EXISTENCE AND STABILIZATION OF A KIRCHHOFF MOVING STRING WITH A DELAY IN THE BOUNDARY OR IN THE INTERNAL FEEDBACK

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Abstract. In this paper, we study the effect of an internal or boundary time-delay on the stabilization of a moving string. The models adopted here are nonlinear and of “Kirchhoff” type. The well-posedness of the systems is proven by means of the Faedo–Galerkin method. In both cases, we prove that the solution of the system approaches the equilibrium in an exponential manner in the energy norm. To this end we request that the delayed term be dominated by the damping term. This is established through the multiplier technique.

1. Introduction. In the last few years a great attention was paid to the axially moving structures due to their appearance in many engineering applications, like: Textile fibres, paper sheets, robot arms, magnetic tapes, conveyor belts, and aerial cables. Unfortunately, these structures often suffer from transverse vibrations produced during their functioning. This has encouraged researchers to explore different ways of controlling these devices. Many studies which lead to interesting results have been established in this area. One of the most efficient ways employed in this direction is the boundary control using a state feedback. Fung and Tseng in [5] showed the exponential stability of a linear model of an axially moving string using a feedback comprising the displacement, the velocity and the slope of the string at one of the endpoints. By using a nonlinear feedback boundary controller coupled with an MDS controller, Fung et al. in [6] proved that the system is exponentially stable. For nonlinear models, a class of moving strings is stabilized by boundary controls with negative feedback in [7, 20, 21] and by a boundary control of memory type in [10].

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The present paper deals with the existence and stabilization of solutions of an axially moving string with a retarded boundary term

$$\begin{cases}
y_{tt} + 2vy_{xt} - (1 - v^2 + b\|y_x\|^2)y_{xx} = 0, \quad x \in (0,1), \quad t > 0, \\
y(0,t) = 0, \quad t \geq 0, \\
(1 + b\|y_x\|^2)y_x(1,t) = -\mu_1(y_t + vy_x)(1,t) - \mu_2(y_t + vy_x)(1,t - \tau), \quad t \geq 0, \\
y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in (0,1), \\
y_1(1,t - \tau) + vy_x(1,t - \tau) = f_0(1,t - \tau), \quad t \in (0,\tau),
\end{cases}$$

(1)

where we denote by $y(x,t)$ the transverse displacement of the string which is axially moving with a constant velocity $v$ such that $0 < v < 1$. The real number $\tau > 0$ indicates the time delay and the functions $y_0(x), y_1(x)$ and $f_0$ are respectively the initial displacement, the initial velocity and the observation of the system in $[-\tau,0]$. The coefficients $\mu_1$ and $\mu_2$ are positive real numbers. The tension in the string is given by the expression $T(t) = 1 + b\|y_x\|^2$ where $b$ is a positive constant and $\|\|_2$ designates the $L^2$-norm.

It was shown, in many papers, that the boundary delay term, which arises in many practical problems, produces some instability effects. To explain one of the main features in this study, we shall set out some interesting results concerning the case of one dimensional wave equation. We consider the simple mathematical model

$$\begin{cases}
y_{tt} - y_{xx} = 0, \quad x \in (0,1), \quad t > 0, \\
y(0,t) = 0, \quad y_x(1,t) = u(t), \quad t \geq 0, \\
y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in (0,1), \quad t \geq 0, \\
w(t) = y_1(1,t), \quad t \geq 0
\end{cases}$$

(2)

where $u(t)$ denotes the control input and $w(t)$ is an observation of the system. If the control input $u(t)$ does not contain delays, it was shown that a simple boundary control of the form $u(t) = -ky_t(1,t), \quad k > 0$ is capable of stabilizing the system exponentially (see, e.g. [14, 23, 22]). However, a small delay in the control input of the form $u(t) = -ky_t(1,t - \epsilon), \quad k > 0, \quad t > \epsilon$, provokes instability as shown in [4]. The control used in [24] is of the form

$$u(t) = \mu f(t) + (1 - \mu)f(t - \tau), \quad \mu \in (0,1), \quad t > 0$$

where $\mu$ is the weight of the time-delay effect, $\mu = 1$ denotes no input delay, and $\mu = 0$ denotes an entire input delay. If $f(t) = -kw(t)$, the control input takes the form

$$\begin{cases}
u(t) = k\mu y_t(1,t) + k(1 - \mu)y_t(1,t - \tau), \quad k > 0, \quad \mu \in (0,1), \quad t > 0, \\
y_t(1,t - \tau) = f(1,t - \tau), \quad t \in (0,\tau).
\end{cases}$$

(3)

The authors in [24], by adopting the spectral analysis approach, proved that the stability of the closed loop system depends on $\mu$. When $\mu > \frac{1}{2}$, the closed loop system is exponentially stable and it is unstable when $\mu < \frac{1}{2}$. In case $\mu = \frac{1}{2}$, it is shown that if $\tau$ is rational the system is unstable and it is asymptotically stable if $\tau$ is irrational. Nicaise et al. [18] examined the case where the delay is a time-varying function. The authors established an exponential decay result under the condition $\mu_2 < \sqrt{1 - d\mu_1}$, where $\mu_1$ and $\mu_2$ are the coefficients of the damping and the delay term, respectively and $d$ is a constant verifying $\tau'(t) \leq d < 1, \quad t > 0$.

Regarding the multi-dimensional case

$$y_{tt} - \Delta y = 0, \quad x \in \Omega, \quad t > 0,$$

(4)
where $\Omega$ is a bounded domain with smooth boundary $\partial \Omega = \Gamma_0 \cup \Gamma_1$, Nicaise and Pignotti in [16] considered the boundary control

$$\frac{\partial y}{\partial n}(x,t) = -\mu_1 y_t(x,t) - \mu_2 y_t(x,t - \tau), \quad x \in \Gamma_1, \quad t > 0.$$ 

They proved, under the assumption $\mu_2 < \mu_1$, that the solution is exponentially stable by using inequalities derived from Carleman estimates (see [13]) combined with compactness-uniqueness arguments. It is also demonstrated that if $\mu_2 \geq \mu_1$ the system is unstable. Note that this problem with dynamic boundary conditions was investigated, first by Nicaise and Pignotti in [17], then Gerbi and Said-Houari in [9].

These works have motivated our present study of the time delay effect on the stability of a moving string. The model adopted here is derived with the help of the Newton’s second law under the smallness assumption of the spatial variation of the tension (see [1]). The exponential stabilization of the system 1 without a delay term was investigated in [7] by considering the boundary control $u(t) = -ky_t(1,t)$, $k > 0$ and by the boundary control $u(t) = -k(t)y_t(1,t) - \theta(t) \sin t$, $k, \theta \in C^1[0, +\infty)$ in [12]. Recently, a similar result was obtained in [10] by considering a much weaker damping, namely a boundary control of memory type.

The present paper deals also with an axially moving string with retarded internal terms

$$
\begin{cases}
    y_{tt} + 2v y_{xt} - (1 - v^2 + b\|y_x\|^2)y_{xx} + \mu_1 (y_t + vy_x)(t) \\
    + \mu_2 (y_t + vy_x)(t - \tau) = 0, \\
    y(0, t) = 0, \quad t \geq 0, \\
    y_t(1, t) - v(y_t(1, t) + vy_x(1, t)) + \left(1 + b\|y_x\|^2\right)y_x(1, t) = f_c(t), \quad t \geq 0, \\
    y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, 1), \\
    y_t(x, t - \tau) + vy_x(x, t - \tau) = g_0(x, t - \tau), \quad t \in (0, \tau).
\end{cases}
$$

(5)

The second boundary condition reflects the fact that the string is controlled by a hydraulic touch-roll actuator at the right endpoint. The function $f_c(t)$ is a control force applied by the touch-roll actuator to dissipate the energy of vibrations and $g_0$ is the observation of the system.

Exponential stability results for the above problem in the case of $\mu_2 = 0$, have been obtained in the linear case in [8] and in the presence of $y_x^2$ instead of $\|y_x\|^2$ in [2]. The case of an internal viscoelastic damping term has been also discussed by the present authors in [11] where they obtained an arbitrary decay result. For the case of immobile string, we can refer the readers to [16] and [17].

To the best of our knowledge, the issue of the stabilization of an axially moving structure with delay described by 1 and 5 has not been addressed previously. The main difficulties encountered in this analysis are the axial movement of the string and the presence of the delay. The axial movement of the string is overcome by utilizing the Reynolds Transport Theorem (for more details, see [25]). The nonlinearity in the Kirchhoff term requires some further manipulations.

The content of the remaining parts of this paper is structured into two parts. The first part is reserved to the first problem 1 and the second part is concerned with the problem 6. In each part we discuss the well-posedness by means of Faedo-Galerkin method and we prove that if the the delay coefficient $\mu_2$ is less than the one of the damping $\mu_1$, the solution decays exponentially to zero.
2. Boundary feedback. In this section we present an existence and uniqueness result for problem 1. In order to deal with the delay feedback term, we introduce the following new dependent variable

\[ z(\rho, t) = y_t(1, t - \tau \rho) + vy_x(1, t - \tau \rho), \quad \rho \in (0, 1), \quad t > 0. \]

Then, it is easy to check that

\[ \tau z_t(\rho, t) + z_\rho(\rho, t) = 0, \quad \rho \in (0, 1), \quad t > 0. \]

So, problem 1 may be rewritten in the form

\[
\begin{cases}
y_{tt} + 2vy_{xt} - \left(1 - v^2 + b\|y_x\|^2\right) y_{xx} = 0, \quad x \in (0, 1), \quad t > 0, \\
\tau z_t(\rho, t) + z_\rho(\rho, t) = 0, \quad \rho \in (0, 1), \quad t > 0, \\
y(0, t) = 0, \quad t \geq 0, \\
(1 + b\|y_x\|^2) y_x(1, t) = -\mu_1 z(0, t) - \mu_2 z(1, t), \quad t \geq 0, \\
z(0, t) = y_t(1, t) + vy_x(1, t), \quad t > 0, \\
y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, 1), \\
z(\rho, 0) = f_0(1 - \tau \rho), \quad \rho \in (0, 1).
\end{cases}
\]  

(6)

Because the string is pulled with a constant speed \(v\), the total derivative operator or the material derivative with respect to time is defined by (see [25] and [19])

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} = (\cdot)_t + v(\cdot)_x.
\]

Then, the first and the second total derivative expressions of \(y\) with respect to time are

\[ \frac{dy}{dt} = y_t + vy_x, \quad \frac{d^2 y}{dt^2} = y_{tt} + 2vy_{xt} + v^2 y_{xx}. \]

Let \(L^2(0, 1)\) be the usual Hilbert space with the inner product \((\cdot, \cdot)\) and the induced norm \(\|\cdot\|\). In order to state our existence result, we introduce

\[ V = \{w \in H^1(0, 1) : w(0) = 0\} \]

and we denote by \(V^*\) its dual.

The following inequalities will be utilized in this paper

**Lemma 2.1.** (Young inequality) Let \((a, b) \in \mathbb{R}^2\), for any \(\eta > 0\), we have

\[ ab \leq \eta a^2 + \frac{b^2}{4\eta}. \]

**Lemma 2.2.** (Poincaré inequality) Let \(w \in C(\mathbb{R}_+; V)\) then the following inequalities hold

\[ w(x, t) \leq \|w_x\|^2, \quad \forall x \in [0, 1], \quad \forall t \in \mathbb{R}_+ \]

and

\[ \|w\|^2 \leq \|w_x\|^2, \quad \forall t \in \mathbb{R}_+. \]
2.1. Existence result.

**Definition 2.3.** Let $T > 0$, a couple $(y, z)$ such that
\[
y \in C ([0, T), V) \cap C^1 ([0, T), L^2(0, 1)), \]
\[
z \in C ([0, T), L^2(0, 1))
\]
is called a weak solution of (6) if for any $(w, u) \in V \times L^2(0, 1)$
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d^2 y}{d\tau^2}, w) = -(1 + b \|y_x\|^2) (y_x, w_x) - w(1) [\mu_1 z(0, t) + \mu_2 z(1, t)] \\
\tau (z_t, u) + (z, u, w) = 0
\end{array} \right.
\end{aligned}
\]
and
\[
y(0) = y_0, \quad y(0) = y_1, \quad z(0) = f_0.
\]

**Theorem 2.4.** Let $(y_0, y_1, f_0) \in V \times L^2(0, 1) \times L^2(0, 1)$. Assume that $\mu_2 \leq \mu_1$. Then, there exists a unique global weak solution of (6) such that
\[
y \in C ([0, T), V), \quad \frac{dy}{d\tau} \in C ([0, T), L^2(0, 1)), \quad z \in C ([0, T), L^2(0, 1)) \quad \text{for any } T > 0.
\]

**Proof.** Let us solve the variational problem (8). For this we consider a complete orthogonal system $\{w^i\}_{i=1}^\infty$ of $V \cap H^2(0, 1)$ which is orthonormal in $L^2(0, 1)$. We denote $W_m = \text{span}\{w^1, w^2, ..., w^m\}$.

Next, we define for $1 \leq i \leq m$ the sequence $u^i(x, \rho)$ by $u^i(x, 0) = w^i(x)$. Then, we may extend $u^i(x, 0)$ to $u^i(x, \rho)$ in $L^2((0, 1), (0, 1))$ and denote $U_m = \text{span}\{u^1, u^2, ..., u^m\}$.

For each $m \in \mathbb{N}$, a solution in the form
\[
\begin{aligned}
y^m(x, t) &= \sum_{i=1}^m c^i_m (t) u^i(x), \quad x \in (0, 1), \quad t \geq 0 \\
z^m(\rho, t) &= \sum_{i=1}^m d^i_m (t) u^i(\rho), \quad \rho \in (0, 1), \quad t \geq 0
\end{aligned}
\]
of the approximate problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d^2 y^m}{d\tau^2}, w) = -(1 + b \|y^m_x\|^2) (y^m_x, w_x) - w(1) [\mu_1 z^m(0, t) + \mu_2 z^m(1, t)] \\
\tau (z^m_t, u) + (z, u, w) = 0 \\
z^m(0, t) = (y^m_t + vy^m_x)(1, t) \\
y^m(0) = y_0^m, \quad y^m_1(0) = y_1^m, \quad z^m(\rho, 0) = z_0^m
\end{array} \right.
\end{aligned}
\]
satisfying
\[
\begin{aligned}
y_0^m &\to y_0 \text{ strongly in } V \cap H^2(0, 1), \\
y_1^m &\to y_1 \text{ strongly in } L^2(0, 1), \\
z_0^m &\to f_0 \text{ strongly in } L^2(0, 1),
\end{aligned}
\]
is sought. We note that the system (10) leads to a system of ODEs for the unknown functions $(c^i_m(t), d^i_m(t))_{i=1}^{m}$ on $(0, t_m)$ where $t_m \in (0, T)$. Based on the standard existence theory for ODEs, one can guarantee the existence of a local solution $(y^m, z^m)$ of (10) on a maximal time interval $[0, t_m)$, for each $m \in \mathbb{N}$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independently of $m$ and $t$.

**A priori estimate:** Let $w = \frac{d}{d\tau} y^m = y_t^m + vy_x^m$ in the first equation, we get
\[
\begin{aligned}
&\frac{1}{2} \frac{d}{d\tau} \left( \|y^m_t + vy^m_x\|^2 + \|y^m_x\|^2 + \frac{b}{2} \|y^m_t\|^4 \right) \\
&= -z^m(0, t) [\mu_1 z^m(0, t) + \mu_2 z^m(1, t)], \quad t \in [0, t_m). \tag{11}
\end{aligned}
\]
For $\xi > 0$ to be chosen later and $u = z^m$ in the second equation of (10) we obtain by an integrating by parts

$$\frac{\xi}{2} \frac{d}{dt} \|z^m\|^2 = -\frac{\xi}{2r} (z^m, z^m) = -\frac{\xi}{2r} \left( [z^m(1,t)]^2 - [z^m(0,t)]^2 \right), \quad t \in [0, t_m). \tag{12}$$

Adding the resulting equations 11 and 12 and using Young inequality, we arrive at

$$\frac{1}{2} \frac{d}{dt} \left( \|y^m + v y^m_x\|^2 + \|y^m_x\|^2 + \frac{b}{2} \|y^m_x\|^4 + \xi \|z^m\|^2 \right) \leq \left( -\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2r} \right) [z^m(0,t)]^2 + \left( \frac{\mu_2}{2} - \frac{\xi}{2r} \right) [z^m(1,t)]^2, \quad t \in [0, t_m). \tag{13}$$

If we choose $\tau \mu_2 \leq \xi \leq \tau (2\mu_1 - \mu_2)$, the expression

$$E^m(t) = \frac{1}{2} \|y^m + v y^m_x\|^2 + \frac{1}{2} \|y^m_x\|^2 + \frac{b}{4} \|y^m\|^4 + \frac{\xi}{2} \|z^m\|^2, \quad t \in [0, t_m)$$

verifies

$$E^m(t) \leq E^m(0) \leq K$$

for some $K > 0$. Having in mind that $(y^m_0), (y^m_1)$ and $(z^m)$ are bounded in $V \cap H^2(0,1), L^2(0,1)$ and $L^2(0,1)$, respectively, it results that, the positive constant $K$ is independent of $m$ and $t$. From this, we conclude that

$$\begin{cases}
    y^m \text{ is uniformly bounded in } L^\infty(0,T; V), \\
y^m_t + vy^m_x \text{ is uniformly bounded in } L^\infty(0,T; L^2(0,1)), \\
z^m \text{ is uniformly bounded in } L^\infty(0,T; L^2(0,1)).
\end{cases} \tag{14}$$

**Passage to the limit**

As a result of the previous considerations, there exists a subsequence of $y^m$ satisfying

$$\begin{cases}
y^\mu &\rightharpoonup y \text{ weakly star in } L^\infty(0,T; V) \text{ and weakly in } L^2(0,T; V), \\
y^\mu_t + vy^\mu_x &\rightharpoonup y_t + v y_x \text{ weakly star in } L^\infty(0,T; L^2(0,1)) \text{ and weakly in } L^2(0,T; L^2(0,1)), \\
z^\mu &\rightharpoonup z \text{ weakly star in } L^\infty(0,T; L^2(0,1)) \text{ and weakly in } L^2(0,T; L^2(0,1)).
\end{cases} \tag{15}$$

Then, from Aubin-Lions theorem (see [14]), for any $T > 0$

$$\begin{align*}
y^\mu &\rightharpoonup y \text{ strongly in } L^2(0,T; L^2(0,1)), \\
y^\mu &\rightarrow y \text{ a.e in } (0,1) \times (0,T).
\end{align*} \tag{16}$$

The above convergences 15 and 16 are sufficient to pass to the limit in the linear terms of 10. Regarding the nonlinear term we can argue as in Lemma 3.1 of [3].

**Uniqueness:** The uniqueness can be proved by a straightforward use of Gronwall’s inequality.

**2.2. Stability result.** We define the energy associated to system 6 by

$$E(t) = \frac{1}{2} \|y_t + vy_x\|^2 + \frac{1}{2} \|y_x\|^2 + \frac{b}{4} \|y_x\|^4 + \frac{\xi}{2} \|z\|^2, \quad t \geq 0$$

where $\xi$ is a positive constant satisfying

$$\tau \mu_2 < \xi < \tau (2\mu_1 - \mu_2).$$

**Lemma 2.5.** The energy functional $E(t)$ satisfies along solutions of 6

$$\frac{d}{dt} E(t) \leq -k_1 z^2(0,t) - k_2 z^2(1,t), \quad t \geq 0 \tag{17}$$

where $k_1 = \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2r}$ and $k_2 = \frac{1}{2} \left( \frac{\xi}{r} - \mu_2 \right).$
Proof. In view of 7, we see that
\[ \frac{d}{dt} E(t) = \int_0^1 (y_t + v y_x) (y_{tt} + 2 v y_{xt} + v^2 y_{xx}) \, dx 
+ \int_0^1 \left( 1 + b \| y_x \|^2 \right) y_x (y_{xt} + v y_{xx}) \, dx + \frac{\xi}{2} \frac{d}{dt} \| z \|^2, \quad t \geq 0. \]
Substituting the second total derivative of \( y \) (see 6) into the previous identity and integrating by parts, we obtain
\[ \frac{d}{dt} E(t) = \left[ (y_t + v y_x) \left( 1 + b \| y_x \|^2 \right) y_x \right]_0^1 + \frac{\xi}{2} \frac{d}{dt} \| z \|^2, \quad t \geq 0. \quad (18) \]
Notice that
\[ \frac{\xi}{2} \frac{d}{dt} \| z \|^2 = -\frac{\xi}{\tau} (z_{\rho}, z) = -\frac{\xi}{2\tau} \left[ z^2(1,t) - z^2(0,t) \right], \quad t \geq 0. \]
Next, taking into account the relation in 18 and using the boundary conditions, we arrive at
\[ \frac{d}{dt} E(t) \leq -z(0,t) \left[ \mu_1 z(0,t) + \mu_2 z(1,t) \right] - \frac{\xi}{2\tau} \left[ z^2(1,t) - z^2(0,t) \right], \quad t \geq 0. \]
The relation 17 follows immediately by applying the Young inequality (with \( \eta = \frac{1}{2} \)) to the first term in this last relation.

In order to prove the main result, we introduce the following functionals
\[ \Phi_1(t) = \int_0^1 x y_x (y_t + v y_x) \, dx, \quad t \geq 0, \]
\[ \Phi_2(t) = \xi \int_0^1 e^{-2\rho \tau} z^2(\rho,t) d\rho, \quad t \geq 0 \]
and the modified functional
\[ L(t) = E(t) + \epsilon (\Phi_1(t) + \Phi_2(t)), \quad t \geq 0 \]
for some \( \epsilon \) positive to be determined later. We shall now establish an equivalence result between \( L(t) \) and \( E(t) \).

**Proposition 1.** There exist positive constants \( \beta_i, i = 1,2 \) such that for small \( \epsilon \)
\[ \beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad t \geq 0. \quad (19) \]
**Proof.** Applying Young inequality to the functional \( \Phi_1(t) \), and considering \( \epsilon \) small enough, the relation 19 follows immediately.

**Theorem 2.6.** Assume that \( \mu_2 < \mu_1 \). Then, there exist two positive constants \( K \) and \( \delta \) independent of \( t \) such that for \( (y,z) \) solution of problem 6, we have
\[ E(t) \leq K e^{-\delta t}, \quad t \geq 0. \]
**Proof.** A differentiation of the functional \( \Phi_1(t) \) gives
\[ \frac{d}{dt} \Phi_1(t) = \int_0^1 x (y_{xt} + v y_{xx})(y_t + v y_x) \, dx + v \int_0^1 y_x (y_t + v y_x) \, dx 
+ \int_0^1 x y_x (y_{tt} + 2 v y_{xt} + v^2 y_{xx}) \, dx, \quad t \geq 0. \quad (20) \]
Substituting the second total derivative (see 6) into 20, we find
\[
\frac{d}{dt}\Phi_1(t) = \int_0^1 x(\frac{y_{xx} + vy_{yxx}}{y_t + vy_x}) dx + v \int_0^1 y_x(y_t + vy_x) dx + \int_0^1 xy_x(1 + b\|y_x\|^2) y_{xx} dx, \quad t \geq 0. \tag{21}
\]
Taking into account the definition of \(z\) and the boundary conditions, performing integration by parts and using Young inequality allow us to estimate the different terms in 21 for \(t \geq 0\) as follows
\[
\int_0^1 x(\frac{y_{xx} + vy_{yxx}}{y_t + vy_x}) dx = \frac{1}{2}z^2(0, t) - \frac{1}{2}\|y_t + vy_x\|^2, \tag{22}
\]
\[
v \int_0^1 y_x(y_t + vy_x) dx \leq \frac{v}{2}\|y_x\|^2 + \frac{v}{2}\|y_t + vy_x\|^2, \tag{23}
\]
and
\[
\left(1 + b\|y_x\|^2 \right)\int_0^1 xy_x y_{xx} dx = \frac{1}{2} \left(1 + b\|y_x\|^2 \right) y^2_y(1) - \frac{1}{2} \left(1 + b\|y_x\|^2 \right)\|y_x\|^2 \leq \mu_1^2z^2(0, t) + \mu_2^2z^2(1, t) - \frac{1}{2} \left(1 + b\|y_x\|^2 \right)\|y_x\|^2. \tag{24}
\]
Gathering the estimates 22-24 in 21, we obtain
\[
\frac{d}{dt}\Phi_1(t) \leq \frac{1}{2} \left(1 + 2\mu_1^2 \right)z^2(0, t) + \mu_2^2z^2(1, t) - \frac{1}{2} \left(1 - v + b\|y_x\|^2 \right)\|y_x\|^2 - \frac{1}{2} (1-v)\|y_t + vy_x\|^2, \quad t \geq 0. \tag{25}
\]
Then, the second equation in 6 allows us to write
\[
\frac{d}{dt}\Phi_2(t) = 2\xi\int_0^1 e^{-2\rho z}(\rho, t)z(t, \rho) d\rho = -\xi\int_0^1 e^{-2\rho z} \frac{\partial}{\partial \rho} z(\rho, t) d\rho, \quad t \geq 0.
\]
Next, by using an integration by parts and the definition of \(z\), the above formula can be rewritten as
\[
\frac{d}{dt}\Phi_2(t) = -\xi\left(e^{-2\rho z}(1, t) - z^2(0, t)\right) - 2\xi\int_0^1 e^{-2\rho z} z^2(\rho, t) d\rho, \quad t \geq 0. \tag{26}
\]
Exploiting the previous estimates 25 and 26, we entail that
\[
\frac{d}{dt}L(t) \leq -\left(k_1 - \frac{e}{2} - \frac{\xi}{\tau} - e\mu_1^2 \right)z^2(0, t) - \left(k_2 + \frac{\xi}{\tau} e^{-2\tau} \right)z^2(1, t) - \frac{e}{2}\left(1 - v + b\|y_x\|^2 \right)\|y_x\|^2 + (1-v)\|y_t + vy_x\|^2
\]
\[
-2\xi\int_0^1 e^{-2\rho z} z^2(\rho, t) d\rho, \quad t \geq 0. \tag{27}
\]
We now pick \(\epsilon\) small enough such that the coefficients in 27 are negative, to obtain
\[
\frac{d}{dt}L(t) \leq -CE(t), \quad t \geq 0. \tag{28}
\]
On the other hand, by virtue of 19, the last inequality 28 takes the form
\[
\frac{d}{dt}L(t) \leq -\frac{C}{\beta_2}L(t), \quad t \geq 0. \tag{29}
\]
An integration of 29 on $(0,t)$ leads to

$$L(t) \leq K^* e^{-\delta t}, \quad t \geq 0$$

for some $K^* > 0$ where $\delta = \frac{C}{\delta_2}$. Using 19 again, we deduce the existence of $K > 0$ such that

$$E(t) \leq Ke^{-\delta t}, \quad t \geq 0.$$

**Remark 1.** Similar arguments apply if the boundary feedback contains more than one delay term, namely if the second boundary condition in 1 takes the form

$$\left(1 + b \|y_x\|^2\right) y_x(1, t)$$

$$= -\mu_0 \left(y_t(1, t) + vy_x(1, t)\right) - \sum_{i=1}^{n} \mu_i \left(y_t(1, t - \tau_i) + vy_x(1, t - \tau_i)\right), \quad t \geq 0$$

where $\mu_0, \mu_i, \tau_i, i = 1, ..., n$, are positive constants. Accordingly, the energy of the system will be

$$E(t) = \frac{1}{2} \|y_t + vy_x\|^2 + \frac{1}{2} \|y_x\|^2 + \frac{b}{4} \|y_x\|^4 + \sum_{i=1}^{n} \frac{\xi_i}{2} \|z_i\|^2, \quad t \geq 0$$

where $z_i = \left(y_t(1, t - \rho \tau_i) + vy_x(1, t - \rho \tau_i)\right)^2$ and $\xi_i, i = 1, ..., n$ are positive constants.

Assuming that $\mu_0 > \sum_{i=1}^{n} \mu_i$ and choosing $\mu_i < \tau_i^{-1} \xi_i, i = 1, ..., n$, and $\sum_{i=1}^{n} \tau_i^{-1} \xi_i < 2\mu_0 - \sum_{i=1}^{n} \mu_i$, the energy $E(t)$ is exponentially decaying to zero.

**Remark 2.** Consider the following delayed axially moving string system in the presence of a strong damping

$$\begin{cases}
y_{tt} + 2vy_{xt} - (1 - v^2 + b \|y_x\|^2)y_{xx} - \alpha(y_t + vy_x)_{xx} = 0, \quad x \in (0, 1), \quad t > 0, \\
y(0, t) = 0, \quad t \geq 0, \\
(1 + b \|y_x\|^2) y_x(1, t) = -\alpha(y_t + vy_x)(1, t) - \mu_1 \left(y_t(1, t) + vy_x(1, t)\right) \\
-\mu_2 \left(y_t(1, t - \tau) + vy_x(1, t - \tau)\right), \quad t \geq 0, \\
y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, 1), \\
y(1, t - \tau) + vy_x(1, t - \tau) = f_0(1, t - \tau), \quad t \in (0, \tau).
\end{cases}$$

(30)

It is shown in [9] for the case of a delayed wave system that, even when the coefficient of the delay is greater than the coefficient of the damping in the boundary conditions, the strong damping term forces the exponential stability of the system. This result can be generalized to the system 30. To this end, we define the energy $E(t)$ associated to 30 by

$$E(t) = \frac{1}{2} \|y_t + vy_x\|^2 + \frac{1}{2} \|y_x\|^2 + \frac{b}{4} \|y_x\|^4 + \frac{\xi}{2} \|z\|^2, \quad t \geq 0$$

with

$$\xi \geq \tau \mu_2$$

and

$$\alpha > \left(\frac{\mu_2}{2} + \frac{\xi}{2\tau} - \mu_1\right).$$

The energy $E(t)$ satisfies, along solutions of 30,

$$\frac{d}{dt} E(t) \leq -k_1 z^2(0, t) - k_2 z^2(1, t) - \alpha \|(y_t + vy_x)_x\|^2$$

$$\leq - (k_1 + \alpha) \|(y_t + vy_x)_x\|^2 - k_2 z^2(1, t), \quad t \geq 0.$$
where \( k_1 = \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{\tau} \) and \( k_2 = \frac{1}{2} \left( \frac{\xi}{\tau} - \mu_2 \right) \).

**Theorem 2.7.** Assume that \( \mu_2 > \mu_1 \) and \( \alpha > \mu_2 - \mu_1 \). Then, there exist two positive constants \( K \) and \( \delta \), independent of \( t \) such that
\[
E(t) \leq Ke^{-\delta t}, \quad t \geq 0.
\]

3. Internal feedback.

3.1. Existence result. In this section, we present an existence and uniqueness result for problem 5. In order to deal with the delay feedback term, we define the new dependent variable \( z \) by
\[
z(x, \rho, t) = y_t(x, t - \tau \rho) + v y_x(x, t - \tau \rho), \quad x \in (0, 1), \quad \rho \in (0, 1), \quad t > 0.
\]
As in the previous section, we need to construct an equation satisfied by \( z \). For this, we set \( t' = t - \tau \rho \), and notice that
\[
\frac{\partial z}{\partial \rho} = \frac{\partial z}{\partial t'} \frac{\partial t'}{\partial \rho} + \frac{\partial z}{\partial x} \frac{\partial x}{\partial \rho} = -\tau \frac{\partial z}{\partial t'} + \frac{\partial z}{\partial x} \frac{\partial v}{\partial \rho} = -\tau \frac{\partial z}{\partial t'} - \tau v \frac{\partial z}{\partial x} \tag{31}
\]
and
\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial t'} + v \frac{\partial z}{\partial x} = \frac{\partial z}{\partial t'} + v \frac{\partial z}{\partial x} \tag{32}
\]
Multiplying 32 by \( \tau \) and adding it to 31, we obtain the required equation
\[
\tau (z_t(x, \rho, t) + vz_x(x, \rho, t)) + z_{\rho}(x, \rho, t) = 0, \quad x \in (0, 1), \quad \rho \in (0, 1), \quad t > 0.
\]
So, problem 5 is equivalent to
\[
\begin{align*}
y_t + 2 vy_{xx} - (1 - v^2 + b \| y_x \|^2) y_{xx} + \mu_1 (y_t + vy_x)(t) \\
+ \mu_2 (y_t + vy_x)(t - \tau) = 0, \\
\tau (z_t(x, \rho, t) + vz_x(x, \rho, t)) + z_{\rho}(x, \rho, t) = 0, \quad x \in (0, 1), \quad \rho \in (0, 1), \quad t > 0, \\
y(0, t) = 0, \quad t \geq 0, \\
y_t(1, t) - v y_t(1, t) + vy_x(1, t)) + \left(1 + b \| y_x \|^2\right) y_x(1, t) = f_c(t), \quad t \geq 0, \\
z(x, 0, t) = y_t(x, t) + vy_x(x, t), \quad x \in (0, 1), \quad t > 0, \\
y(x, 0) = y_0(x), \quad y_t(x, 0) = y_t(x), \quad x \in (0, 1), \\
z(x, \rho, 0) = g_0(x, -\tau \rho), \quad x \in (0, 1), \quad \rho \in (0, 1). \tag{33}
\end{align*}
\]
In order to achieve our objective, we propose the following boundary control force
\[
f_c(t) = -k_f (y_t + vy_x)(1, t) - vy_{xx}(1, t), \quad t \geq 0 \tag{34}
\]
where the coefficient \( k_f \) is a positive constant.

**Definition 3.1.** Let \( T > 0 \), a couple \((y, z)\) such that
\[
y \in C([0, T), V) \cap C^1 ([0, T), L^2(0, 1)), \quad z \in C([0, T), L^2(0, 1))
\]
is called a weak solution of 33, if for any \((w, u) \in V \times L^2 ((0, 1) \times (0, 1))\)
\[
\begin{align*}
\left( \frac{\partial^2}{\partial x^2} w, u \right) = & \left( 1 + b \| y_x \|^2 \right) (y_x, w_x) - (w, \mu_1 z(x, 0, t) + \mu_2 z(x, 1, t)) \\
+ w(1) \left[ f_c(t) - y_t(1, t) + v(y_t + vy_x)(1, t) \right] = 0, \quad t \geq 0, \\
\tau (z_t(x, \rho, t) + vz_x(x, \rho, t)) + (z_{\rho}, u) = 0, \quad t \geq 0 \tag{35}
\end{align*}
\]
We shall deal with the variational problem 35. Let $T > 0$.

**Theorem 3.2.** Let $(y_0, y_1, g_0) \in V \times L^2(0, 1) \times L^2(0, 1)$. Assume that $\mu_2 \leq \mu_1$ and $k_f \geq v$. Then there exists a unique global weak solution of 33 such that for any $T > 0$

$$y \in C([0, T), V), \frac{dy}{dt} \in C([0, T), L^2(0, 1)), z \in C([0, T), L^2((0, 1) \times (0, 1)))$$

**Proof.** We shall deal with the variational problem 35. Let $\{u^i\}_{i=1}^\infty$ be a complete orthogonal system of $V \cap H^2(0, 1)$ assumed orthonormal in $L^2(0, 1)$ and let $W_m = \text{span}\{u^1, u^2, ..., u^m\}$.

For $1 \leq i \leq m$, we define the sequence $u^i(x, \rho)$ by $u^i(x, 0) = w^i(x)$. Then we may define $u^i(x, 0)$ in $L^2((0, 1) \times (0, 1))$ by $u^i(x, \rho)$ and denote $U_m = \text{span}\{u^1, u^2, ..., u^m\}$.

For each $m \in \mathbb{N}$, we look for a solution of the form

$$\begin{cases}
y^m(x, t) = \sum_{i=1}^m c^m_i(t)w^i(x), & x \in (0, 1), \ t \geq 0 \\
z^m(x, \rho, t) = \sum_{i=1}^m d^m_i(t)u^i(x, \rho), & x \in (0, 1), \ \rho \in (0, 1), \ t \geq 0
\end{cases} \quad (36)$$

verifying the approximate problem

$$\begin{cases}
\left(\frac{d^2}{dx^2}y^m, w\right) = -\left(1 + b\|y^m_x\|^2\right)(y^m_x, w_x) - (w, \mu_1 z^m(x, 0, t) + \mu_2 z^m(x, 1, t)) \\
+w(1)\left[f^m_c(t) - y^m_{tt}(1, t) + v(y^m_t + vy^m_x)(1, t)\right] = 0, \ t \geq 0, \\
\tau(z^m_t(x, \rho, t) + v z^m_x(x, \rho, t), u) + (z^m_x, u) = 0, \ t \geq 0, \\
z^m(x, 0, t) = (y^m_t(x, t) + vy^m_x(x, t)), \ t > 0, \\
y^m(0) = y^m_0, \ y^m_t(0) = y^m_1, \ z^m(0, 0) = z^m_0
\end{cases} \quad (37)$$

such that

$$\begin{cases}
y^m_0 \to y_0 \text{ strongly in } V \cap H^2(0, 1), \\
y^m_1 \to y_1 \text{ strongly in } L^2(0, 1), \\
z^m_0 \to g_0 \text{ strongly in } L^2((0, 1) \times (0, 1)).
\end{cases}$$

and

$$f^m_c(t) = -k_f(y^m_t + vy^m_x)(1, t) - vy^m_x(x, 1, t), \ t \geq 0.$$ 

The system 37 leads to a system of ODEs for the unknown functions $(c^m_i(t), d^m_i(t))$, $i = 1, ..., m$, $t_m \in (0, T)$. Based on the standard existence theory for ODE, one can ensure the existence of a local solution $(y^m, z^m)$ of 37 on a maximal time interval $[0, t_m]$, $t_m \in (0, T)$, for each $m \in \mathbb{N}$. The extension of $t_m$ to $T$ is a consequence of the *a priori* estimate below where the $(y^m, z^m)$ is uniformly bounded independent of $m$ and $t$.

**A priori estimate:** In the first equation of 37 we take $w = y^m_t + vy^m_x$. Then, for $t \in (0, t_m)$ we have

$$\frac{1}{2} \frac{d}{dt} \left(\|y^m_t + vy^m_x\|^2 + \|y^m_x\|^2 + \frac{b}{2}\|y^m\|^4\right) = -(z^m(x, 0, t), \mu_1 z^m(x, 0, t) + \mu_2 z^m(x, 1, t)) + (y^m_t + vy^m_x)(1, t) \left[f^m_c(t) - y^m_{tt}(1, t) + v(y^m_t + vy^m_x)(1, t)\right].$$
Considering the expression of $f^m(t)$ in the previous estimate, we obtain $t \in [0, t_m)$

\[
\frac{1}{2} \frac{d}{dt} \left( \|y^m_t + vy^m_x\|^2 + \|y^m_x\|^2 + \frac{b}{2} \|y^m_v\|^4 + (y^m_t(1,t) + vy^m_x(1,t))^2 \right)
\]

\[
= - (z^m(x,0,t), \mu_1 z^m(x,0,t) + \mu_2 z^m(x,1,t)) - (k_f - v) (y^m_t + vy^m_x)^2(1,t),
\]

In the second equation of 37, we set $u = z^m$ and we integrate over $(0,1)$ to get

\[
\frac{\xi}{2} \frac{d}{dt} \int_0^1 \int_0^1 (z^m)^2 \, dx \, d\rho
\]

\[
= - \frac{\xi}{\tau} \int_0^1 (z^m, z^m) \, d\rho
\]

\[
= - \frac{\xi}{2\tau} \left( \int_0^1 [z^m(x,1,t)]^2 \, dx - \int_0^1 [z^m(x,0,t)]^2 \, dx \right), \quad t \in [0, t_m)
\]

where $\xi > 0$ is to be chosen later.

Adding the resulting equations 38 and 39, for $t \in [0, t_m)$ we arrive at

\[
\frac{1}{2} \frac{d}{dt} \left( \|y^m_t + vy^m_x\|^2 + \|y^m_x\|^2 + \frac{b}{2} \|y^m_v\|^4 + \xi \int_0^1 \int_0^1 (z^m)^2 \, dx \, d\rho + (y^m_t(1,t) + vy^m_x(1,t))^2 \right)
\]

\[
= - (z^m(x,0,t), \mu_1 z^m(x,0,t) + \mu_2 z^m(x,1,t)) - (k_f - v) (y^m_t + vy^m_x)^2(1,t)
\]

\[
+ \frac{\xi}{2\tau} \int_0^1 [z^m(x,1,t)]^2 \, dx - \int_0^1 [z^m(x,0,t)]^2 \, dx.
\]

Next we use the Young inequality, the previous identity becomes

\[
\frac{1}{2} \frac{d}{dt} \left( \|y^m_t + vy^m_x\|^2 + \|y^m_x\|^2 + \frac{b}{2} \|y^m_v\|^4 + \xi \int_0^1 \int_0^1 (z^m)^2 \, dx \, d\rho + (y^m_t(1,t) + vy^m_x(1,t))^2 \right)
\]

\[
\leq \left( -\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau} \right) \int_0^1 [z^m(x,0,t)]^2 \, dx + \left( \frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) \int_0^1 [z^m(x,1,t)]^2 \, dx
\]

\[
- (k_f - v) (y^m_t + vy^m_x)^2(1,t)
\]

for $t \in [0, t_m)$. The choice $\tau \mu_2 \leq \xi \leq \tau (2\mu_1 - \mu_2)$ and the assumption $k_f \geq v$, ensure that the expression

\[
E^m(t) = \frac{1}{2} \|y^m_t + vy^m_x\|^2 + \frac{1}{2} \|y^m_x\|^2 + \frac{b}{4} \|y^m_v\|^4 + \frac{\xi}{2} \int_0^1 \int_0^1 (z^m)^2 \, dx \, d\rho
\]

\[
+ \frac{1}{2} (y^m_t(1,t) + vy^m_x(1,t))^2, \quad t \in [0, t_m)
\]

satisfies, for some, $K > 0$

\[
E^m(t) \leq E^m(0) \leq K
\]

where $K$ is independent of $m$ and $t$ as the sequences $(y^m_0)$, $(y^m_t)$ and $(z^m)$ are bounded in $V \cap H^2(0,1)$, $L^2(0,1)$ and $L^2((0,1) \times (0,1))$, respectively. This permits to deduce

\[
\begin{cases}
  y^m \text{ is uniformly bounded in } L^\infty(0,T; V), \\
  y^m_t + vy^m_x \text{ is uniformly bounded in } L^\infty(0,T; L^2(0,1)), \\
  z^m \text{ is uniformly bounded in } L^\infty(0,T; L^2((0,1) \times (0,1))), \\
  (y^m + vy^m_x)(1,t) \text{ is uniformly bounded in } L^2(0,T).
\end{cases}
\]
Passage to the limit

Consequently, there exists a subsequence of \( y^n \) satisfying

\[
\begin{align*}
\begin{cases}
y^n \rightharpoonup^* y & \text{weakly star in } L^\infty(0, T; V) \text{ and weakly in } L^2(0, T; V), \\
y^n + vy^n \rightharpoonup^* y_t + vy & \text{weakly star in } L^\infty(0, T; L^2(0, 1)) \text{ and weakly in } \\
L^2(0, T; L^2(0, 1)), \\
z^n \rightharpoonup^* z & \text{weakly star in } L^\infty(0, T; L^2((0, 1) \times (0, 1))) \text{ and weakly in } \\
L^2(0, T; L^2((0, 1) \times (0, 1))), \\
(y^n + vy^n)(1, t) \rightharpoonup (y_t + vy_x)(1, t) & \text{weakly in } L^2(0, T; L^2(0, 1)).
\end{cases}
\end{align*}
\]

Then, from Aubin–Lions theorem (see \([14]\)), for any \( T > 0 \)

\[
\begin{cases}
y^n \rightarrow y & \text{strongly in } L^2(0, T; L^2(0, 1)), \\
y^n \rightarrow y \ a.e \text{ in } (0, 1) \times (0, T).
\end{cases}
\]

Arguing analogously to the case of boundary feedback, we can pass to the limit in 37 to get a weak solution to problem (33).

**Uniqueness:** The uniqueness can be proved by a straightforward use of Gronwall’s inequality. \( \square \)

3.2. **Stability result.** We define the energy corresponding to the system (33) by

\[
E(t) = \frac{1}{2} \|y_t + vy_x\|^2 + \frac{1}{2} \|y_x\|^2 + \frac{b}{4} \|y_x\|^4 + \frac{\xi}{2} \int_0^1 \int_0^1 z^2(x, \rho, t)d\rho dx \\
+ \frac{1}{2} (y_t(1, t) + vy_x(1, t))^2, \ t \geq 0
\]

where \( \xi \) is a positive constant satisfying

\[
\tau \mu_2 < \xi < \tau (2\mu_1 - \mu_2).
\]

**Lemma 3.3.** The energy functional \( E(t) \) satisfies along solutions of 6

\[
\frac{d}{dt} E(t) \leq -k_1 \int_0^1 z^2(x, 0, t)dx - k_2 \int_0^1 z^2(x, 1, t)dx - (k_f - \nu)(y_t + vy_x)^2(1, t), \ t \geq 0.
\]

where \( k_1 = \mu_1 - \frac{\mu_2}{2} - \frac{\xi}{2\tau} \) and \( k_2 = \frac{1}{2} \left( \frac{\xi}{\tau} - \mu_2 \right) \).

**Proof.** Applying 7 to the functional \( E(t) \), it holds

\[
\begin{align*}
\frac{d}{dt} E(t) &= \int_0^1 (y_t + vy_x) (y_{tt} + 2vy_{xt} + v^2 y_{xx}) dx \\
+ \int_0^1 \left( 1 + b \|y_x\|^2 \right) y_x (y_{xt} + vy_{xx}) dx \\
+ \frac{\xi}{2} \frac{d}{dt} \int_0^1 \int_0^1 z^2(x, \rho, t)d\rho dx + (y_t + vy_x)(1, t)(y_{tt} + vy_{xt})(1, t).
\end{align*}
\]

We substitute \( y_{tt} \) and \( y_{tt}(1, t) \) (see 33) to get

\[
\begin{align*}
\frac{d}{dt} E(t) &= \left[ (y_t + vy_x) \left( 1 + b \|y_x\|^2 \right) y_x \right]_0^1 - \mu_1 (y_t + vy_x)^2 \\
&\quad - \mu_2 (y_t + vy_x)(y_t - \tau + vy_x(t - \tau)) + (y_t + vy_x)(1, t) \times \\
&\quad \left[ v(y_t(1, t) + vy_x(1, t)) - \left( 1 + b \|y_x\|^2 \right) y_x(1, t) + vy_{xt}(1, t) + f_c(t) \right] \\
&\quad + \frac{\xi}{2} \frac{d}{dt} \int_0^1 \int_0^1 z^2(x, \rho, t)d\rho dx.
\end{align*}
\]
Next, the second equation in (33) and the last term in (44) imply
\[ \frac{\xi}{2} \frac{d}{dt} \int_0^1 \int_0^1 z^2(x, \rho, t) d\rho dx = \xi \int_0^1 \int_0^1 z(x, \rho, t) (z_t(x, \rho, t) + vz_x(x, \rho, t)) d\rho dx \]
\[ = -\frac{\xi}{2\tau} \int_0^1 \int_0^1 z(x, \rho, t)z_{\rho}(x, \rho, t) d\rho dx \]
\[ = -\frac{\xi}{2\tau} \int_0^1 (z^2(x, 1, t) - z^2(x, 0, t)) dx, \quad t \geq 0. \]

Taking into account the previous relation and the expression of \( f_c(t) \) in (44) and using the boundary condition, we arrive at
\[ \frac{d}{dt} E(t) \leq -\int_0^1 z(x, 0, t) (\mu_1 z(x, 0, t) + \mu_2 z(x, 1, t)) dx \]
\[ -\frac{\xi}{2\tau} \int_0^1 (z^2(x, 1, t) - z^2(x, 0, t)) dx, \quad t \geq 0. \]

Then, the assertion 43 results directly by using the Young inequality in the first term of the previous identity.

In order to study the stability of our system, we define the modified energy functional
\[ L(t) = E(t) + \epsilon_1 (\Phi_1(t) + \Phi_3(t)) + \epsilon_2 \Phi_2(t), \quad t \geq 0 \]
where
\[ \Phi_1(t) = \int_0^1 y (y_t + vy_x) dx + y(1, t) (y_t + vy_x)(1, t), \quad t \geq 0, \]
\[ \Phi_2(t) = -\int_0^1 xy_x (y_t + vy_x) dx, \quad t \geq 0 \]
and
\[ \Phi_3(t) = \xi \int_0^1 e^{-2\rho \tau} \int_0^1 z^2(x, \rho, t) dxd\rho, \quad t \geq 0. \]
The following equivalence result between \( L(t) \) and \( E(t) \) can be easily obtained by applying Young and Poincaré inequalities.

**Proposition 2.** There exist positive constants \( \beta_i, \ i = 1, 2 \) such that for small \( \epsilon_1 \) and \( \epsilon_2 \)
\[ \beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad t \geq 0, \quad (45) \]

**Theorem 3.4.** Assume that \( \mu_2 < \mu_1 \) and \( k_f > v \). Then, there exist two positive constants \( K \) and \( \delta \) independent of \( t \) such that for \( (y; z) \) solution of problem (33), we have
\[ E(t) \leq Ke^{-\delta t}, \quad t \geq 0. \]

**Proof.** A differentiation of the functional \( \Phi_1(t) \) gives
\[ \frac{d}{dt} \Phi_1(t) = \int_0^1 z^2(x, 0, t) dx + \int_0^1 y (y_t + 2vy_{xt} + v^2y_{xx}) dx \]
\[ + y(1, t) (y_t + vy_{xt})(1, t) + y_t(1, t) (y_t + vy_x)(1, t), \quad t \geq 0. \]
Using the first equation and the second boundary condition in 33, we entail
\[ \frac{d}{dt}\Phi_1(t) = \int_0^1 z^2(x, 0, t)dx + \left(1 + b\|y_x\|^2\right) \int_0^1 yy_x dx \]
\[ - \int_0^1 y \left[ \mu_1 \left(y_t + vy_x(t)\right) + \mu_2 \left(y_t(t - \tau) + vy_x(t - \tau)\right) \right] dx \]
\[ + y(1, t) \left(v(y_t + vy_x)(1, t) - \left(1 + b\|y_x\|^2\right) y_x(1, t) + f_c(t) + vy_{xt}\right)(1, t) \]
\[ + y_t(1, t) \left(y_t + vy_x\right)(1, t), \ t \geq 0. \]

An Integration by parts and the expression of \( f_c(t) \) yield
\[ \frac{d}{dt}\Phi_1(t) = \int_0^1 z^2(x, 0, t)dx - \left(1 + b\|y_x\|^2\right) \|y_x\|^2 \]
\[ - \int_0^1 y \left[ \mu_1 \left(y_t + vy_x(t)\right) + \mu_2 \left(y_t(t - \tau) + vy_x(t - \tau)\right) \right] dx \]
\[ + (y_t + vy_x)(1, t) \left(\left|v - k_f\right| y(1, t) + y_t(1, t)\right), \ t \geq 0. \quad (46) \]

Next, we pass to estimate the different terms. Using Cauchy and Poincaré inequalities, we get for \( \eta > 0 \)
\[ - \int_0^1 y \left[ \mu_1 \left(y_t + vy_x(t)\right) + \mu_2 \left(y_t(t - \tau) + vy_x(t - \tau)\right) \right] dx \]
\[ \leq \eta (\mu_1 + \mu_2) \|y_x\|^2 + \frac{1}{4\eta} \left( \mu_2 \int_0^1 z^2(x, 1, t)dx + \mu_1 \int_0^1 z^2(x, 0, t)dx \right), \ t \geq 0 \quad (47) \]
and
\[ (y_t + vy_x)(1, t) \left(\left|v - k_f\right| y(1, t) + y_t(1, t)\right) \]
\[ \leq \eta \left(\left|v - k_f\right| \|y_x\|^2 + y_t^2(1, t)\right) + \frac{1}{4\eta} \left(\left|v - k_f\right| + 1\right) (y_t + vy_x)^2(1, t), \ t \geq 0. \quad (48) \]

Taking into account the estimates 47 and 48 in 46, the derivative of \( \Phi_1(t) \) is now estimated by
\[ \frac{d}{dt}\Phi_1(t) \leq \left(1 + \frac{\mu_1}{4\eta}\right) \int_0^1 z^2(x, 0, t)dx + \frac{\mu_2}{4\eta} \int_0^1 z^2(x, 1, t)dx \]
\[ - \left[1 + \eta (\mu_1 + \mu_2 + \left|v - k_f\right|) \|y_x\|^2 - b \|y_x\|^4 + \eta y_t^2(1, t) \right] \]
\[ + \frac{1}{4\eta} \left(\left|v - k_f\right| + 1\right) (y_t + vy_x)^2(1, t), \ \eta > 0, \ t \geq 0. \quad (49) \]

For \( \Phi_2(t) \), we have
\[ \frac{d}{dt}\Phi_2(t) = -\int_0^1 x (y_{xt} + vy_{xx})(y_t + vy_x) dx - \int_0^1 y_x (y_t + vy_x) dx \]
\[ - \int_0^1 x y_x (y_{tt} + 2v y_{xt} + v^2 y_{xx}) dx, \ t \geq 0. \]

The substitution of the second total derivative of \( y \) from (see 6) into 20, we find
\[ \frac{d}{dt}\Phi_2(t) = -\int_0^1 x (y_{xt} + vy_{xx})(y_t + vy_x) dx - \int_0^1 y_x (y_t + vy_x) dx \]
\[ - \left(1 + b \|y_x\|^2\right) \int_0^1 x y_x y_{xx} dx \]
A differentiating of \( \Phi \) and

\[
\Phi \leq -\frac{2}{3} v(t) \leq d_t L(t), \quad t \geq 0.
\]  

Next, we proceed to estimate the different terms in the right hand side of (50). Clearly,

\[
- \int_0^1 x (y_{xx} + vy_{x}) (y_t + vy_x) \, dx \\
= -\frac{1}{2} (y_t + vy_x)^2 (1) + \frac{1}{2} \int_0^1 z^2 (x, 0, t) \, dx \\
\leq -\frac{1}{2} (1 - v) y_t^2 (1) + \frac{v}{2} (1 - v) y_x^2 (1) + \frac{1}{2} \int_0^1 z^2 (x, 0, t) \, dx, \quad t \geq 0,
\]

\[
- \int_0^1 y_x (y_t + vy_x) \, dx \leq v \eta \|y_x\|^2 + \frac{v}{4 \eta} \int_0^1 z^2 (x, 0, t) \, dx, \quad t \geq 0
\]  

(51)

and

\[
- \left(1 + b \|y_x\|^2\right) \int_0^1 x y_x y_{xx} \, dx \\
= -\frac{1}{2} \left(1 + b \|y_x\|^2\right) y_t^2 (1) + \frac{1}{2} \left(1 + b \|y_x\|^2\right) \|y_x\|^2, \quad t \geq 0,
\]

(52)

\[
\int_0^1 x y_x [\mu_1 (y_t + vy_x) (t) + \mu_2 (y_t + vy_x) (t - \tau)] \, dx \\
\leq \eta (\mu_1 + \mu_2) \|y_x\|^2 + \frac{1}{4 \eta} \int_0^1 (\mu_2 z^2 (x, 1, t) + \mu_1 z^2 (x, 0, t)) \, dx, \quad t \geq 0.
\]

(53)

Considering (51-53) into (50), we get this estimate

\[
\frac{d}{dt} \Phi_2 (t) \leq \left(\frac{1}{2} + \frac{v}{4 \eta} + \frac{\mu_1}{4 \eta}\right) \int_0^1 z^2 (x, 0, t) \, dx + \frac{\mu_2}{4 \eta} \int_0^1 z^2 (x, 1, t) \, dx \\
+ \left[\eta (v + \mu_1 + \mu_2) + \frac{1}{2} \left(1 + b \|y_x\|^2\right)\right] \|y_x\|^2 \\
- \frac{1}{2} (1 - v) y_t^2 (1) - \frac{1}{2} \left[1 - v (1 - v) + b \|y_x\|^2\right] y_x^2 (1).
\]

(54)

A differentiating of \( \Phi_3 (t) \) and the second equation in (33) lead to

\[
\frac{d}{dt} \Phi_3 (t) = 2 \xi \int_0^1 e^{-2 \rho} \int_0^1 z (x, \rho, t) (z_t (x, \rho, t) + vz_x (x, \rho, t)) \, dx \, d\rho \\
= -\frac{\xi}{\tau} \int_0^1 e^{-2 \rho} \int_0^1 \frac{\partial}{\partial \rho} z^2 (x, \rho, t) \, dx \, d\rho \\
= -\frac{\xi}{\tau} \int_0^1 \left[e^{-2 \tau} z^2 (x, 1, t) - z^2 (x, 0, t)\right] \, dx - 2 \Phi_3 (t).
\]

(55)

The previous estimates (49, 54 and 55) yield

\[
\frac{d}{dt} L(t) \leq - \left\{ k_1 - \epsilon_1 \left[\left(1 + \frac{\mu_1}{4 \eta}\right) + \frac{\xi}{\tau}\right] - \epsilon_2 \left(\frac{1}{2} + \frac{\mu_1}{4 \eta} + \frac{v}{4 \eta}\right)\right\} \int_0^1 z^2 (x, 0, t) \, dx \\
- \left[ k_2 - \epsilon_1 \left(\frac{\mu_2}{4 \eta} - \frac{\xi e^{-2 \tau}}{\tau}\right) - \epsilon_2 \frac{\mu_2}{4 \eta}\right] \int_0^1 z^2 (x, 1, t) \, dx
\]
- \left\{ \epsilon_1 \left[ 1 + \eta (\mu_1 + \mu_2 + |v - k_f|) \right] - \epsilon_2 \left[ \eta (v + \mu_1 + \mu_2) + \frac{1}{2} \right] \right\} \| y_x \|^2 \\
- b \left( \epsilon_1 - \frac{\epsilon_2}{2} \right) \| y_x \|^2 - \left[ -\eta \epsilon_1 + \frac{\epsilon_2}{2} (1 - v) \right] y_t^2(1, t) \\
- \left\{ (k_f - v) - \frac{\epsilon_1}{4\eta} (|v - k_f| + 1) \right\} (y_t + vy_x)^2(1, t) \\
- \frac{\epsilon_2}{2} \left[ 1 - v(1 - v) + b \| y_x \|^2 \right] y_x^2(1) - 2\Phi_3(t). \tag{56}

At first, we take \( \epsilon_1 = \epsilon_2 \) and \( \eta < \min \left\{ \frac{1}{2} (1 - v), 1/ [2 (2\mu_1 + 2\mu_2 + k_f)] \right\} \). Then, we choose \( \epsilon_2 \) small enough such that the coefficients in 56 are negative to obtain

\[
\frac{d}{dt} L(t) \leq -CE(t), \quad t \geq 0.
\]

On the other hand, by virtue of 45, the last inequality implies

\[
\frac{d}{dt} L(t) \leq -\frac{C}{\beta^2} L(t), \quad t \geq 0.
\]

An integration of 29 on \((0, t)\) leads to

\[
L(t) \leq K e^{-\delta t}, \quad t \geq 0.
\]

where \( \delta = \frac{C}{\beta^2} \). Using 19 again, we deduce the existence of \( K > 0 \) such that

\[
E(t) \leq Ke^{-\delta t}, \quad t \geq 0.
\]

\[\square\]

Remark 3. Similar arguments apply if the boundary feedback contains more than one delay term in the internal feedback, namely if the first equation in 1 takes the form

\[
y_{tt} + 2vy_{xt} - (1 - v^2 + b \| y_x \|^2)y_{xx} + \mu_0 \left( y_t(t) + vy_x(t) \right) + \sum_{i=1}^n \mu_i (y_t(t - \tau_i) + vy_x(t - \tau_i)) = 0
\]

where \( \mu_0, \mu_i, \tau_i, i = 1, \ldots, n \), are positive constants. The energy is

\[
E(t) = \frac{1}{2} \| y_t + vy_x \|^2 + \frac{1}{2} \| y_x \|^2 + \frac{b}{4} \| y_x \|^4 + \sum_{i=1}^n \frac{\xi_i}{2} \int_0^1 \int_0^1 z_i^2(x, \rho, t) d\rho dx \\
+ \frac{1}{2} (y_t(1, t) + vy_x(1, t))^2, \quad t \geq 0
\]

where \( z_i = \left( y_t(t - \rho \tau_i) + vy_x(t - \rho \tau_i) \right)^2 \) and \( \xi_i, i = 1, \ldots, n \) are positive constants.

Assuming that \( \mu_0 > \sum_{i=1}^n \mu_i \) and choosing \( \mu_i < \tau_i^{-1} \xi_i \), \( i = 1, \ldots, n \), and \( \sum_{i=1}^n \tau_i^{-1} \xi_i < 2\mu_0 - \sum_{i=1}^n \mu_i \), the energy \( E(t) \) is exponentially decaying to zero.

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