Analytic Formulae for the Off-Center CMB Anisotropy in a General Spherically Symmetric Universe

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The local void model has recently attracted considerable attention because it can explain the apparent accelerated expansion of the present universe without introducing dark energy. However, in order to justify this model as an alternative to the standard \(\Lambda\)CDM cosmology, the model should be tested by various observations, such as the CMB temperature anisotropy, besides the distance-redshift relation of SNIa. For this purpose, we derive analytic formulae for the dipole and quadrupole moments of the CMB temperature anisotropy that hold for any spherically symmetric universe model and can be used to compare consequences of such a model with observations of the CMB temperature anisotropy rigorously. We check that our formulae are consistent with the numerical studies previously made for the CMB temperature anisotropy in the void model. We also update the constraints concerning the location of the observers in the void model by applying our analytic dipole formula with the latest WMAP data.

\ §1. Introduction

In modern cosmology, it is commonly assumed that our universe be isotropic and homogeneous on large scales and accordingly be described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric as a first approximation. Then, together with consequences of various cosmological observations such as the power spectrum of Cosmic Microwave Background (CMB) temperature anisotropy, the distance-redshift relation of type Ia supernovae (SNIa) indicates that the expansion of the present universe is accelerated. We are then led to introduce “dark energy,” which has negative pressure and behaves just like a positive cosmological constant, thus having the standard \(\Lambda\)CDM model. However, there does not appear to be any satisfactory theory that can naturally explain the origin of dark energy and its magnitude required by observations. It is therefore tempting to seek alternative explanations of the apparent cosmic acceleration, more specifically, the SNIa luminosity distance-redshift relation, without invoking dark energy. (For a number of approaches to the dark energy problem, see e.g., 1) and references therein).

One of such attempts is a “local void model” proposed by Tomita,\(^{2,3}\) also independently by Celerier,\(^{4}\) and by Goodwin et al.\(^{5}\) In this model, our universe is no longer assumed to be homogeneous, having instead an under-dense local void in

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the surrounding overdense universe.\footnote{See, e.g., 6) for earlier work along a similar line. For other proposals that also attempt to exploit the effects of inhomogeneities to solve the dark energy problem, interests and criticisms thereof, see e.g., 7)–19), 21), 22).} The isotropic nature of cosmological observations on large scales is realized by assuming the spherical symmetry and demanding that we live close to the center of the void. Furthermore, the model is supposed to contain only ordinary dust like cosmic matter, describing, say, the CDM component. Since such a spacetime can be described by the Lemaitre-Tolman-Bondi (LTB) metric,\footnote{In fact, the LTB metric has been considered in the cosmological context even much before the present form of the dark energy problem was raised by the results of SNIa observations. For example, an early proposal of the void model constructed by the LTB metric was made by Moffat and Tatarski.\footnote{26), 27}} we also call this model the “LTB cosmological model.”\footnote{\textsuperscript{**}} Since the rate of expansion in the void region is larger than that in the outer overdense region, this model can account for the observed dimming of SNIa luminosity. In fact, recent numerical analyses\footnote{\textsuperscript{28)–32),34)–40),42),43)} have shown that the LTB model can accurately reproduce the SNIa distance-redshift relation. For this reason, despite the relinquishment of the widely accepted Copernican/cosmological principle, the local void/LTB model has recently attracted considerable attention.

However, in order to justify the LTB model as a viable alternative to the standard $\Lambda$CDM model, one has to test this model by various observations other than the SNIa distance-redshift relation. A number of papers for this purpose have appeared recently, studying constraints from observations, such as the CMB temperature anisotropy,\footnote{\textsuperscript{31)–33),43)} the baryon acoustic oscillation,\footnote{\textsuperscript{41)–43)} the kinematic Sunyaev-Zeldovich effect,\footnote{\textsuperscript{44), 45)} the galaxy count-redshift relation,\footnote{\textsuperscript{46)–49)) etc. Many LTB models with a small, a few hundreds Mpc size void are already ruled out, but at present, the models with a huge, Gpc size void still remain to be tested. More details of the current status of the LTB models are discussed in 6). Most of these analyses have been performed for various types of LTB models by using numerical methods, and it does not seem to be straightforward to compare analyses for each different model so as to have a coherent understanding of the results. In order to have general consequences of the LTB cosmology and systematically examine its viability, it is desirable to develop some general, analytic methods that can apply, independently of the details of each specific model.

Apart from the quest of alternatives to dark energy, the LTB metric may also serve as a toy model for getting some insights into the dynamics and possible observational effects of non-linear perturbations in the standard $\Lambda$CDM cosmology. The LTB metric can incorporate a cosmological constant in a straightforward manner, hence being able to describe, as an exact solution, highly non-linear inhomogeneities—almost arbitrary in magnitude, as long as being spherically symmetric—in the FLRW universe with dark energy. In view of this, it is also worth attempting to derive some analytic formulae that can be used to make theoretical predictions of the $\Lambda$-LTB spacetimes and rigorous comparison with cosmological observations.

The purpose of this paper is to derive analytic formulae for the dipole and quadrupole moments of the CMB temperature anisotropy in general spherically
symmetric inhomogeneous spacetimes, including the $\Lambda$-LTB spacetime as a particular case. In the standard cosmology, basic properties of the CMB temperature anisotropy are derived by inspecting perturbations of the Einstein and Boltzmann equations in the FLRW background universe. Ideally, it is desirable to do the same thing in the LTB background. However, perturbations in a LTB spacetime, let alone those in a general spherically symmetric spacetime, have not been very well studied mainly because the perturbation equations are much more involved to solve in the spherical but inhomogeneous background, though the linear perturbation formulae themselves have long been available\(^{50),51)*\). In this paper, we are not going to deal with perturbations of the LTB metric. Instead, we will exploit the key requirement of the LTB cosmology that we, observers, are restricted to be around very near the center of the spherical symmetry: Namely, we first note that the small distance between the symmetry center and an off-center observer gives rise to a corresponding deviation in the photon distribution function. Then, by taking ‘Taylor-expansions’ of the photon distribution function at the center with respect to the deviation, we can read off the CMB temperature anisotropy caused by the deviation in the photon distribution function. By doing so, we can, in principle, construct the $l$-th order multiple moment of the CMB temperature anisotropy from the (up to) $l$-th order expansion coefficients, with the help of the background null geodesic equations and the Boltzmann equation. We will do so for the first and second-order expansions to find the CMB dipole and quadrupole moments. We also provide the concrete expression of the corresponding formulae for the LTB cosmological model. Our formulae are then checked to be consistent with the numerical analyses of the CMB temperature anisotropy in the LTB model, previously made by Alnes and Amarzguioui.\(^{53)}\)

We then apply our formulae to place the constraint on the distance between an observer and the symmetry center of the void, by using the latest Wilkinson Microwave Anisotropy Probe (WMAP) data, thereby updating the results of the previous analyses.

In the next section, we derive analytic formulae for the CMB temperature anisotropy in the most general spherically symmetric spacetime. In §3, we obtain analytic formulae for the CMB temperature anisotropy in the LTB model, and some constraints concerning the position of the observer. §4 is devoted to summary. Appendix discusses the regularity of the photon distribution function and the behavior of some geometric quantities at the symmetry center.

§2. The CMB temperature anisotropy in spherically symmetric spacetimes

In this section, we will derive analytic formulae for the dipole and quadrupole moments of the CMB temperature anisotropy. We first briefly discuss the null geodesic equations, the photon distribution function and its relation to the CMB photon temperature in the most general spherically symmetric spacetime. Next, inspecting the first-order derivatives of the photon distribution function at the center of the spher-
ical symmetry and using the null geodesic equations, we derive the dipole formula for the CMB. Then, taking further derivatives, we derive the quadrupole formula of the CMB temperature anisotropy in general spherically symmetric spacetimes.

2.1. The photon distribution function in a general spherically symmetric spacetime

The most general spherically symmetric metric can be written in the following form

$$ds^2 = -N^2(t, r)dt^2 + S^2(t, r)dr^2 + R^2(t, r)(d\theta^2 + \sin^2 \theta d\phi^2).$$  (2.1)

In this background, the relevant geodesic equations are given by

$$\frac{dp^t}{d\lambda} = -\frac{\dot{N}}{N} (p^t)^2 - 2\frac{N'}{N} p^t p^r - \frac{SS'}{N^2} (p^r)^2 - \frac{\dot{R}}{N^2 R} p^r_\perp,$$  (2.2)

$$\frac{dp^r}{d\lambda} = -\frac{N N'}{S^2} (p^t)^2 - 2\frac{S}{S} p^t p^r - \frac{SS'}{S} (p^r)^2 + \frac{R'}{S^2 R} p^r_\perp,$$  (2.3)

where $p^\mu = dx^\mu/d\lambda$, $p_\mu p^\mu = 0$, with $\lambda$ being an affine parameter, and $p^r_\perp$ is given by $p^r_\perp \equiv R^2 \{ (p^\theta)^2 + (p^\phi)^2 \sin^2 \theta \}$. Here and in the rest of the paper, prime and dot denote the derivatives with respect to $r$ and $t$, respectively.

We are concerned with the distribution function, $F(x, p)$, for the CMB photons that leave the “last scattering surface” appropriately defined, say, $t = t_i$ hypersurface, in the universe modelled by the above metric and that eventually reach an “observer” very near the symmetry center. We assume that the photon distribution function $F(x, p)$ is spherically symmetric, respecting the symmetry of the background geometry, so that

$$F(x, p) = F_0(t, r, \omega, \mu),$$  (2.4)

where $\omega \equiv p^t$ and $\mu \equiv Sp^r/(N\omega)$. Note that from the above geodesic equations, we have

$$\dot{r} = \mu \frac{N}{S},$$  (2.5)

$$\dot{\omega} = -\omega \left( \frac{\dot{N}}{N} + 2\mu \frac{N'}{N} + \mu^2 \frac{\dot{S}}{S} + (1 - \mu^2) \frac{\dot{R}}{R} \right),$$  (2.6)

$$\dot{\mu} = (1 - \mu^2) \left\{ \frac{N}{S} \left( \frac{R'}{R} - \frac{N'}{N} \right) + \mu \left( \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right) \right\}.$$  (2.7)

In particular, it follows from the above equations

$$\partial_\omega \dot{r} = \partial_\omega \dot{\mu} = 0, \quad \partial_r \dot{\mu} = 0,$$  (2.8)

where the last one holds for the radial null geodesics, for which $\mu = \pm 1$.

Now, suppose the universe is locally in thermal equilibrium, that is, $F$ is given as the Planck distribution function, $\Phi$, at the last scattering surface, as required in most of the known LTB models. Then, given a photon geodesic $\gamma$, the ratio of the
temperature $T$ of the photon and its energy $\omega$ is preserved along the trajectory $\gamma$. Therefore, $\omega$ comes in $F$ in the form

$$F = \Phi(\omega/T).$$

(2.9)

Then, the CMB temperature anisotropy $\delta T/T$ is generally given by

$$(\delta F)^{(1)} + (\delta F)^{(2)} + \cdots = \left\{-\frac{\delta T}{T} \omega \partial_\omega + \frac{1}{2} \left(\frac{\delta T}{T}\right)^2 (\omega \partial_\omega)^2 + \cdots \right\} \Phi.$$  

(2.10)

Now, suppose that an observer lives at a distance of $\delta x^i$ from the symmetry center. Then, the left-side are written as

$$\left(\delta F\right)^{(1)} = \delta x^i (\partial_i F)_0, \quad \left(\delta F\right)^{(2)} = \frac{1}{2} \delta x^i \delta x^j (\partial_i \partial_j F)_0,$$  

(2.11)

where here and in the following, the subscript ‘0’ implies the value evaluated at the center $(r = 0)$ at the present time $(t = t_0)$. The CMB temperature anisotropy dipole $(\delta T/T)^{(1)}$ and quadrupole $(\delta T/T)^{(2)}$ are therefore given by

$$\left(\frac{\delta T}{T}\right)^{(1)} = -\frac{\delta x^i (\partial_i F)_0}{\omega \partial_\omega F_0},$$  

$$\left(\frac{\delta T}{T}\right)^{(2)} = -\frac{1}{2} \delta x^i \delta x^j (\partial_i \partial_j F)_0 + \frac{1}{2} \left\{\left(\frac{\delta T}{T}\right)^{(1)}\right\}^2 \frac{(\omega \partial_\omega)^2 F_0}{\omega \partial_\omega F_0}.$$  

(2.12)

(2.13)

Since (2.4) implies $\partial_t F = (\partial_t r) \partial_t F_0 + (\partial_t \omega) \partial_\omega F_0 + (\partial_t \mu) \partial_\mu F_0, (\partial_t F)_0, (\partial_t \partial_j F)_0$ in the right-hand side of the above equations are given by $\partial_\alpha F_0$ and $\partial_\alpha \partial_\beta F_0$ $(\alpha, \beta = r, \omega, \mu)$. Therefore, our task is to find the concrete expressions of $\partial_\alpha F_0$ and $\partial_\alpha \partial_\beta F_0$ in terms of relevant geometric quantities.

2.2. The CMB dipole formula

First, we will obtain the CMB temperature dipole formula. For this purpose, we derive the expression for $\partial_\alpha F_0$. Our stating point is the Boltzmann equation for $F_0(t, r, \omega, \mu),$

$$\frac{d}{dt} F_0 = \partial_t F_0 + \dot{r} \partial_r F_0 + \dot{\omega} \partial_\omega F_0 + \dot{\mu} \partial_\mu F_0 = 0,$$  

(2.14)

where $\dot{r}, \dot{\omega}, \dot{\mu}$ are defined for a given null geodesic curve $\gamma$. By differentiating this equation by $\alpha = r, \omega, \mu$, and using the formula (2.8), we obtain the first order differential equations for $\partial_\alpha F,$

$$\frac{d}{dt} \begin{pmatrix} \partial_\omega F_0 \\ \partial_r F_0 \\ \partial_\mu F_0 \end{pmatrix} = - \begin{pmatrix} \partial_\omega \dot{\omega} & 0 & 0 \\ \partial_r \dot{\omega} & \partial_r \dot{r} & 0 \\ \partial_\mu \dot{\omega} & \partial_\mu \dot{r} & \partial_\mu \dot{\mu} \end{pmatrix} \begin{pmatrix} \partial_\omega F_0 \\ \partial_r F_0 \\ \partial_\mu F_0 \end{pmatrix}.$$  

(2.15)

The set of these equations can easily be integrated along the given photon trajectory $\gamma$ to yield the solutions

$$\omega \partial_\omega F_0 = (\omega \partial_\omega F_0)_i,$$  

(2.16)
\[ \partial_t F_0 = e^{-P(t,t_1)}(\partial_t F_0)_{t_1} + (\omega \partial_\omega F_0)_{t_1} \int_{t_1}^t dt_1 e^{-P(t,t_1)} A(t_1), \tag{2.17} \]
\[ \partial_\mu F_0 = e^{-Q(t,t_1)}(\partial_\mu F_0)_{t_1} + (\partial_\lambda F_0)_{t_1} \int_{t_1}^t dt_1 e^{-Q(t,t_1)-P(t,t_1)} \left( -\frac{N}{S} \right)_{t_1} \]
\[ + (\omega \partial_\omega F_0)_{t_1} \int_{t_1}^t dt_1 e^{-Q(t,t_1)} \left\{ B(t_1) - \left( \frac{N}{S} \right)_{t_1} \int_{t_1}^{t_1} dt_2 e^{-P(t_1,t_2)} A(t_2) \right\}, \tag{2.18} \]

where
\[ A \equiv \left( \frac{\dot{N}}{N} \pm 2 \frac{N'}{S} + \frac{\dot{S}}{S} \right)^' \quad \text{and} \quad B \equiv 2 \left\{ \frac{N'}{S} \pm \left( \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right) \right\}, \tag{2.19} \]

and
\[ P(t_1,t_2) \equiv \mp \int_{t_1}^{t_2} dt \left( \frac{N}{S} \right)^' \tag{2.20} \]
\[ Q(t_1,t_2) \equiv 2 \int_{t_1}^{t_2} dt \left\{ \pm \frac{N'}{S} \left( \frac{R'}{R} - \frac{N'}{S} \right) + \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right\} \]
\[ = 2 \left[ \ln \frac{R}{N} \right]_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} dt \left( \frac{\dot{N}}{N} - \frac{\dot{S}}{S} \right), \tag{2.21} \]

and where the subscript ‘i’ denotes the value evaluated at the last scattering surface.

Note that \( \omega \partial_\omega F_0 \) is constant for any null geodesics, as in (2.16). Furthermore, as shown in Appendix, by inspecting the regularity of \( F \) at the center, as well as the behavior of some relevant geometric quantities near the center, we can observe that \( \partial_t F_0 \) does not contribute to the leading behavior of \( \partial F \) in the limit \( r \to 0 \). Therefore, we only need to find the expression of \( \partial_\mu F \).

Using the null geodesic equation for a radial geodesic, \( \mu = \pm 1, dt/dr = \pm S/N \), we find that (2.20) and (2.21) become
\[ P(t_1,t_2) = \left[ \ln \frac{S}{N} \right]_{t_1}^{t_2} + U(t_1,t_2), \quad Q(t_1,t_2) = 2 \left[ \ln \frac{R}{N} \right]_{t_1}^{t_2} + 2U(t_1,t_2), \tag{2.22} \]
where
\[ U(t_1,t_2) \equiv \int_{t_1}^{t_2} dt \left( \frac{\dot{N}}{N} - \frac{\dot{S}}{S} \right). \tag{2.23} \]

Substituting these into (2.18), we have
\[ \partial_\mu F_0 = \frac{R^2}{N^2} \frac{N^2}{R^2} e^{-2U(t_1,t_2)} (\partial_\mu F_0)_{t_1} - \frac{R^2}{N^2} \frac{N}{S_i} (\partial_i F_0)_{t_1} \int_{t_1}^{t_1} dt_1 \left( \frac{N S}{R^2} \right)_{t_1} e^{-2U(t_1,t_1)-U(t_1,t_1)} \]
\[ + \frac{R^2}{N^2} (\omega \partial_\omega F_0)_{t_1} \int_{t_1}^{t_1} dt_1 \left( \frac{N^2}{R^2} \right)_{t_1} e^{-2U(t_1,t_1)} \left\{ B(t_1) - \int_{t_1}^{t_1} dt_2 e^{-U(t_1,t_2)} A(t_2) \right\}. \tag{2.24} \]
Here we note that the second and third terms in the right-side of the above equation have the form

\[ I(r) \equiv \int_{t_i}^{t_f} dt' \frac{V(t, t_1)}{R^2(t_1)} = \pm \int_{r_i}^{r_f} dr' \frac{V(r, r_1)}{R^2(r_1)}, \tag{2.25} \]

where \( V(r, r_1) \) is a function of \( r \) and \( r_1 \) that is regular at \( 0 \leq r \leq r_1 \). For the radial geodesic that reaches \( r = 0 \) at \( t = t_0 \), \( I(r) \) behaves as

\[ I(r) \simeq -\frac{1}{r} S_0 \frac{V(0, 0)}{N_0 (R_0')^2}. \tag{2.26} \]

Substituting the above expression of \( \partial_\mu F_0 \) into

\[ \partial_i F_0 \simeq S_0 \frac{p^i}{p} (f_0) = S_0 \frac{p^i}{p} \left( \frac{\partial_\mu F_0}{r} \right)_0, \quad \text{as } r \to 0, \tag{2.27} \]

which is derived in Appendix (see (A.16)), we have

\[ (\partial_i F)_0 \simeq \frac{S_0^2}{N_0} \frac{p^i}{p} \left[ \frac{N_i}{S_i} e^{-U(t_0, t_i)} (\partial_r F_0)_i + (\omega \partial_\omega F_0)_i \left\{ \int_0^{r_i} d\epsilon e^{-U(t_0, \epsilon)} A(t) - B_0 \right\} \right]. \tag{2.28} \]

Thus, using this expression, we can write (2.12) as

\[ \left( \frac{\delta T}{T} \right)^{(1)} = \mp \frac{\delta L_n \cdot \Omega}{N_0} \left[ \frac{N_i}{S_i} e^{-U(t_0, t_i)} \left( \frac{\partial_\mu F_0}{\omega \partial_\omega F_0} \right)_i + \int_0^{r_i} d\epsilon e^{-U(t_0, \epsilon)} A(t) - B_0 \right], \tag{2.29} \]

where \( \delta L_n \) is the position vector of the observer. This is our dipole formula for the CMB temperature anisotropy in the most general spherically symmetric spacetime. The regularity of the metric, (2.1) at the symmetry center implies that \( N^2 \simeq C_1 + O(r^2) \), \( R/r \simeq C_2 + O(r^2) \), and \( S^2 \simeq (R/r)^2 + O(r^2) \) near the center, with \( C_1, C_2 \) being some constants with respect to \( r \). Using these estimations, we can check that the right-hand side of (2.29) is convergent, hence well-defined.

2.3. The CMB quadrupole formula

Next, we will derive the CMB quadrupole formula by inspecting the second-order derivatives, \( \partial_\alpha \partial_\beta F_0 \), \( (\alpha, \beta = r, \omega, \mu) \). By differentiating the first row of (2.15) with respect to \( \ln \omega, r, \) and \( \mu \), we obtain

\[ \frac{d}{dt} \{ (\omega \partial_\omega)^2 F_0 \} = 0, \tag{2.30} \]

\[ \frac{d}{dt} (\omega \partial_\omega \partial_r F_0) = \mp \left( \frac{N}{S} \right)' \omega \partial_\omega \partial_r F_0 + A(\omega \partial_\omega)^2 F_0, \tag{2.31} \]

\[ \frac{d}{dt} (\omega \partial_\omega \partial_\mu F_0) = 2 \left\{ \pm \frac{N}{S} \left( \frac{R'}{R} - \frac{N'}{N} \right) + \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right\} \omega \partial_\omega \partial_\mu F_0 \]

\[ -\frac{N}{S} \omega \partial_\omega \partial_r F_0 + B(\omega \partial_\omega)^2 F_0, \tag{2.32} \]
for a radial geodesic for which $\mu = \pm 1$. We can easily integrate this set of equations, and get the solutions

\begin{equation}
(\omega \partial_\omega)^2 F_0 = \{(\omega \partial_\omega)^2 F_0\}_i, \tag{2.33}
\end{equation}

\begin{equation}
\omega \partial_\omega \partial_\mu F_0 = e^{-P(t,t_1)}(\omega \partial_\omega \partial_\mu F_0)_i + \{(\omega \partial_\omega)^2 F_0\}_i \int_{t_i}^t dt_1 e^{-P(t,t_1)} A(t_1), \tag{2.34}
\end{equation}

\begin{equation}
\omega \partial_\omega \partial_\mu F_0 = e^{-Q(t,t_1)}(\omega \partial_\omega \partial_\mu F_0)_i - \{(\omega \partial_\omega)^2 F_0\}_i \int_{t_i}^t dt_1 e^{-Q(t,t_1)-P(t,t_1)} \left(\frac{N}{S}\right)_{t_1}
\end{equation}

\begin{equation}
+\{(\omega \partial_\omega)^2 F_0\}_i \int_{t_i}^t dt_1 e^{-Q(t,t_1)} \left\{ B(t_1) - \left(\frac{N}{S}\right)_{t_1} \int_{t_i}^{t_1} dt_2 e^{-P(t_1,t_2)} A(t_2) \right\}. \tag{2.35}
\end{equation}

Similarly, we can get the ordinary differential equations

\begin{equation}
\frac{d}{dt}(\partial^2_{r} F_0) = \mp 2 \left(\frac{N}{S}\right)' \partial^2_{r} F_0 \mp \left(\frac{N}{S}\right)'' \partial_\mu F_0 + A'(\omega \partial_\omega F_0)_i + 2A\omega \partial_\omega \partial_r F_0, \tag{2.36}
\end{equation}

\begin{equation}
\frac{d}{dt}(\partial_r \partial_\mu F_0) = \left[\mp \left(\frac{N}{S}\right)' + 2 \left\{ \pm \frac{N}{S} \left(\frac{R'}{R} - \frac{N'}{N} \right) + \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right\} \right] \partial_r \partial_\mu F_0 - \frac{N}{S} \partial^2_\mu F_0
\end{equation}

\begin{equation}
+ B\omega \partial_\omega \partial_r F_0 + A\omega \partial_\omega \partial_\mu F_0 - \left(\frac{N}{S}\right)' \partial_r F_0 + C \partial_\mu F_0 + B'(\omega \partial_\omega F_0)_i, \tag{2.37}
\end{equation}

\begin{equation}
\frac{d}{dt}(\partial^2_\mu F_0) = 4 \left\{ \pm \frac{N}{S} \left(\frac{R'}{R} - \frac{N'}{N} \right) + \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right\} \partial^2_\mu F_0 - 2 \frac{N}{S} \partial_r \partial_\mu F_0 + 2B\omega \partial_\omega \partial_\mu F_0
\end{equation}

\begin{equation}
+ D \partial_\mu F_0 + 2 \left(\frac{\dot{S}}{S} - \frac{\dot{R}}{R}\right) (\omega \partial_\omega F_0)_i, \tag{2.38}
\end{equation}

for a radial geodesic, $\mu = \pm 1$. We can also integrate this set of equations, and obtain the solutions

\begin{equation}
\partial^2_\mu F_0 = e^{-2P(t,t_1)}(\partial^2_\mu F_0)_i
\end{equation}

\begin{equation}
+ \int_{t_i}^t dt_1 e^{-2P(t,t_1)} \left\{ \mp \left(\frac{N}{S}\right)'' \partial_r F_0 + A'(\omega \partial_\omega F_0)_i + 2A\omega \partial_\omega \partial_r F_0 \right\}, \tag{2.39}
\end{equation}

\begin{equation}
\partial_r \partial_\mu F_0 = e^{-P(t,t_1)-Q(t,t_1)}(\partial_r \partial_\mu F_0)_i
\end{equation}

\begin{equation}
+ \int_{t_i}^t dt_1 e^{-P(t,t_1)-Q(t,t_1)} \left\{ -\frac{N}{S} \partial^2_\mu F_0 + B\omega \partial_\omega \partial_r F_0 + A\omega \partial_\omega \partial_\mu F_0
\end{equation}

\begin{equation}
- \left(\frac{N}{S}\right)' \partial_r F_0 + C \partial_\mu F_0 + B'(\omega \partial_\omega F_0)_i \right\}, \tag{2.40}
\end{equation}

\begin{equation}
\partial^2_\mu F_0 = e^{-2Q(t,t_1)}(\partial^2_\mu F_0)_i + \int_{t_i}^t dt_1 e^{-2Q(t,t_1)} \left\{ -2 \frac{N}{S} \partial_r \partial_\mu F_0 + 2 \left(\frac{\dot{S}}{S} - \frac{\dot{R}}{R}\right) (\omega \partial_\omega F_0)_i
\end{equation}
In fact, as we show in Appendix (see (A.30)), the CMB quadrupole formula (2.13) is written as

\[ +2B\omega\partial_\omega\partial_\mu F_0 + D\partial_\mu F_0 \bigg|_{t_1} \]

where

\[ C \equiv 2 \left\{ \pm \frac{N}{S} \left( \frac{R'}{R} - \frac{N'}{N} \right) + \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right\}, \quad D \equiv 2 \frac{N}{S} \left( \frac{R'}{R} - \frac{N'}{N} \right) \pm 6 \left( \frac{\dot{R}}{R} - \frac{\dot{S}}{S} \right). \]

Thus, we have the six solutions, \((\omega\partial_\omega)^2 F_0, \omega\partial_\omega\partial_i F_0, \) etc. However, we note that \((\omega\partial_\omega)^2 F_0\) is just constant. Furthermore, by inspecting the regularity of \(F_0\) at the center, as well as the behavior of some geometric quantities near the center, we can find that only \(\partial^2_r F_0\) becomes relevant to the evaluation of \(\partial_i \partial_j F\) in the limit \(r \to 0\).

In fact, as we show in Appendix (see (A-30)),

\[ \partial_i \partial_j F \to 2 \left( \delta_{ij} - S_0^2 \frac{p_ip_j}{p^2} \right) (f_2)_0 \] 

\[ + \left\{ \frac{a_\perp''}{a_\perp} - \frac{a''}{N''} \right\} \delta_{ij} + C_0 \frac{p_ip_j}{p^2} (\omega\partial_\omega F_0)_i, \]

where \(f_2 = \partial_r^2 F\) (see (A.7) in Appendix), and \(a_\perp \equiv R/r, \) which corresponds to the 'scale factor' perpendicular to the radial direction. Then, in terms of \(f_2\) and \(\partial^2_r F_0, \) the CMB quadrupole formula (2.13) is written as

\[ \left( \frac{\delta T}{T} \right)^{(2)} = \frac{1}{2} \frac{\delta x^i \delta x^j}{(\omega\partial_\omega F_0)_i} \left[ 2(\delta_{ij} - \Omega_i \Omega_j)(f_2)_0 + \Omega_i \Omega_j (\partial^2_r F_0)_0 \right. \]

\[ + \left\{ \frac{a_\perp''}{a_\perp} - \frac{a''}{N''} \right\} \delta_{ij} + C_0 \frac{p_ip_j}{p^2} \left( \omega\partial_\omega F_0 \right)_i \right] \]

\[ + \frac{1}{2} \left\{ \left( \frac{\delta T}{T} \right)^{(1)} \right\}^2 \left( \frac{(\omega\partial_\omega)^2 F_0}_i \right), \]

where \(\Omega_i \equiv \delta_{ij} x^j/r. \) So, the remaining task is to find the expressions of the leading behavior of \(f_2\) and \(\partial^2_r F_0\) at the center \(r \to 0. \)

First, we note from (A.28) that \((f_2)_0\) is given by

\[ (f_2)_0 = \frac{1}{2} \left( \frac{\partial_\mu F_0}{r} + \frac{\partial_\mu F_0}{r^2} \right)_0. \]

In the limit \(r \to 0, \) from (2.18), the second term of this equation can be written as

\[ \frac{\partial_\mu F_0}{r^2} \to \frac{a_\perp^2}{N_0^2} \frac{N_0^2}{R_i^2} e^{-2U_{(t_0, t_i)}} (\partial_\mu F_0)_i \]

\[ + \frac{a_\perp^2}{N_0^2} \int_0^{r_i} dr \frac{N_0^2}{r^2} e^{-2U_{(t_0, t_i)}} \left( -\partial_r F_0 + \frac{S}{N} B \omega \partial_\omega F_0 \right)_i \]

\[ = \frac{a_\perp^2}{N_0^2} \frac{N_0^2}{R_i^2} e^{-2U_{(t_0, t_i)}} (\partial_\mu F_0)_i \]

\[ + \frac{a_\perp^2}{a_\perp^2} \frac{e^{-2U_{(t_0, t_i)}}}{r_i} \left( -\partial_r F_0 + \frac{S}{N} B \omega \partial_\omega F_0 \right)_i. \]
\[ \pm \left\{ \frac{1}{r} \left( -\partial_{r} F_{0} + \frac{S}{N} B \omega \partial_{\omega} F_{0} \right) \right\}_{0} \]
\[ \pm \frac{a_{1} a_{1}^{2}}{N_{0}^{2} r_{i}^{2}} \int_{t_{i}}^{t_{0}} dt N_{0}^{2} \left\{ \frac{e^{-2U(t_{0}, t)}}{a_{1}^{2}} \left( -\partial_{r} F_{0} + \frac{S}{N} B \omega \partial_{\omega} F_{0} \right) \right\} . \] (2.46)

From the second row of (2.15), we have
\[ \frac{d}{dt} \left( \partial_{r} F_{0} e^{-2U(t_{0}, t)} \right) = \left\{ 2 \left( \frac{\dot{S}}{S} - \frac{\dot{R}}{R} - \frac{\dot{N}}{N} \mp \frac{N a_{i}'}{S a_{\perp}} \right) + \left( \frac{N}{S} \right)' \right\} \frac{\partial_{r} F_{0}}{a_{1}^{2}} e^{-2U(t_{0}, t)} \]
\[ + \frac{A}{a_{1}^{2}} e^{-2U(t_{0}, t)} (\omega \partial_{\omega} F_{0})_{i}. \] (2.47)

Thus, \((f_{2})_{0}\) becomes
\[ (f_{2})_{0} = \pm \frac{1}{2} \frac{a_{1} a_{1}^{2} N_{0}^{2}}{2 N_{0}^{2} R_{i}^{2}} e^{-2U(t_{0}, t_{i})} \left( \partial_{\mu} F_{0} \right)_{i} + \left( \frac{1}{r} \frac{S}{N} B \omega \partial_{\omega} F_{0} \right)_{0} \]
\[ - \frac{a_{1} a_{1}^{2}}{2 r_{i}^{2}} \int_{t_{i}}^{t_{0}} dt N_{0}^{2} \left\{ -\partial_{r} F_{0} + \frac{S}{N} B \omega \partial_{\omega} F_{0} \right\}_{i} \]
\[ - \frac{1}{2} \frac{a_{1} a_{1}^{2}}{2 N_{0}^{2}} \int_{t_{i}}^{t_{0}} dt N_{0}^{2} \left\{ -2 \left( \frac{\dot{S}}{S} - \frac{\dot{R}}{R} - \frac{\dot{N}}{N} \mp \frac{N a_{i}'}{S a_{\perp}} \right) + \left( \frac{N}{S} \right)' \right\} \partial_{r} F_{0} \]
\[ + \left\{ -A - 2 \left( \frac{\dot{N}}{N} - \frac{\dot{S}}{S} \right) \frac{S}{N} B \right\} \frac{e^{-2U(t_{0}, t)}}{a_{1}^{2}} \]
\[ + a_{1} a_{1}^{2} \frac{d}{dt} \left( \frac{S B}{N a_{1}^{2}} \right) \left( \omega \partial_{\omega} F_{0} \right)_{i}. \] (2.48)

Next, from (2.39), we obtain
\[ (\partial_{r}^{2} F_{0})_{0} = e^{-2P(t_{0}, t_{i})} (\partial_{r}^{2} F_{0})_{i} \]
\[ + \int_{t_{i}}^{t_{0}} dt e^{-2P(t_{0}, t)} \left\{ 2 A \omega \partial_{\omega} \partial_{r} F_{0} + A' (\omega \partial_{\omega} F_{0})_{i} \pm \left( \frac{N}{S} \right); \right\} \partial_{r} F_{0} \} . \] (2.49)

Thus, substituting (2.48) and (2.49) with (2.17) and (2.34) into (2.44), we finally obtain the quadrupole formula for the CMB temperature anisotropy. As in the dipole formula case, under the assumption that our metric (2.1) is regular at the symmetry center, we can check that the above quadrupole formula is well-defined.

§3. The CMB temperature anisotropy in the LTB model

In this section, the CMB temperature anisotropy formulae obtained in the previous section will be given concrete expressions for the LTB cosmological models, and then will be applied in some specific LTB models considered in the numerical analyses of Alnes and Amarzguioui\(^{[3]}\) to check the consistency of the formulae. Further, the constraint on the location of off-center observers will be derived by our
analytic formulae with the latest WMAP data, thereby updating the previous results numerically obtained. But before doing so, we will first briefly recapitulate the LTB metric.

3.1. The LTB spacetime

A spherically symmetric spacetime with only non-relativistic matter, or dust, is described by the LTB metric, which may be given by setting in (2.1),

$$N^2 = 1, \quad S = \frac{R'(t,r)}{1 - k(r)r^2},$$

with $k(r)$ begin an arbitrary function of $r$. Then, the Einstein equations reduce to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{2GM(r)}{R^3} - \frac{k(r)r^2}{R^2}, \quad 4\pi\rho(t,r) = \frac{M'(r)}{R^2R},$$

where $M(r)$ is an arbitrary function of only $r$, and $\rho(t,r)$ is the energy density of the dust fluid. The solutions to (3.2) depend on the sign of $k(r)$ and can be expressed in parametric form: For $k(r) > 0$, we have

$$R(t,r) = \frac{M(r)}{k(r)r^2}(1 - \cos \eta), \quad t - t_s(r) = \frac{M(r)}{\{k(r)r^2\}^{\frac{1}{2}}}(\eta - \sin \eta),$$

where $t_s(r)$ is an arbitrary function of only $r$. For $k(r) = 0$, we have

$$R(t,r) = \left(\frac{9}{2}\right)^{\frac{1}{3}} M^\frac{4}{3}(r)\{t - t_s(r)\}^{\frac{2}{3}}.$$  

(3.3)

For $k(r) < 0$, we have

$$R(t,r) = \frac{M(r)}{-k(r)r^2}(\cosh \eta - 1), \quad t - t_s(r) = \frac{M(r)}{\{-k(r)r^2\}^{\frac{1}{2}}}(\sinh \eta - \eta).$$

(3.4)

The area radius $R(t,r)$ vanishes at $t = t_s(r)$, so that $t_s(r)$ is called the big-bang time. The solutions admit three arbitrary functions $k(r)$, $M(r)$ and $t_s(r)$, but due to one degree of freedom in rescaling $r$, only two of them are independent. By appropriately choosing the profile of these two arbitrary functions, one can construct LTB cosmological models that can reproduce the observed SNIa distance-redshift relation.

3.2. The analytic formula for the CMB dipole

In the LTB cosmological model, in addition to $a_\perp = R/r$ defined previously, we also introduce the ‘scale factor along the radial direction’ by $a_{\parallel}(t,r) \equiv R'(t,r)$. Accordingly, we also define two Hubble expansion rates in the radial and azimuthal direction, respectively, by

$$H_{\parallel} \equiv \frac{\dot{S}}{S} = \frac{\dot{a}_{\parallel}}{a_{\parallel}}, \quad H_{\perp} \equiv \frac{\dot{R}}{R} = \frac{\dot{a}_{\perp}}{a_{\perp}}.$$
From (2.19) and (2.23), we obtain

\[ U(t_1, t_2) = - \int_{t_1}^{t_2} dt H_{\parallel}, \quad A = H'_{\parallel}, \quad B = \pm 2(H_{\parallel} - H_{\perp}). \]  

(3.7)

Then, the analytic formula for the CMB temperature anisotropy dipole (2.29) takes the form

\[ \left( \frac{\delta T}{T} \right)^{(1)} = \mp \delta L \mathbf{n} \cdot \Omega \left\{ \frac{e^{-U(t_0, t_i)}}{S_i} \left( \frac{\partial_r F_0}{\omega \partial_\omega F_0} \right)_i + \int_0^{r_i} dr H'_{\parallel} e^{-U(t_0, t_i)} \right\}. \]  

(3.8)

Now, using this formula, we derive some constraints concerning the position of off-center observers. In general, the CMB temperature anisotropy can be decomposed in terms of the spherical harmonics \( Y_{lm} \) by

\[ \frac{\delta T}{T} = \sum_{l, m} a_{lm} Y_{lm}. \]

We are interested in \( a_{10} \) as the dipole moment. From (3.8), we obtain

\[ a_{10} = \mp \sqrt{\frac{4 \pi}{3}} \delta L \left\{ \frac{e^{-U(t_0, t_i)}}{S_i} \left( \frac{\partial_r F_0}{\omega \partial_\omega F_0} \right)_i + \int_0^{r_i} dr H'_{\parallel} e^{-U(t_0, t_i)} \right\}. \]  

(3.9)

Assuming that the universe is locally in thermal equilibrium at the last scattering surface, the first term in the bracket is of the order of \( (\partial_r T/T)_i \) because \( F_0 \) is isotropic and depends only on \( \omega/T_i \). In the models we consider in the present paper, this term can be neglected because the void size is sufficiently smaller than the horizon size and therefore, the observed region of the LTB universe is homogeneous with good accuracy on the last scattering surface. Furthermore, in order to estimate this term correctly, we have to specify the universe model before the last scattering. This is beyond the scope of this paper. Therefore, we have only estimated the contribution of the second term numerically. For the LTB model considered in 53), we have found that the induced \( a_{10} \) is about \( 1.23 \times 10^{-3} \) or less, according to WMAP data,\(^5\) which implies that the distance from the observer to the center, \( \delta L \), has to be, \( \delta L \lesssim 16 \text{Mpc} \). This is consistent with the numerical result of 53). We have also applied this formula to various LTB models, and found, for example, \( \delta L \lesssim 14 \text{Mpc} \) in the Garfinkle model,\(^{35}\) and \( \delta L \lesssim 12 \text{Mpc} \) in the GBH model.\(^{39}\)

3.3. **The analytic formula for the CMB quadrupole**

As for the quadrupole moment in the LTB model, from (2.48) and (2.49), we obtain

\[ (f_2)_0 = \mp \frac{1}{2} \frac{a_2^2}{R_i^2} e^{-2U(t_0, t_i)} (\partial_\mu F_0)_i - \frac{a_0^2}{a_{\perp}^2} \frac{e^{-2U(t_0, t_i)}}{2r_i} \left( -\partial_r F_0 + \frac{a_{\parallel}}{\xi} B \omega \partial_\omega F_0 \right)_i \]

\[ - \frac{a_0^2}{2} \int_{t_i}^{t_0} dt \frac{1}{r} \left\{ - \left\{ 2 \left( H_{\parallel} - H_{\perp} + \frac{\xi}{a_{\parallel}} a'_{\perp} \right) + \left( \frac{\xi}{a_{\parallel}} \right)' \right\} \partial_r F_0 \right. \]

\[ + \left\{ -H'_{\parallel} + 2H_{\parallel} \frac{a_{\parallel}}{\xi} B \right. \]

\[ + a_0^2 \frac{d}{dt} \left( \frac{SB}{Na_{\perp}^2} \right) \left( \omega \partial_\omega F_0 \right)_i \right\} \frac{e^{-2U(t_0, t)}}{a_{\perp}^2}, \]  

(3.10)
\[(\partial^2_r F_0)_0 = e^{-2P(t_0,t_i)}(\partial^2_r F_0)_i \]
\[+ \int_{t_i}^{t_0} dt e^{-2P(t_0,t)} \left\{ 2H'_{\parallel} \omega \partial_{\parallel} F_0 + H''_{\parallel} (\omega \partial_{\parallel} F_0)_i + \left( \frac{\xi}{a_{\parallel}} \right)^{''} \partial_r F_0 \right\}, \]
\[\tag{3.11} \]
\[\]
where \(a_0 \equiv S_0 = a_{\parallel 0} = a_{\perp 0},\) and \(\xi \equiv \sqrt{1 - k(r)r^2},\) just for notational simplicity.

\[P(t_1,t_2) = \mp \int_{t_1}^{t_2} dt \left( \frac{\xi}{a_{\parallel}} \right)^{'}, \]
\[\tag{3.12} \]
which is obtained from (2.20). Thus, we now have the analytic formula for the quadrupole moment of the CMB temperature anisotropy in the LTB model: (2.44) together with (3.10) and (3.11).

From (2.44), we derive
\[\]
\[a_{20} = -\sqrt{\frac{16\pi}{45}} \frac{(\delta L)^2}{2a_0^2} \left\{ -\frac{2(f_2)_0}{(\omega \partial_{\omega} F_0)_i} + \frac{(\partial^2_r F_0)_0}{(\omega \partial_{\omega} F_0)_i} + \frac{(S'' - a''_0)}{a_0} \right\} \]
\[+ \frac{(a_{10})^2 ((\omega \partial_{\omega} F_0)_i)}{2\sqrt{15\pi}} (\omega \partial_{\omega} F_0)_i. \]
\[\tag{3.13} \]

If the universe is locally in thermal equilibrium at the beginning, we can set \((\partial_{\alpha} \partial_{\mu} F_0)_i\) to be zero. Further, for the same reason as we explained for the dipole formula, we neglect the term \((\partial_{\alpha} \partial_r F_0)_i\) in the present paper. Under these assumption, we have estimated the quadrupole moment using this formula numerically for the model in 53), and found that \(a_{20} \simeq 8.61 \times 10^{-7}.\) This is consistent with the numerical result of 53). For other models, for example, \(a_{20} \simeq 5.51 \times 10^{-6}\) in the Garfinkle model,35) and \(a_{20} \simeq -9.27 \times 10^{-7}\) in the GBH model.39)

§4. Summary and discussions

In this paper, we have derived the analytic formulae for the dipole (2.29) and quadrupole (2.44) moments of the CMB temperature anisotropy in general spherically symmetric spacetimes, including the LTB cosmological model as a special case. The formulae can be used to compare consequences of the LTB/local void models with observations of the CMB temperature anisotropy rigorously. The formulae also enable us to identify physical origins of the CMB temperature anisotropy in the LTB models. For example, in the CMB dipole formula (2.29), the first term comes from the initial (spherical) inhomogeneity at the last scattering surface, while the second term represents the integrated Sachs-Wolfe effect. Note that the first term also contains a contribution that reduces to the second-order ISW effect in the spatially homogeneous case.

We have checked the consistency of our formulae for both dipole and quadrupole, with the widely-used recent numerical results for the special LTB model by H. Alnes and M. Amarzguioui.53) Furthermore, we applied our formulae to other LTB models, such as those in 35), 39) and in particular, for the dipole moment, we found the
constraints on the distance between the void center and an off-center observer, by
using the latest WMAP data.

We can also utilize our analytic quadrupole formula to discuss the relevance of
the LTB model to observed anomalies. For example, the observed magnitude of the
quadrupole is known to be significantly lower than the ΛCDM model predicts. This
is usually understood as a cosmic variance, i.e., to be produced by a special feature
of our Universe, one particular realization of the statistical ensemble. Because the
local void model is one of such realization with a very low probability in the standard
ΛCDM model, it is tempting to see whether the quadrupole anomaly of the CMB
anisotropy can be explained by a local void model. Unfortunately, however, the above
analysis of the constraint on the observer offset by the dipole moment implies that the
observed anomaly cannot be explained solely by the induced quadrupole moment in
LTB models such as those in 35), 39), 53). Nevertheless, this result is not conclusive.
For example, we have implicitly assumed that the off-center observer stays at a fixed
comoving position. If the observer has a peculiar velocity pointed toward the center
of the void, however, the value of δL could be chosen to be much larger than the
case with no peculiar velocity. If it is the case, then the observed anomaly of the
quadrupole could be explained within the LTB models of 35), 39), 53). Therefore, it
would also be worth attempting to develop other analytic formulae concerning CMB
polarizations, lensing effects, etc. (cf. 55)) that can be used to distinguish the LTB
and FLRW cosmologies.

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Appendix A

The regularity and derivatives of $F_0$ near the center

In this appendix, we discuss the regularity of the distribution function $F$ at
the symmetry center, and find which of $\partial_\alpha F$ (resp. $\partial_\alpha \partial_\beta F$), $\alpha, \beta = r, \omega, \mu$, become
relevant in the leading behavior of the first- (resp. second-) order derivatives of $F$
in the limit $r \to 0$.

Mathematically, the distribution function can become singular at some radius
including at the center. Furthermore, some authors concluded that a $C^2$-class
singularity of the metric should be allowed at the center in order to construct a
model with accelerated expansion exactly at the center.36) However, such a model
can be easily made smooth by appropriate smoothing and the original singular model
can be recovered as a limit such that the smoothing length approaches zero. Hence,
the final formulae for the dipole and quadrupole anisotropy of CMB given in the
present paper can be applied also to models with singularity in the metric or the distribution function if the results are finite. Hence, in this appendix, we assume that the metric and the distribution function are regular and smooth everywhere in Cartesian-type space coordinates on which the rotational symmetry group acts as on the standard Cartesian coordinates of the Euclidean space.

In such a coordinate system \( \mathbf{x} = (x^i) \), the spatial part of the metric \( (2.1) \), denoted here by \( g_{ij} \), can be written as

\[
g_{ij} = S^2 \Omega_i \Omega_j + a_\perp^2 (\delta_{ij} - \Omega_i \Omega_j), \tag{A.1}
\]

where \( \Omega^i \equiv x^i/r \) and \( a_\perp \equiv R/r \). For any smooth and rotationally invariant function \( h \) in this coordinate system, when it is Taylor expanded as

\[
h = h_0 + h_i x^i + h_{ij} x^i x^j + \cdots, \tag{A.2}
\]

each coefficient \( h_{ij} \cdots \) must be a rotationally invariant constant tensor. Hence, these coefficient tensors vanish for odd ranks and can be written as the sum of products of the Kronecker delta \( \delta_{ij} \) for even ranks. This implies that \( h \) is a smooth function of \( r^2 = \delta_{ij} x^i x^j \).

Similarly, a smooth distribution function in this coordinate system can be Taylor expanded as

\[
F(t, x, \mathbf{p}) = b_0(t, \mathbf{p}) + b_i(t, \mathbf{p}) x^i + b_{ij}(t, \mathbf{p}) x^i x^j + \cdots, \tag{A.3}
\]

where \( \mathbf{p} = (p^i) \). When \( F \) is rotationally invariant, each term on the right-hand side is rotationally invariant separately. This implies that each \( b_{i_1 \cdots i_l} \) is an \( \text{SO}(3) \) tensor depending only on the non-trivial vector \( p^i \) and therefore, can be written as the sum of products of the Kronecker delta \( \delta_{ij} \) and the vector \( \mathbf{p} = (p^i) \). This implies that \( F \) can be written

\[
F(t, x, \mathbf{p}) = \bar{f}(t, x_i x^i, |\mathbf{p}|, x_i p^i), \tag{A.4}
\]

where \( x_i x^i = r^2 \) and \( |\mathbf{p}| = (\delta_{ij} p^i p^j)^{1/2} \). Here, from

\[
p^2 \equiv g_{ij} p^i p^j = C(x_i p^i)^2 + a_\perp^2 \delta_{ij} p^i p^j, \tag{A.5}
\]

where \( C \) is a smooth function defined by

\[
C(t, r) \equiv \frac{S^2 - a_\perp^2}{r^2}, \tag{A.6}
\]

it follows that \( |\mathbf{p}| \) is a smooth function of \( t, \omega = p, x_i p^i \) and \( r^2 \). Therefore, the regularity of \( F \) at the center is equivalent to the condition that the corresponding function \( F_0(t, r, \omega, \nu) \) can be written in terms of a smooth function \( f \) with the four arguments \( t, r^2, y \equiv \ln \omega, \nu \equiv r \mu = S x_i p^i/p \)

\[
F_0(t, r, \omega, \nu) = f(t, r^2, y, \nu). \tag{A.7}
\]

Then,

\[
\partial_i F = \partial_i f = \{\partial_i (r^2)\} f_2 + (\partial_i y) f_y + (\partial_i \nu) f_\nu, \tag{A.8}
\]
where \( f_2 \equiv \partial_x f, f_y \equiv \partial_y f, \) and \( f_\nu \equiv \partial_\nu f. \) Here, because the spatial derivative of \( p^2 \) can be written

\[
\partial_i (p^2) = \left\{ \frac{C'}{r} (rp^r)^2 + \frac{(a_j^2)' \delta_{jp^j} p^k}{r} \right\} x^i + 2C(rp^r)p^i, \tag{A.9}
\]

we have

\[
\partial_i (r^2) = 2x^i, \tag{A.10}
\]

\[
\partial_i (\ln \omega) = \left\{ -\frac{N'}{rN} + \frac{a'_i}{ra_{\perp}} + \left( \frac{C'}{2r} - \frac{Ca'_i}{ra_{\perp}} \right) \left( \frac{\mu r}{S} \right)^2 \right\} x^i + C \frac{\mu r p^i}{S p}, \tag{A.11}
\]

\[
\partial_\nu = S \frac{p^i}{p} + rS \frac{p^i}{r} - rSp \frac{\partial_i (p^2)}{2p^3}. \tag{A.12}
\]

In particular, in the limit \( r \to 0, \) we see

\[
\partial_i (p^2), \partial_i (r^2), \partial_i (\ln \omega) \to O(r), \tag{A.14}
\]

\[
\partial_\nu \to S_0 \frac{p^i}{p}. \tag{A.15}
\]

Therefore, from (A.8), we find that the first derivative of \( F_0 \) behaves at the center as

\[
\partial_i F \to S_0 \frac{p^i}{p} (f_\nu)_0 = S_0 \frac{p^i}{p} \left( \frac{\partial_\nu F_0}{r} \right)_0. \tag{A.16}
\]

Next we study the second order derivatives of \( F \) with respect to \( x^i, \) which are written as

\[
\begin{align*}
\partial_i \partial_j F &= \{ \partial_i \partial_j (r^2) \} f_2 + \{ \partial_i \partial_j (r^2) \} \left[ \{ \partial_i (r^2) \} f_{22} + (\partial_i y)f_{2y} + (\partial_i \nu) f_{2\nu} \right] \\
&+ (\partial_i \partial_y) f_y + (\partial_i \nu) \left[ \{ \partial_i (r^2) \} f_{y2} + (\partial_i y)f_{yy} + (\partial_i \nu) f_{y\nu} \right] \\
&+ (\partial_i \partial_\nu) f_\nu + (\partial_i \nu) \left[ \{ \partial_i (r^2) \} f_{\nu2} + (\partial_i y)f_{\nu y} + (\partial_i \nu) f_{\nu \nu} \right]. \tag{A.17}
\end{align*}
\]

We find that

\[
\partial_i \partial_j (r^2) = 2\delta_{ij}, \tag{A.18}
\]

\[
\begin{align*}
\partial_i \partial_j (p^2) &= \left[ \left( \frac{C'}{r} \right) (rp^r)^2 + \left\{ \frac{(a_j^2)' \delta_{jp^j} p^k}{r} \right\} x^i x^j \right] + 2\frac{C'}{r} (rp^r) (p^i x^j + p^j x^i) \\
&+ \left\{ \frac{C'}{r} (rp^r)^2 + \frac{(a_j^2)' \delta_{jp^j} p^k}{r} \right\} \delta_{ij} + 2C p^i p^j, \tag{A.19}
\end{align*}
\]

\[
\begin{align*}
\partial_i \partial_j (\ln \omega) &= \frac{\partial_i \partial_j (p^2)}{2p^2} - \frac{\partial_i (p^2) \partial_j (p^2)}{2p^2} - \left( \frac{N'}{rN} \right) x^i x^j - \frac{N'}{rN} \delta_{ij}, \tag{A.20}
\end{align*}
\]

\[
\begin{align*}
\partial_i \partial_j \nu &= S \frac{r}{p} \left[ \frac{x^i x^j + 2x^i p^j}{p} - \frac{r p^r}{2p^2} \{ x^j \partial_i (p^2) + x^i \partial_j (p^2) \} \right] + \left( \frac{S'}{r} \right) \frac{r p^r}{p} \frac{x^i x^j}{r} \\
&- \frac{S r}{2p^2} \left( \partial_i \partial_j (p^2) + p^i \partial_j (p^2) + p^j \partial_i (p^2) \right) - S \frac{r p^r}{2p^2} \left\{ \partial_i \partial_j (p^2) - \frac{3}{2p^2} \partial_i (p^2) \partial_j (p^2) \right\}. \tag{A.21}
\end{align*}
\]
In particular, in the limit $r \to 0$,

$$\partial_i \partial_j (r^2) \to 2\delta_{ij},$$  \hfill (A.22)

$$\partial_i \partial_j (p^2) \to 2 \left( \frac{a''}{a} \right) p^2 \delta_{ij} + 2C_0 \rho^i \rho^j,$$  \hfill (A.23)

$$\partial_i \partial_j (\ln \omega) \to \left( \frac{a''}{a} - \frac{N''}{N} \right) \delta_{ij} + C_0 \rho^i \rho^j,$$  \hfill (A.24)

$$\partial_i \partial_j (\nu) \to O(r).$$  \hfill (A.25)

From these we find that in the limit $r \to 0$,

$$\partial_i \partial_j F \to 2\delta_{ij} (f_2)_0 + \left\{ \left( \frac{a''}{a} - \frac{N''}{N} \right) \delta_{ij} + C_0 \rho^i \rho^j \right\} (f_y)_0 + S_0 \rho^i \rho^j (f_{\nu \nu})_0.$$ \hfill (A.26)

Now, from (A.7), we find that

$$f_\nu \to \left( \frac{\partial F_0}{r} \right)_0,$$$$

f_2 \to \frac{1}{2} \left( \frac{\partial F_0}{r} - \frac{\partial \mu F_0}{r^2} \right)_0,$$$$

f_{\nu \nu} \to \frac{1}{\mu^2} \left( \partial^2 F_0 - \frac{\partial \mu F_0}{r} + \frac{\mu}{r^2} \partial \mu F_0 \right)_0,$$

$$= \frac{1}{\mu^2} \left( \partial^2 F_0 - 2f_2 \right)_0.$$ \hfill (A.27-29)

Thus, we finally obtain

$$\partial_i \partial_j F \to 2 \left( \delta_{ij} - S_0 \rho^i \rho^j \right) (f_2)_0 + S_0 \rho^i \rho^j (\partial^2 F_0)_0$$

$$+ \left\{ \left( \frac{a''}{a} - \frac{N''}{N} \right) \delta_{ij} + C_0 \rho^i \rho^j \right\} (\omega \partial \omega F_0),$$ \hfill (A.30)

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