BURNSIDE’S THEOREM IN THE SETTING OF GENERAL FIELDS

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Abstract. We extend a well-known theorem of Burnside in the setting of general fields as follows: for a general field $F$ the matrix algebra $M_n(F)$ is the only algebra in $M_n(F)$ which is spanned by an irreducible semigroup of triangularizable matrices. In other words, for a semigroup of triangularizable matrices with entries from a general field irreducibility is equivalent to absolute irreducibility. As a consequence of our result we prove a stronger version of a theorem of Janez Bernik.

1. Introduction

A version of a celebrated theorem of Burnside [2, Theorem on p. 433] asserts that for an algebraically closed field $F$, the matrix algebra $M_n(F)$ is the only algebra in $M_n(F)$ which is spanned by an irreducible semigroup of matrices. We prove a counterpart of Burnside’s Theorem in the setting of general fields as follows: for a general field $F$, the matrix algebra $M_n(F)$ is the only algebra in $M_n(F)$ which is spanned by an irreducible semigroup of triangularizable matrices. In other words, for a semigroup of triangularizable matrices with entries from a general field irreducibility is equivalent to absolute irreducibility.

Throughout, $F$ and $K$ stand for fields and $F$ is a subfield of $K$. We view the elements of the matrix algebra $M_n(F)$ as linear transformations acting on the left of $F^n$, the vector space of all $n \times 1$ column vectors with entries from $F$. A family $\mathcal{F}$ in $M_n(F)$ is said to be irreducible if the orbit of any nonzero $x \in F^n$ under the algebra generated by $\mathcal{F}$, denoted by $\text{Alg}(\mathcal{F})$, is $F^n$. When $n > 1$, this is easily seen to be equivalent to the lack of nontrivial invariant subspaces for the family $\mathcal{F}$; the trivial spaces being $\{0\}$ and $F^n$. Reducible, by definition, means not irreducible. Absolute irreducibility means irreducibility over any.
field extension of the ground field $F$. It follows from Burnside’s Theorem that a family $\mathcal{F}$ in $M_n(F)$ is absolutely irreducible if and only if $\text{Alg}(\mathcal{F}) = M_n(F)$. On the opposite side of irreducibility, we have the notion of triangularizability. More precisely, a family $\mathcal{F}$ in $M_n(F)$ is called triangularizable if there exists a maximal chain 
\[{0} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_n = F^n \]
of subspaces of $F^n$ with $\mathcal{M}_i$’s being invariant under the family $\mathcal{F}$. Any such chain is called a triangularizing chain of subspaces for the family $\mathcal{F}$. It is a standard observation that a family $\mathcal{F}$ in $M_n(F)$ is triangularizable if and only if there exists a basis for $F^n$, called a triangularizing basis, relative to which every element of the family has an upper triangular matrix. This occurs if and only if there exists an invertible matrix $P \in M_n(F)$ such that the family $P^{-1}\mathcal{F}P$ consists of upper triangular matrices.

2. Main Results

The following can be thought of as an extension of Burnside’s Theorem to general fields. In fact, it extends [9, Theorem 2.3] to arbitrary fields.

**Theorem 2.1.** Let $n \in \mathbb{N}$, $F$ be a field, and $\mathcal{S}$ a semigroup of triangularizable matrices in $M_n(F)$. Then the semigroup $\mathcal{S}$ is irreducible iff it is absolutely irreducible.

**Proof.** The “if” implication is trivial. We prove “the only if” implication. Let $\mathcal{S}$ be a semigroup of triangularizable matrices in $M_n(F)$ and let $\mathcal{A} = \text{Alg}(\mathcal{S})$ denote the algebra generated by $\mathcal{S}$ and $r$ the minimal nonzero rank present in $\mathcal{A}$. As shown in [8] (the remark following Theorem 2.9 of that paper), $r$ divides $n$ and $\mathcal{A}$ is simultaneously similar to $M_{n/r}(\Delta)$, where $\Delta$ is an irreducible division algebra in $M_r(F)$, which is necessarily of dimension $r$. This in particular implies that the algebra $\mathcal{A}$ is simple (and semisimple). We prove the assertion by showing that $r = 1$. Let $F_c$ denote the algebraic closure of $F$ and view $\mathcal{A}$ as a simple $F$-algebra in $M_n(F_c)$. Apply a simultaneous similarity to put $\mathcal{A} \subseteq M_n(F_c)$ in block upper triangular form so that the number $k$ of the diagonal blocks is maximal and hence each diagonal block is absolutely irreducible. Note that the diagonal blocks are all nonzero because $\mathcal{A}$ contains the identity matrix. If necessary, using [11, Theorem 1.1] and applying a simultaneous similarity, we may assume that each diagonal block of $\mathcal{A}$ is the full matrix algebra $M_{n_i}(F)$ for
some \( n_i \in \mathbb{N} \) \( (1 \leq i \leq k) \). For each \( 1 \leq i \leq k \), let \( \mathcal{A}_i = M_{n_i}(F) \) denote the \( i \)-th diagonal block of \( \mathcal{A} \). In view of the simplicity of \( \mathcal{A} \), the mapping \( \phi_{ij} : \mathcal{A}_i \to \mathcal{A}_j \) defined by \( \phi_{ij}(A_i) = A_j \) is a well-defined nonzero homomorphism of \( F \)-algebras whose inverse \( \phi_{ji} : \mathcal{A}_j \to \mathcal{A}_i \) is also a homomorphism of \( F \)-algebras. Thus \( n_i = n_j = n/k \), and hence \( \mathcal{A}_i = \mathcal{A}_j = M_{n/k}(F) \) for each \( 1 \leq i, j \leq k \). Since \( F \)-algebra automorphisms of \( M_{n/k}(F) \) are all inner, again if necessary applying another simultaneous similarity, we may assume that the diagonal blocks of each element of \( A \in \mathcal{A} \) are all of the size \( n/k \) and equal. It thus follows from the Noether-Skolem Theorem, \([3, \text{p. 39}]\), that the \( F \)-algebra \( \mathcal{A} \) is similar to the \( k \)-fold inflation of the \( F \)-algebra \( M_{n/k}(F) \) because it is isomorphic to it. This in particular implies \( r = k \) and \( \dim \mathcal{A} = \dim M_{n/r}(\Delta) = \dim M_{n/k}(F) \). This clearly yields \( k^2 = k \), and hence \( r = k = 1 \), proving the assertion. \( \square \)

We used the main result of Bernik \([1, \text{Theorem 1.1}]\) to prove Theorem 2.1. The following shows that Theorem 2.1 implies a stronger version of the main result of \([1]\). Therefore, any independent proof of Theorem 2.1 would provide a new proof of Bernik’s result. (Thus it should be observed that the following proof does not use Bernik’s Theorem.)

**Theorem 2.2.** Let \( n \in \mathbb{N} \), \( F \) and \( K \) be fields with \( F \leq K \), and \( \mathcal{S} \) an irreducible semigroup of triangularizable matrices in \( M_n(K) \) with spectra in \( F \). Then \( \text{Alg}_F(\mathcal{S}) \) is similar to \( M_n(F) \) over \( M_n(K) \).

**Proof.** Since irreducibility implies absolute irreducibility for semigroups of triangularizable matrices, we see that \( \{0\} \neq \text{tr}(\mathcal{S}) \subseteq F \). It thus follows from \([8, \text{Corollary 2.8}]\) and absolute irreducibility of \( \mathcal{S} \) that \( \dim_F \text{Alg}_F(\mathcal{S}) = \dim_K \text{Alg}_K(\mathcal{S}) = n^2 \). Let \( \mathcal{B} \subseteq \mathcal{S} \) be a basis for \( \mathcal{A} := \text{Alg}_F(\mathcal{S}) \) and for \( A \in \mathcal{A} \), \( L_A : \mathcal{A} \to \mathcal{A} \), defined by \( L_A(B) = AB \), be the linear operator of left multiplication by \( A \). It is plain that the mapping \( \phi : \mathcal{A} \to M_{n^2}(F) \) defined by \( \phi(A) = [L_A]_\mathcal{B} \), where \([L_A]_\mathcal{B}\) denotes the matrix representation of \( L_A \) with respect to the basis \( \mathcal{B} \), is an embedding of the \( F \)-algebra \( \mathcal{A} \) in \( M_{n^2}(F) \). Clearly, \( \phi(\mathcal{A}) \) is a simple subalgebra of \( M_{n^2}(F) \). Apply a simultaneous similarity to put \( \phi(\mathcal{A}) \subseteq M_{n^2}(F) \) in block upper triangular form so that the number \( k \) of the diagonal blocks is maximal and hence each diagonal block is irreducible. Note that the diagonal blocks are all nonzero because \( \mathcal{A} \) contains the identity matrix. Also note that \( \mathcal{A} = \text{Alg}_F(\mathcal{S}) \) and \( \mathcal{S} \) consists of triangularizable matrices. But irreducibility implies absolute irreducibility for semigroups of triangularizable matrices. Thus, each diagonal block of \( \phi(\mathcal{A}) \) is an absolutely irreducible \( F \)-algebra in
$M_n(F)$, and hence is equal to the full matrix algebra $M_n(F)$. From this point on, an argument almost identical to that of the proof of Theorem 2.1 shows that the $F$-algebra $\phi(A)$ is similar to the $k$-fold inflation of the $F$-algebra $M_{n^2/k}(F)$ for some $k$ dividing $n^2$. This in particular gives

$$n^2 = \dim_F A = \dim_F \phi(A) = \dim_F M_{n^2/k}(F) = n^4/k^2,$$

which in turn implies $n = k$. Consequently, the $F$-algebra $A$ is isomorphic to the 1st block diagonal of $\phi(A)$, which is $M_n(F)$. It thus follows from the Noether-Skolem Theorem, [3, p. 39], that $A$ is similar to $M_n(F)$, which is the desired result. □

Remark. With this theorem at our disposal, we can prove the counterparts of [4, Theorem B on p. 99] and [6, Theorem 1] for semigroups of triangularizable matrices in $M_n(K)$ with spectra in $F$, see [9, Theorems 2.7 and 2.8]

An extension of Burnside’s Theorem was proved in [7, Theorems 2.1-2] as follows: for an $n > 1$ and a finite field, or more generally a quasi-algebraically closed field $F$, $M_n(F)$ is the only irreducible algebra in $M_n(F)$ that, as a vector space over $F$, is spanned by triangularizable matrices in $M_n(F)$. The following proposition answers a question left open in [7, Remark 1 following Theorem 2.1]. More precisely, it shows that the theorem does not hold for general fields, e.g., for the real field because $M_n(\mathbb{H})$, viewed as a proper irreducible subalgebra of $M_{4n}(\mathbb{R})$, is spanned by the identity matrix and nilpotents as a vector space over $\mathbb{R}$.

Proposition 2.3. Let $n \in \mathbb{N}$ with $n > 1$ and $\mathbb{H}$ denote the division ring of quaternions. Then $M_n(\mathbb{H})$ is spanned by $I$, the identity matrix, and nilpotents as a vector space over $\mathbb{R}$.

Proof. It suffices to prove the assertion for $M_2(\mathbb{H})$. Since

$\begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$,

$\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}$, and

$\begin{pmatrix} p & p \\ -p & -p \end{pmatrix}$

are all nilpotents for all $p, q \in \mathbb{H}$, we only have to show that $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ is spanned by $I$ and nilpotents for all $p \in \mathbb{H}$. Let $p = a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$. Thus it suffices to show that $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$, $\begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}$, and $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$ are all in the desired
span. Now
\[
\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} i & j \\ -j & i \end{pmatrix} + \begin{pmatrix} 0 & -j \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}.
\]
But \( \begin{pmatrix} i & j \\ -j & i \end{pmatrix}^2 = 0 \). This completes the proof. \(\square\)

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