EXISTENCE OF ASYMPTOTIC PAIRS IN POSITIVE ENTROPY GROUP ACTIONS

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Abstract. We provide a definition of a ≺-asymptotic pair in a topological action of a countable amenable group \( G \), where ≺ is an order on \( G \) of type \( \mathbb{Z} \). We then prove that if \((\mathcal{O}, \nu, G)\) is a multiorder on \( G \), then for every topological \( G \)-action of positive entropy there exists an ≺-asymptotic pair for almost every order \( \prec \in \mathcal{O} \). This result is a generalization of the Blanchard-Host-Ruette Theorem for classical topological dynamical systems (actions of \( \mathbb{Z} \)).

1. Introduction

In classical topological dynamical system \((X, T)\), where \( X \) is a compact metric space with a metric \( d_X \) and \( T : X \to X \) is a homeomorphism, an asymptotic pair is defined as a pair of distinct points \( x, y \in X \) such that
\[
\lim_{k \to +\infty} d_X(T^k(y), T^k(y)) = 0.
\]
The set of all asymptotic pairs in \((X, T)\) does not depend on the choice of the metric \( d_X \).

Using the notion of asymptotic pairs it is possible to characterize topological systems with actions of \( \mathbb{Z} \) of entropy zero. Firstly, in 2002, F. Blanchard, B. Host and S. Ruette proved the following theorem.

**Theorem 1.1 (Blanchard-Host-Ruette).** [BHR, Proposition 1] Let \((X, T)\) be an invertible topological dynamical system with positive topological entropy. Then \((X, T)\) has asymptotic pairs.

More precisely, the set of points belonging to an asymptotic pair has measure 1 for any ergodic measure on \( X \) with positive entropy.

Later, in 2010, T. Downarowicz and Y. Lacroix proved that every topological dynamical system of entropy zero is a factor of a system with no asymptotic pairs [DL]. These two results combined give a full characterization of zero entropy systems as factors of systems with no asymptotic pairs. There have been many attempts to generalize this characterization for actions of groups other than \( \mathbb{Z} \). Any such attempt requires providing a definition of an asymptotic pair fitting the more general setup. So far, the best results were obtained by W. Huang, L. Xu and Y. Yi in 2014 [HXY] and by W. Bulat and B. Kamiński and J. Świątkoski in 2016 [BKS]. These two teams of authors independently proved that if \( G \) is an infinite amenable group which is orderable, then for every topological \( G \)-action of positive entropy, there exists a pair which is asymptotic with respect to the invariant order (note that this order need not be of type \( \mathbb{Z} \)). However, none of the definitions provided so
far can be used in the case of arbitrary countable amenable groups (which need not admit an invariant order). Recently T. Downarowicz, P. Oprocha, M. Więcek and G. Zhang considered orders of type $\mathbb{Z}$ on countable groups. This approach opens a possibility of creating a new definition of an asymptotic pair in general systems with actions of countable groups, generalizing the classical one. A multiorder, introduced in [DOWZ] is a collection of orders of type $\mathbb{Z}$ invariant under the specific action of the group and supporting an invariant probability measure $\nu$. With the aid of multiorders, it is possible to generalize the Blanchard-Host-Ruette Theorem to the case of actions of countable amenable groups.

Section 2 of this note contains the definitions of a multiorder and a multiordered dynamical system, as well as some useful theorems concerning the properties of these notions.

Section 3 begins with the definition of a $\prec$-asymptotic (read prec-asymptotic) pair in a topological dynamical system with an action of a countable group $G$, where $\prec$ is an order of type $\mathbb{Z}$ on $G$. It is followed by the formulation and proof of the main theorem of this note being the generalization of the Blanchard-Host-Ruette Theorem. It states that, for every topological $G$-action $(X,G)$ of positive entropy and every multiorder $(\bar{O},\nu,G)$ on $G$, for $\nu$-almost every order $\prec \in \bar{O}$, there exists a $\prec$-asymptotic pair in $X$. Since the natural order $<$ on $\mathbb{Z}$ is invariant under addition of integers, it forms a one-point multiorder on $\mathbb{Z}$. Thus, the classical theorem of Blanchard, Host and Ruette is a specific case of our theorem. Moreover, our theorem implies the existence of asymptotic pairs in positive entropy $\mathbb{Z}$-actions not only along the natural order on $\mathbb{Z}$ but also along many nonstandard ones (examples of such orders are given in [DOWZ]).

2. Preliminaries - Multiorders and multiordered dynamical systems

This section is devoted to familiarize the reader with the definitions of a multiorder and a multiordered dynamical system and with some important properties of these objects. In this section we skip the proofs of all theorems, which can be found in the paper [DOWZ].

**Definition 2.1.** Let $G$ be a countable set. An order of type $\mathbb{Z}$ on $G$ is a total order $\prec$ on $G$ such that all order intervals $[a,b)\prec = \{a,b\} \cup \{g \in G : a \prec g \prec b\}$ ($a,b \in G$, $a \prec b$) are finite and there are no minimal or maximal elements in $G$.

The set $\bar{O}$ of all orders on $G$ of type $\mathbb{Z}$ is a subset of the family of all relations on $G$ which can be viewed as the topological space $\{0,1\}^{G \times G}$. Hence, $\bar{O}$ inherits from $\{0,1\}^{G \times G}$ a natural topological and Borel structure. When $G$ is a countable group, $G$ acts on $\bar{O}$ by homeomorphisms as follows: for $g \in G$ and $\prec \in \bar{O}$ we have $g(\prec) = \prec'$ where $\prec'$ is given by

$$a \prec' b \iff ag \prec bg.$$  

(2.1)

**Definition 2.2.** Let $\nu$ be a $G$-invariant Borel probability measure supported by $\bar{O}$. By a multiorder (on $G$) we mean the measure-preserving $G$-action $(\bar{O},\Sigma_{\bar{O}},\nu,G)$, where $\Sigma_{\bar{O}}$ is the Borel sigma-algebra on $\bar{O}$.

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1By a total order on $G$ we mean a transitive relation $\prec$ such that for every $a,b \in G$ exactly one of the alternatives holds: either $a \prec b$ or $b \prec a$, or $a = b$. 

For brevity, in what follows, we will write \((\hat{\mathcal{O}}, \nu, G)\) instead of \((\hat{\mathcal{O}}, \Sigma_{\hat{\mathcal{O}}}, \nu, G)\). Note that the natural order \(<\) on \(\mathbb{Z}\) is a fixed point of the action of \(G\) given by the additive version of formula (2.1). Hence the system \((\{<\}, \delta_\prec, \mathbb{Z})\), where \(\delta_\prec\) is a Dirac measure centered on \(<\), is a multiorder. However, there exist many more multiorders on \(\mathbb{Z}\) consisting of non-standard orders (see e.g. [DOWZ, Example B.9]).

**Theorem 2.3.** [DOWZ, Theorem 2.6] Let \(G\) be a countable amenable group. There exists a multiorder \((\hat{\mathcal{O}}, \nu, G)\) of entropy zero.

There are other ways of representing multiorders on countable groups. Especially useful is viewing a multiorder as a family of anchored bijections from \(\mathbb{Z}\) to \(G\) (which can be interpreted as a subset of the metrizable and separable topological space \(G\mathbb{Z}\)).

**Definition 2.4.** With each \(\prec \in \hat{\mathcal{O}}\) we associate the bijection \(\operatorname{bi}_\prec : \mathbb{Z} \to G\) which is anchored (i.e. satisfies \(\operatorname{bi}_\prec(0) = e\), where \(e\) is the unit of \(G\)), and on the rest of \(\mathbb{Z}\) is determined by the property:

\[
\operatorname{bi}_\prec(i) = g \iff \operatorname{bi}_\prec(i + 1) = \operatorname{succ}_\prec(g), \quad i \in \mathbb{Z}, \ g \in G,
\]

where \(\operatorname{succ}_\prec(g)\) denotes the successor of \(g\) with respect to \(\prec\).

On the set \(\operatorname{Bi}(\mathbb{Z}, G)\) of anchored bijections \(\operatorname{bi} : \mathbb{Z} \to G\) we define an action of \(G\) by the following formula:

\[
(\operatorname{gbi})(i) = \operatorname{bi}(i + k) \cdot g^{-1}, \text{ where } k \text{ is such that } g = \operatorname{bi}(k).
\]

**Theorem 2.5.** [DOWZ, Proposition 2.11] The assignment \(\psi : \hat{\mathcal{O}} \to \operatorname{Bi}(\mathbb{Z}, G)\), given by \(\psi(\prec) = \operatorname{bi}_\prec\), is a measurable bijection with a continuous inverse, which intertwines the action of \(G\) on \(\hat{\mathcal{O}}\) given by (2.1) with the action of \(G\) on \(\operatorname{Bi}(\mathbb{Z}, G)\) given by (2.3).

By convention, if \(\nu\) is an invariant measure on \(\hat{\mathcal{O}}\) we will denote the push-forward measure on \(\operatorname{Bi}(\mathbb{Z}, G)\) by the same letter \(\nu\). We will denote by \(k^\prec\) the element \(\operatorname{bi}_\prec(k)\).

Using this notation we have:

**Corollary 2.6.** For every order \(\prec \in \hat{\mathcal{O}}\) and every \(i \in \mathbb{Z}\) and \(g \in G\) we have

\[
\psi^\prec(i) = (i + k)^\prec \cdot g^{-1}, \text{ equivalently } i^\prec \cdot g^{-1} = (i - k)^\psi^\prec(i),
\]

where \(k\) is the unique integer such that

\[
g = k^\prec, \text{ equivalently } g^{-1} = (-k)^\psi^\prec(i).
\]

Now we provide the definition of a multiordered dynamical system as well as some important theorems concerning its properties.

**Definition 2.7.** By a multiordered dynamical system, denoted by \((X, \mu, G, \phi)\), we will mean a measure-preserving \(G\)-action \((X, \mu, G)\) with a fixed measure-theoretic factor map \(\phi : (X, \mu, G) \to (\hat{\mathcal{O}}, \nu, G)\) to a multiorder (provided such a factor map exists).

Let \(G\) and \(\Gamma\) be two countable groups acting on the same probability space \((X, \mu)\). These actions are called orbit equivalent if for \(\mu\)-almost every \(x \in X\) we have \(\{g(x) : g \in G\} = \{\gamma(x) : \gamma \in \Gamma\}\).

It turns out that every multiordered \(G\)-action is orbit equivalent to a specific \(\mathbb{Z}\)-action such that the orbit equivalence preserves the multiorder factor, as the theorem below states.
Theorem 2.8. [DOWZ, Theorem 3.5] Suppose \( \varphi : X \to \hat{O} \) is a measure-theoretic factor map from a measure-preserving \( G \)-action \((X, \mu, G)\) to a multiorder \((\hat{O}, \nu, G)\). Then \((X, \mu, G)\) is orbit-equivalent to the \( \mathbb{Z} \)-action generated by the successor map defined as follows:

\[
S(x) = 1^\prec(x), \quad \text{where} \quad \prec = \varphi(x),
\]

i.e. \( S(x) = g(x) \), where \( g = 1^\prec(x) \). Moreover, for any \( k \in \mathbb{Z} \), we have

\[
S^k(x) = k^\prec(x).
\]

Let \( \hat{S} \) denote the transformation on \( \hat{O} \) defined by

\[
\hat{S}(\prec) = 1^{\prec}(\prec),
\]

i.e. \( \hat{S}(\prec) = g(\prec) \), where \( g = 1^{\prec} \) and \( g(\prec) \) is given by the formula (2.1). Then \( \hat{S} \) preserves the measure \( \nu \), the \( \mathbb{Z} \)-action generated by \( \hat{S} \) is orbit equivalent to the \( G \)-action given by (2.1), and \( \varphi \) is a factor map from the \( \mathbb{Z} \)-action \((X, \mu, S)\) to the \( \mathbb{Z} \)-action \((\hat{O}, \nu, \hat{S})\).

Additionally, the above orbit equivalence between \((X, \mu, G)\) and \((X, \mu, S)\) preserves the conditional entropy with respect to the multiorder factor, which is captured by the following theorem.

Theorem 2.9. [DOWZ, Theorem 5.1] Let \((X, \mu, G, \varphi)\) be a multiordered dynamical system and let \( S \) denote the successor map defined by the formula (2.6). Then, for every finite, measurable partition \( P \) of \( X \) we have

\[
h(\mu, G, P|\Sigma) = h(\mu, S, P|\Sigma),
\]

where \( h(\mu, G, P|\Sigma) \) is the conditional (with respect to \( \Sigma \)) entropy of the process \((X, \mu, P, G)\) generated by \( P \) under the action of \( G \) and \( h(\mu, S, P|\Sigma) \) is the analogous conditional entropy of the process \((X, \mu, P, S)\) generated by \( P \) under the action of \( \mathbb{Z} \) given by the iterates of \( S \).

3. Asymptotic pairs in positive entropy actions of countable amenable groups

We begin with providing the definition of a \( \prec \)-asymptotic pair for a topological action of a general countable group \( G \). By a topological action we mean a compact metric space \( X \) (with a metric \( d_X \)) on which \( G \) acts by homeomorphisms.

Definition 3.1. Let \((X, G)\) be a topological action of a countable group \( G \). Let \( \prec \) be an order of type \( \mathbb{Z} \) on \( G \). A pair of distinct points \( x, y \in X \) is called \( \prec \)-asymptotic if it satisfies

\[
\lim_{k \to +\infty} d_X(k^\prec(x), k^\prec(y)) = 0
\]

(recall that \( k^\prec \) denotes the \( k \)-th element of \( G \), counting from \( e \), in the order \( \prec \)).

Now we formulate and prove the main theorem of this note. In case of the one-element multiorder \((\{<\}, \delta <, \mathbb{Z})\) on \( \mathbb{Z} \), our theorem reduces to the Blanchard-Host-Ruette Theorem.

Theorem 3.2. Let \((\hat{O}, \nu, G)\) be a multiorder on a countable amenable group \( G \). Let \((X, G)\) be a topological \( G \)-action with positive entropy. For \( \nu \)-almost every \( \prec \in \hat{O} \), there exists a \( \prec \)-asymptotic pair in \( X \).
Proof. By the variational principle for $G$-actions \cite{MO} Variational Principle 5.2.7, there exists an ergodic measure $\mu$ on $X$ such that the measure-theoretic entropy $h(\mu, G)$ is positive. Recall that $\mathcal{O}$ can be viewed as the set $\text{Bi}(\mathbb{Z}, G)$ of all anchored bijections from $\mathbb{Z}$ to $G$, which is a subset of the metrizable and separable space $G^\mathbb{Z}$. We arrange all elements of $G$ in a sequence $(g_n)_{n \in \mathbb{N}}$ (this arrangement is arbitrary and has nothing to do with the multiorder). The formula

$$
\rho(g_n, g_m) = \begin{cases} 
0, & \text{if } n = m \\
\frac{1}{2\min(n,m)}, & \text{if } n \neq m
\end{cases}
$$

defines a totally bounded metric on $G$ compatible with the discrete topology. Then, we equip $\mathcal{O}$ with the metric

$$(3.2) \quad d_\mathcal{O}(<, <') = \sum_{k \in \mathbb{Z}} \frac{1}{2|k|} \rho(k^<, k^{<'})$$

which is totally bounded and generates the standard product topology inherited from $G^\mathbb{Z}$.

By the standard ergodic decomposition argument it suffices to prove the theorem for an ergodic measure $\nu$. Consider the measurable space $(X \times \mathcal{O}, \Sigma_X \otimes \Sigma_\mathcal{O})$, where $\Sigma_X$ is the Borel sigma-algebra on $X$. Observe that $\Sigma_X \otimes \Sigma_\mathcal{O}$ is the Borel sigma-algebra on $X \times \mathcal{O}$ for the product topology on $X \times \mathcal{O}$ (this follows from the fact that both $X$ and $\mathcal{O}$ are separable metric spaces). For any joining $\mu \lor \nu$ of the measures $\mu$ and $\nu$, in particular for the product joining $\mu \times \nu$, the measure theoretic dynamical system $(X \times \mathcal{O}, \Sigma_X \otimes \Sigma_\mathcal{O}, \mu \lor \nu, G)$ is a multiorder system with the projection on the second coordinate in the role of the factor map $\varphi$. Let $S$ be the successor map on $X \times \mathcal{O}$ induced by the multiorder $\mathcal{O}$. By Theorem 2.9,

$$h(\mu \lor \nu, S) \geq h(\mu \lor \nu, S|\Sigma_\mathcal{O}) = h(\mu \lor \nu, G|\Sigma_\mathcal{O}) = h(\mu, G) > 0.
$$

The product joining $\mu \times \nu$ needs not be ergodic itself, but by the ergodic decomposition argument, there exists an ergodic joining $\mu \lor \nu$ which satisfies $h(\mu \lor \nu, S) > 0$. Therefore, by Lemma 4 in \cite{BHR}, the system $(X \times \mathcal{O}, \Sigma_X \otimes \Sigma_\mathcal{O}, \mu \lor \nu, S)$ admits a partition $\mathcal{P}$ (not necessarily finite or countable) such that any pair of points belonging to the same atom of the sigma-algebra $\mathcal{P}^+ = \bigvee_{k \geq 1} S^{-k}(\mathcal{P})$ is asymptotic (w.r.t. the successor map $S$) and $\mathcal{P}^+$ is a proper (up to the measure $\mu \lor \nu$) sub-sigma-algebra of $\Sigma_X \otimes \Sigma_\mathcal{O}$. \footnote{A joining of two measure-preserving systems $(X, \mu, G)$ and $(Y, \nu, G)$ is any measure on $X \times Y$ (usually denoted as $\mu \lor \nu$) invariant under the product action of $G$, and whose respective marginals are $\mu$ and $\nu$. It is well known that if both $\mu$ and $\nu$ are ergodic then there exists an ergodic joining $\mu \lor \nu$ (see e.g. \cite{R} Proposition 1.4); the same proof applies to amenable group actions.)\footnote{In the paper \cite{BHR}, $\mathcal{P}^+$ is denoted as $\mathcal{P}^-$.
} \footnote{Lemma 4 in \cite{BHR} is stated for the measure theoretic $\mathbb{Z}$-action $(X, \Sigma_X, \mu, T)$, where $X$ is a compact metric space and $T$ is a homeomorphism on $X$. However in the proof of the lemma, continuity of $T$ is not used, while compactness of $X$ is required only to guarantee the existence of a refining sequence of finite partitions $(\mathcal{Q}_n)_{n \in \mathbb{N}}$ such that the diameters of these partitions tend to 0 as $n \to +\infty$. In case of the system $(X \times \mathcal{O}, \Sigma_X \otimes \Sigma_\mathcal{O}, \mu \lor \nu, S)$, the space $X \times \mathcal{O}$ need not be compact but it is totally bounded, hence the desired refining sequence of partitions on $X \times \mathcal{O}$ exists, which makes Lemma 4 in \cite{BHR} valid.} Let $J$ be the set of points in $X \times \mathcal{O}$ which are members of the asymptotic pairs (w.r.t. the map $S$) and for a given pair $(x, <)$ let $A(x, <)$ denote the set of points $(x', <') \in X \times \mathcal{O}$ for which

$$(3.3) \quad \lim_{k \to +\infty} d_{X \times \mathcal{O}}(S^k(x, <), S^k(x', <')) = 0$$

Observe that $\Sigma = \bigcup_{n \in \mathbb{N}} \Sigma_n$ is the Borel sigma-algebra on $X$.
(this set consists of \((x,\prec)\) and all points \((x',\prec')\) asymptotic to \((x,\prec)\)). Clearly, \(J\) is measurable and \(S\)-invariant. By the ergodicity of \(\mu \lor \nu\), either \((\mu \lor \nu)(J) = 1\) or \((\mu \lor \nu)(J) = 0\). Assume the latter case. Then \(A(x,\prec) = \{(x,\prec)\}\) for \((\mu \lor \nu)\)-almost every \((x,\prec)\). Hence, almost all atoms of \(\mathcal{P}^+\) are singletons and consequently \(\mathcal{P}^+ = \Sigma_X \otimes \Sigma_{\mathcal{O}}\) which is a contradiction. We have shown that \((\mu \lor \nu)(J) = 1\).

By Theorem 2.3, \(S^k(x,\prec) = (k^\prec(x),k^\prec(\prec))\) and \(S^k(x',\prec') = (k^\prec(x'),k^\prec(\prec'))\). Henceforth, the convergence (3.5) is equivalent to the following two convergences

\[
(3.4) \quad \lim_{k \to +\infty} d_X(k^\prec(x),k^\prec(x')) = 0,
\]
\[
(3.5) \quad \lim_{k \to +\infty} d_\mathcal{O}(k^\prec(\prec),k^\prec(\prec')) = 0.
\]

Note that the latter condition means that the orders \(\prec\) and \(\prec'\) are either equal or asymptotic under the successor map \(\tilde{S}\) on \(\mathcal{O}\). We suspend the proof in order to give a detailed condition for the orders \(\prec,\prec'\) to be asymptotic w.r.t. \(\tilde{S}\).

**Lemma 3.3.** Two distinct orders \(\prec,\prec'\) are asymptotic w.r.t. the successor map \(\tilde{S}\) (i.e. satisfy (3.5)) if and only there exists \(k_0 \in \mathbb{N}\) and \(g_0 \in G\) such that for every \(k \geq k_0\) we have \(k^\prec = k^{\prec'} \cdot g_0\).

**Proof.** In view of formula (3.2), the convergence (3.5) is equivalent to the following condition: for every finite set \(I \subset \mathbb{Z}\) there exists \(k_0\) such that for every \(k \geq k_0\) and every \(i \in I\) we have \(i^{k^\prec} = i^{k^{\prec'}}\). For \(I = \{1\}\) this means that for all \(k\) larger than or equal to some \(k_0\) we have

\[
(3.6) \quad 1^{k^\prec} = 1^{k^{\prec'}}.
\]

Let \(g_0 = (k_0^{\prec'})^{-1} \cdot k_0^\prec\). We will show by induction that \(g_0\) satisfies the assertion of the lemma. Of course, \(k_0^\prec = k_0^{\prec'} \cdot g_0\). Assume that for some \(k \geq k_0\) we have

\[
k^\prec = k^{\prec'} \cdot g_0.
\]

We need to prove the same for \(k + 1\). By the formulas (2.4) and (2.5) for \(i = 1\) and \(g = k^\prec\), the equality (3.6) takes on the form

\[
(k + 1)^\prec \cdot (k^\prec)^{-1} = (k + 1)^{\prec'} \cdot (k^{\prec'})^{-1}.
\]

Therefore,

\[
(k + 1)^\prec = (k + 1)^{\prec'} \cdot (k^\prec)^{-1} \cdot k^\prec = (k + 1)^{\prec'} \cdot g_0.
\]

\[\square\]

Now we continue the main proof. We will show that if \((x,\prec)\) and \((x',\prec')\) form an asymptotic pair in \(X \times \tilde{O}\) (w.r.t. the map \(S\)), then \((x,g_0^{-1}(x'))\) is a \(\prec\)-asymptotic pair, where \(g_0\) is as in Lemma 3.3 or \(g_0 = e\) in case \(\prec = \prec'\).

Let \(k_0\) be as in Lemma 3.3 (or, if \(\prec = \prec'\), we let \(k_0 = 1\)) and let \(\varepsilon\) be a positive number. By (3.4), there exists \(k_1 \geq k_0\) such that for every \(k \geq k_1\) we have

\[
d_X(k^\prec(x),k^{\prec'}(x')) < \varepsilon.
\]

Since \(k^\prec(g_0^{-1}(x')) = (k^\prec \cdot g_0^{-1})(x') = k^\prec(x')\) (where the latter equality follows from Lemma 3.3), we have

\[
d_X(k^\prec(x),k^\prec(g_0^{-1}(x'))) = d_X(k^\prec(x),k^{\prec'}(x')) < \varepsilon,
\]

which was to be shown.
Since \((\mu \lor \nu)(J) = 1\), the projection of \(J\) to \(\hat{O}\) has \(\nu\)-measure 1. This means that the set of \(\prec \in \hat{O}\) for which there exists \(x \in X\) such that \((x, \prec)\) is a member of an asymptotic pair (w.r.t. the map \(S\)) has \(\nu\)-measure 1. We have just shown that for every asymptotic (w.r.t. \(S\)) pair \((x, \prec), (x', \prec')\), there exists \(g_0 \in G\) such that the pair \((x, g_0^{-1}(x'))\) is \(\prec\)-asymptotic. This means that for \(\nu\)-almost every \(\prec \in \hat{O}\) there exists a \(\prec\)-asymptotic pair in \(X\), which ends the proof. \(\square\)

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