Second-order Lagrangians admitting a first-order Hamiltonian formalism

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Abstract
Second-order Lagrangian densities admitting a first-order Hamiltonian formalism are studied; namely, i) for each second-order Lagrangian density on an arbitrary fibred manifold \( p: E \to N \) the Poincaré-Cartan form of which is projectable onto \( J^1 E \), by using a new notion of regularity previously introduced, a first-order Hamiltonian formalism is developed for such a class of variational problems; ii) the existence of first-order equivalent Lagrangians are discussed from a local point of view as well as global; iii) this formalism is then applied to classical Einstein-Hilbert Lagrangian and a generalization of the BF theory. The results suggest that the class of problems studied is a natural variational setting for GR.

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1 Preliminaries

1.1 Legendre & Poincaré-Cartan forms
Below, a fibred manifold \( p: E \to N \) is considered over a connected \( n \)-dimensional manifold \( N \) oriented by a volume form \( v = dx^1 \wedge \cdots \wedge dx^n \). The bundle of \( k \)-jets
of local sections of $p$ is denoted by $p^k: J^kE \to N$, with natural projections $p^1_k: J^kE \to J^{k-1}E$, $k \geq 1$.

Every fibred coordinate system $(x^j, y^a)$, $1 \leq j \leq n$, $1 \leq a \leq m = \dim E - n$, for the submersion $p$, induces a coordinate system $(x^j, y^a_I)$ ($I = (i_1, \ldots, i_n)$ being a multi-index in $\mathbb{N}^n$ of order $|I| = i_1 + \ldots + i_n \leq r$) on $J^rE$ defined by,

$$y_I^a (j^r x) = \frac{\partial^{i_1}(y_0^a x)}{\partial (x^a_{\alpha_1})_{i_1}}(x),$$

where $s$ is a local section of $p$. We also set $(j) = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n$, $(jk) = (j) + (k)$, etc., and $y_{(j)}^a = y_I^a$.

The Legendre form of a second-order Lagrangian density $\Lambda = L v$, defined on $p: E \to N$, $L \in C^\infty(J^2E)$, is the $V^\ast(p^1)$-valued $p^3$-horizontal $(n - 1)$-form $\omega_\Lambda$ on $J^3E$ locally given by (e.g., see [18], [21], [25]),

$$\omega_\Lambda = (-1)^{i-1}L^i_\alpha v_i \otimes dy^\alpha + (-1)^{i-1}L^j_\alpha v_i \otimes dy^\alpha,$$

where $v_i = dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$, and

$$L^i_\alpha = \frac{1}{2 - \delta_{ij}} \frac{\partial L}{\partial y^\alpha_{i(j)}},\quad L^0_\alpha = \frac{\partial L}{\partial y^\alpha_0} - \frac{1}{2 - \delta_{ij}} D_j \left( \frac{\partial L}{\partial y^\alpha_{i(j)}} \right),$$

and $D_j = \frac{\partial}{\partial x^j} + \sum_{|j|=0}^\infty \sum_{\alpha=1}^m y_I^a \frac{\partial}{\partial y^\alpha_{i(j)}}$ denotes the total derivative with respect to the coordinate $x^j$. The Poincaré-Cartan form (or P-C form for short) attached to $\Lambda$ is the ordinary $n$-form on $J^3E$ given by $\Theta_\Lambda = (p^3_2)^\ast \theta^2 \wedge \omega_\Lambda + \Lambda$ (e.g., see [18], [25]), where $\theta^1$, $\theta^2$ are the first- and second-order structure forms on $J^1E, J^2E$, locally given by (cf. [17], [24]), $\theta^1 = \theta^0 \otimes \frac{\partial}{\partial y^\alpha_0}$, $\theta^2 = \theta^0 \otimes \frac{\partial}{\partial y^\alpha_0} + \theta^1 \otimes \frac{\partial}{\partial y^\alpha_0}$, respectively, and $\theta^0 = dy^\alpha_0 - y^a_{(i)} dx^k, \theta^1_0 = dy^\alpha_0 - y^a_{(ik)} dx^k$, is the standard basis of contact 1-forms, and the exterior product of $(p^3_2)^\ast \theta^2$ and the Legendre form, is taken with respect to the standard pairing $V(p^1) \times J^1E V^\ast(p^1) \to \mathbb{R}$.

### 1.2 Projecting onto $J^2E$ or $J^1E$

The most outstanding difference with a first-order Lagrangian density is that the Legendre and Poincaré-Cartan forms associated with a second-order Lagrangian density are generally defined on $J^3E$, thus increasing by one the order of the Lagrangian density $\Lambda$.

For certain second-order Lagrangian densities it is known that the P-C form is projectable onto $J^2E$; e.g., see [10]. More precisely, the P-C form of a second-order Lagrangian projects onto $J^2E$ if and only if the following system of PDEs holds (cf. [6], [10]):

$$\frac{1}{2 - \delta_{ia}} \frac{\partial^2 L}{\partial y^a_{ia} \partial y^\alpha_{i(a)}} + \frac{1}{2 - \delta_{ic}} \frac{\partial^2 L}{\partial y^a_{ic} \partial y^\alpha_{i(a)}} + \frac{1}{2 - \delta_{ia}} \frac{\partial^2 L}{\partial y^a_{ic} \partial y^\alpha_{i(a)}} = 0,$$

for all indices $1 \leq a \leq b \leq c \leq n$, $\alpha, \beta = 1, \ldots, m$. 

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More surprisingly, there exist second-order Lagrangians for which the associated P-C form projects not only on $J^2E$ but also on $J^1E$. Notably, this is the case of the Einstein-Hilbert Lagrangian in General Relativity.

As is well known (e.g., see [19 (1.3)], [21, 2.1]), $p_{r-1}^*: J^rE \to J^{r-1}E$ admits an affine bundle structure modelled over the vector bundle

$$W^r = (p_{r-1}^{-1})^* S^r T^* N \otimes (p_0^{-1})^* V(p) \to J^{r-1}E.$$  

Proposition 1.1 (cf. [19], [23]). The Poincaré-Cartan form attached to a Lagrangian $L \in C^\infty(J^2E)$ projects onto $J^1E$ if and only if $L$ is an affine function with respect to the affine structure of $p_1^*: J^2E \to J^1E$, namely

$$L = L^{ij}_a y^{a}_{(ij)} + L_0, \quad L^{ij}_0 = L^{ij}_0 \in C^\infty(J^1E), L_0 \in C^\infty(J^1E),$$

and the following equations hold:

$$\frac{\partial L^a_{ih}}{\partial y^a_i} = \frac{\partial L^a_{ih}}{\partial y^a_i}, \quad a, h, i = 1, \ldots, n, \quad \alpha, \beta = 1, \ldots, m.$$  

The equations (5) admit a variational meaning. The Euler-Lagrange (or E-L for short) operator of an arbitrary second-order Lagrangian can be written in terms of the coefficients of the P-C form (see the formulas (1), (2)) as follows:

$$E_\alpha(L) = \sum_{i \leq j} D_i D_j \left( \frac{\partial L^a_{(ij)}}{\partial y^a_i} \right) - D_i \left( \frac{\partial L^a_{ij}}{\partial y^a_i} \right) = \frac{\partial L^a_{ij}}{\partial y^a_i} - D_i \left( L^{a}_0 \right), \quad 1 \leq \alpha \leq m.$$  

The E-L equations for an affine second-order Lagrangian $L$, given as in the formula (4), are of third order and they are of second order if and only if the equations (5) hold (cf. [23, Proposition 2.2]).

As the projection $p_{r-1}^*: J^rE \to J^{r-1}E$ admits an affine-bundle structure, a natural vector-bundle isomorphism is obtained,

$$\Gamma: (p_{r-1}^*)^* W^r = (p_r^{-1})^* S^r T^* N \otimes (p_0^{-1})^* V(p) \cong V(p_{r-1}^*),$$

where the vector bundle $W^r$ is defined in (3). Given an arbitrary vector bundle $W \to N$, there exists an antiderivation

$$d_{E/N}: \Gamma(E, \wedge^r V^* (p) \otimes p^* W) \to \Gamma(E, \wedge^{r+1} V^* (p) \otimes p^* W)$$

doing $+$1—called the fibre differential (e.g., see [19 (1.9)])—such that, $d_{E/N}(fp^* \xi) = df_{|V(p)} \otimes \xi$, for all $f \in C^\infty(E)$ and all $\xi \in \Gamma(E, W)$. (In the previous paragraph, the relevant fact is that the vector bundle $W \to N$ is defined over the base manifold $N$, and not over the fibred manifold $E$.)

In what follows we are mainly concerned with the fibre derivative $d_{L/E/J^0E}$, which will simply be denoted by $d_{10}$ for the sake of simplicity.

A Lagrangian $L \in C^\infty(J^2E)$ is an affine function with respect to the affine structure of $p_1^*: J^2E \to J^1E$ if there exists a linear form $w_L: W^2 \to \mathbb{R}$, which
is unique, such that, \( L(\tau + J^2 s) = w_L(\tau) + L(J^2 s), \forall \tau \in S^2 T^* N \otimes V_{s(\tau)}(p) \) and \( \forall J^2 s \in J^E. \)

By using the \( (W^2)^* \equiv (p_1^*)^* S^2 T N \otimes (p_0^*)^* V^*(p) \), the linear form \( w_L \) defines a section of the vector bundle \( (p_1^*)^* S^2 T N \otimes (p_0^*)^* V^*(p) \rightarrow J^1 E \). If \( L \) is locally given by the formula (4), then \( w_L = L^h_i \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial y^h} \otimes dy^\alpha \big|_{V(p)} \), where the symbol \( \otimes \) denotes symmetric product.

If \( i^2 : (W^2)^* \rightarrow (p_1^*)^* \otimes^2 T N \otimes (p_0^*)^* V^*(p) \) is the natural embedding, then we consider the section

\[
(7) \quad w_L = \frac{1}{2} (I^1 \circ i^2 \circ w_L) : J^1 E \rightarrow (p_1^*)^* T N \otimes V^*(p_0^*)
\]

obtained by composing the following mappings:

\[
J^1 E \xrightarrow{w_L} (p_1^*)^* S^2 T N \otimes (p_0^*)^* V^*(p) = (W^2)^* \xrightarrow{i^2} (p_1^*)^* \otimes^2 T N \otimes (p_0^*)^* V^*(p) = ((p_1^*)^* T N \otimes (p_0^*)^* V^*(p)) \xrightarrow{I^1} (p_1^*)^* T N \otimes V^*(p_0^*)
\]

where \( I^1 = 1_{(p_1^*)^* T N \otimes (p_0^*)^* V^*(p)} \) is the isomorphism deduced from (4) for \( r = 1 \). As \( I^1(dx^a \otimes \partial \gamma^a) = \partial / \partial y^a \), dually we obtain \( (I^1)^*(d_{10} y^a) = \partial / \partial x^a \otimes dy^a \big|_{V(p)} \).

Hence \( w_L^h = L^h_i d_{10}(y^h) \otimes \frac{\partial}{\partial y^i} \).

Remark 1.1. The equations (5) simply means that for every index \( h \) the form \( \eta^h = L^h_i dy^i \) is \( d_{10} \)-closed, namely \( d_{10} \eta^h = 0 \). Hence, there exist functions \( L^i \in C^\infty(J^1 E) \) such that locally,

\[
(8) \quad (i) \quad L^h_i = \frac{\partial L^i}{\partial y_h}, \quad (ii) \quad \frac{\partial L^i}{\partial y_h} = \frac{\partial L^h_i}{\partial y_h}, \quad 1 \leq \alpha \leq m, \ h, i = 1, \ldots, n,
\]

the equations (ii) above being a consequence of the symmetry \( L^h_i = L^i_h \).

Letting \( W = T N \) in the definition of the fibre differential above, recalling that the Poincaré lemma also holds for fibre differentiation (e.g., see [19]) and recalling that the fibres of \( p_0^* : J^1 E \rightarrow E \) are simply connected as they are diffeomorphic to \( \mathbb{R}^m \), the following global characterization of second-order variational problems with a P-C form projecting onto \( J^1 E \), is obtained:

**Proposition 1.2.** (see [23 Proposition 3.1]) *The Poincaré-Cartan form of a Lagrangian \( L \in C^\infty(J^2 E) \) projects onto \( J^1 E \) if and only if \( L \) is an affine function with respect to the affine structure of \( p_1^* : J^2 E \rightarrow J^1 E \) and the T N-valued 1-form \( w'_L \) defined in the formula (7) is \( d_{10} \)-closed. In this case, for every global (smooth) section \( \sigma : E \rightarrow J^1 E \) of \( p_0^* \), there exists a unique globally defined section \( w_L^\sigma \in \Gamma(J^1 E, (p_1^*)^* T N) \) such that, \( d_{10}(w_L^\sigma) = w'_L, \ w_L^\sigma (\sigma(e)) = 0, \forall e \in E. \)

**Remark 1.2.** A general procedure to obtain global sections \( \sigma : E \rightarrow J^1 E \) of \( p_0^* \) is to use Ehresmann (or non-linear) connections, i.e., to use a differential 1-form \( \gamma \) on \( E \) taking values in the vertical sub-bundle \( V(p) \) such that \( \gamma(X) = X, \forall X \in V(p) \); hence, locally (cf. [22]), \( \gamma = (dy^a + \gamma^a_j dx^j) \otimes \frac{\partial}{\partial y^a} \), \( \gamma^a_j \in C^\infty(E) \). The vertical differential of a section \( s : U \rightarrow E \) (defined on a neighbourhood \( U \) of
$x \in N$) at $e = s(x)$ is defined to be the linear mapping $(d^v s)_e : T_e E \rightarrow V_e(p)$, $(d^v s)_e X = X - s_\ast p_e(X), \forall X \in T_e E$. We claim that for every $e \in E$, there exists a unique $j^1_1 s \in J^1 E$ such that, i) $s(x) = e$, where $x = p(e)$, and ii) $(d^v s)_e = e$. In fact, one has $\partial (y^\alpha \circ s) / \partial x^j (x) = -\gamma_j^\alpha (e)$, and the section $\sigma^\gamma$ attached to $\gamma$ is defined by, $\sigma^\gamma (e) = j^1_1 s$.

1.3 Summary of contents

Bearing the previous definitions and notations in mind, the paper is organized as follows: In section 2 the Hamiltonian function, the momenta, and the Hamilton-Cartan equations attached to each of the aforementioned Lagrangians are introduced as a consequence of a normal form for their P-C form. This section also deals with the notion of regularity for the class of second-order variational problems with a P-C form that projects to first-order jet bundle. Although the Hessian metric vanishes identically for the Lagrangians of such class, a suitable notion of regularity is introduced for them.

In [23] the study of the formal integrability of the field equations of second-order Lagrangians with projectable P-C form to first order in their Hamiltonian form is devoted. In the real analytic case, this allows one to solve the Cauchy initial value problem for this class of Lagrangians. The previous sections are then applied to GR in section 3, thus showing how the theory developed fits very well to the standard Lagrangians in this setting. Specifically, section 3.1 studies Einstein-Hilbert Lagrangian from this point of view, proving its regularity and giving a new statement for the initial problem. Similarly, section 3.2 provides a strong generalization of the classical Lagrangians in BF-theory, again showing that the results obtained above can naturally be applied to these new Lagrangians. In section 4 the existence of first-order Lagrangians variationally equivalent to a second-order Lagrangian admitting a first-order Hamiltonian formalism is studied, both from local and global point of view. This generalizes previous results obtained for the E-H Lagrangian in [3]. Section 5 introduces the notions of symmetry and Noether invariant for the class of variational problems dealt with throughout the paper and section 6 discusses in particular such concepts for the E-H Lagrangian. Finally, in section 7 the notion of a Jacobi field along an extremal is introduced and the presymplectic structure attached to a variational problem is defined. Several explicit examples are also developed in detail.

2 Regularity and Hamiltonian formalism

In the usual (i.e., first-order) calculus of variations, a section $s$ is an extremal of the Lagrangian density $\Lambda$ on $J^1 E$ if and only if it satisfies the so-called "Hamilton-Cartan equations" (or H-C for short; e.g., see [11 (3.8)], [10 (1)]), namely, if and only if the following equation holds: $(j^1 s) \ast (i_X d\Theta_\Lambda) = 0$ for every $p^1$-vertical vector field $X$ on $J^1 E$. 

5
If $\Lambda = Lv$ is an arbitrary second-order Lagrangian density on $E$, then the following formula holds (e.g., see [15]):

$$d\Theta_\Lambda = \mathcal{L}_\alpha (L) \theta^\alpha \wedge v + \eta_{n+1} (L),$$

where $\eta_{n+1} (L) = (-1)^i \eta_i^2 (L) \wedge v_i$ and $\eta_i^2 (L)$ is the 2-contact 2-form given by,

$$\eta_i^2 (L) = \frac{\partial L^0}{\partial y^i} \theta^0 \wedge \theta^3 + \left( \frac{\partial L^0}{\partial y^j} \right) \theta^0 \wedge \theta^j + \sum_{j < k} \left( \frac{\partial L^0}{\partial y^{jk}} \right) \theta^0 \wedge \theta^{jk} + \sum_{i < k} \left( \frac{\partial L^0}{\partial y^{(i)}} \right) \theta^0 \wedge \theta^{(i)kj} + \sum_{k \leq l} \left( \frac{\partial L^0}{\partial y^{kl}} \right) \theta^0 \wedge \theta^{kl}.$$ 

From the formula (9) it follows that the H-C equations also characterize critical sections for a second-order density $\Lambda$; i.e., $s$ is an extremal for $\Lambda$ if and only if, $(j^3 s)^* (i_X d\Theta_\Lambda) = 0$ for every $j^3$-vertical vector field $X$ on $J^3 E$.

**Remark 2.1.** If the P-C form of a second-order density $\Lambda$ projects onto $J^1E$, then its H-C equations have the same formal expression of a first-order density (see the formula (13) below), although there is no first-order density having $\Theta_\Lambda$ as its P-C form. In fact, the P-C form of a first-order Lagrangian density $\tilde{\Lambda} = \tilde{L}v$, $\tilde{L} \in C^\infty (J^1 E)$, is given by,

$$\Theta_{\tilde{\Lambda}} = (-1)^{i-1} \frac{\partial L}{\partial y^i} dy^0 \wedge v_i + \tilde{H} v, \quad \tilde{H} = \tilde{L} - \frac{\partial L}{\partial y^i} y_i^0.$$ 

If $\Theta_\Lambda = \Theta_{\tilde{\Lambda}}$, then the following three equations are obtained:

1) $L^i_{\alpha} = 0$,
2) $L_0 - y^0 L^0_{\alpha} = \tilde{L} - \frac{\partial L}{\partial y^i} y_i^0$,
3) $L^0_{\alpha} = \frac{\partial \tilde{L}}{\partial y_i^0}.$

From (11) and 1) it follows $L = L_0$; hence $L$ is of first order.

Moreover, taking (2) into account, the formulas 2) and 3) above are respectively rewritten as $L_0 - \tilde{L} = y^0 \frac{\partial (L_0 - \tilde{L})}{\partial y^0}$, $\frac{\partial (L_0 - \tilde{L})}{\partial y^i} = 0$. Hence $\tilde{L} = L$.

**Theorem 2.1.** (see [23, Theorem 4.1]) If $\Lambda = Lv$ is a second-order Lagrangian density on $E$ whose Poincaré-Cartan form projects onto $J^1 E$, then letting

$$p_{\alpha}^i = L^0_{\alpha} - \frac{\partial L}{\partial x^i}, \quad H = L_0 - y^0 L^0_{\alpha} - \frac{\partial L}{\partial x^i},$$

where the functions $L^1$ are defined by the formulas (11)-(i), the following formula holds:

$$d\Theta_\Lambda = (-1)^{i-1} dp_{\alpha}^i \wedge dy^0 \wedge v_i + dH \wedge v.$$
Furthermore, if the linear forms $d_{10}(p^i): V(p^0_1) \to \mathbb{R}, 1 \leq \alpha \leq m, 1 \leq i \leq n,$ are linearly independent, then a section $s: N \to E$ is an extremal for $\Lambda$ if and only if it satisfies the following equations:

$$
\begin{align*}
0 &= \frac{\partial (p^i \circ j^1 s)}{\partial x^j} - \frac{\partial H}{\partial y^\alpha \circ j^1 s}, \quad 1 \leq \alpha \leq m, \\
0 &= \frac{\partial (y^\alpha \circ s)}{\partial p^i} + \frac{\partial H}{\partial p^i} \circ j^1 s, \quad 1 \leq \alpha \leq m, 1 \leq i \leq n.
\end{align*}
$$

(14)

As is well known (e.g., see [11]), if the Hessian metric $\text{Hess}(L)$ of a first-order density $\Lambda = Lv$ is non-singular, then every section $s^1: N \to J^1E$ of the projection $p^1: J^1E \to N$ that satisfies the P-C equation for $\Lambda$ is holonomic; i.e., $s^1$ coincides with the 1-jet extension of the section $s = p^0 \circ s^1$ of the projection $p$. Namely, $(s^1)^*(i_X d\Theta_\Lambda) = 0$ for every $p^1$-vertical vector field $X$ on $J^1E$, implies $s^1 = j^1 s$.

In the case of a second-order density with a P-C form projecting onto $J^1E$, the following result holds:

**Proposition 2.2** (23). If $\Lambda = Lv$ is a second-order Lagrangian on $E$ such that, (i) its Poincaré-Cartan form $\Theta_\Lambda$ projects onto $J^1E$, (ii) the linear forms $d_{10}(p^i): V(p^0_1) \to \mathbb{R}, 1 \leq \alpha \leq m, 1 \leq i \leq n,$ where the functions $p^i$ are introduced in (11), are linearly independent, then every solution to its H-C equations, is holonomic.

As $p^0_1: J^1E \to E$ is an affine bundle modelled over $W^1 = p^*(T^*N) \otimes V(p)$ (cf. [3]), there is a canonical isomorphism $I: (p^0_1)^*W^1 \cong V(p^0_1)$ locally given by $I(j^1_s, (dx^i)_x \otimes (\partial/\partial y^\alpha)(s(x))) = (\partial/\partial y^\alpha(j^1_s))_x$.

According to the previous lemma, we can define a bilinear form

$$
\begin{align*}
0 &= \frac{\partial (p^i \circ j^1 s)}{\partial x^j} - \frac{\partial H}{\partial y^\alpha \circ j^1 s}, \quad 1 \leq \alpha \leq m, \\
0 &= \frac{\partial (y^\alpha \circ s)}{\partial p^i} + \frac{\partial H}{\partial p^i} \circ j^1 s, \quad 1 \leq \alpha \leq m, 1 \leq i \leq n.
\end{align*}
$$

(15)

where $\phi^k_v$ is the isomorphism defined by

$$
\phi^k_v: \wedge^k T_x N \to \wedge^{n-k} T_x N
$$

for every $1 \leq k \leq n - 1$, obtained by contracting with $v$, namely

$$
\phi^k_v(X_1 \wedge \cdots \wedge X_k) = i_{X_1} \cdots i_{X_k} v, \quad \forall X_1, \ldots, X_k \in T_x N.
$$

If $w_0 = (dx^i)_x$ and $Y_0 = (\partial/\partial y^\alpha)(s(x))$, then one readily obtains,

$$
i_{Y_0} i_Y (d\Theta_\Lambda) = (-1)^{i-1} \left( \frac{\partial L^0_\alpha}{\partial y^\beta j^1_s} (j^1_s) - \frac{\partial L^j_\beta}{\partial y^\alpha j^1_s} (j^1_s) \right) (v_i)_x,
$$

$$
\langle w_0, (\phi^1_v)^{-1} (i_{Y_0} i_Y (d\Theta_\Lambda)) \rangle = \frac{\partial L^0_\alpha}{\partial y^\beta j^1_s} (j^1_s) - \frac{\partial L^j_\beta}{\partial y^\alpha j^1_s} (j^1_s).
$$
In other words,
\[ b_\Lambda \left( j^1_x, (dx^i)_x \otimes \left( \frac{\partial}{\partial y^\alpha} \right)_{s(x)} \right) \left( j^1_x \right)_x = \frac{\partial L^0_i}{\partial y^j} (j^1_x)_x - \frac{\partial L^i_j}{\partial y^\alpha} (j^1_x)_s. \]

Hence, the next result follows:

**Corollary 2.3.** Let \( \Lambda \) be a second-order density on \( E \) whose P-C form projects onto \( J^1 E \). If the bilinear form defined in (15) is non-singular, then every solution to the H-C equations for \( \Lambda \) is holonomic.

**Proposition 2.4.** (see Proposition 5.4) The bilinear form \( b_\Lambda \) defined in (15) is symmetric.

In fact, if \( L \) is the Lagrangian defined by
\[ L = L_0 - \frac{\partial L_i^j}{\partial x^i} - y_i^\alpha \frac{\partial L^i_j}{\partial y^\alpha}, \]
then, as a calculation shows,
\[ p_i^\alpha = \frac{\partial L}{\partial y^i} \]

### 3 Applications to GR

#### 3.1 Einstein-Hilbert Lagrangian

Below, we follow [23]. Let \( p_M : M = M(N) \to N \) be the bundle of pseudo-Riemannian metrics of a given signature \((n^+, n^-)\), \( n^+ + n^- = n \). Every coordinate system \((x^i)_{i=1}^n\) on an open domain \( U \subseteq N \) induces a coordinate system \((x^i, y_{jk})\) on \((p_M)^{-1}(U)\), where the functions \( y_{jk} = y_{kj} \) are defined by,
\[ g_x = y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x = \sum_{i \leq j} \frac{1}{i + j} y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \quad \forall g_x \in (p_M)^{-1}(U). \]

Following the notations in [15], the Ricci field attached to the symmetric connection \( \Gamma \) is given by
\[ S^i (X, Y) = \text{trace}(Z \mapsto R^F (Z, X) Y), \]
where \( R^F \) denotes the curvature tensor field of the covariant derivative \( \nabla^F \) associated to \( \Gamma \) on the tangent bundle; hence \( S^i = (R^F)^{jl} dx^l \otimes dx^j \), where \( (R^F)^{jl} = (R^F)^{kl}_j \), and \( (R^F)^{ijkl} = \partial \Gamma^i_{jl} / \partial x^k - \partial \Gamma^i_{jk} / \partial x^l + \Gamma^m_{jk} \Gamma^i_{ml} - \Gamma^m_{jm} \Gamma^i_{lk} \).

The E-H Lagrangian density is given by
\[ (\Lambda_{EH})^j_{zg} = g^{ij}(x) (R^g)^h_{ij}(x) v_g(x) = L_{EH}(j^2 g) v_x, \]
where \( \psi \) is the standard volume form, \( R^g \) is the curvature tensor of the Levi-Civita connection \( \Gamma^g \) of the metric \( g \), and \( v_g \) denotes the Riemannian volume form attached to \( g \); i.e., in coordinates, \( v_g = \sqrt{\det((g_{ij})_{a,b=1}^n)} |v| \). Hence,
\[ L_{EH} \circ j^2 g = (\rho \circ g) (j^i \circ g) (R^g)^h_{ij}, \quad \rho = \sqrt{\det((g_{ab})_{a,b=1}^n)}. \]
The local expression for $L_{EH}$ is readily seen to be

$$L_{EH} = \rho \sum_{a,b} \sum_{c,d} \sum_{e,f} \left( y_{ac}^{\alpha} y_{bd}^{\beta} - y_{ab}^{\alpha} y_{cd}^{\beta} \right) \partial y_{ab,cd} + (L_{EH})_0,$$

where

$$(L_{EH})_0 = \frac{\rho}{2} \sum_{r \leq s} \sum_{k \leq l} \sum \frac{1}{(1+\delta_{r,k})(1+\delta_{s,l})} \left( 2y^{rs} (y^{ki} y^{jl} + y^{ki} y^{jl}) - 2y^{kl} y^{rs} y^{ij} \right)$$

$$+ 2y^{kl} (y^{ir} y^{si} + y^{is} y^{ri}) + 3y^{ij} (y^{kr} y^{ls} + y^{ks} y^{lr})$$

$$- y^{ir} (y^{ks} y^{il} + y^{is} y^{jr}) - y^{is} (y^{kr} y^{il} + y^{ir} y^{jk})$$

$$- 2y^{kl} (y^{il} y^{jr} + y^{ir} y^{js}) - 2y^{ij} (y^{kl} y^{jr} + y^{kr} y^{js}) \right) y_{kl,i} y_{rs,j}.$$ 

Hence $L_{EH}$ is an affine function and according to Proposition 1.1 its P-C form projects onto $J^1 M$ if and only if the following equations hold:

$$0 = 2 \frac{\partial (L_{EH})_{rs}^{ij}}{\partial y_{st,a}} - \frac{\partial (L_{EH})_{st}^{ij}}{\partial y_{rs,h}} - \frac{\partial (L_{EH})_{sh}^{ij}}{\partial y_{rs,i}},$$

where

$$(L_{EH})_{rs}^{ij} = \frac{1}{2\delta_{r,s}} \frac{\partial L_{EH}}{\partial y_{rs,ij}}$$

$$= \frac{1}{1+\delta_{r,s}} \rho \left( y^{ir} y^{js} + y^{is} y^{jr} - 2y^{rs} y^{ij} \right).$$

and the result follows immediately as $(L_{EH})_{rs}^{ij}$ does not depend on the variables $y_{ij,k}$.

In the present case, one has

$$p_{kl}^i = \sum_{r \leq s} \left( \frac{\partial^2 L_{0}}{\partial y_{rs,j} \partial y_{kl,i}} - \frac{\partial (L_{EH})_{rs}^{ij}}{\partial y_{rs}} - \frac{\partial (L_{EH})_{kl}^{ij}}{\partial y_{kl}} \right) y_{rs,j}$$

$$= \sum_{r \leq s} Y_{rs,j}^{i;kl} y_{rs,j},$$

$$Y_{kl}^{rs,j} = \frac{\rho}{(1+\delta_{r,l})(1+\delta_{s,k})} \left[ 2y^{rs} y^{kl} y^{ij} - (y^{rk} y^{ls} + y^{rl} y^{sk}) y^{ij} \right.$$

$$+ (y^{sk} y^{lj} + y^{sl} y^{kj}) y^{ri} + (y^{rk} y^{lj} + y^{rl} y^{kj}) y^{si}$$

$$- (y^{lk} y^{ij} + y^{il} y^{kj}) y^{rs} - (y^{ri} y^{sj} + y^{rj} y^{si}) y^{kl} \right],$$

and

$$H = \rho \sum_{k \leq l} \sum_{r \leq s} \sum_{(1+\delta_{r,l})(1+\delta_{s,k})} \left[ -y^{ij} y^{kl} y^{rs} \right.$$

$$+ y^{kl} (y^{ir} y^{js} + y^{is} y^{jr}) + y^{ij} (y^{ks} y^{lr} + y^{kr} y^{ls})$$

$$- y^{ir} (y^{il} y^{ks} + y^{ij} y^{lr}) - y^{is} (y^{il} y^{ks} + y^{ij} y^{lr}) y_{rs,j} y_{kl,i}.$$
Remark 3.1. As a calculation shows, from the expression in (24) for the Hamiltonian of the E-L Lagrangian, for every $j \in J$ the following formula holds true: $H(j) = \rho(x)g^{ij}(x)(\Gamma^g)^{r}_{hi}(x) - (\Gamma^g)^{h}_{ri}(x)(\Gamma^g)^{r}_{hi}(x)$. Hence the function $H$—considered as a first-order Lagrangian—not only provides the H-C equations for $\Lambda_{EH}$ but also its own E-L equations, e.g., see [4, 3.3.1].

Theorem 3.1 (cf. [3], [7], [23]). We have

(i) With the natural identification $V(p_M) \cong p_* M S^2 T^* N$, the bilinear form $b_{L_{EH}}$ is defined on $p_* M (T^* N \otimes S^2 T^* N)$.

(ii) The Lagrangian function $\bar{L}_{EH}$ defined in (17) coincides with the opposite to the Hamiltonian function.

(iii) The E-H Lagrangian satisfies the regularity condition of Corollary 2.3.

Proof. (i) From the formula

$$\frac{\partial p^i_{\alpha}}{\partial y^j} = \frac{\partial L^0_{\alpha}}{\partial y^j} - \frac{\partial L^h_{\alpha}}{\partial y^j},$$

and (22), (23) it follows that the matrix of $b_{L_{EH}}$ in the basis $(dx^i) \otimes (\partial/\partial y^j g)e$, $g \in p^{-1}(x)$, $1 \leq i \leq n$, $1 \leq j \leq k \leq n$, at a point $j g$ is

$$(\partial p^j_{mr}/\partial y_{cd,h})g)_{c,d,h}^m \leq r,j = (Y_{mr,j}^{cd,h}(g)_c d,h)_{c,d,h}^m \leq r,j,$$

and one can conclude.

(ii) It follows from the formulas (18), (22), (24) by means of a simple calculation.

(iii) The proof is similar to that of Proposition 5.1 in [3], as

$$\frac{\partial p^i_{\alpha}}{\partial y_{cd,h}} = \frac{\partial^2 L_{EH}}{\partial y_{mr,j} \partial y_{cd,h}} = \frac{\partial^2 L^\gamma}{\partial y_{mr,j} \partial y_{cd,h}},$$

where $L^\gamma$ is the first-order Lagrangian variationally equivalent to $L_{EH}$ introduced in [3].

In the present case, the equations (14) become

$$\begin{align*}
0 &= \frac{\partial (p^i_{kl} \circ j^1 s)}{\partial x^l} - \frac{\partial H}{\partial y_{kl}} \circ j^1 s, \quad 1 \leq k \leq l \leq n, \\
0 &= \frac{\partial (y_{kl} \circ s)}{\partial x^l} + \frac{\partial H}{\partial p_{kl}} \circ j^1 s, \quad 1 \leq i \leq n, 1 \leq k \leq l \leq n.
\end{align*}$$

Remark 3.2. By using the previous theorem, in [23 Theorem 6.2] the following result has been obtained:

“Given symmetric scalars $\gamma^i_{jk} = \gamma^i_{kj}, i, j, k = 1, \ldots, n$, there exists a Ricci-flat (pseudo-)Riemannian metric $g$ of signature $(n^-, n^+)$ defined on a neighbourhood of $x_0 \in N$ such that, $g_{ij}(x_0) = \delta_{ij}, (\Gamma^g)^{j}_{ik}(x_0) = \gamma^i_{jk}$, for all $i, j, k$.”
3.2 BF field theory

In this section we consider a new approach to BF Lagrangians (cf. [2], [5], [8], [14], [15]) generalizing the E-H functional.

Let \( \pi: F(N) \to N \) be the principal \( Gl(n, \mathbb{R}) \)-bundle of linear frames on \( N \). Given a metric \( g \) on \( N \), let \( \pi^*_g: F_g(N) \subset F(N) \to N \) be the subbundle of orthonormal linear frames with respect to \( g \), i.e., \( u = (X_1, \ldots, X_n) \) belongs to \( F_g(N) \) if and only if, \( g(X_i, X_j) = \varepsilon \delta_{ij} \), with \( \varepsilon = +1 \) for \( 1 \leq i \leq n^+ \) and \( \varepsilon = -1 \) for \( 1 + n^+ < i \leq n \). This is a principal bundle with structure group the orthogonal group \( O(n^+), n^+ + n^- = n \), associated to the quadratic form \( q(x) = \sum_{a=1}^{n^+} (x^a)^2 - \sum_{b=n^++1}^{n^++n^-} (x^b)^2 \).

By virtue of the symmetries of the curvature tensor \( R^g \) of the Levi-Civita connection of a metric \( g \), for every \( X, Y \in T_xN \) the endomorphism \( R^g(X, Y) \) takes values in the vector subspace of skew-symmetric linear operators (with respect to \( g_x \)) in \( \text{End}(T_xN) \). More generally, let \( p_M: M \to N \) be the bundle of pseudo-Riemannian metrics of signature \((n^+, n^-)\), and let

\[
\mathcal{A}(TN) \subset (p_M)^* \text{End}(TN) = M \times_N \text{End}(TN)
\]

be the vector subbundle of the pairs \((g_x, A), g_x \in (p_M)^{-1}(x) \) and \( A \in \text{End}(T_xN) \), such that \( g_x(AX, Y) + g_x(X, AY) = 0 \), \( \forall X, Y \in T_xN \); i.e., \( A \) is skew-symmetric with respect to \( g_x \). Pulling \( \mathcal{A}(TN) \) back along a metric \( g \), understood as a smooth section of \( p_M: M \to N \), one obtains the adjoint bundle of the bundle of orthonormal frames with respect to \( g \), i.e., the bundle associated to \( F_g(N) \) under the adjoint representation of \( O(n^+, n^-) \) on its Lie algebra \( \mathfrak{o}(n^+, n^-) \), i.e., \( g^*\mathcal{A}(TN) = \text{ad} F_g(N) = (F_g(N) \times \mathfrak{o}(n^+, n^-))/O(n^+, n^-) \).

If \( \beta \) is an \( \mathcal{A}(TN) \)-valued \( p_M \)-horizontal \((n-2)\)-form on \( M \), then a second-order Lagrangian density \( \Lambda_\beta \) is defined on \( J^2M \) by setting,

\[
(\Lambda_\beta)_{j^2_xg} = L_\beta(j^2_xg)\psi(x) = \text{trace } (\beta(g_x) \wedge R^g(x)),
\]

where \( R^g \) is considered as a \( \text{ad} F_g(N) \)-valued 2-form on \( N \). Locally,

\[
R^g \equiv \sum_{k<l} (R^g)^{j}_{kl} dx^k \wedge dx^l \otimes \frac{\partial}{\partial x^j},
\]

\[
\beta = \sum_{k<l} \beta^{i}_{kl,j} v^j_{kl} \otimes dx^j \otimes \frac{\partial}{\partial x^i}, \quad \beta^{i}_{kl,j} \in C^\infty(M),
\]

where \( v^j_{kl} = dx^1 \wedge \cdots \wedge \hat{dx^k} \wedge \cdots \wedge \hat{dx^l} \wedge \cdots \wedge dx^n \). Here and below, we identify the vector space \( \text{End}(T_xN) \) to \( T_x^*N \otimes T_xN \) by agreeing that \( w \otimes X \) is identified to the endomorphism given by, \( (w \otimes X)(Y) = w(Y)X, \forall X, Y \in T_xN, w \in T_x^*N \).

Hence

\[
L_\beta(j^2_xg) = \sum_{k<l} (-1)^{k+l+1} \beta^{i}_{kl,j} (g_x)(R^g)^{j}_{ikl}(x).
\]
If we set $\beta^i_{kl,i} = -\beta^i_{lk,i}$ for $k \geq l$, then, as a calculation shows, the following local expression holds:

\[
L_\beta = (-1)^{k+l+1} \beta^i_{kl,i} y^i y_{hl,jk} + L^0_\beta
\]

with

\[
L^0_\beta = \sum_{k \leq l} \sum_{r \leq s} \frac{-1}{4(1 + 6s + 1 + 5r)} \left\{ \left[ (-1)^s \beta^r_{sl} y^{tr} + (-1)^r \beta^r_{rt} y^{ts} \right] y^{ij} \right. \\
+ \left[ (-1)^l \beta^{li}_{jl} y^{lj} + (-1)^j \beta^{li}_{jl} y^{lj} \right] y^{ks} + \left[ (-1)^j \beta^{li}_{jl} y^{lj} + (-1)^l \beta^{li}_{jl} y^{lj} \right] y^{ls} \\
+ \left[ (-1)^i \beta^{li}_{jl} y^{lj} + (-1)^j \beta^{li}_{jl} y^{lj} \right] y^{kr} + \left[ (-1)^j \beta^{li}_{jl} y^{lj} + (-1)^i \beta^{li}_{jl} y^{lj} \right] y^{lr} \\
- \left[ (-1)^s \beta^{sl}_{jl} y^{lj} + (-1)^l \beta^{sl}_{jl} y^{lj} \right] y^{kj} - \left[ (-1)^l \beta^{sl}_{jl} y^{lj} + (-1)^s \beta^{sl}_{jl} y^{lj} \right] y^{ij} \\
- \left[ (-1)^l \beta^{li}_{jl} y^{lj} + (-1)^i \beta^{li}_{jl} y^{lj} \right] y^{is} - \left[ (-1)^i \beta^{li}_{jl} y^{lj} + (-1)^l \beta^{li}_{jl} y^{lj} \right] y^{ir} \bigg\
\right.
\]

where $\beta^i_{kl} = (-1)^k \beta^i_{kl,t} + (-1)^l \beta^i_{lk,t}$, and the equations $\beta^d_{ac,i} y^a + \beta^d_{bc,i} y^b = 0$ have been used, which hold because $\beta$ takes values in $\mathcal{A}(TN)$.

**Remark 3.3.** Attached to each $\mathcal{A}(TN)$-valued $p_M$-horizontal $(n - 2)$-form $\beta$ on $M$ there exists a section $\tilde{\beta}$ of the vector bundle $(p_M)^*(\wedge^2 TN) \otimes \mathcal{A}(TN)$, given by

\[
\tilde{\beta}(g_x) = \beta(g_x) \circ (\phi^T_\gamma \otimes \text{id}_{\mathcal{A}(TN)})^{-1}, \quad \forall g_x \in M,
\]

where $\phi^T_\gamma$ is the isomorphism defined in (16). If $\beta$ is locally given as in (20), then

\[
\tilde{\beta}(g_x) = \sum_{k \leq l} (-1)^{k+l+1} \beta^i_{kl,i}(g_x) \left( \frac{\partial}{\partial x^i} \right)_x \wedge \left( \frac{\partial}{\partial x^j} \right)_x \otimes \left( dx^i \right)_x \otimes \left( \frac{\partial}{\partial x^j} \right)_x,
\]

\[
\forall g_x \in M.
\]

If $\text{sym}_{14} : \otimes^4 T_x N \to \otimes^4 T_x N$ is the symmetrization operator of the arguments $1$ and $4$, i.e., $\text{sym}_{14}(X_1 \otimes X_2 \otimes X_3 \otimes X_4) = X_1 \otimes X_2 \otimes X_3 \otimes X_4 + X_4 \otimes X_3 \otimes X_1 \otimes X_2$ for all $X_i \in T_x N$, $1 \leq i \leq 4$, and for every $p \geq 0$, $q \geq 1$, the symbol $\xi$ denotes the isomorphism $\otimes^{p+1} T_x^* N \otimes^q T_x N \to \otimes^p T_x^* N \otimes^q T_x N$ induced by the metric $g_x$, then

\[
\text{sym}_{14} \left( \beta^i_{kl}(g_x) \right) = (-1)^l \beta^{ik}_{lj}(g_x) y^j(x) \left( \frac{\partial}{\partial x^i} \right)_x \otimes \left( \frac{\partial}{\partial x^l} \right)_x \otimes \left( \frac{\partial}{\partial x^j} \right)_x \otimes \left( \frac{\partial}{\partial x^k} \right)_x,
\]

and the formula (28) can be rewritten as, $L_\beta = (-1)^{k+l+1} \beta^{ab}_{cd} y^a y^b y^c y^d + L^0_\beta$.

**Theorem 3.2.** Let $\Lambda_\beta$ be the Lagrangian density attached to a $\mathcal{A}(TN)$-valued $p_M$-horizontal $(n - 2)$-form $\beta$ as defined in (25). Then

(i) The Lagrangian function (27) coincides with the E-H Lagrangian (i.e., $\Lambda_\beta = \Lambda_{EH}$) if and only if the form $\beta$ is given by

\[
(\beta_{EH})^i_{kl,i} = (-1)^{k+l+1+\rho} \left( \delta^{ik} y^j - \delta^{il} y^k \right),
\]

where the function $\rho$ is defined in (19).
With the natural identification \( V(p_M) \cong p_M^* S^2 T^* N \), the bilinear form \( b_{\Lambda, \beta} \) is defined on \( p_M^* (T^* N \otimes S^2 T^* N) \).

(iii) The E-L equations for the Lagrangian density \( \Lambda_{\beta} \) are the following:

\[
(30) \quad g^*(d_{M/N} \beta) \wedge R^g + \text{sym}_{12} \circ d^{V^g} (\omega_{n-1}(g, \beta)) = 0,
\]

where,

- \( \nabla^g \) is the covariant differentiation with respect to the Levi-Civita connection of a section \( g \) of the bundle \( p_M : M \to N \).
- The fibre differential \( d_{M/N} \beta \) is understood to be a section of the vector bundle \( (p_M)^* ((S^2 T^* N) \otimes \Lambda^{n-2} T^* N \otimes \text{End}(T N)) \), taking the isomorphism \( V^g(p_M) \cong (p_M)^* (S^2 T^* N) \) into account.
- \( g^*(d_{M/N} \beta) \wedge R^g \) is the \( S^2 T^* N \)-valued \( n \)-form on \( N \) defined by,

\[
(g^*(d_{M/N} \beta) \wedge R^g) (w_1, w_2, X_1, \ldots, X_n)
= \sum_{k<\ell} (-1)^{k+l+1} \\text{trace} \{ g^*(d_{M/N} \beta) (w_1, w_2, X_1, \ldots, \hat{X}_k, \ldots, \hat{X}_\ell, \ldots, X_n) \circ R^g(X_k, X_\ell) \},
\]

\( \forall X_1, \ldots, X_n \in T_x N, \forall w_1, w_2 \in T^*_x N. \)

- \( \omega_{n-1}(g, \beta) \) is the \( (T N \otimes T N) \)-valued \( (n-1) \)-form on \( N \) given by,

\[
\omega_{n-1}(g, \beta) = \left( (\phi^1_v)^{-1} \otimes \text{id}_{T N} \otimes \phi^1_v \right) \left( d^{V^g} (g^* \beta)^2 \right),
\]

\( \phi^1_v \) being defined in the formula (16).
- \( \text{sym}_{12} : \otimes^2 T N \to S^2 T N \) denotes the symmetrization operator.

Proof. (i) By comparing the formula (27) with the following:

\[
L_{EH}(j_{x}^2 g) = \sum_{k<l} \rho(x) \left( \delta^{ik} g^{jl}(x) - \delta^{il} g^{jk}(x) \right) (R^g)^i_{jkl}(x),
\]

we obtain (29) directly.
(ii) As a calculation shows, the matrix of \( b_{\lambda \rho} \) is given as follows:

\[
(F_\beta)_{r \leq s, i, a \leq b, j} = \frac{\partial^2 L_\beta^{ai}}{\partial y_{rs} \partial y_{ab}} - \frac{\partial L_\beta^{ai}}{\partial y_{rs}} \frac{\partial y_{ab}}{\partial y_{rs}} = \frac{1}{2} \frac{1}{1 + \delta_{rs}} \frac{1}{1 - \delta_{rs}} \sum_{b} \left\{ \left[ (-1)^a \beta^{rs}_{ab} y^{tb} + (-1)^b \beta^{rs}_{at} y^{ta} \right] y^{ij} + \left[ (-1)^i \beta^{ab}_{it} y^{js} + (-1)^j \beta^{ab}_{rt} y^{is} \right] y^{ia} + \left[ (-1)^i \beta^{ab}_{it} y^{js} + (-1)^j \beta^{ab}_{rt} y^{is} \right] y^{jb} \right\} + \frac{1}{1 + \delta_{rs}} \left( (-1)^a \frac{\partial \beta^{rs}_{ab}}{\partial y_{rs}} y^{tb} + (-1)^b \frac{\partial \beta^{rs}_{ab}}{\partial y_{ab}} y^{ta} \right) \right.

\[
+ \left. (1 + \delta_{rs}) \left( (-1)^a \frac{\partial \beta^{rs}_{ab}}{\partial y_{rs}} y^{tb} + (-1)^b \frac{\partial \beta^{rs}_{ab}}{\partial y_{ab}} y^{ta} \right) \right\} ,
\]

thus proving the statement.

(iii) The E-L equations for the Lagrangian density \( \Lambda_\beta = L_\beta v \) are straightforwardly computed, thus obtaining,

\[
\mathcal{E}^{ab}(L_\beta) \circ \partial g = \frac{1}{2} (-1)^{k+l+1} \left[ \Phi^{i}_{ab} \right] (R^g)_{jkl} \frac{\partial \beta^{ij}_{kl}}{\partial y_{kl}} \quad \text{(3.1)}
\]

\[
- \frac{1}{1 + \delta_{rs}} \left\{ \frac{\partial}{\partial x_r} \left[ (-1)^a \Phi^{rb}_a + (-1)^b \Phi^{ra}_a \right] + (-1)^j \left[ \Phi^{rb}_a \Lambda^{i}_{rt} + \Phi^{ra}_a \Lambda^{i}_{rt} \right] \right\},
\]

for \( 1 \leq a \leq b \leq n \), where

\[
\Phi^{rb}_a = \sum_k (-1)^k \frac{\partial}{\partial x_k} \left( \beta \circ g \right)^{rb}_{ka} + \left( \beta \circ g \right)^{rb}_{ka} \left( \Lambda^m_{ki} + \Lambda^m_{ki} - \Lambda^m_{ki} \right) \quad \text{(3.2)}
\]

Moreover, the following local expressions are deduced:

\[
\tilde{g}^i (d_{M/N} \beta) \circ R^g = \frac{1}{2} (-1)^{k+l+1} \sum_{a \leq b} \left[ \frac{\partial \beta^{ij}_{kl}}{\partial y_{kl}} \right] (R^g)_{jkl} \frac{\partial}{\partial x_a} \circ \frac{\partial}{\partial x_b} \circ v,
\]

\[
\left( d v \right)^{\gamma \beta}_a = \sum_l \Phi^{ab}_{vl} \circ \frac{\partial}{\partial x_a} \circ \frac{\partial}{\partial x_b},
\]

from which the result follows. \( \square \)
Corollary 3.3. A flat metric \( g \) is a solution to the equations \( \Box g = 0 \) if and only if the form \( \beta \) in \( \Box g = 0 \) satisfies the following equation:

\[
\epsilon_{12}^{23} \left[ (\nabla g)^2 \left\{ \text{sym}_{14} \left( \tilde{\beta} \circ g \right) \right\} \right] = 0,
\]

where \( \epsilon_{12}^{23} : \bigotimes^2 T^*M \otimes \bigotimes^4 TM \to \bigotimes^2 TM \) denotes the contraction operator of the first covariant index with the second contravariant one, and the second covariant index with the third contravariant one.

Remark 3.4. The geometric construction of the form \( \beta \) as \( \beta(x_1, \ldots, x_{n-2}) = 0 \) is as follows: Given an arbitrary system \( X_1, \ldots, X_{n-2} \in T_x N \), we must define a skew-symmetric (with respect to \( g_x \)) endomorphism \( \beta(x_1, \ldots, X_{n-2}) : T_x N \to T_x N \).

If the given system is linearly dependent, then \( \beta(X_1, \ldots, X_{n-2}) = 0 \). We assume: i) The system \( (X_1, \ldots, X_{n-2}) \) is linearly independent. Hence its orthogonal \( \Pi = (X_1, \ldots, X_{n-2})^\perp \) is a subspace of dimension 2 in \( T_x N \), ii) the subspace \( (X_1, \ldots, X_{n-2}) \) is not singular with respect to \( g_x \). Hence

\[ T_x N = \Pi \oplus (X_1, \ldots, X_{n-2}), \]

and \( \Pi \) is also non-singular. Let \( (\pi^+(\Pi), \pi^-(\Pi)) \in \{(2,0),(1,1),(0,2)\} \) be its signature and let

\[
\left( \begin{array}{cc}
\varepsilon_1(\Pi) & 0 \\
0 & \varepsilon_2(\Pi)
\end{array} \right), \quad \langle \varepsilon_1(\Pi), \varepsilon_2(\Pi) \rangle \in \{(1,1),(1,-1),(-1,-1)\},
\]

be the matrix of \( g_x \) in an orthonormal basis \( (Y_1, Y_2) \) of \( \Pi \), which, in addition, is assumed to satisfy the following: \( v(X_1, \ldots, X_{n-2}, Y_1, Y_2) > 0 \). If \( Z_j = b_i Y_i, i, j = 1, 2 \), is another orthonormal basis with \( v(X_1, \ldots, X_{n-2}, Z_1, Z_2) > 0 \), then \( \det(b_i^j) = 1 \). Hence \( (b_i^j) \) belongs to \( SO(n^+(\Pi), n^-(\Pi)) \), and the endomorphism \( J^\Pi_x : \Pi \to \Pi \) given by \( J^\Pi_x (Y_1) = \varepsilon_1(\Pi)Y_2, J^\Pi_x (Y_2) = -\varepsilon_2(\Pi)Y_1, \) is independent of the basis chosen (as \( SO(n^+(\Pi), n^-(\Pi)) \) is commutative) and skew-symmetric.

We define \( J^\Pi_x : T_x N \to T_x N \) by setting, \( J^\Pi_x |_\Pi = J^\Pi_x, J^\Pi_x |_{(x_1, \ldots, x_{n-2})} = 0 \). Finally,

\[
(\beta_{EH})(x_1, \ldots, x_{n-2}) = \det (g(Y_a, Y_b))_{a,b=1}^2 v_{g_x} (X_1, \ldots, X_{n-2}, Y_1, Y_2) J^\Pi_x.
\]

Remark 3.5. The bilinear form \( b_{\lambda, \beta} \) is identified to a section of the vector bundle \( p_M^* ((TN \otimes S^2TN)) \), and the following formula holds:

\[
b_{\lambda, \beta} = \frac{1}{2} \text{sym}_{45}(\text{alt}_{16}(\text{sym}_{12}(\tilde{\beta}) + \text{alt}_{13}(\tilde{\beta}) - \tilde{\beta})) - \frac{1}{2} \text{sym}_{12}(\tilde{\beta}) + \text{alt}_{13}(\tilde{\beta}) - \tilde{\beta}
\]

\[
+ \frac{1}{2} \text{sym}_{12}(\text{sym}_{14}(\tilde{\beta}^i))
\]

where the operators \( \text{alt}_{ij}^3, \text{sym}_{ij}^3, \text{sym}_{12}(4,5) : \bigotimes^6 T_x N \to \bigotimes^6 T_x N, 1 \leq i < j \leq 6, \)
are defined as follows:

\[
\text{alt}_{ij}(X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6) = \\
X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6 - X_1 \otimes \cdots \otimes X_j \otimes \cdots \otimes X_i \otimes \cdots \otimes X_6,
\]

\[
\text{sym}_{ij}(X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6) = \\
X_1 \otimes \cdots \otimes X_i \otimes \cdots \otimes X_j \otimes \cdots \otimes X_6 + X_1 \otimes \cdots \otimes X_j \otimes \cdots \otimes X_i \otimes \cdots \otimes X_6,
\]

\[
\text{sym}_{(1,2)(4,5)}(X_1 \otimes \cdots \otimes X_6) = \\
X_1 \otimes \cdots \otimes X_6 + X_4 \otimes X_5 \otimes X_3 \otimes X_1 \otimes X_2 \otimes X_6,
\]

\[
X_1, \ldots, X_6 \in T_x N,
\]

and the contravariant 6-tensor \( \hat{\beta} \) is given by,

\[
\hat{\beta} = \text{sym}_{15} \left[ \text{sym}_{23} \left( \text{sym}_{14} \left( \tilde{\beta} \right) \right) \otimes (g^i)^j \right] - \text{sym}_{23} \left( \text{sym}_{14} \left( \tilde{\beta} \right) \right) \otimes (g^i)^j.
\]

**Remark 3.6.** If \( \beta = \beta_{EH} \) in Theorem 3.2-(iii), then the functions \( \Phi^{rb}_a \) (appearing in the proof) vanish, and the equations (30) reduce to Einstein’s vacuum equations for arbitrary signature.

### 4 First-order equivalent Lagrangians

**Theorem 4.1.** Let \( \Lambda = LV \) be a second-order Lagrangian density on \( p: E \to N \) whose Poincaré-Cartan form projects onto \( J^1 E \). We have

(i) The H-C equations of the first-order Lagrangian \( \bar{L}V \) given in (17) coincide locally with the H-C equations of \( \Lambda \). Furthermore, if \( \bar{L}' \) is another first-order Lagrangian fulfilling this property, then \( \bar{L}'V - \bar{L}V = D\alpha_{n-1} \), where \( D \) denotes the horizontal exterior derivative and \( \alpha_{n-1} \) is a \( p \)-horizontal \((n-1)\)-form on \( E \).

(ii) The E-L equations of \( \Lambda \), considered as a second-order partial differential system, satisfy the Helmholtz conditions.

(iii) The E-L equations of the first-order Lagrangian \( \bar{L}V \) above coincide with E-L equations of \( \Lambda \).

(iv) Let \( \phi^k_v \) be the isomorphism defined in (10) for \( k = 1 \) and let \( w^0_{L,\sigma} \) be the \( TN \)-valued section on \( J^1 E \) defined as in Proposition 1.2. The composite mapping \( \phi^1_v \circ w^0_{L,\sigma} \) can be viewed as a \( p^1 \)-horizontal \((n-1)\)-form on \( J^1 E \) and the difference \( L_{\sigma}V = L_{\sigma} - D(\phi^1_v \circ w^0_{L,\sigma}) \) determines a globally defined first-order Lagrangian which is variationally equivalent to \( LV \), but this is not canonically attached to \( LV \) as it depends on the section \( \sigma \).

**Proof.** (i) Locally, the Hamiltonian and the momenta associated to \( \bar{L} \) are given respectively by (cf. formula (10) in Remark 2.1),

\[
\bar{H} = \bar{L} - y^{\alpha}_i \frac{\partial \bar{L}}{\partial y^\alpha_i}, \quad \bar{p}^\alpha = \frac{\partial \bar{L}}{\partial y^\alpha_i}.
\]
By comparing the H-C equations for $\bar{L}$ with the H-C equations for $L$ given in \((14)\), one obtains, $H = \bar{H}$ and $p_i^\alpha = \bar{p}_i^\alpha$. Hence

\[
\begin{align*}
L_0 - y_i^\alpha L_0^\alpha - \frac{\partial L_i^i}{\partial x^i} & = \bar{L} - y_i^\alpha \frac{\partial \bar{L}}{\partial y_i^\alpha}, \\
L_0^\alpha - \frac{\partial L_i^i}{\partial y^\alpha} & = \frac{\partial \bar{L}}{\partial y_i^\alpha}.
\end{align*}
\]

Replacing \((32)\) into \((31)\), one concludes that $\bar{L}$ is given as in the formula \((17)\). Moreover, if $L'$ is the first-order Lagrangian associated to other primitive functions $L_i^\alpha = L_i^\alpha + A_i$, $A_i \in C^\infty(E)$, according to Proposition 1.2, then $\bar{L}' = \bar{L} - D_i A_i$.

(ii) As a simple—although rather long—computation shows, the second-order differential operator $E_{\sigma}^\alpha(L)$ satisfies the equations (1.5a), (1.5b), and (1.5c) in \([1]\). In fact, by using the formulas (1), (2), and (8), the following equations are checked:

\[
\begin{align*}
(1.5a) \quad 0 & = \frac{\partial E_{\sigma}^\alpha(L)}{\partial y_{(ij)}^\sigma} - \frac{\partial E_{\sigma}(L)}{\partial y_{(ij)}^\sigma}, \\
(1.5b) \quad 0 & = \frac{\partial E_{\sigma}^\alpha(L)}{\partial y_i^\sigma} + \frac{\partial E_{\sigma}(L)}{\partial y_i^\sigma} - (1 + \delta_{ij})D_j \left( \frac{\partial E_{\sigma}(L)}{\partial y_{(ij)}^\sigma} \right), \\
(1.5c) \quad 0 & = \frac{\partial E_{\sigma}^\alpha(L)}{\partial y^\sigma} - \frac{\partial E_{\sigma}(L)}{\partial y^\alpha} + D_i \left( \frac{\partial E_{\sigma}(L)}{\partial y_i^\alpha} \right) - \sum_{i \leq j} D_i D_j \left( \frac{\partial E_{\sigma}(L)}{\partial y_{(ij)}^\alpha} \right).
\end{align*}
\]

(iii) From the formula \((17)\), it follows that the Lagrangian $\bar{L}$ can also be written as $\bar{L} = L - D_i L_i$, thus proving that $L$ and $\bar{L}$ differ on a total divergence and, hence $E_{\sigma}(L) = E_{\sigma}(\bar{L})$.

(iv) Locally, $w_{0,\sigma}^i = L_i^0 \partial / \partial x^i$; hence $\phi_0 \circ w_{0,\sigma}^i = (-1)^{i-1} L_i^0 \psi_i$, and consequently, $D(\phi_0 \circ w_{0,\sigma}^i) = (D_i L_i^0) \psi_i$. The result thus follows from $\bar{L}_\sigma = L - D_i L_i$ in item (iii).

\[\square\]

Remark 4.1. As is known (e.g., see \([12]\) (2.21)–(2.25))), the Vainberg-Tonti Lagrangian $L_{VT}$ attached to a second-order affine Lagrangian as in \([4]\) is also affine, say $L_{VT} = (L_{VT})_0 + (L_{VT})_1$, with $(L_{VT})_1 = (L_{VT})_1^0 y_{(ij)}^\alpha$. Then, as a computation shows, one has

\[
L_{VT} - \bar{L} = -D_h \left( \int_0^1 y^\alpha \left( \frac{\partial \bar{L}}{\partial y_h^\alpha} \circ \chi_\lambda \right) d\lambda \right),
\]

where $\chi_\lambda(x^i, y^\alpha, y_h^\alpha) = (x^i, \lambda y^\alpha, \lambda y_h^\alpha)$, but it should be noted that the Vainberg-Tonti Lagrangian is of second order in the general case; e.g., if $L(x, y, \dot{y}, \ddot{y}) =$
\[ L_1(x, y, \dot{y}) \dot{y} + L_0(x, y, \dot{y}), \text{ then } L_{VT} = (L_{VT})_0 + (L_{VT})_1 \dot{y}, \text{ with} \]

\[
(L_{VT})_1 = y \int_0^1 \left\{ 2\lambda \frac{\partial L_1}{\partial y}(x, \lambda y, \lambda \dot{y}) + \lambda \frac{\partial^2 L_1}{\partial x \partial \dot{y}}(x, \lambda y, \lambda \dot{y}) \\
+ \lambda^2 \frac{\partial^2 L_1}{\partial y \partial \dot{y}}(x, \lambda y, \lambda \dot{y}) - \lambda \frac{\partial L_0}{\partial y^2}(x, \lambda y, \lambda \dot{y}) \right\} \, d\lambda, \\
(L_{VT})_0 = y \int_0^1 \left\{ \frac{\partial L_0}{\partial y}(x, \lambda y, \lambda \dot{y}) + \frac{\partial^2 L_1}{\partial x^2}(x, \lambda y, \lambda \dot{y}) \\
+ \lambda^2 (\dot{y}) \frac{\partial^2 L_1}{\partial y^2}(x, \lambda y, \lambda \dot{y}) + 2\lambda \dot{y} \frac{\partial^2 L_1}{\partial x \partial \dot{y}}(x, \lambda y, \lambda \dot{y}) \\
- \frac{\partial^2 L_0}{\partial y^2}(x, \lambda y, \lambda \dot{y}) - \lambda \frac{\partial L_0}{\partial y \partial \dot{y}}(x, \lambda y, \lambda \dot{y}) \right\} \, d\lambda.
\]

Therefore \( L_{VT} \) is of second order, except when \( (L_{VT})_1 = 0 \), and this latter condition is seen to be equivalent to the following:

\[ 0 = 2 \frac{\partial L_1}{\partial y} + \frac{\partial^2 L_1}{\partial x \partial \dot{y}} + \dot{y} \frac{\partial^2 L_1}{\partial y \partial \dot{y}} - \frac{\partial^2 L_0}{\partial y^2}. \]

In the particular case of the bundle of metrics, there exists a more specific way to obtain a section \( \sigma \) of \( p_1^0: M \to J^1 M \) than the procedure suggested in Remark \[ \[1,2 \] \] which depends on a linear connection only rather than a non-linear connection; namely,

**Lemma 4.2.** Let \( p_M: M \to N \) be the bundle of pseudo-Riemannian metrics of a given signature \((n^+, n^-)\), \( n^+ + n^- = n \), and let \( \nabla \) be a symmetric linear connection on \( N \). For every \( g_x \in (p_M)^{-1}(x) \), there exists a unique 1-jet of metric \( J^1_x \tilde{g} \in J^1_x M \) such that, 1) \( \tilde{g}_x = g_x \), and 2) \( (\nabla \tilde{g})_x = 0 \). The mapping \( \sigma_\nabla : M \to J^1 M \) given by \( \sigma_\nabla (g_x) = J^1_x \tilde{g} \) is a section of \( p_1^0 : J^1 M \to M \).

**Proof.** If \( \Gamma^j_{ik} \) are the local symbols of \( \nabla \) in a coordinate system, then as a calculation shows, the condition 2) — assuming 1) — of the statement is equivalent to,

\[
\frac{\partial \tilde{g}_{ij}}{\partial x^k}(x) = \Gamma^h_{ik}(x)g_{ih}(x) + \Gamma^h_{jk}(x)g_{hi}(x),
\]

thus proving that \( \sigma_\nabla \) makes sense.

**Proposition 4.3** (cf. [3] II). Let \( p_M: M \to N \) be as in Lemma 4.2. For the E-H Lagrangian, the density \( (L_{EH})_{\sigma v} \) introduced in Theorem 4.1(iv) is given by, \( (L_{EH})_{\sigma v} (j^2\tilde{g}) v_x = c \left( \text{alt}_{23} (\nabla^g T^g) \right)^2 (v_g)_x \) for all \( j^2\tilde{g} \in J^2 M \), where \( \text{alt}_{23} : \otimes^3 T^* M \otimes TM \to \otimes^3 T^* M \otimes TM \) denotes the alternating operator of the second and third covariant indices, and

\[
\cdot : \otimes^3 T^* M \otimes TM \to \otimes^2 T^* M \otimes^2 TM
\]

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is the isomorphism induced by $g$, i.e.,

$$w_1 \otimes w_2 \otimes w_3 \otimes X \mapsto w_1 \otimes w_2 \otimes (w_3)^s \otimes X,$$

and $c: \otimes^2 T^*M \otimes^2 TM \to \mathbb{R}$ is the total contraction of the first (resp. second) covariant index with the first (resp. second) contravariant one.

5 Symmetries and Noether invariants

Given fibred manifolds $p: E \to N$, $p': E' \to N'$, every morphism $\Phi: E \to E'$ for which the associated map on the base manifolds $\phi: N \to N'$ is a diffeomorphism, induces a map

$$\Phi^{(r)}: J^r E \to J^r E',$$

$$\Phi^{(r)}(j^r_s) = j^r_{\phi(s)}(\Phi \circ s \circ \phi^{-1}).$$

If $\Phi_t$ is the flow of a $p$-projectable vector field $X$, then $\Phi_t^{(r)}$ is the flow of a vector field $X^{(r)}(t) \in \mathfrak{X}(J^r E)$, called the infinitesimal contact transformation of order $r$ associated to the vector field $X$. The mapping $X \mapsto X^{(r)}$ is an injection of Lie algebras. For $r = 1, 2$, the general prolongation formulas read as follows:

$$X = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha},$$

$$u^i \in C^\infty(N), v^\alpha \in C^\infty(E),$$

$$X^{(1)} = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v^\alpha \frac{\partial}{\partial y^i_{\alpha}},$$

$$v^\alpha_i = D_i (v^\alpha - u^h y^\alpha_{hi});$$

$$X^{(2)} = u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v^\alpha \frac{\partial}{\partial y^i_{\alpha}} + \sum_{i,j} v^\alpha_i \frac{\partial}{\partial y^i_{\alpha j}},$$

$$v^\alpha_{ij} = D_i D_j (v^\alpha - u^h y^\alpha_{hi});$$

Theorem 5.1. Let $\Lambda = L v$ be a second-order Lagrangian density on $p: E \to N$ with P-C form projectable onto $J^1 E$. If $X$ is a $p$-projectable vector field on $E$, then the P-C form of the second-order Lagrangian density $N' = L' v = L_{X^{(2)}(1)} \Lambda$ also projects onto $J^1 E$ and the following formula holds:

$$\Theta_{L_{X^{(2)}(1)} \Lambda} = L_{X^{(1)}}(1) \Theta_{\Lambda}.$$ 

Therefore, if $s: N \to E$ is an extremal for $\Lambda$ and $X$ is an infinitesimal symmetry (i.e., $L_{X^{(2)}(1)} \Lambda = 0$), then the $(n - 1)$-form $(J^1 s)^* i_{X^{(1)}}(1) \Theta$ is closed. (The $(n - 1)$-form $i_{X^{(1)}}(1) \Theta$ is called the Noether invariant associated to $X$.)

Proof. We have $L' = X^{(2)}(L) + \text{div}(X') L$, $X'$ being the projection of $X$ onto $N$ and $\text{div}(X')$ the divergence of $X'$ with respect to $v$. According to Proposition 1.1 we must prove the existence of functions $L'_0$, $L'^{\alpha}_{\alpha i} = L'^{\alpha}_{\alpha i}$ on $J^1 E$ such that,

$$L' = L'^{\alpha}_{\alpha i} y^\alpha_{(ij)} + L'_0,$$

$$\frac{\partial L'^{\alpha}_{\alpha i}}{\partial y^h_{ai}} = \frac{\partial L'^{\alpha}_{\alpha i}}{\partial y^h_{ai}}, \quad a, h, i = 1, \ldots, n, \quad \alpha, \beta = 1, \ldots, m.$$
As \( L \) satisfies such formulas by virtue of the hypothesis, and \( X^{(2)} \) projects onto \( X^{(1)} \), we have

\[
L' = \left[ X^{(1)} \left( L^0_{ab} \alpha + \frac{\partial v^b}{\partial y^a} L^b_{ij} - 2 \frac{\partial u^a}{\partial x^b} L^b_{ij} + \text{div}(X') L^b_{ij} \right) \right] + X^{(1)}(L_0) + \text{div}(X')L_0.
\]

Hence

\[
L'^{ab}_{\alpha} = X^{(1)} \left( L^0_{ab} \alpha + \frac{\partial v^b}{\partial y^a} L^b_{ij} - 2 \frac{\partial u^a}{\partial x^b} L^b_{ij} + \text{div}(X') L^b_{ij} \right)\]

\[
+ X^{(1)}(L_0) + \text{div}(X')L_0.
\]

From the formula (35) it also follows:

\[
L'^{ij} = L^0_{ij} v^0_{ij} - \left( L^0_{ij} \frac{\partial v^0}{\partial y^j} - 2 \frac{\partial u^0}{\partial x^j} L^0_{ij} \right) v^{(ij)} + X^{(1)}(L_0) + \text{div}(X'L_0).
\]

Replacing \( \frac{\partial(X^{(1)}(L^{ij}_{(i)))}}{\partial y^k_h} \) into the formula for \( \frac{\partial L'^{ij}_{(i)}}{\partial y^k_h} \), we obtain

\[
\frac{\partial L'^{ij}_{(i)}}{\partial y^k_h} = X^{(1)} \left( \frac{\partial L^0_{ij}}{\partial y^k_h} + \frac{\partial v^0}{\partial y^j} \frac{\partial L^0_{ij}}{\partial y^k_h} - \frac{\partial u^0}{\partial x^j} \frac{\partial L^0_{ij}}{\partial y^k_h} + \frac{\partial v^1}{\partial y^j} \frac{\partial L^1_{ij}}{\partial y^k_h} - \frac{\partial u^1}{\partial x^j} \frac{\partial L^1_{ij}}{\partial y^k_h} + \text{div}(X') \frac{\partial L^0_{ij}}{\partial y^k_h} \right)
\]

and similarly,

\[
\frac{\partial L'^{ih}_{(i)}}{\partial y^j_h} = X^{(1)} \left( \frac{\partial L^0_{ih}}{\partial y^j_h} + \frac{\partial v^0}{\partial y^j} \frac{\partial L^0_{ih}}{\partial y^j_h} - \frac{\partial u^0}{\partial x^j} \frac{\partial L^0_{ih}}{\partial y^j_h} + \frac{\partial v^1}{\partial y^j} \frac{\partial L^1_{ih}}{\partial y^j_h} - \frac{\partial u^1}{\partial x^j} \frac{\partial L^1_{ih}}{\partial y^j_h} + \text{div}(X') \frac{\partial L^0_{ih}}{\partial y^j_h} \right),
\]

and taking the formulas (35) into account, we can conclude that \( \frac{\partial L'^{ij}_{(i)}}{\partial y^k_h} = \frac{\partial L'^{ij}_{(i)}}{\partial y^k_h} \).

Moreover, from the formula (8) we know

\[
\Theta_A = (-1)^{i-1} \left( L^0_{\alpha} \partial y^\alpha + L^h_{\alpha} \partial y^\alpha \right) \wedge \nu + \left( L - y^\alpha \partial L^0_{\alpha} - y^\alpha L^0_{\alpha} \right) \nu,
\]

where

\[
L - y^\alpha \partial L^0_{\alpha} - y^\alpha L^0_{\alpha} = L_0 - y^\alpha \partial L^0_{\alpha},
\]

\[
L^0_{ij} = \frac{1}{\partial y^k_h} \partial L^0_{ij},
\]

\[
L^0_{ij} = \frac{1}{\partial y^k_h} \partial L^0_{ij} - y^k_h \partial L^0_{ij}.
\]
the third equation again being a consequence of (5). Hence

\[
L_{X^{(1)}} \Theta_{A} = (-1)^{i-1} \left( X^{(1)} (L_{0}^{i0}) dy^{\alpha} + X^{(1)} (L_{0}^{ih}) dy_{h}^{0} \right) \wedge v_{i}
+ \left( L_{0}^{i0} \frac{\partial v^{\alpha}}{\partial x^{i}} + L_{0}^{ih} \frac{\partial v_{h}^{0}}{\partial x^{i}} \right) v_{i} + (-1)^{i-1} \frac{\partial v^{\alpha}}{\partial y^{i}} L_{0}^{0} dy^{\beta} \wedge v_{i}
+ (-1)^{i-1} L_{0}^{ih} \left( \frac{\partial v^{\alpha}}{\partial y_{i}} dy^{\beta} + \frac{\partial v_{h}^{0}}{\partial y_{i}} dy_{j}^{\beta} \right) \wedge v_{i}
+ (-1)^{i-1} \left( L_{0}^{i0} dy^{\alpha} + L_{0}^{ih} dy_{h}^{0} \right) \wedge L_{X^{(1)}} (v_{i})
+ \left[ X^{(1)} (L_{0} - y_{i}^{0} L_{0}^{i0}) \right] v + \text{div}(X^{(1)} (L_{0} - y_{i}^{0} L_{0}^{i0})) v.
\]

Expanding the right-hand side above we obtain

\[
L_{X^{(1)}} \Theta_{A} = (-1)^{i-1} \left( X^{(1)} (L_{0}^{i0}) + \frac{\partial v^{\beta}}{\partial y_{i}} L_{0}^{i0} + \frac{\partial v_{h}^{0}}{\partial y_{i}} L_{0}^{ih} \right) dy^{\alpha} \wedge v_{i}
+ (-1)^{i-1} \left( X^{(1)} (L_{0}^{ih}) + \frac{\partial v^{\beta}}{\partial y_{i}} L_{0}^{ij} \right) dy_{h}^{0} \wedge v_{i}
+ (-1)^{i-1} \left( L_{0}^{i0} dy^{\alpha} + L_{0}^{ih} dy_{h}^{0} \right) \wedge L_{X^{(1)}} (v_{i})
+ \left[ X^{(1)} (L_{0} - y_{i}^{0} L_{0}^{i0}) + L_{0}^{i0} \frac{\partial v^{\alpha}}{\partial x^{i}} + L_{0}^{ih} \frac{\partial v_{h}^{0}}{\partial x^{i}} \right] v
+ \text{div}(X^{(1)} (L_{0} - y_{i}^{0} L_{0}^{i0})) v.
\]

Moreover, by applying the formula (36) to the density \( \Lambda' \) we have

\[
\Theta_{\Lambda'} = (-1)^{i-1} \left( L_{0}^{i0} dy^{\alpha} + L_{0}^{ih} dy_{h}^{0} \right) \wedge v_{i} + (L_{0}' - y_{i}^{0} L_{0}^{i0}) v.
\]

We first compute \( L_{0}^{i0} \). From (2), (33), (34), and (35) we deduce

\[
L_{0}^{i0} = X^{(1)} (L_{0}^{i0}) + \text{div}(X') L_{0}^{i0} + \frac{\partial v^{\beta}}{\partial y^{i}} L_{0}^{i0} - \frac{\partial u^{i}}{\partial x^{i}} L_{0}^{0} + \frac{\partial v^{\beta}}{\partial y^{i}} L_{0}^{0}.
\]

Furthermore,

\[
L_{0}^{ij} = X^{(1)} (L_{0}^{ij}) + \text{div}(X') L_{0}^{ij} + A_{0}^{ij},
L_{0}' = X^{(1)} (L_{0}') + \text{div}(X') L_{0}' + T_{0}^{\beta} L_{0}^{\beta},
\]

with

\[
A_{0}^{ij} = \frac{\partial v^{\beta}}{\partial y^{i}} L_{0}^{ij} - \frac{\partial u^{i}}{\partial x^{i}} L_{0}^{ij} + \frac{\partial u^{i}}{\partial x^{i}} L_{0}^{ij},
T_{0}^{\beta} = \frac{\partial^{2} \nu^{\beta}}{\partial x^{i} \partial x^{j}} y_{i}^{j} + \frac{\partial^{2} \nu^{\beta}}{\partial y^{i} \partial x^{j}} y_{i}^{j} + \frac{\partial^{2} \nu^{\beta}}{\partial y^{i} \partial y^{j}} y_{i}^{j} - \frac{\partial^{2} \nu^{\beta}}{\partial x^{i} \partial x^{j}} y_{i}^{j}.
\]

Hence

\[
L_{0}' - y_{i}^{0} L_{0}^{i0} = X^{(1)} (L_{0} - y_{i}^{0} L_{0}^{i0}) + \text{div}(X') (L_{0} - y_{i}^{0} L_{0}^{i0}) + \frac{\partial v^{\beta}}{\partial x^{i}} L_{0}^{0} + \frac{\partial v^{\beta}}{\partial x^{i}} L_{0}^{0}.
\]
and we obtain
\[
\Theta_{\mathcal{A}} = (-1)^{i-1} \left( X^{(1)}(L_0^0) + \frac{\partial h^a}{\partial y^i} L_0^0 \right) dy^a \wedge v_i
+ (-1)^{i-1} \left( \text{div}(X') L_0^0 - \frac{\partial h^a}{\partial x^i} L_0^0 \right) dy^a \wedge v_i
+ (-1)^{i-1} \left( X^{(1)}(L_0^i) + \frac{\partial h^a}{\partial y^i} L_0^i - \frac{\partial h^a}{\partial y^i} L_0^i \right) dy^a \wedge v_i
- (-1)^{i-1} \frac{\partial h^a}{\partial x^i} L_0^i dy^a \wedge v_i
+ \left( X^{(1)}(L_0 - y^0 L_0^0) + \text{div}(X') \left( L_0 - y^0 L_0^0 \right) \right) \left( \frac{\partial h^a}{\partial y^i} L_0^0 + \frac{\partial h^a}{\partial y^i} L_0^i \right) v.
\]

By using the formula \( L_{X'}(v_i) = \text{div}(X') v_i + \sum_{h=1}^n (-1)^{h+i-1} \frac{\partial h^a}{\partial x^i} v_i \), we can thus conclude that \( \Theta_{\mathcal{A}} = L_{X'}(\Theta) \). Finally, if \( L_{X'}(\Lambda) = 0 \), then \( \Theta_{L_{X'}}(\Lambda) = 0 \) and by virtue of the formula in the first part of the statement we deduce \( L_{X'}(\Theta) = 0 \). Hence \((j^1 s)^* \left( \partial X_{(1)}(\Theta) \right) + (j^1 s)^* \left( \partial X_{(1)}(d\Theta) \right) = 0\), and we can conclude recalling that the second term in the left-hand side vanishes, as follows from the H-C equations in Theorem 2.1.

\[\Box\]

6 Symmetries of the E-H Lagrangian density

Example 6.1. In the particular case of the bundle of pseudo-Riemannian metrics of a given signature \( p_M : M \to N \) (cf. section 3.1), the natural lift of a vector field \( X' = u^i \frac{\partial}{\partial x^i} \) in \( \mathfrak{X}(N) \) is given as follows (cf. [22 section 2.2]):
\[
X'_{\mathcal{M}} = u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial h^a}{\partial x^j} y_{ij} + \frac{\partial h^a}{\partial y_j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M),
\]
and from the geometric properties of the scalar curvature the E-H Lagrangian density \( \Lambda_{EH} \) admits \( X'_{\mathcal{M}} \) as an infinitesimal symmetry for every \( X' \in \mathfrak{X}(N) \). Let us compute its Noether invariant \((j^1 g)^* \left( X'_{\mathcal{M}} \right) (\Theta_{EH})\) along an Einstein metric \( g \). From the formulas
\[
(X'_{\mathcal{M}})^{(1)} = u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left( \frac{\partial h^a}{\partial x^j} y_{ij} + \frac{\partial h^a}{\partial y_j} y_{ih} \right) \frac{\partial}{\partial y_{ij}}
- \sum_{i \leq j} \left( \frac{\partial^2 h^a}{\partial x^i \partial x^j} y_{ij} + \frac{\partial^2 h^a}{\partial x^i \partial y_j} y_{ih} + \frac{\partial^2 h^a}{\partial y_i \partial x^j} y_{iij} + \frac{\partial^2 h^a}{\partial y_i \partial y_j} y_{iij} \right) \frac{\partial}{\partial y_{ij}},
\]
\[
\Theta_{EH} = \left( -1 \right)^{i-1} \sum_{k \leq l} \left( (L_{EH})_{kl}^0 dy_{kl} + (L_{EH})_{kl}^i dy_{kl,h} \right) \wedge v_i
+ \left( (L_{EH})_{0} - \sum_{k \leq l} y_{kl,i} (L_{EH})_{kl}^0 \right) v,
\]
where \((L_{EH})_{kl}^0\), \((L_{EH})_{kl}^i\), and \((L_{EH})_{0}\) are given in [2], [21], and [20], respectively, by using a normal coordinate system \((x^i)^{\alpha}_{\alpha=1} \) centred at \( x \in N \) we even-
The E-H Lagrangian is the null vector field. The statement is equivalent to saying that the only infinitesimal symmetry of the Lagrangian density \( \Lambda_{EH} \) is its projection onto \( N \), where \( N = \text{dim} \, N \geq 3 \) the vector fields of the form \( X_M \), \( X' \in \mathfrak{X}(N) \), are the only infinitesimal symmetries of the Lagrangian density \( \Lambda_{EH} \).

**Theorem 6.1.** For \( n = \text{dim} \, N \geq 3 \) the vector fields of the form \( X_M \), \( X' \in \mathfrak{X}(N) \), are the only infinitesimal symmetries of the Lagrangian density \( \Lambda_{EH} \).

**Proof.** Let \( p_M: M \to N \) be the bundle of pseudo-Riemannian metrics of a given signature. If \( X \) is an infinitesimal symmetry of \( \Lambda_{EH} \) and \( X' \) is its \( p_M \)-projection onto \( N \), then \( X - X_M \) is a \( p_M \)-vertical symmetry of \( \Lambda_{EH} \). Hence, the statement is equivalent to saying that the only \( p_M \)-vertical symmetry \( X \) of the E-H Lagrangian is the null vector field.
Let \( p: E \to N \) be a submersion. If \( X = V^\alpha \frac{\partial}{\partial y^\alpha} \), \( V^\alpha \in C^\infty(E) \) is an infinitesimal symmetry of a second-order Lagrangian \( L \) with P-C form projectable onto \( J^1E \), then \( X(2)(L) = 0 \), where

\[
X(2) = V^\alpha \frac{\partial}{\partial y^\alpha} + D_i(V^\alpha) \frac{\partial}{\partial y_i} + \sum_{h \leq i} D_h D_i(V^\alpha) \frac{\partial}{\partial y_{hi}}.
\]

As

\[
D_i(V^\alpha) = \frac{\partial V^\alpha}{\partial x^i} + y^\beta \frac{\partial V^\alpha}{\partial y^\beta},
\]

\[
D_h D_i(V^\alpha) = \frac{\partial^2 V^\alpha}{\partial x^h \partial x^i} + y^\beta \frac{\partial^2 V^\alpha}{\partial x^i \partial y^\beta} + y^\beta \frac{\partial V^\alpha}{\partial y^\beta} + y^\lambda \left( \frac{\partial^2 V^\alpha}{\partial x^h \partial y^\beta} + y^\gamma \frac{\partial^2 V^\alpha}{\partial y^\gamma \partial y^\beta} \right),
\]

it follows:

\[
X(2)(L) = V^\alpha \frac{\partial L_0}{\partial y^\alpha} + \frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial y^\alpha} \frac{\partial L}{\partial y^\beta} + L_{\alpha} \frac{\partial L}{\partial y^\alpha}
\]

\[
+ \frac{\partial^2 V^\alpha}{\partial x^i \partial x^j} + y^\beta \frac{\partial^2 V^\alpha}{\partial x^j \partial y^\beta} + y^\beta \left( \frac{\partial^2 V^\alpha}{\partial x^i \partial y^\beta} + y^\gamma \frac{\partial^2 V^\alpha}{\partial y^\gamma \partial y^\beta} \right).
\]

Hence, the coefficient of \( y^\beta \) must vanish and we obtain the following system of partial differential equations:

\[
0 = V^\alpha \frac{\partial L}{\partial y^\alpha} + \frac{\partial V^\alpha}{\partial x^i} + \frac{\partial V^\alpha}{\partial y^\beta} \frac{\partial L}{\partial y^\beta} + L_{\alpha} \frac{\partial V^\alpha}{\partial y^\alpha},
\]

\[
0 = V^\alpha \frac{\partial L}{\partial x^i} + \frac{\partial V^\alpha}{\partial x^i} + \frac{\partial V^\alpha}{\partial y^\beta} \frac{\partial L}{\partial y^\beta} + \sum_{h \leq i} L_{hi} \left( \frac{\partial^2 V^\alpha}{\partial x^h \partial x^i} + y^\beta \frac{\partial^2 V^\alpha}{\partial x^i \partial y^\beta} + y^\beta \left( \frac{\partial^2 V^\alpha}{\partial x^h \partial y^\beta} + y^\gamma \frac{\partial^2 V^\alpha}{\partial y^\gamma \partial y^\beta} \right) \right).
\]

In the case of the E-H Lagrangian, we obtain

\[
[i] \quad 0 = \sum_{a \leq b} \left( \frac{\partial (L_{EH})}{\partial y_{ab}} V^{ab} + (L_{EH})_{ab} \frac{\partial V^{ab}}{\partial y_{ab}} \right), \quad j \leq k, s \leq t,
\]

\[
[ii] \quad 0 = \sum_{a \leq b} \left( \frac{\partial (L_{EH})}{\partial y_{ab}} V^{ab} + \frac{\partial V^{ab}}{\partial x^i} + \sum_{u \leq v} y_{uv,i} \frac{\partial V^{ab}}{\partial y_{uv}} \right) \frac{\partial (L_{EH})}{\partial y_{ab,i}}
\]

\[
+ \sum_{h \leq i} (L_{EH})_{ab} \left[ \frac{\partial^2 V^{ab}}{\partial x^h \partial x^i} + \sum_{s \leq t} y_{st,h} \frac{\partial^2 V^{ab}}{\partial x^i \partial y_{st}} \right]
\]

\[
+ \sum_{s \leq t} y_{st,i} \left( \frac{\partial^2 V^{ab}}{\partial x^i \partial y_{st}} + \sum_{u \leq v} y_{uv,h} \frac{\partial^2 V^{ab}}{\partial y_{uv} \partial y_{st}} \right)igr) \biggr],
\]

as \( \frac{\partial (L_{EH})}{\partial y_{ab,i}} = 0 \), by virtue of (21), with \( V = \sum_{a \leq b} V^{ab} \frac{\partial}{\partial y_{ab}}, V^{ab} \in C^\infty(M) \).

Collecting the terms of degrees 2, 1, and 0 in the variables \( y_{ab,c}, a \leq b \), on the
Moreover, as a calculation shows, we have

\[
0 = \sum_{a \leq b} \left\{ \frac{\partial A_{b,i;r,s,j}^{kl,rs,j}}{\partial y_{ab}} V_{ab} + \left( A_0^{ab,i;r,s,j} + A_0^{s,i;ab,j} \right) \frac{\partial V_{ab}}{\partial y_{ab}} \right\} + \frac{1}{2-\delta_{ij}} (L_{EH})_{ab}^{ij} \frac{\partial^2 V_{ab}}{\partial y_{rs} \partial y_{st}},
\]

where we have used the notations below,

\[
(L_{EH})_{ab}^{ij} = \sum_{h \leq i, a, b \leq h} (L_{EH})_{ab}^{ij},
\]

\[
A_0^{kl,rs,j} = \sum_{r \leq s, k \leq l} \frac{1}{(1+y_{kl})(1+y_{rs})} (2y_{ir}^s (y_{kl}^s + y_{ij}^j) - 2y_{ij}^i y_{ir}^s + y_{ij}^j y_{ir}^s) + 2y_{ik} (y_{ij}^j + y_{ir}^s) + 3y_{ij}^i (y_{ik}^k + y_{is}^s) - y_{ir}^s (y_{ik}^k + y_{is}^s) - 2y_{ik} (y_{ir}^s + y_{is}^s).
\]

Moreover, as a calculation shows, we have

\[
\det \left( (L_{EH})_{rs}^{ij} \right)_{1 \leq r \leq s \leq n} = -(n-1) \rho^{(n+1)(n+4)},
\]

where \( \rho \) is defined in (37). If \( \Lambda = \left( A_{ab}^{kl,rs,j} \right)_{1 \leq a \leq b \leq n}^{1 \leq j \leq k \leq n} \) is the inverse matrix of \( (L_{EH})_{ab}^{ij} \), then from (37)-[ii] for \( h \leq i \), it follows:

\[
\frac{\partial V_{ab}}{\partial y_{st}} = - \sum_{c \leq d} \sum_{p \leq q} A^{ab}_{pq} \frac{\partial (L_{EH})_{rs}^{pq}}{\partial y_{cd}} V_{cd}, \quad a \leq b, s \leq t,
\]

and by imposing the integrability conditions to these equations we obtain

\[
0 = \sum_{a \leq b} \sum_{j \leq k} \left[ \left( \frac{\partial A^{bi}_{ab}}{{\partial y}_{uv}} \frac{\partial (L_{EH})_{rs}^{pq}}{\partial y_{ab}} - \frac{\partial A^{bi}_{ab}}{{\partial y}_{st}} \frac{\partial (L_{EH})_{rs}^{pq}}{\partial y_{ab}} \right) + \Lambda_{bi}^{jk} \left( \frac{\partial^2 (L_{EH})_{rs}^{pq}}{{\partial y}_{ab}{\partial y}_{cd}} - \frac{\partial^2 (L_{EH})_{rs}^{pq}}{{\partial y}_{st}{\partial y}_{cd}} \right) \right] V_{ab} + \Lambda_{bi}^{jk} \left( \frac{\partial (L_{EH})_{rs}^{pq}}{{\partial y}_{ab}} \frac{\partial V_{ab}}{\partial y_{uv}} - \frac{\partial (L_{EH})_{rs}^{pq}}{{\partial y}_{st}} \frac{\partial V_{ab}}{\partial y_{st}} \right).
\]
and substituting (38) in the previous equation, we eventually have

\[
0 = \sum_{c \leq d} \sum_{j \leq k} \left[ \frac{\partial \Lambda_{hi}^{jk}}{\partial y_{uv}} - \frac{\partial \Lambda_{hi}^{jk}}{\partial y_{cd}} \right] + \Lambda_{hi}^{jk} \sum_{a \leq b} \sum_{p \leq q} \Lambda_{ab}^{pq} \left( \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{uv}} - \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{cd}} \right)
\]

Furthermore, from

\[
\frac{\partial \Lambda_{hi}^{jk}}{\partial y_{pq}} = -\Lambda_{hi}^{jk} \frac{\partial L}{\partial y_{pq}} \cdot \Lambda,
\]

it follows:

\[
\frac{\partial \Lambda_{hi}^{jk}}{\partial y_{uv}} = -\sum_{\zeta \leq \eta \leq \rho \leq \sigma} \Lambda_{hi}^{jk} \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{uv}} \Lambda_{pq}^{st}.
\]

Hence

\[
\frac{\partial \Lambda_{hi}^{jk}}{\partial y_{uv}} = -\sum_{\zeta \leq \eta \leq \rho \leq \sigma} \Lambda_{hi}^{jk} \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{uv}} \Lambda_{pq}^{st}.
\]

and letting

\[
\Phi_{st,uv,c}^{jk} = \frac{\partial^2 (L_{EH})^{ik}_{st}}{\partial y_{uv} \partial y_{cd}} - \frac{\partial^2 (L_{EH})^{ik}_{st}}{\partial y_{cd} \partial y_{ct}} + \sum_{a \leq b \leq p \leq q} \Lambda_{ab}^{pq} \left( \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{uv}} - \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{cd}} \right) + \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{uv}} \frac{\partial (L_{EH})^{pq}_{st}}{\partial y_{cd}},
\]

the equations (39) transforms into the following:

\[
0 = \sum_{c \leq d} (\Lambda \cdot \Phi_{st,uv,c})_{cd} V^{cd}, \quad h \leq i, s \leq t, u \leq v,
\]

where \( \Phi_{st,uv} \) is the matrix \( (\Phi_{st,uv})_{cd}^{jk} = \Phi_{st,uv,c}^{jk} \), for every \( s \leq t, u \leq v \). As \( \dim \Phi_{11,23} \neq 0 \) for \( n = 3 \) and \( \det \Phi_{12,34} \neq 0 \) for \( n \geq 4 \), it follows \( V^{cd} = 0 \). □

**Remark 6.1.** For \( n = 2 \) the E-H Lagrangian density is known to be a conformally invariant 2-form; hence \( \Lambda_{EH} \) admits—in this dimension—the Liouville vector field as a vertical infinitesimal symmetry.

### 7 Jacobi fields and presymplectic structure

Let \( V(p) \subset TE \) be the sub-bundle of \( p \)-vertical tangent vectors for the submersion \( p: E \to N \). The infinitesimal variation of a one-parameter variation \( S_t \) of
an extremal \( J \) defined on \( s \in S \) linearized Hamilton-Cartan equation. Precisely, a Jacobi field along a n extremal \( T \) denote it by \( \forall s \) satisfies the Jacobi equation. Hence we think of the Jacobi fields a long \( \leq 1 \) is the first-order infinitesimal contact transformation on \( s \) the tangent space at \( \forall s \) is an extremal for every \( t \) is a section \( \in S \) sections to the Hamilton-Cartan equation; that is, \( \forall s \) (e.g., see \([17], [19], [24]\)). If \( X \) is the extremals can be characterized as the solutions to the linearized Hamilton-Cartan equation. Precisely, a Jacobi field along an extremal \( s \in S(U) \) is a p-vertical vector field defined along \( s, X \in \Gamma(U, s^*V(p)) \), satisfying the Jacobi equation \( (j^1s)^*(i_\gamma d\Theta_\Lambda) = 0 \) for all \( Y \in \mathfrak{X}(J^1E) \). Jacobi fields are the solutions to the \( X \) (see \([17], [19], [24]\)). If \( X = V^\alpha(x) \left( \frac{\partial}{\partial y^\alpha} \right)_{s(x)} \), then

\[
\left( X^{(1)} \right)_{j^1s} = V^\alpha(x) \left( \frac{\partial}{\partial y^\alpha} \right)_{j^1s} + \frac{\partial V^\alpha}{\partial x^\gamma}(x) \left( \frac{\partial}{\partial y^\gamma} \right)_{j^1s}.
\]

In fact, it is readily checked that if \( S_t \) is a one-parameter variation of \( s \) and \( S_t \) is an extremal for every \( t \), then the infinitesimal variation \( X \) of \( S_t \) (see \([10]\)) satisfies the Jacobi equation. Hence we think of the Jacobi fields along \( s \) as being the tangent space at \( s \) to the “manifold” \( S(U) \) of extremals and accordingly we denote it by \( T_sS(U) \). Let \( s : N \to E \) be an extremal of a Lagrangian density \( \Lambda \) defined on \( J^1E \).

In a fibred coordinate system \((x^i, y^\alpha)\) a vector field \( X \in \Gamma(U, s^*V(p)) \) along an extremal \( s \) is a Jacobi field if and only if \( (j^1s)^*(i_\gamma d\Theta_\Lambda) = 0 \), for \( 1 \leq \alpha \leq m \) (see \([20], \text{section 3.5}\)). By using the formulas \([11], [23], \text{and } [18]\), we obtain

\[
L_{X^{(1)}}d\Theta_\Lambda = L_{X^{(1)}} \left\{ (-1)^{i-1} dp^i_\alpha \land dy^\alpha \land v_i + dH \land v \right\} = (-1)^{i-1} d \left( X^{(1)} p^i_\alpha \right) \land dy^\alpha \land v_i + (-1)^{i-1} dp^i_\alpha \land dV^\alpha \land v_i + d \left( X^{(1)} H \right) \land v.
\]

Hence

\[
i_\partial/\partial y^\alpha L_{X^{(1)}}d\Theta_\Lambda = (-1)^{i-1} \frac{\partial X^{(1)}(p^\alpha_\gamma)}{\partial y^\gamma} dy^\beta \land v_i - \frac{\partial (X^{(1)}(p^\alpha_\gamma))}{\partial x^\gamma} v_i - (-1)^{i-1} \frac{\partial (X^{(1)}(p^\gamma)}{\partial y^\gamma} dy^\beta \land v_i - (-1)^{i-1} \frac{\partial (X^{(1)}(p^\gamma))}{\partial y^\gamma} v_i + \frac{\partial V^\alpha}{\partial x^\gamma} v + \frac{\partial H}{\partial x^\gamma} v,
\]

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and finally,

\[
0 = \frac{\partial x^\mu}{\partial t^i} \left\{ \left( \frac{\partial X^{(1)}(p^a_i)}{\partial y^a} \circ j^1 s \right) - \left( \frac{\partial (X^{(1)}(p^a_i))}{\partial y^a} \circ j^1 s \right) \right\} \\
+ \left( \frac{\partial X^{(1)}(H)}{\partial y^a} \circ j^1 s \right) - \left( \frac{\partial (X^{(1)}(p^a_i))}{\partial x^a} \circ j^1 s \right) \\
- \left( \frac{\partial (X^{(1)}(p^a_j))}{\partial y^j} \circ j^1 s \right) \frac{\partial^2 s^a}{\partial x^a \partial x^j} + \frac{\partial V^a}{\partial x^a} \frac{\partial p^a_i}{\partial y^a} \circ j^1 s \right). \\
\]

Expanding,

\[
\frac{\partial^2 V^{\gamma}}{\partial x^i \partial x^j} \left( \frac{\partial p^i}{\partial y^j} \circ j^1 s \right) = V^{\gamma} \left\{ \frac{\partial x^\mu}{\partial x^i} \left( \frac{\partial^2 p^a_i}{\partial y^a \partial y^a} \circ j^1 s \right) - \frac{\partial^2 p^a_i}{\partial y^a \partial y^a} \circ j^1 s \right\} \\
+ \frac{\partial^2 H}{\partial y^a \partial y^a} \circ j^1 s - \frac{\partial^2 p^i}{\partial y^a \partial y^a} \circ j^1 s \\
+ \frac{\partial V^{\gamma}}{\partial x^a} \left\{ \frac{\partial x^\mu}{\partial y^a} \left( \frac{\partial^2 p^a_i}{\partial y_i \partial y_h} \circ j^1 s \right) - \frac{\partial^2 p^i}{\partial y_i \partial y_h} \circ j^1 s \right\} \\
- \frac{\partial^2 H}{\partial y^a \partial y^a} \circ j^1 s + \frac{\partial^2 p^i}{\partial y^a \partial y_h} \circ j^1 s \right), \\
\]

(41)

\[1 \leq \alpha \leq m.\]

**Remark 7.1.** In the case of the E-H Lagrangian density, Greek indices of the general case transform into a pair of non-decreasing Latin indices: \( \alpha = (a, b), \)

\(1 \leq a \leq b \leq n,\) and a Jacobi vector field along \( g \) can locally be written as follows:

\[
X_x = \sum_{a \leq b} V^{ab}(x) \left( \frac{\partial}{\partial y_{ab}} \right) g(x) \\
= \sum_{a \leq b} V^{ab}(x)(dx^a)_x \circ (dx^b)_x, \quad \forall x \in N,
\]

with \( V^{ab} = V^{ba} \) for \( a > b. \) Moreover, in this case, the general equations (41) for Jacobi fields can also be written as follows:

\[
0 = \frac{1}{2} \left[ (\delta_{ab} \delta_{jk} + \delta_{ak} \delta_{bj}) g^{ab} - g^{ij} \delta_{ab} \delta_{ji} + \delta_{ab} \delta_{jk} \frac{\partial^2 V^{\gamma}}{\partial x^i \partial x^j} \right] \\
+ \frac{1}{2} \left[ \frac{\partial g^{ab}(\Gamma^g)_{ij}^\lambda}{\partial y_j} - \frac{\partial g^{ij}(\Gamma^g)_{ab}^\lambda}{\partial y_j} \right] - \frac{\partial^2 \gamma_{ab}}{\partial y^a \partial y^b} \\
+ \frac{1}{2} \left[ \frac{\partial g^{ij}(\Gamma^g)_{ab}^\lambda}{\partial y^a} - \frac{\partial g^{ij}(\Gamma^g)_{ab}^\lambda}{\partial y^b} \right], \quad 1 \leq \mu \leq \nu \leq n,
\]

(42)

where \( \Gamma^g \) denotes the Levi-Civita connection of \( g, \) and \( R^a \) its curvature tensor.
Example 7.1. If \((N, g)\) is a flat 4-dimensional Lorentzian manifold, then locally, 
\[ g = \varepsilon_1(dx^2) + \varepsilon_2(dx^3) + \varepsilon_3(dx^4) + \varepsilon_4(dx^5), \]
with \(\varepsilon_1 = -1, \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = +1\), and the equations \(\text{(2)}\) of the Jacobi fields along \(g\) are as follows:

\[
\sum_{B=1}^{10} P_B^A(D)U_B = 0, \quad 1 \leq A \leq 10,
\]
where

\[
\begin{align*}
U^1 &= V^{11}, & U^2 &= V^{12}, & U^3 &= V^{13}, & U^4 &= V^{14}, & U^5 &= V^{22}, \\
U^6 &= V^{23}, & U^7 &= V^{24}, & U^8 &= V^{33}, & U^9 &= V^{34}, & U^{10} &= V^{44},
\end{align*}
\]

\[
\begin{align*}
P_1 &= \frac{1}{2} \sum_{i=2}^{4} (D^i)^2, & P_2 &= D^1 D^2, & P_3 &= D^1 D^3, & P_4 &= D^1 D^4, \[2pt]
P_5 &= 0, & P_6 &= \frac{1}{4} (D^1)^2 - \frac{1}{2} (D^1)^2, & P_7 &= \frac{1}{2} D^2 D^3, & P_8 &= \frac{1}{6} D^2 D^4, \[2pt]
P_9 &= 0, & P_{10} &= \frac{1}{4} (D^2)^2 - \frac{1}{2} (D^2)^2, & P_{11} &= \frac{1}{2} D^3 D^4, & P_{12} &= \frac{1}{6} D^3 D^4, \[2pt]
P_{13} &= \frac{3}{2} D^2 D^3, & P_{14} &= D^2 D^4, & P_{15} &= \frac{1}{2} D^2 D^4, & P_{16} &= \frac{1}{6} D^2 D^4, \[2pt]
P_{17} &= \frac{1}{2} D^3 D^4, & P_{18} &= \frac{1}{3} D^3 D^4, & P_{19} &= \frac{1}{2} D^3 D^4, & P_{20} &= \frac{1}{6} D^3 D^4, \[2pt]
P_{21} &= \frac{1}{2} D^4 D^4, & P_{22} &= \frac{1}{4} D^4 D^4, & P_{23} &= \frac{1}{6} D^4 D^4, & P_{24} &= \frac{1}{6} D^4 D^4, \[2pt]
P_{25} &= 0, & P_{26} &= \frac{1}{2} D^4 D^4, & P_{27} &= \frac{1}{6} D^4 D^4, & P_{28} &= \frac{1}{6} D^4 D^4, \[2pt]
P_{29} &= 0, & P_{30} &= \frac{1}{2} D^4 D^4, & P_{31} &= \frac{1}{6} D^4 D^4, & P_{32} &= \frac{1}{6} D^4 D^4, \[2pt]
P_{33} &= \frac{3}{2} D^4 D^4, & P_{34} &= \frac{3}{4} D^4 D^4, & P_{35} &= \frac{3}{6} D^4 D^4, & P_{36} &= \frac{3}{6} D^4 D^4, \[2pt]
P_{37} &= \frac{3}{2} D^4 D^4, & P_{38} &= \frac{3}{4} D^4 D^4, & P_{39} &= \frac{3}{6} D^4 D^4, & P_{40} &= \frac{3}{6} D^4 D^4, \[2pt]
P_{41} &= \frac{3}{2} D^4 D^4, & P_{42} &= \frac{3}{4} D^4 D^4, & P_{43} &= \frac{3}{6} D^4 D^4, & P_{44} &= \frac{3}{6} D^4 D^4,
\end{align*}
\]

1 \leq i \leq 4.
If a solution \((U^A)_{A=1}^{10}\) is expanded in power series up to second order, i.e., \(U^A = \lambda^A + \sum_{1 \leq j \leq 4} \lambda^A_j x^j + \sum_{1 \leq j \leq k \leq 4} \lambda^A_{jk} x^j x^k + \) terms of order \(\geq 3\), then evaluating it at \(x^1 = \ldots = x^4 = 0\), we obtain

\[
\begin{align*}
\lambda^1_{22} &= \lambda^5_{22} + \lambda^6_{33} + \lambda^5_{44} - \lambda^6_{22} - \lambda^7_{22} + \lambda^8_{22} + \lambda^9_{22}, \\
\lambda^1_{23} &= \lambda^5_{13} + \lambda^5_{12} - 2\lambda^6_{11} + 2\lambda^6_{14} - \lambda^7_{13} + \lambda^8_{13} + \lambda^9_{13}, \\
\lambda^1_{24} &= \lambda^5_{14} + \lambda^5_{12} - \lambda^6_{14} - 2\lambda^6_{11} + 2\lambda^7_{13} + \lambda^8_{14} - \lambda^9_{13}, \\
\lambda^1_{33} &= \lambda^5_{33} + \lambda^5_{32} + \lambda^6_{31} + \lambda^8_{33} + \lambda^9_{33}, \\
\lambda^1_{34} &= \lambda^5_{13} + \lambda^5_{12} + \lambda^6_{13} - \lambda^6_{12} - \lambda^7_{11} + 2\lambda^8_{12}, \\
\lambda^1_{44} &= \lambda^5_{14} - \lambda^6_{22} - 2\lambda^7_{13} + 2\lambda^8_{14} - 2\lambda^9_{22} - \lambda^7_{12} + \lambda^6_{12} - \lambda^9_{13}, \\
\lambda^1_{32} &= \lambda^5_{32} + 2\lambda^6_{31} + \lambda^7_{13} + \lambda^8_{12} - \lambda^9_{13} - \lambda^9_{14} + \lambda^1_{13}, \\
\lambda^1_{33} &= \lambda^5_{33} + \lambda^5_{32} + \lambda^6_{31} + \lambda^7_{13} + \lambda^8_{12} - \lambda^9_{13} - \lambda^9_{14} - \lambda^1_{13}, \\
\lambda^1_{34} &= -2\lambda^5_{33} - 2\lambda^6_{31} + 2\lambda^7_{13} - 2\lambda^8_{14} + 2\lambda^9_{22} - 2\lambda^9_{13} - 2\lambda^9_{14} - \lambda^1_{13}.
\end{align*}
\]

Hence the space of quadratic Jacobi fields along \(g\) is a vector space of dimension 90, with basis

\[
\begin{align*}
&(x^1)^2 E_1, \quad x^1 x^2 E_1, \quad x^1 x^3 E_1, \quad x^1 x^4 E_1, \quad (x^1)^2 E_2, \quad (x^2)^2 E_2, \\
&(x^1)^2 E_3, \quad x^1 x^2 E_2, \quad (x^1)^2 E_3, \quad x^2 x^4 E_3, \quad (x^1)^2 E_4, \quad (x^2)^2 E_4, \\
&(x^3)^2 E_4, \quad (x^1)^2 E_4, \quad x^1 x^2 E_5, \quad (x^1)^2 E_5, \quad x^2 x^4 E_5, \quad x^1 x^4 E_5, \\
&(x^1)^2 E_6, \quad x^1 x^2 E_6, \quad (x^1)^2 E_6, \quad x^1 x^3 E_7, \quad (x^1)^2 E_7, \quad (x^1)^2 E_8, \\
&(x^1)^2 E_9, \quad x^1 x^2 E_9, \quad (x^1)^2 E_9, \quad x^3 x^4 E_8, \quad x^1 x^2 E_9, \quad (x^1)^2 E_9, \\
&(x^1)^2 E_{10}, \quad x^1 x^4 E_{10}, \quad x^3 x^4 E_{10}, \quad (x^2)^2 E_{10}, \\
&(x^1)^2 E_1 + x^1 x^2 E_2, \quad x^2 x^4 E_1 + x^1 x^4 E_2, \quad x^2 x^4 E_1 + x^1 x^4 E_3, \quad x^2 x^4 E_1 + x^1 x^4 E_3, \\
&(x^1)^2 E_1 + x^1 x^2 E_2, \quad x^2 x^4 E_1 + x^2 x^4 E_2, \quad (x^1)^2 E_1 + x^1 x^2 E_3, \quad (x^1)^2 E_1 + x^1 x^2 E_3, \\
&(x^1)^2 E_1 + x^1 x^2 E_2, \quad x^1 x^4 E_1 + x^1 x^4 E_3, \quad (x^1)^2 E_1 + x^1 x^4 E_3, \quad (x^1)^2 E_1 + x^1 x^4 E_3, \\
&- (x^1)^2 E_1 + (x^1)^2 E_5, \quad x^1 x^4 E_1 + x^2 x^4 E_5, \quad ((x^1)^2 - (x^2)^2) (E_1 + E_5), \quad ((x^1)^2 - (x^2)^2) (E_1 + E_5), \\
&2x^1 x^2 E_3 + (x^1)^2 E_6, \quad -2x^1 x^3 E_3 + (x^1)^2 E_6, \quad -2x^1 x^2 E_4 + x^1 x^4 E_6, \quad -2x^1 x^2 E_4 + x^1 x^4 E_6, \\
&-x^1 x^2 E_4 + x^1 x^4 E_6, \quad -x^2 x^3 E_1 + x^3 x^4 E_7, \quad -2x^2 x^3 E_1 + (x^1)^2 E_7, \quad -2x^2 x^3 E_1 + (x^1)^2 E_7, \\
&2x^2 x^3 E_1 + (x^1)^2 E_7, \quad -x^1 x^2 E_4 + x^2 x^4 E_7, \quad x^2 x^4 E_1 + x^2 x^4 E_8, \quad x^2 x^4 E_1 + x^2 x^4 E_8, \\
&-x^2 x^3 E_1 + x^2 x^4 E_9, \quad x^2 x^4 E_1 + x^1 x^4 E_9, \quad x^2 x^4 (E_1 + E_9), \quad x^2 x^4 (E_1 + E_9), \\
&(x^1)^2 E_9 + (x^1)^2 E_{10}, \quad 2x^2 x^3 E_2 + (x^2)^2 E_3, \quad 2x^2 x^3 E_2 + (x^2)^2 E_3, \quad 2x^2 x^3 E_2 + (x^2)^2 E_3, \\
&(x^2)^2 E_3 + x^2 x^4 E_3, \quad -x^2 x^4 E_2 + x^3 x^4 E_3, \quad -x^2 x^3 E_2 + x^1 x^4 E_4, \quad -x^2 x^3 E_2 + x^1 x^4 E_4, \\
&2x^2 x^4 E_2 + (x^2)^2 E_4, \quad 2x^2 x^4 E_2 + (x^2)^2 E_4, \quad (x^2)^2 E_2 + x^2 x^4 E_4, \quad (x^2)^2 E_2 + x^2 x^4 E_4, \\
&x^2 x^3 E_2 + x^1 x^4 E_5, \quad x^2 x^4 E_2 + x^1 x^4 E_5, \quad -x^2 x^3 E_2 + x^1 x^4 E_6, \quad -x^2 x^3 E_2 + x^1 x^4 E_6, \\
&(x^1)^2 E_2 + x^1 x^3 E_6, \quad -x^1 x^3 E_2 + x^1 x^3 E_6, \quad (x^1)^2 E_2 + x^1 x^3 E_6, \quad (x^1)^2 E_2 + x^1 x^3 E_6, \\
&x^2 x^4 E_2 + x^1 x^4 E_6, \quad -x^1 x^3 E_2 + x^1 x^3 E_6, \quad x^2 x^4 E_2 + x^1 x^4 E_6, \quad -x^1 x^3 E_2 + x^1 x^3 E_6, \\
&-x^2 x^4 E_2 + x^1 x^4 E_6, \quad x^1 x^3 E_2 + x^1 x^3 E_6, \quad -x^2 x^4 E_2 + x^1 x^4 E_6, \quad -x^1 x^3 E_2 + x^1 x^3 E_6, \\
&(x^1)^2 E + x^2 x^3 E_6, \quad -(x^1)^2 E + x^2 x^3 E_6, \quad -(x^1)^2 E + x^2 x^3 E_6, \quad -(x^1)^2 E + x^2 x^3 E_6 \quad \end{align*}
\]
\[(x^3)^2 - (x^4)^2)E_1 + (x^3)^2E_5 + x^2x^4E_7,\]
\[(x^4)^2 - (x^2)^2)E_1 - (x^3)^2E_5 + (x^2)^2E_8,\]
\[(x^2)^2 - (x^4)^2)E_1 + (x^3)^2E_5 + x^3x^4E_9,\]
\[(x^3)^2 - (x^2)^2)E_1 - (x^3)^2E_5 + (x^3)^2E_{10},\]
\[(x^4)^2 - (x^3)^2)E_1 - (x^3)^2E_5 + (x^2)^2E_{10},\]

where

\[\begin{align*}
E_1 &= \frac{\partial}{\partial y_{11}}, & E_2 &= \frac{\partial}{\partial y_{12}}, & E_3 &= \frac{\partial}{\partial y_{13}}, & E_4 &= \frac{\partial}{\partial y_{14}}, & E_5 &= \frac{\partial}{\partial y_{22}}, \\
E_6 &= \frac{\partial}{\partial y_{23}}, & E_7 &= \frac{\partial}{\partial y_{24}}, & E_8 &= \frac{\partial}{\partial y_{33}}, & E_9 &= \frac{\partial}{\partial y_{34}}, & E_{10} &= \frac{\partial}{\partial y_{44}}.
\end{align*}\]

More generally, if \(U^A = \sum_{|J| = r} \lambda_j^A (x^1)^{j_1} \cdots (x^4)^{j_4}\), where \(1 \leq A \leq 10\), and \(J = (j_1, \ldots, j_4)\), is a homogeneous Jacobi field along \(g\) of order \(r \geq 3\), then for every multi-index \((i_1, \ldots, i_4)\) of order \(i_1 + \ldots + i_4 = r - 2\), the functions 
\[
[(D^1)^{i_1} \circ \cdots \circ (D^4)^{i_4}] (U^A) = \sum_{a < b} \lambda_{a+b}^A (x^a)^{i_1} \cdots (x^a)^{i_4} (x^b)^{i_1} \cdots (x^b)^{i_4}
\]
and consequently, the functions
\[
\lambda_{ab}^A = \lambda_{a+b}^A (x^a)^{i_1} \cdots (x^a)^{i_4} (x^b)^{i_1} \cdots (x^b)^{i_4},
\]
\[
\lambda_{aa}^A = \frac{1}{2} \lambda_{a}^A (x^a)^{i_1} \cdots (x^a)^{i_4} (x^a)^{i_1} \cdots (x^a)^{i_4},
\]
must satisfy the equations \([14]\) for \(1 \leq a < b \leq 4\), and every multi-index \((i_1, \ldots, i_4)\) of order \(r - 2\).

**Example 7.2.** If \(N = (\mathbb{R}/2\pi\mathbb{Z})^4\) is a 4-dimensional torus with Lorentzian metric \(g = \varepsilon_1 (dx^1)^2, \quad \varepsilon_1 = -1, \quad \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = +1\), as in the Example \([7.1]\) then we can obtain the global solutions to Jacobi equations \([43]\) by expanding in Fourier series; namely,
\[
U^A = \sum_{(k_1, \ldots, k_4) \in \mathbb{Z}^4} U^A_{k_1, \ldots, k_4} \exp(ik_jx^j), \quad U^A_{k_1, \ldots, k_4} \in \mathbb{C},
\]
so that \(\frac{\partial^2 U^A}{\partial x^i \partial x^j} = -\sum_{(k_1, \ldots, k_4) \in \mathbb{Z}^4} k_i k_a U^A_{k_1, \ldots, k_4} \exp(ik_jx^j)\) and the equations \([43]\) transform into the following:
\[
0 = \frac{1}{2} ((k_2)^2 + (k_3)^2 + (k_4)^2) U^1_{k_1, \ldots, k_4} - k_1 k_2 U^2_{k_1, \ldots, k_4} - k_1 k_3 U^3_{k_1, \ldots, k_4} - k_1 k_4 U^4_{k_1, \ldots, k_4} - k_2 k_3 U^5_{k_1, \ldots, k_4} - k_2 k_4 U^6_{k_1, \ldots, k_4} - k_3 k_4 U^7_{k_1, \ldots, k_4} - k_1 k_2 U^8_{k_1, \ldots, k_4} - k_1 k_3 U^9_{k_1, \ldots, k_4} - k_1 k_4 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = -(k_2)^2 U^2_{k_1, \ldots, k_4} + k_2 k_3 U^3_{k_1, \ldots, k_4} + k_2 k_4 U^4_{k_1, \ldots, k_4} + k_2 k_3 U^5_{k_1, \ldots, k_4} + k_2 k_4 U^6_{k_1, \ldots, k_4} + k_2 k_3 U^7_{k_1, \ldots, k_4} + k_2 k_4 U^8_{k_1, \ldots, k_4} + k_2 k_3 U^9_{k_1, \ldots, k_4} + k_2 k_4 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = k_2 k_3 U^2_{k_1, \ldots, k_4} - ((k_2)^2 + (k_4)^2) U^3_{k_1, \ldots, k_4} + k_2 k_3 U^4_{k_1, \ldots, k_4} + k_2 k_3 U^5_{k_1, \ldots, k_4} + k_2 k_3 U^6_{k_1, \ldots, k_4} + k_2 k_3 U^7_{k_1, \ldots, k_4} + k_2 k_3 U^8_{k_1, \ldots, k_4} + k_2 k_3 U^9_{k_1, \ldots, k_4} + k_2 k_3 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = \frac{1}{2} (k_2)^2 U^2_{k_1, \ldots, k_4} - k_1 k_2 U^3_{k_1, \ldots, k_4} + \frac{1}{2} ((k_1)^2 - (k_3)^2 - (k_4)^2) U^4_{k_1, \ldots, k_4} + k_2 k_3 U^5_{k_1, \ldots, k_4} + k_2 k_4 U^6_{k_1, \ldots, k_4} + k_2 k_3 U^7_{k_1, \ldots, k_4} + k_2 k_4 U^8_{k_1, \ldots, k_4} + k_2 k_3 U^9_{k_1, \ldots, k_4} + k_2 k_4 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = k_2 k_3 U^3_{k_1, \ldots, k_4} - k_1 k_3 U^4_{k_1, \ldots, k_4} - k_1 k_2 U^5_{k_1, \ldots, k_4} + ((k_1)^2 - (k_4)^2) U^6_{k_1, \ldots, k_4} + k_2 k_3 U^7_{k_1, \ldots, k_4} + k_2 k_4 U^8_{k_1, \ldots, k_4} + k_2 k_3 U^9_{k_1, \ldots, k_4} + k_2 k_4 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = k_2 k_3 U^4_{k_1, \ldots, k_4} - k_1 k_3 U^5_{k_1, \ldots, k_4} - k_1 k_2 U^6_{k_1, \ldots, k_4} + ((k_1)^2 - (k_3)^2) U^7_{k_1, \ldots, k_4} + k_2 k_3 U^8_{k_1, \ldots, k_4} + k_2 k_4 U^9_{k_1, \ldots, k_4} + k_2 k_3 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = \frac{1}{2} k_2 (k_1)^2 U^1_{k_1, \ldots, k_4} - k_1 k_3 U^2_{k_1, \ldots, k_4} + k_2 k_3 U^3_{k_1, \ldots, k_4} + k_2 k_4 U^4_{k_1, \ldots, k_4} + k_2 k_3 U^5_{k_1, \ldots, k_4} + k_2 k_4 U^6_{k_1, \ldots, k_4} + k_2 k_3 U^7_{k_1, \ldots, k_4} + k_2 k_4 U^8_{k_1, \ldots, k_4} + k_2 k_3 U^9_{k_1, \ldots, k_4} + k_2 k_4 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = \frac{1}{2} k_2 (k_3)^2 U^3_{k_1, \ldots, k_4} - k_1 k_3 U^4_{k_1, \ldots, k_4} + k_2 k_3 U^5_{k_1, \ldots, k_4} + k_2 k_4 U^6_{k_1, \ldots, k_4} + k_2 k_3 U^7_{k_1, \ldots, k_4} + k_2 k_4 U^8_{k_1, \ldots, k_4} + k_2 k_3 U^9_{k_1, \ldots, k_4} + k_2 k_4 U^{10}_{k_1, \ldots, k_4},
\]
\[
0 = \frac{1}{2} k_2 (k_4)^2 U^4_{k_1, \ldots, k_4} - k_1 k_3 U^5_{k_1, \ldots, k_4} + k_2 k_3 U^6_{k_1, \ldots, k_4} + k_2 k_4 U^7_{k_1, \ldots, k_4} + k_2 k_3 U^8_{k_1, \ldots, k_4} + k_2 k_4 U^9_{k_1, \ldots, k_4} + k_2 k_3 U^{10}_{k_1, \ldots, k_4},
\]

for every system \((k_1, \ldots, k_4) \in \mathbb{Z}^4\). Solving these equations for \(k_2 \neq 0\), we obtain

\[
U^1_{k_1, \ldots, k_4} = U^3_{k_1, \ldots, k_4} = U^4_{k_1, \ldots, k_4} = U^8_{k_1, \ldots, k_4} = U^9_{k_1, \ldots, k_4} = U^{10}_{k_1, \ldots, k_4} = 0,
\]
\[
U^2_{k_1, \ldots, k_4} = \frac{1}{2} k_1 k_2 U^5_{k_1, \ldots, k_4},
\]
\[
U^6_{k_1, \ldots, k_4} = \frac{1}{2} k_3 k_2 U^5_{k_1, \ldots, k_4},
\]
\[
U^7_{k_1, \ldots, k_4} = \frac{1}{2} k_4 k_2 U^5_{k_1, \ldots, k_4},
\]

and the unknowns \(U^6_{k_1, \ldots, k_4}\) remain undetermined. If \(k_2 = 0\) but \(k_4 \neq 0\), then the solutions to the previous equations are

\[
U^1_{k_1, \ldots, k_4} = U^3_{k_1, \ldots, k_4} = U^4_{k_1, \ldots, k_4} = U^5_{k_1, \ldots, k_4} = U^7_{k_1, \ldots, k_4} = 0,
\]
\[
U^2_{k_1, \ldots, k_4} = \frac{k_1}{k_4} U^7_{k_1, \ldots, k_4}, U^6_{k_1, \ldots, k_4} = \frac{k_3}{k_4} U^7_{k_1, \ldots, k_4}.
\]
the unknowns $U^7_{k_1,\ldots,k_4}$ remaining undetermined. If $k_2 = k_4 = 0$ but $k_3 \neq 0$, then
\[
U^4_{k_1,\ldots,k_4} = U^7_{k_1,\ldots,k_4} = U^9_{k_1,\ldots,k_4} = U^{10}_{k_1,\ldots,k_4} = 0,
\]
\[
U^5_{k_1,\ldots,k_4} = -\frac{k_1}{(k_3)^2} (k_1 U^8_{k_1,\ldots,k_4} - 2k_3 U^3_{k_1,\ldots,k_4}),
\]
\[
U^2_{k_1,\ldots,k_4} = \frac{k_1}{k_3} U^6_{k_1,\ldots,k_4},
\]
the unknowns $U^7_{k_1,\ldots,k_4}$, $U^8_{k_1,\ldots,k_4}$, and $U^9_{k_1,\ldots,k_4}$ remaining undetermined. Finally, if $k_2 = k_3 = k_4 = 0$, then $U^A_{k_1,\ldots,k_4} = 0$, $5 \leq A \leq 10$, and the unknowns $U^A_{k_1,\ldots,k_4}$, $1 \leq A \leq 4$ remain undetermined. Therefore
\[
U^1 = -\sum_{k_2=k_4=0,k_3 \neq 0} \frac{k_1}{(k_3)^2} \left( k_1 U^8_{k_1,\ldots,k_4} - 2k_3 U^3_{k_1,\ldots,k_4} \right) \exp(ik_j x^j) + U^1_{k_1,0,0,0} \exp(ik_1 x^1),
\]
\[
U^2 = -\frac{1}{2} \sum_{k_2 \neq 0} \frac{k_1}{k_3} U^8_{k_1,\ldots,k_4} \exp(ik_j x^j) + \sum_{k_2=0,k_4 \neq 0} \frac{k_1}{k_3} U^7_{k_1,\ldots,k_4} \exp(ik_j x^j)
\]
\[+ \sum_{k_2=k_4=0,k_3 \neq 0} \frac{k_1}{k_3} U^6_{k_1,\ldots,k_4} \exp(ik_j x^j) + U^2_{k_1,0,0,0} \exp(ik_1 x^1),
\]
\[
U^3 = \sum_{k_2=k_4=0} U^3_{k_1,\ldots,k_4} \exp(ik_j x^j),
\]
\[
U^4 = U^4_{k_1,0,0,0} \exp(ik_1 x^1),
\]
\[
U^5 = \sum_{k_2 \neq 0} U^5_{k_1,\ldots,k_4} \exp(ik_j x^j),
\]
\[
U^6 = -\frac{1}{2} \sum_{k_2 \neq 0} \frac{k_1}{k_3} U^8_{k_1,\ldots,k_4} \exp(ik_j x^j) + \sum_{k_2=0,k_4 \neq 0} \frac{k_1}{k_3} U^7_{k_1,\ldots,k_4} \exp(ik_j x^j)
\]
\[+ \sum_{k_2=k_4=0,k_3 \neq 0} \frac{k_1}{k_3} U^6_{k_1,\ldots,k_4} \exp(ik_j x^j),
\]
\[
U^7 = -\frac{1}{2} \sum_{k_2 \neq 0} \frac{k_1}{k_3} U^8_{k_1,\ldots,k_4} \exp(ik_j x^j) + \sum_{k_3=0} U^7_{k_1,\ldots,k_4} \exp(ik_j x^j),
\]
\[
U^8 = \sum_{k_2=k_4=0,k_3 \neq 0} U^8_{k_1,\ldots,k_4} \exp(ik_j x^j),
\]
\[
U^9 = 0,
\]
\[
U^{10} = 0.
\]
Hence, by using the formulas \((45)\) we obtain
\[
\sum_{A=1}^{10} U^A E_A = U^8_{k_1,0,k_3,0} \exp(i(k_1 x^1 + k_3 x^3)) \left\{ E_6 - \left( \frac{k_3}{(k_3)^2} \right) E_4 \right\}
\]
\[+ U^7_{k_1,0,k_3,4} \exp(i(k_1 x^1 + k_3 x^3 + k_4 x^4)) \left\{ \frac{k_4}{(k_3)^2} E_2 + \frac{k_1}{k_3} E_6 + E_7 \right\}
\]
\[+ U^6_{k_1,0,k_3,0} \exp(i(k_1 x^1 + k_3 x^3)) \left\{ \frac{k_3}{(k_3)^2} E_2 + E_6 \right\}
\]
\[+ U^5_{k_1,\ldots,k_4} \exp(ik_j x^j) \left\{ \frac{1}{2} \frac{k_3}{k_3} E_2 + E_5 + \frac{1}{2} \frac{k_1}{k_3} E_6 + \frac{1}{2} \frac{k_4}{k_3} E_7 \right\}
\]
\[+ U^4_{k_1,0,0,0} \exp(ik_1 x^1) Y_4 + U^3_{k_1,0,k_3,0} \exp(i(k_1 x^1 + k_3 x^3)) \left\{ 2k_3 E_1 + E_3 \right\}
\]
\[+ U^2_{k_1,0,0,0} \exp(ik_1 x^1) E_2 + U^1_{k_1,0,0,0} \exp(ik_1 x^1) E_1,
\]
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and the vector fields

\[
X_1^k = \exp\left[ i(k_1 x^1 + k_3 x^3) \right] \left\{ \frac{\partial}{\partial y_{13}} - \frac{(k_1)^2}{(k_3)^2} \frac{\partial}{\partial y_{11}} \right\}, \quad k_3 \neq 0,
\]

\[
X_2^k = \exp\left[ i(k_1 x^1 + k_3 x^3 + k_4 x^4) \right] \left\{ \frac{k_1}{k_3} \frac{\partial}{\partial y_{12}} + \frac{k_1}{k_4} \frac{\partial}{\partial y_{23}} + \frac{\partial}{\partial y_{14}} \right\}, \quad k_4 \neq 0,
\]

\[
X_3^k = \exp\left[ i(k_1 x^1 + k_3 x^3) \right] \left\{ \frac{k_1}{k_3} \frac{\partial}{\partial y_{12}} + \frac{\partial}{\partial y_{13}} \right\}, \quad k_3 \neq 0,
\]

\[
X_4^k = \exp\left[ i(k_1 x^1 + k_2 x^2 + k_3 x^3 + k_4 x^4) \right] \left\{ \frac{1}{2} k_2 \frac{\partial}{\partial y_{12}} + \frac{1}{2} k_2 \frac{\partial}{\partial y_{22}} + \frac{1}{2} k_2 \frac{\partial}{\partial y_{14}} + \frac{1}{2} k_2 \frac{\partial}{\partial y_{24}} \right\}, \quad k_2 \neq 0,
\]

\[
X_5^k = \exp(ik_1 x^1) \frac{\partial}{\partial y_{14}},
\]

\[
X_6^k = \exp\left[ i(k_1 x^1 + k_3 x^3) \right] \left\{ 2k_3 \frac{\partial}{\partial y_{11}} + \frac{\partial}{\partial y_{13}} \right\},
\]

\[
X_7^k = \exp(ik_1 x^1) \frac{\partial}{\partial y_{11}},
\]

\[
X_8^k = \exp(ik_1 x^1) \frac{\partial}{\partial y_{14}},
\]

with \( k \in \mathbb{Z}^4 \), span \( T_s \mathbb{S}(\mathbb{R}/2\pi \mathbb{Z})^4 \) topologically.

Let \( \Lambda \) be a Lagrangian density on an arbitrary fibred manifold \( p: E \to N \) and let \( \Theta_\Lambda \) be the P-C form associated to \( \Lambda \). Let \( X, Y \in T_s \mathbb{S}(N) \) be Jacobi vector fields defined along an extremal \( s \in \mathbb{S}(N) \) for the Lagrangian density \( \Lambda \). Then, \( d(1^s)\ast(i_{Y(1)}i_{X(1)}d\Theta_\Lambda) = 0 \) (e.g., see [H]); i.e., the \( (n-1) \)-form \( i_{Y(1)}i_{X(1)}d\Theta_\Lambda \) is closed along \( j^i s \).

The alternate bilinear mapping taking values in the space \( Z^{n-1}(N) \) of closed \( (n-1) \)-forms, defined by

\[
(\omega_2)_s: T_s \mathbb{S}(N) \times T_s \mathbb{S}(N) \to Z^{n-1}(N),
\]

\[
(\omega_2)_s(X, Y) = (j^1 s)\ast (i_{Y(1)}i_{X(1)}d\Theta_\Lambda)
\]

is called the presymplectic structure associated to \( \Lambda \).

\[\textbf{Theorem 7.1.}\] Let \( s \) be an extremal of a second-order Lagrangian density \( \Lambda = L_\ast \) on \( p: E \to N \) with Poincaré-Cartan form projectable onto \( J^1 E \). Assume that the variational problem defined by \( \Lambda \) is regular in the sense of Proposition 2.2. For every \( x \in N \), let \( R_2^s(\Lambda) \subseteq J^2_z(s^\ast V(p)) \) be the vector subspace of 2-jets of \( J^2_z X \) of p-vertical vector fields along \( s \) that satisfy the Jacobi equations at \( x \). If the natural projection \( p_2^s: R_2^s(\Lambda) \to J^1_z(s^\ast V(p)) \) is surjective for every \( x \in N \), then the radical of the valued 2-form \( \omega_2)_s \) vanishes.

\[\text{Proof.}\] According to [13], we have \( d\Theta_\Lambda = (-1)^{i-1} dp_i^\ast \wedge dy^\ast \wedge v_i + dH \wedge v \). If \( X^{(1)} = V^\ast \frac{\partial}{\partial y^\ast} + \frac{\partial V^\ast}{\partial x^\ast} \frac{\partial}{\partial y^\ast} \), \( Y^{(1)} = W^\ast \frac{\partial}{\partial y^\ast} + \frac{\partial W^\ast}{\partial x^\ast} \frac{\partial}{\partial y^\ast} \), with \( V^\ast, W^\ast \in C^\infty(N) \), then

\[
(\omega_2)_s(X, Y) = (-1)^{i-1} \left\{ \left( V^\ast W^\ast - V^\ast W^\ast \right) \left( \frac{\partial p_i^\ast}{\partial y^\ast} \circ j^1 s \right) + \left( \frac{\partial V^\ast}{\partial x^\ast} W^\ast - V^\ast \frac{\partial W^\ast}{\partial x^\ast} \right) \left( \frac{\partial p_i^\ast}{\partial y^\ast} \circ j^1 s \right) \right\}_{j^1 s} v_i.
\]

If we assume the vector field \( X \) belongs to \( \text{rad}(\omega_2)_s \), then by evaluating at \( x \) the equation \( (\omega_2)_s(X, Y) = 0, \forall Y \in T_s \mathbb{S}(N) \), we obtain

\[
0 = \left[ V^\ast(x) W^\ast(x) - V^\ast(x) W^\ast(x) \right] \frac{\partial p_i^\ast}{\partial y^\ast}(j^1 s) + \left[ \frac{\partial V^\ast}{\partial x^\ast}(x) W^\ast(x) - V^\ast(x) \frac{\partial W^\ast}{\partial x^\ast}(x) \right] \frac{\partial p_i^\ast}{\partial y^\ast}(j^1 s), \quad 1 \leq i \leq n.
\]
The assumption in the statement implies that given arbitrary values for $W^\beta(x)$ and $\frac{\partial W^\beta}{\partial x^\alpha}(x)$, there exists an element $j^1_s \in R^2_s(\Lambda)$ projecting under the natural mapping $p^1_s : R^2_s(\Lambda) \to J^1_s(s^* V(p))$ onto the 1-jet at $x$ the coordinates of which coincide with these values. Accordingly, the coefficients of $W^\beta(x)$ and $\frac{\partial W^\beta}{\partial x^\alpha}(x)$ in (46) must vanish, i.e.,

$$0 = V^\alpha \left( \frac{\partial p^1_s}{\partial y^\alpha} \circ j^1_s \right) - V^\alpha \left( \frac{\partial p^1_s}{\partial x^\alpha} \right) + \frac{\partial V^\alpha}{\partial x^\alpha} \left( \frac{\partial p^1_s}{\partial y^\alpha} \circ j^1_s \right),$$

$$1 \leq i \leq n, \ 1 \leq \beta \leq m,$$

$$0 = V^\alpha \left( \frac{\partial p^1_s}{\partial y^\alpha} \circ j^1_s \right), \quad h, i = 1, \ldots, n, \ 1 \leq \beta \leq m,$$

as the point $x$ is arbitrary. Hence the formulas (47) are the equations for the radical of $(\omega)_s$. If we set

$$V = (V^1, \ldots, V^m), \quad O_m = (0, \ldots, 0), \quad \Upsilon = \left( \frac{\partial p^1_s}{\partial y^\alpha} \right)_{1 \leq i \leq n, 1 \leq \alpha \leq m},$$

then the second group of equations in (47) can matricially be written as

$$(V, \ldots, V) \cdot (\Upsilon \circ j^1_s) = (O_m, \ldots, O_m).$$

If the variational problem defined by the density $\Lambda$ is regular in the sense of Proposition 2.2 then $\det \Upsilon \neq 0$; Hence $V = 0$. \qed

**Criterion 7.1.** Next, we give a criterion in order to ensure that the condition of Theorem 7.4 holds. According to (13) we have $p^1_s = \frac{\partial L}{\partial y^\alpha} s$, where $L$ is the first-order Lagrangian defined by (17), also see Theorem 4.1. As is known, the Hessian metric of $L$ is the section of the vector bundle $S^2 V^* (p^1_s)$ locally given by,

$$\text{Hess}(L) = \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} d_{10} y^\alpha_{10} \otimes d_{10} y^\beta_{10}.$$ As mentioned in section 2 there is a canonical isomorphism

$$I : (p^1_s)^* (p^* (T^*_x N) \otimes V(p)) \to V(p^1_s),$$

$$I \left( j^1_s : (dx^1) \otimes \left( \frac{\partial}{\partial y^\alpha} \right)_{s(x)} \right) = \left( \frac{\partial}{\partial y^\alpha} \right)_{j^1_s},$$

and dually,

$$I^* : V^*(p^1_s) \to (p^1_s)^* (p^* (T_x N) \otimes V^*(p)),$$

$$I^* \left( j^1_s : (dy^\alpha) \otimes \left( \frac{\partial}{\partial x^i} \right) \right) = (d_{10} y^\alpha_{10} j^1_s).$$

Hence the Hessian metric can be viewed as a symmetric bilinear form

$$\text{Hess}(L)_{j^1_s} : V_{j^1_s} (p^1_s) \times V_{j^1_s} (p^1_s) \equiv [(T^*_x N) \otimes V_{s(x)}(p)] \times [(T^*_x N) \otimes V_{s(x)}(p)] \to \mathbb{R},$$

and we can define a linear map as follows:

$$\text{Hess}(L)_{j^1_s} : (T^*_x N) \otimes (T^*_x N) \otimes V_{s(x)}(p) \to V_{s(x)}(p),$$

$$\text{Hess}(L)_{j^1_s} (w_1, w_2, X_1)(X_2) = \text{Hess}(L)_{j^1_s} (w_1 \otimes X_1, w_2 \otimes X_1),$$

$$\forall w_1, w_2 \in T^*_x N, \quad \forall X_1, X_2 \in V_{s(x)}(p).$$

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The matrix of \( \text{Hess}(L)_{j_1 s}^{\alpha} \) is \( \Upsilon^\alpha \) in the standard basis. Moreover, letting \( v^\gamma_i = \partial^\gamma v^i \), and denoting by \( E^\alpha \) the right-hand side of the formula (41), this formula, evaluated at \( m \), reads as follows: \( v^\gamma_i \frac{\partial E^\alpha}{\partial y^j} (j_1 s) = E^\alpha (x) \), which is a linear system with \( m \) equations in the \( \frac{m}{2} n(n+1) \) unknowns \( v^\gamma_i \), \( 1 \leq i \leq j \leq n \), and the matrix of this system is precisely \( \Upsilon^\alpha \). Consequently, if \( \text{Hess}(L)_{j_1 s}^{\alpha} \) is assumed to be surjective, then the previous system is compatible.

**Corollary 7.2.** The radical of the valued 2-form \( (\omega_2)_g \) corresponding to the E-H Lagrangian density along an arbitrary extremal metric \( g \), vanishes.

**Proof.** According to Theorem [7](#), in order to prove the corollary above, we need only to verify that the projection \( p^2_1 : R^2_x (A) \rightarrow J^1_x (s^* V(p)) \) is surjective for every \( x \in N \). By considering a system of normal coordiantes for the metric \( g \) at the point \( x \), and letting \( v^{ab} = V^{ab} (x) \), \( v_i^a = \varphi^V (x) \partial_p \), evaluated at \( x \), are written as follows:

\[
0 = \frac{1}{2} \left[ \varepsilon_i \left( \delta_{ab} \delta_{ij} + \delta_{ai} \delta_{bj} \right) \partial^b - \varepsilon_i \delta_{ab} \delta_{ai} \delta_{bj} - \delta_{ab} \delta_{aj} \delta_{bi} \right] v_i^a + \varepsilon_i (R^g)^a_{\mu \nu b} (x) v_i^a,
\]

\[
1 \leq \mu \leq \nu \leq n,
\]

which is a system with \( \frac{m}{2} n(n+1) \) equations in the \( \frac{m}{2} n^2 (n+1)^2 \) unknowns \( v_{\mu \nu}^{ij} \), \( 1 \leq i \leq j \leq n \), \( 1 \leq \mu \leq \nu \leq n \), with \( v_{\mu \nu}^{ij} = v_{ij}^{\mu \nu} \), and where the scalars \( v^{ab} \), \( 1 \leq a \leq b \leq n \), can take arbitrary values. A particular solution to this system is obtained by letting,

\[
\begin{align*}
(\text{i}) & \quad \varepsilon_i v_{\mu \nu}^{ij} = \varepsilon_i v_{\mu \nu}^{ij} = 0 \\
(\text{ii}) & \quad \varepsilon_i v_{\mu \nu}^{ij} = \varepsilon_i v_{\mu \nu}^{ij} = -\varepsilon_i (R^g)^a_{\mu \nu b} (x) v_i^a,
\end{align*}
\]

\[
1 \leq \mu \leq \nu \leq n.
\]

The equations (48)-(i) hold by setting \( v_{\mu \nu}^{ij} = v_{\mu \nu}^{ij} \), \( \forall i, \mu, \nu = 1, \ldots, n, \mu \leq \nu \), while the equations (48)-(ii) hold by setting

\[
\begin{align*}
& v_{\mu}^{\mu} = v_{\mu}^{\mu} = 0, \\
& v_{1 \nu}^{1 \nu} = v_{1 \nu}^{1 \nu} = -\varepsilon_1 (R^g)^a_{\mu \nu b} (x) v_1^a, \\
& v_{1 \nu}^{1 \nu} = v_{1 \nu}^{1 \nu},
\end{align*}
\]

\[
2 \leq i \leq n,
\]

\[
1 \leq \mu \leq \nu \leq n.
\]

\( \Box \)

**Example 7.3.** Below, we compute the presymplectic structure associated to Example 7.2 i.e., we compute \( (\omega_2)_g \) for the E-H Lagrangian density when \( N = (\mathbb{R}/2\pi \mathbb{Z})^4 \) and \( g = \varepsilon_1 (dx^i)^2 \), \( \varepsilon_1 = -1 \), \( \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = +1 \) by using the basis \( X^k_h \), \( 1 \leq h \leq 8 \), \( k \in \mathbb{Z}^4 \) of that example. We follow some ideas in [26, Section 7] for our particular case.
According to the previous notations and calculations, we have
\[
(\omega_2)_g(X, Y) = (-1)^{i-1} \left\{ \left( V^{kl} W^{ab} - V^{ab} W^{kl} \right) \left( \frac{\partial p_{kl}}{\partial y_{ab}} \circ j^1 g \right) \right. \\
+ \left. \left( \frac{\partial V^{kl}}{\partial x^a} W^{ab} - V^{ab} \frac{\partial W^{kl}}{\partial x^a} \right) \left( \frac{\partial p_{kl}}{\partial y_{ab,j}} \circ j^1 g \right) \right\} \bigg|_{\varepsilon^1} v_i,
\]
and from the formulas (22), (23) it follows:
\[
\frac{\partial p^i}{\partial y_{uv}} \circ j^1 g = 0.
\]
Therefore
\[
(\omega_2)_g(X, Y) = (-1)^{i-1} \omega_2^i(X, Y) v_i,
\]
where
\[
\omega_2^i(X, Y) = \sum_{k \leq i} \sum_{a \leq b} \left( \frac{\partial p_{ab}}{\partial y_{kl,j}} \circ j^1 g \right) \left( \frac{\partial W^{kl}}{\partial x^a} W^{ab} - V^{ab} \frac{\partial W^{kl}}{\partial x^a} \right),
\]
and, as a computation shows, the scalar differential forms \( \omega_2^i \) are given by
\[
\omega_2^i = \frac{1}{2} \left( \frac{\partial W^{13}}{\partial x^a} + \frac{\partial W^{14}}{\partial x^a} + \frac{\partial W^{12}}{\partial x^a} \right) V^{11} - \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x^a} + \frac{\partial W^{33}}{\partial x^a} + \frac{\partial W^{22}}{\partial x^a} \right) V^{12}
+ \frac{1}{2} \left( \frac{\partial W^{12}}{\partial x^a} + \frac{\partial W^{13}}{\partial x^a} - \frac{\partial W^{23}}{\partial x^a} \right) V^{13} + \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x^a} + \frac{\partial W^{33}}{\partial x^a} + \frac{\partial W^{22}}{\partial x^a} \right) V^{14}
+ \frac{1}{2} \left( \frac{\partial W^{12}}{\partial x^a} + \frac{\partial W^{13}}{\partial x^a} - \frac{\partial W^{23}}{\partial x^a} \right) V^{23} + \frac{1}{2} \left( \frac{\partial W^{33}}{\partial x^a} + \frac{\partial W^{44}}{\partial x^a} + \frac{\partial W^{12}}{\partial x^a} \right) V^{22}
+ \frac{1}{2} \left( \frac{\partial W^{13}}{\partial x^a} + \frac{\partial W^{14}}{\partial x^a} - \frac{\partial W^{34}}{\partial x^a} \right) V^{24} + \frac{1}{2} \left( \frac{\partial W^{33}}{\partial x^a} + \frac{\partial W^{22}}{\partial x^a} + \frac{\partial W^{13}}{\partial x^a} \right) V^{33}
+ \frac{1}{2} \left( \frac{\partial W^{14}}{\partial x^a} + \frac{\partial W^{13}}{\partial x^a} - \frac{\partial W^{34}}{\partial x^a} \right) V^{34} + \frac{1}{2} \left( \frac{\partial W^{33}}{\partial x^a} + \frac{\partial W^{22}}{\partial x^a} + \frac{\partial W^{14}}{\partial x^a} \right) V^{44}
+ \frac{1}{2} \left( \frac{\partial W^{12}}{\partial x^a} + \frac{\partial W^{14}}{\partial x^a} \right) W^{11} + \frac{1}{2} \left( \frac{\partial W^{22}}{\partial x^a} + \frac{\partial W^{44}}{\partial x^a} \right) W^{12}
+ \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x^a} + \frac{\partial W^{33}}{\partial x^a} \right) W^{13} + \frac{1}{2} \left( \frac{\partial W^{22}}{\partial x^a} + \frac{\partial W^{44}}{\partial x^a} \right) W^{14}
+ \frac{1}{2} \left( \frac{\partial W^{13}}{\partial x^a} + \frac{\partial W^{12}}{\partial x^a} \right) W^{23} - \frac{1}{2} \left( \frac{\partial W^{33}}{\partial x^a} + \frac{\partial W^{44}}{\partial x^a} \right) W^{22}
+ \frac{1}{2} \left( \frac{\partial W^{14}}{\partial x^a} + \frac{\partial W^{13}}{\partial x^a} \right) W^{24} - \frac{1}{2} \left( \frac{\partial W^{33}}{\partial x^a} + \frac{\partial W^{44}}{\partial x^a} \right) W^{33}
+ \frac{1}{2} \left( \frac{\partial W^{34}}{\partial x^a} + \frac{\partial W^{14}}{\partial x^a} \right) W^{34} - \frac{1}{2} \left( \frac{\partial W^{22}}{\partial x^a} + \frac{\partial W^{44}}{\partial x^a} \right) W^{44},
\]
\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{12}}{\partial x_1} - \frac{\partial W^{23}}{\partial x_1} + \frac{\partial W^{44}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_1} - \frac{\partial W^{24}}{\partial x_1} \right) V^{11} + \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} - \frac{\partial W^{44}}{\partial x_2} \right) W^{12} + \frac{1}{2} \left( \frac{\partial W^{12}}{\partial x_1} + \frac{\partial W^{23}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{13} + \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) W^{14} + \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{12} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{12} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{13} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{13} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{14} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{14} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{22} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{22} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{23} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{23} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{24} + \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{24} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{33} + \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{33} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{34} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{34} \]

\[ \omega_2^3 = \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x_1} + \frac{\partial W^{22}}{\partial x_1} - \frac{\partial W^{33}}{\partial x_2} + \frac{\partial W^{44}}{\partial x_2} \right) V^{44} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x_1} + \frac{\partial V^{22}}{\partial x_1} - \frac{\partial V^{33}}{\partial x_2} - \frac{\partial V^{44}}{\partial x_2} \right) W^{44} \]
\[ \omega_4^2 = \frac{1}{2} \left( \frac{\partial W^{33}}{\partial x^4} - \frac{\partial W^{34}}{\partial x^4} + \frac{\partial W^{22}}{\partial x^4} - \frac{\partial W^{24}}{\partial x^4} - \frac{\partial W^{14}}{\partial x^4} \right) V^{11} \]

\[ + \frac{1}{2} \left( \frac{\partial V^{13}}{\partial x^4} - \frac{\partial V^{34}}{\partial x^4} + \frac{\partial V^{22}}{\partial x^4} - \frac{\partial V^{24}}{\partial x^4} - \frac{\partial V^{14}}{\partial x^4} \right) W^{11} \]

\[ + \left( \frac{\partial W^{12}}{\partial x^4} + \frac{\partial W^{24}}{\partial x^4} + \frac{\partial W^{14}}{\partial x^4} \right) V^{12} - \left( \frac{\partial V^{12}}{\partial x^4} + \frac{\partial V^{24}}{\partial x^4} + \frac{\partial V^{14}}{\partial x^4} \right) W^{12} \]

\[ + \left( \frac{\partial W^{13}}{\partial x^4} + \frac{\partial W^{34}}{\partial x^4} + \frac{\partial W^{14}}{\partial x^4} \right) V^{13} - \left( \frac{\partial V^{13}}{\partial x^4} + \frac{\partial V^{34}}{\partial x^4} + \frac{\partial V^{14}}{\partial x^4} \right) W^{13} \]

\[ + \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x^4} - \frac{\partial W^{22}}{\partial x^4} - \frac{\partial W^{33}}{\partial x^4} + \frac{\partial W^{44}}{\partial x^4} \right) V^{14} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x^4} - \frac{\partial V^{22}}{\partial x^4} - \frac{\partial V^{33}}{\partial x^4} + \frac{\partial V^{44}}{\partial x^4} \right) W^{14} \]

\[ + \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x^4} + \frac{\partial V^{12}}{\partial x^4} + \frac{\partial V^{13}}{\partial x^4} \right) V^{22} - \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x^4} + \frac{\partial V^{22}}{\partial x^4} + \frac{\partial V^{33}}{\partial x^4} \right) W^{22} \]

\[ + \left( \frac{\partial W^{13}}{\partial x^4} - \frac{\partial W^{24}}{\partial x^4} + \frac{\partial W^{34}}{\partial x^4} \right) V^{23} - \left( \frac{\partial V^{13}}{\partial x^4} - \frac{\partial V^{24}}{\partial x^4} + \frac{\partial V^{34}}{\partial x^4} \right) W^{23} \]

\[ + \frac{1}{2} \left( \frac{\partial W^{11}}{\partial x^4} + \frac{\partial W^{22}}{\partial x^4} + \frac{\partial W^{33}}{\partial x^4} \right) V^{24} + \frac{1}{2} \left( \frac{\partial V^{11}}{\partial x^4} + \frac{\partial V^{22}}{\partial x^4} + \frac{\partial V^{33}}{\partial x^4} \right) W^{24} \]

\[ + \frac{1}{2} \left( \frac{\partial W^{14}}{\partial x^4} - \frac{\partial W^{24}}{\partial x^4} + \frac{\partial W^{34}}{\partial x^4} \right) V^{33} - \frac{1}{2} \left( \frac{\partial V^{14}}{\partial x^4} - \frac{\partial V^{24}}{\partial x^4} + \frac{\partial V^{34}}{\partial x^4} \right) W^{33} \]

\[ + \frac{1}{2} \left( \frac{\partial W^{22}}{\partial x^4} + \frac{\partial W^{33}}{\partial x^4} - \frac{\partial W^{11}}{\partial x^4} + \frac{\partial W^{44}}{\partial x^4} \right) V^{34} + \frac{1}{2} \left( \frac{\partial V^{22}}{\partial x^4} + \frac{\partial V^{33}}{\partial x^4} - \frac{\partial V^{11}}{\partial x^4} + \frac{\partial V^{44}}{\partial x^4} \right) W^{34} \]

\[ + \frac{1}{2} \left( \frac{\partial W^{14}}{\partial x^4} + \frac{\partial W^{34}}{\partial x^4} + \frac{\partial W^{24}}{\partial x^4} \right) V^{44} - \frac{1}{2} \left( \frac{\partial V^{14}}{\partial x^4} + \frac{\partial V^{34}}{\partial x^4} + \frac{\partial V^{24}}{\partial x^4} \right) W^{44} . \]

Hence

\[ \omega_2^4 (X_1^2, X_1^3) = 0, \quad \omega_2^4 (X_2^1, X_2^3) = 0, \]

\[ \omega_2^4 (X_3^1, X_3^3) = 0, \quad \omega_2^4 (X_4^1, X_4^3) = 0, \quad 1 \leq i \leq 4, \]

\[ \omega_2^4 (X_5^1, X_5^3) = 0, \quad \omega_2^4 (X_6^1, X_6^3) = 0, \quad i \neq 4, \]

\[ \omega_2^4 (X_7^1, X_7^3) = -ik_3(k_1 + l_1) \exp \left( i \left[ (k_1 + l_1)x^1 + k_3x^3 \right] \right), \]

\[ \omega_2^4 (X_8^1, X_8^3) = -\frac{i(k_1 + l_1)}{2} \exp \left( i \left[ (k_1 + l_1)x^1 \right] \right), \]

\[ \omega_2^4 (X_9^1, X_9^3) = 0, \quad \omega_2^4 (X_1^1, X_1^3) = 0, \quad \omega_2^4 (X_2^1, X_2^3) = 0, \quad \omega_2^4 (X_3^1, X_3^3) = 0, \quad \omega_2^4 (X_4^1, X_4^3) = 0, \quad \omega_2^4 (X_5^1, X_5^3) = 0, \quad \omega_2^4 (X_6^1, X_6^3) = 0, \quad \omega_2^4 (X_7^1, X_7^3) = 0, \quad \omega_2^4 (X_8^1, X_8^3) = 0, \quad \omega_2^4 (X_9^1, X_9^3) = 0, \]

\[ \omega_2^4 (X_1^2, X_1^4) = c_2 \exp \left( i \left[ (k_1 + l_1)x^1 + (k_3 + l_3)x^3 + k_4x^4 \right] \right), \]

\[ c_2 = \frac{1}{2} \frac{(k_1)^2 + k_1l_1 - k_3l_3)((l_1)^2 - (l_3)^2) + ((k_3)^2 + (k_4)^2)((l_1)^2 + (l_4)^2)}{k_4(k_3)^2}, \]
\[ \omega_2^2(X^k, X^l_1) = c_{31} \exp\left( i \left[ (k_1 + l_1) x^1 + (k_3 + l_3) x^3 \right] \right), \]
\[ c_{31} = \frac{i}{2} \left( k_3 l_1 - k_1 l_3 \right) \left[ \left( l_1^2 - l_2^2 \right)^2 + \left( k_1^2 + k_3^2 \right) \left( l_1^2 + l_3^2 \right) \right] \]
\[ \omega_2^2(X^k, X^l_2) = 2i l_1 \exp\left( i \left[ (k_1 + l_1) x^1 + l_3 x^3 + l_4 x^4 \right] \right), \]
\[ \omega_2^2(X^k, X^l_3) = c_{62} \exp\left( i \left[ (k_1 + l_1) x^1 + (k_3 + l_3) x^3 + l_4 x^4 \right] \right), \]
\[ c_{62} = -\frac{i}{4} \left( k_3 l_1 - (k_1 l_3 + k_3 l_1 + k_4 l_2) \right). \]
\[ \omega_3^2(X^k_1, X^l_1) = c_{63} \exp\left( i \left[ (k_1 + l_1) x^1 + (k_3 + l_3) x^3 \right] \right), \]
\[ c_{63} = -\frac{i}{4} \left( k_3 l_1 - (k_1 l_3 - 2l_2 l_4 + k_3 l_2 + k_4 l_1) \right). \]
\[ \omega_3^2(X^k_1, X^l_2) = i(k_1 + l_1) \exp\left( i \left[ (k_1 + l_1) x^1 + l_3 x^3 \right] \right), \]
\[ \omega_3^2(X^k_1, X^l_3) = -\frac{i}{2} \left( k_4 l_1 - (k_1 l_3 - 2l_2 l_4 + k_3 l_2 + k_4 l_1) \right). \]
\[ \omega_4^2(X^k, X^l_1) = c_{41} \exp\left( i \left[ (k_1 + l_1) x^1 + k_2 x^2 + (k_3 + l_3) x^3 + k_4 x^4 \right] \right), \]
\[ c_{41} = \frac{-i}{2} \left( k_3 l_1 + k_4 l_2 - 2l_1 l_2 \right). \]
\[ \omega_4^2(X^k, X^l_2) = -\frac{i}{4} \left( k_4 l_1 - (k_1 l_3 - 2l_2 l_4 + k_3 l_2 + k_4 l_1) \right). \]
\[ \omega_4^2(X^k, X^l_3) = -\frac{i}{2} \left( k_4 l_1 - (k_1 l_3 - 2l_2 l_4 + k_3 l_2 + k_4 l_1) \right). \]
\[ \omega_4^2(X^k_1, X^l_2) = -\frac{i}{4} \left( k_4 l_1 - (k_1 l_3 - 2l_2 l_4 + k_3 l_2 + k_4 l_1) \right). \]
\[ \omega_4^2(X^k_1, X^l_3) = -\frac{i}{2} \left( k_4 l_1 - (k_1 l_3 - 2l_2 l_4 + k_3 l_2 + k_4 l_1) \right). \]
\[ \omega_4^2(X^k, X^l_4) = 0. \]
\[ \omega_i^j (X_k^j, X_l^j) = c_{ij} \exp \left( i \left[ (k_1 + l_1) x^1 + (k_2 + l_2) x^2 + (k_3 + l_3) x^3 + (k_4 + l_4) x^4 \right] \right), \]
\[ 1 \leq i \leq 3, \]
\[ \omega_i^j (X_k^j, X_l^j) = 0, \quad 1 \leq i \leq 3, \]
\[ \omega_i^j (X_k^j, X_l^j) = \frac{i}{4} \frac{(l_1)^3 + k_3 (l_3)^2 - l_3 (l_5)^2}{(l_5)^2} \exp \left( i \left[ (k_1 + l_1) x^1 + l_3 x^3 \right] \right), \]
\[ \omega_i^j (X_k^j, X_l^j) = \frac{i}{4} \frac{(k_3)^2 - (l_3)^2}{(l_5)^2} \exp \left( i \left[ (k_1 + l_1) x^1 + (k_3 + l_3) x^3 \right] \right), \]
\[ 1 \leq i \leq 4, \]
\[ c_{ij}^1 = \frac{i}{4} (k_1 l_1 + k_3 - l_3), \]
\[ c_{ij}^2 = \frac{i}{4} \frac{k_1 (l_1)^2 + k_3 (l_3)^2 - 2k_1 l_1 + k_3 (l_3)^2 + k_3 (l_5)^2}{(l_5)^2}, \]
\[ c_{ij}^3 = \frac{i}{4} \frac{k_3 l_1 - (k_3)^2}{(l_5)^2}, \]
\[ c_{ij}^4 = \frac{i}{4} \frac{k_3 l_1 - (k_3)^2}{(l_5)^2}, \]
\[ \omega_i^j (X_k^j, X_l^j) = i \left[ (k_3)^2 - (l_3)^2 \right] \exp \left( i \left[ (k_1 + l_1) x^1 + (k_3 + l_3) x^3 \right] \right), \]
\[ \omega_i^j (X_k^j, X_l^j) = 0, \]
\[ \omega_i^j (X_k^j, X_l^j) = -i (k_1 + l_1) (k_3 - l_3) \exp \left( i \left[ (k_1 + l_1) x^1 + (k_3 + l_3) x^3 \right] \right), \]
\[ \omega_i^j (X_k^j, X_l^j) = 0, \]
\[ \omega_i^j (X_k^j, X_l^j) = -\frac{i}{4} l_2 \exp \left( i \left[ (k_1 + l_1) x^1 + l_3 x^3 + l_4 x^4 \right] \right), \]
\[ \omega_i^j (X_k^j, X_l^j) = \frac{i (k_1 + l_1)}{4} \exp \left( i \left[ (k_1 + l_1) x^1 + l_2 x^2 + l_3 x^3 + l_4 x^4 \right] \right), \]
\[ \omega_i^j (X_k^j, X_l^j) = 0, \]
\[ \omega_i^j (X_k^j, X_l^j) = 0, \]
\[ \omega_i^j (X_k^j, X_l^j) = c_{ij}^4 \exp \left( i \left[ (k_1 + l_1) x^1 + l_2 x^2 + l_3 x^3 + l_4 x^4 \right] \right), \]
\[ c_{ij}^j = \frac{i}{4}, \quad j = 1, 3, 4, \]
\[ c_{ij}^2 = -\frac{i (k_1 l_1 + (l_1)^2 + (l_3)^2 + (l_5)^2)}{4 l_5}. \]
From the previous formulas it follows the closed 3-form $\omega_2(X^k, X^l_i)$ is exact except in the following cases, where $[\omega_3]$ denotes the cohomology class of a closed 3-form $\omega_3$:

\[
\begin{align*}
[\omega_2(X^k, X^l_i)] &= ik_1 [v_4], & k_1 + l_1 = l_2 = l_3 = 0, \\
[\omega_2(X^k, X^l_i)] &= -\frac{i(k_3^2 - k_1)}{2} [v_2], & k_1 + l_1 = k_3 + l_3 = l_4 = 0, l_2 \neq 0, \\
[\omega_2(X^k, X^l_i)] &= \frac{i(2k_1 + k_3 l_3 - 2(k_3)^2)}{2} [v_3], & k_1 + l_1 = l_2 = l_4 = 0, \\
[\omega_2(X^k, X^l_i)] &= \frac{ik_1}{2} [v_4], & k_1 + l_1 = k_3 + l_3 = l_2 = 0, \\
[\omega_2(X^k, X^l_i)] &= \frac{ik_1}{2} [v_1], & l_2 = l_3 = l_4 = 0, \\
[\omega_2(X^k, X^l_i)] &= \frac{ik_1}{2} [v_3], & k_1 + l_1 = l_2 = l_4 = 0, \\
[\omega_2(X^k, X^l_i)] &= \frac{ik_1}{2} [v_4], & k_1 + l_1 = l_2 = l_3 = 0, \\
[\omega_2(X^k, X^l_i)] &= 8i\pi^3 \frac{(k_1)^2}{(k_3)^2} [v_2], & k_2 \neq 0, \\
[\omega_2(X^k, X^l_i)] &= 8\pi^3 c_4^1 [v_1], & k_1 + l_1 \neq 0, k_2 = k_4 = 0, \\
c_4^1 &= -\frac{4}{3} \frac{k_1(l_1)^2 + k_1(k_3)^2 - 2l_1(k_3)^2}{(k_3)^2}, \\
[\omega_2(X^k, X^l_i)] &= 8\pi^3 c_4^3 [v_3], & k_1 + l_1 = k_2 = k_4 = 0, \\
c_4^3 &= -\frac{4}{3} \frac{k_1(l_1)^2 + 2l_1(k_3)^2 l_1 + k_3(k_3)^2}{(k_3)^2}, \\
[\omega_2(X^k, X^l_i)] &= 8\pi^3 c_4^4 [v_4], & k_2 = 0, k_4 \neq 0, \\
c_4^4 &= -\frac{4}{3} \frac{k_1(l_1)^2 - (k_3)^2}{(k_3)^2}.
\end{align*}
\]

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