Abstract. Given a framed quiver, i.e. one with a frozen vertex associated to each mutable vertex, there is a concept of green mutation, as introduced by Keller. Maximal sequences of such mutations, known as maximal green sequences, are important in representation theory and physics as they have numerous applications, including the computations of spectrums of BPS states, Donaldson-Thomas invariants, tilting of hearts in the derived category, and quantum dilogarithm identities. In this paper, we study such sequences and construct a maximal green sequence for every quiver mutation-equivalent to an orientation of a type A Dynkin diagram.

1. Introduction

A very important problem in cluster algebra theory, with applications to polyhedral combinatorics and the enumeration of BPS states in string theory, is to determine when a given quiver has a maximal green sequence. In particular, it is open to decide which cluster algebras from surfaces admit a maximal green sequence, although progress for surfaces with boundary has been made in the physics literature [1]. Therein, they give heuristics for exhibiting maximal green sequences for cluster algebras from surfaces with boundaries. For other surfaces, such as the once-punctured torus, maximal green sequences do not necessarily exist, and it is still unknown the exact set of surfaces such that the corresponding cluster algebra admits a maximal green sequence.

Additionally, even for cases where the existence of maximal green sequences is known, the problem of exhibiting, classifying or counting maximal green sequences, even for type A is still open. (By a quiver of type A, we mean any quiver mutation-equivalent to an orientation of a type A Dynkin diagram.) See [4] for work on this for cluster algebras of type A for small rank. In the case where $Q$ is acyclic, one can find a maximal green sequence of length $|Q_0|$ by mutating at sources and iterating until all vertices have been mutated exactly once. However, even in the smallest non-acyclic case, e.g. $A_3$, the smallest maximal green sequence is of length 4 and describing...
the collection of maximal green sequences is mysterious. See Section 8.4 for more details. With an eye towards better understanding such sequences, in this paper we explicitly construct a maximal green sequence for every quiver of type \( \mathbb{A} \).

In Section 2 we begin with background on quivers and their mutations. Section 3 describes how to decompose quivers into direct sums of strongly connected components, which we call irreducible quivers. As shown in Propositions 3.7 and 3.13 to construct a maximal green sequence of a quiver, it suffices to construct maximal green sequences for each of its irreducible components.

For type \( \mathbb{A} \) quivers, irreducible quivers have an especially nice form as a tree of 3-cycles, as described by Corollary 4.2. This allows us to restrict our attention to embedded irreducible quivers of type \( \mathbb{A} \), which are constructed in detail in Section 4. With this construction in mind, we construct a special maximal green sequence associated to \( \mathbb{P} \). Following references such as [4, Section 2.3], given a quiver \( \mathbb{P} \) are maximal green sequences for each of its irreducible components. Propositions 3.7 and 3.13, to construct a maximal green sequence of a quiver, it suffices to construct maximal green sequences for any quiver of type \( \mathbb{A} \).

The proof of Theorem 5.7 is broken into two parts. Section 6 includes technical Lemmas and introduces notation that is used in Section 7 for the proof. In particular, in Section 6.1 we exactly describe the permutations (with respect to the frozen vertices) that are induced by applying a maximal green sequence to an embedded irreducible quiver of type \( \mathbb{A} \). This is a result that may be of independent interest and is used in our proof of Theorem 5.7. Finally, Section 8 ends with further remarks and ideas for future directions, including extensions to cluster algebras of other surfaces.

Acknowledgements. The authors would like to thank T. Brüstle, M. Del Zotto, B. Keller, R. Patrías, N. Reading, V. Reiner, D. Speyer, and H. Thomas for useful discussions. The authors were supported by NSF Grants DMS-1067183 and DMS-1148634.

2. Preliminaries and Notation

The reader may find excellent surveys on the theory of cluster algebras and maximal green sequences in [4]. For our purposes, we recall a few of the relevant definitions.

A quiver is a finite directed graph with no 2-cycles nor loops. By convention, we assume the vertices of \( \mathbb{P} \), denoted as \( (\mathbb{P})_0 = [N] = \{1, 2, \ldots , N\} \) and the edges of \( \mathbb{P} \), denoted as \( (\mathbb{P})_1 \), are referred to as arrows. The mutation of a quiver (at vertex \( k \)), denoted \( \mu_k \), produces a new quiver \( \mu_k \mathbb{P} \) by the three step process:

1. For every 2-path \( i \to k \to j \), adjoin a new arrow \( i \to j \).
2. Reverse the direction of all arrows incident to \( k \).
3. Delete any 2-cycles created during the first two steps.

Following references such as [4, Section 2.3], given a quiver \( \mathbb{P} \) on vertices \( [N] \), we form the framed quiver associated to \( \mathbb{P} \) by adjoining, to \( \mathbb{P} \), the vertices \( [N'] = \{1', 2', \ldots , N'\} \) and the arrows \( i \to i' \) for \( 1 \leq i \leq N \).

We denote this as \( \hat{\mathbb{P}} \). Since we never allow mutation at this second set of vertices, we refer to the elements of \( [N'] \) as frozen vertices. We also let \( \hat{\mathbb{Q}} \) denote the analogous quiver on vertices \( [N] \cup [N'] \) with arrows \( i' \to i \) instead. This is known as the co-framed quiver associated to \( \mathbb{P} \). For any other quiver on vertices \( [N] \cup [N'] \) with more complicated incidences of arrows, we will use the notation \( \hat{\mathbb{Q}} \), reserving \( \hat{\mathbb{Q}} \) and \( \hat{\mathbb{Q}} \) for the above two situations.

Given a mutation sequence \( \nu \) of a quiver \( \mathbb{P} \) (or \( \hat{\mathbb{P}} \)) we define the support of \( \nu \), denoted \( \text{supp}(\nu) \), to be the set of vertices of \( \mathbb{P} \) appearing in \( \nu \). Also, we define the support of a vector \( \mathbf{v} \in \mathbb{Z}^N \) to be the set of indices \( i \in [N] \) such that the \( i \)th entry of \( \mathbf{v} \) is nonzero. More generally, we define the support of an element \( \mathbf{v} \) of a finitely generated, free \( \mathbb{Z} \)-module \( M \) (with a fixed basis) to be the set of indices \( i \in [N] \) where \( N \) is the number of generators of \( M \) such that the \( i \)th entry of \( \mathbf{v} \) is nonzero. Additionally, given a subset \( S \subset (\mathbb{P})_0 \), define \( \mathbf{S} := \{x' : x \in S\} \) to be the set of corresponding frozen vertices.

A vertex \( i \) of \( \mathbb{Q} \subset \hat{\mathbb{Q}} \) is said to be green (resp. red) if all arrows from \( [N'] \) incident to \( i \) point away from (resp. towards) \( i \). Note that all vertices of \( \hat{\mathbb{Q}} \) are green and all vertices of \( \hat{\mathbb{Q}} \) are red. By the Sign-Coherence of \( \mathbb{E} \) and \( \mathbf{g} \)-vectors for cluster algebras [6, Theorem 1.7], it follows that even after mutation, each vertex of \( \hat{\mathbb{Q}} \) is either red or green.

Let \( \mu := \mu_i_1 \circ \cdots \circ \mu_i_k \) be a mutation sequence of \( \hat{\mathbb{Q}} \). Define \( \{\mathbb{Q}(k)\}_{k \geq 0} \) to be the sequence of quivers where \( \mathbb{Q}(0) = \hat{\mathbb{Q}} \) and \( \mathbb{Q}(j) = (\mu_i_1 \circ \cdots \circ \mu_i_j) (\hat{\mathbb{Q}}) \). (In particular, throughout this paper, we apply a sequence mutations in order from right-to-left.) A green sequence of \( \hat{\mathbb{Q}} \) is a mutation sequence \( \mu \) with \( \text{supp}(\mu) \subseteq \mathbb{Q} \) such that \( \mu_i_j = i \) is a green vertex of \( \hat{\mathbb{Q}}(j-1) \) for each \( 1 \leq j \leq k \).
Sequence $\mu$ is a \textbf{maximal green sequence} if it is a green sequence such that in the final quiver $Q^{(d)}$, vertices 1, 2, \ldots, $N$ are all red. In other words, $Q^{(d)}$ contains no green vertices. Following [4], we let $\text{green}(Q)$ denote the set of maximal green sequences for $Q$.

Proposition 2.10 of [4] shows that any maximal green sequence $\mu$ permutes the vertices of $Q$, i.e. $\hat{Q}^{(d)} = \overline{\hat{\sigma}}$ for some permutation $\sigma \in S_N$. We call this the \textbf{permutation induced} by $\mu$.

3. \textbf{Direct Sums of Quivers}

In this section, we define a direct sum of quivers. We also show that, under certain restrictions, if a quiver $Q$ can be written as a direct sum of quivers where each summand has a maximal green sequence, then the maximal green sequences of the summands can be concatenated in some way to give a maximal green sequence for $Q$. Throughout this section, we let $Q_1$ and $Q_2$ be finite quivers with $N_1$ and $N_2$ vertices, respectively. Furthermore, we assume $(Q_1)_0 = [N_1]$ and $(Q_2)_0 = [N_1 + 1, N_1 + N_2]$.

\textbf{Definition 3.1.} Let $\{a_1, \ldots, a_k\}$ denote an ordered $k$-multiset on $(Q_1)_0$ and $\{b_1, \ldots, b_k\}$ a $k$-multiset on $(Q_2)_0$. We define the \textbf{direct sum} of $Q_1$ and $Q_2$, denoted $Q_1 \oplus (b_1, \ldots, b_k)_{(a_1, \ldots, a_k)} Q_2$, to be the quiver with vertices

$$\left( Q_1 \oplus (b_1, \ldots, b_k)_{(a_1, \ldots, a_k)} Q_2 \right)_0 := (Q_1)_0 \sqcup (Q_2)_0 = [N_1 + N_2]$$

and arrows

$$\left( Q_1 \oplus (b_1, \ldots, b_k)_{(a_1, \ldots, a_k)} Q_2 \right)_1 := (Q_1)_1 \sqcup (Q_2)_1 \cup \left\{ a_i \overset{\alpha}{\rightarrow} b_i : i \in [k] \right\}.$$

We say that $Q_1 \oplus (b_1, \ldots, b_k)_{(a_1, \ldots, a_k)} Q_2$ is a \textbf{$t$-colored direct sum} if $t = \#\{\text{distinct elements of } \{a_1, \ldots, a_k\}\}$ and there does not exist $i$ and $j$ such that

$$\#\{a_i \overset{\alpha}{\rightarrow} b_j\} \geq 2.$$

\textbf{Remark 3.2.} The direct sum of two quivers is a non-associative operation as is shown in Example 3.4.

\textbf{Definition 3.3.} We say that a quiver $Q$ is \textbf{irreducible} if

$$Q = Q_1 \oplus (b_1, \ldots, b_k)_{(a_1, \ldots, a_k)} Q_2$$

for some ordered $k$-multiset $\{a_1, \ldots, a_k\}$ on $(Q_1)_0$ and some $k$-multiset $\{b_1, \ldots, b_k\}$ on $(Q_2)_0$ implies that $Q_1 = 0$ or $Q_2 = 0$.

\textbf{Example 3.4.} Let $Q$ denote the quiver shown below. Define $Q_1$ to be the full subquiver of $Q$ on the vertices 1, \ldots, 4, $Q_2$ to be the full subquiver of $Q$ on the vertices 6, \ldots, 11, and $Q_3$ to be the full subquiver of $Q$ on the vertex 5. Note that $Q_1$, $Q_2$, and $Q_3$ are each irreducible. Then

$$Q = Q_1 \oplus (5,8,11,8,9,11)_{(1,1,1,3,4,4)} Q'_2$$

where $Q'_2 = Q_2 \oplus (5) Q_3$ so $Q$ is a 3-colored direct sum. On the other hand, we could write

$$Q = Q'_1 \oplus (5,5)_{(1,6)} Q_3$$

where $Q'_1 = Q_1 \oplus (8,11,8,9,11)_{(1,1,3,4,4)} Q_2$ so $Q$ is a 2-colored direct sum. Additionally, note that

$$Q_1 \oplus (5,8,11,8,9,11)_{(1,1,1,3,4,4)} Q'_2 = Q_1 \oplus (5,8,11,8,9,11)_{(1,1,1,3,4,4)} (Q_2 \oplus (5) Q_3) \neq (Q_1 \oplus (5,8,11,8,9,11)_{(1,1,1,3,4,4)} Q_2) \oplus (5) Q_3$$
where the last equality does not hold because $Q_1 \oplus_{(5,8,11,8,9,11)}^{(5,8,11,8,9,11)} Q_2$ is not defined as 5 is not a vertex of $Q_2$. This shows that the direct sum of two quivers, in the sense of this paper, is not associative.

For the remainder of this section, we will consider a quiver $Q$ with summands $Q_1$ and $Q_2$. Let $\mu = \mu_{i_1} \circ \cdots \circ \mu_{i_1}$ be a mutation sequence of $\hat{Q}_i$, and $\{Q^{(k)}\}_{k \geq 0}$ be the sequence of quivers as above. We will also write $Q_i^{(k)}$ with $i = 1, 2$ to denote the full subquiver of $Q^{(k)}$ on the vertices of $Q_1$. Additionally, define

$$\alpha(x, y, k) := \#\{x \xrightarrow{\mu} y \in (Q^{(k)})_1\} - \#\{y \xrightarrow{\mu} x \in (Q^{(k)})_1\}.$$  

3.1. 1-colored Direct Sums. In this subsection, we let $Q = Q_1 \oplus_{(a, \ldots, a)}^{(b_1, \ldots, b_r)} Q_2$ and assume that the $b_i$‘s are pairwise distinct.

**Lemma 3.5.** Let $\mu$ be a mutation sequence of $Q$ supported on $Q_1$. For any $x \in (Q_1^{(k)})_0$, $\alpha(x, b_i, k) = \alpha(x, b_j, k)$ for all $i, j \in [r]$.

**Proof.** We proceed by induction on $k$. If $k = 0$, no mutations have been applied so we have the result.

Suppose the result holds for $Q^{(k-1)}$ and we will show that the result also holds for $Q^{(k)}$. We can write $Q^{(k)} = \mu_y Q^{(k-1)}$ for some $y \in (Q_1)_0$. Let $x$ be any vertex of $Q_1$. If $x$ is not connected to $y$ in $Q^{(k-1)}$, then we have $\alpha(x, b_i, k) = \alpha(x, b_i, k-1)$ for all $i \in [r]$ by induction.

If $x = y$, then $\alpha(x, b_i, k) = -\alpha(x, b_i, k-1)$ for all $i \in [r]$ so by induction we have the result.

Suppose $x$ is connected to $y$ in $Q^{(k-1)}$. The number $\alpha(x, b_i, k-1)$ will only differ from $\alpha(x, b_i, k)$ if in $Q^{(k-1)}$, the local configuration

\[
\begin{array}{c}
\alpha(x, b_i, k-1) \\
\alpha(x, y, k-1) \\
x
\end{array}
\xrightarrow{\mu} \begin{array}{c}
y \alpha(y, b_i, k-1) \\
b_i
\end{array}
\]

appears with $\alpha(x, y, k-1), \alpha(y, b_i, k-1)$ both strictly positive or both strictly negative. Assume this is the case. Then, by induction, if the above configuration appears for some $i \in [r]$, then it will appear for all $i \in [r]$. In other words, by induction, we will have that $m(x, k-1) = \alpha(x, b_i, k-1)$ for all $i \in [r]$ and $m(y, k-1) = \alpha(y, b_i, k-1)$ for all $i \in [r]$ for some integers $m(x, k-1)$ and $m(y, k-1)$. Thus in $Q^{(k)}$, we have $\alpha(x, b_i, k) = m(x, k-1) + \alpha(x, y, k-1)m(y, k-1)$ for all $i \in [r]$. Since the right hand side of this equation does not depend on $i$, we have obtained the desired result. \[\square\]

Lemma 3.5 implies that for any $x \in (Q_1^{(k)})_0$ and any $k \geq 0$, there exists an integer $m(x, k)$ such that $\alpha(x, b_i, k) = m(x, k)$ for all $i \in [r]$.

The next Lemma gives a “Sign-Coherence” result for the arrows connecting $Q_1$ to $Q_2$ in $Q$.

**Lemma 3.6.** Let $x, a \in (Q_1)_0$. Recall that $a' \in (\hat{Q}_1)_{0}$ is the frozen vertex corresponding to $a$ and let $\mu$ be a mutation sequence supported on $Q_1$. For any $k \geq 0$ and any $x \in (Q_1^{(k)})_0$, $\alpha(x, a', k) = m(x, k)$.

**Proof.** By Lemma 3.5 it is enough to prove that $\alpha(x, a', k) = m(x, k) = \alpha(x, b_1, k)$.

The desired result holds for $k = 0$ since we assume $Q$ is a 1-colored direct sum with all $b_i$‘s distinct. Now suppose that the result holds for all $k < k$ where $k$ is given. Let $x$ be a vertex of $Q_1^{(k)}$ with $\alpha(x, a', k) = m$. Suppose $Q^{(k)} = \mu_y Q^{(k-1)}$ for some $y \in (Q_1)_0$. There are three cases to consider:
Case i) $x = y$

Case ii) $x$ is connected to $y$ and $\text{sgn}(\alpha(x, y, k - 1)) = \text{sgn}(\alpha(y, b_1, k - 1))$

Case iii) $x$ is connected to $y$ and $\text{sgn}(\alpha(x, y, k - 1)) \neq \text{sgn}(\alpha(y, b_1, k - 1))$ or $x$ is not connected to $y$.

Thus we have

$$\alpha(x, a', k) = \begin{cases} 
-\alpha(y, a', k - 1) & : \text{Case i) } \\
\alpha(x, b_1, k - 1) + \alpha(x, y, k - 1)\alpha(y, b_1, k - 1) & : \text{Case ii) } \\
\alpha(x, b_1, k - 1) & : \text{Case iii) }
\end{cases}$$

$$= \begin{cases} 
-m(y, k - 1) & : \text{Case i) } \\
m(x, k - 1) + \alpha(x, y, k - 1)m(y, k - 1) & : \text{Case ii) } (\text{by induction}) \\
m(x, k - 1) & : \text{Case iii) }
\end{cases}$$

$$= \alpha(x, b_1, k).$$

\[\square\]

**Proposition 3.7.** If $\mu_1 \in \text{green } (Q_1)$ and $\mu_2 \in \text{green } (Q_2)$, then $\mu_2 \circ \mu_1 \in \text{green } \left(Q_1 \oplus_{(a, a, \ldots, a)} b_1, b_2, \ldots, b_k \right) Q_2.$

We need some additional preparation before presenting the proof of Proposition 3.7. The proof of Proposition 3.7 can be found at the end of the next section.

3.2. **t-colored Direct Sums.** Now suppose we have a subset $\{a_1, \ldots, a_s\} \subset (Q_1)_0$ and for each $j \in [s]$ we have subsets $\{b_{1j}, \ldots, b_{sj}\} \subset (Q_2)_0$. Let $Q$ denote the direct sum quiver obtained by adding one arrow from $a_i$ to $b_{ij}$ for each $i \in [s]$ and all $\ell \in [r_i]$. For each $i \in [s], \ell \in [r_i]$ assign what we will call a color to the arrows $\alpha(i, \ell) : a_i \to b_{ij}$ by defining a function $f$ on the arrows connecting $Q_1$ to $Q_2$ into the natural numbers such that $f(\alpha(i, \ell)) = f(\alpha(j, r_j))$ if and only if $i = j$. Thus we will say that the arrows $\alpha(i, \ell) : a_i \to b_{ij}$ have color $f_i$ and we will refer to $f$ as a **coloring function** on $Q$. It is easy to see that a quiver $Q$ is a $t$-colored direct sum if and only if there exists a coloring function $f$ associated to $Q$ that takes on exactly $t$ distinct values.

The following more general results about $t$-colored direct sums are general enough to encompass all quivers coming from triangulated surfaces because as the following Lemma shows we will not have a double arrow connecting two summands and because a vertex in such a quiver has at most two outgoing arrows.

**Lemma 3.8.** Except for the Kronecker quiver, if $Q$ is defined by a triangulated surface (with 1 connected component) and $a \xrightarrow{\alpha_1} b \xrightarrow{\alpha_2}$ appears in $Q$, then there exists a path from $b$ to $a$.

**Proof.** Since $Q$ comes from a triangulated surface, there exists a block decomposition $\{R_1\}_{i \in [m]}$ of $Q$ by Theorem 13.3 in [FST]. By definition of the blocks, $\alpha_1$ and $\alpha_2$ come from distinct blocks. Without loss of generality, $\alpha_1$ is an arrow of $R_1$ and $\alpha_2$ is an arrow of $R_2$. Furthermore, in $R_i$ with $i = 1, 2$ we must have that $s(\alpha_1)$ and $t(\alpha_1)$ are outlets. Thus $R_i$ with $i = 1, 2$ is of type I, II, or IV, but by assumption $R_1$ and $R_2$ are not both of type I. When we glue the $R_1$ to $R_2$ to using the identifications associated with $Q$, a case by case analysis shows that there exists a path of length 2 from $b$ to $a$. Furthermore, the vertices corresponding to $a$ and $b$ are no longer outlets. Thus attaching the remaining $R_j$’s will not delete any arrows from this path.

\[\square\]

**Corollary 3.9.** If $Q$ is defined by a triangulated surface (with 1 connected component) and $Q$ is not irreducible, then $Q$ is a $t$-colored direct sum for some $t \in \mathbb{N}$.

In what follows we let $Q = Q_1 \oplus_{(a_1, \ldots, a_1, \ldots, a_s)} b_1, b_2, \ldots, b_k \oplus Q_2$. Let $\mu$ be a mutation sequence of $Q$ supported on $Q_1$, and let $\{Q^{(k)}\}_{k \geq 0}$ denote the sequence of quivers appearing from mutations of $\mu$. Now fix a coloring of the $f$ of the edges connecting $Q_1$ to $Q_2$ as described above. Additionally, define

$$\alpha(x, b_{ij}, f_j, k) := \#\{x \xrightarrow{\alpha} b_{ij} \in \left(Q^{(k)}_1\right)_{f_j}\}.$$

Thus $Q$ is an $s$-colored quiver with $\alpha(a_j, b_{ij}, f_j, 0) = 1$ for all $\ell \in [r_j]$ and for all $j \in [s]$.

**Example 3.10.** Using the notation from Example 3.4 and writing

$$Q = Q_1 \oplus_{(1,1,1,3,4,4)} \left(Q_2 \oplus_{(5)} Q_3\right),$$

then
we have \( a_1 = 1 \) and \( b_{i_1}^{(1)} = 5, b_{i_2}^{(1)} = 8, b_{i_1}^{(1)} = 11, a_2 = 3 \) and \( b_{i_1}^{(2)} = b_{i_2}^{(2)} = 8, \) and \( a_3 = 4 \) and \( b_{i_1}^{(3)} = 9, b_{i_3}^{(3)} = 11. \) Also, the figure below shows how a coloring function \( f \) on \( Q \) can be defined.

The proof of the following Lemma is analogous to that of Lemma 3.5 so we omit it.

**Lemma 3.11.** For any \( k \geq 0, \) for any \( x \in \left( Q_1^{(k)} \right)_0 \) and for any \( j \in [s], \) there is an integer \( m(x, f_j, k) \) such that \( \alpha(x, b_{i_t}^{(j)}, f_j, k) = m(x, f_j, k) \) for all \( t \in [r_j]. \)

The proof of the following “Sign-Coherence” Lemma for \( t \)-colored quivers is essentially the same as that of Lemma 3.6 so we omit it.

**Lemma 3.12.** For any \( k \geq 0, \) for any \( x \in \left( Q_1^{(k)} \right)_0 \) and for any \( j \in [s], \) we have \( \alpha(x, a_j', k) = m(x, f_j, k) \) if and only if \( \alpha(x, b_{i_t}^{(j)}, f_j, k) = m(x, f_j, k) \) for all \( t \in [r_j]. \)

**Proof of Prop. 3.7.** Let \( \sigma_i \) denote the permutation of the vertices of \( Q_i \) determined by \( \mu_{i}. \) Now observe that
\[
\mu_2 \circ \mu_1 \left( Q_1 \oplus_{(a,\ldots,a)} Q_2 \right) = \mu_2 \circ \mu_1 \left( \widehat{Q}_1 \oplus_{(a,\ldots,a)} \widehat{Q}_2 \right) = \mu_2 \left( \widehat{Q}_2 \oplus_{(b,\ldots,b)} \widehat{Q}_1 \sigma_1 \right)
\]
where the last equality holds because \( \mu_1 \widehat{Q}_1 = \widehat{Q}_1 \sigma_1 \cong \widehat{Q}_1 \) (as \( \mu_1 \in \text{green} (Q_1) \)) and because by Lemma 3.6 all arrows from \( Q_1 \) to \( Q_2 \) must point to \( Q_1 \) in \( \mu_1 \widehat{Q}. \) Notice that \( \widehat{Q}_2 \oplus_{(b,\ldots,b)} \widehat{Q}_1 \sigma_1 \) is a \( k \)-colored direct sum. Also, notice that since \( \mu_1 \) is supported on \( Q_1 \) and \( \mu_1 \in \text{green} (Q_1), \) we have only performed mutations at green vertices in the process of applying \( \mu_1. \)

Next, \( \mu_2 \widehat{Q}_2 = \widehat{Q}_2 \sigma_2 \cong \widehat{Q}_2 \) (as \( \mu_2 \in \text{green} (Q_2) \)) and by Lemma 3.12 all arrows from \( Q_2 \) to \( Q_1 \) must point to \( Q_2 \) in \( \left( \mu_2 \circ \mu_1 \right) \left( \widehat{Q} \right) \). Also, note that by Lemma 3.12 any mutation \( \mu_2 \) appearing in \( \mu_2 \) will not add any arrows from \( Q_1 \) to \( Q_2 \) and thus in the process of applying \( \mu_2 \) to \( \widehat{Q}_2 \oplus_{(b,\ldots,b)} \widehat{Q}_1 \sigma_1 \) no vertices in \( Q_1 \) become green. Thus we have
\[
\mu_2 \left( \widehat{Q}_2 \oplus_{(b,\ldots,b)} \widehat{Q}_1 \sigma_1 \right) = \widehat{Q}_1 \sigma_1 \oplus_{(a,\ldots,a)} \widehat{Q}_2 \sigma_2.
\]
Notice that \( \mu_2 \) is supported on \( Q_2 \) and \( \mu_2 \in \text{green} (Q_2) \) so we have only performed mutations at green vertices in the process of applying \( \mu_2. \) Thus \( \mu_2 \circ \mu_1 \) is a green mutation sequence of \( Q_1 \oplus_{(a,\ldots,a)} Q_2. \)

Now it is easy to see that
\[
\widehat{Q}_1 \sigma_1 \oplus_{(a,\ldots,a)} \widehat{Q}_2 \sigma_2 = \left( Q_1 \sigma_1 \oplus_{(a,\ldots,a)} \widehat{Q}_2 \sigma_2 \right) \cong \left( Q_1 \oplus_{(a,\ldots,a)} Q_2 \right) \sigma_2.
\]
Thus \( \mu_2 \circ \mu_1 \in \text{green} \left( Q_1 \oplus_{(a,\ldots,a)} Q_2 \right). \)
Proposition 3.13. If $\mu_1 \in \text{green} (Q_1)$ and $\mu_2 \in \text{green} (Q_2)$, then $\mu_3 \circ \mu_1 \in \text{green} \left( Q_1 \oplus_{(a_1, \ldots, a_z, \ldots, a_z)} (k_1^{(1)}, \ldots, k_t^{(1)}, \ldots, k_t^{(z)}) Q_2 \right)$. 

The proof of Proposition 3.13 is nearly identical to that of Proposition 3.7 so we omit it.

4. Embedded, Irreducible Type $A$ Quivers

In this section, we define irreducible type $A$ quivers and present some basic facts about them. We will then define and discuss embedded, irreducible type $A$ quivers.

4.1. Properties of Irreducible Type $A$ Quivers. We will use the following result.

Lemma 4.1. [5, Prop. 2.4] A quiver $Q$ is of type $A$ if and only if $Q$ satisfies the following:

i) All non-trivial cycles in the underlying graph of $Q$ are oriented and of length 3.

ii) Any vertex has at most four neighbors.

iii) If a vertex has four neighbors, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle.

iv) If a vertex has exactly three neighbors, then two of its adjacent arrows belong to a 3-cycle, and the third arrow does not belong to any 3-cycle.

Corollary 4.2. Besides the quiver of type $A_1$, the irreducible quivers of type $A$ are exactly those quivers $Q$ obtained by gluing together a finite number of Type II blocks $\{S_i\}_{i \in [n]}$ in such a way that the only cycles in $Q$ are those oriented 3-cycles determined by a block $S_i$.

Definition 4.3. Let $Q$ be an irreducible type $A$ quiver with at least one 3-cycle. Define a leaf 3-cycle in $Q$ to be a 3-cycle in $Q$ that is connected to at most one other 3-cycle in $Q$. We define a root 3-cycle to be a chosen leaf 3-cycle.

Lemma 4.4. Suppose $Q$ is an irreducible type $A$ quiver with at least one 3-cycle. Then $Q$ has a leaf 3-cycle.

Proof. If $Q$ has exactly one 3-cycle $R$, then $Q = R$ is a leaf 3-cycle. If $Q$ is obtained from the Type II blocks $\{S_i\}_{i \in [n]}$, consider the block $S_{i_1}$. If $S_{i_1}$ is connected to only one other 3-cycle, then $S_{i_1}$ is a leaf 3-cycle. If $S_{i_1}$ is connected to more than one 3-cycle, let $S_{i_2}, \ldots, S_{i_r}$ denote one of the 3-cycles to which $S_{i_1}$ is connected. If $S_{i_2}$ is only connected to $S_{i_1}$, then $S_{i_2}$ is a leaf 3-cycle. Otherwise, there exists a 3-cycle $S_{i_3} \neq S_{i_1}$ connected to $S_{i_2}$. By Lemma 4.1 there are no non-trivial cycles in the underlying graph of $Q$ besides those determined by the blocks $\{S_i\}_{i \in [n]}$ so this process will end. Thus $Q$ has a leaf 3-cycle. □

Let $(Q, S)$ denote an irreducible type $A$ quiver $Q$ with at least one 3-cycle and $S$ denotes a leaf 3-cycle in $Q$. We will define an embedding $\rho$ of $Q$ in the plane, which is defined once we choose $S$ as the root 3-cycle. We will call the data $(Q, S, \rho)$ an embedded, irreducible type $A$ quiver with respect to $S$, which we will denote as $Q$. The quiver $Q$ is determined by the number of 3-cycles it has and by the identifications made to glue those 3-cycles together. To define this embedding, we will explain how to embed any number of 3-cycles allowing only identifications such that the result will be an irreducible type $A$ quiver.

Let $T$ be a labelled quiver that is an oriented 3-cycle with exactly one arrow between any two vertices. We consider the following two embeddings of $T$ in the plane.

In the sequel, we will always be referring to an oriented 3-cycle when we use the word 3-cycle. If we embed $T$ in the plane as the quiver on the left (respectively, right), we say $T_i$ is upward-pointing (downward-pointing).

In what follows we will often suppress the vertex labels of the quivers we are considering. To obtain an embedded, irreducible type $A$ quiver $Q$ with exactly one 3-cycle we define $Q := T_1$ with $T_1$ upward-pointing. Thus in what follows we assume that the desired irreducible type $A$ quiver has more than one 3-cycle.

Begin with the abstract quiver $Q$ embedded as shown below with respect to the root 3-cycle $T_1$. As discussed above we write $Q$ to denote this embedded quiver.
The quiver $Q$ has 3-cycles labeled by $T_1$ and $T_2$ and where $a_1$ and $a_2$ denote outlets in the sense of [7, Section 13]. By convention, we assume the vertices of $T_1$ are no longer outlets. We will build up an irreducible type $\mathbb{A}$ quiver from $Q$. We can attach additional 3-cycles at the outlets subject to the following rules.

If we have a downward-pointing 3-cycle $T_i$, for $i \geq 2$,

we can attach an upward-pointing 3-cycle $T_{i+1}$ to outlet $a_1$ to get

or we can attach a downward-pointing 3-cycle $T_{i+1}$ to outlet $a_2$ to get

Similarly, if we have an upward-pointing 3-cycle $T_i$, for $i \geq 3$,

with outlets $a_1$ and $a_2$ we can attach an upward-pointing 3-cycle $T_{i+1}$ to $a_2$ to obtain
In this situation, we say that $R$ has a single branch. We observe that the 3-cycles that make up $R$ inherit a labeling by $[n]$ from the construction of $R$. Thus $R$ consists of 3-cycles $T_1, \ldots, T_n$.

It will be helpful to define an ordering on the vertices of $R$. If $v \in (R)_0$, then define $i(v) = j$ where $v$ is a vertex of $T_j$. Note that given $u, v \in (R)_0$, $u$ and $v$ belong to the same 3-cycle of $R$ if and only if $i(u) = i(v)$. If $T_j$ is an upward-pointing 3-cycle in $R$ we set $x_j < y_j < z_j$. If $T_j$ is a downward-pointing 3-cycle in $R$ we set $x_j < z_j < y_j$. Now given $u, v \in (R)_0$, we say that $u < v$ if $i(u) < i(v)$ or $i(u) = i(v)$ and $u < v$ as elements of the same 3-cycle. We will call this ordering of the vertices of $R$ the standard ordering.

Suppose we have obtained a quiver $R$ with $n$ 3-cycles built up from $Q$ using rules described above. For the moment, we will only attach 3-cycles in such a way that no 3-cycle has 3-cycles attached to all three of its vertices.

Remark 4.5. In Section 7, we will define matrices in $\mathbb{Z}^{N\times 2N}$ that will describe connections between vertices of an embedded, irreducible type $A$ quiver $Q$ with $N$ vertices after applying certain sequences of mutation to $Q$. These matrices will have rows indexed by the mutable vertices of $Q$ and columns indexed by the mutable vertices and frozen vertices of $Q$. We will use the standard ordering to order the rows and columns of these matrices.

Let $a_1, \ldots, a_\ell$ denote the outlets of $R$ where we have indexed the outlets so that $i < j$ if and only if $a_i > a_j$ in the standard ordering. We can describe the outlets of $R$ explicitly by considering separately the cases where $T_n$ is upward-pointing and where $T_n$ is downward-pointing. Let $J_n$ be the set of elements $i \in [2, n-1]$ where $T_{i+1}$ is upward-pointing. More generally, we define $J_{r,s}$ to be the set of elements $i \in [r+1, s-1]$ where $T_{i+1}$ is upward-pointing. Then $J_n = J_{1,n}$. Now we have the following two cases describing the outlets of $R$. If $T_n$ is upward-pointing

$$a_1 = z_n, a_2 = y_n, a_3 = z_{\max} J_n, \ldots, a_j = z_{\max} J_n \setminus \{i(a_3), \ldots, i(a_{j-1})\}, \ldots, a_\ell = z_{\min} J_n.$$  

If $T_n$ is downward-pointing

$$a_1 = y_n, a_2 = z_n, a_3 = z_{\max} J_n, \ldots, a_j = z_{\max} J_n \setminus \{i(a_3), \ldots, i(a_{j-1})\}, \ldots, a_\ell = z_{\min} J_n.$$  

In other words, the outlets of $R$ are $y_n$ and the degree two $z_i$ vertices. See Example 4.6

4.2. Embedded, Irreducible Type $A$ Quivers with Multiple Branches. We now describe the process of attaching 3-cycles to an embedded, irreducible type $A$ quiver $R$ with one branch so that we will obtain a quiver with at least one 3-cycle $T$ having a 3-cycle attached to each vertex of $T$. We will refer to such 3-cycles as branching 3-cycles. We will also refer to this process of creating a branching 3-cycle as creating a new branch of $R$. If we attach a 3-cycle to $R$ in such way that we do not create a new branch of $R$, we say that the attachment continues the current branch of $R$. It is clear from the discussion above that we can only create a new branch of $R$ by attaching a downward-pointing 3-cycle to $R$ at $z_i$ where $i < n$.  

4
Suppose that $\mathcal{R}$ has outlets $a_1, \ldots, a_\ell$ where, as above, $i < j$ if and only if $a_i > a_j$ in the standard ordering. Let $\{a_i, \ldots, a_j\}$ denote the outlets where $a_i = z_i$ and $z_{ij}$ is from some downward-pointing 3-cycle $T_{ij} \neq T_n$. Note that $\{a_i, \ldots, a_j\} = (a_3, \ldots, a_\ell)$. Then creating a new branch of $\mathcal{R}$ by attaching a downward-pointing 3-cycle at $a_j$ with $j \in [3, \ell]$ gives the quiver $\mathcal{R} \cup_{a_j} T_{n+1}$ whose outlets are

$$b_1 := y_{n+1}, b_2 := z_{n+1}, b_3 := a_{j+1}, \ldots, b_{\ell-j+2} := a_\ell$$

when $a_j \neq a_\ell(= z_2)$ and are

$$b_1 := y_{n+1}, b_2 := z_{n+1}$$

when $a_j = a_\ell$. Roughly speaking, creating a new branch kills any outlets “northeast” of where the new branch was created.

**Example 4.6.** Here we give an example of creating a new branch and how this affects the outlets of a quiver with a single branch. Let $Q$ denote the (embedded) full subquiver on the vertices of $T_1, \ldots, T_{10}$ shown below. Then the outlets of $Q$ are $a_1 = 20, a_2 = 21, a_3 = 13, a_4 = 9, a_5 = 7, a_\ell = a_6 = 5$. To illustrate we have circled the outlets below. By attaching $T_1$ to $Q$ to obtain $Q \cup T_{11}$, we are creating a new branch of $Q$. The outlets of $Q \cup T_{11}$ are $b_1 = 22, b_2 = 23, b_{\ell-j+2} = b_3 = a_6 = 5$. Note that as another option we could have attached $T_1$ at 9, but to draw $Q \cup T_{11}$ in such a way that $T_{11}$ does not overlap with $T_6$ we would need to make $T_{11}$ more narrow than the other 3-cycles in $Q \cup T_{11}$.

We can continue to attach 3-cycles to $\mathcal{R} \cup_{a_j} T_{n+1}$. If we attach a 3-cycle $T_{n+2}$ to $b_1$ or $b_2$, we are continuing the current branch of $\mathcal{R} \cup_{a_j} T_{n+1}$. Thus the quiver $\mathcal{S}$ obtained from $\mathcal{R} \cup_{a_j} T_{n+1}$ by attaching finitely many 3-cycles $T_{n+2}, \ldots, T_m$ such that each 3-cycle continues its current branch will have outlets on the new branch given by

$$c_1 := z_m, c_2 := y_m, c_3 := z_{\max J_{n+1,m}}, \ldots, c_k := z_{\max J_{n+1,m} \setminus \{t(c_2), \ldots, t(c_k-1)\}}, \ldots, c_{\ell_1} := z_{\min J_{n+1,m}}$$

for some $\ell_1$ if $T_m$ is upward-pointing and will have outlets

$$c_1 := y_m, c_2 := z_m, c_3 := z_{\max J_{n+1,m}}, \ldots, c_k := z_{\max J_{n+1,m} \setminus \{t(c_2), \ldots, t(c_k-1)\}}, \ldots, c_{\ell_1} := z_{\min J_{n+1,m}}$$

for some $\ell_1$ if $T_m$ is downward-pointing. If $a_j \neq a_\ell$, then $\mathcal{S}$ will also have the outlets

$$c_{\ell_1+1} := b_3, \ldots, c_{\ell_1+s-2} := b_s, \ldots, c_{\ell_1+\ell-j+2} := b_{\ell-j+2}$$

outside of the new branch. We can also create a new branch of $\mathcal{R} \cup_{a_j} T_{n+1}$ and obtain $\mathcal{S} = (\mathcal{R} \cup_{a_j} T_{n+1}) \cup_{b_j} T_{n+2}$ with $j_1 \in [3, \ell - j + 2]$ and with outlets $c_1 := y_{n+2}$ and $c_2 := z_{n+2}$ and with additional outlets

$$c_3 := b_{j_1+1}, \ldots, c_{\ell-j+2-j_1+j+2} := b_{\ell-j+2}$$

when $j_1 \neq \ell - j + 2$.

Now suppose we have an embedded quiver $\mathcal{R}$ with $n$ 3-cycles built according to these rules. As above, we observe that the 3-cycles that make up $\mathcal{R}$ inherit a labeling by $[n]$ from the construction of $\mathcal{R}$. Thus $\mathcal{R}$ consists of $3$-cycles $T_1, \ldots, T_n$. For such a quiver $\mathcal{R}$ we will call this labeling of the 3-cycles of $\mathcal{R}$ the **standard labeling**. Unless otherwise stated, we will assume that we are using the standard labeling. Note that now we can naturally extend the standard ordering on vertices to embedded, irreducible type $A$ quivers with multiple branches.

We can partition any embedded, irreducible type $A$ quiver $Q$ into a collection of **branches** where only the last 3-cycle of each branch is allowed to be a branching 3-cycle and each branch ends in this way or with a leaf 3-cycle. More precisely, let $\{T_{i_1}, \ldots, T_{i_k}\}$ be the branching 3-cycles of $Q$ where $s < t$ if and only if $i_s < i_t$. A **branch** of $Q$ is a full subquiver on the vertices of a collection of 3-cycles $\{T_{j_1}, \ldots, T_{j_l}\}$ where $s < t$ if and only if $i_s < i_t$. If $i_s < i_t$ and $i_s < i_r$ and $s < t$ then $s < r$.
$j_s < j_t$ that satisfies one of the following

* $j_1 = 1, j_2 = 2, \ldots, j_\ell = i_1$
* $j_1 = 1, j_2 = 2, \ldots, j_\ell = n$
* $j_1 = i_j + 1, j_2 = i_j + 2, \ldots, j_\ell = m$
* $j_1 = i_j + 1, j_2 = i_j + 2, \ldots, j_\ell = i_j + 1$
* $j_1 = j, j_2 = j + 1, \ldots, j_\ell = i_m$
* $j_1 = j, j_2 = j + 1, \ldots, j_\ell = m$

if $Q$ has a single branch

for some $j$ and $m$ where $T_m$ is a leaf 3-cycle

for some $j$

for some $j$ and $m$ where $T_j$ is attached to a branching

3-cycle, call it, $T_{i_\ell}$ such that $z_{i_{j_0}} = x_j$

for some $j$ and $m$ where $T_j$ is attached to a branching

3-cycle, call it, $T_{i_\ell}$ such that $z_{i_{j_0}} = x_j$ and $T_m$ is a leaf

3-cycle

and such that only $T_{i_\ell}$ can be a branching 3-cycle. It is easy to see that the branches of $Q$ form a partition of the set of 3-cycles of $Q$, although adjacent branches will share vertices.

We order the branches of $Q$ in the following way. If $S$ and $S'$ are branches of $Q$ we say $S < S'$ if

$$\min\{i \in [n] : T_i is a 3-cycle in S\} < \min\{i \in [n] : T_i is a 3-cycle in S'\}.$$  

We will refer to this ordering as the **standard ordering** on the branches of $Q$. Henceforth, we define $\mathcal{B}$ to be the set of branches of $Q$ and we will write $\mathcal{B} = \{S(1), S(2), \ldots, S(m)\}$ where $i < j$ if and only if $S(i) < S(j)$ in the standard ordering and in this case we will say that $S(i)$ is a **higher branch** than $S(j)$. We will refer to $S(j)$ as the $j$th branch of $Q$.

Denote the outlets of $R$ by $a_1, \ldots, a_\ell$ where $i < j$ if and only if $a_i > a_j$ in the standard ordering. We can attach additional 3-cycles to $R$ using the above rules and the resulting quiver $R \cup_{\alpha_i} T_{n+1}$ will have outlets $b_1 := z_{n+1}, b_2 := y_{n+1}$ if $T_{n+1}$ is upward-pointing and $b_1 := y_{n+1}, b_2 := z_{n+1}$ if $T_{n+1}$ is downward-pointing. If $i \neq \ell$, $R \cup_{\alpha_i} T_{n+1}$ will have additional outlets

$$b_3 := a_{j+1}, \ldots, b_{\ell-j+2} := a_\ell.$$ 

**Example 4.7.** Let $R$ denote the quiver below. In $R$, $n = 15$ and $T_4$ and $T_{10}$ are branching 3-cycles. Here $S(1)$ is the full subquiver of $R$ on the vertices of $T_1, T_2, T_3$, and $T_4$, $S(2)$ is the full subquiver of $R$ on the vertices of $T_5$ and $T_6$, $S(3)$ is the full subquiver on the vertices of $T_7, T_8, T_9, T_{10}$, $S(4)$ is the full subquiver on the vertices of $T_{11}, T_{12}, T_{13}, T_{14}$ and $S(5)$ is the full subquiver on the vertices of $T_{15}$. The outlets of $R$ are circled.

![Quiver diagram](attachment:quiver_diagram.png)

In addition, consider $R \cup_5 T_{16}$ obtained from $R$ by attaching $T_{16}$ at outlet 5 in $R$. The outlets of $R \cup_5 T_{16}$ will be $y_{16}$ and $z_{16}$ with all other outlets removed.
Remark 4.8. If $Q$ is an embedded, irreducible type $A$ quiver with more than one 3-cycle, then once a leaf 3-cycle is chosen to be the root 3-cycle, the planar embedding is completely determined by the rules given above.

5. Associated Mutation Sequences

In this section we work with a given embedded, irreducible type $A$ quiver $Q$ with respect to a fixed root 3-cycle $T$. We construct a mutation sequence of $Q$ that we will call the associated mutation sequence of $Q$. Later we state some results regarding associated mutation sequences.

5.1. Definition of Associated Mutation Sequences. Here we will denote the associated mutation sequence of $Q$ by $\mu$ or by $\mu^Q$ if it is not clear from context which quiver is being mutated. We assume that $Q$ consists of the 3-cycles $T_1, \ldots, T_n$ under the standard labeling. Before defining associated mutation sequence of $Q$, we present some terminology.

Notation 5.1. Let $T_k$ be a 3-cycle of $Q$. Define the special upward-pointing 3-cycle before $T_k$, denoted $T_{r(k)}$, to be $T_k$ if $T_k$ is upward-pointing. If $T_k$ is downward-pointing, then there exists a sequence $(T_{i_1}, \ldots, T_{i_d})$ of downward-pointing 3-cycles with $T_{i_1} = T_k$ such that locally $Q$ looks like

$$
\begin{array}{c}
\text{x}_{r(k)} \quad \text{y}_{r(k)} \\
\text{x}_{i_d} \quad \text{T}_{r(k)} \\
\text{y}_{i_d} \quad \text{x}_{i_{d-1}} \\
\end{array}
$$

where $\deg(y_{i_j}) \in \{2, 4\}$ for all $j \in [d]$. We define $T_{r(k)}$ to be the unique upward-pointing 3-cycle appearing in this local picture. We call the sequence $(T_{i_1}, \ldots, T_{i_d})$ the path between $T_k$ and $T_{r(k)}$. Since the path between $T_k$ and $T_{r(k)}$ depends on $k$, we will often write it as $(T_{i(k)_1}, \ldots, T_{i(k)_{d(k)}})$.

Notation 5.2. If $T_k$ is upward-pointing, define

$$v(k) := x_k.$$  

If $T_k$ is downward-pointing, $(T_{i_1}, \ldots, T_{i_d})$ is the path between $T_k$ and $T_{r(k)}$, $\deg(y_{r(k)}) = 2$, and $\deg(y_{i_j}) = 2$ for all $j \in [2, d]$, then

$$v(k) := x_{r(k)}.$$  

If not all $y_{r(k)}, y_{i_2}, \ldots, y_{i_d}$ are degree 2 vertices, consider $y_s$ where

$$s = \max \{ j \in \{r(k), i_2, \ldots, i_d\} : \deg(y_j) = 4 \}.$$  

When $s$ exists, $Q$ locally looks like
if \( s = i_j \) for some \( j \in [2, d] \) and where \( \text{deg}(y_{i_0}) = 2 \) for all \( j_0 \in [2, j - 1] \) or \( Q \) locally looks like

if \( s = r(k) \) and where \( \text{deg}(y_{i_0}) = 2 \) for all \( j_0 \in [2, d] \). Note that in both diagrams above, we have not considered \( \text{deg}(y_i) \). Even if \( \text{deg}(y_i) = 4 \), this will not affect the following definition.

In either case, we define

\[ v(k) := z_{s_k}. \]

Since \( z_{s_k} \) depends on \( k \), we will often write \( z_{s_k}^{(i)} \) to emphasize this. We will refer to the sequence \( (T_{s_1}, T_{s_2}, \ldots, T_{s_k}) \) as the twig above \( T_k \). Note that in the second diagram we made the 3-cycles \( T_{i_j} \) smaller than the other 3-cycles shown so that the diagram wouldn’t intersect itself.

We now use the above notation to define mutation sequences associated to embedded, irreducible type \( \mathbb{A} \) quivers.

**Definition 5.3.** Let \( Q \) be an embedded, irreducible type \( \mathbb{A} \) quiver with respect to root 3-cycle \( T \) whose 3-cycles are \( T_1, \ldots, T_n \) in the standard labeling. Define \( \mu_0 := \mu_{x_1} \). For \( k \in [n] \) we define the associated mutation sequence of \( T_k \), denoted \( \mu_{s_k} \), as follows. Note that when we write \( \emptyset \) below we mean the empty mutation sequence. We define
where $\mu_A, \mu_B, \mu_C,$ and $\mu_D$ are mutation sequences defined in the following way

\[
\begin{align*}
\mu_D & := \mu_k \circ \mu_y \circ \mu_k \\
\mu_C & := \begin{cases} 
\mu_{x_i(k),\mu(k)} \circ \cdots \circ \mu_{x_i(k)} : & \text{if } T_k \text{ is downward-pointing} \\
\emptyset & \text{if } T_k \text{ is upward-pointing}
\end{cases} \\
\mu_B & := \begin{cases} 
\mu_{\mu(r(k)-1)} : & \text{if } r(k) \neq 1 \\
\emptyset & \text{if } r(k) = 1
\end{cases} \\
\mu_A & := \mu_v(k).
\end{align*}
\]

Now define the associated mutation sequence of $Q$ to be $\underline{\mu} := \mu_n \circ \cdots \circ \mu_1 \circ \mu_0$.

At times it will be useful to write $\mu_k = \mu_A(k) \circ \mu_B(k) \circ \mu_C(k) \circ \mu_D(k)$.

**Example 5.4.** Continuing with Example 4.7, we describe $\mu_i$ for each $0 \leq i \leq 15$:

| $i$ | $\mu_i$ |
|-----|---------|
| 0   | $\mu_1$ |
| 1   | $\mu_1 \circ \mu_3 \circ \mu_2$ |
| 2   | $\mu_1 \circ \mu_3 \circ \mu_5 \circ \mu_4$ |
| 3   | $\mu_4 \circ \mu_1 \circ \mu_7 \circ \mu_6$ |
| 4   | $\mu_4 \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_8$ |
| 5   | $\mu_8 \circ \mu_4 \circ \mu_{11} \circ \mu_{10}$ |
| 6   | $\mu_8 \circ \mu_4 \circ \mu_{13} \circ \mu_{12}$ |
| 7   | $\mu_{13} \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_{15} \circ \mu_{14}$ |
| 8   | $\mu_{13} \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_{17} \circ \mu_{16}$ |
| 9   | $\mu_{13} \circ \mu_1 \circ \mu_7 \circ \mu_9 \circ \mu_{15} \circ \mu_{17} \circ \mu_{18}$ |
| 10  | $\mu_{18} \circ \mu_{13} \circ \mu_21 \circ \mu_{20}$ |
| 11  | $\mu_{20} \circ \mu_{18} \circ \mu_{23} \circ \mu_{22}$ |
| 12  | $\mu_{22} \circ \mu_{20} \circ \mu_{25} \circ \mu_{24}$ |
| 13  | $\mu_{22} \circ \mu_{20} \circ \mu_{25} \circ \mu_{27} \circ \mu_{26}$ |
| 14  | $\mu_{22} \circ \mu_{20} \circ \mu_{25} \circ \mu_{27} \circ \mu_{29} \circ \mu_{28}$ |
| 15  | $\mu_{23} \circ \mu_{13} \circ \mu_{21} \circ \mu_{31} \circ \mu_{30}$ |

Thus, we would obtain the associated mutation sequence $\mu_{15} \circ \mu_{14} \circ \cdots \circ \mu_1 \circ \mu_0$ corresponding to the $Q$ of Example 4.7.

The following lemma is immediate from the definition of associated mutation sequences.

**Lemma 5.5.** Fix $k \in [n]$. Suppose $T_k$ is downward-pointing in $Q$ and that $(T_{i(k)}, \ldots, T_{i(d(k))})$ is the path between $T_k$ and $T_r(k)$ in $Q$. Then $\mu_{B(k)} = \mu_{B(i(k))}$ for all $j \in [d(k)]$ and $\mu_{B(k)} = \mu_{B(r(k))}$.

**Lemma 5.6.** If $Q$ is an embedded, irreducible type $A$ quiver with associated mutation sequence $\underline{\mu} = \mu_n \circ \cdots \circ \mu_0$, then the mutation sequence $\mu_{D(1)} \circ \mu_{D(2)}$ appearing in $\mu_1 \circ \mu_0$ is a green mutation sequence. Furthermore, for each $k \in [2, n]$, $\mu_{D(k)}$ is a green mutation sequence of $\left(\mu_{k-1} \circ \cdots \circ \mu_0 \right)$.

**Proof.** The sequence $\mu_{D(1)} \circ \mu_{D(2)}$ is green because in $\hat{Q}$, no vertices have been mutated yet. Similarly, in $\left(\mu_{k-1} \circ \cdots \circ \mu_0 \right)$, neither $y_k$ nor $z_k$ have been mutated yet so $\mu_{D(k)}$ is green.

**Main Theorem 5.7.** If $Q$ is an embedded, irreducible type $A$ quiver with respect to a root 3-cycle $T_1$ with associated mutation sequence $\underline{\mu}$, we have $\underline{\mu} \in \text{green}(Q)$.

To prove Theorem 5.7 we need to show that all mutations of $\underline{\mu}$ take place at green vertices and that $\underline{\mu} \hat{Q}$ has only red mutable vertices. Lemma 5.6 shows that many of the mutations appearing in $\underline{\mu}$ take place at green vertices. We will need some additional tools to show that the remaining mutations occur at green vertices and to show that $\underline{\mu} \hat{Q}$ has only red vertices.

**Remark 5.8.** For a given irreducible type $A$ quiver, the length of $\underline{\mu}$ depends on the choice of leaf 3-cycle. Let $Q$ denote the following irreducible type $A$ quiver

```
1 ← 2 ← 3 ← 4 ← 5 ← 6 ← 7.
```

Embedding $Q$ with respect to the 3-cycle $1, 2, 3$ gives the quiver $Q_1$ shown below on the left and embedding $Q$ with respect to the 3-cycle $5, 6, 7$ gives the quiver $Q_2$ shown below on the right.
Then the associated mutations of $Q_1$ and $Q_2$ are

$$
\mu_{Q_1} = \mu_1 \circ \mu_3 \circ \mu_5 \circ \mu_7 \circ \mu_6 \circ \mu_1 \circ \mu_3 \circ \mu_5 \circ \mu_4 \circ \mu_1 \circ \mu_3 \circ \mu_2 \circ \mu_1
$$

$$
\mu_{Q_2} = \mu_3 \circ \mu_6 \circ \mu_2 \circ \mu_1 \circ \mu_4 \circ \mu_3 \circ \mu_6 \circ \mu_5 \circ \mu_7 \circ \mu_6.
$$

Furthermore, the maximal green sequence produced by Theorem 5.7, i.e., the associated mutation sequence of an embedded, irreducible type $\ast$ quiver, is not a minimal length maximal green sequence. For example, it is easy to check that $\nu = \mu_3 \circ \mu_1 \circ \mu_4 \circ \mu_7 \circ \mu_6 \circ \mu_2 \circ \mu_5 \circ \mu_1 \circ \mu_4 \circ \mu_7$ is a maximal green sequence of $Q$, which is of length less than $\mu Q_1$ or $\mu Q_2$.

5.2. Mutation sequences for general type $\ast$ quivers. We note that if $Q$ is a general type $\ast$ quiver, then by Corollary 3.9, we may decompose $Q$ as a $t$-colored direct sum of irreducible type $\ast$ quivers $\{Q_1, Q_2, \ldots, Q_k\}$. In other words,

$$
Q = Q_1 \oplus_{(a_{11}, a_{21}, \ldots, a_{(1,1)})} Q_2 \text{ where } Q'_j = Q_j \oplus_{(a_{12}, a_{22}, \ldots, a_{(j,2)})} Q'_j \text{ for } 2 \leq j \leq k - 1, \text{ and } Q'_k = Q_k.
$$

Choose an embedding in the plane, $Q_i$, for each such $Q_i$ and by Theorem 5.7, there is a maximal green sequence $\mu_i$ for $Q_i$. By applying Proposition 3.13 iteratively, we obtain $\mu = \mu_k \circ \cdots \circ \mu_3 \circ \mu_1$ is a maximal green sequence of $Q$. See examples of such decompositions, for quivers of more general surfaces, in Section 8.1.

6. Preparations for the Proof of Theorem 5.7

Thus far in this paper, we have discussed quiver mutation. For every quiver $Q$, one can also associate a skew-symmetric matrix $B_Q$ and define a matrix mutation that is an algebraic phrasing of quiver mutation. First off, we define $B_Q$ by $(B_Q)_{ij} = \#\{x \dashv y \in (Q)_i \} - \#\{y \dashv x \in (Q)_i \}$ for all $1 \leq i, j \leq N$. If $Q$ has $M$ frozen vertices, then it is sufficient to define the $(N + M)$-by-$N$ submatrix $B_0$ whose rows are indexed by $[N]$ and whose columns are indexed by $[N + M]$. Note that $B_Q$ is the block matrix $[B_Q I]$, where $I$ is the $N$-by-$N$ identity matrix. Similarly, $B_{\bar{Q}} = [B_Q, -I]$. We call the left-hand side of matrices $B_{\bar{Q}}$ the exchangeable part and the right-hand side the extended part.

For non-frozen vertex $k \in (Q)_0$, we then define the mutation of $B_{\bar{Q}} = [b_{ij}]$ as $\mu_k (B_{\bar{Q}}) = [b'_{ij}]$ where

$$
b'_{ij} = \begin{cases} 
- b_{ij} & \text{if } i = k \text{ or } j = k \\
\frac{b_{ij} + |b_{ik}|b_{kj} + |b_{jk}|b_{ki}|}{2} & \text{otherwise}.
\end{cases}
$$

6.1. Associated Permutations. In this section, we work with a fixed embedded, irreducible type $\ast$ quiver $Q$ with $N$ vertices and with respect to a root 3-cycle $T_1$. We write $\mu$ for the mutation sequence associated to $Q$. As we mutate $Q$ along $\mu$, we will see in Section 7 that for each $i \in [n]$ the sequence $\mu_i \circ \cdots \circ \mu_0$ is a maximal green sequence of the full subquiver of $Q$ on the vertices of the 3-cycles $T_1, \ldots, T_i$, denoted $Q_i$. As such, the sequence $\mu_i \circ \cdots \circ \mu_0$ induces a permutation of the vertices of $Q_i$. In this section, we associate to each initial segment $\mu_i \circ \cdots \circ \mu_0$ in $\mu$ a permutation $\sigma_i$ and we will see in Section 7 that this permutation is induced by the maximal green sequence $\mu_i \circ \cdots \circ \mu_0$ of $Q_i$. In this section, we also prove some results related to these permutations that will be useful in the proof of our main result.

For each $\mu_i$ appearing in $\mu$ we define a permutation $\tau_i \in \mathfrak{S}_{(Q)_0} \equiv \mathfrak{S}_N$ associated to $\mu_i$ where $\mathfrak{S}_{(Q)_0}$ denotes the symmetric group on the vertices of $Q$. In the special case where $i = 0$, we let $\tau_0 := 1_{\mathfrak{S}_{(Q)_0}}$ where $1_{\mathfrak{S}_{(Q)_0}}$ denotes the identity permutation. Then for $i \geq 1$ where $\mu_i = \mu_{i_d} \circ \cdots \circ \mu_{i_1}$, we define $\tau_i := (i_2, \ldots, i_d)$ in cycle notation. Note that $i_1 = y_k$. We also define

$$
\sigma_i := \tau_i \cdots \tau_1 \tau_0 \quad = \quad \tau_i \cdots \tau_1
$$

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as \(\tau_0\) is the identity permutation. We call \(\tau_i\) the permutation associated to \(\mu_i\), and we call \(\sigma_i\) the permutation associated to \(\mu_i \circ \cdots \circ \mu_0\). Some examples of the action of \(\tau_k\) on vertices are shown below. Note that we have suppressed the orientations of the 3-cycles.

We now show how the permutations \(\sigma_i\) act on vertices of \(Q\). We begin by showing that elements of \(\text{supp} \left(\mu_{C(k)} \circ \mu_{D(k)}\right) \setminus \{y_k\}\) (Lemma 6.1) and elements of \(\text{supp} \left(\mu_{B(k)}\right)\) (Lemma 6.3) are fixed by \(\tau_\ell\) for certain \(\ell \in [n]\). Using these Lemmas, we then show how \(\sigma_k\) acts on \(\mu_k \setminus \{y_k\}\) (Lemma 6.4) and how it acts on \(y_k\) (Corollary 6.6).

The proof of Lemma 6.1 (respectively, Lemma 6.3) amounts to showing that when \(T_k\) is downward-pointing \(x_{i(k)} \neq v(\ell - 1), v(\ell)\) for all \(j \in [d(k)]\) (respectively, \(v(r(k) - 1) \neq v(\ell - 1), v(\ell)\) for all \(j \in [d(k)]\)) where \(T_\ell\) is “northeast” of the path between \(T_k\) and \(T_{r(k)}\). The following notation makes the notion of “northeast” more precise.

Let \(k \in [n]\) where \(\deg(z_k) = 2\). We define \(A(k)\) to be the set of \(s \in [n]\) such that there exists \(j \in [d(k)]\) and a sequence of 3-cycles \(T_{i(k)}), T_{i(k)} + 1, \ldots, T_s\) or a sequence \(T_{r(k)}, T_{r(k)} + 1, \ldots, T_s\) such that the full subquiver of \(Q\) on the vertices of these 3-cycles is connected.

Lemma 6.1. Let \(k \in [n]\) where \(\deg(z_k) = 2\). Then for any \(v \in \text{supp} \left(\mu_{C(k)} \circ \mu_{D(k)}\right)\) and any \(\ell \in (A(k) \cup [k]) \setminus \{r(k)\} \cup \text{supp} \left(\mu_{C(k)}\right)\) we have that \(v \cdot \tau_\ell = v\).

Proof. It is clear that \(z_k \cdot \tau_\ell = z_k\) for all \(\ell \in (A(k) \cup [k]) \setminus \{r(k)\} \cup \text{supp} \left(\mu_{C(k)}\right)\). This proves the result when \(T_k\) is upward-pointing so we can assume \(T_k\) is downward-pointing.

If \(\ell < r(k)\), then \(x_{i(k)} \cdot \tau_\ell = x_{i(k)}\) because \(\text{supp} \left(\mu_{\ell}\right) \cap \{x_{i(k)}, \ldots, x_{i(k)}\} = \emptyset\). Thus it is enough to prove that \(x_{i(k)} \cdot \tau_\ell = x_{i(k)}\) for all \(\ell \in (A(k) \cup [r(k), k]) \setminus \{r(k), i(k) \cup [d(k)]\}\) and all \(j \in [d(k)]\).

It is clear that \(x_{i(k)} \cdot \tau_\ell = x_{i(k)}\) so we need to show that \(x_{i(k)} \neq v(\ell - 1), v(\ell)\) for all \(j \in [d(k)]\). If \(r(\ell - 1) \neq \{i(k), \ldots, i(k)\} \cup [r(k), k]\), then \(v(\ell - 1) \neq x_{i(k), \ldots, i(k)}\) so we can assume \(r(\ell - 1) \in \{i(k) \cup [d(k)]\}\). In this situation,

\[v(\ell - 1) = x_{r(k)} \neq x_{i(k)}\]

for all \(j \in [d(k)]\).

Similarly, since \(\ell \neq \{i(k) \cup [r(k), k]\}\), we have that \(v(\ell) \neq x_{i(k), \ldots, i(k)}\). Thus \(x_{i(k)} \cdot \tau_\ell = x_{i(k)}\) and \(z_{i(k)} \cdot \tau_\ell = z_{i(k)}\) for all \(\ell \in (A(k) \cup [k]) \setminus \{r(k), i(k) \cup [d(k)]\}\) and all \(j \in [d(k)]\).

Remark 6.2. Lemma 6.1 says that any vertex of \(Q\) of the form \(x_{i(k)}\) or \(z_{i(k)}\) for some \(k \in [n]\) and \(j \in [d(k)]\) will be fixed by any \(\tau_{\ell}\) where \(T_\ell\) is “northeast” of the path determined between \(T_k\) and \(T_{r(k)}\).

Lemma 6.3. Let \(k \in [n]\) where \(\deg(z_k) = 2\). Suppose \(T_k\) is downward-pointing in \(Q\) and \((T_{i(k)}), \ldots, T_{i(d(k))}\) is the path between \(T_k\) and \(T_{r(k)}\). Then \(v(r(i(k)) - 1) \cdot \tau_\ell = v(r(i(k)) - 1)\) for all \(\ell \in (A(k) \cup [r(k), k]) \setminus \{r(k), i(k) \cup [d(k)]\}\) and all \(j \in [d(k)]\).
Proof. We know that \( v(r(i(k)_j) - 1) = v(r(k) - 1) \) for all \( j \in [d(k)] \) so it is enough to show that \( v(r(k) - 1) \notin \text{supp} \left( \mu_k \right) \) for all 
\[ \ell \in (\mathcal{A}(k) \cup [r(k), k]) \setminus \{r(k), i(k)_1, \ldots, i(k)_{d(k)}\} \].
Also note that for any \( \ell \in (\mathcal{A}(k) \cup [r(k), k]) \setminus \{r(k), i(k)_1, \ldots, i(k)_{d(k)}\} \), the vertex \( v(r(k) - 1) \notin \text{supp} \left( \mu_{C(\ell)} \circ \mu_{D(\ell)} \right) \)
so it is enough to show that \( v(r(k) - 1) \notin \text{supp} \left( \mu_{A(\ell)} \circ \mu_{B(\ell)} \right) \). In other words, we need to show that if we have \( \ell \in (\mathcal{A}(k) \cup [r(k), k]) \setminus \{r(k), i(k)_1, \ldots, i(k)_{d(k)}\} \), then \( v(r(k) - 1) \neq v(r(\ell) - 1) \) and \( v(r(k) - 1) \neq v(\ell) \).

To see that \( v(r(k) - 1) \neq v(r(\ell) - 1) \), note that \( r(k) < r(\ell) \). Note that this implies that \( r(k) < r(\ell) \) because the closest that \( T_{r(\ell)} \) can be to the path \( (T_{i(k)_1}, \ldots, T_{i(k)_{d(k)}}) \) is when \( T_{r(\ell)} \) is one of the dotted 3-cycles shown below (more precisely, the closest that \( T_{r(\ell)} \) can be to the path is when \( T_{r(\ell)} = T_{x_{i(k)_s}+1} \) for some \( s \in [d(k)] \) where \( \text{deg}(y_{i(k)_s}) = 4 \) or \( T_{r(\ell)} = T_{r(k)+1} \) where \( \text{deg}(y_{r(k)}) = 4 \). Thus \( r(k) - 1 < r(\ell) \leq r(\ell) - 1 \) so \( v(r(\ell) - 1) \) belongs to \( T_s \) where \( s \geq r(k) \) and \( v(r(k) - 1) \) belongs to \( T_t \) where \( t \leq r(k) - 1 \). If \( s = r(k) \) and \( t = r(k) - 1 \), then \( v(r(\ell) - 1) = x_{r(k)} \neq x_{r(k) - 1} = v(r(k) - 1) \).

To see that \( v(r(k) - 1) \neq v(\ell) \), note that \( v(r(k) - 1) \) belongs to \( T_t \) where \( t \leq r(k) - 1 \) and \( v(\ell) \) belongs to \( T_s \) where \( s \geq r(k) + 1 \) where this last inequality holds because we showed that \( r(\ell) > r(k) \) in the previous paragraph.

The next Lemma describes how \( \sigma_k \) acts on vertices in \( \text{supp} \left( \mu_k \right) \setminus \{y_k\} \). The following two figures show some examples of the action of \( \sigma_k \) described in Lemma 6.4.
Lemma 6.4. Fix $k \in [n]$. Then

1) If $r(k) = 1$, we have $z_k \cdot \sigma_k = x_1$ and $x_1 \cdot \sigma_k = z_k$.

2) If $r(k) > 1$, $z_k \cdot \sigma_k = z_{r(k)-1}$.

3) If $r(k) > 1$ and $T_k$ is downward-pointing, $x_k \cdot \sigma_k = x_{r(k)}$.

4) We have that $v(k) \cdot \sigma_k = z_k$ (Recall that $v(k) = z_{r(k)}$).

5) If $r(k) = 1$ and $T_k$ is downward-pointing, $x_{i(k)} \cdot \sigma_k = x_{i(k)+1-j}$ and $x_{i(k)+1-j} \cdot \sigma_k = x_{i(k)}$.

6) If $r(k) > 1$ and $T_k$ is downward-pointing, $j \in \left[1, \left\lfloor \frac{d(k)+1}{2} \right\rfloor \right]$ we have $x_{i(k)} \cdot \sigma_k = x_{i(k)+1-j}$ and $x_{i(k)+1-j} \cdot \sigma_k = x_{i(k)}$.

Proof. For part 1), we have

$$z_k \cdot \sigma_k = z_k \cdot \sigma_i(k) = z_k \cdot \sigma_i(k)_2$$

(by the definition of $\tau_i$ and (possibly) Lemma 6.1)

$$= x_k \cdot \sigma_i(k)$$

(3-cycle containing the vertex $1$)

$$= x_i(k) \cdot \sigma_i(k) \cdot \sigma_i(k+1) \cdot \sigma_i(k+2)$$

(using Lemma 6.1 repeatedly and the definition of $\tau_i(k)$)

$$= \begin{cases} z_1 \cdot (z_1, x_1) : r(k) = 1 \\ z_r(k) \cdot (z_r(k), v(r(k)-1), x_r(k)) \cdot \sigma_i(k-1) : r(k) > 1 \end{cases}$$

$$= \begin{cases} x_1 : r(k) = 1 \\ v(r(k)-1) \cdot \sigma_i(k-1) : r(k) > 1 \end{cases}$$

We know from the definition of associated mutation sequence that $\mu_{v(r(k)-1)} = \mu_{A(r(k)-1)}$, so

$$v(r(k)-1) \cdot \sigma_{r(k)-1} = z_{r(k)-1} \cdot \sigma_{r(k)-2} = v(r(k)-1)$$

where the last equality holds because $z_{r(k)-1} \notin \text{supp}(\mu_{v(r(k)-1)} \circ \cdots \circ \mu_0)$.

Also, $x_1 \cdot \sigma_k = z_k \cdot \sigma_k-1 = z_k$ where the last equality holds because $z_k \notin \text{supp}(\mu_{k-1} \circ \cdots \circ \mu_0)$.

For part 2), we have

$$x_k \cdot \sigma_k = x_i(k) \cdot \tau_i(k) \cdot \sigma_i(k) \cdot \sigma_i(k+1) \cdot \sigma_i(k+2)$$

(using Lemma 6.1 and the definition of $\tau_i(k)$)

$$= v(r(k)-1) \cdot \sigma_i(k)$

(3-cycle containing the vertex $1$)

$$= v(r(k)-1) \cdot \tau_i(k) \cdot \sigma_i(k-1)$$

(using Lemma 6.1 and $\mu_{r(k)} \neq \emptyset$ as $r(k) > 1$.)

$$= x_{r(k)} \cdot \tau_i(k) \cdot \sigma_i(k-2)$$

(since $T_i(k)$ is upward-pointing and $v(r(k)) = x_{r(k)}$)

$$= x_{r(k)} \cdot \sigma_i(k-2)$$

$$= x_{r(k)}$$

For part 3), we have $v(r(k)-1) \cdot \sigma_k = v(k) \cdot \sigma_k-1$. We need to consider the case when $k-1 = i(k)$ and the case when $k-1 > i(k)$. If $k-1 = i(k)$, we have

$$v(k) \cdot \sigma_{k-1} = v(k-1) \cdot \sigma_{k-2} = v(k) \cdot \sigma_{k-2} = x_k \cdot \sigma_{k-2}$$

where the last equality holds because $z_k \notin \text{supp}(\mu_{k-2} \circ \cdots \circ \mu_0)$.

Now suppose $k-1 > i(k)$. For simplicity of notation we let $T_v$ denote the 3-cycle containing the vertex $v(k)$. Then we can also write $v_k = v(k)$. Now we have

$$z_k \cdot \sigma_{k-1} = z_k \cdot \sigma_k$$

(by repeatedly using Lemma 6.1)

$$= z_{r(k)-1}$$

(using Lemma 6.4 i)

$$= z_{i(k)}$$

$$= x_k$$

To prove part iv), observe that $v(k) \cdot \sigma_k = z_k \cdot \sigma_k-1 = z_k$ where the last equality holds because $z_k \notin \text{supp}(\mu_{k-1} \circ \cdots \circ \mu_0)$.

Next, we prove part v) when $r(k) > 1$. The proof when $r(k) = 1$ is analogous. We have
We will prove the result for $\sigma_k$ by proving the following two cases:

1. When $i(k)_{2+d(k)−j} = i(k)_{3+d(k)−j}$ and
2. When $i(k)_{2+d(k)−j} < i(k)_{3+d(k)−j}$.

Using Lemma 6.5, we have

$$x(i(k)_{j}) \cdot \sigma_k = x(i(k)_{d(k)}) \cdot \tau_i(k)_{1+d(k)−j} \cdot \sigma(i(k)_{1+d(k)−j}) = v(r(k)−1) \cdot \sigma(i(k)_{1+d(k)−j}) = v(r(k)−1) \cdot \sigma_i(k)_{2+d(k)−j} = v(i(k)_{2+d(k)−j}) \cdot \sigma(i(k)_{2+d(k)−j}) \cdot \sigma_k(i(k)_{2+d(k)−j})^{-1}.$$  

For the second case, we have

$$v(i(k)_{2+d(k)−j}) \cdot \sigma(i(k)_{2+d(k)−j}) = z_i(k)_{3+d(k)−j} \cdot \sigma(i(k)_{3+d(k)−j})^{-1} = x(i(k)_{2+d(k)−j}) \cdot \sigma(i(k)_{2+d(k)−j}) = x(i(k)_{2+d(k)−j}).$$

If $i(k)_{2+d(k)−j} > i(k)_{3+d(k)−j}$, we have

$$v(i(k)_{2+d(k)−j}) = z_s \cdot \sigma_k = x(i(k)_{2+d(k)−j}) \cdot \sigma_k(i(k)_{2+d(k)−j})^{-1}.$$

The proof that $x(i(k)_{d(k)+2−j}) \cdot \sigma_k = x(i(k)_{j})$ is similar.

The next two results describe the action of $\sigma_k^{-1}$ on $y_k(k)\cdot y_i(k), \ldots, y_i(k)_{d(k)}$. Stating these results using inverses of associated permutations will make it more clear how we apply them in the proof of Lemma 7.1.

**Lemma 6.5.** Fix $k \in [n]$. Suppose $T_k$ is downward-pointing in $Q$ and that $(T_i(k)_{d(k)}, \ldots, T_i(k))$ is the path between $T_k$ and $T_r(k)$. Then for each $j \in [d(k)]$ there exists $K \in [n]$ such that

$$y_i(k) \cdot \sigma_k^{-1} = \begin{cases} y_k(k) \quad : \quad \text{deg}(y_i(k)) = 2 \\ x_r(k) \quad : \quad \text{deg}(y_i(k)) = 4, \text{deg}(z_r(k)) = 2 \\ x_K \quad : \quad \text{deg}(y_i(k)) = \text{deg}(z_r(k)) = 4 \end{cases}$$

and

$$y_k(k) \cdot \sigma_k^{-1} = \begin{cases} y_i(k) \quad : \quad \text{deg}(y_i(k)) = 2 \\ v(r(i(k))) - 1) \quad : \quad \text{deg}(z_i(k)) = 2 \\ x_K \quad : \quad \text{deg}(y_i(k)) = \text{deg}(z_i(k)) = 4 \end{cases}$$

**Proof.** We will prove the result for $y_i(k)$, where $j \in [d(k)]$ and the proof for $y_i(k)$ is very similar. If $\text{deg}(y_i(k)) = 2$, then $y_i(k) \cdot \sigma_k^{-1} = y_i(k)$, because, in general, $y_i \cdot \sigma_k^{-1} = y_i$ for all $i, s \in [n]$ when $\text{deg}(y_i) = 2$.

Next, assume that $\text{deg}(y_i(k)) = 4$ and $\text{deg}(z_i(k)) = 2$. Then we have

$$y_i(k) \cdot \sigma_k^{-1} = \begin{cases} y_i(k) \quad : \quad \text{deg}(y_i(k)) = 2 \\ v(r(i(k))) - 1) \quad : \quad \text{deg}(z_i(k)) = 2 \\ x_K \quad : \quad \text{deg}(y_i(k)) = \text{deg}(z_i(k)) = 4 \end{cases}$$

Now assume that $\text{deg}(y_i(k)) = 4$ and $\text{deg}(z_i(k)) = 4$, there exists $K \in [n]$ such that $\text{deg}(z_K) = 2$ and $i(k)_{j+1} = r(K)$. Thus we have

$$y_i(k) \cdot \sigma_k^{-1} = \begin{cases} x_k(k) \cdot \sigma_K^{-1} \quad : \quad K = k - 1 \\ x_r(k) \cdot \sigma_K^{-1} \quad : \quad K < k - 1 \\ x_K \quad : \quad K = k - 1 \\ x_K \cdot \tau_K^{-1} \quad : \quad K < k - 1 \end{cases}$$

where the last equality follows from Lemma 6.4. Now Lemma 6.1 implies that in both cases we have $y_i(k) \cdot \sigma_k^{-1} = x_K$.

To prove the result for $y_i(k)$, the integer $K$ is defined analogously.

**Corollary 6.6.** Fix $k \in [n]$. Suppose $T_k$ is downward-pointing and that $(T_i(k), \ldots, T_i(k)_{d(k)})$ is the path between $T_k$ and $T_r(k)$. Then $y_i(k) \cdot \sigma_k^{-1} = y_i(k) \cdot \sigma_k^{-1}$ and $y_i(k) \cdot \sigma_k^{-1} = y_i(k)_{d(k)} \cdot \sigma_k^{-1}$ for all $j \in [d(k)]$. 

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Proof. Using the identities proved in Lemma 6.5 and the fact that $\tau_k^{-1}$ fixes $y_{r(k)}, x_{r(k)}, x_K, y_{i(k)}, v(r(i(k))) - 1)$, and $x_K$ for all $j \in [d(k)]$, we obtain the desired result.

6.2. $T_k$-Matrices and $R_k$ Subquivers. Let $Q$ denote an embedded, irreducible type $\mathbb{A}$ quiver. It may happen that in $\left(\mu_k \circ \cdots \circ \mu_1 \circ \mu_0\right)(\hat{Q})$ some of the 3-cycles of $Q$ besides $T_{k+1}$ have been partially mutated. That is, there may exist 3-cycles $T_{m_1}, \ldots, T_{m_s}$ such that in $\left(\mu_k \circ \cdots \circ \mu_1 \circ \mu_0\right)(\hat{Q})$ the vertex $x_m$, has been mutated, but no other vertices of $T_{m_1}$ have been mutated.

With this in mind, define a set $\mathcal{T}$ of 3-cycles of $Q$ by

$$\mathcal{T} := \{T_m : T_m \text{ is downward-pointing and is connected to some branching 3-cycle } T_k \text{ such that } z_k = x_m\}$$

and for each $k \in [n]$ define

$$\mathcal{T}_k := \{T_m \in \mathcal{T} : x_m \in \text{supp}(\mu_k \circ \cdots \circ \mu_0), y_m, z_m \notin \text{supp}(\mu_k \circ \cdots \circ \mu_0)\}.$$ 

Write $\mathcal{T}_k = \{T_{m_1}, \ldots, T_{m_s}\}$ where we label the elements of $\mathcal{T}_k$ in the way that agrees with the standard labeling of 3-cycles.

Let $\mathcal{R}_k$ denote the full, embedded subquiver of $Q$ on the vertices of the 3-cycles in the set $\mathcal{R}_k := \{T_{k+2}, \ldots, T_n\} \setminus \mathcal{T}_k$. Next, define $\mathcal{Q}_k := \{T_1, \ldots, T_k\}$, $\Omega := \mathcal{Q}_n$, and $\mathcal{Q}_k$ to be the full, embedded subquiver of $Q$ on the vertices of $\mathcal{Q}_k$. The quivers $\mathcal{Q}_k$ will be useful in the proof of Lemma 7.1. Observe that for each $k \in [n]$ we have that $\mathcal{Q}_k$, the set of 3-cycles of $Q$, decomposes as

$$\mathcal{Q} = \mathcal{Q}_k \sqcup (\mathcal{T}_k \cup \{T_{k+1}\}) \sqcup \mathcal{R}_k$$

where $\mathcal{T}_k$ may or may not already include $T_{k+1}$.

Now fix $k \in [n]$ and consider $T_{m_1} \in \mathcal{T}_k$. We let $T_{b(m_1)}$ denote the branching 3-cycle attached to $T_{m_1}$ such that $s_{b(m_1)} = x_{m_1}$. Let $(T_{m_1}, \ldots, T_{s_{t(m_1)}})$ denote the twig above $T_{m_1}$. Recall that the twig will be “northeast” of $T_{b(m_1)}$. Note that $s_1^{m_1} = b(m_1) + 1$ and that if $T_{b(m_1)}$ is upward-pointing $T_{b(m_1)} = T_{r(m_1)}$. Next, define $t(m_1, k) := \max\left(k \cap \{s_1^{m_1}, \ldots, s_{s_{t(m_1)}}^{m_1}\}\right)$, if it exists. Additionally, if $T_i$ is a 3-cycle in $\mathcal{Q}_k$ we define $v|_{\mathcal{Q}_k}(i)$ to be $v(i)$ restricted to the embedded, irreducible subquiver $\mathcal{Q}_k$.

Remark 6.7. Notice that no such integer $t(m_1, k)$ exists if and only if $b(m_1) = k$. In this case, $s_1^{m_1} = k + 1$.

With the above notation, for each $k \in [n]$ we associate to $\mathcal{Q}_k$ a skew-symmetric matrix $M(\mathcal{Q}_k) \in \mathbb{Z}^{N \times N}$ whose rows are indexed by $(Q)_0$ and whose columns are indexed by $(\hat{Q})_0$. Since the matrix $M(\mathcal{Q}_k)$ will be skew-symmetric, we define it by only writing down the entries below the diagonal. Fix $k \in [n]$ and let $T_{m_1} \in \mathcal{Q}_k$ be given. Then exactly one of the following cases holds.

Case 1: We have $s_1^{m_1} > b(m_1) + 1$, $t(m_1, k)$ exists and is less than $t(m_1)$ (the twig above $T_{m_1}$ has been partially mutated).

Case 2: We have $s_1^{m_1} > b(m_1) + 1$, $t(m_1, k)$ exists and equals $t(m_1)$ (the twig above $T_{m_1}$ has been fully mutated).

Case 3: We have $s_1^{m_1} \geq b(m_1) + 1$ and $t(m_1, k)$ does not exist. Equivalently, by Remark 6.7 we have $s_1^{m_1} \geq b(m_1) + 1$ and $b(m_1) = k$ (the twig above $T_{m_1}$ has not been mutated).

The matrix $M(\mathcal{Q}_k)$ is defined by

$$M(\mathcal{Q}_k)_{i,j} = \begin{cases} 1 & : (i,j) = (y_{m_1}, v|_{\mathcal{Q}_k}(b(m_1))) \text{ for } T_{m_1} \in \mathcal{T}_k \text{ or } \\
 & (i,j) = (y_{k+1}, v(k+1)) \\
-1 & : (i,j) = (z_{m_1}, x_{m_1}) \text{ for } T_{m_1} \in \mathcal{T}_k \text{ or } \\
 & (i,j) = (y_{m_1}, z_{r(m_1,k+1)}) \text{ and } T_{m_1} \in \mathcal{T}_k \text{ satisfies Case 1 or } \\
 & (i,j) = (y_{m_1}, b_{m_1}(1)) \text{ and } T_{m_1} \in \mathcal{T}_k \text{ satisfies Case 2 or } \\
 & (i,j) = (z_{k+1}, x_{k+1}) \text{ and } T_{k+1} \text{ is downward-pointing or } \\
 & (i,j) = (z_{k+1}, v(k)), T_{k+1} \text{ is upward-pointing, and } r(k+1) \neq 1 \\
0 & : \text{otherwise}. \end{cases}$$

We will refer to this matrix as the $\mathcal{T}_k$-matrix. We will see later that $M(\mathcal{Q}_k)$ describes the connections between vertices in the set $\{y_{k+1}, z_{k+1}, y_{m_1}, z_{m_1}, \ldots, y_{m_s}, z_{m_s}\}$ and other mutable vertices in $(\mu_k \circ \cdots \circ \mu_0)(\hat{Q})$. Below
we show four examples of the connections described by $M(\mathfrak{T}_k)$. The 3-cycle $T_{m_\ell}$ in the upper-left (resp. upper-right) example satisfies Case 1 (resp. Case 3).

Example 6.8. Let $\mathcal{Q}$ be the quiver shown above. Here we have $\mathfrak{T} = \{T_{11}, T_{12}, T_{16}\}$ and

| $k$         | $\mathfrak{T}_k$ | $\mathfrak{T}_{m_\ell}$ | $\mathfrak{T}_{b(m_\ell)}$ | $\mathfrak{T}_{s(m_\ell)}$ |
|-------------|------------------|--------------------------|-----------------------------|-----------------------------|
| 1,2,3,16    | $\emptyset$      | $\mathfrak{T}_{16}$     | $\mathfrak{T}_{16}$        | $\mathfrak{T}_{16}$        |
| 4,12,13,14,15| $\{T_{16}\}$    | $T_{11}$                 | $T_{16}$                   | $T_{16}$                   |
| 5,6,7,11    | $\{T_{12}, T_{16}\}$ | $T_{12}$                 | $T_{16}$                   | $T_{16}$                   |
| 8,9,10      | $\{T_{11}, T_{12}, T_{16}\}$ | $T_{11}$                 | $T_{12}$                   | $T_{16}$                   |
We show below what cases the elements of $\mathcal{I}_k$ satisfy for different values of $k$ as we vary $T_m$.

| $T_{11}$ | $T_{12}$ | $T_{16}$ |
|---|---|---|
| $k$ | Case | $k$ | Case | $k$ | Case |
| 8 | 3 | 5 | 3 | 4 | 3 |
| 9 | 1 | 6,\ldots, 11 | 2 | 5,\ldots, 13 | 1 |
| 10 | 2 | | | 14, 15 | 2 |

7. Proof of Theorem 5.7

In this section, we prove that given an embedded, irreducible type $\Lambda$ quiver $Q$, $\underline{\mu}$ is a maximal green sequence of of $Q$. Before doing so, we prove a technical lemma (Lemma 7.1) describing the intermediate quivers appearing while applying $\mu$ to $\tilde{Q}$. As above, we assume that $N = \#(Q)_0$. Recall that $Q_k$ denotes the full, embedded subquiver of $Q$ on the vertices of $T_1,\ldots, T_k$. We define $Q_0$ to be the empty quiver. Also, we define the permutation matrix of $\sigma$ with a right action on $[N]$ to be $M(\sigma) \in \mathbb{Z}^{N \times N}$ where

$$M(\sigma)_{i,j} = \begin{cases} 1 & : \text{if } i \cdot \sigma = j \\ 0 & : \text{otherwise.} \end{cases}$$

Given a matrix $B \in \mathbb{Z}^{N \times N}$, we define $B\sigma$ to be the matrix with entries $(B\sigma)_{i,j} = B_{i,j} \sigma_{j,i}$.

Below we define some matrices that will be used to describe the extended $B$-matrices of $\tilde{Q}$ appearing while applying the associated mutation sequence.

Now given $k \in [n+1]$, we define a $3 \times 6$ block matrix $M_{k-1} \in \mathbb{Z}^{N \times 2N}$ whose rows are indexed by $(Q)_0$ and whose columns are indexed by $\left( \mathfrak{Q} \right)_0$. Within the first row and column of blocks of $M_{k-1}$, the rows and columns are indexed by the vertices $A := \{x_1, y_1, z_1, y_2, z_2, \ldots, y_k, z_k\}$ and ordered using the standard ordering. Within the second row and column of blocks, the rows and columns are indexed by the vertices in $B := \{y_i, z_i : T_i \in \mathcal{I}_{k-1} \cup \{T_k\}\}$ ordered using the standard ordering restricted to such $y_i$ and $z_i$. Within the third row and column of blocks, the rows and columns are indexed by the vertices in $C := \{y_i, z_i : i \geq k+1, T_i \in \mathcal{I}_{k-1}\}$ and ordered using the standard ordering restricted to such $y_i$ and $z_i$. The fourth, fifth, and sixth columns of blocks are indexed by $A' = \{x_1', y_1', \ldots, y_{k-1}', z_{k-1}'\}$, $B' = \{y_i', z_i' : T_i \in \mathcal{I}_{k-1} \cup \{T_k\}\}$, and $C' = \{y_i', z_i' : i \geq k+1, T_i \in \mathcal{I}_{k-1}\}$, respectively. The fourth, fifth, and sixth columns of blocks are ordered such that the column indexed by $u'$ is to the left of the column corresponding to $v'$ if and only if the column corresponding to $u$ is to the left of the column corresponding to $v$. We define

$$M_{k-1} := \begin{bmatrix} B_{\mathcal{I}_{k-1}} \sigma_{k-1} & M(\mathcal{I}_{k-1})_{A,B} & 0 & -M(\sigma_{k-1}) & 0 & 0 \\ M(\mathcal{I}_{k-1})_{B,A} & M(\mathcal{I}_{k-1})_{B,B} & M(\mathcal{I}_{k-1})_{B,C} & M(\tilde{c}(v)) & I & 0 \\ 0 & B_{\mathcal{I}_{k-1}} & B_{\mathcal{I}_{k-1}} & 0 & 0 & I \end{bmatrix}$$

where $M(\mathcal{I}_{k-1})_{I,J}$ denotes the matrix $M(\mathcal{I}_{k-1})$ restricted to the rows indexed by $I$ and the columns indexed by $J$. Note that $M(\mathcal{I}_{k-1})_{I,I} = -M(\mathcal{I}_{k-1})_{I,I}^t$. The matrix $M(\tilde{c}(v))$ is made up of row vectors $\tilde{c}(v)$ defined for each vertex $v \in B$ by the formula

$$\tilde{c}(v) := \begin{cases} e_{x_{r(v)}} + \chi_{(r(v)>1)}e_{x_{r(v)-1}} + \left( \sum_{u \in \text{supp}(\mathfrak{H}_{r(v)})} e_{u^\prime} \right) & : v = z_i \text{ for some } z_i \in B \\ 0 & : v = y_i \text{ for some } y_i \in B. \end{cases}$$

where $\chi_{(r(v)>1)}$ denotes an indicator function. With this notation, we define for any vertex $v \in B$ the $c$-vector associated to $v$ to be $c(v) := \tilde{c}(v) + e_{v^\prime}$.

Also, note that by definition

$$M_n := \left[ B_{\mathcal{I}_n} \sigma_n \quad -M(\sigma_n) \right]$$

since $\mathcal{R}_n$ and $\mathcal{I}_n$ are empty and thus the quiver corresponding to $M_n$ is isomorphic to $\tilde{Q}$.

We will see that as we mutate $\tilde{Q}$ along $\underline{\mu}$, the matrix $M(\mathcal{I}_{k-1})_{A,B}$ will encode connections between green and red vertices in $\left( \mu_{k-1} \circ \cdots \circ \mu_0 \right)\left( \tilde{Q} \right)$, while the matrices $M(\mathcal{I}_{k-1})_{B,B}$ and $M(\mathcal{I}_{k-1})_{B,C}$ will encode connections between pairs of green vertices not both belonging to $\mathcal{R}_{k-1}$ in $\left( \mu_{k-1} \circ \cdots \circ \mu_0 \right)\left( \tilde{Q} \right)$. 

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**Lemma 7.1.** Given an embedded, irreducible type $\mathbb{A}$ quiver $Q$ with associated mutation sequence $\mu = \mu_n \circ \cdots \circ \mu_0$. Then for all $k \in [0, n]$, the matrix $\left( \mu_k \circ \cdots \circ \mu_0 \right) (B_Q) = M_k$.

**Remark 7.2.** Let $v$ be any mutable vertex of $(Q)_0$ and let $R \in \text{Mut} (\hat{Q})$ be given. By Lemma 4.1, if we regard $v$ as a mutable vertex in $R$, then

$$
\# \{ x \overset{\alpha}{\rightarrow} v \in (R)_1 \text{ for some } x \in (R)_0 \} \leq 2
$$

$$
\# \{ x \overset{\alpha}{\rightarrow} v \in (R)_1 \text{ for some } x \in (R)_0 \} \leq 2.
$$

**Proof of Lemma 7.1.** For $k = 0$, a direct calculation shows that $\mu_0 B_{\hat{Q}} = M_0$. Now suppose the statement holds for $0, \ldots, k - 1$ and we will show that it holds for $k$.

So we have $\left( \mu_{k-1} \circ \cdots \circ \mu_0 \right) (B_Q) = M_{k-1}$. Suppose $T_k$ is downward-pointing with $r(k) > 1$ and that locally $Q$ looks like the quiver below with $d(k) \geq 2$. Although, $T_{r(k)-1}$ is upward-pointing, we will see that the proof does not depend on this.

It is possible that any of $z_{r(k)-1}, y_k,$ or $z_k$ have degree 2. In these cases the computation that must be done is a specialization of that which we show below so it is enough to assume that $z_{r(k)-1}, y_k,$ and $z_k$ have degree 4. There are no additional degree restrictions on vertices in this figure. It will also be useful to set $k_1 := i(k)_2$ so that we can avoid writing $i(i(k)_j)$ where $j \in [d(i(k)_2)]$.

We include several local configurations of quivers below. In these quivers, we will color mutable vertices green or red accordingly.

The first mutation appearing in $\mu_k$ occurs at $y_k$. We claim that the quiver of $M_{k-1}$ locally looks like the left quiver shown below (in the picture below we are not imposing the usual embedding) and that it is transformed as is shown below by $\mu_{y_k}$. To see this, note that the 3-cycle $y_k, y_{k+1}, z_{k+1}$ exists and there are no other arrows between any of these vertices by referring to the matrices $M(\Sigma_{k-1})_{C,B}$ and $B_{R_{k+1}}$. The matrix $M(\Sigma_{k-1})_{B,A}$ tells us that $y_k$ has exactly the connections shown below with vertices in $A$. The matrix $M(\hat{Q}(v))$ tells us that $y_k$ is only connected to $y'_k$ as shown below. Lastly, since $v(k) \in A$ and $y_{k+1}, z_{k+1} \in C$, there are no arrows connecting $v(k)$ with either of these vertices. It is clear that the quiver of $M_{k-1}$ is transformed as desired.
Next, we mutate at $z_k$. We claim that the quiver of $\mu_{y_k}(\mathcal{M}_{k-1})$ locally looks like the left quiver shown below and that $\mu_{z_k}$ transforms the quiver of $\mu_{y_k}(\mathcal{M}_{k-1})$ in the following way. Below, we use the notation $\text{supp}(c(z_k))\backslash \{z_{\tau(k)-1}'\}$ to emphasize that the each element in the set $\text{supp}(c(z_k))\backslash \{z_{\tau(k)-1}'\}$ has the same connections to $z_k$ in the left diagram and the same connections $x_k$ and $z_{(i)0}$ (and the same connections to $z_k$ but with the arrows reversed) in the right diagram.

![Quiver Diagram](image)

To see that locally the quiver of $\mu_{y_k}(\mathcal{M}_{k-1})$ looks like the left quiver above, first, observe that the 3-cycle $x_k, z_k, y_{m_r}$ exists and that there are no other arrows between any of these vertices by referring to the matrices $M(\Sigma_{k-1})_{B,B}$ and $M(\Sigma_{k-1})_{B,B'}$. Next, the 3-cycle $z_k, y_{(i)0}, z_{(i)0}$ exists and there are no other arrows between any of these vertices by referring to the matrices $M(\Sigma_{k-1})_{C,B}$ and $B_{R_{k-1}}$. Next, by referring to the matrix $M(\hat{G}(v))$ and by the fact that we have not mutated $z_k$ in the quiver of $\mu_{y_k}(\mathcal{M}_{k-1})$, we see that $z_k$ has exactly the connections with frozen vertices shown in the left quiver above. Lastly, there is exactly one arrow from $z_{\tau(k)-1}'$ to $x_k$ in the quiver of $\mu_{y_k}(\mathcal{M}_{k-1})$. To see this, we show by the following computation that the automorphism of $\mathcal{Q}_{k-1}$ determined by $\sigma_{k-1}$ sends $x_k$ to $z_{\tau(k)-1}'$.

$$x_k \cdot \sigma_{k-1} = z_{\tau(k)-1}' \quad \text{(by Lemma 6.4(ii))}$$

Observe that $\text{supp}(c(z_{(i)0})) = \text{supp}(c(z_k)) \cup \{z_{(i)0}\}$ and no other mutations appearing in $\mu_k$ will affect the connections between $z_{(i)0}$ and elements of $\text{supp}(c(z_k)) \backslash \{z_{\tau(k)-1}'\}$ shown in the right diagram so in the quiver of $\mu_k(\mathcal{M}_{k-1})$ the vertex $z_{(i)0}$ will have the desired associated $c$-vector. Also, note that $T_{m_r} T_{(i)0} \in \mathcal{Q}_k$ and by referring to the right diagram we see that the 3-cycle $\{y_{m_r}, z_k, z_{(i)0}\}$ is as described in $M(\Sigma_k)$.

To compute $M_j := (\mu_{x_k} \circ \mu_{z_k} \circ \mu_{y_k})(\mathcal{M}_{k-1})$, we must first determine the connections between mutable vertices and $x_k$ in the quiver of $(\mu_{z_k} \circ \mu_{y_k})(\mathcal{M}_{k-1})$. As an aside, we will denote the intermediate matrices by a “roman” $M_j$ with $j \in \{d(k)\} \cup \{v(r(k)-1), v(k)\}$ from here on out. By Lemma 6.4 we have

$$x_k \cdot \sigma_{k-1} = z_{\tau(k)-1}' \quad \text{(by Lemma 6.4(ii))}$$

From this computation we see that $y_{(r(k)-1)', \sigma_{k-1}^{-1}}$ and $x_{(r(k)-1)', \sigma_{k-1}^{-1}}$ are the only vertices of $(\mathcal{Q}_{k-1})_0$ connected to $x_k$ in the quiver of $(\mu_{z_k} \circ \mu_{y_k})(\mathcal{M}_{k-1})$. Next, we have

$$y_{(r(k)-1)', \sigma_{k-1}^{-1}} = x_{(r(k)-1), \sigma_{k-1}^{-1}}$$

Thus the quiver of $(\mu_{z_k} \circ \mu_{y_k})(\mathcal{M}_{k-1})$ locally looks like the left diagram below and is transformed by $\mu_{x_k}$ in the following way.
By doing analogous computations, we have that for any \( s \in [2, d(k) - 1] \), the quiver of \( M_{s-1} := \mu_{x_{i(k)}_{s-1}} M_{s-2} \) is transformed by \( \mu_{x_{i(k)}_s} \) as follows.
where \( t \) ranges over \([0, s - 2]\). Furthermore, by analogous computations, we have that the quiver of \( M_{d(k) - 1} := \mu x_{i(k)_{d(k) - 1}} M_{d(k) - 2} \) is transformed by \( \mu x_{i(k)_{d(k)}} \) in the following way

\[
\begin{align*}
\text{supp}(c(z_k)) \setminus \{z'_{r(k) - 1}, y'_{r(k) - 1}, x'_{i(k)_{d(k) - t}} \} & \quad \text{supp}(c(z_k)) \setminus \{z'_{r(k) - 1}, y'_{r(k) - 1}, x'_{i(k)_{d(k) - t}} \}
\end{align*}
\]

where \( t \) ranges over \([0, d(k) - 2]\). To see that \( v(r(k) - 1) \) is connected to \( x_{i(k)_{d(k)}} \), as shown in the left diagram, observe that

\[
x_{i(k)_{2}} \cdot \sigma_{k-1}^{-1} = x_{i(k)_{1}} \cdot \sigma_{k-1}^{-1} = x_{i(k)_{1}} \cdot \sigma_{k-1}^{-1} \cdot \tau_{k-1}^{-1} \cdot \tau_{k-1}^{-1} = v(r(k) - 1) \cdot \tau_{k-1}^{-1} \cdot \tau_{k-1}^{-1} \quad \text{(by Lemma 6.4 iii)}
\]

Next, we compute \( M_{v(r(k) - 1)} := \mu v(r(k) - 1) M_{d(k)} \). Before doing so, we show that \( x_k \cdot \sigma_{k-1}^{-1} = v(k) \). If \( k_1 = k - 1 \), then \( x_k \cdot \sigma_{k-1}^{-1} = z_{k_1} \cdot \sigma_{k-1}^{-1} = v(k_1) = v(k) \). Now suppose \( k_1 < k - 1 \). Here there exists \( K \in [n] \) such that \( \deg(z_K) = 2 \) and \( r(K) = k_1 + 1 \). Now we have

\[
x_k \cdot \sigma_{k-1}^{-1} = \begin{cases} 
z_{r(k) - 1} \cdot \sigma_{K}^{-1} & : K = k - 1 
z_{r(k) - 1} \cdot \sigma_{K+1}^{-1} \cdot \tau_{k-1}^{-1} & : K < k - 1 
z_k & : K = k - 1 
z_k \cdot \tau_{K+1}^{-1} \cdot \tau_{k-1}^{-1} & : K < k - 1 \end{cases}
\]

where the last equality holds by Lemma 6.4 ii). Now Lemma 6.1 implies that in both cases we have \( x_k \cdot \sigma_{k-1}^{-1} = v(k) \) as \( z_K = v(k) \).

Also, note that for \( t \) ranging over \([0, d(k) - 1]\) we have

\[
\text{supp}(c(z_k)) \setminus \{z'_{r(k) - 1}, y'_{r(k) - 1}, x'_{i(k)_{d(k) - t}} \} = \{z'_t \}
\]

Now we see that the quiver of \( M_{d(k)} \) is transformed by \( \mu v(r(k) - 1) \) in the following way.
Next, we compute $\mu_k(M_{k-1}) = \mu_{\nu(k)}(M_{\nu(r(k)-1)})$. By the computation of $M_{\nu(r(k)-1)}$ and $\mu_{y_k}(M_{k-1})$, we have the following local configuration in the quiver of $M_{\nu(r(k)-1)}$ that is transformed by $\mu_{\nu(k)}$ as follows.

By examining the diagrams above and using Corollary 6.6, one sees that $\mu_k(M_{k-1}) = M_k$. This completes the proof when $T_k$ is downward-pointing with $r(k) > 1$, $d(k) \geq 2$, and $\deg(z_{r(k)-1}) = \deg(y_k) = \deg(z_k) = 4$. The other cases can be checked by a similar, easier computation (Remark 7.3 outlines the other cases that need to be checked).

*Proof of Theorem 5.7.* It is clear from the proof of Lemma 7.1 that all mutations of $\hat{Q}$ appearing in $\mu$ occur at green vertices. By Lemma 7.1, we know $\mu(B_Q) = M_n$ so $\mu\hat{Q}$ has only red vertices. Thus $\mu \in \text{green}(Q)$.

**Remark 7.3.** The left quiver above shows the type of quivers to which we would apply $\mu_k$ in the proof of Lemma 7.1 if $T_k$ was downward-pointing and $r(k) = 1$. The right quiver above shows the type of quivers to which we would apply $\mu_k$ in the proof of Lemma 7.1 if $T_k$ was upward-pointing. In this case, we have drawn $T_{k-1}$ so that it is downward-pointing, but the proof when $T_k$ is upward-pointing is independent of this.
8. Additional Questions

In this section, we give an example to show how our results provide explicit maximal green sequences for quivers that are not of type $A$. We also mention ideas we have for further research.

8.1. Maximal Green Sequences for Cluster Algebras from Surfaces. The following example shows how our formulas for maximal green sequences for type $A$ quivers can be used to give explicit formulas for maximal green sequences for cluster algebras from surfaces arising from certain types of triangulations.

**Example 8.1.** Consider the surface $S$ with triangulation $\Delta$ shown below on the left. The surface is a once-punctured pair of pants with triangulation $\Delta_1 \cup \Delta_2 \cup \{\eta, \epsilon, \zeta\}$ where $\alpha_1, \alpha_2, \alpha_3 \in \Delta_1$ and $\beta_1, \beta_2, \beta_3, \nu \in \Delta_2$. We assume that the boundary arcs $b_i$ with $i \in [5]$ contain no marked points except for those shown below. The other boundary arcs may contain any number of marked points. Let $Q$ denote the quiver determined by $\Delta$ and let $v_\delta \in (Q)_0$ denote the vertex corresponding to arc $\delta \in \Delta$.

We can think of the surface $S_1$ determined by $b_1, \beta_1, c_1, \alpha_2, c_2, \alpha_3, c_3, \beta_2, b_2, c_4, \beta_3, b_3, c_5, \beta_3, b_4, c_6, b_5$ as an $M_1$-gon and we can think of $\Delta_1$ as a triangulation of $S_1$. Similarly, we can think of the surface $S_2$ determined by $\alpha_1, c_3, \eta, b_5, c_6, \alpha_3, c_5, b_2, \alpha_2, c_4, b_1$ as an $M_2$-gon and we can think of $\Delta_2$ as a triangulation of $S_2$. Thus quiver $Q_i$ determined by $\Delta_i$ is a type $A$ quiver for $i = 1, 2$. Furthermore, we have

$$Q = Q_1 \oplus_{(v_{\beta_1}, v_{\beta_2}, v_{\beta_3})} Q_2 \oplus_{(v_\nu)} R$$

where

$$R = v_\eta$$

As in Sections 5.1 and 5.2, $Q_1$ and $Q_2$ each have a maximal green sequence, i.e. $\mu^{Q_i}$ for $i = 1, 2$. Since $R$ is acyclic, we can define $\mu^R$ to be any sequence of mutations where each mutation occurs at a source (for instance, put $\mu^R = \mu_{v_\epsilon} \circ \mu_{v_\zeta} \circ \mu_{v_\eta}$). Then $\mu^R$ is clearly a maximal green sequence of $R$. Thus $\mu^R \circ \mu^{Q_2} \circ \mu^{Q_1}$ is a maximal green sequence of $Q$.

Suppose that $\Delta_1$ and $\Delta_2$ are given by the triangulations shown above in the figure on the right. Then we have that $Q_1$ and $Q_2$ are the quivers shown below where we think of the irreducible parts of

$$Q_1 = x_1^{(1)} \xrightarrow{T_1^{(1)}} y_1^{(1)} \xrightarrow{T_2^{(1)}} z_1^{(1)} \xrightarrow{T_3^{(1)}} y_2^{(1)} \xrightarrow{T_3^{(1)}} z_2^{(1)} \xrightarrow{T_3^{(1)}} w_1 \leftarrow w_2 \leftarrow v_{\alpha_1}$$

$$Q_2 = x_2^{(1)} \xrightarrow{T_1^{(1)}} y_2^{(1)} \xrightarrow{T_2^{(1)}} z_2^{(1)} \xrightarrow{T_3^{(1)}} y_3^{(1)} \xrightarrow{T_3^{(1)}} z_3^{(1)} \xrightarrow{T_3^{(1)}} w_1 \leftarrow w_2 \leftarrow v_{\alpha_1}$$
Q₁ and Q₂ as embedded, irreducible type A quivers with respect to the root 3-cycles T₁(1) and T₁(2), respectively. In this situation, Q₁ and Q₂ have maximal green sequences

\[ \mu^{Q₁} = \mu_{w₁} \circ \mu_{w₂} \circ \mu_{vₙ₁} \circ \mu_{x₁(1)} \circ \mu_{y₁(1)} \circ \mu_{z₁(1)} \circ \mu_{y₁(1)} \circ \mu_{x₂(2)} \circ \mu_{xₙ₁(1)} \circ \mu_{zₙ₁(1)} \circ \mu_{yₙ₁(1)} \]

\[ \mu^{Q₂} = \mu_{w₄} \circ \mu_{w₃} \circ \mu_{y₅(2)} \circ \mu_{x₅(2)} \circ \mu_{z₅(2)} \circ \mu_{y₅(2)} \circ \mu_{x₄(2)} \circ \mu_{x₃(2)} \circ \mu_{z₃(2)} \circ \mu_{y₃(2)} \]

respectively where

\[ \mu^{Q₁}_{x₁(1)} = \mu_{x₁(1)} \]

\[ \mu^{Q₁}_{z₁(1)} = \mu_{z₁(1)} \circ \mu_{x₁(1)} \circ \mu_{y₁(1)} \]

\[ \mu^{Q₁}_{y₁(1)} = \mu_{y₁(1)} \circ \mu_{x₁(1)} \circ \mu_{z₁(1)} \circ \mu_{y₁(1)} \]

\[ \mu^{Q₁}_{zₙ₁(1)} = \mu_{zₙ₁(1)} \circ \mu_{xₙ₁(1)} \circ \mu_{yₙ₁(1)} \]

\[ \mu^{Q₂}_{x₅(2)} = \mu_{x₅(2)} \circ \mu_{x₄(2)} \circ \mu_{y₄(2)} \]

\[ \mu^{Q₂}_{z₅(2)} = \mu_{z₅(2)} \circ \mu_{z₄(2)} \circ \mu_{y₄(2)} \]

\[ \mu^{Q₂}_{y₅(2)} = \mu_{y₅(2)} \circ \mu_{z₅(2)} \circ \mu_{y₅(2)} \]

and \( \mu^{R} \circ \mu^{Q₂} \circ \mu^{Q₁} \) is a maximal green sequence of Q. In general, if we have a quiver Q that can be realized as a direct sum of type A quivers and acyclic quivers, we can write an explicit formula for a maximal green sequence of Q.

**Problem 8.2.** Find explicit formulas for maximal green sequences for cluster algebras arising from surfaces.

Using Corollary 3.9 we can reduce Problem 8.2 to the problem of finding explicit formulas for maximal green sequences of irreducible quivers that arise from a triangulated surface. In [1], the authors sketch an argument showing the existence of maximal green sequences of quivers arising from triangulated surfaces. However, we would like to prove the existence of maximal green sequences by giving explicit formulas for maximal green sequences of such quivers.

**8.2. Trees of Cycles.** Our study above of embedded, irreducible type A quivers was made somewhat simple by the fact that such quivers consist entirely of 3-cycles glued together. Let\( C(3,n,T) \) denote an embedded, irreducible type A quiver with 3-cycles \( T₁, \ldots, Tₙ \) and root 3-cycle T. We can think of \( C(3,n,T) \) as a tree of 3-cycles with a certain embedding. If we replace 3 with k to obtain an embedded (irreducible) **tree of k-cycles** with root k-cycle T denoted \( C(k,n,T) \), we could try a similar technique to that which we used above to find maximal green sequences of \( C(k,n,T) \).

**Problem 8.3.** Find explicit formulas for maximal green sequences of \( C(k,n,T) \).

**8.3. Enumeration of Maximal Green Sequences.** By Remark 4.8 we know that for a given embedded, irreducible type A quiver Q with root 3-cycle T and with at least two 3-cycles our algorithm produces a maximal green sequence \( \mu = \mu(T) \) of Q for each leaf 3-cycle in Q. It would be interesting to see how many maximal green sequences of Q can be obtained from the maximal green sequences \( \mu \) as the choice of the root 3-cycle T varies.

**Problem 8.4.** Determine what maximal green sequences of Q can be obtained via commutation relations and Pentagon Identity relations applied to the maximal green sequences in \( \{ \mu(T) : T \text{ is a leaf 3-cycle} \} \).

Additionally, in [1] there are several tables giving the number of maximal green sequences of certain quivers by length. These computations may be useful for making progress on the problem of enumerating maximal green sequences of quivers.

As discussed in Remark 5.8 the associated mutation sequences constructed here are not necessarily the shortest possible maximal green sequences. This motivates the following problem.

**Problem 8.5.** Provide a construction of the maximal green sequences of minimal length, possible by showing how to apply Pentagon Identity relations to the associated mutation sequences.
8.4. Classification of Maximal Green Sequences. Note that maximal green sequences of a quiver $Q$ can be thought of as maximal chains (from the unique source to the unique sink) in the oriented exchange graph \cite{3} Section 2). In the case that $Q$ is of type $A$, the exchange graph is an orientation of the 1-skeleton of the associahedron. The oriented exchange graph is especially nice in the case when $Q$ is acyclic. For example, it is the Hasse graph of the Tamari lattice in the case $Q$ is linear and equioriented and it is the Hasse graph of a Cambrian lattice (in the sense of Reading \cite{12}) otherwise \cite{10} Section 3). In particular, this means that we consider the finite Coxeter group $G$ whose Dynkin Diagram is the unoriented version of $Q$ and a choice of Coxeter element $c$ compatible with the orientation of $Q$ and then maximal green mutation sequences are in bijection with maximal chains in the Cambrian lattice, a quotient of the weak Bruhat order on $G$. Note, that this bijection is studied further in \cite{11} where each c-sortable word is shown to correspond to a green sequence.

To indicate the difficulty of describing the set of maximal green sequences once we consider quivers with cycles, we focus on the $A_3$ case here. In the case where $Q$ is a 3-cycle (with $1 \to 2 \to 3 \to 1$), there is not a corresponding Cambrian congruence that one can apply to the weak Bruhat order on the symmetric group $G = S_4$ to obtain the desired Hasse diagram. In particular, the corresponding Cambrian lattice is constructed from the geometry of the affine $A_2$ root system instead of from a finite Coxeter group. Intersecting this coarsening of the Coxeter Lattice with the Tits Cone yields 11 regions rather than the 14 we obtain in the acyclic case \cite{13}.

Nonetheless, we can still compute maximal green sequences in this case, and see that they are indeed the set of oriented paths through a certain orientation of the 1-skeleton of the associahedron. There are six possible maximal green sequences of length 4: $s_1 \circ s_2 \circ s_3 \circ s_4$, $s_1 \circ s_2 \circ s_3 \circ s_4$, $s_1 \circ s_2 \circ s_3 \circ s_4$, $s_1 \circ s_2 \circ s_3 \circ s_4$, $s_1 \circ s_2 \circ s_3 \circ s_4$, and $s_1 \circ s_2 \circ s_3 \circ s_4$. Applying the relation $s_3 \circ s_4 \to s_3 \circ s_4$ to the first three of these, we get three maximal green sequences of length 5: $s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5$, $s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5$, and $s_1 \circ s_2 \circ s_3 \circ s_4 \circ s_5$. As in Figure 22 of \cite{4}, there are no other maximal green sequences of this quiver.

In an attempt to understand this example in terms of the Coxeter group of type $A_3$, i.e. $S_4$, we consider the presentation described in \cite{3} for quivers with cycles. In this case, if we let $s_1 = (14)$, $s_2 = (24)$, $s_3 = (34)$, we obtain $S_4 = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_1)^3 = (s_1 s_2 s_3)^2 = 1 \rangle$.

Unlike the acyclic $A_3$ case where the permutation in $S_4$ corresponding to the longest word, i.e. 4321, is the only element of $S_4$ whose length as a reduced expression, e.g. $s_1 s_2 s_3 s_1 s_2 s_3$, is of length 6, in the Barot-Marsh presentation, the permutuations 4321, 3412, 2143, 1342, and 1432 all have reduced expressions of maximal length, namely 4.

Further, if we visualize the order complex of $S_4$ under this presentation, we obtain a torus (see Example 3.1 of \cite{2}) rather than a simply-connected surface like the acyclic case and there are no permutations with reduced expression of length 5, hence reduced expressions in this presentation cannot correspond to maximal green sequences. We thank Vic Reiner for bringing his paper with Eric Babson to our attention. Hence, understanding the full collection of maximal green sequences for other quivers with cycles, even those of type $A$, appears to require more than an understanding of the associated Coxeter groups.

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