COIDEALS, QUANTUM SUBGROUPS AND IDEMPOTENT STATES

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Abstract. We establish a one to one correspondence between idempotent states on a locally compact quantum group \( G \) and integrable coideals in the von Neumann algebra \( L^\infty(G) \) that are preserved by the scaling group. In particular we show that there is a one to one correspondence between idempotent states on \( G \) and \( \psi_G \)-expected left-invariant von Neumann subalgebras of \( L^\infty(G) \). We characterize idempotent states of Haar type as those corresponding to integrable normal coideals preserved by the scaling group. We also establish a one to one correspondence between open subgroups of \( G \) and central idempotent states on the dual \( \hat{G} \). Finally we characterize coideals corresponding to open quantum subgroups of \( G \) as those that are normal and admit an atom. As a byproduct of this study we get a number of universal lifting results for Podleś condition, normality and regularity and we generalize a number of results known before to hold under the coamenability assumption.

1. Introduction

Locally compact quantum groups theory is formulated in terms of operator algebras. The system of axioms becomes particularly simple when written in the language of von Neumann algebras [16]. In this case a locally compact quantum group is given by a von Neumann algebra equipped with a comultiplication and a pair of (left and right invariant) weights. Given a von Neumann quantum group, its \( C^\ast \)-algebraic version, which fits the \( C^\ast \)-system of axioms as formulated in [15] and [17], may be recovered. Conversely, a \( C^\ast \)-quantum group yields a von Neumann version, making \( C^\ast \) and von Neumann approaches equivalent.

Yet another face of a locally compact quantum group is given by its universal \( C^\ast \)-counterpart [14] which is directly linked with representation theory of the dual locally compact quantum group. The duality here extends the famous Pontryagin duality discovered in the context of abelian locally compact groups. In what follows a quantum group will be denoted by \( G \), its von Neumann algebra by \( L^\infty(G) \), its reduced \( C^\ast \)-algebra by \( C_0(G) \) and the universal \( C^\ast \)-algebra by \( C_u^\ast(G) \).

A locally compact quantum group can be studied through its representation theory and its actions on \( C^\ast \)-algebras and von Neumann algebras. A distinguished class of actions is given by taking the quantum quotient of a locally compact quantum group \( G \) by its closed quantum subgroup \( H \). This class was considered in [25]: it is worth mentioning that the von Neumann quotient \( L^\infty(G/H) \) can always be easily formed whereas the existence of the \( C^\ast \)-quotient \( C_0(G/H) \) is more subtle issue and in general it was proved under the regularity assumption on \( G \). If \( H \subset G \) is compact \( C_0(G/H) \) can always be formed, [13].

In this paper we study the actions of locally compact quantum groups that correspond to idempotent states (see [21]). The latter can be viewed as a generalization of the quantum quotient by a compact quantum subgroup. In particular, an idempotent state \( \omega \) on \( G \) gives rise to a von Neumann coideal \( N \subset L^\infty(G) \) which in the subgroup case is the quotient \( L^\infty(G/H) \) by a compact quantum subgroups \( H \subset G \). We give a von Neumann characterization of \( N \subset L^\infty(G) \) corresponding to an idempotent state in terms of the integrability of the \( G \)-action on \( N \). We also extend beyond the coamenable case the characterization of \( C^\ast \)-subalgebras \( X \subset C_u^\ast(G) \) corresponding to compact quantum subgroups \( H \subset G \) by taking the quotient \( X = C_0^\ast(G/H) \) (see [19]) and we formulate the von Neumann counterpart of this result. Finally we characterize subalgebras \( N \subset L^\infty(G) \) which are of the form \( L^\infty(G/H) \) with \( H \subset G \) being an open quantum subgroup. As a byproduct of study

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The multiplicative unitary $W$ of a compact quantum subgroup. Our techniques are very similar to those developed in [19] but we used throughout the paper (see [15, Proposition 6.8]).

In Section 6 we establish a 1-1 correspondence between open quantum subgroups of $G$ being a compact quantum subgroup. In Section 6 we establish a 1-1 correspondence between idempotent states on $G$ and central idempotent states on $G$. For the theory of locally compact quantum groups we refer to [14, 15, 16]. Let us recall that a closed quantum subgroup $H$ will be denoted by $\hat{H}$.

In Section 3 we lift some results that hold for regular quantum groups from the reduced to the universal level. As an application we describe the universal lift $C_0(G/H)$ of the $C^*$-quotient $C_0(G/H)$ for a closed quantum subgroup $H \subset G$. In Section 4 a 1-1 correspondence between idempotent states on a locally compact quantum group $G$ and integrable coideals in the von Neumann algebra $L^\infty(G)$ that are preserved by the scaling group is established. Using this result we were able to weaken the assumptions of [21, Theorem 1] and show that there is a 1-1 correspondence between idempotent states on $G$ and $\psi_G$-expected left-invariant von Neumann subalgebras of $L^\infty(G)$ (see Remark 4.4). Section 5 is divided into two parts. The first one is the characterization of $C^*$-subalgebras $X \subset C_0^\infty(G)$ which are of the form $C_0^\infty(G/H)$ where $H \subset G$ is a compact quantum subgroup. Our techniques are very similar to those developed in [19] but we were able to drop the coamenability assumption. In the second part of Section 5 we characterize von Neumann subalgebras $N \subset L^\infty(G)$ which are of the form $N = L^\infty(G/H)$ still with $H \subset G$ being a compact quantum subgroup. In Section 6 we establish a 1-1 correspondence between open quantum subgroups of $G$ and central idempotent states on $\hat{G}$. In Section 7 we characterize coideal subalgebras $N \subset L^\infty(G)$ which are of the form $N = L^\infty(G/H)$ for an open quantum subgroup $H \subset G$. In the Appendix we extend beyond the coamenable case the result proved in [6], stating that a closed quantum subgroups $H \subset G$ has a Haagerup property if $G$ has it.

2. Preliminaries

We will denote the minimal tensor product of $C^*$-algebras with the symbol $\otimes$. The ultraweak tensor product of von Neumann algebras will be denoted by $\bar{\otimes}$. For a $C^*$-subalgebra $B$ of a $C^*$-algebra $A$ the multipliers $M(A)$ of $A$, the closed linear span of the set $\{ba \mid b \in B, a \in A\}$ will be denoted by $BA$. A morphism between two $C^*$-algebras $A$ and $B$ is a $*$-homomorphism $\pi$ from $A$ into the multiplier algebra $M(B)$, which is non-degenerate, i.e. $\pi(A)B = B$. We will denote the set of all morphisms from $A$ to $B$ by $\text{Mor}(A,B)$. The non-degeneracy of a morphism $\pi$ yields its natural extension to the unital $*$-homomorphism $M(A) \to M(B)$ also denoted by $\pi$. Let $B$ be a $C^*$-subalgebra of $M(A)$. We say that $B$ is non-degenerate if $BA = A$. In this case $M(B)$ can be identified with a $C^*$-subalgebra of $M(A)$. The symbol $\sigma$ will denote the flip morphism between $*$-homomorphisms of tensor product of operator algebras. If $X$ is a subset of topological vector space $Y$, by $X^{\text{cl}}$ we mean the closed linear span of $X$. In particular if $X \subset A$, where $A$ is a $C^*$-algebra then $X^{\text{norm-cl}}$ denotes the norm closure of the linear span of $X$; if $X \subset M$, where $M$ is a von Neumann algebra then $X^{\sigma\text{-weak cl}}$ denotes the $\sigma$-weak closure of the linear span of $X$. For a $C^*$-algebra $A$, the space of all functionals on $A$ and the state space of $A$ will be denoted by $A^*$ and $S(A)$ respectively. The predual of a von Neumann algebra $N$ will be denoted by $N_*$. For a Hilbert space $H$ the $C^*$-algebras of compact operators on $H$ will be denoted by $K(H)$. The algebra of bounded operators acting on $H$ will be denoted by $B(H)$. For $\xi, \eta \in H$, the symbol $1_{\xi,\eta} \in B(H)_*$ is the functional $T \mapsto (\xi, T\eta)$.

For the theory of locally compact quantum groups we refer to [14, 15, 16]. Let us recall that a von Neumann algebra locally compact quantum group is a quadruple $G = (L^\infty(G), \Delta_G, \varphi_G, \psi_G)$, where $L^\infty(G)$ is a von Neumann algebra with a coassociative comultiplication $\Delta_G : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$, and $\varphi_G$ and $\psi_G$ are, respectively, normal semifinite faithful left and right Haar weights on $L^\infty(G)$. The GNS Hilbert space of the right Haar weight $\psi_G$ will be denoted by $L^2(G)$ and the corresponding GNS map will be denoted by $\eta_G$. The antipode, the scaling group and the unitary antipode will be denoted by $S, (\tau_t)_{t \in \mathbb{R}}$ and $R$. We will denote $(\sigma_t)_{t \in \mathbb{R}}$ and $(\sigma'_t)_{t \in \mathbb{R}}$ the modular automorphism groups assigned to $\varphi_G$ and $\psi_G$ respectively. The following relation will be used throughout the paper (see [15, Proposition 6.8])

$$\Delta_G \circ \tau_t = (\sigma_t \otimes \sigma'_{-t}) \circ \Delta_G. \quad (2.1)$$

The multiplicative unitary $W^G \in B(L^2(G) \otimes L^2(G))$ is a unique unitary operator such that

$$W^G(\eta_G(x) \otimes \eta_G(y)) = (\eta_G \otimes \eta_G)(\Delta_G(x)(\mathbb{1} \otimes y))$$
for all \(x, y \in D(\eta_G); W^G\) satisfies the pentagonal equation \(W^G_{12}W^G_{13}W^G_{23} = W^G_{23}W^G_{12}\) [2][27]. Using \(W^G\), \(G\) can be recovered as follows:

\[
L^\infty(\hat{G}) = \{\omega \in B(L^2(G)) \}_{\omega}\text{-weak cls},
\]

\[
\Delta_G(x) = W^G(x \otimes 1)W^G*.
\]

A locally compact quantum group admits a dual object \(\hat{G}\). It can be described in terms of \(W^\hat{G}\)

\[
L^\infty(\hat{G}) = \{\omega \in B(L^2(G)) \}_{\omega}\text{-weak cls},
\]

\[
\Delta_G(x) = W^G(x \otimes 1)W^G*.
\]

where \(W^\hat{G} = \sigma(W^G)^*\). Note that \(W^\hat{G} \in L^\infty(\hat{G}) \otimes L^\infty(G)\). The modular element of \(G\) will be denoted by \(\delta\).

**Definition 2.1.** A von Neumann subalgebra \(N\) of \(L^\infty(G)\) is called

- *Left coideal* if \(\Delta_G(N) \subseteq L^\infty(G) \otimes N\);
- *Invariant subalgebra* if \(\Delta_G(N) \subseteq N \otimes N\);
- *Baaï-Vaes subalgebra* if \(N\) is an invariant subalgebra of \(L^\infty(\hat{G})\) which is preserved by the unitary antipode \(R\) and the scaling group \((\tau_t)_{t \in R}\) of \(G\);
- *Normal* if \(W^G(G \otimes N)W^G* \subseteq L^\infty(\hat{G}) \otimes N\);
- *Integrable* if the set of integrable elements with respect to the right Haar weight \(\psi_G\) is dense in \(N^+\); in other words, the restriction of \(\psi_G\) to \(N\) is semifinite.

Using terminology of [21] a left coideal is nothing but a left-invariant von Neumann subalgebra of \(L^\infty(G)\). In what follows a left coideal will be called a coideal. If \(N\) is a coideal of \(L^\infty(G)\), then \(\tilde{N} = N' \cap L^\infty(\hat{G})\) is a coideal of \(L^\infty(\hat{G})\) called the codual of \(N\); it turns out that \(\tilde{N} = N\) (see [13] Theorem 3.9).

The C*-algebraic version \((C_0(G), \Delta_G)\) of a given quantum group \(G\) is recovered from \(W^G\) as follows

\[
C_0(G) = \{\omega \in B(L^2(G)) \}_{\omega}\text{-norm cls},
\]

\[
\Delta_G(x) = W^G(x \otimes 1)W^G*.
\]

The comultiplication can be viewed as a morphism \(\Delta_G \in \text{Mor}(C_0(G), C_0(G) \otimes C_0(G))\) and we have \(W^\hat{G} \in M(C_0(G) \otimes C_0(G))\).

**Definition 2.2.** A non-degenerate C*-subalgebra \(B\) of \(M(C_0(G))\)

- is called *left-invariant* if \((\mu \otimes \text{id})\Delta_G(B) \subseteq B\) for all \(\mu \in C_0(G)^*\);
- is called *symmetric* if \(W^G(G \otimes B)W^G* \subseteq M(C_0(\hat{G}) \otimes B);\)
- satisfies *Podleš condition* if \(\Delta_G(B)(C_0(G) \otimes 1) = C_0(G) \otimes B;\)
- satisfies *weak Podleš condition* if \((\tilde{C_0(G)} \otimes 1)\Delta_G(B)(C_0(G) \otimes 1) = C_0(G) \otimes B\)

Let us note that Podleš condition \(\implies\) weak Podleš condition \(\implies\) left-invariance.

We adopt the following terminology from [21 Section 1].

**Definition 2.3.** Let \(B\) be a C*-subalgebra of \(C_0(G)\). We say that

(i) \(B\) is \(\varphi_G\)-expected if there exists a \(\varphi_G\)-preserving conditional expectation \(E\) from \(C_0(G)\) onto \(B\);

(ii) \(B\) is \(\psi_G\)-expected if there exists a \(\psi_G\)-preserving conditional expectation \(E\) from \(C_0(G)\) onto \(B\);

(iii) \(B\) is expected if there exists a conditional expectation \(E\) from \(C_0(G)\) onto \(B\), which preserves \(\psi_G\) and \(\varphi_G\).

Let \(N\) be a von Neumann subalgebra of \(L^\infty(G)\). We say that

(i) \(N\) is \(\varphi_G\)-expected if there exists a \(\varphi_G\)-preserving conditional expectation \(E\) from \(L^\infty(G)\) onto \(N\);
(ii) \( N \) is \( \psi_G \)-expected if there exists a \( \psi_G \)-preserving conditional expectation \( E \) from \( L^\infty(G) \) onto \( N \);

(iii) \( N \) is expected if there exists a conditional expectation \( E \) from \( L^\infty(G) \) onto \( N \), which preserves \( \psi_G \) and \( \varphi_G \).

We will show (see Proposition 3.1) that a non-zero \( C^* \)-subalgebra \( B \subset M(C_0(G)) \) such that \( (\mu \otimes \id)\Delta_G(b) \in B \) for all \( \mu \in C_0(G)^* \) and \( b \in B \) is automatically non-degenerate. In particular a non-zero \( C^* \)-subalgebra \( B \subset M(C_0(G)) \) satisfying Podleś condition is non-degenerate. Proposition 2.6 provides a link between weak Podleś condition discussed in [3 Section 5] (called weak continuity) and weak Podleś condition introduced in Definition 2.2. Let us recall the definition of an action of a quantum group \( G \).

**Definition 2.4.** A (left) action of quantum group \( G \) on

- von Neumann algebra \( N \) is a unital injective normal *-homomorphism \( \alpha : N \to L^\infty(G) \otimes N \) s.t. \( (\Delta_G \otimes \id) \circ \alpha = (\id \otimes \alpha) \circ \alpha \).
- \( C^* \)-algebra \( B \) is an injective morphism \( \alpha \in \mathrm{Mor}(B, C_0(G) \otimes B) \) s.t. \( (\Delta_G \otimes \id) \circ \alpha = (\id \otimes \alpha) \circ \alpha \).

Let us remark, that some authors define \( C^* \)-actions as (not necessarily injective) morphisms \( \alpha \in \mathrm{Mor}(B, C_0(G) \otimes B) \) satisfying \( (\Delta_G \otimes \id) \circ \alpha = (\id \otimes \alpha) \circ \alpha \) and Podleś condition (see Definition 2.5). For a nice discussion of Podleś condition see [22].

In the course of this paper we shall use the action \( \beta : C_0(G) \to M(C_0(\hat{G}) \otimes C_0(\hat{G})) \) of \( \hat{G} \) on \( C_0(G) \), where

\[
\beta(x) = W^G(1 \otimes x)W^{G^*}.
\]

Note that \( \beta \) admits a von Neumann extension (which we shall also denote by \( \beta \)):

\[
\beta(x) = W^G(1 \otimes x)W^{G^*} \in L^\infty(\hat{G}) \otimes L^\infty(G)
\]

for all \( x \in L^\infty(G) \).

**Definition 2.5.** Let \( \alpha \in \mathrm{Mor}(B, C_0(G) \otimes B) \) be an action of \( G \) on a \( C^* \)-algebra \( B \). We say that \( B \) satisfies

- **\( \alpha \)-Podleś condition** if \( \alpha(B)(C_0(G) \otimes 1) = C_0(G) \otimes B \);
- **\( \alpha \)-weak Podleś condition** if \( B = \{ (\omega \otimes \id)\alpha(B) \mid \omega \in B(L^2(G))_* \} \) norm-cls.

Let us note that \( \alpha \)-Podleś condition \( \implies \) \( \alpha \)-weak Podleś condition.

**Proposition 2.6.** Let \( \alpha \) be an action of \( G \) on a \( C^* \)-algebra \( B \). Then \( B \) satisfies \( \alpha \)-weak Podleś condition if and only if

\[
(C_0(G) \otimes 1)\alpha(B)(C_0(G) \otimes 1) = C_0(G) \otimes B.
\]

**Proof.** The “if” part is clear. In order to get the “only if” we compute

\[
(C_0(G) \otimes 1)\alpha(B)(C_0(G) \otimes 1) = \{ (\omega \otimes \id)\alpha(B)(C_0(G) \otimes 1) \mid \omega \in B(L^2(G))_* \} \text{ norm-cls}
\]

\[
= \{ (\omega \otimes \id)(\alpha(B))_1(\alpha(B))_2(\alpha(B))_3(\omega \otimes x \otimes 1) \mid \omega \in B(L^2(G))_* \}
\]

\[
= \{ (\omega \otimes \id)(y \otimes x \otimes 1) \alpha(B)(y' \otimes x' \otimes 1) \mid \omega \in B(L^2(G))_* \}
\]

\[
= \{ (\omega \otimes \id)(y \otimes x \otimes 1) \alpha(B)(y' \otimes x' \otimes 1) \mid \omega \in B(L^2(G))_* \}
\]

\[
= C_0(G) \otimes B.
\]

A locally compact quantum group \( G \) is assigned with a universal version [14]. The universal version \( C_0^u(G) \) of \( C_0(G) \) is equipped with a comultiplication \( \Delta^u \in \mathrm{Mor}(C_0^u(G) \otimes C_0^u(G) \otimes C_0^u(G)) \) satisfying (see [14] Proposition 6.1)

\[
\Delta^u(C_0^u(G))(C_0^u(G) \otimes 1) = C_0^u(G) \otimes C_0^u(G) = \Delta^u(C_0^u(G))(1 \otimes C_0^u(G))
\]
which will be also referred to as Podleś condition. The counit is a $*$-homomorphism $\varepsilon : C^*_0(G) \to \mathbb{C}$ satisfying $(id \otimes \varepsilon) \circ \Delta^u_0 = id = (\varepsilon \otimes id) \circ \Delta^u_0$. Multiplicative unitary $W^G \in M(C_0(G) \otimes C_0(G))$ admits the universal lift $W^G \in M(C^u_0(\hat{G}) \otimes C^u_0(\hat{G}))$. The reducing morphisms for $G$ and $\hat{G}$ will be denoted by $A_G \in \text{Mor}(C^u_0(\hat{G}), C_0(G))$ and $\hat{A}_G \in \text{Mor}(C^u_0(\hat{G}), C_0(\hat{G}))$ respectively. We have $(A_G \otimes id)(W^G) = W^G$. We shall also use the half-lifted versions of $W^G$, $W^G = (id \otimes \hat{A}_G)(W^G) \in M(C^u_0(\hat{G}) \otimes C_0(G))$ and $W^G = (\hat{A}_G \otimes id)(W^G) \in M(C_0(\hat{G}) \otimes C^u_0(\hat{G}))$. They satisfy the appropriate versions of pentagonal equation

$$W^G_{12}W^G_{13}W^G_{23} = W^G_{23}W^G_{12},$$

$$W^G_{12}W^G_{13}W^G_{23} = W^G_{23}W^G_{12}.$$ 

The half-lifted versions of comultiplications will be denoted by $\Delta^u_r \in \text{Mor}(C^u_0(\hat{G}), C_0(G) \otimes C^u_0(\hat{G}))$ and $\hat{\Delta}^u_r \in \text{Mor}(C_0(\hat{G}), C_0(G) \otimes C^u_0(\hat{G}))$, e.g.

$$\Delta^u_r(x) = W^G(x \otimes 1)W^G, \quad x \in C_0(G).$$

We have

$$(A_G \otimes id) \circ \Delta^u_0 = \Delta^u_r \circ A_G,$$

$$(\hat{A}_G \otimes id) \circ \Delta^u_0 = \hat{\Delta}^u_r \circ A_G. \quad (2.4)$$

The following forms of Podleś conditions are satisfied

$$\Delta^u_r(C_0(G))(C_0(G) \otimes 1) = C_0(G) \otimes C^u_0(G),$$

$$\hat{\Delta}^u_r(C_0(\hat{G}))(C_0(\hat{G}) \otimes 1) = C_0(\hat{G}) \otimes C^u_0(\hat{G}).$$

We shall consider $C^*$-subalgebras $B$ of $M(C^u_0(\hat{G}))$ and the following terminology.

**Definition 2.7.** A non-degenerate $C^*$-subalgebra $B$ of $M(C^u_0(\hat{G}))$

- is called *left-invariant* if $(\mu \otimes id)\Delta^u_0(B) \subset B$ for all $\mu \in C^u_0(\hat{G})$;
- is called *symmetric* if $W^G(1 \otimes B)W^G \subset M(C_0(\hat{G}) \otimes B)$;
- satisfies *Podleś condition* if $\Delta^u_0(B)(C_0(\hat{G}) \otimes 1) = C_0(\hat{G}) \otimes B$;
- satisfies *weak Podleś condition* if $(C^u_0(G) \otimes 1)\Delta^u_0(B)(C^u_0(G) \otimes 1) = C^u_0(G) \otimes B$.

Furthermore we adopt the following

**Definition 2.8.** Let $G$ be a locally compact quantum group, $B$ a $C^*$-algebra and let $\alpha \in \text{Mor}(B, C^u_0(G) \otimes B)$ be such that $(id \otimes \alpha) \circ \Delta^u_0 = (\Delta^u_0 \otimes id) \circ \alpha$. We say that $B$ satisfies

- *$\alpha$-Podleś condition* if $\alpha(B)(C^u_0(G) \otimes 1) = C^u_0(G) \otimes B$;
- *$\alpha$-weak Podleś condition* if $B = \{(w \otimes id)\alpha(B) \mid w \in C^u_0(G)^*\}^{\text{norm-cls}}$.

Let us note that $\alpha$-Podleś condition $\implies$ $\alpha$-weak Podleś condition.

Given a locally compact quantum group $G$, the comultiplications $\Delta_G$ and $\Delta_G^u$ induce Banach algebra structures on $L^\infty(G)_*$ and $C_u^G(G)^*$ respectively. The corresponding multiplications will be denoted by $\cdot$ and $\bar{\cdot}$. We shall identify $L^\infty(G)_*$ with a subspace of $C_u^G(G)^*$ when convenient. Under this identification, $L^\infty(G)_*$ forms a two sided ideal in $C_u^G(G)^*$. Following [14], for any $\mu \in C_u^G(G)^*$ we define a normal map $L^\infty(G) \to L^\infty(G)$ such that $x \mapsto (id \otimes \mu)((W^G(x \otimes 1)W^G)^*)$ for all $x \in L^\infty(G)$. We shall use a notation $\mu \bar{x} = (id \otimes \mu)((W^G(x \otimes 1)W^G)^*)$.

A state $\omega \in S(C_u^G(G))$ is said to be an *idempotent state* if $\omega \bar{x} \omega = \omega$. For a nice survey describing the history and motivation behind the study of idempotent states see [21]. For the theory of idempotent state we refer to [21]. We shall use [21] Proposition 4] which in particular states that an idempotent state $\omega \in S(C_u^G(G))$ is preserved by the universal scaling group $\tau^u_t$:

$$\omega \circ \tau^u_t = \omega \quad (2.5)$$

for all $t \in \mathbb{R}$. An idempotent state $\omega \in S(C_u^G(G))$ yields a conditional expectation $E : C_0(G) \to C_0(G)$ (see [21])

$$E(x) = \omega \bar{x} x$$
for all \( x \in C_0(\mathbb{G}) \). Using \([9.6]\) we easily get
\[
\tau_t(E(x)) = E(\tau_t(x)).
\] (2.6)
Conditioned expectation extends to \( E : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \) and clearly \([2.0]\) holds for all \( x \in L^\infty(\mathbb{G}) \).

The image of \( B = E(C_0(\mathbb{G})) \) forms a \( C^* \)-subalgebra of \( C_0(\mathbb{G}) \). Let us note that \( E \) admits the universal version \( E^u : C_u^0(\mathbb{G}) \to C_u^0(\mathbb{G}) \)
\[
E^u = (\text{id} \otimes \omega) \circ \Delta^u_{\mathbb{G}}.
\]
In particular \( B \) admits the universal version \( B^u = E^u(C_u^0(\mathbb{G})) \). Let \((e_i)_{i \in I}\) be an approximate unit for \( C_u^0(\mathbb{G}) \). Then \( \lim_i E^u(e_i) = 1 \) strictly. Since \( E^u(e_i) \in B^u \) we see that \( B^u \) is a non-degenerate \( C^* \)-subalgebra of \( C_u^0(\mathbb{G}) \). Similarly, \( B \) is non-degenerate \( C^* \)-subalgebra of \( C_0(\mathbb{G}) \). It is easy to check that
\[
M(B) = \{ x \in M(C_0(\mathbb{G})) : E(x) = x \}
\]
and
\[
M(B^u) = \{ x \in M(C_u^0(\mathbb{G})) : E^u(x) = x \}. \quad (2.7)
\]
Since \( E \) is \( \Delta_{\mathbb{G}} \)-covariant, i.e.
\[
\Delta_{\mathbb{G}} \circ E = (\text{id} \otimes E) \circ \Delta_{\mathbb{G}}
\]
we conclude that
\[
\Delta_{\mathbb{G}}(B) \subset M(C_0(\mathbb{G}) \otimes B).
\]
Moreover \( B \) satisfies Podleś condition. Indeed
\[
\Delta_{\mathbb{G}}(B)(C_0(\mathbb{G}) \otimes 1) = (\text{id} \otimes E)(\Delta_{\mathbb{G}}(C_0(\mathbb{G})))(C_0(\mathbb{G}) \otimes 1)) = C_0(\mathbb{G}) \otimes B.
\]
Similarly, \( B^u \) satisfies Podleś condition.

A locally compact quantum group \( \mathbb{G} \) is called coamenable if \( \Lambda_{\mathbb{G}} \in \text{Mor}(C_u^0(\mathbb{G}), C_0(\mathbb{G})) \) is an isomorphism. In this case we shall identify \( C_u^0(\mathbb{G}) = C_0(\mathbb{G}) \). It can be shown that \( \mathbb{G} \) is coamenable if and only if \( C_0(\mathbb{G}) \) admits count \([3, \text{Theorem 3.1}]\). A quantum group \( \mathbb{G} \) is compact if the \( C^* \)-algebra \( C_0(\mathbb{G}) \) is unital. In this case we write \( \mathbb{C}(\mathbb{G}) \) and \( \mathbb{C}^u(\mathbb{G}) \) instead of \( C_0(\mathbb{G}) \) and \( C_u^0(\mathbb{G}) \) respectively.

A locally compact quantum group \( \mathbb{G} \) is said to be
- regular if
  \[
  \{ (\text{id} \otimes \omega)(\Sigma \mathbb{W}^{\mathbb{G}}) \big| \omega \in B(L^2(\mathbb{G})), \}^{\text{norm-cls}} = \mathcal{K}(L^2(\mathbb{G})), \quad (2.8)
  \]
- semi-regular if
  \[
  \{ (\text{id} \otimes \omega)(\Sigma \mathbb{W}^{\mathbb{G}}) \big| \omega \in B(L^2(\mathbb{G})), \}^{\text{norm-cls}} \supset \mathcal{K}(L^2(\mathbb{G}))
  \]
where \( \Sigma : L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \to L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \) is the Hilbert space flip. It can be shown that \( \mathbb{G} \) is regular if and only if only \( \mathbb{G} \) is regular if and only if \( \mathbb{G} \) satisfies \( \beta \)-Podleś condition (for the definition of \( \beta \) see \([2.2]\)). Remarkably, it was proved in \([3, \text{Proposition 5.6}]\) that \( \mathbb{G} \) is regular if and only if \( C_0(\mathbb{G}) \) satisfies \( \beta \)-weak Podleś condition.

Let \( \mathbb{H} \) and \( \mathbb{G} \) be locally compact quantum groups. Then a morphism \( \pi \in \text{Mor}(C_u^0(\mathbb{H}), C_u^0(\mathbb{G})) \) such that
\[
(\pi \otimes \pi) \circ \Delta^u_{\mathbb{G}} = \Delta^u_{\mathbb{H}} \circ \pi
\]
is said to define a homomorphism from \( \mathbb{H} \) to \( \mathbb{G} \). If \( \pi(C_u^0(\mathbb{G})) = C_u^0(\mathbb{H}) \), then \( \mathbb{H} \) is called Woronowicz-closed quantum subgroup of \( \mathbb{G} \). A homomorphism from \( \mathbb{H} \) to \( \mathbb{G} \) admits the dual homomorphism \( \hat{\pi} \in \text{Mor}(C_u^0(\mathbb{H}), C_u^0(\mathbb{G})) \) such that
\[
(\text{id} \otimes \pi)(\mathbb{W}^{\mathbb{G}}) = (\hat{\pi} \otimes \text{id})(\mathbb{W}^{\mathbb{H}}).
\]
A homomorphism from \( \mathbb{H} \) to \( \mathbb{G} \) identifies \( \mathbb{H} \) as a closed quantum subgroup \( \mathbb{G} \) if there exists an injective normal unital \( \ast \)-homomorphism \( \gamma : L^\infty(\mathbb{H}) \to L^\infty(\mathbb{G})) \) such that
\[
\Lambda_{\mathbb{G}} \circ \hat{\pi}(x) = \gamma \circ \Lambda_{\mathbb{H}}(x)
\]
for all $x \in C_0^\ast(\mathbb{H})$. Let $\mathbb{H}$ be a closed quantum subgroup of $G$, then $\mathbb{H}$ acts on $L^\infty(G)$ (in the von Neumann algebraic sense) by the following formula

$$\alpha : L^\infty(G) \to L^\infty(G) \otimes L^\infty(\mathbb{H}), \ x \mapsto V(x \otimes 1)V^*$$

where

$$V = (\gamma \otimes \text{id})(W^\mathbb{H}). \quad (2.10)$$

The fixed point space of $\alpha$ is denoted by

$$L^\infty(G/\mathbb{H}) = \{ x \in L^\infty(G) \mid \alpha(x) = x \otimes 1 \}$$

and referred to as the algebra of bounded functions on the quantum homogeneous space $G/\mathbb{H}$. If $\mathbb{H}$ is a compact quantum subgroup of $G$ then there is a conditional expectation $E : L^\infty(G) \to L^\infty(G)$ onto $L^\infty(G/\mathbb{H})$ which is defined by

$$E = (\text{id} \otimes \psi_\mathbb{H}) \circ \alpha \quad (2.11)$$

where $\psi_\mathbb{H}$ is the Haar measure of $\mathbb{H}$.

According to [9, Definition 2.2] we say that $\mathbb{H}$ is an open quantum subgroup of $G$ if there is a surjective normal $*$-homomorphism $\rho : L^\infty(G) \to L^\infty(\mathbb{H})$ such that

$$\Delta_\mathbb{H} \circ \rho = (\rho \otimes \rho) \circ \Delta_G.$$ 

Every open quantum subgroup is closed [10, Theorem 3.6]. We recall that a projection $P \in L^\infty(G)$ is a group-like projection if $\Delta_G(P)(1 \otimes P) = P \otimes P$. There is a 1-1 correspondence between (isomorphism classes of) open quantum subgroups of $G$ and central group-like projections in $G$ [10, Theorem 4.3]. The group-like projection assigned to $\mathbb{H}$, i.e. the central support of $\rho$, will be denoted by $1_\mathbb{H}$.

3. LIFTING AND NON-DEGENERACY RESULTS

In this section we shall prove a number of lifting and non-degeneracy results. Not all of them will be used in next sections. The reader who is focused on the theory of idempotent states should get familiar with Proposition 3.1, Proposition 3.2 and Proposition 3.13 and may go directly to next sections but we believe that the results obtained in this section are interesting in its own and have the potential to be used elsewhere.

Let $B$ be a non-zero $C^*$-subalgebra of $M(C_0(G))$. If $B$ is non-degenerate then $M(B)$ can be identified with a subalgebra of $M(C_0(G))$. The next proposition is thus important while considering the symmetry condition $W^G(I \otimes B)W^{G^*} \subset M(C_0(G) \otimes B)$ for a non-zero left-invariant $B \subset M(C_0(G))$ (see Section 5).

**Proposition 3.1.** Let $B$ be a non-zero left-invariant $C^*$-subalgebra of $M(C_0(G))$. Then $B$ is non-degenerate, $B C_0(G) = C_0(G)$.

**Proof.** Let us define

$$X = \text{span}\{ ba \mid b \in B, a \in C_0(G) \}^{\sigma\text{-weak cls}}.$$

$X$ is a $\sigma$-weakly closed right ideal of $L^\infty(G)$, so there exists a projection $p \in L^\infty(G)$ such that $X = p L^\infty(G)$.

Let $b \in B$ and $a \in C_0(G)$. We shall prove that for all $\mu \in B(L^2(G))_*$, $(\mu \otimes \text{id})(\Delta_G(ba)) \in X$. It suffices to check the latter for $\mu = c \cdot \omega$ where $c \in C_0(G)$ and $\omega \in B(L^2(G))_*$. Let us note that

$$(c \cdot \omega \otimes \text{id})(\Delta_G(ba)) = (\omega \otimes \text{id})(\Delta_G(b)\Delta_G(a)(c \otimes 1))$$

and

$$(\omega \otimes \text{id})(\Delta_G(b)\Delta_G(a)(c \otimes 1)) \in \{ (\mu \otimes \text{id})(\Delta_G(b)) \mid b \in B, \mu \in C_0(G)^* \}^{\text{norm-cls}} C_0(G) \subset X.$$ 

Thus we conclude that $\Delta_G(X) \subset L^\infty(G) \otimes X$. In particular $\Delta_G(p) \leq 1 \otimes p$. Using [15, Lemma 6.4] we get $p = 1$ and $X = L^\infty(G)$. Therefore

$$B C_0(G) \supseteq \{ (\omega \otimes \text{id})(W^G(b \otimes 1)W^{G^*})a \mid \omega \in B(L^2(G))_*, b \in B, a \in C_0(G) \}^{\text{norm-cls}} C_0(G) = \{ (\omega \otimes 1)(W^G(b \otimes 1)W^{G^*})a \mid \omega \in B(L^2(G))_*, b \in B, a \in C_0(G), c \in C_0(G) \}^{\text{norm-cls}} C_0(G) = \{ (\omega \otimes \text{id})(W^G(b \otimes 1)) \mid \omega \in B(L^2(G))_*, b \in B \}^{\text{norm-cls}} C_0(G)$$. 

\[
\omega = \{ (\omega \otimes \text{id})(W^G(ba \otimes 1)) \mid \omega \in B(L^2(G))^*, b \in B, a \in C_0(G) \}^{\text{norm-cls}} C_0(G)
\]

where the second last equality holds because the von Neumann algebra generated by \( BC_0(G) \) is \( L^\infty(G) \).

In the next proposition we will prove the counterpart of Proposition 3.1 for \( C^* \)-subalgebras of \( M(C_0''(G)) \).

**Proposition 3.2.** Let \( B \) be a left-invariant \( C^* \)-subalgebra of \( M(C_0''(G)) \) such that \( \Lambda_G(B) \neq 0 \). Then \( B \) is non-degenerate, \( B C_0''(G) = C_0''(G) \).

**Proof.** Let us consider \( D = \Lambda_G(B) \subset M(C_0(G)) \). Then it is easy to check that \( D \) satisfies the assumptions of Proposition 3.1. In particular \( D C_0(G) = C_0(G) \).

Let us emphasize that the assumption \( \Lambda_G(B) \neq 0 \) in Proposition 3.2 is essential. In order to see the relevant example let us consider a non-coamenable locally compact quantum group \( G \) and \( B = \text{ker} \Lambda_G \subset C_0''(G) \). Then \( B \subset C_0''(G) \) is a \( C^* \)-algebra which is left-invariant. Indeed for all \( b \in B \) and \( \mu \in C_0''(G) \) we have

\[
\Lambda_G((\mu \otimes \text{id})(\Delta_G^\mu(b))) = (\mu \otimes \text{id})(((\text{id} \otimes \Lambda_G)\Delta_G^\mu(b))
\]

thus \( (\mu \otimes \text{id})(\Delta_G^\mu(b)) \in B \). But \( B \) being an ideal in \( C_0''(G) \) cannot be non-degenerate.

Let \( \omega \in S(C_0''(G)) \) be an idempotent state on \( G \). It yields a pair of conditional expectations \( E^\omega : C_0''(G) \to C_0''(G) \) and \( E : C_0(G) \to C_0(G) \) and \( C^* \)-subalgebras \( X^\omega = E^\omega(C_0''(G)) \subset C_0''(G) \) and \( X = E(C_0(G)) \subset C_0(G) \) satisfying Podleś conditions. Note that \( \Lambda_G(X^\omega) = X \). In what follows we shall analyze the passage from the reduced level to the universal level in a more general context of a \( C^* \)-subalgebra \( B \subset M(C_0(G)) \) satisfying (weak) Podleś condition. In order to do this we need preparatory results.

Let us recall that \( G \) is a regular quantum group if and only if \( (2.9) \) is satisfied. We shall show that equality \( (2.9) \) implies apparently stronger condition with "half-lifted" multiplicative unitary.

**Lemma 3.3.** A locally compact quantum group \( G \) is regular if and only if

\[
C_0(\widehat{G}) \otimes C_0''(G) = \{(x \otimes 1)W^G(1 \otimes y) \mid x \in C_0(\widehat{G}), y \in C_0''(G)\}^{\text{norm-cls}}.
\]

**Proof.** The "if" part is trivial. To prove the "only if", assume \( G \) is a regular locally compact quantum group. Applying \((\text{id} \otimes \Delta^\omega)(2.9) \) to \( (2.9) \) we obtain

\[
C_0(\widehat{G}) \otimes \Delta^\omega(C_0(G)) = \{(x \otimes 1 \otimes 1)W^{G}_{12}W^{G}_{13}(1 \otimes \Delta^\omega_\mu(y)) \mid x \in C_0(\widehat{G}), y \in C_0(G)\}^{\text{norm-cls}}.
\]

Since \( \Delta^\omega \) satisfies Podleś condition, slicing the second leg of (3.1) we get

\[
C_0(\widehat{G}) \otimes C_0''(G) = \{ (\mu \otimes \text{id})(x \otimes 1 \otimes 1)W^{G}_{12}W^{G}_{13}(1 \otimes \Delta^\omega_\mu(c)(1 \otimes d \otimes 1)) \mid \\
\mu \in C_0(G)\}, x \in C_0(\widehat{G}), c, d \in C_0(G)\}^{\text{norm-cls}}
\]

\[
= \{ (\mu \otimes \text{id})(x \otimes 1 \otimes 1)W^{G}_{12}W^{G}_{13}(1 \otimes e \otimes y) \mid \\
\mu \in C_0(G), x \in C_0(\widehat{G}), c, d \in C_0(G)\}^{\text{norm-cls}}
\]
\[ \mu \in C_0(G)^*, x \in C_0(\hat{G}), e \in C_0(G), y \in C_0^n(G) \]\norm-cls

\[ \{ (id \otimes \mu \otimes id)((x \otimes 1)W^G(1 \otimes e) \otimes 1)W^G_{13}(1 \otimes 1 \otimes y) \mid \mu \in C_0(G)^*, x \in C_0(\hat{G}), e \in C_0(G), y \in C_0^n(G) \}\norm-cls \]

\[ \{ (id \otimes \mu \otimes id)(x \otimes a \otimes 1)W^G_{13}(1 \otimes 1 \otimes y) \mid \mu \in C_0(G)^*, x \in C_0(\hat{G}), a \in C_0(G), y \in C_0^n(G) \}\norm-cls \]

\[ \{ (x \otimes 1)W^G(1 \otimes y) \mid x \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls . \]

\[ \square \]

We shall show that yet stronger condition with the universal bicharacter \( W^G \) is also implied by \( \text{(2.9)} \).

**Lemma 3.4.** A locally compact quantum group \( G \) is regular if and only if

\[ C^u_0(\hat{G}) \otimes C^u_0(G) = \{ (x \otimes 1)W^G(1 \otimes y) \mid x \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls . \]

**Proof.** The "if" part is trivial. In order to prove the "only if" assume \( G \) is a regular. Lemma 3.3 yields

\[ C_0(\hat{G}) \otimes C_0(G) = \{ (x \otimes 1)W^G(1 \otimes y) \mid x \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls . \] (3.2)

Applying \( (\hat{\Delta}^r, u) \otimes id \) to (3.2) we get

\[ \hat{\Delta}^r, u(C_0(\hat{G})) \otimes C_0(G) = \{ (\hat{\Delta}^r, u(x) \otimes 1)W^G_{23}W^G_{13}(1 \otimes 1 \otimes y) \mid x \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls . \] (3.3)

Since \( \hat{\Delta}^r, u \) satisfies Podleś condition, by slicing the first leg of (3.3) we get

\[ C^u_0(\hat{G}) \otimes C^u_0(G) = \{ (\mu \otimes id \otimes id)((\hat{\Delta}^r, u(x) \otimes 1)W^G_{23}W^G_{13}(1 \otimes 1 \otimes y) \mid \mu \in B(L^2(G))_+, x \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls \]

\[ = \{ (\mu \otimes id \otimes id)((a \otimes 1)\hat{\Delta}^r, u(b) \otimes 1)W^G_{23}W^G_{13}(1 \otimes 1 \otimes y) \mid \mu \in B(L^2(G))_+, a, b \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls \]

\[ = \{ (\mu \otimes id \otimes id)(a \otimes b \otimes 1)W^G_{23}W^G_{13}(1 \otimes 1 \otimes y) \mid \mu \in B(L^2(G))_+, a, b \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls \]

\[ = \{ (\mu \otimes id \otimes id)(a \otimes b \otimes 1)W^G_{23}(1 \otimes 1 \otimes y) \mid \mu \in B(L^2(G))_+, a, b \in C_0(\hat{G}), y \in C_0^n(G) \}\norm-cls \]

where in the fourth equality we used Lemma 3.3. \[ \square \]

Let us prove an auxiliary lemma.

**Lemma 3.5.** Let \( G \) be a regular locally compact quantum group, \( D \) a \( C^\ast \)-algebra and \( \alpha \in \text{Mor}(D, C^u_0(G) \otimes D) \) a morphism satisfying

\[ (id \otimes \alpha) \circ \alpha = (\Delta^G_0 \otimes id) \circ \alpha \]

If \( D \) satisfies \( \alpha \)-weak Podleś condition then it satisfies \( \alpha \)-Podleś condition.

**Proof.** In what follows we shall denote \( \alpha_r = (\Delta^G_0 \otimes id) \circ \alpha \). Clearly, \( \alpha \)-weak Podleś condition yields

\[ D = \{ (\omega \otimes id)(\alpha_r(d)) \mid d \in D, \omega \in L^\infty(G)_+ \}\norm-cls . \]
We compute
\[
(C_u^0(G) \otimes I)\alpha(D) = \{(a \otimes I)\alpha((\omega \otimes id)\alpha_r(d)) \mid a \in C_u^0(G), \omega \in L^\infty(G)_*, d \in D\}^{\text{norm-cls}}
\]
\[
= \{(\omega \otimes id \otimes id)((I \otimes a)W^G_{12})_r_{13}(\alpha_r(d))_{12} \mid a \in C_u^0(G), \omega \in L^\infty(G)_*, d \in D\}^{\text{norm-cls}}
\]
\[
= \{(\omega \otimes id \otimes id)(\alpha_r(d)_r_{13}(I \otimes a)W^G_{12}^* \otimes (b \otimes I))_{12} \mid a \in C_u^0(G), b \in C_0(\widehat{G}), \omega \in L^\infty(G)_*, d \in D\}^{\text{norm-cls}}
\]
\[
= C_u^0(G) \otimes D.
\]
where in the fourth equality we use Lemma 3.3.

In the next theorem we shall show that a C*-subalgebra $B$ of $M(C_0(G))$ satisfying Podleś condition admits a unique universal lift under regularity condition on $G$. Then we shall discuss the universal lift for $B \subset M(C_0(G))$ satisfying weak Podleś condition (with regularity condition dropped).

**Theorem 3.6.** Let $G$ be a regular locally compact quantum group. Suppose that $B$ is a non-zero C*-subalgebra of $M(C_0(G))$ satisfying Podleś condition. Then there exists a unique C*-subalgebra $B_u \subset M(C_0(G))$ such that $\Delta_u(B_u) = B$ and $\Delta_u^*(B_u)(C_u^0(G) \otimes I) = C_u^0(G) \otimes B_u$.

**Proof.** Let us define
\[
B_u = \{(\omega \otimes id)(W^G(b \otimes I)W^G) \mid \omega \in B(L^2(G)_*, b \in B\}^{\text{norm-cls}} \subset M(C_0(G)).
\]
Clearly $B_u$ is a *-closed subspace of $M(C_0(G))$. We will show that $B_u$ satisfies all the required conditions. We shall first check the Podleś condition: $\Delta_u^*(B_u)(C_u^0(G) \otimes I) = C_u^0(G) \otimes B_u$. Let us note that we abuse here the terminology concerning Podleś condition since we do not know yet if $B_u$ forms a C*-algebra. This will be checked later. We compute
\[
\Delta_u^*(B_u)(C_u^0(G) \otimes I) = \{\Delta_u^*(\omega \otimes id)(W^G(b \otimes I)W^G_\ast)(y \otimes I) \mid \omega \in B(L^2(G)_*, b \in B, y \in C_u^0(G)\}^{\text{norm-cls}}
\]
\[
= \{(\omega \otimes id \otimes id)(W^G_{12}W^G_{13}(b \otimes I \otimes I)W^G_{15}W^G_{13})(1 \otimes y \otimes I) \mid \omega \in B(L^2(G)_*, b \in B, y \in C_u^0(G)\}^{\text{norm-cls}}
\]
\[
= \{(\omega \otimes id \otimes id)(W^G_{12}W^G_{13}(b \otimes I \otimes I)W^G_{15}W^G_{13})(c \otimes y \otimes I) \mid \omega \in B(L^2(G)_*, b \in B, y \in C_u^0(G), c \in C_0(\widehat{G})\}^{\text{norm-cls}}
\]
\[
= \{(c \cdot \omega \otimes id \otimes id)(W^G_{12}W^G_{13}(b \otimes I \otimes I)W^G_{15})(1 \otimes y \otimes I) \mid \omega \in B(L^2(G)_*, b \in B, y \in C_u^0(G), c \in C_0(\widehat{G})\}^{\text{norm-cls}}
\]
\[
= \{(\omega \otimes id \otimes id)((d \otimes I \otimes I)W^G_{12}(1 \otimes y \otimes I))(W^G_{13}(b \otimes I \otimes I)W^G_{13}) \mid \omega \in B(L^2(G)_*, d \in C_0(\widehat{G}), y \in C_u^0(G), b \in B\}^{\text{norm-cls}}
\]
\[
= \{(\omega \cdot d \otimes id \otimes id)((1 \otimes y \otimes I))(W^G_{13}(b \otimes I \otimes I)W^G_{13}) \mid \omega \in B(L^2(G)_*, d \in C_0(\widehat{G}), y \in C_u^0(G), b \in B\}^{\text{norm-cls}}
\]
\[
= \{(\omega \otimes id \otimes id)(W^G_{13}(b \otimes I \otimes I)W^G_{13}) \mid \omega \in B(L^2(G)_*, c \in C_0(G), b \in B\}^{\text{norm-cls}}
\]
\[
= \{(y \otimes I)((\omega \otimes id \otimes id)W^G_{13}(b \otimes I \otimes I)W^G_{13}) \mid y \in C_u^0(G), \omega \in B(L^2(G)_*, b \in B\}^{\text{norm-cls}}
\]
where in the sixth equality we used Lemma 3.3.

Let us check that $\Delta_u(B_u) = B$:
\[
B = \{(\omega \otimes id)(\Delta_u(b)(c \otimes I)) \mid \omega \in B(L^2(G)_*, b \in B, c \in C_0(G)\}^{\text{norm-cls}}
\]
\[
= \{(c \cdot \omega \otimes id)(W^G(b \otimes I)W^G_\ast) \mid \omega \in B(L^2(G)_*, b \in B, c \in C_0(G)\}^{\text{norm-cls}}
\]
In particular, from techniques of the proof of \([10, \text{Proposition 2.3}]\) we observe that \(C\) finite sets

\[
\Delta^r_u = \{ \omega \otimes \text{id}((\Lambda_G \otimes \text{id})\Delta^u_G)(d) \mid d \in D, \omega \in B(L^2(G))_\ast \}^{\text{norm-cls}}.
\]

Let us prove that \(B^u\) is a \(C^*\)-algebra. By applying \(\Lambda_G \otimes \text{id}\) to Podleš condition satisfied by \(B^u\) we have \(\Delta^r_u(B)\subset C_0(G)\otimes 1 = C_0(G)\otimes B^u\). Taking the conjugate of this equality we obtain

\[
C_0(G) \otimes B^u = (C_0(G) \otimes 1)\Delta^r_u(B).
\]

Let us fix \(\epsilon > 0\) and \(x, x' \in B^u, 0 \neq a \in C_0(G)\). There exist finite sets \(I\) and \(J\) such that

\[
\|aa^* \otimes x - \sum_{i \in I} (y_i \otimes 1)\Delta^r_u(x_i)\| \leq \epsilon
\]

and

\[
\|aa^* \otimes x' - \sum_{j \in J} \Delta^r_u(x'_{j})(y_j \otimes 1)\| \leq \epsilon
\]

for some \(x_i, x'_j, y_i, y'_j \in C_0(G)\) so

\[
\|aa^* \otimes x' - \sum_{i,j} (y_i \otimes 1)\Delta^r_u(x_i, x'_j)(y_j \otimes 1)\| \leq \epsilon'.
\]

where \(\epsilon' = (\|a \otimes x\| + \|a^* \otimes x'\|)\epsilon\). Now choosing a functional \(\mu \in C_0^\ast(G)^\ast\) such that \(\mu(aa^*) = 1\) and applying \(\mu \otimes \text{id}\) to \((3.4)\) we get

\[
\|x x' - \sum_{i,j} (y'_j \cdot y_i \otimes 1)(\Delta^r_u(x_i, x'_j))\| \leq \|\mu\|\epsilon'.
\]

This shows that \(B^u\) is closed under product and completes the existence part of the proof.

In order to prove the uniqueness, suppose \(D \subset M(C_0^\ast(G))\) satisfies the required conditions. Then the Podleš condition yields

\[
D = \{ (\omega \otimes \text{id})((\Lambda_G \otimes \text{id})\Delta^u_G)(d) \mid d \in D, \omega \in B(L^2(G))_\ast \}^{\text{norm-cls}}.
\]

Using \((2.3)\) we get

\[
D = \{ (\omega \otimes \text{id})((\Lambda_G \otimes \text{id})\Delta^u_G)(d) \mid d \in D, \omega \in B(L^2(G))_\ast \}^{\text{norm-cls}}
\]

\[
= \{ (\omega \otimes \text{id})\Delta^r_u(b) \mid b \in B, \omega \in B(L^2(G))_\ast \}^{\text{norm-cls}} = B^u
\]

and we get uniqueness. \(\square\)

**Example 3.7.** Let \(\mathbb{H}\) be a closed quantum subgroup of a regular locally compact quantum group \(G\). In \([23]\) among other things the \(C^*\)-quantum homogeneous space \(C_0(\mathbb{H} \ltimes \mathbb{H}) \subset M(C_0(G))\) was constructed (under the regularity condition on \(G\)). Using Theorem 3.6 we get its universal version \(C_0^\ast(G/\mathbb{H})\). The existence of \(C_0^\ast(G/\mathbb{H})\) was hinted in \([26, \text{Remark 8.6}]\).

**Remark 3.8.** Let \(N \subset L^\infty(G)\) be a von Neumann algebra satisfying Baaj-Vaes conditions. Then there exists a locally compact quantum group \(\mathbb{H}\) such that \(N = L^\infty(\mathbb{H}) \subset L^\infty(G)\). Using the techniques of the proof of \([10, \text{Proposition 2.3}]\) we observe that \(C_0(\mathbb{H})\) embeds into \(M(C_0(G))\). Let us denote the embedding by \(\pi : C_0(\mathbb{H}) \to M(C_0(G))\). Then \(\pi\) satisfies

\[
\Delta_{\mathbb{H}} \circ \pi = (\pi \otimes \pi) \circ \Delta_G.
\]

Using \([18]\) we get the universal lift \(\pi^u : C_0^\ast(\mathbb{H}) \to M(C_0^\ast(G))\) defining a quantum group homomorphism from \(G\) to \(\mathbb{H}\).

Let us note that \(B = \pi(C_0(\mathbb{H}))\) satisfies Podleš condition

\[
\Delta_G(B)(C_0(G) \otimes 1) = C_0(G) \otimes B.
\]

It is easy to verify that \(\pi^u(C_0^\ast(\mathbb{H}))\) satisfies the conditions of Theorem 3.6 i.e. \(B^u = \pi^u(C_0^\ast(\mathbb{H}))\). In particular, \(B^u\) exists without the regularity of \(G\). It may happen that \(\pi^u\) is not injective so we don’t have \(B^u \cong C_0^\ast(\mathbb{H})\) in general. For an example when \(\pi^u\) is not injective we refer to \([15]\).

In the next definition we adopt the terminology introduced in Definition 2.2 and Definition 2.7.
Definition 3.9. Let $B \subset M(C_0(G))$ be a non-zero $C^*$-subalgebra satisfying weak Podleś condition. We say that a $C^*$-subalgebra $B^u \subset M(C_0(G))$ is a weak lift of $B$ if $\Delta_G(B^u) = B$ and $B^u$ satisfies weak Podleś condition.

Let $B \subset M(C_0(G))$ be a non-zero $C^*$-subalgebra satisfying Podleś condition. We say that a $C^*$-subalgebra $B^u \subset M(C_0(G))$ is a lift of $B$ if $\Delta_G(B^u) = B$ and $B^u$ satisfies Podleś condition.

Using Proposition 3.2 we get

Corollary 3.10. Let $B \subset M(C_0(G))$ be a non-zero $C^*$-algebra satisfying weak Podleś condition and admitting a weak lift $B^u$. Then $B^u$ is non-degenerate, $B^u C_0(G) = C_0(G)$.

Remark 3.11. It can be easily proved that if the weak lift $B^u$ of $B$ in the sense of Definition 3.9 exists then it is uniquely given by

$$\{(\omega \otimes \text{id})(\Delta^{G,u}_G(b)) \mid b \in B, \omega \in B(L^2(G))^*\}_{\text{norm-cl}}.$$  \hspace{1cm} (3.5)

Conversely, suppose that $B \subset M(C_0(G))$ is a non-zero $C^*$-subalgebra satisfying weak Podleś condition. Let us consider $B^u$ defined by (3.5). Then $B^u$ is a $*$-closed linear subspace of $M(C_0(G))$. Using the techniques of the first part of the proof of Theorem 3.6 we can prove (not using regularity) that $B^u$ satisfies weak Podleś condition. Note again the abuse of terminology as in the proof of Proposition 5.7. Assuming semi-regularity of $G$ and using the techniques of the proof of [3] we prove that $B^u$ is a $C^*$-subalgebra of $M(C_0(G))$.

We were not able to prove that if $B \subset M(C_0(G))$ satisfies weak Podleś condition then $\Delta_G(B) \subset M(C_0(G) \otimes B)$. Suppose that $\Delta_G(B) \subset M(C_0(G) \otimes B)$ holds and $B$ admits a weak lift $B^u$. In this case we were not able to prove that $\Delta_G(B^u) \subset M(C_0(G) \otimes B^u)$. Note again that the latter holds if $G$ is regular. The above discussion yields the following definition.

Definition 3.12. Let $B \subset M(C_0(G))$ be a non-zero $C^*$-subalgebra satisfying weak Podleś condition and such that $\Delta_G(B) \subset M(C_0(G) \otimes B)$. We say that a $C^*$-subalgebra $B^u \subset M(C_0(G))$ is a strong lift of $B$ if $\Delta_G(B^u) = B$, $B^u$ satisfies weak Podleś condition and $\Delta_G(B^u) \subset M(C_0(G) \otimes B^u)$.

In what follows we shall consider the behavior of symmetry of $B$ under the lift to the universal level $B^u$.

Proposition 3.13. Let $B$ be a non-zero $C^*$-subalgebra of $M(C_0(G))$ satisfying weak Podleś condition and such that $\Delta_G(B) \subset M(C_0(G) \otimes B)$. If $B$ admits strong lift $B^u$ then $B^u$ is symmetric, i.e. $W^G(1 \otimes B^u)_* W^{G,*} \subset M(C_0(\hat{G}) \otimes B^u)$.

Proof. Let us first note that $\Delta_G(B^u) \subset M(C_0(G) \otimes B^u)$ implies that $\Delta^{G,u}_G(B) \subset M(C_0(G) \otimes B^u)$. Thus using $W^G(1 \otimes B)_* W^{G,*} \subset M(C_0(\hat{G}) \otimes B)$ we get

$$W^{G}_1 W^G\otimes_1 (1 \otimes \Delta^{G,u}_G(B)) W^{G,*} \subset M(C_0(\hat{G}) \otimes C_0(G) \otimes B^u).$$

In particular $W^{G}_1 W^G\otimes_1 (1 \otimes \Delta^{G,u}_G(B)) W^{G,*} \subset M(C_0(\hat{G}) \otimes C_0(G) \otimes B^u)$ which when sliced with $(\text{id} \otimes \mu \otimes \text{id})$, where $\mu$ runs over all functionals on $C_0(\hat{G})$ yields $W^G(1 \otimes B^u)_* W^{G,*} \subset M(C_0(\hat{G}) \otimes B^u)$ \hspace{1cm} \(\Box\)

Before formulating the next definition we refer to [22] for a definition of the action $\beta \in \text{Mor}(C_0(G), C_0(\hat{G}) \otimes C_0(G))$ of $\hat{G}$ on $C_0(G)$. Let $B \subset M(C_0(G))$ be a non-degenerate $C^*$-subalgebra such that $\beta(B) \subset M(C_0(\hat{G}) \otimes B)$. We say that $B$ satisfies

- $\beta$-Podleś condition if $\beta(B)(C_0(\hat{G}) \otimes 1) = C_0(\hat{G}) \otimes B$,
- $\beta$-weak Podleś condition if $(C_0(\hat{G}) \otimes 1) \beta(B)(C_0(\hat{G}) \otimes 1) = C_0(\hat{G}) \otimes B$.

Similarly we define $\beta^u : \hat{C}_0^u(G) \rightarrow M(C_0(\hat{G}) \otimes \hat{C}_0^u(G))$

$$\beta^u(x) = (\hat{W}^{G}(1 \otimes x)) \hat{W}^{G,*}$$

for all $x \in \hat{C}_0^u(G)$ and the notion of $\beta^u$-(weak) Podleś condition for a non-degenerate $D \subset M(\hat{C}_0^u(G))$.

Example 3.14. Let $G$ be a regular locally compact quantum group and $\mathbb{H}$ a closed quantum subgroup of $G$. Using [25] Theorem 8.2 we see that $C_0(G/\mathbb{H})$ satisfies $\beta$-Podleś condition.
Lemma 3.15. Let $B \subset M(C_0(G))$ be a symmetric $C^*$-subalgebra satisfying weak Podleś and $\beta$-weak Podleś condition. Suppose that $B$ admits a weak lift $B^u$. Then $B^u$ satisfies $\beta^u$-weak Podleś condition

$$(C_0(\hat{G}) \otimes 1)\beta^u(B^u)(C_0(\hat{G}) \otimes 1) = C_0(\hat{G}) \otimes B^u.$$ 

Proof. $\beta$-weak Podleś condition for $B$ writes

$$(C_0(\hat{G}) \otimes 1)(W^G(1 \otimes B)W^{G^*})(C_0(\hat{G}) \otimes 1) = C_0(\hat{G}) \otimes B$$  \hspace{1cm} (3.6)

Applying $id \otimes \Delta_{r,u}^*$ to both sides of (3.6) and then slicing the second leg by the functionals on $C_0(G)$ we get

$$C_0(\hat{G}) \otimes B^u = \{(id \otimes \mu \otimes id)((y \otimes 1 \otimes 1)W_{12}^G W_{13}^G(1 \otimes \Delta_{r,u}^*(b))W_{13}^G W_{12}^G(y \otimes 1 \otimes 1)) \mid \mu \in C_0(G)^*, b \in B, y, \tilde{y} \in C_0(\hat{G})\}$$

$$= \{(id \otimes \mu \otimes id)((y \otimes 1 \otimes 1)W_{13}^G(1 \otimes \Delta_{r,u}^*(b))W_{13}^G(y \otimes 1 \otimes 1)) \mid \mu \in C_0(G)^*, b \in B, y, \tilde{y} \in C_0(\hat{G})\}$$

$$= \{(id \otimes \mu \otimes id)((y \otimes 1 \otimes 1)W_{13}^G(1 \otimes x \otimes b)W_{13}^G(y \otimes 1 \otimes 1)) \mid \mu \in C_0(G)^*, b \in B, x, \tilde{x} \in C_0(G), y, \tilde{y} \in C_0(\hat{G})\}$$

$$= \{(\tilde{y} \otimes 1)(W^G(1 \otimes b)W^{G^*})(y \otimes 1) \mid y, \tilde{y} \in C_0(\hat{G}), b \in B^u\}$$

$$= (C_0(\hat{G}) \otimes 1)\beta^u(B^u)(C_0(\hat{G}) \otimes 1).$$

$\square$

Definition 3.16. Let $D$ be a non-degenerate $C^*$-subalgebra of $M(C_0(G))$. We say that $D$ is strongly symmetric if $W^G(1 \otimes D)W^{G^*} \subset M(C_0(\hat{G}) \otimes D)$.

In the following lemma we will establish a relation between the concepts of symmetric and strongly symmetric subalgebras of $M(C_0(G))$.

Lemma 3.17. Let $B \subset M(C_0(G))$ be a symmetric subalgebra satisfying weak Podleś condition, $\beta$-weak Podleś condition and such that $\Delta_G(B) \subset M(C_0(G) \otimes B)$. Suppose that $B$ admits strong lift $B^u$. Then $B^u$ is strongly symmetric and

$$(C_0(\hat{G}) \otimes 1)W^G(1 \otimes B^u)W^{G^*}(C_0(\hat{G}) \otimes 1) = C_0(\hat{G}) \otimes B^u$$  \hspace{1cm} (3.7)

Proof. $B$ is symmetric, so using Proposition 3.18 we get

$$W^G(1 \otimes B^u)W^{G^*} \subset M(C_0(\hat{G}) \otimes B^u).$$  \hspace{1cm} (3.8)

Applying $\tilde{\Delta}_{r,u} \otimes id$ to (3.8) we observe that

$$\tilde{\Delta}_{r,u} \otimes id)(W^G(1 \otimes B^u)W^{G^*} \subset M(C_0(\hat{G}) \otimes C_0(\hat{G}) \otimes B^u).$$  \hspace{1cm} (3.9)

By slicing the first leg of (3.9) we obtain a subset of $M(C_0(\hat{G}) \otimes B^u)$ and now we compute the left hand side

$$M(C_0(\hat{G}) \otimes B^u) \supset \{(\mu \otimes id \otimes id)(\tilde{\Delta}_{r,u} \otimes id)W^G(1 \otimes b)W^{G^*} \mid \mu \in C_0(\hat{G})^*, b \in B^u\}$$

$$= \{(\mu \otimes id \otimes id)W_{23}^G W_{13}^G(1 \otimes 1 \otimes b)W_{13}^G W_{23}^G \mid \mu \in C_0(\hat{G})^*, b \in B^u\}$$

$$= \{(\mu \otimes id \otimes id)W_{23}^G(\tilde{y} \otimes 1 \otimes 1)W_{13}^G(1 \otimes 1 \otimes b)W_{13}^G(y \otimes 1 \otimes 1)) \mid \mu \in C_0(\hat{G})^*, \tilde{y}, y \in C_0(\hat{G}), b \in B^u\}$$

$$= W^G(1 \otimes B^u)W^{G^*}$$

COIDEALS, QUANTUM SUBGROUPS AND IDEMPOTENT STATES 13
where in the third equality we use Lemma 3.15. In order to prove (3.7) we compute
\[ C_0^u(\hat{G}) \otimes B^u = \left\{ \left( \mu \otimes id \right) \left( \hat{\Delta}_r^u(y) \otimes 1 \right) W_{23}^G \left( \Delta_r^u(y) \otimes 1 \right) W_{13}^G \left( 1 \otimes 1 \otimes b \right) W_{23}^\ast \left( \hat{G} \right) \mid \mu \in C_0(\hat{G})^\ast \right\} \]
\[ = \left\{ \left( \mu \otimes id \right) \left( \hat{\Delta}_r^u(y) \otimes 1 \right) W_{23}^G \left( \Delta_r^u(y) \otimes 1 \right) W_{13}^G \left( 1 \otimes 1 \otimes b \right) W_{23}^\ast \left( \hat{G} \right) \mid \mu \in C_0(\hat{G})^\ast , y \in C_0(\hat{G}), b \in B^u \right\} \]
\[ = \left\{ \left( \mu \otimes id \right) \left( \hat{\Delta}_r^u(y) \otimes 1 \right) W_{23}^G \left( \Delta_r^u(y) \otimes 1 \right) W_{13}^G \left( 1 \otimes 1 \otimes b \right) W_{23}^\ast \left( \hat{G} \right) \mid \mu \in C_0(\hat{G})^\ast , y \in C_0(\hat{G}), b \in B^u \right\} \]
where in the fourth equality we use Lemma 3.15. Thus we get Equation (3.7). \( \square \)

Using Lemma 3.17, Lemma 3.16, and Theorem 3.6 in the context of \( C_0(\hat{G}/\hat{H}) \) we get

**Proposition 3.18.** Let \( G \) be a regular locally compact quantum group and \( H \) a closed quantum subgroup of \( G \). Then \( C_0(\hat{G}/\hat{H}) \) is strongly symmetric. Moreover it satisfies Podleś condition
\[ \Delta_r^u(\hat{G}/\hat{H}) \mid C_0^u(G/H) \otimes 1 = C_0^u(G) \otimes C_0^u(G/H) \]
and the following holds
\[ (C_0^u(\hat{G}) \otimes 1) W_{23}^\ast \left( \hat{G} \right) \mid c_0^u(\hat{G}) \otimes 1) W_{23}^\ast \left( \hat{G} \right) = C_0^u(\hat{G}) \otimes C_0^u(\hat{G}/\hat{H}) \]

Let \( H \subset G \) be a compact quantum group. In the final part of this section we shall analyze the covariance of the conditional expectation \( E : L^\infty(G) \to L^\infty(G/H) \) under \( \beta : L^\infty(G) \to L^\infty(G/H) \). For the formula for \( E \) we refer to (2.11).

**Definition 3.19.** Let \( G \) be a locally compact quantum group, \( H \) a compact quantum subgroup of \( G \), \( E : L^\infty(G) \to L^\infty(G) \) the conditional expectation onto \( L^\infty(G/H) \) and \( \beta : L^\infty(G) \to L^\infty(G/H) \) the action of \( G \) on \( L^\infty(G) \) defined in (2.3). We say that \( E \) is \( \beta \)-covariant if
\[ (id \otimes E) \circ \beta = \beta \circ E. \]
Let \( H \) be a compact quantum group and let us consider the map \( \Theta : L^\infty(H) \to L^\infty(H) \)
\[ \Theta(x) = (id \otimes \psi_H)(W_{23}^\ast (\hat{G}) \otimes 1) W_{13}^\ast. \]
In [9] Corollary 3.9 it was proved that \( \Theta(x) = \psi_H(x) \mathbb{I} \) if and only if \( H \) is of Kac type. Using the invariance of the Haar measure under the unitary antipode \( \psi_H = \psi_H \circ R_{\hat{H}} \) and the formula \( (R_{\hat{H}} \otimes R_{\hat{H}})(W_{23}^\ast) = W_{23}^\ast \) we get the following result.

**Corollary 3.20.** Let \( H \) be a compact quantum group. Then
\[ (id \otimes \psi_H)(W_{23}^\ast (\hat{G}) \otimes 1) W_{13}^\ast = \psi_H(x) \mathbb{I} \]
for all \( x \in L^\infty(H) \) if and only if \( H \) is of Kac type.

**Proposition 3.21.** Let \( H, G, \beta \) and \( E \) be as in Definition 3.14. Then \( E \) is \( \beta \)-covariant if and only if \( H \) is of Kac type.

**Proof.** Let \( \alpha : L^\infty(G) \to L^\infty(G) \otimes L^\infty(H) \) be the corresponding action of \( H \) on \( L^\infty(G) \). Using Theorem 3.6 (3) we conclude that
\[ L^\infty(H) = \left\{ (\omega \otimes id) (\alpha(x)) : \omega \in L^\infty(G), x \in L^\infty(G) \right\}^{\ast \text{-weak cls}}. \]
(3.10)
The conditional expectation is given by the formula \( E = (id \otimes \psi_H) \circ \alpha \) thus using the formula \( (id \otimes \alpha)(W_{23}^\ast) = W_{13}^\ast V_{13} \), where \( V \) is given by (2.10), we get
\[ (id \otimes E) \circ \beta(x) = (id \otimes E)(W_{23}^\ast (\hat{G}) \otimes 1) W_{13}^\ast \]
\[ = W_{23}^\ast (id \otimes id \otimes \psi_H)(V_{13}(\mathbb{I} \otimes \alpha(x))V_{13}) W_{13}^\ast. \]
On the other hand
\[ \beta \circ E(x) = W^G(1 \otimes (\text{id} \otimes \psi_H) \circ \alpha(x))W^G*. \]
In particular \( E \) is \( \beta \)-covariant if and only if
\[
(id \otimes \psi_H)(W^G_{13}(1 \otimes \alpha(x))W^{G*}_{13}) = 1 \otimes (id \otimes \psi_H) \circ \alpha(x)
\]
which by (3.10) is equivalent with
\[
(id \otimes \psi_H)(W^H(1 \otimes x)W^{H*}) = \psi_H(x) I
\]
for all \( x \in L^\infty(H) \). Using Corollary 3.20 we get the desired equivalence. \( \square \)

Let \( G \) be a regular locally compact quantum group and \( H \) a Kac type compact quantum subgroup of \( G \). In this case the proof that \( C_0(G/H) \) satisfies \( \beta \)-Podle's is easy (see Example 3.14 for the discussion of the general case) and follows from the following computation.
\[
\beta(C_0(G/H))(C_0(G) \otimes 1) = \beta(E(C_0(G)))(C_0(G) \otimes 1)
\]
\[
= (id \otimes E)(\beta(C_0(G))(C_0(G) \otimes 1))
\]
\[
= (id \otimes E)(C_0(G) \otimes C_0(G)) = C_0(G) \otimes C_0(G/H).
\]

4. Integrable coideals and idempotent states on \( G \)

Let \( \omega \in S(C_0^*(G)) \) be an idempotent state on \( G \) and let \( E : L^\infty(G) \to L^\infty(G) \) be the conditional expectation assigned to \( \omega \). Let \( N = E(L^\infty(G)) \) be a coideal in \( L^\infty(G) \) assigned to \( \omega \).

**Proposition 4.1.** Let \( \omega \) and \( N \) be as above. Then

- \( N \) is integrable;
- \( \tau_t(N) = N \) for all \( t \in \mathbb{R} \).

**Proof.** Integrability of \( N \) is equivalent with \( \psi_G|_N \) being semifinite. Let us take \( x \in L^\infty(G), x \geq 0 \). Then we have
\[
\psi_G(x) I = (\psi_G \otimes \text{id})(id \otimes E) (\Delta_G(x))
\]
\[
= (\psi_G \otimes \text{id})(\Delta_G(E(x))
\]
\[
= \psi_G(E(x)) I
\]
which implies the required semifiniteness. Using (2.6) we get
\[
\tau_t(N) = \tau_t(E(L^\infty(G))) = E(\tau_t(L^\infty(G))) = N.
\]

\( \square \)

In what follows we shall prove the converse of Proposition 4.1.

**Theorem 4.2.** Let \( N \) be an integrable coideal of \( L^\infty(G) \) which is \( \tau_t \)-invariant. Then there exists a unique conditional expectation \( E : L^\infty(G) \to L^\infty(G) \) onto \( N \) such that \( \psi_G(x) = \psi_G(E(x)) \) for all \( x \in L^\infty(G)^+ \). Moreover there exists a unique idempotent state \( \omega \in C_0^*(G)^* \) such that for every \( x \in L^\infty(G), E(x) = \omega \overline{\tau} x \) and \( N = \{ x \in L^\infty(G) : \omega \overline{\tau} x = x \} \).

**Proof.** \( N \) is an integrable von Neumann subalgebra of \( L^\infty(G) \) so the restriction of right Haar weight \( \psi_G \) to \( N \) is semifinite. Let us show that since \( N \) is preserved by \( \tau_t \), it must be also preserved by \( \sigma_t' \). For the latter we first use (2.11) and the fact that \( N \) is a coideal, i.e.
\[
(\sigma_t \otimes \sigma_t') \circ \Delta_G(N) = \Delta_G \circ \tau_t(N) = \Delta_G(N) \subset L^\infty(G) \otimes N.
\]

Slicings (4.1) and using [13] Corollary 2.6 we get
\[
N = \sigma_t' \left( \{ (\omega \circ \sigma_t \otimes \text{id})(\Delta_G(N)) \mid \omega \in B(L^2(G)) \}^{\text{\sigma-weak cla}} \right)
\]
\[
= \sigma_t' \left( \{ (\omega \otimes \text{id})(\Delta_G(N)) \mid \omega \in B(L^2(G)) \}^{\text{\sigma-weak cla}} \right) = \sigma_t'(N).
\]
Using [24, Theorem IX.4.2] we conclude that there exists a unique normal conditional expectation $E : L^\infty(G) \to L^\infty(G)$ onto $N$ which preserves $\psi_G$. Using the Kadison inequality

$$E(x)^* E(x) \leq E(x^* x)$$

we can see that $E$ admits the Hilbert space extension, i.e. a projection $P : L^2(G) \to L^2(G)$ such that

$$P_\eta(x) = \eta_\omega(E(x))$$

(4.2)

for all $x \in L^\infty(G)$ such that $\psi_G(x^* x) < \infty$.

Let us note that since $E$ is a conditional expectation onto $N$, the coideal property of $N$ yields

$$(\text{id} \otimes E) \circ \Delta_G \circ E = \Delta_G \circ E$$

(4.3)

Our aim now is to show that $(\text{id} \otimes E) \circ \Delta_G = \Delta_G \circ E$. In order to prove it let us consider the canonical implementation $U \in L^\infty(G) \otimes B(L^2(G))$ of $\Delta_G$ (see [26]). Let us recall that $U$ satisfies

$$\Delta_G(x) = U(1 \otimes x)U^*,$$

$$\left(\Delta_G \otimes \text{id}\right)(U) = U_{23}U_{13}.$$ Using [26, Proposition 2.4] we get the explicit formula defining $U$:

$$\left((\omega_{\xi, \delta} \otimes \text{id}) \circ \Delta_G(x)\right) = \eta_\omega\left((\omega_{\xi, \delta} \otimes \text{id})\Delta_G(x)\right)$$

(4.4)

for all $\xi, \eta \in D(\delta^{1/2})$; here we view the modular element $\delta$ as an unbounded operator acting on $L^2(G)$. Slicing (1.3) with $(\omega_{\xi, \delta} \otimes \text{id})$, applying $\eta_\omega$ to the result, using (1.2) and (1.4) we get $PxP = xP$ for all $x \in M$ where

$$M = \left\{ (\omega_{\xi, \eta} \otimes \text{id})(U) : \xi, \eta \in L^2(G) \right\}^{\sigma\text{-weak clos}}$$

(4.5)

Since $U$ is a representation of $G$, $M$ forms a von Neumann algebra. In particular having $PxP = xP$ satisfied for all $x \in M$ we easily conclude that $Px = xP$. This in turn is equivalent with $(\text{id} \otimes E) \circ \Delta_G = \Delta_G \circ E$.

We conclude by using [24, Theorem 5], which yields the unique idempotent state $\omega \in S(C_0^*(G))$ such that $E(x) = \omega \tau x$. \hfill \Box

Having Proposition 4.1 and Theorem 4.2 and using [21, Theorem 1] we get

Corollary 4.3. There is a 1-1 correspondence between integrable coideals $N \subset L^\infty(G)$ preserved by $\tau_\omega$ and idempotent states on $G$, where denoting the conditional expectation given by $\omega$ with $E : L^\infty(G) \to L^\infty(G)$, we have $N = E(L^\infty(G))$. Moreover $E$ preserves $\psi_G$ and $\varphi_G$.

Remark 4.4. Let us note that in the course of the proof of Theorem 4.2 we could not use the left version of [21, Corollary 3] and immediately conclude the existence of $\omega$. This would be possible knowing that $\varphi_G$ is $E$-invariant. Actually using the techniques of the proof of Theorem 4.2 we can prove Theorem 4.4 which is a strengthened version of [21, Theorem 1, v]: there is a 1-1 correspondence between idempotent states on $G$ and $\psi_G$-expected left-invariant von Neumann subalgebras of $L^\infty(G)$.

Theorem 4.5. Let $N$ be a $\psi_G$-expected coideal in $L^\infty(G)$. Then the conditional expectation $E$ onto $N$ satisfies $(\text{id} \otimes E) \circ \Delta_G = \Delta_G \circ E$. In particular there exists a unique idempotent state $\omega \in C_0^*(G)^*$ such that for all $x \in L^\infty(G)$, $E(x) = \omega \tau x$ and $N = \{ x \in L^\infty(G) : \omega \tau x = x \}$.

Proof. Since $\Delta_G(N) \subset L^\infty(G) \otimes N$ we have $(\text{id} \otimes E) \circ \Delta_G \circ E = \Delta_G \circ E$. Let $P$ be the $L^2(G)$-version of $E$ (it exists since $E$ preserves $\psi_G$). Proceeding as in the second part of the proof of Theorem 4.2 we conclude that $(\text{id} \otimes E) \circ \Delta_G = \Delta_G \circ E$ and we are done. \hfill \Box

Let us formulate a result which may be viewed as strengthen version of [21, Corollary 3].

Theorem 4.6. Let $B \subset C_0(G)$ be a non-zero left-invariant $C^*$-subalgebra. Then the following are equivalent:

(i) $B$ is expected;
(ii) $B$ is $\varphi_G$-expected;
(iii) $B$ is $\psi_G$-expected.
Proof. Using Proposition 3.1 we get that $B$ is non-degenerate. The equivalence of (i) and (ii) is the left-side counterpart of [21 Corollary 3]. Clearly (i) $\Rightarrow$ (iii). Suppose that $E : C_0(G) \to C_0(G)$ is a conditional expectation onto a $C^*$-subalgebra $B$ preserving $\psi_G$. Let $P : L^2(G) \to \mathcal{L}(G)$ be the Hilbert space extension of $E$ (it exists since $E$ preserves $\psi_G$). Using the techniques of the proof of Theorem 4.2 we can show that $Px = xP$ for all $x \in B$ where $B$ was defined in (3.3). In particular $\Delta \circ E = (\text{id} \otimes E) \circ \Delta$. Using [21 Lemma 11] we extend $E$ to a unital normal conditional expectation $E : L^\infty(G) \to L^\infty(G)$ satisfying $\Delta \circ E = (\text{id} \otimes E) \circ \Delta$. Using [21 Theorem 5] we get a unique idempotent state $\omega \in C^*_0(G)^*$ such that $E(x) = \omega x$. Finally using [21 Theorem 1] we conclude that $E$ preserves $\varphi_G$, i.e. $B$ is expected. Thus (iii) $\Rightarrow$ (i) and we are done. $\square$

5. Normal coideals and compact quantum subgroups

In [19], among other results, the characterization of left-invariant $C^*$-subalgebras of $C_0(G)$ corresponding to compact quantum subgroups of $G$ were provided under coamenability assumption on $G$. In this section we give a characterization of left-invariant $C^*$-subalgebras of $C_0^n(G)$ corresponding to compact quantum subgroups with coamenability assumption dropped. Moreover, we characterize coideals of $L^\infty(G)$ corresponding to compact quantum subgroups of $G$.

5.1. Universal $C^*$-version. In this subsection we show that there is a 1-1 correspondence between compact quantum subgroups of a locally compact quantum groups and left-invariant, symmetric $C^*$-subalgebras of $C_0^n(G)$ equipped with a conditional expectation. In order to do it we prove a number of results in the context of $C_0^n(G)$ proved in [19] under the assumption that $C_0(G) = C_0^n(G)$. We do not give the proofs when they are essentially the same as those given in [19] (the main difference in the proof then is that we use $W^G$ in place of $W^G$). We present the proofs in case we were able to find simplifications. We will always assume that a left-invariant subalgebra $X \subset C_0^n(G)$ satisfies $\Lambda_G(X) \neq 0$ (see Proposition 5.2). In particular $X C_0^n(G) = C_0^n(G)$.

Definition 5.1. For a $C^*$-algebra $A$ and left-invariant $C^*$-subalgebra $X \subset C_0^n(G)$, a non-degenerate *-homomorphism $\rho : C_0^n(G) \to M(A)$ \textit{is} called $X$-trivial if for every $x \in X$, $\rho(x) = \varepsilon(x)1_{M(A)}$.

Denoting the set of all equivalence classes of non-degenerate $X$-trivial representations of $C_0^n(G)$ by $T_X$ we define an ideal $I_X = \bigcap_{\rho \in T_X} \ker \rho$.

The proof of the next theorem is essentially the same as the proof of [19 Theorem 2].

Theorem 5.2. Let $G$ be a locally compact quantum group and $X \subset C_0^n(G)$ a left-invariant $C^*$-subalgebra. There exists a compact quantum subgroup $H \subset G$ such that $C^*_n(H) = C_0^n(G)/I_X$.

Notation 5.3. Let $X \subset C_0^n(G)$ be a left-invariant $C^*$-subalgebra. Then the compact quantum subgroup of $G$ assigned to $X$ will be denoted by $H_X$.

Definition 5.4. Let $H$ be a compact quantum subgroup of $G$ and $\pi : C_0^n(G) \to C^*(H)$ the associated homomorphism. Define $F = (\ker \pi)^\perp \cap S(C_0^n(G))$. Then we will consider the following sets $X_{H^1} = \{ x \in C_0^n(G) \mid (\text{id} \otimes \mu)(\Delta^n_B(x)) = x \text{ for every } \mu \in F \}$, $X_{H^2} = \{ x \in C_0^n(G) \mid (\text{id} \otimes \theta)(\Delta^n_B(x)) = x \text{ where } \theta = \varphi_H \pi \}$, $X_{H^3} = \{ x \in C_0^n(G) \mid (\text{id} \otimes \pi)(\Delta^n_B(x)) = x \otimes 1_H \}$.

Lemma 5.5. Adopting the above notations we have $X_{H^1} = X_{H^2} = X_{H^3}$.

Proof. It is trivial that $X_{H^1} \subset X_{H^2}$. In order to prove that $X_{H^3} \subset X_{H^2}$ let us fix $\mu \in F$, then $\mu = \mu' \circ \pi$ where $\mu'$ is a state of $C^*_n(H)$. Thus if $x \in X_{H^3}$, then

$$(\text{id} \otimes \mu)(\Delta^n_B(x)) = x \mu'(1_H) = x$$

and $x \in X_{H^1}$.
Now take \( x \in X_{\mathbb{H}} \), i.e. \((\text{id} \otimes \vartheta)(\Delta_{\mathbb{H}}^u(x)) = x\). Then
\[
(id \otimes \pi)(\Delta_{\mathbb{H}}^u(x)) = (id \otimes \pi)((id \otimes \vartheta)\Delta_{\mathbb{H}}^u(x))
\]
\[
= (id \otimes id \otimes \varphi_{\mathbb{H}})((id \otimes \pi \otimes \pi)(\Delta_{\mathbb{H}}^{\pi \otimes \pi} \otimes id)\Delta_{\mathbb{H}}^u(x))
\]
\[
= (id \otimes id \otimes \varphi_{\mathbb{H}})((id \otimes \Delta_{\mathbb{H}}^u)(id \otimes \pi)\Delta_{\mathbb{H}}^u(x))
\]
\[
= (id \otimes \varphi_{\mathbb{H}})((id \otimes \pi)\Delta_{\mathbb{H}}^u(x)) \otimes 1 = x \otimes 1
\]
which shows that \( x \) is an element of \( X_{\mathbb{H}} \). Summarizing we proved that \( X_{\mathbb{H}}^2 \subset X_{\mathbb{H}} \subset X_{\mathbb{H}}^2 \subset X_{\mathbb{H}} \) thus we have the required equalities. \( \square \)

From now on we will denote these sets by \( X_{\mathbb{H}} \) and freely use Lemma 5.5. Note that \( X_{\mathbb{H}} = C_{\mathbb{H}}^0(G/\mathbb{H}) \). In particular \( X_{\mathbb{H}} \subset C_{\mathbb{H}}^0(G) \) is a non-degenerate left-invariant \( C^* \)-subalgebra.

Lemma 5.6. Let \( X \subset C_{\mathbb{H}}^0(G) \) be a left-invariant \( C^* \)-subalgebra. Then \( X \subset X_{\mathbb{H}} \).

Proof. Since \( \pi \) is X-trivial we have \( \pi((\mu \otimes id)(\Delta_{\mathbb{H}}^u(x))) = \mu(x)1 \) for all \( \mu \in C_{\mathbb{H}}^0(G)^* \) and \( x \in X \). Thus
\[
(id \otimes \pi)(\Delta_{\mathbb{H}}^u(x)) = x \otimes 1
\]
and we see that \( x \in X_{\mathbb{H}} \). \( \square \)

Theorem 5.7. Let \( \mathbb{H} \) be a compact quantum subgroup of a locally compact quantum group \( G \) and \( X_{\mathbb{H}} \subset C_{\mathbb{H}}^0(G) \) a left-invariant \( C^* \)-subalgebra assigned to \( \mathbb{H} \). Then \( X_{\mathbb{H}} \) is symmetric and the map \( E \) is a conditional expectation from \( C_{\mathbb{H}}^0(G) \) onto \( X_{\mathbb{H}} \) such that \( (id \otimes E) \circ \Delta_{\mathbb{H}} = \Delta_{\mathbb{H}} \circ E \).

Proof. The proof is almost the same as [19] Theorem 10] to be aware of differences we only prove that \( X_{\mathbb{H}} \) is symmetric. We compute
\[
(id \otimes id \otimes \pi)((id \otimes \Delta_{\mathbb{H}}^u)(\mathcal{W}(1 \otimes x)\mathcal{W}^*)) = (id \otimes id \otimes \pi)((\mathcal{W}_{12} \mathcal{W}_{13} \mathcal{W}_{14})(1 \otimes \Delta_{\mathbb{H}}^u(x))\mathcal{W}_{12} \mathcal{W}_{13} \mathcal{W}_{14})
\]
\[
= \mathcal{W}_{12} \mathcal{W}_{13}(1 \otimes x \otimes 1)\mathcal{V}_{12} \mathcal{W}_{13}
\]
\[
= \mathcal{W}_{12}(1 \otimes x \otimes 1)\mathcal{W}_{13}
\]
\[
= \mathcal{V}(1 \otimes x)\mathcal{W}^* \otimes 1
\]
where we define \( \mathcal{V} = (id \otimes \pi)\mathcal{W}^G \). Thus \( (id \otimes E)((\mathcal{W}(1 \otimes x)\mathcal{W}^*)) = \mathcal{W}(1 \otimes x)\mathcal{W}^* \) and we conclude by using (2.7). \( \square \)

Definition 5.8. Let \( X \) be a left-invariant subalgebra of \( C_{\mathbb{H}}^0(G) \). Then we define
\[
F_0 = \{ \mu \in S(C_{\mathbb{H}}^0(G)) : (id \otimes \mu)(\Delta_{\mathbb{H}}^u(x)) = x \text{ for every } x \in X \}
\]
The proof of the next lemma is essentially the same as [19] Lemma 3].

Lemma 5.9. (1) \( F_0 = \{ \mu \in S(C_{\mathbb{H}}^0(G)) : \mu = \varepsilon \text{ on } X \} \).

(2) If \( \mu \in F_0 \), then \( \mu(ax) = \mu(a)\mu(x) \) and \( \mu(xa) = \mu(x)\mu(a) \) for every \( a \in C_{\mathbb{H}}^0(G) \) and \( x \in X \).

The second claim of Lemma 5.9 has the following obvious extension.

Lemma 5.10. Let \( H \) be a Hilbert space, \( T \in \text{M}(K(H) \otimes X) \) and \( A \in \text{M}(K(H) \otimes C_{\mathbb{H}}^0(G)) \). Then \( (id \otimes \mu)(TA) = (id \otimes \mu)(T(id \otimes \mu)(A)) \) for all \( \mu \in F_0 \). Similarly \( (id \otimes \mu)(AT) = (id \otimes \mu)(A(id \otimes \mu)(T)) \).

The next lemma is the universal counterpart of [19] Lemma 4]. We give here a simple proof.

Lemma 5.11. Suppose that \( X \subset C_{\mathbb{H}}^0(G) \) is a left-invariant and symmetric \( C^* \)-subalgebra. For every \( \mu \in F_0 \) and \( a \in C_{\mathbb{H}}^0(G) \) such that \( \mu(\ast a) \neq 0 \), the functional \( \mu_{a}(b) := \mu(\ast ba)/\mu(\ast a) \) is in \( F_0 \).

Proof. It suffices to prove that \( \mu(axc) = \mu(x)\mu(ac) \) for all \( x \in X \) and \( a,c \in C_{\mathbb{H}}^0(G) \). Actually it suffices to prove the latter for \( a = (\omega \otimes id)(\mathcal{W}) \). We compute
\[
\mu(axc) = \omega((id \otimes \mu)(\mathcal{W}(1 \otimes xc))
\]
\[
= \omega((id \otimes \mu)(\mathcal{W}(1 \otimes x)\mathcal{W}^*\mathcal{W}(1 \otimes c))
\]
Theorem 5.15. \(\tau\) is by the scaling group.

Theorem 5.13 yields on a compact quantum subgroup 5.2.

Let \(\mu \in S(C^0_u(\mathbb{G}))\) is in our case \(F\) in our case.

Finally let us prove the universal counterpart of [19, Theorem 10].

\(\pi^* : C^u(\mathbb{H})^* \rightarrow C^0_u(\mathbb{G})^*\) is the adjoint of the quotient map \(\pi\).

Theorem 5.12.

Let \(G\) be a locally compact quantum group and \(X \subset C^0_u(G)\) a left-invariant symmetric \(C^*-\)subalgebra. A state \(\mu \in S(C^0_u(\mathbb{G}))\) is in \(F_0\) if and only if its GNS-representation is \(X\)-trivial. Moreover, if \((\mathbb{H}, \pi)\) is the compact subgroup associated to \(X\), then \(F_0 = \pi^*(S(C^u(\mathbb{H})))\) where \(\pi^* : C^u(\mathbb{H})^* \rightarrow C^0_u(\mathbb{G})^*\) is the adjoint of the quotient map \(\pi\).

Using Lemma 5.11 we can prove the universal counterpart of [19, Theorem 5]

\[
\begin{align*}
= \omega((id \otimes \mu)(\mathcal{H}^G(1 \otimes x)\mathcal{H}^{G^*})(id \otimes \mu)(\mathcal{H}^G(1 \otimes c))) \\
= \omega((id \otimes \varepsilon)(\mathcal{H}^G(1 \otimes x)\mathcal{H}^{G^*})(id \otimes \mu)(\mathcal{H}^G(1 \otimes c))) \\
= \varepsilon(x)\omega((id \otimes \mu)(\mathcal{H}^G(1 \otimes c))) \\
= \mu(x)\mu(ac)
\end{align*}
\]

where in the third equality we use symmetry of \(X\) and Lemma 5.10 with \(T = \mathcal{H}^G(1 \otimes x)\mathcal{H}^{G^*}\) and \(A = \mathcal{H}^G(1 \otimes b)\); in the fifth equality we use \((id \otimes \varepsilon)(\mathcal{H}^G) = 1\).

\(\square\)

Using Lemma 5.11 we can prove the universal counterpart of [19, Theorem 10].

Theorem 5.13.

Let \(G\) be a locally compact quantum group and \(X\) a non-zero, symmetric, left-invariant \(C^*-\)subalgebra of \(C^0_u(G)\) such that there is a conditional expectation \(E : C^0_u(G) \rightarrow C^0_u(G)\) onto \(X\) satisfying \((id \otimes E) \circ \Delta^u = \Delta^u \circ E\). Then \(X_{\text{Ha}x} = X\).

Proof. Using Lemma 5.10, we get \(X \subset X_{\text{Ha}x}\). Conversely, let \(a \in X_{\text{Ha}x}\). Clearly \(\varepsilon \circ E \in F_0\) and since in our case \(F_0 = F\) (see Theorem 5.12) we have

\[
a = (id \otimes \varepsilon \circ E)(\Delta^u(a)) = (id \otimes \varepsilon)(\Delta^u(E(a))) = E(a).
\]

So \(a\) is in the image of \(E\) which is \(X\).

\(\square\)

Let \(H\) be a compact quantum subgroup of \(G\). Then \(L^\infty(H)\) is a codual coideal of \(L^\infty(G/H)\).

Since \(L^\infty(G/H) = \Lambda_G(X_H)''\), \(X_H\) can be used to recover \(H\), i.e. the assignment \(H \mapsto X_H\) is injective. Theorem 5.13 yields \(X_H = X_{\text{Ha}x}\) and we conclude that \(H_{\text{Ha}x} = H\). Summarizing we get

Theorem 5.14.

Let \(G\) be a locally compact quantum group. There is a 1-1 correspondence between compact quantum subgroups of \(G\) and symmetric, left-invariant \(C^*-\)subalgebras \(X\) of \(C^0_u(G)\) equipped with a conditional expectation \(E : C^0_u(G) \rightarrow C^0_u(G)\) onto \(X\) such that \((id \otimes E) \circ \Delta^u = \Delta^u \circ E\).

5.2. Normal coideals and quantum subgroups.

In the next theorem we get a 1-1 correspondence between idempotent states of Haar type (i.e. the states corresponding to a Haar measure on a compact quantum subgroup \(H\) of \(G\)) and normal integrable coideals \(N \subset L^\infty(G)\) preserved by the scaling group.

Theorem 5.15. Let \(N\) be a normal integrable coideal von Neumann subalgebra of \(L^\infty(G)\) which is \(\tau_t\)-invariant. Then there exists a unique compact quantum subgroup \(H \subset G\) such that \(N = L^\infty(G/H)\).

Proof. Let \(\omega \in C^0_u(G)^*\) be the idempotent state corresponding to \(N\) as described in Theorem 12.

We define

\[
\begin{align*}
X' &= \{x \in C_0(G) \mid \omega \tau x = x\}^{\text{norm-cls}} \\
X'' &= \{y \in C^0_u(G) \mid (id \otimes \omega)(\Delta^u(y)) = y\}^{\text{norm-cls}}.
\end{align*}
\]

Let \(E : L^\infty(G) \rightarrow L^\infty(G)\) be the conditional expectation assigned to \(\omega\). The normality of \(N\) implies that \(X'\) is symmetric. Indeed

\[
W^G(1 \otimes X')W^{G^*} \subset M(C_0(\hat{G}) \otimes C_0(G))
\]

and

\[
(id \otimes E)(W^G(1 \otimes X')W^{G^*}) = W^G(1 \otimes X')W^{G^*}
\]

i.e.

\[
W^G(1 \otimes X')W^{G^*} \subset M(C_0(\hat{G}) \otimes X').
\]
Using Proposition 5.13 we conclude that $X^n$ is symmetric
$$W^G(1 \otimes X^n)W^G* \subset M(C_0(\widehat{G}) \otimes X^n)$$
so according to Theorem 5.13 there exists a compact quantum group $H$ of $G$ such that $X^n = C_0^n(G/H)$. Since $X^* = \Lambda_G(X^n) = \Lambda_G(C_0^n(G/H)) = C_0(G/H)$ we get $N = L^\infty(G/H)$. 

6. OPEN QUANTUM SUBGROUPS OF $G$ AND IDEMPOTENT STATES ON $\widehat{G}$

In this section we establish a 1-1 correspondence between open quantum subgroups of $G$ and central idempotent states on $\widehat{G}$.

**Theorem 6.1.** Let $H$ be an open quantum subgroup of a locally compact quantum group $G$. Then there exists a conditional expectation $E : L^\infty(\widehat{G}) \rightarrow L^\infty(H)$ such that $(\text{id} \otimes E) \circ \Delta_\widehat{G} = \Delta_H \circ E = (E \otimes \text{id}) \circ \Delta_\widehat{G}$. Conversely for a von Neumann subalgebra $N$ of $L^\infty(\widehat{G})$ equipped with a conditional expectation $E : L^\infty(\widehat{G}) \rightarrow L^\infty(G)$ onto $N$ satisfying

$$(\text{id} \otimes E) \circ \Delta_\widehat{G} = \Delta_H \circ E = (E \otimes \text{id}) \circ \Delta_\widehat{G} \tag{6.1}$$

there exists a unique open quantum subgroup $H$ of $G$ such that $N = L^\infty(H)$.

**Proof.** Since $H$ is an open quantum subgroup of $G$, then it is closed (see [10]) thus $L^\infty(H) \subset L^\infty(\widehat{G})$ is $\tau_\widehat{G}$-invariant. Furthermore the restriction of $\psi_\widehat{G}$ to $L^\infty(H)$ is semifinite [12, Corollary 3.4]. Using Theorem 4.2 there exists a conditional expectation $E : L^\infty(\widehat{G}) \rightarrow L^\infty(\widehat{H})$ onto $L^\infty(\widehat{H})$ such that $(\text{id} \otimes E) \circ \Delta_\widehat{G} = \Delta_\widehat{H} \circ E$. Since $L^\infty(\widehat{H})$ is preserved by $\Delta_\widehat{G}$ we can proceed as in the proof of Theorem 4.2 to get $(E \otimes \text{id}) \circ \Delta_\widehat{G} = \Delta_\widehat{H} \circ E$.

Conversely, suppose $N$ is a von Neumann subalgebra of $L^\infty(\widehat{G})$ equipped with conditional expectation $E$ satisfying (6.1). It is easy to see that in this case $N$ is an invariant subalgebra, i.e. $\Delta_\widehat{G}(N) \subseteq N \otimes N$. The following relations show that the restriction of $\psi_\widehat{G}$ and $\varphi_\widehat{G}$ to $L^\infty(H)$ are semifinite,

$$\psi_\widehat{G}(E(x)) = (\psi_\widehat{G} \otimes \text{id})\Delta_\widehat{G}(E(x)) = (\psi_\widehat{G} \otimes \text{id})(\text{id} \otimes E)\Delta_\widehat{G}(x) = E(1)\psi_\widehat{G}(x) = \psi_\widehat{G}(x),$$

$$\varphi_\widehat{G}(E(x)) = (\text{id} \otimes \varphi_\widehat{G})\Delta_\widehat{G}(E(x)) = (\text{id} \otimes \varphi_\widehat{G})(E \otimes \text{id})\Delta_\widehat{G}(x) = E(1)\varphi_\widehat{G}(x) = \varphi_\widehat{G}(x).$$

In particular $(N, \Delta_\widehat{G}|N, \varphi_\widehat{G}|N)$ is a locally compact quantum group which we shall denote by $\widehat{H}$. Using [10, Theorem 7.5] we see that $H$ can be identified with an open subgroup of $G$. 

The next corollary is the infinite-dimensional version of [8, Theorem 3.2] and also the generalization of [8, Theorem 4.1] and [11].

**Corollary 6.2.** Let $G$ be a locally compact quantum group. There is a 1-1 correspondence between open quantum subgroups of $G$ and central idempotent states $\omega$ on $\widehat{G}$, i.e. idempotent states $\omega \in C_0^n(\widehat{G})^*$ such that $\omega \varpi \mu = \mu \varpi \omega$ for all $\mu \in C_0^n(\widehat{G})^*$.

**Proof.** If $\omega \in C_0^n(\widehat{G})^*$ is an idempotent state satisfying $\omega \varpi \mu = \mu \varpi \omega$ for all $\mu \in C_0^n(\widehat{G})^*$ then the corresponding conditional expectation $E : L^\infty(\widehat{G}) \rightarrow L^\infty(G)$ satisfies (6.1).

Conversely, let $H$ be an open quantum subgroup of $G$, $\omega \in C_0^n(\widehat{G})^*$ the corresponding idempotent state and $E$ the conditional expectation. Then $(\text{id} \otimes E) \circ \Delta_\widehat{G} = (E \otimes \text{id}) \circ \Delta_\widehat{G}$ implies $\omega \varpi \mu = \mu \varpi \omega$ for all $\mu \in L^\infty(\widehat{G})_*$. Since $L^\infty(\widehat{G})_*$ forms a two sided ideal in $C_0^n(\widehat{G})^*$ we get

$$\nu \varpi \mu \varpi \omega = \nu \varpi \omega \varpi \mu = \omega \varpi \nu \varpi \mu$$

for all $\nu \in C_0^n(\widehat{G})^*$ and $\mu \in L^\infty(\widehat{G})_*$. Thus we conclude that

$$(\nu \varpi \omega \otimes \Lambda_\widehat{G}) \circ \Delta_{\widehat{G}}^n = (\omega \varpi \nu \otimes \Lambda_\widehat{G}) \circ \Delta_{\widehat{G}}^n.$$

Using Podleś condition for $\Delta_{\widehat{G}}^n$ we conclude that $\nu \varpi \omega = \omega \varpi \nu$ for all $\nu \in C_0^n(\widehat{G})^*$. 

\qed
Remark 6.3. Let \( \mathbb{H} \subset \mathbb{G} \) be an open quantum subgroup, \( \mathbb{I}_\mathbb{H} \) the corresponding central group-like projection as explained in the last paragraph of Section 2 and let \( \omega \in S(C_0^\ast(\hat{\mathbb{G}})) \) be the corresponding idempotent state. It is observed in [11] that \( \mathbb{I}_\mathbb{H} = (\omega \otimes \text{id})(W^G) \). Thus using Corollary 6.2 and the results of [11] we see that a central projection \( P \in L^\infty(\hat{\mathbb{G}}) \) is a group-like projection if and only if there exists a central idempotent state \( \omega \in C_0(\hat{\mathbb{G}}) \) such that \( P = (\omega \otimes \text{id})(W^G) \).

7. Normal coideals assigned to open quantum subgroups

In this section we characterize normal coideals \( N \subset L^\infty(G) \) corresponding to open quantum subgroups. Roughly speaking \( N = L^\infty(G/\mathbb{H}) \) for \( \mathbb{H} \) open in \( G \) if and only if \( N \) admits an atom.

Theorem 7.1. Let \( N \subset L^\infty(G) \) be a normal coideal admitting a minimal projection \( P \in N \) which is central \( P \in Z(N) \). Suppose that

\[
\mathbb{I} \in \{ (\text{id} \otimes \omega)((\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P)) \mid \omega \in B(L^2(\mathbb{G}))^\ast \}^\ast\text{-weak cls}. \tag{7.1}
\]

Then there exists an open subgroup \( \mathbb{H} \subset G \) such that \( N = L^\infty(G/\mathbb{H}) \). Conversely, if \( \mathbb{H} \subset G \) is an open quantum subgroup given by \( \pi : L^\infty(G) \to L^\infty(\mathbb{H}) \) then the central support \( \mathbb{I}_\mathbb{H} \) of \( \pi \) is a minimal central projection in \( L^\infty(G/\mathbb{H}) \) satisfying (7.1).

Proof. Since \( P \) is minimal and central, there exists \( x \in L^\infty(\hat{\mathbb{G}}) \) such that

\[
W^G(\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P) = x \otimes P
\]

Thus

\[
(\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P) = (\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P)(x \otimes 1). \quad \tag{7.2}
\]

Applying \( (\text{id} \otimes \omega) \) to (7.2), we conclude that

\[
(\text{id} \otimes \omega)((\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P)) = (\text{id} \otimes \omega)((\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P))x.
\]

Thus (7.1) yields \( x = \mathbb{I} \), i.e.

\[
(\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P) = W^G \ast (\mathbb{I} \otimes P). \tag{7.3}
\]

Let \( \omega \in B(L^2(\mathbb{G}))^\ast \). Slicing (7.3) with \( (\omega \otimes \text{id}) \) we get \( PaP = aP \) for all \( a \in L^\infty(G) \). In particular, \( a^*P = Pa^*P = (PaP)^* = (aP)^* = Pa^* \), i.e. \( P \) is central.

Using minimality and centrality of \( P \) again, we see that for all \( x \in N \) there exists \( y \in L^\infty(G) \) such that

\[
\Delta_G(x)(1 \otimes P) = y \otimes P
\]

i.e. \( W^G(x \otimes 1)W^G \ast (1 \otimes P) = y \otimes P \) which implies

\[
(x \otimes 1)W^G \ast (1 \otimes P) = W^G \ast (1 \otimes P)(y \otimes 1).
\]

This in turn implies that

\[
(x \otimes 1)(\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P) = (\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P)(y \otimes 1).
\]

Slicing with \( (\text{id} \otimes \omega) \) and using (7.1) we get \( x = y \). Thus

\[
\Delta_G(x)(1 \otimes P) = x \otimes P \quad \text{for all } x \in N.
\]

In particular, \( P \in N \) is a group-like projection. Let \( \mathbb{H} \) be an open quantum subgroup of \( \mathbb{G} \) assigned to \( P \), i.e. \( P = \mathbb{I}_\mathbb{H} \). Using (7.4) we see that \( N \subset L^\infty(G/\mathbb{H}) \). Using [10] Theorem 3.3 we get the converse containment \( L^\infty(G/\mathbb{H}) \subset N \). Thus \( N = L^\infty(G/\mathbb{H}) \).

For the converse we use [10] Proposition 3.2 which yields minimality of \( \mathbb{I}_\mathbb{H} \in L^\infty(G/\mathbb{H}) \). In order to see that \( \mathbb{I}_\mathbb{H} \) satisfies (7.1) we note that, under the identification \( L^\infty(\mathbb{H}) = \mathbb{I}_\mathbb{H}L^2(\mathbb{G}) \), the bicharacter \( V \in L^\infty(\hat{\mathbb{G}}) \otimes L^\infty(\mathbb{H}) \) is identified with \( W^G(\mathbb{I} \otimes \mathbb{I}_\mathbb{H}) \). In particular

\[
\mathbb{I} \in L^\infty(\hat{\mathbb{H}}) = \{ (\text{id} \otimes \omega)((\mathbb{I} \otimes P)W^G \ast (\mathbb{I} \otimes P)) \mid \omega \in B(L^2(\mathbb{G}))^\ast \}^\ast\text{-weak cls}.
\]

□
Let us give another characterizations of normal coideals of $N \subset L^\infty(G)$ corresponding to open quantum subgroups. In order to formulate it we shall denote $\Delta_G|_N = \alpha$ and $\beta : N \to L^\infty(\hat{G}) \otimes N$ where $\beta(x) = W^G(1 \otimes x)W^{G*}$.

**Theorem 7.2.** Let $N \subset L^\infty(G)$ be a normal coideal. If $N$ admits a normal $*$-homomorphism $\varepsilon : N \to C$ such that

\[
(id \otimes \varepsilon) \circ \beta = 1 \cdot \varepsilon \\
(id \otimes \varepsilon) \circ \alpha = id_N
\]

then there exists an open quantum subgroup $\mathbb{H} \subset G$ such that $N = L^\infty(G/\mathbb{H})$. Conversely, if $\mathbb{H} \subset G$ is open then $L^\infty(G/\mathbb{H})$ admits a normal $*$-homomorphism $\varepsilon : N \to C$ satisfying above conditions.

**Proof.** Let $P$ be the support of $\varepsilon$. Then $(id \otimes \varepsilon) \circ \beta = 1 \cdot \varepsilon$ implies

\[
W^G(1 \otimes P)W^{G*}(1 \otimes P) = (1 \otimes P)
\]

which then implies that $P$ is central. Similarly $(id \otimes \varepsilon) \circ \alpha = id_N$ yields

\[
\Delta_G(x)(1 \otimes P) = x \otimes P
\]

for all $x \in N$. In particular $P$ is a group-like projection corresponding to an open subgroup $\mathbb{H} \subset G$. Proceeding as in the proof of Theorem 7.1 we get the identification $N = L^\infty(G/\mathbb{H})$. For the converse we consider the restriction of $\pi : L^\infty(G) \to L^\infty(\mathbb{H})$ to $L^\infty(G/\mathbb{H})$ which yields the required $\varepsilon$ via $\pi(x) = \varepsilon(x)1$ for all $x \in L^\infty(G/\mathbb{H})$.

\[\square\]

**8. Appendix**

In this section we will show that a Woronowicz-closed quantum subgroup $\mathbb{H}$ of a locally compact quantum group $G$ has the Haagerup property if $G$ has it. Assuming coamenability of $G$ the result was proved in [6] Proposition 5.8.

Let $U \in M(C_0(G) \otimes \mathcal{K}(H))$ be a unitary. We say that $U$ is a unitary representation of a quantum group $G$ on $H$ if $(\Delta_G \otimes id)(U) = U_{13}U_{23}$. A unitary representation $U$ of $G$ admits a unique unitary lift $\tilde{U} \in M(C_0(G) \otimes \mathcal{K}(H))$ such that

\[(\Delta^u_G \otimes id)(\tilde{U}) = U_{13}U_{23}\]

satisfying $(\Delta^u_G \otimes id)(U) = U$.

**Definition 8.1.** A unitary representation $U \in M(C_0(G) \otimes \mathcal{K}(H))$ of $G$ on a Hilbert space $H$ is mixing if for all $\xi, \eta \in H$, $(id \otimes \omega_{\xi,\eta})(U) \in C_0(G)$.

**Lemma 8.2.** Let $U \in M(C_0(G) \otimes \mathcal{K}(H))$ be a mixing representation of $G$. Then $U \in M(C_0(G) \otimes \mathcal{K}(H))$ is mixing, i.e.

\[
\{ (id \otimes \omega)U : \omega \in B(H)_* \} \subset C_0^u(G).
\]

**Proof.** Note that we have $(\Delta^{u*}_G \otimes id)(U) = U_{13}U_{23}$ and for every $\omega \in B(H)_*$,

\[
\Delta_G^{u*}((id \otimes \omega)U) = (id \otimes id \otimes \omega)(U_{13}U_{23})
\]

(8.2)

Since $(id \otimes \omega)(U) \in C_0(G)$ for all $\omega \in B(H)_*$ we can use Podleś condition and get

\[
(\mu \otimes id)((\Delta^{u*}_G(id \otimes \omega)U)) \in C_0^u(G)
\]

for all $\mu \in B(L^2(G))_*$. On the other hand using (8.2) we get

\[
(\mu \otimes id)((\Delta^{u*}_G(id \otimes \omega)U)) = (id \otimes \omega \cdot a)(U) \in C_0^u(G)
\]

(8.3)

where $a = (\mu \otimes id)(U)$. Since $\{ (\mu \otimes id)U : \mu \in B(L^2(G))_* \}$norm-clss forms a $C^*$-algebra acting non-degenerately on $H$ we conclude (8.1) from (8.3).

Using Lemma 8.2 and [6] Remark 5.9 we get

**Corollary 8.3.** Let $\mathbb{H}$ be a Woronowicz-closed quantum subgroup of $G$. If $G$ has the Haagerup property. Then $\mathbb{H}$ has the Haagerup property.
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