Virasoro Action and Virasoro Constraints on Integrable Hierarchies of the $r$-Matrix Type

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Abstract

For a large class of hierarchies of integrable equations admitting a classical $r$–matrix, we propose a construction for the Virasoro algebra action on the Lax operators which commutes with the hierarchy flows. The construction relies on the existence of dressing transformations associated to the $r$-matrix and does not involve the notion of a tau function. The dressing-operator form of the Virasoro action gives the corresponding formulation of the Virasoro constraints on hierarchies of the $r$–matrix type. We apply the general construction to several examples which include KP, Toda and generalized KdV hierarchies, the latter both in scalar and the Drinfeld-Sokolov formalisms. We prove the consistency of Virasoro action on the scalar and matrix (Drinfeld-Sokolov) Lax operators, and make an observation on the difference in the form of the string equation in the two formalisms.
0. Introduction

A striking feature of the completely integrable systems, besides the integrability by itself, is the number and diversity of their physical applications \[^1\]. Yet another one has been discovered recently: completely integrable systems appear in the study of non-perturbative two-dimensional quantum gravity in the formalism of Matrix Models \[^2\].

Integrable equations turn out to govern exact “renormalization-group” evolutions in the space of coupling constants. As there are infinitely many coupling constants in gravitational theories, it is perhaps less surprising that one actually obtains hierarchies of integrable equations.

The fact that really came about as a surprise, that integrable equations appear at all, might seem less unexpected if one notes that orthogonal polynomials, which are a standard technique to work with matrix integrals \[^2\], do in a sense solve non-linear integrable equations.

Further, solutions to integrable hierarchies that do come from a matrix model are by no means general (nor of much independent interest to the experts in non-linear integrable equations!). The special class of solutions related to matrix models is singled out by the highest-weight conditions on the tau function with respect to a certain Virasoro algebra \[^1\]. These conditions are usually called the Virasoro constraints. They were first recognized in the apparently weaker guise of the string equation \[^2\], which is in fact a consequence of the vanishing condition under the $L_{-1}$ Virasoro generator. For a number of cases, the machinery of integrable systems allows one to prove all the $L_{\geq 0}$ vanishing conditions starting from the string equation.

The form of the string equation, however, is tied up with particular properties of the linear system associated to the hierarchy. The Virasoro constraints, on the other hand, appear to have a more universal meaning, as they are of essentially the same form for a wide class of integrable hierarchies. Thus, despite the success of the use of the string equation in applications, \[^1\], we prefer to promote the Virasoro constraints to a “first-principle” of a hierarchial description of gravity-coupled two-dimensional theories. More generally, one should not be limited to the Virasoro constraints alone: Much as in the historical development of conformal field theory in two dimensions, higher symmetry algebras extending Virasoro have been identified in the analysis of matrix models, namely the $W$–algebras \[^1\]. Expected next might be highest-weight conditions with respect to a semi-direct product of Virasoro and Kac-Moody algebras. Such a possibility for the $N$-KdV hierarchies was pointed out in \[^34\].

Thus, let us assume that the constraints with respect to some good algebra imposed on integrable hierarchies may serve as a counterpart of the field-theoretic description. Then it is not only the algebra but also the hierarchy itself, that is optional, and our aim in this paper will be to investigate Virasoro constraints on integrable hierarchies of as general form as possible. We will in fact propose a rather general construction for the Virasoro algebra action on integrable hierarchies and then apply it to the study of the constraints as the invariance conditions under this action.

The formalism which we are going to use, and which allows considerable generality, is

\[^1\] Any list of references appropriate for the present paper, which is not a special review article, would be inadequate.

\[^2\] This new development was initiated by refs. \[^6\], \[^13\], \[^21\]; for a review see, for instance, \[^2\] and references therein.
that of the classical $r$—matrix \([35][36]\) and dressing transformations related to the $r$—matrix. Integrability of essentially all integrable systems can be 'explained' by an underlying $r$—matrix. Our starting point will thus be an abstract $r$—matrix satisfying the classical Yang-Baxter equation.

In contrast with most of the recent papers dealing with Virasoro constraints in Matrix Models, we do not rely on the existence of a tau function. The Virasoro algebra will be represented in our approach on the Lax operators of an abstract hierarchy, i.e., on certain coadjoint orbits. By specializing our general construction to several popular examples we will be able to reproduce the known results on the Virasoro algebra action.

We therefore start in Sect.1 with defining hierarchies of the $r$—matrix type. Further, in Sect. 2, we specialize to those hierarchies which admit a Virasoro algebra action. This, of course, requires making additional assumptions on the ingredients of the $r$—matrix formalism, so we proceed with first stating ”kinematical” axioms and then translate them into a construction of the Virasoro action compatible with the equations of a hierarchy.

In sections 3 and 4 we consider a number of examples of the general construction. The KP case, chosen for its special simplicity, is purely illustrative, while the relation between Virasoro constraints and string equations in two different formalisms for generalized KdV-hierarchies reveals certain subtleties and may be of an independent interest to the experts.

1 Classical $r$—matrices and integrable hierarchies

In this section we recall a number of basic facts about classical $r$—matrices and their use for the construction of integrable systems. The readers familiar with refs.\([35][36]\) may skip to the next section (however, we borrow the presentation partly from \([25]\)).

Let $(\mathfrak{g}, [\ , \ ])$ be a Lie algebra and $\mathfrak{g}^*$ its dual space.

1.1. Definition. The classical $r$—matrix is a linear mapping

$$r : \mathfrak{g}^* \to \mathfrak{g}$$

such that the classical Yang-Baxter equation (CYBE)

$$r(\text{ad}^*_{r\xi}\eta + \text{ad}^*_{r\eta}\xi) + [r\xi, r\eta] = 0$$

is satisfied for all $\xi, \eta \in \mathfrak{g}^*$. Here $^tr$ denotes the transposed mapping $^tr : \mathfrak{g}^* \to \mathfrak{g}$.

1.1.1. Remark. To make contact with the usual ‘tensor’ formulation of the classical Yang-Baxter equation, one uses the isomorphism

$$\text{Hom}(\mathfrak{g}^*, \mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}$$

to identify $r$ with a tensor

$$r = r^{ab}t_a \otimes t_b \in \mathfrak{g} \otimes \mathfrak{g}.$$ 

With the components of $r$ w.r.t. a basis $\{t_a\}$ in $\mathfrak{g}$ thus defined, we have

$$r\xi = \xi_a r^{ab} t_b, \quad ^tr = \xi_a r^{ba} t_b.$$
Now contracting the LHS of the CYBE, which is an element of $\mathfrak{g}$, with a $\zeta \in \mathfrak{g}^*$, we rewrite the result as,

$$- \langle \eta, [r\xi, r\zeta] \rangle - \langle \xi, [r\eta, r\zeta] \rangle + \langle \zeta, [r\xi, ^t r\eta] \rangle$$

where

$$r_{12} = r^{ab} t_a \otimes t_b \otimes 1, \quad r_{13} = r^{ab} 1 \otimes t_a, \quad r_{23} = r^{ab} 1 \otimes t_a \otimes t_b.$$ 

Thus we arrive at the standard form

$$[r_{13}, r_{23}] + [r_{12}, r_{23}] + [r_{12}, r_{13}] = 0$$

1.2. Decomposing $r$ into its symmetric and antisymmetric parts,

$$s = \frac{1}{2}(r + ^t r), \quad a = \frac{1}{2}(r - ^t r),$$

assume that $s$ is invertible and $ad$-invariant:

$$s \circ ad^*_x = ad_x \circ s \quad \forall x \in \mathfrak{g}.$$ 

(We thus invoke a Killing form on $\mathfrak{g}$, which is the case encountered in applications.) Then all the ”degrees of freedom” of an $r-$matrix are contained in its antisymmetric part.

1.3. With the above assumptions on $s$, introduce

$$R = a \circ s^{-1} : \mathfrak{g} \to \mathfrak{g}$$

Then, by virtue of 1.1,

$$[Rx, Ry] - R([Rx, y] + [x, Ry]) = -[x, y]$$

holds for any $x, y \in \mathfrak{g}$. This will be called the modified classical Yang-Baxter equation (mCYBE).

1.4. As a straightforward consequence of the mCYBE note that

$$\mathfrak{g}_- \equiv \frac{1}{2}(1 - R)\mathfrak{g}$$

is a subalgebra in $\mathfrak{g}$. For any $x \in \mathfrak{g}$ we denote

$$x_- = \frac{1}{2}(1 - R)x$$

The mapping $\frac{1}{2}(1 - R) : \mathfrak{g} \to \mathfrak{g}_-$ provides us with an embedding $\mathfrak{g}_+^* \to \mathfrak{g}_+$. We will identify $\mathfrak{g}_+^*$ with its image in $\mathfrak{g}_+^*$. Then for an $m \in \mathfrak{g}_-$ one should clearly distinguish between the representation

$$ad^*_m : \mathfrak{g}_+^* \to \mathfrak{g}_+^*$$

and the properly coadjoint action

$$ad^*_(-)m : \mathfrak{g}_+^* \to \mathfrak{g}_+^*$$
of the $\mathfrak{g}_-$ algebra on its dual space.

Define, formally, $G_- = \exp \mathfrak{g}_-$ to be the Lie group of $\mathfrak{g}_-$. We will call it the group of dressing transformations and its elements, dressing operators.

1.5. Next, let us fix an element

$$\Lambda \in \mathfrak{g}_+$$

such that

$$\text{ad}_{(\cdot)}^\ast \mathfrak{g}_- \Lambda = 0.$$ 

Viewing $\Lambda$ as an element of $\mathfrak{g}^*$, let $Z_\Lambda$ be the maximum commutative subalgebra of the invariance subalgebra of $\Lambda$ in $\mathfrak{g}$, i.e., of

$$\{x \in \mathfrak{g} \mid \text{ad}_{x}^\ast \Lambda = 0\}$$

(In applications to the Kac-Moody algebras, for instance, this invariance subalgebra would by itself be commutative for regular $\Lambda$.)

1.6. The elements of the $\text{ad}^\ast$-orbit of $G_-$ in $\mathfrak{g}^*$ going through $\Lambda$, will be called Lax operators $Q$

$$Q \equiv Q(K) = \text{Ad}_K^\ast \Lambda \in \mathfrak{g}^* \quad K \in G_-$$

1.7. For any $a \in Z_\Lambda$ we introduce an evolution on the $\text{Ad}$- and $\text{Ad}^\ast$-orbits of $G_-$

First, for $X = \text{Ad}_K^\ast x$ we set

$$\frac{\partial}{\partial t} X = -[A_-, X]$$

where

$$A \equiv A(K) = \text{Ad}_K^\ast a$$

Similarly, for $\Xi = \text{Ad}_K^\ast \xi$ being an element of the coadjoint orbit,

$$\frac{\partial}{\partial t} \Xi = -\text{ad}_{A_-}^\ast \Xi$$

1.8. A generic evolution equation of the hierarchy that we will associate to the triple $(\mathfrak{g}, r, \Lambda)$ is the $\text{ad}^\ast$-equation of 1.7 imposed on the Lax operator $Q$:

$$\frac{\partial}{\partial t} Q = -\text{ad}_{A_-}^\ast Q$$

$$Q = \text{Ad}_K^\ast \Lambda \quad A = \text{Ad}_K^\ast a, \quad a \in Z_\Lambda.$$ 

1.9. Lemma. Any two flows of the type 1.8 commute.

Indeed, consider the flows

$$\frac{\partial}{\partial t_i} Q = -\text{ad}_{A_i}^\ast (1-R_i)A_i, Q, \quad i = 1, 2.$$ 

with

$$A_i = \text{Ad}_K^\ast a_i, \quad a_i \in Z_\Lambda.$$ 

\footnote{Some of the evolution flows may happen to be ‘trivial’ (as are, for instance, linear equations). Fewer such ‘trivial’ equations are among those associated to $a \in Z_\Lambda \cap \mathfrak{g}_+$. We will ignore this point, however.}
Then, using 1.7 and 1.8, evaluate
\[ \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} Q = ad^*_{\frac{1}{2}(1-R)A_1,A_2}.Q + ad^*_{\frac{1}{2}(1-R)A_2,A_1}.Q \]
whence the antisymmetrized second derivative equals,
\[ \left( \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_2} - \frac{\partial}{\partial t_2} \frac{\partial}{\partial t_1} \right) Q = ad^*_\omega(1,2).Q \]
with
\[ \omega(1,2) = -\frac{1}{4}(1-R)[(1-R)A_1,A_2] - \frac{1}{4}(1-R)[A_1,(1-R)A_2] + \frac{1}{4}[(1-R)A_1,(1-R)A_2] \]
\[ = -\frac{1}{2}(1-R)[A_1,A_2] \quad \text{(by 1.4)} \]
\[ = -\frac{1}{2}(1-R)Ad_{K}.[a_1,a_2] \]
\[ = 0 \quad \text{(by 1.5)} \]
\]

Thus, picking up different elements of \( Z_\Lambda \), which in applications is usually infinite-dimensional, one can generate an (infinite) set of mutually commuting differential equations which we call the hierarchy associated to \((g,r,\Lambda)\).

At the closing of this section let us present a construction which will be used below to implement symmetries of hierarchies of the above type.

1.10. Every \( v \in g \) defines a vector field \( \hat{v} \) on \( Ad^{-}\) and \( Ad^*\)-orbits of \( G^{-} \). That is, for a function \( f : Ad_{G^{-}}.x \to \mathbb{C} \), the action of \( \hat{v} \) on \( f \) is defined by
\[ (\hat{v}f)(X) = \frac{d}{d\epsilon} f(X - \epsilon \text{ad}_V X) \big|_{\epsilon=0}, \]
\[ X = Ad_K.x \quad V = Ad_K.v \]
(recall that \( (\ldots)_- \) was defined in 1.4). Similarly, for a function \( \phi : Ad_{G^{-}}.\xi \to \mathbb{C} \), we set
\[ (\hat{v}\phi)(\Xi) = \frac{d}{d\epsilon} \phi(\Xi - \epsilon \text{ad}_{V^-}^*.\Xi) \big|_{\epsilon=0}, \]
\[ \Xi = Ad^*_K.\xi. \]
We will denote by \([ [ , ] ]\) the commutator of such vector fields.

1.11. Lemma. The mapping
\[ v \mapsto \hat{v} \]
is a representation of \( g \) in the \([ [ , ] ]\)-algebra of the above vector fields.

Proof follows directly from the mCYBE; the calculation is identical to that of 1.9. □ .
Now we extend the construction of 1.10 to *time-dependent* \( \hat{v} \) that deform the Lax operators.

1.12. **Lemma.** *The condition for a vector field* \( \hat{v} \) *to commute with the hierarchy flows* 1.8, *i.e., with*

\[
\frac{\partial}{\partial t_a} Q = -\text{ad}_{\frac{1}{2}(1-R)\text{Ad}_K.a}^* Q, \quad a \in \mathcal{Z}_\Lambda,
\]

*is*

\[
\text{ad}_{\mathcal{F}_-}^* Q = 0
\]

*where*

\[
\mathcal{F} \equiv \text{Ad}_K. \left( \frac{\partial v}{\partial t_a} + [v, a] \right).
\]

**Proof.** Applying \( \hat{v} \) to deform the equation of motion, we find

\[
\delta \frac{\partial Q}{\partial t} = \text{ad}_{\frac{1}{2}(1-R)\text{Ad}_K.a}^* \mathcal{F}, \quad Q = \text{ad}_{\frac{1}{2}(1-R)\text{Ad}_K.a}^* \mathcal{F} \cdot Q
\]

*where*

\[
\mathcal{V} \equiv \text{Ad}_K.v, \quad \mathcal{A} \equiv \text{Ad}_K.a
\]

On the other hand, we use 1.7 to evaluate \( \frac{\partial}{\partial t} \delta Q \) along the evolution flow. Then in the difference \([\delta, \frac{\partial}{\partial t}] Q\) we again use the mCYBE, and thus demonstrate that it indeed equals \(-\frac{1}{2}\text{ad}_{\mathcal{F}_-}^* Q \). \( \square \).

1.13. *We will actually use the sufficient condition*

\[
\frac{\partial v}{\partial t_a} + [v, a] = 0
\]

*for a time-dependent vector field* \( \hat{v} \) *to commute with a hierarchy equation.*

## 2 Hierarchies with the Virasoro algebra action

Now we specify the construction of the previous section to hierarchies which admit a Virasoro algebra action. This will require adopting certain further postulates concerning the triple \((\mathfrak{g}, r, \Lambda)\): for instance, the Lie algebra \( \mathfrak{g} \) itself has to be 'big' enough for the theory based on it to admit a Virasoro action at all. We start in this section with rather mild assumptions on \((\mathfrak{g}, r, \Lambda)\), and the Virasoro algebra will not appear until 2.5.

The first extension of the data we had in Sect.1 is achieved by allowing \( v \) from 1.10 to take values in the derivation algebra \( \text{Der}\mathfrak{g} \) of \( \mathfrak{g} \).

2.1. **Extend the adjoint action** \( \text{Ad}_G \) to \( \text{Der}\mathfrak{g} \) by

\[
\text{Ad}_g. M = \text{Ad}_g \circ M \circ \text{Ad}_{g^{-1}} M \in \text{Der}\mathfrak{g}
\]

For \( M = \text{ad}_m \) being an inner derivation, this reduces to the \( \text{Ad}_g \) action on \( m \).

Now we start increasing our demands on the objects we are dealing with.
2.2. Assume the mapping $R : g \to g$ defined in 1.3 extends to a linear mapping

$$R : \text{Der} g \to \text{Der} g$$

such that

$$[RM, RN] - R([RM, N] + [M, RN]) = -[M, N]$$

for any $M, N \in \text{Der} g$, and

$$\text{Rad}_x = \text{ad}_{Rx}$$

for inner derivations.

2.3. Comment. For $M$ and $N$ both inner, the above equation will be satisfied by virtue of the mCYBE. When only $N = \text{ad}_n$ is inner, the required relation takes the form of the condition that for any $M \in \text{Der} g$, the mapping

$$[(RM), R] - R \circ M \circ R + M \in \text{End} g$$

maps $g$ into its center. However, an attempt for a characterization of additional data responsible for the desired extension of $R$ to $\mathbf{R}$ does not seem to make much sense unless a considerable extra input is introduced into our formal treatment (note, in particular, that no assumptions have been made on $\text{Der} g / \text{Inn} g$).

On the other hand, recall that outer derivations of a Lie algebra $g$ may be thought of as induced by inner derivations of a ‘bigger’ Lie algebra which contains $g$ as an ideal. Formally, this bigger algebra is nothing but a semidirect product of $g$ with $\text{Der} g$. In some cases this formal construction happens to be itself a ‘good’ Lie algebra, and so all we would need in that case is an $r-$matrix defined on this bigger algebra.

2.4. Given an extension 2.2, one can further generalize the construction of 1.10 to include derivations $M \in \text{Der} g$ instead of $v \in g$. Then 1.11 would still hold with $M \mapsto \hat{M}$ providing a representation of $\text{Der} g$ in the vector fields tangent to the orbits:

$$(\hat{M}f)(\mathcal{X}) = \frac{d}{d\epsilon} f(\mathcal{X} - \epsilon(\text{Ad}_{K.M}) \mathcal{X}) |_{\epsilon=0}.$$ 

In particular, such an $\hat{M}$ can be used to deform the Lax operator $Q$, and, moreover, time-dependent $M$’s can be allowed. Then the sufficient condition for the compatibility of $\hat{M}(t)$ with the hierarchy equations would read

$$\frac{\partial M}{\partial t_a} + \text{ad}_{M,a} = 0$$

Thus only the time-independent part of $M$ can be an outer derivation. However, for notational convenience, we will mostly write $M = \text{ad}_m$ even when $M \notin \text{Inn} g$ and thus $m \notin g$. An appropriate formal setting would again be the semidirect product of $g$ with $\text{Der} g$, but we will not make it explicit. Accordingly, by a slight abuse of notation, we will find it convenient to denote the above vector field as $\hat{m}$ rather than $\hat{M}$.

Now we narrow the class of the triples $(g, r, \Lambda)$ by introducing the Virasoro algebra into the game. To start with, a ‘kinematical’ Virasoro action will be postulated in the form of a Virasoro representation on $g$, while the main point of the construction to follow will consist
in dressing this action, i.e., carrying it over to coadjoint orbits in a way compatible with the hierarchy equations.

2.5. Let there be given a homomorphism $\text{Vir} \to \text{Der}g$ of the Virasoro algebra into the derivation algebra of $g$. Let $\text{ad}_{l_n}$ denote the image of the standard Virasoro generators under this homomorphism. Thus,

$$[\text{ad}_{l_m}, \text{ad}_{l_n}] = (m - n)\text{ad}_{l_{m+n}}.$$ 

2.5.1. Remark. Note that $g$ itself is not required to be big enough to contain the Virasoro algebra as a subalgebra. See 2.4 concerning our notations: we do not require $\text{ad}_{l_j}$ to be inner derivations! Let us note once again (so as not to return to it anymore) that the $l_n$ may be viewed quite rigorously as elements $(0, l_n)$ of the semidirect product of $g$ with Vir.

2.6. Further, assume that the representation of $\text{Vir}$ on $g$ by derivations restricts to $Z\Lambda$. Then we define the derivations $\text{ad}_{l_j}$ with

$$l_j = l_j + \sum_{a \in I} t_a \text{ad}_{l_j.a}$$

where $I$ is a set of (linearly-independent) elements from $Z\Lambda$ and the times $t_a$ label the flows associated to these elements $a$. These $l_j$ are now used to construct vector fields $\hat{l}_j$,

$$(\hat{l}_j \phi)(\Xi) = \frac{d}{d\varepsilon} \phi(\Xi - \varepsilon \left((1 - R) \left(\text{Ad}_K \text{ad}_{l_j}\right)\right) \Xi) \bigg|_{\varepsilon = 0}, \quad \Xi = \text{Ad}_{K^*} \xi, \quad \xi \in g^*.$$ 

(2.4) and $\text{Ad}_K$ defined in 2.1).

2.7. Theorem.

1. Vector fields $\hat{l}_j$, $j \in \mathbb{Z}$, furnish a representation of the Virasoro algebra on the coadjoint orbit $\text{Ad}^*_G \Lambda$ and are compatible with the hierarchy flows associated to $a \in I$, i.e., with the flows

$$\frac{\partial Q}{\partial t_a} = -\text{ad}_{\text{Ad}_K^*.a}Q, \quad a \in I$$

2. Conversely, for fixed $l_j$ and $I \subset Z\Lambda$, the above $l_j$’s are the only time-dependent derivatives $l_j(t)$ on $g$ such that:

(i) $[l_i(t), l_j(t)] = (i - j)l_{i+j}(t)$;

(ii) $l_i(t)$ act as symmetries of the subhierarchy associated to $I \subset Z\Lambda$;

(iii) $l_i(0) = l_j$.

Proof is immediate in view of the above lemmas. 1.11 (together with 2.1 and 2.2, as discussed in 2.4) reduces the commutators $[[l_i, l_j]]$ to the 'bare' commutators $[l_i, l_j]$. Compatibility with the hierarchy equations follows from 1.13 (extended to $\text{Der}g$ as explained in 2.4): the $l_i$’s were constructed so that, indeed,

$$\frac{\partial l_i}{\partial t_a} + \text{ad}_{l_j.a} = 0$$

Conversely, solving the equation of 2.4 for the dependence on $t_a$, we find

$$l_i(t) = \left(\exp \sum_{a \in I} t_a \text{ad}_a\right) l_i(0)$$

\footnote{that is, for a fixed homomorphism $\text{Vir} \to g$ and a fixed subhierarchy that one wishes to consider.}
which gives the above $I_i(t)$ in view of 1.5. □.

2.8. Later on we will encounter the following specialization of 2.7: suppose there exist certain elements $A_m \in \mathcal{Z}_\Lambda$, $m \in \mathbb{Z}\setminus N\mathbb{Z}$, such that

$$\text{ad}_{i_m}.A_n = -\frac{n}{N} A_{n+Nm}.$$ 

Then the time-dependent Virasoro generators take the form

$$I_j = l_j + \sum_{\alpha=1}^{N} \sum_{i=0}^{K} \left( i + \frac{\alpha}{N} \right) t_{Ni+i\alpha} A_{N(i+j)+\alpha}, \quad j \in \mathbb{Z}$$

which involves the hierarchy times $t_{Ni+i\alpha}$ with $i \leq K$ and $\alpha \in \{1, 2, \ldots, N-1\}$, thus guaranteeing the consistency with the corresponding hierarchy flows. Note that to achieve consistency with all the flows one has to make $K$ infinitely large, thus allowing infinite sums of elements of $\mathfrak{g}$.

Also, given the $l_j$’s in this particular case, one can redefine them as

$$l_j \mapsto l_j + \beta(j)A_{Nj}$$

where the Virasoro commutation relations would require

$$j\beta(j) - k\beta(k) = (j - k)\beta(j + k)$$

or,

$$\beta(j) = -j\beta(-1) + (j + 1)\beta(0)$$

which we rewrite by introducing constants $J$ and $q$, as

$$l_j(J,q) = l_j + (Jj + q)A_{Nj}$$

These derivations of $\mathfrak{g}$ still obey the Virasoro algebra and can be used instead of the $l_j$ to construct time-dependent derivations $I_j$.

2.9. Remarks. 1. Let us point out a heuristic construction for the $l_j$ and $A_n$, which, however, works in a number of particular cases. First, find a derivation $\text{ad}_p \in \text{Der}\mathfrak{g}$ such that

$$\text{ad}_{-p}.\Lambda = \Lambda.$$ 

Again, $\text{ad}_p$ is not necessarily an inner derivation. Second, by the use of the Killing form, $\Lambda$ can be considered as an element of $\mathfrak{g}$, and, further, its powers $\Lambda^n \in U\mathfrak{g}$ may (and do in many cases) happen to lie in fact in $\mathfrak{g}$ at least for certain values of $n$ (such as $n \in \mathbb{Z}\setminus N\mathbb{Z}$). Finally, using an appropriate associative structure, one can construct the $l_m$ as

$$l_m = \frac{1}{N} p\Lambda^{Nm}.$$ 

2. To continue the previous remark, note that the choice of $p$ is by no means unique: the above requirements may only determine $p$ modulo $\mathbb{Z}_\Lambda$. A particular choice of $p$ is the choice
of a representative in $\text{Der}g/\text{ad}_{z_A}$. However, we will not discuss this issue in any detail in the present paper, as it would require introducing more special assumptions.

We finally recall that one of our motivations was to build a formalism for dealing with the Virasoro constraints on integrable hierarchies.

2.10. The highest-weight conditions with respect to the Virasoro representation constructed, take the form

$$\mathcal{L}_j \equiv \left( \text{Ad}_K \left( l_j + \sum_{a \in I} t_a \text{ad}_{l_j} a \right) \right)_- = 0, \quad j \geq 0.$$ 

For the reader who prefers an abuse of his patience to the abuse of notations, we rewrite the constraints more rigorously as

$$\mathcal{L}_j \equiv \frac{1}{2} (1 - R) \left( \text{Ad}_K \left( \text{ad}_{l_j} + \sum_{a \in I} t_a [\text{ad}_{l_j}, \text{ad}_a] \right) \right)_- = 0, \quad j \geq 0$$

with $\text{Ad}_K$ defined in 2.1 and only $\text{ad}_a$ being in general an inner derivation of $g$.

These constraints can be imposed “off-shell”, i.e., independently of the hierarchy equations, just as constraints on the dressing operator $K \in G_-$. They in an obvious way carry a dependence on the $r$–matrix chosen, as well as the Lie algebra $g$.

In particular, in the case discussed in 2.8 we find the Virasoro constraints of the form

$$\mathcal{L}_j \equiv \left( \text{Ad}_K \left( l_j + \sum_{a=1}^N K \sum_{i=0}^j i + \frac{\alpha}{N} t_{N_+} A_{N(i+j)+1} \right) \right)_- = 0, \quad j \geq 0.$$ 

Recall that these constraints are consistent with only a finite subset of equations of the hierarchy associated to $(g, r, \Lambda)$. Below, in Sect.3, we will formally allow $K$ to be infinitely large, which will allow us to achieve consistency with all the equations of certain hierarchies.

3 Virasoro generators and Virasoro constraints on the KP, $N$-KdV and Toda hierarchies

KP

3.1. For the KP hierarchy the Lie algebra $g = \psi\text{Diff}$ is that of arbitrary order pseudodifferential operators in the derivation $D = \partial/\partial x$:

$$g = \psi\text{Diff} \ni \left\{ F = \sum_{i=-\infty}^n f_i D^i \right\}$$

where $f_i$ are (scalar) functions of $x$.

3.2. The standard $r$–matrix is of the form

$$RF = \left( \sum_{i \geq 0} f_i D^i - \sum_{i < 0} f_i D^i \right).$$
Then
\[ F_- \equiv \frac{1}{2} (1-R) F = \sum_{i<0} f_i D^i \]
and \( G_- \) is the group of operators of the form
\[ K = 1 + \sum_{n \geq 1} w_n D^{-n}, \]
with its Lie algebra \( g_- \) being the algebra of all negative order pseudodifferential operators.

3.3. The trace functional \([1]\) allows us to identify \( g^*_- \) with \( g^*_+ \), which is the algebra of all differential operators.

3.4. Let us choose \( \Lambda = D \in g_+ \approx g^-_* \)
For this, indeed (see 1.5)
\[ \text{Ad}_{(-)K} D = (KDK^{-1})_+ = D \]
The hierarchy equations read,
\[ \frac{\partial Q}{\partial t_n} = -[(Q^n)_-, Q], \quad Q = KDK^{-1}, \quad n \geq 1. \]

3.5. The Virasoro generators \( l_n \) admit a construction in the spirit of 2.9:
\[ p = xD, \quad [-p, D] = D \]
\[ l_n = pD^n = xD^{n+1} \]
and, further,
\[ A_n = D^n, \]
so that the \( (J,q) \)-dependent generators of 2.8 become,
\[ \mathfrak{l}_n(J,q) = xD^{n+1} + JnD^n + qD^n + \sum_{m \geq 1} t_m m \ D^{m+n} \]

Finally, the extensions, required in 2.1 and 2.2, of the \( \text{Ad}_{G_-} \)-action on all the derivations involved, are achieved most straightforwardly, and so the vector fields \( \hat{l}_n(J,q) \) act on the Lax operator \( Q \) via
\[ \hat{l}_n(J,q).Q = -[\mathfrak{L}_n(J,q), Q] \]
with
\[ \mathfrak{L}_n(J,q) = \left( K \left( xD^{n+1} + JnD^n + qD^n + \sum_{m \geq 1} t_m m \ D^{m+n} \right) K^{-1} \right)_-. \]

This form of Virasoro generators on the KP hierarchy was first arrived at in [29] by a direct calculation from the well-known expression for the Virasoro generators acting on the tau function. A geometric interpretation of \( J \) and \( q \) was also given there.

\[ ^5 \text{In a more recent paper [17] these generators were rederived and also reinterpreted from the infinite-Grassmannian point of view.} \]
3.6. There exists, however, a yet more general expression for \( l_n \) and therefore for the above \( \mathcal{L}'s \). To arrive at it in a systematic way, recall that the representation \( Q = KDK^{-1} \) with \( K \in G_- \) for a pseudodifferential operator \( Q \) such that \( Q_+ = D \), is not unique: \( K \) can be multiplied from the right by pseudodifferential operators from \( G_- \) with constant coefficients (so as to commute with \( D \)). This arbitrariness has no effect on the evolution equations (see 3.4), but does change the terms \( KxD^{n+1}K^{-1} \) as

\[
KxD^{n+1}K^{-1} \mapsto KxD^{n+1}K^{-1} + \sum_{m \leq -1} h_m D^{m+n}
\]

with constant \( h_m \). This evidently can be viewed as a redefinition of \( p \),

\[
p \mapsto xD + \sum h_m D^m,
\]

which is precisely the arbitrariness discussed in 2.9.2: \( p \) is determined unambiguously only as an element of \( \text{Der}g/\text{ad}_z \). We will stay with the above choice for \( p \) with all \( h_m = 0 \).

3.7. As to the Virasoro constraints, they take a very simple form \( \mathcal{L}_n = 0 \) for \( n \geq 0 \). Choosing for simplicity \( q = J \), a generating expression for the constraints may be written as

\[
(K(P + \ell J)e^{\ell D}K^{-1})_- = 0
\]

where \( \ell \) is a parameter and

\[
P = \sum_{r \geq 1} rt_r D^{r-1} = x + 2t_2 D + \ldots
\]

Removing \( (\ldots)_- \) results in replacing zero on the RHS with a differential operator \( S(x,\ell) \):

\[
K(x) \circ (x + \ell J + \sum_{s \geq 2} st_s D^{s-1}) = S \circ K(x + \ell)
\]

This \( S \) parametrizes independent degrees of freedom and therefore ‘solves’ the constraint. By virtue of the KP equations \( S \) satisfies a set of non-local evolution equations \[30\] \[33\]

\[
\frac{\partial S}{\partial t_r} = Q(x)^r_+ S - SQ(x + \ell)^r_+
\]

Evolution equations of this type were studied also in \[1\]. It was suggested there to look at them as the result of “quantizing” the spectral parameter of standard, local, integrable equations. A very similar philosophy was advocated in \[28\] (“quantum” Riemann surfaces) also in relation with the Virasoro constraints.

3.7.1. \textbf{Remark.} The Virasoro constraints on the KP hierarchy give rise to higher constraints, related to the underlying algebra \( W_\infty \) \[28\], which can be written in the form of a generating expression depending on two parameters \( \ell \) and \( z \). For \( J = 0 \) one finds:

\[
\exp \left( \sum_{r \geq 2} t_r \left( \frac{\partial}{\partial \ell} + z \right)^r - \frac{\partial^r}{\partial \ell^r} \right) (Ke^{zx}e^{\ell D}K^{-1})_- = (Ke^{\ell D}K^{-1})_-
\]

\( N \text{-KdV} \)
3.8. Virasoro generators for the generalized KdV hierarchies follow by a reduction from the KP ones. Only the generators with mode numbers \( n = Nj, j \in \mathbb{Z} \) are compatible with the constraint

\[
(KD^N K^{-1})_\leq = 0
\]

defining the \( N \)--reduction, and thus

\[
\mathfrak{L}^{KdV}_j = \frac{1}{N^2} (K \sum_{a=1}^{N-1} (Ni + \alpha) t_{a,i} D^{N(j+i)+\alpha} K^{-1})_\leq
\]

where \( t_{a,i} = t_{N_i+\alpha} \) and \( x \) was identified with the time \( t_{0,1} \). As in the KP case, these \( \mathfrak{L} \)'s are used to define vector fields

\[
\hat{t}^{KdV}_j \cdot f(Q) = \frac{d}{d\epsilon} f(Q - \epsilon[\mathfrak{L}^{KdV}_j, Q])|_{\epsilon=0} \quad Q \equiv KDK^{-1}.
\]

Strictly speaking, one should consider functions of the \( N \)-KdV Lax operator \( L = KDK^{-1} \) (which is differential by virtue of the above constraint), rather than \( Q = KDK^{-1} \). This and similar other book-keeping style corrections are left to the reader.

3.9. Note that adding to \( \hat{t}^{KdV}_j = \frac{1}{N^2} x D^{Nj+1} \) the piece \((Jj + q)D^{Nj}\), as suggested in 2.8, would not change the corresponding vector fields \( \hat{t}^{KdV}_j \); for \( j \geq 0 \) this can be seen already for the operators \( \mathfrak{L}^{KdV}_j \), while for \( j < 0 \) the \(...\)--projection becomes irrelevant, so removing it one is left with \( KD^{Nj}K^{-1} \) which commutes with the Lax operator. On the other hand, \( p = \frac{1}{N} x D \), being a representative of a class from \( \text{Der}_{\mathfrak{g}/\text{ad}_{Z_\lambda}} \), can be changed by arbitrary powers of \( D \), as discussed in 3.6. The commutative subalgebra \( Z_\Lambda \) is the same as in the KP case, \( i.e., \) the one generated by \( D^n, n \in \mathbb{Z} \).

3.10. Remarks.

1. One can make contact with a more familiar formalism that has been used to describe the Virasoro action on the KdV hierarchies, the tau function approach. As shown in [33], by evaluating the residue of the above \( \mathfrak{L}^{KdV}_j \), one finds the Virasoro generators

\[
L_j = \frac{1}{N^2} \sum_{a=1}^{N-1} \sum_{i=0}^{j-1} \partial \partial_{a,i} t_{N-a-j-i-1} + \frac{1}{N} \sum_{a=1}^{N-1} \sum_{i=0}^{j-1} (Ni + \alpha) t_{a,i} \partial_{a,i+j}, \quad j > 0
\]

(and similar expressions for \( L_{j\leq0} \)) which act on the tau function of the \( N \)-KdV hierarchy.

2. Imposing the Virasoro constraints \( \mathfrak{L}^{KdV}_j = 0 \) for \( j \geq -1 \) in the case of the \( N \)-KdV hierarchy generates a system of ‘higher’ constraints as explained in [33]; these are summarized by the equation

\[
\left(Ke^{zP^{D_1-N}} e^{\ell D_N K^{-1}}\right)_\leq = 0
\]

where \( z \) and \( \ell \) are parameters and \( P \) is the same as in the KP case, but with the times \( t_{Nj} \) dropped.

Toda

3.11. For the Toda lattice hierarchy [38, 37] the Lie algebra \( \mathfrak{g} = \mathfrak{gl}(\infty) \) is that of \( \infty \times \infty \) matrices whose matrix elements will be labelled according to

\[
\Phi = \sum_{i \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} \phi_i(s) |s\rangle \langle s| \Lambda^i
\]
where $|s\rangle$ with $s \in \mathbb{Z}$ are the elements of an orthonormal basis and

$$\Lambda |s\rangle = |s - 1\rangle$$

3.12. The $r$–matrix is chosen to be the one associated to the decomposition of $\mathfrak{gl}(\infty)$ into a sum of two subalgebras which consist of (strictly) lower-triangular and (non-strictly) upper-triangular matrices respectively. This can also be expressed as

$$\Phi_+ = \sum_{i \geq 0} \sum_{s \in \mathbb{Z}} \phi_i(s) |s\rangle \langle s| \Lambda^i$$

The dressing operator, denoted now as $W$, therefore has the form

$$W = 1 + \sum_{i \geq 1} \sum_{s \in \mathbb{Z}} \phi_i(s) |s\rangle \langle s| \Lambda^{-i}$$

3.13. Matrix trace allows us to identify $\mathfrak{g}^*$ with the strictly upper-triangular matrices. (We deliberately avoid the discussion, in this illustrative example, of the conditions for the trace to exist.) The hierarchy equations read,

$$\frac{\partial L}{\partial x_n} = -[(L^n)_-, L], \quad L = W \Lambda W^{-1}, \quad n \geq 1.$$

where $x_n$ are the hierarchy times (the $x$-times of the ”two-dimensional” Toda lattice hierarchy $[38]$).

3.14. Now one can construct the ‘bare’ Virasoro generators in the spirit of 2.9 by first defining a matrix $p$ by

$$p |s\rangle = s |s\rangle,$$

whence

$$[\Lambda, p] = \Lambda,$$

and then setting

$$l_n = p \Lambda^n, \quad A_n = \Lambda^n$$

and

$$\mathfrak{l}^{\text{Toda}}_n(J, q) = p \Lambda^n + J n \Lambda^n + q \Lambda^n + \sum_{r \geq 1} r x_r \Lambda^{n+r}.$$ 

The $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ decomposition can be extended by setting $(p)_+ = p$; then

$$\mathfrak{l}^{\text{Toda}}_n(J, q) = \left(W \left\{ [J n + q + p] \Lambda^n + \sum_{r \geq 1} r x_r \Lambda^{n+r} \right\} W^{-1}\right)_{-}.$$ 

These Virasoro generators were first derived in [30] via an explicit calculation starting from the standard expression for the Virasoro generators acting on the Toda tau function.

3.15. Remark. The set of the Virasoro constraints $\mathfrak{l}^{\text{Toda}}_n(J, J) = 0$, $n \geq 0$, have been shown in [31] to undergo a scaling into the KP Virasoro constraints $\mathfrak{l}^{\text{KP}}_p(J, J) = 0$, $p \geq -1$. By scaling we mean blowing up the asymptotic region $s \gg 1$ so that $s$ can be written as a formal expansion in inverse powers of $\epsilon$ for $\epsilon \to 0$. Then the scaling involves certain $\epsilon$-dependent ansatze for the hierarchy times $x_m$ (expressing them through new parameters $t_r$ which become KP-times) and for matrix elements of $W$ (expressing them through what becomes coefficient functions of a KP dressing operator $K$).
4 Matrix versus scalar $N$-KdV Virasoro constraints and string equations

The $N$-KdV hierarchies admit a very important formulation, which is alternative to that used in Sect.3, in terms of first order differential operators with matrix coefficients \[14\]. This formulation transpires both the geometric nature of the equations (hamiltonian reduction) and the Lie- (in fact, Kac-Moody $sl(N)$) algebraic structure, which allows generalizations to Kac-Moody algebras other than $sl(N)$. Specializing the general construction of Sect.2 to the Drinfeld-Sokolov formulation of $N$-KdV should therefore be viewed as an example of what this construction becomes for the generalized KdV’s of Drinfeld and Sokolov. The respective Virasoro constraints would then correspond, in particular, to gravity-coupled minimal models from the $ADE$ classification of the latter. (Note the paper \[39\] which deals with the scalar formulation of these.)

Thus, the $N$-KdV hierarchies that we have met in Sect.3 will now be called the $sl(N)$-KdV hierarchies and will be described in the formalism which explicitly relies on the underlying Kac-Moody $sl(N)$ structure.

4.1. Recall that the $sl(N)$-KdV hierarchy is formulated in terms of a differential operator

$$\mathcal{L} = \frac{\partial}{\partial x} + L$$

where $L$ is of the form

$$L(\zeta) = \Lambda_N(\zeta) + \begin{pmatrix} * & * & * & 0 \\ * & * & \cdot & \cdot \\ * & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 \end{pmatrix} \in sl(N), \quad \Lambda_N(\zeta)^N = \zeta \cdot 1$$

with $\zeta$ being a spectral parameter and

$$\Lambda_N(\zeta) = \begin{pmatrix} 0 & 1 & 0 & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 1 \end{pmatrix} \in sl(N), \quad \Lambda_N(\zeta)^N = \zeta \cdot 1$$

By a gauge transformation

$$\frac{\partial}{\partial x} + L = W \left( \frac{\partial}{\partial x} + L_0 \right) W^{-1},$$

with

$$W = 1 + \sum_{j \geq 1} \sum_{s=0}^{N-1} |s\rangle w_j(s) \langle s | \Lambda_N(\zeta)^{-j}.$$ 

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one can bring $L_0$ to the form

$$L_0 = \Lambda_N(\zeta) + \sum_{i \geq 0} f_i \Lambda_N(\zeta)^{-i}$$

where $f_i$ are scalar functions.

4.2. The hierarchy flows are induced on gauge equivalence classes by the equations

$$\frac{dL}{dt_{\alpha,i}} = -[(\zeta^i A^\alpha)_-, L], \quad i \geq 0, \quad \alpha \in \{1, 2, \ldots, N-1\}$$

in which

$$A = W \Lambda_N(\zeta) W^{-1}$$

and either of the two possibilities can be chosen as the definition of $(\ldots)_-$ (and therefore of the $r$-matrix): projection onto negative powers of $\zeta$ or onto negative powers of $\Lambda_N(\zeta)$. In the latter case one uses the presentation

$$N = \sum_{i \leq n} \sum_{s=0}^{N-1} |s\rangle n_i(s) \langle s| \Lambda_N(\zeta)^i$$

for an arbitrary matrix from the algebra; then

$$N_\ldots = \sum_{i < 0} \sum_{s=0}^{N-1} |s\rangle n_i(s) \langle s| \Lambda_N(\zeta)^i$$

The two definitions lead to the same evolutions on the gauge equivalence classes.

4.3. The Virasoro transformations of $L$ are given by,

$$\hat{i}_j L = [\mathfrak{L}_j, L]$$

with

$$\mathfrak{L}_j = \left(W \left\{ -\zeta \frac{\partial}{\partial \zeta} + \hat{\pi} + \sum_{\alpha=1}^{N-1} \left( \sum_{i \geq 0} \left( i + \frac{\alpha}{N} \right) t_{\alpha,i} \zeta^i \right) \Lambda_N(\zeta)^\alpha \right\} \zeta^j W^{-1} \right)_-$$

where

$$\hat{\pi} = \frac{1}{N} \begin{pmatrix} 1 \\ 2 \\ \ddots \\ N \end{pmatrix} + \sum h_i \Lambda_N(\zeta)^{-i}$$

and $h_i$ are arbitrary scalar functions. This is the same type of arbitrariness as discussed in 3.6 and 3.9.

4.4. Now the Virasoro constraints read

$$\mathfrak{L}_j = 0 \quad \text{for} \quad j \geq -1.$$
In each of these equations we remove the \( (\ldots)_- \)-projection, which results in replacing zero on the \( \text{rhs} \) with a \(-S_+^{(j)} \) (with \( S_+^{(j)} \) being purely \((\ldots)_+ \)-matrices). Further, one readily discovers that
\[
S_+^{(j)} = (j + 1) \zeta^j + \zeta^{j+1} S_+^{(-1)}
\]
and thus the Virasoro constraints lead to a single equation of the form
\[
\frac{\partial W^{-1}}{\partial \zeta} = W^{-1} S_+ + \left( \hat{\pi} + 1 + \sum_{\alpha,i} \left( i + \frac{\alpha}{N} \right) t_{\alpha,i} \zeta^i \Lambda_N(\zeta)^\alpha \right) \zeta^{-1} W^{-1}
\]
where \( S_+ \) is an arbitrary \((\ldots)_+ \)-matrix. Due to obvious reasons, it is advantageous to choose here the definition of \((\ldots)_+ \) as positive powers of the spectral parameter \( \zeta \).

4.4.1. REMARK. As an aside, note that for the matrix
\[
\Psi = W \exp \sum_{\alpha,i} t_{\alpha,i} \zeta^i \Lambda_N(\zeta)^\alpha
\]
the linear equation takes the form (cf. [28])
\[
-\frac{\partial \Psi}{\partial \zeta} = S_+ \Psi + \Psi(\hat{\pi} + 1) \zeta^{-1}
\]
Together with the standard linear problem for the \( sl(N) \)-KdV hierarchy this gives a "generalized linear system" whose integrability leads, in particular, to the string equation.

Now we investigate the consistency between Virasoro generators in the scalar and matrix formalisms for the KdV hierarchy.

4.5. Recall first the standard procedure to rederive the scalar formalism from the matrix one [14]. Denote by \( \mathcal{H} \) a vector space whose elements are columns of height \( N \) consisting of formal Laurent series in \( \zeta \), which might also bear a dependence on \( x \). For a fixed Lax operator \( \mathcal{L} \) one can make \( \mathcal{H} \) into a module over the ring of pseudodifferential operators by
\[
\mathcal{L} : \psi \text{Diff} \times \mathcal{H} \to \mathcal{H},
\]
where on the \text{rhs} the action by the usual matrix multiplication and application of \( D = \partial/\partial x \) are understood. The definition is correct since
\[
D_{\mathcal{L}}(f \eta) = (f D) \eta + (\partial f) \eta = (D \circ f)_{\mathcal{L}} \eta
\]
for a scalar function \( f(x) \), whence
\[
F G_{\mathcal{L}} \eta = F_{\mathcal{L}} (G_{\mathcal{L}} \eta) \quad \text{for} \quad F, G \in \psi \text{Diff}.
\]
4.6. Now, the scalar Lax operator can be recovered starting from the matrix one from the relation

\[ L \dot{\eta}_0 = \zeta \eta_0 \]

\[ \eta_0 \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} \]  

\( (\ast) \)

It turns out that this defines \( L \) uniquely, and it has the form

\[ L = D^N + \sum_{i=0}^{N-1} u_i D^i, \]

where \( u_i \) are certain scalar functions of \( x \). It is now a result of the standard theory that this operator can be represented as

\[ L = KD^N K^{-1} \]

where \( K \) is of the same form as in 3.2.

We want to see whether the above relation between the matrix and scalar Lax operators is preserved by the Virasoro action. That is, viewing the LHS of (\( \ast \)) as a function \( \{ \dot{L} \} : \mathcal{L} \mapsto \{ \dot{L} \}(\mathcal{L}) \), we deform both the function and the argument, by

\[ \hat{\delta}^{sc} L = [\mathfrak{L}^{sc}_j, L] \]

and

\[ (\hat{\delta}^{m} \{ \dot{L} \})(\mathcal{L}) = \frac{d}{d\epsilon} \{ \dot{L} \}(\mathcal{L} - \epsilon [\mathfrak{L}^{m}_j, \mathcal{L}]) \big|_{\epsilon = 0} \]

respectively. Here \( \mathfrak{L}^{sc}_j \) and \( \mathfrak{L}^{m}_j \) are borrowed from 3.8 and 4.3 respectively.

It turns out that this combined action of scalar and matrix Virasoro generators does not leave (\( \ast \)) invariant. Instead, we have

4.7. THEOREM (consistency relation between scalar and matrix Virasoro generators). For the above Virasoro generators it follows

\[ \delta^{\dot{L}}_j (L \dot{\eta}_0 - \zeta \eta_0) = 0, \]

where

\[ \delta^{\dot{L}}_j \mathcal{L} = -[\mathfrak{L}^{m}_j, \mathcal{L}] \]

\[ \delta^{\dot{L}}_j L = -[\mathfrak{L}^{sc}_j, L] \]

\[ \delta^{\dot{L}}_j \zeta = \zeta^{j+1} \]

REMARK (trivial). Note that the bottom line is just the action of reparametrizations \( \zeta^{j+1} \frac{\partial}{\partial \zeta} \) on a complex parameter.

PROOF. We evaluate directly the action of \( \delta^{\dot{L}}_j \) on \( (L \dot{\eta}_0) \):

\[ \delta^{\dot{L}}_j (L \dot{\eta}_0) = (\mathfrak{L}^{sc}_j L) \dot{\eta}_0 - (L \mathfrak{L}^{sc}_j) \dot{\eta}_0 - \left\{ [\mathfrak{L}^{m}_j, \mathcal{L}^N] + \sum_{i=0}^{N-1} u_i [\mathfrak{L}^{m}_j, \mathcal{L}^i] \right\} \eta_0, \]
where \( u_i \) are the above coefficients of the unperturbed scalar Lax operator \( L \). Taking into account that \( [L^m, u_i] = 0 \) and using again equation (\( * \)) of 4.6 (and also the ring homomorphism property 4.5), we bring the last equation to the form,

\[
\delta_{\text{tot}}^j \left( L_{\bar{\zeta}} \eta_0 \right) = L^{sc}_j L_{\bar{\zeta}} \zeta \eta_0 - L^{sc}_j \zeta \eta_0 - L^{sc}_j \delta_{\text{tot}}^j \left( L^{sc}_j L_{\bar{\zeta}} \eta_0 - L^{sc}_j \eta_0 \right)
\]

\[
= \zeta^{j+1} \eta_0 + (\zeta - L_{\bar{\zeta}}) \left\{ L^{sc}_j L_{\bar{\zeta}} \eta_0 - L^{sc}_j \eta_0 \right\}. \tag{**}
\]

Next, the scalar generator \( L^{sc}_j \), which is of course nothing but \( L^K_{KdV} \) we had in 3.8, can be rewritten as,

\[
L^{sc}_j = \frac{1}{N} (K x D^{Nj+1} K^{-1})_+ + \sum_{\alpha=1}^{N-1} \sum_{i \geq 0} \left( i + \frac{\alpha}{N} \right) t_{a,i} \left( L^{i+j+\frac{N}{N}} \right)_-.
\]

(Recall that \( L = KD^N K^{-1} \).)

Now, as shown in [14],

\[
(L^{i+j+\frac{N}{N}})_- \eta_0 = (WA_N(\zeta)^{N(i+j)+\alpha} W^{-1})_- \eta_0.
\]

where \( W \) is defined just by the relation of 4.1 and \( (\ldots)_- \) on the RHS refers to negative powers of \( \Lambda_N(\zeta) \). Therefore, it follows for the \( \sum \sum \)-piece of \( L^{sc}_j \) that

\[
\sum_{\alpha=1}^{N-1} \sum_{i \geq 0} (\ldots)_- = \left( W \left( \sum_{a,i} \left( i + \frac{\alpha}{N} \right) t_{a,i} \right) A_N(\zeta)^{\alpha} \right) \zeta W^{-1} \eta_0,
\]

which coincides with the respective piece in the expression for \( L^{m}_j \) from 4.3. It therefore remains to calculate the \( \delta_{\text{tot}}^j \)-action of the first term in \( L^{sc}_j \).

As shown in [14], the \( (\ldots)_- \)'s are “equivariant”. Thus it suffices to determine an \( X_j \) from the formula

\[
\frac{1}{N} (K x D^{Nj+1} K^{-1}) L_{\bar{\zeta}} \eta_0 = X_j \eta_0.
\]

The pseudodifferential operator from the LHS satisfies the following commutation relation:

\[
\left[ L^{\frac{N}{N}}, \frac{1}{N} (K x D^{Nj+1} K^{-1}) \right] = \frac{1}{N} L^{j+\frac{N}{N}}.
\]

Therefore for \( X_j \) one should have

\[
\left[ WA_N(\zeta) W^{-1}, X_j \right] = \frac{1}{N} W A_N(\zeta)^{Nj+1} W^{-1},
\]

which determines \( X_j \) in the form

\[
X_j = W \left( -\zeta \frac{\partial}{\partial \zeta} + \frac{1}{N} \begin{pmatrix} 1 & 2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & N \end{pmatrix} + \sum_{i} h_i A_N(\zeta)^{-i} \right) A_N(\zeta)^{Nj} W^{-1}
\]

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with $h_i$ arbitrary scalars as in the expression for $\hat{\pi}$ in 4.3. Putting everything together, we thus arrive at

$$L^s_j \xi \eta_0 = \left( W \left( -\zeta \frac{\partial}{\partial \zeta} + \frac{1}{N} \begin{pmatrix} 1 & 2 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots \end{pmatrix} \right) + \sum_{\alpha,i} \left( i + \frac{\alpha}{N} \right) t_{\alpha,i} \zeta^i N(\zeta)^{\alpha} + \ldots \right) \zeta^j W^{-1} \eta_0$$

where the ellipsis denotes the $h-$terms. These can be chosen to coincide with the corresponding ones from the formula in 4.3 and thus the curly bracket in (***) vanishes, which completes the proof. $\square$

4.8. An important difference between the scalar and matrix formulations has to do with the string equation. We have seen that the Virasoro generators and therefore the Virasoro constraints in the two cases agree. However, although the string equation is a consequence of Virasoro constraints, its form in the matrix formalism is not a naive replica of the scalar one. We have seen in 3.8 that, in particular, $\mathcal{L}^{KdV}$, which we now denote as $\mathcal{L}^{sc}_{-1}$, is of the form $\mathcal{L}^{sc}_{-1} = (\mathcal{P}^{sc})_-$, so that the $(\cdot)$-Virasoro constraint takes the form of the condition that $\mathcal{P}^{sc}$ be a differential operator. As such, it satisfies the string equation

$$[L, \mathcal{P}^{sc}] = 1$$

where $L = KD^N K^{-1}$ is the Lax operator. On the other hand we have seen that the matrix counterpart of this equation reads

$$[\zeta, \mathcal{P}^m] = 1,$$

where, similarly to the scalar case, $(\mathcal{P}^m)_- = \mathcal{L}^m_{-1}$, whereas, generally, $[\mathcal{L}, \mathcal{P}^m] \neq 1$. It should be noted also that, contrary to the scalar case, $\zeta$ is not an eigenvalue of the Lax operator $\mathcal{L}$ in the matrix case.

A similar remark would apply also to similar constructions for other Kac-Moody algebras.

5 Concluding remarks

Symmetries of (constrained) integrable hierarchies, ‘higher’ than the Virasoro ones, have been a subject of considerable interest recently [3, 15, 16, 17, 20, 23, 33]. In the formalism we have developed in this paper, the higher symmetries can be generated from the Virasoro ones by essentially algebraic manipulations. Consider, for instance, Virasoro-constrained hierarchies as in 2.10 and assume for simplicity that $1/2(1 - R)$ is a projection (as was the case in all the examples in Sects. 3 and 4). Then,

$$(\text{Ad}_{K^*}(I_i I_j))_- = ((\text{Ad}_{K^*}I_i)(\text{Ad}_{K^*}I_j))_-$$

which is again zero by the Virasoro constraint saying that each factor is a pure $(\ldots)_-$. Thus $I_i I_j$ and, similarly, higher order monomials, give us an infinite set of ‘higher’ constraints whose

and, moreover, put to zero throughout.
algebra is not difficult to derive. Note that one naturally uses in this analysis the associative structure in $\text{End}_g$ to compose derivations and products thereof. It is such an ‘associative mechanism’ that allowed one to derive, for instance, the constraints of 3.7.1 in the KP case. The interested reader would formulate the necessary requirements that the $r$-matrix must satisfy in order that the construction make sense rigorously.

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