CONDITIONAL MEASURES OF DETERMINANTAL POINT PROCESSES

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Abstract. For a class of one-dimensional determinantal point processes including those induced by orthogonal projections with integrable kernels satisfying a growth condition, it is proved that their conditional measures, with respect to the configuration in the complement of a compact interval, are orthogonal polynomial ensembles with explicitly found weights. Examples include the sine-process and the process with the Bessel kernel. The argument uses the quasi-invariance, established in [1], of our point processes under the group of piecewise isometries of $\mathbb{R}$.

1. Formulation of the main result.

1.1. Conditional measures. Let $E$ be a locally compact complete metric space, let $\text{Conf}(E)$ be the space of configurations on $E$. Given a configuration $X \in \text{Conf}(E)$ and a subset $C \subset E$, we let $X|_C$ stand for the restriction of $X$ onto the subset $C$.

A point process on $E$ is a Borel probability measure on $\text{Conf}(E)$. For such a measure $P$, the measure $P(\cdot|X; C)$ on $\text{Conf}(E \setminus C)$ is defined as the conditional measure of $P$ with respect to the condition that the restriction of our random configuration onto $C$ coincides with $X|_C$. More formally, consider the surjective restriction mapping $X \to X|_C$ from $\text{Conf}(E)$ to $\text{Conf}(C)$. Fibres of this mapping can be identified with $\text{Conf}(E \setminus C)$, and conditional measures, in the sense of Rohlin [6], are precisely the measures $P(\cdot|X; C)$. If the point process $P$ admits correlation measures of order up to $l$, then, given distinct points $q_1, \ldots, q_l \in E$, we let $P_{q_1,\ldots,q_l}$ stand for the $l$-th reduced Palm measure of $P$ conditioned at points $q_1, \ldots, q_l$ (here and below we follow the conventions of [1] in working with Palm measures).

The main results of this note can informally be summarized as follows. If the measure $P(\cdot|X; C)$ is supported on the subspace of configurations with precisely $l$ particles and the reduced Palm measures, conditioned at different $l$-tuples of points, are equivalent, then, under certain additional assumptions (see Proposition 3.1 below), the conditional measure $P(\cdot|X; C)$ has the form

$$Z^{-1}(q_1, \ldots, q_l) \frac{dP_{p_1,\ldots,p_l}}{dP_{q_1,\ldots,q_l}} (X|_C) dp_l(p_1, \ldots, p_l),$$
where \( q_1, \ldots, q_l \) is almost any fixed \( l \)-tuple, \( \rho_l \) is the \( l \)-th correlation measure of \( \mathbb{P} \) and \( Z(q_1, \ldots, q_l) \) is the normalization constant. In particular, for one-dimensional determinantal processes induced by projections with integrable kernels satisfying a growth condition and \( C \) the complement of a compact interval, it is proved that \( \mathbb{P}(\cdot|X; C) \) is an orthogonal polynomial ensemble with the weight found explicitly. We proceed to precise formulations.

Given a compact subset \( B \subset E \) and a configuration \( X \in \text{Conf}(E) \), let \( \#_B(X) \) stand for the number of particles of \( X \) lying in \( B \). Given a Borel subset \( C \subset E \), we let \( F_C \) be the \( \sigma \)-algebra generated by all random variables of the form \( \#_B \), \( B \subset C \). Write \( F^p_C \) for the \( \mathbb{P} \)-completion of \( F_C \).

**Definition** (Ghosh and Peres [3], [4]). A point process \( P \) on \( E \) is called **rigid** if for any compact subset \( B \subset E \) the function \( \#_B \) is \( F^p_{E \setminus B} \)-measurable.

For a subset \( C \subset E \) and a natural number \( l \), we write \( \text{Conf}_l(C) \) for the space of \( l \)-particle configurations on \( C \); in other words, the space of all subsets of \( C \) of cardinality \( l \). Rigidity implies that for any precompact set \( B \subset E \) and \( \mathbb{P} \)-almost any \( X \) the conditional measure \( \mathbb{P}(\cdot|X; E \setminus B) \) is supported on the subset \( \text{Conf}_l(B) \), where \( l = \#_B(X) \).

Let \( U \subset \mathbb{R} \) be an open set endowed with the Lebesgue measure \( \text{Leb} \). Let \( \Pi(x, y), x, y \in U \), be a kernel smooth in the totality of variables. Assume that the kernel \( \Pi \) induces an operator of orthogonal projection acting in \( L^2(U, \text{Leb}) \); slightly abusing notation, we keep the same symbol \( \Pi \) for this operator. Let \( L \) be the range of \( \Pi \). By the Macchi-Soshnikov Theorem, the determinantal measure \( P_{\Pi} \) induced by the operator \( \Pi \) is the reduced Palm measure of \( \mathbb{P}_\Pi \) at the point \( p \):

\[
P_{\Pi} = \mathbb{P}_\Pi^p.
\]

**Assumption 1.** Let \( p \in U \). If \( \varphi \in L \) is such that \( \varphi(p) = 0 \), then \( \frac{\varphi(x)}{x - p} \in L \).

Proposition 3.3 in [1] shows that Assumption 1 holds, in particular, for kernels \( \Pi \) having integrable form \( \Pi(x, y) = \frac{A(x)B(y) - B(x)A(y)}{x - y} \).

1.2. **The trace-class case.** In the first theorem, we will make a restrictive

**Assumption 2.** We have \( \int_U \Pi(x, x)dx \left\{ 1 + |x| \right\} < +\infty \).
The Bessel kernel satisfies Assumption\textsuperscript{2}. Under Assumption\textsuperscript{2} the operators \((|x|+1)^{-1}\Pi\) and \((x+i)^{-1}\Pi\) belong to the trace class, and for \(p, q \in U\), the multiplicative functional

\[
\Psi_{p,q}^\Pi(X) = \prod_{x \in X} \left(\frac{x-p}{x-q}\right)^2
\]

exists and belongs to \(L_1(\text{Conf}(U), \mathbb{P}_\Pi^\psi)\). By Corollary 4.12 in \cite{[1]}, we have the \(\mathbb{P}_\Pi^\psi\)-almost sure equality \(\frac{d\mathbb{P}_\Pi^p}{d\mathbb{P}_\Pi^q} = Z_{p,q}^{-1}\Psi_{p,q}^\Pi\), where \(Z_{p,q}\) is the normalization constant. Since, for \(p, q, r \in U\), we have

\[
\int_{\text{Conf}(U)} \Psi_{p,q}^\Pi(X)d\mathbb{P}_\Pi^\psi(X) = \frac{\rho_{p,q}^\Pi}{\rho_{p,q}^\Pi} \Pi(p, q) \Pi(q, q).
\]

If \(\Pi\) is the Christoffel-Darboux kernel of a family of orthogonal polynomials and \(\mathbb{P}_\Pi\) the corresponding orthogonal polynomial ensemble, then \(\rho_{p,q}^\Pi\) is the weight. The function \(\rho_{p,q}^\Pi\) is defined up to a multiplicative constant.

**Theorem 1.1.** Let \(U \subset \mathbb{R}\) be an open set. Let \(\Pi(x, y), x, y \in U\), be a smooth kernel that induces an operator of orthogonal projection acting in \(L_2(U, \text{Leb})\), satisfying Assumptions \textsuperscript{1} \textsuperscript{2} and such that the determinantal point process \(\mathbb{P}_\Pi\) is rigid. Let \(I \subset U\) be a compact interval. Then

1. For almost any \(2\ell\) distinct points \(p_1, \ldots, p_\ell, q_1, \ldots, q_\ell \in U\), we have the \(\mathbb{P}^{p_1, \ldots, p_\ell, q_1, \ldots, q_\ell}\)-almost sure equality

\[
\frac{d\mathbb{P}^{p_1, \ldots, p_\ell, q_1, \ldots, q_\ell}}{d\mathbb{P}^{p_1, \ldots, q_\ell}}(X) = \frac{\det \Pi(q_i, q_j)_{i,j=1,\ldots,\ell} \prod_{1 \leq i < j \leq \ell} \left(\frac{p_i - p_j}{q_i - q_j}\right)^2}{\det \Pi(p_i, p_j)_{i,j=1,\ldots,\ell} \prod_{i=1}^{\ell} \rho_{p_i, q_i}^\Pi(X)};
\]

2. For \(\mathbb{P}_\Pi\)-almost any \(X \in \text{Conf}(U)\), the measure \(\mathbb{P}(-|X; U \setminus I)\) has the form

\[
Z(I, X)^{-1} \prod_{1 \leq i < j \leq \#_I(X)} (t_i - t_j)^2 \prod_{i=1}^{\#_I(X)} \rho_{I,X}^\Pi(t_i),
\]

where \(Z(I, X)\) is the normalization constant and the function \(\rho_{I,X}^\Pi\) satisfies, for any \(p, q \in I\), the relation

\[
\frac{\rho_{I,X}^\Pi(p)}{\rho_{I,X}^\Pi(q)} = \frac{\rho_{I}^\Pi(p)}{\rho_{I}^\Pi(q)} \prod_{x \in X \setminus I} \left(\frac{x-p}{x-q}\right)^2.
\]
Remark. The order of the points in Claim 1 is immaterial: for any permutation $\pi$ on $l$ symbols, by definition, we have $\prod_{i=1}^{l} \Psi_{\pi R_i}^{II} = \prod_{i=1}^{l} \Psi_{\pi R_{\pi(i)}}^{II}$.

Let $U = (0, +\infty)$, take $s > -1$ and consider the Bessel kernel

$$J_s(x, y) = \frac{\sqrt{x} J_{s+1}(\sqrt{x}) J_{s}(\sqrt{y}) - \sqrt{y} J_{s+1}(\sqrt{y}) J_{s}(\sqrt{x})}{2(x - y)}$$

(see, e.g., page 295 in Tracy and Widom [14]). The kernel $J_s$ induces on $L^2((0, +\infty), \text{Leb})$ the operator of orthogonal projection onto the subspace of functions whose Hankel transform is supported in $[0, 1]$ (see [14]).

**Proposition 1.2.** For any $s > -1$, we have $\rho^{Js}(t) = t^s$.

1.3. The Hilbert-Schmidt Case. We now impose a weaker

**Assumption 3.** We have $\int_U \frac{\Pi(x, x) dx}{1 + x^2} < +\infty$.

It follows that the operator $(x + i)^{-1} \Pi$ is Hilbert-Schmidt. The sine-kernel, for example, satisfies Assumption[3] but not Assumption[2].

Let $\lambda(x)$ be a continuous function on $\mathbb{R}$ satisfying

$$\sup_{x \in \mathbb{R}} \left| x^2 \lambda(x) - x \right| < +\infty.$$  

(6)

For example, one can take $\lambda(x) = (x + i)^{-1}$ or $\lambda(x) = \frac{x}{x^2 + 1}$.

We start by formulating an auxiliary

**Proposition 1.3.**  

1. For $p, q \in U$, the limit

$$\Psi_{p, q}^{\Pi, \lambda}(X) = \lim_{R \to \infty} \exp \left( \frac{2(p - q)}{\Pi(p, p)} \int_{[-R, R] \cap U} \Pi(x, x) \lambda(x) dx \right) \prod_{x \in X : |x| \leq R} \left( \frac{x - p}{x - q} \right)^2$$

exists in $L_1(\text{Conf}(U), \mathbb{P}_{\Pi^0})$. Furthermore, for any compact subset $K \subset U$, there exists a subsequence $R_n \to \infty$, along which the almost sure convergence in (7) takes place for all $p, q \in K$.

2. There exists a positive function $\rho^{\Pi, \lambda} : U \to \mathbb{R}$ such that

$$\int_{\text{Conf}(U)} \Psi_{p, q}^{\Pi, \lambda}(X) d\mathbb{P}_{\Pi^0}(X) = \frac{\rho^{\Pi, \lambda}(q) \Pi(p, p)}{\rho^{\Pi, \lambda}(p) \Pi(q, q)}.$$

(8)
If a configuration $X$ is represented in the form $X = \{t_1, \ldots, t_l\} \cup Y$, where $Y \in \text{Conf}(U)$, then, by definition, we have

$$
\Psi_{p,q}^{\Pi,\lambda}(X) = \prod_{i=1}^{l} \left( \frac{t_i - p}{t_i - q} \right)^2 \Psi_{p,q}^{\Pi,\lambda}(Y).
$$

We are now ready to formulate the analogue of Theorem 1.1.

**Theorem 1.4.** Let $U \subset \mathbb{R}$ be an open set. Let $\Pi(x,y)$, $x, y \in U$, be a smooth kernel that induces an operator of orthogonal projection acting in $L_2(U, \text{Leb})$, satisfying Assumptions 1.3 and such that the determinantal point process $\mathbb{P}_\Pi$ is rigid. Let $I \subset U$ be a compact interval. Let $\lambda(x)$ be a continuous function on $\mathbb{R}$ satisfying (6). Then

1. For almost any $2l$ distinct points $p_1, \ldots, p_l, q_1, \ldots, q_l \in U$, we have the $\mathbb{P}_\Pi$-almost sure equality

$$
\frac{d\mathbb{P}_{p_1,\ldots,p_l}}{d\mathbb{P}_{q_1,\ldots,q_l}}(X) = \frac{\det \Pi(q_i, q_j)_{i,j=1,\ldots,l}}{\det \Pi(p_i, p_j)_{i,j=1,\ldots,l}} \prod_{1 \leq i < j \leq l} \left( \frac{p_i - p_j}{q_i - q_j} \right)^2 \prod_{i=1}^{l} \frac{\rho_{\Pi,\lambda}(p_i)}{\rho_{\Pi,\lambda}(q_i)} \Psi_{p,q}^{\Pi,\lambda}(X).
$$

2. For $\mathbb{P}_\Pi$-almost every $X \in \text{Conf}(U)$, the measure $\mathbb{P}_\Pi(\cdot|X; U \setminus I)$ has the form

$$
Z(I, X, \lambda)^{-1} \prod_{1 \leq i < j \leq \#I(X)} (t_i - t_j)^2 \prod_{i=1}^{\#I(X)} \rho_{I,X}^{\Pi,\lambda}(t_i),
$$

where $Z(I, X, \lambda)$ is the normalization constant and the function $\rho_{I,X}^{\Pi,\lambda}$ satisfies, for any $p, q \in I$, the relation

$$
\frac{\rho_{I,X}^{\Pi,\lambda}(p)}{\rho_{I,X}^{\Pi,\lambda}(q)} = \frac{\rho_{\Pi,\lambda}(p)}{\rho_{\Pi,\lambda}(q)} \Psi_{p,q}^{\Pi,\lambda}(X|_{\mathbb{R}\setminus I}).
$$

**Remark.** 1. The order of the points in Claim 1 is of course again immaterial: see the Remark to Theorem 1.1.

2. Different choices of the function $\lambda$ result in the multiplication of $\Psi_{p,q}^{\Pi,\lambda}(X)$ by a constant. More precisely, given continuous functions $\lambda_1$ and $\lambda_2$ satisfying (6), the integral

$$
\beta_{\Pi}(\lambda_1, \lambda_2) = \int_{U} (\lambda_1(x) - \lambda_2(x)) \Pi(x,x) dx
$$

converges absolutely by Assumption 3. From the definitions we now have $\Psi_{p,q}^{\Pi,\lambda_1}(X) = \Psi_{p,q}^{\Pi,\lambda_2}(X) \exp(2(p - q)\beta_{\Pi}(\lambda_1, \lambda_2))$, and, consequently, we have $\rho_{\Pi,\lambda_1}(p) = \rho_{\Pi,\lambda_2}(p) \exp(2(q-p)\beta_{\Pi}(\lambda_1, \lambda_2))$. The expression (9) does not, of course, depend on the specific choice of $\lambda$. 

3. Claim 2 of Theorem 1.4 implies that for \( \mathbb{P}_\Pi \)-almost every \( X \in \text{Conf}(U) \) and any Borel automorphism \( F \) of \( U \) preserving the Lebesgue measure class and acting by the identity in the complement of a compact subset \( V \subset U \), setting \( X \cap V = \{p_1, \ldots, p_l\} \) and keeping the same symbol \( F \) for the natural induced action of \( F \) on the space of configurations, we have

\[
\frac{d\mathbb{P}_\Pi \circ F}{d\mathbb{P}}(X) = \prod_{1 \leq i < j \leq l} \left( \frac{F(p_i) - F(p_j)}{p_i - p_j} \right)^2 \prod_{i=1}^l \frac{\rho_{\Pi,\lambda}(F(p_i))}{\rho_{\Pi,\lambda}(p_i)} d\text{Leb} \circ F(p_i) \Psi_{\Pi,\lambda}(X|U \setminus V).
\]

Let \( S(x, y) = \frac{\sin \pi (x - y)}{\pi (x - y)} \) be the sine-kernel. For \( \lambda_0(x) = \frac{x}{x^2 + 1} \) (any odd function satisfying (6) would work), we have

\[
\Psi_{\mathcal{S},\lambda_0}(X) = \lim_{R \to \infty} \prod_{|x| \leq R} \left( \frac{x - p}{x - q} \right)^2.
\]

Convergence in (12) is in \( L_1 \) and almost sure along a subsequence, for instance, \( R_n = n^4 \). Approximating the sine-kernel by Christoffel-Darboux kernels of Hermite polynomials in the usual way, we obtain \( \rho_{\mathcal{S},\lambda_0} = 1 \).

Theorem 1.4 now yields

**Corollary 1.5.** Let \( I \) be a compact interval on \( \mathbb{R} \). For \( \mathbb{P}_\mathcal{S} \)-almost any configuration \( X \in \text{Conf}(\mathbb{R}) \), the conditional measure \( \mathbb{P}_\mathcal{S}(\cdot|X; \mathbb{R} \setminus I) \) has the form

\[
Z(I, X)^{-1} \prod_{1 \leq i < j \leq \#_i(X)} (t_i - t_j)^2 \prod_{i=1}^{\#_i(X)} \rho_{\mathcal{S},X}(t_i),
\]

where \( Z(I, X) \) is the normalization constant and the function \( \rho_{\mathcal{S},X} \) satisfies, for any \( p, q \in I \), the relation

\[
\frac{\rho_{\mathcal{S},X}(p)}{\rho_{\mathcal{S},X}(q)} = \lim_{R \to \infty} \prod_{x \in X \setminus I; |x| \leq R} \left( \frac{x - p}{x - q} \right)^2.
\]

2. **Multiplicative functionals and Palm measures.**

2.1. **Proof of Proposition 1.3** Let \( D_2 \Pi \) stand for the Hessian of the kernel \( \Pi \). The symbol \( || \cdot || \) stands for the Euclidean norm of a vector or a matrix.
Lemma 2.1. For any $\varepsilon > 0$ and compact subset $K \subset U$, there exists a positive constant $C(\varepsilon, K)$ such that for any $p, q \in K$ we have

$$\sup_{R \in \mathbb{R}} \left| \int_{[-R,R] \cap U} \left( \left( \frac{x-p}{x-q} \right)^2 - 1 \right) \Pi^q(x,x) + 2(p-q)\Pi(x,x)\lambda(x) \right| dx \leq$$

$$\leq C(\varepsilon,K) \left( 1 + \max_{|x-q| \leq \varepsilon, |y-q| \leq \varepsilon} (||D_2\Pi|| + ||\Pi||) + \int_U \Pi(x,x)dx \frac{1}{1+x^2} \right).$$

The proof of Lemma 2.1 is routine. We represent the integral from $-R$ to $R$ as a sum of two: first, the integral from $q - \varepsilon$ to $q + \varepsilon$, and, second, the integral over the remaining arcs. The first integral is estimated above by $C(\varepsilon,K) \max_{|x-q| \leq \varepsilon, |y-q| \leq \varepsilon} (||D_2\Pi|| + ||\Pi||)$, the second, in view of (6), by $C(\varepsilon,K) \int_U \Pi(x,x)dx \frac{1}{1+x^2}$. The lemma is proved.

The result of [1] on the regularization of multiplicative functionals can be reformulated as follows:

Lemma 2.2. For $p, q \in U$, the limit

$$\lim_{R \to \infty} \exp \left( - \int_{[-R,R] \cap U} \left( \frac{x-p}{x-q} \right)^2 - 1 \right) \Pi^q(x,x)dx \prod_{x \in X : |x| \leq R} \left( \frac{x-p}{x-q} \right)^2$$

exists in $L_1(\text{Conf}(U), \mathbb{P}_\Pi)$. Furthermore, for any compact subset $K$ of $U$, there exists a subsequence $R_n \to \infty$, along which the almost sure convergence takes place for all $p, q \in K$.

Lemmas 2.1 and 2.2 imply Proposition 1.3.

2.2. The function $\rho^{\Pi,\lambda}$. By Proposition 1.3 we have $\Psi_{p,q}^{\Pi,\lambda}(X) \in L_1(\text{Conf}(U), \mathbb{P}_{\Pi^q})$. Assumption [1] implies the relation

$$L(p) = \frac{x-p}{x-q} L(q).$$

By Corollary 4.12 in [1], for any $p, q \in U$ there exists a positive constant $C_\lambda(p,q)$ such that for $\mathbb{P}^q$-almost every $X \in \text{Conf}(U)$ we have

$$(15) \quad \frac{d\mathbb{P}^p}{d\mathbb{P}^q}(X) = C_\lambda(p,q)\Psi_{pq}^{\Pi,\lambda}(X).$$

For $p, q, r \in U$, we have $\Psi_{pq}^{\Pi,\lambda} \Psi_{qr}^{\Pi,\lambda} = \Psi_{pr}^{\Pi,\lambda}$ and $C_\lambda(p,q)C_\lambda(q,r) = C_\lambda(p,r)$. 

We now introduce a positive function $\rho^{\Pi,\lambda}$ on $U$ by setting

$$C_{\lambda}(p, q) = \frac{\rho^{\Pi,\lambda}(p)\Pi(q, q)}{\rho^{\Pi,\lambda}(q)\Pi(p, p)},$$

and (8) is established. The function $\rho^{\Pi,\lambda}$ is of course defined up to a multiplicative constant.

In the case when the kernel $\Pi$ satisfies the stronger assumption (2), we can simply take $\lambda = 0$ (even though $\lambda = 0$ does not satisfy (6)): the operator $(x - q)^{-1}\Pi^q$ belongs to the trace class (since so does $(x + i)^{-1}\Pi$), and we arrive at (2).

2.3. Relation between Radon-Nikodym derivatives of Palm measures of different orders. As before, let $P$ be a point process on a locally compact metric space $E$ endowed with a sigma-finite measure $\mu$ without atoms. As usual, we assume that for any $l$ the process $P$ admits the $l$-th correlation measure of the form $\mu_l(p_1, \ldots, p_l)d\mu(p_1)\ldots d\mu(p_l)$.

**Proposition 2.3.** Assume that for any natural number $l$ and $\mu^{\otimes l}$-almost any two $l$-tuples $(p_1, \ldots, p_l), (q_1, \ldots, q_l)$ of distinct points in $E$, the reduced Palm measures $P^{p_1, \ldots, p_l}$ and $P^{q_1, \ldots, q_l}$ are equivalent. Then for $\mu^{\otimes 2l}$-almost any $2l$-tuple $(p_1, \ldots, p_l, q_1, \ldots, q_l)$ of distinct points in $E$ we have

$$\frac{\mu_l(p_1, \ldots, p_l)dP^{p_1, \ldots, p_l}(X)}{\mu_l(q_1, \ldots, q_l)dP^{q_1, \ldots, q_l}(X)} = \prod_{i=1}^l \frac{\mu_l(p_i)}{\mu_l(q_i)} \cdot \frac{dP^{q_1, \ldots, q_l}}{dP^{q_1, \ldots, q_l}}(X \cup q_1 \cup \ldots \cup q_{i-1} \cup p_{i+1} \cup \ldots \cup p_l).$$

**Proof.** For $\mu$-almost any distinct $p, q, r_1, \ldots, r_m \in E$, we clearly have

$$\frac{\mu_{m+1}(p, r_1, \ldots, r_m)dP^{p, r_1, \ldots, r_m}}{\mu_{m+1}(q, r_1, \ldots, r_m)dP^{q, r_1, \ldots, r_m}}(X) = \frac{\mu_1(p)}{\mu_1(q)} \cdot \frac{dP^p}{dP^q}(X \cup r_1 \cup \ldots \cup r_m).$$

The proposition is now proved by induction. For $l = 2$ and $\mu$-almost any $p_1, p_2, q_1, q_2$, we have

$$\frac{\mu_2(p_1, p_2)dP^{p_1, p_2}}{\mu_2(q_1, q_2)dP^{q_1, q_2}}(X) = \frac{\mu_2(p_1, p_2)dP^{p_1, p_2}}{\mu_2(q_1, p_2)dP^{q_1, p_2}}(X) \cdot \frac{\mu_2(q_1, p_2)dP^{q_1, p_2}}{\mu_2(q_1, q_2)dP^{q_1, q_2}}(X) =$$

$$= \frac{\mu_1(p_1)}{\mu_1(q_1)} \cdot \frac{dP^{p_1}}{dP^{q_1}}(X \cup p_2) \cdot \frac{\mu_1(p_2)}{\mu_1(q_2)} \cdot \frac{dP^{p_2}}{dP^{q_2}}(X \cup q_1).$$

For the induction step, we write

$$\frac{\mu_l(p_1, \ldots, p_l)dP^{p_1, \ldots, p_l}}{\mu_l(q_1, \ldots, q_{l-1}, p_l)dP^{q_1, \ldots, q_{l-1}, p_l}}(X) = \frac{\mu_l(p_1, \ldots, p_{l-1})dP^{p_1, \ldots, p_{l-1}}}{\mu_l(q_1, \ldots, q_{l-1})dP^{q_1, \ldots, q_{l-1}}}(X \cup p_l),$$

whence, using the induction hypothesis, we conclude

$$\frac{\mu_l(p_1, \ldots, p_l)dP^{p_1, \ldots, p_l}}{\mu_l(q_1, \ldots, q_l)dP^{q_1, \ldots, q_l}}(X) =$$
then, for \( \rho_l \) under the natural projection map \( \pi \), the proposition is proved completely.

**Corollary 2.4.** Let \( \mathbb{P} \) be a point process satisfying all assumptions of Proposition 2.3. If there exists a positive Borel function \( \Psi : E \times E \times \text{Conf}(E) \to \mathbb{R}_+ \) and a positive Borel function \( \Phi : \text{Conf}_2(E) \to \mathbb{R}_+ \) such that

1. for \( \mu \)-almost any \( p, q \in E \), for \( \mathbb{P}^q \)-almost any \( X \in \text{Conf}(E) \), any \( l \in \mathbb{N} \) and any distinct particles \( r_1, \ldots, r_l \in X \), we have

\[
\Psi(p, q, X) = \frac{\Phi(p, r_1)}{\Phi(q, r_1)} \cdot \frac{\Phi(p, r_2)}{\Phi(q, r_2)} \cdot \cdots \cdot \frac{\Phi(p, r_l)}{\Phi(q, r_l)} \times \Psi(p, q, X \setminus \{r_1, \ldots, r_l\});
\]

2. for any \( p, q, r \in E \) and \( \mathbb{P}^q \)-almost any \( X \in \text{Conf}(E) \) we have

\[
\Psi(p, q, X) \cdot \Psi(q, r, X) = \Psi(p, r, X);
\]

3. for \( \mu \)-almost any \( p, q \in E \) and \( \mathbb{P}^q \)-almost any \( X \in \text{Conf}(E) \) we have

\[
\frac{\rho_l(p) d\mathbb{P}_l}{\rho_l(q) d\mathbb{P}_l}(X) = \Psi(p, q, X),
\]

then, for \( \mu^q \)-almost any \( (p_1, \ldots, p_l) \in \text{Conf}_l(E) \), \( (q_1, \ldots, q_l) \in \text{Conf}_l(E) \) and \( \mathbb{P}^{q_1, \ldots, q_l} \)-almost any \( X \in \text{Conf}(E) \), we have

\[
\frac{\rho_l(p_1, \ldots, p_l) d\mathbb{P}_l^{p_1, \ldots, p_l}}{\rho_l(q_1, \ldots, q_l) d\mathbb{P}_l^{q_1, \ldots, q_l}}(X) = \prod_{1 \leq i < j \leq l} \Phi(p_i, p_j) \cdot \prod_{i=1}^{l} \Psi(p_i, q_i, X).
\]

Proposition 8, together with Proposition 2.3 and Corollary 2.4, applied to our functional \( \Psi_{p,q}^{\Pi,\lambda} \) satisfying (16) with \( \Phi(p, q) = |p - q|^2 \), directly implies the first claim of Theorems 1.1, 1.4. We proceed to proving the second one.

### 2.4. Conditional Campbell measures.

The following Proposition 2.5 will not be used in the proof and is included to clarify the context.

Let \( \mathbb{P} \) be a point process with locally finite intensity (in other words, admitting the first correlation measure) on \( E \). Write \( \xi_{\mathbb{P}} \) for the first correlation measure of \( \mathbb{P} \). Let \( C \subset E \) be a Borel subset. Let \( \overline{\mathbb{P}}_C \) stand for the image of \( \mathbb{P} \) under the natural projection map \( \pi : \text{Conf}(E) \to \text{Conf}(C) \).

**Proposition 2.5.** Assume that for \( \mathbb{P} \)-almost every \( X \) the intensity \( \xi_{\mathbb{P}}^{\Pi,\lambda} \) of the conditional process is absolutely continuous with respect to \( \xi_{\mathbb{P}} \). Then

1. for \( \mathbb{P}^q \)-almost every \( q \in E \) and \( \overline{\mathbb{P}}_C \)-almost every \( Y \in \text{Conf}(C) \) we have

\[
(\mathbb{P}^q)_{\cdot|Y,C} = (\mathbb{P}_{\cdot|Y,C})^q;
\]
(2) for $\xi\mathbb{P}$-almost every $q \in E$ we have

\[ \mathbb{P}^q = \int_{\text{Conf}(C)} \mathbb{P}^q(\cdot|Y;C) \cdot \frac{d\xi\mathbb{P}(\cdot|Y;C)}{d\xi\mathbb{P}}(q) \cdot d\mathbb{P}_C(Y). \]

**Proof.** Recall that the Campbell measure $\mathcal{C}_\mathbb{P}$ of the point process $\mathbb{P}$ is defined, for a compact subset $B \subset E$ and a Borel subset $Z \subset \text{Conf}(E)$, by the formula

\[ \mathcal{C}_\mathbb{P}(B \times Z) = \int_Z \#_B(X) \cdot d\mathbb{P}(X). \]

By definition, we have $\mathcal{C}_\mathbb{P} = \int_{\text{Conf}(C)} \mathcal{C}_{\mathbb{P}(\cdot|Y;C)} d\mathbb{P}_C(Y)$. Let $\hat{\mathbb{P}}^q$ stand for the non-reduced Palm measure of $\mathbb{P}$ at the point $q$. We have $\mathcal{C}_\mathbb{P} = \int_{E} \hat{\mathbb{P}}^q d\xi\mathbb{P}(q)$ and, similarly, $\mathbb{P}(\cdot|Y;C) = \int_{E} \hat{\mathbb{P}}^q(\cdot|Y;C) d\xi\mathbb{P}(\cdot|Y;C)(q)$. Removing the point at $q$ and passing to reduced Palm measures, we arrive at (17).

**Corollary 2.6.** Let $\mathbb{P}$ be a point process on $E$ such that for $\mathbb{P}$-almost every $X$ the intensity $\xi\mathbb{P}(\cdot|X;C)$ of the conditional process is absolutely continuous with respect to $\xi\mathbb{P}$ and for $\xi\mathbb{P}$-almost any $p, q \in E$ the reduced Palm measures $\mathbb{P}^p$ and $\mathbb{P}^q$ are equivalent. Then for $\xi\mathbb{P}$-almost any $p, q \in E$ and $\mathbb{P}$-almost any $X \in \text{Conf}(E)$ we have

\[ \frac{d\mathbb{P}^p}{d\mathbb{P}^q}(X) = \frac{d\xi\mathbb{P}(\cdot|X;C)}{d\xi\mathbb{P}}(p) \cdot \frac{d\xi\mathbb{P}(\cdot|X;C)}{d\mathbb{P}^p}(q) \cdot d\mathbb{P}_C(X|E\setminus C). \]

Corollary 2.6 is insufficient for our purposes: we need relation (18) to hold on a fixed subset of $\text{Conf}(E)$ of full measure and for $\xi\mathbb{P}$-almost any $p, q \in E$. To check this, we use the quasi-invariance of our point processes under the group of compactly supported piecewise isometries of $E$.

### 3. Palm Measures and Conditional Measures

#### 3.1. Characterization of Conditional Measures

In this subsection, a general result is formulated describing conditional measures of point processes in terms of Radon-Nikodym derivatives of Palm measures of the same order.

Let $E$ be an open subset of $\mathbb{R}^d$, endowed with the Lebesgue measure $dv = dv_1 \ldots dv_d$. Let $\mathbb{P}$ be a point process on $E$ satisfying the following.
Assumption 4. The point process $\mathbb{P}$ admits correlation measures of all orders. For any $l > 0$, the $l$-th correlation measure of $\mathbb{P}$ has the form

$$\rho_l(p_1, \ldots, p_l)dp_1 \ldots dp_l,$$

where $\rho_l$ is a symmetric continuous function on $E^l$.

Recall that the tail sigma-algebra on $\text{Conf}(E)$ is the intersection of all sigma-algebras $\mathcal{F}_{E^lB}$ over all compact $B \subset E$.

Assumption 5. There exists a Borel subset $\mathcal{W} \subset \text{Conf}(E)$, belonging to the tail sigma-algebra of $\text{Conf}(E)$, and, for any $l > 0$, a Borel measurable function $\Psi(p_1, \ldots, p_l; q_1, \ldots, q_l; X)$, defined for $X \in \mathcal{W}$ and any two distinct $l$-tuples of points not containing particles of the configuration $X$, such that the following holds:

1. $\mathbb{P}(\mathcal{W}) = 1$;
2. for fixed $X$, the function $\Psi(p_1, \ldots, p_l; q_1, \ldots, q_l; X)$ is continuous in $(p_1, \ldots, p_l) \in \text{Conf}(E \setminus X), (q_1, \ldots, q_l) \in \text{Conf}(E \setminus X)$;
3. for fixed $X$ and any three $l$-tuples $(p_1, \ldots, p_l), (q_1, \ldots, q_l), (r_1, \ldots, r_l)$ in $\text{Conf}(E \setminus X)$, we have

$$\Psi(p_1, \ldots, p_l; q_1, \ldots, q_l; X) = \Psi(p_1, \ldots, p_l; r_1, \ldots, r_l; X) \Psi(r_1, \ldots, r_l; q_1, \ldots, q_l; X).$$

4. for $\mathbb{P}$-almost any $Y \in \mathcal{W}$, any $l$ distinct particles $(p_1, \ldots, p_l) \in Y$ and $\mu^E$-almost any $l$-tuple $(q_1, \ldots, q_l) \in \text{Conf}(E \setminus Y)$, we have

$$\frac{d\mathbb{P}_{p_1, \ldots, p_l}}{d\mathbb{P}_{p_1, \ldots, q_l}}(Y) = \Psi(p_1, \ldots, p_l; q_1, \ldots, q_l; Y \setminus \{p_1, \ldots, p_l\}).$$

Proposition 3.1. Let $\mathbb{P}$ be a rigid point process on $E$ satisfying Assumptions 4 and 5. Let $I \subset E$ be a precompact open subset. Let $l \in \mathbb{N}$ be such that

$$\mathbb{P}(\{X : \#_I(X) = l\}) > 0.$$

Then, for $\mathbb{P}$-almost every $X \in \text{Conf}(E)$ such that $\#_I(X) = l$ and almost any distinct points $q_1, \ldots, q_l \in E$, the conditional measure $\mathbb{P}(-|X, E \setminus I)$ has the form

$$Z_{q_1, \ldots, q_l}^{-1} \Psi(p_1, \ldots, p_l; q_1, \ldots, q_l; X|_{E \setminus I})\rho_l(p_1, \ldots, p_l)dp_1 \ldots dp_l,$$

where $Z_{q_1, \ldots, q_l}$ is the normalization constant.

Remark. The reference $l$-tuple $q_1, \ldots, q_l \in E$ can be chosen arbitrarily; a different choice results in a change of the normalization constant.

3.2. Quasi-invariance under piecewise isometries. We endow $\mathbb{R}^d$ with the norm $||v|| = \max_{i=1,\ldots,d} |v_i|$ and the corresponding metric. The balls in this metric are cubes. We take distinct points $p_1, \ldots, p_l, q_1, \ldots, q_l \in E$, take $\delta_1 > 0, \delta_2 > 0, \ldots, \delta_l > 0$ sufficiently small in such a way that the balls of radius $\delta_i$ centred at $p_1, \ldots, p_l, q_1, \ldots, q_l$ do not intersect, and
consider the piecewise isometry of $E$ that sends the closed ball of radius $\delta_i$ centred at $p_i$ to the corresponding ball centred at $q_i$, $i = 1, \ldots, l$, leaving the complement to the union of the closed balls fixed. The group generated by such piecewise isometries is denoted $\mathcal{G} = \mathcal{G}(E)$. For example, if $E = \mathbb{R}$, then the resulting group is the group of all interval exchange transformations on $\mathbb{R}$, while in higher dimension we arrive at the group of all interval exchange transformations. The countable subgroup $G_0 = \mathcal{G}_0(E)$ generated by transformations of the above form such that the centres of all the balls have rational coordinates and the radii of the balls are rational. For a subset $C \subset E$, let $\mathcal{G}(C)$ and $\mathcal{G}_0(C)$ be the subgroups of maps acting as the identity on $E \setminus C$. For brevity, we write $\mathbf{p} = (p_1, \ldots, p_l)$, $d\mathbf{p} = dp_1 \ldots dp_l$, $T \mathbf{p} = (Tp_1, \ldots, Tp_l)$, etc.

**Proposition 3.2.** Let $I \subset \mathbb{R}^d$ be a bounded open set, let $l \in \mathbb{N}$. Let $F : \text{Conf}_I(I) \to \mathbb{R}_+$ be a positive continuous function. Let $\mu$ be a Borel probability measure on $\text{Conf}_I(I)$ such that the equality

\begin{equation}
\frac{d\mu \circ T}{d\mu}(\mathbf{p}) = \frac{F(T\mathbf{p})}{F(\mathbf{p})}.
\end{equation}

holds $\mu$-almost surely for all $T \in \mathcal{G}_0(I)$. Then (20) holds for all $T \in \mathcal{G}(I)$ and $d\mu(\mathbf{p}) = F(\mathbf{p})d\mathbf{p}$.

**Proof.** We first show that $\mu$ assigns mass zero to boundaries of balls:

**Lemma 3.3.** For any $p \in I$ we have $\mu(\{r \in \text{Conf}_I(I) : p \in r\}) = 0$.

**Remark.** The continuity of $F$ is essential, since any atomic measure with atoms of positive mass at all rational points in $\text{Conf}_I(I)$ is quasi-invariant under $\mathcal{G}_0(I)$.

**Proof of Lemma 3.3.** First, we note that the measure $\mu$ cannot have atoms: if $\mu(\mathbf{p}) = \delta_0 > 0$, then, since the orbit of the configuration $\mathbf{p}$ under $\mathcal{G}_0$ is dense in $\text{Conf}_I(I)$ and (20) implies that there exists $\delta_1 > 0$ depending on $\delta_0$ and $F$ such that the set $\{q \in \text{Conf}_I(I) : \mu(q) \geq \delta_1\}$ is infinite; but then the measure $\mu$ cannot be finite. Next, for any $i \leq d$ and any distinct points $p_1, \ldots, p_i \in I$ we show

$$\mu(\{r \in \text{Conf}_I(I) : p_1, \ldots, p_i \in r\}) = 0.$$  

We argue by induction on $i = d, d - 1, \ldots, 1$. The case $i = d$ is precisely the absence of atoms already established. For the induction step, assume $\mu(\{r : p_1, \ldots, p_i \in r\}) > 0$. Then there exist points $q_1, \ldots, q_i \in I$ and $\delta > 0$, $\varepsilon > 0$ and a ball $B(\varepsilon)$ of radius $\varepsilon$ in $\text{Conf}_I(I)$ such that distances between distinct $q_k$ all exceed $2\varepsilon$ and we have $\mu(\{r : q_1, \ldots, q_i \in r\} \cap B(\varepsilon)) > \delta$. Write $D = \{r : q_1, \ldots, q_i \in r\} \cap B(\varepsilon)$. By continuity of $F$, there exists $\delta_1 > 0$ such that the set of the “shifts” $TD$ of the set $D$ by elements $T \in \mathcal{G}_0(I)$ satisfying $\mu(TD) > \delta_1$ is infinite. The induction
hypothesis implies $\mu(D \cap TD) = 0$. It follows that the measure $\mu$ cannot be finite, and Lemma 3.3 is proved completely.

We proceed with the proof of Proposition 3.2. A ball of radius $r$ centred at a configuration $p \in \text{Conf}_I(I)$ will be called proper if the distances between the distinct $p_i$ are all less than $r/2$.

Take two finite collections $B_1, \ldots, B_k, B'_1, \ldots, B'_k$ of disjoint isometric proper balls and let $T$ be a piecewise isometry interchanging $B_i$ and $B'_i$, $i = 1, \ldots, k$. To establish Proposition 3.2, it suffices to establish (20) for piecewise isometries $T$ of this form.

Take an exhausting sequence $B_{n,i} \subset B_i, B'_{n,i} \subset B'_i$ of isometric balls with rational centres and radii. Define $T_n \in \mathcal{G}_0$ as the map that interchanges $B_{n,i}$ and $B'_{n,i}$, $i = 1, \ldots, k$. Lemma 3.3 implies that the sequence $\mu \circ T_n$ weakly converges to $\mu \circ T$ and also that the sequence $F(T_n p) \mu$ weakly converges to the limit $F(T p) \mu$ as $n \to \infty$.

Take $\varepsilon > 0$. Set $\text{Conf}_{I,\varepsilon}(I) = \{p \in \text{Conf}_I(I) : \min_{i,j=1,\ldots,d} |p_i - p_j| \geq \varepsilon\}$. Let $\varphi$ be a bounded continuous function on $\text{Conf}_I(I)$ supported on $\text{Conf}_{I,\varepsilon}(I)$. The function $\varphi(p)/F(p)$ is then bounded and continuous, and we have

$$\lim_{n \to \infty} \int_{\text{Conf}_I(I)} \varphi(p) \cdot \frac{F(T_n p)}{F(p)} d\mu(p) = \int_{\text{Conf}_I(I)} \varphi(p) \cdot \frac{F(T p)}{F(p)} d\mu(p),$$

whence the sequence of probability measures $\frac{F(T_n p)}{F(p)} \mu = \mu \circ T_n$ vaguely converges, as $n \to \infty$, to the measure $\frac{F(T p)}{F(p)} \mu$. Since the sequence $\mu \circ T_n$ weakly converges to $\mu \circ T$, the equality (20) is proved for all $T \in \mathcal{G}$.

To conclude the proof of Proposition 3.2, consider the measure $\eta$ given by $d\eta(p) = d\mu(p)/F(p)$. By continuity and positivity of $F$, for any $\varepsilon > 0$, the measure $\eta$ is finite in restriction to $\text{Conf}_{I,\varepsilon}(I)$. Since $\eta$ is $\mathcal{G}$-invariant, the measure $\eta$, in restriction to $\text{Conf}_{I,\varepsilon}(I)$, coincides with the Lebesgue measure. Since $\varepsilon$ is arbitrary, Proposition 3.2 is proved completely.

3.3. **Completion of the proof of Proposition 3.1.** Let $S$ be a standard Borel space, let $\mu$ be a Borel probability measure on $S$. Let $\mathcal{F}$ be a $\sigma$-algebra of Borel subsets of $S$, let $\pi$ be the corresponding measurable partition. We let $\bar{\mu}$ be the quotient measure of $\mu$ under the partition $\pi$, and, for an element $\xi$ of the partition $\pi$, we let $\mu^\xi$ be the corresponding conditional measure. Finally, let $T$ be a Borel transformation of the space $S$ such that every set of $\mathcal{F}$ is $T$-invariant and the measure $\mu$ is $T$-quasi-invariant. The definitions directly imply
Proposition 3.4. Let $F$ be a Borel function such that the equality
\[ \frac{d\mu \circ T}{d\mu} = F \]
holds $\mu$-almost surely. Then for $\bar{\mu}$-almost every element $\xi$ of the partition $\pi$ we have the $\mu^\xi$-almost sure equality
\[ \frac{d\mu^\xi \circ T}{d\mu^\xi} = F. \]

Proposition 2.9 in [1] claims that for a piecewise isometry $T \in \mathcal{G}$ acting as the identity beyond a compact set $V$ and a configuration $X \in \text{Conf}(E)$ such that $X \cap V = \{p_1, \ldots, p_l\}$, we have, $\mathbb{P}$-almost surely, the equality
\[ (21) \quad \frac{d\mathbb{P} \circ T}{d\mathbb{P}}(X) = \frac{\rho_1(Tp_1, \ldots, Tp_l)}{\rho_1(p_1, \ldots, p_l)} \frac{d\mathbb{P}T^{p_1}, \ldots, T^{p_l}}{d\mathbb{P}^{p_1}, \ldots, p_l} (X \setminus \{p_1, \ldots, p_l\}). \]

Let $I \subset E$ be precompact and open. By Proposition 3.4, for $\mathbb{P}$-almost any $X \in \text{Conf}(E)$ and any $T \in \mathcal{G}_0$, the measure $\mathbb{P}(\cdot | X, E \setminus I)$ satisfies the equality (21) (in which one must, of course, substitute $\mathbb{P}(\cdot | X, E \setminus I)$ for $\mathbb{P}$).

By Proposition 3.2, the same equality holds for all $T \in \mathcal{G}$ and the measure $\mathbb{P}(\cdot | X, E \setminus I)$ has the form (19). Proposition 3.1 is proved completely.

4. Continuity of the functions $\rho^\Pi$, $\rho^{\Pi,\lambda}$ and the proofs of Proposition 1.2, Corollary 1.5

4.1. The trace class case. Let $D_1 \Pi$ stand for the Jacobi matrix of the kernel $\Pi$. Our definitions immediately imply the following important continuity property of the function $\rho^\Pi$.

Proposition 4.1. Let $\Pi_n$ be a sequence of smooth kernels, each inducing an operator of orthogonal projection in $L_2(U, \text{Leb})$, each satisfying Assumptions 1 and 2. Assume that, as $n \to \infty$, we have
\begin{enumerate}
  \item $\Pi_n \to \Pi, D_1 \Pi_n \to D_1 \Pi, D_2 \Pi_n \to D_2 \Pi$ uniformly on compact subsets of $U \times U$;
  \item $(|x| + 1)^{-1} \Pi_n \to (|x| + 1)^{-1} \Pi$ in the space of trace class operators acting in $L_2(U, \text{Leb})$.
\end{enumerate}

Then, for any any $p, q \in U$, we have
\[ \lim_{n \to \infty} \frac{\rho^{\Pi_n}(p)}{\rho^{\Pi_n}(q)} = \frac{\rho^{\Pi}(p)}{\rho^{\Pi}(q)}. \]

4.2. The Bessel kernel: computation of the function $\rho^{J_s}$. Let $s > -1$. Let $P_n^{(s)}$ be the standard Jacobi orthogonal polynomials corresponding to the weight $(1 - u)^s$. Let $\tilde{K}_n^{(s)}(u_1, u_2)$ the $n$-th Christoffel-Darboux kernel of the Jacobi orthogonal polynomial ensemble. Recall that the classical Heine-Mehler asymptotics for Jacobi orthogonal polynomials (see e.g. Chapter 8 in Szegö [11]) implies that for any $s > -1$, as $n \to \infty$, the kernel $\tilde{K}_n^{(s)}$...
converges to the kernel \( J_s \) uniformly in the totality of variables on compact subsets of \((0, +\infty) \times (0, +\infty)\), indeed, on arbitrary simply connected compact subsets of \((\mathbb{C} \setminus 0) \times (\mathbb{C} \setminus 0)\). Our next aim is to justify the limit transition

\[
\lim_{n \to \infty} \frac{\rho_{\Pi_n}^s(p)}{\rho_{\Pi_n}^s(q)} = \lim_{n \to \infty} \frac{(1 - (1 - p/2n^2))^s}{(1 - (1 - q/2n^2))^s} = \frac{p^s}{q^s} = \rho_{J_s}^s(p) / \rho_{J_s}^s(q).
\]

By Proposition 4.1, it remains to prove that \((1 + x)^{-1} \tilde{K}_n^{(s)} \to (1 + x)^{-1} J_s\) in the space of trace class operators acting in \(L_2(\mathbb{R}_+^2)\). For \(s > 0\), this trace class convergence directly follows from standard inequalities for Jacobi polynomials, see as e.g. Theorem 7.3.2 in Szegő [11]. To treat the case \(s \in (-1, 0]\), note that for any \(s > -1\) we have the recurrence relations

\[
\tilde{K}_n^{(s)}(u_1, u_2) = \frac{s + 1}{2s + 1} \rho_{n-1}^{(s+1)}(u_1)(1 - u_1)^s/2 \rho_{n-1}^{(s+1)}(u_2)(1 - u_2)^s/2 + \tilde{K}_n^{(s+2)}(u_1, u_2)
\]

\[
J_s(x, y) = J_{s+2}(x, y) + \frac{s + 1}{\sqrt{xy}} J_{s+1}(\sqrt{x}) J_{s+1}(\sqrt{y}).
\]

Relations (23), (24) imply the convergence \((1 + x)^{-1} \tilde{K}_n^{(s)} \to (1 + x)^{-1} J_s\) in trace class norm for any \(s > -1\). Proposition 4.2 is proved completely.

4.3. The Hilbert-Schmidt Case. Our definitions directly imply

**Proposition 4.2.** Let \(\Pi_n\) be a sequence of smooth kernels, each inducing an operator of orthogonal projection acting in \(L_2(U, \text{Leb})\), each satisfying Assumptions 1 and 3. If, as \(n \to \infty\), we have

1. \(\Pi_n \to \Pi, D_1 \Pi_n \to D_1 \Pi, D_2 \Pi_n \to D_2 \Pi\) uniformly on compact subsets of \(U \times U\);
2. \((x + i)^{-1} \Pi_n \to (x + i)^{-1} \Pi\) in the Hilbert-Schmidt norm,

then, for any continuous \(\lambda\) satisfying (6) and any \(p, q \in U\) we have

\[
\lim_{n \to \infty} \frac{\rho_{\Pi_n}^\lambda(p)}{\rho_{\Pi_n}^\lambda(q)} = \frac{\rho_{\Pi}^\lambda(p)}{\rho_{\Pi}^\lambda(q)}.
\]

**Proposition 4.3.** If \(\Pi_n \to \Pi\) uniformly on compact subsets of \(U\) and there exists \(\alpha, 0 \leq \alpha < 1/2\), such that

\[
\sup_{n \in \mathbb{N}, x \in U} \frac{\Pi_n(x, x)}{1 + |x|^\alpha} < +\infty.
\]

Then \((x + i)^{-1} \Pi_n \to (x + i)^{-1} \Pi\) in Hilbert-Schmidt norm.
Proof. Indeed, by Gr"umm’s theorem (see e.g. Simon [9]), it suffices to check the relation

\[ \lim_{n \to \infty} \int_{-\infty}^{\infty} \frac{\Pi_n(x, x) dx}{1 + x^2} = \int_{-\infty}^{\infty} \frac{\Pi(x, x) dx}{1 + x^2}. \]

For any \( R_0 > 0 \), the uniform convergence of our kernels on compact subsets implies the convergence

\[ \lim_{n \to \infty} \int_{-R_0}^{R_0} \frac{\Pi_n(x, x) dx}{1 + x^2} = \int_{-R_0}^{R_0} \frac{\Pi(x, x) dx}{1 + x^2}. \]

Condition (26), in turn, immediately implies, for any \( \varepsilon > 0 \), the existence of \( R_0 > 0 \) such that

\[ \sup_{n \in \mathbb{N}} \int_{|x| > R_0} \frac{\Pi_n(x, x)}{1 + x^2} < +\infty, \]

convergence (27) follows, and Proposition 4.3 is proved.

4.4. The sine-kernel. Let \( \lambda_0(x) = x(x^2 + 1)^{-1} \) so that (12) holds. Since

\[ \text{Var}_{\mathcal{P}} \left( \sum_{x \in X, |x| \geq R} \left( \log |x - p| - \log |x - q| \right) \right) = O\left( R^{-2} \right). \]

the Borel-Cantelli lemma implies convergence in (12), for example, along the sequence \( R_n = n^4 \). Let \( \widetilde{K}_n^{(H)} \) be the Christoffel-Darboux kernel of the standard Hermite polynomials and set

\[ K_n^{(H)}(x, y) = \frac{\pi}{\sqrt{2n}} K_n^{(H)} \left( \frac{x}{\sqrt{2n}}, \frac{y}{\sqrt{2n}} \right). \]

We have \( \lim_{n \to \infty} K_n^{(H)}(x, y) = \mathcal{I}(x, y) \). Convergence is uniform with all derivatives as long as \( x, y \) range over compact subsets of the complex plane. The Plancherel-Rotach asymptotic for Hermite polynomials, see e.g. Theorem 8.22.9 in Szegö [11], implies (26) for \( \Pi_n = K_n^{(H)} \), and Proposition 4.3 implies the Hilbert-Schmidt convergence \( (x + i)^{-1} K_n^{(H)} \to (x + i)^{-1} \mathcal{I} \).

Since \( \lambda_0 \) is odd and \( K_n^{(H)}(x, x) \) is even, similarly to (12), we have

\[ \Psi_{K_n^{(H)}}, \lambda_0(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left( \frac{x_i - p}{x_i - q} \right)^2. \]

Since \( \lim_{n \to \infty} \exp\left( -p^2/2n + q^2/2n \right) = 1 \), we conclude \( \rho_{\mathcal{P}, \lambda_0}(p) = 1 \).

Remark. The Airy kernel satisfies all assumptions of Theorem 1.4; the explicit constants will be given in the sequel to this paper.
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