Efficient Inner-product Algorithm
for Stabilizer States

Héctor J. García∗
hjgarcia@eecs.umich.edu
Igor L. Markov∗
imarkov@eecs.umich.edu
Andrew W. Cross†
adrew.w.cross@ibm.com

∗University of Michigan – EECS Department
2260 Hayward Street, Ann Arbor, MI, 48109-2121
†IBM T. J. Watson Research Center
Yorktown Heights, NY 10598

Abstract

Large-scale quantum computation is likely to require massive quantum error correction (QEC). QEC codes and circuits are
described via the stabilizer formalism, which represents stabilizer states by keeping track of the operators that preserve them.
Such states are obtained by stabilizer circuits (consisting of CNOT, Hadamard and Phase only) and can be represented compactly on
conventional computers using $\Omega(n^2)$ bits, where $n$ is the number of qubits. Although techniques for the efficient simulation
of stabilizer circuits have been studied extensively [1], [9], [10], techniques for efficient manipulation of stabilizer states are not
currently available. To this end, we leverage the theoretical insights from [1] and [16] to design new algorithms for: (i) obtaining
canonical generators for stabilizer states, (ii) obtaining canonical stabilizer circuits, and (iii) computing the inner product between
stabilizer states. Our inner-product algorithm takes $O(n^3)$ time in general, but observes quadratic behavior for many practical
instances relevant to QEC (e.g., GHZ states). We prove that each $n$-qubit stabilizer state has exactly $4(2^n - 1)$ nearest-neighbor
stabilizer states, and verify this claim experimentally using our algorithms. We design techniques for representing arbitrary quantum
states using stabilizer frames and generalize our algorithms to compute the inner product between two such frames.

I. INTRODUCTION

Gottesman [9] and Knill showed that for certain types of non-trivial quantum circuits known as stabilizer circuits, efficient
simulation on classical computers is possible. Stabilizer circuits are exclusively composed of stabilizer gates – Controlled-NOT,
Hadamard and Phase gates (Figure 1) – followed by measurements in the computational basis. Such circuits are applied to
a computational basis state (usually $|00\ldots0\rangle$) and produce output states called stabilizer states. The case of purely unitary
stabilizer circuits (without measurement gates) is considered often, e.g., by consolidating measurements at the end. Stabilizer
circuits can be simulated in poly-time by keeping track of a set Pauli operators that stabilize the quantum state. Such stabilizer
operators uniquely represent a stabilizer state up to an unobservable global phase. Equation 1 shows that the number of $n$-qubit
stabilizer states grows as $2^{n^2/2}$, therefore, describing a generic stabilizer state requires at least $n^2/2$ bits. Despite their compact
representation, stabilizer states can exhibit multi-qubit entanglement and are often encountered in many quantum information
applications such as Bell states, GHZ states, error-correcting codes and one-way quantum computation. To better understand the
role stabilizer states play in such applications, researchers have designed techniques to quantify the amount of entanglement [8],
[18], [12] in such states and studied relevant properties such as purification schemes [7]. Bell inequalities [11] and equivalence
classes [17]. Efficient algorithms for the manipulation of stabilizer states (e.g., computing the angle between them), can help
lead to additional insights related to linear-algebraic and geometric properties of stabilizer states.

In this work, we describe in detail algorithms for the efficient computation of the inner product between stabilizer states.
We adopt the approach outlined in [11], which requires the synthesis of a unitary stabilizer circuit that maps a stabilizer state
to a computational basis state. The work in [11] shows that, for any unitary stabilizer circuit, there exists an equivalent block-structured
canonical circuit that applies a block of Hadamard ($H$) gates only, followed by a block of CNOT ($C$) only, then
a block of Phase ($P$) gates only, and so on in the 7-block sequence $H\cdot C\cdot P\cdot C\cdot P\cdot H$. Using an alternate representation
for stabilizer states, the work in [16] proves the existence of a $(H\cdot C\cdot P\cdot C)$-canonical circuit, where the $CZ$ block consists of
Controlled-Z (CPHASE) gates. However, no algorithms are known to synthesize such smaller 4-block circuits given an
arbitrary stabilizer state. In contrast, we describe an algorithm for synthesizing $(H\cdot C\cdot CZ\cdot P\cdot H)$-canonical circuits given any
input stabilizer state. We prove that any $n$-qubit stabilizer state $|\psi\rangle$ has exactly $4(2^n - 1)$ nearest-neighbors – stabilizer states
$|\varphi\rangle$ such that $\langle\psi|\varphi\rangle$ attains the largest possible value $\neq 1$. Furthermore, we design techniques for representing arbitrary
quantum states using stabilizer frames and generalize our algorithms to compute the inner product between two such frames.

This paper is structured as follows. Section II reviews the stabilizer formalism and relevant algorithms for manipulating
stabilizer-based representations of quantum states. Section III describes our circuit-synthesis and inner-product algorithms. In
Section IV we evaluate the performance of our algorithms. Our findings related to geometric properties of stabilizer states are

1An operator $U$ is said to stabilize a state iff $U|\psi\rangle = |\psi\rangle$. 
described in Section [VI] we discuss stabilizer frames and how they can be used to represent arbitrary states and extend our algorithms to compute the inner product between frames. Section VII closes with concluding remarks.

II. BACKGROUND AND PREVIOUS WORK

Gottesman [10] developed a description for quantum states involving the Heisenberg representation often used by physicists to describe atomic phenomena. In this model, one describes quantum states by keeping track of their symmetries rather than explicitly maintaining complex vectors. The symmetries are operators for which these states are 1-eigenvectors. Algebraically, symmetries form group structures, which can be specified compactly by group generators. It turns out that this approach, also known as the stabilizer formalism, can be used to represent an important class of quantum states.

A. The stabilizer formalism

A unitary operator $U$ stabilizes a state $|\psi\rangle$ if $|\psi\rangle$ is a 1-eigenvector of $U$, i.e., $U|\psi\rangle = |\psi\rangle$ [9], [14]. We are interested in operators $U$ derived from the Pauli matrices shown in Figure 1. The following table lists the one-qubit states stabilized by the Pauli matrices.

\[
\begin{align*}
X : & \quad (|0\rangle + |1\rangle)/\sqrt{2} \quad -X : \quad (|0\rangle - |1\rangle)/\sqrt{2} \\
Y : & \quad (|0\rangle + i|1\rangle)/\sqrt{2} \quad -Y : \quad (|0\rangle - i|1\rangle)/\sqrt{2} \\
Z : & \quad |0\rangle \quad -Z : \quad |1\rangle
\end{align*}
\]

Observe that $I$ stabilizes all states and $-I$ does not stabilize any state. As an example, the entangled state $(|00\rangle + |11\rangle)/\sqrt{2}$ is stabilized by the Pauli operators $X \otimes X$, $-Y \otimes Y$, $Z \otimes Z$ and $I \otimes I$. As shown in Table I it turns out that the Pauli matrices along with $I$ and the multiplicative factors $\pm 1$, $\pm i$, form a closed group under matrix multiplication [14]. Formally, the Pauli group $\mathcal{G}_n$ on $n$ qubits consists of the $n$-fold tensor product of Pauli matrices, $P = i^k P_1 \otimes \cdots \otimes P_n$ such that $P_j \in \{I, X, Y, Z\}$ and $k \in \{0, 1, 2, 3\}$. For brevity, the tensor-product symbol is often omitted so that $P$ is denoted by a string of $I$, $X$, $Y$ and $Z$ characters or Pauli literals and a separate integer value $k$ for the phase $i^k$. This string-integer pair representation allows us to compute the product of Pauli operators without explicitly computing the tensor products [11]. Since $|\mathcal{G}_n| = 4^n+1$, $\mathcal{G}_n$ can have at most $\log_2 |\mathcal{G}_n| = \log_2 4^n+1 = 2(n+1)$ irredundant generators [14]. The key idea behind the stabilizer formalism is to represent an n-qubit quantum state $|\psi\rangle$ by its stabilizer group $S(|\psi\rangle)$ – the subgroup of $\mathcal{G}_n$ that stabilizes $|\psi\rangle$. As the following theorem shows, if $|S(|\psi\rangle)| = 2^n$, the group uniquely specifies $|\psi\rangle$.

**Theorem 1.** For an $n$-qubit pure state $|\psi\rangle$ and $k \leq n$, $S(|\psi\rangle) \cong \mathbb{Z}_2^k$. If $k = n$, $|\psi\rangle$ is specified uniquely by $S(|\psi\rangle)$ and is called a stabilizer state.

**Proof:** (i) To prove that $S(|\psi\rangle)$ is commutative, let $P, Q \in S(|\psi\rangle)$ such that $PQ|\psi\rangle = |\psi\rangle$. If $P$ and $Q$ anticommute, $-QP|\psi\rangle = -Q(P|\psi\rangle) = -Q(|\psi\rangle) = -|\psi\rangle \neq |\psi\rangle$. Thus, $P$ and $Q$ cannot both be elements of $S(|\psi\rangle)$.
(ii) To prove that every element of $S(|\psi\rangle)$ is of degree 2, let $P \in S(|\psi\rangle)$ such that $P|\psi\rangle = |\psi\rangle$. Observe that $P^2 = i^l I$ for $l \in \{0, 1, 2, 3\}$. Since $P^2|\psi\rangle = P(P|\psi\rangle) = P|\psi\rangle = |\psi\rangle$, we obtain $i^l = 1$ and $P^2 = I$.
(iii) From group theory, a finite Abelian group with $a^2 = a$ for every element must be $\cong \mathbb{Z}^k_2$.
(iv) We now prove that $k \leq n$. First note that each independent generator $P \in S(|\psi\rangle)$ imposes the linear constraint $P|\psi\rangle = |\psi\rangle$ on the $2^n$-dimensional vector space. The subspace of vectors that satisfy such a constraint has dimension $2^{n-1}$, or half the space. Let $\text{gen}(|\psi\rangle)$ be the set of generators for $S(|\psi\rangle)$. We add independent generators to $\text{gen}(|\psi\rangle)$ one by one and impose their

---

**TABLE I**

MULTIPLICATION TABLE FOR PAULI MATRICES. SHIMED CELLS INDICATE ANTICOMMUTING PRODUCTS.

|   | I | X | Y | Z |
|---|---|---|---|---|
| I | I | X | Y | Z |
| X | X | I | -iZ | -iY |
| Y | Y | -iZ | I | iX |
| Z | Z | iY | -iX | I |

---

2This holds true due to the identity: $(A \otimes B)(C \otimes D) = (AC \otimes BD)$. 2
linear constraints, to limit $|\psi\rangle$ to the shared 1-eigenvector. Thus the size of $\text{gen}(|\psi\rangle)$ is at most $n$. In the case $|\text{gen}(|\psi\rangle)| = n$, the $n$ independent generators reduce the subspace of possible states to dimension one. Thus, $|\psi\rangle$ is uniquely specified.

The proof of Theorem [1] shows that $S(|\psi\rangle)$ is specified by only $\log_2 2^n = n$ irredundant stabilizer generators. Therefore, an arbitrary $n$-qubit stabilizer state can be represented by a stabilizer matrix $\mathcal{M}$ whose rows represent a set of generators $g_1, \ldots, g_n$ for $S(|\psi\rangle)$. (Hence we use the terms generator set and stabilizer matrix interchangeably.) Since each $g_i$ is a string of $n$ Pauli literals, the size of the matrix is $n \times n$. The phases of each $g_i$ are stored separately using a vector of $n$ integers. Therefore, the storage cost for $\mathcal{M}$ is $\Theta(n^2)$, which is an exponential improvement over the $O(2^n)$ cost often encountered in vector-based representations.

Theorem [1] suggests that Pauli literals can be represented using only two bits, e.g., $00 = I$, $01 = Z$, $10 = X$ and $11 = Y$. Therefore, a stabilizer matrix can be encoded using an $n \times 2n$ binary matrix or tableau. The advantage of this approach is that this literal-to-bits mapping induces an isomorphism $\mathbb{Z}^{2n}_2 \rightarrow G_2$ because vector addition in $\mathbb{Z}^{2n}_2$ is equivalent to multiplication of Pauli operators up to a global phase. The tableau implementation of the stabilizer formalism is covered in [1], [14].

**Proposition 2.** The number of $n$-qubit pure stabilizer states is given by

$$N(n) = 2^n \prod_{k=0}^{n-1} (2^{n-k} + 1) = 2^{(\frac{5}{2} + o(1))n^2}$$

(1)

The proof of Proposition 2 can be found in [1]. An alternate interpretation of Equation 1 is given by the simple recurrence relation $N(n) = 2(2^n + 1)N(n - 1)$ with base case $N(1) = 6$. For example, for $n = 2$ the number of stabilizer states is 60, and for $n = 3$ it is 1080. This recurrence relation stems from the fact that there are $2^n + 1$ ways of combining the generators of $N(n - 1)$ with additional Pauli matrices to form valid $n$-qubit generators. The factor of 2 accounts for the increase in the number of possible sign configurations. Table II and Appendix A list all two-qubit and three-qubit stabilizer states, respectively.

**Observation 3.** Consider a stabilizer state $|\psi\rangle$ represented by a set of generators of its stabilizer group $S(|\psi\rangle)$. Recall from the proof of Theorem [1] that, since $S(|\psi\rangle) \cong \mathbb{Z}^{n}_2$, each generator imposes a linear constraint on $|\psi\rangle$. Therefore, the set of generators can be viewed as a system of linear equations whose solution yields the $2^n$ basis amplitudes that make up $|\psi\rangle$. Thus, one needs to perform Gaussian elimination to obtain the basis amplitudes from a generator set.

**Canonical stabilizer matrices.** Although stabilizer states are uniquely determined by their stabilizer group, the set of generators may be selected in different ways. For example, the state $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is uniquely specified by any of the following stabilizer matrices:

$$\mathcal{M}_1 = \begin{bmatrix} XX \\ ZZ \end{bmatrix} \quad \mathcal{M}_2 = \begin{bmatrix} XX \\ -YY \end{bmatrix} \quad \mathcal{M}_3 = \begin{bmatrix} -YY \\ ZZ \end{bmatrix}$$

One obtains $\mathcal{M}_2$ from $\mathcal{M}_1$ by left-multiplying the second row by the first. Similarly, one can also obtain $\mathcal{M}_3$ from $\mathcal{M}_1$ or $\mathcal{M}_2$ via row multiplication. Observe that, multiplying any row by itself yields II, which stabilizes $|\psi\rangle$. However, II cannot be

**TABLE II**

SIXTY TWO-QUBIT STABILIZER STATES AND THEIR CORRESPONDING PAULI GENERATORS. SHorthand notation represents a stabilizer state as $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ where $\alpha_i$ are the normalized amplitudes of the basis states. The basis states are emphasized in bold. The first column lists states whose generators do not include an upfront minus sign, and other columns introduce the signs. A sign change creates an orthogonal vector. Therefore, each row of the table gives an orthogonal basis. The cells in dark gray indicate stabilizer states with four non-zero basis amplitudes, i.e., $\alpha_i \neq 0 \forall i$. The $\angle$ column indicates the angle between that state and $|00\rangle$, which has 12 nearest-neighbor states (light gray) and 15 orthogonal states (⊥).

| STATE | GEN TORS | $\angle$ | STATE | GEN TORS | $\angle$ | STATE | GEN TORS | $\angle$ | STATE | GEN TORS |
|-------|----------|---------|-------|----------|---------|-------|----------|---------|-------|----------|
| 1,1,1 | IX, XI   | 0       | 1,1,0 | IX, ZI   | 0       | 1,0,1 | IX, ZI   | 90°     | 1,0,0 | IX, ZI   |
| 1,1,0 | IX, XI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,0,0 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 180°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,1,1 | IX, XI   | 0       | 1,1,0 | IX, ZI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   |
| 1,1,0 | IX, XI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,0,0 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 180°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |

| STATE | GEN TORS | $\angle$ | STATE | GEN TORS | $\angle$ | STATE | GEN TORS | $\angle$ | STATE | GEN TORS |
|-------|----------|---------|-------|----------|---------|-------|----------|---------|-------|----------|
| 1,1,1 | IX, XI   | 0       | 1,1,0 | IX, ZI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   |
| 1,1,0 | IX, XI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,0,0 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 180°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,1,1 | IX, XI   | 0       | 1,1,0 | IX, ZI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   |
| 1,1,0 | IX, XI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,0,0 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 180°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |

| STATE | GEN TORS | $\angle$ | STATE | GEN TORS | $\angle$ | STATE | GEN TORS | $\angle$ | STATE | GEN TORS |
|-------|----------|---------|-------|----------|---------|-------|----------|---------|-------|----------|
| 1,1,1 | IX, XI   | 0       | 1,1,0 | IX, ZI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   |
| 1,1,0 | IX, XI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,0,0 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 180°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,1,1 | IX, XI   | 0       | 1,1,0 | IX, ZI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   |
| 1,1,0 | IX, XI   | 90°     | 1,0,1 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |
| 1,0,0 | IX, ZI   | 270°    | 1,0,0 | IX, ZI   | 180°    | 1,0,0 | IX, ZI   | 0       | 1,0,0 | IX, ZI   |

[3]
used as a stabilizer generator because it is redundant and carries no information about the structure of $|\psi\rangle$. This also holds true in general for $\mathcal{M}$ of any size. Any stabilizer matrix can be rearranged by applying sequences of elementary row operations in order to obtain a particular matrix structure. Such operations do not modify the stabilizer state. The elementary row operations that can be performed on a stabilizer matrix are transposition, which swaps two rows of the matrix, and multiplication, which left-multiplies one row with another. Such operations allow one to rearrange the stabilizer matrix in a series of steps that resemble Gauss-Jordan elimination. Given an $n \times n$ stabilizer matrix, row transpositions are performed in constant time while row multiplications require $O(n)$ time. Algorithm 1 rearranges a stabilizer matrix into a row-reduced echelon form that contains: (i) a minimum set of generators with $X$ and $Y$ literals appearing at the top, and (ii) generators containing a minimum set of $Z$ literals only appearing at the bottom of the matrix. This particular stabilizer-matrix structure, shown in Figure 2, defines a canonical representation for stabilizer states [6], [10]. The algorithm iteratively determines which row operations to apply based on the Pauli (non-$I$) literals contained in the first row and column of an increasingly smaller submatrix of the full stabilizer matrix. Initially, the submatrix considered is the full stabilizer matrix. After the proper row operations are applied, the dimensions of the submatrix decrease by one until the size of the submatrix reaches one. The algorithm performs this process twice, once to position the rows with $X(Y)$ literals at the top, and then again to position the remaining rows containing $Z$ literals only at the bottom. Let $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$ be the index of the first row and first column, respectively, of submatrix $\mathcal{A}$. The steps to construct the upper-triangular portion of the row-echelon form shown in Figure 2 are as follows.

1. Let $k$ be a row in $\mathcal{A}$ whose $j^{th}$ literal is $X(Y)$. Swap rows $k$ and $i$ such that $k$ is the first row of $\mathcal{A}$. Decrease the height of $\mathcal{A}$ by one (i.e., increase $i$).
2. For each row $m \in \{0, \ldots, n\}$, $m \neq i$ that has an $X(Y)$ in column $j$, use row multiplication to set the $j^{th}$ literal in row $m$ to $I$ or $Z$.
3. Decrease the width of $\mathcal{A}$ by one (i.e., increase $j$).

To bring the matrix to its lower-triangular form, one executes the same process with the following difference: (i) step 1 looks for rows that have a $Z$ literal (instead of $X$ or $Y$) in column $j$, and (ii) step 2 looks for rows that have $Z$ or $Y$ literals (instead of $X$ or $Y$) in column $j$. Observe that Algorithm 1 ensures that the columns in $\mathcal{M}$ have at most two distinct types of non-$I$ literals. Since Algorithm 1 inspects all $n^2$ entries in the matrix and performs a constant number of row multiplications each time, the runtime of the algorithm is $O(n^3)$. An alternative row-echelon form for stabilizer generators along with relevant algorithms to obtain them were introduced in [2]. However, their matrix structure is not canonical as it does not guarantee a minimum set of generators with $X/Y$ literals.

Stabilizer circuit simulation. The computational basis states are stabilizer states that can be represented using the following stabilizer-matrix structure.

**Definition 4.** A stabilizer matrix is in basis form if it has the following structure.

$$
\begin{pmatrix}
\pm Z & I & \cdots & I \\
\pm I & Z & \cdots & I \\
\vdots & \vdots & \ddots & \vdots \\
\pm I & I & \cdots & Z
\end{pmatrix}
$$

In this matrix form, the $\pm$ sign of each row along with its corresponding $Z$-literal designates whether the state of the $j^{th}$ qubit is $|0\rangle$ $+$ or $|1\rangle$ $-$. Suppose we want to simulate circuit $\mathcal{C}$. Stabilizer-based simulation first initializes $\mathcal{M}$ to specify some basis state $|\psi\rangle$. To simulate the action of each gate $U \in \mathcal{C}$, we conjugate each row $g_i$ of $\mathcal{M}$ by $U$. We require that

---

3Since Gaussian elimination essentially inverts the $n \times 2n$ matrix, this could be sped up to $O(n^{2.376})$ time by using fast matrix inversion algorithms. However, $O(n^3)$-time Gaussian elimination seems more practical.

4Storing pointers to rows facilitates $O(1)$-time row transpositions—one simply swaps relevant pointers.

5Since $g_i |\psi\rangle = |\psi\rangle$, the resulting state $U |\psi\rangle$ is stabilized by $U_{g_i} U_{g_i}^{-1}$ because $(U_{g_i} U_{g_i}^{-1}) U |\psi\rangle = U_{g_i} |\psi\rangle = U |\psi\rangle$. 

---

Fig. 2. Canonical (row-reduced echelon) form for stabilizer matrices. The $X$-block contains a minimal set of rows with $X/Y$ literals. The rows with $Z$ literals only appear in the $Z$-block. Each block is arranged so that the leading non-$I$ literal of each row is strictly to the right of the leading non-$I$ literal in the row above. The number of Pauli (non-$I$) literals in each block is minimal.
$U_g U^\dagger$ maps to another string of Pauli literals so that the resulting stabilizer matrix $M'$ is well-formed. It turns out that the Hadamard, Phase and CNOT gates (Figure 1) have such mappings, i.e., these gates conjugate the Pauli group onto itself [10, 14]. Table III lists the mapping for each of these gates.

For example, suppose we simulate a CNOT operation on $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ using $M$,

$$M = \begin{bmatrix} XX & CNOT \end{bmatrix}, M' = \begin{bmatrix} XI & IZ \end{bmatrix}$$

One can verify that the rows of $M'$ stabilize $|\psi\rangle \overset{CNOT}{\mapsto} (|00\rangle + |10\rangle)/\sqrt{2}$ as required.

Since Hadamard, Phase and CNOT gates are directly simulated using stabilizers, these gates are commonly called **stabilizer gates**. They are also called **Clifford gates** because they generate the Clifford group of unitary operators. We use these names interchangeably. Any circuit composed exclusively of stabilizer gates is called a **unitary stabilizer circuit**. Table III shows that at most two columns of $M$ are updated when one simulates a stabilizer gate. Thus, such gates are simulated in $\Theta(n)$ time.

**Theorem 5.** An $n$-qubit stabilizer state $|\psi\rangle$ can be obtained by applying a stabilizer circuit to the $|0\rangle^\otimes n$ basis state.

**Proof:** The work in [1] represents the generators using a tableau, and then shows how to construct a unitary stabilizer circuit from the tableau. We refer the reader to [1, Theorem 8] for details of the proof. ■

**Corollary 6.** An $n$-qubit stabilizer state $|\psi\rangle$ can be transformed by stabilizer gates into the $|0\rangle^\otimes n$ basis state.

**Proof:** Since every stabilizer state can be produced by applying some unitary stabilizer circuit $C$ to the $|0\rangle^\otimes n$ state, it suffices to reverse $C$ to perform the inverse transformation. To reverse a stabilizer circuit, reverse the order of gates and replace every $P$ gate with $PPP$.

The stabilizer formalism also admits one-qubit measurements in the computational basis [10]. However, the updates to $M$ for such gates are not as efficient as for stabilizer gates. Note that any qubit in a stabilizer state is either in a superposition of both. The former case is called a **deterministic outcome** and the latter a **random outcome**. We can tell these cases apart in $\Theta(n)$ time by searching for $X$ or $Y$ literals in the $j$th column of $M$. If such literals are found, the qubit must be in a superposition and the outcome is random with equal probability ($p(0) = p(1) = .5$); otherwise the outcome is deterministic ($p(0) = 1$ or $p(1) = 1$).

---

**Algorithm 1** Canonical form reduction for stabilizer matrices

**Input:** Stabilizer matrix $M$ for $S(|\psi\rangle)$ with rows $R_1, \ldots, R_n$

**Output:** $M$ is reduced to row-echelon form

1. $i \leftarrow 1$
2. for $j \in \{1, \ldots, n\}$ do
3. $k \leftarrow$ index of row $R_k \in \{i, \ldots, n\}$ with $j^{th}$ literal set to $X(Y)$
4. if $k$ exists then
5. ROWSWAP($M$, $i$, $k$)
6. for $m \in \{0, \ldots, n\}$ do
7. if $j^{th}$ literal of $R_m$ is $X$ or $Y$ and $m \neq i$ then
8. $R_m = ROWMULT(M, R_i, R_m)$
9. end if
10. end for
11. $i \leftarrow i + 1$
12. end if
13. end for
14. for $j \in \{1, \ldots, n\}$ do
15. $k \leftarrow$ index of row $R_k \in \{i, \ldots, n\}$ with $j^{th}$ literal set to $Z$
16. if $k$ exists then
17. ROWSWAP($M$, $i$, $k$)
18. for $m \in \{0, \ldots, n\}$ do
19. if $j^{th}$ literal of $R_m$ is $Z$ or $Y$ and $m \neq i$ then
20. $R_m = ROWMULT(M, R_i, R_m)$
21. end if
22. end for
23. $i \leftarrow i + 1$
24. end if
25. end for

---

$^6$ An arbitrary state $|\psi\rangle$ with computational basis decomposition $\sum_{\lambda} \lambda |k\rangle$ is said to be **unbiased** if for all $\lambda_i \neq 0$ and $\lambda_j \neq 0$, $|\lambda_i|^2 = |\lambda_j|^2$. Otherwise, the state is **biased**. One can verify that none of the stabilizer gates produce biased states.
Theorem 7. Let \( S(|\psi\rangle) \) and \( S(|\varphi\rangle) \) be the stabilizer groups for \(|\psi\rangle\) and \(|\varphi\rangle\), respectively. If there exist \( P \in S(|\psi\rangle) \) and \( Q \in S(|\varphi\rangle) \) such that \( P = -Q \), then \(|\psi\rangle \perp |\varphi\rangle\).

Proof: Since \(|\psi\rangle\) is a 1-eigenvector of \( P \) and \(|\varphi\rangle\) is a \((-1)\)-eigenvector of \( P \), they must be orthogonal.

Theorem 8. \( \square \) Let \(|\psi\rangle\), \(|\varphi\rangle\) be non-orthogonal stabilizer states. Let \( s \) be the minimum, over all sets of generators \( \{P_1, \ldots, P_n\} \) for \( S(|\psi\rangle) \) and \( \{Q_1, \ldots, Q_n\} \) for \( S(|\varphi\rangle) \), of the number of different \( i \) values for which \( P_i \neq Q_i \). Then, \(|\langle\psi|\varphi\rangle| = 2^{-s/2}\).

Proof: Since \(|\langle\psi|\varphi\rangle|\) is not affected by unitary transformations \( U \), we choose a stabilizer circuit such that \( U|\psi\rangle = |b\rangle \), where \(|b\rangle\) is a basis state. For this state, select the stabilizer generators \( M \) of the form \( I \ldots IZ1 \ldots I \). Perform Gaussian elimination on \( M \) to minimize the incidence of \( P_i \neq Q_i \). Consider two cases. If \( U|\varphi\rangle \neq |b\rangle \) and its generators contain only \( IZ \) literals, then \( U|\varphi\rangle \perp U|\psi\rangle \), which contradicts the assumption that \(|\psi\rangle\) and \(|\varphi\rangle\) are non-orthogonal. Otherwise, each generator of \( U|\varphi\rangle \) containing \( XY \) literals contributes a factor of \( 1/\sqrt{2} \) to the inner product.

### Synthesizing canonical circuits

A crucial step in the proof of Theorem 5 is the computation of a stabilizer circuit that brings an \( n \)-qubit stabilizer state \(|\psi\rangle\) to a computational basis state \(|b\rangle\). Consider a stabilizer matrix \( M \) that uniquely identifies \(|\psi\rangle\). \( M \) is reduced to basis form (Definition 3) by applying a series of elementary row and column operations. Recall that row operations (transposition and multiplication) do not modify the state, but column (Clifford) operations do. Thus, the column operations involved in the reduction process constitute a unitary stabilizer circuit \( C \) such that \( C|\psi\rangle = |b\rangle \), where \(|b\rangle\) is a basis state. Algorithm 2 reduces an input stabilizer matrix \( M \) to basis form and returns the circuit \( C \) that performs such a mapping.

Definition 9. Given a finite sequence of quantum gates, a circuit template describes a segmentation of the circuit into blocks where each block uses only one gate type. The blocks must correspond to the sequence and be concatenated in that order. For example, a circuit satisfying the \( H-C-P \) template starts with a block of Hadamard (\( H \)) gates, followed by a block of CNOT (\( C \)) gates, followed by a block of Phase (\( P \)) gates.

Definition 10. A circuit with a template structure consisting entirely of CNOT, Hadamard and Phase blocks is called a canonical stabilizer circuit.

Canonical forms are useful for synthesizing stabilizer circuits that minimize the number of gates and qubits required to produce a particular computation. This is particularly important in the context of quantum fault-tolerant architectures that are based on stabilizer codes. Given any stabilizer matrix, Algorithm 2 synthesizes a 5-block canonical circuit with template \( H-C-CZ-P-H \) (Figure 3a), where the \( CZ \) block consists of Controlled-Z (CPHASE) gates. Such gates are stabilizer gates since \( \text{CPhase}_{i,j} = H_j \text{CNOT}_{i,j} H_j \) (Figure 3b). In our implementation, such gates are simulated directly on the stabilizer. The work
operations run in \( \Theta(\ldots) \) on the number of gates. As in our approach, such circuits are dominated by the size of the CNOT blocks, which contain \( O(g) \) gates. The work in [15] shows that any CNOT circuit has an equivalent CNOT circuit with only

Our approach for computing the inner product between these two states is shown in Algorithm 3. Following the proof of Theorem 5 can apply the techniques from [15] directly to CPHASE blocks, which would facilitate similar optimizations to our proposed one simply applies such techniques to each of the CNOT blocks in the canonical circuit. It is an open problem whether one

Algorithm 1. Finally, each generator in \( M \), commutes with \( Q \), and check whether \( Q \) is in the stabilizer group generated by \( M \). This is accomplished by multiplying the appropriate generators in \( M \), such that we create Pauli operator \( R \), which has the same literals as \( Q \), and check whether \( R \) has an opposite sign to \( Q \). If this is the case, then, by Theorem 7, the states are orthogonal. Clearly, the most time-consuming step of Algorithm 3 is the call to Algorithm 2, therefore, the overall runtime is \( O(n^3) \). However, as we show in Section IV, the performance of our algorithm depends strongly on the stabilizer matrices considered and exhibits quadratic behaviour for certain stabilizer states.

IV. EMPIRICAL VALIDATION

We implemented our algorithms in C++ and designed a benchmark set to validate the performance of our inner-product algorithm. Recall that the runtime of Algorithm 2 is dominated by the two nested for-loops (lines 20-35). The number of times

\[ 2 \times \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \]

\[ 7 \text{-step canonical procedure. However, the last four steps pertain to reducing destabilizer rows, which we do not consider in our approach.} \]
Algorithm 2 Synthesis of basis normalization circuit

**Input:** Stabilizer matrix \( \mathcal{M} \) for \( S(\lvert \psi \rangle) \) with rows \( R_1, \ldots, R_n \)

**Output:** (i) Unitary stabilizer circuit \( C \) such that \( C \lvert \psi \rangle \) equals basis state \( \lvert b \rangle \), and (ii) reduce \( \mathcal{M} \) to basis form

\[ \Rightarrow \text{GAUSS}(\mathcal{M}) \text{ reduces } \mathcal{M} \text{ to canonical form (Figure 2)} \]

\[ \Rightarrow \text{ROWSWAP}(\mathcal{M}, i, j) \text{ swaps rows } R_i \text{ and } R_j \text{ of } \mathcal{M} \]

\[ \Rightarrow \text{ROWMULT}(\mathcal{M}, i, j) \text{ left-multiplies rows } R_i \text{ and } R_j, \text{ returns updated } R_i \]

\[ \Rightarrow \text{CONJ}(\mathcal{M}, \alpha_j) \text{ conjugates } j^{th} \text{ column of } \mathcal{M} \text{ by Clifford sequence } \alpha \]

1: \( \text{GAUSS}(\mathcal{M}) \) \hspace{1cm} \( \triangleright \) Set \( \mathcal{M} \) to canonical form
2: \( C \leftarrow \emptyset \)
3: \( i \leftarrow 1 \)
4: for \( j \in \{1, \ldots, n\} \) do \hspace{1cm} \( \triangleright \) Apply block of Hadamard gates
5: \( k \leftarrow \text{index of row } R_{k \in \{i, \ldots, n\} \text{ with } j^{th} \text{ literal set to } X \text{ or } Y} \)
6: if \( k \) exists then
7: \( \text{ROWSWAP}(\mathcal{M}, i, k) \)
8: else \( k_2 \leftarrow \text{index of last row } R_{k_2 \in \{i, \ldots, n\} \text{ with } j^{th} \text{ literal set to } Z} \)
9: if \( k_2 \) exists then
10: \( \text{ROWSWAP}(\mathcal{M}, i, k_2) \)
11: if \( R_i \) has \( X \), \( Y \) or \( Z \) literals in columns \( \{j+1, \ldots, n\} \) then
12: \( \text{CONJ}(\mathcal{M}, H_j) \)
13: \( C \leftarrow C \cup H_j \)
14: end if
15: end if
16: end if
17: end if
18: \( i \leftarrow i + 1 \)
19: end for
20: for \( j \in \{1, \ldots, n\} \) do \hspace{1cm} \( \triangleright \) Apply block of CNOT gates
21: for \( k \in \{j+1, \ldots, n\} \) do
22: if \( k^{th} \) literal of row \( R_j \) is set to \( X \) or \( Y \) then
23: \( \text{CONJ}(\mathcal{M}, \text{CNOT}_{j,k}) \)
24: \( C \leftarrow C \cup \text{CNOT}_{j,k} \)
25: end if
26: end for
27: end for
28: for \( j \in \{1, \ldots, n\} \) do \hspace{1cm} \( \triangleright \) Apply a block of Controlled-Z gates (Figure 3)
29: for \( k \in \{j+1, \ldots, n\} \) do
30: if \( k^{th} \) literal of row \( R_j \) is set to \( Z \) then
31: \( \text{CONJ}(\mathcal{M}, \text{CPHASE}_{j,k}) \)
32: \( C \leftarrow C \cup \text{CPHASE}_{j,k} \)
33: end if
34: end for
35: end for
36: for \( j \in \{1, \ldots, n\} \) do \hspace{1cm} \( \triangleright \) Apply block of Phase gates
37: if \( j^{th} \) literal of row \( R_j \) is set to \( Y \) then
38: \( \text{CONJ}(\mathcal{M}, P_j) \)
39: \( C \leftarrow C \cup P_j \)
40: end if
41: end for
42: for \( j \in \{1, \ldots, n\} \) do
43: if \( j^{th} \) literal of row \( R_j \) is set to \( X \) then
44: \( \text{CONJ}(\mathcal{M}, H_j) \)
45: \( C \leftarrow C \cup H_j \)
46: end if
47: end for
48: for \( j \in \{1, \ldots, n\} \) do \hspace{1cm} \( \triangleright \) Eliminate trailing \( Z \) literals to ensure basis form (Definition 4)
49: for \( k \in \{j+1, \ldots, n\} \) do
50: if \( j^{th} \) literal of row \( R_k \) is set to \( Z \) then
51: \( R_k \leftarrow \text{ROWMULT}(\mathcal{M}, R_j, R_k) \)
52: end if
53: end for
54: end for
55: return \( C \)
Algorithm 3 Inner product for stabilizer states

Input: Stabilizer matrices (i) \( \mathcal{M}^\psi \) for \(|\psi\rangle\) with rows \( P_1, \ldots, P_n \), and (ii) \( \mathcal{M}^\phi \) for \(|\phi\rangle\) with rows \( Q_1, \ldots, Q_n \).

Output: Inner product between \(|\psi\rangle\) and \(|\phi\rangle\).

\[
\begin{align*}
1: & \quad C \leftarrow \text{BASISNORMCIRC}(\mathcal{M}^\psi) \\
2: & \quad \text{CONJ}(\mathcal{M}^\phi, C) \\
3: & \quad \text{GAUSS}(\mathcal{M}^\phi) \\
4: & \quad k \leftarrow 0 \\
5: & \quad \textbf{for each} \text{ row } Q_i \in \mathcal{M}^\phi \text{ \textbf{do}} \\
6: & \quad \quad \textbf{if} \ Q_i \text{ contains } X \text{ or } Y \text{ literals \textbf{then}} \\
7: & \quad \quad \quad k \leftarrow k + 1 \\
8: & \quad \quad \textbf{else} \\
9: & \quad \quad \quad \quad R \leftarrow I^\otimes n \\
10: & \quad \quad \quad \quad \textbf{for each} \ Z \text{ literal in } Q_i \text{ found at position } j \text{ do} \\
11: & \quad \quad \quad \quad \quad R \leftarrow \text{LEFTMULT}(P_j, R) \\
12: & \quad \quad \quad \quad \textbf{end for} \\
13: & \quad \quad \quad \textbf{if} \ R = -Q_i \text{ then} \\
14: & \quad \quad \quad \quad \textbf{return} \ 0 \\
15: & \quad \quad \textbf{end if} \\
16: & \quad \textbf{end if} \\
17: & \quad \textbf{end for} \\
18: & \quad \textbf{return} \ 2^{-k/2}
\end{align*}
\]

\( \triangleright \) Apply Algorithm 2 to \( \mathcal{M}^\psi \)

\( \triangleright \) Compute \( C \mid \phi \rangle \)

\( \triangleright \) Set \( \mathcal{M}^\phi \) to canonical form

\( \triangleright \) Check orthogonality, i.e., \( Q_i \notin S(b) \).

\( \triangleright \) By Theorem 7

\( \triangleright \) By Theorem 8

these loops execute depends on the amount of entanglement in the input stabilizer state. In turn, the number of entangled qubits depends on the number of CNOT gates in the circuit \( \mathcal{C} \) used to generate the stabilizer state \( \mathcal{C} |0^\otimes n \rangle \) (Theorem 5).

By a simple heuristic argument [11], one generates highly entangled stabilizer states as long as the number of CNOT gates in \( \mathcal{C} \) is proportional to \( n \log n \). Therefore, we generated random \( n \)-qubit stabilizer circuits for \( n \in \{20, 40, \ldots, 500\} \) as follows: fix a parameter \( \beta > 0 \); then choose \( \beta \lfloor n \log_2 n \rfloor \) unitary gates (CNOT, Phase or Hadamard) each with probability \( 1/3 \). Then, each random \( \mathcal{C} \) is applied to the \( |00\ldots0\rangle \) basis state to generate random stabilizer matrices (states). The use of randomly generated benchmarks is justified for our experiments because (i) our algorithms are not explicitly sensitive to circuit topology and (ii) random stabilizer circuits are considered representative [13].

For each \( n \), we applied Algorithm 3 to pairs of random stabilizer matrices and measured the number of seconds needed to compute the inner product. The entire procedure was repeated for increasing degrees of entanglement by ranging \( \beta \) from 0.6 to 1.2 in increments of 0.1. Our results are shown in Figure 4-a.

The runtime of Algorithm 3 appears to grow quadratically in \( n \) when \( \beta = 0.6 \). However, when the number of unitary gates is doubled (\( \beta = 1.2 \)), the runtime exhibits cubic growth. Therefore, Figure 4-a shows that the performance of Algorithm 3 is highly dependent on the degree of entanglement in the input stabilizer states. Figure 4-b shows the average size of the basis-normalization circuit returned by the calls to Algorithm 2. As expected (Proposition 11), the size of the circuit grows.

![Fig. 4. Average runtime for Algorithm 3 to compute the inner product between two random \( n \)-qubit stabilizer states. The stabilizer matrices that represent the input states are generated by applying \( \beta n \log_2 n \) unitary gates to \( |0^\otimes n \rangle \).](image-url)
quadratically in \( n \). Figure 5 shows the average runtime for Algorithm 3 to compute the inner product between: (i) the all-zeros basis state and random \( n \)-qubit stabilizer states, and (ii) the \( n \)-qubit GHZ state \( \ket{\text{GHZ}} \) and random stabilizer states. GHZ states are maximally entangled states that have been realized experimentally using several quantum technologies and are often encountered in practical applications such as error-correcting codes and fault-tolerant architectures. Figure 5 shows that, for such practical instances, Algorithm 3 can compute the inner product in roughly \( O(n^2) \) time (e.g. \( \langle \text{GHZ} | \phi \rangle \)). However, without apriori information about the input stabilizer matrices, one can only say that the performance of Algorithm 3 will be somewhere between quadratic and cubic in \( n \).

V. NEAREST-NEIGHBOR STABILIZER STATES

We used Algorithm 3 to compute the inner product between \( \ket{0^\otimes n} \) and all two-qubit stabilizer states. Our results are shown in Table II. We leveraged these results to formulate the following properties related to the geometry of stabilizer states.

**Definition 12.** Given an arbitrary state \( \ket{\psi} \) with \( \| \psi \| = 1 \), a stabilizer state \( \ket{\varphi} \) is a nearest stabilizer state to \( \ket{\psi} \) if \( \langle \psi | \varphi \rangle \) attains the largest possible value \( \neq 1 \).

**Proposition 13.** Consider two orthogonal stabilizer states \( \ket{\alpha} \) and \( \ket{\beta} \) whose unbiased superposition \( \ket{\psi} \) is also a stabilizer state. Then \( \ket{\psi} \) is a nearest stabilizer state to \( \ket{\alpha} \) and \( \ket{\beta} \).

**Proof:** Since stabilizer states are unbiased, \( |\langle \psi | \alpha \rangle| = |\langle \psi | \beta \rangle| = \frac{1}{\sqrt{2}} \). By Theorem 6 this is the largest possible value \( \neq 1 \). Thus, \( \ket{\psi} \) is a nearest stabilizer state to \( \ket{\alpha} \) and \( \ket{\beta} \).

**Lemma 14.** For any two stabilizer states, the numbers of nearest-neighbor stabilizer states are equal.

**Proof:** By Corollary 6 any stabilizer state can be mapped to another stabilizer state by a stabilizer circuit. Since the operators effected by these circuits are unitary, inner products are preserved.

**Lemma 15.** Let \( \ket{\psi} \) and \( \ket{\varphi} \) be orthogonal stabilizer states such that \( \ket{\varphi} = P \ket{\psi} \) where \( P \) is an element of the Pauli group. Then \( \frac{\ket{\psi} + \ket{\varphi}}{\sqrt{2}} \) is a stabilizer state.

**Proof:** Suppose \( \ket{\psi} = (g_k)_{k=1,2,\ldots} \) is generated by elements \( g_k \) of the \( n \)-qubit Pauli group. Let

\[
\begin{align*}
f(k) &= \begin{cases} 0 & \text{if } P \cdot g_k = 0 \\ 1 & \text{otherwise} \end{cases}
\end{align*}
\]

and write \( \ket{\varphi} = \langle (-1)^{f(k)} g_k \rangle \). Conjugating each generator \( g_k \) by \( P \) we see that \( \ket{\varphi} \) is stabilized by \( \langle (-1)^{f(k)} g_k \rangle \). Let \( Z_k \) (respectively \( X_k \)) denote the Pauli operator \( Z(X) \) acting on the \( k \)-th qubit. By Corollary 6 there exists an element \( L \) of the \( n \)-qubit Clifford group such that \( L \ket{\psi} = \ket{0^\otimes n} \) and \( L \ket{\varphi} = (LPL^t) \ket{\psi} = i^t \ket{f(1)^2 \ldots f(n)} \). The second equality follows from the fact that \( LPL^t \) is an element of the Pauli group and can therefore be written as \( i^t X(u) Z(u) \) for some \( t \in \{0, 1, 2, 3\} \) and \( u, v \in \mathbb{Z}_2^n \). Therefore,

\[
\frac{\ket{\psi} + \ket{\varphi}}{\sqrt{2}} = L^t (\ket{0^\otimes n} + i^t \ket{f(1)^2 \ldots f(n)})
\]

An \( n \)-qubit GHZ state is an equal superposition of the all-zeros and all-ones states, i.e., \( \frac{\ket{0^\otimes n} + \ket{1^\otimes n}}{\sqrt{2}} \).

\( \xi \)
The state in parenthesis on the right-hand side is the product of an all-zeros state and a GHZ state. Therefore, the sum is stabilized by $S' = L^t(S_{\text{zero}}, S_{\text{ghz}})L$ where $S_{\text{zero}} = \{Z_i, i \in \{k|f(k) = 0\}\}$ and $S_{\text{ghz}}$ is supported on $\{k|f(k) = 1\}$ and equals $((-1)^{t/2}XX\ldots X,\forall i Z_iZ_{i+1})$ if $t = 0 \mod 2$ or $((-1)^{(t-1)/2}YY\ldots Y,\forall i Z_iZ_{i+1})$ if $t = 1 \mod 2$.

**Theorem 16.** For any $n$-qubit stabilizer state $|\psi\rangle$, there are $4(2^n - 1)$ nearest-neighbor stabilizer states, and these states can be produced as described in Lemma 15.

**Proof:** The all-zeros basis amplitude of any stabilizer state $|\psi\rangle$ that is a nearest neighbor to $\{0\}^n$ must be $\propto 1/\sqrt{2}$. Therefore, $|\psi\rangle$ is an unbiased superposition of $|0\rangle^n$ and one of the other $2^n - 1$ basis states, i.e., $|\psi\rangle = |0\rangle^n + P|0\rangle^n \sqrt{2}$, where $P \in G_n$ such that $P|0\rangle^n \neq \alpha|0\rangle^n$. As in the proof of Lemma 15 we have $|\psi\rangle = |0\rangle^n + t|\phi\rangle$, where $|\phi\rangle$ is a basis state and $t \in \{0, 1, 2, 3\}$. Thus, there are 4 possible unbiased superpositions, and a total of $4(2^n - 1)$ nearest stabilizer states. Since $|0\rangle^n$ is a stabilizer state, all stabilizer states have the same number of nearest stabilizer states by Lemma 14.

Table II shows that $|0\rangle$ has 12 nearest-stabilizer states. We computed inner products between all pairs of 2-qubit stabilizer states and confirmed that each had exactly 12 nearest neighbors. We used the same procedure to verify that all 3-qubit stabilizer states have 28 nearest neighbors. We verified the correctness of our algorithm by comparing against inner product computations based on explicit basis amplitudes.

**VI. STABILIZER FRAMES**

Given an $n$-qubit stabilizer state $|\psi\rangle$, there exists an orthonormal basis including $|\psi\rangle$ and consisting entirely of stabilizer states. Using Theorem 1, one can generate such a basis from the stabilizer representation of $|\psi\rangle$. Observe that, one can create a state $|\varphi\rangle$ that is orthogonal to $|\psi\rangle$ by changing the signs of an arbitrary non-empty subset of generators of $S(|\psi\rangle)$, i.e., by permuting the phase vector of the stabilizer matrix for $|\psi\rangle$. Moreover, selecting two different subsets will produce two mutually orthogonal states. Thus, one can produce $2^n - 1$ additional orthogonal stabilizer states. Such states, together with $|\psi\rangle$, form an orthonormal basis. This is illustrated by Table II, where each row constitutes an orthonormal basis.

**Definition 17.** A stabilizer frame $F$ is a set of $k \leq 2^n$ stabilizer states that forms an orthonormal basis $\{|\psi_1\rangle,\ldots,|\psi_k\rangle\}$ and spans a subspace of the $n$-qubit Hilbert space. $F$ is represented by a pair consisting of (i) a stabilizer matrix $M$ and (ii) a set of $k$ distinct phase vectors $\sigma(M), j \in \{1,\ldots,k\}$. The size of the frame, which we denote by $|F|$, is equal to $k$.

Stabilizer frames are useful for representing arbitrary quantum states and for simulating the action of stabilizer circuits on such states. Let $\alpha = (\alpha_1,\ldots,\alpha_k) \in \mathbb{C}^k$ be the decomposition of the arbitrary $n$-qubit state $|\phi\rangle$ onto the basis $\{|\psi_1\rangle,\ldots,|\psi_k\rangle\}$ defined by $F$, i.e., $|\phi\rangle = \sum_{i=1}^k \alpha_k |\psi_k\rangle$. Furthermore, let $U$ be a stabilizer gate. To simulate $U|\phi\rangle$, one simply rotates the basis defined by $F$ to get the new basis $\{U|\psi_1\rangle,\ldots,U|\psi_k\rangle\}$. This is accomplished with the following two-step process: (i) update the stabilizer matrix $M$ associated with $F$ as per Section II-A, (ii) iterate over the phase vectors in $F$ and update each accordingly (Table III). The second step is linear in the number of phase vectors as only a constant number of elements in each vector needs to be updated. Also, $\alpha$ may need to be updated, which requires the computation of the global phase of each $U|\psi_i\rangle$. Since the stabilizer does not maintain global phases directly, each $\alpha_i$ is updated as follows:

1. Use Gaussian elimination to obtain a basis state $|b\rangle$ from $M$ (Observation 5) and store its non-zero amplitude $\beta$. If $U$ is the Hadamard gate, it may be necessary to sample a sum of two non-zero (one real, one imaginary) basis amplitudes.
2. Compute $U\beta|b\rangle = \beta'|b\rangle$ directly using the state-vector representation.
3. Obtain $|b\rangle'$ from $U^tMU^t$ and store its non-zero amplitude $\gamma$.
4. Compute the global-phase factor generated as $\alpha_i = (\alpha_i \cdot \beta')/\gamma$.

Observe that, all the above processes take time polynomial in $k$, therefore, if $k = \text{poly}(n)$, $U|\phi\rangle$ can be simulated efficiently on a classical computer via frame-based simulation.

**Inner product between frames.** We now discuss how to use our algorithms to compute the inner product between arbitrary quantum states. Let $\langle \phi |$ and $| \varphi \rangle$ be quantum states represented by the pairs $< F^\phi, \alpha = (\alpha_1,\ldots,\alpha_k) >$ and $< F^\varphi, \beta = (\beta_1,\ldots,\beta_l) >$, respectively. The following steps compute $|\langle \phi | \varphi \rangle|$.  

1. Apply Algorithm 2 to $M^\phi$ (the stabilizer matrix associated with $F^\phi$) to obtain basis-normalization circuit $C$.
2. Rotate frames $F^\phi$ and $F^\varphi$ by $C$ as outlined in our previous discussion.
3. Reduce $M^\phi$ to canonical form (Algorithm 1) and record the row operations applied. Apply the same row operations to each phase vector $\sigma_j, i \in \{1,\ldots,k\}$ in $F^\phi$. Repeat this step for $M^\varphi$ and the phase vectors in $F^\varphi$.
4. Let $M^\phi_j$ denote the leading-phases of the rows in $M^\phi$ are set equal to $\sigma_j^\phi$. Similarly, $M^\varphi_j$ denotes that the phases of $M^\varphi$ are equal to $\sigma_j^\varphi$. Furthermore, let $\delta(M^\phi_j, M^\varphi_j)$ be the function that returns 0 if the orthogonality check from Algorithm 3 (lines 9–15) returns 0, and 1 otherwise. The inner product is computed as:

$$|\langle \phi | \varphi \rangle| = \frac{1}{2^{\ell/2}} \sum_{i=1}^k \sum_{j=1}^l |\alpha_i^\phi \beta_j|^2 \delta(M^\phi_j, M^\varphi_j)$$
where $s$ is the number of rows in $M_{\varphi}$ that contain $X$ or $Y$ literals.

Prior work on representation of arbitrary states using the stabilizer formalism can be found in [1]. The authors propose an approach that represents a quantum state as a sum of density-matrix terms. Our frame-based technique offers more compact storage ($|F| \leq 2^n$ whereas a density matrix may have $4^n$ non-zero entries) but requires more sophisticated book-keeping.

VII. CONCLUSION

The stabilizer formalism facilitates compact representation of stabilizer states and efficient simulation of stabilizer circuits. Stabilizer states appear in many different quantum-information applications, and their efficient manipulation via geometric and linear-algebraic operations may lead to additional insights. To this end, we study algorithms to efficiently compute the inner product between stabilizer states. A crucial step of this computation is the synthesis of a canonical circuit that transforms a stabilizer state into a computational basis state. We designed an algorithm to synthesize such circuits using a 5-block template structure and showed that these circuits contain $O(n^2)$ stabilizer gates. We analysed the performance of our inner-product algorithm and showed that, although its runtime is $O(n^3)$, there are practical instances in which it runs in linear or quadratic time. Furthermore, we proved that an $n$-qubit stabilizer state has exactly $4(2^n - 1)$ nearest-neighbor states and verified this result experimentally. Finally, we designed techniques for representing arbitrary quantum states using stabilizer frames and generalize our algorithms to compute the inner product between two such frames.

REFERENCES

[1] S. Aaronson and D. Gottesman, “Improved simulation of stabilizer circuits,” Phys. Rev. A, vol. 70, no. 052328 (2004).
[2] K. M. R. Audenaert and M. B. Plenio, “Entanglement on mixed stabiliser states: normal norms and reduction procedures,” New J. Phys., vol. 7, no. 170 (2005).
[3] A. Calderbank, E. Rains, P. Shor, and N. Sloane, “Quantum error correction via codes over GF(4),” IEEE Trans. Inf. Theory, vol. 44, pp. 1369–1387 (1998).
[4] R. Cleve and D. Gottesman, “Efficient computations of encodings for quantum error correction,” Phys. Rev. A, vol. 56, no. 1 (1997).
[5] J. Dehaene and B. De Moor, “Clifford group, stabilizer states, and linear and quadratic operations over GF(2),” Phys. Rev. A, vol. 68, no. 042318 (2003).
[6] I. Djordjevic, “Quantum information processing and quantum error correction: an engineering approach,” Academic press (2012).
[7] W. Dur, H. Aschauer and H.J. Briegel, “Multiparticle entanglement purification for graph states,” Phys. Rev. Lett., vol. 91, no. 10 (2003).
[8] D. Fattal, T. S. Cubitt, Y. Yamamoto, S. Bravyi and I. L. Chuang, “Entanglement in the stabilizer formalism,” arXiv:0406168 (2004).
[9] D. Gottesman, “Stabilizer codes and quantum error correction,” Caltech Ph.D. thesis (1997).
[10] D. Gottesman, “The Heisenberg representation of quantum computers,” arXiv:9807006v1 (1998).
[11] O. Guehne, G. Toth, P. Hyllus and H.J. Briegel, “Bell inequalities for graph states,” Phys. Rev. Lett., vol. 95, no. 120405 (2005).
[12] M. Hein, J. Eisert, and H.J. Briegel, “Multi-party entanglement in graph states,” Phys. Rev. A, vol. 69, no. 6 (2004).
[13] E. Knill, et. al., “Randomized benchmarking of quantum gates,” Phys. Rev. A, vol. 77, no. 1 (2007).
[14] M. A. Nielsen and I. L. Chuang, “Quantum Computation and Quantum Information,” Cambridge University Press (2000).
[15] K. N. Patel, I. L. Markov and J. P. Hayes, “Optimal synthesis of linear reversible circuits,” Quant. Inf. Comp., vol. 8, no. 3 (2008).
[16] M. Van den Nest, “Classical simulation of quantum computation, the Gottesman-Knill theorem, and slightly beyond,” Quant. Inf. Comp., vol. 10, pp. 0258–0271 (2010).
[17] M. Van den Nest, J. Dehaene and B. De Moor, “On local unitary versus local Clifford equivalence of stabilizer states,” Phys. Rev. A, vol. 71, no. 062323 (2005).
[18] H. Wunderlich and M. B. Plenio, “Quantitative verification of fidelities and entanglement from incomplete measurement data”, J. Mod. Opt., vol. 56, pp. 2100–2105 (2009).
