Recolouring weakly chordal graphs and the complement of triangle-free graphs

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Abstract

For a graph $G$, the $k$-recolouring graph $R_k(G)$ is the graph whose vertices are the $k$-colourings of $G$ and two colourings are joined by an edge if they differ in colour on exactly one vertex. We prove that for all $n \geq 1$, there exists a $k$-colourable weakly chordal graph $G$ where $R_{k+n}(G)$ is disconnected, answering an open question of Feghali and Fiala. We also show that for every $k$-colourable $3K_1$-free graph $G$, $R_{k+1}(G)$ is connected with diameter at most $4|V(G)|$.

1 Introduction

Let $G$ be a finite simple graph with vertex-set $V(G)$ and edge-set $E(G)$. For a positive integer $k$, a $k$-colouring of $G$ is a mapping $\alpha : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $\alpha(u) \neq \alpha(v)$ whenever $uv \in E(G)$. The $k$-recolouring graph, denoted $R_k(G)$, is the graph whose vertices are the $k$-colourings of $G$ and two colourings are joined by an edge if they differ in colour on exactly one vertex. We say that $G$ is $k$-mixing if $R_k(G)$ is connected. If $G$ is $k$-mixing, the $k$-recolouring diameter of $G$ is the diameter of $R_k(G)$. We say that $G$ is quadratically $k$-mixing if the $k$-recolouring diameter of $G$ is $O(|V(G)|^2)$.

Bonamy, Johnson, Lignos, Patel, and Paulusma [4] showed that a $k$-colourable chordal or chordal bipartite graph is quadratically $(k+1)$-mixing. The authors also asked whether this statement holds more generally for perfect graphs. This was answered negatively by Bonamy and Bousquet [3] using an example of Cereceda, van den Heuvel, and Johnson [5] who showed that for all $k \geq 3$, there exists a bipartite graph that is not $k$-mixing. This started an investigation into other classes of perfect graphs which have this special property: chordal and chordal bipartite [4], $P_4$-free [3], distance-hereditary [2], $P_4$-sparse [1], co-chordal, and 3-colourable ($P_5$, $P_6$, $C_5$)-free [7].

The property of being $(k+1)$-mixing does not extend to the class of weakly chordal graphs. Feghali and Fiala [7] showed that for all $k \geq 3$, there exists a $k$-colourable weakly chordal graph that is not $(k+1)$-mixing. The authors left as an open problem whether there exists an integer $l \geq k+1$ for which every $k$-colourable weakly chordal graph is $l$-mixing. We answer this question in the negative with the following theorem.

Theorem 1. For all $n \geq 1$, there exists a $k$-colourable weakly chordal graph that is not $(k+n)$-mixing.

This question has also been investigated for the class of graphs defined by forbidding an induced path. That is, determining the values of $t$ for which a $k$-colourable $P_t$-free graph is $(k+1)$-mixing. Bonamy and Bousquet [3] showed that a $k$-colourable $P_4$-free graph is $(k+1)$-mixing, and using the same example of Cereceda, van den Heuvel, and Johnson [5], showed [7].
that for all $k \geq 3$ and $t \geq 6$, there is a $k$-colourable $P_t$-free graph that is not $(k + 1)$-mixing. It was also mistakenly reported in [3] that there exists a $k$-colourable $P_5$-free graph that is not $(k + 1)$-mixing (see [9]). This leaves $t = 5$ as the last open case.

In this paper, we investigate this question for a subclass of $P_5$-free graphs, namely $3K_1$-free graphs. This class of graphs also includes the perfect class of co–bipartite graphs.

**Theorem 2.** If $G$ is a $k$-colourable $3K_1$-free graph, then $G$ is $(k + 1)$-mixing and the $(k + 1)$-recolouring diameter of $G$ is at most $4|V(G)|$.

The proof of Theorem 2 leads to a polynomial time algorithm to find a path of length at most $4|V(G)|$ between any two $(k + 1)$-colourings of $G$ in the recolouring graph.

The rest of the paper is organized as follows. In Section 2 we give definitions and notation used throughout the paper. We prove Theorem 1 in Section 3 and we prove Theorem 2 and in Section 4. We end with some discussion on future work in Section 5.

## 2 Preliminaries

For a graph $G$, a **clique** of $G$ is a set of pairwise adjacent vertices and a **stable set** is a set of pairwise non-adjacent vertices. A graph $G$ is $3K_1$-free if the maximum number of vertices in a stable set of $G$ is at most 2. The **clique number** of $G$, denoted by $\omega(G)$, is the maximum number of vertices in a clique of $G$. The **chromatic number** of $G$, denoted by $\chi(G)$, is the minimum $k$ such that $G$ is $k$-colourable. Clearly, $\chi(G) \geq \omega(G)$. A graph $G$ is perfect if for all induced subgraphs $H$ of $G$, $\chi(H) = \omega(H)$.

The **complement** of $G$, denoted $\overline{G}$, is the graph with vertex-set $V(G)$ such that $uv \in E(G)$ exactly when $uv \notin E(G)$. A graph is **bipartite** if its vertices can be partitioned into two stable sets and a graph is **co–bipartite** if it is the complement of a bipartite graph. A **hole** is a chordless cycle on at least five vertices and an **antihole** is the complement of a hole. A hole is **even** or **odd** if it has an even or odd number of vertices, respectively. For a set of graphs $\mathcal{H}$, we say that $G$ is $\mathcal{H}$-free if $G$ does not contain an induced subgraph isomorphic to any graph in $\mathcal{H}$. A graph is **perfect** if and only if it is (odd hole, odd antihole)-free [6]. A graph is **weakly chordal** if it is (hole, antihole)-free. Clearly, a graph $G$ is weakly chordal if and only if $\overline{G}$ is weakly chordal.

For a vertex $v \in V(G)$, the **open neighbourhood** of $v$ is the set of vertices adjacent to $v$ in $G$. The **closed neighbourhood** of $v$ is the set of vertices adjacent to $v$ in $G$ together with $v$. For $X, Y \subseteq V(G)$, we say that $X$ is **complete** to $Y$ if every vertex in $X$ is adjacent to every vertex in $Y$. If no vertex of $X$ is adjacent to a vertex of $Y$, we say that $X$ is **anticanonically** to $Y$. Let $G$ and $H$ be vertex-disjoint graphs and let $v \in V(G)$. By **substituting** $H$ for the vertex $v$ of $G$, we mean taking the graph $G - v$ and adding an edge between every vertex of $H$ and every vertex of $G - v$ that is adjacent to $v$ in $G$.

For a colouring $\alpha$ of $G$ and $X \subseteq V(G)$, we say that the colour $c$ **appears** in $X$ if $\alpha(x) = c$ for some $x \in X$. A $k$-colouring of a graph $G$ is called **frozen** if it is an isolated vertex in the recolouring graph $R_k(G)$. In other words, for every vertex $v \in V(G)$, each of the $k$ colours appears in the closed neighbourhood of $v$.

## 3 Frozen colourings of weakly chordal graphs

In this section we prove Theorem 1. One technique to prove that a graph $G$ is not $k$-mixing is to exhibit a frozen $k$-colouring of $G$. We construct a family of graphs $\{G_n \mid n \geq 1\}$ such that $G_n$ is a $k$-colourable weakly chordal graph that has a frozen $(k + n)$-colouring. See Figure 1 for a 3-colouring and a frozen 4-colouring of $G_1$. For $n \geq 2$, we recursively construct $G_n$ by substituting $G_{n-1}$ into four vertices of $G_1$ (see Figure 2).
We first prove that substituting a weakly chordal graph for some vertex of a weakly chordal
graph results in a weakly chordal graph. We note that there might be a proof of this in the
literature, and for example, Lovász proved an analogous theorem for perfect graphs [8].

**Theorem 3.** Substituting a weakly chordal graph for some vertex of a weakly chordal graph
results in a weakly chordal graph.

**Proof.** Let $G_1$ and $G_2$ be vertex-disjoint weakly chordal graphs and let $v \in V(G_1)$. Let $G$ be
the graph obtained by substituting $G_2$ for the vertex $v$ of $G_1$.

By contradiction, suppose $G$ contains a hole $H$. Then $H$ must contain at least 2 vertices
$v_1, v_2$ of $G_2$ since $G_1$ is a weakly chordal graph. Furthermore, since $G_2$ is a weakly chordal
graph, $H$ must contain at least one vertex $x$ in $G_1$ that is either adjacent to $v_1$ or $v_2$ in $G$. But
any vertex of $G - G_2$ that has a neighbour in $G_2$ is complete to $G_2$. So $x$ must be adjacent to
both $v_1$ and $v_2$. Since $x$ can have at most two neighbours in $H$ and since $H$ is a hole, $H$ cannot
contain any more neighbours of $x$. Then $H$ cannot contain another vertex from $G_2$ since $x$ is
complete to $G_2$. But any other vertex of $H$ adjacent to $v_1$ or $v_2$ must be adjacent to both $v_1$
and $v_2$, so $H$ cannot be a hole, a contradiction.

Now suppose that $G$ contains an antihole. Note that $\overline{G}$ is obtained by substituting the weakly
chordal graph $\overline{G_2}$ into the vertex $v$ of the weakly chordal graph $\overline{G_1}$. But since $G$ contains
an antihole, $\overline{G}$ contains a hole, a contradiction. \hfill \Box

**Lemma 1.** For all $n \geq 1$, $G_n$ is a weakly chordal graph.

**Proof.** The proof is by induction on $n$. It is easy to verify that $G_1$ is weakly chordal and so the
statement holds for $n = 1$. By the induction hypothesis, $G_{n-1}$ is a weakly chordal graph. The
graph $G_n$ is constructed by substituting $G_{n-1}$ into 4 vertices of $G_1$. Since $G_1$ and $G_{n-1}$ are
both weakly chordal graphs, it follows from Theorem 3 that $G_n$ is a weakly chordal graph. \hfill \Box

We are now ready to prove Theorem 1, which follows from Lemma 2 and 3. Recalling the
notation used in Figure 2, note that in $G_n$ and for $v \in \{w, x, y, z\}$, $v$ is complete to exactly
three copies of $G_{n-1}$ and anticomplete to the other copy of $G_{n-1}$. For $v \in \{w, x, y, z\}$, let $G^{w}_{n-1}$
denote the copy of $G_{n-1}$ in $G_n$ that is anticomplete to $v$.

**Lemma 2.** For all $n \geq 1$, $\chi(G_n) = \omega(G_n) = 2n + 1$.

**Proof.** The proof is by induction on $n$. The statement holds for $n = 1$ since $G_1$ is 3-colourable
and contains a clique of size 3 (see Figure 1). By the induction hypothesis, $\chi(G_{n-1}) = \omega(G_{n-1}) = 2n - 1$. Fix a $(2n - 1)$-colouring $\alpha$ of $G_{n-1}$. We show how to extend $\alpha$ to a
$(2n + 1)$-colouring of $G_n$. Since each copy of $G_{n-1}$ is pairwise anticomplete, we can colour each
copy of $G_{n-1}$ identically using $\alpha$. To complete this colouring of $G_n$, we make $\alpha(w) = \alpha(z) = 2n$
and $\alpha(x) = \alpha(y) = 2n + 1$. Since $wz, xy \notin E(G)$, this gives a proper $(2n + 1)$-colouring of

![Figure 1: A 3-colouring and frozen 4-colouring of $G_1$.](image)
G_n. To find a clique of size 2n + 1 in G_n, take a clique K of size 2n - 1 in G^z_{n-1}. Then since wx \in E(G) and since \{w, x\} is complete to G^w_{n-1}, it follows that K \cup \{w, x\} is a clique of size 2n + 1 in G_n.

Lemma 3. For all n \geq 1, G_n has a frozen (3n + 1)-colouring.

Proof. The proof is by induction on n. The statement holds for n = 1 since G_1 has a frozen 4-colouring (see Figure 1). By the induction hypothesis, G_{n-1} has a frozen (3n - 2)-colouring. To construct a frozen (3n + 1)-colouring \alpha of G_n, we take a frozen (3n - 2)-colouring of each copy of G_{n-1} in G_n using a different set of colours.

For v \in \{w, x, y, z\}, let \alpha_v v denote the colouring of G_n restricted to the subgraph G^w_{n-1}. Let \alpha^w be a frozen (3n - 2)-colouring of G^w_{n-1} using the colours \{1, 2, \ldots, 3n - 2\}. Let \alpha^x, \alpha^y, \alpha^z be frozen (3n - 2)-colourings of G^x_{n-1}, G^y_{n-1}, G^z_{n-1} using the colours \{1, 2, \ldots, 3n - 3, 3n - 1\}, \{1, 2, \ldots, 3n - 3, 3n\}, \{1, 2, \ldots, 3n - 3, 3n + 1\}, respectively. Since each each copy of G_{n-1} is pairwise anticomplete, this creates no conflicts. To complete this colouring of G_n, make \alpha(w) = 3n - 2, \alpha(x) = 3n - 1, \alpha(y) = 3n, and \alpha(z) = 3n + 1. Note that for each v \in \{w, x, y, z\}, \alpha(v) only appears on v and in G^v_{n-1}. Since v is anticomplete to G^w_{n-1}, this creates no conflicts. Therefore, \alpha is a proper (3n + 1)-colouring of G_n.

To see that \alpha is a frozen colouring, first examine a vertex u in G^w_{n-1} for v \in \{w, x, y, z\}. By construction, there are 3n - 2 colours appearing on the closed neighbourhood of u in G^w_{n-1}. Also by construction, the remaining 3 colours are used to colour \{w, x, y, z\} \setminus \{v\}. Since each of \{w, x, y, z\} \setminus \{v\} is complete to G^w_{n-1}, all 3n + 1 colours appear on the closed neighbourhood of u and it cannot be recoloured. Now examine vertex v \in \{w, x, y, z\}. Since v is complete to each G^w_{n-1} for u \in \{w, x, y, z\} \setminus \{v\}, there are 3n colours appearing on the open neighbourhood of v. Since \alpha is a proper colouring, the last colour is being used to colour v and so it cannot be recoloured.

4 Recolouring the complement of triangle-free graphs

In this section we prove Theorem 2. Note that in any colouring of a 3K_1-free graph at most two vertices share the same colour. With this in mind, it is not hard to see that an optimal colouring of a 3K_1-free graph can be found in polynomial time by finding a maximum matching in the complement. We begin by proving the following lemma.

Lemma 4. Let G be a k-colourable 3K_1-free graph. In any (k + 1)-colouring of G, there exists a colour c that either does not appear in G or is used to colour exactly one vertex of G.
Proof. Let $G$ be as in the statement of the lemma and fix some $(k+1)$-colouring of $G$. We can assume that all $k+1$ colours appear on the vertices of $G$ since, if not, the first condition is satisfied. Now by contradiction assume that all $k+1$ colours appear twice on the vertices of $G$. We know that $|V(G)| \leq 2\chi(G)$ since we can partition the vertices of $G$ into at most $\chi(G)$ stable sets, each having at most two vertices. But since all $k+1$ colours appear twice on the vertices of $G$, we have $|V(G)| \geq 2(k+1) > 2\chi(G)$, a contradiction. \qed

Let $\gamma$ be a $\chi(G)$-colouring of $G$ and let $\mathcal{C}$ be the partition of the vertices of $G$ given by the colour classes of $\gamma$. Given two colourings $\alpha$ and $\beta$ of $G$, our strategy is to first recolour each to a $\chi(G)$-colouring $\alpha'$ and $\beta'$ whose colour classes correspond exactly to the partition $\mathcal{C}$, and then use the following Renaming Lemma.

Lemma 5 (Renaming Lemma [3]). If $\alpha'$ and $\beta'$ are two $k$-colourings of $G$ that induce the same partition of vertices into colour classes, then $\alpha'$ can be recoloured into $\beta'$ in $\mathcal{R}_{k+1}(G)$ by recolouring each vertex at most 2 times.

Proof of Theorem 2. Let $G$ be a $k$-colourable $3K_1$-free graph and let $\alpha$ and $\beta$ be two $(k+1)$-colourings of $G$. Fix a $\chi(G)$-colouring $\gamma$ of $G$ and let $\mathcal{C}$ be the partition of $V(G)$ given by the colour classes of $\gamma$. Note that $|\mathcal{C}| = \chi(G)$ and each colour class $C \in \mathcal{C}$ has one or two vertices.

Claim 1. The colouring $\alpha$ can be recoloured into a $\chi(G)$-colouring $\alpha'$ of $G$ such that $\alpha'$ and $\gamma$ partition the vertices of $G$ into the same colour classes by recolouring each vertex at most once.

We prove the claim by induction on $\chi(G)$. For $\chi(G) = 1$ the claim is trivial. Now assume the statement holds for $\chi(G) - 1$. By Lemma 4, there exists some colour $c$ of $\alpha$ that either does not appear in $G$ or appears on exactly one vertex of $G$.

First suppose the colour $c$ appears in $G$ and let $u$ be the vertex coloured $c$. Let $\mathcal{C}$ be the colour class of $\gamma$ which contains $u$. If $\mathcal{C}$ contains some other vertex $v$ then, from $\alpha$, recolour $v$ with $c$. If instead $c$ does not appear in $G$, we select $u$, $v$, and $\mathcal{C}$ as follows. Take some colour class of $\alpha$ that is not a colour class of $\gamma$ (if no such colour class exists we are done) and some vertex $u$ in this colour class. From $\alpha$, recolour $u$ with the colour $c$. Let $\mathcal{C}$ be the colour class of $\gamma$ which contains $u$. If there is another vertex $v \in \mathcal{C}$ then recolour $v$ to the colour $c$. This can be done since $uv \notin E$ and no other vertex is coloured $c$.

Let $\alpha_C$ be the current colouring of $G$ restricted to $G - C$ with $c$ taken out of its set of colours. Let $\gamma_C$ be the colouring $\gamma$ restricted to $G - C$.

Since $\gamma$ is a $\chi(G)$-colouring of $G$, it follows that $\chi(G - C) = \chi(G) - 1$. Then $\alpha_C$ is a $k$-colouring of $G - C$ (since we removed the colour $c$) and $k \geq \chi(G - C) + 1$. By the induction hypothesis, $\alpha_C$ can be recoloured into a $(\chi(G - C) - 1)$-colouring $\alpha'_C$, of $G - C$ such that $\alpha'_C$ and $\gamma_C$ partition the vertices of $G$ into the same colour classes by recolouring each vertex at most once. Since the colour of $u$ and $v$ are never used again, this recolouring sequence from $\alpha_C$ to $\alpha'_C$ can be extended to a recolouring sequence between $\alpha$ and $\alpha'$. Since $u$ and $v$ are recoloured at most once, each vertex of $G$ is recoloured at most once. This completes the proof of the claim.

Similarly, $\beta$ can be recoloured into a $\chi(G)$-colouring $\beta'$ such that $\beta'$ and $\gamma$ partition the vertices of $G$ into the same colour classes by recolouring each vertex at most once. By Lemma 5, we can recolour $\alpha'$ into $\beta'$ by recolouring each vertex at most twice. This gives us a recolouring sequence from $\alpha$ to $\beta$ by recolouring each vertex at most 4 times. \qed

5 Conclusion

In this paper, we answered an open question of Feghali and Fiala by showing that for all $n \geq 1$, there exists a $k$-colourable weakly chordal graph with a frozen $(k+n)$-colouring. We also showed that every $k$-colourable $3K_1$-free graph is $(k+1)$-mixing with a linear $(k+1)$-recolouring diameter. It is an open problem whether a $k$-colourable $P_3$-free graph is $(k+1)$-mixing [9]. This
question has been answered for several subclasses of $P_5$-free graphs. These include when $k = 2$ [4], for co–chordal graphs, for $(P_5, P_5, C_5)$-free graphs and $k = 3$ [7], for $P_4$-sparse graphs [1], and now for $3K_1$-free graphs. It may be hard to answer this question for the entire class of $P_5$-free graphs and so it would be interesting to continue studying subclasses of $P_5$-free graphs for which this question can be answered.

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