Two paths towards circulation time derivative
(Maxwell’s $\mathcal{E}$ revisited)

D V Redžić

1Faculty of Physics, University of Belgrade, PO Box 44, 11000 Beograd, Serbia
E-mail: redzic@ff.bg.ac.rs

Abstract. The time derivative of the circulation of a vector field $A$ over a moving and deforming closed curve, $\frac{d}{dt} \oint A \cdot dr$, is computed in two ways, with and without bringing the time derivative under the integral sign. As a by-product, the computations reveal that the conceptualization of Faraday’s law of electromagnetic induction may depend on which of the two methods is employed. The discussion presented provides an unexpected argument in favor of Maxwell’s mysterious choice for his electromotive intensity $\mathcal{E}$, made in Article 598 of his Treatise.

1. Introduction

Recently, we expounded how Maxwell had arrived, through an ingenious analysis of Faraday’s law of electromagnetic induction given in Article 598 of his Treatise [1], at a general expression for his electromotive intensity $\mathcal{E}$ in a moving medium:

$$\mathcal{E} = v \times B - \frac{\partial A}{\partial t} - \nabla \Psi;$$ (1)

here $v$ is the velocity of an infinitesimal portion (‘particle’) of the medium, $B = \nabla \times A$ is the magnetic flux density, $A$ is the vector potential, and a scalar field $\Psi$ is Maxwell’s electric potential [2]. We recalled that various authors claimed that Maxwell should have included a term $-\nabla (A \cdot v)$ in expression (1), as is strongly suggested by his derivation of Article 598. Namely, in Maxwell’s computation of the negative time derivative of the circulation of $A$, $-\frac{d}{dt} \oint A \cdot dr$, two terms of his result are expressed through $-\oint d(A \cdot v) = -\oint \nabla (A \cdot v) \cdot dr$. However, Maxwell mysteriously leaves out the gradient term $-\nabla (A \cdot v)$ in his final version of the integrand, noting simply that it vanishes when integrated round a closed curve, and introduces a brand-new term $-\nabla \Psi$, ‘for the sake of giving generality’ to the expression (1) for $\mathcal{E}$. The situation is even more curious, taking into account that the alternative expression for the electromotive intensity,

$$\mathcal{E}_{\text{HWT}} = v \times B - \frac{\partial A}{\partial t} - \nabla \Psi - \nabla (A \cdot v),$$ (2)

as proposed by Helmholtz [3], Watson [4], and J J Thomson [1] (vol 2, p 260), see also [5, 6], complies perfectly with Maxwell’s general principle of relativity applied to the Faraday’s law, as is demonstrated in [7].
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Perhaps surprisingly, it turns out that the appearance of the term $\nabla(A \cdot v)$ is an artefact of the specific path employed by Maxwell for computing $\frac{d}{dt} \oint A \cdot dr$. Namely, in an alternative computation path, the controversial term simply does not appear. This fact which seems to be little known, unfortunately, had escaped our attention during the writing of [2, 7], while it was implicit in [8].

In the present note, we first outline Maxwell’s computation of $\frac{d}{dt} \oint A \cdot dr$ over a moving and deforming closed curve, given in Article 598, which involves the non-obvious step of bringing the time derivative under the integral sign. Then we give the alternative, simpler computation of $\frac{d}{dt} \oint A \cdot dr$, applying the Kelvin-Stokes theorem twice, which avoids bringing the time derivative under the integral sign, and which is free from the term $\nabla(A \cdot v)$. Both computations could be useful from didactic point of view. Also, it could be inspiring for the student to learn that the conceptualization of Faraday’s law may depend on the specific path chosen for computing $\frac{d}{dt} \oint A \cdot dr$. Moreover, the simpler computation appears to provide an unexpected vindication of Maxwell’s happy and controversial choice for $E$, one of the key concepts of his electromagnetic theory and the progenitor of the Lorentz force expression.

2. Two paths for the computation of circulation time derivative

2.1. Maxwell’s path

For the sake of completeness, we outline Maxwell’s computation of the total time derivative of the circulation of an arbitrary, continuous and differentiable vector field $A(r, t)$ over a moving and deforming closed curve $C(t)$ at the instant $t$. Contrary to Maxwell, who writes everything in the Cartesian form, we employ the modern vector notation, benefiting from Hamilton’s operator $\nabla$, keeping, however, the spirit of Maxwell’s argument.

Maxwell writes the circulation as, in modern notation,

$$
\oint_{C(t)} A \cdot dr = \int_{0}^{s_{\text{max}}(t)} \left( A_x \frac{\partial x}{\partial s} + A_y \frac{\partial y}{\partial s} + A_z \frac{\partial z}{\partial s} \right) ds,
$$

where $r = r(s, t)$ is the position vector of a point of the contour, parameter $s$ is the arc length of the point considered at the instant $t$, and $s_{\text{max}}(t)$ is the total length of the contour at that instant. Since the circulation of $A$ refers to the fixed instant $t$, $dr$ is the partial differential of $r$ with respect to $s$, that is $dr = \frac{\partial r}{\partial s} ds \equiv d_s r$. Note that Maxwell takes tacitly that the parametrization which refers to the fixed instant $t$ suffices for describing the moving and deforming contour also in subsequent instants so that $s$ is time-independent. (This is of course correct; as can be seen, the fact that the total length of the moving contour is time-dependent is irrelevant, there is a bijection between the corresponding two sets of points.) Thus, a point $r(s, t)$ at the instant $t + dt$ becomes

† Maxwell’s original argument, free from $\nabla$, is presented in full detail in [2], and also, almost literally, and in a somewhat complemented form, in [9].
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\[ r(s, t + dt) = r(s, t) + v(s, t)dt \] where \( v(s, t) = \frac{\partial r(s, t)}{\partial t} \) is the instantaneous velocity of the point relative to the Cartesian coordinate system chosen.

To compute the time derivative of the circulation in the case of a moving and deforming contour \( C(t) \), Maxwell takes the time derivative inside the integral sign and thus

\[
\frac{d}{dt} \oint_{C(t)} A \cdot dr = \oint_{C(t)} \frac{dA}{dt} \cdot dr + \oint_{C(t)} (v \cdot \nabla)A \cdot dr + \oint_{C(t)} A \cdot dv.
\] (4)

The differentiations yield

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + (v \cdot \nabla)A,
\] (5)

and

\[
\frac{d}{dt} dr = dv,
\] (6)

where \( dv = \frac{\partial v}{\partial s} ds \equiv d_s v \), since \( s \) is time-independent. Maxwell thus obtains

\[
\frac{d}{dt} \oint_{C(t)} A \cdot dr = \oint_{C(t)} \frac{\partial A}{\partial t} \cdot dr + \oint_{C(t)} [(v \cdot \nabla)A] \cdot dr + \oint_{C(t)} A \cdot dv.
\] (7)

Equation (7) is, basically, Maxwell’s equation (2) of Art. 598, written in compact form, employing the modern vector notation.

Now express \( (v \cdot \nabla)A \) via the well-known vector identity

\[
v \times (\nabla \times A) = \nabla(v \cdot A^*) - (v \cdot \nabla)A,\]
(8)

where the asterisk in the expression \( \nabla(v \cdot A^*) \) indicates that \( \nabla \) operates only on \( A \). Employing also equation

\[
\nabla(v \cdot A^*) \cdot dr = d(v \cdot A^*) = v \cdot dA,
\] (9)

one obtains

\[
[(v \cdot \nabla)A] \cdot dr = [(\nabla \times A) \times v] \cdot dr + v \cdot dA,
\] (10)

Inserting (10) into (7) yields

\[
\frac{d}{dt} \oint_{C(t)} A \cdot dr = \oint_{C(t)} \left[ \frac{\partial A}{\partial t} + (\nabla \times A) \times v \right] \cdot dr + \oint_{C(t)} A \cdot dv.
\] (11)

\[\frac{\partial}{\partial t} \oint_{C(t)} A \cdot dr \] is nothing but Maxwell’s \( \left[ \frac{\partial}{\partial s} \left( \frac{\partial r}{\partial s} \right) \right] ds \) (clearly implicit in Art. 598), since \( s \) is time-independent.

§ Note that our expression \( \frac{\partial}{\partial t} \cdot dr \) is not very obvious and a proof is given in the appendix of [2], arriving at equation (7) directly from the definition of \( \frac{\partial}{\partial s} \oint_{C(t)} A \cdot dr \). An alternative proof, involving a renormalization of the variable \( s \) at each instant \( t \) is presented in [9]. While the renormalization procedure is mathematically expedient, it is not indispensable, the parametrization at one instant suffices, cf the appendix of [2]. As can be seen, another way of vindicating this step would be to invoke the Leibniz rule for differentiating an integral function, cf, e.g., [10], taking into account that \( s \) is time-independent.
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or, equivalently,

\[
\frac{d}{dt} \oint_{C(t)} A \cdot dr = \oint_{C(t)} \left[ \frac{\partial A}{\partial t} + (\nabla \times A) \times v + \nabla (A \cdot v) \right] \cdot dr. \tag{12}
\]

Finally, noting that the last integral in eq. (11) vanishes since it is taken round the closed curve, Maxwell arrives at

\[
\frac{d}{dt} \oint_{C(t)} A \cdot dr = \oint_{C(t)} \left[ \frac{\partial A}{\partial t} + (\nabla \times A) \times v \right] \cdot dr. \tag{13}
\]

Equation (13) is a purely mathematical and general result valid for an arbitrary moving and deforming closed curve \(C(t)\) that remains continuous and closed during its motion, and for arbitrary, continuous and differentiable vector field \(A(r, t)\) and velocity field \(v(r, t)\). Note that the appearance of the controversial term \(\nabla (A \cdot v)\) in eq. (12) is a consequence of computing \(\frac{d}{dt}(A \cdot dr)\).

2.2. The simpler path

Now we present a simpler computation of \(\frac{d}{dt} \oint_{C(t)} A \cdot dr\), which avoids bringing the time derivative under the integral sign, and avoids (explicit) parametrization of the curve \(C(t)\).

The time derivative of the circulation of \(A\) is by definition:

\[
\frac{d}{dt} \oint_{C(t)} A(r, t) \cdot dr = \frac{\oint_{C(t+dt)} A(r, t+dt) \cdot dr - \oint_{C(t)} A(r, t) \cdot dr}{dt}. \tag{14}
\]

A Taylor series expansion in the first integral on the right hand side of eq. (14) yields

\[
\oint_{C(t+dt)} A(r, t+dt) \cdot dr = \oint_{C(t+dt)} A(r, t) \cdot dr + \oint_{C(t+dt)} \frac{\partial A(r, t)}{\partial t} dt \cdot dr, \tag{15}
\]

and applying the Kelvin-Stokes theorem to the second integral on the right hand side of eq. (14) one has

\[
\oint_{C(t)} A(r, t) \cdot dr = \int_{S[C(t)]} [\nabla \times A(r, t)] \cdot dS, \tag{16}
\]

where \(S[C(t)]\) is any open surface bounded by the closed curve \(C(t)\). Choosing for \(S[C(t)]\) a surface which consists of a ribbon swept by the moving contour during the time interval \(dt\) and a surface \(S[C(t+dt)]\) (any open surface bounded by the closed curve \(C(t+dt)\)), the surface integral becomes

\[
\int_{S[C(t)]} [\nabla \times A(r, t)] \cdot dS = \oint_{C(t)} [\nabla \times A(r, t)] \cdot (dr \times v dt) + \int_{S[C(t+dt)]} [\nabla \times A(r, t)] \cdot dS, \tag{17}
\]

where \(v\) is the instantaneous velocity of the circuit element \(dr\) at the instant \(t\). Transforming the right hand side of eq. (17), rearranging terms in the first integral
through a cyclic permutation and applying the Kelvin-Stokes theorem to the second integral, one obtains
\[ \oint_{C(t)} A(r, t) \cdot dr = \oint_{C(t)} dr \cdot \{ vdt \times [\nabla \times A(r, t)] \} + \oint_{C(t+dt)} A(r, t) \cdot dr. \] (18)

Finally, inserting expressions (15) and (18) into the right-hand side of eq. (14), taking into account that
\[ \lim_{dt \to 0} \oint_{C(t+dt)} \frac{\partial A(r, t)}{\partial t} \cdot dr = \oint_{C(t)} \frac{\partial A(r, t)}{\partial t} \cdot dr, \] (19)
the result (13) follows.

3. Concluding comments

The above discussion reveals that the conceptualization of Faraday’s induction law may depend on the specific path employed for computing \( \frac{d}{dt} \oint A \cdot dr \). The simpler computation path, applying the Kelvin-Stokes theorem twice, which avoids bringing the time derivative under the integral sign, does not yield the controversial term \( \nabla (A \cdot v) \). Thus the issue of its inclusion into Maxwell’s original expression for the electromotive intensity \( \mathcal{E} \) is basically a pseudo-problem. Namely, it seems reasonable to take that a quantity whose appearance depends on the specific path chosen for computing the physical quantity, \( \frac{d}{dt} \oint A \cdot dr \), may have but a spurious physical meaning. Consequently, the present note provides an unexpected argument in favor of Maxwell’s mysterious choice for \( \mathcal{E} \).

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References

[1] Maxwell J C 1891 A Treatise on Electricity and Magnetism 3rd edn (Oxford: Clarendon) (reprinted 1954: New York: Dover)
[2] Redžić D V 2018 Maxwell’s inductions from Faraday’s induction law Eur. J. Phys. 39 025205
[3] Helmholtz H V 1874 Ueber die Theorie der Elektrodynamik. Dritte Abhandlung. Die elektrodynamischen Kräfte in bewegten Leitern. J. Reine Angew. Math. 78 273-324.
[4] Watson H W 1888 Note on the electromotive force in moving conductors Phil. Mag. 25 271–3
[5] Thomson J J 1893 Notes on Recent Researches in Electricity and Magnetism (Oxford: Clarendon) pp 534–43
[6] Buchwald J Z 2005 An error within a mistake? in Buchwald J Z and Franklin A, editors, 2005 Wrong for the Right Reasons (Dordrecht: Springer) pp 185-208
[7] Redžić D V 2018 On an episode in the life of \( \mathcal{E} \) Eur. J. Phys. 39 055206
[8] Redžić D V 2007 Faraday’s law via the magnetic vector potential Eur. J. Phys. 28 N7–N10
[9] Yaghjian A D 2020 Maxwell’s derivation of the Lorentz force from Faraday’s law Progress In Electromagnetics Research M 93 35–42
[10] Benedetto E 2017 Some remarks about flux time derivative Afr. Mat. 28 23–7