The geodesic total curvature of spherical curves

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Abstract. The geodesic total curvature of rectifiable spherical curves is analyzed. We extend to the case of high dimension spheres the explicit formula that holds true for curves supported into the 2-sphere. For this purpose, we take advantage of some new integral-geometric formulas concerning both the Euclidean and geodesic total curvature of spherical curves.

Keywords : Spherical curves; geodesic curvature; integral-geometric formulas; non-smooth curves

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Introduction

The total curvature of curves in Euclidean spaces is defined by J. W. Milnor \cite{6,7} as the supremum of the rotation $k^*(P)$ computed among all the inscribed polygonals $P$. Following J. M. Sullivan \cite{10}, we recall that a curve in $\mathbb{R}^{N+1}$ with compact support and finite total curvature is rectifiable. Therefore, its arc-length parameterization $c$ is a Lipschitz-continuous function, hence it is differentiable a.e., by Rademacher’s theorem. Furthermore, denoting by $t = \dot{c}$ the tangent indicatrix (or tantrix) of a rectifiable curve $c$, it turns out that $c$ has finite total curvature $TC(c)$ if and only if its tantrix is a function of bounded variation. In this case, moreover, the explicit formula

$$TC(c) = \text{Var}_{S^N}(t), \quad t = \dot{c}$$

holds, where the definition of essential variation $\text{Var}_{S^N}(t)$ of the tantrix involves the geodesic distance $d_{S^N}$ in the Gauss $N$-sphere $S^N$ instead of the Euclidean distance in $\mathbb{R}^{N+1}$, see \cite{1,2}.

In this paper, we deal with rectifiable curves $c$ supported in the unit hyper-sphere $S^N$ of $\mathbb{R}^{N+1}$,

$$S^N := \{x \in \mathbb{R}^{N+1} : \|x\| = 1\}, \quad N \geq 2.$$

Notice that we shall make use of a different notation in order to distinguish between the sphere $S^N$ where the curve is supported and the Gauss sphere $S^N$ where the tantrix of the curve takes value.

The geodesic rotation $k_{S^N}(P)$ of a spherical polygonal $P$ in $S^N$ is the sum of the turning angles between its consecutive geodesic arcs. However, since $S^N$ has positive sectional curvature, the expected monotonicity formula for the geodesic rotation of inscribed spherical polygonals fails to hold, see \cite{3} and Example 2.1.

Therefore, the good definition of geodesic total curvature, say $TC_{S^N}(c)$, turns out to be the one introduced by Alexandrov-Reshetnyak \cite{1}, see Definition 2.2. It involves the modulus of spherical polygonals inscribed in $c$, compare e.g. \cite{5}. Since for polygonals $P$ in $S^N$ one has $k^*(P) = k_{S^N}(P) + \mathcal{L}(P)$, for a rectifiable curve $c$ in $S^N$ one infers that

$$TC_{S^N}(c) < \infty \iff TC(c) < \infty.$$

Referring to Secs. 3.1–3.2 of \cite{2} for the notation and properties of one-dimensional BV functions, we remark here that the Cantor component of the distributional derivative of the tantrix is orthogonal to $c$, see \cite{1,2}. Moreover, one gets $\dot{t}(s) \cdot c(s) = -1$ for a.e. $s \in I_L := (0, L)$, where $L = \mathcal{L}(c)$, whence the vector

$$\dot{t}^T(s) := \dot{t}(s) + c(s)$$

is tangential to $S^N$ at $c(s)$ for a.e. $s$, and for smooth curves $c$ its modulus is equal to the geodesic curvature.

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The above facts lead us to introduce the geodesic curvature energy functional

\[ F(c) := \int_{I_L} |\dot{t}^T| \, ds + |D^C t|(I_L) + \sum_{s \in \Delta_k} d_{SN}(t(s+), t(s-)), \quad t = \dot{c} \]

of a rectifiable curve \( c \) in \( S^N \) satisfying \( TC_{SN}(c) < \infty \). Notice that on account of formula (1.2) one has:

\[ F(c) = \text{Var}_{SN}(t) - \mathcal{L}(c). \]

In Theorem 3.1 the main result of this paper, we prove in any dimension \( N \geq 2 \) and for any rectifiable curve \( c \) in \( S^N \) satisfying \( TC_{SN}(c) < \infty \) the following representation formula:

\[ TC_{SN}(c) = F(c). \]

When \( N = 2 \), the latter formula has been obtained in our paper [8], where we exploited the existence of a weak parallel transport along the curve \( c \) whose angle function \( \Theta \) has bounded variation and satisfies \( |D\Theta|(I_L) = F(c) \). The main feature, that actually holds true for curves supported into (Riemannian) surfaces, is that the angle function has distributional derivative equal to the “weak” signed geodesic curvature of \( c \), whereas a generalized Gauss-Bonnet theorem holds true in this framework. Therefore, our argument from [8] fails to hold for curves supported in high dimension Riemannian manifolds \( M \).

In the proof of Theorem 3.1 we exploit its validity in the case \( N = 2 \), by making use of some new integral-geometric formulas that we now present.

Denote by \( G_{j+1}R^{N+1} \) the Grassmannian of the unoriented \( (j + 1) \) -planes \( p \) in \( R^{N+1} \), by \( \mu_{j+1} \) the corresponding Haar measure, and by \( \pi_p \) the orthogonal projection of \( R^{N+1} \) onto some \( p \) in \( G_{j+1}R^{N+1} \). Also, denote by \( \eta_p(x) \) the nearest point to \( x \in S^N \) onto the \( j \)-dimensional sphere \( S_p^j := S^N \cap p \), see [24].

For any integers \( 2 \leq j \leq N - 1 \) and any rectifiable curve \( c \) in \( S^N \) with finite total curvature, in Propositions 3.2 and 3.3 we prove the average formulas:

\[ TC(c) = \int_{G_{j+1}R^{N+1}} TC(\eta_p(c)) \, d\mu_{j+1}(p) \]
\[ TC_{SN}(c) = \int_{G_{j+1}R^{N+1}} TC_{S_p^j}(\eta_p(c)) \, d\mu_{j+1}(p). \]

We expect that the previous representation formula for the geodesic (or intrinsic) total curvature holds true for curves supported in high dimension Riemannian manifolds \( M \), provided that in the expression of the curvature energy functional \( F(c) \) we take \( \dot{t}^T(s) \) equal to the projection of the approximate derivative of the tantrix onto the tangent space to \( M \) at \( c(s) \). However, its validity cannot be checked by means of averaging arguments as in Theorem 3.1 when \( M \) fails to be an \( N \)-sphere.

We conclude the introduction by describing the content of this paper. In Sec. 11 we recall the main properties concerning the (Euclidean) total curvature, and the related integral-geometric formulas. In Sec. 4 we introduce the notion of geodesic total curvature and collect some relevant average formulas concerning the length of rectifiable curves in \( S^N \) and the geodesic rotation of polygonal curves, Proposition 2.5. In Sec. 5 we finally prove our main results, Theorem 3.1 and Propositions 3.2 and 3.3.

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1 Total curvature

In this section, we recall some properties concerning the (Euclidean) total curvature of curves in \( \mathbb{R}^{N+1} \).

Total curvature. The rotation \( k^*(P) \) of a polygonal curve \( P \) in \( \mathbb{R}^{N+1} \) is the sum of the exterior angles between consecutive segments. If \( P' \) is a polygonal inscribed in \( P \), one has \( k^*(P') \leq k^*(P) \). Therefore, the (Euclidean) total curvature \( TC(c) \) of a curve \( c \) in \( \mathbb{R}^{N+1} \) is defined, following Milnor [6, 7], as the supremum of the rotation \( k^*(P) \) computed among all the polygonals \( P \) in \( \mathbb{R}^{N+1} \) which are inscribed in \( c \). Then,
Integral-geometric formulas. Several properties concerning curves in $\mathbb{R}^{N+1}$ can be obtained once they are checked for e.g. planar curves, by making use of averaging arguments based on the validity of suitable integral-geometric formulas that we now recall.

For $0 \leq j \leq N - 1$ integer, denote by $G_{j+1}\mathbb{R}^{N+1}$ the Grassmannian of the unoriented $(j + 1)$-planes $p$ in $\mathbb{R}^{N+1}$. It is a compact group, and it can be equipped with a unique rotationally invariant probability measure, that will be denoted by $\mu_{j+1}$. For $p \in G_{j+1}\mathbb{R}^{N+1}$, we denote by $\pi_p$ the orthogonal projection of $\mathbb{R}^{N+1}$ onto $p$.  

Example 1.1 If $c$ is a rectifiable curve in $\mathbb{R}^{N+1}$, the following integral-geometric formula for the length $L(c)$ holds true for any $j = 0, \ldots, N - 1$:

$$L(c) = \frac{\sigma_j}{\sigma_N} \int_{G_{j+1}\mathbb{R}^{N+1}} L(\pi_p(c)) \, d\mu_{j+1}(p)$$

where $\sigma_j$ and $\sigma_N$ are positive constants only depending on $j$ and $N$, respectively, see e.g. [11, Sec. 4.8].

Also, a classical result that goes back to Fáry [4] shows that the total curvature of a curve (with finite total curvature) is the average of the total curvatures of all its projections onto $(j + 1)$-planes:

$$TC(c) = \int_{G_{j+1}\mathbb{R}^{N+1}} TC(\pi_p(c)) \, d\mu_{j+1}(p) \quad \forall j = 0, \ldots, N - 1.$$  

(1.1)

Following [10, Prop. 4.1], it suffices to prove the average formula for an angle, hence for the rotation $k^*(P)$ of a polygonal $P$, and then use the monotone convergence theorem.

We e.g. readily check that if a curve $c$ in $\mathbb{R}^{N+1}$ has compact support and finite total curvature, then $c$ is a rectifiable curve. In fact, if $d$ is the diameter of $c$, one has $L(\pi_p(c)) \leq d \,(TC(\pi_p(c)) + 1)$ for $\mu_1$-a.e. $p \in G_1\mathbb{R}^{N+1}$. Therefore, the previous average formulas (with $j = 0$) yield

$$L(c) = \frac{\sigma_0}{\sigma_N} \int_{G_1\mathbb{R}^{N+1}} L(\pi_p(c)) \, d\mu_1(p) \leq \frac{\sigma_0}{\sigma_N} \int_{G_1\mathbb{R}^{N+1}} (TC(\pi_p(c)) + 1) \, d\mu_1(p) = \frac{\sigma_0}{\sigma_N} \,(TC(c) + 1) < \infty.$$

Essential variation in the Gauss sphere. On account of the previous remark, since we are interested in curves with support in the unit hyper-sphere of $\mathbb{R}^{N+1}$, we deal with rectifiable curves $c$ in $\mathbb{R}^{N+1}$. We then tacitly assume that $c$ is parameterized by arc-length, so that $c = c(s)$, with $s \in [0, L] = \mathcal{T}_L$, where $I_L := (0, L)$ and $L = L(c)$. Since $c$ is a Lipschitz function, by Rademacher’s theorem (cf. [2, Thm. 2.14]) it is differentiable $L^1$-a.e. in $I_L$. Denoting by $\dot{f} := \frac{d}{ds}f$ the derivative w.r.t. the arc-length parameter $s$, one has $\dot{t}(s) := \dot{c}(s) \in \mathbb{S}^N$ for a.e. $s$, where $\mathbb{S}^N$ is the Gauss $N$-sphere. If $c$ is smooth and regular, one recovers the formula $TC(c) = \int_{I_L} |\dot{c}| \, ds$, where $\dot{c}(s) := \ddot{c}(s)$ is the curvature vector. In general, if $TC(c) < \infty$ the tantrix $t = \ddot{c} : I_L \to \mathbb{R}^{N+1}$ is a function of bounded variation. More precisely, a rectifiable curve $c$ has finite total curvature if and only its tantrix $t$ is a function of bounded variation, see [10, Prop. 3.1].

We refer to Secs. 3.1–3.2 of [2] for the notation and properties of one-dimensional BV functions, see also [8]. We only recall that the essential variation $\text{Var}_{\mathbb{R}^{N+1}}(v)$ of a function $v \in BV(I_L, \mathbb{R}^{N+1})$ agrees with the total variation of the distributional derivative $Dv$, namely:

$$\text{Var}_{\mathbb{R}^{N+1}}(v) = |Dv|(I_L) = \int_{I_L} |v| \, ds + \sum_{s \in J_v} |v(s+) - v(s-)| + |D^Cv|(I_L)$$

where $v \in L^1(I_L, \mathbb{R}^{N+1})$ is the approximate derivative, $v(s \pm)$ denote the right and left limits of $v$ at each point $s \in I_L$, the Jump set $J_v$ is the at most countable set of points in $I_L$ where $v(s-) \neq v(s+)$, and $D^Cv$ is the Cantor component of the distributional derivative, see Example 1.2.
If $TC(c) < \infty$, we thus have $t \in BV(I_L, S^N)$. The essential variation $Var_S(t)$ of $t \in BV(I_L, S^N)$ in $S^N$ differs from $Var_{S+1}(t)$, as its definition involves the geodesic distance $d_{S^N}$ in $S^N$ instead of the Euclidean distance in $\mathbb{R}^{N+1}$, and one has:

\[
Var_{S^N}(t) = \int_{I_L} |t| ds + \sum_{s \in J_t} d_{S^N}(t(s+), t(s-)) + |D^C t|(I_L).
\] (1.2)

Therefore, in general $Var_{S+1}(t) \leq Var_S(t)$, and equality holds if and only if $t$ has a continuous representative. Moreover, if a sequence $\{t_h\} \subset BV(I_L, S^N)$ weakly-* converges in $BV$ to $t \in BV(I_L, S^N)$, the following lower semicontinuity inequalities hold:

\[
Var_{S+1}(t) \leq \liminf_{h \to \infty} Var_{S+1}(t_h), \quad Var_S(t) \leq \liminf_{h \to \infty} Var_S(t_h).
\] (1.3)

Notice that the Cantor component of the derivative of the tantrix is orthogonal to $c$. In fact, using that $c$ is Lipschitz-continuous, and that $\bullet \cdot c \equiv 0$, where $\bullet$ is the scalar product in $\mathbb{R}^{N+1}$, we get

\[
0 = D^C (t \cdot c) = t \cdot D^C c + c \cdot D^C t = c \cdot D^C t.
\] (1.4)

**Example 1.2** If $P$ is a polygonal curve, the tantrix $t_P$ is a pure Jump function. At each corner point of $P$ one has $d_{S^N}(t(s+), t(s-)) = \theta$, the turning angle, whereas $|t(s+) - t(s-)| = 2\sin(\theta/2)$, so that $k^*(P) = Var_S(t_P)$.

Notice that the Cantor component $D^C t$ is non-trivial, in general. In fact, let e.g. $\gamma : T \to \mathbb{R}^2$, where $I = (0,1)$, denote the Cartesian curve $\gamma(t) := (t, u(t))$ in $\mathbb{R}^2$ given by the graph of the primitive $u(t) := \int_0^t v(\lambda) d\lambda$ of the classical Cantor-Vitali function $v : T \to \mathbb{R}$ associated to the “middle thirds” Cantor set. It turns out that $t = (1 + v^2)^{-1/2}(1, v)$, whence $t$ is a Cantor function, i.e., $D^n t = D^j t = 0$, and

\[
Dt(I) = D^C t(I) = \int_I \frac{1}{(1 + v^2)^{3/2}} (-v, 1) dD^C v.
\]

In particular, one gets $Var_{S^2}(t) = Var_S(t) = |D^C t|(I) = TC(\gamma) = \pi/4$.

**A representation formula.** Let $c$ be a rectifiable curve in $\mathbb{R}^{N+1}$ with finite total curvature, and let $t = \hat{c}$. For any polygonal $P$ inscribed in $c$ one has

\[
Var_{S^N}(t_P) \leq Var_{S^N}(t).
\] (1.5)

In fact, assume that $P$ is generated by the consecutive vertexes $c(s_i)$, where $0 = s_0 < s_1 < \cdots < s_n = L$, and let $v_i$ be the oriented segment of $P$ from $c(s_{i-1})$ to $c(s_i)$. When $c$ is a planar curve (i.e., when $N = 1$), the value of $t_P \in S^1$ on the segment $v_i$ is equal to one of the values of the tantrix $t$ in the interval $[s_{i-1}, s_i]$, when $t$ is completed to a continuous curve in $S^1$ by connecting with geodesic arcs the points $t(s-)$ and $t(s+)$ for each $s \in J_t$. Therefore, when $N \geq 2$ the value of the tantrix $t_P$ in $v_i$ is an average of the values of the restriction to $(s_{i-1}, s_i)$ of the completed tantrix $t$, compare [4].

As a consequence of (1.3), one obtains the following representation formula:

\[
TC(c) = Var_{S^N}(t), \quad t = \hat{c}.
\] (1.6)

In fact, assume that $P$ is an inscribed sequence satisfying mesh($P_h$) $\to 0$, and $P_h$ is parameterized with (piecewise) constant velocity in $I_L$, then $\{t_h\}$ converges to $t$ weakly-* in $BV(I_L, \mathbb{R}^{N+1})$. The upper bound (1.5) and the lower semicontinuity inequality in (1.3) yield the strict convergence $Var_{S^N}(t_h) \to Var_{S^N}(t)$. Using that $Var_{S^N}(t_h) = k^*(P_h) \to TC(c)$, one concludes with (1.6).

We finally notice that on account of (1.6), one can re-write the integral-geometric formula (1.1) in terms of the tantrix $t$, namely:

\[
Var_{S^N}(t) = \int_{G_{j+1}\mathbb{R}^{N+1}} Var_{S^p}(t_{(p)}) \, d\mu_{j+1}(p) \quad \forall j = 0, \ldots, N - 1
\] (1.7)

where $S^p_{j} := S^N \cap p$ and $t_{(p)}$ denotes the tantrix of the projected curve $\pi_p(c)$. 


2 Geodesic total curvature of spherical curves

In the sequel we deal with rectifiable curves supported in the unit hyper-sphere $\mathbb{S}^N$ of $\mathbb{R}^{N+1}$,

$$\mathbb{S}^N := \{ x \in \mathbb{R}^{N+1} : \|x\| = 1 \}, \quad N \geq 2.$$

**Geodesic Total Curvature.** For a general curve $c$ contained in $\mathbb{S}^N$, we shall denote by $\mathcal{P}_{\mathbb{S}^N}(c)$ the class of polygons in $\mathbb{S}^N$ which are inscribed in $c$. The *geodesic rotation* $k_{\mathbb{S}^N}(P)$ of a polygonal $P$ in $\mathbb{S}^N$ is the sum of the turning angles between the consecutive geodesic arcs of $P$. One is tempted to define the geodesic total curvature of $c$ as in the Euclidean case, i.e., by taking the supremum of the geodesic rotation $k_{\mathbb{S}^N}(P)$ computed among all the polygonals $P$ in $\mathcal{P}_{\mathbb{S}^N}(c)$. However, as observed in [3], the latter definition does not work. In fact, if $P, P' \in \mathcal{P}_{\mathbb{S}^N}(c)$, and $P$ is obtained by adding a vertex in $c$ to the vertexes of $P'$, then the monotonicity inequality $k_{\mathbb{S}^N}(P') \leq k_{\mathbb{S}^N}(P)$ is violated, since $\mathbb{S}^N$ has positive sectional curvature.

**Example 2.1** If e.g. $N = 2$ and $c$ is a parallel which is not a great circle, then the opposite inequality $k_{\mathbb{S}^2}(P') \geq k_{\mathbb{S}^2}(P)$ holds, and for any $P \in \mathcal{P}_{\mathbb{S}^2}(c)$ one has $k_{\mathbb{S}^2}(P) > \int_c |\tilde{r}_g| \, ds$, where $\tilde{r}_g$ is the geodesic curvature of the parallel $c$.

Actually, the good definition turns out to be the one introduced by Alexandrov-Reshetnyak [1]. For this purpose, compare e.g. [3], we recall that the *modulus* $\mu_{\mathbb{S}^N}(P)$ of a polygonal $P$ in $\mathcal{P}_{\mathbb{S}^N}(c)$ is the maximum of the geodesic diameter of the arcs of $c$ determined by the consecutive vertexes in $P$. For $\varepsilon > 0$, we also let

$$\Sigma_\varepsilon(c) := \{ P \in \mathcal{P}_{\mathbb{S}^N}(c) \mid \mu_{\mathbb{S}^N}(P) < \varepsilon \}.$$

**Definition 2.2** The *geodesic total curvature* of a curve $c$ in $\mathbb{S}^N$ is

$$\text{TC}_{\mathbb{S}^N}(c) := \lim_{\varepsilon \to 0^+} \sup \{ k_{\mathbb{S}^N}(P) \mid P \in \Sigma_\varepsilon(c) \}.$$

**Properties.** Since for polygonals $P$ in $\mathbb{S}^N$ one has $k^*(P) = k_{\mathbb{S}^N}(P) + L(P)$, for a rectifiable curve $c$ in $\mathbb{S}^N$ one infers the equivalence:

$$\text{TC}_{\mathbb{S}^N}(c) < \infty \iff \text{TC}(c) < \infty. \quad (2.1)$$

**Remark 2.3** In [1] Thm. 6.3.1] it was erroneously stated that a curve with finite geodesic total curvature has finite Euclidean total curvature, too. This is true if the spherical diameter of the curve is smaller than a dimensional constant $c_N$. In this case, in fact, for polygonal curves in $\mathbb{S}^N$ one has $k^*(P) \leq \pi + 2k_{\mathbb{S}^N}(P)$. Therefore, the previous statement holds true provided that the curve can be divided in a finite number of arcs each one with spherical diameter smaller than $c_N$. However, the latter property is false, in general, if the curve fails to be rectifiable. If e.g. one takes a curve that winds around an equator of $\mathbb{S}^N$ infinitely many times, its total Euclidean curvature is zero but its length and total geodesic curvature are both infinite.

If $c$ is smooth (and parameterized in arc-length), since $|t| = 1$ the curvature vector $k(s) := \dot{t}(s)$ is orthogonal to $t(s)$ and decomposes as $k(s) = -c(s) + \tilde{r}_g(s)u(s)$, where $u(s)$ is a tangent unit vector to $\mathbb{S}^N$ at $c(s)$ and $\tilde{r}_g(s)$ is the geodesic curvature at $c(s)$. In fact, one has $k \cdot c = \dot{c} \cdot \dot{c} = -\dot{c} \cdot \dot{c} = -|t|^2 = -1$.

If $\text{TC}_{\mathbb{S}^N}(c) < \infty$, we have seen that $t \in \text{BV}(I_L,\mathbb{S}^N)$, and in a similar way one gets $\dot{t}(s) \cdot c(s) = -1$ for a.e. $s$, whence the vector

$$\dot{t}^\top(s) := \dot{t}(s) + c(s)$$

is tangential to $\mathbb{S}^N$ at $c(s)$ for a.e. $s \in I_L$, and in the smooth case one has $|\dot{t}^\top(s)| = \tilde{r}_g(s)$ for every $s$.

More precisely, the polar decomposition $Dt = u|Dt|$, where $u$ is the Radon-Nikodym derivative of the measure $Dt$ with respect to its total variation, implies that $u : I_L \to \mathbb{S}^N$ is a Borel function satisfying

$$u(s) = \frac{\dot{t}(s)}{|\dot{t}(s)|} \quad \text{and} \quad u(s) = \frac{t(s+) - t(s-)}{|t(s+) - t(s-)|}.$$
at $L^1$-a.e. point $s \in I_L$ and at any point $s \in J_k$, respectively. Also, due to the orthogonality condition \( \eta \), it turns out that $u$ is tangential to $\mathbb{S}^N$ at $c(s)$ at $|D^C t|\cdot \text{-a.e.} \ s \in I_L$ and actually:

$$D^C t = (u \cdot D^C t) u.$$  

The above facts lead us to introduce the geodesic curvature energy functional

$$F(c) := \int_{I_L} |\dot{t}|^2 \, ds + |D^C t|(I_L) + \sum_{s \in J_k} d_{\mathbb{S}^N}(t(s+), t(s-)), \quad t = \dot{c}$$  \hspace{1cm} (2.2)

for rectifiable curves $c$ in $\mathcal{S}^N$ satisfying $\text{TC}_{\mathbb{S}^N}(c) < \infty$. Notice that since $|\dot{t}| = |\dot{t}|^2 + 1$, by (1.2) one has:

$$F(c) = \text{Var}_{\mathbb{S}^N}(t) - \mathcal{L}(c).$$  \hspace{1cm} (2.3)

For piecewise smooth curves, the geodesic curvature energy functional $F(c)$ agrees with the sum of the integral of the geodesic curvature (computed separately outside the corner points of $c$) plus the sum of the turning angles of the tangent vector to $c$ at the corner points $c(s)$, that correspond to the values of the parameter $s$ in the finite set $J_k$ of the discontinuity points of $t$. In particular, for polygonal curves $P$ of $\mathcal{S}^N$ one has $F(P) = k_{\mathbb{S}^N}(P)$. Therefore, one expects the validity of the formula

$$\text{TC}_{\mathbb{S}^N}(c) = F(c).$$

**Curves into the 2-sphere.** When $N = 2$, the previous representation formula has been obtained in our paper [3], where we exploited the existence of a weak parallel transport along the curve $c$ whose angle function $\Theta$ has bounded variation and satisfies $|D \Theta|(I_L) = F(c)$. The main feature, that actually holds true for curves supported into (Riemannian) surfaces, is that the angle function has weak derivative equal to the “weak” signed geodesic curvature of $c$. The validity of the Gauss-Bonnet theorem in this framework, in fact, allowed us to prove the strict convergence of the angle function along sequences of inscribed approximating polygons, yielding to $k_{\mathbb{S}^2}(P_k) \rightarrow F(c)$. Since we also know that $k_{\mathbb{S}^2}(P_k) \rightarrow \text{TC}_{\mathbb{S}^2}(c)$ provided that $\{P_h\} \subset \mathcal{P}_{\mathbb{S}^2}(c)$ satisfies $\mu_c(P_h) \rightarrow 0$, the previous representation formula holds true for any rectifiable curve $c$ in $\mathcal{S}^2$ satisfying $\text{TC}_{\mathbb{S}^2}(c) < \infty$. Notice that the above strategy fails to hold for curves supported in high dimension Riemannian manifolds $\mathcal{M}$, since we do not have a unique way to enclose a loop in $\mathcal{M}$ and hence we cannot rely on arguments based on the Gauss-Bonnet theorem.

In order to extend the representation formula to the case of high dimension $N \geq 3$, we shall make use of some new integral-geometric formulas relating the geodesic and Euclidean total curvature of a curve $c$ in $\mathcal{S}^N$ with the geodesic and Euclidean total curvature of its projections onto 2-dimensional spheres, respectively. For this purpose, we collect some more notation, a formula for the length of rectifiable curves, and a formula for the geodesic rotation of polygonal curves.

**Averages on spheres.** Following [1], for $j = 1, \ldots, N$ and $p \in G_{j+1} \mathbb{R}^{N+1}$, we denote by $\eta_p(x)$ the nearest point to $x \in \mathbb{S}^N$ onto the $j$-dimensional sphere $\mathcal{S}^j_p := \mathbb{S}^N \cap p$. It is well-defined by

$$\eta_p(x) := \frac{\pi_p(x)}{|\pi_p(x)|}$$  \hspace{1cm} (2.4)

provided that $x$ is not orthogonal to the $(j+1)$-plane $p$, i.e., if $x \in \mathbb{S}^N \setminus \mathcal{S}^j_p$, where $\mathcal{S}^j_p$ is the $(N - j - 1)$-sphere given by the *polar* to $\mathcal{S}^j_p$ in $\mathbb{S}^N$.

The average formula concerning the length of spherical curves was proved in [1] Thm. 4.8.3, p. 108].

**Proposition 2.4** Given a rectifiable curve $c$ in $\mathbb{S}^N$, for any integer $1 \leq j \leq N - 1$ one has

$$\mathcal{L}(c) = \int_{G_{j+1} \mathbb{R}^{N+1}} \mathcal{L}(\eta_p(c)) \, d\mu_{j+1}(p).$$

The following integral-geometric formula for the geodesic rotation of spherical polygonal curves was proved in [1] Thm. 6.2.2, p. 190 for $j = 1$. Actually, it holds true for all the ranges of values of $j$. For the sake of completeness, we report here the proof in the case $j > 1$ taken from [9].
Proposition 2.5 Given a polygonal curve $\gamma$ in $S^N$, for any integer $1 \leq j \leq N - 1$ one has

$$k_{S^N}(\gamma) = \int_{G_{j+1}R^{N+1}} k_{S^p}(\eta_p(\gamma))\,d\mu_{j+1}(p).$$

Proof: Assume $j > 1$. For the sake of simplicity we denote here by $K_p(\gamma)$ the geodesic rotation of a polygonal $\gamma$ in a unit sphere of generic dimension. For $\mu_{j+1}$ a.e. $p \in G_{j+1}R^{N+1}$, the cited integral-geometric formula from [1] implies that the geodesic rotation of the projected curve $K_p(\eta_p(\gamma))$ is equal to the averaged integral of the geodesic rotation of the projection of the curve $\eta_p(\gamma)$ onto the unit circles corresponding to the 2-planes $q$ of $R^{N+1}$ that are contained in $p$, i.e.,

$$K_p(\eta_p(\gamma)) = \int_{G_2R^j+1} K_p(\eta^p_q(\eta_p(\gamma)))\,d\mu^j_2(q)$$

where $\mu^j_2$ is the probability measure corresponding to the Grassmannian $G_{2R^j+1}$, with $R^{j+1} = p$, and $\eta^p_q$ is the nearest point projection from $S^j_p$ onto the 1-circle $S^j_p \cap q$. Therefore, we have:

$$\int_{G_{j+1}R^{N+1}} K_p(\eta_p(\gamma))\,d\mu_{j+1}(p) = \int_{G_{j+1}R^{N+1}} \left( \int_{G_2R^j+1} K_p(\eta^p_q(\eta_p(\gamma)))\,d\mu^j_2(q) \right)\,d\mu_{j+1}(p) =: I.$$

Moreover, the iterated integral $I$ on the right-hand side is equal to

$$I = \int_{G_2R^{N+1}} K_p(\eta_p(\gamma))\,d\mu_2(r)$$

and hence, by applying again the formula from [1], we get $I = K_p(\gamma)$, as required.

As a consequence, since $TC(\eta_p(\gamma)) = L(\eta_p(\gamma)) + k_{S^j_p}(\eta_p(\gamma))$, for a polygonal curve $\gamma$ in $S^N$ one also gets:

$$TC(\gamma) = \int_{G_{j+1}R^{N+1}} TC(\eta_p(\gamma))\,d\mu_{j+1}(p), \quad j = 1, \ldots, N - 1.$$

3 The explicit formula

In this section, we extend the previous representation formula to the case $N > 2$, by proving the following

Theorem 3.1 Let $N \geq 3$ and let $c$ be a rectifiable curve in $S^N$ with finite geodesic total curvature, $TC_{S^N}(c) < \infty$. Then

$$TC_{S^N}(c) = F(c)$$

where $F(c)$ is the geodesic curvature energy functional given by [22].

The above formula is obtained by exploiting its validity in the case $N = 2$, and by means of the representation result (1.6) for the total Euclidean curvature, through the following new integral-geometric formulas (where we choose $j = 2$), the proof of which is postponed. In both the statements we assume that $c$ is a rectifiable curve in $S^N$ and that $2 \leq j \leq N - 1$ are integers.

Proposition 3.2 If $TC(c) < \infty$, then $TC(c) = \int_{G_{j+1}R^{N+1}} TC(\eta_p(c))\,d\mu_{j+1}(p)$.

Proposition 3.3 If $TC_{S^N}(c) < \infty$, then $TC_{S^N}(c) = \int_{G_{j+1}R^{N+1}} TC_{S^j}(\eta_p(c))\,d\mu_{j+1}(p)$.

Remark 3.4 We recall that in the previous propositions, for rectifiable curves the boundedness of the Euclidean and geodesic total curvature are equivalent properties, see [23]. Moreover, from the proof of Proposition 3.3 it turns out that if $c$ is a rectifiable curve in $S^N$ with finite geodesic total curvature, then $TC_{S^N}(c)$ is equal to the limit of $k_{S^N}(P_h)$ for any sequence $\{P_h\} \subset P_{S^N}(c)$ satisfying $\mu_c(P_h) \to 0$.  

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Proof of Theorem 3.1: We already know that \( c \) has finite Euclidean total curvature \( \text{TC}(c) \), see \( (2.4) \), hence \( t \in \text{BV}(I_L, S^N) \), where \( t = \dot{c} \). On account of \( (2.3) \) and of the representation formula \( (1.6) \) for the Euclidean total curvature, it then suffices to show that
\[
\mathcal{L}(c) + \text{TC}_{S^N}(c) = \text{TC}(c) .
\]
(3.1)

By Propositions \( (2.4) \) and \( (3.3) \) with \( j = 2 \), we can write
\[
\mathcal{L}(c) + \text{TC}_{S^N}(c) = \int_{G_3 \mathbb{R}^{N+1}} [\mathcal{L}(\eta_p(c)) + \text{TC}_{S^2}(\eta_p(c))] \, d\mu_3(p)
\]
where \( \mathcal{L}(\eta_p(c)) + \text{TC}_{S^2}(\eta_p(c)) < \infty \) for \( \mu_3 \text{-a.e. } p \in G_3 \mathbb{R}^{N+1} \). For any such good projection, on account of the result from \( (3) \) for curves into the 2-sphere we infer that
\[
\text{TC}_{S^2}(\eta_p(c)) = \mathcal{F}(\eta_p(c)) , \quad \mathcal{F}(\eta_p(c)) = \text{TC}(\eta_p(c)) - \mathcal{L}(\eta_p(c)) .
\]
Therefore, we can write
\[
\mathcal{L}(c) + \text{TC}_{S^N}(c) = \int_{G_3 \mathbb{R}^{N+1}} \text{TC}(\eta_p(c)) \, d\mu_3(p)
\]
and hence equality \( (3.1) \) follows from Proposition \( (3.2) \) \( \square \).

It remains to prove the integral-geometric formulas stated in Propositions \( (3.2) \) and \( (3.3) \).

Proof of Proposition \( (3.2) \): If \( P \) is a polygonal in \( \mathbb{R}^{N+1} \) inscribed in \( c \), we know that \( k^*(P) = \mathcal{L}(t_P) \), where \( t_P \) is the completion of the tantrix of \( P \) in \( \mathbb{S}^N \). By Proposition \( (2.4) \) we thus have:
\[
k^*(P) = \int_{G_{j+1} \mathbb{R}^{N+1}} \mathcal{L}(\eta_p(t_P)) \, d\mu_{j+1}(p) .
\]
We now observe that for \( \mu_{j+1} \text{-a.e. } p \in G_{j+1} \mathbb{R}^{N+1} \), the projection of the completed tantrix \( \eta_p(t_P) \) agrees with the completion of the tantrix of the polygonal \( Q = Q(P, p) \) of \( p \) inscribed in the spherical curve \( \eta_p(c) \), where the ordered vertexes are taken in correspondence to the ordered vertexes of \( P \) in \( c \). Therefore, we have \( \mathcal{L}(\eta_p(t_P)) = k^*(Q(P, q)) \) and we can thus write
\[
k^*(P) = \int_{G_{j+1} \mathbb{R}^{N+1}} k^*(Q(P, p)) \, d\mu_{j+1}(p) .
\]
Now, taking a sequence \( \{P_h\} \) with \( \text{mesh}(P_h) \to 0 \), we have \( k^*(P_h) \ngtr \text{TC}(c) \) and for a.e. \( p \) as above we correspondingly have \( \text{mesh}(Q(P_h, p)) \to 0 \), whence \( k^*(Q(P_h, p)) \ngtr \text{TC}(\eta_p(c)) \), as \( h \to \infty \). The assertion then follows from the monotone convergence theorem. \( \square \).

Proof of Proposition \( (3.3) \): We first deal with the case \( j = 2 \). If \( P \in \mathcal{P}_{S^N}(c) \), by Proposition \( (2.4) \) we have
\[
k_{S^N}(P) = \int_{G_3 \mathbb{R}^{N+1}} k_{S^2}(\eta_p(P)) \, d\mu_3(p) .
\]
(3.2)
Moreover, for \( \mu_3 \text{-a.e. } p \in G_3 \mathbb{R}^{N+1} \) we have \( \eta_p(P) \in \mathcal{P}_{S^2}(\eta_p(c)) \).

Denoting again by \( Q = Q(P, p) \) the Euclidean polygonal in \( p \) inscribed in the spherical curve \( \eta_p(c) \), where the ordered vertexes are taken in correspondence to the ordered vertexes of \( P \) in \( c \), the geodesic rotation of \( \eta_p(P) \) is lower than the rotation of \( Q(P, p) \), which is (by definition) lower than the Euclidean total curvature of the spherical curve \( \eta_p(c) \).

We now show that for \( \mu_3 \text{-a.e. } p \in G_3 \mathbb{R}^{N+1} \) the Euclidean total curvature of \( \eta_p(c) \) is lower than the Euclidean total curvature of the curve \( \pi_p(c) \) where, we recall, \( \pi_p \) is the orthogonal projection onto \( p \). In fact, let \( \tilde{P} \) denote a polygonal in \( \mathbb{R}^{N+1} \) inscribed in \( c \) and generated by the consecutive vertexes \( c(s_i) \), where \( 0 = s_0 < s_1 < \cdots < s_n = L \), and assume that both \( c \) and \( \tilde{P} \) do not intersect the \((N-3)\)-sphere \( S^2_p \) given
by the polar to $S^2_S$ in $S^N$. Notice that since $c$ is a rectifiable curve, for any choice of the vertexes $c(s_i)$ this is a condition verified for $\mu_3$-a.e. $p \in G_3\mathbb{R}^{N+1}$, compare \[\text{(1.1)}.\]

Denoting by $P_1$, $P_2$ the polygons in $p$ inscribed in $\eta_p(c)$ and $\pi_p(c)$ and generated by the consecutive vertexes $\eta_p(c(s_i))$ and $\pi_p(c(s_i))$, respectively, since $\eta_p(c(s_i)) = \pi_p(c(s_i))/|\pi_p(c(s_i))|$ for each $i$, see \[2.4\], it turns out that $k^*(P_1) \leq k^*(P_2)$.

We have definitely obtained for $\mu_3$-a.e. $p \in G_3\mathbb{R}^{N+1}$,
\[
k_{S^2_S}(\eta_p(P)) \leq k^*(Q(P,p)) \leq TC(\eta_p(c)) \leq TC(\pi_p(c)) =: g(p) \tag{3.3}
\]
whereas by the integral-geometric formula \[\text{(1.1)}\]
\[
\int_{G_3\mathbb{R}^{N+1}} g(p) \mu_3(p) = \int_{G_3\mathbb{R}^{N+1}} TC(\pi_p(c)) \mu_3(p) = TC(c) < \infty,
\]
whence $g(p)$ is a non-negative summable function in $L^1(G_3\mathbb{R}^{N+1}, \mu_3)$.

Choose now a sequence $\{P_h\} \subset \mathcal{P}_{S^N}(c)$ with $\mu_3(P_h) \to 0$. For a.e. $p$ as above we have $\{\eta_p(P_h)\} \subset \mathcal{P}_{S^2_S}(\eta_p(c))$, with $\mu_3(\eta_p(P_h)) \to 0$. On account of Propositions \[2.4\] and \[3.2\] we have $L(\eta_p(c)) < \infty$ and $TC_{S^2_S}(\eta_p(c)) \leq TC_{S^2_S}(\eta_p(c)) < \infty$. Therefore, by the cited result from \[8\], we know that $k_{S^2_S}(\eta_p(P_h)) \to TC_{S^2_S}(\eta_p(c))$ as $h \to \infty$ for a.e. $p$, whereas $k_{S^2_S}(\eta_p(P_h)) \leq g(p)$ for each $h$ and a.e. $p$, see \[3.3\]. As a consequence, by the dominated convergence theorem we get from \[3.2\]
\[
\lim_{h \to \infty} k_{S^N}(P_h) = \int_{G_3\mathbb{R}^{N+1}} \lim_{h \to \infty} k_{S^2_S}(\eta_p(P_h)) \mu_3(p) = \int_{G_3\mathbb{R}^{N+1}} TC_{S^2_S}(\eta_p(c)) \mu_3(p).
\]
Since the above limit holds true for any sequence $\{P_h\}$ as above, the assertion in the case $j = 2$ readily follows from the definition of geodesic total curvature $TC_{S^N}(c)$.

Finally, when $N \geq 4$, the case $j > 2$ of the integral-geometric formula for $TC_{S^N}(c)$ can be obtained from the case $j = 2$, by using the same argument that we followed in the proof of Proposition \[2.4\], this time taking $\gamma = c$ and $K_p(\gamma) = k(p)$ equal to the geodesic total curvature of a curve $\gamma$ in a unit sphere of generic dimension. We omit any further detail. \[\Box\]

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