Wasserstein Contraction Bounds on Closed Convex Domains with Applications to Stochastic Adaptive Control

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Abstract—This paper is motivated by the problem of quantitatively bounding the convergence of adaptive control methods for stochastic systems to a stationary distribution. Such bounds are useful for analyzing statistics of trajectories and determining appropriate step sizes for simulations. To this end, we extend a methodology from (unconstrained) stochastic differential equations (SDEs) which provides contractions in a specially chosen Wasserstein distance. This theory focuses on unconstrained SDEs with fairly restrictive assumptions on the drift terms. Typical adaptive control schemes place constraints on the learned parameters and their update rules violate the drift conditions. To this end, we extend the contraction theory to the case of constrained systems represented by reflected stochastic differential equations and generalize the allowable drifts. We show how the general theory can be used to derive quantitative contraction bounds on a nonlinear stochastic adaptive regulation problem.

I. INTRODUCTION

Adaptive control has a rich history in the controls literature [1], [2], [3], and [4]. It has wide applications in areas such as robotics [5], aerospace systems [1], and electromechanical systems [6]. The typical approach utilizes Lyapunov-based design to update the parameters while guaranteeing stability.

In recent years, there has been a drive to connect adaptive control methods with techniques from reinforcement learning [7]–[10]. In parallel, methods from reinforcement learning have seen an explosion of work on linear giving precise optimality guarantees [11]–[13]. These works rely on precise convergence bounds that are fairly straightforward for linear systems, but substantially more complex in stochastic nonlinear systems.

Convergence of stochastic nonlinear systems is a vast area with numerous approaches, e.g. [14]–[17]. In order to derive convergence guarantees analogous to those available in linear systems, explicit quantitative bounds are required.

The motivation behind this paper is to derive quantitative convergence guarantees for stochastic adaptive control methods. To this end, we build upon methodologies at the intersection of stochastic differential equations and optimal transport [18], [19]. However, the existing methods in this area are too restrictive to be applied directly to common adaptive control schemes. In particular, these focus unconstrained processes with strong Lipschitz-like conditions on the drift term. However, in adaptive control, the parameters are typically constrained and their update rules often contain quadratic terms that violate the drift conditions.

Our primary contribution to stochastic convergence theory is an extension of the methodology from [18] to constrained processes with less restrictive drift conditions. We derive an explicit exponential contraction bound under a specially constructed Wasserstein distance. The result implies exponential convergence to a unique stationary distribution under a variety of measures, including total variation distance and Euclidean Wasserstein distances. We then show how this result can be used to prove exponential convergence in a feedback-linearizable stochastic adaptive regulation problem. Additionally, we show how a projection method based on reflected stochastic differential equations can be used to constrain the parameters to an arbitrary closed convex set. This provides a flexible alternative to handling constraints, which contrasts with more specialized projection operators commonly employed in adaptive control [1], [2], [4].

The remaining parts of the paper are organized as follows. Section II presents preliminary notation. Section III presents the main contraction results, while Section IV presents the application to stochastic adaptive control. Section V presents numerical results and we provide closing remarks in Section VI. The main contraction theorem is proved in the appendix.

II. NOTATION

Random variables are denoted in bold, e.g. $x$. Time indices are denoted by subscripts, e.g. $x_t$ denotes a stochastic process. We equip $\mathbb{R}^n$ with an inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. We interpret $x, y \in \mathbb{R}^n$ as column vectors and let $x^*$ denote the dual row vector such that $\langle x, y \rangle = x^* y$. (Since $\langle \cdot, \cdot \rangle$ is not necessarily the Euclidean inner product, we may have $x^* \neq x^T$.) More generally, for a matrix $G$, let $G^*$ denote its conjugate with respect to the inner product. If $A$ is a square matrix, its trace is denoted by $\text{tr}(A)$.

If $\mathcal{X}$ is a closed convex set in $\mathbb{R}^n$, then at any $y \in \mathbb{R}^n$ the normal cone of $\mathcal{X}$ is defined as

$$N_{\mathcal{X}}(y) = \{ v \in \mathbb{R}^n \mid \langle x - y, v \rangle \leq 0 \ \forall x \in \mathcal{X} \} ,$$

and the convex projection on $\mathcal{X}$ is $\Pi_{\mathcal{X}}(y) = \arg \min_{x \in \mathcal{X}} \| y - x \|$.

We use the shorthand notations $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}. \mathbb{I}$ denotes the indicator function.

III. CONTRACTION FOR REFLECTED STOCHASTIC DIFFERENTIAL EQUATIONS

This section gives a general convergence result for reflected stochastic differential equations (RSDEs) over closed convex domains. See Fig. 1. In this paper, we consider RSDEs to handle the constraints that arise in adaptive parameter tuning rules.
The results build upon the unconstrained contraction theory of [20], but substantial novel work is required to enable the adaptive control applications in Section IV.

A. Problem Setup

Let $\mathcal{X}$ be a closed convex subset of $\mathbb{R}^n$. We will examine contravariant properties of reflected stochastic differential equations of the form:

$$d x_t = \mathcal{H}(x_t)dt + Gdw_t - \psi_t d\mu(t),$$

where $G$ is an invertible $n \times n$ matrix, $w_t$ is a standard Brownian motion, and $\psi_t = -\int_0^t v_s d\mu(s)$ is a bounded variation reflection process that enforces that $x_t \in \mathcal{X}$ for all $t \geq 0$ whenever $x_0 \in \mathcal{X}$. In this case, $w_t$ has mean zero and $\mathbb{E}[w_t w_t^*] = tI$.

When $\mathcal{H}$ is Lipschitz, it can be shown that $\psi_t$ is the unique bounded variation process such that for all $t \geq 0$ and $v_t \in N_X(x_t)$, $\|v_t\| \in \{0,1\}$, $\mu$ is a random measure with $\mu([0,t]) < \infty$, and the solution has $x_t \in \mathcal{X}$. See [21], [22]. The Lipschitz condition can be relaxed to $\mathcal{H}$ being locally Lipschitz, provided that the process is non-explosive. See Section 2.4 of [23].

Reflected stochastic differential equations can be simulated numerically via a projected Euler method:

$$x_{t+\eta} \approx \Pi_X(x_t + \eta \mathcal{H}(x_t) + G(w_{t+\eta} - w_t)).$$

See [21], [24]. In other words, the effect of $\psi_t$ is the continuous time limit of a projection operation.

B. Assumptions

The first requirement is a one-sided growth condition. We assume that there is a function $\kappa(r) : (0, \infty) \rightarrow (0, \infty)$ with $\int_0^1 sk(s)ds < \infty$ and a non-negative number $\alpha$ such that for all $x, y \in \mathcal{X}$ with $r = \|x - y\|$ the following bound holds:

$$\langle x - y, \mathcal{H}(x) - \mathcal{H}(y) \rangle \leq \kappa(r)r^2 + \alpha r(\|x\| + \|y\|)$$

This one-sided growth condition generalizes the one-sided Lipschitz condition from [20], which corresponds to the special case with $\alpha = 0$. The extra terms are required to handle the application to adaptive control in Section IV.

Let $A$ denote the generator associated with the process $x_t$. Specifically, for any function $g : \mathcal{X} \rightarrow \mathbb{R}$

$$(Ag)(x) = \lim_{h \downarrow 0} h^{-1}(\mathbb{E}[g(x_h)|x_0 = x] - g(x)).$$

We assume that there is a twice continuously differentiable Lyapunov function $V : \mathcal{X} \rightarrow [0, \infty)$ and positive numbers $\lambda$ and $C$ such that for all $x \in \mathcal{X}$ and all $t \geq 0$:

$$(AV)(x) \leq C - \lambda V(x).$$

We assume that $V(x)$ increases monotonically with $\|x\|$. Specifically, there is a strictly monotonically increasing function $\phi$ such that $V(x) = \phi(\|x\|)$. We will further assume that $\phi$ grows at least linearly with $\|x\|$.

Let $R_1$ be the diameter of the set $\{(x, y) \in \mathcal{X} \mid V(x) + V(y) \leq 4C/\lambda\}$. Linear growth implies that $R_1$ is finite. By construction, if $\|x - y\| > R_1$, then

$$(AV)(x) + (AV)(y) \leq -2C(\|x\| + \|y\|).$$

Let $M$ be a positive number such that $M \geq R_1$ and for all $x$ with $\|x\| \geq M$, the following bound holds:

$$V(x) \geq \max \left\{ \frac{2}{\lambda} (\alpha \|x\| + C), \frac{4C}{\lambda} (2\|x\| + 1) \right\}.$$ Let $R_2$ be the diameter of $\{(x, y) \in \mathcal{X} \mid \|x\| \leq M \text{ and } \|y\| \leq M\}$. Note that $R_2 \leq 2M$ by the triangle inequality.

In Section IV we take $V(x) = \|x\|^2 + 1$, so the growth conditions are automatically satisfied.

C. Background on Wasserstein Distances

Our main theory is describe convergence in a Wasserstein distance. To state the result, some basic concepts from optimal transport are required. See [25] for a general reference. Let $P$ and $Q$ be probability measures over $\mathcal{X}$ with respect to the standard Borel sigma algebra. A measure, $\Gamma$, over $\mathcal{X} \times \mathcal{X}$ is called a coupling of $P$ and $Q$ if its marginals are $P$ and $Q$, respectively. In other words, for any Borel set $S$, we have $\Gamma(S \times \mathcal{X}) = P(S)$ and $\Gamma(\mathcal{X} \times S) = Q(S)$. Let $C(P, Q)$ denote the set of all couplings of $P$ and $Q$.

If $\rho : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$ is a metric the induced $q$-Wasserstein distance between $P$ and $Q$ is defined by:

$$W_\rho^q(P, Q) = \inf_{\Gamma \in C(P, Q)} \left( \int_{\mathcal{X} \times \mathcal{X}} \rho(x, y)^q d\Gamma(x, y) \right)^{1/q}.$$ For simple notation, we follow the convention that $W_\rho^q := \rho$ for general $\rho$ and for the norm, $W^q := W^q_{\|\cdot\|}$.

D. Main Contraction Result

The idea behind [20] is to construct a new metric for which convergence of $W_\rho$ can be tractably analyzed, and then use the result to examine more standard measures such as $W_2^q$ and the total variation distance. We follow a similar approach, but the metric must be modified to account for the more general growth condition.

Our metric will have the form:

$$\rho(x, y) = [f(||x - y||) + \gamma V(x) + \gamma V(y) + \gamma V(x) \vee \phi(M) + V(y) \vee \phi(M)] I(x \neq y).$$
Recall that $\phi$ is a function such that $V(x) = \phi(\|x\|)$ and the $I$ is the indicator function. Here $\gamma$ is a positive number defined below.

The function $f : [0, \infty) \rightarrow [0, \infty)$ is defined via the following chain of definitions:

$$h(r) = \frac{1}{\sigma_{\min}(G)^2} \left( \frac{1}{2} \int_0^r s \kappa(s) ds + \alpha M r \right) \tag{9a}$$

$$\varphi(r) = e^{-h(r)} \tag{9b}$$

$$\Phi(r) = \int_0^r \varphi(s) ds \tag{9c}$$

$$\xi^{-1} = \int_{R^1} \varphi(s)^{-1} ds \tag{9d}$$

$$\beta^{-1} = \int_{R^2} (\Phi(s)/\varphi(s)) ds \tag{9e}$$

$$g(r) = 1 - \frac{\xi}{4} \int_0^{r \wedge R_1} \varphi(s)^{-1} ds - \frac{\beta}{4} \int_0^{r \wedge R_2} (\Phi(s)/\varphi(s)) ds \tag{9f}$$

$$f(r) = \int_0^{r \wedge R_2} \varphi(s) g(s) ds. \tag{9g}$$

Here $\sigma_{\min}(G) > 0$ is the smallest singular value of $G$.

Additionally, we set

$$\gamma = \frac{\xi \sigma_{\min}(G)^2}{4C}. \tag{10}$$

The general contraction result is given below. The proof is available in [26].

**Theorem 1:** The function $\rho(x, y)$ is a metric over $X$. Let $x_t$ and $y_t$ be two solutions to (2) with respective laws $P_t$ and $Q_t$. If the initial laws satisfy $\int_X V(x) dP_0(x) < \infty$ and $\int_X V(x) dQ_0(x)$, then

$$W_p(P_t, Q_t) \leq e^{-\alpha t} W_p(P_0, Q_0).$$

where $\alpha = \min\{\lambda, \xi \sigma_{\min}(G)^2/2, \beta \sigma_{\min}(G)^2/2\}/2$.

A corollary which then establishes convergence to a stationary distribution in total variation distance and norm-based $q$-Wasserstein distances, is also available in [26].

**IV. APPLICATION TO ADAPTIVE REGULATION**

**A. Problem Setup**

We analyze a stochastic variation of a model reference adaptive control problem examined in Chapter 9 of [1].

The basic plant model has the form

$$dx_t = [A x_t + B (\bar{\Omega}_t \Psi(x_t) + u_t)] dt + G_x dw_x^t. \tag{11}$$

Here $\bar{A}$ is an unknown state matrix, $B$ is a known input matrix, $\Psi$ is a known vector of feature functions (which could be linear or nonlinear), $\bar{\Omega}$ is an unknown matrix of parameters, and $G_x$ is an unknown matrix scaling the Brownian motion. The state is $x_t$ and the inputs are $u_t$. The setup in [1] also includes an unknown scaling factor on $B$, which we have omitted for simplicity.

We focus on the problem of adaptive regulation, while [1] examines tracking problems. In our setup, the matching assumption is that there is a known Hurwitz matrix, $A$, and an unknown feedback gain, $\bar{K}$, such that

$$\bar{A} + B \bar{K} = A.$$ 

Here $A$ is the state matrix for the reference system.

It appears to be possible to extend the contraction theory to tracking problems, but this is left for future work.

If we knew $\bar{K}$ and $\bar{\Omega}$, we could set $u_t = \bar{K} x_t - \bar{\Omega}_t \Psi(x_t)$ and render the system stochastically stable:

$$dx_t = A x_t + G_x dw_x^t.$$ 

The challenge is that we do not know $\bar{K}$ or $\bar{\Omega}$. So instead, we use $u_t = K_t x_t - \Omega_t \Psi(x_t)$, where $K_t$ and $\Omega_t$ are estimates.

We derive rules for their computation later.

To simplify notation, we set

$$\tilde{\Theta} = \begin{bmatrix} -\bar{K}^\top \\ \bar{\Omega} \end{bmatrix}, \quad \Theta_t = \begin{bmatrix} -K_t^\top \\ \Omega_t \end{bmatrix}, \quad \Lambda(x_t) = \begin{bmatrix} x_t \\ \Psi(x_t) \end{bmatrix}.$$ 

Then the dynamics of (11) with $u_t = K_t x_t - \Omega_t \Psi(x_t)$ can be written as

$$dx_t = (A x_t + B (\tilde{\Theta} - \Theta_t)^\top \Lambda(x_t)) dt + G_x dw_x^t. \tag{12}$$

We assume that the function $\Lambda : \mathbb{R}^n \rightarrow \mathbb{R}^L$ is Lipschitz:

$$\|\Lambda(x) - \Lambda(y)\|_2 \leq \mathcal{L} \|x - y\|_2,$$

for some $\mathcal{L} > 0$, where $\|\cdot\|_2$ is the Euclidean norm.

Assume that $\hat{\Theta}$ and $\Theta_t$ are $L \times L$ matrices, and set $m = L \ell$.

Let $S : \mathbb{R}^m \rightarrow \mathbb{R}^{L \times L}$ be the reshaping function defined by $S(v)_{i,j} = v_{(i-1)\ell+j}$ for $i = 1, \ldots, L$ and $j = 1, \ldots, \ell$. Then $S$ is an invertible linear function.

Let $\hat{\theta} = S^{-1}(\hat{\Theta})$ be the unknown parameters and let $K$ be a compact convex subset of $\mathbb{R}^m$, containing $\hat{\theta}$. We assume that $K$ is known. Let $D$ be the diameter of $K$.

Now let $X = \mathbb{R}^n \times K$ be the closed convex subset of $\mathbb{R}^{n+m}$ containing the combined state $z_t = [x_t^\top \theta_t] \top$, and assume that $\theta_t$ has dynamics of the form

$$d \theta_t = R(z_t) dt + G_\theta dw_\theta^t - \psi_\theta^t d \mu_\theta(t), \tag{13}$$

where $R : X \rightarrow \mathbb{R}^m$, $w_\theta^t \in \mathbb{R}^m$ is standard Brownian motion with invertible coefficient matrix $G_\theta \in \mathbb{R}^{m \times m}$, and $\psi_\theta^t = -\int_0^t \psi_\theta^s d \mu_\theta(s)$ is a bounded variation reflection process that enforces $\theta_t \in K$ for all $t \geq 0$ whenever $\theta_0 \in K$. Here we assume that the reflections are computed with respect to the standard Euclidean inner product for simplicity.

**Remark 1:** Our parameter tuning rule from (13) differs from typical methods from adaptive control in how it forces $\theta_t$ to remain in the constraint set, $K$. Indeed, our method uses reflection processes, which can be approximated in discrete time by convex projections. For simple sets such as Euclidean balls and boxes, the convex projections have simple analytic formulas. More generally, for any convex set, the convex projection can be computed via optimization. In contrast, the parameters of adaptive control laws are commonly constrained using specialized projection operators that are designed for specific classes of convex sets [1]–[4].
B. Lyapunov-Based Adaptation

Here we follow a Lyapunov-based design procedure to design the update rule, (13), similar to the method from [1]. The main differences are that we examine stochastic problems and the constraints are enforced by reflection.

First we construct the Lyapunov function candidate. Fix a positive definite $Q \in \mathbb{R}^{n \times n}$. Then since $A$ is Hurwitz, there is a unique positive definite $P$ such that

$$A^T P + PA = -Q.$$  

For $x^T = [x^T, \theta^T]$, we define the Lyapunov function candidate by:

$$V(x) = x^T P x + (\theta - \bar{\theta})^T (\theta - \bar{\theta}) + 1.$$  

Define a norm over $\mathbb{R}^{n+m}$ by $\|z\|^2 = x^T P x + \theta^T \theta$.

Theorem 1 requires that the Lyapunov function be a monotonic function of a norm. This can be attained using the affine coordinate transformation $\hat{z}^T = [x^T, (\theta - \bar{\theta})^T]$. In that case we can set $\tilde{V}(\hat{z}) = V(z) = \|\hat{z}\|^2 + 1$. In the analysis below, we will work in the original coordinates.

To derive the required decrease condition, we use Itô’s formula:

$$dV(x_t) = 2 x_t^T P x_t dt + 2 (\theta_t - \bar{\theta})^T d\theta_t + \mu_t^T d\mu_t$$

$$= x_t^T (A^T P + PA) x_t dt$$

$$- 2 x_t^T PB (\Theta_t - \bar{\Theta})^T (\Lambda(x_t)) dt + 2 x_t^T P G_x d\mu_t$$

$$+ dx_t^T P dx_t + 2 \mu_t^T d\mu_t.$$

where the last equation uses the definition of $Q, P$. Additionally, we use the fact that

$$x_t^T PB (\Theta_t - \bar{\Theta})^T (\Lambda(x_t)) = \mu_t^T (\Theta_t - \bar{\Theta})^T (\Lambda(x_t)) x_t^T P B,$$

and the Itô rule $d\mu_t^T (d\mu_t^T) = dt I$.

By examining the second term in (15), we see that the $\Lambda(x_t) x_t^T P B$ term is canceled if $\tilde{R}$ in (13) is defined as

$$\tilde{R}(z_t) = S^{-1} (\Lambda(x_t) x_t^T P B) ,$$

where $S^{-1}$ is the inverse shaping function.

Plugging (13) into (15), as $d\Theta_t = S(d\theta_t)$, then gives

$$dV = -x_t^T Q x_t dt$$

$$+ 2 \mu_t^T d\mu_t$$

$$= -x_t^T Q x_t + 2 (\theta_t - \bar{\theta})^T v_t^0 dt$$

$$+ 2 x_t^T P G_x^T d\mu_t$$

where again we use that $d\mu_t^T (d\mu_t^T) = dt I$.

Now since $v_t^0 \in N_K(\theta_t)$ and $\mu_t^T$ is a nonnegative measure, (1) implies

$$- (\theta_t - \bar{\theta})^T v_t^0 d\mu_t(t) \leq 0 .$$

Therefore, the generator of the Lyapunov function $\tilde{V}(z)$ satisfies

$$\tilde{A} \tilde{V}(z) \leq -x^T Q x + \mu_t^T (G_x^T P G_x + 2 x_t^T P G_x d\mu_t) dt + \mu_t^T (2 (\theta_t - \bar{\theta})^T G_x^T d\mu_t + 2 (\theta_t - \bar{\theta})^T G_x^T d\mu_t) dt \leq 0 .$$

C. One-Sided Growth for Adaptive Regulation

The final task needed to apply Theorem 1 to the adaptive regulation problem is ensuring that the one-sided growth conditions from (3) holds. Note that the combination of (12), (13), (16) leads to a special case of (2) with:

$$\hat{R}(z_t) = \begin{bmatrix} A x_t + B (\Theta - \bar{\Theta})^T (\Lambda(x_t)) \\ S^{-1} (\Lambda(x_t) x_t^T P B) \end{bmatrix},$$

$$G = \begin{bmatrix} G_x & 0 \\ 0 & G_\theta \end{bmatrix}, \quad v_t = v_t^0 d\mu_t = \begin{bmatrix} 0 \\ v_t^0 d\mu_t(t)^0 \end{bmatrix}.$$}

Set $z^T = [x^T, \theta^T]$ and $\hat{z}^T = [x^T, \theta^T]$. Direct calculation using the specialized inner product shows that

$$\langle z - \hat{z}, \hat{R}(z) - \hat{R}(\hat{z}) \rangle = (x - \bar{x})^T P A_X (x - \bar{x})$$

$$- (x - \bar{x})^T P B (\Theta - \bar{\Theta})^T (\Lambda(x) x^T - \Lambda(\bar{x} x^T)) P B$$

$$+ (x - \bar{x})^T P B (\Theta - \bar{\Theta})^T (\Lambda(x) x^T - \Lambda(\bar{x} x^T)) P B$$

$$+ \mu_t^T (\Theta - \bar{\Theta})^T (\Lambda(x) x^T - \Lambda(\bar{x} x^T)) P B$$

$$\leq -x^T Q x + \mu_t^T (G_x^T P G_x + 2 x_t^T P G_x d\mu_t) dt + \mu_t^T (2 (\theta_t - \bar{\theta})^T G_x^T d\mu_t + 2 (\theta_t - \bar{\theta})^T G_x^T d\mu_t) dt \leq 0 .$$

Therefore, the generator of the Lyapunov function $\tilde{V}(z)$ satisfies

$$\tilde{A} \tilde{V}(z) \leq -x^T Q x + \mu_t^T (G_x^T P G_x + 2 x_t^T P G_x d\mu_t) dt + \mu_t^T (2 (\theta_t - \bar{\theta})^T G_x^T d\mu_t + 2 (\theta_t - \bar{\theta})^T G_x^T d\mu_t) dt \leq 0 .$$

Now using standard quadratic form bounds, followed by the diameter condition on $K$ gives:

$$\tilde{x}^T Q \tilde{x} \geq \lambda_{\min}(Q) x^T P x$$

$$\geq \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \tilde{V}(z) - \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} (D^2 + 1) .$$

Here $\lambda_{\min}(Q)$ is the minimum eigenvalue of $Q$ and $\lambda_{\max}(P)$ is the maximum eigenvalue of $P$. Plugging this bound into (18) shows $\tilde{V}$ satisfies (4) with

$$C = \text{tr}(G_x^T P G_x) + \text{tr}(G_\theta^T G_\theta) + \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} (D^2 + 1)$$

$$\lambda = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} .$$

In particular, $\tilde{V}$ satisfies all of the required conditions to apply the general theory from Section III.
Set
\[ y = PB(\Theta - \tilde{\Theta})^T(\Lambda(x) - \Lambda(\tilde{x})) \]
\[ \tilde{y} = PB(\tilde{\Theta} - \tilde{\Theta})^T(\Lambda(x) - \Lambda(\tilde{x})). \]

Then note that
\[ \|y - \tilde{y}\|_2 = \|PB(\Theta - \tilde{\Theta})^T(\Lambda(x) - \Lambda(\tilde{x}))\|_2 \]
\[ \leq \|PB\|_2\|\Theta - \tilde{\Theta}\|_F\|x - \tilde{x}\|_2 \]
\[ \leq \|PB\|_2\|\Theta - \tilde{\Theta}\|_F\|x - \tilde{x}\|_2 \]
\[ \leq \|PB\|_2 D\mathcal{L}\|x - \tilde{x}\|_2. \]

Here \( \cdot \|_2 \) applied to matrices refers to the induced 2-norm. Then the first inequality uses submultiplicativity followed by the Frobenius norm on \( \Lambda \). Next we note that the induced 2-norm is bounded above by the Frobenius norm, and that \( \|\Theta - \tilde{\Theta}\|_F = \|\tilde{\Theta} - \hat{\Theta}\|_2 \). So, the final inequality follows from the diameter bound.

An analogous derivation shows that
\[ \|\tilde{y}\|_2 \leq \|PB\|_2 D\mathcal{L}\|x - \tilde{x}\|_2. \]

Now the right side of (19) can be upper bounded by:
\[ -x^T\tilde{y} + \tilde{x}^Ty = (\tilde{x} - x)^T\tilde{y} + \tilde{x}^T(y - \tilde{y}) \]
\[ \leq \|\tilde{x} - x\|_2\|\tilde{y}\|_2 + \|\tilde{x}\|_2\|y - \tilde{y}\|_2 \]
\[ \leq \|PB\|_2 D\mathcal{L}\|x - \tilde{x}\|_2^2 + \|\tilde{x}\|_2\|x - \tilde{x}\|_2. \]

Thus using the bound \( \|x\|_2 \leq \frac{1}{\sqrt{\lambda_{\min}(P)}}\|\tilde{x}\| \) gives a special case of (3) with \( \kappa(r) = \alpha = \frac{\|PB\|_2 D\mathcal{L}}{\lambda_{\min}(P)}. \)

The results are summarized in the following theorem:

**Theorem 2:** The controller \( u_t = \Theta_n^T\Psi(x_t) \) with parameters by (13) and (16) renders drives the closed-loop system to a unique stationary distribution. Convergence is exponential with respect to total variation distance and \( W^2 \), with convergence rates described by Theorem 1 and the follow-on Corollary.

**V. Numerical Results**

We simulate\(^1\) the plant system from Sec. 11.5 of [1], with \( n = 2, \ell = 1, N = 3, L = 7 \), and thus \( m = 14 \) (flattening of \( \Omega \) and \( \tilde{K} \)). The \( G \) matrices were \( G_x = I_n \) and \( G_{\theta} = I_m \). The Euler method was run over time \( t \in [0, 50] \) seconds and with timestep \( \eta = 0.001 \) seconds. The compact set \( \mathcal{K} \) was a separate 2-dimensional polygon applied to each row of \( \Theta_n \) (a parameter for each control input). For the \( N = 3 \) rows of \( \Omega_t \) (6 total parameters), the same polygon was used, and is shown in Figure 4. Figure 2 shows the reflection coupling of the system states \( x_t \) and \( \tilde{x}_t \), and their coupling time \( \tau \). Figure 3 shows the respective Lyapunov functions \( V(z_t) \) and \( V(\tilde{z}_t) \), also with the coupling time. Figure 4 shows the results of the convex projection on the polygon for each 2-dimensional space corresponding to a row of \( \Omega_t \) and \( \tilde{\Omega}_t \).

\(^1\)All code available at: https://github.com/tylerlekang/CDC2021
VI. CONCLUSION
A. Discussion on Practical Applications

In regards to practical application of the results, the authors would like to highlight two key factors: 1) the flexibility afforded by the projection method in the various geometries that can constrain the parameter estimates, as an improvement over existing methods, and 2) opening the door for analysis of Langevin Algorithms on more general state spaces (see [27]).

B. Closing Remarks

In this paper we introduced a novel extension of contraction methods for SDEs which enables restrictions to closed convex domains. We utilized this theory to prove convergence for stochastic versions of adaptive controllers from [1]. Future work includes expanding the class of systems for which this theory holds, including exogenous input tracking.

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