Abstract

We show that the volumes of certain hyperbolic A-adequate links can be bounded (above and) below in terms of two diagrammatic quantities: the twist number and the number of certain alternating tangles in an A-adequate diagram. We then restrict our attention to plat closures of certain braids, a rich family of links whose volumes can be bounded in terms of the twist number alone. Furthermore, in the absence of special tangles, our volume bounds can be expressed in terms of a single stable coefficient of the colored Jones polynomials. Consequently, we are able to provide a new collection of links that satisfy a Coarse Volume Conjecture.

1 Introduction

One of the current aims of knot theory is to strengthen the relationships among the hyperbolic volume of the link complement, the colored Jones polynomials, and data extracted from link diagrams. Recently, Futer, Kalfagianni, and Purcell ([7], [8]) showed that, for sufficiently twisted negative braid closures and for certain Montesinos links, the volume of the link complement can be bounded above and below in terms of the twist number of an A-adequate link diagram. Similar results for alternating links were found in [12] and improved upon in the appendix of [12] and in [4]. The volume of many families of link complements has also been expressed in terms of coefficients of the colored Jones polynomial ([4], [8], [9], [10], [11], [13]).

In this paper, we begin with a study of the structure of A-adequate link diagrams whose all-A states satisfy a certain two-edge loop condition. We use this study to express a lower bound on the volume of the link complement in terms of two diagrammatic quantities: the twist number and the number of certain alternating tangles (called special tangles) in the A-adequate diagram. This result complements the work of Agol and D. Thurston ([12], Appendix), in which the volume is bounded above in terms of the twist number alone. It should also be noted that the recent work of Futer, Kalfagianni, and Purcell in [6] shows that the links considered in this paper must be hyperbolic. Let \( t(D) \) denote the twist number of \( D(K) \) and let \( st(D) \) denote the number of special tangles in \( D(K) \). The main result of this paper is stated below:

**Theorem 1.0.1 (Main Theorem).** Let \( D(K) \) be a connected, prime, A-adequate link diagram that satisfies the two-edge loop condition and contains \( t(D) \geq 2 \) twist regions. Then \( K \) is hyperbolic and the complement of \( K \) satisfies the following volume bounds:

\[
\frac{v_8}{3} \cdot |t(D) - st(D)| \leq \text{vol}(S^3\backslash K) \leq 10v_3 \cdot (t(D) - 1),
\]

where \( t(D) \geq st(D) \). If \( t(D) = st(D) \), then \( D(K) \) is alternating and the lower bound of \( \frac{v_8}{2} \cdot (t(D) - 2) \) from Theorem 2.2 of [12] may be used. Recall that \( v_8 = 3.6638 \ldots \) and \( v_3 = 1.0149 \ldots \) denote the volumes of a regular ideal octahedron and tetrahedron, respectively.

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Figure 1: A crossing neighborhood of a link diagram (middle), along with its A-resolution (right) and B-resolution (left).

Note that the coefficients of $t(D)$ in the upper and lower bounds differ by a multiplicative factor of $8.3102 \ldots$, a factor that we would like to reduce by studying specific families of links. Therefore, we will later restrict attention to A-adequate plat closures of certain braids (which we call strongly negative plat diagrams and mixed-sign plat diagrams). By studying the structure of these two families of link diagrams, we can provide volume bounds that are usually sharper than those given by the Main Theorem.

Furthermore, we are able to translate the volume bounds of the Main Theorem so that they may be expressed in terms of $s t(D)$ and a single stable coefficient, $\beta'_K$, of the colored Jones polynomial. In many cases, the volume of the strongly negative and mixed-sign plats can be bounded in terms of $\beta'_K$ alone. Results of this nature can be viewed as providing families of links that satisfy a Coarse Volume Conjecture ([8], Section 10.4).

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2 Preliminaries

Let $D(K) \subseteq S^3$ denote a diagram of a link $K \subseteq S^3$. To smooth a crossing of the link diagram $D(K)$, we may either A-resolve or B-resolve this crossing according to Figure 1. By A-resolving each crossing of $D(K)$ we form the all-A state of $D(K)$, which is denoted by $H_A$ and consists of a disjoint collection of all-A circles and a disjoint collection of dotted line segments, called A-segments, that are used record the locations of crossing resolutions. We will adopt the convention throughout this paper that any unlabeled segments are assumed to be A-segments. We call a link diagram $D(K)$ A-adequate if $H_A$ does not contain any A-segments that join an all-A circle to itself, and we call a link $K$ A-adequate if it has a diagram that is A-adequate.

Remark 2.0.1. While we will focus exclusively on A-adequate links, our results can easily be extended to semi-adequate links by reflecting the link diagram $D(K)$ and obtaining the corresponding results for B-adequate links.

From $H_A$ we may form the all-A graph, denoted $G_A$, by contracting the all-A circles to vertices and reinterpreting the A-segments as edges. From this graph we can form the reduced all-A graph, denoted $G'_A$, by replacing all multi-edges with a single edge. For an example of a diagram $D(K)$, its all-A resolution $H_A$, its all-A graph $G_A$, and its reduced all-A graph $G'_A$, see Figure 2. Let $v(G)$ and $e(G)$ denote the number of vertices and edges, respectively, in a graph $G$. Let $-\chi(G) = e(G) - v(G)$ denote the negative Euler characteristic of $G$.

Remark 2.0.2. Note that $v(G'_A)$ is the same as the number of all-A circles in $H_A$ and that $e(G_A)$ is the same as the number of A-segments in $H_A$. From a graphical perspective, A-adequacy of $D(K)$ can equivalently be defined by the condition that $G_A$ contains no one-edge loops that connect a vertex to itself.

Definition 2.0.1. Define a twist region of $D(K)$ to be a longest possible string of bigons in the projection graph of $D(K)$. Denote the number of twist regions in $D(K)$ by $t(D)$ and call $t(D)$ the twist number of $D(K)$. Note that it is possible for a twist region to consist of a single crossing of $D(K)$.
Figure 2: A link diagram $D(K)$, its all-A resolution $H_A$, its all-A graph $G_A$, and its reduced all-A graph $G'_A$.

Figure 3: Long and short resolutions of a twist region of $D(K)$.

**Definition 2.0.2.** If a given twist region contains two or more crossings, then the A-resolution of a left-handed twist region will be called a long resolution and the A-resolution of a right-handed twist region will be called a short resolution. See Figure 3 for depictions of these resolutions. We will call a twist region long if its A-resolution is long and short if its A-resolution is short.

**Definition 2.0.3.** A link diagram $D(K)$ satisfies the two-edge loop condition (TELC) if, whenever two all-A circles share a pair of A-segments, these segments correspond to crossings from the same short twist region of $D(K)$.

**Definition 2.0.4.** Call an alternating tangle in $D(K)$ a special tangle if, up to planar isotopy, it consists of exactly one of the following:

1. a tangle sum of a vertical one-crossing twist region and a vertical short twist region
2. a tangle sum of two vertical short twist regions
3. a tangle sum of a horizontal long twist region and a vertical short twist region

To look for such tangles in $D(K) \subseteq S^2$, we look for simple closed curves in the plane that intersect $D(K)$ exactly four times and that contain a special tangle on one side of the curve. Equivalently, special tangles of $D(K)$ can be found in the all-A state $H_A$ by looking for all-A circles that are incident to A-segments from a pair of twist regions from the tangle sums mentioned above. We will call these all-A circles special circles of $H_A$. See Figure 4 for depictions of special tangles and special circles. Let $st(D)$ denote the number of special tangles in $D(K)$ (or, equivalently, the number of special circles in $H_A$).

**Remark 2.0.3.** The advantage to looking for special circles in $H_A$, as opposed to looking for special tangles in $D(K)$, is that special circles are necessarily disjoint. Special tangles, on the other hand, can share one or both twist regions with another special tangle.

By combining results from [6], [8], and [12], we get the following key result:

**Theorem 2.0.2** (Corollary 1.4 of [6], Theorem from Appendix of [12]). Let $D(K)$ be a connected, prime, A-adequate link diagram that satisfies the TELC and contains $t(D) \geq 2$ twist regions. Then $K$ is hyperbolic and:

$-v_8 \cdot \chi(G'_A) \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1)$. 
3 Volume Bounds for A-Adequate Links

3.1 Twist Regions, State Circles, and $\mathbb{G}'_A$

We begin with a study of the twist regions of an A-adequate link diagram $D(K)$ that satisfies the TELC. Because long and short resolutions are not distinguishable when there is only one crossing in a twist region, we will begin by considering the case of one-crossing twist regions. See Figure 1 for a one-crossing twist region and its A-resolution. Let $C_1$ and $C_2$ denote the (portions of the) relevant all-A circles in $H_A$. Since $D(K)$ is A-adequate, then $C_1 \neq C_2$. Since $D(K)$ satisfies the TELC, then there can be no other additional A-segments between $C_1$ and $C_2$. Thus, the edge of $\mathbb{G}_A$ corresponding to this one-crossing twist region can never be a redundant parallel edge and, therefore, will always appear in $\mathbb{G}'_A$.

Remark 3.1.1. Let $t_1(D)$ denote the number of one-crossing twist regions in $D(K)$. By what was said in the above paragraph, $t_1(D)$ is also the number of edges in $\mathbb{G}'_A$ that come from the one-crossing twist regions of $D(K)$.

Let us now consider twist regions that have at least two crossings (the short and long twist regions). See the right side of Figure 1 for a twist region and its short resolution. Again using $C_1$ and $C_2$ to denote the (portions of the) relevant all-A circles, the A-adequacy of $D(K)$ implies that $C_1 \neq C_2$ and the TELC implies that there can be no other A-segments between $C_1$ and $C_2$ (besides those of the short resolution). Furthermore, note that a short twist region will always create redundant parallel edges in $\mathbb{G}_A$ since the parallel A-segments of $H_A$ join the same pair of state circles. Thus, all but one of these edges is removed when forming $\mathbb{G}'_A$. Said another way, there will be one edge of $\mathbb{G}'_A$ per short twist region of $D(K)$.

Remark 3.1.2. Let $t_s(D)$ denote the number of short twist regions in $D(K)$. By what was said in the above paragraph, $t_s(D)$ is also the number of edges in $\mathbb{G}'_A$ that come from the short twist regions in $D(K)$.

See the left side of Figure 3 for a twist region and its long resolution. We will use $C_1$ and $C_2$ to denote upper and lower (portions of the) relevant state circles. If there are three or more crossings in the twist region being considered, then it must necessarily be the case that none of the corresponding edges in $\mathbb{G}_A$ are lost in the reduction to form $\mathbb{G}'_A$. If there are two crossings in the twist region, then the TELC implies that
because, otherwise, we would have a two-edge loop in $G_A$ coming from a long twist region. As a result, we have that no edges of $G_A$ coming from long resolutions are removed when forming $G'_A$.

Recall that a long resolution will consist of (portions of) two state circles joined by a path of A-segments and (small) all-A circles.

**Definition 3.1.1.** We call each (small) all-A circle in the interior of the long resolution a *small inner circle (SIC)*. The remaining all-A circles in the rest of $H_A$ will simply be called *other circles (OCs)*.

**Notation:** Let $t_l(D)$ denote the number of long twist regions in $D(K)$ and let $e_l(G'_A)$ denote the number of edges in $G'_A$ coming from long twist regions.

By inspection, it can be seen that the number of A-segments in the long resolution is always one greater than the number of small inner circles in the long resolution. Since this phenomenon occurs for each long resolution, then we have that:

$$
\# \{\text{SICs}\} = e_l(G'_A) - t_l(D).
$$

3.2 Computation of $-\chi(G'_A)$

**Lemma 3.2.1.** Let $D(K)$ be a connected $A$-adequate link diagram that satisfies the TELC. Then we have that:

$$-\chi(G'_A) = t(D) - \# \{\text{OCs}\}.$$ 

**Proof.** By Remark 2.0.2 and Definition 3.1.1 we get:

$$-\chi(G'_A) = e(G'_A) - v(G'_A) = e(G'_A) - \# \{\text{all-A state circles}\} = e(G'_A) - \# \{\text{SICs}\} - \# \{\text{OCs}\}.$$ (2)

Looking at how the twist regions of $D(K)$ were partitioned in Section 3.1 we get:

$$t(D) = t_1(D) + t_s(D) + t_l(D).$$ (3)

Next, Remark 3.1.1 and Remark 3.1.2 imply that:

$$e(G'_A) = t_1(D) + t_s(D) + e_l(G'_A).$$ (4)

By substituting Equation 4 and Equation 1 into Equation 2 and then using Equation 3 we get:

$$-\chi(G'_A) = e(G'_A) - \# \{\text{SICs}\} - \# \{\text{OCs}\}$$

$$= [t_1(D) + t_s(D) + e_l(G'_A)] - [e_l(G'_A) - t_l(D)] - \# \{\text{OCs}\}$$

$$= t(D) - \# \{\text{OCs}\}.$$ 

\[\square\]

3.3 Special Circles and Special Tangles

**Lemma 3.3.1.** Let $D(K)$ be a connected, prime, $A$-adequate link diagram that satisfies the TELC and contains $t(D) \geq 2$ twist regions. Furthermore, assume that $D(K)$ is not the link diagram depicted in Figure 5. Then:

(1) there are no other circles of $H_A$ that are incident to $A$-segments from zero or one twist region resolutions, and

(2) the other circles of $H_A$ that are incident to $A$-segments from exactly two twist region resolutions are precisely the special circles of $H_A$. 

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Remark 3.3.1. Given the lemma above, notice that \( st(D) = 0 \) for link diagrams all of whose other circles are incident to at least three twist region resolutions.

**Proof of Lemma.** See Figure 6 for schematic depictions of the A-resolutions of the twist regions of \( D(K) \). Let \( C \) denote an other circle of the all-A state \( H_A \). Such a circle must exist because, otherwise, we would have that \( H_A \) is a cycle of small inner circles and A-segments. Since this all-A state corresponds to the standard \((2,p)\)-torus link diagram, then we would get a contradiction of the assumption that \( t(D) \geq 2 \).

Suppose \( C \) is an other circle with no incident twist region resolutions. Then \( C \) corresponds to a standard unknotted component of \( D(K) \). Thus, either \( D(K) \) is not connected or \( D(K) \) is the standard unknot diagram. In either case we get a contradiction, given the assumptions that \( D(K) \) is connected and contains \( t(D) \geq 2 \) twist regions.

Next, suppose \( C \) is an other circle with a single incident twist region resolution. If this resolution starts and ends at \( C \), then we again get a contradiction of the assumption that \( D(K) \) is connected and contains \( t(D) \geq 2 \) twist regions. If the resolution joins \( C \) to a different state circle \( C' \), then consider the portion of \( H_A \) corresponding to Figure 6 and recall that \( C \) and \( C' \) are closed curves. If \( C' \) were incident to no other additional twist region resolutions, then we again get a contradiction of the assumption that \( D(K) \) is connected and contains \( t(D) \geq 2 \) twist regions. If \( C' \) were incident to additional twist region resolutions, then we get a contradiction of the primeness of the corresponding diagram \( D(K) \). This proves assertion (1) of the lemma.

Now suppose that \( C \) is an other circle with exactly two incident twist region resolutions. If one of the twist region resolutions starts and ends at \( C \), then (recalling Figure 1 and Figure 3 if needed) this twist region resolution can only correspond to a long resolution. This is because, otherwise, we would get a contradiction of the assumption that \( D(K) \) is A-adequate.

Consider the case where both (long) twist region resolutions start and end at \( C \). The first three possibilities are depicted in Figure 7. As the rectangular dashed closed curves in the figure indicate, we get a contradiction of the primeness of the corresponding diagram \( D(K) \). The fourth and final possibility is depicted in Figure 8. Translating back to \( D(K) \), we get the exceptional link diagram depicted in Figure 5. Note that, by the TELC, it must be the case that there are at least three crossings per (long) twist region. This is because, otherwise, we would have a two-edge loop whose edges do not correspond to crossings of a short twist region. Recall that the link diagram of Figure 5 has been excluded from consideration.

Next, consider the case where one (long) twist region resolution starts and ends at \( C = C_1 \) and the other twist region resolution connects \( C_1 \) to another circle \( C_2 \). The three possibilities are depicted in Figure 9. As
Figure 7: Three possibilities for an other circle $C$ with two incident long resolutions that start and end at $C$.

Figure 8: The fourth possibility for an other circle $C$ with two incident long resolutions that start and end at $C$.

Finally, consider the case where both twist region resolutions connect $C$ to different other circles $C_1$ and $C_2$, where $C_1 = C_2$ is possible in some cases. The first possibility, that $C_1$ and $C_2$ are on opposite sides of $C$, is depicted in Figure 10. As the rectangular dashed closed curve indicates, we get a contradiction of the primeness of the corresponding diagram $D(K)$.

Remark 3.3.2. Note that, by the TELC, it must be the case that $C_1 \neq C_2$ in the left and middle diagrams of Figure 12. However, because a long resolution involves a path of at least two A-segments, then it is possible that $C_1 = C_2$ in the right diagram of Figure 12. It is also important to note that, in all three diagrams, the remaining twist region resolutions and all-A circles (not depicted) must somehow join $C_1$ to $C_2$ in a second way. This is because, otherwise, there would exist a simple closed curve that cuts $C$ in half and separates $C_1$ from $C_2$, a contradiction of the primeness of the corresponding diagram $D(K)$.

Assuming the conditions laid out in the above remark are satisfied, notice that the three possibilities in Figure 12 do not give a contradiction of the assumptions of Lemma 3.3.1. Equally as important, notice that these three possibilities correspond exactly to the three types of special circles depicted on the right side of Figure 4 (and to the three types of special tangles depicted on the left side of Figure 4).

Figure 9: Three possibilities for an other circle $C_1$ with two incident twist region resolutions, one resolution from a long twist region that starts and ends at $C_1$ and the other resolution connecting to a different state circle $C_2$. 

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3.4 Volume Bounds in Terms of $t(D)$ and $st(D)$ (The Main Theorem)

In this section, we will shift perspective from the link diagram $D(K)$ and its all-A state $H_A$ to the reduced all-A graph $G'_A$. By combining some graph theory with both our previous computation of $-\chi(G'_A)$ (Lemma 3.2.1) and our newly acquired knowledge about special circles (Lemma 3.3.1), we will prove the Main Theorem (Theorem 1.0.1).

**Definition 3.4.1.** Let $G$ be a graph. We call $G$ **simple** if it contains neither one-edge loops connecting a vertex to itself nor multiple edges connecting the same pair of vertices.

**Notation:** For $G$ a simple graph, let $V(G)$ denote its vertex set and let $E(G)$ denote its edge set. Furthermore, let $\deg(v)$ denote the degree of the vertex $v$, that is, the number of edges incident to $v$.

**Proof of the Main Theorem.** We will begin by using Theorem 2.1 of [2] which states that, for $G$ a simple graph:

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$ (5)

Our strategy will be to apply this result to the reduced all-A graph $G'_A$. We can do this because $A$-adequacy of $D(K)$ implies that $G'_A$ will not contain any loops and the fact that $G'_A$ is reduced implies that $G'_A$ will not contain any multiple edges.

By Remark 2.0.2, Definition 3.1.1 and Lemma 3.3.1 we may partition $V(G'_A)$ into three types of vertices:

1. those corresponding to small inner circles (SICs),
2. those corresponding to special circles (SCs), which are other circles that are incident to exactly two twist region resolutions, and
3. those corresponding to the other circles (remaining OCs) that are incident to three or more twist region resolutions.
Recall that, as said in the paragraph following Remark 3.1.2, all edges corresponding to a long resolution survive the reduction to $G'_A$. Thus, we have that $\deg(v) = 2$ for $v$ corresponding to a small inner circle. (See the left side of Figure 3.) Also recall that, as said in the paragraph preceding Remark 3.1.1, the edge corresponding to a one-crossing twist region survives the reduction to $G$. Finally, as said in the paragraph following Remark 3.1.1, only a single edge coming from a short twist region survives the reduction to $G$. By applying this knowledge to Figure 12, we see that $\deg(v) = 2$ for $v$ corresponding to a special circle. Similarly, we can see that $\deg(v) \geq 3$ for $v$ corresponding to a remaining other circle. By translating Equation 6 to our setting, we get:

$$2 \cdot e(G'_A) = \sum_{\text{SICs}} \deg(v) + \sum_{\text{SCs}} \deg(v) + \sum_{\text{remaining OCs}} \deg(v)$$

$$= 2 \cdot [\# \text{SICs}] + 2 \cdot st(D) + \sum_{\text{remaining OCs}} \deg(v).$$

Substituting Equation 4 and Equation 1 into Equation 6, we get:

$$2 \cdot t_1(D) + 2 \cdot t_s(D) + 2 \cdot e_l(G'_A) = 2 \cdot [e_l(G'_A) - t_l(D)] + 2 \cdot st(D) + \sum_{\text{remaining OCs}} \deg(v).$$

By canceling, rearranging terms, and using Equation 3, we end up with the following:

$$2 \cdot st(D) + \sum_{\text{remaining OCs}} \deg(v) = 2 \cdot t_1(D) + 2 \cdot t_s(D) + 2 \cdot t_l(D) = 2t(D).$$

Recall that $\deg(v) \geq 3$ for $v$ corresponding to a remaining other circle. Thus, we get:

$$2 \cdot st(D) + 3 \cdot [\# \text{remaining OCs}] \leq 2t(D).$$

Adding $st(D)$, the number of other circles that are not remaining other circles, to both sides allows us to write the above inequality in terms of the total number of other circles as:

$$3 \cdot [\# \text{OCs}] = 3 \cdot st(D) + 3 \cdot [\# \text{remaining OCs}] \leq 2t(D) + st(D).$$

Combining this inequality with Lemma 3.2.1 gives:

$$- \chi(G'_A) = t(D) - [\# \text{OCs}] \geq t(D) - \left[\frac{2}{3} \cdot t(D) + \frac{1}{3} \cdot st(D)\right] = \frac{1}{3} \cdot [t(D) - st(D)].$$

Finally, by applying Inequality 9 to Theorem 2.0.2, we get the desired volume bounds.

Furthermore, notice that Equation 8 implies that $t(D) \geq st(D)$. Thus, we have that the lower bound on volume is always nonnegative and is positive precisely when there exists at least one remaining other circle. Looking at Equation 8 from another perspective, note that if $t(D) = st(D)$, then there are can be no remaining other circles in the all-A state $H_A$. Hence, the only types of other circles in this case are special circles. Since each special circle is incident to exactly two twist region resolutions (and since the conditions mentioned in Remark 3.3.2 must be satisfied), then the all-A state $H_A$ must form a cycle alternating between special circles and twist region resolutions. But recall that special tangles (which correspond to special circles) are alternating tangles. Hence, by cyclically fusing these tangles together, we form an alternating link diagram. Consequently, in the case that $t(D) = st(D)$ (which forces the lower bound of the Main Theorem to be zero), Theorem 2.2 of [1] can be used to provide a lower bound of $\frac{v_8}{2} \cdot (t(D) - 2)$ on volume.}

**Corollary 3.4.1.** Let $D(K)$ satisfy the hypotheses of the Main Theorem (Theorem 1.0.1). Furthermore, assume that each other circle of $H_A$ has at least $m \geq 3$ incident twist region resolutions. Then $K$ is hyperbolic, $st(D) = 0$, and the complement of $K$ satisfies the following volume bounds:

$$\frac{m - 2}{m} \cdot v_8 \cdot t(D) \leq \text{vol}(S^3\setminus K) < 10v_3 \cdot (t(D) - 1).$$
Figure 13: A schematic depiction of a special 2n-plat diagram with \( m = 2k + 1 \) rows of twist regions, where the entry in the \( i^{th} \) row and \( j^{th} \) column is a twist region containing \( a_{i,j} \) crossings (counted with sign). The twist regions depicted above are negative twist regions. Having \( a_{i,j} > 0 \) instead will reflect the crossings in the relevant twist region.

Remark 3.4.1. Notice that, as \( m \to \infty \), the lower bound above approaches \( v_8 \cdot t(D) \). Hence, the coefficients of \( t(D) \) in the upper and lower bounds differ by a multiplicative factor of 2.7701... (in the limit).

Proof. We will prove this result by modifying what needs to be modified in the above proof of the Main Theorem. First, the assumption that each other circle has at least \( m \geq 3 \) incident twist region resolutions implies, by Lemma 3.3.1, that special circles cannot exist, so \( st(D) = 0 \). This assumption also implies that \( \deg(v) \geq m \geq 3 \) for \( v \) corresponding to an other circle (which must be a remaining other circle). By incorporating these conditions into Equation 8, we get:

\[
m \cdot \# \{ \text{OCs} \} \leq 2 \cdot st(D) + \sum_{\text{remaining OCs}} \deg(v) = 2t(D).
\]

Combining this inequality with Lemma 3.2.1 gives:

\[
-\chi(G'_A) = t(D) - \# \{ \text{OCs} \} \geq t(D) - \frac{2}{m} \cdot t(D) = \frac{m-2}{m} \cdot t(D).
\]  

Finally, by applying the above inequality to Theorem 2.0.2 we get the desired volume bounds. \( \square \)

4 Volume Bounds for A-Adequate Plats

To provide collections of links that satisfy the hypotheses of the Main Theorem (Theorem 1.0.1) and to seek to improve the lower bounds on volume, we now investigate certain families of A-adequate plat diagrams.

4.1 Background on Plat Closures

Definition 4.1.1. Given a braid \( \beta \) in the even-stringed braid group \( B_{2n} \), we can form the plat closure of \( \beta \) by connecting string position \( 2i-1 \) with string position \( 2i \) for each \( 1 \leq i \leq n \) by using trivial semicircular arcs at the top and bottom of these string positions. See Figure 13 for a schematic depiction of the type of plat closure, call it a special plat closure, that we will consider in this paper.
Notation: Let the special plat closure of $\beta \in B_{2n}$ have $m = 2k + 1$ rows of twist regions. Specifically, if we number the rows of twist regions from the top down, then there are $k + 1$ odd-numbered rows, each of which contains $n - 1$ twist regions, and $k$ even-numbered rows, each of which contains $n$ twist regions. Index the twist regions according to row and column (where by column we really mean the left-to-right ordering of twist regions in a given row). Denote the number of twist regions in row $i$ and column $j$ (counted with sign) by $a_{i,j}$, where $1 \leq i \leq m$ and:

$$
\begin{cases}
1 \leq j \leq n - 1 & \text{if } i \text{ is odd} \\
1 \leq j \leq n & \text{if } i \text{ is even}.
\end{cases}
$$

Refer back to Figure 13 to see this notation in use.

Remark 4.1.1. For the remainder of this paper, the term “for all $i$ and $j$” will be assumed to apply to $i$ and $j$ that satisfy the above conditions.

Definition 4.1.2. Let $D(K)$ denote a special plat closure of a braid $\beta \in B_{2n}$, where $n \geq 3$, that contains $2k + 1$ rows of twist regions, where $k \geq 1$. Then we call $D(K)$ a strongly negative plat diagram if $a_{i,j} \leq -3$ in odd-numbered rows and $a_{i,j} \leq -2$ in even-numbered rows. Similarly, we call $D(K)$ a mixed-sign plat diagram if $a_{i,j} \leq -3$ or $a_{i,j} \geq 1$ in odd-numbered rows and $a_{i,j} \leq -2$ in even-numbered rows. See the left side of Figure 14 for an example of a strongly negative plat diagram and see the left side of Figure 15 for an example of a mixed-sign plat diagram.

Remark 4.1.2. When $n = 2$ we have that $D(K)$ represents a two-bridge link $K$. Using the fact that two-bridge links are alternating, let $D = D_{\text{alt}}(K)$ denote a reduced alternating diagram of $K$. It will be shown later that the plats considered in this work are all hyperbolic. Therefore, by Theorem B.3 of [5], we get the following volume bounds:

$$
2v_3 \cdot t(D) - 2.7066 < \text{vol}(S^3 \setminus K) < 2v_8 \cdot (t(D) - 1) .
$$

Note that the coefficients of $t(D)$ in the upper and lower bounds above differ by a multiplicative factor of 3.6100.... Since we have the above (better) volume bounds when $n = 2$, then will assume for the remainder of this paper (as we have done with the definitions of strongly negative and mixed-sign plat diagrams) that $n \geq 3$.

4.2 Volume Bounds for Strongly Negative Plats in Terms of $t(D)$

Theorem 4.2.1. Let $D(K)$ be a strongly negative plat diagram. Then $D(K)$ is a connected, prime, A-adequate diagram that satisfies the TELC, contains $t(D) \geq 7$ twist regions, and contains $st(D) = 0$ special tangles. Furthermore, $K$ is hyperbolic and the complement of $K$ satisfies the following volume bounds:

$$
\frac{4v_8}{5} \cdot (t(D) - 1) + \frac{v_8}{5} \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1) .
$$

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Figure 15: An example of a mixed-sign plat diagram and its all-A state. Note that the diagram above is obtained from the strongly negative plat diagram of Figure 14 by changing the first negative twist region with three crossings to a positive twist region with a single crossing and changing the last negative twist region with three crossings to a positive twist region with three crossings. These changes create a “secret” small inner circle and a special circle, respectively.

Proof of Theorem 4.2.1. Since \( a_{i,j} \neq 0 \) for all \( i \) and \( j \), then \( D(K) \) must be a connected link diagram. See Figure 13 for visual support.

Since we have that \( k \geq 1 \), that \( n \geq 2 \), and that \( a_{i,j} \neq 0 \) for all \( i \) and \( j \), then by careful and methodical inspection we get that \( D(K) \) is prime. To see this, let \( C \) denote a simple closed curve in the plane that intersects \( D(K) \) twice transversely and let \( p \) be an arbitrary base point for \( C \). Considering the possible locations of \( p \) in \( S^2 \setminus D(K) \) (perhaps using Figure 14 to assist in visualization), it can be seen that it is impossible for \( C \) to both close up and contain crossings on both sides. Thus, \( D(K) \) is indeed a prime link diagram.

By inspecting \( H_A \), we get that \( D(K) \) is A-adequate since \( a_{i,j} \leq -2 \) in odd-numbered rows. To see this, first notice that the vertical A-segments between (the necessarily distinct) other circles can never contribute to non-A-adequacy. Second, notice that the vertical A-segments within a given other circle either connect distinct small inner circles or connect an other circle to a small inner circle. Therefore, since no A-segment connects a circle to itself, then \( D(K) \) is A-adequate.

The assumptions that \( a_{i,j} \leq -3 \) in odd-numbered rows and \( a_{i,j} \leq -2 \) in even-numbered rows guarantee that \( D(K) \) satisfies the TELC. To be specific, having \( a_{i,j} \leq -2 \) in even-numbered rows forces there to always be at least one small inner circle to act as a buffer between adjacent other circles, making it impossible for two given other circles to share any (let alone two) A-segments. Furthermore, notice that a small inner circle from an even-numbered row must always connect to a pair of distinct circles. Next, having \( a_{i,j} \leq -3 \) in odd-numbered rows guarantees that there are at least two inner circles for each odd-rowed twist region, which prevents an other circle from connecting to an interior small inner circle and then back to itself along another A-segment. Finally, by construction, it is impossible for a pair of small inner circles to share more than one A-segment. Since we have just shown that no two all-A circles share more than one A-segment, then the TELC is trivially satisfied.

Since \( n \geq 3 \), \( k \geq 1 \), and \( a_{i,j} \neq 0 \) for all \( i \) and \( j \), then \( t(D) \geq 7 \geq 2 \). Combining this with what was shown above and using Theorem 2.0.2, we can conclude that \( K \) is hyperbolic. Inspection also shows that \( st(D) = 0 \) because each other circle is incident to at least five twist region resolutions.

It remains to show that \( K \) satisfies the desired volume bounds. Since there is one other circle (OC) of \( H_A \) corresponding to each odd row of twist regions in \( D(K) \), then we have that \( \# \{ OCs \} = k + 1 \). Applying
Lemma 3.2.1 gives \(-\chi(G'_A) = t(D) - \#\{OCs\} = t(D) - k - 1\). We would now like to eliminate the dependence of \(-\chi(G'_A)\) on \(k\). Expand \(t(D)\) as:

\[
t(D) = \#(\text{odd-numbered rows}) \cdot \#(\text{twist regions per odd row}) + \#(\text{even-numbered rows}) \cdot \#(\text{twist regions per even row}) = (k + 1)(n - 1) + kn.
\]

(11)

Since \(n \geq 3\), then \(t(D) = (k + 1)(n - 1) + kn = 2kn - k + n - 1 \geq 5k + 2\), which implies that \(k \leq \frac{t(D) - 2}{5}\). Thus, we get the following:

\[
-\chi(G'_A) = t(D) - k - 1 \geq t(D) - \left(\frac{t(D) - 2}{5}\right) - 1 = \frac{4}{5} \cdot (t(D) - 1) + \frac{1}{5}.
\]

(12)

By applying Theorem 2.0.2 we get the desired volume bounds.

Remark 4.2.1. It can be shown that, since \(t(D) \geq 7\), then the lower bound found in Theorem 4.2.1 is always sharper than the lower bound provided by applying Corollary 3.4.1 to strongly negative plats.

4.3 Volume Bounds for Mixed-Sign Plats in Terms of \(t(D)\)

Starting from a strongly negative plat diagram, we can form a mixed-sign plat diagram by iteratively replacing any of the negative twist regions in the odd-numbered rows with positive twist regions (which need only contain at least one crossing). For an example of this process, see how Figure 14 turns into Figure 15. Notice that changing an arbitrary negative twist region of an odd-numbered row to a positive twist region will break the relevant other circle into two all-A circles. This is because a long twist region is changed to a one-crossing or short twist region. In the relevant part of the new all-A state, all but one of the new horizontal A-segments correspond to redundant parallel edges of \(G_A\). Thus, this entire new positive twist region corresponds to a single edge of \(G'_A\). These remarks hold true during every iteration of the procedure mentioned above.

Notation: Let \(t^+(D)\) and \(t^-(D)\) denote the number of positive and negative twist regions in \(D(K)\), respectively.

Theorem 4.3.1. Let \(D(K)\) be a mixed-sign plat diagram. Then \(D(K)\) is a connected, prime, A-adequate diagram that satisfies the TELC, contains \(t(D) \geq 3\) twist regions, and contains \(st(D) \leq 4\) special tangles. Furthermore, \(K\) is hyperbolic and the complement of \(K\) satisfies the following volume bounds:

\[
\frac{v_8}{3} \cdot (2t^-(D) - 1) - \frac{2v_8}{3} \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).
\]

If we also have that \(D(K)\) contains at least as many negative twist regions as it does positive twist regions, then:

\[
\frac{v_8}{3} \cdot (t(D) - 1) - \frac{2v_8}{3} \leq \text{vol}(S^3 \setminus K) < 10v_3 \cdot (t(D) - 1).
\]

Proof of Theorem. The proofs of the connectedness and primeness of \(D(K)\) are the same as those found in the proof of Theorem 4.2.1 and the proof that \(D(K)\) is A-adequate is very similar. The only new observation that is needed is that any horizontal A-segments coming from positive twist regions necessarily connect distinct all-A circles. The proof that \(D(K)\) satisfies the TELC is also similar to that found in the proof of Theorem 4.2.1, but two-edge loops may now exist. The new possibility that \(a_{i,j} \geq 1\) in odd-numbered rows will give rise to two-edge loops whenever \(a_{i,j} \geq 2\). These two-edge loops come from the same short twist region and are, therefore, allowed by the TELC.

Inspection of \(H_A\) shows that the mixed-sign plat diagrams contain at least \(7 - 4 = 3\) twist regions. This is because having \(a_{i,j} = 1\) in any of the four corners of \(D(K)\) means that the corresponding state circles of
$H_A$ in those corners will be "secret" small inner circles rather than other circles and, consequently, we may have to absorb at most four twist regions into existing negative (long) twist regions. See Figure 15 for an example.

Using what was shown above, we can apply Theorem 2.0.2 to conclude that $K$ is hyperbolic. Inspection of $H_A$ shows that special circles can actually occur in mixed-sign plat diagrams. However, special circles can only possibly occur at the four corners of the link diagram. This is because, by the assumption that $a_{i,j} \neq 0$ for all $i$ and $j$, all but at most the four corner other circles must be incident to three or more twist region resolutions. See Figure 15 for an example. Therefore, we have that $st(D) \leq 4$. It remains to show that $K$ satisfies the desired volume bounds.

Recall the observation that we may start with a strongly negative plat diagram and iteratively change any of the negative twist regions in the odd-numbered rows to positive twist regions. This creates either a new other circle (OC) or a new small inner circle. Thus, after changing any odd-rowed negative twist regions to positive twist regions, we have that $\# \{\text{OCs}\} \leq k + 1 + t^+(D)$. Applying Lemma 3.2.1 gives:

$$-\chi(G'_A) = t(D) - \# \{\text{OCs}\} \geq t(D) - k - 1 - t^+(D) = t^-(D) - k - 1.$$ 

Recall that, by construction, we can only have positive twist regions in odd-numbered rows. Thus, all of the even-numbered rows must still contain only negative twist regions. Said another way:

$$t^-(D) \geq \#(\text{even-numbered rows}) \cdot \#(\text{twist regions per even row}) = k \cdot n. \quad (13)$$

Since $t^-(D) \geq kn$, then the assumption that $n \geq 3$ gives $k \leq \frac{t^-(D)}{n} \leq \frac{t^-(D)}{3}$. Therefore, we get:

$$-\chi(G'_A) \geq t^-(D) - k - 1 \geq t^-(D) - \frac{t^-(D)}{3} - 1 = \frac{1}{3} \cdot (2t^-(D) - 1) - \frac{2}{3}.$$ 

Now suppose that $D(K)$ contains at least as many negative twist regions as it does positive twist regions, so that we have $t^-(D) \geq t^+(D)$. This implies that:

$$2t^-(D) = t^-(D) + t^+(D) \geq t^-(D) + t^+(D) = t(D),$$

which then implies that:

$$-\chi(G'_A) \geq \frac{1}{3} \cdot (2t^-(D) - 1) - \frac{2}{3} \geq \frac{1}{3} \cdot (t(D) - 1) - \frac{2}{3}. \quad (14)$$

By applying Theorem 2.0.2 we have the desired volume bounds. \qed

**Remark 4.3.1.** Note that the lower bounds on volume in terms of $t^-(D)$ will be sharper than those in terms of $t(D)$ in the case that $D(K)$ is a mixed-sign plat with more negative twist regions than positive twist regions. Furthermore, as the disparity between the number of positive and negative twist regions increases, the lower bound on volume in terms of $t^-(D)$ will continue to improve over the bound in terms of $t(D)$.

To conclude our study of mixed-sign plats, we would like to find a sufficient condition to guarantee that such a plat contains at least as many negative twist regions as positive twist regions.

**Proposition 4.3.1.** If a mixed-sign plat contains $m \geq 2n - 1$ rows of twist regions, then this plat contains at least as many negative twist regions as positive twist regions.

**Proof.** Since the process to change a strongly negative plat into a mixed-sign plat may create a situation where seemingly different twist regions are actually part of a single twist region, then Equation 11 for strongly negative plats becomes the inequality $t(D) \leq (k+1)(n-1) + kn$ for mixed-sign plats. By Inequality 13 we also have that $t^-(D) \geq kn$. Combining this information, we get that:

$$t^-(D) + t^+(D) = t(D) \leq (k+1)(n-1) + kn \leq (k+1)(n-1) + t^-(D),$$

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which implies that:

\[ t^+(D) \leq (k + 1)(n - 1) = kn + n - k - 1 \leq t^- (D) + n - k - 1. \]

Thus, to guarantee that \( t^- (D) \geq t^+ (D) \), we need that \( n - k - 1 \leq 0 \). But this condition is equivalent to \( k \geq n - 1 \) is equivalent to \( m = 2k + 1 \geq 2n - 1 \). □

**Remark 4.3.2.** It can be shown that, for mixed-sign plats that contain at least as many negative twist regions as positive twist regions, the lower bound found in Theorem 4.3.1 is always slightly sharper than the lower bound provided by applying the Main Theorem (Theorem 1.0.1).

## 5 Volume Bounds in Terms of the Colored Jones Polynomial

**Theorem 5.0.2** ([8], [13]). Denote the \( n \)-th colored Jones polynomial of a link \( K \) by:

\[ J^m_K(t) = \alpha_n t^{m_n} + \beta_n t^{m_n-1} + \cdots + \beta'_n t^{r_n} + \alpha'_n t^r. \]

Let \( D(K) \) be a connected \( A \)-adequate link diagram. Then \( |\beta'_n| \) is independent of \( n \) for \( n > 1 \). Specifically, for \( n > 1 \), we have that:

\[ |\beta'_n| = 1 - \chi (G'_A). \]

**Remark 5.0.3.** By combining the above result with Theorem 2.0.2, we get that:

\[ v_8 \cdot (|\beta'_K| - 1) \leq \text{vol}(S^3 \setminus K) \]

for the links considered in this paper.

By applying Equation 15 and Inequalities 9, 10, 12, and 14 respectively, to Theorem 2.0.2 we get the following respective results:

**Proposition 5.0.2.** Let \( D(K) \) be a connected, prime, \( A \)-adequate link diagram that satisfies the TELC and contains \( t(D) \geq 2 \) twist regions. Then \( K \) is hyperbolic and:

\[ \text{vol}(S^3 \setminus K) < 30v_3 \cdot (|\beta'_K| - 1) + 10v_3 \cdot \text{(st}(D) - 1). \]

**Proposition 5.0.3.** Let \( D(K) \) satisfy the hypotheses of the Main Theorem (Theorem 1.0.1). Furthermore, assume that each other circle of \( H_A \) has at least \( m \geq 3 \) incident twist region resolutions. Then \( K \) is hyperbolic and:

\[ \text{vol}(S^3 \setminus K) < \frac{m}{m - 2} \cdot 10v_3 \cdot (|\beta'_K| - 1) - 10v_3. \]

**Proposition 5.0.4.** Let \( D(K) \) be a strongly negative plat diagram. Then \( K \) is hyperbolic and:

\[ \text{vol}(S^3 \setminus K) < \frac{25v_3}{2} \cdot (|\beta'_K| - 1) - \frac{5v_3}{2}. \]

**Proposition 5.0.5.** Let \( D(K) \) be a mixed-sign plat diagram that contains at least as many negative twist regions as it does positive twist regions. Then \( K \) is hyperbolic and:

\[ \text{vol}(S^3 \setminus K) < 30v_3 \cdot (|\beta'_K| - 1) + 20v_3. \]

**Remark 5.0.4.** The results in this section show that the links of the Main Theorem (including the strongly negative and mixed-sign plats) and the links of Corollary 3.4.1 satisfy a Coarse Volume Conjecture ([8], Section 10.4).
References

[1] Ian Agol, Peter A. Storm, and William P. Thurston. Lower bounds on volumes of hyperbolic Haken 3-manifolds. *J. Amer. Math. Soc.*, 20(4):1053–1077, 2007. With an appendix by Nathan Dunfield.

[2] Gary Chartrand and Ping Zhang. *A First Course in Graph Theory*. Dover Publications, 2012.

[3] Oliver T. Dasbach and Xiao-Song Lin. On the head and the tail of the colored Jones polynomial. *Compos. Math.*, 142(5):1332–1342, 2006.

[4] Oliver T. Dasbach and Xiao-Song Lin. A voluminous theorem for the Jones polynomial of alternating knots. *Pacific J. Math.*, 231(2):279–291, 2007.

[5] David Futer and François Guéritaud. On canonical triangulations of once-punctured torus bundles and two-bridge link complements. *Geom. Topol.*, 10:1239–1284, 2006.

[6] David Futer, Efstratia Kalfagianni, and Jessica Purcell. Hyperbolic semi-adequate links. Arxiv:1311.3008v1.

[7] David Futer, Efstratia Kalfagianni, and Jessica Purcell. Jones polynomials, volume, and essential knot surfaces: a survey. In: Proceedings of Knots in Poland III, Banach Center Publications, to appear, arXiv:1110.6388v2.

[8] David Futer, Efstratia Kalfagianni, and Jessica Purcell. *Guts of surfaces and the colored Jones polynomial*, volume 2069 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2013.

[9] David Futer, Efstratia Kalfagianni, and Jessica S. Purcell. Dehn filling, volume, and the Jones polynomial. *J. Differential Geom.*, 78(3):429–464, 2008.

[10] David Futer, Efstratia Kalfagianni, and Jessica S. Purcell. Symmetric links and Conway sums: volume and Jones polynomial. *Math. Res. Lett.*, 16(2):233–253, 2009.

[11] David Futer, Efstratia Kalfagianni, and Jessica S. Purcell. Cusp areas of Farey manifolds and applications to knot theory. *Int. Math. Res. Not. IMRN*, 2010(23):4434–4497, 2010.

[12] Marc Lackenby. The volume of hyperbolic alternating link complements. *Proc. London Math. Soc. (3)*, 88(1):204–224, 2004. With an appendix by Ian Agol and Dylan Thurston.

[13] Alexander Stoimenow. Coefficients and non-triviality of the Jones polynomial. *J. Reine Angew. Math.*, 657:1–55, 2011.