BRIDGE NUMBER AND CONWAY PRODUCTS

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Abstract. Schubert proved that, given a composite link $K$ with summands $K_1$ and $K_2$, the bridge number of $K$ satisfies the following equation:

$$\beta(K) = \beta(K_1) + \beta(K_2) - 1.$$ 

In “Conway Products and Links with Multiple Bridge Surfaces”, Scharlemann and Tomova proved that, given links $K_1$ and $K_2$, there is a Conway product $K_1 \times_c K_2$ such that

$$\beta(K_1 \times_c K_2) \leq \beta(K_1) + \beta(K_2) - 1.$$ 

In this paper, we define the generalized Conway product $K_1 \ast_c K_2$ and prove the lower bound $\beta(K_1 \ast_c K_2) \geq \beta(K_1) - 1$ where $K_1$ is the distinguished factor of the generalized product. We go on to show this lower bound is tight for an infinite class of links with arbitrarily high bridge number.

Introduction

Bridge number was introduced by Schubert in his paper “Uber eine Numerische Knoteninvariante.” Here Schubert proves that, given a composite knot $K$ with summands $K_1$ and $K_2$, the bridge number of $K$ satisfies the following equation:

$$\beta(K) = \beta(K_1) + \beta(K_2) - 1.$$ 

The techniques used in this paper are inspired by Schultens’ more modern proof of the same equality [5].

In this paper $K$, will be a tame link embedded in $S^3$ and $h : S^3 \to \mathbb{R}$ is a height function with level sets consisting of 2-spheres and two exceptional points corresponding to $+\infty$ and $-\infty$. We require that $h$ restricts to a Morse function on $K$.

Date: February 2, 2008.

* Research partially supported by an NSF grant.
Definition 1. If the maxima of $h|_K$ occur above all of the minima then $K$ is in bridge position. The fewest number of maxima of $h|_K$ over all embeddings of $K$ is the bridge number of $K$, denoted $β(K)$.

Definition 2. A sphere $C$ embedded in $S^3$ which meets a link $K$ transversely in four points is called a Conway sphere.

Let $K_1 ⊂ S^3_1$ and $K_2 ⊂ S^3_2$ be links embedded in distinct 3-spheres. For each $i = 1, 2$ let $τ_i$ be arcs in $S^3_i$ such that $∂τ_i ⊂ K_i$ but $τ_i$ is otherwise disjoint from $K_i$. Let $η(τ_i)$ be a regular closed neighborhood of $τ_i$, then $η(τ_i) ∩ K_i$ is a trivial tangle and $∂(η(τ_i))$ is a Conway sphere for $K_i$. Let $B_i = S^3_i − int(η(τ_i))$.

Definition 3. Let $K_1 *_c K_2$ (the generalized Conway product of $K_1$ and $K_2$) denote the link in $S^3$ formed by removing $int(η(τ_1))$ from $S^3$ and gluing $∂(B_1)$ to $∂(B_2)$ via a homeomorphism which sends $K_1 ∩ ∂(B_1)$ to $K_2 ∩ ∂(B_2)$.

The image $C$ of $∂(η(τ_1))$ and $∂(η(τ_2))$ after their identification is the Conway sphere of the generalized Conway product.

We call $K_1 *_c K_2$ a rational completion of $K_1$ if $(B_2, K_2 ∩ B_2)$ is a rational tangle.

It is also important to note that the link type of $K_1 *_c K_2$ is dependent on $K_1$, $K_2$, $τ_1$, $τ_2$, and the gluing homeomorphism.

Note that nowhere do we require that the Conway sphere in the a generalized Conway product be incompressible. If the Conway sphere is compressible and $K_1 *_c K_2$ is prime, then one of the factor links is a 1 or 2 bridge link. For a further discussion of this special case, see Example 1.

The classical Conway sum and Conway product were originally defined in [1] as operations which received as input two tangle diagrams and produced as output a new tangle diagram. This original operation has inspired several related constructions. In [2], Lickorish studies a method of producing prime links by identifying together the boundaries of prime tangles. Scharlemann’s and Tomova’s operation takes two links, evacuates untangles from the links’ complements to form two tangles, and identifies together the boundaries of these two tangles to form a new link[3]. The definition of generalized Conway product used in this paper encapsulates the construction in [3]. By carefully choosing $τ_1$, $τ_2$ and the gluing map, Scharlemann and Tomova showed the existence of a generalized Conway product which respects bridge surfaces. They go on to prove that the following inequality holds for such a product

$$β(K_1 *_c K_2) ≤ β(K_1) + β(K_2) − 1$$
However, it is also shown in [3] (via a construction by the author) that the above inequality is not always an equality, so a lower bound is needed.

The main goal of this paper is to present a lower bound on the bridge number of the generalized Conway product in terms of the bridge number of the factor links.

**Theorem A. (Main Theorem)** Let $K_1 \ast_c K_2$ be a generalized Conway product and $K_1$ be the distinguished factor, then

$$\beta(K_1 \ast_c K_2) \geq \beta(K_1) - 1$$

In addition, there is an infinite family of generalized Conway products with arbitrarily high bridge number for which $\beta(K_1 \ast_c K_2) = \beta(K_1) - 1$.

The term "distinguished factor" which appears in the above theorem will be defined later in the paper.

I am grateful to Martin Scharlemann for suggesting that I investigate the relationship between Conway products and bridge number and for many helpful conversations.

**Conway Spheres**

This section is devoted to generalizing work of Schultens [5] on companion tori in link complements to the case of Conway spheres.

For the remainder of the paper, $K$ will be the generalized Conway product $K_1 \ast_c K_2$ embedded in $S^3$ with Conway sphere $C$.

We adopt the following notation from [3]. A (punctured) disk will denote a disk embedded in $S^3$ which is disjoint from $K$ or meets $K$ transversely in a single point. A simple closed curve in a Conway sphere $C$ is **c-inessential** if it bounds a (punctured) disk in $C$.

**Definition 4.** Let $F_C$ be the singular foliation on the Conway sphere $C$ induced by $h\mid_C$. Let $\sigma$ be a leaf corresponding to a saddle singularity (by general position we can assume every such $\sigma$ is disjoint from $K$). Then $\sigma$ consists of two circles $s_1$ and $s_2$ wedged at a point. If either $s_1$ or $s_2$ is c-inessential in $C$, then we say $\sigma$ is a c-inessential saddle. Otherwise, $\sigma$ is c-essential.

The following lemma and its proof are immediate generalizations of Schultens’ Lemma 1 [5]. We need alter the statement and proof only slightly to account for punctures in the Conway sphere.

**Lemma 1.** Let $h, K, F_C, C$ be as above. If $F_C$ contains c-inessential saddles then after an isotopy of $C$ that does not change the number of
maxima of $h|_K$ there is a c-inessential saddle $\sigma$ for which the following properties hold:

1) $s_1$ bounds a (punctured) disk $D_1 \subset C$ such that $\partial C$ restricted to $D_1$ contains only disjoint circles and one maximum or minimum.

2) For $L$ the level sphere of $h$ containing $\sigma$, $D_1$ co-bounds a 3-ball $B$ with a disk $\bar{D} \subset L - s_1$, such that $B$ does not contain $+\infty$ or $-\infty$, and such that $s_2$ does not meet $B$.

Proof: Choose a c-inessential saddle $\sigma = s_1 \vee s_2$ to be innermost in $C$. Up to relabeling, $s_1$ bounds a (punctured) disk $D_1 \subset C$ satisfying the first property. $s_1$ cuts the level sphere $L$ containing $\sigma$ into two disks $\bar{D}_1$ and $\bar{D}_2$. $D_1 \cup \bar{D}_1$ and $D_2 \cup \bar{D}_2$ bound 3-balls $\bar{B}_1$ and $\bar{B}_2$ respectively. Up to relabeling, $\bar{B}_1$ contains $+\infty$ or $-\infty$ and $\bar{B}_2$ contains neither. If $s_2$ does not meet $\bar{B}_2$ then property 2 is satisfied and we are done.

Suppose $s_2 \subset \bar{D}_2 \subset \bar{B}_2$. Let us assume $D_1$ contains a single maximum $p$ and $\bar{B}_1$ contains $+\infty$ (the other situation is proved analogously). By general position, we can assume $h|_C$ does not have local maxima or minima at $K \cap C$. Choose $\alpha$ to be a monotone arc with end points $p$ and $+\infty$ which intersects $C$ only at local maxima. Label the points of $C \cap \alpha$ starting at $p$ and increasing toward $+\infty$ as $p, p_1, p_2, \ldots, p_n$. See Fig. 1. Let $S_+$ be a level sphere contained in a small neighborhood of $+\infty$ such that $S_+$ does not meet $C$ or $K$. Let $\beta_n$ be a subarc of $\alpha$ with endpoints $p_n$ and $+\infty$. Enlarge $\beta_n$ slightly to be a vertical solid cylinder $V$ such that $\partial(V)$ consists of a small disk in $D_1$ a small disk in $S_+$ and an annulus $A$ with $F_A$ a collection of circles. Replacing $C$ with the Conway sphere $(C - V) \cup A \cup (S_+ - V)$ represents an isotopy of $C$ in $S^3 - K$ which does not change the number of minima or maxima of $h|_C$.

By induction on $n$, we can assume $\alpha$ is disjoint from $C$ except at the point $p$. By isotoyping $D_1$ to a new disk $D_1^*$ in the manner described above, we have enlarged $\bar{B}_2$ to contain $+\infty$ and shrunk $\bar{B}_1$ so that it is disjoint from $+\infty$. After a small tilt so that $h$ again restricts to a Morse function on $D_1^*$, $F_{D_1^*}$ is a collection of circles and one maximum. By choosing $D_1^*$, $\bar{B}_1$, and $\bar{D}_1$ to be $D_1$, $B$, and $\bar{D}$ respectively we achieve property 2. □

**Definition 5.** Following [5], say a Conway sphere $C$ is taut with respect to $\beta(K)$ if the number of saddles of $\partial C$ is minimal subject to the condition that $h|_K$ has $\beta(K)$ maxima.

**Lemma 2.** Let $h, K, C, F_C$ be as above. If $C$ is taut with respect to $\beta(K)$, then there are no c-inessential saddles in $F_C$. 
Proof: Suppose there is a c-inessential saddle. We can assume there exists a c-inessential saddle $\sigma$ in $F_C$ satisfying the conclusions of Lemma 1. Up to relabeling, $s_1$ bounds a (punctured) disk $D_1 \subset C$. If $D_1$ is a 0-punctured disk, then the conclusion follows from Schultens Lemma 2 [5].

Assume $D_1$ is a 1-punctured disk containing a single maximum $p$ and lying above $L$, the level sphere containing $\sigma$ (the other possible situation, a reflection through $L$, is proved analogously). Let $k = K \cap D_1$ and $\gamma$ be the strand of $K \cap B$ that contains $k$ as an endpoint. The following isotopy was originally described on page 5 of [5].

If $\gamma$ is monotone with respect to $h$ or the closest critical point on $\gamma$ is a minimum, we can skip ahead to the isotopy described in the next paragraph. Otherwise, let $r$ be the maximum of $h\vert_\gamma$ closest to $k$ along $\gamma$. Let $\alpha$ be a monotone arc contained in $B$ starting at $r$ and ending at $p$, the maximum of $D_1$. Let $\beta$ be an arc in $D_1$ transverse to $F_C$ starting at $k$ and ending at $p$. $\alpha$ together with $\beta$ and $\gamma'$ (the segment of $\gamma$ connecting $k$ to $r$) bound a disk $E$ with interior contained in $B$. $K$ intersects $E$ in $\gamma'$ and transversely in points $q_1, ..., q_n$. Let $q_i$ be the highest such point of intersection. Let $\rho \subset (K \cap B)$ be the arc containing $q_i$ and $\tau$ a small monotone sub-arc of $\rho$ containing $q_i$. Replace $\tau$ with a monotone arc which starts at an end point of $\tau$, runs parallel to $E$ until it nearly reaches $D_1$, travels along $D_1$ until it returns to the other side of $E$, travels parallel to $E$ (now on the opposite side) and connects to the other end point of $\tau$. The result is isotopic to

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Figure 1.}
\end{figure}


does not change the number of maxima of $h|_K$ and reduces $n$. By induction on $n$, we may assume that $K \cap E = \gamma'$. Isotope $\gamma'$ along $E$ until it lies just outside of $D_1$ except where it intersects $D_1$ exactly at the point $p$. This isotopy of $K$ does not change the number or nature of the maxima of $h|_K$. See Fig. 2.

At this point $(K \cup C) \cap \text{int}(B)$ can be shrunk horizontally and lowered to lie just below $\tilde{D}$. This isotopy produces a monotone arc connecting $p$ to the image of $K \cap \text{int}(B)$ under the isotopy and does not change the number or nature of critical points of $h|_C$ or $h|_K$.

Since $D_1 \cup \tilde{D}$ bounds a ball minus an unknotted arc, we can isotope $D_1$ to $\tilde{D}$ to create $\tilde{C}$. After a small tilt, we have produced a new Conway sphere $\tilde{C}$ which is isotopic to $C$ while preserving the number of maxima of $h|_K$. See Fig. 3. Since the number of saddles of $\tilde{C}$ is one less than the number of saddles of $F_C$, we have a contradiction to the assumption that $C$ is taut. □

Let $\sigma$ be a saddle in $F_C$. The bicollared neighborhood of $\sigma$ in $C$ has three boundary components $c_1$, $c_2$, and $c_3$ where $c_1$ and $c_2$ are parallel to $s_1$ and $s_2$ respectively. By the above lemma, if $C$ is taut then neither $c_1$ nor $c_2$ bound (punctured) disks. Since $C$ is a 4-punctured sphere, both $c_1$ and $c_2$ bound twice-punctured disks to each side. Consequently, $c_3$ bounds a disk to one side and a 4-punctured disk to the other. Thus, the saddles of a taut Conway sphere are stacked as illustrated in Fig. 4.
C decomposes $S^3$ into two 3-balls $B_1$ and $B_2$. We may assume $c_1$ and $c_2$ are contained in the same level surface $L$. $L \setminus (c_1 \cup c_2)$ is composed of two disks and an annulus $A$. If a collar of $\partial(A)$ in $A$ is contained in $B_1$, then we say $\sigma$ is unnested with respect to $B_1$. If not, we say $\sigma$ is nested with respect to $B_1$. We define nested and unnested with respect to $B_2$ similarly. Note that nested with respect to $B_1$ is the same as unnested with respect to $B_2$ and nested with respect to $B_2$ is unnested with respect to $B_1$.

**Lemma 3.** Let $h, K, C, F_C$ be as above. If $C$ is taut with respect to $\beta(K)$, then all saddles of $F_C$ are c-essential and nested with respect to $B_i$, $i = 1, 2$.

**Proof:** Suppose $\sigma_1$ and $\sigma_2$ are a saddles of $F$ such that $\sigma_i$ is nested with respect to $B_i$ for $i = 1, 2$. We can assume $\sigma_1$ and $\sigma_2$ are adjacent in $C$. If $\sigma_1$ is the circles $s_1^1$ and $s_1^2$ wedged at a point and $\sigma_2$ is the circles $s_2^1$ and $s_2^2$ wedged at a point, then, up to labeling, $s_1^1$ and $s_2^1$ co-bound an annulus in $C$ which is disjoint form all other saddles and does not meet $K$. Here we invoke Schultens’ Lemma 3 where she constructs an isotopy of $C$ which eliminates one saddle of $F_C$ while preserving the number of maxima and minima of $h|_K$. This contradicts the tautness of $C$. □

Summerizing the previous lemmas: if $C$ is taut with respect to $\beta(K)$, then we may assume all saddles of $F_C$ are c-essential and nested with respect to $B_1$ (up to labeling). At this point, $B_1$ can be visualized as a neighborhood of a knotted arc embedded in $S^3$. This useful embedding of $B_1$ allows us to bound $\beta(K)$ in terms of $\beta(K_1)$. Hence, we call $K_1$ the **distinguished factor**. It is relevant to note that $B_1$ and $B_2$ are simultaneously realized as neighborhoods of knotted arcs iff $F_C$ contains no saddles.
Inequalities

Let \( \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) be the set of saddles in \( \mathcal{F}_C \). If \( C \) is taut, then let \( D_1 \) and \( D_2 \) be the two twice punctured disks in \( C - \bigcup_{i=1}^{n} \sigma_i \). We use the following labeling convention: \( \{x_1^i, x_2^i\} = K \cap D_i \) and \( h(x_1^i) \geq h(x_2^i) \) for \( i = 1, 2 \). We will want to keep track of the following properties:

1) Is \( x_j^i \) a local minimum or maximum of \( h|_{K \cap B_1} \) for \( i = 1, 2 \) and \( j = 1, 2 \)?

2) Does \( h|_{D_i} \) have a unique local minimum or maximum for \( i = 1, 2 \)? (i.e. Is \( D_i \) a cap or a cup?)

To accomplish this we define a 3-tuple labeling \( (x, y, z) \in \{m, M\}^3 \) for each \( D_i \) where where \( x = m \) (resp. \( M \)) if \( x_1^i \) is a minimum (resp. maximum) of \( h|_{K \cap B_1} \), where \( y = m \) (resp. \( M \)) if \( x_2^i \) is a minimum (resp. maximum) of \( h|_{K \cap B_1} \), and \( z = m \) (resp. \( M \)) if \( h|_{D_i} \) has a unique local minimum (resp. maximum).

As an example, the disk in Fig. 5 is labeled \( (M, m, m) \).

![Figure 5](image)

**Lemma 4.** Given \( h, K, C, \mathcal{F}_C \) as above. There is an isotopy of \( K \) preserving the number of maxima of \( h|_K \) and resulting in \( h|_K \) having at least one maximum or minimum in \( B_2 \).

Proof: We assume \( C \) is taut. If \( \mathcal{F}_C \) contains saddles then \( D_1 \) and \( D_2 \) are defined as in the above discussion. If \( \mathcal{F}_C \) has no saddles then let \( s \) be a level curve in \( \mathcal{F}_C \) which separates two points in \( C \cap K \) from two others. The two components of \( C - s \) are the twice-punctured disks \( D_1 \) and \( D_2 \).

We will proceed by cases using the 3-tuple labeling of \( D_1 \) and \( D_2 \). An underscore in a coordinate of a labeling will indicate \( m \) or \( M \). (i.e. \( (m, \_ M) \) represents \( (m, m, M) \) or \( (m, M, M) \)).

**Claim:** Neither \( D_1 \) nor \( D_2 \) is labeled \( (M, M, M) \) or \( (m, m, m) \).

Suppose to get a contradiction that \( D_1 \) is labeled \( (M, M, M) \). Let \( \partial(D_1) = s_1 \) and \( \sigma \) be the saddle in \( \mathcal{F}_C \) containing \( s_1 \). Let \( L \) be the level surface containing \( \sigma \). Let \( \{x_1, x_2\} = \{x_1^i, x_2^i\} = K \cap D_1 \).
By appealing to the proof of Lemma 1, we assume $D_1$ co-bounds a 3-ball $B$ with a disk $\tilde{D} \subset L - C$, such that $B$ does not contain $+\infty$ or $-\infty$, and such that $s_2$ does not meet $B$.

$K \cap \text{int}(B)$ can be shrunk horizontally and lowered to lie just below $\tilde{D}$. This isotopy produces two monotone arcs in $B$ connecting $x_1$ and $x_2$ to the image of $K \cap \text{int}(B)$ under the isotopy and does not change the number or nature of critical points of $h|_C$ or $h|_K$.

Since $D_1 \cup \tilde{D}$ bounds a ball minus two monotone unknotted arcs, we can isotope $D_1$ to $\tilde{D}$ to create $\tilde{C}$. After a small tilt, we have produced a new Conway sphere $\tilde{C}$ which is isotopic to $C$ while preserving the number of maxima of $h|_K$. See Fig. 7. Since the number of saddles of $\tilde{F}_C$ is one less than that of $F_C$, we have a contradiction to the assumption that $C$ is taut. The other possibilities in this case are proved analogously.

**Figure 6.**

**Case 1:** One of $D_i$ for $i = 1, 2$ is labeled $(m, \underline{M})$ or $(\underline{M}, m)$.

Up to renaming of the disks, let $D_1$ have the 3-tuple label $(m, \underline{M})$. Let $\partial(D_1) = s_1$ and $\sigma$ be the saddle in $F_C$ containing $s_1$. Let $L$ be the level surface containing $\sigma$. Let $\{x_1, x_2\} = \{x_1^i, x_2^i\}$ and $\gamma$ be the strand of $K \cap B_1$ that contains $x_1$ as an endpoint, so $\gamma$ ascends from $x_1$ into $B_1$.

By appealing to the proof of Lemma 6, we assume $D_1$ co-bounds a 3-ball $B$ with a disk $\tilde{D} \subset L - C$, such that $B$ does not contain $+\infty$ or $-\infty$, and such that $s_2$ does not meet $B$.

We proceed as in the proof of Lemma 8. Let $r$ be the maximum of $h|_\gamma$ closest to $x_1$ along $\gamma$. Let $\alpha$ be a monotone arc contained in $B$ starting at $r$ and ending at $p$, the maximum of $D_1$. Let $\beta$ be an arc in $D_1$ transverse to $F_C$ starting at $x_1$ and ending at $p$. $\alpha$ together with $\beta$ and $\gamma'$ (the segment of $\gamma$ connecting $x_1$ to $r$) bound a disk $E$ with interior contained in $B$. $K$ intersects $E$ in $\gamma'$ and transversely in points $q_1, \ldots, q_n$. Let $q_i$ be the highest such point of intersection. Let $\rho \subset (K \cap B)$ be the arc containing $q_i$ and $\tau$ a small monotone sub-arc of $\rho$ containing $q_i$. Replace $\tau$ with a monotone arc which starts at an
end point of \( \tau \) runs parallel to \( E \) until it nearly reaches \( D_1 \), travels along \( D_1 \) until it returns to the other side of \( E \), travels parallel to \( E \) (now on the opposite side) and connects to the other end point of \( \tau \). Since \( h(x_1) \geq h(x_2) \), then \( h(q_i) \geq h(x_2) \) for \( i = 1, ..., n \) and the link resulting from the above arc replacement is isotopic to \( K \). See Fig. 6.

As in Lemma 8, this isotopy does not change the number of maxima of \( h|_K \) but does reduce \( n = |K \cap \text{int}(E)| \). By induction on \( n \), we may assume that \( K \cap E = \gamma' \). Isotope \( \gamma' \) along \( E \) until it lies just outside \( D_1 \) except where it intersects \( D_1 \) exactly at the point \( p \). Again, this isotopy of \( K \) does not change the number of maxima of \( h|_K \) nor does it alter the tautness of \( C \). We conclude \( h|_K \) has at least one maximum in \( B_2 \). The proof if \( D_i \) is labeled \((m, m, M)\) is analogous.

**Figure 7.**

**Case 2:** The labels of \( D_1 \) and \( D_2 \) are both chosen from the set \( \{(M, m, m), (M, m, M)\} \).

The disks corresponding to these two possible labelings are depicted in Fig. 8.

**Figure 8.**

Suppose \( D_1 \) is labeled \((M, m, M)\) and \( D_2 \) is labeled \((M, m, m)\). Let \( \alpha \) be the component of \( K \cap B_2 \) with an end point \( x_1 \). If \( \alpha \) contains a maximum or minimum of \( h|_K \), then we are done. If not, then \( \alpha \) is monotone and the other endpoint of \( \alpha \) must be \( x_2 \). This leaves \( x_3 \) and \( x_1 \) connected by \( \beta \), the other component of \( K \cap B_2 \). The monotonicity of \( \alpha \) ensures \( h(x_3^2) \geq h(x_1^1) \). Since \( h(x_1^1) \geq h(x_3^1) \), \( h(x_3^2) \geq h(x_2^2) \) and \( h(x_3^2) \geq h(x_1^1) \), then \( h(x_2^2) \geq h(x_1^1) \). However, \( x_2 \) is labeled \( M \) and \( x_1 \) is labeled \( m \), so there must be both a minimum and a maximum of \( h|_K \).
in $\beta \subset B_2$. See Fig 9. This result follows analogously for the other possible labelings of $D_1$ and $D_2$.

\[ \beta(\mathcal{K}) \geq \beta(K_1) - 1 \]

\textbf{Theorem A.} Let $h$, $K$, $C$, $\mathcal{F}_C$ be as above. Then the following inequality holds:

\[ \beta(K) \geq \beta(K_1) - 1 \]

Where $K_1$ is the distinguished factor.

Proof: By the previous lemmas, we can assume that $C$ has no inessential saddles, $C$ is nested with respect to $B_2$, and $h|_K$ has at least one maximum in $B_2$ (the case where $h|_K$ has one minimum in $B_2$ is proved analogously). To prove the theorem, we need only prove that the number of maxima of $h|_K$ in $B_1$ is greater than or equal to $\beta(K_1) - 2$.

The theorem will then follow since $\beta(K) = (\text{number of maxima of } h|_K \text{ in } B_1) + (\text{number of maxima of } h|_K \text{ in } B_2) \geq \beta(K_1) - 2 + 1 = \beta(K_1) - 1$.

First, we analyze the case where $\mathcal{F}_C$ contains no saddles. If $C$ contains no saddles, there is a level preserving isotopy of $S^3$ taking $C$ to a standard round 2-sphere. Such an isotopy preserves the number and nature of maxima of $h|_K$ in $B_1$. As in Lemma 10, a point in $K \cap C$ is labeled with an $m$ if it is a local minimum of $h|_{K \cap B_1}$ and is labeled with an $M$ if it is a local maximum of $h|_{K \cap B_1}$. The link $K_1$ can be recovered from $K \cap B_1$ by taking a rational completion of $K_1$ using a rational tangle $T$. If more points of $K \cap C$ are labeled with an $M$, take $T$ to lie above $C$. If more are labeled with an $m$, take $T$ to lie below $C$. See Fig. 10. Since the portion of the rational tangle lying in the region labeled $R$ can be taken to be monotone with respect to $h$, this rational completion causes the creation of at most two new maxima. The number of maxima of the resulting embedding of $K_1$ is at most
two more than the number of maxima of $h|_K$ in $B_1$. Hence, the number of maxima of $h|_K$ in $B_1$ is greater than or equal to $\beta(K_1) - 2$.

(Note: If $F_C$ has no saddles, we get the analogous estimate that the number of maxima of $h|_K$ in $B_2$ is greater than or equal to $\beta(K_2) - 2$. Hence, in this special case, we get the additional inequality $\beta(K) \geq \beta(K_1) + \beta(K_2) - 4$.)

We now assume $F_C$ contains saddles. To establish the desired inequality in this general setting, we build an isotopy of $S^3$ which takes $B_1$ to a standard round 3-ball and preserves the number and nature of critical points of $h|_K$ in $B_1$. This isotopy, however, does not preserve the number of critical points of $h|_K$ in $B_2$. Let $D_1$ be one of the twice punctured disks in $C - \bigcup_{i=1}^n \sigma_i$. Let $\partial(\bar{D}_1) = s_1$ and $\sigma$ be the saddle in $F_C$ containing $s_1$. $F_{D_1}$ is a collection of circles and one point corresponding to a maximum of $h|_C$ (if the point is a minimum, the case is analogous). Let $L$ be the level surface containing $\sigma$ and $\bar{D}$ be the disk component of $L - s_1$ which does not meet $s_2$. $D_1$ and $\bar{D}$ co-bound a 3-ball $B$. By appealing to the proofs of Lemma 6, we can assume $B$ does not meet $+\infty$. Let $x_1, x_2 = K \cap D_1$. Each point $x_i$ receives a label as described above. Since $h|_{D_1}$ has a maximum as the unique critical point, we can horizontally shrink and vertically lower $B \cup D_1$ until $D_1$ lies just above $\bar{D}$. Let $D_1^*$ be the image of $D_1$ under this isotopy and let $p$ be the unique maximum of $h|_{D_1^*}$. Let $J$ be the level surface containing $p$. By general position, $J \cap C$ consists of the point $p$ and a collection of circles. One such circle $c_2$ is parallel in $C$ to $s_2$. By picking $D_1^*$ close enough to $\bar{D}$, we can choose a point $b$ in $c_2$ and an arc $\alpha$ in $J$ which
is disjoint from $C$ except at its boundary $\{b, p\}$. Choose another arc $\beta$ in $C$ which does not meet $K$, has boundary $\{b, p\}$ and is transverse to $\tilde{f}_C$ everywhere except where it passes through $s_1 \cap s_2$. Having made $D_1^+$ sufficiently close to $\tilde{D}$ we can assume $\alpha$ and $\beta$ co-bound a disk $F$ which is vertical with respect to $h$, disjoint from $K$, and disjoint from $C$ except along $\beta$. Isotope $C$ along $F$ to effectively cancel a saddle with a maximum. See Fig. 11.

![Figure 11](image)

Repeat this process to produce an isotopy $\Phi : S^3 \to S^3$ so that $f_{\Phi(C)}$ contains no saddles. By the previous argument, $f_{\Phi(C)}$ has no saddles implies the number of maxima of $h|_{\Phi(K)}$ in $\Phi(B_1)$ is greater than or equal to $\beta(K_1) - 2$. However, $\Phi$ was constructed so that the number of maxima of $h|_{\Phi(K)}$ in $\Phi(B_1)$ is equal to the number of maxima of $h|_K$ in $B_1$. Hence, the number of maxima of $h|_K$ in $B_1$ is greater than or equal to $\beta(K_1) - 2$. This completes the proof of the theorem. □

**Examples**

**Example 1**

It is important to note that nowhere in the proof of Theorem 1 do we need incompressibility of our Conway sphere $C$. One might ask how we can reconcile Theorem 1 with the fact that there exist rational completions of the unknot with arbitrarily high bridge number. In fact, any Whitehead double of a knot is an example of such a link. In such cases, the distinguished factor is always a rational link. See Fig. 12. Hence, $K_1$ has bridge number at most 2. If we now employ Theorem 1, we get the following trivial inequality $\beta(K_1 * e K_2) \geq \beta(K_1) - 1 \geq 2 - 1 \geq 1$.

**Example 2**

In Fig. 13, $K_1$ is the connect sum of four trefoils and $K_2$ is a satellite link with a trefoil as companion. Schubert’s seminal work on bridge number tells us that $\beta(K_1) = 5$ and $\beta(K_2) \geq 4 \[4\]$. Since Fig. 13
gives a presentation of $K_2$ with exactly 4 maxima we conclude that $\beta(K_2) = 4$. The link $K = K_1 \ast_c K_2$ depicted in Fig. 5 is a satellite link also with a trefoil as companion. Again Schubert’s results tell us that $\beta(K) \geq 4$ and again we have a presentation of $K$ with exactly 4 maxima. Hence, $\beta(K) = 4 = \beta(K_1) - 1$.

To extend this particular example to an infinite family of links where $\beta(K) = \beta(K_1) - 1$ simply take $K_2$ to be a $(p, 2)$ cable link with an n-bridge knot as companion and $K_1$ to be the connect sum of $2n$ copies of a 2-bridge link. After a construction analogous to that in Fig. 13, $K_1 \ast_c K_2$ will be a satellite link with bridge number $2n$. Hence, $\beta(K_1 \ast_c K_2) = 2n = (2n + 1) - 1 = \beta(K_1) - 1$. We conclude that the bound given in the main theorem is tight for an infinite family of generalized Conway products with arbitrarily high bridge number.
Figure 13.

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