Planar orthogonal polynomials as Type II multiple orthogonal polynomials

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Abstract
We show that the planar orthogonal polynomials with logarithmic singularities in the potential are the multiple orthogonal polynomials (Hermite–Padé polynomials) of Type II with measures. We also find the ratio between the determinant of the moment matrix corresponding to the multiple orthogonal polynomials and the determinant of the moment matrix from the original planar measure.

Keywords: planar orthogonal polynomials, Riemann–Hilbert problem, multiple orthogonal polynomials

(Some figures may appear in colour only in the online journal)

1. Main result

Let \( p_n(z) \) be the monic polynomial of degree \( n \) satisfying the orthogonality:

\[
\int_{\mathbb{C}} p_n(z) \overline{p_m(z)} e^{-|z|^2} |W(z)|^2 dA(z) = h_n \delta_{nm}, \quad n, m \geqslant 0,
\]

where \( dA \) is the Lebesgue area measure of the complex plane and \( h_n \) is the positive norming constant. We define, for \( I \geqslant 1 \), the multi-valued function \( W \) by

\[
W(z) = \prod_{j=1}^{I} (z - a_j)^{c_j}, \quad z \in \mathbb{C},
\]

where \( \{c_1, \cdots, c_I\} \) are positive real numbers and \( \{a_1, \cdots, a_I\} \) are distinct points in \( \mathbb{C} \).

The orthogonal polynomial whose measure is supported on the plane is called planar orthogonal polynomial. Such polynomial has been of interest due to its connection to two-dimensional Coulomb gas [1]. Moreover the polynomial defined above appears [2] in the quantized version of Hele-Shaw flow, a type of growth model in the two-dimensional plane.
These connections to physical system, Coulomb gas and Hele-Shaw flow, motivate one to study the large degree behavior of the polynomials. We recommend the recent paper [3] for an important progress in this regard and for the related history. Still lacking, until now, is the understanding of the limiting zero distribution when the degree of the polynomial goes to infinity. Several case studies [4–9] have shown that the zeros tend to certain one-dimensional set. In all of these cases the planar orthogonal polynomial in question turns out to be also either a classical orthogonal polynomial or a multiple orthogonal polynomial [10, 11], of which the asymptotic behavior is possible to study [13] due to rich algebraic structure such as finite term recurrence relation.

The main result of the paper is that our polynomials \( \{ p_n \} \) are multiple orthogonal polynomials of Type II. To introduce the main theorem, let us prepare several notations. To remove the unnecessary complication, we assume that \( a_j \)'s are all nonzero and the arguments of \( a_j \)'s are all different. Without loss of generality, we may assume:

\[
0 \leq \arg a_1 < \cdots < \arg a_l < 2\pi.
\]  

(3)

To determine the branch of the multi-valued function \( W \), we define the union of contours,

\[
B = \bigcup_{j=1}^l B_j, \quad B_j = \{ a_j t : t \geq 1 \},
\]

(4)

where the contours are directed towards the infinity. In the rest of the paper, we define \( W : \mathbb{C} \setminus B \to \mathbb{C} \) be an analytic branch of (2). Let \( B^* \) and \( B_j^* \) be the complex-conjugate images of \( B \) and \( B_j \). Let \( W^* : \mathbb{C} \setminus B^* \to \mathbb{C} \) be defined by

\[
W(z) = W(z) \prod_{j=1}^l (z - \bar{a}_j)^{\delta}.
\]

(5)

Let \( k = (k_1, \ldots, k_l) \) with non-negative integers \( k_j \)'s. When \( \arg z \not\in \{ \arg a_1, \ldots, \arg a_l \} \), we define

\[
\chi_k(z) = W(z) \int_0^{\infty} \prod_{j=1}^l (s - \bar{a}_j)^{k_j} W(s)e^{-zs} ds.
\]

(6)

where the represented integration contour is \( \{ |z| t \geq 0 \} \).

**Definition 1.** Let \( \Gamma \) be a simple closed curve with counterclockwise orientation, that connects \( \{ a_1, \ldots, a_l \} \), encloses the origin, and does not intersect \( B \setminus \{ a_1, \ldots, a_l \} \). Explicitly, we may choose \( \Gamma = \overline{a_1a_2} \cup \cdots \cup \overline{a_{l-1}a_l} \cup \overline{a_la_1} \), by the union of \( l \) line segments.

**Definition 2.** Let \( n = (n_1, \ldots, n_l) \) with non-negative integers \( n_j \)'s. We define \( p_n(z) \) to be the monic polynomial of degree \( |n| = n \) satisfying the orthogonality condition:

\[
\int_{\Gamma} p_n(z) z^k \chi_{n-e}(z) dz = 0, \quad 0 \leq k \leq n_j - 1, \quad 1 \leq j \leq l.
\]

(7)

Here \( e_j \) is the unit vector with one at the \( j \)th entry and zeros at all the other entries. We define \( q_n^{(i)}(z) \) to be the monic polynomial of degree \( |n| - 1 \) satisfying the orthogonality condition:

\[
\int_{\Gamma} q_n^{(i)}(z) z^k \chi_{n-e}(z) dz = 0, \quad 0 \leq k \leq n_j - 1 - \delta_{ij}, \quad 1 \leq i, j \leq l.
\]
The polynomials $p_n(z)$ and $q_n^{(j)}(z)$ are multiple orthogonal polynomials of type II.

Multiple orthogonal polynomials are related to Hermite–Padé approximation to a system of Markov functions [12]. For type II Hermite–Padé approximation, we look for rational functions approximating Markov functions near infinity, which consists of finding a polynomial $P_n$ of degree $|\mathbf{n}|$ and polynomials $Q_{nj}$ ($j = 1, \cdots, l$) of degree less than $|\mathbf{n}|$ such that

$$P_n(z) f_j(z) - Q_{nj}(z) = O\left(\frac{1}{z^{\kappa+1}}\right), \quad z \to \infty, \quad j = 1, \cdots, l,$$

where $f_1, \cdots, f_l$ are $l$ Markov functions given, in our context, by

$$f_j(z) = \int_{\Gamma} \frac{\chi_{n-s}(s)}{z-s} \, ds, \quad z \notin \Gamma, \quad j = 1, \cdots, l.$$

Then $Q_{nj}(z)$ is given by

$$Q_{nj}(z) = \int_{\Gamma} \frac{(P_n(z) - P_n(s)) \chi_{n-s}(s)}{z-s} \, ds.$$

In our context, $P_n = p_n$. We now state the main results:

**Theorem 1.** Given positive integers $n$ and $l$, we define a non-negative integer $\kappa$ and a non-negative integer $0 \leqslant r < l$ such that $n = \kappa l + r$. Then,

$$p_n(z) = p_n(z),$$

where $\mathbf{n} = (n, l) = (\kappa + 1, \cdots, \kappa + 1, \kappa, \cdots, \kappa)$.

The next theorem is an immediate consequence; see [11] for a reference.

**Theorem 2.** Let $n$, $l$, $\kappa$, $r$ and $\mathbf{n}$ be given as in theorem 1. Let the $(l+1)$ by $(l+1)$ matrix function $Y$ be given by

$$Y(z) = \begin{bmatrix}
    p_n(z) & \frac{1}{2\pi i} \int_{\Gamma} \frac{p_n(w) \chi_{n-s}(w)}{w-z} \, dw & \cdots & \frac{1}{2\pi i} \int_{\Gamma} \frac{p_n(w) \chi_{n-s}(w)}{w-z} \, dw \\
    \vdots & \vdots & \ddots & \vdots \\
    \frac{\gamma_j}{2\pi i} \int_{\Gamma} \frac{q_n^{(j)}(w) \chi_{n-s}(w)}{w-z} \, dw & \cdots & \frac{\gamma_j}{2\pi i} \int_{\Gamma} \frac{q_n^{(j)}(w) \chi_{n-s}(w)}{w-z} \, dw \\
    \vdots & \vdots & \ddots & \vdots
\end{bmatrix} \left( j+1 \right) \text{th row},$$

where the constant $\gamma_j$ in the $(j+1)$th row is given by

$$\gamma_j = -\left(\frac{1}{2\pi i} \int_{\Gamma} q_n^{(j)}(w) w^m \chi_{n-s}(w) \, dw\right)^{-1}, \quad m = \begin{cases} \kappa & \text{for } 1 \leqslant j \leqslant r; \\
\kappa - 1 & \text{for } r+1 \leqslant j \leqslant l.
\end{cases}$$

Then the matrix function $Y$ is the unique solution to the Riemann–Hilbert problem given below.
\[
\begin{align*}
Y : \mathbb{C} \setminus \Gamma & \to \mathbb{C}^{(l+1) \times (l+1)} \text{ is holomorphic matrix function;} \\
Y_+ (z) & = Y_- (z) J(z) \text{ on } \Gamma; \\
Y(z) & = (I + O(\frac{1}{z})) \left[ \begin{array}{ccc}
\frac{z^n}{n!} & 0 & 0 \\
0 & z^{-(\kappa+1)} I_{r \times r} & 0 \\
0 & 0 & z^{-n} I_{(l-r) \times (l-r)}
\end{array} \right], \text{ as } z \to \infty.
\end{align*}
\]

Above, the subscript ± in \( Y_\pm \) represents the limiting value when approaching \( \Gamma \) from the corresponding sides of the directed contour, and
\[
J(z) = \left[ \begin{array}{ccc}
1 & \chi_{n-e_1}(z) & \cdots & \chi_{n-e_l}(z) \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{array} \right].
\]

**Remark.** For \( l = 1 \), the contour \( \Gamma \) is a closed curve around the origin passing through \( a_1 \). After a little computation one can see that the jump contour \( \Gamma \) can be deformed to enclose the line segment \([0, a_1]\), to match the one in [6].

Let us define the moments,
\[
\begin{align*}
\nu_{jk}^{(i)} & := \frac{1}{2i} \int_{\Gamma} z^{j+k} \chi_{n-e_i}(z) \, dz = \frac{1}{2i} \int_{\Gamma} z^{j+k} \chi_{n-e_i}(z) \, dz, \\
\mu_{jk} & := \frac{1}{2i} \int_{\Gamma} z^{j+k} \chi_{n-e_i}(z) \, dz = \int_{\mathbb{C}} z^{j+k} e^{-|z|^2} |W(z)|^2 \, d\lambda(z).
\end{align*}
\]  

(8)

**Theorem 3.** Let \( n, l, \kappa, r \) and \( \mathbf{n} = (\kappa+1, \cdots, \kappa+1, \kappa, \cdots, \kappa) \) be given as in theorem 1.

For \( \nu_{jk}^{(i)} \) and \( \mu_{jk} \) given above, set the \( n \) by \( n \) matrices of moments \( d_n \) and \( D_n \) by
\[
d_n = \begin{bmatrix}
\nu_{0,0} & \nu_{0,1} & \cdots & \nu_{0,n-1} \\
\nu_{1,0} & \nu_{1,1} & \cdots & \nu_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\nu_{n-1,0} & \nu_{n-1,1} & \cdots & \nu_{n-1,n-1}
\end{bmatrix}, \quad D_n = \begin{bmatrix}
\mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0} \\
\mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n-1,1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1,n-1} & \mu_{n-1,n-1} & \cdots & \mu_{n-1,n-1}
\end{bmatrix},
\]

where
\[
n_i = \begin{cases}
\kappa + 1 & \text{for } 1 \leq i \leq r; \\
\kappa & \text{for } r + 1 \leq i \leq l.
\end{cases}
\]

Then there exists a unique constant matrix \( A_n \) such that \( d_n = A_n D_n \). Moreover it satisfies
\[
\det A_n = (-1)^n(n-1/2) \left( \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} (c_i + j)^i \right) \prod_{i<j} (\bar{a}_j - \bar{a}_i)^{n^2}
\]

\[
= (-1)^n(n-1/2) \left( \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} (c_i + j)^i \right) \prod_{i<j} (\bar{a}_j - \bar{a}_i)^{n^2}
\]

\[
\times \prod_{1 \leq i < j < r} (\bar{a}_j - \bar{a}_i)^{2n+1} \prod_{j=1}^{r} (\bar{a}_j - \bar{a}_i)^{n^2}.
\] (9)

**Remark.** The determinants of \(d_n\) and \(D_n\) both have integral representations that come from Heine’s formula.

\[
\det D_n = \frac{1}{n!} \int \cdots \int \left( \prod_{i<j} (z_i - z_j)^2 \cdot \prod_{j=1}^{n} \left( e^{-|z_j|^2} |W(z_j)|^2 d\Lambda(z_j) \right) \right),
\]

\[
\det d_n = \frac{1}{n!} \int \cdots \int \left( \prod_{i<j} (z_i - z_j) \cdot \det [\eta_j(z_j)]_{j,k=1}^{n} \prod_{j=1}^{n} dz_j,
\]

where

\[
\eta_j(z) = \begin{cases} 
\frac{1}{2\pi} z^{j-1} \chi_{\mathbf{n} - e_j}(z), & j = 1, \ldots, n_1, \\
\frac{1}{2\pi} z^{j-n_1-1} \chi_{\mathbf{n} - e_j}(z), & j = n_1 + 1, \ldots, n_1 + n_2, \\
\vdots \\
\frac{1}{2\pi} z^{j-(n_1+\cdots+n_{l-1}+1)} \chi_{\mathbf{n} - e_j}(z), & j = n_1 + \cdots + n_{l-1} + 1, \ldots, n.
\end{cases}
\]

Let us explain how our results can apply to some problems in physics. (See, for example, [14] for the physical background and the references.) The orthogonal polynomials that we consider are related to two dimensional Coulomb system, i.e. \(n\) number of interacting point particles in the plane subject to the external potential given by \(|z|^2 - \sum_{j=1}^{n} 2c_j \log |z - a_j|\). It means we consider \(n\) point charges of charge +1 interacting with \(l\) point charges of charges given by \(\{c_j\}\) fixed at the points \(a_j\)’s. The polynomial of degree \(n\) whose roots are at the \(n\) point charges is called the characteristic polynomial, due to its connection to the random matrix theory. The planar orthogonal polynomial \(p_n(z)\) is the averaged characteristic polynomial over all the (weighted) configurations of \(n\) point charges, i.e.

\[
E[|C_n(z)|^2] = p_n(z)
\]

where \(C_n\) stands for the characteristic polynomial of degree \(n\).

Our work is partly motivated to study the averaged moments of the characteristic polynomials, such as

\[
E[|C_n(a_1)|^2|C_n(a_2)|^2 \cdots |C_n(a_l)|^2].
\]

This quantity, in the limit of large \(n\), describes the behavior of the \(l\) point particles surrounded by infinite number of background particles. (In physical terminology, it describes the effective potential between the pseudo-particles). It is also the partition function of the corresponding
Coulomb system and, in particular, \( \det D_n \) that we considered in theorem 3. The behavior of this averaged moment has also been conjectured [16].

To study \( D_n \), since theorem 3 relate it to \( d_n \), which is the moment matrix of the multiple orthogonal polynomials, the asymptotic analysis of the multiple orthogonal polynomial is in order. Theorem 2 provides a way to do the analysis by the nonlinear steepest descent analysis of matrix Riemann–Hilbert problem (see [5, 6, 8, 9]). We also note that the moment determinant is the tau-function from the Riemann–Hilbert problem [15]. We will present this direction in a separate publication.

2. Proof of theorem 1

2.1. Area integral via contour integral

The following definitions will be useful.

\[
\chi_m(z) := W(z) \int_0^z s^m W(s) e^{-zs} ds, \\
\chi_\infty^m(z) := W(z) \int_0^{\infty} s^m W(s) e^{-zs} ds.
\]

Both are well defined if \( \arg z \neq \arg a_j \) for all \( j \). They satisfy the following lemma.

**Lemma 1.** Let \( S = \bigcup_{j=1}^l S_j \) where \( S_j = \{a_j t : 0 \leq t \leq 1\} \). \( \chi_\infty^m(z) - \chi_m(z) \) has continuous extension in \( \mathbb{C} \setminus S \) and, given \( k > 0 \), there exists \( C > 0 \) such that

\[
|z|^k |\chi_\infty^m(z) - \chi_m(z)| \leq C e^{-\sqrt{|z|-1}^2}
\]

for all \( z \) such that \( |z| > 2 \).

**Proof.** It is enough to check the continuity on \( B_1 \setminus \{a_1\} \). The piecewise analytic functions, \( W \) and \( \overline{W} \), satisfy the following jump conditions,

\[
W_+(z) = e^{-2\pi i c} W_-(z), \quad z \in B_1^*, \\
\overline{W}_+(z) = e^{-2\pi i c} \overline{W}_-(z), \quad z \in B_1^*.
\]

Here the subscripts plus stand for the boundary values taken from plus sides of \( B \); we assign plus sides on each point of \( B \setminus \{a_1, a_2, \ldots, a_l\} \) and \( B^* \setminus \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_l\} \) in a standard way, see figure 1.

Let \( p \in B_1 \setminus \{a_1\} \). Note that when \( z \) approaches \( p \) from plus side of \( B_1 \), \( \bar{z} \) approaches \( B_1^* \) from minus side. Then we get

\[
[\chi_\infty^m(p) - \chi_m(p)]_+ = [W(p)]_+ \int_0^{\infty} s^m \overline{W(s)} e^{-ps} ds \\
= [W(p)]_- \int_0^{\infty} s^m \overline{W(s)} e^{-ps} ds \\
= [\chi_\infty^m(p) - \chi_m(p)]_-.
\]

where we used (12) at the second equality. This proves the continuity statement. To prove the statement about the bound, we use the elementary estimate that, given \( k > 0 \), there exists \( C > 0 \) such that
\[ |z| |W(z)| \leq Ce^{|z|} \]

for all \( z \in \mathbb{C} \). Then, for some \( C > 0 \) and \( |z| > 2 \), we get
\[
|z^k (\chi^\infty_m(z) - \chi_m(z))| = |z^k W(z) \int_{z}^{\infty} s^m \overline{W(s)} e^{-sz} ds| \\
\leq Ce^{|z|} \int_{z}^{\infty} e^{s-|z|} ds \\
\leq Ce^{|z|} \int_{|z|}^{\infty} e^{|s|} e^{-|z|} ds = Ce^{-|s|} e^{-|s|-1} = Ce^{-|z|^2 + |z|^2}.
\] (14)

**Proposition 1.** For an arbitrary polynomial \( p(z) \) we have the following identity:
\[
\int_{C} p(z) \overline{z^m W(z)}^2 e^{-|z|^2} |W(z)|^2 \ dA(z) = \frac{1}{2i} \int_{\Gamma} p(z) \chi^\infty_m (z) \ dz.
\] (15)

**Proof.** We apply Green’s theorem to change the integral over \( \mathbb{C} \) to the integral over a contour. First we observe that
\[
\overline{z}^m |W(z)|^2 e^{-|z|^2} = \frac{\partial \chi_m(z)}{\partial \overline{z}}, \quad z \in \mathbb{C} \setminus B.
\] (16)

Therefore, defining \( D_R := \{ z \ | \ |z| < R \} \), we get
\[
\int_{C} p(z) \overline{z^m W(z)}^2 e^{-|z|^2} dA(z) = \lim_{R \to +\infty} \int_{D_R} p(z) \overline{z^m W(z)}^2 e^{-|z|^2} dA(z) \\
= \lim_{R \to +\infty} \int_{D_R \setminus B} p(z) \frac{\partial \chi_m(z)}{\partial \overline{z}} \ dA(z) \\
= \lim_{R \to +\infty} \frac{1}{2i} \left( \int_{\partial D_R} p(z) \chi_m(z) \ dz + \sum_{j=1}^{m} \int_{B_j \cap \partial D_R} p(z) [\chi_m(z)]^- \ dz \right).
\] (17)
where we use Green’s theorem at the last equality.

Since \( \chi_m^\infty(z) \) is analytic in \( \mathbb{C} \setminus (S \cup B) \), by deformation of contour we get the identity

\[
\int_{\Gamma} p(z) \chi_m^\infty(z) \, dz = \int_{\partial D_n} p(z) \chi_m^\infty(z) \, dz + \sum_{j=1}^{m} \int_{B_j \cap D_n} p(z) \chi_m^\infty(z) \, dz. \tag{18}
\]

Using this identity, the right hand side of (17) becomes

\[
\lim_{R \to \infty} \frac{1}{2\pi i} \int_{\partial D_n} p(z) \left( \chi_m(z) - \chi_m^\infty(z) \right) \, dz + \frac{1}{2\pi i} \int_{\Gamma} p(z) \chi_m^\infty(z) \, dz = \frac{1}{2\pi i} \int_{\Gamma} p(z) \chi_m^\infty(z) \, dz, \tag{19}
\]

where the last equality holds because of (11) in lemma 1. This proves proposition 1. \( \square \)

2.2. Several lemmas

**Definition 3.** All the vectors in this paper have only non-negative entries. For two vectors, \( k \) and \( s \), we say \( k \preceq s \) if \( k - s \) has only non-negative entries. If, in addition, \( k \neq s \) then we say \( k > s \). The \( j \)th entry of \( k \) is denoted by \( [k]_j \). We define the length of a vector by \( |k| = |k|_1 + \cdots + |k|_j \).

**Lemma 2.** For any \( n \geq 1 \) we have

\[
\text{span} \{ \chi_j^\infty : 0 \leq j < n \} = \text{span} \{ \chi_k : |k| \leq n \}. \tag{20}
\]

**Proof.** For \( n = 0 \), the lemma holds because \( \chi_0^\infty(z) = \chi_0(z) \). Assume that the lemma holds for \( n = n_0 \). If \( |k| = n_0 + 1 \) we get

\[
\chi_k(z) - \chi_{n_0 + 1}^\infty(z) = W(z) \int_0^{\infty} \prod_{j=1}^{l} (s - \bar{a})^b W(s) e^{-zs} \, ds - W(z) \int_0^{\infty} \times_{n_0 + 1} W(s) e^{-zs} \, ds
\]

\[
= W(z) \int_0^{\infty} \{ \text{polynomial in } s \text{ of degree } \leq n_0 \} \times W(s) e^{-zs} \, ds. \tag{21}
\]

Since the last term belongs to both spans in (20) for \( n = n_0 + 1 \), \( \chi_k \) belongs to the left span in (20) with \( n = n_0 + 1 \) and \( \chi_{n_0 + 1}^\infty \) belongs to the right span in (20) with \( n = n_0 + 1 \). \( \square \)

To prove \( p_n = p_n \), one may try to show that

\[
\text{span} \{ \chi_j^\infty(z) : 0 \leq j < n \} = \text{span} \{ z^k \chi_n - e | 0 \leq |n|, 1 \leq j \leq l \}. \tag{22}
\]

In fact, it is enough to show that the above equality up to functions \( \psi \) that satisfies \( \langle p, \psi \rangle = 0 \) for all polynomial \( p \). For example, we have \( \langle p, \psi \rangle = 0 \) for

\[
\psi(z) = W(z) \int_0^{\infty} \prod_{j=1}^{l} (s - \bar{a})^b W(s) e^{-zs} \, ds. \tag{23}
\]

Since \( \psi \) is analytic in \( \mathbb{C} \setminus B \) and, therefore, the integration contour in \( \int_{\Gamma} p(z) \psi(z) \, dz \) is contractible to a point. This allows us to consider, instead of \( \chi_k \) in (22),
\[ \tilde{\chi}_k := \chi_k - W(z) \int_0^{a_k} \prod_{j=1}^l (s - \tilde{a}_j)^k W(s)e^{-zs} \, ds. \]

As a result, using lemma 2, the proof of theorem 1 is reduced to proving the following proposition.

**Proposition 2.** For any \( n \geq 1 \) and \( l \geq 1 \) let \( n \) be given as in theorem 1. Then the following holds:

\[
\text{span} \{ \tilde{\chi}_k(z) : |k| < n \} = \text{span} \{ z^j \tilde{\chi}_{n-j}(z) \mid 0 \leq k < |n|, 1 \leq j \leq l \}. \quad (24)
\]

The proof of this proposition will be in the next subsection. The following lemma is why it is useful to use \( \tilde{\chi}_k \) instead of \( \chi_k \).

**Lemma 3.**

\[
z \tilde{\chi}_k(z) = \sum_{j=1}^l (c_j + k_j) \tilde{\chi}_{k-e_j}(z). \quad (25)
\]

**Proof.** Taking the integral of the total derivative as following, we have

\[
0 = W(z) \int_{a_k}^{\infty} \partial_s \left[ \prod_{j=1}^l (s - \tilde{a}_j)^k e^{-zs} \right] \, ds
\]

\[
= W(z) \int_{a_k}^{\infty} \left( \sum_{j=1}^l \frac{c_j + k_j}{s - \tilde{a}_j} - z \right) \prod_{j=1}^l (s - \tilde{a}_j)^k e^{-zs} \, ds
\]

\[
= \sum_{j=1}^l (c_j + k_j) \tilde{\chi}_{k-e_j}(z) - z \tilde{\chi}_k(z).
\]

**Corollary 1.** Let \( k = (k_1, k_2, \cdots, k_l) \) and \( s \leq \min \{ k_j \}_{j=1}^l \) be a positive integer. Then \( z^s \tilde{\chi}_k(z) \) can be represented as a linear combination of \( \{ \tilde{\chi}_{k-s}(z) \mid |s| = s \} \). Furthermore, the coefficient of \( \tilde{\chi}_{k-m}(z) \) is nonzero for all \( 1 \leq m \leq l \).

**Proof.** From lemma 3, the corollary is true when \( s = 1 \). Assume, for some \( 1 \leq s < \min \{ k_j \}_{j=1}^l \), that \( z^s \tilde{\chi}_k(z) \) is a linear combination of \( \tilde{\chi}_{k-s}(z) \) for \( |s| = s \) and the coefficient of \( \{ \tilde{\chi}_{k-m}(z) \}_{m=1}^l \) are all non-vanishing.

Then \( z^{s+1} \tilde{\chi}_k(z) \) is a linear combination of \( z \tilde{\chi}_{k-s}(z) \) and, therefore, of \( \tilde{\chi}_{k-s-e_n}(z) \) with \( |s| = s \) and \( 1 \leq m \leq l \). Since the term \( \tilde{\chi}_{k-(s+1)e_n}(z) \) comes only from \( z \tilde{\chi}_{k-s-e_n}(z) \) and since the coefficient of \( \tilde{\chi}_{k-m}(z) \) is non-zero, the coefficient of \( \tilde{\chi}_{k-(s+1)e_n}(z) \) is non-zero. Note that all the coefficients in the right hand side of (25) are non-zero. By induction, this ends the proof.

**Lemma 4.** For \( n \neq m \), we have

\[
\tilde{\chi}_{k+e_n}(z) - \tilde{\chi}_{k+e_m}(z) + (a_n - a_m) \tilde{\chi}_k(z) = 0. \quad (26)
\]
Proof. Since
\[(s - \bar{a}_n) - (s - \bar{a}_m) + (\bar{a}_n - \bar{a}_m) = 0,
\]
we obtain,
\[0 = W(z) \int_a^\infty \left[ (s - \bar{a}_n) - (s - \bar{a}_m) + (\bar{a}_n - \bar{a}_m) \right] \prod_{j=1}^l (s - \bar{a}_j)^{e_j} e^{-sz} \, ds.
\]
By the definition of $\hat{\chi}_k(z)$, (26) holds.

2.3. Proof of proposition 2

By corollary 1, we get $\supset$. To prove $\subset$, we note that any vector $k$ can be uniquely represented as
\[k = n + m - s,
\]
where $|m|, |s| = 0$, i.e. $m$ and $s$ cannot be both non-vanishing in any of the entries. It is then enough to show the following claim.

Claim. For all $s \leq n$ and $m$ satisfying $|n + m - s| < n$, 
\[\tilde{\chi}_{n+m-s} \in \text{span}\{\chi_{n-e_j}(z) \mid 0 \leq k < |n|, 1 \leq j \leq l\}.
\]

We prove this claim in two steps.

Step 1: For all $0 < s \leq n$, $\tilde{\chi}_{n-s} \in \text{span}\{\chi_{n-e_j}(z) \mid 0 \leq k < |n|, 1 \leq j \leq l\}$. If $|s| = 1$ then the inclusion is immediate. Let the inclusion holds for $|s| < m - 1$ for some $m < n$. (If $m \geq n$ then the proof is done.) Below we claim that the inclusion holds for $|s| = m$, which proves Step 1 by induction.

1. If $s$ has more than one non-zero entries, i.e. $|s|_i \neq 0$ and $|s|_j \neq 0$,
\[\tilde{\chi}_{n-s}(z) = \frac{1}{\bar{a}_i - \bar{a}_j} \left( \tilde{\chi}_{n-s+e_i}(z) - \tilde{\chi}_{n-s+e_j}(z) \right).
\]
The left hand side belongs to the span in Claim since the right hand side does by assumption.

2. If $s$ has exactly one non-zero entry, i.e. $s = me_j$ for some $j$. From $s < n$ we have $m \leq |n|$. Since $z^{m-1}\tilde{\chi}_{n-s}(z)$ is a linear combination of $\{\chi_{n-e_j} \mid |s| = m\}$ where the term $\tilde{\chi}_{n-me}$ appears with non-zero coefficient (see corollary 1), and since all the other terms in the linear combination belongs to the span by item 1, $\tilde{\chi}_{n-me}$ also belongs to the span in Claim.

Step 2: Step 1 showed Claim for $|m| = 0$. Assume that Claim is true when $|m| \leq k - 1$. We will show that Claim holds when $|m| \leq k$, i.e. $\tilde{\chi}_{n+m-s}$ belongs to the span in Claim for $|m| = k$.

Let $m$ satisfy $|m| = k \geq 1$. There exists $j$ such that $|m|_j > 0$. Then $\tilde{\chi}_{n+(m-e_j)-s}$ belongs to the span in the claim by the assumption. Since $|n + (m - e_j) - s| < n - 1$ we have $|s| > 0$ and there exists $i \neq j$ such that $|s|_i > 0$. Then $\tilde{\chi}_{n+(m-e_i)-(s-e_j)}$ also belongs to the span by the assumption. Since, by lemma 4, we have
\[ \tilde{\chi}_{n+m-s} = \tilde{\chi}_{n+(m-e_j)-(s-e_i)} + (\tilde{a}_i - \tilde{a}_j)\tilde{\chi}_{n+(m-e_i)-s}, \]

the left hand side belongs to the span. This ends the proof of proposition 2 and theorem 1.

3. Proof of theorem 3

Since \( \det D_n = \prod_{j=0}^{n-1} h_j > 0 \) where \( h_j \) is defined in (1), \( D_n \) is an invertible matrix and this proves the existence and the uniqueness of \( A_n \). In the remainder of the proof, we will construct \( A_n \) using induction. Let us consider the \( j \)th column of \( d_n \),

\[
\begin{bmatrix}
\nu_{j,0}^{(1)} \\
\nu_{j,1}^{(1)} \\
\vdots \\
\nu_{j,n-1}^{(1)} \\
\vdots \\
\nu_{j,0}^{(l)} \\
\nu_{j,1}^{(l)} \\
\vdots \\
\nu_{j,n-1}^{(l)}
\end{bmatrix} = \frac{1}{2i} \int_{\Gamma} z^j V_n(z) \, dz,
\]

where \( V_n = V_n(z) \) is given by

\[
\nu_{j,0} = \frac{1}{2i} \int_{\Gamma} z^j \chi_n(z) \, dz.
\]

The matrix \( B_n \) can be obtained by three successive linear transformations on \( V_n+e_{j+1} \) that we describe below.
Above, each arrow means the linear transformation given by

\[ B_n^{(1)} \text{ LHS of (A)} = \text{ RHS of (A)}, \]

\[ B_n^{(2)} \text{ LHS of (B)} = \text{ RHS of (B)}, \]

\[ B_n^{(3)} \text{ LHS of (C)} = \text{ RHS of (C)}, \]

\[ ... 

\[ 1 \]
\[B_0^{(1)} = \begin{bmatrix} \frac{I_{n+1}}{a_{i-1}} & \cdots & 0 & \frac{I_{n+1}}{a_{i-1} - a_{j+1}} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{I_{n+1}}{a_{i-1}} & \frac{I_{n+1}}{a_{i-1} - a_{j+1}} \\ \end{bmatrix} \]

\[B_0^{(2)} = \begin{bmatrix} \frac{I_{n+1}}{a_{i-1}} & \cdots & 0 & \frac{I_{n+1}}{a_{i-1} - a_{j+1}} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{I_{n+1}}{a_{i-1}} & \frac{I_{n+1}}{a_{i-1} - a_{j+1}} \\ \end{bmatrix} \]

\[B_0^{(3)} = \begin{bmatrix} \frac{I_{n+1}}{a_{i-1}} & \cdots & 0 & \frac{I_{n+1}}{a_{i-1} - a_{j+1}} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{I_{n+1}}{a_{i-1}} & \frac{I_{n+1}}{a_{i-1} - a_{j+1}} \\ \end{bmatrix} \]

where \(I_m\) is the \(m\) by \(m\) identity matrix and \(0_{jk}\) is the zero matrix of size \(j\) by \(k\). We used lemma 4 in the transformation (A) and lemma 3 in (B). This gives \(B_n = B_n^{(3)}B_n^{(2)}B_n^{(1)}\).

Using \(d_n = A_nD_n\) we obtain that
\[
B_n d_{n+1} = B_nA_{n+1}D_{n+1} = \begin{bmatrix} C_0 & \cdots & C_{n-1} & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_n & 1 \\ \end{bmatrix} D_{n+1}. \tag{27}
\]

The identity at the first row is obtained by
\[
\nu_{j,0} = \frac{1}{2i} \int_\Gamma z^j \chi_n(z) \, dz = \frac{1}{2i} \int_\Gamma z^j \sum_{k=0}^n C_k \chi_k^\infty(z) \, dz = \sum_{k=0}^n C_k \mu_{jk},
\]
where \(C_k\) is given by \(\prod_{a=1}^n (z - \bar{a})^n = \sum_{k=0}^n C_k z^k\). We also used that the upper \(n\) by \(n\) diagonal submatrix of \(D_{n+1}\) is \(D_n\).

Taking the determinant of (27) and using \(B_n = B_n^{(3)}B_n^{(2)}B_n^{(1)}\), we get
\[
\det A_{n+1} = (-1)^{n+2} \left( \det B_n^{(1)} \det B_n^{(2)} \det B_n^{(3)} \right)^{-1} \det A_n
\]
\[
= (-1)^{n+2+\sum_{i<j} n_i \left( \prod_{i<j+1} (\bar{a}_i - \bar{a}_{i+1}) \right)^n} \left( \prod_{i<j+1} (\bar{a}_{i+1} - \bar{a}_i)^n \right) (c_{r+1} + \kappa)^n \det A_n. \tag{28}
\]

Now we prove (9) by induction. When \(n = (1, 0, \ldots, 0)\) (i.e. \(\kappa = 0\) and \(r = 1\)), by the definition of \(\nu_{j,0}^{(1)}\) and \(\mu_{jk}\), we observe \(\nu_{0,0}^{(1)} = \mu_{0,0}\). This proves \(d_1 = D_1\) with \(\det A_1 = 1\). If (9) holds up to \(n \leq N\) then (9) holds for \(n = N + 1\) by (28). Remember that if \(n(N, l) = (n_1, \cdots, n_l)\) and \(N = nl + r\) then \(n(N+1, l) = (n_1, \cdots, n_{l+1} + 1, \cdots, n_l)\), increasing only the \((r+1)\)th entry by one. This ends the proof of theorem 3.

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