A complementary relation between classical bits and randomness in local part in the simulating singlet state

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Abstract
Recently, simulating the statistics of the singlet state with non-quantum resources has generated much interest. The singlet state statistics can be simulated by 1 bit of classical communication without using any further non-local correlation. But, interestingly, the singlet state statistics can also be simulated with no classical cost if a non-local box is used. In the first case, the output is completely deterministic whereas in the second case, outputs are completely random. We suggest a (possibly) signaling correlation resource which successfully simulates the singlet statistics, and subsequently leads to a complementary relation between the required classical bits and randomness in the local output involved in the simulation. Our result reproduces the above two models of simulation as extreme cases. We also discuss some important features in Leggett’s non-local model and the model presented by Groblacher et al.

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1. Introduction

The violation of Bell’s inequality \cite{1} by the quantum statistics generated from the singlet state implies the impossibility of reproducing all quantum results by local hidden variable theory. Then Leggett proposed a non-local hidden variable model with some constraint on the local statistics and showed that this model is incompatible with quantum mechanics \cite{2,3}. The result was further generalized by Branciard et al \cite{4}. All these results have generated a new interest in simulating the singlet statistics by some non-local correlation. In this context, it should be mentioned that if 1 cbit of communication is allowed, the singlet statistics can be...
simulated \[5\]. After this work, quite interestingly, the singlet statistics was simulated without communication by using the Popescu–Rorlich (PR) box \[6\]. Recently Colbeck and Renner \[7\] proved a general result by showing that no non-signaling non-local model can generate the statistics of the singlet state if the model has a non-trivial local part and this result is deeply related to the simulation problem. This result was further supported by the work of Branciard \textit{et al} \[4\].

Here, in this work, we suggest a general (possibly) signaling correlation which can be seen as a convex combination of a correlation with the communication capacity of 1 bit and a PR box. We show that with this type of signaling correlation, singlet statistics can be generated. Our result suggests a complementary relation between the amount of classical communication required and randomness in the local binary output in the task of the simulating singlet correlation with classical communication which is limited to 1 cbit.

2. Correlation embracing classical communication

In order to produce our result, we consider the following correlation (hereafter designated by \(S^p\)), with binary input \(x, y \in \{0, 1\}\) and binary output \(a, b \in \{0, 1\}\):

\[
P(ab|x y) = (x y \oplus \delta_{ab})(a \oplus 1)p + a(1 - p).
\]  

(1)

where \(P(ab|x y)\) is the probability of obtaining outputs \(a\) and \(b\) for corresponding inputs \(x\) and \(y\), and \(1/2 \leq p \leq 1\). Here and from now on, \(\oplus\) represents addition modulo 2 and \(\delta_{ab} = a \oplus b \oplus 1\).

Interestingly for \(p \neq \frac{1}{2}\), the \(S^p\) correlation violates the no-signaling condition. In particular, for \(p = 1\), this correlation designated by \(S^1\) can be used to communicate 1 cbit from Alice to Bob; one can easily verify that for this correlation: \(P(00|01) = 1\) and \(P(01|11) = 1\), so if Bob chooses the setting \(y = 1\) then Alice can signal to him via her choice of setting \(x\).

For \(p = \frac{1}{2}\), equation (1) represents a PR correlation designated as \(P^{NL}\). Then it can easily be shown that

\[
S^p = (2p - 1)S^{1\text{cbit}} + 2(1 - p)P^{NL}.
\]  

(2)

3. Simulation of singlet by the correlation \(S^p\)

The protocol for simulating the singlet state statistics by the correlation \(S^p\) is the same as given in [6]. For completeness, we briefly describe the protocol. Alice and Bob share the correlation \(S^p\) along with shared randomness in the forms of pairs of normalized vectors \(\hat{\lambda}_1\) and \(\hat{\lambda}_2\), randomly and independently distributed over the Poincaré sphere. When simulating the singlet, in one turn Alice and Bob are asked to provide the result of measurements along (unit) vectors \(\hat{A}\) and \(\hat{B}\), respectively. The protocol runs as follows. Alice calculates the following quantity:

\[
x = \text{sgn}(\hat{A} \cdot \hat{\lambda}_1) \oplus \text{sgn}(\hat{A} \cdot \hat{\lambda}_2),
\]  

(3)

and inserts it as an input to the machine (\(S^p\) correlation), where

\[
\text{sgn}(z) = \begin{cases} 
1 & \text{if } z \geq 0; \\
0 & \text{if } z < 0.
\end{cases}
\]

As a result of measurement, Alice provides the following quantity:

\[
v(\hat{A}) = a \oplus \text{sgn}(\hat{A} \cdot \hat{\lambda}_1),
\]  

(4)

with \(a\) being the output from the machine. Bob calculates the following quantity:

\[
y = \text{sgn}(\hat{B} \cdot \hat{\lambda}_+) \oplus \text{sgn}(\hat{B} \cdot \hat{\lambda}_-),
\]  

(5)
where \( \vec{\lambda}_\pm = \hat{\lambda}_1 \pm \hat{\lambda}_2 \), and inserts this as an input to the machine. After receiving the bit \( b \) from the machine, he provides the following as the measurement result:

\[
v(\hat{B}) = b \oplus \text{sgn}(\hat{B} \cdot \vec{\lambda}_\pm) \oplus 1.
\]

Equation (6)

Armed with the correlation \( S^p \), one can easily apply the same strategy for simulating the singlet correlation

\[
E[v(\hat{A}) \oplus v(\hat{B})|\hat{A}, \hat{B}] = \frac{1 + \hat{A} \cdot \hat{B}}{2}
\]

in the same line as in [6]. To see how it works one should observe that, from

\[
v(\hat{A}) \oplus v(\hat{B}) = a \oplus b \oplus \text{sgn}(\hat{A} \cdot \hat{\lambda}_1) \oplus \text{sgn}(\hat{B} \cdot \vec{\lambda}_\pm) \oplus 1,
\]

using the correlation \( S^p \) we obtain

\[
v(\hat{A}) \oplus v(\hat{B}) = [(2p - 1)xy + 2(1 - p)xy] \oplus \text{sgn}(\hat{A} \cdot \hat{\lambda}_1) \oplus \text{sgn}(\hat{B} \cdot \vec{\lambda}_\pm) \oplus 1
\]

\[
= xy \oplus \text{sgn}(\hat{A} \cdot \hat{\lambda}_1) \oplus \text{sgn}(\hat{B} \cdot \vec{\lambda}_\pm) \oplus 1,
\]

which is identical to equation (10) in [6] and the result immediately follows.

4. A complementary relation

The \( S^p \) correlation used in our model for simulating the singlet introduces a biasness in the local output \( R(p) \) which is quantified by the Shannon entropy of the outputs for a given input,

\[
R(p) = H(p) = -p \log p - (1 - p) \log (1 - p).
\]

Equation (10)

On the other hand, the amount of bits \( C(p) \) that can be communicated from Alice to Bob using a correlation with binary inputs and binary outputs is quantified by the maximal mutual information between Alice’s inputs and Bob’s outputs for a single choice of Bob’s input. Therefore,

\[
C(p) = \max_{p_x} I(x : b) = 1 - H(p),
\]

Equation (11)

where \( I(x : b) = H(x) + H(b|y = 1) - H(xb|y = 1) \) and \( p_x \) is the probability of Alice’s input \( x \in \{0, 1\} \). For Bob’s input 0, the corresponding mutual information vanishes. Hence we see that in simulating the singlet statistics, as the communication capacity of the correlation resource increases, the randomness of the local output decreases and vice versa. The complementary relation for this model of simulation where the classical communication is limited by 1 cbit can be expressed by

Randomness in local output + Communication capacity of the resource in use = \( R(p) + C(p) = 1 \).

Equation (12)

Obviously one extreme point \( (p = 1) \) generates the Toner–Bacon model [5] and the other extreme point \( (p = \frac{1}{2}) \) generates the model presented by Cerf et al [6]. One should also note that the protocol for the simulation of the singlet by the \( S^p \) correlation does not depend on the value of \( p \). Hence for the simulation, in general one can also use randomly chosen \( S^p \) boxes from an ensemble \( \{ S^p : \frac{1}{2} \leq p \leq 1 \} \), where boxes with arbitrary \( p \) labels appear according to some probability distribution \( \rho(p) \). In this general picture, the complementary relation of the form (13) still holds. Here the average randomness and average communication are given by

\[
\bar{R} = \int H(p) \rho(p) \, dp \quad \text{and} \quad \bar{C} = \int C(p) \rho(p) \, dp,
\]

respectively. Then, using relation (13) we obtain

\[
\bar{R} + \bar{C} = \int \{ H(p) + C(p) \} \rho(p) \, dp = 1.
\]

Equation (13)
One must observe that in this model of simulation, the biasness of the measurement results for any given observable depends only on the parameter of the non-local resources. If this is not the case, i.e. if in some model, biasness is a function of the direction of the observable, then \( R(p) \) can be taken as an average of the Shannon entropy of the measurement results over all possible measurements. Then the local randomness is given by \( R = \langle H(a_i) \rangle \), \( H(a_i) \) being the Shannon entropy for the outcome for the measurement along the direction \( a_i \) on either Alice’s side or Bob’s side (in all the models considered, \( R \) has been taken to be the same on both sides).

Now in the context of following results: (i) simulating the singlet state without communication requires complete randomness for local outcomes [4, 7], (ii) the absence of any less than 1 cbit protocol for simulating the singlet state non-asymptotically using classical communications as the only non-local resource, and (iii) the complementary relation obtained in this work; we conjecture that if there is a model for simulating the statistics of the singlet state with the help of classical communication of \( C \) bit on average, then the complementary relation \( R + C \geq 1 \) holds as a necessary condition.

5. Implications for Leggett’s model

Next, we apply our result to Leggett’s model [2, 3]. In Leggett’s non-local hidden variable model, the local statistics for a given value of hidden variable has been considered to be the same as generated by some completely polarized state and it has been shown that this model does not reproduce the singlet statistics. This result has been generalized in [4] where the local statistics could also be generated by some mixed polarized state and this also does not work for singlet simulation. In both models, the local randomness is not uniform and \( R \) has to be calculated by taking average of the Shannon entropy of outcomes over all possible measurements performed on a pure polarized state or a mixed polarized state on either side. For a general mixed state \( \rho = \frac{1}{2} [I + \mu \hat{n} \cdot \hat{\sigma}] \) with \( 0 \leq \mu \leq 1 \), the average entropy of the output \( R \) over all possible polarization measurement is obtained as

\[
R = \langle H(a_i) \rangle = 1 - \frac{(1 + \mu)^2 \ln (1 + \mu) - (1 - \mu)^2 \ln (1 - \mu) - 2\mu}{4\mu \ln 2}.
\]  

(14)

From the above expression, one can easily check that for \( \mu \neq 0 \), \( R < 1 \) and complementary relation (13) tells why Leggett’s model [2, 3] should fail to reproduce the statistics of the singlet state.

Still, one may question why there is a successful (non-signaling) non-local model which reproduces the singlet statistics for restricted choice of observable [8]. This is possible because, for a given pure polarized state, one can always choose the measurements in a plane which is orthogonal to the direction of polarization, and in that case \( R = 1 \). Using our model of simulation of the singlet, we can extend this result for the choice of observables \( \hat{a} \) and \( \hat{b} \) restricted on two cones \( \hat{a} \cdot \hat{u} = \cos \theta \) and \( \hat{b} \cdot \hat{v} = \cos \theta \) respectively where \( \hat{u} \) and \( \hat{v} \) represent the directions of the polarization of local states in Leggett’s model. But then the model would necessarily require \( (1 - H(\cos^2 \frac{\theta}{2})) \) bits of classical communication. With the average local randomness \( R \neq 1 \) and further satisfying Malu’s law [2, 3] for arbitrary choices of observable for given polarized states on both sides, whether the singlet statistics can be simulated with the assistance of \( (1 - R) \) classical bit or even with finite amount of bits remains open. We think that this is a qualitatively severe constraint (Malu’s law) on the local statistics and even communication of finite amount of bits may not work.
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Note added. After we finished this work, we saw a similar conjecture proposed by Hall [9].

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