Yang-Mills Equation and Bures Metric

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Abstract

It is shown that the connection form (gauge field) related to the generalization of the Berry phase to mixed states proposed by Uhlmann satisfies the source-free Yang-Mills equation *
\[ *D *D \omega = 0, \]
where the Hodge operator is taken with respect to the Bures metric on the space of finite dimensional nondegenerate density matrices.

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1 Connection form and Bures metric

In the last years the Riemannian Bures metric - the quantum analog of the Fisher information in classical statistics - became an interesting object of geometrical investigations as well as of applications. In this paper we show a further interesting feature of this metric, namely, that the gauge field defining this metric on the background of purifications of mixed states fulfills the source-free Yang-Mills equation.

Let \( \mathcal{H} \) be a Hilbert space of finite dimension \( n \). The concept of purification of mixed states leads in the case of the algebra \( \mathcal{A} := \mathcal{B}(\mathcal{H}) \) to the following principal \( U(\mathcal{H}) \)-bundle: The bundle space \( \mathcal{P}^1 \) is the manifold of invertible normalized (\( \text{Tr} W^* W = 1 \)) Hilbert-Schmidt operators and the base space \( \mathcal{D}^1 \) is the manifold of faithful states on \( \mathcal{A} \). Define the bundle projection \( \pi : \mathcal{P}^1 \to \mathcal{D}^1 \) by \( \pi(W)(a) := \langle W, aW \rangle_{HS} = \text{Tr} WW^* a, a \in \mathcal{A} \). Actually, by the polar decomposition of the operators \( W \) we get a \( U(\mathcal{H}) \)-principal bundle. \( W \) is called a purification of \( \pi(W) \) because it represents a pure state of the algebra \( \mathcal{B}(\mathcal{S}_2(\mathcal{H})) \supset \mathcal{A} \) and reduces to the state \( \pi(W) \) of \( \mathcal{A} \) \((10) \). \( \mathcal{S}_2(\mathcal{H}) \) denotes the space of Hilbert-Schmidt operators.

Although the space of Hilbert-Schmidt operators coincides with \( \mathcal{A} \) for finite dimension we used this terminology to emphasize the underlying hermitian product \( \langle \cdot, \cdot \rangle_{HS} \) on \( \mathcal{P}^1 \). Its real part defines an \( U(\mathcal{H}) \)-invariant Riemannian metric \( g \) on the bundle space \( \mathcal{P}^1 \) and gives rise to a connection on the principal bundle, declare the horizontal spaces to be the orthogonal complements of the vertical directions. The corresponding connection form (gauge field) we denote by \( \omega \). It was proposed by Uhlmann generalizing the Berry phase to mixed states \((10)\). Its curvature form \( D\omega \) we denote by \( \Omega \). Moreover, the Riemannian metric and the connection on \( \mathcal{P}^1 \) induce a Riemannian metric \( g^B \) on the base space \( \mathcal{D}^1 \), define the length of a vector tangent to \( \mathcal{D}^1 \) as the length of any of its horizontal lifts, see formula \((11)\). It
turns out that this Riemannian metric first given in [4] is just the Riemannian version of the Bures distance \( \varrho, \mu \mapsto d(\varrho, \mu) := (2 - 2\text{Tr}\ (\varrho^{\frac{1}{2}} \mu \varrho^{\frac{1}{2}}) )^\frac{1}{2} \) (see [4],[8]). More precisely, \( g^B_\varrho = \frac{1}{2}\text{Hessian}\{\mu \mapsto d(\varrho, \mu)^2\}\}_{\mu=\varrho} \) and in local coordinates \( \{\mu_i\} \) it is represented by a half of the matrix of second order partial derivatives of \( \mu \mapsto d(\varrho, \mu)^2 \) at \( \mu_i = \varrho_i \).

Finally, note that the bundle projection takes the form

\[
\pi : P^1 \rightarrow D^1
\]

where \( D \) is the manifold of all faithful positive linear forms on \( \mathcal{A} \) and \( P \) is the dense in \( S_2(\mathcal{H}) \) subspace of invertible operators. The connection and the Riemannian metrics are defined analogously. We use the same symbols for the projection, connection, curvature and metrics on both bundles, which are by construction Riemannian submersions (comp. e.g. [2]).

The aim of this paper is to prove the following

**Theorem:**

The curvature form \( \Omega \) on the principal bundle \( P^1(D^1, U(\mathcal{H}), g^B) \) (resp. \( P(D, U(\mathcal{H}), g^B) \)) fulfills the source-free Yang-Mills equation \( \ast D \ast \Omega = 0 \), where \( \ast \) is the horizontal lift of the Hodge star w.r. to the Bures metric on the base space.

In [3] it was shown, that the Yang-Mills equation holds for \( \dim \mathcal{H} = 2 \) in the case of states. The conjecture that this is true for general \( n \) is due to G. Rudolph. Moreover, for \( n = 2 \) the Einstein-Yang-Mills equation is fulfilled with a certain cosmological constant ([4]). These observations can be understood from the general point of view: The connection on \( P^1 \) is reducible to a connection on the reduced \( SU(n) \)-subbundle \( P^r := \{W \mid \text{Tr} WW^* = 1, \text{Det} W > 0\} \subset P^1 \) and one verifies that for \( n = 2 \) (only) the fibers are isometric to \( SU(n) \) and totally geodesic (or in the language of field theory: the scalar fields on \( D^1 \) describing the vertical part of the bundle metric are constant, compare e.g. [4], 9.56/64).

Our theorem gives, essentially, a solution invariant under the natural left \( U(n) \)-action of the YM-equation on the principal bundle \( \text{Gl}(n, \mathbb{C}) \to \text{Gl}(n, \mathbb{C})/U(n) \) (resp. \( S^{2n^2-1} \supset P^1 \to D^1 \)), where the base space is equipped with the Bures metric. If we regard these solutions as gauge fields on \( D^{(1)} \) they are invariant under the induced \( U(n) \)-conjugation. By inspection one can see, that the bundles are not Einstein-Yang-Mills systems for \( n > 2 \).

## 2 Notations and preliminary formulae

First we identify \( \mathcal{H} \) with \( \mathbb{C}^n \) and express the above structures in terms of matrices. The faithful positive linear forms are represented by nonsingular density matrices;

\[
P^{(1)} = \{ D \in M_{nn}(\mathbb{C}) \mid D > 0 \ (\text{Tr} D = 1) \ \}
\]

and the bundle space becomes

\[
P^{(1)} = \{ W \in M_{nn}(\mathbb{C}) \mid \text{Det} W \neq 0 \ (\text{Tr} WW^* = 1) \ \}
\]

The bundle projection takes the form \( \pi(W) = WW^* \) and the bundle metric is \( g(T_1, T_2) = \text{Re} \langle T_1, T_2 \rangle_{HS} = \frac{1}{2}(\text{Tr} T_1 T_2 + T_1 T_2^\dagger), \ T_i \in TP \). The vertical spaces are generated by the \( U(n) \)-action so that

\[
\ker \pi_{\ast W} = \{ WA \mid A = -A^* \in u(n) \subset M_{nn}(\mathbb{C}) \}\)

For their orthogonal complements one obtains in the not normalized case

\[
T_{W}^{\text{hor}} P := (\ker \pi_{\ast W})^\perp = \{ GW \mid G = G^* \in M_{nn}(\mathbb{C}) \}.
\]
From now on $G$ will be an hermitian and $A$ an antihermitian matrix. Before we express the further geometric quantities we introduce some notations. Let $D := WW^*$ and $\tilde{D} := W^*W$. By $L, R, \tilde{L}, \tilde{R}$ we denote the operators (depending on $W$) of left (resp. right) multiplication by $D$ (resp. $\tilde{D}$). Moreover, we use the notations $x := LR^{-1} = \text{Ad} D$ and analogously for $\tilde{x}$. Note that all these operators are positive, especially the spectrum of $L, \tilde{L}, R, \tilde{R}$ equals the n-fold spectrum of $D$ whereas the spectrum of $x, \tilde{x}$ consists of all quotient of eigenvalues of $D$. Of course, left and right multiplication operators commute.

Since the proof of the theorem will be a calculation on the bundle space we do not need the Bures metric $g^B$ on the base space explicitly. For completeness we remember that

$$g^B = \frac{1}{2} \text{Tr} d g \frac{1}{L+R}(d g), \quad \text{i.e.} \quad g^B(X,Y) = \frac{1}{2} \text{Tr} XG,$$

where $DG + GD = Y; X, Y \in T_D P$ (see [11]). But note, that in affine coordinates (e.g. using the Pauli matrices for $n=2$) the metric becomes very complicated for $n > 2$ and no good parametrization seems to be available for general $n$.

$\nabla$ will be the covariant derivative related to the flat metric $g$ on $P$. Later on we will need e.g.

$$\nabla_T W = T, \quad \nabla_T W^* = T^*.$$  (6)

Using the Leibniz rule and the parallelity of the matrix multiplication this implies

$$(\nabla GW x)(T) = (\nabla GW D)TD^{-1} - DTD^{-1}(\nabla GW D)D^{-1} = [G + x(G), x(T)].$$  (7)

Finally, for the covariant derivative $\nabla^1$ of the submanifold $P^1$ we have

$$\nabla^{1}_{T} T' = \nabla_{T} T' - g(\nabla_{T} T', N)N,$$  (8)

where $N$ is the (normalized) vector field normal to $P^1$ given by $N_W = W$.

The connection form of the above described connection equals

$$\omega(T) = \frac{1}{L+R} (W^*T - T^*W), \quad T \in T_W P.$$  (9)

Indeed, if $T$ is vertical, $T = WA$, then $\omega(T) = A$ and if $T$ is horizontal, $T = GW$, then $\omega(T) = 0$. The curvature form takes the following value if its first argument is horizontal:

$$\Omega(GW, T) = 2W^* \frac{1}{1+x} \left( [G, \frac{1}{1+x} (TW^{-1} + x(W^{-1}T^*))] \right) W^{-1}.$$  (10)

In fact, if $T = WA$ is vertical then $\Omega(GW, WA) = 0$. If $T = G'W$ is horizontal define the horizontal vector fields $T, T'$ by $T_W = GW$ and $T'_W = G'W$. Then

$$\Omega(GW, G'W) = \Omega(T, T') = D\omega(T, T') = -\omega([T, T']) = -\omega(\nabla_T T' - \nabla_{T'} T)$$

$$= \omega([G, G']W) = 2W^* \frac{1}{L+R} (W^*[G, G']W) = 2W^* \frac{1}{1+x} ([G, G']) W^{-1}$$  (11)

But the last term equals the right hand side of (10) in this case.
3 Proof of the theorem

First we show the assertion in the not normalized case. We have to show \(*D* \Omega(T) = 0\) for all horizontal vectors \(T\) at \(W_\alpha \in \mathcal{P}\). Let \(T_\alpha := G_\alpha W_\alpha, \alpha = 1, \ldots, n^2\) be an orthonormal basis of horizontal vectors at \(W_\alpha\). Then

\[
* D * \Omega(T) = -\sum_\alpha (\nabla_{T_\alpha} \Omega)(T_\alpha, T)
\]

and we will first deal with the summands on the right hand side. For this purpose fix \(G\) and the \(G_\alpha\)-s and define horizontal vector fields by \(\mathcal{T}_W = G_\alpha W, \mathcal{T}_W = GW\). By \(\Box, \Box, \Box, \Box, \Box\) and obvious derivation rules we obtain

\[
\Omega(\nabla_{G_\alpha W} G_\alpha W, GW) = \Omega(G_\alpha^2 W, GW) = 2W^* \left( \frac{1}{1+x} [G_\alpha^2, G] \right) W^{*-1} \tag{13}
\]

\[
\nabla_{G_\alpha W} \Omega(G_\alpha W, GW) = \nabla_{G_\alpha W} \Omega(G_\alpha W, G G_\alpha W)
\]

\[
= 2W^* \left( \frac{1}{1+x} [G_\alpha, \frac{1}{1+x} (G G_\alpha + x(G_\alpha G))] \right) W^{*-1} \tag{14}
\]

\[
\nabla_{G_\alpha W} \Omega(G_\alpha W, GW) = \nabla_{G_\alpha W} \left( 2W^* \frac{1}{1+x} ([G_\alpha, G]) W^{*-1} \right)
\]

\[
= 2W^* \left( [G_\alpha, \frac{1}{1+x} [G_\alpha, G]] + \left( \nabla_{G_\alpha W} \frac{1}{1+x} [G_\alpha, G] \right) W^{*-1} \right)
\]

\[
= 2W^* \left( \frac{x}{1+x} \left( [G_\alpha, \frac{1}{1+x} [G_\alpha, G]] \right) W^{*-1} \right). \tag{15}
\]

Using these formulae we get

\[
(\nabla_{T_\alpha} \Omega)(\mathcal{T}_\alpha, T) = (\nabla_{G_\alpha W} \Omega)(G_\alpha W, GW)
\]

\[
= \nabla_{G_\alpha W} \Omega(G_\alpha W, GW) - \Omega(\nabla_{G_\alpha W} G_\alpha W, GW) - \Omega(G_\alpha W, \nabla_{G_\alpha W} GW)
\]

\[
= W^* \frac{1}{1+x} \left( [G_\alpha, \frac{1-x}{1+x} [G_\alpha, G]] - \frac{1}{1+x} (G G_\alpha + x(G_\alpha G)) \right) - [G_\alpha^2, G]) W^{*-1}. \tag{16}
\]

By a suitable choice of the basis of \(\mathcal{H}\) at the very beginning we may suppose that \(W_\alpha = \Lambda U\) with diagonal \(\Lambda\) and unitary \(U\). Moreover, by the equivariance of \(*D* \Omega\) it is sufficient to prove the assertion for \(U = I\). Hence, let \(W_\alpha\) be diagonal, say \(W_\alpha = \Lambda = \sum_i \lambda_i E_{ii}, \lambda_i \in \mathbb{R}_+, \) and put

\[
H_{ij} := \frac{1}{\sqrt{\lambda_i + \lambda_j}} (E_{ij} + E_{ji}), \quad H_i := \frac{1}{\sqrt{2}} H_{ii}, \quad \overline{H}_{ij} := \frac{i}{\sqrt{\lambda_i + \lambda_j}} (E_{ij} - E_{ji}), \quad \tag{17}
\]

where the \(E_{ij}\)-s are the standard \(n \times n\)-matrices. We define the hermitian matrices \(G_\alpha\)-s as the elements of the set

\[
\{ H_i \mid 1 \leq i \leq n \} \cup \{ H_{ij} \mid 1 \leq i < j \leq n \} \cup \{ \overline{H}_{ij} \mid 1 \leq i < j \leq n \}. \tag{18}
\]

It is easy to check that the horizontal vector fields \(\mathcal{T}_\alpha\) are really orthonormal at \(\Lambda\). Our next observation facilitates the summation in \(\Box\).

\[
\sum_\alpha (\nabla_{T_\alpha} \Omega)(T_\alpha, T)
\]

\[
= \sum_i (\nabla_{H_i} \Omega)(H_i W, T) + \sum_{i < j} (\nabla_{H_{ij}} \Omega)(H_{ij} W, T) + \sum_{i < j} (\nabla_{\overline{H}_{ij}} \Omega)(\overline{H}_{ij} W, T)
\]

\[
= \frac{1}{2} \sum_{i,j} (\nabla_{H_{ij}} \Omega)(H_{ij} W, T) + (\nabla_{\overline{H}_{ij}} \Omega)(\overline{H}_{ij} W, T). \tag{19}
\]
Inserting (16), (17) into (19) and taking into account that the operator \( \text{Ad} W^* (1 + x)^{-1} \) has a trivial kernel it remains to show that the sum

\[
\sum_{i,j,\pm} \pm \frac{1}{\lambda_i + \lambda_j} \left\{ - [(E_{ij} \pm E_{ji})^2, G] + \left[ E_{ij} \pm E_{ji}, \frac{1-x}{1+x} [E_{ij} \pm E_{ji}, G] - \frac{1}{1+x} (G(E_{ij} \pm E_{ji}) + x((E_{ij} \pm E_{ji}) G)) \right] \right\}
\]

vanishes at the point \( \Lambda \) for all hermitean \( G \). Using \( x(E_{ij}) |_{W=\Lambda} = \frac{\lambda_i}{\lambda_j} E_{ij} \) and similar formulae for the other operators involving \( x \) we get after a straightforward calculation that both partial sums related to the different signs \( \pm \) in (20) vanish individually at \( \Lambda \) even for every \( G = E_{kl} \). This finishes the proof in the not normalized case.

To proof the assertion in the case of states let \( \Lambda \in \mathcal{P}^1, \sum \lambda_i^2 = 1 \). Of course, formula (12) is valid independently of the choice of the orthonormal basis \( \{T_\alpha\} \). Thus we may also set \( G_1 = \mathds{1} \) and complete the list of the \( G_\alpha \) suitable. Then \( T_1 \Lambda = G_1 \Lambda = \Lambda \) is normal to the submanifold \( \mathcal{P}^1 \subset \mathcal{P} \) and the remaining obtained vectors \( G_\alpha \Lambda \) are tangent to \( \mathcal{P}^1 \) and constitute an orthonormal basis of horizontal vectors of \( \mathcal{P}^1 \) at \( \Lambda \). From (19) we see that the vector \( T_1 \Lambda = \Lambda \) \( (G_\alpha = G_1 = \mathds{1}) \) does not contribute to (12). Moreover, for the remaining vectors \( G_\alpha \Lambda, \alpha > 1 \), holds \( (\nabla_{G_\alpha} \Lambda) (G_\alpha \Lambda, GA) = (\nabla_{G_\alpha} \Lambda) (G_\alpha \Lambda, GA) \). Indeed, using (8) we obtain

\[
(\nabla_{G_\alpha} \Lambda)(G_\alpha \Lambda, GA) = (\nabla_{G_\alpha} \Lambda)(G_\alpha \Lambda, GA)
- g(\nabla_{G_\alpha} \Lambda G_\alpha \Lambda, \Lambda) \Omega(GA, \Lambda) + g(\nabla_{G_\alpha} \Lambda GA, \Lambda) \Omega(G_\alpha \Lambda, \Lambda).
\]

But the last two additional terms vanish by (11) (set \( G' = \mathds{1} \) in (11)). Thus the vanishing of (12) in the not normalized case implies the vanishing in the case of states. This finishes the proof of the theorem.

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