Forbidden regimes in distribution of bipartite quantum correlations due to multiparty entanglement

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Monogamy is a nonclassical property that limits the amount of quantum correlation which can be shared among different parties in a multiparty quantum system. We show that the monogamy scores for the different quantum correlation measures are bounded above by the functions of genuine multiparticle entanglement for a large majority of multiqubit pure states. We analytically show that the bound is universal for three-qubit states and identify the conditions for its validity in higher number of qubits. Moreover, we derive a set of necessary conditions to characterize the extremely small set of states that may violate the multiparty entanglement bound on monogamy score, and numerically observe that the bound is satisfied by almost all Haar uniformly generated multiqubit pure states. The results show that the distribution of bipartite quantum correlations in a multiparty system is restricted by its genuine multiparticle entanglement content.

I. INTRODUCTION

A fundamental property that distinguishes quantum correlation [1, 2], which is a staple resource for exotic quantum information processing and computational tasks [3], from classical correlation is its distribution among several parties in a quantum system. Classical correlation can be distributed among any number of parties, with each pair attaining the maximum possible correlation. However, for quantum correlations, there exist strong constraints on its sharability among the different parties of a multiparty system. Such constraints form the basis for the concept of monogamy of quantum correlations – among three parties, if two possess maximal quantum correlations, they cannot have any quantum correlation with the third party. All quantum correlation measures obey this qualitative monogamy relation while classical correlations do not [4–8] (cf. [9]). In general, monogamy implies that if two systems are strongly correlated with respect to a nonclassical quantity, they can only be weakly correlated with respect to the same quantity to a third system. This property is satisfied by a host of nonclassical quantities including those related to Bell [10] and contextual inequalities [11], quantum steering witnesses [12], and dense-coding capacities [13]. The quantitative version of the monogamy constraint [6] is satisfied by entanglement monotones such as squared-concurrence [6, 8], squared entanglement of formation [14], squared-negativity [15, 16], and squashed entanglement [17, 18] (also see [19]).

The monogamy property is an important feature in quantum information theory [3] and in essence captures the “trade-off” between various quantities of quantum and classical properties [17]. It plays a significant role in the security of quantum key distribution [20] and in the considerations leading to quantum advantage in dense-coding [21]. Further, it has been used to characterize three-qubit genuinely multiparty entangled states [22, 23] and distinguish Bell-like orthonormal bases [24]. See also Ref. [25].

From the perspective of quantum correlation, quantification of the monogamy property in multiparty systems is not very straightforward. This is in part due to the fact that quantum correlation shared among arbitrary parties in a multiparty system is not always computable, making the study of its distribution among several parties extremely difficult. However, there are ongoing efforts to overcome this constraint [26–29]. Further, characterization of both bipartite and multiparticle quantum correlations in higher-dimensional mixed states is not well-developed [1, 2]. Nevertheless, various attempts have been made to systematically quantify the monogamy of quantum correlations. A seminal result was obtained in [6] for the monogamy of squared concurrence [30] in three-qubit states. It was demonstrated that, for a three-qubit state, $\rho_{ABC}$, the sum of the squared concurrence between qubits $A$ and $B$, and that between qubits $A$ and $C$, is bounded above by the squared concurrence between qubit $A$ and the joint subsystem $BC$. Hence, the monogamy property was captured in the form of an inequality, known as the “monogamy inequality”. An advantage of this inequality is that it is a multiparty property expressed in terms of bipartite quantum correlation measures which are well-understood at least for two-qubit systems. The inequality was shown to hold for squared concurrence in multiqubit quantum states [8]. However, it does not satisfy for some entanglement measures [16, 31], such as entanglement of formation [32] and logarithmic negativity [33, 34]. Further, information-theoretic measures of quantum correlation, such as quantum discord [35, 36] and quantum work-deficit [37, 38], are also known to violate the monogamy inequality for three-qubits [22, 39–41]. Recent results on monogamy of quantum correlation have shown that the monogamy inequality is always satisfied for increasing powers of any quantum correlation measure [42] or when large number of parties are considered [43].

In this paper, we establish a relation between monogamy of quantum correlation, quantified by using the concept of “monogamy score” [22, 23, 44], and genuine multiparty entanglement measures for $n$-qubit pure states. The connection holds irrespective of the number of parties and is independent of the choice of the bipartite quantum correlation measure used in the conceptualization of monogamy. We show that for a large majority of pure multiqubit states, the monogamy score for a broad range of quantum correlation measures is upper-bounded by a function of genuine multiparty entanglement in the system, as quantified by the generalized geometric measure [45, 46] (see also [47]). Considering the squared concurrence and squared negativity as measures from the entanglement-separability paradigm, and quantum discord and quantum work-deficit as information-theoretic measures of quantum correlation, we analytically show that the bound
is universally satisfied for all pure three-qubit states \cite{44} and
find conditions for its validity in arbitrary number of qubits.
We also identify a set of necessary conditions to be satisfied
in order to violate the bound for more than three qubits. We
numerically observe that these conditions are only satisfied
for an extremely small set of \( n \)-qubit quantum states. In fact,
by numerically generating random 4- and 5-qubit pure quan-
tum states, using a uniform Haar distribution, we find that the
bound is never violated. The results show that the sharability
of arbitrary bipartite quantum correlations in multisite quan-
tum states is nontrivially limited by the multiparty entangle-
ment content of the states irrespective of the number of par-
ties.

This paper is organized as follows. In Sec. II, we review
the definitions of the monogamy score of a quantum corre-
lation and the generalized geometric measure of an \( n \)-qubit
state. In Sec. III, we consider the multiparty entanglement
bounds on monogamy score in terms of the genuine multi-
party entanglement. In Sec. IV, we analytically show how
the bound is satisfied for a host of \( n \)-qubit \(( n > 3)\) symmetric
and many-body ground states. In Sec. V, we numerically find
that the bound is satisfied for randomly generated four- and
five-qubit quantum states. We conclude in Sec. VI.

II. DEFINITIONS

In this section, we discuss definitions of monogamy of
quantum correlations and genuine multipartite entanglement.
For an \( n \)-qubit pure state, \(|\Psi\rangle\), the monogamy inequality \cite{6}
of a bipartite quantum correlation measure, \( Q \), with respect to
a nodal qubit (say \( j \)), can be written as

\[
Q(\rho_{j:\text{rest}}) \geq \sum_{k \neq j} Q(\rho_{jk}^{(2)}),
\]

(1)

where \( Q(\rho_{j:\text{rest}}) \) is the bipartite quantum correlation between
the nodal qubit and the rest of the qubits taken as a single
party, and \( Q(\rho_{jk}^{(2)}) \) is the bipartite quantum correlation mea-
sure between the nodal qubit \( j \) and the qubit \( k \), obtained
from the two-qubit reduced density matrix \( \rho_{jk}^{(2)} \). A multi-
party state which satisfies the monogamy relation is said
to be monogamous, and otherwise it is non-monogamous.
Quantum correlation measures that satisfy the above inequal-
ity for all multipartite states are termed monogamy-satisfying
or monogamous. Entanglement monotones such as squared
concurrency, squared entanglement of formation, and squared
negativity are known to be monogamous, whereas en-
taglement of formation and logarithmic negativity, along with
information-theoretic quantum correlation measures such as
quantum discord and quantum work-deficit are, in general,
non-monogamous. The definitions of these measures are pro-
vided in the Appendix.

Rewriting Eq. (1), one can define monogamy score, given
by

\[
\delta_j^Q(|\Psi\rangle) = Q(\rho_{j:\text{rest}}) - \sum_{k \neq j} Q(\rho_{jk}^{(2)}),
\]

(2)

with qubit \( j \) as the node. It is non-negative for all the states
that satisfy Eq. (1), while for non-monogamous states, it pos-
sess negative values. On the other hand, all monogamous
quantum correlation measures have a positive monogamy
score, for all multipartite states. An important aspect in defin-
ning both the monogamy inequality and the score is the role of
the nodal qubit. For \( n = 3 \), the monogamy score with respect
to squared concurrence is independent of the choice of a nodal
qubit \cite{6}. However, for other quantum correlation measures
and for \( n > 3 \), this invariance is lost for general multiqubit
states. A logical step is to consider the monogamy score after
finding the minimal value across all choices of nodal qubits,
\( i.e., \delta_j^Q(|\Psi\rangle) = \min_{j \in \{1, n\}} \{\delta_j^Q(|\Psi\rangle)\} \), where \( j \in \{1, n\} \)
indicates that \( j \) is chosen from among \( \{1, 2, \ldots , n\} \).

The genuine multiparty entanglement of an \( n \)-qubit pure state,
\(|\Psi\rangle\), can be conceptualized by using the generalized ge-
ometric measure (GGM) \cite{45, 46} (cf. \cite{47}). \(|\Psi\rangle\) is said to be
genuinely multipartite entangled if it cannot be expressed as a product
across any bipartition of the state. The Greenberger-
Horne-Zeilinger (GHZ) \cite{48} and the W \cite{49, 50} states are the
quintessential examples of genuinely multipartite entangled
states. The GGM \((\mathcal{G})\) of the state \(|\Psi\rangle\) can be reduced to

\[
\mathcal{G}(|\Psi\rangle) = 1 - \max_{\{|\Phi\rangle\}} \langle \Phi |\Psi\rangle^2,
\]

(3)

where the maximization is performed over the set of states \(\{|\Phi\rangle\}\)
that are not genuinely multipartly entangled. From the
definition, it follows that the quantity \(\mathcal{G}(|\Psi\rangle)\) vanishes for all
states that are biseparable across any partition and is non-zero
otherwise. Further, it is a valid entanglement monotone that is
non-increasing under local operations and classical commu-
nication. The optimization in defining GGM can be simpli-
fied in terms of the maximization of the Schmidt coefficients
across all possible bipartitions, allowing the quantity to be cal-
culated for arbitrary pure states in arbitrary dimensions. In
terms of the Schmidt coefficients, the GGM for \(|\Psi\rangle\) can be
defined as \cite{45, 46}

\[
\mathcal{G}(|\Psi\rangle) = 1 - \max \left\{ \lambda_{A:B}^2 |A \cup B = \{1, 2, \ldots , n\}, A \cap B = \emptyset \right\},
\]

(4)

where, \( \lambda_{A:B} \) is the maximal Schmidt coefficient across
the bipartition \( A : B \) of the state \(|\Psi\rangle\). This allows one to com-
pute \(\mathcal{G}(|\Psi\rangle)\) in terms of the eigenvalues of its different reduced
density matrices.

III. MONOGAMY SCORE AND GENUINE MULTIPLICITY
ENTANGLEMENT

In this section, we connect the monogamy score with gen-
uine multiparty entanglement measure. In particular, we show
that the monogamy score of a quantum correlation measure
for any multiqubit pure state is upper-bounded by the genu-
ine multiparty entanglement of the state, quantified using
the generalized geometric measure \cite{45, 46}. Let us consider
an \( n \)-qubit pure state \(|\Psi\rangle\). The corresponding \( k \)-qubit redu-
ced states are given by \( \rho^{(k)} = Tr_{n-k}(|\Psi\rangle\langle\Psi|) \), where \( n - k \)
parties have been traced out. From the definition of GGM,
we know that \( G = 1 - \max_{k \in [1, n/2]} \left\{ \xi_m (\rho^{(k)}) \right\} \), where \( \{ \xi_m (\rho^{(k)}) \} \) is the set of maximum eigenvalues corresponding to all possible \( k \)-qubit reduced states of the \( k = n - k \) bipartitions, for \( k \) ranging from 1 to \( n/2 \). Let us consider the squares of concurrence (\( C^2 \)) and negativity (\( N^2 \)), which are from the entanglement-separability paradigm, quantum discord (\( D \)) and quantum work-deficit (\( \Delta \)), from the information-theoretic paradigm, as quantum correlation measures. Let us now establish the connection between GGM and monogamy of bipartite measures \( Q \).

**Theorem 1.** For all multiqubit pure states, \( |\Psi\rangle \), the monogamy score, \( \delta_Q (|\Psi\rangle) \), of a quantum correlation measure, \( Q \), based on entanglement-separability (information-theoretic) criteria, is bounded above by a quadratic (entropic) function of the generalized geometric measure, \( G (|\Psi\rangle) \), provided the maximum eigenvalue in obtaining \( G \) emerges from any single-qubit reduced density matrix.

**Proof:** Let the generalized geometric measure, \( G = 1 - a \). Here \( a = \max \{ \xi_m (\rho^{(1)}) \} \) is the maximum eigenvalue corresponding to all possible single-qubit reduced density matrices, \( \rho^{(1)} \) of \( |\Psi\rangle \), obtained from states. The monogamy score for node \( j \) is given by \( \delta^Q_j = Q(\rho_{j:rest}) - \sum_{k \neq j} Q(\rho_{jk}^{(2)}) \). Therefore, one obtains \( \delta^Q_j \leq Q(\rho_{j:rest}) \). Now, for \( C^2 \) and \( N^2 \), the quantity \( Q(\rho_{j:rest}) \) reduces to the von Neumann entropy, \( S(\rho_j^{(1)}) \), for pure states. Therefore, we obtain

\[
\begin{align*}
\delta^Q_j &= z \det(\rho_j^{(1)}) = z(a(1-a)) = z f(a) = z (1-G) G = z f(G), \\
S(\rho_j^{(1)}) &= -a \log_2 a - (1-a) \log_2 (1-a) = h(a) = \log_2 (1-G) - G \log_2 G = h(G),
\end{align*}
\]

where \( z = 4 \) and 1 for \( C^2 \) and \( N^2 \), respectively. \( f(x) = x(1-x) \) and \( h(x) \) is the Shannon entropy of the variable \( x \). Hence, from the above relations, we obtain \( \delta^Q_j \leq F^Q (G) \), where \( F^Q (G) \) is equal to \( z f(G) \) or \( h(G) \) depending on whether the quantum correlation is entanglement-based or information-theoretic. Now the monogamy score, \( \delta^Q \), is defined as the minimum score over all possible nodes. Hence, \( \delta^Q \leq \delta^Q_j \) and thus we obtain an upper-bound on the monogamy score in terms of a quadratic or an entropic function of generalized geometric measure, as given by, \( \delta^Q (|\Psi\rangle) \leq F^Q (G (|\Psi\rangle)) \).

**Note that Theorem 1 holds for all bipartite quantum correlation measures like entanglement of formation [32] and distillable entanglement [51], which reduce to the von Neumann entropy of the local density matrices for pure states.**

The applicability of the bound obtained in Theorem 1 is limited, in the sense that it is only valid for genuinely multipartite entangled pure states for which the maximum Schmidt coefficient contributing to the GGM of the state comes from the \( j \): rest bipartition, where \( j \) is the single qubit. However, as we shall observe in the following analysis, the bound in Theorem 1 holds for a large number of randomly generated multiqubit states. There is only a small fraction of randomly generated states for which the maximal Schmidt coefficient comes from other bipartitions than the bipartition containing the single qubit. Nevertheless, the results of Theorem 1 can be extended to other states with specific restrictions.

**Proposition 1.** For a large majority of \( n \)-qubit pure states \( |\Psi\rangle \), where \( n > 3 \) and where the maximum eigenvalue in calculating the generalized geometric measure \( G (|\Psi\rangle) \) emerges from a reduced density matrix, \( \rho^{(k)} \), with \( k \neq 1 \), the upper-bound of monogamy score, \( \delta^Q (|\Psi\rangle) \), is a quadratic (an entropic) function of \( G \), for a quantum correlation, \( Q \), based on entanglement-separability (information-theoretic) criterias.

**Proof:** Let the generalized geometric measure, \( G = 1 - b \), where \( b = \{ \xi_m (\rho^{(k)}) \} \) is the maximum eigenvalue corresponding to all possible non-single-qubit bipartitions, i.e., \( k \neq 1 \). Further, let \( a = \{ \xi_m (\rho^{(1)}) \} \) be the maximum eigenvalue corresponding to all possible single-qubit reduced density matrices. The premise implies that \( b > a \). Let us define the quantity, \( \beta = b - a > 0 \). Using the notations in Theorem 1, we have

\[
\sum_{k \neq j} Q(\rho_{jk}^{(2)}) = z f(a) \neq z f(G),
\]

and

\[
S(\rho_j^{(1)}) = h(a) \neq h(G).
\]

Thus, \( \delta^Q \leq F^Q (b) - \mathcal{R} (b, \beta) = F^Q (G) - \mathcal{R} (b, \beta) \).

Therefore, \( \delta^Q \leq F^Q (b) - \mathcal{R} (b, \beta) = F^Q (G) - \mathcal{R} (b, \beta) \).

Therefore, we obtain the bound \( \delta^Q \leq F^Q (G) \), provided \( \mathcal{R} (b, \beta) \geq 0 \). Table 1 provides the specific form of the functions \( F^Q (b) \) and \( \mathcal{R} (b, \beta) \) for the quantum correlation measures, \( C^2, N^2, D \) and \( \Delta \).

Let us now consider the case when \( \mathcal{R} (b, \beta) < 0 \). In such instances, we look at the function \( H^Q (|\Psi\rangle) = \sum_{k \neq j} Q(\rho_{jk}^{(2)}) + \mathcal{R} (b, \beta) \), where \( j \) corresponds to the node for which \( \delta^Q_j \) is minimal. The monogamy score can be written as

\[
\delta^Q = F^Q (b) - \mathcal{R} (b, \beta) - \sum_{k \neq j} Q(\rho_{jk}^{(2)}) = F^Q (G) - H^Q (|\Psi\rangle).
\]

Therefore, we once again obtain the bound, \( \delta^Q \leq F^Q (G) \), provided \( H^Q (|\Psi\rangle) \geq 0 \).

Hence, to violate the bound on monogamy score, an \( n \)-qubit pure state \( |\Psi\rangle \), where \( n > 3 \), must simultaneously satisfy the following necessary conditions: \( -\beta < 0 \), \( \mathcal{R} (b, \beta) < 0 \), and \( H^Q (|\Psi\rangle) < 0 \). Analytical and numerical analysis of symmetric and random \( n \)-qubit pure states show that the fraction of states that satisfy the above conditions, and thus may violate the bound on monogamy score, is extremely small.
TABLE I. Expressions of $\mathcal{F}_Q(b)$ and $\mathcal{R}_Q(b, \beta)$ for concurrence-square ($C^2$), negativity-square ($N^2$), quantum work-deficit ($D, \Delta$), respectively. See Appendix for the definitions of the single-qubit reduced density matrices $\rho^{(1)}$. The maximum eigenvalue comes from the well-known $W$ state $\ket{\Psi}$. | State | $-\beta \mathcal{R}_{\mathcal{Q}}^{(N^2)}$ | $\mathcal{R}_{\mathcal{Q}}^{(C^2)}$ | $\mathcal{R}_{\mathcal{Q}}^{(D, \Delta)}$ |
|---|---|---|---|
| $|G^1_n\rangle$ | 99.87 | 99.87 | 99.87 |
| $|G^2_n\rangle$ | 91.65 | 91.65 | 91.65 |
| $|G^3_n\rangle$ | 57.04 | 57.04 | 57.04 |
| $|G^4_n\rangle$ | 97.58 | 97.58 | 97.58 |
| $|G^5_n\rangle$ | 60.77 | 60.77 | 60.77 |
| $|\psi^k_{gen}\rangle$ | 4.97 | 4.97 | 4.97 |
| $|\psi^k_{gen}\rangle$ | 0.26 | 0.26 | 0.26 |

Table II indicates the percentages of states that satisfy each of these conditions for different classes of states. It is evident that a large majority of states satisfy the bound on monogamy score.

**Corollary.** For all three-qubit pure states $\ket{\Psi}$ the monogamy score, $\delta_{\mathcal{Q}}(\ket{\Psi})$, is upper-bounded by a quadratic (an entropic) function of the generalized geometric measure, $\mathcal{G}(\ket{\Psi})$, for a quantum correlation, $\mathcal{Q}$, based on entanglement-separability (information-theoretic) criteria.

**Proof:** For any three-qubit pure state $\ket{\Psi}$, for all bipartitions, the relevant reduced density matrices are the single-qubit reduced density matrices $\{\rho^{(1)}\}$. Hence, the maximum eigenvalue contributing to the generalized geometric measure, $\mathcal{G}(\ket{\Psi})$, always comes from $\{\rho^{(1)}\}$, thus satisfying the premise of Theorem 1.

We refer to the bounds obtained on the monogamy scores as the multipartite entanglement bounds.

**IV. ANALYZING THE BOUNDS FOR SPECIAL MULTIQUBIT STATES**

In this section, we study some important classes of multipartite states for which the multiparty entanglement bound on monogamy score holds. If in the evaluation of GGM, the maximum eigenvalue is obtained from the $1:rest$ bipartition, then monogamy score is always bounded above by the GGM via Theorem 1. However, Proposition 1 holds when a state obeys certain conditions. We consider several paradigmatic states for which we check whether the criteria required for Proposition 1 to hold are satisfied.

**A. Dicke States**

Let us consider an $n$-qubit Dicke state [53] with $r$ excitations, given by the equation

$$|\Psi_r^n\rangle_D = \left(\begin{array}{c} n \\ r \end{array}\right)^{-1} \sum_{(\beta)} P(0)_{(n-r)\otimes |1\rangle\langle r|},$$

where the summation is over all possible permutations $(P)$ of the product state having $r$ qubits in the excited state, $|1\rangle$, and $n-r$ qubits in the ground state, $|0\rangle$. The state $|\Psi_r^n\rangle_D$ is the well-known $W$ state $|\Psi\rangle$. Since, the Dicke state is symmetric, all $k:rest$ bipartitions are equivalent, and the reduced density matrix can be written as

$$\rho_D^{(k)} = \frac{1}{\binom{n}{r}} \sum_{i=0}^{k} \binom{k}{i} \binom{n-k}{r} |\Psi_0^k\rangle D / |\Psi_0^k\rangle D.$$
be written as [43]

\[ C^2(\rho_{ij}^{(2)}) = 4(v - \sqrt{uw})^2, \] (12)

\[ \mathcal{N}^2(\rho_{ij}^{(2)}) = \frac{1}{4}(u + w) - \sqrt{(u-w)^2 + 4\alpha^2}, \] (13)

\[ \mathcal{D}(\rho_{ij}^{(2)}) = S' - S'' + h(l), \] (14)

where \( l = \frac{1}{2} \left( 1 + \sqrt{1 - 4(uw + vw + vw)} \right), S' = -(u + v) \log_2(u + v) - (w + v) \log_2(u + w), S'' = -u \log_2 u - 2v \log_2 2v - w \log_2 w, \) \( u = (n - r)(n - r - 1)/(n^2 - n), \) \( v = r(n - r)/(n^2 - n), \) and \( w = r(r - 1)/(n^2 - n). \)

For the Dickey state with \( r = n/2, \) all these quantities become functions of a single parameter, the size of the state, \( n. \) It can be easily shown that the quantity, \( \mathcal{H}^Q(|\Psi_{n/2}^n\rangle_D) = (n - 1)^2 \mathcal{Q}(\rho_{ij}^{(2)}) + \mathcal{R}^Q \geq 0, \) for the quantum correlation measures \( C^2, \mathcal{N}^2, \) and \( \mathcal{D}. \) Thus, from Proposition 1, the GGM is the upper bound on monogamy scores for these states.

**B. Generalized superposition of GHZ and W states**

Let us consider the permutationally invariant states defined by a superposition of generalized Greenberger-Horne-Zeilinger (GHZ) [48] state and W state [49, 50], given by

\[ |\Psi_{\alpha, \beta, \gamma}^{n}\rangle = \tilde{\alpha}|0\rangle^\otimes n + \tilde{\beta}|1\rangle^\otimes n + \tilde{\gamma}|W^n\rangle, \] (15)

where \( (\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) \in \mathbb{C} \) and \( |\tilde{\beta}| = \sqrt{1 - |\tilde{\alpha}|^2 - |\tilde{\gamma}|^2}. \) \( |W^n\rangle \) is the \( n \)-qubit W state, and for \( \tilde{\gamma} = 0, |\Psi_{\alpha, \beta, \gamma}^{n}\rangle \) is the generalized GHZ state. To obtain the reduced density matrices, one can notice that the state can be rewritten in the form

\[ |\Psi_{\alpha, \beta, \gamma}^{n}\rangle = \tilde{\alpha}|0\rangle^\otimes k|0\rangle^\otimes n-k + \tilde{\beta}\sqrt{\frac{n-k}{n}}|0\rangle^\otimes k|W^n-k\rangle + \tilde{\gamma}\sqrt{\frac{k}{n}}|W^k\rangle|0\rangle^\otimes n-k + \gamma|1\rangle^\otimes k|1\rangle^\otimes n-k. \] (16)

Therefore, the reduced \( k \)-qubit density matrix can be written as

\[ \rho_{\alpha, \beta, \gamma}^{(k)} = \begin{pmatrix} |\tilde{\alpha}|^2 + |\tilde{\beta}|^2 \frac{n-k}{n} & \tilde{\alpha}^* \tilde{\beta} \frac{\sqrt{\frac{k}{n}}}{\sqrt{\frac{n-k}{n}}} & 0 \\ \tilde{\alpha} \tilde{\beta}^* \frac{\sqrt{\frac{k}{n}}}{\sqrt{\frac{n-k}{n}}} & |\tilde{\beta}|^2 \frac{k}{n} & 0 \\ 0 & 0 & |\tilde{\gamma}|^2 \end{pmatrix}. \] (17)

in the orthogonal basis formed by \( |0\rangle^\otimes k, |W^k\rangle, \) and \( |1\rangle^\otimes k. \) By evaluating the eigenvalues of the above matrix, we find that the maximum eigenvalue corresponds to the \( 1: rest \) \( (k = 1) \) bipartition and is given by

\[ a = \frac{1}{2} \left( 1 + \sqrt{1 - 4|\tilde{\alpha}|^2|\tilde{\gamma}|^2 + 4(n-1)/n |\tilde{\beta}|^2|\tilde{\gamma}|^2 + \frac{|\tilde{\beta}|^2}{n}. \right) \]

Hence for \( |\Psi_{\alpha, \beta, \gamma}^{n}\rangle \), the multipartite entanglement bound on monogamy score is satisfied via Theorem 1.

**C. The Majumdar-Ghosh model**

Let us now consider a physical system that is useful in studying quantum phenomena in strongly-correlated quantum spin systems. The Majumdar-Ghosh (MG) model [55] is a one-dimensional, antiferromagnetic frustrated system, with a Hamiltonian given by

\[ H_{MG} = J_1 \sum_{\langle i,j \rangle} \sigma_i^1 \cdot \sigma_j^1 + J_2 \sum_{\langle (i,j) \rangle} \sigma_i^1 \cdot \sigma_j^2, \] (18)

where \( \langle i,j \rangle \) and \( \langle (i,j) \rangle \) refer to the nearest and the next-nearest neighbors interactions respectively. \( \sigma \) are the Pauli spin operators. Here, we assume that the number of spins, \( n \), is even, and the chain is periodic. The MG model is a special case of the more general \( J_1 - J_2 \) model for which the ground state is exactly known for \( J_2 = J_1/2 \) [55]. The \( n \)-qubit ground state is doubly degenerate and frustrated. The ground state space is spanned by

\[ |\psi_{\pm}^{n}\rangle = \frac{1}{2^{n/4}} \sum_{i=1}^{n/2} (|02i, 12i+1\rangle - |12i, 02i+1\rangle). \] (19)

Let us consider the ground state

\[ |\Psi_{MG}^{n}\rangle = |\psi_{+}^{n}\rangle + |\psi_{-}^{n}\rangle. \] (20)

It is known to be genuinely multipartite entangled and is rotationally invariant [46, 56]. For \( n \geq 4 \), the maximum eigenvalue is known to come from the \( 2: rest \) nearest-neighbor bipartition, where the reduced two-qubit density matrix is the rotationally invariant Werner state. The maximum eigenvalue from the \( 1: rest \) bipartition is \( a = 1/2 \). The maximum eigenvalue from the nearest-neighbor \( 2: rest \) bipartition is \( b = (1 + 3p)/4 \), where \( p \) is the Werner parameter, given by

\[ p = \frac{1 + 2\frac{3p-1}{2}}{1 + 2\frac{3p-1}{2}}. \] (21)

Hence for \( n > 4 \), we have \( p > 1/3 \), which implies \( b = \frac{b}{a} = (3p-1)/4 > 0 \). For the reduced two-site density matrix, the exact analytical forms for \( C^2, \mathcal{N}^2, \) and \( \mathcal{D} \) are known in terms of the Werner parameter \( p \), and can be written as

\[ C^2(\rho_{ij}^{(2)}) = \max \left[ 0, \frac{3p-1}{2} \right]^2, \]

\[ \mathcal{N}^2(\rho_{ij}^{(2)}) = \left| \frac{1 - 3p}{4} \right|^2, \]

\[ \mathcal{D}(\rho_{ij}^{(2)}) = \frac{p_-^4}{4} \log_2(p_-^4) - \frac{p_+^4}{2} \log_2(p_+^4) + \frac{p'_+^4}{4} \log_2(p'_+^4), \] (22)

where \( p_\pm = 1 \pm p, p' = 1 + 3p, \) and \( i \) and \( j \) are the nearest-neighbors. To prove that the multipartite entanglement bounds on monogamy scores hold for these quantum correlations for the ground states of the MG model, we need to show that either of the quantities, \( \mathcal{R}^Q \) or \( \mathcal{H}^Q(|\Psi_{MG}^{n}\rangle) \), is positive. Using Eq. (21), we can derive that \( \mathcal{R}^2(\mathcal{N}^2) = -z \left( \frac{1}{4}\frac{3p}{4} \right)^2 < \)

\[ \]
FIG. 1. (Color.) Genuine multiparty entanglement versus quantum monogamy scores for the SLOCC inequivalent classes. The figure exhibits plots of quantum monogamy scores (δΩ), as the abscissae, against the generalized geometric measure (G), as the ordinates. Monogamy scores for squared-concurrence (red dots) and squared-negativity (maroon dots) are shown in the first row (Figs. 1(a)-1(f)), and quantum discord (blue dots) and quantum work-deficit (green dots) are shown in the second row (Figs. 1(g)-1(h)). Each of the six columns represents plots for 2,5 × 10^5 random states, generated through uniform Haar distribution, for the normal-form representatives of the six four-qubit SLOCC inequivalent classes ((G^n) (a) and (g)) through (G^n) (f) and (l)) given in Table III. The multiparty entanglement bounds on the monogamy scores are given by the equation, δΩ(Ψ) = FQ(G(Ψ)), as proposed in Sec. III. Monogamy scores for quantum discord and quantum work-deficit, in the second row, can be negative but are not bounded by the negative of the entropic function, i.e., the mirror image of the equation δΩ = FQ(G), for δΩ > 0, about the δΩ = 0 axis. The abscissae are measured in ebits. The ordinates are measured in ebits for Figs. 1(a)-1(f) and in bits for Figs. 1(g)-1(h).

0, for p > 1/3. Similarly, one can show that R^D(Δ) < 0. Hence, we need to show that the quantity H(Ψ^{n}_{MG}) > 0. For the ground state, |Ψ^{n}_{MG}\rangle, the nearest-neighbor spins are entangled and C^2(ρ^{(2)}_{ij}) = N^2(ρ^{(2)}_{ij}) = 0, for j ≠ i ± 1. D(ρ^{(2)}_{ij}) for non-nearest-neighbor qubits is finite but two orders of magnitude lower than the nearest-neighbor values. Hence, H(Ψ^{n}_{MG}) = R^Q + Q(ρ^{(2)}_{i(i+1)}) + Q(ρ^{(2)}_{i(i-1)}), which for Q = D is an approximation. For C^2 and N^2, H(C^2(N^2)) = δ/10(1 - 3p)^2 > 0. Similarly, for D and Δ, one can show that H(D(Δ)) > 0, for all n. Hence, the monogamy score bound is satisfied via Proposition 1.

D. The Ising model

In this section, we consider two paradigmatic Hamiltonians belonging to the Ising group of models [57] that give us multipartite entangled ground states. We first consider the highly frustrated Ising model with long-range antiferromagnetic interactions, also called the Ising gas model. The Hamiltonian for an n-spin Ising gas is given by

\[ H_{\text{gas}}(x) = \frac{J}{n} (S - nx)^2, \quad J > 0, \quad (23) \]

where \( S = \sum_i \sigma_i^z \), \( J > 0 \), and \( 0 \leq x \leq 1 \). The quenched unnormalized ground state of the Ising gas Hamiltonian is given by [55]

\[ |\Psi^n_{\text{gas}}\rangle = \sum_{\{0,1\}} |0\rangle^{\otimes n/2(1+x)} \otimes |1\rangle^{\otimes n/2(1-x)}, \quad (24) \]

where \( \{0,1\} \) indicates that the summation is over all possible combinations of \( |0\rangle \) and \( |1\rangle \) that satisfy the density \( (1 + x)/(1 - x) \). For maximally frustrated ground states, the density is unity \( (x = 0) \), and the ground state reduces to the Dicke state, given by Eq. (9), for \( r = n/2 \). For these states, as discussed in Sec. IV A, the multiparty entanglement always gives the upper bound on the monogamy of quantum correlation.

We next consider the weakly frustrated, periodic Ising spin chain with nearest-neighbor interactions, also called the Ising ring. All the nearest-neighbor interactions are ferromagnetic, except one that is antiferromagnetic. The Hamiltonian is given by

\[ H_{\text{ring}} = -J \sum_{i=1}^{n-1} \sigma_i^z \sigma_{i+1}^z + J \sigma_n^z \sigma_1^z, \quad J > 0. \quad (25) \]

The quenched ground state of the Hamiltonian is given by [55]

\[ |\Psi^n_{\text{ring}}\rangle = \sum_{k=0}^{n-1} \left( |0\rangle^{\otimes n-k} |1\rangle^{\otimes k} + |0\rangle^{\otimes k} |1\rangle^{\otimes n-k-1} \right). \quad (26) \]

For the ground state given in Eq. (26), the reduced density matrix from the 2 : rest bipartition can be written as

\[ \rho^{(2)}_{\text{ring}} = \frac{1}{2n} \begin{pmatrix} n-1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & n-1 \end{pmatrix}. \quad (27) \]

The maximum eigenvalue for the 2 : rest bipartition is given by \( b = \frac{1}{3}(n + 2 + \sqrt{n^2 + 16}) \), and for the 1 : rest bipartition is given by \( a = (1/2)(1 + 1/n) \). These eigenvalues are highest among all bipartitions. For any finite number of spins \( n, a \geq b \). Therefore the bound on monogamy is satisfied via Theorem 1. Interestingly, for \( n \to \infty \), the maximum eigenvalues, \( a = b = 1/2 \), and maximum GGM is achieved.
V. NUMERICAL RESULTS

The $n$-qubit states, considered in the analytical study of the bound on quantum monogamy in the previous section, constitute some special classes of multiparty state of arbitrary number of qubits. To visualize the multiparty entanglement bound obtained in Theorem 1 and Proposition 1, we now randomly generate four- and five-qubit states. The random states are chosen using Haar uniform distribution.

Fig. 1 depicts the behavior of genuine multiparty entanglement with respect to the quantum monogamy scores for $C^2$ and $N^2$ (Figs. 1(a)-1(f)) and for $D$ and $\Delta$ (Figs. 1(g)-1(l)) for the randomly generated four-qubit states corresponding to the parametrized six SLOCC inequivalent classes ($G^i$, for $i = 1$ to 6). The nine SLOCC inequivalent classes of four-qubit states are discussed in Appendix 1 and their exact forms are given in Table III. Fig. 1 shows that the quantum monogamy scores are bounded by the quadratic and entropic functions of generalized geometric measure for the set of states belonging to the SLOCC inequivalent classes for four-qubits. It is known that quantum discord and quantum work-deficit can have negative monogamy scores for certain states, i.e., the measures are not monogamous [22, 39]. This is observed by the negative regions Figs. 1(g)-1(l).

Fig. 2 shows the bound on monogamy scores for randomly generated symmetric four- and five-qubit states. The symmetric states are generated using a random superposition of Dicke states, with different excitations, as shown in Appendix 2 by using a uniform Haar distribution. The figure shows that the generated symmetric four- and five-qubit states satisfy the multiparty entanglement bound on quantum monogamy in terms of the functions of the generalized geometric measure. Fig. 3 exhibits the bound for randomly generated four- and five-qubit states using a uniform Haar distribution.

From the analytical and numerical results obtained in the preceding and this sections, it is observed that the bound on quantum monogamy scores for the quantum correlation measures $C^2$, $N^2$, $D$, and $\Delta$, in terms of derived functions of the generalized geometric measure, is satisfied for a large majority of multiqubit quantum states.

VI. CONCLUSION

Monogamy is an intrinsic feature of quantum correlation that distinguishes it from classical correlations and plays an important role in applications of quantum information theory such as quantum cryptography and other multiparty communication protocols.

In this work, we find that the monogamy score of any quantum correlation measure, for almost all $n$-qubit quantum states, is bounded above by certain simple functions of the generalized geometric measure, which quantifies the amount of genuine multipartite entanglement present in the system. We find that the bound is a quadratic function for the entanglement-based measures considered and an entropic function for the information-theoretic measures, and is universally satisfied for all three-qubit states. We show that such upper bound holds also for an arbitrary number of qubits provided the states satisfy certain conditions. We derive a set of necessary conditions to characterize the set of states that may violate the bound, and numerically observe that the set is extremely small. Moreover, we analytically investigate several important classes of multiparty quantum states with arbitrary number of parties for which we show that the conditions required to have the upper bound on monogamy scores of computable bipartite measures are satisfied.

The obtained monogamy score bound due to the genuine multiparty entanglement in the system shows a forbidden regime in the distribution of bipartite quantum correlation measures among different parties in a multiparty system and limits the amount by which the monogamy inequality can be
satisfied. The results provide a unifying framework to study monogamy relations in both entanglement and information-theoretic quantum correlations and thus provide an interesting direction for further investigation.

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APPENDIX

1. Four-qubit SLOCC classes

For three-qubit pure states, there exist two inequivalent classes of states under stochastic local operation and classical communication (SLOCC), namely the GHZ and the W class of states [50]. However, for four-qubits, there exist infinitely many inequivalent SLOCC classes of states [58]. A useful classification into nine classes for four-qubit was obtained in [52]. It was observed that up to permutation of the qubits, any four-qubit pure state can be transformed into one of the nine classes of states \{52\}, as shown in Table III.

2. Symmetric States

A general symmetric state can be written as a linear combination of the Dicke states as

$$|\Psi_S\rangle = \sum_{r=0}^{n} a_r |\Psi^n_r\rangle_D,$$

where $|\Psi^n_r\rangle_D$ is an $n$-qubit Dicke state [53] with $r$ excitations, given in Eq. (9). The normalization condition is satisfied by demanding $\sum_{r=0}^{n} |a_r|^2 = 1$. Any general symmetric state can be generated by randomly choosing a set of coefficients $a_r$ that satisfy the normalization.

3. Quantum correlation measures

Concurrence: Concurrence [30] is a useful measure of entanglement for general two-qubit states. The concurrence of any two-qubit state, $\rho_{AB}$, is given by,

$$C(\rho_{AB}) = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\},$$

where $\lambda_i$’s are the square roots of the eigenvalues of $\rho_{AB}$. The concurrence for arbitrary two-qubit states can be used to derive a closed form of the entanglement of formation [32], as shown in [30].

Negativity: Another important and computable measure of entanglement is negativity [33]. $N(\rho_{AB})$, of a bipartite state $\rho_{AB}$ is defined as the sum of the absolute values of the negative eigenvalues of $\rho_{AB}^{T_A}$, which is the partial transpose of $\rho_{AB}$ with respect to subsystem $A$. Mathematically, $N(\rho_{AB})$ can be expressed as

$$N(\rho_{AB}) = \frac{\|\rho_{AB}^{T_A}\|_1 - 1}{2},$$

where $\|\rho_{AB}^{T_A}\|_1$ is the trace-norm of the matrix $\rho_{AB}^{T_A}$. For two-qubit states, zero negativity implies that the state is separable.

Quantum discord: In classical information theory [59], the mutual information between two random variables is given by the following two equivalent expressions:

$$I(A : B) = H(A) + H(B) - H(A, B),$$

$$J(A : B) = H(B) - H(B|A),$$

where $H(\cdot)$ is the Shannon entropy [60]. For quantum systems, using von Neumann entropy [61] instead of Shannon entropy, one obtains, for a bipartite state $\rho_{AB}$, the expressions [35, 36, 62]

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}),$$

$$J(\rho_{AB}) = S(\rho_B) - S(\rho_B|A),$$

where the conditional entropy, $S(\rho_{B|A}) = \min_{\{A_i\}} \sum_{i} p_i S(\rho_{B|i})$, for the state $\rho_{AB}$, with $\rho_{B|i} = \frac{1}{p_i} Tr_A(A_i \otimes \mathbb{I}_B) \rho(A_i \otimes \mathbb{I}_B)$, $\mathbb{I}$ being the identity operator on the Hilbert space of $B$, and $\{A_i\}$ forms a rank-one projection measurement on the system held by $A$. The difference between $I(\rho_{AB})$ and $J(\rho_{AB})$, for a bipartite state $\rho_{AB}$, gives us a measure of quantum correlation of $\rho_{AB}$. Quantum discord is defined as [35, 36]

$$D(\rho_{AB}) = I(\rho_{AB}) - J(\rho_{AB}).$$

Quantum work-deficit: Another information-theoretic measure of quantum correlation is quantum work-deficit [37], which is defined, for a bipartite quantum state $\rho_{AB}$, as the difference between the quantity of pure states that can be extracted under allowed “closed global operations” (CGO) and pure product states that can be extracted under “closed local operations and classical operations” (CLOCC).

For a given state $\rho_{AB}$, the class of CGO are any allowed sequences of unitary operations and dephasing using a set of projectors $\{\Pi_i\}$, i.e., $\rho \rightarrow \sum_i \Pi_i \rho \Pi_i$, where $\Pi_i \Pi_j = \delta_{ij} \Pi_i$, $\sum_i \Pi_i = \mathbb{I}$. The number of pure qubits that can be extracted from $\rho_{AB}$ by CGO is

$$I_G(\rho_{AB}) = N - S(\rho_{AB}),$$
TABLE III. Normal-form representatives of the nine four-qubit SLOCC inequivalent classes defined in [52] (see Appendix 1). Here \(a, b, c, d\) are complex parameters with non-negative real parts. The first six classes are parameterized.

\[
\begin{align*}
|G^1_{abcd}\rangle &= \frac{a + b}{\sqrt{2}}(0000) + \frac{c}{\sqrt{2}}(1111) + \frac{d}{\sqrt{2}}(0011) + |1100\rangle,

|G^2_{abc}\rangle &= \frac{a + b}{\sqrt{2}}(0000) + \frac{c}{\sqrt{2}}(1111) + \frac{d}{\sqrt{2}}(0011) + \frac{e}{\sqrt{2}}(1100) + |0101\rangle,

|G^3_{ab}\rangle &= a(0000) + b(0011) + |1010\rangle + |0110\rangle + |0111\rangle,

|G^4_{a}\rangle &= a(0000) + b(0011) + c(0101) + |0110\rangle + |0111\rangle + |1001\rangle,

|G^5_{ab}\rangle &= a(0000) + b(0011) + i(0001) - |1111\rangle + |0110\rangle,

|G^6_{ab}\rangle &= a(0000) + b(0011) + |0110\rangle + |0111\rangle + |1110\rangle,

|G^7\rangle &= |0000\rangle + |0101\rangle + |0110\rangle + |1110\rangle,

|G^8\rangle &= |0000\rangle + |0111\rangle + |1110\rangle + |1111\rangle,

|G^9\rangle &= |0000\rangle + |1111\rangle.
\end{align*}
\]

where \(N = \log_2(\dim \mathcal{H})\). The CLOCC class consists of local unitary, local dephasing, and exchange of dephased states between \(A\) and \(B\). The amount of qubits that can be extracted under CLOCC is given by

\[
I_L(\rho_{AB}) = N - \inf_{\Lambda \in \text{CLOCC}} \{S(\rho'_{AB})\},
\]

where \(\rho'_{AB} = \sum_i p_i (A_i \otimes I_B) \rho_{AB} (A_i \otimes I_B)\) if one restricts to one-way CLOCC. Quantum work-deficit is then defined as

\[
\Delta(\rho_{AB}) = I_G(\rho_{AB}) - I_L(\rho_{AB}).
\]

For such instances, the work-deficit is equal to quantum discord for bipartite states with maximally mixed marginals.

\[\text{References}\]

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).
[2] K. Modi, A. Brodutch, H. Cable, T. Patrek, and V. Vedral, Rev. Mod. Phys. 84, 1655 (2012).
[3] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[4] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[5] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996).
[6] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A 61, 052306 (2000).
[7] B. M. Terhal, IBM J. Res. Dev. 48, 71 (2004).
[8] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. 96, 220503 (2006).
[9] A. Biswas, A. Sen(De), and U. Sen, Phys. Rev. A 89, 032331 (2014).
[10] B. Toner, Proc. R. Soc. A 465, 59 (2009); B. Toner and F. Verstraete, arXiv:quant-ph/0611001.
[11] R. Ramanathan, A. Soeda, P. Kurzynski, and D. Kaszlikowski, Phys. Rev. Lett. 109, 050404 (2012); P. Kurzynski, A. Cabello, and D. Kaszlikowski, Phys. Rev. Lett. 112, 100401 (2014).
[12] M. D. Reid, Phys. Rev. A 88, 062108 (2013); A. Milne, S.Jeffvic, D. Jennings, H. Wiseman, and T. Rudolph, New J. Phys. 16, 083017 (2014).
[13] R. Prabhu, A. K. Pati, A. Sen(De), and U. Sen, Phys. Rev. A 87, 052319 (2013).
[14] Y-K. Bai, Y-F. Xu, and Z. D. Wang, Phys. Rev. Lett. 113, 100503 (2014); T. R. de Oliveira, M. F. Cornelio, and F. F. Fan-chini, Phys. Rev. A 89, 034303 (2014).
[15] Y.-C. Ou and H. Fan, Phys. Rev. A 75, 062308 (2007).
[16] A. Kumar, arXiv:1409.8632 [quant-ph].
[17] M. Koashi and A. Winter, Phys. Rev. A 69, 022309 (2004).
[18] M. Christandl and A. Winter, J. Math. Phys. (N.Y.) 45, 829 (2004).
[19] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A 73, 032345 (2006); T. Hiroshima, G. Adesso, and F. Illuminati, Phys. Rev. Lett. 98, 050503 (2007); M. Seevinck, Phys. Rev. A 76, 012106 (2007); S. Lee and J. Park, Phys. Rev. A 79, 054309 (2009); A. Kay, D. Kaszlikowski, and R. Ramanathan, Phys. Rev. Lett. 103, 050501 (2009); M. Hayashi and L. Chen, Phys. Rev. A 84, 012325 (2011), and references therein.
[20] M. Pawlowski, Phys. Rev. A 82, 032313 (2010); J. Barrett, L. Hardy and A. Kent, Phys. Rev. Lett. 95, 010503 (2005).
[21] M. Horodecki and M. Piani, J. Phys. A 45, 105306 (2012); R. Nepal, R. Prabhu, A. Sen(De), and U. Sen, Phys. Rev. A 87, 032336 (2013).
[22] R. Prabhu, A. K. Pati, A. Sen(De), and U. Sen, Phys. Rev. A 85, 040102(R) (2012).
[23] M. N. Bera, R. Prabhu, A. Sen(De), and U. Sen, Phys. Rev. A 88, 032301 (2013).
S. Hill and W. K. Wootters, Phys. Rev. Lett. 107, 060405 (2011); K. K. Rao, H. Katiyar, T. S. Mahesh, A. Sen(De), U. Sen, and A. Kumar, Phys. Rev. A 88, 022312 (2013). L. Suskind, arXiv:1301.4505 [quant-ph]; S. Lloyd and J. Preskill, J. High Energy Phys. 8, 126 (2014).

C. Eltschka, A. Osterloh, and J. Siewert, Phys. Rev. A 80, 032313 (2009); M. F. Cornelio, Phys. Rev. A 87, 032330 (2013).

B. Regula, S. D. Martino, S. Lee, and G. Adesso, Phys. Rev. Lett. 113, 110501 (2014).

J. S. Kim, Phys. Rev. A 90, 062306 (2014).

Y.-K. Bai, Y.-F. Xu, and Z. D. Wang, Phys. Rev. A 90, 062343 (2014).

S. Hill and W. K. Wootters, Phys. Rev. Lett. 78, 5022 (1997); W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).

F. F. Fanchini, M. C. de Oliveira, L. K. Castelano, and M. F. Cornelio, Phys. Rev. A, 87, 032317 (2013).

C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A, 54, 3824 (1996).

G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002).

J. Lee, M. S. Kim, Y. J. Park, and S. Lee, J. Mod. Opt. 47, 2151 (2000); M. B. Plenio, Phys. Rev. Lett. 95, 090503 (2005).

L. Henderson and V. Vedral, J. Phys. A 34, 6899 (2001).

H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2002).

J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 89, 180402 (2002); M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), and U. Sen, ibid. 90, 100402 (2003); M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen(De), U. Sen, and B. Synak-Radtke, Phys. Rev. A 71, 062307 (2005).

J. Devetak, Phys. Rev. A 71, 062303 (2005).

G. L. Giorgi, Phys. Rev. A 84, 054301 (2011).

F. F. Fanchini, M. F. Cornelio, M. C. de Oliveira, and A. O. Caldeira, Phys. Rev. A 84, 012313 (2011).

A. Streltsov, G. Adesso, M. Piani, and D. Bruß, Phys. Rev. Lett. 109, 050503 (2012).

K. Salini, R. Prabhu, A. Sen(De), and U. Sen, Ann. Phys. 348, 297 (2014).

A. Kumar, R. Prabhu, A. Sen(De), and U. Sen, Phys. Rev. A 91, 012341 (2015).

R. Prabhu, A. K. Pati, A. Sen (De), and U. Sen, Phys. Rev. A 86, 052337 (2012).

A. Sen (De) and U. Sen, Phys. Rev. A 81, 012308 (2010); A. Sen (De) and U. Sen, arXiv:1002.1253 [quant-ph].