On graded stable derived categories of isolated Gorenstein quotient singularities

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Abstract
We show the existence of a full exceptional collection in the graded stable derived category of a Gorenstein isolated quotient singularity using a result of Orlov [Orl09]. We also show that the equivariant graded stable derived category of a Gorenstein Veronese subring of a polynomial ring with respect to an action of a finite group has a full strong exceptional collection, even if the corresponding quotient singularity is neither isolated nor Gorenstein.

1 Introduction

Let \( A = \bigoplus_{d=0}^{\infty} A_d \) be a \( \mathbb{N} \)-graded Noetherian ring over a field \( k \). The ring \( A \) is said to be connected if \( A_0 = k \). A connected ring \( A \) is Gorenstein with parameter \( a \) if \( A \) has finite injective dimension as a right module over itself and

\[
\mathbb{R}\text{Hom}_A(k, A) = k(a)[-n].
\]

Here \( \bullet(a) \) denotes the shift of \( \mathbb{Z} \)-grading and \( \bullet[-n] \) is the shift in the derived category. The graded stable derived category is the quotient category

\[
D^b_{\text{sing}}(\text{gr } A) = D^b(\text{gr } A)/D^b_{\text{perf}}(\text{gr } A)
\]

of the bounded derived category \( D^b(\text{gr } A) \) of finitely-generated \( \mathbb{Z} \)-graded right \( A \)-modules by the full triangulated subcategory \( D^b_{\text{perf}}(\text{gr } A) \) consisting of perfect complexes. Here, an object of \( D^b(\text{gr } A) \) is perfect if it is quasi-isomorphic to a bounded complex of projective modules. Stable derived categories are introduced by Buchweitz [Buc87] motivated by the theory of matrix factorizations by Eisenbud [Eis80]. Stable derived categories are also known as triangulated categories of singularities, introduced by Orlov [Orl04] based on an idea of Kontsevich to study B-branes on Landau-Ginzburg models.

Let \( R = k[x_1, \ldots, x_{n+1}] \) be a polynomial ring in \( n+1 \) variables over a field \( k \). We equip \( R \) with a \( \mathbb{Z} \)-grading such that \( \deg x_i = 1 \) for all \( i \). Let \( G \) be a finite subgroup of \( SL_{n+1}(k) \) whose order is not divisible by the characteristic of \( k \). Assume that the natural action of \( G \) on the affine space \( \mathbb{A}^{n+1} = \text{Spec } R \) is free outside of the origin. This assumption is equivalent to the condition that the invariant subring \( A = R^G \) has an isolated singularity at the origin [IY08 Corollary 8.2]. Two examples of the stable derived categories of \( A \) are studied by Iyama and Yoshino [IY08] and Keller, Murfet and Van den Bergh [KMVdB11]. The general case is studied by Iyama and Takahashi [IT].

Let \( d \in \mathbb{N} \) be a divisor of \( n+1 \) and \( B = \bigoplus_{i=0}^{\infty} A_{id} \) be the \( d \)-th Veronese subring of \( A \). We prove the following in this paper:
Theorem 1.1. The stable derived category $D^b_{\text{sing}}(\gr B)$ has a full exceptional collection.

The full exceptional collection given in Theorem 1.1 is strong when $d = n + 1$. One the other hand, a result of Iyama and Takahashi [IT, Theorem 1.7] gives a full strong exceptional collection for $d = 1$. The proof of Theorem 1.1 is based on the existence of a full strong exceptional collection in the derived category of coherent sheaves on the stack $\text{Proj} B = [(\text{Spec} B \setminus 0)/G_m]$ and a result of Orlov [Orl09, Theorem 2.5.(i)].

Next we discuss equivariant graded stable derived categories. Let $A$ be an $N$-graded connected Gorenstein ring with parameter $a > 0$ and $G$ be a finite group acting on $A$ whose order is not divisible by the characteristic of $k$. The crossed product algebra $A \rtimes G$ is the vector space $A \otimes k[G]$ equipped with the ring structure

$$(a_1 \otimes g_1) \cdot (a_2 \otimes g_2) = a_1 \cdot g_1(a_2) \otimes g_1 \circ g_2,$$

where $k[G]$ is the group ring of $G$. A right $A \rtimes G$-module is often called a $G$-equivariant $A$-module. The crossed product algebra $A \rtimes G$ inherits a grading from $A$ so that the degree zero part is given by the group ring; $(A \rtimes G)_0 = k[G]$. This graded ring is not connected if $G$ is non-trivial.

Let $\gr^G A$ be the abelian category of finitely-generated $\mathbb{Z}$-graded right $A \rtimes G$-modules and $\tor^G A$ be its Serre subcategory consisting of finite-dimensional modules. The quotient abelian category is denoted by $\qgr^G A = \gr^G A/\tor^G A$. If $A$ is commutative, then $\qgr^G A$ is equivalent to the abelian category $\coh^G(\text{Proj} A)$ of $G$-equivariant coherent sheaves of the stack $\text{Proj} A = [(\text{Spec} A \setminus 0)/G_m]$. Let $\text{Irrep}(G) = \{\rho_0, \ldots, \rho_r\}$ be the set of irreducible representations of $G$ where $\rho_0$ is the trivial representation. For any $k \in \mathbb{Z}$, the image of the graded $A \rtimes G$-module $A(k) \otimes \rho_i$ by the projection functor $\pi : \gr A \rtimes G \rightarrow qgr^G A$ will be denoted by $\mathcal{O}(k) \otimes \rho_i$. The following is a straightforward generalization of [Orl09, Theorem 2.5.(i)]:

**Theorem 1.2.** There is a full and faithful functor $\Phi : D^b_{\text{sing}}(\gr^G A) \rightarrow D^b(\qgr^G A)$ and a semiorthogonal decomposition

$$D^b(\qgr^G A) = \langle \mathcal{O} \otimes \rho_0, \ldots, \mathcal{O} \otimes \rho_r, \mathcal{O}(1) \otimes \rho_0, \ldots, \mathcal{O}(1) \otimes \rho_r, \ldots, \mathcal{O}(a - 1) \otimes \rho_0, \ldots, \mathcal{O}(a - 1) \otimes \rho_r, \Phi D^b_{\text{sing}}(\gr^G A) \rangle.$$

Let $R = k[x_1, \ldots, x_{n+1}]$ be a polynomial ring in $n + 1$ variables and $A = \bigoplus_{d=0}^{\infty} R_d$ be the $d$-th Veronese subring. We assume that $d$ is a divisor of $n + 1$ so that $A$ is Gorenstein with parameter $a = (n + 1)/d$. Let $G$ be any finite subgroup of $GL_{n+1}(k)$ whose order is not divisible by the characteristic of $k$. We have the following corollary of Theorem 1.2.

**Theorem 1.3.** The stable derived category $D^b_{\text{sing}}(\gr^G A)$ has a full strong exceptional collection.

The organization of this paper is as follows: In Section 2, we study $\text{Proj} B$ for the Veronese subring $B$ of the invariant ring and prove Theorems 1.1. We prove Theorem 1.2 in Section 3 which immediately gives Theorem 1.3. We discuss a few examples in Section 4.

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2 Invariant subrings

Let $\mathcal{D}$ be a triangulated category and $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. The right orthogonal to $\mathcal{N}$ is the full subcategory $\mathcal{N}^\perp \subset \mathcal{D}$ consisting of objects $M$ such that $\text{Hom}(N, M) = 0$ for any $N \in \mathcal{N}$. The left orthogonal to $\mathcal{N}$ is defined similarly by $\text{Hom}(M, N) = 0$ for any $N \in \mathcal{N}$. A full triangulated subcategory $\mathcal{N}$ of a triangulated category $\mathcal{D}$ is left admissible if any $X \in \mathcal{D}$ sits inside a distinguished triangle $N \to X \to M \to N$ such that $N \in \mathcal{N}$ and $M \in \mathcal{N}^\perp$. Right admissible subcategories are defined similarly. A sequence $(\mathcal{N}_1, \ldots, \mathcal{N}_n)$ of full triangulated subcategories is a weak semiorthogonal decomposition if there is a sequence $\mathcal{N}_1 = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \cdots \subset \mathcal{D}_n = \mathcal{D}$ of left admissible subcategories such that $\mathcal{N}_p$ is left orthogonal to $\mathcal{D}_{p-1}$ in $\mathcal{D}_p$. The decomposition is orthogonal if $\text{Hom}(N, M) = 0$ for any $N \in \mathcal{N}_i$ and $M \in \mathcal{N}_j$ with $i \neq j$.

Let $k$ be a field and $\mathcal{D}$ be a $k$-linear triangulated category. An object $E$ of $\mathcal{D}$ is exceptional if $\text{Hom}(E, E')$ is spanned by the identity morphism and $\text{Ext}^i(E, E) = 0$ for $i \neq 0$. A sequence $(E_1, \ldots, E_r)$ of exceptional objects is an exceptional collection if $\text{Ext}^i(E_j, E_\ell) = 0$ for any $i$ and any $1 \leq \ell < j \leq r$. An exceptional collection is strong if $\text{Ext}^i(E_j, E_\ell) = 0$ for any $i \neq 0$ and any $1 \leq j \leq \ell \leq r$. An exceptional collection is full if the smallest full triangulated subcategory of $\mathcal{D}$ containing it is the whole of $\mathcal{D}$.

Let $G$ be a finite subgroup of $\text{SL}_{n+1}(k)$ acting freely on $\mathbb{A}^{n+1} \setminus \{0\}$. We assume that the order of $G$ is not divisible by the characteristic of the base field $k$. The set of irreducible representations of $G$ will be denoted by $\text{Irrep}(G) = \{\rho_0, \ldots, \rho_r\}$ where $\rho_0$ is the trivial representation. Let further $R = k[x_1, \ldots, x_{n+1}]$ be the coordinate ring of $\mathbb{A}^{n+1}$ and $A = R^G$ be the invariant subring. Equip $R$ with the $\mathbb{N}$-grading such that $\deg x_i = 1$ for all $i = 1, \ldots, n+1$, which induces an $\mathbb{N}$-grading on $A$. This defines a $\mathbb{G}_m$-action on $\text{Spec} A$, and let

$$Y := \text{Proj} A = [(\text{Spec} A \setminus \{0\})/\mathbb{G}_m] = [(\text{Spec} R \setminus \{0\}/G)/\mathbb{G}_m]$$

be the quotient stack. The abelian category $\text{coh}^G \mathbb{P}^n$ of $G$-equivariant coherent sheaves on $\mathbb{P}^n$ is equivalent to the abelian category $\text{coh} Y$ of coherent sheaves on $Y$, which in turn is equivalent to the quotient category

$$\text{qgr} A = \text{gr} A/\text{tor} A$$

of the abelian category $\text{gr} A$ of finitely-generated $\mathbb{Z}$-graded $A$-modules by the Serre subcategory consisting of finite-dimensional modules [Orl09 Proposition 2.17]. Note that $G$-action on $\mathbb{P}^n$ may not be free.

The following theorem is due to Beilinson:

**Theorem 2.1** (Beilinson [Be˘ı78]). $D^b \text{coh} \mathbb{P}^n$ has a full strong exceptional collection

$$(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n))$$

consisting of line bundles.

As an immediate corollary to Theorem 2.1, we have the following:

**Corollary 2.2.** $D^b \text{coh}^G \mathbb{P}^n$ has a full strong exceptional collection

$$(\mathcal{O}_{\mathbb{P}^n} \otimes \rho_0, \ldots, \mathcal{O}_{\mathbb{P}^n} \otimes \rho_r, \mathcal{O}_{\mathbb{P}^n}(1) \otimes \rho_0, \ldots, \mathcal{O}_{\mathbb{P}^n}(1) \otimes \rho_r, \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_0, \ldots, \mathcal{O}_{\mathbb{P}^n}(n) \otimes \rho_r).$$
Let $d$ be a divisor of $n+1$ and $B = \bigoplus_{i \in \mathbb{Z}} A_{id}$ be the $d$-th Veronese subring of $A = R^G$. Let further $G_d = G/T_d$ be the quotient of $G$ by the diagonal subgroup $T_d = \{ \zeta \cdot \text{id}_{k^{n+1}} \in G \mid \zeta^d = 1 \}$ consisting of $d$-th roots of unity. Then $B$ is the invariant subring $(R^{(d)})^{G_d}$ of the $d$-th Veronese subring $R^{(d)}$ of $R = k[x_1, \ldots, x_{n+1}]$, and one has
\[ X := \text{Proj } B = \left[ (\text{Spec } R^{(d)}) \setminus \{ 0 \} / G_d \right]/\mathbb{G}_m. \]

The group $T_d$ is a cyclic group whose order $e$ is a divisor of $d$. If $T_d$ is non-trivial, then $G_d$ is not a subgroup of $SL_{n+1}(k)$ but a subgroup of its quotient $SL_{n+1}(k)/T_d$, and the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ does not have a $G_d$-linearization. On the other hand, the line bundle $\mathcal{O}_{\mathbb{P}^n}(e)$ does have a $G_d$-linearization and descends to a line bundle $\mathcal{O}_X(e)$ on $X$.

Recall that the root stack $\sqrt{\mathcal{L}/X}$ of a line bundle $\mathcal{L}$ on a stack $X$ is the stack whose object over $\varphi : T \to X$ is a line bundle $\mathcal{M}$ on $T$ together with an isomorphism $\mathcal{M}^\otimes e \cong \varphi^* \mathcal{L}$ [AGV08, Cad07]. The morphism $G \to G_d$ of finite groups induces a morphism $p : Y \to X$ of quotient stacks, and the isomorphism $\phi : \mathcal{O}_Y(1)^\otimes e \cong \pi^* \mathcal{O}_X(e)$ of line bundles gives an identification of $Y$ with the root stack $\sqrt{\mathcal{O}_X(e)/X}$. It follows that there is an orthogonal decomposition
\[ D^b \text{coh } Y = \langle p^* D^b \text{ coh } X, \mathcal{O}_Y(1) \otimes p^* D^b \text{ coh } X, \cdots, \mathcal{O}_Y(e-1) \otimes p^* D^b \text{ coh } X \rangle \quad (2.1) \]
of the derived category [IU Lemma 4.1].

The invariant ring $A$ is Gorenstein with parameter $\deg x_1 + \cdots + \deg x_{n+1} = n+1$ by Watanabe [Wat74, Theorem 1], and its Veronese subring $B$ is Gorenstein with parameter $a = (n+1)/d$ by Goto and Watanabe [GW78, Corollary 3.1.5]. The following theorem is due to Orlov:

**Theorem 2.3** ([Orl09 Theorem 2.5.(i)]). If $B$ is a Gorenstein ring with parameter $a > 0$, then there is a full and faithful functor $\Phi : D^b_{\text{sing }}(\text{gr } B) \to D^b(\text{qgr } B)$ and a semiorthogonal decomposition
\[ D^b(\text{qgr } B) = \langle \pi B, \ldots, \pi(B(a-1)), \Phi D^b_{\text{sing }}(\text{gr } B) \rangle, \]
where $\pi : \text{gr } B \to \text{qgr } B$ is the natural projection functor.

Now we prove Theorem [1.1] First consider the case $d = 1$. Recall that the right mutation of an exceptional collection is given by
\[ (E, F) \mapsto (F, R_F E) \]
where $R_F E$ is the mapping cone
\[ R_F E = \{ E \to \text{hom}(E, F)^\vee \otimes F \}. \]
See [Rud90] and references therein for more about mutations of exceptional collections. Write $E_{i,j} = \mathcal{O}_Y(i) \otimes \rho_j$ and perform successive right mutations

$$(E_{0,0}, \ldots, E_{0,r}, E_{1,0}, \ldots, E_{1,r}, \ldots, E_{n,0}, \ldots, E_{n,r})$$

$$\mapsto (E_{0,0}, \ldots, E_{0,r-1}, E_{1,0}, R_{E_1}, E_{0,r}, E_{1,1}, \ldots, E_{1,r}, \ldots, E_{n,0}, \ldots, E_{n,r})$$

$$\mapsto (E_{0,0}, \ldots, E_{0,r-2}, E_{1,0}, R_{E_1}, E_{0,r-1}, \ldots, R_{E_{1},0}, E_{1,1}, \ldots, E_{1,r}, \ldots, E_{n,0}, \ldots, E_{n,r})$$

$$\mapsto \ldots$$

$$\mapsto (E_{0,0}, E_{1,0}, R_{E_1}, E_{0,1}, \ldots, R_{E_{1},0}, E_{0,r}, E_{1,1}, \ldots, E_{1,r}, \ldots, E_{n,0}, \ldots, E_{n,r})$$

$$\mapsto \ldots$$

$$\mapsto (E_{0,0}, E_{1,0}, E_{2,0}, R_{E_2}, E_{1,0}, E_{2,1}, \ldots, E_{2,r}, \ldots, E_{n,0}, \ldots, E_{n,r})$$

$$R_{E_2} E_{1,1}, \ldots, R_{E_2} E_{1,r}, E_{2,1}, \ldots, E_{n,r})$$

$$\mapsto \ldots$$

$$\mapsto (E_{0,0}, E_{1,0}, \ldots, E_{n,0}, R_{E_{n,0}} \cdots R_{E_{1,0}}, E_{0,1}, \ldots, R_{E_{n,0}} \cdots R_{E_{1,0}}, E_{0,r}, E_{1,1}, \ldots, R_{E_{n,0}}, E_{1,r}, E_{2,1}, \ldots, E_{n,r})$$

$$= (E_{0,0}, E_{1,0}, \ldots, E_{n,0}, F_{0,1}, \ldots, F_{0,r}, \ldots, F_{n,1}, \ldots, F_{n,r})$$

where

$$F_{i,j} = R_{E_{n,0}} R_{E_{n-1,0}} \cdots R_{E_{1,0}} E_{i,j}.$$

Since $\pi(A(i)) = \mathcal{O}_Y(i) \otimes \rho_0 = E_{i,0}$ for any $i \in \mathbb{Z}$, it follows that $D^b_{\text{sing}}(\text{gr} A)$ is equivalent to the full triangulated subcategory of $D^b(\text{gr} A)$ generated by the exceptional collection

$$(F_{0,1}, \ldots, F_{0,r}, F_{1,1}, \ldots, F_{1,r}, \ldots, F_{n,1}, \ldots, F_{n,r}).$$

This proves Theorem 1.1 in the case $d = 1$.

Now we discuss the case $d > 1$. Since an exceptional is indecomposable and the decomposition in (2.1) is not only semiorthogonal but orthogonal, each exceptional object in the full strong exceptional collection $(E_{0,0}, \ldots, E_{n,r})$ on $Y$ belongs to one of orthogonal summands in (2.1). It follows that the exceptional collection in Corollary 2.2 is divided into $e$ copies of an exceptional collection, each of which is pulled-back from $X$ and tensored with $\mathcal{O}_Y(i)$ for $i = 0, \ldots, e - 1$. Let $(E_{i,j})_{(i,j) \in \Lambda}$ be the exceptional collection generating the summand $p^* D^b \text{coh} X$ in the orthogonal decomposition in (2.1). Since $e$ divides $d$, the collection $(\mathcal{O}_Y, \mathcal{O}_Y(d), \ldots, \mathcal{O}_Y((a-1)d)) = (p^* \mathcal{O}_X, p^* \mathcal{O}_X(d), \ldots, p^* \mathcal{O}_X((a-1)d))$ is a part of this collection. On the other hand, one has $\pi(B(i)) = \mathcal{O}_X(d_i)$ for any $i \in \mathbb{Z}$ since $B$ is the $d$-th Veronese subring. Now one can move these objects to the left by mutation, and Theorem 1.1 follows from Theorem 2.3 just as in the $d = 1$ case.

When $d = n + 1$, then $B$ is Gorenstein with parameter 1, and one does not need any mutation, so that $D^b_{\text{sing}}(\text{gr} B)$ has a full strong exceptional collection.

One can generalize the story to the case with arbitrary weights $\deg x_i = a_i$ and a finite subgroup $G \subset SL_{n+1}(k)$ with a free action on $\mathbb{A}^{n+1} \setminus 0$ commuting with the $\mathbb{G}_m$-action. The category $\text{gr} A$ is equivalent to the category of coherent sheaves on the weighted projective space $\mathbb{P}(a_1, \ldots, a_{n+1})$, the Beilinson collection is given by $(\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(a_1 + \cdots + a_{n+1}))$, and the Gorenstein parameter of the polynomial ring is $a = a_1 + \cdots + a_{n+1}$. The case $d = a_1 + \cdots + a_{n+1}$ and $G = 1$ is discussed in [Ued08].
Proof of Theorem 1.2. We need to show the existence of a full and faithful functor $\Phi : D^b_{\text{sing}}(\text{gr} A \times G) \to D^b(qgr A \times G)$ and a semiorthogonal decomposition

$$D^b(qgr A \times G) = \langle \mathcal{O} \otimes \rho_0, \ldots, \mathcal{O} \otimes \rho_r, \ldots, \mathcal{O}(a-1) \otimes \rho_0, \ldots, \mathcal{O}(a-1) \otimes \rho_r, \Phi D^b_{\text{sing}}(\text{gr} A \times G) \rangle.$$ 

Since $A$ is Gorenstein, $A$ has finite injective dimension as left and right module over itself. It follows that $A \times G$ also has finite injective dimension as left and right module over itself, and one has mutually inverse equivalences

$$D = \mathbb{R} \text{Hom}_{A \times G}(\bullet, A \times G) : D^b(\text{gr} G A)^\circ \to D^b(\text{gr} G A^\circ),$$

$$D^\circ = \mathbb{R} \text{Hom}_{(A \times G)^\circ}(\bullet, A \times G) : D^b(\text{gr} G A^\circ)^\circ \to D^b(\text{gr} G A).$$

of triangulated categories, where $\bullet^\circ$ denotes the opposite rings and categories.

For an integer $i$, let $\mathcal{S}_{< i}$ be the full subcategory of $D^b(\text{gr} G A)$ consisting of complexes of torsion modules concentrated in degrees less than $i$. In other words, it is the full triangulated subcategory of $D^b(\text{gr} G A)$ generated by $k(e) \otimes \rho$ for $e < -i$ and $\rho \in \text{Irrep}(G)$, where $k(e) \otimes \rho$ is the $e$-shift of the $A \times G$-module which is isomorphic to $\rho$ as a $G$-module and annihilated by $A_+ = \bigoplus_{i=1}^{\infty} A_i$. One can show just as in [Orl09, Lemma 2.3] that $\mathcal{S}_{< i}$ is left admissible in $D^b(\text{gr} G A)$ and the left orthogonal is the derived category $D^b(\text{gr} G A_{\geq i})$ of graded $G \times G$ modules $M$ such that $M_p = 0$ for any $p < i;

$$D^b(\text{gr} G A) = \langle \mathcal{S}_{< i}, D^b(\text{gr} G A_{\geq i}) \rangle. \tag{3.1}$$

Let further $\mathcal{P}_{< i}$ be the full subcategory of $D^b(\text{gr} G A)$ generated by projective modules $A(m) \otimes \rho$ for $m > -i$ and $\rho \in \text{Irrep}(G)$. One can also show

$$D^b(\text{gr} G A) = \langle D^b(\text{gr} G A_{\geq i}), \mathcal{P}_{< i} \rangle \tag{3.2}$$

just as in [Orl09, Lemma 2.3]. The proof of [Orl09, Lemma 2.4] carries over verbatim to the $G$-equivariant case, and gives weak semiorthogonal decompositions

$$D^b(\text{gr} G A_{\geq i}) = \langle \mathcal{D}_i, \mathcal{S}_{\geq i} \rangle, \tag{3.3}$$

$$D^b(\text{gr} G A_{\leq i}) = \langle \mathcal{P}_{\leq i}, \mathcal{T}_i \rangle \tag{3.4}$$

where $\mathcal{D}_i$ and $\mathcal{T}_i$ are equivalent to $D^b(qgr G A)$ and $D^b_{\text{sing}}(\text{gr} G A)$ respectively. (3.1) and (3.3) shows that $\mathcal{S}_{\geq i}$ is right admissible in $D^b(\text{gr} G A)$. The functor $D$ takes the subcategory $\mathcal{S}_{\geq i}(A)$ to the subcategory $\mathcal{S}_{< -i-a+1}(A^\circ)$, so that the right orthogonal $\mathcal{S}_{\leq i}(A)$ is sent to the left orthogonal $\mathcal{P}_{< -i-a+1}(A^\circ)$. The latter subcategory coincides with the right orthogonal $\mathcal{P}_{\leq -i-a+1}(A^\circ)$ by (3.1) and (3.2). The functor $D^\circ$ takes the right orthogonal $\mathcal{P}_{\leq -i-a+1}(A^\circ)$ to the left orthogonal $\mathcal{P}_{\geq i+a}(A)$, so that one has an equality

$$\mathcal{S}_{\geq i} = \mathcal{P}_{\geq i+a} \tag{3.5}$$

3 Crossed product algebras

Let $A$ be an $\mathbb{N}$-graded connected Gorenstein ring with parameter $a > 0$ and $G$ be a finite group acting on $A$. We assume that the characteristic of the base field $k$ does not divide the order of $G$. The set of irreducible representations of $G$ will be denoted by $\text{Irrep}(G) = \{\rho_0, \ldots, \rho_r\}$ where $\rho_0$ is the trivial representation.
of subcategories of $D^b(\text{gr}^G A)$. One has a weak semiorthogonal decomposition

$$D^b(\text{gr}^G A) = \langle S_{<i}, D_i, S_{\geq i} \rangle$$

by (3.1) and (3.3), which gives

$$D^b(\text{gr}^G A) = \langle P_{\geq i+a}, S_{\geq i}, D_i \rangle$$

by (3.5). Since Gorenstein parameter $a$ is positive, the subcategory $P_{\geq i+a}$ is not only right orthogonal but also left orthogonal to $S_{<i}$, and one obtains a weak semiorthogonal decomposition

$$D^b(\text{gr}^G A) = \langle S_{\geq i}, P_{\geq i+a}, D_i \rangle.$$  \hspace{1cm} (3.6)

On the other hand, (3.1) and (3.4) gives a weak semiorthogonal decomposition

$$D^b(\text{gr}^G A) = \langle S_{\geq i}, P_{\geq i}, T_i \rangle.$$ \hspace{1cm} (3.7)

By combining (3.6), (3.7) and

$$P_{\geq i} = \langle P_{\geq i+a}, A(-i-a+1) \otimes \rho_0, \ldots, A(-i-a+1) \otimes \rho_r, \ldots, A(-i) \otimes \rho_0, \ldots, A(-i) \otimes \rho_r \rangle,$$

one obtains

$$D_i = \langle A(-i-a+1) \otimes \rho_0, \ldots, A(-i-a+1) \otimes \rho_r, \ldots, A(-i) \otimes \rho_0, \ldots, A(-i) \otimes \rho_r, T_i \rangle,$$

and Theorem 1.2 follows by setting $i = -a + 1$. \hfill \Box

Let $A = \bigoplus_{i \in \mathbb{Z}} R_{id}$ be the $d$-th Veronese ring of $R = k[x_1, \ldots, x_{n+1}]$ for a divisor $d$ of $n+1$, and $G$ be a finite subgroup of $GL_{n+1}(k)$ whose order is not divisible by the characteristic of $k$. Theorem 1.3 is an immediate consequence of Theorem 1.2.

**Proof of Theorem 1.3.** The graded ring $A$ is Gorenstein with parameter $a = (n+1)/d$, and one has an equivalence

$$qgr^G A \cong \text{coh}^G \mathbb{P}^n$$

of abelian categories. The derived category $D^b(\text{coh}^G \mathbb{P}^n)$ has a full strong exceptional collection

$$(O_{\mathbb{P}^n} \otimes \rho_0, \ldots, O_{\mathbb{P}^n} \otimes \rho_r, O_{\mathbb{P}^n}(1) \otimes \rho_0, \ldots, O_{\mathbb{P}^n}(1) \otimes \rho_r, \ldots, O_{\mathbb{P}^n}(n) \otimes \rho_0, \ldots, O_{\mathbb{P}^n}(n) \otimes \rho_r).$$

Theorem 1.2 shows that the full subcategory of $D^b(\text{coh}^G \mathbb{P}^n)$ generated by

$$(O_{\mathbb{P}^n}(a) \otimes \rho_0, \ldots, O_{\mathbb{P}^n}(a) \otimes \rho_r, O_{\mathbb{P}^n}(a+1) \otimes \rho_0, \ldots, O_{\mathbb{P}^n}(a+1) \otimes \rho_r, \ldots, O_{\mathbb{P}^n}(n) \otimes \rho_0, \ldots, O_{\mathbb{P}^n}(n) \otimes \rho_r)$$

is equivalent to $D^b_{\text{sing}}(\text{gr}^G A)$, and Theorem 1.3 is proved. \hfill \Box
4 Examples

We discuss a few examples in this section. Let us first consider the case when \( G \subset SL_2(\mathbb{C}) \) is the binary dihedral group of type \( D_4 \). The invariant subring \( A = \mathbb{C}[x_1, x_2]^G \) is generated by three elements \( u, v \) and \( w \) of degrees 4, 8 and 10 satisfying \( u^5 + uv^2 + w^2 = 0 \). One has \( \text{Irrep}(G) = \{ \rho_0, \rho_1, \rho_2, \rho_3, \rho_4 \} \) and the quiver describing the total morphism algebra of the full strong exceptional collection \( (\mathcal{O} \otimes \rho_0, \ldots, \mathcal{O}(1) \otimes \rho_4) \) is given as follows:

\[
\begin{align*}
\mathcal{O} \otimes \rho_0 & \quad \mathcal{O} \otimes \rho_1 & \quad \mathcal{O} \otimes \rho_2 & \quad \mathcal{O} \otimes \rho_3 & \quad \mathcal{O} \otimes \rho_4 \\
\mathcal{O}(1) \otimes \rho_0 & \quad \mathcal{O}(1) \otimes \rho_1 & \quad \mathcal{O}(1) \otimes \rho_2 & \quad \mathcal{O}(1) \otimes \rho_3 & \quad \mathcal{O}(1) \otimes \rho_4
\end{align*}
\]

Since the Gorenstein parameter of \( A \) is two, we have to remove \( \mathcal{O} \otimes \rho_0 \) and \( \mathcal{O}(1) \otimes \rho_0 \) from the left. The object \( \mathcal{O} \otimes \rho_0 \) can be removed without any mutation, and when we remove \( \mathcal{O}(1) \otimes \rho_0 \), only \( \mathcal{O} \otimes \rho_2 \) will be affected, which will be turned into

\[ R_{\mathcal{O} \otimes \rho_2} \mathcal{O} \otimes \rho_0 = \{ \mathcal{O} \otimes \rho_2 \to \mathcal{O}(1) \otimes \rho_0 \}. \]

The resulting quiver is given as follows:

\[
\begin{align*}
\mathcal{O} \otimes \rho_1 & \quad R_{\mathcal{O} \otimes \rho_2} \mathcal{O}(1) \otimes \rho_0 & \quad \mathcal{O} \otimes \rho_3 & \quad \mathcal{O} \otimes \rho_4 \\
\mathcal{O}(1) \otimes \rho_1 & \quad \mathcal{O}(1) \otimes \rho_2 & \quad \mathcal{O}(1) \otimes \rho_3 & \quad \mathcal{O}(1) \otimes \rho_4
\end{align*}
\]

The resulting full exceptional collection is strong in this case, and the corresponding quiver is a disjoint union of two Dynkin quivers of type \( D_4 \).

Now let us take a Veronese subring of \( A \). Since the Gorenstein parameter of \( A \) is two, only the second Veronese subring \( B = \bigoplus_{i \in \mathbb{Z}} A_{2i} \) is Gorenstein, which has Gorenstein parameter one. Since \( A \) has no odd components, \( B \) is isomorphic to \( A \) as an algebra, and only the grading is changed. The stack \( \text{Proj} B = \left( \text{Spec} B \setminus 0 \right)/\mathbb{G}_m \) is a weighted projective line \( \mathbb{X}_{2,2,2} \) in the sense of Geigle and Lenzing [GL87] with three orbifold points of order 2, which is obtained from \( \text{Proj} A \) by the inverse root construction (i.e. by removing the generic stabilizer). It follows that \( D^b \text{qgr} A \) is equivalent to the direct sum of two copies of \( D^b \text{qgr} B \), and \( D^b \text{qgr} B \) is equivalent to the full subcategory of \( D^b \text{qgr} A \) generated by half of the full strong exceptional collection in \( D^b \text{qgr} A \) shown below:

\[
\begin{align*}
\mathcal{O} \otimes \rho_0 & \quad \mathcal{O} \otimes \rho_1 & \quad \mathcal{O} \otimes \rho_3 & \quad \mathcal{O} \otimes \rho_4 \\
\mathcal{O}(1) \otimes \rho_2 & \quad \mathcal{O}(1) \otimes \rho_2 & \quad \mathcal{O}(1) \otimes \rho_3 & \quad \mathcal{O}(1) \otimes \rho_4
\end{align*}
\]

Since the Gorenstein parameter of \( B \) is one, \( D^b_{\text{sing}}(\text{gr} B) \) is equivalent to the full subcategory of \( D^b(\text{qgr} B) \) generated by the exceptional collection obtained from the above
collection by removing $\mathcal{O} \otimes \rho$, which gives a Dynkin quiver of type $D_4$:

$$
\begin{array}{ccc}
\mathcal{O} \otimes \rho_1 & \mathcal{O} \otimes \rho_3 & \mathcal{O} \otimes \rho_4 \\
\downarrow & \downarrow & \downarrow \\
\mathcal{O}(1) \otimes \rho_2 & \\
\end{array}
$$

On the other hand, the crossed product algebra $R \rtimes G$ with $R = \mathbb{C}[x_1, x_2]$ is regular, so that $D^b_{\text{sing}}(\text{gr} R^{(2)})$ is zero. The graded stable derived category $D^b_{\text{sing}}(\text{gr} R^{(2)})$ of the second Veronese subring $R^{(2)} \rtimes G$ is equivalent to the full subcategory of $D^b(\text{gr} R^{(2)}) \cong D^b \text{coh}^G \mathbb{P}^1$ generated by the strong exceptional collection

$$(\mathcal{O}(1) \otimes \rho_0, \mathcal{O}(1) \otimes \rho_1, \ldots, \mathcal{O}(1) \otimes \rho_4)$$

by Theorem 1.3, which is just the direct sum of five copies of the derived category of finite-dimensional vector spaces.

Next we consider the case when $G = \langle \exp(2\pi \sqrt{-1}/3) \cdot \text{id}_{\mathfrak{a}^3} \rangle$ is a cyclic subgroup of $SL_3(k)$ of order three. The total morphism algebra of the full strong exceptional collection $(\mathcal{O} \otimes \rho_0, \ldots, \mathcal{O}(2) \otimes \rho_2)$ in $D^b \text{coh}^G \mathbb{P}^2$ is given as follows:

$$
\begin{array}{ccc}
\mathcal{O} \otimes \rho_0 & \mathcal{O}(1) \otimes \rho_1 & \mathcal{O}(2) \otimes \rho_2 \\
\mathcal{O} \otimes \rho_1 & \mathcal{O}(1) \otimes \rho_2 & \mathcal{O}(2) \otimes \rho_0 \\
\mathcal{O} \otimes \rho_2 & \mathcal{O}(1) \otimes \rho_0 & \mathcal{O}(2) \otimes \rho_1 \\
\end{array}
$$

Note that this is the disjoint union of three copies of the Beilinson quiver for $\mathbb{P}^2$. The full exceptional collection in $D^b_{\text{sing}}(\text{gr} A)$ is obtained from the above collection by removing $\mathcal{O} \otimes \rho_0$, $\mathcal{O}(1) \otimes \rho_0$ and $\mathcal{O}(2) \otimes \rho_0$. To remove the second and the third object, we can mutate the above collection as

$$
\begin{array}{ccc}
\mathcal{O} \otimes \rho_0 & \mathcal{O}(1) \otimes \rho_1 & \mathcal{O}(2) \otimes \rho_2 \\
\mathcal{O}(2) \otimes \rho_0 & \mathcal{O}(3) \otimes \rho_1 & \mathcal{O}(4) \otimes \rho_2 \\
\mathcal{O}(1) \otimes \rho_0 & \mathcal{O}(2) \otimes \rho_1 & \mathcal{O}(3) \otimes \rho_2 \\
\end{array}
$$

so that the three objects $\mathcal{O} \otimes \rho_0$, $\mathcal{O}(1) \otimes \rho_0$ and $\mathcal{O}(2) \otimes \rho_0$ can safely be removed from the left to obtain three copies of the generalized Kronecker quiver

$$
\bullet \equiv \equiv \equiv \bullet
$$
with three arrows. On the other hand, the third Veronese subring $B = \bigoplus_{i \in \mathbb{Z}} A_{3i}$ is Gorenstein with parameter one and satisfies $\text{Proj} B = \mathbb{P}^3$, so that $D^b_{\text{sing}}(\text{gr} B)$ is equivalent to the derived category of modules over the generalized Kronecker quiver with three arrows. These results are in complete agreement with the works of Iyama and Yoshino [IY08], Keller, Murfet and Van den Bergh [KMvdBI11], and Iyama and Takahashi [IT].

The stable derived category of $R \rtimes G$ for the above $G$ and $R = \mathbb{C}[x_1, x_2, x_3]$ is zero again, and that of its third Veronese subring $R^{(3)} \rtimes G$ is equivalent to the full subcategory of $D^b_{\text{qgr}} G^{(3)} \cong D^b_{\text{coh}} \mathbb{P}^2$ generated by the strong exceptional collection

$$(O(1) \otimes \rho_0, O(1) \otimes \rho_1, O(1) \otimes \rho_2, O(2) \otimes \rho_0, O(2) \otimes \rho_1, O(2) \otimes \rho_2)$$

which happens to be equivalent to $D^b_{\text{sing}}(\text{gr} A)$ above.

Theorem 1.2 can be useful also in other contexts. An integer $n \times n$ matrix $(a_{ij})_{i,j=1}^n$ defines a polynomial

$$W = \sum_{i=1}^n x_1^{a_{i1}} \cdots x_n^{a_{in}},$$

which is called invertible if the origin is an isolated singularity. They play a pivotal role in transposition mirror symmetry of Berglund and Hübsch [BH93], which attracts much attention recently. See e.g. [Kra] and references therein for more on invertible polynomials and mirror symmetry.

Any invertible polynomial is weighted homogeneous, and the choice of a weight is unique up to multiplication by a constant. The quotient ring $A = k[x_1, \ldots, x_n]/(W)$ is Gorenstein with parameter

$$a = \deg x_1 + \cdots + \deg x_n - \deg W.$$  

If $a$ is positive, then for any group $G$ of symmetries of $W$, one has a semiorthogonal decomposition in Theorem 1.2. One can also prove an analogue of [Orl09, Theorem 2.5.(ii),(iii)] for $a \leq 0$ just as in Theorem 1.2. A typical example is the case when $G$ is a subgroup of the group

$$G^{\text{max}} = \{(\alpha_1, \ldots, \alpha_n) \in (k^\times)^n \mid \alpha_1^{a_{11}} \cdots \alpha_n^{a_{1n}} = \cdots = \alpha_1^{a_{n1}} \cdots \alpha_n^{a_{nn}} = 1\}$$

of maximal diagonal symmetries of $W$, but one can also deal with other cases such as the action of the symmetric group $\mathfrak{S}_n$ on the Fermat polynomial $W = x_1^m + \cdots + x_n^m$.

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