On general closure operators and quasi factorization structures

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Abstract. In this article the notions of quasi mono (epi) as a generalization of mono (epi), (quasi weakly hereditary) general closure operator $C$ on a category $\mathcal{X}$ with respect to a class $\mathcal{M}$ of morphisms, and quasi factorization structures in a category $\mathcal{X}$ are introduced. It is shown that under certain conditions, if $(\mathcal{E}, \mathcal{M})$ is a quasi factorization structure in $\mathcal{X}$, then $\mathcal{X}$ has a quasi right $\mathcal{M}$-factorization structure and a quasi left $\mathcal{E}$-factorization structure. It is also shown that for a quasi weakly hereditary and quasi idempotent QCD-closure operator with respect to a certain class $\mathcal{M}$, every quasi factorization structure $(\mathcal{E}, \mathcal{M})$ yields a quasi factorization structure relative to the given closure operator; and that for a closure operator with respect to a certain class $\mathcal{M}$, if the pair of classes of quasi dense and quasi closed morphisms forms a quasi factorization structure, then the closure operator is both quasi weakly hereditary and quasi idempotent. Several illustrative examples are provided.

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1 Introduction and preliminaries

Closure operators have been around for almost one century in the context of categories of topological spaces and lattices. In [27], Salbany introduces a particular closure operator in the category of topological spaces. This idea was later transformed to an arbitrary category, which led to the general concept of categorical closure operators, see [8–10]. Weakly hereditary and idempotent closure operators play an important role, as they arise from factorization structures. In [22], quasi right factorization structures were introduced and their connection with closure operators was investigated, while quasi left factorization structures appear in [19].

There are many important structures that are not factorization structures nor even weak factorization structures; however they are quasi factorization structures, as introduced in this article. Many examples of such structures are provided and the connections between quasi right factorization structures, quasi left factorization structures, quasi factorization structures, and closure operators are investigated.

In Section 2, to develop some theory related to closure operators in the more general context of a quasi right factorization structure $\mathcal{M}$ on a category $\mathcal{X}$, the notions of quasi mono and quasi epi are given. Then we will study some preliminary results and we will provide some examples of these notions. The strong point of these examples is to provide epimorphisms which are quasi mono and monomorphisms which are quasi epi. In Section 3, the definition of a general closure operator on a category $\mathcal{X}$ with respect to the class $\mathcal{M}$ of morphisms is introduced, some related results and several examples are also given. In Section 4, after defining quasi weakly hereditary closure operator, we prove that for a quasi idempotent closure operator we have a quasi right factorization structure and for a quasi weakly hereditary closure operator under some conditions we have a quasi left factorization structure. In Section 5, for morphism classes $\mathcal{E}$ and $\mathcal{M}$, the notion of $(\mathcal{E}, \mathcal{M})$-quasi factorization structure is introduced and examples of quasi factorization structures which are not weak factorization structures are furnished. It is shown that if $(\mathcal{E}, \mathcal{M})$ is a quasi factorization structure in $\mathcal{X}$, then $\mathcal{X}$ has quasi right $\mathcal{M}$-factorization structure provided that $\mathcal{M}$ has $\mathcal{X}$-pullbacks and it has quasi left $\mathcal{E}$-factorization structure provided that $\mathcal{M} \subseteq \text{Mon}(\mathcal{X})$, the class of monos, and $\mathcal{E}$ has $\mathcal{X}$-pushouts. It is also shown that for a quasi weakly hereditary and quasi idempotent QCD-closure oper-
ator with respect to a class $\mathcal{M}$ that is contained in the class of quasi monos and is closed under composition, every quasi factorization structure $(\mathcal{E}, \mathcal{M})$ yields a quasi factorization structure relative to the given closure operator. Finally it is proved that for a closure operator with respect to a class $\mathcal{M}$ that is contained in the class of strongly quasi monos and is a codomain, if the pair of classes of quasi dense and quasi closed morphisms forms a quasi factorization structure, then the closure operator is both quasi weakly hereditary and quasi idempotent.

To this end we will give some basic definitions and results which will be used in the following sections.

**Definition 1.1.** [22] Let $\mathcal{M}$ be a class of morphisms in $\mathcal{X}$ and for every object $X$ of $\mathcal{X}$, $\mathcal{M}/X$ be the class of all morphisms with codomain $X$. We say that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations or $\mathcal{M}$ is a quasi right factorization structure in $\mathcal{X}$, whenever for every morphisms $Y \xrightarrow{f} X$ in $\mathcal{X}$, there exists $M \xrightarrow{m_f} X \in \mathcal{M}/X$ such that

(a) $f = m_f g$ for some $g$;

(b) if there exists $m \in \mathcal{M}/X$ such that $f = mg$ for some $g$, then $m_f = mh$ for some $h$. $m_f$ is called a quasi right part of $f$.

With $\langle m \rangle = \{ mh : h \in \mathcal{X} \text{ and } mh \text{ is defined} \}$ denoting the sieve generated by $m$, see [21], (a) is equivalent to

(a') $\langle f \rangle \subseteq \langle m_f \rangle$;

and (b) is equivalent to:

(b') if there exists $m \in \mathcal{M}/X$ such that $\langle f \rangle \subseteq \langle m \rangle$, then $\langle m_f \rangle \subseteq \langle m \rangle$.

Note that right $\mathcal{M}$-factorizations, as defined in [10], are quasi right $\mathcal{M}$-factorizations.

**Lemma 1.2.** [22] Suppose that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations. Let $f$ be a morphism in $\mathcal{X}$ and $m_f$ be a quasi right part of $f$.

(a) If $f \in \mathcal{M}$, then $\langle m_f \rangle = \langle f \rangle$.

(b) $m$ is a quasi right part of $f$ if and only if $m \in \mathcal{M}$ and $\langle m \rangle = \langle m_f \rangle$.

(c) If $\langle f \rangle \subseteq \langle g \rangle$, then $\langle m_f \rangle \subseteq \langle m_g \rangle$.

(d) If $\langle g \rangle = \langle f \rangle$, then $m_f$ is a quasi right part of $g$.

The class of all isomorphisms in $\mathcal{X}$ is denoted by $\text{Iso}(\mathcal{X})$. 
Proposition 1.3. Suppose $\mathcal{M}$ is closed under composition with isomorphisms on the left, that is, $m \in \mathcal{M}$, $\alpha \in \text{Iso}(\mathcal{X})$, and $\alpha m$ defined, yields that $\alpha m \in \mathcal{M}$. If $f$ is a morphism in $\mathcal{X}$ and $m_f$ is a quasi right part of $f$, then $\alpha m_f$ is a quasi right part of $\alpha f$.

Proof. This follows directly from the definition. 

Notation 1.4. For each composite $Z \xrightarrow{g} X \xrightarrow{f} Y$ in $\mathcal{X}$ we will denote by $f(g) : f(Z) \to Y$ a chosen quasi right part of a quasi right $\mathcal{M}$-factorization of $fg$. Note that if $m'$ is another quasi right part of $fg$, by Lemma 1.2(b), we have $\langle f(g) \rangle = \langle m' \rangle$.

Remark 1.5. Suppose that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations.

(a) For each $M \xrightarrow{m} X \in \mathcal{M}$ we have $\langle m(1_M) \rangle = \langle m \rangle$.

(b) For each $X \xrightarrow{f} Y$, $T \xrightarrow{g} Y$ and $Y \xrightarrow{h} Z$ in $\mathcal{X}$ if $\langle f \rangle \subseteq \langle g \rangle$, then $\langle h(f) \rangle \subseteq \langle h(g) \rangle$.

Proposition 1.6. Suppose that $\mathcal{M}$ is closed under composition with isomorphisms on the left. For each morphism $X \xrightarrow{f} Y$ and isomorphism $Y \xrightarrow{\alpha} Z$ in $\mathcal{X}$ we have $\langle \alpha(f) \rangle = \langle \alpha(f(1_X)) \rangle$.

Proof. Obvious. 

The notion of a cosieve is dual to that of a sieve. A principal cosieve generated by $f$ is denoted by $\langle f \rangle$. Also the notion of a quasi left $\mathcal{E}$-factorization is dual of quasi right $\mathcal{M}$-factorization, see [19].

2 Quasi mono and quasi epi

In this section, quasi monos and quasi epis as a generalization of monos and epis will be defined and some of their properties will be studied. Then we will provide some examples of these notions. The significant point of these examples is to provide epimorphisms which are quasi mono and monomorphisms which are quasi epi. Since these notions, especially “quasi mono”, are used in the study of some kinds of general closure operators, some examples of quasi right $\mathcal{M}$-factorization structures are given, in which the class $\mathcal{M}$ is contained in the class of quasi monos.
Definition 2.1. (a) A morphism $f$ in $\mathcal{X}$ is called \emph{quasi mono}, whenever for each morphism $a, b \in \mathcal{X}$ if $fa = fb$, then $\langle a \rangle = \langle b \rangle$. The class of all quasi monos in $\mathcal{X}$ is denoted by $QM(\mathcal{X})$.

(b) A morphism $f$ is called \emph{quasi epi}, whenever for each morphism $a, b \in \mathcal{X}$ if $af = bf$, then $\langle a \rangle = \langle b \rangle$. The class of all quasi epis in $\mathcal{X}$ is denoted by $QE(\mathcal{X})$.

Proposition 2.2. We have the following:

(a) $f$ is a quasi mono if and only if for all morphisms $u$ and $v$ with the same codomain if $\langle fu \rangle \subseteq \langle fv \rangle$, then $\langle u \rangle \subseteq \langle v \rangle$.

(b) If $gf$ is a quasi mono, then $f$ is a quasi mono.

(c) If $f$ and $g$ are quasi monos and $gf$ is defined, then $gf$ is a quasi mono.

(d) If $A \xrightarrow{f} C$ is a quasi mono, and $B \xrightarrow{g} C$ is a mono and the diagram

\[
\begin{array}{ccc}
A \times_C B & \xrightarrow{\pi_2} & B \\
\pi_1 \downarrow & & \downarrow p.b \\
A \xrightarrow{f} C
\end{array}
\]

is a pullback in $\mathcal{X}$, then $\pi_2$ is a quasi mono.

Proof. The proof is straightforward. \qed

Proposition 2.3. We have the following:

(a) If $gf$ is a quasi epi, so is $g$.

(b) If $f$ is an epi and $g$ is a quasi epi and $gf$ is defined, then $gf$ is a quasi epi.

Proof. The proof is straightforward. \qed

Lemma 2.4. We have the following:

(a) If $f$ is a quasi mono and a split epi, then $f$ is an isomorphism.

(b) If $f$ is a quasi epi and a split mono, then $f$ is an isomorphism.

Proof. (a) Since $X \xrightarrow{f} Y$ is a split epi, we have $Y \xrightarrow{s} X$ such that $fs = 1_Y$. Thus $fsf = f$, and so $\langle sf \rangle = \langle 1_X \rangle$, because $f$ is a quasi mono.
Therefore there exists a morphism $X \xrightarrow{t} X$ such that $sft = 1_X$. Hence $s$, and thus $f$, is an isomorphism.

(b) Similar to (a).

**Corollary 2.5.** We have the following:

(a) If the class of quasi monos in $\mathcal{X}$ is pullback stable, then quasi monos are monos.

(b) If the class of quasi epis is pushout stable, then quasi epis are epis.

**Proof.** (a) Suppose that a quasi mono $X \xrightarrow{f} Y$ is given. Consider the pullback diagram

There exists a unique morphism $g : X \longrightarrow X \times_Y X$ in $\mathcal{X}$ such that $\pi_1g = 1_X$, $\pi_2g = 1_X$. Thus $\pi_2$ and $\pi_1$ are quasi monos and split epis, and so, by Lemma 2.4(a), they are isomorphisms. Therefore $f$ is a mono.

(b) Similar to (a).

**Remark 2.6.** In the categories, **Set** of sets, **Top**, of topological spaces and **R-Mod**, of left $R$-modules, quasi monos (quasi epis) are monos (epis).

To give examples of quasi monos which are not monos, we need the following definitions.

Recall that, [4, p.72], a submodule $K$ of $M$ is

(i) superfluous in $M$, abbreviated $K \ll M$, in case for every submodule $L \leq M$, $K + L = M$ implies $L = M$.

(ii) is essential in $M$, abbreviated $K \lhd M$, in case for every submodule $L \leq M$, $K \cap L = 0$ implies $L = 0$. It is easy to see that $K$ is essential in $M$ if and only if, for every nonzero $m \in M$, there is $r \in R$ with $rm \in K$ and $rm \neq 0$. 


For a module $M$ the socle of $M (= \text{Soc}(M))$ is defined as the sum of all simple (minimal) submodules of $M$, (see [30, p.174]).

An $R$-module $M$ is called:

(i) uniform if every non-zero submodule of $M$ is essential in $M$ (see [30, 19.9]).

(ii) quasi-projective in case $M$ is $M$-projective, (see [15]).

(iii) cocyclic if there is an $m_0 \in M$ with the property: every morphism $h : M \rightarrow N$ with $m_0 \notin K_h$ is monic, where $K_h$ is the kernel of $h$, (see [30, p.115]).

Let $R$ be a domain and $M$ be an $R$-module. Then $M$ is called torsion-free, whenever there are no nonzero $x \in M$ and $r \in R$ with $rx = 0$. Note that projective modules are torsion free (see [7, Proposition 1.1]).

**Example 2.7.** Let $\mathbb{Q}P$ be the full subcategory of $R$-$\text{Mod}$ whose objects are quasi-projective modules and let $R$ be a domain.

(a) Let $P$ be a uniform projective module. By [30, 21.1], $\text{Soc}(P)$ is the intersection of essential submodules of $P$ and since $P$ is uniform, every non-zero submodule of $P$ contains $\text{Soc}(P)$ (and hence it is minimum proper submodule of $P$). Consider the following commutative diagram

$$
\begin{array}{ccc}
P & \xrightarrow{p} & P/\text{Soc}(P) \\
\downarrow{f} & & \downarrow{q} \\
R^{(\Lambda)} & & \\
\end{array}
$$

where $P/\text{Soc}(P)$ is homomorphic image of a free $R$-module $R^{(\Lambda)}$ for some set $\Lambda$, $p$ is the canonical projection and $qf = p$. Therefore $f$ is a non-zero map. We show that $f$ is quasi mono in $\mathbb{Q}P$. To this end let the diagram,

$$
\begin{array}{ccc}
N & \xrightarrow{g} & P \\
\downarrow{h} & & \downarrow{f} \\
 & & R^{(\Lambda)}
\end{array}
$$

in $\mathbb{Q}P$ be given such that $fg = fh$. So we have,

$$K_f + \text{Im} g = K_f + \text{Im} h \quad (2.1)$$

If $\text{Im} h = 0$, then $\text{Im} g = 0$. Because, otherwise, if $\text{Im} g \neq 0$, then $\text{Im} g \subseteq P$. Since $f \neq 0$ there exists $0 \neq p_0 \in P$ such that $f(p_0) \neq 0$. This implies that
there exists $r \in R$ such that $0 \neq rp_0 \in \text{Img}$. Thus $rf(p_0) = f(g(t)) = f(h(t)) = 0$. This is a contradiction, because $R^{(\Lambda)}$ is torsion-free. So $\text{Img} = 0$. Thus we have two cases:

(i) $\text{Img} = \text{Im} h = 0$, and so $\langle g \rangle = \langle h \rangle = \{0\}$;

(ii) $\text{Img} \neq 0$ and $\text{Im} h \neq 0$. Thus equality (1) implies that $\text{Img} = \text{Im} h$.

Since $N$ is quasi-projective, there exist morphisms $\alpha, \beta : N \longrightarrow N$ such that the diagram

\[
\begin{array}{c}
N \\
\downarrow \alpha \\
\downarrow \beta \\
\text{Img} = \text{Im} h
\end{array}
\]

commutes, where for each $x \in N$, $\bar{g}(x) = g(x)$ and $\bar{h}(x) = h(x)$. Therefore $\langle g \rangle = \langle h \rangle$ and hence $p$ is quasi mono.

(b) Let $P$ be a cocyclic projective module and $L$ be the intersection of all non-zero submodules of $P$, hence, by [30, 14.8(c)], $L$ is a non-zero minimum submodule of $P$. Since every non-zero projective module contains a maximal submodule (see [4, Proposition 17.14]), $L$ is a proper submodule of $P$. Consider the following commutative diagram

\[
\begin{array}{c}
P \\
\downarrow f \\
\downarrow q \\
\approx R^{(\Lambda)}
\end{array}
\]

where $P/L$ is homomorphic image of a free $R$-module $R^{(\Lambda)}$ for some set $\Lambda$, $p$ is the canonical projection and $qf = p$. As in (a), we can see that $f$ is a quasi mono in $QP$.

Recall that, [30, p.348], for a submodule $U$ of a left $R$-module $M$, a submodule $V \leq M$ is called a supplement or addition complement of $U$ in $M$ if $V$ is a minimal element in the set of submodules $L \leq M$ with $U + L = M$. A module is called supplemented if every submodule has a supplement (see [30, p.349]).

A projective left (right) module over a ring $R$ will be called left hereditary in case every left sub-module is projective (see [16]).
Example 2.8. Let $M$ be a supplemented left hereditary module and $0 \neq K \ll M$. Let $p : M \rightarrow M/K$ be the canonical projection. We show that $p$ is quasi mono in $R$-Mod. To this end, let the diagram

$$
\begin{array}{ccc}
N & \xrightarrow{g} & M \\
\xrightarrow{h} & & \xrightarrow{p} M/K
\end{array}
$$

with $pg = ph$ be given. Thus $K + \text{Img} = K + \text{Im}h$. Since $K + \text{Img}$ has a supplement in $M$, by [30, 41.1(5)] we have $K = K \cap (K + \text{Img}) \ll (K + \text{Img})$. Therefore $\text{Im}h = K + \text{Img}$ and hence $\text{Img} \subseteq \text{Im}h$. Similarly $\text{Im}h \subseteq \text{Img}$, so $\text{Img} = \text{Im}h$. Since $M$ is hereditary, $\text{Im}g$ is a projective submodule of $M$. So there exist $\iota_1 : \text{Img} \rightarrow N$ and $\iota_2 : \text{Im}h \rightarrow N$ such that $g\iota_1 = 1_{\text{Img}}$ and $h\iota_2 = 1_{\text{Im}h}$. Thus $g = h(\iota_2 g)$ and $h = g(\iota_1 h)$, hence $\langle g \rangle = \langle h \rangle$. Therefore $p$ is quasi mono.

Example 2.9. Let $\mathcal{QP}$ be the category of quasi-projective left $R$-modules ($R$ need not be a domain). Define the subcategory $\mathcal{C}$ of $\mathcal{QP}$ to have $\text{Obj}(\mathcal{QP})$ as objects and for all $P, Q \in \mathcal{C}$ a morphism $f : P \rightarrow Q$ in $\mathcal{C}$ is a homomorphism in $R$-$\text{Mod}$ satisfying the condition

$$\forall K \leq P, f(K) \ll Q \Leftrightarrow K \ll P. \quad \text{(C1)}$$

Every isomorphism in $\mathcal{QP}$ satisfies in the condition (C1). Note that for each morphism $f : P \rightarrow Q$ in $\mathcal{C}$

(i) if $\text{Im}f \ll Q$, then $P \ll P$ and hence $P = \{0\}$;

(ii) $Kf \ll P$.

Let $p : P \rightarrow Q$ be a non-zero morphism in $\mathcal{C}$ satisfying

$$\forall A \leq P, K_p \preceq A \Rightarrow K_p \ll A. \quad \text{(C2)}$$

Thus for each morphism $g : L \rightarrow P$, $pg = 0$ implies that $L = \{0\}$. We show that $p$ is quasi mono. To this end, let the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{g} & P \\
\xrightarrow{h} & & \xrightarrow{p} Q
\end{array}
$$

in $\mathcal{C}$ such that $pg = ph$ and $L \neq 0$ be given. So we have

$$K_p + \text{Img} = K_p + \text{Im}h. \quad \text{(2.4)}$$
Since \( \text{Im} g \not\leq K_p \) and \( \text{Im} h \not\leq K_p \), \( K_p \leq K_p + \text{Im} g \) and hence, by (C2), \( K_p \ll K_p + \text{Im} g \). Thus equality (4) implies that \( \text{Im} h = K_p + \text{Im} g \) and so \( \text{Im} g \leq \text{Im} h \). Similarly \( \text{Im} h \leq \text{Im} g \). Therefore \( \text{Im} g = \text{Im} h \). Since \( L \) is quasi-projective, there exist morphisms \( \alpha, \beta : L \rightarrow L \) such that the diagram

\[
\begin{array}{ccc}
L & \xrightarrow{\alpha} & L \\
\downarrow{\bar{g}} & & \downarrow{\bar{h}} \\
\text{Im} g = \text{Im} h & & \\
\end{array}
\]

commutes, where for each \( x \in L \), \( \bar{g}(x) = g(x) \) and \( \bar{h}(x) = h(x) \). It is easy to see that \( \alpha \) and \( \beta \) are morphisms in \( C \) and \( \langle g \rangle = \langle h \rangle \). Therefore \( p \) is quasi mono in \( C \).

Recall that a semiperfect ring is one for which every finitely generated module has a projective cover, (see [25, p.179]). Semiperfect rings \( T \), whose indecomposable, projective left and right modules have simple and essential socles, are called QF-2 rings (see [30, p.557]).

**Example 2.10.** Let \( R \) be a QF-2 ring and \( P \) be a projective cover of a simple module. Since every simple module is trivially quasi-projective and indecomposable, \( P \) is indecomposable, (see [4, Exercises 17, p.203]). Thus \( P \) has a simple and essential socle and hence each non-zero submodule of \( P \) contains \( \text{Soc}(P) \). Furthermore \( \text{Soc}(P) \) is a fully invariant submodule of \( P \), ( [28, p.16]), and so \( P/\text{Soc}(P) \) is quasi-projective, (see [26, Lemma 4.2]). Now consider the canonical projection \( p : P \rightarrow P/\text{Soc}(P) \). Let \( K \leq P \), \( p(K) \ll P/\text{Soc}(P) \) and \( K + L = P \) for some \( L \leq P \). Thus \( p(K) + p(L) = P/\text{Soc}(P) \) and hence \( p(L) = P/\text{Soc}(P) \). Therefore \( p^{-1}p(L) = L \). Let \( x \in p^{-1}p(L) \), so there exists \( l \in L \) such that \( p(x) = p(l) \). Thus \( x - l \in K_p = \text{Soc}(P) \). Since \( \text{Soc}(P) \leq L \), \( x - l \in L \). This implies that \( x \in L \) and so \( p^{-1}p(L) = L \). Therefore \( L = P \) and hence \( K \ll P \). Since \( p \) is an epimorphism and \( K_p \) is simple and essential, it fulfills the conditions (C1) and (C2) of 2.9 and hence \( p \) is quasi mono in \( C \).

**Example 2.11.** (1) (see [6, p.36] and [29, Example 2.18]). Let \( R = \mathbb{Z}_2[x_1, x_2, \ldots] \) where \( x_i^2 = 0 \) for all \( i \), \( x_ix_j = 0 \) for all \( i \neq j \) and \( x_i^2 = x_j^2 = m \neq 0 \) for all \( i \) and \( j \). Then \( R \) is a commutative local ring
and \( R \) has a simple essential socle \( J(R)^2 = \mathbb{Z}_2 m \) as \( R \)-module. In particular, \( R \) is uniform. Note that \( \text{Soc}(R) \subseteq J(R) \). Thus \( R \) has no non-zero semisimple direct summand, hence every simple \( R \)-submodule of \( R \) is superfluous, (see [23, Lemma 2.4]). Therefore \( \text{Soc}(R) \ll R \). As 2.10 the canonical projection \( p : R \rightarrow R/\text{Soc}(R) \) fulfills the conditions (C1) and (C2) of 2.9 and hence \( p \) is quasi mono in \( C \).

(2) In (1), if we replace \( \mathbb{Z}_2 \) by an ordered field, the results are still valid (see [3, Example 20]).

(3) Since for all \( n \in \mathbb{Z} \), \( n\mathbb{Z} \) is a left \( \mathbb{Z} \)-right \( \text{End}(\mathbb{Z}\mathbb{Z}) \) submodule of \( \mathbb{Z}\mathbb{Z} \), \( \mathbb{Z}/n\mathbb{Z} \) is a quasi-projective \( \mathbb{Z} \)-module (see [4, Exercises 16(3), p.203]). Therefore \( \mathbb{Z}_8/\text{Soc}(\mathbb{Z}_8) \) is a quasi-projective \( \mathbb{Z} \)-module and \( \text{Soc}(\mathbb{Z}_8) = \{ \bar{0}, \bar{4} \} \). Thus \( \text{Soc}(\mathbb{Z}_8) \) is a simple and essential \( \mathbb{Z} \)-submodule of \( \mathbb{Z}_8 \). Also, \( \mathbb{Z}_8/\text{Soc}(\mathbb{Z}_8) \cong \mathbb{Z}_4 \) and hence \( \mathbb{Z}_8/\text{Soc}(\mathbb{Z}_8) \) is a quasi-projective \( \mathbb{Z} \)-module. As in 2.10, the canonical projection \( \mathbb{Z}_8 \rightarrow \mathbb{Z}_8/\text{Soc}(\mathbb{Z}_8) \) fulfills the conditions (C1) and (C2) of 2.9 and hence \( p \) is quasi mono in \( C \). Note that the lattice of submodules of \( \mathbb{Z}_8 \) is a totally ordered set by inclusion. This gives other examples of quasi mono.

**Example 2.12.** Let \( X \) and \( X_0 \) be two sets with \( X_0 \subsetneq X \) and \( \mathcal{C} \) be a subcategory of \( \text{Set} \), such that \( X_0 \in \mathcal{C} \) and \( X \notin \mathcal{C} \). Now define the subcategory \( \mathcal{D} \) of \( \text{Set} \) to have \( \text{obj}(\mathcal{C}) \cup \{ X \} \) as objects and for all \( A, B \in \mathcal{D} \), \( \text{Hom}_{\mathcal{D}}(A, B) = \)

\[
\begin{cases}
\text{Hom}_{\mathcal{C}}(A, B) & A, B \in \mathcal{C} \\
\text{Hom}_{\text{Set}}(A, B) & A = X, B \in \mathcal{C} \\
\emptyset & A \in \mathcal{C}, B = X \\
\{ X \xrightarrow{f} X \mid f(X_0) \subseteq X_0, X - X_0 \xrightarrow{f|_{X - X_0}} X - X_0 \} & A = X, B = X
\end{cases}
\]

where \( X - X_0 \xrightarrow{f|_{X - X_0}} X - X_0 \) is the restriction of \( f \) to \( X - X_0 \) and is assumed to be a bijection to \( X - X_0 \). Let \( X_0 \xrightarrow{j} X \) be an inclusion map and so there exists \( X \xrightarrow{r} X_0 \) such that \( rj = 1_{X_0} \). Then \( r \) is a quasi mono in \( \mathcal{D} \). To prove this, suppose the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{g} & & \downarrow{r} \\
X & \xrightarrow{r} & X_0
\end{array}
\]
is given in \( \mathcal{D} \) such that \( rf = rg \). Define \( h : X \rightarrow X \) by
\[
h(x) := \begin{cases} 
  x & x \in X_0 \\
  t & x \in X - X_0 \text{ and } g \upharpoonright (X - X_0)(t) = f \upharpoonright (X - X_0)(x). 
\end{cases}
\]

It is easy to see that \( f = gh \). Thus \( f \in \langle g \rangle \). Similarly \( g \in \langle f \rangle \) and hence \( \langle f \rangle = \langle g \rangle \).

In the following examples we give a quasi right \( \mathcal{M} \)-factorization structure in which each \( m \) in \( \mathcal{M} \) is a quasi mono.

**Example 2.13.** Consider the category \( R\text{-Mod} \), where \( R \) is a semisimple ring. Now define the class \( \mathcal{M} \) as
\[
\mathcal{M} = \{ m \mid m \text{ is mono or superfluous epi} \}.
\]
Then \( R\text{-Mod} \) has quasi right \( \mathcal{M} \)-factorizations.

Now let \( M \) be a supplemented module and \( 0 \neq K \ll M \). Let \( p : M \rightarrow M/K \) be the canonical projection. Consider the commutative diagram
\[
\begin{array}{ccc}
R(\Lambda) & \xrightarrow{q} & M/K \\
\downarrow{f} & & \downarrow{p} \\
M & \xrightarrow{p} & M/K
\end{array}
\]
where \( M/K \) is the homomorphic image of a free \( R \)-module \( R(\Lambda) \) for some set \( \Lambda \), \( p \) is the canonical projection and \( q = pf \). Since \( pf \) is an epi and \( p \) is a superfluous epi, \( f \) is epi, (see \([4, Corollary~5.15.]\)). As we have shown in 2.8, \( M \rightarrow M/K \) is a quasi mono. Note that since \( R \) is semisimple, \( M \) is hereditary. Now consider the unbroken commutative diagram in \( R\text{-Mod} \), where \( n \in \mathcal{M} \).
\[
\begin{array}{ccc}
R(\Lambda) & \xrightarrow{u} & L \\
\downarrow{f} & & \downarrow{n} \\
M & \xrightarrow{p} & M/K
\end{array}
\]
where \( pf = nu \) and so \( n \) is epi. Since \( M \) is projective, there exists a morphism \( d \) such that \( nd = p \). Thus the factorization \( q = pf \) is a quasi right \( \mathcal{M} \)-factorization and \( p \) is a quasi mono.
Example 2.14. Recall that a ring $R$ is completely hereditary if its class of quasi-projective modules is closed under taking submodules, (see [14]). Let $\mathcal{C}$ be as in Example 2.9 and

$$\mathcal{M} = \{ m \in \mathcal{C} \mid m \text{ is mono or split epi in } \mathcal{C} \}.$$ 

Let $f : P \to Q$ be a morphism in $\mathbb{Q}\mathbb{P}$. Since $P/Kf$ is isomorphic to a submodule of $Q$, $P/Kf$ is quasi-projective. Let $p : P \to Q$ be a morphism in $\mathcal{C}$. Consider the factorization

$$P \xrightarrow{p} Q = P \xrightarrow{\bar{p}} P/Kp \xrightarrow{q} Q$$

in $\mathbb{R}\text{-Mod}$, where $\bar{p}$ is the canonical projection and for each $x \in P$, $q(x + Kp) = p(x)$. By [4, Proposition 5.17], the morphisms $\bar{p}$ and $q$ are in $\mathcal{C}$ and satisfy the condition (C2) of 2.9. Thus the factorization $p = q\bar{p}$ is a quasi right $\mathcal{M}$-factorization of $p$.

Now let $R$ also be left artinian ring and $R/J$ be projective $R$-module, where $J = J(R)$ is the Jacobson radical of $R$. Thus there is a complete set of pairwise orthogonal primitive idempotents $e_1, \ldots, e_n$ such that $R = Re_1 \oplus \cdots \oplus Re_n$ and each $Re_i/Je_i$ is simple and $Re_i \xrightarrow{p_i} Re_i/Je_i$ is a projective cover, (see [4, Exercises 20, p.203]). Therefore for each $i$, $J/e_i$ is a superfluous maximal submodule of $Re_i$. Thus we have the commutative diagram

$$\begin{array}{ccc}
R & \xrightarrow{q} & Re_i/Je_i \\
| & \downarrow{p_i} & | \\
Re_i & \xrightarrow{f_i} & \\
\end{array}$$

where for each $r \in R$, $q(r) = re_i + Je_i$. By [30, Exercises 21.17(3), p.183], for each $i$, $Je_i = \text{Rad}(Re_i)$ and by [30, 21.6(2)], it is a fully invariant submodule of $Re_i$. So $Re_i/Je_i$ is a quasi-projective module. Since $q$ is epi and $p_i$ is a superfluous epi, $f_i$ is epi, (see [4, Corollary 5.15.]). Note that [4, Proposition 5.17.] implies that $q$ is a morphism in $\mathcal{C}$. Now we show that for each $i$, $f_i$ is a morphism in $\mathcal{C}$. Let $N \ll R$. Since $f_i$ is an epimorphism, $f_i(N) \ll Re_i$. Now, if $f_i(N) \ll Re_i$, then $q(N) = p_i(f_i(N)) \ll Re_i/Je_i$ and so $N \ll R$. So the above diagram is in $\mathcal{C}$. Also for each $i$, since $Je_i$ is superfluous and maximal submodule of $Re_i$, $p_i$ satisfies the condition (C2) and hence is a
quasi mono. Since \( R/J \cong Re_1/Je_1 \oplus \cdots Re_n/Je_n \) and \( R/J \) is a projective \( R \)-module, for each \( i \), \( Re_i/Je_i \) is a projective \( R \)-module and hence \( p_i \) is split epi. Thus there exists \( s_i : Re_i/Je_i \rightarrow Re_i \) such that \( p_is_i = 1_{Re_i/Je_i} \). Now we show that \( s_i \) is a morphism in \( C \). Let \( L/Je_i \triangleleft Re_i/Je_i \) and \( s_i(L/Je_i)+K = Re_i \). Thus \( L/Je_i+p_i(K) = Re_i/Je_i \) and so \( (K+Je_i)/Je_i = p_i(K) = Re_i/Je_i \). Therefore \( K+Je_i = Re_i \) and hence \( K = Re_i \). This implies that \( s_i(L/Je_i) \triangleleft Re_i \) and hence \( s_i \) is a morphism in \( C \). Therefore for each \( i \), \( p_i \in \mathcal{M} \). Similar to 2.13, the factorization \( q = p_if_i \) is a quasi right \( \mathcal{M} \)-factorization.

In the following two examples we give a class of quasi epis which are not epics.

**Example 2.15.** Let \( C \) be the category of subrings of real numbers \( \mathbb{R} \). Then the inclusion \( \mathbb{Z} \xrightarrow{j} \mathbb{Q}(\sqrt{p}) \), where \( p \) is a prime number, is quasi epi. To show this, suppose the diagram,

\[
\begin{array}{c}
\mathbb{Z} \xrightarrow{j} \mathbb{Q}(\sqrt{p}) \xrightarrow{f} S \\
\end{array}
\]

is given in \( C \) such that \( fj = gj \). Thus for each \( m \in \mathbb{Z} \), \( f(m) = g(m) \) and thus for each \( a \in \mathbb{Q} \), \( f(a) = g(a) \). On the other hand \( f(\sqrt{p}) = g(\sqrt{p}) \) or \( f(\sqrt{p}) = -g(\sqrt{p}) \). In the first case \( f = g \). In the second case define \( h : \mathbb{Q}(\sqrt{p}) \rightarrow \mathbb{Q}(\sqrt{p}) \) by \( h(a + b\sqrt{p}) = a - b\sqrt{p} \), implying \( f = gh \) and \( g = fh \).

**Example 2.16.** Let \( X \) and \( X_0 \) be in \( \text{Set} \) with \( X_0 \subsetneq X \) and \( C \) be a subcategory of \( \text{Set} \), such that \( X - X_0 \in C \) and \( X \notin C \). Now define the subcategory \( \mathcal{D} \) of \( \text{Set} \) to have \( \text{obj}(C) \cup \{X\} \) as objects and \( \text{Hom}_{\mathcal{D}}(A, B) = \)

\[
\begin{cases}
\text{Hom}_{C}(A, B) & A, B \in C \\
\text{Hom}_{\text{Set}}(A, B) & A \in C, B = X \\
\emptyset & A = X, B \in C \\
\{ X \xrightarrow{f} X \mid X_0 \xrightarrow{f|X_0} X_0 \} & A = X, B = X
\end{cases}
\]

Then the inclusion map \( X - X_0 \xrightarrow{j} X \) is a quasi epi in \( \mathcal{D} \).
3 General closure operator

Consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of morphisms (not necessarily monomorphisms) in $\mathcal{X}$ which we think of as generalized subobjects. For every object $X$ of $\mathcal{X}$, let $\mathcal{M}/X$ be the class of all morphisms in $\mathcal{M}$ with codomain $X$. The relation given by

$$m \leq n \iff \langle m \rangle \subseteq \langle n \rangle$$

is reflexive and transitive, hence $\mathcal{M}/X$ is a preordered class. Note that $m \leq n$ means that there exists a morphism $j$ such that $m = nj$. Since $\mathcal{M}$ is an arbitrary class of morphisms, $j$ is not uniquely determined. Also $m \leq n$ and $n \leq m$ do not imply that $j$ is an isomorphism. If $m \leq n$ and $n \leq m$, then we say $m$ and $n$ are $\langle \rangle$-equal and we write $m \sim n$. Of course, $\sim$ is an equivalence relation, and $\mathcal{M}/X$ modulo $\sim$ is a partially ordered class. In fact, we shall use the notations $\leq$ and $\sim$ for elements of $\mathcal{M}/X$ rather than for their $\sim$-equivalence classes. So, with $\overline{m}$ denoting the $\sim$-class of $m$, we have

$$m \leq n \iff \overline{m} \leq \overline{n}$$

$$m \sim n \iff \overline{m} = \overline{n}.$$ 

**Definition 3.1.** Suppose that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations. A general closure operator $C$ on $\mathcal{X}$ with respect to $\mathcal{M}$ is given by $C = (c_X)_{X \in \mathcal{X}}$, where $c_X : \mathcal{M}/X \longrightarrow \mathcal{M}/X$ is a map satisfying

(a) the extension property: for all $m \in \mathcal{M}/X$, $m \leq c_X(m)$;

(b) the monotonicity property: for all $m, m' \in \mathcal{M}/X$ whenever $m \leq m'$ in $\mathcal{M}/X$, then $c_X(m) \leq c_X(m')$;

(c) the continuity property: for every morphism $f : X \longrightarrow Y$ in $\mathcal{X}$ and for all $m \in \mathcal{M}/X$, $m_{f \circ_X (m)} \leq c_Y(m_{fm})$.

Since $\mathcal{M}$ is an arbitrary class of morphisms and $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations, Definition 3.1 is a generalization of the closure operator that is defined in [10].

**Remark 3.2.** Suppose that $\mathcal{M}$ has $\mathcal{X}$-pullbacks (that is, $m \in \mathcal{M}$ and $f \in \mathcal{X}$ with the same codomains, implies the pullback, $f^{-1}(m)$, of $m$ along
f is in \( \mathcal{M} \) and \( \mathcal{X} \) has quasi right \( \mathcal{M} \)-factorizations. For every object \( X \) in \( \mathcal{X} \) consider the preordered class \( \mathcal{M}/X \). We have the adjunction,

\[
\begin{array}{ccc}
\mathcal{M}/X & \xleftarrow{f(-)} & \mathcal{M}/X \\
& \downarrow & \\
& f^{-1}(-) & 
\end{array}
\]

(3.1)

see [22]. In the presence of (b), by adjunction (3.1), the continuity condition can equivalently be expressed as:

\[ (c') \text{ for every morphisms } f : X \longrightarrow Y \text{ in } \mathcal{X} \text{ and for all } m \in \mathcal{M}/Y, \]

\[ c_X(f^{-1}(m)) \leq f^{-1}(c_Y(m)). \]

**Example 3.3.** (1) Let \( \mathcal{X} \) be a pointed category with finite products, and \( \mathcal{K} \) be a non-empty class of objects of \( \mathcal{X} \) such that for any pair of isomorphic objects either both are in \( \mathcal{K} \) or both are not; and let \( \mathcal{M} \) be the class of all split epis with kernel in \( \mathcal{K} \). Then \( \mathcal{M} \) is a quasi right factorization structure in \( \mathcal{X} \) which is closed under composition with isomorphisms on both sides. A morphism \( f : X \longrightarrow Y \) in \( \mathcal{X} \) can be factored as \( f = \pi_2 \langle 0, f \rangle \), where \( \langle 0, f \rangle : X \longrightarrow K \times Y \) for some \( K \in \mathcal{K} \). It is easy to see that \( \pi_2 \) is a split epi with kernel in \( \mathcal{K} \). Since \( \mathcal{M} \) is a collection of split epis, any family \( \mathcal{C} = (c_X)_{X \in \mathcal{X}} \) where \( c_X \) is a map from \( \mathcal{M}/X \) to itself, forms a general closure operator on \( \mathcal{X} \) with respect to \( \mathcal{M} \).

Examples (2), (3) and (4) below are special cases of this type.

(2) Consider the category \( _R\mathcal{M}_S \) of \( (R,S) \)-bimodules, where \( R \) and \( S \) are commutative rings and suppose that there exists a ring homomorphism \( \sigma : R \longrightarrow S \) such that \( \sigma(1_R) = 1_S \). Thus \( S \) is an \( R \)-module by \( r \cdot s = \sigma(r)s \) and hence \( S \in _R\mathcal{M}_S \). Suppose that \( \mathcal{C} \) is a full subcategory of \( _R\mathcal{M}_S \) whose objects are \( (R,S) \)-bimodules \( M \) such that for each \( r \in R, s \in S \) and \( m \in M \) we have \( s(rm) = (s \cdot r)m \). Let \( \mathcal{M} \) be the class of all split epis in \( \mathcal{C} \). One can easily verify that \( \mathcal{M} \) is a quasi right factorization structure in \( \mathcal{C} \). A morphism \( f : X \longrightarrow Y \) in \( \mathcal{C} \) can be factored as

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\langle 0,f \rangle \downarrow & \downarrow & \downarrow \\
X \oplus Y & & \pi_2
\end{array}
\]
For each morphism \( \varphi : M \to X \) in \( \mathcal{M} \), define its closure \( \bar{\varphi} \) to be the unique map making the diagram

\[
\begin{array}{ccc}
S \times M & \xrightarrow{\otimes_R} & S \otimes_R M \\
\downarrow \psi & & \downarrow \phi \\
X & & X
\end{array}
\]

commute, where the map \( \psi : S \times M \to X \) takes \((s, m)\) to \(s \varphi(m)\).

(3) Let \( \mathcal{C} \) be the category of torsion free modules, \([11]\), and \( \mathcal{M} \) be the class of all split epis. Then \( \mathcal{M} \) is a quasi right factorization structure in \( \mathcal{C} \), \([22]\). Suppose that \( m : M \to X \) is a morphism in \( \mathcal{M} \). There is a torsion free precover \( \varphi : T \to X \). Since \( M \) is torsion free, there is a map \( \psi : M \to T \) such that \( \varphi \psi = m \). Now define the closure of \( m \) to be the map \( \varphi \).

(4) Let \( \mathcal{C} \) be an abelian category with enough injectives, \([13]\). The collection \( \mathcal{M} \) of all epis whose kernels are injective is a quasi right factorization structure. A morphism \( f : X \to Y \) can be factored as \( \pi_2(i, f) \), where \( i : X \to E \) is the mono from \( X \) to an injective object \( E \) and \( \pi_2 : E \times Y \to Y \) is the projection to the second factor. Now for each morphism \( m : M \to X \) define its closure to be the map \( m \pi_2 : K \oplus M \to X \), where \( K = \text{Ker}(m) \).

(5) Let \( \mathcal{C} \) be a closed model category, \([24]\). The collection \( \mathcal{M} \) of fibrations in \( \mathcal{C} \) is a quasi right factorization structure. For each object \( X \in \mathcal{C} \) we have a trivial fibration \( p_X : Q(X) \to X \) with \( Q(X) \) cofibrant. Now for each morphism \( m : M \to X \) in \( \mathcal{M} \) define its closure to be the map \( mp_m : Q(M) \to X \).

(6) As a special case of (5), in the category \textbf{Top}, of topological spaces and continuous maps, the collection \( \mathcal{M} \) of Serre fibrations is a quasi right factorization structure. Now the closure of a morphism \( m : M \to X \) in \( \mathcal{M} \) is as in (5).

(7) Let \( \mathcal{C} \) be a model category. For the category of fibrant objects, \( \mathcal{C}_f \), the collection \( \mathcal{M} \) of fibrations is a quasi right factorization structure. Define the closure of \( m : M \to X \) in \( \mathcal{M} \) to be the projection to the first factor, \( \pi_1 : X \times M \to X \).
As a special case of (7), in the category Top, in which all the objects are fibrant, the collection $\mathcal{M}$ of Serre fibrations is a quasi right factorization structure. Define the closure of $m: M \rightarrow X$ in $\mathcal{M}$ to be the projection to the first factor, $\pi_1: X \times M \rightarrow X$.

In the cofibrant category $(\text{Top}, \text{cofibrations}, \text{homotopy equivalences})$, the collection $\mathcal{M}$ of homotopy equivalences is a quasi right factorization structure. A morphism $f$ can be factored as $X \xrightarrow{f} Y = X \xrightarrow{i_f} Z_f \xrightarrow{r_f} Y$, where $r_f$ is a homotopy equivalence, $i_f$ is a cofibration and $Z_f$ is the mapping cylinder of $f$, [20]. For each morphism $m: M \rightarrow X$ in $\mathcal{M}$ define its closure to be the map, $Z_m \xrightarrow{r_m} X$.

In the fibrant category $(\text{Top}, \text{fibrations}, \text{homotopy equivalences})$, the collection $\mathcal{M}$ of fibrations is a quasi right factorization structure. A morphism $f$ can be factored as $X \xrightarrow{f} Y = X \xrightarrow{q_f} P_f \xrightarrow{k_f} Y$, where $q_f$ is a homotopy equivalence, $k_f$ is a fibration and $P_f$ is the mapping path space of $f$, [20]. For each morphism $m: M \rightarrow X$ in $\mathcal{M}$ define its closure to be the map $k_m: P_m \rightarrow X$.

In the Kleisli category $\text{Set}_P$, where $P$ is the power set monad $\mathbb{P} = (P, \eta, \mu)$, for each morphism $\hat{f}: X \rightarrow Y$ in $\text{Set}_P$, let $f: X \rightarrow P(Y)$ be its associated morphism in $\text{Set}$ and

$$X \xrightarrow{f} P(Y) = X \xrightarrow{f'} I_f \xrightarrow{m_f} P(Y)$$

be the $(\text{Epi}, \text{Mono})$ factorization of $f$. The class $\mathcal{M} = \{\hat{m}_f : \hat{f} \in \text{Set}_P\}$ is a quasi right factorization structure, see [22]. For each morphism $\hat{m}_f: I_f \rightarrow Y$ in $\mathcal{M}$ define its closure to be the map $\mu_Y P(\hat{m}_f): P(I_f) \rightarrow Y$.

Let $\mathcal{X}$ be a category with binary coproducts. Then the class $\mathcal{M} = \{[f, f] \in \mathcal{M} \mid f \text{ is a morphism in } \mathcal{X}\}$ is a quasi right factorization structure. A morphism $X \xrightarrow{f} Y$ can be factored as $f = [f, f] \nu_1$, where $\nu_1$ is...
the injection of the coproduct. For each morphism \( m \) in \( \mathcal{M} \) define its closure to be the map \([m, m]\).

(13) Given any adjunction \( \mathcal{X} \xleftarrow{F} \mathcal{A} \). Let \( \tilde{\mathcal{A}} \) be the full subcategory of \( \mathcal{A} \) consisting of those objects \( A \) such that \( \epsilon_A : FG(A) \to A \) the component of the counit (of the above adjunction) at \( A \), is a split epi. The class \( \mathcal{M} \) consisting of those split epis in \( \mathcal{A} \) whose domain is \( F(X) \) for some object \( X \) in \( \mathcal{X} \) is a quasi right factorization structure on \( \tilde{\mathcal{A}} \). Each morphism \( f \) in \( \tilde{\mathcal{A}} \) factorizes as

\[
A \xrightarrow{f} B = A \xrightarrow{sf} FG(B) \xrightarrow{\epsilon_B} B,
\]

where \( s \) is any splitting of \( \epsilon_B \). If in addition \( \mathcal{X} \) has binary coproducts, for each morphism \( m : F(X) \to A \) in \( \mathcal{M} \) define its closure to be the unique morphism \( c_X(m) : F(X + X) \to A \), corresponding by adjunction to \( X \bar{m} \to G(A) \), where \( X \bar{m} \to G(A) \) is the morphism corresponding by adjunction to \( m \). Now we show that \( c_X(m) \) is a split epi. Let

\[
\text{Hom}_A(F-,-) \xrightarrow{\theta_{X,-}} \text{Hom}_X(-,G-) : \mathcal{X}^{\text{op}} \times \mathcal{A} \to \text{Set}
\]

be the natural isomorphism corresponding to the adjunction \( F \dashv G \). For \( X \xrightarrow{\bar{m}} G(A) \) we have

\[
\begin{array}{ccc}
X & \xrightarrow{\theta_{G,A}} & \text{Hom}_A(FG(A),A) \\
\downarrow \text{Hom}_A(FG(A),A) & \\
G(A), & \xrightarrow{\theta_{X,A}} & \text{Hom}_X(G(A),G(A)) \\
\downarrow \text{Hom}_X(G(A),G(A)) & \\
\epsilon_A & \xrightarrow{\epsilon_A} & 1_{G(A)} \\
\downarrow \epsilon_A \text{Hom}_X(G(A),G(A)) & \\
\epsilon_A F(\bar{m}) & \xrightarrow{\epsilon_A F(\bar{m})} & G(A) \\
\end{array}
\]

Since \( \theta_{X,A}(m) = \bar{m} \) and \( \theta_{X,A} \) is one-to-one, \( m = \epsilon_A F(\bar{m}) \). Suppose that \( \iota_1, \iota_2 : X \to X + X \) are the canonical injections of the coproduct \( X + X \). Since left adjoint preserves coproducts, \( F([\bar{m}, \bar{m}]) \) is the unique morphism such that \( F([\bar{m}, \bar{m}])F(\iota_1) = F([\bar{m}, \bar{m}])F(\iota_2) = F(\bar{m}) \). Thus \( m = \epsilon_A F(\bar{m}) = \epsilon_A F([\bar{m}, \bar{m}])F(\iota_1) = c_X(m)F(\iota_1) \) and hence \( c_X(m) \) is split epi, because \( m \) is so.

(14) As a special case of (13), consider \( \text{Proj}(R\text{-Mod}) \) as the full subcategory of the category \( R\text{-Mod} \), consisting of all projective \( R \)-modules.
The collection $\mathcal{M}$ of all epis with free domains is a quasi right factorization structure, see [22]. For each morphism $m : F \rightarrow P$ in $\mathcal{M}$ define its closure to be the map $[m, m] : F \oplus F \rightarrow P$.

Now on instead of saying $C$ is a general closure operator on the category $\mathcal{X}$ with respect to $\mathcal{M}$ we will say $C$ is a closure operator.

**Lemma 3.4.** (Weak Diagonalization Lemma) Let $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations and $C$ be a closure operator. For every commutative diagram

\[
\begin{array}{c}
M \\ ^m \searrow \\
\downarrow \qquad \downarrow \ \\
X & \overset{v}{\longrightarrow} & Y
\end{array}
\]

with $m, n \in \mathcal{M}$, there is a morphism $w$ rendering the lower square in the diagram

\[
\begin{array}{c}
M \overset{u}{\longrightarrow} N \\
\downarrow \quad \downarrow \\
j_m \quad j_n \\
\text{c}_X(M) \overset{w}{\longrightarrow} \text{c}_Y(N) \\
\downarrow \quad \downarrow \\
\text{c}_X(m) & \overset{///}{\longrightarrow} & \text{c}_Y(n) \\
\downarrow \quad \downarrow \\
X & \overset{v}{\longrightarrow} & Y
\end{array}
\]

commutative.

**Proof.** Let $\text{c}_X(M) \overset{v\text{c}_X(m)}{\longrightarrow} Y = \text{c}_X(M) \overset{e}{\longrightarrow} M' \overset{m\text{c}_X(m)}{\longrightarrow} Y$, be the $\mathcal{M}$-quasi right factorization of $v\text{c}_X(m)$. Since $vm \leq n$, by 1.1(b'), $mvm \leq n$, hence $m\text{c}_X(m) \leq \text{c}_Y(mvm) \leq \text{c}_Y(n)$. Thus there exists $w' : M' \longrightarrow \text{c}_Y(N)$ such that $\text{c}_Y(n)w' = m\text{c}_X(m)$. Put $w = w'e$, so $\text{c}_Y(n)w = v\text{c}_X(m)$.

**Definition 3.5.** Suppose that $\mathcal{M}$ is a class of morphisms in $\mathcal{X}$ and $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations. Also suppose that $C$ is a closure operator and $m \in \mathcal{M}/X$, where $X$ is an object in $\mathcal{X}$. We say $m$ is

(a) quasi $C$-closed in $X$, if $\text{c}_X(m) \sim m$ (see [22]);

(b) quasi $C$-dense in $X$, if $\text{c}_X(m) \sim 1_X$. 

A morphism $f$ in $\mathcal{X}$ is called \textit{quasi} $\mathbf{C}$-\textit{dense}, whenever $m_f$ is quasi $\mathbf{C}$-dense in $\mathcal{X}$. We denote by $\mathcal{E}^{QC}$, the class of all quasi $\mathbf{C}$-dense morphisms in $\mathcal{X}$. Let $\mathcal{M}^{QC}$ be the class of quasi $\mathbf{C}$-closed members of $\mathcal{M}$.

**Remark 3.6.** Let $f$ and $g$ be two morphisms in the category $\mathcal{X}$.

(a) $f \sim 1_\mathcal{X}$ is equivalent to $1_\mathcal{X} \leq f$ which is equivalent to $f$ being a split epi.

(b) If $f \leq g$ and $f$ is a split epi, then $g$ is a split epi.

**Example 3.7.** (i) In Example 3.3, (1), (2), (3), (4), (9), (10) and (11) members of $\mathcal{M}$ are all quasi $\mathbf{C}$-closed. (Example 3.3(1), (2)).

(ii) In Example 3.3, (1), (2), (3), (4), (7) and (9) members of $\mathcal{M}$ are all quasi $\mathbf{C}$-dense.

**Proposition 3.8.** Suppose that $\mathcal{M}$ has $\mathcal{X}$-pullbacks and $\mathbf{C}$ is a closure operator. Then $\mathcal{M}^{QC}$ has $\mathcal{X}$-pullbacks.

**Proof.** Let $m \in \mathcal{M}^{QC}$ and the diagram

$$
\begin{array}{ccc}
M^* & \xrightarrow{f^*} & M \\
m^* & \downarrow & m \\
X & \xrightarrow{f} & Y
\end{array}
$$

be a pullback in $\mathcal{X}$. Thus $m^* \in \mathcal{M}$. By 3.4, there exists $\mathbf{c}_X(M^*) \xrightarrow{w} \mathbf{c}_Y(M)$ such that $\mathbf{c}_Y(m)w = f\mathbf{c}_X(m^*)$. Since $\mathbf{c}_Y(m) \sim m$, there exists $q : \mathbf{c}_Y(M) \xrightarrow{mq} M$ such that $\mathbf{c}_Y(m) = mq$. Therefore

$$f\mathbf{c}_X(m^*) = \mathbf{c}_Y(m)w = m(qw)$$

and hence there exists a unique morphism $\mathbf{c}_X(M^*) \xrightarrow{r} M^*$ such that $m*r = \mathbf{c}_X(m^*)$. Thus $\mathbf{c}_X(m^*) \leq m^* \leq \mathbf{c}_X(m^*)$ and so $m^* \sim \mathbf{c}_X(m^*)$. Therefore $m^* \in \mathcal{M}^{QC}$. 

**Proposition 3.9.** Suppose that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations and $\mathbf{C}$ is a closure operator.

(a) Let $\mathcal{M}$ be closed under composition with isomorphisms on the left. For each morphism $X \xrightarrow{f} Y \in \mathcal{E}^{QC}$ and isomorphism $Y \xrightarrow{\alpha} Z$ in $\mathcal{X}$ we have $\alpha f \in \mathcal{E}^{QC}$.

(b) If $f \leq g$ and $f \in \mathcal{E}^{QC}$, then $g \in \mathcal{E}^{QC}$. 

Proof. (a) By Propositions 1.3 and 1.6 and the continuity property of $C$, we have $\alpha(c_Y(f(1_X))) \leq c_Z((\alpha f)(1_X))$. Since $c_Y(f(1_X)) \sim 1_Y$ and $\alpha(1_Y) \sim 1_Z$, $\alpha f \in \mathcal{E}^{QC}$.

(b) Obvious. \qed

Remark 3.10. Suppose that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations and $C$ is a closure operator.

(a) For each $m, n \in \mathcal{M}$ if $m \sim n$ and $m$ is quasi $C$-dense, then $n$ is quasi $C$-dense.

(b) If $\mathcal{M}$ is a class of monos, then $m$ is quasi $C$-closed (dense) if and only if $m$ is $C$-closed (dense).

4 Quasi idempotent and quasi weakly hereditary closure operator

In this section we define quasi idempotent and quasi weakly hereditary closure operators and we show which one of the examples in the previous section have these properties. Finally we prove under what conditions on the closure operator we have another quasi right(left) factorization structure.

Definition 4.1. Suppose that $\mathcal{X}$ has quasi right $\mathcal{M}$-factorizations and $C$ is a closure operator. $C$ is called

(a) quasi idempotent, if for each $X \in \mathcal{X}$ and $m \in \mathcal{M}/X$, $c_X(c_X(m)) \sim c_X(m)$ (see [22]).

(b) quasi weakly hereditary, if for each $X \in \mathcal{X}$ and $m : M \longrightarrow X$ in $\mathcal{M}$, there exists a quasi $C$-dense morphism $j_m : M \longrightarrow c_X(M)$ such that $c_X(m)j_m = m$.

Example 4.2. (1) In Example 3.3, (1), (2), (3), (4), (7), (9), (10) and (11) the closure operator is quasi idempotent.

(2) In Example 3.3, (1), (2), (3), (4), (9), (10) and (11) the closure operator is quasi weakly hereditary.

With $(\mathcal{E}, \mathcal{M})$-factorization structure as defined in [2], we have

Theorem 4.3. [22] Suppose that $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorization structure and $C$ is a quasi idempotent closure operator. Then $\mathcal{M}^{QC}$ is a quasi right factorization structure for $\mathcal{X}$. 
Theorem 4.4. Let $X$ has quasi right $M$-factorizations. For a quasi idempotent closure operator $C$, $X$ has quasi right $M^{QC}$-factorizations.

Proof. For a given morphism $f$ in $X$, $f \leq m_f \leq c(m_f)$ which is quasi closed. That is $c(m_f) \in M^{QC}$. If $f \leq n$ with $n$ quasi closed, then $m_f \leq n$. Thus $c(m_f) \leq c(n) \sim n$. \qed

Remark 4.5. For a closure operator $C$ we have $M^{QC} \cap E^{QC} = \{ f \in M \mid f \sim 1 \}$. If $M \subseteq QM(X)$, then $M^{QC} \cap E^{QC} = Iso(X) \cap M$.

Proposition 4.6. Suppose $M \subseteq QM(X)$ is closed under composition. Also suppose $X$ has quasi right $M$-factorizations and $C$ is a closure operator. If $X \longrightarrow Y = X \xrightarrow{e} M \xrightarrow{m_f} Y$ is a quasi right $M$-factorization of $f$, then we have $e \in E^{QC}$ and $e(1_X), c_M(e(1_X)) \in Iso(X)$.

Proof. Since $f \leq m_fm_e$, by Lemma 1.2 (a) and (c), $m_f \leq m_fm_e$. By Proposition 2.2(a), we have $1_M \leq m_e$ and so $e \in E^{QC}$. Since $1_M \leq m_e \leq c_M(m_e), 1_M \sim m_e \sim c_M(m_e)$. Thus $m_e$ and $c_M(m_e)$ are split epis. Since $m_e$ and $c_M(m_e)$ are quasi monos, by Lemma 2.4 they are isomorphisms. \qed

Proposition 4.7. Suppose that $X$ has quasi right $M$-factorizations and $M \subseteq QM(X)$. If $X \longrightarrow M$ is quasi $C$-dense, then $c_M(m_e)$ is an isomorphism.

Proof. Since $e \in E^{QC}$ and $c_M(m_e) \sim 1_M$, there exist morphisms $f$ and $g$ such that $f = 1_Mf = c_X(m_e)$ and $c_M(m_e)g = 1_M$. Hence $fg = 1_M$ and $fgf = f$. Since $f = c_X(m_e) \in M, gf \sim 1_{c_M(e(X))}$ and there exists a morphism $h$ such that $gfh = 1_{c_M(e(X))}$. Therefore $f$ is an isomorphism. \qed

Notation 4.8. We write, $e' \xleftarrow{c} m$, whenever in the unbroken commutative diagram

\[
\begin{array}{ccc}
  & e & \\
  e' \downarrow & & \downarrow m \\
  & d & \\
\end{array}
\]

there exists a morphism $d$ such that $e = de'$. 


Remark 4.9. \( e' \mathrel{\sqsubseteq} e \) is equivalent to: if \( \langle m \rangle \subseteq \langle e' \rangle \), then \( \langle e \rangle \subseteq \langle e' \rangle \).

Now we can define the class \( \mathcal{E} \) as follows:
\[
\mathcal{E} : \overset{\text{def}}{=} \{ e' \in \mathcal{E} \mid e' \mathrel{\sqsubseteq} e, \text{mm} \forall e \in \mathcal{E} \text{ and } \forall m \in \mathcal{M} \}.
\]

For a closure operator \( C \) consider the following property:

Definition 4.10. Suppose \( \mathcal{M} \) is a class of morphisms, closed under composition, \( \mathcal{X} \) has quasi right \( \mathcal{M} \)-factorizations and \( C \) is a closure operator. We say \( C \) satisfies the property (QCD) if compositions of quasi \( C \)-dense morphisms are quasi \( C \)-dense.

Theorem 4.11. Suppose that \( \mathcal{X} \) has quasi right \( \mathcal{M} \)-factorizations and \( C \) is a closure operator. If \( \mathcal{M} \subseteq \mathcal{Q} \mathcal{M}(\mathcal{X}) \) is closed under composition and \( \mathcal{E} \subseteq \mathcal{M} \), then \( \mathcal{X} \) has quasi left \( \mathcal{E} \)-factorization structures.

Proof. Suppose that \( f = m_f e \) is a quasi right \( \mathcal{M} \)-factorization of \( f \). We will show that this factorization is also a quasi left \( \mathcal{E} \)-factorization of \( f \). By Proposition 4.6, we have \( e \in \mathcal{E} \). Given \( e' \in \mathcal{E} \) such that \( \langle f \rangle \subseteq \langle e' \rangle \), since \( \mathcal{E} \subseteq \mathcal{M} \), \( \langle e \rangle \subseteq \langle e' \rangle \). Therefore \( f = m_f e \) is a quasi left \( \mathcal{E} \)-factorization of \( f \). \( \square \)

Theorem 4.12. Suppose that \( \mathcal{X} \) has quasi right \( \mathcal{M} \)-factorizations and \( C \) is a quasi weakly hereditary closure operator. If \( \mathcal{M} \subseteq \mathcal{Q} \mathcal{M}(\mathcal{X}) \) is closed under composition, \( \mathcal{E} \subseteq \mathcal{M} \) and \( \mathcal{E} \subseteq \mathcal{Q} \mathcal{E}(\mathcal{X}) \), then \( \mathcal{M} = \mathcal{M} \).

Proof. We only need to prove that \( \mathcal{M} \subseteq \mathcal{M} \). Let \( m \) be an element of \( \mathcal{M} \) and \( j_m \) be its quasi \( C \)-dense morphism. Since \( 1_M \in \mathcal{E} \subseteq \mathcal{M} \), \( \langle 1_M \rangle \subseteq \langle j_m \rangle \). So we have \( c(m) \leq m \), because \( \mathcal{E} \subseteq \mathcal{Q} \mathcal{E}(\mathcal{X}) \). Thus \( m \in \mathcal{M} \). \( \square \)

5 Quasi factorization structures

In this section the notations \( \mathcal{H}^\Lambda \) and \( \mathcal{V} \mathcal{H} \) are introduced and after studying some of their properties, the notion of quasi factorization structure in a
category $\mathcal{X}$ is given. We will see that weak factorization structures as defined in [1] are quasi factorization structures, but the converse is not true as we will show by some examples. Finally we state the relation between a quasi factorization structure and a quasi idempotent and quasi weakly hereditary closure operator.

**Notation 5.1.** (a) Given $E \xrightarrow{e} X \in \mathcal{X}$ and $M \xrightarrow{m} X \in \mathcal{X}$, $e \triangledown m$ means that in every commutative triangle,

$$
\begin{array}{ccc}
E & \xrightarrow{u} & M \\
\downarrow{e} & \diagdown & \downarrow{m} \\
X & & 
\end{array}
$$

there exists $w : X \rightarrow M$ such that $mw \sim 1_X$. (Note that $e \triangledown m$ means that $e \leq m \Rightarrow 1_X \leq m$.)

(b) Given $M \xrightarrow{e} E \in \mathcal{X}$ and $M \xrightarrow{m} X \in \mathcal{X}$, $e \triangledown m$ means that in every commutative triangle,

$$
\begin{array}{ccc}
M & \xrightarrow{e} & E \\
\downarrow{m} & \diagdown & \downarrow{v} \\
X & & X
\end{array}
$$

there exists $w : E \rightarrow M$ such that $mw \sim v$.

Let $\mathcal{H}$ be a class of morphisms. We denote by $\mathcal{H}^\Delta$ the class of all morphisms $m$ with

$$h \triangledown m \quad \text{for all } \ h \in \mathcal{H}$$

and similarly, by $\triangledown \mathcal{H}$ the class of all morphisms $e$ with

$$e \triangledown h \quad \text{for all } \ h \in \mathcal{H}.$$  

Saying $\mathcal{H}$ has $X$-pushouts if the pushout of each morphism in $\mathcal{H}$ exists and is in $\mathcal{H}$, we have:

**Proposition 5.2.** For each classes $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ we have:

(i) If $\mathcal{H}_1 \subseteq \mathcal{H}_2$, then $\triangledown \mathcal{H}_2 \subseteq \triangledown \mathcal{H}_1$ and $\mathcal{H}_2^\Delta \subseteq \mathcal{H}_1^\Delta$.

(ii) $\text{Ret}(X) \subseteq \triangledown \mathcal{H}$. 


(iii) If $H \subseteq QE(\mathcal{X})$, then $\text{Sec}(\mathcal{X}) \subseteq H^{\Delta}$.
(iv) If $H$ has $\mathcal{X}$-pullbacks, then $\nabla H$ is closed under composition. Dually, if $H$ has $\mathcal{X}$-pushouts, then $H^{\Delta}$ is closed under composition.

Proof. The proofs of (i) and (ii) follow directly from the definition.

(iii) Suppose that the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{h} & H \\
\downarrow{s} & & \downarrow{v} \\
X & & X
\end{array}
\]

such that $s$ is a section and $h \in H$ is given. Thus, $h$ is a section and since $h$ is a quasi epi, by 2.4(b), $h \in \text{Iso}(\mathcal{X})$. Put $w = h^{-1}$ and so $sw = v$.

(iv) Suppose that $E \xrightarrow{e_2} E_1 \xrightarrow{e_1} X$ are composable morphisms in $\nabla H$ and the following commutative triangle is given

\[
\begin{array}{ccc}
E & \xrightarrow{u} & M \\
\downarrow{e_1e_2} & & \downarrow{h} \\
X & & X
\end{array}
\]

such that $h \in H$. Since $H$ has $\mathcal{X}$-pullbacks, we have

\[
\begin{array}{ccc}
E & \xrightarrow{u} & M \\
\downarrow{e_1} & & \downarrow{h} \\
N & \xrightarrow{e_2} & E_1
\end{array}
\]

Thus, there exists a morphism $E_1 \xrightarrow{w_1} N$ such that $e_1^{-1}(h)w_1 \sim 1_{E_1}$ and hence $e_1^{-1}(h)w_1g = 1_{E_1}$ for some morphism $g : E_1 \rightarrow E_1$. Therefore we have the commutative triangle

\[
\begin{array}{ccc}
E_1 & \xrightarrow{e_1^*w_1g} & M \\
\downarrow{e_1} & & \downarrow{h} \\
X & & X
\end{array}
\]
and so there exists a morphism $X \xrightarrow{w} M$ such that $hw \sim 1_X$. This implies that $e_1 e_2 \in \mathcal{H}$. The proof of the dual is similar.

**Proposition 5.3.** Suppose that $X$ has quasi right $\mathcal{M}$-factorizations and $\mathcal{M} \subseteq QM(X)$. If $C$ is a closure operator, then $\mathcal{E}^{QC} \subseteq \mathcal{V}(\mathcal{M}^{QC})$.

**Proof.** Suppose that $mu = e$, where $e \in \mathcal{E}^{QC}$ and $m \in \mathcal{M}^{QC}$. Consider the quasi right $\mathcal{M}$-factorization of $e$ as

$$E \xrightarrow{e} X = E \xrightarrow{e_1} e(E) \xrightarrow{m_e} X.$$  

Since $e \in \mathcal{E}^{QC}$, by Proposition 4.7, we have $c_X(m_e)$ is an isomorphism and so $c_X(m_e) \in \mathcal{M}^{QC}$. We have $m_e = c_X(m_e) j$ and so $e = c_X(m_e)(je_1)$. If $e \leq n$ with $n$ quasi closed, then $m_e \leq n$. Therefore $c_X(m_e) \leq c_X(n) \sim n$. Thus the factorization $e = c_X(m_e)(je_1)$ is a quasi right $\mathcal{M}^{QC}$-factorization of $e$. So we have the commutative diagram.

$$
\begin{array}{ccc}
E & \xrightarrow{u} & M \\
\downarrow{j e_1} & & \downarrow{m} \\
c_X(e(E)) & \xrightarrow{c_X(m_e)} & X
\end{array}
$$

Put $w := w_1 c_X(m_e)^{-1}$, so $mw = 1_X$. Therefore $\mathcal{E}^{QC} \subseteq \mathcal{V}(\mathcal{M}^{QC})$. □

**Proposition 5.4.** Suppose that $X$ has quasi right $\mathcal{M}$-factorizations and $\mathcal{M} \subseteq QM(X)$ is closed under composition. If $C$ is a quasi weakly hereditary closure operator such that $\mathcal{E}^{QC} \subseteq QE(X)$ and $\mathcal{E}^{QC} \subseteq \mathcal{V}(\mathcal{M})$, then $\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^\Delta$.

**Proof.** Suppose that $m = ve$, where $e \in \mathcal{E}^{QC}$ and $m \in \mathcal{M}^{QC}$. First we claim that $M \xrightarrow{m} X = M \xrightarrow{j m} c_X(M) \xrightarrow{c_X(m)} X$ is a quasi right $\mathcal{M}$-factorization of $m$. For this reason suppose that the unbroken commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{u} & N \\
\downarrow{j m} & & \downarrow{n} \\
c_X(M) & \xrightarrow{c_X(m)} & X
\end{array}
$$
with \( n \in \mathcal{M} \), is given. By Theorem 4.11, the quasi right \( \mathcal{M} \)-factorization

\[
M \xrightarrow{u} N = M \xrightarrow{e_1} E \xrightarrow{m_u} N
\]

of \( u \) is a quasi left \( \mathcal{E}^{QC} \)-factorization of \( u \) and so \( e_1 \in \mathcal{E}^{QC} \). Now consider the unbroken commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{e_1} & E \\
\downarrow{j_m} & & \downarrow{nm_u} \\
c_X(M) & \xrightarrow{c_X(m)} & X
\end{array}
\]

Since \( \mathbf{C} \) is a quasi weakly hereditary closure operator, \( j_m \in \mathcal{E}^{QC} \) and since \( \mathcal{M} \) is closed under composition, \( nm_u \in \mathcal{M} \). Thus \( \mathcal{E}^{QC} \subseteq \mathcal{Q} \mathcal{M} \) implies that there exists a morphism \( d_1 : c_X(M) \rightarrow E \) such that \( d_1j_m = e_1 \). Now define \( w' \defeq m_ud_1 \), so we have \( nw'j_m = nm_ud_1j_m = nm_ue_1 = c_X(m)j_m \).

Since \( j_m \in \mathcal{E}^{QC} \) and \( \mathcal{E}^{QC} \subseteq \mathcal{Q}E(X) \), we have \( nw' \sim c_X(m) \). Therefore there exists a morphism \( w'' \) such that \( c_X(m) = nw'w'' \). Define \( w \defeq w'w'' \).

Thus \( c_X(m) \leq n \) and the claim is proved. Therefore, by Theorem 4.11, the factorization \( m = c_X(m)j_m \) is a quasi left \( \mathcal{E}^{QC} \)-factorization of \( m \). So we have the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{e} & E \\
\downarrow{j_m} & & \downarrow{v} \\
c_X(M) & \xrightarrow{d'} & X
\end{array}
\]

such that \( j_m = d'e \). Since \( m \in \mathcal{M}^{QC} \), we have \( c_X(m) \sim m \). Thus there exists \( g : c_X(M) \rightarrow M \) such that \( c_X(m) = mg \) and hence \( mgj_m = m \).

Put \( d \defeq gd' \). Thus \( md = mgd' = c_X(m)d' \) and so \( mde = c_X(m)d'e = ve \). Since \( \mathcal{E}^{QC} \subseteq \mathcal{Q}E(X) \), \( md \sim v \).

**Proposition 5.5.** Suppose that \( X \) has quasi right \( \mathcal{M} \)-factorizations. Then \( \mathcal{N} \mathcal{M} \subseteq \mathcal{E}^{QC} \).

**Proof.** Let \( e \in \mathcal{N} \mathcal{M} \). Since \( e\mathcal{N}m_e \), \( 1 \leq m_e \). Thus \( 1 \leq c(m_e) \).
Let $E$ and $M$ be classes of morphisms in $X$. We say that $E$ and $M$ are closed under composition with isomorphisms

(i) if $\alpha \in \text{Iso}(X)$ and $e \in E$, and $\alpha e$ exists, then $\alpha e \in E$;
(ii) if $\alpha \in \text{Iso}(X)$ and $m \in M$, and $m\alpha$ exists, then $m\alpha \in M$.

**Proposition 5.6.** Let $E$ and $M$ be classes of morphisms in $X$ and for each morphism $f$ there exist $m \in M$ and $e \in E$ such that $f = me$ and $M \subseteq QM(X)$.

(i) If $E$ is closed under composition with isomorphisms, then $\nabla M \subseteq E$.
(ii) If $E^\Delta \subseteq QM(X)$ and $M$ is closed under composition with isomorphisms, then $E^\Delta \subseteq M$.

**Proof.** Consider the factorization $X \xrightarrow{f} Y = X \xrightarrow{e} M \xrightarrow{m} Y$ of $f$ where $m \in M$ and $e \in E$.

(i) If $f \in \nabla M$, then there exists $w : Y \xrightarrow{} M$ such that $mw \sim 1_Y$. Thus, $m$ is a split epi. Since $m$ is a quasi mono, by 2.4(a), $m \in \text{Iso}(X)$ and hence $f \in E$.

(ii) If $f \in E^\Delta$, then by 2.2(b), $e$ is a quasi mono and there exists $w : M \xrightarrow{} X$ such that $fw \sim m$. Thus, $mew \sim m$. Since $m$ is a quasi mono, by 2.2(a), $ew \sim 1_M$ and hence $e$ is a split epi. Therefore, by 2.4(a), we have $e \in \text{Iso}(X)$ and hence $f \in M$. □

In the following definition $X$ need not have quasi right $M$-factorizations.

**Definition 5.7.** A quasi factorization structure in a category $X$ is a pair $(E, M)$ of classes of morphisms such that;

(a) every morphism $f$ has a factorization as

$$X \xrightarrow{f} Y = X \xrightarrow{e} M \xrightarrow{m} Y,$$

where $e \in E$ and $m \in M$;
(b) $E \subseteq \nabla M$ and $M \subseteq E^\Delta$.

**Remark 5.8.** (i) If $(E, M)$ is a weak factorization structure in a category $X$, then it is a quasi factorization structure.

Suppose that $(E, M)$ is a quasi factorization structure in $X$ and $M \subseteq QM(X)$. By 5.6, we have
(ii) $E \cap M \subseteq \text{Iso}(X)$. To show this, let $f \in E \cap M$ and

$$X \xrightarrow{f} Y = X \xrightarrow{e} M \xrightarrow{m} Y$$

where $e \in E$ and $m \in M$. Since $f \in E \subseteq \nabla M$, there exists $Y \xrightarrow{w} M$ such that $mw \sim 1_Y$ and hence $m \in \text{Iso}(X)$. On the other hand, $f \in M \subseteq E^\Delta$ and so there exists $w' : M \xrightarrow{} X$ such that $fw' \sim m \sim 1_M$. Therefore $f$ is a split epi and hence $f$ is an isomorphism.

(iii) If $E$ is closed under composition with isomorphisms, then $E = \nabla M$ and so, by 5.2(ii), $\text{Ret}(X) \subseteq E$. Moreover, if $M$ has $X$-pullbacks, then by 5.2(iv) $E$ is closed under composition.

(iv) If $E^\Delta \subseteq QM(X)$ and $M$ is closed under composition with isomorphisms, then $M = E^\Delta$. Also if $E \subseteq QE(X)$, then by 5.2(iii) $\text{Sec}(X) \subseteq M$. Moreover, if $E$ has $X$-pushouts, then 5.2(iv) implies that $M$ is closed under composition.

In the following example $(E, M)$ is a quasi factorization structure which is not a weak factorization structure.

**Example 5.9.** (1) Let $C$ be a closed model category whose objects are cofibrant. The pair $(E, M)$ of morphisms in $C$, where $E$ is the class of weak equivalences and $M$ is the class of fibrations form a quasi factorization structure. To prove this, first note that every morphism $f$ in $C$ has a factorization $f = pj$, where $j$ is a trivial cofibration and $p$ is a fibration, [17, Definition 7.1.3]. Now assume $E \xrightarrow{e} X \in E$ and $M \xrightarrow{m} X \in M$ and let $e = mu$ for some $u \in C$. By [17, Proposition 7.2.6], $E \xrightarrow{e} X = E \xrightarrow{i} W \xrightarrow{p} X$, where $i$ is a trivial cofibration and $p$ is a trivial fibration. Thus, [17, Definition 7.1.3] implies that there exists $W \xrightarrow{d} M$ such that $p = md$. By [17, Proposition 7.6.11] there exists a morphism $X \xrightarrow{s} W$ such that $ps = 1_X$. Put $w = ds$, so $mw = 1_X$. Therefore $E \subseteq \nabla M$. Similarly, we can show that $M \subseteq E^\Delta$. Since $E \cap M \notin \text{Iso}X$, the system is not weak.

(2) As a special case of (1), in the category $\text{Top}$, in which all the objects are cofibrant, the collections $E$ of homotopy equivalences and $M$ of Serre fibrations form a quasi factorization structure.

(3) In Example 3.3 (11) above, let $E = \{\hat{e}_f : \hat{f} \in \text{Set}_r\}$, where $e_f = \eta_{f'} f'$. Then $(E, M)$ is a quasi factorization structure. To show $(E, M)$ is not
a weak factorization structure, let $X = \{x, x', x''\}$ and consider the map $f : X \rightarrow P(X)$ taking all the points to $\{x\}$. Let $f = \hat{m}_h \hat{k}$ be an $(\mathcal{E}, \mathcal{M})$ factorization of $\hat{f}$. As proved in [22] we can see the following commutative diagram has no diagonal:

\[
\begin{array}{ccc}
X & \xrightarrow{\hat{u}} & I_g \\
\downarrow{\hat{k}} & & \downarrow{\hat{m}_g} \\
I_h & \xrightarrow{\hat{v}\hat{m}_h} & X
\end{array}
\]

(4) Let $\mathcal{X}$ be a category with binary products in which projections are split epis. Let $\mathcal{E} = \text{Sec}$ and $\mathcal{M} = \text{Ret}$, where $\text{Sec}$ and $\text{Ret}$ denote the collection of all sections and split epis, respectively. Then $(\mathcal{E}, \mathcal{M})$ is a quasi factorization structure. To show $(\text{Sec}, \text{Ret})$ is not generally a weak factorization structure, let $\text{Top} - \{\emptyset\}$ be the full subcategory of $\text{Top}$ consisting of the non-empty topological spaces and consider the following commutative diagram

\[
\begin{array}{ccc}
\{0\} & \xrightarrow{u} & \{0, 1, 2\} \\
\downarrow{s} & & \downarrow{r} \\
\{0, 1\} & \xrightarrow{\tau} & \{0, 1\}
\end{array}
\]

where $u$ sends 0 to 1, with codomain having $\{1\}$ open; $s$ is the inclusion with codomain having $\{1\}$ open; $r$ sends 0 to 0 and sends 1 and 2 to 1 with codomain having indiscrete topology; and $\tau$ is the twist map. It is easy to see that $s$ is a section and $r$ is a split epi. The square has no diagonal, since otherwise if $d$ is a diagonal, then $ds = u$ and $rd = \tau$. It follows that $d(0) = 1$ and $d(1) = 0$. Since $d^{-1}(\{1\}) = \{0\}$, $d$ is not continuous.

(5) Let $\mathcal{X}$ be a category with coproducts,

\[
\mathcal{E} = \{A \xrightarrow{\nu_1} A \coprod B \mid \nu_1 \text{ is the coproduct inclusion to the first factor}\}
\]

and $\mathcal{M}$ be any collection of split epis. Then $(\mathcal{E}, \mathcal{M})$ is a quasi factorization structure. Since $\text{Iso}(\mathcal{X}) \not\subseteq \mathcal{E}$, the system is not weak.

(6) Let $\mathcal{X}$ be an abelian category. Define

\[
\mathcal{E} = \{A \xrightarrow{(0,f)} A \times B \mid A \xrightarrow{f} B \in \mathcal{X}\},
\]
\( M = \{ A \times B \xrightarrow{\pi_2} B \mid \pi_2 \text{ is the second projection} \} \). Then \((\mathcal{E}, M)\) is a quasi factorization structure. Since \(\text{Iso}(\mathcal{X}) \not\subseteq M\), the system is not weak.

**Theorem 5.10.** Suppose that \( \mathcal{X} \) has a quasi right \( M \)-factorizations such that \( M \) is closed under composition, \( M \subseteq QM(\mathcal{X}) \) and for each \( f \in QM(\mathcal{X}) \), \( f \) is an isomorphism whenever \( m_f \) is an isomorphism. Then there is a class \( \mathcal{E} \) such that \((\mathcal{E}, M)\) is a quasi factorization structure in \( \mathcal{X} \).

**Proof.** Let \( \mathcal{E} \) be the class \( \downarrow M \) and \( E \xrightarrow{f} Y = E \xrightarrow{e} X \xrightarrow{m_f} Y \) be a quasi right \( M \)-factorization of an arbitrary morphism \( f \) in \( \mathcal{X} \). We show that \( e \in \mathcal{E} \).

For this reason let the commutative triangle

\[
\begin{array}{ccc}
E & \xrightarrow{u} & M \\
\downarrow{e} & \downarrow{m} & \\
X & \xrightarrow{m} & \\
\end{array}
\]

such that \( m \in M \) is given. Thus \( f = m_f e = m_f m u \) and so there exists a morphism \( X \xrightarrow{d} M \) such that \( m_f m d = m_f \). Therefore \( \langle md \rangle = \langle 1_M \rangle \) and hence \( m \) is a split epi. Since \( m \) is a quasi mono, \( m \) is an isomorphism. This implies that \( e \in \downarrow M = \mathcal{E} \). Now we prove that \( M \subseteq \mathcal{E}^\Delta \). Let \( m \in M \) and the commutative triangle

\[
\begin{array}{ccc}
M & \xrightarrow{m} & X \\
\downarrow{e} & \downarrow{m} & \\
E & \xrightarrow{v} & \\
\end{array}
\]

such that \( e \in \mathcal{E} \) be given. Thus, by 2.2(b) we have \( e \in QM(\mathcal{X}) \). Let \( M \xrightarrow{e} E = M \xrightarrow{e'} M_1 \xrightarrow{m_e} E \) be a quasi right \( M \)-factorization of \( e \).

Therefore there exists a morphism \( E \xrightarrow{w} M_1 \) such that \( m_e w \sim 1_E \) and hence \( m_e \) is a split epi. Since \( m_e \in QM(\mathcal{X}) \), \( m_e \) is an isomorphism and so by hypothesis we have \( e \) is an isomorphism. Let \( e^{-1} \) be the inverse of \( e \), so \( m e^{-1} = v \). Thus, \( m \in \mathcal{E}^\Delta \).

**Theorem 5.11.** Suppose that \((\mathcal{E}, M)\) is a quasi factorization structure in \( \mathcal{X} \).
(a) If $\mathcal{M}$ has $\mathcal{X}$-pullbacks, then $\mathcal{X}$ has a quasi right $\mathcal{M}$-factorization structure.

(b) If $\mathcal{M} \subseteq \text{Mon}(\mathcal{X})$ and $\mathcal{E}$ has $\mathcal{X}$-pushouts, then $\mathcal{X}$ has a quasi left $\mathcal{E}$-factorization structure.

Proof. (a) Let $f : X \rightarrow Y$ be a morphism in $\mathcal{X}$ and consider the quasi factorization of $f$ as

$$X \xrightarrow{f} Y = X \xrightarrow{e_f} M \xrightarrow{m_f} Y,$$

where $e_f \in \mathcal{E}$ and $m_f \in \mathcal{M}$. Suppose the unbroken square,

$$\begin{array}{c}
X \xrightarrow{u} N \\
\downarrow^{e_f} \quad \downarrow^{n} \\
M \xrightarrow{m_f} Y
\end{array}$$

is commutative, where $n \in \mathcal{M}$. So there is a morphism $t$ such that the triangles in the following diagram commute:

Since $m_f^{-1}(n) \in \mathcal{M}$ and $\mathcal{E} \subseteq \vee \mathcal{M}$, there exists $w : M \xrightarrow{} m_f^{-1}(N)$ such that $m_f^{-1}(n)w \sim 1_M$. Thus there exists a morphism $\alpha : M \xrightarrow{} M$ such that $m_f^{-1}(n)w\alpha = 1_M$. Now define $d \overset{\text{def}}{=} m'w\alpha$. So we have $nd = nm'w\alpha = m_fm_f^{-1}(n)w\alpha = m_f$. 

(b) Consider the quasi factorization of $f$ as

$$X \xrightarrow{f} Y = X \xrightarrow{e_f} M \xrightarrow{m_f} Y.$$
where $e_f \in \mathcal{E}$ and $m_f \in \mathcal{M}$. Suppose the unbroken square

\[
\begin{array}{ccc}
X & \xrightarrow{e} & E \\
\downarrow e_f & & \downarrow v \\
M & \xrightarrow{m_f} & Y
\end{array}
\]

is commutative, where $e \in \mathcal{E}$. So there exists a morphism $t'$ such that the triangles in the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{e} & E \\
\downarrow e_f & & \downarrow v \\
M & \xrightarrow{m_f} & E'
\end{array}
\]

Since $e'' \in \mathcal{E}$ and $\mathcal{M} \subseteq \mathcal{E}^\Delta$, there exists $w' : E' \longrightarrow M$ such that $mfw' \sim t'$. Thus there exists $\beta : E' \longrightarrow E'$ such that $mfw'\beta = t'$. Now define $d' \triangleq w'\beta e'$. So we have $mfw'\beta e'e = mfw'\beta e'e_f = t'e''e_f = mfe_f$. Therefore $d'e = e_f$. □

Calling $\mathcal{M}$, $\sim$-closed, if whenever $m \in \mathcal{M}$ and $f \sim m$, then $f \in \mathcal{M}$, we have:

**Corollary 5.12.** Suppose that $\mathcal{X}$ has a quasi right $\mathcal{M}$-factorization and $\mathcal{M} \subseteq \mathcal{Q}M(\mathcal{X})$ is closed under composition and is $\sim$-closed, the closure operator $\mathcal{C}$ is quasi weakly hereditary and quasi idempotent, and (QCD) holds for every $X \in \mathcal{X}$. If $\mathcal{E}_{QC} \subseteq \mathcal{Q}E(\mathcal{X})$ and $\mathcal{E}_{QC} \subseteq \mathcal{V} \mathcal{M}$, then $(\mathcal{E}_{QC}, \mathcal{M}_{QC})$ is a quasi factorization structure in $\mathcal{X}$.

**Proof.** Let $f = mfe_f$ be a quasi right $\mathcal{M}$-factorization of $f$. By Theorems 4.4 and 4.11, the factorization $f = c_Y(m_f)(j_m e_f)$ is both quasi right $\mathcal{M}_{QC}$-factorization and quasi left $\mathcal{E}_{QC}$-factorization of $f$. By Propositions 5.3 and 5.4, $(\mathcal{E}_{QC}, \mathcal{M}_{QC})$ is a quasi factorization structure in $\mathcal{X}$. □
In [10], it is proved that if the category $\mathcal{X}$ has $(\mathcal{E}, \mathcal{M})$-factorization structures and $\mathcal{C}$ is a closure operator on $\mathcal{X}$, then $\mathcal{C}$ is idempotent and weakly hereditary if and only if $\mathcal{X}$ has $(\mathcal{E}^C, \mathcal{M}^C)$-factorizations. In the following we prove a similar result under weaker conditions.

**Theorem 5.13.** Suppose that $\mathcal{M} \subseteq QM(\mathcal{X})$ has $\mathcal{X}$-pullbacks, is closed under composition and $(\mathcal{E}, \mathcal{M})$ is a quasi factorization structure.

(i) Let $\mathcal{M}$ be closed under composition with isomorphisms and the closure operator $\mathcal{C}$ be quasi weakly hereditary and quasi idempotent, and $(QCD)$ holds for every $X \in \mathcal{X}$. Then $(\mathcal{E}^{QC}, \mathcal{M}^{QC})$ is a quasi factorization structure if and only if $\mathcal{M}^{QC} \subseteq (\overline{\mathcal{M}^{QC}})^\Delta$.

(ii) If $(\mathcal{E}^{QC}, \mathcal{M}^{QC})$ is a quasi factorization structure, then $\mathcal{M}^{QC}$ is closed under composition and $\mathcal{C}$ satisfies the property $(QCD)$.

**Proof.** (i) Let $f = me$ be a quasi factorization of $f$, where $e \in \mathcal{E}$ and $m \in \mathcal{M}$. By Propositions 5.11 and 5.5, $e \in \mathcal{E}^{QC}$. Since $\mathcal{C}$ is quasi weakly hereditary, there exists a quasi $\mathcal{C}$-dense morphism $j_m : M \longrightarrow c_Y(M)$ such that $c_Y(m)j_m = m$. Also

$$X \xrightarrow{f} Y = X \xrightarrow{j_mE} c_Y(M) \xrightarrow{c_Y(m)} Y,$$

where $c_Y(m) \in \mathcal{M}^{QC}$. Put $d \overset{def}{=} j_mE$. Since $\mathcal{E}^{QC}$ is closed under composition, $d \in \mathcal{E}^{QC}$. Thus every morphism $f$ has a factorization such that its left part is in $\mathcal{E}^{QC}$ and its right part is in $\mathcal{M}^{QC}$. Now we show that $\mathcal{E}^{QC} = \overline{\mathcal{M}^{QC}}$. By 5.3, $\mathcal{E}^{QC} \subseteq \overline{\mathcal{M}^{QC}}$. Let $h \in \overline{\mathcal{M}^{QC}}$. Thus there exist morphisms $e_1 \in \mathcal{E}^{QC}$ and $n_1 \in \mathcal{M}^{QC}$ such that

$$N \xrightarrow{h} E = N \xrightarrow{e_1} h(N) \xrightarrow{n_1} E.$$

Thus there is a morphism $w_1$ as in the following diagram

![Diagram](attachment:diagram.png)

and hence $n_1w_1 \sim 1_E$. This implies that $n_1$ is a split epi and since $n_1 \in \mathcal{M}$, $n_1$ is an isomorphism. Thus by 3.9(a), $h \in \mathcal{E}^{QC}$ and hence

$$\mathcal{E}^{QC} = \overline{\mathcal{M}^{QC}}. \quad (5.1)$$
Now, if \((\mathcal{E}^{QC}, \mathcal{M}^{QC})\) is a quasi factorization structure, since \(\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^\Delta\) by equality (6), \(\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^{\Delta}\). Converely, if \(\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^{\Delta}\), then by equality (6), \(\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^\Delta\) and so \((\mathcal{E}^{QC}, \mathcal{M}^{QC})\) is a quasi factorization structure.

(ii) Consider morphisms \(M \xrightarrow{m_1} X \xrightarrow{m_2} Y\) such that \(m_1, m_2 \in \mathcal{M}^{QC}\) and

\[
M \xrightarrow{m_2m_1} Y = M \xrightarrow{e} N \xrightarrow{n} Y
\]

is a \((\mathcal{E}^{QC}, \mathcal{M}^{QC})\)-factorization of \(m_2m_1\). Consider the following pullback diagram

\[
\begin{array}{ccc}
M & \xrightarrow{m_1} & X \\
\downarrow{e} & \nearrow{s} & \downarrow{n*} \\
L & \xrightarrow{m_2} & N \\
\end{array}
\]

3.8 implies that \(m_2^* \in \mathcal{M}^{QC}\) and since \(\mathcal{E}^{QC} \subseteq (\mathcal{M}^{QC})^{\Delta}\), there exists \(s_1 : N \rightarrow L\) such that \(m_2^*s_1 \sim 1_N\). Therefore \(m_2^*\) is a split epi and since \(m_2^* \in \mathcal{M}\), \(m_2^*\) is an isomorphism. Also \(n^*(m_2^*)^{-1}e = n^*(m_2^*)^{-1}m_2s = n*s = m_1\) and since \(\mathcal{M}^{QC} \subseteq (\mathcal{E}^{QC})^\Delta\), we have the following diagram

\[
\begin{array}{ccc}
N & \xrightarrow{n^*(m_2^*)^{-1}} & X \\
\downarrow{e} & & \downarrow{m_1} \\
M & & \\
\end{array}
\]

such that \(m_1w_1' \sim n^*(m_2^*)^{-1}\) and hence there exists \(l : N \rightarrow N\) such that \(m_1w_1'l = n^*(m_2^*)^{-1}\). Thus \(m_2m_1(w_1'l) = m_2n^*(m_2^*)^{-1} = n\) and hence \(n \leq m_2m_1\). Since \(m_2m_1 = ne, m_2m_1 \leq n\) and so \(m_2m_1 \sim n\). Therefore \(m_2m_1 \in \mathcal{M}^{QC}\).

Now we prove that \(C\) satisfies \((QCD)\). Let \(X \xrightarrow{e_1} Y \xrightarrow{e_2} Z\) be morphisms such that \(e_1, e_2 \in \mathcal{E}^{QC}\). We show that \(e_2e_1 \in (\mathcal{E}^{QC})^{\Delta}\). Let \(X \xrightarrow{e_1} Y \xrightarrow{e_2} Z = X \xrightarrow{u} M \xrightarrow{m} Z\) such that
Consider the pullback diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & M \\
\searrow & & \swarrow \\
& K & \\
\downarrow & & \downarrow \\
Y & \xrightarrow{e_2} & Z
\end{array}
\]

Thus \( m^* \in \mathcal{M}^{QC} \) and since \( E^{QC} \subseteq \downarrow (\mathcal{M}^{QC}) \), there exists \( t_1 : Y \to K \) such that \( m^* \circ t_1 \sim 1_Y \). Therefore \( m^* \) is a split epi and since \( m^* \in \mathcal{M}, m^* \) is an isomorphism. Thus we have the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{k(m^*)^{-1}} & M \\
\searrow & & \nearrow \\
& Z & \\
\downarrow & & \uparrow \\
& & t_2
\end{array}
\]

such that \( mt_2 \sim 1_Z \). Thus \( e_2 e_1 \in \downarrow (\mathcal{M}^{QC}) \). It is easy to see that \( \mathcal{M}^{QC} \) is closed under composition with isomorphisms and since \( \mathcal{M}^{QC} \) has \( \mathcal{X} \)-pullbacks, by 5.11(a), \( \mathcal{X} \) has quasi right \( \mathcal{M}^{QC} \)-factorization. So, by 3.9(a), \( E^{QC} \) is closed under composition with isomorphisms. Thus, by 5.6(i), \( \downarrow (\mathcal{M}^{QC}) \subseteq E^{QC} \) and hence \( e_2 e_1 \in E^{QC} \). Therefore \( \mathcal{C} \) satisfies the property (QCD).

**Definition 5.14.** (a) A nonempty class \( \mathcal{M} \) is called a codomain if \( m \in \mathcal{M} \) and \( \langle m \rangle \subseteq \langle a \rangle \), yields \( a \in \mathcal{M} \).

(b) A morphism \( f \) is called a strongly quasi mono, whenever for every morphisms \( a, b \in \mathcal{X} \) if \( fa = fb \), then \( \langle a \rangle = \langle b \rangle \) and \( \langle a \rangle = \langle b \rangle \).

**Example 5.15.** Let \( \mathcal{C} \) be a subcategory of \( \text{Set} \) and \( X \) be an object in \( \text{Set} \) which is not in \( \mathcal{C} \). Now define the subcategory \( \mathcal{D} \) of \( \text{Set} \) to have \( \text{obj}(\mathcal{C}) \cup \{ X \} \) as objects and for all \( A, B \in \mathcal{D} \), \( \text{Hom}_\mathcal{D}(A, B) = \)


\[
\begin{align*}
\text{Hom}_C(A, B) & \quad A, B \in C \\
\text{Hom}_{\text{Set}}(A, B) & \quad A = X, B \in C \\
\emptyset & \quad A \in \mathcal{C}, B = X \\
\{ X \xrightarrow{f} X \mid f \text{ is a bijective map} \} & \quad A = X, B = X
\end{align*}
\]

For all \( A \in \mathcal{D} \), the morphisms \( f : X \to A \) are strongly quasi monos in \( \mathcal{D} \).

**Notation 5.16.** The class of all strongly quasi monos is denoted by \( \text{SQM}(\mathcal{X}) \).

Note that \( \text{Mon}(\mathcal{X}) \subseteq \text{SQM}(\mathcal{X}) \).

**Remark 5.17.** (a) If \( m \in \text{Mon}(\mathcal{X}) \) and \( \{ m \} \subseteq \{ a \} \), then \( a \in \text{Mon}(\mathcal{X}) \). Therefore \( \text{Mon}(\mathcal{X}) \) is a codomain.

(b) If \( \mathcal{M} \) is a codomain, then it is closed under composition with isomorphisms on the left.

(c) If \( \mathcal{M} \) is a codomain and \( M \xrightarrow{m} X \in \mathcal{M} \), then \( 1_M \in \mathcal{M} \). Note that for each \( X \xrightarrow{f} Y \) since \( f = f1_X \), \( \{ f \} \subseteq \{ 1_X \} \). Therefore if \( \{ M \mid M \text{ is a domain of an element of } \mathcal{M} \} = \text{Obj}(\mathcal{X}) \), then \( \mathcal{M} \) contains all the identities.

**Theorem 5.18.** Suppose that \( \mathcal{M} \subseteq \text{SQM}(\mathcal{X}) \) and it is a codomain. If \( \mathcal{C} \) is a closure operator such that \( (\mathcal{E}^{QC}, \mathcal{M}^{QC}) \) is a quasi factorization structure, then \( \mathcal{C} \) is quasi weakly hereditary and quasi idempotent.

**Proof.** Consider an \( (\mathcal{E}^{QC}, \mathcal{M}^{QC}) \) quasi factorization structure

\[
\begin{align*}
M & \xrightarrow{d} N \xrightarrow{n} X \\
\text{of } m = nd \in \mathcal{M} \text{, where } d \in \mathcal{E}^{QC} \text{ and } n \in \mathcal{M}^{QC}. \text{ So } m \leq n \text{ and hence } c_X(m) \leq n. \text{ Since } \{ m \} \subseteq \{ d \}, d \in \mathcal{M}. \text{ Consider the diagram}
\end{align*}
\]
Since $d \in \mathcal{E}^{QC} \cap \mathcal{M}$, by Proposition 4.7, $\mathfrak{c}_N(d)$ is an isomorphism. By 3.4, there exits $w : \mathfrak{c}_N(M) \longrightarrow \mathfrak{c}_X(M)$ such that $n\mathfrak{c}_N(d) = \mathfrak{c}_X(m)w$. Since $\mathfrak{c}_N(d)$ is an isomorphism, $n \leq \mathfrak{c}_X(m)$. Therefore $\mathfrak{c}_X(m) \sim n$, and hence $\mathfrak{c}_X(\mathfrak{c}_X(m)) \sim \mathfrak{c}_X(m)$. Therefore $\mathcal{C}$ is quasi idempotent. Since $\mathfrak{c}_X(m) \leq n$, there exists $\mathfrak{c}_X(M) \xrightarrow{d'} N$ such that $\mathfrak{c}_X(m) = nd'$. Therefore $n = nd'w(\mathfrak{c}_N(d))^{-1}$ and hence $\mathfrak{c}_X(m) = \mathfrak{c}_X(m)w(\mathfrak{c}_N(d))^{-1}d'$. Thus $d'w(\mathfrak{c}_N(d))^{-1} = 1_N$ and $w(\mathfrak{c}_N(d))^{-1}d' = 1_{\mathfrak{c}_X(M)}$. These equivalences imply that $w(\mathfrak{c}_N(d))^{-1}$ is an isomorphism. It follows that $w$ is an isomorphism. It is easy to see that $wjd \leq jm$. Thus, by Proposition 3.9(b), we have $jm \in \mathcal{E}^{QC}$.

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