EQUIVALENCE OF SPECTRAL PROJECTIONS IN SEMICLASSICAL LIMIT
AND A VANISHING THEOREM FOR HIGHER TRACES IN $K$-THEORY

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Abstract. In this paper, we study a refined $L^2$ version of the semiclassical approximation of projectively invariant elliptic operators with invariant Morse type potentials on covering spaces of compact manifolds. We work on the level of spectral projections (and not just their traces) and obtain an information about classes of these projections in $K$-theory in the semiclassical limit as the coupling constant $\mu$ goes to zero. An important corollary is a vanishing theorem for the higher traces in cyclic cohomology for the spectral projections. This result is then applied to the quantum Hall effect. We also give a new proof that there are arbitrarily many gaps in the spectrum of the operators under consideration in the semiclassical limit.

Introduction

Let $M$ be a closed Riemannian manifold and $\tilde{M}$ be its universal cover. Let $\omega$ be a closed 2-form on $M$ and $B$ be its lift to $\tilde{M}$, so that $B$ is a $\Gamma$-invariant closed 2-form on $\tilde{M}$ where $\Gamma$ denotes the fundamental group of $M$ acting on $\tilde{M}$ by the deck transformations. We assume that $B$ is exact. Choose a 1-form $A$ on $\tilde{M}$ such that $dA = B$. As in geometric quantization we may regard $A$ as defining a Hermitian connection $\nabla_A = d + iA$ on the trivial line bundle $L$ over $\tilde{M}$, whose curvature is $iB$. Physically we can think of $A$ as the electromagnetic vector potential for a magnetic field $B$. Suppose that $E$ is a Hermitian vector bundle on $M$ and $\tilde{E}$ the lift of $E$ to the universal cover $\tilde{M}$. Let $\tilde{\nabla}^E$ denote a $\Gamma$-invariant Hermitian connection on $\tilde{E}$. Then consider the Hermitian connection $\nabla = \tilde{\nabla}^E \otimes 1 + 1 \otimes \nabla_A$ on $\tilde{E} \otimes L = \tilde{E}$. It is no longer $\Gamma$-invariant, but it is invariant under a projective action of $\Gamma$, as will be explained below.

Using the Riemannian metric on $\tilde{M}$ and the Hermitian metric on $\tilde{E}$, consider the elliptic self-adjoint differential operator given by

$$H(\mu) = \mu \nabla^* \nabla + B + \mu^{-1}V,$$

where $B, V$ are $\Gamma$-invariant self-adjoint endomorphisms of the bundle $\tilde{E}$, $\mu$ is the coupling constant and where $V$ satisfies in addition the following Morse type condition:

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For all \( x \in \tilde{M} \), \( V(x) \geq 0 \). Moreover, if the matrix \( V(x_0) \) is degenerate for some \( x_0 \) in \( \tilde{M} \), then \( V(x_0) = 0 \) and there is a positive constant \( c \) such that \( V(x) \geq c|x - x_0|^2 I \) for all \( x \) in a neighborhood of \( x_0 \), where \( I \) denotes the identity endomorphism of \( \tilde{E} \).

We will also assume that \( V \) has at least one zero point. We remark that all functions \( V = |df|^2 \), where \( |df| \) denotes the pointwise norm of the differential of a \( \Gamma \)-invariant Morse function \( f \) on \( \tilde{M} \), are examples of Morse type potentials.

We will analyse the qualitative aspects of the spectrum of \( H(\mu) \) acting on the Hilbert space

\[
\mathcal{H} = L^2(\tilde{M}, \tilde{E}).
\]

An important feature of the elliptic operator \( H(\mu) \) is that it commutes with a projective \((\Gamma, \sigma)\)-action of the fundamental group \( \Gamma \). Here \( \sigma \) denotes the multiplier or \( U(1) \)-valued 2-cocycle on \( \Gamma \) defining this projective action.

Associated to \( H(\mu) \), there is a model operator \( K(\mu) \) (cf. section 2) which is obtained as a direct sum of quadratic parts of \( H(\mu) \) near the degenerate points of \( V \) in a fundamental domain. It acts on the Hilbert space

\[
\mathcal{H}_K = L^2(\mathbb{R}^n, C^k)^N,
\]

where \( n \) is the dimension of \( M \), \( C^k \) is the fibre of \( E \) and \( N \) denotes the number of zeroes of \( V \) that lie in a fundamental domain. The model operator \( K(\mu) \) is a Hilbert direct sum of harmonic oscillators. If we take a direct sum of all these harmonic oscillators over \( \tilde{M} \) (and not only in a fundamental domain) then we will get another version of the model operator which has the same spectrum and represents the Hamiltonian for the crystal obtained with perfectly isolated atoms.

Note that \( K(\mu) \) is obtained by a simple scaling from the operator \( K = K(1) \) and has a discrete spectrum independent of \( \mu \). Therefore the spacing of its eigenvalues is bounded below. Then we have the following

**Theorem 1** (Existence of spectral gaps). In the notation above, let \( V \) be a Morse type endomorphism of the vector bundle \( \tilde{E} \) over \( \tilde{M} \) and \( H(\mu) \) as in (1) be the elliptic self-adjoint operator acting on the Hilbert space \( \mathcal{H} \) as in (2). If \([a, b]\) is an interval in \( \mathbb{R} \) that does not intersect the spectrum of the model operator \( K \) acting on the Hilbert space \( \mathcal{H}_K \), cf. (3), then there exists \( \mu_0 > 0 \) such that for all \( \mu \in (0, \mu_0) \), the interval \([a, b]\) also does not intersect the spectrum of \( H(\mu) \). There exists arbitrarily large number of gaps in the spectrum of \( H(\mu) \) provided the coupling constant \( \mu \) is sufficiently small.

The physical explanation for the appearance of gaps in the spectrum \( H(\mu) \) is that the potential wells get deeper as \( \mu \to 0 \) and the atoms get (asymptotically) isolated, so that the energy levels of \( H(\mu) \) are approximated by those of the corresponding model operator \( K \).

The special case of Theorem 1 in the absence of a magnetic field was established in [Sh] using a different method, and the special case of this result in the presence of a magnetic field but in the scalar case was established in [MS] using the same method as in [Sh].
Theorem 2 below is a significant refinement of Theorem 1 above and will be established using a refinement of the $L^2$ semiclassical asymptotics used there. We first set some notation.

If $\mathcal{H}$ is a Hilbert space, then $\mathcal{K}(\mathcal{H})$ denotes the algebra of compact operators in $\mathcal{H}$, and $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Let $\mathcal{A}$ be a unital $*$-algebra with the unit $1_A$, and let $\text{Proj}(\mathcal{A})$ be its set of self-adjoint projections. Two projections $P, Q \in \text{Proj}(\mathcal{A})$ are said to be Murray-von Neumann equivalent if there is an element $V \in \mathcal{A}$ such that $P = V^*V$ and $Q =VV^*$. Denote $M_n(\mathbb{C}) \otimes \mathcal{A} = \mathcal{M}_n(\mathcal{A})$, where $M_n(\mathbb{C})$ denotes the square matrices of size $n$ over $\mathbb{C}$. Then $M_n(\mathcal{A})$ is also a $*$-algebra. Let $M_\infty(\mathcal{A}) = \lim_{n \to \infty} M_n(\mathcal{A})$ be the direct limit of the embeddings of $M_n(\mathcal{A})$ in $M_{n+1}(\mathcal{A})$ given by $A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Let $V(\mathcal{A}) = \text{Proj}(M_\infty(\mathcal{A}))/\sim$ denote the Murray-von Neumann equivalence classes of projections in $M_\infty(\mathcal{A})$. Then $V(\mathcal{A})$ is an Abelian semi-group under with the operation induced by the direct sum, and the associated Abelian group is called the Grothendieck group $\mathbb{K}_0(\mathcal{A})$.

The homomorphism $\pi : \mathbb{C} \to \mathcal{A}$ given by $\lambda \mapsto \lambda \cdot 1_A$ induces a homomorphism $\pi_* : \mathbb{K}_0(\mathbb{C}) \cong \mathbb{Z} \to \mathbb{K}_0(\mathcal{A})$. Then the reduced $K$-group $\tilde{\mathbb{K}}_0(\mathcal{A})$ is defined as $\tilde{\mathbb{K}}_0(\mathcal{A}) = \text{coker} \, \pi_* \cong \mathbb{K}_0(\mathcal{A})/\mathbb{Z}$.

Suppose that $\mathcal{A}$ is a non-unital $*$-algebra. Let $\mathcal{A} = \{(a, \lambda) : a \in \mathcal{A}, \lambda \in \mathbb{C}\}$. Then $\mathcal{A}$ is a unital $*$-algebra containing $\mathcal{A}$, with product given by $(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu)$. By definition, the $K$-group $\mathbb{K}_0(\mathcal{A})$ is the reduced $K$-group $\tilde{\mathbb{K}}_0(\mathcal{A})$ of $\mathcal{A}$.

Recall that the Morita invariance of $K$-theory asserts that there is a natural isomorphism $\mathbb{K}_0(\mathcal{A}) \cong \mathbb{K}_0(M_n(\mathcal{A}))$ which is induced by the standard algebra homomorphism $\mathcal{A} \to M_n(\mathcal{A})$ which maps $a \in \mathcal{A}$ to a matrix with the left-upper corner matrix element $a$, the rest matrix elements being $0$. If $\mathcal{A}$ is a $C^*$-algebra, then the Morita invariance of $K$-theory asserts that there is a natural isomorphism $\mathbb{K}_0(\mathcal{A}) \cong \mathbb{K}_0(\mathcal{A} \otimes \mathcal{K})$ which is induced by the similar algebra homomorphism map $\mathcal{A} \to \mathcal{A} \otimes \mathcal{K}$.

Finally, we denote by $C_r^*(\Gamma, \sigma)$ the reduced twisted group $C^*$-algebra of the group $\Gamma$. We will assume that the algebra $C_r^*(\Gamma, \sigma)$ acts on $\ell^2(\Gamma)$ by left twisted convolutions.

Theorem 2 (Semiclassical vanishing theorem in $K$-theory for spectral projections). In the notation above, let $V$ be a Morse type endomorphism of the vector bundle $\mathcal{E}$ over $M$ and $H(\mu)$ as in (1) be the elliptic self-adjoint operator acting on the Hilbert space $\mathcal{H}$ as in (2). Let $\lambda \in \mathbb{C}$ be such that $\lambda$ is not in the spectrum of the model operator $K$ acting on the Hilbert space $\mathcal{H}_K$, cf. (2). Let $E(\lambda) = \chi_{(-\infty, \lambda]}(H(\mu))$ and $E^0(\lambda) = \chi_{(-\infty, \lambda]}(K(\mu))$ denote the spectral projections.

(1) There exists a $(\Gamma, \sigma)$-equivariant isometry $U : \mathcal{H} \to \ell^2(\Gamma) \otimes \mathcal{H}_K$ (see section 2) and a constant $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, the spectral projections $UE(\lambda)U^*$ and $\text{id} \otimes E^0(\lambda)$ are in $C_r^*(\Gamma, \sigma) \otimes \mathcal{K}(\mathcal{H}_K)$ and are Murray-von Neumann equivalent in $C_r^*(\Gamma, \sigma) \otimes \mathcal{K}(\mathcal{H}_K)$. In particular,

\[ [UE(\lambda)U^*] = [\text{id} \otimes E^0(\lambda)] \in K_0(C_r^*(\Gamma, \sigma) \otimes \mathcal{K}(\mathcal{H}_K)) \cong K_0(C_r^*(\Gamma, \sigma)); \]

\[ [E(\lambda)] = 0 \in \tilde{K}_0(C_r^*(\Gamma, \sigma)). \]
There is a smooth subalgebra $\mathcal{B}(\Gamma, \sigma)$ of $C^*_r(\Gamma, \bar{\sigma}) \otimes K(H_K)$, cf. section 3, such that the spectral projections $UE(\lambda)U^*$ and $\text{id} \otimes E^0(\lambda)$ are in $\mathcal{B}(\Gamma, \sigma)$ and are also Murray-von Neumann equivalent in $\mathcal{B}(\Gamma, \sigma)$. That is, for all $\mu \in (0, \mu_0)$, one has

$$[UE(\lambda)U^*] = [\text{id} \otimes E^0(\lambda)] \in K_0(\mathcal{B}(\Gamma, \sigma)).$$

Let $\text{Tr}_\Gamma$ denote the trace on $C^*_r(\Gamma, \bar{\sigma}) \otimes K(H_K)$, which is the tensor product of the canonical finite trace $\text{tr}_\Gamma$ on $C^*_r(\Gamma, \bar{\sigma})$ and the standard trace $\text{Tr}$ on $K(H_K)$. As an immediate consequence of Theorem 2, we get the following

**Corollary 3** (Semiclassical asymptotics of the trace of spectral projections). In the notation of Theorem 2 one has,

$$\text{Tr}_\Gamma(UE(\lambda)U^*) = \text{rank}(E^0(\lambda)) \quad \text{for all} \quad \mu \in (0, \mu_0).$$

The following corollary uses in addition the Rapid Decay property (RD) for discrete groups. This property is related with the Haagerup inequality, which estimates the convolution norm in terms of the word lengths. Groups that are either virtually nilpotent or word hyperbolic have property (RD). For these groups, it is also known that every group cohomology class can be represented by a group cocycle $c \in Z^j(\Gamma, \mathbb{R})$ that is of polynomial growth, cf. [Gr].

**Corollary 4** (Semiclassical vanishing of the higher traces of spectral projections). Let $\Gamma$ be a discrete group that has property (RD). Let $c \in Z^j(\Gamma, \mathbb{R})$ (j even $> 0$) be a normalised group cocycle that is of polynomial growth, and $\tau_c$ the induced cyclic cocycle on the twisted group algebra $C(\Gamma, \bar{\sigma})$. Then the tensor product cocycle $\tau_c \# \text{Tr}$ extends continuously to $\mathcal{B}(\Gamma, \sigma)$, and in the notation of Theorem 2 one has, for all $\mu \in (0, \mu_0)$

$$\tau_c \# \text{Tr}(UE(\lambda)U^*, \ldots, UE(\lambda)U^*) = \tau_c \# \text{Tr}(\text{id} \otimes E^0(\lambda), \ldots, \text{id} \otimes E^0(\lambda)) = 0.$$
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It is quite possible that the technique developed in these papers can also be applied to the problems studied in this paper.

We believe, however, that our use of the polar decomposition and closed image technique to establish $C^*$-algebra equivalence of spectral projections is new and may lead to further important results. For instance, in [Kor04], the technique of this paper was applied to prove existence of arbitrarily large number of gaps in the spectrum of the magnetic Schrödinger operators $H^h = (ihd + A)^*(ihd + A)$ with the periodic magnetic field $B = dA$ on covering spaces of compact manifolds under some Morse type assumptions on $B$ in the semiclassical limit of strong magnetic field $h \to 0$.

We remark that as an application of Theorems 1 and 2, one obtains a new proof of the $L^2$-Morse inequalities, [NovSh, Sh]. We also obtain an application to the quantum Hall effect which we will now describe.

The Kubo formula for the Hall conductance both in the usual model of the integer quantum Hall effect on the Euclidean plane and in the model of the fractional quantum Hall effect on the hyperbolic plane can be naturally interpreted as a (densely defined) cyclic 2-trace $\text{tr}_K$ on the algebra of observables $\mathcal{B}(\Gamma, \sigma)$, [Bel, CHMM, MM]. The Hall conductance cocycle $\text{tr}_K$ can also be shown to be given by a quadratically bounded group cocycle. Moreover, it is well-known that $\mathbb{Z}^2$ and cocompact Fuchsian groups have property (RD). Therefore we have the following consequence of Corollary 4. When $\mathcal{E}$ is trivial with trivial connection, the endomorphism $B$ is zero, and the Morse type potential $V$ is a scalar valued function, we get the magnetic Schrödinger operator $H_{A,V}(\mu) = \mu^{-1}H(\mu) = \nabla^*\nabla + \mu^{-2}V$.

**Corollary 5** (Semiclassical vanishing of the Hall conductance on low energy bands). Let $\widetilde{M}$ be either the Euclidean plane $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}$, $V$ a Morse type potential. Let $\lambda \in \mathbb{R}$ be the Fermi level, $\lambda \notin \text{spec}(K)$. Let $P_\lambda = \chi_{(-\infty, \mu^{-1}\lambda]}(H_{A,V}(\mu))$ denote the spectral projection of the magnetic Schrödinger operator $H_{A,V}(\mu)$. Then there exists $\mu_0 > 0$ such that for all values of the coupling constant $\mu \in (0, \mu_0)$, the Hall conductance vanishes,

$$\sigma_\lambda = \text{tr}_K(P_\lambda, P_\lambda, P_\lambda) = 0.$$  

That is, the low energy bands do not contribute to the Hall conductance.

In the case of the Euclidean plane and when the magnetic field is uniform, this result was established by a different method in [Nak+Bel].

The physical explanation for this semiclassical vanishing theorem for the Hall conductance is as follows. The Hall conductance for the model operator vanishes, since it is the Hamiltonian of a crystal with perfectly isolated atoms as mentioned earlier, therefore there can be no current flowing through it, which remains valid for small perturbations of the model operator.

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1. Preliminaries

Let $M$ be a compact connected Riemannian manifold, $\Gamma$ be its fundamental group and $\tilde{M}$ be its universal cover, i.e. one has the principal bundle $\Gamma \to \tilde{M} \to M$. To make the paper self-contained, we include preliminary material, some of which may not be new, cf. [Bel], [BrSu], [CHMM], [MM], [Ma].

1.1. Projective action, or magnetic translations. Let $\omega$ be a closed real-valued 2-form on $M$ such that $B = p^* \omega$ is exact. So $B = dA$ where $A$ is a 1-form on $\tilde{M}$. We will assume without loss of generality that $A$ is real-valued. Define $\nabla_A = d + iA$. Then $\nabla_A$ is a Hermitian connection on the trivial line bundle over $\tilde{M}$ with the curvature $(\nabla_A)^2 = iB$. Suppose that $E$ is a Hermitian vector bundle on $M$ and $\tilde{E}$ the lift of $E$ to the universal cover $\tilde{M}$. The connection $\nabla_A$ defines a projective action of $\Gamma$ on $L^2$ sections of $\tilde{E}$ as follows.

Observe that since $B$ is $\Gamma$-invariant, one has $0 = \gamma^*B - B = d(\gamma^*A - A) \quad \forall \gamma \in \Gamma$. So $\gamma^*A - A$ is a closed 1-form on the simply connected manifold $\tilde{M}$, therefore

$$\gamma^*A - A = d\psi_\gamma, \quad \forall \gamma \in \Gamma,$$

where $\psi_\gamma$ is a smooth function on $\tilde{M}$. It is defined up to an additive constant, so we can assume in addition that it satisfies the following normalization condition:

- $\psi_\gamma(x_0) = 0$ for a fixed $x_0 \in \tilde{M}$, $\forall \gamma \in \Gamma$.

It follows that $\psi_\gamma$ is real-valued and $\psi_e(x) \equiv 0$, where $e$ denotes the neutral element of $\Gamma$. It is also easy to check that

- $\psi_\gamma(x) + \psi_{\gamma'}(\gamma x) - \psi_{\gamma'\gamma}(x)$ is independent of $x \in \tilde{M}$, $\forall \gamma, \gamma' \in \Gamma$.

Then $\sigma(\gamma, \gamma') = \exp(-i\psi_\gamma(\gamma' \cdot x_0))$ defines a multiplier on $\Gamma$ i.e. $\sigma : \Gamma \times \Gamma \to U(1)$ satisfies

- $\sigma(\gamma, e) = \sigma(e, \gamma) = 1, \quad \forall \gamma \in \Gamma$;
- $\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1 \gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3), \quad \forall \gamma_1, \gamma_2, \gamma_3 \in \Gamma$ (the cocycle relation).

It follows from these relations that $\sigma(\gamma, \gamma^{-1}) = \sigma(\gamma^{-1}, \gamma)$.

The complex conjugate multiplier $\bar{\sigma}(\gamma, \gamma') = \exp(i\psi_\gamma(\gamma' \cdot x_0))$ also satisfies the same relations.

For $u \in L^2(\tilde{M}, \tilde{E})$ and $\gamma \in \Gamma$ define

$$U_\gamma u = (\gamma^{-1})^*u, \quad S_\gamma u = \exp(-i\psi_\gamma)u.$$  

Then the operators $T_\gamma = U_\gamma \circ S_\gamma$ satisfy

$$T_e = id, \quad T_{\gamma_1}T_{\gamma_2} = \sigma(\gamma_1, \gamma_2)T_{\gamma_1\gamma_2},$$

for all $\gamma_1, \gamma_2 \in \Gamma$. In this case one says that the map $T : \Gamma \to U(L^2(\tilde{M}, \tilde{E})), \gamma \mapsto T_\gamma$, is a projective $(\Gamma, \sigma)$-unitary representation, where for any Hilbert space $\mathcal{H}$ we denote by $U(\mathcal{H})$ the group of all
unitary operators in $\mathcal{H}$. In other words one says that the map $\gamma \mapsto T_\gamma$ defines a $(\Gamma, \sigma)$-action in $\mathcal{H}$.

It is also easy to check that the adjoint operator to $T_\gamma$ in $L^2(\tilde{M}, \tilde{E})$ (with respect to a smooth $\Gamma$-invariant measure on $\tilde{M}$ and a $\Gamma$-invariant Hermitian structure on $\tilde{E}$) is

$$T_\gamma^* = \bar{\sigma}(\gamma, \gamma^{-1}) T_{\gamma^{-1}}.$$ 

The operators $T_\gamma$ are also called magnetic translations.

1.2. **Twisted group algebras.** Denote by $\ell^2(\Gamma)$ the standard Hilbert space of complex-valued $L^2$-functions on the discrete group $\Gamma$. We will use a left $(\Gamma, \bar{\sigma})$-action on $\ell^2(\Gamma)$ (or, equivalently, a $(\Gamma, \bar{\sigma})$-unitary representation in $\ell^2(\Gamma)$) which is given explicitly by

$$T_\gamma^L f(\gamma') = f(\gamma^{-1} \gamma') \bar{\sigma}(\gamma, \gamma^{-1} \gamma'), \quad \gamma, \gamma' \in \Gamma.$$ 

It is easy to see that this is indeed a $(\Gamma, \bar{\sigma})$-action, i.e.

$$T_e^L = \text{id} \quad \text{and} \quad T_{\gamma_1}^L T_{\gamma_2}^L = \bar{\sigma}(\gamma_1, \gamma_2) T_{\gamma_1 \gamma_2}^L, \quad \forall \gamma_1, \gamma_2 \in \Gamma.$$ 

Also

$$(T_\gamma^L)^* = \sigma(\gamma, \gamma^{-1}) T_{\gamma^{-1}}^L.$$ 

Let

$$\mathcal{A}^R(\Gamma, \sigma) = \left\{ A \in \mathcal{B}(\ell^2(\Gamma)) : [T^L_\gamma, A] = 0, \quad \forall \gamma \in \Gamma \right\}$$

be the commutant of the left $(\Gamma, \bar{\sigma})$-action on $\ell^2(\Gamma)$. Here by $\mathcal{B}(\mathcal{H})$ we denote the algebra of all bounded linear operators in a Hilbert space $\mathcal{H}$. By the general theory, $\mathcal{A}^R(\Gamma, \sigma)$ is a von Neumann algebra and is known as the *(right) twisted group von Neumann algebra*. It can also be realized as follows. Let us define the following operators in $\ell^2(\Gamma)$:

$$T_\gamma^R f(\gamma') = f(\gamma' \gamma) \sigma(\gamma', \gamma), \quad \gamma, \gamma' \in \Gamma.$$ 

It is easy to check that they form a right $(\Gamma, \sigma)$-action in $\ell^2(\Gamma)$ i.e.

$$T_e^R = \text{id} \quad \text{and} \quad T_{\gamma_1}^R T_{\gamma_2}^R = \sigma(\gamma_1, \gamma_2) T_{\gamma_1 \gamma_2}^R, \quad \forall \gamma_1, \gamma_2 \in \Gamma,$$

and also

$$(T_\gamma^R)^* = \bar{\sigma}(\gamma, \gamma^{-1}) T_{\gamma^{-1}}^R.$$ 

This action commutes with the left $(\Gamma, \bar{\sigma})$-action defined above i.e.

$$T_{\gamma'}^L T_\gamma^R = T_{\gamma'}^R T_\gamma^L, \quad \forall \gamma, \gamma' \in \Gamma.$$ 

It can be shown that the von Neumann algebra $\mathcal{A}^R(\Gamma, \sigma)$ is generated by the operators $\{T_\gamma^R\}_{\gamma \in \Gamma}$ (see e.g. a similar argument in [Sh2]).

Similarly we can introduce a von Neumann algebra

$$\mathcal{A}^L(\Gamma, \bar{\sigma}) = \left\{ A \in \mathcal{B}(\ell^2(\Gamma)) : [T_\gamma^R, A] = 0, \quad \forall \gamma \in \Gamma \right\}.$$ 

We will refer to it as *(left) twisted group von Neumann algebra*. It is generated by the operators $\{T_\gamma^L\}_{\gamma \in \Gamma}$, and it is the commutant of $\mathcal{A}^R(\Gamma, \sigma)$. 
Let us define a twisted group algebra $\mathbb{C}(\Gamma, \sigma)$ which consists of complex valued functions with finite support on $\Gamma$ and with the twisted convolution operation

$$(f * g)(\gamma) = \sum_{\gamma_1, \gamma_2: \gamma_1 \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) \sigma(\gamma_1, \gamma_2).$$

The basis of $\mathbb{C}(\Gamma, \sigma)$ as a vector space is formed by $\delta$-functions $\{\delta_\gamma \}_{\gamma \in \Gamma}$, $\delta_\gamma(\gamma') = 1$ if $\gamma = \gamma'$ and 0 otherwise. We have

$$\delta_{\gamma_1} \delta_{\gamma_2} = \sigma(\gamma_1, \gamma_2) \delta_{\gamma_1 \gamma_2}.$$ 

Associativity of this multiplication is equivalent to the cocycle condition.

Note also that the $\delta$-functions $\{\delta_\gamma \}_{\gamma \in \Gamma}$ form an orthonormal basis in $\ell^2(\Gamma)$. It is easy to check that

$$T^L_\gamma \delta_{\gamma'} = \bar{\sigma}(\gamma, \gamma') \delta_{\gamma \gamma'}, \quad T^R_\gamma \delta_{\gamma'} = \sigma(\gamma' \gamma^{-1}, \gamma) \delta_{\gamma' \gamma^{-1}}.$$ 

It is clear that the correspondences $\delta_\gamma \mapsto T^L_\gamma$ and $\delta_\gamma \mapsto T^R_\gamma$ define representations of $\mathbb{C}(\Gamma, \bar{\sigma})$ and $\mathbb{C}(\Gamma, \sigma)$ respectively. In both cases the weak closure of the image of the twisted group algebra coincides with the corresponding von Neumann algebra $(\mathcal{A}^L(\Gamma, \bar{\sigma})$ and $\mathcal{A}^R(\Gamma, \sigma)$ respectively). The corresponding norm closures are so called reduced twisted group $C^*$-algebras which are denoted $C^*_r(\Gamma, \bar{\sigma})$ and $C^*_r(\Gamma, \sigma)$ respectively.

The von Neumann algebras $\mathcal{A}^L(\Gamma, \bar{\sigma})$ and $\mathcal{A}^R(\Gamma, \sigma)$ can be described in terms of the matrix elements. For any $A \in \mathcal{B}(\ell^2(\Gamma))$ denote $A_{a,\beta} = (A_{\bar{\beta} \bar{\gamma}}, \delta_{a \beta})$ (which is a matrix element of $A$). Then repeating standard arguments (given in a similar situation e.g. in [Sh2]) we can prove that for any $A \in \mathcal{B}(\ell^2(\Gamma))$ the inclusion $A \in \mathcal{A}^R(\Gamma, \sigma)$ is equivalent to the relations

$$A_{x,\gamma y} = \bar{\sigma}(\gamma, x) \sigma(\gamma, y) A_{x, y}, \quad \forall x, y, \gamma \in \Gamma.$$ 

In particular, we have for any $A \in \mathcal{A}^R(\Gamma, \sigma)$

$$A_{x, \gamma y} = A_{x, y}, \quad \forall x, \gamma \in \Gamma.$$ 

Similarly, for any $A \in \mathcal{B}(\ell^2(\Gamma))$ the inclusion $A \in \mathcal{A}^L(\Gamma, \bar{\sigma})$ is equivalent to the relations

$$A_{x \gamma, y} = \bar{\sigma}(x, \gamma) \sigma(y, \gamma) A_{x, y}, \quad \forall x, y, \gamma \in \Gamma.$$ 

In particular, we have

$$A_{x \gamma, \gamma} = A_{x, \gamma}, \quad \forall x, \gamma \in \Gamma,$$

for any $A \in \mathcal{A}^L(\Gamma, \bar{\sigma})$.

A finite von Neumann trace $\text{tr}_{\Gamma, \bar{\sigma}} : \mathcal{A}^L(\Gamma, \bar{\sigma}) \to \mathbb{C}$ is defined by the formula

$$\text{tr}_{\Gamma, \bar{\sigma}} A = (A \delta_{\epsilon \gamma}, \delta_{\epsilon \gamma}).$$

We can also write $\text{tr}_{\Gamma, \bar{\sigma}} A = A_{\gamma, \gamma} = (A \delta_{\gamma}, \delta_{\gamma})$ for any $\gamma \in \Gamma$ because the right hand side does not depend of $\gamma$.

A finite von Neumann trace $\text{tr}_{\Gamma, \sigma} : \mathcal{A}^R(\Gamma, \sigma) \to \mathbb{C}$ is defined by the same formula, so we will denote by $\text{tr}_{\Gamma}$ any of these traces.

Let $\mathcal{H}$ denote an infinite dimensional complex Hilbert space. Then the Hilbert tensor product $\ell^2(\Gamma) \otimes \mathcal{H}$ is both $(\Gamma, \bar{\sigma})$-module and $(\Gamma, \sigma)$-module under the actions $\gamma \mapsto T^L_\gamma \otimes \text{id}$ and $\gamma \mapsto T^R_\gamma \otimes \text{id}$.
respectively. Let \( A^L_H(\Gamma, \overline{\sigma}) \) and \( A^R_H(\Gamma, \sigma) \) denote the von Neumann algebras in \( \ell^2(\Gamma) \otimes \mathcal{H} \) which are commutants of the \( (\Gamma, \sigma) \)- and \( (\Gamma, \overline{\sigma}) \)-actions respectively. Clearly \( A^L_H(\Gamma, \overline{\sigma}) \cong A^L(\Gamma, \sigma) \otimes \mathcal{B}(\mathcal{H}) \) and \( A^R_H(\Gamma, \sigma) \cong A^R(\Gamma, \sigma) \otimes \mathcal{B}(\mathcal{H}) \) in the usual sense of von Neumann algebra tensor products. Moreover, we have the following

**Lemma 1.1.** Any operator \( A \in A^L_H(\Gamma, \overline{\sigma}) \) can be represented as

\[
A = \sum_{\gamma \in \Gamma} T^L_{\gamma} \otimes A(\gamma),
\]

where \( A(\gamma) \in \mathcal{B}(\mathcal{H}) \), and the series in the right-hand side of this identity converges in the strong operator topology.

**Proof.** Let \( A \in A^L_H(\Gamma, \overline{\sigma}) \). Define a bounded operator \( A(\gamma) \) in \( \mathcal{H} \) by the formula

\[
A(\delta_e \otimes v) = \sum_{\gamma \in \Gamma} \delta_\gamma \otimes A(\gamma)v, \quad v \in \mathcal{H}.
\]

Take any \( x = \sum_{\gamma \in \Gamma} \delta_\gamma \otimes x_\gamma \in \ell^2(\Gamma) \otimes \mathcal{H} \). Then we have

\[
Ax = \sum_{\gamma_1} A(\delta_{\gamma_1} \otimes x_{\gamma_1}) = \sum_{\gamma_1} \sigma(\gamma_1, \gamma_1^{-1})^{-1} A(T^R_{\gamma_1} \otimes \text{id})(\delta_e \otimes x_{\gamma_1}) = \sum_{\gamma_1} \sigma(\gamma_1, \gamma_1^{-1})^{-1} (T^R_{\gamma_1} \otimes \text{id})A(\delta_e \otimes x_{\gamma_1}) = \sum_{\gamma_1} \sigma(\gamma_1, \gamma_1^{-1})^{-1} (T^R_{\gamma_1} \otimes \text{id}) \sum_{\gamma_2} \delta_{\gamma_2} \otimes A(\gamma_2)x_{\gamma_1} = \sum_{\gamma_1,\gamma_2} \sigma(\gamma_1, \gamma_1^{-1})^{-1} \sigma(\gamma_2\gamma_1, \gamma_1^{-1}) \delta_{\gamma_2\gamma_1} \otimes A(\gamma_2)x_{\gamma_1}.
\]

By the cocycle identity, we have \( \sigma(\gamma_2, \gamma_1)\sigma(\gamma_2\gamma_1, \gamma_1^{-1}) = \sigma(\gamma_2, e)\sigma(\gamma_1, \gamma_1^{-1}) \), that implies

\[
\sigma(\gamma_1, \gamma_1^{-1})^{-1} \sigma(\gamma_2\gamma_1, \gamma_1^{-1}) = \overline{\sigma}(\gamma_2, \gamma_1)
\]

and, finally, gives

\[
Ax = \sum_{\gamma_1,\gamma_2} \overline{\sigma}(\gamma_2, \gamma_1) \delta_{\gamma_2\gamma_1} \otimes A(\gamma_2)x_{\gamma_1} = \sum_{\gamma_1,\gamma_2} T^L_{\gamma_2} \delta_{\gamma_1} \otimes A(\gamma_2)x_{\gamma_1},
\]

that completes the proof. \( \square \)

Let us note the following elementary lemma.

**Lemma 1.2.** Let \( A \in A^L_H(\Gamma, \overline{\sigma}) \), \( A = \sum_{\gamma \in \Gamma} T^L_{\gamma} \otimes A(\gamma) \), where \( A(\gamma) \in \mathcal{B}(\mathcal{H}) \). Then

\[
\sup_{\gamma \in \Gamma} \|A(\gamma)\| \leq \|A\| \leq \sum_{\gamma \in \Gamma} \|A(\gamma)\|,
\]

where the righthand side of the inequality is not necessarily finite.
Define the semifinite tensor product trace $\text{Tr}_T = \text{tr}_T \otimes \text{Tr}$ on each of the algebras $A^L_{\tilde{\mathcal{H}}}(\Gamma, \bar{\sigma})$ and $A^L_{\tilde{\mathcal{H}}}(\Gamma, \sigma)$. Here $\text{Tr}$ denotes the standard (semi-finite) trace on $\mathcal{B}(\mathcal{H})$.

The $C^*$-tensor product $C^*_r(\Gamma, \bar{\sigma}) \otimes \mathcal{K}(\mathcal{H})$ is the norm closure of the algebraic tensor product $C^*_r(\Gamma, \bar{\sigma}) \otimes \mathcal{K}(\mathcal{H}) \subset A^L_{\tilde{\mathcal{H}}}(\Gamma, \bar{\sigma}) \cong A^L(\Gamma, \bar{\sigma}) \otimes \mathcal{B}(\mathcal{H})$ in $\mathcal{B}(\ell^2(\Gamma) \otimes \mathcal{H})$. One can give the following sufficient conditions for an operator $A \in A^L_{\tilde{\mathcal{H}}}(\Gamma, \bar{\sigma})$ to belong to the algebra $C^*_r(\Gamma, \bar{\sigma}) \otimes \mathcal{K}(\mathcal{H})$.

**Lemma 1.3.** If $A \in A^L_{\tilde{\mathcal{H}}}(\Gamma, \sigma)$, $A = \sum_{\gamma \in \Gamma} T^L_{\gamma} \otimes A(\gamma)$ is such that $A(\gamma) \in \mathcal{K}(\mathcal{H})$ and also satisfies

$$\sum_\gamma \|A(\gamma)\| < \infty,$$

then $A \in C^*_r(\Gamma, \bar{\sigma}) \otimes \mathcal{K}(\mathcal{H})$ and $\|A\| \leq \sum_\gamma \|A(\gamma)\|$.

**Proof.** Let $K_1 \subset K_2 \subset \cdots$ be a sequence of finite subsets of $\Gamma$ which is an exhaustion of $\Gamma$, i.e.

$$\bigcup_{j \geq 1} K_j = \Gamma.$$  

For all $j \in \mathbb{N}$, define $A_j \in A^L_{\tilde{\mathcal{H}}}(\Gamma, \bar{\sigma})$ as $A_j = \sum_{\gamma \in \Gamma} T^L_{\gamma} \otimes A_j(\gamma)$, where

$$A_j(\gamma) = \begin{cases} A(\gamma) & \text{if } \gamma \in K_j; \\ 0 & \text{otherwise}. \end{cases}$$

Then in fact $A_j \in C(\Gamma, \bar{\sigma}) \otimes \mathcal{K}(\mathcal{H})$ by definition. Using Lemma 1.2, we have

$$\|A - A_j\| \leq \sum_\gamma \|A - A_j(\gamma)\| = \sum_\gamma \|A(\gamma) - A_j(\gamma)\| = \sum_{\gamma \in \Gamma \setminus K_j} \|A(\gamma)\|.$$  

By hypothesis, $\sum_\gamma \|A(\gamma)\| < \infty$, therefore $\sum_{\gamma \in \Gamma \setminus K_j} \|A(\gamma)\| \to 0$ as $j \to \infty$, since $K_j$ is an increasing exhaustion of $\Gamma$. This proves that $A \in C^*_r(\Gamma, \bar{\sigma}) \otimes \mathcal{K}(\mathcal{H})$. \qed

1.3. Projectively invariant elliptic operators. As before, let $M$ be a closed Riemannian manifold and $\tilde{M}$ be its universal cover. Let $\mathcal{E}$ be a Hermitian vector bundle on $M$ and $\tilde{\mathcal{E}}$ the lift of $\mathcal{E}$ to the universal cover $\tilde{M}$. Let $\nabla^{\tilde{\mathcal{E}}}$ denote a $\Gamma$-invariant Hermitian connection on $\tilde{\mathcal{E}}$. Then consider the Hermitian connection $\nabla = \nabla^{\tilde{\mathcal{E}}} \otimes \text{id} + \text{id} \otimes \nabla_\mathcal{A}$ on $\tilde{\mathcal{E}} \otimes \mathcal{L} = \tilde{\mathcal{E}}$, where $\mathcal{L}$ is the trivial line bundle on $\tilde{M}$, $\nabla_\mathcal{A} = d + i\mathcal{A}$.

Using the Riemannian metric on $\tilde{M}$ and the Hermitian metric on $\tilde{\mathcal{E}}$, consider the elliptic self-adjoint differential operator given by

$$(6) \quad H(\mu) = \mu \nabla^* \nabla + B + \mu^{-1} V,$$

where $B, V$ are $\Gamma$-invariant self-adjoint endomorphisms of the bundle $\tilde{\mathcal{E}}$, $\mu$ is the coupling constant and where $V$ satisfies in addition the Morse type condition. Then $H(\mu)$ acts on $L^2(\tilde{M}, \tilde{\mathcal{E}})$ and is a self-adjoint second order elliptic differential operator. It commutes with the magnetic translations $T_\gamma$ (for all $\gamma \in \Gamma$), i.e. with the $(\Gamma, \sigma)$-action which was defined above. To see this note first that the operators $U_\gamma = (\gamma^{-1})^* \otimes S_\gamma$ (the multiplication by $\exp(-i\psi_\gamma)$) are defined not only on sections of $\tilde{\mathcal{E}}$ but also on $\tilde{\mathcal{E}}$-valued 1-forms (and actually on $\tilde{\mathcal{E}}$-valued $p$-forms for any $p \geq 0$) on
Proof. Given \( \phi \in L^2(\tilde{M}, E) \), we compute

\begin{align*}
\mathbf{U}(\phi) &= \sum_{\gamma \in \Gamma} \delta_\gamma \otimes i^* T_\gamma \phi, \quad \phi \in L^2(\tilde{M}, E).
\end{align*}

\textbf{Lemma 1.4.} The map \( \mathbf{U} : L^2(\tilde{M}, E) \to \ell^2(\Gamma) \otimes L^2(F, \tilde{E}|_F) \) defined above in (8) is a \((\Gamma, \sigma)\)-equivariant unitary operator, where the \((\Gamma, \sigma)\)-action on \( \ell^2(\Gamma) \otimes L^2(F, \tilde{E}|_F) \) is given by the operators \( T_\gamma^R \otimes \text{id} \).

\textbf{Proof.}
\[ \mathbf{U}(T_{\gamma}\phi) = \sum_{\gamma' \in \Gamma} \delta_{\gamma'} \otimes i^*(T_{\gamma'}T_{\gamma}\phi) = \sum_{\gamma' \in \Gamma} \sigma(\gamma', \gamma) \delta_{\gamma'} \otimes i^*(T_{\gamma'\gamma}\phi) \]
\[ = \sum_{\gamma' \in \Gamma} \sigma(\gamma'\gamma^{-1}, \gamma) \delta_{\gamma'} \otimes i^*(T_{\gamma'}\phi) = (T^R_{\gamma} \otimes \text{id}) \mathbf{U}\phi, \]

which proves that \( \mathbf{U} \) is a \( (\Gamma, \sigma) \)-equivariant map. It is straightforward to check that the operator \( \mathbf{U} \) is unitary. \( \square \)

Since \( \mathcal{U}_{\mathcal{D}}(\Gamma, \bar{\sigma}) \) is the commutant of \( \{T_{\gamma}\gamma \in \Gamma\} \), and \( \mathcal{A}_H^L(\Gamma, \bar{\sigma}) \) is the commutant of \( \{T^R_{\gamma} \otimes \text{id}\gamma \in \Gamma\} \), we see that \( \mathbf{U} \) induces an isomorphism of von Neumann algebras \( \mathcal{U}_{\mathcal{D}}(\Gamma, \bar{\sigma}) \) and \( \mathcal{A}_H^L(\Gamma, \bar{\sigma}) \). Therefore we can transfer the trace \( \text{Tr}_{\Gamma} \) from \( \mathcal{A}_H^L(\Gamma, \bar{\sigma}) \) to \( \mathcal{U}_{\mathcal{D}}(\Gamma, \bar{\sigma}) \). The result will be a semifinite \( \Gamma \)-trace on \( \mathcal{U}_{\mathcal{D}}(\Gamma, \bar{\sigma}) \) which we will still denote \( \text{Tr}_{\Gamma} \).

Note that the trace \( \text{Tr}_{\Gamma} \) is faithful. This follows by well known arguments from the theory of von Neumann algebras, which are reproduced for instance, in [Ta] on pages 316-317, in the proof of Proposition V.2.14. The result in [Ta] is directly applicable to the case of the trivial multiplier only, but the arguments easily work in the general case. In the case of non-trivial multiplier the trace was briefly considered by Br"uning and Sunada [BrSu], who also observed the faithfulness of the trace.

It is easy to check that for any \( \mathcal{Q} \in \mathcal{U}_{\mathcal{D}}(\Gamma, \bar{\sigma}) \) with a finite \( \Gamma \)-trace and a continuous Schwartz kernel \( k_{\mathcal{Q}} \) we have

\[ \text{Tr}_{\Gamma} \mathcal{Q} = \int \text{tr} k_{\mathcal{Q}}(x, x) dx \]

where \( dx \) denotes the \( \Gamma \)-invariant measure and \( \text{tr} \) the pointwise trace. An important particular case is a spectral projection \( E(\lambda) \) of the elliptic self-adjoint operator \( H(\mu) \), which has a smooth Schwartz kernel and so a finite \( \Gamma \)-trace. Therefore we can define a spectral density function

\[ N_{\Gamma}(\lambda; H) = \text{Tr}_{\Gamma} E(\lambda), \]

which is finite for all \( \lambda \in \mathbb{R} \). It is easy to see that \( \lambda \mapsto N_{\Gamma}(\lambda; H(\mu)) \) is a non-decreasing function, and the spectrum of \( H(\mu) \) can be reconstructed as the set of its points of growth, i.e.

\[ \text{spec}(H(\mu)) = \{\lambda \in \mathbb{R} : N_{\Gamma}(\lambda + \varepsilon; H(\mu)) - N_{\Gamma}(\lambda - \varepsilon; H(\mu)) > 0, \forall \varepsilon > 0\}. \]

### 2. Refined semiclassical approximation principle and existence of spectral gaps

The main goal of this section is to prove Theorem 1 and the first part of Theorem 2. We start with an abstract operator-theoretic setting, where similar results can be stated. Then these results are applied to projectively invariant elliptic operators with invariant Morse type potentials on covering spaces of compact manifolds.
2.1. **General results on equivalence of projections and existence of spectral gaps.** Let $\mathfrak{A}$ be a $C^*$-algebra, $\mathcal{H}$ a Hilbert space equipped with a faithful $*$-representation of $\mathfrak{A}$, $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$. For simplicity of notation, we will often identify the algebra $\mathfrak{A}$ with its image $\pi(\mathfrak{A})$.

Consider Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ equipped with inner products $(\cdot, \cdot)_1$ and $(\cdot, \cdot)_2$. Assume that there are given unitary operators $V_1 : \mathcal{H}_1 \to \mathcal{H}$ and $V_2 : \mathcal{H}_2 \to \mathcal{H}$. Using the unitary isomorphisms $V_1$ and $V_2$, we get representations $\pi_1$ and $\pi_2$ of $\mathfrak{A}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ accordingly, $\pi_l(a) = V_1^{-1} \circ \pi(a) \circ V_l$ for $l = 1, 2, a \in \mathfrak{A}$.

Consider (unbounded) self-adjoint operators $A_1$ in $\mathcal{H}_1$ and $A_2$ in $\mathcal{H}_2$ with the domains $\text{Dom}(A_1)$ and $\text{Dom}(A_2)$ respectively. We will assume that

- the operators $A_1$ and $A_2$ are semi-bounded from below:
  \begin{align}
  (A_1 u, u)_1 & \geq \lambda_{01} \|u\|_1^2, \quad u \in \text{Dom}(A_1), \\
  (A_2 u, u)_2 & \geq \lambda_{02} \|u\|_2^2, \quad u \in \text{Dom}(A_2),
  \end{align}
  \[(9)\]
  \[(10)\]

  with some $\lambda_{01}, \lambda_{02} \leq 0$;

- for any $t > 0$, the operators $e^{-tA_l}$, $l = 1, 2$, belong to $\pi_l(\mathfrak{A})$.

Let $\mathcal{H}_0$ be a Hilbert space, equipped with injective bounded linear maps $i_1 : \mathcal{H}_0 \to \mathcal{H}_1$ and $i_2 : \mathcal{H}_0 \to \mathcal{H}_2$. Assume that there are given bounded linear maps $p_1 : \mathcal{H}_1 \to \mathcal{H}_0$ and $p_2 : \mathcal{H}_2 \to \mathcal{H}_0$ such that $p_1 \circ i_1 = \text{id}_{\mathcal{H}_0}$ and $p_2 \circ i_2 = \text{id}_{\mathcal{H}_0}$. The whole picture can be represented by the following diagram (note that this diagram is not commutative).

Consider a self-adjoint bounded operator $J$ in $\mathcal{H}_0$. We assume that

- the operator $V_2 i_2 J p_1 V_1^{-1}$ belongs to the von Neumann algebra $\pi(\mathfrak{A})''$;
- $(i_2 J p_1)^* = i_1 J p_2$;
- for any $a \in \mathfrak{A}$, the operator $\pi(a) V_2 (i_2 J p_1) V_1^{-1}$ belongs to $\pi(\mathfrak{A})$. 

Since the operators $i_l : \mathcal{H}_0 \to \mathcal{H}_l$, $l = 1, 2$, are bounded and have bounded left-inverse operators $p_l$, they are topological monomorphisms, i.e. they have closed image and the maps $i_l : \mathcal{H}_0 \to \text{Im} i_l$ are topological isomorphisms. Therefore, we can assume that the estimate
\begin{equation}
\rho^{-1} \|i_2 Ju\|_2 \leq \|i_1 Ju\|_1 \leq \rho \|i_2 Ju\|_2, \quad u \in \mathcal{H}_0,
\end{equation}
holds with some $\rho > 1$ (depending on $J$).

Define the bounded operators $J_l$ in $\mathcal{H}_l$, $l = 1, 2$, by the formula $J_l = i_l J p_l$. We assume that

- the operator $J_l$, $l = 1, 2$, maps the domain of $A_l$ to itself;
- $J_l$ is self-adjoint, and $0 \leq J_l \leq \text{id}_{\mathcal{H}_l}$, $l = 1, 2$;
- for $u \in \mathcal{H}_0$, $i_l Ju \in \text{Dom}(A_1)$ iff $i_2 Ju \in \text{Dom}(A_2)$.

Denote $D = \{u \in \mathcal{H}_0 : i_1 Ju \in \text{Dom}(A_1)\} = \{u \in \mathcal{H}_0 : i_2 Ju \in \text{Dom}(A_2)\}$.

Introduce a self-adjoint positive bounded linear operator $J_l'$ in $\mathcal{H}_l$ by the formula $J_l^2 + J_l'^2 = \text{id}_{\mathcal{H}_l}$. We assume that

- the operator $J_l'$, $l = 1, 2$, maps the domain of $A_l$ to itself;
- the operators $[J_l, [J_l, A_l]]$ and $[J_l', [J_l', A_l]]$ extend to bounded operators in $\mathcal{H}_l$, and
\begin{equation}
\max(||[J_l, [J_l, A_l]]||_l, ||[J_l', [J_l', A_l]]||_l) \leq \gamma_l, \quad l = 1, 2.
\end{equation}

Finally, we assume that
\begin{equation}
(A_l J_l' u, J_l' u)_l \geq \alpha_l \|J_l' u\|^2_l, \quad u \in \text{Dom}(A_l), \quad l = 1, 2,
\end{equation}
for some $\alpha_l > 0$, and
\begin{align}
(A_2 i_2 Ju, i_2 Ju)_2 &\leq \beta_1 (A_1 i_1 Ju, i_1 Ju)_1 + \varepsilon_1 \|i_1 Ju\|^2_1, \quad u \in D, \label{14} \\
(A_1 i_1 Ju, i_1 Ju)_1 &\leq \beta_2 (A_2 i_2 Ju, i_2 Ju)_2 + \varepsilon_2 \|i_2 Ju\|^2_2, \quad u \in D, \label{15}
\end{align}
for some $\beta_1, \beta_2 \geq 1$ and $\varepsilon_1, \varepsilon_2 > 0$.

Denote by $E_l(\lambda), l = 1, 2$, the spectral projection of the operator $A_l$, corresponding to the semi-axis $(-\infty, \lambda)$. We assume that there exists a faithful, normal, semi-finite trace $\tau$ on $\pi(\mathfrak{A})''$ such that, for any $t > 0$, the operators $\mathcal{V}_t e^{-t A_l} \mathcal{V}_t^{-1}, l = 1, 2$, belong to $\pi(\mathfrak{A})$ and have finite trace. By standard arguments, it follows that $\mathcal{V}_t E_l(\lambda) \mathcal{V}_t^{-1} \in \pi(\mathfrak{A})''$, and $\tau(\mathcal{V}_t E_l(\lambda) \mathcal{V}_t^{-1}) < \infty$ for any $\lambda, l = 1, 2$.

**Theorem 2.1.** Under current assumptions, let $b_1 > a_1$ and

\begin{align}
\alpha_2 &= \rho \left[ \beta_1 \left( a_1 + \gamma_1 + \frac{(a_1 + \gamma_1 - \lambda_0)^2}{a_1 - a_1 - \gamma_1} \right) + \varepsilon_1 \right], \
\beta_2 &= \frac{\beta_1^{-1} (b_1 \rho^{-1} - \varepsilon_2)(a_2 - \gamma_2) - \alpha_2 \gamma_2 + 2 \lambda_0 \gamma_2 - \lambda_0^2}{a_2 - 2 \lambda_0 + \beta_2^{-1} (b_1 \rho^{-1} - \varepsilon_2)}.
\end{align}

Suppose that $\alpha_1 > a_1 + \gamma_1$, $\alpha_2 > b_2 + \gamma_2$ and $b_2 > a_2$. If the interval $(a_1, b_1)$ does not intersect with the spectrum of $A_1$, then:

1. the interval $(a_2, b_2)$ does not intersect with the spectrum of $A_2$;
(2) for any $\lambda_1 \in (a_1, b_1)$ and $\lambda_2 \in (a_2, b_2)$, the projections $V_1 E_1(\lambda_1) V_1^{-1}$ and $V_2 E_2(\lambda_2) V_2^{-1}$ belong to $A$ and are Murray-von Neumann equivalent in $A$.

Remark 2.2. Since $\rho > 1, \beta_1 \geq 1, \gamma_1 > 0$ and $\varepsilon_1 > 0$, we, clearly, have $a_2 > a_1$. The formula (17) is equivalent to the formula

$$b_1 = \rho \left[ \beta_2 \left( b_2 + \gamma_2 + \frac{(b_2 + \gamma_2 - \lambda_{02})^2}{\alpha_2 - b_2 - \gamma_2} \right) + \varepsilon_2 \right],$$

which is obtained from (17), if we replace $\alpha_1, \beta_1, \gamma_1, \varepsilon_1, \lambda_{01}$ by $\alpha_2, \beta_2, \gamma_2, \varepsilon_2, \lambda_{02}$ accordingly and $a_1$ and $a_2$ by $b_2$ and $b_1$ accordingly. In particular, this implies that $b_1 > b_2$.

2.2. Localization theorem for spectral projections. The goal of this Section is to prove Proposition 2.3, which we need for the proof of Theorem 2.1.

Let $A_1$ be an (unbounded) self-adjoint operator in a Hilbert space $H_1$ with the domain $\text{Dom}(A_1)$. We assume that $A_1$ is semi-bounded from below:

$$\begin{align*}
(18) & \quad (A_1u, u) \geq \lambda_0 \|u\|^2, \quad u \in \text{Dom}(A_1)
\end{align*}$$

with some $\lambda_0 \leq 0$.

Let $J$ be a self-adjoint bounded operator in $H_1$ that maps the domain of $A_1$ into itself, $J : \text{Dom}(A_1) \rightarrow \text{Dom}(A_1)$. We assume that $0 \leq J \leq \text{id}_{H_1}$. Introduce a self-adjoint positive bounded operator $J'$ in $H_1$ by the formula $J^2 + (J')^2 = \text{id}_{H_1}$. We assume that $J'$ maps the domain of $A_1$ into itself, the operators $[J, [J, A_1]]$ and $[J', [J', A_1]]$ extend to bounded operators in $H_1$ and

$$\begin{align*}
(19) & \quad \max(||[J, [J, A_1]]||, ||[J', [J', A_1]]||) \leq \gamma.
\end{align*}$$

Finally, we assume that

$$\begin{align*}
(20) & \quad (A_1J'u, J'u) \geq \alpha \|J'u\|^2, \quad u \in \text{Dom}(A_1)
\end{align*}$$

for some $\alpha > 0$.

Denote by $E(\lambda)$ the spectral projection of the operator $A_1$, corresponding to the semi-axis $(-\infty, \lambda]$. We have

$$\begin{align*}
(21) & \quad (A_1E(\lambda)u, E(\lambda)u) \leq \lambda \|E(\lambda)u\|^2, \quad u \in \text{Dom}(A_1).
\end{align*}$$

Proposition 2.3. If $\alpha > \lambda + \gamma$, then we have the following estimate

$$\begin{align*}
(22) & \quad \|JE(\lambda)u\|^2 \geq \frac{\alpha - \lambda - \gamma}{\alpha - \lambda_0} \|E(\lambda)u\|^2, \quad u \in H_1.
\end{align*}$$

Remark 2.4. Note that in the case $\lambda < \lambda_0$ the statement is trivial. In the opposite case $\lambda \geq \lambda_0$, since $\alpha > \lambda + \gamma$ and $\gamma \geq 0$, the coefficient, entering in the right-hand side of the formula (22), satisfies the estimate

$$\begin{align*}
0 < \frac{\alpha - \lambda - \gamma}{\alpha - \lambda_0} \leq 1.
\end{align*}$$
Proof. We can assume that \( \lambda \geq \lambda_0 \). By the IMS localization formula (see [Sh] and references there), we have

\[
A_1 = JA_1J + J'A_1J' + \frac{1}{2}[J, [J, A_1]] + \frac{1}{2}[J', [J', A_1]].
\]

Applying this formula to \( E(\lambda)u, u \in \text{Dom}(A_1) \), we get

\[
(A_1E(\lambda)u, E(\lambda)u) = (A_1JE(\lambda)u, JE(\lambda)u) + (A_1J'E(\lambda)u, J'E(\lambda)u)
\]

\[
+ \frac{1}{2}([J, [J, A_1]]E(\lambda)u, E(\lambda)u) + \frac{1}{2}([J', [J', A_1]]E(\lambda)u, E(\lambda)u).
\]

Combining (20), (21), (18), (19), we get (24)

\[
\|J'E(\lambda)u\|^2 \leq \frac{1}{\alpha}(A_1J'E(\lambda)u, J'E(\lambda)u)
\]

\[
= \frac{1}{\alpha}((A_1E(\lambda)u, E(\lambda)u) - (A_1JE(\lambda)u, JE(\lambda)u) - \frac{1}{2}([J, [J, A_1]]E(\lambda)u, E(\lambda)u)
\]

\[
- \frac{1}{2}([J', [J', A_1]]E(\lambda)u, E(\lambda)u)
\]

\[
\leq \frac{1}{\alpha}((\lambda + \gamma)\|E(\lambda)u\|^2 - \lambda_0\|JE(\lambda)u\|^2).
\]

Hence, we have

\[
\|JE(\lambda)u\|^2 = \|E(\lambda)u\|^2 - \|J'E(\lambda)u\|^2 \geq \left(1 - \frac{\lambda + \gamma}{\alpha}\right)\|E(\lambda)u\|^2 + \frac{\lambda_0}{\alpha}\|JE(\lambda)u\|^2,
\]

that immediately implies the required estimate.

\[\square\]

Corollary 2.5. If \( \alpha > \lambda + \gamma \), then we have the following estimate:

\[
\|J'E(\lambda)u\|^2 \leq \frac{\lambda + \gamma - \lambda_0}{\alpha - \lambda_0}\|E(\lambda)u\|^2, \quad u \in \mathcal{H}_1.
\]

Proof. This follows immediately from the equality \( \|Jv\|^2 + \|J'v\|^2 = \|v\|^2 \) for any \( v \in \mathcal{H}_1 \).

\[\square\]

Corollary 2.6. If \( \alpha > \lambda + \gamma \), then we have the following estimate

\[
(A_1JE(\lambda)u, JE(\lambda)u) \leq \left(\lambda + \gamma - \lambda_0\frac{\lambda + \gamma - \lambda_0}{\alpha - \lambda_0}\right)\|E(\lambda)u\|^2, \quad u \in \text{Dom}(A_1).
\]

Proof. From (21) and (21), we get

\[
(A_1JE(\lambda)u, JE(\lambda)u) = (A_1E(\lambda)u, E(\lambda)u) - (A_1J'E(\lambda)u, J'E(\lambda)u)
\]

\[
- \frac{1}{2}([J, [J, A_1]]E(\lambda)u, E(\lambda)u) - \frac{1}{2}([J', [J', A_1]]E(\lambda)u, E(\lambda)u)
\]

\[
\leq \left((\lambda + \gamma)\|E(\lambda)u\|^2 - \lambda_0\|J'E(\lambda)u\|^2\right)
\]

\[
\leq \left(\lambda + \gamma - \lambda_0\frac{\lambda + \gamma - \lambda_0}{\alpha - \lambda_0}\right)\|E(\lambda)u\|^2,
\]

as desired.

\[\square\]
2.3. Proof of Theorem 2.1. In this Section, we will use the notation of Section 2.1. We start with the following

Proposition 2.7. If \( \alpha_1 > \lambda_1 + \gamma_1 \) and

\[
\lambda_2 > \rho \left[ \beta_1 \left( \lambda_1 + \gamma_1 + \frac{(\lambda_1 + \gamma_1 - \lambda_0)^2}{\alpha_1 - \lambda_1 - \gamma_1} \right) + \varepsilon_1 \right],
\]

then there exists \( \varepsilon_0 > 0 \) such that

\[
\|E_2(\lambda_2)i_2Jp_1E_1(\lambda_1)u\|_2 ^2 \geq \varepsilon_0 \|E_1(\lambda_1)u\|_1 ^2, \quad u \in H.
\]

Proof. Applying \( (\ref{equation1}) \) to a function \( p_1E_1(\lambda_1)u, u \in \text{Dom}(A_1) \) and taking into account that \( J_1 = i_1Jp_1 \), we get

\[
(A_2i_2Jp_1E_1(\lambda_1)u, i_2Jp_1E_1(\lambda_1)u)_2 \leq \beta_1(A_1J_1E_1(\lambda_1)u, J_1E_1(\lambda_1)u)_1 + \varepsilon_1 \|J_1E_1(\lambda_1)u\|_1 ^2.
\]

Clearly, for any \( \lambda \) and \( l = 1, 2 \) we have the estimate

\[
(A_l(id_{H_l} - E_l(\lambda)))u, (id_{H_l} - E_l(\lambda))u)_l \geq \lambda \|(id_{H_l} - E_l(\lambda))u\|_l ^2, \quad u \in \text{Dom}(A_l).
\]

By \( (\ref{equation2}) \), \( (\ref{equation3}) \) and \( (\ref{equation4}) \), it follows that

\[
(A_2i_2Jp_1E_1(\lambda_1)u, i_2Jp_1E_1(\lambda_1)u)_2 = (A_2E_2(\lambda_2)i_2Jp_1E_1(\lambda_1)u, E_2(\lambda_2)i_2Jp_1E_1(\lambda_1)u)_2
\]

\[
+ (A_2(id_{H_2} - E_2(\lambda_2))i_2Jp_1E_1(\lambda_1)u, (id_{H_2} - E_2(\lambda_2))i_2Jp_1E_1(\lambda_1)u)_2
\]

\[
\geq \lambda_0^2\|E_2(\lambda_2)i_2Jp_1E_1(\lambda_1)u\|_2 ^2 + \lambda_2\|(id_{H_2} - E_2(\lambda_2))i_2Jp_1E_1(\lambda_1)u\|_2 ^2
\]

\[
= \lambda_2\|i_2Jp_1E_1(\lambda_1)u\|_2 ^2 - (\lambda_2 - \lambda_0^2)\|E_2(\lambda_2)i_2Jp_1E_1(\lambda_1)u\|_2 ^2
\]

\[
\geq \lambda_2\rho^{-1}\|J_1E_1(\lambda_1)u\|_1 ^2 - (\lambda_2 - \lambda_0^2)\|E_2(\lambda_2)i_2Jp_1E_1(\lambda_1)u\|_2 ^2.
\]

From the other side, by \( (\ref{equation2}) \), \( (\ref{equation3}) \), we have

\[
(A_2i_2Jp_1E_1(\lambda_1)u, i_2Jp_1E_1(\lambda_1)u)_2 \leq \beta_1(A_1J_1E_1(\lambda_1)u, J_1E_1(\lambda_1)u)_1 + \varepsilon_1 \|J_1E_1(\lambda_1)u\|_1 ^2
\]

\[
\leq \beta_1 \left( \lambda_1 + \gamma_1 - \lambda_0 \frac{\lambda_1 + \gamma_1 - \lambda_0}{\alpha_1 - \lambda_1 - \gamma_1} \right) \|E_1(\lambda_1)u\|_1 ^2 + \varepsilon_1 \|J_1E_1(\lambda_1)u\|_1 ^2.
\]

Combining \( (\ref{equation20}) \) and \( (\ref{equation21}) \), we get

\[
\lambda_2\rho^{-1}\|J_1E_1(\lambda_1)u\|_1 ^2 - (\lambda_2 - \lambda_0^2)\|E_2(\lambda_2)i_2Jp_1E_1(\lambda_1)u\|_2 ^2
\]

\[
\leq \beta_1 \left( \lambda_1 + \gamma_1 - \lambda_0 \frac{\lambda_1 + \gamma_1 - \lambda_0}{\alpha_1 - \lambda_1 - \gamma_1} \right) \|E_1(\lambda_1)u\|_1 ^2 + \varepsilon_1 \|J_1E_1(\lambda_1)u\|_1 ^2,
\]
By assumption, \( T \) performs the Murray-von Neumann equivalence of the projections \( \pi \).
It follows from Proposition 2.7 that the map
\[
\| E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) u \|_2^2 \\
\geq \frac{1}{\lambda_2 - \lambda_0} \left[ (\lambda_2 \rho^{-1} - \varepsilon_1) \| J_1 E_1(\lambda_1) u \|_1^2 - \beta_1 \left( \frac{\lambda_1 + \gamma_1 - \lambda_0}{\alpha_1 - \lambda_0} \right) \| E_1(\lambda_1) u \|_1 \right] \\
\geq \frac{1}{\lambda_2 - \lambda_0} \left[ (\lambda_2 \rho^{-1} - \varepsilon_1) \frac{\alpha_1 - \lambda_1 - \gamma_1}{\alpha_1 - \lambda_0} - \beta_1 \left( \frac{\lambda_1 + \gamma_1 - \lambda_0}{\alpha_1 - \lambda_0} \right) \right] \| E_1(\lambda_1) u \|_1^2
\]
as desired. 

Remark 2.8. Note that we only used estimate (22) (but not (15)) in the proof of Proposition 2.7.

Proof of Theorem 2.7. As above, we will use the notation of Section 2.1. Take arbitrary \( \lambda_1 \in (a_1, b_1) \) and \( \lambda_2 \in (a_2, b_2) \). Consider the bounded operator \( T = \mathcal{V}_2 E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) \mathcal{V}_1^{-1} \) in \( \mathcal{H} \).
By assumption, \( T \) belongs to the von Neumann algebra \( \pi(\mathfrak{A})'' \).

Since
\[
\lambda_2 > a_2 = \rho \left[ \beta_1 \left( a_1 + \gamma_1 + \frac{(a_1 + \gamma_1 - \lambda_0)^2}{a_1 - a_1 - \gamma_1} \right) + \varepsilon_1 \right]
\]
and (see Remark 2.2)
\[
b_1 = \rho \left[ \beta_2 \left( b_2 + \gamma_2 + \frac{(b_2 + \gamma_2 - \lambda_0)^2}{a_2 - b_2 - \gamma_2} \right) + \varepsilon_2 \right] \\
> \rho \left[ \beta_2 \left( \lambda_2 + \gamma_2 + \frac{(\lambda_2 + \gamma_2 - \lambda_0)^2}{a_2 - \lambda_2 - \gamma_2} \right) + \varepsilon_2 \right],
\]
it follows from Proposition 2.7 that the map
\[
E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) = E_2(\lambda_2) i_2 J p_1 E_1(a_1 + 0) : \text{Im} \ E_1(\lambda_1) \rightarrow \text{Im} \ E_2(\lambda_2)
\]
is injective and has closed image, and the map
\[
(E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1))^* = E_1(\lambda_1) i_1 J p_2 E_2(\lambda_2) = E_1(b_1 - 0) i_1 J p_2 E_2(\lambda_2) : \text{Im} \ E_2(\lambda_2) \rightarrow \text{Im} \ E_1(\lambda_1)
\]
is injective. Hence, the map \( E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) : \text{Im} \ E_1(\lambda_1) \rightarrow \text{Im} \ E_2(\lambda_2) \) is bijective.

Let \( T = U S, U, S \in \pi(\mathfrak{A})'' \), be the polar decomposition of \( T \). Since \( \text{Ker} \ T = \mathcal{V}_1(\text{Im} \ E_1(\lambda_1)) = \text{Im} \ \mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1} \) and \( \text{Im} \ T = \mathcal{V}_2(\text{Im} \ E_2(\lambda_2)) = \text{Im} \ \mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1} \), \( U \) is a partial isometry that performs the Murray-von Neumann equivalence of the projections \( \mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1} \) and \( \mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1} \) in the von Neumann algebra \( \pi(\mathfrak{A})'' \).

Since the interval \((a_1, b_1)\) does not intersect with the spectrum of \( A_1 \), the spectral density function \( \tau(\mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1}) \) is constant for any \( \lambda_1 \in (a_1, b_1) \). Using the Murray-von Neumann equivalence of \( \mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1} \) and \( \mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1} \) and the tracial property, we conclude that the spectral density function \( \tau(\mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1}) \) is constant for any \( \lambda_2 \in (a_2, b_2) \). Since the trace \( \tau \) is faithful, the interval \((a_2, b_2)\) does not intersect with the spectrum of \( A_2 \), that completes the proof of the first part of Theorem 2.7.

Note that \( E_1(\lambda_1) = \chi_{[e^{-t\lambda_1}, \infty)}(e^{-tA_1}) \). Using the fact that \( \lambda_1 \) belongs to a gap in the spectrum \( A_1 \) and \( e^{-tA_1} \in \pi_1(\mathfrak{A}) \), one can replace \( \chi_{[e^{-t\lambda_1}, \infty)} \) by a continuous function and obtain that
E_1(\lambda_1) \in \pi_1(\mathfrak{A}) \text{ for any } \lambda_1 \in (a_1, b_1). \text{ Similarly, } E_2(\lambda_2) \in \pi_2(\mathfrak{A}) \text{ for any } \lambda_2 \in (a_2, b_2). \text{ By assumption, this implies that } T \text{ belongs to the C*-algebra } \pi(\mathfrak{A}).

**Lemma 2.9.** Let \( \mathfrak{A} \) be a C*-algebra, \( \mathcal{H} \) a Hilbert space equipped with a faithful *-representation of \( \mathfrak{A} \), \( \pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}) \). If \( P \in \pi(\mathfrak{A}) \) has closed image and \( P = US \) is its polar decomposition, then \( U, S \in \pi(\mathfrak{A}) \).

**Proof.** We will identify \( \mathfrak{A} \) with \( \pi(\mathfrak{A}) \). Since \( P \) has closed image, the operators \( P^* \) and \( P^*P \) also have closed image, hence 0 is an isolated point in the spectrum of \( P^*P \). Then it follows that the projection on the kernel of \( P \) (or, which is the same, the kernel of \( P^*P \)) is in \( \pi(\mathfrak{A}) \). Clearly \( S = \sqrt{P^*P} \) is in \( \mathfrak{A} \) and has 0 its isolated point in the spectrum. Now we define a bounded operator \( S^{(-1)} \) in \( \mathcal{H} \) by the equalities \( S^{(-1)}(Sx) = x, x \in (\text{Ker } S)^\perp \), on \( \text{Im } S \) and \( S^{(-1)}(x) = 0 \) on the orthogonal complement \( (\text{Im } S)^\perp \). Since \( S^{(-1)} \) is given as \( f(S) \), where \( f \) is a continuous function on the spectrum of \( S \) defined as \( f(\lambda) = \lambda^{-1} \) if \( \lambda \neq 0 \), \( f(0) = 0 \), we see that \( S^{(-1)} \) is in \( \mathfrak{A} \). It remains to notice that \( U = PS^{(-1)} \). \( \square \)

Applying Lemma 2.9 to the operator \( T \), we get that the partial isometry \( U \in \pi(\mathfrak{A}) \) performs the desired Murray-von Neumann equivalence of the projections \( \mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1} \) and \( \mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1} \) in the C*-algebra \( \pi(\mathfrak{A}) \).

**Corollary 2.10.** Under assumptions of Theorem 2.9 for any \( \lambda_1 \in (a_1, b_1) \) and \( \lambda_2 \in (a_2, b_2) \), the spectral projections \( \mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1} \) and \( \mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1} \) define the same element in K-theory of \( \mathfrak{A} \):

\[
[\mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1}] = [\mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1}] \in K_0(\mathfrak{A}).
\]

### 2.4. The model operator and equivalence of spectral projections in the C*-algebra.

As above, let \( M \) be a compact connected Riemannian manifold, \( \Gamma \) its fundamental group and \( \tilde{M} \) its universal cover. Let \( \mathcal{E} \) be a Hermitian vector bundle on \( M \) and \( \tilde{\mathcal{E}} \) the lift of \( \mathcal{E} \) to the universal cover \( \tilde{M} \). Let \( \omega \) be a closed real-valued 2-form on \( M \) such that \( \mathcal{B} = p^*\omega \) is exact, \( \mathcal{B} = dA \), where \( A \) is a real-valued 1-form on \( \tilde{M} \).

As above, consider the elliptic self-adjoint differential operator in \( L^2(\tilde{M}, \tilde{\mathcal{E}}) \) given by

\[
H(\mu) = \mu \nabla^* \nabla + B + \mu^{-1} V,
\]

where \( \nabla \) is a Hermitian connection on the vector bundle \( \tilde{\mathcal{E}} \) over \( \tilde{M} \) of the form \( \nabla = \tilde{\nabla}^\mathcal{E} \otimes \text{id} + \text{id} \otimes \nabla_A \), where \( \tilde{\nabla}^\mathcal{E} \) is a \( \Gamma \)-invariant Hermitian connection on \( \tilde{\mathcal{E}} \), \( \nabla_A = d + iA \) is a Hermitian connection on the trivial vector bundle, \( B, V \) are \( \Gamma \)-invariant self-adjoint endomorphisms of the bundle \( \tilde{\mathcal{E}} \), and where \( V \) satisfies in addition the following Morse type condition: \( V(x) \geq 0 \) for all \( x \in \tilde{M} \). Also if the matrix \( V(x_0) \) is degenerate for some \( x_0 \) in \( \tilde{M} \), then \( V(x_0) = 0 \) and there is a positive constant \( c \) such that \( V(x) \geq c|x - x_0|^2 I \) for all \( x \) in a neighborhood of \( x_0 \). We will also assume that \( V \) has at least one zero point.

Choose a fundamental domain \( \mathcal{F} \subset \tilde{M} \) so that there is no zeros of \( V \) on the boundary of \( \mathcal{F} \). This is equivalent to saying that the translations \( \{ \gamma \mathcal{F}, \gamma \in \Gamma \} \) cover the set \( V^{-1}(0) \) (the set of all zeros of \( V \)). Let \( V^{-1}(0) \cap \mathcal{F} = \{ \bar{x}_j | j = 1, \ldots, N \} \) be the set of all zeros of \( V \) in \( \mathcal{F} \); \( \bar{x}_i \neq \bar{x}_j \) if \( i \neq j \).
Let $K$ denote the model operator of $H$ (cf. [Sh]), which is obtained as a direct sum of quadratic parts of $H$ in all points $\bar{x}_1, \ldots, \bar{x}_N$. More precisely, $K$ is an operator in $L^2(\mathbb{R}^n, \mathbb{C}^k)^N$ given by
\[
K = \oplus_{1 \leq j \leq N} K_j,
\]
where $K_j$ is an unbounded self-adjoint operator in $L^2(\mathbb{R}^n, \mathbb{C}^k)$ which corresponds to the zero $\bar{x}_j$. It is a quantum harmonic oscillator and has a discrete spectrum. We assume that we have fixed local coordinates on $\tilde{M}$ and trivialization of the bundle $\tilde{E}$ in a small neighborhood $B(\bar{x}_j, r)$ of $\bar{x}_j$ for every $j = 1, \ldots, N$. We assume that $\bar{x}_j$ becomes zero in these local coordinates. Then $K_j$ has the form
\[
K_j = H^{(2)}_j + \tilde{B}_j + V^{(2)}_j,
\]
where all the components are obtained from $H$ as follows. The second order term $H^{(2)}_j$ is a homogeneous second order differential operator with constant coefficients (without lower order terms) obtained by isolating the second order terms in the operator $H$ and freezing the coefficients of this operator at $\bar{x}_j$. (Note that $H^{(2)}_j$ does not depend on $A_j$.) The zeroth order term $V^{(2)}_j$ is obtained by taking the quadratic part of $V$ in the chosen coordinates near $\bar{x}_j$.

More explicitly,
\[
H^{(2)}_j = -\sum_{i, k = 1}^{n} g^{ik}(\bar{x}_j) \frac{\partial^2}{\partial x_i \partial x_k}, \quad V^{(2)}_j = \frac{1}{2} \sum_{i, k = 1}^{n} \frac{\partial^2 V}{\partial x_i \partial x_k}(\bar{x}_j)x_i x_k,
\]
where $(g^{ik})$ is the inverse matrix to the matrix of the Riemannian tensor $(g_{ik})$.

Finally, $\tilde{B}_j = B(\bar{x}_j)$, $j = 1, \ldots, N$, so $\tilde{B}_j$ is an endomorphism of the fiber of the bundle $\tilde{E}$ over the point $\bar{x}_j$.

We will also need the operator
\[
K(\mu) = \oplus_{1 \leq j \leq N} K_j(\mu),
\]
where
\[
K_j(\mu) = \mu H^{(2)}_j + \tilde{B}_j + \mu^{-1} V^{(2)}_j, \quad \mu > 0.
\]
It is easy to see that $K(\mu)$ has the same spectrum as $K = K(1)$.

We will say that $H$ is flat near $\bar{x}_j$ if $H(\mu) = K_j(\mu)$ for all $\mu$ near $\bar{x}_j$. (In particular, in this case we should have $A = 0$ near $\bar{x}_j$.)

We are going to apply Theorem 2.1 in the following particular setting. Take the $C^*$ algebra $\mathfrak{A}$ to be $C^*_r(\Gamma, \sigma) \otimes \mathcal{K}$. Let $\mathcal{H}$ be the Hilbert space $\ell^2(\Gamma) \otimes \ell^2(\mathbb{N})$. Put $\mathcal{H}_1 = \ell^2(\Gamma) \otimes L^2(\mathbb{R}^n, \mathbb{C}^k)^N$ ($\ell^2(\Gamma) \otimes \mathcal{H}_K$ in the above notation) and $\mathcal{H}_2 = L^2(\tilde{M}, \tilde{E})$ (denoted by $\mathcal{H}$ above). Choose an arbitrary unitary isomorphism $V_1 : L^2(\mathbb{R}^n, \mathbb{C}^k)^N \to \ell^2(\mathbb{N})$ and define an unitary operator $V_1 : \mathcal{H}_1 \to \mathcal{H}$ as $V_1 = \text{id} \otimes V_1$. Similarly, choose an arbitrary unitary isomorphism $V_2 : L^2(\mathcal{F}, \tilde{E}|_{\mathcal{F}}) \to \ell^2(\mathbb{N})$ and define an unitary operator $V_2 : \mathcal{H}_2 \to \mathcal{H}$ as $V_2 = (\text{id} \otimes V_2) \circ U$, where $U$ is the $(\Gamma, \sigma)$-equivariant isometry $\mathfrak{U}$.

Let $\pi$ be the representation of the algebra $\mathfrak{A}$ in $\mathcal{H}$ given by the tensor product of the representation of $C^*_r(\Gamma, \sigma)$ on $\ell^2(\Gamma)$ by left twisted convolutions and the natural representation of $\mathcal{K}$ in
\( \ell^2(\mathbb{N}) \). So we have \( \pi(C_*(\Gamma, \sigma) \otimes K) \subset A_{\ell^2(\mathbb{N})}^L(\Gamma, \sigma) \) and \( \pi(C_*(\Gamma, \sigma) \otimes K)^\prime = A_{\ell^2(\mathbb{N})}^L(\Gamma, \sigma) \). Using the unitary isomorphisms \( V_1 \) and \( V_2 \), we get representations \( \pi_1 \) and \( \pi_2 \) of \( \mathfrak{A} \) in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) accordingly, \( \pi_l(a) = \mathcal{V}_l^{-1} \circ \pi(a) \circ \mathcal{V}_l, l = 1, 2, a \in \mathfrak{A} \).

Consider self-adjoint, semi-bounded from below operators \( A_1 = \id \otimes K(\mu) \) in \( \mathcal{H}_1 \) and \( A_2 = K(\mu) \) in \( \mathcal{H}_2 \). It is clear that \( e^{-tA_1} = \id \otimes e^{-tK(\mu)} \in \pi_1(\mathfrak{A}) \cong C_*^\prime(\Gamma, \sigma) \otimes K(\mathcal{A}_K) \) for any \( t > 0 \). As shown in Lemma 4 below, for any \( t > 0 \), the operator \( e^{-tA_2} \) belongs to \( \pi_2(\mathfrak{A}) \). Remark that, in notation of Theorem 2, \( E_1(\lambda) = \id \otimes E_0(\lambda) \) and \( E_2(\lambda) = E(\lambda) \).

Let \( \mathcal{H}_0 = \ell^2(\Gamma) \otimes \bigoplus_{j=1}^N L^2(\bar{x}_j, r), \bar{E}|_{B(x_j, r)} \bigg) \). An inclusion \( i_1 : \mathcal{H}_0 \to \mathcal{H}_1 \) is defined as \( i_1 = \id \otimes j_1 \), where \( j_1 \) is the inclusion of \( \bigoplus_{j=1}^N L^2(\bar{x}_j, r), \bar{E}|_{B(x_j, r)} \bigg) \) in \( L^2(\mathbb{R}^n, \mathbb{C}^k)^N \) given by the chosen local coordinates and trivializations of the vector bundle \( \bar{E} \). An inclusion \( i_2 : \mathcal{H}_0 \to \mathcal{H}_2 \) is defined as \( i_2 = U^* \circ (\id \otimes j_2) \), where \( j_2 \) is the natural inclusion of \( \bigoplus_{j=1}^N L^2(\bar{x}_j, r), \bar{E}|_{B(x_j, r)} \bigg) \) in \( L^2(F, \bar{E}|_F) \).

The operator \( p_1 : \mathcal{H}_1 \to \mathcal{H}_0 \) is defined as \( p_1 = \id \otimes r_1 \), where \( r_1 \) is the restriction operator \( L^2(\mathbb{R}^n, \mathbb{C}^k)^N \to \bigoplus_{j=1}^N L^2(\bar{x}_j, r), \bar{E}|_{B(x_j, r)} \bigg) \). The operator \( p_2 : \mathcal{H}_1 \to \mathcal{H}_0 \) is defined as \( p_2 = (\id \otimes r_2) \circ U \), where \( r_2 : L^2(F, \bar{E}|_F) \to \bigoplus_{j=1}^N L^2(\bar{x}_j, r), \bar{E}|_{B(x_j, r)} \bigg) \) is the restriction operator.

Fix a function \( \phi \in C_0^\infty(\mathbb{R}^n) \) such that \( 0 \leq \phi \leq 1 \), \( \phi(x) = 1 \) if \( |x| \leq 1 \), \( \phi(x) = 0 \) if \( |x| \geq 2 \), and \( \phi' = (1 - \phi^2)^{1/2} \in C_0^\infty(\mathbb{R}^n) \). Fix a number \( \kappa, 0 < \kappa < 1/2 \), which we shall choose later. For any \( \mu > 0 \) define \( \phi(\mu)(x) = \phi(\mu^{-1} x) \). For any \( \mu > 0 \) small enough, let \( \phi_j = \phi(\mu) \in C_0^\infty(B(\bar{x}_j, r)) \) in the fixed coordinates near \( \bar{x}_j \). Denote also \( \phi_{j, \gamma} = (\gamma^{-1})^* \phi_j \). (This function is supported near \( \gamma \bar{x}_j \).

We will always take \( \mu \in (0, \mu_0) \) where \( \mu_0 \) is sufficiently small, so in particular the supports of all functions \( \phi_{j, \gamma} \) are disjoint.

Let \( \Phi = \bigoplus_{j=1}^N \phi_j \in \bigoplus_{j=1}^N C_0^\infty(B(\bar{x}_j, r)) \subset C_0^\infty(F) \). Consider a \( (\Gamma, \sigma) \)-equivariant, self-adjoint, bounded operator \( J \) in \( \mathcal{H}_0 \) defined as \( J = \id \otimes \Phi \), where \( \Phi \) denotes the multiplication operator by the function \( \Phi \) in the space \( \bigoplus_{j=1}^N L^2(\bar{x}_j, r), \bar{E}|_{B(x_j, r)} \bigg) \).

It is clear that \( V_2 i_2 J p_1 V_1^{-1} = \id \otimes V_2 j_2 J p_1 r_1 V_1^{-1} \), where \( j_2 J p_1 r_1 \) is the multiplication operator by the function \( \Phi \), considered as an operator from \( L^2(\mathbb{R}^n, \mathbb{C}^k)^N \) to \( L^2(F, \bar{E}|_F) \). Hence, one can easily see that the operator \( V_2 i_2 J p_1 V_1^{-1} \) belongs to the von Neumann algebra \( \pi(\mathfrak{A})'' = A_{\mathcal{B}(\mathcal{L}(N))}^L(\Gamma, \sigma) \cong A^L(\Gamma, \sigma) \otimes \mathcal{B}(\ell^2(\mathbb{N})) \), \( (i_2 J p_1)^* = i_1 J p_2 \) and, for any \( a \in \mathfrak{A} \), the operator \( \pi(a) V_2 (i_2 J p_1) V_1^{-1} \) belongs to \( \pi(\mathfrak{A}) \).

We will use local coordinates near \( \bar{x}_j \) such that the Riemannian volume element at the point \( \bar{x}_j \) coincides with the Euclidean volume element given by the chosen local coordinates. Similarly we will fix a trivialization of the bundle \( \bar{E} \) near \( \bar{x}_j \) such that the Hermitian metric becomes trivial in this trivialization. Then the estimate \( \square \) holds with \( \rho = 1 + O(\mu^\kappa) \).

Denote by the same letters \( \phi \) and \( \phi' \) the multiplication operators in \( L^2(\mathbb{R}^n, \mathbb{C}^k) \) by the functions \( \phi \) and \( \phi' \) accordingly. Let \( \Phi_1 \) and \( \Phi_1' \) be the bounded operators in \( L^2(\mathbb{R}^n, \mathbb{C}^k)^N \cong L^2(\mathbb{R}^n, \mathbb{C}^k) \otimes \mathbb{C}^N \) given by \( \Phi_1 = \phi \otimes \id_\mathbb{C} \) and \( \Phi_1' = \phi' \otimes \id_\mathbb{C} \). Then we have \( J_1 = \id \otimes \Phi_1 \) and \( J_1' = \id \otimes \Phi_1' \) in
Let $a^{(2)}_{1,j}, j = 1, 2, \ldots, N,$ be the principal symbol of $K_j$, which is a function on $T^*\mathbb{R}^n$:

$$a^{(2)}_{1,j}(x, \xi) = \sum_{i,k=1}^{n} g^{ik}(x_j)\xi_i\xi_k, \quad (x, \xi) \in T^*\mathbb{R}^n.$$ 

Then the operators $[J_1, [J_1, A_1]], [J_1', [J_1', A_1]]$ are given by $-\mu \text{id} \otimes \left( \oplus_{1 \leq j \leq N} a^{(2)}_{1,j}(x, d\phi(x)) \right)$ and $-\mu \text{id} \otimes \left( \oplus_{1 \leq j \leq N} a^{(2)}_{1,j}(x, d\phi'(x)) \right)$ in $L^2(\Gamma) \otimes L^2(\mathbb{R}^n, \mathbb{C}^k)^N$ accordingly. Similarly, let $a^{(2)}_2$ be the principal symbol of $H(1)$, which is a function on $T^*\tilde{M}$:

$$a^{(2)}_2(x, \xi) = \sum_{i,k=1}^{n} g^{ik}(x)\xi_i\xi_k, \quad (x, \xi) \in T^*\tilde{M}.$$ 

Then $[J_2, [J_2, A_2]], [J_2', [J_2', A_2]]$ are the multiplication operators by functions $-\mu a^{(2)}_2(x, d\Phi_2(x))$ and $-\mu a^{(2)}_2(x, d\Phi'_2(x))$ in $L^2(\tilde{M}, \tilde{E}).$ Therefore,

$$\gamma_1 = \mu \max_{j=1,2,\ldots,N} \max_{x \in \mathbb{R}^n} \left( \sup_{x \in \mathbb{R}^n} a^{(2)}_{1,j}(x, d\phi(x)), \sup_{x \in \mathbb{R}^n} a^{(2)}_{1,j}(x, d\phi'(x)) \right) = O(\mu^{1-2\kappa}),$$

$$\gamma_2 = \mu \max \left( \sup_{x \in \tilde{M}} a^{(2)}_2(x, d\Phi_2(x)), \sup_{x \in \tilde{M}} a^{(2)}_2(x, d\Phi'_2(x)) \right) = O(\mu^{1-2\kappa}).$$

Since there exists $c_0 > 0$ such that $V^{(2)}_j \geq c_0 \mu^{2\kappa}, j = 1, 2, \ldots, N,$ on $\text{supp} \phi'$ and $V \geq c_0 \mu^{2\kappa}$ on $\text{supp} \Phi_2$, the estimates hold with $\alpha_l = c_0^{-1+2\kappa}$ (see [Sh] Lemma 3.3 for more details).

The constants $\lambda_{0l}, l = 1, 2$, can be chosen to be independent of $\mu$. One can take

$$\lambda_{01} = \lambda_0(K(1))_-,$$

where $\lambda_0(K(1))$ is the bottom of the spectrum of the operator $K(1)$ in $L^2(\mathbb{R}^n, \mathbb{C}^k)^N$ ($a_- = \min(a, 0)$) and

$$\lambda_{02} = \inf_{x \in \tilde{M}} \lambda_0(B(x))_-,$$

where $\lambda_0(B(x))_-, x \in \tilde{M}$, is the lowest eigenvalue of the linear map $B(x) : \tilde{E}_x \rightarrow \tilde{E}_x$ given by the action of the endomorphism $B$ in fibres of $\tilde{E}$.

Finally, the estimates hold with $\beta_l = 1 + O(\mu^{\kappa})$ and $\varepsilon_l = O(\mu^{3\kappa-1})$. This is an immediate consequence of the next lemma, which is an easy extension of Lemma 3.4 in [Sh]. We will state this lemma in a slightly more general situation than we need in this paper.

Let $B(0, r)$ denote the open ball in $\mathbb{R}^n$ with radius $r$ centered at the origin. Consider volume elements $\omega_1$ and $\omega_2$ of the form $\omega_l = \sqrt{g_l(x)} \, dx, l = 1, 2$, where $g_l \in C^\infty(B(0, r))$ and $g_l > 0$. Let $(\cdot, \cdot)_l, l = 1, 2$, denote the inner products in $C^\infty_c(B(0, r), \mathbb{C}^k)$ given by the volume elements $\omega_l$.
and the standard Hermitian structure in $\mathbb{C}^k$ and $L^2(B(0,r),\mathbb{C}^k,\omega_l)$ the Hilbert space of square integrable $\mathbb{C}^k$-valued functions on $B(0,r)$ equipped with the inner product $(\cdot,\cdot)_l$.

Consider formally self-adjoint differential operators $T_l, l = 1, 2$, in $L^2(B(0,r),\mathbb{C}^k,\omega_l)$, depending on a small parameter $\mu > 0$, of the form

$$T_l = -\mu D_l + B_l + \mu^{-1}V_l, \quad l = 1, 2.$$  

Here $D_l$ is a second order formally self-adjoint uniformly elliptic differential operator with a negative principal symbol, so that $-D_l$ is semi-bounded from below on $C_c^\infty(B(0,r),\mathbb{C}^k)$; $B_l$ and $V_l$ are zero order formally self-adjoint operators, i.e. the multiplication operators by Hermitian $k \times k$ matrix functions $B_l$ and $V_l$ respectively.

Being formally self-adjoint the operator $D_l$ should have the form

$$D_l = A_l^{(2)} + A_l^{(1)} + A_l^{(0)},$$

where $A_l^{(s)}$ is an operator of order $s$, $s = 0, 1, 2$,

$$A_l^{(2)} = \sum_{1 \leq r, s \leq n} \frac{1}{\sqrt{g_l}} \frac{\partial}{\partial x^r} \sqrt{g_l} A_{l,rs}(x) \frac{\partial}{\partial x^s}, \quad A_l^{(s)} = A_l^{s,r};$$

$$A_l^{(1)} = \sum_{1 \leq r \leq n} A_{l,r}(x) \frac{\partial}{\partial x^r},$$

$A_{l,rs}$ and $A_{l,r}$ are $k \times k$ smooth matrix functions on $B(0,r)$; $A_l^{(0)}$ is just a multiplication by a smooth matrix function $A_l^{(0)}(x)$.

The principal symbol of $-D_l$ is the matrix function on $B(0,r) \times \mathbb{R}^n$

$$a_l^{(2)}(x,\xi) = \sum_{1 \leq r, s \leq n} \xi_r \xi_s A_{l,rs}(x).$$

For the self-adjoint operator $-D_l$ its uniform ellipticity and semi-boundedness from below mean that the matrix $a_l^{(2)}(x,\xi)$ is positive definite for all $(x,\xi) \in B(0,r) \times \mathbb{R}^n$ and

$$a_l^{(2)}(x,\xi) \geq C_l |\xi|^2, \quad (x,\xi) \in B(0,r) \times \mathbb{R}^n,$$

with some constants $C_l > 0$.

Let us assume

$$A_{1,rs}(0) = A_{2,rs}(0), \quad B_1(0) = B_2(0), \quad V_1(0) = V_2(0) = 0,$$

$$\frac{\partial V_1}{\partial x^r}(0) = \frac{\partial V_2}{\partial x^r}(0) = 0, \quad r = 1, 2, \ldots, n,$$

$$\frac{\partial^2 V_1}{\partial x^r \partial x^s}(0) = \frac{\partial^2 V_2}{\partial x^r \partial x^s}(0), \quad r, s = 1, 2, \ldots, n.$$  

Assume also that $g_l(0) = 1, l = 1, 2$, i.e. the volume elements $\omega_l = \sqrt{g_l(x)} \, dx, l = 1, 2$, at the origin coincide with the Euclidean volume element in $\mathbb{R}^n$. 

Finally, as above, let $\phi \in C^\infty_c(\mathbb{R}^n)$ satisfy $0 \leq \phi \leq 1$, $\phi(x) = 1$ if $|x| \leq 1$, $\phi(x) = 0$ if $|x| \geq 2$, and, for any $\mu > 0$ small enough, define $\phi^{(\mu)}(x) = \phi(\mu^{-\kappa}x), x \in B(0, r)$. Denote by the same letter $\phi^{(\mu)}$ the multiplication operator in $C^\infty_c(B(0, r), \mathbb{C}^k)$ by the function $\phi^{(\mu)}$.

**Lemma 2.11.** Let $1/3 < \kappa < 1/2$. There exist $C > 0$ and $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$,

$$
(T_2 \phi^{(\mu)} u, \phi^{(\mu)} u)_2 \leq (1 + C\mu^{\kappa})(T_1 \phi^{(\mu)} u, \phi^{(\mu)} u)_1 + C\mu^{3\kappa - 1}(\phi^{(\mu)} u, \phi^{(\mu)} u)_1
$$

for any $u \in C^\infty_c(B(0, r), \mathbb{C}^k)$.

**Proof.** We want to estimate from above the quadratic form $(T_2 \phi^{(\mu)} u, \phi^{(\mu)} u)_2$ in terms of the form $(T_1 \phi^{(\mu)} u, \phi^{(\mu)} u)_1$ for any $u \in C^\infty_c(B(0, r), \mathbb{C}^k)$. We start with the term

$$
(\mu A_2^{(2)} \phi^{(\mu)} u, \phi^{(\mu)} u)_2 = \mu \int_{|x| \leq 2\mu^\kappa} \sum_{1 \leq r, s \leq n} (A_{2rs}(x) - \partial(\phi^{(\mu)} u), \partial(\phi^{(\mu)} u)) \sqrt{g_2} dx.
$$

Denote by $D(\phi^{(\mu)} u)$ the $k \times n$ matrix of all first partial derivatives of all components of the vector $\phi^{(\mu)} u$. Let $\|D(\phi^{(\mu)} u)\|_0$ be the $L^2$ norm of $D(\phi^{(\mu)} u)$ considered as a vector function:

$$
\|D(\phi^{(\mu)} u)\|_0 = \left( \int_{|x| \leq 2\mu^\kappa} \left( \sum_{1 \leq r, s \leq n} \left| \frac{\partial(\phi^{(\mu)} u)}{\partial x^r} \right|^2 dx \right) \right)^{1/2}.
$$

Similarly denote by $\|\phi^{(\mu)} u\|_0$ the $L^2$-norm of $\phi^{(\mu)} u$ with respect to the Euclidean volume form:

$$
\|\phi^{(\mu)} u\|_0 = \left( \int_{|x| \leq 2\mu^\kappa} \left| \phi^{(\mu)} u \right|^2 dx \right)^{1/2}.
$$

Since $g_l(0) = 1, l = 1, 2$, we have

$$
(1 - C\mu^{\kappa})\|\phi^{(\mu)} u\|_0^2 \leq \|\phi^{(\mu)} u\|_0^2 \leq (1 + C\mu^{\kappa})\|\phi^{(\mu)} u\|_0^2, \quad l = 1, 2,
$$

with some constant $C > 0$. From (32) it follows that

$$
C_1\|D(\phi^{(\mu)} u)\|_0^2 \leq (1 - A_1^{(2)} \phi^{(\mu)} u, \phi^{(\mu)} u)_0 \leq C_2\|D(\phi^{(\mu)} u)\|_0^2, \quad l = 1, 2,
$$

with some constants $C_1, C_2 > 0$. Replacing in (34) $\sqrt{g_2}$ by $\sqrt{g_1}$ and $A_{1rs}(x)$ by $A_{1rs}(x)$, we add terms of similar form but with additional factor $O(\mu^{\kappa})$. Taking into account (35), we get that there exist $C > 0$ and $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0)$

$$
(\mu A_2^{(2)} \phi^{(\mu)} u, \phi^{(\mu)} u)_2 \leq (1 + C\mu^{\kappa})(\mu A_1^{(2)} \phi^{(\mu)} u, \phi^{(\mu)} u)_1.
$$

Now estimate the term $(-\mu A_1^{(2)} \phi^{(\mu)} u, \phi^{(\mu)} u)_2$. Then we obviously have for every $\varepsilon > 0$

$$
\|(\mu A_2^{(1)} \phi^{(\mu)} u, \phi^{(\mu)} u)_2\| \leq C\mu\|D(\phi^{(\mu)} u)\|_0\|\phi^{(\mu)} u\|_0 \leq C\mu\|D(\phi^{(\mu)} u)\|_0^2 + C\mu\varepsilon^{-1}\|\phi^{(\mu)} u\|_0^2.
$$

Taking $\varepsilon = \mu^{\kappa}$ and using (35) we obtain

$$
\|(\mu A_2^{(1)} \phi^{(\mu)} u, \phi^{(\mu)} u)_2\| \leq C\mu^{\kappa}(\mu A_1^{(2)} \phi^{(\mu)} u, \phi^{(\mu)} u)_1 + C\mu^{1-\kappa}(\phi^{(\mu)} u, \phi^{(\mu)} u)_1, \quad \mu \in (0, \mu_0).
$$

Also obviously

$$
\|(\mu A_2^{(0)} \phi^{(\mu)} u, \phi^{(\mu)} u)_2\| \leq C\mu(\phi^{(\mu)} u, \phi^{(\mu)} u)_1.
$$
Therefore we obtain for small $\mu$
\[
(-\mu D_2 u, u) \leq (1 + C \mu^\kappa)(-\mu A_1^{(2)}(a, b)u, \phi(a)u)_1 + C \mu^{1-\kappa}(\phi(a)u, \phi(a)u)_1.
\]
Replacing $B_2(x)$ by $B_1(x)$ and $\sqrt{g_2}$ by $\sqrt{g_1}$ in the quadratic form $(B_2\phi(a)u, \phi(a)u)_2$ contributes a term which can be estimated by $C \mu^\kappa(\phi(a)u, \phi(a)u)_1$:
\[
|(B_2\phi(a)u, \phi(a)u)_2 - (B_1\phi(a)u, \phi(a)u)_1| \leq C \mu^{3\kappa-1}(\phi(a)u, \phi(a)u)_1.
\]
Finally, we have
\[
|\mu^{-1}(V_2\phi(a)u, \phi(a)u)_2 - \mu^{-1}(V_1\phi(a)u, \phi(a)u)_1| \leq C \mu^{\kappa-1}(\phi(a)u, \phi(a)u)_1.
\]
Gathering together all these estimates, we obtain (35).

Now we complete the proofs of Theorem 1, the first part of Theorem 2 and Corollary 3. Assume that the interval $[a, b]$ does not intersect with the spectrum of $A_1$. Then there exists an open interval $(a_1, b_1)$ that contains $[a, b]$ and does not intersect with the spectrum of $A_1$. Using the formulas
\[
\rho = 1 + O(\mu^\kappa), \quad \alpha_l = O(\mu^{-1+2\kappa}), \quad \beta_l = 1 + O(\mu^\kappa),
\]
\[
\varepsilon_l = O(\mu^{3\kappa-1}), \quad \lambda_l = O(1), \quad \gamma_l = O(\mu^{1-2\kappa}),
\]
one can see that, for $a_2$ and $b_2$ given by (16) and (17), we have
\[
(36) \quad a_2 = a_1 + O(\mu^s), \quad b_2 = b_1 + O(\mu^s), \quad \mu \to 0,
\]
where $s = \min\{3\kappa - 1, 1 - 2\kappa\}$. The best possible value of $s$ which is
\[
s = \max_{\kappa} \min\{3\kappa - 1, 1 - 2\kappa\} = 1/5
\]
is attained when $\kappa = 2/5$.

Hence, if $\mu > 0$ is small enough, we have $\alpha_1 > a_1 + \gamma_1$, $\alpha_2 > b_2 + \gamma_2$, $b_2 > a_2$ and the interval $(a_2, b_2)$ contains $[a, b]$. By Theorem 2.1 we conclude that the interval $(a_2, b_2)$ does not intersect with the spectrum of $A_2$, that completes the proof of Theorem 1. Moreover, we have that, for any $\lambda_1 \in (a_1, b_1)$ and $\lambda_2 \in (a_2, b_2)$, the spectral projections $V_1E_1(\lambda_1)V_1^{-1}$ and $V_2E_2(\lambda_2)V_2^{-1}$ are equivalent in $\mathfrak{A}$. Putting $U = V_1^{-1}V_2$, we get the desired Murray-von Neumann equivalence of $E_1(\lambda_1) = \text{id} \otimes E^0(\lambda)$ and $V_1^{-1}V_2E_2(\lambda_2)V_2^{-1}V_1 = UE(\lambda)U^{-1}$ in $\pi_1(\mathfrak{A}) = C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(L^2(\mathbb{R}^n, \mathbb{C}^k)^N)$. This immediately implies (1) (see also Corollary 2.10).

Fix an arbitrary isomorphism $\mathfrak{H}_K \cong l^2(\mathbb{N})$. It induces an isomorphism $\mathcal{K}(\mathfrak{H}_K) \cong \mathcal{K}$ and allows us to write any element $x \in \mathcal{K}(\mathfrak{H}_K)$ as a matrix $(x_{ij}, i, j \in \mathbb{N})$. Recall that the Morita equivalence of $K$-theory $K_0(C^*_r(\Gamma, \sigma)) \cong K_0(C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(\mathfrak{H}_K))$ is induced by the standard algebra homomorphism $\mathcal{A} \to \mathcal{A} \otimes \mathcal{K}$ which maps $a \in \mathcal{A}$ to a matrix with the left-upper corner matrix element $a$, the rest matrix elements being 0. The projection $E^0(\lambda)$ belongs to some matrix algebra $M_n(\mathbb{C}) \subset \mathcal{K}$, and, under the above isomorphism, the element $[\text{id} \otimes E^0(\lambda)] \in K_0(C^*_r(\Gamma, \sigma) \otimes \mathcal{K}(\mathfrak{H}_K))$ corresponds to the element $[E^0(\lambda)] \in K_0(C^*_r(\Gamma, \sigma))$ given by the matrix $1 \otimes E^0(\lambda) \in C^*_r(\Gamma, \sigma) \otimes M_n(\mathbb{C}) = M_n(C^*_r(\Gamma, \sigma))$. It is easy to see that the element $[E^0(\lambda)]$ belongs to the image of
$K_0(\mathbb{C})$ in $K_0(C_r^*(\Gamma, \sigma))$, and the corresponding class in $\tilde{K}_0(C_r^*(\Gamma, \sigma))$ vanishes, that proves \(5\) and completes the proof of the first part of Theorem 2.

Now Corollary follows immediately from the equality \(1\) (see also Lemma 3.8 below).

**Remark 2.12.** If $H$ is flat near all points $\bar{x}_j$, then the estimate \(36\) can be improved as follows: for any $\varepsilon > 0$

$$a_2 = a_1 + O(\mu^{1-\varepsilon}), \quad b_2 = b_1 + O(\mu^{1-\varepsilon}), \quad \mu \to 0.$$  

Indeed, in this case we have (see \[Sh\], Lemma 3.4)

$$\phi_j H(\mu) \phi_j = \phi_j K_j(\mu) \phi_j, \quad \mu > 0,$$

therefore,

$$\beta_l = 1, \quad \varepsilon_l = 0, \quad l = 1, 2.$$  

Taking this into account, we easily get $a_2 = a_1 + O(\mu^{1-2\kappa}), b_2 = b_1 + O(\mu^{1-2\kappa})$ as $\mu \to 0$ for any $0 < \kappa < 1/2$ as desired.

3. Review of smooth algebras and higher traces

3.1. Definition and properties of the $\ast$-algebra $B(\Gamma, \sigma)$. We begin by recalling some generalities on smooth subalgebras of $C^*$-algebras. Let $\mathfrak{A}$ be a $C^*$-algebra and $\tilde{\mathfrak{A}}$ be obtained by adjoining a unit to $\mathfrak{A}$. Let $\mathfrak{A}_0$ be a $\ast$-subalgebra of $\mathfrak{A}$ and $\tilde{\mathfrak{A}}_0$ be obtained by adjoining a unit to $\mathfrak{A}_0$. Then $\mathfrak{A}_0$ is said to be a smooth subalgebra of $\mathfrak{A}$ if the following two conditions are satisfied:

1. $\mathfrak{A}_0$ is a dense $\ast$-subalgebra of $\mathfrak{A}$;
2. $\mathfrak{A}_0$ is stable under the holomorphic functional calculus, that is, for any $a \in \tilde{\mathfrak{A}}_0$ and for any function $f$ that is holomorphic in a neighbourhood of the spectrum of $a$ (thought of as an element in $\tilde{\mathfrak{A}}$) one has $f(a) \in \tilde{\mathfrak{A}}_0$.

Assume that $\mathfrak{A}_0$ is a dense $\ast$-subalgebra of $\mathfrak{A}$ such that $\mathfrak{A}_0$ is a Fréchet algebra with a topology that is finer than that of $\mathfrak{A}$. A necessary and sufficient condition for $\mathfrak{A}_0$ to be a smooth subalgebra is given by the spectral invariance condition cf. \[Schw\], Lemma 1.2:

- $\mathfrak{A}_0 \cap GL(\mathfrak{A}) = GL(\tilde{\mathfrak{A}}_0)$, where $GL(\mathfrak{A}_0)$ and $GL(\mathfrak{A})$ denote the group of invertibles in $\mathfrak{A}_0$ and $\tilde{\mathfrak{A}}$ respectively.

**Remark 3.1.** This fact remains true in the case when $\mathfrak{A}$ is a locally multiplicatively convex (i.e. its topology is given by a countable family of submultiplicative seminorms) Fréchet algebra such that the group $GL(\tilde{\mathfrak{A}})$ of invertibles is open \[Schw\], Lemma 1.2].

One useful property of smooth subalgebras is the following. If $\mathfrak{A}_0$ is a smooth subalgebra of a $C^*$-algebra $\mathfrak{A}$, then the inclusion map $\mathfrak{A}_0 \to \mathfrak{A}$ induces an isomorphism in $K$-theory, \[Co81\], Sect. VI.3], \[Rosi\]. Another useful property of smooth subalgebras is that sometimes there are interesting cyclic cocycles on $\mathfrak{A}_0$ that do not extend to $\mathfrak{A}$.

Let $\Gamma$ be a discrete group and $\sigma$ a multiplier on $\Gamma$, $\ell$ denote the word length function on the group $\Gamma$ with respect to a finite set of generators, i.e. $\ell(\gamma) = d_\Gamma(\gamma, e)$ where $d_\Gamma$ denotes the word
metric. Let $\Delta$ denote the (unbounded) self-adjoint operator on $\ell^2(\mathbb{N})$ defined by $\Delta \delta_j = j \delta_j$ for all $j \in \mathbb{N}$, and $D$ denote the (unbounded) self-adjoint operator on $\ell^2(\Gamma)$ defined by $D \delta_\gamma = \ell(\gamma) \delta_\gamma$ for all $\gamma \in \Gamma$. Consider the unbounded derivation $\partial = \text{ad}(D \otimes \text{id})$ of $\mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(\mathbb{N}))$. Recall that $\partial$ is a closed derivation with the domain $\text{Dom}(\partial)$, which consists of all operators $T \in \mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(\mathbb{N}))$ such that $T$ maps $\text{Dom}(D \otimes \text{id})$ into itself, and the operator $\partial(T) = (D \otimes \text{id}) \circ T - T \circ (D \otimes \text{id})$ defined initially on $\text{Dom}(D \otimes \text{id})$ extends to a bounded operator in $\ell^2(\Gamma) \otimes \ell^2(\mathbb{N})$. Let

$$
\mathcal{B}_\infty(\Gamma, \sigma) = \bigcap_{k \in \mathbb{N}} \text{Dom}(\partial^k) \cap C^*_r(\Gamma, \sigma) \otimes \mathcal{K}
$$

and

$$
\mathcal{B}(\Gamma, \sigma) = \left\{ T \in \mathcal{B}_\infty(\Gamma, \sigma) : \partial^k(T) \circ (\text{id} \otimes \Delta) \text{ is bounded } \forall k \in \mathbb{N} \right\}.
$$

Then $\mathcal{B}(\Gamma, \sigma)$ is a left ideal in $\mathcal{B}_\infty(\Gamma, \sigma)$ - this follows from the observation that, since $\partial$ is a derivation,

$$
\partial^k(T \circ S) \circ (\text{id} \otimes \Delta) = \sum_{m=0}^{k} \binom{k}{m} \partial^m(T) \circ \partial^{k-m}(S) \circ (\text{id} \otimes \Delta).
$$

One has the following sufficient conditions for an operator $A \in \mathcal{A}_{\ell^2(\mathbb{N})}^L(\Gamma, \sigma)$ to belong to the algebra $\mathcal{B}(\Gamma, \sigma)$. For any $T \in \mathcal{B}(\ell^2(\mathbb{N}))$, let $T_{ij} = (T(\delta_j), \delta_i), i, j \in \mathbb{N}$, be the matrix elements of $T$. Let $\mathcal{R}$ be the subalgebra in $\mathcal{K}$, which consists of all compact operators in $\ell^2(\mathbb{N})$, which are given by rapidly decaying matrices,

$$
\mathcal{R} = \left\{ T \in \mathcal{K} : \sup \left\{ i^k j^l | T_{ij} | : i, j \in \mathbb{N} \right\} < \infty, \ \forall k, l \in \mathbb{N} \right\}.
$$

**Lemma 3.2.** If $A \in \mathcal{A}_{\ell^2(\mathbb{N})}^L(\Gamma, \sigma)$, $A = \sum_{\gamma \in \Gamma} T_{\gamma}^L \otimes A(\gamma)$ is such that $A(\gamma) \in \mathcal{R}$ and also satisfies

$$
\sum_{\gamma} \ell(\gamma)^k \| A(\gamma) \Delta \| < \infty,
$$

for all positive integers $k$, then $A \in \mathcal{B}(\Gamma, \sigma)$.

**Proof.** As in the proof of Lemma 3.1, let $K_1 \subset K_2 \subset \cdots$ be a sequence of finite subsets of $\Gamma$ which is an exhaustion of $\Gamma$, i.e. $\bigcup_{j \geq 1} K_j = \Gamma$. For all $j \in \mathbb{N}$, define $A_j \in \mathcal{A}_{\ell^2(\mathbb{N})}^L(\Gamma, \sigma)$ as

$$
A_j(\gamma) = \begin{cases} 
A(\gamma) & \text{if } \gamma \in K_j; \\
0 & \text{otherwise}.
\end{cases}
$$

Then $A_j \in \mathcal{C}(\Gamma, \sigma) \otimes \mathcal{R}$, and, by Lemma 3.1, the sequence $A_j$ converges to $A$ in the norm topology of $\mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(\mathbb{N}))$. 

Let \( T \in \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{K} \). Since \( D\delta_e = 0 \), for any \( v \in \ell^2(\mathbb{N}) \), one has
\[
\partial^k(T)(\delta_e \otimes v) = (D \otimes \text{id})^k \circ T(\delta_e \otimes v) \\
= (D \otimes \text{id})^k \sum_{\gamma \in \Gamma} \delta_{\gamma} \otimes T(\gamma)v \\
= \sum_{\gamma \in \Gamma} \delta_{\gamma} \otimes \ell(\gamma)^k T(\gamma)v.
\]
Therefore, \( T \) belongs to \( \text{Dom}(\partial^k) \) for any \( k \in \mathbb{N} \), and
\[
(39) \quad [\partial^k T](\gamma) = \ell(\gamma)^k T(\gamma), \quad \gamma \in \Gamma.
\]

As in the proof of Lemma 3.2 using Lemma 3.3 and 3.4, one can establish that, for any \( k \in \mathbb{N} \), the sequences \( \partial^k(A_j) \) and \( \partial^k(A_j) \circ (\text{id} \otimes \Delta) \) converge in the norm topology of \( \mathcal{B}(\ell^2(\Gamma) \otimes \ell^2(\mathbb{N})) \). This proves that \( A \) belongs to \( \text{Dom}(\partial^k) \) for any \( k \in \mathbb{N} \), and the operator \( \partial^k(A) \circ (\text{id} \otimes \Delta) \) is bounded in \( \ell^2(\Gamma) \otimes \ell^2(\mathbb{N}) \). Therefore \( A \in \mathcal{B}(\Gamma, \sigma) \). \( \square \)

Following the arguments given in [Co2], III.5.\( \gamma \), we get

**Lemma 3.3.** The \( * \)-algebra \( \mathcal{B}(\Gamma, \sigma) \) is a smooth subalgebra of \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K} \).

**Proof.** By Lemma 3.2 it follows that \( \mathcal{B}(\Gamma, \sigma) \) contains \( \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R} \). Therefore \( \mathcal{B}(\Gamma, \sigma) \) is a dense \( * \)-subalgebra of \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K} \).

It is well-known (see, for instance, [HI] Theorem 1.2]) that \( \mathcal{B}_\infty(\Gamma, \sigma) \) is a smooth subalgebra of \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K} \). Since \( \mathcal{B}(\Gamma, \sigma) \) is a left ideal in \( \mathcal{B}_\infty(\Gamma, \sigma) \), the proof is completed by the following simple algebraic fact: if \( A \) is a spectral invariant subalgebra of an algebra \( B \) and \( I \) is a left ideal in \( A \), then \( I \) is spectral invariant in \( B \). \( \square \)

For any \( k \in \mathbb{N} \) and \( f \in C^*_r(\Gamma, \sigma) \), put
\[
\nu_k(f) = \left( \sum_{\gamma \in \Gamma} \ell(\gamma)^{2k} |f(\gamma)|^2 \right)^{1/2}, \quad k \in \mathbb{N}.
\]
We clearly have that \( \nu_k(f) < \infty \) for any \( k \in \mathbb{N} \) and \( f \in \mathbb{C}(\Gamma, \sigma) \).

Consider any element \( A \) in \( C^*_r(\Gamma, \sigma) \otimes \mathcal{K} \), \( A = \sum_{\gamma \in \Gamma} T_{\gamma}^{L} \otimes A(\gamma) \). Let \( (A_{ij}(\gamma)) \) be the matrix, corresponding to \( A(\gamma) \). Then \( A_{ij} \in C^*_r(\Gamma, \sigma) \) for any \( i, j \in \mathbb{N} \). Put
\[
N_k(A) = \left( \sum_{i,j} \nu_k(A_{ij})^2 \right)^{1/2}, \quad k \in \mathbb{N}.
\]

**Lemma 3.4.** For all \( A \in \mathcal{B}(\Gamma, \sigma) \) and \( k \in \mathbb{N} \), we have \( N_k(A) < \infty \).

**Proof.** One can be easily seen that, for any \( A \in C^*_r(\Gamma, \sigma) \otimes \mathcal{K} \),
\[
A(\delta_e \otimes \delta_j) = \sum_{\gamma \in \Gamma} A_{ij}(\gamma)\delta_\gamma \otimes \delta_i.
\]
Using (39), one has
\[ \partial^k (A) \circ (\text{id} \otimes \Delta)(\delta_e \otimes \delta_j) = j \partial^k (A)(\delta_e \otimes \delta_j) \]
\[ = j \sum_{i \in \mathbb{N}, \gamma \in \Gamma} \ell(\gamma)^k A_{ij}(\gamma) \delta_i \otimes \delta_i. \]

Therefore for \( A \in \mathcal{B}(\Gamma, \sigma) \) and \( k \in \mathbb{N} \), there is a positive constant \( C \) such that
\[ \sum_{i \in \mathbb{N}, \gamma \in \Gamma} \ell(\gamma)^2 k |A_{ij}(\gamma)|^2 < C j^{-2}. \]

Hence,
\[ \sum_{i,j \in \mathbb{N}, \gamma \in \Gamma} \ell(\gamma)^2 k |A_{ij}(\gamma)|^2 < C, \]
that completes the proof. \( \square \)

3.2. **Group cocycles and cyclic cocycles.** The cyclic cocycles that we consider arise from normalised group cocycles on \( \Gamma \). Recall (see, for instance, [Gui]) that a (homogeneous) group \( k \)-cocycle is a map \( h: \Gamma^{k+1} \rightarrow \mathbb{C} \) satisfying the identities
\[ h(\gamma\gamma_0, \ldots, \gamma\gamma_k) = h(\gamma_0, \ldots, \gamma_k); \]
\[ \sum_{i=0}^{k+1} (-1)^i h(\gamma_0, \ldots, \gamma_{i-1}, \gamma_i, \gamma_i+1, \ldots, \gamma_{k+1}) = 0. \]

Then an (inhomogeneous) group \( k \)-cocycle \( c \in Z^k(\Gamma, \mathbb{C}) \) that is associated to such an \( h \) is given by
\[ c(\gamma_1, \ldots, \gamma_k) = h(e, \gamma_1, \gamma_1\gamma_2, \ldots, \gamma_1 \ldots \gamma_k). \]

It can be easily checked that \( c \) satisfies the following identity
\[ c(\gamma_1, \gamma_2, \ldots, \gamma_k) + \sum_{i=0}^{k-1} (-1)^{i+1} c(\gamma_0, \ldots, \gamma_{i-1}, \gamma_i, \gamma_{i+1}, \gamma_{i+2}, \ldots, \gamma_k) + (-1)^{k+1} c(\gamma_0, \gamma_1, \ldots, \gamma_{k-1}) = 0. \]

A group \( k \)-cocycle is said to be normalised (in the sense of Connes), if \( c(\gamma_1, \gamma_2, \ldots, \gamma_k) \) is zero if either \( \gamma_i = e \) for some \( i \) or if \( \gamma_1 \ldots \gamma_k = e \).

Recall that a cyclic \( k \)-cocycle on an algebra \( A \) is a \( k+1 \)-linear functional \( \phi \) on \( A \), satisfying the following identities
\[ \phi(f_k, f_0, \ldots, f_{k-1}) = (-1)^k \phi(f_0, f_1, \ldots, f_k), \]
\[ b\phi(f_0, f_1, \ldots, f_{k+1}) \equiv \sum_{0 \leq j \leq k} (-1)^j \phi(f_0, \ldots, f_j f_{j+1}, \ldots, f_{k+1}) + (-1)^{k+1} \phi(f_{k+1} f_0, f_1, \ldots, f_k) = 0, \]
where \( f_0, f_1, \ldots, f_{k+1} \in A \).
Similarly, we get for $\tau$ from where (41) follows immediately.

Using these identities and (40), we obtain

$$
\tau_c(a_0 \delta_{\gamma_0}, \ldots, a_k \delta_{\gamma_k}) = \begin{cases} 
  a_0 \ldots a_k c(\gamma_1, \ldots, \gamma_k) \text{tr}(\delta_{\gamma_0} \ast \delta_{\gamma_1} \ast \ldots \ast \delta_{\gamma_k}) & \text{if } \gamma_0 \ldots \gamma_k = e; \\
  0 & \text{otherwise},
\end{cases}
$$

where $a_j \in \mathbb{C}$ for $j = 0, 1, \ldots k$.

**Lemma 3.5.** For any normalised group cocycle $c \in Z^k(\Gamma, \mathbb{C})$, $k = 0, \ldots, \dim M$, the functional $\tau_c$ defined by the formula (43) is a cyclic cocycle on $\mathbb{C}(\Gamma, \sigma)$.

**Proof.** It is clearly sufficient to check the identity (41) in the case when $f_j = a_j \delta_{\gamma_j}, j = 0, 1, \ldots, k$, with $a_j \in \mathbb{C}$ and $\gamma_0 \gamma_1 \ldots \gamma_k = e$. Then, since $c$ is normalised, (40) implies that

$$
c(\gamma_1, \gamma_2, \ldots, \gamma_k) + (-1)^{k+1} c(\gamma_0, \gamma_1, \ldots, \gamma_{k-1}) = 0,
$$

from where (41) follows immediately.

To prove the identity (42), we again assume that $f_j = a_j \delta_{\gamma_j}, j = 0, 1, \ldots, k + 1$, with $a_j \in \mathbb{C}$ and $\gamma_0 \gamma_1 \ldots \gamma_{k+1} = e$. Then we have

$$
\tau_c(f_0 f_1, f_2, \ldots, f_{k+1}) = \tau_c(a_0 \delta_{\gamma_0} \ast a_1 \delta_{\gamma_1} \ast a_2 \delta_{\gamma_2} \ast \ldots \ast a_{k+1} \delta_{\gamma_{k+1}}) \\
= \sigma(\gamma_0, \gamma_1) \tau_c(a_0 a_1 \delta_{\gamma_0 \gamma_1} \ast a_2 \delta_{\gamma_2} \ast \ldots \ast a_{k+1} \delta_{\gamma_{k+1}}) \\
= \sigma(\gamma_0, \gamma_1) a_0 \ldots a_{k+1} c(\gamma_2, \ldots, \gamma_{k+1}) \text{tr}(\delta_{\gamma_0} \ast \delta_{\gamma_1} \ast \delta_{\gamma_2} \ast \ldots \ast \delta_{\gamma_{k+1}})
$$

Similarly, we get for $j = 1, 2, \ldots, k$

$$
\tau_c(f_0, \ldots, f_j f_{j+1}, \ldots, f_{k+1}) = a_0 \ldots a_{k+1} c(\gamma_1, \ldots, \gamma_j \gamma_{j+1}, \ldots, \gamma_{k+1}) \text{tr}(\delta_{\gamma_0} \ast \delta_{\gamma_1} \ast \delta_{\gamma_2} \ast \ldots \ast \delta_{\gamma_{k+1}}),
$$

and

$$
\tau_c(f_{k+1} f_0, f_1, \ldots, f_k) = a_0 \ldots a_{k+1} c(\gamma_1, \ldots, \gamma_k) \text{tr}(\delta_{\gamma_0} \ast \delta_{\gamma_1} \ast \delta_{\gamma_2} \ast \ldots \ast \delta_{\gamma_{k+1}}).
$$

Using these identities and (40), we obtain

$$
\begin{align*}
  b \tau_c(f_0, f_1, \ldots, f_{k+1}) &= a_0 \ldots a_{k+1} \text{tr}(\delta_{\gamma_0} \ast \delta_{\gamma_1} \ast \delta_{\gamma_2} \ast \ldots \ast \delta_{\gamma_{k+1}}) \\
  &\quad (c(\gamma_2, \ldots, \gamma_{k+1}) + \sum_{j=1}^{k} (-1)^j c(\gamma_1, \ldots, \gamma_j \gamma_{j+1}, \ldots, \gamma_{k+1}) \\
  &\quad + (-1)^{k+1} c(\gamma_1, \ldots, \gamma_k)) = 0,
\end{align*}
$$

that completes the proof. \qed

Of particular interest is the case $k = 2$, when the formula (43) reduces to

$$
\tau_c(a_0 \delta_{\gamma_0}, a_1 \delta_{\gamma_1}, a_2 \delta_{\gamma_2}) = \begin{cases} 
  a_0 a_1 a_2 c(\gamma_1, \gamma_2) \sigma(\gamma_1, \gamma_2) \sigma(\gamma_2^{-1}, \gamma_2) & \text{if } \gamma_0 \gamma_1 \gamma_2 = e; \\
  0 & \text{otherwise}.
\end{cases}
$$
Lemma 3.6. Let a group $\Gamma$ have property (RD) and a normalised group $k$-cocycle $c$ be polynomially bounded. Then the associated cyclic cocycle $\tau_c \# \text{Tr}$ on $\mathbb{C}(\Gamma, \bar{\sigma}) \otimes \mathcal{R}$ is continuous for the norm $\nu_K$, for $K$ sufficiently large.

**Proof.** For all $f_0, f_1, \ldots, f_k \in \mathbb{C}(\Gamma, \bar{\sigma})$, one has,

$$\tau_c(f_0, f_1, \ldots, f_k) = \sum_{\gamma_0 \gamma_1 \ldots \gamma_k = e} f_0(\gamma_0) \ldots f_k(\gamma_k) c(\gamma_1, \ldots, \gamma_k) \text{tr}_\Gamma(\delta_{\gamma_0} * \delta_{\gamma_1} \ldots * \delta_{\gamma_k}).$$
Using the Haagerup inequality, we get
\[
|τ_c(f_0, f_1, \ldots, f_k)| = |τ_c(f_1, f_2, \ldots, f_k, f_0)|
\leq C \sum_{γ_0 γ_1 \cdots γ_k = e} |f_1(γ_1)| \cdots |f_k(γ_k)||f_0(γ_0)|(1 + ℓ(γ_1))^{α_1}(1 + ℓ(γ_2))^{α_2} \cdots (1 + ℓ(γ_k))^{α_k}
= C|(1 + ℓ)^{α_1} f_1| \cdots |(1 + ℓ)^{α_k} f_k| |f_0|(e)
\leq C(C')^n |f_0| \prod_{j=1}^k ν_N(|(1 + ℓ)^{α_j} f_j|)
= C(C')^n ν_0(f_0) \prod_{j=1}^k ν_{N+a_j}(|f_j|)
= C(C')^n ν_0(f_0) \prod_{j=1}^k ν_{N+a_j}(f_j),
\]
that proves the desired continuity property.

\[\square\]

**Lemma 3.7.** If \(Γ\) has property (RD), and given a polynomially bounded normalized group \(k\)-cocycle \(c\), a cyclic \(k\)-cocycle \(τ_c#\) \(\text{Tr}\) on \(C(Γ, σ) ⊗ R\) given by the formula (44) extends by continuity to \(B(Γ, σ)\).

**Proof.** The proof is a word-by-word repetition of the proof of Lemma 6.4 in [CM90]. Take any \(f_i = \sum_{γ ∈ Γ} δ_γ ⊗ f_i(γ) ∈ C(Γ, σ) ⊗ R, l = 0, 1, \ldots, k.\) Represent any \(f_i(γ) ∈ R\) by a matrix \((f_{ij}^l(γ))\).

Then, for any \(l = 0, 1, \ldots, k, i ∈ N\) and \(j ∈ N\), we have \(f_{ij}^l ∈ C(Γ, σ)\), and one can see that
\[
τ_c# \text{Tr}(f_0, f_1, \ldots, f_k) = \sum_{i_0, i_1, \ldots, i_k} τ_c(f_{i_0 i_1}^0, f_{i_1 i_2}^1, \ldots, f_{i_k i_0}^k).
\]

So, using Lemma 3.6, we have
\[
|τ_c# \text{Tr}(f_0, f_1, \ldots, f_k)| \leq \sum_{i_0, i_1, \ldots, i_k} |τ_c(f_{i_0 i_1}^0, f_{i_1 i_2}^1, \ldots, f_{i_k i_0}^k)|
\leq C \sum_{i_0, i_1, \ldots, i_k} ν_K(f_{i_0 i_1}^0)ν_K(f_{i_1 i_2}^1) \cdots ν_K(f_{i_k i_0}^k)
\]
with some natural \(K\). Then we use the following inequality
\[
\sum_{i_0, i_1, \ldots, i_k} α_{0}^i_0 α_{1}^i_1 \cdots α_{k}^i_k \leq \prod_{l=0}^{k} \left(\sum_{i,j} (α_{ij}^l)^2\right)^{1/2},
\]
which holds for any \(k ≥ 1\), that gives us the estimate
\[
|τ_c# \text{Tr}(f_0, f_1, \ldots, f_k)| \leq C N_K(f_0) N_K(f_1) \cdots N_K(f_k),
\]
and concludes the proof. \[\square\]
3.3. A vanishing lemma for pairing with cyclic cocycles. Let a group $\Gamma$ be a discrete group, $c$ be a normalised group $k$-cocycle on $\Gamma$ ($k$ even), and $\tau_c$ be the associated cyclic cocycle on $C(\Gamma, \hat{\sigma})$. By the pairing theory of [Co], we get an additive map

$$[\tau_c \# \text{Tr}]: K_0(C(\Gamma, \hat{\sigma}) \otimes R) \rightarrow \mathbb{R}.$$ 

Explicitly, $[\tau_c \# \text{Tr}][(\mathbb{1} - [f])] = \overline{\tau}_c(e, \ldots, e) - \overline{\tau}_c(f, \ldots, f)$, where $e, f$ are idempotent matrices with entries in $C(\Gamma, \hat{\sigma}) \otimes R$, which is the unital algebra obtained by adding the identity to $C(\Gamma, \hat{\sigma}) \otimes R$, and $\overline{\tau}_c$ denotes the canonical extension of $\tau_c \# \text{Tr}$ to $C(\Gamma, \hat{\sigma}) \otimes \overline{R} \otimes M_N(\mathbb{C})$ defined as follows;

$$\overline{\tau}_c(f_0 \otimes R_0, \ldots, f_k \otimes R_k) = \text{tr}(R_0 \ldots R_k) \tau_c \# \text{Tr}(f_0, \ldots, f_k),$$

where $f_j \in C(\Gamma, \hat{\sigma}) \otimes \overline{R}, R_j \in M_N(\mathbb{C}), j = 0, 1, \ldots, k$.

Recall that $\text{Tr}_R$ denotes the tensor product of the canonical finite trace $\text{tr}_\Gamma$ on $C(\Gamma, \hat{\sigma})$ and the standard trace $\text{Tr}$ on $R$.

**Lemma 3.8.** Let $c$ be a normalised $k$-cocycle on a discrete group $\Gamma$ ($k$ even) and $\tau_c \# \text{Tr}$ be the associated cyclic $k$-cocycle on the group algebra $C(\Gamma, \hat{\sigma}) \otimes R$. Let $I \otimes P$ be the tensor product of the identity in $C(\Gamma, \hat{\sigma})$ and a projection $P$ in $R$. Then we have,

$$\text{Tr}_R(I \otimes P) = \text{rank}(P); \quad \tau_c \# \text{Tr}(I \otimes P, \ldots, I \otimes P) = 0 \quad \text{for } k > 0.$$

**Proof.** Observe that $I \otimes P = \delta_e \otimes P$. The statement is trivially true when $k = 0$. For $k > 0$, we have

$$\tau_c \# \text{Tr}(I \otimes P, \ldots, I \otimes P) = \tau_c \# \text{Tr}(\delta_e \otimes P, \ldots, \delta_e \otimes P)$$

$$= \text{rank}(P)\tau_c(\delta_e, \ldots, \delta_e)$$

$$= \text{rank}(P)c(e, \ldots, e) \text{tr}_R(\delta_e * \delta_e * \ldots * \delta_e).$$

Since $c$ is a normalised group cocycle, $c(e, \ldots, e) = 0$, and the result follows. \hfill \square

4. Semiclassical vanishing theorems for spectral projections

4.1. General results on equivalence of projections in smooth subalgebras. In the setting of Section 2.1, suppose, in addition, that there is given a smooth subalgebra $\mathfrak{A}_0$ in the $C^*$-algebra $\mathfrak{A}$ such that:

- for any $t > 0$, the operators $e^{-tA_l}$ belong to $\pi_l(\mathfrak{A}_0)$, $l = 1, 2$.

Consider an interval $(a_1, b_1)$, which does not intersect with the spectrum of $A_1$. Let $a_2, b_2$ be given by the formulas (16) and (17). Suppose that $\alpha_1 > a_1 + \gamma_1, \alpha_2 > b_2 + \gamma_2$ and $b_2 > a_2$. By Theorem 2.1, the interval $(a_2, b_2)$ does not intersect with the spectrum of $A_2$.

Note that $E_1(\lambda_1) = \chi_{[e^{-t\lambda_1}, \infty)}(e^{-tA_1})$. Since $\lambda_1 \notin \text{spec}(A_1)$, by the Riesz formula one has,

$$E_1(\lambda_1) = \frac{1}{2\pi i} \int_C (\lambda - e^{-tA_1})^{-1} d\lambda,$$
where $C$ is a contour intersecting the real axis at $e^{-tA_1}$ and at some large positive number not in the spectrum of $e^{-tA_1}$. It follows that $E_1(\lambda_1)$ is a holomorphic function of $e^{-tA_1}$, and therefore one has $E_1(\lambda_1) \in \pi_1(\mathfrak A_0)$. Similarly, for any $\lambda_2 \in (a_2, b_2)$, the spectral projection $E_2(\lambda_2)$ belongs to $\pi_2(\mathfrak A_0)$.

**Theorem 4.1.** The projections $V_1 E_1(\lambda_1) V_1^{-1}$ and $V_2 E_2(\lambda_2) V_2^{-1}$ are Murray-von Neumann equivalent in $\pi(\mathfrak A_0)$ for any $\lambda_1 \in (a_1, b_1)$ and $\lambda_2 \in (a_2, b_2)$.

**Proof.** Consider a bounded operator $T = V_2 E_2(\lambda_2) i_2 J p_1 E_1(\lambda_1) V_1^{-1}$ in $\mathcal H$. As shown in the proof of Theorem 2.1, the operator $T$ belongs to $\pi(\mathfrak A)$ and is invertible as an operator from $V_1(\text{Im } E_1(\lambda_1))$ to $V_2(\text{Im } E_2(\lambda_2))$. Since $\pi(\mathfrak A_0)$ is dense in $\pi(\mathfrak A)$, there exists an operator $T_1$ in $\pi(\mathfrak A_0)$ such that the operator $P = V_2 E_2(\lambda_2) V_2^{-1} T_1 V_1 E_1(\lambda_1) V_1^{-1}$ is invertible as an operator from $V_1(\text{Im } E_1(\lambda_1))$ to $V_2(\text{Im } E_2(\lambda_2))$. Then the operator $P$ as an operator in $\mathcal H$ belongs to $\pi(\mathfrak A_0)$, and its image $\text{Im } P = V_2(\text{Im } E_2(\lambda_2))$ is closed. The desired statement follows from the following lemma.

**Lemma 4.2.** Let $\mathfrak A$ be a $C^*$-algebra, $\mathcal H$ a Hilbert space equipped with a faithful $\ast$-representation of $\mathfrak A$, $\pi : \mathfrak A \to \mathcal B(\mathcal H)$, $\mathfrak A_0$ a smooth subalgebra in $\mathfrak A$. If a bounded operator $P$ in $\mathcal H$ belongs to $\pi(\mathfrak A_0)$ and has closed image, and $P = US$ is its polar decomposition, then $U, S \in \pi(\mathfrak A_0)$.

**Proof.** As shown in the proof of Lemma 2.11, zero is an isolated point in the spectrum of $P^* P$. Since $\mathfrak A_0$ is stable under holomorphic functional calculus, the operator $S = \sqrt{P^* P}$ is in $\mathfrak A_0$ and has zero as an isolated point in its spectrum. Furthermore, the function $f$ introduced in the proof of Lemma 2.10 extends to a holomorphic function in a neighborhood of the spectrum of $S$, that implies that $S(-1) = f(S)$ and $U = PS(-1)$ are also in $\pi(\mathfrak A_0)$.

Applying Lemma 4.2 to the operator $P$ as above, we obtain an isometry $U \in \pi(\mathfrak A_0)$, which gives the desired equivalence of the projections $V_1 E_1(\lambda_1) V_1^{-1}$ and $V_2 E_2(\lambda_2) V_2^{-1}$. □

### 4.2. Proof of the semiclassical vanishing theorem of the higher traces of spectral projections

In this Section, we prove the second part of Theorem 4.1 and Corollary 1. For this, we apply Theorem 1.21 in the setting of Section 2.3. We will use the notation of this section. Here we make a particular choice of the unitary isomorphisms $V_1 : L^2(\mathbb R^n, C^k)^N \to \ell^2(\mathbb N)$ and $V_2 : L^2(\mathcal F, \tilde E|_F) \to \ell^2(\mathbb N)$. Namely, we define $V_1$, using the spectral decomposition for the model operator $K = K(1)$ in $L^2(\mathbb R^n, C^k)^N$. More precisely, suppose that $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ are the eigenvalues of the operator $K$ (counting with multiplicities), and $\phi_j$ the corresponding eigenfunctions, which form a complete orthonormal system in $L^2(\mathbb R^n, C^k)^N$. Then $V_1$ is defined as $V_1 \phi_j = \delta_j, j \in \mathbb N$. Similarly, take any second order self-adjoint $(\Gamma, \sigma)$-invariant elliptic differential operator $P$ in $L^2(\tilde M, \tilde E)$. Define the unitary isomorphism $V_2 : L^2(\mathcal F, \tilde E|_F) \to \ell^2(\mathbb N)$, using the orthonormal basis of eigenvectors for the operator $P$ in $L^2(\mathcal F, \tilde E|_F) \cong L^2(\tilde M, \tilde E)$ with the $(\Gamma, \sigma)$-periodic boundary conditions as above.

Take a smooth subalgebra $\mathfrak A_0$ of the $C^*$-algebra $\mathfrak A = C^*_r(\Gamma, \bar \sigma) \otimes K$ to be $\mathcal B(\Gamma, \sigma)$.

**Lemma 4.3.** For any $t > 0$, the operator $V_1 e^{-tA_1} V_1^*$ belongs to $\mathcal B(\Gamma, \sigma) \subset C^*_r(\Gamma, \bar \sigma) \otimes K$. 
Proof. Since $\mathcal{V}_1 e^{-tA_1} \mathcal{V}_1^* = \text{id} \otimes \mathcal{V}_1 e^{-tK(\mu)} \mathcal{V}_1^*$, it suffices to prove that the operator $V_1 e^{-tK(\mu)} \mathcal{V}_1^* \Delta$ is bounded in $\ell^2(\mathbb{N})$. By definition of $V_1$, the operator $V_1 K \mathcal{V}_1^*$ in $\ell^2(\mathbb{N})$ is given by

$$V_1 K \mathcal{V}_1^* \delta_j = \lambda_j \delta_j, \quad j \in \mathbb{N}.$$  

It is well-known that $\lambda_j \sim C j^{1/n}, j \to \infty$. Therefore, the operator $K^{-n} \mathcal{V}_1^* \Delta \mathcal{V}_1$ is bounded in $L^2(\mathbb{R}^n, \mathbb{C}^k)^N$. Since $V_1 e^{-tK(\mu)} \mathcal{V}_1^* \Delta = V_1 e^{-tK(\mu)} K^{-n} \mathcal{V}_1^* \Delta \mathcal{V}_1$, and the operator $e^{-tK(\mu)} K^n$ is bounded in $L^2(\mathbb{R}^n, \mathbb{C}^k)^N$, this immediately completes the proof. \hfill \Box

Lemma 4.4. For all $t > 0$, $\mathcal{V}_2 e^{-tH(\mu)} \mathcal{V}_2^* \in \mathcal{B}(\Gamma, \sigma) \subset C^*_r(\Gamma, \sigma) \otimes \mathcal{K}$.

Proof. First, recall the following well-known properties of the heat operator $e^{-tH(\mu)}$, cf. [C, Ko]. Let $d$ denote the Riemannian distance function on $\tilde{M}$.

Lemma 4.5. The Schwartz kernel $k(t, x, y)$ of the heat operator $e^{-tH(\mu)}$ is smooth for all $t > 0$. Moreover, for any $t > 0$ and for any $(\Gamma, \sigma)$-invariant differential operators $A = a(x, D_x)$ and $B = b(x, D_x)$ in $C^\infty(\tilde{M}, \tilde{E})$ there are positive constants $C_1, C_2$ depending on $\mu$ such that the following off-diagonal estimate holds

$$|a(x, D_x)b(y, D_y)k(t, x, y)| \leq C_1 e^{-C_2 d(x, y)^2}, \quad x \in \tilde{M}, \quad y \in \tilde{M}.$$  

We have $\mathcal{V}_2 e^{-tH(\mu)} \mathcal{V}_2^* \in \mathcal{A}^L_{\ell^2(\mathbb{N})}(\Gamma, \sigma)$, so we can write the operator $\mathcal{V}_2 e^{-tH(\mu)} \mathcal{V}_2^*$ as

$$\mathcal{V}_2 e^{-tH(\mu)} \mathcal{V}_2^* = \sum_{\gamma \in \Gamma} T^L_{\gamma} \otimes h_{t, \mu}(\gamma)$$

with some $h_{t, \mu}(\gamma) \in \mathcal{B}(\ell^2(\mathbb{N}))$. We will identify the space $L^2(\mathcal{F}, \tilde{E}|_\mathcal{F})$ with the subspace in $L^2(\tilde{M}, \tilde{E})$, which consists of sections from $L^2(\tilde{M}, \tilde{E})$, vanishing outside of $\mathcal{F}$. As above, $i : \mathcal{F} \to \tilde{M}$ denote the inclusion map.

Lemma 4.6. The operator $V_2^* h_{t, \mu}(\gamma) V_2$ is given by the restriction of the operator $i^* T^L_{\gamma} e^{-H(\mu)}$ to $L^2(\mathcal{F}, \tilde{E}|_\mathcal{F})$.

Proof. Recall that a unitary operator $\mathcal{V}_2 : \mathcal{H} \to \mathcal{H}$ is defined as $\mathcal{V}_2 = (\text{id} \otimes \mathcal{V}_2) \circ \mathcal{U}$, where $\mathcal{U}$ is the $(\Gamma, \sigma)$-equivariant isometry $[\mathfrak{S}]$. Therefore, we have

$$\mathcal{U} e^{-tH(\mu)} \mathcal{U}^* = \sum_{\gamma \in \Gamma} T^L_{\gamma} \otimes V_2^* h_{t, \mu}(\gamma) V_2.$$  

Let $\phi \in L^2(\mathcal{F}, \tilde{E}|_\mathcal{F})$. By definition of $\mathcal{U}$, it easily follows that $\mathcal{U}^*(\delta_c \otimes \phi) \in L^2(\tilde{M}, \tilde{E})$ coincides with $\phi$. Therefore, by $[\mathfrak{S}]$, we get

$$\mathcal{U} e^{-tH(\mu)} \mathcal{U}^*(\delta_c \otimes \phi) = \sum_{\gamma \in \Gamma} \delta_{\gamma} \otimes i^* T^L_{\gamma} e^{-H(\mu)} \phi,$$

that immediately completes the proof. \hfill \Box
As in the proof of Lemma 4.3, using the Weyl asymptotic formula \( \alpha_j \sim C_j^2/n, j \to \infty \), for the eigenvalues \( \alpha_j \) of the operator \( P \), one can show that the operator \( P^{-n/2}V^*_2 \Delta V_2 \) is bounded in \( L^2(\mathcal{F}, \mathcal{E}|\mathcal{F}) \). By this fact and Lemma 4.6, it follows that

\[
\|h_{t,\mu}(\gamma)\Delta\| = \|V^*_2 h_{t,\mu}(\gamma)V_2 V^*_2 \Delta V_2\| = \|i^* T_\gamma e^{-tH(\mu)} P^m P^{-m} V^*_2 \Delta V_2\| \leq C_3 \|i^* T_\gamma e^{-tH(\mu)} P^m\|
\]

for any natural \( m > n/2 \) with some positive constant \( C_3 \).

It is well known that

\[
\ell(\gamma) \leq C_4 (\inf_{x,y \in \mathcal{F}} d(\gamma x, y) + 1)
\]

for some positive constant \( C_4 \). From (18) and Lemma 4.5, we get

\[
\|i^* T_\gamma e^{-tH(\mu)} P^m\| \leq C_5 e^{-C_6 \ell(\gamma)^2}.
\]

Observe that one has the estimate

\[
\# \{ \gamma \in \Gamma \mid \ell(\gamma) \leq R \} \leq C_7 e^{C_8 R}
\]

for some positive constants \( C_7, C_8 \), since the growth rate of the volume of balls in \( \Gamma \) is at most exponential. By (47), (49) and (50), it follows that

\[
\sum_\gamma \ell(\gamma)^k \|h_{t,\mu}(\gamma)\Delta\| < \infty
\]

for all positive integers \( k \). By Lemma 4.2, this implies that \( \mathcal{V} e^{-tH(\mu)} \mathcal{V}_2^* \in \mathcal{B}(\Gamma, \sigma) \subset C^*_r(\Gamma, \delta) \otimes \mathcal{K} \) for all \( t > 0 \).

**Remark 4.7.** Since the Schwartz kernel of \( e^{-tH(\mu)} \) is smooth \( \forall \gamma \in \Gamma \) by Lemma 4.3, it follows from the proof of Lemma 4.4 that \( h_{t,\mu}(\gamma) \in \mathcal{R} \forall \gamma \in \Gamma \) (cf. also Lemma 5 in III.4.3 in [Co2]).

By Lemmas 4.3 and 4.4, it follows that, for any \( t > 0 \), the operators \( e^{-tA_1} \) belong to \( \pi_l(\mathcal{A}_0) \), \( l = 1, 2 \). So we can apply Theorem 4.1 that immediately completes the proof of the second part of Theorem 2. Now Corollary 4 follows immediately from the second part of Theorem 2 and Lemma 4.8.

In the case under consideration, we can give a more explicit description of an operator \( \mathcal{U} \) that provides Murray-von Neumann equivalence of the projections \( \mathcal{V}_1 E_1(\lambda_1) \mathcal{V}_1^{-1} \) and \( \mathcal{V}_2 E_2(\lambda_2) \mathcal{V}_2^{-1} \).

This is based on the following

**Lemma 4.8.** For any \( a \in \mathcal{A}_0 = \mathcal{B}(\Gamma, \sigma) \), the bounded operator \( \mathcal{V}_2 i_2 J_1 V_1^{-1} \pi(a) \) in \( \ell^2(\Gamma) \otimes \ell^2(\mathcal{N}) \) belongs to \( \pi(\mathcal{A}_0) \).

**Proof.** We have \( \mathcal{V}_2 i_2 J_1 V_1^{-1} = \text{id} \otimes \mathcal{V}_2 j_2 J_0 r_1 V_1^{-1} \). It follows that, for any \( a \in C^*_r(\Gamma, \delta) \otimes \mathcal{K} \), the operator \( \mathcal{V}_2 i_2 J_1 V_1^{-1} \pi(a) \) belongs to \( \pi(C^*_r(\Gamma, \delta) \otimes \mathcal{K}) \). Moreover, the bounded operator \( \mathcal{V}_2 i_2 J_1 V_1^{-1} \) in \( \ell^2(\Gamma) \otimes \ell^2(\mathcal{N}) \) commutes with \( D \otimes \text{id} \) that implies

\[
\mathcal{V}_2 i_2 J_1 V_1^{-1} \in \bigcap_{k \in \mathbb{N}} \text{Dom} \tilde{\partial}^k.
\]

Since the space \( \{ P \in \bigcap_{k \in \mathbb{N}} \text{Dom} \tilde{\partial}^k : \tilde{\partial}^k(P) \circ (\text{id} \otimes \Delta) \) is bounded \( \forall k \in \mathbb{N} \} \) is a left ideal in \( \bigcap_{k \in \mathbb{N}} \text{Dom} \tilde{\partial}^k \) (cf. [37]), we get \( \mathcal{V}_2 i_2 J_1 V_1^{-1} \pi(a) \in \mathcal{B}(\Gamma, \sigma) \) for any \( a \in \mathcal{B}(\Gamma, \sigma) \) as desired. \( \square \)
By Lemma 4.8, the operator $T = \mathcal{V}_2E_2(\lambda_2)\mathcal{V}_2E_1(\lambda_1)\mathcal{V}_1^{-1}$ in $\mathcal{H}$ belongs to $\pi(\mathfrak{A}_0)$, and the operator $U$ that provides Murray-von Neumann equivalence of the projections $\mathcal{V}_1E_1(\lambda_1)\mathcal{V}_1^{-1}$ and $\mathcal{V}_2E_2(\lambda_2)\mathcal{V}_2^{-1}$ can be taken from the polar decomposition of this operator as in the proof of Theorem 4.1.

5. Quantum Hall effect

Since the results that we have obtained were essentially known for Euclidean space and applied to the Euclidean space model for the integer quantum Hall effect, we will focus on the hyperbolic space model for the fractional quantum Hall effect, [CHMM], [MM].

5.1. The Hamiltonian. We begin by reviewing the construction of the Hamiltonian. First we take as our principal model of hyperbolic space, the hyperbolic plane. This is the upper half-plane $\mathbb{H}$ in $\mathbb{C}$ equipped with its usual Poincaré metric $(dx^2 + dy^2)/y^2$, and symplectic area form $\omega_{2\mathbb{H}} = dx \wedge dy/y^2$. The group $\text{SL}(2, \mathbb{R})$ acts transitively on $\mathbb{H}$ by Möbius transformations

$$x + iy = \zeta \mapsto g \cdot \zeta = \frac{a\zeta + b}{c\zeta + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Any Riemann surface of genus $g$ greater than 1 can be realised as the quotient of $\mathbb{H}$ by the action of its fundamental group realised as a discrete subgroup $\Gamma$ of $\text{SL}(2, \mathbb{R})$.

Choose a 1-form $A$ called a vector potential whose curvature is the uniform magnetic field $B = dA = \theta\omega_{2\mathbb{H}}$, whose flux is $\theta$. As in geometric quantisation we may regard $A$ as defining a Hermitian connection $\nabla = d + iA$ on the trivial line bundle $\mathcal{L}$ over $\mathbb{H}$, whose curvature is $i\theta\omega_{2\mathbb{H}}$. Using the Riemannian metric the Hamiltonian of an electron in this field is given in suitable units by

$$H = H_{A,V} = \nabla^*\nabla + \mu^{-2}V = (d + iA)^*(d + iA) + \mu^{-2}V,$$

where $V$ is an electric potential associated to a real material and $\mu$ is the coupling constant. $V$ is also assumed to be invariant under $\Gamma$ respecting a crystalline type structure. It can be checked that $H$ commutes with the projective $(\Gamma, \sigma)$-action on $L^2(\mathbb{H})$ as defined in the earlier sections.

5.2. Algebra of observables. Let $\mathcal{F}$ be a connected fundamental domain for the action of $\Gamma$ on $\mathbb{H}$. Take any second order self-adjoint $(\Gamma, \sigma)$-invariant elliptic differential operator $P$ in $L^2(\mathbb{H})$, for instance, $P = H_{A,V}(1)$. Define the unitary isomorphism $V_2 : L^2(\mathcal{F}) \to \ell^2(\mathbb{N})$, using the orthonormal basis $\{\varphi_j : j \in \mathbb{N}\}$ of eigenvectors for the operator $P$ in $L^2(\mathcal{F}) \cong L^2(\mathbb{H}/\Gamma)$ with the $(\Gamma, \sigma)$-periodic boundary conditions: $V_2\varphi_j = \delta_j, j \in \mathbb{N}$. Introduce a unitary operator $V_2 : L^2(\mathbb{H}) \to \ell^2(\Gamma) \otimes \ell^2(\mathbb{N})$ as $V_2 = (\text{id} \otimes V_2) \circ U$, where $U : L^2(\mathbb{H}) \cong \ell^2(\Gamma) \otimes L^2(\mathcal{F})$ is the $(\Gamma, \sigma)$-equivariant isometry of $\mathcal{F}$. This operator induces an isomorphism of the algebra $\mathcal{U}_{L^2(\mathbb{H})}(\Gamma, \sigma)$, consisting of operators on $L^2(\mathbb{H})$ that commute with the projective $(\Gamma, \sigma)$-action, with the von Neumann algebra $A_{L^2(\mathbb{N})}(\Gamma, \sigma) \cong A^L(\Gamma, \sigma) \otimes \mathcal{B}(\ell^2(\mathbb{N}))$.

Define the algebra of observables to be $\mathcal{B}(\Gamma, \sigma)$ introduced at the beginning of section 3 which is considered as a $*$-subalgebra of $\mathcal{U}_{L^2(\mathbb{H})}(\Gamma, \sigma)$. Recall that we have established in section 3 that $e^{-tH} \in \mathcal{B}(\Gamma, \sigma)$. The observables of the model include those spectral projections of the
Hamiltonian $H$ corresponding to gaps in the spectrum. The fact that such a projection belongs to $\mathcal{B}(\Gamma, \sigma)$ was established in section 3 by using the Riesz representation for the projection and the fact that $\mathcal{B}(\Gamma, \sigma)$ is closed under the holomorphic functional calculus. This justifies the choice of $\mathcal{B}(\Gamma, \sigma)$ as the algebra of observables.

5.3. Canonical derivations on the algebra of observables. Let $\Sigma_g = \mathbb{H}/\Gamma$ be the Riemann surface determined by quotienting by $\Gamma$. We follow the usual conventions (see for example [GH]) in fixing representative homology generators corresponding to cycles $A_j, B_j, j = 1, 2, \ldots, g$ with each pair $A_j, B_j$ intersecting in a common base point and all other intersection numbers being zero. Let $a_j, j = 1, \ldots, g$ be harmonic 1-forms dual to $A_j, j = 1, \ldots, g$ and $b_j, j = 1, 2, \ldots, g$ be harmonic 1-forms dual to $B_j, j = 1, 2, \ldots, g$. Let $\tilde{a}_j, \tilde{b}_j$ denote the lifts of $a_j, b_j$ to $\mathbb{H}$ respectively.

Define the functions on $\mathbb{H}$ given by,

$$\Omega_j(z) = i \int_u^z \tilde{a}_j, \quad \Omega_{j+g}(z) = i \int_u^z \tilde{b}_j,$$

where $u \in \mathbb{H}$ is a fixed point. Since $\tilde{a}_j, \tilde{b}_j$ are bounded 1-forms on $\mathbb{H}$, one sees that there are positive constants $C_j$ such that

$$|\Omega_j(z)| \leq C_j d(u, z) \quad \text{for all } z \in \mathbb{H}. \quad (51)$$

For any $j = 1, 2, \ldots, 2g$, denote by $\Omega_j$ the operator in $L^2(\mathbb{H})$ of multiplication by the function $\Omega_j$. Define

$$\delta_j(T) = [\Omega_j, T],$$

where $T \in \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R} \subset \mathcal{B}(\Gamma, \sigma)$ is considered as a bounded operator in $L^2(\mathbb{H})$.

**Lemma 5.1.** For any $T \in \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R}$, the operator $\delta_j(T)$ is in $\mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R}$.

**Proof.** Using the arguments given in the proof of Lemma 4.3, one can easily see that, under the isomorphism $T \in A^2_{L^2(\mathbb{H})}(\Gamma, \sigma) \mapsto \mathcal{V}_2^* TV_2 \in \mathcal{U}_{L^2(\mathbb{H})}(\Gamma, \sigma)$, the $\ast$-subalgebra $\mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R} \subset A^2_{L^2(\mathbb{H})}(\Gamma, \sigma)$ corresponds to the subalgebra in $\mathcal{U}_{L^2(\mathbb{H})}(\Gamma, \sigma)$, which consists of all $(\Gamma, \sigma)$-invariant bounded operators $Q$ in $L^2(\mathbb{H})$ whose Schwarz kernels $k_Q$ are smooth and properly supported (i.e. $k_Q(x, y) = 0$ when $d(x, y) > C$ with some constant $C > 0$). Recall that $(\Gamma, \sigma)$-invariance of $Q$ is equivalent to the relation (7) for its Schwarz kernel $k_Q$.

Take any $T \in \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R}$ considered an a bounded operator in $L^2(\mathbb{H})$. Let $k_T$ be its Schwarz kernel. Then the Schwarz kernel $k_{[\Omega_j, T]}$ of the operator $[\Omega_j, T]$ is given by

$$k_{[\Omega_j, T]}(x, y) = (\Omega_j(x) - \Omega_j(y))k_T(x, y), \quad x, y \in \bar{M}.$$

Clearly, $k_{[\Omega_j, T]}$ is smooth and properly supported.

It is easy to see that, for $\gamma \in \Gamma$, the difference $\Omega_j(\gamma, z) - \Omega_j(z)$ is constant independent of $z \in \mathbb{H}$. Using this fact, one can check that $k_{[\Omega_j, T]}$ satisfies (7), that completes the proof. \hfill \Box

Therefore $\delta_j, j = 1, 2, \ldots, 2g$, is a (densely defined) derivation on the algebra of observables $\mathcal{B}(\Gamma, \sigma)$. 
5.4. Hall conductance cyclic cocycle. In this subsection we recall the Kubo formula for the Hall conductivity. The reasoning is that the Hall conductivity is measured by determining the equilibrium ratio of the current in the direction of the applied electric field to the Hall voltage, which is the potential difference in the orthogonal direction. To calculate this mathematically we instead determine the component of the induced current that is orthogonal to the applied potential. The conductivity can then be obtained by dividing this quantity by the magnitude of the applied field. Interpreting the generators of the fundamental group as geodesics on hyperbolic space gives a family of preferred directions emanating from the base point. One of the basic results in [CHMM] is the following: the expectation of the current \( J_k \) is given by

\[
\text{Tr}_\Gamma(P\delta_k H) = i \text{Tr}_\Gamma(P[\delta_k P, \delta_k P]) = -iE_j \text{Tr}_\Gamma(P[\delta_j P, \delta_k P]),
\]

where \( E_j \) is the electric field in the \( j \) direction. Therefore one sees that,

The conductivity for currents in the \( k \) direction induced by electric fields in the \( j \) direction is given by

\[-i \text{Tr}_\Gamma(P[\delta_j P, \delta_k P]).\]

The following is Lemma 12 in [CHMM].

**Lemma 5.2.** For any \( j, k = 1, \ldots, 2g \), the formula

\[c_{j,k}(T_0, T_1, T_2) = \text{Tr}_\Gamma(T_0[\delta_j T_1, \delta_k T_2]) = \text{Tr}_\Gamma(T_0[\Omega_j, T_1][\Omega_k, T_2]), \quad T_0, T_1, T_2 \in \mathbb{C}(\Gamma, \bar{\sigma}) \otimes \mathcal{R},\]

defines a cyclic 2-cocycle on \( \mathbb{C}(\Gamma, \bar{\sigma}) \otimes \mathcal{R} \).

**Definition.** The Kubo formula for the Hall conductance cyclic 2-cocycle \( \text{tr}_K \) on a dense subalgebra \( \mathbb{C}(\Gamma, \bar{\sigma}) \otimes \mathcal{R} \) of the algebra of observables \( \mathcal{B}(\Gamma, \sigma) \) is defined as

\[
\text{tr}_K = \sum_{j=1}^{g} c_{j,j+g}.
\]

There is a symplectic map from \( \mathbb{H} \) to \( \mathbb{R}^{2g} \) given by \( \Xi : z \mapsto (\Omega_1(z), \ldots, \Omega_{2g}(z)) \). It is the lift to \( \mathbb{H} \) of the Abel-Jacobi map on \( \mathbb{H}/\Gamma \). Define a group 2-cocycle \( \Psi : \Gamma \times \Gamma \rightarrow \mathbb{R} \) as follows. Consider the straight-edged triangle \( \Delta(u, \gamma_1, \gamma_2) \) in \( \mathbb{R}^{2g} \) obtained by joining the 3 points \( \Xi(u) = 0, \Xi(\gamma_1^{-1}u) \) and \( \Xi(\gamma_2u) \). Then \( \Psi(\gamma_1, \gamma_2) \) is defined to be the symplectic area of \( \Delta(u, \gamma_1, \gamma_2) \), which is equal to

\[
\sum_{j=1}^{g} (\Omega_j(\gamma_1^{-1}u)\Omega_{j+g}(\gamma_2u) - \Omega_{j+g}(\gamma_1^{-1}u)\Omega_j(\gamma_2u)).
\]

We have seen earlier that \( |\Omega_j(\gamma, u)| \leq C_j \ d(u, \gamma, u) \leq C'_j \ell(\gamma) \). Using the Cauchy-Schwartz inequality and the fact that \( \ell(\gamma) = \ell(\gamma^{-1}) \), we see that there is a positive constant \( C \) such that

\[
|\Psi(\gamma_1, \gamma_2)| \leq C(1 + \ell(\gamma_1))^2(1 + \ell(\gamma_2))^2 \quad \text{for all} \quad \gamma_1, \gamma_2 \in \Gamma.
\]

That is, \( \Psi \) is a polynomially bounded group 2-cocycle on \( \Gamma \). Recall that the group 2-cocycle \( \Psi \) defines the cyclic 2-cocycle \( \tau_\Psi \) on \( \mathbb{C}(\Gamma, \bar{\sigma}) \). By Lemma 5.7, we see that \( \tau_\Psi \# \text{Tr} \) extends by continuity to \( \mathcal{B}(\Gamma, \sigma) \).

One of the main results of [CHMM] is the following.

**Theorem 5.3.** The Hall conductance cyclic 2-cocycle \( \text{tr}_K \) agrees with \( \tau_\Psi \# \text{Tr} \) on \( \mathbb{C}(\Gamma, \bar{\sigma}) \otimes \mathcal{R} \).
Hence by Lemma 3.7, the Hall conductance cyclic 2-cocycle $tr_K$ also extends by continuity to $B(\Gamma, \sigma)$. Therefore we are in a position to apply the Corollary 4 to deduce Corollary 5.

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