A Continuous Theory of Persistence for Mappings Between Manifolds

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Abstract

Using sheaf theory, I introduce a continuous theory of persistence for mappings between compact manifolds. In the case both manifolds are orientable, the theory holds for integer coefficients. The sheaf introduced here is stable to homotopic perturbations of the mapping. This stability result has a flavor similar to that of bottleneck stability in persistence.

1 Introduction

This is a theoretical paper motivated by data analysis where the data is a mapping \( f : \mathbb{M} \rightarrow \mathbb{N} \) between two manifolds. Singularity theory, as introduced by Whitney [13] and developed by Thom [12] and Arnold [1], studies critical points of differentiable mappings. Loosely speaking, a singularity is a point in a level set where the manifold property fails making Poincaré duality unavailable. Singularity theory studies local structures of singularities that are stable to perturbations of the mapping. Unstable singularities are considered unobservable. Often the data \( f \) is an interpolation of a finite collection of noisy observations. The differentiable structure of \( f \) is not clear or simply unavailable. In this paper, I choose to study the homology of level sets and quantify its stability to perturbations of the mapping. A high stability measurement relative to noise indicates relevant information.

Let \( f : \mathbb{M} \rightarrow \mathbb{N} \) be a continuous mapping from an \( m \)-manifold \( \mathbb{M} \) to an \( n \)-manifold with metric \( \mathbb{N} \). Letting \( \mathbb{F} = f^{-1}(U) \), for an open ball \( U \subseteq \mathbb{N} \) or \( U = \mathbb{N} \), I define a homomorphism \( \Psi_f(U) : H^p(\mathbb{F}_U) \rightarrow H_{m-n-p}(\mathbb{F}_U) \) satisfying the following four properties:

1. **Duality.** If \( U \) is a point, then \( \Psi_f(\{x\}) : H^p(\mathbb{M}) \rightarrow H_{m-p}(\mathbb{M}) \) is the Poincaré duality. If \( x \in \mathbb{N} \) is regular, then there is a neighborhood \( U \) of \( x \) such that \( \Psi_f(U) : H^p(\mathbb{F}_U) \rightarrow H_{m-n-p}(\mathbb{F}_U) \) is an isomorphism, for each dimension \( p \). Loosely speaking, \( x \) is regular if it has a closed \( n \)-ball neighborhood \( B \) such that \( \mathbb{F}_B \) is a closed \( n \)-ball bundle over the base space \( f^{-1}(x) \).

2. **Topological degree.** If \( \mathbb{M} \) and \( \mathbb{N} \) are equidimensional and both orientable, then \( \Psi_f(\{x\}) \) encodes the topological degree of \( f \).

3. **Local stability.** Let \( \psi : \mathbb{M} \times [0,1] \rightarrow \mathbb{N} \) be a homotopy connecting \( f = h_0 \) to \( g = h_1 \) such that the distance between \( h_0(x) \) and \( h_t(x) \) is at most \( \varepsilon \cdot t \), for each \( t \in [0,1] \) and \( x \in \mathbb{M} \). Then

\[
\begin{align*}
H^p(\mathbb{F}_{U^c}) \xrightarrow{\Psi_f(U^c)} H_{m-n-p}(\mathbb{F}_{U^c}) \\
\downarrow \quad \uparrow \\
H^p(\mathbb{G}_U) \xrightarrow{\Psi_g(U)} H_{m-n-p}(\mathbb{G}_U)
\end{align*}
\]

commutes, where \( U^c \supseteq U \) is an open ball of points at most \( \varepsilon \) from \( U \) and the vertical homomorphisms are induced by inclusion of spaces.

4. **Global stability.** The assignment of a stable group \( \mathbb{F}^p(U) = H^p(\mathbb{F}(U))/\ker(\Psi_f(U)) \) to each open ball of \( \mathbb{N} \) defines a presheaf. Using the Local Stability result, I argue that the sheaf of sections of the \( \text{étalé} \) space of this presheaf is stable to homotopic perturbations of the mapping \( f \). This stability result has a flavor similar to that of bottleneck stability in persistence.

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The theory of persistence studies the homology of inverses for the case $N = \mathbb{R}$ [7]. The sheaf we introduce in this paper is related to the level set zigzag [3], a version of persistence equivalent to extended persistence [6]. Whereas the level set zigzag contains information for only a finite number of inverses, this sheaf carries information for each open set of $\mathbb{R}$. This method provides for a continuous interpretation of persistence requiring a different set of assumptions on the mapping $f : M \to \mathbb{R}$ than is usually made for persistence. In fact, the only requirement is that $M$ must be a manifold. No assumptions are needed on the tameness of $f$. If $M$ is orientable, then our results hold for integer coefficients. The stability of this sheaf implies a stability result for the interval decomposition of the level set zigzag. The ideas in this paper evolved more directly from previous work [8, 9] on quantifying the stability of inverses.

Section 2 is devoted to the definition of the homomorphism $\Psi_f(U)$. Here I will discuss the duality properties of $\Psi_f(U)$ and how $\Psi_f(N)$ encodes the topological degree of $f$. In Section 3, I argue local and global stability. In Section 4, I discuss persistence in relation to the sheaf of stable groups. I conclude with a discussion in Section 5.

2 The Intersection Homomorphism

Let $f : M \to N$ be a continuous mapping between two compact orientable manifolds where $\dim M = m$ and $\dim N = n$. For an open ball $U \subseteq N$, denote by $F_U$ the inverse $f^{-1}(U)$. I will assume an orientation on both manifolds so that we may use integer coefficients. If both manifolds are not orientable, simply switch to integer modulo two coefficients. To simplify the exposition, I assume both manifolds are connected and without boundary.

**Definition.** The manifolds $M$ and $N$ are orientable, compact, and connected, and therefore $H_m(M) \cong H^0(N) \cong \mathbb{Z}$. The choice of an isomorphism to $\mathbb{Z}$ is a choice of a generator. Choose a generator $\Gamma \in H_m(M)$ and a generator $\Delta \in H^0(N)$. Now for each open ball $U \subseteq N$, consider the following fiber bundle pairs:

$$(N, \overline{U}) \to (F_U \times N, F_U \times \overline{U}) \quad \pi$$

$$(N, \overline{U}) \to (F_U \times N, F_U \times \overline{U}) \quad \rho$$

where $\overline{U}$ is shorthand for the complement of $U$ in $N$. If $U$ is the retract of a larger ball and assuming $(F_U \times N, F_U \times \overline{U})$ is a CW complex, the relative homology group $H_m(F_U \times N, F_U \times \overline{U})$ is isomorphic to $H_m(F_U \times N/F_U \times \overline{U})$ [10] page 441. The mapping $f$ provides a continuous section of the space $F_U \times \overline{N}/F_U \times \overline{U}$ taking each point $a \in F_U$ to $(a, f(a))$ and each point outside $F_U$ to the point $F_U \times \overline{U}$. The section induces a homomorphism $f_U : H_m(M) \to H_m(F_U \times N/F_U \times \overline{U}) \cong H_m(F_U \times N, F \times \overline{U})$. Denote by $[F_U]$ the class $f_U(\Gamma)$. The projection map $\rho$ induces a homomorphism $\rho : H^0(N) \to H^0(F_U \times N, F_U \times \overline{U})$. Denote by $[U]$ the class $\rho(\Delta)$.

For each open ball $U \subseteq N$ that is the retract of a larger ball containing it, define the intersection homomorphism $\Psi_f(U) : H^p(F_U) \to H_{m-n-p}(F_U)$ as the composition of the following maps:

$$H^p(F_U \times N, F_U \times \overline{U}) \sim [U] \to H^{p+n}(F_U \times N, F_U \times \overline{U}) \sim [F_U] \to H_{m-n-p}(F_U \times N)$$

Intuitively the intersection homomorphism takes an $(m-p)$-cycle in the section of $f$ and intersects it with the base space $N$. The resulting intersection is an $(m-n-p)$-cycle counted with a sign. If $U = N$, then the intersection homomorphism is well defined and is invariant to homotopic perturbations of $f$. If $n = 0$, then $\Psi_f(N)(\alpha) = \alpha \cap \Gamma$ is simply the Poincaré duality.

**Commutativity Lemma.** If $V \subseteq U$ is a non-empty open ball contained in $U$, then

$$H^p(F_U) \xrightarrow{\Psi_f(U)} H_{m-n-p}(F_U)$$

$$H^p(F_V) \xrightarrow{\Psi_f(V)} H_{m-n-p}(F_V)$$

2
commutes. Here the unlabeled homomorphisms are induced by inclusion of spaces.

**Proof.** Expand the two intersection homomorphisms to:

\[
\begin{align*}
\text{H}^n(F_U) & \xrightarrow{\pi^*} \text{H}^n(F_\mathbb{U} \times N, \mathbb{N} \times U) \\
& \xrightarrow{\sim} \text{H}^{n+p}(F_\mathbb{U} \times N, \mathbb{N} \times U) \\
& \xrightarrow{\sim} \text{H}_{m-n-p}(F_\mathbb{U} \times N) \\
& \xrightarrow{\pi_*} \text{H}_{m-n-p}(F_U)
\end{align*}
\]

where the vertical homomorphisms are induced by inclusion of spaces. It is clear the rightmost square commutes. Choose a class \(x \in \text{H}^n(F_U)\) and push it forward to \(\text{H}_{m-n-p}(F_\mathbb{U} \times N)\) along the topmost path.

We have

\[
(\pi^*(x)) \sim [F_U]. \tag{1}
\]

Note the chains in \([F_U]\) supported by \(F_U \times \mathbb{U}\) go to zero in Equation (1). Now take \(x\) to the same group via the second path. We have

\[
j(((\pi^* \circ i)(x)) \sim [F_U]). \tag{2}
\]

Similarly, the chains in \([F_U]\) supported by \(F_U \times \mathbb{V}\) go to zero.

Pick a cocycle \(c\) in \(\alpha\) and a cycle \(\sigma\) in \(T \in \text{H}_m(\mathbb{M})\). Now pick a cocycle \(d\) in \(\Delta \in \text{H}^n(\mathbb{N})\) that is non-zero on a single point \(x \in V\). That is, the cocycle \(d\) is non-zero when evaluated on the singular \(n\)-simplex taking an \(n\)-simplex to \(x\). Following Equation (1), the triple \(c, d, \) and \(\sigma\) results in a cycle that is supported by \(F_\mathbb{U} \times N\), where \(F_\mathbb{U} = f^{-1}(U)\). Following Equation (2) we get the same cycle. It follows that class \(1\) is class \(2\).

**Duality.** If the inverse of an open ball \(U \subseteq \mathbb{N}\) is free of singularities, then intersection homomorphism \(\Psi_f(U)\) is an isomorphism.

**Duality Theorem.** Let \(x \in \mathbb{N}\) and \(B \subseteq \mathbb{N}\) a closed \(n\)-ball containing \(x\). If \(\mathbb{F}_B = f^{-1}(B)\) is a closed \(n\)-ball bundle with fiber \(\mathbb{B}\) over the base space \(\mathbb{F}_x = f^{-1}(x)\) such that

\[
\begin{array}{c}
\mathbb{B} \\
\approx
\end{array} \xrightarrow{f} \mathbb{F}_B \xrightarrow{r} \mathbb{F}_x
\]

commutes, then the intersection homomorphism \(\Psi_f(U) : \text{H}^n(F_U) \rightarrow \text{H}_{m-n-p}(F_U)\), where \(U\) is the interior of \(B\), is an isomorphism.

**Proof.** It does not hurt to assume \(\mathbb{F}_x\) is connected. Note the mapping \(f\) gives a homeomorphism from \(\mathbb{F}_x \times B\) to \(\mathbb{F}_x \times \mathbb{B} = \mathbb{F}_B\). The proof follows by showing the following diagram commutes:

\[
\begin{array}{c}
\text{H}^n(F_U) \\
\approx \xrightarrow{r^*} \text{H}^{n+p}(F_\mathbb{U} \times N, \mathbb{N} \times U) \xrightarrow{\pi_*} \text{H}_{m-n-p}(F_U)
\end{array}
\]

\[
\begin{array}{c}
\text{H}^n(F_\mathbb{U}) \\
\xrightarrow{\sim} \text{H}^{n+p}(F_\mathbb{U} \times N, \mathbb{N} \times U) \xrightarrow{\pi_*} \text{H}_{m-n-p}(F_U)
\end{array}
\]

The top composition of homomorphisms is the intersection homomorphism. The isomorphisms labelled with \(r\) are induced by the retraction \(r\) taking \(\mathbb{F}_U\) to \(\mathbb{F}_x\).

Let \(i^* : \text{H}^n(F_\mathbb{U} \times N, \mathbb{N} \times U) \rightarrow \text{H}^n(F_\mathbb{U} \times B, \mathbb{F}_x \times \mathbb{B} B)\) be the homomorphism induced by inclusion of spaces. By the retraction \(r\) and by excision, \(i^*\) is an isomorphism. Define \([B] \in \text{H}^n(F_\mathbb{U} \times B, \mathbb{F}_x \times \mathbb{B} B)\) as \(i^*(U)\). Similarly the homomorphism \(i_* : \text{H}_m(F_\mathbb{U} \times B, \mathbb{F}_x \times \mathbb{B} B) \rightarrow \text{H}_m(F_\mathbb{U} \times N, \mathbb{F}_U \times U)\) induced by inclusion of spaces is also an isomorphism. Define \(\Gamma_B \in \text{H}_m(F_\mathbb{U} \times B, \mathbb{F}_x \times \mathbb{B} B) \cong \text{H}_m(F_B, \mathbb{F}_B)\) as the restriction of the generator \(\Gamma \in \text{H}_m(\mathbb{M})\). Note \(\imath_B(\Gamma_B) = [F_U]\).

The fiber bundle pair:
Consider the mapping \( \text{Topological Degree.} \)

the intersection homomorphism \( \Psi \). Let \( \Gamma \)

\( H_1(B, \text{Bd } B) \rightarrow (F_x \times B, F_x \times \text{Bd } B) \xrightarrow{\pi} F_x \)

is a Thom space [11] page 441] and the class \([B] \in H^n(F_x \times B, F_x \times \text{Bd } B)\) is a Thom class because, when restricted to each fiber, \([B]\) generates \( H^n(B, \text{Bd } B) \cong Z \). Thus the composition \( \sim [B] \circ \pi_* \) is the Thom isomorphism. The cap product, \( \cap \Gamma_B \), is simply the Lefschetz duality. The diagram commutes by our choice of \([B]\) and \( \Gamma_B \).

**Topological Degree.** Now assume \( \dim M = \dim N = m \). Intuitively the degree of a mapping \( f : M \rightarrow N \) is the number of times the mapping \( f \) winds \( M \) around \( N \). Choose a generator \( \Pi \in H_m(N) \cong Z \) such that the Kronecker index [11] page 276] of \( \Delta \) and \( \Pi \), written \( (\Delta, \Pi) \), is one. The mapping \( f \) induces a homomorphism \( f_* : H_m(M) \rightarrow H_m(N) \). The topological degree of the mapping \( f \) is the unique integer \( \deg f \) such that

\[ f_*(\Gamma) = \deg f \cdot \Pi. \]

The degree of \( f \) is encoded in the intersection homomorphism. To show how, I must choose more generators. Let \( \Gamma^* \in H^0(M) \cong Z \) be the Poincaré dual of the class \( \Gamma \in H_m(M) \). In other words, \( \Gamma^* \) is the class such that \( \Gamma^* \cap \Gamma = \Gamma \). Now choose a generator \( \Lambda \in H^m(M) \cong Z \) such that \( \langle \Lambda, \Gamma \rangle = 1 \) and denote by \( \Lambda_* = \Lambda \cap \Gamma \) its Poincaré dual.

**Degree Theorem.** \( \Psi_f(N)(\Gamma^*) = \deg f \cdot \Lambda_* \).

**Proof.** The intersection homomorphism \( \Psi_f(N) : H^0(M) \rightarrow H_0(M) \) is a composition of the following homomorphisms:

\[ H^0(M) \xrightarrow{\pi^*} H^0(M \times N) \xrightarrow{[N]} H^0(M \times N) \xrightarrow{\sim} H^0(M) \xrightarrow{\pi_*} H_0(M). \]

Start with the class \( \Gamma^* \) and push it forward. I use the notation \( \Gamma_{i, \ldots, j} \) to indicate the singular simplices spanned by vertices \( i \) through \( j \) of each singular simplex in a singular cycle representing \( \Gamma \). We have

\[ \langle \pi^*(\Gamma^*) \sim [N], [M]_{i, \ldots, j} \rangle \cdot \pi^*([M]_{m, \ldots, m}) = \langle \Gamma^*, \Gamma_{0, \ldots, 0} \rangle \cdot \langle \Delta, \rho \circ f_0(\Gamma_{0, \ldots, 0}) \rangle \cdot \pi^*([M]_{m, \ldots, m}) = 1 \cdot \langle \Delta, \deg f \cdot \Pi \rangle \cdot \pi^*([M]_{m, \ldots, m}) = \deg f \cdot \pi^*([M]_{m, \ldots, m}) = \deg f \cdot \Gamma_{m, \ldots, m} = \deg f \cdot \Lambda_. \]

**Example A.** Consider the mapping \( f : M \rightarrow S^1 \) as shown in Figure 1 from the torus \( M \) to \( S^1 \). For the open ball \( U \) as indicated, \( H^0(F_U) \cong Z, H^1(F_U) \cong Z \oplus Z, H_1(F_U) \cong Z \oplus Z, \) and \( H_0(F_U) \cong Z \). For \( p = 0 \), the intersection homomorphism \( \Psi_f(U) \) is the composition

\[ H^0(F_U) \xrightarrow{\pi^* \sim [U]} H^1(F_U \times S^1, F_U \times \{U\}) \xrightarrow{\sim [F_U]} H_1(F_U \times S^1) \xrightarrow{\pi_*} H_0(F_U). \]

Choose a cocycle \( d \in [U] \) that assigns a non-zero value to the singular 1-simplex taking a 1-simplex to the point \( x \). Choose a cocycle \( c \) in a non-zero class \( \alpha \in H^0(F_U) \) and a cycle \( \sigma \in [F_U] \). After applying the projection, cup product, and cap product homomorphisms, the result is a 1-cycle supported by \( f^{-1}(x) \). This cycle is a non-zero multiple of cycle \( A + B \). Thus the image of the intersection homomorphism is isomorphic to \( Z \) interpreted as the subgroup of \( H_1(F_U) \) that is supported by every level set over \( U \).

For \( p = 1 \), the intersection homomorphism is the composition

\[ H^1(F_U) \xrightarrow{\pi^* \sim [U]} H^2(F_U \times S^1, F_U \times \{U\}) \xrightarrow{\sim [F_U]} H_0(F_U \times S^1) \xrightarrow{\pi_*} H_0(F_U). \]

Choose, as before, the cocycle \( d \in [U] \) and the cycle \( \sigma \in [F_U] \). Choose a 1-cocycle \( a \) supported by \( F_U \) that sends the 1-cycle \( A \) to one and all other 1-cycles to zero. Similarly, choose a 1-cocycle \( b \) that sends the 1-cycle \( B \) to one and all other 1-cycles to zero. The cocycles \( a \) and \( b \) belong to independent non-zero classes of \( H^1(F_U) \). The composition applied to \( a \) results in a non-trivial 0-cycle. The same is true of \( b \). The composition applied to \( a - b \) results in a trivial 0-cycle. Thus the image of the intersection homomorphism is isomorphic to \( H_0(F_U) \).
The mapping $f$ is the projection of the torus to the 1-sphere as shown. The inverse of the open set $U$ is a pair of pants. The 1-cycles $A$ and $B$ are non-trivial non-homologous cycles of $\mathbb{F}_U$.

## 3 Stability

The definition of the intersection homomorphism is motivated by the desire to quantify the stability of each homology class in a level set. In addition to the assumption that both $M$ and $N$ are compact, connected, and orientable, we assume a metric on $N$. The metric allows for a measurement of stability.

### 3.1 Local Stability

Let $f, g : M \to N$ be two mappings connected by a homotopy $h : M \times [0, 1] \to N$. Denote by $\varepsilon \geq 0$ the minimum value such that the distance between $h_0(x)$ and $h_t(x)$ in $N$ is at most $\varepsilon \cdot t$, for each $t \in [0, 1]$ and $x \in M$. Intuitively $\varepsilon$ is a measure of the wildness in the homotopy. Given an open ball $U$, define the dilation of $U$ by $\varepsilon$ as the open set $U^\varepsilon$ of points at most $\varepsilon$ from $U$. Note $U^\varepsilon$ may not be an open ball. Define $F_{U^\varepsilon} = f^{-1}(U^\varepsilon)$ and $G_{U^\varepsilon} = g^{-1}(U)$ as the inverses of $U^\varepsilon$ and $U$.

**Local Stability Theorem.** If $f, g : M \to N$ are connected by a homotopy $h$ with wildness less than $\varepsilon$, then

$$
\begin{align*}
\Psi_f(U^\varepsilon) &\to H_{p-n}(F_{U^\varepsilon}) \\
\Psi_g(U) &\to H_{p-n}(G_U)
\end{align*}
$$

commutes, where both $U$ and its dilation $U^\varepsilon$ are open balls of $N$ for which the intersection homomorphism is well defined. Here the unlabeled homomorphisms are induced by inclusion of spaces.

**Proof.** The proof follows the proof of the Commutativity Lemma. Expand the two intersection homomorphisms to

$$
\begin{align*}
H_p(F_{U^\varepsilon}) &\xrightarrow{\pi^*} H_p(F_{U^\varepsilon} \times N, F_{U^\varepsilon} \times \mathcal{U}) \\
&\xrightarrow{[U^\varepsilon]} H_{p+n}(F_{U^\varepsilon} \times N, F_{U^\varepsilon} \times \mathcal{U}) \\
&\xrightarrow{[F_{U^\varepsilon}]} H_{p-n}(F_{U^\varepsilon} \times N) \\
&\xrightarrow{\pi^*} H_{p-n}(F_{U^\varepsilon}) \\
H_p(G_U) &\xrightarrow{\pi^*} H_p(G_U \times N, G_U \times \mathcal{U}) \\
&\xrightarrow{[U]} H_{p+n}(G_U \times N, G_U \times \mathcal{U}) \\
&\xrightarrow{[G_U]} H_{p-n}(G_U \times N) \\
&\xrightarrow{\pi^*} H_{p-n}(G_U),
\end{align*}
$$
where the vertical homomorphisms are induced by inclusion of spaces. It is clear the rightmost square commutes. Choose a class \( \alpha \in H^p(\mathbb{F}_{U^{\varepsilon}}) \) and push it forward to \( H_{m-n-p}(\mathbb{F}_{U^{\varepsilon}} \times \mathbb{N}) \) along the topmost path. We have

\[
(\pi^*(\alpha) \sim [U]) \sim [\mathbb{F}_{U^{\varepsilon}}].
\]

Now take \( \alpha \) to the same group via the second path. We have

\[
j((\pi^* \circ i)(\alpha) \sim [U]) \sim [\mathbb{G}_{U}].
\]

Pick a cocycle \( c \) in \( \alpha \) and a cycle \( \sigma \) in \( \Gamma \in H_m(\mathbb{N}) \). Now pick a cocycle \( d \) in \( \Delta \in H^p(\mathbb{N}) \) that is non-zero on a single point \( x \in U \). Following Equation (3) the triple \( c, d, \) and \( \sigma \) results in a cycle \( a \) supported by \( \mathbb{F}_x \), where \( \mathbb{F}_x = f^{-1}(x) \). Following Equation (2) we get a cycle \( b \) supported by \( \mathbb{G}_x \). However, the cycles \( a \) and \( b \) are not necessarily the same, but the homotopy \( h \) provides an \((m - n - p + 1)-\)chain with boundary \( a - b \). It follows that \( \alpha \) class (3) is class (4). \( \square \)

**Stable groups.** For each mapping \( f \) and \( g \), there is a collection \( \mathbb{F}^p(U) = H^p(\mathbb{F}_U)/\ker \Psi_f(U) \) and \( \mathbb{G}^p(U) = H^p(\mathbb{G}_U)/\ker \Psi_g(U) \) of quotient groups. If \( V \subseteq U \) is a non-empty open ball, then there are monomorphisms between the quotient groups as follows:

\[
\begin{align*}
\mathbb{F}^p(U) &\leftarrow q \quad H^p(\mathbb{F}_U) \xrightarrow{\Psi_f(U)} H_{m-n-p}(\mathbb{F}_U) & \mathbb{G}^p(U) &\leftarrow q \quad H^p(\mathbb{G}_U) \xrightarrow{\Psi_g(U)} H_{m-n-p}(\mathbb{G}_U) \\
ds_{U,V} &\quad \downarrow & \quad \downarrow &\quad \downarrow \\
\mathbb{F}^p(V) &\leftarrow q \quad H^p(\mathbb{F}_V) \xrightarrow{\Psi_f(V)} H_{m-n-p}(\mathbb{F}_V) & \mathbb{G}^p(V) &\leftarrow q \quad H^p(\mathbb{G}_V) \xrightarrow{\Psi_g(V)} H_{m-n-p}(\mathbb{G}_V),
\end{align*}
\]

where \( q \) is the quotient homomorphism. Each element \( s \in \mathbb{F}^p(U) \) is \( \alpha + \ker \Psi_f(U) \), for some \( \alpha \in H^p(\mathbb{F}_U) \). The homomorphism \( \text{res}_{U,V}(s) \) restricts \( \alpha \) to \( \mathbb{F}^p(\mathbb{F}_V) \) and then applies the quotient map \( q \). If \( \text{res}_{U,V}(s) = 0 \), then the restriction of \( \alpha \) to \( H^p(\mathbb{F}_V) \) belongs in \( \ker \Psi_f(V) \). By the Commutativity Lemma, \( \alpha \) must belong to \( \ker \Psi_f(V) \) and therefore \( s = 0 \). The two restriction homomorphisms are monomorphisms. The Local Stability Theorem implies a monomorphism between \( \mathbb{F}^p(U^{\varepsilon}) \) and \( \mathbb{G}^p(U) \) as follows:

\[
\begin{align*}
\mathbb{F}^p(U^{\varepsilon}) &\leftarrow q \quad H^p(\mathbb{F}_{U^{\varepsilon}}) \xrightarrow{\Phi_f(U^{\varepsilon})} H_{m-n-p}(\mathbb{F}_{U^{\varepsilon}}) & \mathbb{G}^p(U) &\leftarrow q \quad H^p(\mathbb{G}_U) \xrightarrow{\Psi_g(U)} H_{m-n-p}(\mathbb{G}_U) \\
\Sigma_{f,g}(U) &\quad \downarrow & \quad \downarrow &\quad \downarrow \\
\mathbb{G}^p(U) &\leftarrow q \quad H^p(\mathbb{G}_U) \xrightarrow{\Psi_g(U)} H_{m-n-p}(\mathbb{G}_U).
\end{align*}
\]

For open balls \( V^{\varepsilon} \subseteq U \), the stable group \( \mathbb{F}(U) \) lives in \( \mathbb{G}(U) \) as a subgroup for all mappings \( g \) connected to \( f \) by a homotopy of wildness at most \( \varepsilon \). For this reason, I call the quotient groups \( \mathbb{F}^p(U) \) and \( \mathbb{G}^p(U) \) stable groups or equivalently, stable subgroups of \( H_{m-n-p}(\mathbb{F}_U) \) and \( H_{m-n-p}(\mathbb{G}_U) \).

### 3.2 Global Stability

The Local Stability Theorem implies a stability result for the étalé space of the presheaf of stable groups. The étalé space provides a geometric interpretation of the cohomology of level sets bringing to light some interesting phenomena around the critical values of \( f \) as we will see in Example B.

**Presheaf.** Choose a basis for the topology of \( \mathbb{N} \) that is a collection of open balls for which the intersection homomorphism is well defined. For each \( U \) in the basis, set \( \mathbb{P}_f(U) \) to the direct sum \( \mathbb{F}_U = \mathbb{G}_U \mathbb{F}^p(U) \) of stable groups. The restriction homomorphism \( \text{res}_{U,V} : \mathbb{P}_f(U) \rightarrow \mathbb{P}_f(V) \), applied on each component of the direct sum individually, is an isomorphism. For any triple of open balls \( W \subseteq V \subseteq U \), the Commutativity Lemma implies \( \text{res}_{U,W} = \text{res}_{V,W} \circ \text{res}_{U,V} \). \( \mathbb{P}_f \) is a presheaf.

The intersection homomorphism is not well defined for a single point \( x \in \mathbb{N} \), but one may assign to \( x \) a stable group by considering \( \mathbb{P}_f(U) \) for smaller and smaller open neighborhoods \( U \) of \( x \). The stalk \( \mathbb{P}_f(x) \) at \( x \) is the direct limit

\[
\mathbb{P}_f(x) = \lim_{\rightarrow} \mathbb{P}_f(U),
\]
over all open balls $U$ containing $x$. For an open ball $U$ and a point $x \in U$, the stable group $P_f(U)$ restricts to the stalk $P_f(x)$ at $x$. In fact, this restriction is a monomorphism. For an element $s \in P_f(U)$, the restriction $\text{res}_{U,x}(s)$ of $s$ to $P_f(x)$ is called the germ of $s$ at $x$. Every element in a stalk is a germ. In this sense, every element of a stalk is stable because there is a corresponding element in all nearby stalks. If $x \in \mathbb{N}$ is regular, as defined in the statement of the Duality Theorem, then the stalk $P_f(x)$ is isomorphic to the cohomology of the inverse of $x$.

Sheaf. Associate to the presheaf $P_f$ the sheaf of sections of the étalé space of $P_f$. Let $E_f$ be the disjoint union of the elements in the stalks $P_f(x)$, over all $x \in \mathbb{N}$. For each open set $U \subseteq \mathbb{N}$ in the basis and each $s \in P_f(U)$, the germ of $s$ at each point $x \in U$ defines a basis for the étalé space $E_f$. Let $\pi : E_f \to \mathbb{N}$ be the continuous projection sending each point $(s,x) \in E_f$ to $x$. The sheaf of sections $\mathcal{F}$ of the étalé space is a collection of mappings $s : U \to E_f$, for each open set $U \subseteq \mathbb{N}$, such that $\pi \circ s = \text{id}$. A sheaf satisfies two axioms:

1. If $\{U_i\}$ is an open covering of an open set $U$, and $s,t \in \mathcal{F}(U)$ two sections such that $\text{res}_{U,U_i}(s) = \text{res}_{U,U_i}(t)$ for each $U_i$ in the covering, then $s = t$.

2. Let $\{U_i\}$ be an open covering of an open set $U$. If there is a section $s_i \in \mathcal{F}(U_i)$, for each covering set $U_i$, and $\text{res}_{U_i,U_j}(s_i) = \text{res}_{U_i,U_j}(s_j)$, then there is a section $s \in \mathcal{F}(U)$ such that $\text{res}_{U_i,s}(s) = s_i$.

For an open ball $U$ for which $P_f(U)$ is defined, $P_f(U)$ is isomorphic to $\mathcal{F}(U)$. This follows from the fact that no two elements in $\mathcal{F}(U)$ map to the same germ. In fact, for any open set $U$, one may simply define the stable group $F(U)$ as $\mathcal{F}(U)$ thus completing the presheaf $P_f$. A section $s \in \mathcal{F}(U)$ is maximal if for any open set $W \subseteq U$, there is no section of $\mathcal{F}(W)$ that restricts to $s$. From now on, I will use the letters $s$ and $t$ to refer to sections of the sheaf $\mathcal{F}$.

Example B. Consider a cusp singularity $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as $f_1(x_1,x_2) = x_1x_2 - x_1^3$ and $f_2(x_1,x_2) = x_2$. Although $f$ is defined from the plane, which is not compact, to itself, one may think of $f$ as a restriction of a mapping between the 2-sphere to itself. Shown in Figure 2(left) is the set of critical values of $f$. Notice the cusp singularity at $(0,0)$. The inverse, $\mathcal{F}_U$, of the open set $U$ contains three components. The open set $U$ is contained in a larger open set $W$, which contains the open set $V$. Take a section $s$ in $\mathcal{F}(U)$ representing a single component of $\mathcal{F}_U$. The section $s$ is the restriction of a section $t$ in $\mathcal{F}(W)$, which restricts to a section in $\mathcal{F}(V)$. This process continues around the circle of open sets shown in Figure 2(left) if $s$ represents one of the two of the three components of $\mathcal{F}_U$. The resulting section in $\mathcal{F}(U)$, after having travelled around the circle of open sets, is not $s$. This behavior corresponds to a connected subspace of the étalé space $E_f$ that, when projected to the plane, covers the triangular region twice. See Figure 2(right).

Resolution. For a value $r \in [0,\infty)$ and an open set $U \subseteq \mathbb{N}$, define $U^{1/r}$ as the set of points in $\mathbb{N}$ at a distance at most $1/r$ from $U$. Define the sheaf $\mathcal{F}^r$ by setting $\mathcal{F}^r(U)$ to the image of the restriction $\text{res}_{U^{1/r},U} : \mathcal{F}(U^{1/r}) \to \mathcal{F}(U)$. The definition of $\mathcal{F}^r$ provides an obvious morphism $\Sigma_f : \mathcal{F}^r \to \mathcal{F}$ between the two sheaves taking $\mathcal{F}^r(U)$ to $\text{res}_{U^{1/r},U}(\mathcal{F}(U^{1/r}))$. A morphism between sheaves commutes with the restriction homomorphism. That is

$$\begin{array}{ccc}
\mathcal{F}^r(U) & \xrightarrow{\Sigma_f(U)} & \mathcal{F}(U) \\
\text{res}_{U,V} & \downarrow & \downarrow \text{res}_{U,V} \\
\mathcal{F}^r(V) & \xrightarrow{\Sigma_f(V)} & \mathcal{F}(V)
\end{array}$$

commutes, for every pair of open sets $V \subseteq U$. As the sheaf morphism $\Sigma_f$ is injective, one may think of the sheaf $\mathcal{F}^r$ as a subsheaf of $\mathcal{F}$.

For $r_1 \leq r_2$, the stalk $\mathcal{F}^{r_1}(x)$ of the sheaf $\mathcal{F}^{r_1}$ at $x \in \mathbb{N}$ is a subgroup of the stalk $\mathcal{F}^{r_2}(x)$. Thus the parameter $r$ provides a filtration

$$\mathcal{F}^0(x) \subseteq \mathcal{F}^{r_1}(x) \subset \cdots \subset \mathcal{F}^{r_2}(x) \subseteq \mathcal{F}^\infty(x) = \mathcal{F}(x)$$

of each stalk. The infimum of all values $r$ such that a germ $(s,x) \in \mathcal{F}(x)$ belongs to $\mathcal{F}^r(x)$ is the resolution of the germ $(s,x)$. 
Similarity measure. For two homotopic mappings \( f \) and \( g \), define the similarity measure between their sheaves \( F \) and \( G \) as the maximum value \( r \geq 0 \) such that there is a monomorphism \( F' \to G \) and a monomorphism \( G' \to F \). If \( 1/r \) is the diameter of \( N \), then \( F' = G' \) simply because the intersection homomorphism is homotopy invariant. Thus the similarity measure between \( F \) and \( G \) is at least \( r \), where \( 1/r \) is the diameter of \( N \).

Global Stability Theorem. If \( f, g: M \to N \) are connected by a homotopy \( h \) with wildness \( \varepsilon \), then the similarity measure between \( F \) and \( G \) is at least \( 1/\varepsilon \).

Proof. For a point \( x \in N \), call \( \tau_x \) the supremum of all radii \( \rho \) such that the set of points at a distance less than \( \rho \) from \( x \) is an open ball. Define \( \tau \) as the infimum of \( \tau_x \) over all \( x \in N \). Assuming \( \tau > 0 \), choose a value \( 0 < \sigma < \tau \). Choose, as a basis for the topology of \( N \), a collection of open balls with radius at most \( \sigma \). Construct the presheaves \( P_1 \) and \( P_{\varepsilon} \) of stable groups using this basis.

Set \( r = 1/\varepsilon \). If \( \varepsilon \leq \tau - \sigma \), then the Local Stability Theorem provides a monomorphism \( \Sigma_{f,g}: P_f(U^r) \to P_g(U) \) and symmetrically a monomorphism \( \Sigma_{g,f}: P_g(U^r) \to P_f(U) \). This in turn induces a monomorphism \( \Sigma_{f,g}^*: F^r \to G \) and symmetrically a monomorphism \( \Sigma_{g,f}^*: G^r \to F \). Thus the similarity measure between the two sheaves is at least \( r = 1/\varepsilon \).

If \( \varepsilon > \tau - \sigma \), then divide the interval \([0,1]\) into \( k \) parts

\[
0 = t_0 < t_1 < \ldots < t_k = 1
\]

such that the wildness \( w_i \) of the homotopy between \( h_{t_{i-1}} \) and \( h_{t_i} \), for \( 1 \leq i \leq k \), is at most \( \tau - \sigma \). The sum \( w_1 + w_2 + \ldots + w_k = \varepsilon \). Set \( r_1 = 1/w_i \) and let \( H_i \) be the sheaf of stable groups for the mapping \( h_{t_i} \).

We have:

\[
\begin{array}{c}
 F = H_0 \xrightarrow{\Sigma_0^1} H_1 \xrightarrow{\Sigma_1^2} H_2 \xrightarrow{\Sigma_2^3} \ldots \xrightarrow{\Sigma_{k-1}^k} H_{k-1} \xrightarrow{\Sigma_k^{k+1}} H_k = G.
\end{array}
\]

Restrict the image of \( H_0 \) in \( H_1 \) to sections \( s \in H_1(U) \), where \( s \) is the restriction of a section in \( H_1(U^r) \). This restriction forms a subsheaf of \( H_1 \), which is in the image of \( \Sigma_{1}^{2} \). In fact, this restriction is the sheaf \( F_{1}^{r_1}+r_2 \). Take the inverse of this subsheaf using the morphism \( \Sigma_1^{2} \) resulting in a subsheaf of \( H_1 \). Now take this subsheaf of \( H_1 \) and push it forward, restrict it, and then take its inverse. Continue this process all the way to the sheaf \( G \). The result is the sheaf \( F_{1}^{r_1}+r_2+\ldots+r_k = F^r \) embedded in \( G \). The symmetric argument shows an embedding of \( G^r \) in \( F \). We have:

\[
\text{Figure 2: The triangular region is defined by the set of critical values of } f. \text{ On the left, is a sequence of interleaved open sets around the cusp point. On the right is a connected subspace of the étalé space that, when projected to the plane, covers the triangular region twice.}
\]
The above sheaf morphisms take a germ at a point \( x \in \mathbb{N} \) to a germ at the same point \( x \).

If the resolution \( \rho \) of a germ \((s, x)\) in \( \mathcal{F} \) is at most \( 1/\varepsilon \), then it lives through the homotopy to a germ \((t, x)\) in \( \mathcal{G} \) with resolution between \( \rho - 1/\varepsilon \) and \( \rho + 1/\varepsilon \). No two germs map to the same. In fact, the Global Stability Theorem provides a continuous embedding from the étalé space of \( \mathcal{F}' \) into the étalé space of \( \mathcal{G} \) and visa versa.

## 4 Persistence

In this section, I argue that the sheaf of stable groups is a continuous interpretation of persistence. I introduce persistence not as it was originally \(^7\) defined but as the level set zigzag \(^3\). The Global Stability Theorem implies a stability result for the interval decomposition \(^2\) of the level set zigzag.

**Level set zigzag.** Let \( f : M \to \mathbb{R} \) be a continuous mapping on a compact manifold \( M \) without boundary. A point \( x \in \mathbb{R} \) is regular if it satisfies the definition given in the statement of the Duality Theorem. If \( x \) is not regular, then it is critical. Assume the mapping \( f \) has a finite number of critical values \( c_1 < \ldots < c_l \). Choose a sequence \( r_0 < r_1 < \cdots < r_l \) of regular values such that

\[
0 < c_1 < r_1 < \cdots < r_{i-1} < c_i < r_i < \cdots < r_{l-1} < c_l < r_l.
\]

For \( 0 \leq i \leq l \), define \( H_\ast(i, i+1) \) as the direct sum \( \oplus \delta H_{m-1-p}(f^{-1}(r_i - \delta, r_{i+1} + \delta)) \) of homology groups, where \( \delta > 0 \) is sufficiently small so that the open interval \((r_i - \delta, r_{i+1} + \delta)\) does not contain the critical values \( c_i \) and \( c_{i+1} \). Define \( H_\ast(i) \) as the direct sum \( \oplus \delta H_{m-1-p}(f^{-1}(r_i)) \). The following diagram of \( 2l + 1 \) groups connected by homomorphisms induced by inclusion of spaces is called the **level set zigzag**:

\[
\begin{array}{c}
H_\ast(i-1, i) \quad H_\ast(i, i+1) \quad H_\ast(i+1, i+2) \\
\downarrow f_{i-1,i} \quad \downarrow f_{i,i+1} \quad \downarrow f_{i+1,i+2} \\
H_\ast(i) \quad H_\ast(i+1) \quad \cdots
\end{array}
\]

The even numbered positions (bottom row) are occupied by the homology groups of the level sets, and the odd numbered positions (top row) with the homology groups of the inverses of open intervals.

The level set zigzag admits a compact representation when each group is a finite dimensional vector space \(^2\). We now switch to \( \mathbb{Z}/2\mathbb{Z} \) coefficients for all our co/homology computations so that each group is indeed a vector space. Also assume each such group is finite dimensional. Define the **interval zigzag** as a sequence of \( 2l + 1 \) groups such that the groups in positions \( b \geq 0 \) through \( d \leq 2l \) are copies of \( \mathbb{Z}/2\mathbb{Z} \) and the remaining positions are copies of 0. Now connect the groups in the interval zigzag with homomorphisms in directions consistent with the directions in the level set zigzag. If \( k \) is even and both \( I_{b,d}(k) \) and \( I_{b,d}(k+1) \) are copies of \( \mathbb{Z}/2\mathbb{Z} \), then define the homomorphism \( I_{b,d}(k) \to I_{b,d}(k+1) \) as the identity homomorphism. Set \( I_{b,d}(k) \to I_{b,d}(k+1) \) to the zero homomorphism for the remaining cases. Similarly attach homomorphisms \( I_{b,d}(k) \to I_{b,d}(k-1) \). The level set zigzag decomposes in to the direct sum of a finite number of intervals. That is, there is a finite collection of intervals \( \{I_{b,d}\} \) such that the group in position \( k \) of the level set zigzag is isomorphic to the direct sum \( \oplus \{I_{b,d}(k)\} \) and the homomorphisms \( f_{i,i+1} \) and \( f_{i,i-1} \) are direct sums of the corresponding homomorphisms in the interval zigzags \( \{I_{b,d}\} \). An interval \( I_{b,d} \) spans an open interval \((c_i, c_j) \subset \mathbb{R} \) if for each regular value \( r_k \) in \((c_i, c_j) \), \( I_{b,d}(2k) \) is non-zero. Assuming the span of \( I_{b,d} \) is maximal, which I will from now on, the **persistence** of \( I_{b,d} \) is the value \( c_j - c_i \).
Intervals are maximal sections. Each interval in the level set decomposition of the level set zigzag corresponds to a maximal section of the sheaf $F$ of $f$. The definition of $F$ requires as input a mapping between two compact spaces. For convenience, think of the real line as an open subset of the circle.

**Interval Lemma.** Let $I_{b,d}$ be an interval zigzag of non-zero persistence in the decomposition $\{I_{b,d}\}$ of the level set zigzag. Then there is a maximal section $s \in F(U)$ such that $U$ is the span of $I_{b,d}$.

**Proof.** Observe that no connected subspace of the étalé space $E_f$ covers a point in $\mathbb{R}$ twice. The persistence of $I_{b,d}$ is non-zero implying there is at least one regular value $r_i$ such that $I_{b,d}(2i) \neq 0$. Let $\alpha \in H_*(i)$ be the class such that its projection to each component in the decomposition $\{I_{b,d}(2i)\}$ is zero except for its projection to $I_{b,d}(2i)$. By the Duality Theorem, $H_*(i)$ is isomorphic to the stalk $F(i)$ at the regular value $r_i$. Let $s \in F(U)$ be the maximal section of the sheaf $F$ such that its germ at $r_i$ is $\alpha$. The section $s$ corresponds to an element of the stable group $P_f(U)$. The stable group $P_f(U)$ is, in turn, isomorphic to a subgroup of the homology $H_*(F_U)$. If the interval $(r_i - \delta, r_i + \delta)$ belongs to $U$, then, by the Commutativity Lemma, the restriction of $s$ to $H_*(i, i+1)$ is $f_{i,i+1}(\alpha)$. A continuation of this argument shows that $U$ is a subset of the span of $I_{b,d}$.

Now assume that $I_{b,d}(2i)$ and $I_{b,d}(2(i+1))$ are non-zero and that the regular value $r_i$ belongs to $U$ but $r_{i+1}$ does not. I show that this can not be the case implying that $U$ is exactly the span of $I_{b,d}$. Let $\beta \in H_*(i, i+1)$ such that $f_{i,i+1}(\alpha) = \beta$, and let $\gamma \in H_*(i+1)$ such that $f_{i+1,i}(\gamma) = \beta$. By the Duality Theorem, the stalk $F(i+1)$ is isomorphic to $H_*(i+1)$. Let $t \in F(V)$ be the maximal section of $F$ such that its germ at $r_{i+1}$ is $\gamma$. If both $U$ and $V$ contain the interval $(r_i - \delta, r_i + \delta)$, then, by the Commutativity Lemma and by the definition of the stable group, the sections $s$ and $t$ equal violating the assumption that $s$ is maximal. Now assume $U \cap V = \emptyset$. There is a class $\alpha' \in H^{m-1-*}(i)$ that maps, via the intersection homomorphism, to $\alpha \in H_*(i)$. Similarly there is a class $\gamma' \in H^{m-1-*}(i+1)$ that maps to $\gamma$ via the intersection homomorphism. By Meyer-Vietoris, the class $\beta$ is supported by the inverse of every point in the interval $(r_i - \delta, r_{i+1} + \delta)$. This implies a class $\beta' \in H^{m-1-*}(i, i+1)$ that restricts to $\alpha'$ and $\gamma'$ and whose image, under the intersection homomorphism, is $\beta$. Thus $s$ is not maximal violating the assumption. 

**Stability of intervals.** If $g: \mathbb{M} \rightarrow \mathbb{R}$ is a second mapping such that $\sup_{x \in \mathbb{M}} |f(x) - g(x)| \leq \varepsilon$, then the wildness of the straight line homotopy connecting $g$ to $f$ is $\varepsilon$. Let $F$ and $G$ be the stable group sheaves of the two mappings. Let $I_{b,d}$ be an interval in the interval decomposition of the level set zigzag of $f$ and assume its persistence is at least $\varepsilon$. By the Interval Lemma, $I_{b,d}$ has a corresponding maximal section $s \in F(U^\varepsilon)$. Setting the resolution $r = 1/\varepsilon$, call $s' \in F$ the maximal section that maps to $s$ via the sheaf morphism $\Sigma_f$. By the Global Stability Theorem, $F$ maps injectively to $G$ via the morphism $\Sigma_{f,g}$. Let $t \in G(V)$ be a maximal section that restricts to $\Sigma_{f,g}(s') \in G(U)$. Note $U \subseteq V$. If $V = W^\varepsilon$, for some open set $W$, then the symmetric argument implies $\overline{W} \subseteq U^\varepsilon$. The Hausdorff distance between $U^\varepsilon$ and $V$ is at most $\varepsilon$. Thus for each interval in the level set zigzag decomposition of $f$ with persistence at least $\varepsilon$, there is a corresponding maximal section in $G$. The injectivity of the sheaf morphism $\Sigma_{f,g}$ implies that these maximal sections of $G$ provide a partial interval decomposition of the level set zigzag of $g$.

5 Discussion

I have presented a theory that offers a few generalizations to the theory of persistence. Given a continuous mapping $f: \mathbb{M} \rightarrow \mathbb{N}$, the fundamental unit of persistence is the stable group one for each open set of $\mathbb{N}$. There are no assumptions on $f$ other than that both $\mathbb{M}$ and $\mathbb{N}$ are compact manifolds. In the case both manifolds are orientable, the theory holds for integer coefficients. This is one generalization to persistence, which requires field coefficients. As we saw in Section 4, the vector space requirement for persistence allows for a finite description of the level set zigzag. The stability results in persistence argue the stability of the intervals in the interval decomposition of the zigzag. It is not likely that the sheaf of stable groups admits a finite description without additional assumptions on the mapping $f$. For this reason, I have argued local stability at the class level and global stability using sheaves.

One may think of the sheaf of stable groups $F$ as a tree. Each node of this tree represents an open set of $\mathbb{N}$, and an open set $V$ is the child of $U$ if $V \subseteq U$. Furthermore each node $U$ carries with it the
sections $\mathcal{F}(U)$. The sections $\mathcal{F}(U)$ include injectively into the sections $\mathcal{F}(V)$, for each node $V$ that is descendant of $U$. The sheaf $\mathcal{F}'$ with resolution $r$ is a trimming of this tree because $\mathcal{F}'$ includes into $\mathcal{F}$. However, the trimmed tree remains large. The restriction of this tree to a finite open cover of $\mathbb{N}$ results in a tree with a finite number of nodes rooted at $\mathbb{N}$. One way to discretize the sheaf is to approximate the mapping $f : M \to \mathbb{N}$ with a simplicial mapping $g : K \to L$, where $K$ and $L$ are triangulations of $M$ and $\mathbb{N}$. Assign to collections of top dimensional simplices in $L$ a stable group and call the resulting sheaf of sections $\mathcal{G}$. If the approximation $g$ is homotopic to $f$, then $\mathcal{F}' = \mathcal{G}'$, where $1/r$ is the diameter of $\mathbb{N}$. If the wildness of the homotopy connecting $f$ to $g$ is known, then, using the Global Stability Theorem, it is possible to understand which sections of $\mathcal{F}$ are preserved in $\mathcal{G}$. This is just one idea of how one may think about discretizing the sheaf of sections of stable groups.

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