A New Expression for the Product of Two $\kappa - \mu$ Shadowed Random Variables and its Application to Wireless Communication

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Abstract

In this work, the product of two independent and non-identically distributed (i.n.i.d) $\kappa - \mu$ shadowed random variables is studied. We derive the series expression for the probability density function (PDF), cumulative distribution function (CDF), and moment generating function (MGF) of the product of two (i.n.i.d) $\kappa - \mu$ shadowed random variables. The derived formulation in this work is quite general as they incorporate most of the typically used fading channels. As an application example, outage probability (OP) has been derived for cascaded wireless systems and relay-assisted communications with a variable gain relay. Extensive Monte-Carlo simulations have also been carried out.

Index Terms

$\kappa - \mu$ shadowed fading, Product statistics, Cascade channel, Mellin transformation

I. INTRODUCTION

A wireless channel is governed mainly by two physical phenomena shadowing (which results in long-term signal variation) and multipath (which results in short-term fading). Shadowing is typically modeled using lognormal distribution [1] or sometimes using gamma distribution [2]. In contrast to shadowing, multipath effect is characterized by a broad range of distribution such as Rayleigh, Rician, Nakagami-$m$, Hoyt and more general distribution like $\kappa - \mu$, $\eta - \mu$ [3] and
The $\kappa - \mu$ shadowed distribution introduced in [5] provides a natural generalization of the $\kappa - \mu$ fading where the line-of-sight (LOS) component of received signal is random in nature, i.e., subject to shadowing. This type of fading model is known as the LOS shadow fading model in the literature. In [6] the authors showed that the $\kappa - \mu$ shadowed distribution with an integer value of $\mu$ and $m$ can be represented as a mixture of Gamma distribution.

In a variety of wireless communication applications, such as relay-based communication systems [7], and, intelligent reflecting surfaces (IRS) assisted communication system [8]–[10], the transmitted signal from the source reaches the destination after experiencing a couple of fading environments. To analyze such a communication system’s performance, one needs to know the statistics of the product of corresponding fading distributions. Hence, the statistical characterization of the product of two random variables (RVs) has crucial importance in wireless communication. For example, in bistatic scatter radio communication, the indirect channel between carrier emitter and software-defined radio (SDR) reader through a radio frequency (RF) tag is modeled as the product of two Rayleigh and Rician fading channels in [11] and [12], respectively. The work in [11] and [12] were focused on point to point communication, whereas a multiscatter scenario is considered in [13] where multiple carrier emitters are present, and each channel is modeled as Nakagami–$m$ fading. Hence, the channel between carrier emitter to SDR reader is a product of two Nakagami–$m$ RVs. In [14], authors presented a general result for the product statistics of Rayleigh fading. A generic cascaded channel has been considered in [15], [16] with Nakagami–$m$ and generalized Nakagami–$m$ fading, respectively. Authors in [17] studied the product statistics of two independent and non-identically distributed $\kappa - \mu$ RVs. In [18] the statistical characterization of $\alpha - \mu$ and $\eta - \mu$ RV is done along with $\kappa - \mu$ RV. Recently, the authors in [19] considered an IRS-assisted communication system where each link undergoes $\kappa - \mu$ fading, and hence the link between source and destination via IRS is the product of two $\kappa - \mu$ RVs.

In this work, we are interested in $\kappa - \mu$ shadowed fading since it unites various popular fading models such as one-sided Gaussian, Rician, Rayleigh, $\kappa - \mu$, Nakagami–$m$ and Rician shadowed. Apart from its generalized nature, $\kappa - \mu$ shadowed distribution has good analytical tractability and it found a lot of traction in wireless community in recent literature [20]–[26]. This motivates us to look at the product statistics of $\kappa - \mu$ shadowed fading. We provided the statistical characterization of the product of two independent non-identically distributed (i.n.i.d.) $\kappa - \mu$ shadowed RVs. The closed-form exact expression for PDF and CDF of the considered RV
is derived using Mellin transformation. The contribution and utility of this work are summarized as follows:

- Series expressions for PDF, CDF, and MGF of the product of two, i.n.i.d. \( \kappa - \mu \) shadowed RV are derived using the direct application of Mellin transform \(^1\).
- We presented the performance metrics for a cascaded wireless system and outage probability for a relay-assisted wireless communication system as an application example for the presented fading distribution.

II. PROPOSED STATISTICAL CHARACTERIZATION USING INVERSE MELLIN TRANSFORM

We considered two independent non-identically distributed (i.n.i.d) \( \kappa - \mu \) shadowed RVs, say \( X_i \) with mean \( \bar{\gamma}_i \) and non-negative real shaping parameters \( \kappa_i, \mu_i, m_i \) for \( i = 1, 2 \). Each \( X_i \) follows the distribution given by [5, eq. (4)],

\[
f_{X_i}(x_i) = \frac{\mu_i^{\mu_i} m_i^{m_i} (1 + \kappa_i)^{\mu_i}}{\Gamma(\mu_i) \bar{\gamma}_i^{\mu_i}(\mu_i \kappa_i + m_i)^{m_i}} \left( \frac{x_i}{\bar{\gamma}_i} \right)^{-\mu_i-1} e^{-\mu_i(1+\kappa_i)x_i} \Gamma_{1,F_1}(m_i; \mu_i; \frac{\mu_i \kappa_i (1 + \kappa_i) x_i}{\mu_i \kappa_i + m_i} \bar{\gamma}_i),
\]

where \( i = 1, 2 \). \( \kappa_i \) is the ratio of power contribution from dominant path to scattered waves, \( \mu_i \) is the real extension to number of multipath clusters and \( m_i \) is the shaping parameter for LOS shadowing component, \( \Gamma_{1,F_1}(\cdot;\cdot;\cdot) \) is confluent hypergeometric function [28].

Our objective is to statistically characterize the RV \( Y = X_1 X_2 \). We will now use the technique of Mellin transform to derive the PDF of \( Y \). First, we re-write the PDF of \( X_i \) as follows

\[
f_{X_i}(x_i) = \theta_i x_i^{\mu_i-1} g_i(x_i),
\]

where \( \theta_i = \frac{\mu_i^{\mu_i} b_i}{\Gamma(\mu_i) \bar{\gamma}_i^{\mu_i}(\mu_i \kappa_i + m_i)^{m_i}} \), \( g_i(x_i) = e^{-a_i x_i} \Gamma_{1,F_1}(m_i; \mu_i; a_i c_i x_i) \), \( a_i = \frac{\mu_i (1 + \kappa_i)}{\bar{\gamma}_i} \), \( b_i = \frac{m_i^{m_i}}{(\mu_i \kappa_i + m_i)^{m_i}} \), and \( c_i = \frac{\mu_i \kappa_i}{(\mu_i \kappa_i + m_i)} \). Then, the Mellin transform of \( f_{X_i}(x_i) \) is

\[
\mathcal{M}[f_{X_i}(x_i); s] = \theta_i \mathcal{M}[g_i(x_i); s + \mu_i - 1]
\]

Now, we need to find the Mellin transform of \( g_i(x_i) \) which is

\[
\mathcal{M}[g_i(x_i); s] = \int_{0}^{\infty} x_i^{s-1} g_i(x_i) \, dx_i
\]

\[= \int_{0}^{\infty} x_i^{s-1} e^{-a_i x_i} \Gamma_{1,F_1}(m_i; \mu_i; a_i c_i x_i) \, dx_i\]

\(^1\)Very recently, some statistics of the product of two \( \kappa - \mu \) shadowed random variable has been derived in [27]. However, the method utilized there is totally different from the one we used in this work. Hence, the resulting expression are also different and original. Derived series expression involve simple hypergeometric function hence can be easily computed using popular software as Mathematica.
Using the identity [29, eq. (7.621.4)], we have

\[ \mathcal{M} [g_i(x_i); s] = \Gamma(s) a_i^{-s} 2F_1(m_i, s; \mu_i; c_i) \] (5)

Finally,

\[ \mathcal{M} [f_{X_i}(x_i); s] = \theta_i \Gamma(s + \mu_i - 1) a_i^{s+\mu_i-1} 2F_1(m_i, s + \mu_i - 1; \mu_i; c_i) \]

\[ = b_i \Gamma(s + \mu_i - 1) \Gamma(\mu_i) a_i^{s-1} 2F_1(m_i, s + \mu_i - 1; \mu_i; c_i) \] (6)

It is a well-known fact that the Mellin convolution of individual PDFs gives the PDF of the product of two independent RVs, and the Mellin transform of the said PDF is the product of the Mellin transform of corresponding PDFs [30]. Hence, the Mellin transform of \(Y\) is

\[ \mathcal{M} [f_Y(y); s] = \prod_{i=1}^{2} \mathcal{M} [f_{X_i}(x_i); s] \]

\[ = \frac{b_1 b_2 \Gamma(s + \mu_1 - 1) \Gamma(s + \mu_2 - 1)}{\Gamma(\mu_1) \Gamma(\mu_2) (a_1 a_2)^{s-1}} 2F_1(m_1, s + \mu_1 - 1; \mu_1; c_1) \]

\[ \times 2F_1(m_2, s + \mu_2 - 1; \mu_2; c_2) \] (7)

Now, \(f_Y(y)\) is obtained using the inverse Mellin transform, i.e.,

\[ f_Y(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M} [f_Y(y); s] y^{-s} ds \]

\[ = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{b_1 b_2 \Gamma(s + \mu_1 - 1) \Gamma(s + \mu_2 - 1)}{\Gamma(\mu_1) \Gamma(\mu_2) (a_1 a_2)^{s-1}} 2F_1(m_1, s + \mu_1 - 1; \mu_1; c_1) \right. \]

\[ \times 2F_1(m_2, s + \mu_2 - 1; \mu_2; c_2) y^{-s} \right) ds \] (8)

By definition, \(c_1, c_2 < 1\) and \(\mu_1, \mu_2 > 0\) so both hypergeometric function, present in the integrand of above integral, will be analytic \(\forall s\). Hence, value of the integral will be decided by the position of poles of Gamma function. Based on the value of \(\mu_1, \mu_2\), there are two cases.

I) When \(\mu_2 - \mu_1 \notin \mathbb{Z}\): In this case, the poles of \(\Gamma(s + \mu_1 - 1)\) and \(\Gamma(s + \mu_2 - 1)\) are distinct. Hence, by the virtue of residue theorem and Jordan’s Lemma [30], we have

\[ f_Y(y) = \frac{b_1 b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \sum_{n=0}^{\infty} [R_{1,n} + R_{2,n}] , \] (9)
where
\[
R_{1,n} = \lim_{s \to -n-\mu_1+1} \frac{(s+n+\mu_1-1) \Gamma(s+\mu_1-1) \Gamma(s+\mu_2-1)}{(a_1a_2)^{s-1}} \frac{\psi}{\Gamma(\mu_2)} \frac{y^{-s}}{2F_1(m_1, s+\mu_1-1; \mu_1; c_1)} 
\times 2F_1(m_2, s+\mu_2-1; \mu_2; c_2) y^{-s} 
= \frac{(a_1a_2)^{n+\mu_1} \Gamma(\mu_2-\mu_1-n)}{(-1)^n n!} \frac{\psi}{\Gamma(\mu_2)} \frac{y^{-n}}{2F_1(m_1, n; \mu_1; c_1) 2F_1(m_2, \mu_2 - n; \mu_2; c_2) y^{n+\mu_1-1}} 
\]
and, similarly
\[
R_{2,n} = \frac{(a_1a_2)^{n+\mu_2} \Gamma(\mu_1-\mu_2-n)}{(-1)^n n!} \frac{\psi}{\Gamma(\mu_2)} \frac{y^{-n}}{2F_1(m_1, \mu_1 - \mu_2; \mu_1; c_1) 2F_1(m_2, -n; \mu_2; c_2) y^{n+\mu_2-1}} 
\]
Hence, PDF of \( Y \) is
\[
f_Y(y) = \frac{b_1b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \sum_{n=0}^{\infty} \left[ A_n y^{n+\mu_1-1} + B_n y^{n+\mu_2-1} \right],
\]
where
\[
A_n = \frac{(a_1a_2)^{n+\mu_1} \Gamma(\mu_2-\mu_1-n)}{(-1)^n n!} 2F_1(m_1, n; \mu_1; c_1) 2F_1(m_2, \mu_2 - n; \mu_2; c_2) 
\quad \text{and} \quad
B_n = \frac{(a_1a_2)^{n+\mu_2} \Gamma(\mu_1-\mu_2-n)}{(-1)^n n!} 2F_1(m_1, \mu_1 - \mu_2; \mu_1; c_1) 2F_1(m_2, -n; \mu_2; c_2).
\]
Note that the \( A_n \) and \( B_n \) only depend on the parameters of both \( \kappa - \mu \) shadowed RV \( i.e. \), independent of \( y \). Thus, the PDF of \( Y \) is simply a power series of \( y \).

2) \( When \ mu_2 - \mu_1 \in \mathbb{Z}: \) Without loss of generality, let \( \mu_2 > \mu_1 \) and \( \mu_2 - \mu_1 = N \) then the poles of \( \Gamma(s+\mu_1-1) \) and \( \Gamma(s+\mu_2-1) \) coincides for \( n \geq N \). So, there are \( N \) poles of order one and the remaining poles are of order two. Again, using residue theorem, we have
\[
f_Y(y) = \frac{b_1b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \left( \sum_{n=0}^{N-1} S_{1,n} + \sum_{n=N}^{\infty} S_{2,n} \right) 
\]
The first \( N \) poles are due to \( \Gamma(s+\mu_1-1) \) so we have
\[
S_{1,n} = A_n y^{n+\mu_1-1}, \quad n = 0, 1, \ldots, N - 1.
\]
and
\[
S_{2,n} = \frac{(-1)^N (a_1a_2)^{n+\mu_1} y^{n+\mu_1-1}}{(n-N)! n!} \left\{ 2F_1^{(0,1,0,0)}(m_1, -n; \mu_1; c_1) 2F_1(m_2, -n+N; \mu_2; c_2) 
+ 2F_1(m_1, -n; \mu_1; c_1) 2F_1^{(0,1,0,0)}(m_2, -n+N; \mu_2; c_2) 
+ \left[ \psi(n+1) + \psi(n-N+1) - \ln(y) - \ln(a_1a_2) \right] 2F_1(m_1, -n; \mu_1; c_1) 2F_1(m_2, -n+N; \mu_2; c_2) \right\}.
\]
Now, we derive the expression for MGF using the PDF in (12) and (16).

\[ f_Y(y) = \frac{b_1 b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \left( \sum_{n=0}^{N-1} A_n y^{n+\mu_1} + \sum_{n=N}^{\infty} \left[ C_n - D_n \ln(y) \right] y^{n+\mu_1} \right), \]  

(16)

where

\[ C_n = \frac{(-1)^N (a_1 a_2)^n}{(n-N)! n!} \left\{ 2F_1^{(0,1,0,0)}(m_1, -n; \mu_1; c_1) 2F_1(m_2, -n+N; \mu_2; c_2) 
\right. 
+ \left. 2F_1(m_1, -n; \mu_1; c_1) 2F_1^{(0,1,0,0)}(m_2, -n+N; \mu_2; c_2) 
\right. 
\left. + [\psi(n+1) + \psi(n-N+1) - \ln(a_1 a_2)] 2F_1(m_1, -n; \mu_1; c_1) 2F_1(m_2, -n+N; \mu_2; c_2) \right\}, \]

(17)

and,

\[ D_n = \frac{(-1)^N (a_1 a_2)^{n+\mu_1}}{(n-N)! n!} 2F_1(m_1, -n; \mu_1; c_1) 2F_1(m_2, -n+N; \mu_2; c_2). \]

(18)

In (17), \( \psi(.) \) is the digamma function [31] and \( 2F_1^{(0,1,0,0)}(a, b, c, z) \) is the derivative of the confluent hypergeometric function with respect to the parameter \( b \) [32]

\[ 2F_1^{(0,1,0,0)}(a, b, c, z) = \frac{z a}{c} F_{2,0,1} \left( \begin{array}{c} a+1, b+1; 1; b \\ 2, c + c; -; b+1 \end{array} \right) \]

(19)

where \( F_{l,m,n}^{p,q,k} \left( \begin{array}{c} (a_p); (b_q); (c_k) \\ (\alpha_l); (\beta_m); (\gamma_n) \end{array} \right) x, y \) is the Kampé de Fériet’s Series.

A. Cumulative Distribution Function

Similar to PDF, CDF expression also depends the values of \( \mu_1 \) and \( \mu_2 \). Using PDF in (12) the CDF for the case when \( \mu_2 - \mu_1 \notin \mathbb{Z} \) is given as

\[ F_Y(y) = \frac{b_1 b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \sum_{n=0}^{\infty} \left[ \frac{A_n}{n+\mu_1} y^{n+\mu_1} + \frac{B_n}{n+\mu_2} y^{n+\mu_2} \right] \]

(20)

and the CDF for the other case, using PDF in (16) is as follows

\[ F_Y(y) = \frac{b_1 b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \left( \sum_{n=0}^{N-1} \frac{A_n}{n+\mu_1} y^{n+\mu_1} + \sum_{n=N}^{\infty} \left[ C_n - D_n \ln(y) \right] y^{n+\mu_1} + \sum_{n=N}^{\infty} \frac{D_n}{(n+\mu_1)^2} y^{n+\mu_1} \right) \]

(21)

Now, we derive the expression for MGF using the PDF in (12) and (16).
B. Moment Generating Function

The MGF of $Y$ using the PDF in (12) is given as

$$M_Y(s) = \mathcal{L}[f_Y(y); -s] = \frac{b_1 b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \sum_{n=0}^{\infty} \left[ \frac{A_n \Gamma(n + \mu_1)}{(-s)^{n+\mu_1}} + \frac{B_n \Gamma(n + \mu_2)}{(-s)^{n+\mu_2}} \right]$$

(22)

and the MGF for the case when $\mu_2 - \mu_1 \in \mathbb{Z}$, using PDF in (16) is as follows

$$M_Y(s) = \frac{b_1 b_2}{\Gamma(\mu_1) \Gamma(\mu_2)} \left( \sum_{n=0}^{N-1} \frac{A_n \Gamma(n + \mu_1)}{(-s)^{n+\mu_1}} + \sum_{n=N}^{\infty} \frac{[C_n + D_n (\ln(-s) - \psi(n + \mu_1))] \Gamma(n + \mu_1)}{(-s)^{n+\mu_1}} \right)$$

(23)

C. Moments

The $n$-th order moment of RV $Y$ is given by $\mathbb{E}[Y^n] = \mathbb{E}[X^n_1] \mathbb{E}[X^n_2]$, since $X_1$ and $X_2$ are independent RVs. Also, note that the $\mathbb{E}[Y^n] = \mathcal{M}[f_Y(y); s]|_{s=n+1}$. Hence, from (7) we have

$$\mathbb{E}[Y^n] = \frac{b_1 b_2 \Gamma(\mu_1 + n) \Gamma(\mu_2 + n)}{\Gamma(\mu_1) \Gamma(\mu_2) (a_1 a_2)^n} 2F_1(m_1, \mu_1 + n; \mu_1; c_1) 2F_1(m_2, \mu_2 + n; \mu_2; c_2)$$

$$= \frac{b_1 b_2 (\mu_1)_n (\mu_2)_n}{(a_1 a_2)^n} 2F_1(m_1, \mu_1 + n; \mu_1; c_1) 2F_1(m_2, \mu_2 + n; \mu_2; c_2)$$

(24)

Simplified form of $n$-th order moment for the case of mixed product, i.e., $X_1$ and $X_2$ follows $\kappa - \mu$ shadowed and $\kappa - \mu$ distribution, respectively, is as follows

$$\mathbb{E}[Y^n] = \frac{e^{-\kappa^2 \mu^2} b_1 (\mu_1)_n (\mu_2)_n}{(a_1 a_2)^n} 2F_1(m_1, \mu_1 + n; \mu_1; c_1) 1F_1(\mu_2 + n; \mu_2; \kappa_2 \mu_2)$$

$$\overset{(a)}{=} \frac{b_1 (\mu_1)_n (\mu_2)_n}{(a_1 a_2)^n} 2F_1(m_1, \mu_1 + n; \mu_1; c_1) 1F_1(-n; \mu_2; \kappa_2 \mu_2)$$

(25)

where $(a)$ follows from the functional relation given in [29, Eq. 9.212.1]. In the next section, we give two applications where these expression are useful.

III. Application Examples

In a multiple scattering or “keyholes” scenario, the wireless channel is modeled as the product of multiple fading distribution [33]–[36]. For the case of a double scattered wireless channel, the following are the two application scenario where we considered double $\kappa - \mu$ shadowed fading channel.
A. Cascaded Wireless System

Consider a two-tap cascaded channel as described in [16] where both taps follow $\kappa - \mu$ shadowed fading. This section derives the analytical expression for various important metrics for such a system.

1) Amount of Fading: The amount of fading (AF) measures the severity of any fading channel. It is defined as the ratio of variance to square of the mean of instantaneous SNR [37]. Hence, the AF for product $\kappa - \mu$ shadowed channel is

$$AF = \frac{\text{V}[Y]}{(\text{E}[Y])^2} = \frac{\text{E}[Y^2] - (\text{E}[Y])^2}{(\text{E}[Y])^2}.$$

By substituting values from (24), we have

$$AF = \left(1 + \frac{2\kappa_1 + 1}{\mu_1 (1 + \kappa_1)^2} + \frac{\mu_1 \kappa_1^2}{m_1 (\mu_1 + 1)}\right) \left(1 + \frac{2\kappa_2 + 1}{\mu_2 (1 + \kappa_2)^2} + \frac{\mu_2 \kappa_2^2}{m_2 (\mu_2 + 1)}\right) - 1. \quad (27)$$

The details of the derivation of $AF$ are presented in Appendix B.

2) Channel Quality Estimation Index: In [38], a new performance metric is defined for any wireless channel named as channel quality estimation index (CQEI). By definition, it is the ratio of the variance of the instantaneous SNR to the cube of the mean of instantaneous SNR, i.e.,

$$\text{CQEI} = \frac{\text{V}[Y]}{(\text{E}[Y])^3} = \frac{AF}{\text{E}[Y]}.$$

By substituting the value of AF from (27) in (28), we have

$$\text{CQEI} = \frac{1}{\gamma_1 \gamma_2} \left[ \left(1 + \frac{2\kappa_1 + 1}{\mu_1 (1 + \kappa_1)^2} + \frac{\mu_1 \kappa_1^2}{m_1 (\mu_1 + 1)}\right) \left(1 + \frac{2\kappa_2 + 1}{\mu_2 (1 + \kappa_2)^2} + \frac{\mu_2 \kappa_2^2}{m_2 (\mu_2 + 1)}\right) - 1 \right]. \quad (29)$$

It can be observed from the expression of AF and CQEI that both of these metrics are monotonically decreasing with $\mu_1, m_1, \mu_2$ and $m_2$, i.e., the severity of the channel decreases as these parameter increases. This fact can also be mathematically confirmed by taking the derivative of AF and CQEI. Also, the metrics are are monotonically decreasing with $\kappa_1$ and $\kappa_2$ when $m_1 > \mu_1$ but monotonically increasing otherwise.

3) Outage Probability: In any communication system, outage is an event when the received strength of signal falls below a certain threshold. The outage probability (OP) is defined as $P_{OP}(\gamma_{th}) = \mathbb{P}(Y \leq \gamma_{th})$, hence, from (20) we have

$$P_{OP}(\gamma_{th}) = \frac{b_1 b_2}{\Gamma(\mu_1)\Gamma(\mu_2)} \sum_{n=0}^{\infty} \left[ \frac{K_{1,n} \gamma_{th}^{n+\mu_1}}{n + \mu_1} + \frac{K_{2,n} \gamma_{th}^{n+\mu_2}}{n + \mu_2} \right]. \quad (30)$$
B. Relay with Variable Gain

Consider a relay assisted wireless system with a source node (S) communicating with a destination node (D) using a relay (R). The direct link between source and receiver is assumed be in permanent outage. Hence, the source passes the information signal to relay which amplify-and-forward (AF) it to destination. Here, relay, S and D are equipped with a single antenna. Assume that S – R link experiences $\kappa - \mu$ shadowed fading and the R – D link experiences a cascaded $\kappa - \mu$ shadowed fading. Signal-to-noise-ratio (SNR) at D with AF-based relay and variable gain is given as \[39, 40]\[
\gamma = \frac{\gamma_{sr} \gamma_{rd}}{\gamma_{sr} + \gamma_{rd} + 1} \approx \min (\gamma_{sr}, \gamma_{rd}),
\]
where $\gamma_{sr}, \gamma_{rd}$ represent the SNR of S – R and R – D link respectively. The OP at node D is evaluated as,
\[
P_{OP}^{VGR}(\gamma_{th}) = \mathbb{P} (\min (\gamma_{sr}, \gamma_{rd}) \leq \gamma_{th})
= F_{\gamma_{sr}} (\gamma_{th}) + F_{\gamma_{rd}} (\gamma_{th}) - F_{\gamma_{sr}} (\gamma_{th}) F_{\gamma_{rd}} (\gamma_{th}),
\]
where $F_{\gamma_{sr}} (\cdot), F_{\gamma_{rd}} (\cdot)$ are given by (20) or (21) depending on the values of parameters.

IV. Numerical Results

This section presents the simulation results that show the correctness and utility of the theoretical expression presented in the previous sections. Without loss of generality, we have assumed $\bar{\gamma}_1 = \bar{\gamma}_2 = 1$ for all the plots. In all the figures, We have used solid lines to draw the theoretical values and the dotted markers are used for simulated values. In Figs. 1-3, several plots for the product PDF of two $\kappa - \mu$ shadowed RV for various values of $\kappa, \mu$ and $m$. A wide range of shapes can be obtained via choosing different parameter as confirmed by the Figs. 1-3. Also, one can observe that the simulated PDFs are perfectly matching with the values obtained through theoretical expressions in (12) or (16) when the difference of $\mu_2$ and $\mu_1$ is an integer. It validates the exactness of series expression.

Next, we studied the effect of individual parameters on the OP of a cascaded $\kappa - \mu$ shadowed fading channel. Fig. 4 shows the impact of increasing the $m$ parameter of single link when all other parameters are kept constant. In Fig 4(a), we kept $\kappa_1 = 5.0, \mu_1 = 1.2$ and $\kappa_2 = 2.1, \mu_2 = 3.0, m_2 = 0.8$ then we changed the value of $m_1$. Similarly in Fig. 4(b) we kept $\kappa_1 = 5.0, \mu_1 = 1.2, m_1 = 0.5$ and $\kappa_2 = 2.1, \mu_2 = 3.0$ then we changed the value of $m_2$. It can be observed that as $m$ increases the OP decreases but the effect is not independent of other
parameter as in Fig. 4(a) the impact of increasing \( m \) is more dominant compare to Fig. 4(b) where it saturates for \( m_2 = 4.4 \) only. One reason for this behavior may be that the \( \kappa_1 > \kappa_2 \) and dominates the overall link. We fixed all the parameter except \( \mu_1 \) and \( \mu_2 \) in 5(a) and 5(b), respectively. Again, we can conclude that as \( \mu \) increases the OP decreases. In other words, a fading channel with higher \( \mu \) is more reliable. In the same way, the impact of \( \kappa \) has been
demonstrated in Fig. 6.

Fig. 5: OP of cascaded $\kappa - \mu$ shadowed channel with $(\kappa_1, m_1) = \{0.9, 4\}$, $(\kappa_2, m_2) = \{2.2, 10\}$ and various values of $\mu_1$ and $\mu_2$

Fig. 6: OP of cascaded $\kappa - \mu$ shadowed channel with $(\mu_1, m_1) = \{1.5, 4\}$, $(\mu_2, m_2) = \{2.1, 10\}$ and various values of $\kappa_1$ and $\kappa_2$

Next, in Fig. 7 and 8 we plotted the OP for the relay-assisted wireless communication system with variable gain relay to validate the expression in (32). These figures re-validate the exactness of formulation as the simulated and theoretical values match perfectly. Here, also one can observe that with the increase in parameter values fading channel gets more reliable as the OP decreases.
V. CONCLUSION

This paper presents the series expression for PDF, CDF, and MGF of the product of two $\kappa - \mu$ shadowed RV. The series expression is obtained via a direct application of Mellin transformation and can be easily computed using popular software like Mathematica. A couple of application examples are also provided for the considered cascaded fading channel. An interesting extension for this work can be to consider the product of an arbitrary number of $\kappa - \mu$ shadowed RVs.

APPENDIX A

EVALUATION OF $S_{2,n}$

\[
S_{2,n} = \lim_{s \to -n-\mu_1+1} \frac{d}{ds} \left\{ \frac{(s + n + \mu_1 - 1)^2 \Gamma(s + \mu_1 - 1) \Gamma(s + \mu_2 - 1)}{(a_1 a_2)^{s-1}} \right\} \left(2F_1(m_1, s + \mu_1 - 1; \mu_1; c_1) \times 2F_1(m_2, s + \mu_2 - 1; \mu_2; c_2) y^{-s} \right) \right\}
\]

\[
= \lim_{s \to -n-\mu_1+1} \{ \phi'(s) h(s) + \phi(s) h'(s) \}
\]

\[
= \left( \lim_{s \to -n-\mu_1+1} \phi'(s) \right) \left( \lim_{s \to -n-\mu_1+1} h(s) \right) + \left( \lim_{s \to -n-\mu_1+1} \phi(s) \right) \left( \lim_{s \to -n-\mu_1+1} h'(s) \right) \]

(33)

where,

\[
\phi(s) = (s + n + \mu_1 - 1)^2 \Gamma(s + \mu_1 - 1) \Gamma(s + \mu_2 - 1)
\]

(34)

and

\[
h(s) = \frac{1}{(a_1 a_2)^{s-1}} \left(2F_1(m_1, s + \mu_1 - 1; \mu_1; c_1) \times 2F_1(m_2, s + \mu_2 - 1; \mu_2; c_2) y^{-s}\right)
\]

(35)

Following the procedure as described in [41], we can rewrite $\phi(s)$ as

\[
\phi(s) = \frac{\Gamma^2(s + \mu_1 + n)}{(s + \mu_1 + n - 2)^2 \cdots (s + \mu_1 + N - 2)^2 (s + \mu_1 + N - 2) \cdots (s + \mu_1 - 1)}
\]

(36)

From (36), we have

\[
\lim_{s \to -n-\mu_1+1} \phi(s) = \frac{(-1)^N}{(n-N)! n!}
\]

(37)

To compute the $\phi'(s)$, we take the logarithmic derivative i.e.,

\[
\phi'(s) = \phi(s) \frac{d}{ds} \ln(\phi(s))
\]

(38)
After substituting (37) and (41) in (38), we get

\[
\frac{d}{ds} \ln (\phi (s)) = 2\psi (s + \mu_1 + n) - \frac{2}{(s + \mu_1 + n - 2)} - \cdots - \frac{2}{(s + \mu_1 + N - 1)}
\]

(39)

After some algebraic manipulations, we get

\[
\lim_{s \to -n - \mu_1 + 1} \frac{d}{ds} \ln (\phi (s)) = 2\psi (1) + 2 \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-N}\right) + \left(\frac{1}{n-N+1} + \cdots + \frac{1}{n}\right)
\]

(40)

By using [41, Eq. 1.4.6], we have

\[
\lim_{s \to -n - \mu_1 + 1} \frac{d}{ds} \ln (\phi (s)) = \psi (n + 1) + \psi (n - N + 1)
\]

(41)

After substituting (37) and (41) in (38), we get

\[
\lim_{s \to -n - \mu_1 + 1} \phi' (s) = \frac{(-1)^N}{(n-N)! n!} [\psi (n + 1) + \psi (n - N + 1)]
\]

(42)

As \(h(s)\) is analytic function so we can simply substitute \(s = -n - \mu_1 + 1\) to get

\[
\lim_{s \to -n - \mu_1 + 1} h(s) = y^{n+\mu_1-1} (a_1 a_2)^{n+\mu_1} \frac{1}{2} F_1 (m_1, -n; \mu_1; c_1) \frac{1}{2} F_1 (m_2, -n + N; \mu_2; c_2)
\]

(43)

Now, after taking the derivative of \(h(s)\) and substituting the \(s = -n - \mu_1 + 1\), we get

\[
\lim_{s \to -n - \mu_1 + 1} h'(s) = \left\{ 2 \frac{1}{2} F_1 (0, 1, 0, 0) (m_1, -n; \mu_1; c_1) \frac{1}{2} F_1 (m_2, -n + N; \mu_2; c_2) (a_1 a_2)^{n+\mu_1} y^{n+\mu_1-1}
\]

\[
+ 2 \frac{1}{2} F_1 (0, 1, 0, 0) (m_2, -n + N; \mu_2; c_2) \frac{1}{2} F_1 (m_1, -n; \mu_1; c_1) (a_1 a_2)^{n+\mu_1} y^{n+\mu_1-1}
\]

\[
- \ln (a_1 a_2) \frac{1}{2} F_1 (m_1, -n; \mu_1; c_1) \frac{1}{2} F_1 (m_2, -n + N; \mu_2; c_2) (a_1 a_2)^{n+\mu_1} y^{n+\mu_1-1}
\]

\[
- \frac{1}{2} \ln (y) \frac{1}{2} F_1 (m_1, -n; \mu_1; c_1) \frac{1}{2} F_1 (m_2, -n + N; \mu_2; c_2) (a_1 a_2)^{n+\mu_1} y^{n+\mu_1-1}
\]

(44)

Finally,

\[
S_{2,n} = \left\{ \frac{(-1)^N (a_1 a_2)^{n+\mu_1} y^{n+\mu_1-1}}{(n-N)! n!} \left\{ 2 \frac{1}{2} F_1 (0, 1, 0, 0) (m_1, -n; \mu_1; c_1) \frac{1}{2} F_1 (m_2, -n + N; \mu_2; c_2)
\]

\[
+ 2 \frac{1}{2} F_1 (m_1, -n; \mu_1; c_1) \frac{1}{2} F_1 (0, 1, 0, 0) (m_2, -n + N; \mu_2; c_2)
\]

\[
+ [\psi (n + 1) + \psi (n - N + 1) - \ln (y) - \ln (a_1 a_2)] \frac{1}{2} F_1 (m_1, -n; \mu_1; c_1) \frac{1}{2} F_1 (m_2, -n + N; \mu_2; c_2)
\}
\]

(45)
APPENDIX B
CALCULATION OF AF

From the definition of AF in (26), we have

\[ AF = \frac{\mathbb{E} [Y^2]}{[\mathbb{E} [Y]]^2} - 1 \]  

We have \( \mathbb{E} [Y] = \tilde{\gamma}_1 \tilde{\gamma}_2 \) and

\[ \mathbb{E} [Y^2] = \frac{b_1 b_2 (\mu_1)^2 (\mu_2)^2}{(a_1 a_2)^2} {}_2F_1 (m_1, \mu_1 + 2; \mu_1; c_1) {}_2F_1 (m_2, \mu_2 + 2; \mu_2; c_2) \]  

Using the Euler transformation from [42, Eq. 1.2.2.2], we have

\[ {}_2F_1 (m_1, \mu_1 + 2; \mu_1; c_1) = \frac{1}{(1 - c_1)^{m_1}} {}_2F_1 (m_1, -2; \mu_1; \frac{c_1}{c_1 - 1}) \]

\[ = (1 - c_1)^{-m_1} \left[ 1 - \frac{2m_1}{\mu_1} \frac{c_1}{c_1 - 1} + \frac{m_1 (m_1 + 1)}{\mu_1 (\mu_1 + 1)} \left( \frac{c_1}{c_1 - 1} \right)^2 \right] \]  

\[ = \frac{1}{b_1} \left[ 1 + 2\kappa_1 + \frac{\mu_1 (m_1 + 1)}{m_1 (\mu_1 + 1) \kappa_1^2} \right] \]

\[ \mathbb{E} [Y^2] = \frac{(\mu_1 + 1) (\mu_2 + 1)}{\mu_1 \mu_2 (\kappa_1 + 1)^2 (\kappa_2 + 1)^2} \left[ 1 + 2\kappa_1 + \frac{\mu_1 (m_1 + 1) \kappa_1^2}{m_1 (\mu_1 + 1) \kappa_1^2} \right] \left[ 1 + 2\kappa_2 + \frac{\mu_2 (m_2 + 1) \kappa_2^2}{m_2 (\mu_2 + 1) \kappa_2^2} \right] \]

Finally, the substitution of (50) in (46) and some algebraic manipulation results in (27) and completes the proof.

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