VOLUME OF INTERSECTION OF A CONE WITH A SPHERE

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Abstract. The manuscript provides formulas for the volume of a body defined by the intersection of a solid cone and a solid sphere as a function of the sphere radius, of the distance between cone apex and sphere center, and of the cone aperture angle.

If the sphere center lies on the (extended) cone axis the analysis may be based on cylinder coordinates fixed at the cone axis, and the volume is the sum of the well-known volumes of finite cones and sphere caps.

At the general geometry the sphere center is not on the (extended) cone axis. Our approach calculates the volume by slicing space perpendicular to the cone axis and by integrating the lens areas defined by the sphere-cone intersection. These volume integrals are rephrased with the aid of the Byrd-Friedmann tables to Elliptic Integrals of the First, Second and Third Kind.

1. On-axis. Apex inside Sphere

1.1. Polar coordinates. The volume of the intersection of a cone which contains the sphere center on its (extended) rotation axis is computed in this section in tutorial fashion. The principle parameters are the sphere radius $R$, the distance $d$ between sphere center and cone apex, and $0 \leq \varphi \leq \pi/2$, half the field-of-view angle of the cone. A spherical coordinate system may be defined, centered at the sphere center, with radial coordinate $r$, azimuth angle $\phi$, polar angle $\theta$ and Jacobian

\begin{equation}
\text{Figure 1. Spherical coordinates } r \text{ and } \theta \text{ of a sphere of radius } R \text{ intersecting a cone with apex at a distance } d \text{ from the sphere center.}
\end{equation}
Figure 2. Cylinder coordinates $z$ and $\rho$ of a sphere of radius $R$ intersecting a cone with apex at a distance $d > 0$ from the sphere center. Secondary cone shell in green.

$$r^2 \sin \theta \ [7, 4.603].$$

(1) $x = r \sin \theta \cos \phi;$  
(2) $y = r \sin \theta \sin \phi;$  
(3) $z = r \cos \theta$

(Some authors measure $\theta$ as a latitude from the equator and the factor $\cos \theta$ appears in the Jacobian $[4, (1.7.9.3)].$)

1.2. **Cylinder coordinates.** A cylindrical coordinate system may also be used, centered at the cone apex, with $x$ and $y$ spanning the horizontal plane, radial coordinate $r = \sqrt{x^2 + y^2}$, azimuth angle $\phi$, vertical coordinate $z$ of the cone axis, and Jacobian $r$

(4) $x = r \cos \phi;$  
(5) $y = r \sin \phi;$

1.3. **Volume.** The volume of the intersection of the cone and sphere is essentially the volume of the cone delimited by the horizontal plane where the cone meets the sphere surface, plus the volume of the sphere cap for larger values of $z$. We define the Cartesian coordinate system such that the cone apex is at $z = 0$ and the sphere center at $z = -d; \,$ the sign of $d$ indicates whether the the cone apex is above or below the equatorial plane of the sphere. In Figure 2 the integral starts at $z = 0$ with cone radius $r = 0$ and covers the range up to $z = Z$ where the cone radius has grown linearly to $\rho$, and continues in the range $z = Z$ up to $z = R - d$ with a radius $r$ shrinking from $\rho$ down to zero delimited by the sphere surface. The volume of the cone-sphere intersection is

(6) $V^{(i)}(R, d, \varphi) = \iiint dxdydz = \int_0^{2\pi} d\phi \int_0^\rho rdr \int_0^{R-d} dz = 2\pi \int_0^{R-d} dz \int_0^\rho rdr.$

The transitory value $Z$ is calculated by emitting a ray from the cone apex into the direction of $\varphi$ with coordinates $(t \sin \varphi, t \cos \varphi)$ in the cross section of Figure 3, where $t$ is the Euclidean distance from the cone apex, and equating these with a coordinates of a point on the sphere with polar angle $\theta$, $(R \sin \theta, R \cos \theta - d)$. If the
z-coordinate of the apex is negative like in Figure 3, $d$ is smaller than zero and the cone contains the sphere center. Solving for $t$ gives $t = R\sin\theta/\sin \varphi$, then

$$(7) \quad R \sin(\varphi - \theta) = d \sin \varphi.$$ 

**Remark 1.** This is also the sine equation for the plane triangle with sides $d$, and $R$ and interior angles $\theta$, $\pi - \varphi$ and $\varphi - \theta$ [3, (3.88)] [2, 4.3.148].

Solving for $\theta$ yields

$$(8) \quad \theta = \varphi - \arcsin(d \sin \varphi / R).$$

There are two possible branches of the arcsin in this equation, the principle value and its $\pi$-complement. The one to be taken is the one where the variable $t$ remains positive. The other is associated with the branch of the double cone that points with its axis into the opposite direction, the green lines in Figures 2 and 3. The projection on the vertical axis is

$$(9) \quad Z + d = R \cos \theta = R \cos[\varphi - \arcsin(d \sin \varphi / R)]
= \cos \varphi \sqrt{R^2 - (d \sin \varphi)^2} + d \sin^2 \varphi.$$

For $\varphi < \pi/2$ the intersection of cone and sphere is at $Z > 0$; for the “stretched” cone where $\varphi > \pi/2$ the intersection is at $Z < 0$. The simplest strategy for the stretched cone is to define a complementary volume which is inside the sphere but outside the cone,

$$(10) \quad \bar{V}^{(i)}(R, d, \varphi) \equiv \frac{4}{3} \pi R^3 - V^{(i)}(R, d, \varphi),$$

and to handle the cases $\varphi > \pi/2$ by flipping the cone axis upside-down:

$$(11) \quad V^{(i)}(R, d, \varphi) = \frac{4}{3} \pi R^3 - V^{(i)}(R, -d, \pi - \varphi), \quad \pi/2 < \varphi \leq \pi.$$
Splitting (6) into the two regions \( z \lesssim Z \) yields

\[
V^{(i)}(R, d, \varphi) = 2\pi \int_0^{R-d} dz \int_0^{\tan\varphi} rdr
= \pi \int_0^Z dz \tan^2 \varphi + \pi \int_Z^{R-d} dz (R^2 - (z + d)^2)
= \frac{1}{3} \pi \tan^2 \varphi Z^3 + \frac{1}{3} \pi (2R + Z + d)(R - Z - d)^2, \quad 0 \leq \varphi \leq \pi/2.
\]

Remark 2. Equating \( z \tan \varphi = \sqrt{R^2 - (z + d)^2} \) in the upper limits and solving for \( z \) gives again (9).

Figures 2 and 3 show \( \tan \varphi = \rho/Z \), so the first term is the well-known volume

\[
V_\Delta(\rho, Z) = \frac{1}{3} \pi \rho^2 Z
\]

of the cone with base radius \( \rho \) and height \( Z \) [5, 3.151] in region I of Figure 3, the second term is the well-known volume

\[
V_\land(\rho, h) = \frac{1}{3} \pi h^2 (3R - h)
\]

of the spherical cap of thickness \( h = R - Z - d \) [5, 3.162], region II.

Limiting cases are

- the cone with apex at the center of the sphere with \( Z = R \cos \varphi \):

\[
V^{(i)}(R, 0, \varphi) = \frac{2}{3} \pi R^3 (1 - \cos \varphi)
\]

which is \( R^3/3 \) times the solid angle covered by the cone,

- the volume of the spherical cap [5, 3.162]

\[
V^{(i)}(R, d, \pi/2) = \frac{1}{3} \pi (2R + d)(R - d)^2.
\]

- the apex at the sphere surface with \( Z = R[1 + \cos(2\varphi)] \),

\[
V^{(i)}(R, R, \varphi) = \frac{4}{3} \pi R^3 \sin^2 \varphi (1 + \cos^2 \varphi)
\]

such that for \( \varphi \to \pi/2 \) the sphere is entirely inside the cone and the volume is the entire sphere volume \( 4\pi R^3/3 \),

- the complementary spherical cap

\[
V^{(i)}(R, -d, \pi/2) = \frac{4}{3} \pi R^3 - \frac{1}{3} \pi (2R + d)(R - d)^2 = \bar{V}^{(i)}(R, d, \pi/2).
\]

2. On-axis. Apex outside the Sphere

If the cone apex is outside the sphere, \( d < -R \), the cone (projection) intersects the sphere at a near point characterized by (projected) cylinder coordinates \( Z_1, \rho_1 \) and a far point \( Z_2, \rho_2 \) as sketched in Figure 4 [14]. In the figure the polar angle for the far point is \( \theta < \pi/2 \), but for sufficiently small \( d \) and sufficiently large \( \varphi \) the far point may actually be located below the horizontal line.
We follow the strategy to obtain the volume of the intersection as the sum of the volume of the sphere cap of radius $\rho_1$ at the south pole, a truncated cone of radii $\rho_1$ and $\rho_2$, and a sphere cap of radius $\rho_2$ at the north pole.

The ray from the apex into the direction $\phi$ meets the circle (projection) fixed by the condition that the distance to the sphere center be $R$. The same analysis that led to (9) requires

\[
Z_{1,2} + d = R \cos[\phi - \arcsin(d \sin \phi / R)] \\
= \mp \cos \phi \sqrt{R^2 - (d \sin \phi)^2} + d \sin^2 \phi.
\]

where $\arcsin$ refers to both branches of the inverse trigonometric sine, generating two different signs of their cosines. For sufficiently large $\phi$, $\sin \phi > R / |d|$, the cone walls may be tangential to the sphere or not intersect the sphere at all; then the volume of the intersection is the full $4\pi R^3 / 3$ of the sphere. The corresponding (positive) radii are

\[
\rho_{1,2} = \sqrt{R^2 - (Z_{1,2} + d)^2} = Z_{1,2} \tan \phi.
\]

The thickness of the south polar sphere cap, region I in Figure 4, is $R + Z_1 + d$; the height of the truncated cone, region II, is $Z_2 - Z_1 \geq 0$; the thickness of the north polar sphere cap, region III, is $R - Z_2 - d$. The sum of these 3 positive values is $2R$. The sum of the volumes of the three regions is

\[
V^{(o)}(R, d, \phi) = V_{\sigma}(R, R + Z_1 + d) + V_{\Delta}(\rho_2, Z_2) - V_{\Delta}(\rho_1, Z_1) + V_{\sigma}(R, R - Z_2 - d) \\
= \frac{1}{3} \pi [ (R + Z_1 + d)^2(2R - Z_1 - d) + (Z_2 - Z_1)(\rho_1^2 + \rho_1 \rho_2 + \rho_2^2) + (R - Z_2 - d)^2(2R + Z_2 + d) ].
\]

for $\sin \phi \leq R / d \leq 1$, $Z_1 \leq Z_2$. 

**Figure 4.** Cone apex outside the On-axis Sphere. $d < -R$. 

Figure 5. Cone apex inside the off-axis sphere. Impact parameter $b$ and signed distance $d$ between apex and the sphere’s equatorial plane.

**Remark 3.** If $d \to -\infty$ and $\varphi \to 0$ keeping $d\sin \varphi$ constant, the geometry approaches a sphere intersected with a circular cylinder [3, 8, 13].

### 3. Off-axis. Apex inside Sphere

The more general geometry shows a sphere center that has a nonzero distance $b > 0$, the impact parameter of particle physics, from the cone axis as in Figure 5. The distance $d$ between the apex and the sphere center, measured along the sphere axis orthogonal to $b$, may be positive or negative.

**Remark 4.** One can always find a rectangular coordinate system in which the cone axis and the sphere center are in the $x-z$ plane, such that there are 4 relevant parameters: $R$, $\varphi$, $b$ and $d$. Given a generic set of $\vec{S}$ (location of the sphere center), $\vec{C}$ (location of the cone apex), and $\vec{a}$ (vector from the apex along the cone axis), the reduction to the principal parameters $b$ and $d$ is given by the projection of $\vec{S}$ on $\vec{a}$. $\vec{C}$ is reached from $\vec{S}$ via $\vec{S} - \vec{b} + t\vec{a} = \vec{C}$ with $t$ measuring distances along the cone axis. $\vec{b} \perp \vec{a}$ yields in an intermediate step $t = (\vec{C} - \vec{S}) \cdot \vec{a}/(\vec{a} \cdot \vec{a})$. Then $b$ follows as $b = |\vec{b}| = |\vec{S} - \vec{C} + t\vec{a}|$, and finally $d = t|\vec{a}|$.

The apex is in the sphere while $d^2 + b^2 \leq R^2$; otherwise the analysis of Section 4 takes over. We place the apex at the center of the cylinder coordinates; the cone wall is defined by

$$z = r \cot \varphi; \quad z \geq 0$$

and the sphere surface by

$$ (x + b)^2 + y^2 + (z + d)^2 = R^2. $$
This fixes the sign of \( d \): \( d \) is positive if the cone apex is above the altitude of the sphere center. Section 1.2 turns this into

\[
(24) \quad r^2 + b^2 + 2rb \cos \phi + (z + d)^2 = R^2.
\]

In the projection, Figure 5, a lower value \( Z_1 \) and a higher value \( Z_2 \) of the latitudes appear where sphere and cone intersect.

**Remark 5.** On the curve of the intersection one can eliminate \( z \) from (24) inserting (22):

\[
(25) \quad r^2 + b^2 + 2rb \cos \phi + (r \cot \varphi + d)^2 = R^2.
\]

This is the projection of the intersection curve on the horizontal plane (perpendicular to the cone axis). The first derivative of (25) along the curve of \( r(\varphi) \), with \( r' \equiv \partial r/\partial \varphi \), is

\[
(26) \quad 2rr' + 2r' b \cos \phi - 2rb \sin \phi + 2(r \cot \varphi + d)r' = 0.
\]

The maximum and minimum distance of this planar curve from the cone axis are set by \( r' = 0 \) which leads with this equation to \( \sin \varphi = 0 \); so it is correct to locate the points of \( Z_{1,2} \) in the cross section of the \( x-z \)-plane.

**Remark 6.** We only consider angles \( \varphi \leq \pi/2 \) here. The cases of overstretched cones are forwarded to (11) with flipped sign of \( d \).

The locations of \( Z_{1,2} \) are calculated in spherical coordinates centered at the cone apex sending a ray of parameter \( t \) from the apex into the direction \( \varphi \) along \((x, z)\) coordinates \((t \sin \varphi, t \cos \varphi)\). Another ray from the sphere center to the sphere surface has coordinates \((R \sin \theta - b, R \cos \theta - d)\) where \( \theta \) is the angle measured from the center of the sphere. They meet at

\[
(27) \quad t \sin \varphi = R \sin \theta - b; \quad t \cos \varphi = R \cos \theta - d.
\]

Introducing unitless

\[
(28) \quad \hat{d} \equiv d/R; \quad \hat{b} \equiv b/R
\]

and eliminating \( t \) gives

\[
(29) \quad \sin(\varphi - \theta) = \hat{d} \sin \varphi - \hat{b} \cos \varphi,
\]

\[
(30) \quad \theta = \varphi - \arcsin(\hat{d} \sin \varphi - \hat{b} \cos \varphi)
\]

consistent with the on-axis equation (7) as \( b \to 0 \).

**Remark 7.** There are two branches of the \( \arcsin \), selecting the cone and its twin at the same apex but the axis pointing into the opposite direction.

Back to the cylinder coordinates centered at the apex the projections are

\[
(31) \quad Z_{1,2} + d = R \cos \theta
\]

switching to \(-\varphi\) in (30) to obtain the second value:

\[
(32) \quad Z_1 + d = R \cos[\varphi - \arcsin(\hat{d} \sin \varphi - \hat{b} \cos \varphi)];
\]

\[
(33) \quad Z_2 + d = R \cos[-\varphi - \arcsin(-\hat{d} \sin \varphi - \hat{b} \cos \varphi)].
\]

The limits of the radii of the cylinder coordinates are

\[
(34) \quad \rho_{1,2} = Z_{1,2} \tan \varphi.
\]
Remark 8. An alternative to this vector-algebra is to continue from Remark 5 with an algebraic approach: Since the critical values are where $\phi = 0, \pi$, one may insert $\cos \phi = \pm 1$ into (25), solve the two quadratic equations for $r$ in the range $r > 0$, call them $\rho_{1,2}$ and insert these back into (22) to get $Z_{1,2}$.

Remark 9. In the limit $b = 0$ both solutions of (25) are the same because the term with $\cos \phi$ vanishes from the equation.

The intersecting volume $V^{(i)}(R, d, b, \varphi) = V_I + V_{II} + V_{III}$ contains three subvolumes if the $z$-axis is chopped by horizontal planes into slices.

I) For $0 < z < Z_1$, region I in Figure 5, it is the volume of the cone of height $Z_1$ and base radius $\rho_1 \equiv Z_1 \tan \varphi$; this subvolume has the value $V_I = V_\Delta(\rho_1, Z_1) = \frac{1}{3} \pi \rho_1^2 Z_1$.

II) For $Z_1 < z < Z_2$, region II, the limits of $r$ are partially set by the cone wall and partially by the sphere surface.

III) For $Z_2 < z < R - d$ the limits of $r$, region III, are set by the sphere surface as a function of the azimuth $\phi$. It is a sphere cap of base radius $\rho_2 - b$ and thickness $h = R - d - Z_2$ thinking in terms of polar coordinates at the sphere center. This portion of the volume of the North Polar Cap is $V_{III} = V_{\cap}(R, h)$. For sufficiently small $\varphi$, $Z_2 \tan \varphi < b$, the sphere cap is outside the cone and there is no contribution of region III to the volume. Then the left branch of the red line of the cone in Figure 6 hits the circle before the red arc on the circle reaches the north pole. [This type of argument indicating the presence of sphere caps in the volume of intersection will recur many times in Section 4.]

This leaves to find a formula for the volume $V_{II}$ of the intermediate region (II). For constant $z$ the interior of the cone defines a circle of radius $z \tan \varphi$ at the center, and the interior of the sphere defines a circle of radius $\sqrt{R^2 - (z + d)^2}$ displaced by $b$. In region (2) none of both is entirely inside the other, and the horizontal slice defines an overlap area $A$ as discussed in Appendix A.

The volume integral gathers the cross section areas (60) along the $z$-direction,

$$V_{II} = \int_{Z_1}^{Z_2} A \left( z \tan \varphi, \sqrt{R^2 - (z + d)^2}, b \right) dz,$$

which splits into three terms overlapping two sectors and subtracting one triangle,

$$V_{II} = v_1 + v_2 - v_\Delta.$$

The ordinate sections of (55) are for these $z$-dependent radii

$$x_1 = \frac{1}{2b} \left( R^2 - \frac{z^2}{\cos^2 \varphi} - 2zd - d^2 - b^2 \right),$$

$$x_2 = \frac{1}{2b} \left( R^2 - \frac{z^2}{\cos^2 \varphi} - 2zd - d^2 + b^2 \right).$$

To consolidate the algebraic representation it is useful to introduce the dimensionless variable

$$\hat{z} \equiv z/(R \cos \varphi).$$
3.1. Cone Sector. The contribution $v_1$ to (36) along the vertical direction induced by the area of (60) is

$$v_1 = \int dz \alpha_1 r_1^2 = \int dz r_1^2 \arccos \frac{-x_1}{r_1}$$

$$= \int dz z^2 \tan^2 \varphi \arccos \frac{-1}{z d}(R^2 - \frac{z^2}{\cos^2 \varphi} - 2zd - d^2 - b^2)$$

$$= R^3 \cos \varphi \sin^2 \varphi \int d\zeta \zeta^2 \arccos \frac{-1 + \zeta^2 + 2 \cos \varphi \zeta d + \zeta^2 + b^2}{2b \zeta \sin \varphi}.$$  

With partial integration and $\frac{d}{d\zeta} \arccos x = -1/\sqrt{1-x^2}$ for the outer derivative:

$$\int d\zeta \zeta^2 \arccos \frac{-1 + \zeta^2 + 2 \zeta d + \zeta^2 + b^2}{2b \zeta \sin \varphi}$$

$$= \frac{1}{3} \zeta^3 \arccos[\ldots] - \frac{1}{3} \int d\zeta \zeta^3$$

$$= \frac{1}{3} \zeta^3 \arccos[\ldots] + \frac{1}{3} \int d\zeta \zeta^3$$

$$= \frac{1}{3} \zeta^3 \arccos[\ldots] + \frac{1}{3} \int d\zeta \zeta^3$$

$$= \frac{1}{3} \zeta^3 \arccos[\ldots] + \frac{1}{3} \int d\zeta \zeta^3$$

$$\zeta \sqrt{-\frac{(2 \cos \varphi \zeta d - 1 + \zeta^2 + \zeta d + \zeta^2 + 2 \zeta d + \zeta^2 - b^2 \sin \varphi \zeta)}{2 \cos \varphi \zeta d - 1 + \zeta^2 + \zeta d + \zeta^2 - b^2 \sin \varphi \zeta}}$$

$$= \frac{1}{3} \zeta^3 \arccos[\ldots] + \frac{1}{3} \int d\zeta \zeta^3$$

where the roots of the two quadratic polynomials in the denominator are

$$\zeta_1^\pm = -(\hat{d} \cos \varphi + b \sin \varphi) \pm \sqrt{(\hat{d} \cos \varphi + b \sin \varphi)^2 + 1 - \hat{d}^2 - \hat{b}^2}; \quad \hat{z}_1^- < 0 < \hat{z}_1^+$$

$$\zeta_2^\pm = -(\hat{d} \cos \varphi - b \sin \varphi) \pm \sqrt{(\hat{d} \cos \varphi - b \sin \varphi)^2 + 1 - \hat{d}^2 - \hat{b}^2}; \quad \hat{z}_2^- < 0 < \hat{z}_2^+$$

$\hat{z}_1^+$ is the scaled version of $Z_1$, $\hat{z}_2^+$ is the scaled version of $Z_2$. If $\hat{d}^2 + \hat{b}^2 < 1$ the apex is inside the sphere of radius $R$. The integral is reduced as detailed in Appendix C.

3.2. Sphere Sector. The contribution $v_2$ to (36) along the vertical direction induced by the area of (60) is in the scaled variable (39)

$$v_2 = \int dz \alpha_2 r_2^2 = \int dz r_2^2 \arccos \frac{x_2}{r_2}$$

$$= \int dz [R^2 - (z + d)^2] \arccos \frac{1}{2b} (R^2 - \frac{z^2}{\cos^2 \varphi} - 2zd - d^2 + b^2)$$

$$= R^3 \cos \varphi \int d\zeta [1 - (\zeta \cos \varphi + \hat{d})^2] \arccos \frac{1 - \zeta^2 - 2 \cos \varphi \zeta d + \zeta^2 + \hat{b}^2}{2b \sqrt{1 - (\cos \varphi \zeta + \hat{d})^2}}.$$
By partial integration

\begin{equation}
(45) \quad \cos \varphi \int d\hat{z} \left[ 1 - (\hat{z} \cos \varphi + \hat{d})^2 \right] \arccos \frac{1 - \hat{z}^2 - 2 \cos \varphi \hat{d} \hat{z} - \hat{d}^2 + \hat{b}^2}{2b \sqrt{1 - (\cos \varphi \hat{z} + \hat{d})^2}}
= \cos \varphi \left[ \hat{z} - \frac{(\hat{z} \cos \varphi + \hat{d})^3}{3 \cos \varphi} \right] \arccos[\ldots] + \cos \varphi \int d\hat{z} \left[ \hat{z} - \frac{(\hat{z} \cos \varphi + \hat{d})^3}{3 \cos \varphi} \right]
\times \frac{1}{\sqrt{1 - \frac{(1 - \hat{z}^2 - 2 \cos \varphi \hat{d} \hat{z} - \hat{d}^2 + \hat{b}^2)^2}{4b^2(1 - \cos \varphi \hat{z} + \hat{d})^2}}}
\times \left[ -2\hat{z} - 2\hat{d} \cos \varphi + \frac{(1 - \hat{z}^2 - 2\hat{d} \cos \varphi \hat{z} - \hat{d}^2 + \hat{b}^2)(\cos \varphi \hat{z} + \hat{d}) \cos \varphi}{1 - (\cos \varphi \hat{z} + \hat{d})^2} \right]
\times \left[ \hat{z} \cos \varphi - \frac{(\hat{z} \cos \varphi + \hat{d})^3}{3} \right] \arccos[\ldots] + \int d\hat{z} \left[ \hat{z} \cos \varphi - \frac{(\hat{z} \cos \varphi + \hat{d})^3}{3} \right]
\times \left[ -2(\hat{z} + \hat{d} \cos \varphi) + \cos \varphi \frac{(1 - \hat{z}^2 - 2\hat{d} \cos \varphi \hat{z} - \hat{d}^2 + \hat{b}^2)(\hat{z} \cos \varphi + \hat{d})}{(1 + \cos \varphi \hat{z} + \hat{d})(1 - \cos \varphi \hat{z} - \hat{d})} \right]
\times \frac{1}{\sqrt{(\hat{z} - \hat{z}_1^+)(\hat{z} - \hat{z}^-_1)(\hat{z}^+_2 - \hat{z})(\hat{z} - \hat{z}^-_2)}}
\end{equation}

with \( \hat{z}^\pm_{1,2} \) given by (42)-(43).

**Remark 10.** If the cone apex is on the sphere surface, \( \hat{d}^2 + \hat{b}^2 = 1 \) and two of the four polynomial roots \( z_{1,2}^\pm \) are zero such that a spurious singularity \( \propto 1/z \) arises from the denominator product \( \sqrt{(\hat{z} - \hat{z}_1^+)(\hat{z} - \hat{z}^-_1)(\hat{z}^+_2 - \hat{z})(\hat{z} - \hat{z}^-_2)} \). In this case this singularity can be lifted by writing the kernel of the integral of the previous equation (two brackets and square root) as

\begin{equation}
(46) \quad \int d\hat{z} \left[ \frac{1}{3} \hat{d}(-3 + 2 \cos^2 \varphi + 2 \hat{d}^2) + \frac{1}{3} \cos \varphi(-4 + 2 \cos^2 \varphi + 5 \hat{d}^2)\hat{z} + \frac{4}{3} \cos^2 \varphi \hat{d}^2 \hat{z}^2 + \frac{1}{3} \cos^3 \varphi \hat{d} \hat{z}^3 \right]
\times \frac{\hat{d}^2/2 + 5\hat{d}/6 - \cos^2 \varphi \hat{d} + 1/3 - 2 \cos^2 \varphi/3}{1 + \cos \varphi \hat{z} + \hat{d}}
\times \frac{-\hat{d}^2/2 + 5\hat{d}/6 - \cos^2 \varphi \hat{d} + 1/3 - 2 \cos^2 \varphi/3}{1 - \cos \varphi \hat{z} - \hat{d}}
\times \frac{1}{\sqrt{(\hat{z} - \hat{z}_1^+)(\hat{z} - \hat{z}^-_1)(\hat{z}^+_2 - \hat{z})(\hat{z} - \hat{z}^-_2)}}
\end{equation}
Then (45) contains first a sum of integrals of powers of \( \hat{z} \) divided by the quartic square root manageable by Appendix C. In addition there is an integral with a quadratic polynomial in \( \hat{z} \) in the denominator, which is split into two terms with a linear polynomial by decomposition into partial fractions. To reduce complexity introduce an intermediate \( \hat{z} \equiv \hat{z} \cos \varphi + \hat{d} \), so

\[
(47) \quad \cos^2 \varphi \left[ \hat{z} - \left( \frac{\hat{z} \cos \varphi + \hat{d}}{3 \cos \varphi} \right)^3 \right] \times \left[ -2(\dot{\hat{z}} + \hat{d} \cos \varphi) + \cos \varphi \frac{(1 - \hat{z}^2 - 2\hat{d} \cos \varphi \hat{z} - \hat{d}^2 + \hat{b}^2)(\hat{z} \cos \varphi + \hat{d})}{(1 + \cos \varphi \hat{z} + \hat{d})(1 - \cos \varphi \hat{z} - \hat{d})} \right] \\
= \frac{2}{3} + \hat{d}^2 + \cos^2 \varphi \left[ -\frac{2}{3} - \frac{2}{3} b^2 - \hat{d}^2 + \frac{1}{3} \hat{d}^4 + \frac{1}{3} b^2 \hat{d}^2 \right] \\
+ \frac{1}{3} \hat{d} \cos \varphi \left[ -3 + 2b^2 \cos^2 \varphi + 2\hat{d}^2 + 2\hat{d}^2 \cos^2 \varphi \right] \hat{z} \\
+ \frac{1}{3} \cos^2 \varphi \left[ -4 + \cos^2 \varphi + b^2 \cos^2 \varphi + 5\hat{d}^2 + \hat{d}^2 \cos^2 \varphi \right] \hat{z}^2 \\
+ \frac{4}{3} \hat{d} \cos^3 \varphi \hat{z}^3 + \frac{1}{3} \cos^4 \varphi \hat{z}^4 \\
+ \frac{(-\frac{1}{3} \hat{d}^2 + \frac{1}{3} \hat{d} + \frac{1}{3} \hat{d}^3 - \frac{1}{3}) \sin^2 \varphi + (-\frac{1}{3} \hat{b} \hat{d}^2 + \frac{1}{3} \hat{b}^2 \hat{d}^2) \cos^2 \varphi}{1 - \hat{z} \cos \varphi - \hat{d}} \\
+ \frac{(-\frac{1}{3} \hat{d}^2 - \frac{1}{3} - \frac{1}{3} \hat{d} - \frac{1}{3} \hat{d}^3) \sin^2 \varphi + (\frac{1}{3} \hat{b} \hat{d}^2 + \frac{1}{3} \hat{b}^2 \hat{d}^2) \cos^2 \varphi}{1 + \hat{z} \cos \varphi + \hat{d}}.
\]

This type of integrals is covered by Appendix D.

3.3. **Triangle.** The radius of the lens of the intersecting planar circles is according to (56)

\[
(48) \quad \rho = \frac{1}{2b} \sqrt{-(R^2 - d^2 - b^2 - 2zd - z^2 / \cos^2 \varphi - 2bz \tan \varphi)(R^2 - d^2 - b^2 - 2zd - z^2 / \cos^2 \varphi + 2bz \tan \varphi)}.
\]

Then the third contribution to (36) can be written as

\[
(49) \quad v_\Delta = \int \rho dz = \frac{1}{2} R^3 \cos \varphi \int d\hat{z} \\
\times \sqrt{-(1 + d^2 + b^2 + 2d \cos \varphi \hat{z} + \hat{z}^2 + 2b \hat{z} \sin \varphi)(-1 + d^2 + b^2 + 2d \cos \varphi \hat{z} + \hat{z}^2 - 2b \hat{z} \sin \varphi)}
\]

The square root contains a product of two quadratic polynomials in \( \hat{z} \). Their roots are provided by (42)–(43). Since \( \hat{z} \) will be integrated in these limits, the writeup as an Elliptic Integral is

\[
(50) \quad v_\Delta = \frac{1}{2} R^3 \cos \varphi \int_{\hat{z}_1^+}^{\hat{z}_2^+} d\hat{z} \sqrt{(\hat{z} - \hat{z}_1^+)(\hat{z} - \hat{z}_1^-)(\hat{z}_2^+ - \hat{z})(\hat{z} - \hat{z}_2^-)}.
\]

Details of the evaluation are posted in Appendix B.
4. Off-axis. Apex outside Sphere

Determining the curve of the intersection is of interest for satellite imagery where the sphere represents the Earth and the cone apex an orbiting satellite [12].

Miller [9] considers five major cases of intersections which can be registered by the number of real-valued quantities in Equations (42)–(43).

If all 4 values $z_{1,2}^\pm$ are imaginary, the bodies either do not intersect at all or the sphere is entirely inside the cone with intersecting volume of 0 or $4\pi R^3/3$, respectively.

There are essentially the two-branched case of Figure 6 of 7, where all 4 values $z_{1,2}^\pm$ are real and the lines of intersection are two separate quadrics, and the one-branched case of Figure 8 where the 2 values $z_1^\pm$ are imaginary and the lines of intersection is a single quadric.

The cases with a single tangent point where $z_1^+ = z_1^-$ are just limiting values (the Viviani case, so to speak) and not of special importance to the computation of the volume of the intersection of the two bodies.

Remark 11. $z_1^\pm$ are imaginary if the argument $(\hat{d}\cos\varphi + \hat{b}\sin\varphi)^2 + 1 - \hat{d}^2 - \hat{b}^2 = 1 - (\hat{d}\sin\varphi - \hat{b}\cos\varphi)^2$ of the square root (42) is negative. This is the same criterion which keeps the value of the arcsin argument in (32) outside the interval $[-1, 1]$.

4.1. Two Branches. In Figures 6 and 7, the intersection $V = V_I + V_{II} + V_{III} + V_{IV} + V_V$ comprises

I) for sufficiently large $\varphi$ like in Figure 7 but not in Figure 6, $Z_2^- \tan \varphi > b$,

a sphere cap of thickness $h = R + d + Z_2^-$ at the south pole, with volume $V_I = V_\infty(R, h)$.  

Figure 7. Cone apex outside the off-axis sphere. Two-branch intersection. Real-valued $Z_{1,2}^\pm$ and real-valued reduced $\hat{z}_{1,2}^\pm$. With south pole cap in intersection.

Figure 8. Cone apex outside the off-axis sphere. One-branch intersection. Real-valued $Z_2^\pm$ and real-valued reduced $\hat{z}_2^\pm$.

II)

\begin{equation}
V_{II} = \int_{Z_2^+}^{Z_2^-} A \left( z \tan \varphi, \sqrt{R^2 - (z + d)^2}, b \right) dz
\end{equation}
III) the truncated cone

\[ V_{III} = V_\Delta(\rho_1^+, Z_1^+) - V_\Delta(\rho_1^-, Z_1^-), \]

where \( \rho \) and \( Z \) are correlated by (34);

IV)

\[ V_{IV} = \int_{Z_1^-}^{Z_1^+} A \left( z \tan \varphi, \sqrt{R^2 - (z + d)^2}, b \right) \, dz \]

V) for sufficiently large \( \varphi \), \( Z_2^+ \tan \varphi > b \), a sphere cap of thickness \( h = R - d - Z_2^+ \) at the north pole covering the range \( Z_2^+ \leq z \leq R - d \) with volume \( V_V = V_r(R, h) \). For \( Z_2^+ \tan \varphi < b \) the intersection at altitude \( Z_2^+ \) may be at the horizontal cylinder coordinate smaller than \( b \); then this sphere cap is outside the cone and does not contribute to the volume.

The algebra for regions II and IV is the same as the algebra of computing (35). The only difference is that the numerical order of the four real-valued reduced \( \hat{z} \) values in \( \sqrt{-(\hat{z} - z_1^+)(\hat{z} - z_1^-)(\hat{z} - z_2^+)(\hat{z} - z_2^-)} \) may differ such that the parameters \( a - d \) in the appendices need to be permuted to yield applicable Byrd-Friedman-integrals.

4.2. One Branch. If only \( \hat{z}_2^\pm \) are real-valued the volume of the intersection \( V = V_I + V_{II} + V_{III} \) contains

I) for sufficiently large \( \varphi \), \( Z_2^- \tan \varphi > b \) a sphere cap of thickness \( h = R + d + Z_2^- \) at the south pole, volume \( V_I = V_r(R, h) \). In Figure 8 that south pole cap is not contributing.

II) a subvolume

\[ V_{II} = \int_{Z_2^-}^{Z_2^+} A \left( z \tan \varphi, \sqrt{R^2 - (z + d)^2}, b \right) \, dz \]

as in (35). For the main integral the analysis of Section 3 remains valid, but here \( \hat{z}_1^+ \) and \( \hat{z}_1^- \) are two conjugated complex values, so in the appendices the alternative integrals with complex-conjugated roots of the quartic polynomial are activated.

III) for sufficiently large \( \varphi \), \( Z_1^+ \tan \varphi > b \) a sphere cap of thickness \( h = R - d - Z_1^+ \) for \( Z_2^+ < z < R - d \) up to the north pole, volume \( V_{III} = V_r(R, h) \).

In overview, the one-branch cases have the same criteria and formulas to include the polar caps as the two-branch cases, they have no contribution from a truncated cone in intermediate \( Z \)-regions, and the two integrals that depend on the limits \( Z_1^\pm \) in the two-branch cases are glued into a single integral covering the entire interval \( [Z_2^-, Z_2^+] \).

5. Summary

The volume of the intersection of cone and on-axis sphere is given by (12) if the apex is inside the sphere, and by (21) and the apex is outside the sphere: sums of (truncated) cones and sphere caps.

Answering a question of Shah [15], the volumes with off-axis spheres have been reduced to Elliptic Integrals in Sections 3 and 4 for apexes in- and outside the sphere.
Figure 9. Two intersecting circles of radius \( r_1 \) and \( r_2 \) at a distance \( b \) with base radius \( \rho \) of the red asymmetric lens. \( b^2 > r_1^2 + r_2^2 \). \( x_2 > 0 \), \( x_1 < 0 \).

Appendix A. Two Intersecting Circles

The geometry of two planar intersecting circles of radii \( r_1 \) and \( r_2 \) at distance \( b \) is illustrated in Figures 9 and 10. Only the cases with non-vanishing overlap, \( b < r_1 + r_2 \), are of interest here. We also assume that the circle rims intersect, which means \( b + r_2 > r_1 \) \([11]\).

The (right) circle of radius \( r_1 \) is placed at the center of coordinates:

\[ x^2 + y^2 = r_1^2. \]  

The (left) circle of radius \( r_2 \) is placed at \((-b, 0)\):

\[ (x + b)^2 + y^2 = r_2^2. \]  

Solving the first equation for \( y^2 \), insertion in the second equation and solving for \( x \) gives for the horizontal coordinates of the lens position

\[ x_1 = \frac{r_2^2 - r_1^2 - b^2}{2b} < 0; \quad x_2 = b + x_1 = \frac{r_2^2 - r_1^2 + b^2}{2b}. \]  

\( x_2 \) is positive in Figure 9, negative in Figure 10. The radius \( \rho \geq 0 \) of the lens in the right triangle of sides \( r_1, x_1 \) and \( \rho \) is the associated \( y \)-value from \((53)\),

\[ \rho = \sqrt{r_1^2 - x_1^2} = \sqrt{r_2^2 - x_2^2} = \frac{\sqrt{(b + r_2)^2 - r_1^2} \left[ r_1^2 - (r_2 - b)^2 \right]}{2b}. \]  

Imaginary values of \( \rho \) are numerical indicators for circles with non-intersecting rims, because they either are too far apart or one circle lies entirely within the other.

Remark 12. If the value of \( x_1 \) becomes less than \(-b - r_2 \) (indicating that the intersection would be to the left of the left circle), the factor \((b + r_2)^2 - r_1^2 \) in the discriminant of this radix becomes negative. If the value of \( x_2 \) becomes larger than \( r_2 \) (indicating that the intersection is to the right of the left circle), the factor \( r_1^2 - (r_2 - b)^2 \) in the discriminant becomes negative. Because the polynomial of the discriminant is a symmetric function of \( r_1 \) and \( r_2 \), the equivalent criteria apply also for the right circle.
Remark 13. The discriminant of the square root is a polynomial of order 4 in $b$. Imagine the smaller circle wandering from left to right with decreasing $b$ in front of the larger circle. There are 4 positions associated with the roots of this polynomial where the two circles have only one point in common and where $\rho$ becomes zero: 2 positions where the circles barely touch, and 2 positions where the smaller circle is inside the larger circle.

The angles $\alpha_{1,2}$ at which $\rho$ appears from the centers of the circles are blue in Figures 9 and 10:

\begin{align}
\sin \alpha_1 &= \frac{\rho}{r_1}, \quad \sin \alpha_2 = \frac{\rho}{r_2}, \\
\cos \alpha_1 &= -\frac{x_1}{r_1}, \quad \cos \alpha_2 = \frac{x_2}{r_2}; \quad 0 \leq \alpha_1, \alpha_2 \leq \pi.
\end{align}

In Figure 9 $\alpha_2 < \pi/2$; in Figure 10 $\alpha_2 > \pi/2$.

The area of the intersection delineated in red is the sum of the areas of the two circular segments with radii $r_1$ and $r_2$ [5, 3.76]. Using the principle of inclusion-exclusion, the area of the blue triangle in Figure 9 is the area of the left and right circular sectors minus half the area $A$ of the red lens [10]:

\begin{align}
\frac{1}{2} \rho b &= \frac{1}{2} \alpha_2 r_2^2 + \frac{1}{2} \alpha_1 r_1^2 - \frac{1}{2} A,
\end{align}
\(VOLUME\ \ OF\ \ INTERSECTION\ \ OF\ \ A\ \ CONE\ \ WITH\ \ A\ \ SPHERE\ \ 17\)

\[A(r_1, r_2, b) = \alpha_2 r_2^2 + \alpha_1 r_1^2 - \rho b,\]

where \(\alpha_{1,2}\) are measured in radians.

A closer inspection of 10 shows that the same formula holds.

**Appendix B. Elliptic Integral of the Triangular region**

**B.1. 4 real roots.** Byrd’s reduction of the integral for \(v_\Delta\) to standard form is [6, 256.38]

\[\int_b^y \sqrt{(a-t)(t-b)(t-c)(t-d)}\,dt = (b-c)^2(a-b)(b-d)\alpha^2 g \int_0^{u_1} \frac{\text{sn}^2 u \text{cn}^2 u \text{dn}^2 u}{(1-\alpha^2 \text{sn}^2 u)^2} \,du\]

where

\[\alpha^2 = (a-b)/(a-c) < 1,\]

\[k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)},\]

\[g = 2/\sqrt{(a-c)(b-d)},\]

\[\varphi = \text{am} u_1 = \arcsin \left(\frac{(a-c)(y-b)}{(a-b)(y-c)}\right),\]

\[\text{sn} u_1 = \sin \varphi.\]

[6, 362.25]

\[\int_0^{u_1} \frac{\text{sn}^2 u \text{cn}^2 u \text{dn}^2 u}{(1-\alpha^2 \text{sn}^2 u)^2} \,du\]

\[= \frac{1}{\alpha^6}[-k^2 \Pi(u, \alpha^2) + (3k^2 - \alpha^2 k^2 - \alpha^2) V_2 + (2\alpha^2 k^2 + 2\alpha^2 - 3k^2 - \alpha^4) V_3 + (\alpha^2 - 1)(\alpha^2 - k^2) V_4]\]

with [6, 336]

\[V_0 = F(\varphi, k);\]

\[V_1 = \Pi(\varphi, \alpha^2, k);\]

\[V_2(u) = \frac{1}{2(\alpha^2 - 1)(k^2 - \alpha^2)} \left[\alpha^2 E(u) + (k^2 - \alpha^2) u + (2\alpha^2 k^2 + 2\alpha^2 - \alpha^4 - 3k^2) \Pi(\varphi, \alpha^2, k) - \frac{\alpha^4 \text{sn} u \text{cn} u \text{dn} u}{1-\alpha^2 \text{sn}^2 u}\right];\]

\[V_{m+3} = \frac{1}{2(m+2)(1-\alpha^2)(k^2 - \alpha^2)} \left[(2m+1)k^2 V_m + 2(m+1)(\alpha^2 k^2 + \alpha^2 - 3k^2) V_{m+1} + (2m+3)(\alpha^4 - 2\alpha^2 k^2 - 2\alpha^2 + 3k^2) V_{m+2} + \frac{\alpha^4 \text{sn} u \text{cn} u \text{dn} u}{1-\alpha^2 \text{sn}^2 u} \right];\]

Note that in the application of this manuscript the upper limit \(y\) in (61) equals \(a\), so \(\varphi = \pi/2\) in (65), \(\text{sn} u_1 = 1\), \(\text{cn} u_1 = 0\), the Elliptic Integrals are Complete.
Elliptic Integrals, and the Jacobian Elliptic Functions in (70)–(71) do not need to be evaluated.

B.2. 2 complex-conjugated roots. The case of complex conjugated \( d = c^* \) is expanded as

\[
\int_b^y \sqrt{(a-t)(t-b)(t-c)(t-c^*)} \, dt = \int_b^y \frac{(a-t)(t-b)(t-c)(t-c^*)}{\sqrt{(a-t)(t-b)(t-c)(t-c^*)}} \, dt
\]

\[
= - \int_b^y \frac{t^4}{\sqrt{(a-t)(t-b)(t-c)(t-c^*)}} \, dt + (a+b+2\Re c) \int_b^y \frac{t^2}{\sqrt{(a-t)(t-b)(t-c)(t-c^*)}} \, dt
\]

\[
- (ab + 2a\Re c + 2b\Re c + |c|^2) \int_b^y \frac{t}{\sqrt{(a-t)(t-b)(t-c)(t-c^*)}} \, dt - (2ab\Re c + a|c|^2 + b|c|^2) \int_b^y \frac{1}{\sqrt{(a-t)(t-b)(t-c)(t-c^*)}} \, dt
\]

and relegated to the formulas in App. C.

APPENDIX C. ELLIPTIC INTEGRAL OF CONE SECTOR

C.1. 4 real roots. The integral (41) is from \( t = t_1 \) to \( t = t_3 \), where the argument of the arccos is \(-1\) and \(1\). The following integral needed to be evaluated for \( m = 2 \) and \( m = 4 \) [6, 257.11]:

\[
\int_b^a \frac{t^m}{\sqrt{(a-t)(t-b)(t-c)(t-d)}} \, dt = a^m g \int_0^{u_1} \frac{(1 - \alpha_1^2 \sin^2 u)^m}{(1 - \alpha_2^2 \sin^2 u)^m} \, du
\]

where

\[
\alpha_1^2 = \frac{(b-a)d}{a(b-d)}
\]

\[
\alpha_2^2 = \frac{b-a}{b-d} < 0
\]

\[
\varphi = \am u_1 = \arcsin \sqrt{\frac{(b-d)(a-y)}{(a-b)(y-d)}}
\]

The definitions for \( k^2, g \) and \( \sn u_1 \) are the same as in Appendix B. The right-hand side of (73) is [6, 340.04]

\[
\int \frac{(1 - \alpha_1^2 \sin^2 u)^m}{(1 - \alpha_2^2 \sin^2 u)^m} \, du = \frac{\alpha_1^{2m}}{\alpha_2^{2m}} \sum_{j=0}^{m} \binom{m}{j} \frac{(\alpha_2^2 - \alpha_1^2)^j}{\alpha_1^{2j}} V_j,
\]

and the \( V_j \) given by (68)–(71). Note that in the application of this manuscript the lower limit \( y \) in (73) equals \( b \), so \( \varphi = \pi/2 \) in (76), \( \sn u_1 = 1 \), \( \cn u_1 = 0 \), the Elliptic Integrals are Complete Elliptic Integrals, and the Jacobian Elliptic Functions in (70)–(71) do not need to be evaluated.
C.2. 2 complex-conjugated roots. If \( c \) and \( d = c^* \) are a pair of complex-conjugate values, the applicable entry is [6, 259.03]

(78)
\[
\int_b^y \frac{t^m}{(a-t)(b-t)(c-t)}dt = g (aB + bA)^m \sum_{j=0}^{m} \binom{m}{j} \alpha_2^{m-j} (\alpha - \alpha_2)^j \int_0^{u_1} \frac{du}{(1 + \alpha \text{cn} u)^j},
\]

where \( b_1 = \Re c, a_1 = \Im c, A^2 = (a - b_1)^2 + \alpha_1^2, B^2 = (b - b_1)^2 + \alpha_1^2, g = 1/\sqrt{AB}, \alpha = (A - B)/(A + B), \alpha_2 = (bA - aB)/(aB + bA), \) and [6, 341.05]

(79)
\[
R_m = \int \frac{du}{(1 + \alpha \text{cn} u)^m}.
\]

The \( R_m \) are recursively

(80)
\[
R_{-2} = \frac{1}{k^2} [(k^2 - \alpha^2 k^2)u + \alpha^2 E(u) + 2\alpha k \text{arccos}(\text{dn} u)];
\]

(81)
\[
R_{-1} = u + \frac{\alpha}{k} \text{arccos}(\text{dn} u);
\]

(82)
\[
R_0 = u;
\]

(83)
\[
R_1 = \frac{1}{1 - \alpha^2} \left[ \Pi(\varphi, \frac{\alpha^2}{\alpha^2 - 1}, k) - \alpha f_1 \right]
\]
at \( u = F(\varphi, k) \), \( E(u) = E(\varphi, k) \), and [6, 361.54]

(84)
\[
f_1 = \begin{cases} 
\sqrt{\frac{1 - \alpha^2}{k^2 + k^2 \alpha^2}} \arctan \left[ \sqrt{\frac{k^2 + k^2 \alpha^2}{1 - \alpha^2}} \, \text{sd} u \right], & \alpha^2/(\alpha^2 - 1) < k^2; \\
\frac{1}{2} \sqrt{\frac{\alpha^2 - 1}{k^2 + k^2 \alpha^2}} \ln \frac{\sqrt{k^2 + k^2 \alpha^2} \, \text{du} + \sqrt{1 - \alpha^2} \, \text{sn} u}{\sqrt{k^2 + k^2 \alpha^2} \, \text{du} - \sqrt{1 - \alpha^2} \, \text{sn} u}, & \alpha^2/(\alpha^2 - 1) > k^2;
\end{cases}
\]

[6, 341.05]

(85)
\[
R_m = \frac{1}{(m-1)(\alpha^2 - 1)(k^2 + \alpha^2 k^2)} \left\{ (3 - 2m)[\alpha^2(1 - 2k^2) + 2k^2]R_{m-1} + 2(5-2m)(k^2 R_{m-3} + (m-1)(k^2 + \alpha^2 - 2k^2 \alpha^2)R_{m-2} + (m-3)k^2 R_{m-4} + \frac{\alpha^3 \text{sn} u \, \text{du}}{(1 + \alpha \text{cn} u)^{m-1}} \right\}.
\]

APPENDIX D. ELLIPTIC INTEGRAL OF SPHERES SECTOR

D.1. 4 real roots. Eq. 47 requires the integrals [7, 3.151.7] [6, 257.39]

(86)
\[
\int_y^a \frac{dt}{(p-t)^m \sqrt{(a-t)(b-t)(c-t)(t-d)}} = \frac{g}{(p-a)^m} \int_0^{u_1} \frac{(1 - \alpha^2 \text{sn}^2 u)^m du}{(1 - \alpha_1^2 \text{sn}^2 u)^m},
\]

where \( p \neq a, \alpha_1^2 \equiv (p-d)(a-b)/(a-p)/(b-d), \alpha^2 \) defined in (75), and the integral of the right hand side reduced in (77).
D.2. **2 complex-conjugated roots.** If \( d = c^* \) are complex conjugated in the previous integral \([6, 259.04]\)

\[
\int_b^y \frac{1}{(t-p)^m \sqrt{(a-t)(t-b)(t-c)(t-c^*)}} \, dt
\]

\[
= \frac{(A+B)^m}{[A(b-p)-B(a-p)]^m} \sum_{j=0}^m \binom{m}{j} \alpha_1^{m-j} (\alpha - \alpha_1)^j \int_0^u \frac{du}{(1 + \alpha \operatorname{cn} u)^j}
\]

where \( A \) and \( B \) are defined after (78), \( \alpha \equiv (bA-aB+pB-pA)/(aB+bA-pA-pB) \), \( \alpha_1 \equiv (A-B)/(A+B) \), and the right hand side is evaluated with (79).

D.3. **C++ implementation.** The computation of the volume is implemented in the C++ source code in the ancillary directory, using the GNU scientific library (GSL) to evaluate the elliptic integrals [1]. For Linuxes the minimum package names depend on the distribution. For openSUSE compiler and GSL are retrieved with `zypper install gcc-c++ cpp gsl-devel make`, for example, on Ubuntu like `apt install g++ cpp libgsl-dev make`. The `Makefile` compiles two main programs, `sphereCylVol` and `sphereConeVol`:

- The volume of the intersection of a sphere and a cylinder [8] is calculated with the call
  
  `sphereCylVol r R b`

  with three floating point parameters: \( r \) is the sphere radius, \( R \) the cylinder radius, and \( b \) the impact parameter. Viviani’s volume for example is computed with `sphereCylVol 1.0 0.5 0.5`.

- The volume of intersection of a sphere and a cone is calculated with one of

  `sphereConeVol [-N samples [-v]] [-r radius] [-p phiDegrees] sx sy sz ax ay az dx dy dz`

  `sphereConeVol [-N samples [-v]] [-r radius] [-p phiDegrees] ax ay az dx dy dz`

  The option `-r` followed by a positive floating-point number specifies the sphere radius \( R \). If the argument is not used the radius is assumed to be 1.

  The option `-p` followed by a positive floating-point number specifies the cone half angle \( \varphi \) in degrees. If the argument is not used it is assumed to be 45.

  The 9, 6 or 3 trailing arguments are signed floating point numbers with groups of Cartesian \( x, y \) and \( z \) coordinates. (If at least one of the numbers is negative, a double-dash `--` should be inserted after the options to disambiguate the meaning of their minus-sign and the dashes of the options.) The triple \( sx sy sz \) are the Cartesian coordinates of the sphere center. If the triple is missing, \((0, 0, 0)\) is assumed. The triple \( ax ay az \) are the Cartesian coordinates of the cone apex. The triple \( dx dy dz \) are the Cartesian coordinates of the direction of the cone axis. (The length of that vector does not need to be normalized to unity but must be nonzero.) If \( dx dy dz \) are absent, the direction \((0, 0, 1)\) parallel to the \( z \)-axis is assumed.

  The option `-N` is a debugging option and lets the program compute the approximate (!) volume by slicing the sphere into that many pieces and adding the areas of the circular intersections with the cone in the sense of a Simpson summation. The higher the integer argument `samples`, the
more accurate the result. If the argument is not used, the integrals of
this manuscript are evaluated. For increasingly large samples both results
ought to converge.

If the samples argument is a negative number, the approximate area (1)
of the sphere surface is calculated which is inside the cone. [There is no
equivalent analytical evaluation of areal integrals in this manuscript.] This
samples the front surface and also the back surface. In satellite imaging
the back surface would not be visible; if the option -v is also used, only
the visible area is accumulated (i.e. the parts where the vector from sphere
center to surface and the vector from apex to sphere surface have a dot
product less than zero).

The the validity of the analytical integrals is investigated with

\texttt{sphereConeVol -t [-N samples]}

which runs a triple loop over a finite grid over the three parameters \( \hat{d}, \hat{b} \)
and \( \varphi \) and prints for each point \( \hat{b}, \hat{d}, \varphi \) (in radians), the result obtained by
the elliptic integrals and the result obtained by a Simpson rule. If the two
values of the volume differ by more than \( 10^{-6} \), an additional exclamation
mark and the difference is printed.

The numerical Simpson rule slices the sphere in samples pieces. If the
-N option is not used, a value of 100,000 is assumed.

This is equivalent to a few hundred separate calls of \texttt{sphereConeVol}
with and without the -N option.

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