The lightweight topology optimization of plate/shell structures with different mechanical performance constraints

W W Wang¹, H L Ye*, Y K Sui¹

¹ College of Mechanical Engineering and Applied Electronics Technology, Beijing University of Technology, Beijing, 100124, China

* yehongl@bjut.edu.cn

Abstract. The plate/shell structures are popular among in engineering because they show great potential in lightweight design under pressure loads. This paper develops a lightweight topology optimization method of plate/shell structures considering different mechanical performances based on independent, continuous and mapping (ICM) method. A mathematical formulation is established to describe the plate/shell topology optimization problem with minimum volume or mass as objective, the critical buckling load and nodal displacement as constraints. The finite element model is employed to subdivide the design domain and the existing state of each element is described as a topology design variable. In addition, linear buckling and static analyses are used to get the critical buckling load and nodal displacement of plate/shell structures and relevant information of elements and nodes used in the process of topology optimization. The filter functions are introduced to recognize element properties and map the 0-1 integer programming model into a formulation with continuous variables. Sensitivity analysis and Taylor expansion are applied to achieve a standard quadratic programming model. Finally, numerical examples are provided to test the effectiveness and feasibility of the theory above, and discuss the influence of constraints on optimal structures. This study can provide a theoretical basis for the lightweight topology optimization of plate/shell structures.

1. Introduction

The topology optimization can provide a reference for the conceptual design of structures with the ability of founding the optimal layout of holes within the design domain under given constraints. Comparing with size and shape optimization, the greater economic benefit of topology optimization has attracted many scholars and experts [1-2]. The lightweight design concept is becoming more and more popular with the pursuit of energy conservation, intelligence and multifunction, especially in areas of aerospace and automobile. Naturally, plate/shell structures with light and thin characteristics get favour by designers and become important components in many structural systems. But the appearance of holes in a plate/shell may have a great influence on the stiffness and stability. Therefore, it is necessary to consider the abilities of plate/shell structures to resist deformation and guarantee stability in the process topology optimization.

At present, the topology optimization research has made some progress in the performance design of continuum structures, including the strength topology optimization [3], the stiffness topology optimization [4], the dynamic topology optimization [5], the stability topology optimization [6], the
fatigue topology optimization [7] and others. However, most studies intended to focus on the topology optimization with single structural performance. In this paper, a lightweight topology optimization scheme with different mechanical performance constraints is investigated and discussed.

2. Independent continuous mapping method
Independent continuous mapping (ICM) method topology method was proposed by Sui [8]. A type of independent topology variables was defined to indicate the existence state of element and it made the topology optimization no longer dependent on the material or geometry parameters. And filter functions with the properties of continuous differentiability is introduced to approximate the relationship between the properties and the topology variable because the present mathematical solution algorithms are difficult to get the precise solution of the integer programming model with a large number of design variables [8]. Then the original problem are converted in to a continuous one and solved by a series of approximate analyses and numerical solution algorithms.

3. The lightweight topology optimization formulation

3.1 Establishment of topology optimization formulation
In engineering, the structural performances should firstly meet the corresponding indexes in order to satisfy regular service and the safety. Then a smallest economic indicator is wanted to pursue the maximum economic interest. Therefore the lightweight topology optimization model with performance constraints is proposed to describe the design requirements as follows:

\[
\begin{aligned}
\text{Find} & \quad \mathbf{t} \in E^N \\
\text{make} & \quad O(t) = \sum_{i=1}^{N} o_i(t_i) \rightarrow \min \\
\text{s.t.} & \quad C_j(t) \geq C_{ij} \quad (j=1,\ldots,J) \\
& \quad t_{\min} \leq t_i \leq 1 \quad (i = 1,\ldots,N)
\end{aligned}
\]

(1)

where \( t \) is the topology variable vector, \( N \) is the total number of topology variables and \( J \) is the total number of performance constraints, \( O \) and \( o_i \) are the lightweight indexes of the structure and the \( i \)-th element, \( C \) and \( C_{ij} \) are the performance indexes of the structure and the corresponding lower limit. \( t_{\min} \) is the lower limit of topology variables.

3.2 Standardization of the topology optimization model
The objective and constraints are all related to topology variables and they are usually the implicit functions of topological variables. Therefore, it is necessary to transform the formulation into a solvable mathematical programming model.

The objective function is expanded by second-order Taylor expansion as follows:

\[
O(t) \approx \sum_{i=1}^{N} \frac{\partial O(t)}{\partial t_i^{(o)}} (t_i - t_i^{(o)}) + \frac{1}{2} \sum_{i=1}^{N} \left( \frac{\partial^2 O(t)}{\partial t_i^{(o)}} (t_i - t_i^{(o)}) \right)^2 + O(t^{(o)})
\]

(2)

where \( \nu \) is the iterative number, \( \frac{\partial O(t)}{\partial t_i^{(o)}} \) and \( \frac{\partial^2 O(t)}{\partial^2 t_i^{(o)}} \) are the first-order partial derivative and second-order partial derivative of objective. Constants in objective have no effect on the optimal solution, so the objective can be simplified into a simpler form ignoring constant terms.

\[
O(t) \approx \sum_{i=1}^{N} \frac{\partial^2 O(t)}{\partial^2 t_i^{(o)}} t_i^2 + \sum_{i=1}^{N} \left( \frac{\partial^2 O(t)}{\partial t_i^{(o)}} + \frac{\partial^3 O(t)}{\partial^2 t_i^{(o)}} \right) t_i
\]

(3)
The structural mechanical performance function is expanded by first-order Taylor expansion:

\[ C_j(t) \approx \sum_{i=1}^{N} \frac{\partial C_j(t)}{\partial t_i^{(0)}} (t_i - t_i^{(0)}) + C_j(t_i^{(0)}) \quad (j=0, \ldots, J) \]  

(4)

where \( \frac{\partial C_j(t)}{\partial t_i^{(0)}} \) is the first-order partial derivative of the constraint.

Therefore, the formulation equation (1) can be rewritten as follows:

\[
\begin{aligned}
\text{Find} \quad t & \in \mathbb{E}^N \\
\text{make} \quad O(t) = \sum_{i=1}^{N} a_i \times t_i^2 + b_i t_i \rightarrow \min \\
\text{s.t.} \quad \sum_{i=1}^{N} c_{ij} \times t_i \geq d_j \quad (j = 1, \ldots, J) \\
& \quad t \in \mathbb{R}^N \\
& \quad \text{and} \quad d_j = C_j - C_j(t_i^{(0)}) + \sum_{i=1}^{N} \frac{\partial C_j(t)}{\partial t_i^{(0)}}.
\end{aligned}
\]

(5)

4. Strategies for obtaining explicit functions of mechanical performances

4.1 The explicit expression of the nodal displacement

For linear elastic structures, the equilibrium equation is given as follows:

\[ Ku = F \]  

(6)

where the \( K \) is the structural stiffness matrix, the \( u \) is the nodal displacement vector and the \( F \) is the given nodal force vector.

It is obvious that the nodal displacement is only related to the stiffness matrix for the given loading condition. And the structural stiffness matrix is assembled by the stiffness matrix of all elements as follows:

\[ K(t) = \sum_{i=1}^{N} [k_i(t_i)]^e = \sum_{i=1}^{N} f_e(t_i)[k_i]^e \]  

(7)

where the \( k_i(t_i) \) is the stiffness matrix of element \( i \) with a topology variable \( t_i \) and \( k_i^0 \) is the initial stiffness matrix of element with a topology variable \( t_i = 1 \). \( f_e(t_i) \) is the corresponding stiffness matrix filter function. Superscript \( e \) means the properties belongs to elements.

Applying unit load \( F^V = \{0, \ldots, 1, \ldots, 0\} \) at the node we are focus, then the key nodal displacement can be obtained by the reciprocal theorem of work.

\[ u_i = (F^V)^T u^R = (F^R)^T u^V = \sum_{i=1}^{N} \int_{\Omega} \sigma_e^R \epsilon_e^V d\Omega = \sum_{i=1}^{N} p^R_i u^V \]

(8)

Then the sensitivity analysis is applied to obtain the first-order partial derivative of the nodal displacement and the explicit displacement constraint can be obtained.

4.2 The explicit expression of the critical buckling load
The linear buckling eigenvalue equation of structures is given as follows:

\[(K + \lambda_j G)\phi_j = 0, \quad \lambda_j = -\frac{\phi_j^TK\phi_j}{\phi_j^T G \phi_j}, \quad P_{crj} = \lambda_j P.\] (9)

where \(G\) denotes the structural geometric stiffness matrix. \(\lambda_j\) is the \(j\) th-order critical buckling load factor and \(\phi_j\) represents the corresponding eigenvector of \(\lambda_j\). \(P\) is the given external mechanical load, and the \(P_{crj}\) is the critical buckling load.

Obviously, the critical buckling load can be replaced by the critical buckling load factor constraints owing to that the given external mechanical load \(P\) is a constant in the iterative process [6]. Besides, the critical buckling load factor is dependent on both the stiffness matrix and the geometric stiffness matrix of structures. Therefore, the stiffness matrix and geometric stiffness matrix of element should be recognized for the buckling topology optimization problems as follows:

\[
K(t) = \sum_{i=1}^{N}[k_i(t_i)]' = \sum_{i=1}^{N} f_k(t_i)[k_i^0]' \]
\[
G(t) = \sum_{i=1}^{N}[g_i(t_i)]' = \sum_{i=1}^{N} f_g(t_i)[g_i^0]' \] (10)

where the \(k_i(t_i)\) is the geometric stiffness matrix of element \(i\) with a topology variable \(t_i\) and \(g_i^0\) is the initial geometric stiffness matrix of element with a topology variable \(t_i=1\). \(f_k(t_i)\) and \(f_g(t_i)\) are the geometric stiffness matrix filter function.

Then the constraint of the critical buckling load can be replaced by that of the critical buckling load factor and the explicit buckling constraint can be formed by the sensitivity analysis of the critical buckling load factor.

5. Numerical examples
Two numerical examples are given to demonstrate the effectiveness and feasibility of the theory in this paper and discuss influence of constraints on the optimal results. The material parameters of all examples are \(E = 68.89 \text{GPa}, \quad \mu = 0.3, \quad \rho = 2700 \text{kg/m}^3\).

Example 1
Figure 1 shows a rectangular design domain with the size \(20 \times 40 \times 1 \text{mm}^3\). The lower boundary is fixed and the concentrated force \(P\) acts on the midpoint of the upper boundary. The total mass of the design domain is 2.16g. The critical buckling load of the design domain is \(P_{crj} = 180.76 \text{N}\). The buckling constraint is given in a range of 60N-160N.

![Figure 1. The design domain of a plate.](image-url)
Table 1. The optimal topology configurations with different buckling constraints.

| Buckling constrain/N | 60 | 80 | 100 | 120 | 140 | 160 |
|----------------------|----|----|-----|-----|-----|-----|
| Before inversion     | ![Image](image1.png) | ![Image](image2.png) | ![Image](image3.png) | ![Image](image4.png) | ![Image](image5.png) | ![Image](image6.png) |
| After inversion      | ![Image](image7.png) | ![Image](image8.png) | ![Image](image9.png) | ![Image](image10.png) | ![Image](image11.png) | ![Image](image12.png) |
| After smooth         | ![Image](image13.png) | ![Image](image14.png) | ![Image](image15.png) | ![Image](image16.png) | ![Image](image17.png) | ![Image](image18.png) |

**Figure 2.** Iterative curves of the critical buckling load with different buckling constraints.

**Figure 3.** Iterative curves of mass with different buckling constraints.

Table 1 gives the optimal topological configurations under different buckling constraints. Figures 2 and 3 show the iterative curves of the critical buckling load and structural mass, respectively. Figure 4 shows the curve of mass with different buckling constraint.

From table 1, the optimal topological structure does not a simple increase of material in the original load-transfer path with the increase of buckling constraint value, but changes greatly even under the same working condition. And there are few grey elements on all optimal continuous topological configurations. Figure 2 and 3 show that the all the iterative processes converge steadily under different buckling constraints and the optimal results we obtained are all satisfied the given constraints. From figure 4, we can find that the structural mass increases linearly with the increase of the buckling constraint.
Example 2

In this example, the design domain is a part of cylindrical shell with 2mm in thickness, 520 mm in arc length and 260 mm in generatrix. It is divided into 52x26 shell elements and the finite element model is shown in figure 5. The four corners of the structure are fixed and a concentrated force $P=2500\text{N}$ is applied on the midpoint of the upper boundary. For the design domain, the first-order critical buckling load is 2860.75N, and the displacement in the loading point is 0.117mm. The constraints are given as follows:

Type A: Buckling and displacement constraints $P_{\text{cr1}} \geq 2500\text{N} \& u_{i} \leq 0.15\text{mm}$.

Type B: Buckling constraint $P_{\text{cr1}} \geq 2500\text{N}$.

Type C: Displacement constraint $u_{i} \leq 0.15\text{mm}$.

In this example, the topological optimization problem with different types of performance constraints is analysed. Table 2 gives the optimal topological structures under three types of constraints. Table 3 shows the detail values of optimal results with different types of constraints, including iterative numbers, volume of structures, critical buckling loads and the nodal displacements. Figure 6 shows the iterative history of volume with different types of constraints.

From Table 2, we can find that the optimal topology of the structure is similar, but not same. When the buckling and displacement constraints are both effective, the upper of the optimal topological configuration with multi-performance constraint is more similar with the optimal topological configuration with single buckling constraints, and the lower structure is more similar with single displacement constraints. From Table 3, it can be found that all performances satisfy the given constraints and the volume with multi-performance constraints is the largest when both buckling and displacement constraints are effective. The iterative curves of the volume show that the iteration processes converge steadily for three types of constraints.
Table 2. Optimal topology configurations with different types of constraints.

| Constraint type | Type A | Type B | Type C |
|-----------------|--------|--------|--------|
| The optimal topology configuration |

Table 3. Results of topology optimization with different types of constraints.

| Constraint type     | Type A     | Type B     | Type C     |
|---------------------|------------|------------|------------|
| Iterative number    | 28         | 24         | 34         |
| volume(×10⁵)/mm³    | 1.77       | 1.67       | 1.76       |
| The critical buckling load/N | 2503.25 | 2501.75 | —         |
| Displacement/mm     | 0.150      | —          | -0.150     |

6. Conclusions
A typical lightweight topology optimization model with the mechanical performance constraints based on ICM method is established and solved in this paper. The influence of displacement and buckling is analysed and discussed in the numerical examples. The effectiveness and feasibility are proved by the optimal topology configurations and the stable convergent iterative processes with different types of mechanical constraints. The results of examples also show that both the value and type of constraint would change load-transferring path greatly. Therefore, development of multi-performance constrained topology optimization method and related software is very necessary and meaningful for the structural design in engineering.

Acknowledgments: This work was supported by the National Natural Science Foundation of China (11872080, 11172013), Beijing Natural Science Foundation (3192005) and Beijing Education Committee Development Project (SQKM201610005001).

References
[1] Eschenauer H A and Olhoff N 2001 Appl. Mech. Rev. 54 331–390
[2] Sigmund O 2001 Struct. Multidisc. Optim. 21: 120-127
[3] Zhang W S, Dong L, Zhou J H, Du Z L, Li B J and Guo X 2018 Comput. Method. Appl. M. 334 381-413.
[4] Huang X and Xie Y M 2010 Struct. Multidisc. Optim. 40 409-416
[5] Tsai T D and Cheng C C 2013 Struct. Multidisc. Optim. 47 673-686.
[6] Ye H L, Wang W W, Chen N and Sui Y K 2017 Acta. Mech. Sinica. 33 899–911
[7] Collet M, Bruggi M and Duysinx P 2017 Struct. Multidisc. Optim. 55:839-855.
[8] Sui Y K and Peng XR 2018 Modeling, solving and application for topology optimization of continuum structures ICM method based on step function (Beijing: Tsinghua University Press) p 37.