Lipschitz Homotopy Groups of Contact 3-Manifolds

Daniel Perry
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Abstract
We study contact 3-manifolds using the techniques of sub-Riemannian geometry and geometric measure theory, in particular establishing properties of their Lipschitz homotopy groups. We prove a biLipschitz version of the Theorem of Darboux: a contact \((2n+1)\)-manifold endowed with a sub-Riemannian structure is locally biLipschitz equivalent to the Heisenberg group \(\mathbb{H}^n\) with its Carnot-Carathéodory metric. Then each contact \((2n+1)\)-manifold endowed with a sub-Riemannian structure is purely \(k\)-unrectifiable for \(k > n\). We extend results of Dejarnette et al. [4] and Wenger and Young [16] by showing for any purely 2-unrectifiable sub-Riemannian manifold \((M,\xi,g)\) that the \(n\)th Lipschitz homotopy group is trivial for \(n \geq 2\) and that the set of oriented, horizontal knots in \((M,\xi)\) injects into the first Lipschitz homotopy group. Thus, the first Lipschitz homotopy group of any contact 3-manifold is uncountably generated. Therefore, in the sense of Lipschitz homotopy groups, a contact 3-manifold is a \(K(\pi,1)\)-space for an uncountably generated group \(\pi\). Finally, we prove that each open distributional embedding between purely 2-unrectifiable sub-Riemannian manifolds induces an injective map on the associated first Lipschitz homotopy groups. Therefore, each open subset of a contact 3-manifold determines an uncountable subgroup of the first Lipschitz homotopy group of the contact 3-manifold.

1 Introduction.
In this paper, we use metric geometry to show a sense in which each connected contact 3-manifold is a \(K(\pi,1)\)-space for an uncountably generated group \(\pi\). After we endow a contact 3-manifold with a metric structure sensitive to the distribution, we probe the space with Lipschitz maps. The metric is the Carnot-Carathéodory metric of sub-Riemannian geometry. Our results are phrased in terms of Lipschitz homotopy groups.

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In contact topology, the classification of contact 3-manifolds is of active interest. The primary tools used for better understanding contact 3-manifolds have been Reeb vector fields and singular foliations. For an introduction to contact geometry, see [7]. For a thorough overview of techniques and results in contact topology, see [8]. For results classifying contact 3-manifolds, see [5] or [13].

In this paper, we instead apply the techniques of sub-Riemannian geometry to study contact 3-manifolds. Though contact 3-manifolds do not have an inherent sub-Riemannian structure, they can be endowed with such structure. Once the sub-Riemannian structure is fixed, the underlying manifold inherits a Carnot-Carathéodory metric structure which is sensitive to the contact distribution. For an overview of sub-Riemannian geometry, see [14].

Our primary tool for studying the metric structure of a contact manifold (of any dimension) is Lipschitz homotopy groups. Dejarnette et al. [4], first introduced Lipschitz homotopy groups in order to study Sobolev mappings into the sub-Riemannian manifold $\mathbb{H}^1$. Since Lipschitz homotopy groups were introduced, they have been calculated for various Heisenberg groups in [4], [9], [11], [10], and [16].

By the Theorem of Darboux, the distributional structure of contact $(2n+1)$-manifolds is locally modeled by the contact structure of the $n$th Heisenberg group $\mathbb{H}^n$ [3]. So, we are able to apply the strategies and approaches of sub-Riemannian geometry used to study $\mathbb{H}^n$ to study contact manifolds. Indeed, we prove a biLipschitz version of the Theorem of Darboux which says that the metric structure of a contact $(2n+1)$-manifold (after being endowed with a Carnot-Carathéodory metric) is locally modeled by the metric space $\mathbb{H}^n$ (Corollary 2.23).

Among the metric properties of $\mathbb{H}^1$, we make use of $\mathbb{H}^1$ being purely 2-unrectifiable in the sense of [1]. Indeed, for any $k > n$, the $n$th Heisenberg group $\mathbb{H}^n$ is purely $k$-unrectifiable [2]. Using the biLipschitz version of the Theorem of Darboux, any contact $(2n+1)$-manifold (with Carnot-Carathéodory metric) is also purely $k$-unrectifiable for $k > n$ (Theorem 2.27).

Once shown that contact 3-manifolds are purely 2-unrectifiable, the properties of the associated Lipschitz homotopy groups listed in Theorem 1.1 follow from similar tools and strategies employed in [4] and [16].

**Theorem 1.1.** Let $(M, \xi)$ be a contact 3-manifold. Endow $(M, \xi)$ with a sub-Riemannian structure and consider the resulting Carnot-Carathéodory metric $d_{CC}^M$. Then,

1. $\pi_1^{Lip}(M, d_{CC}^M)$ is uncountably generated, and
2. $\pi_n^{Lip}(M, d_{CC}^M) = 0$ for $n \geq 2$.

Furthermore, let $(M', \xi')$ be a contact 3-manifold which is endowed with a sub-Riemannian structure and let $\varphi : (M, \xi) \to (M', \xi')$ be an open distributional embedding.
3. The homomorphism induced by $\varphi$ on first Lipschitz homotopy groups

$$\varphi^\#: \pi_1^{Lip}(M, d^M_C) \rightarrow \pi_1^{Lip}(M', d^{M'}_C)$$

is injective.

The paper is organized as follows. In section 2, we introduce necessary background on contact manifolds and sub-Riemannian manifolds. We then focus on distributional maps between sub-Riemannian manifolds. These are smooth maps whose derivative carries the distribution of the domain into the distribution of the codomain. We show that, with respect to the Carnot-Carathéodory metrics, any distributional map is locally Lipschitz. That every distributional embedding is locally biLipschitz is an immediate consequence, as is the biLipschitz Darboux theorem. Finally, we extend the result in [2] to all contact manifolds by showing that a contact $(2n+1)$-manifold is purely $k$-unrectifiable for $k > n$.

In section 3, we recall the definition of Lipschitz homotopy groups. We then focus on purely 2-unrectifiable sub-Riemannian manifolds and determine properties of their Lipschitz homotopy groups. First, we show that a Lipschitz homotopy between smooth embeddings of $\mathbb{S}^1$ with distinct images must sweep out positive 2-dimensional Hausdorff measure. Since we are considering purely 2-unrectifiable spaces, we show that no such Lipschitz homotopies exist. Thus, there is a distinct element of the first Lipschitz homotopy group for every distinct horizontal knot. We then make use of a result of Wenger and Young (Theorem 5 in [16]) that says that all Lipschitz maps from a Lipschitz simply connected, quasi-convex space into a purely 2-unrectifiable space factor through a metric tree. An immediate corollary is that all higher Lipschitz homotopy groups are trivial for purely 2-unrectifiable spaces. We also use the result of Wenger and Young to show that a distributional embedding of a purely 2-unrectifiable sub-Riemannian manifold into another induces an injective map on the associated first Lipschitz homotopy groups. Thus, for any contact 3-manifold, there is an uncountable subgroup of the first Lipschitz homotopy group for every connected open neighborhood of the chosen base point.

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2 Contact 3-manifolds are purely 2-unrectifiable.

Any contact 3-manifold is locally modeled by a purely 2-unrectifiable space. Indeed, by the Theorem of Darboux, contact 3-manifolds locally look like copies of $\mathbb{H}^1$ [3] and $\mathbb{H}^1$ is a purely 2-unrectifiable space [1]. As will be shown, for any point in a contact 3-manifold, there is an open neighborhood of the point such that the distributional embedding guaranteed by Darboux restricted to the
open neighborhood is a biLipschitz map with respect to the associated Carnot-Carathéodory metrics. Thus, these biLipschitz maps will carry this metric condition on $\mathbb{H}^1$ to the contact 3-manifold.

In fact, we will show a more general result: since a contact $(2n+1)$-manifold is locally modeled by the $n$th Heisenberg group $\mathbb{H}^n$ (again by the Theorem of Darboux) and $\mathbb{H}^n$ is purely $k$-unrectifiable for all $k > n$ [2], contact $(2n+1)$-manifolds are purely $k$-unrectifiable for $k > n$ when endowed with a Carnot-Carathéodory metric.

This result will be achieved by inspecting the interplay of distributional maps and the Carnot-Carathéodory lengths of paths. After covering some background material, we will show that the length of the image of a horizontal path under distributional map, which again is a horizontal path, is bounded. Thus, the distributional embedding guaranteed by the Theorem of Darboux can only distort lengths of paths, and thus distances between points, by a manageable amount.

Next, we will account for subsets of contact manifolds not necessarily being geodesically convex. We will show that for any open ball in a sub-Riemannian manifold, there is a bounded open subset containing the ball in which distances between points in the ball can be well-approximated by horizontal paths that remain in the new bounded open subset. These tools will be enough to restrict a distributional embedding to a neighborhood such that the map is also biLipschitz with respect to the Carnot-Carathéodory metrics.

### 2.1 Contact manifolds, horizontal paths, and distributional embeddings.

**Convention 2.1.** Throughout this paper, the term “manifold” will refer to a smooth connected manifold, and the term “distribution” will refer to a smooth vector subbundle of a tangent bundle. The tangent bundle of a manifold $M$ will be denoted $TM$. The derivative of a smooth map $f$ will be denoted $Df$.

**Definition 2.2.** A manifold with a distribution is a pair $(M, \xi)$ composed of a manifold $M$ and a distribution $\xi \subset TM$. If the distribution is bracket-generating, the pair $(M, \xi)$ is called a Carnot manifold. Additionally, if the manifold $M$ is of odd-dimension $2n+1$ and the bracket-generating distribution $\xi$ is co-dimension 1, the pair $(M, \xi)$ is called a contact $(2n+1)$-manifold.

Contact manifolds are the primary interest of this paper. For a more thorough discussion of Carnot manifolds and bracket-generating distributions, see [12].

The following example, called the $n$th Heisenberg group, is the quintessential contact manifold in that all contact manifolds are locally modeled by a Heisenberg group.

**Example 2.3** ($n$th Heisenberg group). Let $M = \mathbb{R}^{2n+1}$ with coordinates denoted by $x_1, \ldots, x_n, y_1, \ldots, y_n, t$. Define a co-dimension 1 distribution on $\mathbb{R}^{2n+1}$ by

$$\xi^{\text{std}} := \text{span}(X_1, \ldots, X_n, Y_1, \ldots, Y_n),$$
where, for \( i = 1, \ldots, n \), the vector fields \( X_i \) and \( Y_i \) are defined by

\[
X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t} \quad \text{and} \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}.
\]

A calculation verifies that the \( n \)th Heisenberg group \( \mathbb{H}^n := (\mathbb{R}^{2n+1}, \xi^{std}) \) is indeed a contact manifold.

We now describe means of probing the distrubtional structure of a manifold with a distribution.

**Definition 2.4.** Let \((M, \xi)\) be a manifold with a distribution and let \(N\) be a manifold with boundary. A smooth map \( f : N \rightarrow M \) is horizontal, denoted

\[
f : N \rightarrow (M, \xi)
\]

if it is tangent to the distribution:

\[
\exists \xi \quad \xi \quad \xi
\]

\[
TN \xrightarrow{Df} TM
\]

\[
\downarrow \quad \downarrow
\]

\[
N \xrightarrow{f} M.
\]

If \( N \) is a closed interval, the map \( f \) is a horizontal path. If the horizontal map is an embedding, the map \( f \) is a horizontal embedding.

**Remark 2.5.** Legendrian knots are examples of horizontal embeddings of \( S^1 \) into a contact 3-manifold.

**Definition 2.6.** Let \((M', \xi')\) and \((M, \xi)\) be manifolds with distributions. A smooth map \( f : M' \rightarrow M \) is a distributional map, denoted

\[
f : (M', \xi') \rightarrow (M, \xi),
\]

if its derivative maps one distribution into the other:

\[
\xi' \xrightarrow{\exists} \xi
\]

\[
TM' \xrightarrow{Df} TM
\]

\[
\downarrow \quad \downarrow
\]

\[
M' \xrightarrow{f} M.
\]
If the distributional map is an embedding, the map \( f \) is a **distributional embedding**.

Now that the contact structure on \( \mathbb{H}^n \) has been established and we have a means of embedding this structure into other contact manifolds via distributional embeddings, we can make precise that contact manifolds are locally modeled by the Heisenberg group.

**Theorem 2.7** *(Theorem of Darboux)*. Let \((M,\xi)\) be a contact \((2n+1)\)-manifold. For any point \( p \in M \), there exists an open distributional embedding

\[
\varphi : \mathbb{H}^n \to (M,\xi)
\]

of the \( n \)th Heisenberg group into the contact manifold \((M,\xi)\) such that \( \varphi(0) = p \).

The Theorem of Darboux was originally proved by Darboux in [3]. For a modern statement and proof, see Theorem 2.24 in [7].

In the notation of Theorem 2.7, the neighborhood \( \varphi(\mathbb{H}^n) \), along with the associated distributional embedding \( \varphi \), will be referred to as a **Darboux neighborhood**.

### 2.2 Sub-Riemannian manifolds and the Carnot-Carathéodory metric.

Rather than probing the distributional structure of contact manifolds, we will endow each contact manifold with a metric structure sensitive to its distribution and then probe the metric structure.

**Definition 2.8.** A **sub-Riemannian manifold** is a triple \((M,\xi, g)\) consisting of a Carnot manifold \((M,\xi)\) and a smooth map

\[
g : \xi \times M \xi \to \mathbb{R},
\]

such that, for each \( p \in M \), the map \( g : \xi_p \oplus \xi_p \to \mathbb{R} \) is an inner product on the vector space \( \xi_p \). Such a map \( g \) is referred to as a **sub-Riemannian metric** on \((M,\xi)\).

**Remark 2.9.** Any Carnot manifold can be endowed with a sub-Riemannian structure. Indeed, as any manifold can be endowed with a Riemannian metric, restricting the Riemannian metric to the bracket-generating distribution yields a sub-Riemannian metric. Going forward, we will assume that each Carnot manifold is endowed with a sub-Riemannian structure.

**Example 2.10.** Continuing Example 2.3, the \( n \)th Heisenberg group \( \mathbb{H}^n \) is naturally endowed with a sub-Riemannian structure. Indeed, define a sub-Riemannian metric \( g \) such that, for each \( p \in \mathbb{H}^n \), the vectors

\[
X_1(p), \ldots, X_n(p), Y_1(p), \ldots, Y_n(p)
\]

form an orthonormal basis for \( \xi_p^{std} \). Going forward, it will be assumed that \( \mathbb{H}^n \) has this sub-Riemannian structure.
To ensure that the metric structure imposed on a Carnot manifold is sensitive to its distribution, the metric will be defined as a path metric where only the lengths of horizontal paths are considered.

**Definition 2.11.** Let \((M, \xi, g)\) be a sub-Riemannian manifold. The **Carnot-Carathéodory length** of a horizontal path \(\gamma : [a, b] \to (M, \xi)\) is

\[
l^M(\gamma) := \int_a^b \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt.
\]

The **Carnot-Carathéodory metric** on \(M\) is

\[
d^{MC}_M(p, p') := \inf \left\{ l^M(\gamma) \mid \gamma : [a, b] \to (M, \xi) \text{ with } \gamma(a) = p \text{ and } \gamma(b) = p' \right\},
\]

for any \(p, p' \in M\).

**Remark 2.12.** The Carnot-Carathéodory metric \(d^{MC}_M\) is in fact a metric on the manifold \(M\). Indeed, via Chow-Rashevskii Theorem (Theorem 42 in [14]), as the associated distribution is bracket-generating, any two points in the space can be joined by a horizontal path that has finite Carnot-Carathéodory length. As such, for any points \(p, p' \in M\), the metric \(d^{MC}_M(p, p')\) is defined.

**Convention 2.13.** Going forward, we will assume that any sub-Riemannian manifold is endowed with the Carnot-Carathéodory metric. The pair \((M, d^{MC}_M)\) will be used to identify the sub-Riemannian manifold \((M, \xi, g)\) as a metric space endowed with the Carnot-Carathéodory metric. The open ball centered at \(p \in M\) of radius \(R > 0\) with respect to the Carnot-Carathéodory metric will be denoted by \(B^{MC}_M(p, R)\).

Having endowed Carnot manifolds with a metric, we examine how horizontal and distributional maps interact with the Carnot-Carathéodory structure. First, we inspect how a distributional map can affect Carnot-Carathéodory length. The image of a horizontal path under a distributional map is horizontal and thus the Carnot-Carathéodory length of the image is defined. The next result articulates a sense in which the Carnot-Carathéodory length of a horizontal path is scaled by the derivative of a distributional map.

**Lemma 2.14.** Let \(f : (M', \xi') \to (M, \xi)\) be a distributional map between sub-Riemannian manifolds \((M', \xi', g')\) and \((M, \xi, g)\). Let \(A \subset M'\) be a compact subset of \(M'\). Then, there exists a value \(B \geq 0\) such that, for any horizontal path \(\gamma : [a, b] \to (M', \xi')\) mapping into \(A\), the following inequality holds:

\[
l^M(f \circ \gamma) \leq B \, l^{M'}(\gamma).
\]
Proof. First, consider the operator norm evaluated on the derivative of the map \( f \) on the subset \( A \subset M' \):

\[
\|D_{(-)}f\| : A \longrightarrow [0, \infty).
\]

This map is a continuous, real-valued map with compact domain. Thus, the map has an upper bound \( B \geq 0 \).

Now, let \( \gamma \) be an horizontal map into the manifold with distribution \((M', \xi')\) such that its image lies in the compact subset \( A \). As \( \gamma \) is a horizontal path and \( f \) is a distributional map, the composition \( f \circ \gamma \) is a horizontal path in \((M, \xi)\). So, the Carnot-Carathéodory length of \( f \circ \gamma \) is defined.

By the definition of Carnot-Carathéodory length of a horizontal path and the chain rule, the length of the horizontal path \( f \circ \gamma \) can be rewritten as follows:

\[
l^{M}(f \circ \gamma) = \int_{a}^{b} \sqrt{g \left( \frac{d}{dt}(f \circ \gamma(t)), \frac{d}{dt}(f \circ \gamma(t)) \right)} \, dt
\]

By definition of the operator norm, for any point \( p \in M' \) and any vector \( v \in \xi'_p \),

\[
\sqrt{g(D_p f(v), D_p f(v))} \leq \|D_p f\| \sqrt{g'(v,v)}.
\]

Thus, we have the inequality

\[
\int_{a}^{b} \sqrt{g \left( D_{\gamma(t)} f(\dot{\gamma}(t)), D_{\gamma(t)} f(\dot{\gamma}(t)) \right)} \, dt \leq \int_{a}^{b} \|D_{\gamma(t)} f\| \sqrt{g'(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt.
\]

Finally, using the upperbound \( B \) of the map \( \|D_{(-)}f\| \), the following inequality holds:

\[
\int_{a}^{b} \|D_{\gamma(t)} f\| \sqrt{g'(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt \leq \int_{a}^{b} B \sqrt{g'(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt = B l^{M'}(\gamma).
\]

Therefore, the desired inequality holds:

\[
l^{M}(f \circ \gamma) \leq B l^{M'}(\gamma).
\]

We now describe means of probing the metric structure. Let \( X \) and \( Y \) be metric spaces with metrics \( d^X \) and \( d^Y \) respectively.

**Definition 2.15.** A map \( f : X \longrightarrow Y \) is Lipschitz if there exists \( L \geq 0 \) such that for all \( x, x' \in X \)

\[
d^Y(f(x), f(x')) \leq L \, d^X(x, x').
\]

Denote the set of all Lipschitz maps from \( X \) to \( Y \) by \( \text{Map}^{\text{Lip}}(X, Y) \). Furthermore, if \( x_0 \in X \) and \( y_0 \in Y \) are base points, the set of all based Lipschitz maps from \( X \) to \( Y \) is denoted by \( \text{Map}^{\text{Lip}}_{x_0}(X, Y) \).
Often, we will consider a slightly weaker class of maps; locally Lipschitz maps.

**Definition 2.16.** A map \( f : X \to Y \) is locally Lipschitz if, for all \( p \in X \), there exists an open neighborhood \( p \in U \subset X \) such that \( f|_U \) is Lipschitz.

Lipschitz and locally Lipschitz maps are a natural choice to substitute for smooth maps. Smooth maps between Riemannian manifolds are locally Lipschitz with respect to the associated path metrics. As will be shown in Lemma 2.20, distributional maps between sub-Riemannian manifolds are locally Lipschitz with respect to the associated Carnot-Carathéodory metrics.

We now define a notion of an embedding between metric spaces. BiLipschitz bijections, for the purposes of this paper, are the appropriate notion of equivalence between a metric space and its image.

**Definition 2.17.** A Lipschitz map \( \phi : X \to Y \) is biLipschitz if \( \phi \) is injective and its inverse map \( \phi^{-1} : \phi(X) \to X \) is also Lipschitz with respect to the metric \( d_Y \) restricted to \( \phi(X) \).

As with Lipschitz maps, we have a parallel notion of locally biLipschitz mappings.

**Definition 2.18.** A locally Lipschitz map \( \phi : X \to Y \) is locally biLipschitz if \( \phi \) is injective and for all \( p \in X \), there exists an open neighborhood \( p \in U \subset X \) such that \( \phi|_U \) is biLipschitz.

### 2.3 BiLipschitz Darboux theorem.

As a consequence of the Theorem of Darboux, all contact \((2n + 1)\)-manifolds are locally modeled on \( \mathbb{H}^n \), which, when endowed with the metric \( d_{CC}^{\mathbb{H}^n} \), is purely \( k \)-unrectifiable for \( k > n \) [2]. In order to relay this metric quality on \( \mathbb{H}^n \) to a contact \((2n + 1)\)-manifold \((M, \xi)\), we require an adjustment of the Theorem of Darboux. It will be shown that the distributional embeddings of \( \mathbb{H}^n \) into \((M, \xi)\) guaranteed by Darboux can be restricted such that they are biLipschitz with respect to the associated Carnot-Carathéodory metrics.

It is worth noting that it is not immediate that the distributional embeddings guaranteed by Darboux are locally biLipschitz with respect to Carnot-Carathéodory metrics. The distributional embeddings are smooth and thus, assuming there are Riemannian metrics on the associated manifolds, locally biLipschitz with respect to the path metrics. But, these path metrics are not necessarily biLipschitz equivalent to the Carnot-Carathéodory metrics. Indeed, it is known for any sub-Riemannian manifold that these two metrics are not biLipschitz equivalent (Theorem 2.10 in [14]).

Thus, to guarantee that these distributional embeddings are taken to be locally biLipschitz, or even locally Lipschitz, we must better understand the Carnot-Carathéodory metric, in particular, where horizontal curves approximating the distance between two points live. As the Carnot-Carathéodory metric is defined in terms of lengths of horizontal curves, it is desirable to know how a...
distributional map can distort these lengths. Lemma 2.14 is a tool for bounding
lengths of paths that live in a given compact set. Choosing an open subset of
the domain that is bounded then becomes the focus.

As Lemma 2.14 yields a bound for horizontal paths that remain in a compact
subset, it is important that the bounded open subset contains horizontal paths
that well-approximate the Carnot-Carathéodory distance between some set of
points. In practice, we cannot expect that the set containing the points and the
set containing the horizontal paths to be equal. Given an arbitrary open subset
of a contact manifold, it is unlikely that it is geometrically convex, i.e., contains
all length-minimizing horizontal paths between all of its points. Indeed, it is
known that the only geometrically convex open subset of $\mathbb{H}^1$ is itself [15].

So, we should not expect to be able to well-approximate Carnot-Carathéodory
distance between points in a given bounded open subset via horizontal paths
that remain in the open subset. Rather, given a bounded open subset, there
is a larger but still bounded open subset of the ambient space in which the
Carnot-Carathéodory distance between points in the former open subset can be
well-approximated via horizontal paths that map into the latter.

**Lemma 2.19.** Let $(M,\xi,g)$ be a sub-Riemannian manifold. Consider the open
ball $B^M_{CC}(p, R) \subset M$. The set

$$GH(p, R) := \bigcup_{q \in B^M_{CC}(p, R)} B^M_{CC}(q, 2R)$$

satisfies the following properties:

1. $GH(p, R)$ is an open subset of $M$, bounded with respect to $d^M_{CC}$, and con-
tains $B^M_{CC}(p, R)$.

2. Let $x, y \in B^M_{CC}(p, R)$ and let $0 < \varepsilon < 2R - d^M_{CC}(x, y)$. Then there exists a
horizontal path

$$[0, 1] \xrightarrow{\gamma_\varepsilon} (M, \xi)$$

$$(GH(p, R), \xi)$$

from $x$ to $y$ such that

$$d^M_{CC}(x, y) < t^M(\gamma_\varepsilon) \leq d^M_{CC}(x, y) + \varepsilon < 2R.$$

Property (2) is what is meant by Carnot-Carathéodory distance between
two points being well-approximated via horizontal paths.

**Proof.** (1) The subset $GH(p, R)$ is the union of open balls and is thus an open
subset. From the definition, it is obvious that $GH(p, R)$ contains the open ball
$B^M_{CC}(p, R)$. 

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To see that $GH(p,R)$ is bounded, consider the distance between $p$ and an arbitrary element $x \in GH(p,R)$. By definition of $GH(p,R)$, there exists $q \in B^{M}_{CC}(p,R)$ such that $x \in B^{M}_{CC}(q,2R)$. By triangle inequality,

$$d^{M}_{CC}(x,p) \leq d^{M}_{CC}(x,q) + d^{M}_{CC}(q,p) < 2R + R = 3R$$

and thus $x \in B^{M}_{CC}(p,3R)$. So, $GH(p,R)$ is contained in the open ball $B^{M}_{CC}(p,3R)$ and is therefore bounded.

(2) Let $0 < \varepsilon < 2R - d^{M}_{CC}(x,y)$. By the infimum definition of $d^{M}_{CC}$, there exists a horizontal path $\gamma_{\varepsilon} : [0,1] \to (M,\xi)$ such that $\gamma_{\varepsilon}(0) = x$, $\gamma_{\varepsilon}(1) = y$, and

$$d^{M}_{CC}(x,y) < l^{M}(\gamma_{\varepsilon}) \leq d^{M}_{CC}(x,y) + \varepsilon < 2R.$$

It remains to be verified that $\gamma_{\varepsilon}$ maps into $GH(p,R)$. It is sufficient to see that $\gamma_{\varepsilon}$ maps into $B^{M}_{CC}(x,2R) \subset GH(p,R)$.

Take $t \in [0,1]$. The restriction $\gamma_{\varepsilon}|_{[0,t]}$ is a horizontal path in $(M,\xi)$ connecting $x$ and $\gamma_{\varepsilon}(t)$. By the infimum definition of $d^{M}_{CC}$, the Carnot-Carathéodory distance between $x$ and $\gamma_{\varepsilon}(t)$ is no more than the length of this restriction:

$$d^{M}_{CC}(x,\gamma_{\varepsilon}(t)) \leq l^{M}(\gamma_{\varepsilon}|_{[0,t]}).$$

Obviously, the length of $\gamma_{\varepsilon}|_{[0,t]}$ is no more than the length of $\gamma_{\varepsilon}$. Thus the following inequality holds,

$$d^{M}_{CC}(x,\gamma_{\varepsilon}(t)) \leq l^{M}(\gamma_{\varepsilon}|_{[0,t]}) \leq l^{M}(\gamma_{\varepsilon}) \leq d^{M}_{CC}(x,y) + \varepsilon < 2R$$

and therefore $\gamma_{\varepsilon}(t) \in B^{M}_{CC}(x,2R)$. 

So, given an open ball with respect to the Carnot-Carathéodory metric, there is a bounded open subset that contains all horizontal paths that well-approximate the Carnot-Carathéodory distance between points in the ball. We will use the bound guaranteed by Lemma 2.14 on this larger bounded set to guarantee that distributional maps are locally Lipschitz.

**Lemma 2.20.** Let $(M^{'},\xi^{'},g^{'})$ and $(M,\xi,g)$ be sub-Riemannian manifolds and let $\varphi : (M^{'},\xi^{'}) \to (M,\xi)$ be a distributional map. Then, the map

$$\varphi : (M^{'},d^{M^{'}}_{CC}) \to (M,d^{M}_{CC})$$

is locally Lipschitz with respect to the Carnot-Carathéodory metrics.

**Proof.** Fix a point $p \in M^{'}.$. Take a radius $R > 0$ such that the closed ball $B^{M^{'}}_{CC}(p,R)$ is compact. Let $V^{'} = B^{M^{'}}_{CC}(p,R/4)$.

Let $q,q^{'} \in V^{'}$. By Lemma 2.19, there exists an open and bounded subset $GH(p,R/4) \subset M^{'}$ containing $V^{'}$ in which the Carnot-Carathéodory distance between $q$ and $q^{'}$ can be well-approximated by lengths of horizontal paths in $GH(p,R/4)$. Also, the subset $GH(p,R/4) \subset B^{M^{'}}_{CC}(p,3R/4) \subset B^{M^{'}}_{CC}(p,R)$ is contained in the compact subset $B^{M^{'}}_{CC}(p,R)$. 

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Let $\varepsilon > 0$ be given. Then, there exists a horizontal path $\gamma_\varepsilon$ contained in the open subset $GH(p,R/4)$ connecting $q$ and $q'$ such that
\[ l^{M'}(\gamma_\varepsilon) \leq d_{CC}^{M'}(q,q') + \varepsilon. \]

Now, the open subset $GH(p,R/4)$ is contained in the compact subset $B^{M'}_{CC}(p,R)$. By Lemma 2.14, for the distributional map $\varphi$, there exists a value $B \geq 0$ independent of $\varepsilon$ and $\gamma_\varepsilon$ such that
\[ l^{M}(\varphi \circ \gamma_\varepsilon) \leq B l^{M'}(\gamma_\varepsilon). \]

Since $\varphi \circ \gamma_\varepsilon$ is a horizontal path in $M$ connecting the points $\varphi(q)$ and $\varphi(q')$, by the infimum definition of the metric,
\[ d_{CC}^{M}(\varphi(q),\varphi(q')) \leq l^{M}(\varphi \circ \gamma_\varepsilon). \]

Stringing these inequalities together, we get the following:
\[ d_{CC}^{M}(\varphi(q),\varphi(q')) \leq l^{M}(\varphi \circ \gamma_\varepsilon) \leq B l^{M'}(\gamma_\varepsilon) \leq B (d_{CC}^{M'}(q,q') + \varepsilon). \]

As $\varepsilon$ can be taken to be arbitrarily small, $d_{CC}^{M}(\varphi(q),\varphi(q')) \leq B d_{CC}^{M'}(q,q')$. Therefore, the map $\varphi$ is Lipschitz on the neighborhood $V'$ of the point $p$. Since $p$ was arbitrary, $\varphi$ is locally Lipschitz.

\[ \square \]

**Remark 2.21.** Since Riemannian manifolds are sub-Riemannian manifolds where the distribution is taken to be the entire tangent bundle, Lemma 2.20 also implies that smooth maps between Riemannian manifolds and horizontal maps from a Riemannian manifold into a sub-Riemannian manifold are locally Lipschitz with respect to the associated path metrics.

This strategy can be used to guarantee that any distributional embedding is locally biLipschitz with respect to the Carnot-Carathéodory metrics. On the image of such a distributional embedding, the inverse map is also a distributional map. Indeed, as will be shown in the following argument, Lemma 2.20 yields that this inverse map is locally Lipschitz and thus the original map is locally biLipschitz.

**Lemma 2.22.** Let $(M',\xi',g')$ and $(M,\xi,g)$ be sub-Riemannian manifolds and let $\varphi : (M',\xi') \to (M,\xi)$ be an open distributional embedding. Then the map
\[ \varphi : (M',d_{CC}^{M'}) \to (M,d_{CC}^{M}) \]
is locally biLipschitz with respect to the associated Carnot-Carathéodory metrics.

**Proof.** Let $p \in M'$ be a point in $M'$. By Lemma 2.20, there exists an open neighborhood $p \in V' \subset M'$ such that the restriction $\varphi_{\mid V'}$ is Lipschitz.

Now, since the map $\varphi$ is invertible on its image,
\[ \varphi^{-1}_{\mid \varphi(V')} : \varphi(V'),\xi_{\mid \varphi(V')} \to (V',\xi_{\mid V'}) \]
is also a distributional map. Again by Lemma 2.20, there exists an open neighborhood $\varphi(p) \in U \subset \varphi(V')$ on which $\varphi^{-1}$ is Lipschitz. Then, $V = \varphi^{-1}(U) \subset V'$ is an open subset on which $\varphi|_V$ is Lipschitz and invertible. Since the inverse is also Lipschitz,

$$\varphi|_V : (V, d_{CC}^M) \to (\varphi(V), d_{CC}^M)$$

is biLipschitz.

With this more general result established, a biLipschitz Darboux Theorem is an immediate corollary.

**Corollary 2.23 (BiLipschitz Theorem of Darboux).** Let $(M, \xi)$ be a contact $(2n+1)$-manifold. For every $p \in M$, there exists an open neighborhood $0 \in V \subset \mathbb{H}^n$ and an open biLipschitz distributional embedding

$$\varphi : (V, d_{CC}^M) \to (M, d_{CC}^M)$$

such that $\varphi(0) = p$.

Such a neighborhood $\varphi(V)$, along with the associated biLipschitz distributional embedding $\varphi$, will be referred to as a biLipschitz Darboux neighborhood.

**Proof of Corollary 2.23.** By the Theorem of Darboux (Theorem 2.7), for any point $p \in M$, there exists an open distributional embedding of $\mathbb{H}^n$,

$$\varphi : \mathbb{H}^n \to (M, \xi)$$

where $\varphi(0) = p$. Now, by Lemma 2.22, there is an open neighborhood $p \in V \subset \mathbb{H}^n$ such that the restriction $\varphi|_V$ is biLipschitz.

2.4 Unrectifiability of contact manifolds.

**Convention 2.24.** Here and going forward, $\mathbb{R}^k$, and any subset thereof, is endowed with the Lebesgue measure $L^n$ and the Euclidean metric $d^k$ unless otherwise mentioned.

**Definition 2.25.** Let $k \geq 1$ be a positive integer. A metric space $X$ is purely $k$-unrectifiable if, for all Borel sets $A \subset \mathbb{R}^k$ and all Lipschitz maps $f : A \to X$, the $k$-dimensional Hausdorff measure of the image vanishes:

$$\mathcal{H}^k(\text{Im}(f)) = 0.$$ 

Informally, a space is purely $k$-unrectifiable if Lipschitz maps from $k$-dimensional Euclidean space into the space cannot sweep out any of the $k$-dimensional Hausdorff measure.

**Theorem 2.26 (Theorem 1.1 in [2]).** The $n$th Heisenberg group $\mathbb{H}^n$ is purely $k$-unrectifiable for $k > n$. 

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As contact \((2n+1)\)-manifolds are locally modeled by \(\mathbb{H}^n\), such spaces are the union of biLipschitz Darboux neighborhoods. Since \(\mathbb{H}^n\) is purely \(k\)-unrectifiable for \(k > n\), we will show that the union of the biLipschitz Darboux neighborhoods is as well.

**Theorem 2.27.** Any contact \((2n+1)\)-manifold \((M,\xi)\), endowed with the Carnot-Carathéodory metric, is purely \(k\)-unrectifiable for \(k > n\).

**Proof.** Fix a positive integer \(k > n\). Construct a cover of \((M,\xi)\) by biLipschitz Darboux neighborhoods. By Corollary 2.23, each point in \((M,\xi)\) has a biLipschitz Darboux neighborhood. \(M\) can be covered by such neighborhoods and, since \(M\) is a manifold, it can be reduced to a countable cover.

Let \(\{\varphi_{\alpha} : (V_{\alpha}, d_{CC}^{\mathbb{H}^n}) \to (M, d_{CC}^M)\}_{\alpha \in J}\) denote a countable collection of open biLipschitz distributional embeddings where \(V_{\alpha}\) is open for each \(\alpha \in J\), such that \(\{\varphi_{\alpha}(V_{\alpha})\}_{\alpha \in J}\) is a countable cover of \(M\).

Let \(f : (A, d^A) \to (M, d_{CC}^M)\) be a Lipschitz map whose domain \(A \subset \mathbb{R}^k\) is a Borel set. To verify that \((M, d_{CC}^M)\) is purely \(k\)-unrectifiable, it is enough to show that \(H^k(\text{Im}\ f) = 0\).

Fix an \(\alpha \in J\) and consider \(f\) restricted to the open subset \(f^{-1}(\varphi_{\alpha}(V_{\alpha})) \subset A\). By Corollary 2.23, \(\varphi^{-1}_{\alpha} : (\varphi_{\alpha}(V_{\alpha}), d_{CC}^{d_{\mathbb{H}^n}}) \to (V_{\alpha}, d_{CC}^{\mathbb{H}^n})\) is a Lipschitz map. As \(f|_{f^{-1}(\varphi_{\alpha}(V_{\alpha}))}\) maps into \(\varphi_{\alpha}(V_{\alpha})\),

\[
\varphi^{-1}_{\alpha} \circ f|_{f^{-1}(\varphi_{\alpha}(V_{\alpha}))} : (f^{-1}(\varphi_{\alpha}(V_{\alpha})), d^A) \to (V_{\alpha}, d_{CC}^{\mathbb{H}^n})
\]

is defined and is Lipschitz;

As \(\varphi_{\alpha}(V_{\alpha}) \subset M\) is open and \(f\) is continuous, \(f^{-1}(\varphi_{\alpha}(V_{\alpha})) \subset A\) is open, \(f^{-1}(\varphi_{\alpha}(V_{\alpha}))\) is an open subset of a Borel set and is thus Borel.

Since \(\mathbb{H}^n\) is purely \(k\)-unrectifiable (Theorem 2.26),

\[
H^k(\text{Im}(\varphi^{-1}_{\alpha} \circ f|_{f^{-1}(\varphi_{\alpha}(V_{\alpha}))})) = 0.
\]

As \(\varphi_{\alpha}\) is Lipschitz and the Lipschitz image of a \(H^k\)-measure zero set is a \(H^k\) measure zero set,

\[
H^k(\text{Im}(f|_{f^{-1}(\varphi_{\alpha}(V_{\alpha}))})) = H^k(\varphi_{\alpha}(\text{Im}(\varphi^{-1}_{\alpha} \circ f|_{f^{-1}(\varphi_{\alpha}(V_{\alpha}))})) = 0.
\]
Now, note that $\text{Im } f = \bigcup_{\alpha \in J} \text{Im}(f|_{f^{-1}(\varphi_\alpha(V_\alpha))})$. By subadditivity of the outer measure $\mathcal{H}^k$,

$$0 \leq \mathcal{H}^k(\text{Im } f) \leq \sum_{\alpha \in J} \mathcal{H}^k(\text{Im}(f|_{f^{-1}(\varphi_\alpha(V_\alpha))})).$$

Since $\alpha \in J$ above was arbitrary, the right hand side of the inequality is zero and $\mathcal{H}^k(\text{Im } f) = 0$. \qed

3 Lipschitz homotopy groups of purely 2-unrectifiable sub-Riemannian manifolds.

3.1 Lipschitz homotopy groups.

Having endowed Carnot manifolds with a Carnot-Caratheodory metric structure, we report the probing of the metric structure by Lipschitz maps via Lipschitz homotopy groups. Going forward, let $I = [0,1]$ be the unit interval.

Definition 3.1. Let $s_0 \in S^n$ be a base point for the $n$-sphere. For a based metric space $(X,d)$ with basepoint $x_0 \in X$, the $n$th Lipschitz homotopy group is

$$\pi_n^{\text{Lip}}((X,d),x_0) := \text{Map}^{\text{Lip}}(S^n,X)/\sim,$$

where two based Lipschitz maps in $f_0,f_1 \in \text{Map}^{\text{Lip}}(S^n,X)$ are equivalent $f_0 \sim f_1$ if there exists a Lipschitz homotopy $H \in \text{Map}^{\text{Lip}}(I \times S^n,X)$ such that

- $H|_{\{0\} \times S^n} = f_0$,
- $H|_{\{1\} \times S^n} = f_1$, and
- $H|_{I \times \{s_0\}} = x_0$.

This definition agrees with the definition of Lipschitz homotopy groups provided in Definition 4.1 in [4]. Provided that $X$ is a Riemannian manifold with the associated path metric, the Lipschitz homotopy groups of $X$ agree with the classical homotopy groups (Theorem 4.3 in [4]).

The base point will often be suppressed when it is not of utmost importance. In fact for any sub-Riemannian manifold, by Chow-Rashevskii theorem, the $n$th Lipschitz homotopy group is the same no matter the choice of base point. See Theorem 4.2(2) in [4].

3.2 Cardinality of first Lipschitz homotopy groups of purely 2-unrectifiable sub-Riemannian manifolds.

We will now identify properties of the first Lipschitz homotopy group for an arbitrary purely 2-unrectifiable sub-Riemannian manifold. First, we will consider the cardinality of this group. We will focus on horizontal embeddings of $S^1$ into the purely 2-unrectifiable sub-Riemannian manifold $(M,\xi)$. By Remark 2.21,
horizontal maps are Lipschitz with respect to the Carnot-Carathéodory metric. So, horizontal embeddings of $S^1$ are representatives of elements in $\pi_1^{\text{Lip}}(M,d^M_{CC})$. It will then be shown that the image of a Lipschitz homotopy between such embeddings has positive $\mathcal{H}^2$-measure. Since $(M,d^M_{CC})$ is purely 2-unrectifiable, the image of any Lipschitz homotopy into $(M,d^M_{CC})$ cannot have positive $\mathcal{H}^2$-measure and, thus, there will be no Lipschitz homotopies between distinct horizontal embeddings of $S^1$. Therefore, each distinct horizontal embedding of $S^1$ into $(M,\xi)$ will yield a distinct element in the group $\pi_1^{\text{Lip}}(M,d^M_{CC})$.

In the case that $(M,\xi)$ is a contact 3-manifold, there are uncountably many horizontal embeddings of $S^1$ into $(M,\xi)$. Thus, the group $\pi_1^{\text{Lip}}(M,d^M_{CC})$ will be shown to be uncountable and therefore uncountably generated.

Before proceeding, the statement of a Lipschitz version of Stokes’ theorem is given.

**Lemma 3.2 (Lemma 4.9 in [4]).** If $f : D^{n+1} \to \mathbb{R}^k$, $k \geq n$, is Lipschitz and $\omega$ is a smooth $n$-form on $\mathbb{R}^k$, then

$$ \int_{\partial D^{n+1}} f^* \omega = \int_{D^{n+1}} f^*(d\omega). $$

We will now shown that a Lipschitz homotopy $H$ between smooth embeddings of $S^1$ into $\mathbb{R}^k$ must sweep out “area,” i.e., $\mathcal{H}^2(\text{Im}(H)) > 0$.

In Lemma 3.3 and Corollary 3.4, assume $[1,2] \times S^1$ is embedded in $\mathbb{R}^2$ via polar coordinates as the standard annulus centered at the origin of inner radius 1 and outer radius 2. $\{1\} \times S^1$ is associated to the unit circle and $\{2\} \times S^1$ is associated to the circle of radius 2 centered at the origin.

The case where a smooth embedding is Lipschitz null homotopic is covered in Proposition 4.8 in [4].

**Lemma 3.3.** Let $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^k$, $k \geq 2$ be smooth embeddings of $S^1$ such that their images are distinct: $\text{Im}(\gamma_1) \neq \text{Im}(\gamma_2)$. Let $H : [1,2] \times S^1 \to \mathbb{R}^k$ be a Lipschitz homotopy between $\gamma_1$ and $\gamma_2$. Then, the 2-dimensional Hausdorff measure of the image of $H$ is positive:

$$ \mathcal{H}^2(\text{Im}(H)) > 0. $$

**Proof.** Take a 1-form $\omega \in \Omega^1(\mathbb{R}^k)$ such that

$$ \int_{S^1} \gamma_1^* \omega = 0 \quad \text{and} \quad \int_{S^1} \gamma_2^* \omega \neq 0. $$

Such a 1-form can be constructed using local coordinates and a partition of unity.

Next, it will be shown that the integral $\int_{[1,2] \times S^1} H^* d\omega$ is nonzero. Lemma 3.2 will be of significant importance to the calculation. As this result only applies to maps with domain isomorphic to $D^2$, it is necessary that the Lipschitz map $H$ be extended to a closed ball, in particular, the closed ball of radius 2 centered at the origin in $\mathbb{R}^2$, denoted $D^2(2)$. Since a Lipschitz map defined on a subset of Euclidean space can be extended to a Lipschitz map on the entire ambient space,
space, the Lipschitz map $H$ can be extended to a Lipschitz map defined on all of $\mathbb{R}^2$:

$$\tilde{H} : \mathbb{R}^2 \to \mathbb{R}^k,$$

where $\tilde{H}^{|[1,2] \times S^1} = H$.

As there is an equality of sets $D^2(2) = ([1,2] \times S^1) \cup D^2$ where the intersection of the subsets $[1,2] \times S^1$ and $D^2$ is $\{1\} \times S^1$, there is an equality of integrals,

$$\int_{D^2(2)} H^* d\omega = \int_{[1,2] \times S^1} H^* d\omega + \int_{D^2} H^* d\omega.$$

Lemma 3.2 will be used to show that $\int_{D^2(2)} H^* (d\omega) \neq 0$ and $\int_{D^2} H^* (d\omega) = 0$. Once these properties are established, since $H$ is an extension of $H$, it will be immediate that the integral

$$\int_{[1,2] \times S^1} H^* d\omega = \int_{[1,2] \times S^1} H^* d\omega = \int_{D^2(2)} H^* d\omega - \int_{D^2} H^* d\omega \neq 0$$

is non-zero.

First, consider the integral $\int_{D^2} H^* d\omega$. Via Lemma 3.2, we have an equality of integrals

$$\int_{D^2} H^* d\omega = \int_{[1,2] \times S^1} H^* d\omega,$$

as $\partial D^2 = \{1\} \times S^1$. Since the restriction of the extension $\tilde{H}^{|[1,1] \times S^1} = H^{|[1,1] \times S^1} = \gamma_1$ agrees with the embedding $\gamma_1$, by the construction of $\omega$, this integral vanishes,

$$\int_{[1,1] \times S^1} H^* d\omega = \int_{[1,1] \times S^1} \gamma_1^* d\omega = 0.$$

Now, consider $\int_{D^2(2)} H^* d\omega$. Again, via Lemma 3.2, we have an equality of integrals

$$\int_{D^2(2)} H^* d\omega = \int_{[2,2] \times S^1} H^* d\omega,$$

as $\partial D^2(2) = \{2\} \times S^1$. Since the restriction of the extension $\tilde{H}^{|[2,2] \times S^1} = H^{|[2,2] \times S^1} = \gamma_2$ agrees with the embedding $\gamma_2$, by construction of $\omega$, this integral is non-zero,

$$\int_{[2,2] \times S^1} H^* d\omega = \int_{[2,2] \times S^1} \gamma_2^* d\omega \neq 0.$$

Therefore, we have shown that the integral

$$\int_{[1,2] \times S^1} H^* d\omega = \int_{[2,2] \times S^1} \gamma_2^* d\omega - \int_{[1,1] \times S^1} \gamma_1^* d\omega \neq 0$$

is non-zero. Since the integral of the pullback of the 2-form $d\omega$ by the Lipschitz map $H$ is non-zero, $H$ has rank 2 on a set of positive measure.

Thus, the Jacobian map $J_H : [1,2] \times S^1 \to \mathbb{R}$, defined by

$$J_H(p) := \sqrt{\det((D_p H)^T (D_p H))} > 0.$$
for $p \in [1, 2] \times S^1$, is positive on a set of positive measure. Thus, its integral is positive,
\[ \int_{[1,2] \times S^1} J_H \, dA > 0, \]
where $dA$ is an area form on $\mathbb{R}^2$.

By the Area Formula (Theorem 3.8 in [6]), we have an equality of integrals
\[ \int_{\text{Im}(H)} H^0(H^{-1}(y)) \, dH^2(y) = \int_{[1,2] \times S^1} J_H \, dA > 0, \]
where $H^0$ is the counting measure. Since a measurable function integrates against the measure $H^2$ to a non-negative value, the 2-dimensional Hausdorff measure of the image of $H$ is positive: $H^2(\text{Im}(H)) > 0$.

Now that we have verified that Lipschitz homotopies between distinct embeddings of $S^1$ into $\mathbb{R}^k$ sweep out a region of positive $H^2$-measure, we will show if $(M, d^M_{CC})$ is purely 2-unrectifiable, then there are no Lipschitz homotopies between horizontal embeddings.

**Corollary 3.4.** Let $\gamma_1, \gamma_2 : S^1 \to (M, \xi)$ be horizontal embeddings of $S^1$ into a purely 2-unrectifiable sub-Riemannian manifold $(M, d^M_{CC})$ such that $\text{Im}(\gamma_1) \neq \text{Im}(\gamma_2)$. Then $\gamma_1$ and $\gamma_2$ are not Lipschitz homotopic.

**Proof.** Assume that such a Lipschitz homotopy $H : [1, 2] \times S^1 \to (M, d^M_{CC})$ exists between $\gamma_1$ and $\gamma_2$. It will be shown that $H$ yields a Lipschitz map from an annulus in $\mathbb{R}^2$ to some high dimensional $\mathbb{R}^k$. Since the sub-Riemannian manifold $(M, d^M_{CC})$ is assumed to be purely 2-unrectifiable, this map into $\mathbb{R}^k$ will have 2-dimensional Hausdorff measure 0. This will then contradict Lemma 3.3 and the proof will be complete.

First, the Lipschitz map $H$ is used to find a Lipschitz map from an annulus into $\mathbb{R}^k$. To begin, the Lipschitz map $H$ is used to find a Lipschitz map into a Riemannian manifold. Let $\tilde{g}$ be any Riemannian metric on $M$. By Theorem 2.10 in [14], the identity map from the sub-Riemannian manifold $(M, d^M_{CC})$ to the Riemannian manifold $(M, \tilde{g})$ is locally Lipschitz. Thus, the composite of locally Lipschitz maps $[1,2] \times S^1 \xrightarrow{H} (M, d^M_{CC}) \xrightarrow{\text{Id}} (M, d^\tilde{g}) \xrightarrow{\text{Id}} (\mathbb{R}^k, d^\tilde{g})$ is a locally Lipschitz map into $M$ with respect to the path metric $d^\tilde{g}$ associated to the Riemannian metric $\tilde{g}$.

By Whitney embedding theorem, there exists $k$ such that there is a smooth embedding $\iota : M \to \mathbb{R}^k$ of the manifold $M$ into $k$-dimensional Euclidean space. Since smooth maps between Riemannian manifolds are locally Lipschitz, the embedding $\iota$ is a locally Lipschitz map with respect to the associated metrics. Thus, the composite of locally Lipschitz maps $[1,2] \times S^1 \xrightarrow{H} (M, d^M_{CC}) \xrightarrow{\text{Id}} (M, d^\tilde{g}) \xrightarrow{\iota} (\mathbb{R}^k, d^\tilde{g})$ is a locally Lipschitz map. Since the domain $[1,2] \times S^1$ is compact, the composite map is Lipschitz.

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Proceeding, it will be argued that the image of this composite map has Hausdorff 2-measure zero with respect to the Euclidean metric on \( \mathbb{R}^k \). This will yield a contradiction as Lemma 3.3 states that a Lipschitz homotopy between smooth embeddings of \( S^1 \) must have positive Hausdorff 2-measure.

First, since \( H \) is a Lipschitz map with domain contained in \( \mathbb{R}^2 \) and \( (M, d^M_{CC}) \) is purely 2-unrectifiable, the Hausdorff 2-measure of the image of \( H \) in \( M \) with respect to the Carnot-Carathéodory metric is 0: \( H^2_{CC}(\text{Im}(H)) = 0 \).

Now, as cited above, the identity map on \( M \),
\[
\mathbb{1}_M : (M, d^M_{CC}) \rightarrow (M, d^\emptyset),
\]
is locally Lipschitz. Also, the smooth embedding \( \iota \) of \( M \) into \( \mathbb{R}^k \) is locally Lipschitz. Thus, the composition \( \iota \circ \mathbb{1}_M \) is locally Lipschitz. Since the image of compact sets with Hausdorff measure zero under locally Lipschitz maps have Hausdorff measure zero, the Hausdorff 2-measure of the image of \( \iota \circ \mathbb{1}_M \circ H \) is zero:
\[
H^2_\delta(\text{Im}(\iota \circ \mathbb{1}_M \circ H)) = 0.
\]

On the other hand, the map \( \iota \circ \mathbb{1}_M \circ H \) is a Lipschitz homotopy between the embeddings \( \iota \circ \gamma_1 \) and \( \iota \circ \gamma_2 \) in \( \mathbb{R}^k \). Since the images of these embeddings differ, by Lemma 3.3, the Hausdorff 2-measure of the image of the composition is positive,
\[
H^2_\delta(\text{Im}(\iota \circ \mathbb{1}_M \circ H)) > 0,
\]
a contradiction.

Corollary 3.4 will be used to construct an injective map of sets from the set of horizontal knots into \( \pi^{\text{Lip}}_1(M, d^M_{CC}) \).

**Corollary 3.5.** Let \( (M, d^M_{CC}) \) be a purely 2-unrectifiable sub-Riemannian manifold. There is an injective map of sets
\[
\left\{ \text{Based, Oriented, Horizontal Knots in } (M, \xi) \right\} \xrightarrow{\pi^{\text{Lip}}_1(M, d^M_{CC})} K \rightarrow [\gamma]
\]
where \( \gamma : S^1 \rightarrow (M, \xi) \) is a based, orientation-preserving horizontal embedding parametrizing the knot \( K \).

**Proof.** First, the indicated map will be shown to be well-defined. Let \( K \) be a based, oriented, horizontal knot in \( (M, \xi) \). By definition, there is at least one horizontal embedding \( \gamma : S^1 \rightarrow (M, \xi) \) parametrizing \( K \). By Remark 2.21, any horizontal embedding of \( S^1 \) into \( (M, \xi) \) is a Lipschitz map with respect to the Carnot-Carathéodory metric. Thus, \([\gamma]\) is indeed an element in the group \( \pi^{\text{Lip}}_1(M, d^M_{CC}) \).
Now, it will be shown that different parametrizations of the same knot determine the same element in $\pi_1^{\text{Lip}}(M, d^M_{\text{CC}})$. Suppose that $\gamma, \psi : S^1 \to (M, \xi)$ are horizontal embeddings such that $\text{Im}(\gamma) = K = \text{Im}(\psi)$ and each parametrization agrees with the orientation on $K$. To verify that this map is well-defined, it will be shown that $\gamma$ and $\psi$ are Lipschitz homotopic.

As $\psi$ maps diffeomorphically onto the submanifold $K$,

$$\psi^{-1} : K \to S^1$$

is a smooth map that preserves orientation, as is the map $\psi^{-1} \circ \gamma : S^1 \to S^1$. Thus, each arrow in the following commutative diagram is an orientation-preserving diffeomorphism,

$$\begin{array}{ccc}
S^1 & \xrightarrow{\gamma} & K \\
\psi^{-1} \circ \gamma \downarrow & & \downarrow \\
S^1 & \xrightarrow{\psi} & S^1
\end{array}$$

As $\psi^{-1} \circ \gamma$ is an orientation-preserving automorphism of $S^1$, it is smoothly homotopic to the identity map on $S^1$. Indeed, the degree of any orientation-preserving diffeomorphism is $+1$ and thus the Hopf degree theorem guarantees that the self-maps $\psi^{-1} \circ \gamma$ and $1_{S^1}$ of $S^1$ are smoothly homotopic. Let

$$h : [1, 2] \times S^1 \to S^1$$

be a smooth homotopy witnessing $\psi^{-1} \circ \gamma$ homotopic to $1_{S^1}$. As $h$ is a smooth map between Riemannian manifolds whose domain is compact, $h$ is Lipschitz. Then,

$$\psi \circ h : [1, 2] \times S^1 \to (M, d^M_{\text{CC}})$$

is a Lipschitz homotopy between the Lipschitz maps $\gamma = \psi \circ (\psi^{-1} \circ \gamma)$ and $\psi = \psi \circ 1_{S^1}$.

So, the orientation-preserving maps $\gamma$ and $\psi$ parametrizing the knot $K$ are Lipschitz homotopic. Since $\gamma$ and $\psi$ were arbitrary orientation-preserving parametrizations of $K$, the same Lipschitz homotopy class of $\pi_1^{\text{Lip}}(M, d^M_{\text{CC}})$ is determined no matter the choice of orientation-preserving parametrization of a knot $K$. Therefore, the map in the statement of the Corollary is well-defined.

To see that the map in the statement of the Corollary is injective, take distinct, based, oriented, horizontal knots $K \neq K'$ in $(M, \xi)$. Take parametrizations $\gamma, \gamma' : S^1 \to (M, \xi)$ associated to the knots $K$ and $K'$, respectively. Thus, the images of the chosen parametrizations $\text{Im}(\gamma) = K \neq K' = \text{Im}(\gamma')$ are distinct and, by Corollary 3.4, each parametrization determines distinct Lipschitz homotopy classes, i.e., $[\gamma] \neq [\gamma']$. Therefore, the map in the statement of the Corollary is injective. 

$\square$
**Proof of Theorem 1.1.1.** There are uncountably many based, oriented, horizontal knots in the contact 3-manifold (See Lemma A.1). Thus, the domain of the injective map in Corollary 3.5 is uncountable and the map’s target, $\pi_1^{\mathrm{Lip}}(M, d^M_{\mathrm{CC}})$, is therefore uncountable. Groups with uncountable cardinality are uncountably generated. \( \square \)

### 3.3 A distributional open embedding induces an injective map on $\pi_1^{\mathrm{Lip}}$.

In [16], Wenger and Young showed that certain Lipschitz maps into a purely 2-unrectifiable space factor through metric trees.

**Theorem 3.6** (Theorem 5 in [16]). *Let $X$ be a quasi-convex metric space with $\pi_1^{\mathrm{Lip}}(X) = 0$. Let furthermore $Y$ be a purely 2-unrectifiable metric space. Then every Lipschitz map from $X$ to $Y$ factors through a metric tree.*

Since metric trees are Lipschitz contractible, any Lipschitz map with appropriate domain and purely 2-unrectifiable target is Lipschitz null-homotopic. For example, Corollary 3.7 covers the case that the domain is an $n$-sphere with $n \geq 2$. This result is stated in [16] as a corollary to Theorem 3.6. Theorem 1.1.2 then follows immediately.

**Corollary 3.7.** *Let $Y$ be a purely 2-unrectifiable metric space. If $n \geq 2$ and $\alpha : S^n \to Y$ is a Lipschitz map, then $\alpha$ is Lipschitz null-homotopic. That is, $\pi_1^{\mathrm{Lip}}(Y) = 0$.***

**Proof.** The $n$-sphere $S^n$, with its standard Riemannian metric, is quasi-convex and is Lipschitz simply connected. Thus, by Theorem 3.6, the Lipschitz map $\alpha$ factors through a metric tree $T$.

\[
\begin{array}{ccc}
S^n & \xrightarrow{\alpha} & Y \\
\downarrow{\psi} & & \downarrow{\phi} \\
T & \xrightarrow{\phi} & Y \\
\end{array}
\]

The maps $\psi : S^n \to T$ and $\phi : T \to Y$ are Lipschitz as well.

Since $T$ is a metric tree, $T$ is contractible by a Lipschitz homotopy $h : I \times T \to T$. Therefore, the homotopy $H : I \times S^n \to Y$ given by $H(p, t) = \phi(h(\psi(p), t))$ is a Lipschitz null-homotopy of the map $\alpha$. \( \square \)

**Proof of Theorem 1.1.2.** Since the contact 3-manifold is purely 2-unrectifiable (Theorem 2.27), the result follows immediately from Corollary 3.7. \( \square \)

In the remainder of this paper, we will apply Theorem 3.6 to argue that an open distributional embedding of a purely 2-unrectifiable sub-Riemannian manifold into another induces an injective homomorphism between the respective first Lipschitz homotopy groups. First, we will show that the Lipschitz null homotopy of a Lipschitz null homotopic loop can be taken such that the loop shrinks to a point along its image.
**Lemma 3.8.** Let $Y$ be a purely 2-unrectifiable metric space. Let $H_0 : \mathbb{D}^2 \to Y$ be a Lipschitz map. Then, the map $H_0$ is Lipschitz homotopic to a Lipschitz map $H_1 : \mathbb{D}^2 \to Y$ such that the image of $H_1$ is contained in the image of $H_0$ restricted to the boundary of the 2-disk: $H_1 \left( \mathbb{D}^2 \right) \subset H_0 \left( \partial \mathbb{D}^2 \right)$. Furthermore, the homotopy is relative to the boundary of $\mathbb{D}^2$.

**Proof.** Let $H_0 : \mathbb{D}^2 \to Y$ be a Lipschitz map. By Theorem 3.6, since $\mathbb{D}^2$ is quasi-convex and Lipschitz simply-connected, the map $H_0$ factors through a metric tree $T$:

$$
\mathbb{D}^2 \xrightarrow{H_0} Y \xrightarrow{T} \mathbb{D}^2
$$

Both maps $\psi$ and $\phi$ are Lipschitz. Since $\psi$ is then continuous, the image $\psi(\partial \mathbb{D}^2) \subset T$ is connected and compact. So, $\psi(\partial \mathbb{D}^2)$ is a subtree of the metric tree $T$. Thus, there exists a Lipschitz deformation retract $F : I \times T \to T$ of the tree $T$ onto the subtree $\psi(\partial \mathbb{D}^2)$.

Consider the Lipschitz map

$$
\phi \circ F \circ (1_I \times \psi) : I \times \mathbb{D}^2 \to Y.
$$

We will argue that this map is a homotopy between $H_0$ and

$$
H_1 := \phi \circ F \circ (1_I \times \psi) \circ (1 \times 1_{\mathbb{D}^2}) : \mathbb{D}^2 \to Y
$$

satisfying the desired properties.

Since the map $F$ is a deformation retract of the metric tree $T$, we have an equality of maps $F \circ (0 \times 1_T) = 1_T$. Thus, precomposing $\phi \circ F \circ (1_I \times \psi)$ by the natural inclusion of $\{0\} \times \mathbb{D}^2$ into $I \times \mathbb{D}^2$ yields the original map $H_0 = \phi \circ \psi$. This equality is indicated in the filled diagram in Figure 1.

$$
\begin{array}{c}
\{1\} \times \mathbb{D}^2 \xrightarrow{1 \times \psi} \{1\} \times T \xrightarrow{\psi(\partial \mathbb{D}^2)} \phi \circ \psi(\partial \mathbb{D}^2) \xrightarrow{H_0} H_0(\partial \mathbb{D}^2) \\
I \times \mathbb{D}^2 \xrightarrow{1 \times \psi} I \times T \xrightarrow{F} T \xrightarrow{\phi} Y \\
\{0\} \times \mathbb{D}^2 \xrightarrow{H_0} \end{array}
$$

**Figure 1**
We now argue that there are factorizations of the maps \( \mathbb{1} \times \psi, F, \) and \( \phi \) as is indicated by the dashed arrows in Figure 1. For the Lipschitz map \( \mathbb{1} \times \psi : I \times \mathbb{D}^2 \to I \times T \), precomposing by the natural inclusion of \( \{1\} \times \mathbb{D}^2 \) yields the Lipschitz map \( 1 \times \psi : \{1\} \times \mathbb{D}^2 \to \{1\} \times T \). Next, since the map \( F \) is a deformation retract of the metric tree \( T \) onto the subtree \( \psi(\partial \mathbb{D}^2) \), we have that \( F \circ (1 \times \mathbb{1}_T) : \{1\} \times T \to \psi(\partial \mathbb{D}^2) \) maps onto the subtree \( \psi(\partial \mathbb{D}^2) \). The third dashed arrow comes from restricting the Lipschitz map \( \phi \) to the subset \( \psi(\partial \mathbb{D}^2) \subset T \). Finally, since \( H_0 \) factors into the composition \( \phi \circ \psi \), when restricted to the boundary of the 2-disk there is an equality of sets \( \phi \circ \psi(\partial \mathbb{D}^2) = H_0(\partial \mathbb{D}^2) \).

Therefore, precomposing the map \( \phi \circ F \circ (\mathbb{1} \times \psi) \) by the natural inclusion of \( \{1\} \times \mathbb{D}^2 \) into \( I \times \mathbb{D}^2 \) yields a map \( H_1 \) that has image contained in \( H_0(\partial \mathbb{D}^2) \subset Y \). Moreover, since the map \( F \) is a deformation retract onto \( \psi(\partial \mathbb{D}^2) \), the Lipschitz homotopy \( \phi \circ F \circ (\mathbb{1} \times \psi) \) is constant on \( \partial \mathbb{D}^2 \) for all time \( t \in I \).

We now show that each open distributional embedding from a purely 2-unrectifiable sub-Riemannian manifold into another induces an injective map on their respective first Lipschitz homotopy groups.

Before proceeding, note that an open distributional embedding \( \varphi : (M, \xi) \to (M', \xi') \) between sub-Riemannian manifolds does induce a homomorphism between Lipschitz homotopy groups. Indeed, via Lemma 2.20, the map \( \varphi \) is locally Lipschitz and thus, for any Lipschitz map \( \alpha : \mathbb{S}^n \to (M, d_{CC}^M) \), the map \( \varphi \circ \alpha \) is Lipschitz since its domain is compact.

**Theorem 3.9.** Let \( (M, \xi, g) \) and \( (M', \xi', g') \) be purely 2-unrectifiable sub-Riemannian manifolds. Let \( \varphi : (M, \xi) \to (M', \xi') \) be an open distributional embedding. Then the homomorphism induced by \( \varphi \) on first Lipschitz homotopy groups

\[
\varphi_# : \pi_1^{Lip}(M, d_{CC}^M) \to \pi_1^{Lip}(M', d_{CC}^{M'})
\]

is injective.

**Proof.** As \( \varphi_# \) is a homomorphism, we can show that the map is injective by showing that the kernel of the map is trivial.

Let \( \alpha : \mathbb{S}^1 \to (M, d_{CC}^M) \) be a Lipschitz map that represents an element of the kernel of \( \varphi_# \). So, there exists a Lipschitz map \( H : \mathbb{D}^2 \to (M', d_{CC}^{M'}) \) such that \( H \) restricted to the boundary is the Lipschitz map \( \varphi \circ \alpha \):

\[
H|_{\partial \mathbb{D}^2} = \varphi \circ \alpha.
\]

Since \( \varphi \circ \alpha : \mathbb{D}^2 \to (M', d_{CC}^{M'}) \) is the composition of Lipschitz functions, \( \varphi \circ \alpha \) is Lipschitz.

By Lemma 3.8, the Lipschitz homotopy \( H \) can be taken such that the image of \( H \) is contained in the image of the Lipschitz map \( \varphi \circ \alpha \). Thus, \( H \) takes image entirely in the image of \( \varphi \):

\[
\text{Im}(H) \subset \text{Im}(\varphi \circ \alpha) \subset \text{Im}(\varphi).
\]
Since the inverse $\varphi^{-1} : \text{Im} \varphi \to (M, \xi)$ is a distributional diffeomorphism, the map given by composition

$$\varphi^{-1} \circ H : \mathbb{D}^2 \to (M, d_M^{\text{CC}})$$

is Lipschitz and, when the map is restricted to the boundary of $\mathbb{D}^2$ equals the map $\alpha$. Thus, $\alpha$ is Lipschitz null homotopic. Therefore, the only element in the kernel of $\varphi_{\#}$ is the trivial homotopy class.

Proof of Theorem 1.1.3. Since contact 3-manifolds are purely 2-unrectifiable (Theorem 2.27), the result follows immediately from Theorem 3.9.

Remark 3.10. Theorem 3.9 indicates that the cardinality of $\pi_1^\text{Lip}(M, d_M^{\text{CC}})$ is extremely large for any purely 2-unrectifiable sub-Riemannian manifold $(M, \xi, g)$. For a base point in $M$, any connected, open neighborhood $(U, \xi|_U)$ is a purely 2-unrectifiable sub-Riemannian manifold that openly and distributionally embeds into $(M, \xi)$. Thus, a copy of the set $\pi_1^\text{Lip}(U, d_U^{\text{CC}})$ is a subgroup of $\pi_1^\text{Lip}(M, d_M^{\text{CC}})$. Additionally, from Theorem 1.1.1, if $(M, \xi)$ is a contact 3-manifold, the subgroup $\pi_1^\text{Lip}(U, d_U^{\text{CC}})$ in $\pi_1^\text{Lip}(M, d_M^{\text{CC}})$ is of uncountable cardinality.

Appendices

A Cardinality of horizontal knots in contact 3-manifolds.

We will show that the set of based, oriented, horizontal knots in a contact 3-manifold is uncountable. Per the Theorem of Darboux (Theorem 2.7), a contact 3-manifold is locally modeled by the first Heisenberg group $\mathbb{H}^1$. As such, it is sufficient to show that the set of oriented, horizontal knots based at the origin in $\mathbb{H}^1$ is uncountable.

A calculation using the coordinate system for the first Heisenberg group yields a formula to completely recover a horizontal path from its projection onto the $xy$-plane, as is illustrated below. Using this formula, an uncountable family of horizontal knots in $\mathbb{H}^1$ is readily constructed in the proof of Lemma A.1.

Let

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) : [a, b] \to \mathbb{H}^1$$

be a horizontal path in the first Heisenberg group. Here, $\gamma_i$ is the projection of the path $\gamma$ onto the $i$th coordinate of $\mathbb{R}^3$.

Recall that the associated distribution $\xi^\text{std}$ is spanned globally by the vector fields

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t} \quad \text{and} \quad Y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}$$
where $\mathbb{R}^3$ has global coordinates given by $x$, $y$, and $t$. As the path $\gamma$ is horizontal, its derivative $D\gamma$ must map into the contact distribution $\xi^{std}$:

Fixing the standard Euclidean metric on $[a, b]$, the unit vector $1_s \in T_s[a, b]$ is identified in the tangent space of each point $s \in [a, b]$. As $D\gamma$ is point-wise a linear map from the 1-dimensional tangent space spanned by the unit vector, $D_s\gamma$ is determined by how it acts on $1_s$ for each $s \in [a, b]$.

Denote the coordinate-wise derivatives of $\gamma$ with respect to $s$ by $\gamma_i'$ for $i = 1, 2, 3$ where, for each $s \in [a, b],$

$$(\gamma_1'(s), \gamma_2'(s), \gamma_3'(s)) = D_s\gamma(1_s).$$

As $(\gamma_1'(s), \gamma_2'(s), \gamma_3'(s))$ yields a vector in $\xi^{std}_{\gamma(s)} = \text{span}\{X_{\gamma(s)}, Y_{\gamma(s)}\}$ for each $s \in [a, b]$, and as $(\gamma_1', \gamma_2', \gamma_3')$ smoothly-varies over $[a, b]$, there exists smooth coefficient functions $C_1, C_2 : [a, b] \to \mathbb{R}$ such that

$$(\gamma_1'(s), \gamma_2'(s), \gamma_3'(s)) = C_1(s)X_{\gamma(s)} + C_2(s)Y_{\gamma(s)}$$

for each $s \in [a, b]$. Thus, the following vectors are equal in

$$\xi^{std}_{\gamma(s)} = \text{span}\{X_{\gamma(s)}, Y_{\gamma(s)}\}$$

for each $s \in [a, b]$:

$$(\gamma_1'(s), \gamma_2'(s), \gamma_3'(s)) = C_1(s)X_{\gamma(s)} + C_2(s)Y_{\gamma(s)} = (C_1(s), C_2(s), 2(\gamma_2(s)C_1(s) - \gamma_1(s)C_2(s))).$$

In fact, these coordinate functions are determined completely by this equality as

$$C_1(s) = \gamma_1'(s) \quad \text{and} \quad C_2(s) = \gamma_2'(s)$$

and therefore, $\gamma_3'$ is determined by $\gamma_1$ and $\gamma_2$:

$$\gamma_3'(s) = 2(\gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s)).$$

Thus, given a smooth path $(\gamma_1, \gamma_2) : [a, b] \to \mathbb{R}^2$ and a specified initial $t$-coordinate $\gamma_3(a)$, there is a unique lift to a horizontal path

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) : [a, b] \to \mathbb{H}^1.$$
Namely, \( \gamma_3 \) is

\[
\gamma_3(s) = \gamma_3(a) + 2 \int_a^s \gamma_2(u)\gamma_1'(u) - \gamma_1(u)\gamma_2'(u) \; du.
\]  

(\#)

If we additionally ask that \( \gamma \) is a closed curve, that is \( \gamma(a) = \gamma(b) \), then (\#) yields that

\[
0 = \frac{1}{2} (\gamma_3(b) - \gamma_3(a)) = \int_a^b \gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s) \; ds.
\]

Let \( A \) denote the region bounded by the closed curve \( (\gamma_1, \gamma_2) \) in \( \mathbb{R}^2 \) running from time \( a \) to time \( b \). Then the signed area of \( A \) is

\[
\int_A dx \wedge dy = \frac{1}{2} \int_{\partial A} ydx - ydx = -\frac{1}{2} \int_a^b \gamma_2(s)\gamma_1'(s) - \gamma_1(s)\gamma_2'(s) \; ds = 0.
\]

Here, the first equality is guaranteed by Stokes' theorem and the second by change of variables. Thus, the region enclosed by the closed path \( (\gamma_1, \gamma_2) \) must have signed area zero. Furthermore, horizontal loops in \( \mathbb{H}^1 \) are generated by closed loops in \( \mathbb{R}^2 \) that bound signed area zero.

We are now ready to prove the main result of this appendix.

**Lemma A.1.** For a contact 3-manifold \((M, \xi)\) and any choice of base point, there are uncountably many based, oriented, horizontal knots.

**Proof.** Let \( p \in M \) denote the base point. First, it will be shown that there is an injective map of sets from horizontal knots in \( \mathbb{H}^1 \) based at 0 to horizontal knots in \((M, \xi)\) based at \( p \).

By the Theorem of Darboux, there is a distributional embedding \( \varphi: \mathbb{H}^1 \to (M, \xi) \) mapping base point to base point, \( \varphi(0) = p \). Since \( \varphi \) is a distributional embedding, \( \varphi \) maps horizontal knots in \( \mathbb{H}^1 \) to horizontal knots in \((M, \xi)\). Thus, there is a map of sets,

\[
\begin{align*}
\{ \text{Based at 0, Oriented, Horizontal Knots in } \mathbb{H}^1 \} & \to \{ \text{Based at } p, \text{ Oriented, Horizontal Knots in } (M, \xi) \} \\
K & \to \varphi(K).
\end{align*}
\]

This map will be shown to be injective. Since \( \varphi \) is a smooth embedding, there is a smooth inverse defined on its image,

\[
\varphi^{-1}: \varphi(\mathbb{H}^1) \to \mathbb{H}^1.
\]

For horizontal knots \( K \) and \( K' \) in \( \mathbb{H}^1 \), if their images under \( \varphi \) are equal in \((M, \xi)\), that is \( \varphi(K) = \varphi(K') \), then

\[
K = \varphi^{-1} \circ \varphi(K) = \varphi^{-1} \circ \varphi(K') = K'.
\]
Therefore, the defined map is injective.

So, in order to show that there are uncountably many based, oriented, horizontal knots in \((M, \xi)\), it is enough to show that there are uncountably many based at 0, oriented, horizontal knots in \(H^1\). An uncountable family of such horizontal knots will be generated to verify this claim.

As was shown at the beginning of this appendix, horizontal paths in \(H^1\) are completely determined by their projections into the \(xy\)-plane. For a smooth path \((\gamma_1, \gamma_2) : [0, 2\pi] \rightarrow \mathbb{R}^2\) in \(\mathbb{R}^2\) and an initial \(t\)-coordinate \(\gamma_3(0)\), the \(t\)-coordinate \(\gamma_3 : [0, 2\pi] \rightarrow H^1\) of the horizontal path \(\gamma = (\gamma_1, \gamma_2, \gamma_3) : [0, 2\pi] \rightarrow H^1\) is uniquely determined:

\[
\gamma_3(t) = \gamma_3(0) + 2 \int_0^t \gamma_1'(s)\gamma_2(s) - \gamma_2'(s)\gamma_1(s) \, ds.
\]  

(1)

In order to guarantee that \(\gamma\) is a loop based at 0, set \(\gamma_3(0) = 0\) and demand that

\[
\int_0^{2\pi} \gamma_1'(s)\gamma_2(s) - \gamma_2'(s)\gamma_1(s) \, ds = 0,
\]  

(2)

that is that the loop \((\gamma_1, \gamma_2)\) in the \(xy\)-plane bounds zero signed area.

A family of loops satisfying (2) is given by

\[
(\gamma^a_1, \gamma^a_2) : [0, 2\pi] \rightarrow \mathbb{R}^2
\]

for any \(a > 0\), which is verified by straightforward integration. Since

\[
\frac{d^n\gamma_i^a}{ds^n}(0) = \frac{d^n\gamma_i^a}{ds^n}(2\pi) \text{ for } i = 1, 2,
\]

\((\gamma^a_1, \gamma^a_2)\) yields a smooth map with domain \(S^1\). Thus, for each \(a > 0\), there is a horizontal loop

\[
\gamma^a = (\gamma^a_1, \gamma^a_2, \gamma^a_3) : S^1 \rightarrow H^1
\]

where \(\gamma^a_3\) is determined by (1).

Let \(a > 0\) be fixed. The horizontal map \(\gamma^a\) will be shown to be an embedding of \(S^1\).

To see that the map \(\gamma^a\) is injective, note that the projection of \(\gamma^a\) onto its first two coordinates only fails to be injective because \((\gamma^a_1, \gamma^a_2)(0) = (\gamma^a_1, \gamma^a_2)(\pi)\). But (1) and straightforward integration gives that the \(t\)-coordinate of \(\gamma^a(0)\) is not equal to the \(t\)-coordinate of \(\gamma^a(\pi)\):

\[
\gamma^a_3(\pi) = -\frac{8}{3} a^2 \neq 0 = \gamma^a_3(0).
\]
So, $\gamma^a(0) \neq \gamma^a(\pi)$ and, more generally, $\gamma^a(s) \neq \gamma^a(s')$ for all distinct $s$ and $s'$ in $S^1$.

Now, $\gamma^a$ is shown to be an immersion. Note that the derivative is never zero, even for the projection $(\gamma^a_1, \gamma^a_2)$. Indeed,

$$\left( \frac{d\gamma^a_1}{ds}, \frac{d\gamma^a_2}{ds} \right) (s) = (a \cos s, -2a \cos 2s)$$

and thus $\frac{d\gamma^a_2}{ds}(s)$ only vanishes when $s = \frac{\pi}{2}, \frac{3\pi}{2}$. But the derivatives of $\gamma^a_2$ for these $s$-values are non-zero,

$$\frac{d\gamma^a_2}{ds} \left( \frac{\pi}{2} \right) = 0 \neq \frac{d\gamma^a_2}{ds} \left( \frac{3\pi}{2} \right).$$

So, the derivative $D_s \gamma^a$ is never the zero map for any $s \in S^1$. Since $S^1$ is one dimensional, the map $\gamma^a$ is an immersion.

Thus, $\gamma^a(S^1)$ is a horizontal knot based at 0 in $\mathbb{H}^1$ that inherits an orientation from the smooth embedding $\gamma^a$.

Now, we note for distinct positive real numbers $a \neq b$ that $\gamma^a \neq \gamma^b$. Indeed, the point $(a,0) = (\gamma^a_1, \gamma^a_2)(\frac{\pi}{2})$ appears nowhere on the loop $(\gamma^b_1, \gamma^b_2)$ in $\mathbb{R}^2$. As their projections are distinct, the knots $\gamma^a(S^1)$ and $\gamma^b(S^1)$ are distinct. Since the set $(0, \infty)$ is uncountable, the family of horizontal knots $\{\gamma^a\}_{a \in (0, \infty)}$ is uncountable. Therefore, there are uncountably many based, oriented, horizontal knots in $\mathbb{H}^1$.

As was shown at the beginning of this argument, there is an injective map from based, oriented, horizontal knots in $\mathbb{H}^1$ to based, oriented, horizontal knots in $(M, \xi)$. Since the domain of this injective map is uncountable, so is its target.

\[ \square \]

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