On the stability of small-scale ballooning modes in mirror traps

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(Dated: December 1, 2020)

It is shown that steepening of the radial plasma pressure profile leads to a decrease in the critical value of beta, above which small-scale balloon-type perturbations in a mirror trap become unstable. This means that small-scale ballooning instability leads to a smoothing of the radial plasma profile. This fact seems to have received little attention in the available publications. The critical beta values for the real magnetic field of the gas-dynamic trap was also calculated. In the best configuration the critical beta 0.72 is obtained for a plasma with a parabolic radial pressure profile.

I. INTRODUCTION

There are a number of publications, in one way or another, related to the stability of ballooning MHD perturbations in open traps [1–35]. Most of them were published in the 80s of the last century. In the next decade, interest in the problem of ballooning instability in open traps significantly weakened (in contrast to what is happening in tokamaks, see e.g. [36–38]), which was a consequence of the termination of the TMX (Tandem Mirror eXperiment) and MFTF-B (Mirror Fusion Test Facility B) projects in the USA in 1986 [39]. However, the achievement of a high electron temperature and high beta (β, the ratio of the plasma pressure to the magnetic field pressure) in the Gas-Dynamic Trap (GDT) at the Budker Institute of Nuclear Physics in Novosibirsk [40–48] and the emergence of new ideas [49] and new projects [50, 51] makes us rethink old results.

We are planning to prepare a series of three articles and, following the logic of previous research, in the first article we turn to the study of the stability of small-scale ballooning disturbances, which are characterized by large values of the azimuthal wave number m, m ≫ 1. In this first article, we restrict ourselves to the paraxial approximation, assuming that the transverse size a of the plasma in the trap is small compared to its length L, a ≪ L. In the next article, the effects of non-paraxiality will be taken into account, and in the third article, the stability of the rigid ballooning mode m = 1 will be examined.

We will use the canonical method for studying ballooning instability, historically associated with the energy principle in the approximation of one-fluid ideal magnetohydrodynamics [9]. It should be understood that this method has limited applicability. In particular, it assumes that in equilibrium the plasma is stationary, that is, it does not rotate around its axis and does not flows out through magnetic mirrors. In addition, the energy principle ignores the effects of a finite Larmor radius (FLR effects) [20–22], which should stabilize small-scale flute and ballooning perturbations with m > a^2/(Lρi) (ρi is the ion Larmor radius) [23]. Nevertheless, even with the listed limitations, the energy principle allows important conclusions to be drawn. In particular, this article will show that small-scale ballooning instability leads to a smoothing of the radial plasma pressure profile.

First part of the study undertaken in the present paper is based on a simplified derivation of the equation for ballooning oscillations from the energy principle performed by Dmitri Ryutov and Gennady Stupakov [1, 2]. The simplification includes (a) the use of the paraxial approximation, and (b) the use of the low beta approximation, β ≪ 1, which allows us to assume that the magnetic field both inside and outside the plasma column is close to the vacuum one.

A more accurate calculation, which used only the first (paraxial) approximation (a), was made by Ol’ga Bushkova and Vladimir Mirnov [3, 4]. These authors used the balloon equation that was derived by William Newcomb in [5]. The same equation with a short derivation is given in [8] for the case of isotropic plasma. It has been studied in [6], and the effect of strong anisotropy created by a rigid ring of electrons has been included into consideration in [7].

In the classical paper [9] by Ira Bernstein et al, where the energy principle was formulated in its modern form, in the last section a relatively simple expression for the potential energy of flute and ballooning perturbations in a non-paraxial axially symmetric open trap is given in the approximation of infinite azimuthal number m → ∞, when the disturbance is like a knife blade. Attempts to calculate the equilibrium and stability of nonparaxial systems were made by Vladimir Arsenin et al. [14–19], however, no significant progress seems to have been made.

It is commonly believed that FLR effects [20–22] are capable to stabilize all modes with the azimuthal numbers m ≥ 2 and impose on the m = 1 mode the shape of a rigid ("solid-state") displacement without deformation of the plasma density profile in each section of the plasma column. More accurate treatment is given in [23]. It leads the conclusion that ballooning modes with m > a^2/Lρi are stable for any β. The stability of ballooning perturbations with m = 1 was considered in Refs. [24–35]. These and some other papers on the stability of the m = 1 ballooning mode and wall stabilization will be reviewed in a forthcoming paper.
In what follows, we will adhere to the following order of presentation. The equation for ballooning oscillations in the low-pressure plasma limit ($\beta \ll 1$) is written in section II. Its solutions for some model magnetic field and pressure profiles are presented in section III. Here the critical value of beta $\beta_{\text{crit}}$ is calculated, above which ballooning disturbances become unstable. It turned out that the calculated values of $\beta_{\text{crit}}$ in some cases exceed one. This means that the assumption $\beta \ll 1$, made when deriving the simplified equation, does not hold. Nevertheless, the low-pressure plasma approximation gives a quantitatively correct result for a plasma with a radial pressure profile close to a stepwise one, for which the critical beta is computed for realistic magnetic field of the gasdynamic trap. Main results are summarized in Section VI.

II. EQUATION OF BALLOONING MODES IN LOW-PRESSURE PLASMA

In contrast to flute perturbations, ballooning modes are localized in the region of unfavorable curvature $\kappa$ of magnetic field lines (where $\kappa < 0$), distorting the magnetic field approximately as shown in Fig. 1, approaching zero amplitude in the region of favorable curvature (where $\kappa > 0$). Deformation of magnetic field lines requires energy expenditure, the source of which is the internal plasma energy $p/(\gamma - 1)$, where $p$ is the plasma pressure and $\gamma$ is the specific heat ratio; therefore ballooning-type disturbances are unstable only at a finite value of relative plasma pressure $\beta = 8\pi p / B^2$, exceeding some limit $\beta_{\text{crit}}$:

$$\beta > \beta_{\text{crit}} \quad \text{(unstable).}$$

The goal of exact theory is to calculate the $\beta_{\text{crit}}$ parameter, but we will restrict ourselves to simple reasoning and order of magnitude estimates. In particular, for simplicity, we assume (as it was done in [1, 2] and many other papers) that ideally conducting ends are placed directly in the plugs in order to provide stability of the flute modes due to “line-tying” [52]. Thus, we narrow the class of possible perturbations allowed to compete while minimizing the potential energy of perturbations, so the found values of $\beta_{\text{crit}}$ will actually give an upper bound for this value. At the same time, the difference between the true values of $\beta_{\text{crit}}$ from this upper boundary should be small, since behind the plug there is a deep (and plasma-filled) magnetic well (a region of favorable curvature), as a result of which the magnetic field lines are actually rigidly fixed in end plugs, returning us to the condition of being frozen into the ends.

In accordance with the real situation for devices of the GDT type, we will assume (following again Refs. [1, 2] and many subsequent papers) that the magnetic field lines make a small angle with the magnetic axis of the system (paraxial approximation) and that the distance between the magnetic plugs $2L$ is greater than the total plugs length $2L_m$, which can be separated by a quasi-uniform part. The small parameter characterizing the accuracy of the paraxial approximation is $a/L_m$, where $a$ is the transverse plasma size in the homogeneous part of the trap and $L_m < L$ is the plug length. In the paraxial approximation, integration over the field line (over $dl$) can be replaced by integration over the trap axis (over $dz$). Let us agree that the coordinates $z = \pm L$ correspond to the magnetic mirrors, and the homogeneous part of the trap occupies the region $|z| < L - L_m$. Assuming also that $\beta \ll 1$ and the magnetic field even inside the plasma is close to vacuum field, we will assume that it is described with sufficient accuracy by the function $B = B(z)$, which specifies the axial profile of the magnetic field on the trap axis. In this approximation, the reduced (that is, divided by $2\pi$) magnetic flux at a distance $r$ from the axis is equal to $\Psi = r^2 B(z)/2$, and the field line equation

$$r(z) = r_0 q(z), \quad q(z) = \sqrt{B_0 / B(z)} \quad (2)$$

is obtained from the condition of constancy of the magnetic flux inside the flux tube $\Psi = r_0^2 B_0/2 = \text{const}$, where $r_0$ and $B_0$ have the meaning of the radius of the flux lines and values of the magnetic field in the median plane $z = 0$.

For perturbations of the flute type in an axially symmetric mirror cell, for the projection of the displacement vector $\xi$ normal to the field line, the following relation holds:

$$\xi_s r B = \xi_0 r_0 B_0 = \text{const}. \quad (3)$$

Hence, in the paraxial approximation, we find

$$\xi_s(z) = \xi_0 q(z) \equiv \xi_0(z), \quad (4)$$

where the amplitude of the flux tube displacement $\xi_0 = \xi_0(\Psi, \theta)$ is assumed to be a function of the magnetic flux $\Psi$ and the azimuthal angle $\theta$. In the case of a small-scale perturbation, the $\xi_0$ amplitude is sharply dived (localized) near some values of $\Psi_s$ and $\theta_s$, but for the time being this fact is not essential for subsequent calculations.

The displacement (4) does not satisfy the condition of freezing $\xi_s(\pm L) = 0$ at the ends, therefore instead of (4) one should consider a displacement of a more general form

$$\xi_s(z) = \alpha(z) \xi_0(z), \quad (5)$$
where \( \alpha(z) \) is a still unknown dimensionless function. We impose two boundary conditions on it:

\[
\alpha(\pm L) = 0, \quad \alpha(0) = 1. \tag{6}
\]

The first of them ensures the fulfillment of the condition of being frozen into the ends, and the second has the meaning of the normalization condition, it fixes the displacement at the center of the trap. Taking into account the symmetry of the problem, we assume that the sought function \( \alpha(z) \) is even, that is, \( \alpha(-z) = \alpha(z) \), so it suffices to find a solution for only one half of the trap, say, for \( 0 < z < L \). An odd solution \( \alpha(-z) = -\alpha(z) \) also exists. It is obtained for the boundary conditions \( \alpha(\pm L) = \alpha(0) = 0 \), but it corresponds to a larger value of \( \beta_{\text{crit}} \).

As shown in [53, \S 29] (see also [1, 2]), the energy of the ballooning-type perturbation is

\[
W_F = \left( \frac{1}{8\pi} \int d\theta \int d\xi \xi_0^2 B_0 \right) \left( \int dz \left\{ (\alpha')^2 + 2\beta q^3 q'' \alpha^2 \right\} \right), \tag{7}
\]

where

\[
\beta = \int d\theta \int d\xi \xi_0^2 \rho(\Psi) \left/ \int d\theta \int d\xi \xi_0^2 \right., \tag{8}
\]

\[
\alpha = -\frac{8\pi \Psi}{B^2} \frac{d\rho}{d\Psi}, \tag{9}
\]

and the prime denotes the derivative with respect to \( z \).

According to the energy principle [9], the system is stable if the minimum value of the integral (7) is greater than zero. The factor inside the first pair of parentheses in (7) is certainly greater than zero, so the sign of \( W_F \) is determined by the sign of the integral inside the second pair of parentheses. To find the minimum of \( W_F \), we take the first variation of the integral (7),

\[
\delta \int_0^L dz \left\{ (\alpha')^2 + 2\beta q^3 q'' \alpha \right\} =
2 \int_0^L dz \left\{ \alpha' \delta \alpha' + 2\beta q^3 q'' \alpha \delta \alpha \right\},
\]

and set it to zero. In this way we obtain an equation that the extremal must satisfy, that is, the function \( \alpha(z) \), which minimizes the perturbation energy \( W_F \). Since the values of the function \( \alpha(z) \) at the ends of the integration interval are given by the boundary conditions (6), calculating the variation, we must assume that \( \delta \alpha(0) = \delta \alpha(L) = 0 \). Taking this fact into account and integrating by parts, we get the integral

\[
2 \int_0^L dz \left\{ -\alpha'' + 2\beta q^3 q'' \alpha \right\} \delta \alpha
\]

which vanishes for any variation of \( \delta \alpha \) if

\[
\alpha'' - 2\beta q^3 q'' \alpha = 0. \tag{10}
\]

As one can see, the calculation of the critical value \( \beta_{\text{crit}} \) of the parameter \( \beta \), for which there is a nonzero solution to Eq. (10)

![Figure 2](image_url)

Figure 2. Magnetic field (12), effective potential \( q'q'' \) and eigenfunction \( \alpha(z) \) in the Sturm-Liouville problem for the case \( K = 15 \), \( L = 6 \), \( \Delta L = 1 \); the dotted line shows the graph of the function (13) for \( L_m = 2.2 \Delta L \).

with the above boundary conditions (6), is reduced to the quantum mechanical problem of determining the conditions for the occurrence of a zero energy level in the potential

\[
V(z) = \begin{cases} \infty, & \text{if } z < 0; \\ 2\beta q^3 q'', & \text{if } 0 < z < L; \\ \infty, & \text{if } z > L. \end{cases} \tag{11}
\]

We can also say that \( \beta_{\text{crit}} \) is the eigenvalue of the classical Sturm-Liouville problem for Eq. (10) with boundary conditions (6), and the solution to this problem \( \alpha(z) \) is an eigenfunction. It is easy to prove that the substitution of the eigenfunction and the eigenvalue in the integral (7) makes the energy of the ballooning disturbance to zero, which, in accordance with the energy principle, corresponds to the marginal state between stability and instability.

The value of \( \beta_{\text{crit}} \) is calculated in Section III, and the actual value of the \( \bar{\beta} \) parameter to be compared with the critical value depends on the spacial profile of function \( \xi_0 = \xi_0(\Psi, \theta) \) and pressure profile \( \rho(\Psi) \).

### III. CRITICAL BETA IN LOW PRESSURE PLASMA

For the first example, let us choose the dependence of the magnetic field on the coordinate \( z \) on the trap axis in the form

\[
B(z)/B_0 = 1 + (K - 1) \left( e^{-|z|L^2/\Delta L^2} + e^{-(z+L)^2/\Delta L^2} \right), \tag{12}
\]

where \( K \) has the meaning of the mirror ratio, and \( 2L \) is the distance between the magnetic mirrors. For such a field, the effective potential function \( q'q'' \) is approximately equal to zero outside the plug (for \( 0 < |z| < L - L_m \)) and rapidly decreases deep into the plug along with the growth of the mirror ratio \( B/B_0 = 1/q^2 \), as shown in Figure 2. There is a hole at the entrance to the magnetic mirror on the \( q'q'' \) graph. Its width is approximately equal to \( L_m \), and the well itself is located in the region of the plug, adjacent to the uniform magnetic field, where \( B/B_0 \sim 1 \) and \( z \approx L - L_m \). Outside the region of the
potential well, the equation (14) reduces to the trivial equality \( \alpha' = 0 \), so there \( \alpha(z) \) will be approximately a linear function of the coordinate \( z \). On both sides of the potential well, the derivative \( \alpha' \) is a constant, but the constant changes in the region of the well.

For the magnetic field (12), the calculated value \( \beta_{\text{crit}} = 0.580 \) only slightly differs from the exact eigenvalue \( \beta_{\text{crit}} = 0.597 \). The graph of the eigenfunction \( \alpha(z) \) is shown in Fig. 2, where the dashed line shows the test function \( \alpha_1(z) \).

Calculations show that the critical beta is very sensitive to the magnetic field profile. This fact is confirmed by the second example with a magnetic field

\[
\frac{B(z)}{B_0} = 1 + (K - 1) \left[ \frac{b^3}{(b^2 + (z - L)^2)^{3/2}} + \frac{b^3}{(b^2 + (L + z)^2)^{3/2}} \right],
\]

which simulates a mirror cell composed of two coils. In this example, the critical value \( \beta_{\text{crit}} = 1.18 \) is formally greater than one.

Let us discuss the results obtained. The calculated value of \( \beta_{\text{crit}} \) turned out to be close to one or even more than one, whereas, starting to derive the equation of ballooning oscillations in Section II, we used the approximation \( \beta \ll 1 \) and assumed the magnetic field to be vacuum. Therefore, this hypothesis of ours was not entirely correct. Nevertheless, our work was not entirely useless.

First, note that the \( \beta \) parameter, like the \( \beta(\Psi) \) function, which are defined by the formulas (8) and (9), are not equal to the parameter beta in the traditional sense

\[
\beta_0 = \frac{8\pi p_0}{B_0^2},
\]

where \( p_0 \) has the meaning of the plasma pressure on its axis at \( \Psi = 0 \). In the particular case when the pressure has a power-law profile

\[
p(\Psi) = p_0 (1 - \Psi^\delta / \Psi_a^\delta)
\]

for \( \Psi < \Psi_a \), and \( p(\Psi) = 0 \) for \( \Psi > \Psi_a \), the maximum value of \( \beta(\Psi) \) is reached on the boundary field line \( \Psi = \Psi_a \), where \( \beta(\Psi_a) = k\beta_0 \). If \( k = 1 \) (pressure profile parabolic in \( r \)), the equality \( \beta(\Psi_a) = \beta_0 \) holds, but for a steeper profile with \( k > 1 \) the value \( \beta(\Psi_a) \) is greater than \( \beta_0 \). In the same way, for a steep pressure profile close to a step, the value of \( \beta \) can be much larger than \( \beta_0 \), so a situation is possible where \( \beta \sim 1 \) whereas \( \beta_0 \ll 1 \) and the vacuum magnetic field approximation can be completely valid.

The stepped pressure profile can be considered as the limit of a trapezoidal distribution close to a step with a small but finite width of the boundary layer \( \Delta \Psi \). For definiteness, let us assume that the plasma pressure is equal to \( p_0 \) at \( \Psi < \Psi_a \) and decreases linearly to zero in the boundary layer with the width \( \Delta \Psi \). The result of calculating \( \beta \) depends on the ratio of \( \Delta \Psi \) and the size \( \delta \Psi \) of the perturbation with respect to the variable \( \Psi \).

If the size of the flute tube is small compared to the width of the border, \( \delta \Psi \ll \Delta \Psi \), the local parameter (9) in the integral (8) can be considered constant across the size of the tube, so that \( \beta \approx \beta(\Psi) \), where \( \Psi \) is the value of \( \Psi \) on the field line, near which the disturbance is localized. Local value

\[
\beta(\Psi) \approx \beta_0 \frac{\Psi_a}{\Delta \Psi}
\]

in the boundary layer on the field line, where the disturbance is localized, is large for a small boundary width and can easily exceed the critical value. Consequently, small-scale perturbations will be unstable even if the plasma pressure is much lower than the magnetic field pressure, ie, \( \beta_0 \ll 1 \).

However, small-scale oscillations are unlikely to cause great damage to the equilibrium plasma configuration. Large-scale disturbances are more dangerous. If \( \delta \Psi \gg \Delta \Psi \), performing integration over \( \Psi \) in the formula (8), we can assume that

\[
\beta(\Psi) \approx \beta_0 \Psi_a \delta(\Psi - \Psi_a).
\]

Substitution of (19) into the formula (8) results in the estimation

\[
\bar{\beta} \approx \beta_0 \frac{\Psi_a}{\delta \Psi}.
\]

It can be seen from it that \( \bar{\beta} \) decreases with an increase in the scale of the disturbance; therefore, the largest-scale perturbations for a given value of \( \beta_0 \) can be stable, while small-scale disturbances will be unstable. Thus, the development
of ballooning instability will result in the establishment of a “smooth” radial pressure profile.

There is also a second conclusion, which can be drawn from the fact established in the above, that in the approximation of a vacuum magnetic field it is formally $\beta_{\text{crit}} \gtrsim 1$. Now we can assert that the critical $\beta_0$ in an axially symmetric mirror cell with a smooth pressure profile is not small in comparison with unity. To correctly calculate the critical $\beta_0$ it is necessary to abandon the vacuum field approximation, which is also called the low-pressure approximation. The necessary calculations are relatively easy to perform for paraxial mirror traps. These are described in the next section.

IV. PARAXIAL MIRROR TRAP

In the paraxial approximation, the potential energy of ballooning perturbations is found in the article by William Newcomb [5]. For an axially symmetric open trap with anisotropic plasma, Newcomb gives the following expression (his formula 177)

$$ W_F = \frac{1}{2} \int \int \int d\psi d\Omega \int d\zeta \left\{ \frac{Q}{B} \left( \frac{X^2}{r^2 B^2} + r^2 Y^2 \right) - \frac{2\zeta}{r B^2 \partial \psi} X^2 \right\} . $$

(21)

It is assumed here that the prime denotes the partial derivative with respect to $\zeta$, and all functions except for $B_{\text{vac}} = B_{\text{vac}}(\zeta)$, $X = X(\theta, \Psi, \zeta)$, and $Y = Y(\theta, \Psi, \zeta)$, depend on $\Psi$ and $\zeta$.

$$ r^2 = 2 \int_0^\Psi \frac{d\Psi}{B(\Psi, \zeta)}, \quad B^2 = B_{\text{vac}}(\zeta) - 8\pi p_\perp, \quad (22a) $$

$$ Q = B^2 + 4\pi (p_\perp - p_\parallel), \quad \bar{p} = \frac{p_\perp + p_\parallel}{2}, \quad (22b) $$

$$ X = \xi \cdot \nabla \Psi = r B \xi_\psi, \quad Y = \xi \cdot \nabla \theta = \frac{\xi_\theta}{r}, \quad (22c) $$

$$ \xi = r^{''}. \quad (22d) $$

Newcomb pointed out that the function $Q$ is positive under the condition of stabilization of the hose instability

$$ \beta_\parallel > 2 + \beta_\perp, \quad (23) $$

which was always assumed to be fulfilled. Otherwise, the first term in square brackets in the integrand (21) becomes negative (destabilizing), like the second term.

Minimizing $W_F$ over $Y$ gives $Y^\prime = 0$, which allows us to consider the term with $Y^2$ in (21) as zero. The calculations for this case were performed by Bushkova and Mirnov [3, 4]. They investigated the limit of isotropic plasma, when $p_\perp = p_\parallel = p(\psi)$ and rewrote Eq. (21) in the dimensionless form convenient for numerical calculations:

$$ W_F = \frac{\Delta \psi}{2\pi} \int d\zeta \left[ \frac{X^2}{r^2 B^2} - \beta_0 \frac{d^2 f}{r B^2 \partial \psi} X^2 \right], \quad (24) $$

where $\psi = \Psi / \Psi_\epsilon$,

$$ \beta_0 = 8\pi p_0 / B_{\text{vac}}^2, \quad f(\psi) = p(\psi) / p_0, \quad (25a) $$

$$ b = B / B_{\text{vac}}, \quad B_{\text{vac}} = B_{\text{vac}} / B_{\text{vac}0}, \quad (25b) $$

$$ b(\psi, \zeta) = \sqrt{B_{\text{vac}}(\zeta) - \beta_0 f(\psi)}, \quad (25c) $$

$$ r^2(\psi, \zeta) = 2 \int_0^\Psi \frac{d\psi}{\sqrt{B_{\text{vac}}(\zeta) - \beta_0 f(\psi)}}. \quad (25d) $$

Varying the integral (24) leads to the equation

$$ \frac{d}{d\zeta} \left( \frac{1}{r B} \frac{1}{d\zeta} \right) + \beta_0 \frac{d^2 f}{r B^2 \partial \psi} X = 0. \quad (26) $$

The same equation can be deduced from Eq. (18) in Ref. [8] if we put zero frequency, $\omega = 0$. Calculations were performed for the pressure profile

$$ f(\psi) = 1 - \psi^2, \quad (27) $$

which allows one to calculate the integral (25d) and write the equation of the field line:

$$ r^2(\psi, \zeta) = \frac{2}{\sqrt{\beta_0}} \log \frac{\sqrt{B_{\text{vac}}(\zeta)}^2 - \beta_0 (1 - \psi^2) + \sqrt{\beta_0} \psi}{\sqrt{B_{\text{vac}}(\zeta) - \beta_0}}. \quad (28) $$

The critical beta was calculated for the boundary field line at $\psi = 0.99$ for a family of axial profiles of the vacuum magnetic field

$$ b_{\text{vac}}(\zeta) = \left[ 1 - \left( 1 - K^{-\gamma/2} \right)^2 \right]^{-2/\gamma} \quad (29) $$

with mirror ratio $K = 100$ and a set of parameters $\mu$ and $\gamma$. In order to check our code, we recalculate their results assuming the boundary conditions

$$ X(0) = 1, \quad X(1) = 0 \quad (30) $$

and obtained similar results which are listed in Table I.

The performed calculations show that the profile (29), which is optimal for stabilizing flute modes at $\mu = 1, \gamma = 2$, becomes unstable with respect to ballooning vibrations at $\beta_0 > 0.359$. To increase the critical values of $\beta_0$, Bushkova and Mirnov proposed to go to steeper axial profiles of the magnetic field described by Eq. (29) with $\mu, \gamma \approx 5/6$. In this way, one can raise $\beta_{\text{crit}}$ to the values above 0.7.

To study the effect of the radial plasma pressure profile, we calculated the critical beta values for a smoother pressure profile, described by the function $f(\psi) = 1 - \psi$, and a steeper pressure profile, described by $f(\psi) = 1 - \psi^4$, with the same set of values for the parameters $\mu$ and $\gamma$ as in Table I. The calculation results are presented respectively in Tables II and III. As would be expected from the discussion in Section III, the smooth plasma pressure profile is more robust against small-scale ballooning disturbances. The calculation for a parabolic radial pressure profile (Table II) gave very high values of beta approaching unity. These values are obtained for large parameters $\mu$ and $\gamma$. They correspond to the axial profile of the magnetic field that grows very steeply near the magnetic mirror, which can hardly be formed in a real mirror trap. We made the calculation for such values of $\mu$ and $\gamma$ just to compare our results with the results of [3, 4].
Table I. Critical beta for local perturbations on the extreme field line $\psi = 1$ for $f(\psi) = 1 - \psi^2$, mirror ratio $K = 100$, and different values of the parameters $\mu$ and $\gamma$.

| $\gamma \setminus \mu$ | 1         | 2         | 3         | 4         | 5         | 6         |
|------------------------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.5                    | 0.189     | 0.332     | 0.383     | 0.409     | 0.424     | 0.434     |
| 2                      | 0.359     | 0.486     | 0.526     | 0.545     | 0.556     | 0.564     |
| 6                      | 0.595     | 0.682     | 0.706     | 0.718     | 0.724     | 0.729     |

Table II. Same as in Table I for more smooth radial plasma profile, $f(\psi) = 1 - \psi$.

| $\gamma \setminus \mu$ | 1         | 2         | 3         | 4         | 5         | 6         |
|------------------------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.5                    | 0.365     | 0.618     | 0.701     | 0.741     | 0.764     | 0.780     |
| 2                      | 0.646     | 0.832     | 0.882     | 0.905     | 0.918     | 0.927     |
| 6                      | 0.911     | 0.982     | 0.995     | 0.999     | 1.00      | 1.00      |

Table III. Same as in Table I for more sharp radial plasma profile, $f(\psi) = 1 - \psi^2$.

| $\gamma \setminus \mu$ | 1         | 2         | 3         | 4         | 5         | 6         |
|------------------------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.5                    | 0.0971    | 0.175     | 0.204     | 0.218     | 0.227     | 0.253     |
| 2                      | 0.193     | 0.269     | 0.294     | 0.306     | 0.313     | 0.318     |
| 6                      | 0.356     | 0.418     | 0.437     | 0.445     | 0.450     | 0.454     |

Figure 4. Axial profile of vacuum magnetic field in gas-dynamic trap.

V. GAS-DYNAMIC TRAP

The calculations performed in Section IV for the model field (29) can be repeated for more realistic profiles of the vacuum magnetic field along the trap axis. As the last example, let us calculate the critical beta for the Gas-Dynamic Trap [54, 55].

The vacuum magnetic field in the GDT is symmetric about the equatorial plane $z = 0$. Fig. 4 shows the axial profile of the magnetic field in one half of the GDT for two options for switching magnetic coils: the so-called standard version of the GDT and the GDT with a short pit (short mirror cell). In the second variant, a shallow local minimum of the magnetic field is formed near the stopping point of fast ions ($z \approx 140$ cm). A population of fast ions arises upon oblique injection of beams of neutral atoms at the angle of 45° to the axis of the trap.

Due to the presence of fast ions, the plasma pressure in the GDT is anisotropic, i.e., $p_r \neq p_\perp$, and the function $p_\perp$ depends not only on the magnetic flux $\psi$, but also on the magnitude of the magnetic field $B$. In the central section of the GDT (i.e., between the magnetic mirrors), function $p_\perp(\psi, B)$ has a maximum near the stopping point of fast ions, and in the expander (i.e., outside the central section) it decreases roughly proportional to $B$. As a result of all this, it turns out that Eq. (22a) cannot be solved analytically for $B$ and $r$, as in the case of an isotropic plasma. Thus, calculating the critical beta in anisotropic plasma requires much more complex calculations than the method described in Section IV allows. To simplify our task, we ignore the fact of anisotropy and still use the method from Section IV, taking into account that the plasma pressure varies only slightly along the magnetic field lines in the region $|z| \leq 400$ cm, where the magnetic field is not less than in the middle of the central section at $z = 0$ cm.

Since the axial profile of the magnetic field along the GDT axis is rigidly set by the configuration of the magnetic system, the only free parameter that can affect the critical beta is the coordinate $z_{end}$ of an imaginary or real conducting plasma diaphragm, on which the perturbation $X$ vanishes. In other words, we calculated the limit for beta by solving Eq. (26) with the boundary conditions

$$X(0) = 1, \quad X(z_{end}) = 0.$$  \hspace{1cm} (31)

The results of calculations are presented in Table IV for two magnetic field configurations and three values of $z_{end}$. The coordinate $z_{end} = 400$ cm corresponds to the placement of the plasma receiver in the expander, and $z_{end} = 350$ cm in the magnetic plug. Finally, $z_{end} = 329.5$ cm marks the coordinate of an actually existing conducting diaphragm, which is actually capable of playing the role of a stabilizer of small-scale flute disturbances on the lateral surface of the plasma column. Analysis of the data in the table shows that there is a very strong dependence of the critical value of beta on the steepness of the radial plasma pressure profile, but the dependence on the coordinate $z_{end}$ does not seem to be very significant. The second point to pay attention to is a noticeable decrease in critical beta in a GDT configuration with a short pit compared to the standard version of GDT. This is the expected result.

Table IV. Critical beta for local perturbations on the extreme field line $\psi = 1$ for three radial pressure profiles $f(\psi)$ and three positions $z_{end}$ of plasma limiter or receiver.

| $f(\psi) \setminus z_{end}$ | Standard GDT | GDT with Short Pit |
|-----------------------------|--------------|--------------------|
| $1 - \psi^2$                | 0.411        | 0.354              |
| $1 - \psi$                  | 0.719        | 0.690              |
| $1 - \psi^4$                | 0.229        | 0.218              |

$B$ (T)

| $z$ (cm) |
|----------|
| 0        |
| 1        |
| 5        |
| 10       |
| 50       |
| 100      |
| 200      |
| 300      |
| 400      |
| 500      |

GDT with Short Pit -- Standard GDT
On the other hand, comparison of the calculation results in table IV for $z_{end} = 350$ cm (they are in bold) with the calculation results for the model field (29) leads to an unexpected conclusion. When designing the GDT, the magnetic field was tuned so as to minimize the destabilizing contribution of the central section to the Rosenbluth-Longmire criterion of flute stability [56]. Such a minimum is provided by a magnetic field with the profile that is described by Eq. (29) for $\mu = 1$ and $\gamma = 2$. The corresponding critical beta values in tables I, II, III are also shown in bold. Due to the discrete structure of the magnetic system, the real magnetic field in the GDT is slightly rippled and only approximately follows the formula (29). This is clearly seen in Fig. 5, where it is shown that the curvature of the field line oscillates, whereas according to the formula (29) it should be a smooth function. We expected that curvature ripples would result in a decrease in critical beta, but according to table IV, the critical beta turned out to be slightly higher than in the tables I, II and III. We currently have no explanation for this fact, although it worth noting that the magnetic field ripples are known to improve stability of the $m = 1$ rigid ballooning mode [33].

VI. DISCUSSION

We have shown that steepening of the radial plasma pressure profile leads to a decrease in the critical value of beta, above which small-scale balloon-type disturbances in a mirror trap become unstable. It means that small-scale ballooning instability leads to a smoothing of the radial plasma profile. This fact does not seem to have received due attention in the available publications. It is known that the study of the stability of the global ballooning mode leads to the opposite conclusion. For example, Ref. [31] states that wall stabilization of the global ballooning mode is more effective for plasma with a hollow pressure profile.

We also calculated the critical beta values for the real magnetic field of the GDT in two configurations: a standard GDT and the GDT with a short well. For the version of the standard GDT, the calculation result turned out to be close to the results of calculations in the model used earlier by Bushkova and Mirnov [3, 4]. In the best case, the critical beta is 0.719. It is obtained for a plasma with a parabolic radial pressure profile, which was not analyzed in Refs. [3, 4]. The maximum beta, which has been achieved in experiments at the GDT to date, is 0.6 [41].

ACKNOWLEDGMENTS

The work was financially supported by the Ministry of Education and Science of the Russian Federation. This study was also supported by Chinese Academy of Sciences President’s International Fellowship Initiative (PIFI) under the Grant No. 2019VMA0024 and Chinese Academy of Sciences International Partnership Program under the Grant No. 116134KYSB20200001. The authors are grateful to Vladimir Mirnov for valuable explanations of the method he used to solve Eq. (26), and to Dmitry Yakovlev for providing the results of calculating the magnetic field in the GDT.

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