Global Padé approximations of the generalized Mittag-Leffler function and its inverse

Caibin Zeng\textsuperscript{a,}, YangQuan Chen\textsuperscript{b}

\textsuperscript{a}School of Sciences, and School of Automation Science and Engineering, South China University of Technology, Guangzhou 510640, China
\[b\]MESA (Mechatronics, Embedded Systems and Automation) LAB, School of Engineering, University of California, Merced, 5200 N Lake Road, Merced, CA 95343, USA

Abstract

This paper reports the finding of a global Padé approximation of the generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ with $x \in [0, +\infty)$. This uniform approximation can account for both the Taylor series for small arguments and asymptotic series for large arguments. More precisely, we work out the global Padé approximation with degree 2 for the particular cases $\{0 < \alpha < 1, \beta > \alpha\}$, $\{0 < \alpha = \beta < 1\}$, and $\{\alpha = 1, \beta > 1\}$, respectively. Moreover, these approximations are inverted to yield a global Padé approximation of the inverse generalized Mittag-Leffler function $-L_{\alpha,\beta}(x)$ with $x \in (0, 1/\Gamma(\beta)]$. We also provide several examples with selected values $\alpha$ and $\beta$ to compute the errors from the approximations. Finally, we point out the possible applications using our established approximations in the ordinary and partial time-fractional differential equations in the sense of Riemann-Liouville.

Keywords:
Mittag-Leffler function, complete monotonicity, fractional approximations, Padé approximants.

2010 MSC: 26A33, 33E12, 35S10, 45K05
1. Introduction

The Mittag-Leffler function and its generalizations are very important functions that find widespread use in the framework of fractional calculus. Similar to the exponential function frequently used in the solutions of integer-order systems, the (generalized) Mittag-Leffler functions play an analogous role in the solution of fractional-order systems. In fact, the exponential function itself is a very specific form, one of an infinite set, of these seemingly ubiquitous functions.

The standard definition of Mittag-Leffler function is defined as

\[ E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \quad \alpha \gt 0. \]  

where \( \alpha \gt 0 \). The Mittag-Leffler function with two parameters (sometimes also called the generalized Mittag-Leffler function), appears most frequently and has the following form

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \quad \alpha \gt 0 \text{ and } \beta \gt 0. \]

For \( \beta = 1 \), we have \( E_{\alpha}(z) = E_{\alpha,1}(z) \). Also, \( E_{1,1}(z) = e^z \). Later on, some studies contribute to several properties and applications of the (generalized) Mittag-Leffler functions. Nowadays it is worth of being referred to as the Queen Function of Fractional Calculus.

In this respect we also recommend some classical books on fractional calculus.

Despite a wealth of analytical information about the (generalized) Mittag-Leffler function, its behavior as a holomorphic function and dependence upon the parameters are largely unexplored, because there seem to be no numerical algorithms available to compute the function accurately for all parameters. Gorenflo et al. studied the computation of the generalized Mittag-Leffler function of a complex variable for different positions of argument \( z \) in the complex plane. The Taylor series definition and asymptotic series serve the cases where the absolute value \( |z| \) is small and large, respectively. Hilfer and Seybold reported a simpler algorithm to compute the generalized Mittag-Leffler function.
than the one in [15]. It’s also worth pointing out that Podlubny [17] provided a MATLAB routine for evaluating the generalized Mittag-Leffler function with desired accuracy.

Recently, Starovotov and Starovotova [18] discussed the Padé approximations for the Mittag-Leffler function and showed that the approximations serve uniformly on the compact set \(|z| \leq 1\). Mainardi [19] also used the Padé approximation to provide lower and upper bounds to the Mittag-Leffler function \(E_\alpha(-t^\alpha)\) for \(t > 0\). The Padé approximation is better than a truncated Taylor series, but it is not necessarily compatible with the asymptotic behavior for large arguments. Fortunately, Winitzki [20] provided the so-called global Padé approximation, which constructs uniform approximations to analytic transcendental functions. These uniform approximations are built from elementary functions using both Taylor and asymptotic series of the given transcendental function. The authors applied it to find good approximations of several functions including the elliptic function, the error function of real and imaginary arguments, the Bessel functions, and the Airy function. Also, Atkinson and Osseiran [21] applied this method to find a uniform rational approximation of the Mittag-Leffler function \(E_\alpha(-z)\) with \(0 < \alpha < 1\) and \(z \in (0, \infty)\).

However, to the best of our knowledge, it is still open to find the global Padé approximation for the generalized Mittag-Leffler function \(E_{\alpha,\beta}(z)\) for all parameters \(\alpha, \beta\) and \(z\).

On the other hand, the completely monotone functions are known to play an important role in different branches of mathematics and especially in the probability theory (see [22] for example). In particular, Pollard [23] proved that the Mittag-Leffler function \(E_\alpha(-x)\) with \(x \geq 0\) was completely monotonic, that is

\[ (-1)^m \frac{d^m}{dx^m} E_\alpha(-x) \geq 0 \]

for all \(m, m = 0, 1, 2, \ldots\), if \(0 \leq \alpha \leq 1\). Based on the usage of the corresponding probability measures and the Hankel contour integration, Schneider [24] proved that the generalized Mittag-Leffler function \(E_{\alpha,\beta}(-x)\) with \(x \geq 0\) was completely
monotonic if and only if $0 < \alpha \leq 1$ and $\beta \geq \alpha$. In other word, it yields
\[
(-1)^m \frac{d^m}{dx^m} E_{\alpha,\beta}(-x) \geq 0
\]
for all $m$, $m = 0, 1, 2, \ldots$, if $0 < \alpha \leq 1, \beta \geq \alpha$. This result was also proved in a simpler way [25]. This property is essential for the discussion of the inverse generalized Mittag-Leffler function below.

In this paper we focus on the generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ and its inverse with $0 < \alpha \leq 1$ and $\beta \geq \alpha$. More precisely, we will divide into three particular cases $\{0 < \alpha < 1, \beta > \alpha\}$, $\{0 < \alpha = \beta < 1\}$, and $\{\alpha = 1, \beta > 1\}$, for which one can find good approximations by using both Taylor and asymptotic series.

The paper is organized as follows. In Section 2 we develop the global Padé approximations of the generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ for particular cases $\{0 < \alpha < 1, \beta > \alpha\}$, $\{0 < \alpha = \beta < 1\}$, and $\{\alpha = 1, \beta > 1\}$, respectively. In Section 3 we study the inverse generalized Mittag-Leffler function $-L_{\alpha,\beta}(x)$ and obtain its uniform approximations. Finally, we give some concluding discussions in Section 4 and close the paper.

2. Global Padé approximations of $E_{\alpha,\beta}(-x)$

We now consider the global Padé approximations of the generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ in the domain $[0, +\infty)$. Note that the function $E_{\alpha,\beta}(-x)$ is finite everywhere in the interval $[0, +\infty)$ for $0 < \alpha \leq 1$ and $\beta \geq \alpha$. So we can apply the idea for nonsingular functions by Winitzki [20] to obtain a global Padé approximation of the generalized Mittag-Leffler function. Also, we notice the fact that $E_{\alpha,\beta}(-\infty) = 0$, which motivates us to choose the series at $x = +\infty$ starting with a higher power of $x^{-1}$.

For the case $\{0 < \alpha < 1, \beta > \alpha\}$, it follows from the definition (2) that
\[
\Gamma(\beta - \alpha) x E_{\alpha,\beta}(-x) = \Gamma(\beta - \alpha) x \sum_{k=0}^{m-2} \frac{(-x)^k}{\Gamma(\beta + ak)} + O(x^m) \equiv a(x) + O(x^m),
\]
\[ \Gamma(\beta - \alpha) x E_{\alpha,\beta}(-x) = -\Gamma(\beta - \alpha) x \sum_{k=1}^{n} \frac{(-x)^{-k}}{\Gamma(\beta - \alpha k)} + \mathcal{O}(x^{-n}) \equiv b(x) + \mathcal{O}(x^{-n}), \quad (5) \]

at \( x = 0 \) and \( x = +\infty \), respectively. The multiplication by \( \Gamma(\beta - \alpha) x \) ensures that the first coefficient of the asymptotic series (5) is 1.

We now look for a rational approximation of the form

\[ \Gamma(\beta - \alpha) x E_{\alpha,\beta}(-x) \approx \frac{p(x)}{q(x)} = \frac{p_0 + p_1 x + \ldots + p_\nu x^\nu}{q_0 + q_1 x + \ldots + q_\nu x^\nu}, \quad (6) \]

where \( \nu \) is an appropriately chosen integer. The problem is to find the coefficients \( p_i \) and \( q_i \) such that (6) has the correct expansions at \( x = 0 \) and \( x = +\infty \). Since the leading term of (6) at \( x = +\infty \) is \( p_\nu / q_\nu \), we can set \( p_\nu = q_\nu = 1 \).

This formulation is similar to the problem of Hermite-Padé interpolation with two anchor points, except that one of the points is at infinity where we use an expansion in \( x^{-1} \). The unknown coefficients \( p_i \) and \( q_i \) are found from the system of linear equations written compactly as

\[ p(x) - q(x) a(x) = \mathcal{O}(x^m) \text{ at } x = 0, \quad (7) \]

\[ \frac{p(x)}{x^\nu} - \frac{q(x)}{x^\nu} b(x^{-1}) = \mathcal{O}(x^{-n}) \text{ at } x = +\infty. \quad (8) \]

Here it is implied that the surviving polynomial coefficients in \( x \) or \( x^{-1} \) are equated. This assumes that \( p(x) \) and \( q(x) \) have no common polynomial factors. Moreover, these two equations form an inhomogeneous linear system of \( (m + n - 1) \) equations for \( 2\nu \) unknowns \( p_i, q_i, 0 \leq i \leq \nu - 1 \). So, when a solution exists, it is unique if \( m + n \) is odd.

We work out the case \( \nu = 2 \), noting that it is easily generalized to a higher order approximation. Higher order approximations, i.e., values of \( \nu \) greater than 2, can be computed to gain greater accuracy. We are thus searching for an approximation for the function \( \Gamma(\beta - \alpha) x E_{\alpha,\beta}(-x) \) of the form

\[ \Gamma(\beta - \alpha) x E_{\alpha,\beta}(-x) \approx \frac{p_0 + p_1 x + x^2}{q_0 + q_1 x + x^2}, \quad (9) \]

where the two series (4) and (5) are truncated to orders \( m = 3 \) and \( n = 2 \), yielding the functions

\[ a(x) = \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta)} x - \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta + \alpha)} x^2, \quad (10) \]
$$b(x^{-1}) = 1 - \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta - 2\alpha)} x^{-1}.$$  \hspace{1cm} (11)

Substituting (9)-(11) into (7) and (8) and collecting equal powers of $x$ through $O(x^m)$ and $O(x^{-n})$, we obtain

$$\begin{cases} 
  p_0 = 0, \\
  p_1 - \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta)} q_0 = 0, \\
  1 + \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta + \alpha)} q_0 - \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta)} q_1 = 0, \\
  p_1 - q_1 + \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta - 2\alpha)} = 0.
\end{cases}$$

Its solution can be expressed by

$$\begin{cases} 
  p_0 = 0, \\
  p_1 = \frac{\Gamma(\beta) \Gamma(\beta + \alpha) - \Gamma(\beta - \alpha) \Gamma(\beta - \alpha)}{\Gamma(\beta + \alpha) \Gamma(\beta - \alpha) - \Gamma(\beta) \Gamma(\beta - 2\alpha)} q_0, \\
  q_0 = \frac{\Gamma(\beta) \Gamma(\beta + \alpha) - \Gamma(\beta - \alpha) \Gamma(\beta - \alpha)}{\Gamma(\beta + \alpha) \Gamma(\beta - \alpha) - \Gamma(\beta) \Gamma(\beta - 2\alpha)} q_1, \\
  q_1 = \frac{\Gamma(\beta) \Gamma(\beta + \alpha) - \Gamma(\beta - \alpha) \Gamma(\beta - \alpha)}{\Gamma(\beta + \alpha) \Gamma(\beta - \alpha) - \Gamma(\beta) \Gamma(\beta - 2\alpha)}.
\end{cases}$$  \hspace{1cm} (12)

Therefore, we obtain a global Padé approximation from (9) and (12) that

$$E_{\alpha,\beta}(-x) \approx \frac{1}{\Gamma(\beta - \alpha)x} \left( p_0 + p_1 x + x^2 \right) = \frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta - \alpha) q_0} x = \frac{1}{\Gamma(\beta)} + \frac{1}{q_0} x + \frac{1}{q_0} x^2.$$  \hspace{1cm} (13)

As an example, for $\alpha = 1/2$ and $\beta = 3/2$, it follows from the properties\(^1\) of the generalized Mittag-Leffler function that

$$E_{\frac{1}{2}, \frac{3}{2}}(-x) = \begin{cases} 
  \frac{2}{\sqrt{\pi}}, & \text{if } x = 0, \\
  \frac{1 - \exp(x^2) \text{erfc}(x)}{x}, & \text{if } x > 0,
\end{cases}$$

where erfc($x$) is complementary to the error function. Also, using the fact $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$, we get its global Padé approximations with degree 2:

$$E_{\frac{1}{2}, \frac{3}{2}}(-x) \approx \frac{2}{\sqrt{\pi}} + \frac{4 - \pi x}{\frac{\pi - 2}{2} x + \frac{\pi - 5}{2} x^2}.$$  \hspace{1cm} 

The evolutions of the generalized Mittag-Leffler function $E_{1/2,3/2}(-x)$ and its global Padé approximation with degree 2 are drawn in Figure 1. In this situation, the maximum error is 0.0034 over all positive $x$.

\(^1\) $E_{1/2}(z) = \exp(z^2) \text{erfc}(-z); \ E_{\alpha,\beta}(z) = z E_{\alpha,\alpha+\beta}(z) + 1/\Gamma(\beta).$
As a special situation, say, $\beta = 1$, we reduce (13) to

$$E_\alpha(-x) \approx \frac{1 + \frac{1}{\Gamma(1-\alpha)q_0^2}x}{1 + \frac{q_1}{q_0}x + \frac{1}{q_0^2}x^2}, \tag{14}$$

where

$$q_0^* = \frac{\Gamma(1+\alpha) - \Gamma(1-\alpha)}{\Gamma(1+\alpha)\Gamma(1-\alpha)}, \quad q_1^* = \frac{\Gamma(1+\alpha) - \Gamma(1-\alpha)}{\Gamma(1+\alpha)\Gamma(1-\alpha)-1}. \tag{15}$$

It should be pointed out that this special case was studied by \[21\].

As an example, for $\alpha = 1/2$, it follows from the properties of the generalized Mittag-Leffler function that

$$E_{1/2}(-x) = \exp(x^2)\erfc(x).$$

Also, using the fact $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$, we get its global Padé approximations with degree 2:

$$E_{1/2}(-x) \approx \frac{1 + \frac{x^2}{\sqrt{\pi}x}}{1 + \sqrt{\pi}x + (\pi - 2)x^2}.$$ 

The evolutions of the generalized Mittag-Leffler function $E_{1/2}(-x)$ and its global Padé approximation with degree 2 are drawn in Figure 2. In this situation, the maximum error is 0.0079 over all positive $x$. 

Figure 1: The evolutions of the generalized Mittag-Leffler function $E_{1/2,3/2}(-x)$ and its global Padé approximation with degree 2, and the maximum error is 0.0034 over all positive $x$. 

Figure 2: The evolutions of the generalized Mittag-Leffler function $E_{1/2}(-x)$ and its global Padé approximation with degree 2 are drawn in Figure 2. In this situation, the maximum error is 0.0079 over all positive $x$. 

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Figure 2: The evolutions of the generalized Mittag-Leffler function $E_{1/2}(-x)$ and its global Padé approximation with degree 2, and the maximum error is 0.0079 over all positive $x$.

For the case $\{0 < \alpha = \beta < 1\}$, the generalized Mittag-Leffler function $E_{\alpha,\alpha}(-x)$ admits the two series

$$
\frac{\Gamma(1-\alpha)}{\alpha} x^2 E_{\alpha,\alpha}(-x) = \frac{\Gamma(1-\alpha)}{\alpha} x^2 \sum_{k=0}^{m-3} \frac{(-x)^k}{\Gamma(\alpha + \alpha k)} + \mathcal{O}(x^m), \quad (16)
$$

$$
\frac{\Gamma(1-\alpha)}{\alpha} x^2 E_{\alpha,\alpha}(-x) = -\frac{\Gamma(1-\alpha)}{\alpha} x^2 \sum_{k=2}^{n+1} \frac{(-x)^{-k}}{\Gamma(\alpha - \alpha k)} + \mathcal{O}(x^{-n}), \quad (17)
$$
at $x = 0$ and $x = +\infty$, respectively. The multiplication by $\Gamma(1-\alpha)x^2/\alpha$ ensures that the first coefficient of the asymptotic series (17) is 1.

In the same approach, we can obtain a global Padé approximations with degree 2 for $E_{\alpha,\alpha}(-x)$ of the form

$$
E_{\alpha,\alpha}(-x) \approx \frac{\frac{\Gamma(\alpha)}{1 + \frac{2\Gamma(1-\alpha)^2}{\Gamma(1+\alpha)(1-2\alpha)} x + \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} x^2}}{\Gamma(1-\alpha)} \quad (18)
$$

As an example, for $\alpha = \beta = 1/2$, it follows from the properties of the generalized Mittag-Leffler function that

$$
E_{\frac{1}{2},\frac{1}{2}}(-x) = \frac{1}{\sqrt{\pi}} - x \exp(x^2) \text{erfc}(x).
$$
Figure 3: The evolutions of the generalized Mittag-Leffler function $E_{1/2,1/2}(-x)$ and its global Padé approximation with degree 2, and the maximum error is 0.1349 over all positive $x$.

Also, using the fact $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$, we get its global Padé approximations with degree 2:

$$E_{1/2,1/2}(-x) \approx \frac{1}{1 + 2x^2}.$$

The evolutions of the generalized Mittag-Leffler function $E_{1/2,1/2}(-x)$ and its global Padé approximation with degree 2 are drawn in Figure 3. In this situation, the maximum error is 0.1349 over all positive $x$.

For the case \(\{\alpha = 1, \beta > 1\}\), the generalized Mittag-Leffler function $E_{1,\beta}(-x)$ admits the two series

$$\Gamma(\beta - 1)x E_{1,\beta}(-x) = \Gamma(\beta - 1)x \sum_{k=0}^{m-2} \frac{(-x)^k}{\Gamma(\beta + k)} + O(x^m), \quad (19)$$

$$\Gamma(\beta - 1)x E_{1,\beta}(-x) = -\Gamma(\beta - 1)x \sum_{k=1}^{n} \frac{(-x)^{-k}}{\Gamma(\beta - k)} + O(x^{-n}), \quad (20)$$

at $x = 0$ and $x = +\infty$, respectively. The multiplication by $\Gamma(\beta - 1)x$ ensures that the first coefficient of the asymptotic series \(20\) is 1.
In the same approach, we can obtain a global Padé approximations with degree 2 for $E_{1,\beta}(-x)$ of the form

$$E_{1,\beta}(-x) \approx \frac{1}{\Gamma(\beta)} + \frac{1}{\Gamma(\beta+1)} x + \frac{1}{\beta(\beta-1)} x^2.$$  \hspace{1cm} (21)

As an example, for $\beta = 1$, we know that

$$E_{1,1}(-x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1-\exp(-x)}{x}, & \text{if } x > 0, \end{cases}$$

and its global Padé approximations with degree 2:

$$E_{1,2}(-x) \approx \frac{1 + \frac{1}{2} x}{1 + x + \frac{1}{2} x^2}.$$  

The evolutions of the generalized Mittag-Leffler function $E_{1,2}(-x)$ and its global Padé approximation with degree 2 are drawn in Figure 4. In this situation, the maximum error is 0.0352 over all positive $x$.

More precisely, we collect the above global Padé approximations with degree $\nu = 2$ of the generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$, $x \in [0, +\infty)$ for particular cases, respectively. See Table 1 for more details.
3. Global Padé approximations of $L_{\alpha,\beta}(x)$

Hifer and Seybold [16] introduced and studied the inverse generalized Mittag-Leffler function $L_{\alpha,\beta}(x)$ defined as the solution of the equation

$$L_{\alpha,\beta}(E_{\alpha,\beta}(x)) = x.$$ 

In the complex plane, the authors have succeeded in determining the principal branch of $L_{\alpha,\beta}(x)$ provided that three conditions are satisfied: (1) the function $L_{\alpha,\beta}(x)$ is single valued and well defined on its principal branch; (2) its principal branch reduces to the principal branch of the logarithm for $\alpha \to 1$; (3) its principal branch is a simply connected subset of the complex plane.

Spencer et al. [26] provided also the asymptotic formulas for the zeros of the function $L_{\alpha,\beta}(x)$. In particular, Atkinson and Osseiran [21] got a global rational approximation of the inverse Mittag-Leffler function $-L_{\alpha,1}(x) = -L_{\alpha}(x)$ with $x \in (0, 1]$ and $0 < \alpha < 1$.

It is obvious from the complete monotonicity of the generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$, described in [3] for $x \in [0, +\infty)$, that $E_{\alpha,\beta}(-x)$ is a decreasing and continuous function on the interval $[0, +\infty)$. This means that the corresponding inverse Mittag-Leffler function $-L_{\alpha,\beta}(x)$ is well defined on the interval $(0, 1/\Gamma(\beta)]$ using these properties and the fact that $E_{\alpha,\beta}(0) = 1/\Gamma(\beta)$ by definition.

Our ability to calculate the global Padé approximation of $E_{\alpha,\beta}(-x)$ allows us to evaluate also the inverse generalized Mittag-Leffler function $-L_{\alpha,\beta}(x)$. In fact, the uniform approximations [13], [18], and [21] can be inverted to yield a global Padé approximation of the inverse Mittag-Leffler function $-L_{\alpha,\beta}(x)$ for particular cases \{0 < $\alpha < 1, \beta > \alpha$\}, \{0 < $\alpha = \beta < 1$\}, and \{$\alpha = 1, \beta > 1$\}.

For the case \{0 < $\alpha < 1, \beta > \alpha$\}, by rearranging the approximation [13], solving the resulting quadratic equation, and rearranging, we find that a global Padé approximation of $-L_{\alpha,\beta}(x)$ with degree 2:

$$-L_{\alpha,\beta}(x) \approx \frac{1}{2\Gamma(\beta - \alpha)x} - \frac{q_1}{2} + \sqrt{\left(\frac{q_1}{2} - \frac{1}{2\Gamma(\beta - \alpha)x}\right)^2 - q_0 \left(1 - \frac{1}{\Gamma(\beta)x}\right)}, \quad (22)$$
where \(-L_{\alpha,\beta}(x)|_{x\to 0^+} = +\infty\) and \(-L_{\alpha,\beta}(x)|_{x\to 1/\Gamma(\beta)} = 0\).

When \(\beta = 1\), the approximation (22) reduces to

\[-L_{\alpha}(x) \approx \frac{1}{2\Gamma(1-\alpha)x} - \frac{q_1^2}{2} + \sqrt{\left(\frac{q_1^2}{2} - \frac{1}{2\Gamma(1-\alpha)x}\right)^2 - \frac{q_0}{\Gamma(1-\alpha)}\left(1 - \frac{1}{x}\right)}, \tag{23}\]

where \(-L_{\alpha}(x)|_{x\to 0^+} = +\infty\) and \(-L_{\alpha}(x)|_{x\to 1} = 0\).

For the case \(\{0 < \alpha = \beta < 1\}\), by rearranging the approximation (18), solving the resulting quadratic equation, and rearranging, we find that a global Padé approximation of \(-L_{\alpha,\alpha}(x)\) with degree 2:

\[-L_{\alpha,\alpha}(x) \approx -\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)x} + \sqrt{\frac{\Gamma(1-\alpha)^2}{2\Gamma(1-2\alpha)^2x^2} - \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)}\left(1 - \frac{1}{\Gamma(\alpha)x}\right)}, \tag{24}\]

where \(-L_{\alpha,\alpha}(x)|_{x\to 0^+} = +\infty\) and \(-L_{\alpha,\alpha}(x)|_{x\to 1/\Gamma(\alpha)} = 0\).

For the case \(\{\alpha = 1, \beta > 1\}\), by rearranging the approximation (21), solving the resulting quadratic equation, and rearranging, we find that a global Padé approximation of \(-L_{1,\beta}(x)\) with degree 2:

\[-L_{1,\beta}(x) \approx \frac{1}{2\Gamma(\beta-1)x} - \beta + 1 + \sqrt{\left(\beta - 1 - \frac{1}{2\Gamma(\beta-1)x}\right)^2 - \beta(\beta - 1)\left(1 - \frac{1}{\Gamma(\beta)x}\right)}, \tag{25}\]

where \(-L_{1,\beta}(x)|_{x\to 0^+} = +\infty\) and \(-L_{1,\beta}(x)|_{x\to 1/\Gamma(\beta)} = 0\).

More precisely, we collect the above global Padé approximations with degree \(\nu = 2\) of the inverse generalized Mittag-Leffler function \(-L_{\alpha,\beta}(x)\), \(x \in (0, 1/\Gamma(\beta)]\) for particular cases, respectively. See Table 1 for more details.

4. Concluding discussions

On the basis of the Taylor series and the asymptotic series of generalized Mittag-Leffler function, we have constructed a global Padé approximation of the function \(E_{\alpha,\beta}(-x)\) with \(x \in [0, +\infty)\) for the particular cases \(\{0 < \alpha < 1, \beta > \alpha\}\), \(\{0 < \alpha = \beta < 1\}\), and \(\{\alpha = 1, \beta > 1\}\), respectively. These uniform approximations pick up the initial exponential-type behavior of the generalized Mittag-Leffler function as well as its asymptotic power laws for large arguments. Moreover, these approximations were inverted to yield a global Padé approximation of the inverse generalized Mittag-Leffler function \(-L_{\alpha,\beta}(x)\).
Table 1: Global Padé approximations with degree \( \nu = 2 \) of the generalized Mittag-Leffler function and its inverse

| Parameters | Function | Global Padé approximation |
|------------|----------|---------------------------|
| \( 0 < \alpha < 1, \beta > \alpha \) | \( E_{\alpha,\beta}(-x) \) | \( \frac{\frac{1}{\Gamma(\beta)} + \frac{1}{1-\alpha}x}{\frac{1}{\Gamma(\beta)} + \frac{1}{1-\alpha}x + \frac{1}{q_0}x^2} \) |
| \(-L_{\alpha,\beta}(x)\) | \( \frac{\frac{1}{2!}(\beta-\alpha)x}{\frac{1}{2!}(\beta-\alpha)x + \frac{1}{q_0}x^2} \) | \( \frac{\frac{1}{\Gamma(\beta)} + \frac{1}{\beta-1}x}{\frac{1}{\Gamma(\beta)} + \frac{1}{\beta-1}x + \frac{1}{q_0}x^2} \) |
| \( 0 < \alpha < 1, \beta = 1 \) | \( E_{\alpha}(-x) \) | \( \frac{\frac{1}{\Gamma(\beta-1)} + \frac{1}{1-\alpha}x}{\frac{1}{\Gamma(\beta-1)} + \frac{1}{1-\alpha}x + \frac{1}{q_0}x^2} \) |
| \(-L_{\alpha}(x)\) | \( \frac{\frac{1}{2!}(1-\alpha)x}{\frac{1}{2!}(1-\alpha)x + \frac{1}{q_0}x^2} \) | \( \frac{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x}{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x + \frac{1}{q_0}x^2} \) |
| \( 0 < \alpha = \beta < 1 \) | \( E_{\alpha,\alpha}(-x) \) | \( \frac{\frac{1}{\Gamma(\beta-1)} + \frac{1}{1-\alpha}x}{\frac{1}{\Gamma(\beta-1)} + \frac{1}{1-\alpha}x + \frac{1}{q_0}x^2} \) |
| \(-L_{\alpha,\alpha}(x)\) | \( \frac{\frac{1}{1!}(1-\alpha)x}{\frac{1}{1!}(1-\alpha)x + \frac{1}{q_0}x^2} \) | \( \frac{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x}{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x + \frac{1}{q_0}x^2} \) |
| \( \alpha = 1, \beta > 1 \) | \( E_{1,\beta}(-x) \) | \( \frac{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x}{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x + \frac{1}{q_0}x^2} \) |
| \(-L_{1,\beta}(x)\) | \( \frac{\frac{1}{2!}(\beta-1)x}{\frac{1}{2!}(\beta-1)x + \frac{1}{q_0}x^2} \) | \( \frac{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x}{\frac{1}{\Gamma(\beta-1)} + \frac{1}{\beta-1}x + \frac{1}{q_0}x^2} \) |

Note: the values of \( q_0, q_1, q_0^\ast \) and \( q_1^\ast \) are listed in [12] and [15].
Atkinson and Osseiran \cite{21} reported the rational solution of time-fractional diffusion equation under Caputo definition by using the global Padé approximation of the Mittag-Leffler function \( E_{\alpha}(-x) \) with \( 0 < \alpha < 1 \). However, it do not suffice to derive the rational solution of the ordinary and partial time-fractional differential equations in the sense of Riemann-Liouville. To demonstrate the advantage of our established results, let us consider the following illustrative examples.

For example, an ordinary linear fractional differential equation reads
\[
0 D_t^\alpha f(t) + \lambda f(t) = 0, \quad t \geq 0, \quad \left[0 D_t^{-\alpha} f(t)\right]_{t=0} = C_1,
\]
where \( 0 < \alpha < 1, \lambda > 0, \) and \( 0 D_t^\alpha \) is the Riemann-Liouville fractional derivative of order \( \alpha \). It admits the following solution \cite{27}
\[
f(t) = C_1 t^{-\alpha} E_{\alpha,\alpha}(-\lambda t^\alpha).
\]
Using the approximation \cite{13}, we have the global Padé approximation with degree 2:
\[
f(t) \approx \frac{C_1}{\Gamma(\alpha)t^\alpha + \frac{2\Gamma(1-\alpha)^2}{\alpha^2(1-2\alpha)} t^{2\alpha} + \frac{\lambda^2\Gamma(1-\alpha)}{t^{4\alpha}}.}
\]
We consider another example of the form
\[
0 D_t^\alpha g(t) + 0 D_t^\beta g(t) = \delta(t), \quad t \geq 0,
\]
where \( 0 < \alpha < \beta < 1 \) and \( \delta(t) \) stands for the the Dirac delta function. It yields the following solution \cite{11}
\[
g(t) = (C_2 + 1) t^{\beta-1} E_{\beta-\alpha,\beta}(-t^{\beta-\alpha}),
\]
where \( C_2 = \left[0 D_t^\alpha g(t) + 0 D_t^\beta g(t)\right]_{t=0} \). Using the approximation \cite{13}, we have the global Padé approximation with degree 2:
\[
g(t) \approx \frac{C_2 + 1}{\Gamma(\beta)) t^{\beta-1} + \frac{C_2 + 1}{\Gamma(\alpha) q_0^1} t^{2\beta-1-\alpha} + \frac{1}{q_0^1} t^{2(\beta-\alpha)},
\]
where
\[
q_0^1 = \frac{\Gamma(\beta) \Gamma(2\beta-\alpha)}{\Gamma(\alpha) \Gamma(2\beta-\alpha)}, \quad q_1^1 = \frac{\Gamma(\beta) \Gamma(2\beta-\alpha)}{\Gamma(\alpha) \Gamma((2\beta-\alpha)^2)}.
\]
From the above discussions, our constructed global Padé approximations of the generalized Mittag-Leffler function are quite effective in constructing the rational solution to fractional differential equations across an infinite range of the argument.

Acknowledgements

This work was partly supported by the National Natural Science Foundation of China (No. 11301090, No. 11271139, No. 61104138), Guangdong Natural Science Foundation (No. S2011040001704). The authors thank Francesco Mainardi (Professor of Mathematical Physics, University of Bologna, Italy) for his critical comments and helpful suggestions.

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