A New and Elegant Argument that $P \neq NP$

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Abstract: In this note, we present a new and elegant argument that $P \neq NP$ by demonstrating that the Meet-in-the-Middle algorithm must have the fastest running-time of all deterministic and exact algorithms which solve the SUBSET-SUM problem on a classical computer.

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“This one's from The Book!” - Paul Erdős (1913-1996)

1 The File-Searching Problem

You are given a file with a huge number $N$ of unsorted records. There is one special record in the file, and you want to find this record. One way to do this might be to have your computer search each of the records, one after another, until it finds the special record. But since $N$ is huge, this could take a very long time. Another way to do this might be to use $N$ computer processors: In parallel, each of the computer processors reads one of the $N$ records, and the processor that reads the special record then outputs the contents of the special record. But since $N$ is huge, $N$ computer processors could be very costly.

Is there a better way to find the special record without taking too much time and without spending too much money on computer processors? Yes, it is possible to find the special record using only $\Theta(\sqrt{N})$ processors and $\Theta(\sqrt{N})$ time: Each processor reads $\Theta(\sqrt{N})$ records until one of the processors finds the special record and then outputs the contents of the special record. And this is the best we can do, as $\Theta(\sqrt{N})$ processors and $\Theta(\sqrt{N})$ time is the asymptotic solution (for large $N$) to the following optimization problem:

\[
\begin{align*}
\text{Minimize} & \quad P + T \\
\text{subject to} & \quad PT \geq \Theta(N), \\
& \quad P, T \geq 0.
\end{align*}
\]

$P$ is the number of processors and $T$ is the amount of time.

2 The SUBSET-SUM Problem

Now, let us consider the following problem: You are given a set $A = \{a_1, ..., a_n\}$ of $n$ integers and another integer $b$. You want to find a subset of $A$ for which the sum of its elements (we shall call this quantity a subset-sum) is equal to $b$. We shall consider the sum of the elements of the empty set to be zero. This problem is called the SUBSET-SUM problem [2, 4]. It turns out that our observation from the previous section, that using $\Theta(\sqrt{N})$ processors and $\Theta(\sqrt{N})$ time is the most efficient way to solve the file-searching problem, can actually help us to understand the SUBSET-SUM problem, as we shall soon see.

Let us suppose that we only have one computer processor for solving the SUBSET-SUM problem. Then one might think that the fastest way to solve the SUBSET-SUM problem would be to program the computer to compute the subset-sum of every possible subset of $A$ (there are $2^n$ subsets of $A$) until it finds a subset-sum that matches $b$, just like the file-searching problem. But this is a mistake, as one must keep in mind that the SUBSET-SUM problem is a problem with an inherent mathematical structure, unlike the problem of searching a generic file of $N$ records for a special record. Because of this, it turns out that we can solve the SUBSET-SUM problem in $\Theta(\sqrt{2^n})$ time with only one computer processor - consider the following algorithm for solving the SUBSET-SUM problem:

Meet-in-the-Middle Algorithm - First, partition the set $A$ into two subsets, $A^+ = \{a_1, ..., a_{\lceil n/2 \rceil}\}$ and $A^- = \{a_{\lceil n/2 \rceil + 1}, ..., a_n\}$. Let us define $S^+$ and $S^-$ as the sets of
subset-sums of $A^+$ and $A^-$, respectively. Sort sets $S^+$ and $b - S^-$ in ascending order. Compare the first elements in both of the lists. If they match, then output the corresponding solution and stop. If not, then compare the greater element with the next element in the other list. Continue this process until there is a match, in which case there is a solution, or until one of the lists runs out of elements, in which case there is no solution.

This algorithm takes $\Theta(\sqrt{2^n})$ time, since it takes $\Theta(\sqrt{2^n})$ steps to sort sets $S^+$ and $b - S^-$ and $O(\sqrt{2^n})$ steps to compare elements from the sorted lists $S^+$ and $b - S^-$. Are there any faster algorithms for solving SUBSET-SUM? It turns out that no deterministic and exact algorithm with a better worst-case running-time has ever been found since Horowitz and Sahni discovered this algorithm in 1974 [3, 5]. And the reason for this is because it is impossible for such an algorithm to exist. Why?

**Explanation:** As we discussed earlier, in order to search a set of size $N$ in a more efficient way than a brute-force search by a single computer processor, it is necessary for many processors to search the set in parallel. But the Meet-in-the-Middle algorithm for solving the SUBSET-SUM problem involves only one computer processor, so how can it run in $\Theta(\sqrt{2^n})$ time when the size of the search space of the SUBSET-SUM problem is $\Theta(2^n)$? The answer is that the Meet-in-the-Middle algorithm essentially creates $\Theta(\sqrt{2^n})$ processors by sorting the two lists, $S^+$ and $b - S^-$, of size $\Theta(\sqrt{2^n})$: For example, suppose that $n = 10$ and the Meet-in-the-Middle algorithm discovers that some element $a_1 + a_3$ in list $S^+$ is less than some other element $b - a_6 - a_8 - a_9$ in list $b - S^-$. Then since list $b - S^-$ is sorted in ascending order, the algorithm has also immediately proven that $a_1 + a_3$ is less than every element after $b - a_6 - a_8 - a_9$ in list $b - S^-$, just as if there were many computer processors comparing $a_1 + a_3$ to every element after $b - a_6 - a_8 - a_9$ in list $b - S^-$ in parallel; hence, it is reasonable to say that the Meet-in-the-Middle algorithm creates imaginary processors by sorting the two lists, $S^+$ and $b - S^-$. Notice that it is only because the SUBSET-SUM problem has an inherent mathematical structure, unlike the problem that we discussed earlier of searching a generic file of $N$ records for a special record, that it is possible for an algorithm solving SUBSET-SUM to create imaginary processors. And also notice that imaginary processors do not come for free; the Meet-in-the-Middle algorithm has to pay $\Theta(\sqrt{2^n})$ units of time (by sorting lists $S^+$ and $b - S^-$) to create $\Theta(\sqrt{2^n})$ imaginary processors, i.e., at least $P$ steps are required for an algorithm that solves SUBSET-SUM to create $P$ imaginary processors. But imaginary processors are worth it, since they enable one computer processor to solve the SUBSET-SUM problem in $\Theta(\sqrt{2^n})$ time via the Meet-in-the-Middle algorithm, instead of the $O(2^n)$ time of a brute-force search.

It is now possible to solve the problem of finding a nontrivial lower-bound for the running-time of a deterministic and exact algorithm that solves the SUBSET-SUM problem by thinking of such a lower-bound as a solution to the following optimization problem:

$$\begin{align*}
\text{Minimize} & \quad T \\
\text{subject to} & \quad PT \geq \Theta(2^n), \\
& \quad T \geq P \geq 0.
\end{align*}$$

$P$ is the number of imaginary processors and $T$ is the amount of time. Since the size of the search space of the SUBSET-SUM problem is $N = 2^n$, we have the constraint that $PT \geq \Theta(2^n)$. And since at least $P$ steps are required to create $P$ imaginary processors when solving the SUBSET-SUM problem, we have the constraint that $T \geq P \geq 0$.

Because the running-time of $T = \Theta(\sqrt{2^n})$ is the asymptotic solution to this minimization problem (for large $n$) and because the Meet-in-the-Middle algorithm achieves this running-time, we can conclude that $\Theta(\sqrt{2^n})$ is a tight lower-bound for the running-time of any deterministic and exact algorithm which solves the SUBSET-SUM problem. And this conclusion implies that $P \neq NP$ [1, 2].

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**References**

[1] P.B. Bovet and P. Crescenzi, *Introduction to the Theory of Complexity*, Prentice Hall, 1994.

[2] T.H. Cormen, C.E. Leiserson, and R.L. Rivest, *Introduction to Algorithms*, McGraw-Hill, 1990.

[3] E. Horowitz, and S. Sahni, “Computing Partitions with Applications to the Knapsack Problem”, *Journal of the ACM*, vol. 21, no. 2, April 1974, pp 277-292.

[4] A. Menezes, P. van Oorschot, and S. Vanstone, *Handbook of Applied Cryptography*, CRC Press, 1996.

[5] G.J. Woeginger, “Exact Algorithms for NP-Hard Problems”, *Lecture Notes in Computer Science*, Springer-Verlag Heidelberg, Volume 2570, pp. 185-207, 2003.