On the Seventh Power Moment of $\Delta(x)$

Jinjiang Li

Department of Mathematics,
China University of Mining and Technology,
Beijing 100083, P. R. China
Email: jinjiang.li.math@gmail.com

Abstract: Let $\Delta(x)$ be the error term of the Dirichlet divisor problem. In this paper, we establish an asymptotic formula of the seventh-power moment of $\Delta(x)$ and prove that

$$
\int_2^T \Delta^7(x)\,dx = \frac{7(5s_{7,3}(d) - 3s_{7,2}(d) - s_{7,1}(d))}{2816\pi^7} T^{11/4} + O(T^{11/4-\delta_7+\varepsilon})
$$

with $\delta_7 = 1/336$, which improves the previous result.

Keywords: Dirichlet divisor problem; higher-power moment; asymptotic formula.

1 Introduction and main result

Throughout this paper, let $d(n)$ denote the Dirichlet divisor function. In 1838, Dirichlet proved that the error term

$$
\Delta(x) := \sum_{n \leq x} d(n) - x \log x - (2\gamma - 1)x,
$$

with $x \geq 2$, satisfies $\Delta(x) \ll x^{1/2}$. Here $\gamma$ is Euler’s constant. Since then the determination of the exact order of $\Delta(x)$ has been called Dirichlet’s divisor problem. Many writers have sharpened Dirichlet’s bound for $\Delta(x)$. The latest result is due to Huxley [5], who proved that

$$
\Delta(x) \ll x^{131/416} (\log x)^{26957/8320}.
$$

For a survey of the history of this problem, see Krätzel [10].

In the opposite direction, Hardy [3] proved that

$$
\Delta(x) = \begin{cases} 
\Omega_+ \left( x^{1/4} (\log x)^{1/4} \log \log x \right), \\
\Omega_- \left( x^{1/4} \right).
\end{cases}
$$

The best results in this direction to date are

$$
\Delta(x) = \Omega_+ \left( x^{1/4} (\log x)^{1/4} (\log \log x)^{(3+\log 4)/4} \exp(-c\sqrt{\log \log \log x}) \right)
$$

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and

\[ \Delta(x) = \Omega - \left( x^{1/4} \exp(c \log \log x)^{1/4} (\log \log \log x)^{-3/4} \right) \]

for some constant \( c > 0 \), due to Hafner [2] and [1] respectively. It is conjectured that \( \Delta(x) \ll x^{1/4+\varepsilon} \) is true for every \( \varepsilon > 0 \). The evidence in support of this conjecture has been given by Tong [14] and Ivic [6], who proved, respectively, that

\[ \int_2^T \Delta^2(x) \, dx = \frac{\zeta(3)^2}{6\pi^2} T^{3/2} + O(T \log^5 T) \tag{1.1} \]

and

\[ \int_2^T |\Delta(x)|^4 \, dx \ll x^{1+4\varepsilon}, \quad \text{for } 0 \leq A \leq \frac{35}{4} \tag{1.2} \]

and any \( \varepsilon > 0 \). On the other hand, Voronoï [16] proved that

\[ \int_2^T \Delta(x) \, dx = T/4 + O(T^{3/4}), \tag{1.3} \]

which in conjunction with (1.1) and (1.2) shows that \( \Delta(x) \) has a lot of sign changes and cancellations between the positive and negative portions.

Tsang [15] first studied the third and fourth-power moments of \( \Delta(x) \). He proved that

\[ \int_2^T \Delta^3(x) \, dx = \frac{3c_1}{28\pi^3} T^{7/4} + O(T^{7/4-\delta_3+\varepsilon}), \tag{1.4} \]

\[ \int_2^T \Delta^4(x) \, dx = \frac{3c_2}{64\pi^4} T^2 + O(T^{2-\delta_4+\varepsilon}), \tag{1.5} \]

where \( \delta_3 = 1/14, \delta_4 = 1/23 \) and

\[ c_1 := \sum_{\alpha, \beta, h \in \mathbb{N}} (\alpha \beta (\alpha + \beta))^{-3/4} h^{-9/4} |\mu(h)| d(\alpha^2 h) d(\beta^2 h) d((\alpha + \beta)^2 h), \]

\[ c_2 := \sum_{n, m, \ell, k \in \mathbb{N}} (n m k \ell)^{-3/4} d(n) d(m) d(k) d(\ell), \]

and \( \mu(h) \) is the Möbius function.

In [18], Zhai proved that (1.4) holds for \( \delta_3 = 1/4 \). Ivic and Sargos [8] proved that (1.4) holds for \( \delta_3 = 7/20 \). Following the approach of Tsang [15], Zhai [18] proved that the equation (1.5) holds for \( \delta_4 = 2/41 \). This approach used the method of exponential sums. In particular, if the exponent pair conjecture is true, namely, if \( (\varepsilon, 1/2 + \varepsilon) \) is an exponent pair, then the equation (1.5) holds for \( \delta_4 = 1/14 \). Moreover, in [8], Ivic and Sargos proved a substantially better result that the equation (1.5) holds for \( \delta_4 = 1/12 \). Later, combining the method of [8] and a deep result of Robert and Sargos [12], Zhai [20] proved that the equation (1.5) holds for \( \delta_4 = 3/28 \). Recently, Kong [9] proved that \( \delta_4 = 1/8 \).

By a unified approach, Zhai [19] proved that the asymptotic formula

\[ \int_1^T \Delta^k(x) \, dx = C_k T^{1+k/4} + O \left( T^{1+k/4-\delta_k+\varepsilon} \right) \tag{1.6} \]
holds for $3 \leq k \leq 9$, where $C_k$ and $0 < \delta_k < 1$ are explicit constants. He gives $\delta_5 = 1/64, \delta_6 = 35/4742, \delta_7 = 17/6312, \delta_8 = \sqrt{9433}, \delta_9 = 13/75216$. The asymptotic formula (1.6) improved the result of Heath-Brown [4]. When $k = 5$, the asymptotic formula (1.6) holds for $\delta_5 = 1/64$, which improved an earlier exponent $\delta_5 = 5/816$ proved in [18] by the approach of Tsang [15]. In [21], Zhang and Zhai improved the previous result of the case $k = 5$ and proved $\delta_5 = 3/80$. Meanwhile, Wang [17] studied the case $k = 6$ and proved $\delta_6 = 3/248$, which improved the result of Zhai [19], i.e. $\delta_6 = 35/4742$.

The aim of this paper is to improve the value of $\delta_7 = 17/6312$, which is achieved by Zhai [19]. The main result is the following

**Theorem 1.1** We have

$$
\int_2^T \Delta^7(x)dx = \frac{7(5s_{7,3}(d) - 3s_{7,2}(d) - s_{7,1}(d))}{2816\pi^7}T^{11/4} + O(T^{11/4-\delta_7+\epsilon})
$$

with $\delta_7 = 1/336$, where

$$
s_{7,3}(d) = \sum_{n,m,k,r,q\in\mathbb{N}^*} \frac{d(n)d(m)d(k)d(r)d(s)d(q)}{(nmktrsq)^{3/4}},
$$

$$
s_{7,2}(d) = \sum_{n,m,k,r,q\in\mathbb{N}^*} \frac{d(n)d(m)d(k)d(r)d(s)d(q)}{(nmktrsq)^{3/4}},
$$

$$
s_{7,1}(d) = \sum_{n,m,k,r,q\in\mathbb{N}^*} \frac{d(n)d(m)d(k)d(r)d(s)d(q)}{(nmktrsq)^{3/4}}.
$$

**Notations.** Throughout this paper, $\|x\|$ denotes the distance from $x$ to the nearest integer, i.e., $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$. $[x]$ denotes the integer part of $x$; $n \sim N$ means $N < n \leq 2N$; $n \asymp N$ means $C_1 N \leq n \leq C_2 N$ with positive constants $C_1, C_2$ satisfying $C_1 < C_2$. $\epsilon$ always denotes an arbitrary small positive constant which may not be the same at different occurrences. We shall use the estimates $d(n) \ll n^{\epsilon}$. Suppose $f : \mathbb{N} \rightarrow \mathbb{R}$ is any function satisfying $f(n) \ll n^{\epsilon}$, $k \geq 2$ is a fixed integer. Define

$$
s_{k,\ell}(f) := \sum_{n_1, \ldots, n_k \in \mathbb{N}^*} \frac{f(n_1) f(n_2) \cdots f(n_k)}{(n_1 n_2 \cdots n_k)^{3/4}}, \quad 1 \leq \ell < k.
$$

We shall use $s_{k,\ell}(f)$ to denote both of the series (1.7) and its value. Suppose $y > 1$ is a large parameter, and we define

$$
s_{k,\ell}(f; y) := \sum_{n_1, \ldots, n_k \in \mathbb{N}^*} \frac{f(n_1) f(n_2) \cdots f(n_k)}{(n_1 n_2 \cdots n_k)^{3/4}}, \quad 1 \leq \ell < k.
$$

## 2 Preliminary Lemmas

**Lemma 2.1** Suppose $k \geq 3, (i_1, i_2, \ldots, i_{k-1}) \in \{0, 1\}^{k-1}$ such that

$$
\sqrt{n_1} + (-1)^{i_1} \sqrt{n_2} + (-1)^{i_2} \sqrt{n_3} + \cdots + (-1)^{i_{k-1}} \sqrt{n_k} \neq 0.
$$
Then we have
\[|\sqrt{n_1} + (-1)^{n_1} \sqrt{n_2} + (-1)^{n_2} \sqrt{n_3} + \cdots + (-1)^{n_k} \sqrt{n_k}| \gg \max(n_1, n_2, \ldots, n_k)^{-(2^{k-2} - 2^{-1})} .\]

**Proof.** See Lemma 2.2 of [19].

**Lemma 2.2** If \( g(x) \) and \( h(x) \) are continuous real-valued functions of \( x \) and \( g(x) \) is monotonic, then
\[\int_a^b g(x)h(x)dx \ll \left( \max_{a \leq x \leq b} |g(x)| \right) \left( \max_{a \leq u < v \leq b} \left| \int_u^v h(x)dx \right| \right) .\]

**Proof.** See Lemma 1 of [15].

**Lemma 2.3** Suppose \( A, B \in \mathbb{R}, A \neq 0 \). Then
\[\int_T^{2T} \cos(A\sqrt{T} + B)dt \ll T^{1/2}|A|^{-1} .\]

**Proof.** It follows from Lemma 2.2 easily.

**Lemma 2.4** Suppose \( K \geq 10, \alpha, \beta \in \mathbb{R}, 2K^{-1/2} \leq |\alpha| \ll K^{1/2} \) and \( 0 < \delta < 1/2 \). Then we have
\[#\{ k \sim K : \| \beta + \alpha \sqrt{t} \| < \delta \} \ll K\delta + K^{1/2+\epsilon} .\]

**Proof.** See Lemma 4 of [20].

**Lemma 2.5** Let \( a, \delta \) be real numbers, \( 0 < \delta < a/4 \), and let \( k \) be a positive integer. There exists a function \( \varphi(y) \) which is \( k \) times continuously differentiable and such that
\[
\begin{align*}
\varphi(y) &= 1, & \text{for } |y| \leq a - \delta, \\
0 < \varphi(y) < 1, & \text{for } a - \delta < |y| < a + \delta, \\
\varphi(y) &= 0, & \text{for } |y| \geq a + \delta,
\end{align*}
\]
and its Fourier transform
\[\Phi(x) = \int_{-\infty}^{+\infty} e(-xy)\varphi(y)dy\]
satisfies the inequality
\[|\Phi(x)| \leq \min \left( 2a, \frac{1}{\pi|x|}, \frac{1}{\pi|x|} \left( \frac{k}{2\pi|x|\delta} \right)^k \right) .\]

**Proof.** See [11] or [13].

**Lemma 2.6** Let \( d(n) \) denote the divisor function. Then we have
\[|s_{k,\ell}(d) - s_{k,\ell}(d; y)| \ll y^{-1/2+\epsilon}, \quad 1 \leq \ell < k .\]

**Proof.** See Lemma 3.1 of [19].
Lemma 2.7 Suppose \( k \geq 3; (i_1, i_2, \ldots, i_{k-1}) \in \{0, 1\}^{k-1}, (i_1, i_2, \ldots, i_{k-1}) \neq (0, 0, \ldots, 0) \); \( 1 < N_1, N_2, \ldots, N_k; 0 < \Delta \ll E^{1/2}, E = \max(N_1, N_2, \ldots, N_k) \). Let
\[
\mathcal{A} = \mathcal{A}(N_1, N_2, \ldots, N_k; i_1, i_2, \ldots, i_k; \Delta)
\]
denote the number of solutions of the inequality
\[
|\sqrt{n} + (-1)^{i_1} \sqrt{m_2} + (-1)^{i_2} \sqrt{m_3} + \cdots + (-1)^{i_{k-1}} \sqrt{m_k}| < \Delta
\]
with \( n_j \sim N_j, j = 1, 2, \ldots, k \). Then
\[
\mathcal{A} \ll \Delta E^{-1/2} N_1 N_2 \cdots N_k + E^{-1} N_1 N_2 \cdots N_k.
\]

Proof. See Lemma 2.4 of [19].

Lemma 2.8 Suppose \( 1 \leq N \leq M \leq K \leq L, 1 \leq R \leq S \leq Q, L = Q \) and \( 0 < \Delta \ll Q^{1/2} \). Let \( \mathcal{A}_1(N, M, K, L, R, S, Q; \Delta) \) denote the number of solutions of the inequality
\[
0 < |\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{r} - \sqrt{s} - \sqrt{q}| < \Delta
\]
(2.1)
with \( n \sim N, m \sim M, k \sim K, \ell \sim L, r \sim R, s \sim S, q \sim Q \). Then
\[
\mathcal{A}_1(N, M, K, L, R, S, Q; \Delta) \ll \Delta Q^{1/2} N M K L R S + N M K R S L^{1/2+\varepsilon}.
\]

In particular, if \( \Delta Q^{1/2} \gg 1 \), then
\[
\mathcal{A}_1(N, M, K, L, R, S, Q; \Delta) \ll \Delta Q^{1/2} N M K L R S.
\]

Proof. If \((n, m, k, \ell, r, s, q)\) satisfies (2.1), then
\[
\left(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}\right) + \sqrt{\ell} = q + \theta \Delta
\]
for some \( 0 < |\theta| < 1 \). Thus, we have
\[
2^{1/2}\left(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}\right) + \left(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}\right)^2 + \ell = q + u
\]
with \( |u| = 2q^{1/2} |\theta \Delta + \theta^2 \Delta^2| \leq 2q^{1/2} \Delta + \Delta^2 \ll \Delta Q^{1/2} \). Then we have
\[
q = 2^{1/2}\left(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}\right) + \left(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}\right)^2 + \ell - u
\]
with \( |u| \leq C \Delta Q^{1/2} \) for some absolute positive constant \( C > 0 \). Hence the quantity of \( \mathcal{A}_1(N, M, K, L, R, S, Q; \Delta) \) does not exceed the number of solutions of
\[
\left|2^{1/2}\left(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}\right) + \left(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}\right)^2 + \ell - q\right| < C \Delta Q^{1/2}
\]
(2.2)
with \( n \sim N, m \sim M, k \sim K, \ell \sim L, r \sim R, s \sim S, q \sim Q \).

If \( \Delta Q^{1/2} \gg 1 \), then for fixed \((n, m, k, \ell, r, s)\), the number of \( q \) for which (2.2) holds is \( \ll 1 + \Delta Q^{1/2} \ll \Delta Q^{1/2} \). Hence
\[
\mathcal{A}_1(N, M, K, L, R, S, Q; \Delta) \ll \Delta Q^{1/2} N M K L R S.
\]
Now suppose $\Delta Q^{1/2} \leq 1/4C$. Then for fixed $(n, m, k, \ell, r, s)$, there is at most one $q$ such that (2.2) holds. If such a $q$ exists, then we have
\[
\left\| 2^{\ell/2}(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}) + (\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s})^2 \right\| < C\Delta Q^{1/2}.
\]
(2.3)

We shall use Lemma 2.4 to bound the number of solutions of (2.3) with $\alpha = 2(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}), \beta = (\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s})^2$. Let $\mathcal{D}_1$ denote the number of solutions of (2.3) with $|\alpha| \geq 2L^{-1/2}$, and $\mathcal{D}_2$ the number of solutions of (2.3) with $|\alpha| < 2L^{-1/2}$. By Lemma 2.4, we get
\[
\mathcal{D}_1 \ll (\Delta Q^{1/2} L + L^{1/2+\epsilon}) NM KLRS
\ll \Delta Q^{1/2} NM KLRS + N M K R S L^{1/2+\epsilon}.
\]

Now we estimate $\mathcal{D}_2$. From $|\alpha| < 2L^{-1/2}$, we can get $K \approx R$. If $\sqrt{n} + \sqrt{m} + \sqrt{k} = \sqrt{r} + \sqrt{s}$, then from (2.2), we get $\ell = q$. This contradicts the fact that $|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} - \sqrt{s}| > 0$. Therefore, we have $\sqrt{n} + \sqrt{m} + \sqrt{k} \neq \sqrt{r} + \sqrt{s}$. By Lemma 2.1, we have $|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} + \sqrt{s}| \gg S^{-15/2}$ for any such $(n, m, k, r, s)$. By a splitting argument and Lemma 2.7, there exists a $\delta$ satisfying $S^{-15/2} \ll \delta \ll L^{-1/2}$, which holds
\[
\mathcal{D}_2 \ll \log 2S \cdot \sum_{\delta < |\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{r} + \sqrt{s}| \leq 2\delta} 1
\ll \log 2S \cdot (\delta S^{1/2} NM KL RS + N M K R S)
\ll L^{-1/2} S^{1/2+\epsilon} N M K R S + N M K R S^\epsilon
\ll N M K R S^\epsilon,
\]
which can be absorbed into the estimate of $\mathcal{D}_1$. This completes the proof of Lemma 2.8.

**Lemma 2.9** Suppose $1 \leq N \leq M \leq K \leq L, 1 \leq R \leq S \leq Q, L \asymp Q$ and $0 < \Delta \ll Q^{1/2}$. Let $\mathcal{A}_{2, \pm}(N, M, K, L, R, S, Q; \Delta)$ denote the number of solutions of the inequality
\[
0 < |\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{r} + \sqrt{s} + \sqrt{R} - \sqrt{S} - \sqrt{T}| \leq \Delta
\]
with $n \sim N, m \sim M, k \sim K, r \sim R, s \sim S, q \sim Q$. Then
\[
\mathcal{A}_{2, \pm}(N, M, K, L, R, S, Q; \Delta) \ll \Delta Q^{1/2} N M K R S + N M K R S L^{1/2+\epsilon}.
\]
In particular, if $\Delta Q^{1/2} \gg 1$, then
\[
\mathcal{A}_{2, \pm}(N, M, K, L, R, S, Q; \Delta) \ll \Delta Q^{1/2} N M K R S.
\]

**Proof.** The proof of Lemma 2.9 is similar to that of Lemma 2.8, so we omit the details.

**Lemma 2.10** Suppose $N_j \geq 2$ $(j = 1, 2, 3, 4, 5, 6, 7)$ are real numbers, $\Delta > 0$. let $\mathcal{A}_{\pm}(N_1, N_2, N_3, N_4, N_5, N_6, N_7; \Delta)$ denote the number of solutions of the inequality
\[
0 < |\sqrt{n_1} + \sqrt{n_2} + \sqrt{n_3} + \sqrt{n_4} + \sqrt{n_5} + \sqrt{n_6} - \sqrt{n_7}| \leq \Delta
\]
with $n_j \sim N_j, (j = 1, 2, 3, 4, 5, 6, 7), n_j \in \mathbb{N}^+$. Then we have
\[
\mathcal{A}_{\pm}(N_1, N_2, N_3, N_4, N_5, N_6, N_7; \Delta) \ll \prod_{j=1}^{7} \left( \Delta^{1/7} N_j^{13/14} + N_j^{5/7} \right) N_j^\epsilon.
\]
Proof. Taking \( a = 6\Delta/5, \delta = \Delta/5 \) in Lemma 2.5, there exists a function \( \varphi_1(y) \), which is \( \ell = [7 \log(N_1N_2N_3N_4N_5N_6N_7)] \) times continuously differentiable such that

\[
\left\{
\begin{array}{ll}
\varphi_1(y) = 1, & \text{if } |y| \leq \Delta, \\
0 < \varphi_1(y) < 1, & \text{for } \Delta < |y| < 7\Delta/5, \\
\varphi_1(y) = 0, & \text{for } |y| \geq 7\Delta/5.
\end{array}
\right.
\]

Let

\[
\Phi_1(x) = \int_{-\infty}^{\infty} e(-xy)\varphi_1(y)dy,
\]

then it satisfies

\[
|\Phi_1(x)| \leq \min \left( \frac{12\Delta}{5}, \frac{1}{|x|}, \frac{1}{2\pi|x|} \left( \frac{5\ell}{2\pi|x|\Delta} \right)^\ell \right), \tag{2.4}
\]

and

\[
\varphi_1(y) = \int_{-\infty}^{+\infty} e(xy)\Phi_1(x)dx. \tag{2.5}
\]

Set

\[
R_{\pm,\pm} = \sum_{n_j \sim N_j} \varphi_1(\sqrt{n_1 + \sqrt{n_2} + \sqrt{n_3} + \sqrt{n_4} \pm \sqrt{n_5} \pm \sqrt{n_6} \pm \sqrt{n_7}).
\]

By the definition of \( \varphi_1(y) \), we get

\[
\omega_{\pm,\pm}(N_1, N_2, N_3, N_4, N_5, N_6, N_7; \Delta) \leq R_{\pm,\pm}. \tag{2.6}
\]

We estimate \( R_{-, -} \) first. By (2.5), we have

\[
R_{-, -} = \sum_{n_j \sim N_j} \varphi_1(\sqrt{n_1 + \sqrt{n_2} + \sqrt{n_3} + \sqrt{n_4} - \sqrt{n_5} - \sqrt{n_6} - \sqrt{n_7})
\]

\[
= \sum_{n_j \sim N_j} \int_{-\infty}^{+\infty} e(x(\sqrt{n_1 + \sqrt{n_2} + \sqrt{n_3} + \sqrt{n_4} - \sqrt{n_5} - \sqrt{n_6} - \sqrt{n_7}) \Phi_1(x)dx.
\]

Let \( S(x; N) := \sum_{n \sim N} e(x\sqrt{n}) \), we have

\[
R_{-, -} = \int_{-\infty}^{+\infty} S(x; N_1)S(x; N_2)S(x; N_3)S(x; N_4)S(x; N_5)S(x; N_6)S(x; N_7)\Phi_1(x)dx,
\]

if we notice that \( S(x; N) = \sum_{n \sim N} e(-x\sqrt{n}) \). Applying Hölder’s inequality, we get

\[
R_{-, -} \leq \prod_{j=1}^{7} \left( \int_{-\infty}^{+\infty} |S(x; N_j)|^7 |\Phi_1(x)|dx \right)^{1/7}. \tag{2.7}
\]

Let

\[
T(N) := \int_{-\infty}^{+\infty} |S(x; N)|^7 |\Phi_1(x)|dx.
\]
It is sufficient to estimate $T(N)$, where $N = N_j$ for some $j \in \{1, 2, 3, 4, 5, 6, 7\}$. Let
\[ K := \frac{100\ell^2}{\Delta}, \quad \ell = [7\log(N_1N_2N_3N_4N_5N_6N_7)]; \quad K_0 := N^{1/2}. \]

Using the trivial estimate $S(x; N) \ll N$ and the estimate
\[ |\Phi_1(x)| \leq \frac{1}{\pi|x|} \left( \frac{5\ell}{2\pi|x|\Delta} \right)^\ell, \]
we have
\[ \int_K^\infty |S(x; N)|^7|\Phi_1(x)|dx \ll N^7 \int_K^\infty |\Phi_1(x)|dx \ll N^7 \left( \frac{5\ell}{2\pi\Delta} \right)^\ell \int_K^\infty \frac{1}{x^{\ell+1}}dx \ll \frac{N^75^\ell}{2\pi K \Delta} \ll \frac{N^7(N_1 \cdots N_7)^{7\log 5}}{(N_1 \cdots N_7)^{7\log 7}(N_1 \cdots N_7)^{7\log \log(N_1 \cdots N_7)}} \ll 1. \quad (2.8) \]

For the mean square of $S(x; N)$, we have
\[ \int_0^{K_0} |S(x; N)|^2dx = \int_0^{K_0} \sum_{n \sim N} \sum_{m \sim N} e(x(\sqrt{n} - \sqrt{m}))dx \ll \frac{N^3}{2} \log N. \quad (2.9) \]
If we notice $|\Phi_1(x)| \ll \Delta$ from (2.4) and the trivial estimate $S(x; N) \ll N$, then

$$
\int_0^{K_0} |S(x; N)|^7 |\Phi_1(x)| \, dx 
\ll \Delta N^5 \int_0^{K_0} |S(x; N)|^2 \, dx
\ll \Delta N^{13/2} \log N.
$$

(2.10)

If $K \leq K_0$, then from (2.8) and (2.10) we get

$$
T(N) \ll \Delta N^{13/2} \log N.
$$

(2.11)

Now suppose $K < K_0$. By a splitting argument, we have

$$
\int_{K_0}^K |S(x; N)|^7 |\Phi_1(x)| \, dx \ll \Delta \log K \times \max_{K_0 \leq U \leq K} \int_U^{2U} |S(x; N)|^7 \, dx.
$$

(2.12)

From Lemma 2.1, we can get $\Delta^{-1} \ll \max(N_1, N_2, N_3, N_4, N_5, N_6, N_7)^{63/2}$ and thus $\log K \ll \ell^2$. On the other hand, we have

$$
\int_U^{2U} |S(x; N)|^7 \, dx \ll \max_{U \leq x \leq 2U} |S(x; N)|^2 \times \int_U^{2U} |S(x; N)|^5 \, dx
\ll N^2(N^{9/2} + UN^3)N^\varepsilon
\ll (N^{13/2} + UN^5)N^\varepsilon,
$$

(2.13)

which is derived from equation (2.18) of Zhang and Zhai [21].

From (2.12) and (2.13) and noticing $\Delta K \ll \ell^2$, we get

$$
\int_{K_0}^K |S(x; N)|^7 |\Phi_1(x)| \, dx \ll \Delta \log K \times (N^{13/2} + UN^5)N^\varepsilon
\ll (\Delta N^{13/2} + N^5)N^\varepsilon,
$$

which combining (2.8) and (2.10) gives

$$
T(N) = \int_0^\infty |S(x; N)|^7 |\Phi_1(x)| \, dx \ll (\Delta N^{13/2} + N^5)N^\varepsilon.
$$

(2.14)

From (2.6), (2.7) and (2.14), we get the result of Lemma 2.10 for the case “−, −”. By noting the properties of conjugation, the estimates of other cases are exactly the same as that of the case “−, −”. This completes the proof of Lemma 2.10.

3 Proof of Theorem

In this section, we shall prove the theorem. We begin with the following truncated form of the Voronoï’s formula ([7], equation (2.25)), i.e.

$$
\Delta(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \leq N} \frac{d(n)}{n^{3/4}} x^{1/4} \cos(4\pi \sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon} N^{-1/2}),
$$

(3.1)
for $1 \leq N \ll x$. Set $\Delta(x) := R_1 + R_2$, where

$$R_1 := R_1(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \leq y} \frac{d(n)}{n^{3/2}} x^{1/4} \cos(4\pi \sqrt{nx} - \pi/4), \quad R_2 := R_2(x) = \Delta(x) - R_1.$$  

Take $y = T^{1/4}$. By the elementary estimate $(a + b)^7 - a^7 \ll |b| a^6 + |b|^7$, we have

$$\int_1^T \Delta^7(x) dx = \int_1^T R_1^7(x) dx + O \left( \int_1^T |R_1|^6 |R_2| dx + \int_1^T |R_2|^7 dx \right).$$

By a splitting argument, it is sufficient to prove the result in the interval $[T, 2T]$. We will divide the process of the proof of the theorem into two parts.

**Proposition 3.1** For fixed $T \geq 10, N = T, y = T^{1/4}$, we have

$$\int_T^{2T} R_1^7 dx = \frac{7(5s_7,3(d) - 3s_7,2(d) - s_7,1(d))}{2816\pi^7} T^{11/4} + O(T^{11/4 - 1/336 + \varepsilon}). \quad (3.2)$$

**Proof.** Let

$$g := g(n, m, k, \ell, r, s, q) = \begin{cases} \frac{(dn_1)(dn_2)(dn_3)(dn_4)(dn_5)(dn_6)(dn_7)}{(nmkrsq)^{3/4}}, & \text{if } n, m, k, \ell, r, s, q \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

According to the elementary formula

$$\cos a_1 \cos a_2 \cdots \cos a_h = \frac{1}{2^{k-1}} \sum_{(i_1, i_2, \ldots, i_{k-1}) \in \{0, 1\}^{k-1}} \cos \left( a_1 + (-1)^i a_2 + \cdots + (-1)^{i_{k-1}} a_h \right),$$

we can write

$$R_1^7 = S_1(x) + S_2(x) + S_3(x) + S_4(x) + S_5(x) + S_6(x) + S_7(x),$$

where

\begin{align*}
S_1(x) &:= \frac{35}{64} \cos \frac{\pi}{4} \sum_{n, m, k, \ell, r, s, q \leq y} g x^{7/4}, \\
S_2(x) &:= \frac{35}{64} \sum_{n, m, k, \ell, r, s, q \leq y} g x^{7/4} \\
&\quad \times \cos \left( \frac{4\pi}{\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{\ell} - \sqrt{r} - \sqrt{s} - \sqrt{q}} \sqrt{x} - \frac{\pi}{4} \right), \\
S_3(x) &:= \frac{21}{64} \cos \frac{3\pi}{4} \sum_{n, m, k, \ell, r, s, q \leq y} g x^{7/4}, \\
S_4(x) &:= \frac{21}{64} \sum_{n, m, k, \ell, r, s, q \leq y} g x^{7/4} \\
&\quad \times \cos \left( \frac{4\pi}{\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{\ell} + \sqrt{r} - \sqrt{s} - \sqrt{q}} \sqrt{x} - \frac{3\pi}{4} \right),
\end{align*}
Now we consider the contribution of $S_5(x)$:

\[
S_5(x) := \frac{7}{64} \cos \left(\frac{5\pi}{4} \sum_{n,m,k,l,r,s<q\leq y} g x^{7/4} \right).
\]

Next, we consider $S_6(x)$:

\[
S_6(x) := \frac{7}{64} \sum_{n,m,k,l,r,s<q\leq y} g x^{7/4} \times \cos \left(4\pi \left(\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l} + \sqrt{s} - \sqrt{\eta} \right) \sqrt{x} - \frac{5\pi}{4} \right)
\]

Similarly, we get

\[
S_7(x) := \frac{1}{64} \sum_{n,m,k,l,r,s<q\leq y} g x^{7/4} \times \cos \left(4\pi \left(\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l} + \sqrt{s} + \sqrt{\eta} \right) \sqrt{x} - \frac{7\pi}{4} \right).
\]

By Lemma 2.6, we get

\[
\int_T^{2T} S_1(x) dx = \frac{35\sqrt{2}}{128} \int_T^{2T} s_{7,3}(d;y) x^{7/4} dx
\]

Similarly, we can get

\[
\int_T^{2T} S_3(x) dx = \frac{21\sqrt{2}}{128} \int_T^{2T} s_{7,2}(d) x^{7/4} dx + O \left(T^{11/4 - 1/8 + \varepsilon} \right)
\]

and

\[
\int_T^{2T} S_5(x) dx = \frac{7\sqrt{2}}{128} \int_T^{2T} s_{7,1}(d) x^{7/4} dx + O \left(T^{11/4 - 1/8 + \varepsilon} \right).
\]

We now proceed to consider the contribution of $S_7(x)$. Applying Lemma 2.3, we have

\[
\int_T^{2T} S_7(x) dx \ll \sum_{n,m,k,l,r,s<q\leq y} g T^{3/4} \sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l} + \sqrt{s} + \sqrt{\eta}
\]

\[
\ll T^{9/4 + \varepsilon} \sum_{n\leq m\leq k\leq l\leq r\leq s\leq q\leq y} \frac{1}{(nmklmrqs)^{3/4}y^{1/2}} \ll T^{9/4 + \varepsilon} g y^{5/4} \ll T^{11/16 + \varepsilon}.
\]

Now we consider the contribution of $S_2(x)$.

By the first derivative test, we get

\[
\int_T^{2T} S_2(x) dx \ll \sum_{n,m,k,l,r,s<q\leq y} g \times \min \left(\frac{T^{11/4}}{\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l} - \sqrt{s} - \sqrt{\eta}}, \frac{T^{9/4}}{\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l} + \sqrt{s} + \sqrt{\eta}} \right)
\]

\[
\ll x^2 G(N, M, K, L, R, S, Q).
\]
where
\[
G(N, M, K, L, R, S, Q) = \sum_{\sqrt{m} + \sqrt{k} + \sqrt{t} \neq \sqrt{y} + \sqrt{q}} \sum_{\substack{n \sim N, m \sim M, k \sim K, t \sim L, r \sim R, s \sim S, q \sim Q \\ 1 \leq R \leq S \leq L \leq y \\ 1 \leq n \leq M \leq K \leq L \leq y}} g
\times \min \left( \frac{T^{11/4}}{\sqrt{m} + \sqrt{k} + \sqrt{t} - \sqrt{y} - \sqrt{q}}, \frac{T^{9/4}}{\sqrt{m} + \sqrt{k} + \sqrt{t} - \sqrt{y} - \sqrt{q}} \right).
\]

If \(L \geq 100Q\), then \(|\sqrt{m} + \sqrt{k} + \sqrt{t} - \sqrt{y} - \sqrt{q}| \approx L^{1/2}\), so the trivial estimate yields
\[
G(N, M, K, L, R, S, Q) \ll \frac{T^{9/4 + \varepsilon} N M K L R S (Q^5 L^{1/2})}{(N M K L R S Q)^{5/4}} \ll T^{9/4 + \varepsilon} y^{5/4} \ll T^{41/16 + \varepsilon}.
\]

If \(Q \geq 100L\), we can get the same estimate. So later we always suppose that \(L \asymp Q\). Let \(\eta = \sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{t} - \sqrt{y} - \sqrt{s} - \sqrt{q}\). Write
\[
G(N, M, K, L, R, S, Q) = G_1 + G_2 + G_3,
\]
where
\[
G_1 := T^{11/4} \sum_{0 < |\eta| \leq T^{-1/2}} g,
G_2 := T^{9/4} \sum_{T^{-1/2} < |\eta| \leq 1} g|\eta|^{-1},
G_3 := T^{9/4} \sum_{|\eta| > 1} g|\eta|^{-1}.
\]

We estimate \(G_1\) first. From \(|\eta| \leq T^{-1/2}\), we get \(Q \gg T^{1/63}\) via Lemma 2.1. By Lemma 2.8, we get
\[
G_1 \ll \frac{T^{11/4 + \varepsilon}}{(N M K L R S Q)^{5/4}} \xi_1(N, M, K, L, R, S, Q; T^{-1/2})
\ll \frac{T^{11/4 + \varepsilon}}{(N M K L R S Q)^{5/4}} \left( T^{-1/2} Q^{1/2} N M K L R S + N M K L R S L^{1/2} \right)
\ll T^{9/4 + \varepsilon} y^{5/4} + T^{11/4 + \varepsilon} (N M K L R S)^{1/4} L^{-1}
\ll T^{41/16 + \varepsilon} + T^{11/4 + \varepsilon} (N M K L R S)^{1/4} L^{-1}.
\]

By Lemma 2.10, we get
\[
G_1 \ll \frac{T^{11/4 + \varepsilon}}{(N M K L R S Q)^{5/4}} \xi_{-\varepsilon}(N, M, K, L, R, S, Q; T^{-1/2})
\ll \frac{T^{11/4 + \varepsilon}}{(N M K L R S Q)^{5/4}} \left( T^{-1/4} N^{13/14} + N^{5/7} \right) (T^{-1/4} M^{13/14} + M^{5/7})
\times \left( T^{-1/4} K^{13/14} + K^{5/7} \right) (T^{-1/4} R^{13/14} + R^{5/7}) (T^{-1/4} S^{13/14} + S^{5/7}) (T^{-1/7} Q^{13/7} + Q^{10/7})
\ll T^{11/4 + \varepsilon} Q^{-1/14} (N M K L R S)^{-1/28} (T^{-1/7} Q^{3/7} + 1) (T^{-1/4} N^{3/4} + 1)
\times (T^{-1/4} M^{3/4} + 1) (T^{-1/4} K^{3/4} + 1) (T^{-1/4} R^{3/4} + 1) (T^{-1/4} S^{3/4} + 1)
\ll T^{11/4 + \varepsilon} Q^{-1/14} (N M K L R S)^{-1/28}
\]
(3.10)
by noting that $T^{-1/7}D^{4/7} \ll 1$ for $D = Q, N, M, K, R, S$. From (3.9) and (3.10), we get

$$G_1 \ll T^{41/16+\varepsilon} + T^{11/4+\varepsilon} \cdot \min\left(\frac{(NMKLRS)^{1/4}}{L}, \frac{1}{Q^{1/14}(NMKLRS)^{1/28}}\right)^{1/8} \left(\frac{1}{Q^{1/14}(NMKLRS)^{1/28}}\right)^{7/8}$$

$$\ll T^{41/4+\varepsilon} + T^{11/4+\varepsilon} Q^{-5/16}$$

if we notice the fact that $Q \gg T^{1/63}$.

Now we estimate $G_2$. We also suppose $K \leq R$ and the other cases are the same. By a splitting argument, we get the estimate

$$G_2 \ll \frac{T^{9/4+\varepsilon}}{(NMKLRS)^{3/4}} \sum_{\delta < |\eta| \leq 2\delta} 1$$

for some $T^{-1/2} \leq \delta \leq 1$. By Lemma 2.8, we get

$$G_2 \ll \frac{T^{9/4+\varepsilon}}{(NMKLRS)^{3/4}} \phi_1(N, M, K, L, R, S; 2\delta)$$

$$\ll \frac{T^{9/4+\varepsilon}}{(NMKLRS)^{3/4}} \left(\delta Q^{1/2}NMKLRS + NMKRSL^{1/2}\right)$$

$$\ll T^{9/4+\varepsilon} \delta^{-1}(\delta^1/N^{5/28} + N^{-1/28})(\delta^1/M^{5/28} + M^{-1/28})(\delta^1/K^{5/28} + K^{-1/28})$$

$$\times(\delta^1/R^{5/28} + R^{-1/28})(\delta^1/S^{5/28} + S^{-1/28})(\delta^2/Q^{5/14} + Q^{-1/14})$$

$$\ll T^{9/4+\varepsilon} \delta^{-1}Q^{-1/14}(NMKLRS)^{-1/28}(\delta^2/Q^{3/7} + 1)\cdot\phi(NMKLRS)^{3/14} + \delta^2/(KLRS)^{3/14} + \delta^2/(RS)^{3/14} + \delta^2/(S)^{3/14} + 1)$$

On the other hand, by Lemma 2.10, we have

$$G_2 \ll \frac{T^{9/4+\varepsilon}}{(NMKLRS)^{3/4}} \phi_1(N, M, K, L, R, S; 2\delta)$$

$$\ll \frac{T^{9/4+\varepsilon}}{(NMKLRS)^{3/4}} \phi(NMKLRS)^{-1/28}(\delta^2/Q^{3/7} + 1)\cdot\phi(NMKLRS)^{3/14} + \delta^2/(KLRS)^{3/14} + \delta^2/(RS)^{3/14} + \delta^2/(S)^{3/14} + 1)$$
\[
\begin{align*}
\ll T^{9/4+\epsilon} y^{5/4} &+ T^{9/4+\epsilon} T^{1/7} y^{23/28} + T^{9/4+\epsilon} \delta^{-1} Q^{-1/14} (N M K R S)^{-1/28} (\delta^{2/7} Q^{3/7} + 1) \\
&\times (\delta^{1/7} Q^{6/7} + \delta^{3/7} Q^{9/14} + \delta^{2/7} Q^{3/7} + \delta^{1/7} Q^{1/14} + 1) \\
\ll T^{291/112+\epsilon} &+ T^{9/4+\epsilon} \delta^{-1} Q^{-1/14} (N M K R S)^{-1/28} (\delta^{6/7} Q^{9/7} + 1). \quad (3.13)
\end{align*}
\]

From (3.12) and (3.13), we get
\[
\begin{align*}
G_2 \ll T^{291/112+\epsilon} + T^{9/4+\epsilon} \delta^{-1} \min \left( \frac{(N M K R S)^{1/4}}{L} \right) \left( \frac{\delta^{6/7} Q^{9/7} + 1}{Q^{1/14} (N M K R S)^{1/28}} \right)^{7/8} \\
\ll T^{291/112+\epsilon} + T^{9/4+\epsilon} \delta^{-1} Q^{-3/16} (\delta^{3/4} Q^{9/8} + 1). \quad (3.14)
\end{align*}
\]

If \( \delta \gg Q^{-3/2} \), then \( \delta^{3/4} Q^{9/8} \gg 1 \) and (3.14) implies (recall \( \delta \gg T^{-1/2} \))
\[
\begin{align*}
G_2 \ll T^{291/112+\epsilon} + T^{9/4+\epsilon} \delta^{-1} Q^{15/16} &+ T^{9/4+\epsilon} \delta^{-1} Q^{15/16} \\
\ll T^{167/64+\epsilon}. \quad (3.15)
\end{align*}
\]

If \( \delta \ll Q^{-3/2} \), then \( \delta^{3/4} Q^{9/8} \ll 1 \) and (3.14) implies
\[
\begin{align*}
G_2 \ll T^{291/112+\epsilon} + T^{9/4+\epsilon} \delta^{-1} Q^{-3/16}. \quad (3.16)
\end{align*}
\]

By Lemma 2.1, we have \( 2\delta \gg |\eta| \gg Q^{-63/2} \), which implies \( \delta^{-1} \ll Q^{63/2} \). Therefore, we get
\[
\delta^{-1} \ll \min \left( Q^{63/2}, T^{1/2} \right).
\]

From the above estimate and (3.16), we get
\[
\begin{align*}
G_2 \ll T^{291/112+\epsilon} + \min \left( T^{9/4+\epsilon} Q^{501/16}, T^{111/4+\epsilon} Q^{-3/16} \right) \\
\ll T^{291/112+\epsilon} + \left( T^{9/4+\epsilon} Q^{501/16} \right)^{1/168} \left( T^{111/4+\epsilon} Q^{-3/16} \right)^{167/168} \\
\ll T^{11/4-1/336+\epsilon}. \quad (3.17)
\end{align*}
\]

For \( G_3 \), by a splitting argument and Lemma 2.8 (notice \( |\eta| \gg 1 \)), we get
\[
\begin{align*}
G_3 \ll T^{9/4+\epsilon} \sum_{\delta \ll |\eta| \ll 25} \frac{1}{|\delta|} \\
\ll T^{9/4+\epsilon} Q^{11/2} N M K R S \\
\ll T^{9/4+\epsilon} y^{5/4} \\
\ll T^{41/16+\epsilon}. \quad (3.18)
\end{align*}
\]

Combining (3.7), (3.8), (3.11), (3.17) and (3.18), we get
\[
\int_T^{2T} S_2(x) dx \ll T^{11/4-1/336+\epsilon}. \quad (3.19)
\]

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In the same way, we can prove that
\[
\int_T^{2T} S_4(x)dx \ll T^{11/4 - 1/336 + \varepsilon} \tag{3.20}
\]
and
\[
\int_T^{2T} S_6(x)dx \ll T^{11/4 - 1/336 + \varepsilon}, \tag{3.21}
\]
if we use Lemma 2.9 instead of Lemma 2.8.

Now Proposition 3.1, i.e. equation (3.2), follows from (3.3)-(3.6) and (3.19)-(3.21).

**Proposition 3.2** For \( T^\varepsilon \ll y \ll T^{1/4} \), we have
\[
\int_T^{2T} |R_2|^7dx \ll T^{11/4 - 1/336 + \varepsilon} \tag{3.22}
\]
and
\[
\int_T^{2T} |R_1|^6|R_2|dx \ll T^{11/4 - 1/336 + \varepsilon}. \tag{3.23}
\]

**Proof.** First, for \( T \leq x \leq 2T \), we have
\[
R_2 = \frac{1}{\sqrt{2\pi}} \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} 2^{1/4} \cos(4\pi \sqrt{nx} - \pi/4) + O(x^{1/2+\varepsilon}T^{-1/2})
\]
\[
\ll x^{1/4} \left| \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{nx} - \pi/4) \right| + T^\varepsilon
\]
\[
\ll x^{1/4} \left| \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} (2\sqrt{nx}) \right| + T^\varepsilon.
\]

Therefore, one has
\[
\int_T^{2T} |R_2|^2dx \ll T^{1+\varepsilon} + \int_T^{2T} x^{1/2} \left| \sum_{y < n \leq T} \frac{d(n)}{n^{3/4}} (2\sqrt{nx}) \right|^2 dx
\]
\[
\ll T^{1+\varepsilon} + T^{1/2} \int_T^{2T} \sum_{y < n \leq T} \sum_{y < m \leq T} \frac{d(n)d(m)}{(nm)^{3/4}} e \left( 2(\sqrt{n} - \sqrt{m})\sqrt{x} \right) dx
\]
\[
= T^{1+\varepsilon} + T^{1/2} \sum_{y < n \leq T} \sum_{y < m \leq T} \frac{d(n)d(m)}{(nm)^{3/4}} \int_T^{2T} e \left( 2(\sqrt{n} - \sqrt{m})\sqrt{x} \right) dx
\]
\[
\ll T^{1+\varepsilon} + T^{3/2} \sum_{y < n \leq T} \frac{d^2(n)}{n^{3/2}} + T \sum_{y < n \leq T} \sum_{y < m \leq T} \frac{d(n)d(m)}{(nm)^{3/4}|\sqrt{n} - \sqrt{m}|}
\]
\[
\ll T^{1+\varepsilon} + T^{3/2} \sum_{n > y} \frac{d^2(n)}{n^{3/2}} + T \sum_{m < n \leq T} \frac{d(n)d(m)}{(nm)^{3/4}|\sqrt{n} - \sqrt{m}|}
\]
\[
\ll T^{1+\varepsilon} + T^{3/2} y^{-1/2} \log^3 y + T \sum_{m < n \leq T} \frac{d(n)d(m)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})}
\]
\[
\ll T^{3/2} y^{-1/2} \log^3 T + T \sum_{m < n \leq T} \frac{d(n)d(m)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})}. \tag{3.24}
\]
For the last sum in (3.24), we have
\[ \sum_{m<n \leq T} \frac{d(n)d(m)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})} := \Sigma_1 + \Sigma_2, \tag{3.25} \]
where
\[ \Sigma_1 := \sum_{m<n \leq T} \frac{d(n)d(m)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})}, \]
\[ \Sigma_2 := \sum_{m<n \leq T} \frac{d(n)d(m)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})}. \]

For \( \Sigma_1 \), one has
\[ \Sigma_1 \ll \sum_{n \leq T} \sum_{m<n} \frac{d(n)d(m)}{nm} \ll \sum_{n \leq T} \sum_{m<n} \frac{d(n)}{n} \sum_{m<n} \frac{d(m)}{m} \ll \log^4 T. \tag{3.26} \]

For \( \Sigma_2 \), we can write
\[ \Sigma_2 \ll \sum_{m<n \leq T} \frac{d^2(n) + d^2(m)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})} := \Sigma_{21} + \Sigma_{22}, \tag{3.27} \]
where
\[ \Sigma_{21} := \sum_{0<\sqrt{n} - \sqrt{m} \leq \frac{1}{10}(nm)^{1/4}} \frac{d^2(n)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})}, \]
\[ \Sigma_{22} := \sum_{0<\sqrt{n} - \sqrt{m} \leq \frac{1}{10}(nm)^{1/4}} \frac{d^2(m)}{(nm)^{3/4}(\sqrt{n} - \sqrt{m})}. \]

By the mean value theorem and taking \( n - m = r \), we get
\[ \Sigma_{21} \ll \sum_{n \leq T} \sum_{m<n} \frac{d^2(n)}{n(n-m)} \ll \sum_{n \leq T} \frac{d^2(n)}{n} \sum_{1 \leq r \leq T} \frac{1}{r} \ll \log^5 T, \tag{3.28} \]
\[ \Sigma_{22} \ll \sum_{n \leq T} \sum_{m<n} \frac{d^2(m)}{m(n-m)} \ll \sum_{m \leq T} \frac{d^2(m)}{m} \sum_{1 \leq r \leq T} \frac{1}{r} \ll \log^5 T. \tag{3.29} \]

By the estimate (3.24)-(3.29), we get
\[ \int_T^{2T} |R_2|^2 \, dx \ll T^{3/2} y^{-1/2} \log^3 T. \tag{3.30} \]

Taking \( y = T^{1/4} \), we get \( R_1 \ll T^{1/4} y^{1/4} \ll T^{131/416} \), and therefore \( R_2 \ll T^{131/416+\varepsilon} \). Using Ivić’s large-value technique directly to \( R_2 \) without modifications, one can get that the estimate
\[ \int_T^{2T} |R_2|^4 \, dx \ll T^{1+\frac{5}{4}+\varepsilon}. \tag{3.31} \]
holds with $A_0 = \frac{184}{19}, T^\varepsilon \ll y \ll T^{1/4}$. Then, for $2 < A < A_0$, by (3.30), (3.31) and Hölder’s inequality, we have

$$
\int_T^{2T} |R_2|^A \, dx = \int_T^{2T} |R_2|^{\frac{2(A_0-A)}{A_0-2}} \, dx \\
\ll \left( \int_T^{2T} |R_2|^2 \, dx \right)^{\frac{A_0-A}{A_0-2}} \left( \int_T^{2T} |R_2|^{A_0} \, dx \right)^{\frac{A_0-2}{A_0-2}} \\
\ll T^{1+\frac{A_0-A}{2(A_0-2)}+\varepsilon} y^{-\frac{A_0-A}{2(A_0-2)}}.
$$

Especially, we take $A = 7, y = T^{1/4}$ and get

$$
\int_T^{2T} |R_2|^7 \, dx \ll T^{11/4+\varepsilon} y^{-51/292} \\
\ll T^{11/4-51/1168+\varepsilon} \\
\ll T^{11/4-1/336+\varepsilon}.
$$

By Hölder’s inequality, we have

$$
\int_T^{2T} |R_1|^6 |R_2| \, dx \ll \left( \int_T^{2T} (|R_1|^6)^{7/6} \, dx \right)^{6/7} \left( \int_T^{2T} |R_2|^7 \, dx \right)^{1/7} \\
\ll \left( \int_T^{2T} |R_1|^7 \, dx \right)^{6/7} \left( \int_T^{2T} |R_2|^7 \, dx \right)^{1/7} \\
\ll (T^{11/4})^{6/7} \cdot (T^{11/4-51/1168+\varepsilon})^{1/7} \\
\ll T^{11/4-1/336+\varepsilon}.
$$

Now Proposition 3.2 follows from (3.32) and (3.33).

From Proposition 3.1 and Proposition 3.2, we get the seventh-power moment of $\Delta(x)$.

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