ON THE SIZE OF CERTAIN SUBSETS OF INVARIANT BANACH SEQUENCE SPACES

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ABSTRACT. The essence of the notion of lineability and spaceability is to find linear structures in somewhat chaotic environments. The existing methods, in general, use ad hoc arguments and few general techniques are known. Motivated by the search of general methods, in this paper we extend recent results of G. Botelho and V.V. Fávaro on invariant sequence spaces to a quite general setting. Our main results show that, under very weak assumptions, some subsets of invariant sequence spaces contain, up to the null vector, a closed infinite-dimensional subspace.

1. Introduction

The notion of invariant sequence spaces, as we investigate in this note, was introduced in [8] although it seems to have its roots in [2, 7]. Our main results are extensions of recent results of G. Botelho and V.V. Fávaro [9]. We show, among other results, that some very special invariant sequence spaces used in [9] can be replaced by quite general invariant sequence spaces. Let us first recall the notion of invariant sequence space.

Definition 1.1. ([8]) Let $X \neq \{0\}$ be a Banach space.
(a) Given $x \in X^\mathbb{N}$, $x^0$ is defined as: if $x$ has only finitely many non-zero coordinates, then $x^0 = 0$; otherwise, $x^0 = (x_j)_{j=1}^\infty$ where $x_j$ is the $j$-th non-zero coordinate of $x$.
(b) An invariant sequence space over $X$ is an infinite-dimensional Banach or quasi-Banach space $E$ of $X$-valued sequences enjoying the following conditions:
(b1) For $x \in X^\mathbb{N}$ such that $x^0 \neq 0$, $x \in E$ if and only if $x^0 \in E$, and $\|x\| \leq K\|x^0\|$ for some constant $K$ depending only on $E$.
(b2) $\|x_j\| \leq \|x\|$ for every $x = (x_j)_{j=1}^\infty \in E$ and every $j \in \mathbb{N}$.

Example 1.2. As mentioned in [8], usual sequence spaces are invariant sequence spaces. For instance
(a) For every $0 < p \leq \infty$, the spaces
$$\ell_p(X) = \left\{ (x_j)_{j=1}^\infty \in X^\mathbb{N} : \| (x_j)_{j=1}^\infty \| := \left( \sum_{j=1}^\infty \|x_j\|^p \right)^\frac{1}{p} < \infty \right\},$$
$$\ell_w^u(X) = \left\{ (x_j)_{j=1}^\infty \in X^\mathbb{N} : \| (x_j)_{j=1}^\infty \|_{w,p} := \left( \sum_{j=1}^\infty |\varphi(x_j)|^p \right)^\frac{1}{p} < \infty, \varphi \in X' \right\},$$
$$\ell_p^w(X) = \left\{ (x_j)_{j=1}^\infty \in \ell_p(X) : \lim_{n \to \infty} \| (x_j)_{j=n}^\infty \|_{w,p} = 0 \right\},$$
$$c_0(X) = \left\{ (x_j)_{j=1}^\infty \in X^\mathbb{N} : \lim_{j \to \infty} x_j = 0 \right\},$$
$$c(X) = \left\{ (x_j)_{j=1}^\infty \in X^\mathbb{N} : \lim_{j \to \infty} x_j \text{ exists} \right\}.$$
Proposition 2.3. spaces is not needed in the next proposition. Also, in the forthcoming Remark 2.4 we stress that even the full definition of invariant sequence spaces is either empty or spaceable in the particular sequence spaces we do not need the notions of non-contractive/strongly non-contractive mappings and, moreover the particular sequence spaces $\ell_p(Y), \ell^w_q(Y)$ and $c_0(Y)$ can be replaced by general invariant sequence spaces. Also, in the forthcoming Remark 2.4 we stress that even the full definition of invariant sequence spaces is not needed in the next proposition.

2. Preliminary results: invariant sequence spaces and lineability

The spirit of the concept of lineability and spaceability is to look for linear structures in nonlinear settings. There are few general methods (see, for instance, [5] [4]) to prove lineability and spaceability and in general particular problems need \textit{ad hoc} arguments. For more details on the subject we refer to [1] [3] [6] and the references therein.

The following definition is a natural extension of [9] Definition 2.2:

**Definition 2.1.** Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set and $E$ be an invariant sequence space over $X$. If $E_l$, for all $l \in \Gamma$, are invariant sequence spaces over $Y$ and $f : X \to Y$ is any map, we define the set

$$G(E, f, (E_l)_{l \in \Gamma}) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{l \in \Gamma} E_l \right\}.$$  

According to [9] Definition 2.3 a map $f : X \to Y$ between normed spaces is said to be:

(a) **Non-contractive** if $f(0) = 0$ and for every scalar $\alpha \neq 0$ there is a constant $K(\alpha) > 0$ such that

$$(\text{1}) \quad \|f(\alpha x)\|_Y \geq K(\alpha) \cdot \|f(x)\|_Y$$

for every $x \in X$.

(b) **Strongly non-contractive** if $f(0) = 0$ and for every scalar $\alpha \neq 0$ there is a constant $K(\alpha) > 0$ such that

$$|\varphi(f(\alpha x))| \geq K(\alpha) \cdot |\varphi(f(x))|$$

for all $x \in X$ and $\varphi \in Y'$.

The following result was recently proved by Botelho and Fávaro (see [9] Theorem 2.5):

**Theorem 2.2.** ([9]) Let $X$ and $Y$ be Banach spaces, $E$ be an invariant sequence space over $X$, $f : X \to Y$ a function and $\Gamma \subseteq (0, \infty]$.

(a) If $f$ is non-contractive, then

$$C(E, f, \Gamma) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{q \in \Gamma} \ell_q(Y) \right\},$$

$$C(E, f, 0) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin c_0(Y) \right\}$$

are either empty or spaceable in $E$ (i.e., the union of the set with $\{0\}$ contains a closed infinite-dimensional subspace of $E$).

(b) If $f$ is strongly non-contractive, then

$$C^w(E, f, \Gamma) = \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{q \in \Gamma} \ell^w_q(Y) \right\}$$

is either empty or spaceable in $E$.

Let us recall that a subset $A$ of a topological vector space $V$ is said to be $c$-lineable in $V$ (where $c$ denotes the cardinality of the continuum) if $A \cup \{0\}$ contains a subspace of dimension $c$ of $V$. The following proposition shows that in the above result (at least to prove the $c$-lineability of the sets) we do not need the notions of non-contractive/strongly non-contractive mappings and, moreover the particular sequence spaces $\ell_q(Y), \ell^w_q(Y)$ and $c_0(Y)$ can be replaced by general invariant sequence spaces. Also, in the forthcoming Remark 2.4 we stress that even the full definition of invariant sequence spaces is not needed in the next proposition.

**Proposition 2.3.** Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set, $E$ be an invariant sequence space over $X$, and $E_l$ be invariant sequence spaces over $Y$ for all $l \in \Gamma$. If $f : X \to Y$ is a function with $f(0) = 0$, then $G(E, f, (E_l)_{l \in \Gamma})$ is either empty or $c$-lineable in $E$. 


Proof. Let \( \alpha = (\alpha_n)_{n=1}^{\infty} \in \mathbb{K}^N \), \( w \in X \) and denote
\[
w \otimes \alpha := (\alpha_n w)_{n=1}^{\infty} \in X^N.
\]
Assume that \( G(E, f, (E_l)_{l \in \Gamma}) \) is non-void and consider \( x = (x_j)_{j=1}^{\infty} \in G(E, f, (E_l)_{l \in \Gamma}) \). We shall first show that
\[
x^0 \in G(E, f, (E_l)_{l \in \Gamma}).
\]
Let \( U := \bigcup_{l \in \Gamma} E_l \). We know that \( (f(x_j))_{j=1}^{\infty} \notin U \), and thus \( [(f(x_j))_{j=1}^{\infty}]^0 \notin U \), since \( E_l \), for each \( l \), is an invariant sequence space. Denote \( x^0 = (x_{jk})_{k=1}^{\infty} \), where \( x_{jk} \) is the \( k \)-th non null coordinate of \( x \). Then, we shall show that \( (f(x_{jk}))_{k=1}^{\infty} \notin U \). Since \( f(0) = 0 \), it follows that
\[
[(f(x_j))_{j=1}^{\infty}]^0 = [(f(x_j))_{j=1}^{\infty}]^0 \notin U.
\]
Hence, \( (f(x_{jk}))_{k=1}^{\infty} \notin U \) and thus \( x^0 \in G(E, f, (E_l)_{l \in \Gamma}) \). As usual, let us split \( \mathbb{N} \) as a countable union of pairwise disjoint subsets \( (\mathbb{N}_i)_{i=1}^{\infty} \) of \( \mathbb{N} \) and we denote \( \mathbb{N}_i = \{ i_1 < i_2 < \ldots \} \). Consider
\[
y_i = \sum_{k=1}^{\infty} x_{jk} \otimes e_{ik} \in X^N.
\]
Note that the set \( \{ y_1, y_2, \ldots \} \) is linearly independent and observe that \( y_i^0 = x^0 \); we thus have \( 0 \neq y_i^0 \in E \) for all \( i \). Since \( E \) is an invariant sequence space, it follows that \( y_i \in E \) for all \( i \in \mathbb{N} \). Moreover, \( y_i \in G(E, f, (E_l)_{l \in \Gamma}) \). In fact, if \( y_i = (y_{im})_{m=1}^{\infty} \), then
\[
[(f(y_m))_{m=1}^{\infty}]^0 = [(f(x_j))_{j=1}^{\infty}]^0 \notin E_l
\]
for each \( i \in \mathbb{N} \) and \( l \in \Gamma \). Let \( K \) be the constant from Definition 1.1(b1) and consider \( \hat{s} = 1 \) if \( E \) is a Banach space and \( \hat{s} = s \) if \( E \) is an \( s \)-Banach space, \( 0 < s < 1 \). For \( (a_i)_{i=1}^{\infty} \in \ell_{\hat{s}} \),
\[
\sum_{i=1}^{\infty} \| a_i y_i \|_E ^{\hat{s}} = \sum_{i=1}^{\infty} |a_i|^\hat{s} \cdot \| y_i \|_E ^\hat{s} \leq K^\hat{s} \cdot \sum_{i=1}^{\infty} |a_i|^\hat{s} \cdot \| y_i^0 \|_E ^\hat{s} \\
= K^\hat{s} \cdot \| x^0 \|_E ^\hat{s} \cdot \sum_{i=1}^{\infty} |a_i|^\hat{s} = K^\hat{s} \cdot \| x^0 \|_E ^\hat{s} \cdot \sum_{i=1}^{\infty} |a_i|^\hat{s} \leq K^\hat{s} \cdot \| x^0 \|_E ^\hat{s} \cdot \| (a_i)_{i=1}^{\infty} \|_\ell_{\hat{s}} ^{\hat{s}} < \infty.
\]
Then \( \sum_{i=1}^{\infty} \| a_i y_i \|_E < \infty \) and in any case we conclude that \( \sum_{i=1}^{\infty} a_i y_i \) converges in \( E \). Thus the operator
\[
T : \ell_{\hat{s}} \rightarrow E , \ (a_i)_{i=1}^{\infty} = \sum_{i=1}^{\infty} a_i y_i ,
\]
is well-defined, linear and injective. Since \( \ell_{\hat{s}} \) is a vector space with dimension \( \epsilon \), the proof is done. \( \square \)

Remark 2.4. Note that (b2) of Definition 1.1 is not used in the above proposition. So, in some sense it is valid for “weak” invariant sequence spaces (of course, the word “weak” here denotes the lack of property (b2)).

The following straightforward corollary shows that, as mentioned before, the \( \epsilon \)-lineability part of Theorem 2.2 does not depend on the notion of non-contractive maps.

Corollary 2.5. Let \( X \) and \( Y \) be Banach spaces, \( \Gamma \) be an arbitrary set, \( E \) be an invariant sequence space over \( X \), \( \Gamma \subseteq (0, \infty) \) and \( f : X \rightarrow Y \) be a function such that \( f(0) = 0 \). The sets \( C(E, f, \Gamma), C^w(E, f, \Gamma) \) and \( C(E, f, 0) \) are either empty or \( \epsilon \)-lineable.
3. Spaceability and strongly invariant sequence spaces

From now on, if an invariant sequence space $E$ is such that $(x_j)_{j=1}^\infty \in E$ if, and only if, all subsequence of $(x_j)_{j=1}^\infty$ also belong to $E$, we say that $E$ is an strongly invariant sequence space.

**Example 3.1.** If $X$ is a Banach space, then $\ell_q(X), \ell_q^w(X), \ell_q^s(X), c(X), c_0(X)$ are strongly invariant sequence spaces.

The notion of strongly invariant sequence space is quite natural, but the following example shows that there exist invariant sequence spaces which are not strongly invariant sequence spaces:

**Example 3.2.** The Banach space $E = \{ (x_j)_{j=1}^\infty \in \ell_\infty : x_{2n-1} = x_{2n} \text{ for all positive integers } n \}$, with the supremum norm, is an invariant sequence space but is not an strongly invariant sequence space.

The following definition shall be used in the statement of our main result and in Section 4.

**Definition 3.3.** Let $X, Y$ be Banach spaces and $E$ be an invariant sequence space over $Y$. A map from $f : X \to Y$ such that $f(0) = 0$ is said to be compatible to $E$ if for any sequence $(x_j)_{j=1}^\infty$ of elements of $X$, we have

$$(f(x_j))_{j=1}^\infty \notin E \Rightarrow (f(ax_j))_{j=1}^\infty \notin E$$

regardless of the scalars $a \neq 0$.

**Example 3.4.** Any non-contractive mapping $f : X \to Y$ is compatible to $\ell_q(Y)$ and $c_0(Y)$.

Now we state and prove one of the main results of this paper. We show that [1] Theorem 2.5 (a)] can be generalized to a more general setting, under weaker and more general hypotheses. The proof is an abstraction of the proof of Theorem 2.2(a):

**Theorem 3.5.** Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set, $E$ be an invariant sequence space over $X$ and $E_l$ be strongly invariant sequence spaces over $Y$ for all $l \in \Gamma$. If $f : X \to Y$ is compatible with $E_l$ for all $l \in \Gamma$, then $G(E, f, (E_l)_{l \in \Gamma})$ is either empty or spaceable.

**Proof.** From the previous theorem we know that $G(E, f, (E_l)_{l \in \Gamma})$ is $\ell_\infty$-lineable. Let us recall

$$T : \ell_\infty \to E, \quad T((a_i)_{i=1}^\infty) = \sum_{i=1}^\infty a_i y_i.$$  

Let us show that $\overline{T(\ell_\infty)}$ is closed (infinite dimensional) in $E$. Keeping the notation of the proof of Proposition 2.3, $y_i = \sum_{k=1}^\infty x_{jk} \otimes e_{ik} \in X^N$ where $x_{jk}$ is $k$-th non-zero coordinate $x = (x_j)_{j=1}^\infty \in G(E, f, (E_l)_{l \in \Gamma})$. We shall show that if $z = (z_n)_{n=1}^\infty \in \overline{T(\ell_\infty)}$ is a non null sequence then $(f(z_n))_{n=1}^\infty \notin \bigcup_{l \in \Gamma} E_l$. There are sequences $(a_i^{(k)})_{i=1}^\infty \in \ell_\infty, k \in \mathbb{N}$, such that $z = \lim_{k \to \infty} T((a_i^{(k)})_{i=1}^\infty)$ in $E$. Note that, for each $k \in \mathbb{N}$,

$$T((a_i^{(k)})_{i=1}^\infty) = \sum_{i=1}^\infty a_i^{(k)} y_i = \sum_{i=1}^\infty a_i^{(k)} \cdot \sum_{p=1}^\infty x_{jp} e_{ip} = \sum_{i=1}^\infty a_i^{(k)} x_{jp} e_{ip}.$$  

Since $z \neq 0$, let $r \in \mathbb{N}$ be such that $z_r \neq 0$. Since $\mathbb{N} = \bigcup_{j=1}^\infty N_j$, there exist unique $m, t \in \mathbb{N}$ such that $e_{mj} = e_r$. Thus, for each $k \in \mathbb{N}$, the $r$-th coordinate of $T((a_i^{(k)})_{i=1}^\infty)$ is the vector $a_m^{(k)} x_{jt}$. Condition 1.1(b2) assures that convergence in $E$ implies coordinatewise convergence, so

$$z_r = \lim_{k \to \infty} a_m^{(k)} x_{jt} = \left( \lim_{k \to \infty} a_m^{(k)} \right) x_{jt}.$$  

It follows that $a_m := \lim_{k \to \infty} a_m^{(k)} \neq 0$. On the one hand we have

$$a_m x_{jp} = \left( \lim_{k \to \infty} a_m^{(k)} \right) x_{jp} = \lim_{k \to \infty} a_m^{(k)} x_{jp}.$$
for every $p \in \mathbb{N}$. On the other hand, for $p, k \in \mathbb{N}$, the $m_p$-th coordinate of $T \left( \left( a^{(k)}_i \right)_{i=1}^\infty \right)$ is $a^{(k)}_m x_{j_p}$. So, coordinatewise convergence gives $\lim_{k \to \infty} a^{(k)}_m x_{j_p} = z_m$. It follows that $z_m = a_m x_{j_p}$ for every $p \in \mathbb{N}$. As $(f(z_m))_{p=1}^\infty = (f(a_m x_{j_p}))_{p=1}^\infty$ and $(f(x_{j_p}))_{p=1}^\infty \notin E_l$, for all $l \in \Gamma$, by Definition 5.3 it follows that $(f(z_m))_{p=1}^\infty \notin E_l$, for all $l \in \Gamma$. Since $(f(z_m))_{p=1}^\infty$ is a subsequence of $(f(z_n))_{n=1}^\infty$ and $E_l$ for each $l \in \Gamma$ is a strongly invariant sequence space it follows that $(f(z_n))_{n=1}^\infty \notin E_l$,

for all $l \in \Gamma$, and it completes the proof that $z \in G(E, f, (E_l)_{l \in \Gamma})$.

From the previous theorem and Examples 3.1 and 3.4 we have the following corollary that recovers (9, Theorem 2.5 (a)):

**Corollary 3.6.** (9) Let $X$ and $Y$ be Banach spaces, $E$ be an invariant sequence space over $X$, $f : X \to Y$ is a function and $\Gamma \subseteq (0, \infty]$. If $f$ is non-contractive, then $C(E, f, \Gamma)$ and $C(E, f, 0)$ are either empty or spaceable.

The next immediate corollary of Theorem 4.8 shows that the [9, Corollaries 2.7, 2.8 and 2.10] and [8, Theorem 1.3] are all particular cases of the following general result:

**Corollary 3.7.** Let $X$ and $Y$ be Banach spaces. Let $E$ be an invariant sequence space over $X$ and $F$ be an strongly invariant sequence space over $Y$. If $f : X \to Y$ is compatible with $F$ and the set

$$A := \{(x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin F\}$$

is non empty, then $A$ is spaceable in $E$.

4. The “weak” case

In this section we prove an extension of [9, Theorem 2.5(b)], i.e., an extension of Theorem 2.2(b) to general invariant sequence spaces.

**Definition 4.1.** Let $E$ be an invariant sequence space over $\mathbb{K}$. For any Banach space $Y$ we define

$$E^w(Y) := \{(x_j)_{j=1}^\infty \in Y^\mathbb{N} : (\varphi(x_j))_{j=1}^\infty \in E \text{ for all } \varphi \in Y'\}.$$ 

**Proposition 4.2.** If $E$ is an invariant sequence space over $\mathbb{K}$, then

$$\sup_{\|\varphi\| \leq 1} \| (\varphi(x_j))_{j=1}^\infty \|_E < \infty$$

for all $(x_j)_{j=1}^\infty \in E^w(Y)$.

**Proof.** Let $(x_j)_{j=1}^\infty \in E^w(Y)$. Define

$$u : Y' \to E$$

$$\varphi \mapsto (\varphi(x_j))_{j=1}^\infty.$$ 

Let us use a general version of the Closed Graph Theorem to topological vector spaces (see [10, page 51]) to prove that $u$ is continuous. Suppose that

$$\varphi_n \to \varphi_0 \text{ and } u(\varphi_n) \to (z_j)_{j=1}^\infty \in E.$$ 

We shall show that $(z_j)_{j=1}^\infty = u(\varphi_0).$ Since $\varphi_n \to \varphi_0$ we have $\varphi_n(x_j) \to \varphi_0(x_j)$ for all $j$. On the other hand, since $u(\varphi_n) \to (z_j)_{j=1}^\infty$ in $E$, i.e., $(\varphi_n(x_j))_{j=1}^\infty \to (z_j)_{j=1}^\infty$ in $E$, we also have $\varphi_n(x_j) \to z_j$ for all $j$. Therefore

$$\varphi_0(x_j) = z_j$$

for all $j$, and hence

$$(z_j)_{j=1}^\infty = u(\varphi_0).$$

□
Remark 4.3. If $E$ is a $p$-normed invariant sequence space, it is simple to verify that
$$\|(x_j)_{j=1}^\infty\|_p := \sup_{\|\varphi\| \leq 1} \|(\varphi(x_j))_{j=1}^\infty\|_E$$
is a $p$-norm in $E^w(Y)$. It is also simple to verify that $E^w(Y)$ is an invariant sequence space over $Y$.

Example 4.4. For the invariant sequence spaces $E = \ell_p, c, c_0$, the respective $E^w(Y)$ are the well-known invariant sequence spaces $\ell^w_p(Y), \ell^w(Y), c^w_0(Y)$.

Definition 4.5. Let $X$ and $Y$ be Banach spaces, and $E$ be an invariant sequence space over $\mathbb{K}$. A map $f : X \to Y$ such that $f(0) = 0$ is strongly compatible with $E^w(Y)$ if $\varphi \circ f$ is compatible with $E$ for all continuous linear functionals $\varphi : Y \to \mathbb{K}$.

Example 4.6. Any strongly non-contractive mapping $f : X \to Y$ is strongly compatible to $\ell_q^w(Y)$ and $c^w_q(Y)$.

Definition 4.7. Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set and $E$ be an invariant sequence space over $X$. If $F_l$, for all $l \in \Gamma$, are invariant sequence spaces over $\mathbb{K}$, and $f : X \to Y$ is any map, we define the set
$$G^w(E, f, (F_l)_{l \in \Gamma}) = \left\{(x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{l \in \Gamma} F^w_l(Y) \right\}.$$

The following theorem is a generalization of Theorem 2.2(b). The proof follows the lines of the previous proofs but we present the details for the sake of completeness:

**Theorem 4.8.** Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set, $E$ be an invariant sequence space over $X$ and $F_l$ be an invariant sequence spaces over $\mathbb{K}$ such that $F^w_l(Y)$ are strongly invariant sequence spaces for all $l \in \Gamma$. If $f : X \to Y$ is strongly compatible with $F^w_l(Y)$ for all $l \in \Gamma$, then $G^w(E, f, (F_l)_{l \in \Gamma})$ is either empty or spaceable.

**Proof.** Assume that $G^w(E, f, (F_l)_{l \in \Gamma})$ is non-void and consider $x = (x_j)_{j=1}^\infty \in G^w(E, f, (F_l)_{l \in \Gamma})$. As in the previous proofs, we shall begin by showing that
$$x^0 \in G^w(E, f, (F_l)_{l \in \Gamma}).$$
Denote $U = \bigcup_{l \in \Gamma} F^w_l(Y)$. We know that for each $l$ there is a $\varphi_l$ such that
$$(\varphi_l \circ f(x_j))_{j=1}^\infty \notin F_l,$$
and thus
$$(\varphi_l \circ f(x_j))_{j=1}^\infty \notin F_l,$$
for all $l$, because $F_l$, for each $l$, is an invariant sequence space. Denote $x^0 = (x_{jk})_{k=1}^\infty$, where $x_{jk}$ is the $k$-th non null coordinate of $x$. Now, we shall show that
$$(\varphi_l \circ f(x_{jk}))_{k=1}^\infty \notin F_l$$
for all $l$. Suppose that
$$(\varphi_{l_0} \circ f(x_{jk}))_{k=1}^\infty \in F_{l_0}$$
for some $l_0$. From (b), since $F_{l_0}$ is an invariant sequence space, we have $[(\varphi_{l_0} \circ f(x_{jk}))_{k=1}^\infty]_0 \in F_{l_0}$. But, since $\varphi_{l_0} \circ f(0) = 0$, it follows from (a) that
$$[(\varphi_{l_0} \circ f(x_{jk}))_{k=1}^\infty]_0 = [(\varphi_{l_0} \circ f(x_{j}))_{j=1}^\infty]_0 \notin F_{l_0}.$$Since $F_{l_0}$ is an invariant sequence space we have
$$(\varphi_{l_0} \circ f(x_{jk}))_{k=1}^\infty \notin F_{l_0}$$
and this contradicts (b).
Therefore we have (b), i.e.,
$$(f(x_{jk}))_{k=1}^\infty \notin F^w_l(Y).$$
for all $l$, and thus

$$x^0 \in G^w(E, f, (F_i)_{i \in \Gamma}).$$

Again, let us separate $\mathbb{N}$ as a countable union of pairwise disjoint subsets $(\mathbb{N}_i)_{i=1}^\infty$ de $\mathbb{N}$ and as usual, for all $i$, we represent $\mathbb{N}_i = \{i_1 < i_2 < \ldots\}$. Consider

$$y_i = \sum_{k=1}^\infty x_{jk} \otimes e_k \in X^N.$$ 

It is plain that $\{y_1, y_2, \ldots\}$ is linearly independent and $y_i^0 = x^0$; we thus have $0 \neq y_i^0 \in E$ for all $i$. Since $E$ is an invariant sequence space, it follows that $y_i \in E$ for all $i \in \mathbb{N}$. Moreover, $y_i \in G^w(E, f, (F_i)_{i \in \Gamma})$. In fact, if $y_i = (y^i_m)_{m=1}^\infty$, then

$$([f(y^i_m)]_{m=1}^\infty)^0 = ([f(x_j)]_{j=1}^\infty)^0 \notin F^w_l(Y)$$

for each $i$. Therefore

$$([f(y^i_m)]_{m=1}^\infty)^0 \notin F^w_l(Y)$$

for each $i \in \mathbb{N}$ and $l \in \Gamma$. Let $\tilde{s} = 1$ if $E$ is a Banach space and $\tilde{s} = s$ if $E$ is an $s$-Banach space, $0 < s < 1$. Proceeding as in the proof of Proposition 2.3 we know that the operator

$$T: \ell_{\tilde{s}} \rightarrow E, \quad T((a^i)_{i=1}^\infty) = \sum_{i=1}^\infty a_i y_i,$$

is well-defined, linear and injective. It remains to show that $T(\ell_{\tilde{s}})$ is closed (infinite dimensional) in $E$. We shall show that if $z = (z_n)_{n=1}^\infty \in T(\ell_{\tilde{s}})$ is a non null sequence then $(f(z_n))_{n=1}^\infty \notin \bigcup_{l \in \Gamma} F^w_l(Y)$. Let $\left(a^i(k)\right)_{i=1}^\infty \in \ell_{\tilde{s}}, k \in \mathbb{N}$, be such that $z = \lim_{k \rightarrow \infty} T\left(a^i(k)_{i=1}^\infty\right)$ in $E$. Note that, for each $k \in \mathbb{N}$,

$$T\left(a^i(k)_{i=1}^\infty\right) = \sum_{i=1}^{\infty} a^i(k) y_i = \sum_{i=1}^{\infty} a^i(k) \cdot \sum_{p=1}^{\infty} x_{jp} e_{ip} = \sum_{i=1}^{\infty} \sum_{p=1}^{\infty} a^i(k) x_{jp} e_{ip}.$$ 

Fix $r \in \mathbb{N}$ such that $z_r \neq 0$. Since $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathbb{N}_j$, there are (unique) $m, t \in \mathbb{N}$ such that $e_m = e_r$. Thus, for each $k \in \mathbb{N}$, the $r$-th coordinate of $T\left(a^i(k)_{i=1}^\infty\right)$ is the vector $a^i_m x_j$, From the Definition (b2) we know that convergence in $E$ implies coordinatewise convergence, and thus

$$z_r = \lim_{k \rightarrow \infty} a^i_m x_j = \left(\lim_{k \rightarrow \infty} a^i_m\right) x_j.$$ 

It follows that $\alpha_m := \lim_{k \rightarrow \infty} a^i_m \neq 0$ and

$$\alpha_m x_{jp} = \left(\lim_{k \rightarrow \infty} a^i_m\right) x_{jp} = \lim_{k \rightarrow \infty} a^i_m x_{jp}$$

for every $p \in \mathbb{N}$. Besides, for $p, k \in \mathbb{N}$, the $m_p$-th coordinate of $T\left(a^i(k)_{i=1}^\infty\right)$ is $a^i_m x_{jp}$. So, coordinatewise convergence gives

$$\lim_{k \rightarrow \infty} a^i_m x_{jp} = z_{m_p}$$

and hence

$$z_{m_p} = \alpha_m x_{jp}.$$ 

for every $p \in \mathbb{N}$. Therefore

$$\phi_l \circ f(z_{m_p})_{p=1}^\infty = (\phi_l \circ f(a_m x_{jp}))_{p=1}^\infty$$

and from (5) we have

$$(\phi_l \circ f(x_{jp}))_{p=1}^\infty \notin F_l,$$

for all $l$. Since $f$ is strongly compatible with $F^w_l(Y)$ for all $l$, we conclude that $\phi_l \circ f$ is compatible with $F_l$ and hence

$$(\phi_l \circ f(a_m x_{jp}))_{p=1}^\infty \notin F_l.$$
It follows from [7] and [9] that
\begin{equation}
(\varphi_l \circ f(z_{m_p}))_{p=1}^{\infty} \notin F_l,
\end{equation}
for all $l$. Therefore
\begin{equation}
(f(z_{m_p}))_{p=1}^{\infty} \notin F_l^w(Y).
\end{equation}
Since $(f(z_{m_p}))_{p=1}^{\infty}$ is a subsequence of $(f(z_n))_{n=1}^{\infty}$ and $F_l^w(Y)$ is a strongly invariant sequence space, it follows from (11) that
\begin{equation}
(f(z_n))_{j=1}^{\infty} \notin F_l^w(Y),
\end{equation}
for all $l \in \Gamma$, and we finally conclude that $z \in G^w(E, f, (F_l)_{l \in \Gamma})$.
\[\square\]

Let $X$ and $Y$ be Banach spaces, $E$ be an invariant sequence space over $X$ and $F_l = \ell_l$, with $l \geq 1$. If $f: X \to Y$ is strongly non-contractive then $f$ is strongly compatible to $\ell_l^w(Y)$ and from the previous theorem we conclude that $G^w(E, f, (F_l)_{l \in \Gamma})$ is either empty or spaceable. But
\[G^w(E, f, (F_l)_{l \in \Gamma}) = \left\{ (x_j)_{j=1}^{\infty} \in E : (f(x_j))_{j=1}^{\infty} \notin \bigcup_{l \in \Gamma} \ell_l^w(Y) \right\}\]
and we recover Theorem 2.2(b) for $\Gamma \subset [1, \infty]$.

Remark 4.9. It is interesting to note that in all the results presented in this note (and in the respective versions from [2],[9]) the lineability/spaceability results satisfy a slightly stronger condition, in the following sense: given any point $x$ of the set $G(E, f, (E_l)_{l \in \Gamma})$ it is proved here that there is a $c$-dimensional (or closed infinite dimensional, depending on the case) vector space $V$ such that $x \in V \subset G(E, f, (E_l)_{l \in \Gamma}) \cup \{0\}$. This leads to the following extension of the notion of lineability that may be interesting to investigate in different contexts: a subset $A$ of a vector space $W$ is uniformly $\lambda$-lineable if for any $x \in A$ there is a $\lambda$-dimensional vector space $V$ such that $x \in V \subset A \cup \{0\} \subset W$. The same definition can be adapted to the notion of spaceability. It is not difficult to verify that in general these concepts are strictly stronger than just lineability/spaceability. For instance, let $W = \ell_2$ and
\[A = (\text{span}\{e_1\}) \cup (\text{span}\{e_2, e_3\}) \cup (\text{span}\{e_4, \ldots, e_6\}) \cup \ldots\]
It is plain that $A$ is $n$-lineable for all positive integer $n$, but $A$ is not uniformly 2-lineable.

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