Quadratic Models for Engineered Control of Open Quantum Systems

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We introduce a framework to model the evolution of a class of open quantum systems whose environments periodically undergo an instantaneous non-unitary evolution stage. For the special case of quadratic models, we show how this approach can generalise the formalism of repeated interactions to allow for the preservation of system-environment correlations. Furthermore, its continuous zero-period limit provides a natural description of the evolution of small systems coupled to large environments in negligibly perturbed steady states. We explore the advantages and limitations of this approach in illustrative applications to thermalisation in a simple hopping ring and to the problem of initialising a qubit chain via environmental engineering.

I. INTRODUCTION

Open quantum systems, which evolve coupled to one or more large environments, are a central concept for both the development of quantum technologies and investigation into the very foundations of quantum mechanics. In practice, any working quantum device can never be completely isolated from its environment, which makes these systems ubiquitous in experiments and applications. Depending on the context, environmental coupling can either be undesirable, causing dissipation and loss of coherence and entanglement [1, 2] or may provide a valuable resource for engineered control on quantum devices [3, 4].

Although in principle the dynamics of open quantum systems follows directly from unitary evolution of the system-environment pair according to the Schrödinger equation, a direct approach is impractical due to the computational complexity associated with the high dimensionality of any large environment’s Hilbert space. Whilst there is already a great wealth of frameworks and approximate methods for describing the evolution of open quantum systems [5–11], finding novel approaches which complement existing ones remains an important avenue of investigation [12]. Instead of coarse-graining the environmental degrees of freedom to reduce the complexity of the problem, an interesting indirect approach to take into account the effect of large quantum environments is the framework of repeated interactions: whereby the environment is modelled by an infinite collection of independent quantum systems which take turns interacting with the system of interest [13]. Although this prescription is only physically exact for the type of scenarios where the system of interest interacts with particle beams or with repeatedly-reset (measuring) devices [14], it can be more generally viewed as a model for a class of open environments whose behaviour is well approximated by tractable non-unitary dynamics [14–16].

The obvious limitation of the repeated interactions approach is that by construction it cannot apply when a considerable amount of system-environment correlations persists for significant periods of time. Overcoming this limitation should enable progress in our understanding of fundamental problems such as the emergence of thermalisation [17], as well as technological challenges such as fast qubit initialisation protocols [18–21].

In this work, we introduce a more general approach to modelling non-unitary effects of large environments, based on periodically resetting the state of the environment – which reduces to repeated interactions in one of two physically relevant limits on which we focus here. Crucially, the other relevant limit of this technique allows for system-environment correlations to evolve freely. We show how exact results can be analytically obtained, for finite times between iterations and in the continuous limit, for the special case of quadratic Hamiltonian models (in which repeated interactions can be solved [22–24]). In Section II we review some well-known results on quadratic models and explore in what sense they can be said to thermalise. In Section III we introduce the resetting procedure for quadratic models, and show how it can be analytically solved (up to a system of linear equations). In Section IV we illustrate this approach by applying it to specific situations of interest, while comparing the repeated interactions limit to one in which system-environment correlations evolve freely. Finally, in Section V we take stock of the advantages and limitations of this approach, and discuss future directions for its applications.

II. QUADRATIC MODELS

A. Exact solutions

In this work we focus on quadratic models, described by Hamiltonians of the form

$$\hat{H} = \sum_{\alpha \beta} \hat{a}_\alpha \hat{M}_{\alpha \beta} \hat{a}_\beta,$$  \hspace*{1cm} (1)

where the indices $\alpha$ and $\beta$ run between 0 and some (large) integer $N - 1$; $\hat{a}_\alpha$ and $\hat{a}_\beta$ denote either fermionic or bosonic creation and annihilation operators, respectively;
and $M_{\alpha\beta}$ are the elements of a Hermitian square matrix. This Hamiltonian can be thought of as modelling particle hopping in a lattice (the topology of which is set by the connectivity of $M_{\alpha\beta}$), and in the fermionic case can also be mapped (by a Jordan-Wigner transformation [25]) into a quadratic hard-core boson Hamiltonian of the form

$$\hat{H} = \sum_{\alpha\beta} \hat{b}_\alpha^\dagger M_{\alpha\beta} \hat{b}_\beta,$$

(2)

where $\hat{b}_\alpha^\dagger$ and $\hat{b}_\beta$ respectively denote creation and annihilation operators satisfying both the canonical commutation relation $[\hat{b}_\alpha^\dagger, \hat{b}_\beta] = \delta_{\alpha\beta}$ and the hard-core condition $(\hat{b}_\alpha^\dagger)^2 = (\hat{b}_\beta)^2 = 0$.

Such models, by virtue of their quadratic form and of the canonical commutation relations their operators obey, are exactly and straightforwardly solvable as shown in detail in Appendix A. In particular, states are fully determined by $\{\hat{a}_\alpha^\dagger \hat{a}_\beta\}$, whose time evolution is fully determined by $M_{\alpha\beta}$. Importantly, many-body expectation values involving more than two operators $a_\alpha, a_\beta^\dagger$ are also uniquely defined from the single-particle density matrix via Wick’s theorem (see e.g. [26]).

### B. Pseudo-thermalisation in open quadratic systems

Quadratic models present a unique opportunity for the study of open quantum systems. Interpreting $\hat{a}_\alpha^\dagger$ as creating a particle with a well-defined position at the $(\alpha + 1)^{\text{th}}$ site in a lattice, we can refer to a small (typically contiguous) part of this lattice as belonging to a system ($\alpha \in \mathcal{S}$) and to the rest of it as being in an environment ($\alpha \in \mathcal{E}$). Therefore we can treat this small system as a model of an open system as long as we can solve the quadratic problem for a sufficiently large environment. Owing to the simple dynamics of quadratic models as well as to it being possible to encode the state of these systems in a relatively small $N \times N$ matrix (the single-particle density matrix; as opposed to requiring a $2^N \times 2^N$ density matrix in the fermionic case, and a potentially infinite one in the bosonic case), this approach permits us to computationally reach larger environment sizes than using more complex models.

It is well known [27] that most integrable models (of which quadratic models are a simple example) equilibrate after long times under unitary evolution in the thermodynamic limit – i.e., time-averaged quantities tend to a constant in the infinite future. In fact, for all quadratic models without specific relations between energy levels [28–30], the time-averaged single-particle density matrix in this limit reduces to the diagonal ensemble result

$$\bar{\rho}_{\alpha\beta} \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_{\alpha\beta}(t) \, dt = \sum_{\alpha} \psi^*_{\alpha n} \psi_{n\beta} \bar{\rho}_{nn}(0) = \sum_{\alpha\beta} \psi^*_{\alpha n} \psi_{n\beta} \psi^*_{\beta n} \rho_{\alpha\beta}(0),$$

(3)

where $\psi_{n\alpha}$ is defined in Eq. (A1) and in which $n$ indexes Hamiltonian eigenstates (see Eq. (A2) and the discussion surrounding it). This corresponds to dropping all off-diagonal entries in the initial single-particle density matrix in the (single-particle) energy eigenbasis, $\bar{\rho}_{nm}$ (see Eq. (A5)).

Generally, we are interested in the long-time expectation values of operators whose support is contained in the system (which can be expressed as combinations of terms involving $\bar{\rho}_{\alpha\beta}$ for $\alpha, \beta \in \mathcal{S}$). In the more realistic case of non-integrable models, the generic expectation [17] is that such operators not just equilibrate in this sense [31] but also thermalise in such a way that, in the thermodynamic limit (i.e., when $\mathcal{E}$ is arbitrarily large), $\bar{\rho}_{\alpha\beta}$ will generically be arbitrarily close to a thermal (Gibbs) state. For integrable (and in particular free, such as here) systems, this is not the case [32] unless a special initial state is chosen. As such, one cannot generally speak of thermalisation occurring in quadratic models for open systems.

Nevertheless, we can sometimes speak of the occurrence of a weaker phenomenon, which we call pseudo-thermalisation: when, in the thermodynamic limit, the long-time state of the system (i.e., the section of $\psi^{\dagger}(\mathcal{E})$ corresponding only to system correlations) does not depend on the initial state of the system. This is the case that we will in fact consider in the present work. From Eq. (3), it is straightforward to verify that pseudo-thermalisation takes place whenever

$$\lim_{N \to \infty} \sum_n \psi^*_{n\alpha} \psi_{n\beta} \psi_{n\gamma} \psi^*_{n\delta} = 0 \ \forall \alpha, \beta, \gamma, \delta \in \mathcal{S},$$

(4)

where the limit for large $N$ is imposed by the thermodynamic limit [33].

Since the eigenvectors $\psi_{n\alpha}$ are orthonormal by construction, most models will obey Eq. (4) [34]. Intuitively, this follows from the expectation that a typical element of such orthonormalised eigenvectors will have a magnitude of order $O\left(1/\sqrt{N}\right)$, and thus the sum in Eq. (4) will be of order $O(1/N) \to 0$. In particular, any translation-invariant quadratic model must pseudo-thermalise in this way [35] (unless its energy levels are fine-tuned to prevent this [36]). Nevertheless, there are several couterexamples, such as models with broken translational invariance leading to localised states (for example in Anderson localization) and thus a finite term in Eq. (4).
III. RESETTING PROCESS

Ideally we would like to find settings in which open quadratic systems pseudo-thermalise more generically. Bearing in mind that most real-world systems are not integrable (and certainly not quadratic), we focus on situations where our quadratic environment is connected to some more complicated super-environment. In order to circumvent the mathematical difficulties that would come with actually calculating the evolution of such a complex system, the effect of this super-environment on our quadratic universe can be captured by some class of non-unitary evolution. Naturally, the exact form of this non-unitary evolution depends on the exact nature of the super-environment and its coupling with the quadratic sector.

A well-motivated example of such non-unitary evolution is found in the framework of repeated interactions [13–16], where, in between periods of purely unitary evolution, the environment is periodically reset to its initial state. This comes about when the system interacts with periodically externally (and non-unitarily) reset devices (when the super-environment is made up of whatever mechanism is being used to reset the device) or when it periodically comes into contact with identical copies of the same environment (when the super-environment is made up of all the copies of the environment that are not currently interacting with the system [37]).

In this section we expand this framework to allow for more general resetting processes; which may, for example, preserve correlations between system and environment. In what follows, we consider a quadratic model as described in Section II, which we can generally interpret as relating to particles in a lattice which can be divided into a (small) system $S$ and an environment $E$ (as discussed in Subsection II B). This model is allowed to evolve unitarily for periods of duration $\tau$ separated by instants in which some elements of the full single-particle density matrix are reset to their initial values.

Mathematically, the only restriction on which elements can be reset and on their values after resetting is that this process should not violate the Hermiticity of $\rho$. Physically, these choices are determined by exactly what physical process one seeks to model. In this work, we will focus especially on two physically relevant examples: repeated interactions and the case of environments which periodically and (approximately) instantaneously thermalise with some arbitrarily large super-environment. In the former, only the system-system block of $\rho$ is not reset (and the system-environment blocks must be reset to zero). Conversely, in the latter only the environment-environment block is reset (to some appropriate thermal distribution).

A. Evolution and long-time limit

The unitary part of the evolution of the single-particle density matrix is simply given by Eq. (A8). Defining $R$ as the set of pairs of indices whose corresponding entries in $\rho$ get periodically reset, the evolved single-particle density matrix after one time interval $\tau$ is thus

$$\rho_{\alpha\beta}(t + \tau) = \sum_{\alpha'\beta'} U^*_{\alpha\alpha'}(\tau) U_{\beta\beta'}(\tau) \rho_{\alpha'\beta'}(t) \tag{5}$$

for $(\alpha, \beta) \notin R$ ($U$ being a matrix representation of the evolution operator – see Eq. (A9)) and

$$\rho_{\alpha\beta}(t + \tau) = \rho_{\alpha\beta}(t) \tag{6}$$

for $(\alpha, \beta) \in R$.

Given the periodic nature of this setting, we focus on the stroboscopic evolution of the model at discrete times of the form $t = n\tau$ (for integer $n$). Then we can simply ignore the evolution of those elements $\rho_{\alpha\beta}$ for which $(\alpha, \beta) \in R$ and organise the remaining entries in a state vector

$$V_i[n] \equiv \rho_{\alpha_i\beta_i}(n\tau), \quad (7)$$

where $(\alpha_i, \beta_i) \notin R$ is a sequence that covers each pair of indices not in $R$ exactly once. Eq. (5) can then be rewritten as the discrete evolution equation

$$V_i[n + 1] = \sum_j D_{ij} V_j[n] + C_i, \quad (8)$$

where we have introduced the square (non-Hermitian) matrix

$$D_{ij} = U^*_{\alpha_i\alpha_j}(\tau) U_{\beta_i\beta_j}(\tau) \quad (9)$$

and the constant vector

$$C_i = \sum_{(\alpha', \beta') \in R} U^*_{\alpha_i\alpha'}(\tau) U_{\beta_i\beta'}(\tau) \rho_{\alpha'\beta'}(0). \quad (10)$$

The matrix $D$ only depends on the model’s Hamiltonian (through Eqs. (9) and (A9)), whereas the constant vector $C$ additionally depends on the manner in which resetting is implemented (through $\rho_{\alpha'\beta'}(0)$ in Eq. (10)). We therefore intuitively expect that the rate of convergence to any attractor will be mostly set by properties of the Hamiltonian (which makes no reference to the super-environment), whilst the state to which $R$ is reset will play a role in shifting these attractors away from the null vector [38].

In Appendix B we show that there is usually a unique and attractive fixed point of Eq. (8), so that at long times the system pseudo-thermalises, completely forgetting its initial state.
B. High-frequency resetting and the continuous resetting limit

One interesting limit for this procedure is when \( \tau \) is much smaller than all other relevant characteristic time scales – particularly as \( \tau = 0 \) corresponds to the problem of the continuous evolution of a small system coupled to an environment in a static state (which in particular can be taken to be a Gibbs state).

In this continuous limit, the repeated interactions framework becomes equivalent to the Born approximation which is typically the starting point for the weak coupling limit. This approximation states that the density matrix for the universe can be approximately expressed as a direct product between the system and the environment reduced density matrices, with the latter being constant. Therefore, the case we consider in which the environment is kept in a constant (thermal) state while allowing for evolving system-environment correlations can be seen as describing a less stringent version of the weak coupling limit.

In this limit, we can expand the evolution operator to leading order in \( \tau \), yielding

\[
U_{\alpha\beta}(\tau) = \langle \alpha | \left( 1 - i\frac{\hat{H}_\tau}{\hbar} + \mathcal{O}(\tau^2) \right) | \beta \rangle \\
= \delta_{\alpha\beta} - i\frac{\tau}{\hbar} M_{\alpha\beta} + \mathcal{O}(\tau^2),
\]

(11)

Analogously to the discrete case (see Appendix C), the continuous time evolution can then be written in the form

\[
\frac{dV_j}{dt}(t) = \sum_j D_{ij} V_j(t) + C_i,
\]

(12)

where the quantities \( V, D \) and \( C \) are continuous versions of \( V, D \) and \( C \) in Eq. (8). Mutatis mutandis, one can then show (as in Appendix C) that in this continuous limit the system also typically possesses a single (attractive) steady state, thus pseudo-thermalising.

IV. EXAMPLES OF APPLICATION

A. Thermalisation

We shall now focus on situations in which the environment is reset to (or kept at) a thermal state, which offer us relatively simple physical interpretations. Where system-environment correlations are reset to zero (i.e., in the repeated interactions limit) this corresponds to the system interacting with consecutive identical independent environments at the same temperature. Where such correlations are allowed to evolve, this corresponds to a situation where the environment is actually part of a bigger "super-environment" with which it periodically interacts and thermalises much faster than any other relevant time scales. In the continuous limit, the latter can be seen as an approximate model of a system interacting with a thermal environment which isn’t noticeably perturbed by this interaction.

Models for which these interactions lead to thermal attractors are of particular interest as they can be taken as toy models for thermalisation processes. Although that is not a generic property of this type of resetting procedure, we are able to find and study a few simple cases in which this takes place.

The simplest example is when the environment is repeatedly reset to an infinite temperature state, which generically induces the system to settle at a thermal attractor state with infinite temperature in both scenarios we consider (as shown in Appendix D).

This result suggests that if one is interested in finding scenarios in which quadratic models thermalise with an environment at finite temperatures, then a high-temperature limit is a natural regime to look into. As thermalisation is not a generic feature of these models away from \( \beta = 0 \), we consider the special case of one of the simplest and most well-known quadratic models: a fermionic periodic hopping ring, whose Hamiltonian is given by

\[
\hat{H} = -J \sum_{\ell=0}^{N-1} \left( \hat{a}_\ell \hat{a}_{\ell+1} + \hat{a}_{\ell+1}^\dagger \hat{a}_\ell^\dagger \right),
\]

(13)

where \( J \) is the hopping amplitude and \( \hat{a}_{\ell+N} \equiv \hat{a}_\ell \).

To illustrate our results we focus on a ring with \( N = 100 \) where the system is defined as a segment of eight consecutive sites, in a scenario with \( \tau = 0.01 \hbar/J \) so as to comfortably be in the high-frequency limit [39]. The evolution operator corresponding to Eq. (13) (which can be straightforwardly computed using Eq. (A9)) can be used in Eq. (9) and Eq. (10) to determine the components of \( D \) and \( C \), respectively; which in turn can be used in Eq. (B6) (following an appropriate change of basis) to yield the elements of the system’s steady-state single-particle density matrix, \( \rho^S_{\alpha\beta} \).

If the steady state of the system is approximately thermal, we should expect the corresponding single-particle density matrix to be consistent with a Fermi-Dirac distribution: being approximately diagonal in the system Hamiltonian’s eigenbasis, with diagonal entries in said basis approximately corresponding to

\[
\rho^S_{\alpha\alpha} = \frac{1}{e^{\beta E_\alpha^S} + 1},
\]

(14)

where \( \beta = 1/k_B T \) is the inverse thermal energy (\( k_B \) standing for the Boltzmann constant and \( T \) for the temperature) and \( E_\alpha^S \) are eigenvalues of the system Hamiltonian.

Figure 1 contrasts the sums of (the modulus of) the diagonal and off-diagonal entries in \( \rho^S_{\alpha\beta} \) for the case of repeated interactions and evolving system-environment correlations, as functions of the temperature of the thermal state to which the environment is reset, \( \beta \). Even for
relatively high values of $\beta$ (low temperatures), we can see that the weight of off-diagonal elements is negligible (especially for repeated interactions), consistently with a thermal steady state. As expected from the exact result for $\beta = 0$, the higher the environment temperature the more negligible off-diagonal elements are (although there appears to be a residual constant contribution of these elements for low $\beta$, which should come from numerical errors dealing with very small numbers).

To verify the thermal nature of these steady states, it is necessary to further show that the diagonal elements follow Eq. (14). This can be done by computing an effective inverse Boltzmann temperature $\beta_\alpha$ corresponding to each diagonal entry of the steady-state single-particle density matrix in the system Hamiltonian’s eigenbasis — as calculated from Eq. (B6) for a system of eight sites in a 100-sites-long hopping ring coupled to an environment reset to a thermal state at temperature $\beta$ with period $\tau = 0.01$, both in the repeated interactions regime (RI, dashed) and allowing system-environment correlations to evolve (EC, solid). Since an exactly thermal state has vanishing off-diagonal terms, the smallness of these terms is consistent with an approximately thermal attractor for all high temperatures and even some intermediate temperatures (or even low temperatures in the case of repeated interactions).

A curious feature of these plots is that they describe approximately thermal steady states above relatively low environment temperatures $\beta J \approx 0.1$, which assuming a coupling between sites of order $J/\hbar \approx 10$ MHz (e.g. as in Ref. [41] for qubit-qubit coupling) corresponds to a temperature of about $T \approx 100$ K. Whilst these temperatures are still too high to be of relevance for many of the current quantum technology components, such as superconducting qubits, it is remarkable that our simple model seems able to describe thermalisation for a fairly wide range of temperatures which include room temperature. There are actually recent proposals of qubits that can operate at elevated temperatures up to ambient temperature [42].

### B. Spontaneous qubit initialisation

Initialising a chain of qubits at its ground state is a key example of a control problem of practical importance for quantum technologies, particularly for the implementation of quantum correction codes required for large-scale gate-based quantum computation [43–45]. One promising approach to this problem is to tune the coupling of this chain to a specially engineered dissipative environment so as to induce fast relaxation to the required state [19, 20]. Whilst studies of the speed and fidelity of such protocols have been carried out within a weak-coupling approximation, it has been shown that

![FIG. 2. Effective inverse Boltzmann temperatures $\beta_\alpha$ corresponding to each diagonal entry of the steady-state single-particle density matrix in the system Hamiltonian’s eigenbasis as calculated from Eq. (B6) and Eq. (15) for a system of eight sites in a 100-sites-long hopping ring coupled to an environment reset to a thermal state at temperature $\beta$ with period $\tau = 0.01$, both in the repeated interactions regime (RI, blue) and allowing system-environment correlations to evolve (EC, red). Small differences between related lines indicate a Fermi-Dirac distribution of diagonal elements, consistently with a thermal steady state.](image-url)
system-environment interactions play an important role that warrants more accurate analysis [18]. Therefore, where these environments are to be kept at thermal states throughout this procedure, our method (in the continuous limit) may help to refine studies of their performance.

To completely address this goal is beyond the scope of this paper. However, we can illustrate how this technique is relevant to such studies by applying it to a simplified version of the problem. As such, in what follows we merely ask whether, given a system of linearly coupled qubits, there exists an environment composed of linearly coupled qubits to which it can be linearly coupled so that, if the environment is maintained at some undefined state, the system evolves towards a pure state where all qubits are in their ground state.

Modelling qubits as lattice sites populated by spinless fermions (or, equivalently, hard-core bosons [25]), whose two eigenstates correspond to the presence or absence of a particle, the Hamiltonian that captures the dynamics in which we are interested can be written as in Eq. (2).

Then the Jordan-Wigner transformation defined by

\[
\hat{b}_\alpha^\dagger = \prod_{\beta < \alpha} \exp \left( \hat{a}_\beta^\dagger \hat{a}_\beta \right) \hat{a}_\alpha^\dagger,
\]

transforms this Hamiltonian into the one in Eq. (1) in the case where \( \hat{a} \) is a fermionic operator. Moreover, if system indices are chosen to always be lower than environment indices, this implies that states where the system is empty correspond to states where the system is empty. We therefore need only apply the formalism described above to look for attractor solutions corresponding to the system being empty.

In the language of Eq. (C4), we want to find settings where \( V_i = 0 \) \( \forall i \) is an attractor; which is equivalent to finding settings where \( C_i = 0 \) \( \forall i \). Without resetting system-environment correlations, Eq. (C6) can be written as

\[
C_i = \frac{i}{\hbar} \sum_{(\alpha', \beta') \in \mathcal{E}^2} \rho_{\alpha', \beta'}(0) \left( M_{\alpha', \alpha}^* \delta_{\beta', \beta} - \delta_{\alpha', \alpha} M_{\beta', \beta} \right).
\]

Here at least one term inside parenthesis must always be zero (as \( \alpha_i \) and \( \beta_i \) cannot both be environment indices simultaneously), and therefore we have

\[
C_i = \begin{cases} 
\frac{i}{\hbar} \sum_{\alpha' \in \mathcal{E}} \rho_{\alpha', \alpha}(0) M_{\alpha', \alpha}^* & \text{if } \alpha_i \in \mathcal{S} \land \beta_i \in \mathcal{E}; \\
\frac{i}{\hbar} \sum_{\beta' \in \mathcal{E}} \rho_{\alpha, \beta'}(0) M_{\beta', \beta} & \text{if } \alpha_i \in \mathcal{E} \land \beta_i \in \mathcal{S}; \\
0, & \text{c.c.}
\end{cases}
\]

Since \((\alpha_i, \beta_i)\) runs over all possible pairs of indices outside of \( \mathcal{E}^2 \), we want to find when

\[
\sum_{\beta' \in \mathcal{E}} \rho_{\alpha, \beta'}(0) M_{\beta, \beta'} = 0 \forall \alpha \in \mathcal{E} \forall \beta \in \mathcal{S}.
\]

Choosing to work in the basis where the single-particle density matrix of the environment is diagonal (i.e., \( \rho_{\alpha, \beta'}(0) \equiv \delta_{\alpha, \beta'} \delta_{\alpha', \alpha} \)), this boils down to requiring that

\[
\rho_{\alpha, \beta'}(0) = 0 \forall \alpha \in \mathcal{E} \forall \beta \in \mathcal{S}.
\]

This means that the attractor can only be \( V_i = 0 \) \( \forall i \) if the resetting procedure empties all environment modes which connect to the system (i.e., \( n_{\alpha \in \mathcal{E}} = 0 \lor \exists \beta \in \mathcal{S} \in \mathcal{S} \)). In other words, the only way this procedure leaves the system exactly empty is if (in this basis) all nodes adjacent to the system are kept empty "by hand" – or equivalently (as can most easily be seen if those nodes’ indices are chosen to be the lowest in the environment), if all corresponding qubits that couple to the system are kept in their ground states. Therefore this simplified approach merely manages to shift the problem to the system’s boundary.

V. SUMMARY AND CONCLUSIONS

In this work we introduce a formalism to study a system in contact with environments that are periodically reset to a fixed arbitrary state, allowing for system-environment correlations to be preserved through the reset. We also study the special limit where this period vanishes, which describes an open system with time-independent environments. We obtain the conditions under which the system eventually loses memory of the initial state, even when the final state is not thermal.

We apply this formalism to the analysis of a simple system coupled to an environment which is brought to a thermal state at a fixed temperature. In this context we compare the limit where system-environment correlations are erased by the reset (equivalent to the formalism of repeated interactions) to the one where they are allowed to evolve freely; finding that both limits lead to thermal system steady states – although crucially not for the low temperatures which are relevant for existing quantum technologies. However, only when such correlations evolve freely does the system end up at the same temperature as the environment.

We further explore the possibilities opened up by this formalism by examining whether a specific environment could be devised that would drive the system to a particular desired state (here corresponding to an initialised chain of qubits). Whilst the answer for this specific example is negative, it illustrates how this approach can naturally contribute to more general problems in environment engineering.

Two possible extensions of this work immediately present themselves. Firstly, extending our approach to interacting systems, which would be required to take advantage of it in all but the simplest applications. The formal analysis is easily extended, but the system-environment separation would then be less transparent as it would be defined in a Fock space rather than a lattice space (probably requiring non-trivial mathematical mapping to tackle problems of interest). Secondly, as demonstrated in our second example, this approach may...
be useful for informing the design of protocols to lead systems to specific states of interest. This way of achieving it is appealing but it is unclear how general steady states can be achieved and what the limitations are. This then appears to be a fruitful direction to follow in future work.

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[1] F. Verstraete, M. Wolf, and J. Ignacio Cirac, Quantum computation and quantum-state engineering driven by dissipation, *Nature Phys* 5, 633 (2009).

[2] T. Yu and J. H. Eberly, Sudden death of entanglement, *Science* 323, 598 (2009), https://science.sciencemag.org/content/323/5914/598.full.pdf.

[3] M. Wiedmann, J. T. Stockburger, and J. Ankerhold, Non-markovian dynamics of a quantum heat engine: out-of-equilibrium operation and thermal coupling control, *New Journal of Physics* 22, 033007 (2020).

[4] W. H. Zurek, Quantum Darwinism, *Nature Physics* 5, 181 (2009), arXiv:0903.5082 [quant-ph].

[5] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, 2002).

[6] H.-P. Breuer, J. Gemmer, and M. Michel, Non-markovian quantum dynamics: Correlated projection superoperators and hilbert space averaging, *Phys. Rev. E* 73, 016139 (2006).

[7] H.-P. Breuer, B. Kappler, and F. Petruccione, Stochastic wave-function method for non-markovian quantum master equations, *Phys. Rev. A* 59, 1633 (1999).

[8] P. Haikka and S. Maniscalco, Non-markovian dynamics of a damped two-state system, *Phys. Rev. A* 81, 052103 (2010).

[9] S. Nakajima, On Quantum Theory of Transport Phenomena: Steady Diffusion, *Progress of Theoretical Physics* 20, 948 (1958), https://academic.oup.com/ptp/article-pdf/20/6/948/5440766/20-6-948.pdf

[10] A. Rivas and S. F. Huelga, *Open Quantum Systems: An Introduction*, SpringerBriefs in Physics. ISBN 978-3-642-23353-1. The Author(s), 2012.

[11] R. Zwanzig, Ensemble method in the theory of irreversibility, *The Journal of Chemical Physics* 33, 1338 (1960), https://doi.org/10.1063/1.1731409.

[12] S. Alipour, A. T. Rezakhani, A. P. Babu, K. Mølmer, M. Möttönen, and T. Ala-Nissila, Correlation-picture approach to open-quantum-system dynamics, *Phys. Rev. X* 10, 041024 (2020).

[13] S. Attal and Y. Pautrat, From repeated to continuous quantum interactions, *Ann. Henri Poincaré* 7, 59 (2006).

[14] S. Attal and C. Pellegrini, Stochastic Master Equations in Thermal Environment, arXiv e-prints, arXiv:1004.3559 (2010), arXiv:1004.3559 [math-ph].

[15] D. Grimmer, D. Layden, R. B. Mann, and E. Martín-Martínez, Open dynamics under rapid repeated interaction, *Phys. Rev. A* 94, 032126 (2016).

[16] P. Strasberg, G. Schaller, T. Brandes, and M. Esposito, Quantum and information thermodynamics: A unifying framework based on repeated interactions, *Phys. Rev. X* 7, 021003 (2017).

[17] S. Popescu, A. J. Short, and A. Winter, Entanglement and the foundations of statistical mechanics, *Nature Physics* 2, 754 (2006), arXiv:quant-ph/0511225 [quant-ph].

[18] J. Tuorila, J. Stockburger, T. Ala-Nissila, J. Ankerhold, and M. Möttönen, System-environment correlations in qubit initialization and control, *Phys. Rev. Research* 1, 013004 (2019).

[19] P. J. Jones, J. A. M. Huhtamäki, J. Salmilehto, K. Y. Tan, and M. Möttönen, Tunable electromagnetic environment for superconducting quantum bits, *Sci Rep* 3 (2013).

[20] J. Tuorila, M. Partanen, T. Ala-Nissila, and M. Möttönen, Efficient protocol for qubit initialization with a tunable environment, *npj Quantum Inf* 3 (2017).

[21] M. Silveri, H. Grabert, S. Masuda, K. Y. Tan, and M. Möttönen, Theory of quantum-circuit refrigeration by photon-assisted electron tunneling, *Phys. Rev. B* 96, 094524 (2017).

[22] S. Andréys, Repeated interaction processes in the continuous-time limit, applied to quadratic fermionic systems, *Ann. Henri Poincaré* 21, 115 (2020).

[23] D. Karevski and T. Platini, Quantum nonequilibrium steady states induced by repeated interactions, *Phys. Rev. Lett.* 102, 207207 (2009).

[24] T. Platini and R. J. Low, Quantum repeated interactions and the chaos game, *Journal of Physics A: Mathematical and Theoretical* 51, 395301 (2018).

[25] X.-G. Wen, *Parafermion Chain with 2π/k Floquet Edge Modes*, npj Quantum Inf 3 (2017).

[26] A. Lazarides, A. Das, and R. Moessner, Peri-od thermodynamics of isolated quantum systems, *Phys. Rev. Lett.* 112, 150401 (2014).

[27] M. A. Cazalilla, A. Iucci, and M. Chung, Thermalization and quantum correlations in exactly solvable models, *Phys Rev E Stat Nonlin Soft Matter Phys.* 85, 10.1103/PhysRevE.85.011133 (2012).

[28] V. Khemani, A. Lazarides, R. Moessner, and S. L. Sondhi, Phase structure of driven quantum systems, *Phys. Rev. Lett.* 116, 250401 (2016).

[29] G. J. Sreejith, A. Lazarides, and R. Moessner, Parafermion chain with 2π/k floquet edge modes, *Phys. Rev. B* 94, 045127 (2016).

[30] A. Lazarides and R. Moessner, Fate of a discrete time crystal in an open system, *Phys. Rev. B* 95, 195135 (2017).
[31] P. Reimann, Foundation of statistical mechanics under experimentally realistic conditions, Phys. Rev. Lett. 101, 190403 (2008).

[32] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Relaxation in a completely integrable many-body quantum system: An ab initio study of the dynamics of the highly excited states of 1d lattice hard-core bosons, Phys. Rev. Lett. 98, 050405 (2007).

[33] Mathematically, one can always interpret the thermodynamic limit as a continuous limit of such lattice models, for example by defining a sequence of Hamiltonians $\hat{H}[N]$ such that the size of the environment relative to the size of the system grows with increasing $N$ – and then the limit in Eq. (4) can be taken literally.

[34] I.e., most sets of orthonormal $\psi_{n\alpha}$ can be seen to verify this property – e.g., by checking the entries in normal matrices randomly generated by taking the imaginary exponential of uniformly randomly picked Hermitian matrices, or by QR-decomposing such Hermitian matrices.

[35] As $\Sigma_n |\psi_{n\alpha}|^2 = 1$ implies $|\psi_{n\alpha}| = 1/\sqrt{N}$ if all sites are equivalent. In the special case of single-site systems, the resulting relation $\Sigma_n |\psi_{n\alpha}|^4 = 1/N$ can additionally be shown to correspond to the fastest possible convergence to pseudo-thermalisation with increasing $N$.

[36] A notable such counterexample being the case of Bethe lattices, whose energy levels are highly degenerate.

[37] Technically, in this case, the definition of the environment depends on exactly which copy is currently interacting with the system.

[38] Note that the case where the null vector is an attractor solution to Eq. (8) is an interesting one from the point of view of environment engineering, as discussed in some detail in Subsection IV B.

[39] The smallest typical time scale in this ring being of order $\pi\hbar/J$. Note we deliberately avoid working in the continuous limit in this particular case due to additional mathematical complications arising from a null determinant in the result of Eq. (C5).

[40] I.e., evolving towards a thermal state at approximately the same temperature as the environment.

[41] S. A. Caldwell, N. Didier, C. A. Ryan, E. A. Sete, A. Hudson, P. Karalekas, R. Manenti, M. P. da Silva, R. Sinclair, E. Acala, N. Alidoust, J. Angeles, A. Bestwick, M. Block, B. Bloom, A. Bradley, C. Bui, L. Capeluto, R. Chilcott, J. Cordova, G. Crossman, M. Curtis, S. Deshpande, T. E. Bouayadi, D. Girshovich, S. Hong, K. Kuang, M. Lenihan, T. Manning, A. Marchenkov, J. Marshall, R. Maydra, Y. Mohan, W. O’Brien, C. Osborn, J. Otterbach, A. Papageorge, J.-P. Paquette, M. Pelstring, A. Polloreno, G. Prawiroatmodjo, V. Rawat, M. Reagor, R. Renzas, N. Rubin, D. Russell, M. Rust, D. Scarabelli, M. Scheer, M. Selvanayagam, R. Smith, A. Staley, M. Suska, N. Tezak, D. C. Thompson, T.-W. To, M. Vahidpouri, N. Vodrashalli, T. Whyland, K. Yadav, W. Zeng, and C. Rigetti, Parametrically activated entangling gates using transmon qubits, Phys. Rev. Applied 10, 034050 (2018).

[42] S. Arroyo-Camejo, A. Lazariev, S. W. Hell, and G. Balasubramanian, Room temperature high-fidelity holonomic single-qubit gate on a solid-state spin, Nature communications 5, 4870 (2014).

[43] A. G. Fowler, Two-dimensional color-code quantum computation, Phys. Rev. A 83, 042310 (2011).

[44] A. G. Fowler, A. M. Stephens, and P. Groszkowski, High-threshold universal quantum computation on the surface code, Phys. Rev. A 80, 052312 (2009).

[45] P. W. Shor, Scheme for reducing decoherence in quantum computer memory, Phys. Rev. A 52, R2493 (1995).

[46] $D$ must be diagonalisable at least for sufficiently small $\tau$ due to it being Hermitian for $\tau = 0$ and continuous in $\tau$. 
Appendix A: Quadratic models

In general, quadratic models described by Eq. (1) can be exactly solved following a change of variables

$$\hat{a}_\alpha = \sum_n \psi_{n\alpha} \tilde{c}_n, \quad (A1)$$

where $\psi_{n\alpha}$ is the $\alpha^{th}$ element of the $n^{th}$ eigenvector of the matrix $M$ (chosen so that $\sum_\beta M_{\alpha\beta} \psi_{n\beta} = \varepsilon_n \psi_{n\alpha}$ for some eigenvalue $\varepsilon_n$ and $\psi_{n\alpha}$ defines a unitary matrix), after which the Hamiltonian displays the simple diagonal form

$$\hat{H} = \sum_n \varepsilon_n \tilde{c}_n \tilde{c}_n^\dagger. \quad (A2)$$

Owing to the unitarity of $\psi_{n\alpha}$, Eq. (A1) can be inverted to yield

$$\tilde{c}_n = \sum_\alpha \psi_{n\alpha}^* \hat{a}_\alpha, \quad (A3)$$

where $\tilde{c}_n^\dagger$ and $\tilde{c}_n$ are, respectively, creation and annihilation operators of the same kind as their counterparts in Eq. (1) — and therefore evolution according to the Hamiltonian in Eq. (A2) can be linearly mapped to the dynamics of $N$ free particles of energies given by $\varepsilon_n$.

The state of a system described by the Hamiltonian in Eq. (1) is fully determined by the single-particle density matrix $\rho_{\alpha\beta}$, which can be written as

$$\rho_{\alpha\beta} \equiv \langle \hat{a}_\alpha^\dagger \hat{a}_\beta \rangle = \sum_{nm} \psi_{n\alpha}^* \psi_{m\beta} \tilde{\rho}_{nm}, \quad (A4)$$

where $\tilde{\rho}_{nm}$ is the single-particle density matrix defined with respect to the diagonalised Hamiltonian in Eq. (A2),

$$\tilde{\rho}_{nm} \equiv \langle \tilde{c}_n^\dagger \tilde{c}_m \rangle = \sum_{\alpha\beta} \psi_{n\alpha} \psi_{m\beta}^* \rho_{\alpha\beta}. \quad (A5)$$

The unitary dynamics of $\tilde{\rho}_{nm}$ is just given by

$$\tilde{\rho}_{nm} (t) = \tilde{\rho}_{nm} (0) e^{i(\varepsilon_n - \varepsilon_m) t / \hbar}, \quad (A6)$$

where $t$ represents time and $\hbar$ is the reduced Planck constant. Changing back to the original variables this translates to the relation

$$\rho_{\alpha\beta} (t) = \sum_{\alpha'\beta'} \rho_{\alpha'\beta'} (0) \sum_{nm} \psi_{n\alpha}^* \psi_{m\beta} \psi_{n\alpha'} \psi_{m\beta'}^* e^{i(\varepsilon_n - \varepsilon_m) t / \hbar}, \quad (A7)$$

which can be equivalently written as

$$\rho_{\alpha\beta} (t) = \sum_{\alpha'\beta'} U_{\alpha\alpha'} (t) U_{\beta'\beta} (t) \rho_{\alpha'\beta'} (0), \quad (A8)$$

where we have introduced the components of the evolution operator

$$U_{\alpha\beta} (t) \equiv \langle \alpha | \hat{U} (t) | \beta \rangle \equiv \langle \alpha | e^{-i\hat{H}t / \hbar} | \beta \rangle = \left( e^{-iEt / \hbar} \right)_{\alpha\beta} = \sum_n e^{-i\varepsilon_n t / \hbar} \psi_{n\alpha} \psi_{n\beta}^*, \quad (A9)$$

$|\alpha\rangle \equiv \hat{a}_\alpha^\dagger |0\rangle$ being an element of this system’s Fock state, and $|0\rangle$ its vacuum.

Appendix B: Fixed point of Eq. (8)

Equation (8) can always be solved if the matrix $P$ that diagonalises $D$ (i.e., such that $\sum_{k} P_{ik} D_{k\alpha} P_{\alpha\gamma}^{-1} = \lambda_i \delta_{\gamma\alpha}$, where $\lambda_i$ are the eigenvalues of $D$) is known [46]. Changing to the eigenbasis of $D$,

$$\tilde{V}_i [n + 1] = \sum_{j} P_{ij} V_j [n] + C_k,$$

and straightforwardly manipulating Eq. (8)

$$\tilde{V}_i [n + 1] = \sum_{j} P_{ij} D_{kj} V_j [n] + C_k,$$

we find the simple relation

$$\tilde{V}_i [n + 1] = \lambda_i \tilde{V}_i [n] + \tilde{C}_i,$$ (B3)

where

$$\tilde{C}_i = \sum_{k} P_{ik} C_k. \quad (B4)$$

Solutions to Eq. (B3) are of the form

$$\tilde{V}_i [n] = \begin{cases} \left( \tilde{V}_i [0] + \frac{\tilde{C}_i}{\lambda_i - 1} \right) \lambda_i^n - \frac{\tilde{C}_i}{\lambda_i - 1}, & \text{if } \lambda_i \neq 1; \\ \tilde{V}_i [0] + n \tilde{C}_i, & \text{if } \lambda_i = 1, \end{cases} \quad (B5)$$

from which we can conclude that physical solutions must have $|\lambda_i| \leq 1$ and $\lambda_i = 1 \Rightarrow \tilde{C}_i = 0$ for all $i$ (lest the correlation functions diverge in the infinite future) and that if $\lambda_i \neq 1$ then the only fixed point is the state vector

$$\tilde{V}_i = -\frac{\tilde{C}_i}{\lambda_i - 1}, \quad (B6)$$

which is actually an attractor if $|\lambda_i| < 1$ (and an attractor for the time average after long times for all physical solutions). Therefore, we can also see that the issue of pseudo-thermalisation depends solely on properties of the Hamiltonian – as these models necessarily pseudo-thermalise if $\lambda_i \neq 1$ for $i$. 
Appendix C: Fixed point in the continuous limit

For small \( \tau \), Eq. (11) can be substituted into Eqs. (9) and (10) to find

\[
D_{ij} = \delta_{ij} + \frac{i\tau}{\hbar} \left( M^*_{\alpha_i, \alpha_j} \delta_{\beta_i, \beta_j} - \delta_{\alpha_i, \alpha_j} M_{\beta_i, \beta_j} \right) + \mathcal{O} \left( \tau^2 \right),
\]

(C1)

and

\[
C_i = \frac{i\tau}{\hbar} \sum_{(\alpha', \beta') \in \mathcal{R}} \rho_{\alpha', \beta'} \left( 0 \right) \left( M^*_{\alpha_i, \alpha_j} \delta_{\beta_i, \beta_j} - \delta_{\alpha_i, \alpha_j} M_{\beta_i, \beta_j} \right) + \mathcal{O} \left( \tau^2 \right).
\]

(C2)

In the extreme version of this limit, when \( \tau = 0 \) and thus each iterative transformation is only allowed to differ from identity infinitesimally, a continuous approach is required. This can be straightforwardly done by defining the continuous variable

\[
\mathcal{V}_i \left( t \right) \equiv \rho_{\alpha_i, \beta_i} \left( t \right) = \lim_{\tau \to 0} V_i \left[ \frac{t}{\tau} \right],
\]

(C3)

and applying Eq. (8) to write the linear system of ordinary differential equations

\[
\frac{d\mathcal{V}_i}{dt} \left( t \right) = \sum_j D_{ij} \mathcal{V}_j \left( t \right) + C_i,
\]

(C4)

where we have introduced the new matrix

\[
D_{ij} \equiv \lim_{\tau \to 0} \frac{D_{ij} - \delta_{ij}}{\tau} = \frac{i}{\hbar} \left( M^*_{\alpha_i, \alpha_j} \delta_{\beta_i, \beta_j} - \delta_{\alpha_i, \alpha_j} M_{\beta_i, \beta_j} \right),
\]

and the new vector

\[
C_i \equiv \lim_{\tau \to 0} \frac{C_i}{\tau} = \frac{i}{\hbar} \sum_{(\alpha', \beta') \in \mathcal{R}} \rho_{\alpha', \beta'} \left( 0 \right) \left( M^*_{\alpha_i, \alpha_j} \delta_{\beta_i, \beta_j} - \delta_{\alpha_i, \alpha_j} M_{\beta_i, \beta_j} \right).
\]

(C5)

Since it follows from Eq. (C5) that any matrix \( P \) that diagonalises \( D \) must also diagonalise \( D \), we can solve Eq. (C4) analogously to Eq. (8), finding

\[
\tilde{V}_i \left( t \right) = \begin{cases} 
\frac{e^{\sigma_i t}}{\sigma_i} \left( \tilde{V}_i \left( 0 \right) + \tilde{C}_i / \sigma_i \right) - \tilde{C}_i / \sigma_i, & \text{if } \sigma_i \neq 0; \\
\tilde{V}_i \left( 0 \right) + \tilde{C}_i t, & \text{if } \sigma_i = 0,
\end{cases}
\]

(C7)

where, as before, tilde denotes multiplication by \( P \), and \( \sigma_i = d\lambda_i / d\tau \bigg|_{\tau=0} = \lim_{\tau \to 0} \tau^{-1} \ln \lambda_i \) are the eigenvalues of \( D \) (so that \( \sum_k \rho_{\alpha_k, \alpha_k} = \sigma_i \delta_{\alpha_i, \alpha_c} \)). Also analogously to the discrete case, we can conclude that physical solutions must have \( \text{Re} \left( \sigma_i \right) \leq 0 \) and \( \sigma_i = 0 \Rightarrow \tilde{C}_i = 0 \) for all \( i \) and that if \( \sigma_i \neq 0 \) then the only fixed point is the initial vector

\[
\mathcal{V}_i \left( 0 \right) = -\frac{\tilde{C}_i}{\sigma_i},
\]

(C8)

which is an attractor as long as \( \text{Re} \left( \sigma_i \right) < 0 \).

Appendix D: Infinite temperature attractor

Consider an arbitrary quadratic model described by a Hamiltonian of the form in Eq. (1), where the indices in some set \( \mathcal{S} \) are interpreted as corresponding to a system of interest and the indices in its complement \( \mathcal{E} \) are interpreted as corresponding to an environment. At each iteration the state of the environment is reset to a thermal state at infinite temperature, i.e.,

\[
\rho_{\alpha \beta} \left( N \tau \right) = \frac{1}{2} \delta_{\alpha \beta} \forall \alpha, \beta \in \mathcal{E} \forall N \in \mathbb{Z}.
\]

(D1)

Regardless of whether system-environment correlations are reset to zero or allowed to evolve, Eq. (10) can then be written as

\[
C_i = \frac{1}{2} \sum_{\alpha' \in \mathcal{E}} U_{\alpha_i, \alpha'}^* \left( \tau \right) U_{\beta_i, \alpha'} \left( \tau \right),
\]

(D2)

which allows us to see that one of the fixed points of the system must always correspond to the infinite temperature state (with vanishing system-environment correlations), i.e.,

\[
\mathcal{V}_i = \frac{1}{2} \delta_{\alpha_i, \beta_i} \forall i,
\]

(D3)

as then substitution into Eq. (9) yields

\[
\sum_j D_{ij} \mathcal{V}_j = \frac{1}{2} \sum_{\alpha' \in \mathcal{S}} U_{\alpha_i, \alpha'}^* \left( \tau \right) U_{\beta_i, \alpha'} \left( \tau \right),
\]

(D4)

and thus

\[
\sum_j D_{ij} \mathcal{V}_j + C_i = \frac{1}{2} \sum_{\alpha'} U_{\alpha_i, \alpha'}^* \left( \tau \right) U_{\beta_i, \alpha'} \left( \tau \right) = \frac{1}{2} \delta_{\alpha_i, \beta_i} = V_i.
\]

(D5)

Therefore generic quadratic models are expected to thermalise with any environment kept at infinite temperature.