INEQUALITIES OF HERMITE–HADAMARD TYPE FOR EXTENDED HARMONICALLY \((s,m)\)-CONVEX FUNCTIONS

CHUN-YING HE, BO-YAN XI*, AND BAI-NI GUO*

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Abstract. In the paper, the authors introduce a new notion “extended harmonically \((s,m)\)-convex function” and establish some integral inequalities of the Hermite–Hadamard type for extended harmonically \((s,m)\)-convex functions in terms of hypergeometric functions.

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1. INTRODUCTION

We recall some definitions on diverse convex functions in the literature.

**Definition 1.** A function \(f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}\) is said to be convex if the inequality
\[
f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]
holds for all \(x, y \in I\) and \(t \in [0,1]\).

**Definition 2 ([2,11]).** Let \(s \in (0,1]\) be a real number. A function \(f : \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}\) is said to be \(s\)-convex in the second sense if the inequality
\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]
holds for all \(x, y \in I\) and \(t \in [0,1]\).

**Definition 3 ([25]).** For \(f : [0,b] \rightarrow \mathbb{R}, b > 0,\) and \(m \in (0,1],\) if the inequality
\[
f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)
\]
is valid for all \(x,y \in [0,b]\) and \(t \in [0,1]\), then we say that \(f\) is an \(m\)-convex function on \([0,b]\).

*Corresponding author.

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**Definition 4 ([28]).** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended $s$-convex if the inequality
\[
f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)
\]
holds for all $x, y \in I$ and $t \in (0, 1)$ and for some fixed $s \in [-1, 1]$.

**Definition 5 ([13]).** Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $I \to \mathbb{R}$ is said to be harmonically convex if the inequality
\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq tf(x) + (1-t)f(y) \tag{1.1}
\]
holds for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.1) is reversed, then $f$ is said to be harmonically concave.

**Definition 6 ([17, Definition 2.6]).** A function $I \subseteq \mathbb{R}_+ = (0, \infty) \to \mathbb{R}$ is said to be harmonically $s$-convex if the inequality
\[
f \left( \frac{xy}{tx + (1-t)y} \right) \leq t^s f(x) + (1-t)^s f(y)
\]
is valid for $x, y \in I$, $t \in (0, 1)$, and $s \in [-1, 1]$.

**Definition 7 ([29]).** Let $f : (0, b] \to \mathbb{R}$ and let $m \in (0, 1]$ be a constant. If the inequality
\[
f \left( \frac{t}{x} + \frac{1-t}{y} \right)^{-1} \leq tf(x) + m(1-t)f(y)
\]
is valid for all $x, y \in (0, b]$ and $t \in [0, 1]$, then $f$ is said to be an $m$-harmonic-arithmetically convex function or, simply speaking, an $m$-HA-convex function.

**Definition 8 ([9]).** Let $f : (0, b] \to \mathbb{R}$ and let $\alpha, m \in (0, 1]$ be constants. If the inequality
\[
f \left( \frac{t}{x} + m \frac{1-t}{y} \right)^{-1} \leq t^\alpha f(x) + m(1-t^\alpha)f(y)
\]
is valid for all $x, y \in (0, b]$ and $t \in [0, 1]$, then $f$ is said to be an $(\alpha, m)$-harmonic-arithmetic convex function or, simply speaking, an $(\alpha, m)$-HA-convex function.

In recent decades, establishing integral inequalities of the Hermite–Hadamard type for diverse convex functions has been an active direction in mathematics. Some of these results can be reformulated as follows.

**Theorem 1 ([5, Theorem 2.2]).** Let $f : I^0 \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^0$ and $a, b \in I^0$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.
\]
Theorem 2 ([20, Theorems 1 and 2]). Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be differentiable on \( I' \) and \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is convex on \([a, b]\) and \( q \geq 1 \), then
\[
\frac{|f(a) + f(b)|}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq b-a \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}
\]
and
\[
\frac{|f\left(\frac{a+b}{2}\right)|}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq b-a \left[ \frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{1/q}.
\]

Theorem 3 ([6]). Let \( m \in (0, 1] \) and \( f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R} \) be \( m \)-convex. If \( f \in L_1([a, b]) \) for \( 0 \leq a < b < \infty \), then
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq \min \left\{ \frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2} \right\}.
\]

Theorem 4 ([14]). Let \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R} \) be differentiable on \( I' \) and \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1] \) and \( q \geq 1 \), then
\[
\frac{|f(a) + f(b)|}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq b-a \left[ \frac{2 + 1/2^s}{(s+1)(s+2)} \right]^{1/q} \left[ |f'(a)|^q + |f'(b)|^q \right]^{1/q}.
\]

Theorem 5 ([12]). Let \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R} \) be differentiable on \( I' \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1] \) and \( q > 1 \), then
\[
\frac{|f\left(\frac{a+b}{2}\right)|}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq b-a \left[ \frac{1}{(s+1)(s+2)} \right]^{1/q} \left( \frac{1}{2} \right)^{1/p} \times \left\{ \left[ |f'(a)|^q + (s+1) \left| f'\left(\frac{a+b}{2}\right)\right| \right]^{1/q} + \left[ |f'(b)|^q + (s+1) \left| f'\left(\frac{a+b}{2}\right)\right| \right]^{1/q} \right\},
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Theorem 6 ([24]). Let \( f : I \subseteq \mathbb{R}_0 \to \mathbb{R} \) be differentiable on \( I' \) and \( f' \in L[a, b] \). If \( |f'| \) is \( s \)-convex on \([a, b]\) for some \( s \in (0, 1] \) and \( p > 1 \), then
\[
\frac{1}{6} \left[ |f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)| - \frac{1}{b-a} \int_a^b f(x)dx \right] \leq \frac{(s-4)6^{s+1} + 2 \times 5^{s+2} - 2 \times 3^{s+2} + 2}{6^{s+2}(s+1)(s+2)} (|f'(a)| + |f'(b)|),
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).
For more information developed in recent decades on this topic, please refer to the papers [1, 3, 4, 7, 8, 10, 15, 16, 18, 19, 21–23, 26, 27, 30] and closely related references.

In this paper, we will introduce a new notion “extended harmonically \( (s,m) \)-convex function” and establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically \( (s,m) \)-convex functions.

2. A DEFINITION AND A LEMMA

Now we introduce the notion “extended harmonically \( (s,m) \)-convex function”.

**Definition 9.** For \( m \in (0, 1] \) and \( s \in [-1, 1] \), a function \( f : (0, b] \to \mathbb{R} \) is said to be extended harmonically \( (s,m) \)-convex on \( (0, b] \) if the inequality

\[
f \left( \frac{t}{x} + m \frac{1-t}{y} \right)^{-1} \leq t^s f(x) + m(1-t)^s f(y)
\]

holds for all \( x, y \in (0, b] \) and \( t \in (0, 1) \).

**Example 1.** Let \( s \in [-1, 1] \) and \( f(x) = \frac{1}{x^r} \) for \( x \in \mathbb{R}_+ \) and \( r \geq 1 \). Since

\[
f \left( \frac{t}{x} + m \frac{1-t}{y} \right)^{-1} \leq \frac{ty^r + (1-t)(mx)^r}{(xy)^r} \leq t^s f(x) + m(1-t)^s f(y)
\]

for all \( x, y \in \mathbb{R}_+ \) and \( t \in (0, 1) \), the function \( f(x) = \frac{1}{x^r} \) is extended harmonically \( (s,m) \)-convex on \( \mathbb{R}_+ \).

To establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically \( (s,m) \)-convex functions, we need the following lemma.

**Lemma 1.** Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R} \) be a differentiable function on \( I \) and \( a, b \in I \) with \( a < b \). If \( f' \in L_1([a, b]) \) and \( 0 \leq \lambda, \mu \leq 1 \), then

\[
\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f(H(a, b)) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2}
\]

\[
= \frac{b-a}{4ab} \int_0^1 \left[ (1-\lambda-t) \left( \frac{t}{a} + \frac{1-t}{H(a, b)} \right)^{-2} f' \left( \frac{t}{a} + \frac{1-t}{H(a, b)} \right) \right] dt.
\]

**In particular, if** \( \lambda = \mu = 0 \), then

\[
f(H(a, b)) - \frac{ab}{b-a} \int_a^b f(x) \frac{dx}{x^2}
\]

\[
= \frac{b-a}{4ab} \int_0^1 \left[ (1-t) \left( \frac{t}{a} + \frac{1-t}{H(a, b)} \right)^{-2} f' \left( \frac{1+t}{2a} + \frac{1-t}{2b} \right) \right] dt.
\]
\[-(1-t)\left(\frac{t}{b} + \frac{1-t}{H(a,b)}\right)^{-2}f'\left(\frac{t}{a} + \frac{1-t}{H(a,b)}\right)\] 
\[dt,\]

where \(H(a,b) = \frac{2ab}{a+b}\).

**Proof.** Putting \(x = (ta^{-1} + (1-t)[H(a,b)]^{-1})^{-1}\) for \(t \in [0,1]\) gives

\[\int_0^1 (1-\lambda-t)\left(\frac{t}{a} + \frac{1-t}{H(a,b)}\right)^{-2}f'\left(\frac{t}{a} + \frac{1-t}{H(a,b)}\right)dt \tag{2.3}\]

\[= \frac{2ab}{b-a}[\lambda f(a) + (1-\lambda)f(H(a,b))] - \left(\frac{2ab}{b-a}\right)^2 \int_a^b f(x) x^2 dx.\]

Similarly, letting \(x = (tb^{-1} + (1-t)[H(a,b)]^{-1})^{-1}\) for \(t \in [0,1]\) results in

\[\int_0^1 (\mu-1+t)\left(\frac{t}{b} + \frac{1-t}{H(a,b)}\right)^{-2}f'\left(\frac{t}{b} + \frac{1-t}{H(a,b)}\right)dt \tag{2.4}\]

\[= \frac{2ab}{b-a}[\mu f(b) + (1-\mu)f(H(a,b))] - \left(\frac{2ab}{b-a}\right)^2 \int_a^b f(x) x^2 dx.\]

Adding the equalities (2.3) and (2.4) leads to the equality (2.1). The proof of Lemma 1 is thus complete. \(\square\)

### 3. INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE

Now we start out to establish some new integral inequalities of the Hermite–Hadamard type for extended harmonically \((s,m)\)-convex functions.

**Theorem 7.** Let \(f : (0,d] \rightarrow \mathbb{R}\) be differentiable, \(a,b \in (0,d]\) with \(a < b\), \(f' \in L_1([a,b])\), and \(0 \leq \lambda,\mu \leq 1\). If \(|f'|^q\) for \(q \geq 1\) is extended harmonically \((s,m)\)-convex on \((0,d]\) for some fixed \(m \in (0,1]\) and \(s \in [-1,1]\), then

1. When \(-1 < s \leq 1\),

\[\left|\frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f(H(a,b)) - \frac{ab}{b-a} \int_a^b f(x) x^2 dx\right| \tag{3.1}\]

\[\leq \frac{(b-a)^{(2-q)/q}}{2^{(q+1)/q}(ab)^{1/q}} \left\{ |T(a,b,\lambda)|^{1-1/q} (mK(a, H(a,b), s, \lambda)) |f'(mH(a,b))|^q \right.\]

\[+ K(H(a,b), a, s, 1-\lambda) |f'(a)|^q \left. + |T(b,a,\mu)|^{1-1/q} (mK(b, H(a,b), s, \mu) \right.\]

\[\times |f'(mH(a,b))|^q + K(H(a,b), b, s, 1-\mu) |f'(b)|^q \right\}^{1/q};\]

2. When \(s = -1\),

\[\left|f(H(a,b)) - \frac{ab}{b-a} \int_a^b f(x) x^2 dx\right| \leq \frac{(ab)^{(q-2)/q}(a+b)^{(1-q)/q}}{2^{1/q}(b-a)^{2-1/q}} \left\{ |(a+b)(\ln a\right.\]

\[\left. - (a+b)\ln b)|\right\}.\]
\[-\ln H(a, b) + (b - a)^{1 - 1/q} \left( (2b^2(\ln(2a) - \ln H(a, b) - bH(a, b)) |f'(a)|^q + m\ln H(a, b)|f'(mb)|^q \right)^{1/q} + \left[ m(2a^2(\ln(2b) - \ln H(a, b)) - aH(a, b)) |f'(mb)|^q + bH(a, b)|f'(a)|^q \right]^{1/q} \right] \]

where

\[ T(a, b, \lambda) = 2ab(\ln a + \ln H(a, b)) - 2\ln[(1 - \lambda)H(a, b) + \lambda a] + (b - a)[(1 - \lambda)H(a, b) - \lambda a] \]

and

\[ K(a, u, s, \lambda) = \frac{2\lambda^s+2a^2}{(s+1)(s+2)} _2F_1 \left( 2, s+1, s+3, \frac{\lambda(u-a)}{u} \right) \]

\[ - \frac{\lambda a^2}{s+1} _2F_1 \left( 2, s+1, s+2, \frac{u-a}{u} \right) + \frac{a^2}{s+2} _2F_1 \left( 2, s+2, s+3, \frac{u-a}{u} \right) \]

with the hypergeometric function

\[ _2F_1(c, d, e; z) = \frac{\Gamma(e)}{\Gamma(d)\Gamma(e-d)} \int_0^1 t^{d-1}(1-t)^{e-d-1}(1-zt)^{-c} dt \]  

(3.2)

for \( e > d > 0, |z| < 1, c \in \mathbb{R}, \) and \( u > 0. \)

**Proof.** When \(-1 < s \leq 1,\) by virtue of Lemma 1 and the extended harmonic \((s, m)\)-convexity of \(|f'|^q,\) we obtain

\[ \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f(H(a, b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \]

\[ \leq \frac{b-a}{4ab} \left[ \left( \int_0^1 |1 - \lambda - t| \left( \frac{t}{a} + \frac{1-t}{2H(a, b)} \right)^{-2} dt \right)^{1-1/q} \right. \]

\[ \times \left( \int_0^1 |1 - \lambda - t| \left( \frac{t}{a} + \frac{1-t}{2H(a, b)} \right)^{-2} \left| f' \left( \left( \frac{t}{a} + \frac{1-t}{2H(a, b)} \right)^{-1} \right) \right| dt \right)^{1/q} \]

\[ + \left( \int_0^1 |1 - \mu - t| \left( \frac{t}{b} + \frac{1-t}{2H(a, b)} \right)^{-2} dt \right)^{1-1/q} \]

\[ \times \left( \int_0^1 |1 - \mu - t| \left( \frac{t}{b} + \frac{1-t}{2H(a, b)} \right)^{-2} \left| f' \left( \left( \frac{t}{b} + \frac{1-t}{2H(a, b)} \right)^{-1} \right) \right| dt \right)^{1/q} \]

\[ \leq \frac{b-a}{4ab} \left( \int_0^1 |1 - \lambda - t| \left( \frac{t}{a} + \frac{1-t}{2H(a, b)} \right)^{-2} dt \right)^{1-1/q} \left( \int_0^1 |1 - \lambda - t| \right). \]
The inequality (3.1) is thus proved.

where we used the facts

\[ \int_0^1 |1 - \lambda - t| \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt = \frac{2ab}{(b-a)^2} T(a,b,\lambda), \]

\[ \int_0^1 |1 - \lambda - t| t^\mu \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt = K(H(a,b),a,s,1-\lambda), \]

and

\[ \int_0^1 (1 - \lambda - t)(1 - t)^\mu \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt = K(a,H(a,b),s,\lambda). \]

The inequality (3.1) is thus proved.

When \( s = -1, \) by the identity (2.2) and the extended harmonic \((s,m)\)-convexity of \( |f'|^q \), we have

\[
\left| f(H(a,b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
\leq \frac{b-a}{2^{2-1/a}ab} \left[ \left( \int_0^1 (1 - t) \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt \right)^{1-1/q} \\
\times \left( \int_0^1 (1 - t) \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} [(1 + t)^{-1} |f'(a)|^q] dt \right)^{1/q} \\
+ m(1 - t)^{-1} |f'(mb)|^q dt \right]^{1/q} + \left( \int_0^1 (1 - t) \left( \frac{t}{b} + \frac{1 - t}{H(a,b)} \right)^{-2} dt \right)^{1-1/q} \\
\times \left( \int_0^1 (1 - t) \left( \frac{t}{b} + \frac{1 - t}{H(a,b)} \right)^{-2} [(1 - t)^{-1} |f'(a)|^q] dt \right)^{1/q} \\
+ m(1 + t)^{-1} |f'(mb)|^q dt \right]^{1/q} \\
= \frac{(ab)^{(q-2)/q} (a+b)^{(1-q)/q}}{2^{1/q}(b-a)^{q-2/q}} \left[ [(a+b)(\ln a - \ln H(a,b)) + (b-a)]^{1-1/q} \\
- \left( \int_0^1 (1 - t) \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt \right)^{1-1/q} \\
\times \left( \int_0^1 (1 - t) \left( \frac{t}{b} + \frac{1 - t}{H(a,b)} \right)^{-2} [(1 + t)^{-1} |f'(a)|^q] dt \right)^{1/q} \\
\times \left( \int_0^1 (1 - t) \left( \frac{t}{b} + \frac{1 - t}{H(a,b)} \right)^{-2} [(1 - t)^{-1} |f'(a)|^q] dt \right)^{1/q} \\
+ m(1 + t)^{-1} |f'(mb)|^q dt \right]^{1/q} \right],
\]

where we used the facts

\[ \int_0^1 |1 - \lambda - t| \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt = \frac{2ab}{(b-a)^2} T(a,b,\lambda), \]

\[ \int_0^1 |1 - \lambda - t| t^\mu \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt = K(H(a,b),a,s,1-\lambda), \]

and

\[ \int_0^1 (1 - \lambda - t)(1 - t)^\mu \left( \frac{t}{a} + \frac{1 - t}{H(a,b)} \right)^{-2} dt = K(a,H(a,b),s,\lambda). \]
\[
\times \left[ (2b^2(\ln(2a) - \ln H(a, b)) - bH(a, b)) |f'(a)|^q + maH(a, b)|f'(mb)|^q \right]^{1/q} \\
+ \left[ m(2a^2(\ln(2b) - \ln H(a, b)) - aH(a, b)) |f'(mb)|^q + bH(a, b)|f'(a)|^q \right]^{1/q} \\
\times [(a + b)(\ln b - \ln H(a, b)) - (b - a)]^{1-1/q} \right].
\]

The proof of Theorem 7 is thus complete. \hfill \square

**Corollary 1.** Under conditions of Theorem 7, when \( q = 1 \),

1. If \(-1 < s \leq 1 \), then
   \[
   \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f(H(a, b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
   \leq \frac{b-a}{4ab} \left\{ K(H(a, b), a, s, 1-\lambda)|f'(a)| + K(H(a, b), b, s, 1-\mu)|f'(b)| \\
   + mK(a, H(a, b), s, \lambda) + K(b, H(a, b), s, \mu)]|f'(mH(a, b))| \right\} ;
   \]

2. If \( s = -1 \), then
   \[
   \left| f(H(a, b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
   \leq \frac{b-a}{ab} \left( b^2|\ln(2a) - \ln H(a, b)||f'(a)| + ma^2|\ln(2b) - \ln H(a, b)||f'(mb)| \right). \]

**Corollary 2.** Under conditions of Theorem 7, when \( q = s = 1 \) and \( \lambda = \mu = \frac{1}{2} \), we have

\[
\left| \frac{f(a) + 2f(H(a, b)) + f(b)}{4} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{1}{4(b-a)^2} \\
\times \left\{ (20ab^2 + 12a^2b) \left( \ln \sqrt{aH(a, b)} - \ln H(a, H(a, b)) \right) - a(b-a)^2 \right| f'(a) | \\
+ 4am \left[ (7b+a) \left( \ln H(a, H(a, b)) - \ln \sqrt{aH(a, b)} \right) \\
+ (7a+b) \left( \ln \sqrt{bH(a, b)} - \ln H(b, H(a, b)) \right) \right] |f'(mH(a, b))| \\
+ \left[(20a^2b + 12ab^2) \left( \ln H(b, H(a, b)) - \ln \sqrt{bH(a, b)} \right) + b(b-a)^2 \right] |f'(b)| \right\} .
\]

**Theorem 8.** Let \( f : (0, d] \to \mathbb{R} \) be differentiable, \( a, b \in (0, d] \) with \( a < b \), \( f' \in L_1([a, b]) \), and \( 0 \leq \lambda, \mu \leq 1 \). If \( |f'|^q \) for \( q > 1 \) is extended harmonically \((s, m)\)-convex on \((0, d]\) for some fixed \( s \in [-1, 1]\) and \( 0 < m \leq 1 \), then

1. If \(-1 < s \leq 1 \),
   \[
   \left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f(H(a, b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
   \leq \frac{b-a}{4ab[(s+1)(s+2)]^{1/q}} \left[ Q^{1-1/q}(a, H(a, b), \lambda) \left( (2(1-\lambda)^{s+2} + \lambda(s+2) \right) \right]
   \]
\begin{equation}
\left| f(H(a,b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{b-a}{2^{(2q-1)/q} qab} \tag{3.6}
\end{equation}

(2) when $s = -1,$

\begin{align*}
&\left| f(H(a,b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
&\leq \frac{b-a}{4ab} \left[ \int_0^1 \left| 1 - \lambda - t \left( \frac{t}{a} + \frac{1-t}{H(a,b)} \right) \right|^{-2q/(q-1)} \, dt \right]^{-1-1/q} \\
&\times \left( \int_0^1 \left| 1 - \lambda - t \right| \left[ \left| t^s \left( f'(a) \right| + m(1-t)^s \left| f'(mH(a,b)) \right| \right] \, dt \right]^{1/q} \\
&+ \left( \int_0^1 \left| 1 - \mu - t \right| \left( \frac{t}{b} + \frac{1-t}{H(a,b)} \right)^{-2q/(q-1)} \, dt \right)^{1-1/q} \\
&\times \left( \int_0^1 \left| 1 - \mu - t \right| \left[ \left| t^s \left( f'(b) \right| + m(1-t)^s \left| f'(mH(a,b)) \right| \right] \, dt \right]^{1/q} \right),
\end{align*}

where, for $a > 0$ and $u \neq a$

\begin{align*}
Q(a,u,\lambda) &= \frac{a(u-a)^{2q/(q-1)}}{(q+1)(u-a)} \left\{ (1-\lambda)a^{-(q+1)/(q-1)} - \lambda a^{-(q+1)/(q-1)} \right. \\
&\left. - \frac{q-1}{2(u-a)} \left[ a^{2/(q-1)} - 2[(1-\lambda)(u-a) + a]^{-2/(q-1)} + u^{2/(q-1)} \right] \right\}.
\end{align*}

\textbf{Proof.} If $-1 < s \leq 1,$ by the inequality (3.3) and the extended harmonic $(s,m)$-convexity of $|f'|^q,$ we derive

\begin{align*}
\left| \frac{\lambda f(a) + mf(b)}{2} + \frac{2 - \lambda - \mu}{2} f(H(a,b)) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \leq \frac{b-a}{4ab} \left[ \int_0^1 \left| 1 - \lambda - t \left( \frac{t}{a} + \frac{1-t}{H(a,b)} \right) \right|^{-2q/(q-1)} \, dt \right]^{-1-1/q} \\
&\times \left( \int_0^1 \left| 1 - \lambda - t \right| \left[ \left| t^s \left( f'(a) \right| + m(1-t)^s \left| f'(mH(a,b)) \right| \right] \, dt \right]^{1/q} \\
&+ \left( \int_0^1 \left| 1 - \mu - t \right| \left( \frac{t}{b} + \frac{1-t}{H(a,b)} \right)^{-2q/(q-1)} \, dt \right)^{1-1/q} \\
&\times \left( \int_0^1 \left| 1 - \mu - t \right| \left[ \left| t^s \left( f'(b) \right| + m(1-t)^s \left| f'(mH(a,b)) \right| \right] \, dt \right]^{1/q} \right),
\end{align*}

where

\( \int_0^1 \left| 1 - \lambda - t \right| \left( \frac{t}{a} + \frac{1-t}{H(a,b)} \right)^{-2q/(q-1)} \, dt = Q(a,H(a,b),\lambda), \)

\( \int_0^1 \left| 1 - \lambda - t \right| t^s \, dt = \frac{2(1-\lambda)s^2 + \lambda(s+2) - 1}{(s+1)(s+2)}, \)
Combining (3.7) with (3.8) gives the required inequality (3.5). Similarly, by the inequality (3.4), we can prove the inequality (3.6). The proof of Theorem 8 is complete. □

**Corollary 3.** Under assumptions of Theorem 8,

1. If \(-1 < s \leq 1\), then

\[
\left| \frac{\lambda f(a) + \mu f(b)}{2} + \frac{2 - \lambda - \mu}{2} f(H(a, b)) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{b - a}{24ab} \left( \frac{6}{(s + 1)(s + 2)} \right)^{1/q} \left\{ \left[ (2(1 - \lambda)^{s+2} + \lambda(s + 2) - 1) |f'(a)|^q \right]^{1/q} \left[ (2(1 - \lambda)^3 + 3\lambda - 1) a^{2q/(q-1)} + 2H^{2q/(q-1)}(a, b) \right]^{1-1/q} + \left[ (2(1 - \mu)^{s+2} + \mu(s + 2) - 1) |f'(b)|^q \right]^{1/q} \left[ b^{2q/(q-1)} + 2H^{2q/(q-1)}(a, b) \right]^{1-1/q} \right\};
\]

2. If \(s = -1\), then

\[
\left| f(H(a, b)) - \frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \right| \\
\leq \frac{b - a}{2 \times 12(q-1)/q} \left( \frac{2(1 - 2\ln 2) |f'(a)|^q}{ab} \right)^{1/q} \left[ a^{2q/(q-1)} + 2H^{2q/(q-1)}(a, b) \right]^{1-1/q} + \left[ |f'(a)|^q \right]^{1/q} \left[ b^{2q/(q-1)} + 2H^{2q/(q-1)}(a, b) \right]^{1-1/q}.
\]

**Proof:** Substituting

\[
Q(a, H(a, b), \lambda) \leq \frac{2(1 - \lambda)^3 + 3\lambda - 1}{6} a^{2q/(q-1)} + \frac{2\lambda^3 - 3\lambda + 2}{6} H^{2q/(q-1)}(a, b)
\]

into (3.7) yields Corollary 3. □

**Theorem 9.** Let \(f : (0, d] \to \mathbb{R}\) is extended harmonically \((s, m)\)-convex for some fixed \(m \in (0, 1]\), \(s \in (-1, 1]\), and \(a, b \in (0, d]\) with \(a < b\). If \(f \in L_1([a, b])\), then

\[
2^s f(H(a, b)) \leq \frac{ab}{b - a} \int_a^b \frac{f(x) + mf(mx)}{x^2} \, dx
\]

and

\[
\frac{ab}{b - a} \int_a^b \frac{f(x)}{x^2} \, dx \leq \min \left\{ \frac{f(a) + mf(mb)}{s + 1}, \frac{mf(ma) + f(b)}{s + 1} \right\}.
\]
In particular, when \( m = 1 \), we have
\[
2^{s-1} f(H(a,b)) \leq \frac{ab}{b-a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{s+1}.
\] (3.10)

**Remark 1.** The inequality (3.10) appeared in [17].

**Proof.** From the extended harmonic \((s,m)\)-convexity \( f \), it follows that
\[
f(H(a,b)) = \int_{0}^{1} \frac{2}{(ta^{-1}+(1-t)b^{-1}+tb^{-1}+(1-t)a^{-1})} \, dt
\leq \frac{1}{2^s} \int_{0}^{1} \left[ f \left( \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \right) + mf \left( m \left( \frac{t}{b} + \frac{1-t}{a} \right)^{-1} \right) \right] \, dt.
\]
Let \( x = \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \) for \( t \in [0,1] \). Then
\[
\int_{0}^{1} f \left( \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \right) \, dt = \frac{ab}{b-a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx.
\] (3.11)

Similarly, we have
\[
\int_{0}^{1} f \left( \left( \frac{t}{b} + \frac{1-t}{a} \right)^{-1} \right) \, dt = \frac{ab}{b-a} \int_{a}^{b} \frac{f(mx)}{x^2} \, dx.
\] (3.12)
From (3.11) and (3.12), the inequality (3.9) follows immediately.

Let \( x = \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \) for \( t \in [0,1] \). Then
\[
\frac{ab}{b-a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx = \int_{0}^{1} f \left( \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \right) \, dt
\leq \int_{0}^{1} [t^s f(a) + m(1-t)^s f(mb)] \, dt = \frac{f(a) + mf(mb)}{s+1}.
\]
The proof of Theorem 9 is complete. \( \square \)

**Corollary 4.** Under assumptions of Theorem 9, if \( s = m = 1 \), then
\[
f(H(a,b)) \leq \frac{ab}{b-a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx \leq \frac{f(a) + f(b)}{2}.
\]

**Theorem 10.** For \( m \in (0,1] \) and \( s \in (-1,1] \), let \( f : (0,d] \rightarrow \mathbb{R} \) is extended harmonically \((s,m)\)-convex and \( a, b \in (0,d) \) with \( a < b \). If \( f \in L_{1}([a,b]) \), then
\[
2^{s-2} H^{2}(a,b) f(H(a,b)) \leq \frac{ab}{b-a} \int_{a}^{b} \left[ f(x) + mf(mx) \right] \, dx
\]
and
\[
\frac{ab}{b-a} \int_{a}^{b} f(x) \, dx \leq \frac{a^2}{s+1} \left[ m \times_{2} F_{1}(2,1,s+2,1-ab^{-1}) \right] f(ma)
\]
where $\,_{2}F_{1}(a,b;c;z)$ is the hypergeometric function defined by (3.2).

**Proof.** By the extended harmonic $(s,m)$-convexity of $f$, we have

$$H(a,b)f(H(a,b)) \leq 2^{2-s} \int_{0}^{1} \left[ \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-2} \right] f \left( \left( \frac{t}{a} + \frac{1-t}{b} \right)^{-1} \right) dt$$
$$+ 2^{2-s}m \int_{0}^{1} \left[ \left( \frac{t}{b} + \frac{1-t}{a} \right)^{-2} \right] f \left( \left( \frac{t}{b} + \frac{1-t}{a} \right)^{-1} \right) dt$$
$$\leq \frac{ab}{b-a} \int_{a}^{b} f(x) + mf(mx) \, dx$$

and

$$\frac{ab}{b-a} \int_{a}^{b} f(x) \, dx = \int_{0}^{1} \left( \frac{1-t}{a} + \frac{t}{b} \right)^{-2} f \left( \left( \frac{1-t}{a} + \frac{t}{b} \right)^{-1} \right) dt$$
$$\leq \int_{0}^{1} \left( \frac{1-t}{a} + \frac{t}{b} \right)^{-2} \left[ m(1-t)^{s} f(ma) + t^{s}f(b) \right] dt$$
$$= \frac{a^{2}}{s+1} \left[ m \times _{2}F_{1}(2,1,s+2,1-ab^{-1}) f(ma) \right.$$  
$$+ _{2}F_{1}(2,s+1,s+2,1-ab^{-1}) f(b) \right].$$

The proof of Theorem 10 is thus complete. $\square$

**Corollary 5.** Under assumptions of Theorem 10, if $s = m = 1$, then

$$\frac{[H(a,b)]^{2}f(H(a,b))}{4} \leq \frac{ab}{b-a} \int_{a}^{b} f(x) \, dx$$
$$\leq \frac{a^{2}b[b \ln(a^{-1}b) - (b-a)]}{(b-a)^{2}} f(a) + \frac{ab^{2}[(b-a) - a \ln(a^{-1}b)]}{(b-a)^{2}} f(b).$$

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**Authors’ addresses**

**Chun-Ying He**  
College of Mathematics and Statistics, Hulunbuir University, Hailaer 021008, Inner Mongolia, China  
College of Mathematics and Physics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China  
*E-mail address:* hechunying9209@qq.com

**Bo-Yan Xi**  
College of Mathematics and Physics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China  
*E-mail address:* baoyintu78@qq.com, baoyintu78@imun.edu.cn

**Bai-Ni Guo**  
School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454010, Henan, China  
*E-mail address:* bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com