ON CERTAIN MODULI SPACES OF IDEAL SHEAVES AND DONALDSON-THOMAS INVARIANTS

SHELDON KATZ\textsuperscript{1}, WEI-PING LI\textsuperscript{2}, AND ZHENBO QIN\textsuperscript{3}

Abstract. We determine the structure of certain moduli spaces of ideal sheaves by generalizing an earlier result of the first author. As applications, we compute the (virtual) Hodge polynomials of these moduli space, and calculate the Donaldson-Thomas invariants of certain 3-folds with trivial canonical classes.

1. Introduction

Donaldson-Thomas theory was introduced in [DT, Tho] via integrals over the moduli spaces of semistable sheaves and via the theory of virtual fundamental cycles. It was further developed by Maulik-Pandharipande [MP] and Jun Li. In [LQ1], the second and third authors constructed rank-2 stable vector bundles over certain Calabi-Yau manifolds, and calculated the corresponding Donaldson-Thomas invariants. Recently, a rich interplay among Donaldson-Thomas theory, Gromov-Witten theory and Gopakumar-Vafa invariants has been proposed, and in certain cases, verified in [MNOP1, MNOP2, Kat, OP] (see the references there for other papers). For this interplay, Donaldson-Thomas theory is defined via the moduli spaces parametrizing ideal sheaves of 1-dimensional closed subschemes in a 3-fold, while Gromov-Witten theory is based on the moduli spaces of stable morphisms to the same 3-fold. A complete mathematical theory of Gopakumar-Vafa invariants has not yet been developed.

In this paper, we study the moduli spaces of ideal sheaves in certain 3-folds $X$, and compute the corresponding Donaldson-Thomas invariants of $X$. Let $S$ be a smooth projective surface, and $\mu : X \to S$ be a Zariski-locally trivial fibration whose fibers are smooth irreducible curves of genus-$g$. Let $\beta \in H_2(X; \mathbb{Z})$ be the class of a fiber. For two nonnegative integers $m$ and $n$, the moduli space $\mathcal{I}_{m(1-g)+n}(X, m\beta)$ parametrizes ideal sheaves $I_Z \subset \mathcal{O}_X$ where $Z$ denotes 1-dimensional closed subschemes of $X$ with $\chi(\mathcal{O}_Z) = m(1 - g) + n$ and $[Z] = m\beta$. Here $[Z]$ is the class associated to the dimension-1 component (weighted by their intrinsic multiplicities) of $Z$. When $n = 0$, the moduli space $\mathcal{I}_{m(1-g)}(X, m\beta)$ is naturally identified

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with the Hilbert scheme \( S^{[m]} \) which parametrizes the length-\( m \) 0-dimensional closed subschemes of \( S \). Our main result is to determine \( \mathcal{I}_{m(1-g)+1}(X, m\beta) \).

**Theorem 1.1.** Identify \( S^{[m]} \) and \( \mathcal{I}_{m(1-g)}(X, m\beta) \). Let \( \widetilde{S^{[m]} \times X} \) be the blow-up of \( S^{[m]} \times X \) along the universal curve. Then, we have an isomorphism

\[
\mathcal{I}_{m(1-g)+1}(X, m\beta) \cong \widetilde{S^{[m]} \times X}.
\]

Moreover, the moduli space \( \mathcal{I}_{m(1-g)+1}(X, m\beta) \) is smooth.

As an application to Theorem 1.1, we compute the (virtual) Hodge polynomials of the moduli space \( \mathcal{I}_{m(1-g)+1}(X, m\beta) \) using the results of Cheah and Ellingsrud-Strømme [Che, ES]. As another application, we show that when \( X = S \times C \) with \( K_S = 0 \) and \( K_C = 0 \), the Donaldson-Thomas invariant of the 3-fold \( X \) corresponding to the moduli space \( \mathcal{I}_{m(1-g)+1}(X, m\beta) \) is equal to zero.

Theorem 1.1 is proved in Sect. 2. The key step is to establish Lemma 2.2 which provides a relation between the two moduli spaces \( \mathcal{I}_{1-g}(X, \beta) \) and \( \mathcal{I}_{2-g}(X, \beta) \). By assuming a canonical isomorphism between the blow-up along the universal curve and the projectivization associated to the ideal sheaf defining the universal curve, our Lemma 2.2 generalizes an earlier result of the first author (see Remark 2.3). Indeed, the proof of Lemma 2.2 follows the same argument as in the proof of the Lemma 1 in [Kat]. The only difference is that instead of using local arguments, we apply the universal properties of various constructions.

In Sect. 3, we discuss some relation between the two moduli spaces \( \mathcal{I}_{1-g}(X, \beta) \) and \( \mathcal{I}_{3-g}(X, \beta) \). We show that \( \mathcal{I}_{3-g}(X, \beta) \) is not smooth in general.

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### 2. The moduli space \( \mathcal{I}_{2-g}(X, \beta) \) and main results

#### 2.1. The definition of the moduli space \( \mathcal{I}_n(X, \beta) \).

Let \( X \) be a smooth projective complex variety of dimension \( r \). For a 1-dimensional closed subscheme \( Z \) of \( X \), let \([Z] \in H_2(X; \mathbb{Z})\) be the class associated to the dimension-1 component (weighted by their intrinsic multiplicities) of \( Z \). As in [MNOP1, MNOP2], we make the following definition.

**Definition 2.1.** For a fixed homology class \( \beta \in H_2(X; \mathbb{Z}) \) and a fixed integer \( n \), we define \( \mathcal{I}_n(X, \beta) \) to be the moduli space parametrizing the ideal sheaves \( I_Z \) of 1-dimensional\(^1\) closed subschemes \( Z \) of \( X \) satisfying the conditions:

\[
\chi(O_Z) = n, \quad [Z] = \beta.
\]

Notice that it is also convenient to regard \( \mathcal{I}_n(X, \beta) \) as the moduli space parametrizing the corresponding closed subschemes \( Z \).

\(^{1}\)If \( \beta = 0 \), the closed subschemes \( Z \) are actually 0-dimensional. To avoid repeatedly having a separate discussion of this case, we abuse terminology slightly in this paper by speaking only of 1-dimensional closed schemes.
The degree-0 moduli space $\mathcal{I}_n(X, 0)$ is isomorphic to the Hilbert scheme $X^{[n]}$ parametrizing length-$n$ 0-dimensional closed subschemes of $X$. In general, when $\beta \neq 0$, the moduli space $\mathcal{I}_n(X, \beta)$ is only part of the Hilbert scheme defined in terms of certain degree-1 Hilbert polynomial (see [Gro]). The Zariski tangent space of $\mathcal{I}_n(X, \beta)$ at a point $[Z]$ is canonically isomorphic to

$$\text{Hom}(I_Z, O_Z).$$

By the Lemma 1 in [MNOP2], when $\dim(X) = 3$, the perfect obstruction theory on $\mathcal{I}_n(X, \beta)$ defined in [Tho] has virtual dimension equal to

$$-(\beta \cdot K_X)$$

where $K_X$ stands for the canonical class of $X$.

2.2. A relation between $\mathcal{I}_{1-g}(X, \beta)$ and $\mathcal{I}_{2-g}(X, \beta)$.

Let $g \geq 0$. Assume that all the closed subschemes parametrized by $\mathcal{I}_{1-g}(X, \beta)$ are curves of arithmetic genus $g$, and that the 1-dimensional components of all the closed subschemes parametrized by $\mathcal{I}_{2-g}(X, \beta)$ are supported on the curves parametrized by $\mathcal{I}_{1-g}(X, \beta)$. It follows that if $[Z] \in \mathcal{I}_{2-g}(X, \beta)$, then there exists a unique $[Z'] \in \mathcal{I}_{1-g}(X, \beta)$ together with an exact sequence:

$$0 \to I_Z \to I_{Z'} \to O_x \to 0 \quad (2.4)$$

where $x$ is some point in $X$. Let $\mathcal{I}_{1-g}$ be the universal ideal sheaf over $\mathcal{I}_{1-g}(X, \beta) \times X$,

and let $\mathcal{C}_{1-g} \subset \mathcal{I}_{1-g} \times X$ be the universal curve. Then, $\mathcal{I}_{1-g} = I_{\mathcal{C}_{1-g}}$.

Let $\mathbb{P}(\mathcal{I}_{1-g})$ be the projectivization of the sheaf $\mathcal{I}_{1-g}$, and

$$\tilde{\pi} : \mathbb{P}(\mathcal{I}_{1-g}) \to \mathcal{I}_{1-g}(X, \beta) \times X \quad (2.5)$$

be the natural projection. Then there exists a universal quotient

$$\tilde{\pi}^*\mathcal{I}_{1-g} \to O_{\mathbb{P}(\mathcal{I}_{1-g})}(1) \to 0 \quad (2.6)$$

over $\mathbb{P}(\mathcal{I}_{1-g})$. Let

$$\pi : \mathcal{I}_{1-g}(X, \beta) \times X \to \mathcal{I}_{1-g}(X, \beta) \times X \quad (2.7)$$

be the blow-up of $\mathcal{I}_{1-g}(X, \beta) \times X$ along $\mathcal{C}_{1-g}$, and $E$ be the exceptional divisor. Then there exists a surjection over $\mathcal{I}_{1-g}(X, \beta) \times X$:

$$\pi^*\mathcal{I}_{1-g} \to O_{\mathcal{I}_{1-g}(X, \beta) \times X}(E) \to 0. \quad (2.8)$$

By the universal property of the projectivizations, there exists a canonical morphism $\phi : \mathcal{I}_{1-g}(X, \beta) \times X \to \mathbb{P}(\mathcal{I}_{1-g})$ making a commutative diagram:

$$\begin{array}{ccc}
\mathcal{I}_{1-g}(X, \beta) \times X & \xrightarrow{\phi} & \mathbb{P}(\mathcal{I}_{1-g}) \\
\downarrow \pi & & \Downarrow \tilde{\pi} \\
\mathcal{I}_{1-g}(X, \beta) \times X & &
\end{array} \quad (2.9)$$
such that the pull-back of (2.6) via $\phi^*$ is (2.8). In particular,
\[
\phi^* \mathcal{O}_{\mathcal{I}_1} (1) = \mathcal{O}_{\mathcal{J}_1} (X, \beta) (-E).
\] (2.10)

**Lemma 2.2.** Assume that the closed subschemes parametrized by $\mathcal{J}_{1-g}(X, \beta)$ are curves of arithmetic genus $g$, and that the 1-dimensional components of the closed subschemes parametrized by $\mathcal{J}_{2-g}(X, \beta)$ are supported on the curves parametrized by $\mathcal{J}_{1-g}(X, \beta)$. If the canonical morphism $\phi$ in (2.4) is an isomorphism, then
\[
\mathcal{J}_{2-g}(X, \beta) \cong \mathcal{J}_{1-g}(X, \beta) \times X.
\] (2.11)

**Proof.** For convenience, let $\mathcal{J}_i = \mathcal{J}_i (X, \beta)$. For $i = 2$ or $3$, let
\[
\pi_i : \mathcal{J}_{1-g} \times X \times X \to \mathcal{J}_{1-g} \times X
\]
be the projection to the first and $i$-th factors. Let $\Delta_X$ be the diagonal in $X \times X$. Over $\mathcal{J}_{1-g} \times X \times X$, we have the composition of the natural morphisms:
\[
\pi_{13} : \mathcal{J}_{1-g} \to \mathcal{O}_{\mathcal{J}_{1-g} \times X \times X} \to \mathcal{O}_{\mathcal{J}_{1-g} \times \Delta_X}.
\] (2.12)

Regarding (2.12) as a family of morphisms parametrized by $\mathcal{J}_{1-g} \times X$, we see that (2.12) vanishes precisely along $\mathcal{C}_{1-g}$. So there is an induced morphism:
\[
(\pi \times \text{Id}_X)^* \pi_{13} \mathcal{I}_{1-g} \to (\pi \times \text{Id}_X)^* \mathcal{O}_{\mathcal{J}_{1-g} \times \Delta_X} (-\pi_{12}^* E)
\] (2.13)
over $\mathcal{J}_{1-g} \times X \times X$, where $\pi_{12} : \mathcal{J}_{1-g} \times X \times X \to \mathcal{J}_{1-g} \times X$ is the natural projection. Note that we have a commutative diagram of morphisms:
\[
\begin{array}{ccc}
\mathcal{J}_{1-g} \times X \times X & \xrightarrow{\pi_{12}} & \mathcal{J}_{1-g} \times X \\
\downarrow \pi \times \text{Id}_X & & \downarrow \pi \\
\mathcal{J}_{1-g} \times X \times X & \xrightarrow{\pi_{12}} & \mathcal{J}_{1-g} \times X.
\end{array}
\] (2.14)

Therefore we see from (2.8) that (2.13) is surjective since
\[
(\pi \times \text{Id}_X)^* \pi_{13} \mathcal{I}_{1-g} | (\pi \times \text{Id}_X)^{-1} \mathcal{J}_{1-g} \times \Delta_X = (\pi \times \text{Id}_X)^* \pi_{12} \mathcal{I}_{1-g} | (\pi \times \text{Id}_X)^{-1} \mathcal{J}_{1-g} \times \Delta_X
\]
\[
= \pi_{12}^* \mathcal{I}_{1-g} | (\pi \times \text{Id}_X)^{-1} \mathcal{J}_{1-g} \times \Delta_X
\]
\[
= \mathcal{O}_{\mathcal{J}_{1-g} \times X} (-E)
\]
\[
= \phi^* \mathcal{O}_{\mathcal{J}_{1-g} \times X} (-\pi_{12}^* E).
\]

Let $\hat{\mathcal{I}}$ be the kernel of (2.13). Then we have an exact sequence
\[
0 \to \hat{\mathcal{I}} \to (\pi \times \text{Id}_X)^* \pi_{13} \mathcal{I}_{1-g} \to (\pi \times \text{Id}_X)^* \mathcal{O}_{\mathcal{J}_{1-g} \times \Delta_X} (-\pi_{12}^* E) \to 0
\] (2.15)
of $\mathcal{J}_{1-g} \times X \times X$. Now $\hat{\mathcal{I}}$ is flat over $\mathcal{J}_{1-g} \times X$ since the other two terms in (2.15) are. Also, the fibers of $\hat{\mathcal{I}}$ over $\mathcal{J}_{1-g} \times X$ are ideal sheaves parametrized by $\mathcal{J}_{2-g}$.

Next, we check that $\hat{\mathcal{I}} \subset \mathcal{O}_{\mathcal{J}_{1-g} \times X}$ is the universal ideal sheaf. Let $\mathcal{J} \subset \mathcal{O}_{T \times X}$ be a flat family of ideal sheaves in $\mathcal{J}_{2-g}$ parametrized by $T$. Let $\mathcal{J}'$ be the saturation
of $\mathcal{J} \subset \mathcal{O}_{T \times X}$ (see Definition 1.1.5 in [HL]). Then, $\mathcal{J}'$ is a flat family of ideal sheaves in $\mathcal{J}_{1-g}$, and fits in an exact sequence

$$0 \to \mathcal{J} \to \mathcal{J}' \to (\text{Id}_T \times f_1)_* \mathcal{L} \to 0$$

(2.16)

for some morphism $f_1 : T \to X$ and some line bundle $\mathcal{L}$ on $T$. The flat family $\mathcal{J}'$ over $T \times X$ induces a morphism $f_2 : T \to \mathcal{J}_{1-g}$ such that

$$(f_2 \times \text{Id}_X)^* \mathcal{I}_{1-g} = \mathcal{J}'$$.

(2.17)

By base-change, we have the commutative diagram:

$$
\begin{array}{ccc}
P(\mathcal{J}') & \longrightarrow & P(\mathcal{I}_{1-g}) \\
\downarrow & & \downarrow \\
T \overset{\text{Id}_T \times f_1}{\longrightarrow} T \times X & f_2 \times \text{Id}_X \longrightarrow & \mathcal{J}_{1-g} \times X.
\end{array}
$$

(2.18)

By (2.16), we get a surjection $(\text{Id}_T \times f_1)^* \mathcal{J}' \to (\text{Id}_T \times f_1)^*(\text{Id}_T \times f_1)_* \mathcal{L} \to 0$. Since the natural morphism $(\text{Id}_T \times f_1)^*(\text{Id}_T \times f_1)_* \mathcal{L} \to \mathcal{L}$ is surjective, we obtain

$$(\text{Id}_T \times f_1)^* \mathcal{J}' \to \mathcal{L} \to 0$$

over $T$. By the universal property of $P(\mathcal{J}')$, we obtain a commutative diagram

$$
\begin{array}{ccc}
P(\mathcal{J}') & \longrightarrow & P(\mathcal{I}_{1-g}) \\
\downarrow & & \downarrow \\
T \overset{\text{Id}_T \times f_1}{\longrightarrow} T \times X & f_2 \times \text{Id}_X \longrightarrow & \mathcal{J}_{1-g} \times X.
\end{array}
$$

(2.19)

Thus the morphism $f_2 \times f_1 : T \to \mathcal{J}_{1-g} \times X$ can be lifted to a morphism

$$f : T \longrightarrow P(\mathcal{I}_{1-g}) \overset{\phi^{-1}}{\longrightarrow} \mathcal{J}_{1-g} \times X.$$

Finally, we apply the pull-back $(f \times \text{Id}_X)^*$ to (2.15). Using (2.10) and the property of the morphism $f$, we obtain a commutative diagram:

$$
\begin{array}{ccc}
0 & \to & (f \times \text{Id}_X)^* \mathcal{I} \to \mathcal{J}' \to (f \times \text{Id}_X)^*(\pi \times \text{Id}_X)^* \mathcal{O}_{\mathcal{C}_{1-g}}(-\mathcal{E}) \to 0 \\
0 & \to & \mathcal{J} \to \mathcal{J}' \to (\text{Id}_T \times f_1)_* \mathcal{L} \to 0
\end{array}
$$

over $T \times X$. In particular, there exists an injection

$$0 \to (f \times \text{Id}_X)^* \mathcal{I} \overset{\psi}{\rightarrow} \mathcal{J}.$$ 

(2.20)

Since both $(f \times \text{Id}_X)^* \mathcal{I}$ and $\mathcal{J}$ are flat families of ideal sheaves in $\mathcal{I}_{2-g}(X, \beta)$, we conclude that the morphism $\psi$ is an isomorphism. □

**Remark 2.3.** If the universal curve $\mathcal{C}_{1-g} \subset \mathcal{J}_{1-g} \times X$ is a local complete intersection, then $(\mathcal{I}_{1-g})^n \cong \text{Sym}^n(\mathcal{I}_{1-g})$, and so the canonical morphism $\phi$ in (2.9) is an isomorphism. Hence Lemma 2.2 indeed generalizes the Lemma 1 in [Kat].
2.3. Applications.

In this subsection, we adopt the following basic assumptions.

Assumption 2.4. We assume that $X$ admits a Zariski-locally trivial fibration

$$\mu : X \to S$$

(2.21)

where $S$ is a smooth surface, the fibers are smooth irreducible curves of genus-$g$, and $\beta \in H_2(X; \mathbb{Z})$ is the class of a fiber. For $m, n \geq 0$, we let

$$\mathcal{I}_{m, n} := \mathcal{I}_{m(1-g)+n}(X, m\beta).$$

(2.22)

Note that the elements in $\mathcal{I}_{m, 0}$ correspond to the ideal sheaves of the form $\mu^*I_\xi$ for some $\xi \in S^{[m]}$. Hence there exists a bijective morphism $S^{[m]} \to \mathcal{I}_{m, 0}$. It is well-known that the Hilbert scheme $S^{[m]}$ is smooth. Combining with (2.2), one checks that the moduli space $\mathcal{I}_{m, 0}$ is smooth and that

$$\mathcal{I}_{m, 0} \cong S^{[m]}.$$  

(2.23)

Theorem 2.5. Identify $S^{[m]}$ and $\mathcal{I}_{m, 0}$ by (2.23). Let $\widetilde{S^{[m]} \times X}$ be the blow-up of $S^{[m]} \times X$ along the universal curve. Then, we have an isomorphism

$$\mathcal{I}_{m, 1} \cong \widetilde{S^{[m]} \times X}.$$  

Moreover, the moduli space $\mathcal{I}_{m, 1} = \mathcal{I}_{m(1-g)+1}(X, m\beta)$ is smooth.

Proof. We will apply Lemma 2.2 to the present situation. First of all, note that the 1-dimensional components of the closed subschemes parametrized by $\mathcal{I}_{m, 1}$ are supported on the curves parametrized by $\mathcal{I}_{m, 0}$. Next, let $\mathcal{Z} \subset S^{[m]} \times S$ be the universal codimension-2 subscheme. Set-theoretically,

$$\mathcal{Z} = \{(\xi, x) \in S^{[m]} \times S | x \in \text{Supp}(\xi)\}.$$  

(2.24)

Let $\widetilde{S^{[m]} \times S}$ be the blow-up of $S^{[m]} \times S$ along $\mathcal{Z}$. By the results in [ES], $\widetilde{S^{[m]} \times S}$ is smooth and there exists a canonical isomorphism:

$$\widetilde{S^{[m]} \times S} \cong \mathbb{P}(I_{\mathcal{Z}}).$$  

(2.25)

Now the universal curve $\mathcal{C} \subset S^{[m]} \times X$ is $(\text{Id}_{S^{[m]} \times X})^*\mathcal{Z}$. Since $\mu$ is Zariski-locally trivial, we obtain canonical isomorphisms:

$$\widetilde{S^{[m]} \times X} \cong (\widetilde{S^{[m]} \times S}) \times_{S^{[m]} \times S} (S^{[m]} \times X)$$  

$$\cong \mathbb{P}(I_{\mathcal{Z}}) \times_{S^{[m]} \times S} (S^{[m]} \times X)$$  

$$\cong \mathbb{P}(I_{\mathcal{C}}).$$  

(2.26)

By Lemma 2.2 we have an isomorphism $\mathcal{I}_{m, 1} \cong \widetilde{S^{[m]} \times X}$. Note from the isomorphism (2.26) that $\widetilde{S^{[m]} \times X}$ is smooth. Hence $\mathcal{I}_{m, 1}$ is smooth as well. \qed
Corollary 2.6. Let $e(\cdot; s, t)$ denote the (virtual) Hodge polynomial. Then,
\[
\sum_{m=0}^{+\infty} e(\mathcal{J}_{m,1}; s, t)q^m = \frac{q}{1 - stq} \cdot e(X; s, t) \cdot \prod_{k=1}^{+\infty} \prod_{i,j} \left( \frac{1}{1 - s^{i+k-1}t^{j+k-1}q^{k}} \right)^{e_{i,j}(S)}
\]
where $e_{i,j}(S) = (-1)^{i+j} h^{i,j}(S)$ and $h^{i,j}(S)$ denotes the Hodge numbers of $S$.

Proof. Since $µ$ is Zariski-locally trivial, \( \mathcal{J}_{m,1} \) implies that the natural projection
\[
\overline{S^{[m]}} \times X \to \overline{S^{[m]}} \times S
\]
is a Zariski-locally trivial fibration with fibers being isomorphic to the fibers $C$ of $µ$. Thus, $e(S^{[m]} \times X; s, t) = e(C; s, t) \cdot e(S^{[m]} \times S; s, t)$. By Theorem 2.5,
\[
\sum_{m=0}^{+\infty} e(\mathcal{J}_{m,1}; s, t)q^m = e(C; s, t) \cdot \sum_{m=0}^{+\infty} e(S^{[m]} \times S; s, t)q^m. \tag{2.27}
\]

By the Proposition 2.2 in [ES], $\overline{S^{[m]}} \times S$ is isomorphic to the incidence variety:
\[
S_{m,m+1} = \{ (\eta, \xi) \in S^{[m]} \times S^{[m+1]} | \eta \subset \xi \}. \tag{2.28}
\]
The (virtual) Hodge polynomial of $S_{m,m+1}$ has been computed by Cheah:
\[
\sum_{m=0}^{+\infty} e(S_{m,m+1}; s, t)q^m = \frac{q}{1 - stq} \cdot e(S; s, t) \cdot \prod_{k=1}^{+\infty} \prod_{i,j} \left( \frac{1}{1 - s^{i+k-1}t^{j+k-1}q^{k}} \right)^{e_{i,j}(S)}
\]
(see p.485 in [Che]). Combining this with (2.27) completes the proof. \qed

Remark 2.7. It is well-known that $e(\cdot; 1, 1)$ is equal to the topological Euler number $\chi(\cdot)$. Therefore, we see from Corollary 2.6 that
\[
\sum_{m=0}^{+\infty} \chi(\mathcal{J}_{m,1})q^m = \frac{q}{1 - q} \cdot \chi(X) \cdot \prod_{k=1}^{+\infty} \left( \frac{1}{1 - q^{k}} \right)^{\chi(S)}. \tag{2.29}
\]
It is also interesting to note from (2.23) and Göttsche’s formula in [Got] that
\[
\sum_{m=0}^{+\infty} \chi(\mathcal{J}_{m,0})q^m = \sum_{m=0}^{+\infty} \chi(S^{[m]})q^m = \prod_{k=1}^{+\infty} \left( \frac{1}{1 - q^{k}} \right)^{\chi(S)}. \tag{2.30}
\]

Let $X = S \times C$ where $C$ is an elliptic curve and $S$ is a smooth surface with $K_S = 0$. Let $µ : X \to S$ be the first projection. Then, $K_X = 0$. By (2.3), the virtual dimension of the moduli space $\mathcal{J}_{m,n}$ is zero. The corresponding Donaldson-Thomas invariant is an integer. We denote this Donaldson-Thomas invariant by
\[
N_{m,n}. \tag{2.31}
\]
Corollary 2.8. Let $X = S \times C$ where $C$ is an elliptic curve and $S$ is a smooth surface with $K_S = 0$. Then, $N_{m,1} = 0$ for every $m \geq 0$. 

Proof. Since $K_X = 0$ and the moduli space $\mathcal{I}_{m,1}$ is smooth, the obstruction bundle over $\mathcal{I}_{m,1}$ is the dual of the tangent bundle of $\mathcal{I}_{m,1}$ (see [Tho]). Hence

$$N_{m,1} = (-1)^{\dim \mathcal{I}_{m,1}} \cdot \chi(\mathcal{I}_{m,1}).$$

Since $\chi(X) = 0$, we see from (2.29) that $\chi(\mathcal{I}_{m,1}) = 0$. Therefore, $N_{m,1} = 0$. \hfill \Box

Conjecture 2.9. Let $X = S \times C$ where $C$ is an elliptic curve and $S$ is a smooth surface with $K_S = 0$. Then, $N_{m,n} = 0$ for all $m \geq 0$ and $n \geq 1$.

3. The moduli space $\mathcal{I}_{3-g}(X, \beta)$

In this section, we make a few comments about the moduli space $\mathcal{I}_{3-g}(X, \beta)$. As in Subsect. 2.2, we assume that all the closed subschemes parametrized by $\mathcal{I}_{1-g}(X, \beta)$ are curves of arithmetic genus $g$, and that the 1-dimensional components of all the closed subschemes parametrized by $\mathcal{I}_{3-g}(X, \beta)$ are supported on the curves parametrized by $\mathcal{I}_{1-g}(X, \beta)$. Therefore, if $[Z] \in \mathcal{I}_{3-g}(X, \beta)$, then there exists a unique $[Z'] \in \mathcal{I}_{1-g}(X, \beta)$ together with an exact sequence:

$$0 \to I_Z \to I_{Z'} \to Q \to 0 \quad (3.1)$$

where $Q$ is a torsion sheaf on $X$ with $\ell(Q) = 2$.

Lemma 3.1. Let $Q$ be a torsion sheaf on $X$ from (3.1). Then, either $Q \cong \mathcal{O}_x$ for some $x \in X^{[2]}$, or $Q \cong \mathcal{O}_x \oplus \mathcal{O}_x$ for some point $x \in Z'$.

Proof. Our lemma follows immediately from the following claim.

Claim. Let $Q$ be a length-2 torsion sheaf supported on at most two points of $X$. Then, either $Q \cong \mathcal{O}_x \oplus \mathcal{O}_x$ for some $x \in X^{[2]}$, or $Q \cong \mathcal{O}_x$ for some $x \in X$.

Proof. If $\text{Supp}(Q) = \{x_1, x_2\}$ with $x_1 \neq x_2$, then $Q \cong \mathcal{O}_{x_1} \oplus \mathcal{O}_{x_2}$ and we are done. In the following, we assume that $\text{Supp}(Q) = \{x\}$ for some $x \in X$. We further assume that $X = \text{Spec}(A)$ is affine and $Q$ is an $A$-module by abuse of notation.

Take a nonzero $v \in Q$ and define an $A$-module homomorphism $\varphi : A \to Q$ by sending 1 to $v$. For the ideal $J = \ker \varphi$, the induced homomorphism $\overline{\varphi} : A/J \to Q$ is injective. If $\overline{\varphi}$ is also surjective, then $Q \cong \mathcal{O}_x$ for some $x \in X^{[2]}$. If $\overline{\varphi}$ is not surjective, then we have an exact sequence

$$0 \to A/J \to Q \to Q' \to 0$$

where $Q'$ and $A/J$ must be of length one. Therefore $A/J \cong \mathcal{O}_x$ and $Q' \cong \mathcal{O}_x$.

It follows that the minimal number of generators of $Q$ is at most two.

If $Q$ is generated by a single element, say $v_0$, then by replacing the above $v \in Q$ by $v_0 \in Q$, we conclude that $Q \cong \mathcal{O}_{x_0}$ for some $x_0 \in X^{[2]}$.

Now we are left with the case when the minimal number of generators for $Q$ is two. Assume that $v_1$ and $v_2$ generate $Q$. We define two homomorphisms:

$$\varphi_1 : A \to Q, \quad \varphi_1(1) = v_1;$$

$$\varphi_2 : A \to Q, \quad \varphi_2(1) = v_2.$$ 

Then $\varphi_1$ and $\varphi_2$ induce injective homomorphisms:

$$\overline{\varphi}_1 : A/J_1 \to Q \quad \text{and} \quad \overline{\varphi}_2 : A/J_2 \to Q$$
respectively. Note that both \( A/J_1 \) and \( A/J_2 \) must be of length-1. So \( A/J_1 \cong O_x \) and \( A/J_2 \cong O_x \). Define \( \psi: A/J_1 \oplus A/J_2 \to Q \) by putting
\[
\psi(a, b) = av_1 + bv_2.
\]
Then \( \psi \) is surjective. Since both \( Q \) and \( A/J_1 \oplus A/J_2 \) have length two, \( \psi \) must be an isomorphism. Therefore, we see that \( Q \cong O_x \oplus O_x \). \( \square \)

Conversely, we can show that if \([Z'] \in J_{1-g}(X, \beta)\) and \(Z'\) is smooth at a point \(x \in X\), then both types of \(\mathcal{O}_x\) can occur in \([Z]\).

**Proposition 3.2.** Let \([Z'] \in J_{1-g}(X, \beta)\) and \(Z'\) be smooth at a point \(x \in X\). Then,

(i) there exists a unique \([Z] \in J_{3-g}(X, \beta)\) sitting in the exact sequence:
\[
0 \to I_Z \to I_{Z'} \to O_x \oplus O_x \to 0; \tag{3.2}
\]

(ii) \( \dim \text{Hom}(I_Z, O_Z) = 10 + \dim \text{Hom}(I_{Z'}, O_{Z'}) \).

(iii) the moduli space \(J_{3-g}(X, \beta)\) is not smooth at \([Z]\).

**Proof.** (i) It suffices to show the existence and uniqueness of \(Z\) in an analytic neighborhood \(U_x\) of \(x\). Let \(w_1, w_2, w_3\) be the coordinates of \(U_x\) centered at \(x\) such that \(Z'\) is given by \(w_1 = w_2 = 0\). Then \(I_{Z'} = (w_1, w_2) \subset \mathbb{C}[w_1, w_2, w_3]\). Note that
\[
(w_1, w_2) \otimes \mathbb{C}[w_1, w_2, w_3] \cong (w_1, w_2) / (w_1, w_2) \cdot (w_1, w_2, w_3) \cong \mathbb{C} \oplus \mathbb{C}. \tag{3.3}
\]

Hence tensoring \(\mathbb{C}[w_1, w_2, w_3] \to \mathbb{C}[w_1, w_2, w_3] / (w_1, w_2, w_3) \to 0\) by \(I_{Z'}\) yields
\[
I_{Z'} \to O_x \oplus O_x \to 0. \tag{3.4}
\]

The kernel of the surjection \(I_{Z'} \to O_x \oplus O_x\) defines an element \([Z] \in J_{3-g}(X, \beta)\) satisfying the exact sequence \((3.2)\). Note that in \(U_x\), we have
\[
I_Z = (w_1, w_2, w_3) \cdot (w_1, w_2) = (w_1^2, w_1w_2, w_2^2, w_1w_3, w_2w_3). \tag{3.5}
\]

For the uniqueness of \(Z\), note that specifying a surjection
\[
I_{Z'} \to O_x \oplus O_x \to 0
\]
is equivalent to specifying a surjection \(I_{Z'} \otimes O_x \to O_x \oplus O_x \to 0\), i.e.,
\[
O_x \oplus O_x \to O_x \oplus O_x \to 0
\]
in view of \((3.3)\). Now the surjection \(O_x \oplus O_x \to O_x \oplus O_x\) must be an isomorphism. Therefore, there is only one quotient class \([I_{Z'} \to O_x \oplus O_x]\) up to isomorphisms of \(O_x \oplus O_x\). This proves the uniqueness of \([Z] \in J_{3-g}(X, \beta)\) satisfying \((3.2)\).

(ii) We cover \(X\) by the (analytic) open subsets \(U_x\) and \(X \setminus \{x\}\). Regard \(\text{Hom}(I_{Z'}, O_{Z'})\) as obtained from \(\text{Hom}(I_{Z'}|_{U_x}, O_{Z'}|_{U_x})\) and
\[
\text{Hom}(I_{Z'}|_{X \setminus \{x\}}, O_{Z'}|_{X \setminus \{x\}}) \tag{3.6}
\]
by gluing along \(U_x \cap (X \setminus \{x\})\). Since \(I_{Z'}|_{U_x} = (w_1, w_2)\) and
\[
O_{Z'}|_{U_x} = \frac{\mathbb{C}[w_1, w_2, w_3]}{(w_1, w_2)} \cong \mathbb{C}[w_3],
\]

we see that the homomorphisms \( f \in \text{Hom}(I_{Z'}|_{U_x}, \mathcal{O}_{Z'}|_{U_x}) \) are of the form

\[
f(w_1) = \sum_{i=0}^{+\infty} a_{1i} w_3^i, \quad f(w_2) = \sum_{i=0}^{+\infty} a_{2i} w_3^i \tag{3.7}
\]

where \( a_{1i}, a_{2i} \) are independent complex parameters.

Similarly, \( \text{Hom}(I_Z, \mathcal{O}_Z) \) is obtained from \( \text{Hom}(I_{Z'}|_{U_x}, \mathcal{O}_{Z'}|_{U_x}) \) and

\[
\text{Hom}(I_Z|_{x-(x)}, \mathcal{O}_Z|_{x-(x)}) \tag{3.8}
\]

by gluing along \( U_x \cap (X - \{x\}) \). Note that (3.7) and (3.8) are identical.

Next, we compare \( \text{Hom}(I_{Z'}|_{U_x}, \mathcal{O}_{Z'}|_{U_x}) \) and \( \text{Hom}(I_Z|_{U_x}, \mathcal{O}_Z|_{U_x}) \). By (3.5), we check that the homomorphisms \( h \in \text{Hom}(I_{Z'}|_{U_x}, \mathcal{O}_{Z'}|_{U_x}) \) are of the form

\[
\begin{align*}
  h(w_1^2) &= b_{11} w_1 + b_{12} w_2, \\
  h(w_1 w_2) &= b_{21} w_1 + b_{22} w_2, \\
  h(w_2^2) &= b_{31} w_1 + b_{32} w_2, \\
  h(w_1 w_3) &= b_{41} w_1 + b_{42} w_2 + \sum_{i=0}^{+\infty} a_{1i} w_3^i, \\
  h(w_2 w_3) &= b_{51} w_1 + b_{52} w_2 + \sum_{i=0}^{+\infty} a_{2i} w_3^i \tag{3.9}
\end{align*}
\]

where \( b_{ij}, a_{ij}' \) are independent complex parameters. By (3.5) again,

\[
I_{Z'}|_{U_x \cap (X - \{x\})} = (w_1, w_2). \tag{3.11}
\]

So \( w_1 = w_2 = 0 \) in \( \mathcal{O}_{Z'}|_{U_x \cap (X - \{x\})} \), and the restrictions of the homomorphisms \( h \in \text{Hom}(I_{Z'}|_{U_x}, \mathcal{O}_{Z'}|_{U_x}) \) to \( U_x \cap (X - \{x\}) \) are of the form

\[
\begin{align*}
  h(w_1 w_3) &= \sum_{i=0}^{+\infty} a_{1i}' w_3^i, \\
  h(w_2 w_3) &= \sum_{i=0}^{+\infty} a_{2i}' w_3^i.
\end{align*}
\]

Combining this with (3.11), we see that the restrictions of the homomorphisms \( h \in \text{Hom}(I_{Z'}|_{U_x}, \mathcal{O}_{Z'}|_{U_x}) \) to \( U_x \cap (X - \{x\}) \) are of the form

\[
\begin{align*}
  h(w_1) &= \sum_{i=0}^{+\infty} a_{1i}' w_3^i, \\
  h(w_2) &= \sum_{i=0}^{+\infty} a_{2i}' w_3^i \tag{3.12}
\end{align*}
\]

which is precisely of the form (3.7). Therefore, we conclude that

\[
\dim \text{Hom}(I_Z, \mathcal{O}_Z) = \#\{b_{ij} | 1 \leq i \leq 5, 1 \leq j \leq 2\} + \dim \text{Hom}(I_{Z'}', \mathcal{O}_{Z'})
\]

\[
= 10 + \dim \text{Hom}(I_{Z'}', \mathcal{O}_{Z'}).
\]

(iii) Let \( \mathcal{J}_{1-g} \) be an irreducible component of the moduli space \( \mathcal{J}_{1-g}(X, \beta) \) containing the point \( [Z'] \). Let \( \mathfrak{U} \) be the open subset of the moduli space \( \mathcal{J}_{3-g}(X, \beta) \) consisting of all the closed subschemes of \( X \) of the form:

\[
C \cup \{x_1, x_2\}
\]
where $[C] \in \mathcal{I}_{1-g}$ and $x_1, x_2$ are distinct points not contained in the curve $C$. Then $[Z] \in \overline{U}$ since $Z$ is the flat limit as $t$ approaches 0 of the subschemes

$$Z' \cup \{x_1(t), x_2(t)\}$$

where $x_1(t) = (t, 0, 0) \in U_x \subset X$ and $x_2(t) = (0, t, 0) \in U_x \subset X$. Hence $\overline{U}$ is an irreducible component of the moduli space $\mathcal{I}_{3-g}(X, \beta)$ containing the point $[Z]$. By (2.2), the Zariski tangent space to $\mathcal{I}_{3-g}(X, \beta)$ at $[Z]$ is $\text{Hom}(I_Z, \mathcal{O}_Z)$, and the Zariski tangent space to $\mathcal{I}_{1-g}(X, \beta)$ at $[Z']$ is $\text{Hom}(I_{Z'}, \mathcal{O}_{Z'})$. Since

$$\dim \text{Hom}(I_Z, \mathcal{O}_Z) = 10 + \dim \text{Hom}(I_{Z'}, \mathcal{O}_{Z'})$$

$$\geq 10 + \dim \mathcal{I}_{1-g}$$

$$= 4 + \dim \overline{U},$$

we conclude that the moduli space $\mathcal{I}_{3-g}(X, \beta)$ is not smooth at $[Z]$. □

Similarly, assuming the same conditions as in Proposition 3.2 and assuming $\xi \in X[2]$ with $\text{Supp}(\xi) = \{x\}$, we can prove that if $\xi$ points to the tangent direction of $Z'$ at $x$, then the set of surjections $I_{Z'} \to \mathcal{O}_x \to 0$ up to isomorphisms has dimension 2. If $\xi$ is transverse to $Z'$ at $x$, then the set of surjections $I_{Z'} \to \mathcal{O}_x \to 0$ up to isomorphisms has dimension 1. However, it is not clear how to globalize these local data into a global description of $\mathcal{I}_{3-g}(X, \beta)$ in terms of $\mathcal{I}_{1-g}(X, \beta)$.

Remark 3.3.

(i) By Proposition 3.2 (iii), the moduli space $\mathcal{I}_{3-g}(X, \beta)$ is not smooth in general. Hence there is no guarantee that the corresponding Donaldson-Thomas invariant (when $K_X = 0$) is equal to the topological Euler number of $\mathcal{I}_{3-g}(X, \beta)$ up to sign. Note however that the equality does occur in some important cases with a singular moduli space. This is most notably the case for the degree-0 Donaldson-Thomas invariants. These degree-0 invariants have been conjectured in [MNOP1, MNOP2], and computed by Jun Li [Li].

(ii) Under Assumption 2.4, the topological Euler number of the moduli space

$$\mathcal{I}_{1,2} = \mathcal{I}_{3-g}(X, \beta)$$

(see (2.22)) has been computed in [LQ2] by using virtual Hodge polynomials.

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Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801

*E-mail address: katz@math.uiuc.edu*

Department of Mathematics, HKUST, Clear Water Bay, Kowloon, Hong Kong

*E-mail address: mawpli@ust.hk*

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

*E-mail address: zq@math.missouri.edu*