Supersymmetry constraints on the $\mathcal{R}^4$ multiplet in type IIB string theory on $T^2$

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Abstract

We consider a class of eight derivative interactions in the effective action of type IIB string theory compactified on $T^2$. These $1/2$ BPS interactions have moduli-dependent couplings. We impose the constraints of supersymmetry to show that each of these couplings satisfies a first order differential equation on moduli space which relate it to other couplings in the same supermultiplet. These equations can be iterated to give second order differential equations for the various couplings. The couplings which only depend on the $SO(2)\backslash SL(2, \mathbb{R})$ moduli satisfy the Laplace equation on moduli space and are given by modular forms of $SL(2, \mathbb{Z})$. On the other hand the ones that only depend on the $SO(3)\backslash SL(3, \mathbb{R})$ moduli satisfy the Poisson equation on moduli space where the source terms are given by other couplings in the same supermultiplet. The couplings of the interactions which are charged under $SU(2)$ are not automorphic forms of $SL(3, \mathbb{Z})$. Among the interactions that we consider the $\mathcal{R}^4$ coupling depends on all the moduli.

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1. Introduction

Constructing the low energy effective action of string theory in a certain background yields detailed information about the various symmetries of the theory. The degrees of freedom of the effective action are the various massless modes of the theory. Though in general, it is difficult to calculate the effective action, there are certain cases where a class of terms can be calculated exactly. Of course, this turns out to be possible because of the large amount of symmetry the theory possesses. In this paper, we are concerned with a particular example of this class of theories. We will consider type II string theory compactified on $T^3$, which has maximal supersymmetry and is conjectured to have an exact $SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z})$ symmetry [1, 2]. The $SL(3, \mathbb{Z})$ symmetry is a symmetry of the action, while the $SL(2, \mathbb{Z})$ symmetry is a symmetry of the equations of motion. Considering M theory on $T^3$, the $SL(3, \mathbb{Z})$ has a geometric origin as the group of large diffeomorphisms of the $T^3$, and $SL(2, \mathbb{Z})$ arises from the modular transformations of the complexified volume of the $T^3$. This theory is a particular example of toroidal compactifications of type II string theory, which preserves maximal supersymmetry. The moduli space is a coset space $H \backslash G$, where $G$ is a non-compact group, and $H$ is the maximal compact subgroup of $G$ [3, 4].1 Non-perturbative effects break

1 Only for $d = 8$, the moduli space factorizes into $(H_1 \backslash G_1) \otimes (H_2 \backslash G_2)$, where each factor satisfies this property.
2
the continuous symmetry to a discrete subgroup of $H$, which is the $U$-duality symmetry of the theory.

The motivation for studying the theory in eight dimensions arises from the fact that a certain class of terms in the effective action is known in ten dimensions, and explicit forms of the couplings and their non-renormalization properties have been analyzed [5–14]. The eight-dimensional case is the next one in order of complexity. Hence, this is the starting point for going down to lower dimensions. Our aim is to begin with the action of $N = 2, d = 8$ supergravity which is obtained by dimensionally reducing $d = 11$ supergravity on $T^3$. We then want to construct a certain set of terms among the various higher derivative corrections to the supergravity action. The set of terms we want to consider are $1/2$ BPS and satisfy non-renormalization properties, in particular, they receive perturbative contributions only up to one loop. These terms arise at the eight-derivative level in the effective action. We would like to use the constraints coming from supersymmetry to obtain equations satisfied by the moduli-dependent couplings of these interactions in the effective action. Various aspects of higher derivative corrections in eight and lower dimensions with maximal supersymmetry have been analyzed in [15–21], and various properties of the couplings have been deduced. This has led to explicit expressions for the $R^4$ coupling in lower dimensions in [22, 23].

The effective action that we will construct is one particle irreducible, and hence has infrared divergences. However, the equations of motion are duality invariant and coupled with the constraints of supersymmetry, certain terms are amenable to a detailed analysis. In order to construct a class of such terms in the effective action, we will implement the Noether procedure to the required order in the derivative expansion, also taking into account the corrected supersymmetry transformations, generalizing the work of [8]. In particular, we will use the invariance of the action under supersymmetry. We write the action and supersymmetry transformations as

$$S = S^{(0)} + \sum_{n=3}^{\infty} S^{(n)}, \quad \delta = \delta^{(0)} + \sum_{n=3}^{\infty} \delta^{(n)},$$

where $S^{(0)}$ and $\delta^{(0)}$ are the supergravity action and the supersymmetry transformations of the various fields at the two-derivative level, respectively. There are arguments to suggest that $S^{(1)}$ and $S^{(2)}$ vanish, and consequently so does $\delta^{(1)}$ and $\delta^{(2)}$. Thus, the first correction to the supergravity action is given by $S^{(3)}$, and this is the term we want to focus on. Our convention is that $S^{(n)}$ carries $2n + 2$ derivatives. Thus, using the Noether procedure, one has to implement the relations

$$\delta^{(0)} S^{(n)} + \delta^{(n)} S^{(0)} + \sum_{p+q=n} \delta^{(p)} S^{(q)} = 0$$

up to a total derivative for $n = 0$ and $n \geq 3$.

We begin with an analysis of the field content and the action of $N = 2, d = 8$ supergravity. This is followed by a discussion of the supersymmetry transformations at the two-derivative level. In the next section, we discuss the issue of gauge fixing the local symmetries of the moduli space, as well as supersymmetry. We then construct the transformations of the moduli under $U$-duality. After that, we focus on the issue of constructing the effective action beyond the two-derivative level. To begin with, we construct a set of higher derivative terms in the effective action which are $1/2$ BPS, starting from on-shell linearized superspace. We then consider the role of supersymmetry in constraining these higher derivative couplings. We look at a set of couplings which involves only the $U(1)\backslash SL(2, \mathbb{R})$ moduli, and another set which involves only the $SO(3)\backslash SL(3, \mathbb{R})$ couplings. We also consider a coupling which involves all the moduli. We briefly discuss the systematics of the analysis for lower dimensions very schematically.
For the couplings which depend only on the $U(1) \backslash SL(2, \mathbb{R})$ moduli, we show that each coupling satisfies a first order differential equation on the moduli space which relates it to another coupling. From the explicit structure of the equations, we conclude that each coupling satisfies the Laplace equation on moduli space. The couplings are given by automorphic forms of $SL(2, \mathbb{Z})$ with non-trivial weights, which is determined by the $U(1)$ charges of the corresponding interactions. The couplings depend only on the $SO(3) \backslash SL(3, \mathbb{R})$ moduli, and so we conclude that each coupling satisfies the Poisson equation on moduli space, with source terms given by couplings in the same supermultiplet. Furthermore, the couplings also satisfy first order differential equations on moduli space which relate to other couplings. However, the structure of the equations is such that it follows that each coupling satisfies the Poisson equation on moduli space, with source terms given by couplings in the same supermultiplet. Furthermore, the couplings for the interactions which carry non-trivial $SU(2)$ charges are not automorphic forms of $SL(3, \mathbb{Z})$, but transform in a complicated way.

It follows that supersymmetry does impose very strong constraints on the structure of the higher derivative corrections. It would be interesting to generalize the analysis to lower dimensions, and also to look at interactions in the effective action which preserve less supersymmetry.

2. The field content and the action of $N = 2, d = 8$ supergravity

Let us first consider the field content of $N = 2, d = 8$ supergravity [24], which is obtained by dimensionally reducing $d = 11$ supergravity [25] on $T^3$.

2.1. The bosonic degrees of freedom

The bosonic fields are given by

$$
e^a_\mu, \quad L_m^i, \quad L_U^a, \quad A^a_{\mu i}, \quad B^m_{\mu \nu}, \quad C_{\mu \nu \rho}.
\tag{2.1}
$$

In (2.1), $e^a_\mu$ is the vielbein, and $\mu, a = 0, \ldots, 7$, where $\mu$ is the world index and $a$ is the local frame index. There are seven scalars in the theory which are parametrized by $L_m^i$ and $L_U^a$ which satisfy

$$
det L_m^i = 1, \quad det L_U^a = 1.
\tag{2.2}
$$

Here $L_m^i$ parametrizes elements of the coset space $SO(3) \backslash SL(3, \mathbb{R})$, and so $m = 1, 2, 3$ transforms as the 3 of $SL(3, \mathbb{R})$ while $i = 1, 2, 3$ transforms as the 3 of $SO(3)$. Also $L_U^a$ parametrizes elements of the coset space $SO(2) \backslash SL(2, \mathbb{R})$, and so $U = 1, 2$ transforms as the 2 of $SL(2, \mathbb{R})$ while $a = 1, 2$ transforms as the 2 of $SO(2)$. Thus, under $SL(3, \mathbb{R})$ and $SO(3)$ transformations, $L_m^i$ which carries five degrees of freedom transforms as

$$
L_m^i(x) \rightarrow O_j^i(x)L_m^j(x)R_m^i,
\tag{2.3}
$$

where $O \in SO(3)$ and $R \in SL(3, \mathbb{R})$. Similarly, under $SL(2, \mathbb{R})$ and $SO(2)$ transformations, $L_U^a$ which carries two degrees of freedom transforms as

$$
L_U^a(x) \rightarrow N^a_v(x)L_U^v(x)S_U^a,
\tag{2.4}
$$

where $N \in SO(2)$ and $S \in SL(2, \mathbb{R})$. Thus, the classical moduli space is

$$
(U(1) \backslash SL(2, \mathbb{R})) \otimes (SO(3) \backslash SL(3, \mathbb{R})).
\tag{2.5}
$$

There are six Abelian gauge fields $A^a_{\mu i}$ which transform as the $(3, 2)$ of $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$, and three two-forms $B^m_{\mu \nu}$ which transform as the $(3, 1)$ of $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$. Finally, the
$U(1)$ invariant (anti)self-dual field strength $F_4^\pm$, where $F_4 = dC + \cdots$, and $G_4^\pm$ which is defined by

$$\frac{ie}{4!} G_{4\mu\nu\rho\lambda} = \pm \frac{\partial L}{\partial F_4^{\mu\nu\rho\lambda\pm}}$$

form a doublet

$$F_4^{U,\pm} = \begin{pmatrix} F_4^\pm \\ G_4^\pm \end{pmatrix}$$

under $SL(2, \mathbb{R})$, and is uncharged under $SL(3, \mathbb{R})$. Thus, the action is invariant under $SL(3, \mathbb{R})$, while only the equations of motion are invariant under $SL(2, \mathbb{R})$. The theory has 128 bosonic degrees of freedom.

### 2.2. The fermionic degrees of freedom

Now let us consider the 256 fermionic degrees of freedom in the theory. The fermions of $N = 2, d = 8$ supergravity are charged under $H$, but are uncharged under $G$ of the coset spaces $H\backslash G$. Since we are now considering spinors, $H$ is now $U(1) \times SU(2)$. We use $U(1)$ rather than $SO(2)$ for simplicity of manipulations because we will consider Weyl fermions, and $SU(2)$ because spin(3) = $SU(2)$. In order to understand the dimensional reduction in the fermionic sector under

$$\text{spin}(10, 1) \to \text{spin}(7, 1) \times SU(2),$$

consider a 32-component Majorana fermion $\eta$ in $d = 11$, which we decompose as

$$\eta = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \begin{pmatrix} \sigma_3 \psi_L^* \end{pmatrix},$$

where $\psi_L$ and $\psi_R$ are each eight-component chiral fermions of spin(7, 1) in the 2 of $SU(2)$, and $(\sigma_i)_{AB}$ are the Pauli matrices. Thus, explicitly,

$$\eta = \begin{pmatrix} \psi_{LA} \\ \psi_{RA} \end{pmatrix} = \begin{pmatrix} \psi_{LA} \\ -ie_{AB} \psi_{LB} \end{pmatrix},$$

where we have defined

$$(\psi_{LA})^* = \psi_{LA}^A$$

and

$$\epsilon_{12} = -\epsilon_{21} = 1, \quad \epsilon_{12} = -\epsilon_{21} = 1.$$
where $\Gamma_9$ is the $d = 8$ chirality matrix defined by
\[
\Gamma_9 = i\Gamma^0\Gamma^1 \cdots \Gamma^7.
\]
(2.17)

For the $\Gamma^a$ matrices, we consider a chiral basis given by
\[
\Gamma^a = \begin{pmatrix} 0 & \gamma^a \\ \bar{\gamma}^a & 0 \end{pmatrix},
\]
(2.18)
where
\[
\gamma^a \bar{\gamma}^b + \gamma^b \bar{\gamma}^a = 2\eta^{ab},
\]
(2.19)
and $\eta^{ab} = \text{diag}(-, +, \ldots, +)$. We consider an explicit basis for the $\gamma^a$ and $\bar{\gamma}^a$ matrices given by
\[
\begin{align*}
\gamma^0 &= 1 \otimes 1 \otimes 1, \\
\gamma^1 &= \sigma^2 \otimes \sigma^2 \otimes \sigma^2, \\
\gamma^2 &= 1 \otimes \sigma^3 \otimes \sigma^2, \\
\gamma^3 &= \sigma^3 \otimes \sigma^2 \otimes 1, \\
\gamma^4 &= \sigma^2 \otimes 1 \otimes \sigma^3, \\
\gamma^5 &= 1 \otimes \sigma^1 \otimes \sigma^2, \\
\gamma^6 &= \sigma^1 \otimes \sigma^2 \otimes 1, \\
\gamma^7 &= \sigma^2 \otimes 1 \otimes \sigma^1,
\end{align*}
\]
(2.20)
and
\[
\bar{\gamma}^0 = -\gamma^0, \quad \bar{\gamma}^I = \gamma^I
\]
for $I = 1, \ldots, 7$. Thus, $\Gamma^0$ is anti-Hermitian, while $\Gamma^I$ is Hermitian. Also
\[
\bar{\gamma}^a = -\gamma^a T, \quad \gamma^a \ast = -\bar{\gamma}^a.
\]
(2.22)

Note that $\eta$ in (2.9) satisfies the $d = 11$ Majorana condition
\[
\eta = C_{11} \bar{\eta} T,
\]
(2.23)
where $C_{11}$ is the $d = 11$ charge conjugation matrix given by
\[
C_{11} = 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2,
\]
(2.24)
where the factor of $\sigma_2$ acts on the $SU(2)$ indices. $C_{11}$ satisfies
\[
C_{11} \hat{\Gamma} \hat{C}_{11}^{-1} = -\hat{\Gamma} T,
\]
(2.25)
as expected.

Now, it is natural to ask, what kind of fermion is $\eta$ from the $d = 8$ point of view? Thus, we now think of it as a 16-component constrained Dirac spinor, with the $SU(2)$ indices coming from the extended supersymmetry. To analyze this, we first need to consider a constrained Dirac spinor of $N = 1, d = 8$ supersymmetry. This is given by
\[
\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.
\]
(2.26)
This satisfies the $d = 8$ Majorana condition
\[
\psi = C \bar{\psi} T,
\]
(2.27)
\[\text{It follows from the discussion below that } \psi_L \text{ does transform as } \psi_R.\]
where $C_8$ is the $d = 8$ charge conjugation matrix given by

$$C_8 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

(2.28)

which satisfies

$$C_8 \Gamma^\alpha C_8^{-1} = \Gamma^\alpha T.$$  (2.29)

Thus, $\psi$ in (2.26) is a pseudo-Majorana fermion. Now, for the $N = 2$ theory

$$\psi_A = \begin{pmatrix} \psi_{LA} \\ \psi_{RA} \end{pmatrix} = \begin{pmatrix} \psi_{LA} + i \epsilon_{AB} \psi^B_L \\ \psi_{RA} \end{pmatrix}$$

(2.30)

satisfies the condition

$$\psi_A = -i \epsilon_{AB} \gamma_5 C_8 \bar{\psi}^B_T.$$  (2.31)

Thus, (2.30) is an $SU(2)$ pseudo-Majorana fermion in the Weyl basis (see [28] for example).

To show that we are indeed working in a chiral basis for the $N = 2$ fermions, consider an infinitesimal Lorentz transformation in $d = 11$, under which a Majorana fermion transforms as

$$\delta \eta = \frac{1}{2} \xi_{\tilde{A} \tilde{B}} \tilde{\eta} \eta.$$  (2.32)

This leads to

$$\delta \psi_A = \frac{1}{2} (\xi_{ab} \Gamma^{ab} + i \xi_i \sigma^i) \psi_B,$$

(2.33)

where

$$\xi_i = \epsilon_{ijk} \xi_{jk}.$$  (2.34)

Now, as before, label

$$\psi_A = \begin{pmatrix} \psi_{LA} \\ \psi_{RA} \end{pmatrix},$$

(2.35)

which satisfies

$$\psi_R = N \psi_L^*.$$  (2.36)

The fact that $\psi_L$ and $\psi_R$ have opposite chiralities leads to

$$N (\xi_{ab} \gamma^{ab} + i \xi_i \sigma^i) \gamma^{-1} = \xi_{ab} \gamma^{ab} + i \xi_i \sigma^i.$$  (2.37)

Thus,

$$N = M \sigma_2,$$

(2.38)

where $M$ satisfies

$$M (\gamma^a \tilde{\gamma}^b - \gamma^b \tilde{\gamma}^a)^* M^{-1} = \tilde{\gamma}^a \gamma^b - \tilde{\gamma}^b \gamma^a.$$  (2.39)

Thus, $N$ factorizes into the $SU(2)$ part and the spacetime part, and (2.39) is solved by

$$M = \gamma^0 = -\tilde{\gamma}^0 = 1,$$

(2.40)

and so

$$\psi_R = \sigma_2 \psi_L^*.$$  (2.41)

3 This is called ‘pseudo’ because $C_8$ satisfies (2.29) with a $+$ sign on the right-hand side, and not $-$ [26, 27].

4 In general, $\epsilon_{AB}$ can be replaced by the symplectic form $\omega_{\alpha \beta}$, leading to $Sp(2n)$ pseudo-Majorana fermions. This structure for extended supersymmetry is exactly as in $N = 2$, $d = 4$ where one has Majorana, instead of pseudo-Majorana fermions.
as before. Also note that

$$\Gamma_0 \psi_L = -\psi_L, \quad \Gamma_9 \psi_R = \psi_R.$$  

(2.42)

Let us now tabulate the fermions of the $N = 2, d = 8$ theory. The fermions of negative chirality are given by

$$\psi_{\mu L}, \chi^i_{L A}, \lambda_{L A}. \quad \text{(2.43)}$$

In (2.43), the spin 3/2 gravitini $\psi_{\mu L}$ transform as the 2 of $SU(2)$, while the spin 1/2 fermions $\chi^i_{L A}$ and $\lambda_{L A}$ transform as the 4 and 2 of $SU(2)$, respectively, and so

$$\sigma_{\chi_{L A}} = 0. \quad \text{(2.44)}$$

Under $U(1)$, $\psi_{\mu L}$, $\chi^i_{L A}$ and $\lambda_{L A}$ carry charges 1/2, $-1/2$, and 3/2, respectively. The positive chirality fermions are denoted by $\psi_{\mu RA}$, $\chi^i_{RA}$ and $\lambda_{RA}$, and are the conjugates of the negative chirality ones, according to the discussion above. They carry $U(1)$ charges $-1/2, 1/2$, and $-3/2$, respectively. The supersymmetry transformation parameter $\epsilon_{L A}$ is in the 2 of $SU(2)$, and carries $U(1)$ charge 1/2, while $\epsilon_{RA}$ carries $U(1)$ charge $-1$. Thus, $\psi_{\mu L}, \chi^i_{L A}$ and $\lambda_{L A}$ carry 160, 64 and 32 degrees of freedom, respectively.

For arbitrary fermions $\psi_1$ and $\psi_2$, we define conjugation by

$$\psi_1 \psi_2^\dagger = -\psi_2^\dagger \psi_1^\dagger. \quad \text{(2.45)}$$

We also make use of the relations

$$\frac{i}{2} \Gamma^\mu \partial_\mu \psi = \bar{\psi} \tilde{\Gamma}^\mu \partial_\mu \psi \equiv \bar{\psi} \Gamma^\mu \partial_\mu \psi, \quad \text{(2.46)}$$

on using (2.22), and ignoring total derivatives. In (2.46), we have also defined

$$\bar{\psi} = \psi^\dagger \Gamma^0, \quad \bar{\psi}_L = \psi_L^\dagger, \quad \bar{\psi}_R = -\psi_R^\dagger. \quad \text{(2.47)}$$

2.3. Relevant terms in the action

Now let us consider some of the terms in the action which are relevant for our purposes. They are given by

$$e^{-1} \mathcal{L}^{(0)} = R - 2 \mathcal{P}_{\mu \nu}^\mu R - P_{\rho ij} \mathcal{P}^{\rho ij} - \frac{1}{4} M_{MN} M_{nm} F_{2 \nu}^{mn} F_2^{\mu \nu} + \frac{1}{12} M_{mn} F_{3 \mu \nu \rho \sigma} F_3^{\mu \nu \rho \sigma} + \frac{i}{48} \left( L^2 \mathcal{P}_{4 \mu \nu \rho}^{\mu \nu \rho} F_4^{\mu \nu \rho} - L^2 \mathcal{P}_{4 \mu \nu \rho}^{\mu \nu \rho} F_4^{\mu \nu \rho} \right) \right)

+ 4 \bar{\psi}_{\mu L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_{RA} + 4 \bar{\chi}_{\lambda L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \chi^i_{RA} + 2 \bar{\lambda}_{\lambda L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \lambda_{RA} - \frac{1}{48 \sqrt{2}} \left[ F_{\mu \nu \rho \sigma}^{\mu \nu \rho \sigma} \left( \bar{\psi}_{\mu L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_{RA} + \bar{\psi}_{R} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_{RA} \right) \right]

+ \frac{F_{\mu \nu \rho \sigma}^{\mu \nu \rho \sigma}}{L^2} \left( \bar{\psi}_{\mu L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_{RA} + \bar{\psi}_{R} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_{RA} - \bar{\psi}_{\mu L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \lambda_{RA} - \bar{\psi}_{R} \tilde{\Gamma}^{\mu \rho} \partial_\rho \lambda_{RA} \right), \quad \text{(2.48)}$$

where the covariant derivatives are defined later. Thus, we get that

$$G_{4 \mu \nu \rho \sigma}^+ = \frac{L^2}{L^4} \mathcal{P}_{4 \mu \nu \rho}^+ F_4^{\mu \nu \rho} + \frac{i}{2 \sqrt{2} L^2} \left( \bar{\psi}_{\mu L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_{RA} \right) \psi_{RA} + \bar{\psi}_{R} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_{RA} \right) \psi_{RA} - \bar{\psi}_{\mu L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \lambda_{RA} - \bar{\psi}_{R} \tilde{\Gamma}^{\mu \rho} \partial_\rho \lambda_{RA} \right), \quad \text{(2.49)}$$

where $L^2$ is defined shortly.

This argument also goes through for $N = 1$ supersymmetry by looking at $d = 8$ Lorentz transformations, as there are no internal indices coming from the extended supersymmetry.

Thus, for example, $(\psi_L \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_L) = \bar{\psi}_{\mu L} \tilde{\Gamma}^{\mu \rho} \partial_\rho \psi_L$ up to a total derivative.
In (2.48), the various field strengths are defined by
\[ \frac{1}{2} F^{μν}_{2\nu'λ'} = \tilde{∂}_{μ} A^{μν}_{λ'\nu'} , \]
\[ \frac{1}{2} F_{3μνλμ} = \tilde{∂}_{μ} B^{μνλ}_κ + \frac{1}{2} \epsilon_{μνρσ} A^{μνλ}_κ F^{ρσ}_{λμν} , \]
\[ \frac{1}{2} F_{4μνλμνλ} = \tilde{∂}_{μ} C^{μνλ}_λ + \frac{1}{2} (B^{μμ}_κ - \epsilon_{μμρσ} A^{μμλ}_κ F^{λσ}_{μμλ}) F^{λμ}_{2μν} , \]
(2.50)

We have also defined the (anti)self-dual parts of any 4-form \( X_4 \) by
\[ X_4^± = \frac{1}{2} (X_4 ± i \ast X_4) , \]
(2.51)

where \( \ast \)
\[ \ast X_{abcd} = \frac{1}{4!} \epsilon_{abcd}^{efgh} X_{efgh} . \]
(2.53)

Note that
\[ \frac{1}{4!} \epsilon_{abcd}^{efgh} Y_{abcd} = \gamma^{efgh} , \]
(2.54)

leading to
\[ \gamma^{μνλκ} X_{4μνλκ} = \gamma^{μνλκ} X_{4μνλκ} , \]
(2.55)

Also the \( SO(3) \backslash SL(3, \mathbb{R}) \) and \( SO(2) \backslash SL(2, \mathbb{R}) \) moduli are contained in
\[ M_{mn} = L_{μ} L_{ν} δ_{μν} , \]
(2.56)

and
\[ M_{UV} = L_{μ}' L_{ν}' δ_{μν} , \]
(2.57)

respectively.

To understand the structure of the kinetic terms of the moduli parametrizing \( SO(3) \backslash SL(3, \mathbb{R}) \), consider \( L \) which parametrizes the elements of the coset space \( H \backslash G \). We use the Cartan decomposition (see [29] for example)
\[ -LdL^{-1} = P + Q , \]
(2.58)

where \( Q \) is in \( H \), while \( P \) is in \( H \backslash G \). Thus, \( P \) and \( Q \) are invariant under the global transformation \( L \rightarrow L G \). Then, the kinetic terms for the moduli can be expressed in terms of \( P \), while \( Q \) gives rise to the composite \( H \) gauge field. Thus, for the coset space \( SO(3) \backslash SL(3, \mathbb{R}) \), we have that
\[ -L_{μ} \partial_{μ} L_{μ}' = P_{μ} + Q_{μ} , \]
(2.59)

where \( P_{μ} \) is the symmetric, and \( Q_{μ} \) is the anti-symmetric part of the left-hand side of (2.59).

\( P_{μ} \) is automatically traceless because \( \det L_{μ}' = 1 \).

It is also useful for our purposes to write the kinetic terms for the moduli only in terms of the matrix \( M \). This is done by noting that
\[ P_{μ} = -\frac{1}{2} L_{μ} L_{μ}' \partial_{μ} M_{μ}^{μ} , \]
(2.60)

\[ \square \] Square brackets are normalized with unit weight.

\[ \square \] We have that \( \epsilon_{0123} = 1 \)
(2.52)

for the frame indices.
which leads to
\[-P_{\mu ij} P^{\mu ij} = \frac{1}{4} \partial_\mu M_{mn} \partial^\mu M^{mn} = \frac{1}{4} \text{Tr}(\partial_\mu M \partial^\mu M^{-1}). \tag{2.61}\]

To understand the structure of the moduli fields parametrizing $SO(2) \backslash SL(2, \mathbb{R})$, it is convenient to consider the complex basis
\[L_\pm = \frac{1}{\sqrt{2}} (L_1 V \pm i L_2 V), \tag{2.62}\]
where the subscripts in $L_\pm$ label the $U(1)$ charges. Thus, we also have that
\[(L_\pm)^* = L_\mp, \quad \epsilon_{UV} L_\pm V L_\mp U = i. \tag{2.63}\]

The kinetic term for the moduli can be expressed in terms of the $SL(2, \mathbb{R})$ invariant combination $P_\mu$ defined by (it carries $U(1)$ charge 2)
\[P_\mu = -\epsilon_{UV} L_\pm U \partial_\mu L_\pm V, \tag{2.64}\]
while the composite $U(1)$ gauge field is given by the $SL(2, \mathbb{R})$ invariant combination $Q_\mu = -\epsilon_{UV} L_\pm U \partial_\mu L_\pm V$. \tag{2.65}

Let us consider the transformations of the various fields under infinitesimal $U(1) \times SU(2)$ gauge transformations. Under a $U(1)$ gauge transformation, a field $\Phi_q$ carrying $U(1)$ charge $q$ transforms as
\[\delta \Phi_q(x) = iq \Sigma(x) \Phi_q(x), \tag{2.66}\]
and thus the gauge field $Q_\mu$ transforms as
\[\delta Q_\mu = \partial_\mu \Sigma. \tag{2.67}\]

Thus, the covariant derivative
\[D_\mu \Phi_q = \partial_\mu \Phi_q - iqQ_\mu \Phi_q \tag{2.68}\]
transforms as
\[\delta D_\mu \Phi_q = \partial_\mu D_\mu \Phi_q. \tag{2.69}\]

To consider $SU(2)$ gauge transformations, let us define
\[A_{\mu i} = \frac{1}{2} \epsilon_{ijk} Q_\mu jk. \tag{2.70}\]

Thus, under a gauge transformation\(^9\)
\[\delta L_i^m = -\epsilon_{ijk} \theta_j L_k^m, \tag{2.71}\]
we have that
\[\delta A_{\mu i} = \partial_\mu \theta_i + \epsilon_{ijk} A_{\mu j} \theta_k. \tag{2.72}\]

So, for $\Psi = (\psi_\mu, \lambda)$, the gauge transformation
\[\delta \Psi = \frac{i}{2} \sigma^i \Psi \tag{2.73}\]
leads to the covariant derivative
\[D_\mu \Psi = \partial_\mu \Psi - \frac{i}{2} A_\mu \sigma^i \Psi, \tag{2.74}\]
\[^9\] This also leads to
\[\delta L_i^m = -\epsilon_{ijk} \theta_j L_k^m, \tag{2.75}\]
on using $\delta(L_i^m L_m^n) = 0$. 

10
which transforms as
\[
\delta D_\mu \Psi = \frac{i}{2} \theta^\prime \sigma^\prime D_\mu \Psi. 
\] (2.76)

For \(\chi^i\) satisfying \(\sigma_i \chi^i = 0\), we also have that
\[
\chi^i = i \epsilon^{ijk} \chi^j \chi^k, 
\] (2.77)

leading to
\[
\delta \chi^i = - \epsilon^{ijk} \theta^j \chi^k + \frac{3i}{2} \theta^\prime \sigma^\prime \chi^i, 
\] (2.78)
on using
\[
\sigma_i \chi_j - \sigma_j \chi_i = -i \epsilon^{ijk} \chi_k. 
\] (2.79)

So the covariant derivative
\[
D_\mu \chi^i = \partial_\mu \chi^i + i A_\mu^i \sigma^i \chi^i - \frac{3i}{2} A_\mu^i \sigma^\prime \chi^i, 
\] (2.80)
transforms as
\[
\delta D_\mu \chi^i = -i \theta^\prime \sigma^\prime D_\mu \chi^i + \frac{3i}{2} \theta^\prime \sigma^\prime D_\mu \chi^i, 
\] (2.81)
on using the Schouten identity
\[
\epsilon^{ijk} \chi^j + \epsilon^{lij} \chi^j + \epsilon^{jlk} \chi^i + \epsilon^{kli} \chi^j = 0. 
\] (2.82)

Thus, for the various fermionic interactions in (2.48), we have that
\[
D_\mu \psi_{iL} = D_\mu \psi_{iL} - \frac{i}{2} A_\mu^i \sigma^i \psi_{iL} - \frac{i}{2} Q_\mu \psi_{iL}, 
\]
\[
D_\mu \lambda_L = D_\mu \lambda_L - \frac{i}{2} A_\mu^i \sigma^i \lambda_L - \frac{3i}{2} Q_\mu \lambda_L, 
\]
\[
D_\mu \chi^i_L = D_\mu \chi^i_L + i A_\mu^i \sigma^i \chi^i_L - \frac{3i}{2} A_\mu^i \sigma^\prime \chi^i_L + i Q_\mu \chi^i_L, 
\] (2.83)
\[
D_\mu \psi_{iR} = D_\mu \psi_{iR} - \frac{i}{2} A_\mu^i \sigma^i \psi_{iR} + \frac{1}{2} Q_\mu \psi_{iR}, 
\]
\[
D_\mu \lambda_R = D_\mu \lambda_R - \frac{i}{2} A_\mu^i \sigma^i \lambda_R + \frac{3i}{2} Q_\mu \lambda_R, 
\]
\[
D_\mu \chi^i_R = D_\mu \chi^i_R + i A_\mu^i \sigma^i \chi^i_R - \frac{3i}{2} A_\mu^i \sigma^\prime \chi^i_R - \frac{i}{2} Q_\mu \chi^i_R, 
\]
where \(D_\mu\) is the ordinary covariant derivative.

It is convenient for our purposes to redefine field strengths that are invariant under \(G\), and carry specific charges under \(H\). For the 2-form field strengths, we define
\[
F^i_{\mu \nu} = \epsilon_{UV} F^i_{2mu} L^V_{\nu} L^i_{m}, 
\]
\[
F^{ij}_{\mu \nu} = \epsilon_{UV} F^{ij}_{2mu} L^V_{\nu} L^i_{m}, 
\] (2.84)
which carry charges 1 and \(-1\), respectively, under \(U(1)\), and are in the 3 of \(SU(2)\). For the 3-form field strengths, we define
\[
F^i_{3\mu \nu \rho} = L^m_{\mu} F^i_{3\mu \nu \rho m}, 
\] (2.85)
which is uncharged under \( U(1) \), and is in the 3 of \( SU(2) \). Finally, for the 4-form field strengths, we define the self-dual field strength
\[
T^{\mu
u
rho
sigma}_{+} = \epsilon_{UVL} L_{+} F^{-V}_{4\mu
u
rho
sigma},
\]
which carries \( U(1) \) charge \( -1 \), and the anti-self-dual field strength
\[
T^{\mu
u
rho
sigma}_{-} = \epsilon_{UVL} L_{+} F^{-V}_{4\mu
u
rho
sigma},
\]
which carries \( U(1) \) charge 1. Both \( T^{\pm} \) are uncharged under \( SU(2) \).

3. Deriving the supergravity action and the supersymmetry transformations

In order to construct the relevant terms in the \( d = 8 \) action as well as the supersymmetry transformations of the various fields, we start from the action and the supersymmetry transformations of the \( d = 11 \) supergravity theory. The action is given by \([25]\)
\[
V^{-1} \mathcal{L}_{11} = \frac{R(\alpha')}{4} - \frac{1}{48} F_{MNPQ} F^{MNPQ} + \frac{1}{2} \tilde{\eta}_M F^{MNPQ} D_N (\frac{\alpha + \dot{\alpha}}{2}) \eta_P
\]
\[
- \frac{1}{192} \tilde{\eta}_M F^{RSMNPQ} \eta_N + 12 \eta^M (\frac{\alpha}{2} - \frac{\dot{\alpha}}{2} + \eta_P F^{MNPQ} + \tilde{F}_{MNPQ})
\]
\[
+ \frac{2}{(144)^2} V^{-1} \epsilon_{M_1 M_2 M_3} F_{M_1 M_2} F_{M_3 M_4} C_{M_4 M_5 M_6},
\]
(3.1)

We denote the local frame and world indices by \( \hat{A}, \hat{B}, \ldots \) and \( M, N, \ldots \), respectively. In (3.1), \( V^4, C_{MNP} \) and \( \eta_M \) are the vielbein, the 3-form potential and the gravitino, respectively. We also have that
\[
D_M \left( \frac{\alpha + \dot{\alpha}}{2} \right) \eta_N = \partial_M \eta_N + \frac{1}{8} (\alpha + \dot{\alpha}) M^{\hat{A} \hat{B}} \Gamma_{\hat{A} \hat{B}} \eta_N.
\]
(3.2)

We work in the second order formalism where \( \omega_M^{\hat{A} \hat{B}} \) is an independent field which satisfies its equation of motion (see [30] for example), leading to
\[
\omega_M^{\hat{A} \hat{B}} = \omega_M^{\hat{A} \hat{B}} (V) + K_M^{\hat{A} \hat{B}}.
\]
(3.3)

In (3.3), \( \omega_M^{\hat{A} \hat{B}} (V) \) is the standard spin connection of pure gravity given by
\[
\omega_M^{\hat{A} \hat{B}} (V) = \frac{1}{4} V^{\hat{A} \hat{B}} (\partial_M V^\hat{C} - \partial_N V^\hat{C} \gamma_{\hat{A} \hat{B}} - \partial_{\hat{D}} V^\hat{C} \gamma_{\hat{A} \hat{B}}) - \frac{1}{2} V^{\hat{A} \hat{B}} (\tilde{\partial}_M V^\hat{C} - \tilde{\partial}_N V^\hat{C} \gamma_{\hat{A} \hat{B}} - \tilde{\partial}_{\hat{D}} V^\hat{C} \gamma_{\hat{A} \hat{B}})
\]
\[
- \frac{1}{2} V^{\hat{A} \hat{B}} (\tilde{\partial}_M V^\hat{C} - \tilde{\partial}_N V^\hat{C} \gamma_{\hat{A} \hat{B}} - \tilde{\partial}_{\hat{D}} V^\hat{C} \gamma_{\hat{A} \hat{B}}),
\]
(3.4)

while the contorsion tensor \( K_M^{\hat{A} \hat{B}} \) is given by
\[
K_M^{\hat{A} \hat{B}} = \frac{1}{2} [\tilde{\eta}^{\hat{C} \hat{D}} \hat{I}_{\hat{M} \hat{A} \hat{B} \hat{C} \hat{D}} \eta^\hat{C} + 2 (\tilde{\eta}^{\hat{C} \hat{D}} \hat{I}_{\hat{M} \hat{A} \hat{B} \hat{C} \hat{D}} \eta^\hat{C} + \tilde{\eta}^{\hat{C} \hat{D}} \hat{I}_{\hat{M} \hat{A} \hat{B} \hat{C} \hat{D}} \eta^\hat{C})].
\]
(3.5)

The supercovariant spin connection in (3.1) is given by
\[
\tilde{\omega}_M^{\hat{A} \hat{B}} = \omega_M^{\hat{A} \hat{B}} + \frac{1}{4} \tilde{\eta}^{\hat{C} \hat{D}} \hat{I}_{\hat{M} \hat{A} \hat{B} \hat{C} \hat{D}} \eta^\hat{C} \tilde{\eta}^{\hat{C} \hat{D}} \hat{I}_{\hat{M} \hat{A} \hat{B} \hat{C} \hat{D}} \eta^\hat{C}
\]
\[
= \omega_M^{\hat{A} \hat{B}} (V) + \frac{1}{2} (\tilde{\eta}^{\hat{C} \hat{D}} \hat{I}_{\hat{M} \hat{A} \hat{B} \hat{C} \hat{D}} \eta^\hat{C} + \tilde{\eta}^{\hat{C} \hat{D}} \hat{I}_{\hat{M} \hat{A} \hat{B} \hat{C} \hat{D}} \eta^\hat{C}).
\]
(3.6)
For the 3-form potential, it is easier to work with the frame indices to perform the dimensional reduction, and so the supercovariant 4-form field strength $F_{ABCD}$ is given by

$$
\hat{F}_{ABCD} = F_{ABCD} + 3\eta_A \hat{\Gamma}^B \gamma^C \eta_D, 
$$

(3.7)

where

$$
F_{ABCD} = 4\delta_{\{A} C_{B|CD\}} + 12\alpha_{\{A} \hat{\epsilon} (V) C_{|D\}E\}}. 
$$

(3.8)

Let us now consider the supersymmetry transformations of the various fields. Apart from the fermionic trilinear terms in the supervariation of the fermions, the other transformations can be directly obtained from [24], as mentioned in detail below. In order to obtain the fermionic trilinear terms in the supervariation of the fermions, we consider maximal supergravity in $d = 11$ and obtain them by dimensional reduction, given the complete supervariations of the theory. We mention only those steps which are relevant for our manipulations. The supersymmetry transformations and the local Lorentz transformations of the $d = 11$ theory are

$$
\begin{align*}
V^M & \delta^{(0)} V_{M\lambda} = -\bar{\xi} \hat{\Gamma}_\lambda^{\hat{\lambda}} \gamma_{\hat{\lambda}} + \lambda_{\hat{\lambda} M}, \\
\delta^{(0)} C_{AB} &= -\frac{1}{2} \bar{\xi} \hat{\Gamma}_B^{\hat{B}} C_{\hat{A}\hat{B}C} - 3 C_{\hat{A}B\hat{E}} \gamma^C \delta^{(0)} V_{M\hat{E}}, \\
\delta^{(0)} \eta_A &= \hat{D}_A \xi = \frac{1}{16} (\hat{\Gamma}_{\hat{A}\hat{B}DE} - 8 \hat{\Gamma}_{\hat{C}\hat{D}E} \delta^{\hat{A}}_\hat{B} \hat{\Gamma}_{\hat{C}DE}) \xi \hat{F}_{\hat{B}\hat{C}DE} - V_A^M (\delta^{(0)} V_{M\hat{B}}) \eta_{\hat{B}} + \frac{1}{2} \lambda_{\hat{B} M} \hat{\Gamma}_{\hat{B}DE} \eta_A.
\end{align*}
$$

(3.9)

where $\xi$ is the supersymmetry parameter and $\lambda_{\hat{A} M}$ is the parameter for local Lorentz transformations. The supercovariant derivative $\hat{D}_M \xi$ is given by

$$
\hat{D}_M \xi = (\partial_M + \frac{1}{2} \hat{\omega}_M \hat{\Gamma}_{\hat{A}B}) \bar{\xi}. 
$$

(3.10)

Let us briefly mention the relation between the various fields we have and those used by [24]. While they work directly in terms of the two moduli of $U(1) \setminus SL(2, \mathbb{R})$ and thus the $SL(2, \mathbb{R})$ covariance is not manifest, we maintain the explicit covariance by working in terms of $L_\lambda^U$. In fact, we will later gauge fix the $U(1)$ transformation, which will force the $SL(2, \mathbb{R})$ transformations to be realized nonlinearly on the various fields. Our formulas then exactly reduce to the ones obtained by [24]. We now mention the relations needed to go from their formulas (SS) to ours. We set $\kappa = 1$, as well as

$$
e^{2\eta_M} = U_2, \quad B_{SS} = -\frac{U_1}{2}, \quad e^{aSS}_\mu = e^{a}_\mu , \quad L_{SS} = L_{\mu}^i, 
$$

(3.11)

and so

$$
P_{SS}^{\mu ij} = P_{\mu ij}, \quad Q_{SS}^{\mu ij} = -Q_{\mu ij}. 
$$

(3.12)

For the two 1-form potentials, we set

$$
A_{SS}^\mu = \frac{A^1_\mu}{2}, \quad B_{SS}^\mu = \frac{A^2_\mu}{2}, 
$$

(3.13)

leading to

$$
F_{SS}^{\mu \nu} = \frac{1}{2} F_{2\mu \nu}, \quad G_{SS}^{\mu \nu} = \frac{1}{2} (F_{2\mu \nu}^m - U_1 F_{2\mu \nu}^m). 
$$

(3.14)

For the 2-form potential, we set

$$
2B_{SS}^{\mu \nu \rho} = B_{\mu \nu \rho} - \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} (A^2_\sigma A^1_\rho - A^1_\sigma A^2_\rho), 
$$

(3.15)

leading to

$$
G_{SS}^{\mu \nu \rho} = \frac{1}{2} F_{3\mu \nu \rho}. 
$$

(3.16)

10 Correcting several typos.
For the 3-form potential, we set
\[ B_{\mu\nu\lambda}^{SS} = \frac{1}{2} C_{\mu\nu\lambda} \]  
(3.17)
leading to
\[ G_{\mu\nu\rho}^{SS} = \frac{1}{2} F_{\mu\nu\rho} \]  
(3.18)
We will later need to construct certain quartic fermion couplings in the \( d = 8 \) theory starting from (3.1). For that, we will need to know the relations between the fields \( F_{A\hat{b}\hat{c}\hat{d}} \) and the fields \( G_{A}^{SS}, \ G_{A}^{SS} \) and the scalar \( B^{SS} \). They are
\[ G_{\mu\nu\omega}^{SS} = e^{-ia\psi/3} e_{a} e_{b} e_{c} e_{\rho} \frac{F_{abcd}}{2}, \]
\[ G_{\mu\nu\rho}^{SS} = e^{-a\psi/3} e_{a} e_{b} e_{c} e_{\rho} G_{abc}, \]
\[ G_{\mu\nu\lambda}^{SS} = \frac{1}{2} e^{2a\psi/3} e_{m} e_{n} e_{a} e_{\lambda} F_{mn} F_{abij}, \]
\[ \partial_{\mu} B^{SS} = \frac{1}{2} e^{a\psi/3} e_{a} e_{b} e_{\mu} F_{bijk}. \]  
(3.19)

To obtain the fermions, we set
\[ \psi_{\mu} = \psi_{\mu}, \quad \chi_{i} = \chi_{i} + \sigma_{i} \Gamma \frac{\lambda}{3}, \]  
(3.20)
where \( \psi_{\mu}, \chi_{i} \) and \( \lambda \) are \( SU(2) \) pseudo-Majorana fermions in the Weyl basis as discussed before. Thus, for the action in (2.48), this gives us
\[ e^{-1} L^{(0)} = 4(e^{SS})^{-1} L^{SS}. \]  
(3.21)

Note that the complete set of Chern–Simons terms in the action is given by
\[ 4L^{SS} = \frac{1}{12\pi} e^{a\psi/6} \left[ 3B_{SS} G_{\mu\nu\rho}^{SS} G_{\lambda\mu\nu\rho}^{SS} - 8 e^{i\lambda} G_{\mu\nu\rho}^{SS} G_{\mu\nu\rho}^{SS} G_{\mu\nu\rho}^{SS} B_{\mu\nu\rho}^{SS} 
+ 288 F_{\mu\nu\rho}^{SS} G_{\mu\nu\rho}^{SS} \left( (\partial_{\mu} B_{\mu\nu\rho}) B_{\mu\nu\rho} - \frac{1}{2} G_{\mu\nu\rho}^{SS} B_{\mu\nu\rho}^{SS} \right) 
- 96 B_{\mu\nu\rho}^{SS} \partial_{\mu} B_{\mu\nu\rho}^{SS} G_{\mu\nu\rho}^{SS} \right], \]  
(3.22)
where we have repeated integrally by parts, and used the Bianchi identities
\[ \partial_{\mu} G_{\mu\nu}^{SS} = 4 F_{\mu\nu} G_{\mu\nu}, \quad \partial_{\mu} G_{\mu\nu\rho}^{SS} = 3 \epsilon_{\mu\nu\rho} F_{\mu\nu} G_{\mu\nu\rho}^{SS}. \]  
(3.23)
In (2.48), we have only kept the first term in (3.22). The other terms in (3.22) are independent of \( F_{\mu\nu\rho}^{SS} \), and so do not contribute to \( G_{\mu\nu\rho}^{SS} \).

Of course, the values of \( L_{c}^{(i)} \) have to be substituted using the ones obtained later in (4.6). Some of these calculations have an overlap with [31], who work in a covariant formalism.

Note that the relation between the \( d = 11 \) fermionic fields and the \( d = 8 \) fermionic fields which are relevant for our purposes are given in equations (29) and (34) of [24]. In particular, the \( d = 8 \) fermions are given by \[ \eta_{\mu} = e^{\phi_{(\psi)/6}} \left( \psi_{\mu} - \frac{1}{6} \Gamma_{\mu} \right), \]  
(3.25)
\[ \eta_{i} = e^{\phi_{(\psi)/6}} \left( \chi_{i} + \sigma_{i} \Gamma_{\phi} \frac{\lambda}{3} \right), \]  
where we have also used (3.20).

\[ \xi = e^{-\psi/6}. \]  
(3.24)
This is useful in constructing the only other term in the action (2.48) which contributes to the
deformation of $G_{\mu
u\rho\sigma}$ apart from those already mentioned before. This term is given by
\begin{equation}
\epsilon_{SS}' \mathcal{L}_{SS} = -\frac{1}{2} \bar{\eta}_i \Gamma^i \bar{\delta}^{abcd} \eta_j \eta_k + 12 \bar{\eta} \Gamma^{ij \lambda} \eta^j \mathcal{F}_{abcd},
\end{equation}
where $\mathcal{F}_{abcd}$ is a particular component of $F_{ABCD}$. Let us now mention the local Lorentz transformation parameters in $d = 8$ which are obtained directly from $d = 11$, as described by [24]. The dimensional reduction on $T^3$ breaks the $SO(10, 1)$ symmetry of the frame indices to $SO(3) \times SO(7, 1)$, which is implemented by a gauge choice $V_{\mu} = 0$. Preserving this gauge choice, as well as requiring that $L^{-1} \delta^{(0)} L$ is in $SO(3) \backslash SL(3, \mathbb{R})$ fixes $\lambda_{ia}$, and the local Lorentz transformations parameters $\lambda'_{ab}$ and $\lambda'_{ij}$ of $SO(7, 1)$ and $SO(3)$ respectively in terms of $\lambda_{ab}$ and $\lambda_{ij}$. These relations are given by
\begin{equation}
\begin{align*}
\lambda_{ab} &= -\bar{\epsilon} \Gamma_a \left( \chi_i + \sigma_i \Gamma_0 \frac{\lambda}{3} \right), \\
\lambda'_{ab} &= \lambda_{ab} + \frac{1}{6} \bar{\epsilon} \Gamma_{ab} \lambda, \\
\lambda'_{ij} &= \lambda_{ij} - \frac{1}{2} \bar{\epsilon} \Gamma_0 (\sigma_i \chi_j - \sigma_j \chi_i) - \frac{i}{3} \epsilon_{ijk} \bar{\epsilon} \sigma^k \lambda.
\end{align*}
\end{equation}
We will construct the complete supervariations of the various fermions from (3.9), remembering to parametrize the residual local Lorentz transformations by $\lambda'_{ab}$ and $\lambda'_{ij}$. Thus, the $d = 8$ supersymmetry and local Lorentz transformations are given by
\begin{equation}
\begin{align*}
\delta^{(0)} \eta_i &= \cdots + \frac{i}{2} (\lambda'_{ab} \Gamma^{ab} + \lambda'_{ij} \sigma^{ij}) \eta_i + \lambda'_{ij} \eta_j, \\
\delta^{(0)} \eta_a &= \cdots + \frac{i}{2} (\lambda'_{ab} \Gamma^{bc} + \lambda'_{ij} \sigma^{ij}) \eta_a + \lambda'_{ab} \eta_b,
\end{align*}
\end{equation}
where the $\cdots$ are the supersymmetry transformations given in (C.1).

It should be noted that the supersymmetry transformations for all the fields given in (C.1) are not simply obtained from those in [24] by substituting the various expressions above. This is because they have already gauge fixed the $U(1)$ transformations, while our transformations are manifestly gauge covariant. Thus, their transformations are the same as what we have only for the fields that are $U(1)$ invariant. For the other fields, their transformations have extra terms, which we will describe later when we fix the $U(1)$ gauge symmetry. These extra terms, which are not gauge invariant, take a very simple form at the end, though they look very complicated to start with. Various cancellations which are a consequence of supersymmetry are responsible for this simplification. In appendix D, we outline the nature of these cancellations, which is quite intricate. Thus, the complete supersymmetry transformations of the $d = 8$ theory are given by (C.1).

4. Gauge fixing the local symmetry transformations

We have two sets of moduli, each of which parametrizes a coset space $H \backslash G$. We first gauge fix $H$ to obtain the physical degrees of freedom in $L$. We then show how the $G$ symmetry is realized nonlinearly on the moduli. Finally, we consider the gauge-fixed supersymmetry transformations.

4.1. Gauge fixing $L$

So far we have parametrized the elements of the coset space $H \backslash G$ in terms of $L$. We will now gauge fix $L$ to obtain the physical degrees of freedom. To do so, we use the Iwasawa decomposition to represent $L$ as
\begin{equation}
L = HKN,
\end{equation}
\begin{equation}
H = \hat{H},
\end{equation}
\begin{equation}
K = \hat{K},
\end{equation}
\begin{equation}
N = \hat{N}.
\end{equation}
where $H$ is a matrix of gauge transformations, $K$ is a diagonal matrix of unit determinant and $N$ is an upper triangular matrix with diagonal entries unity. Thus, to work in a fixed gauge, we simply remove the factor of $H$ in (4.1).

Let us now mention the moduli of type IIB string theory on $T^2$. The complex structure $U$ of the $T^2$ parametrizes the moduli space $SO(2)\backslash SL(2, \mathbb{R})$. The five degrees of freedom parametrizing the moduli space $SO(3)\backslash SL(3, \mathbb{R})$ include the complexified coupling $\tau$ obtained from ten dimensions, and the Kahler structure $T$ of the $T^2$ defined by

$$T = B_N + iV,$$

where $V$ is the volume of the $T^2$ in the string frame. The remaining modulus is $B_R$, where $B_N$ ($B_R$) is obtained from the NS–NS (R–R) 2-form in ten dimensions. Thus, the $SO(2)\backslash SL(2, \mathbb{R})\tau$ of $S$-duality, and the $SO(2)\backslash SL(2, \mathbb{R})\tau$ of $T$-duality are intertwined in the $SO(3)\backslash SL(3, \mathbb{R})$ moduli space. We now express the components of $L$ in terms of these degrees of freedom.

For the $U(1)\backslash SL(2, \mathbb{R})$ moduli space, we take

$$L^\nu = \left( \sqrt{\frac{U_2}{2}} 0 0 -U_1/U_2 \right) \frac{1}{\sqrt{U_2}} \left( \begin{array}{cc} U_2 & -U_1 \\ 0 & 1 \end{array} \right),$$

which leads to

$$L^\nu = \frac{1}{\sqrt{U_2}} \left( \begin{array}{cc} U_1 \\ 0 \end{array} \right)$$

and

$$M_{UV} = \frac{1}{U_2} \left( \begin{array}{cc} |U|^2 & -U_1 \\ -U_1 & 1 \end{array} \right).$$

Thus, we get that

$$\left( \begin{array}{cc} L^1_+ & L^1_- \\ L^2_+ & L^2_- \end{array} \right) = \frac{1}{\sqrt{2U_2}} \left( \begin{array}{cc} U_1 \\ 0 \end{array} \right),$$

yielding

$$P_\mu = \frac{\partial_\mu U}{2U_2},$$

$$Q_\mu = \frac{\partial_\mu U_1}{2U_2}.$$  \hfill (4.7)

So in the action, we get that

$$-2P^\mu P^\nu = -\frac{\partial_\mu U \partial^\nu \bar{U}}{2U_2^2}.$$  \hfill (4.8)

For the $SO(3)\backslash SL(3, \mathbb{R})$ moduli space, we take

$$L^i_m = \left( \begin{array}{ccc} \nu^{-1/3} & 0 & 0 \\ 0 & \sqrt{2} \nu^{1/6} & 0 \\ 0 & 0 & \nu^{1/6}/\sqrt{2} \end{array} \right) \left( \begin{array}{ccc} 1 & -\frac{\nu}{\sqrt{2} \tau_2} \text{Im} B & \frac{\nu}{\tau_2} \text{Re} B \\ 0 & 1 & -\tau_1/\tau_2 \\ 0 & 0 & 1 \end{array} \right)$$

$$= \nu^{1/6} \left( \begin{array}{ccc} \sqrt{\frac{\tau_2}{\nu}} & -\text{Im} B & \text{Re} B \\ 0 & \tau_2 & -\tau_1 \\ 0 & 0 & 1 \end{array} \right),$$

where $\nu = (\tau_2 V)^{-1}$ and $B = B_R + \tau B_N$.  \hfill (4.9)
This leads to
\[ M_{\mu\nu} = \frac{\nu^{1/3}}{\tau_2} \begin{pmatrix} \tau_2/\nu + |B|^2 & -\text{Re}(\bar{\tau} B) & \text{Re} B \\ -\text{Re}(\bar{\tau} B) & |\tau_1|^2 & -\tau_1 \\ \text{Re} B & -\tau_1 & 1 \end{pmatrix}. \] (4.10)

It is useful to see how the \( U(1) \langle SL(2, \mathbb{R}) \rangle_1 \) and \( U(1) \langle SL(2, \mathbb{R}) \rangle_2 \) subspaces of (4.9) are intertwined. To see the \( U(1) \langle SL(2, \mathbb{R}) \rangle_1 \) subspace, we drop the \( B \) dependence for simplicity, and focus on \( \tau \) and \( V \), leading to
\[ L^i_m = \nu^{1/6} \begin{pmatrix} \nu^{-1/2} & 0 & 0 \\ 0 & \sqrt{\tau_2} & -\tau_1/\sqrt{\tau_2} \\ 0 & 0 & 1/\sqrt{\tau_2} \end{pmatrix}, \] (4.11)
where \( \nu \) is \( S \)-duality invariant. To see the \( U(1) \langle SL(2, \mathbb{R}) \rangle_2 \) subspace, we drop the \( B \) and \( \tau_1 \) dependence for simplicity, and focus on \( T, \tau_2 \) and \( V \), leading to
\[ L^i_m = e^{-\varphi/3} \begin{pmatrix} \sqrt{T_2} & -T_1/\sqrt{T_2} & 0 \\ 0 & 1/\sqrt{T_2} & 0 \\ 0 & 0 & e^{\varphi/2} \end{pmatrix}, \] (4.12)
where
\[ e^{-2\varphi} = \tau_2^2 V \] (4.13)
is the \( T \)-duality invariant \( d = 8 \) dilaton.

4.2. Nonlinearly realized \( G \) symmetry

Having gauge fixed \( H \), let us now see how the \( G \) symmetry is realized nonlinearly on the moduli. First consider the \( U(1) \langle SL(2, \mathbb{R}) \rangle_1 \) moduli space, where
\[ L^U_{\pm} \rightarrow e^{i\alpha \Sigma} L^V_{\pm} S^U_{V}. \] (4.14)
It is sufficient for our purposes to look at infinitesimal transformations. Thus, taking
\[ S^U_{V} = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 - \alpha \end{pmatrix}, \] (4.15)
where \( \alpha, \beta, \gamma \) and \( \delta \) are infinitesimal real parameters, requiring reality of \( L^1 \) in (4.6), we get that
\[ \Sigma = -\gamma U_2. \] (4.16)
Also including the constraints coming from \( L^2 \), we get that
\[ \delta U = \beta - 2aU - \gamma U^2. \] (4.17)
It is also easy to write down the finite transformations. Taking
\[ S^U_{V} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \] (4.18)
where \( a, b, c, d \in \mathbb{R} \) and \( ad - bc = 1 \), the above analysis leads to
\[ \tan \Sigma = -\frac{cU_2}{cU_1 + d}, \] (4.19)
thus leading to the \( SL(2, \mathbb{R}) \) transformation
\[ U \rightarrow \frac{aU + b}{cU + d}. \] (4.20)

12 This is the same as [15] on sending \( \tau_1 \rightarrow -\tau_1 \) and \( B_N \rightarrow -B_N \).
For the $SO(3) \backslash SL(3, \mathbb{R})$ moduli, in (2.3), we take

$$R_m^n = \begin{pmatrix}
1 + \alpha & b & c \\
d & 1 + \beta & f \\
g & h & 1 - \alpha - \beta
\end{pmatrix}, \quad (4.21)$$

where $\alpha, \beta, b, c, d, f, g,$ and $h$ are infinitesimal real parameters, and

$$O^j_i = \delta_{ij} + \epsilon_{ijk}\theta_k. \quad (4.22)$$

Thus, preserving the gauge choice $L_2^1 = L_3^1 = L_3^2 = 0$ in (4.9), we get that

$$\theta_1 = c \text{Im} B - f \tau_2, \quad \theta_2 = c \sqrt{\tau_2 \nu}, \quad \theta_3 = -\frac{1}{\sqrt{\nu}}(b + c \tau_1). \quad (4.23)$$

Thus, the transformations of the remaining non-vanishing elements of $L_m^n$ lead to

$$\delta \nu = -3 \nu \left( \alpha - c \text{Re} B + \frac{(b + c \tau_1)}{\tau_2} \text{Im} B \right),$$

$$\delta \tau = (\alpha + 2\beta - cB)\tau - hb - h + f \tau^2,$$

$$\delta B = (2\alpha + \beta)B - i \frac{(b + c \tau)}{\nu} + g - d \tau + f \tau B - cB^2. \quad (4.24)$$

One can consider finite transformations as well. In order to do so, we use the definition of the finite form of the matrix $O$ in (4.22) given by

$$O^j_i = \frac{1}{2} \text{Tr}(g^{-1} \sigma_i g \sigma_j), \quad (4.25)$$

where $g$ is an element of the $SU(2)$ group, and the trace is in the fundamental representation of $SU(2)$. This follows from the defining equation for the transformation matrices $D$ of the adjoint representation of any group, given by

$$g^{-1} T_a g = D^b_a (g) T_b, \quad (4.26)$$

where $g$ is an element of the group, and $T_a$ are the generators in the fundamental representation. Thus, (4.25) follows for $SU(2)$, where

$$g = \cos \frac{\vec{\theta}}{2} + i \frac{\vec{\sigma} \cdot \vec{\theta}}{2|\vec{\theta}|} \sin \frac{|\vec{\theta}|}{2}, \quad (4.27)$$

and we have chosen the normalization in (4.25) to obtain (4.22) in the infinitesimal limit. This leads to

$$O^j_i (\theta) = \delta_{ij} \cos |\vec{\theta}| + \epsilon_{ijk}\theta_k \frac{\sin |\vec{\theta}|}{|\vec{\theta}|} + \frac{2 \theta_i \theta_j}{|\vec{\theta}|^2} \sin^2 \frac{|\vec{\theta}|}{2}. \quad (4.28)$$

This representation is called the axis-angle representation in the literature. It consists of an $SO(3)$ rotation about an axis given by

$$\vec{n} = \frac{\vec{\theta}}{|\vec{\theta}|}, \quad (4.29)$$

by an angle $|\vec{\theta}|$. One can then proceed exactly along the lines of the above discussion to obtain the finite transformations. Thus, the finite transformations are given by the solution of a set of involved equations, which are difficult to manipulate. Thus, to obtain the finite transformations, we use a different representation of $SO(3)$ rotation matrices.
We parametrize an arbitrary $SO(3)$ rotation by successive rotations around the 2, 3 and 1 axes by angles $-\phi_2$, $\phi_3$ and $\phi_1$, respectively, and so

$$O = R_{23}(\phi_1)R_{12}(\phi_3)R_{13}(-\phi_2),$$

(4.30)

where

$$R_{23}(\phi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi_1 & \sin \phi_1 \\ 0 & -\sin \phi_1 & \cos \phi_1 \end{pmatrix},$$

(4.31)

for example. These angles are called the Tait–Bryan angles in the literature\(^\text{13}\). Thus, we have that

$$O_j = \begin{pmatrix} \cos \phi_2 \cos \phi_3 & \sin \phi_3 & -\sin \phi_2 \cos \phi_3 \\ -\cos \phi_1 \cos \phi_2 \sin \phi_3 + \sin \phi_1 \sin \phi_2 & \cos \phi_1 \cos \phi_3 & \cos \phi_1 \sin \phi_2 \sin \phi_3 + \sin \phi_1 \cos \phi_2 \\ \sin \phi_1 \cos \phi_2 \sin \phi_3 + \cos \phi_1 \sin \phi_2 & -\sin \phi_1 \cos \phi_3 & -\sin \phi_1 \sin \phi_2 \sin \phi_3 + \cos \phi_1 \cos \phi_2 \end{pmatrix}. $$

(4.32)

Thus, for small angles,

$$\phi_j = 0. $$

(4.33)

We further define

$$\mu^j = L_j/R^j. $$

(4.34)

Thus, preserving the gauge choices $L_3^1 = L_3^1 = 0$, we get that

$$\sin \phi_2 = \frac{\mu_3^1 \mu_2^2 - \mu_1^1 \mu_2^3}{\sqrt{(\mu_3^1 \mu_2^2 - \mu_1^1 \mu_2^3)^2 + (\mu_2^1 \mu_3^3 - \mu_3^1 \mu_2^3)^2}}, $$

\[
\cos \phi_2 = \frac{\mu_2^3 \mu_3^1 - \mu_1^1 \mu_2^3}{\sqrt{(\mu_3^1 \mu_2^2 - \mu_1^1 \mu_2^3)^2 + (\mu_2^1 \mu_3^3 - \mu_3^1 \mu_2^3)^2}},
\]

(4.35)

$$\sin \phi_3 = \frac{\mu_3^1 \mu_2^1}{\sqrt{(\mu_3^1 \mu_2^2 - \mu_1^1 \mu_2^3)^2 + (\mu_2^1 \mu_3^3 - \mu_3^1 \mu_2^3)^2 + (\mu_1^1 \mu_3^3 - \mu_3^1 \mu_3^3)^2}}, $$

\[
\cos \phi_3 = \frac{\sqrt{(\mu_3^1 \mu_2^2 - \mu_1^1 \mu_2^3)^2 + (\mu_2^1 \mu_3^3 - \mu_3^1 \mu_2^3)^2 + (\mu_1^1 \mu_3^3 - \mu_3^1 \mu_3^3)^2}}{\mu_3^1 \mu_2^1 \mu_3^3 - \mu_3^1 \mu_3^3 - \mu_1^1 \mu_3^3 + \mu_3^1 \mu_3^3},
\]

Finally, preserving the gauge choice $L_3^2 = 0$, we get that

$$\sin \phi_1 = -\frac{\mu_2^3 \sec \phi_3}{\sqrt{(\mu_2^3 \sec \phi_3)^2 + (\mu_3^1 \sin \phi_2 + \mu_3^3 \cos \phi_2)^2}}, $$

(4.36)

$$\cos \phi_1 = \frac{\mu_3^1 \sin \phi_2 + \mu_3^3 \cos \phi_2}{\sqrt{(\mu_2^3 \sec \phi_3)^2 + (\mu_3^1 \sin \phi_2 + \mu_3^3 \cos \phi_2)^2}},$$

where we have used the relation

$$\mu_3^1 \cos \phi_2 - \mu_3^3 \sin \phi_2 = -\mu_2^3 \tan \phi_3.$$ 

(4.37)

There is an overall sign ambiguity in obtaining (4.35) and (4.36), which is fixed by taking the small $\phi_i$ limit, and matching with (4.23). Thus, the angles $\phi_1$, $\phi_2$ and $\phi_3$ are fixed, which

\(^{13}\) Also called yaw, pitch and roll. These angles are not to be confused with the Euler angles, which involve three rotations as well, but the first and third are about the same axis.
when inserted into the expressions given below determine the complete set of transformations. The remaining expressions are obtained by varying the non-vanishing elements of the vielbein leading to

\[ v' = (O^i_j \mu^j_1)^{-3}, \]
\[ \tau' = -\frac{(O^3_j - iO^2_j)\mu^j_2}{O^3_j \mu^j_3}, \]
\[ B' = \frac{(O^3_j - iO^2_j)\mu^j_1}{O^3_j \mu^j_3}, \]

(4.38)

where we have used

\[ (O^i_j \mu^j_1)(O^2_j \mu^j_2)(O^3_j \mu^j_3) = 1. \]

(4.39)

It is straightforward to write down these explicit expressions. We keep them implicit as they are quite complicated and will be calculated explicitly later.

The coordinates \( v, \tau \) and \( B \) in (4.24) that we have chosen to parametrize the \( SO(3) \setminus SL(3, \mathbb{R}) \) moduli space are a natural choice from the \( U(1) \setminus SL(2, \mathbb{R}) \) point of view. To see this, we take the corresponding \( SL(2, \mathbb{R}) \) subspace in (4.21) given by setting \( \alpha = b = c = d = \varpi = 0 \). Then, (4.24) yields that \( \delta \nu = 0 \) as expected. Also we get that

\[ \tau \rightarrow \frac{\hat{a} \tau + b}{\hat{c} \tau + d}, \quad B \rightarrow \frac{B}{\hat{c} \tau + d}, \]

(4.40)

where \( \hat{a} = 1 + \beta, \hat{b} = -h, \hat{c} = -f \) and \( \hat{d} = 1 - \beta \) in the infinitesimal limit, as required.

Another useful parametrization of the \( SO(3) \setminus SL(3, \mathbb{R}) \) moduli space is by the coordinates \( T, \xi \) and \( e^{-2\delta} \), where

\[ \xi = -B_R + i\tau_1 V, \]

(4.41)

which is natural from the \( U(1) \setminus SL(2, \mathbb{R}) \) point of view which follows from the discussion below. Thus, the coordinates \( \phi, T \) and \( \xi \) are related to \( v, B \) and \( \tau \) by (4.13) and

\[ T = \frac{1}{\tau_2} (\text{Im} B + i\tau_2 V), \]
\[ \xi = \frac{1}{\tau_2} (\text{Im}(B\overline{T}) + i\tau_1 \overline{\tau_2} V). \]

(4.42)

This leads to

\[ L^i_m = \begin{pmatrix} e^{-\phi/3} \sqrt{T_2} & -e^{-\phi/3} T_1 / \sqrt{T_2} & e^{2\phi/3} \text{Im}(\xi \overline{T}) / T_2 \\ 0 & e^{-\phi/3} / \sqrt{T_2} & -e^{2\phi/3} \text{Im} \xi / T_2 \\ 0 & 0 & e^{2\phi/3} \end{pmatrix}, \]

(4.43)

and thus

\[ M_{mn} = \frac{e^{4\phi/3}}{T_2} \begin{pmatrix} e^{-2\delta} |T|^2 + (\text{Im}(\xi \overline{T}))^2 / T_2 & -(e^{-2\delta} T_1 + \xi_2 \text{Im}(\xi \overline{T}) / T_2) & \text{Im}(\xi \overline{T}) \\ -(e^{-2\delta} T_1 + \xi_2 \text{Im}(\xi \overline{T}) / T_2) & (\xi_2^2 + T_2 e^{-2\phi}) / T_2 & -\xi_2 \\ \text{Im}(\xi \overline{T}) & -\xi_2 & T_2 \end{pmatrix}. \]

(4.44)

Proceeding as before, for infinitesimal transformations, we get that

\[ \theta_1 = -\frac{e^{-\phi} (c T_1 - f)}{\sqrt{T_2}}, \]
\[ \theta_2 = -\frac{e^{-\phi} T_2}{\sqrt{T_2}}, \]
\[ \theta_3 = -\frac{e^{-\phi} \xi_2}{\sqrt{T_2}}, \]

(4.45)
as well as
\[ \delta \hat{\phi} = \frac{3}{2} \left( - (\alpha + \beta) + c \frac{\text{Im} \xi T_{\tilde{T}}}{T_2} - f \frac{\text{Im} \xi}{T_2} \right), \]
\[ \delta T = (\alpha - \beta)T - d - f \xi + b T^2 + c T \xi, \] \quad (4.46)
\[ \delta \xi = -g - h T + (\beta + 2 \alpha) \xi + c \xi + b T^2 + i c T e^{-2 \hat{\phi}} - i f e^{-2 \hat{\phi}}. \]

The finite transformations are given by
\[ e^{2 \hat{\phi}} = (O^i_j \mu^j_1)^3, \]
\[ T' = -\frac{(O^2_j - i O^1_j) \mu^j_1}{O^2_j \mu^j_2}, \] \quad (4.47)
\[ \xi' = (O^1_i \mu^i_1) [O^3_j \mu^j_1 (O^2_k \mu^k_1) - O^3_j \mu^j_1 (O^2_k \mu^k_2) - i (O^1_j \mu^j_1) (O^3_k \mu^k_2)] \].

Consider the SL(2, \mathbb{R})_T subspace in (4.21) given by setting \( \alpha = -\beta, c = f = g = h = 0 \).

Then, (4.46) yields that \( \delta \hat{\phi} = 0 \) as expected. Also we get that
\[ T \rightarrow \frac{\hat{\xi} T + \hat{\phi}}{\hat{c} T + \hat{d}}, \quad \xi \rightarrow \frac{\xi}{\hat{c} T + \hat{d}}, \] \quad (4.48)
where \( \hat{\alpha} = 1 + \alpha, \hat{\beta} = -d, \hat{c} = -b \) and \( \hat{d} = 1 - \alpha \) in the infinitesimal limit. The fermions also transform accordingly by gauge transformations given by the \( \theta_t \).

A similar analysis for \( N = 8, d = 4 \) supergravity has been carried out in [33].

### 4.3. Gauge-fixed supersymmetry transformations

Now let us consider the gauge-fixed supersymmetry transformations of the moduli. In order to maintain the choices of gauge (4.6) and (4.9), we have to make additional gauge transformations with field-dependent parameters. First consider the moduli for the \( SO(2) \backslash SL(2, \mathbb{R}) \) coset [34].

From (C.1), preserving the reality of \( L_{\pm}^{-1} \) gives us that
\[ \Sigma_e = -\frac{i}{2} (\xi_{\tilde{L}} \lambda_{\tilde{R}} - \bar{\xi}_{\tilde{L}} \lambda_{\tilde{R}}). \] \quad (4.49)

Also including the effect of this compensating gauge transformation on \( L_{\pm}^{-1} \), we get that
\[ \delta^{(0)} U = -2i U_2 \bar{\xi}_{\tilde{L}} \lambda_{\tilde{R}} - U_2 \bar{\xi}_{\tilde{L}} \lambda_{\tilde{R}}. \] \quad (4.50)

We next consider the moduli for the \( SO(3) \backslash SL(3, \mathbb{R}) \) coset, where including compensating gauge transformations, we get that
\[ \delta^{(0)} L_{m}^{-1} = \Lambda_{ij} L_{m}^{-1} - \epsilon^{ijk} \delta_{m}^{ij} L_{m}^{-1}, \] \quad (4.51)
where
\[ \Lambda_{ij} = \frac{1}{2} (\xi_{\tilde{L}} (\sigma_i \chi_{jR} + \sigma_j \chi_{iR}) - \bar{\xi}_{\tilde{R}} (\sigma_i \chi_{jL} + \sigma_j \chi_{iL})). \] \quad (4.52)

Thus, \( \Lambda_{ij} \) is real and satisfies \( \Lambda_{ij} = \Lambda_{ji} \), as well as
\[ \Lambda_{11} + \Lambda_{22} + \Lambda_{33} = 0. \] \quad (4.53)

So \( \Lambda_{ij} \) has five independent components which we can take to be \( \Lambda_{11}, \Lambda_{12}, \Lambda_{13}, \Lambda_{22} \) and \( \Lambda_{23} \). For our purposes, it is very convenient to explicitly use
\[ \Lambda_{ij} = \frac{1}{2} \delta_{ij} + \Lambda_{kk} \] \quad (4.54)
in place of \( \Lambda_{ij} \) in (4.51). While it gives no extra information, the tracelessness is automatic, and we do not have to choose an explicit basis for the five independent components.
Proceeding as before, preserving the gauge choice $L_2^1 = L_3^1 = L_3^2 = 0$ in (4.9) or (4.43), we get that
\[
\theta_1^r = -\Lambda_{23},
\theta_2^r = \Lambda_{13},
\theta_3^r = -\Lambda_{12},
\]
which together with the remaining supervariations in (4.9) gives us
\[
\delta^{(0)} v = -2 \left( \Lambda_{11} - \frac{1}{2} (\Lambda_{22} + \Lambda_{33}) \right) v,
\]
\[
\delta^{(0)} \tau = i r_2 (\Lambda_{22} - \Lambda_{33} + 2i \Lambda_{23}),
\]
\[
\delta^{(0)} B = 2 \left( \Lambda_{13} \sqrt{\frac{r_2}{v}} - \Lambda_{23} \text{Im} B \right) + i \left( (\Lambda_{22} - \Lambda_{33}) \text{Im} B - 2 \Lambda_{12} \sqrt{\frac{r_2}{v}} \right).
\]
In the coordinate system given by (4.43), we get that
\[
\hat{\delta} \phi = \left( \frac{1}{\Lambda_{33}} - \frac{1}{2} (\Lambda_{11} + \Lambda_{22}) \right),
\]
\[
\delta^{(0)} T = T_2 (-2 \Lambda_{12} + i (\Lambda_{11} - \Lambda_{22})),
\]
\[
\delta^{(0)} \xi = -2 \left( \sqrt{\frac{T_2}{\Lambda_{13}}} e^{-\phi} + \Lambda_{12} \text{Im} \xi \right) + i \left( (\Lambda_{11} - \Lambda_{22}) \text{Im} \xi - 2 \Lambda_{23} \sqrt{T_2} e^{-\phi} \right).
\]
This also leads to extra terms in the supersymmetry transformations for the other fields, of which the fermions are relevant for our purposes. For the fermions, the extra terms in the supervariation which have to be added to (C.1) are
\[
\hat{\delta} \lambda_L = \frac{3i}{2} \Sigma \epsilon \lambda_L + \frac{i}{2} \theta_j^r \sigma^j \lambda_L,
\]
\[
\hat{\delta} \lambda_R = - \frac{3i}{2} \Sigma \epsilon \lambda_R + \frac{i}{2} \theta_j^r \sigma^j \lambda_R,
\]
\[
\hat{\delta} \psi_{\mu L} = \frac{i}{2} \Sigma \epsilon \psi_{\mu L} + \frac{1}{2} \theta_j^r \sigma^j \psi_{\mu L},
\]
\[
\hat{\delta} \psi_{\mu R} = - \frac{i}{2} \Sigma \epsilon \psi_{\mu R} + \frac{i}{2} \theta_j^r \sigma^j \psi_{\mu R},
\]
\[
\hat{\delta} \chi_L = - \frac{i}{2} \Sigma \epsilon \chi_L - i \theta_j^r \sigma^j \chi_L + \frac{3i}{2} \theta_j^r \sigma^j \chi_L,
\]
\[
\hat{\delta} \chi_R = \frac{i}{2} \Sigma \epsilon \chi_R - i \theta_j^r \sigma^j \chi_R + \frac{3i}{2} \theta_j^r \sigma^j \chi_R,
\]
where $\Sigma$ and $\theta_j^r$ are given by (4.49) and (4.55), respectively. Thus, from now onward, the complete supersymmetry transformations of the supergravity theory will be denoted as $\delta^{(0)}$.

In particular, $\delta^{(0)}$ for the various fermions is given by the sum of (C.1) and (4.58).

5. Transformations of the moduli under $U$-duality

In the above discussion, we have gauge fixed the $H$ symmetry transformations to obtain the physical degrees of freedom. This led to the transformations of the various fields where the $G$ symmetry is realized nonlinearly.

We now focus on these transformations of the moduli in detail. While the transformations of the moduli parametrizing the coset space $SO(2)/SL(2, \mathbb{R})$ were easy to write down explicitly as given above, this was not the case for the moduli parametrizing the coset space...
$SO(3)/SL(3, \mathbb{R})$ due to the gauge redundancy. So for this purpose it is easier to use the matrix $M$, which is gauge invariant. In fact, under $L \rightarrow OLS$, we have that
\[ M_{mn} = L_m L_n^i \delta_{ij} \rightarrow (S^T MS)_{mn}. \] (5.1)
Thus, we obtain the transformations of the moduli using (5.1), and we also change the notation slightly from before for later convenience.

5.1. Transformation under $SL(2, \mathbb{Z})$

For the $SL(2, \mathbb{R})$ transformations, taking
\[ S = \begin{pmatrix} A & -C \\ -B & D \end{pmatrix}, \] (5.2)
where $A, B, C$ and $D$ are real numbers satisfying $AD - BC = 1$, (5.1) gives us that
\[ U'_1 = \frac{1}{2} \left( \frac{AU + B}{CU + D} + \frac{A\bar{U} + B}{C\bar{U} + D} \right), \quad U'_2 = \frac{U_2}{|CU + D|^2}, \] (5.3)
leading to
\[ U'' = \frac{AU + B}{CU + D} \] (5.4)
as before.

5.2. Transformation under $SL(3, \mathbb{Z})$

For the $SL(3, \mathbb{R})$ transformations, we take
\[ S = \begin{pmatrix} A & -C & J \\ -B & D & -F \\ H & -E & G \end{pmatrix}, \] (5.5)
where $A, B, C, D, E, F, G, H$ and $J$ are real numbers satisfying $\det S = 1$.
Again, using (5.1), we get that
\[ v' = \frac{1}{v^2 \tau_2} \left[ \frac{v^2}{2} (C\bar{Z}_3 - J\bar{Z}_2)^2 + \frac{v}{2} \text{Im}(\bar{Z}_2 \bar{Z}_3) \right]^{3/2} \]
\[ = \frac{1}{v^2 \tau_2} \left[ \frac{v^2}{2} (\bar{\nu}_1 \tau + \bar{\nu}_3 \bar{\nu} - \nu^2 (\bar{\nu}_1 \text{Im}(\bar{\nu}) - \bar{\nu}_2 \text{Im} \bar{\nu} - \bar{\nu}_3 \bar{\nu})^2) \right]^{3/2}, \]
\[ \tau'_1 = \frac{CJ \tau + v \text{Re} (\bar{Z}_2 \bar{Z}_3)}{\bar{J}^2 \tau_2 + v |Z_3|^2}, \]
\[ \tau'_2 = \frac{[v^2 (C\bar{Z}_3 - J\bar{Z}_2)^2 + v^2 \text{Im}(\bar{Z}_2 \bar{Z}_3)]^{1/2}}{\bar{J}^2 \tau_2 + v |Z_3|^2} \]
\[ = \frac{[v^2 (\bar{\nu}_1 \tau + \bar{\nu}_3 \bar{\nu})^2 + v^2 (\bar{\nu}_1 \text{Im}(\bar{\nu}) - \bar{\nu}_2 \text{Im} \bar{\nu} - \bar{\nu}_3 \bar{\nu})^2]^{1/2}}{\bar{J}^2 \tau_2 + v |Z_3|^2}, \]
\[ \text{Re} B' = \frac{AJ \tau + v \text{Re} (\bar{Z}_2 \bar{Z}_3)}{(\bar{J}^2 \tau_2 + v |Z_3|^2)^2}, \]
\[ \text{Im} B' = \frac{\tau_2 v \text{Re} (J \bar{Z}_2 + C \bar{Z}_3) + v^2 \text{Im}(\bar{Z}_2 \bar{Z}_3) \text{Im}(\bar{Z}_2 \bar{Z}_3)}{(\bar{J}^2 \tau_2 + v |Z_3|^2)^3 [v^2 (C\bar{Z}_3 - J\bar{Z}_2)^2 + v^2 \text{Im}(\bar{Z}_2 \bar{Z}_3)]^{1/2} \bar{J}^2 \tau_2 + v |Z_3|^2} \]
\[ = \frac{\tau_2 \text{Im}(\bar{Z}_2 \bar{Z}_3)}{(\bar{J}^2 \tau_2 + v |Z_3|^2)^2 [v^2 (\bar{\nu}_1 \tau + \bar{\nu}_3 \bar{\nu})^2 + v^2 (\bar{\nu}_1 \text{Im}(\bar{\nu}) - \bar{\nu}_2 \text{Im} \bar{\nu} - \bar{\nu}_3 \bar{\nu})^2]^{1/2}}. \] (5.6)
where
\[ Y_1 = CF - DJ, \]
\[ Y_2 = CG - EF, \]
\[ Y_3 = DG - EF, \]
\[ Y_4 = AF - BJ, \]
\[ Y_5 = AG - JH, \]
\[ Y_6 = BG - HF, \]
\[ \Omega_1 = (J \Xi_1 - A \Xi_2)(J \Xi_3 - C \Xi_3) + (J \Xi_1 - A \Xi_2)(J \Xi_2 - C \Xi_2) \]
\[ = (Y_1 \tau + Y_2)(Y_4 \tau + Y_5) + (Y_1 \tau + Y_2)(Y_4 \tau + Y_5), \]
\[ \Omega_2 = (\Xi_1 \Xi_3 - \Xi_2 \Xi_3)(\Xi_2 \Xi_3 - \Xi_3 \Xi_2) \]
\[ = -4(Y_1 \text{Im}(B \tau) + Y_2 \text{Im}B + Y_5 \text{Im} \tau)(Y_4 \text{Im}(B \tau) + Y_5 \text{Im}B + Y_6 \text{Im} \tau) \]
for brevity.

Thus, we have that
\[ \tau' = \left[ \frac{C^2 \tau_2 + v|E_3|^2}{J^2 \tau_2 + v|E_3|^2} \right]^{1/2} e^{i \phi}, \]
\[ B' = \frac{e^{i \phi}}{[J^2 \tau_2 + v|E_3|^2][v \tau_2|C \Xi_3 - J \Xi_1|^2 + v \text{Im}(E_2 \Xi_3)]^{1/2}} \times \left[ [v \tau_2|C \Xi_3 - J \Xi_1|^2 + v \text{Im}(E_2 \Xi_3)](A \tau_2 + v \text{Re}(E_1 \Xi_3))^2 \right. \]
\[ + (v \tau_2 \text{Re}(J \Xi_1 - A \Xi_3)(J \Xi_2 - C \Xi_3) + v^2 \text{Im}(E_1 \Xi_3) \text{Im}(E_2 \Xi_3))^2 \]^{1/2},
\[ \tan \theta_\tau = \frac{v \tau_2 \text{Re}(J \Xi_1 - A \Xi_3)(J \Xi_2 - C \Xi_3) + v^2 \text{Im}(E_1 \Xi_3) \text{Im}(E_2 \Xi_3)}{C \tau_2 + v \text{Re}(E_2 \Xi_3)} \]
\[ \tan \theta_B = \frac{v \tau_2 \text{Re}(J \Xi_1 - A \Xi_3)(J \Xi_2 - C \Xi_3) + v^2 \text{Im}(E_1 \Xi_3) \text{Im}(E_2 \Xi_3)}{[A \tau_2 + v \text{Re}(E_1 \Xi_3)][v \tau_2|C \Xi_3 - J \Xi_1|^2 + v \text{Im}(E_2 \Xi_3)]^{1/2}}. \]

All this can be written compactly as
\[ v' = \frac{1}{v \tau_2} \left( \xi_{23}^2 - \xi_{33}^2 \right)^{3/2}, \]
\[ \tau' = \frac{\xi_{23} + i \sqrt{\xi_{22} \xi_{33} - \xi_{23}^2}}{\xi_{33}}, \]
\[ B' = \frac{\xi_{13} + i (\xi_{12} \xi_{33} - \xi_{13} \xi_{23}) / \sqrt{\xi_{22} \xi_{33} - \xi_{23}^2}}{\xi_{33}}, \]
where
\[ \xi_{12} = \mathcal{A} \tau_2 + v \text{Re}(\Xi_1 \Xi_2), \]
\[ \xi_{13} = \mathcal{A} \eta_2 + v \text{Re}(\Xi_1 \Xi_3), \]
\[ \xi_{23} = C \eta_2 + v \text{Re}(\Xi_2 \Xi_3), \]
\[ \xi_{22} = C^2 \tau_2 + v|\Xi_2|^2, \]
\[ \xi_{33} = \mathcal{J}^2 \tau_2 + v|\Xi_3|^2. \]

(5.13)

One can also write down the expressions for the transformed moduli in terms of the coordinates \( e^{2\phi}, T \) and \( \xi \) on moduli space. This gives
\[ e^{2\phi} = e^{\phi} \eta_{33}^{3/2}, \]
\[ T' = \frac{(\eta_{12} \eta_{33} - \eta_{13} \eta_{23}) + \sqrt{\eta_{33}} e^{-2\phi}}{\eta_{22} \eta_{33} - \eta_{23}^2}, \]
\[ \xi' = \frac{(\eta_{12} \eta_{23} - \eta_{13} \eta_{22}) + i\eta_{23} e^{-2\phi}/\sqrt{\eta_{33}}}{\eta_{22} \eta_{33} - \eta_{23}^2}. \]

(5.14)

where
\[ \eta_{12} = \frac{e^{-2\phi}}{T_2} \text{Re}(\mathcal{A} T + \mathcal{B} (\mathcal{C} \mathcal{T} + \mathcal{D})) + \left( A \text{Im}(\xi \mathcal{T}) + B \frac{\xi_2}{T_2} + \mathcal{H} \right) \left( e^{\text{Im}(\xi \mathcal{T})} + \mathcal{D} \frac{\xi_2}{T_2} + \mathcal{E} \right), \]
\[ \eta_{13} = \frac{e^{-2\phi}}{T_2} \text{Re}(\mathcal{A} T + \mathcal{B} (\mathcal{C} \mathcal{T} + \mathcal{D})) + \left( A \text{Im}(\xi \mathcal{T}) + B \frac{\xi_2}{T_2} + \mathcal{H} \right) \times \left( e^{\text{Im}(\xi \mathcal{T})} + \mathcal{D} \frac{\xi_2}{T_2} + \mathcal{E} \right), \]
\[ \eta_{23} = \frac{e^{-2\phi}}{T_2} \text{Re}(\mathcal{C} T + \mathcal{D}) + \left( e^{\text{Im}(\xi \mathcal{T})} + \mathcal{D} \frac{\xi_2}{T_2} + \mathcal{E} \right) \left( \mathcal{J} \frac{\text{Im}(\xi \mathcal{T})}{T_2} + \mathcal{F} \frac{\xi_2}{T_2} + \mathcal{G} \right), \]
\[ \eta_{22} = \frac{e^{-2\phi}}{T_2} |\mathcal{C} T + \mathcal{D}|^2 + \left( e^{\text{Im}(\xi \mathcal{T})} + \mathcal{D} \frac{\xi_2}{T_2} + \mathcal{E} \right)^2 \]
\[ \eta_{33} = \frac{e^{-2\phi}}{T_2} |\mathcal{J} T + \mathcal{F}|^2 + \left( \mathcal{J} \frac{\text{Im}(\xi \mathcal{T})}{T_2} + \mathcal{F} \frac{\xi_2}{T_2} + \mathcal{G} \right)^2. \]

(5.15)

It is easy to check that the transformations (5.12) and (5.14) reproduce (4.24) and (4.46) in the infinitesimal limit.

This continuous symmetry of supergravity is broken in string theory by non-perturbative effects. The full theory has a discrete \( U \)-duality symmetry \( SL(2, \mathbb{Z}) \times SL(3, \mathbb{Z}) \). Under this symmetry, the various fields continue to transform as above, the only difference being that (5.2) and (5.5) have integer entries.

6. A class of interactions in the higher derivative action from on-shell linearized superspace

In order to construct higher derivative corrections to supergravity in the low energy effective action, consider \( N = 2, d = 8 \) superspace with superderivatives given by
\[ D_L^A = \frac{\partial}{\partial \theta_{LA}} + i \tilde{W}_L A^\mu \partial_\mu, \quad D_R^A = -\frac{\partial}{\partial \theta_{RA}} + i \tilde{W}_R A^\mu \partial_\mu. \]

(6.1)
The superspace fermionic coordinates \( \theta_L \) and \( \theta_R \) carry \( U(1) \) charges 1/2 and -1/2, respectively, and are in the 2 of \( SU(2) \). The degrees of freedom of the supergravity multiplet are contained in linearized superfields of this superspace. In particular, they are contained in a chiral superfield \( W \) and a linear superfield \( L_{ABCD} \) [35, 36].

### 6.1. Chiral and linear superfields

The chiral superfield \( W \) satisfies

\[
D_{RA} W = 0, \quad D_{LA} \bar{W} = 0.
\]

(6.2)

It carries charge 2 under \( U(1) \) and is uncharged under \( SU(2) \). The linear superfield \( L_{ABCD} \) which satisfies the reality condition \( L_{ABCD} = (L_{ABCD})^* \) is totally symmetric in its \( SU(2) \) indices, and satisfies

\[
D_{LA}(L_{BCDE}) = 0, \quad D_{RA}(L_{BCDE}) = 0.
\]

(6.3)

Thus, \( L_{ABCD} \) is in the 5 of \( SU(2) \) and is uncharged under \( U(1) \).

Since we are dealing with a theory with maximal supersymmetry, it is expected that all the fields will be part of a single supermultiplet. This is indeed the case, because the chiral and linear superfields are not independent, and satisfy the on-shell relations

\[
D_{LAB} \gamma^{\mu
\nu} \delta_{\Gamma} \bar{W} = D_{RA} \gamma^{\mu
\nu} \delta_{\Gamma} \bar{W} = D_{LA} \gamma^{\mu
\nu} \delta_{\Gamma} \bar{W} = D_{RA} \gamma^{\mu
\nu} \delta_{\Gamma} \bar{W} = D_{LA} \gamma^{\mu
\nu} \delta_{\Gamma} \bar{W} = D_{RA} \gamma^{\mu
\nu} \delta_{\Gamma} \bar{W}.
\]

(6.4)

We now write down some of the components of the chiral and linear superfields at low orders in \( \theta_L \), \( \theta_R \) (ignoring various numerical factors), which will be relevant for our purposes. The lowest component of the chiral superfield \( W \) is given by

\[
\epsilon_{UV} \lambda^U \delta \lambda^V = \frac{\delta U}{2U_2}.
\]

(6.5)

which in the gauge (4.6), yields

\[
\epsilon_{UV} \lambda^U \delta \lambda^V = \frac{\delta U}{2U_2}.
\]

(6.6)

Thus, we have that

\[
W = \frac{\delta U}{U_2} + \bar{\theta}_R \lambda_L + [(\bar{\theta}_R \sigma^I \lambda_{\mu \nu} \sigma_{\lambda}) \bar{F}^{I}_{\mu \nu} + (\bar{\theta}_R \gamma^{\mu \nu} \lambda_{\sigma}) \bar{\gamma}^{\lambda \rho}] \\
\quad + (\bar{\theta}_R \sigma^I \lambda_{\mu \nu} \lambda_{\lambda}) (\bar{\theta}_R \sigma^J \lambda_{\nu \lambda}) + (\bar{\theta}_R \gamma^{\mu \nu} \lambda_{\sigma}) (\bar{\theta}_R \gamma^{\lambda \rho} \lambda_{\sigma}) + \cdots.
\]

(6.7)

In order to avoid proliferation of indices while writing down the degrees of freedom in the linear superfield \( L_{ABCD} \), it is convenient for our purposes to define the real superfield \( L_{ij} \) by

\[
L_{ij} = [(\sigma_{I})^{AB}(\sigma_{J})^{CD} - \frac{1}{2} \delta_{ij}(\sigma_{K})^{AB}(\sigma_{K})^{CD}] L_{ABCD} + \text{h.c.}
\]

(6.8)

Thus, \( L_{ij} \) is in the 5 of \( SU(2) \) and is uncharged under \( U(1) \). The lowest component of \( L_{ij} \) is given by

\[
l_{ij} = \frac{1}{2} [L_{m} \delta_{L_{m}} + L_{m} \delta_{L_{m}} - \frac{2}{5} \delta_{ij} L_{m} \delta_{L_{m}}],
\]

(6.9)

leading to

\[
l_{ij} = L_{ij} + [\bar{\theta}_L \sigma_{(i} \lambda_{j)} \bar{\theta}_R \sigma_{(i} \lambda_{j)} L_{m} \delta_{L_{m}} + i \epsilon_{(i} \epsilon_{j)} \bar{\theta}_L \sigma_{(i} \lambda_{j)} \bar{\theta}_R \sigma_{(i} \lambda_{j)} L_{m} \delta_{L_{m}}]
\]

(6.10)

To write down explicit expressions for \( l_{ij} \) in a fixed gauge, we choose coordinates \( \gamma, \xi \) and \( e^{-2\phi} \) on the \( SO(3)/SL(3, \mathbb{R}) \) moduli space. This is because the couplings when expanded at
weak string coupling have perturbative contributions which are functions of only $T$ and $\bar{T}$. Thus, $\xi$ appears only in the non-perturbative part of the various couplings. Working in the gauge (4.43), and using

$$L_0^m = \begin{pmatrix} e^{\phi/3}/\sqrt{T_2} & e^{\phi/3}T_1/\sqrt{T_2} & e^{\phi/3}\xi_1/\sqrt{T_2} \\ 0 & e^{\phi/3}/\sqrt{T_2} & e^{\phi/3}\xi_2/\sqrt{T_2} \\ 0 & 0 & e^{-2\phi/3} \end{pmatrix},$$

(6.11)

we see that the five independent components of $l_{ij}$ are given by

$$l_{11} = \frac{\delta T_2}{2T_2} - \frac{\delta \phi}{3},$$

$$l_{22} = -\frac{\delta T_2}{2T_2} - \frac{\delta \phi}{3},$$

$$l_{12} = -\frac{\delta T_1}{2T_2},$$

$$l_{13} = \frac{e^{-\phi/3}}{2T_2^2} (\xi_2 \delta T_1 - T_2 \delta \xi_1),$$

$$l_{23} = -\frac{e^{-\phi/3}}{2T_2} \delta \left( \frac{\xi_2}{T_2} \right).$$

(6.12)

In order to construct a class of terms in the higher derivative effective action, we need to construct superspace actions using the chiral and linear superfields.

6.2. Superactions

First let us construct terms in the effective action involving the chiral superfield $W$. It is given by

$$\int d^8x e\int d^{16}\bar{\theta}_R f(W) + \text{h.c.},$$

(6.13)

where

$$d^{16}\bar{\theta}_R = \epsilon^{a_1\cdots a_8} e^{b_1\cdots b_8} (d\bar{\theta}_{R(a_1} \cdots d\bar{\theta}_{Rb_8)} (d\bar{\theta}_{Rb_1)} \cdots d\bar{\theta}_{Rb_8}).$$

(6.14)

Using the Schouten identity

$$\epsilon^{a_1\cdots a_8} e^{b_1\cdots b_8} = 0,$$

(6.15)

it follows that (6.14) is proportional to $(D_{+L})^8 (D_{-L})^8$, where $(D_{+L})^8$ and $(D_{-L})^8$ are both spacetime scalars. Thus, (6.13) is invariant under supersymmetry transformations using $\delta W \sim D_{+L}W$.

In order to construct terms in the effective action using the linear superfield $L_{ABCD}$, let us consider the combination

$$\tilde{L} = L_{++++} + 4\xi L_{+++} + 6\xi^2 L_{+++} + 4\xi^3 L_{+++} + \xi^4 L_{+\ldots},$$

(6.17)

14 All such discussions are true up to a total derivative. The $\pm SU(2)$ indices are defined by

$$V_{\pm} = V_1 \pm iV_2.$$
where \( \zeta \) is a complex parameter. Now from (6.3), we get that
\[
\begin{align*}
D_{++}L_{++++} &= 0, \\
D_{--}L_{++++} + 4D_{++}L_{++--} &= 0, \\
2D_{++}L_{++++} + 3D_{++}L_{++--} &= 0, \\
2D_{--}L_{++--} + 3D_{--}L_{++--} &= 0, \\
D_{++}L_{++--} + 4D_{--}L_{++--} &= 0, \\
D_{--}L_{++--} &= 0,
\end{align*}
\]
and similarly for \( D_R \), leading to
\[
\begin{align*}
D_{+}Lg(\tilde{L}) &= -\zeta D_{-}Lg(\tilde{L}), \\
D_{+}Rg(\tilde{L}) &= -\zeta D_{-}Rg(\tilde{L}).
\end{align*}
\]
This ability to interchange \( D_{\pm} \) when acting on \( g(\tilde{L}) \) is useful to write down a superspace action involving the linear superfield.

Such a superspace action is given by [36]
\[
\int d^8x e \int d^8\theta_R d^8\bar{\theta} \left[ \oint_0 d\zeta g(\tilde{L}, \zeta) + \text{h.c.} \right],
\]
where
\[
d^8\theta_R d^8\bar{\theta} \left[ \oint_0 d\zeta g(\tilde{L}, \zeta) \right] = \epsilon^{a_1 \cdots a_8} \epsilon^{b_1 \cdots b_8} (d\bar{\theta}_{\alpha_1} \cdots d\bar{\theta}_{\alpha_8}) (d\theta_{\beta_1 A_1} \cdots d\theta_{\beta_8 A_8}),
\]
and the contour integral in the complex \( \zeta \) plane is around the origin. In order to show that (6.20) is invariant under supersymmetry, we note that (6.21) gives 8 powers of \( D_L \) and 8 powers of \( D_R \), while a supervariation of \( \tilde{L} \) yields one more factor of \( D_L \) and \( D_R \). However, using (6.19), it follows that \( D_{++} \) and \( D_{--} \) (and also \( D_{+} \) and \( D_{-} \)) can be interchanged, and so finally we end up with 9 powers of the same superderivative which vanishes, and thus the action is invariant. Thus, using the linear superfield, we get the action
\[
\int d^8x e \int d^8\theta_R d^8\bar{\theta} \left[ g(L_{ABCD}) + \text{h.c.} \right],
\]
where
\[
g(L_{ABCD}) = \oint_0 d\zeta g(\tilde{L}, \zeta).
\]
We finally redefine
\[
g(L_{ij}) \equiv g(L_{ABCD}) + \text{h.c.},
\]
giving us the superspace action
\[
\int d^8x e \int d^8\theta_R d^8\bar{\theta} Lg(L_{ij})
\]
involving the linear superfield.

We now consider higher derivative corrections to supergravity coming from the superspace Lagrangian
\[
e^{-1} \mathcal{L} = \left[ \int d^8\theta_R f(W) + \text{h.c.} \right] + \int d^8\theta_R d^8\bar{\theta} Lg(L_{ij}).
\]
7. Higher derivative corrections and supersymmetry constraints

We next consider the role of supersymmetry in constraining the various couplings which arise as the coefficients involving the moduli of the various interactions in the effective action. We first consider a set of couplings which involve only the $\text{SO}(2) \backslash \text{SL}(2, \mathbb{R})$ moduli, and then the ones which involve only the $\text{SO}(3) \backslash \text{SL}(3, \mathbb{R})$ moduli. Finally, we consider a coupling which involves all the moduli.

7.1. Couplings involving only the $\text{SO}(2) \backslash \text{SL}(2, \mathbb{R})$ moduli

The set of couplings that we will consider are the ones obtained from linearized superspace. Thus, let us consider interactions in (6.26), involving the chiral superfield. We will see that these couplings are automorphic forms of $\text{SL}(2, \mathbb{Z})$. In order to construct these interactions, we make use of the definitions

$$\lambda_{L}^{16} = \frac{1}{8!9!} \epsilon^{a_{1}...a_{8}} \epsilon^{b_{1}...b_{9}} \epsilon_{A_{1}B_{1}}...\epsilon_{A_{8}B_{8}} (\lambda_{L_{1}}^{A_{1}}...\lambda_{L_{8}}^{A_{8}})(\lambda_{L_{1}}^{B_{1}}...\lambda_{L_{8}}^{B_{8}})$$

$$= (\lambda_{L_{1}}^{1} \lambda_{L_{2}}^{2})... (\lambda_{L_{8}}^{1} \lambda_{L_{8}}^{2}),$$

$$\lambda_{L}^{15} \epsilon_{B_{1}} = \frac{2!}{7!9!} \epsilon^{a_{1}...a_{7}} \epsilon^{b_{1}...b_{9}} \epsilon_{A_{1}B_{1}}...\epsilon_{A_{7}B_{9}} (\lambda_{L_{1}}^{A_{1}}...\lambda_{L_{7}}^{A_{7}})(\lambda_{L_{1}}^{B_{1}}...\lambda_{L_{9}}^{B_{9}}),$$

$$\lambda_{L}^{14} \epsilon_{B_{1}} = \frac{3!}{6!9!} \epsilon^{a_{1}...a_{6}} \epsilon^{b_{1}...b_{9}} \epsilon_{A_{1}B_{1}}...\epsilon_{A_{6}B_{9}} (\lambda_{L_{1}}^{A_{1}}...\lambda_{L_{6}}^{A_{6}})(\lambda_{L_{1}}^{B_{1}}...\lambda_{L_{9}}^{B_{9}}),$$

(7.1)

such that

$$\lambda_{L}^{15} \epsilon^{A_{1}B_{1}} = \delta^{A_{1}}_{B_{1}} \delta^{A_{2}}_{B_{2}} \delta^{A_{3}}_{B_{3}},$$

$$\lambda_{L}^{14} \delta^{A_{1}B_{1}} = (\lambda_{L}^{15})^{A_{1}}_{B_{1}} \delta^{A_{2}}_{B_{2}} \delta^{A_{3}}_{B_{3}} - (\lambda_{L}^{15})^{A_{2}}_{B_{2}} \delta^{A_{1}}_{B_{1}} \delta^{A_{3}}_{B_{3}} + (\lambda_{L}^{15})^{A_{3}}_{B_{3}} \delta^{A_{1}}_{B_{1}} \delta^{A_{2}}_{B_{2}},$$

(7.2)

leading to

$$\lambda_{L}^{15} \epsilon^{A_{1}B_{1}} = 16 \lambda_{L}^{16},$$

$$\lambda_{L}^{14} \delta^{A_{1}B_{1}} = 21 \lambda_{L}^{15},$$

$$\lambda_{L}^{14} \epsilon^{A_{1}B_{1}} \epsilon^{A_{2}B_{2}} \epsilon^{A_{3}B_{3}} = \lambda_{L}^{16} (\delta^{A_{1}}_{B_{1}} \delta^{A_{2}}_{B_{2}} - \delta^{A_{2}}_{B_{2}} \delta^{A_{1}}_{B_{1}})(\delta^{C_{3}}_{D_{3}} B^{D} + \delta^{C_{3}}_{B_{3}} D^{D}).$$

(7.3)

Note that the interactions in (7.1) are particular examples of the general form

$$\lambda_{L}^{8+r} \epsilon^{a_{1}...a_{r}} \epsilon^{b_{1}...b_{r}} \epsilon_{A_{1}B_{1}}...\epsilon_{A_{r}B_{r}} (\lambda_{L_{1}}^{A_{1}}...\lambda_{L_{r}}^{A_{r}})(\lambda_{L_{1}}^{B_{1}}...\lambda_{L_{r}}^{B_{r}})$$

(7.4)

for $0 \leq r \leq 8$, and are the only ones we need.

We now consider interactions involving 16 fermions in $S^{(3)}$. In particular, we consider interactions of the form $\lambda_{L}^{16}$ and $\psi_{L_{a}} \bar{\psi}^{\mu} \lambda_{L}^{15}$. These interactions mix with no other interactions in $S^{(3)}$ under the supersymmetry transformations $\delta^{(0)}$ of the type discussed below. To consider these terms in the effective action, we take a subset of terms in (6.13) given by

$$S^{(3)} = \int d^{8}x \left[ f^{(12, -12)} (U, \bar{U}) \lambda_{L}^{16} + f^{(11, -11)} (U, \bar{U}) \bar{F} \lambda_{L}^{14} \right] + \cdots,$$

(7.5)
This leads to
\[
\mathcal{L}^{(3)} = e \left[ f^{(12,-12)}(U, \bar{U}) \lambda^L_{16} + f^{(11,-11)}(U, \bar{U}) \bar{\psi}_{\mu L} \bar{\gamma}^\mu \lambda^L_{15} + \ldots \right],
\] (7.9)
where we have used \( \hat{F}_2 \sim \bar{\psi}_L \lambda_L \) from (C.14) in the expression for \( \hat{F}_{14}^L \). We have also rescaled \( f^{(11,-11)} \), and used the identity
\[
\bar{\gamma}_L \gamma^\mu = -7 \bar{\gamma}^\mu.
\] (7.10)

Now consider the supervariation under \( \delta^{(0)} \) of (7.9) into terms of the form \( \lambda^{16}_L \epsilon_L \psi_R \). Thus,
\[
\delta^{(0)} \mathcal{L}^{(3)} = \delta^{(0)} f^{(12,-12)} \lambda^L_{16} + e \left[ f^{(12,-12)} \delta^{(0)} \lambda^L_{16} + (\delta^{(0)} f^{(11,-11)}) \bar{\psi}_{\mu L} \bar{\gamma}^\mu \lambda^L_{15} + f^{(11,-11)} \delta^{(0)} (\bar{\psi}_{\mu L} \bar{\gamma}^\mu \lambda^L_{15}) \right] + \ldots .
\] (7.11)

These supervariations can be evaluated using the supersymmetry transformations (C.1)\(^16\) leading to
\[
\begin{align*}
\delta^{(0)} f^{(12,-12)} \lambda^L_{16} & = -ef^{(12,-12)} \lambda^L_{16} (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}), \\
\delta^{(0)} f^{(11,-11)} (\bar{\psi}_{\mu L} \bar{\gamma}^\mu \lambda^L_{15}) & = -2U \frac{\partial f^{(11,-11)}}{\partial U} \lambda^L_{16} (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}), \\
f^{(11,-11)} \delta^{(0)} (\bar{\psi}_{\mu L} \bar{\gamma}^\mu \lambda^L_{15}) & = 11i f^{(11,-11)} \lambda^L_{16} (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}), \\
f^{(12,-12)} \delta^{(0)} \lambda^L_{16} & = \left( \frac{21}{4} + \frac{35}{8} \right) f^{(12,-12)} \lambda^L_{16} (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}).
\end{align*}
\] (7.12)

The two contributions to the last equation in (7.12) come from \( \delta^{(0)} \lambda_L \sim \hat{F}_2 \epsilon_L \) and \( \delta^{(0)} \lambda_L \sim \bar{T}^\epsilon_L \), respectively. Thus, (7.11) gives us
\[
\delta^{(0)} \mathcal{L}^{(3)} = e \left[ 2iD_{11} f^{(11,-11)} + \frac{69i}{16} f^{(12,-12)} \lambda^L_{16} (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}) \right],
\] (7.13)
where \( D_{11} \) is given by (G.2).

Let us consider possible supervariations \( \delta^{(3)} \) which acting on terms in \( \mathcal{L}^{(0)} \), the supergravity action, might also contribute to \( \lambda^{16}_L \epsilon_L \psi_R \). Based on the \( U(1) \) invariance of \( \mathcal{L}^{(0)} \), it is easy to see that there can be no such terms in \( \mathcal{L}^{(0)} \). Thus, (7.13) does not receive any more contributions and we get that
\[
D_{11} f^{(11,-11)} = \frac{69i}{16} f^{(12,-12)}.
\] (7.14)

Next consider the supervariation under \( \delta^{(0)} \) of (7.9) into terms of the form \( \lambda^{16}_L \epsilon_R \lambda_R \). This gives us
\[
\delta^{(0)} \mathcal{L}^{(3)} = e \left[ (\delta^{(0)} f^{(12,-12)}) \lambda^L_{16} + f^{(12,-12)} \delta^{(0)} \lambda^L_{16} + f^{(11,-11)} (\delta^{(0)} \bar{\psi}_{\mu L} \bar{\gamma}^\mu \lambda^L_{15}) \right] + \ldots .
\] (7.15)

\(^15\) Note that the contribution of the type
\[
\hat{T}^\lambda_{14} = \hat{T}^{\mu \nu \rho} (\gamma^{\mu \nu \rho})_{\mu \nu \rho} \epsilon^{AB} (\lambda^A_L)_{AB}
\] (7.6)
vanishes because
\[
\epsilon^{AB} (\lambda^A_L)_{AB} = 0.
\] (7.7)

\(^16\) The \( U(1) \) violating terms due to gauge fixing also have to be added, as discussed before.
Again using (C.1), we get

\[
(\delta^{(0)f^{(12, -12)}})\lambda_R^{16} = 2\text{i}U_{22} \frac{\partial f^{(11, -11)}}{\partial U} \lambda_R^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
f^{(12, -12)} \delta^{(0)} \lambda_R^{16} = 12 f^{(12, -12)} \lambda_R^{16} (\bar{\epsilon}_L \lambda_R),
\]

\[
f^{(11, -11)} \delta^{(0)} \bar{\psi}_{\mu L} \tilde{\phi}^{\mu} \lambda_R^{15} = 14 i f^{(11, -11)} \lambda_R^{16} (\bar{\epsilon}_L \lambda_R).
\]

The last equation in (7.16) involves many contributions coming from (C.1). It can be deduced using the identities

\[
\frac{i}{27} (\bar{\gamma}_\mu \sigma^i \lambda_L) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \sigma^i \lambda_R) = 0,
\]

\[
- \frac{11i}{54} (\sigma^i \lambda_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \sigma^i \lambda_L) = \frac{44i}{9} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
\frac{i}{54} (\bar{\gamma}_\mu \sigma^i \epsilon_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \sigma^i \lambda_R) = - \frac{28i}{9} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
\frac{5i}{54} (\sigma^i \epsilon_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \sigma^i \lambda_R) = \frac{20i}{9} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
- \frac{1}{12} (\bar{\gamma}_\mu \epsilon_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \lambda_R) = \frac{21}{3} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
- \frac{i}{144} (\bar{\gamma}_\mu \lambda_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \lambda_R) = \frac{7i}{3} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
\frac{i}{144} (\bar{\gamma}_\mu \lambda_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \lambda_R) = \frac{32i}{3} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
- \frac{i}{36} (\sigma^i \bar{\gamma}_\mu \lambda_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \lambda_R) = \frac{35i}{3} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R),
\]

\[
- \frac{i}{432} (\sigma^i \bar{\gamma}_\mu \lambda_R) A^{\dagger} (\tilde{\phi}_\mu) a^b (\lambda_L^{15})_{A^b} (\bar{\epsilon}_L \lambda_R) = \frac{35i}{3} \lambda_L^{10} (\bar{\epsilon}_L \lambda_R).
\]

Thus, (7.15) gives

\[
\delta^{(0)} L^{(3)} = \epsilon \{- 2\bar{D}_{-2} f^{(12, -12)} + 14 f^{(11, -11)}\} \lambda_R^{16} (\bar{\epsilon}_L \lambda_R).
\]

One might think there can be a term of the form \(\lambda_L^{15} \chi_L\), which might contribute for \(\delta^{(0)} \chi_L \sim \epsilon_R \lambda_L \lambda_R\). Such a term, which does not follow from linearized superspace, would have to be of the form

\[
(\lambda_L^{14})_{A^b} (\sigma^i \lambda_L^4 A_{L^b})
\]

which vanishes using (7.2).

Once again, we consider modified supersymmetry transformations \(\delta^{(3)}\) which acting on terms in \(L^{(0)}\) might contribute to \(\lambda_L^{16} \epsilon_R \lambda_R\). The only possibility is a term of the form \(\lambda_L^{12} \lambda_R^2\) in \(L^{(0)}\). This term in the supergravity action is given by

\[
L^{(0)} = \frac{1}{36} \epsilon [30 (\bar{\lambda}_R Y^{\mu} \lambda_R) (\bar{\lambda}_R Y^{\nu} \lambda_R) + (\bar{\lambda}_R Y^{\mu \nu} \lambda_R) (\bar{\lambda}_R Y^{\nu \nu} \lambda_R)]
\]

as deduced in (E.13). Now (7.20) can vary into \(\lambda_L^{16} \epsilon_R \lambda_R\) for \(\delta^{(3)} \lambda_R \sim \lambda_L^{14} \epsilon_R\). In general, it is difficult to write down complete expressions for the corrected supersymmetry transformations.
\(\delta^{(3)}\) for any field. For the case we need, let us consider the supervariation given by\(^{17}\)

\[
\delta^{(3)}\lambda_{\text{tree}} = (\lambda_{\text{tree}}^{(3)})_{AB} \left[ g_1(U, \tilde{U})(\gamma_{\mu})_A (\gamma^\mu \epsilon_R)_B + g_2(U, \tilde{U})(\gamma_{\mu\nu})_A (\gamma^{\mu\nu})_B \right] + g_3(U, \tilde{U})(\gamma_{\mu})_A (\gamma_\mu \epsilon_R)_B.
\]

(7.22)

Though (7.22) looks complicated, it is the simplest set of terms that one can try based on the complexity. Of course, we will not consider the set of all possible supervariables due to its general considerations. So from now onward, our aim will be to obtain the structure of the equations, and not bother about the coefficients. Acting on (7.20), we note that it gives

\[
\delta^{(3)} \mathcal{L}^{(2)} = 252i \epsilon g L^{16} (\bar{\epsilon}_L \lambda_R),
\]

(7.23)

where

\[ g = g_1 - 6g_2 + 4g_3. \]

(7.24)

It seems difficult to make stronger statements given that there are three undetermined moduli-dependent coefficients in (7.22). However, we will now see that the constraints imposed by the closure of the superalgebra prove strong enough to determine what is needed for our purpose.

Because we are dealing with a theory of maximal supersymmetry, there exists no off-shell formulation of the theory. In fact, the closure of the superalgebra is only up to the equations of motion of the various fields, and various local symmetry transformations. This is also true for \(\lambda_R\). Thus,

\[
[\delta_1, \delta_2] \lambda_R = \left( [\delta_1^{(0)}, \delta_2^{(0)}] + [\delta_1^{(3)}, \delta_2^{(3)}] + \ldots \right) \lambda_R
\]

(7.25)
closes only up to the equation of motion of \(\lambda_R\), and other local symmetries. We will use this to our advantage and use the equation of motion of \(\lambda_R\) to constrain \(g_1, g_2\) and \(g_3\).

Let us first consider closure at the level of supergravity. From the various expressions in (C.1), we get that

\[
[\delta_1^{(0)}, \delta_2^{(0)}] \lambda_R = -\bar{\gamma}^\mu \epsilon_{L2} (\bar{\epsilon}_L \lambda_R) + \frac{1}{2} \bar{\gamma}^{\mu\nu} \sigma^\epsilon \epsilon R_1 \gamma^\epsilon \sigma^\mu \lambda_R
\]

(7.26)

Using the Fierz and Schouten identities repeatedly, we rewrite (7.26) as

\[
[\delta_1^{(0)}, \delta_2^{(0)}] \lambda_R = (\bar{\epsilon}_L \gamma^\mu \epsilon_{L2} \partial_\mu \lambda_R + \frac{1}{16} (\bar{\epsilon}_L \gamma^\mu \epsilon_{L2}) \gamma_\mu + \frac{1}{16} (\bar{\epsilon}_L \gamma^{\mu\nu} \epsilon_{L2}) \gamma_{\mu\nu}) \lambda_R - \frac{1}{4} \epsilon R_1 (\bar{\epsilon}_L \gamma_{\mu\nu}) \lambda_R - \frac{1}{2} \bar{\gamma}^{\mu\nu} \epsilon R_1 (\bar{\epsilon}_L \gamma_{\mu\nu}) \lambda_R
\]

(7.27)

While the first term on the right-hand side of (7.27) is the standard derivative term, the remaining terms must vanish, leading to the free equation of motion for \(\lambda_R\).

Now let us focus on the first corrections to (7.27) obtained from (7.25) on using only the terms given in (7.22), along with the ones which yield the \(SL(2, \mathbb{C})\) covariant derivative. Considering only the \(O(\epsilon R_1 \gamma_{\mu\nu})\) term, we get that

\[
\left( [\delta_1^{(0)}, \delta_2^{(3)}] + [\delta_1^{(3)}, \delta_2^{(0)}] \lambda_R = \left( 2U_2 \frac{\partial}{\partial U} + \frac{45}{4} \right) \left[ -\frac{7}{2} (\bar{\epsilon}_L \gamma^\mu \epsilon_{L2}) \gamma_\mu \lambda_{L1}^{15} + \frac{1}{12} (\bar{\epsilon}_L \gamma^{\mu\nu} \epsilon_{L2}) \gamma_{\mu\nu} \lambda_{L1}^{15} - \frac{1}{2} \bar{\gamma}^{\mu\nu} \epsilon R_1 (\bar{\epsilon}_R \gamma_{\mu\nu}) \lambda_{L1}^{15} \right]
\]

(7.21)

\(^{17}\) A contribution of the type

\[ g_1(U, \tilde{U}) (\lambda_{L1}^{14})_{AB} (\gamma_{\mu\nu})_B (\bar{\gamma}^{\mu\nu}) \epsilon R_1^B \]

vanishes because \(\gamma_{\mu\nu}^{\mu\nu} = \gamma_{\mu\nu} \epsilon R_1^B\).
\[
\begin{align*}
&= (2D_{11}g) \left[ -\frac{7}{2} (\bar{\epsilon}_{L1} \bar{\gamma}^\mu \epsilon_{L2}) \bar{\gamma}^\nu \lambda_L^{15} + \frac{1}{12} (\bar{\epsilon}_{L1} \bar{\gamma}^\mu \epsilon_{L2}) \bar{\gamma}^\nu \lambda_L^{15} \\
&\quad - \frac{1}{2} \bar{\gamma}^{\mu \nu} \epsilon_{R1} (\bar{\epsilon}_{R2} \gamma_{\mu \nu} \lambda_L^{15}) \right] + \frac{1}{4} \delta^{(3)} \lambda_R, \\
\end{align*}
\]

where \( \delta^{(3)} \) is the supersymmetry transformation (7.22) with the parameter
\[
\hat{e} = (\bar{\epsilon}_{R2} \lambda_L \epsilon_{R1}).
\]

The choice (7.29) is uniquely determined once the appropriate \( SL(2, \mathbb{Z}) \) weight has been assigned to \( g \).

In (7.28), exactly the same linear combination of \( g_1, g_2 \) and \( g_3 \) given by (7.24) appears as the one in (7.23). Thus, the closure of the superalgebra on \( \lambda_R \) is good enough to provide us precisely the information we need. Thus, up to a local supersymmetry transformation, considering the terms of the form \( (\bar{\epsilon}_{L1} \bar{\gamma}^\mu \epsilon_{L2}), (\bar{\epsilon}_{L1} \bar{\gamma}^\mu \lambda_L^{15}) \) and \( \epsilon_{R1} (\bar{\epsilon}_{R2} \gamma_{\mu \nu} \ldots) \) in (7.27) and (7.28), we get the equation of motion
\[
\delta \lambda_R + 16(D_{11}g) \lambda_L^{15} + \cdots = 0,
\]

which we match with the equation of motion obtained from the action (2.48) and (7.9)
\[
\delta \lambda_R - \frac{1}{2} e^{(12, -12)} \lambda_L^{15} = 0.
\]

Note that this cannot be the complete analysis. This is because (7.28) does not contribute to the free equation of motion obtained from the term \( \epsilon_{R1} (\bar{\epsilon}_{R2} \ldots) \) in (7.27). Thus, there must be other supervariations which will also contribute, and which will yield the final equation we need. Even without worrying about the other possible contributions, from (7.30) and (7.31) we get that
\[
16D_{11}g + \cdots = -\frac{1}{2} e^{(12, -12)},
\]

where the \( \cdots \) denote the other contributions. Based on \( SL(2, \mathbb{Z}) \) covariance, we get that
\[
g \sim e^{(11, -11)},
\]

which must also be true of the other contributions.

Note that there are more constraints that can be obtained from imposing the closure of the superalgebra acting on \( \lambda_R \). We looked at those terms that involve the \( \lambda_L \) equation of motion. There are several other such terms, for example, the gravitino equation of motion also arises from the same closure. This leads to very strong constraints on the couplings.

Now, from (7.14), (7.18), (7.23) and (7.33), we get that
\[
D_{11}f^{(11, -11)} \sim f^{(12, -12)}, \quad D_{-12}f^{(11, -11)} \sim f^{(11, -11)},
\]

leading to
\[
4D_{-12}D_{11}f^{(11, -11)} = af^{(11, -11)}, \quad 4D_{11}D_{-12}f^{(12, -12)} = af^{(12, -12)},
\]

Though we have not determined the coefficient \( a \), clearly it can be determined based on the arguments we have made, and taking into account all the terms. Thus, (7.35) is completely fixed by supersymmetry.

Equations (7.35) have a unique solution on the fundamental domain of \( SL(2, \mathbb{Z}) \) given the boundary condition that the couplings have a power law behavior in \( U_2 \) for large \( U_2 \) based on physical considerations. In fact, the solutions must be given by (G.7) in appendix G.1 for \( m = 11 \) and 12 for some choice of \( s \). Thus, the value of \( a \) is also determined by the value of \( s \).
In order to determine $s$, we simply use data from string perturbation theory. The $U$-dependent one-loop amplitude for the $\mathcal{R}^4$ interaction is known to be given by (G.11). The $\mathcal{R}^4$ interaction is obtained in the effective action from linearized superspace using (6.7). Thus, based on the $SL(2, \mathbb{Z})$ covariance of the various couplings, they must be related to one another by the action of the $SL(2, \mathbb{Z})$ covariant derivatives (G.2). Thus $s = 1$, which fixes

$$a = -121.$$  \hspace{1cm} (7.36)

Thus, supersymmetry completely determines the moduli-dependent couplings of some of the interactions in the effective action, which we have obtained using linearized superspace. These interactions were obtained using the chiral superfield; hence, the couplings are independent of the $SO(3) \setminus SL(3, \mathbb{R})$ moduli. In fact, the couplings which have non-zero weights under $SL(2, \mathbb{Z})$ transformations are the coefficient functions of interactions charged under $U(1)$ and so cannot receive contributions from interactions constructed from the linear superfield, which is neutral under $U(1)$. This will also be true the other way round when we will consider couplings which transform non-trivially under $SL(3, \mathbb{Z})$ transformations, which are coefficient functions of interactions that carry $SU(2)$ charge. They will depend only on the $SO(3) \setminus SL(3, \mathbb{R})$ moduli.

Thus, the only interactions which can have couplings that depend on both the $U(1) \setminus SL(2, \mathbb{R})$ and $SO(3) \setminus SL(3, \mathbb{R})$ moduli are those that are uncharged under $U(1)$ as well as $SU(2)$. Among the interactions that follow from linearized superspace, one such interaction is the $\mathcal{R}^4$ interaction. Thus, our above discussion fixes only the $U, \bar{U}$ moduli dependence of this coupling.

Note that all the other couplings which we have determined have some striking differences from the $\mathcal{R}^4$ coupling. These couplings receive contributions only from one loop in string theory, and there are no other perturbative or non-perturbative contributions. Also, unlike the $\mathcal{R}^4$ coupling, they do not have an infrared logarithmic divergence at one loop, because of the absence of the $\ln U_2$ term in its expression.

There is a direct relationship between the $U(1)$ charge of a specific interaction, and the weight of its coupling. The coupling of an interaction which carries $U(1)$ charge $q$ is an $SL(2, \mathbb{Z})$ automorphic form of weight $(q/2, -q/2)$.

### 7.2. Couplings involving only the $SO(3) \setminus SL(3, \mathbb{R})$ moduli

We next consider a set of couplings that involve only the $SO(3) \setminus SL(3, \mathbb{R})$ moduli. To begin with, we will consider a set of 16 fermion interactions arising from the part of the action involving the linear superfield in (6.26). We will see that compared to the discussion above, the analysis is considerably more complicated.

We look at a small subset of interactions in $\mathcal{S}^{(3)}$ which mix with no other interactions under supersymmetry transformations $\delta^{(0)}$ of the type that we will consider. This will lead to a coupled set of linear differential equations for the various couplings we consider, which will lead to Poisson equations on moduli space for the various individual couplings.

As before, the equations obtained from $\delta^{(0)} \mathcal{S}^{(3)}$ will also receive contributions from $\delta^{(1)} \mathcal{S}^{(0)}$. However, unlike the above analysis involving only the chiral superfield, there are several possible terms in the supergravity action $\mathcal{S}^{(0)}$ which can contribute. This is because the superaction involving the linear superfield involves the integral $\int d\theta^a_R d\bar{\theta}^a_L$ which is real and yields those 16 fermion interactions in $\mathcal{S}^{(3)}$ such that there are several contributions from $\delta^{(1)} \mathcal{S}^{(0)}$. Of course, the procedure to calculate them is exactly the same as above. We will only constrain the structure of the equations using supersymmetry, but we will not fix the various coefficients. In particular, we will be very schematic in our discussion of the $\delta^{(1)} \mathcal{S}^{(0)}$.
contributions. In principle, they can all be fixed using only supersymmetry, but the calculations get very tedious.

In order to avoid the large number of indices associated with these interactions, we will adopt a simple notation. The SU(2) spin \(3/2\) fermions \(\chi_L\) and \(\chi_R\) in the various interactions will always arise in the combination \(\sigma_i(\chi_L)\) and \(\sigma_i(\chi_R)\). We will simply denote them

\[
\sigma_i(\chi_L) = \chi_L, \quad \sigma_i(\chi_R) = \chi_R.
\] (7.37)

and drop the various \(i, j\) indices when there is no scope for confusion.

For these fermions, we define

\[
\chi^8_{L,R} = e^{\alpha_1 \cdots \alpha_8} \epsilon^\beta_1 \ldots \epsilon^\beta_8 \epsilon_{A_1 B_1} \cdots \epsilon_{A_8 B_8} \left( (\sigma_i(\chi_L))_{A_1}^{B_1} \cdots (\sigma_i(\chi_L))_{A_8}^{B_8} \right)
\]

\[
\times \left( (\sigma_i(\chi_R))_{R_1}^{B_1} \cdots (\sigma_i(\chi_R))_{R_8}^{B_8} \right).
\]

\[
\chi^7_{L,R} = e^{\alpha_1 \cdots \alpha_7} \epsilon^\beta_1 \ldots \epsilon^\beta_7 \epsilon_{A_1 B_1} \cdots \epsilon_{A_7 B_7} \left( (\sigma_i(\chi_L))_{A_1}^{B_1} \cdots (\sigma_i(\chi_L))_{A_7}^{B_7} \right)
\]

\[
\times \left( (\sigma_i(\chi_R))_{R_1}^{B_1} \cdots (\sigma_i(\chi_R))_{R_7}^{B_7} \right).
\]

\[
\chi^6_{L,R} = e^{\alpha_1 \cdots \alpha_6} \epsilon^\beta_1 \ldots \epsilon^\beta_6 \epsilon_{A_1 B_1} \cdots \epsilon_{A_6 B_6} \left( (\sigma_i(\chi_L))_{A_1}^{B_1} \cdots (\sigma_i(\chi_L))_{A_6}^{B_6} \right)
\]

\[
\times \left( (\sigma_i(\chi_R))_{R_1}^{B_1} \cdots (\sigma_i(\chi_R))_{R_6}^{B_6} \right).
\]

First let us consider interactions of the form

\[
\chi^8_{L,R} \bar{\chi}^8_{L,R}, \quad \bar{\psi}_{L,R} \gamma^\mu \chi^7_{L,R}, \quad (\bar{\psi}_{L,R} \gamma^\mu \gamma^\nu \chi^7_{R,L})^2, \quad (\bar{\psi}_{L,R} \gamma^\mu \gamma^\nu \chi^6_{L,R})^2 \chi^8_{L,R} \chi^8_{R,L} \] (7.39)

in \(S^{(3)}\). These interactions can all be obtained from (6.26), on using (6.10). In order to write down these interactions, we consider a subset of the interactions given by (6.25). They are given by

\[
\mathcal{L}^{(3)} = \epsilon \left[ g(1,8) \chi^8_{L,R} \bar{\chi}^8_{L,R} + g(1,8) \chi^7_{L,R} \bar{\chi}^7_{L,R} (\bar{F}_3 + \bar{F}_2) + g(1,7) \chi^6_{L,R} \bar{\chi}^6_{L,R} (\bar{F}_3 + \bar{F}_2)^2 \right] + \text{h.c.}
\] (7.40)

In (7.40), the SO(3), SL(3, \(\mathbb{R}\)) moduli dependence of the various couplings has not been denoted for brevity. Thus, for example,

\[
g(1,8) \equiv g(1,8) \left( T, \bar{T}, \xi, \bar{\xi}, \epsilon, e^{-2\delta} \right). \] (7.41)

Explicitly, the first term in (7.40) which is of the form \(\chi^8_{L,R} \bar{\chi}^8_{L,R}\) is given by

\[
g^{(6)}(\tau_i \ldots \tau_{6}) (\tau_{i} \ldots \tau_{6}) \epsilon^{\alpha_1 \cdots \alpha_6} \epsilon^\beta_1 \ldots \epsilon^\beta_6 \epsilon_{A_1 B_1} \cdots \epsilon_{A_6 B_6} \left( (\sigma_i(\chi_L))_{A_1}^{B_1} \cdots (\sigma_i(\chi_L))_{A_6}^{B_6} \right)
\]

\[
\times \left( (\sigma_i(\chi_R))_{R_1}^{B_1} \cdots (\sigma_i(\chi_R))_{R_6}^{B_6} \right),
\] (7.42)

while the second term is given by

\[
g^{(5)}(\tau_i \ldots \tau_{6}) (\tau_{i} \ldots \tau_{6}) \epsilon^{\alpha_1 \cdots \alpha_5} \epsilon^\beta_1 \ldots \epsilon^\beta_5 \epsilon_{A_1 B_1} \cdots \epsilon_{A_6 B_6} \left( (\sigma_i(\chi_L))_{A_1}^{B_1} \cdots (\sigma_i(\chi_L))_{A_5}^{B_5} \right)
\]

\[
\times \left( (\sigma_i(\chi_R))_{R_1}^{B_1} \cdots (\sigma_i(\chi_R))_{R_5}^{B_5} \right) (\gamma^\mu \gamma^\nu)_{\beta \gamma} (\sigma_i(\chi_L))_{A_6}^{B_6} \bar{F}_{\gamma 2 \mu \nu},
\] (7.43)

and similarly for the other terms.
Now (7.43) will lead to an interaction of the form \( g \tilde{\psi}_{R} \gamma^\mu \chi^b_8 \chi^c_8 \gamma^\nu \tilde{\chi}^d_8 \) using \( \tilde{F}_2 \sim \tilde{\psi}_R \chi_R \) as we will show below. The index structure of this coupling \( g \) in (7.43) has been assigned based on the structure of this interaction we want to consider, which will be evident from the discussion below. The conjugate interaction yields a term of the form \( g^2 \tilde{\psi}_{L} \gamma^\mu \chi^c_8 \gamma^\nu \tilde{\chi}^d_8 \).

Some of the remaining terms in (7.40) also yield interactions of the form \( \tilde{\psi}_{R} \gamma^\mu \chi^b_8 \chi^c_8 \tilde{\chi}^d_8 \) and \( \tilde{\psi}_{L} \gamma^\mu \chi^b_8 \chi^c_8 \tilde{\chi}^d_8 \), on using \( \tilde{F}_3 \sim \tilde{\psi}_L \chi_R + \tilde{\psi}_R \chi_L \) and \( \tilde{F}_4 \sim \tilde{\psi}_L \chi_R + \tilde{\psi}_R \chi_L \). The interactions of the form \( \tilde{\psi}_{L} \gamma^\mu \chi^b_8 \gamma^\nu \tilde{\chi}^d_8 \) are obtained from \( \chi^b_8 \chi^c_8 \tilde{\chi}^d_8 \) on using \( \tilde{F}_3 \sim \tilde{\psi}_R \chi_L + \tilde{\psi}_L \chi_R \). Similarly we can work out the various relevant interactions arising from the remaining terms in (7.40). The analysis is exactly along the lines of the one we do below. We should mention that as in the discussion above, there are often several terms in the superaction which contribute to the same interaction. In such cases, we expect the couplings coming from the various contributions to be the same because they follow from the same superfield.

Before we proceed further, we also need to know how the interactions corresponding to the couplings \( g_{(1..8)(1..8)} \), \( g_{(1..8)(1..7)} \), \( g_{(1..7)(1..7)} \) and \( g_{(1..8)(1..6)} \) in (7.40) transform under \( SU(2) \). We can only talk about the \( SU(2) \) transformation properties of the various interactions, and not the couplings themselves. This is because \( SU(2) \) is only a symmetry of supergravity and is broken by the higher derivative corrections. Actually, these couplings transform non-trivially under \( SL(3, \mathbb{Z}) \), and we should denote them by their \( SL(3, \mathbb{Z}) \) transformation properties. However, unlike the previous case, we will see later that the couplings for the interactions which have non-trivial \( SU(2) \) charges do not transform as automorphic forms of \( SL(3, \mathbb{Z}) \), and transform in a complicated way. Thus, we find it easier to simply denote the couplings by the \( SU(2) \) charges that the corresponding interactions carry. We will often loosely denote it by the \( SU(2) \) charges of the couplings themselves, but this is only for brevity.

First let us consider the case of \( g_{(1..8)(1..8)} \). As discussed before, every factor of \( \sigma((i)(j)) \) (and consequently every factor of \( (ij) \) in \( g_{(1..8)(1..8)} \)) transforms in the spin 2 representation of \( SU(2) \). Thus, the spacetime interaction in (7.42) involves the product of 16 spin 2 representations of \( SU(2) \). Expressing this as a sum of irreducible representations, we choose the interaction to project onto the spin 32 representation of \( SU(2) \). Thus, the coupling is symmetric under the interchange of any pair of \( (ij) \) indices. While this is not necessary for our analysis, and one can focus on any irreducible representation of \( SU(2) \), we choose the highest spin representation for simplicity, as the symmetry under interchange of the various indices simplifies our calculations considerably. Similarly, we take the interaction corresponding to the \( g_{(1..8)(1..7)} \) coupling to transform in the spin 30 representation of \( SU(2) \) (this follows from the analysis below because the \( (k_l f) \) term in \( g \) gets coupled to \( \sigma((i)(j)) \), and those corresponding to the \( g_{(1..7)(1..7)} \) and \( g_{(1..8)(1..6)} \) couplings to transform in the spin 28 representation of \( SU(2) \). We always consider interactions which transform as the highest spin representation of \( SU(2) \), and this will be always implicit in the discussion below.

We will find it convenient to write the couplings in the form \( g_{(i)j} \) where the indices in the two parentheses are the number of \( \chi_L \) and \( \chi_R \) fields in the interaction. Obviously, this division is artificial, as only the \( SU(2) \) representation matters. We will later shift to the convention where the division between \( \chi_L \) and \( \chi_R \) is removed, and thus an arbitrary interaction can be analyzed.

Now let us simplify the structure in (7.43) to obtain the interaction of the type we want. The \( \chi^b_8 \chi^c_8 \tilde{\chi}^d_8 \) part is already of the form we want. Focusing on the rest, we use

\[
(\sigma_{i_1} \gamma^{b_1} b_1)_{R}^A (\chi_L)^{b_1}_R = e^{AB} (\sigma_{i_2} \chi_L)^{b_2}_R + (\sigma_{i_3} \gamma^{b_3} A)_{R}^A (\chi^b_8 \chi^c_8 \tilde{\chi}^d_8)_{R_R}^R,
\]  
(7.44)
which follows from using the Schouten identity. In (7.44), we note that the first term gives a spacetime contribution of the kind we want, and so we consider this contribution to (7.43). Finally, we use
\[ (x^a_R)_A^B x^c_R \sim \delta^c_A [(x^a_R)_B^a] \delta^e_B - (x^a_R)_B^a \delta^e_B + \cdots \] (7.45)
to get a spacetime contribution of the form \[ \bar{\psi}_{R;\mu} \gamma^\mu \chi_L \chi_L^R, \]
where
\[ \bar{\psi}_{R;\mu} \gamma^\mu \chi_L \chi_L^R = (\bar{\psi}_{R;\mu} \gamma^\mu)_{\alpha} \chi_L^R. \] (7.46)
We perform a similar analysis for the other couplings in (7.40) to obtain the relevant terms in the action, where we have used the notations
\[ \bar{\psi}_{L;\mu} \gamma^\mu \chi_L \chi_L^R = (\bar{\psi}_{L;\mu} \gamma^\mu)_{\alpha} \chi_L^R. \] (7.47)
Thus in \[ \mathcal{L}^{(3)}, \] we consider interactions of the form
\[ \mathcal{L}^{(3)} = e \left[ g_{(1..8)(1..6)} \chi_L^8 \chi_L^8 + g_{(1..8)(1..7)} \bar{\psi}_{R;\mu} \gamma^\mu \chi_L \chi_L^R + g_{(1..7)(1..8)} \bar{\psi}_{L;\mu} \gamma^\mu \chi_L \chi_L^R \right. \]
\[ \left. + g_{(1..8)(1..7)} (\bar{\psi}_{L;\mu} \gamma^\mu \chi_L) (\bar{\psi}_{R;\nu} \gamma^\nu \chi_L) + g_{(1..6)(1..8)} (\bar{\psi}_{L;\mu} \gamma^\mu \chi_L)^2 + \cdots \right]. \] (7.48)
Note that
\[ g_{(1..8)(1..7)} = g_{(1..7)(1..8)}. \] (7.51)
Also to project onto the spacetime structure of the terms we want, we use the relation
\[ (x_L)_A^B x^e_B \sim \frac{1}{6} \delta^e_A \delta^B_A \chi_L^8 + \cdots. \] (7.52)
Let us first consider various contributions coming from \[ \delta^{(0)} \mathcal{S}^{(3)} \]. To begin with, let us consider the variation of (7.48) into terms of the form \( \mathcal{L}^R \chi_L \). The first term gives
\[ \delta^{(0)} (e g_{(1..8)(1..6)} \chi_L^8 \chi_L^8) \sim e (\delta g_{(1..8)(1..6)}) (\bar{\epsilon}_R \chi_L) \chi_L^8 + \cdots. \] (7.53)
Explicitly, the right-hand side of (7.53) is given by
\[ \partial (\epsilon_{(a_1}) g_{(a_2 \ldots a_n)(b_1 \ldots b_m)} (b_1 \ldots b_m) \bar{\epsilon}_R \sigma_{b_1 b_2 b_3} + \cdots. \] (7.54)
The term we have written down in (7.53) obviously comes from taking the derivatives with respect to the moduli on using (4.57). We would like to know if there are more contributions to (7.53), which promote the ordinary derivative to a generalized derivative, and acts on the space of couplings, like the \( U(1) \), \( SL(2, \mathbb{R}) \) case.

In fact there are such contributions to (7.53) which we now calculate. Let us first focus on the contribution from \( \delta^{(0)} \chi_L \sim \epsilon_L \chi_L^2 \). In order to contribute to (7.53), we only need terms of the form \( \sigma_{(a_1} \delta^{(0)} \chi_L^{b_1} \sim \sigma_{(a_1} \chi_L^{b_1)} \), and similarly for \( \chi_L \). There are no such terms from the \( SU(2) \) covariant expression for \( \delta^{(0)} \chi_L \) in (C.1), and they only arise from the extra terms from gauge
\( 18 \) Unlike the expression in (7.2), note that (7.45) has more terms represented by \( \cdots \), because \( \chi_L \) has 32 degrees of freedom.
\( 19 \) We suitably rescale the various couplings.
\( 20 \) We will soon consider the convention where the couplings are characterized only by their \( SU(2) \) spins. Thus, we will denote
\[ g_{(a_1 \ldots a_n)(b_1 \ldots b_m)} = g_{(a_1 \ldots a_n)} = g^{(e)}. \] (7.49)
So, for example,
\[ g_{(a_1 \ldots a_n)(b_1 \ldots b_m)} = g^{(16)} \cdot g_{(a_1 \ldots a_n)(b_1 \ldots b_m)} = g^{(14)}, \]
where (7.51) implies that \( g^{(13)} \) is real.
fixing the supersymmetry transformations given by (4.58). Thus, these terms, as expected, are not SU(2) covariant. From (4.58), we get the relevant term
\[ \sigma_i \delta^{(0)} \chi_{jL} = -\frac{1}{2} \varepsilon_{i}^{ \mu \nu \rho} \sigma_j \chi_{L}^\nu. \]  
(7.55)

Using (4.55), (2.79) and the Fierz and Schouten identities repeatedly, we get that
\[ \sigma_i \delta^{(0)} \chi_{jL} = \frac{1}{10} \left[ \sigma_i \chi_{jL} (\tilde{\epsilon}_R \sigma_3 \chi_{L}) - \frac{1}{2} \gamma^{\mu \nu} \sigma_i \chi_{jL} (\tilde{\epsilon}_R \gamma_{\mu \nu \rho \sigma} \sigma_3 \chi_{L}) + \frac{1}{12} \gamma^{\mu \nu \rho \sigma} \sigma_i \sigma_3 \chi_{jL} (\tilde{\epsilon}_R \gamma_{\mu \nu \rho \sigma} \sigma_3 \chi_{L}) + \sigma_i \sigma_3 \epsilon L (\chi_R \chi_{jL}) - \frac{1}{2} \gamma^{\mu \nu} \sigma_i \sigma_3 \chi_{jL} (\tilde{\epsilon}_R \gamma_{\mu \nu \rho \sigma} \sigma_3 \chi_{L}) \right] - (3 \rightarrow 1) \]
- \frac{1}{3} \delta \sigma_i \sigma_j (2 \chi_{L} (\tilde{\epsilon}_R \sigma_3 \chi_{jL}) + 2 \chi_{L} (\tilde{\epsilon}_R \sigma_3 \chi_{L}) + i \chi_M (\tilde{\epsilon}_R \chi_{2L}) - i \chi_{2L} (\tilde{\epsilon}_R \chi_{M})) 
+ \delta \sigma_i \sigma_j (2 \chi_{L} (\tilde{\epsilon}_R \sigma_3 \chi_{jL}) - 2 \chi_{L} (\tilde{\epsilon}_R \sigma_3 \chi_{L}) + i \chi_M (\tilde{\epsilon}_R \chi_{2L}) - i \chi_{2L} (\tilde{\epsilon}_R \chi_{M})) 
- i \delta \sigma_i \sigma_j (\chi_{M} (\tilde{\epsilon}_R \chi_{3L}) + \chi_M (\tilde{\epsilon}_R \chi_{L})). \]
(7.56)

where we have kept only the terms proportional to \( \epsilon L \). This gives a contribution
\[ \sigma_i \delta^{(0)} \chi_{jL} = \frac{1}{10} \sigma_i \chi_{jL} (\tilde{\epsilon}_R \sigma_3 \chi_{L} - \tilde{\epsilon}_R \sigma_3 \chi_{L}), \]  
(7.57)

exactly of the kind we need. The coefficient in (7.57) can receive additional contributions from \( \delta^{(3)} L^{(0)} \), as we will schematically describe later. However, it is not difficult to write down the final answer which is
\[ \sigma_i \delta^{(0)} \chi_{jL} = 2 \sigma_i \chi_{jL} (\tilde{\epsilon}_R \sigma_3 \chi_{L} - \tilde{\epsilon}_R \sigma_3 \chi_{L}), \]  
(7.58)

and similarly for \( \chi_R \). This can be seen by considering the expression (G.21), and noting that
\[ D^{(1)}_{\mu \nu} D^{(0)}_{\nu \lambda} = 2 \Delta, \]  
where \( \Delta \) is the Laplacian on SO(3)?SL(3, \mathbb{R}) given by (F.10). Now \( D^{(0)}_{\mu \nu} \) which contains only the derivatives comes from varying the \( \mathcal{R}^4 \) coupling. While the derivative terms in \( D^{(1)}_{\mu \nu} \) have a similar origin, the remaining terms come from the supervariation of one factor of \( \sigma_i \chi_{jL} \) in the spacetime interaction, which directly leads to (7.58).

Thus, the right-hand side of (7.53) is given by
\[ D^{(10)}_{\mu \nu \lambda} (\tilde{\epsilon}_R \sigma_i \chi_{jL}) \chi_{L} \chi_{R}, \]  
(7.59)

where the expression for \( D^{(10)}_{\mu \nu \lambda} \) is given in (G.21).

The second term in (7.48) gives contributions from the supervariation of \( \psi_L \). Since we only want to see the specific spacetime structure emerge, it is good enough to do the analysis for any term in \( \delta^{(0)} \psi_L \). From (C.1), we thus consider
\[ \delta^{(0)} \psi_L = -\frac{1}{\xi} (\tilde{\epsilon}_R \sigma_i \chi_{jL}) \sigma_y \chi_{R}, \]  
(7.60)

which gives
\[ \delta^{(0)} \left( \bar{\psi} \tilde{\mu} R \gamma^\mu X_L^R \right) = \frac{4 i}{3} (\tilde{\epsilon}_R \sigma_i \chi_{jL}) \chi_L^R \bar{\sigma}_i \chi_{R}. \]  
(7.61)

Thus, we get that
\[ \delta^{(0)} \left( \epsilon_{\nu (1, 2)} \gamma^\nu X_L^R \right) \sim \epsilon_{\nu (1, 2)} \gamma^\nu X_L^R \]  
(7.62)

The right-hand side of (7.62) contains
\[ \left[ R \left( \chi_{(1, 2)} \right) \left( \chi_{(1, 2)} \right) \right] \delta_{\nu (1, 2)} \delta_{\nu (1, 2)}. \]  
(7.63)

Similarly, the third term in (7.48) gives contributions from the supervariation of \( \psi_R \). As before, from (C.1), we consider
\[ \delta^{(0)} \psi_R = \frac{1}{\xi} (\tilde{\epsilon}_R \sigma_i \chi_{jL}) \sigma_y \chi_{R}, \]  
(7.64)

21 This is unlike the U(1)\( \times \)SL(2, \mathbb{R}) analysis done before.

22 Right now, we are looking at only the minimal set of terms in the effective action which provide the necessary supervariations in an obvious way. At least this much structure is needed to see that a set of equations relating the various couplings arises. We will talk about possible additional terms that could modify the equations later.
which gives
\[ \delta^{(0)} \left( \bar{\psi}_{\mu L} \tilde{\gamma}^\mu \chi^R_{L R} \right) = -\frac{4i}{3} (\bar{\epsilon}_R \sigma_{i J L}) (\chi^i_{L} \sigma_{i J L}) \chi^8_R. \] (7.65)
leading to
\[ \delta^{(0)} \left( \epsilon_R^{(1 \ldots 7)(1 \ldots 8)} \bar{\psi}_{\mu L} \tilde{\gamma}^\mu \chi^R_{L R} \right) \sim \epsilon \delta_R^{(1 \ldots 7)(1 \ldots 8)} (\bar{\epsilon}_R \chi^R_{L}) \chi^8_R. \] (7.66)
The right-hand side of (7.66) contains
\[ [g_{(i_1 j_1) \ldots (i_n j_n)}(k_1 j_1) \ldots (k_1 j_1)] \delta_{i_1 j_1} \delta_{i_2 j_2}. \] (7.67)
The remaining terms in (7.48) do not give a contribution of the type we want.

Thus, (7.53), (7.62) and (7.66), on using (7.49), we get that
\[ D^{(16)} \delta_{(i_1 j_1) \ldots (i_n j_n)} = c_1 g_{(i_1 j_1) \ldots (i_n j_n)} \delta_{i_1 j_1} \delta_{i_2 j_2}. \] (7.68)
In short, we write (7.68) schematically as
\[ D^{(16)} \delta^{(16)} = c_1 g^{(15)}. \] (7.69)
Thus, \( D^{(16)} \) acts as a spin 2 operator which acting on a coupling corresponding to an interaction in the spin 32 representation gives the coupling corresponding to an interaction in the spin 30 representation of \( SU(2) \).

Note that there can be a term in the effective action of the type
\[ e^{(16)}_{i L} (\chi^R_{L})_{\alpha \beta} (\sigma_{i J L})_{\alpha} (\sigma_{i J L})_{\beta} \] (7.70)
which can vary into \( e^{(16)}_{i L} \chi^R_{L} \) for
\[ \delta^{(0)} \chi^R_{L} \sim (\bar{\epsilon}_R \sigma_{i L R}) \sigma_{i L R}. \] (7.71)
This term cannot be obtained from (6.26) using linearized superspace. However, the contribution of this term simply changes the value of \( c_1 \) in (7.69).

Now let us consider the supervariation of (7.48) under \( \delta^{(0)} \) into terms of the form \( \chi^R_{L} \chi^R_{L} e_L \psi_R \). We will give some of the details of the calculations for the first couple of terms, and then simply give the answers. The first term in (7.48) gives three contributions involving the supervariations of \( e^{(0)}_{i L}, \chi_R \) and \( \chi_R \), respectively. The metric variation yields
\[ \delta^{(0)} e^{(16)}_{i L} \chi^R_{L} \chi^R_{L} = -\epsilon (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}) \chi^8_R. \] (7.72)

The contribution from the supervariation of \( \chi_L \) is obtained from the \( \hat{F}^+_{\mu} \) term in (C.1) on using \( \hat{F}^+_{\mu} \sim \psi_R \chi_L \). To obtain it, we use
\[ (\sigma_{i L R})_{\alpha} = \frac{1}{4} (\bar{\psi}_{\mu L} \tilde{\gamma}^\mu \psi^R) [\epsilon_{B C} (\sigma_{i L R})_{\alpha} + \epsilon_{D B} (\sigma_{i L R})_{\alpha}], \] (7.73)
which can be obtained on using the Schouten identity. Note that the first term in (7.73) yields a contribution of the type we want on using (7.52), leading to
\[ \delta^{(0)} \chi^R_{L} \chi^R_{L} \sim (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}) \chi^8_R. \] (7.74)
Similarly, the term involving the supervariation of \( \chi_R \) gives
\[ \chi^R_{L} \delta^{(0)} \chi^R_{L} \sim (\bar{\epsilon}_R \gamma^\mu \psi_{\mu R}) \chi^8_R. \] (7.75)
on using \( \hat{F}_{\mu} \sim \psi_R \chi_L \) and \( \hat{F}^+_{\mu} \sim \psi_R \chi_L \), and
\[ \gamma_{\nu \sigma} \gamma^{\mu \nu \sigma} = -42 \gamma^\mu. \] (7.76)

23 The sum of the supervariations does vanish for the contributions (7.61) and (7.65) we have considered. However, this should not vanish when all the contributions are taken into account.

24 One can obtain such a term by manipulating the corresponding terms in (C.1).
Thus, we get that
\[ \delta^{(0)}(e^{g(1\ldots8)(1\ldots8)}\chi_L^8\bar{\chi}_R^8) \sim e^{g(1\ldots8)(1\ldots8)}(\bar{\psi}_R^{\mu\nu}\psi_{\mu\nu})\chi_L^8\bar{\chi}_R^8. \] (7.77)

The second term in (7.48) can give a possible contribution using
\[ \delta^{(0)}\psi_{\mu\nu}L = \frac{1}{24}\sigma^i(Y_{\mu\nu\dot{\lambda}} - 6g_{\mu\nu}Y_{\dot{\lambda}\rho})\epsilon_{L}(\bar{\psi}_L^{\mu\nu}\bar{\psi}_R^{\dot{\lambda}\rho}) - \frac{1}{4}\bar{\sigma}_i\epsilon_L(\bar{\psi}_L^{\mu\nu}\chi_R^8) + \frac{1}{2}\bar{\sigma}_i\epsilon_R(\bar{\psi}_R^{\mu\nu}\bar{\chi}_L^8), \] (7.78)

which follows from (C.1). However, this does not yield a contribution of the type we want, and so it vanishes. Thus,
\[ \delta^{(0)}(e^{g(1\ldots8)(1\ldots8)}\bar{\psi}_R^{\mu\nu}\chi_L^8\bar{\chi}_R^8) = 0. \] (7.79)

The third term in (7.48) yields contributions coming from the supervariations of \( g(1\ldots7)(1\ldots8), \chi_L \) and \( \chi_R \) leading to
\[ e^{g(1\ldots8)(1\ldots8)}(\bar{\psi}_L^{\mu\nu}\bar{\psi}_R^{\mu\nu})\chi_L^8\bar{\chi}_R^8 = \frac{i}{16}(D^{(15)}g(1\ldots7)(1\ldots8))(\bar{\psi}_R^{\mu\nu}\psi_{\mu\nu})(\chi_L^8\bar{\chi}_R^8), \] where
\[ D^{(15)}g(1\ldots7)(1\ldots8) \equiv D^{(15)}g_{(i_1i_2\ldots)(j_1j_2\ldots)(k_1k_2\ldots)}(i_1i_2\ldots)(j_1j_2\ldots)(k_1k_2\ldots). \] (7.80)

Note that there is no contribution to \( D^{(15)} \) from the supervariation of \( \psi_{R\mu} \), which is not difficult to see from (4.58). This is what is expected, because the spacetime interaction has all its \( SU(2) \) indices contracted except the spin 2 indices carried by the \( \sigma_i(\chi) \) factors. This continues to hold in our discussion below.

Exactly similarly the contribution from the supervariation of the terms involving \( g^{(14)} \) in (7.48) comes from \( \delta^{(0)}\psi_{\mu\nu}L \) and \( \delta^{(0)}\bar{\psi}_R^{\mu\nu}L \) on using (C.1), and yields
\[ \delta^{(0)}g(1\ldots7)(1\ldots8)(\bar{\psi}_L^{\mu\nu}\bar{\psi}_R^{\mu\nu})\bar{\chi}_L^8\bar{\chi}_R^8 \sim e\delta g(1\ldots7)(1\ldots8)(\bar{\psi}_R^{\mu\nu}\psi_{\mu\nu})\chi_L^8\bar{\chi}_R^8. \] (7.82)

From (7.77), (7.79), (7.80) and (7.82), we get that
\[ D^{(15)}g(1\ldots7)(1\ldots8) = d_1g_{(i_1j_1\ldots)(j_1j_2\ldots)(k_1k_2\ldots)} + d_2g_{(i_1j_1\ldots)(j_2j_3\ldots)(k_1k_2\ldots)}\delta_{i_1j_1}\delta_{j_2j_3}, \] (7.83)

where we have absorbed the contribution from the term involving \( g(1\ldots6)(1\ldots8) \) into that from \( g(1\ldots7)(1\ldots7) \) using (7.49). Using (7.49), (7.83) can be schematically written as
\[ D^{(15)}g^{(15)} = d_1g^{(16)} + d_2g^{(14)}, \] (7.84)

which is explicitly
\[ D^{(15)}g_{(i_1j_1\ldots)(j_1j_2\ldots)(k_1k_2\ldots)} = d_1g_{(i_1j_1\ldots)(j_1j_2\ldots)(k_1k_2\ldots)} + d_2g_{(i_1j_1\ldots)(j_1j_2\ldots)(k_1k_2\ldots)}\delta_{i_1j_1}\delta_{j_2j_3}\delta_{j_3j_4}. \] (7.85)

In (7.84), \( D^{(15)} \) acts as a spin 2 operator acting on a coupling corresponding to an interaction in the spin 30 representation gives couplings corresponding to interactions in the spin 32 and spin 28 representations of \( SU(2) \). In spite of the complexity of the calculations, the structure of the final answer is quite simple, due to supersymmetry.

Thus, the supervariation of the terms in the effective action we have considered gives us (7.69) and (7.84). Does this pattern continue?

It is not difficult to see that it does. In fact, the next equation is given by
\[ D^{(14)}g^{(14)} = e_1g^{(15)} + e_2g^{(13)}. \] (7.86)

Now, (7.86) can be obtained by starting with the term \( g^{(14)}\chi_L^8\bar{\chi}_R^8\bar{\psi}_L^2 \) in the effective action which can be obtained from \( g^{(14)}\chi_L^8\bar{\chi}_R^8\bar{\psi}_L^2 \), and varying it into \( D^{(14)}g^{(14)}\epsilon_1\chi_L^8\bar{\chi}_R^8\bar{\psi}_L^2 \). This supervariation can also be obtained from other terms in the effective action which contribute to (7.86). These
terms are, for example, of the form $g^{(15)} \chi_{L} \chi_{R} \psi_{L}$ and $g^{(13)} \chi_{L} \chi_{R} \psi_{R}$, which can be obtained from $g^{(15)} \chi_{L} \chi_{R} \tilde{F}_{3}$ and $g^{(13)} \chi_{L} \chi_{R} \tilde{F}_{3}$, respectively. It is crucial that there are no other couplings with any other spin that contribute to (7.86). This generalizes easily all the way up to

$$D^{(9)} g^{(9)} \sim g^{(10)} + \delta^{(9)},$$

(7.87)

using the interactions $g^{(16-n)} \chi_{L} \chi_{R} \psi_{n}^{(8-n)} \tilde{F}_{3}$ for $0 \leq n \leq 8$, and repeating the above logic. To obtain the analogous equation for $D^{(8)} g^{(8)}$, while $g^{(8)}$ is obtained as above, none of the terms that give a contribution involving $g^{(7)}$ arise from the linearized superfield (6.10). This is not surprising because the linearized superspace gives only a small subset of interactions in the effective action. However, it is not difficult to write down a term involving $g^{(7)}$ that gives the relevant supersuperposition. Such a term of the form $\chi_{L} \psi_{R} \psi_{L}$ is given by

$$\epsilon_{a_{1}} \cdots \epsilon_{a_{n}} \epsilon_{b_{1}} \cdots \epsilon_{b_{n}} \left(\sigma_{(i)} \chi_{j_{1}}\right)_{a_{1}} \cdots \left(\sigma_{(i)} \chi_{j_{n}}\right)_{a_{n}} \left(\psi_{R_{i}} Y_{\mu_{1}} \bar{Y}_{\mu_{1}}\right)_{b_{1}} \cdots \left(\psi_{R_{i}} Y_{\mu_{1}} \bar{Y}_{\mu_{1}}\right)_{b_{n}},$$

(7.88)

which gives the required supersuperposition for

$$\delta^{(0)} \psi_{R} \sim \gamma_{0} \sigma_{i} \chi_{j} (\bar{e}_{R} \sigma_{i} \chi_{j}).$$

(7.89)

Now the analysis goes through all the way up to

$$D^{(8)} g^{(8)} \sim g^{(1)},$$

(7.90)

using the interactions $g^{(8-n)} \chi_{L} \psi_{R} \psi_{L}$ for $0 \leq n \leq 8$. These interactions are explicitly given by

$$\epsilon_{a_{1}} \cdots \epsilon_{a_{n}} \epsilon_{b_{1}} \cdots \epsilon_{b_{n}} \left(\sigma_{(i)} \chi_{j_{1}}\right)_{a_{1}} \cdots \left(\sigma_{(i)} \chi_{j_{n}}\right)_{a_{n}} \left(\psi_{R_{i}} Y_{\mu_{1}} \bar{Y}_{\mu_{1}}\right)_{b_{1}} \cdots \left(\psi_{R_{i}} Y_{\mu_{1}} \bar{Y}_{\mu_{1}}\right)_{b_{n}}.$$ 

(7.91)

Thus, we obtain the sequence of equations

$$D^{(16)} g^{(16)} = g^{(15)},$$

$$D^{(15)} g^{(15)} = g^{(16)} + g^{(14)},$$

$$D^{(14)} g^{(14)} = g^{(15)} + g^{(13)},$$

$$\vdots$$

$$D^{(1)} g^{(1)} = g^{(2)} + g^{(0)},$$

$$D^{(0)} g^{(0)} = g^{(1)}.$$ 

(7.92)

for the various couplings in the supermultiplet, which follows as a consequence of supersymmetry. We have set the various undetermined coefficients in (7.92) to 1 for brevity, but they can all be completely determined by supersymmetry. It follows that if any one of the couplings in (7.92) at either end of the sequence can be determined, then all the other couplings can be determined recursively. The structure of the equations (7.92) is consistent with the fact that for the kind of couplings we have considered, on any spin 2 index, $D^{(n)}$ acts as

$$D^{(n)}_{ij} : A_{(ij)} \rightarrow A_{(i(j)} + A_{i} \delta_{j}.$$

(7.93)

Thus, it reduces the spin by 2 and increases it by 2. This occurs with every coupling in (7.92), except for the first and last couplings in the sequence, because there are no couplings to vary into for some spins.

So far we have considered only the contributions coming from $\delta^{(0)} \mathcal{L}^{(3)}$. Let us now focus on the contributions coming from $\delta^{(3)} \mathcal{L}^{(0)}$ very schematically. To be specific, we consider the added contributions to (7.69). These terms are obtained from the supersupervariation of the action
into $\delta_{L,X_l}$ and $X_l X_R^8$. There are several terms in $L^{(0)}$ which can give this supervariation for appropriate $\delta^{(3)}$. They are

$$\lambda_L X_l^3, \lambda_R X_R^3, \lambda_L X_R^2, \lambda_L X_R \Psi_L, \lambda_R X_L \Psi_R.$$  \hspace{1cm} (7.94)

We focus on only the $X_l X_R^2$ term, and consider the supervariation

$$\delta^{(3)} X_l \sim f^{(15)} L^{(0)} L^{(0)} L^{(0)} \chi_L^8 + f^{(14)} L^{(0)} L^{(0)} L^{(0)} \chi_L^8,$$  \hspace{1cm} (7.95)

where the $\SU(2)$ indices on the new couplings have been assigned anticipating the answer. Note that the $f^{(16)}$ term is non-covariant, and $X_L \cdot X_R \equiv (\sigma^0 X_L^1)(\sigma^0 X_R^1)$. Now let us consider the contributions from $\delta^{(3)} (X_L^2 X_R^2)$ given (7.95). The contribution of the $f^{(16)}$ term is of the form required to extend $\theta \rightarrow D$ in (7.69), and thus $f^{(16)} \sim g^{(16)}$. For the subset of terms we are considering, we do not expect the $f^{(14)}$ term (as well as similar terms) to contribute to the final answer because of (7.93).\(^{25}\) Thus, we get that

$$\delta^{(3)} (X_L^2 X_R^2) \sim (f^{(15)} + g^{(16)}) \epsilon L^{(0)} L^{(0)} L^{(0)} \chi_L^8.$$  \hspace{1cm} (7.96)

To get more information, we now impose the constraint of closure of the superalgebra on $X_L$ which gives, among other things, the $X_R$ equation of motion. We get that

$$\left[ \delta^{(0)}_1, \delta^{(3)}_1 \right] X_L \sim \left( D^{(15)} f^{(15)} \epsilon L^{(0)} L^{(0)} L^{(0)} \chi_L^8 + f^{(14)} \epsilon L^{(0)} L^{(0)} L^{(0)} \chi_L^8 \right.$$  \hspace{1cm} (7.97)

which is not expected to contribute because of (7.93).\(^{26}\) The terms in the first line of (7.97), and (7.98) arise in an obvious way, while the terms in the second line of (7.97) involving $f^{(16)}$ and $f^{(15)}$ are there to provide the correct coefficients to extend $\theta \rightarrow D$, while the $f^{(14)}$ term must arise in a suitable way that all the terms in the second line of (7.97) can be represented as $\delta^{(3)}_L X_L$ for $\hat{\theta} \sim \epsilon L^{(0)} L^{(0)} L^{(0)} \chi_L$. Thus, this entire contribution in the closure can be absorbed as a local symmetry transformation.

Along with

$$\left[ \delta^{(0)}_1, \delta^{(3)}_2 \right] X_L \sim \epsilon L^{(0)} L^{(0)} L^{(0)} \chi_L.$$  \hspace{1cm} (7.100)

(7.97) leads to the equation of motion

$$\hat{\theta} \chi_L + \left( D^{(15)} f^{(15)} \epsilon L^{(0)} L^{(0)} L^{(0)} \chi_L^8 + f^{(14)} \epsilon L^{(0)} L^{(0)} L^{(0)} \chi_L^8 + \cdots \right) = 0,$$  \hspace{1cm} (7.101)

leading to

$$\left( D^{(15)} f^{(15)} \right) \sim \hat{g}^{(15)} + f^{(14)},$$  \hspace{1cm} (7.102)

using the terms in the action. On the other hand, (7.98) would have yielded a term in the action which cannot exist, and hence the total contribution must vanish. Thus, (7.102) yields $f^{(15)} \sim g^{(15)}$, $f^{(14)} \sim g^{(14)}$ on using (7.84) self-consistently, and so from (7.96) we see that the structure of (7.69) is unchanged, with the coefficients receiving corrections. We expect this analysis to go through for the other couplings.

\(^{25}\) We will see below that there can be other terms that can contribute to (7.92), and then such terms will contribute. For example, a non-covariant $f^{(14)}$ term in (7.95) is needed to send $\theta f^{(14)} \rightarrow D f^{(14)}$.

\(^{26}\) Such a contribution vanishes because one has 16 pairs of indices to contract at the end, while this gives 15, the remaining pair gives 0 on tracing.
Thus, we are left with the set of equations (7.92), which can be analyzed further to give Poisson equations on moduli space for every coupling. The source terms in each equation for a specific coupling are given by some other couplings in the same supermultiplet. The structure of the source terms, however, is quite complicated as can be seen from (7.92).

Are there corrections to (7.92)? We now address this issue because so far we have only looked at the minimal set of terms needed to get this structure. There are other terms that can possibly contribute to (7.92). Looked at the minimal set of terms needed to get this structure. There are other terms that can be added on the right-hand side of (7.92) which on iteration shows that each coupling satisfies Poisson equation on moduli space, where the source terms are complicated.

Let us analyze in detail the two equations which lead to the Poisson equation for \( g^{(0)} \). From (7.92), we get that

\[
D_{(ij)}^{(0)}g_{(ij)}^{(0)} = \lambda_1 g_{(ij)}^{(1)} + \lambda_2 g_{(ij)}^{(2)},
\]

which leads to

\[
\Delta g^{(0)} = \mu g^{(0)},
\]

where \( \lambda_1 = 12a_1a_3 \) and \( \lambda_2 = a_1a_2 \). Thus, the coupling \( g^{(0)} \), which is the coefficient of the \( R^4 \) interaction satisfies (7.104), which is completely determined by the constraints of supersymmetry.

However, our analysis based on supersymmetry is not strong enough to solve (7.104) for \( g^{(0)} \), because of the presence of the unknown source term. Now, we simply make some plausible arguments to determine \( g^{(0)} \). We have that \( \Delta g^{(0)} \) and \( g^{(0)} \) are both \( SL(3, \mathbb{Z}) \) invariant automorphic forms, and thus so is \( g^{(0)} \). The tree level amplitude which contributes to \( g^{(0)} \) is known to be proportional to \( \zeta(3) e^{-2\phi} \). Because \( \zeta(3) \) is not factorizable, it is plausible that \( \lambda_2 = 0 \), or if non-zero, then \( g^{(2)} \sim g^{(0)} \) itself. In that case, \( g^{(0)} \) satisfies the Laplace equation

\[
\Delta g^{(0)} = \mu g^{(0)}.
\]

Based on the boundary conditions, \( g^{(0)} \) is therefore uniquely given by an Eisenstein series defined by (G.17) for some choice of \( s \). From the tree level amplitude, we see that \( s = 3/2 \) and so \( \mu = 0 \) [15]. However, as discussed in appendix G.2, \( E_{3/2}(M) \) is divergent and has to be regularized.

We should mention that the generalized derivative \( D_{(ij)}^{(n)} \) has appeared in the supervariations, and consequently, in equations (7.92). It is clear from the way they appear that the value of \( n \) gives the \( SU(2) \) spin of the interaction. However, since the couplings involved are not automorphic forms of \( SL(3, \mathbb{Z}) \), we do not understand its role, if any, as some covariant derivative of the \( U \)-duality group.

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27 This factorizability has been crucial for the \( D^4R^4 \) coupling in ten dimensions [12]. The coupling, call it \( f \), satisfies

\[
\Delta f = 12f - 6E_3, \tag{7.105}
\]

on the fundamental domain of \( SL(2, \mathbb{Z}) \). Here, the tree level contribution \( \sim \zeta(3)f^2 \), which also matches that from \( E_{3/2} \), using (G.4).
7.3. A coupling involving all the moduli

As discussed before, a coupling which involves all the moduli is the $\mathcal{R}^4$ coupling. This is given by [15]

$$\hat{E}_{3/2}(M) + 2\hat{E}_1(U, \bar{U}).$$  \hspace{1cm} (7.107)

The relative coefficient between the two terms is fixed to satisfy the $U \leftrightarrow T$ symmetry of the perturbative part of the amplitude, which interchanges the type IIA and type IIB theories.

7.4. Some plausible generalizations

Although the calculations are complicated and it is difficult to fix the coefficients, the structure of the first order differential equations that emerge as a consequence of supersymmetry is quite simple. It suggests that this procedure should be generalizable to lower dimensions, for example, to $N = 8, d = 4$ supergravity, where the classical moduli space is $(SU(8)/\mathbb{Z}_2) \backslash E_{7(7)}(\mathbb{R})$, and the U-duality group is $E_{7(7)}(\mathbb{Z})$. In that case, in order to construct the relevant 16 fermion terms in the effective action, one should again use the linearized superspace. A candidate 1/2 BPS superaction is given by

$$\int d^4x \int d^{16}\theta f(W),$$  \hspace{1cm} (7.108)

where the fermionic integral over the chiral part of the superspace is given by

$$\int d^{16}\theta \equiv \int \epsilon_{i_1,\ldots,i_{16}}\epsilon_{j_1,\ldots,j_{16}}(d\theta^{i_1\alpha_1}\ldots d\theta^{i_{16}\alpha_{16}})(d\bar{\theta}_{j_1}^{\alpha_1}\ldots d\bar{\theta}_{j_{16}}^{\alpha_{16}}).$$  \hspace{1cm} (7.109)

In (7.109), $\theta_i$ is in the 8 of $SU(8)$, and we have used the two-component chiral spinor notation. In (7.108), the chiral superfield $W_{ij\ell}$ which satisfies [37, 38]

$$W_{ij\ell} = \frac{1}{4!}\epsilon^{ij\ell mno}W_{mno}$$  \hspace{1cm} (7.110)

is in the 70 of $SU(8)$ and is given by [39]

$$W_{ij\ell} = \phi_{ij\ell} + \theta_{\ell}\chi_{ij\ell} + (\theta_{\ell}\sigma^{\mu\nu}\theta_{\mu})(\bar{\psi}_{\ell}^{m}\delta_{\ell}\sigma_{\mu\nu}\chi_{m}) + \cdots.$$  \hspace{1cm} (7.111)

In (7.111), the unitary gauge has been used and so $\phi_{ij\ell}$ which also satisfies (7.110) are the 70 scalars, and the spin 1/2 fermions $\chi_{ij\ell}$ and the gravitini $\bar{\psi}_{ij}$ are in the 56 and 8 of $SU(8)$, respectively. Thus, from (7.109) one immediately obtains the $\chi^16$ and $\bar{\psi}\chi^{15}$ interactions in $S^{31}$, which should be the starting point of the analysis as we have done. Decompactifying the degrees of freedom in the couplings to higher dimensions must produce the couplings in the higher dimensions as well.

In fact we can see very schematically what structure to expect. The interactions $\chi^16$ and $\bar{\psi}\chi^{15}$ will involve the tensor product of 16 and 15 of the 70 representations of $SU(8)$. As we have done in the $d = 8$ case, we can project onto the completely symmetric product among the irreducible representations that arise in the tensor product. Thus, the $\chi^16$ and $\bar{\psi}\chi^{15}$ interactions are in the 715, 536, 058, 545 and 313, 203, 587, 004 representations of $SU(8)$, respectively, and we can carry out the analysis. Thus, given the structure of the interactions, the generalized derivative $D_{(i,j,k)}$ in the 70 of $SU(8)$ should act as

$$D_{(i,j,k)} : \mathcal{A}_{[mno]} \rightarrow \mathcal{A}_{[i,j,k],[mno]} + \mathcal{A}\delta_{m\delta_{jn}}\delta_{k\delta_{lp}},$$  \hspace{1cm} (7.112)

where the symmetrization is implicit. This is enough to lead to a set of equations like (7.92), along with corrections of the type discussed before. Thus, we get a repetition of the structure we have for $d = 8$, and this should be true for $d = 5, 6, 7$.

Among the theories where the classical moduli space involves finite-dimensional groups, the only other case is $N = 16, d = 3$, where the moduli space is $SO(16)\backslash E_{6(6)}(\mathbb{R})$, and the
The entire degrees of freedom are contained in the 128 scalars and the 128 Majorana spin 1/2 particles [40], where they are in the two inequivalent spinor representations of $SO(16)$. The $\mathcal{R}^4$ interaction vanishes in $d = 3$ because it only involves the Weyl tensor which follows from perturbative string computations for maximal supersymmetry in any dimension. It is plausible that all the interactions of the type we have considered which result from the supermultiplet vanish, and there is nothing to consider.

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**Appendix A. Fierz transformations and various gamma matrix identities**

We need Fierz transformations involving eight-dimensional chiral fermions in our calculations. They are given by

\[
\begin{align*}
\chi L \bar{\lambda} R \beta &= -\frac{i}{2} \delta_{\alpha \beta} (\bar{\chi} \bar{R} \lambda L) - \frac{1}{16} (\gamma^{\mu \nu})_{\alpha \beta} (\bar{\chi} \bar{R} Y_{\mu \nu} \lambda L) - \frac{1}{16} (\gamma^{\mu \nu \lambda \rho})_{\alpha \beta} (\bar{\chi} \bar{R} Y_{\mu \nu \lambda \rho} \lambda L), \\
\chi L \bar{\sigma} L \beta &= -\frac{i}{2} \delta_{\alpha \beta} (\bar{\chi} \bar{R} \lambda L) - \frac{1}{16} (\gamma^{\mu \nu})_{\alpha \beta} (\bar{\chi} \bar{L} Y_{\mu \nu} \lambda R) - \frac{1}{16} (\gamma^{\mu \nu \lambda \rho})_{\alpha \beta} (\bar{\chi} \bar{L} Y_{\mu \nu \lambda \rho} \lambda R), \\
\chi R \bar{\gamma} R \beta &= -\frac{i}{2} \delta_{\alpha \beta} (\bar{\chi} \bar{L} \lambda R) - \frac{1}{16} (\gamma^{\mu \nu})_{\alpha \beta} (\bar{\chi} \bar{L} Y_{\mu \nu} \lambda R) - \frac{1}{16} (\gamma^{\mu \nu \lambda \rho})_{\alpha \beta} (\bar{\chi} \bar{L} Y_{\mu \nu \lambda \rho} \lambda R), \\
\chi R \bar{\delta} R \beta &= -\frac{i}{2} (\bar{\gamma}^{\mu})_{\alpha \beta} (\bar{\chi} \bar{L} \lambda R) + \frac{1}{16} (\bar{\gamma}^{\mu \nu \lambda})_{\alpha \beta} (\bar{\chi} \bar{L} Y_{\mu \nu \lambda} \lambda R),
\end{align*}
\]

where $\alpha, \beta = 1, \ldots, 8$ are spinor indices.

We also make use of the relations involving the gamma matrices

\[
\begin{align*}
\gamma^{\mu \nu \lambda \rho} Y_{\mu \nu \lambda \rho} &= 1680, & \gamma^{\mu \nu \lambda \rho} Y_{\sigma \theta \nu \lambda} &= -240 \gamma_{\sigma \theta}, & \gamma^{\mu \nu \lambda \rho} \gamma_{\sigma \theta \kappa \lambda} &= 144 \gamma_{\sigma \theta \kappa}, \\
\gamma^{\mu \nu \rho} \bar{\eta}_{\mu \nu \rho} &= -336, & \gamma^{\mu \nu \rho} \bar{\eta}_{\nu \mu \rho} &= 84 \gamma_{\nu}, & \gamma^{\mu \nu \rho} \bar{\eta}_{\mu \nu \rho} &= 24 \gamma_{\nu}, \\
\gamma^{\mu \nu \rho} \bar{\eta}_{\lambda \nu \rho} &= -36 \gamma_{\lambda \nu}, & \gamma^{\mu \nu \rho} \bar{\eta}_{\lambda \mu \rho} &= 0, & \gamma^{\mu \nu} \eta_{\lambda \nu} &= -56, & \gamma^{\mu \nu} \eta_{\lambda \nu} &= -28 \gamma_{\lambda}, \\
\gamma^{\mu \nu \lambda \rho} Y_{\mu \nu \lambda \rho} &= 8 \gamma_{\rho}, & \gamma^{\mu \nu \lambda \rho} Y_{\mu \nu \rho} &= 4 \gamma_{\nu \rho}, & \gamma^{\mu \nu} Y_{\lambda \rho \sigma} &= 8 \gamma_{\lambda \rho \sigma}, & \gamma^{\mu \nu} \bar{\gamma}_{\mu} &= 8, \\
\gamma^{\mu \nu} \bar{\gamma}_{\nu} &= -6 \gamma_{\nu}, & \gamma^{\mu \nu} \bar{\gamma}_{\nu} &= 4 \gamma_{\nu}, & \gamma^{\mu \nu} \bar{\gamma}_{\lambda \rho} &= -2 \gamma_{\lambda \rho}, & \gamma^{\mu \nu} \bar{\gamma}_{\lambda \rho \sigma} &= 0. \quad (A.2)
\end{align*}
\]

**Appendix B. The fermions from $d = 10$**

Let us now express the fermionic degrees of freedom in terms of the ten-dimensional fermions of type IIB supergravity. In order to do so, it is sufficient to consider the covariant derivatives in (2.83), and look at the $U(1)$ charges carried by the various fields, which can be calculated using (4.11). For this purpose, instead of the constrained field $X^A_{\mu L}$, it is more convenient to consider the unconstrained field $\eta_{\lambda A}$ ($A = 1, 2, 3, 4$) in the 4 of $SU(2)$. Thus, under $U(1) \backslash SL(2, \mathbb{R})$, we have that

\[
\begin{align*}
D_{\mu} \psi_{\nu L} &= D_{\mu} \psi_{\nu L} + \frac{i}{2} Q_{\mu}^{\sigma} \sigma^1 \psi_{\nu L} + \cdots, \\
D_{\mu} \lambda_{\nu L} &= D_{\mu} \lambda_{\nu L} + \frac{i}{2} Q_{\mu}^{\sigma} \sigma^1 \lambda_{\nu L} + \cdots, \\
D_{\mu} \eta_{\nu L} &= D_{\mu} \eta_{\nu L} + \frac{i}{2} Q_{\mu}^{T} T^1 \eta_{\nu L} + \cdots, \quad (B.1)
\end{align*}
\]
where
\[
T^1 = \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
\sqrt{3} & 0 & 2 & 0 \\
0 & 2 & 0 & \sqrt{3} \\
0 & 0 & \sqrt{3} & 0 \\
\end{pmatrix}
\] (B.2)
and
\[
Q^r_\mu = -\frac{\partial_{\mu} \tau_1}{2\tau_2}. 
\] (B.3)
Thus, we see that the combinations
\[
\Omega_1 = \left\{ \psi_{\mu L} - \psi_{\mu R}, \lambda_{1L} - \lambda_{2L}, \eta_{1L} - \eta_{2L} - \frac{\eta_{3L} + \eta_{4L}}{\sqrt{3}} \right\} 
\] (B.4)
have \(U(1)\) charge 1/2, while the combination
\[
\Omega_2 = -\eta_{1L} + \sqrt{3}\eta_{2L} - \sqrt{3}\eta_{3L} + \eta_{4L}
\] (B.5)
has \(U(1)\) charge 3/2. Also the combinations
\[
\Omega_3 = \left\{ \psi_{\mu L} + \psi_{\mu R}, \lambda_{1L} + \lambda_{2L}, -\eta_{1L} + \frac{\eta_{2L}}{\sqrt{3}} + \frac{\eta_{3L}}{\sqrt{3}} + \eta_{4L} \right\} 
\] (B.6)
and
\[
\Omega_4 = \eta_{1L} + \sqrt{3}\eta_{2L} + \sqrt{3}\eta_{3L} + \eta_{4L}
\] (B.7)
carry \(U(1)\) charges \(-1/2\) and \(-3/2\), respectively. So, \(\{\Omega_1, \Omega_4\}\) descend from the \(d = 10\) gravitino \(\Psi_\mu\) which has \(U(1)\) charge 1/2, while \(\{\Omega_2, \Omega_3\}\) descend from the \(d = 10\) dilatino \(\hat{\lambda}\) which has \(U(1)\) charge 3/2. Of course, this precise decomposition of the degrees of freedom depends on the choice of gauge. However, from the gauge covariance of the equations, it follows that the degrees of freedom in \(\psi_{\mu L} (160), \lambda_L (32)\) and half of those in \(\chi_L (32)\) descend from the gravitino (224), while the remaining half of the degrees of freedom in \(\chi_L (32)\) descend from the dilatino (32). Thus, the fermionic degrees of freedom intermingle in a complicated way.

Intuitively, one can also deduce it from the supersymmetry transformations (C.1). It follows that \(\lambda_L\) (because its supervariation involves the \(U(1)\) \(\text{SL}(2, \mathbb{R})\) moduli), \(\psi_{\mu L}\) and a part of \(\chi_L\) (because its supervariation involves the \(SU(2)\) \(\text{SL}(3, \mathbb{R})\) moduli) descend from the gravitino, while the remaining part of \(\chi_L\) descends from the dilatino. The rest follows from simply counting the degrees of freedom.

One can also calculate the \(U(1)_T\) charges of the various fields in (2.83) using (4.12) in a straightforward manner.

Appendix C. Supersymmetry transformations

At the two-derivative level, the supersymmetry transformations of the various fields of \(N = 2, d = 8\) supergravity are given by
\[
\delta^{(0)} e_\mu^a = - (\bar{e}_L T^a e_{\mu L} + \bar{e}_R T^a e_{\mu R}),
\]
\[
\delta^{(0)} L_+^U = L_+^U \bar{e}_R \lambda_L,
\]
\[
\delta^{(0)} L_-^U = L_-^U \bar{e}_L \lambda_R,
\]
\[
L_+^a \delta^{(0)} L_{mj} = -\frac{1}{2} \{ \bar{e}_L (\sigma_i x_{jr} + \sigma_j x_{ir}) - \bar{e}_R (\sigma_i x_{jl} + \sigma_j x_{il}) \}.
\]
\[ \delta^{(0)} A_{\mu}^{ml} = \sqrt{2} L^+ \left[ L^l \left( \bar{\epsilon}_{RL} \sigma^i \psi_{\mu L} + \frac{1}{2} \bar{\epsilon}_{RL} \tilde{\gamma}_i \sigma^\lambda L - \bar{\epsilon}_{RY} \tilde{\chi}_R \right) \right. \\
\left. - L^l \left( \bar{\epsilon}_{RL} \sigma^i \psi_{\mu R} + \frac{1}{2} \bar{\epsilon}_{RY} \sigma^\lambda R + \bar{\epsilon}_{LY} \tilde{\chi}_L \right) \right]. \]

\[ \delta^{(0)} B_{\mu v m} = -\epsilon_{mn\rho} (\delta^{(0)} A_{\mu}^{ml}) A_{\rho v} + 2 L^m (\bar{\epsilon}_{RL} \sigma^i \tilde{\gamma}_i \psi_{\mu L} - \bar{\epsilon}_{RY} \sigma^i \psi_{\mu R}) \\
- L^m (\bar{\epsilon}_{RL} \tilde{\gamma}_i \psi_{\mu L} + \bar{\epsilon}_{RY} \tilde{\gamma}_i \psi_{\mu R}). \]

\[ \delta^{(0)} C_{\mu v} = -3 (\delta^{(0)} A_{\mu}^{ml}) (B_{v \mu m} - \epsilon_{mn\rho} A_{\rho v}^{A^1}) - 3 \sqrt{2} L^l \left( \bar{\epsilon}_{RY} \sigma^i \psi_{\mu L} - \frac{1}{6} \bar{\epsilon}_{LY} \tilde{\chi}_R \right) \\
- 3 \sqrt{2} L^l \left( \bar{\epsilon}_{LY} \psi_{\mu L} - \frac{1}{6} \bar{\epsilon}_{RY} \psi_{\mu R} \right). \]

\[ \delta^{(0)} \lambda_{L} = i \gamma^\mu \bar{\psi}_{\mu} \epsilon_{RL} - \frac{i}{4 \sqrt{2}} \hat{F}_{2\mu v} \gamma^\mu \sigma^i \epsilon_{L} - \frac{i}{96 \sqrt{2}} \hat{F}_{\mu \nu \lambda \rho} \gamma^{\mu \nu \lambda \rho} \epsilon_{L} \\
+ \frac{1}{6} \epsilon_{\gamma_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \chi_{RL}) + \frac{1}{3} \bar{\gamma}_{RL} \chi_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \sigma^\lambda L) - \frac{1}{12} \sigma_{\lambda} \lambda (\bar{\epsilon}_{RL} \chi_{RL}) \\
- \frac{1}{2} \epsilon_{\epsilon_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) - \frac{1}{24} \gamma^\mu \sigma_{\epsilon_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{3} \bar{\epsilon}_{RL} \psi_{\mu L} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) + \frac{1}{12} \sigma_{\lambda} \lambda (\bar{\epsilon}_{RL} \chi_{RL}) \\
+ \frac{1}{8} \epsilon_{R} (\bar{\epsilon}_{RL} \chi_{RL} - \frac{1}{2} \epsilon_{R} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \mu \lambda) - \frac{1}{2} \epsilon_{R} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{16} \gamma^\mu \sigma_{\epsilon_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}). \]

\[ \delta^{(0)} \lambda_{R} = -i \gamma^\mu \bar{\psi}_{\mu} \epsilon_{RL} - \frac{i}{4 \sqrt{2}} \hat{F}_{2\nu \mu} \gamma^\mu \sigma^i \epsilon_{R} + \frac{i}{96 \sqrt{2}} \hat{F}_{\mu \nu \lambda \rho} \gamma^{\mu \nu \lambda \rho} \epsilon_{R} \\
- \frac{1}{6} \epsilon_{\gamma_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \chi_{RL}) - \frac{1}{3} \bar{\gamma}_{RL} \chi_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \sigma^\lambda L) + \frac{1}{12} \sigma_{\lambda} \lambda (\bar{\epsilon}_{RL} \chi_{RL}) \\
+ \frac{1}{2} \epsilon_{\epsilon_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) + \frac{1}{24} \gamma^\mu \sigma_{\epsilon_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) - \frac{1}{3} \bar{\epsilon}_{RL} \psi_{\mu L} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) + \frac{1}{12} \sigma_{\lambda} \lambda (\bar{\epsilon}_{RL} \chi_{RL}) \\
- \frac{1}{8} \epsilon_{R} (\bar{\epsilon}_{RL} \chi_{RL} - \frac{1}{2} \epsilon_{R} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \mu \lambda) - \frac{1}{2} \epsilon_{R} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{16} \gamma^\mu \sigma_{\epsilon_{RL}} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}). \]

\[ \delta^{(0)} \chi_{R} = -\frac{1}{2} \gamma^\mu \bar{\psi}_{\mu} \chi_{R} - \frac{i}{4 \sqrt{2}} \hat{F}_{2\nu \mu} \gamma^\mu \sigma^i \epsilon_{R} - \frac{i}{96 \sqrt{2}} \hat{F}_{\mu \nu \lambda \rho} \gamma^{\mu \nu \lambda \rho} \epsilon_{R} - \frac{1}{2} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \chi_{RL}) \\
+ \frac{1}{16} \Delta_{ij} \gamma^\mu \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{2} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) - \frac{1}{12} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) \\
- \frac{1}{16} \Delta_{ij} \gamma^\mu \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{2} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) - \frac{1}{12} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) \\
+ \frac{1}{4} \Delta_{ij} \gamma^\mu \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{4} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}). \]

\[ \delta^{(0)} \chi_{R} = -\frac{1}{2} \gamma^\mu \bar{\psi}_{\mu} \chi_{R} - \frac{i}{4 \sqrt{2}} \hat{F}_{2\nu \mu} \gamma^\mu \sigma^i \epsilon_{R} - \frac{i}{96 \sqrt{2}} \hat{F}_{\mu \nu \lambda \rho} \gamma^{\mu \nu \lambda \rho} \epsilon_{R} - \frac{1}{2} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \chi_{RL}) \\
+ \frac{1}{16} \Delta_{ij} \gamma^\mu \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{2} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) - \frac{1}{12} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) \\
- \frac{1}{16} \Delta_{ij} \gamma^\mu \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{2} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \tilde{\gamma}_i \lambda) - \frac{1}{12} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) \\
+ \frac{1}{4} \Delta_{ij} \gamma^\mu \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}) + \frac{1}{4} \Delta_{ij} \epsilon_{RL} (\bar{\epsilon}_{RL} \psi_{\mu L}). \]
\[
\delta^{(0)} \psi_{\mu L} = D_{\mu} e_{\rho} + \frac{i}{24\sqrt{2}} \tilde{F}^{\mu \nu \rho} \sigma^i (\gamma_{\mu \nu \chi L} - 10 g_{\mu \nu} \gamma_{\chi L} e_{\rho} + \frac{1}{72} \tilde{F}^{\nu \rho \lambda} \sigma^i (\gamma_{\nu \rho \lambda} - 6 g_{\nu \rho} \gamma_{\lambda L} e_{\mu L})
- \frac{i}{192 \sqrt{2}} \tilde{F}^{\nu \rho \lambda} \sigma^i \gamma_{\nu \rho \lambda} e_{\mu L} - \frac{1}{18} (8 g_{\mu \nu} - \gamma_{\mu \nu}) \lambda_{\sigma \lambda L} (\tilde{e}_L \tilde{\gamma}^\nu \chi L + \chi_L (\tilde{e}_L \tilde{\gamma}^\nu \sigma \lambda L))
+ \frac{1}{72} \left[ \gamma^{\nu} (\tilde{e}_R \gamma_{\nu \mu} \lambda_L) + \frac{1}{2} \gamma_{\nu \mu \nu} (\tilde{e}_R \tilde{\gamma}^\nu \nu \lambda_L) \right] \sigma \epsilon_R
+ \frac{1}{6} \left[ \epsilon_R (\tilde{\psi}_{\mu R} \chi L) - \tilde{\psi}_{\mu L} (\tilde{e}_R \chi L - \tilde{e}_L \chi R) \right] - \chi_R (\tilde{e}_R \sigma \epsilon_R)
- \frac{i}{2} Y^\nu (\tilde{e}_R \gamma_{\nu \mu} \lambda_L) + \frac{1}{6} \left[ (\tilde{e}_L \gamma_{\nu \mu}) \gamma_R + (\tilde{e}_L \chi R) \right] \gamma_R e_R - \frac{1}{48} \gamma_{\mu R} (\tilde{e}_R \chi R)
+ \frac{1}{6} (5 g_{\mu \nu} - \gamma_{\mu \nu}) \chi_L (\tilde{e}_R \gamma^\nu \chi L) + \frac{1}{6} \left[ (\tilde{e}_R \gamma_{\nu}) \gamma_R - \gamma_{\mu R} (\tilde{e}_R \chi_R) \sigma \lambda L \right] \sigma \epsilon_L
- \frac{1}{6} \left[ \sigma R (\tilde{e}_R \gamma_{\nu \mu}) + (\tilde{e}_R \gamma_{\nu \mu}) \chi_L \right] \gamma_R e_R - \frac{1}{27} \gamma_{\mu R} (\tilde{e}_R \sigma \epsilon_R)
- \frac{1}{54} (11 g_{\mu \nu} - \gamma_{\mu \nu}) \lambda_{\sigma \lambda L} (\tilde{e}_R \gamma_{\nu \mu}) + \frac{1}{54} \left[ (5 \lambda_R \gamma_{\mu}) \sigma \lambda L + \gamma_{\nu R} (\tilde{e}_R \gamma^\nu \sigma \lambda L) \right] \sigma \epsilon_L
- \frac{1}{12} (\tilde{e}_R \gamma_{\nu \mu}) + \frac{\gamma_{\nu R}}{12} (\tilde{e}_R \gamma_{\nu \mu} \lambda_R) \lambda_L + \frac{1}{12} g_{\mu \nu} (\tilde{e}_R \gamma_{\mu \lambda L} - \frac{1}{3} (\tilde{e}_R \gamma_{\nu \lambda L}) \gamma^\nu \lambda R
- \frac{1}{6} \left[ \sigma R \gamma_{\nu \mu} (\tilde{e}_R \gamma^\nu \sigma \lambda L) - \gamma_{\nu R} (\tilde{e}_R \chi_R) \gamma^\nu \sigma \lambda L \right] + \frac{1}{3} \sigma \lambda L (\tilde{e}_R \sigma \epsilon_R)
\delta^{(0)} \psi_{\mu R} = D_{\mu} e_R + \frac{i}{24 \sqrt{2}} \tilde{F}^{\mu \nu \rho} \sigma^i (\tilde{\gamma}_{\mu \nu \chi R} - 10 g_{\mu \nu} \gamma_{\chi R} e_R + \frac{1}{72} \tilde{F}^{\nu \rho \lambda} \sigma^i (\gamma_{\nu \rho \lambda} - 6 g_{\nu \rho} \gamma_{\lambda R} e_R)
+ \frac{i}{192 \sqrt{2}} \tilde{F}^{\nu \rho \lambda} \sigma^i \gamma_{\nu \rho \lambda} e_R + \frac{1}{18} (8 g_{\mu \nu} - \gamma_{\mu \nu}) \lambda_{\sigma \lambda R} (\tilde{e}_L \tilde{\gamma}^\nu \chi R + \chi_R (\tilde{e}_L \tilde{\gamma}^\nu \sigma \lambda R))
+ \frac{1}{72} \left[ \gamma^{\nu} (\tilde{e}_R \gamma_{\nu \mu} \lambda R) + \frac{1}{2} \gamma_{\nu \mu \nu} (\tilde{e}_R \tilde{\gamma}^\nu \nu \lambda R) \right] \sigma \epsilon_R
- \frac{1}{6} \left[ \epsilon_R (\tilde{\psi}_{\mu R} \chi R - \tilde{\psi}_{\mu L} (\tilde{e}_R \chi R - \tilde{e}_L \chi L) \right] - \chi_L (\tilde{e}_R \sigma \epsilon_R)
+ \frac{i}{2} Y^\nu (\tilde{e}_R \gamma_{\nu \mu} \lambda R) + \frac{1}{6} \left[ (\tilde{e}_L \gamma_{\nu \mu}) \gamma_R + (\tilde{e}_L \chi R) \right] \gamma_R e_R - \frac{1}{48} \gamma_{\mu R} (\tilde{e}_R \chi R)
+ \frac{1}{6} (5 g_{\mu \nu} - \gamma_{\mu \nu}) \chi_R (\tilde{e}_R \gamma^\nu \chi R) + \frac{1}{6} \left[ (\tilde{e}_R \gamma_{\nu}) \gamma_R - \gamma_{\mu R} (\tilde{e}_R \chi_R) \sigma \lambda R \right] \sigma \epsilon_R
- \frac{1}{6} \left[ \sigma R \gamma_{\nu \mu} (\tilde{e}_R \gamma^\nu \sigma \lambda R) - \gamma_{\nu R} (\tilde{e}_R \chi_R) \gamma^\nu \sigma \lambda R \right] + \frac{1}{3} \sigma \lambda R (\tilde{e}_R \sigma \epsilon_R)
\]
where
\[ \Delta_{ij} = \delta_{ij} - \frac{1}{2} \sigma_i \sigma_j, \]
and the various supercovariant expressions are defined below. We also have that
\[ D_\mu \epsilon_L = D_\mu (\hat{\omega}) \epsilon_L - \frac{i}{2} \hat{Q}_\mu \epsilon_L - \frac{i}{2} A_\mu' \sigma' \epsilon_L, \]
\[ D_\mu \epsilon_R = D_\mu (\hat{\omega}) \epsilon_R + \frac{i}{2} \hat{Q}_\mu \epsilon_R - \frac{i}{2} A_\mu' \sigma' \epsilon_R, \]
where
\[ D_\mu (\hat{\omega}) \epsilon = (\partial_\mu + \frac{i}{2} \hat{\omega}^{ab} \Gamma_{ab}) \epsilon. \]  

In (C.4), \( \hat{\omega}^{ab}_\mu \) is the d = 8 supercovariant spin connection given by
\[ \hat{\omega}^{ab}_\mu = \omega^{ab}_\mu - \left( \psi_{\mu R} \gamma^a \psi^b_R + \bar{\psi}_{\mu L} \bar{\gamma}^a \psi^b_L + \bar{\psi}_R \gamma^a \psi_{\mu R}^b \right). \]

Note that in (C.1), the spacetime structure of the \( \hat{T}^\pm \) terms in the supervariation of the gravitini matches that in [24] on using the relation
\[ \Gamma_{\mu \nu \rho \sigma} = \Gamma_{\nu \rho \sigma \mu} + g_{\mu \nu} \Gamma_{\rho \sigma} - g_{\mu \nu} \Gamma_{\rho \sigma} - g_{\rho \sigma} \Gamma_{\nu \mu}. \]

At various places, we have used the identities
\[ \tilde{\psi}_R \chi_L = \bar{\chi}_R \psi_L, \]
\[ \tilde{\psi}_R Y_{\mu \nu} \chi_L = \bar{\chi}_R Y_{\nu \mu} \psi_L, \]
\[ \tilde{\chi}_R \sigma_i \psi_L = - \bar{\chi}_R \sigma_i \chi_L, \]
\[ \tilde{\psi}_L \tilde{\gamma}_i \chi_L = - \bar{\psi}_L \gamma_i \psi_R, \]
\[ \tilde{\psi}_L \tilde{\gamma}_{\mu \nu} \chi_L = - \bar{\psi}_L \tilde{\gamma}_{\mu \nu} \psi_R, \]
\[ \tilde{\psi}_R \tilde{\gamma}_{\mu \nu} \chi_L = - \bar{\psi}_R \gamma_{\mu \nu} \psi_L, \]
\[ \tilde{\psi}_R \tilde{\gamma}_{\nu \mu \rho} \chi_L = - \bar{\psi}_R \tilde{\gamma}_{\nu \mu \rho} \psi_L, \]
\[ \tilde{\psi}_L \tilde{\gamma}_{\nu \mu \rho} \chi_L = - \bar{\psi}_L \tilde{\gamma}_{\nu \mu \rho} \psi_R, \]
\[ \tilde{\psi}_R \tilde{\gamma}_{\mu \nu \rho \sigma} \chi_L = - \bar{\psi}_R \gamma_{\mu \nu \rho \sigma} \chi_L, \]
\[ \tilde{\psi}_L \tilde{\gamma}_{\mu \nu \rho \sigma} \chi_L = - \bar{\psi}_L \tilde{\gamma}_{\mu \nu \rho \sigma} \psi_R, \]
where \( \psi \) and \( \chi \) are arbitrary SU(2) pseudo-Majorana fermions in a chiral basis. We also make use of the relations
\[ (\tilde{\psi}_L \chi_L)^{\dagger} = \bar{\psi}_R \chi_L, \quad (\tilde{\psi}_L \tilde{\gamma}_i \chi_L)^{\dagger} = \bar{\psi}_R \gamma_i \chi_L, \]
\[ (\tilde{\psi}_L \tilde{\gamma}_{\mu \nu} \chi_L)^{\dagger} = \bar{\psi}_R \gamma_{\mu \nu} \chi_L, \quad (\tilde{\psi}_L \tilde{\gamma}_{\nu \mu \rho} \chi_L)^{\dagger} = \bar{\psi}_R \tilde{\gamma}_{\nu \mu \rho} \chi_L, \]
\[ (\tilde{\psi}_L \tilde{\gamma}_{\nu \mu \rho \sigma} \chi_L)^{\dagger} = \bar{\psi}_R \tilde{\gamma}_{\nu \mu \rho \sigma} \chi_L. \]

In obtaining the supersymmetry transformations, it is useful to note that the kinetic term for \( F^{\mu \nu}_{2 \mu \nu} \) can also be written as
\[ F^{i \mu \nu}_{2 \mu \nu} F^{* \mu \nu}_{2 \mu \nu} = \frac{1}{2} M_{\mu \nu} M_{\mu \nu} F^{B \mu \nu} F^{B \mu \nu}, \]
on using
\[ M_{\mu \nu} = \frac{1}{2} (\epsilon_{\mu \nu} \epsilon_{\rho \sigma} + \epsilon_{\rho \sigma} \epsilon_{\mu \nu}) (L^\mu_{+} L^\nu_{-} + L^\nu_{+} L^\mu_{-}). \]
Let us outline in brief how to fix the coefficients of the various terms involving the supercovariant expressions in the supersymmetry transformations of $\lambda_L$ in (C.1). The same method works for the other fermions as well. We use the relations
\[ \tilde{\gamma}^{\mu
u} \partial_\mu F_{2\nu}^{\rho} = 0, \]
\[ \tilde{\gamma}^{\mu
u} \partial_\mu F_{3\nu \lambda \rho} = \frac{1}{2} \epsilon_{\mu
u\rho} \epsilon_{UV} \tilde{\gamma}^{\nu\lambda} \partial_\mu F_{2^{\mu} U}^{\rho}, \]  
resulting from the Bianchi identities. They lead to the relations
\[ \tilde{\gamma}^{\mu} \gamma^{\nu\rho} \partial_\mu F_{2\nu}^{\rho} = 2 \tilde{\gamma}^{\nu\lambda} \partial_\mu F_{3\lambda \rho}^{\mu}, \]
\[ \tilde{\gamma}^{\mu} \gamma^{\nu\rho} \partial_\mu F_{3\nu \lambda \rho} = 3 \tilde{\gamma}^{\nu\lambda} \partial_\mu F_{4\nu \lambda \rho}^{\mu}, \]  
which are needed in canceling the supervariation of the fermion kinetic terms against the supervariation of the kinematic terms of the various gauge potentials, on using (C.7). Of course, all these terms directly follow from [24], and the above statements provide a cross check of the calculations directly in $d = 8$.

In order to derive the form of the various supercovariant expressions in (C.1), such that all the partial $\partial_\mu$ terms vanish in the supervariation, we note that
\[ \delta^{(0)} F_\mu = i(\partial_\mu \bar{\epsilon}_R) \lambda_L + \cdots, \]
\[ \delta^{(0)} F_{\mu j} = -\frac{i}{2} ((\partial_\mu \bar{\epsilon}_L) (\sigma_i \gamma_{jR} + \sigma_j \gamma_{iR}) - (\partial_\mu \bar{\epsilon}_R) (\sigma_i \gamma_{jL} + \sigma_j \gamma_{iL})) + \cdots, \]
\[ \delta^{(0)} F_{\mu i j} = 2 \sqrt{2} ((\partial_\mu \bar{\epsilon}_R) \sigma_i \gamma_{j} \lambda_L + \frac{1}{2} (\partial_\mu \bar{\epsilon}_L) \bar{\gamma}_i \gamma_j \lambda_R - (\partial_\mu \bar{\epsilon}_R) \gamma_i \gamma_j \lambda_R) + \cdots. \]

Thus, given the gravitino supervariation in (C.1), this leads to the supercovariant expressions given by
\[ \tilde{P}_\mu = P_\mu - i \bar{\psi}_R \lambda_L, \]
\[ \tilde{P}_\mu^\dagger = P_\mu^\dagger + i \bar{\psi}_R \lambda_L, \]
\[ \tilde{F}_{\mu j}^\dagger = F_{\mu j}^\dagger + (\psi_{\nu L} \sigma_{(j}) \gamma_{iR} - \bar{\psi}_R \sigma_{(j)} \gamma_{iL}), \]
\[ \tilde{F}_{\mu i j}^\dagger = F_{\mu i j}^\dagger + \sqrt{2} [\tilde{\psi}_{R[\mu} \sigma_i \gamma_{j]} \lambda_L + \tilde{\psi}_{R[\mu} \bar{\gamma}_i \gamma_j \lambda_R - 2 \tilde{\psi}_{R[\mu} \gamma_i \gamma_j \lambda_R], \]
\[ \tilde{F}_{\mu i j}^\dagger = F_{\mu i j}^\dagger + \sqrt{2} [\tilde{\psi}_{R[\mu} \sigma_i \gamma_{j]} \lambda_L + \tilde{\psi}_{R[\mu} \gamma_i \gamma_j \lambda_R + 2 \tilde{\psi}_{R[\mu} \bar{\gamma}_i \gamma_j \lambda_R], \]
\[ \tilde{F}_{\mu i j k l} = F_{\mu i j k l} + 3 (2 \tilde{\psi}_{R[\mu} \gamma_{i} \gamma_{j k} \lambda_L + \tilde{\psi}_{R[\mu} \bar{\gamma}_i \gamma_j \gamma_{k L} + \tilde{\psi}_{R[\mu} \gamma_i \gamma_{j k} \lambda_R) + \frac{1}{6} \bar{\psi}_{R} \gamma_{i j k l} \lambda_L, \]
\[ \tilde{F}_{\mu i j k l} = F_{\mu i j k l} + 6 \sqrt{2} \left[ L_{1}^{\dagger} \left( \tilde{\psi}_{R[\mu} \gamma_{i} \gamma_{j k l} \lambda_L - \frac{1}{3} \tilde{\psi}_{R[\mu} \bar{\gamma}_i \gamma_j \gamma_{k l} \lambda_R \right) \right] \]
\[ + L_{1}^{\dagger} \left( \tilde{\psi}_{R[\mu} \bar{\gamma}_i \gamma_j \gamma_{k l} \lambda_R - \frac{1}{3} \tilde{\psi}_{R[\mu} \gamma_i \gamma_{j k l} \lambda_R \right) \]
\[ = \tilde{F}_{\mu i j k l}^+ + \tilde{F}_{\mu i j k l}^-, \]
where
\[ \tilde{F}_{\mu i j k l}^+ = F_{\mu i j k l}^+ + \frac{1}{2 \sqrt{2}} \left[ L_{1}^{-1} (\tilde{\psi}_{R} \gamma_{i j k} \gamma_{l} \lambda_L + \tilde{\psi}_{R} \lambda_L \bar{\gamma}_{i j k} \lambda_R) \right. \]
\[ - L_{1}^{\dagger} (\tilde{\psi}_{R} \lambda_L \bar{\gamma}_{i j k} \lambda_R) + \tilde{\psi}_{R} \gamma_{i j k} \gamma_{l} \lambda_R \right]. \]
\[ \hat{F}_{\mu \nu \lambda \rho} = F_{\mu \nu \lambda \rho} + \frac{1}{2\sqrt{2}} [L^{-1}_+ (\bar{\psi}_\sigma L \tilde{\gamma}^0 \gamma_{\mu \nu \lambda \rho} \gamma^1 \psi_{TR} + \bar{\psi}_\sigma R \gamma_{\mu \nu \lambda \rho} \gamma^0 \lambda_R)] \]

\[-L^{-1}_- (\bar{\psi}_\sigma R \gamma_{\mu \nu \lambda \rho} \gamma^0 \gamma^1 \psi_{TL} + \bar{\psi}_\sigma L \tilde{\gamma}^0 \gamma_{\mu \nu \lambda \rho} \lambda_L)]. \quad (C.15)\]

In obtaining (C.15) from (C.14), we have used the identities:

\[\begin{align*}
\bar{\psi}_{R[\mu} \gamma_{\nu \rho \sigma]} \lambda_R &= \frac{i}{4} (\bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_L - \bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_L), \\
\bar{\psi}_{L[\mu} \gamma_{\nu \rho \sigma]} \lambda_L &= \frac{i}{4} (\bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_L - \bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_L), \\
\bar{\psi}_{R[\mu} \gamma_{\nu \rho \sigma]} \lambda_R &= \frac{i}{4} (\bar{\psi}_{R[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_R - \bar{\psi}_{R[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_R), \\
\bar{\psi}_{L[\mu} \gamma_{\nu \rho \sigma]} \lambda_R &= \frac{i}{4} (\bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_R - \bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_R), \\
\end{align*}\]

(C.16)

to decompose \( \hat{F}_5 \) into \( \hat{F}_{\pm} \).

Thus, we have that

\[\begin{align*}
\hat{F}^+_{\mu \nu \rho \sigma} &= -i L^+ \hat{T}^+_{\mu \nu \rho \sigma} + \frac{L^+}{\sqrt{2}} \hat{k}_{L} \bar{\gamma}_{\mu \nu \rho \sigma} \lambda^L, \\
\hat{F}^-_{\mu \nu \rho \sigma} &= i L^- \hat{T}^-_{\mu \nu \rho \sigma} + \frac{L^-}{\sqrt{2}} \hat{k}_{R} \gamma_{\mu \nu \rho \sigma} \lambda^R, \quad (C.17)\end{align*}\]

where

\[\begin{align*}
\hat{T}^+_{\mu \nu \rho \sigma} &= T^+_{\mu \nu \rho \sigma} - \frac{i}{\sqrt{2}} (\bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_R + \bar{\psi}_{R[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_R), \\
\hat{T}^-_{\mu \nu \rho \sigma} &= T^-_{\mu \nu \rho \sigma} + \frac{i}{\sqrt{2}} (\bar{\psi}_{R[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_L + \bar{\psi}_{L[\lambda} \gamma_{\mu \nu \rho \sigma]} \lambda_L). \quad (C.18)\end{align*}\]

are supercovariant field strengths.

Of course, the term not involving the gravitino in (C.14) is not fixed by this argument and one need not include it in the definition. We have, however, included it because the supersymmetry transformations look simpler\(^2\). The structure of these terms, in particular, the extra term that does not involve the gravitino in (C.14), follows naturally from dimensionally reducing the supercovariant 4-form field strength, and the supercovariant spin connection in \( d = 11 \) using (3.7) and (3.6), and inserting them in the supersymmetry transformation (3.9) for the \( d = 11 \) gravitino. The relevant expressions obtained from (3.7) are given by

\[\begin{align*}
\hat{G}^{SS}_{\mu \nu \lambda \rho} &= G_{\mu \nu \lambda \rho} + 3e^{-\phi_0} \left[ -\frac{1}{2} \bar{\psi}_{R[\mu} \gamma_{\nu \lambda \rho]} \lambda_R + \frac{1}{8} \bar{\psi}_{L[\mu} \gamma_{\nu \lambda \rho]} \lambda_L \right] \\
&\quad + \left( \bar{\psi}_{L[\mu} \gamma_{\nu \lambda \rho]} \lambda_R - \frac{1}{3} \bar{\psi}_{R[\mu} \gamma_{\nu \lambda \rho]} \lambda_R \right) - \frac{1}{36} (\lambda_R \gamma_{\mu \nu \lambda \rho} \lambda_L + \lambda_L \gamma_{\mu \nu \lambda \rho} \lambda_R), \\
\hat{G}^{SS}_{\mu \nu \rho \lambda} &= G_{\mu \nu \rho \lambda} + 3 \left( 2 \bar{\psi}_{L[\mu} \gamma_{\nu \rho \lambda]} \lambda_R + \bar{\psi}_{L[\mu} \gamma_{\nu \rho \lambda]} \lambda_R \right) + \frac{1}{12} (\lambda_R \gamma_{\mu \nu \rho} \lambda_L), \\
&\quad + \frac{1}{4} (\bar{\psi}_{R[\mu} \gamma_{\nu \rho \lambda]} \lambda_R + \bar{\psi}_{R[\mu} \gamma_{\nu \rho \lambda]} \lambda_R). \end{align*}\]

\(^2\) A previous example of such a definition involves the definition of \( \hat{F}_5 \) in type IIB supergravity in \( d = 10 \) [34], where there is an extra term that does not involve the gravitino.
\( \hat{G}_{\mu\nu}^{SS} = G_{\mu\nu}^{SS} + i e^{\phi_S} \left( \frac{1}{2} \left( \hat{\psi}_{RL} \sigma^i \psi_{e,R} + \hat{\psi}_{LR} \sigma^i \psi_{e,L} \right) - \left( \hat{\psi}_{RL} \gamma_5 \chi^R_{\mu} + \hat{\psi}_{LR} \sigma^i \chi_{\mu} \right) \right) + \frac{1}{2} \left( \hat{\psi}_{LJ} \sigma^i \lambda_L + \hat{\psi}_{JR} \gamma_5 \chi^R_{\mu} \right) + \frac{1}{4} \left( \tilde{\chi}_{JL} \gamma_\mu \sigma^i \chi_{JR} + \tilde{\chi}_{JL} \gamma_\mu \sigma^i \chi_{JR} \right) + \frac{1}{24} \left( \lambda_L \sigma^i \tilde{\chi}_{LJ} \lambda_R + \tilde{\chi}_{LJ} \gamma_\mu \sigma^i \chi_{JR} \right) \right]. 

\( \partial_{\mu} \hat{B}^{SS} = \partial_{\mu} B^{SS} + \frac{i}{2} e^{\phi_S} \left( \left( \hat{\psi}_{\mu L} \lambda_R - \hat{\psi}_{\mu R} \lambda_L \right) - \frac{1}{3} \tilde{\lambda}_R \gamma_\mu \lambda_R + \tilde{\chi}_{LR} \gamma_\mu \chi_{LR} \right) \right), \quad \text{(C.19)} 

while the relevant expressions obtained from (3.6) are given by\(^{29}\)

\[ \hat{\omega}_{abc} = e^{\phi_S/3} \left( \omega_{abc} + \frac{1}{3} \eta_{ab} \partial_c \phi_{SS} - \frac{1}{3} \eta_{ac} \partial_b \phi_{SS} \right) \]

\[ + \frac{e^{\phi_S/3}}{2} \left( -2 \left( \tilde{\psi}_{aR} \gamma_\mu \psi_{c,J} + \tilde{\psi}_{aL} \gamma_\mu \psi_{c,L} + \tilde{\psi}_{aL} \gamma_\mu \psi_{c,R} \right) \right) + \frac{1}{3} \left( \tilde{\psi}_{aR} \gamma_\nu \lambda_L + \tilde{\psi}_{aL} \gamma_\nu \lambda_R - 2 \eta_{ab} \tilde{\psi}_{cL} \gamma_\mu \lambda_R - 2 \eta_{ab} \tilde{\psi}_{cR} \gamma_\mu \lambda_R + \frac{1}{18} \tilde{\lambda}_R \gamma_\mu \lambda_R \right) \right] \]

\[ \hat{\omega}_{abi} = e^{\phi_S/3} F_{ab}^{SS} - \frac{e^{\phi_S/3}}{2} \left( 2 \left( \tilde{\psi}_{R(aL} \gamma_\mu \chi_{bR)} + \tilde{\psi}_{L(aR} \gamma_\mu \chi_{bL)} \right) \right) \]

\[ + \tilde{\psi}_{R(aL} \gamma_\nu \sigma^i \lambda_L + \tilde{\psi}_{L(aR} \gamma_\nu \sigma^i \lambda_R \right) + \frac{1}{3} \eta_{ab} \left( \tilde{\chi}_{JR} \lambda_R + \tilde{\chi}_{JL} \lambda_L \right) \right] \]

\[ - \frac{1}{36} \left( \lambda_L \sigma^i \tilde{\chi}_{ab} \lambda_R + \tilde{\chi}_{JL} \gamma_\mu \gamma_\nu \lambda_R \right) \right] \]

\[ \hat{\omega}_{aij} = e^{\phi_S/3} \left( \omega_{aij} + \frac{2i}{3} \eta_{ij} \left( \tilde{\psi}_{aL} \gamma_\mu \sigma^k \lambda_L + \tilde{\psi}_{aR} \gamma_\mu \sigma^k \lambda_R \right) + 2 \left( \tilde{\psi}_{aR} \gamma_\mu \chi_{iJ} - \tilde{\psi}_{aL} \gamma_\mu \chi_{iJ} \right) \right) \]

\[ - 2 \tilde{\chi}_{R(i} \gamma_\mu \chi_{J)} + \left( \tilde{\lambda}_L \gamma_\mu \sigma_i \chi_{J} + \tilde{\lambda}_R \gamma_\mu \sigma_i \chi_{J} \right) - \frac{4i}{9} \epsilon_{ijk} \tilde{\lambda}_R \gamma_\mu \sigma^k \lambda_R \right) \]

\[ \hat{\omega}_{ia} = - e^{\phi_S/3} F_{ia}^{SS} + \frac{e^{\phi_S/3}}{2} \left( -2 \left( \tilde{\psi}_{R(aL} \gamma_\mu \chi_{iR)} + \tilde{\psi}_{L(aR} \gamma_\mu \chi_{iL)} \right) \right) \]

\[ + \tilde{\psi}_{R(aL} \gamma_\nu \sigma^i \lambda_L + \tilde{\psi}_{L(aR} \gamma_\nu \sigma^i \lambda_R \right) + \frac{1}{3} \eta_{ia} \left( \tilde{\chi}_{JR} \lambda_R + \tilde{\chi}_{JL} \lambda_L \right) \right] \]

\[ \hat{\omega}_{ij} = e^{\phi_S/3} \left( p_{ij}^{SS} + \frac{2}{3} \delta_{ij} \partial_a \phi_{SS} \right) + \frac{e^{\phi_S/3}}{2} \left[ 2 \tilde{\chi}_{R(i} \gamma_\mu \chi_{J)} + \frac{2i}{9} \epsilon_{ijk} \tilde{\lambda}_R \gamma_\mu \sigma^k \lambda_R \right] \]

\[ + \frac{2}{3} \delta_{ij} \left( \lambda_R \tilde{\psi}_{aL} + \tilde{\chi}_{JL} \gamma_\mu \chi_{aR} \right) + 2 \left( \tilde{\psi}_{aR} \gamma_\mu \sigma_i \chi_{J} - \tilde{\psi}_{aL} \gamma_\mu \sigma_i \chi_{J} \right) \]

\[ + \frac{1}{3} \left( \tilde{\chi}_{R(i} \gamma_\mu \sigma_i \chi_{J)} + \tilde{\chi}_{JL} \gamma_\mu \sigma_i \chi_{aR} \right) - \frac{2}{3} \left( \tilde{\chi}_{R(i} \gamma_\mu \sigma_i \chi_{J)} + \tilde{\chi}_{JL} \gamma_\mu \sigma_i \chi_{aR} \right) \right]. \]

\[ \hat{\omega}_{ijk} = \frac{e^{\phi_S/3}}{2} \left[ \frac{1}{9} \epsilon_{ijk} \left( \lambda_R \lambda_R + \tilde{\chi}_{LJ} \sigma_i \sigma_j \chi_{L} + \tilde{\lambda}_L \gamma_\mu \sigma_i \chi_{L} \right) \right] \]

\[ \text{Note that} \]

\[ F_{ab}^{SS} = e^{\phi_S} \epsilon_{abc} F_{c\mu}^{SS} = e^{\phi_S} \epsilon_{a\mu} F_{b\nu}^{SS} \]
\[ + (\bar{\chi}_L \gamma^i \chi_R - \bar{\chi}_R \gamma^i \chi_L) + \frac{2}{3} (\delta_{ij} (\bar{\lambda}_L \gamma^i \chi_R + \bar{\lambda}_R \gamma^i \chi_L) - \delta_{il} (\bar{\lambda}_L \gamma^l \chi_R + \bar{\lambda}_R \gamma^l \chi_L)) \\
- i \epsilon_{jl} (\bar{\chi}_L \gamma^j \lambda_R + \bar{\chi}_R \gamma^j \lambda_L) \] 

where \( \sigma^{\mu}_{\nu} \) is the d = 8 spin connection constructed out of the vielbein \( e_\mu^\nu \).

**Appendix D. Various fermionic relations**

The supersymmetry transformations given in (C.1) for the fields charged under \( U(1) \) are different from the ones given in [24]. The extra non-gauge-invariant contributions add to give very simple contributions, as we briefly explain. First consider the supervariation of \( \lambda_L \). We have that

\[ \delta^{SS} \lambda_L = \delta^{(0)} \lambda_L + \delta^{NG} \lambda_L, \quad \text{(D.1)} \]

where \( \delta^{NG} \lambda_L \) is the non-gauge-invariant contribution, and \( \delta^{(0)} \lambda_L \) is given in (C.1).

**D.1. Calculating \( \delta^{NG} \lambda_L \)**

Now, \( \delta^{NG} \lambda_L \) is the sum of the following seven contributions, which we also evaluate as follows:

(i) \( O(\bar{\lambda}_L \gamma^i \chi_R) \):

\[ \frac{1}{576} \gamma^{\mu i-j} \bar{e}_L (\bar{\lambda}_R \gamma^i \gamma^j \chi_R) - \frac{1}{72} \bar{e}_L (\bar{\lambda}_R \gamma^i \chi_R) + \frac{5}{72} \lambda_L (\bar{\epsilon}_R \lambda_R) \]

\[ - \frac{1}{32} \gamma^{\mu i-j} \bar{e}_L (\bar{\lambda}_R \gamma^i \gamma^j \chi_R) - \frac{1}{24} \gamma^{\mu i-j} \lambda_L (\bar{\epsilon}_R \gamma^i \chi_R) + \frac{1}{18} \sigma^i \lambda_L (\bar{\epsilon}_R \gamma^j \chi_R) = \frac{3}{2} \lambda_L (\bar{\epsilon}_R \chi_R). \quad \text{(D.2)} \]

(ii) \( O(\bar{\lambda}_L \gamma^i \chi_R) \):

\[ \frac{1}{2} \gamma^{\mu i-j} \bar{e}_R (\bar{\lambda}_L \gamma^i \gamma^j \chi_R) - \frac{1}{2} \sigma^i \gamma^j \bar{e}_R (\bar{\lambda}_L \gamma^j \gamma^i \chi_R) + \frac{5}{12} \lambda_L (\bar{\epsilon}_R \lambda_R) \]

\[ - \frac{1}{24} \gamma^{\mu i-j} \bar{e}_R (\bar{\lambda}_L \gamma^i \gamma^j \chi_R) - \frac{1}{18} \gamma^{\mu i-j} \lambda_L (\bar{\epsilon}_R \chi_R) + \frac{1}{12} \sigma^j \lambda_L (\bar{\epsilon}_R \chi_R) = \frac{3}{2} \lambda_L (\bar{\epsilon}_R \chi_R). \quad \text{(D.3)} \]

(iii) \( O(\bar{\chi}_L \gamma^i \chi_L) \):

\[ - \frac{1}{576} \gamma^{\mu i-j} \bar{e}_L (\bar{\chi}_R \gamma^i \gamma^j \chi_R) - \frac{1}{32} \bar{e}_L (\bar{\chi}_R \chi_R) + \frac{1}{12} \chi_L (\bar{\epsilon}_L \chi_R) \]

\[ + \frac{1}{24} \gamma^{\mu i-j} \bar{e}_L (\bar{\chi}_R \gamma^i \gamma^j \chi_R) + \frac{1}{18} \gamma^{\mu i-j} \chi_L (\bar{\epsilon}_L \gamma^j \chi_R) = 0. \quad \text{(D.4)} \]

(iv) \( O(\bar{\lambda}_R \gamma^i \chi_L) \):

\[ \frac{1}{6} \sigma_i \gamma^j \bar{e}_R (\bar{\lambda}_L \gamma^j \chi_R) + \frac{1}{12} \chi_L (\bar{\epsilon}_L \sigma_i \chi_R) + \frac{1}{6} \gamma^{\mu i} \sigma_i \lambda_R (\bar{\epsilon}_L \gamma^j \chi_R) = 0. \quad \text{(D.5)} \]

(v) \( O(\bar{\lambda}_R \gamma^i \chi_L) \):

\[ \frac{1}{144} \gamma^{\mu i-j} \sigma_i \bar{e}_L (\bar{\chi}_R \gamma^i \gamma^j \chi_R) + \frac{1}{18} \gamma^{\mu i-j} \sigma_i \lambda_R (\bar{\epsilon}_R \gamma^j \chi_R) = 0. \quad \text{(D.6)} \]

(vi) \( O(\bar{\chi}_R \gamma^i \chi_L) \):

\[ - \frac{1}{2} \sigma_i \bar{e}_L (\bar{\chi}_R \gamma^i \chi_L) - \frac{1}{24} \gamma^{\mu i-j} \sigma_i \bar{e}_L (\bar{\chi}_R \gamma^i \gamma^j \chi_R) + \frac{1}{18} \gamma^{\mu i-j} \sigma_i \lambda_R (\bar{\epsilon}_R \gamma^j \chi_R) = 0. \quad \text{(D.7)} \]
(vii) $O(\chi_L, \chi_R e^\epsilon_R)$:
\[
-\frac{1}{2} \gamma^{\mu} \epsilon_R (\bar{\chi}_R \gamma_{\mu} \chi_R) + \frac{i}{2} \epsilon_{i \ell} \gamma^{\mu} \epsilon_R (\bar{\chi}_R \gamma_{\mu} \chi_R) - \frac{1}{2} \gamma^{\mu} \chi_R (\bar{\epsilon}_{i \ell} \gamma_{\mu} \chi_L) \\
- \frac{1}{2} \chi_L (\bar{\epsilon}_{i \ell} \chi_R) - \frac{1}{2} \gamma_{\mu} \chi_R (\sigma_i \chi_J + \sigma_J \chi_R) = 0.
\] (D.8)

In obtaining these results and the ones that follow, we make heavy use of (A.1), (A.2) and the Schouten identity
\[
\theta_A \epsilon_{BC} + \theta_B \epsilon_{CA} = 0,
\] (D.9)
where $\theta_A$ is a fermion, $(\sigma_i \lambda)_A$ or $(\sigma_i \sigma_j)_D$. We also use the relation
\[
(\sigma_i)_A \lambda^B (\sigma_i)_c^D = 2 (\delta_i^A \delta_k^B - \frac{1}{2} \delta_i^B \delta_k^D),
\] (D.10)
and other relations like
\[
\tilde{\lambda}_{RA} \psi^B_L = - \tilde{\psi}^B_R \lambda_{LA}, \\
\tilde{\lambda}_{L A} \gamma^\mu \psi^B_L = \tilde{\psi}^B_R \gamma^\mu \lambda_{RA},
\] (D.11)
\[
\lambda^\mu \epsilon^i (\bar{\lambda}_{R} \gamma^i \lambda_L) = -2 \gamma^\mu \epsilon^i (\bar{\epsilon}_{R} \gamma^i \lambda_L) + 30 \gamma^\mu \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L) - \frac{1}{2} \gamma^\mu \epsilon^i (\bar{\lambda}_{R} \gamma^i \lambda_L),
\]
\[
\epsilon_L (\bar{\gamma}_{R} \lambda_L) = - \frac{1}{2} \gamma^\mu \gamma^i \epsilon^i (\bar{\lambda}_{R} \gamma^i \lambda_L) - \frac{1}{2} \gamma^\mu \gamma^i \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L) - \frac{1}{3} \gamma^\mu \gamma^i \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L),
\]
\[
\gamma^\mu \gamma^i \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L) = 14 \lambda_L (\bar{\epsilon}_{R} \lambda_L) + \gamma^\mu \gamma^i \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L) - \frac{1}{2} \gamma^\mu \gamma^i \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L),
\]
\[
\sigma_i \lambda_L (\bar{\epsilon}_{R} \sigma_i \chi_J) = - \frac{1}{2} \gamma^\mu \gamma^i \epsilon^i (\bar{\epsilon}_{R} \gamma^i \lambda_L) + \frac{1}{2} \gamma^\mu \gamma^i \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L) - \frac{1}{3} \gamma^\mu \gamma^i \epsilon^i (\bar{\gamma}_{R} \gamma^i \lambda_L),
\] (D.13)
and adding the various contributions, we get (D.2). Similar is the analysis for $\delta^{NG} \lambda_R$.

D.2. Calculating $\delta^{NG} \chi_L$

The various non-gauge-invariant contributions are given by

(i) $O(\chi_L, \lambda_L, \epsilon_L)$:
\[
\frac{1}{2} \Delta_{ij} \gamma^{\mu} \epsilon_L (\bar{\chi}_{Rj} \gamma_{\mu} \lambda_L) + \frac{1}{2} \Delta_{ij} \gamma^{\mu} \epsilon_L (\bar{\chi}_{Rj} \gamma_{\mu} \lambda_L) - \frac{1}{2} \Delta_{ij} \epsilon_L (\bar{\epsilon}_{Rj} \chi_{iL}) + \frac{1}{2} \chi_{L} (\bar{\epsilon}_{Rj} \lambda_L) \\
- \frac{1}{2} \gamma^\mu \chi_L (\bar{\epsilon}_{Rj} \gamma_{\mu} \lambda_L) + \frac{1}{2} \sigma_i \lambda_L (\bar{\epsilon}_{Rj} \sigma_i \chi_{iL}) \\
- \frac{1}{2} \Delta_{ij} \chi_{L} (\bar{\epsilon}_{Rj} \lambda_L) = \frac{1}{2} \lambda_{L} (\bar{\epsilon}_{Rj} \lambda_L).
\] (D.14)

(ii) $O(\chi_L \lambda_R \epsilon_R)$:
\[
\frac{1}{2} \gamma^\mu \sigma_i \epsilon_R (\bar{\chi}_{Rj} \gamma_{iR} \lambda_R) - \frac{1}{4} \Delta_{ij} \gamma^\mu \epsilon_R (\bar{\chi}_{Rj} \gamma_{nR} \lambda_R) - \frac{1}{6} \Delta_{ij} \gamma^\mu \epsilon_R (\bar{\chi}_{Rj} \gamma_{iR} \lambda_R) \\
+ \frac{1}{2} \chi_{L} (\bar{\epsilon}_{Rj} \lambda_R) - \frac{1}{2} \gamma^\mu \chi_{L} (\bar{\epsilon}_{Rj} \lambda_R) + \frac{1}{2} \Delta_{ij} \gamma^\mu \lambda_R (\bar{\epsilon}_{iR} \lambda_R) \\
- \frac{1}{2} \Delta_{ij} \sigma_i \chi_{L} (\bar{\epsilon}_{Rj} \lambda_R) = - \frac{1}{2} \chi_{L} (\bar{\epsilon}_{Rj} \lambda_R).
\] (D.15)
(iii) $O(\chi_{L}\chi_{R}\varepsilon_{L})$:
\[
-\frac{1}{16}\Delta_{ij}^{\mu\nu}e_{L}(\bar{\chi}_{Lj}\sigma_{i}\bar{\chi}_{j}\lambda_{L}) - \frac{1}{48}\Delta_{ij}^{\mu\nu\rho}e_{R}(\bar{\chi}_{Lj}\bar{\chi}_{i}\lambda_{L}) + \frac{1}{8}\Delta_{ij}\sigma_{i}e_{R}(\bar{\chi}_{Lj}\sigma_{j}\lambda_{L}) \\
- \frac{1}{4}\sigma_{i}\lambda_{L}(\bar{e}_{L}\sigma_{i}\chi_{R}) - \frac{1}{2}\Delta_{ij}^{\mu\nu}\sigma_{i}\chi_{R}(\bar{e}_{L}\bar{\chi}_{i}\sigma_{j}\lambda_{L}) - \frac{1}{4}\Delta_{ij}\lambda_{L}e_{L}(\bar{e}_{L}\chi_{R}) \\
= 0. \tag{D.16}
\]

(iv) $O(\chi_{R}\chi_{R}\varepsilon_{L})$:
\[
-\frac{1}{16}\Delta_{ij}^{\mu\nu}e_{L}(\bar{\chi}_{Lj}\sigma_{i}\bar{\chi}_{j}\lambda_{R}) + \frac{1}{8}\Delta_{ij}^{\mu\nu\rho}e_{R}(\bar{\chi}_{Lj}\sigma_{j}\lambda_{R}) \\
+ \frac{1}{2}\Delta_{ij}^{\mu\nu}\sigma_{i}\chi_{R}(\bar{e}_{R}\bar{\chi}_{i}\lambda_{R}) - \frac{1}{4}\Delta_{ij}\lambda_{L}e_{L}(\bar{\chi}_{j}\chi_{R}) \\
= 0. \tag{D.17}
\]

(v) $O(\bar{\chi}_{L}\lambda_{R}\varepsilon_{L})$:
\[
-\frac{1}{32}\Delta_{ij}^{\mu\nu}e_{L}(\bar{\chi}_{Lj}\sigma_{i}\bar{\chi}_{j}\lambda_{R}) + \frac{1}{2}\Delta_{ij}^{\mu\nu}\lambda_{R}(\bar{e}_{R}\sigma_{j}\lambda_{L}) = 0. \tag{D.18}
\]

(vi) $O(\bar{\chi}_{L}\lambda_{L}\varepsilon_{L})$:
\[
\frac{1}{16}\Delta_{ij}^{\mu\nu}e_{L}(\bar{\chi}_{R}\bar{\chi}_{j}\sigma_{i}\lambda_{L}) + \frac{1}{4}\Delta_{ij}\lambda_{L}(\bar{e}_{R}\sigma_{j}\lambda_{L}) = 0. \tag{D.19}
\]

Thus,
\[
\delta^{NG}_{\chi_{L}} = \frac{1}{2}\chi_{L}(\bar{e}_{R}\lambda_{R} - \bar{e}_{L}\lambda_{R}). \tag{D.20}
\]

To prove (D.14), for example, we use the relations
\[
\Delta_{ij}\varepsilon_{L}(\bar{\chi}_{R}\chi_{L}) = -\frac{1}{2}\left[\chi_{R}\sigma_{i}\varepsilon_{L}\varepsilon_{L} + \frac{1}{4}\gamma^{\mu\nu}\chi_{R}\varepsilon_{L}(\bar{\chi}_{R}\sigma_{j}\varepsilon_{L})\varepsilon_{L}\right] \\
+ \frac{1}{2}\chi_{R}^{\mu\nu\rho}\chi_{R}(\bar{\chi}_{R}\sigma_{i}\varepsilon_{L}) + \frac{1}{2}\varepsilon_{L}(\bar{\chi}_{R}\chi_{R}),
\]

\[
\gamma^{\mu\nu}\Delta_{ij}\varepsilon_{L}(\bar{\chi}_{R}\chi_{L}) = -\frac{1}{2}\left[\chi_{L}\sigma_{i}\varepsilon_{L}\varepsilon_{L} + \frac{1}{4}\gamma^{\mu\nu}\chi_{L}\varepsilon_{L}(\bar{\chi}_{R}\sigma_{j}\varepsilon_{L})\varepsilon_{L}\right] \\
- \frac{1}{2}\chi_{L}^{\mu\nu\rho}\chi_{L}(\bar{\chi}_{R}\sigma_{i}\varepsilon_{L}) - \frac{1}{2}\varepsilon_{L}(\bar{\chi}_{R}\chi_{L}),
\]

\[
\frac{1}{2}\Delta_{ij}\lambda_{L}(\bar{e}_{R}\chi_{L}) + \frac{1}{2}\sigma_{i}\lambda_{L}(\bar{e}_{R}\varepsilon_{L}) = \frac{1}{2}\varepsilon_{L}(\bar{\chi}_{R}\lambda_{L}) + \frac{1}{2}\varepsilon_{L}(\bar{\chi}_{R}\lambda_{L}),
\]

\[
+ \frac{1}{2}\chi_{R}^{\mu\nu\rho}\chi_{R}(\bar{\chi}_{R}\sigma_{i}\varepsilon_{L}) + \frac{1}{2}\varepsilon_{L}(\bar{\chi}_{R}\chi_{R}),
\]

\[
\Delta_{ij}\varepsilon_{L}(\bar{\chi}_{R}\sigma_{L}) - \Delta_{ij}\sigma_{i}\chi_{L}(\bar{e}_{R}\chi_{L}) = -\Delta_{ij}\varepsilon_{L}(\bar{\chi}_{R}\chi_{L}) + \chi_{L}(\bar{e}_{R}\lambda_{L}) \\
- \frac{1}{2}\left[\chi_{L}\varepsilon_{L}(\bar{\chi}_{R}\sigma_{j})\varepsilon_{L}\varepsilon_{L} + \frac{1}{4}\gamma^{\mu\nu}\varepsilon_{L}(\bar{\chi}_{R}\sigma_{j}\varepsilon_{L})\varepsilon_{L}\right] \\
- \frac{1}{2}\chi_{L}^{\mu\nu\rho}\varepsilon_{L}(\bar{\chi}_{R}\sigma_{j}\varepsilon_{L}) + \frac{1}{2}\varepsilon_{L}(\bar{\chi}_{R}\lambda_{L}) \\
+ \frac{1}{2}\chi_{R}^{\mu\nu\rho}\varepsilon_{L}(\bar{\chi}_{R}\sigma_{j}\varepsilon_{L}) + \frac{1}{2}\varepsilon_{L}(\bar{\chi}_{R}\lambda_{L}), \tag{D.21}
\]

where
\[
\hat{\Delta}_{ij} = \delta_{ij} + \frac{1}{2}\delta_{ij}. \tag{D.22}
\]

Adding the various contributions, we get (D.14).
D.3. Calculating $\delta^{NG}_{\gamma} \psi_{\mu L}$

The various non-gauge-invariant contributions are given by

(i) $O(\psi_L^R, \lambda_L^R, \epsilon_L)$:

$$\begin{align*}
- \frac{1}{114} \Psi_{\mu L}(\bar{\epsilon}_L^R \lambda_L^R) + \frac{1}{22} \gamma^{\nu \rho} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_L^R) - \frac{1}{5} \gamma^L e_L(\bar{\psi}_\nu^R \lambda_L^R) + \frac{1}{2} \delta e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_L^R) \\
- \frac{1}{52} \gamma^{\nu \rho} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_L^R) + \frac{1}{6} \delta e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_L^R) - \frac{1}{3} \frac{1}{\delta} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_L^R) \\
= - \frac{1}{2} \Psi_{\mu L}(\bar{\epsilon}_L^R \lambda_L^R).
\end{align*}$$  \hspace{1cm} (D.23)

(ii) $O(\psi_L^R, \lambda_R^R, \epsilon_L)$:

$$\begin{align*}
- \frac{1}{114} \Psi_{\mu L}(\bar{\epsilon}_L^R \lambda_R^R) - \frac{1}{6} \delta e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) - \frac{1}{22} \gamma^{\nu \rho} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) \\
- \frac{1}{52} \gamma^{\nu \rho} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) - \frac{1}{3} \frac{1}{\delta} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) = \frac{1}{114} \Psi_{\mu L}(\bar{\epsilon}_L^R \lambda_R^R).
\end{align*}$$  \hspace{1cm} (D.24)

(iii) $O(\lambda_L^R, \chi_R^L, \epsilon_L)$:

$$\begin{align*}
\frac{1}{114} \delta(\gamma_{\mu \nu} \psi_{\rho}) e_L(\bar{\chi}_L^R \gamma^{\nu \rho} \lambda_L^R) - \frac{7}{2} \delta e_L(\bar{\lambda}_L^R \chi_R^L) + \frac{1}{2} \delta e_L(\bar{\lambda}_L^R \chi_R^L) \\
- \frac{1}{52} \gamma^{\nu \rho} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_L^R) - \frac{1}{3} \frac{1}{\delta} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_L^R) = 0.
\end{align*}$$  \hspace{1cm} (D.25)

(iv) $O(\lambda_R^R, \chi_R^L, \epsilon_L)$:

$$\begin{align*}
\frac{1}{114} \delta(\gamma_{\mu \nu} \psi_{\rho}) e_L(\bar{\chi}_L^R \gamma^{\nu \rho} \lambda_R^R) - \frac{7}{2} \delta e_L(\bar{\lambda}_L^R \chi_R^L) + \frac{1}{2} \delta e_L(\bar{\lambda}_L^R \chi_R^L) \\
+ \frac{1}{52} \gamma^{\nu \rho} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) + \frac{1}{3} \frac{1}{\delta} e_L(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) = 0.
\end{align*}$$  \hspace{1cm} (D.26)

(v) $O(\lambda_L^R, \lambda_R^R, \epsilon_L)$:

$$\begin{align*}
\frac{1}{114} \gamma^{\nu \rho} \psi_{\rho} e_R(\bar{\lambda}_R^R \gamma^{\nu \rho} \lambda_L^R) + \frac{1}{2} \gamma^L e_R(\bar{\lambda}_R^R \lambda_L^R) - \frac{1}{2} \delta e_R(\bar{\lambda}_R^R \gamma^{\nu \rho} \lambda_L^R) \\
+ \frac{1}{52} \gamma^{\nu \rho} e_R(\bar{\lambda}_L^R \gamma^{\nu \rho} \lambda_R^R) + \frac{1}{3} \frac{1}{\delta} e_R(\bar{\lambda}_L^R \gamma^{\nu \rho} \lambda_R^R) = 0.
\end{align*}$$  \hspace{1cm} (D.27)

(vi) $O(\chi_L^R, \lambda_L^R, \epsilon_L)$:

$$\begin{align*}
\frac{1}{114} \gamma^{\nu \rho} \psi_{\rho} e_R(\bar{\lambda}_R^R \gamma^{\nu \rho} \lambda_L^R) + \frac{1}{2} \gamma^L e_R(\bar{\lambda}_R^R \lambda_L^R) - \frac{1}{2} \delta e_R(\bar{\lambda}_R^R \gamma^{\nu \rho} \lambda_L^R) \\
+ \frac{1}{52} \gamma^{\nu \rho} e_R(\bar{\lambda}_R^R \gamma^{\nu \rho} \lambda_L^R) + \frac{1}{3} \frac{1}{\delta} e_R(\bar{\lambda}_R^R \gamma^{\nu \rho} \lambda_L^R) = 0.
\end{align*}$$  \hspace{1cm} (D.28)

(vii) $O(\lambda_R^R, \lambda_R^R, \epsilon_L)$:

$$\begin{align*}
\frac{1}{114} \gamma^{\nu \rho} \psi_{\rho} e_R(\bar{\lambda}_R^R \lambda_R^R) - \frac{1}{72} \gamma^L e_R(\bar{\lambda}_R^R \lambda_R^R) - \frac{1}{22} \gamma^{\nu \rho} e_R(\bar{\lambda}_R^R \gamma_{\nu \rho} \lambda_R^R) \\
- \frac{1}{52} \gamma^{\nu \rho} e_R(\bar{\lambda}_R^R \gamma_{\nu \rho} \lambda_R^R) - \frac{1}{3} \frac{1}{\delta} e_R(\bar{\lambda}_R^R \gamma_{\nu \rho} \lambda_R^R) = 0.
\end{align*}$$  \hspace{1cm} (D.29)

(viii) $O(\psi_R^L, \lambda_R^R, \epsilon_L)$:

$$\begin{align*}
\frac{1}{114} \gamma^{\nu \rho} e_R(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) + \frac{1}{3} \gamma^L e_R(\bar{\psi}_\nu^R \lambda_R^R) + \frac{1}{2} \delta e_R(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) \\
- \frac{1}{3} \frac{1}{\delta} e_R(\bar{\psi}_\nu^R \gamma_{\nu \rho} \lambda_R^R) = 0.
\end{align*}$$  \hspace{1cm} (D.30)
Similarly, the ones in \( \epsilon \)

\[
\begin{align*}
\frac{1}{18} \sigma_+ Y^\nu \epsilon_R(\hat{\chi}_L \gamma_\mu \lambda_R) + \frac{1}{18} \sigma_\gamma Y_\mu \gamma_\nu \epsilon_R(\hat{\chi}_L \gamma^\rho \lambda_R) \\
- \frac{1}{18} \sigma_\gamma Y_\mu \lambda_R(\hat{\epsilon}_L \chi_R) - \frac{1}{18} Y_\mu \chi_R(\hat{\epsilon}_L \sigma_\lambda R) = 0,
\end{align*}
\]

(D.31)

(x) \( O(\psi_R \chi_L) \):

\[
-\frac{1}{2} \sigma_+ Y_\nu \epsilon(\hat{\psi}_L \gamma^\nu \chi_L) + \chi_L(\hat{\epsilon}_L \gamma_\nu \psi_R) + \frac{1}{2} \sigma_\gamma Y_\nu \psi_R(\hat{\epsilon}_L \gamma^\nu \chi_L) = 0.
\]

(D.32)

Thus,

\[
\delta^\text{NG} \psi_{\mu L} = -\frac{1}{4} \psi_{\mu L}(\hat{\epsilon}_R \gamma_L - \hat{\epsilon}_L \gamma_R).
\]

(D.33)

In proving (D.23), for example, we have used the relations

\[
\begin{align*}
\lambda_L(\hat{\epsilon}_R \psi_{\mu L}) &= -\frac{1}{4} \epsilon(\psi_R \gamma_\mu \lambda_L) + \frac{1}{16} Y^{\nu \rho} \epsilon_L(\hat{\psi}_R \gamma_\nu \gamma_\rho \lambda_L) - \frac{1}{16} Y^{\nu \rho \sigma} \epsilon_L(\hat{\psi}_R \gamma_\nu \gamma_\rho \gamma_\sigma \lambda_L) \\
+ \frac{1}{16} \psi_R(\hat{\epsilon}_R \lambda_L) + \frac{1}{16} Y^{\nu \rho} \epsilon_R(\hat{\psi}_R \gamma_\nu \gamma_\rho \lambda_L) - \frac{1}{16} Y^{\nu \rho \sigma} \epsilon_R(\hat{\psi}_R \gamma_\nu \gamma_\rho \gamma_\sigma \lambda_L),
\end{align*}
\]

(D.34)

Adding the various contributions, we get (D.23).

\section*{D.4. The expressions for the gauge-invariant contributions}

The various gauge-invariant contributions can be rewritten in several different ways. For example, the ones in \( \delta^{(1)} \lambda_L \) can also be written as

\[
\begin{align*}
\frac{1}{8} \sigma_+ Y_\mu \epsilon(\hat{\lambda}_R Y^\mu \chi_R) + \frac{1}{8} Y_\mu \chi_R(\hat{\epsilon}_L \gamma_\nu \sigma_\lambda L) - \frac{1}{16} \sigma_\gamma Y_\mu \lambda_L(\hat{\epsilon}_L \chi_R) \\
= \frac{1}{16} \left[ 15 Y_\mu \chi_R(\hat{\epsilon}_L \gamma^\nu \sigma_\lambda L) + \frac{1}{2} Y_\mu \chi_R(\hat{\epsilon}_L \gamma^\nu \sigma_\lambda L) \right],
\end{align*}
\]

(D.35)

Similarly, the ones in \( \delta^{(0)} \chi_L \) can be rewritten as

\[
\begin{align*}
\frac{1}{24} \Delta_{ij} Y^{\mu \nu} \epsilon_L(\hat{\chi}_L \gamma_\mu \gamma_\nu \lambda_R) + \frac{1}{6} \Delta_{ij} \sigma_\lambda L(\hat{\chi}_L \gamma_\mu \gamma_\nu \lambda_R) - \frac{1}{6} \Delta_{ij} \epsilon_L(\hat{\chi}_L \gamma_\mu \gamma_\nu \lambda_R) \\
+ \frac{1}{6} \Delta_{ij} Y^{\mu \nu} \epsilon(\hat{\chi}_L \gamma_\mu \gamma_\nu \lambda_R) + \frac{1}{3} \Delta_{ij} Y^{\mu \nu} \lambda_R(\hat{\epsilon}_L \gamma_\mu \gamma_\nu \lambda_R) \\
= \frac{1}{24} \Delta_{ij} \epsilon_L(\hat{\chi}_L \gamma_\nu \lambda_R) + \frac{1}{2} \Delta_{ij} Y^\mu \lambda_R(\hat{\epsilon}_L \gamma_\mu \gamma_\nu \lambda_R) - \frac{1}{16} \Delta_{ij} \epsilon_L(\hat{\chi}_L \gamma_\nu \lambda_R) \\
+ \frac{1}{4} \Delta_{ij} \epsilon_L(\hat{\chi}_L \gamma_\nu \lambda_R) - \frac{1}{4} \Delta_{ij} \epsilon_L(\hat{\chi}_L \gamma_\nu \lambda_R) - \frac{1}{4} \Delta_{ij} \epsilon_L(\hat{\chi}_L \gamma_\nu \lambda_R) \\
= \frac{1}{24} \delta^{(0)} \chi_L(\hat{\epsilon}_R \gamma_\mu \sigma_\lambda L) + \frac{17}{2} Y^{\mu \nu} \lambda_R(\hat{\epsilon}_R \gamma_\mu \sigma_\lambda L) \\
+ \frac{7}{24} \gamma^{\mu \nu \rho} \chi_R(\hat{\epsilon}_R \gamma_\mu \gamma_\nu \gamma_\rho \sigma_\lambda L).
\end{align*}
\]

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\[
\begin{align*}
&\frac{-i}{18}\epsilon^{ijk}\Delta_{ij}\sigma_{k}\psi_{\mu}\psi_{\nu}\tilde{\lambda}_{R}\psi_{\sigma}\lambda_{R} + \frac{1}{9}\Delta_{ij}\lambda_{L}(\tilde{\psi}_{L}\psi_{\mu}\lambda_{R}) - \frac{1}{9}\Delta_{ij}\psi_{\mu}\lambda_{R}(\tilde{\psi}_{L}\psi_{\nu}\lambda_{L}) \\
&= \frac{1}{6}\Delta_{ij}\lambda_{L}(\tilde{\psi}_{L}\psi_{\mu}\lambda_{R}) - \frac{1}{24}\Delta_{ij}\psi_{\mu}\lambda_{L}(\tilde{\psi}_{L}\psi_{\nu}\sigma_{j}\lambda_{R}), \\
&- \frac{1}{2}\Delta_{ij}\chi_{\mu}\sigma_{k}\psi_{\nu}(\tilde{\psi}_{R}\psi_{\nu}\chi_{\lambda}) + \Delta_{ij}\chi_{\mu}(\tilde{\psi}_{L}\psi_{\nu}\chi_{R}) - \frac{1}{2}\Delta_{ij}\psi_{\mu}\chi_{R}(\tilde{\psi}_{L}\psi_{\nu}\chi_{L}) \\
&= \frac{1}{4}\Delta_{ij}\psi_{\mu}\chi_{R}(\tilde{\psi}_{L}\psi_{\nu}\chi_{L}) + \frac{1}{6}\Delta_{ij}\chi_{\mu}(\tilde{\psi}_{L}\psi_{\nu}\chi_{R}) - \frac{1}{12}\Delta_{ij}\psi_{\mu}(\tilde{\psi}_{L}\psi_{\nu}\chi_{L}), \\
&+ 2\gamma^{\mu\nu}\chi_{L}(\tilde{\psi}_{L}\psi_{\nu}\chi_{R}) + \frac{1}{6}\gamma^{\mu\nu}\chi_{R}(\tilde{\psi}_{L}\psi_{\nu}\chi_{L}) - 8\gamma^{\mu\nu}\chi_{L}(\tilde{\psi}_{L}\psi_{\nu}\chi_{R}) + \frac{1}{6}\gamma^{\mu\nu}\chi_{R}(\tilde{\psi}_{L}\psi_{\nu}\chi_{L}).
\end{align*}
\]

and the same holds for the various terms in the expression for \(\delta^{(0)}\psi_{\mu L}\). Of course, which way of representing the same expression is best depends on what is being asked for. We use the above formulas to rewrite the various expressions only when the resulting expressions are considerably shorter than the previous ones; the rest we leave as they are. Same is the analysis for the various gauge-invariant terms in \(\delta^{(0)}\psi_{\mu L}\) because there is not much simplification.

The terms in the expression for \(\delta^{(0)}\lambda_{R}, \delta^{(0)}\chi_{R}\) and \(\delta^{(0)}\psi_{\mu R}\) are obtained by conjugation.

**Appendix E. The \(O(\lambda_{R}^{2}, \chi_{R}^{2})\) term in the supergravity action**

We need the \(O(\lambda_{R}^{2}, \chi_{R}^{2})\) term in the \(d = 8\) supergravity action. In order to get it, we write down the quartic fermion terms in the \(d = 11\) supergravity action. We start from (3.1) and use the expressions for the fermion bilinears appearing in the definitions of \(\omega, \tilde{\omega}\) and \(F\) in (3.3), (3.6) and (3.7), respectively. The \(F\) term contributes

\[
V^{-1}\mathcal{L}_{11} = -\frac{1}{64}\tilde{\eta}_{MNP\eta}(\tilde{\eta}_{R}\tilde{\Gamma}^{RSMNPQ}\eta_{S} + 12\eta^{M}\tilde{\Gamma}^{NP}\eta^{0}),
\]

while the \(R\) and \(\tilde{\eta}\partial\eta\) terms together contribute

\[
V^{-1}\mathcal{L}_{11} = \frac{1}{8}\tilde{K}_{MAB}(\tilde{\eta}_{R}\tilde{\Gamma}^{M}\eta^{0}) + \frac{1}{4}(\tilde{\omega}_{MAB}\tilde{\eta}^{A}\tilde{\eta}^{B} + \tilde{\omega}_{MAB}\tilde{\eta}^{A}\tilde{\eta}^{B} + \tilde{\omega}_{MAB}\tilde{\eta}^{A}\tilde{\eta}^{B} + \tilde{\omega}_{MAB}\tilde{\eta}^{A}\tilde{\eta}^{B})
\]

to the action (3.1). In (E.2), \(\tilde{K}_{MAB}\) is the part of the contorsion (3.5) which does not involve the fermionic part of \(\tilde{\omega}\). Thus,

\[
\tilde{K}_{MAB} = -\frac{1}{4}\tilde{\eta}^{A}\tilde{\Gamma}_{MABC}\tilde{\eta}^{B},
\]

while \(\tilde{\omega}_{MAB}\) is the fermionic part of \(\tilde{\omega}\). Thus,

\[
\tilde{\omega}_{MAB} = \frac{1}{4}(\tilde{\eta}_{M}\tilde{\eta}_{B}^{A}\tilde{\eta}^{B} - \tilde{\eta}_{M}\tilde{\eta}_{A}^{B}\tilde{\eta}^{B} + \tilde{\eta}_{M}\tilde{\eta}_{A}^{B}\tilde{\eta}^{B}).
\]

Adding (E.1) and (E.2), the total contribution from all the four fermion terms is given by

\[
V^{-1}\mathcal{L}_{11} = \frac{1}{16}\tilde{\eta}_{MNPQ}(-\tilde{\eta}^{M}\tilde{\eta}^{N}\tilde{\eta}^{PQ} + 2\tilde{\eta}^{M}\tilde{\Gamma}^{NP}\eta^{0} + 4\eta^{M}\tilde{\eta}^{N}\tilde{\eta}^{PQ} + \frac{1}{2}\tilde{\eta}_{R}\tilde{\Gamma}^{RSMNPQ}\eta_{S}).
\]

We now calculate the \(O(\lambda_{R}^{2}, \chi_{R}^{2})\) term that is obtained from the action (3.1). The contribution from (E.1) is given by

\[
-\frac{1}{576}(\tilde{\lambda}_{R}\tilde{\lambda}^{\mu\nu}\sigma^{i}\lambda_{R})(\tilde{\lambda}_{R}\tilde{\lambda}^{\mu\nu}\sigma^{i}\lambda_{R}) + \frac{1}{16}(\tilde{\lambda}_{R}\tilde{\lambda}^{\mu}\lambda_{R})(\tilde{\lambda}_{R}\tilde{\lambda}^{\mu}\lambda_{R}).
\]

This contribution is gauge invariant.
The contribution from (E.2) is given by\(^{30}\)
\[
\frac{1}{54}(\tilde{\lambda}_R \sigma^I \gamma^\nu \lambda_R)(\tilde{\lambda}_R \sigma_I \gamma^\mu \lambda_R) + \frac{1}{288}(\tilde{\lambda}_R Y^{\mu \nu} \lambda_R)(\tilde{\lambda}_R Y_{\mu \nu} \lambda_R)
\]
\[
- \frac{1}{108}(\tilde{\lambda}_R \sigma^I \gamma^\mu \lambda_R - \tilde{\lambda}_R \sigma_I \gamma^\nu \lambda_R)(\tilde{\lambda}_R \sigma^I \gamma^\mu \lambda_R - \tilde{\lambda}_R \sigma_I \gamma^\nu \lambda_R)
\]
\[
+ \frac{1}{24}((\tilde{\lambda}_R \lambda_R - \tilde{\lambda}_R \lambda_R) (\tilde{\lambda}_R \lambda_R - \tilde{\lambda}_R \lambda_R)).
\]
(E.8)

Naively, this contribution does not look \(U(1)\) gauge invariant, because of the \(O(\lambda_R^0)\) and \(O(\lambda_R^1)\) terms in the last two lines of (E.8).

However, the non-gauge-invariant terms in (E.8) are given by
\[
\frac{1}{54}[\tilde{\lambda}_L \lambda_L]^2 - \frac{1}{288}((\tilde{\lambda}_R \lambda_L)\lambda_L)^2 - \frac{1}{288}((\tilde{\lambda}_R \lambda_L)\lambda_L)^2\]
But (E.9) vanishes using the relations
\[
(\tilde{\lambda}_L \sigma^I \gamma^\mu \lambda_R)^2 = 14(\tilde{\lambda}_L \lambda_L)^2 - \frac{1}{12}(\tilde{\lambda}_L Y_{\mu \nu} \lambda_L)^2,
\]
(\tilde{\lambda}_L \lambda_L)^2 = \frac{1}{12}(\tilde{\lambda}_L Y_{\nu \mu} \lambda_L)^2,
(E.10)

and their conjugates, which can be deduced using the Fierz identities (A.1). Thus, (E.8) is gauge invariant as well.

In order to simplify the remaining contributions from (E.6) and (E.8), we use the relations
\[
(\tilde{\lambda}_L \sigma^I \gamma^\mu \lambda_R)(\tilde{\lambda}_R \sigma_I \gamma^\nu \lambda_R) = -7(\tilde{\lambda}_R \gamma^\mu \lambda_R)^2 + \frac{1}{2}(\tilde{\lambda}_R Y^{\mu \nu} \lambda_R)^2,
\]
(\tilde{\lambda}_R \sigma^I \gamma^\nu \lambda_R)(\tilde{\lambda}_R \sigma_I \gamma^\mu \lambda_R) = \frac{1}{2}(\tilde{\lambda}_R \gamma^\mu \lambda_R)^2 - \frac{1}{12}(\tilde{\lambda}_R Y^{\nu \mu} \lambda_R)^2,
(\tilde{\lambda}_R \sigma^I \gamma^\nu \lambda_R)(\tilde{\lambda}_R \sigma_I \gamma^{\mu \nu} \lambda_R) = -21(\tilde{\lambda}_R \gamma^\mu \lambda_R)^2 - \frac{7}{2}(\tilde{\lambda}_R Y^{\mu \nu} \lambda_R)^2.
(E.11)

Thus, (E.6) and (E.8) add to give
\[
\frac{1}{54}[30(\tilde{\lambda}_R \gamma^\mu \lambda_R)(\tilde{\lambda}_R Y_{\mu \nu} \lambda_R) + (\tilde{\lambda}_R Y^{\mu \nu} \lambda_R)(\tilde{\lambda}_R Y_{\mu \nu} \lambda_R)],
\]
leading to the quartic fermion term
\[
e^{-1}L^{(0)} = \frac{1}{54}[30(\tilde{\lambda}_R \gamma^\mu \lambda_R)(\tilde{\lambda}_R Y_{\mu \nu} \lambda_R) + (\tilde{\lambda}_R Y^{\mu \nu} \lambda_R)(\tilde{\lambda}_R Y_{\mu \nu} \lambda_R)]\]
(E.12)
in the \(d = 8\) action.

Appendix F. Constructing the Laplacians on the moduli spaces

We calculate the Laplacians on the moduli spaces \(SO(2) \backslash SL(2, \mathbb{R})\) and \(SO(3) \backslash SL(3, \mathbb{R})\). Let the matrix \(M\) parametrize the elements of the coset space \(H \backslash G\). Then, the metric on the moduli space \(H \backslash G\) is defined by \([41, 42]\)
\[
-\frac{1}{2} Tr(M dM^{-1}) = g_{\alpha \beta} dz^\alpha d\bar{z}^\beta.,
\]
where \(z^\alpha\) are the coordinates on the moduli space \(H \backslash G\). Then, the Laplacian \(\Delta\) is given by
\[
\Delta = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} g^{\alpha \beta} \partial_\beta).
\]
(F.2)

\(^{30}\) We use the relations
\[
2 \Gamma^C d \bar{e} d a = 2 \Gamma^C d \bar{e} d a + [\eta^C \Gamma^a d \bar{e} d a + \eta^C \Gamma^a d \bar{e}] + \eta^A (\Gamma^C \Gamma^a d \bar{e} + \Gamma^C d \bar{e})
\]
\[
- \eta^C (\Gamma^a d \bar{e} + \Gamma^a d \bar{e}) - (A \leftrightarrow B),
\]
\[
\Gamma^C d \bar{e} = \Gamma^C d \bar{e} - \eta^C \Gamma^a d \bar{e} + \eta^C \Gamma^a d \bar{e}.,
\]
(E.7)
For $SO(2)\backslash SL(2, \mathbb{R})$, from (4.5) we obtain
\[ M^{-1} = \frac{1}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \] (F.3)
leading to
\[ -\frac{1}{2} \text{Tr}(dM^2 M^{-1}) = \frac{dU}{U_2^2}. \] (F.4)
Thus, we have that
\[ \Delta = 4U_2^2 \frac{\partial^2}{\partial U \partial U}. \] (F.5)

**F.2. The Laplacian on $SO(3)\backslash SL(3, \mathbb{R})$**

If we directly use the matrix $M$ in (4.10) to calculate the metric on the $SO(3)\backslash SL(3, \mathbb{R})$ moduli space using (F.1), the calculation gets very involved. The calculation is considerably simplified if we express $M$ in terms of $L$. Thus,
\[ (dL_{m}^m)(dL_{m}^m) - M^{mn} (dL_m^m)(dL_n^m) = -g_{AB} \, dz^A \, dz^B, \] (F.6)
where we use (4.43),
\[ L^m \equiv \begin{pmatrix} e^{\delta/3} \sqrt{T_2} & e^{\delta/3} T_1 / \sqrt{T_2} & e^{\delta/3} \xi_1 / \sqrt{T_2} \\ 0 & e^{\delta/3} \sqrt{T_2} & e^{\delta/3} \xi_2 / \sqrt{T_2} \\ 0 & 0 & e^{-2\delta/3} \end{pmatrix}, \] (F.7)
and
\[ M^{-1} = \frac{e^{2\delta/3}}{T_2} \begin{pmatrix} T_1 & \xi_1 \\ \xi_1 & \text{Re}(\xi \bar{T}) \end{pmatrix} e^{-2\delta/3} T_2 + |\xi|^2. \] (F.8)

This leads to
\[ -\frac{1}{2} \text{Tr}(dMdM^{-1}) = \frac{1}{3} e^{4\delta} (de^{-2\delta} + \frac{|dT|^2}{T_2^2} + \frac{e^{2\delta} |T_2 d\xi - \xi_2 dT|^2}{T_2^2} = \frac{1}{3} \left( \frac{du}{v} \right)^2 + \frac{|d\tau|^2}{\tau_2^2} + \frac{v}{\tau_2^2} |d\tau - B_2 d\tau|^2. \] (F.9)
Thus, we have that
\[ \Delta = 3e^{-4\delta} \frac{\partial^2}{\partial (e^{-2\delta})^2} + 4e^{-2\delta} T_2 \frac{\partial^2}{\partial \xi \partial \bar{\xi}} + 2 \left( T_2 \frac{\partial}{\partial T} \bar{\xi} \frac{\partial}{\partial \xi} + \frac{i}{2} \left( T_2 \frac{\partial}{\partial \bar{T}} \xi \frac{\partial}{\partial \xi} - \frac{i}{2} \left( T_2 \frac{\partial}{\partial \bar{T}} \xi \frac{\partial}{\partial \xi} \right) \right) \right) + 3 \frac{\partial}{\partial v} \left( v^2 \frac{\partial}{\partial v} \right) + 4v^{-1} t_2 \frac{\partial^2}{\partial B \partial \bar{B}} + \frac{2}{t_2} \left( t_2 \frac{\partial}{\partial \bar{\tau}} + B_2 \frac{\partial}{\partial \bar{B}} \right) + \frac{2}{t_2} \left( t_2 \frac{\partial}{\partial \bar{\tau}} + B_2 \frac{\partial}{\partial \bar{B}} \right). \] (F.10)
Appendix G. Automorphic forms of the $U$-duality groups

G.1. Automorphic forms of $SL(2, \mathbb{Z})$

We need various details about automorphic forms of $SL(2, \mathbb{Z})$. Under an $SL(2, \mathbb{Z})$ transformation \((5.4)\), an automorphic form \(\Phi^{(m,n)}(U, \bar{U})\) of weight \((m, n)\) transforms as
\[
\Phi^{(m,n)}(U, \bar{U}) \rightarrow \Phi^{(m,n)}(U', \bar{U}') = (CU + D)^m (\bar{C}\bar{U} + \bar{D})^n \Phi^{(m,n)}(U, \bar{U}).
\] (G.1)

The automorphic covariant derivatives of $SL(2, \mathbb{Z})$ are defined by
\[
D_m = i \left( U_2 \frac{\partial}{\partial U} - \frac{in}{2} \right), \quad \bar{D}_n = -i \left( U_2 \frac{\partial}{\partial \bar{U}} + \frac{in}{2} \right).
\] (G.2)

Their actions on \(\Phi^{(m,n)}\) are given by
\[
D_m \Phi^{(m,n)} \rightarrow \Phi^{(m+1,n-1)}, \quad \bar{D}_n \Phi^{(m,n)} \rightarrow \Phi^{(m-1,n+1)}.
\] (G.3)

First let us consider a class of automorphic forms of weight \((0, 0)\). These are given by the non-holomorphic Eisenstein series of $SL(2, \mathbb{Z})$ of order \(s\), defined by
\[
E_s(U, \bar{U}) = \sum_{(p,q)\neq(0,0)} \frac{U_2^p}{|p + qU|^2} = 2\xi(2s)U_2^2 + 2\sqrt{\pi}U_2^{1-s} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{1}{\xi(2s-1)} + 2\pi \sqrt{U_2} \sum_{m \neq 0, n \neq 0} \frac{|m|^{s-1/2}}{n} K_{s-1/2}(2\pi|mn|U_2) e^{2\pi in m/\lambda},
\] (G.4)

which satisfy
\[
4\bar{D}_s D_s E_s(U, \bar{U}) = 4D_{s-1} \bar{D}_s E_s(U, \bar{U}) = \Delta E_s(U, \bar{U}) = s(s-1)E_s(U, \bar{U}),
\] (G.5)

where the Laplacian is given by \((F.5)\).

We will call them
\[
f_s^{(0,0)}(U, \bar{U}) \equiv E_s(U, \bar{U}).
\] (G.6)

Then, we define automorphic forms of weight \((m, -m)\) as
\[
f_s^{(m,-m)}(U, \bar{U}) \equiv D_{m-1} \ldots D_0 f_s^{(0,0)}(U, \bar{U}) = \frac{\Gamma(s+m)}{2^n \Gamma(s)} \sum_{(p,q)\neq(0,0)} \left( \frac{p + q\bar{U}}{p + qU} \right)^m \frac{U_2^p}{|p + qU|^2},
\] (G.7)

which satisfy
\[
4\bar{D}_{s-1} D_{s-1} f_s^{(m,-m)} = 4\bar{D}_{s-m} D_{s-m} f_s^{(m,-m)} = (s+m)(s-m-1)f_s^{(m,-m)}.
\] (G.8)

The Eisenstein series defined by \((G.4)\) diverges for \(s = 1\), and it has to be properly regularized. This is done by setting \(1 - s = \epsilon\), and noting that as \(\epsilon \rightarrow 0\), the divergence appears as a simple pole in \(\Gamma(\epsilon)\) on using
\[
\Gamma(s-1/2) \xi(2s-1) = \pi^{2s-3/2}\Gamma(1-s)\xi(2-2s)
\] (G.9)

and
\[
\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon).
\] (G.10)
We then perform an $\bar{M}S$ kind of regularization where we remove the $1/\epsilon$ pole as well as the $O(1)$ terms including the Euler constant, leading to the regularized Eisenstein series

$$E_1(U, \tilde{U}) = -\pi \ln(U_2|\eta(U)|^4).$$  \hspace{1cm} (G.11)

In obtaining (G.11), we have used

$$K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x},$$  \hspace{1cm} (G.12)

the definition of the Dedekind eta function

$$\eta(U) = e^{i\pi U/12} \prod_{k=1}^{\infty} (1 - e^{2\pi ikU}),$$  \hspace{1cm} (G.13)

and

$$\xi(2) = \frac{\pi^2}{6}, \hspace{0.5cm} \xi(0) = -\frac{1}{2}.$$  \hspace{1cm} (G.14)

Thus, $\hat{E}_1(U, \tilde{U})$ satisfies

$$\Delta \hat{E}_1(U, \tilde{U}) = \pi.$$  \hspace{1cm} (G.15)

However, the modular forms of non-zero weights for $\epsilon = 1$ which are constructed using (G.7) do not have to be regularized because $D_0$ removes the divergence of the offending term. In particular,

$$f^{(-1)}_{1} (U, \tilde{U}) = D_0 E_1(U, \tilde{U}) = -\frac{\pi}{2} - 2\pi i U_2 \frac{\partial \eta(U)}{\partial U},$$  \hspace{1cm} (G.16)

which satisfies (G.8).

G.2. Automorphic forms of SL($3, \mathbb{Z}$)

We consider a family of automorphic forms of SL($3, \mathbb{Z}$) which are invariant under the SL($3, \mathbb{Z}$) transformations given by (5.14). They are given by the Eisenstein series of order $s$ defined by

$$E_s(M) = \sum_{m_0} (m_0 M_{m_0} m_n)^{-s}$$

$$= \sum_{m_0} e^{-4\phi/3} \left[ e^{-2\phi} \left( \frac{m_1 T + m_2}{T_2} \right)^2 + \frac{1}{T_2^2} \left( m_1 \ln \xi T + m_2 \xi_2 + m_3 \xi T \right)^2 \right]^{-s}$$

$$= \sum_{m_0} \nu^{-s/3} \left[ \frac{1}{\tau_2} \left( \frac{m_0 + m_3 B}{m_0 + m_3 B} \right)^2 + \frac{m_3^2}{\nu} \right]^{-s},$$  \hspace{1cm} (G.17)

where $m_0$ are integers, and the sum excludes $[m_1, m_2, m_3] = [0, 0, 0]$, which satisfies

$$\Delta E_s(M) = \frac{2s}{3}(2s - 3) E_s(M),$$  \hspace{1cm} (G.18)

where the Laplacian is given by (F.10).

The Eisenstein series (G.17) can be expanded for the weak string coupling as

$$E_s(M) = 2\zeta(2s)(e^{-2\phi})^{2s/3} + \frac{\sqrt{\pi} \Gamma(s-1/2)}{\Gamma(s)} (e^{-2\phi})^{1/2-s/3} E_{s-1/2}(T, \tilde{T})$$

$$+ \frac{2\pi e^{-2\phi}}{\Gamma(s) \sqrt{T_2}^{1/2-s/2}} \sum_{m_0 \neq 0, \nu \neq 0} \frac{|m_0|^{s-1/2}}{|m_0|} K_s-1/2(2\pi |m_0| \tau_2) e^{2\pi i m_0 \nu}$$

$$+ \frac{2\pi e^{-2\phi}}{\Gamma(s) \sqrt{T_2}^{1/2-s/3}} \sum_{m_0 \neq 0, \nu \neq 0} \frac{|n - m \tau|^{s-1}}{p} K_s-1(2\pi |p(n - m \tau)| \tau_2) e^{2\pi i (n \tau - m \xi_1)},$$  \hspace{1cm} (G.19)
In the main text, we consider interactions in the effective action which transform non-trivially under $SU(2)$, and we denote their couplings as

$$f_{(i_1,\ldots,i_n)}(T, \bar{T}, \xi, \bar{\xi}, e^{-2\phi}),$$

where every $(ij)$ index is symmetrized and traceless, and is in the spin 2 representation of $SU(2)$. Furthermore, the coupling is symmetric under the interchange of any pair of $(ij)$ indices. Thus, the couplings we consider are the coefficient functions of interactions which are in the spin 2 representation of $SU(2)$.

The various couplings are related to each other by the action of generalized derivatives defined by

$$D\!^{(n)}_{(ij)} = -2 \left[ \delta_{3i} \delta_{j3} - \frac{1}{2} (\delta_{1i} \delta_{j1} + \delta_{2i} \delta_{j2}) \right] \left( e^{-2\phi} \frac{\partial}{\partial e^{-2\phi}} - n \right)$$

and

$$2D\!^{(1)}_{(ij)} D\!^{(0)}_{(ij)} = \Delta.$$  

These generalized derivatives transform in a complicated way under $SL(3,\mathbb{Z})$ transformations given by (5.14). Thus, for example, acting on an $SL(3,\mathbb{Z})$ invariant automorphic form $g^{(0)}(M)$ it gives $D\!^{(n)}_{(ij)} g^{(0)}(M)$, which is not an automorphic form of any definite weight. This is evident from (G.21) because the various components transform differently. Of course, though the explicit form of $D\!^{(n)}_{(ij)}$ depends on how the $SU(2)$ has been gauge fixed, the fact that it does not produce automorphic forms is a gauge-invariant statement.

The Eisenstein series defined by (G.17) diverges for $s = 3/2$ for the same reason as the divergence in appendix G.1, and is regularized in the same way. Thus, we define

$$\hat{E}_3/2(M) = 2\xi (3) e^{-2\phi} + 2\hat{E}_1(T, \bar{T}) + 4\pi e^{-\phi} \sqrt{T_2} \sum_{m \neq 0, n \neq 0} \frac{m}{n} K_1(2\pi |mn| \tau_2) e^{2\pi imn \tau_1}$$

and

$$\Delta \hat{E}_3/2(M) = 2\pi.$$  

where the explicit mapping from one coupling to another using (G.21) is given in the main text. Note that

$$(D\!^{(n)}_{(ij)})^\dagger = D\!^{(n)}_{(ij)}$$

and

$$2D\!^{(1)}_{(ij)} D\!^{(0)}_{(ij)} = \Delta.$$  

This diverges for $s = 3/2$ for the same reason as the divergence in appendix G.1, and is regularized in the same way. Thus, we define

$$\hat{E}_3/2(M) = 2\xi (3) e^{-2\phi} + 2\hat{E}_1(T, \bar{T}) + 4\pi e^{-\phi} \sqrt{T_2} \sum_{m \neq 0, n \neq 0} \frac{m}{n} K_1(2\pi |mn| \tau_2) e^{2\pi imn \tau_1}$$

which satisfies

$$\Delta \hat{E}_3/2(M) = 2\pi.$$  

where the explicit mapping from one coupling to another using (G.21) is given in the main text. Note that

$$(D\!^{(n)}_{(ij)})^\dagger = D\!^{(n)}_{(ij)}$$

and

$$2D\!^{(1)}_{(ij)} D\!^{(0)}_{(ij)} = \Delta.$$  

This diverges for $s = 3/2$ for the same reason as the divergence in appendix G.1, and is regularized in the same way. Thus, we define

$$\hat{E}_3/2(M) = 2\xi (3) e^{-2\phi} + 2\hat{E}_1(T, \bar{T}) + 4\pi e^{-\phi} \sqrt{T_2} \sum_{m \neq 0, n \neq 0} \frac{m}{n} K_1(2\pi |mn| \tau_2) e^{2\pi imn \tau_1}$$

and

$$2D\!^{(1)}_{(ij)} D\!^{(0)}_{(ij)} = \Delta.$$
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