Koszul duality between Higgs and Coulomb categories $\mathcal{O}$

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Abstract. We prove a Koszul duality theorem between the category of weight modules over the quantized Coulomb branch (as defined by Braverman, Finkelberg and Nakajima) attached to a group $G$ and representation $V$ and a category of $G$-equivariant D-modules on the vector space $V$. This is proven by relating both categories to an explicit, combinatorially presented category.

These categories are related to generalized categories $\mathcal{O}$ for symplectic singularities. Letting $\mathcal{O}_{\text{Coulomb}}$ and $\mathcal{O}_{\text{Higgs}}$ be these categories for the Coulomb and Higgs branches associated to $V$ and $G$, we obtain a functor $\mathcal{O}_{\text{Coulomb}}^! \to \mathcal{O}_{\text{Higgs}}$ from the Koszul dual of one to the other. This functor is an equivalence in the special cases where the hyperkähler quotient of $T^*V$ by $G$ is a Nakajima quiver variety or smooth hypertoric variety. We also show that this equivalence intertwines so-called twisting and shuffling functors. This confirms the most important components of the symplectic duality conjecture of Braden, Licata, Proudfoot and the author.

1. Introduction

Let $V$ be a complex vector space, and let $G$ be a connected reductive algebraic group with a fixed faithful linear action on $V$. Attached to this data, we have two interesting spaces, which physicists call the Higgs and Coulomb branches (of the associated 3-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory).

- The Higgs branch is well-known to mathematicians: it is given by an algebraic symplectic reduction of the cotangent bundle $T^*V$. That is, we have

$$\mathcal{M}_H := \mu^{-1}(0)/G = \text{Spec}(C[\mu^{-1}(0)]^G)$$

where $\mu : T^*V \to \mathfrak{g}$ is the moment map.

- The Coulomb branch has only been precisely defined in a recent series of papers by Nakajima, Braverman and Finkelberg. It is defined as the spectrum of a ring constructed as a convolution algebra in the homology of the affine Grassmannian. The choice of representation $V$ is incorporated as certain “quantum corrections” to convolution in homology, which are kept track of by an auxiliary vector bundle. To readers unhappy with the terms that appear in the sentences above: in this paper, we will use a purely algebraic description of the Coulomb branch; the geometric description given above will be only used to

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show that this algebraic presentation is correct, so readers can safely set the affine Grassmannian to one side if they desire.

A conjecture of Braden, Licata, Proudfoot and the author suggests a surprising relationship between these spaces: they should be symplectic dual [BLPW]. This conjecture requires a number of different geometric and representation theoretic properties, the most important of which is a Koszul duality between generalizations of category $\mathcal{O}$ over quantizations of these varieties. The existence of such a duality has been proven in several special cases (see [BLPW12, Webb]) but in this paper, we will give a general construction of this Koszul duality.

First, let us be a bit more precise about what we mean by Koszul duality. For any algebra $A$ over a field $\mathbb{k}$ graded by the non-negative integers with $A_0$ finite dimensional and semi-simple, we can define a Koszul dual $A^!$ which is a quadratic algebra with the same properties. By [MOS09, Thm. 30], we have that $A \cong (A^!)^!$ if and only if $A$ is Koszul in the usual sense. For a graded category $\mathcal{C}$ equivalent to $A$-gmod for $A$ as above, the category $\mathcal{C}^! \cong A^!$-gmod only depends on $\mathcal{C}$ up to canonical equivalence.

In order to construct category $\mathcal{O}$'s, we need to choose auxiliary data, which determine finiteness conditions: we must choose a flavor $\phi$ (a $\mathbb{C}^*$-action on $\mathcal{M}_H$ with weight 1 on the symplectic form and commuting with the $\mathbb{C}^*$-action induced by scaling), and a stability parameter $\xi \in (\mathfrak{g}^*)^G$. Note that the choice of $\xi$ allows us to define the GIT quotient $\mathcal{M}_{H,\xi}$ with $\xi$ as stability condition. Taking the unique closed orbit in the closure of a semi-stable orbit defines a map $\mathcal{M}_{H,\xi} \to \mathcal{M}_H$. In many cases, this is a resolution of singularities, but $\mathcal{M}_{H,\xi}$ may not be smooth, or may be a resolution of a subvariety of $\mathcal{M}_H$. The variety $\mathcal{M}_{H,\xi}$ has a natural quantization obtained from the Hamiltonian reduction of microlocal differential operators on $T^*V$ (for the usual moment map sending $X \in \mathfrak{g}$ to the action vector field $X_V$), as defined in [BLPW, §3.4].

Associated to the data of $(G, V, \phi, \xi)$, we have two versions of category $\mathcal{O}$:

1. We let $\mathcal{O}_{\text{Higgs}}$ be the geometric category $\mathcal{O}$ over the quantized structure sheaf on $\mathcal{M}_{H,\xi}$ discussed above, associated to the flavor $\phi$.
2. We let $\mathcal{O}_{\text{Coulomb}}$ be the algebraic category $\mathcal{O}$ for the quantization of $\mathcal{M}_C$ defined by the flavor $\phi$ with integral weights. The element $\xi$ induces an inner grading on this algebra which we use to define the category $\mathcal{O}$.

While there is a small asymmetry here since one of these categories is a category of sheaves, and the other a category of modules, the difference is smaller than it may appear. By [BLPW, Cor. 3.19], we can compare algebraic and geometric category $\mathcal{O}$'s and express $\mathcal{O}_{\text{Higgs}}$ as an algebraic category $\mathcal{O}$ at the cost of requiring more care regarding parameters. The category $\mathcal{O}_{\text{Higgs}}$ has an intrinsically defined graded lift $\tilde{\mathcal{O}}_{\text{Higgs}}$, which uses the category of mixed Hodge modules on $V$; the category $\mathcal{O}_{\text{Coulomb}}$ has a graded lift which we'll give an explicit algebraic definition of below.
**Theorem A** There is a functor $\tilde{\mathcal{O}}_{\text{Coulomb}} \rightarrow \tilde{\mathcal{O}}_{\text{Higgs}}$. If $\mathcal{M}_H$ is a Nakajima quiver variety or smooth hypertoric variety, then this functor is an equivalence.

There is a general geometric property (†) which assures the equivalences above. We expect this holds in all cases where $\mathcal{M}_H$ is smooth and is proven in the quiver and smooth hypertoric cases in [Webb], but at the moment, we lack general tools to prove it in full generality. For hypertoric varieties, Theorem A is proven in [BLPW12]. For the quiver cases, the connection to Coulomb branches was only recently made precise, so this version of the theorem was not proved before, but the results of [SVV, Webb] were very suggestive for the affine type A case. Since the case of finite-type quiver varieties is the most novel and interesting case of this result, we’ll discuss it in more detail in Section 4.4.

In certain other cases, such as non-smooth hypertoric varieties, this functor is an equivalence onto a block of $\mathcal{O}_{\text{Higgs}}$. One can also strengthen this theorem to include the case where the flavor $\phi$ is a vector field which does not integrate to a $\mathbb{C}^*$ action or we allow non-integral weights. In this case, we have an analogous functor from $\tilde{\mathcal{O}}_{\text{Coulomb}}$ to the category $\mathcal{O}$ attached to a Higgs branch, but one associated to a subspace of $V$ as a representation over a Levi of $G$. This phenomenon is a generalization of the theorem proved in [Webe, Webc] relating blocks of the Cherednik category $\mathcal{O}$ to weighted KLR algebras (see also Section 4.4).

This result depends on an explicit calculation. For arbitrary $(G, V, \phi, \xi)$, we give two explicit presentations of the endomorphisms of the projective generators in $\mathcal{O}_{\text{Coulomb}}$; one of these is more natural from a geometric perspective, but the other has the advantage of being graded, and thus allowing us to define the the graded lift $\tilde{\mathcal{O}}_{\text{Coulomb}}$. After this paper circulated as a preprint, H. Nakajima pointed out to us that the connection between these presentations has a geometric explanation, using the concentration map to the fixed points of a complexified cocharacter, as in the work of Varagnolo and Vasserot [VV10, §2], which concerns the case of the adjoint representation in connection with double affine Hecke algebras. This will be explained in more detail in forthcoming work of his [Naka].

This second presentation also appears naturally in the Ext algebra of certain semi-simple $G$-equivariant D-modules on $V$, which makes the functor $\tilde{\mathcal{O}}_{\text{Coulomb}} \rightarrow \tilde{\mathcal{O}}_{\text{Higgs}}$ manifest. If instead of category $\mathcal{O}$, we consider the category $\mathcal{W}_{\text{Coulomb}}$ of all integral weight modules, which has a graded lift $\tilde{\mathcal{W}}_{\text{Coulomb}}$ defined using the same presentation. We obtain a fully faithful functor $\tilde{\mathcal{W}}_{\text{Coulomb}} \rightarrow \mathcal{D}(V/G)$-mod to the category of strongly $G$-equivariant D-modules on $V$, independent of any properties of $V$ or $G$. The functor $\tilde{\mathcal{O}}_{\text{Coulomb}} \rightarrow \tilde{\mathcal{O}}_{\text{Higgs}}$ is induced by this functor, and the hypertoric or quiver hypothesis is needed to assure that the quotient functor from $\mathcal{D}(V/G)$-mod to modules over the quantization of $\mathcal{M}_{H,\xi}$ has the correct properties.

Thus, Theorem A can be strengthened to not just give an equivalence between these categories, but in fact a combinatorial description of both of them. The algebras that
appear are an interesting generalization of (weighted) KLR algebras. Considering the richness of the theory developed around KLR algebras, there is reason to think these new algebras will also prove quite interesting from the perspective of combinatorial representation theory.

Particularly interesting context in which consider these is when the Coulomb branch is considered over a field of characteristic $p$. In this case, there is a natural relationship between quantizations, tilting bundles and coherent sheaves, which we will consider in more detail in future work.

Because of the nature of our proof of Theorem A, it extends easily to show that these equivalences are compatible with certain natural autoequivalences of derived categories, called shuffling and twisting functors. See [BLPW, §8] for more on these functors.

**Theorem B** Under the hypothesis (†), the functor of Theorem A induces an equivalence of graded derived categories $D^b(\mathcal{O}_{\text{Coulomb}}) \to D^b(\mathcal{O}_{\text{Higgs}})$ which intertwines twisting functors with shuffling functors and vice versa.

This verifies two of the most important predictions of the conjecture that Higgs and Coulomb branches of a single theory are symplectic dual to each other in the sense of [BLPW, Def. 10.1]; it remains to confirm the more geometric aspects of this duality, such as a bijection between special strata.

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2. The Higgs side

Let $V$ be a complex vector space, and let $G$ be a connected reductive algebraic group with a fixed faithful linear action on $V$ with no trivial summands. Let $H = \text{Aut}_G(V)$ and let $Z = H \cap G = Z(G)$.

Let $\mathbb{T}$ be a copy of $\mathbb{C}^*$ acting on $T^*V \cong V \oplus V^*$, commuting with the action of $G$, and acting with weight 1 on the symplectic form $\Omega$. Note that this means we have a perfect pairing between the $k$ weight space on $V$ and the $-k - 1$ weight space on $V^*$; this action is necessarily faithful. Let $\tilde{G}$ be the subgroup in $GL(T^*V)$ generated by $\mathbb{T}$ and $G$.

Our constructions will only depend on the representation of $\tilde{G}$ on $T^*V$, and not on the choice of invariant Lagrangian subspace $V$. However, making a distinguished choice will be useful moving forward. The reader might prefer to consider symplectic representations of $G$ with a commuting action of $\mathbb{T}$ that has weight 1 on the symplectic form, without a choice of Lagrangian subspace. However, in this situation, one will
always exist, since the non-negative weight spaces for the action of $\mathbb{T}$ form a $G$-invariant Lagrangian subspace.

**Remark 2.1.** This depends very sensitively on the fact that $\mathbb{T}$ has weight 1 on the symplectic form. Every symplectic representation has a commuting $\mathbb{T}$ action with all weights negative (the inverse scalar multiplication) which has weight 2 on $\Omega$.

The group $\mathbb{T}$ acts naturally on the Higgs branch $\mathcal{M}_{H,\gamma}$ for any character $\gamma$. If $\mathcal{M}_{H,\gamma}$ is smooth then considering the action of $\mathbb{T}$ on the tangent space at any point of $\mathcal{M}_{H,\gamma}^T$, we see that the fixed subspace $\mathcal{M}_{H,\gamma}^T$ is isotropic (in the Poisson sense), and the set

$$\mathcal{M}_{H,\gamma}^+ = \{ m \in \mathcal{M}_{H,\gamma} | \lim_{t \to 0} t \cdot m \text{ exists} \}$$

is Lagrangian (in the Poisson sense). Let $\mathcal{D}$ be a quantization of the structure sheaf compatible with a conical $\mathbb{C}^*$ action, as in [BPW, §3.2]. Note that there is a subtlety here: we have to choose a conical $\mathbb{C}^*$-action (we usually denote the corresponding copy of $\mathbb{C}^*$ by $\mathbb{S}$) in order to make sense of this category, but this geometric category $\mathcal{O}$ will not depend on the choice (since the underlying sheaves are unchanged). For simplicity, we will fix this action to let $\mathbb{S}$ be the action induced by the scaling action of $T^*V$.

Recall that a **good** $\mathcal{D}$ module is one which admits a coherent $\mathcal{D}(0)$-lattice. We wish to define a special category of $\mathcal{D}$-modules based on the structure of the action of the flavor $\phi$. This is a generalization of the geometric category $\mathcal{O}$ defined in [BLPW]: the key difference is that our $\mathbb{T}$-action has weight 1 on the symplectic form, rather than weight 0 as in [BLPW]. However, by correctly writing this definition, we can give a consistent definition for both cases.

We endow $\mathcal{M}_{H,\gamma}^+$ with the scheme structure defined by the ideal generated by all global functions on $\mathcal{M}_{H,\gamma}$ with positive weight under the action of $\mathbb{T}$.

**Definition 2.2** A good $\mathcal{D}$-module $\mathcal{M}$ on $\mathcal{M}_{H,\gamma}$ lies in category $\mathcal{O}_\phi$ for the flavor $\phi$ if it has a $\mathcal{D}(0)$-lattice $\mathcal{M}(0)$ such that $\mathcal{M}(0)/h\mathcal{M}(0)$ is scheme-theoretically supported on $\mathcal{M}_{H,\gamma}^+$.

**Lemma 2.3** If $\phi_0 \colon \mathbb{T} \to T_H$, the the category $\mathcal{O}_\phi$ for $\phi_0$ in [BLPW, Def. 3.15] is the same as that of Definition 2.2 for the pointwise product of $\phi_0^\ell$ and the action of $\mathbb{S}$ for $\ell \gg 0$.

**Proof.** This follows immediately from [BLPW, Prop. 3.18]: the functions with positive weight under this pointwise product for $\ell \gg 0$ are those where $\mathbb{T}$ has positive weight, or $\mathbb{T}$-weight 0 and positive $\mathbb{S}$-weight. Note that only the constant functions have $\mathbb{S}$-weight 0 and no functions have $\mathbb{S}$-weight 1 since $V \oplus V^*$ has no $G$-invariants, so all of these functions must have $\mathbb{S}$ weight $\geq 2$. Thus, these are precisely the functions in the ideal $J$ defined in [BLPW, §3.1].

If $\mathcal{M}_{H,\gamma}$ is not smooth then this is still true, but this requires a more careful argument. Since we don’t need this fact, we will not include this argument; see [Webb, Proposition 2.11].
2.1. Lifts and chambers. In this section, we make some combinatorial definitions needed in order to understand this category \( \mathcal{O} \).

We have a natural character \( \nu: \tilde{G} \to \mathbb{T} \) splitting the inclusion of \( \mathbb{T} \); this is induced by the action of a group element on the symplectic form: \( g \cdot \Omega = \nu(g)\Omega \). We call a splitting of this character \( \gamma: \mathbb{T} \to \tilde{G} \) a lift of \( \mathbb{T} \). This is the same as a choice of linear \( \mathbb{T} \)-action on \( V \) such that the Hamiltonian reduction is \( \mathbb{T} \)-equivariant. A rational (real, etc.) lift is a splitting of the derivative of \( \nu \) on the rational Lie algebras \( \mathfrak{g}_\mathbb{Q} \to \mathfrak{g} \).

Pick a maximal torus \( \tilde{T} \subset \tilde{G} \), and let \( T = \tilde{T} \cap G \). Let \( X_\ast(T) \) denote the space of real splittings. The affine space \( \mathfrak{t}_1 \) is naturally equipped with a cooriented affine hyperplane arrangement, defined by the vanishing sets of the weights of \( T^\ast V \). Let \( \{ \varphi_1, \ldots, \varphi_d \} \) be an enumeration of the weights of \( V \), with multiplicity. Thus, we can choose a decomposition \( V \cong \bigoplus V_{\varphi_i} \) into 1-dimensional subspaces, such that every \( V_{\varphi_i} \) has weight \( \varphi_i \) under \( \tilde{T} \), and generates a simple \( \tilde{G} \)-representation which is a direct sum

\[
U(\mathfrak{g}) \cdot V_{\varphi_i} = \bigoplus_{j \sim i} V_{\varphi_j}
\]

for \( j \sim i \) the equivalence relation \( V_{\varphi_j} \subset U(\mathfrak{g}) \cdot V_{\varphi_i} \).

Example 2.i. We’ll use the example of \( GL(2) \) with \( V \cong \mathbb{C}^2 \oplus \mathbb{C}^2 \) as our standard example throughout. In this case, \( d = 4 \), with \( \varphi_1 = \varphi_3 = \gamma_1 \) and \( \varphi_2 = \varphi_4 = \gamma_2 \). We choose the relation so that \( 1 \sim 2 \) and \( 3 \sim 4 \). We let \( \mathbb{T} \) be the \( \mathbb{C}^\ast \) action with weight 1 on the spaces \( V_{\varphi_1} \) and \( V_{\varphi_2} \) and weight \(-1\) on the spaces \( V_{\varphi_3} \) and \( V_{\varphi_4} \). The symplectic condition forces it to have weight \(-2\) and \( 0 \) on the duals of these spaces. Thus, in this basis, \( \widetilde{GL(2)} \cong GL(2) \times \mathbb{T} \) acts on \( V \) by the matrices

\[
\begin{bmatrix}
tA & 0 \\
0 & t^{-1}A \\
\end{bmatrix}
\]

\( A \in GL(2), t \in \mathbb{T} \cong \mathbb{C}^\ast \)

Definition 2.4 For a sign sequence \( \sigma \in \{+,0,-\}^d \), we let \( V_\sigma \) be the sum of the subspaces \( V_{\varphi_i} \) with \( \sigma_i = + \). We let \((T^\ast V)_\sigma\) be the sum of \( V_{\varphi_i} \) with \( \sigma_i = + \) and \( V_{\varphi_i}^\ast \) with \( \sigma_i = - \).

If we have any other set \( \mathcal{I} \) equipped with a map \( \iota: \mathcal{I} \to \{+,0,-\}^d \), then we can denote \( V_x = V_{\iota(x)} \) and \((T^\ast V)_x = (T^\ast V)_{\iota(x)} \) for any \( x \in \mathcal{I} \). This notation leaves \( \iota \) implicit, but in all examples we will consider, this map will be unambiguous.

Definition 2.5 We call \( \sigma \) compatible with a Borel \( \tilde{B} \) containing \( \tilde{T} \) if \((T^\ast V)_\sigma\) is \( \tilde{B} \)-invariant. We let \( \mathcal{K} \) be the set of pairs of sign vectors in \( \{+,0,-\}^d \) and compatible Borels.

If we fix a preferred Weyl chamber of \( \tilde{G} \) (and thus a standard Borel \( \tilde{B} \)), we have a bijection of all the Weyl chambers with the Weyl group \( W \) of \( G \) (which is also the Weyl group of \( \tilde{G} \)), and thus can think of \( \mathcal{K} \) as a subset of \( \{+,0,-\}^d \times W \).
Example 2.ii. In our running example, if \( \tilde{B} \) is the standard Borel, then the non-compatible sign vectors are of the form

\[
(-, +, *, *) \quad (-, 0, *, *) \quad (0, +, *, *) \quad (*, *, *, -) \quad (*, *, 0, +)\]

If we consider the opposite Borel (the only other), then +’s and −’s exchange places.

Now, we let

\[
\varphi_i^+ = \varphi_i \quad \varphi_i^- = -\varphi_i - \nu.
\]

Together, these give the weights of \( \tilde{t} \) acting on \( T^*V \).

Example 2.iii. In our example, \( \tilde{t} \) is 3 dimensional, and identified with the diagonal matrices of the form \( \text{diag}(a + t, b + t, a - t, b - t) \); passing to \( t_1 \) means considering these with \( t = 1 \). The weights \( \varphi_i^+ \) are the entries of this diagonal matrix; the weights \( \varphi_i^- \) are the weights on \( V^* \), which come from the matrix \( \text{diag}(-a - 2t, -b - 2t, -a, -b) \).

Definition 2.6 For a sign sequence \( \sigma \in \{+,-\}^d \), we let

\[
c_\sigma = \{ \gamma \in X_*(T)_1 | \varphi_i^{\sigma}(\gamma) \geq 0 \} \quad C_\sigma = \{ \gamma \in t_1 | \varphi_i^{\sigma}(\gamma) \geq 0 \}.
\]

We let \( C_{\sigma,w} \) be the intersection of \( C_\sigma \) with the open Weyl chamber attached to \( w \). Note that if \( C_{\sigma,w} \neq \emptyset \), then \( \sigma \) is compatible with \( wBw^{-1} \).

We can extend this notation to sequences in \( \{+,-,0\}^d \) by requiring \( \varphi_i^+(\gamma) \in (-1,0) \) if \( \sigma_i = 0 \).

Example 2.iv. Thus, if we use \( a \) and \( b \) as our coordinates on \( t_1 \), we obtain the hyperplane arrangement:

\[
\begin{array}{cccc}
\varphi_3^+ = 0 & \varphi_1^+ = 0 \\
C_{-,+,+,+} & C_{-,+,0,+} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,+,+} & C_{-,+,0,+} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,+,+} & C_{-,+,0,+} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
C_{-,+,-,+} & C_{-,+,0,-} & C_{-,+,+,+} & C_{+,+,+,+} \\
\varphi_3^- = 0 & \varphi_1^- = 0
\end{array}
\]
The side of a hyperplane carrying a fringe indicates the positive side (which thus includes the hyperplane itself).

Given a lift \( \gamma \in X_*(T)_1 \), let

\[ V_\gamma = \{ x \in V \mid \lim_{t \to 0} \gamma(t) \cdot x \text{ exists} \} \]

be the sum of the non-negative weight spaces for \( \gamma \), and \((T^*V)_\gamma\) be the corresponding sum for \( T^*V \). Using the notation above, we have \( V_\gamma = V_\sigma \) for all \( \gamma \in C_\sigma \).

The space \((T^*V)_\gamma\) is Lagrangian and thus the conormal to \( V_\gamma \) for any (integral) lift; for a real or rational lift, this space may be isotropic (and not Lagrangian) if \( \varphi^\pm(\gamma) \in (-1,0) \), or equivalently, if \( \gamma \notin C_\sigma \) for any \( \sigma \in \{+,-\}^d \). Furthermore, \((T^*V)_\gamma = (T^*V)_{\gamma'}\) for lifts \( \gamma \) and \( \gamma' \) if and only if both lie in \( C_\sigma \) for some \( \sigma \in \{+,-,0\}^d \), in which case, both are equal to \((T^*V)_\sigma\).

In the diagram below, we’ve marked the chambers corresponding to sign vectors in \{+,−\}^d\) with \( V_\sigma \) (represented in terms of which coordinates are non-zero).

2.2. The Steinberg algebra. For each pair \((\sigma, w) \in \mathcal{K}\), we have an attached space \( X_{\sigma,w} = G \times_{wBw^{-1}} V_\sigma \), with the induced map \( p_{\sigma,w} : X_{\sigma,w} \to V \) sending \((g,v)\) to \(gv\).

For any collection of these pairs \( I \subset \mathcal{K}\), we can define a Steinberg variety by taking the fiber product of each pair of them over \( V\):

\[ X_I := \bigsqcup_{(\sigma,w) \in I} \bigsqcup_{(\sigma',w') \in I} X_{\sigma,w} \times_V X_{\sigma',w'} \]

with a natural \( G \) action.
**Definition 2.7** The $G$-equivariant Borel-Moore homology $H^{BM,G}_*(X_I)$ equipped with its convolution multiplication is called the **Steinberg algebra** in $[\text{Sau}]$.

Equivalently, we can think of the **Steinberg category** $\mathcal{X}_I$ whose objects are elements of $I$ and where morphisms $(\sigma', w') \to (\sigma, w)$ are given by $H^{BM,G}_*(X_{\sigma, w} \times_V X_{\sigma', w'})$, with composition given by convolution. The Steinberg algebra is simply the sum of all the morphisms in this category; modules over the Steinberg algebra are naturally equivalent to the category of modules over the category $\mathcal{X}_I$ (that is, functors from this category to the category of $\mathbb{k}$-vector spaces).

This category has a sheaf-theoretic interpretation as well. By $[\text{CG97}, \text{Thm. 8.6.7}]$, we have that

$$H^{BM,G}_*(X_{\sigma, w} \times_V X_{\sigma', w'}) \cong \text{Ext}^\bullet((p_{\sigma, w})_* \mathcal{O}_{X_{\sigma, w}}, (p_{\sigma', w'})_* \mathcal{O}_{X_{\sigma', w'}})$$

with convolution product matching Yoneda product. The argument in $[\text{CG97}]$ in fact shows that that this can be enhanced to a dg-functor $\mathcal{X}_I \to D_{dg}^b(V)$, where $\mathcal{X}_I$ is made into a dg-category by replacing $H^{BM,G}_*(X_{\sigma, w} \times_V X_{\sigma', w'})$ with the Borel-Moore chain complex on $X_{\sigma, w} \times_V X_{\sigma', w'}$. As argued in $[\text{Webb}, \text{Prop. 2.19}]$, this induced dg-structure on $\mathcal{X}_I$ is formal (and thus, can essentially by ignored).

Of course, we can define the same space, algebra or category when $I$ is a set with a map to $\mathcal{K}$. The Steinberg category $\mathcal{X}_I$ attached to a set with such a map is equivalent to the category attached to its image (so the corresponding algebras are Morita equivalent). Furthermore, the spaces $X_{\sigma, 1}$ and $X_{w \cdot \sigma, w}$ are isomorphic via the action of any lift of $w$ to $\tilde{G}$, so the graph of this isomorphism provides an isomorphism between the objects $(\sigma, 1)$ and $(w \cdot \sigma, w)$ in the Steinberg category.

### 2.3. A presentation of the Steinberg category.

We will give an explicit presentation of Steinberg algebras for certain sets which generalize both the KLR algebras of $[\text{KL09}, \text{Rou}]$ and the hypertoric algebras of $[\text{BLPW10}, \text{BLPW12}]$.

To give a simpler presentation, we will make an auxiliary modification to our chambers. Choose $\epsilon_i \in (-1, 0)$ generically with respect to the constraint that $\epsilon_i = \epsilon_j$ if $i \sim j$.

Now, consider the larger chambers

$$C'_\sigma = \{ \gamma \in t_1 \mid \varphi_i(\gamma) > \epsilon_i \text{ if } \sigma_i = + \text{ and } \varphi_i(\gamma) < \epsilon_i \text{ if } \sigma_i = -, \}$$

with $C'_{\sigma, w}$ defined as in Definition 2.6.
\textit{Example 2.v.} We'll choose $\epsilon_1 = \epsilon_2 = -1/3$ and $\epsilon_3 = \epsilon_4 = -2/3$ in our running example. Thus, our arrangement becomes

\[
\begin{array}{ccc}
\varphi_3 &=& -2/3 \\
C'_{-,+,+,+} & C'_{-,++,+} & C'_{+,++,+} \\
\varphi_2 &=& -1/3 \\
C'_{-,+,+-} & C'_{-,++,} & C'_{+,++,} \\
\varphi_4 &=& -2/3 \\
C'_{-,+,--} & C'_{-,--,} & C'_{+,--,} \\
\end{array}
\]

We'll call the hyperplane $H_i = \{ \gamma \mid \varphi_i(\gamma) = \epsilon_i \}$ a \textbf{matter hyperplane}, and $h_\alpha = \{ \gamma \mid \alpha(\gamma) = 0 \}$ a \textbf{Coxeter hyperplane}. We'll draw matter hyperplanes with solid lines and Coxeter hyperplanes with dotted lines in diagrams. Just as the chambers $C_\sigma$ capture the behavior of the subspace $V_\gamma$ as $\gamma$ changes, the chambers $C'_\sigma$ give the corresponding subspaces for the action of $t_i$, which is the same subspace as $t_1$, but with a different action on $T^*V$: we let $t$ act on $V_{\varphi_i}$ by $\varphi_i - \epsilon_i$ and on the dual space by $-\varphi_i + \epsilon_i$. Note that $C_{\sigma,w} \subset C'_{\sigma,w}$.

\textbf{Definition 2.8} We let $I$ (resp. $I'$) be the set of sign vectors $\sigma \in \{+,-\}^d$ such that there exists a choice of flavor $\phi$ such that $c_{\sigma,1} \neq 0$ (resp. $C'_{\sigma,1} \neq 0$); note that $I' \supseteq I$. If we fix the flavor $\phi$, we denote the corresponding sets $I_\phi$ and $I'_\phi$.

\textit{Example 2.vi.} In our running example, we have

$I' = I = \{(+,+,+,+), (+,-,+,+), (+,-,-,+), (-,-,+,+), (-,-,+,+), (-,-,+,+), (-,-,-,+)\}.$

This can change if we choose a different $\phi$: if $\phi$ acts trivially on $V$ and $\epsilon_1 > \epsilon_2$ then $I'$ is unchanged but

$I = \{(+,+,+,+), (+,-,+,+), (-,-,-,+)\}.$

If $C_\sigma \neq \emptyset$, then there is a unique sign vector $w\sigma$ such that $C_{w\sigma} = w \cdot C_\sigma$. This is the unique permutation of $\sigma$ such that each $\varphi_i$ is switched with $\varphi_j = w\varphi_i$ such that $i \sim j$. This is well-defined since if $\varphi_i = \varphi_k$ and $i \sim k$, then these have the same sign (since $C_\sigma \neq \emptyset$). In particular, if $\sigma \in I'$, the translate $w\sigma$ is well-defined.

Given a pair $(\sigma, \sigma')$, we let $\varphi(\sigma, \sigma')$ be the product of the weights $\varphi_i$ such that $\sigma_i = +$ and $\sigma'_i = -$. Given a triple $(\sigma, \sigma', \sigma'')$, we let $\varphi(\sigma, \sigma', \sigma'')$ be the product of the weights $\varphi_i$ such that $\sigma_i = \sigma''_i = -\sigma'_i$.
Let $\partial_\alpha(f) = \frac{s_\alpha f - f}{s_\alpha}$ be the usual BGG-Demazure operator on $S := \text{Sym}(t^*) \cong H^*_G(G/B)$.

**Definition 2.9** We let $A_I$ denote the free category with objects given by the sign vectors $\sigma \in I'$, and morphisms generated by

- An action of $S$ on each object $\sigma$.
- Wall-crossing elements $w(\sigma; \sigma') : \sigma' \to \sigma$.
- Elements $\psi_\alpha(\sigma) : \sigma \to \sigma$ for roots $\alpha$ such that $s_\alpha \cdot \sigma = \sigma$.

subject to the “codimension 1” relations:

(2.1a) \[ w(\sigma, \sigma') w(\sigma', \sigma'') = \varphi(\sigma, \sigma', \sigma'') w(\sigma, \sigma'') \]

(2.1b) \[ \mu w(\sigma, \sigma') = w(\sigma, \sigma') \mu \]

(2.1c) \[ \psi_\alpha(\sigma)^2 = 0 \]

(2.1d) \[ \psi_\alpha(\sigma)(\mu) - (s_\alpha \mu) \psi_\alpha(\sigma) = \partial_\alpha(\mu) \]

with $\sigma, \sigma', \sigma'' \in I', \mu \in t^*$ and $\alpha$ a root with $s_\alpha \cdot C_\sigma = C'_\sigma$ and the “codimension 2” relations (2.1e–2.1i) below. We get one of these for every codimension 2 intersection of hyperplanes which forms a face of $C'_\sigma$. There are 3 possible types of these intersections, which in each case below, we represent by drawing a transverse neighborhood to the codimension 2 intersection:

1. The codimension 2 subspace is the intersection of 2 Coxeter hyperplanes $h_\alpha$ and $h_\beta$. For any chamber $C_{\sigma,1}$ adjacent to these hyperplanes, we have the usual Coxeter relations for $m = \alpha^\vee(\beta) \cdot \beta^\vee(\alpha)$:

   \[
   \psi_\alpha(\sigma) \psi_\beta(\sigma) \psi_\alpha(\sigma) \cdots = \psi_\beta(\sigma) \psi_\alpha(\sigma) \psi_\beta(\sigma) \cdots
   \]

   (2.1e)

2. The codimension 2 subspace is the intersection of a Coxeter hyperplane $h_\alpha$ and $H_\beta$. In this case, the codimension 2 subspace lies in 1 other hyperplane: the one corresponding to the Weyl translate $\varphi_k = s_\alpha \varphi_j$, given by $H_k$. Both these hyperplanes have a multiplicity $w$, given by dimension of the corresponding weight space in $V$. We label the adjacent chambers $\rho, \sigma, \tau$ as shown, with $\rho$ on
the positive side of both hyperplanes (and thus $\tau$ on the negative side of both).

We then have the relations

\begin{align}
(2.1f) & \quad \psi_\alpha(\rho)w(\rho, \sigma)w(\sigma, \tau) = w(\rho, \sigma)w(\sigma, \tau)\psi_\alpha(\tau) \\
(2.1g) & \quad \psi_\alpha(\tau)w(\tau, \sigma)w(\sigma, \rho) = w(\tau, \sigma)w(\sigma, \rho)\psi_\alpha(\rho) \\
(2.1h) & \quad w(\sigma, \tau)\psi_\alpha(\tau)w(\tau, \sigma) = w(\sigma, \rho)\psi_\alpha(\rho)w(\rho, \sigma) - \partial_\alpha(\varphi^w_i)e(\sigma)
\end{align}

(3) The codimension 2 subspace is the intersection of two hyperplanes $\varphi_i(\gamma) = \epsilon_i$ and $\varphi_j(\gamma) = \epsilon_j$. The resulting relation here is a consequence of (2.1a), but we include it for completeness. In this case, the codimension 2 subspace lies in no other hyperplanes by the genericity of $\epsilon_i$. We label the adjacent chambers $\pi, \rho, \sigma, \tau$ as shown.

We then have the relation

\begin{align}
(2.1i) & \quad w(\pi, \rho)w(\rho, \sigma) = w(\pi, \tau)w(\tau, \sigma)
\end{align}

Example 2.vii. In our running example, the resulting algebra is well-known: we can represent the positive Weyl chamber as a pair of points on the real line giving the coordinates $(a, b)$. Since we are in the positive Weyl chamber $a > b$, there is no ambiguity. We cross a hyperplane when these points meet, or when they cross $x = 2/3$ or $x = -5/3$. Thus, if we add red points at $x \in \{2/3, -5/3\}$, we'll obtain a bijection between chambers and configurations of points up to isotopy leaving the red points in
We’ll represent morphisms $\sigma \to \sigma'$ by Stendhal diagrams (as defined in [Weba, §4]) that match $\sigma$ at the bottom and $\sigma'$ at the top (with composition given by stacking, using isotopies to match the top and bottom if possible). We send the

- identity on $\sigma$ to a diagram with all strands vertical,
- the action of $\mathbb{C}[\gamma_1, \gamma_2]$ to a polynomial ring placing dots on the two strands,
- $w(\sigma; \sigma')$ to diagram with straight lines interpolating between the top and bottom
- $\psi_\alpha(\sigma)$ is only well-defined if there is no red line separating the two black lines; we send this to a crossing of the two black strands.

The relations (2.1a–2.1h) exactly match those of $\tilde{T}_2^2$ as defined in [Weba, Def. 2.3] (a special case of the algebras defined in [Weba, §4]). This is a special case of a much more general result, which we will discuss in Sections 2.5 and 4.4

Let $G_i \subset G/B \times G/B$ be the preimage of the diagonal in $G/P_i \times G/P_i$. Given a $P_i$-representation $Q$, we let $L_{P_i}(Q)$ be the pullback of the associated bundle on $G/P_i$ to $G_i$, and if $Q$ is a representation of the Borel, then let $L(Q)$ be the associated vector bundle on the diagonal.

**Theorem 2.10** We have a natural equivalence $A_{\mathcal{I}'} \cong \mathcal{X}_{\mathcal{I}'}$ which matches objects in the obvious way, and sends
(1) $\mu: \sigma \to \sigma$ to the Euler class $e(L(\mu))$ of the associated bundle on the diagonal copy of $X_{\sigma,1}$ in $X_I$.

(2) $w(\sigma, \sigma')$ to the fundamental class of the associated variety $[L(V_\sigma \cap V_{\sigma'})]$ embedded naturally in $X_{\sigma,1} \times_X X_{\sigma',1} \subset G/B \times G/B \times V$.

(3) $\psi_\alpha(\sigma)$ to the fundamental class of the associated variety $[L_{P,\sigma}(V_\sigma)]$ embedded naturally in $X_\sigma \times_X X_\sigma \subset G/B \times G/B \times V$.

We will prove this theorem below, once we have developed some of the theory of these algebras.

**Lemma 2.11** The algebra $A_I$ has a natural representation $Y$ which sends each object $\sigma$ to the polynomial ring $S$. The action is defined by the formulae:

\[
\begin{align*}
    w(\sigma, \sigma') \cdot f &= \varphi(\sigma, \sigma') f \\
    \psi_\alpha(\sigma) \cdot f &= \partial_\alpha(f) \\
    \mu \cdot f &= \mu f
\end{align*}
\]

For each pair $(\sigma, \sigma') \in I' \times I'$, and $w \in W$, we fix a path of minimal length (i.e. crossing a minimal number of hyperplanes) from $C'_{\sigma,1}$ to $C'_{w\sigma,w}$. Now, fold this path so that it lies in the positive Weyl chamber: the first time it crosses a root hyperplane, apply the corresponding simple reflection to what remains of the path. Then follow this new path until it strikes another wall, and apply that simple reflection to the remaining path, etc. The result is a sequence $\beta_1, \beta_2, \ldots, \beta_p$ of simple root hyperplanes and sign vectors $\sigma_1, \ldots, \sigma_p$ corresponding to the chambers where we reflect. Now, consider the product

\[
\tilde{w}(\sigma, \sigma', w) = w(\sigma, \sigma_p)\psi_{\beta_p}(\sigma_p)w(\sigma_p, \sigma_{p-1})\psi_{\beta_{p-1}}(\sigma_{p-1}) \cdots \psi_{\beta_1}(\sigma_1)w(\sigma_1, \sigma').
\]

**Example 2.viii.** In our running example, this is given by the diagrams without dots which join the black strands with no crossing if $w = 1$ and with a crossing if $w = s_\alpha$, and a minimal number of red/black crossings possible.
In the diagram below, we show one possible path \( \sigma' = (+, -, +, -) \to s_\alpha \sigma = (-, +, +, +) \), and its reflection.

\[ \varphi_1 = -1/3 \quad \varphi_2 = -1/3 \quad \varphi_3 = -2/3 \quad \varphi_4 = -2/3 \]

The resulting element \( \tilde{w}((+, -, +, +), (+, -, +, -), s_\alpha) \) is given by:

\[ \tilde{w}((+, -, +, +), (-, -, +, +)) \psi_\alpha((- -, +, +)) \tilde{w}((- -, +, +), (+, -, +, -)) \]

and represented by the diagram

\[ \begin{array}{c}
\varphi_1 = -1/3 \\
\varphi_2 = -1/3 \\
\varphi_3 = -2/3 \\
\varphi_4 = -2/3 \\
\end{array} \]

\[ \alpha = 0 \]

\[ \begin{array}{c}
\varphi_1 = -1/3 \\
\varphi_2 = -1/3 \\
\varphi_3 = -2/3 \\
\varphi_4 = -2/3 \\
\end{array} \]

**Theorem 2.12** The elements \( \tilde{w}(\sigma, \sigma', w) \) are a basis of the morphisms in \( A_\Pi \) as a right module over \( S \).

**Proof.** First, we note that they span. For this, it suffices to show that their span contains the identity of each object, which is \( \tilde{w}(\sigma, \sigma, 1) \) and is closed under right multiplication by the generators \( \tilde{w}(-, -) \) and \( \psi(-) \). Note that

\[ \tilde{w}(\sigma, \sigma', w) \psi_\alpha(\sigma) = \begin{cases} 
\tilde{w}(\sigma, \sigma', ws_\alpha) & ws_\alpha > w \\
0 & ws_\alpha < w 
\end{cases} \]

so this shows that these vectors span.

Now, consider the action of these operators in the representation \( Y \) localized over the fraction field of \( S \). The action of \( \tilde{w}(\sigma, \sigma', w) \) is given by the element \( w \), times a non-zero rational function, plus elements which are shorter in Bruhat order. Thus, the operators \( \sigma \to \sigma' \) span the twisted group algebra of \( W \) over rational functions. Since this group algebra is a vector space of dimension \( \#W \) over the fraction field, this is only possible if the elements \( \tilde{w}(\sigma, \sigma', w) \) are linearly independent over \( S \). \( \square \)
Proof of Theorem 2.10. The functor described in the statement matches the action of $A_I$ on $Y$ with that of $H^{BM,G}_*(X_I)$ on $H^{BM,G}_*(X_I)$, as simple computations with push-forward and pullback confirm (for example, as in [VII]). The action on the latter is faithful following the argument in [SW, Proposition 4.7], so this shows we have a faithful functor $A_I \to H^{BM,G}_*(X_I)$.

Let $\mathcal{X}(w)$ be the subset of the space $\mathcal{X}_I$ where the relative position of the two flags is $w \in W$. The surjectivity follows from the fact that $\tilde{w}(\sigma, \sigma'; w)$ is supported on $\mathcal{X}(w)$, and pulls back to the fundamental class on $\mathcal{X}(w)$. The intersection of this space with $X_{\sigma'} \times_V X_\sigma$ is an affine bundle with fiber given by a conjugate of $V_{w\sigma'} \cap V_{\sigma}$. If a weight is positive or negative for both $w\sigma'$ and $\sigma$, then a minimal length path does not cross the corresponding hyperplane, whereas it will cross it once if the signs are different. □

2.4. Variations. As earlier, we can generalize these algebras by taking any set $P$ with a map $\iota: P \to I'$, and considering the category $\mathcal{Z}_P$ with objects given by $P$ where

$$\text{Hom}_{\mathcal{Z}_P}(p, p') := \text{Hom}_{\mathcal{Z}_{I'}}(\iota(p), \iota(p')).$$

We let $J$ (resp. $J'$) be the subset of $\mathcal{K}$ such that $C_{\sigma, w} \neq 0$ (resp. $C'_{\sigma, w} \neq 0$); note that $J' \supseteq J$. In this case, the map $J' \to I'$ is given by $(\sigma, w) \mapsto (w^{-1}\sigma, 1)$. The algebra $\mathcal{Z}_P$ is Morita equivalent to $\mathcal{Z}_{I'}$. However, it is a convenient framework for understanding this category, because we can define certain special elements of it. We let $w: (\sigma, w') \to (w\sigma, ww')$ to be the image of the identity on $(w^{-1}\sigma, 1)$ under the isomorphism

$$\text{Hom}_{\mathcal{Z}_{I'}}((w\sigma, ww'), (\sigma, w')) := \text{Hom}_{\mathcal{Z}_{I'}}(((w')^{-1}\sigma, 1), ((w')^{-1}\sigma, 1)).$$

These obviously satisfy the relations of $W$. It’s more natural to think of the $S$ action on $(\sigma, w)$ to be the conjugate by $w$ of that on $(w^{-1}\sigma, 1)$. For each pair of pairs $(\sigma, w)$ and $(\sigma', w')$, we have a well-defined element of this algebra $w(\sigma', w'; \sigma, w)$ defined using the folding of a minimal path from $C'_{w^{-1}\sigma, 1}$ to $C'_{w^{-1}\sigma', w^{-1}w'}$ (using the same notation as (2.3)) by

$$w(\sigma', w'; \sigma, w) = w'w(\sigma', \sigma_p)s_{\beta_p}w(\sigma_p, \sigma_{p-1})\cdots s_{\beta_1}w(\sigma_1, \sigma)w^{-1}.$$

When we extend the polynomial representation $Y$ to this category, we thus still send every object to a copy of $S$ with the action given by

$$w(\sigma, w; \sigma', w') \cdot f = \varphi(\sigma, \sigma')f$$

(2.4)

$$\psi_\alpha(\sigma', w') \cdot f = \partial_\alpha(f)$$

$$w \cdot f = f^w$$

$$\mu \cdot f = \mu f$$

Note that if a sign vector $\sigma$ is compatible with $w$ and $w'$, then $w(\sigma, w; \sigma, w')$ gives an isomorphism between these objects. Thus, we can reduce the size of our category by only choosing one object per sign vector $\sigma$, and identifying it any others via the elements $w(\sigma, w; \sigma, w')$. This is the algebra $\mathcal{Z}_K$ attached to the set $K$ of sign vectors
with $C_\sigma \neq 0$ for some $\phi$ (similarly, we can define $K'$), with the map to $I'$ associating a sign vector to the unique Weyl translate compatible with $1 \in W$. Note that in this category, if $s_\alpha \sigma = \sigma$, then $s_\alpha$ is an endomorphism of this object, and computation in the polynomial representation confirms the relation $s_\alpha = \alpha \psi_\alpha + 1$.

In $\mathcal{X}_K$, we have morphisms $w(\sigma, \sigma'), \psi_\alpha(\sigma), w \in W, \mu \in t'$ as above, labeled by feasible sign vectors $\sigma, \sigma'$, and these act as in (2.4). This algebra contains as a subcategory $\mathcal{X}_K^{ab}$, the category attached to the representation $V$ and the torus $\tilde{T} \subset \tilde{G}$. This is generated over $S$ by the elements $w(\sigma, \sigma')$.

We can also consider the set $X_*(T)_1$ of lifts. Every element of this set lies in the one of the chambers in $K$. Thus, we have a map $X_*(T)_1 \rightarrow \{+,-\}^d$ sending each lift to its chamber. We have an associated Steinberg category $\mathcal{X}_X = \mathcal{X}_{X_*(T)_1}$.

Finally, we consider the extended arrangement on $t_\epsilon$ defined by the hyperplanes $\varphi_i(\xi) = n + \epsilon_i$ for $n \in \mathbb{Z}$. Note that if we use the isomorphism $t_\epsilon \cong t'$, with $\xi'$ denoting the image of $\xi$, we have that $\varphi_i(\xi) = \varphi_i(\xi') - \epsilon_i$. The chambers $C'$ of this arrangement are defined not by sign vectors, but rather by integer vectors: associated to $a = (a_1, \ldots, a_d)$ we have the chamber

$$C'_a = \{ \xi \in t_\epsilon \mid a_i < \varphi_i(\xi) < a_i + 1 \text{ for all } i \}. \quad (2.5)$$

As usual, we call $a$ feasible if this set is non-empty. Considering the inclusion of chambers induces a map $\eta: C \rightarrow K'$, which gives us a category $\mathcal{X}_C$. Since the map $C \rightarrow K'$ is surjective, $\mathcal{X}_C$ is equivalent to $\mathcal{X}_{K'}$, but it will be useful to have this category for comparison to the Coulomb case. As before, we can generate the morphisms of this category with morphisms $w(a, a')$ and copies of $\mathcal{X}_C$.

**Definition 2.13** Given two subsets $P, P' \subset I'$, we define the $A_{P'} - A_P$-bimodule $P \rightarrow A_P$ (or similarly a $\mathcal{X}_{P'} - \mathcal{X}_P$-bimodule $P \rightarrow \mathcal{X}_P$) by simply associating to the pair $(p', p) \in P' \times P$ the vector space $\text{Hom}_{A_{P'}}(p, p')$.

This extends in an obvious way to $P, P'$ simply mapping to $I'$ (or to $K'$, etc.).

### 2.5. The quiver and hypertoric cases

If $G$ is abelian, then all relations involving $\psi$ do not occur, since there are no Coxeter hyperplanes. We are left with the relations (2.1a, 2.1b, 2.1i), which appeared in [BLPW10, BLPW12]. The result is the algebra $A_{pol}(\vartheta, -)$ from [BLPW12, §8.6].
Now we fix a quiver $\Gamma$ and let $V = \bigoplus_{i \rightarrow j} \text{Hom}(C^d_i, C^d_j)$ as a module over $G = \prod GL_{d_i}$ as usual. In this case we obtain the relations of a weighted KLR algebra as defined in [Webi].

Let $\mathfrak{t}$ act on $T^*V$ by cotangent scaling and write the lifts in $t_1$ as this action, plus an element of $t$. The Lie coalgebra $t^*$ is generated by the weights of the defining representations $z_{i,k}$ for $i \in V(\Gamma)$ and $k = 1, \ldots, d_i$. In this case, the chambers $C_{\sigma}$ are bounded by the inequalities $z_{i,k} \geq z_{j,m}$ or $z_{i,k} \leq z_{j,m} - 1$ if we have an arrow $j \rightarrow i$.

Recall that the KLR category of a graph $\Gamma$ is an category whose objects are lists $i \in V(\Gamma)^n$ and morphisms are certain string diagrams carrying dots. The KLR algebra is the formal sum of all morphisms in this category.

We’ll use a slightly unorthodox generating set for this category:

- The dots acting as a polynomial ring on each object.
- Given $i, j \in V(\Gamma)^n$, there is a unique diagram $i1_j: i \rightarrow j$, as defined in the proof of [KL09, Thm. 2.5], which connects these objects with a minimal number of crossings.
- If $i_k = i_{k+1}$, then $\psi_k: i \rightarrow i$ switches these strands (in many other sources, $\psi_k$ is used for the morphism switching these strands no matter what the label; we have absorbed those with different labels into the diagrams $i1_j$ above).

Given a list $i \in V(\Gamma)^n$ where $n = \sum d_i$, we let $\xi_i$ be the unique coweight where $z_{j,1} < \cdots < z_{j,d_j}$ and $z_{j,k} \in [1, n]$ satisfies $i_{z_{j,k}} = j$ for all $k$. Note that switching two entries of $i$ which are not connected will not change the underlying chamber. Let $I$ be the set of coweights occurring this way, with the obvious map $I \rightarrow K$ just remembering the chamber where each coweight lies. By comparing the representation (2.2) with [Rou, 3.12], we see immediately that:

**Proposition 2.14** We have an equivalence of $A_J$ to the KLR algebra of $\Gamma$ for the dimension vector $d$, sending:

1. $z_{j,k}$ to the dot on the $k$th strand from the right labeled $j$,
2. $w(i, j)$ to $i1_j$, and
3. $\psi_{\alpha_{j,k}}(\xi)$ to the element $\psi$ crossing the $k$th and $k+1$st strands from the left with label $j$ (these must be adjacent for $\psi_{\alpha_{j,k}}(\xi)$ to be defined).

Note that this quite similar to the isomorphism discussed in Example 2.vii.

We can simplify this a bit in the case where $\Gamma$ is bipartite, composed of “odd” and “even” vertices; we choose an orientation pointing from odd vertices to even vertices, and reindex by adding $1/2$ to all the weights for odd vertices. In this case, our inequalities become $z_{i,k} \geq z_{j,m} + 1/2$ independent of orientation.

The existence of a non-trivial flavor complicates the situation. For each edge $e: i \rightarrow j$, we have a weight $\phi_e$ and a choice of $\epsilon_{i,j} \in (0, -1)$. In this case, the chambers $C'_{\alpha}$ are
defined by inequalities of the form
\[ z_{i,k} \leq z_{j,m} + \epsilon_e + w_e \]
\[ z_{i,k} \geq z_{j,m} + \epsilon_e + w_e \]
\[ \sigma_{i,j;k,m} = + \]
\[ \sigma_{i,j;k,m} = - . \]

Thus, the chambers $C'_{\sigma,w}$ are precisely the equivalence classes of loadings for the KLR algebra with the weighting $\vartheta_e = \epsilon_e + \phi_e$ by [Webf, 2.12].

Given a set $B$ of loadings, we have a map $B \to K'$ sending a loading to the corresponding chamber as above. Comparing [Webf, 2.7] to (2.2) shows that:

**Proposition 2.15** We have an isomorphism $W_B^\vartheta \cong A_B$.

**2.6. Category $\mathcal{O}$**. In this section, we’ll assume that $\mathbb{k} = \mathbb{C}$; furthermore, we’ll fix an index set $Q$ with a map $Q \to I'$, and let $A := A_Q$. In this case, the Steinberg algebra has an interpretation in terms of $G$-equivariant D-modules on $V$. This corresponds to the sheaf theoretic interpretation discussed before by the Riemann-Hilbert correspondence.

Consider the union $X = \bigsqcup_{\sigma \in I} X_{\sigma}$ and let $p: X \to V$ be the projection to the second factor. Let $L = p_\ast \mathcal{G}_X$ be the D-module pushforward of the structure sheaf on $X$ by this proper map and $L_{\sigma} = p_\ast \mathcal{G}_{X_{\sigma}}$.

As discussed earlier, [CG97, Thm. 8.6.7] together with the Riemann-Hilbert correspondence shows that:

**Proposition 2.16** We have a quasi-isomorphism of dg-algebras $A \cong \text{Ext}^\bullet(L, L)$ where the left hand side is thought of as a dg-algebra with trivial differential.

This isomorphism induces an equivalence between the dg-subcategory of bounded complexes of D-modules on $V/G$ generated by $L$, and the category of dg-modules over $A$. Alternatively, it shows that the dg-subcategory $D_L^b(\text{MHM}(V/G))$ of mixed Hodge modules on $V/G$ generated by Tate twists of $L$ is equivalent to the category of dg-category of complexes of graded $A$-modules. This equivalence sends simple mixed Hodge modules to complexes with a single indecomposable projective term satisfying $\text{Hom}(P, A) \cong P$, and intertwines Tate twist with simultaneous shift of internal and homological grading grading on $A_I$-modules. This leads us to:

**Theorem 2.17** This equivalence induces an equivalence of categories between the abelian category $\text{MHM}_L$ of mixed Hodge modules lying in $D_L^b(\text{MHM}(V/G))$ and the graded category $\text{LPC}(A)$ of linear projective complexes over $A$.

**Proof.** We need only show that a complex in $D^b(A\text{-gmod})$ is quasi-isomorphic to a linear projective complex if and only if it is the image of a mixed Hodge module.

A linear projective complex has a filtration where the subquotients are single term linear projective complexes. These correspond to Tate twists of the mixed Hodge modules which are shifts of summands of $L$. Thus, the corresponding complex in
$D^b_\mathcal{L}(\text{MHM}(V/G))$ has a filtration whose subquotients are these mixed Hodge modules, and thus is itself a mixed Hodge module.

Now, we will show the converse by induction on the length of the mixed Hodge module. We have already discussed the length 1 case. If $X$ is a mixed Hodge module in $\text{MHM}_L$ then it has a simple submodule $K$ which is a shift of a summand of $L$. By assumption, the complex corresponding to $X/K \oplus K$ is linear projective, and $X$ has a corresponding complex with the same underlying module, and a different differential (we take the cone of the corresponding element of $\text{Ext}^1(X/K, K)$), which is thus also linear projective. 

Now, we consider how this construction behaves under reduction. For a given character $\xi$, we can define a GIT quotient $\mathcal{M}_{H,\xi} = T^*V/\!/G$, which is a quasi-projective variety. We wish to study modules over a quantization of $\mathcal{M}_{\xi}$ as in [BPW, BLPW] (also called DQ-modules in the terminology of Kashiwara and Schapira [KS12]). As introduced in [BLPW, §3.3], we have a category $\mathcal{O}_g$ attached to any quantization on this variety. In our case, we can construct these quantizations as noncommutative Hamiltonian reductions by the standard non-commutative moment map, sending $X \in \mathfrak{g} \mapsto X_V$, the corresponding vector field on $V$. The category $\mathcal{O}_g$ is a quotient of a category called $p\mathcal{O}_g$ defined in [Webb, 2.8], which contains $L$ by [Webb] Thm. 2.18 if $Q \mapsto I'_\phi \subset I'$. We let $D_{\mathcal{O}_g}$ and $D_{p\mathcal{O}_g}$ be the subcategories of the derived category generated by these abelian categories.

For a given character $\xi$, we call a sign vector $\sigma$ unsteady if there is a cocharacter $\nu$ with $\langle \xi, \nu \rangle \geq 0$, and $(T^*V)_\nu \supset (T^*V)_\sigma$. Let $\mathcal{I} \subset A$ be the ideal in $A$ generated by all morphisms factoring through the object $\sigma$ given by an unsteady sign vector. Since $r(L_\sigma) = 0$ by [Webb] Thm. 2.18, we have an induced functor $r(L)^L_{\otimes A/\mathcal{I}}: A/\mathcal{I} \text{-dg-mod} \to \mathcal{O}_g$.

In order to have the strongest version of our results, we need to make some assumptions introduced in [Webb, §2.6]. The strongest of these is:

$(\dagger)$ Each simple module in $p\mathcal{O}_g$ is a summand of a shift of $L$, and every simple with unstable support is a summand of $L_\sigma$ for $\sigma$ unsteady.

Note that whether $(\dagger)$ holds depends on the choice of $Q$. If it holds for any $Q$, then it holds for $Q = I_\phi'$. In this case, this assumption holds for quiver varieties and smooth hypertoric varieties, as shown in [Webb]. A slightly weaker assumption is:

$(\dagger')$ Each simple module $M$ in $p\mathcal{O}_g$ such that $\text{Ext}^*(L, M) \neq 0$ is a summand of a shift of $L$, and every such simple with unstable support is a summand of $L_\gamma$ for $\gamma \in B$ unsteady.

This holds for all hypertoric varieties, and seems likely to be the correct statement when $\mathcal{M}_{H,\xi}$ is not smooth. We know of no situation where $(\dagger)$ fails, but it seems to be a quite difficult statement to prove; it is not simple to describe the condition of being $p\mathcal{O}_g$ using only the geometry of $V/G$, since it is a microlocal property.
As argued in [Webb, Thm 2.25], we have that:

**Theorem 2.18** If (†) holds, then the functor $\tau(L)^L_{A/\mathcal{I}} - : A/\mathcal{I}$-dg-mod $\rightarrow D_{Og}$ is fully faithful. If (†) holds, then it is an equivalence.

We can define a graded version of category $O$ by considering the right adjoint $\tau_!$. A **grading** on an object $M \in O_g$ is a mixed Hodge structure on $\tau_!(M)$. Let $\tilde{O}_g$ be the category of graded objects in $O_g$ with morphisms given by $\text{Hom}_{\text{MHM}}(\tau(M), \tau(N))$.

**Theorem 2.19** If (†) holds, then the functor $\tau(L)^L_{A/\mathcal{I}} - : LCP(A/\mathcal{I}) \rightarrow \tilde{O}_g$ is fully faithful. If (†) holds, then it is an equivalence.

For our purposes, we wish to have a more user friendly characterization of instability.

**Lemma 2.20** If $C'_\sigma \neq \emptyset$, then the sign vector $\sigma$ is unsteady if $\xi$ does not attain a maximum on a bounded subset of $C'_\sigma$.

*Proof.* The condition that $\langle \xi, \nu \rangle > 0$ is equivalent to the claim that $\xi$ does not attain a unique maximum on any ray parallel to $\nu$ and $(T^*V)_\nu \supset (T^*V)_\sigma$ if and only if $C_\sigma$ contains a ray parallel to $\nu$. By a standard result of linear programming, we have a maximum on a bounded subset of the chamber if and only if we have a unique maximum on each ray in the chamber. \hfill \Box

Thus, if $Q = I'_\phi$, we could also define $\mathcal{I}$ as the ideal generated by projections to $L_\sigma$ with $\xi$ not attaining a maximum on a bounded subset of $C'_\sigma$.

### 3. The Coulomb side

The Coulomb side of our correspondence is given by a remarkable recent construction of Braverman, Finkelberg and Nakajima [Nakb, BFNb]. As we mentioned in the introduction, a more algebraic minded reader could ignore this geometric construction and take Theorem 3.6 as a definition. We’ll wish to modify this construction somewhat, so let us describe it in some detail. As before, $G$ be a reductive algebraic group over $\mathbb{C}$, with $G((t)), G[[t]]$ its points over $\mathbb{C}((t)), \mathbb{C}[[t]]$. For a fixed Borel $B \subset G$, we let $I$ be the associated Iwahori subgroup

$$I = \{ g(t) \in G[[t]] \mid g(0) \in B \} \subset G[[t]].$$

The **affine flag variety** $\mathcal{F} = G((t))/I$ is just the quotient by this Iwahori.

Let $V$ be a $G$-representation fixed in the previous section, and $U \subset V((t))$ a subspace invariant under $I$. We equip $V((t))$ with a loop $\mathbb{C}^*$-action such that $vt^a$ has weight $a$. This is compatible with the standard loop action on $G((t))$. We’ll be interested in the infinite-dimensional vector bundle on $\mathcal{F}$ given by $\mathcal{X}_U := (G((t)) \times U)/I$. Note that we have a natural $G((t))$-equivariant projection map $\mathcal{X}_U \rightarrow V((t))$. 

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Definition 3.1 The flag BFN space is the fiber product $X_{V[[t]]} \times_{V((t))} X_{V[[t]]}$.

We’ll consider this as a set of triples $X_{V[[t]]} \times_{V((t))} X_{V[[t]]} \subset V((t)) \times \mathcal{F} \times \mathcal{F}$. This space has a natural action of $G((t))$ by the diagonal action, as well as an action of $H$ and a loop action of $\mathbb{C}^*$ induced by that on $V((t))$ and $G((t))$. Let $\tilde{G}((t))$ be the subgroup of $\tilde{G}((t)) \times \mathbb{C}^*$ generated by $G((t))$, and the image of $\tilde{G} \hookrightarrow \tilde{G} \times \mathbb{C}^*$ included via the identity times $\nu$.

We’ll want to consider the equivariant homology $H^{BM,G((t))}_*(X_{V[[t]]} \times_{V((t))} X_{V[[t]]})$. Defining this properly is a finicky technical issue, since the space $X_{V[[t]]} \times_{V((t))} X_{V[[t]]}$ can be thought of as a union of affine spaces which are both infinite dimensional and infinite codimensional, making it hard to define their degree in homology. First, we note that it is technically more convenient to consider the space

$$V[[t]]X_{V[[t]]} = \{(g, v(t)) \in G((t)) \times V[[t]] \mid g \cdot v(t) \in V[[t]]\} / I$$

Basic properties of equivariant homology lead us to expect that

$$H^{BM,G((t))}_*(X_{V[[t]]} \times_{V((t))} X_{V[[t]]}) \cong H^{BM,\tilde{T}}_*(V[[t]]X_{V[[t]]});$$

we will use this as a definition of the left hand side. The preimage in $V[[t]]X_{V[[t]]}$ of a Schubert cell in $\mathcal{F}$ is a cofinite dimensional affine subbundle of $V((t))$; thus, using both the dimension of the Schubert cell, and the codimension of the affine bundle, we can make sense of the difference between the dimensions of these cells. With a bit more work, this allows us to make precise the notion of this homology, as in [BFNB] §2(ii)]. For our purposes, we can use their construction as a black-box, only knowing that basic properties of pushforward and pullback operate as expected.

Definition 3.2 The BFN Steinberg algebra $\mathcal{A}$ is the equivariant Borel-Moore homology $H^{BM,G((t))}_*(X_{V[[t]]} \times_{V((t))} X_{V[[t]]})$.

An important special case of this algebra has also been considered in [BEF] §4, when the representation is of quiver type (as discussed in Section 2.5). As usual, we let $h$ be the equivariant parameter corresponding to the character $\nu$. Note that this algebra contains a copy of $S_h = S[h] \cong \mathbb{C}[\mathfrak{t}]$, the coordinate ring of $\mathfrak{t}$, embedded as $H^{BM,G((t))}_*(X_{V[[t]]}) \cong H_0(\mathfrak{t})$. The algebra $\mathcal{A}$ also possesses a natural action on this cohomology ring.

The original BFN algebra $\mathcal{A}^{sph}$ is defined in essentially the same way, using $\mathcal{Y}_{V[[t]]} := (G((t)) \times V[[t]])/G[[t]]$. Pullback by the natural map $X_{V[[t]]} \rightarrow \mathcal{Y}_{V[[t]]}$ defines a homomorphism $\mathcal{A}^{sph} \rightarrow \mathcal{A}$.

Theorem 3.3 The algebras $\mathcal{A}^{sph}$ and $\mathcal{A}$ are Morita equivalent. In fact, the latter is a matrix algebra over the former of rank $\# W$. 

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Proof. This is a standard result that holds whenever we have a fiber bundle $X \to Y$ such that the pushforward of $\mathbb{L}X$ to $Y$ is a sum of constant sheaves and any map $Y \to Z$: the convolution algebra $H^*(X \times_Z X)$ is a matrix algebra over $H^*(Y \times_Z Y)$ with rank given by the sum of Betti numbers of the fiber.

We have a natural homomorphism $H_{BM,\tilde{T}}(V([t]) \times \mathcal{X}_V([t]) \times \mathcal{X}_V([u])) \to \mathcal{A}$. The map $\mathcal{X}_V([t]) \to V([t])$ is a fiber bundle with fiber $G/B$, and equivariance shows that the pushforward is a sum of constant sheaves. Thus, the former convolution algebra is a matrix algebra over $H_{BM,\tilde{T}}(V([t])) \cong S_h$ of rank $\#W$. The image of any primitive idempotent in $H_{BM,\tilde{T}}(\mathcal{X}_V([t]) \times \mathcal{X}_V([u]))$ gives an idempotent $e \in \mathcal{A}$ such that $\mathcal{A}e\mathcal{A} = \mathcal{A}$ and $\mathcal{A}^{ph} \cong e\mathcal{A}e$, thus these algebras are Morita equivalent. \qed

Note that $\mathcal{A}$ contains as a subalgebra $\mathcal{A}_{ab}$, the BFN Steinberg algebra for the subgroup $\tilde{T}((t))$. If we identify the Steinberg algebra with the homology $\mathcal{A} \cong H^*_{BM,\tilde{T}}(V[[t]]\mathcal{X}_V[[t]])$ then $\mathcal{A}_{ab}$ is the image of pushforward $\pi_{ab}$ from

$$\mathcal{A}_{ab} \cong H^*_{BM,\tilde{T}}\left(\{(g, v(t)) \in T((t)) \times V[[t]] \mid g \cdot v(t) \in V[[t]]\} / T[[t]]\right).$$

3.1. The extended category. While the Coulomb branch is our focus, it is easier to study it in a larger context: there is an extended category in which it appears as the endomorphisms of one object.

Given any $\eta \in \mathfrak{t}_c$, we can consider the induced action on the vector space $V((t))$.

- Let $I_\eta$ be the subgroup whose Lie algebra is the sum of positive weight spaces for the adjoint action of $\eta$. This only depends on the alcove in which $\eta$ lies, i.e. which chamber of the arrangement given by the hyperplanes $\{\alpha(\eta) = n \mid n \in \mathbb{Z}\}$ contains $\eta$; the subgroup $I_\eta$ is an Iwahori if $\eta$ does not lie on any of these hyperplanes.

- Let $U_\eta$ be the subspace of elements of non-negative weight under $\eta$. This subspace is closed under the action of $I_\eta$. This only depends on the vector $a$ such that $\eta \in C'_a$, as defined in (2.5).

We call $\eta$ generic if does not lie on the hyperplanes $\{\varphi_i(\eta) = n + \epsilon_i \mid n \in \mathbb{Z}\}$ or $\{\alpha(\eta) = n \mid n \in \mathbb{Z}\}$; we'll call these hyperplanes the unrolled hyperplane arrangement.
Example 3.i. We illustrate this arrangement in the case of our running example:

\[
\begin{array}{cccc}
\phi_1 = \frac{5}{3} & \phi_3 = -\frac{2}{3} & \phi_2 = \frac{5}{3} & \phi_4 = \frac{2}{3} \\
\phi_1 = \frac{5}{3} & \phi_3 = -\frac{2}{3} & \phi_2 = \frac{5}{3} & \phi_4 = \frac{2}{3} \\
\phi_1 = \frac{5}{3} & \phi_3 = -\frac{2}{3} & \phi_2 = \frac{5}{3} & \phi_4 = \frac{2}{3} \\
\phi_1 = \frac{5}{3} & \phi_3 = -\frac{2}{3} & \phi_2 = \frac{5}{3} & \phi_4 = \frac{2}{3} \\
\end{array}
\]

The spaces \(U_\chi\) which arise will be of the form:

\[
U_\chi = \{ (f_1 t^a, f_2 t^b, f_3 t^{a-\delta_1}, f_4 t^{b-\delta_2}) \mid f_i \in \mathbb{C}[[t]] \}
\]

for \(a, b \in \mathbb{Z}, \delta_i \in \{0, 1\}\) are uniquely characterized by the inequalities:

\[
-a + \epsilon_1 \chi(\gamma_1) < -a + 1 + \epsilon_1 \quad \quad -b + \epsilon_2 \chi(\gamma_2) < -b + 1 + \epsilon_2 \\
-a + \delta_1 + \epsilon_3 \chi(\gamma_1) < -a + \delta_1 + 1 + \epsilon_3 \quad \quad -b + \delta_2 + \epsilon_4 < \chi(\gamma_2) < -b + \delta_2 + 1 + \epsilon_4
\]

Tracing these definitions through, we see that:

- The case \(\delta_1 = \delta_2 = 0\) corresponds to the larger squares in the diagram above.
- The case \(\delta_1 = \delta_2 = 1\) corresponds to the smaller squares.
- The case \(\delta_1 = 1, \delta_2 = 0\) corresponds to the tall rectangles.
- The case \(\delta_1 = 0, \delta_2 = 1\) corresponds to the fat rectangles.

The region where \(a = b = 0\) is shaded in the diagram above.

For any generic \(\eta \in t_e\), we can consider \(X_\eta := X_{U_\eta} := G((t)) \times_{I_\eta} U_\eta\), the associated vector bundle. The space \(t_e\) has a natural adjoint action of \(\tilde{W} = N_{G((t))}(T)/T\), and of course, \(U_{w \cdot \eta} = w \cdot U_\eta\).

We let

\[
\eta X_{\eta'} = \{ (g, v(t)) \in G((t)) \times U_\eta \mid g \cdot v(t) \in U_{\eta'} \} / I_{\eta'}.
\]

**Definition 3.4** Let the extended BFN category \(\mathcal{B}\) be the category whose objects are generic cocharacters \(\eta \in t_e\), and such that

\[
\text{Hom}(\eta, \eta') = H^*_{BM,\widetilde{G((t))}}(X_\eta \times_{V((t))} X_{\eta'}) \cong H^*_{BM,\tilde{T}}(\eta X_{\eta'}).
\]
 As before, this homology is defined using the techniques in [BFN][§2(ii)].

Note that any lift of \( w \in \tilde{W} \) to \( G((t)) \) induces an isomorphism \( X_\eta \cong X_{w \cdot \eta} \) given by \((g, v(t)) \mapsto (wgw^{-1}, w \cdot v(t))\). We denote the homology class of the graph of this isomorphism by \( y_w \) (note that this class is independent of the choice of lift).

In this case, we have \( U_0 = V[[t]] \). Then, we have that \( \mathcal{A} = \text{Hom}_{\mathfrak{g},\mathfrak{b}}(0, 0) \). Thus, this extended category encodes the structure of \( \mathcal{A} \). Furthermore, the category of representations of \( \mathcal{A} \) is closely related to that of \( \mathcal{B} \). Let \( M \) be a representation of \( \mathcal{B} \), that is, a functor from \( \mathcal{B} \) to the category of \( \mathbb{K} \)-vector spaces. The vector space \( N := M(0) \) has an induced \( \mathcal{A} \)-module structure. Since \( \text{Hom}(\eta, 0) \) and \( \text{Hom}(0, \eta) \) are finitely generated as \( \mathcal{A} \)-modules, this functor preserves finite generation, and is in fact a quotient functor, with left adjoint given by

\[
N \mapsto \mathcal{B} \otimes_{\mathcal{A}} N(\eta) := \text{Hom}(\eta, 0) \otimes_{\mathcal{A}} N.
\]

Note that there is a natural subcategory \( \mathcal{B}_{ab} \) (with the same objects), where the morphisms are given by

\[
\text{Hom}_{ab}(\eta, \eta') \cong H^{BM,T \times \mathbb{C}^*}_* \left( \{(g, v(t)) \in T((t)) \times U_\eta \mid g \cdot v(t) \in U_{\eta'} \} / T[[t]] \right).
\]

The inclusion is induced by pushforward in homology.

### 3.2. A presentation of the extended category

Let

\[
r(\eta', \eta) = \left[ \{(e, v(t)) \in T((t)) \times U_\eta \mid v(t) \in U_{\eta'} \} / T[[t]] \right] \in \text{Hom}_{ab}(\eta, \eta').
\]

If \( I_\eta = I_{\eta'} \) (that is, the chambers are in the same alcove), this is sent to the class in \( \text{Hom}(\eta, \eta') \) of the space

\[
Y(\eta, \eta) = \{(e, v(t)) \in G((t)) \times U_\eta \mid v(t) \in U_{\eta'} \} / I_\eta
\]

but this is not the case for \( \eta, \eta' \) in different alcoves. We also have a morphism \( y_\zeta \in \text{Hom}_{ab}(\eta, \eta + \zeta) \) for \( \zeta \in t_\mathbb{Z} \) (thought of as a translation in the extended affine Weyl group. Let \( \Phi(\eta, \eta') \) be the product of the terms \( \varphi_i^+ - nh \) over pairs \((i, n) \in [1, d] \times \mathbb{Z}\) such that we have the inequalities

\[
\varphi_i(\eta) > n + \epsilon_i \quad \varphi_i(\eta') < n + \epsilon_i
\]

hold. Let \( \Phi(\eta, \eta', \eta'') \) be the product of the terms \( \varphi_i^+ - nh \) over pairs \((i, n) \in [1, d] \times \mathbb{Z}\) such that we have the inequalities

\[
\varphi_i(\eta'') > n + \epsilon_i \quad \varphi_i(\eta') < n + \epsilon_i \quad \varphi_i(\eta) > n + \epsilon_i
\]

or the inequalities

\[
\varphi_i(\eta'') < n + \epsilon_i \quad \varphi_i(\eta') > n + \epsilon_i \quad \varphi_i(\eta) < n + \epsilon_i.
\]

These terms correspond to the hyperplanes that a path \( \eta \rightarrow \eta' \rightarrow \eta'' \) must cross twice. Note that if \( h \) is specialized to 0, then we just get each weight \( \varphi_i \) raised to a power given by the number of corresponding unrolled hyperplanes crossed.
Proposition 3.5 The morphisms $\text{Hom}_{ab}(\eta, \eta')$ have a basis over $S_h$ of form $y_\zeta \cdot r(\eta' - \zeta, \eta)$ for $\zeta \in \mathfrak{t}_Z$, with the relations in the category $\mathcal{B}_{ab}$ generated by:

\begin{align}
(3.1a) & \quad y_\zeta \cdot y_{\zeta'} = y_{\zeta + \zeta'} \\
(3.1b) & \quad y_\zeta \cdot r(\eta', \eta) \cdot y_{-\zeta} = r(\eta' + \zeta, \eta + \zeta) \\
(3.1c) & \quad y_\zeta \cdot \mu \cdot y_{-\zeta} = \mu + \zeta \\
(3.1d) & \quad r(\eta'', \eta') r(\eta', \eta) = \delta_{\eta'', \eta'} \Phi(\eta'', \eta', \eta) r(\eta'', \eta)
\end{align}

Proof. This is just a restatement of [BFN jailed Section 4(i–iii)].

If we draw $r(\eta', \eta)$ as a straightline path in $\mathfrak{t}$, and thus compositions of these elements as piecewise linear paths, with the unrolled arrangement drawn in, we can visualize the relation (3.1d) as saying that when we remove two crossings of the hyperplane $\varphi_i(\eta) = n + \epsilon_i$ from the path, we do so at the cost of multiplying by $\varphi_i^+ - nh$. We can thus represent elements of $\text{Hom}_{ab}(\eta, \eta)$ as paths which start at $\eta$ and go to any other chamber of the form $\eta - \zeta$ (we implicitly follow these with translation $y_\zeta$). Composition of two paths $p$ and $q$ is thus accomplished by translating $p$ so its start matches the end of $q$, and then straightening using the relation (3.1d).

Example 3.ii. In our running example, let us fix $\eta = 0$. We let $\xi_1, \xi_2$ be the usual coordinate cocharacters of the diagonal $2 \times 2$ matrices. The algebra $\mathcal{A}_{ab} = \text{Hom}_{ab}(0, 0)$ is generated over $S_h$ by

\begin{align*}
w_1 &= y_{-\xi_1} \cdot r(\xi_1, 0) \quad w_2 = y_{-\xi_2} \cdot r(\xi_2, 0) \quad z_1 = y_{\xi_1} \cdot r(-\xi_1, 0) \quad z_2 = y_{\xi_2} \cdot r(-\xi_2, 0)
\end{align*}

with the relations

\begin{align*}
[z_1, w_1] &= [z_1, w_2] = [z_2, w_1] = [z_2, w_2] = 0 \\
z_1 w_1 &= \gamma_1 (\gamma_1 - 2h) \quad w_1 z_1 = (\gamma_1 + h)(\gamma_1 - h) \\
z_2 w_2 &= \gamma_2 (\gamma_2 - 2h) \quad w_2 z_2 = (\gamma_2 + h)(\gamma_2 - h)
\end{align*}

since

\begin{align*}
\varphi_1^+ &= \gamma_1 + h \quad \varphi_2^+ = \gamma_2 + h \quad \varphi_1^- = \gamma_1 - h \quad \varphi_2^- = \gamma_2 - h.
\end{align*}
In terms of our path description:

\[
\begin{array}{cccccc}
\psi_1 = \frac{1}{3} & \psi_2 = \frac{1}{3} & \psi_3 = \frac{1}{3} & \psi_4 = \frac{1}{3} & \psi_1 = -\frac{1}{3} & \psi_2 = -\frac{1}{3} & \psi_3 = -\frac{1}{3} & \psi_4 = -\frac{1}{3} \\
\psi_1 = \frac{2}{3} & \psi_2 = \frac{2}{3} & \psi_3 = \frac{2}{3} & \psi_4 = \frac{2}{3} & \psi_1 = -\frac{2}{3} & \psi_2 = -\frac{2}{3} & \psi_3 = -\frac{2}{3} & \psi_4 = -\frac{2}{3} \\
\psi_1 = \frac{4}{3} & \psi_2 = \frac{4}{3} & \psi_3 = \frac{4}{3} & \psi_4 = \frac{4}{3} & \psi_1 = -\frac{4}{3} & \psi_2 = -\frac{4}{3} & \psi_3 = -\frac{4}{3} & \psi_4 = -\frac{4}{3} \\
\end{array}
\]

Now, we turn to generalizing this presentation to the nonabelian case. We can easily check that the relations (3.1a–3.1c) hold in \( B \) for all elements of the extended affine Weyl group:

\[
\begin{align*}
(3.2a) & \quad y_w \cdot y_w' = y_{ww'} \\
(3.2b) & \quad y_w r(\eta', \eta) y_w^{-1} = r(w \cdot \eta', w \cdot \eta) \\
(3.2c) & \quad y_w \mu y_w^{-1} = w \cdot \mu
\end{align*}
\]

Finally, if \( \alpha(\eta) = 0 \) for some affine root \( \alpha \) but no other weights or roots vanish, then we can make this generic in two different ways: \( \eta_\pm := \eta \pm \epsilon \alpha^\vee \). Let \( I_\pm \) be the corresponding Iwahoris. Note that for \( \epsilon \ll 1 \), we have \( U_\eta = U_{\eta_\pm} \). Let

\[
X_\alpha(\eta) = \{ (g v(t), g I_\pm, g I_\mp) \in X_{\eta_\pm} \times_{V((t))} X_{\eta_\mp} \mid g \in G((t)), v(t) \in U_\eta \}.
\]

Let \( u_\alpha(\eta) = [X_\alpha(\eta)] \in \text{Hom}(\eta_\pm, \eta_\mp) \).

**Theorem 3.6** The morphisms in the extended BFN category are generated by

1. \( y_w \) for \( w \in \widehat{W} \),
2. \( r(\eta, \eta') \) for \( \eta, \eta' \in \mathfrak{t} \) generic,
3. the polynomials in \( S_h \),
4. \( u_\alpha(\eta) \) for \( \eta_\pm \) affine chambers adjacent across \( \alpha(\eta) = n \).

These act in the polynomial representation by:

\[
\begin{align*}
(3.3a) & \quad w f = w \cdot f \\
(3.3b) & \quad r(\eta, \eta') f = \Phi(\eta, \eta') f \\
(3.3c) & \quad \mu f = \mu \cdot f \\
(3.3d) & \quad u_\alpha f = \partial_\alpha(f)
\end{align*}
\]
The relations between these operators are given by (3.1a [3.2c]) and the further relations

\[(3.4a)\]
\[u_\alpha^2 = 0\]
\[(3.4b)\]
\[u_\alpha u_s \beta u_{s \beta} \cdot \cdots = u_\beta u_{s \beta} \cdot u_{s \beta} \cdot \cdots\]
\[(3.4c)\]
\[wu_u v^{-1} = u_{w \alpha}\]
\[(3.4d)\]
\[u_\alpha \mu - (s_\alpha \cdot \mu) u_\alpha = \partial_\alpha (\mu)\]

whenever these morphisms are well-defined and finally, if \( \eta_+^k \) and \( \eta_-^k \) are two pairs of chambers opposite across \( \alpha(\eta) = 0 \) which both lie on minimal length paths from \( \eta \) to \( s_\alpha \eta \), then

\[(3.4d)\]
\[r(s_\alpha \eta, \eta_+^k) u_\alpha r(\eta_+^k, \eta) - r(s_\alpha \eta, \eta_-^k) u_\alpha r(\eta_-^k, \eta)\]
\[= (\Phi(s_\alpha \eta, \eta_+^k) \partial_\alpha (\Phi(\eta_+^k, \eta)) - \Phi(s_\alpha \eta, \eta_-^k) \partial_\alpha (\Phi(\eta_-^k, \eta))) s_\alpha.\]

Let \( \eta, \eta' \) be two affine chambers, and let \( (\eta_\pm^i, \beta_i) \) be the list of Coxeter hyperplanes with the corresponding opposite chambers crossed by some minimal length path \( \eta \rightarrow \eta' \). Let

\[\tilde{r}(\eta', \eta) = r(\eta', \eta_\pm^k) u_{\beta_{k-1}} r(\eta_\pm^k, \eta_\pm^{(k-1)}) u_{\beta_{k-2}} \cdots u_{\beta_1} r(\eta_1, \eta).\]

As in the abelian case, we can represent morphisms in our category by paths, but now we have to insert Demazure operators every time that we cross an unrolled root hyperplane.

**Proof.** The verification of the action is straightforward using the formula [BFNa A.2]. Given the representation and its faithfulness, the reader readily verify that the relations (3.1a [3.4d]) are satisfied. The most interesting of these relations is (3.4d), so let us verify this relation in more detail. The action of the LHS in the polynomial rep on a polynomial \( f \) is:

\[(3.5)\]
\[\Phi(s_\alpha \eta, \eta_\pm^k) \partial_\alpha (\Phi(\eta_\pm^k, \eta)) f - \Phi(s_\alpha \eta, \eta_-^k) \partial_\alpha (\Phi(\eta_-^k, \eta)) f\]
\[= \Phi(s_\alpha \eta, \eta_\pm^k) \Phi(\eta_\pm^k, \eta) \partial_\alpha (f) + \Phi(s_\alpha \eta, \eta_-^k) \Phi(\eta_-^k, \eta) \partial_\alpha (f)\]
\[- \Phi(s_\alpha \eta, \eta_-^k) \Phi(\eta_-^k, \eta) \partial_\alpha (f) - \Phi(s_\alpha \eta, \eta_-^k) \partial_\alpha (\Phi(\eta_-^k, \eta)) f_{s_\alpha}.\]

Since \( \eta' \) and \( \eta'' \) are both on the minimal length paths, neither is separated from both \( s_\alpha \eta \) and \( \eta \) by any given unrolled hyperplane. Thus, we have that

\[\Phi(s_\alpha \eta, \eta_\pm^k) \Phi(\eta_\pm^k, \eta) = \Phi(s_\alpha \eta, \eta_-^k) \Phi(\eta_-^k, \eta) = \Phi(s_\alpha \eta, \eta).\]

It follows that the first positive and negative terms in (3.5) cancel, and we obtain the RHS of (3.4d). This confirms the relation.

Using the action of the elements of \( \widehat{W} \) we can reduce to the case where \( \eta \) and \( \eta' \) are in the same alcove. The space \( \text{Hom}(\eta, \eta') \) has a filtration by the length of the relative position of the two affine flags. Let \( \text{Hom}^\leq_w (\eta, \eta') \) be the homology classes supported on the pairs of relative distance \( \leq w \). By basic algebraic topology, \( \text{Hom}^\leq_w (\eta, \eta') / \text{Hom}^w (\eta, \eta') \)
is a free module of rank 1 over $S_h$, since this space is isomorphic to the $I$-equivariant Borel-Moore homology of a (infinite dimensional) affine space.

We'll prove that
\[ (\ast) \text{ the } S_h\text{-module } \text{Hom}^{\leq w}(\eta, \eta')/\text{Hom}^{< w}(\eta, \eta') \text{ is generated by the element } \tilde{r}(\eta', w\eta)w. \]

The element $\tilde{r}(\eta', w\eta)w$ is the pushforward of the fundamental class by the map
\[
Y(\eta', \eta_{\pm}) \times I(\eta_{(k)}) X(\beta) \times I(\eta_{(k-1)}) Y(\eta_{(k-1)}, \eta_{\pm}) \times I(\eta_{(k-2)}) \times I(\eta_{(1)}) Y(\eta_{(1)}, \eta) \rightarrow X(\xi) \times V((t)) X(\xi).
\]

This map is an isomorphism on the set of affine flags of relative position $w$. Thus, these elements give a free basis of the associated graded for this filtration. This implies that they are a basis of the original module; in particular, this implies that the elements from the list above are generators.

On the other hand, we can also easily show that the relations displayed are enough to bring any element into the form of a sum of elements $\tilde{r}(\eta', w\eta)w$. We can pull all elements of the Weyl group to the right using \(3.2b, 3.2c, 3.4c\), all elements of $S_h$ to the right using the relation \(3.4d\), and rewrite any crossing of a Coxeter hyperplane by $r(-, -)$ using the relation $r(\eta_{\pm}, \eta_{\pm}) = s_{\alpha} - \alpha_w w^{-1}$. This shows these relations suffice, since there can be no further relations between our basis.

Let $\tilde{H}$ be the group generated by the action on $V[[t]]$ of $G((t))$, $H$ and the loop rotation $\mathbb{C}^*; \text{ let } T_H$ be the torus of the group generated by $G$ and $H$. Note that $\widehat{G((t))} \subset \tilde{H}$.

**Definition 3.7** The deformed extended BFN category is the category with the same objects as $\mathcal{B}$ and
\[
\text{Hom}(\eta, \eta') = H_{BM, \tilde{H}}^*(\mathbb{X}_{\eta} \times_{V((t))} \mathbb{X}_{\eta'}) \cong H_{BM, \tilde{T}_H}^*(\eta \mathbb{X}_{\eta'}). 
\]

The results above, such as Theorem 3.6, carry over in a straightforward way to this category. The only difference is that we must interpret the products of weights $\Phi(-, -)$ as weights of $\tilde{T}_H$, and rather than an action of $S_h$, we have one of $\mathbb{k}[\tilde{t}_H]$.

This is naturally a subcategory inside the extended BFN category for the group $GT_H$ acting on $V$. It is the subcategory where we only consider cocharacters in $t_\mathbb{C}$ as objects, and only allow ourselves to use $t_\mathbb{Z}$ in the extended affine Weyl group, rather than all of $(t_H)_{\mathbb{Z}}$. The category attached to $GT_H$ has a natural action of the dual torus $(T_H/T)^\vee$ on the morphisms between any two objects with $r(-, -), \mathbb{k}[\tilde{t}_H]$ and $\tilde{W}$ having weight 0, and the copy of $\mathbb{k}[T_H]$ having the obvious action. The classes of weight $\nu$ (which is a coweight of $T_H/T$) correspond to homology classes concentrated on the components of the affine flag variety whose corresponding loop has a homotopy class hitting $\nu$ under the map $\pi_1(\mathcal{F}) \rightarrow X(T_H/T)$. This shows that:
Lemma 3.8  The deformed extended BFN category equivalent to the subcategory of \( \mathcal{B}(G_{TH}) \) where we only allow \((T_H/T)^{-}\)-invariant morphisms and objects lying in \( t_{\epsilon} \).

Of course, if we instead fix \( \nu \in X(T_H/T) \) and look only at morphisms with this weight, we obtain a bimodule over the extended BFN category, which we denote \( \mathcal{F}(\nu) \).

The cocharacter lattice \( X(T_H/T) \) acts by pointwise multiplication on the space of flavors into \( T_H \) up to different choices of lift. Thus, given one choice of flavor \( \phi \) and the associated \( t_1 \), we have \( t_1 + \nu \) is the subspace associated to flavor \( \phi + \bar{\nu} \) (where \( \bar{\nu} \) is any lift of \( \nu \) to \( X(T_H) \)).

If we think of a weight \( \mu \) of \( T_H/T \) as a morphism in the extended BFN category (i.e. as an equivariant cohomology class), then its left action on \( \mathcal{T}(\nu) \) is equal to the action of \( \mu + \langle \nu, \mu \rangle \) on the right. Since the ideal \( I(t_1) \) cutting out \( t_1 \) are of this form, we have that the image of the right action of \( I(t_1) \) is the same as the image of the left action of \( I(t_1 + \nu) \).

Definition 3.9  The \( \mathcal{B}_{\phi+\nu} - \mathcal{B}_{\phi} \) bimodule \( \mathcal{B}_{\phi} \) is the quotient of \( \mathcal{F}(\nu) \) by \( I(t_1) \) acting on the right or \( I(t_1 + \nu) \) acting on the left.

Let \( \mathcal{B}_{\phi+\nu} \) be the corresponding bimodule over \( \mathcal{A}_{\phi+\nu} \) and \( \mathcal{A}_{\phi} \).

3.3. Representation theory. Throughout this section, we specialize \( h = 1 \); we let \( S_1 = S_h/(h-1) \cong \mathbb{C}[t_1] \). We call a \( \mathcal{B} \)-module \( M \) (resp. \( \mathcal{A} \)-module \( N \)) a weight module if for every \( \eta \), we have that \( M(\eta) \) (resp. \( N \)) is locally finite as a module over \( S_1 \) with finite dimensional generalized weight spaces. Obviously, if \( M \) is a weight module, then \( N = M(0) \) is as well. The adjoint \( \mathcal{B} \otimes \mathcal{A} \) also sends weight modules to weight modules, since the adjoint action of \( S_1 \) on \( \text{Hom}(\eta,0) \) is semi-simple with eigenspaces finitely generated over \( S_1 \).

For each \( \nu \in t_1, \eta \in t_\epsilon \), we can consider the functor \( W_{\nu,\eta} : \mathcal{B}_{-}\text{-mod} \to \mathcal{K}_{-}\text{-mod} \) defined by

\[
W_{\nu,\eta} = \{ m \in M(\eta) \mid m \mathfrak{m}^N = 0 \text{ for } N \gg 0 \},
\]

with \( \mathfrak{m}_v \) for \( v \in t_1 \) be the corresponding maximal ideal in \( S_1 \). These functors are exact, and prorepresentable. If we let \( \mathcal{A}_\eta := \text{Hom}(\eta,\eta) \), then they are represented by the projective limit

\[
P_{\nu,\eta} := \varprojlim \mathcal{B} \otimes \mathcal{A}_\eta \left( \mathcal{A}_\eta / \mathcal{A}_\eta \mathfrak{m}_v^N \right)
\]

as \( N \to \infty \).

Thus, as in [MV98], we can present the category of weight modules as modules over \( \text{End}(\oplus P_{\nu,\eta}) \). If we restrict the weights we allow in our modules, then the result will be the representations of a subquotient of this ring.

The morphisms \( \text{Hom}(P_{\nu,\eta}, P_{\nu',\eta'}) \) are relatively easy to understand: up to completion, such a morphism is given by right multiplication by a morphism \( f \in \text{Hom}(\eta',\eta) \) such
that $m_{v'} f = f m_v$. The space of such morphisms is spanned by $w \cdot r(w^{-1} \eta, \eta')$ for $w \in \hat{W}$ satisfying $w \cdot v' = v$. In particular:

- If $v \notin \hat{W} \cdot v'$, then $\text{Hom}(P_v, \eta, P_{v'}, \eta') = 0$.
- If $v \in \hat{W} \cdot v'$, then $\text{Hom}(P_v, \eta, P_{v'}, \eta')$ has rank equal to $\text{Stab}_{\hat{W}}(v)$ over the completion of $S_1$ at $v$.

**Definition 3.10** Let $\hat{B}$ be the category whose objects are the set $\mathcal{J}$ of pairs of generic $\eta \in t_\epsilon$ and any $v \in t_1$, such that

$$\text{Hom}_{\hat{B}}((\eta', v'), (\eta, v)) = \lim_{\leftarrow} \text{Hom}(\eta', \eta)/(m_v^N \text{Hom}(\eta', \eta) + \text{Hom}(\eta', \eta)m_v^N).$$

We let $\hat{B}_{v'}$ be the subcategory where we only allow objects with $v \in v' + t_\mathbb{Z}$.

It might seem more natural to consider the larger category where we allow $v \in \hat{W} \cdot v'$, but the resulting categories are equivalent, since $(\eta, v) \cong (w\eta, wv)$ for $w \in W$ in the the finite Weyl group.

The results above establish:

**Lemma 3.11** The category of weight modules over $B$ is equivalent to the category of representations of $\hat{B}$ in the category of finite dimensional vector spaces. The category of weight modules over $B$ with weights in $\hat{W} \cdot v'$ is equivalent to the category of representations of $\hat{B}_{v'}$ in the category of finite dimensional vector spaces.

The category $\hat{B}_{v'}$ contains a subcategory $\hat{A}_{v'}$ given by the objects of the form $(0, v)$ for $v \in v' + t_\mathbb{Z}$. Let $A$-$\text{mod}_{v'}$ denote the representations of this category, which are equivalent to the category of weight modules over $A$ with weights in $\hat{W} \cdot v'$.

4. **Higgs and Coulomb**

4.1. **The isomorphism.** Assume that $k$ has characteristic 0; we are still specializing $h = 1$. Consider $\rho \in t_1$, and let $J = \rho + t_\mathbb{Z}$. We’ll call a weight $\varphi_i$ of $V$ or root $\alpha_i$ of $g$ relevant if it has integral value on $\rho$, and irrelevant if it does not. The relevant roots form the root system of a Levi subalgebra $l = l_I \subset g$, and the sum $V_I$ of relevant weight spaces carries an $l$ action.

Note that $W_l$ is isomorphic to the stabilizer in $\hat{W}$ of any element of $J$, and in particular, $\#W_l$ is the rank of the Hom space between any two objects of $\hat{B}_{\rho}$.

We now turn to considering the constructions of Section 2 for the group $L$ acting on the vector space $V_J$. We can consider the categories $X_{\alpha}, X_X$ as discussed in Section 2.4. Let $\hat{X}_{\varphi}, \hat{X}_X$ be the completions of these categories with respect to their gradings.

Let $\Phi_0(\eta, \eta', \rho)$ be the product of the terms $\varphi_i^+ - n$ over pairs $(i, n) \in [1, d] \times \mathbb{Z}$ such that we have the inequalities

$$\varphi_i(\eta) > n + \epsilon_i \quad \varphi_i(\eta') < n + \epsilon_i \quad \langle \varphi_i^+ , \rho \rangle \neq n.$$
Koszul duality between Higgs and Coulomb categories \( \mathcal{O} \)

Note that if \( \mu \in t^* \), then \( \mu \) does not have a canonical extension to \( t_1 \), but the expression \( \mu - \langle \mu, \rho \rangle \) is well-defined on \( t_1 \), giving the same answer for any extension of \( \mu \) to \( \tilde{t} \).

**Definition 4.1** Let \( \gamma \) be the functor

\[
\gamma : \hat{\mathcal{X}}_e \to \hat{\mathcal{B}}_\rho
\]

which sends each \( a \in \mathcal{C} \) to \((\xi_a, \rho)\), for \( \xi_a \) a generic element of \( C'_a \), and which acts on morphisms by

\[
\begin{align*}
\gamma(w(a,b)) &= \frac{1}{\Phi_0(\eta_a, \eta_b, \rho)} r(\eta_a, \eta_b) \\
\gamma(w) &= w \\
\gamma(\psi_i(a)) &= s_\alpha_i u_\alpha_i \\
\gamma(\mu) &= \mu - \langle \mu, \rho \rangle
\end{align*}
\]

As mentioned in the introduction, these formulae can be explained geometrically. In particular, \( \Phi_0(\eta_a, \eta_b, \rho) \) can be interpreted as the Euler class of a normal bundle, just as in [VV10, Proposition 2.4.7].

**Theorem 4.2** The functor \( \gamma \) is an equivalence \( \hat{\mathcal{X}}_e \cong \hat{\mathcal{B}}_\rho \) which induces an equivalence \( \hat{\mathcal{X}} \cong \hat{\mathcal{A}}_\rho \).

**Proof.** First, we must check that this functor is well-defined. We let \( S_1^v = \lim_{\leftarrow} S_1 / m_v^N \). There is a natural faithful representation of the category \( \hat{\mathcal{B}}_\rho \) sending \((\eta, v)\) to \( S_1^v \). As in many previous proofs, we’ll prove the equivalence by comparing this with the polynomial representation of \( \hat{\mathcal{X}}_e \) given in (2.6). We consider the completion of this polynomial representation with respect to the grading.

Consider the induced isomorphism \( s_\rho : \mathbb{C}[[t]] \to S_\rho \) via shifting, that is, the image of an linear function \( \mu \in t^* \) is \( \mu - \langle \rho, \mu \rangle \). The match for (4.1b–4.1d) is clear, but perhaps we should say a bit more about (4.1a). The image of \( \phi(a, b) \) is the product of the linear factors \( \varphi_i^+ - \langle \rho, \varphi_i \rangle \) over the \( \varphi_i^+ \) which are relevant and which have different signs on \( a \) and \( b \). This always divides \( \Phi(\xi_a, \xi_b) \), and the remaining factors are precisely \( \Phi_0(\xi_a, \xi_b, \rho) \), which shows the compatibility of (2.6) and (4.1a) with (3.3b).

The functor \( \gamma \) is clearly full, since all but one of the generators is an explicit image and \( r(\xi_a, \xi_b) = \gamma^{-1}(\Phi_0(\xi_a, \xi_b, \rho)) w(a, b) \). Thus, the map \( \text{Hom}_{\hat{\mathcal{X}}_e}(a, b) \to \text{Hom}_{\hat{\mathcal{B}}_\rho}(\xi_a, \xi_b) \) is surjective. These are both free modules over \( \mathbb{C}[[t]] \) with rank equal to \#\( W_t \), so a surjective map between them must be an isomorphism.

Finally, we note that if \( C'_a \) contains a point of \( X^*_1(T) \), then we can take this to be \( \xi_a \) and in \( \hat{\mathcal{B}}_\rho \), we have an isomorphism \((\xi_a, \rho) \cong (0, \rho - \xi_a)\). The latter is an object in \( \mathcal{A}_\rho \), and every one is of this form. \( \square \)
As before, let $I$ be the set of sign vectors $\sigma$ whose chamber $C_{\sigma,1}$ contains an element of $X$. The sum of morphisms in the category $A_I$ gives a finite dimensional algebra; we’ll abuse notation and let the same symbol denote this finite dimensional algebra. The identity $1_\sigma$ on $\sigma$ can be thought of as an idempotent in this algebra.

**Corollary 4.3** The category $A$-$\text{mod}_\rho$ of weight modules with weights in $\hat{W} \cdot \rho$ is equivalent to the category of representations of $A_I$ in finite dimensional vector spaces where $S$ acts nilpotently; this functor matches the weight space for any weight in $C_{\sigma,1}$ with the image of the idempotent $1_\sigma$.

This isomorphism also allows us to define a graded lift of the category of weight modules given by modules with a grading on their weight spaces such that $\mathcal{X}_\phi$ acts homogeneously.

We can easily extend this isomorphism to the bimodule $\phi + \nu$. This has a natural completion $\hat{\mathcal{F}}_\phi$ to a bimodule over the categories $\hat{\mathcal{B}}$ associated to the flavors $\phi + \nu$ and $\phi$. Applying Theorem 4.2 to the action of $GT_H$ on $V$, we find an isomorphism:

**Corollary 4.4** $I_{\phi + \nu} \mathcal{A}_I \phi \cong \phi + \nu \mathcal{A}_I \phi$.

### 4.2. Koszul duality

Assume that $A$ is an algebra over a field $\mathbb{k}$ graded by the non-negative integers with $A_0$ finite dimensional and semi-simple. The **Koszul dual** of $A$ is, by definition, the algebra $A^! \cong T_{A_0} A^*_1 / R^\perp$ where $R \subset A_1 \otimes_{A_0} A_1$ is the space of quadratic relations, the kernel of the map to $A_2$. The representation category of $A^!$ is equivalent to the abelian category $LCP(A)$ of linear complexes of projectives over $A$. If an abelian category is equivalent to the modules over an algebra $A$ as above, then the Koszul dual of the category is the category of representations of $A^!$.

Let

$$M := \bigoplus_{C_{\sigma,w} \neq \emptyset} (p_{\sigma,w})^* \mathcal{S}_{X_{\sigma,w}}.$$

Combining Theorems 2.17 & 4.2 shows that:

**Proposition 4.5** The Koszul dual of the category $A$-$\text{mod}_I$ is the category $\text{MHM}_M$ of $L_2$-equivariant mixed Hodge modules of type $M$.

**Remark 4.6.** In general, $D^b(\text{MHM}_M)$ may not be a full subcategory of the derived category of $D^b(\text{MHM})$, so $\text{Ext}^*_\text{MHM}(M,M) \cong A_I$ is not the same as the Ext-algebra in the category $\text{MHM}_M$. For example, in the “pure gauge field” case of $V = 0$, we have that $A_I$ is the cohomology of $BL_3$, which we can think of as symmetric functions on $t$ for the action of the integral Weyl group $W_3$. The algebra $A$ in this case is just the smash product $S_h \rtimes W$, and the subcategory $A$-$\text{mod}_I$ corresponds to the modules which are the sum of their weight spaces over $S_1$ for the $W$-translates of $\rho$.
The $\rho$ weight space has a natural action of the stabilizer $W_\rho$, and considering the $W_\rho$ invariants defines a functor from $\mathcal{A}\text{-mod}_I$ to $H^*(BL_I)$-modules where we let $H^*(BL_I) \cong \mathbb{C}[t]^{W_\rho}$ act by the $\rho$-shifted action. This gives the equivalence induced by Theorem 4.2.

In this case, since $H^*(BL_I)$ has no elements of degree 1, the only linear complexes over this ring are those with trivial differentials. Thus, the category $\mathcal{M}_{\mathcal{H}M_M}$ is equivalent to the category of vector spaces.

**Example 4.i.** Consider the case of $V = \mathbb{C}$ with $G = \mathbb{C}^*$ acting naturally, and flavor $\phi$ giving weight $a$ on $\mathbb{C}^*$ and $-a - 1$ on its dual space. In this case $S_1 \cong \mathbb{C}[t]$ with $t$ the natural cocharacter. The algebra $\mathcal{A} = \mathcal{A}^{\text{ph}}$ has generators $r^+$ and $r^-$ with

$$r^-r^+ = t - a - 1 \quad r^+r^- = t - a.$$ 

Note that $r^\pm$ give an isomorphism between the $k$ and $k + 1$ weight spaces unless $k = a$. Thus, if the weights of $t$ are not in $a + \mathbb{Z}$, then all weight spaces are isomorphic, and we are equivalent to the pure gauge situation.

If we take weight spaces of the form $a + \mathbb{Z}$, then there are two isomorphism classes, represented by $a$ and $a + 1$, with $r^\pm$ giving morphisms in both directions between them, with the composition in either direction acting by the nilpotent part of $t$. Thus, we obtain the completed path algebra of an oriented 2-cycle as $\text{End}(P_a \oplus P_{a+1})$. The Koszulity of this path algebra algebra is easily verified directly (since every simple has a length 2 linear projective resolution).

Since this path algebra has no quadratic relations, its quadratic dual is given by imposing all (two) possible quadratic relations: it is the path algebra of an oriented 2-cycle with all length-2 paths set to 0. This is the endomorphism ring of the projective generator in the category of strongly $\mathbb{C}^*$-equivariant D-modules on $\mathbb{A}^1$ generated by the functions and the $\delta$-functions at the origin. The two indecomposable projective D-modules in this category are $D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(z\frac{\partial}{\partial z})$ and $D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(\frac{\partial}{\partial z})$; their sum has the desired endomorphism algebra. The untruncated path algebra appears as the Ext-algebra of the sum of simple D-modules $D_{\mathbb{A}^1}/D_{\mathbb{A}^1}z \oplus D_{\mathbb{A}^1}/D_{\mathbb{A}^1}\frac{\partial}{\partial z}$.

Let $\mathcal{O}_I$ be the intersection of $\mathcal{A}\text{-mod}_I$ with category $\mathcal{O}$ for $\xi \in (\mathfrak{g}^*)^G$, that is the modules such that the eigenspaces for $\xi$ are finite dimensional and bounded above. We have a graded lift $\mathcal{O}_I$ of this category, defined as modules in $\mathcal{O}_I$ endowed with a grading on which the induced action of $\mathcal{X}_\xi$ is homogeneous.

This is a subcategory of $\mathcal{A}\text{-mod}_I$, consisting of all modules whose composition factors are all killed by $e(\sigma)$ if $\xi$ does not attain a maximum on a bounded subset of $C_\sigma$. Its Koszul dual is thus a quotient of $\mathcal{M}_{\mathcal{H}M_M}$ by the subcategory of modules whose composition factors all appear as summands of $L_\xi$ such that $\xi$ does not attain a maximum on a bounded subset of $C_\sigma$. By Lemma 2.20, this is the same as the quotient by the unsteady sign vectors. Thus, we have that:
Theorem 4.7 If \( (†) \) holds, then the Koszul dual of the category \( \mathring{\mathcal{O}}_\mathfrak{j} \) for the character \( \xi \) and flavor \( \phi \) is equivalent to a block of \( \mathring{\mathcal{O}}_\mathfrak{k}^! \) for the flavor \( \phi \) on \( \mathcal{M}_\xi = T^*V_l \) for the integral quantization.

If \( (\dagger) \) holds, then the Koszul dual of the category \( \mathring{\mathcal{O}}_\mathfrak{j} \) for the character \( \xi \) is equivalent to \( \mathring{\mathcal{O}}_\mathfrak{k}^! \) for the flavor \( \phi \) on \( \mathcal{M}_\xi = T^*V_l \) for the integral quantization.

4.3. Twisting and shuffling functors. Throughout this section, we assume \( (†) \) holds for simplicity. Recall the category \( \mathcal{O}'s \) for the varieties \( \mathcal{M}_H \) and \( \mathcal{M}_C \) are each endowed with actions of two collections of functors: twisting and shuffling functors. We refer the reader to [BPW, BLPW] for a more thorough discussion of these functors. In this paper, we will only consider pure shuffling and twisting functors for simplicity; it will be more natural to discuss the impure functors after a longer discussion of the Namikawa Weyl group of a Higgs branch.

Let us describe the form these functors take in the cases we are considering. Throughout the description below, we let \( \star \in \{!, \ast \} \). On \( \mathcal{M}_H \):

- The pure twisting functors are generated by functors \( \mathfrak{r}^\xi \circ \mathfrak{r}^{\xi'} \) composing the reduction functor \( \mathfrak{r}_\xi \cong D_{C_{\phi}} \rightarrow D_{C_{\phi}} \) to the category \( \mathcal{O} \) on \( \mathcal{M}_\xi \) with the left or right adjoint of this functor.
- The pure shuffling functors are generated by composing the inclusion functor \( i^\phi \) of \( D_{C_{\phi}} \) into \( D^b(\mathcal{D} \text{-mod}) \) with its left or right adjoint \( i^\phi_! \).

On \( \mathcal{M}_C \):

- The pure twisting functors are generated by tensor product with \( \phi / T_\phi \) for \( \phi \) and \( \phi' \) both generic in \( \mathcal{I} \), and its adjoint.
- The pure shuffling functors are generated by composing the inclusion \( i^\xi \) functor of \( \mathcal{O}_{\text{Coulomb}} \) into \( \mathcal{A} \text{-mod} \) with its left or right adjoint \( i^\xi_! \) in the derived category (i.e. the derived functor of taking the largest quotient or submodule in category \( \mathcal{O} \)).

Theorem 4.8 The Koszul duality of Theorem 4.7 switches pure twisting and shuffling functors matching \( \mathfrak{r}^{\xi'} \circ \mathfrak{r}_{\ast}^\xi \) with \( \mathfrak{r}_{\ast}^{\xi'} \circ \mathfrak{r}^\xi \) and \( \phi / T_\phi \otimes A_\phi - \) with \( i^{\phi'}_! \circ i^\phi \).

Proof. The proof of this fact is roughly the same as in [BLPW12, 8.24]. The shuffling functors come from inclusion of an projection to a subcategory, and the twisting functors come from projection to and adjoint inclusion of a quotient category; these naturally interchange under Koszul duality.

Now, let use be more precise. Let

\[
A_\xi^\xi := A_P / \mathcal{I} \quad \quad p^* A_\xi^\xi := p^* A_P / (p^* A_P A_\xi^\xi + p^* A_\xi A_\xi^\xi),
\]

where \( p^* A_P \) is the bimodule defined in Definition 2.13.

- Under the equivalence of \( D_{C_{\phi}} \) to \( A_{I_{\phi}} \text{-mod} \) and \( \mathcal{O}_{\text{Higgs}} \) with \( A_{I_{\phi}}^\xi \text{-mod} \), the functor \( \mathfrak{r}_\ast \) is intertwined with inflation of a \( A_{I_{\phi}}^\xi \) to an \( A_{I_{\phi}} \) module, and thus \( \mathfrak{r} \)
Koszul duality between Higgs and Coulomb categories \( \mathcal{O} \)

with its left adjoint \( A^\xi_{I_\phi} L_{A_1} - \). Since \( r \circ r_* \) is an equivalence, its left and right adjoints agree and \( r \circ r_! \) is intertwined with \( \text{RHom}_{A_{I_\phi}^\xi} (A^\xi_{I_\phi}, -) \).

- The categories \( \mathcal{O}_{\text{Coulomb}} \) for different choices of \( \xi \) are equivalent to the modules over \( A^\xi \), and the inclusion \( i^\xi \) corresponds to the pullback of \( A^\xi \)-modules to \( A \)-modules by the quotient map. Thus, the shuffling functors are \( i^\xi_{I_\phi'} \circ \xi \) is intertwined with \( A^\xi_{I_\phi} L_{A_1} \otimes_{A_{I_\phi}} - \) and \( i^\xi \circ \xi \) with its adjoint.

This shows the first desired match of functors.

- Under the equivalence \( D_{\mathcal{O}_k} \) with \( A/\mathcal{I}_\xi \)-dg-mod, the shuffling functors are determined by taking Ext of \( r(M) \) and \( r(M') \) for the different flavors \( \phi \) and \( \phi + \nu \) respectively. These are summands of \( M'' \), the corresponding sheaf for \( GT_H \) and any flavor, so ultimately, we find that

\[
\text{Ext}^\bullet (r(M'), r(M)) \cong \text{Ext}^\bullet (r(M'), i^{\phi+\nu}_* \circ i^\phi \circ r(M))) \cong I_{\phi+\nu} A^\xi_{I_\phi}
\]

Thus, we have that \( i^{\phi+\nu}_* \circ i^\phi \) corresponds to \( I_{\phi+\nu} A^\xi_{I_\phi} L_{A_1} \otimes_{A_{I_\phi}} - \) and \( i^\phi \circ i^{\phi+\nu}_* \) to \( \text{RHom}_{A_{I_\phi}^\xi} (I_{\phi+\nu} \xi A^\xi_{I_\phi}, -) \).

- Under the isomorphism of Theorem 4.2, the tensor product with \( \phi ^T \) corresponds to \( I_{\phi}^\xi A^\xi_{I_\phi} L_{A_1} - \).

This shows the second desired match. \( \square \)

As usual, this applies to the quiver and smooth hypertoric cases, since \( \square \) holds there.

4.4. Quiver varieties. The most important examples for us are hypertoric and quiver varieties. In the hypertoric case, we just recover the results of [BLPW12] (in fact, the arguments given here have already been given in the hypertoric case in [Webb, §3]), so there is no need for a detailed discussion. Interestingly, Theorem 4.7 gives a new proof of the Koszul duality discussed in [BLPW10] (of course, this duality is fairly easy to prove algebraically).

The quiver variety case is much more rich and interesting. Here we mean that we have a quiver \( \Gamma \), and dimension vectors \( v, w \) and

\[
V = \bigoplus_{i \rightarrow j} \text{Hom}(C^{vi}, C^{vj}) \bigoplus \bigoplus_i \text{Hom}(C^{vi}, C^{wi}) \quad G = \prod GL(v_i).
\]

The Higgs side of this case is studied in [Webb, §4–5]. In particular, the Steinberg algebras in this case are reduced weighted KLR algebras \( \tilde{W}^\vartheta \) as shown in [Webb, Cor. 4.11]; they are reduced since we do not include the action of the \( C^* \) attached to the Crawley-Boevey vertex. As in Proposition 2.15, we can let \( B \) be any set of loadings, and let \( \xi \) be the sum of the trace characters on \( sl(v_i) \) for all \( i \). For any set of loadings \( B \) with the induced map \( B \rightarrow I' \):
**Proposition 4.9** We have an isomorphism $\bar{W}_B^\varnothing \cong A_B$; the ideal $\mathcal{J}_\xi$ is precisely that generated by the idempotents for unsteady loadings.

Regarding the Coulomb branch, the resulting categories are closely related to the truncated shifted Yangians introduced by the author, Kamnitzer, Weekes and Yacobi in [KWWY]. For quivers of type ADE, the algebra $\mathcal{A}^\text{ph}$ is a truncated shifted Yangian by [BFNa, Cor. B.26].

Applying Theorem 4.2 in this case requires a little care, however, since we must throw out irrelevant weights and roots. We chose a flavor $\varphi$. That is, up to conjugacy, we must choose of weight $\varphi_e$ for each edge, and a cocharacter into $\prod GL(\mathbb{C}^w_i)$, that is, $\varphi_i,1,\ldots,\varphi_i,w_i$ for each $i \in V(\Gamma)$. This is the same data as a weighting of the Crawley-Boevey graph of $\Gamma$.

Now, we consider a coset of $t_\mathbb{Z}$ in $t_1$; this is given by fixing the class in $C/\mathbb{Z}$ for each $z_{i,m}$. Considering only relevant roots means expanding the vertex set of our graph to form a new graph $\Gamma_{z,\phi}$. Its vertex set is

$$V(\Gamma_{z,\phi}) = \{(i, [z]) \in \Gamma \times C/\mathbb{Z} \mid z \equiv z_{i,m} \pmod{\mathbb{Z}} \text{ for some } m \in [1,v_i]\}.$$ 

The edges $(i, [z]) \to (j, [u])$ are in bijection with edges $e : i \to j$ with $\varphi_e \equiv z-u \pmod{\mathbb{Z}}$. Note that we can lift paths in $\Gamma$ to $\Gamma_{z,\phi}$, and that a closed path will lift to a closed path if and only if it lies in the kernel of the homomorphism $\pi_1(\Gamma) \to H_1(\Gamma; \mathbb{Z}) \to C/\mathbb{Z}$ with the last map induced by the weighting $\varphi_e$ reduced (mod $\mathbb{Z}$), thought of as a cohomology class in $H^1(\Gamma; C/\mathbb{Z})$. Thus, $\Gamma_{z,\phi}$ is a subgraph of the union of some number of copies of the cover of $\Gamma$ corresponding to this kernel. We have dimension vectors given by

$$v_{(i,[z])} = \#\{m \in [1,v_i] \mid z \equiv z_{i,m} \pmod{\mathbb{Z}}\} \quad w_{(i,[z])} = \#\{m \in [1,w_i] \mid \varphi_{i,m} \equiv z \pmod{\mathbb{Z}}\}.$$ 

**Lemma 4.10** The subspace $V_{z,\phi}$ and group $L_{z,\phi}$ attached to this choice of flavor and coset are isomorphic to

$$V_{z,\phi} \cong \bigoplus_{e \in E(\Gamma')} \text{Hom}(C^v_{(i,[z])}, C^v_{(j,[u])}) \oplus \bigoplus_{(i,[z]) \in V(\Gamma')} \text{Hom}(C^w_{(i,[z])}, C^w_{(i,[z])})$$

$$L_{z,\phi} = \prod GL(d_{(i,[z])})$$

Thus, Theorem 4.2 shows that:

**Theorem 4.11** The category $\mathcal{A} \text{-mod}_\rho$ is equivalent to the representations of a reduced weighted KLR algebra (associated to a set of loadings) for the Crawley-Boevey quiver of $\Gamma_{z,\phi}$, and the intersection $\mathcal{O}_\rho$ of this subcategory with category $\mathcal{O}$ is equivalent to representations of its steadied quotient.

Note that this is not necessarily equivalent to the full reduced weighted KLR algebra, since we may not have $I'_\varnothing = I_\varnothing$, but its representation category is always a quotient of...
the representation category for the full algebra. The difference between the sets $I'_\phi$ and $I_\phi$ is closely related to the monomial crystal (whose connection to shifted Yangians is discussed in [KTW]). This theorem is particularly interesting in the case where the quiver $\Gamma$ is of type ADE; in this case, the reduced weighted KLR algebra appearing is $\tilde{T}_\lambda$ as defined in [Weba, §4], and $O_\rho$ is a tensor product categorification in the sense of [LW15].

Another important special case is when $\Gamma$ is a single loop; in this case our dimension vectors are just integers $v, w$, and $V \cong (C^v)^{\oplus w} \oplus \mathfrak{gl}_v$ as a representation of $GL_v$. In this case, the algebra $A^{sp}_{\phi}$ is isomorphic to the spherical rational Cherednik algebra for the group $S_v \wr \mathbb{Z}/w\mathbb{Z}$ (assuming $w > 0$), as recently shown by Kodera-Nakajima [KN], and expanded upon by Braverman-Etingof-Finkelberg [BEF] and the author [Webc]. This isomorphism matches the weight of the loop to the parameter $k$ in the Cherednik algebra. The reader might find it a bit dissatisfying that in the case where $w = 1$, the algebra $A$ is an algebra containing the spherical Cherednik algebra as a submodule and of rank $(v!)^2$ as a module over it, but it is not the full Cherednik algebra $H(S_n)$. Rather, it’s the matrix algebra $M_{v!(v)}(eH(S_n)e)$, as Theorem 3.3 shows. The algebra $H(S_n)$ appears as the endomorphisms of another object in $\mathcal{B}$; the convolution description is as $H_{BM,G(t)}(X_U \times V(t))$ where $U = C^v[[t]] \oplus i$ and $i$ is the Lie algebra of the standard Iwahori. This matches the presentation of the Cherednik algebra in [Webc], which also appeared in [Gri]. This is a rational version of the K-theoretic description of the double affine Hecke algebra in [VV10, Thm. 2.5.6].

In this loop case, the map induced by the cohomology class is $\pi_1(\Gamma) \cong \mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ sending $1 \mapsto k$. In particular, if $k \notin \mathbb{Q}$, then the corresponding cover is an $A_{\infty}$ graph, and $\Gamma_{z,\phi}$ is a union of segments. If $k = a/e \in \mathbb{Q}$, then the corresponding cover is an $e$-cycle, and so $\Gamma_{z,\phi}$ is a union of segments and $e$-cycles. The most interesting case is when $\phi_{i,m}, z_{i,k} \in \frac{1}{e}\mathbb{Z}$, so $\Gamma_{z,\phi}$ is a single $e$-cycle (assuming every $\mathbb{Z}$-coset contains at least one $z_{i,k}$). The resulting equivalence between modules over the Cherednik algebra and weighted KLR algebras was developed first in [Webd], and is extended in [Webc].

References

[BEF] Alexander Braverman, Pavel Etingof, and Michael Finkelberg, Cyclotomic double affine Hecke algebras, personal communication.

[BFNa] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima, Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian, (with appendices by Alexander Braverman, Michael Finkelberg, Joel Kamnitzer, Ryosuke Kodera, Hiraku Nakajima, Ben Webster, and Alex Weekes), arXiv:1604.03625.

[BFNb] ________, Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II, arXiv:1601.03586.

[BLPW] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, Quantizations of conical symplectic resolutions II: category $\mathcal{O}$, arXiv:1407.0964.

[BLPW10] ________, Gale duality and Koszul duality, Adv. Math. 225 (2010), no. 4, 2002–2049.

[BLPW12] ________, Hypertoric category $\mathcal{O}$, Adv. Math. 231 (2012), no. 3-4, 1487–1545.
Ben Webster

[BPW] Tom Braden, Nicholas J. Proudfoot, and Ben Webster, *Quantizations of conical symplectic resolutions I: local and global structure*, arXiv:1208.3863.

(CG97) Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR 98i:22021

[Gri] Stephen Griffeth, *Unitary representations of rational Cherednik algebras, II*, arXiv:1106.5094.

(KL09) Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups, I*, Represent. Theory 13 (2009), 309–347.

(KN) Ryosuke Kodera and Hiraku Nakajima, *Quantized Coulomb branches of Jordan quiver gauge theories and cyclotomic rational Cherednik algebras*, [arXiv:1608.00875](https://arxiv.org/abs/1608.00875).

(KS12) Masaki Kashiwara and Pierre Schapira, *Deformation quantization modules*, Astérisque (2012), no. 345, xii+147. MR 3012169

(KTW⁺) Joel Kamnitzer, Peter Tingley, Ben Webster, Alex Weekes, and Oded Yacobi, *Highest weights for truncated shifted Yangians and product monomial crystals*, arXiv:1511.09131.

(KWWY) Joel Kamnitzer, Ben Webster, Alex Weekes, and Oded Yacobi, *Yangians and quantizations of slices in the affine Grassmannian*, arXiv:1209.0349.

(LW15) Ivan Losev and Ben Webster, *On uniqueness of tensor products of irreducible categorifications*, Selecta Math. (N.S.) 21 (2015), no. 2, 345–377. MR 3338680

(MOS09) Volodymyr Mazorchuk, Serge Ovsienko, and Catharina Stroppel, *Quadratic duals, Koszul dual functors, and applications*, Trans. Amer. Math. Soc. 361 (2009), no. 3, 1129–1172.

(MV98) Ian M. Musson and Michel Van den Bergh, *Invariants under tori of rings of differential operators and related topics*, Mem. Amer. Math. Soc. 136 (1998), no. 650, viii+85.

[Naka] Hiraku Nakajima, *Modules of quantized Coulomb branches*, forthcoming preprint.

[Nakb] Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I, arXiv:1503.03676.

[Rou] Raphael Rouquier, *2-Kac-Moody algebras*, arXiv:0812.5023.

[Sau] Julia Sauter, *Generalized quiver Hecke algebras*, arXiv:1306.3892.

[SVV] Peng Shan, Michela Varagnolo, and Eric Vasserot, *Koszul duality of affine Kac-Moody algebras and cyclotomic rational DAHA*, arXiv:1107.0146.

[SW] Catharina Stroppel and Ben Webster, *Quiver Schur algebras and q-Fock space*, arXiv:1110.1115.

[VV10] Michela Varagnolo and Eric Vasserot, *Double affine Hecke algebras and affine flag manifolds, I*, Affine flag manifolds and principal bundles, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010, pp. 233–289. MR 3013034

[VV11] Cannonical bases and KLR-algebras, J. Reine Angew. Math. 659 (2011), 67–100.

[Webb] Ben Webster, *Knot invariants and higher representation theory*, to appear in the Memoirs of the American Mathematical Society; arXiv:1309.3796.

[Webb] On generalized category $\mathcal{O}$ for a quiver variety, arXiv:1409.4461.

[Webe] Representation theory of the cyclotomic Cherednik algebra via the Dunkl-Opdam subalgebra, arXiv:1609.05494.

[Webd] Rouquier’s conjecture and diagrammatic algebra, arXiv:1306.0074.

[Webe] Tensor product algebras, Grassmannians and Khovanov homology, arXiv:1312.7357.

[Webf] Weighted Khovanov-Lauda-Rouquier algebras, arXiv:1209.2463.