Consistent Histories of Systems and Measurements in Spacetime

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Abstract

Traditional interpretations of quantum theory in terms of wave function collapse are particularly unappealing when considering the universe as a whole, where there is no clean separation between classical observer and quantum system and where the description is inherently relativistic. As an alternative, the consistent histories approach provides an attractive “no collapse” interpretation of quantum physics. Consistent histories can also be linked to path-integral formulations that may be readily generalized to the relativistic case. A previous paper described how, in such a relativistic spacetime path formalism, the quantum history of the universe could be considered to be an eigenstate of the measurements made within it. However, two important topics were not addressed in detail there: a model of measurement processes in the context of quantum histories in spacetime and a justification for why the probabilities for each possible cosmological eigenstate should follow Born’s rule. The present paper addresses these topics by showing how Zurek’s concepts of einselection and envariance can be applied in the context of relativistic spacetime and quantum histories. The result is a model of systems and subsystems within the universe and their interaction with each other and their environment.

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I. INTRODUCTION

In recent years, work on quantum gravity and quantum cosmology, in particular, has made it impossible to avoid the interpretational issues of quantum mechanics. When considering the universe as a whole, there is no longer any clean separation between the observer and the observed system or between the classical and quantum worlds. Further, any complete cosmological theory must be fully relativistic. In this arena, traditional interpretations in terms of wave function collapse are particularly unappealing.

The consistent or decoherent histories approach provides an attractive “no collapse” interpretation of quantum physics [1–4]. The basic idea of the approach is to assign probabilities to histories, which are time-ordered sequences of quantum properties of a system. When a family of histories are chosen in such a way that they are consistent, they decohere, and classical probabilities can be assigned to them as alternative histories of the system.

Consistent histories can also be linked to path-integral formulations of quantum mechanics [5–7]. If the quantum properties under consideration can be expressed in terms of particle positions, then a quantum history can be considered to be a superposition of those paths in which the particle passes through positions with the required properties at the required times. Looked at another way, a particle path can be considered to be a fine-grained history in which the particle position is exactly determined at every time, while a path integral represents a coarse-grained history as a superposition of all paths meeting some more general criteria. When the criteria are properly chosen, the states for these coarse-grained histories decohere and can have classical probabilities assigned to them [8–10].

The path-integral formulation can also be readily generalized to the relativistic case by replacing paths in space parameterized by time with paths in spacetime parameterized by an invariant path evolution parameter [10–19]. The same consistent histories interpretation of path integrals can then be carried over from the non-relativistic to the relativistic case, where “position” is now understood to mean four-position in spacetime. (For other approaches to relativistic generalization of the consistent histories approach, see [20–23].)

In [24] I presented an approach, based on a relativistic spacetime path formalism [25, 26], in which entire coarse-grained histories of the universe decohere for all time. Each coarse-grained history of all of spacetime is represented by a cosmological state that is constrained by correlations introduced by the measurement-like processes that occur within the history of
the universe. The cosmological state is essentially an eigenstate of the operators representing those processes, and each such state is orthogonal to the rest.

It is only necessary to consider one of these cosmological states to be the “real” history of the actual universe, though, of course, we have only very partial information on which history this actually is. Nevertheless, it is shown in [24] that, if this “real” history is selected with a probability determined by the normal Born rule for the cosmological states, then, from within the history, all observations made at the classical level can be expected to be distributed according to the statistical rules of quantum theory.

Two important topics were not addressed in [24], however. First, no detailed model was given of how a measuring apparatus, as a part of the universe being measured, becomes correlated with some other part of the universe and itself decoheres into non-interfering states. Second, no justification of Born’s rule was given for cosmological states; it was simply shown that the assumption of Born’s rule for cosmological states leads to the proper statistical distribution for repeated experiment results (avoiding the circularity problem with some previous arguments based on relative frequencies [27, 28]).

In this paper, I will address both of these topics (though, for the present paper, I will take a somewhat restricted view of what a “subsystem” is, as described in Sec. IV). In doing so, it is important to not implicitly presuppose the results of [24], but, rather, to independently support the assumptions made there, in order not to re-introduce circularity problems. As noted in [24], the arguments of Zurek on environment-induced superselection (einselection) and entanglement-assisted invariance (envariance) are particularly relevant in this regard.

Zurek has written extensively on einselection and envariance in the non-relativistic context (see, for example, [29–34]). Here, I extend these concepts to the context of a relativistic, spacetime formalism for quantum histories. Einselection addresses the first of the topics introduced above, while envariance addresses the second.

Section II provides a brief overview of the the consistent histories approach to non-relativistic quantum mechanics and motivates the relativistic generalization developed subsequently. This generalization is grounded in the spacetime path formalism, but the mathematics of path integration is not actually required for the results discussed in this paper. The underlying mathematics can instead be packaged in the familiar notations of relativistic states and fields. However, the traditional quantum field theory formalism still presents some difficulties for a straightforward description of relativistic quantum histories. Sec-
tion III addresses these issues through a modified spacetime formalism. The grounding of this formalism in the spacetime path approach is discussed in the appendix, making the connection to previous work in [24–26].

Once the formalism is established, Sec. IV describes the foundational concepts of systems and subsystems used in subsequent sections. The core of the paper comprises Sec. V, which addresses the topic of measurements, and Sec. VI, which address the topic of Born’s rule. Section VII then applies these concepts to the paradigmatic thought experiment of Schrödinger’s cat. Finally, Sec. VIII presents some concluding remarks on the assumptions underlying Zurek’s envariance arguments in relation to the spacetime approach discussed here.

Throughout, I will use a spacetime metric signature of (− + + +) and take $\hbar = c = 1$.

II. CONSISTENT HISTORIES

The consistent histories approach to non-relativistic quantum mechanics assigns probabilities to quantum histories. Such a history is a sequence of quantum properties at a succession of times $t_0 < t_1 < \ldots < t_n$. At each time $t_i$, the properties of interest are represented by a set of projection operators $\hat{P}_i^{\alpha_i}$, where the $\alpha_i$ label different properties the system might have at time $t_i$. These operators satisfy

$$\sum_{\alpha_i} \hat{P}_i^{\alpha_i} = 1 \quad (1)$$

and

$$\hat{P}_i^{\alpha_i} \hat{P}_i^{\beta_i} = \delta_{\alpha \beta} \hat{P}_i^{\alpha_i}$$

Each possible history is then completely identified by a sequence of labels

$$\mathbf{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

and the set of all such histories is known as a family of histories for the system.

Now consider the time evolution of a non-relativistic quantum system from an initial state $|\Phi(t_0)\rangle$ under a generically time-dependent Hamiltonian operator $\hat{H}(t)$:

$$|\Phi(t)\rangle = T(t, t_0) |\Phi(t_0)\rangle \, , \quad (2)$$
where
\[ T(t_f, t_i) \equiv e^{-i \int_{t_i}^{t_f} \text{d}t \hat{H}(t)} \, . \]

Insert Eq. (1) into the time evolution of Eq. (2) at each of the times \( t_i \):
\[
|\Phi(t_n)\rangle = \sum_{\alpha_n} \hat{P}^\alpha_n T(t_n, t_{n-1}) \sum_{\alpha_{n-1}} \hat{P}^{\alpha_{n-1}} T(t_{n-1}, t_{n-2}) \cdots \sum_{\alpha_1} \hat{P}^{\alpha_1} T(t_1, t_0) |\Phi(t_0)\rangle
= \sum_{\alpha} \hat{C}^\alpha |\Phi(t_0)\rangle ,
\]
where
\[ \hat{C}^\alpha \equiv \hat{P}^\alpha_n T(t_n, t_{n-1}) \hat{P}^{\alpha_{n-1}} T(t_{n-1}, t_{n-2}) \cdots \hat{P}^{\alpha_1} T(t_1, t_0) . \] (3)

The operator \( \hat{C}^\alpha \) defined by Eq. (3) is the *chain* or *class* operator for the history \( \alpha \) and the state \( \hat{C}^\alpha |\Phi(t_0)\rangle \) is the *branch* of the system state \( |\Phi(t_n)\rangle \) associated with the history \( \alpha \). We are interested in families of histories such that all branches are mutually orthogonal, that is,
\[ \langle \Phi(t_0) | \hat{C}^{\alpha'} \hat{C}^\alpha |\Phi(t_0)\rangle = \delta_{\alpha' \alpha} p(\alpha) . \] (4)

These are *consistency* or *decoherence* conditions and a family of histories that satisfies them is known as a *consistent* or *decoherent* family. \( p(\alpha) \) then gives the probability for the history \( \alpha \).

The left-hand side of Eq. (4) is known as the *decoherence functional* and is often written in the form
\[ D(\alpha, \alpha') \equiv \langle \Phi(t_0) | \hat{C}^{\alpha'} \hat{C}^\alpha |\Phi(t_0)\rangle = \text{Tr}(\hat{C}^\alpha \rho_0 \hat{C}^{\alpha'}\dagger) , \] (5)
where
\[ \rho_0 \equiv |\Phi(t_0)\rangle \langle \Phi(t_0)| . \]
In this form, the decoherence functional can be generalized to density matrices \( \rho_0 \) for non-pure initial states, but this generality will not be needed here.

Also, the chain operators as defined by Eq. (3) satisfy
\[ \sum_{\alpha} \hat{C}^\alpha = T(t_n, t_0) . \] (6)

It is perhaps more common to define chain operators in the form
\[ \hat{K}^\alpha \equiv \hat{P}^\alpha_n (t_n) \hat{P}^{\alpha_{n-1}}(t_{n-1}) \cdots \hat{P}^{\alpha_1}(t_1) , \]
where the $\hat{P}_i^{\alpha_i}(t_i)$ are the Heisenberg form of the projection operators defined by

$$\hat{P}_i^{\alpha_i}(t_i) \equiv T(t_0, t_i) \hat{P}_i^{\alpha_i} T(t_i, t_0),$$

so that

$$\sum_{\alpha} \hat{K}^\alpha = 1.$$ 

However, the two forms of chain operators are related by

$$\hat{K}^\alpha = T(t_0, t_n) \hat{C}^\alpha,$$

and the extra propagator factor cancels out if $\hat{K}$ is used instead of $\hat{C}$ in the definition of the decoherence functional, Eq. (5). Thus, the two forms of chain operators are essentially equivalent as far as consistent histories are concerned. But the form satisfying the condition of Eq. (6) is more closely analogous to the form of the similar operators to be defined in the following.

As outlined above, a quantum history applies to a single quantum system. However, such a system may be considered to have subsystems in the usual way, by taking the Hilbert space for the system to be the product space of the Hilbert spaces of the subsystems. Projection operators in the history for the system may then represent properties of the the system as a whole or either of its subsystems.

In the interesting cases, of course, the Hamiltonian dynamics for such a system will result in the entanglement of its subsystems. But consider the simple situation in which there are two subsystems: an apparatus $\mathcal{A}$ designed to measure a quantum system $\mathcal{B}$. This has the basic structure of systems that will be of interest in the following.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are initially both in their own identifiable initial states so that

$$|\Phi(t_0)\rangle = |\Phi_A(t_0)\rangle|\Phi_B(t_0)\rangle.$$

Further, suppose that the time development of the system from $t_0$ to $t_1$ does not affect $\mathcal{A}$, so that

$$T(t_1, t_0)|\Phi_A(t_0)\rangle|\Phi_B(t_0)\rangle = |\Phi_A(t_0)\rangle|\Phi_B(t_1)\rangle,$$

while the time development from $t_1$ to $t_2$ does not affect $\mathcal{B}$, so that

$$T(t_2, t_1)|\Phi_A(t_0)\rangle|\Phi_B(t_1)\rangle = |\Phi_A(t_2)\rangle|\Phi_B(t_1)\rangle.$$
Then the only properties of interest at $t_1$ will be those about $\mathcal{B}$, and those of interest at $t_2$ will be about $\mathcal{A}$. That is, the chain operators will have the form

$$C^{(\alpha\beta)} = \hat{P}_A^\alpha T(t_2, t_1) \hat{P}_B^\beta T(t_1, t_0),$$

where the projection operators $\hat{P}_A^\alpha$ act only on the component of the system state for subsystem $\mathcal{A}$ while the $\hat{P}_B^\beta$ act only on the component for $\mathcal{B}$.

Now, for an ideal measurement process, the dynamics correlates the pointer states of the apparatus with the states of the measured subsystem. That is, if $\alpha$ in Eq. (7) enumerates pointer states of $\mathcal{A}$ corresponding to similarly enumerated measured states $\beta$ of $\mathcal{B}$, then the time evolution $T(t_2, t_1)$ ensures that $C^{(\alpha\beta)}$ is zero unless $\alpha = \beta$. Note that this does not change that fact that the $\hat{P}_A^\alpha$ act only on $\mathcal{A}$ states and the $\hat{P}_B^\beta$ act only on $\mathcal{B}$ states.

The above simple analysis motivates the following approach for moving from non-relativistic to relativistic consistent histories.

Consider the subsystems $\mathcal{A}$ and $\mathcal{B}$ to occupy physical three-dimensional volumes within the overall combined system (e.g., the physical space occupied by the apparatus $\mathcal{A}$, etc.). Then take the time interval $[t_1, t_2]$ along with the 3-volume for $\mathcal{A}$, forming a four-dimensional hypervolume within which all interesting dynamics happens for $\mathcal{A}$. Similarly, take the time interval $[t_0, t_1]$ with the 3-volume for $\mathcal{B}$ to form a hypervolume of interest for $\mathcal{B}$.

Heuristically, what is desired is to recast chain operators of the form of Eq. (7) into a form something like

$$C^{(\alpha\beta)} = \hat{P}_A^\alpha \hat{G}_A \hat{P}_B^\beta \hat{G}_B,$$

where the operators $\hat{P}_A^\alpha$ represent properties of interest about the hypervolume associated with $\mathcal{A}$ and $\hat{G}_A$ represents the dynamical interactions that occur in that hypervolume, and similarly for $\hat{P}_B^\beta$, $\hat{G}_B$ and $\mathcal{B}$. The point of this is to develop a spacetime formulation for the chain operator that is manifestly Lorentz invariant.

The use of spacetime hypervolumes here has some similarity to previous analyses of the probabilities for a particle to enter a specific spacetime region in timeless quantum theories [18, 19, 35]. However, in the present case there will generally be many particles in each hypervolume (i.e., the particles that physically make up the subsystem in that hypervolume) and these particles will be interacting within and across the hypervolumes. Thus, to handle multiple, interacting particles in spacetime, we turn to the formalisms of quantum field theory.
III. SPACETIME FORMALISM

A (Heisenberg picture) quantum field is an operator-valued function $\hat{\psi}$ on spacetime satisfying the Klein-Gordon equation

$$\left( \frac{\partial^2}{\partial x^2} - m^2 \right) \hat{\psi}(x) = 0. \quad (9)$$

The operator $\hat{\psi}(x)$ acts on the Fock space of particle position states, destroying a particle at position $x$, while its adjoint $\hat{\psi}^\dagger(x)$ acts to create a particle at $x$ [36–38].

Distinguish particle fields $\hat{\psi}_+$ from antiparticle fields $\hat{\psi}_-$. Each kind of field has a specific nonzero commutator with its adjoint:

$$[\hat{\psi}_\pm(x'), \hat{\psi}_\pm^\dagger(x)] = \Delta_\pm(x' - x), \quad (10)$$

where

$$\Delta_\pm(x' - x) \equiv (2\pi)^3 \int d^3 p \frac{\delta[\mp \omega_p(x'^0 - x^0) + p \cdot (x' - x)]}{2\omega_p},$$

with $\omega_p \equiv \sqrt{p^2 + m^2}$. (For simplicity I will only consider scalar fields here. The generalization to non-scalar fields and fermionic anticommutation rules is straightforward and does not substantially effect the following discussion.)

Let $|0\rangle$ be the vacuum state of the Fock space. Then single particle and antiparticle position states are given by

$$|x_\pm\rangle = \hat{\psi}_\pm^\dagger(x)|0\rangle. \quad (11)$$

While these states represent a particle or antiparticle localized at a specific position $x$, they are not orthogonal and states for different positions overlap. This is due to the fact that they are constrained to be on-shell by Eq. (9). Indeed, the commutation relations from Eq. (10) imply that

$$\langle x'_\pm | x_\pm \rangle = \delta_\pm(x' - x).$$

They do obey a completeness relation, but only over any spacelike hypersurface $\Sigma$, not over all of spacetime [18]:

$$i \int_\Sigma d^3 x \left( \langle x_+ | \hat{\partial}_0 \hat{\psi}_+ | x_+ \rangle - |x_- \rangle \hat{\partial}_0 \langle x_- \rangle \right) = 1.$$

These properties make the $|x_\pm\rangle$ states inconvenient for constructing projection operators. Instead, one would like to have an orthogonal basis for spacetime position states, analogous to the three-dimensional spacial position states familiar from non-relativistic quantum
mechanics. Let $|x\rangle_0$ be such position states, where

$$o(x'|x\rangle_0 = \delta^4(x' - x). \quad (12)$$

Extend these states to the multiparticle position Fock space by defining fields $\hat{\psi}_0(x)$ so that

$$|x\rangle_0 = \hat{\psi}_{0\dagger}(x)|0\rangle.$$

The commutation rule

$$[\hat{\psi}_0(x'), \hat{\psi}_{0\dagger}(x)] = \delta^4(x' - x)$$

then leads to the desired orthogonality relation of Eq. (12).

Of course, the $\hat{\psi}_0(x)$ field does not satisfy the Klein-Gordon equation, Eq. (9), so the $|x\rangle_0$ states are off-shell. However, the $\hat{\psi}_0$ field can be used as a basis for redefining the on-shell $\hat{\psi}_\pm$ fields as

$$\hat{\psi}_\pm(x) \equiv \int d^4x_0 \Delta_\pm(x - x_0) \hat{\psi}_0(x_0).$$

The fields defined in this way do satisfy the Klein-Gordon equation, but they no longer follow the commutation rule of Eq. (10). Instead, they follow a similar commutation rule with $\hat{\psi}_0(x)$:

$$[\hat{\psi}_\pm(x'), \hat{\psi}_{0\dagger}(x)] = \Delta_\pm(x' - x).$$

Using the new fields to create the states $|x_\pm\rangle$ as in Eq. (11) then implies that

$$\langle x'_\pm|x\rangle_0 = \Delta_\pm(x' - x). \quad (13)$$

Actually, more useful in the following will be the field operator

$$\hat{\psi}(x) \equiv \int d^4x_0 \Delta(x - x_0) \hat{\psi}_0(x_0),$$

where $\Delta(x - x_0)$ is the Feynman propagator

$$\Delta(x - x_0) \equiv -i(2\pi)^{-4} \int d^4p \frac{e^{ip\cdot(x-x_0)}}{p^2 + m^2 - i\epsilon},$$

with the commutation relationship

$$[\hat{\psi}(x'), \hat{\psi}_{0\dagger}(x)] = \Delta(x' - x).$$

and the corresponding position states

$$|x\rangle = \hat{\psi}_{0\dagger}(x)|0\rangle = \int d^4x_0 \Delta(x - x_0)|x_0\rangle_0$$
such that

$$\langle x'|x \rangle_0 = \Delta(x' - x).$$

(14)

The $|x\rangle$ states are again off-shell. However, using the well-known relation [36–38]

$$\Delta(x - x_0) = \theta(x^0 - x_0^0)\Delta^+ (x - x_0) + \theta(x^0_0 - x^0)\Delta^- (x - x_0)$$

in Eq. (14), along with Eq. (13), gives

$$\langle x'|x \rangle_0 = \theta(x^0 - x^0_0)\langle x'_+|x \rangle_0 + \theta(x^0_0 - x^0)\langle x'_-|x \rangle_0.$$  (15)

In a path integral approach, the probability amplitude $\langle x'|x \rangle_0$ can be interpreted as the superposition of the probability amplitudes for each possible spacetime path from $x$ to $x'$. One can think of the $|x\rangle_0$ states as representing the position $x$ at which the paths start, while the $|x'\rangle$ states represent the position $x'$ at which the paths end. The difference between the states reflects the directionality of the paths—propagation is always from the start of the path to the end of the path. (The appendix covers in more detail the underlying spacetime path derivation of the formalism used in the main body of the text.)

The decomposition of Eq. (15) separates the case in which $x'$ is in the future of $x$ from that in which $x'$ is in the past of $x$. This shows that, while normal particles propagate into the future, antiparticles can effectively be considered to propagate backwards in time, into the past [11, 39, 40]. This division into particle and antiparticle paths depends, of course, on the choice of a specific coordinate system in which to define the time coordinate. However, if we take the time limit of the end point of the path to infinity for particles and negative infinity for antiparticles, then the particle/antiparticle distinction will be coordinate system independent [25].

The off-shell states $|x\rangle$ are particularly useful because they essentially represent virtual particles and the probability amplitudes $\langle x'|x \rangle_0$ are the propagation amplitudes for these particles on the inner edges of Feynman diagrams. They can therefore be used to construct a convenient representation for the scattering amplitudes of interacting particles.

To do this, it is first necessary to consider multiple fields corresponding to different types of particles that may interact with each other. Then, an individual interaction vertex can be considered an event at which some number of incoming particles are destroyed and some number of outgoing particles are created. Note that the qualifiers “incoming” and “outgoing”
are being used here in the sense of particle paths in spacetime, not in the sense of time—that is, the position states $|x\rangle$ are *not* being separated into particle and antiparticle states.

Such an interaction can be modeled using a *vertex operator* constructed from the appropriate number of annihilation and creation operators. For example, consider the case of an interaction with two incoming particles, one of type $a$ and one of type $b$, and two outgoing particles of the same types. The vertex operator for this interaction is

$$\hat{V} \equiv g \int d^4x \hat{\psi}_0^{(a)\dagger}(x)\hat{\psi}_0^{(b)\dagger}(x)\hat{\psi}^{(a)}(x)\hat{\psi}^{(b)}(x)$$

where the coefficient $g$ represents the relative probability amplitude of the interaction.

In the following, it will be convenient to use the special adjoint $\hat{\psi}^\dagger$ defined by

$$\hat{\psi}^\dagger(x) = \hat{\psi}_0^\dagger(x) \text{ and } \hat{\psi}_0^\dagger(x) = \hat{\psi}^\dagger(x).$$

With this notation, the expression for $\hat{V}$ becomes

$$\hat{V} = g \int d^4x \hat{\psi}_0^{(a)\dagger}(x)\hat{\psi}_0^{(b)\dagger}(x)\hat{\psi}^{(a)}(x)\hat{\psi}^{(b)}(x). \quad (16)$$

To account for the possibility of any number of interactions, we just need to sum up powers of $\hat{V}$ to obtain the *interaction operator*

$$\hat{G} \equiv \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \hat{V}^m = e^{-i\hat{V}}, \quad (17)$$

where the $1/m!$ factor accounts for all possible permutations of the $m$ identical factors of $\hat{V}$. The $-i$ factors are introduced so that $\hat{G}$ is unitary relative to the special adjoint (that is, $\hat{G}^\dagger \hat{G} = \hat{G} \hat{G}^\dagger = 1$), so long as $\hat{V}$ is self-adjoint relative to it (that is, $\hat{V}^\dagger = \hat{V}$).

The self-adjointness of $\hat{V}$ implies that an interaction must have the same number of incoming and outgoing particles, of the same types, at least when only one possible type of interaction is involved (as is the case with the example of Eq. (16)). The formalism can be easily extended to allow for multiple types of interactions by adding additional terms to the definition of $\hat{V}$. In this case, only the overall operator $\hat{V}$ needs to be self-adjoint, not the individual interaction terms.

As mentioned above, normal particle states are obtained in the $+\infty$ time limit, while antiparticle states are obtained in the $-\infty$ time limit. Moving to a momentum representation then results in (multiparticle) on-shell scattering in and out states. These states can be used with the interaction operator $\hat{G}$ to compute multipoint interaction amplitudes. Expanding
\( \hat{G} \) as in Eq. (17) gives a sum of Feynman diagrams for each possible number of interactions. The time-limit momentum states give the correct amplitudes for the truncated external legs of the diagrams [25].

Unfortunately, an interaction operator  \( \hat{G} \) of the form given in Eq. (17) with a vertex operator  \( \hat{V} \) of the form shown in Eq. (16) cannot actually generate all Feynman diagrams. For example, the vertex operator  \( \hat{V} \) from Eq. (16) necessarily has  \( \hat{V}|0\rangle = 0 \). This means that  \( \hat{G} \) cannot generate vacuum bubble diagrams. Indeed, in general,  \( \hat{V} \) cannot construct a vertex unless all incoming particles already exist in the incoming state, which prevents the construction of loops involving particles incoming from vertexes constructed “later”.

The problem is that the directionality of propagation implied by  \( \langle x'|x_0 \rangle \) is essentially arbitrary. We could just as well have defined “reverse” particle states  \( |\bar{x}\rangle \) and  \( |\bar{x}_0\rangle \) such that

\[
\langle \bar{x}'|\bar{x}\rangle_0 = \langle x|x'\rangle_0 = \Delta(x - x').
\]

That is, in terms of the spacetime paths,  \( |\bar{x}\rangle \) now represents the start of the paths, while  \( |\bar{x}_0\rangle \) represents the end.

To properly construct all possible Feynman diagrams, it is necessary to include such reverse particle states. This can be easily done in the formalism by doubling the Fock space through the addition of a reverse particle type  \( \bar{n} \) corresponding to each particle type  \( n \), whose field operators have the commutation rule

\[
[\hat{\psi}^{(n)}(x'), \hat{\psi}^{(n)}\dagger(x)] = \Delta(x - x').
\]

Then, define

\[
\hat{\psi}^{(n')}(x) \equiv \hat{\psi}^{(n)}(x) + \hat{\psi}^{(n)}\dagger(x).
\]

That is, using this operator, the destruction of an  \( n \) particle is treated as equivalent to the creation of an  \( \bar{n} \) particle (and vice versa). This is similar to the way particle destruction is related to antiparticle creation in the traditional field theory formalism, but the distinction for reverse particles is based on direction of propagation along particle paths (which is Lorentz invariant), not the direction of propagation in time.

Now use the new operators  \( \hat{\psi}^{(n')} \) instead of  \( \hat{\psi}^{(n)} \) in the construction of the vertex operator  \( \hat{V} \). For example, an interaction of the form given in Eq. (16) becomes

\[
\hat{V} = \int d^4x \, \hat{V}(x),
\]
where
\[ \hat{V}(x) \equiv g : \hat{\psi}^{(a')}\dagger(x)\hat{\psi}^{(b')}\dagger(x)\hat{\psi}^{(a')}\dagger(x)\hat{\psi}^{(b')}\dagger(x) : \quad (18) \]

with : \ldots : representing normal ordering, that is, placing all \( \hat{\psi} \) operators to the left of all \( \hat{\psi} \) operators in any product. The interaction operator \( \hat{G} = \exp(-i\hat{V}) \) in this vertex operator will then properly include all loops. Since the \( \bar{n} \) reverse particle types are only included in the formalism for this purpose, physical states, such as in and out scattering states, can be constructed using only the original particle types (that is, constructed using the original creation operators \( \hat{\psi}^{(n)\dagger} \)).

The need to include reverse propagation does complicate a bit the spacetime formalism for interaction presented here. However, the great advantage of the result is that Eq. (17) defining \( \hat{G} \) does not involve time-ordering, as is found in the Dyson expansion of the usual scattering operator \( \hat{S} \) [36–38]. This will be critical when considering the decomposition of the interactions within a system into a number of distinct subsystems, a topic to which we turn next.

**IV. SYSTEMS AND SUBSYSTEMS**

As reiterated by Zurek, “The Universe consists of systems” [34]. Quantum processes, measuring apparatuses and observers are all systems, and all subsystems of the system that is the universe as a whole. We make no \textit{a priori} distinction between “quantum” systems and “classical” systems.

We do, however, need to have a crisp definition of how to delineate what the systems of interest are in any given analysis of some or all of the universe. As discussed in Sec. II, it will be convenient here to define a system as being contained in a well-defined hypervolume \( \mathcal{V} \) of spacetime, disjoint from the hypervolumes occupied by all other systems of interest. This hypervolume does not have to be continuous or connected, though it must have measure greater than zero.

The effective identification of systems and subsystems with regions of spacetime is somewhat more restrictive than the typical generic definition used in quantum mechanics, in which a system is taken to be any portion of the universe whose state can be represented by a vector in an appropriate Hilbert space. For example, identifying subsystems with hypervolumes does not allow an overall set of degrees of freedom of a system to be divided between
two subsystems occupying the same physical space (such as when treating the microscope states of the molecules of a gas as the “environment” for the decoherence of the aggregate macroscopic properties of the gas). However, the identification works remarkably well for the situations considered in this paper, and it should be possible to subsequently extend the conclusions made here to more abstract concepts of systems.

The main benefit of identifying a system with a specific hypervolume is that it allows a straightforward definition of what interactions take place “within” the system. Define the restricted interaction operator

\[ \hat{V}_V \equiv \int_V d^4x \hat{V}(x), \]

with \( \hat{V}(x) \) being the vertex operator at position \( x \) (as in Eq. (18)). Then

\[ \hat{G}_V \equiv e^{-i\hat{V}_V} \]

generates interactions only within the hypervolume \( V \).

Let \( \bar{V} \) be the hypervolume of all spacetime with \( V \) removed. Define in and out states \( |\Psi_{\text{in}}\rangle_0 \) and \( |\Psi_{\text{out}}\rangle \) on \( \bar{V} \), that is, as superpositions of states with positions only within \( \bar{V} \) and thus outside \( V \). Then \( \langle \Psi_{\text{out}} | \hat{G}_V | \Psi_{\text{in}} \rangle_0 \) is the scattering amplitude for particles to enter \( V \) (from the in state in \( \bar{V} \)), interact only within \( V \) and then leave \( V \) (to the out state in \( \bar{V} \)). (Note that, while all interaction vertices in this process are restricted to be within \( V \), the paths of particles between such vertices are not so restricted.)

Suppose the system contained in \( V \) is now further divided into a set of \( N \) subsystems with corresponding disjoint hypervolumes \( V_i \) such that \( V = \bigcup_i V_i \). Then

\[ \hat{G}_V = \exp \left( -i \sum_{i=1}^N \hat{V}_{V_i} \right) = \prod_{i=1}^N e^{-i\hat{V}_{V_i}} = \prod_{i=1}^N \hat{G}_{V_i}. \]

(19)

Of course, the second equality above requires that all the \( \hat{V}_{V_i} \) commute. This makes sense conceptually, since the ordering of the subsystems should not effect the generation of the full set of interactions for the complete system. It can also be shown by explicit calculation that \( \hat{V}(x_1) \) and \( \hat{V}(x_2) \) commute for \( x_1 \neq x_2 \), so that \( \hat{V}_{V_i} \) and \( \hat{V}_{V_j} \) commute for disjoint \( V_i \) and \( V_j \).

Note that the easy decomposition of the \( \hat{G} \) operator given in Eq. (19) is only possible because its definition does not require the time-ordering of interactions, as is embodied in the usual scattering operator \( \hat{S} \). Indeed, the \( \hat{S} \) operator may only be decomposed into commuting
factors in the limiting case of widely separated, non-interacting clusters of particles (the so-called “cluster decomposition principle” [37, 41]), which is not very useful for the analysis of interacting subsystems. But note also that the commutivity of the interaction operators \( \hat{V}(x) \) necessary for Eq. (19) requires the inclusion of reverse particle types (as discussed in Sec. III).

We can convert the product of operators in Eq. (19) into a product of matrix elements using the usual trick of inserting resolutions of the identity. For the extended Fock space, such resolutions have the form

\[
1 = \int d\chi d\bar{\chi} \langle \chi, \bar{\chi} | 0 \rangle 00 \langle \chi, \bar{\chi} | 0 ,
\]

where \( \chi, \bar{\chi} \) represent complete configurations of particles (\( \chi \)) and reverse particles (\( \bar{\chi} \)). That is, \( |\chi, \bar{\chi}\rangle_0 \) is a position state of the entire universe, with \( \chi \) symbolically representing the positions of all particles of normal particle types and \( \bar{\chi} \) representing the positions of all particles of reverse particle types. (For particles of non-zero spin, appropriate spin indices should also be considered included.) The integration measure \( \int d\chi d\bar{\chi} \) is intended to not only include integration of particle positions over all spacetime, but also summation over all configurations with all possible numbers of particles of each type (and summation over spin indices, as appropriate).

Inserting the resolution of the identity from Eq. (20) between the operators in Eq. (19) then gives:

\[
\hat{G}_V = \left[ \prod_{i=0}^{N} \int d\chi_i d\bar{\chi}_i \right] |\chi_N, \bar{\chi}_N\rangle_0 \left[ \prod_{i=1}^{N} \langle \chi_i, \bar{\chi}_i | \hat{G}_{V_i} | \chi_{i-1}, \bar{\chi}_{i-1}\rangle_0 \right] \langle \chi_0, \bar{\chi}_0 | .
\]

Each \( \langle \chi_i, \bar{\chi}_i | \hat{G}_{V_i} | \chi_{i-1}, \bar{\chi}_{i-1}\rangle_0 \) factor is the scattering amplitude for the interaction of subsystem \( V_i \) with the rest of the universe. The interfaces on each “side” are bidirectional: on the \( i-1 \) side, the particles given by \( \chi_{i-1} \) are incoming into \( V_i \), but the reverse particles in \( \bar{\chi}_{i-1} \) are effectively outgoing from \( V_i \). Similarly, on the \( i \) side, the particles in \( \chi_i \) are outgoing, but the reverse particles in \( \bar{\chi}_i \) are effectively incoming. Note again that “incoming” and “outgoing” are used in the sense of the propagation along a particle path, not time—incoming particles are thus not necessarily “initial” and outgoing particles are not necessarily “final”.

It is important that the integrations in the resolution of the identity in Eq. (20), and hence also in Eq. (21), cover all of spacetime, not just the hypervolume of any one system.
or subsystem. The expansion of each interaction operator $\hat{G}_{\mathcal{V}_i}$ includes the identity operator and hence the possibility of particles passing through $\mathcal{V}_i$ (from either “side”) with no interactions. As a result, in the absence of any other restrictions, it is possible for a particle created in an interaction in any of the subsystems (or, for that matter, outside the system all together) to be annihilated in an interaction in any of the other subsystems, not just the ones on either “side”. This reflects the commutativity of the operators $\hat{G}_{\mathcal{V}_i}$, such that the actual ordering of the subsystems is irrelevant.

The resolution of the identity in Eq. (20) can be considered a superposition of the most fine-grained position projection operators $\ket{\chi, \bar{\chi}}_0 \bra{\chi, \bar{\chi}}$ that assert that the universe is in the configuration $\chi, \bar{\chi}$. It is, of course, generally more useful to make more course-grained assertions about the state of the universe. Such an assertion can be denoted by a general projection operator $\hat{P}$ on the extended Fock space.

In particular, it is generally useful to define a complete set of projection operators $\hat{P}^\alpha$ such that

$$\sum_\alpha \hat{P}^\alpha = 1 \quad (22)$$

and

$$\hat{P}^\alpha \hat{P}^\beta = \delta_{\alpha\beta} \hat{P}^\alpha,$$

where, for simplicity, we consider the cardinality of the set to be finite, or, at least, countable. It should be kept in mind that such operators define propositions on all of spacetime. Some care needs to be taken when considering propositions on specific systems within limited hypervolumes of spacetime.

Given static Minkowski spacetime, an identified hypervolume $\mathcal{V}$ will always exist. However, there will be many configurations of the universe in which $\mathcal{V}$ will not actually contain anything we are interested in—that is, in these configurations the system of interest presumed to be contained in $\mathcal{V}$ will essentially not exist. For example, suppose that the system (or subsystem) of interest is a measuring instrument, and the projection operators in Eq. (22) represent pointer states of that instrument. But this presumes that the measuring instrument is actually there (and turn on and operating, etc.) in the expected hypervolume $\mathcal{V}$. There will be many possible configurations of the universe in which this is just not the case.

Take the projection operator $\hat{P}_0^\mathcal{V}$ to select all configurations in which a system of interest does not exist in the hypervolume $\mathcal{V}$. The complementary operator $\hat{P}_0^\mathcal{V}$ then asserts that the
system exists in $\mathcal{V}$—and presumably has some additional interesting finer-grained states, such that
\[
\hat{P}_V^0 + \hat{\bar{P}}_V^0 = 1
\]
and
\[
\hat{P}_V^0 = \sum_{\alpha > 0} \hat{P}_V^\alpha ,
\]
where the projection operators for actually interesting system states are denoted by $\hat{P}_V^\alpha$ for $\alpha > 0$. By definition, the non-existence assertion $\hat{P}_V^0$ is never truly interesting (at least for the cases considered here), but it is only when this operator is included that the set of $\hat{P}_V^\alpha$ is actually complete and sums up to the identity as in Eq. (22).

Now, given a system contained in $\mathcal{V}$ divided into $N$ subsystems contained in $\mathcal{V}_i$, for $i = 1, \ldots, N$, suppose we define a complete set of projection operators $\hat{P}_V^\alpha$ corresponding to each of the subsystems, as well as a set $\hat{P}_i^\alpha$ defined on the in state for the system. Inserting Eq. (22) for these operators between the scattering operators for each subsystem in Eq. (19) gives
\[
\hat{G}_V = \left( \sum_{\alpha_N} \hat{P}_V^{\alpha_N} \right) \hat{G}_{\mathcal{V}_N} \left( \sum_{\alpha_{N-1}} \hat{P}_V^{\alpha_{N-1}} \right) \cdots \left( \sum_{\alpha_1} \hat{P}_V^{\alpha_1} \right) \hat{G}_{\mathcal{V}_1} \left( \sum_{\alpha_0} \hat{P}_i^{\alpha_0} \right)
\]
\[= \sum_{\alpha} \hat{C}^\alpha , \quad (23)\]
where
\[
\hat{C}^\alpha = \hat{P}_V^{\alpha_N} \hat{G}_{\mathcal{V}_N} \hat{P}_V^{\alpha_{N-1}} \cdots \hat{P}_V^{\alpha_1} \hat{G}_{\mathcal{V}_1} \hat{P}_i^{\alpha_0} , \quad (24)
\]
for $\alpha = (\alpha_0, \ldots, \alpha_N)$. This equation now has the essential form of Eq. (8) desired for chain operators at the end of Sec. II and Eq. (23) is analogous to the summation of Eq. (6) in the non-relativistic case.

The intent here is that the projection operators $\hat{P}_i^{\alpha_i}$ represent the propositions that either subsystem $i$ does not exist (represented by $\hat{P}_i^0$) or that there is some “interesting” outcome $\alpha_i$ as a result of the interactions generated by $\hat{G}_{\mathcal{V}_i}$ (represented by $\hat{P}_i^{\alpha_i}$ for $\alpha_i > 0$). We will thus always assume in the following that the assertion made by each $\hat{P}_i^{\alpha_i}$, for $\alpha_i > 0$, depends only on outgoing particles from $\mathcal{V}_i$ that have actually interacted within $\mathcal{V}_i$. That is, for all $i$ and any $\alpha_i > 0$,
\[
\hat{P}_i^{\alpha_i} |\chi, \bar{\chi}\rangle_0 = \hat{P}_i^{\alpha_i} |\chi', \bar{\chi}'\rangle_0 , \quad (25)
\]
for all $\bar{\chi}$ and $\chi'$ and any $\chi$ and $\chi'$ that differ only in positions outside of $\mathcal{V}_i$. (Note that this assumption now presumes a specific ordering of the subsystems, but that will actually be convenient in the following, as further discussed in Sec. VIII.)

Given this assumption, each $\hat{P}^{\alpha_i}_{\mathcal{V}_i} \hat{G}_{\mathcal{V}_i} \hat{P}^{\alpha_{i-1}}_{\mathcal{V}_{i-1}}$ effectively defines the probability amplitude for the outcome $\alpha_i$ for subsystem $i$ given the outcome $\alpha_{i-1}$ for subsystem $i - 1$. This cannot exactly be called a “transition” amplitude, since there is not necessarily any ordering in time. Nevertheless, subsystem outcomes of interest will generally be chosen such that certain outcomes of subsystem $i - 1$ preclude the interactions required to generate certain other outcomes of subsystem $i$, in which case $\hat{P}^{\alpha_i}_{\mathcal{V}_i} \hat{G}_{\mathcal{V}_i} \hat{P}^{\alpha_{i-1}}_{\mathcal{V}_{i-1}}$ will be identically zero. Indeed, in the following it is usually the case that outcome $\alpha_{i-1}$ for subsystem $i - 1$ implies a specific outcome $A_i(\alpha_{i-1})$ for subsystem $i$. That is

$$\hat{P}^{\alpha_i}_{\mathcal{V}_i} \hat{G}_{\mathcal{V}_i} \hat{P}^{\alpha_{i-1}}_{\mathcal{V}_{i-1}} = \delta_{\alpha_i, A_i(\alpha_{i-1})} \hat{P}^{A_i(\alpha_{i-1})}_{\mathcal{V}_i} \hat{G}_{\mathcal{V}_i} \hat{P}^{\alpha_{i-1}}_{\mathcal{V}_{i-1}} \quad (26)$$

The situation represented in Eq. (26) is essentially a dynamic process, in which the cross-correlation terms $\hat{P}^{\alpha_i}_{\mathcal{V}_i} \hat{G}_{\mathcal{V}_i} \hat{P}^{\alpha_{i-1}}_{\mathcal{V}_{i-1}}$ for $\alpha_i \neq A_i(\alpha_{i-1})$ are driven to zero. (Such a process may actually only take the cross-correlations to approximately zero, not identically zero, but I will always assume they are zero in the following.) The dynamics in time are not explicit, of course, in the spacetime formalism used here. However, the hypervolume $\mathcal{V}_i$ for any subsystem will generally have a finite temporal extent and it can be arranged for the projection operators $\hat{P}_i$ to represent states at the upper bound of that extent. Thus, when looked at from a time-evolution viewpoint, various processes may be taking place within the hypervolume $\mathcal{V}_i$ leading to zero cross-correlations for outgoing particle states.

We will also assume in the following that either all subsystems of interest “exist”, or none do. That is, $\hat{P}^{\alpha_0}_{\mathcal{V}_1} \hat{G}_{\mathcal{V}_1} \hat{P}^{\alpha_{N-1}}_{\mathcal{V}_{N-1}} \cdots \hat{P}^{\alpha_1}_{\mathcal{V}_1} \hat{G}_{\mathcal{V}_1} \hat{P}^{\alpha_0}_{\mathcal{V}_0}$ is identically zero if some, but not all, of the $\alpha_i$ are zero. Which of the terms for other values of the $\alpha_i$ are non-zero depends on the specific physical situation under consideration. Clearly, with a full theory of interactions, all such dependencies should be determinable from first principles. However, for the purposes of the following sections, it will be enough to simply assert the physical dependencies between subsystems required in each situation considered.
V. MEASUREMENTS

To model a measurement process, first consider a hypervolume $\mathcal{V}$ bounded by hyperplanes at $t = T_I$ and $t = T_F > T_I$. Given these temporal bounds, we can reasonably define a truly initial state $\ket{\Psi_I}_0$ as a superposition of position states for which all positions have $t < T_I$. Similarly, define a final state $\ket{\Psi_F}$ as a superposition of position states for which all positions have $t > T_F$.

Per the discussion in Sec. III, in and out states such as $\ket{\Psi_I}_0$ and $\ket{\Psi_F}$ are required to have no particles of reverse types. Further, the temporal bounding of $\mathcal{V}$ means that, by construction, in the frame of the time $t$, all initial and final particles are regular particles, not antiparticles.

Now divide $\mathcal{V}$ into two subsystems: a measured system $\mathcal{S}$ and a measuring apparatus $\mathcal{A}$. Define a complete set of projection operators $\hat{P}^\alpha_{\mathcal{S}}$ representing the outcomes for $\mathcal{S}$ and another set of operators $\hat{P}^\alpha_{\mathcal{A}}$ representing the pointer outcomes of $\mathcal{A}$. Presuming a given initial state $\ket{\Psi_I}_0$, histories are then given by $(\alpha_{\mathcal{S}}, \alpha_{\mathcal{A}})$, with corresponding chain operators

$$\hat{C}^{(\alpha_{\mathcal{S}}, \alpha_{\mathcal{A}})} = \hat{P}^\alpha_{\mathcal{A}} \hat{G}_{\mathcal{A}} \hat{P}^\alpha_{\mathcal{S}} \hat{G}_{\mathcal{S}}.$$  

Further, assume that, in the given initial state, subsystems exist in both $\mathcal{A}$ and $\mathcal{S}$, so that

$$\hat{C}^{(00)} \ket{\Psi_I}_0 = \hat{P}^0_{\mathcal{A}} \hat{G}_{\mathcal{A}} \hat{P}^0_{\mathcal{S}} \hat{G}_{\mathcal{S}} \ket{\Psi_I}_0 = 0.$$  

Now, for $\mathcal{A}$ to be a proper measuring apparatus for $\mathcal{S}$, the pointer outcomes for $\mathcal{A}$ must be correlated with the outcomes of $\mathcal{S}$. That is,

$$\hat{P}^\alpha_{\mathcal{A}} \hat{G}_{\mathcal{A}} \hat{P}^\alpha_{\mathcal{S}} = \delta_{\alpha_{\mathcal{A}} \alpha_{\mathcal{S}}} \hat{P}^\alpha_{\mathcal{A}} \hat{G}_{\mathcal{A}} \hat{P}^\alpha_{\mathcal{S}},  \quad (27)$$

for $\alpha_{\mathcal{A}}, \alpha_{\mathcal{S}} > 0$. Thus,

$$\hat{G}_{\mathcal{S}} \ket{\Psi_I}_0 = \sum_{\alpha_{\mathcal{S}} > 0} \hat{C}^{(\alpha_{\mathcal{S}}, \alpha_{\mathcal{S}})} \ket{\Psi_I}_0.$$  

Of course, this decomposition suffers from the equivalent of a basis ambiguity. Let

$$\hat{P}^\alpha_{\mathcal{S}} = \sum_{\beta} a_{\alpha \beta} \hat{P}^\beta_{\mathcal{S}},$$

for an alternate set of projection operators $\hat{P}^\beta_{\mathcal{S}}$ and coefficients $a_{\alpha \beta}$ such that

$$\sum_{\alpha} a_{\alpha \beta} = \sum_{\beta} a_{\alpha \beta} = 1.$$
Then
\[
\sum_{\alpha} \hat{P}_A^{\alpha} \hat{G}_A \hat{P}_A^{\alpha} = \sum_{\alpha} \hat{P}_A^{\alpha} \hat{G}_A \sum_{\beta} a_{\alpha\beta} \hat{P}_S^{\beta} \\
= \sum_{\beta} \sum_{\alpha} a_{\alpha\beta} \hat{P}_A^{\alpha} \hat{G}_A \hat{P}_S^{\beta} \\
= \sum_{\beta} \hat{P}_A^{\beta} \hat{G}_A \hat{P}_S^{\beta},
\]
where the \( \hat{P}_A^{\beta} \equiv \sum_{\alpha} a_{\alpha\beta} \hat{P}_A^{\alpha} \) are an alternate set of pointer outcomes for \( A \) correlated with the outcomes represented by the \( \hat{P}_S^{\beta} \).

To resolve this, note that, if \( V \) really represents the entire universe between the times \( T_I \) and \( T_F \), then \( A \) and \( S \) will together typically only be a small part of this. Outside of these subsystems, there will be an environment \( E = V \setminus S \setminus A \). There are thus three relevant subsystems of \( V \), such that
\[
\hat{G}_V |\Psi_I\rangle_0 = \hat{G}_E \hat{G}_A \hat{G}_S |\Psi_I\rangle_0.
\]

Now, suppose that a measurement by \( A \) leaves a record in the environment \( E \) and, further, that this record is independent of any interaction of the environment with \( S \). That is, there are outcomes of \( E \) represented by operators \( \hat{P}_E^{\alpha_E} \) such that
\[
\hat{P}_E^{\alpha_E} \hat{G}_E \hat{P}_A^{\alpha_A} = \delta_{\alpha_E\alpha_A} \hat{P}_E^{\alpha_E} \hat{G}_E \hat{P}_A^{\alpha_A}.
\]
In other words, the environment measures the apparatus. Then
\[
\hat{G}_V |\Psi_I\rangle_0 = \sum_{\alpha_S>0} \hat{C}^{(\alpha_S,\alpha_S,\alpha_S)} |\Psi_I\rangle_0,
\]
where
\[
\hat{C}^{(\alpha_S,\alpha_A,\alpha_E)} = \hat{P}_E^{\alpha_E} \hat{G}_E \hat{P}_A^{\alpha_A} \hat{P}_S^{\alpha_S} \hat{G}_S.
\]
Such a decomposition no longer suffers from basis ambiguity. This is a generalization to the relativistic spacetime path formalism of Zurek’s concept of einselection [30, 34].

Next consider that
\[
\hat{P}_S^{\alpha_S} \hat{G}_S |\Psi_I\rangle_0 = \psi_S^{\alpha_S}(\Psi_I) |s^{\alpha_S}(\Psi_I)\rangle_0,
\]
where \( |s^{\alpha_S}(\Psi_I)\rangle_0 \) is a unit eigenstate of \( \hat{P}_S^{\alpha_S} \) and \( \psi_S^{\alpha_S}(\Psi_I) \) is the magnitude of \( \hat{P}_S^{\alpha_S} \hat{G}_S |\Psi_I\rangle_0 \). \( |s^{\alpha_S}(\Psi_I)\rangle_0 \) represents the outcome \( \alpha_S \) of the interaction \( \hat{G}_S \), given the initial state \( |\Psi_I\rangle_0 \). (If
\( \psi_{S}^{\alpha S}(\Psi_{I}) = 0 \), then \( |s^{\alpha S}(\Psi_{I})\rangle_{0} \) can be chosen arbitrarily from the eigenspace of \( \hat{P}_{S}^{\alpha S} \). Then, because of Eq. (27),

\[
\hat{G}_{A} \hat{P}_{S}^{\alpha S} = \left( \sum_{\alpha A} \hat{P}_{A}^{\alpha A} \right) \hat{G}_{A} \hat{P}_{S}^{\alpha S} = \hat{P}_{S}^{\alpha S} \hat{G}_{A} \hat{P}_{S}^{\alpha S}. \tag{31}
\]

Thus,

\[
\hat{G}_{A} \hat{P}_{S}^{\alpha S} \hat{G}_{S}|\Psi_{I}\rangle_{0} = \hat{G}_{A} \hat{P}_{S}^{\alpha S} \hat{G}_{S}|\Psi_{I}\rangle_{0} = \psi_{S}^{\alpha S} |s^{\alpha S}, a^{\alpha S}\rangle_{0}. \tag{32}
\]

where \( |s^{\alpha S}, a^{\alpha S}\rangle_{0} = \hat{G}_{A}|s^{\alpha S}\rangle_{0} \) (and the explicit dependence on \( \Psi_{I} \) has been dropped for simplicity of notation). Because of Eq. (28), a similar relationship to Eq. (31) holds between \( \mathcal{E} \) and \( \mathcal{A} \). Therefore, using this and Eq. (32) in Eq. (29) gives

\[
\hat{G}_{V}|\Psi_{I}\rangle_{0} = \sum_{\alpha S > 0} \psi_{S}^{\alpha S} |s^{\alpha S}, a^{\alpha S}, e^{\alpha S}\rangle_{0}, \tag{33}
\]

where \( |s^{\alpha S}, a^{\alpha S}, e^{\alpha S}\rangle_{0} = \hat{G}_{E}|s^{\alpha S}, a^{\alpha S}\rangle_{0} \).

Equation (33) is essentially the form assumed for a measurement state in [24]. Each \( |s^{\alpha S}, a^{\alpha S}, e^{\alpha S}\rangle_{0} \) is a state of the overall system \( \mathcal{V} \) with outcome \( \alpha_{S} \) for \( S \) and correlated outcomes for \( \mathcal{A} \) and \( \mathcal{E} \). And these state are orthogonal, so Eq. (33) certainly represents a consistent family of decoherent histories. Thus, one clearly wants to interpret \( |\psi_{S}^{\alpha S}|^{2} \) as the probability for \( |s^{\alpha S}, a^{\alpha S}, e^{\alpha S}\rangle_{0} \) according to the Born rule.

The next section turns, then, to establishing the usual Born-rule probability interpretation. Note, though, that the derivation of Eq. (33) is independent of this interpretation.

**VI. BORN’S RULE**

Consider now a hypervolume \( \mathcal{V} \) bounded by times \( T_{I} \) and \( T_{F} \) and divided into a system \( S \) and its environment \( \mathcal{E} \), such that

\[
\hat{G}_{V} = \hat{G}_{E} \hat{G}_{S} = \sum_{\alpha} \hat{P}_{E}^{\alpha} \hat{G}_{E} \hat{P}_{S}^{\alpha} \hat{G}_{S},
\]

for appropriate projection operators \( \hat{P}_{E}^{\alpha} \) and \( \hat{P}_{S}^{\alpha} \). Then, following a similar argument to Sec. V, given an initial state \( |\Psi_{I}\rangle_{0} \),

\[
\hat{G}_{V}|\Psi_{I}\rangle_{0} = \sum_{\alpha} \psi_{S}^{\alpha} |s^{\alpha}, e^{\alpha}\rangle_{0}. \tag{34}
\]
Suppose that the hypervolumes $\mathcal{S}$ and $\mathcal{E}$ both extend to the final time $T_F$ and that the state of the system $\mathcal{S}$ is unchanged by its interaction with the environment. Then it will be the case that both

$$\hat{P}^{\alpha'}_S |s^{\alpha}, e^{\alpha}\rangle_0 = \delta_{\alpha'\alpha} |s^{\alpha}, e^{\alpha}\rangle_0 \quad (35)$$

and

$$\hat{P}^{\alpha'}_E |s^{\alpha}, e^{\alpha}\rangle_0 = \delta_{\alpha'\alpha} |s^{\alpha}, e^{\alpha}\rangle_0 . \quad (36)$$

Define the unitary operator

$$\hat{U}_S \equiv \sum_\alpha e^{i\sigma_\alpha} \hat{P}^{\alpha}_S .$$

Given Eqs. (34) and (35), the effect of this operator is

$$\hat{U}_S \hat{G}_V |\Psi_1\rangle_0 = \sum_\alpha e^{i\sigma_\alpha} \psi_{S}^{\alpha} |s^{\alpha}, e^{\alpha}\rangle_0$$

Because of the correlation of the environment with the system, as reflected in Eq. (36), the effect of the operator $\hat{U}_S$ can the undone by the operator

$$\hat{U}_E \equiv \sum_\alpha e^{i\varepsilon_\alpha} \hat{P}^{\alpha}_E ,$$

such that $\varepsilon_j = 2\pi\ell_j - \sigma_j$ for some integer $\ell_j$. That is,

$$\hat{U}_E \hat{U}_S \hat{G}_V |\Psi_1\rangle_0 = \hat{G}_V |\Psi_1\rangle_0 .$$

Now, the action of $\hat{U}_S$ is solely on $\mathcal{S}$. On the other hand, $\hat{U}_E$ acts solely on $\mathcal{E}$. That is, a transformation applied to $\mathcal{S}$ can be undone by a transformation applied to $\mathcal{E}$. This is a kind of symmetry that Zurek calls *entanglement-assisted envariance*, or simply *envariance* [32]. (Zurek earlier referred to this as “environment-assisted invariance” [30, 32].)

The transformations $\hat{U}_S$ and $\hat{U}_E$ do not effect the interaction of the system and the environment that takes place within the overall hypervolume $\mathcal{V}$. And we have presumed that the system and environment no longer interact outside that hypervolume. Therefore, as argued by Zurek, we would not expect it to be possible to undo an action on the system by an action on the causally disconnected environment. The conclusion, then, is that any description of the system in $\mathcal{S}$ should not depend on the phases of the $\psi^\alpha_S$, since such phases can be removed by an action on the environment.
Since the $\hat{P}_E^{\alpha E}$ depend only on positions in $E$, while the $\hat{P}_S^{\alpha S}$ depend only on positions in $S$, $\hat{P}_S^{\alpha S} \hat{P}_E^{\alpha E} = \hat{P}_E^{\alpha E} \hat{P}_S^{\alpha S}$, for all $\alpha_S$ and $\alpha_E$. Therefore, it is possible to conceive in general of joint eigenstates of the $\hat{P}_E^{\alpha E}$ and $\hat{P}_S^{\alpha S}$ with uncorrelated outcomes such that

$$\hat{P}_S^\alpha |s^{\alpha S}, e^{\alpha E}\rangle_0 = \delta_{\alpha_S} |s^{\alpha S}, e^{\alpha E}\rangle_0$$

and

$$\hat{P}_E^\alpha |s^{\alpha S}, e^{\alpha E}\rangle_0 = \delta_{\alpha_E} |s^{\alpha S}, e^{\alpha E}\rangle_0$$

Further, we can choose these states so that the $|s^{\alpha}, e^{\alpha}\rangle_0$ with correlated outcomes are just the states that appear in Eq. (34).

Consider now the unitary operator

$$\hat{U}_E^{(\beta \leftrightarrow \gamma)} \equiv \sum_{\zeta} \left( |s^{\beta}, e^{\zeta}\rangle_0 \langle s^{\gamma}, e^{\zeta}| + |s^{\gamma}, e^{\zeta}\rangle_0 \langle s^{\beta}, e^{\zeta}| + \sum_{\alpha \neq \beta, \gamma} |s^{\alpha}, e^{\zeta}\rangle_0 \langle s^{\alpha}, e^{\zeta}| \right).$$

This operator has no effect on the outcomes for the environment relative to the $\hat{P}_E^\alpha$, but it swaps the $\beta$ and $\gamma$ outcomes for the system:

$$\hat{U}_E^{(\beta \leftrightarrow \gamma)} \hat{S}_V |\Psi_I\rangle_0 = \psi^{\beta}_S |s^{\gamma}, e^{\gamma}\rangle_0 + \psi^{\gamma}_S |s^{\beta}, e^{\gamma}\rangle_0 + \sum_{\alpha \neq \beta, \gamma} \psi^{\alpha}_S |s^{\alpha}, e^{\gamma}\rangle_0.$$

Note that the resulting state represents a different “universe” than what would be expected from normal interaction based on the initial state $|\Psi_I\rangle_0$. It can be effectively considered to be the result of the same basic interaction, but proceeding from a different initial state

$$|\Psi_I'\rangle_0 = \hat{G}_V^{-1} \hat{U}_E^{(\beta \leftrightarrow \gamma)} \hat{G}_V |\Psi_I\rangle_0.$$

So, a priori, one cannot assume that the intrinsic properties of the universe represented by the swapped state will be the same as those of the universe represented by the original state.

However, suppose that $\psi^{\beta}_S = \psi^{\gamma}_S$. Then we can apply a unitary “counterswapping” operator for the environment,

$$\hat{U}_E^{(\beta \leftrightarrow \gamma)} \equiv \sum_{\zeta} \left( |s^{\zeta}, e^{\beta}\rangle_0 \langle s^{\zeta}, e^{\gamma}| + |s^{\zeta}, e^{\gamma}\rangle_0 \langle s^{\zeta}, e^{\beta}| + \sum_{\alpha \neq \beta, \gamma} |s^{\zeta}, e^{\alpha}\rangle_0 \langle s^{\zeta}, e^{\alpha}| \right),$$

which swaps the $\beta$ and $\gamma$ outcomes of the environment, but leaves the outcomes of the system unchanged. Swapping first system outcomes and then environment outcomes gives

$$\hat{U}_S^{(\beta \leftrightarrow \gamma)} \hat{U}_E^{(\beta \leftrightarrow \gamma)} \hat{G}_V |\Psi_I\rangle_0 = \psi^{\beta}_S |e^{\gamma}, s^{\gamma}\rangle_0 + \psi^{\gamma}_S |e^{\beta}, s^{\beta}\rangle_0 + \sum_{\alpha \neq \beta, \gamma} \psi^{\alpha}_S |s^{\alpha}, e^{\alpha}\rangle_0.$$
Clearly, if $\psi_\beta^S = \psi_\gamma^S$,
\[
\hat{U}_S^{(\beta \leftrightarrow \gamma)} \hat{U}_E^{(\beta \leftrightarrow \gamma)} \hat{G}_V |\Psi_I\rangle_0 = \hat{G}_V |\Psi_I\rangle_0 .
\]
That is, a swap carried out on the system can be “counterswapped” by acting only on the environment, leaving the overall state unchanged: the state is envariant under swapping.

Suppose that some physical property of $S$ was observably different in the state $\hat{U}_S^{(\beta \leftrightarrow \gamma)} \hat{G}_V |\Psi_I\rangle_0$ than in $\hat{G}_V |\Psi_I\rangle_0$. Then this difference could be removed by acting only on the environment using $\hat{U}_E^{(\beta \leftrightarrow \gamma)}$. But this violates the assumption that the outcome of $S$ does not depend on that of $E$—that is, that information is flowing from $S$ to $E$, but not vice versa.

In particular, as discussed in [24], the statistics for the results of a repeated experiment directly depend on the probability by which the cosmological eigenstate for a given set of outcomes is expected to be selected. Thus, if $S$ includes such a statistical measurement, any difference in the probabilities for measurement outcomes in $\hat{U}_S^{(\beta \leftrightarrow \gamma)} \hat{G}_V |\Psi_I\rangle_0$ from $\hat{G}_V |\Psi_I\rangle_0$ will be physically detectable. If $\hat{G}_V |\Psi_I\rangle_0$ is envariant with respect to swaps, however, this should not be the case.

Therefore, we can conclude, similarly to Zurek [30, 32, 34], that envariant swapping cannot effect the probabilities assigned to the system-interaction eigenstates being swapped. That is, outcomes $\beta$ and $\gamma$ such that $\psi_\beta^S = \psi_\gamma^S$ must be equally likely. Indeed, since we showed previously that the phases of the $\psi_\alpha^S$ can be disregarded, the real requirement is only that $|\psi_\beta^S| = |\psi_\gamma^S|$.

Given this, we can follow an approach analogous to Zurek’s to obtain Born’s rule. To start, assume that the $\psi_\alpha^S$ are all rational numbers of the form
\[
\psi_\alpha^S = \sqrt{m_\alpha/M} ,
\]
where $\sqrt{M}$ is a common denominator of the $\psi_\alpha^S$, so that all the $m_\alpha$ are natural numbers. Further, the normalization $\sum_\alpha (\psi_\alpha^S)^2 = 1$ gives $M = \sum_\alpha m_\alpha$.

Next, further divide the environment projection operators $\hat{P}_E^\alpha$ into a finer-grained set $\hat{P}_E^{\alpha \beta}$, such that
\[
\hat{P}_E^\alpha = \sum_{\beta=1}^{m_\alpha} \hat{P}_E^{\alpha \beta}
\]
and
\[
\hat{P}_E^{\alpha \beta} |s_\alpha^\beta\rangle_0 = |s_\alpha^\beta\rangle_0 / \sqrt{m_\alpha} ,
\]
\[
\hat{P}_E^{\alpha \beta} |s_\alpha^\beta\rangle_0 = |s_\alpha^\beta\rangle_0 / \sqrt{m_\alpha} ,
\]
where the $|s^\alpha\rangle_0$ are defined as in Eq. (30) and the $|s^\alpha, e^{\alpha\beta}\rangle_0$ are unit eigenstates. Since, in the present formalism, all states are ultimately defined on the infinite-dimensional space of fine-grained continuous position states, such a discrete subdivision is always possible. Then

$$|s^\alpha, e^\alpha\rangle_0 = \hat{P}_E^\alpha |s^\alpha\rangle_0 = \sum_{\beta=1}^{m_\alpha} |s^\alpha, e^{\alpha\beta}\rangle_0 / \sqrt{m_\alpha},$$

which respects the unit normalization of the $|s^\alpha, e^\alpha\rangle_0$.

Introduce an ancillary system in a hypervolume $C$ separate from $V$, but able to interact with $E$ without influencing the interaction of $E$ with $S$. The interaction between $C$ and $E$ is such that

$$\hat{G}_C \hat{P}_E^{\alpha\beta} = \delta_{\alpha\gamma} \delta_{\beta\zeta} \hat{P}_C^\gamma \hat{G}_C \hat{P}_E^{\alpha\beta},$$

for an appropriate set of projection operators $\hat{P}_E^{\alpha\beta}$ indexed parallel to the $\hat{P}_E^{\alpha\beta}$. Then, using Eqs. (37), (38) and (39) with Eq. (34),

$$\hat{G}_C \hat{G}_V |\Psi_I\rangle_0 = \sum_\alpha \sqrt{m_\alpha/M} \sum_{\beta=1}^{m_\alpha} \hat{G}_C |s^\alpha, e^{\alpha\beta}\rangle_0 / \sqrt{m_\alpha} = \sum_{\alpha\beta} \sqrt{1/M} |s^\alpha, e^{\alpha\beta}, e^{\alpha\beta}\rangle_0,$$

where

$$\hat{G}_C |s^\alpha, e^{\alpha\beta}\rangle_0 = |s^\alpha, e^{\alpha\beta}, e^{\alpha\beta}\rangle_0,$$

for unit eigenstates $|s^\alpha, e^{\alpha\beta}, e^{\alpha\beta}\rangle_0$.

The terms in Eq. (40) all now have equal coefficients, so we take the corresponding states to all be equally likely. Since there a total of $M$ terms, the probability of any one of the states is $1/M$. Further, since, for each $\alpha$, $m_\alpha$ of the overall system/environment/ancilla states correspond to the system outcome $\alpha$, the probability for this outcome is

$$p_\alpha = m_\alpha/M = |\psi^\alpha_S|^2,$$

which is just Born’s law. By continuity, the same conclusion can be extended to all real $\psi^\alpha_S$.

(Note that the derivation here also assumes the additivity of probabilities, but it is possible to come to the same conclusion without making this assumption [32].)

Of course, this argument only establishes the Born rule for the overall states $|s^\alpha, e^\alpha\rangle_0$. But [24] establishes that, if Born’s rule holds for such states, then it follows that the statistics of repeated measurement experiments would be expected to also follow this rule.
VII. SCHRÖDINGER’S CAT

It is instructive to use the formalism of subsystems developed here to analyze the classic example of macroscopic entanglement: Schrödinger’s Cat. The Schrödinger’s Cat thought experiment can be divided into five subsystems:

- \( \mathcal{R} \), a radioactive atom, with projection operators \( \hat{P}_R^{\text{yes}} \) and \( \hat{P}_R^{\text{no}} \) indicating that it has or has not decayed.
- \( \mathcal{D} \), a detector/poison gas apparatus, with projection operators \( \hat{P}_D^{\text{yes}} \) and \( \hat{P}_D^{\text{no}} \) indicating that a decay product has been detected, with a consequent release of poison gas, or not.
- \( \mathcal{C} \), the cat, with projection operators \( \hat{P}_C^{\text{alive}} \) and \( \hat{P}_C^{\text{dead}} \) indicating that the cat is alive or dead.
- \( \mathcal{B} \), the box (consisting of just the bounding container but not its interior), with projection operators \( \hat{P}_B^{\text{open}} \) and \( \hat{P}_B^{\text{closed}} \) indicating that the box is open or closed.
- \( \mathcal{E} \), the environment with projection operators \( \hat{P}_E^{\text{closed}} \), \( \hat{P}_E^{\text{alive}} \) and \( \hat{P}_E^{\text{dead}} \) indicating that either that the box is closed or it is open and the cat is alive or dead.

The experiment is presumed to have a finite duration, so that, as before, the complete hypervolume \( \mathcal{V} = \mathcal{E} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{R} \) has both upper and lower time bounds.

Let \( |\Psi_I\rangle_0 \) be an initial state in which the box already exists, with the cat, atom and detector sealed inside it. That is,

\[
\hat{P}_R^{\text{no}} |\Psi_I\rangle_0 = \hat{P}_D^{\text{no}} |\Psi_I\rangle_0 = \hat{P}_C^{\text{alive}} |\Psi_I\rangle_0 = \hat{P}_B^{\text{open}} |\Psi_I\rangle_0 = |\Psi_I\rangle_0.
\]

Consider first the interior of the box, consisting of \( \mathcal{I} = \mathcal{R} \cup \mathcal{D} \cup \mathcal{C} \). Clearly,

\[
\hat{G}_I = \hat{P}_C^{\text{alive}} \hat{G}_C \hat{P}_D^{\text{no}} \hat{G}_D \hat{P}_R^{\text{no}} \hat{G}_R |\Psi_I\rangle_0 + \hat{P}_C^{\text{dead}} \hat{G}_C \hat{P}_D^{\text{yes}} \hat{G}_D \hat{P}_R^{\text{yes}} \hat{G}_R |\Psi_I\rangle_0
= |\Psi_R^{\text{no}, r^{\text{no}}, c^{\text{alive}}\rangle_0 + |\Psi_R^{\text{yes}, r^{\text{yes}}, d^{\text{yes}}, c^{\text{dead}}\rangle_0.
\]

The key issue in the Schrödinger’s Cat scenario is, of course, whether opening the box has any relevance to the state of the interior of the box (i.e., by “collapsing the wave function”). Initially, the box is closed. However, at some time during the course of the experiment, the box may be opened, presumably by an experimenter who is part of the environment. But,
since the experimenter cannot see inside the box while it is closed, opening the box is done with no knowledge of what has happened within the interior $\mathcal{I}$ of the box. If we take the interactions necessary to open the box to be captured by $\hat{G}_B$, then this must commute with $\hat{G}_I$ determined above.

Now, opening the box is a macroscopic, classical act which can be presumed to either happen (with probability 1) or not. Whether the box is opened can thus be considered to be fully determined by the initial state, which includes the intention of the experimenter whether to open the box or not. Suppose in the initial state $|\Psi_1\rangle_0$ the experimenter does, in fact, open the box some time within the hypervolume $\mathcal{B} \cup \mathcal{I}$. Then

$$\hat{G}_B \hat{G}_I |\Psi_1\rangle_0 = \hat{G}_I \hat{G}_B |\Psi_1\rangle_0 = \hat{G}_I \hat{P}_B^{\text{open}} \hat{G}_B |\Psi_1\rangle_0 = \hat{P}_B^{\text{open}} \hat{G}_B \hat{G}_I |\Psi_1\rangle_0.$$  

Once the box is open, the interior of the box can interact with the environment and it becomes known in the environment whether the cat is alive or dead in the interior of the box. So

$$\hat{G}_V |\Psi_1\rangle_0 = \hat{G}_\epsilon \hat{G}_B \hat{G}_I |\Psi_1\rangle_0 = \hat{G}_\epsilon \hat{P}_B^{\text{open}} \hat{G}_B \hat{G}_I |\Psi_1\rangle_0$$

$$= \hat{P}_\epsilon^{\text{alive}} \hat{G}_\epsilon \hat{P}_B^{\text{open}} \hat{G}_B \hat{P}_C^{\text{alive}} \hat{G}_I |\Psi_1\rangle_0 + \hat{P}_\epsilon^{\text{dead}} \hat{G}_\epsilon \hat{P}_B^{\text{open}} \hat{G}_C^{\text{dead}} \hat{G}_I |\Psi_1\rangle_0.$$  

Thus, using Eq. (41),

$$\hat{G}_V |\Psi_1\rangle_0 = \hat{P}_\epsilon^{\text{alive}} \hat{G}_\epsilon \hat{P}_B^{\text{open}} \hat{G}_B \hat{P}_C^{\text{alive}} \hat{G}_I |\Psi_1\rangle_0 + \hat{P}_\epsilon^{\text{dead}} \hat{G}_\epsilon \hat{P}_B^{\text{open}} \hat{G}_C^{\text{dead}} \hat{G}_I |\Psi_1\rangle_0$$

$$= \psi_\epsilon^{\text{no}} |r, d, c\rangle_0 + \psi_\epsilon^{\text{yes}} |r, d, c\rangle_0, \quad \text{where the environment records whether the cat is alive or dead.}$$

However, now assume a different initial state $|\Psi_1\rangle_0$ that is the same as $|\Psi_1\rangle_0$ except that it does not lead to the experimenter opening the box during the time period covered by $\mathcal{V}$. In this case

$$\hat{G}_B \hat{G}_I |\Psi_1\rangle_0 = \hat{P}_B^{\text{closed}} \hat{G}_B \hat{G}_I |\Psi_1\rangle_0.$$  

and, with the box closed, the interior cannot interact with the environment:

$$\hat{G}_V |\Psi_1\rangle_0 = \hat{G}_\epsilon \hat{G}_B \hat{G}_I |\Psi_1\rangle_0 = \hat{P}_\epsilon^{\text{closed}} \hat{G}_\epsilon \hat{P}_B^{\text{closed}} \hat{G}_B \hat{G}_I |\Psi_1\rangle_0.$$ 

The change in initial state does not effect what happens in the interior of the box, so, using Eq. (41) again,

$$\hat{G}_V |\Psi_1\rangle_0 = \hat{P}_\epsilon^{\text{closed}} \hat{G}_\epsilon \hat{P}_B^{\text{closed}} \hat{G}_B \hat{G}_I |\Psi_1\rangle_0$$

$$= \hat{P}_\epsilon^{\text{closed}} \hat{G}_\epsilon \hat{P}_B^{\text{closed}} \psi_\epsilon^{\text{no}} |r, d, c\rangle_0 + \psi_\epsilon^{\text{yes}} |r, d, c\rangle_0, \quad \text{(42)}$$

$$= \psi_\epsilon^{\text{no}} |r, d, c\rangle_0 + \psi_\epsilon^{\text{yes}} |r, d, c\rangle_0.$$
The environment is now not correlated with the alternatives inside the box. Nevertheless there are still two orthogonal eigenstates, representing alternative decoherent histories of the full system, in one of which the cat is alive and in the other of which the cat is dead. There is no alternative in which only the cat is in a superposition of alive and dead.

As discussed in [24], we can consider the state of our actual universe to be one or the other of the alternatives in Eq. (42), selected with probabilities given by $|\psi_{\text{no}}^R|^2$ and $|\psi_{\text{yes}}^R|^2$. But, even though one or the other alternative may be chosen as “the” state of the universe—and the cat certainly knows which one it is!—if the box is closed, this information is simply unavailable to the environment outside the box. The outcome for $E$ is thus the same regardless of what happens inside the box.

Note that the above analysis is not changed if we presume that the intent to open the box is formulated in the brain of the experimenter sometime after the initiation of the experiment. Or if the experimenter is replaced with, say, a device that may randomly open the box during the run of the experiment. In all cases, by the end of the experimental period, the box will be either open or still closed.

Whether the box is opened or remains closed, this example illustrates how a microscopic quantum event with orthogonal outcomes can be amplified to determine orthogonal eigenstates for an entire macroscopic system and its environment—and, conceptually, the entire universe. These orthogonal eigenstates represent a consistent set of alternative histories, in one of which the cat is alive and in the other of which the cat is dead. The alternatives of the cat being dead and alive are thus clear and classical. Indeed, the composite subsystem in the interior $I$ of the box is already sufficient to provide the necessary decoherence of alternatives, regardless of whether this information can get outside of the box to its external environment.

VIII. CONCLUDING REMARKS

As with any derivation, Zurek’s derivation of Born’s rule is based on a number of basic assumptions, as nicely elucidated by Schlosshaur and Fine [42] and further addressed by Zurek himself [34]. These assumptions condition the interpretation of the non-relativistic formalism used by Zurek, where two entangled systems are represented as evolving into a Schmidt state in which the individual states of the two systems are correlated. Clearly, sim-
ilar assumptions also underlie the approach I have used here—but the relativistic, spacetime formalism provides a rather interesting new viewpoint on them.

The fundamental difference is that the state $G_V|\Phi_I\rangle_0$ is not a Schmidt state of a cross-product Hilbert space for systems $S$ and $E$ but, rather, a superposition of joint eigenstates $|s^\alpha, e^\alpha\rangle_0$ of correlated outcomes $s^\alpha$ and $e^\alpha$ for the two systems. Therefore, it does not really make sense to speak of separate probabilities for the outcomes $s^\alpha$ and $e^\alpha$. There is only the probability of whether the actual universe is a specific joint eigenstate of these outcomes or not. The correlation of the outcomes for $S$ and $E$ for any such eigenstate is completely determined by the initial state $|\Phi_I\rangle_0$ and the allowed interactions within $V = S \cup E$.

As discussed in Sec. VI, the effect of a system-outcome swap operator is to transform one state of the universe into another. The new state can be considered as having a different effective initial state, starting from which, interactions in the system result in swapped outcomes. Assuming the swapped outcomes have coefficients with the same absolute values, counterswapping the corresponding environment outcomes then results, envariantly, in the original state.

The key assumption that Schlosshauer and Fine find most troubling is that the probability for the system outcomes in the swapped state remain unchanged when the counterswapping operation is applied. From the present point of view, this assumption means the probabilities of system outcomes in a universe based on the new effective initial state resulting from the swap operation should be the same as the probabilities of the system outcomes of the universe based on the original initial state.

However, these probabilities have physically observable consequences within each of the respective possible universes. But the fact that the eigenstate representing one universe can be transformed into the state of the other by applying an operator that effects only the environment would indicate that the physically observable properties of the system should be the same in both universes. The specific environment outcomes with which the system outcomes are correlated are largely arbitrary (at least when the system outcome coefficients have equal absolute values). They are a result of the reaction of the environment to the system based on the initial state of the environment, not an intrinsic property of the system.

A similar statement to the above could, of course, be made about the probabilities of environment outcomes, since the envariance argument is symmetrical between the system and the environment. However, there is a deeper assumption that distinguishes the system
from its environment, which comes out clearly in the formalism: in Eq. (34), it is the initial interaction of the system with the initial state that determines the coefficients $\psi_S^\alpha$ and the decoherence of the cosmological state into a superposition of eigenstates of system outcomes. The further interaction of the environment with the system simply acts to correlate the environment with those already established eigenstates (per the discussion on Eq. (32) and following).

The basic assumption is that information flows from the system to its environment, not vice versa. This assumption is captured in Eq. (25) which states that the outcomes of interest for each subsystem depend only on particles outgoing “to the left” from each subsystem interaction operator in Eq. (23). As a result, we have been able to conveniently order subsystem interaction right to left (e.g, $G_E G_S$ in Sec. VI, $G_E G_A G_S$ in Sec. V and $G_E G_B G_C G_D G_R$ in Sec. VII) such that a subsystem is affected by the outcomes of subsystems to its right, but not by those to its left. That is, information effectively flows from right to left.

This conception is directly related to Zurek’s observation regarding einselection on “the direction of information flow in decoherence, from the decohering apparatus and to the environment…” [30]. This is opposite to the information flow of noise. In an idealized measurement situation, the desired information flow is that necessary for decoherence and the noise effect of the environment on the apparatus is ignored.

With the convention of Eq. (25), the desired (“right to left”) information flow is carried by particles passing from one subsystem to the other on normal particle paths. In contrast, the undesired noise flows “backwards” (“left to right”), carried by particles along reverse particle paths. In both cases, however, the flow of information is along spacetime paths in the direction of increase in the path evolution parameter, regardless of whether this is forward or backward in time.

This interesting connection between particle propagation along spacetime paths and the flow of information is a promising topic for future exploration.

Appendix A: Spacetime Path Formalism

This appendix summarizes the full spacetime path formalism that is developed in detail in [25]. This full formalism provides a more rigorous underpinning for the more familiar
quantum field theoretic approach described in Sec. III. The formalism presented here can also be extended to particles of non-zero spin [26], but, for simplicity, this will not explicitly be considered here, since the introduction of spin indices does not fundamentally affect the points to be made in this paper. However, note that the introduction of reverse particles in Sec. III is an extension to the formalism presented in [25] that is necessary to fully reproduce the results of Sec. V and Sec. VI.

A spacetime path is specified by four functions \( q^\mu (\lambda) \), for \( \mu = 0, 1, 2, 3 \), of a path parameter \( \lambda \). Note that such a path is not constrained to be timelike or even to maintain any particular direction in time. The only requirement is that it must be continuous. And, while there is no a priori requirement for the paths to be differentiable, we can, as usual, treat them as differentiable within the context of a path integral (see the discussion in [25].)

It is well known that a spacetime path integral of the form
\[
\Delta(x - x_0) = \eta \int_{\lambda_0}^{\infty} d\lambda_1 \int D^4q \, \delta^4(q(\lambda_1) - x) \delta^4(q(\lambda_0) - x_0) \exp \left( i \int_{\lambda_0}^{\lambda_1} d\lambda L(\dot{q}^2(\lambda)) \right),
\]
for an appropriate normalization constant \( \eta \) and the Lagrangian function
\[
L(\dot{q}^2) = \frac{1}{4} \dot{q}^2 - m^2,
\]
gives the free-particle Feynman propagator [12, 14, 18, 25]. In the path integral above, the notation \( D^4q \) indicates that the integral is over the four functions \( q^\mu (\lambda) \) and the delta functions constrain the starting and ending points of the paths integrated over. (See also [25] for a justification of Eq. (A1) from a small number of physically motivated postulates.)

Consider, however, that Eq. (A1) can be written
\[
\Delta(x - x_0) = \int_{\lambda_0}^{\infty} d\lambda_1 \Delta(x - x_0; \lambda_1 - \lambda_0),
\]
where
\[
\Delta(x - x_0; \lambda_1 - \lambda_0) \equiv \eta \int D^4q \, \delta^4(q(\lambda_1) - x) \delta^4(q(\lambda_0) - x_0) \exp \left( i \int_{\lambda_0}^{\lambda_1} d\lambda L(\dot{q}^2(\lambda)) \right). \tag{A2}
\]
The value \( \lambda - \lambda_0 \) in \( \Delta(x - x_0; \lambda - \lambda_0) \) can be thought of as fixing a specific intrinsic length for the paths being integrated over. Equation (A2) now has a similar path integral form as the usual non-relativistic propagation kernel [5, 6], except with paths parametrized by \( \lambda \) rather than time. We can, therefore, use the relativistic kernel of Eq. (A2) to define parametrized wave function in a similar fashion to the non-relativistic case:
\[
\psi(x; \lambda) = \int d^4x_0 \Delta(x - x_0; \lambda - \lambda_0) \psi(x_0; \lambda_0). \tag{A3}
\]
These wave functions are parametrized probability amplitude functions in the sense first defined by Stueckelberg [39, 40]. In this sense, the $\psi(x; \lambda)$ represent the probability amplitude for a particle to reach position $x$ at the point along its path with parameter value $\lambda$. (For other related approaches using an invariant “fifth parameter”, though not necessarily a path evolution parameter, see [43–53].)

The functions defined in Eq. (A3) form a Hilbert space over four dimensional spacetime, parametrized by $\lambda$, in the same way that traditional non-relativistic wave functions form a Hilbert space over three dimensional space, parametrized by time. We can therefore define a consistent family of position state bases $|x; \lambda\rangle$, such that

$$\psi(x; \lambda) = \langle x; \lambda | \psi \rangle , \quad (A4)$$

given a single Hilbert space state vector $| \psi \rangle$. These position states are normalized such that

$$\langle x'; \lambda | x; \lambda \rangle = \delta^4(x' - x) .$$

for each value of $\lambda$. Further, it follows from Eqs. (A3) and (A4) that

$$\Delta(x - x_0; \lambda - \lambda_0) = \langle x; \lambda | x_0; \lambda_0 \rangle . \quad (A5)$$

Thus, $\Delta(x - x_0; \lambda - \lambda_0)$ effectively defines a unitary transformation between the various Hilbert space bases $|x; \lambda\rangle$, indexed by the parameter $\lambda$.

The overall state for propagation from $x_0$ to $x$ is given by the superposition of the states for paths of all intrinsic lengths. If we fix $q^\mu(\lambda_0) = x_0^\mu$, then $|x; \lambda\rangle$ already includes all paths of length $\lambda - \lambda_0$. Therefore, the overall state $|x\rangle$ for the particle to arrive at $x$ should be given by the superposition of the states $|x; \lambda\rangle$ for all $\lambda > \lambda_0$:

$$|x\rangle \equiv \int_{\lambda_0}^{\infty} d\lambda \ |x; \lambda\rangle . \quad (A6)$$

Then, using Eq. (A5),

$$\langle x|x_0; \lambda_0 \rangle = \int_{\lambda_0}^{\infty} d\lambda \Delta(x - x_0; \lambda - \lambda_0) = \int_0^{\infty} d\lambda \Delta(x - x_0; \lambda) = \Delta(x - x_0) .$$

Since $\Delta(x - x_0; \lambda - \lambda_0)$ only depends on the difference $\lambda - \lambda_0$, the actual starting value $\lambda_0$ of the path parameter can be shifted arbitrarily. (This can be viewed as a gauge invariance of the path parameter $\lambda$ [14, 25].) Nevertheless, it is convenient to consistently denote the starting value for $\lambda$ as $\lambda_0$. The position states $|x; \lambda_0\rangle$ can then be identified with the states
denoted $|x\rangle_0$ in Sec. III, with the states denoted $|x\rangle$ there being the same as those defined in Eq. (A6).

The position states $|x\rangle$ as defined above make no distinction based on the time-direction of propagation of particles. Normally, particles are considered to propagate from the past to the future. Therefore, we can define normal particle states $|x_+\rangle$ such that

$$\langle x_+ | x_0; \lambda_0 \rangle = \theta(x^0_0 - x^0_0) \Delta(x - x_0),$$  \hspace{1cm} (A7)

On the other hand, antiparticles may be considered to propagate from the future into the past [11, 39, 40]. Therefore, antiparticle states $|x_-\rangle$ are such that

$$\langle x_- | x_0; \lambda_0 \rangle = \theta(x^0_0 - x^0_0) \Delta(x - x_0).$$  \hspace{1cm} (A8)

Note that the states $|x_{\pm}\rangle$ defined here differ from the definitions of the similarly notated states in Sec. III in that the Heaviside theta functions are included in the definitions of the states in Eqs. (A7) and (A8) but not in the definitions in Sec. III. This means that the states $|x_{\pm}\rangle$ defined here are not actually on-shell, but, on the other hand, they clearly capture the fact that particles propagate only into the future and antiparticles propagate only into the past. Nevertheless, as noted in Sec. III, we can recover on-shell states by going to the infinite-time limit.

In taking the infinite-time limit of a spacetime path, one cannot expect to hold the 3-position of the path end point constant. For a free particle, though, it is reasonable to take the particle 3-momentum as being fixed. In [25] it is shown that, at the time limit of infinity (for particles) or negative infinity (for antiparticles), such 3-momentum states do indeed become on-shell. Thus, the momentum shell constraint is not imposed arbitrarily but, rather, is a natural consequence of the infinite-time limit for free particles—but only holds approximately, otherwise.

For the purposes of this paper, the 4-dimensional position states $|x_0; \lambda_0 \rangle$ (or $|x\rangle_0$, as they are denoted in the main body) are more useful than the on-shell particle and antiparticle states. It is, of course, straightforward to construct corresponding momentum states:

$$|p; \lambda_0 \rangle \equiv (2\pi)^{-2} \int d^4x e^{ip \cdot x} |x; \lambda_0 \rangle.$$  

But such states are inherently off shell, with no restriction on the value of the energy $p^0$ relative to the 3-momentum $p$. Nevertheless, keep in mind that in any scattering-like
interaction process (as, e.g., captured in the interaction operator $\hat{G}$ defined in Sec. III) one can consider incoming and outgoing particles to be on-shell sufficiently far outside the interaction area [25].

Multiple particle states can be straightforwardly introduced as members of a Fock space over the Hilbert space of position states $|x; \lambda\rangle$. First, in order to allow for multiparticle states with different types of particles, extend the position state of each individual particle with a particle type index $n$, such that

$$\langle x', n'; \lambda | x, n; \lambda \rangle = \delta_{n'n} \delta^4(x' - x).$$

Then, construct a basis for the Fock space of multiparticle states as symmetrized products of $N$ single particle states:

$$| x_1, n_1, \lambda_1; \ldots; x_N, n_N, \lambda_N \rangle \equiv (N!)^{-1/2} \sum_{\text{perms } \mathcal{P}} | x_{\mathcal{P}1}, n_{\mathcal{P}1}; \lambda_{\mathcal{P}1} \rangle \cdots | x_{\mathcal{P}N}, n_{\mathcal{P}N}; \lambda_{\mathcal{P}N} \rangle,$$

where the sum is over all permutations $\mathcal{P}$ of $1, 2, \ldots, N$. (When including Fermions, one needs to, of course, antisymmetrize rather than symmetrize the products [26].)

It is then convenient to introduce a creation field operator $\hat{\psi}^\dagger(x, n; \lambda)$ such that

$$\hat{\psi}^\dagger(x, n; \lambda) | x_1, n_1, \lambda_1; \ldots; x_N, n_N, \lambda_N \rangle = | x, n, \lambda; x_1, n_1, \lambda_1; \ldots; x_N, n_N, \lambda_N \rangle,$$

with the corresponding annihilation field $\hat{\psi}(x, n; \lambda)$ having the commutation relation

$$[\hat{\psi}(x', n'; \lambda), \hat{\psi}^\dagger(x, n; \lambda_0)] = \delta_{n'n} \Delta(x' - x; \lambda - \lambda_0).$$

Further, define

$$\hat{\psi}(x, n) \equiv \int_{\lambda_0}^\infty d\lambda \, \hat{\psi}(x, n; \lambda),$$

so that

$$[\hat{\psi}(x', n'), \hat{\psi}^\dagger(x, n; \lambda_0)] = \delta_{n'n} \Delta(x' - x).$$

Identifying the field operators $\hat{\psi}(x, n; \lambda_0)$ and $\hat{\psi}(x, n)$ defined here with $\hat{\psi}_0^{(n)}(x)$ and $\hat{\psi}^{(n)}(x)$ defined in Sec. III then completes the grounding of the formalism used in the main text.

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