We review our recent theoretical results about inequivalence between passive gravitational mass and energy for a composite quantum body at a macroscopic level. In particular, we consider macroscopic ensembles of the simplest composite quantum bodies - hydrogen atoms. Our results are as follows. For the most ensembles, the Einstein’s Equivalence Principle is valid. On the other hand, we discuss that for some special quantum ensembles - ensembles of the coherent superpositions of the stationary quantum states in the hydrogen atoms (which we call Gravitational demons) - the Equivalence Principle between passive gravitational mass and energy is broken. We show that, for such superpositions, the expectation values of passive gravitational masses are not related to the expectation values of energies by the famous Einstein’s equation, i.e., \( m_g \neq E \). Possible experiments at the Earth’s laboratories are briefly discussed, in contrast to the numerous attempts and projects to discover the possible breakdown of the Einstein’s Equivalence Principle during the space missions.

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INTRODUCTION

The Galileo Galilei’s Equivalence Principle between gravitational and inertial masses in a combination with the local Lorentz invariance of spacetime amount of the so-called Einstein’s Equivalence Principle. It is known to be a keystone of the classical General Relativity \([1,2]\). Validity of this principle for ordinary matter has been established so far with great accuracy, \( |m_i - m_g| \leq 10^{-17} - 10^{-16} \), in the recent space mission ”MICROSCOPE” (see Refs. \([3,4]\)), where \( m_g \) and \( m_i \) are gravitational and inertial masses, respectively. In literature, there are widely discussed possible new space missions, ”Galileo Galilei” \([5]\) and ”STEP” \([6]\), which may increase the above mentioned accuracy up to \( |m_i - m_g| \leq 10^{-19} \).

The quantum theory of gravity has not been developed yet, but the numerous speculation on this topic predict that the Einstein’s Equivalence Principle may be broken at extremely high energies, \( E \sim 10^{28} \text{eV} \), which will never be accessible for our experimental studies. Nevertheless, recently we have shown (see Refs.\([7-9]\)) that even semiclassical variant of General Relativity, where the field is not quantized but the matter is quantized, predicts breakdown of the Einstein’s Equivalence Principle at low enough experimentally accessible energies. In particular, we have shown \([7,9]\) that the Einstein’s Equivalence principle is broken for passive gravitational mass at a microscopic level for a composite quantum body. Indeed, electron in the hydrogen atom with constant stationary energy, \( E_n \), is not characterized by constant passive gravitational mass \([7-9]\). According to the above cited works, there exists a small (but non-zero) probability that the quantum measurement of the electron mass gives the value \( m_g \neq m_e + \frac{E_n}{c^2} \). The situation with active gravitational mass is, as shown \([8,9]\), even more interesting since it breaks the Einstein’s Equivalence Principle even at a macroscopic level. Indeed, in the above mentioned papers, the Equivalence Principle is considered for different macroscopic ensembles of the hydrogen atoms. It is shown to survive for the majority of the quantum ensembles, with the important exceptions, which are macroscopic ensembles of the coherent quantum superpositions of the stationary electron states, which we call Gravitational demons. For such states, as demonstrated \([8,9]\), the expectation values of the mass can oscillate with time even in the case where the expectation values of energy are constant.

GOAL

In Sec. 4, we discuss in detail the breakdown of the Einstein’s Equivalence Principle between passive gravitational mass and energy (i.e., inertial mass) at a macroscopic level, which was first suggested by us in Ref. \([9]\). We show that...
the accepted by majority of physicists accuracy of the validity of the Equivalence Principle, $|m_i - m_a| \leq 10^{-17} - 10^{-16}$, is overestimated since they experimentally studied only usual condensed matter samples. Below, we discuss behavior of several different macroscopic ensembles of the simplest quantum composite bodies - hydrogen atoms. In agreement with the above mentioned experiments, we show that the Einstein’s Equivalence Principle is valid for almost all of them (see Sec. 3). We demonstrate that this equivalence survives for macroscopic ensembles of the stationary quantum electron states in a hydrogen atom due to the so-called quantum virial theorem [10]. On the other hand, we construct such quantum ensemble, which we call "Gravitational demon" (in analogy with "Maxwell demon"), which breaks the Einstein’s Equivalence Principle at a macroscopic level [9]. The Gravitational demon, by definition, is a macroscopic ensemble of the coherent superpositions of the stationary quantum states, where the above mentioned equivalence is not survived due to the so-called quantum virial term.

Note that the virial term was first suggested in Ref.[11] (see also Refs. [12] and [13]) for the classical model of a hydrogen atom. In particular, it was shown that an external gravitational field is coupled not with the total energy, $E = K + P$, but with the following combination: $E + V$, where the virial term, $V = 2K + P$, with $K$ and $P$ being kinetic and potential energies, respectively. Nevertheless, it was claimed [12,13] that the virial term disappears, if we choose the local proper coordinates in the gravitational field. Therefore, we suggest in the review two methods to calculate passive gravitational mass: one, using gravitational field as a perturbation in the Minkowski’s metric [9] (see Sub-sec. 4.1), and another one - using the local proper coordinates (see Sub-sec. 4.2). We show that both methods for macroscopic ensembles of the coherent superpositions of quantum states in hydrogen atoms (i.e., for Gravitational demons) give the same result - the breakdown of the Einstein’s Equivalence Principle [9]. In Sec. 5, some experimental aspects of the above mentioned breakdown of the Equivalence Principle are discussed. In particular, we pay attention that it is not necessary to conduct very expensive experiments in space. It is possible to create the Gravitational demons in the Earth’s laboratories and, thus, to discover the breakdown of the Einstein’s Equivalence Principle in the laboratories. Although the experiments are expected to be rather difficult, the effect of the breakdown of the Equivalence Principle may be very large and, in principle, even may be of the order of unity.

**EINSTEIN’S EQUIVALENCE PRINCIPLE FOR THE STATIONARY QUANTUM STATES**

For further calculations, we use the textbook weak field approximation [1,2] to describe spacetime outside the isotropic gravitating body (e.g., the Earth),

$$ds^2 = -\left(1 + \frac{2\phi}{c^2}\right)(cdt)^2 + \left(1 - \frac{2\phi}{c^2}\right)(dx^2 + dy^2 + dz^2), \quad \phi = -\frac{GM}{R},$$

(1)

where $c$ is the velocity of light, $G$ is the gravitational constant, $M$ is the Earth’s mass, and $R$ is a distance from center of the Earth. Then, in accordance with the Einstein’s Equivalence principle (which includes the local Lorentz invariance), we can introduce the local proper spacetime coordinates,

$$x' = \left(1 - \frac{\phi}{c^2}\right)x, \quad y' = \left(1 - \frac{\phi}{c^2}\right)y, \quad z' = \left(1 - \frac{\phi}{c^2}\right)z, \quad t' = \left(1 + \frac{\phi}{c^2}\right)t,$$

(2)

where spacetime is the Minkowski’s one:

$$(ds')^2 = -(c dt')^2 + [(dx')^2 + (dy')^2 + (dz')^2].$$

(3)

In the local proper spacetime coordinates (2), the Schrödinger equation for the electron wave functions in a hydrogen atom can be approximately written in the following standard form:

$$i\hbar \frac{\partial \Psi(r', t')}{\partial t'} = \hat{H}(\hat{p}', r')\Psi(r', t'),$$

(4)

with $\hat{H}(\hat{p}', r')$ being the standard Hamiltonian for a hydrogen atom. It is important that, in Eq.(4) and below, we disregard all the so-called tidal effects. In other words, we consider the atom as a point-like body and do not differentiate the gravitational potential with respect to the relative electron coordinates, $r$ and $r'$. It is easy to demonstrate that the disregarded tidal terms in the electron Hamiltonian (4) are very small and are of the relative order of $(r_B/R_0)|\phi/c^2| \sim 10^{-17}|\phi/c^2| \sim 10^{-26}$ in the Earth’s gravitational field. [In this estimation, $r_B$ is the typical "size" of the hydrogen atom (i.e., the Bohr’s radius) and $R_0$ is the Earth’s radius.]
Non-relativistic case

Let us first consider the most important and principle case, where we take account only the kinetic and Coulomb potential energies in the the non-relativistic Schrödinger equation for electron wave functions in a hydrogen atom:

\[ i\hbar \frac{\partial \Psi(r', t')}{\partial r'} = \hat{H}_0(\hat{p}', r') \Psi(r', t'), \quad \hat{H}_0(\hat{p}', r') = m_e c^2 + \frac{\hat{p}'^2}{2m_e} - \frac{e^2}{r'}, \]  

(5)

[In Eq.(5), as usual, \( e \) is the electron charge, \( r' \) is a distance between electron and proton, and \( \hat{p}' = -i\hbar \partial/\partial r' \) is electron momentum operator in the local proper spacetime coordinates.] Below, we consider inertial coordinate system, associated with the spacetime coordinates \((t, x, y, z)\) in Eq.(2) and treat the weak gravitational field (1) as a perturbation. As a result, we obtain the following Hamiltonian \([7, 9]\):

\[ \hat{H}_0(\hat{p}, r) = m_e c^2 + \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + m_e^g \phi, \]  

(6)

From Eq.(6), we rewrite the Hamiltonian in the following more convenient form:

\[ \hat{H}_0(\hat{p}, r) = m_e c^2 + \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + \hat{m}_e^g \phi, \]  

(7)

where the passive gravitational mass operator of electron, \( \hat{m}_e^g \), is introduced by the equation:

\[ \hat{m}_e^g = m_e + \left( \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right)/c^2 + \left( \frac{2\hat{p}^2}{2m_e} - \frac{e^2}{r} \right)/c^2, \]  

(8)

which is equal to electron’s weight operator in the weak field (1). It is important that, in Eq.(8), only the first term corresponds to the bare electron mass, \( m_e \). To the bare electron mass, there exist two corrections: the expected second term, which corresponds to the electron energy contribution, and the non-trivial third term, which is the so-called virial contribution to the passive gravitational mass operator. As shown in Ref.[9], Eqs.(7) and (8) can be directly obtained from the Dirac equation in a weekly curved spacetime (1) [see, for example, Eq.(3.24) in Ref. [14], if we disregard all tidal terms.

Let us discuss one important consequence of Eqs.(7) and (8). It is easy to prove that the operator (8) does not commute with the electron energy operator, taken in the absence of the field (1). Therefore, from the beginning, it seems that the equivalence between electron passive gravitational mass and its energy is broken even for macroscopic ensemble of stationary quantum states. But we show here that it is not so. In particular, to demonstrate this equivalence at a macroscopic level, we consider a macroscopic ensemble of hydrogen atoms being in a stationary quantum state with a definite energy \( E_n \). In this case, we calculate the expectation value of the electron passive gravitational mass operator (per unit atom) from Eq.(8) in the following way:

\[ \langle \hat{m}_e^g \rangle = m_e + \frac{E_n}{c^2} + \left( 2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right)/c^2 = m_e + \frac{E_n}{c^2} \]  

(9)

[Note that the third (virial) term in Eq.(9) is zero, according to the so-called quantum virial theorem [10].] As a result of the calculations, using the quantum virial theorem, we can conclude that the equivalence between passive gravitational mass and energy survives at a macroscopic level for stationary quantum states in the non-relativistic approximation. We stress the important difference between our quantum result \([7, 9]\) of Eq.(9) and the corresponding result in classical case \([12]\) is that the expectation value of the passive gravitational mass corresponds to averaging procedure over a macroscopic ensemble of the atoms, whereas, in classical case, one averages over time.

Relativistic corrections

In this Subsection, we introduce a more general Hamiltonian, which takes into account the so-called relativistic corrections to electron wave functions in a hydrogen atom. It is well known \([15]\), that there are three correction terms, which can be derived from relativistic Dirac equation and which have different physical meanings. As a result, the total relativistic Hamiltonian in the absence of gravitational field can be represented as:

\[ \hat{H}(\hat{p}, r) = \hat{H}_0(\hat{p}, r) + \hat{H}_1(\hat{p}, r), \]  

(10)
with the following corrections,

\[
\hat{H}_1(\hat{p},r) = \alpha \hat{p}^4 + \beta \delta^3(r) + \frac{\hat{S} \cdot \hat{L}}{r^3},
\]

(11)

where the parameters \(\alpha\), \(\beta\), and \(\gamma\) are:

\[
\alpha = -\frac{1}{8m_e^2c^2}, ~ \beta = \frac{\pi e^2\hbar^2}{2m_e^2c^2}, ~ \gamma = \frac{e^2}{2m_e^2c^2}.
\]

(12)

Let us discuss here the physical meaning of the relativistic corrections. Note that the first contribution in Eq.(11) is called the kinetic term, which follows from the relativistic relation between energy and momentum. In the second correction, which has a complicated physical meaning and is called Darwin’s term, \(\delta^3(r) = \delta(x)\delta(y)\delta(z)\) is a three dimensional Dirac’s delta-function. And finally, the third relativistic correction is the spin-orbital interaction, where \(\hat{L} = -i\hbar[\mathbf{r} \times \partial/\partial \mathbf{r}]\) is electron angular momentum operator. Now, in the weak gravitational field (1), the Schrödinger equation for electron wave functions in the local proper spacetime coordinates (2), in the absence of all tidal effects, can be approximately written as:

\[
i\hbar \frac{\partial \Psi(r', t')}{\partial t'} = [\hat{H}_0(\mathbf{p}', r') + \hat{H}_1(\mathbf{p}', r')] \Psi(r', t').
\]

(13)

Using the coordinates transformation (2), the corresponding relativistic Hamiltonian in the inertial coordinate system \((t, x, y, z)\) can be expressed as:

\[
\hat{H}(\mathbf{p}, r) = [\hat{H}_0(\mathbf{p}, r) + \hat{H}_1(\mathbf{p}, r)] \left(1 + \frac{\alpha}{c^2}\right) + \left(\frac{2\hat{p}^2}{2m_e} - \frac{e^2}{r} + 4\alpha \hat{p}^4 + 3\beta \delta^3(r) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{r^3}\right) \frac{1}{c^2}.
\]

(14)

The operator of passive gravitational mass of electron, for the relativistic Hamiltonian (14), can be written in more complicated form than that in Eq.(8):

\[
\hat{m}_e^g = m_e + \left(\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + 4\alpha \hat{p}^4 + 3\beta \delta^3(r) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{r^3}\right) /c^2.
\]

(15)

Here, we consider one more time a macroscopic ensemble of the hydrogen atoms, with each of them being in a stationary quantum state with a definite energy \(E_n^*\). Note that \(E_n^*\) takes into account the relativistic corrections (11) to electron energy. In this case, the expectation value of the relativistic electron mass operator (15) per atom is:

\[
\langle \hat{m}_e^g \rangle = m_e + \frac{E_n^*}{c^2} + \left\langle \frac{2\hat{p}^2}{2m_e} - \frac{e^2}{r} + 4\alpha \hat{p}^4 + 3\beta \delta^3(r) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{r^3}\right\rangle /c^2.
\]

(16)

We stress that the Einstein’s Equivalence Principle will survive at a macroscopic level if the expectation value of the third (virial) term in Eq.(16) is zero. Below, we demonstrate, therefore, that the Einstein’s equation, related the expectation value of passive gravitational mass and energy, can be applied to the stationary quantum states. To this end, we define the so-called virial operator [10],

\[
\hat{G} = \frac{1}{2}(\hat{p} \mathbf{r} + \mathbf{r} \hat{p}),
\]

(17)

and make use of the standard equation of motion for its expectation value:

\[
\frac{d}{dt} \langle \hat{G} \rangle = \frac{i}{\hbar} \langle [\hat{H}_0(\mathbf{p}, r) + \hat{H}_1(\mathbf{p}, r), \hat{G}] \rangle,
\]

(18)

where \([\hat{A}, \hat{B}]\) denotes a commutator of two operators, \(\hat{A}\) and \(\hat{B}\). Note that, in Eq.(18), the derivative \(d < \hat{G} > /dt\) has to be zero, since we consider the stationary quantum state with a definite energy, \(E_n^*\). Therefore,

\[
\langle [\hat{H}_0(\mathbf{p}, r) + \hat{H}_1(\mathbf{p}, r), \hat{G}] \rangle = 0,
\]

(19)
where the Hamiltonian $\hat{H}_0(\hat{p}, \hat{r}) + \hat{H}_1(\hat{p}, \hat{r})$ is defined by Eq.(13). Now, using rather lengthy but straightforward calculations, we show that

$$\frac{[\hat{H}_0(\hat{p}, \hat{r}), \hat{G}]}{-i\hbar} = \frac{1}{2m_e} \hat{p}^2 - \frac{e^2}{r}, \quad \frac{[\alpha \hat{P}^4, \hat{G}]}{-i\hbar} = 4\alpha \hat{r}^4,$$

$$\frac{[\beta \delta^3(\hat{r}), \hat{G}]}{-i\hbar} = 3\beta \delta^3(\hat{r}), \quad \frac{1}{-i\hbar} \left[ \hat{S} \cdot \hat{L}, \hat{G} \right] = 3\gamma \hat{S} \cdot \hat{L} \frac{r^3}{\hbar}, \quad (20)$$

where we take into account the following equality:

$$x_i \frac{d[\delta(x_i)]}{dx_i} = -\delta(x_i). \quad (21)$$

From Eqs.(19) and (20), it is directly follows that

$$\left\langle 2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + 4\alpha \hat{r}^4 + 3\beta \delta^3(\hat{r}) + 3\gamma \frac{\hat{S} \cdot \hat{L}}{\hbar} \right\rangle = 0, \quad (22)$$

and, thus, Eq.(16) can be represented in the Einstein’s form:

$$\langle \hat{m}_e^g \rangle = m_e + \frac{E_{\text{m}}}{\hbar^2}. \quad (23)$$

Let us discuss the status of the Einstein’s Equivalence Principle for the stationary quantum states, considered in this Section. Note that Eq.(23) directly establishes the equivalence between the expectation value of electron passive gravitational mass and its energy in a hydrogen atom, including the relativistic corrections. Therefore, we can say that the Equivalence Principle survives for quantum macroscopic bodies, which contains quantum composite bodies in the stationary quantum states, which first shown in Refs.[7] and [9]. On the other hand, for stationary quantum states the Einstein’s Equivalence Principle is broken at microscopic level. Indeed, as first shown in Refs. [7] and [9], the quantum measurement of passive gravitational mass in state with a definite energy, $E_{\text{m}}$, can give with small probability the value $m_e^g \neq m_e + \frac{E_{\text{m}}}{\hbar^2}$. Although, we consider above and below the simplest quantum composite body - a hydrogen atom, we speculate that our results survive also for more complicated quantum systems, including many-body systems with arbitrary interactions of particles. These and further results reveal and establish the physical meaning of a coupling of a macroscopic quantum test body with a weak gravitational field.

### INEQUIVALENCE BETWEEN PASSIVE GRAVITATIONAL MASS AND ENERGY AT A MACROSCOPIC LEVEL

In this section, we discuss the so-called Gravitational demon - such a macroscopic ensemble of quantum states, which breaks the Einstein’s Equivalence Principle. We show that the Gravitational demon can be created, for example, as a macroscopic ensemble of the coherent superpositions of two quantum states in a hydrogen atom. To demonstrate that the expectation value of gravitational mass of the above mentioned ensemble is not equivalent to the expectation value of its energy, we use below two different methods. In Sub-sec. 4.1, we apply the traditional time-dependent quantum mechanical perturbation method [10] to calculate electron wave functions in suddenly switched on gravitational field and to show that the calculated energy in gravitational field contains the unexpected contribution from the virial term. In Sub-sec. 4.2, we use consideration of the problem in the local proper coordinates (2) in gravitational field (1) and derive the same unexpected virial contribution to electron gravitational mass, in contract to classical physics, where the virial term disappear in the proper local coordinates [12,13]. In the same way, as above, in this Section, we disregard small probabilities of the order of $\frac{\delta^2}{\hbar^2}$ [see Refs.[7] and [9]] and, thus, ignore mass quantization phenomenon.

Inequivalence between passive gravitational mass and energy for a macroscopic ensemble of the coherent superpositions of two stationary quantum states (the first method)

Let us first discuss the accepted in this review procedure of the quantum measurements of passive gravitational mass for a macroscopic ensemble. It is obvious that the expectation values of energy and gravitational mass have to be calculated at the same moment of time, $t = t' = 0$. We suggest that, in the beginning (i.e., at $t < 0$), where
the gravitational field is absent, we have a macroscopic ensemble of coherent superpositions of two wave functions, corresponding to the ground state (1S) wave function, Ψ₁(r), and the first excited energy level (2S) wave function, Ψ₂(r), in a hydrogen atom:

\[ \Psi(r,t) = \frac{1}{\sqrt{2}} \exp\left(\frac{-im_e c^2 t}{\hbar}\right) \left[ \exp\left(\frac{-i E_1 t}{\hbar}\right) \Psi_1(r) + \exp\left(\frac{-i E_2 t}{\hbar}\right) \Psi_2(r) \right], \]  

(24)

where we omit the 2P wave function since the wave functions with different parity do not mix in the gravitational field (1) \[7,9\]. It is important that a macroscopic coherent ensemble of such wave functions, where the difference between phases of functions Ψ₁(r) and Ψ₂(r) is fixed, is difficult but possible to create by means of some laser technique [16]. Note that the expectation value of energy in a macroscopic ensemble (24) in the absence of gravitational field is equal to

\[ <E(t < 0)> = \frac{(E_1 + E_2)}{2}. \]  

(25)

We perform the following Gedanken experiment: we suddenly switch on the gravitational field (1) at \( t \geq 0 \) [see the corresponding perturbation (7) and (8) to the free electron Hamiltonian, \( H_0 \)]. In this case, we can write the following time-dependent electron Hamiltonian in the field:

\[ U_1(r,t) = \frac{\phi}{c^2} [m_e c^2 + \hat{H}_0(r) + \hat{V}(r)] \Theta(t), \]  

(26)

where \( \Theta(t) \) is the step-function and, as shown in Eq.(8), the virial term can be represent as

\[ \hat{V}(r) = 2 \frac{\hbar^2}{2m} \frac{e^2}{r}. \]  

(27)

As we have already mentioned, we disregard all small probabilities of the order of \( \frac{e^2}{r} \) for electron to be in the gravitational field (1) in energy level with \( n > 2 \) (see also Ref. [7] and [9]). Therefore, in a hydrogen atom we can consider only two levels with \( n = 1 \) and \( n = 2 \) and apply to them the two-level variant of the time-dependent perturbation theory [10]. According to this variant, in the gravitational field (1), the wave function can be represented as:

\[ \Psi^1(r,t) = \exp\left(\frac{-im_e c^2 t}{\hbar}\right) \left[ \exp\left(\frac{-i E_1 t}{\hbar}\right) a_1(t) \Psi_1(r) + \exp\left(\frac{-i E_2 t}{\hbar}\right) a_2(t) \Psi_2(r) \right]. \]  

(28)

By means of the standard quantum time-dependent perturbation theory, we can obtain the following equations to determine the functions \( a_1(t) \) and \( a_2(t) \) in Eq.(28):

\[ \frac{da_1(t)}{dt} = -i U_{11}(t) a_1(t) - i U_{12}(t) \exp\left[\frac{-i(E_2 - E_1)t}{\hbar}\right] a_2(t), \]  

\[ \frac{da_2(t)}{dt} = -i U_{22}(t) a_2(t) - i U_{21}(t) \exp\left[\frac{-i(E_1 - E_2)t}{\hbar}\right] a_1(t), \]  

(29)

with the matrix elements of the perturbation (26),(27) being:

\[ U_{11}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi^*_1(r)[m_e c^2 + \hat{H}_0(r) + \hat{V}(r)]\Psi_1(r) d^3r = \Theta(t) \frac{\phi}{c^2}(m_e c^2 + E_1), \]

\[ U_{12}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi^*_1(r)[m_e c^2 + \hat{H}_0(r) + \hat{V}(r)]\Psi_2(r) d^3r = \Theta(t) \frac{\phi}{c^2} V_{12}, \]

\[ U_{22}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi^*_2(r)[m_e c^2 + \hat{H}_0(r) + \hat{V}(r)]\Psi_2(r) d^3r = \Theta(t) \frac{\phi}{c^2}(m_e c^2 + E_2), \]

\[ U_{21}(t) = \Theta(t) \frac{\phi}{c^2} \int \Psi^*_2(r)[m_e c^2 + \hat{H}_0(r) + \hat{V}(r)]\Psi_1(r) d^3r = \Theta(t) \frac{\phi}{c^2} V_{21}, \]  

(30)
where $V_{ij}$ are the matrix elements of the virial operator (27). After solving Eqs.(29) and (30), it possible to find that the function (28) is

$$\Psi^1(r, t) = \exp \left( \frac{-imc^2t}{\hbar} \right) \left[ \Psi^1_1(r, t) + \Psi^1_2(r, t) \right],$$

where

$$\Psi^1_1(r, t) = \frac{1}{\sqrt{2}} \exp \left[ -i \frac{(mc^2 + E_1)t}{c^2\hbar} \right] \exp \left( -i \frac{E_1 t}{\hbar} \right) \left[ 1 - \frac{\phi V_{12}}{c^2(E_2 - E_1)} \right] \Psi_1(r)$$

$$+ \frac{1}{\sqrt{2}} \exp \left( -i \frac{E_2 t}{\hbar} \right) \frac{\phi V_{12}}{c^2(E_2 - E_1)} \Psi_1(r),$$

and

$$\Psi^1_2(r, t) = \frac{1}{\sqrt{2}} \exp \left[ -i \frac{(mc^2 + E_2)t}{c^2\hbar} \right] \exp \left( -i \frac{E_2 t}{\hbar} \right) \left[ 1 - \frac{\phi V_{21}}{c^2(E_1 - E_2)} \right] \Psi_2(r)$$

$$+ \frac{1}{\sqrt{2}} \exp \left( -i \frac{E_1 t}{\hbar} \right) \frac{\phi V_{21}}{c^2(E_1 - E_2)} \Psi_2(r).$$

It is easy to show that with accuracy to the first order of the small parameter, $|\phi| \ll 1$, the wave function (31)-(33) can be written in the following more convenient way:

$$\Psi^1(r, t) = \frac{1}{\sqrt{2}} \exp \left[ -i \frac{(mc^2 + E_1)(1 + \phi/c^2)t}{\hbar} \right] \left\{ \left[ 1 - \frac{\phi V_{12}}{c^2(E_2 - E_1)} \right] \Psi_1(r) \right. \right.$$  

$$\left. + \frac{\phi V_{21}}{c^2(E_1 - E_2)} \Psi_2(r) \right\}$$

$$+ \frac{1}{\sqrt{2}} \exp \left[ -i \frac{(mc^2 + E_2)(1 + \phi/c^2)t}{\hbar} \right] \left\{ \left[ 1 - \frac{\phi V_{21}}{c^2(E_1 - E_2)} \right] \Psi_2(r) \right. \right.$$  

$$\left. + \frac{\phi V_{12}}{c^2(E_2 - E_1)} \Psi_1(r) \right\},$$

where the wave function (34), taken with the same accuracy, is normalized:

$$\int |\Psi^1(r, t)|^2 \Psi^1(r, t) d^3r = 1 + O \left( \frac{\phi^2}{c^4} \right).$$

As we wrote before, wave functions in a macroscopic ensemble of the quantum coherent superpositions at $t = t' = 0$ [see Eq.(24)] are characterized by the constant phase difference and, therefore, can be written as

$$\Psi(r, t) = \frac{1}{\sqrt{2}} \exp \left( \frac{-imc^2t}{\hbar} \right) \left[ \exp \left( -i \frac{E_1 t}{\hbar} \right) \Psi^0_1(r) \right. \right.$$  

$$\left. + \exp(i\tilde{\alpha}) \exp \left( -i \frac{E_2 t}{\hbar} \right) \Psi^0_2(r) \right].$$

with $\Psi^0_1(r)$ and $\Psi^0_2(r)$ being the corresponding real functions and relative phase $\tilde{\alpha}$ being constant. In this case, after some simple calculations, it is possible to show that energy of the state (24) in the weak gravitational field (1) is

$$< E(t \geq 0) > = \int |\Psi^1(r, t)|^2 \left( i\hbar \frac{\partial}{\partial t} \right) \Psi^1(r, t) d^3r$$

$$= mc^2 \left( 1 + \frac{\phi}{c^2} \right) \left( \frac{E_1 + E_2}{2} \right) \left( 1 + \frac{\phi}{c^2} \right) + \tilde{V}_{12} \frac{\phi}{c^2} \cos(\tilde{\alpha}),$$

where

$$\tilde{V}_{12} = \int \Psi^0_1(r) \tilde{V}(r) \Psi^0_2(r) d^3r = \int \Psi^0_1(r) \left( \frac{2\tilde{c}^2}{2m} - \frac{c^2}{r} \right) \Psi^0_2(r) d^3r.$$
From Eq.(37), it is clear that macroscopic ensemble of the coherent superpositions of quantum states (24),(36) (i.e., Gravitational demon) is characterized by the following expectation value of electron mass per one hydrogen atom:

\[
< m_e^q > = m_e + \frac{(E_1 + E_2)}{2e^2} + \frac{\nabla_{12}}{e^2} \cos(\tilde{\alpha}) ,
\]

(39)

where \( \tilde{\alpha} = \text{const} \). Note that here \( m_e \) is the bare electron mass, the second term is the expected kinetic and potential energy contributions to gravitational mass, whereas the third virial term is non-trivial virial contribution to electron mass. Therefore, Eq.(39) directly demonstrates inequivalence between the expectation value of energy (25) and gravitational mass of a macroscopic ensemble of the coherent superpositions of the stationary states. Note that, if we have the incoherent ensemble (where the phase \( \tilde{\alpha} \) is not fixed), then the gravitational mass (39) quickly oscillates with oscillating phase and the equivalence between the expectation values (25) and (39) restores. For the coherent macroscopic ensemble, Eq.(39) crucially depends on the ensemble preparation procedure (i.e., on the phase difference \( \tilde{\alpha} \)). For instance, the expectation value of the gravitational mass can be both larger and smaller than the expected value from the Einstein’s Equivalence Principle and for the simple cases \( \tilde{\alpha}_1 = 0, \tilde{\alpha}_2 = \pi/2, \) and \( \tilde{\alpha}_3 = \pi \) is equal:

\[
< m_e^q > = m_e + \frac{(E_1 + E_2)}{2e^2} + \frac{\nabla_{12}}{e^2} , \quad \tilde{\alpha}_1 = 0 ,
\]

(40)

\[
< m_e^q > = m_e + \frac{(E_1 + E_2)}{2e^2} , \quad \tilde{\alpha}_2 = \pi/2 ,
\]

(41)

and

\[
< m_e^q > = m_e + \frac{(E_1 + E_2)}{2e^2} - \frac{\nabla_{12}}{e^2} , \quad \tilde{\alpha}_3 = \pi .
\]

(42)

To make sure that the suggested effect is not zero, we have calculated the virial matrix element for a hydrogen atom and found that

\[
\nabla_{12} = 0.56 \,(E_2 - E_1) .
\]

(43)

Inequivalence between passive gravitational mass and energy for macroscopic ensemble of the coherent superpositions of two stationary quantum states (the second method)

Below, we make use of different method to obtain Eq.(39). As shown in Refs. [12] and [13], in classical case the virial term disappear if we introduce the proper local coordinates \( (x', y', z') \) [see Eq.(2)]. Therefore, it is important to show that Eq.(39) survives in quantum case if we consider the problem in the proper local coordinates (2). Here, to measure gravitational mass, we perform the same Gedanken experiment as in Sub-sec. 4.1. In particular, at \( t < 0 \), there is no gravitational field and we have a macroscopic ensemble of the coherent superposition of two stationary wave functions [see Eq.(24)]. At \( t = 0 \), we switch on gravitational field (1), which is equivalent to a change of geometry of the space. According, to the local Lorentz invariance, which is a part of the Einstein’s Equivalence Principle, the solution of the Schrödinger equation at \( t \geq 0 \) can be written in the local proper coordinates (2) as

\[
\Psi_2(r', t') = \exp\left(\frac{-im_e c^2 t'}{\hbar} \right) \left[ A \exp\left(\frac{-iE_1 t'}{\hbar} \right) \Psi_1^0(r') + B \exp\left(\frac{-iE_2 t'}{\hbar} \right) \Psi_2^0(r') \right] ,
\]

(44)

where complex coefficients \( A \) and \( B \) are not necessarily equal to \( \frac{1}{\sqrt{2}} \) and take into account the gravitational field (1). On the other hand, in the proper local coordinates, we can represent the wave function (24) at \( t = 0 \) as

\[
\Psi_3(r', t') = \frac{1}{\sqrt{2}} \exp\left[ -\frac{i m_e c^2 t'(1 - \phi/c^2)}{\hbar} \right] \left( 1 + \frac{\phi}{c^2} \right)^{3/2}
\times \left\{ \exp\left[ -\frac{i E_1 t'(1 - \phi/c^2)}{\hbar} \right] \Psi_1^0[r'(1 + \phi/c^2)] + \exp(i\tilde{\alpha}) \exp\left[ -\frac{i E_2 t'(1 - \phi/c^2)}{\hbar} \right] \Psi_2^0[r'(1 + \phi/c^2)] \right\} .
\]

(45)
[Note that the wave function (44) is normalized in the proper local coordinates (2).] It is important that, at \( t = t' = 0 \), the above discussed wave functions (44) and (45) have to be equal to each other:

\[
\Psi_2(r', t' = 0) = \Psi_3(r', t' = 0). \tag{46}
\]

Using Eq.(46) and the following orthogonality condition for real functions \( \Phi_1(r') \) and \( \Phi_2(r') \),

\[
\int_0^{\infty} [\Psi_1^0(r')]^2 d^3r' = \int_0^{\infty} [\Psi_2^0(r')]^2 d^3r' = 1, \quad \int_0^{\infty} \Psi_1^0(r') \Psi_2^0(r') d^3r' = 0, \tag{47}
\]

it is possible to define the coefficients \( A \) and \( B \) in Eq.(44):

\[
A = \frac{1}{\sqrt{2}} [\Delta_{11} + \exp(i\alpha)\Delta_{12}], \quad B = \frac{1}{\sqrt{2}} [\Delta_{21} + \exp(i\alpha)\Delta_{22}], \tag{48}
\]

where

\[
\Delta_{ij} = \left( 1 + \frac{\phi}{c^2} \right) \frac{3}{2} \int_0^{\infty} \psi_1^0(r') \psi_j^0 \left[ r' \left( 1 + \frac{\phi}{c^2} \right) \right] d^3r'. \tag{49}
\]

Let us calculate the matrix elements of the matrix \( \Delta \) with the accepted in this review accuracy - to the first order of the small parameter \( |\hat{\phi}| \ll 1 \). By the definition (49),

\[
\Delta_{11} = \left( 1 - \frac{\phi}{c^2} \right) \frac{3}{2} \int_0^{\infty} \psi_1^0(r') \psi_1^0 \left[ r' \left( 1 - \frac{\phi}{c^2} \right) \right] d^3r'. \tag{50}
\]

From Eq.(50), it follows that \( \Delta_{11} \) is an even function of the variable \( \frac{\phi}{c^2} \), therefore, in our approximation

\[
\Delta_{11} = \int_0^{\infty} [\psi_1^0(r')]^2 d^3r' = 1. \tag{51}
\]

Using the same method, it is easy to show that

\[
\Delta_{22} = \int_0^{\infty} [\psi_2^0(r')]^2 d^3r' = 1. \tag{52}
\]

Calculations of non-diagonal matrix elements of the matrix \( \Delta \) is more complicated procedure. Let us start from calculation of the matrix element \( \Delta_{12} \). From definition of the matrix (49), we have

\[
\Delta_{12} = \left( 1 + \frac{\phi}{c^2} \right) \frac{3}{2} \int_0^{\infty} \psi_1^0(r') \psi_2^0 \left[ r' \left( 1 + \frac{\phi}{c^2} \right) \right] d^3r'
\]

\[
\approx \left( 1 - \frac{\phi}{c^2} \right) \frac{3}{2} \int_0^{\infty} \psi_1^0 \left[ r - \frac{\phi}{c^2} \right] \psi_2^0 d^3r
\]

\[
\approx \left( 1 - \frac{\phi}{c^2} \right) \frac{3}{2} \int_0^{\infty} \psi_1^0(r) \psi_2^0(r) d^3r - \frac{\phi}{c^2} \int_0^{\infty} \psi_1^0(r) \psi_2^0(r) d^3r = -\frac{\phi}{c^2} \Delta, \tag{53}
\]

where

\[
\Psi_1^0(r) = \frac{d\Psi_1^0}{dr}, \quad \Delta = \int_0^{\infty} \psi_1^0(r) \psi_2^0(r) d^3r. \tag{54}
\]

Using the same way, we can find that

\[
\Delta_{21} = \frac{\phi}{c^2} \Delta. \tag{55}
\]
For further development, it is necessary to calculate quantity $\Delta$ in Eqs. (53)-(55) in terms of the virial term. In particular, let us show that

$$\Delta = \int_0^\infty \Psi_1'(r)r\Psi_2^0(r)dr = \frac{\tilde{V}_{12}}{E_2 - E_1},$$  

where the matrix elements of the virial operator (27), $V_{1n}$, are defined by the standard equation:

$$\tilde{V}_{12} = \int_0^\infty \Psi_1^0(r)\tilde{V}(r)\Psi_2^0(r)dr.$$  

To this end, we rewrite the Schrödinger equation in gravitational field (5) in terms of the unperturbed spacetime coordinates $(t, x, y, z)$ (2):

$$(m_ec^2 + E_1)\Psi_1^0\left[(1 - \frac{\phi}{c^2})r\right] = \left[m_ec^2 - \frac{1}{(1 - \phi/c^2)^2}\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{e^2}{r} + \frac{\phi}{c^2}\tilde{V}(r)\right] \Psi_1^0\left[(1 - \frac{\phi}{c^2})r\right].$$

Then, using the accepted weak field approximation and, thus, keeping only terms of the first order with respect to the small parameter $|\frac{\phi}{c^2}| \ll 1$, we obtain:

$$E_1\Psi_1^0(r) - \frac{\phi}{c^2}E_1r\Psi_1'(r) = \left[\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{e^2}{r} + \frac{\phi}{c^2}\tilde{V}(r)\right] \Psi_1^0(r) = \tilde{V}(r)\Psi_1^0(r).$$

As follows from Eq. (59),

$$-E_1r\Psi_1'(r) = \left[\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{e^2}{r}\right] r\Psi_1'(r) + \tilde{V}(r)\Psi_1^0(r).$$

Let us multiply Eq. (60) on $\Psi_2^0(r)$ and integrate,

$$-E_1\int_0^\infty \Psi_2^0(r)r\Psi_1'(r)dr = \int_0^\infty \Psi_2^0(r)\left[\frac{\hbar^2}{2m_e}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + \frac{e^2}{r}\right] r\Psi_1'(r)dr + \int_0^\infty \Psi_2^0(r)\tilde{V}(r)\Psi_2^0(r)dr.$$  

If we make use of the fact that the Hamiltonian and virial term are the Hermitian operators, we can rewrite Eq. (61) as

$$E_1\int_0^\infty \Psi_2^0(r)r\Psi_1'(r)dr = E_2\int_0^\infty \Psi_2^0(r)r\Psi_1'(r)dr - \int_0^\infty \Psi_1'(r)\tilde{V}(r)\Psi_2^0(r)dr.$$  

Then, Eq. (56) directly follows from Eq. (62):

$$\Delta = \int_0^\infty \Psi_1'(r)r\Psi_2(r)dr = \frac{\tilde{V}_{1,2}}{E_2 - E_1}.$$  

Firstly, let us check that the wave function (44) is normalized in the proper local coordinates (2) with the accepted accuracy of our calculations. To this end, we calculate:

$$\int_0^\infty |\Psi_2(0, r')|^2d^3r' = \int_0^\infty [A^*\Psi_1^0(r') + B^*\Psi_2^0(r')][A\Psi_1^0(r') + B\Psi_2^0(r')]d^3r' = |A|^2 + |B|^2,$$
where

\[ |A|^2 + |B|^2 = \frac{1}{2} \left\{ \left[ 1 - \frac{\phi}{c^2} \Delta \exp(-i\hat{\alpha}) \right] \left[ 1 - \frac{\phi}{c^2} \Delta \exp(+i\hat{\alpha}) \right] + \left[ \exp(-i\hat{\alpha}) + \frac{\phi}{c^2} \Delta \right] \left[ \exp(+i\hat{\alpha}) + \frac{\phi}{c^2} \Delta \right] \right\} \approx 1 \]  

(65)

[see Eqs.(48),(53), and (55)].

Second, let us calculate the expectation value of energy for wave function (44) in the gravitational field (1):

\[ < E >= |A|^2 (E_1 + m_e c^2) \left( 1 + \frac{\phi}{c^2} \right) + |B|^2 (E_2 + m_e c^2) \left( 1 + \frac{\phi}{c^2} \right) \]

\[ = \frac{1}{2} \left\{ (E_1 + m_e c^2) \left( 1 + \frac{\phi}{c^2} \right) \left[ 1 - \frac{\phi}{c^2} \Delta \exp(-i\hat{\alpha}) \right] \left[ 1 - \frac{\phi}{c^2} \Delta \exp(+i\hat{\alpha}) \right] \right\} + \left\{ (E_2 + m_e c^2) \left( 1 + \frac{\phi}{c^2} \right) \left[ \exp(-i\hat{\alpha}) + \frac{\phi}{c^2} \Delta \right] \left[ \exp(+i\hat{\alpha}) + \frac{\phi}{c^2} \Delta \right] \right\} \]

\[ \approx m_e c^2 + \frac{1}{2} (E_1 + E_2) + m_e \phi + \frac{1}{2c^2} (E_1 + E_2) \phi + \frac{\sqrt{12}}{c^2} \cos(\hat{\alpha}) \phi. \]  

(66)

The equation for the expectation value of gravitational electron mass (39) directly follows from Eq.(66) and contains the discussed above virial term. So in this Subsection, we have derived Eq.(39), using the local proper coordinates (2), in contrast to the classical case statements [12,13].

**SOME EXPERIMENTAL ASPECTS.**

From our results, discussed in the review, it follows that in order to discover violations of the Einstein’s Equivalence Principle it is not necessary to perform long and expensive experiments in the space. What we actually need is to create special macroscopic ensemble of the coherent superpositions of two or several stationary quantum states (which we call Gravitational demon) and measure its weight. Such macroscopic ensemble, it is possible to create using laser technology (see Ref. [16]). Then, it is necessary to measure the Gravitational demon’s energy and compare the above mentioned two quantities. To evaluate energy in the considered case, it is a good idea just to count a number of the emitted photons from the macroscopic ensemble of the atoms. Let us discuss some obvious difficulties. It is evident that weight of the created Gravitational demon has to be measured at the moment of time which is very close to its creation. In the review, we have used for introduction of the gravitational field the so-called step-like function, \( \Theta(t) \). Of course, this does not mean a motion in the gravitational field (1) (or a motion of the gravitational source) with speed higher than the speed of light. We can use step-like function, if significant change of the gravitational field happens quicker than the characteristic period of quasiclassical rotation of electrons in a hydrogen atom. More strictly speaking, in our case of superposition of two atomic levels, we need the time about \( \delta t \leq t_0 = \frac{2\pi \hbar}{E_2 - E_1} \sim 10^{-15}s \).

To conclude, we have demonstrated, using Gedanken experiments at the Earth’s laboratory, that breakdown of the Einstein’s Equivalence Principle is possible for some special macroscopic quantum states. Finding the best concrete realizations for the corresponding real experiments are the topic for the future investigations.

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