AFFINE KRICHEVER-NOVIKOV ALGEBRAS, THEIR REPRESENTATIONS AND APPLICATIONS

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Dedicated to Professor S.P. Novikov in honour of his 65th Birthday

ABSTRACT. The survey of the current state of the theory of Krichever-Novikov algebras including new results on local central extensions, invariants, representations and casimir operators.

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1. Introduction

In 1987, the soliton theory investigations led Krichever and Novikov to the new fundamental notion in the Lie algebra theory. They introduced the two-dimensional algebraic-geometrical counterpart of the celebrated Virasoro and Kac-Moody algebras. Their definition is based on the well-known in soliton theory type of algebraic-geometrical data, namely Riemann surfaces with punctures and fixed (jets of) coordinates in their neighborhoods. In case of two punctures (the only one considered by them) this enabled them to introduce an almost graded structure, distinguish a unique local central extension among the variety of others and, finally, develop the secondary quantization formalism (mathematically, construct modules generated by vacuum vectors).

The Krichever-Novikov algebras contain affine Kac-Moody and Virasoro algebras as a subclass. This distinguishes them among other (and later) approaches to generalization of the current algebras on two dimensions.

Krichever-Novikov algebras have numerous relations to the fundamental problems of geometry, analysis and mathematical physics. The problem of classification of their co-adjoint orbits turns out to be a version of the Riemann-Hilbert problem. In essential, these orbits are classified by the monodromy representations of fundamental group of the corresponding (punctured) Riemann surface. All known representations of those algebras are parameterized by the holomorphic vector bundles on the Riemann surface. Thus, for the Krichever-Novikov algebras, there exists the correspondence between orbits and representations (originally proposed as a general principle by A.A.Kirillov). As an essential part, this correspondence includes the well-known correspondence between monodromy representations and holomorphic bundles. It is interesting that, for Krichever-Novikov algebras, the correspondence between orbits and representations has also the form of global geometric Langlands correspondence, i.e. the correspondence between representations of the fundamental group of the Riemann surface of a function field (function algebra, in our case) and of the matrix Lie algebra over this field (affine Krichever-Novikov algebra). Below, we try to demonstrate these relations. We also show that the the well-known Hitchin integrals appear as co-adjoint invariants of Krichever-Novikov algebras, and their quantization is closely related to the casimir-type operators which we call semi-casimirs.

There is a fundamental relation, based on the Kodaira-Spencer theory, between the Krichever-Novikov algebras and the moduli spaces of Riemann surfaces with punctures. This relation is as follows: the tangent space to the moduli space of Riemann surfaces with an arbitrary number of punctures and arbitrary orders of fixed jets of local coordinates at punctures is isomorphic to the direct sum of certain homogeneous subspaces of Krichever-Novikov Virasoro-type algebra (see for the 1-puncture situation (respectively, the Virasoro algebra), and for the 2-puncture situation).
In turn, the relations between Krichever-Novikov algebras and the moduli spaces of Riemann surfaces provide us with a basis for certain applications in Conformal Field Theory [27, 26] which we do not concern in this publication, except for the application to the Hitchin integrals.

The present paper is a survey of the current state of the theory of Krichever-Novikov algebras, their invariants, representations and some applications, including recent results.

2. The algebras of Krichever-Novikov type

Let Σ be a compact Riemann surface of genus \( g \), or in terms of algebraic geometry, a smooth projective curve over \( \mathbb{C} \) respectively. Let

\[ I = (P_1, \ldots, P_N), \quad N \geq 1 \]

be a tuple of ordered, distinct points (“marked points”, “punctures”) on \( \Sigma \), \( P_\infty \) a distinguished point on \( \Sigma \) different from \( P_i \) for every \( i \), and \( A = I \cup \{ P_\infty \} \). The points in \( I \) are called the in-points, and the point \( P_\infty \) the out-point. The more general case of an arbitrary finite set of out-points is considered in [19], [22].

2.1. The Lie algebras \( A, \overline{g}, L, D^1 \) and \( D^1_\theta \). Let \( A := \mathcal{A}(\Sigma, I, P_\infty) \) be the associative algebra of meromorphic functions on \( \Sigma \) which are regular except at the points \( P \in A \). Let \( \mathfrak{g} \) be a complex finite-dimensional reductive Lie algebra. Then

\[ \mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} A \]

is called the Krichever-Novikov current algebra [12, 28]. The Lie bracket on \( \overline{\mathfrak{g}} \) is given by the relations

\[ [x \otimes A, y \otimes B] = [x, y] \otimes AB. \]

We will often suppress the symbol \( \otimes \) in our notation.

Let \( \mathcal{L} \) denote the Lie algebra of meromorphic vector fields on \( \Sigma \) which are allowed to have poles only at the points \( P \in A \) [12, 13, 14].

For the Riemann sphere \( (g = 0) \) with quasi-global coordinate \( z \), \( I = \{0\} \) and \( P_\infty = \infty \), \( \mathcal{A} \) is the algebra of Laurent polynomials, the current algebra \( \overline{\mathfrak{g}} \) is the loop algebra, and \( \mathcal{L} \) is the Witt algebra.

The algebra \( \mathcal{L} \) operates on the elements of \( \mathcal{A} \) by the (Lie) derivative. This allows to define the Lie algebra of differential operators \( D^1 \) of degree \( \leq 1 \) as their semi-direct sum. As a vector space, \( D^1 = \mathcal{A} \oplus \mathcal{L} \). The Lie structure is defined by

\[ [(g, e), (h, f)] := (e.h - f.g, [e, f]), \quad g, h \in \mathcal{A}, \ e, f \in \mathcal{L}. \]

Here \( e.f \) denotes the (Lie) derivative.

Similarly, we define the Lie algebra \( D^1_\theta \). As a vector space, \( D^1_\theta = \overline{\mathfrak{g}} \oplus \mathcal{L} \). The Lie structure on \( \overline{\mathfrak{g}}, \mathcal{L} \) is as above, and additionally

\[ [e, x \otimes A] := -[x \otimes A, e] := x \otimes (e.A). \]
In particular, for $g = \mathfrak{gl}(1)$ one obtains $D^1_g = D^1$ as a special case.

2.2. Meromorphic forms of weight $\lambda$ and Krichever-Novikov duality. Let $K$ be the canonical line bundle. For every $\lambda \in \mathbb{Z}$ we consider the bundle $K^\lambda := K^{\otimes \lambda}$. Here we use the usual convention: $K^0 = \mathcal{O}$ is the trivial bundle, and $K^{-1} = K^*$ is the holomorphic tangent line bundle. Indeed, after fixing a theta characteristics, i.e. a bundle $S$ with $S^{\otimes 2} = K$, it is possible to consider $\lambda \in \frac{1}{2}\mathbb{Z}$. Denote by $F^\lambda$ the (infinite-dimensional) vector space of global meromorphic sections of $K^\lambda$ which are holomorphic on $\Sigma \setminus A$. The elements of $F^\lambda$ are called (meromorphic) forms or tensors of weight $\lambda$.

The cases of a special interest are as follows: functions ($\lambda = 0$), vector fields ($\lambda = -1$), 1-forms ($\lambda = 1$), and quadratic differentials ($\lambda = 2$). The space of functions is already denoted by $A$, and the space of vector fields by $L$.

Denote by $C_S$ any cycle homologous to a small circle surrounding $P_\infty$. Following [12], we refer to any $C_S$ as to a separating cycle. The notion of separating cycle is introduced by Krichever and Novikov and closely related to their conceptions of locality and almost-grading.

**Definition 2.1.** The Krichever-Novikov pairing (KN pairing) is the pairing between $F^\lambda$ and $F^{1-\lambda}$ given by

$$\langle f, g \rangle := \frac{1}{2\pi i} \int_{C_S} f \otimes g = \sum_{P \in I} \text{res}_P(f \otimes g) = -\text{res}_{P_\infty}(f \otimes g),$$

where $C_S$ is any separating cycle.

The last equality follows from the residue theorem. Observe that in (2.5) the integral does not depend on the separating cycle chosen.

2.3. Krichever-Novikov bases. In [12], Krichever and Novikov introduced special bases for the spaces of meromorphic tensors on Riemann surfaces with two marked points. For $g = 0$ the Krichever-Novikov bases coincide with the Laurent bases. The multi-point generalization of these bases is given in [19, 22] (see also [17]). We define here the Krichever-Novikov type bases for the tensors of an arbitrary weight $\lambda$ on Riemann surfaces with $N$ marked points as introduced in [19, 22].

For fixed $\lambda$ and for every $n \in \mathbb{Z}$, and $p = 1, \ldots, N$ we exhibit a certain element $f^\lambda_{n,p} \in F^\lambda$. The basis elements are chosen in such a way that they fulfill the duality relation

$$\langle f^\lambda_{n,p}, f^{1-\lambda}_{m,r} \rangle = \delta^m_{-n} \cdot \delta^r_p,$$

with respect to the KN pairing (2.5). In particular, this implies that the KN pairing is non-degenerate. Additionally, the elements fulfill

$$\text{ord}_P(f^\lambda_{n,p}) = (n + 1 - \lambda) - \delta^P_i, \quad i = 1, \ldots, N.$$
choosing local coordinates $z_p$ at the points $P_p$ the scalar can be fixed by requiring

$$f^\lambda_{n,p}(z_p) = z_p^{n-\lambda}(1 + O(z_p))(dz_p)^\lambda, \quad p = 1, \ldots, N.$$  

To give an impression about requirement at $P_\infty$ let us consider the case $g \geq 2$, $\lambda \neq 0,1$ and a generic choice for the points in $A$ (or $g = 0$ without any restriction). Then we require

$$\text{ord}_{P_\infty}(f^\lambda_{n,p}) = -N \cdot (n + 1 - \lambda) + (2\lambda - 1)(g - 1).$$  

By Riemann-Roch type arguments, it is shown in [20] that there exists only one such element.

Explicit expressions for the basis elements $f^\lambda_{n,p}$ in terms of rational functions ($g = 0$), the Weierstraß functions ($g = 1$), prime forms and theta functions ($g \geq 1$) are given in [12], [21], [13], [23]. For $g = 0$ and $g = 1$, such expressions can be found also in [27, §§2,7]. One can see from the explicit expressions that the basis elements “analytically” depend on the complex structure of the Riemann surface.

For the following cases we introduce a special notation:

$$A_{n,p} := f^0_{n,p}, \quad e_{n,p} := f^{-1}_{n,p}, \quad \omega^{n,p} := f^1_{n,p}, \quad \Omega^{n,p} := f^2_{n,p}.$$  

For $g = 0$ and $N = 1$ the basis elements (2.10) coincide with the standard generators of the Laurent polynomials, the Witt algebra and their dual spaces, respectively. For $g \geq 1$ and $N = 1$ these elements coincide, up to a shift of index, with those given by Krichever and Novikov [12, 13, 14].

2.4. Almost graded structure, triangular decompositions. For $g = 0$ and $N = 1$ the Lie algebras introduced in Section 2.1 are graded. A grading is a necessary tool for developing their structure theory and the theory of their highest weight representations. For the higher genus case (and for the multipoint situation for $g = 0$) there is no grading. It is a fundamental observation due to Krichever and Novikov [12] that a weaker concept, an almost grading, is sufficient to develop a suitable structure and representation theory in this more general context.

An (associative or Lie) algebra is called almost-graded if it admits a direct decomposition as a vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, where (1) $\dim V_n < \infty$ and (2) there are constants $R$ and $S$ such that

$$V_n \cdot V_m \subseteq \bigoplus_{h = n+m-R}^{n+m+S} V_h, \quad \forall n, m \in \mathbb{Z}.$$  

The elements of $V_n$ are called homogeneous elements of degree $n$. Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be an almost-graded algebra and $M$ an $V$-module. The module $M$ is called an almost-graded $V$-module if it admits a direct decomposition as a vector space $M = \bigoplus_{m \in \mathbb{Z}} M_m$, where (1) $\dim M_m < \infty$ and (2) there are
constants $T$ and $U$ such that
\begin{equation}
V_n . M_m \subseteq \bigoplus_{h=n+m-T}^{n+m+U} M_h, \quad \forall n, m \in \mathbb{Z}.
\end{equation}

The elements of $M_n$ are called \textit{homogeneous elements of degree} $n$.

For the space $\mathcal{F}_n^\lambda$ of Section 2.2, its homogeneous degree $n$ subspace $\mathcal{F}_n^\lambda$ is defined as the subspace generated by the elements $f_{n,p}^\lambda$, $p = 1, \ldots, N$. Then $\mathcal{F}_n^\lambda = \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_n^\lambda$.

**Proposition 2.2.** \cite{19, 22} With respect to the introduced degree, the vector field algebra $L$, the function algebra $A$, and the differential operator algebra $D^1$ are almost-graded and the $\mathcal{F}_n^\lambda$ are almost-graded modules over them.

For particular algebras, the almost graded structure has a more special description.

**Proposition 2.3.** \cite{19, 22} There exist constants $K, L, M \in \mathbb{N}$ such that for all $n, m \in \mathbb{Z}$
\begin{equation}
A_{n,p} . A_{m,r} = \delta_p^r A_{n+m,p} + \sum_{h=n+m+1}^{n+m+K} \sum_{s=1}^{N} \alpha_{(n,p),(m,r)}^{(h,s)} A_{h,s},
\end{equation}

\begin{equation}
[e_{n,p}, e_{m,r}] = \delta_p^r (m-n) e_{n+m,p} + \sum_{h=n+m+1}^{n+m+L} \sum_{s=1}^{N} \gamma_{(n,p),(m,r)}^{(h,s)} e_{h,s},
\end{equation}

\begin{equation}
e_{n,p} . A_{m,r} = \delta_p^r m A_{n+m,p} + \sum_{h=n+m+1}^{n+m+M} \sum_{s=1}^{N} \beta_{(n,p),(m,r)}^{(h,s)} A_{h,s},
\end{equation}

with suitable coefficients $\alpha_{(n,p),(m,r)}^{(h,s)}, \beta_{(n,p),(m,r)}^{(h,s)}, \gamma_{(n,p),(m,r)}^{(h,s)} \in \mathbb{C}$.

The constants $K, L$ and $M$ depend on the genus $g$ and the number of points $N$. Explicit expressions can be found in \cite{27, §2}.

As a vector space, the algebra $A$ can be decomposed as follows:

\begin{equation}
A = A_+ \oplus A_{(0)} \oplus A_-,
\end{equation}

$A_+ := \langle A_{n,p} \mid n \geq 1, p = 1, \ldots, N \rangle$, $A_- := \langle A_{n,p} \mid n \leq -K - 1, p = 1, \ldots, N \rangle$, $A_{(0)} := \langle A_{n,p} \mid -K \leq n \leq 0, p = 1, \ldots, N \rangle$.

and the Lie algebra $\mathcal{L}$ as follows:

\begin{equation}
\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_{(0)} \oplus \mathcal{L}_-,
\end{equation}

$\mathcal{L}_+ := \langle e_{n,p} \mid n \geq 1, p = 1, \ldots, N \rangle$, $\mathcal{L}_- := \langle e_{n,p} \mid n \leq -L - 1, p = 1, \ldots, N \rangle$, $\mathcal{L}_{(0)} := \langle e_{n,p} \mid -L \leq n \leq 0, p = 1, \ldots, N \rangle$.

We call (2.14), (2.15) the \textit{triangular decompositions}. In a similar way we obtain a triangular decomposition of $D^1$. 
Due to the almost-grading the subspaces $A_\pm$ and $L_\pm$ are subalgebras but the subspaces $A_{(0)}$ and $L_{(0)}$, in general, are not. We use the term critical strip for the latters.

Observe that $A_\pm$ and $L_\pm$ can be described as the algebra of functions (respectively, vector fields) having a zero of, at least, order one (respectively, two) at the points $P_i, i = 1, \ldots, N$. These algebras can be enlarged by adding all elements which are regular at all $P_i$'s (these are $\{A_{0,p}, p = 1, \ldots, N\}$, respectively $\{e_{0,p}, e_{-1,p}, i = 1, \ldots, N\}$). We denote the enlarged algebras by $A^*_\pm$, resp. by $L^*_\pm$.

On the other hand $A_-$ and $L_-$ could also be enlarged such that they contain all elements which are regular at $P_\infty$. This is explained in detail in [27]. We obtain $A^{(p)}_-$ and $L^{(p)}_-$, respectively. For every $p \in \mathbb{N}_0$, let $L^{(p)}_-$ be the subalgebra of vector fields of order $\ge p + 1$ at the point $P_\infty$, and $A^{(p)}_-$ be the subalgebra of functions of order $\ge p$ at the point $P_\infty$. We obtain a decomposition

\[ L = L_+ \oplus L^{(p)}_0 \oplus L^{(p)}_-, \quad \text{for } p \ge 0 \quad \text{and} \quad A = A_+ \oplus A^{(p)}_0 \oplus A^{(p)}_, \quad \text{for } p \ge 1, \]

with “critical strips” $L^{(p)}_0$ and $A^{(p)}_0$, which are only subspaces. The case of $p = 1$ is of particular interest. We call $L^{(1)}_0$ the reduced critical strip. For $g \ge 2$ its dimension is equal to

\[ \dim L^{(1)}_0 = N + N + (3g - 3) + 1 + 1 = 2N + 3g - 1. \]

On the right hand side, the first two terms correspond to $L_0$ and $L_{-1}$. The intermediate term comes from the vector fields in the basis which have poles at the $P_i, i = 1, \ldots, N$ and $P_\infty$. The $1 + 1$ corresponds to the basis vector fields with exact order zero (one) at $P_\infty$.

On the higher genus current algebra $\mathfrak{g}$, the almost-grading is introduced by setting $\deg(x \otimes A_{n,p}) := n$. As above, we obtain a triangular decomposition

\[ \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_-, \quad \text{with} \quad \mathfrak{g}_\beta := \mathfrak{g} \otimes A_\beta, \quad \beta \in \{-, (0), +\}, \]

In particular, $\mathfrak{g}_\pm$ are subalgebras. The corresponding is true for the enlarged subalgebras. Among them, $\mathfrak{g}_{\text{reg}} := \mathfrak{g} \otimes A_{(1)}$ is of special importance. It is called the regular subalgebra.

The finite-dimensional Lie algebra $\mathfrak{g}$ can be naturally considered as a subalgebra of $\mathfrak{g}_0$. It is a subspace of $\mathfrak{g}_0$. To see this, make use of the relation $1 = \sum_{p=1}^N A_{0,p}$ [27] Lemma 2.6].

2.5. Local central extensions. Let $\mathcal{V}$ be a Lie algebra and $\gamma$ a 2-cocycle on $\mathcal{V}$, i.e. an antisymmetric bilinear form obeying

\[ \gamma([f, g], h) + \gamma([g, h], f) + \gamma([h, f], g) = 0, \quad \forall f, g, h \in \mathcal{V}. \]

Given $\gamma$, we can define a Lie algebra structure on $\hat{\mathcal{V}} = \mathcal{V} \oplus \mathbb{C} t$ by

\[ [f, g]^* := [f, g] + \gamma(f, g) \cdot t, \quad [t, \hat{\mathcal{V}}]^* = 0 \]
where $t$ is a formal central generator. Up to equivalence, central extensions are classified by the elements of $H^2(V, \mathbb{C})$, the second Lie algebra cohomology space with values in the trivial module $\mathbb{C}$. In particular, two cocycles $\gamma_1, \gamma_2$ define equivalent central extensions if and only if there exist a linear form $\phi$ on $V$ such that

$$
(2.21) \quad \gamma_1(f, g) = \gamma_2(f, g) + \phi([f, g]).
$$

**Definition 2.4.** Let $V = \bigoplus_{n \in \mathbb{Z}} V_n$ be an almost-graded Lie algebra. A cocycle $\gamma$ for $V$ is called local (with respect to the almost-grading) if there exist $M_1, M_2 \in \mathbb{Z}$ with

$$
(2.22) \quad \forall n, m \in \mathbb{Z} : \quad \gamma(V_n, V_m) \neq 0 \implies M_2 \leq n + m \leq M_1.
$$

Assuming $\deg(t) = 0$, the central extension $\hat{V}$ is almost-graded if and only if it is given by a local cocycle $\gamma$. In this case we call $\hat{V}$ an almost-graded central extension or a local central extension.

For $g = 0$ the description of 2-cocycles is due to I.Gelfand-D.Fuchs, V.Kac, E.Arbarello-C.DeConcini-V.Kac-C.Procesi. For $g > 0$ the problem admits a direct generalization, namely, the problem of description of the universal central extensions. For different Lie algebras on Riemann surfaces such description is given by I.Frenkel, E.Frenkel, P.Etingof, Ch.Kassel, M.Bremner, D.Millionschikov. As it is mentioned above, only local central extensions enable one to develop the highest weight (physically, the secondary quantization) formalism. The problem of classification of the local central extensions reduces to the description of local 2-cocycles. This problem originates in the work of I.Krichever and S.Novikov [12] where the authors introduced what we call below the geometric cocycles for $\mathcal{A}$ and $\mathcal{L}$ and, for $\mathcal{L}$, outlined the proof of the 1-dimensionality of the space of local cocycles.

The full classification of the local cocycles for $\mathfrak{g}$ and $\mathcal{D}_0^1$ with a reductive $\mathfrak{g}$, including full proofs, is given by M.Schlichenmaier [24, 18]. For the Lie algebras of Section 2 he has proven that any local 2-cocycle is cohomologous to a linear combination of the basic geometric cocycles. Below, we give the full list of them. The more precise result of [24, 18] for the Lie algebra $\mathfrak{g}$ is as follows: any local 2-cocycle is cohomologous to an $\mathcal{L}$-invariant one, and the latter is geometric (by definition, a cocycle $\gamma$ on $\mathfrak{g}$ is $\mathcal{L}$-invariant if $\gamma(x(e.A), yB) = \gamma(x(A), y(e.B))$ for any $e \in \mathcal{L}, x, y \in \mathfrak{g}, A, B \in \mathcal{A}$).

The list of basic geometric cocycles is as follows:

— for $\mathcal{A}$ [12]:

$$
(2.23) \quad \gamma^{(A)}(g, h) := \frac{1}{2\pi i} \int_{C_s} g dh;
$$

— for $\mathcal{L}$ [12]:

$$
(2.24) \quad \gamma^{(L)}(e, f) := \frac{1}{2\pi i} \int_{C_S} \left( \frac{1}{2} (\tilde{e}^m \tilde{f} - \tilde{e} \tilde{f}^m) - R \cdot (\tilde{e} \tilde{f} - \tilde{e} \tilde{f}) \right) dz
$$
where \( e = \tilde{e} \frac{d}{dz} \) and \( f = \tilde{f} \frac{d}{dz} \) are the local representations of the involved vector fields, \( \tilde{R} \) be a global holomorphic projective connection (see [12] for details). A different choice of the projective connection (even if we allow meromorphic projective connections with poles only at the points in \( A \)) yields a cohomologous cocycle, hence an equivalent central extension;

— for \( D^1 \): \eqref{2.23} \eqref{2.24} and the following mixing cocycle

\[
\gamma^{(m)}(e,g) := -\gamma^{(m)}(g,e) := \frac{1}{2\pi i} \int_{C_x} (\tilde{e} \cdot g'' + T \cdot (\tilde{e} \cdot g')) \, dz
\]

where \( T \) is an affine connection (see details in \([22], [32], [24]\)) which is holomorphic outside \( A \) and has at most a pole of order one at \( P_\infty \). Again, the cohomology class does not depend on the chosen affine connection;

— for \( g \):

\[
\gamma^{(\alpha)}(x \otimes f, y \otimes g) = \alpha(x,y) \gamma^{(A)}(g, h)
\]

where \( x, y \in \mathfrak{g} \), \( f, g \in \mathcal{A} \), \( \alpha \) is a bilinear form. We give here the full list of basic geometric cocycles only for \( \mathfrak{g} = \mathfrak{sl}(n) \) and \( \mathfrak{g} = \mathfrak{gl}(n) \). In the first case, it consists from only one cocycle of the form \eqref{2.26} corresponding to the (Cartan-Killing) form \( \alpha(x,y) = \text{tr}(ad x \cdot ad y) \). In the second case, there are two independent cocycles corresponding to \( \alpha_1(x,y) = \text{tr}(xy) \) and \( \alpha_2(x,y) = \text{tr}(x) \text{tr}(y) \), respectively;

— for \( D^1_\mathfrak{g} \), \( \tilde{g} = \mathfrak{gl}(n) \):

\[
\gamma^{(m,\mathfrak{g})}(e, x \otimes g) = \text{tr}(x) \gamma^{(m)}(e, g)
\]

where again \( x \in \mathfrak{g} \), \( e \in \mathcal{L} \), \( g \in \mathcal{A} \). Hence, \( \gamma^{(m,\mathfrak{g})} \) for \( \mathfrak{g} = \mathfrak{sl}(n) \). The absence of mixing cocycles in semi-simple case was observed already in \([32]\).

For the estimation of boundaries of locality and other details we refer to \([24, 18]\).

2.6. Affine algebras. Now we are in position to define affine Krichever-Novikov algebras. Let \( \mathfrak{g} \) be a reductive finite-dimensional Lie algebra. By affine Krichever-Novikov algebra we mean the central extension of \( \mathfrak{g} \) given by a cocycle of the form \eqref{2.26}.

Thus, as a linear space, an affine Krichever-Novikov algebra is \( \hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathcal{A} \oplus \mathbb{C} t \), and the Lie bracket is given by

\[
[x \otimes f, y \otimes g] = [x, y] \otimes (fg) + \alpha(x,y) \cdot \gamma^{(A)}(f, g) \cdot t, \quad [t, \hat{\mathfrak{g}}] = 0,
\]

where \( \gamma^{(A)} \) is given by \eqref{2.23}, \( x, y \in \mathfrak{g} \), \( f, g \in \mathcal{A} \), \( t \) is a formal central generator. Certainly, \( \hat{\mathfrak{g}} \) depends on the bilinear form \( \alpha \). For \( \mathfrak{g} = \mathfrak{sl}(n) \) and \( \mathfrak{g} = \mathfrak{gl}(n) \) we take \( \alpha(x,y) = \text{tr}(xy) \).

The cocycle defining the central extension is local, hence \( \hat{\mathfrak{g}} \) is almost-graded (we set \( \text{deg} t := 0 \) and \( \text{deg}(x \otimes A_{n,p}) := n \)). Again we obtain a triangular decomposition

\[
\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_{(0)} \oplus \hat{\mathfrak{g}}_- \quad \text{with} \quad \hat{\mathfrak{g}}_\pm \cong \mathfrak{g}_\pm \quad \text{and} \quad \hat{\mathfrak{g}}_{(0)} = \mathfrak{g}_{(0)} \oplus \mathbb{C} \cdot t.
\]
The corresponding is true for the enlarged algebras. Among them,

$$\hat{g}_{\text{reg}} := \hat{g}^{(1)} = g \otimes A^{(1)}, \quad \hat{g}^{*, \text{ext}}_{+} = \mathbb{C}t = (g \otimes A^{*}_{+}) \oplus \mathbb{C}t,$$

are of special interest.

### 3. Orbits and invariants

In this section, following the lines of [28, 29, 31], we give a description of the co-adjoint orbits and invariants of an affine Krichever-Novikov algebra $\hat{g}$. For simplicity, we consider the case of the Riemann surface with two marked points. Thus, $\hat{g} = g \otimes \mathbb{C}A \oplus \mathbb{C}t$ where $A = A(\Sigma, P_{\pm})$, $\Sigma$ is a compact algebraic curve over $\mathbb{C}$ with two marked points $P_{\pm}$ (genus $\Sigma > 0$), $A(\Sigma, P_{\pm})$ stays for the algebra of meromorphic functions on $\Sigma$ which are regular out the points $P_{\pm}$, $g$ is a complex reductive Lie algebra.

Let $\omega$ be a meromorphic $g^{*}$-valued 1-form on $\Sigma$ regular outside the points $P_{\pm}$, $b \in \mathbb{C}$. We identify the dual space $\hat{g}^{*}$ with the space of operators of the form $bd + \omega$ which take the sections of the $l$-dimensional trivial bundle on $\Sigma$ to the sections of the tensor product of the same bundle by $K$. Here, $d$ denotes the differential on $\Sigma$.

Consider the mappings $\Sigma \to \exp g$ holomorphic outside $P_{\pm}$. Such mappings, with the operation of a point-wise multiplication, form the group which we denote $G$. We call $G$ the current group, and its elements the group currents. The current group acts on the elements of $\hat{g}^{*}$ by gauge transformations: $\omega \mapsto g\omega g^{-1} - b \cdot dg \cdot g^{-1}$, $b \mapsto b$. We want to describe the space of orbits of this action. Since we made no assumption on the behavior of the group currents at $P_{\pm}$, in general, $\hat{g}^{*}$ is not invariant with respect to the gauge action. There are two ways out. The first is to consider orbits in some bigger space. It is an interesting question what space it should be. Our conjecture is that this is a space of matrix Backer-Achiezer functions. Here, we choose another approach. By orbit we mean the intersection of the true orbit with $\hat{g}^{*}$.

**Theorem 3.1.** [28, 29] The space of orbits in generic position which are in an invariant affine hyperplane $b = \text{const}$ ($b \neq 0$) in $\hat{g}^{*}$ is in a one-to-one correspondence with the space of equivalence classes of representations $\tau : \pi_{1}(\Sigma \setminus P_{\pm}) \to \exp g$ such that $\tau(\gamma)$ is a semisimple element (where $\gamma$ is a homotopy class of the separating contour $C_{S}$, see Section 2.3).

We call $\tau$ the monodromy representation of $\pi_{1}(\Sigma \setminus P_{\pm})$. The correspondence between the orbits and the monodromy representations is established as follows [28, 29]: each element $bd + \omega \in \hat{g}^{*}$ is assigned with the monodromy equation

$$E: \text{monoe} \quad (bd + \omega)g = 0,$$

and, further on, with the monodromy representation of this equation. For gauge equivalent elements, this procedure results in the equivalent monodromy representations and vise versa.
To complete the classification of the orbits (at least, in a generic position), we should check that any monodromy representation (of a certain class) corresponds to some equation of the form (3.1), i.e., resolve the Riemann-Hilbert problem. Under assumption of semi-simplicity of $\tau(\gamma)$, this problem can be resolved which follows from the results of [1], see also [2, Proposition 1.5.2].

Define co-adjoint invariants of $\hat{g}$ as such functions on $\hat{g}^*$ that are constant on the co-adjoint orbits, hence can be pushed down to the orbit space. It is our next step to construct a full system of independent co-adjoint invariants. We will do it for $\mathfrak{g} = \mathfrak{gl}(n)$.

Following [2], each monodromy representation coupled with a set of highest weights of $\mathfrak{g}$ assigned to the marked points can be associated with a smooth vector bundle on $\Sigma$ and a flat logarithmic (see below for the definition) connection $\nabla$ on this bundle. We interpret this as an assignment of a bundle (and a connection) to each co-adjoint orbit of $\hat{g}$. It is our conjecture that the additional parameters (highest weights) correspond to the above mentioned hypothetical extension of the dual space $\mathfrak{g}^*$.

Going over to the construction of the invariants, take an arbitrary dominant weight $\lambda_+ = (\lambda_1, \ldots, \lambda_n)$ ($\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$) at the point $P_+$ and a dominant weight $\lambda_-$ of the form $(*, 0, \ldots, 0)$ or $(0, \ldots, 0, *)$ at the point $P_-$. Denote by $B_{\lambda_+ \lambda_- \tau}$ the corresponding vector bundle on $\Sigma$. In accordance with [2], the weight $\lambda_-$ of such form is uniquely determined by the triple consisting of $\lambda_+$, $\tau$ and the degree of the bundle. Let us require the degree to be equal to $gn$, $(g = \text{genus}(\Sigma)$, $n = \text{rank} \mathfrak{g})$, then we can suppress $\lambda_-$ in the notation of the bundle and write it down as $B_{\lambda \tau}$ where $\lambda = \lambda_+$.

According to the Donaldson theorem [4,7], $\nabla$ is gauge equivalent to the connection of the form $d + A + \phi + \phi^*$ where $(A, \phi)$ is a solution to the self-duality equations $d_A^* \phi = 0$, $F(A) + [\phi, \phi^*] = 0$, $F(A)$ is the curvature of the connection $A$, $*$ is the Hermitian conjugation (for $\mathfrak{g} = \mathfrak{gl}(n)$), $d + A = d_A^* + d_A^\tau$ is the standard notation for a connection in complex coordinates, and the form of the connection $A$ has no singularities at the marked points. Define the complex structure on $B_{\lambda \tau}$ by means of the $d_A^\tau$ as a $\overline{\nabla}$-operator. According to [7], the bundle $B_{\lambda \tau}$ with this complex structure is a holomorphic bundle on $\Sigma$ (in particular this implies $\phi \in \Omega^{1,0}(\Sigma \setminus P_\pm)$, $\phi^* \in \Omega^{0,1}(\Sigma \setminus P_\pm)$, hence, both these forms can be retrieved from the canonical form of the connection). The just constructed holomorphic bundle $B_{\lambda \tau}$ has the rank $n$ and the degree $gn$. We assume that $B_{\lambda \tau}$ is generic in sense of [11] (see also [30]), hence the space of its meromorphic sections which are holomorphic out the marked points $P_\pm$ is isomorphic to the space $\mathcal{F}$ of meromorphic vector-functions $f = (f_1, \ldots, f_l)^T$ on the same Riemann surface which are holomorphic out the points $P_\pm$ and the divisor $D = \Sigma_{i=1}^n \gamma_i$ of degree $gn$, have at most simple poles at the points of the latter and satisfy the following relations

$$(\text{res}_{\gamma_i} f_j)\alpha_{ik} = (\text{res}_{\gamma_i} f_k)\alpha_{ij}, \quad i = 1, \ldots, gl, \quad j = 1, \ldots, l,$$

where $\alpha_{ik}$ are constants. The points of the divisor $D$ and the numbers $\alpha_{ik}$ are called the Tjurin parameters of the bundle.
Our goal is to find the invariants of the gauge action on \( \hat{\mathfrak{g}}^* \). To this end, first, take \( \lambda = 0 \). A self-dual pair \( A, \phi \) has the well-known invariants called Hitchin integrals \([6]\), namely the coefficients of the expansion of \( \text{tr} \, \phi^k \) over the basis holomorphic tensors of weight \( k \), \( k = 1, \ldots, n \). For \( \mathfrak{g} = \mathfrak{gl}(n) \), it follows from \([6]\) that their number is equal to \( n^2(g-1) + 1 \). Provided the value of \( \tau \) on the class of the separating contour is nontrivial, \( \phi \), generically, has simple poles at the points \( P_\pm \). The space of such forms has the dimension \( n^2(g+1) \) (the contribution of holomorphic forms is equal to \( n^2g \), and \( n^2 \) is contributed by residues of \( \phi \) at \( P_\pm \), taking into account the relation \( \text{res}_{P_+} \phi + \text{res}_{P_-} \phi = 0 \)), hence, any such form is determined by \( n^2(g+1) \) invariants.

To obtain these invariants, the basis holomorphic tensors should be supplemented with tensors that have poles up to the order \( n^2 \) at \( P_\pm \). The number of independent main parts of such tensors is \( 2n^2(n-k) \) (\( n-k \), \( n-k+1, \ldots, n \)). These \( k \)'s have orders \( 1, \ldots, n-k \) at \( P_\pm \). They span the subspace of dimension \( 2(n-k) \) which gives \( 2(n-k) \) additional (singular) integrals. In total, we have \( 2n \) singular integrals for any \( k = 1, \ldots, n \), thus, \( 2n^2 \) integrals in addition to the regular (Hitchin) integrals. Observe that in the original Hitchin situation (only holomorphic tensors are considered) we should introduce only global holomorphic functions (i.e. constants), thus obtain no new integrals. By the same reason, for \( k = n \) all integrals come from \( \text{tr} \, \phi^n \). Since \( \text{res}_{P_+} \phi + \text{res}_{P_-} \phi = 0 \), the coefficients in the terms of degree \( n \) of the Laurent expansions of the tensor \( \text{tr} \, \phi^n \) at the marked points either are equal or differ by a sign (depending on parity of \( n \)). By this, one parameter is eliminated. Thus, we obtain \( n^2(g+1) \) generalized Hitchin invariants.

One more set of invariants, which depend (via the holomorphic structure) on the component \( A \) of the self-dual pair, is the tuple of above introduced Tjurin parameters of the bundle \( B_{0,\tau} \) \([34,11]\). There is \( n^2(g-1) + 1 \) independent ones among them. In addition, there is one ”trivial” invariant \( b \) (see above). In total, we have constructed \( 2n^2g + 2 \) independent invariants. As it is easy to show, this number equals to the dimension of the orbit space. Thus the generalized Hitchin invariants and the Tjurin invariants form the the full family of independent invariants of the gauge action on \( \hat{\mathfrak{g}}^* \).

4. Representations of Krichever-Novikov algebras

Let \( B \) be a holomorphic vector bundle on \( \Sigma \), \( \nabla \) a meromorphic connection on \( B \), \( \tau \) an irreducible representation of \( \mathfrak{g} \). For this section, our goal is to assign each triple \( (B, \nabla, \tau) \) with a wedge representation of \( \mathcal{D}^1_{\mathfrak{g}} \) and investigate the properties of this representation as a \( \hat{\mathfrak{g}} \)-module.
4.1. Krichever–Novikov bases for the holomorphic vector bundles. Actually, these are the bases in the function space \( \mathcal{F} \) (see Section 3) which is isomorphic to the space of meromorphic sections of \( B \). The bases in question were introduced in [15] (for \( N = 1 \)) and then applied to the construction of wedge (or fermion) representations in [30]. Here, combining the approaches of [15] and Section 2.3 (going back to [20, 21]) we introduce these bases for an arbitrary \( N \).

From now on, let \( n \) stay for the degree of the Krichever-Novikov basis element (to be defined below), \( l \) stay for the rank of \( g \) and \( r \) stay for the rank of \( B \).

For any tryple of integers \( n \in \mathbb{Z}, j = 0, \ldots, r - 1 \) and \( p = 1, \ldots, N \), let \( \psi_{n,j,p} = (\psi_{n,j,p}^i) \) be a vector function \( (i = 0, 1, \ldots, r - 1) \) in \( \mathcal{F} \). This function is specified by its asymptotic behavior at the points of \( A \) which is assumed to be as follows:

\[ \psi_{n,j,p}^i(z_q) = z_q^{n+1-\delta_{pj}}(\xi_{npj}^i + O(z_q)) \]

where \( z_q \) is a local coordinate at \( P_q \), \( q \in \{1, \ldots, N\} \), \( \delta_{pj} \) is the Kronecker symbol, \( \xi_{npj}^i \) are the complex numbers such that \( \xi_{npj}^i \xi_{npq}^j = 1, \xi_{npj}^i \xi_{npq}^j = 0, i > j \);

\[ \psi_{n,j,p}^i(z_\infty) = z_\infty^{-nN-N+1}(\xi_{npi}^i + O(z_\infty)) \]

where \( \xi_{npi}^i = 0, i < j \).

For a given \( n \), the space spanned by vector-valued functions \( \psi_{n,j,p} \) is referred to as the space of degree \( n \) functions.

**Proposition 4.1.** 1. There exists a unique vector-valued function \( \psi_{n,j,p} \) satisfying conditions (4.1) and (4.2).

2. For a given \( n \), the dimension of the space spanned by the vector-valued functions \( \psi_{n,j,p} \) is equal to \( rN \).

**Proof.** Introduce the divisor \( D_{n,p} = D + nP_p + \Sigma_{q \neq p}(n+1)P_q + (-nN-N+1)P_\infty \), hence \( \deg D_{n,p} = \deg D = -rg \). Let \( (D_{n,p}) \) denote the space of (scalar-valued) meromorphic functions with a divisor not less than \( D_{n,p} \), then \( \dim(D_{n,p}) = rg - g + 1 \). For the analogous space of functions taking values in an \( r \)-dimensional vector space (denote it by \( (D_{n,p})_r \)), we have \( \dim((D_{n,p})_r) = r(rg - g + 1) \). Such functions in \( \mathcal{F} \) satisfy the \((r-1)gr\) Tjurin relations. Therefore, \( \dim((D_{n,p})_r \cap \mathcal{F}) = r(gr - g + 1) - (r-1)gr = r \).

Observe that \( \psi_{n,j,p} \in (D_{n,p})_r \cap \mathcal{F} \) for any \( j \). For a given \( j \), this function is distinguished in \( (D_{n,p})_r \), by exactly \( r \) (normalizing) conditions on matrices \( \xi \). Thus, \( \psi_{n,j,p} \) is uniquely defined and the assertion 1 is proven.

Further on, observe that, for a given \( n \), the space of all degree \( n \) functions in \( \mathcal{F} \) is exactly a direct sum of its subspaces \( (D_{n,p})_r \cap \mathcal{F}, p = 1, \ldots, N \). Hence, its dimension equals to \( rN \). \( \square \)

4.2. Fermion representations.

In this paragraph, we briefly outline the construction of a fermion \( D^1_{\mathfrak{g}} \)-module. It was introduced in [30, 32] for the two-point situation \( (N = 1) \).
For the multi-point situation considered here, the exposition is similar up to a different definition of Krichever-Novikov basis (Section 4.1).

Let \( B \) be a rank \( r \) degree \( rg \) holomorphic vector bundle on \( \Sigma \), \( \nabla \) a meromorphic connection on \( B \) having simple poles at most at the points of \( A \) (hence, logarithmic — see Section 3), \( \tau \) an irreducible representation of \( g \) in the finite-dimensional vector space \( V_\tau \). Let \( \Gamma = \Gamma(B) \) denote the space of meromorphic sections of \( B \) holomorphic except at \( P_1, \ldots, P_N, P_\infty \). Introduce \( \Gamma_{B,\tau} := \Gamma(B) \otimes V_\tau \).

Proposition 4.2. Relations (4.3), (4.4) define a Lie algebra representation of \( \mathcal{D}_1^1 \) in \( \Gamma_{B,\tau} \).

Proof. For the \( g \) the claim is directly verified using (4.3).

By assumption, \( \nabla \) is meromorphic, hence, flat. By flatness, \( \nabla_{[e,f]} = [\nabla_e, \nabla_f] \) for all \( e, f \in \mathcal{L} \). Hence, \( \nabla \) defines a representation of \( \mathcal{L} \) in \( \Gamma(B) \).

By definition of a connection, for any \( s \in \Gamma(B), e \in \mathcal{L} \) and \( A \in \mathcal{A} \), we have \( \nabla_e(As) = e.As + A\nabla_es \) where \( e.A \) is the Lie derivative. Hence, \( [\nabla_e, A] = e.A \), i.e. the mapping \( e + A \to \nabla_e + A \) gives rise to a representation of \( \mathcal{D}_1^1 \) in \( \Gamma(B) \). \( \square \)

Choose a Krichever-Novikov base \( \{\psi_{n,p,j}\} \) in \( \Gamma(B) \) (see Section 4.1) and a weight base \( \{v_a|1 \leq a \leq \dim V_\tau\} \) in \( V_\tau \). Introduce \( \psi_{n,p,j,a} := \psi_{n,p,j} \otimes v_a \). Enumerate the elements \( \psi_{n,p,j,a} \) linearly in the ascending lexicographical order of the quadruples \( (n, p, j, a) \). We write down \( \psi_M = \psi_{n,p,j,a} \) if \( M = M(n, p, j, a) \in \mathbb{Z} \) is the number of the quadruple \( (n, p, j, a) \). Introduce the degree of \( \psi_M \) by \( \deg \psi_M := M \).

Proposition 4.3. With respect to the just introduced degree, \( \Gamma_{F,\tau} \) is an almost-graded \( \mathcal{D}_1^1 \)-module.

For the two-point situation \( (N = 1) \) the proof is given in [30, 32]). In case of general \( N \) the proof is similar.

Now we are in position to do the final step of the construction, namely, to introduce the fermion space corresponding to the pair \( (B, \tau) \) and define the \( \mathcal{D}_1^1 \)-action in this space.

Consider the vector space \( \mathcal{H}_{F,\tau} \) generated over \( \mathbb{C} \) by the formal expressions (semi-infinite wedge monomials) of the form \( \Phi = \psi_{N_0} \wedge \psi_{N_1} \wedge \ldots \) where the \( \psi_{N_i} \) are the above introduced basis elements of \( \Gamma_{F,\tau} \), the indices are strictly increasing, i.e. \( N_0 < N_1 < \ldots \), and \( N_k = k + m \) for a suitable \( m \) and all sufficiently large \( k \). The integer \( m \) depends on the monomial and following [9]
is called its charge. For a monomial $\Phi$ of charge $m$, the degree of $\Phi$ is defined as follows:

\[
\deg \Phi = \sum_{k=0}^{\infty} (N_k - k - m).
\]

Observe that there is an arbitrariness in the enumeration of the $\psi_{n,p,j,a}$'s for a fixed $n$; the just defined degree of a monomial does not depend on this arbitrariness.

The monomials without requirement $N_0 < N_1 < \ldots$ are also considered. It is assumed that they are antisymmetric with respect to the order of their wedge co-multipliers.

We want to extend the action of $D^1_\mathfrak{g}$ on $\Gamma_{B,\tau}$ to an action in $\mathcal{H}_{F,\tau}$. To this end, introduce the Lie algebra $\mathfrak{a}_\infty$ of the infinite matrices with "finitely many diagonals" and assign each element of $D^1_\mathfrak{g}$ with the matrix of its operator in $\Gamma(B)$ with respect to the basis $\psi_M$. Thus, we have obtained the embedding of $D^1_\mathfrak{g}$ into $\mathfrak{a}_\infty$. Then we use the standard representation of $\mathfrak{a}_\infty$ in $\mathcal{H}_{F,\tau}$ due to V.Kac (for example, see [9]). The last step is absolutely standard and was used in similar situations in [13, 14, 22]. We follow here the lines of [30, 32].

In turn, the construction due to V.Kac consists of two main steps, namely applying the "Leibnitz rule" and the "regularization" if the result is not well-defined. The latter gives rise to a representation of some central extension $\widehat{D}^1_\mathfrak{g}$ of the $D^1_\mathfrak{g}$. Restrict the cocycle of this central extension to $\mathfrak{g}$ and denote so obtained cocycle by $\gamma$.

\textbf{cokac} Proposition 4.4. For $\mathfrak{g} = \mathfrak{gl}(l)$ and any $x, y \in \mathfrak{g}$, $f, g \in A$

\[
\gamma(xf, yg) = \alpha \cdot \text{tr}(xy) \gamma(A)(f, g)
\]

(compare to (2.27), (2.23)) where $\alpha \in \mathbb{C}$.

The proof is the same as for Kac-Moody algebras [9, p.97, relation (9.16)].

Let $\mathcal{H}_{F,\tau}^{(m)}$ be the subspace of $\mathcal{H}_{F,\tau}$ generated by the semi-infinite monomials of charge $m$. These subspaces are invariant under the action of $\widehat{D}^1_\mathfrak{g}$. Hence, for every $m$ the space $\mathcal{H}_{F,\tau}^{(m)}$ itself is a $\widehat{D}^1_\mathfrak{g}$-module, and $\mathcal{H}_{F,\tau} = \bigoplus_{m \in \mathbb{Z}} \mathcal{H}_{F,\tau}^{(m)}$ as $\widehat{D}^1_\mathfrak{g}$-module.

\textbf{P:fermalm} Proposition 4.5. Let $\mathcal{H}_{F,\tau}^{(m)}$ be the submodule of $\mathcal{H}_{F,\tau}$ of charge $m$.

(a) With respect to the degree (4.5), the homogeneous subspaces $(\mathcal{H}_{F,\tau}^{(m)})_k$ of degree $k$ are finite-dimensional. If $k > 0$ then $(\mathcal{H}_{F,\tau}^{(m)})_k = 0$.

(b) The cocycle $\gamma$ for $D^1_\mathfrak{g}$ defined by the projective representation is local.

(c) The $\mathcal{H}_{F,\tau}^{(m)}$ is an almost-graded $\widehat{D}^1_\mathfrak{g}$-module.

For $N = 1$ the Proposition 4.5 is proven in [30, 32]. For the general case we refer to [26].
4.3. Sugawara representation.

Let \( \hat{\mathfrak{g}} \) be an affine algebra. A \( \hat{\mathfrak{g}} \)-module \( V \) is called admissible if any element of \( V \) is annihilated by all elements of \( \hat{\mathfrak{g}}_+ \) of a sufficiently high degree.

Any admissible \( \hat{\mathfrak{g}} \)-module can be canonically turned into an \( \hat{\mathfrak{L}} \)-module by the Sugawara construction. The corresponding representation of \( \hat{\mathfrak{L}} \) is called a Sugawara representation. For the conventional presentation of Sugawara construction for Kac-Moody algebras, see \([9]\). A generalization onto the Krichever-Novikov algebras is given in \([13, 3, 25, 27]\).

Going over to the Sugawara construction, consider an admissible \( \hat{\mathfrak{g}} \)-module \( V \) such that the central element \( t \) operates as a multiplication by a scalar \( c \) (which is called a level of the representation).

For any \( u \in \mathfrak{g}, A \in \mathcal{A} \) denote by \( u(A) \) the representation operator of \( u \otimes A \). Choose a basis \( u_i, i = 1, \ldots, \dim \mathfrak{g} \) of \( \mathfrak{g} \) and the corresponding dual basis \( u^i, i = 1, \ldots, \dim \mathfrak{g} \) with respect to the (fixed in advance) invariant non-degenerate bilinear form \( \langle .. | .. \rangle \). We also denote \( u(n, p) \) by \( u_n^p \) and \( \sum_i u_i(n, p)u^i(m, q) \) by \( u(n, p)u(m, q) \) for short.

Define the higher genus Sugawara operator (also called Segal operator or energy-momentum tensor) by

\[
T(P) := \frac{1}{2} \sum_{n,m} \sum_{p,s} :u(n, p)u(m, s): \omega^{n,p}(P)\omega^{m,s}(P),
\]

where \( :..: \) denotes some normal ordering, \( \omega^{n,p} \) is the basis in the space of the 1-forms on \( \Sigma \) introduced in Section 2.3. Here, the summation indices \( n, m \) run over \( \mathbb{Z} \), and \( p, s \) over \( \{1, \ldots, N\} \). The precise form of the normal ordering is of no importance here. For example, take the following “standard normal ordering” \( (x, y) \in \mathfrak{g} \)

\[
x(n, p)y(m, r) := \begin{cases} x(n, p)y(m, r) & n \leq m \\ y(m, r)x(n, p) & n > m \end{cases}
\]

(for the discussion of normal orderings in case \( g > 0 \) see \([13, 27]\)).

The expression \( T(P) \) is considered as a formal series of quadratic differentials on \( \Sigma \) with operator-valued coefficients. Expanding it over the basis \( \Omega^{k,r} \) of the quadratic differentials (Section 2.3) we obtain

\[
T(P) = \sum_k \sum_r L_{k,r} \cdot \Omega^{k,r}(P),
\]

with

\[
L_{k,r} = \frac{1}{2\pi i} \int_{C_S} T(P)e_{k,r}(P) = \frac{1}{2} \sum_{n,m} \sum_{p,s} :u(n, p)u(m, s): l_{(k,r)}^{(n,p)(m,s)},
\]

where

\[
l_{(k,r)}^{(n,p)(m,s)} := \frac{1}{2\pi i} \int_{C_S} \omega^{n,p}(P)\omega^{m,s}(P)e_{k,r}(P).
\]

A priori, the operators \( L_{k,r} \) are infinite double sums. But for given \( k \) and \( m \), the coefficient \( l_{(k,r)}^{(n,p)(m,s)} \) will be non-zero only for finitely many \( n \). This can
be seen by checking the residues of the elements appearing under the integral. After applying the remaining infinite sum to a fixed element \( v \in V \), by the normal ordering and admissibility of the representation only finitely many of the operators will operate non-trivially on this element.

**Theorem 4.6.** [25] Let \( \mathfrak{g} \) be a finite dimensional either abelian or simple Lie algebra and \( 2k \) be the eigenvalue of its Casimir operator in the adjoint representation. Let \( V \) be an admissible almost-graded \( \widehat{\mathfrak{g}} \)-module of level \( c \). If \( c + k \neq 0 \) then the rescaled modes

\[
L_{k,p}^* = \frac{-1}{2(c + k)} \sum_{n,m} \sum_{p,s} :u(n,p)u(m,s): i_{(k,r)}^{(n,p)(m,s)},
\]

of the Sugawara operator are well-defined operators on \( V \) and define an admissible representation of \( \widehat{\mathfrak{g}} \).

**Proposition 4.7.** [27] The \( V \) is an almost-graded \( \widehat{\mathfrak{L}} \)-module under the Sugawara action.

We call the \( L_{k,r}^* \), resp. the \( L_{k,r} \) the Sugawara operators too. For \( e = \sum_{n,p} a_{n,p} e_{n,p} \in \mathcal{L}'(a_{n,p} \in \mathbb{C}) \) we set \( T[e] = \sum_{n,p} a_{n,p} L_{n,p}^* \) and obtain the representation \( T \) of \( \mathcal{L}' \). It is called the Sugawara representation of the Lie algebra \( \mathcal{L}' \) corresponding to the given admissible representation \( V \) of \( \widehat{\mathfrak{g}} \).

By the Krichever-Novikov duality the Sugawara operator \( T[e] \) assigned to the vector field \( e \in \mathcal{L}' \) can be given as

\[
T[e] = \frac{-1}{c + k} \cdot \frac{1}{2\pi i} \int_{C_{\mathfrak{g}}} T(P)e(P).
\]

The following proposition expresses a fundamental property of the Sugawara representation.

**Proposition 4.8.** For any reductive \( \mathfrak{g} \), \( x \in \mathfrak{g} \), \( A \in \mathcal{A} \), \( e \in \mathcal{L} \) we have

\[
[T[e], x(A)] = x(e.A).
\]

**Proof.** In case of a semi-simple or abelian \( \mathfrak{g} \), we refer to [25, 27] for a proof. Here, we give the proof for \( \mathfrak{g} = \mathfrak{gl}(l) \) which is only considered below. Actually, this is a general proof in case of reductive \( \mathfrak{g} \).

In our case, \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) where \( \mathfrak{g}_0 \) is the center of \( \mathfrak{g} \) consisting of diagonal matrices and \( \mathfrak{g}_1 = \mathfrak{sl}(l) \). Further on, \( \overline{\mathfrak{g}} = \overline{\mathfrak{g}_0} \oplus \overline{\mathfrak{g}_1} \). Denote by \( x_0(A) \), \( x_1(A) \) the restrictions of the representation \( x(A) \) onto \( \overline{\mathfrak{g}_0} \), \( \overline{\mathfrak{g}_1} \), respectively. For \( x = x_0 + x_1 \), \( x_0 \in \mathfrak{g}_0 \), \( x_1 \in \mathfrak{g}_1 \) we can write \( x(A) = x_0(A) + x_1(A) \) without any conflict of notation. Moreover, for any \( A, B \in \mathcal{A} \), \( x_0(A) \) and \( x_1(B) \) commute because \( [x_0(A), x_1(B)] = [x_0(x_1)A + \text{tr}(x_0x_1)c \cdot id = 0 \).

Let \( T_k, k = 0, 1 \) be the Sugawara representation corresponding to the representation \( x_k \) of \( \overline{\mathfrak{g}}_k \). Define

\[
T = T_0 + T_1
\]

(in our case, this is equivalent to the definition [8, 7] Rem. 10.2/10.3). Since operators of the representations \( T_0 \) are expressed via \( x_0(A) \)'s and operators...
of $T_1$ via $x_1(A)$’s, $T_0$ and $T_1$ commute, hence $T$ is a representation of $\hat{\mathcal{L}}$ and, moreover, for any $x_0 \in \mathfrak{g}_0$, $x_1 \in \mathfrak{g}_1$, $A, B \in \mathcal{A}$, $e \in \mathcal{L}$

$$[T_0[e], x_1(A)] = [T_1[e], x_0(B)] = 0.$$ 

For simple and abelian $\mathfrak{g}$, the (4.12) is proven in [25] (see also [27]). Hence,

$$[T_0[e], x_0(A)] = x_0(e.A), \quad [T_1[e], x_1(A)] = x_1(e.A).$$

Finally, we have

$$[T[e], x(A)] = [T_0[e] + T_1[e], x_0(A) + x_1(A)] = [T_0[e], x_0(A)] + [T_1[e], x_1(A)]$$

$$= x_0(e.A) + x_1(e.A) = x(e.A).$$

\[\square\]

5. Casimirs, semi-casimirs, Hitchin integrals

Casimir operators (casimirs, laplacians) are the most important invariant operators of Lie algebras. They are closely related to all applications of Lie algebras and their representations including integrable systems, the theory of special functions, and many others. Description of casimirs is one of the central problems of the representation theory. The second order casimirs are of special interest in all these questions. In what follows, “casimir” always means “second order casimir”.

Following the lines of [32], we classify here the casimirs for Krichever-Novikov algebras in case $N = 1$ and $\mathfrak{g} = \mathfrak{gl}(r)$ ($\mathfrak{g} = \mathfrak{sl}(r)$). We introduce the more general operators which we call semi-casimirs, investigate their relation to the moduli space of Riemann surfaces and interpret them as a quantization of the Hitchin integrals. We delay the case $N > 1$ for the future publication [26].

5.1. Classification of casimirs.

Let $V$ be any admissible representation of $\hat{\mathcal{D}}^1_\mathfrak{g}$. Its restriction to $\hat{\mathfrak{g}}$ is an admissible representation of the latter.

Let $\hat{e}$ denote the operator of representation of an $e \in \mathcal{L}$, and $T[e]$ be the Sugawara operator of $e$ as introduced above. Consider operators of the form

$$\Delta_e := \hat{e} - T[e].$$

In the following, we consider also the completed vector field algebra $\overline{\mathcal{L}}$ consisting of the infinite sums of the form

$$e = \sum_{n=n_0}^{\infty} a_n e_n, \quad a_n \in \mathbb{C},$$

where, for $N = 1$, $\{e_n = e_{n,1}\}$ is the Krichever-Novikov base in $\mathcal{L}$; similarly $\{A_n = A_{n,1}\}$ is the Krichever-Novikov base in $\mathcal{A}$. By admissibility, for any fixed vector $v$ both $\hat{e}v$ and $T[e]v$ are well-defined elements of $V$ even if $e$ is of the form (5.1).

Let $\overline{\mathcal{A}}_- \subset \mathcal{A}$ be the subspace spanned by all $A_k$, $k < 0$. 

\[S:cas\]
Definition 5.1.  
(a) We call an operator of the form $\Delta_e$ a *casimir* if $[\Delta_e, x(A)] = 0$ for any $A \in \mathcal{A}$ and $x \in \mathfrak{g}$. 
(b) We call an operator of the form $\Delta_e$ a *semi-casimir* if $[\Delta_e, x(A)] = 0$ for any $A \in \tilde{\mathcal{A}}_-$ and $x \in \mathfrak{g}$.

Proposition 5.2. Let $\mathfrak{g} = \mathfrak{gl}(r)$ and $V$ be an admissible almost-graded $\hat{D}_g$-module such that the restriction of its cocycle on $\mathfrak{g}$ is $\mathcal{L}$-invariant. Then for any $x \in \mathfrak{gl}(r)$ and $e \in \mathcal{L}$

$$(5.2) \quad [\Delta_e, x(A)] = \lambda(x) \gamma^{(m)}(e, A) \cdot \text{id},$$

where $\gamma^{(m)}$ is given by (2.25) and $\lambda(x) = r^{-1} \text{tr}(x)$.

The Proposition 5.2 was first formulated in [32] under different assumptions on cocycles. The requirement of $\mathcal{L}$-invariance (see Section 2.5) is proposed by M. Schlichenmaier.

Proof. Since $V$ is an admissible almost-graded $\hat{D}_g$-module, its cocycle is local. By the classification of local $\mathcal{L}$-invariant cocycles (Section 2.5) we have

$$(5.3) \quad [\hat{e}, x(A)] = x(e.A) + \lambda(x) \gamma^{(m)}(e, A) \cdot \text{id}.$$ 

Applying the Proposition 4.8 completes the proof. \qed

It follows from the Proposition 5.2 that for any $x \in \mathfrak{sl}(r)$ (and $A \in \mathcal{A}$) $[\Delta_e, xA] = 0$. Thus, the cocycle $\gamma^{(m)}$ is the only obstacle for $\Delta_e$ to be a casimir; $\Delta(e)$ is a casimir if and only if

$$(5.4) \quad \gamma^{(m)}(A_k, e) = 0, \quad \text{for any} \quad k \in \mathbb{Z}.$$ 

Replacing here $e$ by its expression (5.1), obtain the following linear system of equations on the coefficients $\{a_n\}$:

$$(5.5) \quad \sum_{m \geq m_0} a_m \gamma^{(m)}(A_{-k}, e_m) = 0, \quad \text{for all} \quad k \in \mathbb{Z}, \ k \neq 0.$$ 

The further investigation of casimirs is based on the fact that the system (5.5) is triangular and, for its diagonal elements, in a generic situation, we have $\gamma(A_{-k}, e_k) \neq 0$, for any $k \in \mathbb{Z}$, $k \neq 0$. Since the equation for $k = 0$ is missing, we obtain the 1-dimensional space of solutions $\{a_n\}$ where $a_n = 0$, $n < 0$ and $a_n$ express via $a_0$, $n > 0$. This way, the description of casimirs can be completed in case of the fermion modules. We formulate here only a final result and refer to [32] for the proofs.

Theorem 5.3. For any fermion representation such that its cocycle $\gamma$ satisfies the above condition of genericity, and any connection $\nabla$ (involved via (4.4)), there exists exactly one (up to a scalar factor) casimir. The corresponding vector field has a simple zero at $P_+$. 
5.2. Semi-casimirs, coinvariants, moduli spaces.

Observe that for a vector field \( e \) giving a semi-casimir one has the system of linear equations similar to (5.5) but only for \( k > 0 \):

\[
\gamma(A_{-k}, e) = 0, \quad \text{for any } k \in \mathbb{Z}, \ k > 0.
\]

Thus, the coefficients \( a_m \) with \( m \leq 0 \) turn out to be independent and all the others express via them. Let \( \mathcal{L}_- \subset \mathcal{L} \) be the subspace spanned by \( \{e_k: k \leq 0\} \).

Introduce the map \( \Gamma: \mathcal{L}_- \to \mathcal{L} \) as follows: take \( e \in \mathcal{L}_- \) and represent it in the form (5.1); then substitute the corresponding \( a_m \) (\( m \leq 0 \)) into (5.6) and calculate \( a_m \), \( m > 0 \). Denote by \( \Gamma(e) \) the vector field which corresponds to the full set of \( a_m \)’s.

**Lemma 5.4.** \[32\] The space of semi-casimirs coincides with \( \Delta(\Gamma(\mathcal{L}_-)) \). It is spanned by the elements \( \Delta(\Gamma(e_k)) \), where \( k \leq 0 \). It is isomorphic to \( \mathcal{L}_- \) as a linear space.

Denote the subalgebra \( \hat{g}_{reg} \) introduced in Section 2.6 by \( g_r \), for short, and call it the regular subalgebra. The space of co-invariants of \( g_r \) is defined as a quotient space \( V/U(g_r)V \).

The semi-casimirs are defined in such way that they commute with \( U(g_r) \), hence they are well-defined on the space of coinvariants of the subalgebra \( g_r \).

For \( e \in \mathcal{L}_- \), let \( \Delta(e) \) be the operator induced by \( \Delta(\Gamma(e)) \) on coinvariants. The map \( \Delta \) is defined on \( \mathcal{L}_- \) and by Lemma 5.4 its image is the space \( C_2^s \) of semi-casimirs considered as operators on the space of coinvariants.

Our next step is to show that only a finite number of basis semi-casimirs are nonzero on coinvariants and, for a proper moduli space of Riemann surfaces, establish the correspondence between its tangent space and the space of semi-casimirs (considered on coinvariants).

**Lemma 5.5.** \[32\] For a fermion representation \( V \) there exists such \( p \in \mathbb{Z}_+ \) that \( \mathcal{L}_{\mathcal{L}}^p \subseteq \ker \Delta \).

*Proof.* It is proven in course of the proof of [32, Lemma 4.9] that such \( p \in \mathbb{Z} \) exists that \( \Delta(e)V \subseteq U(g_r)V \) for any \( e \in \mathcal{L}_-^p \). Actually, we need the same for \( \Delta(\Gamma(e)) \). Here, we only want to complete the proof of [32, Lemma 4.9] with this correction.

Let \( V^{(q)} \subset V \) denote the subspace generated by all elements of degree less or equal to \( q \). In fact, it is proven in course of the proof of [32, Lemma 4.9] that for any \( q \in \mathbb{Z} \) such \( p \) exists that \( \Delta(e)V \subseteq U^{(q)} \) for any \( e \in \mathcal{L}_-^p \). Let \( U_- \) be the subspace of the universal enveloping algebra of \( \hat{g} \) generated by the basis elements of the non-positive degree. Choose such \( q \) that \( U_-V^{(q)} \subseteq U(g_r)V \).

Take an arbitrary \( e \in \mathcal{L}_-^p \). Observe that \( \Gamma(e) = e + e_+ \) where \( e_+ \in \mathcal{L}_+ \). For any \( v \in V \), we have \( v = uv_0 \) where \( u \in U_- \) and \( v_0 \) is the vacuum vector. Further on, \( \Delta(e + e_+) \) is a semi-casimir, hence \( \Delta(e + e_+)v = u\Delta(e + e_+)v_0 \).

By [32, Lemmas 3.2, 3.4], \( \Delta(e_+)v_0 = 0 \), hence \( u\Delta(e + e_+)v_0 = u\Delta(e)v_0 \).
Since $\Delta(e)v_0 \subseteq V^{(q)}$ and $uV^{(q)} \subseteq U(\mathfrak g_r)V$, we have $u\Delta(e)v_0 \subseteq U(\mathfrak g_r)V$, hence $\Delta(\mathfrak g_r)u \subseteq U(\mathfrak g_r)V$.

Let $\mathcal M_{g,2}^{(p)}$ be the moduli space of curves of genus $g$ with two marked points $P_\pm$, fixed 1-jet of local coordinate at $P_+$ and fixed $p$-jet of local coordinate at $P_-$. There is a canonical mapping $\theta : \mathcal L \to T_{\Sigma}\mathcal M_{g,2}^{(p)}$ which goes back to $\mathfrak D_{\mathfrak g}$ and is based on the Kodaira–Spencer theory. The cohomological and geometrical versions of this mapping are given in $[27]$ and $[5]$, respectively (see Introduction).

Let $\tilde\theta$ denote the restriction of $\theta$ onto the subspace $\tilde\mathcal L_-$. Let $V$ be a fermion representation of $\mathfrak D_{\mathfrak g}$ and $\gamma_V$ be its cocycle. Let also $C_2 = C_2^V(V)$ denote the second order semi-casimirs of $\hat\mathfrak g$ in the representation $V$. We assume semi-casimirs to be restricted onto coinvariants. The following theorem establishes a natural mapping of the tangent space at $\Sigma \in \mathcal M_{g,2}^{(p-1)}$ onto the space of semi-casimirs in the coinvariants on $\Sigma$ considered as a punctured Riemann surface.

**Theorem 5.6.** $[32]$ Take $p$ as in Lemma $[7,5]

1°. The mapping $\tilde\theta : \tilde\mathcal L_- \to T_{\Sigma}\mathcal M_{g,2}^{(p-1)}$ is surjective and $\ker \tilde\theta = \mathcal L_\hat\theta^{(p)}$.

2°. For such $V$ that $\gamma_V(A_{-k}, e_k) \neq 0$ for any $k \in \mathbb Z_+$, the mapping $\tilde\Sigma : \tilde\mathcal L_- \to C_2^V(V)$ is surjective and $\mathcal L_\hat\theta^{(p)} \subseteq \ker \tilde\Sigma$.

3°. The mapping $\tilde\Sigma \circ \tilde\theta^{-1} : T_{\Sigma}\mathcal M_{g,2}^{(p-1)} \to C_2^\Sigma(V)$ is well-defined and surjective.

### 5.3. Quantization of the second order Hitchin integrals.

In this section, we show how the semi-casimirs appear in course of operator quantization of the second order Hitchin integrals. We do not give any mathematical setting the problem of quantization here. We only show what happens if one follows some conventional recipes.

Let $\phi$ is a Higgs field (mathematically, an arbitrary $\mathfrak g$-valued Krichever-Novikov 1-form on the Riemann surface in question) and $\{\Omega^i\}$ is a base of the cotangent space to $\mathcal M_{g,2}^{(p-1)}$ realized as a certain space of Krichever-Novikov quadratic differentials. We introduce the second order Hitchin integrals $\chi_i$’s by the expansion $\text{tr} \phi^2 = \sum \chi_i \Omega^i$. Observe that this is only a part of generalized second order Hitchin integrals of Section $[3]$ namely the part which contains no additional functional factors.

As a first step of quantization, we replace $\phi$ by its operator, thus we obtain the current $I$ which is an arbitrary Krichever-Novikov 1-form (on the Riemann surface) having values in the representation operators of $\hat\mathfrak g$. Therefore, $I = \sum u_k \omega^k$ where $\omega^k$ are the basis Krichever-Novikov 1-forms, $u_k$ are the operator-valued coefficients ($k \in \mathbb Z$). Further on, the $\phi^2$ should be replaced by $I^2$. By definition of the Wess-Zumino-Witten-Novikov theory, $\text{tr} : I^2$: exactly equals to the energy-momentum tensor $T$ (introduced by $[1,6]$). The trace remains to be ”finite-dimensional”, like in the classic situation, which means that it is linear over the function algebra $\mathcal A$. The expansion $T = \sum L_i \Omega^i$ (cf. $[1,8]$) is the quantum analog of the above expansion $\text{tr} \phi^2 = \sum \chi_i \Omega^i$. We will...
consider the normalized form $-T(e_i)$ of an operator $L_i$ (see Section 4.3). What is usually being done to compensate a normal ordering, is adding certain cartanian elements to the normal ordered quantity. For example, for Kac-Moody algebras the vector field $e_0 = z \frac{\partial}{\partial z}$ is considered and $\hat{e}_0$ is being added which leads to the casimir $z \frac{\partial}{\partial z} - T(z \frac{\partial}{\partial z})$. Applying this idea to an arbitrary $e_i$, we come to the operators of the form $\Delta_i = \hat{e}_i - T(e_i)$. Since there is only a finite number of (independent) Hitchin integrals and the infinite set of $\Delta_i$’s, we formulate a selection rule for them. First, we propose to consider certain linear combinations of $\Delta_i$’s, namely those which are semi-casimirs. Thus, we replace $e_i$ by $\Gamma(e_i)$ (where $\Gamma$ is introduced in Section 5.2). Since there is no canonical choice for the base $\{\Omega_i\}$, the replacement $e_i$ by $\Gamma(e_i)$ can be achieved by adjustment of this base. Second, we select only those semi-casimirs which induce nontrivial operators on conformal blocks. Thus, for each $i$ we consider $\Delta(\Gamma(e_i))$ as a quantization of $\chi_i$. Then by Theorem 4.2 we obtain the natural mapping of the $\chi_i$’s to the $\Delta(\Gamma(e_i))$’s.

Since the classic Hitchin integrals are in involution, the corresponding quantum quantities must commute, at least, up to a scalar. Let us show that this holds for our quantization of Hitchin integrals.

**Proposition 5.7.** In the space of a fermion representation, for any $e, f \in \mathcal{L}$, 

$$[\Delta(e), \Delta(f)] = \lambda(e, f) \cdot \text{id}$$

where $\lambda$ is a bilinear form on $\mathcal{L}$.

**Proof.** For any $u \in \mathfrak{g}$, $A \in \mathcal{A}$ we have 

$$[[\Delta(e), \Delta(f)], u(A)] = [\Delta(e), u(A)] \cdot \Delta(f) + [\Delta(e), \Delta(f), u(A)]$$

$$= [\text{tr} u \gamma(e, A) \cdot \text{id}, \Delta(f)] + [\Delta(f), (\text{tr} u \gamma(f, A) \cdot \text{id}] = 0,$$

where $\gamma$ is a mixing cocycle. Thus, $[\Delta(e), \Delta(f)]$ is an automorphism of the $\hat{\mathfrak{g}}$-module. For modules with a unique highest vector (for example, the fermion modules) all endomorphisms are scalar operators. \qed

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