Non-Hermitian Hamiltonians with real and complex eigenvalues in a Lie-algebraic framework

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Abstract

We show that complex Lie algebras (in particular sl(2,\mathbb{C})) provide us with an elegant method for studying the transition from real to complex eigenvalues of a class of non-Hermitian Hamiltonians: complexified Scarf II, generalized Pöschl-Teller, and Morse. The characterizations of these Hamiltonians under the so-called pseudo-Hermiticity are also discussed.

PACS: 02.20.Sv; 03.65.Fd; 03.65.Ge

Keywords: Non-Hermitian Hamiltonians; PT symmetry; Pseudo-Hermiticity; Lie algebras

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1 Introduction

Some years ago, it was suggested [1] that PT symmetry might be responsible for some non-Hermitian Hamiltonians to preserve the reality of their bound-state eigenvalues provided it is not spontaneously broken, in which case their complex eigenvalues should come in conjugate pairs. Following this, several non-Hermitian Hamiltonians (including the non-PT-symmetric ones [2, 3, 4]) with real or complex spectra have been analyzed using a variety of techniques, such as perturbation theory, semiclassical estimates, numerical experiments, analytical arguments, and algebraic methods. Among the latter, one may quote those connected with supersymmetrization [4, 5, 6, 7, 8, 9, 10], or some generalizations thereof [11], quasi-solvability [3, 12, 13, 14, 15, 16], and potential algebras [4, 17].

Recently, it has been shown that under some rather mild assumptions, the existence of real or complex-conjugate pairs of eigenvalues can be associated with a class of non-Hermitian Hamiltonians distinguished by either their so-called (weak) pseudo-Hermiticity [i.e., such that \( \eta H \eta^{-1} = H^\dagger \), where \( \eta \) is some (Hermitian) linear automorphism] or their invariance under some antilinear operator [18, 19]. In such a context, pseudo-Hermiticity under imaginary shift of the coordinate has been identified as the explanation of the occurrence of real or complex-conjugate eigenvalues for some non-PT-symmetric Hamiltonians [20].

In the course of time, there has been a growing interest in determining the critical strengths of the interaction at which PT symmetry (or some generalization) becomes spontaneously broken, i.e., they appear regular complex-energy solutions, where by regular we mean eigenfunctions satisfying the asymptotic boundary conditions \( \psi(\pm \infty) \to 0 \), so that they are normalizable in a generalized sense [18, 20, 21, 22]. Some analytical results have been obtained both for PT-symmetric potentials [22, 23, 24, 25] and for potentials that are pseudo-Hermitian under imaginary shift of the coordinate [20].

In the present Letter, we wish to show that complex Lie algebras provide us with an easy and elegant method for studying the transition from real to complex eigenvalues, corresponding to regular eigenfunctions, of (PT-symmetric or non-PT-symmetric) pseudo-Hermitian and non-pseudo-Hermitian Hamiltonians.
2 Non-Hermitian Hamiltonians in an \( \text{sl}(2, \mathbb{C}) \) framework

The generators \( J_0, J_+, J_- \) of the complex Lie algebra \( \text{sl}(2, \mathbb{C}) \), characterized by the commutation relations

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = -2J_0, \tag{1}
\]

can be realized as differential operators \[4\]

\[
J_0 = -i \frac{\partial}{\partial \phi}, \quad J_\pm = e^{\pm i\phi} \left[ \pm \frac{\partial}{\partial x} + \left( i \frac{\partial}{\partial \phi} \mp \frac{1}{2} \right) F(x) + G(x) \right], \tag{2}
\]

depending upon a real variable \( x \) and an auxiliary variable \( \phi \in [0, 2\pi) \), provided the two complex-valued functions \( F(x) \) and \( G(x) \) in (2) satisfy coupled differential equations

\[
F' = 1 - F^2, \quad G' = -FG. \tag{3}
\]

Here a prime denotes derivative with respect to spatial variable \( x \).

The solutions of Eq. (3) fall into the following three classes:

I : \( F(x) = \tanh(x - c - i\gamma), \quad G(x) = (b_R + ib_I) \text{sech}(x - c - i\gamma), \)

II : \( F(x) = \coth(x - c - i\gamma), \quad G(x) = (b_R + ib_I) \text{cosech}(x - c - i\gamma), \tag{4}\)

III : \( F(x) = \pm 1, \quad G(x) = (b_R + ib_I)e^{\mp x}, \)

where \( c, b_R, b_I \in \mathbb{R} \) and \(-\frac{\pi}{4} \leq \gamma < \frac{\pi}{4}\), thus providing us with three different realizations of \( \text{sl}(2, \mathbb{C}) \). For \( b_I = \gamma = 0 \), the latter reduce to corresponding realizations of \( \text{sl}(2, \mathbb{R}) \simeq \text{so}(2, 1) \), for which \( J_0 = J_0^\dagger \) and \( J_- = J_+^\dagger \) [20].

The \( \text{sl}(2, \mathbb{C}) \) Casimir operator corresponding to the differential realizations of type (4) can be written as

\[
J^2 \equiv J_0^2 \mp J_0 J_\mp = \frac{\partial^2}{\partial x^2} - \left( \frac{\partial^2}{\partial \phi^2} + \frac{1}{4} \right) F' + 2i \frac{\partial}{\partial \phi} G' - G^2 - \frac{1}{4}. \tag{5}
\]

In this work, we are going to consider the \( \text{sl}(2, \mathbb{C}) \) irreducible representations spanned by the states

\[
|km\rangle = \Psi_{km}(x, \phi) = \psi_{km}(x) e^{im\phi} \sqrt{2\pi}, \tag{6}
\]
with fixed $k$, for which
\[ J_0|km\rangle = m|km\rangle, \quad J^2|km\rangle = k(k-1)|km\rangle, \tag{7} \]
and
\[ k = k_R + ik_I, \quad m = m_R + im_I, \quad m_R = k_R + n, \quad m_I = k_I, \tag{8} \]
where $k_R, k_I, m_R, m_I \in \mathbb{R}$ and $n \in \mathbb{N}$. The states with $m = k$ or $n = 0$ satisfy the equation $J_-|kk\rangle = 0$, while those with higher values of $m$ (or $n$) can be obtained from them by repeated applications of $J_+$ and use of the relation $J_+|km\rangle \propto |k m + 1\rangle$.

When the parameter $m$ is real, i.e., $m_I = 0$, we can get rid of the auxiliary variable $\phi$ by extending the definition of the pseudo-norm with a multiplicative integral over $\phi$ from 0 to $2\pi$. In the case $m$ is complex, i.e., $m_I \neq 0$, a similar result can be obtained through an appropriate change of the integral over $\phi$. In the former (resp. latter) case, $J_0$ is a Hermitian (resp. non-Hermitian) operator.

From the second relation in Eq. (7), it follows that the functions $\psi_{km}(x)$ of Eq. (8) obey the Schrödinger equation
\[ -\psi''_{km} + V_m \psi_{km} = -\left(k - \frac{1}{2}\right)^2 \psi_{km}, \tag{9} \]
where the family of potentials $V_m$ is defined by
\[ V_m = \left(\frac{1}{4} - m^2\right) F' + 2m G' + G^2. \tag{10} \]
Since the irreducible representations of $\text{sl}(2, \mathbb{C})$ correspond to a given eigenvalue in Eq. (3) and the corresponding basis states to various potentials $V_m$, $m = k, k+1, k+2, \ldots$, it is clear that $\text{sl}(2, \mathbb{C})$ is a potential algebra for the family of potentials $V_m$ (see [26] and references quoted therein).

To the three classes of solutions of Eq. (3), given in Eq. (4), we can now associate three classes of potentials:

I : \[ V_m = \left[(b_R + ib_I)^2 - (m_R + im_I)^2 + \frac{1}{4}\right] \sech^2 \tau \]
\[ - 2(m_R + im_I)(b_R + ib_I) \sech \tau \tanh \tau, \quad \tau = x - c - i\gamma, \tag{11} \]

II : \[ V_m = \left[(b_R + ib_I)^2 + (m_R + im_I)^2 - \frac{1}{4}\right] \cosech^2 \tau \]
\[ - 2(m_R + im_I)(b_R + ib_I) \cosech \tau \coth \tau, \quad \tau = x - c - i\gamma, \tag{12} \]

III : \[ V_m = (b_R + ib_I)^2 e^{\mp 2x} \mp 2(m_R + im_I)(b_R + ib_I)e^{\mp x}. \tag{13} \]
It is worth stressing that in the generic case, such complex potentials are not invariant under PT symmetry.

Equation (9) can also be rewritten as

\[-\psi_n^{(m)''} + V_m \psi_n^{(m)} = E_n^{(m)} \psi_n^{(m)},\]

with \(\psi_{km}(x) = \psi_n^{(m)}(x)\) and

\[E_n^{(m)} = -(m_R + im_I - n - \frac{1}{2})^2.\]

Real (resp. complex) eigenvalues therefore correspond to \(m_I = 0\) (resp. \(m_I \neq 0\)).

To be acceptable solutions of Eq. (14), the functions \(\psi_n^{(m)}(x)\) have to be regular, i.e., such that \(\psi_n^{(m)}(\pm \infty) \to 0\). It is straightforward to determine under which conditions there exist acceptable solutions of Eq. (14) with \(n = 0\). The functions \(\psi_0^{(m)}(x)\) are indeed easily obtained by solving the first-order differential equation \(J_\Psi_{mm}(x, \phi) = 0\). For the three classes of potentials (11) – (13), the results read

I : \[\psi_0^{(m)}(x) \propto (\text{sech} \tau)^{m_R + im_I - \frac{1}{2}} \exp\left[\left(b_R + ib_I\right) \arctan(\sinh \tau)\right],\]

II : \[\psi_0^{(m)}(x) \propto (\sinh \frac{\tau}{2})^{b_R + im_R - im_I - \frac{1}{2}} (\cosh \frac{\tau}{2})^{-b_R - ib_I - m_R - im_I + \frac{1}{2}},\]

III : \[\psi_0^{(m)}(x) \propto \exp\left[-(m_R + im_I - \frac{1}{2})x - (b_R + ib_I)e^{-x}\right].\]

Such functions are regular provided \(m_R > \frac{1}{2}\) and \(b_R > 0\), where the second condition applies only to class III.

In the remainder of this letter, we shall illustrate the general theory developed in the present section with some selected examples.

3 Complexified Scarf II potential

The potential

\[V(x) = -V_1 \text{sech}^2 x - iV_2 \text{sech} x \tanh x, \quad V_1 > 0, \quad V_2 \neq 0,\]

(19)
which belongs to class I defined in Eq. (11), is a complexification of the real Scarf II potential [27]. It is not only invariant under PT symmetry but also P-pseudo-Hermitian. Comparison between Eqs. (11) and (19) shows that it corresponds to $c = \gamma = 0$ and

$$b_R^2 - b_I^2 - m_R^2 + m_I^2 + \frac{1}{4} = -V_1,$$

$$b_R b_I - m_R m_I = 0,$$

$$m_R b_R - m_I b_I = 0,$$

$$2(m_R b_I + m_I b_R) = V_2,$$

where we may assume $b_I \neq 0$ since otherwise the sl(2, \mathbb{C}) generators (2) would reduce to sl(2, \mathbb{R}) ones.

To be able to apply the results of the previous section, the only thing we have to do is to solve Eqs. (20) – (23) in order to express the sl(2, \mathbb{C}) parameters $b_R, b_I, m_R, m_I$ in terms of the potential parameters $V_1, V_2$. Equations (22) and (23) yield

$$m_R = \frac{V_2 b_I}{2(b_R^2 + b_I^2)}, \quad m_I = \frac{V_2 b_R}{2(b_R^2 + b_I^2)}.$$

On inserting these results into Eqs. (20) and (21), we get the relations

$$\left(b_R^2 - b_I^2\right) \left(1 + \frac{V_2^2}{4(b_R^2 + b_I^2)^2}\right) = -V_1 - \frac{1}{4},$$

$$b_R b_I \left(1 - \frac{V_2^2}{4(b_R^2 + b_I^2)^2}\right) = 0.$$

The latter is satisfied if either $b_R = 0$ or $b_R \neq 0$ and $b_R^2 + b_I^2 = \frac{1}{2}|V_2|$. It now remains to solve Eq. (25) in those two possible cases.

If we choose $b_R = 0$, then Eq. (25) reduces to a quadratic equation for $b_I^2$, which has real positive solutions

$$b_I^2 = \frac{1}{4} \left(\sqrt{V_1 + \frac{1}{4} + V_2 + \epsilon_I \sqrt{V_1 + \frac{1}{4} - V_2}}\right)^2, \quad \epsilon_I = \pm 1,$$

provided $|V_2| \leq V_1 + \frac{1}{4}$. Equation (27) then yields for $b_I$ the possible solutions

$$b_I = \frac{1}{2} \epsilon_I' \left(\sqrt{V_1 + \frac{1}{4} + V_2 + \epsilon_I \sqrt{V_1 + \frac{1}{4} - V_2}}\right), \quad \epsilon_I, \epsilon_I' = \pm 1,$$
while Eq. (24) leads to \( m_R = V_2/(2b_I) \) and \( m_I = 0 \).

From the regularity condition \( m_R > \frac{1}{2} \) of \( \psi_0^{(m)}(x) \), given in Eq. (14), it then follows that \( b_I \) must have the same sign as \( V_2 \), which we denote by \( \nu \). Furthermore, we must choose \( \epsilon_I = +1 \) or \( \epsilon_I = -\epsilon \) according to whether \( \nu = +1 \) or \( \nu = -1 \).

The first set of solutions of Eqs. (20) – (23), compatible with the regularity condition of \( \psi_0^{(m)}(x) \), is therefore given by

\[
\begin{align*}
2b_R &= 0, \quad b_I = \frac{1}{2} \nu \left( \sqrt{V_1 + \frac{1}{4} + |V_2|} - \epsilon \sqrt{V_1 + \frac{1}{4} - |V_2|} \right), \\
m_R &= \frac{1}{2} \left( \sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon \sqrt{V_1 + \frac{1}{4} - |V_2|} \right), \quad m_I = 0, \quad \epsilon = \pm 1, \quad (29)
\end{align*}
\]

where \( \epsilon = -\epsilon \), provided \( |V_2| \leq V_1 + \frac{1}{4} \) and \( \sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon \sqrt{V_1 + \frac{1}{4} - |V_2|} > 1 \).

On inserting these results into Eq. (15), we get two series of real eigenvalues

\[
E_{n,\epsilon} = -\left[ \frac{1}{2} \left( \sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon \sqrt{V_1 + \frac{1}{4} - |V_2|} \right) - n - \frac{1}{2} \right]^2, \quad \epsilon = \pm 1. \quad (30)
\]

By studying the regularity condition of the associated eigenfunctions obtained by successive applications of \( J_+ \) on \( \psi_0^{(m)}(x) \), it can be shown that \( n \) is restricted to the range \( n = 0, 1, 2, \ldots < \frac{1}{2} \left( \sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon \sqrt{V_1 + \frac{1}{4} - |V_2|} - 1 \right) \).

If, on the contrary, we choose \( b_R \neq 0 \) and \( b_R^2 + b_I^2 = \frac{1}{2} |V_2| \), then Eq. (25) leads to

\[
\begin{align*}
b_R &= \frac{1}{2} \epsilon_R \sqrt{|V_2| - V_1 - \frac{1}{4}}, \quad b_I = \frac{1}{2} \epsilon_I \sqrt{|V_2| + V_1 + \frac{1}{4}}, \quad \epsilon_R, \epsilon_I = \pm 1, \quad (31)
\end{align*}
\]

provided \( |V_2| > V_1 + \frac{1}{4} \).

On inserting such results into Eq. (24) and imposing the regularity condition \( m_R > \frac{1}{2} \), we obtain \( \epsilon = \nu \). The second set of solutions of Eqs. (20) – (23), compatible with the regularity condition of \( \psi_0^{(m)}(x) \), is therefore given by

\[
\begin{align*}
b_R &= \frac{1}{2} \nu \epsilon \sqrt{|V_2| - V_1 - \frac{1}{4}}, \quad b_I = \frac{1}{2} \nu \sqrt{|V_2| + V_1 + \frac{1}{4}}, \\
m_R &= \frac{1}{2} \sqrt{|V_2| + V_1 + \frac{1}{4}}, \quad m_I = \frac{1}{2} \epsilon \sqrt{|V_2| - V_1 - \frac{1}{4}}, \quad \epsilon = \pm 1, \quad (32)
\end{align*}
\]

where we have set \( \epsilon = \nu \epsilon_R \). Here we must assume \( |V_2| > V_1 + \frac{1}{4} \) and \( |V_2| + V_1 + \frac{1}{4} > 1 \).
This set of solutions is associated with a series of complex-conjugate pairs of eigenvalues

\[ E_{n,\epsilon} = -\left[ \frac{1}{2} \left( \sqrt{|V_2| + V_1 + \frac{1}{4}} + i\sqrt{|V_2| - V_1 - \frac{1}{4}} \right) - n - \frac{1}{2} \right]^2, \quad \epsilon = \pm 1, \quad (33) \]

where it can be shown that \( n \) varies in the range \( n = 0, 1, 2, \ldots \) < \( \frac{1}{2} \left( \sqrt{|V_2| + V_1 + \frac{1}{4}} - 1 \right) \).

We conclude that for increasing values of \( |V_2| \), the two series of real eigenvalues \( (30) \) merge when \( |V_2| \) reaches the value \( V_1 + \frac{1}{4} \), then disappear while complex-conjugate pairs of eigenvalues \( (33) \) make their appearance, as already found elsewhere by another method \( [22] \).

Had we chosen the parametrization \( V_1 = B^2 + A(A + 1), V_2 = -B(2A + 1) \), with \( A \) and \( B \) real, as we did in Ref. \( [1] \), we would obtain that the condition \( |V_2| \leq V_1 + \frac{1}{4} \) is always satisfied, thus only getting the two series of real eigenvalues \( (30) \).

### 4 Complexified generalized Pöschl-Teller potential

We next consider the complexification of the generalized Pöschl-Teller potential \( [27] \), namely

\[ V(x) = V_1 \text{cosech}^2 \tau - V_2 \text{cosech} \tau \text{coth} \tau, \quad \tau = x - c - i\gamma, \quad V_1 > -\frac{1}{4}, \quad V_2 \neq 0. \quad (34) \]

It is easy to recognize \( (34) \) to belong to class II defined in Eq. \( (12) \). Note that the above potential is PT-symmetric as well as P-pseudo-Hermitian. Comparing with \( (12) \), we get

\[ b_I^2 - b_R^2 + m_R^2 - m_I^2 - \frac{1}{4} = V_1, \quad (35) \]

\[ b_R b_I + m_R m_I = 0, \quad (36) \]

\[ 2(m_R b_R - m_I b_I) = V_2, \quad (37) \]

\[ m_R b_I + m_I b_R = 0. \quad (38) \]

This time there is no reason to assume that \( b_I \neq 0 \), since the presence of \( \gamma \neq 0 \) in the generators \( (2) \) ensures that we are dealing with sl(2, \( \mathbb{C} \)).

On successively considering the cases where \( b_I = 0 \) or \( b_I \neq 0 \) and proceeding as in the previous section, we are led to the two following sets of solutions of Eqs. \( (35) - (38) \):

\[ b_R = \frac{1}{2} \nu \left( \sqrt{V_1 + \frac{1}{4} + |V_2|} - \epsilon \sqrt{V_1 + \frac{1}{4} - |V_2|} \right), \quad b_I = 0, \quad (39) \]

\[ m_R = \frac{1}{2} \left( \sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon \sqrt{V_1 + \frac{1}{4} - |V_2|} \right), \quad m_I = 0, \quad \epsilon = \pm 1. \]
provided $|V_2| \leq V_1 + \frac{1}{4}$ and $\sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \frac{1}{4} - |V_2|} > 1$, and

$$b_R = \frac{1}{2}\nu\sqrt{|V_2| + V_1 + \frac{1}{4}}, \quad b_I = -\frac{1}{2}\nu\epsilon\sqrt{|V_2| - V_1 - \frac{1}{4}},$$

$$m_R = \frac{1}{2}\sqrt{|V_2| + V_1 + \frac{1}{4}}, \quad m_I = \frac{1}{2}\epsilon\sqrt{|V_2| - V_1 - \frac{1}{4}}, \quad \epsilon = \pm 1,$$

provided $|V_2| > V_1 + \frac{1}{4}$ and $|V_2| + V_1 + \frac{1}{4} > 1$. In both cases, $\nu$ denotes the sign of $V_2$.

Comparison with Eq. (15) shows that the first type solutions (39) lead to two series of real eigenvalues

$$E_{n,\epsilon} = -\left[\frac{1}{2} \left(\sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \frac{1}{4} - |V_2|}\right) - n - \frac{1}{2}\right]^2, \quad \epsilon = \pm 1,$$

while the second type solutions (40) correspond to a series of complex-conjugate pairs of eigenvalues

$$E_{n,\epsilon} = -\left[\frac{1}{2} \left(\sqrt{|V_2| + V_1 + \frac{1}{4}} + i\epsilon\sqrt{|V_2| - V_1 - \frac{1}{4}}\right) - n - \frac{1}{2}\right]^2, \quad \epsilon = \pm 1. \quad (41)$$

In the former (resp. latter) case, it can be shown that $n$ varies in the range $n = 0, 1, 2, \ldots < \frac{1}{2} \left(\sqrt{V_1 + \frac{1}{4} + |V_2|} + \epsilon\sqrt{V_1 + \frac{1}{4} - |V_2|} - 1\right)$ [resp. $n = 0, 1, 2, \ldots < \frac{1}{2} \left(\sqrt{|V_2| + V_1 + \frac{1}{4}} - 1\right)$].

For increasing values of $|V_2|$, we observe a phenomenon entirely similar to that already noted for the complexified Scarf II potential: disappearance of the real eigenvalues and simultaneous appearance of complex-conjugate ones at the threshold $|V_2| = V_1 + \frac{1}{4}$. In this case, however, only partial results were reported in the literature. In Ref. [4], we obtained the two series of real eigenvalues (11) using the parametrization $V_1 = B^2 + A(A + 1)$, $V_2 = B(2A + 1)$, with $A$ and $B$ real, so that the condition $|V_2| \leq V_1 + \frac{1}{4}$ is automatically satisfied. Furthermore, Lévi and Znojil considered both the real [8] and the complex [24] eigenvalues in a parametrization $V_1 = \frac{1}{4}[2(\alpha^2 + \beta^2) - 1]$, $V_2 = \frac{1}{4}(\beta^2 - \alpha^2)$, wherein $\alpha$ and $\beta$ are real or one of them is real and the other imaginary, respectively. Their results, however, disagree with ours in both cases.

5 Complexified Morse potential

The potential

$$V(x) = (V_{1R} + iV_{1I})e^{-2x} - (V_{2R} + iV_{2I})e^{-x}, \quad V_{1R}, V_{1I}, V_{2R}, V_{2I} \in \mathbb{R}, \quad (43)$$
is the most general potential of class III for the upper sign choice in Eq. (13) and is a complexification of the standard Morse potential [27]. Comparison with Eq. (13) shows that

\[ b_R^2 - b_I^2 = V_{1R}, \]  \hspace{1cm} (44)

\[ 2b_Rb_I = V_{1I}, \]  \hspace{1cm} (45)

\[ 2(m_Rb_R - m_Ib_I) = V_{2R}, \]  \hspace{1cm} (46)

\[ 2(m_Rb_I + m_Ib_R) = V_{2I}, \]  \hspace{1cm} (47)

where we may assume \( b_I \neq 0 \).

On solving Eq. (45) for \( b_R \) and inserting the result into Eq. (44), we get a quadratic equation for \( b_I^2 \), of which we only keep the real positive solutions. The results for \( b_R \) and \( b_I \) read

\[ b_R = \frac{1}{\sqrt{2}} \epsilon_I \nu (V_{1R} + \Delta)^{1/2}, \quad b_I = \frac{1}{\sqrt{2}} \epsilon_I (-V_{1R} + \Delta)^{1/2}, \quad \Delta = \sqrt{V_{1R}^2 + V_{1I}^2}, \quad \epsilon_I = \pm 1, \]  \hspace{1cm} (48)

where \( V_{1I} \neq 0 \) if \( V_{1R} \geq 0 \) and \( \nu \) denotes the sign of \( V_{1I} \). On introducing Eq. (48) into Eqs. (46) and (47) and solving for \( m_R \) and \( m_I \), we then obtain

\[ m_R = \frac{\epsilon_I \nu}{2\sqrt{2}\Delta} \left[ (V_{1R} + \Delta)^{1/2}V_{2R} + \nu(-V_{1R} + \Delta)^{1/2}V_{2I} \right], \]  \hspace{1cm} (49)

\[ m_I = \frac{\epsilon_I \nu}{2\sqrt{2}\Delta} \left[ (V_{1R} + \Delta)^{1/2}V_{2I} - \nu(-V_{1R} + \Delta)^{1/2}V_{2R} \right]. \]  \hspace{1cm} (50)

From the regularity conditions \( b_R > 0 \) and \( m_R > \frac{1}{2} \) of \( \psi_0^{(m)}(x) \), given in Eq. (48), it follows that we must choose \( \epsilon_I = \nu, \) \( V_{1I} \neq 0 \) if \( V_{1R} < 0 \), and

\[ (V_{1R} + \Delta)^{1/2}V_{2R} + \nu(-V_{1R} + \Delta)^{1/2}V_{2I} > \sqrt{2}\Delta. \]  \hspace{1cm} (51)

We conclude that \( V_{1I} \neq 0 \) must hold for any value of \( V_{1R} \).

Real eigenvalues are associated with \( m_I = 0 \) and therefore occur whenever the condition

\[ (V_{1R} + \Delta)^{1/2}V_{2I} = \nu(-V_{1R} + \Delta)^{1/2}V_{2R} \]  \hspace{1cm} (52)

is satisfied. In such a case, \( V_{2I} \) can be expressed in terms of \( V_{1R}, V_{1I}, \) and \( V_{2R} \), so that the real eigenvalues are given by

\[ E_n = -\left[ \frac{V_{2R}}{\sqrt{2}|V_{1I}|} (-V_{1R} + \Delta)^{1/2} - n - \frac{1}{2} \right]^2. \]  \hspace{1cm} (53)
It can be shown that regular eigenfunctions correspond to $n = 0, 1, 2, \ldots < (V_{2R}/\sqrt{2}|V_{11}|)(-V_{1R} + \Delta)^{1/2} - \frac{1}{2}$.

Furthermore, when condition (52) is not fulfilled but condition (51) holds, we get complex eigenvalues associated with regular eigenfunctions,

$$E_n = -\left\{\frac{1}{2\sqrt{2}\Delta} \left[(V_{1R} + \Delta)^{1/2} - i\nu(-V_{1R} + \Delta)^{1/2}\right] (V_{2R} + iV_{2I}) - n - \frac{1}{2}\right\}^2,$$

where $n = 0, 1, 2, \ldots < \frac{1}{2\sqrt{2}\Delta} \left[(V_{1R} + \Delta)^{1/2}V_{2R} + \nu(-V_{1R} + \Delta)^{1/2}V_{2I}\right] - \frac{1}{2}$.

It should be stressed that contrary to what happens for the two previous examples, here the real eigenvalues, belonging to a single series, only occur for a special value of the parameter $V_{2I}$, while the complex eigenvalues, which do not appear in complex-conjugate pairs (since $E_n^*$ corresponds to $V^*(x)$), are obtained for generic values of $V_{2I}$.

To interpret such results, it is worth choosing the parametrization $V_{1R} = A^2 - B^2$, $V_{1I} = 2AB$, $V_{2R} = \gamma A$, $V_{2I} = \delta B$, where $A$, $B$, $\gamma$, $\delta$ are real, $A > 0$, and $B \neq 0$. The complexified Morse potential (43) can then be expressed as

$$V(x) = (A + iB)^2 e^{-2x} - (2C + 1)(A + iB)e^{-x}, \quad C = \frac{(\gamma - 1)A + i(\delta - 1)B}{2(A + iB)}.$$

Its (real or complex) eigenvalues can be written in a unified way as $E_n = -(C - n)^2$, while the regularity condition (51) amounts to $(\gamma - 1)A^2 + (\delta - 1)B^2 > 0$.

For $\delta = \gamma > 1$, and therefore $C = \frac{1}{2}(\gamma - 1) \in \mathbb{R}^+$, the potential (55) coincides with that considered in our previous work [4]. Such a potential was shown to be pseudo-Hermitian under imaginary shift of the coordinate [20]. We confirm here that it has only real eigenvalues corresponding to $n = 0, 1, 2, \ldots < C$, thus exhibiting no symmetry breaking over the whole parameter range. For the values of $\delta$ different from $\gamma$, the potential indeed fails to be pseudo-Hermitian. In such a case, $C$ is complex as well as the eigenvalues. The eigenfunctions associated with $n = 0, 1, 2, \ldots < \text{Re}C$ are however regular. The existence of regular eigenfunctions with complex energies for general complex potentials is a phenomenon that has been known for some time (see e.g. [28]).
6 Conclusion

In the present Letter, we have shown that complex Lie algebras (in particular sl(2, C)) provide us with an elegant tool to easily determine both real and complex eigenvalues of non-Hermitian Hamiltonians, corresponding to regular eigenfunctions. For such a purpose, it has been essential to extend the scope of our previous work [4] to those Lie algebra irreducible representations that remain nonunitary in the real algebra limit (namely those with $k_I \neq 0$).

We have illustrated our method by deriving the real and complex eigenvalues of the PT-symmetric complexified Scarf II potential, previously determined by other means [22]. In addition, we have established similar results for the PT-symmetric generalized Pöschl-Teller potential, for which only partial results were available [4, 8, 24]. We have shown that in both cases symmetry breaking occurs for a given value of one of the potential parameters.

Finally, we have considered a more general form of the complexified Morse potential than that previously studied [4, 13, 20]. For a special value of one of its parameters, our potential reduces to the former one and becomes pseudo-Hermitian under imaginary shift of the coordinate. We have proved that here no symmetry breaking occurs, the complex eigenvalues being associated with non-pseudo-Hermitian Hamiltonians.
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