Hölder estimates for resolvents of time-changed
Brownian motions

Kouhei Matsuura
Institute of Mathematics, University of Tsukuba,
1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan
kmatsuura@math.tsukuba.ac.jp

Abstract. In this paper, we study time changes of Brownian motions
by positive continuous additive functionals. Under a certain regularity
condition on the associated Revuz measures, we prove that the resolvents
of the time-changed Brownian motions are locally Hölder continuous in
the spatial components. We also obtain lower bounds for the indices of
the Hölder continuity.

Keywords: Brownian motion, time change, Hölder continuity, resol-
vent, coupling

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1 Introduction

Let $B = \{\{B_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d}\}$ be a Brownian motion on the $d$-dimensional Eu-
clidean space $\mathbb{R}^d$. Let $A = \{A_t\}_{t \geq 0}$ be a positive continuous additive func-
tional (PCAF in abbreviation) of $B$. Then, the time-changed Brownian motion
$\tilde{B} = \{\{\tilde{B}_t\}_{t \geq 0}, \{\tilde{P}_x\}_{x \in F}\}$ by the PCAF $A$ is defined as

$$\tilde{B}_t = B_{\tau_t}, \quad \tilde{P}_x = P_x, \quad (t, x) \in [0, \infty) \times F.$$  

Here, we denote by $\{\tau_t\}_{t \geq 0}$ the right continuous inverse of $\{A_t\}_{t \geq 0}$, and $F$ stands
for the support of $A$ (see (2) below for the definition). From the Revuz corre-
spondence (see (2)), the PCAF $A$ induces a Borel measure $\mu$ on $\mathbb{R}^d$, which is
called the Revuz measure of $A$. It is known that the time-changed Brownian mo-
tion $\tilde{B}$ becomes a $\mu$-symmetric right process on $F$ (see, e.g. [1] Theorem 5.2.1]).
On account of this fact, in what follows, we use the symbol $B^\mu$ ($A^\mu$ and $F^\mu$, respectively) to denote $\tilde{B}$ ($A$ and $F$, respectively).
A typical example of Revuz measures is of the form \( f \, dm \). Here, \( f : \mathbb{R}^d \to [0, \infty) \) is a locally bounded Borel measurable function, and \( m \) stands for the Lebesgue measure on \( \mathbb{R}^d \). Then, we have \( A^f_t \, dm = \int_0^t f(B_s) \, ds, \ t \geq 0 \). However, a Revuz measure \( \mu \) can be singular with respect to \( m \). Then, the behavior of \( B^\mu \) would be quite different from that of the standard Brownian motion. Nevertheless, if \( F^\mu = \mathbb{R}^d \), we can simply describe the Dirichlet form \((\mathcal{E}, F^\mu)\) of \( B^\mu \) by using the extended Dirichlet space \( H^1_e(\mathbb{R}^d) \) of \( B \). If \( d \in \{1, 2\} \), we see from [4, Theorem 2.2.13] that \( H^1_e(\mathbb{R}^d) \) is identified with \( \{ f \in L^2_{\text{loc}}(\mathbb{R}^d, m) | |\nabla f| \in L^2(\mathbb{R}^d, m) \} \).

Here, \( L^2_{\text{loc}}(\mathbb{R}^d, m) \) is the space of locally square integrable functions on \( \mathbb{R}^d \) with respect to \( m \), and \( \nabla f \) denotes the distributional gradient of \( f \). Even if \( d > 2 \), the extended Dirichlet space is characterized with distributional derivatives ([4, Theorem 2.2.12]). From these facts and [4, (5.2.17)], we find that the Dirichlet form \((\mathcal{E}, F^\mu)\) is identified with

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} (\nabla f(x), \nabla g(x)) \, dm(x), \quad f, g \in F^\mu, \tag{1}
\]

\[
F^\mu = \{ \tilde{u} \in H^1_e(\mathbb{R}^d) | \tilde{u} \in L^2(\mathbb{R}^d, \mu) \}.
\]

Here, we denote by \((\cdot, \cdot)\) the standard inner product on \( \mathbb{R}^d \), and \( \tilde{u} \) is the quasi-continuous version of \( u \in H^1_e(\mathbb{R}^d) \). See [5, Lemma 2.1.4 and Theorem 2.1.7] for the existence and the uniqueness. However, even in this setting, it is generally difficult to write down other analytical objects associated with \( B^\mu \), such as the semigroup and the resolvent. Therefore, it is non-trivial to clarify how these objects depend on \( \mu \).

In this paper, we study the continuity of the resolvent of \( B^\mu \) in the spatial component. Even though this kind of problem can be formulated for other Markov processes, the current setting allows us to quantitatively clarify how the continuity depends on \( \mu \). In the main theorem of this paper (Theorem 1), we prove that the resolvent of \( B^\mu \) is Hölder continuous in the spatial component under a certain condition on \( \mu \). The condition is given in (5) below, and the index there represents a regularity of \( \mu \). This also describes a lower bound for the index of the Hölder continuity of the resolvent. In particular, we see that the resolvent is \((1 - \varepsilon)\)-Hölder continuous if the index is sufficiently large. Condition (5) can be regarded as a generalized concept of the \( d \)-measure. We refer the reader to [6] for basic facts on time-changed Hunt processes by PCAFs associated with \( d \)-measures. We also note that the Liouville measure is one of examples which satisfies (5). The reader is referred to [7,8] and references therein for more details and the time changed planar Brownian motion by the PCAF associated with the Liouville measure.

If \( F^\mu = \mathbb{R}^d \), it is not very hard to see that the resolvent of \( B^\mu \) is just Hölder continuous. In fact, we see from [1] that any bounded harmonic function \( h \) on \( B(z, 2r) \) \((z \in \mathbb{R}^d, r > 0)\) with respect to \( B^\mu \) is also harmonic with respect to the standard Brownian motion. Here, \( B(z, r) \) denotes the open ball centered at...
z with radius \( r > 0 \). Then, from [1, Chapter II. (1.3) Proposition], there exists a positive constant \( C \) independent of \( z \) and \( r \) such that
\[
|h(x) - h(y)| \leq C \sup_{z \in \mathbb{R}^d} |h(z)| \frac{|x - y|}{r}, \quad x, y \in B(z, r).
\]
Furthermore, since \( A^\mu \) is a homeomorphism on \([0, \infty)\) (here, we used the assumption that \( F^\mu = \mathbb{R}^d \)), \( A^\mu_{\tau B(x,r)} \) is identified with the exit time of \( B^\mu \) from \( B(x, r) \), where \( \tau_{B(x,r)} \) denotes the first exit time of \( B \) from \( B(x, r) \). This observation and the regularity condition (5) lead us to a mean exit time estimate for \( B^\mu \) (see also Lemma 2). Then, the same argument as in [2, (5.6) Proposition, Section VII] shows that the resolvent of \( B^\mu \) is Hölder continuous. We note that this kind of argument is applicable to several situations (see, e.g., [3, Proposition 3.3 and Theorem 3.5]); however, even if \( \mu = m \), this only implies that the index of the Hölder continuity is greater than or equal to \( 2/3 \). This estimate is not sharp because \( B^m \) is the standard Brownian motion and the resolvent is Lipschitz continuous. Thus, even if \( F^\mu = \mathbb{R}^d \), our result does not directly follow from the method stated above, and implies rather sharp result.

For the proof of Theorem 1 we use the mirror coupling of \( d \)-dimensional Brownian motions. The key to our proof is an inductive argument based on the strong Markov property (of the coupling) and some estimates of the coupling time (Lemmas 3 and 5). Since mirror couplings of stochastic processes are universal concepts, our arguments may be useful for estimating the indices of Hölder continuity of resolvents for other time-changed Markov processes.

The remainder of this paper is organized as follows. In Section 2, we set up a framework and state the main theorem (Theorem 1). In Section 3, we provide some preliminary estimates for PCAFs of the \( d \)-dimensional Brownian motion. In Section 4, we introduce some lemmas on the mirror coupling of Brownian motions, and prove Theorem 1.

**Notation.** In the paper, we use the following symbols and conventions.

- \((\cdot, \cdot)\) and \(|\cdot|\) denote the standard inner product and norm of \( \mathbb{R}^d \), respectively.
- For \( x \in \mathbb{R}^d \) and \( r > 0 \), \( B(x, r) \) (resp. \( \overline{B}(x, r) \)) denotes the open (resp. closed) ball in \( \mathbb{R}^d \) with center \( x \) and radius \( r \).
- For a subset \( S \subset \mathbb{R}^d \) and \( f : S \to [-\infty, \infty] \), we set \( \|f\|_\infty := \sup_{x \in S} |f(x)| \).
- For a topological space \( S \), we write \( \mathcal{B}_b(S) \) for the space of bounded Borel measurable functions on \( S \).
- For \( a, b \in [-\infty, \infty] \), we write \( a \vee b = \max\{a, b\} \) and \( a \wedge b = \min\{a, b\} \).
- \( \inf \emptyset = \infty \) by convention.

2 Main results

Let \( B = \{B_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d} \) be a Brownian motion on \( \mathbb{R}^d \). The Dirichlet form is identified with
\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} (\nabla f(x), \nabla g(x)) \, dm(x), \quad f, g \in H^1(\mathbb{R}^d).
\]
Here, $H^1(\mathbb{R}^d) := H^1_0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, m)$ denotes the first-order Sobolev space on $\mathbb{R}^d$. For an open subset $U \subset \mathbb{R}^d$ and for a subset $A \subset \mathbb{R}^d$, we define

$$\text{cap}(U) = \inf \left\{ \mathcal{E}(f, f) + \int_{\mathbb{R}^d} f^2 \, dm \mid f \in H^1(\mathbb{R}^d), \ f \geq 1, \text{ m.a.e. on } U \right\},$$

$$\text{Cap}(A) = \inf \left\{ \text{cap}(U) \mid A \subset U \text{ and } U \text{ is an open subset of } \mathbb{R}^d \right\}.$$

A non-negative Radon measure $\mu$ on $\mathbb{R}^d$ is said to be smooth if $\mu(A) = 0$ for any $A \subset \mathbb{R}^d$ with $\text{Cap}(A) = 0$. For a smooth measure $\mu$, by [4, Theorem 4.1.1], there exists a unique PCAF $A^\mu = \{A^\mu_t\}_{t \geq 0}$ of $B$ such that for any non-negative functions $f, g \in \mathcal{B}_0(\mathbb{R}^d)$ and $\alpha > 0$,

$$\int_{\mathbb{R}^d} E_x \left[ \int_0^\infty e^{-\alpha t} f(B_t) \, dA^\mu_t \right] g(x) \, dm(x) = \int_{\mathbb{R}^d} E_x \left[ \int_0^\infty e^{-\alpha t} g(B_t) \, dt \right] f(x) \, d\mu(x),$$

where $E_x$ denotes the expectation under $P_x$. See [4, Section 4] and [5, Section 5] for the definition and further details on PCAFs. We also note that the exceptional set of $A^\mu$ can be taken to be empty (see [4, Theorem 4.1.1]).

Let $\{\tau^\mu_t\}_{t \geq 0}$ be the right continuous inverse of $A^\mu$. We define

$$B^\mu_t = B_{\tau^\mu_t}, \ P^\mu_x = P_x, \ (t, x) \in [0, \infty) \times F^\mu,$$

where $F^\mu$ denotes the support of $A^\mu$:

$$F^\mu = \{x \in \mathbb{R}^d \mid \inf \{t > 0 \mid A^\mu_t > 0\} = 0, \ P_x\text{-a.s.}\}. \quad (3)$$

We note that $F^\mu$ is a nearly Borel subset with respect to $B$ (see the paragraph after [4, (A.3.11)]). The support $F^\mu$ is also regarded as a topological subspace of $\mathbb{R}^d$. By [4, Theorems 5.2.1 and A.3.11], $B^\mu = \{\{B^\mu_t\}_{t \geq 0}, \ P^\mu_x, x \in F^\mu\}$ is a $\mu$-symmetric right process on $F^\mu$. The resolvent $\{G_\alpha^\mu\}_{\alpha > 0}$ is given by

$$G_\alpha^\mu f(x) = E_x \left[ \int_0^\infty e^{-\alpha t} f(B^\mu_t) \, dt \right], \quad \alpha > 0, \ f \in \mathcal{B}_0(F^\mu), \ x \in F^\mu.$$

Let $\mathcal{B}_0(\mathbb{R}^d)$ denote the space of bounded universally measurable functions on $\mathbb{R}^d$. That is, any $f \in \mathcal{B}_0(\mathbb{R}^d)$ is bounded and measurable with respect to the $\sigma$-field $\mathcal{B}^\mu(\mathbb{R}^d)$; the family of universally measurable subsets of $\mathbb{R}^d$: $\mathcal{B}^\mu(\mathbb{R}^d) := \bigcap_{\mu \in \mathcal{P}(\mathbb{R}^d)} \mathcal{B}^\mu(\mathbb{R}^d)$. Here, $\mathcal{P}(\mathbb{R}^d)$ denotes the family of all probability measures on $\mathbb{R}^d$ and $\mathcal{B}^\mu(\mathbb{R}^d)$ is the completion of the Borel $\sigma$-field $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$ with respect to $\mu \in \mathcal{P}(\mathbb{R}^d)$. For $\alpha > 0, f \in \mathcal{B}_0(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$, we define

$$V_\alpha^\mu f(x) = E_x \left[ \int_0^\infty e^{-\alpha A^\mu_t} f(B_t) \, dA^\mu_t \right].$$

We see from [4, Exercise A.1.29] that $F^\mu$ is a universally measurable subsets of $\mathbb{R}^d$. By noting this fact and using [4, Lemma A.3.10], we have

$$G_\alpha^\mu f(x) = V_\alpha^\mu f(x). \quad (4)$$
In particular, we have

\[ B \]

non-negative

\[ f \]

The Green function of

Lemma 1.

Let

\[ \mu(B(x,r)) \leq Kr^\kappa \]

for any \( r \leq R \) and \( x \in \mathbb{R}^d \) with \( |x - p| \leq r \). Then, for any \( \alpha > 0 \) and \( \varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa)) \), there exists \( C > 0 \) depending on \( d, p, \kappa, K, R, \varepsilon, \) and \( \alpha \) such that

\[ |V^\mu_a f(x) - V^\mu_a f(y)| \leq C\|f\|_{\infty}|x - y|^{(\frac{(2-d+\kappa)}{1})-\varepsilon} \]

for any \( f \in B^*_0(\mathbb{R}^d) \) and \( x, y \in B(p, 2^{-C'}R) \), where \( C' \) is a positive number depending on \( d, \varepsilon \) and \( \kappa \). In particular, we have

\[ |C^\mu_a f(x) - C^\mu_a f(y)| \leq C\|f\|_{\infty}|x - y|^{(\frac{(2-d+\kappa)}{1})-\varepsilon} \]

for any \( f \in B_0(F^\mu) \) and \( x, y \in F^\mu \cap B(p, 2^{-C'}R) \).

3 Preliminary lemmas

For an open subset \( U \subset \mathbb{R}^d \), we denote by \( U \cup \{\partial U\} \) the one-point compactification. We set \( \tau_U = \inf\{t \in [0, \infty) \mid B_t \notin U\} \). Then, the absorbing Brownian motion

\[ B^U = \{\{B^U_t\}_{t \geq 0}, \{P_x\}_{x \in U}\} \]

on \( U \) is defined as

\[ B^U_t = \begin{cases} B_t, & t < \tau_U, \\ \partial_U, & t \geq \tau_U. \end{cases} \]

We write \( p^U_t = p^U_t(x,y) : (0,\infty) \times U \times U \to [0,\infty) \) for the transition density of \( B^U \). That is, \( p^U_t \) is the jointly continuous function such that

\[ P_x(B^U_t \in dy) = p^U_t(x,y) dm(y), \quad t > 0, \ x \in U. \]

The Green function of \( B^U \) is defined by

\[ g_U(x,y) = \int_0^\infty p^U_t(x,y) dt, \quad x, y \in U. \]

Lemma 1. Let \( U \subset \mathbb{R}^d \) be an open subset. Then, for any \( x \in U, t > 0, \) and non-negative \( f \in B_0(U) \),

\[ E_x \left[ \int_0^{t \wedge \tau_U} f(B_s) \, dA^a_s \right] = \int_U \left( \int_0^t p^U_s(x,y) \, ds \right) f(y) \, d\mu(y). \]

In particular, we have

\[ E_x \left[ \int_0^{\tau_U} f(B_s) \, dA^a_s \right] = \int g_U(x,y)f(y) \, d\mu(y). \]
Proof. We fix $t > 0$ and non-negative functions $f, g \in B_{0}(U)$. We may assume that $f$ is compactly supported. By [4] Proposition 4.1.10, we have

$$
\int_{U} E_{z} \left[ \int_{0}^{t \wedge \tau_{U}} f(B_{s}) \, dA_{s}^{U} \right] g(z) \, dm(z) = \int_{0}^{t} \left( \int_{U} (P_{s}^{U} g)(x) \, d\mu(x) \right) \, ds. \quad (8)
$$

We use Fubini’s theorem to obtain that

$$
\int_{0}^{t} \left( \int_{U} (P_{s}^{U} g)(x) \, d\mu(x) \right) \, ds = \int_{U} \left( \int_{0}^{t} \left( \int_{U} p_{s}^{U}(x, z) \, d\mu(x) \right) \, ds \right) g(z) \, dm(z). \quad (9)
$$

Because $\mu$ is a Radon measure, by letting $g = 1_{U}$ in (8) and (9), we see that

$$
E_{\cdot} \left[ \int_{0}^{t \wedge \tau_{U}} f(B_{s}) \, dA_{s}^{U} \right] \quad \text{and} \quad \int_{0}^{t} \left( \int_{U} p_{s}^{U}(x, \cdot) \, d\mu(x) \right) \, ds
$$

are integrable on $U$ with respect to $m$. Moreover, because $g$ is arbitrarily taken, (8) and (9) imply that for $m$-a.e. $z \in U$,

$$
E_{z} \left[ \int_{0}^{t \wedge \tau_{U}} f(B_{s}) \, dA_{s}^{U} \right] = \int_{0}^{t} \left( \int_{U} p_{s}^{U}(x, z) \, d\mu(x) \right) \, ds. \quad (10)
$$

By following the convention that $f(\partial U) = 0$, we see that the left-hand side of (10) is equal to $E_{z} \left[ \int_{0}^{t} f(B_{s}^{U}) \, dA_{s \wedge \tau_{U}}^{U} \right]$. Hence, we have for $m$-a.e. $z \in U$,

$$
E_{z} \left[ \int_{0}^{t} f(B_{s}^{U}) \, dA_{s \wedge \tau_{U}}^{U} \right] = \int_{0}^{t} \left( \int_{U} p_{s}^{U}(x, z) \, d\mu(x) \right) \, ds. \quad (11)
$$

We see from [4] Exercise 4.1.9 (iii)] that $\{A_{s \wedge \tau_{U}}^{U}\}_{s \geq 0}$ is the PCAF of $B_{s}^{U}$. By using the additivity, the Markov property of $B_{s}^{U}$, and (11), we obtain that for any $x \in U$,

$$
E_{x} \left[ \int_{0}^{t \wedge \tau_{U}} f(B_{s}) \, dA_{s}^{U} \right] = \lim_{u \uparrow 0} E_{x} \left[ \int_{u}^{t} f(B_{s}^{U}) \, dA_{s \wedge \tau_{U}}^{U} \right]
$$

$$
= \lim_{u \uparrow 0} E_{x} \left[ E_{B_{u}^{U}} \left[ \int_{0}^{t-u} f(B_{s}^{U}) \, dA_{s \wedge \tau_{U}}^{U} \right] \right]
$$

$$
= \lim_{u \uparrow 0} \int_{U} p_{u}^{U}(x, z) \left( \int_{0}^{t-u} \left( \int_{U} p_{s}^{U}(y, z) \, d\mu(y) \right) \, ds \right) \, dm(z)
$$

$$
= \lim_{u \uparrow 0} \int_{U} \left( \int_{u}^{t} \left( \int_{U} p_{s}^{U}(x, y) \, d\mu(y) \right) \, ds \right) \, dm(z)
$$

which completes the proof. “In particular” part immediately follows from the monotone convergence theorem. \( \square \)
Let $d \geq 2$, $r \in (0, 1)$, and $x, y \in \mathbb{R}^d$. Then, by [5, Example 1.5.1],

$$g_{B(x, r)}(x, y) = \begin{cases} \frac{-1}{\pi} \log |x - y|, & d = 2, \\ \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}} |x - y|^{2-d}, & d \geq 3. \end{cases}$$

(12)

Here, $\Gamma$ denotes the gamma function. If $d = 1$, we see from [9, Lemma 20.10] that for any $a, b \in \mathbb{R}$ with $a < b$,

$$g_{(a, b)}(x, y) = \frac{2(x \wedge y - a)(b - x \vee y)}{b - a}, \quad x, y \in (a, b).$$

(13)

For $s \in (0, 1]$ and $t > 0$, we define

$$\zeta_d(s, t) = \begin{cases} s^{2-d+t}, & d \geq 3 \text{ or } d = 1, \\ -s^t \log s, & d = 2. \end{cases}$$

**Lemma 2.** Let $p \in \mathbb{R}^d$ and take constants $\kappa > d - 2$, $R \in (0, 1]$, and $K > 0$ so that (5) holds. Then, there exists $C \in (0, \infty)$ depending on $d$, $p$, $\kappa$, $R$, and $K$ such that for any $r \in (0, R/2]$ and $x \in B(p, r)$

$$\int_{B(x, r)} g_{B(x, r)}(x, y) \, d\mu(y) \leq C \zeta_d(r, \kappa).$$

In particular, we have $E_x[A_{r_B(x, r)}^\mu] \leq C \zeta_d(r, \kappa)$.

**Proof.** In view of (5), we have for any $r \in (0, R/2]$ and $x \in B(p, r)$,

$$\mu(B(x, r)) \leq Kr^\kappa.$$  \hspace{1cm} (14)

Therefore, when $d = 1$, we use (13) to obtain that

$$\int_{B(x, r)} g_{B(x, r)}(x, y) \, d\mu(y) \leq 4Kr^{\kappa+1}.$$  

Equation (12) implies that for any $k \in \mathbb{N}$,

$$\sup_{y \in \mathbb{R}^d \setminus B(x, r2^{-k})} g_{B(x, r)}(x, y) \leq \begin{cases} \frac{-1}{\pi} \log(r2^{-k}), & d = 2, \\ \frac{\Gamma(d/2 - 1)}{2\pi^{d/2}} (r2^{-k})^{2-d}, & d \geq 3. \end{cases}$$

Thus, for $d = 2$, we obtain from (13) that

$$\int_{B(x, r)} g_{B(x, r)}(x, y) \, d\mu(y) = \sum_{k=1}^{\infty} \int_{B(x, r2^{-(k-1)2}) \setminus B(x, r2^{-k})} g_{B(x, r)}(x, y) \, d\mu(y)$$
Here, $C$ is a positive constant depending on $\kappa$ and $K$. For $d \geq 3$, we similarly use (14) to obtain that
\begin{equation}
\int_{B(x,r)} g_{B(x,r)}(x,y) \, d\mu(y) \leq\frac{K \Gamma(d/2 - 1)}{2\pi^{d/2}} \sum_{k=1}^{\infty} (r2^{-(k-1)})\kappa (r2^{-k})^{2-d} \leq \left( \frac{K \Gamma(d/2 - 1)}{2\pi^{d/2}} \right) 2^{-(2-d+\kappa)} \sum_{k=1}^{\infty} 2^{-((2-d+\kappa)k)} r^{2-d+\kappa}.
\end{equation}
Because $\kappa > d - 2$, we have
\begin{equation}
\int_{B(x,r)} g_{B(x,r)}(x,y) \, d\mu(y) \leq C r^{2-d+\kappa}.
\end{equation}
Here, $C$ is a positive constant depending on $d$, $K$ and $\kappa$. “In particular” part immediately follows from Lemma 1. \hfill \Box

\section{Proof of Theorem 1}

Let $x, y \in \mathbb{R}^d$ and $\{W_t\}_{t \geq 0}$ a $d$-dimensional Brownian motion starting at the origin. The mirror coupling $(Z_{x}^x, \tilde{Z}_{y}^y) = (\{Z_{t}^{x}\}_{t \geq 0}, \{\tilde{Z}_{t}^{y}\}_{t \geq 0})$ of $d$-dimensional Brownian motions is defined as follows:

- For any $t < \inf\{s > 0 \mid Z_{y}^x = \tilde{Z}_{s}^y\}$,
  \[ Z_{t}^{x} = x + W_{t}, \]
  \[ \tilde{Z}_{t}^{y} = y + W_{t} - 2 \int_{0}^{t} \frac{Z_{s}^{x} - \tilde{Z}_{s}^{y}}{|Z_{s}^{x} - \tilde{Z}_{s}^{y}|^2} (Z_{s}^{x} - \tilde{Z}_{s}^{y}, dW_{s}). \] \hspace{1cm} (15)

- For any $t \geq \inf\{s > 0 \mid Z_{y}^x = \tilde{Z}_{s}^y\}$, we have $Z_{t}^{x} = \tilde{Z}_{t}^{y}$.

\textbf{Remark 1.} (1) The mirror coupling $(Z_{x}^x, \tilde{Z}_{y}^y)$ is a special case of couplings for diffusion processes studied in \cite{10} Section 3.

(2) For $x, y \in \mathbb{R}^d$ with $x \neq y$ and $t < \inf\{s > 0 \mid Z_{y}^x = \tilde{Z}_{s}^y\}$, we have
\[ Z_{t}^{x} - \tilde{Z}_{t}^{y} = x - y + 2 \int_{0}^{t} \frac{Z_{s}^{x} - \tilde{Z}_{s}^{y}}{|Z_{s}^{x} - \tilde{Z}_{s}^{y}|^2} (Z_{s}^{x} - \tilde{Z}_{s}^{y}, dW_{s}). \]

This implies that the random vector $Z_{t}^{x} - \tilde{Z}_{t}^{y}$ is parallel to $x - y$. We then see from (15) that $\tilde{Z}_{t}^{y}$ coincides with the mirror image of $Z_{t}^{x}$ with respect to the hyperplane $H_{x,y} = \{z \in \mathbb{R}^d \mid (z - (x + y)/2, x - y) = 0\}$. Further, $\inf\{s > 0 \mid Z_{s}^x = \tilde{Z}_{s}^y\} = \inf\{s > 0 \mid Z_{s}^x \in H_{x,y}\}$. Then, it is easy to see that $(Z_{x}^x, \tilde{Z}_{y}^y)$ is a strong Markov process on $\mathbb{R}^d \times \mathbb{R}^d$. 

For \( x, y \in \mathbb{R}^d \), we define \( \xi_{x,y} = \inf\{t \geq 0 \mid Z_t^x = Z_t^y \} \). We denote by \( P_{x,y} \) the distribution of \((Z_t^x, Z_t^y)\). For \( t \geq 0 \), we set
\[
A_t^{\mu,x} = A_t^\mu(Z_t^x), \quad \tilde{A}_t^{\mu,y} = A_t^\mu(\tilde{Z}^y),
\]
where we regard \( A_t^\mu \) as \([0, \infty)\)-valued functions on \( C([0, \infty), \mathbb{R}^d) \), the space of \( \mathbb{R}^d \)-valued continuous functions on \([0, \infty)\). Then, \( \{A_t^{\mu,x}\}_{t \geq 0} \) and \( \{\tilde{A}_t^{\mu,y}\}_{t \geq 0} \) become PCAFs of \( Z^x \) and \( \tilde{Z}^y \), respectively. Furthermore, \( A_t^{\mu,x} \) and \( \tilde{A}_t^{\mu,y} \) can be regarded as PCAFs of the coupled process \((Z^x, \tilde{Z}^y)\) in the natural way.

For \( x, y \in \mathbb{R}^d \), we define
\[
\mathcal{I}_{x,y} = E_{x,y}[A_{\xi_{x,y}}^{\mu,x} \wedge 1], \quad \tilde{\mathcal{I}}_{x,y} = E_{x,y}[\tilde{A}_{\xi_{x,y}}^{\mu,y} \wedge 1],
\]
(16)
where \( E_{x,y} \) denotes the expectation under \( P_{x,y} \). At the end of this section (see [14] below), we will show that for any \( f \in \mathcal{B}_b(\mathbb{R}^d) \) with \( \|f\|_\infty \leq 1 \),
\[
|V_t^\mu f(x) - V_t^\mu f(y)| \leq 2(1 + \alpha^{-1})(\mathcal{I}_{x,y} + \tilde{\mathcal{I}}_{x,y}), \quad \alpha > 0, \ x, y \in \mathbb{R}^d.
\]

We now introduce some lemmas to estimate the expectations in (16).

**Lemma 3.** Let \( x, y \in \mathbb{R}^d \), and \( \tau \) be a stopping time of \((Z_t^x, \tilde{Z}_t^y)\). Then,
\[
E_{x,y} \left[ \left| Z_{\xi_{x,y} \wedge \tau}^x - \tilde{Z}_{\xi_{x,y} \wedge \tau}^y \right|^\theta \right] \leq |x - y|^\theta
\]
for any \( \theta \in (0, 1] \).

**Proof.** We fix \( t \geq 0 \) and \( x, y \in \mathbb{R}^d \) with \( x \neq y \). To simplify the notation, we write \( Z_t \) (resp. \( \tilde{Z}_t, L_t, \tilde{L}_t, \xi_t \)) for \( Z_t^x \) (resp. \( Z_t^y, L_t^y, \tilde{L}_t^y, \xi_{x,y} \)). We also fix \( n \in \mathbb{N} \) such that \( |x - y| \geq 1/n \), and set \( \xi_n = \inf\{s > 0 \mid |Z_s - \tilde{Z}_s| \leq 1/n\} \).

For \( s < \xi_n \wedge \tau \), we define
\[
\alpha_s = (Z_s - \tilde{Z}_s)(Z_s - \tilde{Z}_s)^T / |Z_s - \tilde{Z}_s|^2.
\]
Here, \((Z_s - \tilde{Z}_s)^T\) denotes the transpose of \( Z_s - \tilde{Z}_s \). From Itô formula,
\[
|Z_{t \wedge \xi_n \wedge \tau} - \tilde{Z}_{t \wedge \xi_n \wedge \tau}|^2 - |x - y|^2 = 2 \int_0^{t \wedge \xi_n \wedge \tau} (Z_s - \tilde{Z}_s, \alpha_s dW_s) + t \wedge \xi_n \wedge \tau
\]
and for any \( \theta \in (0, 1] \),
\[
|Z_{t \wedge \xi_n \wedge \tau} - \tilde{Z}_{t \wedge \xi_n \wedge \tau}|^\theta - |x - y|^\theta
= \theta \int_0^{t \wedge \xi_n \wedge \tau} |Z_s - \tilde{Z}_s|^{\theta-2}(Z_s - \tilde{Z}_s, \alpha_s dW_s)
+ \{\theta/2 + \theta(\theta/2 - 1)\} \int_0^{t \wedge \xi_n \wedge \tau} |Z_s - \tilde{Z}_s|^{\theta-2} ds.
\]
(17)
Since the first term above is a martingale and the second one is non-positive, by taking the expectations of both sides of (17), we arrive at

$$E_{x,y} \left[ |Z_{t \wedge \xi_n \wedge \tau} - \tilde{Z}_{t \wedge \xi_n \wedge \tau}|^\theta \right] \leq |x - y|^\theta. \quad (18)$$

Letting $n \to \infty$ in (18), we complete the proof. \(\square\)

**Lemma 4.** It holds that

$$P_{x,y}(t < \xi_{x,y}) \leq |x - y|/\sqrt{2\pi t}$$

for any $t > 0$ and $x, y \in \mathbb{R}^d$.

**Proof.** Let $t \geq 0$ and $x, y \in \mathbb{R}^d$ with $x \neq y$. We take $n \in \mathbb{N}$ such that $|x - y| \geq 1/n$.

Letting $\theta = 1$ in (17), we have

$$|Z_{t \wedge \xi_n} - \tilde{Z}_{t \wedge \xi_n}| - |x - y| = \int_0^{t \wedge \xi_n} |Z_s - \tilde{Z}_s|^{-1}(Z_s - \tilde{Z}_s, \alpha_s dW_s). \quad (19)$$

The quadratic variation of the right-hand side of (19) equals to $t \wedge \xi_n$, $t \geq 0$. Hence, by the Dambis–Dubins–Schwartz theorem, there is a one-dimensional Brownian motion $\beta = \{\beta_s\}_{s \geq 0}$ such that

$$\beta_{t \wedge \xi_n} = \int_0^{t \wedge \xi_n} |Z_s - \tilde{Z}_s|^{-1}(Z_s - \tilde{Z}_s, \alpha_s dW_s). \quad (20)$$

By using (19), (20), and the reflection principle of the Brownian motion, we have

$$P_{x,y}(\tau_{x,U} \leq t) \leq P_{x,y} \left( -|x - y| \leq \inf_{0 \leq s \leq t} \beta_s \right)$$

$$= 1 - 2 \int_{-\infty}^{-|x - y|} \frac{1}{\sqrt{2\pi t}} \exp(-u^2/2t) du$$

$$= \int_{-|x - y|}^{0} \frac{1}{\sqrt{2\pi t}} \exp(-u^2/2t) du \leq \frac{|x - y|}{\sqrt{2\pi t}}. \quad (21)$$

Letting $n \to \infty$ in (21) completes the proof. \(\square\)

For $x, y \in \mathbb{R}^d$ and an open subset $U \subset \mathbb{R}^d$, we define

$$\tau^x_U = \tau_U(Z^x), \quad \tau^y_U = \tau_U(\tilde{Z}^y)$$

where we regard $\tau_U$ as $[0, \infty]$-valued function on $C([0, \infty) \times \mathbb{R}^d)$. We note that $\tau^x_U = \inf\{t > 0 \mid (Z^x_t, \tilde{Z}^y_t) \notin U \times \mathbb{R}^d\}$. Hence, $\tau^x_U$ and $\tau^y_U$ are exit times of $(Z^x, \tilde{Z}^y)$. We also see from [11, Lemma II.1.2] that there exists $C > 0$ depending on $d$ such that

$$P_{x,y}(\tau^x_U \leq t) \leq C \exp(-r^2/Ct) \quad (22)$$

for any $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty)$ and $r \in (0, \infty)$. 

Lemma 5. Let $\chi, \varepsilon \in (0,1]$, $R > 0$ and $n \geq 1$ be positive numbers. Then, there is a positive constant $C$ depending on $\varepsilon$, $R$, and $n$ such that
\[
P_{x,y}(\tau^x_B(x,2^{-n}R|x-y|^\chi) \leq \xi_{x,y}) \leq C|x-y|^{1-\chi-\varepsilon}
\]
for any $x, y \in \mathbb{R}^d$ with $|x-y| \in (0,1]$.

Proof. We fix $\chi, \varepsilon \in (0,1]$, $R > 0$, $n \geq 1$. Let $x, y \in \mathbb{R}^d$ with $|x-y| \in (0,1]$. By \cite{22} and Lemma \cite{4} there exists $C > 0$ such that for any $t > 0$,
\[
P_{x,y}(\tau^x_B(x,2^{-n}R|x-y|^\chi) \leq t) + P_{x,y}(\xi_{x,y} > t) \leq C \exp(-2^{-2n}R^2|x-y|^{2\chi}/Ct) + |x-y|/\sqrt{2\pi t}.
\]
(23)

Then, letting $t = |x-y|^{2\chi+2\varepsilon}(\in (0,1])$ in (23), we have
\[
P_{x,y}(\tau^x_B(x,2^{-n}R|x-y|^\chi) \leq \xi_{x,y}) \leq C \exp(-2^{-2n}R^2|x-y|^{2\chi}/C) + |x-y|^{1-\chi-\varepsilon}.
\]
(24)

For any $\delta \in (0,1]$ and $c \in (0,\infty)$ there exists $c_\delta \in (0,\infty)$ depending on $\delta$ and $c$ such that for any $r \in (0,\infty)$,
\[
\exp(-cr^{-\delta}) \leq c_\delta r.
\]
(25)

By using \cite{24} and (25), we obtain the desired inequality. \hfill \Box

From now on, we fix $p \in \mathbb{R}^d$, and take constants $\kappa > d-2$, $R \in (0,1]$, and $K > 0$ so that \cite{5} holds. Theorem \cite{11} is proved by an inductive argument. The following lemma is the first step.

Lemma 6. Let $\varepsilon \in (0,(2-d+\kappa)/(3-d+\kappa))$. There exists $C > 0$ depending on $d$, $\varepsilon$, $p$, $\kappa$, $R$, and $K$ such that
\[
\mathcal{I}_{x,y} \leq C|x-y|^{(2-d+\kappa)/(3-d+\kappa)-\varepsilon}
\]
for any $x, y \in B(p,R/2)$.

Proof. Let $\varepsilon \in (0,(2-d+\kappa)/(3-d+\kappa))$. We fix $x, y \in B(p,R/2)$ and set
\[
r = R|x-y|^{\chi}/2,
\]
where $\chi \in (0,1]$ is a positive number which will be chosen later. Because $|x-y| \leq 1$, we have $r \leq R/2$. A straightforward calculation gives
\[
\mathcal{I}_{x,y} \leq E_{x,y} \left[ A_{\tau^x_B(x,r)}^\mu \right] + P_{x,y}(\tau^x_B(x,r) \leq \xi_{x,y}).
\]
(26)

By applying Lemmas \cite{2} and \cite{4} to (26), we obtain that
\[
\mathcal{I}_{x,y} \leq C\{\zeta_d(r,\kappa) + |x-y|^{1-\chi-\varepsilon}\}.
\]
(27)
Here, $C > 0$ is a positive constant depending on $d$, $\varepsilon$, $\chi$, $\kappa$, $R$, and $K$.

Next, we optimize the right-hand side of (27) in $\chi$. Note that we have for any $a, b > 0$ with $b \leq a$,

$$-s^a \log s \leq (1/b)s^{a-b}, \quad s \in (0, 1].$$

Thus, if $d = 2$, we have

$$\zeta_d(r, \kappa) \leq \frac{\chi}{\varepsilon} \left( \frac{R}{2} \right)^{\kappa-(\varepsilon/\chi)} |x-y|^{\kappa\chi-\varepsilon}$$

provided that $\varepsilon/\chi \leq \kappa$. Let $\chi$ be the solution to $\kappa\chi - \varepsilon = 1 - \chi - \varepsilon$. Then, $\chi = 1/(\kappa + 1)$. Further, $\varepsilon/\chi \leq \kappa$ and

$$I_{x,y} \leq C' |x-y|^{\frac{1}{3\kappa+1}-\varepsilon},$$

where $C'$ is a positive constant depending on $\varepsilon$, $\chi$, $p$, $\kappa$, $R$, and $K$.

If $d \geq 3$ or $d = 1$, we have

$$\zeta_d(r, \kappa) \leq r^{2-d+\kappa-(\varepsilon/\chi)} = \left( \frac{R|x-y|^{\chi}}{2} \right)^{2-d+\kappa-(\varepsilon/\chi)}$$

$$= \left( \frac{R}{2} \right)^{2-d+\kappa-(\varepsilon/\chi)} |x-y|^{(2-d+\kappa)-(\varepsilon/\chi)}.$$

Let $\chi$ be the solution to $\chi(2-d+\kappa)-\varepsilon = 1-\chi-\varepsilon$. Then, $\chi = 1/(3-d+\kappa) \in (0, 1]$ and

$$I_{x,y} \leq C'' |x-y|^{(2-d+\kappa)/(3-d+\kappa)-\varepsilon}.$$

Here, $C''$ is a positive constant depending on $d$, $\varepsilon$, $\chi$, $p$, $\kappa$, $R$, and $K$.

For $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, we set

$$q_{n,\kappa,\varepsilon} = \frac{r_n - \varepsilon r_{n-1}}{r_n + 1} - \varepsilon,$$

where $r_n$ is a positive number defined by

$$r_n = (2 - d + \kappa)(r_{n-1} + 1), \quad r_0 = 0.$$

We then find that $r_n = \sum_{l=1}^{n} (2 - d + \kappa)^l$ and $\{r_n\}_{n=1}^\infty$ is increasing. For any $n \in \mathbb{N}$,

$$q_{n,\kappa,\varepsilon} > 0 \iff \frac{r_n}{r_n + r_{n-1} + 1} > \varepsilon$$

$$\iff \frac{(2 - d + \kappa)r_n}{(2 - d + \kappa)r_n + (2 - d + \kappa)(r_{n-1} + 1)} > \varepsilon$$

$$\iff \frac{2 - d + \kappa}{3 - d + \kappa} > \varepsilon. \quad (29)$$
Lemma 7. Let \( n \in \mathbb{N} \) and \( \varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa)) \). Then, there exists a positive constant \( C_1 \) depending on \( d, \varepsilon, p, \kappa, R, K, \) and \( n \) such that

\[
I_{x,y} \leq C_1 |x - y|^{q_{n,\varepsilon}}
\]

for any \( x, y \in B(p, 2^{-n}R) \).

Proof. If \( n = 1 \), the conclusion follows from Lemma 4. In what follows, we suppose that (30) holds for some \( n \in \mathbb{N} \). Then, for any \( \varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa)) \), there exists \( C_1 > 0 \) depending on \( d, \varepsilon, p, \kappa, R, K, \) and \( n \) such that

\[
I_{x,y} \leq C_1 |x - y|^{q_{n,\varepsilon}}
\]

for any \( x, y \in B(p, 2^{-n}R) \).

Let \( \chi \in (0,1] \) be a positive number which will be chosen later. We fix \( \varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa)) \), and \( x, y \in B(p, 2^{-n-1}) \). To simplify the notation, we write

\[
\tau = \tau^x_{B(x, 2^{-n-1}R|x-y|^\chi)}, \quad \tilde{\tau} = \tau^y_{B(y, 2^{-n-1}R|x-y|^\chi)}, \quad \xi = \xi_{x,y}.
\]

In view of Remark 1(2), we have \( \tau = \tilde{\tau} \). It is straightforward to show that

\[
I_{x,y} \leq E_{x,y} \left[ A^\mu_{\tau^x} \wedge 1 : \xi \leq \tau \right] + E_{x,y} \left[ A^\mu_{\tau} \wedge 1 : \tau < \xi \right]
\]

\[\quad + E_{x,y} \left[ (A^\mu_{\xi} - A^\mu_{\tau}) \wedge 1 : \tau < \xi \right]
\]

\[= E_{x,y} \left[ A^\mu_{\xi} \wedge 1 \right] + E_{x,y} \left[ (A^\mu_{\xi} - A^\mu_{\tau}) \wedge 1 : \tau < \xi \right]
\]

\[= I_1 + I_2.
\]

On the event \( \{ \xi > \tau \} \), we have \( A^\mu_{\xi} - A^\mu_{\tau} = A^\mu_{\xi - \tau} \circ \theta_\tau \leq A^\mu_{\xi - \tau} \circ \theta_\tau \), where \( \{ \theta_t \}_{t \geq 0} \) denotes the shift operator of the coupled process \((Z^x, \tilde{Z}^y)\). We know from Remark 1(2) that \((Z^x, \tilde{Z}^y)\) is a strong Markov process on \( \mathbb{R}^d \times \mathbb{R}^d \). Therefore, we obtain that

\[
I_2 = E_{x,y} \left[ (A^\mu_{\xi - \tau} \circ \theta_\tau) \wedge 1 : \tau < \xi \right]
\]

\[\leq E_{x,y} \left[ E_{Z^x, \tilde{Z}^y} \left[ A^\mu_{\xi - \tau} \wedge 1 : \tau < \xi \right] \right] = E_{x,y} \left[ I_{Z^x, \tilde{Z}^y} : \tau < \xi \right].
\]

Observe that \( Z^x \in B(x, 2^{-n-1}R) \) and \( \tilde{Z}^y \in B(y, 2^{-n-1}R) \). Furthermore, by noting that \( x, y \in B(p, 2^{-n-1}R) \), we have \( |p - Z^x_\tau| < 2^{-n} \) and \( |p - \tilde{Z}^y_\tau| < 2^{-n} \). Then, we use (31) to obtain that

\[
E_{x,y} \left[ I_{Z^x, \tilde{Z}^y} : \tau < \xi \right] \leq C_1 E_{x,y} \left[ |Z^x_\tau - \tilde{Z}^y_\tau|^{q_{n,\varepsilon}} : \tau < \xi \right].
\]

Let \( a_n = (r_n + 1)/r_n \) and \( b_n = r_n + 1 \). Then, \( a_n^{-1} + b_n^{-1} = 1 \). Because \( \varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa)) \), (29) implies that \( 0 < a_n q_{n,\varepsilon} \) and

\[
a_n q_{n,\varepsilon} = \frac{r_n + 1}{r_n} \left( \frac{r_n - \varepsilon r_{n-1}}{r_n + 1} - \varepsilon \right) \leq 1.
\]
By using Hölder’s inequality, Lemmas 3 and 5 we obtain that
\[
E_{x,y} \left[ |Z^\tau - \tilde{Z}^\tau|^\alpha_{n,\kappa,\varepsilon} : \tau < \xi \right] \\
\leq E_{x,y} \left[ |Z_{\tau \wedge \xi} - \tilde{Z}_{\tau \wedge \xi}|^{\alpha_n \alpha_{n,\kappa,\varepsilon}} \right]^{1/\alpha_n} P_{x,y}(\tau < \xi)^{1/b_n} \leq |x-y|^{\alpha_n \alpha_{n,\kappa,\varepsilon}/\alpha_n} P_{x,y}(\tau < \xi)^{1/b_n} \leq C_2|x-y|^{\alpha_{n,\kappa,\varepsilon}+(1-\chi-\varepsilon)/b_n},
\] (35)
where \(C_2\) is a positive constant depending on \(\varepsilon, R,\) and \(n\). Therefore, (33), (34), and (35) imply
\[
I_2 \leq C_3|x-y|^{\alpha_{n,\kappa,\varepsilon}+(1-\chi-\varepsilon)/b_n}.
\] (36)
Here, \(C_3\) is a positive constant depending on \(d, \varepsilon, p, \kappa, R, K,\) and \(n\).

On the other hand, Lemma 2 yields
\[
I_1 \leq E_{x,y} \left[ A_\varepsilon^{\mu,x} \right] \leq C_4 \zeta_d(2^{-n-1}R|x-y|^{\chi,\kappa}),
\] (37)
where \(C_4\) is a positive constant depending on \(d, p, \kappa, R,\) and \(K\). If \(d = 2\), we use (28) to obtain that
\[
\zeta_d(2^{-n-1}R|x-y|^{\chi,\kappa}) \leq (\varepsilon/\chi)(2^{-n-1}R)^{\kappa-(\varepsilon/\chi)}|x-y|^\kappa - \varepsilon.
\] (38)
provided that \(\varepsilon/\chi \leq \kappa\). If \(d \geq 3\) or \(d = 1\),
\[
\zeta_d(2^{-n-1}R|x-y|^{\chi,\kappa}) = (2^{-n-1}R|x-y|^{\chi})^{2-d+\kappa} \leq (2^{-n-1}R)^{2-d+\kappa}|x-y|^{\chi(2-d+\kappa)-\varepsilon}.
\] (39)
Therefore, if \(\varepsilon \leq (2-d+\kappa)\chi\), regardless of the value of \(d\), we obtain from (37), (38), and (39) that
\[
I_1 \leq C_5|x-y|^{(2-d+\kappa)\chi-\varepsilon}.
\] (40)
Here, \(C_5\) is a positive constant depending on \(d, \varepsilon, p, \kappa, R, K,\) and \(n\).

Let \(\eta\) be the solution to
\[
(2-d+\kappa)\eta - \varepsilon = q_{n,\kappa,\varepsilon} + (1-\eta-\varepsilon)/b_n.
\]
Then, a direct calculation and the definition of \(b_n\) imply that
\[
\eta = \frac{b_n q_{n,\kappa,\varepsilon} + 1 + \varepsilon(b_n - 1)}{b_n(2-d+\kappa) + 1} = \frac{(r_n + 1)q_{n,\kappa,\varepsilon} + 1 + \varepsilon r_n}{(2-d+\kappa)(r_n + 1) + 1}.
\]
From the relation \(r_{n+1} = (2-d+\kappa)(r_n + 1)\) and the definition of \(q_{n,\kappa,\varepsilon}\),
\[
(2-d+\kappa)\eta - \varepsilon = \frac{(2-d+\kappa)(r_n + 1)q_{n,\kappa,\varepsilon} + (2-d+\kappa)(1+\varepsilon r_n)}{r_{n+1} + 1} - \varepsilon
\]
Because \( q_{n+1, \kappa, \varepsilon} > 0 \), we have \((2 - d + \kappa)\eta > \varepsilon\). Noting the fact that \( \{r_n\}_{n=1}^\infty \) is increasing, we have

\[
0 < \eta = \frac{r_{n+1} - \varepsilon r_n}{(2 - d + \kappa)(r_{n+1} + 1)} \leq \frac{r_{n+1}}{(2 - d + \kappa)(r_{n+1} + 1)} = \frac{r_{n+1}}{r_{n+2}} \leq 1.
\]

Therefore, we can set \( \chi = \eta \). By combining (32), (30), and (14), we see (30) holds for \( n + 1 \).

Because \( \{r_n\}_{n=1}^\infty \) is increasing, we have for any \( n \in \mathbb{N} \) and \( \varepsilon \in (0, 1) \),

\[
q_{n, \kappa, \varepsilon} = \frac{r_n - \varepsilon r_{n-1}}{r_n + 1} - \varepsilon \geq \frac{r_n}{r_n + 1} - 2\varepsilon.
\]

If \( 2 - d + \kappa \geq 1 \), \( \lim_{n \to \infty} r_n = \infty \). If \( 2 - d + \kappa \in (0, 1) \),

\[
\lim_{n \to \infty} r_n = \frac{2 - d + \kappa}{1 - (2 - d + \kappa)}.
\]

Therefore, we obtain that \( \lim_{n \to \infty} q_{n, \kappa, \varepsilon} \geq (2 - d + \kappa) \land 1 - 2\varepsilon \). Since the same estimate as Lemma 4 holds for \( \mathcal{I}_{x,y} \), we have the following corollary.

**Corollary 1.** For any \( \varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa)) \), there exists a positive constants \( C \) depending on \( d, \varepsilon, p, \kappa, R, \) and \( K \) such that

\[
\mathcal{I}_{x,y} + \tilde{\mathcal{I}}_{x,y} \leq C|x - y|^{(2-d+\kappa)\land 1-\varepsilon}
\]

for any \( x, y \in B(p, 2^{C}R) \), where \( C' \) is a positive constant depending on \( d, \varepsilon \) and \( \kappa \).

We now prove Theorem 1.

**Proof (of Theorem 1).** Let \( \alpha > 0, x, y \in \mathbb{R}^d \) and \( f \in B_b^p(\mathbb{R}^d) \). Without loss of generality, we may assume that \( \|f\|_\infty \leq 1 \). We write \( \xi = \xi_{x,y} \) for the simplicity. Then, we have

\[
V^\mu_{f}(x) - V^\mu_{f}(y) = E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t,x}) f(Z^\mu_t) \, dA^\mu_{t,x} \right] - E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t,y}) f(Z^\mu_t) \, dA^\mu_{t,y} \right] - E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t,x}) f(Z^\mu_t) \, dA^\mu_{t,x} \right] - E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t,y}) f(Z^\mu_t) \, dA^\mu_{t,y} \right] =: J_1 - J_2 - J_3 - J_4.
\]

Because \( \{A^\mu_{t,x}\}_{t \geq 0} \) is a PCAF of \((Z^\mu, \tilde{Z}^\mu)\), we have \( A^\mu_{t+\xi} = A^\mu_{t+\xi} + A^\mu_{t} \circ \theta_\xi \) and \( dA^\mu_{t+\xi} = dA^\mu_{t} \circ \theta_\xi \). By using these equations and the strong Markov property of \((Z^\mu, \tilde{Z}^\mu)\), we obtain that

\[
J_1 = E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t+\xi}) f(Z^\mu_{t+\xi}) \, dA^\mu_{t+\xi} \right] = E_{x,y} \left[ \exp(-\alpha A^\mu_{t+\xi}) V^\mu_{f}(Z^\mu_{t+\xi}) \right],
\]

\[
J_2 = E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t,y}) f(Z^\mu_t) \, dA^\mu_{t,y} \right] = E_{x,y} \left[ \exp(-\alpha A^\mu_{t,y}) V^\mu_{f}(Z^\mu_t) \right],
\]

\[
J_3 = E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t,x}) f(Z^\mu_t) \, dA^\mu_{t,x} \right] = E_{x,y} \left[ \exp(-\alpha A^\mu_{t,x}) V^\mu_{f}(Z^\mu_t) \right],
\]

\[
J_4 = E_{x,y} \left[ \int_0^\infty \exp(-\alpha A^\mu_{t,y}) f(Z^\mu_t) \, dA^\mu_{t,y} \right] = E_{x,y} \left[ \exp(-\alpha A^\mu_{t,y}) V^\mu_{f}(Z^\mu_t) \right].
\]
\[ J_2 = E_{x,y} \left[ \exp(-\alpha \tilde{A}_\xi^{\mu,y}) V^\mu f(\tilde{Z}_\xi^y) \right]. \]

Since \( Z^x_\xi = \tilde{Z}_\xi^y \), we have

\[ J_1 - J_2 = E_{x,y} \left[ \exp(-\alpha A^{\mu,x}_\xi) V^\mu f(Z^x_\xi) \right] - E_{x,y} \left[ \exp(-\alpha \tilde{A}_\xi^{\mu,x}) V^\mu f(\tilde{Z}_\xi^x) \right] \]
\[ + E_{x,y} \left[ \exp(-\alpha A^{\mu,x}_\xi) V^\mu f(\tilde{Z}_\xi^y) \right] - E_{x,y} \left[ \exp(-\alpha \tilde{A}_\xi^{\mu,y}) V^\mu f(\tilde{Z}_\xi^y) \right] \]
\[ = 0 + E_{x,y} \left[ \{ \exp(-\alpha A^{\mu,x}_\xi) - \exp(-\alpha \tilde{A}_\xi^{\mu,y}) \} V^\mu f(\tilde{Z}_\xi^y) \right]. \]

Because \( |\alpha V^\mu f(\tilde{Z}_\xi^y)| \leq \|f\|_\infty = 1 \) and the function \( s \mapsto e^{-\alpha s} \) is \( \alpha \)-Lipschitz continuous on \([0, \infty)\), we obtain that

\[
|J_1 - J_2| \leq E_{x,y} \left[ \left| A^x_\xi - \tilde{A}^y_\xi \right| \wedge \alpha^{-1} \right] 
\leq (1 + \alpha^{-1})(\mathcal{I}_{x,y} + \tilde{\mathcal{I}}_{x,y}). \tag{42}
\]

From Jensen’s inequality,

\[
|J_3 - J_4| \leq \alpha^{-1} E_{x,y} \left[ 1 - \exp(-\alpha A^x_\xi) \right] + \alpha^{-1} E_{x,y} \left[ 1 - \exp(-\alpha \tilde{A}^y_\xi) \right] 
\leq E_{x,y} \left[ A^x_\xi \wedge \alpha^{-1} \right] + E_{x,y} \left[ \tilde{A}^y_\xi \wedge \alpha^{-1} \right] 
\leq (1 + \alpha^{-1})(\mathcal{I}_{x,y} + \tilde{\mathcal{I}}_{x,y}). \tag{43}
\]

By using (41), (42), and (43), we arrive at

\[
|V^\mu f(x) - V^\mu f(y)| \leq 2(1 + \alpha^{-1})(\mathcal{I}_{x,y} + \tilde{\mathcal{I}}_{x,y}) \tag{44}
\]

Corollary 1 and (44) yield the desired estimate. “In particular” part immediately follows from (44). \( \square \)

References

1. R. F. Bass, Probabilistic techniques in analysis, Probability and its Applications (New York), Springer-Verlag, New York, 1995.
2. R. F. Bass, Diffusions and elliptic operators, Probability and its Applications (New York), Springer-Verlag, New York, 1998.
3. R. F. Bass, M. Kassmann and T. Kumagai, Symmetric jump processes: localization, heat kernels and convergence, Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), 59–71.
4. Z.-Q. Chen and M. Fukushima, Symmetric Markov processes, time change, and boundary theory, London Mathematical Society Monographs Series, vol. 35, Princeton University Press, Princeton, NJ, 2012.
5. M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov processes, De Gruyter Studies in Mathematics, vol. 19, Second revised and extended edition, Walter de Gruyter & Co., Berlin, 2011.
6. M. Fukushima and T. Uemura, *Capacitary bounds of measures and ultracontractivity of time changed processes*, J. Math. Pures Appl. (9) **82** (2003), 553–572.
7. C. Garban, R. Rhodes, and V. Vargas, *Liouville Brownian motion*, Ann. Probab. **44** (2016), 3076–3110.
8. C. Garban, R. Rhodes, and V. Vargas, *On the heat kernel and the Dirichlet form of Liouville Brownian motion*, Electron. J. Probab. **19** (2014), no. 96, 25.
9. O. Kallenberg, *Foundations of modern probability*, 2nd ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002.
10. T. Lindvall and L. C. G. Rogers, *Coupling of multidimensional diffusions by reflection*, Ann. Probab. **14** (1986), 860–872.
11. D. W. Stroock, *Diffusion semigroups corresponding to uniformly elliptic divergence form operators*, Séminaire de Probabilités, XXII, Lecture Notes in Math., vol. 1321, Springer, Berlin, 1988, 316–347.