Negative modes and decay-rate transition

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Abstract

We investigate a relationship between the number of the negative modes around periodic instanton solution and the type of the decay-rate transition. It is shown that for the case of first-order decay-rate transition the lowest positive mode at low energy periodic instanton becomes additional negative mode at high energy regime, while in the second-order case there is only one negative mode in the full range of energy. This kind of analysis on the negative modes makes it possible to derive the criterion for the first-order transition.
I. INTRODUCTION

Recently, the decay-rate transition have attracted much attention in various fields from condensed matter physics [1] to particle physics [2,3] and cosmology [4]. This implies that it is a general phenomenon which appears frequently in nature.

Since the decay rate of metastable state is expressed as

\[ \Gamma \propto e^{-S[\phi]}, \] (1)

where \( S[\phi] \) is the Euclidean action, the classical solution in Euclidean space makes a dominant contribution to the decay rate. At zero temperature the classical localized solution is a bounce which has infinite Euclidean period [5]. As temperature increases, due to thermal assistance a periodic solution called periodic instanton [6] plays an important role in the decay rate, and then at higher temperature thermal activation becomes dominated, which is described by static sphaleron [7]. The decay-rate transition means the transition from the periodic instanton-dominated to the sphaleron-dominated regimes.

The decay-rate transition was first argued to be smooth in quantum mechanics by Affleck [8]. After then, Chudnovsky [9] has shown that it is possible for the transition to be sharp first-order as well as smooth second-order, and the type of the transition depends on the shape of the potential barrier. He has also shown that the order of the transition is easily conjectured by \( \beta \)-vs-\( E \) plot, where \( \beta \) and \( E \) are Euclidean period and energy, respectively. The sharp first-order transition takes place when \( \beta(E) \) curve possesses a minimum at \( E = E_c \) lower than the energy of sphaleron \( E_s \). Based on Chudnovsky’s observation the sharp first-order transitions are found at spin tunneling systems with [10] and without [11] external magnetic field.

Using Chudnovsky’s idea a sufficient criterion for the first-order transition is obtained by carrying out the nonlinear perturbation near the sphaleron in two-dimensional string model [12]. Inspired by spin-tunneling problem, this criterion is subsequently extended to the quantum mechanical model when mass is position-dependent [13].
Recently, it is conjectured that the number of the negative modes of the fluctuation operator may change at the bifurcation point [3]. This means it is possible to determine the type of the transition by counting the number of the negative mode near sphaleron. The purpose of the present paper is to address this issue by analyzing the properties of the negative mode, especially near bifurcation point, explicitly with a concrete quantum mechanical example and to derive the criterion for the sharp first-order or smooth second-order transition in this context. As will be shown, the criterion derived from this viewpoint exactly coincides with that of Ref. [12], in which the criterion is derived by computing the variation of the period near sphaleron. In Sec.II we examine the conjecture of Ref. [3] by making use of the simple quantum mechanical model. It is shown that the change of the number of negative modes at bifurcation point is really realized in this model. Getting some physical insight from this simple model, we explore the mechanism of this phenomena at Sec.III. We will show that the number of negative modes are conjectured by considering some relations between the classical solutions which have same period and different action values. Based on the anlysis of Secs.II and III, we derive the criterion for the first-order transition in Sec.IV, and in the final section a brief conclusion will be given.

II. NUMBER OF NEGATIVE MODES AND DECAY-RATE TRANSITION: QUANTUM MECHANICAL EXAMPLE

In this section we calculate numerically periodic instantons and eigenvalues of their fluctuation operators for a quantum mechanical potential

\[ V(\phi) = \frac{4 + \alpha}{12} - \frac{1}{2} \phi^2 - \frac{\alpha}{4} \phi^4 + \frac{\alpha + 1}{6} \phi^6 - \frac{\gamma}{3} \phi^3, \]  

(2)

where \( \alpha \) and \( \gamma \) are some real parameters. Using the criterion of Ref. [12], it is easily shown that when \( \gamma = 0 \) the transition from classical to quantum tunneling is first order for positive \( \alpha \), and in the range of \(-1 < \alpha < 0\) the second-order transition takes place. When \( \gamma \neq 0 \), in the range of \( \gamma^2 < \frac{9}{16} \alpha \) the first-order transition occurs, and we find that in the range
of $\gamma^2 > \frac{5}{10}\alpha$ there are two types of second-order transition; one is usual second-order and the other is unusual second-order transition accompanied by a first-order transition within quantum tunneling region.

Since the Euclidean action is

$$S[\phi] = \int_0^\beta d\tau \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right],$$

the equation of motion reads

$$\ddot{\phi} = V'(\phi),$$

where the prime denotes the derivative with respect to $\phi$, which implies energy conservation in Euclidean space

$$\frac{1}{2} \dot{\phi}^2 = V(\phi) - E.$$

From this equation the period is expressed as

$$\beta = 2 \int_{\phi_1(E)}^{\phi_2(E)} \frac{d\phi}{\sqrt{2(V(\phi) - E)}},$$

where $\phi_1(E)$ and $\phi_2(E)$ are turning points.

Fig. 1 shows plots of potential $V(\phi)$, period $\beta(E)$, and action $S(\beta)$ for the three types of decay-rate transition. One can see that the bifurcation points of periodic instantons correspond to the points of $\frac{d\beta}{dE} = 0$. The periodic instantons in the range of $\frac{d^2S}{d\beta^2} > 0$ or , equivalently, $\frac{dE}{d\beta} > 0$, does not contribute to the decay rate since there are other classical solutions with the same period and lower action.

Now consider a small fluctuation from the periodic instanton:

$$\phi(\tau) = \phi_p(\tau) + \eta(\tau).$$

Then, the action of Eq.(3) becomes upto the second order of $\eta(\tau)$

$$S[\phi] = S[\phi_p] + \frac{1}{2} \int_0^\beta d\tau \eta(\tau) \ddot{\eta}(\tau),$$

where $\ddot{\eta}(\tau)$ is the second derivative of $\eta(\tau)$ with respect to $\tau$. The action $S[\phi_p]$ is the action of the periodic instanton. The last term represents the contribution of the fluctuation $\eta(\tau)$ to the action.
where the fluctuation operator is
\[ \hat{M} = -\frac{\partial^2}{\partial \tau^2} + V''(\phi_p). \] (9)

Let us consider the eigenvalue equation of \( \hat{M} \)
\[ \hat{M}x_n = \epsilon_n x_n. \quad (n = -1, 0, 1, 2, 3 \cdots). \] (10)

For convenience we fix the zero mode as \( x_0 \), so that \( \epsilon_0 = 0 \).

The eigenvalues and eigenfunctions of the fluctuation operator \( \hat{M} \) can be numerically calculated by the diagonalization method of the matrix mainly composed of Fourier coefficients of \( V''(\phi_p) \). Several eigenvalues from the lowest one are plotted in Fig. 2 for the three different types of decay-rate transition. It is clear that the zero mode (\( \dot{\phi}_p(\tau) \)) originated from the time translational symmetry always exists and at the bifurcation points another zero mode occurs, which is relevant not to the symmetry of system, but to the symmetry in the vicinity of the periodic instanton at the bifurcation point. The periodic instantons in the range of \( \frac{d^2S}{d\beta^2} > 0 \) have two negative modes, which means that along the directions of the negative modes in function space the periodic instanton gives maximum action value.

In the next section we will explore the mechanism on the change of the negative modes by considering the classical solutions in function space.

III. ANALYSIS OF NEGATIVE MODES IN FUNCTION SPACE

In the double well potential like Eq.(2) the action has a lower bound in the function space with a fixed period \( \beta \). Since the periodic instanton has one or two negative modes, it is not a minimum in the function space but a saddle point having one or two negative mode directions along which the action value decreases approaching other saddle point or some minimum. Fortunately in this quantum mechanical model one can find all possible saddle points and minima at a given period \( \beta \) all of which are solutions of Euler-Lagrange equation. Therefore, it is possible to expect what solution one can meet ultimately if one follows a particular negative mode direction.
The number of negative modes around spaleron depends on the period \( \beta \). This fact is easily shown as follows. The curvature of barrier top has a relation

\[
V''(\phi_s) = 2\pi/\beta_s, \tag{11}
\]

where \( \beta_s \) is the period with which the periodic instanton meets the sphaleron. Then the eigenvalue equation for the fluctuation is

\[
(-\frac{d^2}{d\tau^2} - (\frac{2\pi}{\beta_s})^2)\chi_n(\tau) = \lambda_n\chi_n(\tau) \tag{12}
\]

where

\[
\chi_n(\tau) = \cos(\frac{2\pi n}{\beta} \tau), \sin(\frac{2\pi n}{\beta} \tau) \tag{13}
\]

\[
\lambda_n = (\frac{2\pi n}{\beta})^2 - (\frac{2\pi}{\beta_s})^2, \quad (n = 0, 1, 2, \ldots). \tag{14}
\]

Except for \( n = 0 \) case, the states are two-fold degenerated. The number of negative modes increases with \( \beta \); for example, for \( 0 < \beta < \beta_s \), there is only one negative mode, and for \( \beta_s < \beta < 2\beta_s \) three negative modes and so on. The increase of negative modes will be shown to be natural in the followings.

Let us consider the sphaleron solution \( \phi_s \) with a period \( \beta_1 < \beta_s \) as shown in Fig.1 (c). Note that at this period there is no any other non-trivial solution. Two trivial stable solutions are \( \phi_+ \) and \( \phi_- \) staying at well minima, respectively (see Fig.1 (a)). The sphaleron has one negative mode and infinite number of positive modes which are two-fold degenerated.

In Fig.3 all solutions with period \( \beta_1 \) and several lowest eigenfunctions are described. It is clear that the sphaleron solution can go to \( \phi_+ \) or \( \phi_- \) by adding or subtracting the negative eigenfunction \( x_{-1} \). So, this situation can be described graphically in function space as Fig.4. It is certain that one classical solution can flow along the negative mode directions into the other solutions with lower action or to the vicinity of those.

Now, consider the periodic instanton \( \phi_p \) with a period \( \beta_2 > \beta_s \) in Fig.1(c). Fig.5 shows all possible solutions with period \( \beta_2 \) and several eigenfunctions of the fluctuation operator at the periodic instantion. The periodic instanton has one negative mode \( (x_{-1}) \), one zero mode \( (x_0) \),
and infinite number of positive modes \((x_n, n = 1, 2, 3, \ldots)\). Since the sphaleron solution \(\phi_s\) flows along the negative mode direction \((x_{-1})\) to \(\phi_+\) or \(\phi_-\), this periodic instanton can not reach exactly \(\phi_+\) or \(\phi_-\) along the negative mode direction, but go through the vicinity of those stable trivial solutions. By the zero mode the periodic instanton is shifted without any variation of action value, which is the time translational symmetry. The lowest positive mode \(x_1\) connects \(\phi_p\) with the sphaleron solution and the \(\beta_2/2\)-shifted periodic instanton \(\phi_p^*\). Combining all these facts one can summarize the situation as Fig.6. From this figure it is clear that the sphaleron with period \(\beta_2\) has three negative modes; one is \(x_{-1}\) which connects \(\phi_s\) with \(\phi_+\) and \(\phi_-\) like Fig.4 (a) and the others are \(x_0\) and \(x_1\) which connect with the \(\beta_2/4\)-shifted periodic instanton \(\phi_p^{**}\) and \(\phi_p\), respectively. The sphaleron with the period of \(2\beta_s < \beta < 3\beta_s\) has five negative modes. This is explained by considering the solution composed of two periodic instanton with period \(\beta/2\) as shown in Fig.7. Thus, the additional two negative mode \(x_2\) and \(x_3\) are the direction to the periodic instanton with half period, \(\phi_{p,\beta/2}\) and the \(\beta/4\)-shifted half periodic instanton \(\phi_{p,\beta/2}^{**}\) as shown Fig.7.

In the case of first-order transition the periodic instanton can have two negative modes. Let us consider the periodic instanton \(\phi_{p,u}\) with period \(\beta_3\) (see Fig.1 (f)) which has two negative modes \(x_{-1}\) and \(x_1\), one zero mode \(x_0\), and infinite number of positive modes. In Fig.8 (a) all possible solutions with \(\beta = \beta_s\) are drawn. All non-trivial solutions can be connected along \(x_1\) direction, which is shown in Fig.8 (b). This figure shows that \(\phi_{p,u}\) has negative eigenvalue and \(\phi_s\) and \(\phi_{p,d}\) have positive eigenvalue in \(x_1\) direction. As \(\beta_3\) decreases, the \(\phi_{p,u}\) and \(\phi_{p,d}\) approach each other, and at the bifurcation point they have the same action value and then disappear. When \(\beta_3\) increases through \(\beta_s\), \(\phi_s\) and \(\phi_{p,u}\) are merging each other and then creating a maximum \(\phi_s\).

Finally, consider the case of unusual second-order transition. All possible solutions with period \(\beta_4\) (see Fig.1 (i)) and their positions along \(x_1\) direction in function space are described in Fig.9. In this case the \(\phi_{p,u}\) and \(\phi_{p,d}\) merge each other at bifurcation point \(\beta_{c1}\) and \(\phi_{p,m}\) and \(\phi_{p,u}\) do at bifurcation point \(\beta_{c2}\), and \(\phi_s\) has negative eigenvalues in \(x_0\) and \(x_1\) directions as well as in \(x_{-1}\) direction.
From this analysis we can conclude that additional negative modes require existence of other solutions with lower action value, which is equivalent to the situation that occurrence of new maximum always accompanies new minima.

IV. CRITERION OF FIRST-ORDER TRANSITION

In this section we derive the criterion for the first-order transition from the fact that in the first-order transition the periodic instanton near sphaleron has two negative modes while in the second-order case it can have only one negative mode. This fact can be applied to the field theoretical as well as quantum mechanical models. In this paper we will concentrate on scalar field theories. However, the extensions to the other kinds of field theories such as non-linear $O(3)$ model with [14] and without [15] Skyrme term are straightforward, which will be addressed in the separate publication.

We start with Euclidean action for a scalar field $\phi$

\[
S[\phi] = \int_0^\beta d\tau \int d\vec{x} \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi)\right].
\]  

(15)

Consider a small fluctuation $\eta(\vec{x}, \tau)$ from a periodic instanton $\phi_p(\vec{x}, \tau)$, then the action is

\[
S[\phi] = S[\phi_p] + \frac{1}{2} \int_0^\beta \int d\vec{x} \eta(\vec{x}, \tau) \hat{M} \eta(\vec{x}, \tau),
\]  

(16)

where

\[
\hat{M} = -\frac{\partial^2}{\partial \tau^2} - \nabla^2 + V''(\phi_p).
\]  

(17)

Since the periodic instanton near the sphaleron can be expressed as [12]

\[
\phi_p(\vec{x}, \tau) = \phi_s(\vec{x}) + ag_0(\vec{x}) \cos \omega \tau + a^2 (g_1(\vec{x}) + g_2(\vec{x}) \cos 2\omega \tau),
\]  

(18)

where $a$ is a small amplitude and $g_0(\vec{x}), g_1(\vec{x}),$ and $g_2(\vec{x})$ will be determined shortly, the eigenvalue equation becomes up to the order of $a^2$

\[
(-\hat{l} + \hat{h} + \hat{H'})\eta(\vec{x}, \tau) = \epsilon \eta(\vec{x}, \tau),
\]  

(19)
\[
\hat{l} = \frac{\partial^2}{\partial \tau^2},
\]
\[
\hat{h} = -\nabla^2 + V''(\phi_s),
\]
\[
\hat{H}' = aV'''(\phi_s)g_0(\vec{x}) \cos \omega \tau + a^2[V''(\phi_s)(g_1(\vec{x}) + g_2(\vec{x}) \cos 2\omega \tau)
\]
\[
+ \frac{1}{2} V''''(\phi_s)g_0^2(\vec{x}) \cos^2 \omega \tau].
\]

We treat the operator \((-\hat{l} + \hat{h})\) as an unperturbed part. The eigenspectrum of \(\hat{l}\) is
\[
\hat{l}(|\sin n \omega \tau >, |\cos n \omega \tau >) = -(n\omega)^2(|\sin n \omega \tau >, |\cos n \omega \tau >).
\]

Let us assume the eigenspectrum of \(\hat{h}\) as
\[
\hat{h}|u_n(\vec{x})> = \alpha_n|u_n(\vec{x})>, \ (n = 0, 1, 2, \cdots).
\]

The \(\hat{h}\) operator has only one negative mode \(|u_0(\vec{x})>\). From the fact that \(\dot{\phi}_p\) and \(\partial \phi_p / \partial x_i\) are zero modes due to the time and space translational symmetry, one can determine \(\omega^2 = -\alpha_0\) and
\[
g_0(\vec{x}) = u_0(\vec{x}),
\]
\[
g_1(\vec{x}) = -\frac{1}{4} \hat{h}^{-1} u_0^2(\vec{x}) V'''(\phi_s),
\]
\[
g_2(\vec{x}) = -\frac{1}{4} (\hat{h} + 4\omega^2)^{-1} u_0^2(\vec{x}) V''''(\phi_s).
\]

The zero modes of the unperturbed part or the fluctuation operator at sphaleron are \(|u_0 \sin \omega \tau >, |u_0 \cos \omega \tau >, \) and \(|u_m >, \ (m = 1, 2, \cdots D)\). When the perturbation \(\hat{H}'\) turns on, only the second zero mode \(|u_0 \cos \omega \tau >\) can be shifted from zero while other zero modes does not affected by the perturbation because these zero modes are relevant to system’s symmetries, i.e., the time and space translational symmetries.

Therefore, using the standard perturbation theory, we can determine the number of negative modes in the vicinity of sphaleron. Let \(\epsilon_0\) and \(\epsilon_1\) be the perturbed eigenvalues of \(|u_0 \sin \omega \tau >\) and \(|u_0 \cos \omega \tau >, \) respectively. Then, when \(\epsilon_1 - \epsilon_0 < 0\) the number of negative
mode becomes two, so that the transition is first order. The criterion for the first-order transition is, then, straightforwardly derived as

\[ < u_0|g_2V'' + \frac{1}{4}V'''g_0^2|u_0 > - \frac{1}{2} < u_0|g_0V'''\hat{h}^{-1}g_0V''|u_0 > < 0. \] (24)

One can easily show that Eq.(24) coincides with the criterion for the first-order transition in Ref. [12], which is obtained from a different point of view.

V. CONCLUSION

We investigate a relationship between the number of the negative modes around periodic instanton solution and the type of the decay-rate transition. It is shown explicitly by calculating the several eigenvalues numerically in quantum mechanical model that the number of the negative modes is changed at the bifurcation points. The mechanism of this phenomenon in the various types of phase diagram is discussed at Sec.III by considering the projection of the compact action manifold in function space. This kind of analysis makes it possible to derive the criterion for the sharp first-order or smooth second-order transition from classical to quantum tunneling regimes. Although our derivation of criterion is exactly same with that of Ref. [12], in which the criterion is derived from completely different point of view, the viewpoint used in this paper seems to provide a more profound understanding on the tunneling phenomena. We hope the relation of negative modes of hessian and the type of the transition explored in this paper furnishes some physical insight to understand more complicated tunneling phenomena in electroweak theory and cosmology.
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FIGURES

FIG. 1. Plots of $V(\phi)$, $\beta(E)$, and $S(\beta)$ for various types of transition. (a,b,c) usual second-order transition at $\alpha = -0.9$ and $\gamma = 0$. (d,e,f) first-order transition at $\alpha = 20$ and $\gamma = 0$. (g,h,i) unusual second-order transition at $\alpha = 10$ and $\gamma = 4.7$. In this case $\beta(E)$ and $S(\beta)$ are calculated using a potential $V(\phi) - V(\phi_-)$.

FIG. 2. Plots of several lowest eigenvalues of $\hat{M}$. (a) Usual second-order transition case. There is only one negative mode in the full range of energy. (b) First-order transition case. One can see that $\epsilon_1$ becomes additional negative mode when $E$ is larger than $E_c$ which corresponds to the bifurcation point. (c) Unusual second-order transition case. One can see that $\epsilon_1$ becomes additional negative mode at $E_{c1} < E < E_{c2}$. $E_{c1}$ and $E_{c2}$ correspond to two different bifurcation points.

FIG. 3. (a) Plots of three trivial solutions sitting at barrier top($\phi_s$) and bottoms($\phi_+,-\phi_-$). (b) Plots of eigenfunctions $x_n (n = -1,0,1,2,3)$ around sphaleron solution when $\beta = \beta_1$ which is smaller than $\beta_s$.

FIG. 4. Schematic description of the action value along the (a) negative and (b) positive mode directions in function space with $\beta = \beta_1$.

FIG. 5. (a) Plots of all possible solutions whose period is $\beta_2 > \beta_s$. $\phi^*_p$ is obtained from $\phi_p$ by changing $\tau \rightarrow \tau - \beta_2/2$. (b) Plots of several eigenfunctions around $\phi_p$.

FIG. 6. Graphical description of action manifold projected on $(x_0,x_1)$ plane when $\beta = \beta_2 > \beta_s$.

FIG. 7. Graphical description of action manifold projected on $(x_2,x_3)$ plane when $2\beta_s < \beta < 3\beta_s$.

FIG. 8. (a) Plots of all possible solutions with $\beta = \beta_3$. (b) Graphical description of action manifold projected on $(x_0,x_1)$ plane.
FIG. 9. (a) Plots of all possible solutions with $\beta = \beta_4$. (b) Graphical description of action manifold projected on $(x_0, x_1)$ plane.
Fig. 1 (a), (b), (c)
Fig. 1 (d), (e), (f)
Fig. 1 (g), (h), (i)
Fig. 2

(a) 

(b) 

(c)
Fig. 3 (a)
\( \chi_2 \)

\( (x_2) \)

\( \chi_1 \)

\( (x_1) \)

\( \chi_0 \)

\( (x_0) \)

\( \chi_{-1} \)

\( (x_{-1}) \)

\( 0 \)

\( \tau \)

\( \beta_1 \)
$S_x \phi_s \phi_+ \phi_- \phi_s$

Fig. 4

(a)

(b) $x_n \ (n > -1)$
Fig. 5 (a)
Fig. 5 (b)
Fig. 6
Fig. 7
Fig. 8 (b)
Fig. 9 (a)
Fig. 9 (b)