Approximate Analytic Solution to the Three Species Lotka – Volterra Differential Equation Model

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Abstract

This paper provides an approximate analytic solution to the three species Lotka – Volterra differential equations by symbolic regression. The approximate analytic solution through symbolic regression is made as close as desired to the actual analytic solution by using the Jacobian system. This is proposed as the equilibrium will be stabilized if and only if the real parts of each of the eigenvalues are negative. As a result, the symbolic regression approach is found to provide an approximation to the faster convergence that can be expected with a more refined Euler numerical approach.

Keywords: approximation, analytic solution, Lotka-Volterra, differential equation model

1.0 Introduction

The famous Lotka – Volterra model is a pair of differential equations representing the populations of predator-prey species that interact with each other. The model was individually proposed in 1925 by Lotka (1925) and Voltera (1926). Suppose that the model extends the two-species model and considers the three-species model with one species at the bottom of the food chain, one in the middle, and one at the top Chauvet et al. (2002). Let these species be x, y, and z and denote the differential equations modeling their behavior by:

\[
\begin{align*}
\frac{dx}{dt} &= ax - \beta xy \\
\frac{dy}{dt} &= -\gamma y + \delta xy - \sigma yz \\
\frac{dz}{dt} &= -\varphi z + \mu yz
\end{align*}
\] (1)

where \(x(t)\) represents the population of the species at the bottom of the food chain the producer and \(y(t)\) represents the population of the species in the middle of the primary consumer and \(z(t)\) represents the population of the top of the food chain or the predator. Equation (1) is independent in the sense that the right-hand side is not explicitly expressed in terms of time. The Lotka–Volterra predator-prey model (Volterra, 1926), is used to study a differential equation system with diffusion and time delays which model the dynamics of predator-prey interactions within three biological species (Wei, 2007). Pekalski and Stauffer (1998) studied of the dynamics of three-species predator-prey models with time delays. Early studies such as those by Guerrero et al. (2009) focused on obtaining analytical solutions for the
transport equations, whereas recently numerical solutions have become increasingly popular using widely available computer software.

Regardless of the rising number of Lotka–Volterra applications, this model is a minor generalization of the two-species model: it adds a third species, but it only allows for a chain of predation. Still, considerable interest remains in analytical solutions because they may be used as relatively simple screening models in regulatory decision making by Devireddy (2016); Regalado and Castillano (2019), and because analytical solutions may serve as benchmarks for testing numerical solutions.

Since 1970s, there have been some interesting and impressive results to the analytic solution on the dynamics of three species predator-prey systems. In the past years, Hsu et al. (2015) analyzed the three species Lotka–Volterra food web model with omnivore which was defined as feeding on more than one trophic level. Pontedeiro et al. (2007) obtained solutions on a finite domain using a semi-analytical approach to perform sensitivity analyses and risk assessments of technologically enhanced, naturally occurring radioactive material (TENORM) disposed in an industrial landfill. The paper solved the decay chain problem using a semi-analytic approach known as the Generalized Integral Transform Technique (GITT), which is a generalization of the Classic Integral Transform Technique, or CITT (Cotta & Mikhailov, 1993). Recently, Cassol et al. (2009) combined GITT and CITT techniques with Laplace transforms and matrix diagonalization methods to develop an analytical solution for transient two-dimensional atmospheric pollutant dispersion on a semi-infinite spatial domain.

There are many papers focused on obtaining analytical solutions for the transport equations, whereas more recent numerical solutions have become increasingly popular using widely available computer software. However, there is no closed-form shown that analytic solution to this three-species predator-prey model exists in the literature. The analytic solutions are still a problem for the three species Lotka–Volterra Equation. Hence, this paper intends to find approximate analytic solutions to the Extended Lotka–Volterra equations.

2.0 Three-Species Model

Suppose that Lotka–Volterra model would be extended from the two-species model into a three-species model with one species at the bottom of the food chain, one in the middle, and one at the top. Let these species be \( x \), \( y \), and \( z \) and denote the differential equations modeling their behavior by equation (1).

The three – species model is a very minor generalization of the two-species model: it adds a third species, but it only agrees for a chain of predation (i.e. this model does not agree us to consider the case where both \( y \) and \( z \) prey on \( x \) or where \( x \) and \( y \) are at the bottom of the food chain and \( z \) preys on both).

Simplified Solution of the Three-Species Model

To find the balances of this model, the researcher considers the values \( x \), \( y \), \( z \) for which

\[
\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0.
\]

Thus:

\[
\begin{align*}
\frac{dx}{dt} &= x(a - \beta y) \\
\frac{dy}{dt} &= y(\delta x - \gamma - \sigma z) \\
\frac{dz}{dt} &= z(\mu y - \varphi)
\end{align*}
\] (2)

This model has related parameters as the classic model: \( x \) has a growth rate denoted by \( ax \).
while \( y \) and \( z \) have a natural death rate denoted by \(-\gamma y\) and \(-\varphi z\), respectively. \( \delta xy \) and \( \mu yz \) denote the growth rate of \( y \) and \( z \) from hunting their prey and \(-\beta xy\) and \(-\sigma yz\) denote the death rate of the prey as a result of being hunted. It is interesting to note that the authors chose to give species \( y \) a death rate (the negative sign in front of \(-\gamma y\)) instead of a growth rate in the absence of a predator \( z \).

### Qualitative Solutions Using Equilibrium Points

Without expressly solving the differential equations, the equilibrium points can be analyzed. Equilibrium points are points for which the derivatives with respect to time are zero. Thus, for the classical Lotka-Volterra model illustrates:

\[
x(\alpha - \beta y = 0) \text{ and } -y(y - \delta x) = 0
\]

which yields:

\[
x = 0, y = 0 \text{ and } x = \frac{\alpha}{\delta}, y = \frac{\alpha}{\beta}
\]

The first solution comes when the two species become nonexistent. The second solution represents a set point at which the populations sustain their present number indefinitely. The Jacobian matrix of the LV model is given by

\[
J = \begin{bmatrix}
\alpha - \beta y & -\beta x & 0 \\
\delta x & -\gamma + \delta x - \sigma z & -\sigma z \\
0 & \mu z & \mu y - \varphi
\end{bmatrix}
\]

The system will have an asymptotical order solution if and only if all the real parts of the eigenvalues of the Jacobian are negative. Consider the roots of the characteristic equation of the above Jacobian, given by

\[
det(J(x, y, z) - \lambda I) = [\alpha - \beta y - \lambda - \beta x \delta y \\
- \gamma + \delta x - \sigma z - \lambda \\
- \sigma y 0 \mu z \mu y - \varphi - \lambda]
\]

Likewise, for the trivial equilibrium position \((0, 0, 0)\), of the three species Lotka-Volterra model, we consider the case where \((\alpha - \beta y) = (\mu y - \varphi) = (\delta x - \gamma - \sigma z) = 0\). The \( y = \frac{\alpha}{\beta} = \frac{\varphi}{\mu} \). Therefore, a second equilibrium position exists only when the constant parameters \( \mu \alpha = \varphi \beta \).

The Jacobian of the system was proposed. Taking the appropriate partial derivatives, it has been found that:

\[
\frac{dx}{dt} = ax \left(1 - \frac{x}{k}\right) - \frac{\beta xyz}{1+\beta x}
\]

\[
\frac{dy}{dt} = \gamma y \left(1 - \frac{y}{kx}\right) + \left(\frac{\delta xy}{1+\delta y}\right) - \left(\frac{\sigma yz}{1+\sigma y}\right)
\]

\[
\frac{dz}{dt} = \varphi z \left(1 - \frac{z}{kxy}\right)
\]

where \( \beta xyz \) is the interaction rate between the species, \( \frac{\beta xyz}{1+\beta x} \) is the effective rate of eating bottom species, \( \gamma \) is the mortality rate of the predators, \( K \) and \( k \) are the carrying capacitance of each population. \( x \) has a growth rate denoted by \( ax \) while \( y \) and \( z \) have a natural death rate denoted...
by \(-\gamma y\) and \(-\varphi z\), respectively, \(\delta xy\) and \(\sigma yz\) denote the growth rate of \(y\) and \(z\) from hunting their prey and \(-\beta xy\) and \(-\sigma yz\) denote the death rate of the prey as a result of being hunted.

Example: Let \(x=100, y=100, z=100\). Then, \(a=15, \gamma=0.05, \delta=0.09, K=500, k=0.82, \beta=0.01, \varphi=1.9, \sigma=0.5, x(0)=100, y(0)=100, z(0)=100\)

The Euler’s recursive relation is,

\[
x_i = x_{i-1} + xs_{i-1}\Delta t
\]

\[
y_i = y_{i-1} + yss_{i-1}\Delta t
\]

\[
z_i = z_{i-1} + zss_{i-1}\Delta t
\]

where \(s_{(i-1)}\) in the equation (8) represent of the slope of every species and \(\Delta t\) represent of the change of time.

This leads to the recurrence:

\[
x_i = x_{i-1} + \left(15x_{i-1} \left(1 - \frac{x_{i-1}}{300}\right) - \frac{0.01x_{i-1}y_{i-1}z_{i-1}}{0.01+1x_{i-1}}\right)
\]

\[
y_i = y_{i-1} + 0.05y_{i-1} \left(1 - \frac{y_{i-1}}{0.82x_{i-1}}\right)
\]

\[
- \left(0.09y_{i-1}z_{i-1}\right) + \frac{0.5y_{i-1}z_{i-1}}{1+0.09y_{i-1}}\right)
\]

\[
z_i = z_{i-1} + 1.9z_{i-1} \left(1 - \frac{z_{i-1}}{0.82x_{i-1}y_{i-1}}\right)
\]

3.0 Three-Species Model using Symbolic Regression Approach

The symbolic regression approach is a type of regression analysis that searches the space of mathematical expressions to find the model that best fits a given data set. No model is provided as a starting point to the algorithm. Instead, initial expressions are formed by randomly combining mathematical building blocks such as mathematical operators, analytic functions, constant, and state variables. New equations are then formed by recombining previous equations, using genetic programming.

The values are then entered into symbolic regression software with generated ordered pairs \(\{(t_i, x_i)\}_{i=0}^n\) for the population of the species at the bottom, \(\{(t_i, y_i)\}_{i=0}^n\) for the population of the species at the middle and \(\{(t_i, z_i)\}_{i=0}^n\) for the population of the species at the top. Let \(t_i = ih, i=0, 1,2,...,n\), and step size \(h\), where \(h = 0.001\).

### Table 1. The result of the first iteration

| T   | Producer | Primary Consumer | Top Predator |
|-----|----------|------------------|--------------|
| 0   | 100      | 100              | 100          |
| 0.001 | 96.2    | 99.95870636      | 100.187628   |
| 0.002 | 92.45503286 | 99.89916516      | 100.3756209  |
| 0.003 | 88.76825685 | 99.82163294      | 100.563807   |
| 0.004 | 85.14280942 | 99.72638952      | 100.752337   |
| 0.005 | 81.58179248 | 99.61373724      | 100.9408929  |
| 0.006 | 78.08825643 | 99.4840001       | 101.1297755  |
| 0.007 | 74.66518313 | 99.33752279      | 101.3188717  |
| 0.008 | 71.31546787 | 99.17466962      | 101.5081706  |
| 1.499 | 14.99960432 | 15.1997386       | 15.23944928  |

| T   | Producer | Primary Consumer | Top Predator |
|-----|----------|------------------|--------------|
| 1.49 | 15.23944928 | 15.26392331      | 15.26392331  |
Table 1 shows the first iteration of the simulated data using equation (9). These data are used to approximate the analytic solution to the extended Lotka–Volterra differential equation model.

Table 2 shows summary statistics for the symbolic regression analysis. The Bottom species (producer) was more fluctuate than the Middle species (primary consumer) and the Top Species (predator).

Table 2. Result Statistics for the Symbolic Regression Analysis

| Species | MSE     | $R^2$ | Equation                                      |
|---------|---------|-------|-----------------------------------------------|
| x(t)    | 138.28192 | 0.9889 | $x = 0.922t^{1.4} \sin(0.014t^2) + 1.0t^2 \exp(8.09t \cos(0.92t^4))$ |
| y(t)    | 12.671713 | 0.9954 | $y = 6.33 + 11.1t + 11.1t \sin(\sin(2.86 + 2.26t + 2.26t^2))$  
|         |         |       | $+ 39.6 \exp(3.78 \sin(2.86 + 2.26t + 2.26t^2))$  
|         |         |       | $- 3.78 \sin(2.86 + 2.26t + 2.26t^2)$  
|         |         |       | $- 33.77t \exp(3.78 \sin(2.86 + 2.26t + 2.26t^2))$ |
| z(t)    | 65.35765 | 0.8906 | $z = 153 + 1416.86t^2 + 377.65t \sin(3.18t) - 1076t - 446.4t^3$  
|         |         |       | $- 251.7t^2 \sin(3.18t) - 85.5t^2 \sin(3.18t)^2$ |

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Figure 1 shows the graph of the solutions $x(t)$, $y(t)$ and $z(t)$. The best fitting of the symbolic regression curve of the $x(t)$ gives an $r$–squared value of 98.89% with mean squared error (MSE) of 138.28, $y(t)$ gives an $r$–squared value of 99.54% with mean squared error (MSE) of 12.67 and for the $z(t)$ gives an $r$–squared value of 89.06% with mean squared error (MSE) of 65.36. This implies that the approximate solution of the $x(t)$, $y(t)$ and $z(t)$ is the best analytic solution to the extended Lotka–Volterra differential equations model.

Figure 2 shows the plot of $x(t)$ vs $y(t)$ vs $z(t)$. Note the closed trajectory as expected.

4.0 Conclusion and Recommendation

The symbolic regression analysis provides a proper means to determine an approximate analytic solution to the Three Species Lotka-Volterra non-linear differential equations. Therefore, the researcher recommends that the approximate analytic solution be made as close as the desired analytic solution of the three species Lotka-Volterra differential equation model problem. Likewise, since the finite difference approach is one method for arriving at a numerical solution of the differential equation, faster convergence can be expected with more sophisticated Euler numerical methods.
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