PROPAGATION AND NEUTRINO OSCILLATIONS IN THE BASE OF A HIGHLY MAGNETIZED GAMMA-RAY BURST FIREBALL FLOW

N. Fraija
Instituto de Astronomía, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., A. Postal 70-264, 04510 México D.F., Mexico

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ABSTRACT

Neutrons play an important role in the dynamics of gamma-ray bursts. The presence of neutrons in the baryon-loaded fireball is expected. If the neutron abundance is comparable to that of protons, important features may be observed, such as quasi-thermal multi-GeV neutrinos in coincidence with a subphotospheric γ-ray emission, nucleosynthesis at later times, and rebrightening of the afterglow emission. Additionally, thermal MeV neutrinos are created by electron–positron annihilation, electron (positron) capture on protons (neutrons), and nucleonic bremsstrahlung. Although MeV neutrinos are difficult to detect, quasi-thermal GeV neutrinos are expected in cubic kilometer detectors and/or DeepCore and IceCube. In this paper, we show that neutrino oscillations have outstanding implications for the dynamics of the fireball evolution and also that they can be detected through their flavor ratio on Earth. For that, we derive the resonance and charged-neutrality conditions as well as the neutrino self-energy and effective potential up to the order of $m_W^{-4}$ at strong, moderate, and weak magnetic field approximations to constrain the dynamics of the fireball. We found important implications: (1) resonant oscillations are suppressed for high baryon densities as well as neutron abundance larger than that of protons, and (2) the effect of magnetic field is to decrease the proton-to-neutron ratio aside from the number of multi-GeV neutrinos expected in the DeepCore detector. Also, we estimate the GeV neutrino flavor ratios along the jet and on Earth.

Key words: gamma-ray burst: general – magnetic fields – neutrinos – stars: jets

Online-only material: color figures

1. INTRODUCTION

Gamma-ray bursts (GRBs) are brief events occurring at an average rate of every few days throughout the universe. These short, energetic bursts of gamma-rays mark the most violent, catastrophic explosions in the universe, likely associated with the birth of stellar-size black holes or rapidly spinning, highly magnetized neutron stars. One of the most successful theories in terms of explaining GRBs and their afterglows is the fireball model (see Mészáros 2006; Zhang & Mészáros 2004 for recent reviews). This model predicts an expanding ultrarelativistic shell that moves into the external surrounding medium. The collision of the expanding shell with another shell (internal shocks) or the interstellar medium (external shocks) gives rise to radiation emission through the leptonic (synchrotron and synchrotron self-Compton (SSC) radiation) or hadronic (photopion decay and inelastic proton–neutron collision) processes. In addition, when the expanding relativistic shell encounters the external medium, two shocks are involved: an outgoing, or forward, shock (Rees & Mészáros 1994; Paczynski & Rhoads 1993) and another one that propagates back into the ejecta, the reverse shock (Mészáros & Rees 1994, 1997a).

The base of the fireball flow is connected to the GRB central engine, a black hole (BH)–torus system or a rapidly rotating magnetar. It is endowed with magnetic fields and is formed by free nucleons and $e^\pm$ pairs in thermal equilibrium of 1–10 MeV inside the initial scale of $\sim 10^7$ cm assuming a $10 M_\odot$ BH. In the initial state, the baryons are essentially at rest with respect to the central engine (Zhang & Mészáros 2004).

In recent years, two observational facts have supported the idea that the central engine could be magnetized: (1) the strong gamma-ray polarization (Coburn & Boggs 2003; Boggs & Coburn 2003; Rutledge & Fox 2004) and (2) a stronger magnetic field in reverse than in forward shock (magnetization discovered in the jet), as a result of a full description of GRBs 980923, 990123, 021004, 021211, 041219A, and 090926A, among others (Zhang & Mészáros 2002; Zhang et al. 2003; Fan et al. 2005; Fraija et al. 2012a, 2012b; Sacahui et al. 2012), so that the fireball may be endowed with a primordial magnetic field (Zhang et al. 2003; Zhang & Kobayashi 2005). Recently, a relevant result based on the magnetization of outflow and the nondetection of neutrinos from GRBs by IceCube (Abbasi et al. 2012b) was found by Zhang & Kumar (2013). They proposed that the suppression of the neutrino flux in the gamma-ray emission region is a consequence of the degree of magnetization of the outflow.

The evolution of the fireball with magnetic fields has been explored by various authors (Usos 1999; Wheeler et al. 2000; Blandford 2002; Lyutikov & Blandford 2002; Spruit & Drenkhahn 2003; Vlahakis & Königl 2003a, 2003b; Zhang & Yan 2011). In these models, an electromagnetic component is introduced in the dynamics of the standard fireball. This extra component is taken into account through the magnetization parameter ($\sigma$), defined as the ratio of Poynting flux (electromagnetic component) to matter energy (internal plus kinetic component). Depending on the dissipation process, part of the dissipated magnetic energy is converted into internal energy, and the other part is used to accelerate the fireball (Michel 1969; Mészáros 2012).

The energy requirement (isotropy-equivalent luminosities $L_{\gamma} \geq 10^{52}$ erg s$^{-1}$) demands magnetic fields at the base in excess of $B \sim 10^{15}$ G, which can be produced by shear and instabilities in an accreting torus around the BH. The energy source can be either the accretion energy or magnetic coupling between the disk and BH, with extraction of angular momentum from the latter occurring via the Blandford–Znajek mechanism.
(Blandford & Zhnejek 1977; Mészáros & Rees 1997b; Usov 1992; Lee et al. 2000; Putten et al. 2001).

On the other hand, the presence of neutrons and their abundance in the fireball GRB play an important role in the dynamics of the jet and its relation with the observed photon and neutrino flux emission (Koers & Giannios 2007; Metzger et al. 2008; Razzaque & Mészáros 2006a; Dermer & Atoyan 2006). Among the more relevant consequences are (1) the generation of quasi-thermal GeV neutrinos and subphotospheric $\gamma$-ray emission via inelastic collisions between streaming protons and neutrons in the fireball; (2) the effects on the development, strength, and energy range of internal shocks (Rossi et al. 2006; Xue et al. 2008; Fan et al. 2005; Fan & Wei 2004); (3) nucleosynthesis (Metzger et al. 2008; Beloborodov 2003a); and (4) rebrightening of the afterglow emission. Above the pion production threshold ($\sim 140$ MeV), the n-p decoupling is followed by inelastic n-p scattering, leading to $\pi^+$ and $\pi^0$. As is known, $\pi^0$ decays to two photons ($\gamma + \gamma$), and $\pi^+$ decays to four particles (three neutrinos; $\gamma^+ + v_e + v_{\mu}$, and then $\gamma^+ + v_e + v_{\mu}$, and then $\gamma^+ + v_e + v_{\mu}$), which is the first signature in photons and neutrinos (Derishev et al. 1999a; Bahcall & Mészáros 2000; Mészáros & Rees 2000; Razzaque & Mészáros 2006b; Murase et al. 2013). Neutrons decay after 881.5 $s$; hence, they play an important role at large radii of $\sim 10^{16} - 10^{17}$ cm, where external shocks happen as even an exponentially small number of surviving neutrons carry an energy much larger than the rest energy of the circumburst medium (Derishev et al. 1999b; Beloborodov 2003a, 2003b; Rossi et al. 2006).

Furthermore, in the initial stage and during the expansion of the fireball, thermal neutrinos will be produced by electron-positron annihilation ($e^- + e^+ \rightarrow Z \rightarrow v_e + \bar{v}_e$), processes of electron capture on protons ($e^- + p \rightarrow n + v_e$) and positron capture on neutrons ($e^+ + n \rightarrow p + v_e$), and nucleon–nucleon bremsstrahlung ($NN \rightarrow NN + v_e + \bar{v}_e$) for $j = e, v, \tau$. Also, neutrinos of similar energies are naturally expected in the accretion disk during the collapse or merger. As is known, the properties of neutrinos get modified when they propagate in this magnetized fireball and, depending on the neutrino flavor, would get modified when they propagate in this magnetized fireball. The results are also discussed in terms of magnetization degree.

Hereafter we use $Q_x = Q/10^x$ in cgs units and $e = \hbar = k = 1$ in natural units.

2. NEUTRINO EFFECTIVE POTENTIAL

We use the finite-temperature field theory formalism to study the effect of a heat bath on the propagation of elementary particles (D’Olivo & Nieves 1996a; Tututi et al. 2002). The effect of magnetic field is taken into account through Schwinger’s proper-time method (Schwinger 1951). The effective potential of a particle is calculated from the real part of its self-energy diagram. The neutrino field equation in a magnetized medium is

$$[k - \Sigma(k)]\Psi_L = 0,$$

where the neutrino self-energy operator $\Sigma(k)$ is a Lorentz scalar that depends on the characterized parameters of the medium, for instance, chemical potential, particle density, temperature, magnetic field, etc. For our purpose, $\Sigma(k)$ can be formed by

$$\Sigma(k) = \mathcal{R}(a_1 k_1 + a_1 k_\perp + bh + cb)\mathcal{L},$$

where $k^\mu = (k^0, k^3)$, $k^\perp = (k^1, k^2)$, and $u^\mu$ stands for the four-velocity of the center of mass of the medium given by $u^\mu = (1, \mathbf{0})$. The projection operators are conventionally defined as $\mathcal{R} = 1/2(1 + \gamma_5)$ and $\mathcal{L} = 1/2(1 - \gamma_5)$. The effect of the magnetic field enters through the four-vector $b^\mu$, which is given by $b^\mu = (0, \mathbf{b})$. The background classical magnetic field vector is along the $z$ axis, and consequently, $b^\mu = (0, 0, 0, 1)$. Thus, using the four-vectors $u^\mu$ and $b^\mu$, we can express

$$k_\parallel = k_0u_0 - k_3b_3,$$

and the self-energy can be expressed in terms of three independent four-vectors, $k_\parallel$, $u^\mu$, and $b^\mu$. Therefore, we can write

$$\Sigma = \mathcal{R}\mathcal{L}(a_1 k_\perp + bh + cb).$$

The neutrino self-energy in a magnetic background can be found from Equation (1):

$$det[k - \Sigma(k)] = 0.$$ 

Using the Dirac algebra, the dispersion relation, $V_{\text{eff}} = k_0 - |k|$, as a function of Lorentz scalars can be written as

$$V_{\text{eff}} = b - c \cos \phi - a_1 |k| \sin^2 \phi,$$

where $\phi$ is the angle between the neutrino momentum and the magnetic field vector. Now the Lorentz scalars $a$, $b$, and $c$, which are functions of neutrino energy, momentum, and magnetic field, can be calculated from the neutrino self-energy due to CC and NC interactions of neutrinos with the background particles.
2.1. One-loop Neutrino Self-energy

Let us consider one-loop corrections to the neutrino self-energy in the presence of a magnetic field. The one-loop neutrino self-energy comes from three components (Bravo & Sahu 2007; Elizalde et al. 2004; Erdas et al. 1998; Sahu et al. 2009a, 2009b), one from the \( W \)-exchange diagram, which we will call \( \Sigma_W(k) \) (Figure 1(a)); one from the \( Z \)-exchange diagram, which will be denoted by \( \Sigma_Z(k) \) (Figure 1(b)); and one from the tadpole diagram, which we will designate by \( \Sigma_t(k) \) (Figure 1(c)). The total neutrino self-energy in a magnetized medium then becomes

\[
\Sigma(k) = \Sigma_W(k) + \Sigma_Z(k) + \Sigma_t(k). \tag{7}
\]

The \( W \)-exchange diagram for the one-loop self-energy is

\[
-i \Sigma_W(k) = \mathcal{R} \left[ \int \frac{d^4p}{(2\pi)^4} \frac{-ig}{\sqrt{2}} \gamma_\mu i S_\ell(p) \times \left( \frac{-ig}{\sqrt{2}} \gamma_\nu i W^{\mu\nu}(q) \right) \right] \mathcal{L}, \tag{8}
\]

where \( g^2 = 4\sqrt{2}G_Fm_W^2 \) is the weak-coupling constant and \( W^{\mu\nu} \) depicts the \( W \)-boson propagator, which in the \( eB \ll m_W^2 \) limit and in unitary gauge is given by (Erdas et al. 1998; Sahu et al. 2009b)

\[
W^{\mu\nu}(q) = \frac{g^{\mu\nu}}{m_W^2} \left( 1 + \frac{q^2}{m_W^2} - \frac{q^{\alpha\beta}}{M_W^2} + \frac{3ie}{2m_W^2} F^{\alpha\beta} \right), \tag{9}
\]

where \( m_W \) is the \( W \)-boson mass, \( g^{\mu\nu} \) is the metric tensor, and \( F^{\mu\nu} \) is the electromagnetic field tensor. \( S_\ell(p) \) stands for the charged lepton propagator, which can be separated into to two charged propagators, one in the presence of a uniform background magnetic field \( (S^U_\ell(p)) \) and the other in a magnetized medium \( (S^A_\ell(p)) \), and can be written as

\[
S_\ell(p) = S^U_\ell(p) + S^A_\ell(p). \tag{10}
\]

Assuming that the \( z \) axis points in the direction of the magnetic field \( B \), we can express the charged lepton propagator in the presence of a uniform background magnetic field as

\[
i S^U_\ell(p) = \int_0^\infty e^{i(p,x)} G(p,s) ds, \tag{11}
\]

where the functions \( \phi(p,s) \) and \( G(p,s) \) are given by

\[
\phi(p,s) = is\left(p_0^2 - m_l^2\right) - is\left[p_0^2 + \tan \zeta \right] p_1^2,
\]

\[
G(p,s) = \sec^2 \zeta [A + iB m_s + m_s (s^2 - i\Sigma^3 \sin \zeta)], \tag{12}
\]

where \( m_l \) is the mass of the charged lepton, \( p_0^2 = p_0^2 - p_1^2 \) and \( p_1^2 = p_1^2 + p_2^2 \) are the projections of the momentum on the magnetic field direction, and \( \zeta = eBs \), with \( e \) being the magnitude of the electron charge. Additionally, the covariant vectors are given as follows: \( A = p_\mu - \sin^2 \zeta (p \cdot u u_\mu - p \cdot b b_\mu) \), \( B_\mu = \sin \zeta \cos \zeta (p \cdot u b_\mu - p \cdot b u_\mu) \), and \( \Sigma^3 = \gamma_5 b u \).

On the other hand, the charged lepton propagator in a magnetized medium is given by

\[
S^A_\ell(p) = i \eta_F(p \cdot u) \left[ e^{i(p \cdot s)} G(p, s) ds \right], \tag{13}
\]

where \( \eta_F(p \cdot u) \) contains the distribution functions of the particles in the medium, which can be written as

\[
\eta_F(p \cdot u) = e^{i(\beta \cdot p \cdot u) + 1} + e^{-i(\beta \cdot p \cdot u) + 1}, \tag{14}
\]

where \( \beta \) and \( \mu \) are the inverse of the medium temperature and the chemical potential of the charged lepton, respectively.

The \( Z \)-exchange diagram for the one-loop self-energy is

\[
-i \Sigma_Z(k) = \mathcal{R} \left[ \int \frac{d^4p}{(2\pi)^4} \frac{-ig}{\sqrt{2} \cos \theta_W} \gamma_\mu i S_\nu(p) \times \left( \frac{-ig}{\sqrt{2} \cos \theta_W} \gamma_\nu i Z^{\mu\nu}(q) \right) \right] \mathcal{L}, \tag{15}
\]

where \( \theta_W \) is the Weinberg angle, \( Z^{\mu\nu}(q) \) is the \( Z \)-boson propagator in vacuum, and \( S_\nu \) is the neutrino propagator in a thermal bath of neutrinos.

The tadpole diagram for the one-loop self-energy is

\[
i \Sigma_t(k) = \mathcal{R} \left[ \left( \frac{g}{2 \cos \theta_W} \right)^2 \gamma_\mu i Z^{\mu\nu}(0) \times \int \frac{d^4p}{(2\pi)^4} \text{Tr} [\gamma_\nu (C_V + C_A \gamma_5) i S_\ell(p)] \right] \mathcal{L}, \tag{16}
\]

where the quantities \( C_V \) and \( C_A \) are the vector and axial-vector coupling constants that come from the neutral-current interaction of electrons, protons \( (p) \), neutrons \( (n) \), and neutrinos with the \( Z \) boson. Their forms are as follows:

\[
C_V = \begin{pmatrix}
\frac{1}{2} & 2 \sin^2 \theta_W & e \\
0 & 1 & \nu \\
\frac{1}{2} - 2 \sin^2 \theta_W & p & \frac{1}{2}
\end{pmatrix}, \tag{17}
\]

and

\[
C_A = \begin{pmatrix}
\frac{1}{2} & \nu & p \\
\frac{1}{2} & e & n
\end{pmatrix}. \tag{18}
\]
By evaluating Equation (8) explicitly, we obtain

\[ Re \Sigma_w(k) = \mathcal{R} \left[ a_w k_\perp + b_w u + c_w b \right] \mathcal{L}, \] (19)

where the Lorentz scalars are given by

\[
a_w = -\frac{\sqrt{2} G_F}{m_W^2} \left[ \frac{m^2}{2 m_W^2} + \frac{E^2_{\nu e}}{m_W^2} \right] (n_e - \bar{n}_e) + \frac{e B}{2 \pi^2} \int_0^\infty dp \left( \sum_{n=0}^\infty (2 - \delta_{n,0}) \right) \left( \frac{m^2}{2 m_W^2} \right) (n_e - \bar{n}_e) \times \left( E_n - \frac{m^2}{2 E_n} \right) \left( f_{e,n} + \bar{f}_{e,n} \right),
\]

\[ b_w = \sqrt{2} G_F \left[ \left( 1 + \frac{3}{2} \frac{m^2}{m_W^2} - \frac{k^2}{m_W^2} \right) (n_e - \bar{n}_e) + \frac{e B}{2 \pi^2} \int_0^\infty dp \left( \sum_{n=0}^\infty (2 - \delta_{n,0}) \right) \left( 2 k_3 E_n \delta_{n,0} + 2 E_{\nu e} \right) \times \left( E_n - \frac{m^2}{2 E_n} \right) \left( f_{e,n} + \bar{f}_{e,n} \right) \right.
\]

\[ c_w = \sqrt{2} G_F \left[ \left( 1 + \frac{3}{2} \frac{m^2}{m_W^2} - \frac{k^2}{m_W^2} \right) (n_e - \bar{n}_e) + \frac{e B}{2 \pi^2} \int_0^\infty dp \left( \sum_{n=0}^\infty (2 - \delta_{n,0}) \right) \left( 2 E_{\nu e} \left( E_n - \frac{m^2}{2 E_n} \right) \delta_{n,0} + 2 k_3 \left( E_n - \frac{3 m^2}{2 E_n} \right) \frac{H}{E_n} \right) \left( f_{e,n} + \bar{f}_{e,n} \right) \right].
\]

where the electron number density and electron distribution function are

\[ n_e(\mu, T, B) = \frac{e B}{2 \pi^2} \sum_{n=0}^\infty (2 - \delta_{n,0}) \int_0^\infty dp \frac{E_{\nu e}}{e^\theta(E_{\nu e} - \mu) + 1}, \] (23)

and

\[ f(E_{e,n}, \mu) = \frac{1}{e^\theta(E_{e,n} - \mu) + 1}, \] (24)

respectively, with \( \bar{f}_{e,n}(\mu, T) = f_{e,n}(\mu, T) \) and \( E_{e,n} = \sqrt{p^2_\perp + m^2_e + H} \), where \( H = 2neB \). We can also express Equation (15) (Z exchange) as

\[ Re \Sigma_c(k) = \mathcal{R}(a_Z k + b_Z u) \mathcal{L}, \] (25)

and explicit evaluation gives (D’Olivo & Nieves 1994)

\[ a_Z = \sqrt{2} G_F \left[ \frac{E_{\nu e}}{m_Z^2} (n_{\nu e} - \bar{n}_{\nu e}) + \frac{2}{3} m_Z^2 \left( \langle E_{\nu e} \rangle |n_{\nu e} + \langle \bar{E}_{\nu e} \rangle |n_{\bar{\nu} e} \right) \right], \] (26)

and

\[ b_Z = \sqrt{2} G_F \left[ (n_{\nu e} - \bar{n}_{\nu e}) - \frac{8 E_e}{3m_Z^2} \left( \langle E_{\nu e} \rangle |n_{\nu e} + \langle E_{\nu e} \rangle |n_{\bar{\nu} e} \right) \right]. \] (27)

where the four-vector \( k \) can be decomposed into the four-vectors \( h \) and \( b \) in accordance with Equation (3).

From the tadpole diagram, Equation (16), we obtain

\[ Re \Sigma_c(k) = \sqrt{2} G_F \mathcal{R} \left[ C_{V'} (n_e - \bar{n}_e) + C_{V'} (n_p - \bar{n}_p) + C_{V'} (n_n - \bar{n}_n) + (n_{\nu e} - \bar{n}_{\nu e}) + (n_{\nu e} - \bar{n}_{\nu e}) \right] \mathcal{L}, \] (28)

For antineutrinos, we must change \((n_e - \bar{n}_e)\) to \(-(n_e - \bar{n}_e)\). The different contributions to the neutrino self-energy up to the order of \(1/m_W^2\) have been calculated in a background of \(\gamma, e^\pm,\) free baryons, neutrinos, and antineutrinos. The effective potential that is applicable to neutrino oscillations in matter is \(V_{\text{eff}} = V_e - V_{\mu,\tau}\), which depends only on electron density (Wolfenstein 1978; D’Olivo et al. 1992); assuming that neutrinos propagate in the same direction of the magnetic field (\(\phi = 0\)) and with \(k_3 = E_{\nu e}\), we derive the neutrino effective potential in all the regimes of the magnetic field: strong, \(\Omega_B = eB/m^2_e \gg 1\); moderate, \(\Omega_B = eB/m^2_e > 1\) (above \(B_t\)) and \(\Omega_B = eB/m^2_e \ll 1\) (below \(B_t\)); and weak, \(\Omega_B = eB/m^2_e \ll 1\), where \(B_t = m^2_e/e = 4.414 \times 10^{13} \text{ G}\) is the critical magnetic field. In the following subsections, we will show the neutrino effective potential in all the regimes (strong, moderate, and weak magnetic field limits; Erdas 2009); the calculations are explicitly shown in Appendix A, B, and C. Using the typical fireball values at the initial stage and the phase of acceleration and temperature in the range of \(1–10 \text{ MeV}\) at initial radius \(r_0 \approx 10^{8.5}–10^{9.5} \text{ cm}\) for thermal MeV neutrinos (Beloborodov 2003a; Ruffert & Janka 1999; Koers & Wijers 2005; Janka et al. 2012) and in the range of \(0.1–1 \text{ MeV}\) at radius \(r \approx 10^{8.5}–10^{9.5} \text{ cm}\) for quasi-thermal GeV neutrinos (Bahcall & Mészáros 2000; Mészáros & Rees 2000; Razzue & Mészáros 2006a, 2006b; Murase et al. 2013), we plot the effective potential in each limit as shown in Figures 2, 3, 4, and 5. Although we are going to do a full analysis of neutrino oscillations in a fireball endowed with moderate magnetic fields, which is more favored for GRB central engines (Beloborodov 2003a; Mészáros 2012), a brief and additional description in the strong and weak field limits as well as a comparison of effective potentials in all the regimes will be given in the following.

2.2. Strong Magnetic Field: \(\Omega_B \gg 1\)

In the strong magnetic field approximation \(m^2_e \ll T^2 \ll \Omega_B m^2_e\), leptons are all confined to the lowest Landau level \((n = 0)\); then only this level will contribute to the potential, and the energy of these leptons will be independent of the magnetic field. The neutrino effective potential at the strong magnetic field is given by

\[ V_{\text{eff, S}} = A\left[ \sum_{i=0}^\infty (-1)^i \sinh \alpha_l K_1(\alpha_l) \left( \frac{m^2_e}{m_W^2} \left( 1 + \frac{E_e^2}{m_e^2} \right) \right) - 3 \frac{m^2_e}{m_W^2} m_e \sum_{i=0}^\infty (-1)^i \cosh \alpha_l K_0(\alpha_l) \right]. \] (29)

In Figure 2, we plot the effective potential at the strong field limit (Equation (29)) as a function of temperature (top left panel),
Figure 2. Neutrino effective potential in the strong magnetic field regime as a function of temperature (T; top left), magnetic field (ΩB; top right), chemical potential (μ; bottom), and neutrino energy (Eν). All plots are obtained for a neutrino energy of 10 MeV.

(A color version of this figure is available in the online journal.)

magnetic field (top right panel), and chemical potential (bottom panel). For these plots, we take into account the neutrino energy of 10 MeV and the values of temperature, magnetic field, and chemical potential in the ranges from 1 to 10 MeV, 10^3 to 10^5 Bc, and 10^{-4} eV to 4.5 keV, respectively. As shown, the effective potential is a quasi-constant function of temperature and an increasing function of magnetic field and chemical potential.

2.3. Moderate Magnetic Field: ΩB > 1 and ΩB ≤ 1

In the moderate field approximation (m_e^2 < ΩB m_e^2 ≤ T^2), leptons start to occupy the next Landau levels (n = 1, 2, 3, ...), which have a separation that is directly proportional to the magnetic field. For this case each of these levels will contribute to the effective potential, and the energy of the leptons is directly proportional to the magnetic field. The neutrino effective potential in the moderate magnetic field is written as

\[
V_{\text{eff}} = A_e \left[ \sum_{l=0}^{\infty} (-1)^l \sinh \alpha_l \left\{ \frac{m_e^2}{m_W^2} \left( 1 + \frac{E_{\nu}^2}{m_e^2} \right) K_1(\alpha_l) \right\} + \sum_{n=1}^{\infty} \lambda_n \left( 2 + \frac{m_e^2}{m_W^2} \left( 3 - 2\Omega_B + \frac{E_{\nu}^2}{m_e^2} \right) \right) K_1(\alpha_l \lambda_n) \right],
\]

with

\[
\lambda^2 = \begin{cases} 
2 n \Omega_B & \text{for } \Omega_B > 1 \\
1 + 2 n \Omega_B & \text{for } \Omega_B \leq 1
\end{cases}
\]

In Figure 3, we have plotted the neutrino effective potential at the moderate field limit above Bc (Equation (30)) as a function of temperature (top left panel), magnetic field (top right panel), and chemical potential (bottom panel). For these plots we take into account the neutrino energy of 10 MeV and the values of temperature, magnetic field, and chemical potential in the ranges from 1 to 10 MeV, Bc < B ≤ 10^2 Bc, and 10^{-4} eV to 4.5 keV, respectively. As shown, the effective potential as a function of temperature has two different behaviors. First, it is an increasing function of temperature in the range from 1 to 3 MeV for values of magnetic field of 5 Bc and 10 Bc and in the range from 1 to 7 MeV for 50 Bc and 100 Bc, and second, it tends to be constant for values of temperature greater than 3 MeV for B = 5 Bc and B = 10 Bc and 7 MeV for B = 50 Bc and
$B = 100B_c$. Figure 3 also shows that the effective potential is an increasing function for both the magnetic field and chemical potential.

In Figure 4, we have plotted the neutrino effective potential at the moderate field limit below $B_c$ (Equation (30)) as a function of temperature (top panel), magnetic field (middle panel), and chemical potential (bottom panel) for neutrino energies $E_\nu = 10$ MeV (left column) and $E_\nu = 10$ GeV (right column). For these plots we take into account the values of the magnetic field and chemical potential in the ranges of $10^{-3}B_c < B \leq B_c$ and $10^{-4}$ eV to 4.5 keV, respectively, and two ranges of temperatures, $0.1$ MeV $\leq T \leq 1$ MeV (right column) and $1$ MeV $\leq T \leq 10$ MeV (left column). In the top panels, the behavior of the effective potential as a function of temperature has a multifunctional dependence that depends on the values of the magnetic field. For instance, in the right panel, for $B = 0.5B_c$, it is a dramatically increasing function, for $B = 10^{-2}B_c$, it is just a steadily increasing function, and for the smaller values of B, it becomes a decreasing function; in the left panel, for $B = 0.5B_c$, it is a steadily increasing function, and as B decreases, the effective potential gradually becomes a decreasing function. In the middle panels, the effective potential is a dramatically (left) and steadily (right) increasing function of the magnetic field regardless of the values of temperature. In the bottom panels, the effective potential represented by means of an increasing function of the chemical potential shows the same behavior but at different energy ranges.

2.4. Weak Magnetic Field: $\Omega_B \ll 1$

In the weak field approximation ($\Omega_B m_e^2 \ll m_\nu^2$), all Landau levels are occupied and overlap each other; therefore, a good description of these levels is to take the approximation $\sum_n \to \int d n$. For this case, the neutrino effective potential is

$$V_{\text{eff}, W} = A \left[ \sum_{l=0}^{\infty} (-1)^l \sinh \alpha_l \left\{ \left( 2 + \frac{m_e^2}{m_W^2} \left( 3 + 4 \frac{E_\nu^2}{m_\nu^2} \right) \right) \times \left( \frac{K_0(\sigma_l)}{\sigma_l} + 2 \frac{K_1(\sigma_l)}{\sigma_l^2} \right) \right. \right]$$

$$\times \left. \frac{\Omega_B^{-1}}{\Omega_B^{-1}} \right]$$

$$- 4 \frac{m_e^2 E_\nu}{m_W^2} \sum_{l=0}^{\infty} (-1)^l \cosh \alpha_l \left\{ \left( \frac{2}{\sigma_l^2 \Omega_B} - 1 \right) \frac{K_0(\sigma_l)}{\sigma_l} \right. \right]$$

$$+ \left( 1 + 4 \frac{\sigma_l}{\sigma_l^2} \right) \frac{K_1(\sigma_l)}{\sigma_l} \Omega_B^{-1} \right]. \quad (31)$$

Figure 5 shows the neutrino effective potential at the weak field limit (Equation (31)) as a function of temperature (top panels) and chemical potential (bottom panels) for neutrino...
energies $E_\nu = 10$ MeV (left panels) and $E_\nu = 10$ GeV (right panels). Because of the strength of the magnetic field it is quite small, and any variation will produce insignificant changes in the effective potential; we only show the magnetic field contribution, i.e., by subtracting the effective potential with $B = 0$, which shows that the potential is a decreasing function of temperature and an increasing function of chemical potential and that the magnetic contribution is the opposite compared to the medium contribution. The effective potential in this regime differs from that calculated by Sahu et al. (2009b) because the authors took the solution of dispersion relation $k_3 = -E_\nu$ instead of $k_3 = E_\nu$.

2.5. Comparison of Effective Potentials in All the Regimes

We plot the neutrino effective potentials as a function of temperature (Figure 6, left) and chemical potential (Figure 6, right) in strong, moderate, and weak magnetic field approximations, which are given in Equations (29), (30), and (31), respectively. As shown in Figure 6, the effective potential lies between $\sim 10^{-11}$ and $\sim 10^{-8}$ eV for a temperature in the range

Figure 4. Neutrino effective potential in the moderate below-critical magnetic field regime as a function of temperature (T; top), magnetic field ($\Omega_B$; middle), and chemical potential ($\mu$; bottom). In the left column a neutrino energy $E_\nu = 10$ MeV was used, whereas the right column was plotted for $E_\nu = 10$ GeV.

(A color version of this figure is available in the online journal.)
from 1 to 8 MeV. One can see several features from Figure 6. (1) At $T \sim 1$ MeV, the effective potentials at the moderate (above and below $B_c$) field limit are quite close at $\sim 5 \times 10^{-10}$ eV; however, as temperature increases, the separation between them also increases, and for the strong and weak field limits, the distance between the effective potentials is more than two orders of magnitude. (2) As temperature increases, the effective potentials at the strong and moderate (above $B_c$) field limits become closer to each other. Dividing Figure 6 into two regions, $T < 3$ MeV and $T \geq 3$ MeV, we can argue that for the given values of the magnetic field and temperature, in the first region $m_c^2 \Omega_B \simeq T^2$, as temperature becomes larger than the magnetic
Equations (29), (30), and (31), there are two common terms that depend on $m$. Is important and dominant over the $m$ field, $T^2$, a difference between both effective potentials for each value of temperature is observed. This difference, which is of the same order of magnitude for both potentials, comes from the contribution of the excited Landau levels ($n=1, 2, \ldots$). (3) The effective potential at the weak field limit is a dramatically increasing function of temperature that tends to be at the moderate limit below $B_c$. (4) The effective potential at the strong field limit decreases very gradually (quasi-constantly) as temperature increases; in fact, for the range of temperature considered, the effective potential is almost invariant to any thermal contribution. Also, it is an increasing function of the chemical potential in the range of $0.1–4 \times 10^3$ eV for all the regimes of the magnetic field. By comparing the effective potentials in the strong (maximum value) and weak (minimum value) field approximations, one can see that the latter approximation is smaller than the former by three orders of magnitude.

Additionally, in Figure 7, we plot the contribution of the $m_n^2$ terms to the neutrino effective potential. As can be seen in Equations (29), (30), and (31), there are two common terms that depend on $m_n^2$ and that contribute to the effective potentials: $(-1)^l \cos (\alpha) K_l(\sigma)$ and $4 E_{\nu}^2/m_n^2$. In the first case, we compare the terms $(-1)^l \sin (\alpha) K_l(\sigma)$ and $(-1)^l \cos (\alpha) K_0(\sigma)$ (left panel), where the first term comes from particle-antiparticle asymmetry ($n_e - \bar{n}_e$) and the second one comes from ($n_e + \bar{n}_e$). From this plot, one can observe that the $(n_e - \bar{n}_e)/(n_e + \bar{n}_e)$ ratio is directly proportional to the chemical potential, and for $\mu \leq 10^2$ eV the $m_n^2$ term begins to be dominant, achieving a minimum value of seven orders of magnitude for $\mu = 10^{-3}$ eV. In other words, as $\mu$ decreases, the correction of order $m_n^2$ is important and dominant over the $m_{\nu}^2$ term, even though one can deduce that for $\mu = 0$, $n_e - \bar{n}_e = 0$, and then the only contribution would come from the term $O(m_{\nu}^4)$. In the second case, we plot $4 E_{\nu}^2/m_n^2$ as a function of $E_{\nu}$ (right panel). From this plot, you can notice that this term in comparison with unity starts to contribute for neutrinos with energies of some tens of GeV. As shown above, these two terms with the values of temperature, chemical potential, and neutrino energy are relevant to the neutrino effective potential.

From the previous analysis, it can be observed that regardless of the magnetic field limit or relativistic temperature or chemical potential, the neutrino effective potential is positive; therefore, because of its positivity ($V_{\text{eff},k} > 0$ for $k = S, M,$ and $W$), MeV–GeV neutrinos can oscillate resonantly.

3. NEUTRINO MIXING AND RESONANCE CONDITION

In the following subsections, we are going to consider the best fit values of the neutrino oscillation parameters for two-neutrino mixing (solar, atmospheric, and accelerator) and three-neutrino mixing.

3.1. Two-Neutrino Mixing

Here we consider the neutrino oscillation process $\nu_e \leftrightarrow \nu_{\mu, \tau}$. The evolution equation for the propagation of neutrinos in the medium is given by (FraiJa 2014)

$$i \begin{pmatrix} \dot{\nu}_e \\ \dot{\nu}_\mu \\ \dot{\nu}_\tau \end{pmatrix} = \begin{pmatrix} V_{\text{eff}} - \Delta \cos 2\theta & 0 & 0 \\ \frac{\Delta}{2} \sin 2\theta & \Delta \cos 2\theta & 0 \\ 0 & 0 & \Delta \cos 2\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix},$$

where $\Delta = \delta m^2/2 E_{\nu}$, $\delta m^2$ is the mass difference, $V_{\text{eff}}$ is the neutrino effective potential between $V_{\nu_e}$ and $V_{\nu_{\mu, \tau}}$ (Equation (30)), $E_{\nu}$ is the neutrino energy, and $\theta$ is the neutrino mixing angle. The conversion probability for the above process at a time $t$ is given by

$$P_{\nu_e \rightarrow \nu_{\mu, \tau}}(t) = \frac{\Delta^2 \sin^2 2\theta}{\omega^2} \sin^2 \left(\frac{\omega t}{2}\right),$$

with

$$\omega = \sqrt{(V_{\text{eff}} - \Delta \cos 2\theta)^2 + \Delta^2 \sin^2 2\theta}.$$  

The oscillation length for the neutrino is given by

$$l_{\text{osc}} = \frac{l_v}{\sqrt{\cos^2 2\theta (1 - \frac{V_{\text{eff}}}{\Delta \cos 2\theta})^2 + \sin^2 2\theta}},$$

where $l_v = 2\pi/\Delta$ is the vacuum oscillation length. Applying the resonance condition given by

$$V_{\text{eff}} = \Delta \cos 2\theta = 5 \times 10^{-7} \text{ eV} \frac{\delta m^2_{\nu_v}}{E_{\nu, \text{MeV}}} \cos 2\theta,$$
we obtain that the resonance length \( l_{\text{res}} \) can be written as

\[
    l_{\text{res}} = \frac{l_v}{\sin 2\theta},
\]

In Equation (36), the neutrino effective potential depends on the chemical potential, the temperature, the neutrino energy, and the oscillation parameters (mass differences and mixing angles). We will use the following parameters for this analysis:

**Solar Neutrinos.** A two-flavor neutrino oscillation analysis yielded \( \delta m^2 = (5.6^{+1.9}_{-0.9}) \times 10^{-5} \text{eV}^2 \) and \( \tan^2 \theta = 0.427^{+0.033}_{-0.029} \) (Aharmin et al. 2011).

**Atmospheric Neutrinos.** Under a two-flavor disappearance model with separate mixing parameters between neutrinos and antineutrinos, the following parameters for the Super-Kamiokande (SK)-I + II + III data were found (Abe et al. 2011): \( \delta m^2 = (2.1^{+0.3}_{-0.4}) \times 10^{-3} \text{eV}^2 \) and \( \sin^2 2\theta = 1.0^{+0.07}_{-0.06} \).

**Accelerator Parameters (Short baselines).** Church et al. (2002) found two well-defined regions of oscillation parameters with either \( \delta m^2 \approx 7 \text{eV}^2 \) or \( \delta m^2 < 1 \text{eV}^2 \) compatible with both the KamLAND and KARMEN experiments for the complementary confidence. In addition, MiniBooNE found evidence of oscillations in the 0.1 to 1.0 eV\(^2\), which are consistent with the Liquid Scintillator Neutrino Detector (LSND) results (Athanassopoulos et al. 1996, 1998).

### 3.2. Three-Neutrino Mixing

To determine the neutrino oscillation probabilities, we have to solve the evolution equation of the neutrino system in matter. In a three-flavor framework, this equation is given by

\[
    \frac{d\nu}{dt} = H\nu, \quad (38)
\]

and the state vector in the flavor basis is defined as

\[
    \nu \equiv (\nu_e, \nu_\mu, \nu_\tau)^T. \quad (39)
\]

The effective Hamiltonian is

\[
    H = U \cdot H^d \cdot U^\dagger + \text{diag}(V_{\text{eff}}, 0, 0), \quad (40)
\]

with

\[
    H^d_0 = \frac{1}{2E_\nu} \text{diag}(-\delta m^2_{21}, 0, \delta m^2_{32}). \quad (41)
\]

Here \( V_{\text{eff}} \) is the effective potential (Equation (30)) and \( U \) is the three-neutrino mixing matrix given by (Gonzalez-Garcia 2002; Akhmedov et al. 2004; Gonzalez-Garcia & Maltoni 2008; Gonzalez-Garcia et al. 2011)

\[
    U = \begin{pmatrix}
        c_{13} c_{12} & s_{12} c_{13} & s_{13} \\
        -s_{12} c_{23} - s_{23} s_{13} c_{12} & c_{23} c_{12} - s_{23} s_{13} s_{12} & s_{23} s_{13} \\
        s_{13} c_{23} s_{12} - s_{23} s_{12} & -s_{23} c_{12} & c_{23} c_{13}
    \end{pmatrix},
\]

where \( s_{ij} = \sin \theta_{ij} \) and \( c_{ij} = \cos \theta_{ij} \). For antineutrinos one has to replace \( U \) with \( U^\dagger \). The different neutrino probabilities are given as

\[
    P_{ee} = 1 - 4s^2_{13} c^2_{13} c^2_{23},
\]

\[
    P_{\mu\mu} = 1 - 4s^2_{13} c^2_{13} c^2_{23} - 4s^2_{13} s^2_{12} c^2_{23} c^2_{32}
\]

\[
    P_{\nu\nu} = 1 - 4s^2_{13} c^2_{13} c^2_{23} - 4s^2_{13} s^2_{12} c^2_{23} S_{32}
\]

where

\[
    \sin 2\theta_{13,m} = \frac{\sin 2\theta_{13}}{\sqrt{(\cos 2\theta_{13} - 2E_\nu V_{\text{eff}}/\delta m^2_{32})^2 + (\sin 2\theta_{13})^2}}, \quad (43)
\]

and

\[
    S_{ij} = \sin^2 \left( \frac{\Delta m^2_{ij} l_{\text{osc}}}{4E_\nu} \right), \quad (44)
\]

with

\[
    \Delta m^2_{21} = \frac{\sin 2\theta_{13}}{2} \left( \sin 2\theta_{13} - 1 \right) - E_\nu V_{\text{eff}}, \quad \Delta m^2_{32} = \frac{\sin 2\theta_{13}}{2} \left( \sin 2\theta_{13} + 1 \right) + E_\nu V_{\text{eff}}, \quad \Delta m^2_{31} = \frac{\sin 2\theta_{13}}{2 \sin 2\theta_{13,m}}, \quad (45)
\]

The oscillation length for the neutrino is given by

\[
    l_{\text{osc}} = \frac{l_v}{\sqrt{\cos^2 2\theta_{13} (1 - \frac{2E_\nu V_{\text{eff}}}{\delta m^2_{32} \cos 2\theta_{13}})^2 + \sin^2 2\theta_{13}}}, \quad (46)
\]

where \( l_v = 4\pi E_\nu/\delta m^2_{32} \) is the vacuum oscillation length. The resonance condition and resonance length are

\[
    V_{\text{eff}} - 5 \times 10^{-7} \frac{\delta m^2_{32} \text{eV}}{E_{\nu,\text{MeV}}} \cos 2\theta_{13} = 0, \quad (47)
\]

and

\[
    l_{\text{res}} = \frac{l_v}{\sin 2\theta_{13,m}}. \quad (48)
\]

Considering the adiabatic condition at the resonance, we can express it as

\[
    \kappa_{\text{res}} \equiv \frac{2}{\pi} \left( \frac{\delta m^2_{32}}{2E_\nu} \sin 2\theta_{13} \right)^2 \left( \frac{dV_{\text{eff}}}{dr} \right)^{-1} \geq 1
\]

\[
    = 3.62 \times 10^{-2} \text{cm}^{-1} \left( \frac{\delta m^2_{32} \text{eV}}{E_{\nu,\text{MeV}}} \sin 2\theta_{13} \right)^2 \frac{1}{V'} \geq 1, \quad (51)
\]

where

\[
    V' = \Omega_B \left[ dV_{\text{eff},1}/dr - 3.16 \times 10^{-10} E_{\nu,\text{MeV}} \frac{dV_{\text{eff},2}}{dr} \right]. \quad (52)
\]
Figure 8. Contour plots of temperature \( (T) \) and chemical potential \( (\mu) \) as a function of thermal neutrino energy for which the resonance condition is satisfied. We have applied the neutrino effective potential with moderate magnetic field \( (\Omega_B = 10) \) and used the best-fit values of the two-neutrino mixing (solar, top left; atmospheric, top right; and accelerator, bottom left) and three-neutrino mixing (bottom right).

(A color version of this figure is available in the online journal.)

\[
V_{\text{eff}, 1} = \sum_{l=0}^{\infty} (-1)^l \sin \alpha_l \left\{ \frac{m^2_{l}}{m_W^2} \left( 1 + 4 \frac{E^2_{\nu}}{m^2_{l}} \right) K_1(\sigma_l) + \sum_{n=1}^{\infty} \lambda_n \left[ 2 + \frac{m^2_{l}}{m_W^2} \left( 3 + 4 \frac{E^2_{\nu}}{m^2_{l}} - 2\Omega_B \right) \right] K_1(\sigma_l \lambda_n) \right\}
\]

\[
V_{\text{eff}, 2} = \sum_{l=0}^{\infty} (-1)^l \cos \alpha_l \left\{ \frac{3}{4} K_0(\sigma_l) + \sum_{n=1}^{\infty} \lambda_n^2 K_0(\sigma_l \lambda_n) \right\}.
\] (53)

We will use the following parameters (Aharmim et al. 2011; Wendell et al. 2010) for this analysis:

for \( \sin^2 13 < 0.053 \): \( \delta m^2_{21} = (7.41^{+0.21}_{-0.19}) \times 10^{-5} \text{ eV}^2 \) and \( \tan^2 \theta_{12} = 0.446^{+0.030}_{-0.029} \)

for \( \sin^2 13 < 0.04 \): \( \delta m^2_{23} = (2.1^{+0.5}_{-0.2}) \times 10^{-3} \text{ eV}^2 \) and \( \sin^2 \theta_{23} = 0.50^{+0.083}_{-0.093} \). (54)

For a complete description of resonant neutrino oscillations at the early fireball stage, we use the values of fireball observables during the initial stage and the phase of acceleration. In the initial stage, we consider a fireball endowed with magnetic field \( B = 10 B_c, \) temperature in the range from 1 to 5 MeV, and thermal neutrino energies \( E_{\nu} = 1, 5, 20, \) and 30 MeV, whereas in the acceleration phase the corresponding values are magnetic field \( B = 10^{-4.3} B_c, \) temperature in the range from 50 to 500 keV, and quasi-thermal GeV neutrino energies \( E_{\nu} = 1, 10, 20, \) and 50 GeV. In both cases we take the best fit of parameters for two-neutrino (solar: \( \delta m^2 = 5.6 \times 10^{-5} \text{ eV}^2 \) and \( \tan^2 \theta = 0.427 \) (Aharmim et al. 2011), atmospheric: \( \delta m^2 = 2.1 \times 10^{-3} \text{ eV}^2 \) and \( \sin^2 2\theta = 1.0 \) (Abe et al. 2011), and accelerator: \( \delta m^2 \sim 0.6 \text{ eV}^2 \) and \( \sin^2 2\theta = 51.2 \times 10^{-3} \) (Church et al. 2002)) and three-neutrino \( (\delta m^2_{32} = 10^{-2.58} \text{ eV}^2 \) and \( \theta_{13} = 11^\circ)) \) mixing in order to analyze the resonance conditions in both phases.

From the resonance conditions (Equations (36) and (49)), we have obtained the contour plots of temperature and chemical potential in the initial stage (Figure 8) and the phase of acceleration (Figure 9). In the initial phase, the chemical potential lies in the range from \( 7.4 \times 10^{-3} \) to 3.01 eV for solar parameters, \( 1.01 \times 10^{-2} \) to 2.84 eV for atmospheric parameters, 0.12 to 3.1 \( \times 10^2 \) keV for accelerator parameters, and
Figure 9. Contour plots of temperature (T) and chemical potential (μ) as a function of quasi-thermal neutrino energy for which the resonance condition is satisfied. We have applied the neutrino effective potential with moderate magnetic field (ΩB = 10^{-4.3}) and used the best-fit values of the two-neutrino mixing (solar, top left; atmospheric, top right; and accelerator, bottom left) and three-neutrino mixing (bottom right). (A color version of this figure is available in the online journal.)

Table 1
Resonance Lengths of Thermal Neutrinos for the Best-fit Parameters of Two- and Three-Neutrino Mixing

| Energy (MeV) | Solar       | Three Flavors |
|-------------|-------------|---------------|
|             | l_{res} (cm) |               |
| 1           | 1.2 × 10^3  | 7.1 × 10^4    |
| 5           | 5.9 × 10^3  | 3.6 × 10^4    |
| 20          | 2.4 × 10^3  | 1.4 × 10^4    |
| 30          | 3.5 × 10^3  | 2.1 × 10^4    |

Note. Resonance lengths of thermal neutrinos for the best-fit parameters of two- and three-neutrino mixing.

0.8 to 50 × 10^2 eV for three-neutrino mixing. One can see that temperature is a decreasing function of chemical potential, which gradually increases as neutrino energy is decreased. In addition, we have computed the resonance lengths, which are shown in Table 1. As shown in Table 1, the resonance lengths lie in the range l_{res} ~ 10^4 to 10^6 cm, which is comparable to the length scale of a fireball. Therefore, depending on the oscillation parameters neutrinos could oscillate resonantly before leaving the fireball. For instance, considering an initial radius of 10 km, only neutrinos with low energy will oscillate resonantly for atmospheric, accelerator, and three-neutrino mixing parameters but not solar parameters. If we assume an initial radius of 100 km and consider atmospheric and accelerator oscillation parameters, then neutrinos would oscillate resonantly regardless of their energies. Once again, assuming a radius of 1000 km, thermal neutrinos will oscillate resonantly. In the phase of acceleration, the chemical potential lies in the range from 2.2 × 10^{-2} to 1 eV for solar parameters, 4.01 × 10^{-2} to 1.3 eV for atmospheric parameters, 1.1 × 10^{-2} to 0.5 keV for accelerator parameters, and 0.5 to 6.1 eV for three-neutrino mixing. One can see that temperature is a decreasing function of the chemical potential, which increases as neutrino energy increases (decreases) for solar and atmospheric (accelerator) oscillation parameters and is doubly degenerate for three-neutrino mixing. Further, we have computed the resonance lengths of GeV neutrinos. As shown in Table 2, the resonance lengths match the length scale where they were created for three-neutrino mixing. In other words, taking into account the former parameters, multi-GeV neutrinos created at ~10^{11}–10^{13} cm will have simultaneously resonant oscillations. Finally, we have studied the survival and
conversion probability for the active-active ($\nu_e,\mu,\tau \leftrightarrow \nu_e,\mu,\tau$) neutrino oscillations and the three-neutrino mixing. Figure 10
plots the survival probability of electron $P_{ee}$, muon $P_{\mu\mu}$, and tau $P_{\tau\tau}$ neutrino and conversion probabilities $P_{e\mu}, P_{e\tau},$ and $P_{\mu\tau}$ as a function of energy for fixed values of length scale, temperature, magnetic field, and neutrino energy. In the left panels, we once again use the fireball values in the initial stage: $T = 5$ MeV, $r_0 = 100$ km (top) and $r_0 = 10$ km (bottom), $B = 10 B_c$, and neutrino energy range from 1 to 30 MeV. In the right panels, we take into account the values already mentioned for the phase of acceleration: $T = 100$ keV, $r = 10^{11}$ cm (top) and $r = 10^{12}$ cm (bottom), $B = 10^{-4.3} B_c$, and neutrino energy range of 1–30 GeV.

Our analysis shows that $P_{ee} = 1$, $P_{e\mu} = P_{e\tau} = 0$, indicating that the propagating electron neutrinos do not oscillate to any other flavor independent of their energies and radii. Instead, it tells us that muon and tau neutrinos oscillate among themselves with equal probability and that the oscillation depends on their energy and distances. As can be seen, the probabilities satisfy the condition

$$\sum_{i=e,\mu,\tau} P_{ei} (\delta m_{32}^2, L) = 1, \quad \sum_{i=e,\mu,\tau} P_{\mu i} (\delta m_{32}^2, L) = 1, \quad \sum_{i=e,\mu,\tau} P_{\tau i} (\delta m_{32}^2, L) = 1.$$  (55)

### 3.3. Neutrino Oscillation from Source to Earth

Between the surface of the star and the Earth the flavor ratio $\phi_{e0} : \phi_{\mu0} : \phi_{\tau0}$ is affected by the full three-description flavor mixing, which is calculated as follows. The probability for a neutrino to oscillate from flavor state $\alpha$ to flavor state $\beta$ in a time starting from the emission of the neutrino at star $t = 0$ is given as

$$P_{\nu_{\alpha} \rightarrow \nu_{\beta}} = |<\nu_{\beta}(t) | \nu_{\alpha}(t = 0) >|^2$$

$$= \delta_{\alpha\beta} - 4 \sum_{j>i} U_{ai} U_{ji} U_{aj} U_{bi} \sin^2 \left( \frac{\delta m_{ij}^2 L_{\text{osc}}}{4 E_{\nu}} \right).$$  (56)
Using the set of parameters given in Equation (54), we can write the mixing matrix

\[ U = \begin{pmatrix} 0.816669 & 0.544650 & 0.190809 \\ -0.504583 & 0.513419 & 0.694115 \\ 0.280085 & -0.663141 & 0.694115 \end{pmatrix}. \]  

(57)

Additionally, averaging the sine term in the probability to changing to a flavor vector \((\nu_e, \nu_\mu, \nu_\tau)_{\text{source}}\) changing to a flavor vector \((\nu_e, \nu_\mu, \nu_\tau)_{\text{Earth}}\) is given as

\[ \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_{\text{Earth}} = \begin{pmatrix} 0.534143 & 0.265544 & 0.200313 \\ 0.265544 & 0.366436 & 0.368020 \\ 0.200313 & 0.368020 & 0.431667 \end{pmatrix} \times \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}_{\text{source}} \]  

(58)

for distances longer than the solar system.

4. DYNAMICS OF THE FIREBALL

Although we have already introduced the general concepts of the fireball model and its dynamics and have also calculated the range of values of chemical potentials and temperatures for which the resonance condition is satisfied, here we are going to quantify and/or estimate the observable quantities in the evolution of the fireball requiring the charged-neutrality condition in addition to the resonance condition. In this manner, we are going to constrain the dynamics of the fireball by means of neutrino oscillations.

4.1. Initial Stage

The initial state of the fireball is magnetized, hot, and dense baryons and \(e^\pm\) pairs in perfect thermodynamic equilibrium with a comoving temperature

\[ T^* = \left( \frac{L_{52}}{4\pi T_{50}^2 a_0} \right)^{1/4} \simeq 3.8 \text{ MeV}, \]  

(59)

where \(a = \pi^2 k^4/15 = 7.6 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}\) is the radiation density constant, \(L\) is the total isotropic-equivalent energy outflow in the jet, and \(r_0\) is the initial radius. Magnetic fields can be estimated by means of the equipartition of the total energy density \((U)\)

\[ B \simeq \sqrt{8\pi \epsilon_B U}, \]  

(60)

which becomes important as the energy density is dominated by baryons (Beloborodov 2003a),

\[ U \simeq \frac{3}{2} T^4, \]  

(61)

and the electrons are degenerated with the charged-neutrality condition, \(n_e(\mu, T, B) = \bar{n}_e(\mu, T, B) = n_p\), where \(n_e(\mu, T, B)\) and \(\bar{n}_e(\mu, T, B)\) are the number densities of electrons and positrons generated in the plasma at temperature \(T\) and magnetic field \(B\) and \(n_p\) is the number density of protons. Taking into account that the number density of neutrinos is comparable to that of protons and using the number density of electrons and positrons at the moderate magnetic field and relativistic temperatures found in Section 2 (Equation (B4)), the charged-neutrality condition is given by

\[ \frac{m_p^3}{\pi^2} \Omega_p^2 \sum_{l=0}^{\infty} (-1)^l \sin \alpha_l \left( K_1(\sigma_l) + 2 \sum_{n=1}^\infty \lambda_n K_1(\sigma_l \lambda_n) \right) = Y_e \frac{\rho}{m_p}, \]  

(62)

where \(\sigma_l\) and \(\alpha_l\) correspond to the values for which the resonance condition is satisfied, \(\rho\) is the baryon density, and \(Y_e = n_p/(n_n + n_p)\) is the proton-to-nucleon ratio. The baryon density is defined by means of the total mass outflow rate in the baryon-loaded jet \(M = 4\pi r_0^2 \rho\) and the dimensionless entropy or baryon load parameter \(\eta = L/M\). The entropy per baryon is conserved in the flow and is related to \(\eta\) and \(T\) as \(s/k_B = 4n_m / 3T\). Initially, because of high temperatures \(T \gg m_e\) and charged current reactions

\[ e^- + p \rightarrow n + v, \quad e^+ + n \rightarrow p + \bar{v}, \]  

(63)

protons convert into neutrons and neutrons into protons, achieving the balance

\[ Y_e = \frac{1}{2} + \frac{7\pi^4}{1350 \xi(5)} \left( \frac{Q/2 - \mu}{T} \right), \]  

(64)

between the rates of \(e^-\) and \(e^+\) capture by \(n_p\) and \(n_n\), with \(Q\) defined through neutron \((m_n)\) and proton \((m_p)\) mass, \(Q = m_n - m_p\) (Beloborodov 2003a). Hence, the charged-neutrality condition can be written as

\[ \sum_{l=0}^{\infty} (-1)^l \sin \alpha_l \left( K_1(\sigma_l) + 2 \sum_{n=1}^\infty \lambda_n K_1(\sigma_l \lambda_n) \right) = \rho \frac{\Omega_p^{-1}}{m_p m_e} \left[ \frac{\pi^2}{2} + \frac{7\pi^6}{1350 \xi(5)} \left( \frac{Q/2 - \mu}{T} \right) \right]. \]  

(65)

It is very important to highlight that the proton-to-nucleon ratio is a function of the temperature and chemical potential of the initial stage of the fireball, when the initial optical depth is extremely high and the phase of acceleration has not yet begun.

Taking into account the range of values of chemical potentials and temperatures found from the resonance conditions (Figures 8 and 9), considering magnetic fields at the moderate field limit, 50 \(B_c\) (left panels) and 0.1 \(B_c\) (right panels), and requiring the charged-neutrality condition (Equations (62) and (65)), we plot contour lines of baryon density \((\lambda_n\) (left panels) and \(0.1\) (right panels), and the proton-to-nucleon ratio \((Y_e\) (bottom) as a function of temperature in Figure 11. From the top panels, one can see that the highest line of baryon density increases gradually, achieving a maximum density of \(7.2 \times 10^8 \text{ g cm}^{-3}\) (left panel) and \(5.6 \times 10^8 \text{ g cm}^{-3}\) (right panel) for 2 MeV \(\leq T \leq 3\) MeV and \(\mu = 160\) keV. This value of chemical potential is the largest one obtained from the resonance condition; hence, resonant oscillations are suppressed for densities larger than these. From the bottom panels one can see that proton-to-nucleon ratio is a decreasing function of temperature, achieving a minimum value of \(\sim 0.52\) and 0.53 for baryon density \(\sim 10^8 \text{ g cm}^{-3}\) (left panel) and \(\leq 10^8 \text{ g cm}^{-3}\) (right panel) and a maximum value of \(\sim 0.87\) and 0.88 for a baryon density of \(10^8 \text{ g cm}^{-3}\) (left panel) and \(10^4 \text{ g cm}^{-3}\) (right panel); therefore, resonant oscillations are only allowed when the number density of protons is at least

\[ \sim 0.87 \text{ g cm}^{-3}. \]
slightly larger than that of neutrons, \( n_p > n_n \). This result could make evident the deneutralization process in the fireball, where \( Y_e \) is less than 0.5 at early times (there are no resonant oscillations) and larger than 0.5 at later times (resonant oscillations are allowed). Another important characteristic from the bottom panels is that the number density of the neutron-to-proton ratio \( n_n/n_p = (1 - Y_e)/Y_e \) is lower for a fireball with more magnetization. For instance, taking a density value of \( \rho = 10^6 \text{ g cm}^{-3} \) in both plots, we can see that \( Y_e(n_n/n_p) = 0.865(0.156) \) for \( B = 50 B_c \) and \( Y_e(n_n/n_p) = 0.751(0.332) \) for \( B = 0.1 B_c \).

### 4.2. Phase of Acceleration

When the fireball starts expanding, the optically thick, hot plasma expands with an increasing bulk Lorentz factor \( \Gamma \propto r/r_0 \) with radius \( r \) following the adiabatic law, and the comoving temperature drops as \( T'(r) \propto r_0/r \). Protons and neutrons remain coupled with each other until the value of the Lorentz factor \( \Gamma \approx \eta \) is larger or smaller than a critical value or the dynamical time \( t_{np} \approx (r/\Gamma) \) is shorter than the elastic scattering time \( t_{np} \approx (n_p^s \sigma_{np})^{-1} \), where \( \sigma_{np} \approx 3 \times 10^{-26} \text{ cm}^2 \) and \( n_p^s \) is the number density of protons. The critical value is defined by

\[
\eta_v = \left( \frac{L \sigma_{np} Y_e}{4\pi m_p r_0^3} \right)^{1/4} \approx 4.6 \times 10^2 L_{52}^{1/4} n_0^{-1/4} Y_{e}^{1/4},
\]

and depending on both \( \eta \) and \( \eta_v \), different processes of decoupling in a p-n outflow take place, leading to varied energy ranges of neutrinos. For \( \eta \geq \eta_v \), neutrinos in the energy range of 5–10 GeV are expected (Bahcall & Mészáros 2000), whereas for \( \eta \leq \eta_v \), neutrino energy lies in the range of 2–25 GeV or higher, depending on the value of \( \eta \) (Mészáros & Rees 2000). Supposing that the decoupling takes place before coasting, then a slowly moving hadron shell (\( \Gamma_s \)) is overtaken by another shell that, in principle, moves faster (\( \Gamma_f = \eta \)) in the flow, producing inelastic collisions at \( r \approx 2\Gamma_s^2 r_0 \). Following Mészáros & Rees (2000), the typical collision Lorentz factor can be written as \( \Gamma_{rel} \approx 1/(\Gamma_s + \Gamma_f/\Gamma_s) \), with \( \Gamma = \sqrt{\Gamma_s \Gamma_f} \approx \Gamma_f \), and the total energy is \( E_{tot} = (2m_p^2 + 2m_p E_{rel})^{1/2} \), whereas the CM threshold energy is \( E_{rel} = 2m_p + m_n \sim 2 \text{ GeV} \). Hence, at depths \( t_{pn} \sim n_p^s \sigma_{pn}(r/\Gamma) > 1 \), each neutron heated by an individual shock could collide \( k_n \sim 2 \) times before its CM energy becomes less than the threshold. From the total number of neutrons
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Figure 12. Contour plots of the critical value of the baryon load parameter ($\eta_\nu$; top) and the number of expected events ($R_{\nu\nu}$; bottom) as a function of temperature ($T$; Mészáros & Rees 2000). We use values of temperature and chemical potential in the resonance condition range and moderate field limit: $10^{-4.3} B_c$ (left) and $10^{-5.3} B_c$ (right). The number of events is calculated for $z = 0.1$.

(A color version of this figure is available in the online journal.)

involved in the shock $N_n = (1 - Y_e) e E / (\eta m_p)$ and assuming a shock dissipation efficiency $\epsilon = 0.2$, the number of expected events per year in the DeepCore detector is

$$R_{\nu\nu} \sim 0.33 E_{53} (1 - Y_e) N_{37.7} R_{65} h_{65}^2 \times \left( \frac{2 - \sqrt{2}}{1 + z - \sqrt{1 + z}} \right)^2 \text{year}^{-1},$$

where we have taken into account the product of the density of ice and the effective volume at 100 GeV as $\rho_{\text{ice}} V_{\text{eff}} \approx 50 \text{ Mt}$ ($V_{\text{eff}} \approx 5.56 \times 10^{-2} \text{ km}^3$ and target protons of $N_n \sim 10^{37.7} N_{37.7}$; Abbasi et al. 2012a), a burst rate out to a Hubble radius of $10^3 R_{10}^2$, and an Einstein–de Sitter universe with a Hubble constant $H = 65 h_{65} \text{ km s}^{-1} \text{ Mpc}^{-1}$. Recently, Murase et al. (2013) showed that for low-luminosity GRBs at $D = 10 \text{ Mpc}$ with $\Gamma = 10$ and neutron luminosity $L_n = 2 \times 10^{56} \text{ erg s}^{-1}$, quasi-thermal neutrinos around 10 GeV are expected in DeepCore.

We plot the critical Lorentz factors ($\eta_\nu$; top panels) and number of neutrinos ($R_{\nu\nu}$; bottom panels) as a function of temperature when the resonance and charged-neutrality conditions are satisfied in Figure 12. From the top panels, one can see that the critical Lorentz factor is a decreasing function of temperature and a decreasing function of baryon density. Also, $\eta_\nu$ lies in the range of $\sim 436$ to 662 for temperatures and baryon densities in the ranges of 50 to $10^3 \text{ keV}$ and $10$ to $10^3 (1 \to 10^2) \text{ gc m}^{-3}$ for $B = 10^{-4.3} B_c$ (left) and $10^{-5.3} B_c$ (right). From these plots, we can see that the effect of the magnetic field is to increase $\eta_\nu$. In the bottom panels, we can observe that the number of expected neutrinos is an increasing function of temperature and baryon density. Comparing both plots, we see that the effect of the magnetic field is to decrease the number of expected neutrinos. From that, we can see the importance of knowing the strength of the magnetic field as it could alter the dynamics.

It is important to notice that independent of the model and considerations assumed (Bahcall & Mészáros 2000; Mészáros & Rees 2000; Murase et al. 2013), multi-GeV neutrinos are expected on Earth; hence, we will estimate their flavor ratio. Considering the flux ratio for $\pi^\pm$ and $\mu^\pm$ decay as $N_{\nu_\pi} \approx 2 N_{\nu_\pi}, N_{\nu_\mu} \approx 2 N_{\nu_\mu}$ (Razzaque & Mészáros 2006a) and using oscillation probabilities at $10^{11}, 10^{12}$, and $10^{13} \text{ cm}$ given in
Section 3, we compute the flavor ratio for neutrino energies of 5, 10, 20, and 50 GeV, as shown in Table 3. From Table 3, one can see an interesting result: although the tau neutrino at GeV energies is not created by p-n decoupling, it appears because of the resonant oscillations of muon neutrinos in the fireball. In addition, from Table 3 and Equation (58), we estimate the flavor ratio expected on Earth for the same range of neutrino energy as shown in Table 4. As shown in Table 4, GeV subphotospheric neutrinos with deviations of this standard flavor ratio are expected.

### 5. RESULTS AND CONCLUSIONS

We have explicitly calculated the neutrino self-energy and neutrino effective potential up to order $m_{ν}^4$ as a function of temperature, chemical potential, magnetic field, and neutrino energy. We have shown that for neutrinos in the GeV energy range as well as small chemical potentials, which is the case for solar and atmospheric neutrino parameters (small particle-antiparticle asymmetry), the contribution of $m_{ν}^4$ terms to the neutrino effective potential is relevant. In the magnetic field framework, we have derived it at the strong, moderate (above and below $B_c$), and weak field limits, taking into account a background composed of pairs of $e^+$, photons, protons, neutrons, and neutrinos. Also, we have derived the resonance and charged-neutrality conditions. By considering the neutrino effective potential at the moderate field limit, using the typical values of a magnetized GRB fireball, and requiring the resonance and charged-neutrality conditions, we have studied the thermal and quasi-thermal neutrino oscillations assuming a neutron abundance that is comparable to that of protons. In the fireball scenario, thermal neutrino oscillations have been studied at the initial stage ($r_0 \approx 10^5-10^7.5$ cm, $B \approx 0.1-50 B_c$, and $T \approx 1-10$ MeV), whereas quasi-thermal GeV neutrinos have been studied in the phase of acceleration ($r \approx 10^{11}-10^{13}$ cm, $B \approx 10^{-4.3}-10^{-5.3} B_c$, and $T \approx 50-700$ keV). This complete analysis has been carried out using the two-neutrino (solar, atmospheric, and accelerator neutrino parameters) and three-neutrino mixing.

The results for the initial stage are as follows.

1. Resonant oscillations are suppressed for baryon densities greater than $10^5$ g cm$^{-3}$ for 50 $B_c$ and $10^6$ g cm$^{-3}$ for 0.1 $B_c$.
2. Neutrinos can hardly oscillate resonantly for a fireball with a number density of neutrons greater than protons $n_n \geq n_p$ or for a proton-to-baryon ratio larger than 0.52 ($X_e \leq 0.52$).
3. The number density of the neutron-to-proton ratio ($n_n/n_p$) is lower for a fireball with more magnetization. This is because positron capture on neutrons is greatly accelerated by the large magnetic phase space factor (Thompson & Gill 2013). Also, baryon densities are larger for a fireball with more magnetization.

From the phase of acceleration, we showed the following.

1. The critical Lorentz factor for neutrino production is limited by the load density, temperature, and magnetic field. The effect of the magnetic field in the emission region is to decrease the expected number of neutrinos.
2. GeV neutrinos created in the subphotospheric region can oscillate resonantly. As a result, we estimate the neutrino flavor ratio, and deviations of standard flavor ratios are expected.

Neutrinos with energies of 1 to 30 MeV are very similar to those produced by a type II supernova, e.g., SN1987A; however, they are of cosmological distance. These cosmological events make the thermal neutrino flux very low on Earth compared to the ones we have seen from supernova SN1987A. Although low-energy neutrinos would be pretty difficult to detect with current neutrino telescopes, such a survey would help us understand the dynamics of the jet as it changes with the content of baryons.

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### APPENDIX A

**STRONG MAGNETIC FIELD: $Ω_B \gg 1$**

In the strong magnetic field approximation, the energy of charged particles is modified to confine the particles to the lowest Landau level ($n = 0$). Thus, the number density of electrons given by Equation (23) becomes

$$n_{e}^0 = \frac{eB}{2\pi^2} \int_0^\infty dp \, f_{e,0}, \quad (A1)$$

where

$$f(E_{e,0}) = \frac{1}{e^{p(E_{e,0}-\mu)}+1}, \quad (A2)$$

and the electron energy in the lowest Landau level is

$$E_{e,0}^2 = \left( p_e^2 + m_e^2 \right). \quad (A3)$$
Assuming that the chemical potentials ($\mu$) of the electrons and positrons are smaller than their energies ($\mu \leq E_e$), the fermion distribution function can be written as a sum given by

$$f(E_{e,0}) = \frac{1}{e^{\beta (E_{e,0} - \mu)} + 1} \approx \sum_{l=0}^{\infty} (-1)^l e^{-\beta (E_{e,0} - \mu)(l + 1)}. \quad (A4)$$

Therefore, the Lorentz scalars in this approximation are reduced to (Elizalde et al. 2004; Sahu et al. 2009a)

$$b_W = \sqrt{2} G_F \left[ \left(1 + \frac{3 m_e^2}{2 m_W^2} + \frac{e B}{m_W^2} + \frac{E_{e,0} k_3}{m_W^2} \right) \right] (N^0_e - \bar{N}^0_e)$$

$$- \frac{e B}{2 \pi^2 m_W^2} \int_0^\infty dp_3 \left[ 2 k_3 E_{e,0} \right]$$

$$+ 2 E_{e,0} \left( E_{e,0} - \frac{m_e^2}{2 E_{e,0}} \right) \left[ f_{e,0} + \bar{f}_{e,0} \right], \quad (A5)$$

and

$$c_W = \sqrt{2} G_F \left[ \left(1 + \frac{m_e^2}{2 m_W^2} + \frac{e B}{m_W^2} - \frac{E_{e,0} k_3}{m_W^2} - \frac{k_3^2}{m_W^2} \right) \right] (N^0_e - \bar{N}^0_e)$$

$$- \frac{e B}{2 \pi^2 m_W^2} \int_0^\infty dp_3 \left[ 2 E_{e,0} \left( E_{e,0} - \frac{m_e^2}{2 E_{e,0}} \right) \right]$$

$$+ 2 k_3 \left( E_{e,0} - \frac{3 m_e^2}{2 E_{e,0}} \right) \left[ f_{e,0} + \bar{f}_{e,0} \right]. \quad (A6)$$

For the Z-exchange diagram, we do not have a magnetic contribution, and for the tadpole diagram only the electron loop will be affected by the magnetic field (Sahu et al. 2009b). The electron number density and other useful quantities at the strong field limit are

$$N^0_e = \frac{m^3}{2 \pi^2} \frac{B}{E_c} \sum_{l=0}^{\infty} (-1)^l e^{\alpha_l} K_1(\sigma_l), \quad (A7)$$

$$N^0_e - \bar{N}^0_e = \frac{B}{E_c} \frac{m^3}{\pi^2} \sum_{l=0}^{\infty} (-1)^l \sinh \sigma_l K_1(\sigma_l), \quad (A8)$$

$$\frac{e B}{2 \pi^2} \int_0^\infty dp_3 E_0 \left( f_{e,0} + \bar{f}_{e,0} \right) = \frac{m^2}{2 \pi^2} \frac{B}{E_c} \sum_{l=0}^{\infty} (-1)^l \cosh \sigma_l$$

$$\times \left( K_0(\sigma_l) + \frac{K_1(\sigma_l)}{\sigma_l} \right), \quad (A9)$$

$$\frac{e B}{2 \pi^2} \int_0^\infty dp_3 \frac{1}{E_0} \left( f_{e,0} + \bar{f}_{e,0} \right) = \frac{m^2}{2 \pi^2} \frac{B}{E_c} \sum_{l=0}^{\infty} (-1)^l \cosh \sigma_l K_0(\sigma_l),$$

where we have defined

$$\alpha_l = \beta \mu (l + 1) \quad \text{and} \quad \sigma_l = \beta m_e (l + 1), \quad (A10)$$

where $K_i$ is the modified Bessel function of integral order $i$. Substituting Equations (A8) and (A9) in the Lorentz scalars (Equations (A5) and (A6)), we obtain

$$b_W = \frac{\sqrt{2} G_F m^3}{\pi^2} \frac{B}{E_c} \left\{ \frac{m^2}{m_W^2} \left( \frac{3}{2} + \frac{2 E_{e,0}^2}{m^2} + \frac{B}{E_c} \right) \right\}$$

$$\times \sum_{l=0}^{\infty} (-1)^l \sinh \sigma_l K_1(\sigma_l)$$

$$- \frac{m^2}{m_W^2} \frac{E_{e,0}}{m_e} \sum_{l=0}^{\infty} (-1)^l \cosh \sigma_l \left\{ 3 K_0(\sigma_l) + \frac{K_1(\sigma_l)}{\sigma_l} \right\}, \quad (A11)$$

$$c_W = \frac{\sqrt{2} G_F m^3}{\pi^2} \frac{B}{E_c} \left\{ \frac{m^2}{m_W^2} \left( \frac{1}{2} - \frac{2 E_{e,0}^2}{m^2} + \frac{B}{E_c} \right) \right\}$$

$$\times \sum_{l=0}^{\infty} (-1)^l \sinh \sigma_l K_1(\sigma_l)$$

$$- 4 \frac{m^2}{m_W^2} \frac{E_{e,0}}{m_e} \sum_{l=0}^{\infty} (-1)^l \cosh \sigma_l \left\{ \frac{3}{4} K_0(\sigma_l) \right\}. \quad (A12)$$

Finally, from Equations (6), (A11), and (A12) and for neutrinos moving along the direction of the magnetic field, the neutrino effective potential in the strong magnetic field regime is

$$V_{\text{eff}} = \frac{\sqrt{2} G_F m^3}{\pi^2} \frac{B}{E_c} \sum_{l=0}^{\infty} (-1)^l \sinh \sigma_l K_1(\sigma_l)$$

$$\times \left\{ \frac{m^2}{m_W^2} \left( \frac{3}{2} + \frac{2 E_{e,0}^2}{m^2} + \frac{B}{E_c} \right) \right\}$$

$$- \left( \frac{m^2}{m_W^2} \left( \frac{1}{2} - \frac{2 E_{e,0}^2}{m^2} + \frac{B}{E_c} \right) \right) \cos \phi \right\}$$

$$- 4 \frac{m^2}{m_W^2} \frac{E_{e,0}}{m_e} \sum_{l=0}^{\infty} (-1)^l \cosh \sigma_l \left\{ \frac{3}{4} K_0(\sigma_l) \right\} \right\}.$$ \quad (A13)

Setting $\Omega_B = B/E_c$, the previous effective potential is written as shown in Equation (29).

**APPENDIX B**

**MODERATE MAGNETIC FIELD: $\Omega_B > 1$ AND $\Omega_B \ll 1$**

The electron energy in the magnetic field is given by

$$E_{e,m}^2 = \left( p_e^2 + m_e^2 + 2 e B \right) = p_e^2 + m_e^2 (1 + 2 n \Omega_B). \quad (B1)$$

In this case, the number density of electrons (Equation (23)) is written as

$$n_e = \frac{\Omega_B m_e^2}{2 \pi^2} \left\{ \int_0^\infty dp_3 f_{e,0} + \sum_{n=1}^{\infty} \int_0^\infty dp_3 f_{e,n} \right\}, \quad (B2)$$

and the electron distribution function is given by

$$f(E_{e,n}) = \frac{1}{e^{\beta(E_{e,n} - \mu)} + 1} \approx \sum_{l=0}^{\infty} (-1)^l e^{-\beta(E_{e,n} - \mu)(l + 1)}. \quad (B3)$$
Calculating useful quantities in this regime,

\[ n_e - \bar{n}_e = \frac{m^2}{\pi^2} \Omega_B \left[ \sum_{l=0}^{\infty} (-1)^l \sinh \alpha_l \right. \]
\[ \times \left. \left\{ K_1(\sigma_l) + 2 \sum_{n=1}^{\infty} \lambda_n K_1(\sigma_l \lambda_n) \right\} \right] \]  

(B4)

\[ n_e + \bar{n}_e = \frac{m^2}{\pi^2} \Omega_B \left[ \sum_{l=0}^{\infty} (-1)^l \cosh \alpha_l \right. \]
\[ \times \left. \left\{ K_1(\sigma_l) + 2 \sum_{n=1}^{\infty} \lambda_n K_1(\sigma_l \lambda_n) \right\} \right] \]  

(B5)

\[ \int_0^\infty dp_3 E_n(f_{n,0} + \tilde{f}_{e,0}) = 2m^2 \sum_{l=0}^{\infty} (-1)^l \cosh \alpha_l \]
\[ \times \left[ \left( \frac{K_0(\sigma_l)}{\sigma_l} + \frac{K_1(\sigma_l)}{\sigma_l^2} \right) \right. \]
\[ \left. + 2 \sum_{n=1}^{\infty} \lambda_n^2 \left( \frac{K_0(\sigma_l \lambda_n)}{\sigma_l} + \frac{K_1(\sigma_l \lambda_n)}{\sigma_l^2} \right) \right] \]  

(B6)

\[ \int_0^\infty dp_3 \frac{1}{E_n} (f_{n,e} + \tilde{f}_{e,n}) = 2 \sum_{l=0}^{\infty} (-1)^l \cosh \alpha_l \]
\[ \times \left[ K_0(\sigma_l) + \frac{1}{2} \sum_{n=1}^{\infty} K_0(\sigma_l \lambda_n) \right] , \]  

where \( \lambda \) is defined by

\[ \lambda^2 = \begin{cases} \frac{2n}{\Omega_B} & \text{for } \Omega_B > 1 \text{ moderately above } \\ 1 + \frac{2n}{\Omega_B} & \text{for } \Omega_B \ll 1 \text{ moderately below, } \end{cases} \]  

for \( Re \sigma_l > 0 \) and we have used the recurrence relation \( K_2(\sigma_l) = 4/\sigma_l K_0(\sigma_l) + (1 + 8/e^2) K_1(\sigma_l) \) and \( K_2(\sigma_l) = K_0(\sigma_l) + 2/\sigma_l K_1(\sigma_l) \). Therefore, from Equations (B4), (B5), and (30), the neutrino effective potential is given by Equation (31), and useful functions of the electron number density can be written as

\[ n_e - \bar{n}_e = \frac{m^2}{\pi^2} \Omega_B \left[ \sum_{l=0}^{\infty} (-1)^l \sinh \alpha_l \right. \]
\[ \times \left. \left\{ \frac{K_0(\sigma_l)}{\sigma_l} + \frac{K_1(\sigma_l)}{\sigma_l^2} \right\} \right] \]  

(C3)

\[ n_e + \bar{n}_e = \frac{m^2}{\pi^2} \Omega_B \left[ \sum_{l=0}^{\infty} (-1)^l \cosh \alpha_l \right. \]
\[ \times \left. \left\{ \frac{K_0(\sigma_l)}{\sigma_l} + \frac{K_1(\sigma_l)}{\sigma_l^2} \right\} \right] \]  

(C4)

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