TORSION GROWTH OF RATIONAL ELLIPTIC CURVES IN SEXTIC NUMBER FIELDS

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Abstract. We classify the possible torsion structures of rational elliptic curves over sextic number fields.

1. Introduction

Let $E/K$ be an elliptic curve defined over a number field $K$. The Mordell-Weil Theorem states that the set of $K$-rational points $E(K)$, is a finitely generated abelian group. Denote by $E(K)_{\text{tors}}$ the torsion subgroup of $E(K)$, which is isomorphic to $C_m \oplus C_n$ for two positive integers $m, n$, where $m$ divides $n$ and where $C_n$ is a cyclic group of order $n$.

For a positive integer $d$,

1. Let $\Phi(d)$ be the set of possible isomorphism classes of groups $E(K)_{\text{tors}}$, where $K$ runs through all number fields $K$ of degree $d$ and $E$ runs through all elliptic curves over $K$.
2. Let $\Phi_\infty(d) \subseteq \Phi(d)$ be the set of possible isomorphism classes of groups $E(K)_{\text{tors}}$, where $K$ runs through all number fields $K$ of degree $d$ and $E$ runs through all elliptic curves defined over $\mathbb{Q}$.
3. Let $\Phi^\infty(d)$ be the subset of isomorphism classes of groups $\Phi(d)$ that occur infinitely often. More precisely, a torsion group $G$ belongs to $\Phi^\infty(d)$ if there are infinitely many elliptic curves $E$, non-isomorphic over $\bar{\mathbb{Q}}$, such that $E(K)_{\text{tors}} \cong G$.
4. Let $R_\infty(d)$ be the set of all primes $p$ such that there exists a number field $K$ of degree $d$, an elliptic curve $E/\mathbb{Q}$ such that there exists a point of order $p$ on $E(K)$.
5. Let $\Phi^{\text{CM}}(d)$ be the set of possible isomorphism classes of groups $E(K)_{\text{tors}}$, where $K$ runs through all number fields $K$ of degree $d$ and $E$ runs through all elliptic curves with CM over $K$.

The set $\Phi(1)$ was determined by Mazur in [25]. Kenku, Momose and Kamienny determined $\Phi(2)$ in [17], [16]. For $d \geq 3$, the set $\Phi(3)$ has been determined by Derickx, Etropolski, Hoeij, Morrow and Zureick-Brown but the result is (at the time of writing this paper) unpublished. In [27], Merel proved that $\Phi(d)$ is finite for all positive integers $d$. It is known that $\Phi(1) = \Phi^\infty(1)$ and $\Phi(2) = \Phi^\infty(2)$.

The set $\Phi^\infty(d)$ has been determined for $d = 3$ by Jeon, Kim and Schweizer [14], for $d = 4$ by Jeon, Kim and Park [15] and for $d = 5, 6$ by Derickx and Sutherland [7].
In [29], Najman has determined the sets $\Phi_Q(2)$ and $\Phi_Q(3)$. $\Phi_Q(4)$ has been determined by Chou [3] and Najman and González-Jiménez [11]. Najman and González-Jiménez also determined $\Phi_Q(p)$, for $p \geq 7$ prime number in [11]. González-Jiménez has determined $\Phi_Q(5)$ in [8]. The set $R_Q(d)$, for $d \leq 3342296$ has been determined by Najman and González-Jiménez in [11]. $\Phi_{CM}(1)$ has been determined by Olson in [31] and $\Phi_{CM}(d)$ for $d = 3, 4$ by Zimmer and his collaborators. The sets $\Phi_{CM}(d)$, for $4 \leq d \leq 13$ have been determined by Clark, Corn, Rice and Stankiewicz in [4].

The main result of this paper is the following theorem.

Theorem 1.1. Let $E/Q$ be an elliptic curve and let $K$ be a sextic number field. Then

$$E(K)_{\text{tors}} \cong \begin{cases} C_m, & m = 1, \ldots, 16, 18, 21, 30, m \neq 11, \\ C_2 \oplus C_{2m}, & m = 1, \ldots, 7, 9, \\ C_3 \oplus C_{3m}, & m = 1, \ldots, 4, \\ C_4 \oplus C_{4m}, & m = 1, 3, \\ C_6 \oplus C_6, \\ C_3 \oplus C_{18}. \end{cases}$$

It is known that all groups mentioned in Theorem 1.1 except $C_3 \oplus C_{18}$ appear for some elliptic curve $E/Q$ and some sextic field $K$. We expect that the group $C_3 \oplus C_{18}$ never occurs in this situation but we’re unable to completely prove it. We give partial result regarding this group in the last theorem of this paper.

In order to prove Theorem 1.1, we heavily rely on results about possible images of Galois representations attached to $E$ by Zywina in [36] (see also [33]) and on possible values of $[Q(P) : Q]$ (classified by Najman and Gonzalez-Jiménez in [11]), where $P \in E(Q)$ is a point of prime order $p$. We will often have information on mod $p$ and mod $q$ Galois representations attached to $E$, where $p$ and $q$ are different prime numbers and we will clasify rational points on the corresponding modular curve.

In our computation, we used Magma [1]. Magma [1] code used in this paper is available on this paper’s arXiv webpage. Many functions we used were taken from Enrique González-Jiménez’s webpage.

2. Notation and auxiliary results

Theorem 2.1 (Mazur, [25]). Let $E/Q$ be an elliptic curve. Then

$$E(Q)_{\text{tors}} \cong \begin{cases} C_m, & m = 1, \ldots, 10, 12, \\ C_2 \oplus C_{2m}, & m = 1, \ldots, 4. \end{cases}$$

Theorem 2.2 (Kenku, Momose, [17], Kamienny [16]). Let $E/F$ be an elliptic curve over a quadratic number field $F$. Then

$$E(F)_{\text{tors}} \cong \begin{cases} C_m, & m = 1, \ldots, 16, 18, \\ C_2 \oplus C_{2m}, & m = 1, \ldots, 6, \\ C_3 \oplus C_{3m}, & m = 1, 2, \\ C_4 \oplus C_4. \end{cases}$$
Let \( E/F \) be an elliptic curve defined over a number field \( F \). There exists an \( F \)-rational cyclic isogeny \( \phi : E \to E' \) of degree \( n \) if and only if \( \ker \phi \) is a \( \text{Gal}(\bar{F}/F) \)-invariant cyclic group of order \( n \); in this case we say that \( E \) has an \( F \)-rational \( n \)-isogeny. When \( F = \mathbb{Q} \), possible degrees \( n \) of elliptic curves over \( \mathbb{Q} \) are known by the following theorem.

**Theorem 2.3** (Mazur [26], Kenku [18], [19], [20], [21]). Let \( E/\mathbb{Q} \) be an elliptic curve with a rational \( n \)-isogeny. Then

\[
\begin{align*}
n \in \{1, \ldots, 19, 21, 25, 27, 37, 43, 67, 163\}.
\end{align*}
\]

There are infinitely many elliptic curves (up to \( \bar{\mathbb{Q}} \)-isomorphism) with a rational \( n \)-isogeny over \( \mathbb{Q} \) for

\[
\begin{align*}
n \in \{1, \ldots, 10, 12, 13, 16, 18, 25\}
\end{align*}
\]

and only finitely many for all the other \( n \).

**Theorem 2.4** (Najman, [29], Theorem 2). Let \( E/\mathbb{Q} \) be an elliptic curve and \( F \) a quadratic field. Then

\[
E(F)_{\text{tors}} \cong \begin{cases}
C_m, & m = 1, \ldots, 10, 12, 15, 16, \\
C_2 \oplus C_{2m}, & m = 1, \ldots, 6, \\
C_3 \oplus C_{3m}, & m = 1, 2, \\
C_4 \oplus C_4.
\end{cases}
\]

**Theorem 2.5** (Najman, [29], Theorem 1). Let \( E/\mathbb{Q} \) be an elliptic curve and \( K \) a cubic field. Then

\[
E(K)_{\text{tors}} \cong \begin{cases}
C_m, & m = 1, \ldots, 10, 12, 13, 14, 18, 21, \\
C_2 \oplus C_{2m}, & m = 1, \ldots, 4, 7.
\end{cases}
\]

**Theorem 2.6** ([4]). Let \( E/K \) be an elliptic curve with CM and let \( K \) be a sextic number field. Then

\[
E(K)_{\text{tors}} \cong \begin{cases}
C_m, & m = 1, 2, 3, 4, 6, 7, 9, 10, 14, 18, 19, 26, \\
C_2 \oplus C_{2m}, & m = 2, 4, 6, 7, \\
C_3 \oplus C_{3m}, & m = 1, 2, 3, \\
C_6 \oplus C_6.
\end{cases}
\]

In [10] Table 1 it was shown that \( \Phi_\mathbb{Q}(6) \) contains the following groups:

\[
C_{30}, \quad C_2 \oplus C_{18}, \quad C_3 \oplus C_9, \quad C_3 \oplus C_{12}, \quad C_4 \oplus C_{12}, \quad C_6 \oplus C_6.
\]

Since \( \Phi_\mathbb{Q}(6) \supseteq \Phi_\mathbb{Q}(2), \Phi_\mathbb{Q}(3) \), we have

\[
\Phi_\mathbb{Q}(6) \supseteq \Phi_\mathbb{Q}(2) \cup \Phi_\mathbb{Q}(3) \cup \{C_{30}, C_2 \oplus C_{18}, C_3 \oplus C_9, C_3 \oplus C_{12}, C_4 \oplus C_{12}, C_6 \oplus C_6\}.
\]

**Lemma 2.7** ([11], Section 5). The set \( R_\mathbb{Q}(6) \) equals \( \{2, 3, 5, 7, 13\} \).

By the previous lemma, we only need to consider groups \( C_m \oplus C_n \) such that the prime factors of \( n \) are in \( R_\mathbb{Q}(6) \).
Galois representations attached to elliptic curves. Let $E/\mathbb{Q}$ be an elliptic curve and $n$ a positive integer. We denote by $E[n]$ the $n$-torsion subgroup of $E(\mathbb{Q})$. The field $\mathbb{Q}(E[n])$ is the number field obtained by adjoining to $\mathbb{Q}$ all the $x$ and $y$-coordinates of the points of $E[n]$. The absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $E[n]$ by its action on the coordinates of the points, inducing a mod $n$ Galois representation attached to $E$:

$$\rho_{E,n} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(E[n]).$$

After fixing a base for the $n$-torsion, we identify $\text{Aut}(E[n])$ with $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$. This means that we can consider $\rho_{E,n}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ as a subgroup of $\text{GL}_2(\mathbb{Z}/n\mathbb{Z})$, uniquely determined up to conjugacy. We shall denote $\rho_{E,n}(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ by $G_E(n)$. Moreover, since $\mathbb{Q}(E[n])$ is Galois extension of $\mathbb{Q}$ and $\ker \rho_{E,n} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(E[n]))$, by the first isomorphism theorem we have $G_E(n) \cong \text{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$.

Rouse and Zureick-Brown [32] have classified all the possible 2-adic images of $\rho_{E,2^n} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{Z}_2)$, and have given explicitly all the 1208 possibilities. We will use the same notation as in [32] for the 2-adic image of a given elliptic curve $E/\mathbb{Q}$. In [23], González-Jiménez and Lozano-Robledo have determined for each possible image the degree of the field of definition of any 2-subgroup. From the results of [9] one can see if a given 2-subgroup is defined over a number field of given degree $d$.

**Division polynomial method.** $E/\mathbb{Q}$ be an elliptic curve and $n$ a positive integer. We denote by $\psi_{E,n}$ the $n$-th division polynomial of $E$ (see [35, Section 3.2]). If $n$ is odd, then the roots of $\psi_{E,n}$ are precisely the $x$-coordinates of points $P \in E[n]$. Similarly, if $n$ is even, then the roots of $\psi_{E,n}/\psi_{E,2}$ are precisely the $x$-coordinates of points $P \in E[n] \setminus E[2]$. $f_{E,n}$ denote the corresponding primitive division polynomial associated to $E$, i.e. its roots are the $x$-coordinates of points $P$ on $E(\mathbb{Q})$ of exact order $n$. We briefly describe the construction of $f_{E,n}$. If $n = p$ is prime, then $f_{E,p} = \psi_{E,p}$. For an arbitrary $n$, we have

$$f_{E,n} = \frac{\psi_{E,n}}{\prod_{d | n, d \neq n} f_{E,d}}.$$

Note that if $E'/\mathbb{Q}$ is a quadratic twist of $E/\mathbb{Q}$, then $\psi_{E,n} = \alpha \psi_{E'/n}$ and $f_{E,n} = \beta f_{E'/n}$, for some rational constants $\alpha, \beta$. Consider the following problem:

**Given a rational number $j$ and $K$ a number field of degree $d$, does there exist an elliptic curve $E/\mathbb{Q}$ such that $j = j(E)$ can $E(K)$ contains a point $P$ of exact order $n$?**

$E_0/\mathbb{Q}$ be any elliptic curve with $j = j(E_0)$. In Magma [1], we compute primitive division polynomial $f_{E_0,n}$. Since every elliptic curve $E/\mathbb{Q}$ with $j(E) = j$ is a quadratic twist of $E_0$, we have $f_{E_0,n} = \beta f_{E,n}$, for some rational number $\beta$. Next, we factor $f_{E_0,n}$ over $\mathbb{Q}[x]$. $d'$ denote the degree of the smallest irreducible factor $f$ of $f_{E_0,n}$ and $x_0$ be a root of $f$. If $d' > d$, then $|\mathbb{Q}(P) : \mathbb{Q}| = |\mathbb{Q}(P) : \mathbb{Q}(x_0) : \mathbb{Q}| = d' > d = [K : \mathbb{Q}]$ and so a point $P$ of exact order $n$ on $E(\mathbb{Q})$ can’t be defined over $K$.

**Notation.** Specific elliptic curves mentioned in this paper will be referred to by
their LMFDB label and a link to the corresponding LMFDB page [22] will be included for the ease of the reader. Conjugacy classes of subgroups of GL₂(ℤ/pℤ) will be referred to by the labels introduced by Sutherland in [33].

3. Classification of \( \Phi_{Q}(6) \)

In this section, we prove Theorem 1.1. From now on, \( K \) denote degree 6 extension of \( Q \).

**Theorem 3.1.** \( E/Q \) be an elliptic curve with CM. Then \( E(K)_{\text{tors}} \) is one of the groups listed in Theorem 1.1.

**Proof.** By Theorem 2.6, we see that the only groups contained in \( \Phi^{\text{CM}}(6) \) that do not appear in

\[
\Phi_{Q}(2) \cup \Phi_{Q}(3) \cup \{C_{30}, C_{2} \oplus C_{18}, C_{3} \oplus C_{9}, C_{3} \oplus C_{12}, C_{4} \oplus C_{12}, C_{6} \oplus C_{6}\}
\]

are \( C_{19} \) and \( C_{26} \). By [5], Proposition 7(A), both of these groups can’t occur. \( \square \)

**Lemma 3.2.** [10] Lemma 2.6, Lemma 2.8, Lemma 2.9] \( E/Q \) be an elliptic curve without CM. Then the following claims hold:

- There are no points of order \( l^{2} \), where \( l \geq 11 \) on an elliptic curve \( E/Q \) over any number field of degree \( d < 55 \).
- There are no points of order 49 on an elliptic curve \( E/Q \) over any number field of degree \( d < 42 \).
- There are no points of order 81 on an elliptic curve \( E/Q \) over any number field of degree \( d < 81 \).

**Lemma 3.3.** [5] Lemma 5] \( E/Q \) be an elliptic curve without CM, \( K/Q \) a sextic field and \( P_{p} \in E(K)_{\text{tors}} \) a point of odd prime order \( p \). Then \( E \) has a rational \( p \)-isogeny, except if \( E \) has LMFDB label \( 2450.y1 \) or \( 2450.z1 \) and \( p = 7 \), where there are not rational 7-isogenies. Moreover, in those last cases, the unique sextic fields where the torsion grows are \( K = \mathbb{Q}(E[2]) \) and \( K' = \mathbb{Q}(P_{7}) \) \( (K'/Q \) is a non-Galois), where \( E(K)_{\text{tors}} \cong C_{2} \oplus C_{2} \) and \( E(K')_{\text{tors}} \cong C_{7} \) respectively.

**Lemma 3.4.** \([K:Q] = 6 \). If \( L, L' \) are cubic subextensions of \( K \) and if \( L/Q \) is Galois, then \( L = L' \).

**Proof.** Assume that \( L \neq L' \). Obviously \( L \cap L' = Q \) and \( LL' = K \). By Galois theory we have \( \text{Gal}(K/L') \cong \text{Gal}(LL'/L') \cong \text{Gal}(L/Q) \). Since \( |\text{Gal}(K/L')| = 2 \) and \( |\text{Gal}(L/Q)| = 3 \), we arrive at the contradiction. \( \square \)

**Theorem 3.5.** \( E/Q \) be an elliptic curve without CM. Then \( E(K)_{\text{tors}} \) can’t contain \( C_{169}, C_{49}, C_{39}, C_{65}, C_{91}, C_{35}, C_{28}, C_{20}, C_{26}, C_{63}, C_{32}, C_{25}, C_{45}, C_{27}, C_{24} \).

**Proof.** If \( E \) has LMFDB label \( 2450.y1 \) or \( 2450.z1 \) this holds by Lemma 3.3. Suppose this isn’t the case. By [3, 2], \( E(K) \) can’t contain \( C_{169} \) and \( C_{49} \). By Lemma 3.3 if \( E(K) \) contains points \( P_{p}, P_{q} \) of odd prime orders \( p \) and \( q \), \( p \neq q \), then \( E(K) \) has a rational \( p \)- and \( q \)-isogenies, so it has a rational \( pq \)-isogeny. When \( pq \in \{39, 65, 91, 35\} \), this cannot happen, because of Theorem 2.3 and so \( E(K) \) can’t contain \( C_{39}, C_{65}, C_{91} \) or \( C_{35} \). In [5], Proposition 6.5), \( k \), it has been proven that \( E(K) \) can’t contain \( C_{28}, C_{20} \) or \( C_{26} \), respectively.

From Lemma 3.2 we conclude that \( E \) has rational 3 and 7 isogenies, so it has a rational 21-isogeny, so \( j(E) \in \{-3^{2} \cdot 5^{6}/2^{3}, 3^{3} \cdot 5^{3}/2, 3^{3} \cdot 5^{3} \cdot 101^{3}/2^{21}, -3^{3} \cdot 5^{3}\}. \)
By Lemma 3.3, \(G\) divides 128. Consequently, \(Q\) must reduce to \(\mathbb{F}_8\) when \(P\) is defined over \(\mathbb{Q}\). Hence, a point \(P\) of order 63 (resp. 42) can’t be defined over \(K\).

By Lemma 3.3, \(E\) has a rational 5-isogeny. By [11] Table 2, we see that \(G_E(5) \in \{5C_s.1.1, 5C_s.1.3, 5C_s.1.4, 5B_s.1.1, 5B_s.1.4, 5B_s.4.1\}\). For each of these possibilities of \(G_E(5)\), we find all subgroups \(G\) of \(\text{GL}_2(\mathbb{Z}/25\mathbb{Z})\) with surjective determinant that reduce to \(G_E(5)\) modulo 5. Then for each vector \(v \in (\mathbb{Z}/25\mathbb{Z})^2\) of order 25 we calculate the index of \(G\) in \(G_e\), where \(G_e\) is stabiliser subgroup corresponding to vector \(v\). By Theorem 2.3, Theorem 2.4 and Theorem 2.5 we have that \([Q(P_{25}) : Q] \notin \{1, 2, 3\}\), so we have \([Q(P_{25}) : Q] = 6\). This means that \([G : G_e] = 6\). Computation in Magma [1] shows that this does not occur. Therefore, \(E\) cannot have a point \(P_{25}\) defined over \(K\).

Since \(E\) has rational 3 and 5 isogenies by Lemma 3.2, \(E\) has a rational 15-isogeny, so \(j(E) \in \{-5^2 \cdot 2, -5^2 \cdot 241^3 / 2^3, -29^3 \cdot 5 / 2^5, 211^3 \cdot 5 / 2^{15^3}\}\). Using exactly the same method as in the \(C_{63}\) and \(C_{42}\) case, we find that a point of order 45 can’t be defined over \(K\).

\(P_{27}\) be a point of order 27 in \(E(K)\) and \(P_{81}\) be a point of order 81 in \(E(\mathbb{Q})\) such that \(3P_{81} = P_{27}\). From [11] Proposition 4.6, we have \([Q(P_{81}) : Q(P_{27})] \leq 9\). Since \([Q(P_{27}) : Q] \leq 6\), we have \([Q(P_{81}) : Q] = [Q(P_{81}) : Q(P_{27})][Q(P_{27}) : Q] \leq 54\). This contradicts Lemma 3.2.

By Lemma 3.3, \(\rho_{E,3}\) is not surjective. \(\rho_{E,8}\) can’t be surjective because a point \(P_{k}\) of order 8 on \(E(\mathbb{Q})\) would satisfy \([Q(P_{k}) : Q] > 6\). By [28] Theorem A (3), we have that \(G_E(8) \subseteq H\), for \(H \in \{H_{30}, H_{31}, H_{39}, H_{45}, H_{47}, H_{50}\}\). Each of these six groups has an order equal to 128. This means that \([Q(E[8]) : Q] = 2^k\) of \(2\) that divides 128. Consequently, \([Q(E[2]) : Q] = 2^k\) of \(2\). Hence, each 2-torsion point on \(E\) is defined over at most quadratic extension of \(Q\). Since \(2^k\)-torsion grows in extensions of degree 1, 2 or 4 ([11] Proposition 4.8)) and since \(E(K) \supseteq C_8\), we need to have point of order 8 on \(E\) defined over at most quadratic extension of \(Q\). Since \(E\) has a rational 3-isogeny, by [11] Table 1 we see that \(E\) must have a point \(P_{3}\) of order 3 such that \([Q(P_{3}) : Q] \notin \{1, 2\}\). Therefore a point \(P_{k} + P_{3}\) of order 24 on \(E\) is defined over a field \(F = Q(P_{3}, P_{k})\) such that \(\text{Gal}(F/Q) \in \{C_1, C_2, C_2 + C_2\}\). This is impossible because of Theorem 2.4, Theorem 2.7 and 3. Theorem 1.4].

**Theorem 3.6.** \(E/\mathbb{Q}\) be an elliptic curve without CM. Then \(E(K)_{tors}\) can’t contain \(C_3 + C_15, C_3 + C_{21}, C_{21}, C_2 + C_{16}, C_4 + C_8, C_6 + C_{12}, C_7 + C_7\) or \(C_9 + C_9\).

**Proof.**

Since \(Q(E[3]) \subseteq K\), we need to have \(G_E(3) \in \{3C_s.1.1, 3B_s.1.1, 3B_s.1.2\}\). If \(G_E(3) = 3C_s.1.1\), Lemma 3.3 gives us that if \(p \in \{5, 7\}\), then \(E\) has a rational \(p\)-isogeny. \(P_{p} \in E(K)\) be a point of order \(p\). \(\{P_{3}, Q_{3}\}\) be a basis for \(E[3]\) such that \(G_E(3) = 3C_s.1.1\) with respect to this basis. This means that \(<P_{3}>\) and \(<Q_{3}>\) are kernels of two independent rational 3-isogenies. Now \(<P_{3}> + P_{3}>\) and \(<Q_{3}>\) are kernels of independent rational 3\(p\) and 3 isogenies, respectively. We conclude that \(E\) is isogenous to \(E'/\mathbb{Q}\) with a rational 9\(p\)-isogeny,
which is impossible by Theorem 2.3. Therefore $G_E(3) \in \{3B.1.1, 3B.1.2\}$. Assume that $C_5 \oplus C_{15} \subseteq E(K)$. We have that $K = \mathbb{Q}(E[3])$ and $\text{Gal}(K/\mathbb{Q}) \cong S_3$. $P_5$ be a point of order 5 in $E(K)$. From [11] Table 1 we see that $[\mathbb{Q}(P_5) : \mathbb{Q}] \in \{1, 2\}$. Denote by $F$ a unique quadratic subextension of $K$. For any possibility for $G_E(3)$, we have that there exists a point $P_3$ of order 3 in $E(K)$ defined over quadratic extension of $\mathbb{Q}$. Therefore we have $\mathbb{Q}(P_3) \subseteq F$, so $E(F) \supseteq C_{15}$. By [29] Theorem 2.c), LMFDB label of $E$ is 50.b3 50.b4 50.a2 or 150.g4 None of these four curves have $C_3 \oplus C_{15}$ torsion over sextic field, by [10].

Assume that $C_3 \oplus C_{21} \subseteq E(K)$. We have that $K = \mathbb{Q}(E[3])$ and $\text{Gal}(K/\mathbb{Q}) \cong S_3$. $P_7$ be a point of order 7 in $E(K)$. If $[\mathbb{Q}(P_7) : \mathbb{Q}] \in \{3, 6\}$, then by [11] Table 1 $\mathbb{Q}(P_7)$ is cyclic Galois over $\mathbb{Q}$. But $K$ is not cyclic and it does not have any Galois cubic subextensions. We conclude that $[\mathbb{Q}(P_7) : \mathbb{Q}] \in \{1, 2\}$. $P_3$ denote a point of order 3 in $E(K)$ defined over at most quadratic extension of $\mathbb{Q}$. Therefore, $\mathbb{Q}(P_3), \mathbb{Q}(P_3) \subseteq F$ so $E(F) \supseteq C_{21}$, but this is impossible, by Theorem 2.3.

$C_{32}, C_2 \oplus C_{16}, C_4 \oplus C_8$. By [23] Corollary 3.5, we get that if $T$ is one of these three groups, then $[\mathbb{Q}(T) : \mathbb{Q}]$ must be divisible by 4, which is impossible since $\mathbb{Q}(T) \subseteq K$.

$C_6 \oplus C_{12}$ If $G_E(3) \in \{3B.1.1, 3B.1.2\}$, we have $K = \mathbb{Q}(E[3])$, $K$ is an $S_3$ extension of $\mathbb{Q}$ and $j(E) = \frac{27(9+4y^2)^2}{(x+y)^2}$, for some $y \in \mathbb{Q}^\times$.

If $G_E(2) = 2C_3$, then $\mathbb{Q}(E[2])$ is cubic Galois over $\mathbb{Q}$ contained in $K$, which is impossible since $K$ is an $S_3$ extension of $\mathbb{Q}$.

If $G_E(2) = \text{GL}_2(\mathbb{F}_2)$, examining the results of [9] we find that if $E$ attains a point of order 4 over sextic field and $G_E(2) = \text{GL}_2(\mathbb{F}_2)$, then the image of 2-adic representation associated to $E$ is contained in $H_{20}$, so $j(E) = \frac{(x^5+3)^2}{(x+1)^2}$.

Taking the fiber product of $X_0(3)$ and $X_20$ we get a singular genus 1 curve $C$ whose normalization is the elliptic curve $E'/\mathbb{Q}$ with LMFDB label 48.a3. Inspecting the rational points on $C$ we get that there are 4 non-singular non-cuspidal points corresponding to the $j$-invariants 109503/64 and $-35937/4$. Additionally, since $\mathbb{Q}(E[2]) \subseteq \mathbb{Q}(E[3])$, by [2] Remark 1.5, we have $j(E) = \frac{2^{10}3^3y^3(1-4y^4)}{(x^2+32x+44)}$. For $a \in \{109503/64, -35937/4\}$ we find that $2^{10}3^3y^3(1-4y^4) - a = 0$ has no rational solutions. Therefore, this case can’t occur.

Consider the case $G_E(2) \in \{2B, 2Cs\}$. There is a unique quadratic extension contained in $K = \mathbb{Q}(E[3])$, namely $\mathbb{Q}(\zeta_3)$ and we have $\mathbb{Q}(E[2]) \subseteq \mathbb{Q}(\zeta_3)$. Since every point $P_2$ of order 2 on $E$ is defined over at most quadratic extension of $\mathbb{Q}$, by [11] Proposition 4.8 we have that a point $P_4$ of order 4 on $E(K)$ satisfies $[\mathbb{Q}(P_4) : \mathbb{Q}] \in \{1, 2\}$. It follows that $C_2 \oplus C_4 \subseteq E(\mathbb{Q}(E[2])) \subseteq E(\mathbb{Q}(\zeta_3))$. Since $G_E(3) \in \{3B.1.1, 3B.1.2\}$, by [11] Table 1 we see that there must exist a point $P_3$ in $E(K)$ such that $[\mathbb{Q}(P_3) : \mathbb{Q}] \in \{1, 2\}$, so $P_3$ is defined over $\mathbb{Q}(E[2]) \subseteq \mathbb{Q}(\zeta_3)$. Finally, we have that $C_2 \oplus C_{12} \subseteq E(\mathbb{Q}(\zeta_3))$, but this is impossible by [30] Theorem 1, iii].

If $G_E(3) = 3C_{1.1} \subseteq 3Ns$, so $j(E) = y^3$, by [36] Theorem 1.1. If $G_E(2) = \text{GL}_2(\mathbb{F}_2)$, then again we get that 2-adic representation associated to $E$ is contained in $H_{20}$. We have that $y^3 = \frac{(x^2-3)^2(-4x^2+32x+44)}{(x+1)^3}$ induces a genus 2 hyperelliptic
curve $C$. In Magma [1], we compute it’s Jacobian $J(C)$ and see that it has rank 0 over $\mathbb{Q}$. Using Chabauty method implemented in Magma [1] we conclude that it does not have affine rational points.

If $G_E(2) = 2C_n$, $j(E) = x^2 + 1728$ and the corresponding fiber product $X_{2C_n \times X_{10}} \times 3N$ is birational to $y^3 = x^2 + 1728$, which is an elliptic curve $E'/\mathbb{Q}$ with LMFDB label 36.a.3. Rational point on $E'$ corresponds to the $j$-invariant 1728, so $E$ would have CM, which contradicts our assumption.

Consider the case $G_E(2) \in \{2B, 2C\}$. Using exactly the same reasoning as before, we conclude that $C_2 \times C_4 \subseteq E(\mathbb{Q}(2))$ and $\mathbb{Q}(E[2])$ is at most quadratic over $\mathbb{Q}$. On the other hand, since $G_E(3) = 3C_{1.1}$, $\mathbb{Q}(E[3])$ is quadratic over $\mathbb{Q}$. Therefore, the composite field $L := \mathbb{Q}(E[2])\mathbb{Q}(E[3])$ is either a $C_2 \times C_2$ or $C_2$ extension of $\mathbb{Q}$ and we have $C_6 \times C_12 \subseteq E(L)$. But this is impossible by [3] Theorem 1.4.]

$C_7 \oplus C_7$. Since $\mathbb{Q}(E[7]) \subseteq K$ we have $|G_E(7)| \leq 6$, but looking at the possible mod 7 images in [1] Table 1 we see that $|G_E(7)| \geq 18$, a contradiction.

$C_9 \oplus C_9$. Since $\mathbb{Q}(E[9]) \subseteq K$ we have $|G_E(9)| = 6$, because otherwise we would have $C_9 \oplus C_9 \in \Phi_9(3)$ or $C_9 \oplus C_9 \in \Phi_9(2)$, which is not true by [29]. Using Magma [1], we find all subgroups $G$ of $\text{GL}_2(\mathbb{Z}/9\mathbb{Z})$ of order 6 such that $det(G) = (\mathbb{Z}/9\mathbb{Z})^\times$. All such groups $G$ are (up to conjugacy) subgroups of the group of upper triangular matrices, so $E$ has a rational 9-isogeny. Additionally, all such groups $G$ reduce modulo 3 to $3C_{1.1}$ (up to conjugacy), which implies that $E$ has two independent rational 3-isogenies. Therefore $E$ has independent rational 9 and 3-isogenies, so it’s isogenous over $\mathbb{Q}$ to $E'/\mathbb{Q}$ with a rational 27-isogeny. It follows that $E'$ has CM and so does $E$, which is a contradiction to our assumption.

Definition 3.7. [6] Definition 3.1] We say that a finite group $G$ is of generalized $S_3$-type if it is isomorphic to a subgroup of a direct product $S_3 \times S_3 \times \ldots \times S_3$.

Theorem 3.8. [6] Lemma 3.2, Corollary 3.4] A finite group $G$ is of Generalized $S_3$-type if and only if

- $G$ is supersolvable
- Sylow subgroups of $G$ are Abelian
- Exponent of $G$ divides 6.

Additionally, if $G$ is of Generalized $S_3$-type, then every subgroup and every quotient group of $G$ is also of Generalized $S_3$-type. If $G_1$ and $G_2$ is of Generalized $S_3$-type, then so is $G_1 \times G_2$.

Theorem 3.9. [6] Theorem 3.5, Theorem 3.6] $L$ be a number field such that $\text{Gal}(\hat{L}/\mathbb{Q})$ is of Generalized $S_3$-type. Then $L \subseteq \hat{L} \subseteq \mathbb{Q}(3^\infty)$. $L$ be a number field in $\mathbb{Q}(3^\infty)$. Then $L \subseteq \mathbb{Q}(3^\infty)$ and $\text{Gal}(\hat{L}/\mathbb{Q})$ is of Generalized $S_3$-type.

It’s easy to see that the groups $S_3$, $C_2$ and $C_3$ are of Generalized $S_3$-type and so are their direct products, $S_3 \times C_2$, $S_3 \times C_3$, $C_2 \times C_3 \cong C_6$.

Theorem 3.10. $E/\mathbb{Q}$ be an elliptic curve without CM. Then $E(K)_{\text{tors}}$ can’t contain $C_{36}$ or $C_2 \oplus C_{30}$.

Proof. $C_{36}, P_9, P_4$ be points of order 9 and 4, such that $[\mathbb{Q}(P_9 + P_4) : \mathbb{Q}] = 6$. If $[\mathbb{Q}(P_9) : \mathbb{Q}] \in \{1, 2, 3\}$, then we have $\mathbb{Q}(P_9) \subseteq \mathbb{Q}(3^\infty)$, since every quadratic
and cubic extension is contained in \(\mathbb{Q}(3^\infty)\). If \([\mathbb{Q}(P_3) : \mathbb{Q}] = 6\), we check using \textbf{Magma} [1] that \(\text{Gal}(\mathbb{Q}(P_3)/\mathbb{Q})\) is isomorphic to one of the following groups: \(C_6, S_3, S_3 \times C_3, S_3 \times C_2\). All these groups are of generalized \(S_3\)-type, so it follows that \(\mathbb{Q}(P_3) \subseteq \mathbb{Q}(3^\infty)\). Similarly, a point \(P_4\) can be defined over extensions of degree 1, 2, 3 or 6. If \([\mathbb{Q}(P_4) : \mathbb{Q}] \in \{1, 2, 3\}\), then we have \(\mathbb{Q}(P_4) \subseteq \mathbb{Q}(3^\infty)\). If \([\mathbb{Q}(P_4) : \mathbb{Q}] = 6\), by the results in [1] we see that if \([\mathbb{Q}(P_4) : \mathbb{Q}] = 6\), then \(\mathbb{Q}(P_4)\) is an \(S_3\) extension of \(\mathbb{Q}\), hence of generalized \(S_3\)-type. We conclude that in any case we have \(\mathbb{Q}(P_4) \subseteq \mathbb{Q}(3^\infty)\). We can now conclude that \(\mathbb{Q}(P_3 + P_4) = \mathbb{Q}(P_3, P_4) \subseteq \mathbb{Q}(3^\infty)\), which is impossible by [6] Theorem 1.8.].

\(C_2 \oplus C_{30}\) By Lemma 3.3 \(E\) has a rational 3 and 5-isogenies, so it has a rational 15-isogeny. If \(E(\mathbb{Q})[2] \supseteq C_2\), then \(E\) has a rational 30-isogeny, which is impossible by Theorem 2.3. If \(G_E(2) = 2Cn\), then \(j(E) = y^2 + 1728\) and since \(E\) has 15-isogeny we \(j(E) \in \{-5^2/2, -5^2 \cdot 21^3/2^3, -29^3 \cdot 5/2^6, 21^3 \cdot 5/2^{15}\}\). \(a\) be one of those 4 values. We have that \(a < 1728\), so \(y^2 + 1728 = a\) does not have a solution in real numbers, so this case is impossible. Consider now the case when \(G_E(2) = \text{GL}_2(\mathbb{F}_2)\). This means that \(E\) attains it’s full 2-torsion over degree 6 extension of \(\mathbb{Q}\). Since \(E(K)[2] = C_2 \oplus C_2\), we have \(K = \mathbb{Q}(E[2]) \subseteq \mathbb{Q}(3^\infty)\), because the Galois group of \(\mathbb{Q}(E[2])\) is of generalized \(S_3\)-type. Therefore, \(C_2 \oplus C_{30} \subseteq E(K) \subseteq E(3^\infty)\). By [6] Theorem 1.8., Table 1 we see that \(j(E) \in \{-121945, 46969655, 32708\}\). For each of these two possibilities, using division polynomial method we calculate a polynomial \(f_{30}\) whose roots are \(j(E)\) of exact order 30. If \(j(E) = -121945\), the smallest irreducible factors of \(f_{30}\) are degree 6. Since \(C_{30} \subseteq E(K) = E(\mathbb{Q}(E[2]))\), one of those polynomials needs to have a root in \(K\), but since \(K\) is Galois, it splits in \(K\). But we check using \textbf{Magma} [1] that the splitting fields of \(f\) and \(g\) are degree 12-extensions of \(\mathbb{Q}\), which is a contradiction. If \(j(E) = -121945\), we do the same as in the previous case. This time, the polynomial \(f_{30}\) does not have irreducible factors of degree \(\leq 6\), so \(C_{30} \not\subseteq E(K)\).

\textbf{Theorem 3.11.} \(E/\mathbb{Q}\) be an elliptic curve without CM. If the 2-adic representation of \(E\) does not equal to \(2B\), then \(E(K)_{\text{tors}}\) can’t contain \(C_3 \oplus C_{18}\).

\textbf{Proof.} We will split the proof into two main cases, depending on \(G_E(3)\).

\(G_E(3) \in \{3B.1.1, 3B.1.2\}\)

We have \(K = \mathbb{Q}(E[3])\) and \(K\) is \(S_3\) extension of \(\mathbb{Q}\).

If \(G_E(2) = 2Cn\), this is shown to be impossible by [5] Proposition 6(m).

If \(G_E(2) = 2Cn\), then \(\mathbb{Q}(E[2])\) is cubic Galois over \(\mathbb{Q}\) contained in \(K\), which is impossible since \(K\) is an \(S_3\) extension of \(\mathbb{Q}\).

Assume first that \(E\) has a rational 9-isogeny. \(G_E(2) = \text{GL}_2(\mathbb{F}_2)\). Since \(K\) is Galois and \(E\) has a point of order 2 in \(K\) and the defining cubic polynomial \(f(x)\) of \(E\) is irreducible and has a root in \(K\), it splits in \(K\). Therefore we have \(K = \mathbb{Q}(E[2]) = \mathbb{Q}(E[3])\). Since \(E\) has a rational 9-isogeny, by [13] Appendix, we have that \(E\) is a twist of elliptic curve

\[E_t : y^2 = x^3 - 3t(t^4 - 24)x + 2(t^6 - 36t^3 + 216),\]

where \(t \in \mathbb{Q} \setminus \{3\}\). We have \(j(E_t) = \frac{3(t^3 - 24)^3}{t^4 - 27}\) and \(\Delta(E_t) = 2^{12}3^6(t^3 - 27)\). It can be easily seen that for \(t \in \{-6, 0\}\) \(E_t\) has CM. Since \(E\) is a twist of some \(E_t\), we
have \( \Delta(E) = u^{12}\Delta(E_1) \), for some \( u \in \mathbb{Q} \). The Weil pairing implies that \( \mathbb{Q}(\zeta_3) \subseteq K \) and since \( K \) is an \( S_3 \) extension of \( \mathbb{Q} \), we conclude that \( \text{Gal}(K/\mathbb{Q}(\zeta_3)) \cong C_3 \), which implies that the discriminant of \( E \) is a square in \( \mathbb{Q}(\zeta_3) \), which is equivalent to

\[
C : y^2 = t^3 - 27, t \in \mathbb{Q} \setminus \{-6, 0, 3\}, y \in \mathbb{Q}(\zeta_3)
\]

having a solution. Using Magma [1] we find that such a solution does not exist. Therefore, in this case there does not exist an elliptic curve with \( C_3 \oplus C_{18} \) torsion defined over \( \mathbb{Q} \).

Assume now that \( E \) does not have a rational 9-isogeny. Obviously, \( E(\mathbb{Q}(3^{\infty})) \) contains a point of order 9, since \( E(\mathbb{Q}(3^{\infty})) \supseteq E(K) \). By [6, Lemma 6.13], we get that \( j(E) = \frac{(x+3)(x^2-3x+9)(x^2+3)^3}{x^3} \).

If \( G_E(2) = 2B \), we have that \( j(E) = \frac{256y(y+1)^3}{y} \). The induced modular curve \( C \) is genus 2 hyperelliptic curve. Computation in Magma [1] shows that points on \( C \) do not correspond to elliptic curves with \( C_3 \oplus C_{18} \) torsion over sextic fields.

If \( G_E(2) = \text{GL}_2(\mathbb{F}_2) \), since \( f(x) \) is irreducible and it has a root in \( K \), it splits in \( K \), since \( K \) is Galois. Therefore, we have \( \mathbb{Q}(E[2]) = \mathbb{Q}(E[3]) = K \). By [2, Remark 1.5], we have \( j(E) = 2^{10}3^{3}y^3(1-4y^3) \), for some \( y \in \mathbb{Q} \). The induced modular curve is an elliptic curve with \( C_3 \) torsion and rank 0 over \( \mathbb{Q} \). None of the points on this curve correspond to elliptic curves with \( C_3 \oplus C_{18} \) torsion over sextic field, which is checked using Magma [1].

\[
G_E(3) = 3\text{Cs.1.1}
\]

Since \( 3\text{Cs.1.1} \subseteq 3\text{Ns} \), we have \( j(E) = y^3 \).

If \( G_E(2) \in \{2\text{Cn}, 2\text{Cs}\} \), this has already been shown to be impossible by Theorem 3.6, in \( C_6 \oplus C_{12} \) case.

If \( G_E(2) = \text{GL}_2(\mathbb{F}_2) \), then we either have \( K = \mathbb{Q}(E[2]) \) or \( K = \mathbb{Q}(\zeta_3) \), where \( L \) is degree 3 extension of \( \mathbb{Q} \) contained in \( \mathbb{Q}(E[2]) \). Obviously, \( E(L)[2] = C_2 \).

If \( K = \mathbb{Q}(\zeta_3) \), then \( \text{Gal}(K/\mathbb{Q}) \cong S_3 \times C_2 \). Using Magma [1] we find that if \( P_9 \) is a point of order 9 defined on \( K \) and \( G_E(3) = 3\text{Cs.1.1} \), then we can’t have \( [\mathbb{Q}(P_9) : \mathbb{Q}] = 6 \), because a Galois closure of \( \mathbb{Q}(P_9) \) over \( \mathbb{Q} \) is one of the following groups: \( C_6, S_3, S_3 \times C_3 \).

Since \( K \) contains only two subfields, \( \mathbb{Q}(\zeta_3) \) and \( L \) (which we check using Magma [1]), we either have \( \mathbb{Q}(P_9) \subseteq \mathbb{Q}(\zeta_3) \), in which case \( E(\mathbb{Q}(\zeta_3)) \supseteq C_3 \times C_5 \) (which is impossible by Theorem 2.4) or \( \mathbb{Q}(P_9) = L \). This means that \( E(\mathbb{Q}) = C_3 \) and \( E(L) = C_{18} \), but this is impossible by [12, Theorem 2]. Therefore, we need to have \( K = \mathbb{Q}(E[2]) \). Since \( \mathbb{Q}(\zeta_3) \subseteq \mathbb{Q}(E[2]) = K \), we need to have \( \mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3}) \). From this equality it follows that \( \sqrt{\Delta} = \alpha + \beta\sqrt{-3} \), for some rational \( \alpha, \beta \). It’s easy to see that \( \alpha = 0 \), so \( \Delta = -3\beta^2 \).

Since \( G_E(3) = 3\text{Cs.1.1} \), by [30, Theorem 1.2], we have that \( E \) is isomorphic to \( y^2 = x^3 - 3(t+1)(t+3)(t^2 + 3)x - 2(t^2 - 3)(t^4 + 6t^3 + 18t^2 + 18t + 9) = x^3 + ax + b \), for some \( t \in \mathbb{Q} \) or a quadratic twist by \(-3\) of such curve. Since twisting does not change 2-division field, we have that \( \Delta = 4a^3 + 27b^2 = 4(-3(t^2 + 3)(t^4 + 3))^3 + 27(2(t^2 - 3)(t^4 + 6t^3 + 18t^2 + 18t)) = (t^2 + 3t + 3)^3 = -3\beta^2 \).

Plugging in \( t = -3t_1 \) and \( \beta = \frac{t_2}{t_1} \) in \( (t^2 + 3t + 3)^3 = -3\beta^2 \) we obtain \( (t_1^2 - 3t_1 + 27)^3 = \beta_1^2 \). Therefore, \( t_1(t_1^2 - 9t_1 + 27) \) must be a square, so \( t_1(t_1^2 - 9t_1 + 27) = \beta_2^2 \), where \( \beta_2^6 = \beta_1^2 \). Finally, put \( t_2 = t_1 - 3 \) to obtain \( t_2^2 + 27 = \beta_2^2 \), which is an elliptic curve \( E' \) with LMFDB label [144.a4] and the only non trivial rational point on \( E' \).
is $(-3,0)$. We have that $\beta_2 = 0$ and so $\beta = 0$, but this is impossible, because $0 \neq \Delta = -3\beta^2$.

We address the issue that occurs in the case $G_E(2) = 2B$. Assume for convenience that $G_E(2) = 3Cs.1.1$. Using Magma [1], we search for possible mod 9 images of $E$ such that $E$ has a point of order 9 defined over sextic number field. For each possibility for $G_E(9)$, we find that it is contained in one of the groups with labels $9H^0 - 9a$, $9H^0 - 9b$ or $9H^0 - 9c$ from [34, Table 1]. The modular curve induced by combining $j$-maps of one of these groups (also available in [34, Table 1]), along with $j$-map of elliptic curve $E$ with $G_E(2) = 2B$, we get a few genus 3 and 4 curves that are not hyperelliptic and which do not have a a nice quotient curve.

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References

1. Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput., 24 (1997), 235265.
2. J. Brau and N. Jones, Elliptic curves with 2-torsion contained in the 3-torsion field, PProc. Amer. Math. Soc., 144 (2016), 925936.
3. M. Chou, Torsion of rational elliptic curves over quartic Galois number fields, J. Number Theory 160 (2016), 603-628.
4. P. L. Clark, P. Corn, A. Rice and J. Stankiewicz, Computation on elliptic curves with complex multiplication, LMS J. Comput. Math. 17 (2014), 509-539.
5. H. B. Daniels and E. González-Jiménez, On the torsion of rational elliptic curves over sextic fields, To appear in Mathematics of Computation data, arXiv: 1808.02887
6. H. B. Daniels, A. Lozano-Robledo, F. Najman and A.V. Sutherland, Torsion subgroups of rational elliptic curves over the compositum of all cubic fields, Math. Comp. 87 (2018), 425-458.
7. M. Derickx and A. V. Sutherland, Torsion subgroups of elliptic curves over quintic and sextic number fields, Proc. Amer. Math. Soc. 145 (2017), 4233-4245.
8. E. González-Jiménez, Compe classification of the torsion structures of rational elliptic curves over quintic number fields, J. Algebra 478 (2017), 484-505.
9. E. González-Jiménez and A. Lozano-Robledo, On the minimal degree of definition of p-primary torsion subgroups of elliptic curves, Mathematical Research ter 24 (2017) 1067-1096.
10. E. González-Jiménez and F. Najman, An Algorithm for determining torsion growth of elliptic curves, preprint.
11. E. González-Jiménez and F. Najman and , Growth of torsion groups of elliptic curves upon base change, Math. Comp. to appear.
12. E. González-Jiménez, F. Najman and J. M. Tornero, Torsion of rational elliptic curves over cubic fields, es over cubic fields, Rocky Mountain J. Math. 46 (2016), 1899-1917.
13. P. Ingram, Diophantine analysis and torsion points on elliptic curves, Proc. London Math. Soc. 94 (2007), 473486.
14. D. Jeon, C. H. Kim and A. Schweizer, On the torsion of elliptic curves over cubic number fields, Acta Arith. 113 (2004) 291-301.
15. D. Jeon, C. H. Kim and E. Park, On the torsion of elliptic curves over quartic number fields, J. London Math. Soc. 74 (2006), 1-12.
16. S. Kamienny, Torsion points on elliptic curves and q-coefficients of modular forms, Invent. Math. 109 (1992), 221-229.
17. M. A. Kenku and F. Momose, Torsion points on elliptic curves defined over quadratic field, Nagoya Math. J. 109 (1988), 125-149.
18. M. A. Kenku, The modular curve $X_0(39)$ and rational isogeny, Math. Proc. Cambridge Philos. Soc. 85 (1979), 21-24.
19. M. A. Kenku, *The modular curves* $X_0(65)$ and $X_0(91)$ and rational isogeny, *Math. Proc. Cambridge Philos. Soc.* 87 (1980), 15-20.
20. M. A. Kenku, *The modular curve* $X_0(169)$ and rational isogeny, *J. London Math. Soc.* (2) 22 (1980), 239-244.
21. M. A. Kenku, *The modular curve* $X_0(125)$, $X_1(25)$ and $X_1(49)$, *J. London Math. Soc.* (2) 23 (1981), 415-427.
22. LMFDB Collaboration, *The L-functions and modular forms database*, available at [http://www.lmfdb.org](http://www.lmfdb.org).
23. A. Lozano-Robledo and E. González-Jiménez, *On the minimal degree of definition of $p$-primary torsion subgroups of elliptic curves*, Math. Res. t. 24 (2017), 1067-1096. (data files [http://matematicas.uam.es/~enrique.gonzalez.jimenez/](http://matematicas.uam.es/~enrique.gonzalez.jimenez/)).
24. A. Lozano-Robledo, *On the field of definition of $p$-torsion points on elliptic curves over the rationals*, Math. Ann 357 (2013), 279-305.
25. B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes tudes Sci. Publ. Math. 47 (1978), 33-186.
26. B. Mazur, *Rational isogenies of prime degree*, Invent. Math. 44 (1978), pp. 129-162.
27. L. Merel, *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*, Invent. Math. 124 (1996), 437-449.
28. J. S. Morrow, *Composite images of Galois for elliptic curves over $Q$ and entanglement fields*, Math. Comp. 88 (2019), 2389-2421.
29. F. Najman, *Torsion of rational elliptic curves over cubic fields and sporadic points on $X_1(n)$*, Math. Res. ters, 23 (2016) 245-272.
30. F. Najman, *Complete classification of torsion of elliptic curves over quadratic cyclotomic fields*, J. Number Theory 130 (2010), 1964-1968.
31. L. D. Olson, *Points of finite order on elliptic curves with complex multiplication*, Manuscripta Math. 14 (1974) 195205.
32. J. Rouse and D. Zywina, *Elliptic curves over $Q$ and 2-adic images of Galois*, Research in Number Theory 1:12, 2015. (Data files and subgroup descriptions available at [http://users.wfu.edu/rouseja/2adic/](http://users.wfu.edu/rouseja/2adic/)).
33. A. V. Sutherland, *Computing images of Galois representations attached to elliptic curves*, Forum Math. Sigma 4 (2016), e4, 79 pp.
34. A. V. Sutherland, D. Zywina, *Modular curves of prime-power level with infinitely many rational points*
35. L. Washington, *Elliptic Curves. Number Theory and Cryptography*, Discrete Mathematics and its Applications (Boca Raton), Chapman and Hall/CRC, Boca Raton, 2003.
36. D. Zywina, *On the possible images of the mod 1 representations associated to elliptic curves over $Q$*, arXiv:1508.07660.