ON THE EXISTENCE OF THREE NON-NEGATIVE SOLUTIONS FOR
(p,q)-LAPLACIAN SYSTEM

DEBANGANA MUKHERJEE AND TUHINA MUKHERJEE

Abstract. The present paper studies the existence of weak solutions for
\[
\begin{aligned}
(\mathcal{P}) \quad & \left\{ \begin{aligned}
(-\Delta)^{s_1}_{p_1} u &= \lambda f_1(x,u,v) + g_1(x,u) \quad \text{in } \Omega, \\
(-\Delta)^{s_2}_{p_2} v &= \lambda f_2(x,u,v) + g_2(x,v) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned} \right.
\end{aligned}
\]
where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain with smooth boundary, \( s_1, s_2 \in (0,1) \), \( 1 < p_i < \frac{N}{s_i} \), \( i = 1, 2 \), \( f_i \) and \( g_i \) has certain growth assumptions for \( i = 1, 2 \). We prove existence of at least three non negative solutions of \( \mathcal{P} \) under restrictive range of \( \lambda \) using variational methods. As a consequence, we also conclude that a similar result can be obtained when we consider a more general non local operator \( \mathcal{L}_{\phi_i} \) instead of \( (-\Delta)^{s_i}_{p_i} \) in \( \mathcal{P} \).

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1. Introduction

In the present article, we study the following non-local system of semilinear elliptic equations:
\[
\begin{aligned}
(\mathcal{P}) \quad & \left\{ \begin{aligned}
(-\Delta)^{s_1}_{p_1} u &= \lambda f_1(x,u,v) + g_1(x,u) \quad \text{in } \Omega, \\
(-\Delta)^{s_2}_{p_2} v &= \lambda f_2(x,u,v) + g_2(x,v) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{aligned} \right.
\end{aligned}
\]
where \( \Omega \subset \mathbb{R}^N \) is a smooth bounded domain in with smooth boundary, \( s_1, s_2 \in (0,1) \), \( 1 < p_i < \frac{N}{s_i} \), \( i = 1, 2 \), the operators \( (-\Delta)^{s_i}_{p_i} \) for \( i = 1, 2 \) are defined as:
\[
(-\Delta)^{s_i}_{p_i} u(x) = \lim_{\varepsilon \to 0} \frac{1}{|\mathcal{B}_\varepsilon(x)|} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(y) - u(x)|^{p_i-2}(u(y) - u(x))}{|x - y|^{N+sp_i}} \, dy, \quad x \in \mathbb{R}^N, \tag{1.1}
\]
and the functions \( f_i, g_i, i = 1, 2 \) are Carathéodory functions, \( f_i : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \), \( g_i : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+ \) satisfying some appropriate conditions which are mentioned in next.

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section. When \( s_1 = s_2 \) the equation reduces to a \((p, q)\) Laplacian problem which appears in a more general reaction-diffusion system

\[
u_t = \text{div}(a(u)\nabla u) + g(x, u) \tag{1.2}
\]

where \( a(u) = |\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u \). Such problems have a wide range of applications in physics and related sciences such as biophysics, plasma physics, and chemical reaction design, etc. where \( u \) describes a concentration, and the first term on the right-hand side of \((1.2)\) corresponds to a diffusion with a diffusion coefficient \( a(u) \); the term \( g(x, u) \) stands for the reaction, related to sources and energy-loss processes. A lot of attention has been given to the study of \((p, q)\)-Laplace equations in the last few years, for instance refer \[12\] \[10\] \[13\] \[15\] \[16\] \[17\].

Recently, the research community of partial differential equations has been attracted towards the study of fractional \((p, q)\)-Laplacian problems and hence, a good amount of literature has been constructed related to this. We start from the article of Chen and Bao \[6\] where they studied existence, nonexistence and multiplicity of the following \((p, q)\)-fractional Laplacian equation over \( \mathbb{R}^N \),

\[
(-\Delta)^s_p u + a(x)|u|^{p-2}u + (-\Delta)^s_q u + b(x)|u|^{q-2}u + \mu(x)|u|^{r-2}u = \lambda h(x)|u|^{m-2}u, \quad x \in \mathbb{R}^N,
\]

with appropriate assumptions on the variables and functions. Next, Bhakta and Mukherjee \[2\] studied the following problem in a bounded domain

\[
(-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u = \theta V(x)|u|^{r-2}u + |u|^{p^*_i - 2}u + \lambda f(x, u) \quad \text{in } \Omega,
\]

\[
\begin{align*}
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*}
\]

where \(0 < s_2 < s_1 < 1 < r < q < p < N/s_1, p^*_i = \frac{Np}{N-s_i p}, s_i \in (0, 1) \) for \( i = 1, 2 \) and \( V \) and \( f \) are some appropriate functions. They proved that there exist weak solutions of the above problem for some range of \( \lambda, \theta \). Also, for \( V(x) \equiv 1, \lambda = 0 \) and assuming certain other conditions on \( n, q, r \), they proved the existence of \( \text{cat}_\Omega(\Omega) \) non negative solutions by using Lusternik-Schnirelmann category theory. Using the Nehari manifold technique, Goel et. al \[11\] proved multiplicity results for the following problem over bounded domain

\[
(-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u = \lambda a(x)|u|^{s_1-2}u + b(x)|u|^{r-2}u \quad \text{in } \Omega,
\]

\[
\begin{align*}
u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{align*}
\]

where \(1 < \delta \leq q \leq p < r \leq p^*_1, 0 < s_2 < s_1 < 1, N > ps_1, \beta > 0 \) and \( a, b \) are sign changing functions. Under appropriate conditions on the parameters, they discuss both sub critical and critical case in the article and also establish that any weak solution of \((1.3)\) belongs to \(L^\infty(\Omega) \cap C^0_{\text{loc}}(\Omega)\) for \( \alpha \in (0, 1) \) when \( 2 \leq q \leq p < r < p^*_1 \). Regularity results for the equation

\[
(-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u = f(x, u) \quad \text{in } \mathbb{R}^N,
\]

where \(0 < s < 1\) and \(2 \leq q \leq p < N/s\) has been studied in \[1\]. Alves, Ambrosio and Isernia dealt with the following class of problems

\[
(-\Delta)^{s_1}_p u + (-\Delta)^{s_2}_q u + V(x)|u|^{p-2}u + |u|^{q-2}u = f(u) \quad \text{in } \mathbb{R}^N,
\]

in \[3\], where \(0 < s < 1\) and \(2 \leq q \leq p < N/s\). Imposing appropriate assumptions on \( V \) and \( f \), authors proved existence of ground state solution concentrating on a minimum point of \( V \), multiplicity of solutions using Lusternik-Schnirelmann category theory and boundedness of solutions. Very recently, an eigenvalue problem for fractional \((p-q)\) Laplace operator has been studied by Nguyen and Vo in \[14\] which is of independent interest. A class of variable exponent \((p, q)\)-fractional Laplacian problems with variable exponents and
indefinite weights has been studied in [8]. We also cite [5, 15, 18] as some recent articles in this context for interested readers.

The essence of our article lies in the fact that we study a system of equation with fractional \((p, q)\)-Laplace operator over a bounded domain which is first of its kind in literature where we establish existence of three non trivial solutions to \((P)\). The approach is variational and inspired by the articles [3, 19] and adapted in the fractional framework. In the same article, we motivate that this problem can be extended to considering more general non local operator \(L_\phi\) instead of \((-\Delta)_p^\alpha\).

This article has been fragmented into five sections- Section 2 contains preliminaries, assumptions on \(f_i\) and \(g_i\) and statements of main results of our article. Proof of Theorem 2.4 is given in Section 3. We have proved Theorem 2.3 in Section 4. Lastly, section 5 comprises proof of Theorem 2.4.

**Notations** - \(a \lor b = \max\{a, b\}, \ a \land b = \min\{a, b\}, \ |u|_r = \left(\int_\Omega |u(x)|^r \, dx\right)^{1/r}\) for any \(r > 1\).

2. Preliminaries

In this section, we define appropriate function spaces which are required for our analysis. Let \(p > 1, \ s \in (0, 1), \ N > ps, \ p_s^* := \frac{Np}{N - sp}\). We denote the standard fractional Sobolev space by \(W^{s,p}(\Omega)\) endowed with the norm

\[
\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy\right)^{1/p}.
\]

We set \(Q := \mathbb{R}^2_+ \setminus (\Omega^c \times \Omega^c)\), where \(\Omega^c = \mathbb{R}^N \setminus \Omega\) and define

\[
X_{s,p}(\Omega) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable} \ \ |u| \in L^p(\Omega) \text{ and } \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy < \infty \right\}.
\]

The space \(X_{s,p}(\Omega)\) is endowed with the norm defined as

\[
\|u\|_{s,p} := |u|_p + \left(\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy\right)^{1/p}.
\]

We note that in general \(W^{s,p}(\Omega)\) is not same as \(X_{s,p}(\Omega)\) as \(\Omega \times \Omega\) is strictly contained in \(Q\). We define the space \(X_{0,s,p}(\Omega)\) as

\[
X_{0,s,p}(\Omega) := \left\{ u \in X_{s,p} : u = 0 \ \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}
\]

or equivalently as \(C_0^\infty(\Omega)_{X_{s,p}(\Omega)}\). It is well-known that for \(p > 1\), \(X_{0,s,p}(\Omega)\) is a uniformly convex Banach space endowed with the norm

\[
\|u\|_{0,s,p} = \left(\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N + sp}} \, dx \, dy\right)^{1/p}.
\]

Since \(u = 0\) in \(\mathbb{R}^N \setminus \Omega\), the above integral can be extended to all of \(\mathbb{R}^N\). The embedding \(X_{0,s,p}(\Omega) \hookrightarrow L^r(\Omega)\) is continuous for any \(r \in [1, p_s^*]\) and compact for \(r \in [1, p_s]\). Moreover, for \(1 < q \leq p\), \(X_{0,s,1}(\Omega) \subset X_{0,s,2,q}(\Omega)\) (see Lemma 2.2 in Section 2 of [2]).

Let us recall some topological tools which will be used to study our problem \((P)\).

**Definition 2.1. (Nehari Manifold):** Let \(J \in C^1(X, \mathbb{R})\) be such that \(J'(0) = 0\), then the

\[
\mathcal{N} := \{ u \in X : \langle J'(u), u \rangle = 0, \ u \neq 0 \}
\]

is called a Nehari manifold of \(X\).
Definition 2.2. Let $X$ be a Banach space, $\eta : X \to \mathbb{R}$ be a function. For $c \in \mathbb{R}$, let us consider the punctured level set of $\eta$ at $c$ by

$$L^c_\eta = \{ u \in X : \eta(u) = c, u \neq 0 \}.$$  

We say $L^c_\eta$ has the sphere property, if the following hypotheses are satisfied:

(i) $\eta \in C(X)$;

(ii) there is a homeomorphic mapping between $L^c_\eta$ and the unit sphere of $X$;

(iii) for any fixed $u \in X \setminus \{0\}$, there exists a unique $t_u \in (0, \infty)$ such that $f(t_u u) = c$;

(iv) $X$ is separated into two open connected subsets by $L^c_\eta$ and the origin is contained in one of these subsets.

Theorem 2.1. Let $X, Y$ be Banach spaces with the following direct sum decomposition:

$$X = X_1 \oplus X_2, Y = Y_1 \oplus Y_2,$$

where $X_1, Y_1$ are finite dimensional subspaces of $X, Y$ respectively. Let $\eta \in C(X), \kappa \in C(Y)$, $c, d \in \mathbb{R}$ and $L^c_\eta, L^d_\kappa$ have the sphere property. Let $(e_x, e_y) \in X_2 \times Y_2$ such that $\eta(e_x) > c$ and $\kappa(e_y) > d$. Let us denote

$$Q^X = \{ u + te_x : u \in X_1 \cap B_{R_1}, t \in [0, 1]\},$$

$$Q^Y = \{ v + se_y : v \in Y_1 \cap B_{R_2}, s \in [0, 1]\},$$

$$N^X_1 = L^c_\eta \cap X_2, N^Y_1 = L^d_\kappa \cap Y_2, Q = Q^X \times Q^Y,$$

and there exist constants $q_i \in (p_i, p_i^*)$ and $C_i > 0$ such that

$$|g_i(x, u)| \leq C_i (|u|^{p_i - 1} + |u|^{q_i - 1})$$

for all $(x, u) \in \Omega \times \mathbb{R} \times \mathbb{R}$.

2.1. Assumptions on $f_i$ and $g_i$, $i = 1, 2$. We assume the following:

(A1) There exists a function $F \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ such that

$$\left( \frac{\partial F}{\partial u}(x, u, v), \frac{\partial F}{\partial v}(x, u, v) \right) = (f_1(x, u, v), f_2(x, u, v))$$

for all $(x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}$.

(A2) For $i = 1, 2$, $g_i \in C(\Omega \times \mathbb{R}^+, \mathbb{R}^+)$ and there exist constants $q_i \in (p_i, p_i^*)$ and $C_i > 0$ such that

$$|g_i(x, u)| \leq C_i (|u|^{p_i - 1} + |u|^{q_i - 1})$$

for all $(x, u) \in \Omega \times R^+$.

(A3) There exists constant $\alpha_i > p_1 \vee p_2$ such that

$$0 < \alpha_i G_i(x, u) \leq u g_i(x, u) \quad \text{for} \quad (x, u) \in \Omega \times (\mathbb{R}^+ \setminus \{0\}),$$

where $G_i(x, u) = \int_0^u g_i(x, \tau) \, d\tau; i = 1, 2$.

(A4) There holds for $i = 1, 2$,

$$\limsup_{u \to 0} \frac{g_i(x, u)}{|u|^{p_i - 2} u} < \lambda_{1, p_i} \quad \text{uniformly w.r.t} \quad x \in \Omega,$$

where $\lambda_{1, p_i}$ are the first eigenvalue of $(-\Delta)_{p_i}^{s_i}$ in $X_{0, s_i, p_i}$, that is,

$$\lambda_{1, p_i} = \inf_{u \in X_{0, s_i, p_i} \setminus \{0\}} \frac{||u||_{p_i}^{p_i}}{||u||_{p_i}^{p_i}}.$$

(A5) There holds,

$$\frac{g_i(x, u)}{|u|^{p_i - 1}}$$

is an increasing function of $u \in \mathbb{R}^+ \setminus \{0\}$.

(A6) There exists constant $q \in (1, p_i^* \wedge p_2^*)$ and $C_3 > 0$ such that

$$|f_1(x, u, v)| + |f_2(x, u, v)| \leq C_3 (|u|^{q - 1} + |v|^{q - 1} + 1).$$
have a unique positive solution, then for any non-negative solutions; among them, one is nontrivial positive solution.

We may consider more general nonlocal operator, for example, we consider $X$ of variational methods in the space $\Lambda$.

An element

\[ f_i(x, u, v) = \frac{\partial F}{\partial u}(x, u, v) = (q - 2)u^{q-3}v^2 + 2uv^{q-2}, \]

for $x \in \Omega$ and $u, v > 0$. Then, we have,

\[ f_1(x, u, v) = \frac{\partial F}{\partial v}(x, u, v) = 2uv^{q-2} + (q - 2)v^{q-3}u^2. \]

Choosing $q_i \in (p_1 \lor p_2, q_1 \land p_2)$ for $i = 1, 2$, we can take $g_1(x, u) = u^{q_1-1}$ and $g_2(x, v) = v^{q_2-1}$ for $x \in \Omega$ and $u, v > 0$. Our first main result in the article is the following:

**Theorem 2.2.** Let $f_i, g_i$ satisfy (A1)-(A7) for $i = 1, 2$, together with the condition

\[ f_i(x, u, 0) = f_i(x, 0, v) = 0, \quad \text{for } i = 1, 2, \quad (2.1) \]

for a.e. $x \in \Omega$, for all $u, v \in \mathbb{R}^+$. Also, we assume $F \in C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ in (A1). Then, there exists $\Lambda > 0$ such that for any $\lambda \in (0, \Lambda)$, system $(P)$ has at least three non-negative solutions. Moreover, if both the problem, for $i = 1, 2$,

\[ (P') \begin{cases} (-\Delta)_{\phi_i}^s u = g_i(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \]

have a unique positive solution, then for any $\lambda \in (0, \Lambda)$, system $(P)$ has at least three non-negative solutions; among them, one is nontrivial positive solution.

We shall prove the existence of weak solutions of $(P)$ for $\lambda = 1$, $g_1 \equiv 0 \equiv g_2$ by means of variational methods in the space $X := X_{0, s_1, p_1}(\Omega) \times X_{0, s_2, p_2}(\Omega)$ endowed with the norm given by $\|(u, v)\| = \|u\|_{0, s_1, p_1} + \|v\|_{0, s_2, p_2}$ for all $(u, v) \in X$.

**Definition 2.3.** An element $(u, v) \in X$ is said to be a weak solution of $(P)$ if $(u, v)$ satisfies

\[ \langle J(u, v), (w, z) \rangle = 0 \quad \text{for all } (w, z) \in X. \]

Our second main out-turn in the article is the following:

**Theorem 2.3.** Let $\lambda = 1$, $g_1 \equiv 0 \equiv g_2$ and $f_1, f_2$ satisfy (A1) with assumption $F \in C^1(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$, (A6), (A8) and $h(x) \leq \lambda$ in $\Omega$ and $h(x) < \lambda$ on a subset of $\Omega$ with positive measure, $\lambda = \min\{\lambda_{1, p_1}, \lambda_{1, p_2}\}$; $\lambda_{i, p_i}$'s are defined in (A4). Then, there exists a weak solution $(u, v) \in (X_{0, s_1, p_1}(\Omega) \times X_{0, s_2, p_2}(\Omega))$ of $(P)$.

**Remark 2.1.** We may consider more general nonlocal operator, for example, we consider the following system of non-local quasilinear elliptic equations:

\[ \begin{cases} -\mathcal{L}_{\phi_1} u = f_1(x, u, v) & \text{in } \Omega, \\ -\mathcal{L}_{\phi_2} v = f_2(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \]
where the operators $L_{\phi_i}, i = 1, 2$ are defined by:

\[
\langle -L_{\phi_1} u, w \rangle = \int_{\mathbb{R}^{2N}} \phi_1(u(x) - u(y))(w(x) - w(y))K_1(x, y) \, dx \, dy,
\]

and

\[
\langle -L_{\phi_2} u, w \rangle = \int_{\mathbb{R}^{2N}} \phi_2(u(x) - u(y))(z(x) - z(y))K_2(x, y) \, dx \, dy,
\]

for all $w, z \in C_c^\infty(\Omega)$, the functions $\phi_i, i = 1, 2$ are assumed to be continuously differentiable satisfying $\phi_i(0) = 0, i = 1, 2$,

the function $t \mapsto t\phi_i(t)$ is convex, \hspace{2cm} (2.2)

and there exists $\gamma_i > 0, i = 1, 2$ such that

\[
\frac{1}{\gamma_i} |t|^{p_i} \leq \phi_i(t) t \leq \gamma_i |t|^{p_i}, \text{ for all } t \in \mathbb{R},
\]

and $K_i : \mathbb{R}^N \to \mathbb{R}$ are assumed to be measurable, symmetric and satisfy for some $\delta_i \geq 1,$

\[
\frac{1}{\delta_i |x - y|^{N + s_i p_i}} \leq K_i(x, y) \leq \frac{\delta_i}{|x - y|^{N + s_i p_i}}, \hspace{2cm} (2.3)
\]

for all $x, y \in \mathbb{R}^N$. With this in hand, we have the following outcome.

**Theorem 2.4.** Let $\lambda = 1, g_1 \equiv 0 \equiv g_2$ and $f_1, f_2$ satisfy (A1),(A6), (A8) and $h(x) \leq \frac{\lambda}{\max\{\gamma_1, \gamma_2\}}$ in $\Omega$ and $h(x) < \frac{\lambda}{\max\{\gamma_1, \gamma_2\}}$ on a subset of $\Omega$ with positive measure, $\lambda = \min\{\lambda_{1, p_i}, \lambda_{2, p_i}\}$; $\lambda_{i, p_i}$’s are defined by

\[
\lambda_{i, p_i} = \inf_{u \in X_{0, s_i p_i} \setminus \{0\}} \frac{\langle -L_{\phi_i} u, u \rangle}{||u||_{p_i}^{p_i}}.
\]

Then, there exists a weak solution $(u, v) \in (X_{0, s_1 p_1}(\Omega) \times X_{0, s_2 p_2}(\Omega))$ of (Q).

### 3. Proof of Theorem 2.2

For $i = 1, 2$, let us first define the functional $J_i : X_{0, s_i p_i} \to \mathbb{R}$ by

\[
J_i(u) = \frac{1}{p_i} ||u||_{p_i}^{p_i} - \int_{\Omega} G_i(x, u) \, dx, \hspace{0.5cm} u \in X_{0, s_i p_i}
\]

where $G_i$’s are defined in (A3). Let us denote the Nehari manifold of $J_i$ on $X_{0, s_i p_i}$ by $N_i$ for $i = 1, 2$ that is

\[
N_i = \left\{ u \in X_{0, s_i p_i} \setminus \{0\} : ||u||_{s_i p_i} = \int_{\Omega} g_i(x, u) u \, dx \right\}.
\]

We prove the following result.

**Theorem 3.1.** Let $g_i$ for $i = 1, 2$ satisfy (A2)-(A5). For $i = 1, 2$, let $e_i \in X_{0, s_i p_i}$ be such that $J_i(e_i) > 0$. Let us define

\[
Q := \left\{ \lambda(e_1, 0) + (1 - \lambda)(0, e_2) : \lambda \in [0, 1] \right\} \subset X
\]

and $N = N_1 \times N_2$, where $N_i$’s are defined above. Then $\partial Q$ links $N$.

**Proof.** We prove this result in two steps.

**Step-1.** The Nehari manifold $N_i$’s both have the sphere property. The proof is similar to the proof of Lemma 4.1 of [9], on page 72, where the functional $\varphi$ is replaced with $J_i$. So we omit it.
Step-2. We note that $J_i$’s satisfy the following for $i = 1, 2$,

$$J_i(u) = \begin{cases} 0, & u = 0, \\ > 0, & u = e_i. \end{cases}$$

By assumption (A3), we can get that $\nabla J_i(x, u) \leq CG_i(x, u)$ for some constant $C > 0$ where $(x, u) \in \Omega \times \mathbb{R}^+$. This gives that $\lim_{t \to \infty} J_i(te_i) = -\infty$ since $\alpha_i \leq \max\{p_1, p_2\}$ from which we note that there exists $t_i \in (0, 1)$ such that

$$\langle J_i'(te_i), e_i \rangle = \begin{cases} 0, & t = t_i, \\ > 0, & 0 < t < t_i, \\ < 0, & t_i < t < \infty. \end{cases}$$

Hence, we have $\langle J_i'(e_i), e_i \rangle < 0$. Using Step-1 along with Theorem 3.1 we conclude that $\partial Q$ links $N$. 

With this machinery in hand, we commence to prove our first main result.

**Proof of Theorem 2.2** We prove this result in three steps. To prove this result, we will use Theorem 3.1 to the functional $J_i$. In the first step, we show that $J$ satisfies Palais-Smale condition. Then, in the next step, for the functional $J_i$, we consider the few critical levels and values corresponding to $J_i$. With these critical values in hand, we prove that infimum over such Nehari manifolds for the functionals $J_1$ and $J_2$ are achieved and they form two solutions to our problem. Using these, we will construct a set $Q$ (defined in Theorem 3.1) such that $Q$ links $N$. Lastly, we prove the existence of our third solution using our step-(1). This yields our result.

**Step-1.** Let us consider the cone $X^+ = \{(u, v) \in X : u, v \geq 0\}$ and define the following

$$J(u, v) = J_1(u) + J_2(v) - \lambda \int_{\Omega} F(x, u, v) \, dx \quad \text{for all } (u, v) \in X^+. \quad (3.1)$$

In this step, we show that $J$ satisfies (PS) condition. Let $\{(u_n, v_n)\} \subset X^+$ be a sequence such that

$$\{J(u_n, v_n)\}_{n \geq 1} \text{ is bounded, } J'(u_n, v_n) \to 0, \quad n \to \infty. \quad (3.2)$$

Then there exists $C_5 > 0$ such that $J(u_n, v_n) \leq C_5$ for all $n \in \mathbb{N}$, that is,

$$J_1(u_n) + J_2(v_n) - \lambda \int_{\Omega} F(x, u_n, v_n) \, dx \leq C_5 \text{ for all } n \in \mathbb{N}, \quad (3.3)$$

and

$$\frac{\|J'(u_n, v_n), (u_n, v_n)\|}{\|(u_n, v_n)\|} < 1 \text{ for large } n \in \mathbb{N},$$

that is,

$$\|J'(u_n, v_n), (u_n, v_n)\| \leq \|(u_n, v_n)\| \text{ for large } n \in \mathbb{N}. \quad (3.4)$$

We first realise that using (A7) and continuity of $f_i$, we get

$$F(x, u, v) = F(x, 0, 0) + \int_0^1 \frac{d}{dt} F(x, tu, tv) \, dt$$

$$= F(x, 0, 0) + \left( \int_0^{\frac{u}{\beta_1}} + \int_{\frac{u}{\beta_1}}^1 \right) \left( f_1(x, tu, tv)u + f_2(x, tu, tv)v \right) dt \quad (3.5)$$

$$\leq F(x, 0, 0) + C' + C_4(|u|^{|\beta_1|} + |v|^{|\beta_2|}), \text{ for } (x, u, v) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^+$$
for some constant $C' > 0$. Let $r \in (p_1 \lor p_2, \alpha_1 \land \alpha_2)$. Then using the above estimate with (3.2), (3.3) and the assumptions (A1)-(A7), we obtain for large $n$,

\[ C_5 + \frac{1}{r} \| (u_n, v_n) \| \geq J(u_n, v_n) - \frac{1}{r} \langle J'(u_n, v_n), (u_n, v_n) \rangle \]

\[ = J_1(u_n) + J_2(u_n) - \lambda \int_{\Omega} F(x, u_n, v_n) \, dx \]

\[ - \frac{1}{r} \left\{ \| u_n \|_{0, s_1, p_1}^{p_1} + \| v_n \|_{0, s_2, p_2}^{p_2} - \lambda \int_{\Omega} \left[ u_n f_1(x, u_n, v_n) + v_n f_2(x, u_n, v_n) \right] \, dx \right\} \]

\[ = \left( \frac{1}{p_1} - \frac{1}{r} \right) \| u_n \|_{0, s_1, p_1}^{p_1} + \left( \frac{1}{p_2} - \frac{1}{r} \right) \| v_n \|_{0, s_2, p_2}^{p_2} \]

\[ + \lambda \left[ \int_{\Omega} \frac{1}{r} u_n f_1(x, u_n, v_n) + v_n f_2(x, u_n, v_n) \, dx - \int_{\Omega} F(x, u_n, v_n) \, dx \right] \]

\[ - \left[ \int_{\Omega} (G_1(x, u_n) - \frac{1}{r} u_n g_1(x, u_n)) \, dx + \int_{\Omega} (G_2(x, v_n) - \frac{1}{r} v_n g_2(x, v_n)) \, dx \right] \]

\[ \geq \left( \frac{1}{p_1} - \frac{1}{r} \right) \| u_n \|_{0, s_1, p_1}^{p_1} + \left( \frac{1}{p_2} - \frac{1}{r} \right) \| v_n \|_{0, s_2, p_2}^{p_2} \]

\[ + \lambda \left[ \int_{\Omega} \frac{1}{r} u_n f_1(x, u_n, v_n) + v_n f_2(x, u_n, v_n) \, dx - \int_{\Omega} F(x, u_n, v_n) \, dx \right] \]

\[ + \frac{(\alpha_1 - r)}{r} \int_{\Omega} G_1(x, u_n) \, dx + \frac{(\alpha_2 - r)}{r} \int_{\Omega} G_2(x, v_n) \, dx \]

\[ \geq \left( \frac{1}{p_1} - \frac{1}{r} \right) \| u_n \|_{0, s_1, p_1}^{p_1} + \left( \frac{1}{p_2} - \frac{1}{r} \right) \| v_n \|_{0, s_2, p_2}^{p_2} + C_6 (|u_n|_{\alpha_1}^{\alpha_1} + |v_n|_{\alpha_2}^{\alpha_2}) \]

\[ - \lambda C_7 (|u_n|_{\beta_1}^{\beta_1} + |v_n|_{\beta_2}^{\beta_2}) - C_8 \]

This implies,

\[ C_5 + \frac{1}{r} \| (u_n, v_n) \| \geq \left( \frac{1}{p_1} - \frac{1}{r} \right) \| u_n \|_{0, s_1, p_1}^{p_1} + \left( \frac{1}{p_2} - \frac{1}{r} \right) \| v_n \|_{0, s_2, p_2}^{p_2} - C_9. \]

(3.6)

It is easy to verify that $a^{p_1} + b^{p_2} \leq 2(a + b)_{\max(p_1, p_2)}$ for any $a, b \in \mathbb{R}^+$ which applies to (3.6) yields that $\{ (u_n, v_n) \}$ is bounded in $X^+$ since $\min \{ p_1, p_2 \} > 1$. Therefore, up to a subsequence, we may assume that there exists $(u, v) \in X^+$ (since $X^+$ is a closed subspace of $X$) such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in $X^+$, $u_n \rightharpoonup u$ strongly in $L_{\gamma_1}(\mathbb{R}^N)$, $v_n \rightharpoonup v$ strongly in $L_{\gamma_2}(\mathbb{R}^N)$ for $\gamma_i \in [1, p_i^*)$, $i = 1, 2$. Also we assume that $(u_n, v_n) \rightharpoonup (u, v)$ as $n \to \infty$ pointwise a.e. in $\Omega$. This immediately implies,

\[ \left| \int_{\Omega} g_1(x, u_n)(u_n - u) \, dx \right| \leq |g_1(\cdot, u_n)|_{q_1'} |u_n - u|_{q_1} \to 0 \text{ as } n \to \infty, \]

where $q_1' = \frac{p_1}{q_1 - 1}$. We also note that,

\[ \left| \int_{\Omega} g_2(x, v_n)(v_n - v) \, dx \right| \leq |g_2(\cdot, v_n)|_{q_2'} |v_n - v|_{q_2} \to 0 \text{ as } n \to \infty, \]

\[ \left| \int_{\Omega} f_1(x, u_n, v_n)(u_n - u) \, dx \right| \leq |f_1(\cdot, u_n, v_n)|_{q} |u_n - u|_{q} \to 0 \text{ as } n \to \infty, \]
Since (2.1) holds, we conclude that \((\bar{u}, v)\) as \(n \to \infty\).

These together with (3.2) implies,

\[
\int_{\Omega} |f_2(x, u_n, v_n)(v_n - v)\,dx| \leq |f_2(\cdot, u_n, v_n)|q|v_n - v|^q \to 0 \text{ as } n \to \infty.
\]

Hence, using (3.7), we obtain,

\[
\langle (u_n, v_n), (u_n, v_n) - (u, v) \rangle_{X', X} \to 0 \text{ as } n \to \infty.
\]

Hence, using (3.7), we obtain,

\[
\|(u_n, v_n) - (u, v)\| \to 0 \text{ as } n \to \infty.
\]

which implies that \((u_n, v_n) \to (u, v)\) strongly in \(X\) as \(n \to \infty\). Therefore, we conclude that \(J\) satisfies the (PS) condition.

**Step-2.** Let us define for \(i = 1, 2,\)

\[
c_i^* := \inf_{u \in \mathcal{N}_i} J_i(u), \quad c_i^{**} := \inf_{u \in X_{0,s_i,p_i} \setminus \{0\}} \max_{t \geq 0} J_i(tu), \quad c_i = \inf_{\gamma \in \Gamma_i} \max_{t \in [0,1]} J_i(\gamma(t))
\]

where

\[
\Gamma_i := \left\{ \gamma \in C([0,1], X_{0,s_i,p_i}) : \gamma(0) = 0, J_i(\gamma(1)) < 0 \right\}
\]

We have the following: \(c_i\) is a critical value of \(J_i\) and \(c_i^* = c_i^{**} = c_i > 0\). The proof is similar to the proof of Theorem 4.2 of [9], on page 73. So we omit it.

**Step-3.** From Step-2, we have that \(c_i^*\)‘s are the critical values of \(J_i\) for \(i = 1, 2\). Therefore, there exists \(\bar{u} \in \mathcal{N}_1\) and \(\bar{v} \in \mathcal{N}_2\) such that

\[
J_1(\bar{u}) = c_1, \quad J_2(\bar{v}) = c_2 \text{ and } J_1'(\bar{u}) = 0 = J_2'(\bar{v}).
\]

Hence, we obtain, \((\bar{u}, 0)\) and \((0, \bar{v})\) as non-negative solutions of

\[
\begin{cases}
(-\Delta)^{s_1} \bar{u} = g_1(x, \bar{u}) \quad &\text{in } \Omega, \\
(-\Delta)^{s_2} \bar{v} = g_2(x, \bar{v}) \quad &\text{in } \Omega, \\
\bar{u} = 0 = \bar{v} \quad &\text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(3.8)

Since (2.4) holds, we conclude that \((\bar{u}, 0)\) and \((0, \bar{v})\) are non-negative solutions of \((P)\) with

\[
J(\bar{u}, 0) = J_1(\bar{u}) = c_1 \text{ and } J(0, \bar{v}) = J_2(\bar{v}) = c_2.
\]
By virtue of (3.9) and since $c$

Hence, applying Theorem 2.1 we assert that

We know from the definition of

Proof of Claim

For $(t, s)$ given by

there exists a $\lambda > 0$ such that,

To obtain the third non-negative solution, let us consider the term $J(tu, sv)$ for $t, s \geq 1$

given by

with some generic constant $C > 0$ (independent of $t, s$ but depends on $\bar{u}, \bar{v}$). Therefore, there exists a $L > 0$ such that,

Let $Q := \{(tL\bar{u}, sLv) \in X : (t, s) \in [0, 1] \times [0, 1]\}$ and $N = N_1 \times N_2$. We observe that,

Hence, applying Theorem 2.21 we assert that $\partial Q$ links $N$.

Claim. For $\lambda > 0$ small enough,

Proof of Claim. We know from the definition of $c_i, c_i^{**}$, as in Step-2, that

By virtue of (3.9) and since $c_i > 0$ from Step-2, we obtain

For $(u, v) \in N$, using (A3) we see that

$$J_1(u) \geq \left( \frac{1}{p_1} - \frac{1}{\alpha_1} \right) \|u\|_{0, s_1, p_1}^{p_1}, \quad J_2(v) \geq \left( \frac{1}{p_2} - \frac{1}{\alpha_2} \right) \|v\|_{0, s_2, p_2}^{p_2}.$$
Using (3.5), we infer that for \((u, v) \in \mathcal{N}\)

\[
\int_{\Omega} F(x, u, v) \, dx \leq C_{10} + C_4 \left( |u|^{\beta_1} |\Omega|^{1-\frac{\alpha_1}{p_1}} + |v|^{\beta_2} |\Omega|^{1-\frac{\alpha_2}{p_2}} \right)
\]

\[
\leq C_{10} + (C_{11}|u|_{\alpha_1} + C_{12}|v|_{\alpha_2}) \text{, using Young's inequality}
\]

\[
\leq C_{10} + \int_{\Omega} G_1(x, u) \, dx + \int_{\Omega} G_2(x, v) \, dx
\]

\[
\leq C_{10} + \frac{1}{\alpha_1} \int_{\Omega} u g_1(x, u) \, dx + \frac{1}{\alpha_2} \int_{\Omega} v g_2(x, v) \, dx \quad \text{from (3.10)}
\]

\[
= C_{10} + \|u\|_{0, s_1, p_1}^{\beta_1} + \|v\|_{0, s_2, p_2}^{\beta_2}
\]

From the above estimates, if we assume \(\lambda < \min \left\{ \frac{\alpha_1 - p_1}{p_1}, \frac{\alpha_2 - p_2}{p_2} \right\} \) then for any \((u, v) \in \mathcal{N}\) we have that

\[
J(u, v) \geq J_1(u) + J_2(u) - \lambda C_{10} + \frac{\lambda |u|^{\beta_1} |\Omega|^{1-\frac{\alpha_1}{p_1}}}{\alpha_1} + \frac{\lambda |v|^{\beta_2} |\Omega|^{1-\frac{\alpha_2}{p_2}}}{\alpha_2}
\]

\[
\geq \left( 1 - \frac{\lambda p_1}{\alpha_1 - p_1} \right) J_1(u) + \left( 1 - \frac{\lambda p_2}{\alpha_2 - p_2} \right) J_2(v) - \lambda \tilde{C}
\]

\[
\geq \left( 1 - \frac{\lambda p_1}{\alpha_1 - p_1} \right) c_1 + \left( 1 - \frac{\lambda p_2}{\alpha_2 - p_2} \right) c_2 - \lambda \tilde{C}
\]

for some \(\tilde{C} > 0\). Let us take

\[
\Lambda = \min \left\{ \frac{\alpha_1 - p_1}{p_1}, \frac{\alpha_2 - p_2}{p_2}, \frac{c_1 + c_2 - (c_1 \lor c_2)}{\frac{p_1 c_1}{\alpha_1 - p_1} + \frac{p_2 c_2}{\alpha_2 - p_2} + \tilde{C}} \right\}
\]

then it is easy to see that whenever \(\lambda \in (0, \Lambda)\),

\[
c_1 \lor c_2 < \left( 1 - \frac{\lambda p_1}{\alpha_1 - p_1} \right) J_1(u) + \left( 1 - \frac{\lambda p_2}{\alpha_2 - p_2} \right) J_2(v) - \lambda \tilde{C} \leq J(u, v), \quad (3.11)
\]

for all \((u, v) \in \mathcal{N}\) which implies,

\[
\inf_{(u, v) \in \mathcal{N}} J(u, v) \geq c_1 \lor c_2. \quad (3.12)
\]

Using (3.12), we have from (3.10) that,

\[
\sup_{(u, v) \in \partial \Omega} J(u, v) \leq \inf_{(u, v) \in \mathcal{N}} J(u, v).
\]

Let us define \(\bar{c} := \inf_{\gamma \in \Gamma} \sup_{(u, v) \in \mathcal{Q}} J(\gamma(u, v))\), where

\[
\Gamma := \left\{ \gamma \in C(\mathcal{Q}, X) : \gamma|_{\partial \mathcal{Q}} = Id|_{\partial \mathcal{Q}} \right\}
\]

\[
\bar{c} \geq \inf_{(u, v) \in \mathcal{N}} J(u, v) > c_1 \lor c_2
\]

and \(\bar{c}\) is a critical point of \(J\). Hence, \((P')\) has at least three non-negative solutions for \(\lambda \in (0, \Lambda)\). Furthermore, if the system \((P')\) have a unique positive solution for \(i = 1, 2\), the third non-negative solution of \((P')\) is positive because if one of the component of third solution is zero, then the value of \(J\) is either \(c_1\) or \(c_2\) which is a contradiction. \(\square\)
4. Proof of Theorem 2.3

This section consists of the study of (P) with \( g_1 \equiv 0 \equiv g_2 \) and \( \lambda = 1 \) that is

\[
(P_0) \begin{cases}
(-\Delta)^{p_1}_u f_1(x, u, v) & \text{in } \Omega, \\
(-\Delta)^{p_2}_v f_2(x, u, v) & \text{in } \Omega,
\end{cases}
\]

where \( f_i \)'s are assumed to satisfy the condition (A1), (A6) and (A8). Moreover, we suppose that \( h(x) \leq \bar{\lambda} \) in \( \Omega \) and \( h(x) < \bar{\lambda} \) on a subset of \( \Omega \) having positive measure, where 

\[
\lambda = \min_{i=1,2} \{ \lambda_{1,p_i} \}.
\]

Now we head to prove Theorem 2.3 so we need the following lemma.

**Lemma 4.1.** Let \( h \in L^\infty(\Omega) \) with the properties that \( h(x) \leq \bar{\lambda} \) and \( h(x) < \bar{\lambda} \) on a subset of \( \Omega \) with positive measure. Then, there exists \( M > 0 \) such that

\[
\|u\|_{0,s_1,p_1} + \|v\|_{0,s_2,p_2}^{p_2} - \int_{\Omega} h(x)(|u|^{p_1} + |v|^{p_2}) \, dx \geq 2M \left( \|u\|_{0,s_1,p_1}^{p_1} + \|v\|_{0,s_2,p_2}^{p_2} \right). \tag{4.1}
\]

**Proof.** Suppose (4.1) is not true. Then, for each \( n \in \mathbb{N} \), there exists \((u_n, v_n) \in X\) such that

\[
\|u_n\|_{0,s_1,p_1}^{p_1} + \|v_n\|_{0,s_2,p_2}^{p_2} - \int_{\Omega} h(x)(|u_n|^{p_1} + |v_n|^{p_2}) \, dx < \frac{1}{n} \text{ for all } n \in \mathbb{N},
\]

and

\[
\|u_n\|_{0,s_1,p_1} + \|v_n\|_{0,s_2,p_2} = 1. \tag{4.2}
\]

Therefore, we have,

\[
\lim_{n \to \infty} \left( \|u_n\|_{0,s_1,p_1}^{p_1} + \|v_n\|_{0,s_2,p_2}^{p_2} - \int_{\Omega} h(x)(|u_n|^{p_1} + |v_n|^{p_2}) \, dx \right) = 0. \tag{4.3}
\]

Since \( \|u_n\|_{0,s_1,p_1}^{p_1} + \|v_n\|_{0,s_2,p_2}^{p_2} = 1 \), so \( \{u_n\} \) is bounded in \( X_{s_1,p_1} \) and \( \{v_n\} \) is bounded in \( X_{s_2,p_2} \). Therefore, up to a subsequence, we may assume that there exists \( u \in X_{s_1,p_1} \) and \( v \in X_{s_2,p_2} \) such that

\[
\begin{align*}
u_n &\to u \text{ weakly in } X_{s_1,p_1}, \\
v_n &\to v \text{ weakly in } X_{s_2,p_2}, \text{and } u_n \to u \text{ strongly in } L^{p_1}(\Omega), \text{ and } v_n \to v \text{ strongly in } L^{p_2}(\Omega).
\end{align*}
\]

As \( h \in L^\infty(\Omega) \), we obtain

\[
\lim_{n \to \infty} \int_{\Omega} h(x)[|u_n|^{p_1} + |v_n|^{p_2}] \, dx = \int_{\Omega} h(x)[|u|^{p_1} + |v|^{p_2}] \, dx. \tag{4.4}
\]

Using (4.2) and (4.4), from (4.3), we have,

\[
1 = \lim_{n \to \infty} \left( \|u_n\|_{0,s_1,p_1}^{p_1} + \|v_n\|_{0,s_2,p_2}^{p_2} \right) = \int_{\Omega} h(x)[|u|^{p_1} + |v|^{p_2}] \, dx. \tag{4.5}
\]

Since \( u_n \to u \) in \( X_{s_1,p_1} \) and \( v_n \to v \) in \( X_{s_2,p_2} \), by the property of weak lower semicontinuity of norm, we get,

\[
\|u\|_{0,s_1,p_1}^{p_1} + \|v\|_{0,s_2,p_2}^{p_2} \leq \liminf_{n \to \infty} \left( \|u_n\|_{0,s_1,p_1}^{p_1} + \|v_n\|_{0,s_2,p_2}^{p_2} \right). \tag{4.6}
\]
Combining (4.5), (4.6) and using the definition of $\lambda_{1,p_1}, \lambda_{1,p_2}$ we have,
\[
\lambda (|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) \leq \lambda_{1,p_1}|u|^{p_1}_{p_1} + \lambda_{1,p_2}|v|^{p_2}_{p_2} \\
\leq \|u\|_{0,s_1,p_1}^{p_1} + \|v\|_{0,s_2,p_2}^{p_2} \\
\leq \int_{\Omega} h(x)(|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) \, dx \\
\leq \lambda (|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}).
\]
Hence, we have,
\[
\int_{\Omega} (\lambda - h(x))(|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) \, dx = 0.
\]
As $h(x) < \lambda$ on a set of positive measure, this yields us, $|u| = 0 = |v|$ a.e. in $\Omega$. This contradicts (4.5). This concludes our result. \( \square \)

**Proof of Theorem 2.3** Let us consider the functional $I : X \to \mathbb{R}$ corresponding to $(P_0)$ defined by
\[
I(u, v) = \frac{1}{p_1} \|u\|_{0,s_1,p_1}^{p_1} + \frac{1}{p_2} \|v\|_{0,s_2,p_2}^{p_2} - \int_{\Omega} F(x, u, v) \, dx, \text{ for all } (u, v) \in X.
\]
We will show that $I$ has a critical point in $X$ using usual variational technique which in turn will be a solution of $(P_0)$.

**Coercivity of $I$.** We note that
\[
I(u, v) \geq \frac{1}{p_1 \vee p_2} (\|u\|_{0,s_1,p_1}^{p_1} + \|v\|_{0,s_2,p_2}^{p_2}) - \int_{\Omega} F(x, u, v) \, dx. \tag{4.7}
\]

By assumption (A8), there exists a function $l \in L^1(\Omega)$ such that
\[
F(x, u, v) \leq (h(x) + \bar{\lambda} M)(|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) + l(x) \text{ for all } (u, v) \in X \tag{4.8}
\]
and $M$ is given in Lemma 4.1. Hence, we have,
\[
\int_{\Omega} F(x, u, v) \, dx \leq \int_{\Omega} h(x)(|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) + \bar{\lambda} M (|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) + \int_{\Omega} l(x) \, dx. \tag{4.9}
\]

Using (4.9), from (4.4) we obtain using Lemma 4.1 that,
\[
I(u, v) \geq \frac{1}{p_1 \vee p_2} (\|u\|_{0,s_1,p_1}^{p_1} + \|v\|_{0,s_2,p_2}^{p_2}) - \int_{\Omega} h(x)(|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) - \bar{\lambda} M (|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) - |l|_1 \\
\geq \frac{2M}{p_1 \vee p_2} (\|u\|_{0,s_1,p_1}^{p_1} + \|v\|_{0,s_2,p_2}^{p_2}) - \bar{\lambda} M (|u|^{p_1}_{p_1} + |v|^{p_2}_{p_2}) - |l|_1 \\
\geq \frac{M}{p_1 \vee p_2} (\|u\|_{0,s_1,p_1}^{p_1} + \|v\|_{0,s_2,p_2}^{p_2}) - |l|_1. \tag{4.10}
\]

Inequality (4.10) implies $I$ is coercive.

**Weak lower semicontinuity of $I$.** By assumption (A16) we have that
\[
F(x, u, v) \leq F(x, 0, 0) + \int_0^1 (f_1(x, tu, tv)u + f_2(x, tu, tv)v) \, dt \\
\leq F(x, 0, 0) + \frac{C_3}{q} (|u|^{q-1}u + |v|^{q-1}v + |u|^{q-1}u + |v|^{q-1}u + u + v)
\]
This implies that for some constant $C > 0$ such that
\[
\int_{\Omega} F(x, u, v) \, dx \leq C \left( 1 + |u|_q + |v|_q + |u|_1 + |v|_1 + \int_{\Omega} (|u|^{q-1}v + |v|^{q-1}u) \, dx \right)
\]
Our next aim is to show that \((u, v) \mapsto \int_{\Omega} F(x, u, v) \, dx\) is weakly lower semicontinuous. In fact, we will show that if \((u_n, v_n) \to (u, v)\) weakly in \(X\) then,
\[
\lim_{n \to \infty} \int_{\Omega} F(x, u_n, v_n) \, dx = \int_{\Omega} F(x, u, v) \, dx.
\]
As \(1 < q < p_1^* \wedge p_2^*\), by compact embedding, up to a subsequence, we know that
\[
(u_n, v_n) \to (u, v) \quad \text{strongly in } L^q(\Omega) \times L^q(\Omega).
\]
Thus, we note that for some constant \(C > 0\) which may vary at each step,
\[
\left| \int_{\Omega} F(x, u_n, v_n) \, dx - \int_{\Omega} F(x, u, v) \, dx \right|
\leq \int_{\Omega} |F(x, u_n, v_n) - F(x, u, v)| \, dx
\leq \int_{\Omega} \left| \int_0^1 \left[ F_u(x, tu_n + (1-t)u, tv_n + (1-t)v)(u_n - u) + F_v(x, tu_n + (1-t)u, tv_n + (1-t)v)(v_n - v) \right] dt \right| \, dx
\leq C \int_{\Omega} \int_0^1 \left( |tu_n + (1-t)u|^{q-1} + |tv_n + (1-t)v|^{q-1} + 1 \right) \left( |u_n - u| + |v_n - v| \right) dt \, dx
\leq C \left( \int_{\Omega} |u_n|^q + |v_n|^q + |u|^q + |v|^q + 1 \right) \left( \int_{\Omega} |u_n - u|^q + |v_n - v|^q \right)^{\frac{1}{q}}
\to 0 \quad \text{as } n \to \infty,
\]
where we have used Hölder inequality in the last step, used the fact that \(u_n, v_n\) are bounded sequences in \(L^q(\Omega)\) and \(u_n \to u\) and \(v_n \to v\) in \(L^q(\Omega)\). Finally, as norm is weakly lower semicontinuous, so we get that \(I\) is weakly lower semicontinuous. From \(X\) being a reflexive Banach space, it follows that \(I\) has a minimum at some point \((u_0, v_0) \in X\), say. We note that by assumption \((A1)\), \(F\) is differentiable. Therefore, \(I\) must continuously differentiable on \(X\). Since \(I\) has a minimum at \((u_0, v_0)\), so we have, \(I'(u_0, v_0) = 0\). Hence, it is noted that
\[
\int_{\mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p_1-2}(u_0(x) - u_0(y))(w(x) - w(y))}{|x - y|^{N+s_1 p_1}} \, dx \, dy
+ \int_{\mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{p_2-2}(v_0(x) - v_0(y))(z(x) - z(y))}{|x - y|^{N+s_2 p_2}} \, dx \, dy
= \int_{\Omega} u_0 f_1(x, u_0, v_0) \, dx + \int_{\Omega} v_0 f_2(x, u_0, v_0) \, dx
\]
which suggests that \((u_0, v_0)\) is a solution of \((P)\). This finishes our proof.

5. Proof of Theorem 2.4

This section is devoted to the proof of Theorem 2.4 which goes hand in hand with the proof of Theorem 2.3. For the sake of completeness, we provide the proof. Let us consider
the energy functional $\tilde{I}: X \to \mathbb{R}$ defined by

$$I(u, v) = \tilde{J}(u, v) - \int_{\Omega} F(x, u, v) dx \text{ for all } (u, v) \in X,$$

where

$$\tilde{J}(u, v) = \int_{\mathbb{R}^{2N}} \phi_1(u(x) - u(y))(u(x) - u(y))K_1(x, y) dxdy$$

$$+ \int_{\mathbb{R}^{2N}} \phi_2(v(x) - v(y))(v(x) - v(y))K_2(x, y) dxdy,$$

for all $(u, v) \in X$. By the condition $(2.2)$ and $(2.3)$ defining $\phi_i$ and $K_i$, $i = 1, 2$, we note that $\tilde{J}$ is convex and lower semicontinuous. By assumption $(A2)$, $F$ is weakly continuous. Hence, $\tilde{I}$ is weakly lower semicontinuous. By assumption $(A1)$, there exists a function $l \in L^1(\Omega)$ such that

$$F(x, u, v) \leq (h(x) + \lambda M)(|u|^{p_1} + |v|^{p_2}) + l(x) \text{ for all } (x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R},$$

and $M$ is given in Lemma 4.1. Hence, we have,

$$\int_{\Omega} F(x, u, v) dx \leq \int_{\Omega} h(x)(\frac{|u|^{p_1} + |v|^{p_2}}{p_1 \vee p_2}) + \lambda \frac{M}{p_1 \vee p_2} \int_{\Omega} |u|^{p_1} + |v|^{p_2} + \int_{\Omega} l(x) dx. \quad (5.1)$$

Using $(5.1)$, $(2.2)$ and $(2.3)$, we obtain, as above,

$$I(u, v) \geq \frac{M}{p_1 \vee p_2} (\|u\|_{0, s_1, p_2}^{p_2} + \|v\|_{0, s_2, p_2}^{p_2}) - |l|_1, \quad (5.2)$$

concluding $I$ is coercive. As $F$ and $\phi_i$’s are continuously differentiable, by assumption $(A1)$, $\tilde{I}$ is continuous. Hence, there exists $(u_0, v_0) \in X$ such that

$$\tilde{I}(u_0, v_0) = \min_{(u, v) \in X} \tilde{I}(u, v) \quad \text{and} \quad \tilde{I}'(u_0, v_0) = 0.$$

which yields,

$$\int_{\mathbb{R}^{2N}} \phi_1(u_0(x) - u_0(y))(w(x) - w(y))K_1(x, y) dxdy$$

$$+ \int_{\mathbb{R}^{2N}} \phi_2(v_0(x) - v_0(y))(z(x) - z(y))K_2(x, y) dxdy \quad (5.3)$$

$$= \int_{\Omega} f_1(x, u_0, v_0)w(x) dx + \int_{\Omega} f_2(x, u_0, v_0)z(x) dx,$$

for all $w, z \in X$. Therefore, $(u_0, v_0)$ is a solution of $(Q)$.

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