Three-Dimensional Vertex Model in Statistical Mechanics, from Baxter-Bazhanov Model

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Abstract

We find that the Boltzmann weight of the three-dimensional Baxter-Bazhanov model is dependent on four spin variables which are the linear combinations of the spins on the corner sites of the cube and the Wu-Kadanoff duality between the cube and vertex type tetrahedron equations is obtained explicitly for the Baxter-Bazhanov model. Then a three-dimensional vertex model is obtained by considering the symmetry property of the weight function, which is corresponding to the three-dimensional Baxter-Bazhanov model. The vertex type weight function is parametrized as the dihedral angles between the rapidity planes connected with the cube. And we write down the symmetry relations of the weight functions under the actions of the symmetry group $G$ of the cube. The six angles with a constrained condition, appeared in the tetrahedron equation, can be regarded as the six spectrums connected with the six spaces in which the vertex type tetrahedron equation is defined.

Keywords: Three-dimensional Baxter-Bazhanov model; Interaction-round-cube (IRC) model; Wu-Kadanoff duality; Cube type tetrahedron equation; Vertex type tetrahedron equation; Vertex type model; Three-dimensional lattice model; Symmetry property; Three-dimensional vertex model; Spherical trigonometry; Vertex type weight function; Three-dimensional star-star relation; Additional constraints; Dihedral angles; symmetry group.

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1 Introduction

Recently big progress has been made in three-dimensional integrable models in statistical mechanics. Bazhanov and Baxter [1] introduced the Interaction-Round-a-Cube (IRC) model which is the generalization of $N = 2$ Zamolodchikov model [2]. Kashaev et al [3] showed that the Boltzmann weights of the Baxter-Bazhanov model satisfy the cube type tetrahedron equation by introducing the star-square relation for which the connection is found [4] with the choral Potts model. The restricted star-triangle relation and the star-star relation of this model have been discussed in detail in Refs [5, 6, 7, 8]. And they connected with the quantum dilogarithm [4] and the shift operator in discrete space-time picture [10, 11]. Then the new series of the three-dimensional integrable lattice models were presented by Mangazeev et al [12] of which the weight functions satisfy modified tetrahedron equation [13]. Recently, Cerchiai et al studied the Baxter-Bazhanov model from the point of link theory and given the representations of the braid group if some suitable spectral limits are taken.

In red. [14] Korepanov got the solution of vertex tetrahedron equation with the spin variables taking $N = 2$ values, which leads to a commuting family of transfer-matrices. From the respect of the scattering process Hietarinta discussed the three corresponding tetrahedron equations in which the Frenkel-Moore equation was fitted
and proposed another vertex solution with 16 nonzero weights \[17\]. And the discrete symmetry groups of vertex models were studied by Boukraa et al. As a generalization of Hietarinta’s solution of tetrahedron equation Mangazeev et al \[18\] proposed another \(N\)-state spin integrable model on a three-dimensional lattice and this model can be reformulated as a vertex model. Now we know that the weight function of this model can be obtained from Baxter-Bazhanov model by taking some limits \[19\]. It is naturally to ask if a three-dimensional vertex model exists corresponding to three-dimensional Baxter-Bazhanov model. One of the aims of this paper is to give a positive answer. And we showed that the weight function of the Baxter-Bazhanov model is dependent on four spin variables which are the linear combinations of the spins located on the corner sites of the elementary cube. The duality between cube and vertex type weight functions is obtained explicitly. The spectrums have the symmetrical property in respect of vertex forms.

This paper is organized as the follows. In section 2 we give a brief describe of the Baxter-Bazhanov model and the duality between the cube and the vertex type tetrahedron equations. The weight functions of the Baxter-Bazhanov model are written as the vertex forms in section 3. Then the duality is obtained explicitly for the Baxter-Bazhanov model. By using the symmetry properties of the weight functions we get the vertex type weight functions for the three-dimensional vertex model. It should be noted that the weight function of model proposed by Mangazeev
et al can be obtained from this vertex type weight function when we take the limit of the spectrum and use the star-triangle relation of the Baxter-Bazhanov model. In section 4 the vertex type weight functions are parametrized as the angles of the spherical triangles by using the methods of the spherical trigonometry parametrization. These angles are the dihedral angles between the “rapidity planes” passing the cubes similarly as in the Zamolodchikov model. In this way, the spectrums appeared in the vertex type tetrahedron equation can be denoted by these angles and they connect with the spaces in which the vertex type tetrahedron equation is defined. In section 5 we discuss the constrained conditions imposed on the tetrahedron equations from the point of the angle variables. Then the symmetry properties of the vertex type weight functions are discussed. They are symmetrical about the transformations of the group $G$ consisting of various rotations, reflections and their combinations of the cube. Finally, some conclusions and remarks are given.

2 Baxter-Bazhanov Model and Duality between Cube and Vertex Type Tetrahedron Equation

2.1 Three-Dimensional Baxter-Bazhanov Model

As is well known, the Baxter-Bazhanov model is an Interaction-Round-a-Cube (IRC) Model. The partition function of it reads

$$Z = \sum_{\text{spins cubes}} \prod W(a|efg|bcd|h)$$

(1)
where $W(a|efg|bcd|h)$ is the Boltzmann weight of the spin configuration $a, \cdots, h$ (see Fig. 1.) and these spin variables take their values in $Z_N$ with $N \geq 2$. The product is over all elementary cubes in the simple cubic lattice $\mathcal{L}$. The Boltzmann weight $W(a|efg|bcd|h)$ can be written as

$$W(a|efg|bcd|h)$$

$$= \frac{w(v_4/v_3, e - c - d + h)s(g, a - g - f + b)}{w(v_4/v_3, a - g - f + b)s(c - h, h - d)}$$

$$\times \left\{ \sum_{\sigma=0}^{N-1} w(v_2, b - f + \sigma)w(v_3, d - h - \sigma)s(\sigma, a)s(\sigma, h) \right\}_0,$$  

with the relation $\omega_{v_1v_4} = v_2v_3$, where the subscript "0" after the curly brackets indicates that the expression in the braces is divided by itself with the zero exterior spins and we have used the notations

$$\frac{w(v, a)}{w(v, 0)} = [\Delta(v)]^a \prod_{j=1}^a (1 - \omega^j v)^{-1}, \quad \Delta(v) = (1 - v^N)^{1/N}$$

$$\omega = \exp(2\pi i/N), \quad \omega^{1/2} = \exp(\pi i/N), \quad s(a, b) = \omega^{ab}$$

Note that the Boltzmann weight function describes a very special type of the interaction of eight spins around the cube as in Fig.1. There are three-spin interactions on the triangles $(a, g, \sigma), (b, f, \sigma), (d, h, \sigma), (c, e, \sigma)$, described by $w(v, a)$ or by $1/w(v, a)$, and two-spin interactions $s(\sigma, a), s(\sigma, c), s(\sigma, h), s(\sigma, f)$ associated with the edges linking $\sigma$ to $a, c, f, h$ in the curly brackets. The factors before the curly brackets denote the spins interactions in the planes $(a, f, b, g)$ and $(c, e, d, h)$. After
introducing an overall normalization and some additional multipliers, the weight function of the Baxter-Bazhanov model is expressed as

$$W_P(a|efg|bcd|h)$$

$$= \frac{\omega^{fb}}{\omega^{ag}} \left[ \frac{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24}|a + d, e + f)}{w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24}|g + h, c + b)} \right]^{1/2}$$

$$\times \left[ \frac{w(x_4, x_{34}, x_3|e + h, d + c)}{w(x_4, x_{34}, x_3|a + b, f + g)} \right]^{1/2}$$

$$\times \left[ \frac{w(x_2, x_{12}, x_1|e + g, a + c)}{w(x_2, x_{12}, x_1|d + b, f + h)} \right]^{1/2} \frac{\omega^{(ag+gb+bh)/2}}{\omega^{(hd+de+ea)/2}}$$

$$\times \left\{ \sum_{\sigma \in \mathbb{Z}} w(x_3, x_{13}, x_1|d, h + \sigma)w(x_4, x_{24}, x_2|a, g + \sigma)w(x_3/\omega, x_{23}, x_2|f, b + \sigma) \right\}_0$$

This satisfies the tetrahedron equation which ensures the commutativity of the layer-to-layer transfer matrices. Here we have used the function

$$w(x, y, z|k, l) = w(x, y, z|k - l)\Phi(l), \quad w(x, y, z|l) = \prod_{j=1}^{i} \frac{y}{z - x\omega^j}, \quad k, l \in \mathbb{Z}_N$$

with the notation

$$x^N + y^N = z^N, \quad \Phi(l) = \omega^{l(l+N)/2}, \quad x_i^N - x_j^N = x_{ij}^N,$$

for $i < j$ and $i, j = 1, 2, 3, 4$. 

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2.2 Duality between Cube and Vertex Type Tetrahedron Equation

The Boltzmann weight function \( W \) in Eq. (3) satisfies the following tetrahedron equation [3]

\[
\sum_d W(a_4|c_2c_1c_3|b_1b_3b_2|d)W'(c_1|b_2a_3b_1|c_4dc_6|b_4)
\times W''(b_1|dcd_3|a_2b_3b_1|c_5)W'''(d|b_2b_1b_3|c_5c_2c_6|a_1)
= \sum_d W'''(b_1|c_1c_4c_3|a_2a_4a_3|d)W''(c_1|b_2a_3a_4|dcd_6|a_1)
\times W'(a_4|c_2d_3|a_2b_3a_1|c_3)W(d|a_1a_3a_2|c_4c_5c_6|b_4)
\]

where \( W, W', W'' \) and \( W''' \) are some four sets of Boltzmann weights. From the respect of the scattering process and using the particle labelling schemes, J. Hietarinta [17] written down the vertex type tetrahedron equation [14, 18]

\[
\sum_{k_1,k_2,k_3,k_4,k_5,k_6} R_{i_1i_2i_3}^{k_1,k_2,k_3} R_{i_4i_5i_6}^{k_4,k_5,k_6} R_{j_1j_2j_3}^{m_{i_1i_2i_3}m_{i_4i_5i_6}m_{j_1j_2j_3}} = \sum_{k_1,k_2,k_3,k_4,k_5,k_6} R_{i_1i_2i_3}^{m_{i_1i_2i_3}m_{i_4i_5i_6}m_{j_1j_2j_3}} R_{i_4i_5i_6}^{k_4,k_5,k_6} R_{j_1j_2j_3}^{k_1,k_2,k_3} R_{i_1i_2i_3}^{k_1,k_2,k_3}
\]

Then we call the relation (8) as the cube type tetrahedron equation. Just as the Wu-Kadanoff duality in the Yang-Baxter equation, the tetrahedron analogue of the Wu-Kadanoff duality between the above two type tetrahedron equations can be constructed by

\[
W(a|efg|bcd|h) = R^{\alpha d+\beta h+\gamma b+\delta f,\alpha h+\beta c+\gamma g+\delta b,\alpha d+\beta e+\gamma c+\delta h}_{a e+\beta c+\gamma g+\delta a,\alpha d+\beta e+\gamma a+\delta f,\alpha f+\beta a+\gamma g+\delta b}
\]
where the constants $\alpha, \beta, \gamma, \delta$ are the parameters of the map $F_{cv} : W = R \circ F_{cv}$ (see Ref. [17]). There are two non-trivial results about the map $F_{cv}$:

$$R_{ijkl} = 0 \text{ unless } \alpha l + \beta i = \beta m + \alpha j \text{ and } \gamma m + \beta j = \beta n + \gamma k$$  \hspace{1cm} (11)

for the case of $\alpha \gamma = \beta \delta$, and

$$R_{ijkl} = 0 \text{ unless } m = i + k \text{ and } j = l + n$$  \hspace{1cm} (12)

for the case of $\alpha = \gamma = 0$ and $\beta = -\delta = 1$. The solution presented in Ref. [18] corresponds to the latter, which can be obtained from the Boltzmann weight of the Baxter-Bazhanov model by taking some limits [19]. In the following section the map $F_{cv}$ will be obtained for the three-dimensional Baxter-Bazhanov model. Then we get a three-dimensional vertex model which corresponds to the IRC model.

3 Three-Dimensional Vertex Model

3.1 The Vertex Type Boltzmann Weight

From the expressions (2) and (5), we know that the Boltzmann weights of the Baxter-Bazhanov model are dependent on the eight spins located on the corner sites of the elementary cube. That is, the Boltzmann weights map as $W : I^8_c \rightarrow \mathcal{C}$ and the cube type tetrahedron equation is defined on $I^{15}_c$. The relations (11) and (12) mean that two labels of $R$ are determined from the others. What is happened about the
three-dimensional Baxter-Bazhanov model? Now we deal with it. Set

\[ r_1 = d - h, \quad r_2 = a - g, \quad r_3 = e - c, \quad r_4 = f - b, \quad r_5 = g + h - b - c \]  \hspace{1cm} (13)

Using the relation (6) we can change the expression (5) into the form:

\[
W_P(a|e f g|b c d|h) = I_\omega \left[ \frac{w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24}|r_1 + r_2 - r_3 - r_4 + r_5)}{w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24}|r_5)} \right.
\times \frac{w(x_4, x_{34}, x_3|r_3 - r_1)w(x_2, x_{12}, x_1|r_3 - r_2)^{1/2}}{w(x_4, x_{34}, x_3|r_2 - r_4)w(x_2, x_{12}, x_1|r_1 - r_4)} \left. \times \left\{ \sum_{\sigma \in Z_N} \omega^{\sigma r_5} w(x_4, x_{14}, x_1|r_3 + \sigma)w(x_3/\omega, x_{23}, x_2|r_4 + \sigma) \right\}_0 \right]
\]

where

\[
I_\omega = (-)^{r_3}(\omega^{1/2})^3 r^2 + r_3 r_4 - r_1 r_3 - r_2 r_3 - r_3 r_5 - r_4 r_5 \left[ \frac{\Phi(r_1)\Phi(r_2)}{\Phi(r_3)\Phi(r_4)} \right]^{1/2} \hspace{1cm} (15)
\]

\( \Phi(r_i), \ i = 1, 2, 3, 4, \) are given in relation (7). By taking account of the property

\[
w(x, y, z|l)w(z, \omega^{1/2}y, \omega x| - l)\Phi(l) = 1, \quad l \in Z_N \hspace{1cm} (16)
\]

the weight function becomes

\[
W_P(a|e f g|b c d|h) = (-)^{k_2}(\omega^{1/2})^{k_1 k_2 + k_2 k_3 + k_1 k_3} \left[ \frac{w(x_1, \omega^{1/2} x_{12}, \omega x_2|k_1)}{w(x_1, \omega^{1/2} x_{12}, \omega x_2|k_4 - k_3)} \right.
\times \frac{w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24}|k_4 - k_1 - k_2 - k_3)w(x_4, x_{34}, x_3|k_3)}{w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24}|k_2)w(x_4, x_{34}, x_3|k_4 - k_1)} \left. \right]^{1/2}
\times \left\{ \sum_{\sigma \in Z_N} \omega^{\sigma k_2} w(x_3, x_{13}, x_1|\sigma)w(x_4, x_{24}, x_2|k_4 + \sigma) \right\}_0
\]

\hspace{1cm} (17)
where
\[ k_1 = r_4 - r_1, \quad k_2 = -r_5, \quad k_3 = r_3 - r_1, \quad k_4 = r_2 - r_1 \]  

(18)

So the Boltzmann weight of the Baxter-Bazhanov model can be reformulated as

\[
R_{i_1 i_2 i_3}^{j_1 j_2 j_3} = (-)^{j_2} (\omega^{1/2})^{j_1 j_2 + j_2 j_3 + j_1 j_3} \\
\times \left[ \frac{w(x_1, \omega^{1/2} x_{12}, \omega x_{23}, j_1) w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24}) - i_2) w(x_4, x_{34}, x_3 | j_3)}{w(x_1, \omega^{1/2} x_{12}, \omega x_{23} | j_1) w(x_{14} x_{23}, x_{12} x_{34}, x_{13} x_{24}) - j_2) w(x_4, x_{34}, x_3 | j_3)} \right]^{1/2} \\
\times \left\{ \sum_{\sigma \in \mathbb{Z}_N} \frac{\omega^{3/2} w(x_{13}, x_{1} | \sigma) w(x_4, x_{24}, x_2 | i_1 + j_3 + \sigma)}{w(x_4, x_{14}, x_{13} | j_3 + \sigma) w(x_{3}/\omega, x_{23}, x_3 | i_1 + \sigma)} \right\} \right|_0 
\]

(19)

where the spin variables \( i_1, i_2, i_3, j_1, j_2, j_3 \) satisfy the conditions \( i_1 + i_2 = j_1 + j_2, i_2 + i_3 = j_2 + j_3 \) and

\[
i_1 = a + c - e - g, \quad i_2 = e + f - a - d, \quad i_3 = a + b - f - g
\]

\[
j_1 = f + h - b - d, \quad j_2 = b + c - g - h, \quad j_3 = e + h - c - d
\]

(20)

Comparing with the relation (10) we know that the parameters of the map \( F_{cv} \) are \( \alpha = -\beta = \gamma = -\delta = -1 \). The expression (19) can be interpreted as the Boltzmann weight of a three-dimensional vertex model. The Boltzmann weight of the vertex model proposed in Ref. [18] can be obtained when we set \( i_3 = j_3 = 0 \) and use the star-triangle relation of the Baxter-Bazhanov model (see Ref. [19]).
3.2 The Spectral Parameters in Weight Function

From the symmetry properties of the Boltzmann weights of the Baxter-Bazhanov model, we have the relation

\[
\left\{ \sum_{\sigma \in \mathbb{Z}_N} \frac{w(x_3, x_{13}, x_1|\sigma + a)w(x_4, x_{24}, x_2|\sigma + c)s(\sigma, n)}{w(x_4, x_{14}, x_1|\sigma + b)w(x_3/\omega, x_{23}, x_2|\sigma + d)} \right\}_0
\]

\[
= \frac{w(x_4, x_{34}, x_3|c - d)}{w(x_4, x_{34}, x_3|b - a)s(a, n)} \times \left\{ \sum_{\sigma \in \mathbb{Z}_N} \frac{w(x_4x_{13}, x_1x_{34}, x_3x_{14}|\sigma - a + b + n)w(x_23, x_34, x_24|\sigma)s(\sigma, d)}{w(x_{13}, \omega x_{34}, \omega x_{14}|\sigma + n)w(x_4x_{23}, x_2x_{34}, x_3x_{24}|\sigma + c - d)s(\sigma, a)} \right\}_0
\]

(21)

where \(a, b, c, d, \sigma, n \in \mathbb{Z}_N\) (see Refs. [3, 7]). By using the above relation the vertex type weight function (19) can be written as

\[
R^{i_1j_2j_3}_{i_1j_2j_3} = (-)^{j_2}\left(\omega^{1/2}\right)^{j_1j_2+j_2j_3+j_3j_1}
\]

\[
\times \left[ \frac{w(x_1, \omega^{1/2}x_{12}, \omega x_2|j_1)w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24}| - i_2)w(x_4, x_{34}, x_3|i_3)}{w(x_1, \omega^{1/2}x_{12}, \omega x_2|1)w(x_{14}x_{23}, x_{12}x_{34}, x_{13}x_{24}| - j_2)w(x_4, x_{34}, x_3|j_3)} \right]^{1/2}
\]

\[
\times \left\{ \sum_{\sigma \in \mathbb{Z}_N} \frac{w(x_4x_{13}, x_1x_{34}, x_3x_{14}|\sigma + j_2 + j_3)w(x_23, x_34, x_24|\sigma)s(\sigma, j_1)}{w(x_{13}, \omega x_{34}, \omega x_{14}|\sigma + j_2)w(x_4x_{23}, x_2x_{34}, x_3x_{24}|\sigma + i_3)} \right\}_0
\]

(22)

Set

\[
u = \frac{x_1}{\omega x_2}, \quad v = \frac{x_4}{x_3}, \quad z = \frac{z_1}{z_2}, \quad z_1 = \frac{x_{13}}{\omega x_{14}}, \quad z_2 = \frac{x_{23}}{x_{24}}
\]

(23)

The Boltzmann weight of the three-dimensional vertex model showed in Fig.2 has the form:

\[
R(u, z, v)^{j_1j_2j_3}_{i_1i_2i_3} = (-)^{j_2}\left(\omega^{1/2}\right)^{j_1j_2+j_2j_3+j_3j_1}\left[ \frac{w(u, j_1)w(z_2/(\omega z_1), -i_2)w(v, i_3)}{w(u, i_1)w(z_2/(\omega z_1), -j_2)w(v, j_3)} \right]^{1/2}
\]
\[
\sum_{\sigma \in \mathbb{Z}_N} \left\{ \sum_{\omega vz, \sigma+j_2+j_3} w(\omega vz, \sigma)s(\sigma, j_1) \right\}_0
\]

where we have used the notation (3). It satisfies the vertex type tetrahedron equation

\[
\sum_{\{k_i\}, i=1, \cdots, 6} R(u_1, u_2, u_3)^{k_1, k_2, k_3}_{i_1, i_2, i_3} R(u_1, u_4, u_5)^{j_1, k_4, k_5}_{i_1, i_4, i_5} R(u_2, u_5, u_6)^{j_2, k_4, k_5}_{i_2, i_4, i_5} R(u_3, u_1, u_6)^{j_3, j_4, j_5}_{i_3, i_4, i_5} = 0
\]

The other constraints on the spectrums will be discussed in the section 5. We can think of each side of the cube type tetrahedron equation as the partition function of the four skewed cubes joined together with a common interior spin \(d\), which forms a rhombic dodecahedron. In this way, we can express both of the two type tetrahedron equations in Figs. 3, 4, graphically. These figures give also the duality between the cube type and vertex type tetrahedron equations.

### 4 The Spectral Parametrization by Using the Spherical Trigonometry

In this section we parametrize the spectrums of the Boltzmann weights as the dihedral angles between the “rapidity planes” passing the cubes similarly as in the
Zamolodchikov model [4]. Following the methods in Refs. [3, 4], we introduce a large sphere (its radius is much larger than the size of the tetrahedra) with a point near the vertices as the center. Consider four great circles on the sphere corresponding to the four “world planes” (see Ref. [2]). A fragment of the stereo-graphic projection of this sphere is shown in Fig. 5. Note that we use the angles which is different from the ones of Zamolodchikov’s. Define

\[
\begin{align*}
  l_1 &= l_{23}/N, \quad l_2 = l_{13}/N, \quad l_3 = l_{12}/N \\
  l'_1 &= l_{45}/N, \quad l'_2 = l_{15}/N, \quad l'_3 = l_{14}/N \\
  l''_1 &= l_{46}/N, \quad l''_2 = l_{26}/N, \quad l''_3 = l_{24}/N \\
  l'''_1 &= l_{56}/N, \quad l'''_2 = l_{36}/N, \quad l'''_3 = l_{35}/N
\end{align*}
\]

(27)

where \(l_{ij}(i, j = 1, \ldots, 6, i < j)\) denotes the length of the segment between \(i\) and \(j\) along the circle. Then we can write down

\[
\begin{align*}
  x_1 &= c_1/s_1, \quad x_2 = \omega^{-1/2}s_1/c_1 \\
  x_3 &= \exp(-il_2)s_3/c_3, \quad x_4 = \omega^{-1/2}\exp(-il_2)c_3/s_3 \\
  x_{12} &= 1/(c_1s_1), \quad x_{13} = \exp[i(l - l_2)]c_2/(c_3s_1) \\
  x_{14} &= \exp[i(l_3 - l)]s_2/(s_1s_3), \quad x_{23} = \omega^{-1/2}\exp[i(l_1 - l)]s_2/(c_1c_3) \\
  x_{24} &= \exp(-il)c_2/(s_3c_1), \quad x_{34} = \exp(-il_2)/(c_3s_3)
\end{align*}
\]

(28)

And the primes added to the \(x\)’s are correspondent to that of the \(c_i, s_i, l_i, l\) with

\[
\begin{align*}
  l &= (l_{12} + l_{13} + l_{23})/(2N), \quad l' = (l_{14} + l_{15} + l_{45})/(2N) \\
  l'' &= (l_{24} + l_{26} + l_{46})/(2N), \quad l''' = (l_{35} + l_{36} + l_{56})/(2N)
\end{align*}
\]

(29)

\[
\begin{align*}
  c'_1 &= c_1 = [\cos(\theta_1/2)]^{1/N}, \quad s'_1 = s_1 = [\sin(\theta_1/2)]^{1/N} \\
  c'_2 &= c_2 = [\cos(\theta_2/2)]^{1/N}, \quad s'_2 = s_2 = [\sin(\theta_2/2)]^{1/N} \\
  c'_3 &= c_3 = [\cos(\theta_3/2)]^{1/N}, \quad s'_3 = s_3 = [\sin(\theta_3/2)]^{1/N} \\
  c''_1 &= c'_2 = [\cos(\theta_4/2)]^{1/N}, \quad s''_1 = s'_2 = [\sin(\theta_4/2)]^{1/N} \\
  c''_2 &= c'_3 = [\cos(\theta_5/2)]^{1/N}, \quad s''_2 = s'_3 = [\sin(\theta_5/2)]^{1/N} \\
  c''_3 &= c'_3 = [\cos(\theta_6/2)]^{1/N}, \quad s''_3 = s'_3 = [\sin(\theta_6/2)]^{1/N}
\end{align*}
\]

(30)
In this way, we have
\[
   u_i = \omega^{-1/2}[ctg(\theta_i/2)]^{2/N}, \quad i = 1, 2, \cdots, 6 \quad (31)
\]

The vertex type tetrahedron equation has the form (see Figs. 3,4):
\[
   \sum_{\{k_i\}, \ i=1, \cdots, 6} R(\theta_1, \theta_2, \theta_3)_{i_1i_2i_3}^{k_1k_2k_3} R(\theta_4, \theta_5, \theta_6)_{i_4i_5i_6}^{k_4k_5k_6} R(\theta_1, \theta_2, \theta_3)_{i_1i_2i_3}^{j_1j_2j_3} R(\theta_4, \theta_5, \theta_6)_{i_4i_5i_6}^{j_4j_5j_6} = \\
   \sum_{\{k_i\}, \ i=1, \cdots, 6} R(\theta_3, \theta_5, \theta_6)_{i_3i_5i_6}^{k_3k_5k_6} R(\theta_2, \theta_4, \theta_6)_{i_2i_4i_6}^{k_2k_4k_6} R(\theta_1, \theta_2, \theta_3)_{i_1i_2i_3}^{j_1j_2j_3} R(\theta_4, \theta_5, \theta_6)_{i_4i_5i_6}^{j_4j_5j_6} \quad (32)
\]
with the angles satisfying the condition
\[
   \left[ \sin \frac{\theta_1 + \theta_2 + \theta_3}{2} \sin \frac{-\theta_1 + \theta_2 + \theta_3}{2} \sin \frac{-\theta_3 + \theta_5 + \theta_6}{2} \sin \frac{\theta_3 + \theta_5 - \theta_6}{2} \right]^{1/2} \\
   - \left[ \sin \frac{\theta_1 - \theta_2 + \theta_3}{2} \sin \frac{\theta_1 + \theta_2 - \theta_3}{2} \sin \frac{\theta_3 - \theta_5 + \theta_6}{2} \sin \frac{\theta_3 + \theta_5 + \theta_6}{2} \right]^{1/2} \\
   = \sin \theta_3 \left[ \sin \frac{\theta_2 + \theta_4 - \theta_6}{2} \sin \frac{-\theta_2 + \theta_4 + \theta_6}{2} \right]^{1/2} \quad (33)
\]
(see Fig. 5). This relation can be obtained from Eq. (3.2) of Ref. [2] by a proper choice of the angles: \( \theta_2 \rightarrow \theta_3, \ \theta_3 \rightarrow \pi - \theta_2, \ \theta_5 \rightarrow \theta_6, \ \theta_6 \rightarrow \pi - \theta_5 \). And the vertex type weight function has the property
\[
   R(\theta_1, \theta_2, \theta_3)_{i_1i_2i_3}^{j_1j_2j_3} = R(\theta_1, \theta_2, \theta_3)_{i_1i_2i_3}^{j_1j_2j_3} \quad (34)
\]
The properties of the others of the weight functions will be given in the following section. These angles \( \theta_1, \theta_2, \cdots, \theta_6 \) can be interpreted as the parameters related to the six spaces in which the vertex tetrahedron equation is defined.
5 The Symmetry Properties of the Vertex Type Weight Function

In this section we first consider the additional constraints imposed on the tetrahedron equations, given by Kashaev et al, from the point of the above angle variables. Then we find that the Boltzmann weights are symmetrical under the transformations of the group \(G\) consisting of various rotations, reflections and their combinations of the cube in the respect of the vertex type weight functions. It can be checked easily that the angle parametrization satisfies the conditions which ensures all the similarity transformation factors \([6]\) to cancel each other. In terms of the ‘coordinated’ parameters the four additional constraints have the form

\[
\omega \frac{x_{23} x_{24}'}{x_3 x_2} = 1, \quad \frac{x_{13} x_{14}'}{x_1 x_4} = 1
\]

\[
\frac{x_{14} x_{14}'}{x_{14} x_4} = 1, \quad \frac{x_{13} x_{13}'}{x_{13} x_1} = 1
\]

By taking account of the expressions (28-30), the above constraints are changed into the relations

\[
l_{12} + l_{24} = l_{14}, \quad l_{13} + l_{35} = l_{15}
\]

\[
l_{23} + l_{36} = l_{26}, \quad l_{45} + l_{56} = l_{46}
\]

Just as showing in Fig. 5, they hold naturally. From Refs. [5, 7], we know that the three-dimensional star-star relation means the transformation \(\xi\):

\[
W(a|efg|bcd|h) \xrightarrow{\xi} W(f|adb|hge|c)
\]
with

\[
\begin{align*}
\frac{x_3}{x_1} & \xrightarrow{\xi} \frac{x_2}{x_3}, \quad \frac{x_4}{x_2} \xrightarrow{\xi} \frac{x_1}{x_4}, \quad \frac{x_4}{x_1} \xrightarrow{\xi} \frac{x_1}{x_4}, \quad \frac{x_3}{x_2} \xrightarrow{\xi} \frac{x_2}{x_3}, \\
\end{align*}
\]

(38)

In terms of the vertex form the star-star relation can be expressed as

\[
R(\theta_1, \theta_2, \theta_3)_{j_1 j_2 j_3} = R(\pi - \theta_3, \pi - \theta_2, \theta_1)^{-j_3 - j_2 j_1}
\]

(39)

as in Fig. 6. Under the transformations \(\tau\) and \(\rho\) of the generating elements of the group \(G\) the weight functions change as [12]

\[
W(a|efg|bcd|h) \xrightarrow{\tau} W(a|feg|cbd|h) \quad (40)
\]

with

\[
\begin{align*}
\frac{x_2}{x_1} \xleftarrow{\tau} & \frac{x_3}{\omega x_4}, \quad \frac{x_4}{x_1} \xleftarrow{\tau} \frac{x_3}{\omega x_2} \\
\end{align*}
\]

(41)

and

\[
W(a|efg|bcd|h) \xrightarrow{\rho} W(g|cab|fhe|d) \quad (42)
\]

with

\[
\begin{align*}
\frac{x_3}{x_1} \xrightarrow{\rho} & \frac{x_{13} x_4}{x_3 x_{14}}, \quad \frac{x_3}{x_2} \xrightarrow{\rho} \frac{x_{23} x_4}{x_3 x_{24}}, \quad \frac{x_4}{x_2} \xrightarrow{\rho} \frac{x_{23}}{x_24}, \quad \frac{x_{14} x_{23}}{x_{13} x_{24}} \xrightarrow{\rho} \frac{x_2}{x_1} \\
\end{align*}
\]

(43)

So the vertex forms of them are

\[
R(\theta_1, \theta_2, \theta_3)_{j_1 j_2 j_3} = R(\theta_3, \theta_2, \theta_1)^{-j_3 - j_2 j_1}
\]

(44)

for transformation \(\tau\), as in Fig. 7, and

\[
R(\theta_1, \theta_2, \theta_3)_{j_1 j_2 j_3} = R(\pi - \theta_3, \theta_1, \pi - \theta_3)^{-j_3 - j_2 j_1}
\]

(45)
for transformation $\rho$ as in Fig. 8. As is Known, the angles $\theta_1, \theta_2, \cdots, \theta_6$ can be interpreted as the dihedral angles between the rapidity planes connected with the cubes. In respect of the vertex model, these angles can be regarded as the parameters related to the spaces on which the vertex type tetrahedron equation is defined. So these parameters and spin variables should be transformed ‘regularly’ under the symmetry group $G$. This is entirely consistent with the above equations. The geometric considerations of them are shown in Figs 6, 7, 8. The above two relations are the “elementary” relations. The others of the transformations of $G$ can be obtained from them. It can be checked easily that the star-star relation (39) can be obtained from relations (44) and (45).

6 Summary

As the conclusions, we get the duality between the cube type weight functions and the vertex type weight functions explicitly for the three-dimensional Baxter-Bazhanov model and find that the Boltzmann weight of the model depends on the four spin variables which are the linear combinations of the spins located on the corner sites of the cube. We can interpreted the vertex type weight function $R(\theta_1, \theta_2, \theta_3)_{i_1i_2i_3}^{j_1j_2j_3}$ as a Boltzmann weight of a three-dimensional vertex model and the spectrums $\theta_1, \theta_2, \theta_3$ are connected to the “line” 1, 2 and 3 (see Fig. 2). In this way, we write down the symmetrical relations of the vertex type Boltzmann weights.
in terms of the angles. We known that these angles are the dihedral angles between the rapidity planes connected with the cubes. Then the weight functions should be transformed ‘regularly’ under the actions of the symmetry group $G$ which is consisted by the various rotations, reflections and their combinations of the cube. The relations (39), (44) and (45) are entirely consistent with it (see Fig. 6,7,8). And the angles $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ with the relation (33) can be interpreted as the spectrums related to the six spaces in which the vertex tetrahedron equation (32) is defined (see Fig. 3 and Fig. 4). When we set $i_3 \equiv j_3 \equiv 0$ and make the specialization of the spectral parameters the Boltzmann weight of the vertex model proposed in Ref. [18] can be obtained from the weight function (19) with the spin assignments [19]. Here we have 16 nonzero weights $R(\theta_1, \theta_2, \theta_3)^{i_1j_2j_3}_{i_1'j_2'j_3'}$ for $N = 2$. So it is a interesting question to find the connection between it and the solutions in Ref. [14].

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Captions

Fig. 1. Arrangements of the spins $a, \cdots, h$ on the corner sites and the spin $\sigma$ in the center of an elementary cube of the simple cubic lattice $\mathcal{L}$.

Fig. 2. The Boltzmann weight of the three-dimensional vertex model corresponding to the IRC model.

Fig. 3. The graph of the left hand sides of the tetrahedron equations.

Fig. 4. The graph of the right hand sides of the tetrahedron equations.

Fig. 5. A fragment of the stereo-graphic projection of the sphere with four great circles.

Fig. 6. The transformation $\xi$ corresponding to the three-dimensional star-star relation.

Fig. 7 The transformation $\tau$ of the weight function.

Fig. 8 The transformation $\rho$ of the weight function.