GONALITY OF THE BRILL-NOETHER CURVE

ABEL CASTORENA, ALBERTO LÓPEZ MARTÍN, AND MONTSERRAT TEIXIDOR I HIGAS

Abstract. For a projective nonsingular genus $g$ curve, the Brill-Noether locus $W_d^r(C)$ parametrizes line bundles of degree $d$ over $C$ with at least $r + 1$ sections. When the curve is generic and the Brill-Noether number $\rho(g, r, d)$ equals 1, one could then talk of the Brill-Noether curve. In this paper, we give a new formula for the genus of this curve and compute its gonality when $C$ has genus 5.

Introduction

Let $C$ be a projective nonsingular genus $g$ curve defined over an algebraically closed field and Pic$^d(C)$ the Picard variety that parametrizes isomorphism classes of degree $d$ line bundles on $C$. Let us denote by $W_d^r(C)$ the subvariety of Pic$^d(C)$ that parametrizes line bundles of degree $d$ over $C$ with at least $r + 1$ linearly independent sections. The expected dimension of $W_d^r(C)$ is given by the Brill-Noether number

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

Over an open set in the moduli space of curves, the expected dimension of $W_d^r(C)$ is actually its dimension. In particular, when $\rho(g, r, d) = 1$ and $C$ is generic, the Brill-Noether locus $W_d^r(C)$ is a curve we will call the Brill-Noether curve. One can then define the rational map

$$\phi : \overline{M}_g \to \overline{M}_{g'}$$

by $\phi([C]) = [W_d^r(C)]$, where $g'$ is the genus $g_{W_d^r}$ of the Brill-Noether curve. Pirola [Pir85] and Eisenbud-Harris [EH87], using the determinantal adjunction formula of Harris-Tu [HT84], calculated this genus, finding that it equals

$$g_{W_d^r} = 1 + \frac{g - d + r}{g - d + 2r + 1} \prod_{i=0}^{r} \frac{i!}{(g - d + r + i)!} \cdot g!$$

While this gives a collection of rational maps between moduli spaces of curves, their images are mostly unknown. Farkas [Far10] and Ortega [Ort13] have recently worked on several questions around or using this map and posed questions about the gonality of the image curve under the map $\phi$. 

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In this note, we provide a new method to compute the genus of the Brill-Noether curve when \( r = 1 \) and compute its gonality in the first interesting case, when \( g = 5 \) and \( r = 1 \). We show that, in this case, the generic Brill-Noether curve has the same gonality as the generic curve of genus \( g' = 11 \). It is sensible to think that, in order to identify the image of the Brill-Noether map, one would need to use tools other than those coming from Brill-Noether theory in \( \overline{M}_{g'} \).

As a result of our computations for the genus, we give a different proof for the Castelnuovo numbers, which are the number of linear series of fixed degree and dimension when this number is finite (see Theorem 1.3).

Our results are obtained by degenerating the given curve to a chain of elliptic components and describing explicitly the Brill-Noether curve in this situation. Our method could be, in principle, extended to values \( r > 1 \) and, in the case of the gonality, to values of \( g \) other than 5. In both situations, however, the combinatorics are more involved.

1. The genus of the Brill-Noether curve

Degeneration methods have been used by many authors. Eisenbud and Harris [EH86] introduced limit linear series on reducible curves of compact type, a technique that allowed to use reducible curves in the solution of several classical and novel problems. We will apply their theory to curves as the ones defined in the following

**Definition 1.1.** Let \( C_i \), for \( i = 1, \ldots, g \), be an elliptic curve and let \( P_i \) and \( Q_i \) be generic points on \( C_i \). Consider the curve obtained from gluing \( Q_i \) to \( P_{i+1} \) for \( i = 1, \ldots, g - 1 \). We will call it a *chain of elliptic curves*.

When a nonsingular curve degenerates to a curve of compact type (a curve whose dual graph has no loops), a linear series on the generic curve degenerates to a limit linear series in the sense of Eisenbud-Harris. A chain of elliptic curves as above, where the nodes are chosen to be generic points on the elliptic components, is generic from the point of view of Brill-Noether; that is, the Brill-Noether loci are of the expected dimension and the limit Petri maps are injective. Therefore, in order to compute the genus of the Brill-Noether curve for a generic curve of genus \( 2a + 1 \), it suffices to compute the genus of the curve of limit linear series for the chain of elliptic curves described above. The genus of the Brill-Noether curve will be written in terms of the \( a^{th} \) Catalan number

\[
c_a = \frac{1}{a + 1} \binom{2a}{a}.\]

**Lemma 1.2.** Assume we have a family of curves for which the generic fiber is a generic genus \( g \) curve \( C \) and the special fiber \( C_0 \) is a chain of elliptic curves as above. Then, on the special fiber, the Brill-Noether curve \( W_{a+2}^1(C) \) degenerates to a reducible curve with \( \nu = g c_a \) irreducible components. These
components are isomorphic to the components of the given curve, there are in fact \(a_i\) components isomorphic to each component of the given curve.

**Proof.** A limit linear series of degree \(d\) and (projective) dimension one is given by a line bundle of degree \(d\) on each component along with a two-dimensional subspace of sections of this line bundle so that, at the nodes, corresponding sections vanish with order adding at least to \(d\). Denote by \(u_i^1 < u_i^2\) the order of vanishing of the sections in this subspace at the point \(P_i\) and by \(v_i^1 > v_i^2\) the order of vanishing of the sections in this subspace at the point \(Q_i\). This means that all the sections on the subspace vanish to order at least \(u_i^1\) at \(P_i\) and \(v_i^1\) at \(Q_i\), while there is at least one section \(s_i\) vanishing to order \(u_i^1\) at \(P_i\) and at least one section \(s_i'\) vanishing to order at least \(v_i^1\) at \(Q_i\). In particular, \(s_i\) vanishes to order \(u_i^1\) at \(P_i\) and \(v_i^1\) at \(Q_i\). As the total order of vanishing of a section of a line bundle cannot be larger than the degree of the line bundle, \(u_i^1 + v_i^1 \leq d\) and equality implies that the line bundle on the curve \(C_i\) is \(\mathcal{O}(u_i^1 P_i + v_i^1 Q_i) = \mathcal{O}(u_i^2 P_i + (d - u_i^1) Q_i)\). Similarly, \(u_i^1 + v_i^1 \leq d\) and equality implies that the line bundle on the curve \(C_i\) is \(\mathcal{O}(u_i^1 P_i + (d - u_i^1) Q_i)\).

Moreover, as \(P_i\) and \(Q_i\) are generic points and \(u_i^1 \neq u_i^2\), \(\mathcal{O}(u_i^1 P_i + (d - u_i^1) Q_i)\) is not isomorphic to \(\mathcal{O}(u_i^2 P_i + (d - u_i^2) Q_i)\). This means that only one of the above equalities can hold. Therefore, \(u_i^1 + v_i^1 + u_i^2 + v_i^2 \leq 2d - 1\) with equality if and only if either the line bundle on the curve \(C_i\) is \(\mathcal{O}(u_i^1 P_i + (d - u_i^1) Q_i)\) or \(\mathcal{O}(u_i^2 P_i + (d - u_i^2) Q_i)\). That is \(u_i^1 + v_i^1 + u_i^2 + v_i^2 \leq 2d - 1 - \epsilon_i\) where \(\epsilon_i = 0\) if the line bundle on the \(i\)th component is of the form given above and is 1 otherwise. Therefore,

\[
\sum_i (u_i^1 + v_i^1 + u_i^2 + v_i^2) \leq (2d - 1)g - \sum \epsilon_i.
\]

Let us now consider the vanishing at the nodes. By definition of limit linear series, \(u_i^{j+1} + v_i^j \geq d, u_i^{j+1} + v_i^j \geq d\) for each node \(i = 1, \ldots, g - 1\), while \(u_i^1 + u_i^2 \geq 1, v_i^1 + v_i^2 \geq 1\). Therefore,

\[
\sum_i (u_i^1 + v_i^1 + u_i^2 + v_i^2) \geq 2(g - 1)d + 2.
\]

From the two inequalities,

\[
\sum \epsilon_i \leq 2d - g - 2.
\]

**Remark 1.** Here we have just been proving the Brill-Noether Theorem for our curve when \(r = 1\). The same proof works for any other value of \(r\) greater than one. The proof does not require characteristic zero. Moreover, a simplified proof of Petri is also possible in the same manner and in all characteristics using these curves (see [Wel85] and [CLT]).

As in our case \(\rho(g, r, d) = 2d - g - 2 = 1\), we find that for a linear series, on each component except for one, the line bundle is predetermined. This shows that the Brill-Noether curve is reducible. A component corresponds to the choice of one of the components of the original curve in which the line
bundle is free to vary and the choice, on the remaining $g - 1$ components of the original curve, of one of the two line bundles: $O(u_1^1 P_i + (d - u_1^1)Q_i)$ or $O(u_2^1 P_i + (d - u_2^1)Q_i)$. We will distinguish each of these possibilities by the subindex of the order of vanishing that appears in the presentation. That way, the line bundle is determined on $g - 1$ components of the original curve, of one of the two line bundles:

$O(u_i^1 P_i + (d - u_i^1)Q_i)$ or $O(u_i^2 P_i + (d - u_i^2)Q_i)$.

We will distinguish each of these possibilities by the subindex of the order of vanishing that appears in the presentation. That way, the line bundle is determined on $g - 1$ of the components, once we choose which of the indices (either 1 or 2) to use. As the set of line bundles on an elliptic curve is isomorphic to the curve itself, this component of the Brill-Noether curve is isomorphic to the component of the original curve on which the line bundle is arbitrary.

Let us see now which choices of subindices (1 or 2) are allowable.

If on one component $j$ we choose the first subindex, that is the line bundle is $O(u_1^j P_j + (d - u_1^j)Q_j)$, then $v_1^j = d - u_1^j$, while $v_2^j = d - u_2^j - 1$. Therefore, $u_1^{j+1} = d - v_1^j = u_1^j$ while $u_2^{j+1} = d - v_2^j = u_2^j + 1$. Similarly, if we choose instead the second subindex for the line bundles on the component $j$, we will have $u_1^{j+1} = u_1^j + 1$, while $u_2^{j+1} = u_2^j$.

It follows from the above discussion (using that $u_1^1 = 0, u_1^2 = 1$) that $u_1^i$ is the number of times we did not choose the line bundle to be $O(u_1^j P_j + (d - u_1^j)Q_j)$ on the components $j < i$, while $u_2^i$ is one more than the number of times we did not choose the line bundle to be $O(u_2^j P_j + (d - u_2^j)Q_j)$ on the components $j < i$. This number of “non-chosen” components includes the case in which we choose the line bundles on one of these components to be generic. As $u_1^1 < u_2^1$ and $u_1^2 = 0, u_2^2 = 1$, this means that, on any subcurve starting at $C_1$, we cannot choose the line bundle to be $O(u_2^j P_j + (d - u_2^j)Q_j)$ more times than we choose it to be $O(u_1^j P_j + (d - u_1^j)Q_j)$.

This problem can be easily formulated in combinatorial terms using the subindices in the expressions of the line bundles: we have to choose occurrences of the numbers 1 and 2 on a list/array of length $2a$ so that both 1 and 2 appear $a$ times each, and the number of occurrences of 2 up to and including any position on the list is always less than or equal to the number of occurrences of 1 up to that same position. We will call an array satisfying these conditions admissible.

Since we are interested in counting how many of these choices for the subindices (line bundles) we have, the solution to this problem is the Catalan number $c_a$ (see [Sta99, Prop. 6.2.1]).

The arguments above could provide a new way to compute the classical Castelnuovo numbers (cf. [ACGH85, VII, Theorem 4.4]).

**Theorem 1.3.** The number of $g_{a+1}^1$ on a curve of genus $2a$ is given by the Catalan number

$$c_a = \frac{1}{a + 1} \binom{2a}{a}.$$
More generally, the number of $g_{r(a+1)}$ on a curve of genus $a(r+1)$ is given by the generalized Catalan number

$$C_{a,r+1} = \frac{(a(r + 1))!}{\prod_{i=0}^{r} i! (a + i)!}.$$ 

**Proof.** The same argument that we used to find the number of components of the Brill-Noether curve could be used to describe $W_{r(a+1)}$ on a curve of genus $a(r + 1)$. In this case the Brill-Noether locus is finite. On our reducible curve, its points correspond to the number of ways to order $a$ copies of each of the numbers $1, \ldots, r + 1$ so that at each position in the corresponding array a larger number does not appear more times than a smaller number. The number of arrays of this form is given by the generalized Catalan numbers. These numbers, that could be considered a generalization of classical Catalan numbers, arise as the solution to the many-candidate ballot problem in combinatorics (see [Zei83]). \hfill \Box

Let us now find the number of nodes of the Brill-Noether curve.

**Lemma 1.4.** The Brill-Noether curve $W_{a+2}^1(C_0)$ (where $C_0$ is as in Lemma 1.2), is a nodal curve with

$$\delta = 2 ((2a + 1) \epsilon_a - \epsilon_{a+1})$$

nodes.

**Proof.** We will count here the number of points of intersection of the components described above. We will see in the next Lemma that there are no triple or worse points.

We want to find conditions for two of the components of the Brill-Noether curve to intersect. Here, by a component of the degenerated Brill-Noether curve, we mean an elliptic curve as described in 1.2. We will also see in Lemma 1.7 that the points of intersection on a given component are all different and that the difference of two points of intersection is a multiple of $P_i - Q_i$, where $P_i, Q_i$ are the nodes on the component $C_i$ of the original curve.

Let $X$ and $X'$ be two components of $W_{a+2}^1(C_0)$ such that their intersection is non-empty. Assume that the bundles on $X$ correspond, via the description in Lemma 1.2, to the ordering of 1’s and 2’s given by the sequence $\alpha = (\alpha_1, \ldots, \alpha_{2a})$ satisfying the conditions mentioned there and with arbitrary line bundle on the $i$-th component. Similarly, assume that the second component $X'$ corresponds to the sequence $\alpha' = (\alpha'_1, \ldots, \alpha'_{2a})$, with arbitrary line bundle on the $i'$-th component, with $i' > i$.

We can now describe the components of the Brill-Noether curve in terms of paths in a 2-dimensional lattice from $(0, 0)$ to $(a, a)$ never rising above the diagonal with steps $(0, 1)$ and $(1, 0)$ (that we will identify with subindices 1
and 2 in Lemma 1.2 respectively) along with a marked vertex\(^1\). The vertex will represent the \(i\)th component of the original curve where the bundle is generic.

As the line bundle on each \(C_j\) other than \(C_i\) (resp. \(C_{i'}\)) is determined by the sequence \(\alpha\), it follows that

\[
\begin{align*}
\alpha_1 &= \alpha'_1, \ldots, \alpha_{i-1} = \alpha'_{i-1} \\
\alpha_i' &= \alpha'_{i'}, \ldots, \alpha_{2a} = \alpha'_{2a}.
\end{align*}
\]

If \(i' = i + 1\), the sequence \(\alpha\) equals \(\alpha'\). Conversely, if \(i' = i + 1\), the line bundle in the first \(i - 1\) and the last \(g - i - 1\) components of the original curve is the same for both choices. The line bundles on the components \(i\) and \(i + 1\) are free to vary for one of the choices and completely determined for the other. Hence, there exists a unique point of intersection between these two components of the Brill-Noether curve. That is, each component of the Brill-Noether curve corresponding to a choice of an ordering \(\alpha\) and a vertex \(i, 1 \leq i \leq g - 1\) of the chain of elliptic curves intersects the component corresponding to the same choice of \(\alpha\) and vertex \(i + 1\). The number of these intersections is \((g - 1)c_a\). Alternatively, one could think of each choice of sequence \(\alpha\) as giving rise to a component of genus \(g\) of the Brill-Noether curve.

Example 1.5. The lattice path for the intersection of the components \(X, X' \subset W_t^2(C_0)\) given by \(\alpha = \alpha' = (1, 2, 1, 2, 1, 1, 2, 1, 2, 2)\), with generic bundles for \(i = 6\) and \(i' = 7\), respectively, is shown in Figure 1.

Assume now that \(i' \neq i + 1\). As the line bundle on \(C_i\) for a point in \(X\) is generic, we have \(u_1^{i+1} = u_1^{i} + 1\) and \(u_2^{i+1} = u_2^{i} + 1\). On the second component of the Brill-Noether curve \(X'\), however, the index corresponding to \(\alpha'_i\) does not increase. Since the line bundle on the curve \(C_{i+1}\) must be the same for one point in \(X\) and one point in \(X'\) (we are assuming the components intersect), this implies \(\alpha'_{i+1} \neq \alpha'_{i}\). Recall now that the line bundle on the component \(C_i\) is generic for the points of \(X\). On the elliptic components of \(X\)

\(^1\)These paths, without the marked point, are classically known as Dyck paths (cf. [Sta99, Cor. 6.2]).
after $C_i$ the line bundles are again determined by $\alpha$. (Note that the indices in $\alpha$ are now off by one, given that the sequence $\alpha$ carries no information for the elliptic component $C_i \subset X$.) We deduce then that $\alpha_i = \alpha'_{i+1}$. As the discrepancy in vanishing between points in $X$ and points in $X'$ persists for the index $\alpha'_{i} \neq \alpha'_{i+1} = \alpha_i$, we have

$$\alpha'_{i+1} = \alpha'_{i+2} = \cdots = \alpha'_{i'-1} = \alpha_i = \alpha_{i+1} = \cdots = \alpha_{i'-2}.$$

Then, necessarily, $\alpha'_{i'-1} = \alpha'_i$.

For example, if $\alpha_i = 1$, this would say that

$$\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i'-2} = 1,$$

$$\alpha'_{i+1} = \alpha'_{i+2} = \cdots = \alpha'_{i'-1} = 1,$$

while

$$\alpha'_i = 2, \alpha'_{i'-1} = 2.$$

Intersections of two components $X$ and $X'$ of the Brill-Noether curve, in this case, correspond to lattice paths given by sequences $\alpha$ and $\alpha'$, respectively, whose entries satisfy (2), i.e. the paths are the same up to the first marked vertex and after the second marked vertex. After the first vertex, the paths become parallel, since $\alpha'_i = 2$, and meet again on the second vertex, since $\alpha'_{i'} = 2$.

**Example 1.6.** An example of an intersection of lattice paths corresponding to the intersection of components of the Brill-Noether curve in this case is represented in Figure 2. There, the first component $X$ is given by the sequence $\alpha = (1, 1, 2, 1, 1, 2, 2, 2)$ and the marked vertex for $i = 4$; the second intersecting component is given by $\alpha' = (1, 1, 2, 2, 1, 1, 1, 2, 2, 2)$ and $i' = 8$.

We are interested in counting how many of these intersections we have. For each pair (path, marked vertex) one can construct a lattice path corresponding to an intersection with the given pair, except when the marked vertex lies on the diagonal. The total number of these pairs is $(a - 1)c_a$, regardless of where the vertex is. The number of paths when the marked vertex is on the diagonal is the product of the number of paths from $(0, 0)$ to this vertex and the number of paths from the vertex to $(a, a)$, i.e. the product of Catalan numbers $c_k c_{a-k}$. The number of nodes on the curve in this case is therefore

$$\alpha_i = 2, \alpha'_{i'-1} = 2.$$

$$\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i'-2} = 1,$$

$$\alpha'_{i+1} = \alpha'_{i+2} = \cdots = \alpha'_{i'-1} = 1,$$

while

$$\alpha'_i = 2, \alpha'_{i'-1} = 2.$$

$$\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i'-2} = 1,$$

$$\alpha'_{i+1} = \alpha'_{i+2} = \cdots = \alpha'_{i'-1} = 1,$$

while

$$\alpha'_i = 2, \alpha'_{i'-1} = 2.$$

$$\alpha_i = \alpha_{i+1} = \cdots = \alpha_{i'-2} = 1,$$

$$\alpha'_{i+1} = \alpha'_{i+2} = \cdots = \alpha'_{i'-1} = 1,$$

while

$$\alpha'_i = 2, \alpha'_{i'-1} = 2.$$
\[
\delta = 2 \left( (a - 1)c_a - \sum_{k=1}^{a-1} c_k c_a - k \right) + (g - 1)c_a.
\]

From the well-known recursion of Catalan numbers
\[
c_{a+1} = \sum_{k=0}^{a} c_k c_{a-k},
\]
and the fact that \(c_0 = 1\), we obtain the expression \(\delta = 2 ((2a + 1)c_a - c_{a+1})\) for the number of nodes of the limit of the Brill-Noether curve, \(W_{1a+2}(C_0)\).

**Lemma 1.7.** Assume we have a family of curves for which the generic fiber is a generic genus \(g\) curve \(C\) and the special fiber \(C_0\) is a chain of elliptic curves. Then,

(a) The points at which a given component \(X\) of the Brill-Noether curve \(W_{1a+2}(C_0)\) intersects any other of the components of \(W_{1a+2}(C_0)\) are all different.

(b) A fixed component \(X\) has at most four points of intersection with other components.

(c) Assume that \(X\) is isomorphic to a component \(C_i\) of \(C_0\) and \(P_i, Q_i\) are the nodes of \(C_i\). If \(R, S\) are two points on \(X\) that are points of intersection with other components of the Brill-Noether curve, then \(R - S = k(P - Q)\) for some integer \(k\).

**Proof.** A component \(X\) of the Brill-Noether curve of \(C_0\) corresponds to the choice of a sequence \(\alpha = (\alpha_1, \ldots, \alpha_{2a})\) of 1’s and 2’s satisfying the conditions in Lemma 1.2 and with arbitrary line bundle on the \(i\)-th component. Assume we have
\[
\alpha_{t-1} \neq \alpha_t = \alpha_{t+1} = \cdots = \alpha_{i-1} \\
\alpha_i = \alpha_{i+1} = \cdots \alpha_{s-1} \neq \alpha_s
\]
Then this component intersects the following components

1. The component corresponding to the same chain \(\alpha\) with arbitrary line bundle on the \((i - 1)\)th component (assuming that \(i > 1\)).
(2) The component corresponding to the same chain $\alpha$ with arbitrary line bundle on the $(i + 1)^{th}$ components (assuming that $i < 2a$)

(3) The component corresponding to the chain

$$\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_s, \alpha_i, \alpha_{i+1}, \ldots, \alpha_{s-1}, \alpha_{s+1}, \ldots, \alpha_{2a})$$

with arbitrary line bundle on the component $s + 1$ (assuming that the chain $\alpha'$ is admissible).

(4) The component corresponding to the chain

$$\alpha'' = (\alpha_1, \ldots, \alpha_{t-2}, \alpha_t, \ldots, \alpha_{i-1}, \alpha_t-1, \alpha_i, \ldots, \alpha_{2a})$$

with arbitrary line bundle on the component $t - 1$ (assuming that the chain $\alpha''$ is admissible).

Choose now another component $X'$ of the Brill Noether curve corresponding to an admissible sequence $\hat{\alpha}$ and the choice of a component $j \neq i$ on which the line bundle is free to vary. Denote by $\hat{u}_1 < \hat{u}_2$, the vanishing at $P_t$ of a linear series in $X'$. Then, the restriction to $C_i$ of the line bundle coming from a linear series on $X'$ is

$$\mathcal{O}(\hat{u}_\alpha P_t + (d - \hat{u}_\alpha)Q_i)$$

If $j > i$

$$\mathcal{O}(\hat{u}_{\alpha_{i-1}} P_t + (d - \hat{u}_{\alpha_{i-1}})Q_i)$$

if $j < i$

It follows that the line bundle on the component $C_i$ in the cases listed above is

(1) $\mathcal{O}((u_{\alpha_{i-1}} + 1)P_t + (d - (u_{\alpha_{i-1}} + 1))Q_i)$  (assuming that $i > 1$).

(2) $\mathcal{O}(u_{\alpha_1} P_t + (d - u_{\alpha_1})Q_i)$  (assuming that $i < 2a$).

(3) $\mathcal{O}(u_{\alpha_2} P_t + (d - u_{\alpha_2})Q_i)$  if $\alpha'$ is admissible where

$$\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_s, \alpha_i, \alpha_{i+1}, \ldots, \alpha_{s-1}, \alpha_{s+1}, \ldots, \alpha_{2a}).$$

This condition is automatically satisfied if $\alpha_s = 1$, since 1 in an admissible sequence can always be pulled to earlier entries. When $\alpha_s = 2$, we need the number of ones in the partial sequence $(\alpha_1, \ldots, \alpha_{i-1})$ to be greater than the number of twos. Therefore, $u_{i} + 1 < u_2$. That is,

$$(*)3 \quad \text{if } \alpha_s = 2, \quad u_{\alpha_i} + 1 = u_1 + 1 < u_2 = u_{\alpha_s}. $$

(4) $\mathcal{O}((u_{\alpha_{t-1}} + 1)P_t) + (d - u_{\alpha_{t-1}} - 1)Q_i)$ assuming that the sequence

$$\alpha'' = (\alpha_1, \ldots, \alpha_{t-2}, \alpha_t, \ldots, \alpha_{i-1}, \alpha_{t-1}, \alpha_i, \ldots, \alpha_{2a})$$

is admissible. This condition is automatically satisfied if $\alpha_{t-1} = 2$ since 2 in an admissible sequence can always be pushed down to later entries. When $\alpha_{t-1} = 1$, the sequence $(\alpha_1, \ldots, \alpha_{t-2}, \alpha_t, \ldots, \alpha_{i-1})$ needs to be the beginning of an admissible sequence, and therefore satisfies that there are at least as many ones as there are twos. The vanishing at $P_{t-1}$ after such list would be $u_1, u_2 - 1$. So when the condition in (4) is satisfied, $u_1 < u_2 - 1$ if $\alpha_{t-1} = 1$. That is,

$$(*)4 \quad \text{if } \alpha_{t-1} = 1, \quad u_{\alpha_{t-1}} + 1 = u_1 + 1 < u_2 = u_{\alpha_{t-1}}.$$
We now need to check that the four values
\[ u_{\alpha_{i-1}+1}, u_{\alpha_i}, u_{\alpha_s}, u_{\alpha_t+1} \]
are different whenever they appear. Note that possible subindices are either 1 or 2 and that, by definition, \( u_1 < u_2 \). We also have \( \alpha_{i-1} \neq \alpha_{t-1}, \alpha_i \neq \alpha_s \).

The only overlaps that could happen are then if one of the entries \( \alpha_{i-1}, \alpha_{t-1} \) were 1 and one of \( \alpha_i, \alpha_s \) were two and \( u_2 = u_1 + 1 \). Conditions (*3) and (*4) exclude all of these options except for \( \alpha_{i-1} = 1, \alpha_i = 2 \) and \( u_2 = u_1 + 1 \). The condition \( u_2 = u_1 + 1 \) tells us that in the sequence \((\alpha_1, \ldots, \alpha_{i-1})\) there are as many ones as there are twos. Therefore, we would not be able to include in this sequence the additional \( \alpha_i = 2 \).

This proves that all points of intersection of a given component of the Brill-Noether curve with any other component are different.

The Brill-Noether curve lives in the Jacobian of the reducible curve, which is isomorphic to the product of elliptic curves \( C_1 \times C_g \). Each component is the product of one fixed point on each of these \( C_i \) except for one in which it is the whole curve. So the intersections are transversal. Moreover, from the proof of the Gieseker-Petri theorem, all points other than points of intersection are non-singular. \( \square \)

**Theorem 1.8.** The genus of \( W^1_{a+2}(C) \) is
\[ g_{W^1_{a+2}} = 1 + \frac{2a(2a+1)}{a+2} \cdot c_a \]

**Proof.** On the special fiber, the Brill-Noether curve degenerates to a reducible curve with a certain number of components \( \nu \). Let us denote by \( X_i \) each of these components and let \( g_{X_i} \) be their genera.

We have then (cf. [HM98, 2.14])
\[ g_{W^1_{a+2}} = \sum_{i=1}^{\nu} g_{X_i} + \delta - \nu + 1. \]

The result follows now from Lemmata 1.2 and 1.4. \( \square \)

### 2. Gonality of the Brill-Noether curve when \( g = 5 \)

The goal of this section is to show that, for a generic genus 5 curve \( C \), the Brill-Noether curve \( W^1_4(C) \) is not 6-gonal. For that, by a semicontinuity argument, it suffices to find one genus 5 curve such that the corresponding \( W^1_4(C) \) is not 6-gonal.

For the sake of completeness, let us recall the definition of gonality of a curve.

**Definition 2.1.** A smooth genus \( g \) curve \( C \) is \( k \)-gonal if there is a nonconstant map \( C \rightarrow \mathbb{P}^1 \) of degree \( k \) and there is no such map of degree \( k-1 \) or less. In other words, \( C \) is said to be \( k \)-gonal if it has a \( g_k^1 \) but no \( g_{k-1}^1 \).
To achieve our goal, we will use a chain of five elliptic curves, as described in Section 1, and take advantage of the description of the corresponding Brill-Noether curve. Note that this Brill-Noether curve is reducible not of compact type. It suffices then to show (see proof of [HM82, Theorem 5]) that this curve does not admit an admissible cover of degree 6.

Recall that by Lemma 1.2, the Brill-Noether curve $W_1^1(C)$ can be described explicitly when $C$ is a chain of elliptic curves. When $a = 2$, the curve $C$ is a chain of five elliptic curves. The discussion in the proof of the aforementioned Lemma, allows us to describe the components of the Brill-Noether curve as lists of length 4 of 1’s and 2’s subject to certain conditions (we referred to the lists satisfying these conditions as admissible). In particular, in our case, the two admissible sequences are $\alpha' = (1, 2, 1, 2)$ and $\alpha'' = (1, 1, 2, 2)$. Each of these will give rise to five components in the Brill-Noether curve, each isomorphic to the five components of the given curve $C$. We summarize the possibilities in the following tables.

| $g_1$ | $C'_1$ | $C'_2$ | $C'_3$ | $C'_4$ | $C'_5$ |
|-------|--------|--------|--------|--------|--------|
|       | L      | 4Q     | 4Q     | 4Q     | 4Q     |
| 1     | $P + 3Q$ | $L$     | $2P + 2Q$ | $2P + 2Q$ | $2P + 2Q$ |
| 2     | $3P + Q$ | $3P + Q$ | $L$     | $P + 3Q$ | $P + 3Q$ |
| 1     | $2P + 2Q$ | $2P + 2Q$ | $2P + 2Q$ | $L$     | $3P + Q$ |
| 2     | $4P$   | $4P$   | $4P$   | $4P$   | $L$    |

| $g_1$ | $C''_1$ | $C''_2$ | $C''_3$ | $C''_4$ | $C''_5$ |
|-------|--------|--------|--------|--------|--------|
|       | L      | 4Q     | 4Q     | 4Q     | 4Q     |
| 1     | $P + 3Q$ | $L$     | $2P + 2Q$ | $2P + 2Q$ | $2P + 2Q$ |
|       | $P + 3Q$ | $P + 3Q$ | $L$     | $3P + Q$ | $3P + Q$ |
| 2     | $4P$   | $4P$   | $4P$   | $L$    | $3P + Q$ |
| 2     | $4P$   | $4P$   | $4P$   | $L$    | $4P$   |

The column in the far left indicates the combinatorial type mentioned above. Each of the remaining entries corresponds to one of the five components of the Brill-Noether curve. An entry $aP + bQ$ on the $i^{th}$ row means that the limit linear series on $C$ on the component $i$ corresponds to the line bundle $\mathcal{O}(aP_i + bQ_i)$. An entry assigned the value $L$ indicates that we are choosing an arbitrary line bundle on this component (and therefore this gives a one dimensional choice).

Two components of the Brill-Noether curve intersect when three of the entries that are completely determined (that is, not an $L$) in both are identical. For example, the component $C'_2$ intersects $C'_1$ at the point $P + 3Q$, $C'_3$ at the point $2P + 2Q$, and $C'_4$ at the point $4Q$. Similarly, the component $C''_2$ intersects $C''_1$ at the point $P + 3Q$, $C''_3$ at $4Q$, and $C''_4$ at the point $2P + 2Q$. The component $C'_4$ intersects $C'_3$ at the point $3P + Q$, $C'_3$ at the point $2P + 2Q$, and $C''_2$ at the point $4P$. Finally, the component $C''_4$...
intersects $C'_5$ at the point $3P + Q$, $C'_3$ at the point $4P$, and $C'_2$ at the point $2P + 2Q$. The remaining components of the Brill-Noether curve intersect only neighboring components in the same chain.

For each of the components with three points of intersection with the rest, we denote by $X'_i$ (resp. $X''_{6-i}$) the line bundle of intersection of the component $C'_i$ (resp. $C''_{6-i}$) with the shortest tail of components of the same sort; denote by $Y'_i$ (resp. $Y''_{6-i}$) the intersection with the longest tail, and $Z'_i$ (resp. $Z''_{6-i}$) the intersection with the component of the different type.

Then $2X' - Y = Z$ and $2X - Y = Z$ where $X, Y,$ and $Z$ have the same indices (from now on, we will omit them whenever it is clear the case we are discussing). As $P_i$ and $Q_i$ were generic on each component $C_i$ of the given curve, the line bundles $X$ and $Y$ are generic on each $C'_i, C''_i$, $i = 2, 4$. Similarly, the points of intersection of $C'_i, C''_i$, $i = 1, 3, 5$ with the remaining components are generic too.

Assume now that the Brill-Noether curve admits a degree 6 admissible cover. Then normalizing the point of intersection of $C''_2$ and $C'_4$, we have a connected curve of compact type. The composition of the normalization map and the admissible cover is an admissible cover. On a curve of compact type, an admissible cover gives rise to a limit linear series of the same degree (see [EH86, §5]). We start by looking at what this limit linear series should look like.

Consider the restriction of the limit linear series to the curve consisting of the components $C'_1, C'_2, \ldots, C'_5$. Reasoning as we did in the previous section, we see that such a limit linear series would have a line bundle of degree 6 on $C'_4$ that vanishes with order at least $(0, 2)$ at $X'_4$ and at least $(0, 4)$ or $(1, 3)$ at $Y'_4$.

By symmetry, a limit linear series would restrict on $C''_2$ to a line bundle of degree 6 with vanishing order at least $(0, 2)$ at $X''_2$ and either $(0, 4)$ or $(1, 3)$ at $Y''_2$.

Let us show first that a linear series of degree 6 and dimension 2 on $C'_4$ that vanishes at $X'_4$ with multiplicity $(0, 2)$ and at $Y'_4$ with multiplicity either $(0, 4)$ or $(1, 3)$ vanishes at $Z'_4$ with multiplicity $(a, b)$ where $a + b \leq 6$. Assume this were not the case. Then we would have such a linear series with vanishing order at $Z'_4$ equal to either $(1, 6)$, $(2, 5)$, or $(3, 4)$.

The case of vanishing $(1, 6)$ at $Z$ and $(1, 3)$ at $Y$ is impossible as the highest vanishing at $Z$ plus the lowest at $Y$ add to more than the degree. It is easy to see that the case of vanishing order $(3, 4)$ at $Z$ and $(0, 4)$ at $Y$ fails using a similar argument.

The remaining cases would correspond (up to interchanging the roles of $Y$ and $Z$ if necessary) to a linear series on $C'_4$ of some degree $k$ and dimension 2 with a section that vanishes to order $k$ at $Z$, a section that vanishes to order $k - 1$ at $Y$, and a section that vanishes to order 2 at $X$. By the genericity of $C'_4, Y_4, Z_4$ and the condition $2X_4 = Y_4 + Z_4$, this is impossible. This can be checked by degenerating the curve to a rational cuspidal curve.
We saw that it is not possible to have vanishing order adding up to 7 at \( Z \). So the vanishing at \( Z \) adds up to at most 6. Let us check that there is a finite number of options that give this vanishing:

- An order of vanishing of \((0, 6)\) at \( Z \) is incompatible with vanishing of \((1, 3)\) at \( Y \), as \(1 + 6 = 7\) is larger than the degree.
- A vanishing order of \((0, 6)\) at \( Z \), \((0, 4)\) at \( Y \), and \((0, 2)\) at \( X \) means that the line bundle is \( \mathcal{O}(6Z) \) and the 2-dimensional series contains a section with vanishing 4 at \( Y \) and one section with vanishing 2 at \( X \) as well as the section with vanishing 6 at \( Z \).

Since

\[
\begin{align*}
\text{h}^0(\mathcal{O}(6Z - 4Y)) &= 2 \\
\text{h}^0(\mathcal{O}(6Z - 2X)) &= 4 \\
\text{h}^0(\mathcal{O}(6Z - 4Y - 2X)) &= 0
\end{align*}
\]

By \( \text{h}^0(\mathcal{O}(6Z - 4Y)) = 2 \) \( \implies \) \( \text{h}^0(\mathcal{O}(6Z)) = \text{h}^0(\mathcal{O}(6Z - 4Y)) \oplus \text{h}^0(\mathcal{O}(6Z - 2X)) \).

Then the section 6\( Z \) can be written in a unique way as a sum of a section of \( \text{h}^0(\mathcal{O}(6Z - 4Y)) \) and a section of \( \text{h}^0(\mathcal{O}(6Z - 2X)) \). This determines the linear series uniquely. The cases of vanishing of \((1, 5)\) at \( Z \), \((1, 3)\) at \( Y \) and \((0, 2)\) at \( X \) or vanishing of \((2, 4)\) at \( Z \), \((0, 4)\) at \( Y \) and \((0, 2)\) at \( X \) are treated similarly.

- A vanishing order of \((1, 5)\) at \( Z \), \((0, 4)\) at \( Y \) and \((0, 2)\) at \( X \) is contained in a linear series in \( Z + |L_5| \), where \( L_5 \) is a line bundle of degree 5 and contains the unique section of \( L_5 \) vanishing at \( Z \) with order 4 and the unique section vanishing at \( Y \) with order 4. For a given \( L_5 \), this determines a unique linear series of dimension 2 and, in general, this 2-dimensional linear series does not contain any section vanishing with order 2 at \( X \). Therefore, there are at most a finite number of line bundles for which these exist and for these line bundles the space of sections is uniquely determined.

- The remaining cases are treated similarly.

The information is summarized in the next three tables.

| \( g^1 \) | \((0, 2), (0, 4), (0, 6)\) | \((0, 2), (0, 4), (1, 5)\) | \((0, 2), (0, 4), (2, 4)\) |
|-----|-----------------|-----------------|-----------------|
| \( C_1 \) | \(4Y + [2Y], \ (0, 1), \ (6, 4)\) | \(4Y + [2Y], \ (0, 1), \ (6, 4)\) | \(4Y + [2Y], \ (0, 1), \ (6, 4)\) |
| \( C_2 \) | \(3Y + [3Y], \ (0, 2), \ (6, 3)\) | \(3Y + [3Y], \ (0, 2), \ (6, 3)\) | \(3Y + [3Y], \ (0, 2), \ (6, 3)\) |
| \( C_3 \) | \(2Y + [4Y], \ (0, 3), \ (6, 2)\) | \(2Y + [4Y], \ (0, 3), \ (6, 2)\) | \(2Y + [4Y], \ (0, 3), \ (6, 2)\) |
| \( C_4 \) | \(6Z, s_{6Z} = s_{4Y} + s_{3X}\) | \(Z + |L_5|, s_{2X} = s_{4Y} + s_{4Z}\) | \(2Z + [4Y], s_{4Y} = s_{2X} + s_{2Z}\) |
| \( C_5 \) | \(4X + [2X], \ (6, 4), \ (0, 1)\) | \(4X + [2X], \ (6, 4), \ (0, 1)\) | \(4X + [2X], \ (6, 4), \ (0, 1)\) |

| \( g^1 \) | \((0, 2), (1, 3), (1, 5)\) | \((0, 2), (1, 3), (2, 4)\) |
|-----|-----------------|-----------------|
| \( C_1 \) | \(4Y + [2Y], \ (0, 1), \ (6, 4)\) | \(4Y + [2Y], \ (0, 1), \ (6, 4)\) |
| \( C_2 \) | \(4Y + [2X], \ (0, 2), \ (5, 4)\) | \(4Y + [2X], \ (0, 2), \ (5, 4)\) |
| \( C_3 \) | \(X + [2Y], \ (1, 2), \ (5, 3)\) | \(X + [2Y], \ (1, 2), \ (5, 3)\) |
| \( C_4 \) | \(Y + [2Z], s_{4Z} = s_{2Y} + s_{2Z}\) | \(Y + [2Z], s_{4Z} = s_{2Y} + s_{2Z}\) |
| \( C_5 \) | \(4X + [2X], \ (6, 4), \ (0, 1)\) | \(4X + [2X], \ (6, 4), \ (0, 1)\) |
The numbers at the top indicate the vanishing at $X, Y, Z$ on $C'_4$. The numbers next to the line bundle on the remaining components indicate the vanishing at the points $X, Y$ (points of intersection with the components before and after respectively). The formula next to the line bundles on $C'_4$ indicates how the sections need to be chosen.

From the proof of [HM82, Theorem 5], the degree of an admissible cover is obtained by adding the degrees of the moving parts of the linear series and subtracting the (common) ramification at the nodes. In order to obtain an admissible cover in the normalization of the Brill-Noether curve at the point of intersection between $C'_2$ and $C''_4$, we will be using limit linear series as above with same ramification at $Z'_2, Z''_4$. In all cases, we obtain an admissible cover of degree 6. In order for this admissible cover to give rise to an admissible cover before normalizing, it should be completely ramified at $Z'_2, Z''_4$. As this is not the case, not such a covering exists and the generic curve is not 6-gonal.

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Centro de Ciencias Matemáticas, Unidad Morelia, Universidad Nacional Autónoma de México, Apartado Postal 61-3 (Xangari), 58089 Morelia, Michoacán
E-mail address: abel@matmor.unam.mx

Department of Mathematics, Tufts University, Bromfield-Pearson Hall, 503 Boston Avenue, Medford, MA 02155
E-mail address: alberto.lopez@tufts.edu
E-mail address: montserrat.teixidoribigas@tufts.edu