Torque or no torque?! The resolution of the paradox using 4D geometric quantities with the explanation of the Trouton-Noble experiment

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In this paper we have resolved the apparent paradox of different mechanical equations for force and torque governing the motion of a charged particle in different inertial frames. The same paradox arises in all usual “explanations” of the Trouton-Noble experiment. It is shown that the real cause of the paradoxes is - the use of three dimensional (3D) quantities, e.g., E, B, F, L, N, their transformations and equations with them. Instead of using 3D quantities we deal with 4D geometric quantities, their Lorentz transformations and equations with them. In such treatment the paradoxes do not appear. The explanation with 4D geometric quantities is in a complete agreement with the principle of relativity and with the Trouton-Noble experiment.

1. Introduction

In a recent paper Jackson [1] discussed the apparent paradox of different mechanical equations for force and torque governing the motion of a charged particle in different inertial frames. Two inertial frames S (the laboratory frame) and S' (the moving frame) are considered. S' moves uniformly in the +x direction with a speed V = cβ. A point charge Q is fixed permanently at the origin in S'. In S' a particle of charge q and mass m experiences only the radially directed electric force caused by Q at the origin. At time t' = 0 in S' the particle of charge q is released at rest with r'(0) = r'_0, see figure 1(a) in [1]. Such initial conditions give that the particle has no angular momentum; it moves radially outward without torque. Thus both the angular momentum L' and the torque N' are zero in S'. (Vectors in the three dimensional (3D) space will be designated in bold-face.) In the laboratory frame S the charge Q is in uniform motion and it produces both an electric field E and a magnetic field B that are given by equations (3a) and (3b) respectively in [1]. The existence of the magnetic field B in S is responsible for the existence of the 3D magnetic force F = qV × B and this force provides a 3D torque on the charged particle relative to the fixed origin in the laboratory N = x × F, see figure 1(b) in [1]. Consequently a nonvanishing 3D angular momentum of the charged particle changes in time in S, N = dL/dt. Here we repeat Jackson’s words [1] about such result: “How can there be a torque and so a time rate of change of angular momentum in one inertial frame, but no angular momentum and no torque in another? Is there a paradox? Some experienced readers will see that there is no paradox - that is just the way things are, ...” (my emphasis) Such reasoning is
considered to be correct by many physicists. However in the considered case the principle of relativity is violated and the “explanation” of the type “that is just the way things are” does not remove the violation of the principle of relativity but only accept that violation as something natural. We consider that such an explanation as in [1] is not natural and not relativistically correct; the paradox remained completely untouched in the approach from [1]. (In the following the paradox examined in [1] will be called Jackson’s paradox.)

In this paper it will be shown that - it is not the way things are, but that there is a simple solution of the above problem which is in a complete accordance with the principle of relativity. The real cause of the paradox is - the use of 3D quantities, e.g., $E, B, F, L, N$, their transformations and equations with them. The 3D quantities are considered as physical, measurable quantities in the 4D spacetime. Instead of using 3D quantities we shall deal from the outset with 4D geometric quantities, their Lorentz transformations (LT) and equations with them. In such treatment the paradox does not appear and the principle of relativity is naturally satisfied. It is considered in our approach that in the 4D spacetime the physical reality, both theoretically and experimentally, is attributed only to the 4D geometric quantities.

The same paradox arises in all usual “explanations” of the Trouton-Noble experiment. Here it will be shown that in the explanation with 4D geometric quantities the Trouton-Noble paradox does not appear and such an explanation is in a complete agreement with the principle of relativity and with experiments.

In section 2 the standard transformations of the 3D $E$ and $B$ are quoted. In sections 3-5 different 4D geometric quantities are introduced and discussed using geometric algebra formalism. This includes the bivector field $F$, section 3, the 4D electric and magnetic fields $E$ and $B$ (1-vectors) and the relations that connect $F$ with $E$ and $B$, section 4, then the 4D Lorentz force $K_L$ (1-vector), the angular momentum $M$ (bivector) and the torque $N$ (bivector), section 5. In section 6 we have presented the Lorentz transformations of the 4D $E$ and $B$ and of other multivectors. In section 7 the standard transformations of the electric and magnetic field are derived and it is shown that they differ from the Lorentz transformations of the 4D $E$ and $B$. The most important sections are sections 8-8.4 and 9.2. The resolution of Jackson’s paradox is presented in four different ways in sections 8-8.4 using 4D geometric quantities. In the same way the resolution of the Trouton-Noble paradox is given in section 9.2. Finally section 10 refers to conclusions.

2. Standard transformations of the 3D $E$ and $B$

Both $E$ and $B$ given by (3a) and (3b) in [1] can be also obtained using the relations that connect the 3D $E$ and $B$ in relatively moving inertial frames. In general they are given by equation (11.149) from [2], which we repeat here

\[
E' = \gamma(E + \beta \times cB) - \left(\gamma^2/\gamma + 1\right)\beta(\beta \cdot E)
\]
\[
B' = \gamma(B + \beta \times E/c) - \left(\gamma^2/\gamma + 1\right)\beta(\beta \cdot B).
\]
The inverse transformations are found by interchanging primed and unprimed quantities and putting \( \beta \to -\beta \). The transformations \( (1) \) are derived by Lorentz [3], Einstein [4], and it seems that according to [5] and [6] Poincaré was the first who gave a mathematically valid derivation of the transformations of the 3D \( \mathbf{E} \) and \( \mathbf{B} \), see two fundamental Poincaré’s papers with notes by Logunov [6]. According to such relations, e.g., the electric field \( \mathbf{E} \) in one inertial frame is expressed by the mixture of \( \mathbf{E}' \) and \( \mathbf{B}' \) from relatively moving inertial frame. They are considered by almost all physicists to be the LT of the 3D \( \mathbf{E} \) and \( \mathbf{B} \), but for the reasons explained below, we shall call them the standard transformations (ST), while the name LT will be reserved for the LT of the 4D quantities.

In our case the relations \( (1) \), when written in components, become

\[
\begin{align*}
E_x &= E'_x, \quad E_y = \gamma(E'_y + \beta c B'_z), \quad E_z = \gamma(E'_z - \beta c B'_y) \\
B_x &= B'_x, \quad B_y = \gamma(B'_y - \beta E'_z/c), \quad B_z = \gamma(B'_z + \beta E'_y/c).
\end{align*}
\]

(2)

Denoting the event of the release of the particle as \( A(0, x'_A, y'_A, z'_A = 0) \). Then the components of \( \mathbf{E}'(t'_A = 0) \) in \( S' \) are \( E'_x(0) = kQx_A/\gamma^3, \quad E'_y(0) = kQy_A/r'_A^3, \quad E'_z = 0 \), where \( k = 1/4\pi\varepsilon_0 \), \( x'_A = r'_A\cos\theta'_A, \quad y'_A = r'_A\sin\theta'_A, \quad r'_A, \quad x'_A, \quad y'_A \) and \( \theta'_A \) are the values of \( r' \), \( x' \), \( y' \) and \( \theta' \) at \( t'_A = 0 \), and also we have \( \mathbf{B}' = 0 \). (In \([1]\) \( r'_A \) is denoted as \( r'_0 \).) The corresponding expressions for \( \mathbf{E} \) and \( \mathbf{B} \) in \( S \) are obtained in all usual approaches to electromagnetism by the use of the ST \( (2) \). They are

\[
\begin{align*}
E_x &= kQ(\gamma x_A - \beta ct_A)/\varsigma^3, \quad E_y = kQ\gamma y_A/\varsigma^3, \quad E_z = 0 \\
B_x &= B_y = 0, \quad B_z = \beta E_y/c
\end{align*}
\]

(3)

where \( \varsigma = [\gamma^2(x_A - \beta ct_A)^2 + y_A^2]^{1/2} \) and the LT of the coordinates of the event \( A(0, x'_A, y'_A, 0) \) are employed, \( ct'_A = 0 = \gamma(ct_A - \beta x_A), \quad x'_A = \gamma(x_A - \beta ct_A), \quad y'_A = y_A, \quad z'_A = z_A = 0 \). Jackson [1] assumed that not only \( t'_A = 0 \) than also \( t_A = 0 \). With such an assumption the relations \( (2) \) become equations \( (3a) \) and \( (3b) \) in \([1]\) but, in fact, it is not correct in this case to take that \( t_A = 0 \) as well. Namely the event of the coincidence of the origins of \( S' \) and \( S \), let it be \( O \), has the coordinates \( O(t'_O = 0, 0, 0, 0) \) in \( S' \) and \( O(t_O = 0, 0, 0, 0) \) in \( S \). Thus the events \( O \) and \( A \) are simultaneous in \( S' \), \( t'_O = t'_A = 0 \) and they cannot be simultaneous at the same time in \( S \), i.e., \( t_A \) must be \( \neq 0 \). However we are not interesting in it since only what is important here is the appearance of \( B_z \neq 0 \) in \( S \). This leads to \( d\mathbf{L}/dt \) and \( \mathbf{N} \) different from zero in \( S \) and thus to the violation of the principle of relativity in the laboratory frame \( S \).

3. The electromagnetic field \( \mathbf{F} \)

Now consider the same problem using geometric 4D quantities. This investigation will be done in the geometric algebra formalism which is presented in \([7-11]\). Physical quantities will be represented by geometric 4D quantities, multivectors that are defined without reference frames, i.e., as absolute quantities (AQs) or, when some basis has been introduced, they are represented as 4D
coordinate-based geometric quantities (CBGQs) comprising both components and a basis. Usually [7-11] one introduces the standard basis. The generators of the spacetime algebra are taken to be four basis vectors \( \{ \gamma_\mu \} , \mu = 0, \ldots, 3 \) (the standard basis) satisfying \( \gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+----) \). This basis is a right-handed orthonormal frame of vectors in the Minkowski spacetime \( M^4 \) with \( \gamma_0 \) in the forward light cone. The \( \gamma_k \) \( (k = 1, 2, 3) \) are spacelike vectors. The basis vectors \( \gamma_\mu \) generate by multiplication a complete basis for the spacetime algebra: \( 1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \gamma_5, \gamma_5 \) (16 independent elements). \( \gamma_5 \) is the pseudoscalar for the frame \( \{ \gamma_\mu \} \), \( \gamma_5 = \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \).

It is worth noting that the standard basis corresponds, in fact, to the specific system of coordinates that we call Einstein’s system of coordinates. In Einstein’s system of coordinates the standard, i.e., Einstein’s synchronization [4] of distant clocks and Cartesian space coordinates \( x^i \) are used in the chosen inertial frame. However different systems of coordinates of an inertial frame are allowed and they are all equivalent in the description of physical phenomena. For example, in [12,13] and in the second and the third paper in [14], two very different, but physically completely equivalent systems of coordinates, Einstein’s system of coordinates and the system of coordinates with a nonstandard synchronization, the everyday (radio) (”r”) synchronization, are exposed and exploited throughout the paper. For the sake of brevity and of clearness of the whole exposition, we shall mainly work with the standard basis \( \{ \gamma_\mu \} \), but remembering that the approach with 4D quantities that are defined without reference frames holds for any choice of basis.

Note that our living arena is the 4D spacetime in which, according to our opinion, physical reality, both theoretically and experimentally, is attributed only to geometric 4D quantities, AQs or CBGQs, and to physical laws expressed by such geometric 4D quantities. When physical laws are written with 4D AQs or 4D CBGQs then there is no room for the preference of any synchronization, standard or nonstandard, or, better to say, of any system of coordinates even in an inertial frame. This is examined in a geometric approach to special relativity (SR), i.e., the invariant SR, which is developed in [12-19] and compared with experiments in [14] and [17-19]. (The name invariant SR comes from the fact that such geometric approach to SR exclusively deals with AQs or with the corresponding CBGQs and every CBGQ is invariant upon the passive LT; the components transform by the LT and the basis by the inverse LT leaving the whole CBGQ unchanged. This will be explained in section 6.) In addition we remark that the usual covariant formalism does not work with geometric quantities but only with components (numbers) taken usually in the \( \{ \gamma_\mu \} \) basis; the basis is only implicit not explicit in the covariant formalism.

Although we shall utilize the geometric algebra formalism in a manner very similar to that one in the above mentioned references [7-11], our results in the electromagnetic field theory markedly differ from all previous results including [1-11]. These results are already published in the tensor formalism [12,13,16] (with tensors as AQs or equivalently as CBGQs) and also presented as e-prints [15], [17-19] both in tensor and geometric algebra formalisms. (In [12] and again in [13] it is found in a manifestly covariant way that there is, contrary to
the generally accepted opinion, a second-order electric field outside stationary superconductor with steady current.) It is important to note that these new results are completely in agreement with the principle of relativity and with experiments that test SR as can be clearly seen, e.g., from [17-19] and particularly [14].

First let us write the bivector field $F(x)$ (or, we shall also call it the electromagnetic field $F(x)$) for a charge $Q$ with constant velocity $u_{Q}$ (1-vector), see, e.g., [10] equation (7.94) or [11] equation (26), or the discussion in [19] section IV.B,

$$F(x) = kQ(x \wedge (u_{Q}/c))/|x \wedge (u_{Q}/c)|^3.$$  \hspace{1cm} (4)

In (4) $F(x)$ is written as an AQ, i.e., it is defined without reference frames. For the charge $Q$ at rest, $u_{Q}/c = \gamma_{0}$, whence

$$F(0) = kQ(x \wedge \gamma_{0})/|x \wedge \gamma_{0}|^3.$$  \hspace{1cm} (5)

All AQs in equations (4) and (5) can be written as CBGQs in some basis. We shall write them in the standard basis \{\gamma_{\mu}\}. In the \{\gamma_{\mu}\} basis $x = x^{\mu}\gamma_{\mu}$, $u_{Q} = u_{Q}^{\mu}\gamma_{\mu}$, $F = (1/2)F^{\alpha\beta}\gamma_{\alpha} \wedge \gamma_{\beta}$ (the basis components $F^{\alpha\beta}$ are determined as $F^{\alpha\beta} = \gamma^{\beta} \cdot (\gamma^{\alpha} \cdot F) = (\gamma^{\beta} \wedge \gamma^{\alpha}) \cdot F$).

4. The relations that connect $F$ with 4D $E$ and $B$

From the given $F$ one can construct electric and magnetic fields represented by different algebraic objects, e.g., 1-vectors or bivectors. Instead of using the spacetime split and the bivectors (relative vectors and relative bivectors) for the representation of the electric and magnetic fields as in [7-11], we shall make an analogy with the tensor formalism [20] and represent the electric and magnetic fields by 1-vectors $E$ and $B$ that are defined without reference frames, i.e., as AQs. Such representation with 1-vectors $E$ and $B$ and their real and complex combination is examined in, e.g., [15] and also in [17-19]. (The formulations of the classical electromagnetism in terms of the 4-vectors (components not geometric quantities) of the electric $E^{\alpha}$ and magnetic $B^{\alpha}$ fields are presented in [21-23] in the usual covariant tensor formalism. In [23] the relativistically correct definition of the electromagnetic 4-momentum with $E^{\alpha}$ and $B^{\alpha}$ is presented and used to resolve the famous “$4/3$” factor appearing in the problem of the electromagnetic mass of the classical electron.) The electric and magnetic fields defined without reference frames, i.e., independent of the chosen reference frame and of the chosen system of coordinates in it, thus as AQs, are given as

$$F = (1/c)E \wedge v + (IB) \cdot v$$

$$E = (1/c)F \cdot v, \quad B = -(1/c^2)I(F \wedge v)$$  \hspace{1cm} (6)

where $I$ is the unit pseudoscalar. ($I$ is defined algebraically without introducing any reference frame, as in [24], section 1.2.) The velocity $v$ and all other quantities entering into the relations (6) are AQs. That velocity $v$ characterizes
some general observer. We can say, as in tensor formalism [20], that $v$ is the
velocity (1-vector) of a family of observers who measures $E$ and $B$ fields. Of
course the relations for $E$ and $B$, equations (6), are coordinate-free relations
and thus they hold for any observer. The relations (6) are manifestly Lorentz
invariant equations. Note that $E \cdot v = B \cdot v = 0$, which yields that only three
components of $E$ and three components of $B$ are independent quantities.

$E$ and $B$ from (6) can be written as CBGQs in the $\{\gamma_\mu\}$ basis and they are

$$
E = E^\mu \gamma_\mu = (1/c) F^{\mu\nu} v_\nu \gamma_\mu \\
B = B^\mu \gamma_\mu = -(1/2c^2) \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} v_\nu \gamma_\mu
$$

where $\varepsilon^{\alpha\beta\mu\nu}$ is the totally skew-symmetric Levi-Civita pseudotensor, $\varepsilon^{0123} = 1$.

When some reference frame is chosen and the standard basis $\{\gamma_\mu\}$ in it and
when $v$ is specified to be in the time direction in that frame, i.e., $v = c \gamma_0$ (the $\gamma_0$
- system), which means that the observers who measure the fields are at rest in
that frame, then results of the classical electromagnetism are recovered in that
$\gamma_0$ - system. Notice that we can select a particular, but otherwise arbitrary,
inertial frame of reference as the $\gamma_0$ - system, to which we shall refer as the
frame of our “fiducial” observers (for this name see [21]). In the $\gamma_0$ - system
equation (6) becomes

$$
F = E_f \wedge \gamma_0 + (\gamma_5 B_f) \cdot \gamma_0 \\
E_f = F \cdot \gamma_0, \ B_f = -(1/c) \gamma_5 (F \wedge \gamma_0)
$$

where in the $\{\gamma_\mu\}$ basis the pseudoscalar $I$ from (6) is $\gamma_5$, $I = \gamma_5$. The subscript
“f” in the above relations stands for “fiducial” and denotes the explicit
dependence of these quantities on the $\gamma_0$ - observer, i.e., “fiducial” - observer.
It can be seen that in the $\gamma_0$ - system $E_f$ and $B_f$ do not have the temporal
components $E_0 = B_0 = 0$. Namely in the $\gamma_0$ - system with the $\{\gamma_\mu\}$ basis $E_f$
and $B_f$, written as CBGQs, are

$$
E_f = E^\mu \gamma_\mu = 0 \gamma_0 + F^{i0} \gamma_i \\
B_f = B^\mu \gamma_\mu = 0 \gamma_0 + (-1/2c) \varepsilon^{0kli} F_{kl} \gamma_i
$$

Thus $E_f$ and $B_f$ actually refer to the 3D subspace orthogonal to the specific
timelike direction $\gamma_0$. It is seen from (9) that the components of $E_f$ and $B_f$
in the $\{\gamma_\mu\}$ basis are

$$
E_0 = F^{i0}, \ B_0 = (-1/2c) \varepsilon^{0kli} F_{kl}
$$

The relation (10) is nothing else than the standard identification of the components
$F^{\mu\nu}$ with the components of the 3D vectors $E$ and $B$, see, e.g., [2]
equation (11.137) and the relations (9) and (10) are, in fact, the spacetime split
as in [7-10].

In Hestenes’ decomposition of $F$, e.g., [9] equations (58)-(60), the bivector
field $F$ is expressed in terms of the sum of a relative vector $E_H$ and a relative
bivector $\gamma B$ by making a spacetime split in the $\gamma_0$ - system

$$F = E_H + c\gamma B_H, \quad E_H = (F \cdot \gamma_0)\gamma_0$$

$$B_H = -(1/c)\gamma_5(F \wedge \gamma_0)\gamma_0. \quad (11)$$

where the subscript $H$ is for “Hestenes.” Both $E_H$ and $B_H$ are, in fact, bivectors. These relations, in the same way as the relations (8), are not manifestly Lorentz invariant equations; they are observer dependent relations. The explicit appearance of $\gamma_0$ in these expressions implies that the spacetime split is observer dependent and thus all quantities obtained by the spacetime split in the $\gamma_0$ - system are observer dependent quantities. The difference between our approach and Hestenes’ one in electromagnetism is that Hestenes deals from the outset with the spacetime split and the decomposition (11), while we start with Lorentz invariant decomposition (6) and introduce the spacetime split specifying the general velocity $v$ to be equal $c\gamma_0$.

This suggests that the relations (11) can also be made manifestly Lorentz invariant equations, as are the equations (12), by replacing $c\gamma_0$, the velocity of observers at rest, with some general velocity $v$. Then the obtained equations are

$$F = E_{Hv} + cIB_{Hv}, \quad E_{Hv} = (1/c^2)(F \cdot v) \wedge v$$

$$B_{Hv} = -(1/c^3)I[(F \wedge v) \cdot v]. \quad (12)$$

(The subscript $Hv$ is for “Hestenes” with $v$ and not, as usual [7-10], with $\gamma_0$.) Now the relations (12) completely correspond to the equations (6). The relations (12) were first presented in [17, 18]. However, it is worth noting that it is much simpler and, in fact, closer to the classical formulation of electromagnetism with the 3D $E$ and $B$ to work with the decomposition of $F$ into 1-vectors $E$ and $B$, as in (9), or in the $\gamma_0$ - system in (6), instead of decomposing $F$ into bivectors $E_{Hv}$ and $B_{Hv}$ (12), or in the $\gamma_0$ - system in (11). Thence we proceed using only the decomposition of $F$ into 1-vectors $E$ and $B$ (6), or (8).

5. $K_L, M, N$ as 4D AQs or 4D CBGQs

All quantities that appear in the problem discussed by Jackson [1] can be written as 4D AQs and equations with them will be manifestly Lorentz invariant equations. Thus the position 1-vector in the 4D spacetime is $x$. Then $x = x(\tau)$ determines the history of a particle with proper time $\tau$ and proper velocity $u = dx/d\tau$. The Lorentz force as a 4D AQ (1-vector) is $K_L = (q/c)F \cdot u$, where $u$ is the velocity 1-vector of a charge $q$ (it is defined to be the tangent to its world line). In the usual geometric algebra approaches [7-10] to SR one makes from the outset the spacetime split and writes the Lorentz force $K_L$ (1-vector) in the Pauli algebra of $\gamma_0$. Since this procedure is observer dependent we express $K_L$ in terms of AQs 1-vectors $E$ and $B$ as

$$K_L = (q/c)F \cdot u = (q/c)[(1/c)E \wedge v + (IB) \cdot v] \cdot u \quad (13)$$
see also [15,17]. (Of course the whole consideration could be equivalently made using $E_{Hv}$ and $B_{Hv}$ from [12] but with more complicated expressions.) The equivalent expression in the tensor formalism, with tensors as AQs, is given, e.g., in [20], by Vanzella, Matsas and Crater. In the general case when charge and observer have distinct worldlines the Lorentz force $K_L$ can be written as a sum of the $v-\perp$ part $K_{L\perp}$ and the $v-\parallel$ part $K_{L\parallel}$, $K_L = K_{L\perp} + K_{L\parallel}$, where

$$K_{L\perp} = \left(\frac{q}{c^2}\right)(v \cdot u)E + \left(\frac{q}{c}\right)((IB) \cdot v) \cdot u \quad (14)$$

$$K_{L\parallel} = \left(-\frac{q}{c^2}\right)(E \cdot u)v \quad (15)$$

respectively. Of course $K_L$, $K_{L\perp}$ and $K_{L\parallel}$ are all 4D quantities defined without reference frames, the AQs, and the decomposition of $K_L$ into $K_{L\perp}$ and $K_{L\parallel}$ is an observer independent decomposition. It can be easily verified that $K_{L\perp} \cdot v = 0$ and $K_{L\parallel} \wedge v = 0$. Particularly from the definition of the Lorentz force $K_L = \left(\frac{q}{c}\right)F \cdot u$ and the relation $E = \left(\frac{1}{c}\right)F \cdot v$ (from (6)) it follows that the Lorentz force ascribed by an observer comoving with a charge, $u = v$, is purely electric $K_L = qE$. Both parts of $K_L$ can be written as CBGQs in the standard basis $\{\gamma_\mu\}$

$$K_{L\perp} = \left(\frac{q}{c^2}\right)(\nu \cdot u_\nu)E^{\nu} \gamma_\mu + \left(\frac{q}{c}\right)\tilde{\varepsilon}_{\nu\rho}u^{\nu}B^{\rho} \gamma_\mu \quad (16)$$

where $\tilde{\varepsilon}_{\nu\rho} \equiv \varepsilon_{\lambda\nu\rho \lambda}v^\lambda$ is the totally skew-symmetric Levi-Civita pseudotensor induced on the hypersurface orthogonal to $v$ and

$$K_{L\parallel} = \left(-\frac{q}{c^2}\right)(E^{\nu} u_\nu) v^{\mu} \gamma_\mu. \quad (17)$$

Speaking in terms of the prerelativistic notions one can say that in the approach with the 1-vectors $E$ and $B$ $K_{L\perp}$ plays the role of the usual Lorentz force lying on the 3D hypersurface orthogonal to $v$, while $K_{L\parallel}$ is related to the work done by the field on the charge. However in our invariant SR only both components together, equations (16) and (17), have physical meaning and they define the Lorentz force both in the theory and in experiments.

Further the angular momentum $M$ (bivector), the torque $N$ (bivector) about the origin for some force $K$ (1-vector) and manifestly Lorentz invariant equation connecting $M$ and $N$ are defined as

$$M = x \wedge p, \quad p = mu,$$

$$N = x \wedge K; \quad N = dM/d\tau \quad (18)$$

where for the Lorentz force $K_L$ the torque $N$ about the origin becomes $N = x \wedge K_L$.

When $M$ and $N$ (for the Lorentz force $K_L$) are written as CBGQs in the $\{\gamma_\mu\}$ basis they become

$$M = \frac{1}{2} M^{\mu\nu} \gamma_\mu \wedge \gamma_\nu, \quad M^{\mu\nu} = m(x^\mu u^\nu - x^\nu u^\mu),$$

$$N = \frac{1}{2} N^{\mu\nu} \gamma_\nu \wedge \gamma_\nu, \quad N^{\mu\nu} = x^\mu K^{\nu}_L - x^\nu K^\mu_L. \quad (19)$$
We see that the components $M^{\mu\nu}$ ($M^{\alpha\beta} = \gamma^{\beta} \cdot (\gamma^{\alpha} \cdot M)$) from (19) are identical to the covariant angular momentum four-tensor given by equation (A3) in Jackson’s paper [1]. However $M$ and $N$ from (18) are geometric 4D quantities, the AQs, which are independent of the chosen reference frame and of the chosen system of coordinates in it, whereas the components $M^{\mu\nu}$ and $N^{\mu\nu}$ that are used in the usual covariant approach, e.g., equation (A3) in [1], are coordinate quantities, the numbers obtained in the specific system of coordinates, Einstein’s system of coordinates, i.e., in the $\{\gamma_\mu\}$ basis. Notice that, in contrast to the usual covariant approach, $M$ and $N$ from (19) are also geometric 4D quantities, the CBGQs, which contain both components and a basis, here bivector basis $\gamma_\mu \wedge \gamma_\nu$.

It is worth noting that the principle of relativity is automatically included in such a theory with geometric 4D quantities, AQs or CBGQs, whereas in the standard approach to SR [4] the principle of relativity is postulated outside the framework of a mathematical formulation of the theory.

6. The LT of 4D $E$ and $B$ and of other multivectors

In the usual Clifford algebra formalism [7-11] the LT are considered as active transformations acting on multivectors as AQs. When AQs are written as CBGQs in some basis then the components of, e.g., some 1-vector relative to a given inertial frame of reference (with the standard basis $\{\gamma_\mu\}$) are transformed by the active LT into the components of a new 1-vector relative to the same frame (the basis $\{\gamma_\mu\}$ is not changed). Furthermore the LT are described with rotors $R, RR = 1$, in the usual way as $p \to p' = RpR$ and it is $p'_\mu = \gamma'^\mu$ when the $\{\gamma_\mu\}$ basis is introduced. To an observer in the $\{\gamma_\mu\}$ basis the vector $p$ appears the same as the vector $p'$ appears to an observer in the $\{\gamma'_\mu\}$ basis. (Reversion is an invariant kind of conjugation, which is defined by $AB = BA$, $\bar{a} = a$ for any vector $a$, and it reverses the order of vectors in any given expression.) But every rotor in spacetime can be written in terms of a bivector as $R = e^{\theta / 2}$. For boosts in arbitrary direction

$$R = e^{\theta / 2} = (1 + \gamma - \gamma^\beta \gamma_0 n)/(2(1 + \gamma))^{1/2}, \quad (20)$$

$\theta = \alpha \gamma_0 n$. $\beta$ is the scalar velocity in units of $c$, $\gamma = (1 - \beta^2)^{-1/2}$, or in terms of an ‘angle’ $\alpha$ we have $\tanh \alpha = \beta$, $\cosh \alpha = \gamma$, $\sinh \alpha = \beta \gamma$, and $n$ is not the basis vector but any unit space-like vector orthogonal to $\gamma_0$; $e^\theta = \cosh \alpha + \gamma_0 n \sinh \alpha$.

One can also express the relationship between the two relatively moving frames $S$ and $S'$ in terms of rotor as $\gamma'_\mu = R \gamma_\mu R$. For boosts in the direction $\gamma_1$ the rotor $R$ is given by the relation (20) with $\gamma_1$ replacing $n$ (all in the standard basis $\{\gamma_\mu\}$). Then for any multivector $M$ the active LT are defined by the relation

$$M' = R M \bar{R}. \quad (21)$$

When the active LT (21) are applied to 1-vectors $E_\mu$ and $B_\mu$ from Eq. (9)
one finds the transformed $E'_f$ as

$$E'_f = R(F \cdot \gamma_0)\tilde{R} = (RF\tilde{R}) \cdot (R\gamma_0 \tilde{R}) = F' \cdot \gamma'_0 = \frac{R(F^{\mu\nu}\gamma_k)\tilde{R}}{R\gamma_0 \tilde{R} = E'_f^{\mu}\gamma_\mu = -\beta \gamma E^{\mu}_1 \gamma_0 + \gamma E^{\mu}_1 \gamma_1 + E^{\mu}_2 \gamma_2 + E^{\mu}_3 \gamma_3}, \quad (22)$$

which is the usual form for the active LT of the 1-vector $E_f = E^\mu_\mu \gamma_\mu$. Similarly we find for $B'_f$

$$B'_f = R \left[ -(1/c^2)\gamma_0 (F \wedge c \gamma_0) \right] \tilde{R} = R \left[ (-1/2c^2)\varepsilon^{0\mu\nu\beta} F_{\mu\nu} \gamma_\beta \right] \tilde{R} = B^{\mu}_f \gamma_\mu = -\beta \gamma B^{\mu}_f \gamma_0 + \gamma B^{\mu}_1 \gamma_1 + B^{\mu}_2 \gamma_2 + B^{\mu}_3 \gamma_3, \quad (23)$$

which is the familiar form for the active LT of the 1-vector $B_f = B^{\mu}_\mu \gamma_\mu$. It is important to note

(i) that $E'_f$ and $B'_f$ are not orthogonal to $\gamma_0$, i.e., they have temporal components $\neq 0$. They do not belong to the same 3D subspace as $E_f$ and $B_f$, but they are in the 4D spacetime spanned by the whole standard basis $\{ \gamma_\mu \}$.

The relations (22) and (23) imply that the spacetime split in the $\gamma_0$ - system is not possible for the transformed $F' = RF\tilde{R}$, i.e., $F'$ cannot be decomposed into $E'_f$ and $B'_f$ as $F$ is decomposed in the relation (5). $F' \neq E'_f \wedge \gamma_0 + c(\gamma_5 B'_f) \cdot \gamma_0$. Notice, what is very important, that

(ii) the components $E^{\mu}_f \ (B^{\mu}_f)$ from equation (22) transform upon the active LT again to the components $E^{\mu}_f \ (B^{\mu}_f)$ from equations (22) (23); there is no mixing of components. Thus by the active LT $E_f$ transforms to $E'_f$ and $B_f$ to $B'_f$.

Actually, as we said, this is the way in which every 1-vector transforms upon the active LT. The LT of the 4D $E$ and $B$ in the tensor formalism are already presented in [16] and in geometric algebra formalism in [17, 18].

The same results can be obtained with the passive LT, either by using a coordinate-free form of the LT (such one as in [12,13,15]), or by using the standard expressions for the matrix of the LT in the Einstein system of coordinates from, e.g., [2], see also the discussion about passive and active LT in Hestenes’ paper [9] and equations (93) - (95) therein. The passive LT always transform the whole 4D quantity, basis and components, leaving the whole 4D quantity unchanged. Thus under the passive LT the field bivector $F$ as a well-defined 4D quantity remains unchanged, i.e., $F = (1/2)F_{\mu\nu} \gamma_\mu \wedge \gamma_\nu = (1/2)F'_{\mu\nu} \gamma'_\mu \wedge \gamma'_\nu$ (all primed quantities are the Lorentz transforms of the unprimed ones). In the same way it holds that, e.g., $E^{\mu}_f \gamma_\mu = E^{\mu}_f \gamma'_\mu$. The invariance of some 4D CBGQ upon the passive LT reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical object for relatively moving observers. Thus in the invariant SR we consider that quantity which does not change upon the passive LT has an independent physical reality, both theoretically and experimentally.

The importance of the concept of sameness of a physical system for different observers is first emphasized in papers by Rohrlich [25] and Gamba [26] and further developed and clarified in, e.g., [12-14], where it is proved that the Lorentz
contraction and the dilatation of time belong to the “apparent” transformations and not to the “true” transformations. The “apparent” transformations do not refer to the same quantity in the 4D spacetime but to the same measurements, whereas the “true” transformations, as are the LT, refer to the same 4D quantity. For example, as explained in [12-14], in the Lorentz contraction the rest spatial length $L_0$ of a rod in its rest frame $S$ and the spatial length $L'_0$ of that rod in relatively moving inertial frame $S'$ do not refer to the same 4D tensor quantity but to two different quantities in 4D spacetime. These quantities are obtained by the same measurements in $S$ and $S'$; the spatial ends of the rod are measured simultaneously at some $t = a$ in $S$ and also at some $t' = b$ in $S'$, and $a$ in $S$ and $b$ in $S'$ are not related by the LT or any other coordinate transformation; see figure 3 in [13] and compare it with figure 1 in [13] for the correct 4D geometric quantity, the spacetime length for a moving rod. The names “apparent” and “true” transformations are introduced in Rohrlich’s paper [25]. The comparisons [14] with well-known experiments that test SR as are the Michelson-Morley experiment, the “muon” experiments, the Kennedy-Thorndike type experiments and the Ives-Stilwell type experiments explicitly show that all these experiments are in a complete agreement with the geometric approach of the invariant SR, whereas, contrary to the general belief, it is not the case for the usual approach that deals with the “apparent” transformations, the Lorentz contraction and the dilatation of time.

7. The derivation of the ST of the electric and magnetic fields

Let us now see how the ST (1) or (2) are obtained in a rigorous mathematical way from the geometric approach to SR, i.e., in the invariant SR. The ST for $E'_{st}$ and $B'_{st}$ (the subscript st is for standard) are derived assuming that the quantities obtained by the active LT of $E_f$ and $B_f$ are again in the 3D subspace of the $\gamma_0$-observer. Thus

(1’) it is supposed that for the transformed $E'_{st}$ and $B'_{st}$ again hold that $E'_{st} = B'_{st} = 0$, i.e., that $E'_{st} \cdot \gamma_0 = B'_{st} \cdot \gamma_0 = 0$ as for $E_f$ and $B_f$.

Hence, in contrast to the LT of $E_f$ and $B_f$, (22) and (23) respectively, it is assumed in all Clifford algebra formalisms, e.g., [7-11], that

$$E'_{st} = (RF\tilde{R}) \cdot \gamma_0 = F' \cdot \gamma_0 = F'^{0i} \gamma_i = E'^{ni} \gamma_i =$$

$$= E_f \gamma_1 + \gamma (E_f^2 - \beta c B_f^3) \gamma_2 + \gamma (E_f^3 + \beta c B_f^2) \gamma_3.$$

(24)

where $F' = R F \tilde{R}$. Similarly we find for $B'_{st}$

$$B'_{st} = -(1/c)\gamma_5 (F' \wedge \gamma_0) = -(1/2c)\varepsilon^{0kl} F'_{kl} \gamma_i = B'^{ni} \gamma_i =$$

$$B_f \gamma_1 + (\gamma B_f^2 + \beta \gamma E_f^3/c) \gamma_2 + (\gamma B_f^3 - \beta \gamma E_f^2/c) \gamma_3.$$

(25)

The ST of, e.g., $E_f$, are given by (24), and this relation shows that only $F$ is transformed while $\gamma_0$ is not transformed. This is the fundamental difference between the LT (22) and (23) and the ST (24) and (25). From the transformations
one simply finds the transformations of the spatial components $E_{st}^i$ and $B_{st}^i$

$$E_{st}^i = F^{i0}, \quad B_{st}^i = (-1/2c)\epsilon^{kli}F_{kl}^i,$$

(26)

which is the relation (10) with the primed quantities. As can be seen from equations (24), (25) and (26) the transformations for $E_{st}^i$ and $B_{st}^i$ are the ST of components of the 3D vectors $E$ and $B$, equation (11), which are quoted in almost every textbook and paper on relativistic electrodynamics including [3-5], see, e.g. Jackson’s book [2] section 11.10. These relations (24), (25) and (26) are explicitly derived and given in the Clifford algebra formalism, e.g., in [7] equation (18.22), [8] chapter 9 equations (3.51a,b), [10] equation (7.33) and in [11] chapter 7 equations (20a,b). Notice that, in contrast to the active LT (22) and (23),

(ii) according to the ST (24) and (25) (i.e., (26)) the transformed components $E_{st}^i$ are expressed by the mixture of components $E_f^i$ and $B_f^i$, and the same holds for $B_{st}^i$.

In all previous treatments of SR the transformations for $E_{st}^i$ and $B_{st}^i$ are considered to be the LT of the 3D electric and magnetic fields. However our analysis shows that the transformations for $E_{st}^i$ and $B_{st}^i$, equation (25), are derived from the transformations (24) and (25), which differ from the LT; the LT are given by the relations (22) and (23).

What is with the concept of sameness when the ST (24) and (25), i.e., (1) or (2), are used. It can be easily shown that $E_{st}^\gamma$ $\neq E_{st}^\gamma'$. This means that, e.g., $E_f^\gamma$ and $E_{st}^\gamma$ are not the same quantity for observers in $S$ and $S'$, and that the ST are also the “apparent” transformations. As far as relativity is concerned the quantities, e.g., $E_f^\gamma$ and $E_{st}^\gamma$, are not related to one another. The fact that they are measured by two observers ($\gamma_0$ - and $\gamma'_0$ - observers) does not mean that relativity has something to do with the problem. The reason is that observers in the $\gamma_0$ - system and in the $\gamma'_0$ - system are not looking at the same 4D physical object but at two different 4D objects. Every observer makes measurement on its own object and such measurements are not related by the LT. Thus the transformations for $E_{st}^i$ and $B_{st}^i$, (24), (25) and (20) or (11) and (24) are not the same as the LT of well-defined 4D quantities, (22) and (23). (All these results are presented in the tensor formalism in [16] and in the geometric algebra formalism in [17-19], where they are also compared with experiments.)

The knowledge of this fundamental difference between the ST and the LT enables us to resolve in a simple way Jackson’s paradox [1] that there is a torque and so a time rate of change of angular momentum in one inertial frame, but no angular momentum and no torque in another.

8. The resolution of the paradox

First let us formulate the problem using AQs. The torque $N$ about the origin as an AQ is $N = x \wedge K_L$, where $K_L$ is the Lorentz force given by (13) or (14), $E$ and $B$ for a charge $Q$ moving with constant velocity $u_Q$ can
be determined from (6) and the expression for the electromagnetic field $F$. They are

\[
E = \frac{D}{c^2} [(u_Q \cdot v)x - (x \cdot v)u_Q]
\]
\[
B = \frac{-D}{c^3} I (x \wedge u_Q \wedge v),
\]

where $D = kQ/|x \wedge (u_Q/c)|^3$ and, as before, $k = 1/4\pi\varepsilon_0$. (The relation (27) is already derived in [19].) All these quantities are AQSs, i.e., they are independent of the chosen reference frame and of the chosen system of coordinates in it. When the world lines of the observer and the charge $Q$ coincide, $u_Q = v$, then (27) yields that $B = 0$ and only an electric field (Coulomb field) remains.

The next step is to write all AQSs as CBGQSs in some conveniently chosen inertial frame with an appropriate basis in it. The main advantage of such geometric approach is that when CBGQSs are determined in a chosen inertial frame they remain unchanged in all other relatively moving inertial frames and they are independent of the chosen system of coordinates in these frames.

In our case one choice for the starting, convenient, frame is the $S'$ frame, in which a point charge $Q$ is fixed permanently at the origin ($u_Q = c\gamma'_0$), and in that frame let the observers who measure the fields are at rest, i.e., in $S'$, $v = c\gamma'_0$ in (27). Thus the $S'$ frame is the frame of our “fiducial” observers or the $\gamma_0$ - system in which results of the classical electromagnetism with the 3D $E$ and $B$ are recovered. However in contrast to the classical electromagnetism we are not concerned with the 3D $E$ and $B$ than by the 4D $E$ and $B$, which have only spatial components in the frame of “fiducial” observers. (Notice that, as already said, the results do not depend on our choice for the $\gamma_0$ - system.) Further in $S'$ we choose Einstein’s system of coordinates, that is, the $\{\gamma_\mu\}$ basis.

When we show that the torque $N = (1/2)N'_{\mu\nu}\gamma'_{\mu} \wedge \gamma'_{\nu} = 0$ in $S'$ then due to the invariance of any CBGQ upon the passive LT $N$ will be zero in all other relatively moving inertial frames, thus in the laboratory frame, the $S$ frame, as well,

\[
N = (1/2)N'_{\mu\nu}\gamma'_{\mu} \wedge \gamma'_{\nu} = (1/2)N_{\mu\nu}\gamma_{\mu} \wedge \gamma_{\nu} = 0. \quad (28)
\]

The paradox does not appear since the principle of relativity is automatically satisfied in such an approach to SR which exclusively deals with geometric 4D quantities, i.e., AQSs or CBGQSs.

8.1. The proof that all $N'_{\mu\nu} = 0$ in $S'$

Now let us show that all components $N'_{\mu\nu}$ are zero in $S'$. $N'_{\mu\nu} = x'_{\nu}K'_{\mu} - x'_{\mu}K'_{\nu}$. The components $x'_{\mu}$ are as in section 2 but we write them without the subscript “$A$,” $x'_{\mu} = (ct', x'_{1}, x'_{2}, 0)$. The components of the Lorentz force $K_L$ are determined from the relations (16) for $K_L\perp$ and (17) for $K_L\parallel$. The electric and magnetic fields, $E$ and $B$ respectively, are determined from the relation (27), taking into account that in $S'$ $v = c\gamma'_0$, which yields that their temporal components are zero in $S'$ (as in (9)). (The $S'$ frame is the frame of “fiducial” observers.) Further in $S'$ $u_Q = c\gamma'_0$ as well, which, from (27), yields,
as already said, that the whole \( B = 0; B = B^\mu \gamma^\mu \). (Notice that due to invariance of any CBGQ upon the passive LT the magnetic field \( B^\mu \gamma^\mu = 0 \) in the laboratory frame \( S \) too.) The electric field is

\[
E = E^\mu \gamma^\mu = D(x^1 \gamma_1^\mu + x^2 \gamma_2^\mu),
\]

where \( D = kQ/r^3 \). Of course, the spatial components of \( E \) are the same as the components of \( E'(t'_A = 0) \) from section 2 as it must. In \( S' \) the velocity 1-vector of the charge \( q \) (at \( t' = 0 \)) is \( u = c\gamma_0', \) i.e., \( u = v(= u_Q) = c\gamma_0' \); in \( S' \) both charges \( Q \) and \( q \) are at rest. This yields that in \( K_L \), which is purely electric, \( K_{L||} = 0 \) and \( K_{L\perp} = q(E^1 \gamma_1^\mu + E^2 \gamma_2^\mu) \). Thus it holds that

\[
K_L = K_{L\perp} = qE = qE^\mu \gamma^\mu
\]

and it is \( qE^\mu \gamma^\mu \) in \( S \). Then the torque \( N \) becomes

\[
N = (x^1 K_L^{2} - x^2 K_L^{1})(\gamma_1^\mu \gamma_2^\mu) = qD(x^1 x^2 - x^2 x^1)(\gamma_1^\mu \gamma_2^\mu) = 0.
\]

Taking into account the relations (28) and (31) we conclude that there is no violation of the principle of relativity and consequently the paradox does not appear in our approach with geometric 4D quantities.

### 8.2. The proof that \( N^{\mu\nu} = 0 \) in \( S \) using the LT of \( K_L \) and \( x \)

Although the relations (28) and (31) complete the proof that the torque \( N \) is zero in all relatively moving inertial frames if it is zero in any one of them we shall, for readers’ convenience, explicitly show that the torque \( N \) is zero in the laboratory frame, the \( S \) frame, if it is zero in the \( S' \) frame. This can be shown in different ways.

One way is to explicitly show that all \( N^{\mu\nu} = 0 \) when \( N'^{\mu\nu} = 0 \) using directly the passive LT of the CBGQs \( K_L^{\mu} \gamma^\mu \) and \( x^{\mu} \gamma^\mu \). The components of \( N \) in the \( S \) frame are \( N^{\mu\nu} = x^\mu K_L^{\nu} - x^\nu K_L^{\mu} \), where the components \( x^\mu \) and \( K_L^{\mu} \)

\[
x^\mu = (\gamma \beta x^1, \gamma x^3, x^2, 0),
\]

\[
K_L^{\mu} = (\gamma \beta K_L^{1}, \gamma K_L^{3}, K_L^{2}, 0),
\]

are obtained by the LT from \( x'^\mu = (0, x^1, x^2, 0) \) and \( K_L'^{\mu} = (0, K_L^{1}, K_L^{2}, 0) \). In fact, the whole CBGQs \( x'^{\mu} \gamma^\mu ' \) and \( K_L'^{\mu} \gamma^\mu ' \) are transformed by the passive LT from \( S' \) to \( S \), and it holds that \( x = x'^{\mu} \gamma^\mu = x^{\mu} \gamma^\mu \) and similarly for \( K_L \). Then it is easy to see that all components \( N^{\mu\nu} \) are zero except \( N^{02} = (-N^{20}) = \gamma \beta(x^1 K_L^{2} - x^2 K_L^{1}) \) and \( N^{12} = (-N^{21}) = \gamma(x^3 K_L^{2} - x^2 K_L^{1}) \), but due to (31) they are also zero, whence it follows that all \( N^{\mu\nu} = 0 \) and consequently

\[
N = (1/2)N^{\mu\nu} \gamma^\mu \wedge \gamma^\nu = 0.
\]

### 8.3. The proof that \( N^{\mu\nu} = 0 \) in \( S \) using the LT of \( E \) and \( B \). The frame of “fiducial” observers is the \( S' \) frame
Another way is, e.g., to use the passive LT corresponding to the active ones for the transformations of CBGQs \( E^{\mu\gamma_\mu} \) and \( B^{\mu\gamma_\mu} \) to \( E^\mu \gamma_\mu \) and \( B^\mu \gamma_\mu \). We suppose, as above, that the observers who measure the fields are at rest in \( S' \), i.e., \( v = c \gamma_0' \mu \) or \( u = c \gamma_0' \mu \) in \( S' \). (It is already mentioned that with this choice \( u = v(=u_Q) = c \gamma_0' \).) The CBGQ \( E^{\mu\gamma_\mu} \) is given by \( \!< \!< \!) \) and all \( B^\mu \) are zero, i.e., \( B = B^\mu \gamma_\mu = 0 \). The CBGQs \( E^\mu \gamma_\mu \) and \( B^\mu \gamma_\mu \) in \( S \) are determined by the passive LT of fields (corresponding to the active LT \( \!< \!) \)), whence the components in \( S \) are

\[
E^\mu = (\gamma \beta E^1, \gamma E^2, 0), \quad B^\mu = 0. \tag{33}
\]

Notice that in \( S \) there is a temporary component \( E^0 = \gamma \beta E^1 \) and there is no magnetic field in relatively moving inertial frame \( S \) if it was zero in the frame of “fiducial” observers, here the \( S' \), i.e., \( \! \!< \!) \), i.e., \( 1 \) or \( 2 \). Remember that upon the passive LT the unit 1-vectors \( \gamma'_\mu \) transform to \( \gamma_\mu \) and it holds that \( E = E^{\mu\gamma_\mu} \gamma_\mu = E^{\mu\gamma_\mu} \) (the same quantity for observers in \( S' \) and \( S \)) and also \( B = B^{\mu\gamma_\mu} \gamma_\mu = B^{\mu\gamma_\mu} \). When \( K_L \) is written as a CBGQ in \( S \) and in the \( \{ \gamma_\mu \} \) basis it is given as the sum of \( \! \!< \!) \) and \( \! \!< \!) \)

\[
K_L = (q/c^2)[(v^\nu u_\nu)E^\mu + \tilde{\varepsilon}^\mu_{\nu\rho} u^\nu \gamma cB^\rho - (E^\nu u_\nu)v^\mu]\gamma_\mu. \tag{34}
\]

Now comes an important point. The CBGQs \( v^\mu \gamma_\mu \) and \( u^\mu \gamma_\mu \) in \( S \) are also determined by the passive LT from those in \( S' \), the observers who were at rest in \( S' \), the “fiducial” observers, are now moving in \( S \), and the charge \( q \) is also moving in \( S \). Thence in \( S \) the components are

\[
v^\mu(=u^\mu) = (\gamma c, \gamma \beta c, 0, 0) \tag{35}
\]

(for the whole CBGQ it again holds \( v = v^\mu \gamma'_\mu = v^\mu \gamma_\mu \) and the same for \( u \)). Equation \( \! \!< \!) \) together with \( \! \!< \!) \) leads to

\[
K_L^0 = q(1 - \gamma^2)E^0 + q\beta\gamma^2 E^1, \\
K_L^1 = q(1 + \beta^2\gamma^2)E^1 - q\beta\gamma^2 E^0, \\
K_L^2 = qE^2, \\
K_L^3 = qE^3. \tag{36}
\]

It is worth noting that the magnetic field \( B \) does not appear in the Lorentz force. Such result for \( B \) is obtained not only in \( S' \), the frame of “fiducial” observers, but in the laboratory frame \( S \) as well. Using the LT of \( E^\mu \) \( \! \!< \!) \) we get

\[
K_L^0 = q\beta\gamma E^1, \quad K_L^1 = q\gamma E^1, \quad K_L^2 = qE^2, \quad K_L^3 = qE^3 = 0. \tag{37}
\]

Then it can be seen that only \( N^{02} = x^0 K^2_L - x^2 K^0_L \) and \( N^{12} = x^1 K^2_L - x^2 K^1_L \) remain. However they are also zero \( N^{02} = q\beta\gamma(x^1 E^2 - x^2 E^1) = 0 \) and \( N^{12} = q\gamma(x^1 E^2 - x^2 E^1) = 0 \), since \( E^{1i} = Dx^{1i} \). Once again it is obtained that \( N = (1/2)N^{\mu\nu} \gamma_\mu \wedge \gamma_\nu = 0 \).

Of course if instead of using the passive LT of \( E \) and \( B \) (corresponding to \( \! \!< \!) \)) we deal with the ST \( \! \!< \!), i.e., \( \! \!< \!) \), then the 3D magnetic field will appear
in the Lorentz force in the $S$ frame. This will cause that in $S$ the 3D torque will be different from zero and the principle of relativity will be violated.

8.4. The proof that $N^{\mu\nu} = 0$ in $S$ using the expressions (27) for $E$ and $B$. The frame of “fiducial” observers is the $S$ frame

Let us now assume that the laboratory frame $S$ is the frame of “fiducial” observers ($v = c\gamma_0$, $v^\mu = (c, 0, 0, 0)$ in $S$) in which the temporal components of the 4D $E$ and $B$ are zero and only their spatial components remain. In the laboratory frame $S$ both charges $Q$ and $q$ are moving and the components in the CBGQs $u_Q^\mu\gamma_\mu$ and $u^\mu\gamma_\mu$ are given as

$$u_Q^\mu = u^\mu = (\gamma c, \gamma\beta c, 0, 0).$$  (38)

The fields $E$ and $B$ as AQs are given by (27) and when they are written as CBGQs in $S$ then $v = c\gamma_0$ and the components of $u_Q$ are determined by (38). The components $E^\mu$ become $E^0 = E^3 = 0$, $E^1 = D\gamma(x^1 - \beta x^0)$, $E^2 = D\gamma x^2$. Taking into account that in $S'$ $t' = 0$, i.e., $x'^0 = \gamma(x^0 - \beta x^1) = 0$, the relation $x^0 = \beta x^1$ is obtained. Inserting this last relations into expressions for $E^\mu$ we find

$$E^0 = E^3 = 0, \quad E^1 = Dx^1/\gamma, \quad E^2 = D\gamma x^2.$$  (39)

The charge $Q$ moves in the $S$ frame (now it is the frame of “fiducial” observers), which yields that the magnetic field $B = B^\mu\gamma_\mu$ is now different from zero. The components $B^\mu$ are

$$B^0 = B^1 = B^2 = 0, \quad B^3 = (1/c)D\gamma\beta x^2 = \beta E^2/c.$$  (40)

The spatial components $E^i$ and $B^i$ from (33) and (40) are the same as the usual expressions for the components of the 3D vectors $E$ and $B$. Inserting (33) and (40) into (34) we find the expression for the Lorentz force $K_L$ in the laboratory frame $S$. The components of $K_L$ in $S$ are

$$K_L^0 = q\gamma\beta E^1, \quad K_L^1 = 0, \quad K_L^3 = K_L^0 = 0,$$

$$K_L^2 = q\gamma E^2, \quad K_L^2 = q\gamma(E^2 - \beta cB^3) = qE^2/\gamma.$$  (41)

We see that in the laboratory frame $S$, when it is the frame of “fiducial” observers, there is the 4D magnetic field (40) which enters into the expression for the total 4D Lorentz force $K_L$. Then using (39), (40), (41) and the relation $x^0 = \beta x^1$ one easily finds all components $N^{\mu\nu}$

$$x^3 = 0, \quad K_L^3 = 0 \Rightarrow N^{03} = N^{13} = N^{23} = 0,$$

$$K_L^0 = \beta K_L^1 \Rightarrow N^{01} = x^1(\beta K_L^1 - K_L^0) = 0,$$

$$K_L^2 = q\gamma(E^2 - \beta cB^3) = qE^2/\gamma \Rightarrow N^{02} = N^{12} = 0.$$  (42)

Thus although in $S$ there is the 4D magnetic field (40) and a part of $K_L$ (in $K_L^2$ in (41)), which corresponds to the magnetic force, it is again obtained that
all components $N^\mu\nu$ are zero, $N^\mu\nu = 0$, and consequently $N = (1/2)N^\mu\nu\gamma_\mu \wedge \gamma_\nu = 0$. This proof is very instructive since it nicely clarifies the fundamental difference between the usual approaches with 3D quantities and our approach with 4D geometric quantities. In the usual approaches the 3D magnetic field $\mathbf{B}$ (which arises from the ST (1) of the 3D $\mathbf{E'}$) yields the 3D magnetic force $q\mathbf{V} \times \mathbf{B}$ and this causes that the 3D torque $\mathbf{N}$ is different from zero in the laboratory frame $S$. On the other hand when geometric 4D quantities are used then the 4D torque $\mathbf{N}$ is zero despite of the fact that in $S$ the charge $Q$, which is moving in $S$, produces both the 4D $\mathbf{E}$ and $\mathbf{B}$ (given by equation (27)). The conclusion that can be drawn from this proof is that the real cause of the violation of the principle of relativity and of Jackson’s paradox is the use of 3D quantities as physical quantities in the 4D spacetime.

We see that always the same result (28) is obtained. This consideration explicitly shows the consistency of the approach with geometric 4D quantities. In addition the proofs from sections 8.1-8.3 once again reveal that the relativistically correct transformations of the 4D electric and magnetic fields, which are in a complete agreement with the principle of relativity, are the LT (22) and (23) and not, as generally believed, the ST (24), (25) and (26) or (1) and (2).

9. Comparison with the Trouton-Noble experiment

The main difference between our geometric approach to the considered problem and the approach in Jackson’s paper [1] is that in the geometric approach the independent physical reality is attributed only to the geometric 4D quantities, AQs or CBGQs, and not, as usual, to the 3D quantities. In [1] even the covariant quantities, e.g., $M^\mu\nu$, $x^\mu$, $u^\nu$, $F^{\alpha\beta}$, etc. are considered as auxiliary mathematical quantities from which “physical” 3D quantities are deduced. However the considerations in the preceding sections and in [12-19] show that the geometric approach is, as already said, in a complete agreement with the principle of relativity and with experiments, see [14] and [16-18].

In this section we shall discuss the Trouton-Noble experiment [27], see also [28], comparing the usual explanations with our geometric approach that explicitly uses AQs or CBGQs. In the experiment they looked for the turning motion of a charged parallel plate capacitor suspended at rest in the frame of the earth in order to measure the earth’s motion through the ether. The explanations, which are given until now (see, e.g., [29-33] and references therein) for the null result of the experiments [27] ([28]) are not correct from the invariant SR viewpoint, since they use quantities and transformations that are not well-defined in the 4D spacetime; e.g., the Lorentz contraction, the nonelectromagnetic forces of undefined nature, the ST for the 3D $\mathbf{E}$ and $\mathbf{B}$ (1) (or (24), (25) and (26)) and for the 3D torque, etc.. In all previous treatments it is found that there is no 3D torque $\mathbf{N}$ for the stationary capacitor since there is only a 3D electric force $q\mathbf{E}$ in the rest frame of the capacitor. However, a 3D torque is always obtained for the moving capacitor that is caused by the 3D magnetic force $q\mathbf{V} \times \mathbf{B}$; the existence of a 3D magnetic field $\mathbf{B}$ in that frame comes from the transformations
Everything happens in the same way as in the above discussed Jackson’s paradox. Then, in order to get the agreement with experiments (and with the principle of relativity), different explanations are offered for the existence of another 3D torque, which is equal in magnitude but of opposite direction giving that the total 3D torque is zero. In our approach the explanation for the null result is very simple and natural; all quantities are invariant 4D quantities, which means that their values are the same in the rest frame of the capacitor and in the moving frame. Thus if there is no torque (but now as a geometric, invariant, 4D quantity) in the rest frame then the capacitor cannot appear to be rotating in a uniformly moving frame. This explanation is the same as that one for Jackson’s paradox which is presented in sections 8-8.4.

We shall not discuss previous explanations given in [29-32] but only the recent “resolution” presented in [33]. It is argued there that the Trouton-Noble paradox is resolved once the electromagnetic momentum of the moving capacitor is properly taken into account. First it is obtained that there is a 3D mechanical torque on the moving capacitor, due to the 3D magnetic force, and then it is shown that the rate of change of the 3D angular electromagnetic field momentum associated with the moving capacitor completely balances the 3D mechanical torque. The consideration with 4D quantities and their LT will reveal that there is no need either for the nonelectromagnetic forces and their torque, [29-32], or for the angular electromagnetic field momentum and its rate of change, i.e., its torque [33]. Therefore we shall examine in more detail the calculation of the 3D torque that is presented in [33], but we do not need to consider the calculation of the 3D angular electromagnetic field momentum. (It is worth noting that the resolution of the the Trouton-Noble paradox using geometric 4D quantities is already presented in [19], but there we dealt with the electromagnetic field $F$.)

### 9.1. Jefimenko’s calculation [33] of the 3D torque

First let us discuss Jefimenko’s calculation [33] of the 3D torque. The rest frame of a thin parallel-plate capacitor is the $S’$ frame. In the $S$ frame the capacitor moves with uniform 3D velocity $V$ in the positive direction of the $x^1$ - axis. (figure 1 from [33] is actually a projection onto the hypersurface $t’ = const.$, which means that $x, y$ and $\Theta$ from that figure 1 would need to be denoted as $x^1, x^2$ and $\Theta'$ respectively.) In the $S’$ frame $A$ denotes the surface area of the capacitor’s plates, $a$ is the distance between the capacitor’s plates and $\Theta’$ is the angle between the line joining the axis of rotation (i.e., the middle of the negative plate) with the middle of the positive plate and the $x^2$ axis. That line is taken to be in the $x^1, x^2$ plane. The uniform surface charge density on the capacitor’s plates is $\sigma$. In the $S’$ frame there is only a 3D electric force and, in the same way as in Jackson’s paradox with two charges, the 3D torque on the stationary capacitor is zero $N’ = r’ \times F’ = 0$; the total 3D force $F’$ and $r’$ are along the same line. In components $N'$ becomes

\[
N_i’ = \varepsilon_{ijk}r_j’F_k’ = 0,
\]  

(43)
where \( F'_k \) are the components of the total 3D force (electric) acting on the positive plate of the stationary capacitor, \( F'_1 = (\sigma^2 A/2\varepsilon_0) \sin \Theta' \), \( F'_2 = -(\sigma^2 A/2\varepsilon_0) \cos \Theta' \) and \( F'_3 = 0 \), and \( r'_i \) are the components of the lever arm joining the axis of rotation with the point of application of the resultant 3D force, i.e., the midpoint of the positive plate,

\[
r'_1 = -a \sin \Theta', r'_2 = a \cos \Theta', r'_3 = 0,
\]

see figure 1 in [33]. (The 3D electric field produced by the negative plate of the capacitor at the location of the positive plate is

\[
E'_x = ( -\sigma / 2\varepsilon_0 a ) ( r'_1 i'_1 + r'_2 j'_1 ),
\]

whence \( F' = qE'_x = (\sigma A)E'_x \), or in components

\[
F'_1 = Cr'_1, F'_2 = Cr'_2, F'_3 = 0, C = -\sigma^2 A / 2\varepsilon_0 a.
\]

\( i' \) and \( j' \) are unit 3D vectors in the direction of the \( x' \) - and \( y' \) - axis, respectively and \( q \) is the total charge residing on the positive plate.) In equations (43)-(46) the components of the 3D vectors \( N'_i \), \( r'_i \) and \( F'_i \) are written with lowered (generic) subscripts, since they are not the spatial components of 4D quantities. This refers to the third-rank antisymmetric \( \varepsilon \) tensor too. The super- and subscripts are used only on components of 4D quantities. Then in [33] the 3D torque experienced by the moving capacitor is determined by using “relativistic” (my quotation-marks) transformation equations for the torque. These “relativistic” transformation equations for the 3D torque given in [33] are

\[
N_1 = N'_1 / \gamma, N_2 = N'_2 + \beta^2 r'_1 F'_3, N_3 = N'_3 - \beta^2 r'_1 F'_2,
\]

where \( \beta = V/c, \gamma = (1 - \beta^2)^{-1/2} \). Equations (47) are equations (1)-(3) in [33].

The transformations (47) of the 3D \( N \) are found, e.g., in Jefimenko’s book [34]. In section 8 in [34], under the title: “From relativistic electromagnetism to relativistic mechanics,” the transformations of different 3D quantities are presented. Among others in section 8-6 in [34] the transformations of a 3D torque are presented. Jefimenko [34], as all others, considers that the transformations (47) are the LT, but we shall call them the ST of the 3D \( N \) (in analogy with the ST of the 3D \( E \) and \( B \) (1) or (2)) since they are not the LT of 4D quantities. The same name, the ST, will be used for the transformations of all other 3D quantities, e.g., the usual transformations of components of the 3D angular momentum \( L \) that are given by equation (11) in [1], then the transformations of the 3D force \( F \) that are given by equations (8-5.1)-(8-5.3) in [34], or by equations (1.56)-(1.58) in [35], then the well-known transformations of the 3D velocity \( V \) given, e.g., by equations (11.31) in [2], or equations (7-2.5)-(7-2.7) in [34], etc. All mentioned transformations of the 3D quantities are, in fact, the “apparent” transformations that are discussed in section 6.

Now let us proceed with the derivation of the 3D torque from [33]. Taking into account in the ST of the 3D \( N \) (47) that \( N'_0 = 0 \) and \( F'_3 = 0 \) Jefimenko [33] finds that \( N_3 \) component is different from zero

\[
N_3 = -(V^2/c^2) r'_1 F'_2.
\]
This result is commented in [33] in the following way: “We have thus obtained a paradoxical result: contrary to the relativity principle, although our stationary capacitor experiences no torque, the same capacitor moving with uniform velocity along a straight line appears to experience a torque. What makes this result especially surprising is that we have arrived at it by using relativistic transformations that are based on the very same relativity principle with which they now appear to conflict.” (my emphasis) Thus again the same paradox arises with the violation of the principle of relativity as in the above discussed Jackson’s paradox. It is assumed in [33], as in many other papers including [29-32], that the transformations (47) are the relativistic transformations, i.e., the LT, that are based on the principle of relativity. Such opinion implicitly supposes that 3D quantities, their transformations and physical laws written in terms of them are physically real in the 4D spacetime and in agreement with the principle of relativity. Actually such opinion prevails already from Einstein’s fundamental work on SR [4].

9.2. Resolution of the Trouton-Noble paradox in the invariant SR

The approach of the invariant SR [12-19] is completely different. There, as already explained, the physical reality in the 4D spacetime is attributed only to geometric 4D quantities, AQs or CBGQs, their LT and physical laws written in terms of them. The principle of relativity is automatically included in such formulation.

Thence in the 4D spacetime we are dealing with the Lorentz force

\[ \mathbf{K} = \left(\frac{q}{c}\right) \mathbf{F} \cdot \mathbf{u}, \]

where \( \mathbf{u} \) is the velocity 1-vector of a charge \( q \). The torque \( \mathbf{N} \), as a 4D AQ, is defined as a bivector

\[ \mathbf{N} = \mathbf{r} \wedge \mathbf{K} = \mathbf{r} \times \mathbf{F}, \]

where \( \mathbf{r} \) is 1-vector associated with the lever arm, \( \mathbf{x}_P \) and \( \mathbf{x}_O \) are the position 1-vectors associated with the spatial point of the axis of rotation and the spatial point of application of the force \( \mathbf{K} \), \( P \) and \( O \) are the events whose position 1-vectors are \( \mathbf{x}_P \) and \( \mathbf{x}_O \).

In general, as in [15], the proper velocity \( u \) for a point particle is \( u = \frac{dx}{d\tau} \), \( \tau \) is the proper time, \( p \) is the proper momentum \( p = mu \), the proper angular momentum of a particle is the bivector \( \mathbf{M} = \mathbf{x} \wedge \mathbf{p} \) and the torque \( \mathbf{N} \) about the origin is the bivector \( \mathbf{N} = d\mathbf{M}/d\tau = \mathbf{x} \wedge \mathbf{K} \), where in this relation \( \mathbf{K} \) is an arbitrary force 1-vector. When \( \mathbf{K} \) is written as a CBGQ in the standard basis \( \{\gamma_\mu\} \) then its components are \( K^\mu = (\gamma_u F_i U_i/c, \gamma_u F_1, \gamma_u F_2, \gamma_u F_3) \), and the components of \( u \) in the \( \{\gamma_\mu\} \) basis are \( u^\mu = (\gamma_u c, \gamma_u U_1, \gamma_u U_2, \gamma_u U_3) \). \( \gamma_u = (1 - U^2/c^2)^{-1/2} \). \( F_i \) are components of the 3D force \( \mathbf{F} \) and \( U_i \) are components of the 3D velocity \( \mathbf{U} \). We see that only when the considered particle is at rest, i.e., \( U_i = 0 \), \( \gamma_u = 1 \) and consequently \( u^\mu = (c, 0, 0, 0) \), then \( K^\mu \) contains only the components \( F_i \), i.e., \( K^\mu = (0, F_1, F_2, F_3) \). However even in that case \( u^\mu \) and \( K^\mu \) are the components of geometric 4D quantities \( u \) and \( K \) in the \( \{\gamma_\mu\} \) basis.
and not the components of some 3D quantities \( U \) and \( F \). The LT correctly transform the whole 4D quantity, which means that there is no physical sense in such transformations like (47); these transformations are not relativistic and they are not based on the principle of relativity. All conclusions derived from such relations as are equations (47) have nothing in common with SR as the theory of the 4D spacetime.

After this digression we go back to the resolution of the Trouton-Noble paradox in the invariant SR. Since we have the same problem as in the above discussed Jackson’s paradox we could use any of the proofs from sections 8.1 - 8.4, but, for simplicity, we shall consider only the proof from section 8.1. As in section 8.1 the \( S' \) frame is the frame of “fiducial” observers and it is the rest frame of the capacitor. In that frame we choose that

\[
\begin{align*}
    r'_{0} &= x'_{0}, \\
    r'_{1} &= x'_{1} - x'_{0}, \\
    r'_{2} &= x'_{2} - x'_{0}, \\
    r'_{3} &= x'_{3} - x'_{0},
\end{align*}
\]

are the same as in (44) (remember the convention about lowered (generic) subscripts for 3D quantities that is mentioned in connection with equation (43)). Further, for the same reasons as in section 8.1, we have that

\[
K'_{\mu} = (0, F'_{1}, F'_{2}, 0),
\]

where \( K'_{1} \) and \( K'_{2} \) are the same as in (46), i.e.,

\[
K' = (\sigma A)(E'_{1} \gamma'_{1} + E'_{2} \gamma'_{2}) = C(r'^{1} \gamma'_{1} + r'^{2} \gamma'_{2}).
\] (50)

This yields that

\[
N'^{10} = N'^{13} = N'^{23} = 0 \text{ and only remains } N'^{12} = r'^{1} K'^{2} - r'^{2} K'^{1}, \text{ which, taking into account (50), becomes}
\]

\[
N'^{12} = C(r'^{1} r'^{2} - r'^{2} r'^{1}) = 0.
\] (51)

Thus all \( N'^{\alpha \beta} \) are zero in the \( S' \) frame in which the capacitor is at rest. Since the CBGQ \((1/2)N'^{\mu \nu} \gamma'_{\mu} \wedge \gamma'_{\nu}\) is an invariant quantity upon the passive LT we have proved, as in section 8.1, that not only the components \( N'^{\alpha \beta} \) are zero but at the same time that the whole torque \( N \) is zero

\[
N = (1/2)N'^{\mu \nu} \gamma_{\mu} \wedge \gamma_{\nu} = (1/2)N^{\mu \nu} \gamma_{\mu} \wedge \gamma_{\nu} = 0.
\] (52)

Thence the torque is zero not only for the stationary capacitor but for the moving capacitor as well. We see that in the approach with geometric 4D quantities there is no Trouton-Noble paradox, as there is no Jackson’s paradox.

10. Conclusions

In both considered paradoxes there is a 3D torque and so a time rate of change of 3D angular momentum in one inertial frame, but no 3D angular momentum and no 3D torque in another. The principle of relativity is violated and also there is no agreement with the Trouton-Noble experiment. In all usual approaches the 3D magnetic field \( B \) arises from the ST \( 1 \) of the 3D \( E' \). This \( B \) field determines the 3D magnetic force \( qV \times B \) and this causes that the 3D torque \( N \) is different from zero in the laboratory frame \( S \), where the charges are moving. However
the proofs from sections 8.1-8.4 and 9.2 reveal that the relativistically correct transformations of the 4D electric and magnetic fields, which are in a complete agreement with the principle of relativity, are the LT (22) and (23) and not, as generally believed, the ST (24), (25) and (26) or (1) and (2). In our geometric approach, i.e., in the invariant SR [12-19], the independent physical reality in the 4D spacetime is attributed only to geometric 4D quantities, AQs or CBGQs, their LT and physical laws written in terms of them and not, as usual, to the 3D quantities. When geometric 4D quantities are used then it is consistently obtained in different manners that the 4D torque $N$ is always zero, see sections 8.1-8.4 and 9.2. The principle of relativity is automatically satisfied with such quantities and there is not either Jackson’s paradox or the Trouton-Noble paradox. The main conclusion that can be drawn from the whole consideration in this paper is that the relativistically correct description of physical phenomena can be achieved with geometric 4D quantities as physical quantities in the 4D spacetime and not, as usual, with 3D quantities. This conclusion is in a full agreement with all other results obtained in [12-19] and [23].

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