Approximate analytical solutions to the condensation-coagulation equation of aerosols

Naftali R. Smitha, Nir J. Shaviva,b, and Henrik Svensmarkc

aRacah Institute of Physics, Hebrew University of Jerusalem, Israel; bThe Institute for Advanced Study, Princeton, New Jersey, USA; cNational Space Institute, Technical University of Denmark, Elektrovej, Lyngby, Denmark

ABSTRACT

We present analytical solutions to the steady state nucleation-condensation-coagulation equation of aerosols in the atmosphere. These solutions are appropriate under different limits but more general than previously derived analytical solutions. For example, we provide an analytic solution to the coagulation limit plus a condensation correction. Our solutions are then compared with numerical results. We show that the solutions can be used to estimate the sensitivity of the cloud condensation nuclei number density to the nucleation rate of small condensation nuclei and to changes in the formation rate of sulfuric acid.

1. Introduction

Aerosols in Earth’s atmosphere are important for the climate systems as they are required to form clouds. As a consequence, different aerosol characteristics translate into different cloud properties, such as different radiative forcing, different life time and precipitation. Since a large part of the characteristics of aerosols can be described by their size distribution, solving for the distribution is of particular interest.

For a spatially homogeneous aerosol distribution, the size distribution can be described with a density function \( n(v, t) \) where \( v \) is the volume of the aerosol, such that \( \int_{v_1}^{v_2} n(v, t) dv \) is the total number of aerosols per unit volume, with volumes between \( v_1 \) and \( v_2 \).

The main equation describing the temporal evolution of \( n(v, t) \) is the coagulation-condensation equation (e.g., Peterson et al. 1978; Seinfeld and Pandis 2006):

\[
\frac{\partial n}{\partial t} + \frac{\partial}{\partial v} [I(v, t)n] = \frac{1}{2} \int_{v_1}^{v_2} \beta(v - \tilde{v}, \tilde{v}) n(v - \tilde{v}, t)n(\tilde{v}, t)d\tilde{v} \\
- \int_0^\infty \beta(v, \tilde{v}) n(v, t)n(\tilde{v}, t)d\tilde{v} \\
- R(v, t) + S(v, t)
\]  

[1]

where \( I(v, t) = \frac{dv}{dt} \) is the growth rate of a particle of volume \( v \), \( \beta(v_1, v_2) \) is the coagulation coefficient for particles with volumes \( v_1 \) and \( v_2 \), \( R(v, t) \) is the rate with which particles are removed from the system, and \( S(v, t) \) is the nucleation rate of new particles. It is of course similar to the Smoluchowski (1916) equation, with the addition of the condensation term.

Klett (1975) provided an analytical solution to the above equation, but without the condensation term, for a few specific cases by means of a Laplace transform. For example, he solved the aerosol distribution for the case of a mass independent coagulation coefficient.

Later, Ramabhadran et al. (1976) provided an analytic solution to the more general problem which includes condensation. Like Klett, they have shown that power laws can provide approximate solutions in different regimes. Peterson et al. (1978) elaborated on the above and provided more general solutions. For example, they solved the time dependent problem. However, they confined themselves to cases in which the particle growth rate is either constant or linearly dependent on the aerosol volume.

It should be noted that an equation similar to the aerosol condensation/coagulation equation appears in other contexts as well. First, the original coagulation equation (with no condensation) was formulated by Smoluchowski (1916) to describe a colloidal fluid. A second example is that of cloud raindrops which behave similarly to the smaller sized aerosols with the main exception that their coagulation equation includes raindrop fragmentation (e.g., see review by Beheng 2010), while outside the terrestrial settings, Birnstiel et al.
(2011) solve a coagulation/fragmentation equation describing dust grains in the interstellar medium.

We begin in Section 2 by writing the coagulation-condensation equation in a dimensionless form and the approximations we assume in the present analysis. This will define the problem that we solve. In Section 3, we first solve the problem while discarding the condensation term, and by doing so we arrive at the same solution found in Klett (1975) by different means. This solution is expanded by treating the condensation term as a perturbation. The new analytic solution is appropriate for large aerosols where the dominant process is coagulation. In Section 4, we solve the opposite limit, when the coagulation term is altogether negligible, and then, when its effects are approximated. This solution is appropriate for small aerosols where the dominant process is condensation. In Section 5 we compare our results to a material, and the nucleation rate of new aerosols.

We begin in Section 2 by writing the coagulation-condensation equation in a dimensionless form and the approximations we assume in the present analysis. This will define the problem that we solve. In Section 3, we first solve the problem while discarding the condensation term, and by doing so we arrive at the same solution found in Klett (1975) by different means. This solution is expanded by treating the condensation term as a perturbation. The new analytic solution is appropriate for large aerosols where the dominant process is coagulation. In Section 4, we solve the opposite limit, when the coagulation term is altogether negligible, and then, when its effects are approximated. This solution is appropriate for small aerosols where the dominant process is condensation. In Section 5 we compare our results to a material, and the nucleation rate of new aerosols.

2. The problem

Although the general problem of aerosol growth, Equation (1), is time dependent, we will concentrate in the present work on the time-independent case, namely, we will assume a steady-state solution \( n(v) \).

As in Peterson et al. (1978), we will assume a constant coagulation coefficient \( \beta = \beta_0 \) and \( I(v, t) = \sigma v^\gamma \) where \( 0 \leq \gamma \leq 1 \), and we will emphasize the case \( \gamma = 1/3 \) (which is appropriate for the “continuum regime”) and \( \gamma = 2/3 \) (which is appropriate for the “kinetic regime”), as we elaborate in Section 6.

We also assume that the timescale for removal of particles from the system is much larger than the time necessary to arrive at the steady-state solution. Therefore, we will discard the removal term altogether.

Finally, we assume a delta function source \( S(v, t) = S_0 \delta(v - v_1) \), again following Peterson et al. (1978). As we shall see below, we will also assume that \( v_1 \) is much smaller than some characteristic volume that will be defined later.

We therefore obtain the following integro-differential equation for the steady-state distribution:

\[
\frac{d}{dv} \left[ \sigma v^\gamma n(v) \right] = \frac{1}{2} \int_0^v \beta_0 n(\bar{v}) n(v - \bar{v}) d \bar{v} \\
- \int_0^\infty \beta_0 n(\bar{v}) n(\bar{v} + S_0 \delta(v - v_1)).
\]  

Our goal in the present work is to provide approximate analytical solutions to the above equation (Equation (2)), at a higher order than presently available in the literature. We will then show in Section 6 how this can be used to estimate the aerosol number density in the atmosphere to changes in the amount of condensible material, and the nucleation rate of new aerosols.

2.1. General kernel

Although we solve in the present work the coagulation equation while assuming a constant coagulation coefficient, we note that the any solution under this assumption can be immediately generalized to coagulations having kernels of the form \( \beta(v, v) = \beta_1 v^\gamma v^\rho \).

Here, the coagulation equation has the form

\[
\frac{d}{dv} \left[ \sigma v^\gamma n(v) \right] = \frac{1}{2} \int_0^v \beta_1 v^\rho (v - \bar{v})^\alpha n(\bar{v}) n(v - \bar{v}) d \bar{v}
- \int_0^\infty \beta_1 v^\rho v^\alpha n(\bar{v}) n(\bar{v}) d \bar{v} + S_0 \delta(v - v_1).
\]  

To reduce it to the fixed kernel form, we define

\[
f(v) \equiv v^\rho n(v),
\]

giving rise to the following equation for \( f \):

\[
\frac{d}{dv} \left[ \sigma v^\gamma - nf(v) \right] = \frac{1}{2} \int_0^v \beta f(\bar{v}) f(v - \bar{v}) d \bar{v}
- \int_0^\infty \beta f(\bar{v}) f(v) d \bar{v} + S_0 \delta(v - v_1).
\]  

This equation is the same as Equation (2), but with a different value of \( \gamma \), that is \( \gamma \to \gamma - \alpha \).

2.2. Dimensionless form

Before we solve Equation (2), we recast it in a dimensionless form. We begin by defining \( \chi_0 \) as the total number of particles in the system, that is, \( \chi_0 \equiv \int_0^\infty n(v) dv \). The differential equation which describes its temporal dependence is (Klett 1975):

\[
\frac{d\chi_0}{dt} = - \frac{1}{2} \beta_0 \chi_0^2 + S_0,
\]

which gives a steady state solution of \( \chi_0 = \sqrt{2 S_0 / \beta_0} \).
Next, we define the characteristic volume as

$$v_2 \equiv \left( \frac{\sigma^2}{S_0 \beta_0} \right)^{1/(2-2\gamma)}$$  \[7\]

and characteristic particle density to be

$$n_0 \equiv \frac{\sqrt{2S_0 \beta_0}}{v_2}.$$  \[8\]

We also defined a dimensionless time as 
$$\tau \equiv t/\sqrt{2/S_0 \beta_0}.$$

Using these characteristic quantities, we can define the dimensionless variables as 
$$x \equiv x/v_2, x_1 \equiv x_1/v_2, \text{ and } y \equiv n/n_0,$$

such that the time dependent version of Equation (2) becomes

$$\frac{dy}{d\tau} + \sqrt{2} \frac{d}{dx} \left[ x^\gamma y(x) \right] = \int_0^x y(\tilde{x})y(x - \tilde{x})d\tilde{x}$$

$$- 2y(x) + \delta(x - x_1).$$  \[9\]

Note the fact that in steady state:

$$\int_0^\infty y(x)dx = 1.$$  \[10\]

So the steady state equation can also be written in the form

$$\sqrt{2} \frac{d}{dx} \left[ x^\gamma y(x) \right] = \int_0^x y(x)y(x - \tilde{x})d\tilde{x}$$

$$- 2y(x) \int_0^\infty y(x)dx + \delta(x - x_1)$$  \[11\]

which is closer to the form of Equation (2). We will assume throughout this article that 
$$v_1 \ll v_2,$$ which translates to 
$$x_1 \ll 1$$ in the dimensionless quantities.

The amount of condensable matter is affected by the processes of nucleation and condensation as well as their production rate from external sources. Most of this matter is sulphuric acid (SA) and organic compounds which condense with it (though over the oceans methanesulfonic acid, MSA, could also be important, as could organic compounds elsewhere, e.g., Jimenez et al. 2009). Thus, although we denote the condensable material as SA, it should include all types of condensable materials with different concentrations, production rates, and condensation parameters. The equation which describes the change of the condensable material is:

$$\frac{dM_{SA}}{dt} = \epsilon_m - \int_{v_3}^{\infty} \sigma v^\gamma n(v)dv - S_0 v_1,$$  \[12\]

where \(\epsilon_m\) is the production rate of sulphuric acid (or MSA) by an external source, together with the organic compounds which condenses with it. Therefore, in steady-state we have:

$$\epsilon_m = \int_{v_1}^{\infty} \sigma v^\gamma n(v)dv + S_0 v_1.$$  \[13\]

Later on we will discard the \(S_0 v_1\) term in Equation (13) as it is much smaller than the first term (under the assumption \(v_1 \ll v_2\)).

3. The coalescence limit

3.1. Full solution with no condensation correction

Klett (1975) solved the fixed coalescence cross-section case while neglecting the condensation equation using Laplace transform, and found that

$$n(v) \approx \sqrt{\frac{\epsilon_m}{2\pi\beta_0}} v^{-3/2},$$  \[14\]

for large values of \(v\). Here we show a shorter and more intuitive way to reach the same solution. In the next subsection, we will add the correction term obtained when adding the condensation at large \(v\)'s (when the condensation term is necessarily small).

We define \(\epsilon(x)\) to be the dimensionless rate of change of the total dimensionless volume of particles of dimensionless size \(x\) or more (with “dimensionless” hereafter omitted), that is:

$$\epsilon(x) \equiv \int_x^{\infty} \frac{\partial}{\partial \tau} y(x)dx.$$  \[15\]

Note that \(\epsilon\) is the dimensionless form of \(\epsilon_m\) which was introduced in Equation (13).

Since we neglect condensation, this increase in the total volume is due to two “types” of particle coalescence. First, two particles with a volume smaller than \(x\) can combine into one particle with a volume larger than \(x\). Second, a particle with a volume larger than \(x\) can combine with a particle with a volume smaller than \(x\). Together, we find that the total volume change due to
coalescence is

\[ \epsilon(x) = \int_{\xi_1 = 0}^{x} \int_{\xi_2 = x - \xi_1}^{x} y(\xi_1)y(\xi_2)(\xi_1 + \xi_2)d\xi_1d\xi_2 \]

\[ + 2 \int_{\xi_1 = 0}^{x} \int_{\xi_2 = x}^{\infty} y(\xi_1)y(\xi_2)\xi_1d\xi_1d\xi_2 \]  \[ \text{[16]} \]

which simplifies to (see schematic diagram in Figure 1)

\[ \epsilon(x) = 2 \int_{\xi_1 = 0}^{x} \int_{\xi_2 = x - \xi_1}^{\infty} y(\xi_1)y(\xi_2)d\xi_1d\xi_2. \]  \[ \text{[17]} \]

Next, we assume a power law solution, i.e., \( y = Bx^{-p} \), and that \( 1 < p < 2 \). The assumption \( p > 1 \) is necessary to ensure a finite total number of particles, while the assumption \( p < 2 \) is necessary to ensure an infinite total volume. The latter is necessary because the total volume is a monotonically increasing function of time. Together with Equation (17), we find after some algebra that

\[ \epsilon(x) = 2B^2x^{3-2p} \frac{1}{p-1} \int_{\eta = 0}^{1} \eta^{1-p}(1-\eta)^{1-p}d\eta. \]  \[ \text{[18]} \]

In steady state, \( \epsilon \) should be independent of \( x \). Therefore \( p = 3/2 \), and

\[ \epsilon = 2B^2 \frac{1}{3/2-1} \int_{\eta = 0}^{1} \eta^{-1/2}(1-\eta)^{-1/2}d\eta = 4\pi B^2. \]  \[ \text{[19]} \]

Thus, the steady-state solution to Equation (9) is

\[ y(x) = \sqrt{\frac{\epsilon}{4\pi}}x^{-3/2}, \]  \[ \text{[20]} \]

We now recall the definition of \( \epsilon_m \) (Equation (13)) as the rate of change of volume in the original equation (Equation (1)). It can easily be seen that

\[ \epsilon_m = S_0 \nu_2 \epsilon. \]  \[ \text{[21]} \]

Therefore, the solution to Equation (2) with no condensation (for large aerosol volume) is

\[ n(\nu) = \frac{\sqrt{\epsilon_m}}{2\pi \beta_0 \nu^{3/2}}, \]  \[ \text{[22]} \]

which is the same as the solution found by (Klett 1975, Equations (23) and (24)).

Incidentally, we can use Equation (21) and the definition of \( \nu_2 \) (Equation (7)) to get

\[ \sqrt{S_0 \beta_0 \left( \frac{\epsilon_m}{S_0 \epsilon} \right)^{(1-\gamma)}} = \sigma. \]  \[ \text{[23]} \]

This result will be used later in order to eliminate \( \sigma \) from our results, by introducing the dimensionless \( \epsilon \) whose value depends on \( \gamma \). This is useful because under most physical scenarios, \( \sigma \) is unknown, but \( \epsilon_m \) is determined from various chemical or physical processes (an example for this is given later in Section 6).

3.2. Full solution with condensation correction for \( \gamma < 1/2 \)

The next step is to generalize the calculation described in Section 3.1 by adding the first-order correction term associated with condensation. With the latter term, the equation describing the increase of the total mass of particles larger than \( x \) now includes
two additional terms
\[ e(x) = 2 \int_x^\infty y(\xi_2) \xi_1 d\xi_1 d\xi_2 + \int_x^\infty \sqrt{2} \xi_2 y(\xi) d\xi + \sqrt{2} x^g y(x) x. \]

The second term in the equation describes volume change due to condensation on particles of volume \( x \) or larger. The third term describes particles of volume slightly less than \( x \), which in a unit time grow to become larger than \( x \) due to condensation.

We will now look for a solution of the type \( y = Bx^{-p} + Dx^{-q} \), where \( q > p \). To our knowledge, this hypothesis for the form of the solution has never been proposed before. It was inspired by numerical solutions. We now plug this solution into Equation (24), and neglect small powers of \( x \). For \( q < 1/2 \), the equation for the highest power of \( x \) is unaffected by the new term \( Dx^{-q} \), so we still have \( p = 3/2 \) and \( B = \sqrt{\epsilon/(4\pi)} \) as before.

The equation we get for the next highest power of \( x \) is (after some algebra):

\[
0 = 2Dx^{3-p-q} \left[ \frac{1}{q-1} + \frac{1}{q-1} \int_0^1 \eta^{1-p} (1-\eta)^{1-q} d\eta \right]
+ \sqrt{2} x^{1+\gamma-p} \left( 1 + \frac{1}{p-1-\gamma} \right). \]

As before, this equation should hold for all values of \( x \) (in the limit \( x \gg 1 \)). Equating the power law indices of \( x \) gives us:

\[ 2 - \gamma = q. \]

This explains why \( \gamma \) should be smaller than \( 1/2 \) for this solution to be valid. Next, we can equate the coefficients.

Once we plug in \( p = 3/2 \) and \( q = 2 - \gamma \), we find

\[
2D \left[ \frac{1}{1-\gamma} + \frac{1}{1/2} \right] \int_0^1 \eta^{-1/2} (1-\eta)^{1-\gamma} d\eta = \sqrt{\pi} \Gamma(1/3)/\Gamma(5/6) \approx 4.20655 \]

such that Equation (27) gives \( D(\gamma = 1/3) = -0.3362 \). For other values of \( \gamma \), see Figure 2.

To summarize, the solution of Equation (9) is of the form (for \( 0 \leq \gamma < 1/2 \)):

\[ y(x) = \sqrt{\frac{\epsilon}{4\pi}} x^{-3/2} + Dx^{-(2-\gamma)}. \]

For the specific case \( \gamma = 1/3 \), we obtain

\[ y(x) = \sqrt{\frac{\epsilon}{4\pi}} x^{-3/2} - 0.336x^{-5/3}. \]

Using the physical quantities, this specific solution becomes

\[ n(u) = \sqrt{\frac{\epsilon_m}{2\pi\beta_0}} u^{-3/2} - 0.475 \frac{\sigma}{\beta_0} u^{-5/3}. \]

In many cases, we are given \( \epsilon_m \), i.e., the sulfuric acid formation rate, instead of \( \sigma \), its equilibrium number density. We can, therefore, use Equation (23) and find

\[ n(u) = \sqrt{\frac{\epsilon_m}{2\pi\beta_0}} u^{-3/2} - 0.475 \beta_0^{-1/2} S_0^{-1/6} \left( \frac{\epsilon_m}{\epsilon} \right)^{2/3} u^{-5/3}. \]

A numerical solution of the equation (described in the appendix) shows that \( \epsilon(\gamma = 1/3) \approx 3.296 \). This is perhaps the most important result in the present work since it describes the leading two terms in the distribution of aerosols in the atmosphere. The analytic form of the second term is described here for the first time.

In principle, it is possible to add higher order corrections as a power series of \( x \), where each term’s power decreases by 1/6.

### 3.3. Full solution with condensation correction for \( \gamma > 1/2 \)

The solution described in Section 3.2 is valid only for \( \gamma < 1/2 \), because it was assumed that \( q > p \) such that \( 2 - \gamma < 1/2 \). However, in a significant number of cases, this assumption is not valid. For example, consider the case where \( \gamma = 2/3 \), which is the most important case where \( \gamma = 1/3 \) has been studied. The analytic form of the solution for this case is

\[ y(x) = \sqrt{\frac{\epsilon}{4\pi}} x^{-3/2} - 0.336x^{-5/3} \]

Using the physical quantities, this specific solution becomes

\[ n(u) = \sqrt{\frac{\epsilon_m}{2\pi\beta_0}} u^{-3/2} - 0.475 \frac{\sigma}{\beta_0} u^{-5/3}. \]
Figure 4. $B$ vs. $\gamma$ for the case $1 > \gamma > 1/2$.

\( \gamma > 3/2 \) (using Equation (26)). Therefore, the larger \( \gamma \) case should be solved separately. As we shall see below, this has a major effect on the solution—the leading power of \( x \) changes.

First, because \( \epsilon(x) \) diverges, instead of working with \( \epsilon(x) \) we will work with \( \epsilon(x, x_3) \) which is the rate of change in the total mass between \( x \) and \( x_3 \), that is,

\[
\epsilon(x, x_3) = 2 \int_{\xi_1}^{x} \int_{\xi_2}^{\infty} y(\xi_1) y(\xi_2) \xi_1 d\xi_1 d\xi_2 + \sqrt{2x^2 y(x)x + \int_{x}^{\infty} \sqrt{2\xi^2} y(\xi)d\xi}. \tag{32}
\]

In steady state, \( \epsilon(x, x_3) \) has to vanish.

Again we assume a power-law solution of the type \( y = Bx^{-\gamma} \), and write the solution in the form

\[
\epsilon(x, x_3) = F(x) - F(x_3). \tag{33}
\]

Using this form, we find after integration that

\[
F(x) = 2B^2 x^{3-2\gamma} \frac{1}{p-1} \int_0^1 \eta^{1-p} (1-\eta)^{1-p} d\eta + \sqrt{2B} x^{1+\gamma-p} - \sqrt{2B} \frac{1}{1+\gamma-p} x^{1+\gamma-p}. \tag{34}
\]

Since \( \epsilon(x, x_3) \) should vanish, we must require \( F(x) \) in Equation (34) to be independent of \( x \). However, since \( p \neq 3/2 \), the only way for \( F(x) \) to be independent of \( x \) is for the \( x \)-dependent terms to cancel each other out. This implies that the exponents are equal, such that, \( 2 - \gamma = p \). Last, we can find the pre-factor \( B \) by requiring the coefficient of \( x^{3-2p} \) (or \( x^{1+\gamma-p} \)) to vanish as well. The result is

\[
y(x) = Bx^{2-\gamma} = \left[ \frac{\sqrt{2} (1-\gamma)^2}{2\gamma-1} \int_0^1 \eta^{\gamma-1} (1-\eta)^{\gamma-1} d\eta \right] x^{2-\gamma}. \tag{35}
\]

For instance, the solution to Equation (9) for \( \gamma = 2/3 \) and large values of \( x \) is approximately given by \( y = 0.230 x^{-4/3} \). In terms of \( \nu \), the solution to Equation (2) for the above case is

\[
n(\nu) = 0.325 \frac{\sigma}{\beta_0} \nu^{-4/3}. \tag{36}
\]

For other values of \( \gamma \) see Figure 4. Unlike the previous case, of \( \gamma < 1/2 \), the condensation term’s contribution to \( \epsilon_m \) formally diverges. This implies that without additional physics introducing a large volume cutoff, such as dry or wet deposition, or a dependence of \( \gamma \) on the aerosol volume, the solution is no longer physical. If we introduce a cutoff volume \( \nu_C \) then we can evaluate \( \epsilon_m \) by calculating the following integral:

\[
\epsilon_m = \int_{\nu_1}^{\nu_C} I(\nu) n(\nu) d\nu. \tag{37}
\]

For the case \( \gamma = 2/3 \), this will yield the following relation between \( \sigma \) and \( \epsilon_m \):

\[
\sigma \approx 1.01 \sqrt{\beta_0 \epsilon_m \nu_C^{-1/3}}. \tag{38}
\]
The solution in terms of \( \epsilon_m \) is then given by:

\[
n(u) = 0.329 \sqrt{\frac{\epsilon_m u_C}{\beta_0}} u^{-4/3}. \tag{39}
\]

### 3.4. Full solution with 2nd condensation correction for \( \gamma > 1/2 \)

As for the small \( \gamma \) case described in Section 3.2, it is possible to derive a higher order correction to the lowest order solution described in Section 3.3 above. To do so, we can repeat Section 3.3 under the assumption that the solution is of the more general form \( y = Bx^{-p} + Dx^{-q} \) (where \( q > p \)), and calculate \( F(x) \) defined in Equation (33).

We then require \( F(x) = 0 \) and compare the leading terms in powers of \( x \). The lowest order term was described above (see Equation (35)). The next order gives,

\[
0 = 2BDx^{3-p-q} \int_0^1 \eta^{1-p} \frac{1}{q-1} (1-\eta)^{1-q} d\eta
+ 2BDx^{3-p-q} \int_0^1 \eta^{1-q} \frac{1}{p-1} (1-\eta)^{1-p} d\eta
+ \sqrt{2} D x^{1+\gamma-q} - \sqrt{2} D \frac{1}{1+\gamma-q} x^{1+\gamma-q}. \tag{40}
\]

The requirement on the exponents gives \( 3 - p - q = 1 + \gamma - q \), however, it follows straightforwardly from \( p = 2 - \gamma \) found in the first order solution. Consequently, we only obtain new information by requiring the sum of the coefficients to vanish. Since \( D \) cancels out, the constraint becomes

\[
\sqrt{2} \frac{1}{1+\gamma-q} - \sqrt{2} = 2B \left[ \frac{1}{q-1} + \frac{1}{p-1} \right] \int_0^1 \eta^{1-p} (1-\eta)^{1-q} d\eta. \tag{41}
\]

This equation can be solved numerically for \( q \), given the previously calculated values of \( p \) and \( B \). For example, the solution for the case \( \gamma = 2/3 \) is \( q \approx 1.47 \). Solutions for other values of \( \gamma \) are described in Figure 3.

The value of \( D \) is found numerically, and it is found to have a rather strong dependence on \( x_1 \). Note that \( D \) can be negative. The numerical simulations also seem to indicate that for \( x_1 \to 0, D(x_1) \to -B \). This can be seen in Figure 5.

In terms of the Equation (2), we have the following solution for the case \( \gamma = 2/3 \):

\[
n(u) = 0.325 \frac{\alpha}{\beta_0} u^{-4/3}
+ D(x_1 = u_1/u_2) \sqrt{\frac{2S_0}{\beta_0 S_0}} (\frac{\alpha^2}{S_0})^{0.706} u^{1.471}. \tag{42}
\]

If we wish to eliminate \( \sigma \) from our results, we get:

\[
\sigma \approx 1.01 \sqrt{\beta_0 \epsilon_m u_C^{-1/3}} - 7.54D(x_1) \sqrt{\beta_0 S_0}
\times \left( \frac{\epsilon_m u_C^{-1/3}}{S_0} \right)^{0.706} u_C^{1/3 - 0.471}, \tag{43}
\]

giving us:

\[
n(u) \approx 0.329 \sqrt{\frac{\epsilon_m u_C}{\beta_0}} u^{-4/3}
+ D(x_1 = u_1/u_2) \sqrt{\frac{S_0}{\beta_0}} \left( \frac{\epsilon_m u_C^{-1/3}}{S_0} \right)^{0.706}
\times \left[ -2.45 u_C^{1/3 - 0.471} u^{-4/3} + 1.44, u^{-1.471} \right]. \tag{44}
\]
4. The condensation limit

Until now we studied the limits where coagulation is much more important than condensation. We now concentrate on the opposite limit, where condensation is much more important. This describes, for example, the growth of small aerosols. We begin by describing the pure condensation limit, and then continue by crudely adding the coagulation as a correction.

4.1. The condensation solution without coagulation corrections

After discarding both coagulation terms in Equation (9), the steady state equation can be solved analytically. Equation (9) simply becomes

\[ \sqrt{2} \frac{d}{dx} [x^r y(x)] = \delta(x - x_1). \]  

[45]

The general solution is \( y = Cx^{-r} \), where \( C \) can be found by the “boundary condition” at \( x = x_1 \), which yields

\[ y = Cx^{-r} = \frac{1}{\sqrt{2}} x^{-r}. \]  

[46]

4.2. The condensation solution with a coagulation correction

The pure condensation solution can be extended by adding the first order correction arising from coagulation. This introduces two terms appearing in Equation (9) which are estimated by using the “zeroth order” solution. The first coagulation term is roughly \( \int_0^x y(x) y(x - \chi) d\chi \approx y(x) y(x) x \approx x^{1-r} \). The second coagulation term is roughly \(-2y(x) \approx x^{-r} \). Therefore, assuming \( r < 1 \), the first coagulation term will be much smaller than the second coagulation term for all \( x \ll 1 \), and we shall not take it into account in the calculation that follows. The steady state dimensionless equation can still be solved analytically. The equation is

\[ \sqrt{2} \frac{d}{dx} [x^r y(x)] = -2y(x) + \delta(x - x_1). \]  

[47]

Its general solution is given by

\[ y = Cx^{-r} \exp \left[ -\sqrt{2} \frac{1}{1-r} x^{1-r} \right]. \]  

[48]

Using the “boundary condition” at \( x = x_1 \), and assuming \( x_1 \ll 1 \), we obtain:

\[ y(x) \approx \frac{1}{\sqrt{2}} x^{-r} \exp \left[ -\sqrt{2} \frac{1}{1-r} x^{1-r} \right], \]  

[49]

or in terms of Equation (2)

\[ n(\nu) \approx \frac{S_0}{\sigma} \nu^{-r} \exp \left[ -\sqrt{2S_0}\frac{1}{\sigma} \frac{1}{1-r} \nu^{1-r} \right]. \]  

[50]

In the case \( r < 1/2 \), we can eliminate \( \sigma \) from this result, and give it instead in terms of \( \epsilon_m \), using Equation (23). This gives

\[ n(\nu) \approx \sqrt{S_0/\beta_0} \left( \frac{\epsilon_m}{\sigma_0 \epsilon} \right)^{1-1/(2-r)} \nu^{-r} \exp \left[ -\sqrt{2S_0}\frac{1}{\sigma_0 \epsilon} \frac{1}{1-r} \nu^{1-r} \right], \]  

[51]

where \( \epsilon \) is \( r \)-dependent, and must be calculated numerically.

4.3. Generalization to the case of particle loss

The solution described in the previous subsection can be generalized to the case where a particle-loss term \( R(\nu, t) = -\lambda n(\nu) \) is added to the RHS of Equation (2). This is done by defining the dimensionless parameters \( x \equiv \nu/\nu_2 \) and \( y(x) \equiv n(\nu)/n_0 \), where:

\[ \nu_2 \equiv \left( \frac{\sigma^2}{S_0 \beta_0} \right)^{1/(2-2r)} \quad n_0 \equiv \frac{S_0}{\beta_0 \nu_2}. \]  

[52]

This process is similar to the one described in Section 2.2. The dimensionless steady-state equation is now given by:

\[ \sqrt{2} \frac{d}{dx} [x^r y(x)] = \int_0^x y(x) y(x - \chi) d\chi \]  

\[ -2 \int_0^\infty y(x) y(x) d\chi \]  

\[ + \delta(x - x_1) - \zeta y(x), \]  

[53]

where

\[ \zeta \equiv \lambda \sqrt{\frac{2}{\beta_0 S_0}}. \]  

[54]
Particle solution. For rescaling the axes according to the values of the typical physical parameters in the atmosphere. For different values of the parameters, the results are obtained by simply approximations obtained above. Figure 6 be obtained numerically, as is described in the Appendix.

Although there is no exact analytic solution for the full coagulation-condensation equation, a full solution can be obtained numerically, as is described in the Appendix.

The full solution can then be used to check the quality of the analytical approximations obtained above. Figure 6 plots the full numerical solution and the analytical approximations obtained for the cases $\gamma = 1/3$ and $\gamma = 2/3$. For $\gamma = 1/3$, the two analytic approximations for the $x \gg 1$ and the $x \ll 1$ limits are relatively accurate. In fact, at $x \sim 0.5$ both solutions are only 30% off the exact solution.

6. Applications to atmospheric aerosols

After having derived the above analytic results, it is now straightforward to apply them and derive the behavior of the aerosol number density in the atmosphere, under steady state conditions. In particular, we are interested in finding how sensitive the aerosol number density is to changes in both the availability of condensible material (which is often sulfuric acid) and the new aerosol nucleation rate.

We begin by noting that there are two limiting behaviors of the condensation term. The continuum and kinetic regimes are defined by the Knudsen number $Kn = \lambda/R$, where $\lambda$ is the mean free path, and $R$ is the radius of the aerosol particle. The continuum regime corresponds to $Kn \ll 1$, and the kinetic regime corresponds to $Kn \gg 1$. The transition between these two regimes actually takes place for $Kn \approx 0.1$ (at which point the interaction rate is half the kinetic regime). It occurs when $R \approx \lambda \approx 0.5\mu m$ (for the earth’s atmosphere at sea level, e.g., see Seinfeld and Pandis 2006).

We assume an approximately constant coagulation coefficient $\beta_0 \approx 2 \times 10^{-5} \text{cm}^3\text{sec}^{-1}$ for small particles (<0.5 $\mu m$ ) and $\beta_0 \approx 6 \times 10^{-10} \text{cm}^3\text{sec}^{-1}$ for large particles (Seinfeld and Pandis 2006). The particles injected to the system are typically of radius 2 nm (Kulmala et al. 2013), and the nucleation rate $S_0$ is between 1 and 100 $\text{particle cm}^{-3}$. In fact, over oceans the rate is lower.
between 0.001 and 0.1 {\text{particle cm}^{-3}}. We now turn to evaluating $\epsilon_m$. The $H_2SO_4$ concentration in the atmosphere is of order $10^6$ cm$^{-3}$ (Eisele and Tanner 1993), the diameter of an $H_2SO_4$ molecule is 0.7 nm (Kurtén et al. 2007), and its typical lifetime is of order 1000 seconds. Therefore, in the atmosphere

$$\epsilon_m \approx 2 \times 10^{-19} \text{ sec}^{-1}.$$ \[57\]

Using Equation (21), we get a typical volume $v_2(\gamma = 1/3)$ which corresponds to particles of radius $r_2(\gamma = 1/3) \approx 2.35$ nm. For the case $\gamma = 2/3$, using Equations (38) and (7), we get:

$$v_2(\gamma = 2/3) \approx \left( \frac{1.03 \epsilon_m v_C^{-1/3}}{S_0} \right)^{3/2},$$ \[58\]

which corresponds to $r_2(\gamma = 2/3) \approx 0.16$ nm in the atmosphere.

Therefore, the value of $D$ in Equation (44) should be $D(\gamma = 2/3, x_1 = 3775) \approx 9.5$. We can now derive an explicit expression for the total number of aerosols above a given size, using Equation (44) (valid for aerosols smaller than 0.5 μm):

$$N^+(v) \equiv \int_{v}^{\infty} n(u)du \approx 0.99 \sqrt{\frac{1}{\beta_0}} \epsilon_m v_C^{-1/3} u^{-1/3}$$

$$+ \frac{D}{0.4} \sqrt{\frac{S_0}{\beta_0}} \epsilon_m v_C^{-1/3} u^{-0.706}$$

$$\times \left[ -2.94 u^{1/3} - 0.471 u^{-1/3} + 1.23, u^{-0.471} \right].$$ \[59\]

We can also differentiate Equation (59) to get the logarithmic derivative:

$$\frac{\partial \ln N^+}{\partial \ln \epsilon_m} \Bigg|_{S_0} = \frac{1}{2} + D \left( \frac{\epsilon_m v_C^{-1/3}}{S_0} \right)^{0.206}$$

$$\times \left( -1.52 v_C^{1/3} - 0.471 + 0.65 v_C^{1/3} - 0.471 \right).$$ \[60\]

This is the relative change in $N^+$ given a relative change in the amount of condensible material. Similarly, the sensitivity to changes in the nucleation rate for fixed condensible material, will be:

$$\frac{\partial \ln N^+}{\partial \ln S_0} \Bigg|_{\epsilon_m} = D \left( \frac{\epsilon_m v_C^{-1/3}}{S_0} \right)^{0.206} \left[ 1.52 v_C^{1/3} - 0.471 - 0.65 v_C^{1/3} - 0.471 \right].$$ \[61\]

Results for these last two sensitivities are shown on the right hand side of Figure 7.

We can repeat the above analysis for the case $\gamma = 1/3$, using Equation (31) which for the real atmosphere corresponds to sizes larger than about 0.5 μm. In this range, we find that the total number of particles above a certain volume is:

$$N^+(v) = \int_{v}^{\infty} n(u)du = 0.8 \sqrt{\frac{\epsilon_m u^{-1/2}}{\beta_0}}$$

$$- 0.32 \beta_0^{-1/2} S_0^{-1/6} \epsilon_m^{1/3} v^{-2/3}.$$ \[62\]

For this limit, we find that the sensitivities to changes in available building material and in the nucleation rate are (using Equation (62)):

$$\frac{\partial \ln N^+}{\partial \ln \epsilon_m} \Bigg|_{S_0} = \frac{1}{2} - 0.066 \left( \frac{\epsilon_m}{S_0} \right)^{1/6}.$$ \[63\]

**Figure 7.** Sensitivities of the total number of particles above a certain volume on the parameters $S_0$ and $\epsilon_m$ for $\gamma = 1/3$ (a) and $\gamma = 2/3$ (b). The parameters in (a) and (b) are the same as the ones in Figure 6a and b, respectively.
and

\[ \frac{\partial \ln N^+}{\partial \ln S_0} \bigg|_{\epsilon_m} = 0.066 \left( \frac{\epsilon_m}{S_0^2} \right)^{1/6}, \quad [64] \]

respectively. These last two sensitivities are plotted on the left hand side of Figure 7.

The last limit we consider is the distribution in the condensation limit of the kinetic regime, which corresponds to particles of volumes much smaller than \( v_0(\gamma = 2/3) \). For typical atmospheric conditions, \( v_0(\gamma = 2/3) \) is too small for this limit to be relevant (as it is smaller the volume of particles injected to the system). However, we include the following results for completeness. Here we use the solution given in Equation (50), after having eliminated \( \sigma \) using Equation (38). Total number of particles above certain volume:

\[ N^+(v) = \sqrt{\frac{S_0}{2\beta_0}} \exp \left[ -4.19 \sqrt{S_0} \epsilon_m^{-1/2} v C^{1/6} v^{1/3} \right]. \quad [65] \]

Now we calculate the sensitivities to changes in available building material and in the nucleation rate. To leading order, these are:

\[ \frac{\partial \ln N^+}{\partial \ln \epsilon_m} \bigg|_{S_0} \approx 2.09 \sqrt{S_0} \epsilon_m^{-1/2} v C^{1/6} v^{1/3} \quad [66] \]

and

\[ \frac{\partial \ln N^+}{\partial \ln S_0} \bigg|_{\epsilon_m} \approx \frac{1}{2} - 2.09 \sqrt{S_0} \epsilon_m^{-1/2} v C^{1/6} v^{1/3}. \quad [67] \]

Finally, we analyze the effective condensation flux (Lehtinen et al. 2007) \( J(\nu) = I(\nu)n(\nu) \), whose physical meaning is the formation rate of particles with sizes above \( \nu \) due to condensation. In the atmospherically relevant kinetic regime \( \gamma = 2/3 \), using the same parameters as in Figure 6, the flux behaves, to leading order, as a power-law \( J(\nu) \propto \nu^{-\gamma/2} \). This rate of decay is significantly faster than in Lehtinen et al. (2007). This is explained by the major difference between our model and the one described in Lehtinen et al. (2007): In their model, the coagulation causes removal of aerosols from the system, while in ours the coagulation is a growth mechanism of the aerosols. The result shown in Lehtinen et al. (2007) is therefore much more similar to our results for the condensation limit, which is not relevant for regular atmospheric parameters.

7. Discussion

One of the interesting implications of the solution to the coagulation/condensation equation is that it helps us understand how the aerosol distribution will change under different perturbations to the CN nucleation rate and to the amount of available condensible gas.

It is known from numerical simulations that perturbing the nucleation rate \( S_0 \) will have an appreciable effect on the density of small CCNs, whose growth is governed by condensation. However, the effect is going to be small for large CCNs once coagulation becomes important (Svensmark et al. 2013).

On the other hand, a perturbation to the amount of available condensible material \( \epsilon_m \), which is usually sulfurous or organic aerosols, will affect both the number density of small CCNs and of the larger CCNs.

With the result of Equations (31) and (50), this behavior can be understood and quantified analytically.

It is probably possible to obtain better theoretical solutions in the coagulation limit by taking the next term in the power series. However, this will probably not introduce any new physical consequences to the solution—unlike the first order correction to the solution (see Equation (31)) which added a dependence on the nucleation rate.

We also solved the case \( \gamma > 1/2 \), which corresponds to high condensation rates for large particles (see Equation (35)). The zeroth order solution was again improved by finding the first correction term. Since the rate of change in the total volume diverges, the solution has no physical meaning without introducing, for example, a large size cutoff.

As mentioned in Section 2.1, another interesting aspect is that the solution can be straightforwardly generalized to describe kernels of the type \( \beta(\nu, \nu) = \beta_1 \nu^\alpha \nu^{\prime} \) through the definition \( \gamma' = \gamma - \alpha \). This allows us to describe more realistic aerosol growth in the real atmosphere. This implies that in the condensation limit the generalized solution is

\[ y(x) \approx \frac{1}{\sqrt{2}} x^{-\alpha - 2\alpha} \exp \left[ -\sqrt{2} \frac{1}{1 - \gamma - \alpha} x^{1 - \gamma - \alpha} \right], \quad [68] \]

while for the coagulation limit we find

\[ y(x) = Bx^{-p - \alpha} + Dx^{-q - \alpha}, \quad [69] \]

where \( B, D, p \) and \( q \) are defined with \( \gamma' \) instead of \( \gamma \). For example, in the limit where coagulation is through hydrodynamic capture of particles in the Stokes regime (Klett 1975), we can expect \( \alpha = 2/3 \).
However, we could not extend our analytic results to kernels which are not of the form given above, such as to Fuchs coagulation kernels (Seinfeld and Pandis 2006). For these kernels, we could only obtain numerical results, from which it is much more difficult to obtain an understanding of their applications to the atmosphere.

Finally, in Section 6, we give quantitative predictions for the effect of changes in available condensable material and in the nucleation rate on the aerosol distribution, for typical atmospheric conditions.

### Funding

This research project was supported by the I-CORE Program of the Planning and Budgeting Committee and The Israel Science Foundation (Center No. 1829/12). NJS also thanks the IBM Einstein Fellowship support by the IAS.

### References

Beheng, K. D. (2010). The Evolution of Raindrop Spectra: A Review of Microphysical Essentials. *Rainfall: State of the Science*American Geophysical Union Geophysical Monograph Series, Washington DC, pp. 29–48.

Birnstiel, T., Ormel, C. W., and Dullemond, C. P. (2011). Dust Size Distributions in Coagulation/Fragmentation Equilibrium: Numerical Solutions and Analytical Fits. *Astron. Astrophys.*, 525(Jan.):A11.

Eisele, F. L., and Tanner, D. J. (1993). Measurement of the Gas Phase Concentration of H2SO4 and Methane Sulfonic Acid and Estimates of H2SO4 Production and Loss in the Atmosphere. *J. Geophys. Res.: Atmos.*, 98(D5):9001–9010.

Jimenez, J. L., Canagaratna, M. R., Donahue, N. M., Prevot, A. S. H., Zhang, Q., Kroll, J. H., DeCarlo, P. F., Allan, J. D., Coe, H., Ng, N. L., Aiken, A. C., Docherty, K. S., Ulbrich, I. M., Grieshop, A. P., Robinson, A. L., Dulldy, J., Smith, J. D., Wilson, K. R., Lanz, V. A., Hueglin, C., Sun, Y. L., Tian, J., Laaksonen, A., Raatikainen, T., Rautiainen, J., Vattovaara, P., Ehn, M., Kulmala, M., Tomlinson, J. M., Collins, D. R., Cubison, M. J., Dunlea, J., Huffman, J. A., Onasch, T. B., Alfarra, M. R., Williams, P. I., Bower, K., Kondo, Y., Schneider, J., Drewnick, F., Borrmann, S., Weimer, S., Demerjian, K., Salcedo, D., Cottrell, L., Griffin, R., Takami, A., Miyoshi, T., Hatakeyama, S., Shimono, A., Sun, J. Y, Zhang, Y. M., Dzepina, K., Kimmel, J. R., Supie, D., Jayne, J. T., Herndon, S. C., Trimborn, A. M., Williams, L. R., Wood, E. C., Middlebrook, A. M., Kolb, C. E., Baltensperger, U., and Worsnop, D. R. (2009). Evolution of Organic Aerosols in the Atmosphere. *Science*, 326 (5959):1525–1529.

Klett, J. D. (1975). A Class of Solutions to the Steady-State, Source-Enhanced, Kinetic Coagulation Equation. *J. Atmos. Sci.*, 32(Feb.):380–389.

Kulmala, M., Kontkanen, J., Junninen, H., Lehtipalo, K., Manninen, H. E., Nieminen, T., Petäjä, T., Sipilä, M., Schobesberger, S., Rantala, P., Fuchsel, A., Jokinen, T., Järvinen, E., Aaltonen, M., Kangasluoma, J., Hakala, J., Aalto, P. P., Paasonen, P., Mikkilä, J., Vanhanen, J., Aalto, J., Hakola, H., Makkonen, U., Ruuskanen, T., Mauldin, Roy L., Duplissy, J., Vehkamäki, H., Bäck, J., Kajos, M., Aalto, P., Rissanen, T., Kurtén, J., Kürten, A., Johnstone, M. V., Smith, James N., Ehn, Mikael, Mentel, Thomas F., Lehtinen, Kari E. J., Laaksonen, A., Kerminen, V-M, and Worsnop, D. R. (2013). Direct Observations of Atmospheric Aerosol Nucleation. *Science*, 339(6122):943–946.

Kurtén, T., Noppel, M., Vehkamäki, H., Salonen, M., and Kulmala, M. (2007). Quantum Chemical Studies of Hydrate Formation of H2SO4 and HSO4-. *Boreal Env. Res.*, 12(3):431–453.

Lehtinen, K. E. J., Maso, M. D., Kulmala, M., and Kerminen, V.-M. (2007). Estimating Nucleation Rates From Apparent Particle Formation Rates and Vice Versa: Revised Formulation of the Kerminen-Kulmala Equation. *J. Aerosol Sci.*, 38:988–994.

Peterson, T. W, Gelbard, F., and Seinfeld, J. H. (1978). Dynamics of Source-Reinforced, Coagulating, and Condensing Aerosols. *J. Colloid Interf. Sci.*, 63(3):426–445.

Ramabadran, T. E., Peterson, T. W., and Seinfeld, J. H. (1976). Dynamics of Aerosol Coagulation and Condensation. *AIChE J.*, 22(5):840–851.

Seinfeld, J. H., and Pandis, S. N. (2006). *Atmospheric Chemistry and Physics*. Wiley, New York.

Smoluchowski, M. V. (1916). Drei Vortrage Uber Diffusion, Brownsche Bewegung und Koagulation von Kolloidteilchen. *Zeitschrift für Physik*, 17:557–585.

Svensmark, Henrik, Enghoff, Martin B., and Pedersen, J. O. P. (2013). Response of Cloud Condensation Nuclei (> 50 nm) to Changes in Ion-Nucleation. *Phys. Lett. A*, 377(37):2343–2347.

### Table 1. Nomenclature

| Symbol | Description | Defining equation(s) and/or section(s) |
|--------|-------------|--------------------------------------|
| $n(v, t)$ | Aerosol density profile | 1 |
| $R(v, t)$ | Particle growth rate + parameters | 1, 2 |
| $\beta(v_1, v_2)$ | Coagulation coefficient + parameters | 1, 2, 3 |
| $\rho$, $\alpha$ | Removal rate of particles + parameters | 1, Sec. 4.3 |
| $S(v, t)$ | Nucleation rate of new particles + parameters | 1, 2 |
| $\chi_0(t)$ | Total number of particles in the System (per unit volume) | Sec. 2.2 |
| $v_0$ | Characteristic volume | 7 |
| $n_0$ | Characteristic particle density | 8 |
| $\tau$ | Dimensionless time | 9 |
| $x$ | Dimensionless volume | 9 |
| $y$ | Dimensionless particle density | 9 |
| $M_{tot}$ | Total condensable material | 12 |
| $i_m$ | Production rate of condensable material by an external source | 12, 13 |
| $z$ | Dimensionless version of $i_m$ | 15 |
| $\beta_D$ | Power-law solutions | Sec. 3.1, Sec. 3.2, Sec. 3.3, Sec. 3.4 |
| $\nu_C$ | Cutoff volume | 39 |
| $\zeta$ | Dimensionless particle loss coefficient | 54 |
| $K_n$ | Knudsen number | Sec. 6 |
| $n^*(v, t)$ | Cumulative density profile | 59 |
Appendix: Numerical solution

Here we describe the numerical solution of Equation (2). Since the equation is the time-independent limit of the full equation (Equation (1)) one method of solving the time independent equation is to solve the time dependent one and let the system relax to its steady state solution. However, we choose a second approach which reduces the CPU usage considerably.

First, we discretize the volume of the particles, as $v_1, v_2, v_3, \ldots$. We arbitrarily choose $v_i = i$ for all $i$, and denote the number of particles of size $i$ by $n_i$. Doing so, we now observe that the time-dependent equation for $n_1$ is:

$$\frac{\partial n_1}{\partial t} + \sigma(v_1)^{1/3} n_1 = 0 - \beta_0 n_1 \chi_0 + S_0.$$

Note that the 0 term denotes the fact that the particles with the smallest volume cannot be formed through the coagulation of smaller particles. For $n_k + 1$, the equation is:

$$\frac{\partial n_{k+1}}{\partial t} - \sigma(k)^{1/3} n_k + \sigma(k + 1)^{1/3} n_{k+1} = \frac{1}{2} \beta_0 \sum_{i=1}^{k} n_i n_{k+1-i} - \beta_0 n_{k+1} \chi_0.$$

Since the coagulation coefficient is assumed to be constant, the dynamics of $n_{k+1}$ depends only on $n_1 \ldots n_k$ and on $\chi_0$ which was found to be $\sqrt{2S_0/\beta_0}$ in Section 2.2. As a consequence, the equations can be solved sequentially from $n_1$.

Note also that the solution can be accelerated by using FFT to compute the convolution term. In this fashion we easily reach $n_{2,000,000}$ in our simulations using just a single CPU.