A note on order-type homogeneous point sets

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Abstract

Let \( \text{OT}_d(n) \) be the smallest integer \( N \) such that every \( N \)-element point sequence in \( \mathbb{R}^d \) in general position contains an order-type homogeneous subset of size \( n \), where a set is order-type homogeneous if all \( (d+1) \)-tuples from this set have the same orientation. It is known that a point sequence in \( \mathbb{R}^d \) that is order-type homogeneous, forms the vertex set of a convex polytope that is combinatorially equivalent to a cyclic polytope in \( \mathbb{R}^d \). Two famous theorems of Erdős and Szekeres from 1935, imply that \( \text{OT}_1(n) = \Theta(n^2) \) and \( \text{OT}_2(n) = 2^{\Theta(n)} \). For \( d \geq 3 \), we give new bounds for \( \text{OT}_d(n) \). In particular:

- We show that \( \text{OT}_3(n) = 2^{\Theta(n)} \), answering a question of Eliáš and Matoušek.
- For \( d \geq 4 \), we show that \( \text{OT}_d(n) = 2^{O(n^{2d-4})} \), where the height of the tower is \( d \).

1 Introduction

In their classic paper \([7]\), Erdős and Szekeres proved the following two well-known results.

**Theorem 1.1.** For \( N = (n-1)^2 + 1 \), let \( P = (p_1, ..., p_N) \subset \mathbb{R} \) be a sequence of \( N \) distinct real numbers. Then \( P \) contains a subsequence \( (p_{i_1}, ..., p_{i_n}) \), \( i_1 < \cdots < i_n \), such that either \( p_{i_1} < p_{i_2} < \cdots < p_{i_n} \) or \( p_{i_1} > p_{i_2} > \cdots > p_{i_n} \).

In fact, there are now at least 6 different proofs of Theorem 1.1 (see \([16]\)). Notice that the point sequence \((p_{i_1}, ..., p_{i_n})\) obtained from Theorem 1.1 has the property that either \( p_{i_k} - p_{i_j} > 0 \) for every pair \( j, k \) such that \( 1 \leq j < k \leq n \), or \( p_{i_k} - p_{i_j} < 0 \) for every pair \( j, k \) such that \( 1 \leq j < k \leq n \). The other well-known result from \([7]\) is the following theorem, which is often referred to as the Erdős-Szekeres cups-caps Theorem (see also \([15]\)).

**Theorem 1.2.** For \( N = \binom{2n-4}{n-2} + 1 \), let \( P = (p_1, ..., p_N) \) be a sequence of \( N \) points in the plane such that no 2 share a common first coordinate, \( P \) is ordered by increasing first coordinate, and no 3 points lie on a line. Then \( P \) contains a subsequence \( (p_{i_1}, ..., p_{i_n}) \), \( i_1 < \cdots < i_n \), such that the slopes of the lines \( p_{i_j}p_{i_{j+1}}, j = 1, 2, ..., n-1 \), are increasing or decreasing.

See Figure 1. Again, notice that the point sequence \((p_{i_1}, ..., p_{i_n})\) obtained from Theorem 1.2 has the property that either every triple has a clockwise orientation, or every triple has a counterclockwise orientation.
The preceding discussion generalizes in a natural way to point sequences in $\mathbb{R}^d$ in general position. A point set $P$ in $\mathbb{R}^d$ is in general position, if no $d+1$ members lie on a common hyperplane, and no $2$ members share the same $i$-th coordinate for $1 \leq i \leq d$.

Let $P = (p_1, ..., p_N)$ be an $N$-element point sequence in $\mathbb{R}^d$ in general position. For $i_1 < i_2 < \cdots < i_{d+1}$, the orientation of the $(d+1)$-tuple $(p_{i_1}, p_{i_2}, ..., p_{i_{d+1}}) \subset P$ is defined as the sign of the determinant of the unique linear mapping $A$ that sends the $d$ vectors $p_{i_2} - p_{i_1}, p_{i_3} - p_{i_1}, ..., p_{i_{d+1}} - p_{i_1}$, to the standard basis $e_1, e_2, ..., e_d$. Geometrically, if $h \subset \mathbb{R}^d$ is the hyperplane spanned by $p_{i_1}, ..., p_{i_d}$, then the orientation of the $(d+1)$-tuple $(p_{i_1}, p_{i_2}, ..., p_{i_{d+1}})$ tells us on which side of the hyperplane $h$ the point $p_{i_{d+1}}$ lies.

The order type of $P = (p_1, p_2, ..., p_N)$ is the mapping $\chi : (P_{d+1}) \to \{+1, -1\}$ (positive orientation, negative orientation), assigning each $(d+1)$-tuple of $P$ its orientation. Hence for $p_i = (a_{i,1}, a_{i,2}, ..., a_{i,d}) \in \mathbb{R}^d$,

$$\chi(\{p_{i_1}, p_{i_2}, ..., p_{i_{d+1}}\}) = \text{sgn det} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{i_1,1} & a_{i_2,1} & \cdots & a_{i_{d+1},1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1,d} & a_{i_2,d} & \cdots & a_{i_{d+1},d} \end{pmatrix}.$$  

Therefore, two $N$-element point sequences $P$ and $Q$ have the same order type if they are “combinatorially equivalent.” We say that a point sequence in $\mathbb{R}^d$ is order-type homogeneous, if all $(d+1)$-tuples have the same orientation. See [13] and [9] for more background on order types.

Order-type homogeneous point sets exhibit several fascinating combinatorial and algebraic properties. Recall that an $n$-vertex cyclic polytope in $\mathbb{R}^d$ is the convex hull of $n$ points on the moment curve $\gamma = \{(t, t^2, ..., t^d) : t \in \mathbb{R}\} \subset \mathbb{R}^d$. A well-known folklore states that a point sequence $P$ in $\mathbb{R}^d$ that is order-type homogeneous, forms the vertex set of a convex polytope which is combinatorially equivalent to the cyclic polytope in $\mathbb{R}^d$ (see [12] Exercise 5.4.3 and [2]). A classic result of McMullen [14] states that among all $d$-dimensional convex polytopes with $n$ vertices, the cyclic polytope maximizes the number of faces of each dimension.

Following the notation of Eliáš and Matoušek [3], we define $OT_d(n)$ to be the smallest integer $N$ such that any $N$-element point sequence in $\mathbb{R}^d$ in general position, contains an $n$-element subsequence that is order-type homogeneous. For $1$-dimension, an order-type homogeneous sequence in $\mathbb{R}$ is just an increasing or decreasing set of real numbers. Hence, Theorem 1.1 implies that $OT_1(n) \leq (n-1)^2 + 1$. On the other hand, a simple construction from [7] shows that $OT_1(n) = (n-1)^2 + 1$. For $2$-dimensions, an order-type homogeneous point sequence in $\mathbb{R}^2$ is a planar point set in convex position, which appear in either clockwise or counterclockwise order along the boundary of their convex hull (see [12]). By combining Theorems 1.1 and 1.2, one can
show\textsuperscript{2} that $OT_2(d) \leq 2^{O(n)}$. On the other hand, a famous construction of Erdős and Szekeres \textsuperscript{3} on point sets in the plane with no large convex subset, shows that $OT_2(n) = 2^{\Theta(n)}$.

For several decades, the best known upper bound on $OT_d(n)$ for fixed $d \geq 3$ was obtained by applying Ramsey numbers\textsuperscript{4} (See \cite{3,11,2} and \cite{10,4,5,6}). This implies

$$OT_d(n) \leq twr_{d+1}(O(n)),$$

where the tower function $twr_k(x)$ is defined by $twr_1(x) = x$ and $twr_{i+1} = 2^{twr_i(x)}$. Recently, Conlon et al. \cite{11} improved this upper bound to $twr_d(n^c_d)$, where $c_d$ is exponential in a power of $d$. Our main result establishes a further improvement on the upper bound of $OT_d(n)$.

**Theorem 1.3.** For fixed $d \geq 2$, we have $OT_d(n) \leq twr_d(O(n))$.

A recent result of Eliáš and Matoušek \cite{3} shows that $OT_3(n) \geq 2^{O(n)}$. Hence as an immediate corollary to Theorem 1.4, we have obtained a reasonably tight bound on $OT_3(n)$.

**Corollary 1.4.** In 3-dimensions, we have $OT_3(n) = 2^{2\Theta(n)}$.

## 2 Proof of Theorem 1.4

In this section, we will prove Theorem 1.4. First we will introduce some notions. For $p, q \in \mathbb{R}^d$ where $p = (a_1, ..., a_d)$ and $q = (b_1, ..., b_d)$, we say that $p$ lies above $q$ if $a_d > b_d$. Likewise, we say that $p$ lies below $q$ if $a_d < b_d$.

Recall the following lemma on arrangements of hyperplanes (see \cite{13}).

**Lemma 2.1.** The number of cells $(d$-faces$)$ in an arrangement of $m$ hyperplanes in $\mathbb{R}^d$ is at most $m^d$.

We now establish the following recursive formula for $OT_d(n)$.

**Lemma 2.2.** For $M = OT_{d-1}(n-1)$ and $d \geq 2$,

$$OT_d(n) \leq 2^{4d^2M \log M}.$$

**Proof.** Let $P = (p_1, ..., p_N)$ be a sequence of $N = 2^{4d^2M \log M}$ points in $\mathbb{R}^d$ in general position, and let $\chi : \binom{P}{d+1} \to \{+1, -1\}$ be the order type of $P$. In what follows, we will recursively construct a sequence of points $q_1, ..., q_r$ from $P$ and a subset $S_r \subset P$, where $r = d - 1, ..., 2M$, such that the following holds.

1. For $i < j$, $q_i$ comes before $q_j$ in the original ordering and every point in $S_r$ comes after $q_r$ in the original ordering.

2. Every $d$-tuple $(q_{i_1}, ..., q_{i_d}) \subset \{q_1, ..., q_r\}$ with $i_1 < i_2 < \cdots < i_d$ has the property that either $\chi(q_{i_1}, ..., q_{i_d}, q) = +1$ for every point $q \in \{q_j : i_d < j \leq r\} \cup S_r$, or $\chi(q_{i_1}, ..., q_{i_d}, q) = -1$ for every point $q \in \{q_j : i_d < j \leq r\} \cup S_r$.

\textsuperscript{2}We write $f(n) = O(g(n))$ if $|f(n)| \leq c|g(n)|$ for some fixed constant $c$ and for all $n \geq 1$; $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$; and $f(n) = \Theta(g(n))$ if both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ holds.

\textsuperscript{3}The Ramsey number $R_k(n)$ is the least integer $N$ such that every red-blue coloring of all unordered $k$-tuples of an $N$-element set contains either a red set of size $n$ or a blue set of size $n$, where a set is called red (blue) if all $k$-tuples from this set are red (blue).
We start by selecting the $d - 1$ points $\{q_1, ..., q_{d-1}\} = \{p_1, ..., p_{d-1}\}$ from $P$ and setting $S_{d-1} = P \setminus \{p_1, ..., p_{d-1}\}$. After obtaining $\{q_1, ..., q_r\}$ and $S_r$, we define $q_{r+1}$ and $S_{r+1}$ as follows. Let $q_{r+1}$ be the smallest indexed element in $S_r$.

In order to obtain (2), we only need to consider the $d$-tuples from $\{q_1, ..., q_r\}$ that include the last point $q_{r+1}$. Notice that each $(d - 1)$-tuple $T = \{q_{i_1}, q_{i_2}, ..., q_{i_{d-1}}\} \subset \{q_1, ..., q_r\}$ gives rise to a hyperplane spanned by the points $T \cup \{q_{r+1}\}$. Let $H_r$ be the set of these $\binom{r}{d-1}$ hyperplanes. By Lemma 2.2, the number of cells in the arrangement of $H_r$ is at most

$$\binom{r}{d-1} \leq r^d.$$ 

By the pigeonhole principle and since $P$ is in general position, there exists a cell $(d$-face) $\Delta \subset \mathbb{R}^d$ that contains at least $(|S_r| - 1)/r^d$ points of $S_r$. Hence, for any fixed $d$-tuple $(q_{i_1}, ..., q_{i_d}) \subset \{q_1, ..., q_{r+1}\}$, we have either

$$\chi(q_1, ..., q_d, p) = +1 \quad \forall p \in \Delta \cap S_r \setminus \{q_{r+1}\}$$

or

$$\chi(q_1, ..., q_d, p) = -1 \quad \forall p \in \Delta \cap S_r \setminus \{q_{r+1}\}.$$ 

Set $S_{r+1} = \Delta \cap S_r \setminus \{q_{r+1}\}$. Now (1) and (2) holds for $\{q_1, ..., q_{r+1}\}$ and $S_{r+1}$. In order to obtain (3), notice that we have the recursive formula

$$|S_{r+1}| \geq \frac{|S_r| - 1}{r^d}.$$ 

Substituting in the lower bound on $|S_r|$, we obtain the desired bound

$$|S_{r+1}| \geq \frac{N}{((r - 1)!)^d r^d} - (r + 1) = \frac{N}{(r!)^d} - (r + 1).$$

This shows that we can construct the sequence $q_1, ..., q_{r+1}$ and the set $S_{r+1}$ with the three desired properties. Since

$$|S_{2M}| \geq \frac{4^{d^2}M \log M}{((2M - 1)!)^d 2M} - 2M \geq 1,$$

this implies that the set $\{q_1, ..., q_{2M}\}$ is well defined for $M = \text{OT}_{d-1}(n - 1)$. By the pigeonhole principle, there exists a subset $Q \subset \{q_1, ..., q_{2M}\}$ such that $|Q| \geq M = \text{OT}_{d-1}(n - 1)$, and $Q$ lies either above or below the point $q_{2M}$. We will only consider the case when $Q$ lies below $q_{2M}$, since the other case is symmetric. We define the hyperplane $h = \{(x_1, ..., x_d) \in \mathbb{R}^d : x_d = c\}$, where $c$ is a constant such that $h$ separates $Q$ and $q_{2M}$. For each point $q_i \in Q$, let $q_i q_{2M}$ be the line in $\mathbb{R}^d$ containing points $q_i$ and $q_{2M}$. Then we define the map $\phi : Q \to Q^*$, where $\phi(q_i) = q_i q_{2M} \cap h$. With a slight perturbation of $Q$ if necessary, $Q^*$ is also in general position in $h = \mathbb{R}^{d-1}$, and the map $\phi$ is bijective. See Figure 2.
By definition of $\text{OT}_{d-1}(n-1)$, there exist $n-1$ points $\{q_1^*, ..., q_{n-1}^*\} \subset Q^*$ such that every $d$-tuple has the same orientation\footnote{With respect to the basis $e_1' = \langle 1, 0, ..., 0, c \rangle, e_2' = \langle 0, 1, 0, ..., 0 \rangle, \ldots, e_{d-1}' = \langle 0, 0, ..., 1, c \rangle$} in $h = \mathbb{R}^{d-1}$. This implies that $q_{2M}$ lies on the same side of each hyperplane spanned by $d$ points from $\{\phi^{-1}(q_1^*), \phi^{-1}(q_2^*), ..., \phi^{-1}(q_{n-1}^*)\}$ in $\mathbb{R}^d$. Hence every $(d+1)$-tuple of the form $\langle \phi^{-1}(q_{i_1}^*), ..., \phi^{-1}(q_{i_d}^*), q_{2M} \rangle$, $1 \leq i_1 < \cdots < i_d \leq n-1$, has the same orientation in $\mathbb{R}^d$. By property (2), every $(d+1)$-tuple in the $n$-element set $\{\phi^{-1}(q_1^*), ..., \phi^{-1}(q_{n-1}^*), q_{2M}\} \subset P$ has the same orientation.

Theorem 1.4 now follows by applying Lemma 2.2 with the fact that $\text{OT}_2(n) = 2^{O(n)}$.

3 Concluding remarks

Let us remark that a simple modification to the construction of Eliáš and Matoušek [3] shows that $\text{OT}_d(n) \geq 2^{2^\Theta(n)}$ for $d \geq 3$. The best known estimates on $\text{OT}_d(n)$ can be summarized in the following table.

| $d$-dimensions | best results | references |
|----------------|--------------|------------|
| $d = 1$        | $\text{OT}_1(n) = (n - 1)^2 + 1$ | Erdős and Szekeres [7] |
| $d = 2$        | $\text{OT}_2(n) = 2^{\Theta(n)}$ | Erdős and Szekeres [7] |
| $d = 3$        | $\text{OT}_3(n) = 2^{2^\Theta(n)}$ | Eliáš-Matoušek [3] and Theorem 1.4 |
| $d \geq 4$     | $2^{2^{\Omega(n)}} \leq \text{OT}_d(n) \leq \text{twr}_d(O(n))$ | Eliáš-Matoušek [3] and Theorem 1.4 |

Hence, there is a significant gap between the known upper and lower bounds on $\text{OT}_d(n)$ for $d \geq 4$. However, we believe that $\text{OT}_d(n)$ is on the order of $\text{twr}_d(\Theta(n))$.

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