On Packing Almost Half of a Square with Anchored Rectangles: A Constructive Approach

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Abstract

In this paper, we consider the following geometric puzzle whose origin was traced to Allan Freedman [13, 22] in the 1960s by Dumitrescu and Tóth [10]. The puzzle has been popularized of late by Peter Winkler [23]. Let \( P_n \) be a set of \( n \) points, including the origin, in the unit square \( U = [0, 1]^2 \). The problem is to construct \( n \) axis-parallel and mutually disjoint rectangles inside \( U \) such that the bottom-left corner of each rectangle coincides with a point in \( P_n \) and the total area covered by the rectangles is maximized. We would term the above rectangles as anchored rectangles. The longstanding conjecture has been that at least half of \( U \) can be covered when anchored rectangles are properly placed. Dumitrescu and Tóth [11] have shown a construction method that can cover at least 0.09121, i.e., roughly 9% of the area.

In our earlier work [2], we have given an existential proof of the conjecture, which gives a lower bound on the optimal value of the area covered by anchored rectangles. Here, we present an algorithm for constructing a packing that covers an area of \( \frac{1}{2} - \epsilon \), where \( \epsilon > 0 \), based on a recent QPTAS for computing a maximum-weight independent set of rectangles as proposed by Adamaszek and Wiese [1].

Keywords packing, constrained packing, anchored rectangles, maximum independent set of rectangles

1 Introduction

In this paper, we consider the following puzzle whose earliest documentation was traced by Peter Winkler [23] to IBM’s puzzle webpage [19]. We quote verbatim: Given \( n \) distinct points in the unit square \([0, 1]^2\), including the origin \((0, 0)\) as one of the \( n \) points. Can you construct \( n \) rectangles, contained in the unit square, with sides parallel to the coordinate axes, pairwise non-intersecting, such that each of our \( n \) given points is the lower-left-hand corner of one of the rectangles, and such that the total area of the rectangles is at least \( 1/2 \)? Later Dumitrescu and Tóth [10] traced the actual origin of the problem.

For brevity, we will call such rectangles to be anchored-rectangles, henceforth.

1.1 Related work

Dumitrescu and Tóth [10], apart from making the first step towards providing a constructive solution of the problem, traced back the history of the problem with the help of Richard Guy to Allen Freedman [13, 22] in the 1960s. Thus, the conjecture has been open for more than 40 years. Of late, Winkler [23, 25, 24] popularized the problem.

Christ et al. [8] and Dumitrescu and Tóth [11] presented a two-player one-round game interpretation of the above problem where Alice chooses an \( n \)-point set \( P \) and Bob chooses axis-parallel, interior-disjoint rectangles whose bottom-left corners are anchored at points chosen by Alice, i.e., Bob chooses the anchored-rectangles. The conjecture states that Bob can choose anchored-rectangles in such a way that their union covers at least half of \( U \). Until recently, there was no known algorithm for constructing such a set of
rectangles; even nothing was known about whether a set of anchored-rectangles can be constructed which
can cover a constant fraction, however small. Christ et al. [8] showed that if Alice can force Bob’s share
to tend to zero (precisely, less than $\frac{1}{r}$), then Alice has to use a large set of points (precisely, $n \geq 2^{3(1/r)}$).
Dumitrescu and Tóth [11] showed that this does not happen for large $r$. Christ et al. [8] even mentioned
that guaranteeing a construction that covers a miniscule fraction (say, even 0.0001%) seems difficult. An
year later, Dumitrescu and Tóth [11] gave a fairly nontrivial construction method, for the first time, which
can cover at least 0.09121, i.e., roughly 9% of the area. Their algorithm processes the points in decreasing
order of the sum of the coordinates and uses an argument that is based on tiling and finding a maximum
area rectangle within a tile. They also pointed out a simple fact that if Alice chooses $P$ to be equally spaced
points along the diagonal of $U$, then Bob can cover an area of at most $\frac{1}{2} + \frac{1}{2n}$.

There exist other important variants of rectangular packing problems with wide applications. For instance
in the classical 2-dimensional bin-packing problem (2BP), given an unlimited number of identical rectangular
bins of certain width and height, the objective is to allocate all the items to the minimum number of bins.
This typical problem finds application in wood or glass industries where rectangular components have to
be cut from large sheets of material; in warehousing, where the containers are to be placed on shelves; in
editing newspaper pages, where the articles have to be accommodated, and so on. In most of the packing
problems, the items to be packed have a fixed orientation with respect to the bin, i.e., one is not allowed to
rotate them. For this particular type of variant, the reader is referred to Lodi et al. [16]. Jansen et al. [14]
gave a 2-approximation for the 2BP problem and Bansal et al. [3] designed a randomized algorithm with
an asymptotic approximation ratio of about 1.525. Recently Bansal and Khan further improved the ratio to
1.405 [5].

There is another variant of packing problem known as 2-dimensional strip packing problem (2SP), in
which a set of rectangular items have to be orthogonally packed, without overlapping, into a strip of a given
width and infinite height by minimizing the overall height of the packing. Various approximation algorithms
for 2SP have been obtained, among others by Sleator [21], Brown [6], Schiermeyer [20]. Recently, Kenyon et
al. [15] proposed an AFPTAS for 2SP. Furthermore, Bansal et al. [4] showed that 2BP does not admit an
AFPTAS.

General surveys on cutting and packing problems can be found in Dyckhoff et al. [12], Dowsland et al.
[9]. For prior work related to packing anchored rectangles and allied areas, we refer an interested reader to
[11] and the references therein.

### 1.2 Problem Definition

Given an axis-parallel unit square $U$ and a finite set of points $P$ in $U$, with the bottom-left corner of $U$
always in $P$, we call an axis-parallel rectangle $R_i$ an anchored-rectangle, anchored at a point $p_i \in P$, if the
following conditions hold (see Figure 1(a))

(i) $p_i$ is the bottom-left corner of $R_i$.

(ii) $R_i$ does not contain any point of $P$, except possibly at the boundaries.

(iii) $R_i$ is contained in $U$.

(iv) $R_i$ is interior disjoint with any other anchored-rectangle $R_j$, $j \neq i$. However, $R_i$ and $R_j$ can share
boundaries.

Let the area of a rectangle $R_i$ be denoted as $A(R_i)$. Let $C(P)$ denote a particular anchored-rectangle
packing for a point set $P$. See Figure 1(b) for an example of a configuration $C(P)$ of a set of anchored-
rectangles. The area covered by any such anchored-rectangle packing configuration for a point set $P$ is
the sum of the areas of the anchored-rectangles and is denoted by $A(C(P)) = \sum_{R_i \in C(P)} A(R_i)$. We would
use the term $A(P)$ instead of $A(C(P))$ when the configuration is obvious in the context of our discussion.
Observe that for any given point set, there can be many possible anchored-rectangle packings. We call an
anchored-rectangle packing maximum for a point set $P$, if it covers the maximum area among all possible
anchored-rectangle packings for \( P \), i.e., among all \( C(P) \)’s. We will denote this configuration of anchored-rectangles that achieves the maximum area for any point set \( P \) in \( U \) as \( R(P) \) and the area covered as \( A(P) \).

1.3 Our contribution

In our earlier paper \[2\] we gave an existential proof of the conjecture that whatever \( P \) be, there always exists a covering of more than half of \( U \) with anchored-rectangles. The outline of the existential proof is as follows. For any given integer \( n \), let \( P_n \) be any \( n \)-point set containing \( p_1 = (0, 0) \). Let \( R(P_n) \) denote the anchored-rectangle packing configuration, which achieves the maximum area covered for \( P_n \), and let \( A(P_n) \) be the maximum area thus covered. We will assume without loss of generality that in \( R(P_n) \), each \( p_i \in P_n \) has a rectangle anchored at it. If a point \( p_i \) exists for which there is no such anchored rectangle, then \( p_i \) must be on the boundary of some other rectangle; we can split that rectangle into two by introducing a rectangle anchored at \( p_i \). Out of all possible \( n \)-point sets \( P_n \) (uncountably infinite), let \( P_n \) be the \( n \)-point set that achieves the minimum of all \( A(P_n) \)’s, i.e., \( P_n = \arg \min_{P_n} A(P_n) \). We call \( P_n \) to be a MIN-MAX point set. To have an existential proof for the long-standing conjecture stated by Winkler \[24, 25\], we prove that \( A(P_n) \geq 1/2 \) where \( U \) is of unit area.

Notice that the existential proof \( A(P_n) \geq 1/2 \) basically gives a lower bound on the maximum area that can be covered with anchored rectangles for any point set \( P \). Recently, Adamaszek and Wiese \[1\] presented the first \((1−\epsilon)\)-approximation algorithm for finding the maximum weight independent set of rectangles among a set of \( n \)-axis parallel rectangles with real positive weights for each rectangle in the 2D-plane; the goal is to select a maximum weight subset of pairwise non-overlapping rectangles. The approximation algorithm of Adamaszek and Wiese runs in quasi-polynomial time, i.e., \((2^{poly(log n/\epsilon)})\). The anchored rectangles in our problem are essentially an independent set of rectangles. We compute all the maximal empty rectangles corresponding to each point in \( P \); the number of all such rectangles is polynomially bounded. Each rectangle is assigned a weight equal to its area. We give this input to the approximation algorithm of Adamaszek and Wiese and argue that their output provides a packing close to \( 1/2 \), i.e., \((1/2−\epsilon)\) in a quasi-polynomial running time \( 2^{poly(log n/\epsilon)} \).

In section 2 we describe the quasi-polynomial construction achieving a packing close to half. Section 3 concludes the paper.

2 A quasi-polynomial construction achieving a packing close to half

We first briefly review the work of Adamaszek and Wiese \[1\] and then show how we can adapt it to provide a quasi-polynomial construction.
2.1 Weighted maximum independent set of axis parallel rectangles

In the problem of Maximum Weight Independent Set of Rectangles (MWISR), the input is a set of \( n \) axis-parallel rectangles \( R = \{ R_1, \ldots, R_n \} \) in the 2D-plane; each rectangle \( R_i \) has a weight \( w(R_i) \in \mathbb{R}^+ \). Any rectangle \( R_i \) is specified as the set \( \{(x, y) \mid (x^L_i < x < x^R_i) \cap (y^L_i < y < y^R_i)\} \) where the bottom-left coordinate is \((x^L_i, y^L_i) \in \mathbb{N}^2\) and the top-right coordinate is \((x^R_i, y^R_i) \in \mathbb{N}^2\). The goal is to select a subset \( R' \subseteq R \) such that the sum of weights of the selected rectangles, \( \sum_{R_i \in R'} w(R_i) \), is maximized and for any two rectangles \( R, R' \in R' \), \( R \cap R' = \emptyset \).

Chan and Har-Peled [7] proposed an \( O(\log n / \log \log n) \)-approximation algorithm for the above problem. Recently,Adamaszek and Weiss [1] made a major breakthrough and achieved a \((1 - \epsilon)\)-approximation algorithm that needs quasi-polynomial time \( 2^{\text{poly}(\log n / \epsilon)} \). We state their result as a Lemma.

**Lemma 1.** [1] The MWISR problem for \( n \) axis-parallel rectangles admits a \((1 - \epsilon)\)-approximation algorithm running in time \( 2^{\text{poly}(\log n / \epsilon)} \).

2.2 The quasi-polynomial construction

We describe next how we use the result of Adamaszek and Weiss. Given a set of \( n \) points \( P_n \) inside \( U \), let \( R_i \) be an anchored rectangle, anchored at the point \( p_i \). Let \((p_x, p_y)\) be the coordinates of the top-right corner of \( R_i \). See Figure 2 for an illustration. We denote the vertical line segment joining the points \((p_x, p_y)\) and \((0, 0)\) as \( h_i \) and the horizontal line segment joining the points \((p_x, p_y)\) and \((0, p_y)\) as \( v_i \). We define an anchored rectangle \( R_i \), anchored at \( p_i \), to be a Maximal Empty Rectangle (MER) if the following conditions are satisfied:

- **(Condition (i))** \( h_i \) has a point from \( P_n \) on it or \( h_i \) lies on the boundary of \( U \).
- **(Condition (ii))** \( v_i \) has a point from \( P_n \) on it or \( v_i \) lies on the boundary of \( U \).

Notice that the MER defined by us is different from the usual maximal empty rectangle as defined in [17, 18]. Recall that an anchored rectangle by definition does not contain any point from \( P_n \) inside it. So, a MER also does not contain any point from \( P_n \) inside it. Notice that the top-right corner \((p_x, p_y)\) of a MER need not be a point of \( P_n \). Let \( \mathcal{L} \) denote the set of all MERs for an \( n \)-point set \( P_n \).

**Lemma 2.** \( |\mathcal{L}| = O(n^3) \).

**Proof.** Any maximal anchored rectangle can have one point each from \( P_n \) on its vertical \((v_i)\) and horizontal \((h_i)\) supporting lines, so there can be \( O(n^2) \) such rectangles for a point of \( P_n \) making the total number of maximal anchored rectangles to be \( O(n^3) \). See Figure 2 for a point set configuration that achieves the \( O(n^3) \) bound.
Figure 3: Maximal anchored rectangles can be \( \sum_{i=2}^{n} i \cdot (n-i) = O(n^3) \).

We next define a \textit{Maximal Anchored Rectangle Packing} (MARP) as an anchored rectangle packing configuration \( C(P_n) \) such that no anchored rectangle \( (R_i \text{ anchored at } p_i) \) in \( C(P_n) \) can be replaced with another anchored rectangle \( (R_i^* \text{ anchored at } p_i) \) resulting in an increase of the packing area. Formally, there does not exist any other rectangle packing \( C^*(P_n) \) such that \( A(C^*(P_n)) > A(C(P_n)) \) where \( C^*(P_n) = C(P_n) \setminus \{R_i\} \cup \{R_i^*\} \); \( R_i \) and \( R_i^* \) are rectangles anchored at \( p_i \) in \( C(P_n) \) and \( C^*(P_n) \), respectively.

\textbf{Lemma 3.} Any anchored rectangle belonging to an instance of MARP is a MER.

\textbf{Proof.} For any anchored rectangle \( R_i \), anchored at point \( p_i \), with the top-right corner \( (p_x, p_y) \), there should be at least two anchored rectangles – one blocking it from above and the other blocking it from right. For the existence of two such rectangles, there should exist points from \( P_n \) on \( h_i \) and \( v_i \). This might even happen with only one rectangle when \( R_i \) shares its top-right corner \( (p_x, p_y) \) with another anchored rectangle. Thus \( R_i \) satisfies Conditions (i) and (ii), and hence, it is a MER. So, any anchored rectangle that belongs to an instance of MARP should be a MER. \( \blacksquare \)

Notice that the converse of the above Lemma is not true, i.e., not all interior disjoint MERs form a MARP instance.

We say, as in \[1\], two rectangles \( R_i, R_j \in L \) intersect, if their interiors intersect. If \( R_i \) and \( R_j \) share boundaries or corners, we say they do not intersect. We assign the area of the rectangle as its weight. Any point \( p_i \in P_n \) (input points of our problem) has real coordinates but the algorithm in \[1\] requires the coordinates of the bottom-left and top-right corners of input rectangles to be positive integers. Draw horizontal and vertical lines through each point \( p_i \in P_n \) and interpret this grid as the coordinate. Notice that with this interpretation of the coordinate system, all MERs in \( L \) have positive integral coordinates.

The approximation algorithm of Adamaszek and Weiss \[1\] obviously returns a maximal independent set. We now relate the maximal independent set with a maximal anchored rectangle packing. To the algorithm of \[1\], we give as input the set \( L \), the set of all MERs.

\textbf{Lemma 4.} MERs that belong to an instance of MARP form a weighted maximal independent set of all MERs in \( L \).

\textbf{Proof.} Consider two anchored rectangles \( R_i \) and \( R_j \) that belong to an instance of MARP. Surely, \( R_i \cap R_j = \emptyset \). So, MERs belonging to an instance of MARP forms an independent set \( S \). If \( S \) is not maximal, then there should exist an MER of the MARP instance (due to Lemma 3 any anchored rectangle in a MARP instance has to be a MER), that does not intersect any of the rectangles in \( S \). This violates the maximality of the MARP instance. So, the lemma holds. \( \blacksquare \)

We next have the reverse implication.

\textbf{Lemma 5.} A maximal weighted independent set of \( L \) forms the MERs of a MARP instance.
Figure 4: The red-colored anchored rectangle in (a) can be replaced with the blue-colored anchored rectangle in (b) to have a MARP instance.

Proof. \( \mathcal{L} \) is formed of MERs. So, any two rectangles in a maximal independent set of \( \mathcal{L} \) are MERs whose interiors do not intersect. These MERs surely form an anchored rectangle packing. Next, we show that these MERs also form an instance of a MARP. Suppose, for a contradiction, there exists a MER \( R_i \) (the red-colored rectangle in Figure 4) that does not allow the packing instance to be a MARP. Then, there exists surely another MER \( R^*_i \) (the blue-colored rectangle in Figure 4) such that the area of \( R^*_i \) is more than the area of \( R_i \). The maximal weighted independent set then should have contained \( R^*_i \) instead of \( R_i \). Thus, we have a contradiction. Hence, the MERs form an instance of MARP. 

Now, to put everything together, given an \( n \)-point set \( P_n \), we form all its MERs with positive integral coordinates as mentioned earlier, and assign the area of each MER as its weight. This forms the set \( \mathcal{L} \). We give this input \( \mathcal{I} \) to the algorithm of Adamaszek and Weiss [1] and interpret their output as stated in the following Lemma. Let \( B \) be the output returned by the algorithm of [1] on the input \( \mathcal{I} \).

Lemma 6. The output \( B \) is a \((1 - \varepsilon)\)-approximation to the optimization version of the anchored rectangle packing problem.

Proof. The result follows from Lemmas 1, 4 and 5.

We now conclude with the final result in the following Theorem.

Theorem 7. For any \( n \)-point set \( P_n \) in the unit square \( U \), an anchored rectangle packing can be constructed that covers an area \( \frac{1}{2} - \varepsilon \) in quasi-polynomial time, where \( \varepsilon > 0 \).

Proof. The input set \( \mathcal{I} \) to the algorithm of [1] is the set \( \mathcal{L} \) coupled with the weights. As per Lemma 2, \( |\mathcal{I}| = O(n^3) \). So, the input is polynomial in \( n \) and hence, our time complexity is quasi-polynomial, i.e., \( 2^{poly(\log n^3/\varepsilon)} \). As to the area of the coverage of the packing, we have the following using Lemma 6 and Theorem 7 of [2]

\[
B \geq (1 - \varepsilon) \cdot OPT \geq (1 - \varepsilon) \frac{1}{2}.
\]

3 Conclusion

In our earlier paper [2], we have presented an existential proof of the conjecture that at least half of an unit square can be covered with anchored rectangles. This bound serves as a lower bound to the optimization version of the anchored rectangle packing problem. This lower bound coupled with a recent QPTAS of Adamaszek and Weiss gives us a quasi-polynomial construction that achieves an anchored rectangle packing close to half. The earlier constructive result led to a 9% packing in polynomial time. We have been able to achieve a much denser packing at the cost of quasi-polynomial time. An immediate open problem is
to bring down this time to polynomial. Then, the optimization version of the anchored rectangle packing problem would admit a PTAS. Whether the optimization version of the anchored rectangle packing problem is NP-hard is another issue that needs to be settled.

A look at our proof suggests that the technique used here may be applicable even if \( U \) is a rectangle. An interesting problem might be to extend the proposed existential proof in \( \mathbb{R}^d \) and seek for a suitable anchored-hyperrectangle packing.

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