Number sense: the result of mathematical experience

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Abstract. Number sense has become an important topic in mathematics education in the last few decades. Number sense, for the most part, refers to an individual general comprehension with regard to numbers and flexibility in using the operations for making mathematical judgments. In general, number sense is not seen as an issue which is directly taught within a sub-chapter or a specific topic. Instead, it is assumed as the result of mathematical experience where students could employ their sense in understanding circumstances involving numbers without exerting standard algorithm. This article entails an in-depth discussion with respect to number sense and its components, the way it can be promoted, and the previous research recording its comparison with students’ achievement.

1. Introduction
When a student is given a question “Is 55 x 0.25 bigger or smaller than 55? or Is there any fraction between 2/5 and 3/5?”, the expected answer would be smaller than 55 and there are infinitely fractions between 2/5 and 3/5. But to some degree, majority students answer these questions incorrectly [1]. What sort of ability do students need to solve these questions? It is called “number sense”, the ability to execute mental calculation without using any standard algorithm. Someone with a satisfactory number sense ability does not require any written calculation in answering a simple calculation, rather he/she is able to process it internally. This is in line with Curriculum and Evaluation Standards for School Mathematics, NCTM [2] which postulates that a person with a good sense of number: (1) have well-understood number meaning, (2) have developed multiple relationships among numbers, (3) recognize the relative magnitude of numbers, (4) know the relative effect of operating on number, and (5) develop a referent for measures of common objects and situations in their environment” (p. 38).

Number sense is the acquired ability of an individual based on the experience of learning mathematics from an early age. The process starts from recognizing numbers and representing them in a meaningful form to recognize the operation and meaning of its use in everyday life. This ability can be well-formed if the learning experience gained is a meaningful learning experience. In other words, number sense is a good intuition of numbers [1][3]. It is not seen as mathematical topic that is explicitly listed in the school curriculum, but it is a knowledge of the numbers obtained and developed continuously throughout the curriculum [3][4].

Number sense involves rational, creative, effective and flexible thinking, and thus with sufficient number sense ability students can solve problems in a creative, analytical and flexible manner [5]. Ekenstam [6] states that lack of understanding of numbers will lead to obstacles that are difficult to overcome in mathematics learning. Students who do not understand that 2.40 isthe representations of 2.4 and 4/5 less than 1 must remember all the rules for dealing with numerical situations in everyday life.
Every facet of life is always associated with numbers. This is supported by Wagner and Davis [7] who professed that "Number permeates the existences of citizens of the modern world, the western world". A system of numbers is used to understand the world from the past, present, and future which is framed and impregnated with numbers. When reading magazines or newspapers, people will always be faced with stock market reports, population increase, the popularity of candidates for a regional head, and an increase of the amount of food consumption by the society at the certain time. In fact, in understanding, all of these things what a person needs most is a good understanding of numbers [7].

Not all students in the future will become a mathematician, but every student who learns math is expected to apply their knowledge that they get in everyday life. The knowledge is not merely vis-a-vis the use of standard algorithms such as the use of complex formulas and written calculations, but intuitions built on meaningful learning experiences. Therefore, this article will first explore the number sense and its components. Secondly, it will be followed by the discussion on how experience constructs knowledge and number sense. Ultimately, the discussion will conclude by examining the link between number sense with achievement and mathematics success of students in school.

2. What Is Number Sense?

Number sense is a holistic concept of the ability to understand quantities, numbers, operations, and relationships among them, which are applied efficiently and flexible in making a mathematical judgment [8]. It refers to a person's general comprehension of numbers and operations along with the ability and propensity to use this understanding flexibly. This reflects the tendency and the ability to use quantitative numbers and methods as a means of communicating, processing and interpreting information to create a positive attitude toward mathematics where numbers are useful and absolute [9].

Number sense is also defined as the ability to quickly understand, estimate and manipulate numerical quantities[10]. Students with qualified number sense skills can find flexible and appropriate ways to solve a numerical problem. What is more, students who have the number sense skill can also be identified from their level of comfort when dealing with the numbers [4]. In solving a mathematical problem, students using number sense are not only rely on the rote and the raw rules explicitly learned in school but also easily identify numerical errors [11]. When performing numerical calculations, students with sufficient number sense skills will assess the truth of the obtained numerical results. When the results are not as expected, the student will review them externally. Students will also be able to transfer their mathematical knowledge not only in school but also the out of school environment.

Number sense not only focuses on the mathematical information system that has been acquired but also serves as the key to individual's ability to complete basic arithmetic calculations. To realize that 17 is greater than 13 owning the same understanding of 9 + 8 and 9 + 4, there are more than 100 basic calculation facts that students have to memorize to automate [12].

Several studies have shown that human representations of numbers and mathematical thinking depend heavily on the sense in estimating numerical quantities [13]. A person with a good number sense ability can see the world in terms of numbers and quantity such as, in considering when the number 100 is worth a lot and a little [14]. While children are not trained to use their senses in handling simple calculations, they tend to get stuck on rules that make their strategies and flexibility undeveloped. As an illustration, Ana, a student of grade 3 still uses her finger for simple calculations 9 + 9. She should know the facts or at least use a more efficient strategy to handle it.

Number sense is more acknowledged as an ability or knowledge, not intrinsic processes [15]. Number sense can appear in various ways if students are confronted with mathematical thinking. Students with number sense prefer to develop computational strategies such as mental calculations, calculator techniques, and estimations, and number sense plays a vital role to the actualization of these strategies at various levels [15].

According to Case [16], someone who has strong number sense able to: 1) comprehend between quantities in the real world and the world of mathematics expressed by numerical expression; 2) create
procedures according to their own way of completing numerical operations; 3) represent numbers in various ways depending in the context and purpose; 4) identify benchmarks and number patterns; 5) identify numerical errors; and 6) understand the general nature of a numerical problem without performing standard calculations. In line with Case, NCTM [2] also describes the criteria of students with number sense ability in Curriculum and Evaluation Standards for School Mathematics: 1) having a good understanding of the meaning of numbers, 2) developing several relationships between numbers, 3) recognizing the relative magnitude of a number, 4) knowing the relative effects of operations on numbers, and 5) developing references to measure objects and general situations in their environment.

In solving high-level math problems, sufficient number sense capability is needed. Sood and Mackey [17] suggest that number sense is the foundation for students to understand formal mathematical concepts. Thus, the ability of a good number sense is very important for students to predict their mathematical success in the future.

3. Components of Number Sense

Number sense components are important parts of meaningful mathematical understanding. The learning process that involves the number sense component is considered to be an essential part and the main topic in the school's mathematics curriculum [15][18][19][20] declares that almost there is no two researchers define the number sense in exactly the same way. This, of course, occurs because cognitive scientists and mathematical education also define numbers in different ways. This discrepancy in definition also has implications for the different number sense components described by the researchers. Based on a literature study conducted by Berch [21], the number sense components are:

1. A faculty permitting the recognition that something has changed in a small collection when, without direct knowledge, an object has been removed or added to the collection.
2. Elementary abilities or intuitions about numbers and arithmetic.
3. Ability to approximate or estimate.
4. Ability to make numerical magnitude comparisons.
5. Ability to decompose numbers naturally.
6. Ability to develop useful strategies to solve complex problems.
7. Ability to use the relationships among arithmetic operations to understand the base-10 number system.
8. Ability to use numbers and quantitative methods to communicate, process, and interpret information.
9. Awareness of various levels of accuracy and sensitivity for the reasonableness of calculations.
10. A desire to make sense of numerical situations by looking for links between new information and previously acquired knowledge.
11. Possessing knowledge of the effects of operations on numbers.
12. Possessing fluency and flexibility with numbers.
13. Can understand number meanings.
14. Can understand multiple relationships among numbers.
15. Can recognize benchmark numbers and number patterns.
16. Can recognize gross numerical errors.
17. Can understand and use equivalent forms and representations of numbers as well as equivalent expressions.
18. Can understand numbers as referents to measure things in the real world.
19. Can move seamlessly between the real world of quantities and the mathematical world of numbers and numerical expressions.
20. Can invent procedures for conducting numerical operations.
21. Can represent the same number in multiple ways depending on the context and purpose of the representation.
22. Can think or talk in a sensible way about the general properties of a numerical problem or expression without doing any precise computation.
23. Engenders an expectation that numbers are useful and that mathematics has a certain regularity.
24. A non-algorithmic feel for numbers.
25. A well-organized conceptual network that enables a person to relate number and operation.
26. A conceptual structure that relies on many links among mathematical relationships, mathematical principles, and mathematical procedures.
27. A mental number line on which analog representations of numerical quantities can be manipulated.
28. A nonverbal, evolutionarily ancient, innate capacity to process approximate numerosities.
29. A skill or kind of knowledge about numbers rather than an intrinsic process.

Based on the components that Berch [21] summarized, the number sense outlines tend to intuition, recognition, awareness, knowledge, and skills or mental calculations. Meanwhile, according to Yang et al., [22], Mohamed & Jhonny [19], Yang, Hsu & Huang [23], Yang & Lin [24] number sense components are:

1. Understand the meaning of numbers and their operations. Students understand the basic concepts of numbers such as place value, base ten system, and number patterns.
2. Recognizing relative number sizes. When students compare the fractions, the students do not use standard algorithms to get them sorted out like matching denominators listed in the curriculum, rather they can use a meaningful way. For example, to compare fractions 3/4 and 6/7, students do not need to equate denominators like standard instructions to find the answer; but they just need to realize that 3/4 + 1/4 = 6/7 + 1/7. 1/4 is larger than 1/7 so 5/6 is less than 6/7.
3. Using different types of numerical representations and operations. Students can use different representation forms, such as image, oral, and symbolic representation in solving numerical problems with flexibility and precision. For example, students can represent 3/4 into different representations.
4. Recognizing the relative effects of operations on numbers. Students know how the four basic operations can affect the calculation result. For example, when students look for the best estimation for 255 x 0.99, students can employ their senses to realize that multiplication does not always produce larger numbers [25].
5. Assess the reasonableness of the calculation results. Students can mentally use the estimation strategy to decide on a reasonable calculation result. For example, when deciding a decimal point place using estimates on the problem 423 x 0.552 = 233496, without using paper pencil calculations, they can directly estimate 400 x 0.555 or around ½ about 200, so the answer is 233,496.

Slightly different from what some other researchers put forward, Faulkner [26] states that algebraic both geometric thinking and proportional reasoning are also included in the important component of number sense in addition to magnitude of numbers, numeration, equality, base ten and form of number. Algebraic and geometric thinking is considered important when students want to develop their number sense and understanding of mathematics. For example in the initial material about the similarity where \( X = Y \) has the same value but \( X \) and \( Y \) are not the same two things. Students must be sensitive to the idea that 1 might be equal to \( 1^3 \) but both are very different forms where the other is a linear measurement and the other is a volume measurement. Proportional reasoning involves a comparison in quantity and comparison between quantities. The important characteristics of proportional reasoning are holistic reasoning which includes quotients, fractions and ratios [27].

To sum up, from the several components that have been proposed by the researchers, it indicates that having a good number sense capability allows an individual to solve complex mathematical
problems; hence, it can be interpreted that component of number sense include higher order thinking [21].

4. **Number sense as result of mathematical experience**

Mathematical discipline consists of three worlds namely the real quantity that exists in space and time, numbers in oral language and formal mathematical symbols [28]. Number sense is a series of relationships between these three worlds. Students have to connect between real quantities and numbers. After students can connect this integrated knowledge into formal mathematical symbols, students will get an understanding of its meaning. To develop the ability of number sense, students need the opportunity to find and build relationships between these three worlds at a higher level of complexity.

There are two sub-systems in number sense, the approximately number system used to determine the approximate magnitude of a number and the exact number system used to determine the exact number of the quantities [29]. The inaccuracy in the representation that increases as the number increases is hallmark feature of ANS. These developments are continuous and vary by individual. Accordingly, Kaminski [4] professes that the ability of a person's number sense develops gradually and continuously. It is a result of numerical exploration, visualization, and is not limited by standard algorithms in various contexts.

Number sense is part of the core knowledge of the individual. Bruer [30] also summarizes that research since the 1970s has produced evidence of a number sense component in children. Numerous tendencies have existed since infants with a reproductive neutral substrate, but this ability will not develop unless accompanied by meaningful mathematical activities and experiences [31]. In other words, the ability of a good sense number is a result of meaningful learning of mathematics. This is also supported by Piaget's influential research summarized in "The Child Conception of Number" [31] he argues that initially, children do not have a stable representation of numbers and that arithmetic knowledge emerges slowly as logical developing construction.

A clear distinction is demonstrated in the arithmetic intuition of a child from 2.5 to 4 years of age [31]. In this age, children begin to recognize the numbers 1 or 2 but this introduction is just to a mere word recognition. After going through several long enough processes, they commence knowing the meaning of the word one [32]. Children slowly learn to chart the word as the age and experience develop from their activities. Upon entering school, they begin to understand half and twice a collection of objects are represented.

This is in line with the tenet demonstrating that the ability of a good number sense is derived from meaningful mathematical experience, and it is significantly affected by the children’s prior knowledge because each child gains different experiences. Some children obtain informal conceptual number sense structures with the family environment before entering school. Other children who have not acquired such knowledge require formal education assistance to do so [30][33]. When children who have gained initial knowledge from their informal environment entering school, they may already know that 9 is greater than 5 and remains 4. Whereas her friend with less developed sense number is only limited to knowing 9 is greater than 5 [11]. It is also reinforced by Antolin and Lipovec [34] in his research on mathematical experience and parental involvement of parents who are and who are not mathematicians. Antolin and Lipovec stated that children with a mathematician background parents’ have a greater chance of gaining higher support and confidence in building strong math experience. The results of Cai's [35] and Ma [36] studies also showed that the involvement of parents at home in childhood math learning has an effect on student learning outcomes at school. Consistent with these findings, LeFevre et al. [37] suggest that children with more direct exposure with mathematical content in the family environment have better numeracy skills. However, this is not about having or not having a number sense, but rather a well-developed process as experience and knowledge gained by each child [38][39].
5. The Effects of Number Sense on Students’ Mathematics Learning Achievement

The ability of calculation includes complex aspects of human cognition abilities, namely linguistic, spatial, memory, body knowledge and executive function abilities [40]. Children with an inability to absorb arithmetic concepts have considerable difficulty in calculations and other numerical skills. There are two hypotheses proposed by Vanbist et al. [41] about the difficulty of learning mathematics. These hypotheses are general domain cognitive deficits and domain specific deficits. In the general domain of cognitive deficits, children with learning disabilities have a deficit in intelligence, working memory, language skills, attention control, executive function, semical memory and data processing where all of these affect mathematical performance. Whereas in domain specific deficits the difficulty of learning mathematics is caused by deficit in number sense.

However, a different result is shown in the relationship between number sense and written calculation abilities. McIntosh et al. [9] claim that high written calculation skills are not always accompanied by good number sense. Yang and Wu [8] also found that rules-based mathematical learning and standard algorithms can reduce students' mathematical thinking and prevent conceptual learning. Interviews by McIntosh on students with moderate and high math skills in their schools show that students are more likely to use computational techniques with standard algorithms on demanding number sense. However, high achiever students are able to escape from rule-based methods when they are asked to answer some question in other ways.

According to Bütüner [5], students with number sense ability are also able to solve problems better than students who rely on rule-based methods. He also argued that the development of mathematical thinking requires the development of number sense. Both are interconnected with each other. In line with Bütüner, another study explains that the mathematics learning faculty encountered by students is due to their less developed sense capabilities [21][45].

The study was conducted by Yang et al. [23] in grade 5 students in Taiwan to identify the relationship between number sense ability and student math success in school. Among the 736 students' mathematics scores were positively correlated with score number sense ability. The same study was conducted by Mohamed and Jonny [19] on the 32 fourth grade students showing that a positive correlation was also found between score of number sense and students' math scores at school. Therefore, it can be concluded that the ability of number sense affects the score of mathematics achievement of students in school, and thus this ability is very crucial to be developed.

6. Conclusion

Number sense refers to good intuition and general understanding of numbers, operations, and relationships between them. A meaningful and gradual learning experience will develop this capability. Research has shown that the ability of the number sense of each person depends on the learning experience that they get in their internal and external environment. In other words, the ability of a sufficient number sense is the result of numerical exploration and a supportive learning experience.

The number sense influences the students' learning outcomes and the success of math in schools. This fact indicates that the ability of number sense is worth developing from an early age. Rule-based learning or non-contextual standard algorithms will cause learning to be meaningless. Hence, curriculum developers and teachers should focus on learning that can support the development of student number sense capabilities whereby this can be a predictor of students' future math success.
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