When does a Steady-State Response Exist in a Periodically Forced Multi-Degree-of-Freedom Mechanical System?

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July 9, 2019

Abstract

While steady-state responses of periodically forced dissipative nonlinear mechanical systems are commonly observed in experiments and numerics, their existence can rarely be concluded in rigorous mathematical terms. This lack of a priori existence criteria for mechanical systems hinders definitive conclusions about periodic orbits from approximate numerical methods, such as harmonic balance. In this work, we establish results guaranteeing the existence of steady-state responses without restricting the amplitude of the forcing or the response. Our results provide previously unavailable a priori justification for the use of numerical methods for the detection of periodic responses. We illustrate on examples that each condition of the existence criterion we discuss is essential.

Keywords: nonlinear oscillations, steady-state response, global analysis, harmonic balance, existence criterion

1 Introduction

Nonlinear mechanical systems are generally assumed to approach a steady-state response under external periodic forcing. While approximately periodic responses are commonly observed in numerical routines (e.g. numerical time integration, numerical continuation or harmonic balance) and experiments, concluding the existence of periodic response rigorously in a nonlinear system is more delicate.

For nonlinear system close to a solvable limit, perturbation methods remain a powerful tool to compute steady-state responses. Among these, the method of averaging requires slowly varying amplitude equations (cf. Sanders et al. [34]), while the method of multiple scales (cf. Nayfeh [30]) assumes evolution on different time scales generated by small parameters. The method of normal forms (cf. Murdock [29]) introduces a series of smooth coordinate changes to approximate and simplify the essential dynamics in a Taylor series in a small enough neighborhood of an equilibrium. Due to the truncation of infinite series arising in these procedures, the approximate dynamics remains valid only for sufficient small values of an underlying perturbation parameter. How small that parameter is required to be is generally unclear, and hence the relevance of the results obtained from perturbation procedures under physically relevant parameter values is a priori unknown.

Rigorous numerics (cf. van den Berg and Lessard [42]), estimating the ignored tail of Taylor expansions, is only applicable to specific numerical examples. In applications, therefore, one typically employs an additional numerical method to verify the predictions of perturbation methods. This is clearly not optimal.

Due to the broad availability of effective numerical packages, numerical time-integration is often used to compute steady-state responses. Modern, lightly damped engineering structures, however, require long integration times to reach a steady-state response. Furthermore, the observed steady-state response depends on the initial condition and unstable branches of the full set of periodic responses cannot be located in this fashion.

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More advanced numerical schemes, such as harmonic balance (cf. Mickens [28]) and numerical collocation (cf. Ascher et al. [2]), reformulate the underlying ordinary differential equations or boundary value problem into a set of algebraic equations by approximating the steady-state response by a set of finite basis functions (e.g., polynomial or Fourier basis). Coupled with numerical continuation schemes, this approach enables the computation of steady-state responses even for high-amplitude oscillations (cf. Dankowicz and Schilder [10]). To justify this procedure a priori and estimate the error due to the truncation of the infinite-dimensional basis-function space, the existence of the periodic orbit would need to be established a priori by an analytic criterion. While this can be guaranteed under small forcing or small nonlinearities by Poincaré map arguments, the problem remains unsolved for general, forced nonlinear mechanical systems.

Recently, Jain et al. [16] have proposed iterative methods to efficiently compute steady-state responses of periodically forced nonlinear mechanical systems without small parameter assumptions. Their existence criterion, derived via the Banach fixed-point theorem, however, fails for high forcing amplitudes and forcing frequencies in resonance with an eigenfrequency of the linearized system.

In the absence of small parameters, general fixed point theorems, such as Brouwer’s or Schauder’s fixed point theorem or the Leray-Schauder principle are powerful tools to prove the existence of periodic orbits (cf. Bobylev et al. [3] or Precup [32] for a summary). Lefschetz [20], for example, proved the existence of a periodic orbit for a specific one-dimensional, forced nonlinear mechanical system. His result, however, requires the damping force to be of the same order as the geometric nonlinearities. Therefore, his argument does not apply to common mechanical systems, such as the classic Duffing oscillator, with linear damping.

This restriction on the damping was relaxed significantly by Lazer [19]. Based on Schauder’s fixed point theorem, Lazer’s result allows for linear damping but requires a growth restriction on the nonlinearity. This result was further strengthened and extended to higher dimensions by Mawhin [24], who required the damping simply to be differentiable. Both results, however, restrict the growth of the nonlinearities to be less than linear for sufficiently high displacements, i.e., exclude polynomial or even linear stiffness forces. As Martelli [23] noted, this restriction can be relaxed to a linear growth with sufficiently low slope, depending on the eigenfrequency of the system and the forcing frequency. Due to the growth restriction on the nonlinearities these results are inapplicable for simple polynomial nonlinearities.

The results of Mawhin and Lazer have been extended to nonsmooth systems by Chu et al. [6] and Torres [39] and to more complex differential operators (cf. Mawhin [25]), relying on an extension of the generalized continuation theorems by Gaines and Mawhin [12] by Manásevich and Mawhin [22]. Furthermore, Antman and Lacabonara [1] give an existence criterion for periodic solutions based on the principle of guiding functions (cf. Krasnosel’skiĭ [18]). This result, however, relies on the specific form of the nonlinearity and external forcing for shells and is restricted to two-degree-of-freedom problems.

A general existence criterion for periodic orbits in second-order differential equations with linear dissipation can be found in the work of Rouche and Mawhin [33]. Their result implies the existence of a periodic response for dissipative nonlinear mechanical systems for arbitrary large forcing amplitudes. It appears, however, that the results in [33] are not known in the mechanical vibrations literature.

In this paper, we refine the Rouche-Mawhin results to be directly relevant for mechanical systems. This gives a general necessary criterion for the existence of periodic orbits in multi-degree-of-freedom forced-damped nonlinear mechanical systems, without any restriction on the magnitude of the forcing or vibration amplitude. To highlight that this criterion is necessary but not sufficient, we also give mechanically relevant examples of periodically forced systems in which a periodic steady-state response does not exist. This proves that the assumptions in our results are essential. Further, we identify a broad family of nonlinear mechanical systems for which our theorem guarantees the existence of a steady-state response. This result enables the rigorous computation of periodic orbits for a large class of strongly nonlinear mechanical systems.

2 Problem statement

We consider a general $N$-degree-of-freedom mechanical system of the form

$$
M \ddot{q} + C \dot{q} + S(q) = f(t), \quad f(t + T) = f(t), \quad q \in \mathbb{R}^N,
$$

(1)
where the mass matrix \( M \in \mathbb{R}^{N \times N} \) is positive definite and \( C \in \mathbb{R}^{N \times N} \) denotes the damping matrix. The vector \( S(q) \) contains all linear and nonlinear stiffness terms and is assumed to be continuous in its arguments. The external forcing \( f \) is assumed to be \( T \)-periodic and continuous.

### 2.1 Motivating example

To address the common belief that periodic forcing always leads to a periodic response of system (1), we start with an example illustrating the contrary. Furthermore, we demonstrate how the popular harmonic balance method yields false results on this example.

In the harmonic balance procedure, the equation of motion of the dynamical system (1) is evaluated along an assumed periodic orbit of the form

\[
q^*(t) = \frac{c_0}{2} + \sum_{k=1}^{K} (s_k \sin(k\Omega t) + c_k \cos(k\Omega t)), \quad s_k, c_k \in \mathbb{R}^N.
\]

Next, the forcing term \( f(t) \) and the nonlinearity evaluated along the postulated periodic orbit, \( S(q^*(t)) \), are projected on the first \( K \) Fourier modes and higher modes are ignored. As a result, one obtains a finite set of nonlinear algebraic equations for the unknown constants \( c_0 \) and \( s_k \), which can be solved e.g., by a Newton-Raphson iteration. For more details, we refer to Mickens [28].

Besides the a priori assumption of the existence of a periodic orbit, the truncation of the periodic orbit (2) at some finite order \( K \) needs to be justified. Classic results (cf. Bobylev et al. [3] or Leipholz [21]) show that this truncation can be justified for a sufficient high \( K \) when a periodic orbit of system (1) actually exists. If the existence of a periodic orbits cannot a priori be guaranteed, conditions derived by Urabe [41] or Stokes [36] might guarantee the existence of a periodic orbit close to the harmonic balance approximation. Their conditions, however, can only be evaluated a posteriori, as they rely on the harmonic balance solution itself. Notably, Kogelbauer et al. [17] strengthen the results of Urabe [41] or Stokes [36] also provide an a priori estimate. In practice, however, their conditions restrict the forcing and response amplitudes to small values.

To illustrate issues that can arise with harmonic balance, we consider the two degree-of-freedom oscillator

\[
\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \mathbf{q} + \kappa \begin{bmatrix} q_1^2 + q_2^2 \\ q_1^2 + q_2^2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \tag{3}
\]

The nonlinearities assumed in system (3) may arise in a Taylor series approximation of a more complex nonlinear forcing vector, which is terminated at second order. Specifically, quadratic nonlinearities arise in the modeling of ship capsize (cf. Thompson [37]), ear drums (cf. Mickens [26]) and shells (cf. Antman and Lacabonara [1]). Touzé et al. [40] study a spring-mass system in which quadratic nonlinearities arise naturally due to the geometry.

We assume forcing in the form of a triangular wave

\[
f_1(t) = -f_2(t) = \frac{f_m}{\pi} \int_0^1 \text{sign}(\cos(\Omega s)) \, ds = \frac{f_m}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\sin((2k+1)\Omega t)}{(2k+1)^2}, \tag{4}
\]

where the parameter \( f_m \) denotes the amplitude and \( \Omega = 2\pi/T \) the excitation frequency. For the remaining parameters, we assume the following non-dimensional numerical values

\[
m_1 = m_2 = 1, \quad k_1 = 1, \quad k_2 = 4, \quad c_1 = 0.001, \quad c_2 = 0, \quad f_m = 0.01178, \quad \Omega = 1, \quad \kappa = 1. \tag{5}
\]

We apply the harmonic balance to system (3) with the parameters (5) and forcing (4). We solve the resulting algebraic system of equations with a Newton-Raphson iteration, whereby we evaluate the nonlinearity in the time domain and transform the time signal to the frequency domain using fast Fourier transforms (cf. Cameron and Griffin [5]). Following common practice (cf. Cochetin and Vergez [8]), we start with a low number of harmonics and successively increase the number \( K \) of harmonics in ansatz (2) until the resulting oscillation amplitude appears converged, i.e., does not change with increasing \( K \). We depict the result of this harmonic balance procedure in Fig. 1.
Figure 1: Amplitudes and time series obtained with the harmonic balance procedure for the mechanical system \((\mathbf{3})\) with parameters \((\mathbf{5})\) and forcing \((\mathbf{4})\).

From the amplitudes depicted in Fig. 1a, one would normally conclude the convergence of the displacement amplitude of the first coordinate to a value of about 0.6. Also, the time series for the choices of \(K\), shown in Fig. 1b, are practically indistinguishable. There is, therefore, every indication that the harmonic balance method has correctly identified a periodic orbit for system \((\mathbf{3})\).

The periodic orbit suggested by the numerical result in Fig. 1, however, does not exist in system \((\mathbf{3})\) for the parameters \((\mathbf{5})\) and the forcing \((\mathbf{4})\). More generally, in Appendix \(B1\) we prove that if the amplitude \(f_m\) of the forcing \((\mathbf{4})\) is above a certain threshold, no periodic orbit exists for system \((\mathbf{3})\). Since the value of the forcing amplitude \((\mathbf{5})\) is above this threshold, the periodic orbit indicated by the harmonic balance procedure in Fig. 1 does not actually exist.

One might argue that due to the discontinuity of the forcing \((\mathbf{4})\), the assumption of a twice differentiable solution is not justified. Indeed, due to the Lipschitz continuity of the forcing \((\mathbf{4})\) just the existence and uniqueness of a local solution can be guaranteed by Picard’s theorem (cf. Coddington and Levinson \([9]\)). Our nonexistence proof, however, relies only on the fact that the amplitudes of the forcing is above a certain threshold. One can therefore easily replace the forcing \((\mathbf{4})\) by a smoother, even analytic alternative and obtain the same conclusion.

As we will see shortly, the crucial reason for the nonexistence of a periodic orbit in the above example is the form of the nonlinearity. Indeed, for a simple system with a single quadratic nonlinearity, Thompson and Stewart \([38]\) were unable to continue a periodic orbit numerically for arbitrarily high forcing amplitudes. Difficulties in applying harmonics balance to systems with quadratic nonlinearities have also lead to the practical guidelines by Mickens \([27]\), who heuristically restricts the harmonics balance procedure to systems with odd nonlinearities.

3 Existence of the steady-state response

With the mean forcing defined as

\[
\bar{f} = \frac{1}{T} \int_0^T f(t) dt,
\]

we will prove the following general result

**Theorem 3.1.** Assume that the forcing \(f(t)\) in system \((\mathbf{1})\) is continuous and the following conditions hold:

(C1) The damping matrix \(\mathbf{C}\) is definite, i.e., there exists a constant \(C_0>0\) such that

\[
|x^T \mathbf{C} x| > C_0 |x|^2, \quad x \in \mathbb{R}^N.
\]

\[
\text{amplitude max}(q_1)
\]

number of harmonics \(K\)

(a) Amplitudes

(b) Harmonic balance solution

\[
0 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.7
\]

\[
0 10 20 30 40 50
\]

\[
0\pi 2\pi
\]
The stiffness terms derive from a potential, i.e., there exists a continuously differentiable scalar function \( V(q) \) such that
\[
S(q) = \frac{\partial V(q)}{\partial q}.
\] (8)

The projection of the quantity \( S(q) - \bar{f} \) onto each degree of freedom has a constant, nonzero sign far enough from the origin. Specifically, there exists a distance \( r > 0 \) and an integer \( 1 \leq n \leq N \) such that
\[
q_j \left( S_j(q) - \bar{f}_j \right) > 0, \quad |q_j| > r, \quad j = 1, \ldots, n,
\]
\[
q_j \left( S_j(q) - \bar{f}_j \right) < 0, \quad |q_j| > r, \quad j = n + 1, \ldots, N.
\] (9)

Then system \( \mathcal{H} \) has a twice continuously differential periodic orbit.

**Proof.** We deduce this theorem from Theorem 6.3 by Rouche and Mawhin [33] after the removal of an unnecessary zero-mean forcing assumption in its original version. We detail the proof in Appendix A1.

**Remark 3.1.** As a consequence of the proof of Theorem 3.1, we obtain an upper bound on the amplitude of the existing periodic orbit. Specifically, with the squared \( L_2 \)-norm
\[
C_f^2 := \int_0^T f^T f dt
\] (10)
of the forcing an estimate for the maximal oscillation amplitude is given by
\[
\sup_{0 \leq t \leq T} |q| \leq \sqrt{N \left( r + \sqrt{T C_f / C_0} \right)},
\] (11)
where the constant \( C_0 \) is defined in equation (7). We detail the derivation of this estimate in Appendix A2.

Our bound (11) is stricter than that obtained by Rouche and Mawhin [33], who have an additional summand of \( \sqrt{T C_f / C_0} \) in equation (11). Further, the bound (11) confirms the intuition arising from linear theory, that the maximal response amplitude is proportional to the quotient of forcing amplitude and minimum damping coefficient. The inequality (11) confirms this intuition for the full nonlinear system without small-parameter assumptions.

For a single degree-of-freedom harmonic oscillator \( N = 1, \ r = 0, \) damping coefficient \( c \) and eigen-frequency \( \omega_0 \) and single harmonic forcing with amplitude \( f \) at resonance, the relationship between the bound (11) and the exact solution \( q_{lin} = f/(2\pi c) \) is as follows:
\[
\sup_{0 \leq t \leq T} |q| \leq \sqrt{N \left( r + \sqrt{T C_f / C_0} \right)} = \frac{f T}{2c} > \frac{f T}{2\pi c} = q_{lin}.
\] (12)
As expected, the bound (12) is conservative, but only by a factor of \( \pi \).

**Theorem 3.2.** If the nonlinearities \( S(q) \) are differentiable, condition \( \text{[C3]} \) holds if
\[(C^*) \quad \text{The Hessian of } V(q) \text{ is definite for } |q| > r^*, \text{ i.e., for some constant } C_v > 0,
\]
\[
|x^T \frac{\partial^2 V(q)}{\partial^2 \dot{q}} x| > C_v |x|^2, \quad x \in \mathbb{R}^N, \quad |q| > r^*.
\] (13)

**Proof.** See Appendix A3 for a proof.

**Remark 3.2.** Condition \( \text{[C3]}^* \) is more restrictive than condition \( \text{[C3]} \). For example, consider the potential
\[
V(q_1, q_2) = k_1 q_1^2 + k_2 q_2^2,
\] (14)
which satisfies \( \text{[C3]}^* \) only if \( k_1 \) and \( k_2 \) have the same sign \( (k_1 k_2 > 0) \), while it satisfies condition \( \text{[C3]} \) for any non-zero \( k_1 \) and \( k_2 \). However, condition \( \text{[C3]}^* \) is more intuitive and generally easier to verify.
Theorems 4.1 and 4.2 can guarantee the existence of periodic orbits for arbitrary large forcing and response amplitudes. These theorems, therefore, enable an a priori justification of the use of otherwise heuristic approaches, such as harmonic balance or numerical collocation. In the following, we demonstrate the use of these theorems on various mechanical systems.

4 Examples

First, we illustrate via mechanically relevant examples that conditions (C1)-(C3) of Theorem 3.1 cannot be weakened. Next, we identify a large class of high-dimensional mechanical systems for which the existence of periodic orbits can be guaranteed by Theorems 4.1 and 4.2.

4.1 The importance of condition (C1)-(C3)

In the following we show that if one of the conditions (C1)-(C3) is violated, one can find mechanical systems with no periodic orbits that nevertheless satisfy the remaining conditions of Theorem 3.1. This underlines the importance of conditions (C1)-(C3).

Example 4.1 (Zero-damping). All solutions in a one-degree-of-freedom, undamped linear oscillator grow unbounded when the oscillator is forced at resonance. As a consequence, no periodic orbits may exist in such a system. Indeed, any undamped linear oscillator violates condition (C1) because damping matrix is identically zero and therefore neither positive nor negative definite.

Example 4.2 (Non-potential nonlinearities). We have seen in section 2.1 that system (3) has no T-periodic orbit. Indeed, trying to apply Theorem 3.1 to this problem, we find that condition (C2) is not satisfied. Condition (C3) is not satisfied either since the stiffness force minus any constant mean forcing is always positive for positions sufficiently far away from the origin:

\[ (q_1^2 + q_2^2) - \bar{f}_j > 0, \quad |q_j| > \sqrt{|f_j|}, \quad j = 1, 2. \]  (15)

Therefore, the quantity \( q_i((q_1^2 + q_2^2) - \bar{f}_j) \) is positive for all \( q_i > \sqrt{|f_j|} \) and negative for \( q_i < -\sqrt{|f_j|} \). This implies that no constant \( r \) exist such that \( q_i((q_1^2 + q_2^2) - \bar{f}_j) \) for all \( |q_i| > r \), i.e. condition (C3) is violated.

We now consider a slight modification of system (3) in the form of

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{q} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \dot{q} + \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} q + \kappa \begin{bmatrix} q_1q_2^* \\ q_2q_1^* \end{bmatrix} = \begin{bmatrix} f_1(t) \\ a \sin(\Omega t) \end{bmatrix},
\]  (16)

which satisfies conditions (C3) for the choice

\[
\kappa > 0, \quad \bar{f}_j \leq 0, \quad \Rightarrow q_1(q_1^2 + \kappa q_2^2 - \bar{f}_j) > 0, \quad \text{for all } q_1, q_2 \in \mathbb{R}. \]  (17)

Condition (C1) is also satisfied for positive damping values \( c_1, c_2 > 0 \). Condition (C2), however, is not satisfied as the stiffness forces of system (10) do not derive from a potential. Accordingly, we will show that system (16) has no T-periodic orbits for an appropriately chosen set of parameters.

Assuming the contrary, we consider a periodic solution \( q^*(t) \) and solve the second equation in (16) to obtain

\[
q_2^*(t) = A(\omega_2, c_2, a, \Omega) \sin(\Omega t - \psi(\omega_2, c_2, a, \Omega)),
\]  (18)

where the amplification factor \( A(\omega_2, c_2, a) \) and the phase shift \( \psi(\omega_2, c_2, a) \) are constants depending on the damping coefficient and eigenfrequency, as well as the forcing amplitude and frequency, as indicated. The exact form of \( A_1 \) and \( \psi \) can be determined from linear theory (see, e.g., Gérardin and Rixen [13]). Substituting equation (18) into the first equation in (16), we obtain

\[
\ddot{q}_1^* + c_1 \dot{q}_1^* + q_1^*(\omega_1^2 + \kappa A^2/2 - \kappa A^2/2 \cos(2\Omega t - 2\psi)) = f_1(t),
\]  (19)

which is a modification of classic forced-damped Matthieu equation (cf., e.g., Guckenheimer and Holmes [14] for the undamped-unforced limit, or Nayfeh and Mook [31] for the unforced limit). Compared to the standard Matthieu equation, an additional term \( q_1^* \kappa A^2/2 \) arises in equation (19). For the unforced Matthieu
equation \((f_1=0)\), a change of stability of the trivial solution is commonly observed for various values of the damping, stiffness and forcing frequency. Utilizing this observation, we can prove the nonexistence of a periodic orbit in system (16) with the following fact:

**Fact 4.1.** If the trivial solution of the system

\[
\ddot{q}_1 + c_1 \dot{q}_1 + q_1 (k_1 + \frac{\kappa A^2}{2} - \frac{\kappa A^2}{2} \cos(2\Omega t - 2\psi)) = 0,
\]

is unstable for some parameter values \(a, \Omega, c_2 > 0, c_1 > 0, \omega_1, \omega_2\) and \(\kappa\), then we can find a \(T\)-periodic forcing \(f_1(t)\) satisfying condition (17), such that system (19) has no periodic orbit.

**Proof.** The proof relies on the fact that a \(T\)-periodic solution to system (19) does not exist if a non-trivial \(T\)-periodic solution exists in the homogeneous system (20) and additional orthogonality conditions between the external forcing and non-trivial \(T\)-periodic solutions are violated (cf. Farkas [11]). In Appendix B2 we show the existence of non-trivial \(T\)-periodic solutions (20) and show that the orthogonality conditions are generally violated for appropriately chosen \(f_1\).

We can use the above fact to establish the nonexistence of a periodic orbit for system (16). To this end, we have to find a set of parameters for which the trivial solution of system (20) is unstable. For the non-dimensional parameters

\[
\omega_2 = 1, \quad c_1 = c_2 = 0.01, \quad \kappa = 1, \quad \Omega = 1,
\]

we calculate the monodromy matrix using numerical integration, covering a parameter range for the forcing amplitude \(a\) and the eigenfrequency \(\omega_1\). We depict the result of the Floquet analysis performed on the monodromy matrix in Fig. 2 where we indicate the system configurations with stable trivial solution in green, while red indicates a system configuration with an unstable trivial solution. The critical system configurations can be found on the stability boundary, which is highlighted in black in Fig. 2. As we prove in Appendix B2, for the black configurations, we can find a continuous \(T\)-periodic forcing \(f_1\) such that the system (19), and hence system (16), has no periodic orbit.

![Figure 2: Stability map of the trivial solution of system (20) with parameters (21). Red denotes the instability of the trivial solution, while green denotes a stable trivial solution. Black lines indicate the stability boundary. At the latter parameter values one of the Floquet multipliers is equal to one in norm.](image)

**Example 4.3** (Stiffness with global extrema). Condition \([\text{C3}]\) requires the sign of the quantities \(q_j(S_j(q) - \bar{f}_j)\) to be constant and non-zero for \(|q_j| > r\). If the stiffness minus the mean forcing have a constant sign outside the region \(|q_j| > r\) for some degree of freedom \((\text{sign}(S_j(q) - \bar{f}_j) = \text{const.} \text{ for } |q_j| > r)\), then the quantities \(q_j(S_j(q) - \bar{f}_j)\) evaluated for \(q_j > r\) and for \(q_j < -r\) have opposite sign.
Therefore, condition \((C3)\) is violated. This is certainly the case, if the nonlinearity has an extremum componentwise, i.e.

\[
S_j(q) > S_{\text{min}}, \quad j = 1, ..., N, \quad \vee \quad S_j(q) < S_{\text{max}}, \quad j = 1, ..., N, \quad q \in \mathbb{R}^N.
\] (22)

Then sign \((S_j(q) - \bar{f}_j)\) is non-constant for \(|q_j| > r\) and for any value of the mean forcing. For system (1) with stiffness terms satisfying (22), we have the following fact:

**Fact 4.2.** If the stiffness terms of system (1) have a global maximum value \((S_{\text{max}})\) or minimum value \((S_{\text{min}})\), and the mean forcing of a single coordinate \(\bar{f}_j\) is larger than the global maximum or less than the global minimum, i.e.

\[
\bar{f}_j < S_{\text{min}}, \quad \vee \quad \bar{f}_j > S_{\text{max}}, \quad 1 \leq j \leq N,
\] (23)

then no periodic orbit exists for system (1).

**Proof.** We detail this proof in Appendix B3.

**Remark 4.1.** In case of constant forcing \((f = \bar{f})\), Fact 4.2 states that no equilibrium position exists if the stiffness terms have a global extremum and the forcing is chosen above respectively below that extreme value.

An example for a nonlinearity satisfying (22) is the simple pendulum, whose nonlinearity \(S(q) = c_p \sin(q)\) has global maxima and minima. Therefore, the damped forced-pendulum

\[
\ddot{q} + c \dot{q} + c_p \sin(q) = f(t), \quad |\bar{f}| > |c_p|,
\] (24)

has no \(T\)-periodic solution.

The previous example indicates that the mean value of the forcing plays a critical role in the existence of periodic orbits. One might wonder if a zero-mean restriction of the forcing, as in the theorem of Rouche and Mawhin [33] (cf. Theorem A.1 in Appendix A1), allows relaxing some of our conditions, notably condition \((C3)\). In the following example, we show that even for zero-mean forcing \((\bar{f}_j = 0)\) condition \((C3)\) cannot be relaxed.

**Example 4.4 (Constant-sign stiffness, zero-mean forcing).** We consider the nonlinear oscillator

\[
\ddot{q} + c \dot{q} + \omega^2 q + \kappa q^2 = f(t) \cos(\Omega t),
\] (25)

the simplest example with a stiffness violating condition \((C3)\). As forcing, we choose simple single harmonic forcing with amplitude \(f\). For system (25), we have then the following fact:

**Fact 4.3.** If the forcing amplitude \(f\) for mechanical system (25) is above the threshold

\[
|f| > \frac{\omega^2}{2|\kappa|} \left( |\Omega^2 + ic\Omega + \omega^2| + 2\omega^2 \right) + |\kappa| \frac{\omega^4}{4\kappa^2},
\] (26)

then no \(T\)-periodic solution to system (25) exist.

**Proof.** We detail the proof in Appendix B4.

Therefore, choosing any forcing amplitude exceeding the threshold (26) will necessarily rule out the existence of a periodic orbit.

### 4.2 Examples with periodic orbits guaranteed by Theorem 3.1

In the following, we give examples in which Theorem 3.1 guarantees the existence of a steady-state response. Since the damping condition \((C1)\) and the assumption \((C2)\) on the stiffness terms derived from a potential are simple to verify, we focus on condition \((C3)\) and \((C3^*)\). We start with the classic Duffing oscillator and proceed with higher-dimensional examples.
Example 4.5 (Duffing oscillator). The forced-damped Duffing oscillator is simple harmonic oscillator with an additional cubic nonlinearity added, i.e.
\[ \ddot{q} + c\dot{q} + \omega^2 q + \kappa q^3 = f \cos(\Omega t), \]  
(27)
where we have chosen single harmonic forcing with amplitude \( f \) and frequency \( \Omega \).

For \( \kappa \geq 0 \), the potential of eq. (27) is positive definite for all \( q \) (cf. Fig. 3a). Therefore, condition \((C3^*)\) is trivially satisfied. Furthermore, condition \((11)\) is satisfied for arbitrarily small radius \( r \), which can be used for the upper bound on the amplitudes \((11)\). Therefore, both Theorems 3.1 and 3.2 apply and guarantee the existence of a steady-state solution without any numerics. We compute the steady-state response with the automated continuation package coco \[10\] and show the amplitudes in Fig. 3b.

Figure 3: Features of the Duffing oscillator \((27)\) for \( \kappa \geq 0 \) (hardening spring stiffness).

For negative values of the coefficient \( \kappa \), the Hessian of the potential is not globally positive (cf. Fig. 4a green curve). However, outside the ball of radius \( r^* = \omega \sqrt{-1/(3\kappa)} \), the second derivative of the potential is negative. Therefore, the existence of a periodic orbit is guaranteed by Theorem 3.2. Furthermore, two nontrivial equilibria arise at \( q_0 = \omega \sqrt{-1/\kappa} \) (cf. Fig. 4a black curve). If we select this \( q_0 \) as the radius \( r \) in Theorem 3.1 then condition \((C3)\) is satisfied. Again, both Theorems 3.1 and 3.2 guarantee the existence of a steady-state response.

We numerically continue all three periodic orbits for increasing forcing amplitude at the fixed forcing frequency \( \Omega = 1 \) and show the amplitudes of the steady-state response in Fig. 4b. For larger forcing amplitudes, numerical continuation becomes more challenging, yet our results continue to imply the existence of a periodic response rigorously.

Example 4.6 (Oscillator chain). We consider the \( N \)-dimensional oscillator chain depicted in Fig. 5. Two adjacent masses are connected via nonlinear springs and linear dampers. The first and last mass are suspended to the wall. The equation of motion of the \( j \)-th mass is given by
\[ m_j \ddot{q}_j - c_j (\dot{q}_{j-1} - \dot{q}_j) + c_{j+1} (\dot{q}_j - \dot{q}_{j+1}) - S_j (q_{j-1} - q_j) + S_{j+1} (q_j - q_{j+1}) = f_j(t), \quad j = 1, ..., N, \]  
(28)
where we set the coordinates \( q_0 \) and \( q_{N+1} \) to zero. Variants of such systems have been investigated by Shaw and Pierre \[35\], Breunung and Haller \[4\] and Jain et al. \[16\]. Physically, the system may represent, e.g., a discretized beam. As we detail in Appendix B5, the following fact holds for the chain system \((28)\):

Fact 4.4. For the parameters
\[ m_j > 0, \quad c_j > 0, \quad \frac{\partial S_j(\delta)}{\partial \delta} > 0, \quad j = 1, ..., N + 1, \]  
(29)
the system \((28)\) satisfies the conditions of Theorem 3.1 and hence must have a steady-state response.
The systems investigated by Shaw and Pierre [35], Breunung and Halle [4], Jain et al. [16] satisfy the conditions (29), as the stiffness forces are of the form 

\[ S_j(\delta) = k_j \delta + \kappa_j \delta^3 \]

with \( k_j > 0 \) and \( \kappa_j \geq 0 \). Therefore, we can guarantee the existence of the steady-state response of these systems for arbitrary large forcing amplitudes.

In the derivations in Appendix B5, we further detail that the assumptions on the parameters (29) can be relaxed to include either the cases \( c_{N+1} = 0 \) and \( S_{N+1}(q_{N+1}) = 0 \) or \( c_1 = 0 \) and \( S_1(q_1) = 0 \). In both cases, the damping matrix \( C \) and the second derivative of the potential remain positive definite. The conditions on the first and the \( N+1 \)-th damping coefficient and stiffness force cannot be relaxed simultaneously. In the case of \( S_1(q_1) = S_{N+1}(q_N) = 0 \), the system is not connected to the walls and hence a non-periodic, free rigid body motion of the whole chain can be initiated with appropriate forcing. For \( c_1 = c_{N+1} = 0 \), this motion is undamped and hence if the springs are linear, then forcing at resonance cannot result in a periodic response.

5 Conclusions

We have discussed the example of a specific mechanical system for which the application of the harmonic balance procedure leads to the wrong conclusion about the existence of a periodic response. This underlines the necessity of rigorous existence criteria for periodic orbits in damped-forced nonlinear mechanical systems. Such existence criteria can give a priori justification for the use of formal perturbation methods and numerical continuation, eliminating erroneous conclusions or wasted computational resources.

To obtain such an existence criterion, we have extend a theorem by Rouche and Mawhin [33] to obtain generally applicable sufficient conditions for the existence of a periodic response in periodically forced, nonlinear mechanical systems. Roughly speaking, these conditions guarantee a periodic orbit under arbitrarily large forcing and response amplitudes, as long as the dissipation acts on all degrees of
freedom, the spring forces are potential, and the potential function is strictly convex or strictly concave outside a neighborhood of the origin.

Since the conditions of our theorem are sufficient but not necessary, the question arises whether they can be relaxed. With mechanically relevant examples, we show that each condition in our theorem is essential, and cannot be relaxed. Based on these results, we identify a large class of nonlinear mechanical systems for which numerical procedures, such as the harmonic balance and the collocation method, are a priori justified. This enables the reliable computation of periodic orbits for large forcing and oscillation amplitudes in this class of systems.

We have limited our discussion to periodic forcing, for which extensive mathematical literature exists. Quasi-periodic forcing is also of interest in engineering applications; indeed, the harmonic balance method has been extended to compute quasi-periodic steady states response of nonlinear mechanical systems (cf. Chua and Ushida [7]). The extension of the present results to quasi-periodic forcing, however, is not immediately clear.

Our discussion is restricted to mechanical equations of motions with geometric nonlinearities, as it is customary in the structural vibrations literature. It is also of interest, however, to extend our conclusions to velocity-dependent nonlinearities.

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In the following, we prove the main Theorems 3.1 and 3.2 and derive an upper bound on the amplitudes of the steady-state responses.

A1 Proof of Theorem 3.1

We base our proof of Theorem 3.1 on a Theorem by Rouche and Mawhin [33], who analyze systems of the following form

\[ \ddot{q} + \dot{C}q + \frac{\partial \overline{V}(q)}{\partial q} = \overline{g}. \]  

(30)

**Theorem A.1.** Assume system (30) satisfies the following conditions:

(RM1) The damping matrix \( \overline{C} \) is positive or negative definite.

(RM2) There exists a distance \( r > 0 \) and an integer \( 1 \leq n \leq N \) such that

\[ q_j \frac{\partial \overline{V}(q)}{\partial q_j} > 0, \quad |q_j| > r, \quad j = 1, \ldots, n, \]

\[ q_j \frac{\partial \overline{V}(q)}{\partial q_j} < 0, \quad |q_j| > r, \quad j = n + 1, \ldots, N. \]  

(31)

(RM3) The mean value of the forcing \( \overline{g} \) is zero, i.e.,

\[ \overline{g} = \frac{1}{T} \int_0^T g(t) dt = 0. \]  

(32)

Then system (30) has at least one \( T \)-periodic solution.

**Proof.** For a proof, we refer to Rouche and Mawhin [33]. \( \Box \)
We transform system (1) such that it is in the form (30) and then show that the conditions of Theorem 3.1 imply that Theorem A.1 applies. First, we absorb the mean forcing into the potential by setting
\[ \hat{V}(q) = V(q) - q^T \bar{f}, \quad \bar{f} = f - \bar{f}. \] (33)
The equation of motion with nonlinearity derived from the potential \( \hat{V} \) and forcing \( \bar{f} \) is equivalent to system (1). Further, we right-multiply equation (1) with the inverse of the mass matrix \( M^{-1} \) and obtain
\[ \ddot{q} + M^{-1} C \dot{q} + M^{-1} \frac{\partial \hat{V}}{\partial q} = M^{-1} f(t). \] (34)
The potential for the stiffness terms of system (34) is given by \( \bar{V} \) and forcing \( \bar{f} \) is equivalent to system (1). The equation of motion with nonlinearity derived from the potential \( \bar{V} \) and forcing \( \bar{f} \) is equivalent to system (1). Therefore, a transformation to equation (30) is not necessary. We derive an upper bound on the solutions of equation (35) with the inverse of the mass matrix \( M^{-1} \)
\[ M \ddot{q} + C \dot{q} + S(q) = f(t), \quad q \in C^2(T), \] (35)
in the \( C^0 \) norm defined by
\[ ||q||_{C^0} = \max_{0 \leq t \leq T} |q|. \] (36)
First, we follow the derivation by Rouche and Mawhin [33] by left-multiplying equation (35) with \( q^T \) and integrating over one period to obtain
\[ \int_0^T \dot{q}^T M \ddot{q} \, dt + \int_0^T \dot{q}^T C \dot{q} \, dt + \int_0^T \dot{q}^T S(q) \, dt = \int_0^T \dot{q}^T f(t) \, dt. \] (37)
Observing that
\[ \int_0^T \dot{q}^T M \ddot{q} \, dt = \int_0^T \frac{d}{dt} \left( \frac{1}{2} \dot{q}^T M \ddot{q} \right) \, dt = 0, \] (38)
where we have used the symmetry of the mass matrix \( (M = M^T) \) and the periodicity of \( q \). Similarly
\[ \int_0^T \dot{q}^T S(q) \, dt = \int_0^T \frac{d}{dt} (V(q)) \, dt = 0, \] (39)
where we have used the fact that the stiffness forces arise from a potential \( S \) and again the periodicity of \( q \). Therefore, from equation (37), we obtain
\[ \left| \int_0^T q^T C \dot{q} \, dt \right| = \left| \int_0^T q^T f \, dt \right|. \] (40)
With the assumption of definite \( C \) matrix (cf. equation (21)), we obtain a lower bound on the left hand side of equation (30) to
\[ C_0 \int_0^T |\dot{q}|^2 \, dt \leq \left| \int_0^T q^T C \dot{q} \, dt \right|. \] (41)
For the right hand side of equation (40), we obtain an upper bound by using the Cauchy-Schwartz inequality
\[ \left| \int_0^T \dot{q}^T f \, dt \right| \leq \left( \int_0^T |\dot{q}|^2 \, dt \right)^{1/2} \left( \int_0^T |f|^2 \, dt \right)^{1/2}. \] (42)
Using the definition (13) of $C_f$ and combining the estimates (41) and (42), we obtain from equation (40), that
\[
\left(\int_0^T |\dot{q}_j|^2 \, dt\right)^{1/2} \leq \frac{C_f}{C_C},
\]
(43)
Equation (43) is an upper bound on the $L_2$-norm of the velocity of the periodic orbit. Rouche and Mawhin [33] derive the same bound. Now we depart from the derivations by Rouche and Mawhin [33] and integrate system (35) for one period, which yields
\[
\int_0^T S(q) \, dt = T\bar{f}, \quad \Leftrightarrow \quad \int_0^T (S(q) - \bar{f}) \, dt = 0,
\]
(44)
where we used the definition (5) of the mean forcing $\bar{f}$. Applying the mean-value theorem to (44), we conclude that
\[
S_j(q(t_j)) - \bar{f}_j = 0, \quad 0 \leq t_j \leq T, \quad j = 1, ..., N.
\]
(45)
From condition (9), we conclude that equation (15) is only satisfied if $|q_j(t_j)| < r$. We conclude
\[
q_j^2(t) = \left( q_j(t_j) + \int_{t_j}^t \dot{q}_j(s) \, ds \right)^2 \leq q_j(t_j)^2 + 2q_j(t_j) \int_{t_j}^t \dot{q}_j(s) \, ds + \left( \int_{t_j}^t \dot{q}_j(s) \, ds \right)^2
\]
\[
\leq r^2 + 2r|t - t_j|^{1/2} \left( \int_{t_j}^t \dot{q}_j^2(s) \, ds \right)^{1/2} + |t - t_j| \int_{t_j}^t \dot{q}_j^2(s) \, ds
\]
\[
\leq r^2 + 2r\sqrt{T} \left( \int_{t_j}^t \dot{q}_j^2(s) \, ds \right)^{1/2} + T \int_{t_j}^t \dot{q}_j^2(s) \, ds \leq \left( r + T \frac{C_f}{C_C} \right)^2.
\]
(46)
Therefore, for the $C^0$-norm of the positions, we obtain
\[
\|q\|_{C^0} < \left( \sum_{j=1}^N \sup_{0 \leq t \leq T} (q_j^2(t)) \right)^{1/2} \leq \sqrt{N} \left( r + \sqrt{T} \frac{C_f}{C_C} \right),
\]
(47)
where we have used the upper bound (43). In contrast, Rouche and Mawhin [33] derive the bound
\[
\|q\|_{C^0} \leq \sqrt{N} \left( r + \sqrt{T} \frac{C_f}{C_C} \right) + \frac{TC_f}{C_0},
\]
(48)
which includes the additional summand $TC_f/C_0$ compared to our bound (47).

### A3 Proof of Theorem 3.2

In the following, we show that condition (C3*) implies that condition (C3) is satisfied. We note that each continuous function $S_j(q)$ has a maximum and a minimum value in a ball of radius $r^*$, which we label with $S_{\max}^j$ and $S_{\min}^j$. Choosing the radius
\[
r_j = r^* + \max(0, -\frac{\bar{f}_j - S_{\min}^j}{c_v}, \frac{S_{\max}^j - \bar{f}_j}{c_v}),
\]
(49)
ensures that the quantity $q_j(S_j(q) - \bar{f}_j)$ has a constant, non-zero sign for all
\[
q \in \mathbb{Q}_j := \{ q \in \mathbb{R} \mid |q_j| > r_j \}.
\]
(50)
First, we assume a positive definite Hessian outside a ball of radius $r^*$. Using a Taylor series expansion of the nonlinearity, we note that outside the $r^*$ ball the following holds:
\[
\mathbf{h}^T (S(q + h) - S(q)) = \int_0^1 \mathbf{h}^T \frac{\partial^2 V(q + sh)}{\partial \mathbf{q}^2} \, \mathbf{h} \, ds \geq c_v |\mathbf{h}|^2 > 0, \quad q, h \in \mathbb{R}^N, \quad 0 \leq s \leq 1, \quad |\mathbf{q} + sh| > r^*.
\]
(51)
For every point \( q \in \mathbb{Q}_j \), we select \( h \) to be the vector pointing from the \( q_j \)-axis to \( q \) with minimal length. Denoting the \( j \)-th unit vector by \( e_j \), we set \( h = q - q_j e_j \) and \( q = q_j e_j \). Since \( |q_j| > r^* \), line connecting \( q_j e_j \) and \( q \) is in the region, where the potential \( V(q) \) is positive definite. From equation (51) we obtain

\[
(q - q_j e_j)^T (S(q) - S(q_j e_j)) = \sum_{n=1}^{N} q_n [S_n(q) - S_n(q_j e_j)] > 0.
\]

Further, we reduce the \( j \)-th coordinate until we reach \( |q_j| = r^* \). The line connecting between the points \( \text{sign}(q_j)r^*e_j \) and \( q_j e_j \) lies in the region with a positive definite Hessian. We evaluate (51) for \( q = \text{sign}(q_j)r^*e_j \) and \( h = q - \text{sign}(q_j)r^*e_j \) to obtain

\[
(q - \text{sign}(q_j)r^*e_j)^T (S(q) - S(\text{sign}(q_j)r^*e_j)) = \sum_{n=1}^{N} q_n [S_n(q) - S_n(\text{sign}(q_j)r^*e_j)] > 0.
\]

For \( q_j > 0 \), equation (49) implies that \( (q_j - \text{sign}(q_j)r^*) \) is positive. Therefore, we obtain from equation (53) that

\[
S_j(q) > c_v(q_j - r^*) + S_j(r^*e_j) > c_v(q_j - r^*) + S_j^\text{min} > c_v(r^* + \frac{\bar{f}_j - S_j^\text{min}}{c_v} - r^*) + S_j^\text{min} = \bar{f}_j, \quad q_j > r_j.
\]

Similarly, for \( q_j < 0 \) the quantity \( S_j(q) - \text{sign}(q_j)r^* \) is negative, therefore equation (53) implies

\[
S_j(q) < c_v(q_j + r^*) + S_j(-r^*e_j) < c_v(q_j + r^*) + S_j^\text{max} < c_v(-r^* + \frac{S_j^\text{max} - \bar{f}_j}{c_v} + r^*) + S_j^\text{max} = \bar{f}_j, \quad q_j^* < -r_j.
\]

Equations (54) and (55) together imply

\[
q_j(S_j(q) - f_j) > 0, \quad q \in \mathbb{Q}_j,
\]

which is equivalent to the upper condition (1), if we set \( r := \max_j(r_j) \).

The same argument can be repeated for potentials having a negative definite Hessian. The sign in equation (51) changes, therefore one obtains

\[
q_j(S_j(q) - f_j) < 0, \quad q \in \mathbb{Q}_j,
\]

which is equivalent to the lower condition (1), if we set \( r := \max_j(r_j) \).

B Derivations for specific examples

B1 Necessary bound on the forcing amplitude for system (3)

In the following, we prove a necessary bound on the forcing amplitude (41) for the existence of periodic solutions for system (3) with parameters (5). Specifically, we assume the existence of a twice continuous differentiable periodic orbit \( q^* \). Transforming system (3) to modal coordinates, we obtain

\[
q_1' = x_1 + x_2, \quad q_2' = x_1 - x_2,
\]

\[
\ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 + 2\kappa x_1^2 = -2\kappa x_2^2,
\]

\[
\ddot{x}_2 + c_1 \dot{x}_2 + (k_1 + 2k_2)x_2 = f_1.
\]

(58a)
The equation of motion of the second modal degree-of-freedom (58a) is linear and therefore the assumed periodic response of the second degree of freedom $x_2$ can be obtained analytically:

$$x_2 = \frac{8f_m}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k \sin(\Omega t - \varphi_k)}{(2k+1)^2 \sqrt{((k+2k_2)-(2k+1)^2\Omega^2)^2 + ((2k+1)c_1\Omega)^2}} = \frac{8f_m}{\pi^2} \sum_{k=0}^{\infty} c_k \sin(\Omega t - \varphi_k),$$

$$\varphi_k = \tan^{-1} \left( \frac{(2k+1)c_1\Omega}{(k_1+2k_2)-(2k+1)^2\Omega^2} \right),$$

(59)

Here we have re-labeled the amplitudes for notational convenience. Next, we integrate (60) over one period and impose periodicity to obtain

$$\int_0^T (k_1 x_1 + 2\kappa x_2^2) dt = -2\kappa \int_0^T x_2^2 dt, = -T\kappa \frac{64f_m^2}{\pi^4} \sum_{k=0}^{\infty} |c_k|^2.$$  

(60)

The infinite sum converges to the limit $c_\infty$, since it can be majorized by $1/k^6$, i.e.

$$c_\infty := \sum_{k=0}^{\infty} |c_k|^2 = \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2((k+2k_2)-(2k+1)^2\Omega^2)^2 + ((2k+1)c_1\Omega)^2} \leq \frac{1}{c_1^2\Omega^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} \leq \frac{1}{c_1^2\Omega^2} \sum_{k=1}^{\infty} 1/k^6.$$  

(61)

For the parameters (5), we compute the value $c_\infty$ numerically and obtain

$$c_\infty := 1371.7577441 > 1371.757744028...$$  

(62)

By the mean-value theorem applied to equation (60), there must be a time instance $t^* \in [0,T]$ at which the integrand on the left-hand side is equal to the infinite sum on the left hand side. Calculating the minimum of the parabola in that integrand and inserting the numerical parameter values (5) yields

$$-\frac{k_1^2}{8\kappa} T \leq (k_1 x_1(t^*) + 2\kappa x_2^2(t^*)) T = -\kappa T \frac{64f_m^2}{\pi^4} c_\infty, \quad 0 \leq t^* < T.$$  

(63)

Solving (63) for the forcing amplitude, we obtain

$$|f_m| < \sqrt{\frac{k_1^2 \pi^4}{512 \kappa^2 c_\infty}} = 0.011777, \quad \kappa > 0.$$  

(64)

Since the forcing amplitude (5) is above the threshold (64), the periodic orbit indicated by the harmonic balance method does not exist.

### B2 Proof of Fact 4.1

In the following, we show that no periodic orbit for system (19) exists, for an appropriately chosen set of parameters. For these sets of parameters, one of the Floquet multipliers of the unforced limit of system (19) equals to one in norm. This introduces the possibility of resonance between the external periodic forcing and the nontrivial solution of the homogeneous part (20), under which no periodic orbit for system (19) can exist.

For further analysis, we introduce the matrices and vectors

$$\mathbf{x} := \begin{bmatrix} q_1^* \\ \dot{q}_1^* \end{bmatrix}, \quad \mathbf{A}(t) := \begin{bmatrix} 0 \\ -k_1 - \kappa \dot{A}_2 + \frac{\kappa^2}{2} \cos(2\Omega t - 2\psi) & 1 \end{bmatrix}, \quad \mathbf{g}(t) := \begin{bmatrix} 0 \\ f_1(t) \end{bmatrix}.$$  

(65)

With the notation (65), we express system (19) in first-order form

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(t), \quad \mathbf{A}(t+T/2) = \mathbf{A}(t), \quad T = 2\pi/\Omega,$$  

(66)

and denote its homogeneous part by

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}.$$  

(67)

Furthermore, we define the adjoint problem,

$$\dot{\mathbf{y}} = -\mathbf{A}^T(t)\mathbf{y}.$$  

(68)

To show the non-existence of a periodic orbit of system (19), we use the following Theorem:
Theorem B.1. Assume that system (67) has \( k \) linearly independent, nontrivial \( T \)-periodic solutions and denote the \( k \) linearly independent solutions to the adjoint system (68) by \( \tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_k \). Then the nonautonomous system (66) has a \( T \)-periodic solution if and only if the orthogonality conditions
\[
\int_0^T \tilde{y}_j^T g(t) dt = 0, \quad j = 1, \ldots, k, \tag{69}
\]
hold.

Proof. For a proof, we refer to Farkas [11]. \( \square \)

First, we note system (67) is periodic with period \( T/2 \) (cf. equation (66)), where \( T \) is determined by the external forcing \( f_2 \) (cf. equation (19)). We denote the complex conjugate Floquet multipliers of system (66) by \( \rho_1 \) and \( \rho_2 \) and further obtain from Liouville’s theorem that
\[
\rho_1 \rho_2 = e^{\int_0^{T/2} \text{Tr}[A(s)] ds} = e^{-\frac{c_1 T}{2}}, \quad \rho_1 = \bar{\rho}_2. \tag{70}
\]
Equation (70) imply that the Floquet multipliers are located either on the circle with radius \( e^{-c_1 T/2} \) (red circle in Fig. 6) or on the real axis with \( \rho_1 = 1/\rho_2 \) (blue line in Fig. 6) in the complex plane.

If the forcing \( f_2 \) is zero, then the parameter \( A \) in system (19) is zero and, due to the positive damping value \( c_1 \), the trivial solution of system (67) stable. Therefore, the Floquet multipliers are located on the red circle in Fig. 6. If we observe instability of the trivial solution to eq. (67) for some non-zero forcing \( (A \neq 0) \), then the Floquet multipliers must have crossed the unit circle in the complex plane. In this critical case, one of the Floquet multipliers is either one or negative one, which we mark with a black square in Fig. 6.

If one of the multipliers, \( \rho_1 \), is one, there exists a non-trivial \( T/2 \)-periodic solution of the homogeneous part of system (66). In the case of a Floquet multiplier of negative one, a non-trivial \( T \)-periodic solution exists (cf. Farkas [11]). As Farkas details further, in both cases, the adjoint system (67) has a non-trivial \( T/2 \) or \( T \)-periodic solution, which we denote by \( \tilde{y}_1 \). Analyzing equation (68), we conclude that a nontrivial \( \tilde{y} \) implies a non-constant value of both coordinates \( \tilde{y}_1(t) \) and \( \tilde{y}_2(t) \). We choose the forcing
\[
f_1(t) = \begin{cases} 
\tilde{y}_2(t), & \text{if } \int_0^T \tilde{y}_2(t) dt < 0, \\
-\tilde{y}_2(t), & \text{if } \int_0^T \tilde{y}_2(t) dt \geq 0,
\end{cases} \tag{71}
\]
which satisfies the negative mean-forcing requirement (17). Further, note that the forcing (71) is \( T/2 \) periodic for the case \( \rho_1 = 1 \) and \( T \) periodic in the case \( \rho_1 = -1 \). Then the orthogonality condition is
\[
\int_0^T \tilde{y} g(t) dt = \pm \int_0^T \tilde{y}_2^2 dt \neq 0, \tag{72}
\]
where the sign depends on the choice of the forcing (17). Clearly, the orthogonality condition (72) is not satisfied and therefore, by theorem B.1, system (19), has no periodic solution.

Figure 6: Locations of the Floquet multipliers of system (67) in the complex plane. The two critical cases, \( \rho_1 = 1 \) and \( \rho_1 = -1 \), are marked with black squares.
B3 Proof of Fact 4.2

In the following, we show, that no periodic orbit for system (1) with stiffness terms possessing a global extremum, and a mean forcing exceeding this value exists. To prove the nonexistence of a \(T\)-periodic orbit, we proceed as in Appendix B1, assuming the existence of a twice continuous differentiable periodic orbit \(q^*\) for system (1). Integrating equation (1) for one period and imposing periodicity yields

\[
\int_0^T S(q^*(t)) \, dt = T \bar{f}, \quad \iff \int_0^T (S(q^*(t)) - \bar{f}) \, dt = 0.
\]  

(73)

By the mean-value theorem, there must be a time instance \(t^*\) within the period at which the integrand in equation (73) is equal to zero. However, due to the choice of the forcing (23), we obtain

\[
S_j(q^*(t^*)) - \bar{f}_j = \begin{cases} 
S_j(q^*(t^*)) - \bar{f}_j > 0, & S_j(q) > S_{\text{min}}, \\
S_j(q^*(t^*)) - \bar{f}_j < 0, & S_j(q) < S_{\text{max}}, 
\end{cases} \quad 0 \leq t^* \leq T,
\]

(74)

which contradicts (73). Therefore, the periodic orbit cannot exist.

B4 Proof of Fact 4.3

In the following, we prove that if the forcing amplitude \(f\) in the oscillator (25) is above the threshold (26), then no periodic solution to system (25) exists. Again, we assume the existence of a twice continuous differentiable periodic orbit \(q^*\) and split the coordinate \(q^*\) into a constant and a purely oscillatory part, i.e.

\[
\bar{q} := \frac{1}{T} \int_0^T q^*(t) \, dt, \quad \bar{q}(t) = q^* - \bar{q}.
\]

(75)

Substituting the definitions (75) into the equation of motion (25), yields

\[
\ddot{\tilde{q}} + c \dot{\tilde{q}} + \omega_0^2 (\bar{q} + \tilde{q}) + \kappa (\bar{q}^2 + 2\bar{q}\tilde{q} + \tilde{q}^2) = f \cos(\Omega t).
\]

(76)

Integrating equation (76) over one period, we obtain

\[
\int_0^T \tilde{q}^2 \, dt = -T \left( \frac{\omega_0^2}{\kappa} \bar{q} + \tilde{q}^2 \right) \leq T \frac{\omega_0^4}{4\kappa^2},
\]

(77)

where we have used that \(\tilde{q}\) has zero mean (cf. definition (75)). Furthermore, we note that the left-hand side of (77) is positive. Since the right-hand side of equation (77) is a parabola which is concave downwards, it is positive on a closed interval. We thus obtain the upper bound on \(\bar{q}\) in the form

\[
|\bar{q}| < \frac{\omega_0^2}{|\kappa|},
\]

(78)

which is independent of the sign of \(\kappa\). Since \(q^*\) is twice continuously differentiable, it can be expressed in a convergent Fourier series. We denote the Fourier coefficients of \(\bar{q}\) by

\[
\tilde{q}_k := \frac{1}{T} \int_0^T q(t) e^{-i\kappa t} \, dt, \quad k \in \mathbb{Z}.
\]

(79)

Using Parseval’s identity and equation (77), we obtain an upper bound on the Fourier coefficients of the assumed periodic orbit as follows

\[
|\tilde{q}_k| \leq \left( \sum_{k \in \mathbb{Z}} |\tilde{q}_k|^2 \right)^{1/2} = \left( \frac{1}{T} \int_0^T \tilde{q}^2 \, dt \right)^{1/2} \leq \frac{\omega_0^2}{2|\kappa|}, \quad k \in \mathbb{Z}.
\]

(80)

Multiplying equation (76) with \(e^{-i\Omega t}\) and integrating over one period yields

\[
\int_0^T (\ddot{\tilde{q}} + c \dot{\tilde{q}} + \omega_0^2 \tilde{q} + 2\kappa \bar{q}\tilde{q}) e^{-i\Omega t} \, dt + \int_0^T \kappa \bar{q}^2 e^{-i\Omega t} \, dt = \frac{f}{2}.
\]

(81)
From equation (81), we obtain

\[
\left| f \right| \leq \int_0^T (\ddot{q} + c\dot{q} + \omega_0^2 q + 2\kappa \dot{q}\ddot{q}) e^{-\alpha t} \, dt + \left| \kappa \right| \int_0^T \left| \ddot{q}^2(t) \right| e^{-\alpha t} \, dt
\]

\[
\leq \left| (-\Omega^2 + ic\Omega + \omega^2 + 2\kappa q) \dot{q}^2 \right| + \left| \kappa \right| \frac{\omega^4}{4\kappa^2} \leq \left| (-\Omega^2 + ic\Omega + \omega^2 + 2\omega_0^2) \right| + \left| \kappa \right| \frac{\omega^4}{4\kappa^2},
\]

where we have used the upper bounds (78) and (80). Equation (82) gives an upper bound for the forcing amplitude \( f \) of the oscillator (25). For forcing amplitudes exceeding this threshold, we obtain a contradiction and therefore no periodic orbit can exist for the oscillator (25).

### B5 Proof of Fact 4.4

We show that the chain system (28) with the parameters (29) satisfies the conditions of Theorem 3.1 and hence a steady-state response exists. First, we show that the conditions [C2] and [C3*] on the stiffness terms are satisfied for the set of parameters (29). The definiteness of the damping matrix (i.e., condition [C1]) can be shown in a fashion similar to the definiteness of the Hessian.

As for condition [C2] the spring forces of system (28) can be derived from the potential

\[
V(q) = \int_0^{q_1} S_1(-p)dp + \sum_{j=2}^{N} \int_0^{q_{j-1}-q_j} S_j(p)dp + \int_0^{q_N} S_{N+1}(p)dp.
\]

(83)

Since the spring forces in of system (28) are continuous by assumption, the integrals in equation (83) exist. Introducing the notation

\[
S_{j,i} := \frac{\partial S_j(q_{j-1}-q_j)}{\partial q_i},
\]

(84)

the Hessian of the potential is given by

\[
H := \frac{\partial^2 V(q)}{\partial q^2} = \begin{bmatrix}
-S_{1,1} + S_{2,1} & S_{2,2} & 0 & \cdots & 0 \\
-S_{1,1} & -S_{2,2} + S_{3,2} & S_{3,3} & \cdots & 0 \\
0 & -S_{3,3} + S_{4,3} & S_{4,4} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -S_{N-1,N-2} & S_{N,N-1} \\
0 & \cdots & 0 & -S_{N-1,N-1} & S_{N,N} \\
-S_{N,N-1} & \cdots & S_{N,N} & S_{N,N} & S_{N,N+1,N}
\end{bmatrix}
\]

(85)

Due to the choice of parameters (29), we have following identities

\[
S_{j,j} < 0, \quad S_{j+1,j} > 0, \quad S_{j,j} = -S_{j,j-1},
\]

(86)

which implies that the main diagonal entries of the Hessian (85) are positive and the off-diagonal elements negative. We define the matrices

\[
H^j := \begin{bmatrix}
-S_{1,1} & S_{2,2} & 0 & \cdots & 0 \\
S_{2,2} & -S_{2,2} - S_{3,2} & S_{3,3} & \cdots & 0 \\
0 & S_{3,3} - S_{4,4} & S_{4,4} & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & S_{j,j} & -S_{j,j} \\
0 & \cdots & 0 & -S_{j,j} & S_{j,j}
\end{bmatrix} \in \mathbb{R}^{j \times j},
\]

(87)

which are equivalent to the leading minors of the Hessian, except for the last term in the main diagonal where the term \(-S_{j+1,j}\) is missing. Therefore, \(H^N\) is not equal to \(H\). The matrices \(H^j\) can be constructed recursively as follows

\[
H^1 = -S_{1,1}, \quad H^{j+1} = H^j \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{j,j} & -S_{j,j} \\ 0 & 0 & S_{j,j} \end{bmatrix}.
\]

(88)
We show that the matrices $H^j$ are positive definite by induction. As a first step, we note that $H^1$ is positive definite. Performing the induction step, we have
\[
x^T H^j x = x^T \begin{bmatrix} H^{j-1} & 0 \\ 0 & 0 \end{bmatrix} x + x^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S_{j,j} & S_{j,j} \\ 0 & S_{j,j} & -S_{j,j} \end{bmatrix} x.
\] (89)

Since the matrix $H^{j-1}$ is positive definite, the first summand in (89) is always positive unless $x$ aligns with the $x_j$-axis, i.e. $x_1 = x_2 = \ldots = x_{j-1} = 0$. Along this axis the first quadratic form is zero, the second quadratic form, however, yields $-S_{j,j}x_j^2$ which is positive. For the case $x_j = 0$ and $|\tilde{x}| = |[x_1, \ldots, x_{j-1}]^T| > 0$, we obtain
\[
x^T H^{j-1} \tilde{x} + x^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & -S_{j,j} & S_{j,j} \\ 0 & S_{j,j} & -S_{j,j} \end{bmatrix} \tilde{x} \geq x^T H^{j-1} \tilde{x}, \quad |\tilde{x}| > 0,
\] (90)

where we have used the fact, that the matrix in the second quadratic form in equation (89) is positive semi definite. Merging both cases
\[
x^T H^j x \geq \begin{cases} -S_{j,j}x_j^2 > 0, & |\tilde{x}| = 0, \quad |x_j| > 0, \\
x^T H^{j-1} \tilde{x} > 0, \quad |\tilde{x}| > 0, \quad x_j = 0, \end{cases}
\] (91)

which implies positive definiteness of all matrices $H^j$. Since the Hessian can be written as the sum of the positive definite matrix $H^N$ and a positive semidefinite the matrix, i.e.
\[
H = H^N + \begin{bmatrix} 0 & 0 \\ 0 & S_{N+1,N} \end{bmatrix},
\] (92)

we conclude that the Hessian is positive definite.

Since the damping matrix is in the form of the Hessian, the positive definiteness proof applies for the damping matrix as well. Therefore, we have verified the remaining condition (C1) of Theorem 3.1, and the existence of a periodic orbit is guaranteed by Theorem 3.1.

We note that in the case of $S_{N+1,N} = 0$, the Hessian $H$ coincides with the matrix $H^N$, which is positive definite. Therefore, the assumptions on the parameters can be relaxed to include this case.