Quantum times of arrival for multiparticle states

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Using the concept of crossing state and the formalism of second quantization, we propose a prescription for computing the density of arrivals of particles for multiparticle states, both in the free and the interacting case. The densities thus computed are positive, covariant in time for time independent hamiltonians, normalized to the total number of arrivals, and related to the flux. We investigate the behaviour of this prescriptions for bosons and fermions, finding boson enhancement and fermion depletion of arrivals.

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I. INTRODUCTION

A long standing issue in the theory and experiment of quantum mechanics has been that of measuring and formalizing time observables. In the last two decades a substantial body of work has been produced clarifying theoretically and measuring experimentally quantities such as dwell times [1], tunneling times, or arrival times [2]. In particular, many recent papers have challenged the classical work of Allcock, who denied the possibility of defining a quantum arrival-time concept [3, 4, 5]. In fact, these theoretical efforts and difficulties concerning arrival times have been essentially decoupled from the daily practice of many laboratories, where time-of-flight (TOF) methods are routinely used. One reason for such a divorce is that, in most cases, a classical analysis of the translational motion and the associated arrival-time distribution is sufficient. It is now the case, however, that the development of laser cooling techniques is bringing the quantum nature of the atomic dynamics to the fore, thus approaching the conditions for testing several proposed time-of-arrival (TOA) theoretical distributions in a regime that differs from the classical approximation.

Yet another difficulty for a comparison and further interaction between experiment and theory is the absence, up to now, of a TOA theory for multiparticle systems. While the possibility of detecting individual atoms with nanosecond time resolution in specific TOF experiments is open [6], in the generic case the TOF spectra are produced by clouds of many particles that may interact with each other or/and with an external field. The aim of this paper xis to provide a quantum TOA theory which is applicable for the generic (one dimensional) multiparticle case using the formalism of second quantization (see for example [7]), together with the crossing states introduced in [8] and developed further in [9]. We shall also portray several numerical examples to illustrate the phenomena of boson enhancement and fermion depletion of common arrivals.

II. TIME OF ARRIVAL OF A SINGLE PARTICLE

One of the major hindrances to the consideration of time observables in the framework of standard quantum mechanics was Pauli’s theorem, which, simply put, states that no self-adjoint operator can exist that has canonical commutation relations with a self-adjoint bounded or semibounded hamiltonian, thus implying that the standard recipe associating self-adjoint operators to observables cannot work for time.

Nonetheless, Aharonov and Bohm considered the motion of free particles as a clock to measure time, and introduced a time operator by symmetrizing the classical expression for the time when a particle, initially at the origin and with momentum \( p \), passes point \( x \), that is, \( t = mx/p \). With a sign change this becomes the time of arrival at the origin of a free particle that, at time \( t = 0 \), is at position \( x \) with momentum \( p \) [10, 11]. If the arrival occurs at \( X \) rather than at the origin, then the corresponding “Aharonov-Bohm time-of-arrival operator” takes the form

\[
\hat{T}_{AB}(X) = \frac{m}{2} \left[ \frac{1}{\hat{p}} (X - \hat{x}) \right].
\]

For all practical purposes, this expression fulfills all the properties one would expect of an operator associated with the observable quantity time-of-arrival for free particles on the line [12] (for more details on this and the following topics, see [3]). It cannot be applied onto states with non vanishing zero momentum, which has been at times regarded as a drawback [13, 14]. In fact, this “difficulty” is perfectly physical, and mirrors the classical divergence of the time...
of arrival when the particle’s momentum tends to zero. It also explains how Pauli’s theorem can be circumvented: Aharonov and Bohm’s time operator $T_{AB}$ is a maximally symmetric operator, therefore not self-adjoint. Other steps had to be taken before maximally symmetric operators and their concomitant POVMs (positive operator valued measures or generalized non-orthogonal resolutions of the identity) were understood physically, however.

Although the faith in Pauli’s theorem could have been slightly shaken by Aharonov and Bohm’s proposal, the issue seemed to be settled after the important series of papers of Alcock [3, 4, 5], which apparently put to rest all hope to obtain a sensible prescription for the quantum prediction of times of arrival. Even so, some adventurous souls kept on searching for alternative formulations within quantum mechanics. Kijowski in 1974 [15] put forward a procedure to compute time-of-arrival probability densities for the free particle case in a purely axiomatic way (see also similar later work by Werner [16]).

With the advent of a better understanding of positive operator valued measures (also known as generalized decompositions of the identity or non-orthogonal measurements) [17, 18, 19, 20], the force of Pauli’s argument was strongly diminished. In fact, it has been possible to show the relation between Aharonov and Bohm’s time-of-arrival operator and Kijowski’s distribution: they follow naturally one from each other [21, 22]. This also suggests a rewriting of the distribution in terms of an operator for the density of arrivals at point $X$.

The fact that $T_{AB}$ is not a self-adjoint operator is clearly identified from the non-orthonality of the complete basis $\{|t, \alpha\}_t,\alpha=\pm$. The eigenvectors are related to each other by means of the relation

$$|t, \alpha\rangle = e^{iH(t-t')/\hbar}|t', \alpha\rangle$$

which assures the invariance of the distribution with respect to time translations. In particular,

$$|t, \alpha\rangle = e^{iHt/\hbar}|v_{\alpha}\rangle,$$

where we have used a special notation for the $t=0$ (generalized) eigenvectors or “crossing states”, $|v_{\alpha}\rangle = |t=0, \alpha\rangle$, where again $\alpha$ stands for either + or − (we will not be denoting explicitly the point of arrival $X$, which is part of the definition of these states, but it is always implied).

In terms of these states we may rewrite Kijowski’s distribution of times of arrival at the point $X$ as

$$\Pi(t, X) = \sum_{\alpha=\pm} |\langle \psi(t)|v_{\alpha}\rangle|^2.$$  

(5)

This also suggests a rewriting of the distribution in terms of an operator for the density of arrivals at point $X$, $\hat{\pi}(X) \equiv \sum_{\alpha} |v_{\alpha}\rangle\langle v_{\alpha}|$,

$$\Pi(t, X) = \langle \psi(t)|\hat{\pi}(X)|\psi(t)\rangle.$$  

(6)

Consider now the explicit form of the states $|v_{\alpha}\rangle$ in momentum representation,

$$\langle p|v_{\alpha}\rangle = \left(\frac{\alpha p}{\hbar m}\right)^{1/2} \Theta(\alpha p)e^{-ipX/\hbar},$$

(7)

where $\Theta(\cdot)$ is Heaviside’s unit step function. The correct correspondence of $\Pi(t, X)$ with the classical case becomes now evident. If the non commutativity of position and momentum operators could be neglected, $\hat{\pi}$ would correspond to the sum of the moduli of the fluxes that cross $X$ from both sides. Also important is the fact that in a classical setting the corresponding dynamical variable provides the arrival distribution irrespective of the dynamics and interaction potentials. In other words, Eq. (4) generalizes the free motion case in a natural and simple way for arbitrary interaction potentials, a task that could not be carried out using the original axiomatic procedure of Kijowski or by quantizing the classical time of arrival for each particular potential (the expressions are not analytically known in general and pose formidable ordering problems).
Underlying this rewriting of the time-of-arrival distribution a change of emphasis is to be found: whereas in Eq. (3) the time-of-arrival distribution is obtained from the overlap of the initial wavefunction with the states associated with arrival at the instant $t$, be it from the left ($\alpha = +1$) or the right ($\alpha = -1$), in Eq. (4) it is obtained as the overlap of the evolved wavefunction with the constant states $|v_n\rangle$ that measure arrivals. The first point of view is, in a way, predictive: given the initial state of the particle, one can predict when the arrivals will occur. In the general case, with interacting potentials, this view may also be adopted with $|t, \alpha\rangle$ given by Eq. (4), where the appropriate Hamiltonian is put in each case. From the second perspective, which could be termed “unconditional”, the arrival or otherwise of a particle at $x = 0$ is directly measured in physical space at every instant, using local definitions that are in no way conditioned by the different potentials in which the particles might be moving. This point of view, inspired by Wigner’s formalization of the time-energy uncertainty relation [23], was advocated in [8, 9], where the properties of the crossing states $|v_n\rangle$ were examined, and Eq. (3) was put forward as an expression of density of arrivals also for the case of interaction. In [8] we rewrote some other distributions that had been proposed in the literature for time-of-arrival distributions of particles in a potential (24), later superseded by (25); see also (26) in terms of crossing states, and showed that those defined in Eq. (3) were the only ones considered that led to classical correspondence with the properties expected of such distributions.

Another particularly relevant aspect of the change of emphasis is that it helps to understand that Eq. (3) need no longer be normalized to unity. In which case $\Pi(t, X)$ is to be understood as a density of arrivals of one particle: there might be a non zero probability for the particle never arriving at $X$, or, if the interacting potential were confining (such as the harmonic oscillator), recurrences would appear corresponding to many different arrivals. Notice that $\Pi(t, X)$ is a density of arrivals, not of first arrivals only.

III. SECOND QUANTIZATION AND TIME OF ARRIVAL

Even though TOF experiments with single atoms might be available in not too distant a future, we need to understand better how to predict time-of-arrival distributions for multiparticle systems. Most suited for such a purpose is the formalism of second quantization. One must first realize that the distribution of arrivals is a property of the same nature as the current density, or the kinetic energy, namely, it is obtained as the sum of “single particle” contributions, irrespective of the external or internal interactions affecting the $N$-particle system. This is a key observation to discard outright, even for free motion, quantizations that would provide two-particle terms.

Let $\hat{a}_p$ and $\hat{a}_p^\dagger$ represent the annihilation and creation operators that respectively eliminate and create a plane wave of momentum $p$. Similarly, $\hat{\psi}(x)$ and $\hat{\psi}^\dagger(x)$ act on the vacuum disposing of and creating a particle at point $x$. The canonical commutation relations read

$$[\hat{a}_p, \hat{a}_q]^\pm = \delta(p - q), \quad \text{and} \quad [\hat{\psi}(x), \hat{\psi}^\dagger(y)]^\pm = \delta(x - y),$$

where, as usual, $[,]^\pm$ stands for the commutator in the case of bosons and for the anticommutator when fermions are involved. The position operator is written as

$$\hat{x} = \int_{-\infty}^{+\infty} dx \, \hat{\psi}(x) \hat{\psi}^\dagger(x),$$

and the inverse of the momentum operator as

$$\hat{p}^{-1} = \int_{-\infty}^{+\infty} dp \, \frac{1}{p} \hat{a}_p^\dagger \hat{a}_p,$$

from which the following form for a generalization of the time-of-arrival operator of Aharonov and Bohm might be inferred ($X = 0$):

$$\hat{T}_{AB}^{(1)} = -\frac{m}{2} (\hat{p}^{-1} + \hat{p}^{-1}\hat{x}) =$$

$$= -\frac{m}{4\pi\hbar} \int_{-\infty}^{+\infty} dx \, dp \, dq \, dr \, \frac{x e^{i(r-q)x/\hbar}}{p} \left[ (\delta(p - q) + \delta(p - r)) \hat{a}_q^\dagger \hat{a}_r + 2\hat{a}_q^\dagger \hat{a}_r \hat{a}_p \right].$$

In this expression one can recognize a one-particle component but also a two-particle one. As pointed out above, this leads us to discard this procedure, because of its unphysicality.

At any rate, $\hat{T}_{AB}$ is only valid for the free particle case, a further limitation of this route. The proper quantization procedure for the multiparticle case starts, as noted above, by recognizing the additive character of the time of arrival in terms of single particle contributions.
The basic trick is that for additive quantities taking the form of a sum of single particle operators,

\[ \hat{G} = \hat{g}_1 + \hat{g}_2 + \ldots + \hat{g}_N, \]

each of which has matrix elements \( g_{ji} = \langle j|\hat{g}|i \rangle \) in a complete (single particle) basis, the multiparticle operator in second quantized form is given by the simple expression

\[ \hat{G} = \sum_{ij} g_{ji} a_j^\dagger a_i, \]

where \( a_i \) and \( a_j^\dagger \) are the i-th annihilation and j-th creation operators. That is, they connect states \( |i\rangle \) and \( |j\rangle \) respectively with the vacuum state \( (a_i|i\rangle = |0\rangle \) and \( |j\rangle = a_j^\dagger |0\rangle \).

In the case of the arrival density operator we can directly apply this procedure in momentum representation. An even more compact expression is obtained by using the crossing states, to generate crossing operators, both annihilation and creation. Consider any (generalized) one particle state \( |\varphi\rangle \). We can write the annihilation and creation operators associated with the state as

\[ \hat{\varphi} = \int_{-\infty}^{\infty} dp \langle \varphi|p\rangle \hat{a}_p; \quad \hat{\varphi}^\dagger = \int_{-\infty}^{\infty} dp \langle p|\varphi\rangle \hat{a}_p^\dagger. \]

On applying this procedure to the crossing states, we obtain the crossing operators

\[ \hat{\varphi}_\alpha(X) = \int_{-\infty}^{\infty} dp \langle v_\alpha|p\rangle \hat{a}_p = \int_{-\infty}^{\infty} dp \left( \frac{\alpha p}{\hbar m} \right)^{1/2} \Theta(\alpha p) e^{ipX/\hbar} \hat{a}_p; \]

\[ \hat{\varphi}_\alpha^\dagger(X) = \int_{-\infty}^{\infty} dp \langle p|v_\alpha\rangle \hat{a}_p^\dagger = \int_{-\infty}^{\infty} dp \left( \frac{\alpha p}{\hbar m} \right)^{1/2} \Theta(\alpha p) e^{-ipX/\hbar} \hat{a}_p^\dagger. \]

Let us now put together Eqs. (12) and (13) with Eq. (11) to write the arrival density operator \( \hat{\Pi}(X) \) for arrivals at \( X \) in second quantized form,

\[ \hat{\Pi}(X) = \sum_{\alpha = \pm} \hat{\varphi}_\alpha^\dagger(X) \hat{\varphi}_\alpha(X) = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \frac{\sqrt{pq}}{\hbar m} \Theta(pq) e^{i(q-p)X/\hbar} \hat{a}_p^\dagger \hat{a}_q. \]

The left and right arrivals density operators \( \hat{\Pi}_L(X) \) and \( \hat{\Pi}_R(X) \) are similarly defined as

\[ \hat{\Pi}_\alpha(X) = \hat{\varphi}_\alpha^\dagger(X) \hat{\varphi}_\alpha(X). \]

We may also write from Eq. (14) the corresponding operator in Heisenberg picture, whose expectation value over the initial state will give us the density of arrivals at point \( X \) at instant \( t \),

\[ \hat{\Pi}(t, X) = \hat{U}^\dagger(t) \hat{\Pi}(X) \hat{U}(t) = \sum_{\alpha = \pm} \hat{\varphi}_\alpha^\dagger(X, t) \hat{\varphi}_\alpha(X, t) \]

\[ = \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \frac{\sqrt{pq}}{\hbar m} \Theta(pq) e^{i(q-p)X/\hbar} \hat{a}_p^\dagger(t) \hat{a}_q(t), \]

where \( \hat{a}_p^\dagger(t) \) and \( \hat{a}_q(t) \) are the time evolved creation and annihilation operators, with evolution operator \( \hat{U}(t) \), i.e.

\[ \hat{a}_p^\dagger(t) = \hat{U}^\dagger(t) \hat{a}_p \hat{U}(t) \]
as similarly \( \hat{a}_p(t) = \hat{U}^\dagger(t) \hat{a}_p \hat{U}(t) \).

The density of arrivals at instant \( t \) at point \( X \) for a generic state \( |\psi\rangle \) may thus be written as

\[ \Pi(t, X; \psi) = \langle \psi(0)|\hat{\Pi}(t, X)|\psi(0)\rangle = \langle \psi(t)|\hat{\Pi}(X)|\psi(t)\rangle. \]

This expression agrees with Eq. (1) whenever \( |\psi\rangle \) is a one particle state. Even though it is not immediately apparent from expression (13) that we are obtaining positive semidefinite distributions, this is indeed the case by construction: \( \hat{\Pi}(t, X) \) is a positive operator because it is a sum of two terms of the form \( \hat{A}^\dagger \hat{A} \).

Furthermore, the one particle operator that is an extension of Aharonov and Bohm’s time-of-arrival operator at position \( X \) for many particles, even in the interacting case, is straightforwardly written as

\[ \hat{T}_X = \int_{-\infty}^{+\infty} dt \hat{\Pi}(t, X) = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq \frac{\sqrt{pq}}{\hbar m} \Theta(pq) e^{i(q-p)X/\hbar} t \hat{a}_p^\dagger(t) \hat{a}_q(t). \]
By construction this is simply a one particle operator, which coincides with $\hat{T}_{AB}$ over states whose content is just one free particle.

If the evolution of the system is governed by the free particle Hamiltonian it is easy to check that the integral over time of the arrival-density operator $\hat{\Pi}^{\text{free}}(t, X)$ sums to the total particle number operator,

$$\int_{-\infty}^{+\infty}dt \hat{\Pi}^{\text{free}}(t, X) = \int_{-\infty}^{+\infty} dp \hat{a}_p^\dagger \hat{a}_p = \hat{N}.$$ 

This is no longer the case whenever the evolution operator is not the free one; anyhow, we deduce from this expression that the arrival density is normalized to the total number of arrivals. Notice that in the interacting case the total number of arrivals need not coincide with the total particle number, it may be smaller or bigger.

A particularly important property of the arrival-density operator is that the density of times of arrival, they are minimal requirements, the lack of which would seriously impair any proposal.

For the sake of completeness, let us note down the flux operator for many particles, in Schrödinger’s picture,

$$\hat{j}(X) = \frac{-i\hbar}{2m} \left\{ \psi^\dagger(X) \partial_X \hat{\psi}(X) - \left[ \partial_X \psi^\dagger(X) \right] \hat{\psi}(X) \right\} = \frac{1}{2\hbar m} \int_{-\infty}^{\infty} dp dq e^{i(q-p)X/\hbar} (p + q) \hat{a}_p^\dagger \hat{a}_q. \quad (17)$$

or in Heisenberg’s picture as

$$\hat{j}(t, X) = \hat{U}(t)\hat{j}(X)\hat{U}(t) = \frac{1}{2\hbar m} \int_{-\infty}^{\infty} dp dq e^{i(q-p)X/\hbar} (p + q) \hat{a}_p^\dagger(t)\hat{a}_q(t), \quad (18)$$

(again assuming that the Hamiltonian is independent of time). Notice that the flux, defined in this standard manner, is a one-particle operator.

A straightforward comparison of Eqs. (16) and (18) reveals the differences and similarities between $\hat{\Pi}$ and the flux. In the former a geometric mean of the momenta takes the place of the arithmetic mean in the latter. Moreover, $\Pi$ counts the case when $p$ and $q$ are both negative as a positive contribution to the arrival density, whereas the same case counts as a negative flux contribution in (18). This means that the quantity that tends classically to the flux is $\Pi_+ - \Pi_-$ rather than $\Pi$ itself.

IV. FREE PARTICLES: BOSON ENHANCEMENT AND FERMION DEPLETION

We have already made out several properties of the proposed arrival-density operator, namely positivity, covariance, one-particle status, classical limit, and normalization to total number of arrivals. There is an obvious missing element yet, in that we have not investigated so far whether the fermionic or bosonic character of the particles involved is somehow reflected in the properties of the distributions of times of arrival, as is to be expected.

In fact, this distinction between fermions and bosons is already present in the proposed arrival-density distributions, as we will be showing in this section. In order to portray this new property it is enough to consider simply two-particle states, of generic form

$$|\psi\rangle = \int_{-\infty}^{\infty} dp_1 dp_2 \psi(p_1, p_2)|p_1, p_2\rangle = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} dp_1 dp_2 \psi(p_1, p_2) \hat{a}_p^\dagger \hat{a}_q^\dagger |0\rangle,$$

both for bosons and fermions, where $|0\rangle$ is the vacuum state, and the normalization condition reads

$$\frac{1}{2} \int_{-\infty}^{\infty} dp_1 dp_2 \left[ \psi(p_1, p_2) \pm \psi(p_2, p_1) \right] \psi(p_1, p_2) = 1,$$
where the upper sign corresponds to bosons and the lower one to fermions. Consider $\psi(p_1, p_2)$ given as

$$
\psi_{\pm}(p_1, p_2) = \frac{1}{\sqrt{2(1 \pm |\langle \chi_a | \chi_b \rangle|^2)}} \left[ \chi_a(p_1)\chi_b(p_2) \pm \chi_a(p_2)\chi_b(p_1) \right],
$$

which fulfills the normalization requirement if $\chi_a$ and $\chi_b$ are normalized one particle wavefunctions. Quite obviously, $\langle \chi_a | \chi_b \rangle$ stands for $\int dp \bar{\chi}_a(p)\chi_b(p)$. In order to compare with the case of distinguishable particles, we shall also be using

$$
\psi_d(p_1, p_2) = \chi_a(p_1)\chi_b(p_2).
$$

Since the arrival-density operator $\hat{\Pi}(t, X)$ is a one-particle operator, the density of arrivals over the state $|\psi_+\rangle$, say, can be reorganized as

$$
\langle \psi_+(0)|\hat{\Pi}(t, X)|\psi_+(0)\rangle = \frac{1}{N^2_+} \sum_{i,j=a,b} \langle \chi_j | \chi_i \rangle \Pi_{ij}(t, X),
$$

where $N_+ = \sqrt{2(1 + |\langle \chi_a | \chi_b \rangle|^2)}$, and

$$
\Pi_{ij}(t, X) = \sum_{\alpha=\pm} \langle \chi_i | \hat{\gamma}^\dagger_{\alpha}(X, t) \hat{\alpha}_\alpha(X, t) | \chi_j \rangle.
$$

Over the fermionic state $|\psi_-\rangle$ the cross terms carry a negative sign in front. The evolved crossing states are given by Eqs. (12) and (13) on substituting $\hat{\alpha}_\alpha$ and $\hat{\gamma}_\alpha$ by $\hat{\alpha}_\alpha(t)$ and $\hat{\gamma}_\alpha(t)$, respectively.

On the other hand, the evaluation of the expectation value of the evolved arrival density operator over the state $|\psi_d\rangle$, which computes the density of arrivals for two distinguishable particles in such a state, produces just the two diagonal terms, i.e.

$$
\langle \psi_d(0)|\hat{\Pi}(t, x)|\psi_d(0)\rangle = \Pi_{aa}(t, x) + \Pi_{bb}(t, x).
$$

It should be observed that these computations are general in that they hold true for the case of interacting particles as well, as long as the states have the form given above. These results indicate that fermions and bosons (antisymmetric and symmetric states) present cross terms in the density of arrivals completely analogous to those that in spatial density signal the statistics of the particles. In fact, the formalism of second quantization carries in itself the fermionic or bosonic character of the particles concerned, through the commutation relations.

As a consistency check one may compute for the above states, $|\psi_{\pm}\rangle$ and $|\psi_d\rangle$, the corresponding reduced one particle density operators $\hat{\rho}^{(j)}$, $j = 1, 2$ ($\rho^{(1)} = \rho^{(2)}$ for $|\psi_{\pm}\rangle$) and note that in all three cases $\Pi(t, X) = \sum_j \Pi^{(j)}(t, X)$, where

$$
\Pi^{(j)}(t, X) = \text{Tr}_j \left[ \hat{\Pi}(t, X)\hat{\rho}^{(j)} \right].
$$

in agreement with the one-particle character of the arrival-time distribution.

The difference between bosons, fermions and distinguishable particles ($|\psi_d\rangle$) is quite apparent in Fig. 1. The one particle states $\chi_a$ and $\chi_b$ are gaussians with a spatial separation between them (in atomic units, 3.5), while their width is 1 (a.u.). Correspondingly, there are two main arrival times (maxima of the arrival densities) for all three cases. Even so, the two sets of principal arrivals for bosons (symmetric state) are much closer together and much less differentiated than for distinguishable particles ($|\psi_d\rangle$), which in turn present closer and less differentiated maxima when compared to the fermionic (antisymmetric) case. It should be noticed that in this situation of free motion the distributions are normalized to 2, as can be readily checked in this numerical simulation.

V. INTERACTING PARTICLES

Consider now a pair of interacting particles, be they distinguishable, bosonic or fermionic, moving in otherwise free space. The two-particle subspace of Fock space can be rewritten in center of mass and relative coordinates, and we shall consider for simplicity factorized states of the form

$$
\psi(p_1, p_2) = \chi(P)\phi(p),
$$
FIG. 1: The solid line corresponds to distinguishable particles (|ψd⟩), the dots show the density of arrivals for bosons (symmetric state |ψ+⟩), and the dashed lines that of fermions (antisymmetric state |ψ−⟩). The states are defined by Eqs. 19 and 20, with χa and χb gaussian states with minimum uncertainty at t = 0, their central positions being ⟨χa|ˆx|χa⟩ = −3.5 and ⟨χb|ˆx|χb⟩ = 0. In both cases, their central positions in momentum space are at 3, and their spatial widths ∆x = 1 (where ∆x is the square root of the spatial variance). The point of observation is X = 3, and the mass m = 1. All magnitudes are expressed in atomic units.

where P = p1 + p2 is the center of mass momentum and p = (p1 − p2)/2 the relative one. Under exchange of the particles P is unchanged, while p flips sign. So in order to ensure that the state is bosonic we are forced to use even functions φ+(p) = φ+(−p), whereas the fermionic case demands odd functions φ−(p) = −φ−(−p). The total mass is 2m, while the reduced mass µ pertaining to the relative system is m/2. The normalization condition is translated into the requirement that φ± and χ(P) be normalized to unity.

In what follows we shall assume that the center of mass function is gaussian with minimum uncertainty product at t = 0. As to the internal states, they will be evolving in a harmonic oscillator potential. We shall consider stationary and coherent internal states.

Figs. 2 and 3 represent two different sets of cases concerning internal stationary states. The ground state and the even excited states are symmetric (bosonic), whereas the odd numbered excited states are antisymmetric (fermionic). The differences between Figs. 2 and 3 are due to the different ratios between internal energy and that of the center of mass motion. In Fig. 2 the internal oscillations are much slower than the center of mass motion, so the humps of the internal spatial wavefunction appear, somewhat distorted for later times because of the spreading, in the arrival density. However those humps are smoothed over in Fig. 3 due to the much slower center of mass motion relative to the internal motion. Correspondingly, the integral of the curves in Fig. 2 is very nearly 2, whereas there is a significant increase of this number in Fig. 3 for the excited states. The higher the excitation the broader the state is, spatially, thus leading to more crossings.

We have also studied a case where the internal motion is time dependent. If it is fast enough with respect to the translational motion, a peak structure corresponding to several oscillations may be observed. Let us consider, at time t = 0, symmetric and antisymmetric combinations of coherent states, of the form (remember that the coherent state |z⟩ is given by exp(−|z|^2/2)Σn=0(2n)!zn/√n)|n⟩, where |n⟩ is the n-th excited state of the harmonic oscillator hamiltonian)

$$|φ±⟩ = \frac{1}{\sqrt{2}} [ |z⟩ ± |\bar{z}⟩] ,$$

where z is the complex conjugate of z. We shall take in particular z = i. In the relative motion space |i⟩ is a minimum uncertainty product gaussian centered at the origin with average momentum (2µωh)^1/2 and spatial variance h/(2ωm). As time progresses it oscillates back and forth along the relative motion coordinate with period T = 2π/ω.

In Fig. 4 the translational motion is faster than the oscillations so we see just one peak for the symmetric case and two maxima for the antisymmetric (fermionic) case. Since there is hardly any component of negative momentum,
FIG. 2: Arrival density of two identical particles in an internal stationary state of the harmonic oscillator. The center of mass state is gaussian, initially centered on \( x = 0 \) and \( p = 4 \) with width \( \Delta x = 0.5 \). Circles correspond to the (internal) ground state, dashes to the first excited state, solid line to the second excited state, dot-dash to the third one. The internal frequency is \( \omega = \sqrt{0.02} \) and the oscillation period is \( T \approx 44.4 \). The point of crossing is \( x = 3 \). All magnitudes in atomic units.

FIG. 3: As in figure 2, with the center of mass central momentum changed to 1, \( \Delta x = 1 \) and the internal frequency to \( \omega = \sqrt{2} \). \( T \approx 4.4 \).

there is no distinction between flux and density of arrivals.

In contrast to Fig. 2, the internal potential is much stronger in the situation depicted in Fig. 3. As opposed to the case of stationary internal states, the tighter binding produces here oscillations that can be clearly seen.
FIG. 4: Arrivals density (solid line for fermions and dashed line for bosons) and flux (triangles for fermions and circles for bosons) for the internal states defined in Eq. (21), with $z = i$ and internal frequency $\omega = \sqrt{0.02}$. The initial center of mass state is a gaussian with central position at $x = 0$ and central momentum $p = 4$. The width is $\Delta x = 0.5$. The point of arrival is $X = 3$. All magnitudes in atomic units.

FIG. 5: Arrivals density (solid line for fermions and dashed line for bosons) for the internal states defined in Eq. (21), with $z = i$ and internal frequency $\omega = \sqrt{2}$. The initial center of mass state is a gaussian with central position at $x = 0$, central momentum $p = 1$, and spatial width $\Delta x = 1$. The point of arrival is $X = 3$. All magnitudes in atomic units.

VI. DISCUSSION

In this work we have proposed a general method for computing densities of arrivals (and related arrival density operators) for multiparticle states, that fulfill a number of quite sensible demands: positivity, covariance (if the evolution is homogeneous in time), related to a one particle operator, normalized to the total number of particles in the free case, related to the flux, and consistent with the classical arrival density. The analysis of the density of arrivals (i.e., the one point function of the arrival density operator) shows consistency with the results one would
expect for bosons, fermions, and distinguishable particles. Numerical computations also show the behaviour expected, both for the free and the interacting case, and reveal a number of physical effects, hitherto unexplored. The proposed distribution is also applicable to the case of external interaction potentials as shown already for the single particle states.

The fact that the arrival-density operator is a one particle operator implies that in fact no distinction is made for the one point function (the density of arrivals) between the bosonic case and the symmetric states of distinguishable particles. The full difference will be seen in two-point and higher order functions, such as the arrival - arrival correlation function. In fact, on computing this two point correlation function, one sees immediately that the one particle component of the two point operator behaves in the natural way and can be substracted from the arrival - arrival correlation function to give the correlation function of pairs of arrivals, \( \langle \psi : \Pi(t) \Pi(0) : \psi \rangle \), where : : stands for normal ordering. The full analysis of this object we will leave for future work.

In this paper we have not been overly concerned with domain problems and the like. We know that the one particle time operator in the free case is only maximally symmetric and cannot be made self-adjoint, and we have no reason to expect that the interacting multiparticle case will be simpler in this respect. Nonetheless, this is not parcticularly relevant for our main interest, which lies in the computation of densities of arrival. Notice furthermore that the fact that Aharonov and Bohm’s operator is not self-adjoint is really no hindrance to a full quantum mechanical analysis of its associated densities.

There might be many other alternative prescriptions for times of arrival. In the present state of knowledge, we do not feel able to discard those outright. However, by pushing to the multiparticle case the definitions used for one particle, this analysis becomes more amenable to experimental test.

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[1] J. G. Muga, in Time in Quantum Mechanics, edited by J. G. Muga, R. Sala-Mayato, and I. L. Egusquiza (Springer Verlag, 2001), to appear, quant-ph/0105081.
[2] J. G. Muga and C. R. Leavens, Phys. Rep. 338, 353 (2000).
[3] G. R. Allcock, Ann. Phys. (N.Y.) 53, 253 (1969).
[4] G. R. Allcock, Ann. Phys. (N.Y.) 53, 286 (1969).
[5] G. R. Allcock, Ann. Phys. (N.Y.) 53, 311 (1969).
[6] A. Robert, O. Sirjean, A. Browaeys, J. Poupard, S. Novak, D. Boiron, C. I. Westbrook, and A. Aspect, Science 292, 461 (2001).
[7] G. Baym, Lectures on Quantum Mechanics (W. A. Benjamin, Reading, Massachusetts, 1974).
[8] A. D. Baute, I. L. Egusquiza, J. G. Muga, and R. Sala Mayato, Phys. Rev. A 61, 052111 (2000), quant-ph/9911088.
[9] A. D. Baute, I. L. Egusquiza, and J. G. Muga, Phys. Rev. A 64, 012501 (2001), quant-ph/0102005.
[10] J. G. Muga, R. Sala Mayato, and J. P. Palao, Superlattices Microstruct. 23, 833 (1998), quant-ph/9801043.
[11] J. G. Muga, C. R. Leavens, and J. P. Palao, Phys. Rev. A 58, 4336 (1998), quant-ph/9807066.
[12] Y. Aharonov and D. Bohm, Phys. Rev. 122, 1649 (1961).
[13] N. Grot, C. Rovelli, and R. S. Tate, Phys. Rev. A 54, 4676 (1996), quant-ph/9603021.
[14] J. S. Briggs and J. M. Rost, Found. Phys. 31, 693 (2001).
[15] J. Kijowski, Rept. Math. Phys. 6, 361 (1974).
[16] R. Werner, J. Math. Phys. 27, 793 (1986).
[17] M. D. Srinivas and R. Vijayalakshmi, Pramana 16, 173 (1981).
[18] A. S. Holevo, Probabilististic and statistical aspects of quantum theory (North Holland, Amsterdam, 1982).
[19] A. Peres, Quantum Theory: Concepts and Methods (Kluwer, Dordrecht, 1993).
[20] P. Busch, M. Grabowski, and P. J. Lahti, Operational quantum mechanics (Springer, Berlin, 1995).
[21] J. G. Muga, J. P. Palao, and C. R. Leavens, Phys. Lett. A253, 21 (1999), quant-ph/9803087.
[22] I. L. Egusquiza and J. G. Muga, Phys. Rev A61, 012104 (2000), see also erratum, Phys. Rev. A 61 (2000) 059901(E), quant-ph/9905023.
[23] E. P. Wigner, in Aspects of quantum theory, edited by A. Salam and E. P. Wigner (Cambridge University Press, London, 1972).
[24] J. León, J. Julve, P. Pitanga, and F. J. de Urrías (1999), quant-ph/9903060.
[25] J. León, J. Julve, P. Pitanga, and F. J. de Urrías, Phys. Rev. A 61, 062101 (2000), quant-ph/0002011.
[26] J. León (2000), quant-ph/0008025.