Schmidt’s game, fractals, and orbits of toral endomorphisms

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Abstract. Given an integer matrix $M \in \text{GL}_n(\mathbb{R})$ and a point $y \in \mathbb{R}^n/\mathbb{Z}^n$, consider the set

$$\tilde{E}(M, y) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : y/ \in \{ M^k x \mod \mathbb{Z}^n : k \in \mathbb{N} \} \}.$$ 

S. G. Dani showed in 1988 that whenever $M$ is semisimple and $y \in \mathbb{Q}^n/\mathbb{Z}^n$, the set $\tilde{E}(M, y)$ has full Hausdorff dimension. In this paper we strengthen this result, extending it to arbitrary $M \in \text{GL}_n(\mathbb{R}) \cap \text{M}_{n \times n}(\mathbb{Z})$ and $y \in \mathbb{R}^n/\mathbb{Z}^n$, and in fact replacing the sequence of powers of $M$ by any lacunary sequence of (not necessarily integer) $m \times n$ matrices. Furthermore, we show that sets of the form $\tilde{E}(M, y)$ and their generalizations always intersect with ‘sufficiently regular’ fractal subsets of $\mathbb{R}^n$. As an application, we give an alternative proof of a recent result [M. Einsiedler and J. Tseng. Badly approximable systems of affine forms, fractals, and Schmidt games. Preprint, arXiv:0912.2445] on badly approximable systems of affine forms.

1. Introduction

Let $\mathbb{T}^n \overset{\text{def}}{=} \mathbb{R}^n/\mathbb{Z}^n$ be the $n$-dimensional torus. Any non-singular $n \times n$ matrix $M$ with integer entries defines a continuous surjective endomorphism $f_M$ of $\mathbb{T}^n$ given by

$$f_M(x + \mathbb{Z}^n) \overset{\text{def}}{=} Mx + \mathbb{Z}^n \quad \forall x \in \mathbb{R}^n,$$

and any continuous surjective endomorphism $f$ of $\mathbb{T}^n$ can be obtained in this way. Criteria for ergodicity of $f$ (with respect to Haar measure on $\mathbb{T}^n$) are well known, and ergodicity implies that $f$-orbits of almost all points are dense in $\mathbb{T}^n$. Also, in many cases, it is known that exceptional sets of points with non-dense orbits are rather big. For example, following the notation used in [14], let us define

$$E(f, y) \overset{\text{def}}{=} \{ x \in \mathbb{T}^n : y/ \notin \{ f^k(x) : k \in \mathbb{N} \} \}$$

for a fixed $y \in \mathbb{T}^n$ and a self-map $f$ of $\mathbb{T}^n$. In 1988, Dani proved the following theorem.

**Theorem 1.1.** [5, Theorem 2.1] For any semisimple $M \in \text{GL}_n(\mathbb{R}) \cap \text{M}_{n \times n}(\mathbb{Z})$ and any $y \in \mathbb{Q}^n/\mathbb{Z}^n$, the set $E(f_M, y)$ is $1/2$-winning.
The above winning property is based on a game, introduced by Schmidt in [26], which is usually referred to as Schmidt’s game. This property implies density and full Hausdorff dimension and is stable with respect to countable intersections; see §2 for more detail.

One of the goals of the present paper is to prove a far-reaching generalization of Theorem 1.1. Namely, we remove the assumptions of $M$ being semisimple and $y$ being rational. Also, we are able to intersect sets $E(f, y)$ with many ‘sufficiently regular’ fractal subsets of $\mathbb{T}^n$. In fact, it will be more convenient to lift the problem to $\mathbb{R}^n$: denote by $\pi$ the quotient map $\mathbb{R}^n \to \mathbb{T}^n$ and, for $M \in M_{n \times n}(\mathbb{R})$ and $y \in \mathbb{T}^n$, consider
\[
\tilde{E}(M, y) \overset{\text{def}}{=} \{ x \in \mathbb{R}^n : y \notin \pi(M^k x) : k \in \mathbb{N} \}.
\]
Clearly,
\[
\tilde{E}(M, y) = \pi^{-1} (E(f, y))
\]
when $M \in \text{GL}_n(\mathbb{R}) \cap M_{n \times n}(\mathbb{Z})$; however, the definition (1.2) makes sense even when $M$ is singular or has non-integer entries.

The ‘sufficient regularity’ of subsets of $\mathbb{R}^n$ will be characterized by their ability to support so-called absolutely decaying measures; see [16] or §3 for a definition. Examples include $\mathbb{R}^n$ itself and limit sets of irreducible families of contracting similarities of $\mathbb{R}^n$ satisfying the open-set condition, such as the Koch snowflake or the Sierpiński carpet. Other interesting examples can be found in [16, 29, 32].

It turns out, as was first observed in [12], that the absolute decay property of a measure can be used for playing Schmidt’s game on its support. Namely, we will say, following [1], that a subset $S$ of $\mathbb{R}^n$ is $\alpha$-winning on a subset $K$ of $\mathbb{R}^n$ if $S \cap K$ is $\alpha$-winning for Schmidt’s game played on the metric space $K$ with the metric induced from $\mathbb{R}^n$. From [26], it immediately follows that the intersection of countably many sets $\alpha$-winning on $K$ is also $\alpha$-winning on $K$. We will say that $S$ is winning on $K$ if it is $\alpha$-winning on $K$ for some $\alpha > 0$. Precise definitions are given in §2. As a trivial consequence of Corollary 3.3, if $S$ is winning on $K = \text{supp } \mu$, where $\mu$ is absolutely decaying, then $S \cap K$ is not contained in a countable union of affine hyperplanes. Furthermore, under some additional assumptions on $\mu$, for example when $K = \mathbb{R}^n$ or one of the self-similar sets mentioned above, one can show that the Hausdorff dimension of $S \cap K$ is equal to dim($K$) whenever $S$ is winning on $K$. See §3 for precise statements.

In this paper we prove a generalization of Theorem 1.1.

**Theorem 1.2.** For every $K \subset \mathbb{R}^n$ which supports an absolutely decaying measure there exists $\alpha = \alpha(K) > 0$ such that for any $M \in \text{GL}_n(\mathbb{R}) \cap M_{n \times n}(\mathbb{Z})$ and any $y \in \mathbb{T}^n$, the set $\tilde{E}(M, y)$ is $\alpha$-winning on $K$.

In particular, for any countable subset $Y$ of $\mathbb{T}^n$, the set
\[
\bigcap_{y \in Y} \bigcap_{M \in \text{GL}_n(\mathbb{R}) \cap M_{n \times n}(\mathbb{Z})} \tilde{E}(M, y)
\]
is also $\alpha$-winning on $K$. It immediately follows that the sets $E(f_M, y)$ discussed in Theorem 1.1 and their countable intersections always intersect those subsets of the torus whose pullbacks to $\mathbb{R}^n$ support absolutely decaying measures. It can also be shown that $\alpha(\mathbb{R}^n) = 1/2$, recovering Dani’s result; see §5.1.
The one-dimensional case of Theorem 1.2 appeared recently in [1], and also, independently and for $K = \mathbb{R}$, in [10]; see also [30]. In other words, the sets

$$\tilde{E}(b, y) \overset{\text{def}}{=} \{ x \in \mathbb{R} : y \notin \{ \pi(b^k x) : k \in \mathbb{N} \} \}$$

(1.3)

were shown to be winning on supp $\mu$ for any absolutely decaying measure $\mu$ on $\mathbb{R}$, any integer $b > 1$, and any $y \in \mathbb{T}$. However, the main result of [1] applies to much more general situations, recovering earlier work [6, 23, 24] by Pollington and de Mathan. In particular, $b$ in (1.3) does not have to be an integer, and one can replace the sequence of powers of $b$ by an arbitrary lacunary sequence $t_k$ of real numbers (we recall that $(t_k)$ is called lacunary if $\inf_{k \in \mathbb{N}} t_{k+1}/t_k > 1$).

We now describe an analogous generalization of Theorem 1.2, which is the main result of the present paper. We are going to fix $m, n \in \mathbb{N}$, consider a sequence $M = (M_k)$ of $m \times n$ matrices and a sequence $Z = (Z_k)$ of subsets of $\mathbb{R}^m$, and define

$$\tilde{E}(M, Z) \overset{\text{def}}{=} \left\{ x \in \mathbb{R}^n : \inf_{k \in \mathbb{N}} d(M_k x, Z_k) > 0 \right\}.$$  

(1.4)

(Here $d(\cdot, \cdot)$ stands for the Euclidean distance on $\mathbb{R}^n$.) The sets $\tilde{E}(M, y)$ defined in (1.2) constitute a special case, with $m = n$, $M = (M_k)$, and $Z_k = \pi^{-1}(y)$.

Some assumptions on $M$ and $Z$ are in order. We will say that a sequence $M$ of non-zero $m \times n$ matrices is lacunary if so is the sequence $(\|M_k\|_{op})$ of the values of their operator norms. A subset $Z$ of $\mathbb{R}^n$ will be called $\delta$-uniformly discrete if $\inf_{x, y \in Z, x \neq y} d(x, y) > \delta$. With some abuse of terminology, we say that a sequence $Z = (Z_k)$ is $\delta$-uniformly discrete if $Z_k$ is $\delta$-uniformly discrete for every $k \in \mathbb{N}$, and that $Z$ is uniformly discrete if it is $\delta$-uniformly discrete for some $\delta > 0$. For example, for an arbitrary sequence $(y_k)$ of points of $\mathbb{T}^m$, the sequence of sets $Z_k = \pi^{-1}(y_k) \subset \mathbb{R}^m$ is 1-uniformly discrete.

We can now formulate our main result, which is proved in §4.

**THEOREM 1.3.** For every $K \subset \mathbb{R}^n$ which supports an absolutely decaying measure there exists a positive $\alpha = \alpha(K)$ such that if $Z$ is a uniformly discrete sequence of subsets of $\mathbb{R}^m$ and $M$ is a lacunary sequence of $m \times n$ matrices with real entries, then $\tilde{E}(M, Z)$ is $\alpha$-winning on $K$.

An important special case is $m = n$ and $M = (M^k)$, where $M$ is an $n \times n$ matrix with spectral radius strictly greater than 1 (not necessarily invertible and not necessarily with integer entries); this is used to derive Theorem 1.2 from Theorem 1.3: see §4. Our main theorem also generalizes results from [2, 22] dealing with a special case where

$$M$$ is a lacunary sequence of $1 \times n$ integer matrices

and $Z = (Z_k)$, where $Z_k = \pi^{-1}(0) \forall k \in \mathbb{N}$.  

(1.5)

It was observed both in [2] and in [22] that the latter set-up can be used to prove the abundance of badly approximable systems of affine forms. Recall that a pair $(A, x)$, interpreted as a function $q \mapsto Aq - x$, $\mathbb{R}^m \to \mathbb{R}^n$ (here $A \in \mathbb{M}_{n \times m}(\mathbb{R})$ and $x \in \mathbb{R}^n$) is said to be badly approximable if

$$\inf_{q \in \mathbb{Z}^m \setminus \{0\}} \|q\|^{m/n} d(Aq - x, \mathbb{Z}^n) > 0.$$  

This is an inhomogeneous analogue of the notion of badly approximable systems of linear forms; see [27, 28]. It was proved in [15] that the set $\text{Bad}(n, m)$ of badly approximable
pairs \((A, x)\) has full Hausdorff dimension. Then a much easier proof was found in \([2]\), where, for fixed \(A \in M_{n \times m}(\mathbb{R})\), the sets
\[ \text{Bad}_A(n, m) = \{x \in \mathbb{R}^n : (A, x) \in \text{Bad}(n, m)\} \]
were considered, and it was shown that \(\dim(\text{Bad}_A(n, m)) = n\) for any \(A\). The latter result was strengthened by Tseng in the case \(m = n = 1\): he proved \([31]\) that \(\text{Bad}_A(1, 1) \subset \mathbb{R}\) is \(\frac{1}{8}\)-winning for any \(a \in \mathbb{R}\). Shortly thereafter, Moshchevitin concluded \([22]\) that the sets \(\text{Bad}_A(n, m)\) are \(\frac{1}{4}\)-winning for any \(m, n\) and any \(A \in M_{n \times m}(\mathbb{R})\). Our main theorem can be used to deduce the following corollary.

**Corollary 1.4.** Let \(K \subset \mathbb{R}^n\) be the support of an absolutely decaying measure, and let \(\alpha\) be as in Theorem 1.3. Then for any \(A \in M_{n \times m}(\mathbb{R})\), \(\text{Bad}_A(n, m)\) is \(\alpha\)-winning on \(K\).

Independently, in a recent preprint \([7]\), Einsiedler and Tseng provided another proof of this result, with a smaller value of \(\alpha\). We derive Corollary 1.4 in \(\S4\). At the end of the paper a remark is made explaining how all our results can be strengthened to replace ‘winning’ with ‘strong winning’, a property introduced recently in \([10, 11, 21]\).

2. **Schmidt’s game**

In this section we describe the game, first introduced by Schmidt in \([26]\). Let \((X, d)\) be a complete metric space. Consider \(\Omega \overset{\text{def}}{=} X \times \mathbb{R}_+\), and define a partial ordering
\[(x_2, \rho_2) \leq_s (x_1, \rho_1) \quad \text{if} \quad \rho_2 + d(x_1, x_2) \leq \rho_1.\]
We associate to each pair \((x, \rho)\) a ball in \((X, d)\) via
\[B(x, \rho) = \{x' \in X : d(x, x') \leq \rho\}.\]
Note that \((x_2, \rho_2) \leq_s (x_1, \rho_1)\) implies (but is not necessarily implied by) \(B(x_2, \rho_2) \subset B(x_1, \rho_1)\). However, the two conditions are equivalent when \(X\) is a Euclidean space.

Schmidt’s game is played by two players, whom we will call Alice and Bob, following a convention used previously in \([1, 18]\). The two players are equipped with parameters \(\alpha\) and \(\beta\), respectively, satisfying \(0 < \alpha, \beta < 1\). Choose a subset \(S\) of \(X\) (a target set). The game starts with Bob picking \(x_1 \in X\) and \(\rho > 0\), hence specifying a pair \(\omega_1 = (x_1, \rho)\). Alice and Bob then take turns choosing
\[\omega'_k = (x'_k, \rho'_k) \leq_s \omega_k \quad \text{and} \quad \omega_{k+1} = (x_{k+1}, \rho_{k+1}) \leq_s \omega'_k\]
respectively satisfying
\[\rho'_k = \alpha \rho_k \quad \text{and} \quad \rho_{k+1} = \beta \rho'_k. \tag{2.1}\]
As the game is played on a complete metric space and the diameters of the nested balls
\[B(\omega_1) \supset B(\omega'_1) \supset \cdots \supset B(\omega_k) \supset B(\omega'_k) \supset \cdots\]
tend to zero as \(k \to \infty\), the intersection of these balls is a point \(x_\infty \in X\). Call Alice the winner if \(x_\infty \in S\). Otherwise, Bob is declared the winner. A strategy consists of specifications for a player’s choices of centers for his or her balls given the opponent’s previous moves.

If for certain \(\alpha, \beta\), and a target set \(S\) Alice has a winning strategy, i.e. a strategy for winning the game regardless of how well Bob plays, we say that \(S\) is an \((\alpha, \beta)\)-winning set. If \(S\) and \(\alpha\) are such that \(S\) is an \((\alpha, \beta)\)-winning set for all possible \(\beta\), we say that \(S\) is an \(\alpha\)-winning set. Call a set winning if such an \(\alpha\) exists.
Intuitively, one expects winning sets to be large. Indeed, every such set is clearly dense in $X$; moreover, under some additional assumptions on the metric space, winning sets can be proved to have positive, and even full, Hausdorff dimension. For example, the fact that a winning subset of $\mathbb{R}^n$ has Hausdorff dimension $n$ is due to Schmidt [26, Corollary 2]. Another useful result of Schmidt [26, Theorem 2] states that the intersection of countably many $\alpha$-winning sets is $\alpha$-winning.

Schmidt himself used the machinery of the game he invented to prove that certain subsets of $\mathbb{R}^n$ or $\mathbb{R}$ are winning, and hence have full Hausdorff dimension. Now let $K$ be a closed subset of $X$. Following an approach initially introduced in [12], we will say that a subset $S$ of $X$ is $(\alpha, \beta)$-winning on $K$ (respectively, $\alpha$-winning on $K$, winning on $K$) if $S \cap K$ is $(\alpha, \beta)$-winning (respectively, $\alpha$-winning, winning) for Schmidt’s game played on the metric space $K$ with the metric induced from $(X, d)$. In the present paper we let $X = \mathbb{R}^n$ and take $K$ to be the support of an absolutely decaying measure. In other words, since the metric is induced, playing the game on $K$ amounts to choosing balls in $\mathbb{R}^n$ according to the rules of a game played on $\mathbb{R}^n$, but with an additional constraint that the centers of all the balls lie in $K$. Since the first appearance of this approach in [12], where it was used to show that sufficiently regular fractals meet with a countable intersection of non-singular affine images of the set of badly approximable vectors in $\mathbb{R}^n$, it has been utilized in [8, 13], and most recently in [1], of which the present paper is a following and a generalization.

3. Absolutely decaying measures

In this section we describe in detail the class of absolutely decaying measures and discuss other related properties and their applications. Following a terminology introduced in [16, 25], we say that a locally finite Borel measure $\mu$ on $\mathbb{R}^n$ is $(C, \gamma)$-absolutely decaying if there exists $\rho_0 > 0$ such that

$$\mu(B(\mathbf{x}, \rho) \cap \mathcal{L}(\varepsilon)) < C(\varepsilon/\rho)^\gamma \mu(B(\mathbf{x}, \rho))$$

for any affine hyperplane $\mathcal{L} \subset \mathbb{R}^n$

and any $\mathbf{x} \in \text{supp } \mu$, $0 < \rho < \rho_0$, $\varepsilon > 0$. (3.1)

Here $B(\mathbf{x}, \rho)$ stands for the closed Euclidean ball in $\mathbb{R}^n$ of radius $\rho$ centered at $\mathbf{x}$, and

$$\mathcal{L}(\varepsilon) \overset{\text{def}}{=} \{ \mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathcal{L}) \leq \varepsilon \}$$

is the closed $\varepsilon$-neighborhood of $\mathcal{L}$. We say that $\mu$ is absolutely decaying if it is $(C, \gamma)$-absolutely decaying for some $C, \gamma > 0$. (This terminology differs slightly from the one introduced in [16], where a less uniform version was considered.) If $\mu$ is $(C, \gamma)$-absolutely decaying, we will denote by $\rho_{C, \gamma}(\mu)$ the supremum of $\rho_0$ for which (3.1) holds.

Another property, which often comes in a package with absolute decay, is the so-called doubling, or Federer, condition. One says that $\mu$ is $D$-Federer if there exists $\rho_0 > 0$ such that

$$\mu(B(\mathbf{x}, 2\rho)) < D\mu(B(\mathbf{x}, \rho)) \quad \forall \mathbf{x} \in \text{supp } \mu, \forall 0 < \rho < \rho_0.$$  

and Federer if it is $D$-Federer for some $D > 0$. Measures which are both absolutely decaying and Federer are called absolutely friendly, a term coined in [25].
Many examples of absolutely friendly measures can be found in \([16, 17, 29, 32]\). The Federer condition is very well studied; it obviously holds when \(\mu\) satisfies a power law, i.e. there exist positive \(\delta, c_1, c_2, \rho_0\) such that

\[
c_1 \rho^\delta \leq \mu(B(x, \rho)) \leq c_2 \rho^\delta \quad \forall x \in \text{supp } \mu, \forall 0 < \rho < \rho_0. \tag{3.3}
\]

Such measures are often referred to as \(\delta\)-Ahlfors regular. However, it is not hard to construct absolutely friendly measures not satisfying a power law; see \([17]\) for an example. Also, when \(n = 1\) the Federer property is implied by the absolute decay, which in its turn is implied by a power law (see \([1]\) for a thorough discussion of equivalent definitions of absolute friendliness in the one-dimensional case). However, these implications fail to hold in higher dimensions. In particular, the volume measures on smooth \(k\)-dimensional submanifolds of \(\mathbb{R}^n\) obviously are \(k\)-Ahlfors regular but not absolutely decaying unless \(k = n\).

The goal of the current work, as well as in several earlier papers \([8, 12, 13, 17, 19]\), is to use measures in order to construct points in their supports with prescribed (dynamical or Diophantine) properties. Our attention will therefore be focused on closed subsets \(K\) of \(\mathbb{R}^n\) which support absolutely decaying and absolutely friendly measures. For example, this is the case when \(K = \mathbb{R}^n\), or when \(K\) is the limit set of an irreducible family of contracting self-similar \([16]\) or self-conformal \([32]\) transformations of \(\mathbb{R}^n\) satisfying the open-set condition. More examples can be found in \([17, 29]\). Note that the paper \([2]\) established full Hausdorff dimension of \(\tilde{E}(\mathcal{M}, \mathcal{Z}) \cap K\) for \(\mathcal{M}, \mathcal{Z}\) as in (1.5) and under an assumption that \(K \subset \mathbb{R}^n\) supports an absolutely decaying, \(\delta\)-Ahlfors regular measure with \(\delta > n - 1\). It is not hard to show, using an elementary covering argument, that (3.3) with \(\delta > n - 1\) implies (3.1) with \(\gamma = \delta - n + 1\). Hence, the sets considered in \([2]\) support absolutely decaying measures.

Recall that the lower point-wise dimension of a measure \(\mu\) at \(x \in \text{supp } \mu\) is defined as

\[
d_\mu(x) \overset{\text{def}}{=} \liminf_{\rho \to 0} \frac{\log \mu(B(x, \rho))}{\log \rho}.
\]

For an open \(U\) with \(\mu(U) > 0\), let

\[
d_\mu(U) \overset{\text{def}}{=} \inf_{x \in \text{supp } \mu \cap U} d_\mu(x). \tag{3.4}
\]

It is well known, see e.g. \([9, \text{Proposition 4.9}]\), that (3.4) constitutes a lower bound for the Hausdorff dimension of \(\text{supp } \mu \cap U\) (where this bound is sharp when \(\mu\) satisfies a power law). It is also easy to see that \(d_\mu(x) \geq \gamma\) for every \(x \in \text{supp } \mu\) whenever \(\mu\) is \((C, \gamma)\)-absolutely decaying: indeed, take \(\rho < \rho_0 < \rho_{C, \gamma}(\mu)\) and \(x \in \text{supp } \mu\); then, using (3.1) and noting that \(B(x, \rho) \subset \mathcal{L}(\rho)\) for some hyperplane \(\mathcal{L}\), one has

\[
\mu(B(x, \rho)) < C \left( \frac{\rho}{\rho_0} \right)^\gamma \mu(B(x, \rho_0)).
\]

Thus, for \(\rho < 1\),

\[
\frac{\log \mu(B(x, \rho))}{\log \rho} \geq \gamma + \frac{\log C - \gamma \log \rho_0 + \log \mu(B(x, \rho_0))}{\log \rho},
\]

and the claim follows.
The following proposition [18, Proposition 5.1] makes it possible to estimate the Hausdorff dimension of sets winning on supports of Federer measures.

**Proposition 3.1.** Let $K$ be the support of a Federer measure $\mu$ on $\mathbb{R}^n$, and let $S$ be winning on $K$. Then for any open $U \subset \mathbb{R}^n$ with $\mu(U) > 0$, one has

$$\dim (S \cap K \cap U) \geq d_\mu (U).$$

In particular, if in addition $\mu$ is $(C, \gamma)$-absolutely decaying, in the above proposition one can replace $d_\mu (U)$ with $\gamma$, and with $\dim (K)$ if $\mu$ satisfies a power law. Note that this generalizes estimates for the Hausdorff dimension of winning sets due to Schmidt [26] for $\mu$ being Lebesgue on $\mathbb{R}^n$, and to Fishman [12, §5] for measures satisfying a power law.

The next lemma exhibits a crucial feature of sets supporting absolutely decaying measures, namely the fact that while playing Schmidt’s game on such a set, Alice can distance herself from hyperplanes ‘efficiently’. This observation is the cornerstone of the proof of our main theorem. The argument has been adapted from the one in [22], where the case $K = \mathbb{R}^n$ was proved with $\alpha = \frac{1}{2}$ (see §5.1 for more detail), and then refined using an observation from [7].

**Lemma 3.2.** For every $C, \gamma > 0$, and

$$\alpha < \frac{1}{2C^{1/\gamma} + 1}, \quad (3.5)$$

there exists $\varepsilon = \varepsilon(C, \gamma, \alpha) \in (0, 1)$ such that if $K$ is the support of a $(C, \gamma)$-absolutely decaying measure $\mu$ on $\mathbb{R}^n$, $0 < \rho < \rho_{C, \gamma}(\mu)$, $x_1 \in K$, $N \in \mathbb{N}$, and $L_1, \ldots, L_N$ are hyperplanes in $\mathbb{R}^n$, there exists $x_2 \in K$ with

$$B(x_2, \alpha \rho) \subset B(x_1, \rho) \quad (3.6)$$

and

$$d(B(x_2, \alpha \rho), L_i) > \alpha \rho \quad \text{for at least } \lceil \varepsilon N \rceil \text{ of the hyperplanes } L_i. \quad (3.7)$$

**Proof.** Let

$$A_i = B(x_1, (1 - \alpha) \rho) \setminus L_i^{(2\alpha \rho)}.$$

By (3.1) and (3.5), for each $1 \leq i \leq N$,

$$\frac{\mu(A_i)}{\mu(B(x_1, (1 - \alpha) \rho))} > 1 - C \left( \frac{2\alpha}{1 - \alpha} \right)^\gamma \equiv \varepsilon > 0.$$

We claim that there exist $j_1, \ldots, j_k$, where $k \geq \lceil \varepsilon N \rceil$, such that

$$K \cap \bigcap_{i=1}^k A_{j_i} \neq \emptyset.$$

To see this, let

$$f(x) = \sum_{i=1}^N \chi_{A_i}(x).$$

Then

$$\int_{B(x_1, (1 - \alpha) \rho)} f(x) \, d\mu(x) \geq N \varepsilon \mu(B(x_1, (1 - \alpha) \rho)),$$

so clearly there exists some $x_2 \in K$ with $f(x_2) \geq N \varepsilon$. Since $f(x_2) \in \mathbb{Z}$, there must exist $j_1, \ldots, j_k$ as above. Hence, $x_2$ satisfies (3.6) and (3.7). \qed
Corollary 3.3. Let \( K \) be the support of a \((C, \gamma)\)-absolutely decaying measure on \( \mathbb{R}^n \), let \( \alpha \) be as in \((3.5)\), let \( S \subset \mathbb{R}^n \) be \( \alpha \)-winning on \( K \), and let \( S' \subset S \) be a countable union of hyperplanes. Then \( S \setminus S' \) is also \( \alpha \)-winning on \( K \).

Proof. In view of the countable intersection property, it suffices to show that for any hyperplane \( \mathcal{L} \subset \mathbb{R}^n \), the set \( \mathbb{R}^n \setminus \mathcal{L} \) is \((\alpha, \beta)\)-winning on \( K \) for any \( \beta \). Let \( \mu \) be a \((C, \gamma)\)-absolutely decaying measure with \( K = \text{supp} \mu \). We let Alice play arbitrarily until the radius of a ball chosen by Bob is less than \( \rho_{C, \gamma}(\mu) \). Then apply Lemma 3.2 with \( N = 1 \) and \( \mathcal{L}_1 = \mathcal{L} \), which yields a ball disjoint from \( \mathcal{L} \). Afterwards, she can keep playing arbitrarily, winning the game. \( \square \)

4. Proofs

Let us now state a more precise version of Theorem 1.3.

Theorem 4.1. Let \( K \) be the support of a \((C, \gamma)\)-absolutely decaying measure on \( \mathbb{R}^n \), and let \( \alpha \) be as in \((3.5)\). Then for any uniformly discrete sequence \( Z \) of subsets of \( \mathbb{R}^m \) and any lacunary sequence \( \mathcal{M} \) of \( m \times n \) real matrices, the set \( \tilde{E}(\mathcal{M}, Z) \) is \( \alpha \)-winning on \( K \).

Proof. Write \( \mathcal{M} = (M_k) \), let \( t_k \overset{\text{def}}{=} \|M_k\|_{\text{op}} \), and let \( v_k \) be a unit vector satisfying \( \|M_k v_k\| = t_k \).

Take \( \delta > 0 \) such that \( Z \) is \( \delta \)-uniformly discrete, and let

\[
\inf_k \frac{t_{k+1}}{t_k} = Q > 1. \tag{4.1}
\]

Now pick an arbitrary \( 0 < \beta < 1 \), take \( \varepsilon \) as in Lemma 3.2, and choose \( N \) large enough that

\[
(\alpha \beta)^{-r} \leq Q^N, \quad \text{where} \quad r = \lfloor \log_{1/(1-\varepsilon)} N \rfloor + 1. \tag{4.2}
\]

We will denote by \( M_k^{-1}(Z) \) the preimage of a set \( Z \subset \mathbb{R}^n \) under \( M_k \). Notice that for each \( k \in \mathbb{N} \), \( M_k^{-1}(Z_k) \) is contained in a countable union of hyperplanes, so, applying Corollary 3.3 a finite number of times, we may assume that \( t_1 \geq 1 \).

By playing arbitrary moves if needed, we may assume without loss of generality that \( B(\omega_1) \) has radius

\[
\rho < \min \left( \frac{\alpha \beta \delta}{4}, \rho_{C, \gamma} \right). \tag{4.3}
\]

Now let

\[
c = \min \left( \rho(\alpha \beta)^{2r-1}, \frac{\delta}{4} \right). \tag{4.4}
\]

We will describe a strategy for Alice to play the \((\alpha, \beta)\)-game on \( K \) and to ensure that for all \( j \in \mathbb{N} \), for all \( x \in B(\omega_{r(j+1)}) \), and for all \( k \) with \( 1 \leq t_k < (\alpha \beta)^{-r_j} \), one has \( d(M_k x, Z_k) > c \). This will imply that

\[
\bigcap_k B(\omega_k') \in \tilde{E}(\mathcal{M}, Z) \cap K,
\]

finishing the proof.
To satisfy the above goal, Alice can choose $\omega_i'$ arbitrarily for $i < r$. Now fix $j \in \mathbb{N}$. By (4.1) and (4.2), there are at most $N$ indices $k \in \mathbb{N}$ for which

$$(\alpha\beta)^{-r(j-1)} \leq t_k < (\alpha\beta)^{-rj}. \quad (4.5)$$

Let $k$ be one of these indices. For any $x \in \mathbb{R}^n$,

$$\|x\| \geq \frac{1}{t_k} \|M_k(x)\|.$$ 

Thus, if $y_1, y_2$ are two different points in $Z_k$, then by (4.3) and (4.5)

$$d(M_k^{-1}(B(y_1, c)), M_k^{-1}(B(y_2, c))) \geq \frac{\delta - 2c}{t_k} \geq \frac{\delta}{2} > (\alpha\beta)^{rj} \geq 2\rho(\alpha\beta)^{r(j-1)}; \quad (4.6)$$

therefore, $B(\omega_{rj})$ intersects with at most one set of the form $M_k^{-1}(B(y, c))$, where $y \in Z_k$. Hence, for each $k$ satisfying (4.5),

$$B(\omega_{rj}) \cap M_k^{-1}(Z_k(c)) \subset M_k^{-1}(B(y, c)) \quad \text{for some } y \in Z_k. \quad (4.7)$$

We will now show that the preimage of such a ball is contained in a ‘small enough’ neighborhood of some hyperplane, so that we can apply the decay condition. Toward this end, let $V \subset \mathbb{R}^m$ be the hyperplane perpendicular to $M_kv_k$ and passing through $0$. Then

$$W \overset{\text{def}}{=} M_k^{-1}(V)$$

is a hyperplane in $\mathbb{R}^n$ passing through $0$.

If $x \not\in W^{(c/t_k)}$, then $x = w + \eta v_k$ for some $\eta > c/t_k$ and $w \in W$; thus,

$$\|M_kx\| = \|M_kw + M_k\eta v_k\| \geq \eta \|M_kv_k\| = t_k\eta > c.$$ 

Hence,

$$M_k^{-1}(B(0, c)) \subset W^{(c/t_k)},$$

which clearly implies that for each $y \in Z_k$,

$$M_k^{-1}(B(y, c)) \subset L^{(c/t_k)}$$

for some hyperplane $L \subset \mathbb{R}^n$. By (4.4) and (4.5),

$$\frac{c}{t_k} \leq (\alpha\beta)^{r(j+1)-1}\rho \overset{\text{def}}{=} \xi.$$ 

Therefore, by (4.7),

$$\bigcup_{t_k \text{ satisfies (4.5)}} B(\omega_{rj}) \cap M_k^{-1}(Z_k(c)) \subset \bigcup_{i=1}^N L_i^{(\xi)}, \quad (4.8)$$

where $L_i$ are hyperplanes. Noticing that by (4.2), $(1 - \varepsilon)^r N < 1$, Alice can utilize Lemma 3.2 $r$ times to distance herself by $\xi$ from each of the hyperplanes $L_i$ after $r$ turns. Thus, for $k$ satisfying (4.5), we have

$$B(\omega_i') \cap M_k^{-1}(Z_k(c)) = \emptyset.$$ 

We conclude that $d(M_kx, Z_k) \geq c$ for any $x \in B(\omega_i')$, which implies the desired statement. \qed

Proof of Theorem 1.2. Recall that we are given $M \in \text{GL}_m(\mathbb{R}) \cap \text{M}_n(\mathbb{Z})$. If all the eigenvalues of $M$ have modulus less than or equal to 1, then obviously every eigenvalue of $M$ must have modulus 1. By a theorem of Kronecker [20], they must be roots of unity,
so there exists an \( N \in \mathbb{N} \) such that the only eigenvalue of \( M^N \) is 1. Let \( J = L^{-1} M^N L \) be the Jordan normal form of \( M^N \), and let \( v_i = L e_i, i = 1, \ldots, n \), be the Jordan basis for \( M^N \). Then, since \( M^N \) is an integer matrix, we have \( v_i \in \mathbb{Q}^n \) for each \( 1 \leq i \leq n \). Hence, letting \( V = \text{span}(v_1, \ldots, v_{n-1}) \), \( V + \mathbb{Z}^n \) is a union of positively separated parallel hyperplanes. Since \( J \) fixes the last coordinate of any vector, if \( a_1, \ldots, a_n \in \mathbb{R} \), then

\[
M^N \left( \sum_{i=1}^n a_i v_i \right) \in a_n v_n + V.
\]

Therefore, for \( x, y \in \mathbb{R}^n \) with \( x - y \not\in V + \mathbb{Z}^n \) and any \( k \in \mathbb{N} \), one has

\[
d(M^N k x, y + \mathbb{Z}^n) \geq c_0 d(x - y, V + \mathbb{Z}^n) > 0,
\]

where \( c_0 \) is a positive constant depending only on \( v_1, \ldots, v_n \). Hence, for any \( y \in \mathbb{T}^n \),

\[
\tilde{E}(M^N, y) \supset \mathbb{R}^n \setminus (\pi^{-1}(y) + V) = \mathbb{R}^n \setminus (y + V + \mathbb{Z}^n),
\]

where \( y \) is an arbitrary vector in \( \pi^{-1}(y) \). Thus, \( \tilde{E}(M^N, y) \) is \( \alpha \)-winning on \( K \) by Corollary 3.3. Hence, \( \tilde{E}(M^N, z) \) is \( \alpha \)-winning on \( K \) whenever \( z \in f^{-i}_M(y) \), where \( 0 \leq i < N \). Thus, the intersection

\[
\tilde{E}(M, y) = \bigcap_{i=0}^{N-1} f^{-i}_M(y)
\]

is also \( \alpha \)-winning on \( K \).

In the case where at least one of the eigenvalues is of absolute value strictly greater than 1, we will show that the sequence (\( \| M^k \|_{\text{op}} \)) is a finite union of lacunary sequences, which will clearly imply that \( \tilde{E}((M^k), Z) \) is \( \alpha \)-winning on \( K \). Let \( J = L^{-1} M L \) be the Jordan normal form of \( M \). Since the operator norm of \( M \) as a real transformation is equal to its operator norm as a complex transformation and

\[
\| J^k \|_{\text{op}} \leq \| L \|_{\text{op}} \| L^{-1} \|_{\text{op}} \| M^k \|_{\text{op}} \quad \text{and} \quad \| M^k \|_{\text{op}} \leq \| L \|_{\text{op}} \| L^{-1} \|_{\text{op}} \| J^k \|_{\text{op}},
\]

letting \( c = \| L \|_{\text{op}} \| L^{-1} \|_{\text{op}} \), we have

\[
1 \leq \frac{1}{c} \| M^k \|_{\text{op}} \leq \| J^k \|_{\text{op}} \leq c \| M^k \|_{\text{op}} \quad \text{for all} \ k \in \mathbb{N}.
\]

Hence, if (\( \| J^k \|_{\text{op}} \)) is eventually lacunary, then there exist \( \ell, N \in \mathbb{N} \), and \( Q > 1 \) such that, for all \( k \geq N \),

\[
\| M^{k+\ell} \|_{\text{op}} \geq \frac{1}{c^2} \| J^{k+\ell} \|_{\text{op}} \geq Q.
\]

Thus, it will suffice to show that (\( \| J^k \|_{\text{op}} \)) is eventually lacunary.

Let \( B \) be an \( m \times m \) block of \( J \) associated to an eigenvalue \( \lambda \) and write \( B^k = (b_{ij}(k)) \). Direct computation shows that, for \( 0 \leq j - i \leq k \),

\[
b_{ij}(k) = \binom{k}{j-i} \lambda^{k-(j-i)},
\]

and \( b_{ij}(k) = 0 \) otherwise. Since \( |b_{ij}(k)| = o(|b_{1m}(k)|) \) as functions of \( k \) for all \( (i, j) \neq (1, m) \),

\[
\lim_{k \to \infty} \frac{\| B^k \|_{\text{op}}}{|b_{1m}(k)|} = 1.
\]
Hence,

$$\lim_{k \to \infty} \frac{\|B^{k+1}\|_{\text{op}}}{\|B^k\|_{\text{op}}} = |\lambda|,$$

(4.11)

so clearly if $|\lambda| > 1$ then $(\|B^k\|_{\text{op}})$ is eventually lacunary. Write $J = B_1 \oplus \cdots \oplus B_s$, where $s \in \mathbb{N}$ and $B_i$ are the Jordan blocks, with associated eigenvalues $\lambda_i$. Let $\lambda_{\text{max}} = \max |\lambda_i|$, and let $B_{\text{max}}$ be a block with associated eigenvalue having absolute value $\lambda_{\text{max}}$ and of maximal dimension among such blocks. By (4.9) and (4.10), for any $i$,

$$\lim_{k \to \infty} \frac{\|B_{\text{max}}^{k+1}\|_{\text{op}}}{\|B_i^k\|_{\text{op}}} \geq 1.$$

Hence, by (4.11),

$$\lim_{k \to \infty} \frac{\|J^{k+1}\|_{\text{op}}}{\|J^k\|_{\text{op}}} = \lim_{k \to \infty} \frac{\|B_{\text{max}}^{k+1}\|_{\text{op}}}{\|B_i^k\|_{\text{op}}} = \lambda_{\text{max}}.$$

Since by assumption $M$ (and therefore $J$) has an eigenvalue with absolute value greater than 1, $(\|J^k\|_{\text{op}})$ is eventually lacunary. \qed

In the remaining part of this section we apply Theorem 1.3 to badly approximable systems of affine forms.

**Proof of Corollary 1.4.** Recall that we need to fix $A \in \mathbb{M}_{m \times m}(\mathbb{R})$ and study the set

$$\text{Bad}_{A}(n, m) = \left\{ x \in \mathbb{R}^n : \inf_{q \in \mathbb{Z}^m \setminus \{0\}} \|q\|^{m/n} d(Aq - x, \mathbb{Z}^n) > 0 \right\}.$$

First observe that the above set is easy to understand in the ‘rational’ case when there exists a non-zero $u \in \mathbb{Z}^m$ such that $A^T u \in \mathbb{Z}^n$ (or, equivalently, when the rank of the group $A^T \mathbb{Z}^n + \mathbb{Z}^m$ is strictly smaller than $m + n$). In this case, by a theorem of Kronecker, see [4, Ch. III, Theorem IV], $\inf_{q \in \mathbb{Z}^m} d(Aq - x, \mathbb{Z}^n)$ is positive if and only if the value of $u \cdot x$ is not an integer. Therefore,

$$\text{Bad}_{A}(n, m) \supset \{ x \in \mathbb{R}^n : u \cdot x \notin \mathbb{Z} \}.$$

Since the right-hand side is the complement of a countable union of hyperplanes, in view of Corollary 3.3 $\text{Bad}_{A}(n, m)$ is $\alpha$-winning on $K$ whenever $K$ is absolutely decaying and $\alpha$ is as in Theorem 1.3.

In the more interesting ‘irrational’ case when $\text{rank}(A^T \mathbb{Z}^n + \mathbb{Z}^m) = m + n$, one can utilize the theory of best approximations to $A$ as developed by Cassels [4, Ch. III] and recently made more precise by Bugeaud and Laurent [3]. In [2, §§5–6], it is shown that if $\text{rank}(A^T \mathbb{Z}^n + \mathbb{Z}^m) = m + n$, then there exists a lacunary sequence of vectors $y_k \in \mathbb{Z}^n$ (a subsequence of the sequence of best approximations to $A$) such that whenever $x \in \mathbb{R}^n$ satisfies

$$\inf_{k \in \mathbb{N}} d(y_k \cdot x, \mathbb{Z}) > 0,$$

it follows that $x \in \text{Bad}_{A}(n, m)$. In other words,

$$\tilde{E}(\mathcal{Y}', \mathcal{Z}) \subset \text{Bad}_{A}(n, m),$$

where $\mathcal{Y} \overset{\text{def}}{=} (y_k)$ and $\mathcal{Z} = (Z_k)$ with $Z_k = \mathbb{Z}$ for each $k$. (See also [22, §2] for an alternative exposition.) Therefore, in this case $\text{Bad}_{A}(n, m)$ is $\alpha$-winning on $K$ by Theorem 1.3. \qed
5. Concluding remarks

5.1. Playing on \( \mathbb{R}^n \) with \( \alpha = 1/2 \). As was mentioned before, the special case \( K = \mathbb{R}^n \) of our main theorem is essentially contained in [22]. In fact, arguing as in §4 and using [22, Lemma 2] (the analogue of our Lemma 3.2) and [22, Lemma 3] (Schmidt’s escaping lemma, cf. [28, Ch. 3, Lemma 1B]), one can show that for \( Z \) and \( M \) as in Theorem 1.3 and any \( \alpha, \beta > 0 \) with \( 1 + \alpha\beta - 2\alpha > 0 \), the sets \( \tilde{E}(M, Z) \) are \( (\alpha, \beta) \)-winning. In particular, this shows that one can take \( \alpha(\mathbb{R}^n) = 1/2 \) in Theorems 1.2 and 1.3.

5.2. Strong winning. Recently in [10, 11] and independently in [21] a modification of Schmidt’s game has been introduced, where condition (2.1) is replaced by

\[
\rho_k' \geq \alpha \rho_k \quad \text{and} \quad \rho_{k+1} \geq \beta \rho_k'.
\]

Following [21], a subset \( S \) of a metric space \( X \) is said to be \( (\alpha, \beta) \)-strong winning if Alice has a winning strategy in the game defined by (5.1). Analogously, one defines \( \alpha \)-strong winning and strong winning sets. It is not hard to verify that strong winning implies winning (see [11] for a proof), and that a countable intersection of \( \alpha \)-strong winning sets is \( \alpha \)-strong winning. Furthermore, this class has stronger invariance properties, e.g. it is proved in [21] that strong winning subsets of \( \mathbb{R}^n \) are preserved by quasisymmetric homeomorphisms.

It is not hard to modify the proofs given above to show that in Theorem 1.3 (and therefore in all of its corollaries), \( \alpha \)-winning may be replaced by \( \alpha \)-strong winning. This is done by adding ‘dummy moves’ in order to accommodate the possibly slower decrease in radii of the chosen balls. Details will appear elsewhere.

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