It Is High Time We Let Go Of The Mersenne Twister

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When the Mersenne Twister made his first appearance in 1997 it was a powerful example of how linear maps on $\mathbb{F}_2$ could be used to generate pseudorandom numbers. In particular, the easiness with which generators with long periods could be defined gave the Mersenne Twister a large following, in spite of the fact that such long periods are not a measure of quality, and they require a large amount of memory. Even at the time of its publication, several defects of the Mersenne Twister were predictable, but they were somewhat obscured by other interesting properties. Today the Mersenne Twister is the default generator in C compilers, the Python language, the Maple mathematical computation system, and in many other environments. Nonetheless, knowledge accumulated in the last 20 years suggests that the Mersenne Twister has, in fact, severe defects, and should never be used as a general-purpose pseudorandom number generator. Many of these results are folklore, or are scattered through very specialized literature. This paper surveys these results for the non-specialist, providing new, simple, understandable examples, and it is intended as a guide for the final user, or for language implementors, so that they can take an informed decision about whether to use the Mersenne Twister or not.

1 INTRODUCTION

"Mersenne Twister" [22] is the collective name of a family of PRNGs (pseudorandom number generators) based on $\mathbb{F}_2$-linear maps.\textsuperscript{1} This means that the state of the generator is a vector of bits of size $n$ interpreted as an $n$-dimensional vector on $\mathbb{F}_2$, the field with two elements, and the next-state function of the generator is an $\mathbb{F}_2$-linear map. Since sum in $\mathbb{F}_2$ is just xor, it is easy to implement such maps so that they can be computed quickly. Several linear PRNGs indeed exist, such as WELL [29] and xorshift [19].

The original paper about the Mersenne Twister was published by Makoto Matsumoto and Takuji Nishimura in 1997 [22]. At that time, the PRNG had several interesting properties. In particular, it was easy to build generators with a very large state space, and the largest version with 19937 bits of state became very popular. More importantly, many techniques used in the Mersenne Twister influenced later development, and helped $\mathbb{F}_2$-linear techniques to recover from the bad fame that followed the "Ferrenberg affaire". [9]

It is difficult for non-specialists to understand the intricacies of PRNGs, but period is easy to understand: the fact that the sequence generated would not repeat before $2^{19937} - 1$ 32-bit integers had been emitted was met with enthusiasm, and quickly the Mersenne Twister was adopted as the standard generator in many environments. For example, the stock PRNG of the gcc compiler and of Python, as well of the Maple mathematical computing framework, is some version of the Mersenne Twister.

Nonetheless, since its very definition a number of problems plagued the Mersenne Twister. First of all, it would have failed statistical tests designed to find bias in linear generator, such as the Marsaglia’s binary-rank test [20] and the linear-complexity test [3, 7] (see Section 2). On the other hand in 1997 few public, easy-to-use implementations of such tests were available, and finding bias at the largest state sizes would have required an enormous amount of computing time.

It was also not clear to practitioners that 19937 bits are ridiculously too much. Even with as little as 256 bits of state (and a period of $\approx 2^{256}$), the fraction of the output that can be accessed by

\textsuperscript{1}Note that the $\mathbb{F}_2$ prefix will be often implied in this paper.
a computation is negligible, and even in a massive parallel computation where processors start from random seeds the chances of overlap are practically zero. In the last decades locality has increasingly become the main factor in performance, and in retrospect using a generator wasting so many bits of the processor cache did not make much sense; it does even less today.

In the following 20 years the Mersenne Twister had many reincarnations: a 64-bit version, a SSE2-based version [30], a version targeted to floating-point numbers [32], and so on. These versions provided more speed, and improved somewhat some measures of quality, but they did not fix the problems in the original version. In the meantime, research went on and several better alternatives were discovered: PRNGs who would not fail statistical tests, had good theoretical properties, and were in fact faster than the Mersenne Twister.

We are conservative with what we do not really understand. Once language implementors decided to use the Mersenne Twister, the choice was doomed to be cemented for a very long time. Also, the problems of the Mersenne Twister are not immediately detectable in everyday applications, and most users really interested in the quality of their PRNG will make an informed choice, rather than relying on the stock PRNG of whichever programming environment they are using. So there was no strong motivation to move towards a better PRNG.

This paper is mostly a survey on the known problems of the Mersenne Twister family written for non-experts. We provide, besides reference to results in the literature, some new, simple but insightful examples that should make even the casual user understand some of the complex issues affecting the Mersenne Twister. It is intended as a guide for casual users, and as well for language implementors, to take an informed decision as to whether to use the Mersenne Twister or not.

## 2 Failures in Statistical Tests for Linearity

Since its inception, the Mersenne Twister was bound to fail two statistical tests that are typically failed by $\mathbb{F}_2$-linear generators: Marsaglia’s binary-rank test [20] and the linear-complexity test [3, 7]. All other linear generators, such as WELL [29] and xorshift [19], fail the same tests.

These two tests exploit two different (but related) statistical biases. In the first case, since the next state of the generator is computed by an $\mathbb{F}_2$-linear map (a square matrix) $M$ applied to a state vector, the resulting bits, when used to fill a large enough matrix on $\mathbb{F}_2$, yield a matrix that is not random, in the sense that its rank is too low. In the second case, since every bit of a linear generator is a linear-feedback shift register (LFSR) [12] defined by the characteristic polynomial of $M$ (the only difference between the bits is that they emit outputs at different points in the orbit of the LFSR) the sequence emitted by such bits can be represented by a linear recurrence of low degree, and this fact can be detected using the Berlekamp–Massey algorithm provided that one uses a large enough degree upper bound (and consequently memory) for determining the linear recurrence.

The two tests are partially orthogonal: that is, it is possible to create PRNGs that fail one test, but not the other, under certain conditions. On the other hand, they are also partially related, as a small matrix gives a low-degree LFSR, which will fail easily the linear-complexity test, and causes the generator to fail the binary-rank test with small matrices. Moreover, some generators might have bits defined by different linear recurrences, so some bits might fail the linear-complexity (and maybe the binary-rank) test while tested in isolation, but not when mixed within in the whole output.

Since the two tests depend on a size (the matrix size, or the upper bound on the degree of the linear recurrence), one should in principle always specify this size when claiming that one of the

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\(^2\)More precisely, any sequence with period length $P$ can be represented with a linear recurrence of degree $P$, but in the case of linear generators the degree is $O(\log P)$.\)
tests is passed or failed: for example, if some bits have higher linear degree than others a larger matrix might be necessary to detect their linearity by means of the binary-rank test.

At this point, a good question is: How come that we are using in so many applications a generator that fails statistical tests? There are two main reasons.

The first reason is that these tests were conceived specifically to find bias in linear generators, using knowledge of their internal structure. In this sense they are different from classical tests, such as the gap test [11], which look for generic undesired statistical regularities (or irregularities).

There is a thin line after which a statistical test becomes an attack, in the sense that it uses too much knowledge about the generator. A good extreme example is that of a PRNG based on a AES in counter mode [33]: we fix a secret, and AES gives us (say) a bijection on 128-bit blocks that is difficult to reverse. We then apply the bijection to 128-bit integers \( k, k + 1, k + 2, \ldots \), and so on, obtaining a strong random generator. Finding bias in such a generator would mean, essentially, finding a weakness in the AES encryption scheme.

However, we can easily design a statistical test that this generator will fail: we simply apply to 128-bit blocks of output the AES decoding function, and then apply a randomness test to the resulting sequence. Since for our PRNG the resulting sequence is \( k, k + 1, k + 2 \ldots \) for some \( k \), it will fail every statistical test.

Does this mean that AES in counter mode fails statistical tests? Really, no, because we used an enormous amount of knowledge about the generator to design the test: basically, we are cheating—nobody can invert AES that easily. One might argue that the same is true of the binary-rank or of the linear-complexity tests, but, once again, we are walking a thin line.

The second reason is that experts in the field have repeatedly stated that failures in these tests are not harmful. For example, Pierre L’Ecuyer and François Panneton in their survey on \( \mathbb{F}_2 \)-linear generators [15] state

All \( \mathbb{F}_2 \)-linear generators fail the tests that look for linear relationships in the sequences of bits they produce, namely, the matrix-rank test for huge binary matrices and the linear complexity tests. […] But whenever the bit sequences are transformed nonlinearly by the application (e.g., to generate real-valued random numbers from non-uniform distributions), the linear relationships between the bits usually disappear, and the linearity is then very unlikely to cause a problem.

The authors describe then a number of ways in which the result can be altered so that linearity disappears.

The inventors of the Mersenne Twister seem to be even less interested in such failures: in the conclusion of the paper presenting the SFMT (SIMD-oriented Fast Mersenne Twister) [30], one of the last incarnations of the Mersenne Twister, Mutsuo Saito and Makoto Matsumoto discuss linear dependencies in the output in terms of equidistribution (see Section 6):

Thus, it seems that \( k(v) \) of SFMT19937 is sufficiently large, far beyond the level of the observable bias. On the other hand, the speed of the generator is observable. Thus, SFMT focuses more on the speed, for applications that require fast generations. (Note: the referee pointed out that statistical tests based on the rank of \( \mathbb{F}_2 \)-matrix is sensitive to the linear relations [9], so the above observation is not necessarily true.)

Failures in statistical tests are not even mentioned in the paper, except for this note, which was added at the request of a referee. In a subsequent paper on the dSFMT (double SIMD-oriented Fast Mersenne Twister) [32], a version of the SFMT generating floating-point numbers with 52 significant bits, they conclude:

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3 Modulo some external entropy regularly introduced into the state of the generator, this is what many secure PRNGs (such as Fortuna [8]) do.
They also passed TestU01 [11] consisting of 144 different tests, except for LinearComp (fail unconditionally) and MatrixRANK tests (fail if the size of dSFMT is smaller than the matrix size). These tests measure the $F_2$-linear dependency of the outputs, and reject $F_2$-linear generators, such as MT, SFMT and WELL.

The failures are not discussed further: they are not deemed relevant.

Are these considerations correct? Certainly, if they are taken into their context: linear generators should not cause problems as long as their output is used to generate uniform reals in the unit interval which are manipulated nonlinearly, as subsequent operations will make the linear artifacts disappear. The scientists making these claims have a background in simulation, where the main purpose of PRNGs is to generate uniform deviates, which are then turned into some other deviate by methods such as inversion or rejection [5], and, indeed, it is difficult to imagine how linearity should influence the result in their case.

However, we are going to argue that, in fact, this is not the case in general.

2.1 An instructive example

Consider a mathematician (or some other kind of scientist) willing to understand the structure of random binary square matrices. For instance, the mathematician might be trying to understand the structure of the adjacency matrix $M$ of a directed Erdős–Rényi graph [6] with probability of a link $p = 1/2$. Or the matrix describing an endorelation of a set were $x$ relates to $y$ with probability $1/2$ (they are the same thing).

As it is customary, for example, in spectral graph theory, the mathematician might want to study some statistical property of the adjacency matrix, such as some property of its characteristic polynomial: she would thus use a PRNG generate a few matrices, compute their characteristic polynomials and have a look at the respective coefficients. As a first raw test, and also as a sanity check, she might want to count the number of odd coefficients, which she should expect to be about $n/2$, where $n$ is the side of the matrix. She would set up a PRNG, use its bits to fill the matrices, duly compute their characteristic polynomials and check the resulting distribution.

We consider the following 64-bit generators for this example:

- AES in counter mode with 128 bits of state, which we take as a reference, as it is a PRNG of cryptographic strength.
- xorshift128+ [36], a small generator based on an $F_2$-linear map with 128 bits of state, followed by a weak scrambler [1] which simply adds the two halves of the state. The resulting carry chains reduce the linearity of the bits as we progress from less significant to more significant bits: the lowest bit is generated by the same LFSR of the linear underlying generator, whereas higher bits see their linear degree increase enormously; moreover, there are no linear relationships between the bits [1].
- xoroshiro128++, a small generator based on an $F_2$-linear map with 128 bits of state, followed by a strong scrambler [1], which should delete all linearity.
- The SFMT (SIMD-oriented Fast Mersenne Twister) [30], one of the latest versions of the Mersenne Twister, with 607 bits of state.
- WELL [29], another $F_2$-linear generator with excellent equidistribution properties (in fact, it is maximally equidistributed; see Section 6), designed to improve the quality of the Mersenne Twister, with 512 bits of state.

Figure 1 shows the results of this simple test using a thousand samples and $n = 1024$: as it is immediate to see, the results of AES in counter mode and xoroshiro128++ are indistinguishable;

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4This is the generator used by the Javascript engine of all major browsers.
Fig. 1. The distribution of the number of odd coefficients in the characteristic polynomial of $1024 \times 1024$ pseudorandom binary matrices generated by a few PRNGs using the output bits to fill the matrix entries.

their averages are 512.9 and 513.5, respectively. The results of the SMFT and of WELL are abysmal: the parity of the coefficients is completely wrong (the averages are 319.4 and 258.1, respectively). Even xorshift128+, despite emitting a few bits of low linear degree, is aligned with AES (average 512.1). If our friend mathematician would be using the SFMT (or WELL) for this purpose, she would be probably looking for some theoretical explanation of the strangely low number of odd terms.

Instead, she would be only witnessing the $F_2$-linearity of the SFMT and WELL influencing heavily the results, even if no $F_2$-linear computation is involved. The mathematically inclined reader will probably have guessed what is happening: the matrices have low rank on $F_2$, which means many eigenvalues are zeroes, which means that the characteristic polynomials on $F_2$ lack several coefficients, and the coefficients on $F_2$ are just the projection on $F_2$ of the coefficients on $\mathbb{Z}$.

One might think that the source of our problem is that we are using the output of the generator directly, rather than generating a uniform number in the unit interval and then manipulating it, as suggested in the references above. But this is not the case: we can perform again the experiment, this time generating a random real number $p$ in the unit interval and filling an entry with a 0 if $p < 1/2$, and with 1 otherwise. Figure 2 was computed in this way and does not look much better, even if we are using the PRNG to generate uniform random numbers in the unit interval. In fact, we even added to the picture the dSFMT (521 bits), which generates floating-point numbers natively (see Section 8), and its results are similar.

Readers acquainted with the representation of floating-point numbers in the IEEE format will immediately recognize that the test $p < 1/2$ is simply using the most significant bit of the output of the generator: in other words, we are using a single bit, rather than all bits, but we are still filling the matrix with bits from the generator.

In fact, there’s a small difference between the two figures: in the first figure, xorshift128+ has a slightly large variance than AES or xoroshiro128++. The variance however is the same in the second figure. This happens because the most significant bit has the largest linear degree—in fact it
is essentially nonlinear. So this example shows also that some care must be taken when evaluating linearity—in principle, not all bits are created equal. Generators such as xorshift128+ yield a nonlinear output when their upper 53 bits are used to generate a uniform real number in the unit interval.

Still, one could think we are cheating, as clearly Panneton and L’Ecuyer speak of nonuniform distributions, and if we set or clear every element of the matrix using the probability threshold \( p = \frac{1}{2} \) we are dealing with a uniform distribution.

Thus, let us try to fill our matrices using the following nonuniform integer distribution:

\[
\Pr(2) = \frac{1}{4} \quad \Pr(5) = \frac{1}{2} \quad \Pr(6) = \frac{1}{4}
\]

We will generate the distribution by a standard inversion, that is, we generate a real number in \([0, 1)\) and emit 2 if \( p < 1/4 \), 5 if \( 1/2 \leq p < 3/4 \), and 6 if \( p \geq 3/4 \). Note that at this point we are not even dealing with matrices with binary entries.

Alas, Figure 3 does not show any improvement on the previous cases. Once again, the reader acquainted with the IEEE representation of floating-point numbers will notice that we are mapping real numbers with the most significant fractional binary digits 00 or 11 into even numbers, and the others into odd numbers: thus, the parity of the entries of the matrix is given by the xor of the two most significant fractional binary digits, and such a xor yields the LSFR associated with the generator: once again, the \( F_2 \)-linear artifacts of the generators surface, even if we are performing integer-based computations using a non-uniform distribution.

The lesson to be learned here is that generators should not fail statistical tests, at least not easily.\(^5\) While a larger-state Mersenne Twister (or any kind of \( F_2 \)-linear generator with larger state) would

\(^5\text{Caveat: “Not easily” needs to be specified as every finite-state generator will fail every statistical test if enough time and space are allocated for the purpose.}\)
need a correspondingly larger matrix to show bias, the deviation from randomness is there, and its importance has been definitely downplayed, as the bias can filter through computations that are not \(F_2\)-linear. Making a linear generator large and space-consuming just to avoid failing tests is not a good strategy.

### 3 Failures in Statistical Tests for Hamming-Weight Dependencies

Linear generators have another known problem: if their next-state maps are represented by sparse matrices, states with few ones or few zeros are usually mapped to states with few ones or few zeros. More generally, there might be hidden dependencies between the number of ones in the sequence of output emitted by a linear PRNG. There are several tests which try to find such dependencies (e.g., [16, 23]), but the Mersenne Twister has no problem with them.

Recently, the author in collaboration with David Blackman has published a stronger test for Hamming-weight dependencies based on a Walsh-like transform [1]. We ran the test against several generators until a petabyte of data has passed, or a \(p\)-value smaller than \(10^{-20}\) was computed: the sooner the low \(p\)-value appears, the stronger the statistical bias. As one can see from Table 1, all small-state members of the Mersenne Twister family fail the test. For the standard Mersenne Twister we used Makoto Matsumoto’s and Takuji Nishimura’s library for dynamic creation of Mersenne Twisters [21], and we found a very wide range of sensitivity to the set: some dynamically generated PRNGs requires a hundred times more data to fail the test than others.

As a reference, we report the same results for a 128-bit xorshift [19] and xoroshiro linear generator [1].\(^6\) The SFMT with 607 bits of state fails using the same amount of data of xorshift128, and one order of magnitude less data than xoroshiro128, both using a state that is five times smaller: this should give a tangible idea of how sparse and pathological the next-state function of the SFMT

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\(^6\)Note that we do not suggest to use these generators without scrambling [1, 36].
is. Linear generator with significantly denser matrices such as WELL [29] do not fail the test, but, as we remarked, they are an order of magnitude slower. Strongly scrambled linear generators such as xoshiro256++ [1] are immune to these issues, they have sub-ns speed, and they have a reasonably sized state.

Once again, larger-state Mersenne Twister PRNGs will not fail the test, as catching dependencies in that case would require a large amount of memory. Nonetheless, as we already remarked, making a generator large and space-consuming just to avoid failing tests is not a good strategy.

4 ESCAPING FROM ZEROLAND AND DECORRELATION

A known weakness of linear generators is the long time it takes, starting from a state in which there are few ones (or few zeroes), to start generating bits which are equally likely zero or one. The reason, once again, is that for the next-state map of a linear generator be computable quickly, its matrix representation must be quite sparse. For example, the original Mersenne Twister (19937 bits) takes hundreds of thousands of iterations, starting from a state in which exactly one bit is set, to get back to normality [29]. This time is called sometimes “escape from zeroland”: it was improved by the WELL family of generators [29], at the expense of a much slower generation (almost an order of magnitude).

While bad initial states are so rare that the probability of hitting such a bad sequence is negligible, for linear generators the escape time from zeroland is the same as the decorrelation time: given a state $S$ and a set $S'$ obtain by flipping a bit (or few bits), how much time it takes so that the two generated sequences are not correlated? Here by “uncorrelated” we mean “empirically uncorrelated”, exactly in the same way we consider a PRNG “random” when it is actually just “empirically random”: to test whether there is correlation, one just interleaves the sequences generate starting from $S$ and $S'$ and uses a number of statistical tests to see whether the resulting sequence looks random.

A short decorrelation time is a good property, because it means that in the eventuality that two initial states are similar, they will produce correlated sequences for a short time. From this viewpoint, linear generators with a large state space, such as the Mersenne Twister with 19937 bits of state, do not behave well—decorrelation takes the same time as escaping from zero, as $V = S \oplus S'$ has just one bit set, so if $M$ is the next-state matrix by linearity $SM^k = S'M^k \oplus VM^k$: as long as $VM^k$ contains mostly zeroes, $SM^k$ and $S'M^k$ will be correlated.

Note that while escaping from zeroland is a problem that is typical of linear generators, decorrelation is a general problem for all fast generators with a large state space. For example, we tested

| Generator                 | $p = 10^{-20}$ @ |
|---------------------------|------------------|
| xorshift128               | $8 \times 10^8$  |
| xoroshiro128              | $1 \times 10^{10}$ |
| Mersenne Twister (521 bits) | $4 \times 10^{10}$ |
| Mersenne Twister (607 bits) | $4 \times 10^8 - 4 \times 10^{10}$ |
| SFMT (607 bits)           | $8 \times 10^8$  |
| dSFMT (521 bits)          | $7 \times 10^{12}$ |

Table 1. Detailed results of the test for Hamming-weight dependencies described in [1]. We report the number of bytes generating a $p$-value smaller than $10^{-20}$.
that Marsaglia’s “Mother-Of-All” PRNG CMWC4096, which has a whopping period of $2^{131104}$, needs about a dozen million iterations to decorrelate.

In general, large periods require a large state, and a large state cannot be perturbed too much by a next-state function if you want speed. So the Mersenne Twister family is caught into a cross fire: members with small state fail miserably linearity tests and tests on Hamming-weight dependencies. On the other hand, members with a large state space might be able to pass some of those test, as the large state space makes their linearity artifacts difficult to detect, but they have very long decorrelation time, so similar states will generate correlated sequences for a large number of iterations, and they occupy a large portion of the processor cache.

5 PROVABLE DETECTABLE DEFICIENCIES
The extremely sparse nature of the next-state matrices of members of the Mersenne Twister family is at the origin at the large “escape from zeroland” time we already discussed, but it induces also other problems. Shin Harase [10] has analyzed in detail the $F_2$-linear relationship induced by the structure of the Mersenne Twister with 19937 bits of state, and reports that subsequences with specific lags fail Marsaglia’s birthday spacing test [18]. This is of course a statistical defect, as in principle any subsequence should be as random as the original sequence, and we could design a statistical test that extract a subsequence with the specified lags; but it is definitely beyond the thin line we discussed in Section 2—the test would uses too much knowledge about the generator.

6 EQUIDISTRIBUTION
One of the motivations for the study and promotion of $F_2$-linear PRNGs is that it is possible to prove that they satisfy equidistribution properties. In fact, the very title of the first Mersenne Twister paper contained the claim that it was 623-dimensionally equidistributed [22].

Equidistribution is a uniformity property of the output of the generator seen as a set of $d$-dimensional points in the unit cube. One fixes a parameter $\ell$, the number of most significant bits considered (if unspecified, the whole output), and a dimension $d$. Then for each possible state of the generator one builds a $d$-dimensional vector whose elements are blocks of upper $\ell$ bits in $d$ consecutive outputs, and checks that all such vectors appear exactly the same number of times, when varying the state in all possible ways [15].

For example, when $\ell = 1$ and $d = 1$ we are just checking that in the output of the generator half of the most significant bits are one and half are zeros. Instead, when $\ell = w$ and $d = 2$, where $w$ is the width of the output of the generator, we are checking that every pair of $w$-bit values appears exactly the same number of times in the output of the generator. For this to happen, of course, $\ell d$ must not be greater than state size in bits. The equidistribution score of a generator is the number of allowable pairs $(\ell, d)$ for which the count is wrong: zero is the best score, whereas the worst score depends on the state and output sizes.

Equidistribution is a property that can be measured for every PRNG, but since it requires examining all the output it can be feasibly computed only on the smallest generators. However, in the case of $F_2$-linear generators the measurement requires just a bit of linear algebra, and can be performed easily even for large-state generators.

This easiness has pushed equidistribution to the forefront of the design of $F_2$-linear generators: for example the WELL family contains generators that are maximally equidistributed, that is, the count is right for all pairs $(\ell, d)$: their equidistribution score is zero. The improvement of equidistribution scores is considered a quality of the SFMT, as we have read in Section 2. Another theoretical measure of quality is the number of terms of the characteristic polynomial of the next-state matrix, which should be close to half the degree [4]. The possibility of computing these theoretical measures is one of the main arguments for linear PRNGs.
In fact, equidistribution scores are often compared with scores in the spectral test, a mathematical test on multipliers for linear congruential generators, which is the basis for finding multipliers of good quality [13]. The argument is that we need to find PRNGs with theoretical good properties, like good equidistribution scores or spectral-test scores, and then test them.

There are several problems with this argument. The first, of general type, is logical: if these scores provide good generators, why should we test them? They should be good, period. The truth is that these scores provide some insight into the quality of the generators, but definitely cannot be used to prove that the generator will pass a battery of tests.

There is one partial exception, however: as Knuth reports in detail [13], Harald Niederreiter has proved that linear congruential generators whose multipliers have a good spectral score will pass a version of the serial test [24–27]. This is, in fact, an incredibly beautiful and strong result, which, in the end, motivates the application of the spectral test.

No such result is known for equidistribution. It is a property that just “looks nice”, but it has no connection with the results of a relevant statistical test. It was inspired by similar properties in the context of quasi Monte–Carlo methods [28], where one, for example, generates sequences of points that fill uniformly a space in order to approximate an integral. But the idea there is that one uses the entire sequence of points, whereas when we use a PRNG we enumerate just a small fraction of its entire sequence of outputs. And, indeed, maximally equidistributed $F_2$-linear generators fail binary-rank and linear-complexity tests (and our test of Section 2) like any other.

The author has also tried to find an empirical correlation between good equidistribution score in [35], by testing extensively the 2200 possible parameterizations of Marsaglia’s xorshift generators and comparing the results of statistical testing with equidistribution score: once one restricts to reasonably good generators, the correlation is extremely weak.

Another observation that suggests that equidistribution does not necessarily lead to particularly good generators is that the equidistribution score is intrinsically unstable. Indeed, if we take a maximally equidistributed sequence, no matter how long, and we flip the most significant bit of a single element of the sequence, the new sequence will have the worst possible score (that is, the count will be wrong for all pairs). This happens because, as it is easy to prove, if $\ell' \geq \ell$, $d' \geq d$ and a generator is not equidistributed with parameters $\ell$ and $d$ then it is not equidistributed with parameters $\ell'$ and $d'$, either: by flipping one bit we make the generator not equidistributed for $\ell = d = 1$, which implies the lack of equidistribution for all possible other pairs. For instance, by flipping the most significant bit of a single chosen value out of the output of WELL1024a we can turn its equidistribution measure from zero to 4143—the worst possible value. But for any statistical or practical purpose the two sequences are indistinguishable—we are modifying one bit out of $2^5(2^{1024} - 1)$. These considerations offer strong evidence that equidistribution will never be linked to the result of a feasible statistical test.

We must also mention a fact that is rarely reported in the literature: while we cannot prove that a generator will pass some statistical test because of a good equidistribution score, we can prove that it will fail one. The collision test [13] generates blocks of bits using the output of a generator and counts how many blocks have appeared before, that is, collide with a previous block. If the blocks are made of $r$ bits, by the birthday paradox collisions start to happen after $O(\sqrt{2^r})$ blocks have been examined, and their approximate distribution is known [13], so we can check whether the empirical and the theoretical distribution do match and obtain a $p$-value.

Consider now a generator with $w$-bit output and $kw$ bits of state that is $\ell = w$, $d = k$ equidis-tributed: that is, every $k$-tuple of consecutive outputs appears exactly once (for example, WELL with 512 bits of state has this property). If we perform a collision test on such tuples (i.e., on blocks of $kw$ bits) using a short sequence of blocks of length $O(\sqrt{2^kw})$ the generator will fail: collisions
will never appear, as they start to appear after $O(2^{kw})$ blocks, instead of $O(\sqrt{2^{kw}})$ blocks, have been examined, because equidistribution for $w$ bits in dimension $k$ implies that blocks will repeat only after the whole period of the generator has been enumerated. While for most realistic generators it would be impossible to execute this test in practice, we can compute analytically its negative result.

These considerations are (willingly) somewhat paradoxical, as a proper collision test using a sequence maximizing the power of the test [34], that is, using $\approx 1.25 \cdot 2^{kw}$ blocks, would be failed with or without equidistribution: there is just not enough state. Moreover, a non-equidistributed generator cannot produce all possible outputs blocks of $kw$ bits, which can be arguably considered a non-random feature. And while non-equidistributed generators might be able to make collisions happen after $O(\sqrt{2^{kw}})$ blocks, in practice the number of collision would have the wrong distribution.

All in all, equidistribution cannot be used, alone, as a reason to forget statistical defects and failures in tests. Richard Brent has made in the past similar considerations [2], but most of the observations above are new.

Equidistribution is the main reason why some researchers in the field do not modify the output of a linear generator using some nonlinear function—it would make their equidistribution-centered design difficult to motivate. But this gives us generators that fail relevant statistical tests.

7 DYNAMIC GENERATION OF PARAMETERS

Parameters for the Mersenne Twister can be generated using a library published by Makoto Matsumoto and Takuji Nishimura [21]. That is, given an integer and a allowable state size in bits, the library provides parameters that can be used to instantiate different Mersenne Twister. Having different parameters for a linear generator is a standard situation: for example, Marsaglia’s xorshift generators with 64 bits of state can be parameterized in $2^{200}$ different ways [19]. However, both theoretical measures as equidistribution and statistical testing show that such generators have a large variation in quality [35].

It is thus puzzling that specifically the Mersenne Twister would make it possible to compute parameters that give generators always of high quality, as touted by the authors. Indeed, this is not true: as we discussed in Section 3, different parameters produced by the library yield generators which fail our test for Hamming-weight dependencies with orders of magnitude of difference in the quantity of data analyzed. Even just using the BigCrush suite from TestU01 [17], some of the generator fail only the usual linearity tests, but other fail even the classical gap test [11].

The conclusion is that the Mersenne Twister is not different from any other linear generator: while we can compute parameters that satisfy some quality criteria and provide full period, we do not have a complete understanding of the interaction between the parameters and the statistical quality of the generator. Dynamic generation of parameters for the Mersenne Twister is just dangerous.

8 DIRECT GENERATION OF FLOATING-POINT NUMBERS

One of the latest addition to the Mersenne Twister family is the already mentioned dSFMT [32], a generator that outputs directly floating-point numbers using a next-state affine function which leaves in the state of an almost exactly the representation of a floating point number in IEEE format.

The dSFMT is very fast (sub-ns, if SSE2 instructions are available), but it only generates 52 significant bits. The IEEE format can express 53 bits of precision (52 in the significand, plus an implicit bit), and thus the dSFMT can only generate half of the possible numbers in the unit interval (assuming that by “uniform generation of reals in the unit intervals” we mean generation of dyadics of the form $j/2^{53}$). Surprisingly, this limitation is never discussed by the authors.
Finally, the dSFMT fails all tests for linear generators we discussed so far: in view of the discussion of Section 2, this case is even more dangerous, because users not acquainted with linearity problems and with the internal of the IEEE representation might think that the bias would not appear. Nothing is farther from truth, as Figure 2 and 3 show.

Using a fast, reliable 64-bit generator and turning its output into a floating-point number is a much better and safer method: for example, xoshiro256+ is a weakly scrambled generators whose upper bits are nonlinear [1]. By taking the upper 53 bits from its output and dividing them by $2^{53}$ one obtains a uniform floating-point number with none of the defects above. The generation is $\approx 20\%$ slower than the dSFMT as the conversion to floating-point takes time, but the advantages are overwhelming: no special instructions are needed, no tests are failed, and one gets 53 significant bits.

9 NONLINEAR MERSENNE TWISTERS

In sharp contrast with what we discussed up to this point, recently Mutsuo Saito and Makoto Matsumoto have introduced a new PRNG, called the Tiny Mersenne Twister [31], with the purpose of providing a reliable generator with a small state (127 bits). Both a 32-bit and a 64-bit exists.

The naming is at best confusing: the Tiny Mersenne Twister contains an addition, that is, a nonlinear operation. It is thus not technically in the same family, and the name ”Mersenne Twister” appear to have been used mainly for marketing purposes. The nonlinear operation is necessary to obtain a generator with a small state that does not fail too many tests.

The Tiny Mersenne Twister passes the BigCrush test suite, but several of its bits are actually of low linear degree or have linear dependencies: the lowest 32 bits of the 64-bit version, for example, or the whole output of the 32-bit version, fail the binary-rank test, and the lowest two bits fail the linear-complexity test (with the same parameters of BigCrush). This is caused by the fact that a single nonlinear operation is not sufficient to delete all linear artifacts.

In fact, the situation is much worse: if we bit-reverse the output of the 32-bit version, the lower bits, which are statistically very weak, are analyzed by BigCrush in detail, resulting in disastrous failures in numerous TestU01 tests such as CollisionOver, SimpPoker, Run, AppearanceSpacings, LongestHeadRun, MatrixRank, LinearComp and Run of bits [17]. It is simply unfathomable why the authors would propose to use a PRNG so flawed.

Finally, the Tiny Mersenne Twister is very slow: the 64-bit version needs almost 4 ns to emit a 64-bit integer, whereas, for example, xoroshiro128++ needs less than a nanosecond, and has none of its defects [1].

10 CONCLUSION

$F_2$-linear generators fail statistical tests which can have an impact on actual applications. They should be used with care, and only in context where there it is certain that strongly nonlinear operations will be applied to their output, so their defects are diluted or completely hidden. In particular, linear generators should never be used as general-purpose generators, unless their output is suitably scrambled by combining it with other, nonlinear generators, or by applying nonlinear maps. The current, dangerous ubiquity of the Mersenne Twister as basic PRNG in many environments is a historical artifact that we, as a community, should take care of. Moreover, several touted advantages of the Mersenne Twister, such as reparameterization, do not actually work properly when examined closely.

There are many available alternative to the Mersenne Twister which do not share its defects. For example, the author has developed in collaboration with David Blackman a number of generators formed by a linear engine and a scrambler—a bijection applied to the state of the linear engine that tries to reduce or eliminate altogether linearity issues [1]. Since such generators do not need large
state spaces to pass statistical tests, one obtains general-purpose generators such as \texttt{xoshiro256++} which have a reasonable state space, sub-ns speed, and do not fail any known statistical test. Of course the idea of scrambling is not new (e.g., it is suggested in [14]), but we use a number of theoretical tools to prove properties of our scramblers. Even \texttt{xoroshiro128++}, which we used in Figure 1, 2 and 3, avoids linearity issues using just 128 bits of state.

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