POSITIVITY OF NLO SPIN-DEPENDENT PARTON DISTRIBUTIONS

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Abstract

We discuss the positivity of the hadron density matrix in QCD. This basic property is shown to be preserved by QCD evolution, provided the relation $|\Delta P_{ij}(z)| \leq P_{ij}$ is valid for all kernels for $z < 1$, and the usual ”+” prescription is used. We comment on the positivity restrictions for the choice of the NLO factorization scheme for the evolution of the spin-dependent parton distributions.

Key-Words : Parton distributions, positivity constraints

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The positivity of spin dependent parton densities is a basic property, which allows a self-consistent partonic interpretation.

The case of next-to-leading order (NLO) is especially interesting, as the parton distribution is no longer a directly observable quantity, moreover it depends on the choice of factorization scheme. For the spin-dependent case, there is also an extra ambiguity due to the choice of $\gamma_5$ prescription. Positivity may be considered as an extra constraint, restricting this ambiguity. The present paper is devoted to a first approach to this problem.

The current next-to-leading (NLO) parametrizations [1, 2] are chosen in such a way, that positivity for all helicity parton distributions is respected at the initial value $Q_0^2$, i.e.

$$|\Delta f(z, Q^2)| \leq f(z, Q^2).$$

One may wonder, to what extent the $Q^2$ evolution is compatible with positivity. The answer becomes clear when one interprets the QCD evolution as a kinetic flow in $x$-space.

In its standard form [1] (for the time being, we shall confine ourselves to the unpolarized non-singlet (NS) case)

$$\frac{dq(x)}{dt} = \frac{\alpha_s}{2\pi} \int_x^1 dy \frac{q(y)}{y} P\left(\frac{x}{y}\right),$$

may be interpreted as a ‘time’ $t = \ln Q^2$ evolution of the ‘particles’ density $q$ in the one dimensional space $0 \leq x \leq 1$ due to the flow from the right to the left, with the probability per unit time equal to the splitting kernel $P$. The key element in such an interpretation is the problem of the infrared (IR) singular terms in $P$, which was considered in detail some years ago [3] (see also [7]). The kinetic interpretation is preserved provided the ‘$+' form of the kernel is written in the following way

$$P_+(z) = P(z) - \delta(1-z) \int_0^1 P(y)dy,$$

leading to the corresponding expression for the evolution equation

$$\frac{dq(x)}{dt} = \frac{\alpha_s}{2\pi} \left[ \int_x^1 dy \frac{q(y)}{y} P\left(\frac{x}{y}\right) - q(x) \int_0^1 P(z)dz \right].$$

1For brevity, the argument $t$ will not be written down explicitly in the parton densities.
The negative second term in (3) cannot change the sign of the distribution because it is 'diagonal' in \(x\), which means that it is proportional to the function at the same point \(x\), as on the l.h.s.. Thus when the distribution gets too close to zero, its stops causing a decrease. This is true for both '+′ and \(\delta(1 - z)\) terms, for any value of their coefficient (if it is positive, it will reinforce the positivity of the distribution).

Let us consider now the spin-dependent case. For simplicity, we postpone the discussion of quark-gluon mixing for a moment, but allow the spin-dependent and spin-independent kernels to be different, as they are at NLO. It is most convenient to write down the equations for definite parton helicities, which was actually the starting point in deriving the equations for the spin-dependent quantities [4]. Although the form, which we shall use, mixes the contributions of different helicities, it makes the positivity properties especially clear. So we have

\[
\frac{dq_+}{dt} = \frac{\alpha_s}{2\pi} (P_{++}(\frac{x}{y}) \otimes q_+(y) + P_{+-}(\frac{x}{y}) \otimes q_-(y)),
\]

\[
\frac{dq_-}{dt} = \frac{\alpha_s}{2\pi} (P_{--}(\frac{x}{y}) \otimes q_+(y) + P_{+-}(\frac{x}{y}) \otimes q_-(y)).
\]

Here \(P_{++}(z) = (P(z) + \Delta P(z))/2, P_{+-}(z) = (P(z) - \Delta P(z))/2\) are the evolution kernels for definite helicities and we have used the fact that \(P_{++} = P_{--}\) and \(P_{+-} = P_{-+}\), and a shorthand notation for the convolution is adopted. As long as \(x < y\), the positivity of the initial distributions \((q_+(x, Q^2_0), q_-(x, Q^2_0)) \geq 0, \text{ or } |\Delta q(x, Q^2_0)| \leq q(x, Q^2_0))\) is preserved, if both kernels \(P_{++}, P_{+-}\) are positive, which is true, and if

\[
|\Delta P(z)| \leq P(z), \ z < 1.
\]

The singular terms at \(z = 1\) do not altering positivity, because they appear only in the diagonal (now in helicities) kernel \(P_{++}\) (only forward scattering is IR dangerous). From the kinetic interpretation again the distributions \(q_+, q_-\) stop decreasing, as soon as they are close to changing sign.

Now to extend the proof to quark gluon mixing is trivial. One should write down the
expressions for the evolutions of quark and gluon distributions of each helicity

\[
\frac{dq_+}{dt} = \frac{\alpha_s}{2\pi} (P_{qq}^{++} \frac{x}{y} \otimes q_+(y) + P_{qq}^{+-} \frac{x}{y} \otimes q_-(y)) \\
+ P_{gg}^{+g} \frac{x}{y} \otimes G_+(y) + P_{gg}^{-g} \frac{x}{y} \otimes G_-(y),
\]

\[
\frac{dq_-}{dt} = \frac{\alpha_s}{2\pi} (P_{qq}^{++} \frac{x}{y} \otimes q_+(y) + P_{qq}^{+-} \frac{x}{y} \otimes q_-(y)) \\
+ P_{gg}^{+g} \frac{x}{y} \otimes G_+(y) + P_{gg}^{-g} \frac{x}{y} \otimes G_-(y),
\]

\[
\frac{dG_+}{dt} = \frac{\alpha_s}{2\pi} (P_{Gq}^{++} \frac{x}{y} \otimes q_+(y) + P_{Gq}^{+-} \frac{x}{y} \otimes q_-(y)) \\
+ P_{GG}^{+G} \frac{x}{y} \otimes G_+(y) + P_{GG}^{-G} \frac{x}{y} \otimes G_-(y),
\]

\[
\frac{dG_-}{dt} = \frac{\alpha_s}{2\pi} (P_{Gq}^{++} \frac{x}{y} \otimes q_+(y) + P_{Gq}^{+-} \frac{x}{y} \otimes q_-(y)) \\
+ P_{GG}^{+G} \frac{x}{y} \otimes G_+(y) + P_{GG}^{-G} \frac{x}{y} \otimes G_-(y)).
\] (6)

If the inequality (5) was valid for each type of partons,

\[
|\Delta P_{ij}(z)| \leq P_{ij}(z), \ z < 1; \ i, j = q, G,
\] (7)

all the kernels, appearing in the r.h.s. of such a system, are positive. Concerning the singular terms, they are again diagonal, now in parton type, and do not affect positivity.

The validity of these equations in LO comes just from the way they were derived, since the (positive) helicity-dependent kernels were in fact calculated first in ref.[4].

But the situation in NLO is more peculiar. Even the spin-averaged quantities do not respect positivity. The most striking example is the gluonic kernel $P_{GG}$ which is negative at low $x$. This is, first of all, a signal, that the NLO contribution comes with a positive (and large) LO one. Moreover, for low $x$ resummation effects are important, coming from the most singular terms at all orders.

We performed a systematic comparison of polarized and unpolarized NLO singlet kernels (see Figure 1).

The result may be described as follows. For quark-quark kernels the inequalities (6) are valid for both singlet and nonsinglet combinations, as well as for gluon-gluon kernels for large $x$, where the positivity of unpolarized NLO kernel is satisfied.
However, the quark-gluon and gluon-quark kernels manifest tiny violations of (7) in the region of large $x \sim 0.7$. This violation is by no means of real importance for evolution, as it is completely screened by the LO contribution satisfying positivity.

At the same time, this violation is of some theoretical interest. Note that both these kernels contain only one $\gamma_5$ matrix and are sensitive to its definitions while performing $\varepsilon-$regularization. Also, the interplay of polarized and unpolarized kernels is very peculiar. Both contain logarithmic and polynomial terms in $x$, not matching each other and only (numerical) addition results in the small violation of inequality (7).

One should note, that the inequality analogous to (7) for the moments of the splitting kernels (anomalous dimensions) is, generally speaking, not sufficient for positivity, since a distribution with all moments positive may be negative for small $x$. Also, moments combine regular and singular at $x = 1$ terms, while only the first are essential for positivity. The case discussed about, however, is free of singular terms and deals with large $x$. Consequently, violation of positivity can be seen in the moments (due to the smallness of the violation, only the case of one of the two kernels is seen in the figures).

To conclude, we have found that NLO evolution (when LO and LO kernels are added) preserves positivity. The small violation for quark-gluon and gluon-quark kernels may be related to the definition of $\gamma_5$ matrix. This is under investigation.

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Figure 1: Unpolarized and polarized kernels as a function of $x$ including LO and NLO contributions.