MOTIVIC ZETA FUNCTION VIA DLT MODIFICATION

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Abstract. Given a smooth variety $X$ and a regular function $f$ on it, by considering the dlt modification, we define the dlt motivic zeta function $Z_{\text{dlt}}^{\text{mot}}(s)$ which does not depend on the choice of the dlt modification.

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1. Introduction

Let $k$ be a field of characteristic 0. Kontsevich invented the concept of motivic integration, which upgrades the $p$-adic integration with the value in a modification of the Grothendieck ring $K_0(\text{Var}_k)$. One main application of motivic integration is to use it to define and study the motivic zeta function (see [DL99]), which is a corresponding upgrade of Igusa’s $p$-adic zeta function. More precisely, let $X$ be a smooth $k$-variety of pure dimension, and let $D = V(f)$ be a Cartier divisor of the zero divisor for a $k$-morphism to the affine line. We define the (naive) motivic zeta function $Z_{\text{mot}}(f,s)$ (or abbreviated as $Z_{\text{mot}}(s)$) of $(X,D)$ by

$$Z_{\text{mot}}(s) = \int_{\mathcal{L}(X)} \left[ L^{-\text{ord}_t(D)} s \right] \in \hat{\mathcal{M}}_k[[L^{-s}]].$$

Here $\text{ord}_t(D)$ is the function $\mathcal{L}(X) \to \mathbb{N} \cup \{\infty\}$ associated to the vanishing order of the jet along the divisor $D$, $\mathcal{M}_k = K_0(\text{Var}_k)(L^{-1})$ where $L = [A_k]$ and $\hat{\mathcal{M}}_k$ is the completion of $\mathcal{M}_k$ with respect to the decreasing filtration $F^m$ ($m \in \mathbb{Z}$) of $\mathcal{M}_k$, where $F^m$ is the subgroup generated by the elements $[S]/L^i$ with $S$ an algebraic variety and $\text{dim}(S) - i \leq -m$.

In [DL99], an explicit formula of motivic zeta function using a log resolution of $(X,D)$ is given: If $h : Y \to (X,D)$ is a log resolution which is isomorphic outside
Denote by $E_i$ ($i \in I$) the irreducible components of the divisor $E = h^{-1}(D)$, and by $(N_i, v_i)$ the corresponding pair of $(\text{mult}_{E_i}(h^*D), a(E_i, X) + 1)$, where $a(E_i, X)$ is the discrepancy. For any non-empty subset $J$ of $I$, we put $E_J = \cap_{i \in J} E_i$ and $E_J^0 = E_J \setminus \bigcup_{i \notin J} E_i$. We denote by $d$ the dimension of $X$. Then

$$Z_{\text{mot}}(s) = \mathbb{L}^{-d} \sum_{J \subseteq I} [E_J^0] \prod_{i \in J} \frac{(\mathbb{L} - 1)\mathbb{L}^{-N_i,s-v_i}}{1 - \mathbb{L}^{-N_i,s-v_i}} \in \mathcal{M}_k[[\mathbb{L}^{-s}]].$$

It is clear that the candidate of poles are of the form $s = -\frac{h_i}{N_i}$. (We remark that $\mathcal{M}_k$ is not a domain, so for the precise meaning of a pole, see Remark 2.1). However, many of them will cancel out. Thus how to determine their poles is a challenging question. In fact, the famous monodromy conjecture predicts that any pole $s$ of the motivic zeta function is indeed a root of Berstein-Sato polynomial $b_s(f)$. A weaker one predicts that $e^{2\pi i s}$ is an eigenvalue of the local monodromy action on the Milnor fiber of $f$ at some point $x \in D = V(f)$.

In this paper, by looking at the divisorial log terminal (dlt) modification of the pair $(X, D_{\text{red}})$, we will define an alternative zeta function with coefficients in a finite extension of $\mathcal{M}_k$. We recall that a dlt modification is a (often non-unique) partial resolution of a pair $(X^{\text{dlt}}, D^{\text{dlt}}) \to (X, D)$, which is introduced in the minimal model program (MMP) theory, and turns out to be very useful in the studying of singularities of pairs.

In fact, for a log canonical pair $(X, \Delta)$, an effective $\mathbb{Q}$-divisor $M$ on $X$ and any dlt pair $(X^{\text{dlt}}, D^{\text{dlt}})$, which admits a birational morphism $g : X^{\text{dlt}} \to X$, we will define an associated motivic zeta function. More precisely, we consider the stratification of $(X^{\text{dlt}}, D^{\text{dlt}})$ by its log canonical centers and write $[D^{\text{dlt}}] = \sum E_i$ ($i \in I$) the sum of reduced divisors. For $J \subseteq I$, we can similarly define $E_J^0 = \cap_{i \in J} E_i \setminus (\bigcup_{i \notin J} E_i)$ as the union of open stratum and write

$$(K_X^{\text{dlt}} + D^{\text{dlt}})|_{E_J^0} = K_{E_J^0} + D_J^0,$$

which is a disjoint union of klt pairs. Let $N_i = \text{ord}_{E_i}(g^*M)$ and $v_i$ be the log discrepancy $a(X, \Delta, E) + 1$ of $(X^{\text{dlt}}, D^{\text{dlt}})$ with respect to $(X, \Delta)$. We know

$$(N_i, v_i) \in \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0}, \quad \text{for any } i.$$

We assume

$$(N_i, v_i) \neq (0, 0)$$

for any divisor $E$. We assume $r$ is the least common multiple of the Cartier index of $(X, \Delta)$ and $M$, i.e., $r$ is the minimal positive integer such that both $r(K_X + \Delta)$ and $rM$ are Cartier.

**Definition 1.1.** We can define the dlt motive zeta function as following:

$$Z_{\text{mot}}^{\text{dlt}}(X^{\text{dlt}}, D^{\text{dlt}}, M, K_X + \Delta; s) := \mathbb{L}^{-d} \sum_{J \subseteq I} \mathcal{E}_{\text{st}}(E_J^0, D_J^0) \prod_{i \in J} \frac{(\mathbb{L} - 1)\mathbb{L}^{-N_i,s-v_i}}{1 - \mathbb{L}^{-N_i,s-v_i}} \in \mathcal{M}_k[[\mathbb{L}^{-s}]].$$

where $\mathcal{M}_k^\flat$ is a finite extension of $\mathcal{M}_k$ by adding a rational power of $\mathbb{L}^\frac{1}{r}$ and $\mathcal{E}_{\text{st}}(E_J^0, D_J^0)$ is the stringy motive (cf. [Bat99], [Vey03]).

Since usually we fix $M$ and $(X, \Delta)$ (but we may change $(X^{\text{dlt}}, D^{\text{dlt}})$), by abuse of notation, we will write it as $Z_{\text{mot}}^{\text{dlt}}(X^{\text{dlt}}, D^{\text{dlt}}; s)$. It has the usual Euler number
specialization
\[ Z_{\text{top}}^{\text{dlt}}(X^{\text{dlt}}, D^{\text{dlt}}; s) := \sum_{J \subset I} \chi_{\text{st}}(E^0_J, D^0_J) \prod_{i \in J} \frac{1}{N_i s + v_i} \in \mathbb{Q}(s). \]

Our main theorem is the following.

**Theorem 1.2.** For a fixed \( M \) and \((X, D)\) as above, if \((X^{\text{dlt}}_1, D^{\text{dlt}}_1)\) and \((X^{\text{dlt}}_2, D^{\text{dlt}}_2)\) are crepant birational equivalent dlt pairs (see (1.4)), we have
\[ Z_{\text{mot}}^{\text{dlt}}(X^{\text{dlt}}_1, D^{\text{dlt}}_1; s) = Z_{\text{mot}}^{\text{dlt}}(X^{\text{dlt}}_2, D^{\text{dlt}}_2; s). \]

Part of the above theorem implicitly implies that \((N_i, v_i) \neq (0, 0)\) for any \( i \in I \) for the components of \( D^{\text{dlt}}_1 \) if and only if the same holds for the components of \( D^{\text{dlt}}_2 \).

Let \( f \) be a regular function on a smooth variety \( X \), if we denote by \( g : (X^{\text{dlt}}, D^{\text{dlt}}) \to (X, D_{\text{red}}) \) a dlt modification of \((X, (f = 0)_{\text{red}}), \Delta = 0, M = g^*(f = 0)\) on \( X^{\text{dlt}} \) and we define the dlt motivic zeta function
\[ Z_{\text{mot}}^{\text{dlt}}(f; s) := Z_{\text{mot}}^{\text{dlt}}((X^{\text{dlt}}, D^{\text{dlt}}), M, K_X; s), \]

an immediate corollary is

**Corollary 1.3.** The dlt motivic zeta function \( Z_{\text{mot}}^{\text{dlt}}(f; s) \) does not depend on the choice of the dlt modification \((X^{\text{dlt}}, D^{\text{dlt}})\).

Since different dlt modifications are crepant birational equivalent to each other, studying it sometimes could yield a good understanding of singularities. For instance, it is proved in [dFKX12] that it can be used to define a finer topological invariant than the dual complex of a singularity. Our note is also motivated by this idea.

The paper is organized in the following way: We prove Theorem 1.2 by studying the stratification provided by the lc centers (see Definition 2.3) and compare two different ones in Section 3.1. This is inspired by the work in [dFKX12]. In Section 3.2, we point out a generalization of Batyrev’s stringy Euler number for non strictly log canonical singularities. In Section 3.3, we also discuss an analogue definition of the case for degenerations of Calabi-Yau manifolds, for which the original global motivic zeta function is studied in [HN11, HN12]. Finally, in Section 4, we post a few questions which we think deserving some further exploring.

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**Convention 1.4.** See [KM98] for the basic definition of the terminologies in birational geometry. Two pairs \((X_i, \Delta_i)\) \((i = 1, 2)\) are called crepant birationally equivalent if there are birational proper morphisms \( f_i : Y \to X_i \) such that
\[ f_i^*(K_{X_i} + \Delta_1) = f_2(K_{X_2} + \Delta_2). \]

2. Preliminary

In this section, we discuss some backgrounds.
2.1. Motivic zeta function. Motive zeta function is the motivic upgrade of Igusa’s \(p\)-adic zeta function which is first introduced in [DL99] using the technique of motivic integration (see [DL99, DL98]). We consider the ring \(\mathcal{M}_k\) which is \(K_0(\text{Var}_k)[[L^{-1}]]\) where \(K_0(\text{Var}_k)\) is the Gorendieck ring of varieties over \(k\) and \(L\) is the Lefschetz motive.

For a smooth variety \(X\) of pure dimension \(m\), and \(f : X \to \mathbb{A}^1\). Let \(L_n(X)\) be the \(n\)-jet of \(X\) and \(f_n\) the induced map on \(n\)-jet spaces by \(f\). We can define an object

\[Z(s) = \sum_{n \geq 0} [X_n] L^{-nm-ns} \in \mathcal{M}_k[[L^{-s}]],\]

where \([X_n] := \{ \gamma \in L_n(X) \mid \text{ord}_f \gamma = n \}\) for \(n \in \mathbb{Z}_{\geq 0}\).

As we mentioned in the introduction, an explicit formula of motivic zeta function can be given provided a log resolution of \((X, D)\): Fix \(h : Y \to (X, D)\) a log resolution which is isomorphic outside \(D\). Denote by \(E_i (i \in I)\) the irreducible components of the divisor \(E = h^{-1}(D)\), and by \((N_i, v_i)\) the corresponding pair of \((\text{mult}_{E_i}(h^*D), a(E_i, X) + 1)\), where \(a(E_i, X)\) is the discrepancy. For any non-empty subset \(J\) of \(I\), we put \(E_J = \bigcap_{i \in J} E_i\) and \(E_J^0 = E_J \setminus \bigcup_{i \notin J} E_i\). We denote by \(d\) the dimension of \(X\). Then

\[Z_{\text{mot}}(s) = L^{-d} \sum_{J \subset I} [E_J^0] \prod_{i \in J} \frac{(L^{-1})L^{-N_i s - v_i}}{1 - L^{-N_i s - v_i}} \in \mathcal{M}_k[[L^{-s}]].\]

**Remark 2.1.** Since the modified Gorendieck ring \(\mathcal{M}_k\) is not a domain, one should specify what is meant by a pole of a rational function over \(\mathcal{M}_k\). The definition we use is the following: if \(Z(L^{-s})\) is an element of

\[\mathcal{M}_k \left[ L^{-s}, \frac{1}{1 - L^{-a - bs}} \right]_{(a,b) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}} \subset \mathcal{M}_k[[L^{-s}]],\]

\(s_0\) is a rational number and \(m\) is a non-negative integer, then we say that \(Z(L^{-s})\) has a pole at \(s_0\) of order at most \(m\) if we find a set \(S\) consisting of multisets in \(\mathbb{Z} \times \mathbb{Z}_{\geq 0}\) such that each element of \(S\) contains at most \(m\) elements \((a, b)\) such that \(a/b = s_0\) and \(Z(L^{-s})\) belongs to the sub-\(\mathcal{M}_k[[L^{-s}]]\)-module of \(\mathcal{M}_k[[L^{-s}]]\) generated by

\[\left\{ \frac{1}{\prod_{(a,b) \in S} (1 - L^{-a - bs})} \mid S \in S \right\}.
\]

The same remark applies to \(\mathcal{M}_k^e\).

Another application of motivic integral is to use it to define the **stringy motive** \(\mathcal{E}_{\text{str}}(X, D)\) for a klt pair \((X, D)\) (see [Bat99, Vey03]).

**Definition 2.2.** For every klt pair \((X, D)\), let \(Y \to (X, D)\) is the log resolution and \(\{E_i\} (i \in I)\) is the set of exceptional divisors and the birational transform of components of \(D\), then we associated with an object

\[\sum_{J \subset I} [E_J^0] \prod_{i \in J} \frac{(1 - L)L^{-a_i}}{L^{-a_i} - 1}\]

where \(a_i = a(E_i, X, D) + 1\), which we call **stringy motive**.
This object does not depend on the choice of the log resolution and it lives in $\mathcal{M}_k$, which is a finite extension of $\mathcal{M}_k$ by adding the element $\mathbb{L}^{+}$, where $N$ is the Cartier index of $K_X + D$. Obviously, if $(X_i, D_i) \ (i = 1, 2)$ are crepant birational equivalent klt pairs, then we have

$$\mathcal{E}_a(X_1, D_1) = \mathcal{E}_a(X_2, D_2).$$

2.2. Dlt pairs. For the reader’s convenience, we give a short account of dlt singularities. For more background, see [KM98, Kol13].

**Definition 2.3** (dlt singularity and stratification). A normal variety $X$ with a $\mathbb{Q}$-divisor $D$ of coefficients belonging to $[0, 1]$ is called dlt if $K_X + D$ is $\mathbb{Q}$-Cartier, and there is an open set $U \subset X$ such that for any divisorial valuation $v$ of center contained in $X \setminus U$, we have the discrepancy $a(v, X, D) > -1$, and on $U$ we have that $D|_U$ only has coefficient 1 and $(U, D|_U)$ is simple normal crossing.

Given a dlt pair $(X, D)$, write $[D] = E = \sum_{i \in I} E_i$. Then we know for any $J \subset I$, the intersection $E_J = \cap_{i \in J} E_i$ is normal, and we call its components log canonical (lc) centers. If we denote a component $W$ of $E^0_J = E_J \setminus \bigcup_{i \notin J} E_i$ and $(K_X + D)|_W = K_W + D_W$, then $(W, D_W)$ is a klt pair. Furthermore, if we denote by $\bar{W}$ the closure, and write $(K_X + D)|_{\bar{W}} = K_{\bar{W}} + D_{\bar{W}}$, then $(\bar{W}, D_{\bar{W}})$ is dlt. We call the stratification of $X$ by components $W$ of $E^0_J$ the lc stratification. We call a component $W$ of $(E^0_J, D^0_J)$ an open strata or simply a strata.

Furthermore, we can define the dual complex $\mathcal{D}R(X, D) = \mathcal{D}R(E)$ as in [dFKX12, Definition 8], which captures the combinatorial intersection information of the stratum.

Obviously, the notion of a dlt pair is a generalization of a simple normal crossing (snc) pair. In fact, it was first defined precisely to characterize the kind of singularities we obtained after running MMP for an snc pair. The log canonical stratification is then a natural correspondence to the snc stratification.

**Definition 2.4** (dlt modification). Let $X$ be a normal variety and $D$ a $\mathbb{Q}$-divisor with coefficients in $[0, 1]$, then we say that $g^{\text{dlt}} : X^{\text{dlt}} \to (X, D)$ is a dlt modification if we write $D^{\text{dlt}}$ as the sum of the divisorial part of $g^{\text{dlt}}$ and the birational transform of $D$, then $(X^{\text{dlt}}, D^{\text{dlt}})$ is dlt and $K_{X^{\text{dlt}}} + D^{\text{dlt}}$ is nef over $X$.

For a pair $(X, D)$, the modification can be constructed by running an relative MMP for a log resolution, where the boundary is chosen to be the sum of the exceptional divisor and the birational transform $D$. When $K_X + D$ is $\mathbb{Q}$-Cartier, a dlt modification always exists by [OX12]. It is not unique in general, but we know that two dlt modifications $(X_i^{\text{dlt}}, D_i^{\text{dlt}}) \ (i = 1, 2)$ of $(X, D)$ are crepant birational equivalent to each other, following the proof of [KM98, 3.52] (see also [dFKX12, 15]).

3. DLT MOTIVIC ZETA FUNCTIONS

3.1. Proof of Theorem 1.2. Now we aim to prove Theorem 1.2 which says that the dlt motivic zeta function does not depend on the choice of the dlt modification $X^{\text{dlt}}$. 
Proposition 3.1. Let \((X_1, D_1)\) and \((X_2, D_2)\) be two crepant birational equivalent \(\text{dlt}\) pairs and the birational map \(X_1 \dashrightarrow X_2\) is isomorphic over an open set containing all the generic points of the stratum, then

\[
Z_{\text{mot}}^\text{dlt}(X_1, D_1, s) = Z_{\text{mot}}^\text{dlt}(X_2, D_2, s).
\]

Proof. By our assumption, there is a one-to-one correspondence between the stratum of \((X_1, D_1)\) and \((X_2, D_2)\). Since we know that \((X_i, D_i)\) are crepant birational equivalent to each other, that means for each strata \(W_i\) a component of \((E^0_s)_{1}\), the correspondence gives a unique strata \(W_2\) such that if we write

\[
(K_{X_i} + D_i)|_{W_i} = K_{W_i} + D_{W_i},
\]

then \((W_i, D_{W_i})\) are crepant birational equivalent. Thus they give the same stringy motive. So each corresponding summand in the expression of \(Z^\text{dlt}(X_i, \Delta_i)\) are equal to each other. \(\square\)

Proposition 3.2. Let \(g : X_1 \rightarrow X_2\) be a morphism of \(\text{dlt}\) modifications. Let \(U \subset X_2\) be an open set containing all lc centers of \((X_2, D_2)\) and satisfies \((U, \Delta_2|_U)\) is snc. Assume \((V = g^{-1}(U), D_1|_V) \rightarrow (U, D_2|_U)\) is a blow up of a strata. Then

\[
Z_{\text{mot}}^\text{dlt}(X_1, D_1, s) = Z_{\text{mot}}^\text{dlt}(X_2, D_2, s).
\]

Proof. Assume \(V \to U\) blow up a closed strata \(D^j_0 = D_j \cap U\) of \(U\).

Let \(J' \supset J\) and \(G \subset D^j_0\) be an open strata of \((X_2, D_2)\), thus

\[
G' := G \cap U \subset D'^j_0 := D^j_0 \cap U \subset D^j_0.
\]

Let \(W\) be the maximal strata of \(X_1\) which is over \(G\). Over a Zariski neighborhood \(G^*\) of the generic point of \(G\),

\[
W \cap g^{-1}(G^*) = G^* \times \mathbb{P}^m = \mathbb{P}^m_{G^*},
\]

where \(m = |J'| - 1\). Furthermore, the birational transform of the divisors \((D_1)_j\) \((i \in J')\) intersecting with \(W\) consists of \(m + 1\) coordinate hyperplane planes sections \((x_j = 0)\) on \(\mathbb{P}^m_{U_{J'}}\).

Write \((K_{X_2} + D_2)|_W = K'_W + D'_W\) and \((K_{X_1} + D_1)|_G = K_G + D_G\). We claim that \((W, D_W)\) is crepant birational equivalent to

\[
(G, D_G) \times (\mathbb{P}^m, T := \sum_{i=1}^m (x_i = 0)) := (G \times \mathbb{P}^m, D_G \times \mathbb{P}^m + G \times T).
\]

In fact, if we take \(p : W' \to W\) and \(q : W' \to G \times \mathbb{P}^m\) a common resolution, then we know that

\[
p^*(K'_W + D'_W) - q^*(K_G \times \mathbb{P}^m + G \times T)
\]

is a vertical divisor, hence it is the pull back of \(D_G\).

To compute \(Z_{\text{mot}}^\text{dlt}(X_2, \Delta_2, s) - Z_{\text{mot}}^\text{dlt}(X_1, \Delta_1, s)\), since the product of a log resolution of \((G, D_G)\) with \((\mathbb{P}^m, T)\) will also give a log resolution of \((G, D_G) \times (\mathbb{P}^m, T)\). We can compare the difference on each piece, and hence reduce to the case that \(G\) is a point.

Write \(J' = J \cup J_1\), where \(J\) and \(J_1\) are disjoint. In \(Z_{\text{mot}}^\text{dlt}(X_1, D_1; s)\) the contribution is

\[
\prod_{j \in J} \frac{(L - 1)_{L - N_{i,s,v_i}}}{1 - L_{-N_{i,s,v_i}}} \cdot \prod_{j \in J_1} \frac{(L - 1)_{L - N_{i,s,v_i}}}{1 - L_{-N_{i,s,v_i}}}.
\]
Let $E_0$ be the exceptional divisor $X_2 \rightarrow X_1$, then its multiplicity along $(f)$ will be $N_0 = \sum_{i \in J} N_i$ and by the log pull back formula $v_0 = \sum_{i \in J} v_i$. In $Z_{\text{mot}}^{\text{dlt}}(X_2, \Delta_2; s)$ the contribution is

$$
\sum_{K \subset J, K \neq J} \frac{(|L| - 1)^{|J| - 1}}{|L - N_0 s - v_0|} \prod_{j \in K} \frac{(L - 1)L^{-N_0 s - v_j}}{1 - L^{-N_0 s - v_j}} \prod_{j \in J} \frac{(L - 1)L^{-N_1 s - v_i}}{1 - L^{-N_1 s - v_i}}.
$$

This just follows from the simple equality

$$
\prod_{i \in J} \frac{1}{t_i - 1} = \frac{1}{\prod_{i \in J} t_i - 1} \cdot \sum_{K \subset J, K \neq J} \prod_{i \in J \setminus K} \frac{1}{t_i - 1}.
$$

Now we can prove Theorem 1.2.

**Proof of (1.2).** We can first take a log resolution $\tilde{X}_1$ of $(X_i, D_i)$. Furthermore, by the weak factorization theorem (see [AKMW02]), we can connect $\tilde{X}_1$ by

$$
\tilde{X}_1 = Y_1 \rightarrow \cdots \rightarrow Y_m = \tilde{X}_2,
$$

where $Y_i \rightarrow Y_{i+1}$ is either a blow up or an inverse, with an admissible center $G_i$. Assume $\psi_i : Y_{i+1} \rightarrow Y_i$ is a blow up, otherwise, we just reverse the arrow. We denote by $E_{Y_i}$ the sum of the birational transform of $D_i$ and the reduced exceptional divisor.

Running the relative MMP of $K_{Y_i} + E_{Y_i}$ over $X$ by [BCHM10], we obtain a dlt modification $X_i^{\text{dlt}} \rightarrow X$. If $Y_i \rightarrow X_i^{\text{dlt}}$ is not an isomorphism around $G_i$, then we know for the exceptional divisor $v_i$ of $\psi$, we have

$$a(v_i, X_i^{\text{dlt}}, D_i^{\text{dlt}}) > -1.
$$

Thus if we run MMP of $(Y_{i+1}, E_{i+1})$ over $X$, we obtain a dlt modification $X_{i+1}^{\text{dlt}}$ which is isomorphic to $X_i^{\text{dlt}}$ near the generic point of all the lc centers. Therefore, we can apply Proposition 3.1 to conclude that they give the same dlt motivic zeta function, i.e.,

$$Z_{\text{mot}}^{\text{dlt}}(X_{i+1}^{\text{dlt}}, \Delta_{i+1}^{\text{dlt}}, s) = Z_{\text{mot}}^{\text{dlt}}(X_i^{\text{dlt}}, \Delta_i^{\text{dlt}}, s).
$$

If $Y_i \rightarrow X_i^{\text{dlt}}$ is an isomorphism around $G_i$, then we know that we can find an open set $U_i \subset X_i^{\text{dlt}}$ such that if we take the blow up of $G_i|_{U_i}$, we get an open set of $X_{i+1}^{\text{dlt}}$ which also contains all the log canonical centers of $(X_{i+1}^{\text{dlt}}, D_{i+1}^{\text{dlt}})$. By [dFKX12, 36], we can extend $U_{i+1} \rightarrow U_i$ to a dlt modification $(X_{i+1}^{\text{dlt}}, (D_{i+1}^{\text{dlt}}))$ of $(X_i^{\text{dlt}}, D_i^{\text{dlt}})$. Thus we apply Proposition 3.2 and conclude that

$$Z_{\text{mot}}^{\text{dlt}}(X_i^{\text{dlt}}, \Delta_i^{\text{dlt}}, s) = Z_{\text{mot}}^{\text{dlt}}((X_{i+1}^{\text{dlt}}), (\Delta_{i+1}^{\text{dlt}}), s).
$$

Since $X_{i+1}^{\text{dlt}} \rightarrow (X_{i+1}^{\text{dlt}})$ is an isomorphism near the generic point of all the lc centers, we again know that

$$Z_{\text{mot}}^{\text{dlt}}(X_{i+1}^{\text{dlt}}, \Delta_{i+1}^{\text{dlt}}, s) = Z_{\text{mot}}^{\text{dlt}}((X_{i+1}^{\text{dlt}}), (\Delta_{i+1}^{\text{dlt}}), s).
$$

$\square$
3.2. Generalization of Batyrev’s stringy invariant. Let \((X, D)\) be a log pair such that \(K_X + D\) is \(\mathbb{Q}\)-Cartier. If we assume that \((X, D)\) does not have strictly log canonical singularities. Let \(g^{\text{dlt}} : (X^{\text{dlt}}, D^{\text{dlt}}) \to (X, D)\) be a dlt modification. We write
\[
K_{X^{\text{dlt}}} + D^{\text{dlt}} + M = (g^{\text{dlt}})^* (K_X + D),
\]
and \(M = \sum_i N_i E_i\), where the sum is over all the exceptional divisors \(E_i\) of \(g^{\text{dlt}}\). Then by our assumption that \((X, D)\) does not have strictly log canonical singularities, we know that the coefficient \(N_i\) of \(M\) along a \(g^{\text{dlt}}\)-exceptional component is always positive.

**Definition–Proposition 3.3.** We define the stringy motive for a log pair \((X, D)\) without strict log canonical singularities to be
\[
\mathcal{E}_{\text{st}}(X, D) = \mathcal{Z}_{\text{mot}}(X^{\text{dlt}}, D^{\text{dlt}}, N, K_{X^{\text{dlt}}} + D^{\text{dlt}}, 1) = L^{-d} \sum_{J \subseteq I} \mathcal{E}_{\text{st}}(E_J, D^0_J) \prod_{i \in J} \frac{(L - 1)L^{-N_i}}{1 - L^{-N_i}},
\]
and its Euler number \(\chi_{\text{st}}(X, D)\).

If \((X_i^{\text{dlt}}, D_i^{\text{dlt}}) \to (X, D)\) \((i = 1, 2)\) are two dlt modifications of \((X, D)\) and \(p_i : Y \to (X_i^{\text{dlt}}, D_i^{\text{dlt}})\) is a common resolution, then
\[
p_1^*(K_{X_1^{\text{dlt}}} + D_1^{\text{dlt}}) = p_2^*(K_{X_2^{\text{dlt}}} + D_2^{\text{dlt}}) \quad \text{and} \quad p_1^*(N_1) = p_2^* N_2.
\]
This is the only property we need for \(M\) being the pull back of \(X\) during the proof Theorem 1.2. Therefore, we conclude that the above definition does not depend on the choice of the dlt modification.

When \((X, D)\) is log terminal, then \(X^{\text{dlt}} = X\) and \(\mathcal{E}_{\text{st}}(X, D)\) is trivially equal to Batyrev’s original definition in [Bat99].

**Remark 3.4.** In [Vey03], Veys also attempted to give a generalization \(Z^V(r)\) of Batyrev’s stringy motive. The possibly most general category of singularities for which Veys can define a stringy invariant is also non strictly lc singularities. In Section 5 of [Vey03], a technical difficulty of defining a numerical invariant in the most general situation is also discussed. It appears since there are possibly log discrepancy 0 valuations even the singularity is not strictly lc. However, this difficulty does not appear in our definition as we only compute on dlt modifications.

3.3. Global setting for degenerations of Calabi-Yau. In [HN11, HN12], a global version of motivic zeta function for a degeneration of smooth Calabi-Yau variety is introduced. And a similar question as monodromy conjecture is asked. We can formulate the dlt motivic zeta function in this global case, which has the only pole being the minimal weight.

We will use the following formula as the definition of the motivic zeta function. For the original conceptual definition, we refer [HN11, HN12].

**Definition 3.5.** Let \(R\) be a DVR, whose function field \(K\) is the function field of a curve over \(k\). Let \(X/K\) be a Calabi-Yau projective smooth variety, i.e., \(K_X\) is trivial. Let \(\mathcal{X}\) be a sncd projective model of \(X\) over \(\text{Spec}(R)\), i.e., \(\mathcal{X} \times_{\text{Spec}(R)} \text{Spec}(K) = X\) and \((\mathcal{X}, (\mathcal{X}_0)_{\text{red}})\) is snc, where \((\mathcal{X}_0)_{\text{red}} = \sum_{i \in J} E_i\) is the special fiber. We fix a gauge form \(\omega\) on \(\mathcal{X}\), whose restriction on \(X\) gives the volume form of \(X\). Write \(\mathcal{X}_0 = \)
\[ \sum_{i \in I} N_i E_i \] and \( \mu_j - 1 \) the vanishing order of \( \omega \) along \( E_i \). Let \( E_J = \cap_{i \in J} E_i \), denote by \( \pi \) an parameter of \( \text{Spec}(R) \). Let \( N = \gcd\{N_i | \in J\} \) and \( \mathfrak{N} \) the normalization of \( \mathfrak{X} \times_{\text{Spec}(R)} \text{Spec}(R)/(x^N = \pi) \),

then we denote by \( \tilde{E}^0_J \) the preimage of \( E^0_J \) under the morphism \( \mathfrak{N} \to \mathfrak{X} \). And we know \( \tilde{E}^0_J \to E^0_J \) is an étale morphism.

Then we define

\[ Z_{X, \omega}(s) = \sum_{\emptyset \neq J \subset I} (L - 1)^{|J| - 1} \mathcal{E}_{st}(\tilde{E}^0_J) \prod_{i \in J} \frac{L - sN_i - \mu_j}{1 - L - sN_i - \mu_i}. \]

Now we can similarly define a dlt motivic zeta function. We call a minimal dlt model \( \mathfrak{X}^{\text{dlt}} \) of \( X/K \) if \( (\mathfrak{X}^{\text{dlt}}, \mathfrak{X}^{\text{dlt}})_{\text{red}} \) is dlt and \( K_{\mathfrak{X}^{\text{dlt}}} + \mathfrak{X}^{\text{dlt}} \sim_{\mathbb{Q}} 0 \). We note that since \( K_{\mathfrak{X}^{\text{dlt}}} \sim 0 \), semi-upper-continuity implies that in fact we have \( K_{\mathfrak{X}^{\text{dlt}}} + \mathfrak{X}^{\text{dlt}} \sim 0 \). It follows from the resolution of singularity and MMP ([BCHM10, HX13]) we know that such a minimal dlt model can be always obtained by running an MMP on \( (\mathfrak{X}, (\mathfrak{X})_{\text{red}}) \) from an sncd model \( \mathfrak{X} \). Write \( (\mathfrak{X}^{\text{dlt}})_{\text{red}} = \sum_{i \in I_0} E_i \) and \( \mathfrak{X}^{\text{dlt}} = \sum_{i \in I_0} N_i E_i \). For any \( J \subset I_0 \), let \( E_J = \cap_{i \in J} E_i \) and \( E^0_J = E_J \setminus \cup_{i \in J} E_i \). Let \( \tilde{E}^0_J \) be the preimage of the morphism \( \mathfrak{N} \to \mathfrak{X}^{\text{dlt}} \), where \( \mathfrak{N} \) is the normalization of \( \mathfrak{X}^{\text{dlt}} \times_{\text{Spec}(R)} \text{Spec}(R)/(x^N = \pi) \),

and \( N = \gcd\{N_i | \in J\} \). Then we know that if we write

\[ (K_{\mathfrak{X}^{\text{dlt}}} + \mathfrak{X}^{\text{dlt}})|_{\tilde{E}^0_J} = K_{\tilde{E}^0_J} + \tilde{D}^0_J, \]

which is a klt pair. Then we define

**Definition 3.6.** The dlt motivic zeta function of \( (X, \omega) \) is

\[ Z_{X, \omega}^{\text{dlt}}(s) = \sum_{\emptyset \neq J \subset I_0} (L - 1)^{|J| - 1} \mathcal{E}_{st}(\tilde{E}^0_J) \prod_{i \in J} \frac{L - sN_i - \mu_j}{1 - L - sN_i - \mu_i}, \]

where \( \mu_i \) is the vanishing order of \( \omega \) on \( E_i \).

Although two minimal dlt models are crepant birational to each other, we can not directly apply Theorem 1.2 because in the definition we use \( \tilde{E}^0_J \) instead of \( E^0_J \) itself. However we can apply the argument in the proof of Theorem 1.2. The only change in the proof is that when we blow up a strata of \( E^0_J \), in the terms appearing in the calculation of the motivic zeta function, the fiber over \( \tilde{E}^0_J \) is not \( (\mathbb{P}^n, T) \) but \( (W, T_W) \) where \( W \) is a weighted projective space and \( T_W \) is still the sum of all coordinate hyperplanes. Since in the motivic ring, we can easily inductively prove that the class given by any weighted projective space is the same as the one given by the projective space, then the rest of the proof will be verbatim. Therefore, we have

**Theorem 3.7.** \( Z_{X, \omega}^{\text{dlt}}(s) \) does not depend on the choice of minimal dlt models.

**Remark 3.8.** Since \( \mu_j - 1 \) is the vanishing order of \( \omega \) along \( E_i \) and \( K_{\mathfrak{X}^{\text{dlt}}} + \mathfrak{X}^{\text{dlt}} \sim 0 \), we know that for every \( i \in I_0, v_i / N_i \) is equal to each other which is the minimal value among \( \{v_i / N_i\} (i \in I) \).
4. Questions

The discussion in the previous sections leaves a few questions.

The first conceptually important one is to give an intrinsic construction of $Z_{\text{mot}}^\text{dlt}(f; s)$ using the motivic integration.

**Question 4.1.** Find a motivic integral definition of $Z_{\text{mot}}^\text{dlt}(f; s)$, without passing to its dlt modification.

Secondly, we are interested in the poles of $Z_{\text{mot}}^\text{dlt}(f; s)$.

We can ask similar questions as Monodromy Conjecture for $Z_{\text{mot}}^\text{dlt}(f; s)$:

**Question 4.2.** For a pole $s$ of $Z_{\text{mot}}^\text{dlt}(f; s)$, we conjecture the following is true:

(weak) $e^{2\pi is}$ is an eigenvalue of the local monodromy action on the Milnor fiber of $f$ at some point $x \in (f = 0)$,

(strong) $s$ is a root of Bernstein-Sato polynomial.

In general, the sets of poles of $Z_{\text{mot}}$ and $Z_{\text{mot}}^\text{dlt}$ could be different by the following simple example.

**Example 4.3.** Consider a general degree $d$ homogenous equation $f_d(x_1, \ldots, x_n) = 0$ and consider its zero locus $D$ in $\mathbb{A}^n$, which has a cone singularity at the original point.

When $d < n$, $(\mathbb{A}^n, D)$ is plt, and the only pole of $Z_{\text{mot}}^\text{dlt}$ is $-1$. To calculate the pole $Z_{\text{mot}}(s)$, we take the log resolution by blowing up the original point. Then we see that there is another pole $-\frac{n}{d}$ for $Z_{\text{mot}}(s)$.

When $d \geq n$, then the dlt modification as well as the log resolution is given by blowing up the original point, and so for both functions the poles are $-1$ and $-\frac{n}{d}$.

In fact, if we denote a dlt modification by

$$g^\text{dlt} : (X^\text{dlt}, D^\text{dlt}) \to (X, D_{\text{red}})$$

where $D = (f = 0)$, then by the definition of the dlt modification,

$$(g^\text{dlt})^*(K_X + D) \geq K_{X^\text{dlt}} + D^\text{dlt},$$

thus we know that the candidates of poles of $Z_{\text{mot}}^\text{dlt}(s)$ are between $[-1, 0]$. So a more precise question which describes the relation between two zeta functions can be asked as following:

**Question 4.4.** For a pole $s$ of $Z_{\text{mot}}^\text{dlt}(f, s)$, we ask whether the following questions are true:

(weak) The poles of the dlt motivic zeta function $Z_{\text{mot}}^\text{dlt}(f; s)$ are *always* poles of $Z_{\text{mot}}(f; s)$.

(strong) The poles of the dlt motivic zeta function $Z_{\text{mot}}^\text{dlt}(f; s)$ are *precisely* the poles of $Z_{\text{mot}}(f; s)$ contained in $[-1, 0]$.

We admit that we do not have many evidences of these questions. In the global setting for a degeneration of Calabi-Yau varieties, the only pole of $Z_{\text{mot}}^\text{dlt}(X, s)$ is the minimal weight, which we know always to be a pole of $Z_{\text{mot}}(X, s)$ (see Remark 3.8)).
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