Testing Bounded Arboricity

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In this article, we consider the problem of testing whether a graph has bounded arboricity. The class of graphs with bounded arboricity includes many important graph families (e.g., planar graphs and randomly generated preferential attachment graphs). Graphs with bounded arboricity have been studied extensively in the past, particularly because for many problems, they allow for much more efficient algorithms and/or better approximation ratios.

We present a tolerant tester in the general-graphs model. The general-graphs model allows access to degree and neighbor queries, and the distance is defined with respect to the actual number of edges. Namely, we say that a graph $G$ is $\epsilon$-close to having arboricity $\alpha$ if by removing at most an $\epsilon$-fraction of its edges, we can obtain a graph $G'$ that has arboricity $\alpha$, and otherwise we say that $G$ is $\epsilon$-far. Our algorithm distinguishes between graphs that are $\epsilon$-close to having arboricity $\alpha$ and graphs that are $c \cdot \epsilon$-far from having arboricity $3\alpha$, where $c$ is an absolute small constant. The query complexity and running time of the algorithm are $\tilde{O}(n^{\frac{1}{\sqrt{\epsilon m}}} + (\frac{1}{c})^{O(\log(1/\epsilon))})$, where $n$ denotes the number of vertices and $m$ denotes the number of edges (we use the notation $\tilde{O}$ to hide poly-logarithmic factors in $n$). In terms of the dependence on $n$ and $m$, this bound is optimal up to poly-logarithmic factors since $\Omega(n^{\frac{1}{\sqrt{m}}})$ queries are necessary.

CCS Concepts: • Theory of computation → Streaming, sublinear and near linear time algorithms;

Additional Key Words and Phrases: Property testing, tolerant testing, graph arboricity, graph degeneracy

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1 INTRODUCTION

The arboricity of a graph is defined as the minimum number of forests into which its edges can be partitioned. This measure is equivalent (up to a factor of 2) to the maximum average degree in
any subgraph \([26, 27, 32]\) and to the degeneracy of the graph.\(^1\) Hence, the arboricity of a graph can be viewed as a measure of its density “everywhere.” The class of graphs with bounded arboricity includes many important families of graphs—for example, all minor-closed graph classes such as planar graphs, graphs of bounded treewidth, and graphs of bounded genus. Furthermore, graphs in this class are not restricted to being minor-free (for some fixed minor). In fact, graphs over \(n\) vertices with arboricity 2 may have a \(K_{\sqrt{n}}\)-minor. In the context of social networks, graphs that are generated according to evolving graph models such as the Barabási-Albert Preferential Attachment model \([4]\) have bounded arboricity. For various graph optimization problems, it is known that better approximation ratios and faster algorithms exist for graphs with bounded arboricity (e.g., \([3, 8, 15, 18]\), and other works \([5, 24]\) in the distributed setting) and several NP-hard problems such as Clique, Independent-Set, and Dominating-Set become fixed-parameter tractable \([1, 15]\).

In this work, we address the problem of testing whether a graph has bounded arboricity. In other words, we are interested in an algorithm that with high constant probability accepts graphs that have arboricity bounded by a given \(\alpha\) and rejects graphs that are relatively far from having slightly larger arboricity (in the sense that relatively many edges should be removed so that the graph will have such arboricity). In fact, as explained precisely next, we solve a tolerant \([30]\) version of this problem in which we accept graphs that are only close to having arboricity \(\alpha\). Furthermore, our result is in what is known as the general-graphs model \([29]\), where there is no upper bound on the maximum degree in the graph and distance to having a property is measured with respect to the number of edges in the graph. As we discuss in more detail in Section 1.4, almost all previous results on testing related bounded graph measures assumed that the graph has bounded degree.

### 1.1 Our Result

Let \(G = (V, E)\) be a graph with \(n\) vertices and \(m\) edges. We assume that for any given vertex \(v \in V\), it is possible to query for its degree, \(d(v)\), as well as query for its \(i\)th neighbor for any \(1 \leq i \leq d(v)\).\(^2\) We say that \(G\) is \(\epsilon\)-close to having arboricity \(\alpha\) if at most \(\epsilon \cdot m\) edges should be removed from \(G\), so that the resulting graph will have arboricity at most \(\alpha\). Otherwise, \(G\) is \(\epsilon\)-far from having arboricity \(\alpha\).

We present an algorithm that given query access to \(G\) together with parameters \(n, \alpha, \text{ and } \epsilon\) distinguishes with high constant probability between the case that \(G\) is \(\epsilon\)-close to having arboricity at most \(\alpha\), as well as the case that \(G\) is \(c \cdot \epsilon\)-far from having arboricity \(3\alpha\) for an absolute constant \(^4\) \(c\). The query complexity and running time of the algorithm are

\[
\tilde{O}\left(\frac{n}{\epsilon \sqrt{m}}\right) + \left(\frac{1}{\epsilon}\right)^{O(\log(1/\epsilon))}
\]

in expectation.

### 1.2 Discussion of the Result

In this section, we discuss several aspects of our result, as well as some variants. The variants are summarized in Table 1.

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\(^1\)The degeneracy of a graph \(G\) is the smallest integer \(k\) such that in every subgraph of \(G\), there is vertex of degree at most \(k\). The arboricity of \(G\) is upper bounded by its degeneracy, and the degeneracy is less than twice the arboricity.

\(^2\)We note that the ordering of the neighbors of vertices is arbitrary and that a neighbor query to vertex \(v\) with \(i > d(v)\) is answered by a special symbol. Observe that a degree query to \(v\) can be replaced by \(O(\log(d(v)))\) neighbor queries.

\(^3\)Usually the general-graphs model also allows for pair queries; however, our algorithm does not require them, and the lower bounds holds also when allowing them.

\(^4\)The constant we achieve is 20. For the sake of simplicity and clarity of the algorithm and its analysis, we did not make an effort to minimize this constant.
Table 1. Unless Explicitly Specified Otherwise, All Results Refer to Two-Sided Error Algorithms in the General-Graphs Model (So That the Distance Is with Respect to the Number of Edges \( m \), and \( m \) is Unknown)

| Summary of Our Result and Variants                  | \( \tilde{O} \left( \frac{n}{\epsilon \sqrt{m}} \right) + \left( \frac{1}{\epsilon} \right)^{O(\log(1/\epsilon))} \) | Theorem 7 |
|----------------------------------------------------|-----------------------------------------------------------------------------------------------------------------|-----------|
| General-graphs model, two-sided error, unknown \( m \), distance w.r.t. \( m \) | \( \Omega \left( \frac{n}{\sqrt{m}} \right) \)                                                                 | Claim 8   |
| Known \( m \) (or \( 1 \pm \epsilon/c \)-estimate) | \( O \left( \frac{m^a}{\epsilon^7 m^b} \right) + \left( \frac{1}{\epsilon} \right)^{O(\log(1/\epsilon))} \) | Theorem 18| Claim 18  |
| Upper bound \( d \) on max-degree                  | \( O \left( \frac{d}{\epsilon^a} \right) + \left( \frac{1}{\epsilon} \right)^{O(\log(1/\epsilon))} \)        | Section 6 |    |
| Upper bound \( d \) on max-degree and distance w.r.t. \( n \cdot d \) | \( \left( \frac{1}{\epsilon} \right)^{O(\log(1/\epsilon))} \)                                               | Section 6 |    |
| Access to random edges                              | \( \left( \frac{1}{\epsilon} \right)^{O(\log(1/\epsilon))} \)                                               | Section 6 |    |
| One-sided error, \( \alpha \geq 2 \)               | \( \Omega(n) \)                                                                                                  | Claim 10  |    |

For each variant we specify the difference(s) in terms of the model (or error type).

The tightness of the complexity bound. If we consider the complexity of the algorithm as a function of \( n \) and \( m \) (ignoring the dependence on \( \epsilon \)), we get that it is \( \tilde{O}(\frac{n}{\sqrt{m}}) \). We observe that this complexity is essentially tight, even for non-tolerant algorithms (i.e., that distinguish between the case that \( G \) has arboricity at most \( \alpha \) and the case that \( G \) is far from having arboricity \( 3\alpha \)). To be precise, for constant \( \epsilon \), \( \Omega(\frac{n}{\sqrt{m}}) \) queries are necessary for any algorithm that is not provided with any information regarding \( m \), or even when it is provided with a constant factor estimate of \( m \) (e.g., a factor-2 estimate). The lower-bound construction is based on two families of graphs, where graphs in one family have arboricity at most \( \alpha \) and graphs in the other family are far from having arboricity at most \( 3\alpha \). The graphs in the second family have a slightly larger number of edges, where, roughly speaking, these edges are the source of the distance to bounded arboricity, and they belong to a relatively small “hidden” subgraph (over \( O(\sqrt{m}) \) vertices). Other than this small subgraph, graphs in the two families have identical structure.

If the algorithm is provided with \( m \) (or a very precise estimate, i.e., within \( (1 \pm \epsilon/c) \) for \( c > 1 \)), then we cannot use this lower-bound construction, as graphs in both families must have the same (or almost the same) number of edges. However, in such a case, we can modify the construction (so that graphs in both families have exactly the same number of edges) and obtain a lower bound of \( \Omega(\frac{n\alpha}{m}) \) (graphs in the two families now differ on subgraphs of size \( O(m/\alpha) \)). Furthermore, we show that this number of queries is sufficient (when the algorithm is provided with a precise estimate of \( m \)). Note that by Nash-Williams [26, 27], \( \alpha \leq \sqrt{m} \) so that \( \frac{n\alpha}{m} \leq \frac{n}{\sqrt{m}} \).

Bounded-degree graphs. Suppose first that we are given an upper bound \( d \) on the maximum degree in \( G \), and let \( \overline{d} = 2m/n \) denote the average degree. Then we can slightly modify the algorithm so that the term \( \tilde{O}(n/\sqrt{m}) \) in the complexity of the algorithm is replaced by \( d/\overline{d} \).

The preceding statement is for the case that distance to having the property is measured (as defined in the general-graphs model) with respect to \( m \) (and we only assume that the algorithm is provided with additional information regarding the maximum degree in the graph). If we consider the bounded-degree model [16], in which not only do we get \( d \) as input but also distance is measured with respect to \( d \cdot n \) (which is an upper bound on \( m \)), then our algorithm can be slightly modified so that its complexity depends only on \( 1/\epsilon \) (and the dependence is quasi-polynomial).
Access to random edges. In the case that the algorithm is given access to uniformly (or almost uniformly) distributed random edges, it can again be slightly modified to run in time quasi-polynomial in $1/\epsilon$. This is true since the term $\tilde{O}(\frac{n}{\epsilon^2 m})$ in the running time arises from sampling edges almost uniformly at random when the number of edges is unknown.

Expected complexity. The reason the query complexity and running time are in expectation is due to the need of the edge sampling procedure to obtain an estimate for the number of edges. If such a (constant factor) estimate is provided to the algorithm, then the upper bound on the complexity of the algorithm always holds.

$\alpha$ vs. $3\alpha$. Our algorithm distinguishes between the case that the graph is close to having arboricity at most $\alpha$ and the case that it is far from having arboricity at most $3\alpha$. The constant 3 can be reduced to $2 + \eta$ at a cost that depends (exponentially) on $1/\eta$, but we do not know how to avoid this cost and possibly go below a factor of 2. However, in some cases, this constant may not be significant. For example, suppose that we want to know whether, after removing a small fraction of the edges, we can obtain a graph $G'$ with bounded arboricity so that we can run an optimization algorithm on $G'$ (or possibly on $G$ itself), whose complexity depends polynomially on the arboricity of $G'$. In such a case, the difference between $\alpha$ and $3\alpha$ is inconsequential.

Two-sided error vs. one-sided error. Our algorithm has two-sided error, and we observe that for $\alpha \geq 2$, every one-sided error algorithm must perform $\Omega(n)$ queries. We note that for $\alpha = 1$, there exists a one-sided error algorithm for testing cycle-freeness [9] that performs $\tilde{O}(\sqrt{n})$ queries (and these many queries are necessary [16]).

Dependence on $\epsilon$. In the second term of the complexity of our algorithm, there is a quasi-polynomial dependence on $1/\epsilon$. It is an open problem whether this dependence can be reduced to polynomial.

1.3 The Algorithm

Challenges of designing algorithms in the general-graphs models and our approach. The general-graphs model presents several challenges. Indeed, relatively few properties were studied in this model. These include bounded diameter and connectivity [29], bipartiteness [23] (or for planar graphs), triangle-freeness [2, 19, 31], cycle-freeness [9] (with one-sided error), and $k$-path-freeness [22] (when random edge queries are also allowed). Since this model allows for unbounded degrees, it could be the case that the large distance to having the property is due to a relatively dense subgraph that resides on a small set of vertices. As a consequence, observing this subgraph might be costly. Furthermore, since the degrees are unbounded, it is not clear how to efficiently explore the graph. Previous techniques in this model include bounded-size BFS [29] (i.e., performing a BFS until a predetermined fixed number of vertices is discovered), random walks [10, 22], and analysis based on color coding [22].

We apply a different technique to meet the preceding challenges. In our analysis, we characterize a “special” set of edges, $S$, such that if the graph is $\epsilon$-close to having bounded arboricity, then $|S|$ is small, and if the graph is far from having bounded arboricity, then $|S|$ is large. The question now is how to decide whether an edge belongs to $S$ or not. We show that given an edge in the graph, we can perform a certain “approximate decision” regarding membership in $S$, which suffices for our purposes. To do so, we use a procedure that recursively samples a subset of the neighbors of a given vertex until it reaches the maximal depth of the recursion. At this point, it returns a deterministic answer based on the degree of the vertex, which propagates up the recursion tree. The resulting queries of this process can also be viewed as a randomized BFS of bounded degree and depth. We next provide some more details on the algorithm.
The algorithm. Our starting point is a simple (non-sublinear and deterministic) algorithm that is similar to the distributed forest decomposition algorithm of Barenboim and Elkin [5]. This algorithm works in $\ell = O(\log(1/\epsilon))$ iterations, where in each iteration it assigns edges to a subset of the vertices, and the vertices that are assigned edges become “inactive.” We show that if the graph has arboricity at most $\alpha$, then, when the algorithm terminates, the number of edges between remaining active vertices, whose set we denote here by $A_{\ell}$, is relatively small. However, if the graph is sufficiently far from having arboricity at most $3\alpha$, then the number of edges between vertices in $A_{\ell}$ is relatively large.

Given this statement regarding the number of remaining edges between vertices in $A_{\ell}$, our algorithm estimates the number of such remaining edges. To this end, we devise a procedure for deciding whether a given vertex $v$ belongs to $A_{\ell}$. This can be done by emulating the deterministic algorithm on the distance-$\ell$ neighborhood of $v$. However, such an emulation may require a very large number of queries (as the maximum degree in the graph is not necessarily bounded). Instead, we perform a certain approximate randomized emulation of the deterministic algorithm, which is much more query efficient. Although this emulation does not exactly answer whether or not $v \in A_{\ell}$, it gives an approximate answer that suffices for our purposes (see Lemma 5 for the precise statement).

The high-level idea is that given a vertex $v$, we select a random subset of its neighbors. We then recursively run the procedure to decide for each of these neighbors whether it belongs to $A_{\ell-1}$ (the vertices that remain active after $\ell - 1$ iterations). This recursive process defines a (random) tree with $\ell$ levels. For each vertex in the tree, we decide if it is active or not according to the fraction of active children it has in the tree, and for the leaf vertices, we simply decide according to their degree.

When analyzing the correctness of this procedure for a vertex $v$, we need to take into account two sources of error. The first is due to a possible bias in the selection of its sample of neighbors. In other words, even if we had an oracle that always answered correctly for a vertex $u$ whether it belongs to $A_{\ell-1}$, we might still err in our decision regarding whether or not $v \in A_{\ell}$. The second source of error is due to incorrect answers on $v$’s neighbors. In other words, we need to analyze how errors propagate (and accumulate) up the recursion tree.

In addition, we upper bound the total size of the tree (which determines the query complexity and running time of the procedure).

1.4 Related Work
In what follows, when we say “testing” we mean “property testing,” as defined earlier—that is, distinguishing between objects that have a property and objects that are far from having the property.

Most of the property testing results related to this work are in the bounded-degree model [16]. Recall that in this model, the algorithm has the same query access to the graph as we consider, but it is also given an upper bound, $d$, on the maximum degree in the graph, and distance is measured with respect to $d \cdot n$ (rather than the actual number of edges, $m$), so that it is less stringent. As noted in Section 1.2, an adaptation of our algorithm to the (“easier”) bounded-degree model achieves complexity that is quasi-polynomial in $1/\epsilon$ (and independent of $n$). As we discuss next, in the bounded-degree model, there are several results on testing whether a graph excludes specific fixed minors, as well as results on testing minor-closed properties in general. In what follows, we assume that $d$ is a constant, as in some of these works, this assumption is made (so that no explicit dependence on $d$ is stated).

Goldreich and Ron [16] provide an algorithm for testing if a graph is cycle-free, namely excluding $C_3$-minors, where the complexity of the algorithm is $O(1/\epsilon^3)$. Yoshida and Ito [33] test outerplanarity (excluding $K_4$-minors and $K_{2,3}$-minors) and if a graph is a cactus (excluding a
diamond-minor) in time that is polynomial in $1/\epsilon$. Benjamini et al. [6] showed that any minor-closed property can be tested in time that depends only on $1/\epsilon$ (where the dependence may be triply exponential). Hassidim et al. [20] introduced a general tool, a partition oracle, for locally partitioning graphs that belong to certain families of graphs into small parts with relatively few edges between the parts. A partition oracle for a family of graphs implies a corresponding (two-sided error) tester for membership in this family. Hassidim et al. [20] designed partition oracles for hyperfinite classes of graphs and minor-closed classes of graphs. One of the implications of their work is improving the running time of testing minor-closed properties from triply exponential in $\sqrt{\epsilon}$ to singly exponential in poly($1/\epsilon$). Levi and Ron [25] later improved the running time of the partition oracle for minor-closed classes of graphs to quasi-polynomial in $1/\epsilon$. Edelman et al. [11] designed a partition oracle for graphs with bounded treewidth whose query and time complexity are polynomial in $1/\epsilon$. Newman and Sohler [28] extended the result of Levi and Ron [20] and showed that every hyperfinite property (i.e., property of hyperfinite graphs) is testable in time that is independent of the size of the graph.

All the aforementioned testing algorithms have two-sided error (and this is also true of our algorithm). Czumaj et al. [9] study the problem of one-sided error testing of $C_k$-minor-freeness and tree-minor freeness. For cycle-freeness ($C_3$-minor-freeness), they give a one-sided error testing algorithm whose complexity is $\tilde{O}(\sqrt{n} \cdot \text{poly}(1/\epsilon))$ (for $k > 3$, there is an exponential dependence on $k$). They show that the dependence on $\sqrt{n}$ is tight for any minor that contains a cycle. However, for tree-minors they give an algorithm whose complexity is $\exp((1/\epsilon)^{O(k)})$, where $k$ is the size of the tree (so that the complexity is independent of $n$).

Finally, we discuss results in the general-graphs model that are related to our result. Czumaj et al. [9] show that their result for cycle-freeness extends to the general-graphs model, where the complexity of the algorithm is $\tilde{O}(\sqrt{n} \cdot \text{poly}(1/\epsilon))$. Iwama and Yoshida [22] consider an augmented model that allows random edge sampling. In this augmented model, they provide several testers for parameterized properties including $k$-path-freeness whose complexity is independent of the size of the graph.

1.5 Organization

Following some basic preliminaries in Section 2, we give the aforementioned “Edge-assignment algorithm” in Section 3. In Section 4, we present our testing algorithm. The lower bounds mentioned in Section 1.2 are provided in Section 5, and the variants of our algorithm (e.g., in the bounded-degree model) appear in Section 6. A procedure for estimating what we refer to as the $\epsilon$-corrected arboricity of a given graph (see Definition 12) appears in Section 7. In the appendix, we describe an improved variant of our algorithm for the case that a precise estimate of the number of edges is given to the algorithm.

2 PRELIMINARIES

For an integer $k$, let $[k] \triangleq \{1, \ldots, k\}$. For an undirected simple graph $G = (V, E)$ let $n = |V|$ and $m = |E|$. For each vertex $v \in V$, let $d(v)$ denote its degree.

We assume that there is query access to the graph in the form of degree queries and neighbor queries. In other words, for any vertex $v \in V$, it is possible to perform a query to obtain $d(v)$, and for any $v$ and $i \in [d(v)]$, it is possible to perform a query to obtain the $i$th neighbor of $v$ (where the order over neighbors is arbitrary). If $i > d(v)$, then a special symbol is returned.

Definition 1 (Distance). For a property $\mathcal{P}$ of graphs, and a parameter $\epsilon \in [0, 1]$, we say that a graph $G$ is $\epsilon$-far from (having) the property $\mathcal{P}$ if more than $\epsilon \cdot m$ edge modifications on $G$ are required so as to obtain a graph that has the property $\mathcal{P}$.
**Definition 2 (Arboricity).** The arboricity of a graph $G = (V, E)$ is the minimum number of forests into which its edges can be partitioned. We denote the arboricity of $G$ by $\alpha(G)$.

By the work of Nash-Williams [26, 27], for every graph $G = (V, E)$,

$$\alpha(G) = \max_{S \subseteq V} \left\lceil \frac{|E(S)|}{|S| - 1} \right\rceil,$$

where $E(S)$ denotes the set of edges in the subgraph induced by $S$.

Let $\exp(x) \triangleq e^x$, and for a random variable $\chi$, we use $\mathbb{E}[\chi]$ to denote its expected value.

We make use of Hoeffding’s inequality [21], stated next. For $i = 1, \ldots, s$, let $\chi_i$ be a 0/1-valued random variable such that $\Pr[\chi_i = 1] = \mu$. Then for any $\gamma \in (0, 1]$,

$$\Pr \left[ \frac{1}{s} \sum_{i=1}^{s} \chi_i > \mu + \gamma \right] < \exp \left( -2\gamma^2 s \right)$$

and

$$\Pr \left[ \frac{1}{s} \sum_{i=1}^{s} \chi_i < \mu - \gamma \right] < \exp \left( -2\gamma^2 s \right).$$

We also apply the following version of the multiplicative Chernoff bound [7]. For $i = 1, \ldots, s$, let $\chi_i$ be a random variables taking values in $[0, B]$ such that $\mathbb{E}[\chi_i] = \mu$. Then for any $\gamma \in (0, 1]$,

$$\Pr \left[ \frac{1}{s} \sum_{i=1}^{s} \chi_i > (1 + \gamma)\mu \right] < \exp \left( -\frac{\gamma^2 \mu s}{3B} \right)$$

and

$$\Pr \left[ \frac{1}{s} \sum_{i=1}^{s} \chi_i < (1 - \gamma)\mu \right] < \exp \left( -\frac{\gamma^2 \mu s}{2B} \right).$$

### 3 A DETERMINISTIC EDGE-ASSIGNMENT ALGORITHM

In this section, we describe a deterministic algorithm that given as input a graph $G = (V, E)$ assigns edges to vertices. The algorithm works iteratively, where in each iteration it assigns edges to a new subset of vertices. The algorithm is provided with parameters that determine an upper bound on the number of edges that are assigned to each vertex (where an edge may be assigned to both of its endpoints). The number of edges assigned to each vertex is at most $3\alpha$ plus a small fraction of its original degree. This fraction is determined by one of the parameters ($\gamma$). When the algorithm terminates, some edges may remain unassigned (and some vertices may not have been assigned any edges). This algorithm (when viewed as a distributed algorithm) is a variant of the algorithm by Barenboim and Elkin [5] for finding a forest decomposition in graphs with bounded arboricity.

**ALGORITHM 1: Assign-Edges($G, \alpha, \epsilon, \gamma$)**

1. $G_0(\gamma) = G$, $A_0(\gamma) = V$.
2. for $i = 1$ to $\ell \triangleq \lceil \log_{\delta(1/\epsilon)} \rceil$ do
3.   let $B_i(\gamma)$ be the set of vertices $v \in V$ whose degree in $G_{i-1}(\gamma)$ is at most $3\alpha + \gamma \cdot d(v)$.
4.   Assign each vertex $v \in B_i(\gamma)$ the edges incident to it in $G_{i-1}(\gamma)$.
5.   let $A_i(\gamma) = A_{i-1}(\gamma) \setminus B_i(\gamma)$, and let $G_i(\gamma)$ be the graph induced by $A_i(\gamma)$.
6. end for

The algorithm **Assign-Edges** is provided with three parameters: $\alpha$, $\epsilon$, and $\gamma$. It might be useful to first consider its execution with $\gamma = 0$. The role of $\gamma$ will become clear subsequently (when we describe our testing algorithm and its relation to **Assign-Edges**). In the case of $\gamma = 0$, in every
iteration, each vertex with degree at most $3\alpha$ in the current graph is assigned all of its incident edges in this graph. The initial graph is $G$, and at the end of iteration $i$, the vertices that are assigned edges, denoted $B_i(y)$ in the algorithm, together with the edges assigned to them, are removed from the graph. Once vertices are assigned edges, we view them as becoming inactive. We use the notation $A_i(y)$ for the vertices that are still active at the end of iteration $i$. Observe that by the definition of the algorithm, for $\gamma_1 \leq \gamma_2$, we have that $A_i(\gamma_2) \subseteq A_i(\gamma_1)$ for every iteration $i$, and hence $G_i(y_2)$ is a subgraph of $G_i(y_1)$.

In the next two lemmas, we upper bound the number of edges in $G_\ell(0)$ when $G$ is close to having arboricity $\alpha$, and we lower bound the number of edges in $G_\ell(y)$ (which is a subgraph of $G_\ell(0)$) when $G$ is far from having arboricity $3\alpha$.

**Lemma 3.** If $G$ is $\epsilon$-close to having arboricity at most $\alpha$, then $|E(G_\ell(0))| \leq 5\epsilon m$.

**Proof.** By the premise of the lemma, $G$ is $\epsilon$-close to having arboricity at most $\alpha$. This implies that for each of its subgraphs, $G'$, there is a subset of at most $\epsilon m$ edges whose removal makes $G'$ have arboricity at most $\alpha$. In particular, this is true for the subgraphs $G_i(0)$ defined by the algorithm, for $i = 1, \ldots, \ell$. Denoting by $m_i$ the number of edges in $G_i(0)$, we have that for every $i \in [\ell],

$$m_i \leq \alpha \cdot |A_i(0)| + \epsilon m.$$  (2)

By the definition of $A_i(0)$, each vertex $v \in A_i(0)$ has degree greater than $3\alpha$ in $G_{i-1}(0)$. It follows that

$$m_{i-1} \geq 3\alpha |A_i(0)|/2.$$  (3)

Suppose that $|A_i(0)| > 4\epsilon m/\alpha$ (so that $\epsilon m < \alpha |A_i(0)|/4$). The upper bound on $m_i$ in Equation (2) implies that $m_i \leq 5\alpha |A_i(0)|/4$. Combining this with the lower bound on $m_{i-1}$ in Equation (3), we get that $\frac{m_i}{m_{i-1}} \leq 5/6$.

Therefore, in every iteration of Assign-Edges in which $|A_i(0)| > 4\epsilon m/\alpha$, the number of edges in the graph decreases by a multiplicative factor of $5/6$. However, if $|A_i(0)| \leq 4\epsilon m/\alpha$, then applying Equation (2) with this upper bound on $|A_i(0)|$, we get that $m_i \leq 5\epsilon m$. Hence, after at most $\lceil \log_{6/5}(1/\epsilon) \rceil$ iterations, there are at most $5\epsilon m$ edges between active vertices. □

**Lemma 4.** If $G$ is $\epsilon'$-far from having arboricity $3\alpha$, then $|E(G_\ell(y))| > (\epsilon' - 2\gamma)y m$.

**Proof.** Assume, contrary to the claim, that $|E(G_\ell(y))| \leq (\epsilon' - 2\gamma)y m$. We shall show that by removing at most $\epsilon' \cdot m$ edges from $G$, we can obtain a graph that has arboricity at most $3\alpha$, thus reaching a contradiction to the premise of the lemma that $G$ is $\epsilon'$-far from having arboricity $3\alpha$. First, we remove all edges in $G_\ell(y)$—that is, all edges in which both endpoints belong to $A_\ell(y)$. We are left with edges that are incident to vertices in the set $V \setminus A_\ell(y)$. For each vertex $v \in V \setminus A_\ell(y)$, let $a(v)$ be the number of edges it is assigned (by Algorithm 1), and recall that $a(v) \leq 3\alpha + \gamma d(v)$. For each vertex $v$ such that $a(v) > 3\alpha$, we remove $a(v) - 3\alpha$ of the edges it is assigned (these edges can be selected arbitrarily), thus leaving it with at most $3\alpha$ assigned edges (recall that some edges may be assigned to both their endpoints). Let $E_R$ denote the subset of edges that were removed. We have the following:

$$|E_R| \leq |E(G_\ell(y))| + \sum_{v \in V \setminus A_\ell(y) : a(v) > 3\alpha} (a(v) - 3\alpha) \leq (\epsilon' - 2\gamma)y m + \sum_{v \in V} \gamma d(v) = \epsilon' \cdot m.$$  (4)

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5Although $B_i(y)$ depends also on $\alpha$, we shall want to refer to these sets when the algorithm is invoked with the same value of $\alpha$ but with different values of $y$. Hence, only $y$ appears explicitly in the notation.

6The term $3\alpha$ can be improved to $(2 + \eta)/\alpha$ for any $\eta > 0$ by increasing the number of iterations by a factor of $1/\eta$. 
4 THE TESTING ALGORITHM

In this section, we present and analyze our algorithm Is-Bounded-Arboricity. We assume that the distance parameter, $\epsilon$, is at most $1/20$ (since otherwise the algorithm can simply accept, as it is required to reject graphs that are $20\epsilon$-far from having arboricity at most $3\alpha$).

We start with a central procedure used by Is-Bounded-Arboricity.

### 4.1 Deciding Whether a Vertex Is Active

In this section, we present a procedure that, roughly speaking, decides whether a given vertex $v$ belongs to the set of active vertices $A_i(0)$ (as defined in the algorithm Assign-Edges from Section 3). This procedure is then used to estimate the number of edges remaining in $G_\ell(0)$ (the subgraph induced by $A_\ell(0)$).

Observe that by the description of the algorithm Assign-Edges, for any vertex $v$, the decision whether $v \in A_i(0)$ can be made by considering the distance-$i$ neighborhood of $v$. However, the size of this neighborhood may be very large, since the maximum degree in the graph is not bounded. Hence, rather than querying for the entire distance-$i$ neighborhood, we query (in a randomized manner) for only a small part of the neighborhood, as detailed in the procedure Is-Active. As stated in Lemma 5, the procedure ensures (with high probability) that its output is correct on $v \in A_\ell(\gamma) \subseteq A_\ell(0)$ and on $v \notin A_\ell(0)$. If $v \in A_\ell(0) \setminus A_\ell(\gamma)$, then the procedure may return any output, and as we shall see subsequently that this suffices for our purposes.

**PROCEDURE 2: Is-Active($v, \ell, \alpha, \gamma, \delta$)**

1. Set the confidence parameter $\rho = \left(\frac{\delta \cdot \gamma}{\ell}\right)^{4\ell}$.
2. Return Recursive-Is-Active($v, \ell, \alpha, \gamma, \rho$).

**PROCEDURE 3: Recursive-Is-Active($v, i, \alpha, \gamma, \rho$)**

1. If $d(v) \leq 3\alpha$, then return No.
2. If $i = 1$ and $d(v) > 3\alpha$, then return Yes.
3. Sample a random multiset, $S_{v, i}$, of $t = \left\lceil\frac{4\ell \log(1/\rho)}{\gamma^2}\right\rceil$ neighbors of $v$.
4. For every $u \in S_{v, i}$, invoke Recursive-Is-Active($u, i - 1, \alpha, \gamma, \rho$) and let $\eta(v, i)$ be the fraction of vertices in $S_{v, i}$ that returned Yes.
5. If $\eta(v, i) \cdot d(v) > 3\alpha + (\gamma/2) \cdot d(v)$, then return Yes, otherwise return No.
LEMMA 5. For \( \delta < \frac{1}{3} \) and \( \gamma < 1 \), the procedure Is-Active(\( v, \ell, \alpha, \gamma, \delta \)) returns a value in \{Yes, No\} such that the following holds:

(1) If \( v \notin A_\ell(0) \), then the procedure returns No with probability at least \( 1 - \delta \).
(2) If \( v \in A_\ell(\gamma) \), then the procedure returns Yes with probability at least \( 1 - \delta \).

The query complexity and running time of Is-Active(\( v, \ell, \alpha, \gamma, \delta \)) are \( O(\frac{6\ell \log(\frac{\ell}{\gamma})}{\gamma^2}) \).

**Proof.** For a vertex \( v \in V \), consider the execution of Is-Active(\( v, \ell, \alpha, \gamma, \delta \)).

For \( 1 \leq i \leq \ell \), define \( S_i \) to be the multiset of vertices on which Recursive-Is-Active is invoked with the parameter \( i \). In particular, \( S_\ell = \{v\} \), and for \( i < \ell \), the vertices in \( S_i \) were selected in invocations of Recursive-Is-Active with \( i + 1 \). For a vertex \( u \), let \( \tilde{\eta}(u, i, \gamma) \) be the fraction of vertices in \( S_{u,i} \) that belong to \( A_{i-1}(\gamma) \), and let \( \tilde{\eta}(u, i, 0) \) be the fraction of vertices in \( S_{u,i} \) that belong to \( A_{i-1}(0) \) (which is a superset of \( A_{i-1}(\gamma) \)). Recall that \( A_0(0) = A_0(\gamma) = V \).

For \( 2 \leq i \leq \ell \), we say that a vertex \( u \in S_i \) is i-successful if one of the following holds:

1. \( u \in A_i(\gamma) \) and \( \tilde{\eta}(u, i, \gamma) \cdot d(u) > 3\alpha + (\gamma/2)d(u) \).
2. \( u \notin A_i(0) \) and \( \tilde{\eta}(u, i, 0) \cdot d(u) \leq 3\alpha + (\gamma/2)d(u) \).
3. \( u \in A_i(0) \setminus A_i(\gamma) \).

Otherwise, it is i-unsuccessful. For \( i = 1 \), every vertex is 1-successful.

Consider a recursive call to Recursive-Is-Active(\( u, i, \alpha, \gamma, \rho \)) on a vertex \( u \in A_i(\gamma) \) for \( 2 \leq i \leq \ell \). Since \( u \in A_i(\gamma) \), we have that \( \mathbb{E}[\tilde{\eta}(u, i, \gamma) \cdot d(u)] > 3\alpha + \gamma d(u) \). By Hoeffding’s inequality, the probability that \( u \) is i-unsuccessful is upper bounded by

\[
\Pr[\tilde{\eta}(u, i, \gamma) \leq \mathbb{E}[\tilde{\eta}(u, i, \gamma)] - \gamma/2] < \exp(-2(\gamma/2)^2 \cdot t) \leq \rho/2.
\]

Consider a vertex \( u \notin A_i(0) \). In other words, \( u \in \bigcup_{i' \leq i} B_{i'}(0) \) (where \( B_{i'}(\cdot) \) is as defined in the algorithm **Assign-Edges**). In this case, we claim that \( \mathbb{E}[\tilde{\eta}(u, i, 0) \cdot d(u)] \leq 3\alpha \).

To verify this claim, observe that since \( u \in B_{i'}(0) \) (for some \( i' \leq i \)), the number of neighbors that \( u \) has in \( A_{i'-1}(0) \) is at most \( 3\alpha \). The claim follows since \( A_{i-1}(0) \subseteq A_{i'-1}(0) \) (for \( i' \leq i \)) and by the definition of \( \tilde{\eta}(u, i, 0) \) (the fraction of vertices in \( S_{u,i} \) that belong to \( A_{i-1}(0) \)). By Hoeffding’s inequality, the probability that \( u \) is i-unsuccessful is upper bounded by

\[
\Pr[\tilde{\eta}(u, i, 0) \geq \mathbb{E}[\tilde{\eta}(u, i, 0)] + \gamma/2] < \exp(-2(\gamma/2)^2 \cdot t) \leq \rho/2.
\]

For \( 2 \leq i \leq \ell \), we say that a vertex \( u \in S_i \) is recursively i-successful if \( u \) is i-successful and all vertices in \( S_{u,i} \) are recursively \((i - 1)\)-successful. For \( i = 1 \), every vertex is defined to be recursively 1-successful. By this definition, if for every \( 2 \leq i \leq \ell \) every \( u \in S_i \) is recursively i-successful, then we also have that for every \( 2 \leq i \leq \ell \) every \( u \in S_i \) is recursively i-successful. By taking a union bound over all \( 2 \leq i \leq \ell \) and \( u \in S_i \), we get that the probability that for some \( i \) there exists \( u \in S_i \) that is not i-successful is at most

\[
\sum_{i=2}^{\ell} t^i \cdot \frac{\rho}{2} \leq t^\ell \cdot \rho = \left[ \frac{4\ell \log(1/\rho)}{\gamma^2} \right]^\ell \cdot \rho \leq \frac{6\ell \ell^\ell \cdot \log^\ell \left( \frac{\gamma}{\delta} \right)}{\gamma^{2\ell}} \cdot \frac{\delta^{4\ell}}{\ell^{4\ell}} \cdot \gamma^{4\ell} < \delta,
\]

where the last inequality is by the assumption that \( \delta < \frac{1}{3} \) and \( \gamma < 1 \).

We next claim that if \( u \in S_i \) is recursively i-successful, then the following holds: if \( u \in A_i(\gamma) \), then Recursive-Is-Active(\( u, i, \alpha, \gamma, \rho \)) returns yes and if \( u \notin A_i(0) \), then Recursive-Is-Active(\( u, i, \alpha, \gamma, \rho \)) returns No. We establish this claim as well by induction on \( i \).
For $i = 1$ (the leaves of the recursion tree), the claim follows by Steps 1 and 2 of the algorithm. For the induction step, assume that the claim holds for $i - 1 \geq 1$, and we prove it for $i$. If $u \in A_i(\gamma)$, then (since $u$ is $i$-successful), $\tilde{\gamma}(u, i, \gamma) \cdot d(u) \geq 3\alpha + (\gamma/2)d(u)$, and since all vertices in $S_{u,i}$ are recursively $(i - 1)$-successful, by induction the algorithm returns yes in Step 5. If $u \notin A_i(0)$, then (again since $u$ is $i$-successful), $\tilde{\gamma}(u, i, 0) \cdot d(u) \leq 3\alpha + (\gamma/2)d(u)$, and since all vertices in $S_{u,i}$ are recursively $(i - 1)$-successful, by induction the algorithm returns No in Step 5.

It remains to bound the complexity of the algorithm. Consider the recursion tree corresponding to the complete execution of Is-Active to their endpoints. Sampling edges (almost uniformly) is done by making use of the following theorem.

**Theorem 6 (Eden and Rosenbaum [14], Rephrased).** Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. There exists an algorithm named Sample-Edge-Almost-Uniformly that is given query access to $G$ and parameters $n$, $\beta$, and $\delta$. The algorithm returns an edge $e \in E$ with probability at least $1 - \delta$, and conditioned on an edge being returned, each edge in the graph is returned with probability in $[\frac{(1-\beta)}{m}, \frac{(1+\beta)}{m}]$. The expected query complexity and running time of the algorithm are $O(\frac{n}{\sqrt{m}}) \cdot \log^2 (n/\delta)$.

We note that it can be shown that the dependence on $\log(n)$ in the complexity of Sample-Edge-Almost-Uniformly can be reduced to a dependence on $\log(n/\sqrt{m})$ (this dependence stems from estimating the average degree up to a constant factor).\(^7\)

**ALGORITHM 4: Is-Bounded-Arboricity**

```
1: Invoke Sample-Edge-Almost-Uniformly($n, 1/4, 1/4$) for $t = \frac{600}{\epsilon}$ times, and let $S$ be the (multi)set of returned edges.
2: Let $s$ be the number of (not necessarily different) edges in $S$. If $s < \frac{300}{\epsilon}$, then return Yes.
3: Set $\ell = \lceil \log_b(s/5) \rceil$.
4: for each edge $(u, v) \in S$ do
5:  Invoke Is-Active($u, \ell, \alpha, \epsilon$, $\epsilon/2$) and Is-Active($u, \ell, \alpha, \epsilon$, $\epsilon/2$).
6:  If Is-Active returned Yes on both invocations, then set $\chi_l = 1$. Otherwise, set $\chi_l = 0$.
7: end for
8: Set $\chi = \frac{1}{s} \sum_{l=1}^{s} \chi_l$.
9: If $\chi < 10\epsilon$, then return Yes. Otherwise, return No.
```

**Theorem 7.** Let $G$ be a graph over $n$ vertices and $m$ edges. If $G$ is $\epsilon$-close to having arboricity at most $\alpha$, then MIs-bounded-arboricity returns yes with probability at least $2/3$, and if $G$ is $20\epsilon$-far from having arboricity at most $3\alpha$, then Is-bounded-arboricity returns No with probability at least $2/3$.

\(^7\)This can be done by applying the algorithm of Eden et al. [12] for estimating the number of edges and slightly modifying their geometric search procedure.
The query complexity and running time of Is-Bounded-Arboricity are
\[ \tilde{O}\left(\frac{n}{\epsilon \sqrt{m}}\right) + \left(\frac{1}{\epsilon}\right)^{O\left(\log(1/\epsilon)\right)} \]
in expectation.

Proof. By Theorem 6, each invocation of Sample-Edge-Almost-Uniformly \((n, 1/4, 1/4)\) succeeds with probability at least 3/4. By the multiplicative Chernoff bound and by the setting of \(t = \frac{600}{\epsilon}\), it follows that with probability at least 5/6, at least 1/2 of the invocations return an edge. Hence, \(s \geq \frac{300}{\epsilon}\) with probability at least 5/6 and the algorithm continues to the following steps. We henceforth condition on this event.

We say that the procedure Is-Active is correct when invoked with a vertex \(v\) in Step 5 of the algorithm if \(v \in A_\ell(\epsilon)\) and Is-Active returns Yes, or if \(v \notin A_\ell(0)\) and Is-Active returns No. For a subgraph \(G'\) of \(G\), we let \(m(G')\) denote the number of edges in \(G'\).

We first consider the case that \(G\) is \(\epsilon\)-close to having arboricity at most \(\alpha\). By Lemma 3, in this case \(m(G_\ell(0)) \leq 5em\). For each \(i \in [s]\) such that the edge \((u_i, v_i)\) does not belong to \(G_\ell(0)\), it holds that either \(u_i\) or \(v_i\) is not in \(A_\ell(0)\). Hence, by Lemma 5, Is-Active returns yes on both vertices with probability at most \(\epsilon/2\) (recall that Is-Active is called in Step 5 with the confidence parameter \(\delta\) set to \(\epsilon/2\)). For each edge \((u_i, v_i) \in G_\ell(0)\), we upper bound the probability that Is-Active returns yes on both vertices by 1. By Theorem 6, when Sample-Edge-Almost-Uniformly is invoked with parameters \(\beta = 1/4\) and \(\delta = 1/4\), if the algorithm returns an edge, then each edge in the graph is returned with probability in \([(3/4)/m, (5/4)/m]\). Therefore, it holds that

\[ \mathbb{E}[\chi_i] \leq \frac{\epsilon}{2} \cdot \left(\frac{5/4 \cdot m}{m}\right) + \frac{(5/4 \cdot m)}{m} \cdot \mathbb{E}[\chi_i] \leq \frac{5 \cdot m(G_\ell(0))/4 + \epsilon m}{m}. \]

Since \(m(G_\ell(0)) \leq 5em\), we get that \(\mathbb{E}[\chi_i] \leq 8\epsilon\) for every \(i \in [s]\).

By the multiplicative Chernoff bound and since \(s \geq \frac{300}{\epsilon}\),

\[ \Pr\left[\frac{1}{s} \sum_{i=1}^{s} \chi_i > \left(1 + \frac{1}{20}\right) \cdot 8\epsilon\right] < \exp\left(- (1/20) \cdot 8\epsilon \cdot s \right) \leq \frac{1}{6}. \]

It follows that if \(G\) is \(\epsilon\)-close to having arboricity at most \(\alpha\), then either the algorithm returns yes in Step 2, or with probability at least 5/6, \(\chi \leq 9\epsilon\), which causes the algorithm to return yes in Step 9.

Now consider the case that \(G\) is 20\(\epsilon\)-far from having arboricity at most 3\(\alpha\). By Lemma 4, setting \(\epsilon' = 20\epsilon\) and \(\gamma = \epsilon\), in this case we get that \(m(G_\ell(\epsilon)) > 18em\).

For each \(i \in [s]\) such that the edge \((u_i, v_i)\) belongs to \(G_\ell(\epsilon)\), we get that \(\chi_i = 1\) if the invocations of Is-Active on both \(u_i\) and \(v_i\) return Yes. Since for \((u_i, v_i) \in G_\ell(\epsilon)\) both \(u_i\) and \(v_i\) belong to \(A_\ell(\epsilon)\), by Lemma 5 and the union bound, Is-Active returns Yes on both vertices with probability at least 1 - \(\epsilon\). Hence (using our assumption that \(\epsilon \leq 1/20\)),

\[ \mathbb{E}[\chi_i] \geq \frac{(1 - \epsilon) \cdot (3/4 \cdot m(G_\ell(\epsilon))}{m} \geq 12\epsilon. \]

By the multiplicative Chernoff bound and since \(s \geq \frac{300}{\epsilon}\),

\[ \Pr\left[\frac{1}{s} \sum_{i=1}^{s} \chi_i < \left(1 - \frac{1}{20}\right) \cdot 12\epsilon\right] \leq \exp\left(- (1/20) \cdot 12\epsilon \cdot s \right) \leq \frac{1}{6}. \]

It follows that (conditioned on \(s \geq \frac{300}{\epsilon}\)), with probability at least 5/6, \(\chi \geq 10\epsilon\), which causes the algorithm to return No in Step 9. Since the probability that \(s < \frac{300}{\epsilon}\) is at most 1/6, if \(G\) is 20\(\epsilon\)-far from having arboricity at most 3\(\alpha\), then the algorithm returns No with probability at least 2/3.
It remains to bound the complexity of the algorithm. By Theorem 6, the $t = \frac{600}{\epsilon} \log^2 n$ invocations of Sample-Edge-Almost-Uniformly with parameters $\beta = 1/4$ and $\delta = 1/4$ take $O\left(\frac{n \log^2 n}{\epsilon \sqrt{m}}\right)$ time. In each step of the for loop, there are at most two invocations of the procedure Is-Active with parameters $\gamma = \epsilon$ and $\delta = \epsilon/2$. By Lemma 5, the query complexity and running time resulting from each of these invocations are $O\left(\frac{6\epsilon^2 \log(\ell/\epsilon)}{\epsilon^2}\right)$. Since $\ell = \lceil \log_{6/5}(1/\epsilon) \rceil$, the total query complexity and running time are $\tilde{O}\left(\frac{n}{\epsilon \sqrt{m}} + \frac{1}{\epsilon^2} O(\log(1/\epsilon))\right)$, as claimed.

5 LOWER BOUNDS

The following lower bounds are quite simple and are proved here for the sake of completeness.

CLAIM 8. For a graph $G$, let $n$ denote the number of vertices in $G$, and let $m$ be a constant factor approximation of the number of edges, $m$, in $G$. Let $A$ be an algorithm that is given query access to a graph $G$ and parameters $n, m, \epsilon < 1/100$, and $\alpha \leq \sqrt{m}$. The algorithm $A$ is required to distinguish with probability at least $2/3$ between the case that $G$ has arboricity at most $\alpha$ and the case that $G$ is $20\epsilon$-far from having arboricity at most $3\alpha$. Then $A$ must perform $\Omega\left(\frac{n}{\epsilon \sqrt{m}}\right)$ queries.

Proof. Consider the following two families of graphs $G_1$ and $G_2$. Every graph in $G_1$ consists of three disjoint subgraphs $G_{11}, G_{12},$ and $G_{13}$ as described next. $G_{11}$ is an independent set of size $n - 2m/\alpha - 2\sqrt{100m}$. $G_{12}$ is a bipartite graph with $m/\alpha$ vertices on each side and $\alpha$ perfect matchings between the two sides, and $G_{13}$ is an independent set of size $2\sqrt{100m}$. The graphs within the family differ from one another only by the labeling of the vertices. The graphs in the second family $G_2$ also consists of three disjoint subgraphs $G_{21}, G_{22}, G_{23}$. Here we have $G_{21} = G_{11}, G_{22} = G_{12},$ and $G_{23}$ is a complete bipartite graph with $\sqrt{100m}$ vertices on each side. As before, the graphs within the family differ only by the labeling of the vertices.

All graphs in $G_1$ have $m$ edges, and all graphs in $G_2$ have $m + 100\sqrt{m} < 2m$ edges. Furthermore, all graphs in $G_1$ have arboricity $\alpha$, whereas in $G_2$, all graphs are $20\epsilon$-far from having arboricity at most $3\alpha$. To verify the latter claim, observe that after removing $20\epsilon m < 40\sqrt{m}$ edges from any graph in $G_2$, the number of edges remaining in the subgraph $G_{i3}$ is greater than $100m - 40\sqrt{m} = 60\sqrt{m}$. Since the number of vertices in $G_{i3}$ is $2\sqrt{100m} = 20\sqrt{m}$, the arboricity of $G_{i3}$ (after the removal of the aforementioned edges) is greater than $3\sqrt{m} > 3\alpha$.

To distinguish between a graph drawn uniformly from the first family and a graph drawn uniformly from the second family, an algorithm must witness a vertex in $G_{i3}$ for $i \in \{1, 2\}$. Since the probability of witnessing such a vertex for both $i = 1$ and $i = 2$ is $\frac{|V(G_{i3})|}{n} = \frac{2\sqrt{100m}}{n}$, at least $\Omega\left(\frac{n}{\epsilon \sqrt{m}}\right)$ queries are required to distinguish between the two families with probability at least $2/3$. 

The proof of Claim 8 relies on the ability to construct a lower bound instance where we “hide” a small set of vertices with very high density. When the algorithm is also given the exact number of edges in the graph, this is no longer possible, and the preceding lower bound does not hold. Instead, for the case where $m$ is known, we prove a weaker lower bound of $\Omega\left(\frac{n\alpha}{m}\right)$.

CLAIM 9. For a graph $G$, let $n$ denote the number of vertices in $G$, and let $m$ denote the number of edges. Let $A$ be an algorithm that is given query access to a graph $G$ and parameters $n, m, \epsilon < 1/100$, and $\alpha \leq \sqrt{m}$. The algorithm $A$ is required to distinguish with probability at least $2/3$ between the case that $G$ has arboricity at most $\alpha$ and the case that $G$ is $20\epsilon$-far from having arboricity at most $3\alpha$. Then $A$ must perform $\Omega\left(\frac{n\alpha}{m}\right)$ queries.

Proof. We describe two families of graphs $G_1$ and $G_2$. Each graph in the first family $G_1$ consists of four disjoint subgraphs, $G_{11}, G_{12}, G_{13},$ and $G_{14}$, which are defined as follows. $G_{11}$ is an independent
set over $n - 2m/\alpha - 2\sqrt{100\epsilon m}$ vertices; $G_2'$ is a bipartite graph with $(1 - 100\epsilon)m/\alpha$ vertices on each side and $\alpha$ perfect matchings between the sides, $G_3'$ is a bipartite graph with $100\epsilon m/\alpha$ vertices on each side and $\alpha$ perfect matchings between the sides, and $G_4'$ is an independent set over $2\sqrt{100\epsilon m}$ vertices. The graphs in the second family $G_2$ also consists of four disjoint subgraphs $G_2^1, G_2^2, G_3^2,$ and $G_4^2$. Here we have $G_2^1 = G_2'$ and $G_2^2 = G_2^2$, $G_3^2$ is an independent set over $200\epsilon m/\alpha$ vertices, and $G_4^2$ is a complete bipartite graph with $\sqrt{100\epsilon m}$ vertices on each side. Within each family, the graphs differ only by the labeling of the vertices. By the preceding description, we get that graphs in both families have exactly $m$ edges. Furthermore, the graphs in $G_1$ have arboricity $\alpha$, and the graphs in $G_2$ are $20\epsilon$-far from having arboricity $3\alpha$ (this follows similarly to what was shown in the proof of Claim 8).

We assume without loss of generality that every neighbor query $(u, i)$ or pair query $(u, v)$ is preceded by one or two degree queries $d(u)$ or $d(u), d(v)$, respectively. Furthermore, assume that whenever the algorithm queries for the degree of some vertex $u$, it also gets an index $j \in \{1, 2, 3, 4\}$ indicating to which of the subgraphs $G_1^j, G_2^j, G_3^j, G_4^j$ it belongs (without revealing the value of $i$). Since $G_1^j = G_2^j$ and $G_2^j = G_2^j$, it is clear that to distinguish between a graph drawn from $G_1$ to a graph drawn from $G_2$, any algorithm must hit either $G_3^j$ or $G_4^j$ for the corresponding $i$ value. Since for both $i = 1$ and $i = 2$ $|G_3^j| + |G_4^j| = O(\frac{\epsilon m}{\alpha})$ (recall that we assume that $\alpha \leq \sqrt{\epsilon m}$), we have that hitting either $G_3^j$ or $G_4^j$ occurs with probability $O(\frac{\epsilon m}{\alpha n^2})$ so that $O(\frac{\alpha m}{\epsilon n})$ queries are required to distinguish the two families with probability at least $2/3$. □

Finally we establish that there is no one-sided error algorithm for bounded arboricity that performs a number of queries that is sublinear in $n$.

**Claim 10.** Let $A$ be an algorithm that is given query access to a graph $G$ and parameters $n, \alpha \geq 2$, and $\epsilon \leq \frac{1}{4}$ (where $n$ is the number of vertices in $G$). It is required to accept $G$ with probability 1 if $G$ has arboricity at most $\alpha$ and reject $G$ with probability at least $2/3$ if $G$ is $\epsilon$-far from having arboricity at most $3\alpha$. Then $A$ must perform $\Omega(n)$ queries.

**Proof.** We shall prove that for any $\alpha \geq 2$, sufficiently large $n$, and $\epsilon \leq 1/4$, there exists a graph $G$ over $n$ vertices for which the following two conditions hold. On one hand, $G$ is $\epsilon$-far from having arboricity $3\alpha$. On the other hand, for any $k \leq n/c$, where $c$ is a sufficiently large constant, every induced subgraph of $G$ over $k$ vertices has arboricity at most $\alpha$. Therefore, for this graph, any one-sided error algorithm must perform $\Omega(n)$ queries.

Consider selecting $G$ according to the distribution $G(n, p)$, where $p = \frac{10\alpha}{n}$. In other words, for each pair of vertices, the probability that we have an edge between these two vertices is $p$ (and the corresponding events for different pairs of vertices are independent). The expected number of edges in $G$ is \(\left(\frac{n}{\alpha}\right) \cdot p = 5\alpha(n - 1)\). By applying the multiplicative Chernoff bound, with very high probability, the number of edges in $G$ is at least $4\alpha(n - 1)$ (i.e., at least 4/5 of the expected value) so that by Equation (1), $G$ is at least $1/4$-far from having arboricity $3\alpha$.

Let $k$ be an integer such that $4 < k \leq n/c$, and let $K$ be a subset of $k$ vertices. We next upper bound the probability that the number of edges in the subgraph induced by $K$, denoted $m(K)$, is more than $\alpha(k - 1)$. For any fixed set $B$ of $\alpha(k - 1)$ pairs of vertices, the probability that we get an edge between every pair in $B$ is $p^{\alpha(k-1)}$. Taking a union bound over all such subsets $B$, and using the inequality \(\left(\frac{k}{\alpha}\right) \leq \left(\frac{e \cdot k \cdot p}{2\alpha}\right)^{\alpha(k-1)} = \left(\frac{5e \cdot k}{n}\right)^{\alpha(k-1)}\).
By taking a union bound over all \( \binom{n}{k} \) subsets of size \( k \), we get that the probability that there exists any such subset of size \( k \) is upper bounded by

\[
\binom{n}{k} \cdot \left( \frac{5e \cdot k}{n} \right)^{\alpha(k-1)} \leq \left( \frac{e \cdot n}{k} \right)^k \cdot \left( \frac{5e \cdot k}{n} \right)^{\alpha(k-1)} \leq \left( 5e^2 \right)^{\alpha(k-1)} \cdot \left( \frac{k}{n} \right)^{\alpha(k-1)-k} \leq \left( 5e^2 \right)^{3} \cdot \left( \frac{k}{n} \right)^{k-2},
\]

deepth and by summing over all \( k \), we get that the probability is bounded away from 1. \( \square \)

As mentioned in Section 1, for the case of \( \alpha = 1 \) (cycle-freeness), there is a one-sided error testing algorithm [9] that performs \( \tilde{O}(\sqrt{n}) \) queries (and these many queries are necessary [16]).

### 6 VARIATIONS AND ADAPTATIONS OF THE ALGORITHM

In this section, we describe several adaptations of our algorithm to other query models and input scenarios. As discussed previously, and as also evident from the description of algorithm Is-Bounded-Arboricity and its analysis, the first term in our complexity stems from sampling \( O(1/e) \) (almost) uniform edges in the graph, and the second term arises from testing whether a significant portion of the sampled edges belongs to \( G_\ell(e) \). As we will describe shortly, other query models or additional input to the tester allow to improve on the first term.

The algorithm is given a precise estimate of the number of edges. In the case that a precise \((1 \pm e/c)\)-estimate of the number of edges \( m \) is given to the algorithm, we can modify the algorithm and improve the query complexity and running time by reducing the first term in the algorithm’s complexity (the term resulting from sampling edges). Here we shortly discuss the ideas behind the variation, and the full details of the modified algorithm and its analysis are provided in Appendix A.

We say that an edge is low if it has at least one endpoint with degree at most \( 2\alpha/e \), and otherwise we say it is high. If a precise estimate for the number of edges is given, then we can rely on the following two facts to reduce the query complexity. First, we shall prove in Claim 15 in Appendix A that if the graph is \( e \)-close to having arboricity at most \( \alpha \), then it cannot have too many high edges. Second, by Eden and Rosenbaum [13], sampling low edges uniformly at random is less costly than sampling edges in the graph uniformly at random. Due to the first fact, if the graph has many high edges, then we can immediately reject (without having to sample uniform edges and check if they belong to \( G_\ell(e) \)). Since we are given a precise estimate for the number of edges in the graph, we can deduce a precise estimate of the number of high edges by estimating the number of low edges (this is not possible when only a rough estimate of the number of edges is given to the algorithm). Specifically, we use the procedure Sample-Light-Edge described in Eden and Rosenbaum [13], which samples a uniform low edge with success probability that is proportional to the number of low edges using \( O(nc/(em)) \) queries (in expectation). By using this procedure, we can estimate the number of low edges, and together with the precise estimate of \( m \), obtain a precise estimate of the number of high edges. If the estimated number of high edges is above the allowed threshold, then we reject. Otherwise, we again use the sampling procedure for low edges (together with Is-Active) to estimate the number of low edges in \( G_\ell(e) \). Hence, we can improve the algorithm’s complexity to \( O(\frac{n\alpha}{\epsilon m} + \left( \frac{1}{\epsilon} \right)^{O(\log(1/\epsilon))}) \).

The algorithm is given an upper bound \( d \) on the maximal degree. If the algorithm is not given the number of edges but is given information on the maximal degree \( d \) in the graph, we can similarly change the algorithm to improve the first term in the running time to \( O(\frac{d}{\epsilon d}) \), where \( d \) is the average degree.
degree of the graph (if \( d > \sqrt{m} \), then it is better to use the original algorithm). In this case, we do the following. We redefine low edges to be edges with at least one endpoint with degree at most \( d \) so that in fact all edges in the graph are low. We then simply invoke Sample-Light-Edge repeatedly with degree threshold \( d \) until we obtain a (multi)set \( S \) of \( \Theta(1/e) \) edges. Since all edges in the graph are low, this process is equivalent to sampling \( \Theta(1/e) \) uniform edges in the graph. We then invoke Is-Active on both endpoints of each edge in \( S \) to verify that the algorithm has indeed relatively few remaining edges in \( G_G(\epsilon) \) (and otherwise we reject).

By Theorem 6, invoking the algorithm Sample-Light-Edge of Eden and Rosenbaum [13] with degree threshold \( d \) returns each edge in the graph with probability \( \frac{1}{n \cdot d} \), so that the expected query complexity of this variant is \( O\left(\frac{n \cdot d}{\epsilon} + \left(\frac{1}{\epsilon}\right)^{O(\log(1/\epsilon))}\right) = O\left(\frac{d}{\epsilon \cdot d} + \left(\frac{1}{\epsilon}\right)^{O(\log(1/\epsilon))}\right) \).

The algorithm is given an upper bound \( d \) on the maximal degree, and the distance is with respect to \( n \cdot d \). Since the distance measure is now with respect to \( n \cdot d \), a graph is \( \epsilon \)-far from having arboricity at most \( \alpha \) if more than \( \epsilon n \cdot d \) edges should be removed to obtain a graph with arboricity at most \( \alpha \), and similarly a graph is \( \epsilon \)-close if at most \( \epsilon n \cdot d \) edges should be removed to obtain the property. In this case, Lemma 3 can be altered to state that if a graph \( G \) is \( \epsilon \)-close to having arboricity at most \( \alpha \), then \( |E(G_G(\epsilon)(0))| \leq 5\epsilon \) and, and Lemma 4 can be altered to state that if a graph \( G \) is \( \epsilon' \)-far from having arboricity at most \( 3\alpha \), then \( |E(G_G(\epsilon)(y))| > (\epsilon' - 2\gamma) n \cdot d \). Since, as in the previous case, we can sample each edge in the graph with probability \( \frac{1}{m} \), it implies that the first term in the upper bound can be improved to \( O(1/\epsilon) \). Hence, the total query complexity is \( \left(\frac{1}{\epsilon}\right)^{O(\log(1/\epsilon))} \).

Access to random edges. Finally, if the algorithm can access random edges (and the distance measure is with respect to \( m \)), then the first term in the complexity of the algorithm is replaced by \( O(1/\epsilon) \) since each edge is sampled with probability \( \frac{1}{m} \), and we are again left with the second term, which is the dominant one.

7 ESTIMATING THE CORRECTED ARBORICITY

In this section, we present a procedure for estimating what we refer to as the \( \epsilon \)-corrected-arboricity of a graph \( G \). The \( \epsilon \)-corrected-arboricity of a graph \( G \) is the minimal arboricity of a graph that \( G \) can be "corrected into." In other words, it is the minimal arboricity over all of the graphs that are \( \epsilon \)-close to \( G \) (see Definition 12). The procedure performs a standard geometric search on the value of the "corrected arboricity" using the testing algorithm Is-Bounded-Arboricity, and we provide the procedure here for the sake of completeness. More precisely, the procedure first obtains an estimate \( \overline{m} \) of \( m \). Then it starts with a guess value \( \overline{\alpha} = 1 \), and for each guess value \( \overline{\alpha} \), it invokes Is-Bounded-Arboricity with \( \overline{\alpha} \) for \( O(\log \log \overline{m}) \) times. If the majority of votes return yes, then the algorithm returns \( \overline{\alpha} \), and otherwise it continues with \( \overline{\alpha} = 2\alpha \). The search ends when \( \overline{\alpha} \) exceeds \( \sqrt{m} \), at which point the algorithm simply returns \( \sqrt{m} \). (Recall that for every graph \( G, \alpha \leq \sqrt{m} \).) We note that the overhead of this procedure, compared to the testing algorithm, is a factor of poly(log \( n \)).

**Theorem 11 (Goldreich and Ron [17], Rephrased).** There exists a procedure that when invoked with a graph \( G \) and a confidence parameter \( \delta \) returns a value \( \overline{m} \) such that with probability at least \( 1 - \delta \), \( \overline{m} \in [m, 2m] \), where \( m \) is the number of edges in \( G \). The expected running time of the procedure is \( O(n \log(n/\delta))/\sqrt{m} \).

**Definition 12 (\( \epsilon \)-Corrected Arboricity).** Let \( G_n \) denote the family of graphs over \( n \) vertices. For a graph \( G \in G_n \), let \( \alpha^*(G, \epsilon) \) denote the minimal value \( \alpha' \) such that \( G \) is \( \epsilon \)-close to a graph \( G' \) with arboricity at most \( \alpha' \). In other words, \( \alpha^*(G, \epsilon) = \min_{G' \in G_n} \{\alpha(G') \mid \text{dist}(G, G') \leq \epsilon \cdot m(G)\} \). We refer to the value \( \alpha^* \) as the \( \epsilon \)-corrected arboricity of the graph \( G \).

**Claim 13.** The algorithm Estimate-Corrected-Arboricity when invoked with a graph \( G \) over \( n \) vertices and parameter \( \epsilon \) returns a value \( \overline{\alpha} \) such that with probability at least \( 2/3 \), \( \alpha^*(G, 20\epsilon)/3 \leq \overline{\alpha} \).
Invoke Goldreich and Ron [17] with parameters $G$ and $1/9$ to obtain an estimate $\overline{m}$ of the number of edges in $G$.

1. Let $\overline{\alpha} = 1$ and $\delta = 1/(9 \log \overline{m})$.

2. while $\overline{\alpha} \leq \sqrt{\overline{m}}$ do

3. Invoke $\text{Is-Bounded-Arboricity}(G, n, \overline{\alpha}, \epsilon)$ for $10 \log (1/\delta)$ times.

4. If more than half of the invocations returned Yes, then return $\overline{\alpha} = \overline{\alpha}$.

5. Let $\overline{\alpha} = 2\overline{\alpha}$.

6. end while

7. return $\sqrt{\overline{m}}$.

$\overline{\alpha} \leq 2\alpha^*(G, \epsilon)$. The expected query complexity and running time of the algorithm are

$$\tilde{O}\left(\frac{n}{\epsilon \sqrt{m}} + \left(\frac{1}{\epsilon}\right)^{O(\log(1/\epsilon))}\right).$$

Proof. By Theorem 11, with probability at least $8/9$, the value $\overline{m}$ is such that $\overline{m} \in [m, 2m]$. We henceforth condition on this event.

For every value $\overline{\alpha}$ such that $\overline{\alpha} < \alpha^*(G, 20\epsilon)/3$, it holds that $G$ is at least $20\epsilon$-far from every graph with arboricity at most $3\overline{\alpha}$. Hence, by Theorem 7, every invocation of $\text{Is-Bounded-Arboricity}(G, n, \overline{\alpha}, \epsilon)$ returns No with probability at least $2/3$, and by a simple amplification argument, the probability that more than half of the invocations return Yes in Step 5 is at most $\delta$. Therefore, with probability at least $1 - \delta$, $\text{Estimate-Corrected-Arboricity}$ will continue to run with a value $2\overline{\alpha}$. Since there are at most $\log(\sqrt{\overline{m}})$ invocations with a value $\overline{\alpha} < \alpha^*(G, 20\epsilon)/3$, by a union bound, the probability that $\text{Estimate-Corrected-Arboricity}$ will return a value $\overline{\alpha} < \alpha^*(G, 20\epsilon)/3$ is at most $1/9$.

Once we reach a value $\overline{\alpha}$ such that $\overline{\alpha} \geq \alpha^*(G, \epsilon)$, then by the definition of $\alpha^*$, $G$ is $\epsilon$-close to having arboricity at most $\overline{\alpha}$, and therefore, by Theorem 7, every invocation of $\text{Is-Bounded-Arboricity}(G, \overline{\alpha}, \epsilon)$ returns Yes with probability at least $2/3$. Hence, with probability at least $1 - \delta$, more than half of the invocations of $\text{Is-Bounded-Arboricity}$ return yes and the algorithm returns $\overline{\alpha}$. Since we increase $\overline{\alpha}$ by a factor 2 at every step, it holds that we will reach a value $\overline{\alpha}$ such that $\overline{\alpha} \in [\alpha^*(G, \epsilon), 2\alpha^*(G, \epsilon)]$, and by the preceding, once we reach such a value, $\text{Estimate-Corrected-Arboricity}$ will return $\overline{\alpha}$ with probability at least $1 - \delta > 8/9$.

By a union bound, with probability at least $2/3$, $\text{Estimate-Corrected-Arboricity}$ returns a value $\overline{\alpha}$ such that $\alpha^*(G, 20\epsilon)/3 \leq \overline{\alpha} \leq 2\alpha^*(G, \epsilon)$.

By Theorem 11, estimating the number of edges in Step 1 takes $O(\frac{n \log^0 n}{\sqrt{m}})$ time in expectation. By Theorem 7, every invocation of the while loop in Step 3 takes $\tilde{O}(\frac{n}{\epsilon \sqrt{m}}) + \left(\frac{1}{\epsilon}\right)^{O(\log(1/\epsilon))}$ time in expectation. Since the number of iterations is at most $\log(\overline{m})$ (and $\overline{m} \leq n^2$), the query complexity and running time are

$$\tilde{O}\left(\frac{n}{\epsilon \sqrt{m}} + \left(\frac{1}{\epsilon}\right)^{O(\log(1/\epsilon))}\right)$$

in expectation.

\textbf{APPENDIX}

\section{Adaptation of the Algorithm When Given a \((1 \pm \epsilon/c)\) Estimate of \(m\)}

In this section, we describe an adaption of the algorithm for the case where it is given a \((1 \pm \epsilon/c)\) estimate of $m$. It will be easier to think of every edge $\{u, v\} \in E$ as two distinct directed edges $(u, v)$ and $(v, u)$. We take advantage of the following definitions and simple claim.
Definition 14. We say that a vertex $v$ is high if $d(v) > 2\alpha/\epsilon$. Otherwise, we say that it is low.

For an edge $(u, v)$, if $u$ is low, then we say that it is a directed low edge, and otherwise we say that it is a directed high edge.

Claim 15. If a graph $G$ has arboricity at most $\alpha$, then it has at most $2\epsilon m$ directed high edges.

Proof. Let $H$ denote the set of high degree vertices in the graph. Then $|H| < 2m/(2\alpha/\epsilon) = \epsilon m/\alpha$, and it follows that $|E(H)| < \alpha |H| = \epsilon m$, implying that the number of directed high edges is at most $2\epsilon m$.

We shall also make use of Lemma 3.1 of Eden and Rosenbaum [13] for sampling directed light edges.\footnote{This theorem appears only in the version presented in Eden and Rosenbaum [13] and not in the final version of the same work [14].}

For a degree threshold $\theta$, let $E_{\leq \theta}$ denote the set of directed edges $(u, v) \in E$ such that $d(u) \leq \theta$.

Lemma 16 (Eden and Rosenbaum [13]). Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. There exists a procedure named \texttt{Sample-Light-Edge} that given $\theta$ returns a directed edge in $E_{\leq \theta}$ with probability $\frac{|E_{\leq \theta}|}{n \theta}$. Furthermore, the procedure performs a constant number of queries, and each directed edge in $E_{\leq \theta}$ is returned with equal probability.

\begin{procedure}
\caption{\texttt{Estimate-High-Edges}$(n, \alpha, \epsilon, \delta, \bar{m})$}
\begin{algorithmic}[1]
\State Set $r = \frac{n\alpha}{\bar{m}} \cdot \frac{200 \ln(1/\delta)}{\epsilon^3}$.
\State Invoke the procedure \texttt{Sample-Light-Edge} for $r$ times with $\theta = 2\alpha/\epsilon$, and let $\chi_i = 1$ if the $i^{th}$ invocation returned an edge, and otherwise let $\chi_i = 0$.
\State Let $\chi = \frac{1}{r} \sum_{i=1}^{r} \chi_i$.
\State Let $\bar{m}_{\text{low}} = \frac{2n\alpha}{\epsilon^2} \cdot \chi$, and let $\bar{m}_{\text{high}} = \bar{m} - \bar{m}_{\text{low}}$.
\State If $\bar{m}_{\text{high}} > 5\epsilon \bar{m}/2$, then return \texttt{Many}. Otherwise, return \texttt{Few}.
\end{algorithmic}
\end{procedure}

Claim 17. Assume that $\bar{m} \in (1 \pm \epsilon/4)m$. If there are more than $3\epsilon m$ directed high edges in the graph, then with probability at least $1 - \delta$, \texttt{Estimate-High-Edges} returns \texttt{Many}, and if there are at most $2\epsilon m$ directed high edges in the graph, then with probability at least $1 - \delta$, \texttt{Estimate-High-Edges} returns \texttt{Few}. The query complexity and running time of the procedure are $O\left(\frac{n\alpha \log(1/\delta)}{\epsilon^3 m}\right)$.

Proof. Let $m_{\text{low}}$ and $m_{\text{high}}$ denote the number of directed low and high edges in the graph, respectively. We first consider the case that $G$ has more than $3\epsilon m$ high edges so that $m_{\text{low}} \leq (1 - 3\epsilon)m$. By the preceding and Lemma 16, $\mathbb{E}[\chi] = \frac{m_{\text{low}}}{(2n\alpha/\epsilon)} \leq \frac{(1 - 3\epsilon)m}{(2n\alpha/\epsilon)}$. Therefore, by the multiplicative Chernoff bound, the setting of $r = \frac{n\alpha}{\bar{m}} \cdot \frac{200 \ln(1/\delta)}{\epsilon^3}$ in the algorithm, the assumption that $\bar{m} \leq (1 + \epsilon/4)m$, and the assumption that $\epsilon \leq 1/20$,

$$\Pr \left[ \chi > \left( 1 + \frac{\epsilon}{4} \right) \cdot \frac{(1 - 3\epsilon)m}{(2n\alpha/\epsilon)} \right] \leq \exp \left( - \frac{\epsilon^2 \cdot (1 - 3\epsilon)m}{16 \cdot 3} \cdot \frac{r}{(2n\alpha/\epsilon)} \right) = \exp \left( \frac{\epsilon^3 (1 - 3\epsilon)m}{96n\alpha} \cdot \frac{n\alpha}{\bar{m}} \cdot \frac{200 \ln(1/\delta)}{\epsilon^3} \right) < \delta. \quad (5)$$

Hence, with probability at least $1 - \delta$,

$$\chi \leq \frac{(1 + \epsilon/4) \cdot (1 - 3\epsilon)m}{2n \cdot \alpha} < \frac{(1 - 5\epsilon/2) \cdot \epsilon \bar{m}}{2n \cdot \alpha}$$
and $m_{low} < (1 - 5\epsilon/2) \cdot m$. It follows that $m_{high} > 5\epsilon m/2$ with probability at least $1 - \delta$ so that the algorithm will return Many.

We now consider the case that $G$ has less than $2\epsilon m$ high edges so that $m_{low} > (1 - 2\epsilon)m$. Hence, by Lemma 16, $\mathbb{E}[\chi] > \frac{(1-2\epsilon)m}{2na/\epsilon}$. By the multiplicative Chernoff bound, the setting of $r$, the assumption that $m \leq (1 + \epsilon/4)m$, and the assumption that $\epsilon \leq 1/20$,

$$\Pr \left[ \chi < \left( 1 - \frac{\epsilon}{4} \right) \cdot \frac{(1 - 2\epsilon)m}{(2n\alpha/\epsilon)} \right] < \exp \left( -\frac{\epsilon^2 \cdot \frac{(1-2\epsilon)m}{(2n\alpha/\epsilon)} \cdot r}{16 \cdot 2} \right) < \delta. \quad (6)$$

It follows that with probability at least $1 - \delta$,

$$\chi \geq \frac{(1 - \epsilon/4)(1 - 2\epsilon)m}{2n \cdot \alpha} > \frac{(1 - 5\epsilon/2) \cdot \epsilon m}{2n \cdot \alpha}.$$ 

Therefore, with probability at least $1 - \delta$, $m_{high} < 5\epsilon m/2$ and the procedure returns Few.

By Lemma 16, the query complexity and running time of each invocation of the procedure Sample-Light-Edge (with $\theta = 2\alpha/\epsilon$) are $O(1)$. Hence, the query complexity and running time of the procedure are $O(r) = O\left( \frac{n\alpha \log(1/\delta)}{\epsilon^3 m} \right)$. $\square$

Consider modifying the algorithm Is-Bounded-Arboricity (from Section 4) as follows. We first check if there are many high edges in the graph. If so, we reply that the graph is far from having arboricity at most $3\alpha$. Otherwise, we sample light edges and check if their endpoints are active.

**Algorithm 7: Is-Bounded-Arboricity-Given-Edges-Estimate** ($G, n, \alpha, \epsilon, \bar{m}$)

1. Invoke Estimate-High-Edges($n, \alpha, \epsilon, 1/12, \bar{m}$), and if the procedure returns Many, then return No.
2. Invoke Sample-Light-Edge with $\theta = 2\alpha/\epsilon$ for $r = \frac{1000n\alpha}{\epsilon^3 m}$ times, and let $S$ be the (multi)set of returned directed edges. Let $s$ be the number of (not necessarily different) edges in $S$.
3. If $s < \frac{800}{\epsilon}$, then return No.
4. Set $\ell = \lceil \log_{5\sqrt{5}}(1/\epsilon) \rceil$.
5. for every directed edge $(u_i, v_i) \in S$ do
   6.   Invoke Is-Active$(u_i, \ell, \alpha, \epsilon, \epsilon/2)$ and Is-Active$(v_i, \ell, \alpha, \epsilon, \epsilon/2)$. If the procedure returned Yes on both invocations, then set $\chi_i = 1$. Otherwise, set $\chi_i = 0$.
7. end for
8. Set $\chi = \frac{1}{s} \sum_{i=1}^{s} \chi_i$.
9. If $\chi < 12\epsilon$, then return Yes. Otherwise, return No.

**Theorem 18.** Assume that $\bar{m} \in (1 \pm \epsilon/4)m$. If $G$ is $\epsilon$-close to having arboricity at most $\alpha$, then Is-Bounded-Arboricity-Given-Edges-Estimate returns Yes with probability at least $2/3$, and if $G$ is $20\epsilon$-far from having arboricity at most $3\alpha$, then Is-Bounded-Arboricity-Given-Edges-Estimate returns No with probability at least $2/3$.

The query complexity and running time of the algorithm are

$$O\left( \frac{n\alpha}{\epsilon^3 m} + \frac{1}{\epsilon} \right)^{O(\log(1/\delta))}$$

in expectation.

**Proof.** We first consider the case that $G$ is $\epsilon$-close to having arboricity at most $\alpha$ and prove that the algorithm returns Yes with probability at least $2/3$. If $G$ is $\epsilon$-close to having arboricity at most $\alpha$, then it follows from Claim 15 that $G$ has less than $2\epsilon m$ directed high edges. Therefore, by Claim 17, the procedure Estimate-High-Edges returns Few with probability at least $11/12$. We henceforth condition on this event.
By Lemma 16, each invocation of Sample-Light-Edge with $\theta = 2\alpha/\epsilon$ returns a directed light edge with probability at least \((1-2\epsilon)m / (2n\alpha/\epsilon)^{12}\). Let us denote this probability by $p_{\text{succ}}$, and let $x_i$ be a random variable that indicates if the $i^{th}$ invocation of Sample-Light-Edge returns an edge. By the multiplicative Chernoff bound, 
\[
\Pr \left[ \sum_{i=1}^{t} x_i < 0.9p_{\text{succ}} \right] < \exp \left( -0.1^2 \cdot p_{\text{succ}} \cdot t \right) \leq \exp \left( -\frac{1}{200} \cdot \frac{(1-2\epsilon) \cdot m}{2n\alpha} \cdot \frac{1000n\alpha}{\epsilon^2 m} \right) \leq \frac{1}{12},
\]
where the last inequality is by the assumption that $\overline{m} \leq (1 + \epsilon/4)m$ and that $\epsilon \leq 1/20$. Therefore, with probability at least $11/12$, $s > 0.9 \cdot p_{\text{succ}} \cdot t > \frac{800}{\epsilon}$. Condition on this event as well.

By Lemma 3, if $G$ is $\epsilon$-close to having arboricity at most $\alpha$, then $m(G_\ell(0)) \leq 5em$ so that $m_{\text{low}}(G_\ell(0)) \leq 10em$, where for a subgraph $G'$ we let $m_{\text{low}}(G')$ denote the number of directed low edges in $G'$. For every $i$ such that $(u_i, v_i)$ is not in $G_\ell(0)$, it holds that either $u_i$ or $v_i$ is not in $A_\ell(0)$. Hence, by Lemma 5, Is-Active returns yes on both vertices with probability at most $\epsilon/2$. For every $i$ such that $(u_i, v_i)$ is in $G_\ell(0)$, we bound the probability that Is-Active returns yes on both vertices by 1. Since by Lemma 16 each directed light edge in the graph is returned with equal probability, it holds that 
\[
\mathbb{E} [\chi_i] \leq \frac{(\epsilon/2) \cdot (m_{\text{low}}(G) - m_{\text{low}}(G_\ell(0)))}{m_{\text{low}}(G)} + \frac{m_{\text{low}}(G_\ell(0))}{m_{\text{low}}(G)} \leq \frac{\epsilon}{2} + \frac{10em}{m_{\text{low}}(G)} \leq 11.7\epsilon,
\]
where the last inequality is by the fact that $m_{\text{low}} \geq (1 - 2\epsilon)m$ and the assumption that $\epsilon < 1/20$. Therefore, by the multiplicative Chernoff bound, and since $s > \frac{800}{\epsilon}$,
\[
\Pr \left[ \frac{1}{s} \sum_{i=1}^{s} \chi_i > \left( 1 + \frac{1}{40} \right) \cdot 11.7\epsilon \right] < \exp \left( -\frac{(1/40)^2 \cdot 11.7\epsilon \cdot s}{3} \right) < 1/6.
\]
Therefore, with probability at least $5/6$, $\chi < 12\epsilon$. By taking a union bound over all “bad” events, it holds that the procedure returns yes with probability at least $2/3$.

Now we consider the case that $G$ is at least $20\epsilon$-far from having arboricity at most $3\alpha$ and prove that with probability at least $2/3$ the algorithm returns No. If $G$ has more than $3em$ high edges, then by Claim 17, with probability at least $11/12$ the procedure Estimate-High-Edges will return Many in Step 1, and therefore the algorithm will return No, and we are done. In addition, if $s < \frac{800}{\epsilon}$, then the procedure returns No in Step 3, and we are done. Therefore, assume that $G$ has at most $3em$ high edges and that $s \geq \frac{800}{\epsilon}$.

By Lemma 4, since $G$ is at least $20\epsilon$-far from having arboricity at most $3\alpha$, it holds that $m(G_\ell(\epsilon)) > 16em$, implying that $m_{\text{low}}(G_\ell(\epsilon)) > 13em$. For every $i$ such that $(u_i, v_i)$ is in $G_\ell(\epsilon)$, both $u_i$ and $v_i$ are in $A_\ell(\epsilon)$, and by Lemma 5 and the union bound, Is-Active returns yes on both vertices with probability at least $1 - \epsilon$. In addition, it follows from Lemma 16 that every directed light edge in the graph is returned with equal probability. Hence, 
\[
\mathbb{E} [\chi_i] \geq \frac{(1 - \epsilon) \cdot m_{\text{low}}(G_\ell(\epsilon))}{m} \geq 12.35\epsilon.
\]
By the multiplicative Chernoff bound and since $s > \frac{800}{\epsilon}$,
\[
\Pr \left[ \frac{1}{s} \sum_{i=1}^{s} \chi_i < \left( 1 - \frac{1}{40} \right) \cdot 12.35\epsilon \right] < \exp \left( -\frac{(1/40)^2 \cdot 12.35\epsilon \cdot s}{2} \right) < 1/6.
\]
Therefore, if $G$ is $20\epsilon$-far from having arboricity at most $3\alpha$, then with probability at least $2/3$, $\chi > 12\epsilon$ and the algorithm returns No.

By Claim 17, the query complexity and running time resulting from the invocation of the procedure Estimate-High-Edges in Step 1 are $O(\frac{n\alpha}{\epsilon^2 m})$. By Lemma 16, the running time and query
complexity of the procedure Sample-Light-Edge are constant, and therefore the query complexity and running time of Step 2 are $O\left(\frac{n\epsilon}{\epsilon m}\right)$. In each step of the for loop, there are two invocations of the procedure Is-Active with parameters $\gamma = \epsilon$ and $\delta = \epsilon/2$. By Lemma 5, the query complexity and running time resulting from these invocations are $O\left(\frac{1}{\epsilon}O(\log(1/\epsilon))\right)$. Therefore, the total query complexity and running time are $O\left(\frac{n\epsilon}{\epsilon m} + \frac{1}{\epsilon}O(\log(1/\epsilon))\right)$. $\Box$

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