb{R}^n) - \dim S^{n-2}(\mathbb{R}^n)$. This proves Theorem 2 stated in the introduction.

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