Infinite square-free self-shuffling words

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Abstract. An infinite word $w$ is called self-shuffling, if $w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i$ for some finite words $U_i, V_i$. Harju [4] recently asked whether square-free self-shuffling words exist. We answer this question affirmatively.

1 Introduction

A self-shuffling word, a notion which was recently introduced by Charlier et al. [2], is an infinite word that can be reproduced by shuffling it with itself. More formally, an infinite word $w \in \Sigma^\omega$, defined over a finite alphabet $\Sigma$, is self-shuffling if $w$ admits factorizations: $w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i$ with $U_i, V_i \in \Sigma^+$. Various well-known words, e.g. the Thue-Morse word or the Fibonacci word, were shown to be self-shuffling.

Harju [4] studied shuffles of both finite and infinite square-free words, i.e. words that have no factor of the form $uu$. More results on square-free shuffles were obtained independently by Harju and Müller [5], and Currie and Saari [3]. However, the question about the existence of an infinite square-free self-shuffling word, posed in [4], remained open. We give a positive answer to this question in this note.

Apart from the usual concepts in combinatorics on words, which can be found for instance in the book of Lothaire [6], we make use of the following notations: For every $k \geq 1$, we denote the alphabet $\{0,1,\ldots,k-1\}$ by $\Sigma_k$. For a word $w = uvz$ we say that $u$ is a prefix of $w$, $v$ is a factor of $w$, and $z$ is a suffix of $w$. We denote these prefix- and suffixrelations by $u \preceq_p w$ and $v \preceq_s w$, respectively.

A prefix code is a set of words with the property that none of its elements is a prefix of another element. Similarly, a suffix code is a set of words where no element is a suffix of another one. A bifix code is a set that is both a prefix code and a suffix code.

A word $w$ is square-free, if it has no factor of the form $uu$, where $u$ is a non-empty finite word. A morphism $h$ is square-free if for all square-free words $w$, the image $h(w)$ is square-free.

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2 A square-free self-shuffling word on 4 letters

Let $h : \Sigma_4^* \rightarrow \Sigma_4^*$ be the morphism defined as follows:

$$
\begin{align*}
    h(0) &= 0121, \\
    h(1) &= 032, \\
    h(2) &= 013, \\
    h(3) &= 0302.
\end{align*}
$$

We will show that the fixpoint $w = h^ω(0)$ is square-free and self-shuffling in the following. Note that $h$ is not a square-free morphism, that is it does not preserve square-freeness, as $h(23) = 0130302$ contains the square 3030.

Lemma 1. The word $w = h^ω(0)$ contains no factor of the form $3u1u3$ for some $u \in \Sigma_4^*$.

Proof. We assume that there exists a factor of the form $3u1u3$ in $w$, for some word $u \in \Sigma_4^*$. From the definition of $h$, we observe that $u$ cannot be empty. Furthermore, we see that every 3 in $w$ is preceded by either 0 or 1. If $1 \leq s u$, then we had an occurrence of the factor 11 in $w$, which is not possible by the definition of $h$, hence $0 \leq s u$. Now, every 3 is followed by either 0 or 2 in $w$ and 01 is followed by either 2 or 3. Since both 3u and 01u are factors of $w$, we must have $2 \leq p u$. This means that the factor 012 appears at the center of $u1u$, which can only be followed by 1 in $w$, thus $21 \leq p u$. However, this results in the factor 321 as a prefix of 3u1u3, which does not appear in $w$, as seen from the definition of $h$. □

Lemma 2. The word $w = h^ω(0)$ is square-free.

Proof. We first observe that $\{h(0), h(1), h(2), h(3)\}$ is a bifix code. Furthermore, we can verify that there are no squares $uu$ with $|u| \leq 3$ in $w$. Let us assume now, that the square $uu$ appears in $w$ and that $u$ is the shortest word with this property. If $u = 02u'$, then $u' = u''03$ must hold, since 02 appears only as a factor of $h(3)$, and thus $uu$ is a suffix of the factor $h(3)u''h(3)u''$ in $w$. As $w = h(w)$, also the shorter square $3h^{-1}(u'')3h^{-1}(u'')$ appears in $w$, a contradiction. The same desubstitution principle also leads to occurrences of shorter squares in $w$ if $u = xu'$ and $x \in \{01, 03, 10, 12, 13, 21, 30, 32\}$.

If $u = 2u'$ then either $03 \leq s u$ or $030 \leq s u$ or $01 \leq s u$, by the definition of $h$. In the last case, that is when $01 \leq s u$, we must have $21 \leq p u$, which is covered by the previous paragraph. If $u' = u''030$, then $uu$ is followed by 2 in $w$ and we can desubstitute to obtain the shorter square $h^{-1}(u'')3h^{-1}(u'')3$ in $w$. If $u = 2u'$ and $u' = u''03$, and $uu$ is preceded by 03 or followed by 2 in $w$, we can desubstitute to $1h^{-1}(u'')1h^{-1}(u'')$ or $h^{-1}(u'')1h^{-1}(u'')1$, respectively. Therefore, assume that $u = 2u''03$ and $uu$ is preceded by 030 and followed by 02 in $w$. This however means that we can desubstitute to get an occurrence of the factor $3h^{-1}(u'')1h^{-1}(u'')3$ in $w$, a contradiction to Lemma 1. □
We now show that \( w = h^{\omega}(0) \) can be written as \( w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i \) with \( U_i, V_i \in \Sigma_3^+ \).

**Lemma 3.** The word \( w = h^{\omega}(0) \) is self-shuffling.

**Proof.** In what follows we use the notation \( x = v^{-1}u \) meaning that \( u = vx \) for finite words \( x, u, v \). We are going to show that the self-shuffle is given by the following:

\[
U_0 = h^2(0), \quad U_1 = 0, \quad \ldots, \quad U_{6i+2} = h^i(0^{-1}h(0)0), \quad U_{6i+3} = h^i(0^{-1}h(3)0),
\]
\[
U_{6i+4} = h^i(0^{-1}h(201)0), \quad U_{6i+5} = h^i(30),
\]
\[
U_{6i+6} = h^i(2h(03)), \quad U_{6i+7} = h^{i+1}(20),
\]
\[
V_0 = h(0)03, V_1 = 2h(2)0, \ldots, V_{6i+2} = h^i(0^{-1}h(1)0), \quad V_{6i+3} = h^i(0^{-1}h(03)0),
\]
\[
V_{6i+4} = h^i(1), \quad V_{6i+5} = h^i(3),
\]
\[
V_{6i+6} = h^{i+1}(0), \quad V_{6i+7} = h^{i+1}(0^{-1}h(2)0).
\]

Now we verify that \( w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i \), from which it follows that \( w \) is self-shuffling. It suffices to show that each of the above products is fixed by \( h \). Indeed, straightforward computations show that

\[
\prod_{i=0}^{\infty} U_i = h^2(0)h^2(121)h^3(121)\ldots
\]

which is fixed by \( h \):

\[
h(\prod_{i=0}^{\infty} U_i) = h[h^2(0)h^2(121)h^3(121)\ldots] = h^3(0)h^3(121)h^4(121)\ldots = \prod_{i=0}^{\infty} U_i,
\]

hence \( \prod_{i=0}^{\infty} U_i \) is fixed by \( h \) and thus \( w = \prod_{i=0}^{\infty} U_i \). In a similar way we show that \( w = \prod_{i=0}^{\infty} V_i = \prod_{i=0}^{\infty} U_i V_i \).

\( \square \)

3 Square-free self-shuffling words on 3 letters

We remark that we can immediately produce a square-free self-shuffling word over \( \Sigma_3 \) from \( h^{\omega}(0) \): Charlier et al. \[2\] noticed that the property of being self-
shuffling is preserved by the application of a morphism. Furthermore, Brandenburg [1] showed that the morphism \( f : \Sigma^* \rightarrow \Sigma^* \), defined by

\[
\begin{align*}
    f(0) &= 01020120210210212, \\
    f(1) &= 010201202102010212, \\
    f(2) &= 010201202120121012, \\
    f(3) &= 010201210201021012,
\end{align*}
\]

is square-free. Therefore, the word \( f(h^\omega(0)) \) is a ternary square-free self-shuffling word, from which we can produce a multitude of others by applying square-free morphisms from \( \Sigma^*_3 \) to \( \Sigma^*_3 \).

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