NON-COMMUTATIVE MARTINGALE TRANSFORMS

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Abstract. We prove that non-commutative martingale transforms are of weak type \((1,1)\). More precisely, there is an absolute constant \(C\) such that if \(\mathcal{M}\) is a semi-finite von Neumann algebra and \((\mathcal{M}_n)_{n=1}^{\infty}\) is an increasing filtration of von Neumann subalgebras of \(\mathcal{M}\) then for any non-commutative martingale \(x = (x_n)_{n=1}^{\infty}\) in \(L^1(\mathcal{M})\), adapted to \((\mathcal{M}_n)_{n=1}^{\infty}\), and any sequence of signs \((\varepsilon_n)_{n=1}^{\infty}\),

\[
\left\| \varepsilon_1 x_1 + \sum_{n=2}^{N} \varepsilon_n (x_n - x_{n-1}) \right\|_{1,\infty} \leq C \|x_N\|_1
\]

for every \(N \geq 2\). This generalizes a result of Burkholder from classical martingale theory to non-commutative setting and answers positively a question of Pisier and Xu. As applications, we get the optimal order of the UMD-constants of the Schatten class \(S_p\) when \(p \to \infty\). Similarly, we prove that the UMD-constant of the finite dimensional Schatten class \(S_{\log n}\) is of order \(\log(n+1)\). We also discuss the Pisier-Xu non-commutative Burkholder-Gundy inequalities.

1. Introduction

Non-commutative (or quantum) probability has developed into an independent field of mathematical research and has received considerable progress in recent years. We refer to the books [1] and [31] for connections between mathematical physics, non-commutative probability and classical probability, the books of Voiculescu, Dykema and Nica [42] and Hiai and Petz [23] for interplay between operator algebras and free probability theory, the work of Biane and Speicher [4] on stochastic analysis and free Brownian motion.

In this paper, our main interest is on non-commutative martingales. Non-commutative martingales have been studied by several authors. For instance, pointwise convergence of non-commutative martingales was considered in [11] and [12]. In [37], Pisier and Xu proved a non-commutative analogue of the Burkholder-Gundy square function inequalities. Shortly after, Pisier [35], using combinatorial method, extended their result to a more general class of sequences called \(p\)-orthogonal sums when \(p\) is an even integer. Very recently, Junge and

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Xu [25] considered the non-tracial case of the main result of [37] along with several related inequalities such as non-commutative analogue of the classical Burkholder inequalities on the conditioned square functions among others. Junge proved in [24] non-commutative versions of Doob’s maximal inequalities. We remark that most inequalities considered in the aforementioned papers were for $p > 1$. We continue this line of research by studying martingale transforms of non-commutative bounded $L^1$-martingales. In the classical probability, the theory of martingale transforms is well-established and has been proven to be a very powerful tool not only in probabilistic situations but also in several parts of analysis. We refer to the survey [7] for discussions on this classical topic. For instance, Burkholder [6] proved that classical martingale transforms are of weak type $(1,1)$. Our main result (see Theorem 3.1 below) is a non-commutative analogue of this classical fact: non-commutative martingale transforms are bounded as maps from non-commutative $L^1$-spaces into the corresponding non-commutative weak-$L^1$-spaces. We should point out that this question was explicitly raised by Pisier and Xu in the recent survey [38] (Problem 7.5) as it is closely related to the main result of [37]. Indeed, combined with general theory of interpolations of operators of weak types, our main result implies that for $p > 1$, martingale difference sequences in non-commutative $L^p$-spaces are unconditional which in turn imply the non-commutative Burkholder-Gundy inequalities. This alternative approach yields constants which are $O(p)$ when $p \to \infty$. This is explained in Sect. 5. Another application of the main result is on UMD-constants of non-commutative $L^p$-spaces. It is now a well known fact that non-commutative $L^p$-spaces on semi-finite von Neumann algebras are UMD-spaces. The UMD-constants of these spaces recorded in the literature thus far seems to be of order $O(p^2)$ when $p \to \infty$. Using the estimates on the constant of unconditionality of non-commutative martingale difference sequences, we can deduce that the UMD-constants for non-commutative $L^p$-spaces are of order $O(p)$ when $p \to \infty$. We refer to Sect. 4 below for more discussion on this along with some related results.

The study of martingales in non-commutative cases often requires additional insights. In fact, most of usual techniques used in the classical case are relaying on stopping times or some other basic truncations which, in many situations, are not available for the non-commutative setting. Our proof is completely self-contained. It is based on a maximal inequality type result from a paper of Cuculescu [11] (see Proposition 2.4 below) which allows ones to reduce the case of bounded $L^1$-martingales to bounded $L^2$-supermartingales. Although, such reduction to supermartingales is standard in classical martingale theory (see for instance [18, Chap. 5]), the non-commutative setting presents considerable additional technical difficulty and therefore requires special care.
The paper is organized as follows: in Sect. 2 below, we set some basic preliminary background concerning non-commutative spaces and martingale theory that will be needed throughout. Sect. 3 is devoted mainly to the statement and proof of the main result. In Sect. 4, we discuss the UMD-constants of non-commutative spaces. As mentioned above, we revisit the non-commutative Burkholder inequalities with special attention given to the order of growths of the constants involved in Sect. 5 and in the last section, we discuss the class $L \log L$ and formulate some related open questions.

Our notation and terminology are standard as may be found in the books [27] and [40].

2. Preliminaries

Let $\mathcal{M}$ be a semi-finite von Neumann algebra with a normal faithful semi-finite trace $\tau$. For $1 \leq p \leq \infty$, let $L^p(\mathcal{M}, \tau)$ be the associated non-commutative $L^p$-space. Note that if $p = \infty$, $L^\infty(\mathcal{M}, \tau)$ is just $\mathcal{M}$ with the usual operator norm; also recall that for $1 \leq p < \infty$, the norm on $L^p(\mathcal{M}, \tau)$ is defined by

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L^p(\mathcal{M}, \tau),$$

where $|x| = (x^*x)^{1/2}$ is the usual modulus of $x$.

In order to describe all the spaces involved in this paper, we recall the general construction of non-commutative spaces as sets of densely defined operators on a Hilbert space. Throughout, $H$ will denote a Hilbert space and $\mathcal{M} \subseteq B(H)$. The identity element of $\mathcal{M}$ is denoted by $1$. A closed densely defined operator $a$ on $H$ is said to be affiliated with $\mathcal{M}$ if $u^*au = a$ for all unitary $u$ in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If $a$ is a densely defined self-adjoint operator on $H$, and if $a = \int_{-\infty}^\infty sde_s a$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(a)$ the corresponding spectral projection $\int_{-\infty}^\infty \chi_B(s)de_s a$. A closed densely defined operator $a$ on $H$ affiliated with $\mathcal{M}$ is said to be $\tau$-measurable if there exists a number $s \geq 0$ such that $\tau(\chi(s,\infty)(|a|)) < \infty$.

The set of all $\tau$-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation [32]. For $x \in \overline{\mathcal{M}}$, the generalized singular value function $\mu(x)$ of $x$ is defined by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(\chi(s,\infty)(|x|)) \leq t\}, \quad \text{for } t \geq 0.$$ 

The function $t \to \mu_t(x)$ from $(0, \tau(1))$ to $[0, \infty)$ is right continuous, non-increasing and is the inverse of the distribution function $\lambda(x)$, where $\lambda_s(x) = \tau(\chi(s,\infty)(|x|))$, for $s \geq 0$. For a complete study of $\mu(\cdot)$ and $\lambda(\cdot)$, we refer to [19]. For the definition below, we refer the reader to [2] and [28] for the theory of rearrangement invariant function spaces.
Definition 2.1. Let $E$ be a rearrangement invariant (quasi-) Banach function space on $(0, \tau(1))$. We define the symmetric space $E(\mathcal{M}, \tau)$ of measurable operators by setting:

$$E(\mathcal{M}, \tau) = \{ x \in \overline{\mathcal{M}} : \mu(x) \in E \}$$

and

$$\|x\|_{E(\mathcal{M}, \tau)} = \|\mu(x)\|_E, \text{ for } x \in E(\mathcal{M}, \tau).$$

It is well known that $E(\mathcal{M}, \tau)$ is a Banach space (resp. quasi-Banach space) if $E$ is a Banach space (resp. quasi-Banach space). The space $E(\mathcal{M}, \tau)$ is often referred to as the non-commutative analogue of the function space $E$ and if $E = L^p(0, \tau(1))$, for $0 < p \leq \infty$, then $E(\mathcal{M}, \tau)$ coincides with the usual non-commutative $L^p$-space associated with $(\mathcal{M}, \tau)$. We refer to [10], [14], [15] and [43] for more detailed discussions about these spaces.

We now recall the general setup for martingales. The reader is referred to [17] and [20] for the classical martingale theory. Let $(\mathcal{M}_n)_{n=1}^{\infty}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n$’s is weak$^*$-dense in $\mathcal{M}$. For each $n \geq 1$, assume that there is a conditional expectation $E_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ satisfying:

(i) $E_n(axb) = aE_n(x)b$ for all $a, b \in \mathcal{M}_n$ and $x \in \mathcal{M}$;

(ii) $\tau \circ E_n = \tau$.

It is clear that for every $m$ and $n$ in $\mathbb{N}$, $E_m E_n = E_n E_m = E_{\min(m,n)}$. Since $E_n$ is trace preserving, it extends to a contractive projection from $L^p(\mathcal{M}, \tau)$ onto $L^p(\mathcal{M}_n, \tau_n)$ for all $1 \leq p \leq \infty$ where $\tau_n$ is the restriction of $\tau$ on $\mathcal{M}_n$. More generally, a simple interpolation argument would prove that if $E$ is a rearrangement invariant Banach function space on $(0, \tau(1))$ then $E_n$ is a contraction from $E(\mathcal{M}, \tau)$ onto $E(\mathcal{M}_n, \tau_n)$.

Remark that if $\mathcal{M}$ is finite, such conditional expectations always exist. Indeed, if $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$. The embedding $\iota : L^1(\mathcal{N}, \tau) \rightarrow L^1(\mathcal{M}, \tau)$ is an isometry and the dual map $\mathcal{E} = \iota^* : \mathcal{M} \rightarrow \mathcal{N}$ yields a conditional expectation (see for instance, [40, Theorem 3.4]).

Definition 2.2. A non-commutative martingale with respect to the filtration $(\mathcal{M}_n)_{n=1}^{\infty}$ is a sequence $x = (x_n)_{n=1}^{\infty}$ in $L^1(\mathcal{M}, \tau)$ such that:

$$E_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1.$$
If additionally \( x \in L^p(\mathcal{M}, \tau) \) then \( x \) is called a \( L^p \)-martingale. In this case, we set
\[
\|x\|_p = \sup_{n \geq 1} \|x_n\|_p.
\]
If \( \|x\|_p < \infty \), then \( x \) is called a bounded \( L^p \)-martingale. The difference sequence of a martingale \( x \) is defined as \( dx = (dx_n)_{n=1}^{\infty} \) with \( dx_1 = x_1 \) and \( dx_n = x_n - x_{n-1} \) for \( n \geq 2 \).

Recall that a subset \( K \) of \( L^1(\mathcal{M}, \tau) \) is said to be uniformly integrable if it is bounded and for every sequence of projections \((p_n)_{n=1}^{\infty}\) with \( p_n \downarrow 0 \), we have \( \lim_{n \to \infty} \sup\{\|p_n h p_n\|_1; h \in K\} = 0 \). It is clear that a martingale \( x = (x_n)_{n=1}^{\infty} \) in \( L^1(\mathcal{M}, \tau) \) is uniformly integrable if and only if there exists \( x_\infty \in L^1(\mathcal{M}, \tau) \) such that \( x_n = \mathcal{E}_n(x_\infty) \) for all \( n \geq 1 \). In this case, the sequence \((x_n)_{n=1}^{\infty}\) converges to \( x_\infty \) in \( L^1(\mathcal{M}, \tau) \). Similarly, if \( 1 < p < \infty \), every bounded \( L^p \)-martingale is of the form \((\mathcal{E}_n(x_\infty))_{n=1}^{\infty}\) for some \( x_\infty \in L^p(\mathcal{M}, \tau) \).

The following decomposition of bounded \( L^1 \)-martingale is the non-commutative extension of the classical Krickeberg’s decomposition of martingales into linear combinations of positive martingales. It will be used in the sequel. A proof for the finite case can be found in [11] but the general case is readily verified with the same techniques.

**Theorem 2.3.** Let \((x_n)_{n=1}^{\infty}\) be a bounded \( L^1 \)-martingale then \((x_n)_{n=1}^{\infty}\) admits the following decomposition:
\[
x_n = (x_n^{(1)} - x_n^{(2)}) + i(x_n^{(3)} - x_n^{(4)})
\]
for all \( n \geq 1 \) where for each \( j \in \{1, 2, 3, 4\} \), the sequence \((x_n^{(j)})_{n=1}^{\infty}\) is a positive martingale. Moreover, if \( x_n = x_n^* \), for all \( n \geq 1 \), then \( \sup_{n \geq 1} \|x_n\|_1 = \tau(x_1^{(1)}) + \tau(x_1^{(2)}) \).

We end this section with a maximal inequality type result. Inspired by Pisier’s vector-valued non-commutative \( L^p \)-spaces, Junge [24] developed an abstract situation that can efficiently describe a non-commutative analogue of the maximal function theory for bounded \( L^p \)-martingales when \( p > 1 \). The proposition below can be viewed as a substitute for the classical weak type \((1,1)\) boundedness of maximal functions. Since it was not presented in the form below and plays a crucial role in the proof of our main result, we will reproduce the proof given in [11].

**Proposition 2.4.** If \((x_n)_{n=1}^{\infty}\) is a positive bounded \( L^1 \)-martingale and \( \lambda > 0 \) then there exists a sequence of decreasing projections \((q_n^{(\lambda)})_{n=1}^{\infty}\) in \( \mathcal{M} \) with:

(i) for every \( n \geq 1 \), \( q_n^{(\lambda)} \in \mathcal{M}_n \);
(ii) \( q_n^{(\lambda)} \) commutes with \( q_n^{(\lambda)} x_n q_n^{(\lambda)} \);
(iii) \( q_n^{(\lambda)} x_n q_n^{(\lambda)} \leq \lambda q_n^{(\lambda)} \).
(iv) $(q_n^{(\lambda)})_{n=1}^{\infty}$ is a decreasing sequence and if we set $q^{(\lambda)} = \bigwedge_{n=1}^{\infty} q_n^{(\lambda)}$ then $\tau(1 - q^{(\lambda)}) \leq \tau(x_1)/\lambda$.

Proof. Let $q_0^{(\lambda)} = 1$ and inductively on $n \geq 1$, define

$$q_n^{(\lambda)} := \chi_{[0,\lambda]}(q_{n-1}^{(\lambda)}x_n q_{n-1}^{(\lambda)}).$$

The above definition makes sense since $q_{n-1}^{(\lambda)}x_n q_{n-1}^{(\lambda)}$ is a positive operator. It is clear (by induction) that for every $n \geq 1$, $q_n^{(\lambda)} \in \mathcal{M}_n$. Moreover, condition (ii) follows directly from the definition of $q_n^{(\lambda)}$ above.

For (iii), $q_n^{(\lambda)} x_n q_n^{(\lambda)} = q_n^{(\lambda)}(q_{n-1}^{(\lambda)}x_n q_{n-1}^{(\lambda)}) q_n^{(\lambda)} = \chi_{[0,\lambda]}(q_{n-1}^{(\lambda)}x_n q_{n-1}^{(\lambda)}).$ $q_{n-1}^{(\lambda)}x_n q_{n-1}^{(\lambda)} \leq \lambda q_n^{(\lambda)}$. For (iv), it is clear that $(q_n^{(\lambda)})_{n=1}^{\infty}$ is decreasing and for every fixed $n \geq 1$,

$$\tau(x_1) = \tau(x_n) = \tau(x_n q_n^{(\lambda)}) + \sum_{k=1}^{n} \tau(x_n(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})) = \tau(x_n q_n^{(\lambda)}) + \sum_{k=1}^{n} \tau(\mathcal{E}_k(x_n)(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})).$$

Since $\tau(q_n^{(\lambda)} x_n q_n^{(\lambda)}) \geq 0$, we have

$$\tau(x_1) \geq \sum_{k=1}^{n} \tau((q_{k-1}^{(\lambda)} - q_k^{(\lambda)})x_k(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})) = \sum_{k=1}^{n} \tau((q_{k-1}^{(\lambda)} - q_k^{(\lambda)})(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)})(q_{k-1}^{(\lambda)} - q_k^{(\lambda}))).$$

From the definition of $q_k^{(\lambda)}$, it is clear that $q_{k-1}^{(\lambda)} - q_k^{(\lambda)} = \chi_{(\lambda,\infty)}(q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)})$ and therefore $(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})q_{k-1}^{(\lambda)}x_k q_{k-1}^{(\lambda)}(q_{k-1}^{(\lambda)} - q_k^{(\lambda)}) \geq \lambda(q_{k-1}^{(\lambda)} - q_k^{(\lambda)})$ hence,

$$\tau(x_1) \geq \lambda \sum_{k=1}^{n} \tau(q_{k-1}^{(\lambda)} - q_k^{(\lambda)}) = \lambda \tau(1 - q_n^{(\lambda)}).$$

Taking the limit as $n$ goes to $\infty$, (iv) follows. This completes the proof.

3. Main Result

In this section, we keep all notations introduced in the preliminaries. In particular, all adapted sequences are understood to be with respect to a fixed filtration of von Neumann subalgebras. The following theorem answers positively a question raised by Pisier and Xu [38, Problem 7.5] and is the main result of this paper.
Theorem 3.1. There is an absolute constant $C$ such that if $x = (x_n)_{n=1}^\infty$ is a bounded $L^1$-martingale and $(\xi_n)_{n=1}^\infty$ is an adapted sequence such that:

(i) for every $n \geq 2$, $\xi_{n-1}$ commutes with $M_n$;
(ii) $\sup_{n \geq 1} \|\xi_n\|_\infty \leq 1$.

Then for every $N \geq 2$,

$$\left\| x_1 + \sum_{k=2}^N \xi_{k-1} dx_k \right\|_{1,\infty} \leq C \|x_N\|_1.$$

Proof of Theorem 3.1: By Theorem 2.3, it is enough to prove the case where $(x_n)_{n=1}^\infty$ is a positive martingale and $(\xi_n)_{n=1}^\infty$ is an adapted sequence of self-adjoint operators satisfying conditions (i) and (ii). In the course of the proof, we will frequently use the tracial property of $\tau$ and the $\tau$-invariance property of the expectations $E_n$'s. For notational purpose, we set $\xi_0 = 1$.

Our goal is to show that there is a constant $C$, independent of $(x_n)_{n=1}^\infty$ and $(\xi_n)_{n=1}^\infty$, such that for every $\lambda > 0$ and $N \geq 2$,

$$\tau \left( \chi_{(\lambda,\infty)} \left( \left\| \sum_{k=1}^N \xi_{k-1} dx_k \right\|_1 \right) \right) \leq \frac{C}{\lambda} \|x_N\|_1.$$

The proof is divided into several steps:

Step 1. (Reduction to bounded difference sequences). Fix $\lambda > 0$ and denote simply by $(q_n)_{n=1}^\infty$ (resp. $q$) the projections $(q^{(\lambda)}_n)_{n=1}^\infty$ (resp. $q^{(\lambda)}$) from Proposition 2.4 and let $N \geq 2$ be fixed throughout the proof.

Lemma 3.2. For every $\alpha \in (0,1)$ and every $\beta \in (0,1),$

$$\tau \left( \chi_{(\lambda,\infty)} \left( \left\| \sum_{k=1}^N \xi_{k-1} dx_k \right\|_1 \right) \right) \leq \alpha^{-1} \tau \left( \chi_{(\beta\lambda,\infty)} \left( \left\| \sum_{k=1}^N q^\lambda \xi_{k-1} dx_k q^\lambda \right\|_1 \right) \right) + \frac{2(1-\alpha)^{-1}}{\lambda} \tau(x_1).$$

Proof. We begin by splitting the operator $S = \sum_{k=1}^N \xi_{k-1} dx_k$ into three parts:

$$S = qSq + (1-q)Sq + S(1-q).$$
Fix $\alpha \in (0,1)$ and $\beta \in (0,1)$. Using properties of the generalized singular value functions $\mu(\cdot)$ from [19],

$$\tau (\chi_{(\lambda,\infty)}(|S|)) = \int_0^\infty x_{(\lambda,\infty)} \{ \mu_t(S) \} \, dt$$

$$\leq \int_0^\infty x_{(\lambda,\infty)} \left\{ \mu_{at}(qS\alpha) + \mu_{(1-a)t/2}(1-q)S + \mu_{(1-a)t/2}(1-q)S(1-q) \right\} \, dt$$

$$= \int_0^\infty x_{(\lambda,\infty)} \left\{ \mu_{at}(qS\alpha) + \mu_{(1-a)t/2}(qS(1-q)) + \mu_{(1-a)t/2}(1-q)S(1-q) \right\} \, dt.$$

As $\mu_{(1-a)t/2}(qS(1-q)) \leq \mu_{(1-a)t/2}(1-q)|S(1-q)|$,

$$\tau (\chi_{(\lambda,\infty)}(|S|)) \leq \int_0^\infty x_{(\lambda,\infty)} \left\{ \mu_{at}(qS\alpha) + 2\mu_{(1-a)t/2}(1-q)|S(1-q)| \right\} \, dt$$

$$\leq \int_0^\infty x_{(\lambda,\infty)} \left\{ \mu_{at}(qS\alpha) \right\} \, dt + \int_0^\infty x_{(\lambda,\infty)} \left\{ \mu_{(1-a)t/2}(qS(1-q)) |S(1-q)| \right\} \, dt$$

$$\leq \int_0^\infty \mu_t \left\{ x_{(\lambda,\infty)}(|S|) \right\} \, dt + \int_0^\infty \mu_{(1-a)t/2} \left\{ x_{(\lambda,\infty)}(|S(1-q)|) \right\} \, dt.$$

Remark that the projection $x_{((1-\beta)(\lambda,\infty))(2|S(1-q)|)}$ is a subprojection of $(1-q)$ so

$$\tau (\chi_{(\lambda,\infty)}(|S|)) \leq \int_0^\infty \mu_t \left\{ x_{(\beta,\lambda,\infty)}(|S(1-q)|) \right\} \, dt \quad \text{and by change of variables,}$$

$$\tau (\chi_{(\lambda,\infty)}(|S|)) \leq \alpha^{-1} \int_0^\infty \mu_t \left\{ x_{(\beta,\lambda,\infty)}(|S(1-q)|) \right\} \, dt + 2(1-\alpha)^{-1} \int_0^\infty \mu_t(1-q) \, dt$$

which shows that $\tau (\chi_{(\lambda,\infty)}(|S|)) \leq \alpha^{-1} \tau (\chi_{(\beta,\lambda,\infty)}(|S(1-q)|)) + 2(1-\alpha)^{-1} \tau (x_1)/\lambda.$

Step 2. (Reduction to difference sequence of a supermartingale in $L^2(\mathcal{M}, \tau)$).

Lemma 3.3. The sequence $(q_k x_k q_k)_{k=1}^\infty$ is a supermartingale in $L^2(\mathcal{M}, \tau)$ and for every $\beta \in (0,1)$,

$$\tau \left( \chi_{(\beta,\lambda,\infty)} \left( \left| \sum_{k=1}^N q\xi_{k-1}dx_k q \right| \right) \right) \leq \frac{1}{\beta^2 \lambda^2} \left\| q_1 x_1 q_1 + \sum_{k=2}^N \xi_{k-1}(q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) \right\|_2^2.$$  

Proof. We remark first that since both sequences $(q_k)_{k=1}^\infty$ and $(x_k)_{k=1}^\infty$ are adapted, it is clear that $(q_k x_k q_k)_{k=1}^\infty$ is adapted. To prove that it is a supermartingale, we need to verify that for every $k \geq 2$, $\mathcal{E}_{k-1}(q_k x_k q_k) \leq q_{k-1} x_{k-1} q_{k-1}$. For this, we remark from Proposition 2.4 that
Likewise, \( \mathcal{E}_{k-1}(q_k x_k q_k) \leq \mathcal{E}_{k-1}(q_{k-1} x_k q_{k-1}) \)
\( = q_{k-1} \mathcal{E}_{k-1}(x_k) q_{k-1} \)
\( = q_{k-1} x_{k-1} q_{k-1} \).

For the second part of the lemma, it is clear that

\[
\tau \left( \chi_{(\beta, \infty)} \left( \sum_{k=1}^{N} q_{k-1} d x_k q \right) \right) \leq \frac{1}{\beta^2 \lambda^2} \tau \left( \sum_{k=1}^{N} q_{k-1} d x_k q \right)^2.
\]

Moreover, \( q_{k-1} d x_k q = q_{k-1} x_k q - q_{k-1} x_{k-1} q = q(q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1})q \). Since \( \xi_{k-1} \) commutes with \( q_k \) and \( q_{k-1} \), we conclude that

\[
q_{k-1} d x_k q = q \left( \xi_{k-1}(q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) \right)q.
\]

Similarly, \( q d x_1 q = q(q_1 x_1 q_1)q \) and therefore,

\[
\tau \left( \sum_{k=1}^{N} q_{k-1} d x_k q \right)^2 = \left\| \sum_{k=1}^{N} q_{k-1} d x_k q \right\|^2_2
\]
\( = \left\| q \left( q_1 x_1 q_1 + \sum_{k=2}^{N} \xi_{k-1}(q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) \right)q \right\|^2_2
\]
\( \leq \left\| q_1 x_1 q_1 + \sum_{k=2}^{N} \xi_{k-1}(q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) \right\|^2_2.
\]

This proves the lemma.

\[\square\]

Step 3. (Change the supermartingale into sum of a martingale and a decreasing sequence of operators). This is very standard: Define

\[
y_k := \begin{cases} 
q_1 x_1 q_1 
& \text{for } k = 1 \\
q_k x_k q_k + \sum_{l=1}^{k-1} q_l x_l q_l - \mathcal{E}_l(q_{l+1} x_{l+1} q_{l+1}) 
& \text{for } k \geq 2.
\end{cases}
\]

Likewise,

\[
z_k := \begin{cases} 
0 
& \text{for } k = 1 \\
\sum_{l=1}^{k-1} \mathcal{E}_l(q_{l+1} x_{l+1} q_{l+1}) - q_l x_l q_l 
& \text{for } k \geq 2.
\end{cases}
\]
It is clear that \((y_k)_{k=1}^\infty\) is a positive martingale. Moreover, for every \(k \geq 1\),
\[
y_k + z_k = q_k x_k q_k
\]
and for every \(k \geq 2\),
\[
z_k \leq z_{k-1} \leq \cdots \leq z_1 = 0.
\]

**Lemma 3.4.** The sequence \((y_k)_{k=1}^\infty\) is a bounded \(L^2\)-martingale with
\[
\|y_N\|_2^2 \leq 6 \lambda \tau (q_1 x_N) - 4 \lambda \tau (q_N x_N) - \|q_1 x_1 q_1\|_2^2 \leq 6 \lambda \tau (x_1).
\]

**Proof.** We will use the identity \(\|y_N\|_2^2 = \|y_1^2 + \sum_{k=2}^{N} (y_k - y_{k-1})^2\|_2^2\). The main idea is to estimate the sum \(\sum_{k=2}^{N} \|y_k - y_{k-1}\|_2^2\) by a telescopic sum. For \(k \geq 2\), we notice first from (3.3) that \(y_k = y_{k-1} + q_k x_k q_k - \mathcal{E}_{k-1}(q_k x_k q_k)\) and therefore
\[
y_k - y_{k-1} = q_k x_k q_k - \mathcal{E}_{k-1}(q_k x_k q_k) = (q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) + (q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k)).
\]
Since \(\| \cdot \|_2^2\) is convex,
\[
\|y_k - y_{k-1}\|_2^2 \leq 2 \left( \|q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}\|_2^2 + \|q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k)\|_2^2 \right)
= 2 \tau \left( (q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1})^2 \right) + 2 \tau \left( (q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k))^2 \right)
= I + II.
\]
We will estimate \(I\) and \(II\) separately. First for \(I\), we use the identity \((a - b)^2 = a^2 - b^2 + b(b-a) + (b-a)b\) for self-adjoint operators. With \(a = q_k x_k q_k\) and \(b = q_{k-1} x_{k-1} q_{k-1}\), we have by taking the trace,
\[
I = 2 \tau \left( (q_k x_k q_k)^2 - (q_{k-1} x_{k-1} q_{k-1})^2 \right) + 4 \tau \left( q_{k-1} x_{k-1} q_{k-1} [q_{k-1} x_{k-1} q_{k-1} - q_k x_k q_k] \right)
= 2 \tau \left( (q_k x_k q_k)^2 - (q_{k-1} x_{k-1} q_{k-1})^2 \right) + 4 \tau \left( q_{k-1} x_{k-1} q_{k-1} [q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k)] \right).
\]
By Proposition 2.4 (iii), \(\|q_{k-1} x_{k-1} q_{k-1}\|_\infty \leq \lambda\). Moreover, as \((q_k x_k q_k)_{k=1}^\infty\) is a supermartingale, \(q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k) \geq 0\). Therefore, we get
\[
I \leq 2 \tau \left( (q_k x_k q_k)^2 - (q_{k-1} x_{k-1} q_{k-1})^2 \right) + 4 \lambda \tau \left( q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k) \right)
= 2 \tau \left( (q_k x_k q_k)^2 - (q_{k-1} x_{k-1} q_{k-1})^2 \right) + 4 \lambda \tau \left( q_{k-1} x_{k-1} q_{k-1} - q_k x_k q_k \right).
\]
For \(II\), again since \(q_{k-1} x_{k-1} q_{k-1} \geq q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k) \geq 0\), we have
\[
\|q_{k-1} x_{k-1} q_{k-1} - \mathcal{E}_{k-1}(q_k x_k q_k)\|_\infty \leq \|q_{k-1} x_{k-1} q_{k-1}\|_\infty \leq \lambda.
\]
Hence, we get

\[ II \leq 2\lambda\tau \left( q_{k-1}x_{k-1}q_k - \varepsilon_{k-1}(q_kx_kq_k) \right) \]

\[ = 2\lambda\tau \left( q_{k-1}x_{k-1}q_k - q_kx_kq_k \right). \]

Combining the preceding estimates on \( I \) and \( II \), we conclude that for every \( k \geq 2 \),

\[ \|y_k - y_{k-1}\|^2 \leq 2 \left( \|q_kx_kq_k\|^2 - \|q_{k-1}x_{k-1}q_{k-1}\|^2 \right) + 6\lambda\tau \left( q_{k-1}x_{k-1}q_k - q_kx_kq_k \right). \]

To conclude the proof of the lemma, we take the summation over \( k \),

\[ \|y_N\|_2^2 = \|q_1x_1\|_2^2 + \sum_{k=2}^{N} \|y_k - y_{k-1}\|^2 \]

\[ \leq \|q_1x_1\|_2^2 + 2\sum_{k=2}^{N} \left( \|q_kx_kq_k\|^2 - \|q_{k-1}x_{k-1}q_{k-1}\|^2 \right) \]

\[ + 6\lambda\sum_{k=2}^{N} \tau \left( q_{k-1}x_{k-1}q_k - q_kx_kq_k \right) \]

\[ = \|q_1x_1\|_2^2 + 2(\|q_Nx_Nq_N\|^2 - \|q_1x_1\|_2^2) + 6\lambda\tau \left(q_1x_1q_k - q_Nx_Nq_N\right) \]

\[ = 2\|q_Nx_Nq_N\|^2 - \|q_1x_1\|_2^2 + 6\lambda\tau \left((q_1 - q_N)x_N\right) \]

\[ \leq 2\lambda\tau \left(q_Nx_N\right) - \|q_1x_1\|_2^2 + 6\lambda\tau \left((q_1 - q_N)x_N\right) \]

\[ = 6\lambda\tau \left(q_Nx_N\right) - \|q_1x_1\|_2^2 \leq 6\lambda\tau (x_1) \]

which completes the proof.

\[ \square \]

Step 4. (Removal of the sequence \( (\xi_n)_{n=1}^{\infty} \) from the estimates). This is done by arguing separately on transforms of the difference sequences of \( (y_k)_{k=1}^{\infty} \) and \( (z_k)_{k=1}^{\infty} \).

**Lemma 3.5.** \( \left\| q_1x_1q_1 + \sum_{k=2}^{N} \xi_{k-1}(q_kx_kq_k - q_{k-1}x_{k-1}q_{k-1}) \right\|_2^2 \leq 4\|y_N\|_2^2. \)

**Proof.** From the definitions of \( (y_k)_{k=1}^{\infty} \) and \( (z_k)_{k=1}^{\infty} \), the convexity of \( \| \cdot \|_2^2 \) implies,

\[ \left\| q_1x_1q_1 + \sum_{k=2}^{N} \xi_{k-1}(q_kx_kq_k - q_{k-1}x_{k-1}q_{k-1}) \right\|_2^2 \]

\[ \leq 2\left\| \sum_{k=1}^{N} \xi_{k-1}dy_k \right\|_2^2 + 2\left\| \sum_{k=2}^{N} \xi_{k-1}(z_k - z_{k-1}) \right\|_2^2 \]

\[ = III + IV. \]
As in Step 3, we will estimate \( III \) and \( IV \) separately. First, since martingale transforms are clearly bounded (with constant=1) in \( L^2(M,\tau) \), it follows that

\[
III \leq 2 \left\| \sum_{k=1}^{N} d y_k \right\|_2^2 = 2 \| y_N \|_2^2
\]

which gives an upper bound of \( III \) that is independent of the sequence \( (\xi_k)_{k=1}^\infty \).

On the other hand, it is clear that

\[
IV = 2 \left\| \sum_{k=2}^{N} \xi_{k-1}(z_k - z_{k-1}) \right\|_2^2
\]

\[
= 2\tau \left( \left\| \sum_{k=2}^{N} \xi_{k-1}(z_{k-1} - z_k) \right\|_2^2 \right)
\]

\[
= 2\tau \left( \sum_{k=2}^{N} \sum_{l=2}^{N} (z_{k-1} - z_k)\xi_{k-1}\xi_{l-1}(z_{l-1} - z_l) \right)
\]

\[
= 2 \sum_{k=2}^{N} \sum_{l=2}^{N} \tau \left( (z_{k-1} - z_k)\xi_{k-1}\xi_{l-1}(z_{l-1} - z_l) \right).
\]

To estimate \( IV \), recall from (3.6) that \( z_{k-1} - z_k \) and \( z_{l-1} - z_l \) are positive operators. Assume for instance that \( k \leq l \) (the case \( l \leq k \) is handled equally) then by assumption, \( \xi_{k-1}\xi_{l-1} \) commutes with \( M_k \) so we have,

\[
(z_{k-1} - z_k)\xi_{k-1}\xi_{l-1} = (z_{k-1} - z_k)^{1/2}\xi_{k-1}\xi_{l-1}(z_{k-1} - z_k)^{1/2} \leq z_{k-1} - z_k.
\]

Therefore by taking the trace,

\[
\tau \left( (z_{k-1} - z_k)\xi_{k-1}\xi_{l-1}(z_{l-1} - z_l) \right) = \tau \left( (z_{l-1} - z_l)^{1/2}[(z_{k-1} - z_k)\xi_{k-1}\xi_{l-1}(z_{l-1} - z_l)^{1/2}] \right)
\]

\[
\leq \tau \left( (z_{l-1} - z_l)^{1/2}(z_{k-1} - z_k)(z_{l-1} - z_l)^{1/2} \right)
\]

\[
= \tau \left( (z_{k-1} - z_k)(z_{l-1} - z_l) \right).
\]
Hence, we get
\[ IV = 2 \sum_{k=2}^{N} \sum_{l=2}^{N} \tau ( (z_{k-1} - z_k) \xi_{k-1} \xi_{l-1} (z_{l-1} - z_l) ) \]
\[ \leq 2 \sum_{k=2}^{N} \sum_{l=2}^{N} \tau ( (z_{k-1} - z_k) (z_{l-1} - z_l) ) \]
\[ = 2 \tau \left( \sum_{k=2}^{N} (z_k - z_{k-1})^2 \right) \]
\[ = 2 \| z_N \|_2^2. \]

By combining the preceding estimates on III and IV, we obtain
\[ \left\| q_1 x_1 q_1 + \sum_{k=2}^{N} \xi_{k-1} (q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) \right\|_2^2 \leq 2 \| y_N \|_2^2 + 2 \| z_N \|_2^2. \]

To conclude the proof of the lemma, note from (3.5) that \( y_N - q_N x_N q_N = -z_N \geq 0 \) so \( y_N \geq -z_N \geq 0 \) which implies \( \| y_N \|_2 \geq \| z_N \|_2 \).

To complete the proof of Theorem 3.1, it is enough, as mentioned above, to verify (3.2).
This is obtained by putting together the four lemmas above. Indeed,
\[ \tau \left( \chi_{(\lambda, \infty)} \left( \sum_{k=1}^{N} \xi_{k-1} dx_k \right) \right) \leq \alpha^{-1} \tau \left( \chi_{(\beta \lambda, \infty)} \left( \sum_{k=1}^{N} q \xi_{k-1} dx_k \right) \right) + \frac{2(1 - \alpha)^{-1}}{\lambda} \tau(x_1) \]
\[ \leq \frac{\alpha^{-1}}{\beta^2 \lambda^2} \left\| q_1 x_1 q_1 + \sum_{k=2}^{N} \xi_{k-1} (q_k x_k q_k - q_{k-1} x_{k-1} q_{k-1}) \right\|_2^2 \]
\[ + \frac{2(1 - \alpha)^{-1}}{\lambda} \tau(x_1) \]
\[ \leq \frac{4\alpha^{-1} \beta^{-2}}{\lambda^2} \| y_N \|_2^2 + \frac{2(1 - \alpha)^{-1}}{\lambda} \tau(x_1) \]
\[ \leq \frac{24\alpha^{-1} \beta^{-2} + 2(1 - \alpha)^{-1}}{\lambda} \tau(x_1). \]

This shows that (3.2) is satisfied with
\[ C = \inf \left\{ 24\alpha^{-1} \beta^{-2} + 2(1 - \alpha)^{-1}; \; \alpha \in (0, 1), \; \beta \in (0, 1) \right\} = \frac{14\sqrt{3}}{3} + 28. \]
Remark 3.6. In the proof above, no significant effort was made to minimize the constant $C$ involved in Theorem 3.1. Recall that in the classical case, the sharp constant $C = 2$ is known and was obtained by Burkholder in [8]. His approach, as expected, is based on a stopping time argument which (at least at the time of this writing) does not seem to have an efficient non-commutative analogue.

Problem 3.7. Find the “sharp” constant $C$ for which Inequality (3.1) holds?

Theorem 3.1 can be extended to transforms of submartingales and supermartingales.

Corollary 3.8. There exists a constant $K$ such that if $(x_n)_{n=1}^\infty$ is either a submartingale or a supermartingale and is bounded in $L^1(M, \tau)$ then for any sequence of signs $(\varepsilon_n)_{n=1}^\infty$,

$$\sup_{N \geq 2} \left\| \varepsilon_1 x_1 + \sum_{k=2}^N \varepsilon_k (x_k - x_{k-1}) \right\|_{1, \infty} \leq K \sup_{n \geq 1} \| x_n \|_1.$$ 

Proof. We will present the proof for submartingale. As in the proof of Theorem 3.1, we split $(x_n)_{n=1}^\infty$ into sum of a martingale and an increasing sequence of positive operators. Let

$$y_k := \begin{cases} x_1 & \text{for } k = 1 \\ x_k + \sum_{l=1}^{k-1} x_l - \mathcal{E}_l(x_{l+1}) & \text{for } k \geq 2. \end{cases}$$

and

$$z_k := \begin{cases} 0 & \text{for } k = 1 \\ \sum_{l=1}^{k-1} \mathcal{E}_l(x_{l+1}) - x_l & \text{for } k \geq 2. \end{cases}$$

The following properties are immediate:

(a) $(y_k)_{k=1}^\infty$ is a martingale;
(b) for every $k \geq 1$, $y_k + z_k = x_k$;
(c) for every $k \geq 2$, $z_k \geq z_{k-1} \geq \cdots \geq z_1 = 0$.

Moreover, for every $k \geq 1$,

$$\| z_k \|_1 = \tau(z_k) = \sum_{l=1}^{k-1} \tau(\mathcal{E}_l(x_{l+1}) - x_l) = \sum_{l=1}^{k-1} \tau(x_{l+1} - x_l) = \tau(x_{k-1} - x_1) \leq 2\| x_k \|_1.$$
As above,
\[
\| x_1 + \sum_{k=2}^{N} \varepsilon_k (x_k - x_{k-1}) \|_{1,\infty} \leq 2 \left\| \sum_{k=1}^{N} \varepsilon_k dy_k \right\|_{1,\infty} + 2 \left\| \sum_{k=2}^{N} \varepsilon_k (z_k - z_{k-1}) \right\|_{1,\infty}
\]
\[
\leq 2C \left\| \sum_{k=1}^{N} dy_k \right\|_1 + 2 \left\| \sum_{k=2}^{N} \varepsilon_k (z_k - z_{k-1}) \right\|_1.
\]

It is easy to see that \(-z_N \leq \sum_{k=2}^{N} \varepsilon_k (z_k - z_{k-1}) \leq z_N\). Therefore \( \| \sum_{k=2}^{N} \varepsilon_k (z_k - z_{k-1}) \|_1 \leq \| z_N \|_1 \) and hence
\[
\| x_1 + \sum_{k=2}^{N} \varepsilon_k (x_k - x_{k-1}) \|_{1,\infty} \leq 2C \| y_N \|_1 + 2 \| z_N \|_1
\]
\[
\leq 2C \| x_N \|_1 + (2C + 2) \| z_N \|_1 \leq K \| x_N \|_1.
\]

The proof is complete. \(\square\)

As in the commutative case, Theorem 3.1 implies that if \(\tau(1) < \infty\), martingale transforms are bounded from \(L^1(\mathcal{M}, \tau)\) into \(L^p(\mathcal{M}, \tau)\) for \(0 < p < 1\).

**Corollary 3.9.** Assume that \(\tau(1) < \infty\). Under the assumption of Theorem 3.1, for every \(0 < p < 1\), there exists a constant \(K_p\) (depending only on \(p\)) such that:
\[
\| x + \sum_{k=2}^{N} \varepsilon_k d\xi_k \|_p \leq K_p \| x_N \|_1.
\]

In [37], Pisier and Xu proved, as a consequence of the non-commutative Burkholder-Gundy inequalities, a non-commutative analogue of Stein’s inequality ([37, Theorem 2.3], [39, Theorem 8 p. 103]) for \(1 < p < \infty\). Their proof reveals that what is needed is the unconditionality of martingale differences in \(L^p(\mathcal{M}, \tau)\). A slightly different proof was given by Junge and Xu ([25]) which yields a better constant. Below, we will adopt their proof together with Theorem 3.1 to get the corresponding result for \(p = 1\).

**Theorem 3.10.** There is a constant \(\gamma > 0\) such that for any finite sequence \((a_k)_{k=1}^{n}\) in \(L^1(\mathcal{M}, \tau)\),
\[
\left\| \left( \sum_{k=1}^{n} \mathcal{E}_k(a_k)^* \mathcal{E}_k(a_k) \right)^{1/2} \right\|_{1,\infty} \leq \gamma \left\| \left( \sum_{k=1}^{n} a_k^* a_k \right)^{1/2} \right\|_1.
\]
Proof. Consider the tensor product \((\mathcal{M}, \tau) \otimes (B(\ell^2_n), \sigma)\) where \(\sigma = n^{-1}tr\) is the usual normalized trace on \(B(\ell^2_n)\). For \(k \geq 1\), let \(\tilde{E}_k = E_k \otimes \text{Id}_{B(\ell^2_n)}\) be the conditional expectation from \(\mathcal{M} \otimes B(\ell^2_n)\) onto the subalgebra \(\mathcal{M}_k \otimes B(\ell^2_n)\).

Let \(A_k = na_k \otimes e_{k,1}\) for \(1 \leq k \leq n\) and \((r_j)_{j \geq 1}\) be the sequence of the Rademacher functions on \([0, 1]\). Then for any \(t \in [0, 1]\),

\[
\sum_{k=1}^{n} E_k(a_k)^* E_k(a_k) \otimes n^2 e_{1,1} = \left| \sum_{k=1}^{n} \tilde{E}_k(r_k(t)A_k) \right|^2
\]

and therefore

\[
\left\| \left( \sum_{k=1}^{n} E_k(a_k)^* E_k(a_k) \right)^{1/2} \right\|_{1, \infty} = \left\| \sum_{k=1}^{n} \tilde{E}_k(r_k(t)A_k) \right\|_{1, \infty}
\]

\[
= \left\| \sum_{k=1}^{n} \tilde{E}_n(r_k(t)A_k) - \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} (\tilde{E}_{j-1} - \tilde{E}_j)(r_k(t)A_k) \right\|_{1, \infty}.
\]

Since \(\|a + b\|_{1, \infty} \leq 2\|a\|_{1, \infty} + 2\|b\|_{1, \infty}\) for every \(a\) and \(b\) in \(L^{1, \infty}(\mathcal{M}, \tau)\),

\[
\left\| \left( \sum_{k=1}^{n} E_k(a_k)^* E_k(a_k) \right)^{1/2} \right\|_{1, \infty} \leq 2 \left\| \sum_{k=1}^{n} \tilde{E}_n(r_k(t)A_k) \right\|_{1, \infty} + 2 \left\| \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} (\tilde{E}_{j-1} - \tilde{E}_j)(r_k(t)A_k) \right\|_{1, \infty}
\]

\[
\leq 2 \left\| \sum_{k=1}^{n} \tilde{E}_n(r_k(t)A_k) \right\|_{1, \infty} + 2 \left\| \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} (\tilde{E}_{j-1} - \tilde{E}_j)(r_k(t)A_k) \right\|_{1, \infty}
\]

\[
\leq 2 \left\| \sum_{k=1}^{n} r_k(t)A_k \right\|_{1, \infty} + 2 \left\| \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} (\tilde{E}_{j-1} - \tilde{E}_j)(r_k(t)A_k) \right\|_{1, \infty}.
\]

Let \(f = \sum_{k=1}^{n-1} r_k A_k\) and consider the filtration \((\mathcal{M}_k \otimes B(\ell^2_n) \otimes L^{\infty}(\mathcal{F}_k))_{k \geq 1}\) where \(\mathcal{F}_k\) is the \(\sigma\)-field generated by \(\{r_1, r_2, \ldots, r_k\}\). Denoting by \((df_j)_{j \geq 1}\) the difference sequence of \(f\) with respect to this filtration, we have:

\[
\sum_{j=1}^{n-1} (\tilde{E}_{j-1} - \tilde{E}_j)(\sum_{k=1}^{j} r_k(t)A_k) = \sum_{j=1}^{n-1} df_{2j+1}.
\]
By Theorem 3.1, we conclude that

\[
\left\| \left( \sum_{k=1}^{n} \mathcal{E}_k(a_k^*) \mathcal{E}_k(a_k) \right)^{1/2} \right\|_{1,\infty} \leq 2 \left\| \left( \sum_{k=1}^{n} a_k^* a_k \right)^{1/2} \right\|_1 + 2C \| f \|_1
\]

\[
\leq (2 + 2C) \left\| \left( \sum_{k=1}^{n} a_k^* a_k \right)^{1/2} \right\|_1.
\]

This shows the theorem with \( \gamma \leq 2 + 2C \).

\[
\square
\]

4. Estimating UMD-constants for non-commutative spaces

In this section, we are primarily interested in UMD-constants of non-commutative spaces. Our main motivation comes mainly from a question of Pisier [33] on the order of the UMD-constants of the Schatten class \( S^p \). To this end, we began by reviewing the relevant background on UMD-spaces (UMD stands for unconditional martingale differences).

**Definition 4.1.** A Banach space \( X \) is said to have the UMD-property if for some \( p \in (1, \infty) \), there exists a constant \( C \), which depends only on \( p \) and \( X \) such that for all \( n \geq 1 \),

\[
\left\| \sum_{j=1}^{n} \varepsilon_j d_j \right\|_{L^p(X)} \leq C \left\| \sum_{j=1}^{n} d_j \right\|_{L^p(X)}
\]

for every \( X \)-valued martingale difference sequence \( (d_j)_{j=1}^{\infty} \) and \( (\varepsilon_j)_{j=1}^{\infty} \in \{-1, 1\}^N \).

Here \( L^p(X) = L^p(\Omega, \Sigma, \mu, X) \) denotes the Bochner space of all strongly measurable functions \( f \) on a probability space \( (\Omega, \Sigma, \mu) \) with values in \( X \) such that:

\[
\| f \|_{L^p(X)} := \left( \int_{\Omega} \| f(\omega) \|^p_X d\mu(\omega) \right)^{1/p} < \infty.
\]

Since we are interested in estimating the constants involved, we will make distinctions between the indices. We will denote the best constant in (4.1) by \( C_p(X) \). By duality, it is clear that \( X \) is a UMD-space if and only if \( X^* \) is a UMD-space. In this case, \( C_p(X) = C_q(X^*) \) with \( 1/p + 1/q = 1 \). For more information on UMD-spaces, we refer to [5] and [9].

**Theorem 4.2.** ([33]) Let \( X \) be a UMD-space then for any \( 1 < p, q < \infty \), there exist positive constants \( \alpha(p, q) \) and \( \beta(p, q) \) depending only on \( p \) and \( q \) such that:

\[
\alpha(p, q) C_p(X) \leq C_q(X) \leq \beta(p, q) C_p(X).
\]

In particular, for any \( p \geq 3 \), we have \( (2\sqrt{3})^{-1} C_2(X) \leq C_p(X) \leq 7pC_2(X) \).
Our main tool in this section is the unconditionality of martingale transforms on $L^p(M, \tau)$ (for $1 < p < \infty$) which follows from our main result. More precisely,

**Theorem 4.3.** Let $1 < p < \infty$. For any finite non-commutative $L^p$-martingale $x$ and any sequence of signs $(\varepsilon_n)_{n=1}^{\infty},$

$$\left\| \sum_{n \geq 1} \varepsilon_n dx_n \right\|_p \leq c_p \|x\|_p,$$

where $c_p \leq C p^2 / (p - 1)$ with $C$ being a universal constant.

The case $1 < p < 2$ follows by interpolation from Theorem 3.1 and the $L^2$-boundedness of martingale transforms. The case $2 < p < \infty$ can be deduced by duality.

**Remark 4.4.** Except for the constants, Theorem 4.3 was obtained in [37]. As $c_p \leq C p^2 / (p - 1)$, it is clear that $c_p = O(p)$ when $p \to \infty$ and $O((p - 1)^{-1})$ when $p \to 1$. These are the optimal order of growths for $c_p$.

We will apply Theorem 4.3 to estimate the UMD-constants of $L^p(M, \tau)$. It is well known that for $1 < p < \infty$, $L^p(M, \tau)$ (and in particular the Schatten class $S^p$) is a UMD-space. This was established as a consequence of the characterization of UMD-spaces due to Burkholder [9] and Bourgain [5] in terms of vector-valued Hilbert transforms ([3], [5]). Such approach gives constants that are $O(p^2)$ when $p \to \infty$. We remark that the UMD property of $L^p(M, \tau)$ also follows from the generalized Riesz projections associated with group representations which was extensively studied by Zsidó [44]. Our next result follows immediately from Theorem 4.3 and the definition of UMD-spaces. It answers positively a question from [33].

**Corollary 4.5.** There exists a constant $C$ such that for every $1 < p < \infty$,

$$C_p(L^p(M, \tau)) \leq C p^2 / (p - 1).$$

In particular, there exists a constant $C'$ such that for $p \geq 2$, $C_p(L^p(M, \tau)) \leq C' p$.

**Proof.** Let $(\Omega, \Sigma, \mu)$ be a probability space and $(d_n)_{n=1}^{\infty}$ be a $p$-integrable $L^p(M, \tau)$-valued martingale difference sequence defined on $(\Omega, \Sigma, \mu)$ relative to an increasing sequence of $\sigma$-subalgebras $(\Sigma_n)_{n=1}^{\infty}$ of $\Sigma$ with conditional expectations $(\mathbb{E}_n)_{n=1}^{\infty}$. Set $\mathcal{N} = L^\infty(\Omega, \Sigma, \mu) \otimes M$ and let $\mathcal{N}_n = L^\infty(\Omega, \Sigma_n, \mu) \otimes M$. Then the conditional expectation $\mathbb{E}_n$ from $\mathcal{N}$ onto $\mathcal{N}_n$ is given by $\mathbb{E}_n \otimes Id$. It is clear that $(d_n)_{n=1}^{\infty}$ is a non-commutative martingale difference sequence in $L^p(\mathcal{N}, \mu \otimes \tau)$ associated to the filtration $(\mathcal{N}_n)_{n=1}^{\infty}$. It is well known that $L^p(\mathcal{N}, \mu \otimes \tau)$ is isometrically isomorphic to the Bochner space $L^p(\mu, L^p(M, \tau))$. By Theorem 4.3, for every
$k \geq 1$ and $\varepsilon_n = \pm 1,$

$$\left\| \sum_{n=1}^{k} \varepsilon_n d_n \right\|_{L^p(\mu, L^p(\mathcal{M}, \tau))} = \left\| \sum_{n=1}^{k} \varepsilon_n d_n \right\|_{L^p(N, \mu \otimes \tau)} \leq c_p \left\| \sum_{n=1}^{k} d_n \right\|_{L^p(N, \mu \otimes \tau)}$$

which shows that $C_p(L^p(\mathcal{M}, \tau)) \leq c_p.$

\[\square\]

**Remarks 4.6.**

1. The preceding corollary shows in particular that $C_p(S^p)$ and $C_2(S^p)$ are $O(p)$ when $p \to \infty$.

2. Replacing $L^\infty(\Omega, \Sigma, \mu)$ by a general non-commutative probability space (in the sense of [34, p. 48]), the proof of Corollary 4.5 shows that the constant for the operator space version of UMD ($UMD_p$ property, [34, Definition 4.8]) of $L^p(\mathcal{M}, \tau)$ is also $O(p)$ when $p \to \infty$.

3. The constants relative to the boundedness of the $L^p(\mathcal{M}, \tau)$-valued Hilbert transforms are also $O(p)$ when $p \to \infty$ but this fact seems to provide only weaker estimates that $C_2(L^p(\mathcal{M}, \tau))$ is $O(p^2)$ when $p \to \infty$.

The above result can be extended to the Haagerup $L^p$-spaces associated to general von Neumann algebras (we refer to [21], [41] for in depth description of such spaces) modulo the following approximation of the Haagerup $L^p$-spaces.

**Theorem 4.7.** ([22]) Let $\mathcal{M}$ be an arbitrary von Neumann algebra and $L^p(\mathcal{M})$ be the Haagerup $L^p$-space associated with $\mathcal{M}$ ($0 < p < \infty$). There exist a Banach space $X$ (a $p$-Banach space if $0 < p < 1$), a directed family $\{ (\mathcal{M}_i, \tau_i) \}_{i \in I}$ of finite von Neumann algebras $\mathcal{M}_i$ (with normal faithful finite traces $\tau_i$), and a family $\{ j_i \}_{i \in I}$ of isometric embeddings $j_i : L^p(\mathcal{M}_i, \tau_i) \to X$ such that:

1. $j_i(L^p(\mathcal{M}_i, \tau_i)) \subset j_k(L^p(\mathcal{M}_k, \tau_k))$ for all $i, k \in I$ with $i \leq k$;
2. $\bigcup_{i \in I} j_i(L^p(\mathcal{M}_i, \tau_i))$ is dense in $X$;
3. $L^p(\mathcal{M})$ is isometric to a (complemented for $1 \leq p < \infty$) subspace of $X$.

Let $\mathcal{M}$ be an arbitrary von Neumann algebra (not necessarily semi-finite) and $p > 1$. If $X$ is the Banach space obtained from the above theorem then $X$ is a UMD-space with $C_p(X) = \sup_{i \in I} C_p(L^p(\mathcal{M}_i, \tau_i))$. In particular, the Haagerup $L^p$-space $L^p(\mathcal{M})$ is a UMD-space with constants equal to those of the finite case.

Let us now consider the case $p = 1$. If $p = 1$ or $p = \infty$ then $S^p$ fails the UMD-property. Let us denote by $S^1(n \times \infty)$ (resp. $S^1(\infty \times n)$) the space of trace class operators for $n \times \infty$.
matrices (resp. $\infty \times n$ matrices). The next result gives an estimate of the UMD-constant of $S^1(n \times \infty)$ when $n \to \infty$. It should be compared with [33, Theorem 6.1].

**Theorem 4.8.** There exists a constant $K$ such that for any $n \geq 1$, we have

$$C_2(S^1(\infty \times n)) \leq K \log(n + 1)$$

and similarly for $S^1(n \times \infty)$.

**Proof.** For every $x \in S^1(\infty \times n)$ and $q \leq 2 \leq p$ with $1/q + 1/p = 1$,

$$\|x\|_q \leq \|x\|_1 \leq n^{1/p} \|x\|_q.$$

Hence

$$C_2(S^1(\infty \times n)) \leq n^{1/p}C_2(S^q)$$

but since $C_2(S^q) = C_2(S^p) \leq 2\sqrt{3}C_p(S^p) \leq 2\sqrt{3}C_p$,

$$C_2(S^1(\infty \times n)) \leq 2\sqrt{3}C_p n^{1/p}.$$

Choosing $p = \max\{2, \log(n)\}$, the theorem follows. \qed

We remark that since $S^1(n \times \infty)$ is the dual of the space of operators $B(\ell^2_n, \ell^2)$ then $C_2(B(\ell^2_n, \ell^2))$ is of order $\log(n)$. In particular, if $M_{n,m}$ is the space of $n \times m$ matrices with the usual norm then $C_2(M_{n,m})$ is of order $\min\{\log(n), \log(m)\}$. The preceding argument also shows that for $N \geq 1$, there exist a constant $K > 0$ such that if $(x_n)_n$ is a finite martingale in $S^1_N$ (as predual of $M_N$) then

$$\left\| \sum_n \varepsilon_n dx_n \right\|_1 \leq K \log(N + 1) \sup_n \|x_n\|_1$$

for all $\varepsilon_n = \pm 1$.

We end this section by considering the general case of rearrangement invariant Banach function spaces. Before proceeding, we need to recall the notion of Boyd indices. Let $E$ be a rearrangement invariant Banach space on $(0, \infty)$. For $s > 0$, the dilation operator $D_s : E \to E$ is defined by setting

$$D_s f(t) = f(t/s), \quad t > 0, \quad f \in E.$$

The **lower and upper Boyd indices** of $E$ are defined by

$$\underline{\alpha}_E := \lim_{s \to 0^+} \frac{\log \|D_s\|}{\log s}, \quad \overline{\alpha}_E := \lim_{s \to \infty} \frac{\log \|D_s\|}{\log s}.$$
It is well known that $0 \leq \underline{\alpha}_E \leq \alpha_E \leq 1$ and if $E = L^p$ for $1 \leq p \leq \infty$ then $\underline{\alpha}_E = \alpha_E = 1/p$. If $0 < \underline{\alpha}_E \leq \alpha_E < 1$, we shall say that $E$ has non-trivial Boyd indices. The next result can be viewed as a martingale analogue of [16, Theorem 4.1] where the existence of generalized Riesz projections where considered.

**Theorem 4.9.** Let $E$ be a rearrangement invariant Banach function space on $(0, \infty)$ with Fatou norm. The following statements are equivalent.

(i) $E$ has non-trivial Boyd indices;

(ii) There exists a constant $c(E)$ depending only on $E$ such that for any semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ and any martingale $(x_n)_{n=1}^{\infty}$ in $E(\mathcal{M}, \tau)$,

$$\left\| \sum_{n=1}^{N} \varepsilon_n dx_n \right\|_{E(\mathcal{M}, \tau)} \leq c(E) \left\| \sum_{n=1}^{N} dx_n \right\|_{E(\mathcal{M}, \tau)}$$

for every $N \geq 2$ and $\varepsilon_n = \pm 1$.

**Proof.** (i) $\implies$ (ii). Choose $1 < p < q < \infty$ such that $1/q < \underline{\alpha}_E \leq \alpha_E < 1/p$ then $E$ is an interpolation space of the pair $(L^p, L^q)$ and therefore $E(\mathcal{M}, \tau)$ is an interpolation space for the pair $(L^p(\mathcal{M}, \tau), L^q(\mathcal{M}, \tau))$. The implication (i) $\implies$ (ii) follows by interpolation from Theorem 4.3.

(ii) $\implies$ (i). Assume first that $\alpha_E = 1$. Choose a filtration of $B(\ell^2)$, a finite martingale $(x_n)_{n=1}^{J}$ in $S^1$ and a sequence $\varepsilon_n = \pm 1$, $1 \leq n \leq J$ such that

$$\|x_J\|_1 = 1, \quad \left\| \sum_{n=1}^{J} \varepsilon_n dx_n \right\|_1 \geq 2c(E).$$

We can assume that $(x_n)_{n=1}^{J}$ are $N \times N$-matrices. Since $\alpha_E = 1$, it follows from [28, Proposition 2.b.6], that there exist non-negative, disjointly supported, equidistributed functions $(f_i)_{i=1}^{N}$ with $\|f_i\|_E = 1$ for $1 \leq i \leq N$, and

$$\frac{2}{3} \sum_{i=1}^{N} |a_i| \leq \left\| \sum_{i=1}^{N} a_i f_i \right\|_E$$

for every choice of scalars $(a_i)_{i=1}^{N} \in \mathbb{C}$. Let $\mathcal{M}$ be $L^\infty(0, \infty) \otimes M_N(\mathbb{C})$ with the trace $\tau$ given by $\lambda \otimes tr$ where $\lambda$ denotes the trace on $L^\infty(0, \infty)$ induced by the Lebesgue measure and $tr$ is the canonical trace on $M_N(\mathbb{C})$. We observe that

$$\| f_1 \otimes A \|_{E(\mathcal{M}, \tau)} = \left\| \sum_{i=1}^{N} s_i(A) f_i \right\|_E$$
for any matrix $A$ in $M_N(\mathbb{C})$, where $(s_i(A))_{i=1}^N$ denotes the singular values of $A$ arranged in decreasing order. In fact, let $A$ be any $N \times N$ matrix and consider $D$ the diagonal matrix with entries $s_1(A), s_2(A), \ldots, s_N(A)$. If $U$ and $V$ are unitary matrices for which $A = UDV$, then for every $t > 0$,

$$
\mu_t(f_1 \otimes A) = \mu_t(1 \otimes U.f_1 \otimes D.1 \otimes V) = \mu_t(f_1 \otimes D) = \mu_t \left( \sum_{i=1}^{N} s_i(A)f_j \right),
$$

where the last equality follows from the fact that $(f_i)_{i=1}^N$ are equidistributed and the definition of the trace on $\mathcal{M}$, and this proves the assertion. It now follows that

$$
\|f_1 \otimes x_J\|_{E(\mathcal{M}, \tau)} = \left\| \sum_{i=1}^{N} s_i(x_J)f_i \right\|_{E} \leq \left( \sum_{i=1}^{N} s_i(x_J) \right) \|f_1\|_{E} = 1.
$$

On the other hand,

$$
\left\| f_1 \otimes \sum_{n=1}^{J} \varepsilon_n dx_n \right\|_{E(\mathcal{M}, \tau)} = \left\| \sum_{i=1}^{N} s_i \left( \sum_{n=1}^{J} \varepsilon_n dx_n \right) f_i \right\|_{E} = \frac{2}{3} \sum_{i=1}^{N} s_i \left( \sum_{n=1}^{J} \varepsilon_n dx_n \right) \geq \frac{4}{3} c(E).
$$

Observe that $(f_1 \otimes x_n)_{n=1}^{J}$ is a finite martingale in $E(\mathcal{M}, \tau)$. Assertion (iii) implies that

$$
\left\| f_1 \otimes \sum_{n=1}^{J} \varepsilon_n dx_n \right\|_{E(\mathcal{M}, \tau)} \leq c(E) \|f_1 \otimes x_J\|_{E(\mathcal{M}, \tau)},
$$

and this yields a contradiction. The same argument can be applied to prove that assertion (iii) implies that $\alpha_E > 0$.

Unlike the case of $L^p(\mathcal{M}, \tau)$, Theorem 4.9 does not lead to UMD-property for $E(\mathcal{M}, \tau)$. Special characterizations that provide ready recognition of UMD-property for rearrangement invariant Banach function spaces on $(0, \infty)$ seem to be unavailable. On the other hand, there are examples of separable rearrangement invariant spaces on $(0, \infty)$ with non-trivial Boyd indices which are not reflexive (see for instance [28, p. 132]), and therefore fail the UMD-property. It is still an open question if $E$ being a UMD-space is sufficient for $E(\mathcal{M}, \tau)$ to be a UMD-space.
5. Non-commutative Burkholder-Gundy inequalities revisited

In this section, we will point out that the weak-type inequality in our main result implies the non-commutative Burkholder-Gundy inequalities proved in [37]. We first recall the two square functions introduced in [37].

Fix $1 \leq p < \infty$ and let $x$ be a bounded $L^p$-martingale. Recall,

$$S_{C,n}(x) = \left( \sum_{k=1}^{n} |dx_k|^2 \right)^{1/2} \quad \text{and} \quad S_{R,n}(x) = \left( \sum_{k=1}^{n} |dx_k^*|^2 \right)^{1/2}.$$ 

For any finite sequence $a = (a_n)_{n \geq 1}$ in $L^p(\mathcal{M}, \tau)$, set

$$\|a\|_{L^p(\mathcal{M}; l^2_C)} = \left\| \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \right\|_p \quad \text{and} \quad \|a\|_{L^p(\mathcal{M}; l^2_R)} = \left\| \left( \sum_{n \geq 1} |a_n^*|^2 \right)^{1/2} \right\|_p.$$ 

The difference sequence $dx$ belongs to $L^p(\mathcal{M}; l^2_C)$ (resp. $L^p(\mathcal{M}; l^2_R)$) if and only if the sequence $(S_{C,n}(x))_{n=1}^{\infty}$ (resp. $(S_{R,n}(x))_{n=1}^{\infty}$) is a bounded in $L^p(\mathcal{M}, \tau)$. In this case, the limits $S_C(x) = (\sum_{k=1}^{\infty} |dx_k|^2)^{1/2}$ and $S_R(x) = (\sum_{k=1}^{\infty} |dx_k^*|^2)^{1/2}$ are elements of $L^p(\mathcal{M}, \tau)$.

For $1 \leq p < \infty$, $\mathcal{H}^p_C(\mathcal{M})$ (resp. $\mathcal{H}^p_R(\mathcal{M})$) is defined as the set of all $L^p$-martingales $x$ with respect to $(\mathcal{M}_n)_{n \geq 1}$ such that $dx \in L^p(\mathcal{M}; l^2_C)$ (resp. $L^p(\mathcal{M}; l^2_R)$), and set

$$\|x\|_{\mathcal{H}^p_C(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M}; l^2_C)} \quad \text{and} \quad \|x\|_{\mathcal{H}^p_R(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M}; l^2_R)}.$$ 

Equipped with the previous norms, $\mathcal{H}^p_C(\mathcal{M})$ and $\mathcal{H}^p_R(\mathcal{M})$ are Banach spaces. The Hardy space of non-commutative martingale is defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}^p_C(\mathcal{M}) + \mathcal{H}^p_R(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \inf \left\{ \|y\|_{\mathcal{H}^p_C(\mathcal{M})} + \|z\|_{\mathcal{H}^p_R(\mathcal{M})} : x = y + z, \ y \in \mathcal{H}^p_C(\mathcal{M}), \ z \in \mathcal{H}^p_R(\mathcal{M}) \right\};$$

and if $2 \leq p < \infty$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}^p_C(\mathcal{M}) \cap \mathcal{H}^p_R(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \max \left\{ \|x\|_{\mathcal{H}^p_C(\mathcal{M})}, \ \|x\|_{\mathcal{H}^p_R(\mathcal{M})} \right\}.$$

The main result of [37] states that:
Theorem 5.1. Let $1 < p < \infty$. Let $x = (x_n)_{n=1}^{\infty}$ be an $L^p$-martingale. Then $x$ is bounded in $L^p(M, \tau)$ if and only if $x$ belongs to $H^p(M)$. If this is the case then

$$(BG_p) \quad \alpha_p^{-1}\|x\|_{H^p(M)} \leq \|x\|_p \leq \beta_p\|x\|_{H^p(M)}.$$ 

The strategy of [36] and [37] for the particular cases of tensor products, Clifford algebras and the Free group von Neumann algebras was to show the unconditionality of martingale differences in $L^p(M, \tau)$ (for $1 < p < \infty$) using transference argument to change non-commutative martingales into commutative vector-valued ones, and then apply non-commutative Khintchine inequalities (which we will recall below) together with a non-commutative analogue of Stein’s inequality. Such approach highlights the fact that non-commutative $L^p$-spaces are UMD-spaces. Their proof for the general case was completely different as they argued inductively on $p = 2^n$ for $n \geq 1$, then used interpolations and duality.

Let us recall the non-commutative Khintchine inequalities for the convenience of the reader. Let $\varepsilon = (\varepsilon_n)_{n \geq 1}$ be a sequence of independent random variables on some probability space $(\Omega, F, P)$ such that $P(\varepsilon_n = 1) = P(\varepsilon_n = -1) = 1/2$ for all $n \geq 1$.

Theorem 5.2. (Non-commutative Khintchine inequalities, [29, 30]) Let $1 \leq p < \infty$. Let $a = (a_n)_{n \geq 1}$ be a finite sequence in $L^p(M, \tau)$.

(i) If $2 \leq p < \infty$,

$$\|a\|_{L^p(M; \ell^2_C) \cap L^p(M; \ell^2_R)} \leq \left( \int_\Omega \| \sum_{n \geq 1} \varepsilon_n a_n \|_p^2 \, dP(\varepsilon) \right)^{1/2} \leq \beta_p \|a\|_{L^p(M; \ell^2_C) \cap L^p(M; \ell^2_R)},$$

(ii) If $1 \leq p < 2$,

$$\alpha \|a\|_{L^p(M; \ell^2_C) + L^p(M; \ell^2_R)} \leq \left( \int_\Omega \| \sum_{n \geq 1} \varepsilon_n a_n \|_p^2 \, dP(\varepsilon) \right)^{1/2} \leq \|a\|_{L^p(M; \ell^2_C) + L^p(M; \ell^2_R)},$$

where $\alpha > 0$ and $\beta > 0$ are absolute constants.

As in the case of unconditionality of martingale difference sequences, the non-commutative Stein’s inequality can also be deduced from Theorem 3.10 above and interpolation. This approach produces better estimate of the constant involved.
Theorem 5.3. Let $1 < p < \infty$. Define the map $Q$ on all finite sequences $a = (a_n)_{n \geq 1}$ in $L^p(\mathcal{M}, \tau)$ by $Q(a) = (E_n(a_n))_{n \geq 1}$. Then

$$(S_p) \quad \|Q(a)\|_{L^p(\mathcal{M}; L^2_{\mathbb{R}})} \leq \gamma_p \|a\|_{L^p(\mathcal{M}; L^2_{\mathbb{R}})}, \quad \|Q(a)\|_{L^p(\mathcal{M}; L^2_{\mathbb{C}})} \leq \gamma_p \|a\|_{L^p(\mathcal{M}; L^2_{\mathbb{C}})},$$

where $\gamma_p \leq Kp^2/(p-1)$ for some absolute constant $K$.

As noted in [37], Theorem 5.3 shows that $Q$ extends to a bounded linear projection on $L^p(\mathcal{M}; l^2_{\mathbb{C}})$ and $L^p(\mathcal{M}; l^2_{\mathbb{R}})$. Consequently, $\mathcal{H}^p(\mathcal{M})$ is complemented in $L^p(\mathcal{M}; l^2_{\mathbb{C}}) + L^p(\mathcal{M}; l^2_{\mathbb{R}})$ or $L^p(\mathcal{M}; l^2_{\mathbb{C}}) \cap L^p(\mathcal{M}; l^2_{\mathbb{R}})$ according to $1 < p \leq 2$ or $2 \leq p < \infty$.

We are now ready to present the proof. **Proof of Theorem 5.1:** Let $1 < p < 2$. By Theorem 4.3 and Theorem 5.2,

$$\alpha\|dx\|_{L^p(\mathcal{M}; l^2_{\mathbb{C}}) + L^p(\mathcal{M}; l^2_{\mathbb{R}})} \leq c_p \|dx\|_p.$$ 

Applying Theorem 4.3 to the martingale difference $(\varepsilon_n dx_n)_{n=1}^{\infty}$ instead of $(dx_n)_{n=1}^{\infty}$, we also have the converse inequality:

$$\|x\|_p \leq c_p \left\| \sum_{n=1}^{\infty} \varepsilon_n dx_n \right\|_p$$

for all $\varepsilon_n = \pm 1$. By Theorem 5.2,

$$\|x\|_p \leq c_p \|dx\|_{L^p(\mathcal{M}; l^2_{\mathbb{C}}) + L^p(\mathcal{M}; l^2_{\mathbb{R}})}$$

and therefore

$$\alpha c_p^{-1} \|dx\|_{L^p(\mathcal{M}; l^2_{\mathbb{C}}) + L^p(\mathcal{M}; l^2_{\mathbb{R}})} \leq \|x\|_p \leq c_p \|dx\|_{L^p(\mathcal{M}; l^2_{\mathbb{C}}) + L^p(\mathcal{M}; l^2_{\mathbb{R}})}.$$ 

By duality, if $2 < p$, then

$$c_p^{-1} \|dx\|_{L^p(\mathcal{M}; l^2_{\mathbb{C}}) \cap L^p(\mathcal{M}; l^2_{\mathbb{R}})} \leq \|x\|_p \leq \alpha^{-1} c_p \|dx\|_{L^p(\mathcal{M}; l^2_{\mathbb{C}}) \cap L^p(\mathcal{M}; l^2_{\mathbb{R}})}.$$ 

This shows $(BG_p)$ for $2 < p < \infty$ with $\alpha_p \leq c_p$ and $\beta_p \leq \alpha^{-1} c_p$.

For the case $1 < p < 2$, remark that $\|x\|_{\mathcal{H}^p(\mathcal{M})} \geq \|dx\|_{L^p(\mathcal{M}; l^2_{\mathbb{C}}) + L^p(\mathcal{M}; l^2_{\mathbb{R}})}$. From $(S_p)$, we conclude that

$$(\gamma_p)^{-1} c_p^{-1} \|x\|_{\mathcal{H}^p(\mathcal{M})} \leq \|x\|_p \leq c_p \|x\|_{\mathcal{H}^p(\mathcal{M})}.$$ 

This proves $(BG_p)$ for $1 < p < 2$ with $\alpha_p \leq \gamma_p c_p$ and $\beta_p \leq c_p$. \qed

Remarks 5.4. As $\gamma_p \leq Kp^2/(p-1)$, $\gamma_p = O(p)$ when $p \to \infty$ and $O((p-1)^{-1})$ when $p \to 1$. These are the optimal orders for $\gamma_p$. Recall that in the commutative case, the optimal order of growths for the constants $\alpha_p$ and $\beta_p$ are (see for instance [7]): $\beta_p$ is bounded when $p \to 1$ and $O(p)$ when $p \to \infty$; $\alpha_p$ is $O((p-1)^{-1})$ when $p \to 1$ and $O(\sqrt{p})$ when $p \to \infty$. The
fact that $\beta_p$ is bounded when $p \to 1$ for the non-commutative case was recovered by Junge and Xu [25, Corollary 4.3]. Pisier showed in [35] that $\beta_p$ is $O(p)$ for $p$ even integers. The proof above also gives $\beta_p$ is $O(p)$ when $p \to \infty$. As for $\alpha_p$, the preceding proof gives $\alpha_p$ is $O((p-1)^{-2})$ when $p \to 1$ and $O(p)$ when $p \to \infty$. For more in depth discussion about the orders of growth of these constants, we refer to a recent paper of Junge and Xu [26].

6. Remarks on the class $L \log L$

Recall first the class $L \log L$. If $L^0(\Omega, \mathcal{F}, P)$ is the space of all (classes) of measurable functions on a given probability space $(\Omega, \mathcal{F}, P)$, the class $L \log L$ is defined by setting

$$L \log L = \left\{ f \in L^0(\Omega, \mathcal{F}, P); \int |f| \log^+ |f| \ dP < \infty \right\}.$$

Set $\|f\|_{L \log L} = \int |f| \log^+ |f| \ dP$. Equipped with the equivalent norm $\|f\| = \int_0^1 f^*(t) \log(1/t) \ dt$, the space $L \log L$ is a rearrangement invariant Banach function space (see for instance [2, Theorem 6.4, pp. 246-247]) so a non-commutative analogue $L \log L(M, \tau)$ is well defined as described in Sect. 2. We remark that if a martingale $x$ is bounded in $L \log L(M, \tau)$ then it is uniformly integrable in $L^1(M, \tau)$ and therefore is of the form $x = (E_n(x_\infty))_{n=1}^\infty$ with $x_\infty \in L \log L(M, \tau)$.

The starting point of this section is the following well known inequality from the classical theory.

**Theorem 6.1.** There is a constant $K$ such that if $(f_k)_{k=1}^\infty$ is a (commutative) martingale then for every $n \geq 1$,

$$(6.1) \quad \mathbb{E} \left( \sup_{1 \leq k \leq n} |f_k| \right) \leq K + K \mathbb{E} \left( |f_n| \log^+ |f_n| \right).$$

By the equivalence of maximal functions and square functions for (commutative) martingales [13], the left hand side of (6.1) can be replaced by $\mathbb{E}(S_n(f))$ where $S_n(f) = (\sum_{k=1}^n |df_k|^2)^{1/2}$. The standard procedure for establishing inequality (6.1) above is to derive first the weak-type inequality for maximal functions by a stopping time argument then integrating from 1 to $\infty$ (see [17, pp. 317-318]; consult also [20, pp. 81-85] for another approach). In a more operator theoretical point of view, inequality (6.1) follows from general theory of interpolation of operators of weak types (see for instance [2, Theorem 6.6, pp. 248-249]). With this observation, the following result follows immediately from Theorem 3.1:
**Theorem 6.2.** There exists a constant $K$ such that if $x = (x_n)_{n=1}^{\infty}$ is a martingale which is bounded in $L \log L(M, \tau)$ then for any sequence of signs $(\varepsilon_n)_{n=1}^{\infty}$,

$$
\sup_{N \geq 1} \left\| \sum_{n=1}^{N} \varepsilon_n dx_n \right\|_1 \leq K + K \| x_{\infty} \|_{L \log L(M, \tau)}.
$$

Using the non-commutative Khintchine inequality, one can deduce

**Corollary 6.3.** There is a constant $K$ such that if $x = (x_n)_{n=1}^{\infty}$ is a martingale that is bounded in $L \log L(M, \tau)$, then

$$
\| dx \|_{L^1(M; P_\infty)} \leq K + K \| x_{\infty} \|_{L \log L(M, \tau)}.
$$

Corollary 6.3 can be viewed as a non-commutative extension of (6.1) above. However, inequality (6.1) is equivalent to: if $f$ is bounded in $L \log L$ then $f \in H^1$. Since $\| x \|_{H^1(M)} \geq \| dx \|_{L^1(M; P_\infty)}$, the following question arises naturally:

**Problem 6.4.** Does there exist a constant $K$ such that for every martingale $x$:

$$
\| x \|_{H^1(M)} \leq K + K \| x_{\infty} \|_{L \log L(M, \tau)}?
$$

An old argument from conjugate function theory together with the fact noted in Remark 5.4 above that $\alpha_p$ is $O((p - 1)^{-2})$ when $p \to 1$ can be used to prove a related inequality. The proof given below is modelled after a presentation in Zygmund’s book ([45, p.119]).

**Proposition 6.5.** There is an absolute constant $K$ such that if $x = (x_n)_{n=1}^{\infty}$ is a martingale that is bounded in $L \log L$ and $\tau(|x_{\infty}|(\log^+ |x_{\infty}|)^2) < \infty$, then

$$
\| x \|_{H^1(M)} \leq K + K \tau \left( |x_{\infty}|(\log^+ |x_{\infty}|)^2 \right).
$$

**Proof.** Let $x = (E_n(x_n))_{n=1}^{\infty}$ be a martingale with $\tau(|x_{\infty}|(\log^+ |x_{\infty}|)^2) < \infty$. Let $a = |x_{\infty}|$ and set $(\varepsilon_n)$ to be the spectral decomposition of $a$. For each $k \in \mathbb{N}$, let $P_k = \chi_{[2^{k-1}, 2^k)}(a)$ be the spectral projection relative to $[2^{k-1}, 2^k)$. Define $a_k = aP_k$ for $k \geq 1$ and $a_0 = a\chi_{[0,1)}(a)$. Clearly $a = \sum_{k=0}^{\infty} a_k$ in $L^1(M, \tau)$.

For every $k \in \mathbb{N}$, consider the martingale $x^{(k)} = (E_n(x_{\infty}P_k))_{n=1}^{\infty}$ then $\| x^{(k)} \|_{H^1(M)} \leq \| x^{(k)} \|_{H^1(M)} \leq \alpha_p \| x^{(k)} \|_p$. So for every $1 < p < 2$, there is a constant $C$ such that,

$$
\| x^{(k)} \|_{H^1(M)} \leq C^2 p(p - 1)^{-2} \| x^{(k)} \|_p.
$$

Since $\| x^{(k)} \|_p = \| a_k \|_p$ and $a_k \leq 2^k P_k$, we get for $1 < p < 2$,

$$
\| x^{(k)} \|_{H^1(M)} \leq 16C^2(p - 1)^{-2} 2^k \tau(P_k)^{\frac{1}{p}}.
$$

If we set $p = 1 + 1/(k + 1)$ and $\eta_k = \tau(P_k)$, we have

$$
\| x^{(k)} \|_{H^1(M)} \leq 16C^2(2^{k+1})^2 \eta_k^{\frac{k+1}{2}}.
$$
Taking the summation over $k$, 
\[
\|x\|_{\mathcal{H}^1(M)} \leq \sum_{k=0}^{\infty} 16C^2(k + 1)^2 2^k \eta_{k+1}^{k+1}.
\]

We note as in [45] that if $J = \{k \in \mathbb{N}; \eta_k \leq 3^{-k}\}$ then
\[
\sum_{k \in J} 16C^2(k + 1)^2 2^k \eta_{k+1}^{k+1} \leq \sum_{k=0}^{\infty} 16C^2(k + 1)^2 2^k (3^{-k})^{k+1} = \alpha < \infty.
\]

On the other hand, for $k \in \mathbb{N} \setminus J$, $\eta_k^{k+1} \leq \eta_k 3^{-2k} \leq \beta \eta_k$ where $\beta = \sup_k 3^{-2k}$. So we get
\[
\|x\|_{\mathcal{H}^1(M)} \leq \alpha + 16C^2 \beta \sum_{k=0}^{\infty} (k + 1)^2 2^k \eta_k
\]
\[
\leq \alpha + 16C^2 \beta (\eta_0 + 8\eta_1) + 16C^2 \beta \sum_{k \geq 2} (k + 1)^2 2^k \eta_k.
\]

Since for $k \geq 2$, $k + 1 \leq 3(k - 1)$, we get
\[
\|x\|_{\mathcal{H}^1(M)} \leq \alpha + 128C^2 \beta + 288C^2 \beta \sum_{k \geq 2} (k - 1)^2 2^{k-1} \eta_k.
\]

To complete the proof, notice that for $k \geq 2$,
\[
(k - 1)^2 2^{k-1} \eta_k = \int_{2^{k-1}}^{2^k} (k - 1)^2 2^{k-1} d\tau(e_t)
\]
\[
\leq \int_{2^{k-1}}^{2^k} \frac{t(\log t)^2}{(\log 2)^2} d\tau(e_t),
\]
as $2^{k-1} \leq t$ and therefore $(k - 1) \log 2 \leq \log t$. Hence if we set
\[
K = \max\{\alpha + 128C^2 \beta, 288C^2 \beta (\log 2)^{-2}\},
\]
then we get:
\[
\|x\|_{\mathcal{H}^1(M)} \leq K + K \tau\left(a(\log^+(a))^2\right).
\]

The proof is complete.

We remark that combining Corollary 6.3 and Theorem 3.10, one can deduce the following: There exists a constant $K'$ such that:
\[
\inf \left\{\|dy\|_{L^1,\infty(M;L^2)}, \|dz\|_{L^1,\infty(M;L^2)} : x = y + z, \ y \in \mathcal{H}^1_C(M), \ z \in \mathcal{H}^1_R(M)\right\}
\]
\[
\leq K' + K' \|x\|_{L^\infty,\infty(M,\tau)}.
\]

The next question corresponds to the weak type boundedness of square functions:
Problem 6.6. Does there exist a constant $K$ such that for every bounded $L^1$-martingale $x$,

$$\inf \left\{ \|dy\|_{L^1(\mathcal{M}; \ell_2^c)} + \|dz\|_{L^1(\mathcal{M}; \ell_2^R)} : x = y + z, \ y \in \mathcal{H}^1_C(\mathcal{M}), \ z \in \mathcal{H}^1_R(\mathcal{M}) \right\} \leq K \|x\|_1$$

We remark that a simple adjustment of the proof of Theorem 3.1 gives: There exists a constant $K$ such that for every $\lambda < 0$,

$$\inf \{ \lambda \tau(\chi_{(\lambda, \infty)}(S_C(y))) + \lambda \tau(\chi_{(\lambda, \infty)}(S_R(z))) : x = y + z \} \leq K \|x\|_1.$$

We conclude by noticing that the proof of Theorem 3.10 combined with Theorem 6.2 yields the following:

**Theorem 6.7.** There exists a constant $K$ such that for any finite sequence $a = (a_k)_{k=1}^n$ in $L \log L(\mathcal{M}, \tau)$, if $Q(a) = (E_k(a_k))_{k=1}^n$ then

$$\|Q(a)\|_{L^1(\mathcal{M}; \ell_2^c)} \leq \|a\|_{L^1(\mathcal{M}; \ell_2^c)} + K + K \tau \left( \frac{1}{n} \sum_{k=1}^n |a_k|^2 \right)^{1/2} \log \left( \frac{1}{n} \sum_{k=1}^n |a_k|^2 \right)^{1/2}.$$

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