RICCI FLOW AND VOLUME RENORMALIZABILITY

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Abstract. Relative to a special boundary defining function, conformally compact asymptotically hyperbolic metrics have an expansion in a collar neighborhood of conformal infinity. When this expansion is even to a certain order and satisfies an additional auxiliary condition, a renormalized volume is defined. We prove such expansions are preserved under the normalized Ricci flow. We study the variation of curvature functionals in this setting. As an application we obtain the following variation formula.

\[ \frac{d}{dt} \text{RenV}(M^n, g(t)) = - \int_{M^n} (S(g(t)) + n(n-1)) \, dV_{g(t)}, \]

where \( S(g(t)) \) is the scalar curvature for the evolving metric \( g(t) \), and \( \text{Riesz renormalization} \) is the Riesz renormalization. This extends our earlier work to a broader class of metrics.

A complete, conformally compactifiable Riemannian manifold \((M, g)\) is called asymptotically hyperbolic if it is conformally isometric to the interior of a smooth compact manifold \( \overline{M} \) with boundary \( \partial M \) with curvatures tending to \(-1\) at \( \partial M \). Equivalently, \( g \) can be written as \( g = \psi^* \rho^{-2} \mathcal{g} \), where \( \mathcal{g} \) is smooth and nondegenerate up to the boundary \( \partial M \) defined by \( \rho = 0 \), and \( |d\rho/\rho|^2_g \rightarrow 1 \) at this boundary. We call \( \rho \) a boundary defining function for \( \partial M \) and call \((\partial M, [h_0])\) the boundary-at-infinity of \((M, g)\), where \([h_0]\) is the conformal class of the metric \( h_0 \) induced on \( \partial M \) by \( \mathcal{g} \). Such metrics provide the setting for many interesting problems and investigations in geometric analysis and physics. It is known [11] that if \( g \) is AH, then there exists another boundary defining function \( x \) such that, in a collar neighborhood of the boundary,

\[ g = \frac{dx^2 + h(x)}{x^{2}}, \]

where \( h \sim \sum x^j h_j \) is a smooth family of metrics on \( \partial M \). Then \( x \) is called a special defining function for \( \partial M \). The ‘special metrics’ in this class are called Poincaré-Einstein (PE); a more general class consists of the asymptotically Poincaré-Einstein (APE) metrics, which are conformally compact and have an Einstein tensor

\[ E(g) = \text{Rc}(g) + (n-1)g \]

that vanishes near \( \partial M \) up to corrections of order \( x^n \) (\( n = \dim M \)). There are slight differences in this description depending on whether \( n \) is even or odd. To simplify the discussion here, and also because it is our primary focus in this paper, we assume henceforth that \( n \) is even.

The normalized Ricci flow in this setting is the system of equations

\[ \partial_t g = -2E(g) , \ t \in [0, T), \quad g(0) = g^0. \]
The normalization factor fixes the conformal infinity in time. PE metrics are the stationary points of this normalized Ricci flow, but this flow also preserves the class of APE metrics \[^3\].

AH metrics have exponential volume growth, but using the regularity of the metric at infinity, i.e., near \(\partial M\), it is sometimes possible to define an Hadamard renormalized volume \(\text{RenV}(M,g)\). This is done by computing the volume of \((M,g)\) in the compact set \(\{x \geq \epsilon\}\), expanding the result in powers of \(\epsilon\) (there can also be \(\log \epsilon\) terms when the bulk dimension \(n\) is odd), and taking the limit at \(\epsilon = 0\) after discarding the singular terms in this expansion. While this may be done for any AH metric with a fixed special defining function \(x\), it turns out \[^{10}\] that this has an invariant meaning which is independent of the choice of \(x\) provided \(g\) satisfies an auxiliary condition concerning the omission of a certain number of odd terms in the expansion of the term \(h(x)\). More precisely, (i) \(h(x)\) must have an even expansion to order \(n - 2\), and (ii) the first odd term in this expansion must have vanishing trace. We call AH metrics which satisfy condition (i) \textit{partially even}. Partially even metrics that satisfy condition (ii) are called \textit{volume renormalizable}. Thus we are asserting that (when \(n\) is even) the renormalized volume of an AH metric \(g\) has a well-defined, invariant meaning if and only if these conditions are satisfied. Significantly, every even dimensional PE metric, as well as every even dimensional APE metric, satisfies these conditions.

AH metrics with various assumptions on the evenness of the expansion have been considered frequently before. Perhaps most important is the appearance of this condition in geometric scattering theory. The paper \[^{14}\] by the second author and Melrose established a general construction in the framework of what is now called geometric microlocal analysis for the resolvent of the scalar Laplacian of an AH metric. More specifically, it is proved there that the family of operators \((\Delta + \zeta(n - 1 - \zeta))^{-1}\), which depends holomorphically on \(\zeta\) in the right half-plane \(\text{Re}\zeta > (n - 1)/2\), extends to a meromorphic function in the punctured plane \(\mathbb{C} \setminus \{(n - 1)/2 - j, j \in \mathbb{N}\}\). That analysis left open the possibility that there could be an accumulation of poles at this sequence of points along the (shifted) negative real axis. An important paper by Guillarmou \[^{13}\] clarified the situation by showing that this accumulation does not happen precisely when the metric \(g\) is even to all orders. This has many significant consequences for the scattering theory. This evenness condition has also appeared in the more recent and quite different approach by Vasy \[^{16}\] for studying the meromorphic continuation of this resolvent family.

In our earlier paper \[^3\], we studied the behaviour of \(\text{RenV}\) under Ricci flow for an evolving family of APE metrics. Our goal in this paper is to extend this analysis by studying this Ricci evolution and its effect on the renormalized volume for the entire class of partially even metrics. The key difference between this larger class of metrics and the APEs is that the coefficient tensors \(h_{2j}, 1 \leq j \leq (n - 2)/2\), and \(\text{tr}h_0h_{n-1}\) in the expansion for \(h\) are formally determined by the leading coefficient \(h_0\) when \(g\) is APE, whereas they are independent of one another when \(g\) is partially even. The interrelationships between these tensors when \(g\) is APE is precisely what forces the Einstein tensor to vanish to order \(n\) in that case, while for partially even metrics, the Einstein tensor need only vanish to second order.

We now state our main theorems.

**Theorem A.** Suppose that \((M,g^0)\), \(\dim M = n = 2m\), is partially even, and let \(g(t)\) be a solution of (0.3) with \(g(0) = g^0\) with maximal interval of existence \([0,T_0)\). Then \((M,g(t))\) remains partially even for \(t < T_0\).
This is proved in several steps. The first is to show that the Ricci-DeTurck flow preserves the property of being partially even relative to a fixed defining function. This depends crucially on the earlier result by the first author [2] that the Ricci-DeTurck and Ricci flows preserve the property of being conformally compact (or polyhomogeneous). We then argue that the transformation which carries this to a solution of the Ricci flow preserves the form of the expansion. We then show that the re-expression of $g(t)$ relative to the appropriate special boundary defining function $x$ (which depends on $t$) still preserves this form of the expansion.

Our second main result computes the variation of the renormalized volume along this flow. We cast this slightly more generally by proving that the variation through volume renormalizable metrics of any curvature functional is just the renormalized first variation of the same form that, on compact manifolds, yields the Euler-Lagrange equations (usually after integration by parts), see Theorem 4.4 below. An immediate application is the

**Theorem B.** Let $(M, g^0)$, $n = 2m$, be volume renormalizable, and $g(t)$ the solution of (0.3) with $g(0) = g^0$. Then

$$
\frac{d}{dt}\text{RenV} = -R\int_M S(g(t)) + n(n-1)dv_{g_t}.
$$

(0.4)

Here $R\int (\cdot)dv_g$ indicates that the integral is not classically convergent, but must be regularized in a way to be made precise below. The analogue of formula (0.4) in [3] appears almost exactly the same except that when $g(t)$ is APE, the integral is convergent and there is no need to renormalize the integral.

There is recent work on the Kähler-Ricci flow in the setting of asymptotically complex hyperbolic metrics. It is proved in [15] that polyhomogeneity is preserved under this flow. There is an analogue of renormalized volume in this setting and it would be interesting to understand how our results could be carried over.

This paper is organized as follows. In Section 1 we review the families of asymptotically hyperbolic metrics considered here and establish some of basic properties involving even expansions. In Section 2 we prove Theorem A that the normalized Ricci flow of a partially even metric remains partially even, and in Section 3 we show that the volume renormalizability condition also persists under the Ricci flow. We defer several long computations needed in this section to the Appendix. Finally in Section 4 we review the Riesz renormalization and apply it to the variation of curvature functionals. We then prove Theorem B.

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1. Fefferman-Graham expansions of AH metrics

1.1. Taxonomy of conformally compactifiable metrics. As in the Introduction, let $\overline{M}$ be a compact manifold with boundary and $M$ its interior.

A metric $g$ on $M$ is called conformally compact if

$$
g = \rho^{-2}\overline{g},
$$

(1.1)

where both $\rho$ and $\overline{g}$ are $C^\infty$ up to $\partial M$, and $\overline{g}$ is a metric on $\overline{M}$. 

Any such metric has a sort of boundary value on \( \partial M \),
\[
\mathcal{c}(g) = \left[ \rho^2 g \right]_{\partial M},
\] (1.2)
called its conformal infinity. Furthermore, a conformally compact metric is called asymptotically hyperbolic, abbreviated as AH, if \( |d\rho|^2_\mathcal{g} = 1 \) at \( \partial M \). Note that \( g \) is unchanged if we replace \( \rho \) and \( g \) by \( a\rho \) and \( a^2 g \) for any smooth positive function \( a \), but nonetheless both \( \mathcal{c}(g) \) and the AH condition are well-defined.

Now, if \( g \) is AH and we select a representative metric \( h_0 \in \mathcal{c}(g) \), then there exists a uniquely determined ‘special’ boundary defining function \( x \) which satisfies
\[
|dx|^2_x g \equiv 1
\]
in a neighborhood of the boundary. With this choice of \( x \), the flow lines of the gradient \( \nabla \mathcal{g} \) identify a collar neighborhood of \( \partial M \) with \( [0, 1) \times \partial M \), and by Gauss’ Lemma
\[
g = \frac{dx^2 + h(x)}{x^2},
\] (1.3)
where \( h(x) \) is a smooth family of metrics on \( \partial M \). This is the Graham-Lee normal form [11]. We shall be primarily concerned with the terms in the asymptotic expansion
\[
h(x) = h_0 + h_1 x + h_2 x^2 + \cdots.
\] (1.4)
For some of the computations below we will also need a full coordinate system near the boundary of \( M \). Let \( \{y^a\} \) be any coordinates on \( \partial M \) extended to be constant along the integral curves of \( \nabla \mathcal{g} \). We use greek indices to index these tangential directions, \( x \) to index \( \partial_x \), and latin indices to include both tangential and normal directions. In the expansion above, the coefficient tensors are functions of the \( y \)-coordinates alone.

**Definition 1.1.** An AH metric \( g \) is called even to order \( 2\ell \) if, in Graham-Lee normal form (1.2), the expansion for \( h(x) \) contains no terms \( h_{2j+1} \) with \( j \leq \ell - 1 \), i.e.,
\[
h(x) \sim h_0 + x^2 h_2 + \cdots + x^{2\ell} h_{2\ell} + x^{2\ell+1} h_{2\ell+1} + \ldots,
\] (1.5)
Our main focus will be on AH metrics \( g \) which are even to order \( n - 2 \), and we call these simply partially even.

**Definition 1.2.** We say that \( g \) is volume renormalizable (VR) if it is partially even and in addition satisfies
\[
\text{tr}^{h_0} h_{n-1} = 0.
\] (1.6)

We now recall one of the most important classes of AH metrics

**Definition 1.3.** An AH metric \( g \) is called Poincaré-Einstein (PE) if its Einstein tensor \( E(g) \), defined in (0.2), vanishes identically.

Since our primary concern is with asymptotic expansions at the boundary, it is also natural to introduce the

**Definition 1.4.** An AH metric \( g \) is called asymptotically Poincaré-Einstein (APE) if \( |E(g)|_g = O(x^n) \).

If \( g \) is APE, or in particular PE, then it is known, see [9], that every odd coefficient \( h_{2j+1}, 0 \leq j \leq (n-4)/2 \), must vanish while the even coefficients \( h_{2j}, 1 \leq j \leq (n-2)/2 \), can be expressed as universal differential operators applied to \( h_0 \), see [9], and finally, \( \text{tr}^{h_0} h_{n-1} = 0 \). This implies that when \( n \) is even, PE \( \implies \) APE \( \implies \) VR \( \implies \) partially even \( \implies \) AH.
We have not included the slightly different results when \( n \) is odd; the key difference in that case is that when \( g \) is PE, the expansion for \( h(x) \) may also include the terms \( x^{n-1+j}(\log x)^j \tilde{h}_{jj}, j \geq 0 \). When \( n = 2m \), if \( g \) is merely \( C^2 \) conformally compact and Einstein, then a regularity theorem due to Chruściel, Delay, Lee and Skinner [6] shows that if \( h_0 \in C^\infty \), then \( h(x) \) is automatically smooth up to \( x = 0 \), i.e., it is not necessary to assume a priori that \( g \) is smoothly conformally compact, so long as its conformal infinity contains a smooth representative.

1.2. Partially even metrics. As before, we assume that \( n = \dim M \) is even, and write \( n = 2m \). To make Definition [11] sensible, we must establish the following

**Proposition 1.5.** If an AH metric \( g \) is even to order \( 2\ell \) in the normal form corresponding to one choice of metric \( h_0 \in \mathfrak{c}(g) \), then it is even to the same order with respect to any other metric \( h_0' \in \mathfrak{c}(g) \).

The proof appears in [13, Lemma 2.1]. We give a proof by an alternative method.

**Proof.** Write \( h_0' = e^{2\omega_0}h_0 \), where \( \omega_0 \in C^\infty(\partial M) \). The special boundary defining function \( x' \) associated to \( h_0' \) is then given by \( x' = e^{\omega_0}x \), where \( |d(e^{\omega_0}x)/(e^{\omega_0}x)|_g^2 \equiv 1 \). Expand this, writing \( \bar{\gamma} = x^2g \), and using that \( |dx/x|^2_\bar{\gamma} \equiv 1 \), to get

\[
2x(dx, dx)_\bar{\gamma} + x^2|d\omega|^2_\bar{\gamma} = 0, \quad \omega|_{\partial M} = \omega_0. \tag{1.7}
\]

As in [11], divide by \( x \) to obtain a nondegenerate Hamilton-Jacobi equation, which means that there is a unique solution \( \omega \in C^\infty \) for \( 0 \leq x < \epsilon \). In the normal form for \( \bar{\gamma} \), this equation becomes

\[
\partial_x\omega + \frac{1}{2}x \left( (\partial_x\omega)^2 + h^{\alpha\beta}(x)\partial_{y^\alpha}\omega\partial_{y^\beta}\omega \right) = 0. \tag{1.8}
\]

There are two different ways to see from this that \( \omega \) must be even to order \( 2\ell + 2 \). We now explain both approaches.

The first involves simply computing the successive coefficients. Thus expand \( \omega = \sum \omega_j(y)x^j \), substitute into the equation, and collect like powers of \( x \), to get the sequence of equations

\[
2j\omega_j + \sum_{2p+q+r+2=j} h^{\alpha\beta}_{2p}(\partial_{y^\alpha}\omega_q)(\partial_{y^\beta}\omega_r) = 0, \quad j \leq 2\ell. \tag{1.9}
\]

If \( j = 1 \), then the second summand is empty, so \( \omega_1 = 0 \). Next, consider any \( \omega_{2j+1} \), \( j \leq \ell \). The terms in the accompanying sum have \( 2p+q+r+2 = 2j+1 \), hence \( p \leq j-1 \), \( q, r \leq 2j-1 \), and precisely one of \( q \) or \( r \) must be odd. By induction, either \( \omega_q = 0 \) or \( \omega_r = 0 \). In any case, each summand has a vanishing factor, so \( \omega_{2j+1} = 0 \). This means that the first possible odd term is \( x^{2\ell+3} \).

For the other approach we need a bit of notation. Let \( \tilde{M} \) denote the double of \( M \) across its boundary. The atlas of \( C^\infty \) functions on \( \tilde{M} \) is determined uniquely once we specify an identification of a collar neighborhood \( \mathcal{U} \) of \( \partial M \) with the product \( \partial M \times [0, \epsilon) \); the double of this neighborhood, \( \tilde{\mathcal{U}} \), is then endowed with the atlas of \( C^\infty \) functions on the product \( \partial M \times (-\epsilon, \epsilon) \). This product decomposition is determined by following the gradient flow lines of \( \nabla \tilde{\gamma}x \); in other words, the choice of \( h_0 \in \mathfrak{c}(g) \) determines \( x \) and \( \bar{\gamma} \) and hence the \( C^\infty \) structure on \( \tilde{M} \).

Now fix the atlas on \( \tilde{M} \) associated to \( h_0 \); it is clear that \( \omega \) is even to order \( 2\ell + 2 \) if and only if its even extension \( \tilde{\omega} \) to \( \tilde{M} \) is \( C^{2\ell+2,\mu} \) for every \( 0 < \mu < 1 \). (Of course, \( \tilde{\omega} \)
is also $C^\infty$ on each ‘side’ of $\partial M$ in $\tilde{M}$.) However, this extended function is a solution of the nondegenerate extended Hamilton-Jacobi equation obtained by extending the coefficients $h^{\alpha\beta}$ of \ref{AHmetric} as even functions and $x$ as an odd function. The extended coefficients $h^{\alpha\beta}(x)$ are in the Hölder class $C^{2\ell,\mu}$ (but not in $C^{2\ell+1}$ if the coefficients of $|x|^{2\ell+1}$ in their expansions do not vanish), so standard theory shows that the solution lies in $C^{2\ell+1,\mu}$. This shows that $\tilde{\omega}$ is even to order $2\ell + 2$ as claimed.

To finish the argument, if $\hat{h}_0 = e^{2\omega} h_0$ and $\hat{x} = e^{2\omega} x$ is the new special boundary defining function, then the metric $\hat{g} = e^{2\omega} \tilde{g} = (\hat{x})^2 g$ is $C^{2\ell,\mu}$ on $\tilde{M}$, so the metric induced by pullback under the map $(\hat{x}, y) \to \exp_\hat{g}(\hat{x} \hat{\nu})$ is $C^{2\ell-1,\mu}$ (here $\hat{\nu} = e^{-\omega_0(y)} \hat{\nu}$ is the $\hat{g}$ unit normal to $\partial M$). But this metric is in normal form and hence is even to order $2\ell$. \hfill $\square$

It follows from this proof that if $g$ is an AH metric which is even to order $2\ell$, then it defines an equivalence class of defining functions $[x]$, where $x \sim x'$ if $x'/x$ is even to order $2\ell + 2$.

We now extend this discussion by considering how to recognize partially even AH metrics which are not written in normal form. Thus fix an AH metric $g$ and suppose that $x$ is some boundary defining function. As before we set $\overline{g} = x^2 g$ and use the gradient lines of $\nabla \overline{g} x$ to define a smooth structure on the double $\tilde{M}$, and finally, consider the extended metric $\tilde{g}$ which restricts to $\overline{g}$ on $M$ and satisfies $I^*\tilde{g} = \tilde{g}$, where $I : \tilde{M} \to M$ is reflection across the submanifold $\partial M \subset \tilde{M}$.

**Definition 1.6.** We say that $g$ is even to order $j$ relative to an arbitrary boundary defining function $x$ if $\tilde{g}$ is $C^j$ on $\tilde{M}$ (still recalling that it is $C^\infty$ on each side of $\partial M$).

If $j = 2\ell$, it is straightforward to see that if $g$ satisfies this condition, then relative to any choice of coordinates $y$ on $\partial M$, the components $\overline{g}_{\alpha\beta}$ and $\overline{g}_{\alpha\beta}$ are even to order $2\ell$ while $\overline{g}_{\alpha\beta}$ is odd to order $2\ell + 1$ (i.e., it is of the form $xm_{\alpha\beta}$ where $m_{\alpha\beta}$ is even to order $2\ell$). To explain this last condition, note that a term of order $2\ell$ in the expansion for $\overline{g}_{\alpha\beta}$ with the correct parity must be of the form $a(y)(\text{sgn } x)|x|^{2\ell}$, and hence is not $C^{2\ell}$. On the other hand, if $j = 2\ell + 1$, then evenness to order $j$ implies that $\overline{g}_{\alpha\beta}$ and $\overline{g}_{\alpha\beta}$ are even to order $j + 1 = 2\ell + 2$ while $\overline{g}_{\alpha\beta}$ is still just odd to order $2\ell + 1$. If the components of $\overline{g}$ satisfy these parity conditions, then by Cramer’s formula, the components $(\overline{g})^{ij}$ of the inverse matrix are also $C^{2\ell}$ across $x = 0$, hence these too satisfy the same parity conditions. Clearly, if $g$ is even to order $2\ell$ in the sense of Definition \ref{AHmetric}, then $g$ is even to order $2\ell$ in the sense of Definition \ref{AHmetric} relative to a special defining function $x$. Conversely, we have the following result.

**Lemma 1.7.** If $g$ is even to order $2\ell$ relative to any fixed boundary defining function in the sense of Definition \ref{AHmetric}, then it is even to order $2\ell$ in the sense of Definition \ref{AHmetric} i.e., it is even to order $2\ell$ with respect to any special boundary defining function.

**Proof.** If $x$ denotes the initial defining function, let $h_0 = x^2 g$ restricted to $T\partial M$. We seek $x' = e^{x'} x$ such that $|dx'/x'|^2_y = 1$, $\omega_0(y) = \omega(0, y) \equiv 0$, or equivalently

$$2(dx, d\omega)_{\overline{g}} + x|d\omega|^2_{\overline{g}} = \frac{1 - |dx|^2_y}{x}, \quad \omega|_{\partial M} = 0. \quad (1.10)$$

In contrast to \ref{AHmetric}, the right hand side of this equation is nonvanishing, but there still exists a unique solution $\omega$ which is $C^\infty$ for $0 \leq x < \epsilon$, and the only issue is the degree of smoothness of its even extension across $x = 0$. However, $x^{-1}(|dx|^2_y - 1) = x^{-1}(\overline{g}xx - 1)$
is odd to order $2\ell - 1$, hence lies in $C^{2\ell - 1}$, and the solution of this equation is one order smoother, i.e. lies in $C^{2\ell}$. This proves that $\omega$ is even to order $2\ell$.

Now set $g' = (x')^2 g = e^{2\omega}\tilde{g}$ and write $\tilde{g}'$ for its $I$-invariant extension. From the paragraph above, $\tilde{g}' \in C^{2\ell}$. In fact, the metric is $C^\infty$ in the tangential direction and its irregularity is only in the direction normal to the boundary (technically, it is polyhomogeneous at the boundary). It is standard to show that the exponential mapping $\Phi : NM \to M$ from the normal bundle of $\partial M$ to $M$, $\Phi(x, y) = \exp_y(xv(y))$ is smooth in the ‘base’ variable $y \in \partial M$ and $C^{2\ell+1}$ in $x$. Hence $\Phi^* \tilde{g}' = d(x')^2 + \tilde{h}'_{\alpha\beta}(x', y)dy^\alpha dy^\beta$, where $\tilde{h}'_{\alpha\beta} \in C^{2\ell}$, or finally, $g'$ is even to order $2\ell$.

\[ \square \]

2. Partially even metrics and Ricci flow

Our aim in this section is to prove the following.

**Proposition 2.1.** Suppose that $g^0$ is an AH metric which is even to order $2\ell$. Let $g(t)$ be the unique solution to the Ricci flow equation \((0.3)\) with initial condition $g^0$. Then $g(t)$ remains even to order $2\ell$ throughout its time of existence.

Notice that we do not use the gauged Ricci flow equation here since we are taking the existence of the solution as given. Indeed, we take from \cite{2} the fact that there is a unique solution to \((0.3)\) and this remains smoothly conformally compact throughout its maximal time interval of existence.

**Proof.** First appeal to the transformation formula for the Ricci tensor under a conformal change of metric, see \cite{3}: if $g = x^{-2}\tilde{g}$, then $E(g) = R^c(g) + (n - 1)g = x^{-2}E(\tilde{g})$, where

\[ E(\tilde{g}) = -(n - 1)\left(\left|dx\right|^2_{\tilde{g}} - 1\right)\tilde{g} + ((n - 2)\text{Hess}_{\tilde{g}}(x) + (\Delta_{\tilde{g}}x)\tilde{g})x + R^c(\tilde{g})x^2. \quad (2.1) \]

In the following computations, we use a fixed defining function $x$, which is a special boundary defining function for the initial metric $g^0$, but of course no longer has this property for $g(t)$, $t > 0$.

We shall analyze the equation

\[ \partial_t \tilde{g} = -2E(\tilde{g}), \quad (2.2) \]

which is obtained if we multiply \((0.3)\) by $x^2$. As a first step, consider the restriction of this equation to $x = 0$:

\[ \partial_t \left(\tilde{g}\big|_{x=0}\right) = 2(n - 1)\left(\left|dx\right|^2_{\tilde{g}} - 1\right)\left(\tilde{g}\big|_{x=0}\right). \quad (2.3) \]

Since $g$ is AH, the leading coefficient $\left|dx\right|^2_{\tilde{g}} - 1$ vanishes at $t = 0$, which shows that the restriction (not just the pullback) of $\tilde{g}$ to the boundary is invariant under the flow. It is thus reasonable to write the evolving solution in the form $g(t) = g^0 + k(t)$, where $|k(t)|^2 = \mathcal{O}(x)$. This was the ansatz in \((1.3)\) in \cite{2}, and a uniqueness theorem in this setting \cite{5} shows that any solution of \((2.2)\) in this quasi-isometry class must agree with the solution in \cite{2}. The proof in \cite{2} also shows that $k(t) \in C^\infty(M)$ for every $t \geq 0$. Consequently, $E(x^2 g(t)) = 0$ at $\partial M$ for all $t \geq 0$.

Now rewrite \((2.2)\) as an equation for the evolving tensor $v(t) = \tilde{g}(t) - \tilde{g}^0$:

\[ \partial_t k(t) = 2(n - 1)\left(\left|dx\right|^2_{\tilde{g}(t)} - 1\right)\tilde{g}(t) - 2\left((n - 2)\text{Hess}_{\tilde{g}(t)}(x) + (\Delta_{\tilde{g}(t)}x)\tilde{g}(t)\right)x - 2R^c(\tilde{g}(t))x^2. \quad (2.4) \]
Define the double $\tilde{M}$ in terms of the fixed defining function $x$ and consider (2.4) on this space. The key point is that $E$ commutes with $I^*$, i.e. if $\hat{g}$ is any extension of $g$ to $\tilde{M}$, then

$$I^*E(\hat{g}) = E(I^*\hat{g}).$$

We shall work with the even extension $\tilde{g}$, where $I^*\tilde{g} = \tilde{g}$, so that $I^*E(\tilde{g}) = E(\tilde{g})$.

We now argue through a sequence of claims:

Claim 1: Let $g$ be even to order $j$. Then $E(\tilde{g}) \in C^j$, or equivalently, $E(\tilde{g})$ is even to order $j$.

To prove this, note that $E$ is a “0-differential operator”, which means that every derivative $\partial_x$ or $\partial_y$ is accompanied by a factor of $x$. When $j$ is even, then the first term in $\tilde{g}$, the even extension of $x^2 g$, which is not $C^\infty$, occurs with the factor $|x|^{j+1}$; furthermore, the normal derivatives which act on this are either $x \partial_x$ or $x^2 \partial_x$, neither of which decrease the order of vanishing. This means that $E(\tilde{g}) \in C^j$ across $x = 0$. Similarly, when $j$ is odd, the leading nonsmooth term for the reflected metric occurs at order $j + 1$ again.

To state the next claim, fix any $\bar{g}$ which is even to order $j \leq 2\ell$, and let $v$ be smooth up to $x = 0$ and denote by $\Pi^{e,j}(x^2 v)$ and $\Pi^{o,j}(x^2 v)$ the even and odd components of $x^j v$ at order $j$. Also, write

$$E(\bar{g} + x^j v) = E(\bar{g}) + D\big|_\bar{g} (x^j v) + O(x^{2j}),$$

and for simplicity, set $D\big|_\bar{g} = \mathcal{L}$.

Claim 2: If $v$ is smooth for $x \geq 0$, then

$$\Pi^{e,j}(\mathcal{L} x^j v) = \mathcal{L}(\Pi^{e,j}(x^j v)).$$

Suppose that $x^j v$ is even to order $j$ and define $\hat{g} = \bar{g} + \Pi^{e,j}(x^j v)$. By Claim 1, $E(\hat{g}) \in C^j$. Furthermore, $E(\hat{g}) \in C^j$ and the quadratic error term vanishes like $x^{2j}$.

Hence $\mathcal{L}(x^j v) \in C^j$, which means that it is even to order $j$ as well. This proves (2.7).

Claim 3: Suppose that $\bar{g}$ is even to order $k$ and $v \in C^\infty(\tilde{M})$; then for $j \leq k$,

$$\Pi^{o,j}(\mathcal{L} x^j v) = \mathcal{L}(\Pi^{o,j}(x^j v)) = 0.$$ (2.8)

Unlike the even case, this seems to require direct calculation. As before, set $\hat{g} = \bar{g} + x^j v$. We shall calculate the leading coefficients of the linearizations of each of the terms $(|dx|^2 - 1)\hat{g}$, $x \text{Hess}_\hat{g}(x)$, $x \Delta_{\hat{g}} x \hat{g}$ and $x^2 \text{Rc}(\hat{g})$ separately (for simplicity we do not bother to assemble these with the correct coefficients into the operator $\mathcal{L}$). This will show immediately that both (2.8) and (2.7) must hold.

In these calculations it is convenient to work in a coordinate system $(x, y)$ around $p \in \partial \tilde{M}$ for which $\bar{g} = dx^2 + \bar{g}_{\alpha\beta} dy^\alpha dy^\beta$ (thus $x$ is the $\bar{g}$-distance to $\partial \tilde{M}$), and so that in addition $\bar{g}_{\alpha\beta}(p) = \delta_{\alpha\beta}$ and all Christoffel symbols of $\bar{g}$ vanish at $p$. The following computations are all carried out at the point $p$ and modulo $x^{j+1}$, and we use $\equiv$ in this sense. Thus, for example, $\hat{g}^{-1} \equiv \delta - v$, etc.

The first term is

$$(|dx|^2 - 1)\hat{g} = (\hat{g}^{00} - 1)\hat{g} \equiv (\delta^{00} - 1 - x^j v_{00})(\bar{g} + v) = -x^j v_{00}.$$ (2.9)

Next,

$$x \text{Hess}_\hat{g}(x)_{ij} = -x \hat{\Gamma}^x_{ij}, \quad \text{and} \quad x \Delta_{\hat{g}} x = -x \hat{g}^{ij} \hat{\Gamma}^x_{ij}.$$ (2.10)
A computation in the appendix shows that
\[ x\widehat{\Gamma}^{x}_{xx} \equiv (j/2)x^{j}v_{00}, \quad x\widehat{\Gamma}^{x}_{x_{\alpha}} \equiv 0, \quad \text{and} \quad x\widehat{\Gamma}^{x}_{\alpha\beta} \equiv -(j/2)x^{j}v_{\alpha\beta}, \quad (2.11) \]
hence in addition,
\[ xg^{ij}\widehat{\Gamma}^{x}_{ij} \equiv (j/2)x^{j}(v_{00} - g^{\alpha\beta}v_{\alpha\beta}). \quad (2.12) \]
As for the final set of terms,
\[ x^{2}Rc(\widehat{g})_{xx} \equiv -x^{2}\partial_{x}x^{\alpha}\widehat{\Gamma}^{x}_{\alpha} \equiv -\frac{1}{2}j(j-1)\delta^{\alpha\beta}x^{j}v_{\alpha\beta} \]
\[ x^{2}Rc(\widehat{g})_{x\alpha} \equiv 0 \]
\[ x^{2}Rc(\widehat{g})_{\alpha\beta} \equiv -\frac{1}{2}j(j-1)/2x^{j}v_{\alpha\beta}. \quad (2.13) \]
It is clear from these calculations that for any \( j \), \( L \circ \Pi^{o,j} = 0 \). On the other hand, since any \( v \equiv \Pi^{e,j}v + \Pi^{o,j}v \) and \( L \Pi^{e,j}v \) is even at order \( j \), we deduce that \( \Pi^{o,j}(Lv) = 0 \) as well. This proves the claim.

We are now in a position to complete the proof. Trivially, \( g(t) \) is even to order 0. Suppose we have shown that \( g(t) \) is even to order \( j \) for any \( j \) with \( 0 \leq j \leq 2\ell - 1 \). Write \( g(t) = g^{0} + v(t) \), and assume by induction we have shown that \( v(t) \) is even to order \( j \). Applying \( \Pi^{o,j} \) to the equation (2.4) and noting that \( \Pi^{o,j}E(g + v) = L(\Pi^{o,j}(v)) = 0 \), we obtain
\[ \partial_{t}\Pi^{o,j}(v(t)) = 0, \quad \Pi^{o,j}v|_{t=0} \equiv 0. \quad (2.15) \]
This is a well-defined evolution equation acting on \( \Pi^{o,j}v \), the first odd term in the expansion for \( v \). This implies of course that \( \Pi^{o,j}v(t) \equiv 0 \), hence \( v \) is in fact even to order \( j + 1 \). This argument works for every \( j \leq 2\ell - 1 \), so \( \overline{g}(t) \) remains even to order \( 2\ell \), as claimed. \( \square \)

3. The volume renormalizability condition

We have now proved that if \( g^{0} \) is even to order \( 2m - 2 \) (where \( 2m = \dim M \)), then \( g(t) \) remains even to order \( 2m - 2 \) throughout the interval of existence. In this section we study the evolution of the first odd term \( h_{2m-1} \). Recall that \( g^{0} \) is volume renormalizable when \( \tr h_{2m-1} = 0 \).

**Proposition 3.1.** If \( g^{0} \) is volume renormalizable, then the solution \( g(t) \) to the Ricci flow with initial condition \( g^{0} \) remains volume renormalizable so long as this solution is defined.

The proof proceeds very much as before. We show that some version of the scalar function \( \tr h_{2m-1} \) satisfies a homogeneous ordinary differential equation in \( t \) with initial condition 0, and hence vanishes for \( t \geq 0 \). The actual quantity we study is slightly more complicated, and is defined below.

Having fixed the initial metric \( g^{0} \) and representative \( h_{0} \in c(g^{0}) \), let \( x \) be the corresponding special boundary defining function. We define the \( t \)-independent metric
\[ G = \frac{dx^{2} + h_{0}}{x^{2}} \quad (3.1) \]
by truncating the expansion of the tangential metric \( h(x) \) in the normal form for \( g^{0} \). We also set \( \overline{G} = x^{2}G \).
Next define the function

\[ F := \text{tr} \frac{\partial}{\partial t} \mathcal{G}(t) = \text{tr} G(t). \]  

(3.2)

Clearly \( F \) is smooth up to \( x = 0 \) for every \( t \geq 0 \). Moreover, since \( G \) and \( g \) are even, \( F \) is even to order \( 2m - 2 \). Our interest is in the first odd term in the expansion of \( F \).

Let us introduce some notation: if \( f \in C^\infty(M) \) then the coefficient of the \( i \)th term in its expansion with respect to the fixed defining function \( x \) will be denoted \( \{ f \}_i \).

**Lemma 3.2.** Let \( \mu = \mu(t, y) = \{ F \}_{2m-1} \). Then

\[ \mu = \{ \mathcal{G}_{xx} \}_{2m-1} + (h_0)^{\alpha\beta} \{ \mathcal{G} \}_{\alpha\beta} \}_{2m-1}. \]  

(3.3)

**Proof.** Note that since \( \mathcal{G}^\alpha_\alpha \equiv 0 \),

\[ F = \mathcal{G}^i_x \mathcal{G}_{xx} + \mathcal{G}^{\alpha\beta} \mathcal{G}_{\alpha\beta}, \]  

(3.4)

where all indices are raised with respect to \( \mathcal{G} \). The result is now straightforward. \( \square \)

**Proposition 3.3.** Setting \( \nu = \{ \mathcal{G}_{xx} \}_{2m-1} \) and \( \mu = \{ \mathcal{G}_{xx} \}_{2m-1} + (h_0)^{\alpha\beta} \{ \mathcal{G}(t) \}_{\alpha\beta} \}_{2m-1} \) as above, and denoting by \( \nu' \) the time-derivative \( \partial_t \), then

\[ \nu' = -(2m - 1)\mu \]  

(3.5)

\[ \nu' = (2m - 3)(2m - 1)\mu. \]

**Proof.** We must compute the first odd term in the expansion of

\[ F' = G^{ij} g_{ij}' = -2G^{ij} E_{ij} = -2G^{ij} \mathcal{G}_{ij}, \]

\[ = -2(2m - 1) (1 - |dx|_g^2) G^{ij} \mathcal{G}_{ij} - 2x G^{ij} [(n-2) \text{Hess}_g(x)_{ij} + (\Delta_g) \mathcal{G}_{ij}] \]

(3.6)

\[ - 2x^2 G^{ij} \text{Re}(\mathcal{G})_{ij}. \]

Let us decompose the expression on the right as the sum of the three terms

\[ A = -2(2m - 1) (1 - |dx|_g^2) G^{ij} \mathcal{G}_{ij}, \]

\[ B = -2x (2m - 2) G^{ij} \text{Hess}_g(x)_{ij} - 2x G^{ij} \text{Hess}_g(x)_{ij} \cdot G^{kl} \mathcal{G}_{ij}, \]  

(3.7)

\[ C = -2G^{ij} (x^2 \text{Re}(\mathcal{G})_{ij}). \]

Powers of \( x \) have been distributed so that these may be expressed in terms of 0-

derivatives \( x \partial_x \) and \( x \partial_{x_n} \), which assists with the parity bookkeeping.

First consider \( A \). Write \( |dx|_g^2 \sim 1 + \sum_{j=1}^{m-1} A_{2j} x^{2j} + A_{2m-1} x^{2m-1} \). Since \( F = G^{ij} \mathcal{G}_{ij} \)

has leading coefficient \( (G^{ij} \mathcal{G}_{ij})_{|x=0} = 2m \) and is even to order \( 2m - 2 \), we have

\[ \{ A \}_{2m-1} = -2(2m - 1) \left\{ (1 - |dx|_g^2) G^{ij} \mathcal{G}_{ij} \right\}_{2m-1} \]

\[ = -2(2m - 1) \left( \{ 1 - |dx|_g^2 \} \{ G^{ij} \mathcal{G}_{ij} \}_{2m-1} + \{ 1 - |dx|_g^2 \}_{2m-1} \{ G^{ij} \mathcal{G}_{ij} \}_{0} \right) \]

\[ = 4m(2m - 1) A_{2m-1} = -4m(2m - 1) \nu, \]  

(3.8)

where the last equality follows from Lemma [A.1] in the appendix.

The calculation for \( B \) is more difficult, and we relegate certain long parity calculations

to the appendix. We begin by studying \( x \text{Hess}_g(x) \), and record the parity of each

component, the leading coefficient and the coefficient of the first odd term.
First,
\[ [x \text{Hess}_g(x)]_{\alpha \beta} = x \left( \partial_{\alpha \beta}^2 x - \Gamma_{\alpha \beta}^k \partial_k x \right) = -x \Gamma_{\alpha \beta}^x. \]  \hspace{1cm} (3.9)

From the appendix, this expression is even to order \(2m - 2\) with leading order 2 and first odd coefficient
\[ \{ [x \text{Hess}_g(x)]_{\alpha \beta} \}_{2m-1} = \frac{1}{2} (2m - 1) \{ \mathcal{F}_{\alpha \beta} \}_{2m-1}. \]  \hspace{1cm} (3.10)

Next,
\[ [x \text{Hess}_g(x)]_{xx} = x \left( \partial_{xx}^2 x - \Gamma_{xx}^k \partial_k x \right) = -x \Gamma_{xx}^x; \]  \hspace{1cm} (3.11)
this term is also even to order \(2m - 2\) with leading order 2 and first odd coefficient
\[ \{ [x \text{Hess}_g(x)]_{xx} \}_{2m-1} = -\frac{1}{2} (2m - 1) j. \]  \hspace{1cm} (3.12)

Finally,
\[ [x \text{Hess}_g(x)]_{x\alpha} = x \left( \partial_{x\alpha}^2 x - \Gamma_{x\alpha}^k \partial_k x \right) = -x \Gamma_{x\alpha}^x. \]  \hspace{1cm} (3.13)

From the appendix, this term has leading order 1 and is odd to order \(2m - 1\), and we do not need more information about it.

Since \(G^{0\alpha} \equiv 0\), the first term of \(\mathcal{B}\),
\[ -2(2m - 2)G^{ij} \cdot x \text{Hess}_g(x)_{ij} \]  \hspace{1cm} (3.14)
equals
\[ \left\{ -2(2m - 2)G^{ij} \cdot x \text{Hess}_g(x)_{ij} \right\}_{2m-1} = -2(2m - 2) \left( \{ x \text{Hess}_g(x)_{xx} \}_{2m-1} + h_0^{\alpha \beta} \{ x \text{Hess}_g(x)_{\alpha \beta} \}_{2m-1} \right) \]  \hspace{1cm} (3.15)
\[ = -2(2m - 2) \left( -\frac{1}{2} (2m - 1) \nu + h_0^{\alpha \beta} \frac{1}{2} (2m - 1) (h_{2m-1})_{\alpha \beta} \right) \]  \hspace{1cm} (3.15)
\[ = - (2m - 1) (2m - 2) \left( -\nu + h_0^{\alpha \beta} \{ \mathcal{F}_{\alpha \beta} \}_{2m-1} \right) \]  \hspace{1cm} (3.15)
\[ = - (2m - 1) (2m - 2) (\mu - \nu) \]  \hspace{1cm} (3.15)
\[ = - (2m - 1) (2m - 2) (\mu - \nu). \]  \hspace{1cm} (3.15)

We consider separately the individual summands in the second term of \(\mathcal{B}\),
\[ -2\bar{g}^{kl} \cdot x \text{Hess}_g(x)_{kl} \cdot \bar{g}^{ij} \bar{g}_{ij} \]  \hspace{1cm} (3.16)
equals
\[ -2 F \left( \bar{g}^{xx} x \text{Hess}_g(x)_{xx} + 2 \bar{g}^{x\alpha} x \text{Hess}_g(x)_{x\alpha} + \bar{g}^{ij} x \text{Hess}_g(x)_{ij} \right) \]  \hspace{1cm} (3.16)
The first of these contributes
\[ \{ -2 F \bar{g}^{xx} x \text{Hess}_g(x)_{xx} \}_{2m-1} = -2 \{ F \bar{g}^{xx} \}_{0} \{ x \text{Hess}_g(x)_{xx} \}_{2m-1} \]  \hspace{1cm} (3.17)
\[ = 2(2m - 1) \frac{1}{2} (2m - 1) \nu = 2m(2m - 1) \nu. \]  \hspace{1cm} (3.17)

Using the analysis of (3.13) and the fact that \(\bar{g}^{x\alpha}\) has leading order \(x\) and is odd to order \(2m - 1\), the middle term, \(-4 F \bar{g}^{x\alpha} x \text{Hess}_g(x)_{x\alpha}\) is even to order \(2m\), hence contributes
nothing at order $2m - 1$. As for the final term,
\[
\left\{-2F \tilde{\gamma}^{\alpha\beta} x \text{Hess}_{\tilde{\gamma}}(x)_{\alpha\beta}\right\}_{2m-1} = -2\left\{ F \tilde{\gamma}^{\alpha\beta} \right\}_0 \left\{ x \text{Hess}_{\tilde{\gamma}}(x)_{\alpha\beta}\right\}_{2m-1} \\
\quad = -2(2m h_0^{\alpha\beta}) \frac{1}{2} (2m - 1) \left\{ \mathcal{G}_{\alpha\beta}\right\}_{2m-1} \\
\quad = -2m(2m - 1) h_0^{\alpha\beta} \left\{ \mathcal{G}_{\alpha\beta}\right\}_{2m-1} \\
\quad = -2m(2m - 1)(\mu - \nu) \\
\quad = 2m(2m - 1)(\nu - \mu).
\]

Altogether, then,
\[
\left\{ \mathcal{B}\right\}_{2m-1} = -(2m - 1)(2m - 2)(\mu - 2\nu) + 2m(2m - 1)(2\nu - \mu) \\
\quad = -2(2m - 1)(2m - 1)\mu + 4(2m - 1)(2m - 1)\nu \\
\quad = 2(2m - 1)^2(2\nu - \mu).
\]

We turn finally to $\mathcal{C}$. The computations are outlined in the appendix. Since $G^{x\alpha} = 0$ we find
\[
\tilde{G}^{ij} \text{Rc}(\tilde{\gamma})_{ij} = \tilde{G}^{xx} \text{Rc}(\tilde{\gamma})_{xx} + \tilde{G}^{\alpha\beta} \text{Rc}(\tilde{\gamma})_{\alpha\beta},
\]
and so from the appendix
\[
\left\{ \mathcal{C}\right\}_{2m-1} = \left\{ -2x^2\tilde{G}^{ij} \text{Rc}(\tilde{\gamma})_{ij}\right\}_{2m-1} \\
\quad = -2\left\{ x^2 \text{Rc}(\tilde{\gamma})_{xx}\right\}_{2m-1} - 2h_0^{\alpha\beta} \left\{ x^2 \text{Rc}(\tilde{\gamma})_{\alpha\beta}\right\}_{2m-1} \\
\quad = -2\left(-2m - 1)(m - 1) h_0^{\alpha\beta} \left\{ \mathcal{G}_{\alpha\beta}\right\}_{2m-1}\right\} - 2h_0^{\alpha\beta} \left(-2m - 1)(m - 1) \left\{ \mathcal{G}_{\alpha\beta}\right\}_{2m-1}\right\} \\
\quad = 4(m - 1)(2m - 1) h_0^{\alpha\beta} \left\{ \mathcal{G}_{\alpha\beta}\right\}_{2m-1} \\
\quad = 4(m - 1)(2m - 1)(\mu - \nu).
\]

Collecting all of this information, we obtain
\[
\mu' = \left\{ \mathcal{A}\right\}_{2m-1} + \left\{ \mathcal{B}\right\}_{2m-1} + \left\{ \mathcal{C}\right\}_{2m-1} \\
\quad = -4m(2m - 1)\nu - (4m - 2)(2m - 1)\mu + 2(2m - 1)(4m - 2)\nu \\
\quad + 4(m - 1)(2m - 1)(\mu - \nu) \\
\quad = -2(2m - 1)\mu.
\]

To obtain the evolution equation for $\nu = \left\{ \mathcal{G}_{xx}\right\}_{2m-1}$, recall that
\[
\partial_t \mathcal{G}_{xx} = x^2 \partial_t g_{xx} = -2\tilde{E}(g)_{xx}.
\]

Now
\[
-2\tilde{E}(g)_{xx} = -2(2m - 1)(1 - |dx|^2) \tilde{\gamma}_{xx} - 2x [(n - 2) \text{Hess}_{\tilde{\gamma}}(x)_{xx} + (\Delta_{\tilde{\gamma}} x) \tilde{\gamma}_{xx}] \\
- 2x^2 \text{Rc}(\tilde{\gamma})_{xx}.
\]

Computing as above gives
\[
\nu' = (2m - 3)(2m - 1)\mu.
\]

Applying the result of Proposition 3.8 we immediately obtain
Corollary 3.4. If \((\mu(0), \nu(0)) = (0, 0)\) then \((\mu(t), \nu(t)) = (0, 0)\) along the flow. In particular, \(h_0^{\alpha \beta} \{\mathcal{g}_{\alpha \beta}\}^{2m-1} = 0\) for \(0 \leq t < T_0\).

We now prove Proposition 3.1.

Proof of Proposition 3.1. Suppose that \(g^0\) is a volume renormalizable metric to order \(2m - 2\). Let \(h_0 \in \mathcal{C}(g^0)\) and \(x\) the corresponding special boundary defining function, so that

\[
g^0 = \frac{dx^2 + h_x}{x^2},
\]

(3.26)

where \(h_x|_{x=0} = h_0\). As usual, fix tangential coordinates \(y^\alpha\) extended to be constant along the integral curves of \(\nabla g\).

Let \(g(t)\) satisfy the Ricci flow equation. We have already shown that it remains even to order \(2m - 2\). We write \(x_t\) for the evolving special boundary defining function corresponding to \(g(t)\) and recall that \(\mathcal{C}(g(t))\) is constant in \(t\), so that

\[
g(t) = \frac{dx_t^2 + h_0 + h_{2} \cdot x + \cdots + h_{2m-2}(t)x^{2m-2} + h_{2m-1}(t)x^{2m-1} + \cdots}{x_t^2}.
\]

(3.27)

We wish to prove that \(\text{tr} h_0 h_{2m-1}(t) = 0\).

Denoting by \(x\) the special bdfr for \(g^0\), we have

\[
x_t = e^{\omega_t(x,y)} x,
\]

(3.28)

where \(\omega_t\) vanishes to second order since the conformal infinity is fixed and is even to order \(2m\). Computing further shows that

\[
dx_t = (x \partial_x \omega + 1)e^{\omega_t} dx + (x \partial_{y^\alpha} \omega_t)e^{\omega_t} dy^\alpha,
\]

(3.29)

and

\[
dx_t^2 = (x \partial_x \omega + 1)^2 e^{2\omega_t} dx^2 + 2(x \partial_x \omega + 1)(x \partial_{y^\alpha} \omega_t)e^{2\omega_t} dy^\alpha dx + (x \partial_{y^\alpha} \omega_t)(x \partial_{y^\beta} \omega_t)e^{2\omega_t} dy^\alpha dy^\beta.
\]

(3.30)

Inserting this into the expression for \(g(t)\) yields

\[
g(t) = (x \partial_x \omega + 1)^2 \frac{dx_t^2}{x^2} + 2(x \partial_x \omega + 1)(x \partial_{y^\alpha} \omega_t) \frac{dy^\alpha dx}{x^2}
\]

\[
+ \left[ e^{-2\omega}(h_0)_{\alpha \beta} + (h_2)_{\alpha \beta} x^2 + \cdots + (h_{2m-2})_{\alpha \beta} e^{(2m-4)\omega} x^{2m-2} \right] + (h_{2m-1})_{\alpha \beta} e^{(2m-3)\omega} x^{2m-1} + \cdots + (x \partial_{y^\alpha} \omega_t)(x \partial_{y^\beta} \omega_t) e^{2\omega} \frac{dy^\alpha dy^\beta}{x^2}.
\]

(3.31)

Now observe the following. By the proof of Lemma 1.1, the coefficient of \(dx^2\) is even to order \(2m - 2\), and its first odd coefficient is \(2(2m - 1)\{\omega\}_{2m-1}\), which vanishes since \(\omega\) is even to order \(2m\). The coefficient of \(dy^\alpha dy^\beta\) is odd to order \(2m - 1\). The entire coefficient of \(dy^\alpha dy^\beta\) in square brackets of is even to order \(2m - 2\). Using the expansion for \(\omega\), its leading term is \(h_0\) while the term at order \(2m - 1\) equals \((h_{2m-1})_{\alpha \beta} - 2\{\omega\}_{2m-1}(h_0)_{\alpha \beta}\). Thus Corollary 3.4 shows that \(\text{tr} h_0 h_{2m-1}(t) = 0\) along the flow.

4. Variation of renormalized curvature functionals

We now take up the proof of Theorem 3. We begin with a quick review of Riesz renormalization, referring to [1] and [7] for more details, and then study the variation of renormalized curvature integrals.
4.1. Regularized and renormalized integrals.

Fix a product decomposition \([0, \epsilon_0) \times \partial M\) of some neighborhood of \(\partial M\), with projection on the first factor a fixed defining function. Recall that a function \(u\) on \(M\) is said to be polyhomogeneous if, in terms of some (and hence any) boundary defining function \(x\), there is an expansion

\[
u \sim \sum_{\ell=0}^{N_j} \sum_{j} u_{j,\ell}(y) x^{\gamma_j}(\log x)^\ell;
\]

here \(\gamma_j\) is a sequence of complex numbers with \(\text{Re}\gamma_j \to \infty\) and the coefficient functions \(u_{j,\ell}\) are \(C^\infty\) on \(\partial M\). For simplicity we assume that \(\text{Re}\gamma_0 \leq \text{Re}\gamma_j\) for all \(j\). This expansion and the differentiated expansion to any order hold in the traditional sense.

**Definition 4.1.** If \(u\) is polyhomogeneous at \(\partial M\), then

\[
z \mapsto I(z) := \int_M x^z u dV_g
\]

extends to a meromorphic function in the entire complex plane. By definition, the Riesz regularized integral of \(u\) is the finite part

\[
\mathcal{R} \int_M u \, dV_g := \text{FP} I(z).
\]

When \(I(z)\) has a simple pole at \(z = 0\), \(\text{FP} I(z) = \lim_{z \to 0} \left( I(z) - \text{Res} I(z) \right)\).

To write this out more carefully, recall first that \(dV_g = x^{-n} \sqrt{\text{det} h} \, dx dV_{h_0} =: x^{-n} J(x, y) \, dx dV_{h_0}\), where the Jacobian factor \(J(x, y)\) has the expansion \(\sum_{k \geq 0} J_k x^k\). Writing the expansion of \(u\) as above, we see that \(I(z)\) is holomorphic on \(\{ z \in \mathbb{C} : \text{Re}(z) > n - 1 - \text{Re}\gamma_0 \}\). Now consider each term

\[
\int_M x^{z-n+i} (\log x)^i u_{j,\ell}(y) J_i(y) \, dx dy = \int_0^\infty \int_{\partial M} x^{z-n+i} (\log x)^i u_{j,\ell}(y) J_i(y) \, dx dy \tag{4.4}
+
\int_{z \geq 0} \int_{\partial M} x^{z-n+i} (\log x)^i u_{j,\ell}(y) J_i(y) \, dx dy,
\]

where \(J(x, y) \sim \sum J_i(y)x^i\). The second term on the right is entire in \(z\), while the first extends meromorphically with a pole of order \(\ell + 1\) at \(z = n - \gamma_j - i - 1\).

As an application, suppose that \(x\) and \(\tilde{x} = xe^{\omega}\) are two boundary defining functions, where \(\omega \in C^\infty(\partial M)\), then \(\tilde{x}^z - x^z = (e^{z\omega} - 1)x^z =: A_j(y, z)x^{j+\omega}\). Suppose also that \(u \sim \sum_{j \geq 0} u_j(y) x^j\), i.e., all \(\gamma_j\) are nonnegative integers and each \(N_j = 0\). A short calculation now shows that if \(I(z)\) and \(\tilde{I}(z)\) are the regularizations with respect to these two defining functions, then near \(z = 0\),

\[
\tilde{I}(z) - I(z) = \int_{\partial M} C(y, 0) \, dV_{h_0} + \mathcal{O}(z),
\]

where \(C(y, z) = \sum_{i+j+k=n-1} u_i(y) A_j(y, z) J_k(y)\). The following is an immediate consequence:
Proposition 4.2. Suppose that \( g \) and \( u \) are both even to order \( n - 2 = 2m - 2 \). Then \( C(y, 0) = \frac{1}{2} u_0 A_0 \text{tr} h_0^{n-1} \). If, on the other hand, the expansion for \( u \) starts with the term \( x^p \) (\( p \) even), then \( C(y, 0) = \frac{1}{2} u_p A_0 \text{tr} h_0^{n-1-p} \).

Proof. By the parity assumptions, \( C(y, 0) = u_0(y) A_0(y, 0) J_{n-1}(y) \); all other terms vanish. It remains to show that

\[
J_{n-1} = \frac{1}{2} \text{tr} h_0^{n-1}.
\] (4.6)

However, by Proposition 1.5, if \( g \) is even to order \( 2m - 2 \) then \( \omega \) is even to order \( 2m \), and by simple calculation \( \sqrt{h}/\sqrt{h_0} \) is even to order \( 2m \) as well. Thus \( (e^{\omega - 1}) \sqrt{h}/\sqrt{h_0} \) is even to order \( 2m - 2 \) and the coefficient of \( x^{2m-1} \) is \( \frac{1}{2} \text{tr} h_0^{n-1} \).

The key example is \( u \equiv 1 \), in which case the renormalized integral is the renormalized volume. If \( g \) is volume renormalizable, then \( \text{tr} h_0^{n-1} = 0 \), which shows that the renormalized volume is well-defined with these hypotheses. A very similar argument shows that

Corollary 4.3. Suppose the conditions of Proposition 4.2 hold and \( \text{tr} h_0^{n-1} = 0 \). Then the renormalized integral of \( u \) is independent of the choice of conformal representative of the boundary metric.

4.2. Variations of renormalized integrals. Let \( g_t \) be a family of metrics on a compact manifold \( Z \), and consider the Riemannian curvature functional

\[
I_Z(g) := \int_Z u(g) \, dV_g,
\] (4.7)

where \( u(g) \) is some scalar quantity defined from the curvature tensor and its covariant derivatives. One is often interested in critical points of this action, which are solutions of the Euler-Lagrange equation. The first step in computing these critical points is the variation of the integrand

\[
\frac{\partial}{\partial t} \bigg|_{t=0} \int_Z u_t \, dV_{g_t} = \int_Z \frac{\partial}{\partial t} \bigg|_{t=0} (u_t \, dV_{g_t}),
\] (4.8)

followed by integration by parts. We are interested in the corresponding calculation on a conformally compact manifold. In particular, we shall only take the action of partially even volume renormalizable metrics and consider variations amongst such metrics. Thus suppose that \( g(t) \) is volume renormalizable for each \( t \) and \( g^0 =: g \); suppose also that the conformal infinity of \( g(t) \) is independent of \( t \), with fixed representative \( h_0 \). Let \( x_t \) be the corresponding family of special boundary defining functions, and write \( x_t = e^{\omega_t} x \). Let \( u_t = u(g_t) \) be the scalar quantity associated to \( g_t \). Set

\[
\mathcal{L}(g) = \left( \frac{\partial}{\partial t} \bigg|_{t=0} u_t + \frac{u_0}{2} \text{tr} h_0 \frac{\partial}{\partial t} \bigg|_{t=0} g_t \right)
\] (4.9)

Writing \( x_0 = x \) and \( u_0 = u \), then we have

Theorem 4.4. With the notation above,

\[
\frac{\partial}{\partial t} \bigg|_{t=0} \int_Z u_t \, dV_{g_t} = \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\text{FP}}{z=0} \int_M x_t^2 u_t \, dV_{g_t} = \frac{\text{FP}}{z=0} \int_M x^2 \mathcal{L}(g) \, dV_g,
\] (4.10)

where the local expression for \( \mathcal{L} \) is exactly the same as in the compact case.
Proof. We begin by computing
\[
\frac{\partial}{\partial t} \bigg|_{t=0} \int_M x^2 u_t \, dV_{g_t} = \int_M x^2 \omega_t \, dV_{g} + \int_M x^2 \frac{\partial}{\partial t} \bigg|_{t=0} (u_t \, dV_{g_t}) \\
= \int_M x^2 \omega_t \, dV_{g} + \int_M x^2 \frac{\partial}{\partial t} \bigg|_{t=0} (u_t \, dV_{g_t})
\]
(4.11)
where \( \dot{x}_t := \frac{\partial}{\partial t} |_{t=0} x_t \), and similarly for \( \dot{\omega}_t \).

Since \( \omega \) is even to order \( 2m \), so is \( \dot{\omega} \). In addition, \( \dot{\omega} \in O(x^2) \) since the conformal infinity is fixed. This produces a shift by 2 in the terms in the expansion of \( \dot{\omega} u \, dV \).

This means that poles arising from integrating this expression appear at odd integers. In particular, there is no pole at \( z = 0 \), which means that
\[
\frac{\partial}{\partial t} \bigg|_{t=0} \int_M x^2 \omega_t \, dV_{g} + \int_M x^2 \mathcal{L}(g) \, dV_{g}.
\]
(4.12)
\[\square\]

Theorem 4.5 (Theorem B). Suppose that \((M^n, g^0), n = 2m, \) is volume renormalizable, and let \( g(t) \) be a solution of (0.3) with \( g(0) = g^0 \). Then along the flow,
\[
\frac{d}{dt} \text{RenV} = - R \int_M S(g(t)) + n(n - 1) dV_{g_t}.
\]
(4.13)

Proof. Proposition 3.1 asserts that \( g(t) \) remains volume renormalizable along the normalized Ricci flow. Consequently the renormalized volume is defined along the flow.

Recall that on a compact manifold the variation of the volume form \( g \mapsto dV_g \) equals \( \frac{1}{2} \text{tr} g(t) \, g(t) \), and for the normalized Ricci flow,
\[
\frac{1}{2} \text{tr} g(t) \partial_t g = - \text{tr} g(t) E_{ij} = -(S(g(t)) + n(n - 1))
\]
(4.14)

Finally, by Theorem 4.4
\[
\frac{d}{dt} \text{RenV}(M, g(t)) = - R \int_M S(g(t)) + n(n - 1) dV_{g_t},
\]
(4.15)
as asserted. \[\square\]

Appendix A. Parity computations

In this appendix we record a number of parity computations used above.

Lemma A.1. Let \( g \) be a metric which is even to order \( 2m - 2 \). Suppose that \( x \) is a defining function for which \( |dx|^2_\mathcal{F} = \mathcal{F}^{xx} \sim 1 + \sum_{j=1}^{m-1} A_{2j} x^{2j} + A_{2m-1} x^{2m-1} \); then completing \( x \) to a coordinate system \((x, y)\), we have that
\[
\mathcal{F}_{xx} = 1 + \sum_{j=1}^{m-1} a_{2j} x^{2j} + a_{2m-1} x^{2m-1},
\]
(A.1)
where \( a_{2m-1} = -A_{2m-1} \).
Proof. We use the formula
\[ \overline{g}^{x\alpha} \overline{g}_{x\alpha} + \overline{g}^{x\alpha} \overline{g}_{x\alpha} = 1, \] (A.2)
recalling that since \( \overline{g} \) is even to order \( 2m - 2 \), the coefficients \( \overline{g}_{x\alpha} \) and \( \overline{g}^{x\alpha} \) are odd to order \( 2m - 1 \), hence may be ignored since their product do not contribute to an odd term before order \( 2m + 1 \). Writing \( \overline{g}_{xx} \sim 1 + \sum_{j=1}^{m-1} a_{2j} x^{2j} + a_{2m-1} x^{2m-1} \), we see that the coefficient at order \( 2m - 1 \) of the product \( \overline{g}^{x\alpha} \overline{g}_{x\alpha} \) equals \( a_{2m-1} + A_{2m-1} \). However, this coefficient vanishes, which shows that the term in \( g_{xx} \) at order \( 2m - 1 \) is \(-A_{2m-1}\).

Suppose that \( f(x, y) \sim a_0(y) + a_1(y)x + \ldots \) is smooth. To simplify notation below, we denote by \( \{f\}_n \) the \( n \)th coefficient function \( a_n(y) \). The following is obvious:

**Lemma A.2.** If \( f \) and \( g \) are smooth functions which are even as functions of \( x \) to order \( 2k \), then the product \( fg \) is even to order \( 2k \), with \( \{fg\}_0 = \{f\}_0 \{g\}_0 \) and \( \{fg\}_{2k+1} = \{f\}_{2k+1} \{g\}_0 + \{f\}_0 \{g\}_{2k+1} \).

Let \( \overline{g} \) be a metric that is even to order \( 2m - 2 \), i.e.,
\[ \overline{g} = \overline{g}_{xx} dx^2 + 2 \overline{g}_{x\alpha} dx dy^\alpha + \overline{g}_{\alpha\beta} dy^\alpha dy^\beta, \] (A.3)
where \( \overline{g}_{xx} = 1 + \{\overline{g}_{xx}\} x^2 + \cdots, \overline{g}_{\alpha\beta} = (h_0)_{\alpha\beta} + \{g_{\alpha\beta}\} x^2 + \cdots \) are even functions to order \( 2m - 2 \), and \( \overline{g}_{x\alpha} \) has leading order \( x \) and is odd to order \( 2m - 1 \).

In the first table we compute the expansions of 0-derivatives (i.e., derivatives with respect to \( x \partial_x \) or \( x \partial_{g^{\alpha\beta}} \)) of the various metric components. Note that because of the prefactor \( x \) in the derivative, all of these terms vanish at \( x = 0 \); this will be important in view of Lemma A.2. An asterisk in this table denotes a coefficient we do not need to calculate.

| Component | Parity to order | Final coefficient |
|-----------|-----------------|-------------------|
| \( x \partial_x \overline{g}_{xx} \) | Even to order \( 2m - 2 \) | \( \{x \partial_x \overline{g}_{xx}\}_{2m-1} = (2m-1) \{\overline{g}_{xx}\}_{2m-1} \) *
| \( x \partial_{x\alpha} \overline{g}_{xx} \) | Odd to order \( 2m - 1 \) | *
| \( x \partial_x \overline{g}_{x\mu} \) | Odd to order \( 2m - 1 \) | *
| \( x \partial_{x\mu} \overline{g}_{xx} \) | Even to order \( 2m \) | *
| \( x \partial_x \overline{g}_{\alpha\beta} \) | Even to order \( 2m - 2 \) | \( \{x \partial_x \overline{g}_{\alpha\beta}\}_{2m-1} = (2m-1) \{\overline{g}_{\alpha\beta}\}_{2m-1} \) *
| \( x \partial_{x\alpha} \overline{g}_{\alpha\beta} \) | Odd to order \( 2m - 1 \) | *

As for the Christoffel symbols, once again each of these vanishes at \( x = 0 \), but it emerges from this computation that any such symbol with an even number of \( x \) components is odd to order \( 2m - 1 \). In subsequent computations for the Ricci tensor such symbols play no role in the first odd term in the expansion of quantities of interest.

| Component | Parity to order | Final coefficient |
|-----------|-----------------|-------------------|
| \( x \Gamma_{xx} \) | Even to order \( 2m - 2 \) | \( \{x \Gamma_{xx}\}_{2m-1} = \frac{1}{2} (2m-1) \{\overline{g}_{xx}\}_{2m-1} \) *
| \( x \Gamma_{\alpha\beta} \) | Even to order \( 2m - 2 \) | \( \{x \Gamma_{\alpha\beta}\}_{2m-1} = -\frac{1}{2} (2m-1) \{\overline{g}_{\alpha\beta}\}_{2m-1} \) *
| \( x \Gamma_{\alpha\beta} \) | Odd to order \( 2m - 1 \) | *
| \( x \Gamma_{\alpha\beta} \) | Even to order \( 2m - 2 \) | \( \{x \Gamma_{\alpha\beta}\}_{2m-1} = \frac{1}{2} (2m-1) h^\beta_0 \{\overline{g}_{\alpha\beta}\}_{2m-1} \) *
| \( x \Gamma_{\alpha\beta} \) | Odd to order \( 2m - 1 \) | *

These results are proved by straightforward calculations.
We now perform a similar analysis on the components of the Ricci curvature. We write this out a bit more carefully. First, we have

\[
x^2 R_{\alpha \beta} = x^2 Rm_{\alpha \beta}^s
\]

\[
= x^2 \partial_\alpha (\Gamma_\alpha^\alpha - x^2 \partial_x (\Gamma_\alpha^\alpha + x) + x \Gamma_\alpha^\alpha x) - x \Gamma_\alpha^\alpha x
\]

\[
= x^2 \partial_\alpha x - x^2 \partial_\alpha x^\alpha + x \Gamma_\alpha^\alpha x + x \Gamma_\alpha^\alpha x - x \Gamma_\alpha^\alpha x
\]

\[
= x^2 \partial_\alpha (x^\alpha x) - x^2 \partial_\alpha x^\alpha + x \Gamma_\alpha^\alpha x + x \Gamma_\alpha^\alpha x - x \Gamma_\alpha^\alpha x
\]

\[
(A.4)
\]

The first and the final four terms in this last expression are even to order 2m and thus do not contribute to the term at order 2m - 1; this uses that each \( x^\alpha \) vanishes at \( x = 0 \). Thus

\[
\{x^2 R_{\alpha \beta} \}_{2m-1} = \{ -x^2 (x \Gamma_\alpha^\alpha + x \Gamma_\alpha^\alpha) \}_{2m-1}
\]

\[
= \frac{1}{2} (2m - 1)^2 + 2m - 1) h_0^{\alpha \beta} \{ \overline{g}_{\alpha \beta} \}_{2m-1}
\]

\[
(A.5)
\]

The other set of components is

\[
x^2 R_{\alpha \beta} = x^2 Rm_{\alpha \beta}^s
\]

\[
= x^2 (\partial_\alpha (\Gamma_\alpha^\alpha - x^2 \partial_x (\Gamma_\alpha^\alpha + x) + x \Gamma_\alpha^\alpha x) - x \Gamma_\alpha^\alpha x
\]

\[
= x^2 (\partial_\alpha \Gamma_\alpha^\alpha - \partial_\alpha \Gamma_\alpha^\alpha - x^2 \partial_x \Gamma_\alpha^\alpha + x \Gamma_\alpha^\alpha x - x \Gamma_\alpha^\alpha x)
\]

\[
(A.6)
\]

This entire expression is even to order 2m - 2, and moreover all terms of the form \((x^\alpha)(x^\alpha)\) are even to order 2m, as are the derivative terms involving tangential partial derivatives. We thus find that

\[
\{x^2 R_{\alpha \beta} \}_{2m-1} = \{x^2 \partial_\alpha (x^\alpha x) \}_{2m-1}
\]

\[
= \frac{1}{2} (2m - 1)^2 \{ \overline{g}_{\alpha \beta} \}_{2m-1} + \frac{1}{2} (2m - 1) \{ \overline{g}_{\alpha \beta} \}_{2m-1}
\]

\[
(A.7)
\]

We do not calculate the components \( x^2 R_{\alpha \beta} \) since these do not come up in our arguments.

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