Some commutative ring extensions defined by almost Bézout condition

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Abstract

In this paper, we study the almost Bézout property in different commutative ring extensions, namely, in bi-amalgamated algebras and pairs of rings. In Section 2, we deal with almost Bézout domains issued from bi-amalgamations. Our results capitalize well known results on amalgamations and pullbacks as well as generate new original class of rings satisfying this property. Section 3 investigates pairs of rings where all intermediate rings are almost Bézout domains. As an application of our results, we characterize pairs of rings \((R, T)\), where \(R\) arises from a \((T, M, D)\) construction to be an almost Bézout domain.

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1. Introduction

Throughout, all rings considered are commutative with unity and all modules are unital. In [2], Anderson and Zafrullah enlarged the class of Bézout domains in the following way: they called a domain \(R\) an almost Bézout domain (\(AB\)-domain for short) if given any two elements \(x, y \in R\), there is a positive integer \(n\) such that the ideal \((x^n, y^n)\) is principal. Among other things, they proved that an integral domain \(R\) is an \(AB\)-domain if and only if the integral closure \(R'\) of \(R\) is a Prüfer domain with torsion class group and \(R \subseteq R'\) is a root extension. Further, they proved that the theory of \(AB\)-domains is closed to the classical one of Bézout domains. In [4], the authors noticed that each \(AB\)-domain is nearly Bézout and a counter-example, using the classical pullback \(K + XL[X]\) was given to disprove the converse. Moreover, They used the same example to illustrate that a Noetherian almost Bézout domain is not necessarily an almost principal ideal-domain (API-domain), although each Noetherian Bézout domain is a principal ideal-domain (PID). In [3], D.D. Anderson and M. Zafrullah showed that a finite intersection of almost valuation domains with the same quotient field is an almost Bézout domain. This generalizes the result that

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a finite intersection of valuation domains with the same quotient field is a Bézout Domain
and they gave a new characterization of Cohen-Kaplansky domains. In [18], Mimouni
studied the transfer of the notion of AB-domain to pullbacks. Later, in [17], the authors
extended the notion of AB-domain defined in [2], to class of rings with zero-divisors. In
particular, they defined almost Bézout rings (AB-rings for short) and they investigated
when this condition is satisfied by an amalgamated algebra and by an idealization (also
called Nagata’s ring). In this paper, we examine when a bi-amalgamation is an AB-ring.
Our results capitalize previous well known results on amalgamations in [17] and on An-
derson and Zafrullah’s paper in [2] as well as generate new original class of rings satisfying
this property. Among other things, we investigate pairs of integral domains where all in-
termediate rings are AB-domains. As a consequence of our results, we provide necessary
and sufficient conditions for a pair \((R, T)\) where \(R\) arises from a \((T, M, D)\) construction,
to be an AB-domain pair.

Section 2 is devoted to the study of AB-ring property in bi-amalgamated algebras.
For this purpose, we recall the definition of bi-amalgamation of rings: Let \(f : A \to B\)
and \(g : A \to C\) be two ring homomorphisms and let \(J\) and \(J’\) be two ideals of \(B\) and \(C,\)
respectively, such that \(I_o := f^{-1}(J) = g^{-1}(J’).\) The bi-amalgamation (or bi-amalgamated
algebra) of \(A\) with \((B, C)\) along \((J, J’)\) with respect to \((f, g)\) is the subring of \(B \times C\) given by

\[
A \bowtie^{f,g} (J, J') := \{(f(a) + j, g(a) + j') \mid a \in A, (j, j') \in J \times J'\}.
\]

This construction was introduced in [15] as a natural generalization of duplications [9,
12,13] and amalgamations [10,11]. In [15], the authors provide original examples of bi-
amalgamations and, in particular, show that Boisen-Sheldon’s CPI-extensions [7] can be
viewed as bi-amalgamations (Notice that [10, Example 2.7] shows that CPI-extensions
can be viewed as quotient rings of amalgamated algebras). They also show how every
bi-amalgamation can arise as a natural pullback (or even as a conductor square) and then
characterize pullbacks that can arise as bi-amalgamations. Then, the last two sections of
[15] deal, respectively, with the transfer of some basic ring theoretic properties to bi-
amalgamations and the study of their prime ideal structures. All their results recover
known results on duplications and amalgamations. Recently in [16], the authors estab-
lished necessary and sufficient conditions for a bi-amalgamation to inherit the arithmetical
property, with applications on the weak global dimension and transfer of the semiheredi-
tary property.

Section 3 is devoted to the study of pairs of AB-domains. When each intermediate ring
\(T\) between \(R\) and \(S\) (that is for each \(T \in [R, S]\)) satisfies a ring theoretic property \(P,\) then
\((R, S)\) is said to be a \(P\)-pair. The notion of \(P\)-pairs was studied for different properties
\(P\) (for instance \(P :=\) Noetherian, Prüfer, almost valuation, treed see [5, 14, 19, 20]). To complete
this circle of ideas, we deal with the property \(P :=\) almost Bézout. Throughout,
for a ring \(R, Spec(R)\) (resp., \(Max(R)\)) will denote the set of all prime (resp., maximal)
ideals of \(R.\) For an integral domain \(R,\) we denote by \(qf(R)\) (resp., \(R’\)) the quotient field
of \(R\) (resp., the integral closure of \(R\) in \(qf(R)\)). For a ring extension \(R \subseteq S,\) we denote
by \([R, S]\) (resp., \([R, S]\)) the set of all rings \(T\) such that \(R \subseteq T \subseteq S\) (resp., \(R \subset T \subseteq S\)).
We shall call a ring \(T\) in \([R, S]\) an \(S\)-overring of \(R.\) Such a ring is said to be a proper
\(S\)-overring of \(R\) if \(T \neq S.\) When \(S = qf(R),\) then each ring \(T \in [R, qf(R)]\) is called an
overring of \(R.\) We denote by \(Jac(R),\) the Jacobson radical of \(R.\) Recall that an extension
of integral domains \(R \subseteq S\) is said to be a root extension if for each \(x \in S,\) there exists a
positive integer \(n\) such that \(x^n \in R.\)
2. Transfer of $AB$–ring property to bi-amalgamated algebras

Let $A$, $B$ and $C$ be three rings, $f : A \rightarrow B$ and $g : A \rightarrow C$ be two ring homomorphisms and let $J$ and $J'$ be two ideals of $B$ and $C$, respectively, such that $f^{-1}(J) = g^{-1}(J') = I_0$. All along this section, $A \triangleright_I^g (J, J')$ will denote the bi-amalgamation of $A$ with $(B, C)$ along $(J, J')$ with respect to $(f, g)$.

Our first result investigates the transfer of $AB$–ring property to bi-amalgamation $A \triangleright_I^g (J, J')$ in case $J$ and $J'$ are proper ideals ($J \neq B$ and $J' \neq C$).

**Theorem 2.1.** Assume that $B$ and $C$ are integral domains and $J$ (resp., $J'$) is a proper ideal of $B$ (resp., $C$). Then the following statements are equivalent:

1. $A \triangleright_I^g (J, J')$ is an $AB$–ring,
2. $f(A) + J$ and $g(A) + J'$ are $AB$–rings and $J = 0$ or $J' = 0$.

**Proof.** (1) $\Rightarrow$ (2) Assume that $A \triangleright_I^g (J, J')$ is an $AB$–ring. Using the fact that the $AB$–ring property is stable under factor ring and in view of the isomorphisms:

$$ \frac{A \triangleright_I^g (J, J')}{{0 \times J'}} \cong f(A) + J $$

and

$$ \frac{A \triangleright_I^g (J, J')}{{J \times 0}} \cong g(A) + J' $$

given by [15, Proposition 4.1(2)]. It follows that $f(A) + J$ and $g(A) + J'$ are $AB$–rings. Next, we claim that $J = 0$ or $J' = 0$; otherwise, for nonzero elements $x \in J$ and $x' \in J'$, we would have $((0, x')^n, (x, 0)^m) = ((0, x^n), (x^n, 0))$ is a principal ideal of $A \triangleright_I^g (J, J')$ for some integer $n \geq 1$. Therefore, there exists $(f(d) + j, g(d) + j') \in A \triangleright_I^g (J, J')$ such that

$$(x^n, 0)A \triangleright_I^g (J, J') + (0, x^m)A \triangleright_I^g (J, J') = (f(d) + j, g(d) + j')A \triangleright_I^g (J, J').$$

And so there exist $(f(a) + k, g(a) + k'), (f(b) + l, g(b) + l')$, $(f(\alpha) + s, g(\alpha) + s')$ and $(f(\beta) + t, g(\beta) + t')$ elements of $A \triangleright_I^g (J, J')$ such that

$$
\begin{align*}
(0, x^m) &= (f(a) + k, g(a) + k')(f(d) + j, g(d) + j') \\
(x^n, 0) &= (f(b) + l, g(b) + l')(f(d) + j, g(d) + j') \\
(f(d) + j, g(d) + j') &= (x^n, 0)(f(\alpha) + s, g(\alpha) + s') + (0, x^m)(f(\beta) + t, g(\beta) + t').
\end{align*}
$$

This implies that

$$
\begin{align*}
0 &= (f(a) + k)(f(d) + j) \quad (i) \\
x^m &= (g(a) + k')(g(d) + j') \quad (ii) \\
0 &= (g(b) + l')(g(d) + j') \quad (iii) \\
x^n &= (f(b) + l)(f(d) + j) \quad (iv) \\
f(d) + j &= x^n(f(\alpha) + s) \quad (v) \\
g(d) + j' &= x^m(g(\beta) + t') \quad (vi)
\end{align*}
$$

From equation $(iv)$, we claim that $f(d) + j \neq 0$. Deny. It follows that $x^n = 0$, making $x = 0$, which is absurd. Since $f(d) + j \neq 0$ and $B$ is an integral domain, then by equation $(i)$, necessarily, $f(a) + k = 0$. Consequently, $a \in f^{-1}(J) = g^{-1}(J')$. By substituting equation $(ii)$ in $(vi)$, it follows that

$$(g(d) + j') = (g(d) + j')(g(\alpha) + k')(g(\beta) + t') \quad (vii).$$

Since $x^n \neq 0$, it follows from equation $(ii)$ that $g(d) + j' \neq 0$. Hence, by equation $(vii)$, $(g(\alpha) + k')(g(\beta) + t') = 1$, which is a contradiction, since $g(\alpha) + k' \in J'$ (as $a \in g^{-1}(J')$) and $J'$ is a proper ideal of $C$. Thus $J = 0$ or $J' = 0$. The converse is straight via the isomorphisms given in [15, Proposition 4.1(2)]. \[\boxtimes\]
We recall here the following result proved by Mahdou and al.:

Recall that the amalgamation of \( A \) with \( B \) along \( J \) with respect to \( f \) is given by

\[
A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J \}.
\]

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since \( A \bowtie^f J = A \bowtie^{id_A f} (f^{-1}(J), J) \).

The following result is an immediate consequence of Theorem 2.1.

**Corollary 2.2.** Under the above notation, assume that \( A \) and \( B \) are integral domains. Then \( A \bowtie^f J \) is an \( AB \)-ring if and only if both \( A \) and \( f(A) + J \) are \( AB \)-rings and \( J = 0 \) or \( f^{-1}(J) = 0 \).

The following result is a particular case of Corollary 2.2 and is [2, Theorem 4.9].

**Corollary 2.3.** Let \( D \) be an integral domain with quotient field \( K \). Then \( D + XK[X] \) is an \( AB \)-domain if and only if \( D \) is an \( AB \)-domain.

**Proof.** Observe that \( A = D, B = K[X], f : D \hookrightarrow K[X] \) is the natural injection and \( J = XK[X] \not= 0 \). Clearly \( f(A) \cap J = D \cap XK[X] = 0 \) and so \( f^{-1}(J) = 0 \) and by [10, Proposition 5.1 (3)], \( A \bowtie^f J = D \bowtie^{f} XK[X] \cong f(A) + J = D + XK[X] \). Hence by Corollary 2.2, the conclusion is trivial. \( \square \)

**Remark 2.4.** We recall here the following result proved by Mahdou and al.: “Let \( A \) and \( B \) be a pair of integral domains and \( f : A \rightarrow B \) be a ring homomorphism. If \( J \) is a nonzero proper ideal of \( B \). Then \( A \bowtie^f J \) is an \( AB \)-ring if and only if \( f \) is injective, \( f(A) + J \) is an \( AB \)-ring and \( f(A) \cap J = (0) \).” [17, Theorem 3.6]. Corollary 2.2 recovers the above result. Indeed, in the case \( J \) is a nonzero proper ideal of \( B \), we show that the following statements are equivalent:

1. “\( A \) and \( f(A) + J \) are \( AB \)-rings and \( f^{-1}(J) = 0 \).”
2. “\( f \) is injective, \( f(A) + J \) is an \( AB \)-ring and \( f(A) \cap J = (0) \).”

Indeed,

(1) \( \implies \) (2) Notice that \( \ker(f) \subseteq f^{-1}(J) = 0 \). So \( f \) is injective, therefore the conclusion is trivial.

(2) \( \implies \) (1) Since \( f(A) + J \) is an \( AB \)-ring and \( f(A) \cong f(A) + J \), then \( f(A) \) is an \( AB \)-ring (as a factor ring of an \( AB \)-ring) and since \( f \) is injective, then \( A(\cong f(A)) \) is also an \( AB \)-ring. Using the fact that \( f(A) \cap J = (0) \), then \( f^{-1}(J) = f^{-1}(\{0\}) = \ker(f) = 0 \), as \( f \) is injective.

Let \( I \) be a proper ideal of \( A \). The (amalgamated) duplication of \( A \) along \( I \) is a special amalgamation given by

\[
A \bowtie^I := A \bowtie^{id_A I} I = \{(a, a + i) \mid a \in A, i \in I \}.
\]

The next corollary is an immediate consequence of Corollary 2.2 on the transfer of \( AB \)-ring property to duplications.

**Corollary 2.5.** Let \( A \) be an integral domain and \( I \) be a proper ideal of \( A \). Then \( A \bowtie^I \) is an \( AB \)-ring if and only if so is \( A \) and \( I = 0 \).

We recall an important characterization of a local Gaussian ring \( A \). Namely, for any two elements \( a \) and \( b \) in the ring \( A \), we have \( (a, b)^2 = (a^2) \) or \( (b^2) \); moreover if \( ab = 0 \) and \( (a, b)^2 = (a^2) \), then \( b^2 = 0 \) (see, [6, Theorem 2.2]).

The following proposition is a partial result about when a bi-amalgamation is an \( AB \)-ring, in case \( B \) and \( C \) are not integral domains.
Proposition 2.6. Assume $(A, m)$ is local Gaussian, $J \times J' \subseteq \text{Jac}(B \times C)$, $J^2 = 0$, $J'^2 = 0$, and for all $a \in m$, $f(a)J = f(a)^2J$, $g(a)J' = g(a)^2J'$. Then $A \bowtie_{\mathcal{J}} (J, J')$ is an $AB$–ring.

**Proof.** Assume $(A, m)$ is local Gaussian, $J^2 = 0$, $J'^2 = 0$ and, for all $a \in m$, $f(a)J = f(a)^2J$, $g(a)J' = g(a)^2J'$. From [15, Proposition 5.3 (b)], $(A \bowtie_{\mathcal{J}} (J, J'), m \bowtie_{\mathcal{J}} (J, J'))$ is local since $A$ is local and $J \times J' \subseteq \text{Jac}(B \times C)$. Let $(f(a) + i, g(a) + i')$ and $(f(b) + j, g(b) + j')$ be elements of $A \bowtie_{\mathcal{J}} (J, J')$. Two cases are then possible:

Case 1: $a$ or $b \notin m$. Then $a$ or $b$ is invertible in $A$. Assume without loss of generality that $a$ is invertible, then $(f(a) + i, g(a) + i')(f(a)^{-1} - f(a)^{-1}i, g(a)^{-1} - g(a)^{-1}i') = (1, 1)$. So $(f(a) + i, g(a) + i')$ is invertible in $A \bowtie_{\mathcal{J}} (J, J')$. Therefore, $((f(a) + i, g(a) + i'), (f(b) + j, g(b) + j'))^2 = ((f(a) + i, g(a) + i')^2) = A \bowtie_{\mathcal{J}} (J, J')$. Thus, it follows that there exists an integer $n = 2$ such that $(f(a) + i, g(a) + i')^n$ and $(f(b) + j, g(b) + j')^n$ are comparable, as desired.

Case 2: $a$ and $b$ in $A$. In the fact that $A$ is local Gaussian, then $(a, b)^2 = (a^2)$ or $(b^2)$. We may assume that $(a, b)^2 = (a^2)$. So we have, $b^2 = ax$ and $ab = ay$ for some $x$ and $y$ in $A$, and so $(f(b))^2 = (f(a))^2f(x), (g(b))^2 = (g(a))^2g(x)$ and $f(a)f(b) = (f(a))^2f(y), g(a)g(b) = (g(a))^2g(y)$. By assumption, $2f(b)j \in f(b)^2J$ and $2f(a)if(x) \in f(x)^2J$. Therefore, there exist $j_1, i_1 \in J$ such that $2f(b)j = f(a)^2f(x)j_1, 2f(a)if(x) = f(a)^2i_1$, and similarly, there exist $j_1', i_1' \in J'$ such that $2g(b)j' = g(a)^2g(x)j_1', 2g(a)^2g(x) = g(a)^2i_1'$. In view of the fact that $J^2 = 0$ and $J'^2 = 0$, one can easily check that $(f(b) + j, g(b) + j')^2 = (f(a) + i, g(a) + i')^2(f(x) + f(x)j_1 - i_1, g(x) + g(x)j_1' - i_1')$. Hence, there exists an integer $n = 2$ such that $(f(a) + i, g(a) + i')^n$ and $(f(b) + j, g(b) + j')^n$ are comparable, as desired.

Thus, in all cases, 

$((f(a) + i, g(a) + i')^2, (f(b) + j, g(b) + j')^2) = ((f(a) + i, g(a) + i')^2)$

which is principal, making $A \bowtie_{\mathcal{J}} (J, J')$ an $AB$–ring.

Proposition 2.6 recovers the special case of amalgamated algebra, as recorded in the next corollary.

Corollary 2.7. Let $(A, m)$ be a local Gaussian ring, $f : A \rightarrow B$ be an injective ring homomorphism and $J$ be an ideal of $B$ such that $J \subseteq \text{Jac}(B)$. If $f(a)J = f(a)^2J$, $af^{-1}(J) = a^2f^{-1}(J)$ for all $a \in m$ and $J^2 = 0$, then $A \bowtie_{\mathcal{J}} J$ is an $AB$–ring.

**Proof.** It is easy to show that $(f^{-1}(J))^2 \subset f^{-1}(J^2)$. Since $f$ is injective and $J^2 = 0$, it follows that $(f^{-1}(J))^2 = 0$. Therefore, by using Proposition 2.6, $A \bowtie_{\mathcal{J}} J = A \bowtie_{\mathcal{J}2} (f^{-1}(J), J)$ is an $AB$–ring.

As an application of Proposition 2.6, we give an explicit example showing the failure of Theorem 2.1, beyond the context $B$ and $C$ are integral domains. Recall that for a ring $A$ and an $A$–module $E$, the trivial ring extension of $A$ by $E$ (also called idealization of $E$ over $A$) is the ring $R := A \times E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', a'e + ea')$.

Example 2.8. Let $(A, m) := (Z/9Z, 3Z/9Z)$ be a local Gaussian Bézout ring with $m^2 = 0$, $K := A/m$, $E$ be a $K$–vector space and $B := A \times E$ be the trivial ring extension of $A$ by $E$. Consider $J : A \rightarrow B$ be the natural injection (defined by $f(a) = (a, 0)$). Let $C := A$ and $g = \text{id}_A$ be the identity of $A$. Let $J = m \times E$ be the maximal ideal of $B$. Let $J' = m$ be the maximal ideal of $C$. Clearly, $f^{-1}(J) = g^{-1}(J') = m, J^2 = J'^2 = 0, f(a)J = f(a)^2J$ and, for all $a \in m$. Then:

1. $A \bowtie_J^g (J, J')$ is an $AB$–ring.
2. $f(A) + J$ and $g(A) + J'$ are $AB$–rings.
3. $J \neq 0$ and $J' \neq 0$.

**Proof.** (1) Since $\text{Jac}(B \times C) = (m \times E) \times (m)$, then by Proposition 2.6, $A \bowtie_J^g (J, J')$ is an $AB$–ring.
Theorem 2.1 enriches the current literature with new original class of $AB$–rings that are not Bézout rings. Recall that if $A$ is a Bézout ring and $I$ is an ideal of $A$, then $A/I$ is a Bézout ring.

**Example 2.9.** Let $F$ be a field of characteristic $p > 0$ (for instance $F = \mathbb{Z}_p$) and let $F \subseteq L$ be a purely inseparable field extension. Consider $A = F + XL[X]$. Observe that the integral closure $A'$ of $A$ in its quotient field $L[X]$ is $L[X]$. One can easily check that for each $q(x)$ in $L[X]$ there exists $n \geq 0$ such that $(q(x))^p^n \in A$. Therefore, $A \subseteq A'$ is a root extension. Since $A' = L[X]$ is a principal ideal domain, then $A'$ is a Bézout domain and so is Prüfer. From [2, Corollary 4.8 (1)], it follows that $A$ is an $AB$-domain. Since $A$ is not integrally closed, then $A$ is not a Bézout domain. Let $f : A \rightarrow B = L(X)[Y]$ be an injective ring homomorphism and let $J := Y L(X)[Y]$ be a maximal ideal of $L(X)[Y]$. Consider the injective ring homomorphism $g : A \rightarrow A \times A$, given by $g(a) = (a, 0)$ and $J' := 0$ be an ideal of $A \times A$. Clearly, $f^{-1}(J) = 0 = I_0 = g^{-1}(J')$. Then:

1. $A \triangleright\triangleleft g (J, J')$ is $AB$–ring.
2. $A \triangleright\triangleleft g (J, J')$ is not a Bézout ring.

**Proof.** (1) By Theorem 2.1, $A \triangleright\triangleleft g (J, J')$ is $AB$–ring since $f(A) + J = A + Y L(X)[Y]$ which is $AB$–ring by Corollary 2.3 and $g(A) + J' = A \times 0 \simeq A$ which is $AB$–domain and $J' = 0$.

(2) Since $A$ is not a Bézout domain, and so by [15, Proposition 4.1 (3)], $A \simeq \frac{A}{I_0} \simeq \frac{A_{\triangleright\triangleleft g (J, J')}}{J \times J'}$ is not a Bézout ring. Hence, $A \triangleright\triangleleft g (J, J')$ is not a Bézout ring (as its quotient is not a Bézout ring).

3. Almost Bézout domain pairs

In this section, we characterize almost Bézout domain pairs.

**Definition 3.1.** Let $R \subseteq S$ be an extension of integral domains. We say that $(R, S)$ is an almost Bézout domain pair (for short $AB$–domain pair) if each ring $T \in [R, S]$ is an $AB$–domain.

It is worth to mention that the proof of the statement (1) of the following theorem is straightforward from [2, Lemma 4.5]. We are very grateful to Anderson and Zafrullah for their result.

**Theorem 3.2.** Let $R \subseteq S$ be an extension of integral domains. Then the following statements hold:

1. If $S$ is an overring of $R$, then $(R, S)$ is an $AB$–domain pair if and only if $R$ is an $AB$–domain.

2. If $(R, S)$ is an $AB$–domain pair, then $R$ is an $AB$–domain and $S$ is algebraic over $R$. The converse holds if $R$ is integrally closed in $S$.

**Proof.** (1) If $(R, S)$ is an $AB$–domain pair, then $R$ is an $AB$–domain. Conversely, let $T \in [R, S]$. Then $T$ is an overring of $R$ since $S$ is an overring of $R$. By [2, Lemma 4.5], it follows that $T$ is an $AB$–domain.

(2) Assume that $(R, S)$ is an $AB$–domain pair. Then it is clear that $R$ is an $AB$–domain. We claim that $R \subseteq S$ is an algebraic extension. Deny. there exists $t \in S$ such that $t$ is transcendental over $R$.
Claim 1. $R$ is a field
Indeed, let \( 0 \neq \alpha \in R \). Using the fact that \( R \subset R[t] \subset S \), then \( R[t] \) is an \( AB \)-domain. Therefore, there exists a positive integer \( l \geq 1 \) such that the ideal \( (\alpha^l, t^l) \) is principal generated by some \( u(t) \in R[t] \). In particular, \( u(t) \) divides \( \alpha^l \). Consequently, \( u(t) = u \) for some constant \( u \in R \). Observe that \( t^l \) is a multiple of \( u \). Then there exists \( u' \in R \) such that \( t^l = uu't^l \) and so \( uu' = 1 \). Hence, \( u \) is a unit of \( R \). On the other hand, \( (\alpha^l, t^l) = (u) = R[t] \). In particular, there exist \( p(t), q(t) \in R[t] \) such that \( 1 = \alpha^l p(t) + t^l q(t) \). By identification, we get \( q(t) = 0 \) and there exists \( \beta \in R \) such that \( p(t) = \beta \) and so it follows \( \alpha^l \beta = 1 \). Hence, \( \alpha \) is a unit of \( R \). Thus, \( R \) is a field, denoted by \( k \).

Claim 2. If \( t \) is a transcendental element over \( k \), then there exists a domain \( T \) that is not an \( AB \)-domain with \( k \subset T \subset k[t] \).

Consider the ring \( T = k + t^3 k[t] \). Clearly, \( k \subset T \subset k[t] \). Consider the elements \( 1 + t^3 \) and \( t^4 \) in \( T \). We claim that \( T \) is not an \( AB \)-domain. Deny. Then there exist an integer \( n \geq 1 \) and \( w(t) \in T \) such that the ideal \( ((1 + t^3)^n, t^4n) = w(t)T \). Since \( w(t) \) divides \( t^4n \), then there exist an element \( w \in k \) and a positive integer \( j \leq 4n \) such that \( w(t) = wt^j \). Next, the fact that \( w(t) \) divides \( (1 + t^3)^n \), then one can easily check that \( j \) must be equal to \( 0 \). And so \( w(t) = w \) for some nonzero \( w \in k \). Observe that \( w \) is a unit of \( T \). Consequently, \( (1 + t^3)^n, t^4n) = T \). In particular, there exist \( p(t), q(t) \in T \) such that \( t^4 = p(t)(1 + t^3)^n + q(t)t^4n \). This implies that \( q(t) = 0 \) and \( p(t) = ct^i \), for some positive integer \( i \leq 3 \) and \( c \in k \), which is absurd, since \( n \geq 1 \). Hence, \( T \) is not an \( AB \)-domain, a contradiction since each proper \( S \)-overrning of \( R = k \) is an \( AB \)-domain. It follows that \( R \subset S \) is an algebraic extension. Conversely, assume that \( R \) is an \( AB \)-domain and \( S \) is algebraic over \( R \) and \( R \) is integrally closed in \( S \). We claim that \( S \subset qf(R) \). Indeed, let \( u \in S \). Then \( u \) is algebraic over \( R \). Therefore, there is \( a \) in \( R \) such that \( v = au \) is integral over \( R \). Hence, \( v \) is an element of \( S \), as \( R \) is integrally closed in \( S \). Consequently, \( u = \frac{v}{a} \) is an element of \( qf(R) \). Hence, \( S \subset qf(R) \), making \( S \) an overring of \( R \). Finally, by statement (1) above, it follows that \( (R, S) \) is an \( AB \)-domain pair, as desired.

Combining Theorem 3.2 and Corollary 2.3 of Section 2, we give a new characterization of almost \( B \)ézout property in pairs of rings.

**Corollary 3.3.** Let \( D \) be an integral domain with quotient field \( K \). Then the following statements are equivalent:

1. \( D \) is an \( AB \)-domain.
2. \( S := D + XK[X] \) is an \( AB \)-domain.
3. \( (D, K) \) is an \( AB \)-domain pair.
4. \( (S, K(X)) \) is an \( AB \)-domain pair.

**Proof.** (1) \( \iff \) (3) and (2) \( \iff \) (4) By statement (1) of Theorem 3.2, we have the desired results.

(1) \( \iff \) (2) By Corollary 2.3, we have the desired result. \( \square \)

Recall that \( R := (T, M, D) \) is a pullback of canonical homomorphisms

\[
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/M
\end{array}
\]

where \( T \) is an integral domain, \( M \) is a maximal ideal of \( T \), \( \phi : T \to T/M \) is the natural projection, \( D \) is a domain contained in \( K = T/M \) and \( R = \phi^{-1}(D) \). Note that \( R := (T, M, D) \) if and only if \( R \) is contained in \( T \) and shares the maximal ideal \( M \) with \( T \) (see [8] for more details).
As an application of Theorem 3.2, we give necessary and sufficient conditions for \((R, T)\) to be an \(AB\)-domain pair when \(R\) arises from a \((T, M, D)\) construction, where \(M\) is a maximal ideal of \(T\).

**Corollary 3.4.** Let \(R := (T, M, D)\). Then the following statements are equivalent:

(i) \((R, T)\) is an \(AB\)-domain pair.

(ii) \(R\) is an \(AB\)-domain.

**Proof.** By \([2\text{, Lemma 4.5}], T\) is an \(AB\)-domain as an overring of \(R\) and so by using statement (1) of Theorem 3.2, the conclusion is trivial. \(\square\)

**Corollary 3.5.** Assume that \((T, M)\) is a local domain, \(k := qf(D)\) and \(K := T/M\) and \(R := (T, M, D)\). Then the following statements are equivalent:

(i) \((R, T)\) is an \(AB\)-domain pair.

(ii) \(R\) is an \(AB\)-domain.

(iii) \(T\) and \(D\) are \(AB\)-domains, and the extension \(k \subseteq K\) is a root extension.

**Proof.** (i) \(\iff\) (ii) From Corollary 3.4.

(ii) \(\iff\) (iii) From [18\text{, Theorem 2.9}], the conclusion is trivial. \(\square\)

As a consequence of our results, we give a necessary and sufficient condition for the polynomial ring and the power series ring to be an \(AB\)-domain.

**Corollary 3.6.** Let \(R\) be an integral domain. The following statements are equivalent:

(i) \(R\) is a field.

(ii) \(R[X]\) is an \(AB\)-domain.

(iii) \(R[[X]]\) is an \(AB\)-domain.

(iv) \((R[X], R(X))\) is an \(AB\)-domain pair.

(v) \((R[[X]], R((X)))\) is an \(AB\)-domain pair.

**Proof.** (ii) \(\iff\) (iv) and (iii) \(\iff\) (v) From statement (1) of Theorem 3.2.

(iii) \(\implies\) (i) and (iii) \(\implies\) (i) Using similar argument as in the proof of claim 1, statement (2) of Theorem 3.2 by replacing \(R[t]\) by \(R[X]\) (resp., \(R[[X]]\)), it follows that \(R\) is a field.

(i) \(\implies\) (ii) Assume that \(R\) is a field. Then \(R[X]\) is a Bézout domain and so is an \(AB\)-domain.

(i) \(\implies\) (iii) If \(R\) is a field, then \(R[[X]]\) is a valuation domain and so is an \(AB\)-domain. \(\square\)

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