Mathematical Abstraction: Constructing Concept of Parallel Coordinates

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Abstract. Mathematical abstraction is an important process in teaching and learning mathematics so pre-service mathematics teachers need to understand and experience this process. One of the theoretical-methodological frameworks for studying this process is Abstraction in Context (AiC). Based on this framework, abstraction process comprises of observable epistemic actions, Recognition, Building-With, Construction, and Consolidation called as RBC + C model. This study investigates and analyzes how pre-service mathematics teachers constructed and consolidated concept of Parallel Coordinates in a group discussion. It uses AiC framework for analyzing mathematical abstraction of a group of pre-service teachers consisted of four students in learning Parallel Coordinates concepts. The data were collected through video recording, students’ worksheet, test, and field notes. The result shows that the students’ prior knowledge related to concept of the Cartesian coordinate has significant role in the process of constructing Parallel Coordinates concept as a new knowledge. The consolidation process is influenced by the social interaction between group members. The abstraction process taken place in this group were dominated by empirical abstraction that emphasizes on the aspect of identifying characteristic of manipulated or imagined object during the process of recognizing and building-with.

1. Introduction

The process of constructing mathematical knowledge, also known as mathematical abstraction, is very crucial for students as well as teachers. Related to the way students learn mathematics, mathematics concepts cannot be simply transferred into students’ minds in the form of ready-made knowledge unless it should be constructed through learning activities. For teachers, information about how the process of constructing mathematical knowledge is essential to understand students’ minds so they can formulate the most appropriate learning strategy. As it is important for mathematics teachers, it needs to be accommodated in pre-service mathematics learning program.

Abstraction in this paper is defined as a process of vertically reorganizing some of the learner’s previous mathematical construct within mathematics and by mathematical means [1]. This mental process is difficult to be expected take place naturally in classroom but this process also can take place in various contexts [2]. The context includes the students’ prior history of learning, learning environment, including possibly available technological and others tools as well as mathematical, curricular, social components which may be an alternation of group work, individual work, and whole
class work[3]. One of the frameworks that specially designed for exploring and analyzing the emergence of mathematical constructs that are new to a learner is Abstraction in Context (AiC). A theoretical-methodological model is a central component of AiC. According to this model, the emergence of a new mathematical construct can be described and analyzed using three observable epistemic actions: recognizing (R), building-with (B), and constructing (C) [4].

Recognizing refers to a situation when learners seeing the relevance of a specific previous construct to the situation or problem at hand. Building-with occurs when they use and combine the recognized constructs in order to achieve a localized goal such as the actualization of a strategy, a justification, an argumentation, or the solution of a problem. Constructing takes place when they produce a new construct by assembling and integrating the previous construct through vertical mathematization. It is can be identified when for the first time the new construct is expressed or used by the learners. During the constructing, the learners probably are not aware yet that they have acquired the new construct; the new construct is often still fragile and context dependent. When the concept becomes freely and flexible for the learners it is related to consolidation.

Consolidation is a never-ending process through these students become aware of their construct, the use of the construct becomes more immediate and self-evident, and the students’ confidence in using the construct increase and the students demonstrate more and more flexibility in using the construct [5]. It can be identified when the learners engage in further activities, whenever they build-with using the new construct. Therefore, consolidation is closely related to the design of simultaneously activities.

One of the topics that can be used to develop mathematical abstraction is concept of Parallel Coordinates [6, 7]. Parallel Coordinates can be categorized as non-conventional mathematics concepts; this concept does not belong to school mathematics. Further, it does not included in advanced mathematics as part of curriculum for pre-service mathematics teachers such as topic of Abstract Algebra, Real Analysis, Complex Analysis, and Numerical Analysis.

This case study is a part of larger study in examining and analyzing the abstraction process of pre-service mathematics teachers in learning concept of Parallel Coordinates in group and whole class context. In this paper, the process of construction and consolidation of a group of pre-service mathematics teachers in learning concept of Parallel Coordinates will be examined.

2. Method
This research was carried out by using case study method (in group) consisted of four pre-service mathematics teachers; three females and one male. Each of the participants is identified with codes S3, S8, S23, and S35. This group consisted of heterogeneous pre-service mathematics teachers from the viewpoint of their achievement in their prior knowledge test in the topic of Cartesian coordinate. This study was held at a public university in Bandung, Indonesia.

In accordance with the AiC framework, two exploratory tasks were designed for the study. The first task dealt with how to construct 2D Parallel Coordinates using the components of Cartesian coordinate as their prior knowledge. The second task dealt with representation of a point and a line in 2D Parallel Coordinate.

In both tasks, the pre-service mathematics teachers were asked to propose a hypothesis regarding the mathematical situation and discussed it with their group and later share with the whole class in the classroom discussion. The activities of the pre-service mathematics teachers were audio-visually recorded during the group discussion as well as the whole class discussion.

![Diagram](image)

**Figure 1.** The Connection between Assumed Three Main Knowledge Elements
In the a priori analysis for the expected abstraction processes, the main knowledge elements were assumed for all tasks. Due to space constraint, this paper will focus only on construction and consolidation of three main knowledge elements. The connections between the knowledge elements are described in Figure 1, and an operational definition was developed for each main element to guide the analysis of pre-service mathematics teachers’ abstraction activities.

\( E_1: \) Construction of 2D Parallel Coordinate system. 2D Parallel Coordinate is constructed by placing axes in parallel position with respect to embedding of 2D Cartesian coordinate system in the plane. The coordinate axes are labeled as \( \overline{X}_1 \) and \( \overline{X}_2 \). Even the orientation of the axes can be chosen freely but in this study vertical layout is preferred. \( \overline{X}_1 \) and \( \overline{X}_2 \) are real lines and equidistant placed and perpendicular to \( x \)-axes in Cartesian coordinate. The distance between \( \overline{X}_1 \) and \( \overline{X}_2 \) is labeled as \( d \). It can be said that the participants have been constructed \( E_1 \) if they can represent 2D Parallel Coordinates with its components.

\( E_2: \) Representation of a point in 2D Parallel Coordinate. A point \( A(x,y) \) in Cartesian coordinate is represented as a line \( \overline{A}(x_1, x_2) \). It can be stated that \( E_2 \) has been constructed by the participants if they can find correspondence between a point in Cartesian coordinate and a line in Parallel coordinate. Based on the design, they have to predict and hypothesize the representation of a point \( A(x,y) \) using concept of relation between variable \( x \) and \( y \) and \( x_1 \) and \( x_2 \). If the participant assumes that a point in Cartesian coordinate is correspondence with a line segment or a line, it can be said that they have constructed the \( E_2 \). It can be considered as a reflection of the approach in this study from participants’ existing perspective rather than from an expert’s perspective.

\( E_3: \) Representation of a line \( \ell \equiv y = mx + b \) with \( m \neq 1 \) and \( m = 1 \). A line \( \ell \equiv y = mx + b \) with \( m \neq 1 \) is represented as a point \( \ell \left( \frac{d}{1-m}, \frac{b}{1-m} \right) \) and \( \ell \equiv y = mx + b \) with \( m = 1 \) is an ideal point in the direction with slope \( \frac{b}{d} \).

3. Result and Discussion

3.1. Episode 1: Constructing \( E_1 \) and \( E_2 \)

In the first task, the participants were asked to predict the representation of a point \( A(3,2) \) in 2D parallel coordinate by producing correspondence between components of 2D Cartesian coordinate and 2D Parallel Coordinates. Based on the students’ worksheet all the members of the group were successful in constructing \( E_1 \) by relating the component of 2D Cartesian coordinate and 2D Parallel Coordinates.

![Figure 2. A Photograph of A Participants’ Worksheet in Constructing \( E_1 \) and \( E_2 \)](image)

From Figure 2 it can be seen they construct the relationship between \( x \)-axes and \( \overline{x}_1 \)-axes together with \( y \)-axes and \( \overline{x}_2 \)-axes. It means that they already construct the \( E_1 \). It also can be seen from the following transcript of group discussion between S3 and S23:

3 S3 : “I think you one [she meant that \( \overline{x}_1 \)-axes] is rotated to \( x \) and \( x \) two [she meant that \( \overline{x}_2 \)-axes] is related to \( y \)”

4 S23: “That is equivalent”

5 S3 : “Yup that’s right”
There were only two participants who were engaged in group discussion related to $E_1$. Two others participants just listened to their friends but all participants successfully constructed $E_1$. It can be seen in the next episode when they consolidated $E_1$ in order to construct $E_2$ and $E_3$.

Based on analysis of participants’ worksheets it can be also identified that from four participants, three of them predict the representation of a point $A(3,2)$ is a line segment, while S35 predict that the representation is a line. From the participants’ worksheet and the video-audio transcript, it can be identified that the participants recognized that a point $A(3,2)$ on Cartesian coordinate has different representation in 2D Parallel Coordinates from the following transcript:

7  S8 : “This coordinate is…”
8  S3 : “Hmm…”[Looking at the problem]
9  S23: “I think it should be straight, like this” [put her hands parallel in front of her face]
10  S8 : “One, two, negative one,…” [looking at his worksheet]
11  S23: “But it is like this”
12  S3 : “Maybe it’s a line” [just guessing]
13  S23: “Do you think it’s a skew line?…”
14  S8 : “Look at this, it is also a point” [cut off the discussion between S3 and S23 and showed his result]
Silent
15  L  : “Please do the task one first” [Talking to the classroom]
16  S23: “Yes, I think the point also like that” [starring the answer of S3]… “But I guess that is a plane”
17  S3 : “How come it can be a plane?”
Silent

Based on the transcript it can be inferred that S3 and S23 had recognized and built-with the notion that a point $A$ will have different representation in 2D Parallel Coordinates but they still failed finding the argumentation. In the process of recognizing, a notion of “transformation” appeared in the discussion in order to build-in $E_2$ (23S8, 24S3).

23S8 : “Look at this axes, if this axes is mapped parallel, this axis will be drawn here, isn’t it? Oh that is the application of transformation” [he meant that drawn the x-axis on 2D Cartesian coordinate so that the x-axis will parallel with y-axis]
24S3 : “Yup transformation?” [as if shouting]
25  : “He..he..he..” [all participants laugh]
26  S3 : “So how?”
27  S8 : “Just draw the x-axis here”
28  S23: “Hm… I see”
29  L  : [The lecturer approached them and asked question] “You do not expected to be right away, just make a prediction first”
30  S23: “Oh… I see if it is rotated from this angle…” [talking to herself]
31  : [all participants working on their worksheet did sketch using ruler]
32  L  : “Could you explain how come you can come to this answer?”
33  S8 : “There is a transformation from a… Cartesian coordinate to parallel coordinate, in Cartesian coordinate the axes are perpendicular while in parallel coordinate the axes are parallel, my presumption is that there is a movement of the point” [Tried to explain his idea]
34S3 : “Yes, a movement”
35  L  : “That’s ok, so what is representation of the point $A$ in 2D parallel coordinate?”
36  S8 : “A line”
37  L  : “A line or a line segment?”
38S3 : “Segment”
39S8 : “A line segment”

As shown on the transcript, S8 constructed the representation of a point $A(3,2)$ in 2D Parallel Coordinates by trying to use concept of rotation of x-axis 90° counterclockwise but he did not mention the translation of x-axis after it is rotated so the x-axis will be parallel and has a certain distance with y-
axis. He realized that there was no more point $A$ represented in 2D Parallel Coordinate except points on both axes, then he connected the two points until got a line segment. His idea triggered S35 and S23 to build-with and localize the situation help by the word “*draw it*”. Finally, all of the participants constructed $E_2$ after they shared their ideas in the group discussion.

### 3.2. Episode 2: Constructing $E_3$

The construction of knowledge elements of $E_3$ was facilitated in Task 2. Task 2 consisted of two activities, first the participants were asked to sketch the graph of $y = -2x + 3$ in 2D Cartesian coordinate, then they were asked to represent graph of $y = -2x + 3$ in 2D parallel coordinate with $d = 3$. They also have to explain the process of getting the hypothesis.

All participants in this group were succeeded in sketching the graph of $y = -2x + 3$ in 2D Cartesian coordinate. Figure 3 shows the result in second activity.

![Figure 3. A Photograph of Representation of $y = -2x + 3$ in 2D Parallel Coordinate in Working Group](image-url)

Their thinking processes started by chosen an arbitrary point on line $= -2x + 3$, for example (0,3) and (1,1), then represented both points on 2D Parallel Coordinate so they got an intersection of two line segments. They imagined that the line was seen from above as a point. In the process of building-with, was also found that there was “a switch of domain” between 2D and 3D unwittingly (99S3, 103S8).

93 S3 : “What is the point?”
94 S8 : “Zero point three”
95 S35: “...And the other one is one and a half point zero, isn’t it?”
96 S8 : “ That is the line”
97 S23: “ How do you think?”
98 S35: “ I think it is a point....”
99 S3 : “ Yup I agree, this line if we see from here, become a point” [she used her hand to demonstrate the situation]
100 S8 : “Wait a minute... hm... yes I see”
101 S3 : “ Right is it?”
102 S23: “Yup”
103 S8 : “ This one if we look from here” [showing his hand on worksheet]
104 S3 : “Yup you are right”

According to the transcript, the process of recognizing and building-with $E_3$ is similar to the process when they were constructing $E_2$. However unlike the previous process, recognizing was not starting from Cartesian coordinate but they directly starting from 2D Parallel Coordinate. The aspect of identifying characteristic of imagined object appeared in episode 2, while the aspect of identifying characteristic of manipulated object appeared in episode 1 [8].

In episode 2 it can be found that sometimes the participants not realized that they worked in second dimensions (99S3, 103S8), as if they worked on 3D Cartesian coordinate so they could manipulate
(by rotating) the line in a space so that the line will be seen as a point from a certain viewpoint. It is well explained by [9] that the y-axis (in 2D Cartesian coordinate) as a second dimension of linear objects functioning as obstacles in students effort to estimate the dimension of various linear objects. In this study probably the similar phenomena also appear to pre-service mathematics teachers in different situation or context. Further study related to this topic can be very valuable to be executed.

4. Conclusion
The study traced the abstraction processes of pre-service mathematics teachers in the context of group discussion. The result of the study shows that abstraction process took place in the group that there was construction of shared knowledge from individual to a group [10]. All participants in the group may benefit from this shared knowledge. From the viewpoint of interaction between participants, the role of S8 and S3 notions’ triggered S23 and S35 to build-with and construct the element of knowledge of E1 and E2. The interactions during discussion process led them to construct “shared knowledge” in their group. The process abstraction occurred in the group can be categorized as empirical mathematical abstraction by identification key common features within real world situations.

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