Abstract. In this work we consider the non local evolution problem
\begin{align*}
\partial_t u(x, t) &= -u(x, t) + g(\beta K(f \circ u)(x, t) + \beta h), \quad x \in \Omega, \; t \in [0, \infty[; \\
 u(x, t) &= 0, \; x \in \mathbb{R}^N \setminus \Omega, \; t \in [0, \infty[; \\
 u(x, 0) &= u_0(x), \; x \in \mathbb{R}^N,
\end{align*}
where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$; $g, f : \mathbb{R} \to \mathbb{R}$ satisfying certain growing condition and $K$ is an integral operator with symmetric kernel, $Kv(x) = \int_{\mathbb{R}^N} J(x, y)v(y)dy$. We prove that Cauchy problem above is well posed, the solutions are smooth with respect to initial conditions, and we show the existence of a global attractor. Furthermore, we exhibit a Lyapunov’s functional, concluding that the flow generated by this equation has a gradient property.

1. Introduction

We consider the non local evolution problem
\begin{align*}
\partial_t u(x, t) &= -u(x, t) + g(\beta K(f \circ u)(x, t) + \beta h), \quad x \in \Omega, \; t \in [0, \infty[ \\
 u(x, t) &= 0, \; x \in \mathbb{R}^N \setminus \Omega, \; t \in [0, \infty[ \\
 u(x, 0) &= u_0(x), \; x \in \mathbb{R}^N,
\end{align*}
where $u(x, t)$ is a real function on $\mathbb{R}^N \times [0, \infty[$, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 1$); $h$ and $\beta$ are nonnegative constants; $f, g : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous satisfying some growth conditions and $K$ is an integral operator with symmetric nonnegative kernel, given by
\begin{equation}
Kv(x) := \int_{\mathbb{R}^N} J(x, y)v(y)dy,
\end{equation}
where $J$ is a symmetric non negative function of class $C^1$, with
\begin{equation*}
\int_{\mathbb{R}^N} J(x, y)dy = \int_{\mathbb{R}^N} J(x, y)dx = 1.
\end{equation*}

The dynamics of non local evolution Equations like in (1.1) has attracted the attention of many researchers in the last years; see for instance [1, 2, 3, 5, 6, 8, 9, 10, 11, 15, 16, 17, 21, 22, 24, 28, 30] and [31]. However, the model considered here presents innovation and generalizes the model considered in [3, 8, 24] and [25], which can be obtained as a particular case of (1.1) with $f$ being the identity, as well as it generalizes the model considered in [21, 24, 28, 30] and [31], which can be obtained as

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a particular case of (1.1) where \( g \) is the identity, \( \beta = 1 \) and the integral operator \( K \) is the convolution product. When \( g \) and \( f \) are identity, \( \beta = 1 \) and the integral operator \( K \) is the convolution product, we also obtain as particular case of (1.1) the model considered in [4].

The approach considered here was motivated by similar approaches in [3, 13] and [27], whose basic idea is to find an abstract way to impose Dirichlet boundary conditions in non local evolution equations.

The paper is organized as follows. In Section 2, assuming a growth condition on the functions \( g \) and \( f \), we prove that (1.1) is well posed with globally defined solution. In Section 3 we prove that (1.1) generates a \( C^1 \) flow in a space \( X \) which is isometric to \( L^p(\Omega) \). In Section 4 we prove existence of a global attractor, and establish some regularity properties for it. In Section 5 we prove comparison and boundedness results for the solutions of (1.1). Finally, in Section 6 we exhibit a continuous Lyapunov’s functional for the flow generated by (1.1), and we use it to prove that the this flow has the gradient property in the sense of [19].

2. Well posedness

In this section, we prove that the Cauchy problem (1.1) is well posed in the suitable phase space

\[
X = \left\{ u \in L^p(\mathbb{R}^N) : u(x) = 0, \text{ if } x \in \mathbb{R}^N \setminus \Omega \right\}
\]

with the induced norm of \( L^p(\mathbb{R}^N) \). For this we assume that the functions \( g \) and \( f \) satisfy the “suitable” following growth conditions: there exist non negative constants \( k_1, k_2, c_1 \) and \( c_2 \) such that

\[
|g(x)| \leq k_1 |x| + k_2, \quad \forall \ x \in \mathbb{R} \quad (2.3)
\]

and

\[
|f(x)| \leq c_1 |x| + c_2, \quad \forall \ x \in \mathbb{R}. \quad (2.4)
\]

The space \( X \) is canonically isomorphic to \( L^p(\Omega) \) and we usually identify the two spaces, without further comment. We also use the same notation for a function in \( \mathbb{R}^N \) and its restriction to \( \Omega \) for simplicity, wherever we believe the intention is clear from the context.

In order to obtain well posedness of (1.1), we consider the Cauchy problem

\[
\begin{aligned}
\partial_t u &= -u + F(u), \\
u(t_0) &= u_0,
\end{aligned} \quad (2.5)
\]

where the map \( F : X \to X \) is defined by

\[
F(u)(x) = \begin{cases}
g(\beta K(f \circ u)(x) + \beta h), & x \in \Omega, \\
0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases} \quad (2.6)
\]

Depending on the properties assumed for \( J \), the map given by (1.2) is well defined as a bounded linear operator in various functions spaces and, in particular, it will be well defined in \( X \).

To prove that \( F \) given in (2.6) is well defined, under the conditions given in (2.3) and (2.4), we need of the estimates below for the map \( K \), which has been proven in [8].

**Lemma 2.1.** Let \( K \) be the map defined by (1.2) and \( \|J\|_r := \sup_{x \in \Omega} \|J(x, \cdot)\|_{L^r(\Omega)}, \ 1 \leq r \leq \infty \). If \( u \in L^p(\Omega), \ 1 \leq p \leq \infty \), then \( (Ku \in L^\infty(\Omega)) \),

\[
|Ku(x)| \leq \|J\|_q \|u\|_{L^p(\Omega)}, \quad \forall \ x \in \Omega, \quad (2.7)
\]

where \( 1 \leq q \leq \infty \) is the conjugate exponent of \( p \), and

\[
\|Ku\|_{L^p(\Omega)} \leq \|J\|_1 \|u\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)}. \quad (2.8)
\]
Moreover, if \( u \in L^1(\Omega) \), then \( Ku \in L^p(\Omega) \), \( 1 \leq p \leq \infty \), and
\[
\|Ku\|_{L^p(\Omega)} \leq \|J\|_p \|u\|_{L^1(\Omega)}. \tag{2.9}
\]

**Definition 2.2.** If \( E \) is a normed space, we say that a function \( F : E \to E \) is locally Lipschitz continuous (or simply locally Lipschitz) if, for any \( x_0 \in E \), there exists a constant \( C \) and a rectangle \( R = \{ x \in E : \|x - x_0\| < b \} \) such that, if \( x \) and \( y \) belong to \( R \), then \( \|F(x) - F(y)\| \leq C\|x - y\| \); we say that \( F \) is Lipschitz continuous on bounded sets if the rectangle \( R \) in the previous definition may be chosen as any bounded rectangle in \( E \).

**Remark 2.3.** The two definitions in (2.2) are equivalent if the normed space \( E \) is locally compact.

**Proposition 2.4.** In addition to the hypotheses from Lemma 2.1, suppose that the functions \( f \) and \( g \) satisfy the two growth conditions (2.3) and (2.4). Then the function \( F \) given by (2.6) is well defined in \( L^p(\Omega) \).

**Proof.** Consider \( 1 \leq p < \infty \) and let \( u \in L^p(\Omega) \). Then, using Hölder inequality (see [18]) and (2.4), we obtain
\[
\|f(u)\|_{L^1(\Omega)} \leq \int_\Omega \left| c_1 |u(x)| + c_2 \right| dx \leq c_1 |\Omega|^{\frac{1}{p}} \|u\|_{L^p(\Omega)} + c_2 |\Omega|, \tag{2.10}
\]
where \( q \) denotes the conjugate exponent of \( p \).

From estimates (2.9) and (2.10), it follows that
\[
\|Kf(u)\|_{L^p(\Omega)} \leq \|J\|_p \|f(u)\|_{L^1(\Omega)} \leq \|J\|_p (c_1 |\Omega|^{\frac{1}{p}} \|u\|_{L^p(\Omega)} + c_2 |\Omega|) = c_1 \|J\|_p |\Omega|^{\frac{1}{p}} \|u\|_{L^p(\Omega)} + \|J\|_p c_2 |\Omega|. \tag{2.11}
\]
Thus, using (2.11), it follows that
\[
\|F(u)\|_{L_p(\Omega)} = \|g(\beta |Kf(u)| + \beta h)\|_{L_p(\Omega)} \leq \left( \int_\Omega [\beta k_1 \|Kf(u)(x)\| + k_1 \beta h + k_2]^p dx \right)^{\frac{1}{p}} \leq \|\beta k_1 |Kf(u)| + (k_1 \beta h + k_2)\|_{L_p(\Omega)} \leq \beta k_1 \|Kf(u)\|_{L_p(\Omega)} + \|k_1 \beta h + k_2\|_{L_p(\Omega)} \leq \beta k_1 (c_1 \|J\|_p |\Omega|^{\frac{1}{p}} \|u\|_{L^p(\Omega)} + \|J\|_p c_2 |\Omega|) + (k_1 \beta h + k_2) |\Omega|^{\frac{1}{p}} \leq \beta k_1 c_1 \|J\|_p |\Omega|^{\frac{1}{p}} \|u\|_{L^p(\Omega)} + \beta k_1 \|J\|_p c_2 |\Omega| + (k_1 \beta h + k_2) |\Omega|^{\frac{1}{p}}, \] showing that, in this case, \( F \) is well defined.

The proof for \( p = \infty \) is straightforward, because if \( u \in L^\infty(\Omega) \), from (2.4) it follows that \( f(u) \in L^\infty(\Omega) \) and, consequently
\[
|K(f(u)(x))| \leq \|J\|_1 \|f(u)\|_{\infty} = \|f(u)\|_{\infty}.
\]
Thus, using (2.4), we obtain
\[
\|Kf(u)\|_{L^\infty(\Omega)} \leq c_1 \|u\|_{\infty} + c_2.
\]
Hence, from (2.3), we have
\[
\|F(u)\|_{L^\infty(\Omega)} \leq k_1 \beta \|Kf(u)\|_{L^\infty(\Omega)} + k_1 \beta h + k_2 \leq \beta k_1 (c_1 \|u\|_{\infty} + c_2) + k_1 \beta h + k_2.
\]
Thus, we conclude the result.

**Proposition 2.5.** Suppose, in addition to the hypotheses from Proposition 2.4, that the functions $g$ and $f$ are Lipschitz continuous on bounded. Then the function $F$ given by (2.6) is Lipschitz continuous on bounded sets of $L^p(\Omega), \ 1 \leq p \leq \infty$.

**Proof.** Suppose $1 \leq p < \infty$ and let $u, v \in L^p(\Omega)$ be such that $\|u\|_{L^p(\Omega)} \leq r$ and $\|v\|_{L^p(\Omega)} \leq r$. Then $\|u\|_{\infty} \leq r|\Omega|^{-\frac{1}{p}}$ and $\|v\|_{\infty} \leq r|\Omega|^{-\frac{1}{p}}$. Let $M$ be the Lipschitz constant of $f$ in the interval $[-r|\Omega|^{-\frac{1}{p}}, r|\Omega|^{-\frac{1}{p}}]$. Then, for all $x \in \Omega$,

$$|f(u(x)) - f(v(x))| \leq M|u(x) - v(x)|.$$ 

From (2.8) it follows that

$$\|Kf(u) - Kf(v)\|_{L^p(\Omega)} \leq \|J\|_1 \|f(u) - f(v)\|_{L^p(\Omega)}$$

$$= \left( \int_{\Omega} |f(u(x)) - f(v(x))|^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \left( \int_{\Omega} MP|u(x) - v(x)|^p \, dx \right)^{\frac{1}{p}}$$

$$= M\|u - v\|_{L^p(\Omega)}.$$ 

Now, if $l = c_1\|J\|_{\Omega}^{\frac{1}{p}} r + \|J\|_{\infty}^{\frac{1}{p}} r \Omega$ and $N$ denotes the Lipschitz constant of $g$ in the interval $[-l, l] \subset \mathbb{R}$, using (2.7), we have that

$$\|F(u) - F(v)\|_{L^p(\Omega)} \leq N\beta\|Kf(u) - Kf(v)\|_{L^p(\Omega)}$$

$$\leq N\beta\|J\|_1 \|Kf(u) - Kf(v)\|_{L^p(\Omega)}$$

$$\leq N\beta M\|u - v\|_{L^p(\Omega)},$$

showing that $F$ is Lipschitz in bounded sets of $L^p(\Omega)$ as claimed. If $p = 1$, the proof is similar, but simpler. Suppose, finally, that $\|u\|_{L^\infty(\Omega)} \leq r$, $\|v\|_{L^\infty(\Omega)} \leq r$, let $M$ be the Lipschitz constant of $f$ and $N$ denotes the Lipschitz constant of $g$ in the interval $[-l, l] \subset \mathbb{R}$, where now $l = c_1\|J\|_{\Omega}^{\frac{1}{p}} r + \|J\|_{\infty}^{\frac{1}{p}} r \Omega$.

Then, using (2.7), we obtain

$$\|Kf(u)\|_{L^\infty(\Omega)} \leq N\|J\|_1 \|f(u)\|_{\infty} = \|f(u)\|_{\infty}.$$ 

Whence, we obtain

$$\|F(u) - F(v)\|_{L^\infty(\Omega)} \leq N\beta M\|u - v\|_{\infty}.$$ 

\[\square\]

From Proposition 2.4 it follows from well known results, on ordinary differential equation in Banach space, that the problem (2.6) has a local solution for arbitrary initial condition in $X$. For the global existence, we need the following result (23 - Theorem 5.6.1).

**Theorem 2.6.** Let $X$ be a Banach space, and suppose that $g : [t_0, \infty[ \times X \to X$ is continuous and $\|g(t, u)\| \leq h(t, \|u\|)$; $\forall \ (t, u) \in [t_0, \infty[ \times X$, where $h : [t_0, \infty[ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $h(t, r)$ is non decreasing in $r \geq 0$, for each $t \in [t_0, \infty[$. Then, if the maximal solution $r(t, t_0, r_0)$ of the scalar initial value problem

$$r' = h(t, r), \ r(t_0) = r_0,$$

exists, it is Lipschitz continuous in the interval $[t_0, \infty[$.
exists throughout \([t_0, \infty]\), the maximal interval of existence of any solution \(u(t, t_0, u_0)\) of the initial value problem
\[
\frac{du}{dt} = g(t, u), \quad t \geq t_0, \quad u(t_0) = u_0,
\]
with \(\|u_0\| \leq r_0\), also contains \([t_0, \infty]\).

**Corollary 2.7.** Suppose, the same hypotheses from Proposition 2.6. Then the problem (2.5) has a unique globally defined solution for arbitrary initial condition in \(X\), which is given, for \(t \geq t_0\), by the “variation of constants formula”
\[
u(t, x) = \begin{cases} 
 e^{-(t-t_0)}u_0(x) + \int_{t_0}^{t} e^{-(t-s)}g(\beta K f(u(s, \cdot))(x) + \beta h)ds, & x \in \Omega, \\
0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

**Proof.** From Proposition 2.6 it follows that the right-hand-side of (2.5) is Lipschitz continuous in bounded sets of \(X\) and, therefore, the Cauchy problem (2.5) is well posed in \(X\), with a unique local solution \(u(t, x)\), given by (2.13) (see [7]).

If \(1 \leq p < \infty\), from (2.12), we obtain that the right-hand-side of (2.5) satisfies
\[
\| - u + F(u) \|_{L^p(\Omega)} \leq (1 + \beta k_1 c_1 \|J\|_p|\Omega|^{\frac{1}{p}})\|u\|_{L^p(\Omega)} + \beta k_1 \|J\|_p c_2 |\Omega| + (k_1 \beta h + k_2) |\Omega|^{\frac{1}{p}}.
\]

If \(p = \infty\), we have that the right-hand-side of (2.5) satisfies
\[
\| - u + F(u) \|_{L^\infty(\Omega)} \leq \beta (1 + k_1 c_1)\|u\|_{\infty} + k_1 (\beta c_2 + \beta h) + k_2.
\]

Hence, defining \(h : [t_0, \infty) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+\), by
\[
h(t, r) = (1 + \beta k_1 c_1 \|J\|_p|\Omega|^{\frac{1}{p}})r + \beta k_1 \|J\|_p c_2 |\Omega| + (k_1 \beta h + k_2) |\Omega|^{\frac{1}{p}},
\]
if \(1 \leq p < \infty\) or by
\[
h(t, r) = \beta (1 + k_1 c_1) r + k_1 (\beta c_2 + \beta h) + k_2,
\]
in the case \(p = \infty\), it follows that (2.5) satisfies the hypotheses from Theorem 2.6 and the global existence follows immediately. The variation of constants formula may be verified by direct derivation. 

**3. Smoothness of the solutions**

In this section, in addtion the hypotheses from previous section, we assume that the functions \(g, f \in \mathcal{C}^1(\mathbb{R})\), and \(g'\) and \(f'\) are locally Lipschitz and there exist non negative constants \(k_3, k_4, c_3\) and \(c_4\), such that
\[
|g'(x)| \leq k_3 |x| + k_4, \quad \forall, \quad x \in \mathbb{R},
\]
\[
|f'(x)| \leq c_3 |x| + c_4, \quad \forall, \quad x \in \mathbb{R}.
\]

The following result has been proven in [20].

**Proposition 3.1.** Let \(X\) and \(Y\) be normed linear spaces, \(F : X \rightarrow Y\) a map and suppose that the Gateaux’s derivative of \(F, DF : X \rightarrow \mathcal{L}(X, Y)\) exists and is continuous at \(x \in X\). Then the Fréchet’s derivative \(F'\) of \(F\) exists and is continuous at \(x\).

Using Proposition 3.1 we have the following result:
Proposition 3.2. Suppose, in addition to the hypotheses of Corollary 3.1 that the function \( g \) and \( f \) have derivative satisfying (3.14) and (3.15), respectively. Then \( F \) is continuously Fréchet differentiable on \( X \) with derivative given by

\[
DF(u)v(x) := \begin{cases} 
-v(x) + g'\left(\beta K f(u)(x) + \beta h\big|Kf'(u(x))\big|v(x)\right), & x \in \Omega, \\
0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

Proof. From a simple computation, using the fact that \( f \) is continuously differentiable on \( \mathbb{R} \), it follows that the Gateaux’s derivative of \( F \) is given by

\[
DF(u)v(x) := \begin{cases} 
-v(x) + g'\left(\beta K f(u)(x) + \beta h\big|Kf'(u(x))\big|v(x)\right), & x \in \Omega, \\
0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}
\]

The operator \( DF(u) \) is clearly a linear operator in \( X \).

Suppose \( 1 \leq p < \infty \) and \( q \) is the conjugate exponent of \( p \). Then, if \( u \in L^p(\Omega) \), using (3.14) and (2.7), we obtain

\[
\|g'\left(\beta K f(u) + \beta h\right)\beta Kf'(u)v\|_{L^p(\Omega)} \leq \left\{ \int_{\Omega} \left| g'\left(\beta K f(u)(x) + \beta h\big|Kf'(u(x))\big|v(x)\right) \right|^p dx \right\}^{\frac{1}{p}}
\]

\[
\leq \left\{ \int_{\Omega} \left[ k_3 \beta|K(f(u)(x))| + k_3 \beta h + k_4 \right]^p \beta^p|K(f'(u(x)))v(x)|^p dx \right\}^{\frac{1}{p}}
\]

\[
\leq \left\{ \int_{\Omega} \left[ k_3 \beta|J|q\|f(u)\|_{L^p(\Omega)} + k_3 \beta h + k_4 \right]^p \beta^p\|J\|q\|f'(u)\|_{L^p(\Omega)}|v(x)|^p dx \right\}^{\frac{1}{p}}.
\]

Thus, from (3.14) and (3.15), we have

\[
\|g'\left(\beta K f(u) + \beta h\right)\beta Kf'(u)v\|_{L^p(\Omega)} \leq
\]

\[
\leq \left\{ \int_{\Omega} \left[ k_3 \beta|J|q\left(c_1\|u\|_{L^p(\Omega)} + c_2|\Omega|^{\frac{1}{p}}\right) + k_3 \beta h + k_4 \right]^p \beta^p\|J\|q(c_3\|u\|_{L^p(\Omega)} + c_4|\Omega|^{\frac{1}{p}})|v(x)|^p dx \right\}^{\frac{1}{p}}
\]

\[
= (k_3 \beta|J|q(c_1\|u\|_{L^p(\Omega)} + c_2|\Omega|^{\frac{1}{p}}) + k_3 \beta h + k_4)\beta\|J\|q\left(c_3\|u\|_{L^p(\Omega)} + c_4|\Omega|^{\frac{1}{p}}\right)\|v\|_{L^p(\Omega)}. \tag{3.16}
\]

From (3.16), we have

\[
\|DF(u)v\|_{L^p(\Omega)} = \left( k_3 \beta|J|q\left(c_1\|u\|_{L^p(\Omega)} + c_2|\Omega|^{\frac{1}{p}}\right) + k_3 \beta h + k_4 \right)\beta\|J\|q\left(c_3\|u\|_{L^p(\Omega)} + c_4|\Omega|^{\frac{1}{p}}\right)\|v\|_{L^p(\Omega)},
\]

showing that \( DF(u) \) is a bounded operator. In the case \( p = \infty \), we have that

\[
\|DF(u)v\|_{L^\infty(\Omega)} = \|g'\left(\beta K f(u) + \beta h\right)\beta Kf'(u)v\|_{\infty}
\]

\[
\leq (k_3 \beta\|Kf(u)\|_{\infty} + k_3 \beta h + k_4)\beta\|K \circ (f'(u))\|_{\infty}\|v\|_{\infty}
\]

\[
\leq \left( k_3 \beta|J|_1\left(c_1\|u\|_{L^\infty(\Omega)} + c_2\right) + k_3 \beta h + k_4\right)\beta\|J\|_1(c_3\|u\|_{L^\infty(\Omega)} + c_4)\|v\|_{\infty}
\]

\[
\leq \left( k_3 \beta(c_1\|u\|_{L^\infty(\Omega)} + c_2) + k_3 \beta h + k_4\right)(c_3\|u\|_{L^\infty(\Omega)} + c_4)\|v\|_{\infty}
\]

showing the boundedness of \( DF(u) \) also in this case.
Suppose now that $u_1, u_2$ and $v$ belong to $L^p(\Omega)$, $1 \leq p < \infty$. Then
\[
\| (DF(u_1) - DF(u_2))v \|_{L^p(\Omega)} =
\leq \| g' (\beta K f(u_1) + \beta h) \beta K f'(u_1) v - g' (\beta K f(u_2) + \beta h) \beta K f'(u_2) v \|_{L^p(\Omega)} + \| f'(u_1) v \|_{L^p(\Omega)}
\leq I + II,
\]
where
\[
I = \| g' (\beta K f(u_1) + \beta h) - g' (\beta K f(u_2) + \beta h) \|_{L^p(\Omega)} \| \beta K f'(u_1) v \|_{L^p(\Omega)}
\]
and
\[
II = \| g' (\beta K f(u_2) + \beta h) \|_{L^p(\Omega)} \| \beta K f'(u_1) v \|_{L^p(\Omega)}
\]

Fixed $u_1 \in L^p(\Omega)$ and letting $u_2 \to u_1$ in $L^p(\Omega)$ follows that $\beta K f(u_2) + \beta h$ is in a ball of $L^\infty$ centered at $\beta K f(u_1) + \beta h$. Then, since $g'$ is locally Lipschitz, there exists $C > 0$, such that
\[
| g' (\beta K f(u_1) + \beta h)(x) - g' (\beta K f(u_2) + \beta h)(x) | \leq C | \beta K f(u_1) - f(u_2) | (x) \leq C | \beta | \| J \|_q | u_1 - u_2 | \| v(x) \|_{L^p(\Omega)}.
\]
Thus, using (2.7), we have that
\[
I \leq \left( \int_{\Omega} | (C \beta | J \|_q \| u_1 - u_2 \|_{L^p(\Omega)})^p | v| f'(u_1)(x) |^p | v(x) |^p \right)^{\frac{1}{p}}
\leq C \beta | J \|_q | u_1 - u_2 | \|_{L^p(\Omega)} \beta \left( \int_{\Omega} | f'(u_1)(x) |^p | v(x) |^p \right)^{\frac{1}{p}}
\leq C \beta^2 | J \|_q | u_1 - u_2 | \|_{L^p(\Omega)} \left( \int_{\Omega} | J \|_q \| f'(u_1) \|_{L^p(\Omega)} | v(x) |^p \right)^{\frac{1}{p}}.
\]
But, from (3.15) follows that
\[
| f'(u_1)(x) | \leq c_3 \| u_1 \|_{L^p(\Omega)} + c_4 | \Omega |^\frac{1}{p}.
\]
Hence,
\[
I \leq C \beta^2 | J \|_q | u_1 - u_2 | \|_{L^p(\Omega)} \beta \left( \int_{\Omega} | J \|_q \| f'(u_1) \|_{L^p(\Omega)} | v(x) |^p \right)^{\frac{1}{p}}.
\]
Now, using (3.14) and (2.7), we obtain
\[
| g' (\beta K f(u_2)(x) + \beta h) | \leq k_5 \beta | K f(u_2)(x) | + k_5 \beta h + k_4
\leq k_5 \beta \| J \|_q \| f'(u_2) \|_{L^p(\Omega)} \beta h + k_4
\leq k_5 \beta \| J \|_q \left( c_1 \| u_2 \|_{L^p(\Omega)} + c_2 | \Omega |^\frac{1}{p} \right) + k_5 \beta h + k_4.
\]
Whence we obtain
\[
II \leq [ k_5 \beta \| J \|_q \left( c_1 \| u_2 \|_{L^p(\Omega)} + c_2 | \Omega |^\frac{1}{p} \right) + k_5 \beta h + k_4 ] \| [ f'(u_1) - f'(u_2) ] \|_{L^p(\Omega)}.
\]
Using (2.9) and Hölder inequality, we have
\[
II \leq [ k_5 \beta \| J \|_q \left( c_1 \| u_2 \|_{L^p(\Omega)} + c_2 | \Omega |^\frac{1}{p} \right) + k_5 \beta h + k_4 ] \| [ f'(u_1) - f'(u_2) ] \|_{L^1(\Omega)} \| v \|_{L^p(\Omega)}.
\]
From (3.17) and (3.18), follow that
\[
\| [DF(u_1) - DF(u_2)]v \|_{L^p(\Omega)} \leq \\
c_\beta |J_1(v)|u_1 - u_2 \|_{L^p(\Omega)} \| J_1(v) \|_q \left( c_3 \| u_1 \|_{L^p(\Omega)} + c_4 |\Omega|^{\frac{1}{p}} \right) \| v \|_{L^p(\Omega)} \\
+ \left[ k_3 \beta \| J_1(v) \|_{L^p(\Omega)} + c_2 \| \Omega |^{\frac{1}{p}} \right] k_3 \beta h + k_4 \| J_1(v) \|_q \| f'(u_1) - f'(u_2) \|_{L^p(\Omega)} \| v \|_{L^p(\Omega)}.
\]
Thus, to prove continuity of the derivative, we only have to show that
\[
\| f'(u_1) - f'(u_2) \|_{L^p(\Omega)} \rightarrow 0
\]
when
\[
\| u_1 - u_2 \|_{L^p(\Omega)} \rightarrow 0.
\]
But, from the growth condition on \( f' \) it follows that
\[
| f'(u_1)(x) - f'(u_2)(x) |^q \leq [c_3 (|u_1(x)| + |u_2(x)|) + 2c_4]^q
\]
and a simple computation show that the right-hand is in \( L^1(\Omega) \). Then the result follows from Lebesgue’s Convergence Theorem.

In the case \( p = \infty \), from (2.8), we obtain
\[
\| [DF(u_1) - DF(u_2)]v \|_{L^\infty(\Omega)} \leq \\
c_\beta \| f'(u_1) - f'(u_2) \|_{L^\infty(\Omega)} \| J_1(v) \|_{L^\infty(\Omega)} \| f'(u_1) \|_{L^\infty(\Omega)} \\
+ \left( k_3 \beta \| f'(u_2) \|_{L^\infty(\Omega)} + k_3 \beta h + k_4 \| f'(u_1) - f'(u_2) \|_{L^\infty(\Omega)} \| v \|_{L^\infty(\Omega)} \\
\right) \leq c_\beta \| J_1(v) \|_{L^\infty(\Omega)} \| f'(u_1) - f'(u_2) \|_{L^\infty(\Omega)} \| J_1(v) \|_{L^\infty(\Omega)} \| f'(u_1) \|_{L^\infty(\Omega)} \\
+ \left( k_3 \beta \| J_1(v) \|_{L^\infty(\Omega)} \right) \| f'(u_1) - f'(u_2) \|_{L^\infty(\Omega)} \| v \|_{L^\infty(\Omega)} \| f'(u_1) \|_{L^\infty(\Omega)} \\
\leq c_\beta \| f'(u_1) - f'(u_2) \|_{L^\infty(\Omega)} \beta ( c_3 \| u_1 \|_{L^\infty(\Omega)} + c_4 ) \| v \|_{L^\infty(\Omega)} \\
+ \left( k_3 \beta \| c_3 \| u_1 \|_{L^\infty(\Omega)} + c_2 \right) \| f'(u_1) - f'(u_2) \|_{L^\infty(\Omega)} \| v \|_{L^\infty(\Omega)}.
\]
And the continuity of \( DF \) follows from the continuity of \( f' \). Therefore, it follows from Proposition 3.3 that \( F \) is Fréchet differentiable with continuous derivative in \( L^p(\Omega) \).

**Remark 3.3.** From Proposition 3.3, it follows that the flow generated by (2.5), given by \( (T(t)u_0)(x) = u(x, t) \), where \( u(x, t) \) is given in (2.13), is \( C^1 \) with respect to initial condition (see [20]).

4. Existence of a global attractor

We prove, in this section, the existence of a global maximal invariant compact set \( A \subset X \equiv L^p(\Omega) \) for the flow of (2.5), which attracts each bounded set of \( X \) (the global attractor, see [19] and [29]).

We recall that a set \( B \subset X \) is an absorbing set for the flow \( T(t) \) if, for any bounded set \( C \subset X \), there is a \( t_1 > 0 \) such that \( T(t)C \subset B \) for any \( t \geq t_1 \).

The following result was proven in [29].

**Theorem 4.1.** Let \( X \) be a Banach space and \( T(t) \) a semigroup on \( X \). Assume that, for every \( t \), \( T(t) = T_1(t) + T_2(t) \), where the operators \( T_1(\cdot) \) are uniformly compact for \( t \) sufficiently large, that is, for every bounded set \( B \) there exists \( t_0 \), which may depend on \( B \), such that
\[
\bigcup_{t \geq t_0} T_1(t)B
\]
is relatively compact in $X$ and $T_2(t)$ is a continuous mapping from $X$ into itself such that the following holds: For every bounded set $C \subset X$, 

$$r_c(t) = \sup_{\varphi \in C} \|T_2(t)\varphi\|_X \to 0 \quad \text{as} \quad t \to \infty.$$ 

Assume also that there exists an open set $\mathcal{U}$ and bounded subset $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}$ is absorbing in $\mathcal{U}$. Then the $\omega$-limit set of $\mathcal{B}$, $A = \omega(\mathcal{B})$, is a compact attractor which attracts the bounded sets of $\mathcal{U}$. It is the maximal bounded attractor in $\mathcal{U}$ (for the inclusion relation). Furthermore, if $\mathcal{U}$ is convex and connected, then $A$ is connected.

Lemma 4.2. Assume that (2.3) and (2.4) hold with $k_1\beta c_1 < 1$. Then, any positive number $\sigma$, the ball of radius 

$$R = (1 + \sigma) \left( \frac{k_1\beta c_2 + k_1\beta h + k_2}{1 - k_1\beta c_1} \right)$$ 

is an absorbing set for the flow $T(t)$ generated by (2.4).

Proof. If $u(\cdot, t)$ is a solution of (2.3) with initial condition $u(\cdot, 0)$ then, for $1 \leq p < \infty$,

$$\frac{d}{dt} \int_{\Omega} |u(x, t)|^p dx = \int_{\Omega} p |u(x, t)|^{p-1} \text{sgn}[u(x, t)] u_t(x, t) dx$$

$$= -p \int_{\Omega} |u(x, t)|^p dx + p \int_{\Omega} |u(x, t)|^{p-1} \text{sgn}[u(x, t)] (\beta K f(u(x, t)) + \beta h) \, dx.$$ 

But, using Hölder inequality, (2.3) and (2.4), it follows that

$$\int_{\Omega} |u(x, t)|^{p-1} \text{sgn}[u(x, t)] (\beta K f(u(x, t)) + \beta h) \, dx \leq$$

$$\leq \left( \int_{\Omega} (|u(x, t)|^{p-1})^q \, dx \right)^{\frac{1}{q}} \left( \int_{\Omega} (\beta K f(u(x, t)) + \beta h)^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \left( \int_{\Omega} |u(x, t)|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} (k_1\beta K f(u(x, t)) + \beta h + k_2)^p \, dx \right)^{\frac{1}{p}}$$

$$\leq \|u(\cdot, t)\|_{L^p(\Omega)}^{p-1} \left( k_1\beta K (f(u(\cdot, t))) \|_{L^p(\Omega)} + k_1\beta h + k_2 \|_{L^p(\Omega)} \right)$$

$$\leq \|u(\cdot, t)\|_{L^p(\Omega)}^{p-1} \left( k_1\beta \|Jh\|_{L^1(\Omega)} + k_1\beta h + k_2 \|\Omega\|^\frac{1}{p} \right)$$

$$\leq \|u(\cdot, t)\|_{L^p(\Omega)}^{p-1} \left( k_1\beta (c_1 \|u(\cdot, t)\|_{L^p(\Omega)} + c_2 \|\Omega\|^\frac{1}{p}) + (k_1\beta h + k_2) \|\Omega\|^\frac{1}{p} \right)$$

Thus, we have that

$$\frac{d}{dt} \|u(\cdot, t)\|_{L^p(\Omega)}^p \leq -p \|u(\cdot, t)\|_{L^p(\Omega)}^p + pk_1\beta c_1 \|u(\cdot, t)\|_{L^p(\Omega)}^p$$

$$+ p \left[ k_1\beta c_2 \|\Omega\|^\frac{1}{p} + (k_1\beta h + k_2) \|\Omega\|^\frac{1}{p} \right] \|u(\cdot, t)\|_{L^p(\Omega)}^{p-1}$$

$$= p \|u(\cdot, t)\|_{L^p(\Omega)}^p \left[ -1 + k_1\beta c_1 + \frac{k_1\beta c_2 + k_1\beta h + k_2}{\|u(\cdot, t)\|_{L^p(\Omega)}^{p-1}} \right].$$
Letting $\varepsilon = 1 - k_1\beta c_1$, when
$$\|u(\cdot,t)\|_{L^p(\Omega)} \geq (1 + \sigma)\left(\frac{k_1\beta c_2 + k_1\beta h + k_2}{\varepsilon}\right)^\frac{1}{p},$$
we have that
$$\frac{d}{dt}\|u(\cdot,t)\|_{L^p(\Omega)}^p \leq p\|u(\cdot,t)\|_{L^p(\Omega)}^p \left(-\varepsilon + \frac{\varepsilon}{1 + \sigma}\right) = -p\frac{\sigma}{1 + \sigma}\|u(\cdot,t)\|_{L^p(\Omega)}^p.$$ 

Therefore when $\|u(\cdot,t)\|_{L^p(\Omega)} \geq (1 + \sigma)\left(\frac{k_1\beta c_2 + k_1\beta h + k_2}{\varepsilon}\right)^\frac{1}{p},$
$$\|u(\cdot,t)\|_{L^p(\Omega)}^p \leq e^{-\frac{\sigma}{1 + \sigma}}\|u(\cdot,0)\|_{L^p(\Omega)} \leq e^{-\frac{\sigma}{1 + \sigma}}\|u(\cdot,0)\|_{L^p(\Omega)}$$
what concludes the proof. \[\square\]

The next result generalizes Theorem 3.3 of [8], Theorem 3.3 of [3] and Theorem 8 of [11].

**Theorem 4.3.** In addition of the hypotheses assumed in Lemma 4.2, suppose that (3.14) holds and lets $\|J_x\| = \sup_{x \in \Omega} \frac{\partial}{\partial x} \|J(x, \cdot)\|_{L^r(\Omega)}$. Then there exists a global attractor $A$ for the flow $T(t)$ generated by (2.5) in $L^p(\Omega)$, which is contained in the ball of radius $\rho$.

**Proof.** If $u(\cdot, t)$ is the solution of (2.5) with initial condition $u(\cdot, 0)$. For $x \in \Omega$ we have, by the variation of constants formula,
$$u(x, t) = e^{-t}u(x, 0) + \int_0^t e^{s-t}g(\beta Kf(u)(x, s) + \beta h)ds. \quad (4.19)$$

Consider
$$T_1(t)u(x) = e^{-t}u(x, 0)$$
and
$$T_2(t)u(x) = \int_0^t e^{s-t}g(\beta Kf(u)(x, s) + \beta h)ds.$$ 

Then, assuming that $u(\cdot, 0) \in C$, where $C$ is a bounded set in $L^p(\Omega)$, (for example $B(0, \rho)$), it follows that
$$\|T_1(t)u\|_{L^2} \quad \text{uniformly in } u.$$

Also, using (4.19), we have that $\|u(\cdot, t)\|_{L^p(\Omega)} \leq L$, for $t \geq 0$, where $L = \max \left\{\rho, \frac{2(k_1\beta c_3 + k_1\beta h + k_2)}{k_1\beta c_1}\right\}$. Therefore, for $t \geq 0$, we have that
$$\frac{\partial T_2(t)u(x)}{\partial x} = \int_0^t e^{s-t}g(\beta Kf(u)(x, s) + \beta h)ds$$
$$= \beta \int_0^t e^{s-t}g'(\beta Kf(u)(x, s) + \beta h)\frac{\partial Kf(u)}{\partial x}(x, s)ds.$$
Thus, using (3.14) and (2.9), we obtain
\[
\left\| \frac{\partial T_2(t)u}{\partial x} \right\|_{L^p(\Omega)} \leq \int_0^t e^{s-t} \| g'(\beta Kf(u)(\cdot, s) + \beta h) \beta \frac{\partial Kf(u)}{\partial x}(\cdot, s) \|_{L^p(\Omega)} ds
\]
\[
\leq \int_0^t e^{s-t} \left[ k_3 \beta \| g(u(\cdot, s)) \|_{L^p(\Omega)} \right. + k_3 \beta + k_4 \beta \| J_x \|_1 (c_1 \| u(\cdot, s) \|_{L^p(\Omega)} + c_2 |\Omega|^{\beta}) ds
\]
\[
+ k_3 \beta h + k_4 \beta \| J_x \|_1 (c_1 \| u(\cdot, s) \|_{L^p(\Omega)} + c_2 |\Omega|^{\beta}) ds
\]
\[
\leq \left[ k_3 \beta (c_1 \| u(\cdot, s) \|_{L^p(\Omega)} + c_2 |\Omega|^{\beta}) \right]
+ k_3 \beta h + k_4 \beta \| J_x \|_1 (c_1 \| u(\cdot, s) \|_{L^p(\Omega)} + c_2 |\Omega|^{\beta})
\]
\[
\leq \left[ k_3 \beta (c_1 + c_2 |\Omega|^{\beta}) + k_3 \beta h + k_4 \beta \| J_x \|_1 (c_1 L + c_2 |\Omega|^{\beta}) \right].
\]

It follows that, for \( t > 0 \) and any \( u \in \mathcal{C} \), the value of \( \left\| \frac{\partial T_2(t)u}{\partial x} \right\|_{L^p(\Omega)} \) is bounded by a constant (independent of \( t \) and \( u \)). Thus, for all \( u \in \mathcal{C} \), we have that \( T_2(t)u \) belongs to a ball of \( W^{1,2}(\Omega) \). From Sobolev’s Imbedding Theorem, it follows that
\[
\bigcup_{t \geq 0} T_2(t)\mathcal{C}
\]
is relatively compact. Therefore, the result follows from Theorem 4.1, the attractor \( \mathcal{A} \) being the set \( \omega \)-limit of the ball \( B(0, R) \).

5. Comparison and Boundedness Results

In this section we prove a comparison result that generalizes the Theorem 2.7 of [25] (where \( g = \tanh \), \( f(x) = x, \forall x \in \mathbb{R} \) and \( h = 0 \)) and Theorem 4.2 of [8] (where \( f(x) = x, \forall x \in \mathbb{R} \)).

**Definition 5.1.** A function \( v(x, t) \) is a subsolution of the Cauchy problem for (2.50) with initial condition \( u(\cdot, 0) \) if \( v(x, 0) \leq u(x, 0) \) for almost all \( x \in \Omega \), \( v \) is continuously differentiable with respect to \( t \) and satisfies
\[
\frac{\partial v(x, t)}{\partial t} \leq -v(x, t) + g(\beta Kf(v)(x, t) + \beta h),
\]
almost everywhere (a.e.).

Analogously, a function \( V(x, t) \) is a super solution if has the same regularity properties as above, satisfies (5.20) with reversed inequality and \( V(x, 0) \geq u(x, 0) \) for almost all \( x \in \Omega \).

**Theorem 5.2.** In addition to the hypotheses of Theorem 4.3, assume that the functions \( g \) and \( f \) are monotonic and Lipschitz continuous on bounded with Lipschitz’s constants \( N \) and \( M \), respectively. Let \( v(u, t) \), \( [V(u, t)] \) be a subsolution [super solution] of the Cauchy problem of (2.50) with initial condition \( u(\cdot, 0) \). Then
\[
v(x, t) \leq u(x, t) \leq V(x, t), \text{ a.e.}
\]
Proof. Define the operator $G$ on $L^\infty(\Omega \times [0,T])$ by

$$G(w)(x,t) = e^{-t}w(x,0) + \int_0^t e^{-(t-s)}g(\beta(Kf(w)(x,s) + h))ds.$$ 

Then $(G(w))(x,0) = w(x,0)$. Also, since $f$ and $g$ are monotonic, it follows that $G$ is monotonic, that is, for any $w_1, w_2 \in L^\infty(\Omega \times [0,T])$ with $w_1 \geq w_2$ (a.e. in $\Omega \times [0,T]$), we have $G(w_1) \geq G(w_2)$ (a.e. in $\Omega \times [0,T]$).

From (2.4), we obtain

$$|G(w)(x,t)| \leq e^{-t}|w(x,0)| + \int_0^t e^{-(t-s)}|g(\beta Kf(w)(x,s) + \beta h)|ds$$

$$\leq e^{-t}|w(x,0)| + \int_0^t e^{-(t-s)}[k_1|\beta Kf(w)(x,s) + \beta h| + k_2]ds$$

$$\leq e^{-t}|w(x,0)| + \int_0^t e^{-(t-s)}k_1|\beta Kf(w)(x,s)|ds + \int_0^t e^{-(t-s)}(k_1\beta h + k_2)ds.$$ 

Since $|Kf(w)(x,s)| \leq \|f\|_1 \|f(w)\|_{\infty} \leq k_1\|w\|_{\infty} + k_2$ a.e. in $\Omega \times [0,T]$, we obtain

$$\|G(w)\|_{\infty} \leq e^{-t}\|w(\cdot,0)\|_{\infty} + \int_0^t e^{-(t-s)}k_1\beta(k_1\|w\|_{\infty} + k_2)ds$$

$$+ \int_0^t e^{-(t-s)}(k_1\beta h + k_2)ds$$

$$\leq \|w\|_{\infty} + k_1\beta(k_1\|w\|_{\infty} + k_2) + (k_1\beta h + k_2).$$

Therefore $G : L^\infty(\Omega \times [0,T]) \to L^\infty(\Omega \times [0,T])$.

Furthermore, if $\beta NMT < 1$, $G$ is a contraction in any subset of functions of $L^\infty(\Omega \times [0,T])$ with the same values at $t = 0$. In fact

$$|G(w_1)(x,t) - G(w_2)(x,t)| = \left| \int_0^t e^{-(t-s)}[g(\beta(Kf(w_1)(x,s) + \beta h) - g(\beta(Kf(w_2)(x,s) + \beta h)]ds \right|$$

$$\leq \int_0^t e^{-(t-s)}N\beta|Kf(w_1)(x,s) - Kf(w_2)(x,s)|ds$$

$$\leq \int_0^t e^{-(t-s)}N\beta|f(w_1) - f(w_2)|ds$$

$$\leq \int_0^t e^{-(t-s)}N\beta|f(w_1) - f(w_2)|_{\infty}ds$$

$$= N\beta T\|f(w_1) - f(w_2)\|_{\infty} \int_0^t e^{-(t-s)}ds$$

$$\leq N\beta MT\|w_1 - w_2\|_{\infty},$$
Remark 5.3. If we add the hypothesis

Therefore, if $\beta NMT < 1$, $G$ is a contraction. Thus, if $u(x, t)$ is a solution of (2.5) with $u^0 = u(x, 0)$, we have

\[
u = \lim_{n \to \infty} G^n(u^0)
\]
on $L^\infty(\Omega \times [0, T])$. The same holds for a solution $\tilde{u}$ with $\tilde{u}^0 = \tilde{u}(x, 0)$. If $\tilde{u}^0 \leq u^0$ a.e., with $g$ and $f$ monotonic, it follows that

\[
G^n(\tilde{u}^0) \leq G^n(u^0), \text{ a.e.}
\]

Now, if $v$ is a subsolution of (2.5), it’s easy to see that

\[
v(x, t) \leq e^{-t}v(x, 0) + \int_0^t e^{-(t-s)}g(\beta(Kf(v)(s) + h))ds, \text{ a.e.}
\]

Therefore $v(x, t) \leq G(v)(x, t)$, a.e., and since $g$ and $f$ are monotonic, it follows that $v(w, t) \leq G^n(v)(x, t)$ a.e. Thus, $v(x, t) \leq z(x, t)$, a.e., where

\[
z = \lim_{n \to \infty} G^{n+1}(v).
\]

Now, from the continuity of $G$, it follows that

\[
G(z) = G\left(\lim_{n \to \infty} G^n(v)\right) = \lim_{n \to \infty} G^{n+1}(v) = z.
\]

Therefore $z$ is a fixed point of $G$, that is, $z$ is a solution of (2.5) in $\Omega \times [0, T]$ with initial condition $z(\cdot, 0) = v(\cdot, 0)$. Thus, if $z(\cdot, 0) \leq u(\cdot, 0)$, a.e., then

\[
v \leq z \leq u, \text{ a.e. in } \Omega \times [0, T],
\]

where $u$ is the solution of (2.5) with initial condition $u(\cdot, 0)$. If $V(x, t)$ is a super solution, we obtain, by the same arguments

\[
u(x, t) \leq u(x, t) \leq V(x, t), \text{ a.e.}
\]
in $\Omega \times [0, T]$.

Since the estimates above do not depend on the initial condition, we may extend the result to $[T, 2T]$ and, by iteration, we can complete the proof of the theorem. \hfill \Box

**Remark 5.3.** If we add the hypothesis $g(x) < \rho$, the comparison result holds in the ball $B = \{L^\infty(\Omega \times [0, T]), \| \cdot \| \infty \leq \rho\}$.

In fact, it is enough to prove that $G|_B : B \to B$. But

\[
|(G|_B(w))(x, t)| \leq e^{-t}w(x, 0) + \rho \int_0^t e^{-(t-s)}ds.
\]

Hence

\[
\|G|_B(w)\|_\infty \leq e^{-t}\|w\|_\infty + \rho \int_0^t e^{-(t-s)}ds \leq e^{-t} + \rho \int_0^t e^{-(t-s)}ds = \rho.
\]

Therefore, $G|_B(w) \in B$.

**Theorem 5.4.** In the same conditions from Theorem 4.3, we have that the attractor $A$ belongs to the ball $\| \cdot \|_\infty \leq \rho$ in $L^\infty(\Omega)$, where $\rho = k_1\beta\|J\|_{q_1}R + k_1\beta\|J\|_{q_2}\|\Omega\|_1^\pm + k_1\beta h + k_2$. 
Proof. From Theorem 4.3 the attractor is contained in the ball $B[0, \rho]$ in $L^p(\Omega)$.

Let $u(x, t)$ be a solution of (2.5) in $A$. Then, for $x \in \Omega$, by the variation of constants formula

$$u(x, t) = e^{-(t-t_0)}u(x, t_0) + \int_{t_0}^t e^{-(t-s)}g(\beta Kf(u(x, s) + \beta h)ds.$$

Since $\|u(\cdot, t)\|_{L^p(\Omega)} \leq R$ for all $u \in A$, we obtain for all $(x, t) \in \Omega \times \mathbb{R}^+$ letting $t_0 \to -\infty$

$$u(x, t) = \int_{-\infty}^t e^{-(t-s)}g(\beta Kf(u(x, s) + \beta h)ds,$$

where the equality above is in the sense of $L^p(\Omega)$. Thus, using (2.3), we have

$$|u(x, t)| \leq \int_{-\infty}^t e^{-(t-s)}|g(\beta Kf(u(x, s) + \beta h)|ds$$

$$\leq \int_{-\infty}^t e^{-(t-s)}|k_1 \beta |Kf(u(x, t)) + \beta h| + k_2|ds$$

$$\leq \int_{-\infty}^t e^{-(t-s)}|k_1 \beta \|J\|_{q}\|f(u(\cdot, t))\|_{L^p(\Omega)} + k_1 \beta h + k_2|ds$$

$$\leq \int_{-\infty}^t e^{-(t-s)}|k_1 \beta \|J\|_{q}(c_1 \|u(\cdot, t)\|_{L^p(\Omega)} + c_2|\Omega|^{1/p}) + k_1 \beta h + k_2|ds$$

$$\leq \int_{-\infty}^t e^{-(t-s)}|k_1 \beta \|J\|_{q}(c_1 R + c_2|\Omega|^{1/p}) + k_1 \beta h + k_2|ds$$

$$\leq \int_{-\infty}^t \rho e^{-(t-s)}ds.$$

Therefore $\|u(\cdot, t)\|_{\infty} \leq \rho$, as claimed \qed

6. Existence of a Lyapunov’s functional

In this section we exhibit a continuous “Lyapunov’s functional” for the flow of (2.5), restricted to the ball of radius \(\rho\) in $L^\infty(\Omega)$, concluding that this flow is gradient, in the sense of [19].

Initially, we claim that $\{L^p(\Omega), \| \cdot \| \leq \rho\}$ is an invariant set for the flow generated by (2.5).\]

In fact, let

$$u(x, t) = e^{-t}u(x, 0) + \int_0^t e^{-(t-s)}g(\beta Kf(u(x, s)) + \beta h)ds$$
be the solution of (2.5) with initial condition \( u(\cdot, 0) \in \{ L^2(\Omega), \| \cdot \|_\infty \leq \rho \} \). Then

\[
|u(x, t)| \leq e^{-t} |u(x, 0)| + \int_0^t e^{-(t-s)} |g(\beta Kf(u(x, s)) + \beta h)| ds
\]

\[
\leq e^{-t} |u(x, 0)| + \int_0^t e^{-(t-s)} |k_1 \beta |Kf(u(x, t)) + \beta h| + k_2| ds
\]

\[
\leq e^{-t} |u(x, 0)| + \int_0^t e^{-(t-s)} |k_1 \beta \| Kf \|_q \| f(u(\cdot, t)) \|_{L^p(\Omega)} + k_1 \beta h + k_2| ds
\]

\[
\leq e^{-t} |u(x, 0)| + \int_0^t e^{-(t-s)} |k_1 \beta \| Kf \|_q (c_1 \| u(\cdot, t) \|_{L^p(\Omega)} + c_2 |\Omega|^{1/p}) + k_1 \beta h + k_2| ds
\]

Whence,

\[
\| u(\cdot, t) \|_\infty \leq e^{-t} \| u(\cdot, 0) \|_\infty + \rho \int_0^t e^{-(t-s)} ds
\]

\[
\leq e^{-t} \rho + \rho \int_0^t e^{-(t-s)} ds
\]

\[
= \rho.
\]

For to exhibit a continuous “Lyapunov’s functional” for the flow of (2.5), we assume that the functions \( f \) and \( g \) satisfy the following conditions:

\[
0 < |g(x)| < \rho, \quad \forall x \in \mathbb{R},
\]

(6.21)

the function \( g^{-1} \) is continuous in \( ]-\rho, \rho[ \) and the function

\[
\theta(m) = \frac{1}{2} f(m)^2 - h f(m) - \beta^{-1} i(m), \quad m \in [-\rho, \rho],
\]

(6.22)

where \( i \) is defined by

\[
i(m) = - \int_0^m g^{-1}(f^{-1}(s)) ds, \quad m \in [-\rho, \rho],
\]

has a global minimum \( \bar{m} \) in \( ]-\rho, \rho[ \).

Note that if (6.21) holds, it follows that (2.5) holds with \( k_1 = 0 \) and \( k_2 = \rho \).

Motivated by functionals that appear in \([8, 12, 14, 22]\) and \([25]\), we define the functional \( \mathcal{F} : \{ L^p(\Omega), \| u \|_\infty \leq \rho \} \rightarrow \mathbb{R} \) by

\[
\mathcal{F}(u) = \int_\Omega [\theta(u(x)) - \theta(m)] dx + \frac{1}{4} \int \int \Omega J(x, y)[f(u(x)) - f(u(y))]^2 dxdy,
\]

(6.23)

where \( \theta \) is given in (6.22), which has been adapted from functions considered in \([25, 8]\).
Note that the functional in (6.23) is defined in the whole space \( L^p(\Omega), \|u\|_\infty \leq \rho \). Furthermore, using the hypotheses on \( f \) and \( g \) and Lebesgue’s Dominated Convergence Theorem, we obtain the following result:

**Theorem 6.1.** In addition to the hypotheses of Theorem 4.3 assume that the hypotheses established in (6.24) and (6.25) hold and that \( f \) is an equilibrium of (2.5). Therefore, we can derive under the integration sign obtaining

\[
\frac{d}{dt}\mathcal{F}(u(\cdot,t)) = -\mathcal{I}(u(\cdot,t)) \leq 0,
\]

where, for any \( u \in L^p(\Omega) \) with \( \|u\|_\infty \leq \rho \),

\[
\mathcal{I}(u(\cdot)) = \int_\Omega [K(f(u(x))) + h - \beta^{-1} g^{-1}(u(x))] [g(\beta K(f(u(x))) + \beta h) - u(x)] f'(u(x)) \, dx.
\]

Furthermore, the integrand in \( \mathcal{I}(u(\cdot)) \) is a non negative function and, \( u \) is a critical point of \( \mathcal{F} \) if only if \( u \) is an equilibrium of (2.5).

**Proof.** From hypotheses on \( g \) and \( f \), it follows that \( \mathcal{F}(u(\cdot,t)) \) is well defined for all \( t \geq 0 \). We assume first that, given \( t > 0 \), there exists \( \varepsilon > 0 \) such that \( \|u(\cdot,s)\|_\infty \leq \rho - \varepsilon \), and, for \( s \in \Delta \) where \( \Delta \) is a closed finite interval containing \( t \). For \( s \in \Delta \) we write

\[
\mathcal{F}(u(\cdot,s)) = \int_\Omega \phi(x,s) \, dx \quad \text{and} \quad \mathcal{I}(u(\cdot,s)) = \int_\Omega u(x,s) \, dx.
\]

As

\[
\frac{\partial \phi(x,s)}{\partial s} = \left[ -f(u(x,s)) - h + \beta^{-1} g^{-1}(u(x,s)) \right] f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} + \frac{1}{2} \int_\Omega J(x,y) \left[ f(u(x,s)) - f(u(y,s)) \right] \left[ f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} - f'(u(y,s)) \frac{\partial u(y,s)}{\partial s} \right] \, dy,
\]

the hypotheses on \( g \), \( f \) and \( f' \) imply that \( \frac{\partial \phi(x,s)}{\partial s} \) is almost everywhere continuous and bounded in \( x \) for \( s \in \Delta \). Thus

\[
\sup_{s \in \Delta} \left\| \frac{\partial \phi(x,s)}{\partial s} \right\|_{L^1} < \infty.
\]

Therefore, we can derive under the integration sign obtaining

\[
\frac{d}{ds}\mathcal{F}(u(\cdot,s)) = \int_\Omega \left[ -f(u(x,s)) - h + \beta^{-1} g^{-1}(u(x,s)) \right] f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} \, dx
\]

\[
+ \frac{1}{2} \int_\Omega \int_\Omega J(x,y) \left[ f(u(x,s)) - f(u(y,s)) \right] \left[ f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} - f'(u(y,s)) \frac{\partial u(y,s)}{\partial s} \right] \, dx \, dy.
\]
But

$$\int_{\Omega} \int_{\Omega} J(x,y)[f(u(x,s)) - f(u(y,s))] \left[ f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} - f'(u(y,s)) \frac{\partial u(y,s)}{\partial s} \right] dxdy$$

$$= \int_{\Omega} \int_{\Omega} J(x,y)f(u(x,s)) f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} dxdy$$

$$- \int_{\Omega} \int_{\Omega} J(x,y)f(u(x,s)) f'(u(y,s)) \frac{\partial u(y,s)}{\partial s} dxdy$$

$$- \int_{\Omega} \int_{\Omega} J(x,y)f(u(y,s)) f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} dxdy$$

$$+ \int_{\Omega} \int_{\Omega} J(x,y)f(u(y,s)) f'(u(y,s)) \frac{\partial u(y,s)}{\partial s} dxdy$$

$$= 2 \int_{\Omega} \int_{\Omega} J(x,y)f(u(x,s)) f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} dxdy$$

$$- 2 \int_{\Omega} \int_{\Omega} J(x,y)f(u(y,s)) f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} dxdy$$

$$= 2 \int_{\Omega} \left( \int_{\Omega} J(x,y)dy \right) f(u(x,s)) f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} dx$$

$$- 2 \int_{\Omega} \left( \int_{\Omega} J(x,y)f(u(y,s))dy \right) f'(u(x,s)) \frac{\partial u(x,s)}{\partial s} dx.$$

Using the fact that

$$\int_{\Omega} J(x,y)dy = \int_{\Omega} J(x,y)dx = 1.$$
it follows that
\[ \frac{d}{ds} \mathcal{F}(u(\cdot, s)) = \int_{\Omega} \left( -f(u(x, s)) - h + \beta^{-1}g^{-1}(u(x, s)) \right) f'(u(x, s)) \frac{\partial u(x, s)}{\partial s} dx \]
\[ + \int_{\Omega} [f(u(x, s)) - Kf(u(x, s))] f'(u(x, s)) \frac{\partial u(x, s)}{\partial s} dx \]
\[ = \int_{\Omega} \left( -f(u(x, s)) - h + \beta^{-1}g^{-1}(u(x, s)) + f(u(x, s)) \right) \]
\[ - Kf(u(x, s)) f'(u(x, s)) \frac{\partial u(x, s)}{\partial s} dx \]
\[ = - \int_{\Omega} \left[ Kf(u(x, s)) + h - \beta^{-1}g^{-1}(u(x, s)) \right] \left[ -u(x, s) \right] \]
\[ + g(\beta Kf(u(x, s)) + \beta h) f'(u(x, s)) dx \]
\[ = -\mathcal{I}(u(\cdot, s)). \]

This proves the first part of the theorem with the additional hypothesis that \( \|u(\cdot, s)\|_{\infty} \leq \rho - \varepsilon \), for \( s \in \Delta \) and some \( \varepsilon > 0 \), where \( \Delta \) is a closed finite interval containing \( t \).

We claim that this hypothesis actually holds for all \( t > 0 \). In fact, let \( \lambda(x, t) \) be the solution of (2.4) such that \( \lambda(x, 0) = \rho \) for any \( x \in \Omega \). Then \( \lambda(x, t) = \lambda(t) \), where
\[ \frac{d\lambda}{dt} = -\lambda(t) + g(\beta(\lambda(t) + h)). \]

Since \( |g(x)| < \rho \), \( \forall x \in \mathbb{R} \), it follows easily that \( \lambda(t) < \rho \) for any \( t > 0 \). As \( u(x, 0) \leq \rho \), we obtain by the Comparison Theorem
\[ u(x, t) \leq \lambda(t) < \rho, \]
for almost every \( x \in \Omega \) and \( t > 0 \). Repeating the same argument, starting from inequality \( u(x, 0) \geq -\rho \), for almost every \( x \in \Omega \), we obtain \( u(x, t) \geq -\lambda(t) > -\rho \), and thus
\[ \|u(\cdot, t)\|_{\infty} \leq \lambda(t) < \rho, \forall \ t > 0 \]
and the claim follows by continuity.

To conclude the proof, it is enough to show that \( u \) is a critical point of \( \mathcal{F} \) if and only if \( u \) is an equilibrium of (2.5). For this, let \( u(x) \) be a critical point of the functional \( \mathcal{F} \), then \( \mathcal{I}(u(\cdot)) = 0 \). Since the integrand is non negative almost everywhere, it follows that
\[ [(Kf(u)(x)) + h - \beta^{-1}g^{-1}(u(x))] f'(u(x))[g(\beta(Kf(u)(x) + h)) - u(x)] = 0 \]
almost everywhere. Since \( f'(u(x)) > 0 \), for all \( x \in \mathbb{R} \), we have that
\[ [(Kf(u)(x)) + h - \beta^{-1}g^{-1}(u(x))] [g(\beta(Kf(u)(x) + h)) - u(x)] = 0 \]
almost everywhere. But the annihilation of any of these factors implies that
\[ g(\beta Kf(u)(x) + \beta h) = u(x). \]
Reciprocally, if \( u \) is a equilibrium of (2.5), it is easy to see that \( \mathcal{I}(u(\cdot)) = 0 \). \( \square \)
As a immediate consequence of the existence of the functional $F$, we obtain the following result.

**Corollary 6.3.** Under the same hypotheses of Theorem 6.2, there are no non trivial recurrent points under the flow of (2.5).

**Remark 6.4.** The integrand in the functional $F$ above is always non negative since $J$ is positive and $m$ is a global minim of $\theta$. Thus, $F$ is lower bounded.

We recall that a $C^r$-semigroup, $T(t)$, is gradient if each bounded positive orbit is precompact and there exists a Lyapunov’s Functional for $T(t)$ (see [19]).

**Proposition 6.5.** Assume the same hypotheses of Theorem 6.2. Then the flow generated by equation (2.5) is gradient.

**Proof.** The precompacity of the orbits follows from the existence of the global attractor (see Theorem 4.3). From Theorems 6.1 and 6.2 and Remark 6.4 we have existence of a continuous Lyapunov’s functional. □

From Proposition 6.5, we have the following characterization of the attractor (see [19] - Theorem 3.8.5).

**Theorem 6.6.** Assume the same assumptions of Proposition 6.5. Then the attractor $A$ is the unstable set of the equilibrium point set of $T(t)$, that is, $A = W^u(E)$.

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