Global Classical Large Solutions to Navier-Stokes Equations for Viscous Compressible and Heat Conducting Fluids with Vacuum

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Abstract

In this paper, we consider the 1D Navier-Stokes equations for viscous compressible and heat conducting fluids (i.e., the full Navier-Stokes equations). We get a unique global classical solution to the equations with large initial data and vacuum. Because of the strong nonlinearity and degeneration of the equations brought by the temperature equation and by vanishing of density (i.e., appearance of vacuum) respectively, to our best knowledge, there are only two results until now about global existence of solutions to the full Navier-Stokes equations with special pressure, viscosity and heat conductivity when vacuum appears (see [13] where the viscosity \( \mu = \text{const} \) and the so-called variational solutions were obtained, and see [1] where the viscosity \( \mu = \mu(\rho) \) degenerated when the density vanishes and the global weak solutions were got). It is open whether the global strong or classical solutions exist. By applying our ideas which were used in our former paper [8] to get \( H^3 \)-estimates of \( u \) and \( \theta \) (see Lemma 3.10, Lemma 3.11, Lemma 3.12 and the corresponding corollaries), we get the existence and uniqueness of the global classical solutions (see Theorem 1.1). In fact, the existence of strong solutions would be done obviously by our estimates if the regularity of the initial data is assumed to be weaker. Like [8], we get \( H^4 \)-regularity of \( \rho \) and \( u \) (see Theorem 1.2). We do not get further regularity of \( \theta \) such as \( H^4 \)-regularity, because of the degeneration and strong nonlinearity brought by vacuum and the term \( (\mu u_x)_x \) in the temperature equation. This can be viewed the first result on global classical solutions to the 1D Navier-Stokes equations for viscous compressible and heat conducting fluids which may be large initial data and contain vacuum.

Key Words: Compressible Navier-Stokes equations, heat conducting fluids, vacuum, global classical solutions.

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1 Introduction

In this paper, we consider the Navier-Stokes equations for viscous compressible and heat conducting fluids (i.e. the full Navier-Stokes equations). The model, describing for instance the motion of gas, plays an important role in applied physics. Mathematically, the model in one dimension can be written as follows in sense of Eulerian coordinates:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \quad \rho \geq 0, \\
(\rho u)_t + (\rho u^2)_x + P_x &= (\mu u)_x, \\
(\rho E)_t + (\rho u E)_x + (P u)_x &= (\mu uu)_x + (\kappa \theta)_x,
\end{align*}
\]

for \((x, t) \in (0, 1) \times (0, +\infty)\). Here \(\rho = \rho(x, t), u = u(x, t), P = P(\rho, \theta), E, \theta\) and \(\kappa = \kappa(\rho, \theta)\) denote the density, velocity, pressure, total energy, absolute temperature and coefficient of heat conduction, respectively. The total energy \(E = e + \frac{1}{2}u^2\), where \(e\) is the internal energy. \(\mu > 0\) is the coefficient of viscosity. \(P\) and \(e\) satisfy the second principle of thermodynamics:

\[
P = \rho^2 \frac{\partial e}{\partial \rho} + \theta \frac{\partial P}{\partial \theta}.
\]

In the present paper, we consider the initial and boundary conditions:

\[
(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0)(x) \text{ in } [0, 1],
\]

and

\[
(u, \theta)|_{x=0,1} = 0, \quad t \geq 0.
\]

Since the model is important, lots of works on the existence, uniqueness, regularity and asymptotic behavior of the solutions were done during the last five decades. While, because of the stronger nonlinearity in \((1.1)\) compared with the Navier-Stokes equations for isentropic fluids (no temperature equation), many known mathematical results mainly focused on the absence of vacuum (vacuum means \(\rho = 0\)), refer for instance to \([17, 18, 24, 25, 29, 30, 34]\) for classical solutions. More precisely, the local classical solutions to the Navier-Stokes equations with heat-conducting fluid in Hölder spaces was obtained respectively by Itaya in \([17]\) for Cauchy problem and by Tani in \([34]\) for IBVP with \(\inf \rho_0 > 0\), where the spatial dimension \(N = 3\). Using delicate energy methods in Sobolev spaces, Matsumura and Nishida showed in \([29, 30]\) that the global classical solutions exist provided that the initial data is small in some sense and away from vacuum and the spatial dimension \(N = 3\). For large initial data and dimension \(N = 1\), Kazhikhov, Shelukhi in \([25]\) (for polytropic perfect gas with \(\mu, \kappa = \text{const.}\))
and Kawohl in [24] (for real gas with \( \kappa = \kappa(\rho, \theta), \mu = \text{const.} \)) respectively got global classical solutions to [11] in Lagrangian coordinates with boundary condition [14] and inf \( \rho_0 > 0 \). The internal energy \( e \) and the coefficient of heat conduction \( \kappa \) in [24] satisfy the following assumptions for \( \rho \leq \overline{\rho} \) and \( \theta \geq 0 \) (we translate these conditions in Eulerian coordinates)

\[
\begin{align*}
\frac{e(\rho, 0)}{\rho} & \geq 0, \quad \nu(1 + \theta^r) \leq \partial_\rho e(\rho, \theta) \leq N(\overline{\rho})(1 + \theta^r), \\
\kappa_0(1 + \theta^q) & \leq \kappa(\rho, \theta) \leq \kappa_1(1 + \theta^q), \\
|\partial_\rho \kappa(\rho, \theta)| + |\partial_{\rho \theta} \kappa(\rho, \theta)| & \leq \kappa_1(1 + \theta^q),
\end{align*}
\]

(1.5)

where \( r \in [0, 1], q \geq 2 + 2r, \) and \( \nu, N(\overline{\rho}), \kappa_0 \) and \( \kappa_1 \) are positive constants. For the perfect gas in the domain exterior to a ball in \( \mathbb{R}^N \) \( (N = 2 \text{ or } 3) \) with \( \mu, \kappa = \text{const.} \), Jiang in [18] got the existence of global classical spherically symmetric large solutions in H"older spaces.

In fact, Kawohl in [24] also considered the case of \( \mu = \mu(\rho) \) for another boundary condition with inf \( \rho_0 > 0 \), where \( 0 < \underline{\mu}_0 \leq \mu(\rho) \leq \overline{\mu}_0 \) for any \( \rho \geq 0 \) and \( \underline{\mu}_0 \) and \( \overline{\mu}_0 \) are positive constant. This result was generalized to the case \( \mu(\rho) = \rho^\alpha \) by Jiang in [19] for \( \alpha \in (0, \frac{1}{4}) \), and by Qin, Yao in [31] for \( \alpha \in (0, \frac{1}{4}) \), respectively.

On the existence, asymptotic behavior of the weak solutions for full Navier-Stokes equations (including the temperature equation) with inf \( \rho_0 > 0 \), please refer for instance to [20, 21, 23] for weak solutions in 1D and for spherically symmetric weak solutions in bounded annular domains in \( \mathbb{R}^N \) \( (N = 2, 3) \), and refer to [12] for variational solutions in a bounded domain in \( \mathbb{R}^N \) \( (N = 2, 3) \).

In the presence of vacuum (i.e. \( \rho \) may vanish), to our best knowledge, the mathematical results about global well-posedness of the full Navier-Stokes equations are usually limited to the existence of weak solutions with special pressure, viscosity and heat conductivity (see [1, 13]). More precisely, Feireisl in [13] got the existence of so-called variational solutions in dimension \( N \geq 2 \). The temperature equation in [13] is satisfied only as an inequality. Anyway, this work in [13] is the very first attempt towards the existence of weak solutions to the full compressible Navier-Stokes equations in higher dimensions, where the viscosity \( \mu \) is constant and

\[
\begin{align*}
\kappa = \kappa(\theta) & \in C^2[0, \infty), \quad \underline{\kappa}(1 + \theta^q) \leq \kappa(\theta) \leq \overline{\kappa}(1 + \theta^q) \quad \text{for all } \theta \geq 0, \\
P = P(\rho, \theta) & = \mathcal{P}_e(\rho) + \theta \mathcal{P}_\theta(\rho) \quad \text{for all } \rho \geq 0 \text{ and } \theta \geq 0, \\
\mathcal{P}_e, \mathcal{P}_\theta & \in C[0, \infty) \cap C^1(0, \infty); \quad \mathcal{P}_e(0) = 0, \ \mathcal{P}_\theta(0) = 0, \\
\mathcal{P}_e(\rho) & \geq a_1 \rho^{\Gamma - 1} - b_1 \quad \text{for all } \rho > 0; \quad \mathcal{P}_e(\rho) \leq a_2 \rho^{\Gamma} + b_1 \quad \text{for all } \rho \geq 0, \\
\mathcal{P}_\theta & \text{ is non-decreasing in } [0, \infty); \quad \mathcal{P}_\theta(\rho) \leq a_3 (1 + \rho^{\Gamma}) \quad \text{for all } \rho \geq 0,
\end{align*}
\]

(1.6)

where \( \Gamma < \frac{\overline{\gamma}}{2} \) if \( N = 2 \) and \( \Gamma = \frac{\overline{\gamma}}{N} \) for \( N \geq 3 \); \( a \geq 2, \ \overline{\gamma} > 1, \) and \( a_1, a_2, a_3, b_1, \underline{\kappa} \) and \( \overline{\kappa} \) are positive constants. Note that the perfect gas equation of state (i.e. \( P = R \rho \theta \) for some constant \( R > 0 \)) is not involved in [16]. In order that the equations are satisfied as equalities in the sense of distribution, Bresch and Desjardins in [1] proposed some different assumptions from [13], and obtained the existence of global weak solutions to the full Navier-Stokes equations with large initial data in \( \mathbb{T}^3 \) or \( \mathbb{R}^3 \). In [1], the viscosity \( \mu = \mu(\rho) \) may vanish.
when vacuum appears, and $\kappa$, $P$ and $e$ are assumed to satisfy

\[
\begin{align*}
\kappa(\rho, \theta) &= \kappa_0(\rho, \theta)(\rho + 1)(\theta^a + 1), \\
P &= r\rho\theta^a + p_c(\rho), \\
e &= C_\nu\theta + e_c(\rho),
\end{align*}
\]

(1.7)

where $a \geq 2$, $r$ and $C_\nu$ are two positive constants, $p_c(\rho) = O(\rho^{-\ell})$ and $e_c(\rho) = O(\rho^{-(\ell-1)})$ (for some $\ell > 1$) when $\rho$ is small enough, and $\kappa_0(\rho, \theta)$ is assumed to satisfy

$$\underline{c_0} \leq \kappa_0(\rho, \theta) \leq \overline{c_0},$$

for $\underline{c_0} > 0$. We have to mention that the smooth solutions in $C^1([0, \infty); H^d(\mathbb{R}^N))$ would blow up when the initial density is of nontrivial compact support (see [36]). On the local existence and uniqueness of strong solutions in $\mathbb{R}^3$, please refer to [4] for the perfect gas with $\mu, \kappa = \text{const.}$

It is still open whether global strong (or classical) solutions exist when vacuum appears (i.e., the density may vanish). Our main concern here is to show the existence and uniqueness of global classical solutions to (1.1)-(1.4) with vacuum and large initial data. In fact, the existence of the strong solutions to this problem is obvious if the regularity of initial data is assumed to be weaker.

For compressible isentropic Navier-Stokes equations (i.e. no temperature equation), there are so many results about the well-posedness and asymptotic behaviors of the solutions when vacuum appears. Refer to [14] [22] [26] [28] and [15] [27] [35] [38] [39] [40] for global weak solutions with constant viscosity and with density-dependent viscosity, respectively. Refer to [6] [10] and [2] [3] [5] [33] for global strong solutions and for local strong (classical) solutions with constant viscosity, respectively. Recently, Huang, Li, Xin in [16] and Ding, Wen, Yao, Zhu in [8] [9] independently got existence and uniqueness of global classical solutions, where the initial energy in [16] is assumed to be small in $\mathbb{R}^3$ and $\rho - \bar{\rho} \in C([0, T]; H^3(\mathbb{R}^3))$, $u \in C([0, T]; D^1(\mathbb{R}^3) \cap D^3(\mathbb{R}^3)) \cap L^\infty([\tau, T]; D^4(\mathbb{R}^3))$ (for $\tau > 0$) which generalized the results in [3], and the initial data in [8] [9] could be large for dimension $N = 1$ and could be large but spherically symmetric for $N \geq 2$, and $(\rho, u) \in C([0, T]; H^4(I))$ ($I$ is bounded in $\mathbb{R}^3$, and is bounded or an exterior domain in $\mathbb{R}^N$).

We would like to give some notations which will be used throughout the paper.

Notations:

1. $I = [0, 1]$, $\partial I = \{0, 1\}$, $Q_T = I \times [0, T]$ for $T > 0$.

2. For $p \in [1, \infty]$, $L^p(I)$ denotes the $L^p$ space with the norm $\| \cdot \|_{L^p}$. For $k \geq 1$ and $p \in [1, \infty]$, $W^{k,p}(I)$ denotes the Sobolev space, whose norm is denoted as $\| \cdot \|_{W^{k,p}}$, $H^k = W^{k,2}(I)$.

3. For an integer $k \geq 0$ and $0 < \alpha < 1$, let $C^{k+\alpha}(I)$ denote the Schauder space of functions on $I$, whose $k$th order derivative is H"{o}lder continuous with exponents $\alpha$, with the norm $\| \cdot \|_{C^{k+\alpha}}$. 
In this paper, our assumptions are the following:

(A1): \( \int_I \rho_0 > 0 \).

(A2): \( \mu = \text{const.} > 0 \), \( e = C_0 Q(\theta) + e_c(\rho) \), \( P = \rho Q(\theta) + P_c(\rho) \), \( \kappa = \kappa(\theta) \), for some constant \( C_0 > 0 \).

(A3): \( P_c(\rho) \geq 0 \), \( e_c(\rho) \geq 0 \), for \( \rho \geq 0 \); \( P_c \in C^2[0, \infty) \); \( \rho \frac{|\partial \rho|}{\theta} \leq C_1 e_c(\rho) \), for some constant \( C_1 > 0 \).

(A4): \( Q(\cdot) \in C^2[0, \infty) \) satisfies

\[
\begin{align*}
C_2 (\beta + (1 - \beta)\theta + \theta^{1+r}) & \leq Q(\theta) \leq C_3 (\beta + (1 - \beta)\theta + \theta^{1+r}), \\
C_4 (1 + \theta^r) & \leq Q'(\theta) \leq C_5 (1 + \theta^r),
\end{align*}
\]

for some constants \( C_i > 0 \) \( (i = 2, 3, 4, 5) \) and \( r \geq 0 \), \( \beta = 0 \) or 1.

(A5): \( \kappa \in C^2[0, \infty) \) satisfies

\[
C_6 (1 + \theta^q) \leq \kappa(\theta) \leq C_7 (1 + \theta^q),
\]

for \( q \geq 2 + 2r \), and some constants \( C_i > 0 \) \( (i = 6, 7) \).

(A6): \( Q, P_c \in C^4[0, \infty) \), and \( \kappa \) satisfies

\[
|\partial_\theta^2 \kappa(\theta)| \leq C_8 (1 + \theta^{q-3}),
\]

for \( \theta > 0 \) and some constant \( C_8 > 0 \).

Remark 1.1

(i) \( (A_1) \) is needed to get the upper bounds of \( \theta \) and \( \theta_t \) in terms of some norms by using mass conservation, Lemma 2.1 and Corollary 2.1.

(ii) The case for the perfect gas (i.e. \( P = R\rho\theta, e = C_v\theta \) for constants \( R > 0 \) and \( C_v > 0 \)) is involved in the above assumptions.

(iii) As it mentioned in [24], the restriction on \( \mu (\mu=\text{const.}, \text{see other restrictions on } \kappa \) and \( e \) in [1.3]) is not physically motivated. Physically, it seems more importantly that the state functions \( e, \mu \) and \( \kappa \) usually depend on both \( \rho \) and \( \theta \). Particularly, the internal energy \( e \) grows as \( \theta^{1+r} \) with \( r \approx 0.5 \), the conductivity \( \kappa \) grows as \( \theta^q \) with \( 4.5 \leq q \leq 5.5 \) and viscosity \( \mu \) increases like \( \theta^p \) with \( 0.5 \leq p \leq 0.8 \) (see [24], [31] and references therein). Because of mathematical technique, in the present paper, we assume \( \mu = \text{const.} \) and \( \kappa = \kappa(\theta) \) as in [1.3] (see [1.6]). From \( (A_2)-(A_5) \), we know that \( e \) and \( \kappa \) grow respectively as \( \theta^{1+r} \) and \( \theta^q \), where \( q \) can be taken as \( q \in [4.5, 5.5] \), and \( r \) can be taken as \( r = 0.5 \) if we consider \( \theta > 0 \).

(iv) The restriction on \( q \) in \( (A_5) \) (i.e. \( q \geq 2 + 2r \)) is same as \( (1.3) \), and is the same as \( (1.6) \) and \( (1.7) \) when we take \( r = 0 \). This assumption plays an important role in the analysis.

Main results:
Theorem 1.1 In addition to (A1)-(A5), we assume \( \rho_0 \geq 0, \rho_0 \in H^2, (\sqrt{\rho_0})_x \in L^\infty, u_0 \in H^3 \cap H^1_0, \theta_0 \in H^3, \partial_x \theta_0 |_{x=0,1} = 0, \) and that the following compatible conditions are valid:

\[
\begin{align*}
\mu u_{0xx} - [P(\rho_0, \theta_0)]_x &= \sqrt{\rho_0} g_1, \\
[k(\theta_0) \theta_{0x}]_x + \mu |u_{0x}|^2 &= \sqrt{\rho_0} g_2, \quad x \in I,
\end{align*}
\]

for some \( g_1, g_2 \in L^2 \), and \( (\sqrt{\rho_0} g_1)_x, (\sqrt{\rho_0} g_2)_x \in L^2 \). Then for any \( T > 0 \) there exists a unique global solution \( (\rho, u, \theta) \) to (1.1)-(1.4) such that

\[
\begin{align*}
\rho &\in C([0,T]; H^2), \quad \rho_t \in C([0,T]; H^1), \quad \sqrt{\rho} \in W^{1,\infty}(Q_T), \\
u &\in L^\infty([0,T]; H^3), \quad \sqrt{\rho} u_t \in L^\infty([0,T]; L^2), \\
\rho u_t &\in L^\infty([0,T]; H^1), \quad u_t \in L^2([0,T]; H^1_0), \quad \sqrt{\rho e_t} \in L^\infty([0,T]; L^2), \\
\rho e_t &\in L^\infty([0,T]; H^1), \quad \theta \in L^\infty([0,T]; H^3), \quad \theta_t \in L^2([0,T]; H^1).
\end{align*}
\]

Remark 1.2 (i) (1.8) was proposed by Cho and Kim in [3] to get local \( H^2 \)-regularity of \( u \) and \( \theta \) for the polytropic perfect gas. The detailed reasons why such conditions were needed can be found in [3]. Roughly speaking, \( g_1 \) and \( g_2 \) are equivalent to \( \sqrt{\rho} u_t \) and \( \sqrt{\rho} e_t \) at \( t = 0 \), respectively.

(ii) By the Sobolev embedding theorems (cf. [7]) and Lemma 2.3 we know from Theorem 1.1 that

\[
\begin{align*}
\rho &\in C \left( [0,T]; C^{1+\frac{1}{2}}(I) \right) \cap C^1 \left( [0,T]; C^\frac{1}{2}(I) \right), \\
u &\in C \left( [0,T]; C^{2+\sigma}(I) \right), \quad (\rho u)_t \in C \left( [0,T]; C^\sigma(I) \right), \\
\theta &\in C \left( [0,T]; C^{2+\sigma}(I) \right), \quad (\rho e)_t \in C \left( [0,T]; C^\sigma(I) \right),
\end{align*}
\]

for any \( T > 0 \) and \( \sigma \in (0, \frac{1}{2}) \). This implies \( (\rho, u, \theta) \) is the classical solution to (1.1)-(1.4).

Theorem 1.2 In addition to (A1)-(A6), we assume \( \rho_0 \geq 0, \rho_0 \in H^4, (\sqrt{\rho_0})_x \in L^\infty, u_0 \in H^4 \cap H^1_0, \theta_0 \in H^3, \partial_x \theta_0 |_{x=0,1} = 0, q > 2 + 2r, \) and that the following compatible conditions are valid:

\[
\begin{align*}
\mu u_{0xx} - [P(\rho_0, \theta_0)]_x &= \rho_0 g_3, \\
[k(\theta_0) \theta_{0x}]_x + \mu |u_{0x}|^2 &= \sqrt{\rho_0} g_2, \quad x \in I,
\end{align*}
\]

for some \( g_3 \in H^1_0, (\sqrt{\rho_0} g_3)_x \in L^2 \), and \( g_2, (\sqrt{\rho_0} g_2)_x \in L^2 \). Then for any \( T > 0 \) there exists a unique global solution \( (\rho, u, \theta) \) to (1.1)-(1.4) satisfying:

\[
\begin{align*}
\rho &\in C([0,T]; H^4), \quad \rho_t \in C([0,T]; H^3), \quad \sqrt{\rho} \in W^{1,\infty}(Q_T), \\
u &\in C([0,T]; H^4) \cap L^2([0,T]; H^5), \quad u_t \in L^\infty([0,T]; H^1_0) \cap L^2([0,T]; H^3), \\
(\rho u)_t &\in C([0,T]; H^2), \quad \sqrt{\rho} u_{txt} \in L^\infty([0,T]; L^2), \quad \sqrt{\rho} e_t \in L^\infty([0,T]; L^2), \\
(\rho e)_t &\in L^\infty([0,T]; H^1), \quad \theta \in L^\infty([0,T]; H^3) \cap L^2([0,T]; H^4), \quad \theta_t \in L^2([0,T]; H^1).
\end{align*}
\]

Remark 1.3 (i) (1.9) was proposed by Cho and Kim in [3] where they consider the local existence of classical solutions for isentropic fluids (no temperature equation). Roughly speaking, \( g_3 \) is equivalent to \( u_t \) at \( t = 0 \).
(ii) We could not get \( \theta \in C([0, T]; H^4) \) (or \( L^\infty([0, T]; H^4) \)) even if (1.9)\(_2\) is changed similarly to (1.9)\(_1\), because of the strong nonlinearity and degeneration brought by \((\mu uu_x)_x\) in the temperature equation and the appearance of vacuum, respectively.

(iii) Using ideas of Cho and Kim in [3], we can also get
\[
L^\infty([\tau, T]; H^4), \quad \theta \in L^\infty([\tau, T]; H^3),
\]
for \( \tau > 0 \).

If we can obtain our estimates in higher dimensions, it will be useful to investigate the local (global) existence of classical solutions to the full Navier-Stokes equations (including the temperature equation) in \( \mathbb{R}^N \) (\( N \geq 2 \)). For example, to guarantee (1.4) in \( \mathbb{R}^N \) (\( N = 2 \) or 3) is valid for all \( t \geq 0 \), it is necessary to get \( \theta \in L^\infty([0, T]; H^3) \). We will consider these problems in the near future.

The constants \( C_0 \) in (A2) and the viscosity \( \mu \) don’t play any role in the analysis, we assume henceforth that \( C_0 = 1 \) and \( \mu = 1 \) for simplicity.

The rest of the paper is organized as follows. In Section 2, we present some useful lemmas which will be used in the next sections. In Section 3, we prove Theorem 1.1 by giving the initial density and the initial temperature a lower bound \( \delta > 0 \), getting a sequence of approximate solutions to (1.1)-(1.4), and taking \( \delta \to 0^+ \) after making some estimates uniformly for \( \delta \).

More precisely, based on Lemma 2.1 and the one-dimensional properties of the equations, we get \( H^2 \)-estimates of the solutions. Using our ideas in [8, 9], we obtain \( H^3 \)-estimates of \( u \) and \( \theta \). In Section 4, using the similar arguments as in Section 3, we prove Theorem 1.2.

2 Preliminaries

**Lemma 2.1** Let \( \Omega = [\alpha, \beta] \) be a bounded domain in \( \mathbb{R} \), and \( \rho \) be a non-negative function such that
\[
0 < M \leq \int_{\Omega} \rho \leq K,
\]
for constants \( M > 0 \) and \( K > 0 \). Then
\[
\|v\|_{L^\infty(\Omega)} \leq \frac{K}{M} \|v_x\|_{L^1(\Omega)} + \frac{1}{M} \left| \int_{\Omega} \rho v \right|,
\]
for any \( v \in H^1(\Omega) \).

**Proof.** For any \( x \in \Omega \), we have
\[
|v(x)| \leq \frac{1}{M} \left| v(x) \int_{\Omega} \rho(y) dy \right|
\]
\[
\leq \frac{1}{M} \left| \int_{\Omega} v(x) \rho(y) dy - \int_{\Omega} \rho(y) v(y) dy \right| + \frac{1}{M} \left| \int_{\Omega} \rho(y) v(y) dy \right|
\]
\[
\leq \frac{1}{M} \left| \int_{\Omega} \int_{y} v_x(\xi) d\xi \rho(y) dy \right| + \frac{1}{M} \left| \int_{\Omega} \rho(y) v(y) dy \right|
\]
\[
\leq \frac{K}{M} \|v_x\|_{L^1(\Omega)} + \frac{1}{M} \left| \int_{\Omega} \rho(y) v(y) dy \right|.
\]
Remark 2.1 The version of higher dimensions for Lemma 2.1 can be found in [12] or [13].

Corollary 2.1 Consider the same conditions in Lemma 2.1, and in addition assume \( \Omega = I \), and
\[
\| \rho v \|_{L^1(I)} \leq \overline{\varepsilon}.
\]
Then for any \( l > 0 \), there exists a positive constant \( C = C(M, K, l, \overline{\varepsilon}) \) such that
\[
\| v^l \|_{L^\infty(I)} \leq C \| (v^l)_x \|_{L^2(I)} + C,
\]
for any \( v^l \in H^1(I) \).

Proof. By Lemma 2.1, we have
\[
\| v^l \|_{L^\infty(I)} \leq C \| (v^l)_x \|_{L^2(I)} + C \int_I \rho |v^l|.
\]
Case 1: \( l \in (0, 1] \).

In this case, we use the Young inequality to get
\[
\| v^l \|_{L^\infty(I)} \leq C \| (v^l)_x \|_{L^2(I)} + C \int_I \rho |v| + C \int_I \rho + C \\
\leq C \| (v^l)_x \|_{L^2(I)} + C.
\]
Case 2: \( l \in (1, \infty) \).

In the case, we use the Young inequality again to get
\[
\| v^l \|_{L^\infty(I)} \leq C \| (v^l)_x \|_{L^2(I)} + C \| v^{l-1} \|_{L^\infty(I)} \int_I \rho |v| \\
\leq C \| (v^l)_x \|_{L^2(I)} + \frac{1}{2} \| v^l \|_{L^\infty(I)} + C.
\]
This gives
\[
\| v^l \|_{L^\infty(I)} \leq C \| (v^l)_x \|_{L^2(I)} + C.
\]

Lemma 2.2 For any \( v \in H^1_0(I) \), we have
\[
\| v \|_{L^\infty(I)} \leq \| v_x \|_{L^1}.
\]
Proof. Since \( v(0) = 0 \), we have for any \( x \in I \)
\[
|v(x)| = |v(x) - v(0)| = \left| \int_0^x v_x \right| \leq \| v_x \|_{L^1(I)}.
\]
This completes the proof.

Lemma 2.3 ([32]). Assume \( X \subset E \subset Y \) are Banach spaces and \( X \hookrightarrow\hookrightarrow E \). Then the following imbedding are compact:
\[(i) \quad \left\{ \varphi : \varphi \in L^q(0,T;X), \frac{\partial \varphi}{\partial t} \in L^1(0,T;Y) \right\} \hookrightarrow\hookrightarrow L^q(0,T;E), \text{ if } 1 \leq q \leq \infty; \]
\[(ii) \quad \left\{ \varphi : \varphi \in L^\infty(0,T;X), \frac{\partial \varphi}{\partial t} \in L^r(0,T;Y) \right\} \hookrightarrow\hookrightarrow C([0,T];E), \text{ if } 1 < r \leq \infty. \]
3 Proof of Theorem 1.1

In this section, we get a global solution to (1.1)-(1.4) with initial density and initial temperature having a respectively lower bound \( \delta > 0 \) by using some \( a \) priori estimates of the solutions based on the local existence. Theorem 1.1 would be got after we make some \( a \) priori estimates uniformly for \( \delta \) and take \( \delta \to 0^+ \).

Denote \( \rho^\delta_0 = \rho_0 + \delta \) and \( \theta^\delta_0 = \theta_0 + \delta \) for \( \delta \in (0, 1) \). Throughout this section, we denote \( c \) to be a generic constant depending on \( \rho_0, u_0, \theta_0, T \) and some other known constants but independent of \( \delta \) for any \( \delta \in (0, 1) \).

Before proving Theorem 1.1, we need the following auxiliary theorem.

**Theorem 3.1** Consider the same assumptions as in Theorem 1.1. Then for any \( T > 0 \) and \( \delta \in (0, 1) \) there exists a unique global solution \( (\rho, u, \theta) \) to (1.1)-(1.4) with initial data replaced by \( (\rho^\delta_0, u_0, \theta^\delta_0) \), such that

\[
\rho \in C([0, T]; H^2), \quad \rho_t \in C([0, T]; H^1), \quad \rho_{tt} \in L^2([0, T]; L^2), \quad \rho \geq \frac{\delta}{c} > 0,
\]

\[
u \in C([0, T]; H^3 \cap H^1_0), \quad u_t \in C([0, T]; H^1) \cap L^2([0, T]; H^2), \quad u_{tt} \in L^2([0, T]; L^2),
\]

\[
\theta \geq c_\delta > 0, \quad \theta \in C([0, T]; H^3), \quad \theta_t \in C([0, T]; H^1) \cap L^2([0, T]; H^2), \quad \theta_{tt} \in L^2([0, T]; L^2),
\]

where \( c_\delta \) is a constant depending on \( \delta \), but independent of \( u \).

**Proof of Theorem 3.1:**

The local solutions as in Theorem 3.1 can be obtained by the successive approximations like in [4]. We omit it here for brevity. The regularities guarantee the uniqueness (refer for instance to [4]). Based on it, Theorem 3.1 can be proved by some \( a \) priori estimates globally in time.

For any given \( T \in (0, +\infty) \), let \( (\rho, u, \theta) \) be the solution to (1.1)-(1.4) as in Theorem 3.1. Then we have the following basic energy estimate.

**Lemma 3.1** Under the conditions of Theorem 3.1, it holds for any \( 0 \leq t \leq T \)

\[
\int_I \rho \left(1 + e_c(\rho) + \theta^{1+r} + u^2\right)(t) \leq c.
\]

**Proof.** Integrating (1.1) and (1.4) over \( I \times [0, t] \), and using (1.4), \( (A_2) \) and \( (A_4) \), we complete the proof of Lemma 3.1.

**Lemma 3.2** Under the conditions of Theorem 3.1, it holds for any \((x, t) \in Q_T\)

\[
\begin{cases}
0 < \rho(x, t) \leq c, \\
\theta(x, t) > 0.
\end{cases}
\]

**Proof.** The proof of the upper bound of \( \rho \) relies on constant viscosity (i.e. \( \mu = \text{const.} \)). It is similar to [37].

Denote

\[
w(x, t) = \int_0^t (u_x - P - \rho u^2) + \int_0^x \rho_0 u_0.
\]
Differentiating (3.1) with respect to $x$, and using (1.1)\textsubscript{2}, we have
\[ w_x = \rho u. \]
This together with Lemma 3.1 and the Cauchy inequality gives
\[ \int_I |w_x| \leq c. \]
It follows from (3.1), (1.2), (A.2), (A.3), (A.4), (1.4), and Lemma 3.1 that
\[ \int_I w \leq c. \]
This gives for any $(x, t) \in Q_T$
\[ |w(x, t)| \leq \int_I |w| + \int_I w \leq |\int_I w_x| + c \leq c, \]
which implies
\[ \|w\|_{L^\infty(Q_T)} \leq c. \] (3.2)
For any $(x, t) \in Q_T$, let $X(s; x, t)$ satisfy
\begin{equation}
\begin{cases}
\frac{dX(s; x, t)}{ds} = u(X(s; x, t), s), & 0 \leq s < t, \\
X(t; x, t) = x.
\end{cases}
\end{equation} (3.3)
Denote
\[ F(x, t) = \exp \{w(x, t)\}. \]
It is easy to verify
\[ \frac{d(\rho F)(X(s; x, t), s)}{ds} = F \left( \rho_s + \frac{\partial \rho}{\partial X} u + \rho \frac{\partial w}{\partial X} u + \rho w_s \right) = -\rho PF. \] (3.4)
Multiplying (3.3) by $\exp (\int_0^s P)$, we have
\[ \frac{d}{ds} \left\{ \rho F \exp \left( \int_0^s P \right) \right\} = 0. \]
Integrating it over $(0, t)$, we have
\[ \rho(x, t) = \frac{F(X(0; x, t), 0)}{F(x, t)} \rho_0^s \exp \left( -\int_0^t P \right), \] (3.5)
which implies
\[ \rho(x, t) > 0, \]
for any $(x, t) \in Q_T$.
By (3.2), (3.5) and $P \geq 0$, we get the upper bound of $\rho$. The lower bound of $\theta$ can be got by (3.7) and the maximum principle for parabolic equations. \hfill \Box
Lemma 3.3 Under the conditions of Theorem 3.1, it holds for any given $\alpha \in (0, 1)$

$$\int_{Q_T} \left( \frac{u_x^2}{\theta^\alpha} + \frac{(1 + \theta^\alpha)\theta_x^2}{\theta^{1+\alpha}} \right) \leq c,$$

where $c$ may depend on $\alpha$.

Remark 3.1 $\alpha$ was usually taken as 1 when the basic energy inequality was done (see [1] and references therein). This depends on $\rho_0 \log \theta_0 \in L^1$ which can not be got under the assumptions of Theorem 1.1 and Theorem 1.2, since $\theta_0$ may vanish.

Proof. From (1.2) and (1.1), we get

$$\rho e \theta t + \rho u e \theta x + \theta P \theta x = u_x^2 + (\kappa(\theta)\theta x).$$

(3.6)

Substituting $e = Q(\theta) + e_c(\rho)$ and $P = \rho Q(\theta) + P_c(\rho)$ into (3.6), we get

$$\rho Q'(\theta)\theta t + \rho u Q'(\theta)\theta x + \rho \theta Q'(\theta)u x = u_x^2 + (\kappa(\theta)\theta x),$$

(3.7)

or

$$(\rho Q)_t + (\rho u)_x + \rho \theta Q'(\theta) u x = u_x^2 + (\kappa(\theta)\theta x).$$

(3.8)

Multiplying (3.7) by $\theta^{-\alpha}$, integrating the resulting equation over $Q_T$, and using integration by parts, we have

$$\int_{Q_T} \left( \frac{u_x^2}{\theta^\alpha} + \frac{\alpha \kappa(\theta) \theta_x^2}{\theta^{1+\alpha}} \right)$$

$$= \int_I \int_0^\theta Q'(\xi) - \int_I \rho_0 \int_0^{\theta_0} \frac{Q'(\xi)}{\xi} + \int_{Q_T} \rho \theta^{1-\alpha} Q'(\theta) u x$$

$$\leq c \int_I \int_0^\theta \frac{1 + \xi^r}{\xi} + c \int_I \rho_0 \int_0^{\theta_0} \xi^{r-\alpha} + c \int_{Q_T} \rho \theta^{1-\alpha} (1 + \theta^r)|u x|$$

$$\leq c \int_I \rho (1 + \theta^{1+r}) + c + \frac{1}{2} \int_{Q_T} \frac{u_x^2}{\theta^\alpha} + c \int_{Q_T} \rho^2 \theta^{2-\alpha+2r}$$

$$\leq c + \frac{1}{2} \int_{Q_T} \frac{u_x^2}{\theta^\alpha} + c \int_0^T \max_{x \in I} \theta^{1+r-\alpha},$$

(3.9)

where we have used (A4), the Cauchy inequality, Lemma 3.1 and Lemma 3.2. Now we estimate the last term of (3.9) as follows:

$$c \int_0^T \max_{x \in I} \theta^{1+r-\alpha} \leq c + \int_0^T \|\theta^{r-\alpha} \theta_x\|_{L^2}$$

$$\leq c + c \int_0^T \left( \int_I \frac{\theta_x^2 \theta^{2r-\alpha+1}}{\theta^{1+\alpha}} \right)^{\frac{1}{2}}$$

$$\leq c + \frac{1}{2} \int_{Q_T} \frac{\alpha \kappa(\theta) \theta_x^2}{\theta^{1+\alpha}},$$

(3.10)

where we have used Corollary 2.1, Lemma 3.1, (A5) and the Cauchy inequality. By (3.9), (3.10) and (A5), we complete the proof. $\square$
Corollary 3.1 Under the conditions of Theorem 3.1, it holds

\[ \int_0^T \| \theta \|_{L_\infty}^{q-\alpha+1} \leq c. \]

Proof. By Corollary 2.1 and Lemma 3.1, we have

\[
\int_0^T \| \theta \|_{L_\infty}^{q-\alpha+1} = \int_0^T \| \theta \|_{L_\infty}^{\frac{q-\alpha+1}{2}} \leq c \int_0^T \left( \frac{\theta^{q-\alpha+1}}{2} \theta_x \right)^2 + c
\]

\[
= c \int_0^T \theta^{q-\alpha-1} \theta_x^2 + c \leq c.
\]

\[ \Box \]

Lemma 3.4 Under the conditions of Theorem 3.1, it holds

\[ \int_{Q_T} u_x^2 \leq c. \]

Proof. From (1.1), we get

\[ \rho u_t + \rho uu_x + P_x = u_{xx} \tag{3.11} \]

Multiplying (3.11) by \( u \), integrating it over \( I \), and using integration by parts, Lemma 2.2, Lemma 3.2 and the Cauchy inequality, we have

\[
\frac{1}{2} \frac{d}{dt} \int_I \rho u^2 + \frac{1}{2} \int_I u_x^2 = \int_I P u_x
\]

\[
\leq \frac{1}{2} \int_I u_x^2 + c \int_I \rho^2 Q^2 + c \int_I P_x^2
\]

\[
\leq \frac{1}{2} \int_I u_x^2 + c \int_I \theta^{q-\alpha+1} + c.
\]

where we have used the Cauchy inequality, \((A_2), (A_3), (A_4)\) and Lemma 3.2. This implies

\[
\frac{d}{dt} \int_I \rho u^2 + \frac{1}{2} \int_I u_x^2 \leq c \sup_{x \in I} \theta^{q-\alpha+1} + c.
\]

Integrating it over \((0, t)\), and using Corollary 3.1, we complete the proof of Lemma 3.4. \( \Box \)

Lemma 3.5 Under the conditions of Theorem 3.1, it holds for any \( 0 \leq t \leq T \)

\[ \int_I \left( u_x^2 + \rho \theta^{q+2+r} \right) + \int_{Q_T} \left( \rho u_t^2 + (1 + \theta)^2 \theta_x^2 \right) \leq c. \]

Proof. Multiplying (3.11) by \( u_t \), integrating it over \( I \), and using integration by parts, Lemma 2.2, Lemma 3.2 and the Cauchy inequality, we have

\[
\int_I \rho u_t^2 + \frac{1}{2} \frac{d}{dt} \int_I u_x^2 = \frac{d}{dt} \int_I P u_x - \int_I \rho u_x u_t - \int_I P_t u_x
\]

\[
\leq \frac{1}{2} \int_I u_x^2 + \frac{1}{2} \int_I \rho u_t^2 + \frac{d}{dt} \int_I P u_x - \int_I P_t u_x
\]

\[
\leq \frac{1}{2} \int_I \rho u_t^2 + c \left( \int_I u_x^2 \right)^2 + \frac{d}{dt} \int_I P u_x - \int_I P_t (u_x - P) - \frac{1}{2} \frac{d}{dt} \int_I P^2,
\]
which implies
\[
\int_I \rho\dot{u}^2 + \frac{d}{dt} \int_I u_x^2 \leq c \left( \int_I u_x^2 \right)^2 + 2 \frac{d}{dt} \int_I Pu_x - \frac{d}{dt} \int_I P^2 - 2 \int_I P_t (u_x - P). \tag{3.12}
\]

We are going to estimate the last term of the right side of (3.12). Using (A2), (3.13), (1.1) and integration by parts, we have
\[
-2 \int_I P_t (u_x - P) = -2 \int_I (\rho Q)_t (u_x - P) - 2 \int_I (P_c)_t (u_x - P)
\]
\[
= -2 \int_I \left[ (\kappa \theta)_x + u_x^2 - (\rho u)_{xx} - \rho \theta Q' (\theta) u_x \right] (u_x - P)
\]
\[
+ 2 \int_I P'_c (\rho u + \rho u_x) (u_x - P)
\]
\[
= 2 \int_I \kappa \theta (u_{xx} - P_x) - 2 \int_I u_x^2 (u_x - P) - 2 \int_I \rho Q (u_{xx} - P_x)
\]
\[
+ 2 \int_I \rho \theta Q' (\theta) u_x (u_x - P) - 2 \int_I P_c u (u_{xx} - P_x) - 2 \int_I P_c u_x (u_x - P)
\]
\[
+ 2 \int_I \rho P'_c (\rho u) u_x (u_x - P).
\]

This, together with (3.11), (A2), (A4), Lemma 2.2, Lemma 3.2, the Cauchy inequality, and
\[W^{1,1}(I) \hookrightarrow L^\infty (I),\]
gives
\[
-2 \int_I P_t (u_x - P) \leq 2 \int_I \kappa \theta (\rho u + \rho u_x) + 2 \left\| u_x - P \right\|_{L^\infty} \int_I u_x^2 - 2 \int_I \rho Q (\rho u_t + \rho u u_x)
\]
\[
+ c \sup_{x \in I} (1 + \theta^{1+r}) \int_I u_x^2 + c \sup_{x \in I} (1 + \theta^{1+r}) \int_I \rho Q^2 + c \sup_{x \in I} \theta^{1+r}
\]
\[
- 2 \int_I P_c u (\rho u_t + \rho u u_x) + c \int_I u_x^2 + c \int_I \rho Q^2 + c
\]
\[
\leq \frac{1}{4} \int_I \rho u_t^2 + c \int_I \kappa^2 \theta^2_x + c \left( \int_I u_x^2 \right)^2 + c \left( \| u_x - P \|_{L^1} + \| \rho u_t + \rho u u_x \|_{L^1} \right) \int_I u_x^2
\]
\[
+ c \int_I u_x^2 \int_I \rho Q^2 + c \sup_{x \in I} (1 + \theta^{1+r}) \int_I (u_x^2 + \rho Q^2) + c sup \theta^{1+r} + c
\]
\[
\leq \frac{1}{4} \int_I \rho u_t^2 + c \int_I \kappa^2 \theta^2_x + c \left( \int_I u_x^2 \right)^2 + \frac{1}{4} \int_I \rho u_t^2 + c \int_I u_x^2 \int_I \rho Q^2
\]
\[
+ c \sup_{x \in I} (1 + \theta^{1+r}) \int_I (u_x^2 + \rho Q^2) + c \sup \theta^{1+r} + c. \tag{3.13}
\]

Substituting (3.13) into (3.12), we have
\[
\int_I \rho u_t^2 + \frac{d}{dt} \int_I u_x^2
\]
\[
\leq c \left( \int_I u_x^2 \right)^2 + 2 \frac{d}{dt} \int_I Pu_x - \frac{d}{dt} \int_I P^2 + c \int_I \kappa^2 \theta^2_x + c \int_I u_x^2 \int_I \rho Q^2
\]
\[
+ c \sup_{x \in I} (1 + \theta^{1+r}) \int_I (u_x^2 + \rho Q^2) + c \sup \theta^{1+r} + c. \tag{3.14}
\]

Integrating (3.14) over (0, t), and using (A2)-(A4), Lemma 3.2, Corollary 3.1, Lemma 3.3 and
the Cauchy inequality, we have

\[
\frac{1}{2} \int_0^t \int_I \rho u_x^2 + \int_I u_x^2 \\
\leq c \int_0^t \left( \int_I u_x^2 \right)^2 + 2 \int_I (\rho Q + P_x)u_x + c \int_0^t \int_I \kappa^2 \theta_x^2 + c \int_0^t \int_I u_x^2 \int_I \rho \theta^{2+2r} \\
+ c \int_0^t \sup_{x \in I} \theta^{1+r} \int_I u_x^2 + c \int_0^t \sup_{x \in I} (1 + \theta^{1+r}) \int_I \rho \theta^{2+2r} + c
\]

\[
\leq c \int_0^t \left( \int_I u_x^2 \right)^2 + \frac{1}{2} \int_I u_x^2 + c \int_I \rho \theta^{2+2r} + c \int_0^t \int_I \kappa^2 \theta_x^2 + c \int_0^t \int_I u_x^2 \int_I \rho \theta^{2+2r} \\
+ c \int_0^t \sup_{x \in I} \theta^{1+r} \int_I u_x^2 + c \int_0^t \sup_{x \in I} (1 + \theta^{1+r}) \int_I \rho \theta^{2+2r} + c.
\]

The second term of the right side can be absorbed by the left. After that, we have

\[
\int_0^t \int_I \rho u_x^2 + \int_I u_x^2 \\
\leq c \int_0^t \left( \int_I u_x^2 \right)^2 + c \int_I \rho \theta^{2+2r} + c \int_0^t \int_I \kappa^2 \theta_x^2 + c \int_0^t \int_I u_x^2 \int_I \rho \theta^{2+2r} \\
+ c \int_0^t \sup_{x \in I} \theta^{1+r} \int_I u_x^2 + c \int_0^t \sup_{x \in I} (1 + \theta^{1+r}) \int_I \rho \theta^{2+2r} + c.
\]

(3.15)

Here, we have used Lemma 3.2 and the Young inequality on the second term of the right side. Note that the terms about \( \theta \) in (3.15) need to be handled. To do this, we make use of (3.7).

Multiplying (3.7) by \( \int_0^t \kappa(\xi)d\xi \), integrating it over \( I \), and using integration by parts, (A4) and (A5), we have

\[
\frac{d}{dt} \int_I \rho \left[ \int_0^\theta Q'(\eta) \int_0^\eta \kappa(\xi)d\xi d\eta \right] + \int_I \kappa^2 \theta_x^2 \\
= \int_I u_x^2 \int_0^\theta \kappa(\xi)d\xi - \int_I \rho \kappa(\xi)d\xi - \int_0^\theta \kappa(\xi)d\xi \\
\leq c\|1 + \theta^2\|_{L^\infty} \int_I u_x^2 + c\|1 + \theta^2\|_{L^\infty} \int_I \rho(1 + \theta^{1+r})|u_x|.
\]

(3.16)

By Corollary 2.1 and (A5), we get

\[
\|1 + \theta^2\|_{L^\infty} \leq c\|\kappa \theta_x\|_{L^2} + c.
\]

(3.17)

Substituting (3.17) into (3.16), and using the H"older inequality, the Cauchy inequality and Lemma 3.2, we get

\[
\frac{d}{dt} \int_I \rho \left[ \int_0^\theta Q'(\eta) \int_0^\eta \kappa(\xi)d\xi d\eta \right] + \int_I \kappa^2 \theta_x^2 \\
\leq c\|\kappa \theta_x\|_{L^2} \int_I u_x^2 + c\int_I u_x^2 + c\|\kappa \theta_x\|_{L^2} \int_I \rho(1 + \theta^{1+r})|u_x| + c\int_I \rho(1 + \theta^{1+r})|u_x| \\
\leq c\|\kappa \theta_x\|_{L^2} \left( \int_I u_x^2 + c\|1 + \theta^{1+r}\|_{L^2}\|u_x\|_{L^2} \right) + c\int_I u_x^2 + c\int_I \rho(1 + \theta^{2+2r}) \\
\leq \frac{1}{2} \int_I \kappa^2 \theta_x^2 + c \left( \int_I u_x^2 \right)^2 + c \int_I \rho \theta^{2+2r} \int_I u_x^2 + c \int_I \rho \theta^{2+2r} + c,
\]
which implies
\[ \frac{d}{dt} \int_I \rho \left[ \int_0^\eta Q'(\eta) \kappa(\xi)d\xi d\eta \right] + \frac{1}{2} \int_I \kappa^2 \theta^2_x \leq c \int_I u_x^2 + c \int_I \rho \theta^{2+2r} x + c. \]

Integrating it over \((0, t)\), and using \((A_4), (A_5), \text{Lemma 3.1} \) and Corollary 3.1, we get
\[ \int_I \rho \theta^{2+2r} + \int_0^t \int_I \kappa^2 \theta^2_x \leq c \int_0^t \left( \int_I u_x^2 \right)^2 + c \int_0^t \left( \int_I \rho \theta^{2+2r} \int_I u_x^2 \right) + c. \tag{3.18} \]

By \((3.15), (3.18), \text{Corollary 3.1} \), Lemma 3.4 and the Gronwall inequality, we complete the proof.

**Lemma 3.6** Under the conditions of Theorem 3.1, it holds for any \(0 \leq t \leq T\)
\[ \int_I (\rho_x^2 + \rho_t^2) + \int_{Q_T} u_{xx}^2 \leq c. \]

*Proof.* Differentiating \((1.1)_1\) with respect to \(x\), we have
\[ \rho_{xt} + \rho_{xx} u + 2\rho_x u_x + \rho u_{xx} = 0. \tag{3.19} \]

Multiplying \((3.19)\) by \(2\rho_x\), integrating it over \(I\) and using integration by parts, we have
\[
\frac{d}{dt} \int_I \rho_x^2 \quad = \quad -3 \int_I \rho_x^2 u_x - 2 \int_I \rho \rho_x u_{xx} \\
\leq \quad c (\|u_x - P\|_{L^2} + \|u_{xx} - P_x \|_{L^2}) \int_I \rho_x^2 + c \int_I u_{xx}^2 + c \int_I \rho_x^2 \tag{3.20}
\]
where we have used \((3.11), \text{the Sobolev inequality, (A2)-(A4), the Cauchy inequality, Lemma 2.2, Lemma 3.2 \) and Lemma 3.5.\]

It follows from \((3.11), \text{Lemma 2.2, Lemma 3.2, (A2)-(A4), Lemma 3.5 \) and the Cauchy inequality that
\[
\int_I u_{xx}^2 \quad \leq \quad c \int_I \rho u_t^2 + c \left( \int_I u_x^2 \right)^2 + c \int_I \rho_x^2 Q^2 + c \int_I \rho^2 [Q'(\theta)]^2 \theta^2_x + c \int_I \rho_x^2 + c \\
\leq \quad c \int_I u_t^2 + c \sup_{x \in I} (1 + \theta^{2+2r}) \int_I \rho_x^2 + c \int_I (1 + \theta^q \theta_x^2 + c \\
\leq \quad c \int_I u_t^2 + c \sup_{x \in I} (1 + \theta^{q-\alpha+1}) \int_I \rho_x^2 + c \int_I (1 + \theta^q \theta_x^2 + c. \tag{3.21}
\]

Substituting \((3.21)\) into \((3.20)\), and using the Gronwall inequality, Corollary 3.1 and Lemma 3.5, we get
\[ \int_I \rho_x^2 \leq c. \tag{3.22} \]
By (3.21), (3.22), Corollary 3.1 and Lemma 3.5 we have
\[ \int_{Q_T} u_{xx}^2 \leq c. \]
It follows from (1.1)\textsubscript{1}, (3.22), Lemma 2.2, Lemma 3.2 and Lemma 3.5 that
\[ \int_I \rho_t^2 \leq c. \]
The proof of the lemma is complete. \(\square\)

**Lemma 3.7** Under the conditions of Theorem 3.1, it holds for any \(0 \leq t \leq T\)
\[ \int_I \left( \rho u_t^2 + \theta_x^2 \right) + \int_{Q_T} \left( u_{xt}^2 + \rho \theta_t^2 \right) \leq c. \]

**Proof.** Differentiating (3.11) with respect to \(t\), we have
\[ \rho u_{tt} + \rho_t u_t + \rho_tuu_x + \rho uu_{x} + \rho uu_{xt} + P_{xt} = u_{xxt}. \] (3.23)

Multiplying (3.23) by \(u_t\), integrating the resulting equation over \(I\), we have
\[
\frac{1}{2} \frac{d}{dt} \int_I \rho u_t^2 + \int_I u_{xt}^2 = -2 \int_I \rho uu_{x}u_t - \int_I \rho uu_{x}u_t - \int_I \rho u_t^2 + \int_0^T u_t \rho u_t
\leq 2 \| \sqrt{\rho} u_t \|_{L^2} \| \sqrt{\rho} \|_{L^\infty} \| u_{x} \|_{L^2} + \| u_t \|_{L^\infty} \| u \|_{L^\infty} \| \rho_t \|_{L^2} \| u_x \|_{L^2} + \| u_x \|_{L^\infty} \int_I \rho u_t^2
+ \| P'_{c}(\rho) \|_{L^\infty} \| \rho \|_{L^2} \| u_x \|_{L^2} + \| Q(\theta) \|_{L^\infty} \| \rho \|_{L^2} \| u_x \|_{L^2} + \| \rho Q'(\theta) \theta_t \|_{L^2} \| u_{x} \|_{L^2}
\leq \frac{1}{2} \int_I u_{x}^2 + c \int_I \rho u_t^2 + c + c \int_I u_{xx}^2 \int_I \rho u_t^2 + c \sup_{x \in I} \theta^{2+2r} + c \int_I \rho (1 + \theta^{q+r}) \theta_t^2.
\]

Here, we have used (1.1)\textsubscript{1}, integration by parts, the Hölder inequality, the Cauchy inequality, the Sobolev inequality, (A\textsubscript{2})-(A\textsubscript{4}), Lemma 2.2, Lemma 3.2, Lemma 3.5 and Lemma 3.6

The first term of the right side can be absorbed by the left. This implies
\[
\frac{d}{dt} \int_I \rho u_t^2 + \int_I u_{xt}^2 \leq c \int_I \rho u_t^2 + c + c \int_I u_{xx}^2 \int_I \rho u_t^2 + c \sup_{x \in I} \theta^{2+2r} + c \int_I \rho (1 + \theta^{q+r}) \theta_t^2. \] (3.24)

Integrating (3.24) over \((0, t)\), and using Corollary 3.1 and Lemma 3.5 we have
\[ \int_I \rho u_t^2 + \int_0^t \int_I u_{xt}^2 \leq \int_I \rho u_t^2(0) + c + c \int_0^t \int_I u_{xx}^2 \int_I \rho u_t^2 + c \int_0^t \int_I \rho (1 + \theta^{q+r}) \theta_t^2. \] (3.25)

Multiplying (3.11) by \(\frac{1}{\sqrt{\rho}}\), taking \(t \to 0^+\) and using (1.8)\textsubscript{1}, we have
\[
|\sqrt{\rho} u_t(x, 0)| \leq \frac{|u_{0xx} - P(\rho_0^5, \theta_0^5)|}{\sqrt{\rho_0^5}} + \sqrt{\rho_0^5}|u_0u_{0x}|
\leq \frac{|u_{0xx} - P(\rho_0, \theta_0)|}{\sqrt{\rho_0^5}} + \frac{|P(\rho_0, \theta_0)x - P(\rho_0^5, \theta_0^5)|}{\sqrt{\rho_0^5}} + \sqrt{\rho_0^5}|u_0u_{0x}|
\leq |g_1| + \frac{\delta}{\sqrt{\rho_0^5}}(|\rho_0| + |\theta_{0x}|) + c,
\]
which implies
\[ \int_I \rho u_t^2(0) \leq c. \] (3.26)

Substituting (3.26) into (3.25), we have
\[ \int_I \rho u_t^2 + \int_0^t \int_I u_{xt}^2 \leq c + c \int_0^t \int_I u_{xx}^2 \int_I \rho u_t^2 + c \int_0^t \int_I \rho (1 + \theta^{q+r}) \theta_t^2. \] (3.27)

Multiplying (3.7) by \( \left( \int_0^t \kappa(\xi) d\xi \right)_t \), integrating the resulting equation over \( I \), and using integration by parts, (A4), (A5), Lemma 2.2, Lemma 3.2, Lemma 3.5 and the Cauchy inequality, we have for any \( \varepsilon > 0 \)
\[ \int_I \rho Q' \kappa(\theta) \theta_t^2 + \frac{d}{dt} \int_I \kappa^2 \theta_x^2 \]
\[ = - \int_I \rho u_t' \kappa \theta_x \theta_t + \int_I \rho \theta Q' \kappa u_x \theta_t + \int_I u_x^2 \left( \int_0^\theta \kappa(\xi) d\xi \right) \]
\[ \leq \frac{1}{2} \int_I \rho Q' \kappa \theta_t^2 + c \int_I \rho u_x^2 Q' \kappa \theta_x^2 + c \int_I \rho Q^2 \kappa u_x^2 \]
\[ + \frac{d}{dt} \left( \int_I u_x^2 \int_0^\theta \kappa(\xi) d\xi \right) - 2 \int_I u_{xt} \int_0^\theta \kappa(\xi) d\xi \]
\[ \leq \frac{1}{2} \int_I \rho Q' \kappa \theta_t^2 + c \int_I (1 + \theta^q) \theta_x^2 + c \left( 1 + \int_I u_{xx}^2 \right) \int_I \rho (1 + \theta^{q+r+2}) \]
\[ + \frac{d}{dt} \left( \int_I u_x^2 \int_0^\theta \kappa(\xi) d\xi \right) + \varepsilon \int_I u_{xt} + c \varepsilon \sup_{x \in I} (1 + \theta^q) \theta_t^2, \]

which combining Lemma 2.2, Lemma 3.2, Lemma 3.5 implies
\[ \int_I \rho Q' \kappa(\theta) \theta_t^2 + \frac{d}{dt} \int_I \kappa^2 \theta_x^2 \]
\[ \leq c \int_I (1 + \theta^q)^2 \theta_x^2 + c \int_I u_{xx} + c + \frac{d}{dt} \left( \int_I u_x^2 \int_0^\theta \kappa(\xi) d\xi \right) + \varepsilon \int_I u_{xt}^2 + c \varepsilon \sup_{x \in I} (1 + \theta^q)^2 \theta_x^2 \]
\[ \leq c \int_I u_{xx} + \frac{d}{dt} \left( \int_I u_x^2 \int_0^\theta \kappa(\xi) d\xi \right) + \varepsilon \int_I u_{xt}^2 + c \varepsilon \int_I (1 + \theta^q)^2 \theta_x^2 + c \varepsilon. \]

Integrating it over \((0,t)\), and using (A4) and (A5), Lemma 2.2, Lemma 3.5, Lemma 3.6 and the Cauchy inequality, we obtain
\[ \int_0^t \int_I \rho \left( 1 + \theta^{q+r} \right) \theta_t^2 + \int_I (1 + \theta^q)^2 \theta_x^2 \]
\[ \leq c \int_I u_x^2 \int_0^\theta \kappa(\xi) d\xi + c \varepsilon \int_0^t \int_I u_{xt}^2 + c \varepsilon \]
\[ \leq c \sup_{x \in I} (1 + \theta^q) \theta_x + c \varepsilon \int_0^t \int_I u_{xt}^2 + c \varepsilon \]
\[ \leq c \|(1 + \theta^{q+r})_{\theta x}\|_{L^2} + c \varepsilon \int_0^t \int_I u_{xt}^2 + c \varepsilon \]
\[ \leq \frac{1}{2} \int_I (1 + \theta^q)^2 \theta_x^2 + c \varepsilon \int_0^t \int_I u_{xt}^2 + c \varepsilon. \]

After the first term of the right side is absorbed by the left, we get
\[ \int_0^t \int_I \rho \left( 1 + \theta^{q+r} \right) \theta_t^2 + \int_I (1 + \theta^q)^2 \theta_x^2 \leq c \varepsilon \int_0^t \int_I u_{xt}^2 + c \varepsilon. \] (3.28)
Multiplying (3.28) by $2c$, adding the resulting inequality to (3.25), taking $\varepsilon = \frac{1}{4c}$, and using the Gronwall inequality and Lemma 3.6, we complete the proof of Lemma 3.7.

From Corollary 2.1, Lemma 3.1 and Lemma 3.7, we get the following corollary immediately.

**Corollary 3.2** Under the conditions of Theorem 3.1, it holds

$$\|\theta\|_{L^\infty(Q_T)} \leq c.$$  

**Corollary 3.3** Under the conditions of Theorem 3.1, it holds for any $0 \leq t \leq T$

$$\|u\|_{W^{1,\infty}(Q_T)} + \int_I u_x^2 + \int_{Q_T} \theta_{xx}^2 \leq c.$$  

**Proof.** It follows from (3.21), Lemma 3.6, Lemma 3.7 and Corollary 3.2 that

$$\int_I u_{xx}^2 \leq c,$$

which, combining Lemma 2.2, Lemma 3.5 and the Sobolev inequality, gives

$$\|u\|_{W^{1,\infty}(Q_T)} \leq c. \quad (3.29)$$

By (3.7), Corollary 3.2, (A_1), (A_5), Lemma 2.2, Lemma 3.2, (3.29), Lemma 3.7, the Hölder inequality, Sobolev inequality and Cauchy inequality, we have

$$\int_I \theta_{xx}^2 \leq c \int_I \theta_x^4 + c \int_I u_x^4 + \int_I \rho \theta_t^2 + c \int_I u^2 \theta_x^2 + c \int_I \theta_{xx}^2 u_x^2$$

$$\leq c \|\theta_x \theta_{xx}\|_{L^1} \int_I \theta_x^2 + \int_I \rho \theta_t^2 + c$$

$$\leq c \|\theta_{xx}\|_{L^2} + \int_I \rho \theta_t^2 + c$$

$$\leq \frac{1}{2} \int_I \theta_{xx}^2 + \int_I \rho \theta_t^2 + c.$$

After the first term of the right side is absorbed by the left, we get

$$\int_I \theta_{xx}^2 \leq \int_I \rho \theta_t^2 + c. \quad (3.30)$$

Integrating (3.30) over $[0, T]$, and using Lemma 3.7 we get

$$\int_{Q_T} \theta_{xx}^2 \leq c.$$

This proves Corollary 3.3.

**Lemma 3.8** Under the conditions of Theorem 3.1, it holds for any $0 \leq t \leq T$

$$\|\rho\|_{W^{1,\infty}(Q_T)} + \|\rho_t\|_{L^\infty(Q_T)} + \int_I (\rho_{xx}^2 + \rho_{xt}^2) + \int_{Q_T} (\rho_{tt}^2 + u_{xx}^2) \leq c.$$
**Proof.** Differentiating (3.19) w.r.t. $x$, we have

$$\rho_{xxt} = -\rho_{xxx}u - 3\rho_{xx}u_x - 3\rho_x u_{xx} - \rho u_{xxx}. \quad (3.31)$$

Multiplying (3.31) by $2\rho_{xx}$, integrating it over $I$, and using integration by parts and the Hölder inequality, we have

$$\frac{d}{dt} \int_I \rho_{xx}^2 = -5 \int_I \rho_{xx}^2 u_x - 6 \int_I \rho_x \rho_{xx} u_{xx} - 2 \int_I \rho \rho_{xx} u_{xxx} \leq 5 \|u_x\|_{L^\infty} \int_I \rho_{xx}^2 + 6 \|\rho_x\|_{L^\infty} \|\rho_{xx}\|_{L^2} \|u_{xx}\|_{L^2} + 2 \|\rho\|_{L^\infty} \|\rho_{xx}\|_{L^2} \|u_{xxx}\|_{L^2}.$$ 

By the Sobolev inequality, Cauchy inequality, Lemma 3.2, Lemma 3.6 and Corollary 3.3, we have

$$\frac{d}{dt} \int_I \rho_{xx}^2 \leq c \int_I \rho_{xx}^2 + c \int_I u_{xxx}^2 + c. \quad (3.32)$$

The next step is to estimate the term $\int_I u_{xxx}^2$. Differentiating (3.11) with respect to $x$, we have

$$u_{xxx} = \rho_x u_t + \rho u_{xt} + \rho_x u u_x + \rho u_x^2 + \rho uu_{xx} + (P_c)_{xx} + (\rho Q)_{xx}. \quad (3.33)$$

By (A₃), (A₄), Lemma 2.2, Lemma 3.2, Lemma 3.6, Corollary 3.2, Corollary 3.3 and the Sobolev inequality, we get

$$\int_I u_{xxx}^2 \leq c \int_I \rho_{xx}^2 + c \int_I \rho_x^2 u_t^2 + c \int_I \rho_{xx}^2 + c \int_I \rho_{xx}^2 + c \int_I \theta_{xx}^2 + c. \quad (3.34)$$

Substituting (3.34) into (3.32), and using the Gronwall inequality, Lemma 3.7 and Corollary 3.3, we get

$$\int_I \rho_{xx}^2 \leq c. \quad (3.35)$$

By (3.35), Lemma 3.2, Lemma 3.6 and the Sobolev inequality, we have

$$\|\rho\|_{W^{1,\infty}(Q_T)} \leq c. \quad (3.36)$$

By (3.34), (3.35), Lemma 3.7 and Corollary 3.3, we get

$$\int_{Q_T} u_{xxx}^2 \leq c.$$ 

The estimates of $\rho_{xt}$ and $\rho_{tt}$ can be obtained directly by (3.19), (1.11), (3.35), (3.36), Lemma 2.2, Lemma 3.2, Lemma 3.6, Lemma 3.7 and Corollary 3.3. The proof of Lemma 3.8 is complete. 

**Lemma 3.9** Under the conditions of Theorem 3.1, it holds for any $0 \leq t \leq T$

$$\int_I \rho \theta_t^2 + \int_{Q_T} |(\kappa \theta_x)_t|^2 \leq c.$$
Differentiating (3.37) w.r.t. $t$, we have

$$
\rho Q'\theta_{tt} + \rho Q''\theta_t^2 + \rho_t\theta_t + (\rho u Q'\theta_x)_t + (\rho \theta Q'u_x)_t = 2u_xu_{xt} + (\kappa\theta_{x})_{xt}.
$$

(3.37)

Multiplying (3.37) by $(\int_0^\theta \kappa(\xi)d\xi)_t$ (i.e. $\kappa(\theta)\theta_t$), integrating it over $I$, and using integration by parts, Corollary 3.2, Lemma 3.2, Corollary 3.3 and the H"{o}lder inequality, we have

$$
\frac{1}{2}\frac{d}{dt}\int_I \rho Q'\kappa \theta_t^2 + \int_I |(\kappa\theta_x)_t|^2 \\
= -\frac{1}{2}\int_I \rho_t\rho Q'\kappa \theta_t^2 - \frac{1}{2}\int_I \rho Q''\theta_t^2\kappa + \frac{1}{2}\int_I \rho Q'\kappa \theta_t^2 - \int_I (\rho u Q'\theta_x)_t\kappa \theta_t \\
- \int_I (\rho Q'u_x)\kappa \theta_t + 2\int_I u_x u_{xt} \kappa \theta_t \\
\leq \frac{1}{2}\int_I (\rho u)_x Q'\kappa \theta_t^2 + c\|\kappa \theta_t\|_L^\infty \int_I \rho \theta_t^2 - \int_I \rho u Q'(\kappa\theta_x)_t \theta_t - \int_I \rho u (Q'' - Q'')\theta_t^2 \theta_x \\
- \int_I (\rho u)_t Q'\theta_x \kappa \theta_t + c\int_I u_x^2 + c\int_I \rho \theta_t^2 - \int_I \rho \theta_t^2 - \int_I \rho u Q'\kappa u_x \kappa \theta_t + c\|\kappa \theta_t\|_L^\infty \|u_{xt}\| \|u_x\| \|u_t\|_L^2.
$$

This, combining integration by parts, (A4), (A5), Lemma 2.1 Lemma 2.2 Corollary 3.2 Corollary 3.3 Lemma 3.7 Lemma 3.8 and the Cauchy inequality, gives

$$
\frac{1}{2}\frac{d}{dt}\int_I \rho Q'\kappa \theta_t^2 + \int_I |(\kappa\theta_x)_t|^2 \\
\leq -\frac{1}{2}\int_I \rho u Q''\kappa \theta_t^2 - \frac{1}{2}\int_I \rho Q'\kappa \theta_t^2 - \int_I \rho u Q'(\kappa\theta_x)_t \theta_t + c\|\kappa \theta_t\|_L^\infty \int_I \rho \theta_t^2 \\
+ c\sqrt{\rho \theta_t}\|_L^\infty \|\kappa \theta_x\|_L^\infty + c\|\kappa \theta_x\|_L^\infty + c\|\kappa \theta_t\|_L^\infty + c\int_I u_x^2 + c\int_I \rho \theta_t^2 \\
+ c\|\kappa \theta_t\|_L^\infty \|u_{xt}\|_L^2 \\
\leq c\|\kappa \theta_x\|_L^2 \int_I \rho \theta_t^2 + c\|\kappa \theta_x\|_L^\infty \int_I \rho \theta_t^2 + c\sqrt{\rho \theta_t}\|_L^\infty \|(\kappa\theta_x)_t\|_L^2 + c\|\kappa \theta_t\|_L^\infty \\
+ c\int_I u_x^2 + c\int_I \rho \theta_t^2 + c\|\kappa \theta_t\|_L^\infty \|u_{xt}\|_L^2 \\
\leq c\|\kappa \theta_x\|_L^2 \int_I \rho \theta_t^2 + c\left( \|(\kappa\theta_x)_t\|_L^2 + \int_I \rho \kappa |\theta_t| \right) \int_I \rho \theta_t^2 + c\sqrt{\rho \theta_t}\|_L^\infty \|(\kappa\theta_x)_t\|_L^2 \\
+ c\|(\kappa \theta_t)_x\|_L^2 + c\int_I \rho \kappa |\theta_t| + c\int_I u_x^2 + c\int_I \rho \theta_t^2 + c\left( \|(\kappa \theta_t)_x\|_L^2 + \int_I \rho \kappa |\theta_t| \right) \|u_{xt}\|_L^2,
$$

which together with the Cauchy inequality, Lemma 3.2 (A5) and Corollary 3.2 gives

$$
\frac{1}{2}\frac{d}{dt}\int_I \rho Q'\kappa \theta_t^2 + \int_I |(\kappa\theta_x)_t|^2 \leq \frac{1}{2}\int_I |(\kappa\theta_x)_t|^2 + c\int_I \rho \theta_t^2 + c\left( \int_I \rho \theta_t^2 \right)^2 + c\int_I u_x^2 + c,
$$

where we have used $(\kappa \theta_t)_x = (\kappa \theta_x)_t$. This gives

$$
\frac{d}{dt}\int_I \rho Q'\kappa \theta_t^2 + \int_I |(\kappa\theta_x)_t|^2 \leq c\int_I \rho \theta_t^2 + c\left( \int_I \rho \theta_t^2 \right)^2 + c\int_I u_x^2 + c.
$$

Integrating it over $(0, t)$, and using (A4), (A5), Lemma 3.7 and Corollary 3.3 we obtain

$$
\int_0^t \int_I |(\kappa\theta_x)_t|^2 \leq c\int_0^t \rho \theta^2 + c\left( \int_0^t \rho \theta^2 \right)^2 + c.
$$

(3.38)
Multiplying (3.7) by \( \frac{1}{Q'(\theta)\sqrt{\rho}} \), taking \( t \to 0^+ \), and using (1.8), we have

\[
|\sqrt{\rho} \theta_t(x,0)| \leq \frac{|u_0^2 + (\kappa(\theta_0^\delta)\theta_{0x})_x|}{Q'(\theta)^0\sqrt{\rho_0^\delta}} + |\sqrt{\rho_0^\delta}u_0\theta_{0x}| + |\sqrt{\rho_0^\delta}\theta_{0x}u_0| \\
\leq \frac{|u_0^2 + (\kappa(\theta_0)\theta_{0x})_x|}{Q'(\theta)^0\sqrt{\rho_0^\delta}} + |(\kappa(\theta_0^\delta)\theta_{0x})_x - (\kappa(\theta_0)\theta_{0x})_x| + c \\
\leq c|g_2| + \frac{c\delta}{\sqrt{\rho_0^\delta}}(1 + |\theta_{0xx}|) + c,
\]

which implies

\[
\int_I \rho \theta_t^2(0) \leq c \int_I g_2^2 + c \int_I \theta_{0xx}^2 + c \leq c.
\]

(3.39)

Substituting (3.39) into (3.38), using the Gronwall inequality and Lemma 3.7, we complete the proof.

\[\square\]

**Corollary 3.4** Under the conditions of Theorem 3.1, it holds

\[
\int_0^T \|\theta_t\|_{L^\infty}^2 \leq c.
\]

**Proof.** By Lemma 3.2, (A5), Corollary 2.1, Corollary 3.2, Lemma 3.9, and \((\kappa\theta_t)_x = (\kappa\theta)_t\), we get

\[
\int_0^T \|\kappa\theta_t\|_{L^\infty}^2 \leq c \int_0^T \|(\kappa\theta_t)_x\|_{L^2}^2 + c \leq c.
\]

This combining (A5) completes the proof. \[\square\]

**Corollary 3.5** Under the conditions of Theorem 3.1, it holds

\[
\int_{Q_T} \theta_{xt}^2 \leq c.
\]

**Proof.** Since

\[
\kappa\theta_{xt} = (\kappa\theta_x)_t - \kappa'\theta_t\theta_x,
\]

we obtain

\[
\int_{Q_T} \theta_{xt}^2 \leq c \int_{Q_T} \kappa^2\theta_{xt}^2 \\
\leq c \int_{Q_T} |(\kappa\theta_x)_t|^2 + c \int_{Q_T} (\kappa')^2\theta_t^2\theta_x^2 \\
\leq c + c \int_0^T \sup_{x \in I} \theta_t^2 \int_I \theta_x^2 \\
\leq c,
\]

where we have used (A5), Lemma 3.7, Lemma 3.9, Corollary 3.2, and Corollary 3.4. \[\square\]
Corollary 3.6 Under the conditions of Theorem 3.1, it holds for any $0 \leq t \leq T$

$$\|\theta\|_{W^{1,\infty}(Q_T)} + \int_0^t \int_{Q_T} \theta_x^2 + \int_{Q_T} \theta_{xxx}^2 \leq c. \quad (3.40)$$

Proof. From (3.30) and Lemma 3.9 we have

$$\int_0^t \theta_{xx}^2 \leq c, \quad (3.40)$$

which, combining Corollary 3.2, Lemma 3.7 and the Sobolev inequality, gives

$$\|\theta\|_{W^{1,\infty}(Q_T)} \leq c. \quad (3.41)$$

Differentiating (3.7) w.r.t. $x$, we have

$$\kappa \theta_{xxx} = -3\kappa' \theta_x \theta_{xx} - \kappa'' \theta_x^3 - 2u_x \theta_{xx} + \rho \theta_x \rho' \theta_t + \rho_x \theta_t + \rho_x \rho' \theta_x + \rho \theta \rho' \theta_x \theta_t + (\rho \theta' u_x) x. \quad (3.42)$$

By (3.40), (3.41), (3.42), (A4), (A5), Lemma 3.8 Lemma 3.9 and Corollary 3.3 we have

$$\int_0^t \theta_{xx}^2 \leq c \int_0^t \theta_x^2 \theta_{xx}^2 + c \int_0^t \theta_x^6 + c \int_0^t u_x^2 \theta_{xx}^2 + c \int_0^t \rho^2 \theta_x^2 + c \int_0^t \rho_x^2 \theta_t^2 + c \int_0^t \rho_x^2 \theta_x^2 \theta_t^2 + c \int_0^t \rho \rho' \theta_x \theta_t + (\rho \theta' u_x) x \leq c \int_0^t \rho^2 \theta_x^2 + c \int_0^t \rho_x^2 \theta_t^2 + c \leq c \int_0^t \theta_x^2 + c \sup_{x \in Q_T} \theta_x^2 + c. \quad (3.43)$$

By (3.43), Corollary 3.4 and Corollary 3.5 we obtain

$$\int_{Q_T} \theta_{xx}^2 \leq c. \quad \square$$

The next lemma, which we used in [8] to get $H^4$-estimates of velocity, plays an important role in getting $H^3$-estimates of $\theta$ in the following.

Lemma 3.10 Under the conditions of Theorem 3.1, it holds

$$\|\sqrt{\rho_x}\|_{L^\infty(Q_T)} + \|\sqrt{\rho_t}\|_{L^\infty(Q_T)} \leq c. \quad (3.44)$$

Proof. Multiplying (1.1) by $\frac{1}{2\sqrt{\rho}}$, we have

$$(\sqrt{\rho})_t + (\sqrt{\rho})_x u + \frac{1}{2\sqrt{\rho}} u_x = 0. \quad (3.44)$$

Differentiating (3.44) with respect to $x$, we get

$$(\sqrt{\rho})_{xt} + (\sqrt{\rho})_{xx} u + \frac{3}{2}(\sqrt{\rho})_{xu} u_x + \frac{1}{2} \sqrt{\rho} u_{xx} = 0.$$
Denote \( h = (\sqrt{\rho})_x \), we have
\[
 h_t + h_x u + \frac{3}{2} h u_x + \frac{1}{2} \sqrt{\rho} u x = 0,
\]
which implies
\[
 \frac{d}{ds} \left\{ h \exp \left( \frac{3}{2} \int_0^s \partial_x u (X(\tau; x, t), \tau) \, d\tau \right) \right\} = - \frac{1}{2} \sqrt{\rho} (\partial_x^2 u) \exp \left( \frac{3}{2} \int_0^s \partial_x u (X(\tau; x, t), \tau) \, d\tau \right),
\]
where \( X(s; x, t) \) is the solution to (3.33).

Integrating (3.45) over \((0, t)\), we get
\[
 h(x, t) = \exp \left( - \frac{3}{2} \int_0^t \partial_x u (X(\tau; x, t), \tau) \, d\tau \right) h(0; x, t, 0) - \frac{1}{2} \exp \left( - \frac{3}{2} \int_0^t \partial_x u (X(\tau; x, t), \tau) \, d\tau \right) \int_0^t \sqrt{\rho} (\partial_x^2 u) \exp \left( \frac{3}{2} \int_0^s \partial_x u (X(\tau; x, t), \tau) \, d\tau \right) \, ds.
\]
This together with Corollary 3.3, Lemma 3.8 and the Sobolev inequality, implies
\[
 \|(\sqrt{\rho})_x\|_{L^\infty(Q_T)} \leq c.
\]  
(3.46)

From (3.44), (3.46), Lemma 3.8 and Corollary 3.3, we get
\[
 \|(\sqrt{\rho})_t\|_{L^\infty(Q_T)} \leq c.
\]
This proves Lemma 3.10.

The next lemma will be used to get \( H^3 \)-estimates of \( \theta \).

**Lemma 3.11** Under the conditions of Theorem 3.1, it holds for any \( 0 \leq t \leq T \)
\[
 \int \rho^2 |(\kappa \theta_x)_t|^2 + \int_{Q_T} \rho^3 \theta^2_{tt} \leq c.
\]

**Proof.** Multiply (3.37) by \( \rho^{\gamma_1} (\kappa \theta_t)_t \) (i.e. \( \rho^{\gamma_1} \kappa \theta_t + \rho^{\gamma_1} \kappa' \theta^2_t \), where \( \gamma_1 \) is to be decided later), and using integration by parts, we have
\[
 \int \rho^{\gamma_1 + 1} \kappa Q' \theta^2_{tt} + \frac{1}{2} \frac{d}{dt} \int \rho^{\gamma_1} |(\kappa \theta_x)_t|^2
 = \frac{\gamma_1}{2} \int \rho^{\gamma_1 - 1} \rho_t |(\kappa \theta_x)_t|^2 - \gamma_1 \int \rho^{\gamma_1 - 1} \rho x \kappa \theta_{tt}(\kappa \theta_x)_t - \gamma_1 \int \rho^{\gamma_1 - 1} \rho x \kappa' \theta^2_t(\kappa \theta_x)_t
 + 2 \int u x u_x (\rho^{\gamma_1} \kappa \theta_t + \rho^{\gamma_1} \kappa' \theta^2_t) - \int \rho^{\gamma_1 + 1} Q' \kappa' \theta^2_t \theta_{tt}
 - \int (\rho Q' \theta^2_t + \rho_t Q' \theta_t + (\rho u Q' u_x)_t) \left( \rho^{\gamma_1} \kappa \theta_{tt} + \rho^{\gamma_1} \kappa' \theta^2_t \right). \quad (3.47)
\]

We are going to look for the minimal of \( \gamma_1 \). It seems that the second term of the right side plays an important role.
\[
 - \gamma_1 \int \rho^{\gamma_1 - 1} \rho x \kappa \theta_{tt}(\kappa \theta_x)_t = -2 \gamma_1 \int \rho^{\gamma_1 - \frac{1}{2}} (\sqrt{\rho})_x \kappa \theta_{tt}(\kappa \theta_x)_t
 \leq \frac{1}{4} \int \rho^{\gamma_1 + 1} \kappa Q' \theta^2_{tt} + c \int \rho^{\gamma_1 - 2} |(\kappa \theta_x)_t|^2, \quad (3.48)
\]
where we have used Lemma 3.10, (A4), (A5), Corollary 3.2, and the Cauchy inequality.

From Lemma 3.9, we know that \( \int_{Q_T} |(\kappa \theta_x)_t|^2 \leq c \). This implies that the minimal of \( \gamma_1 \) should be 2. Substituting \( \gamma_1 = 2 \) into (3.47) and (3.48), and then substituting (3.48) into (3.47), we have

\[
\frac{3}{4} \int_I \rho^3 \kappa Q' \theta_{tt}^2 + \frac{1}{2} \frac{d}{dt} \int_I \rho^2 |(\kappa \theta_x)_t|^2 \\
\leq c \int_I |(\kappa \theta_x)_t|^2 + c \int_I \rho \theta_t^4 + c \int_I u_{xt}^2 + c \int_I \theta_x^2 + c \\
+ c \left( \int_I \rho^3 \kappa Q' \theta_{tt}^2 \right)^\frac{1}{2} \left\{ 1 + \left( \int_I \rho \theta_t^4 \right)^\frac{1}{2} + ||\theta_x||_{L^2} + ||\sqrt{\rho} u_t||_{L^2} + ||u_x||_{L^2} + ||\sqrt{\rho} \theta||_{L^2} \right\} \\
\leq \frac{1}{4} \int_I \rho^3 \kappa Q' \theta_{tt}^2 + c \int_I |(\kappa \theta_x)_t|^2 + c ||\theta_t||_{L^\infty}^2 + c \int_I \theta_x^2 + c \int_I u_{xt}^2 + c,
\]

where we have used (A4), (A5), Lemma 3.7, Lemma 3.8, Lemma 3.9, Corollary 3.3, Corollary 3.6, and the Cauchy inequality. This implies

\[
\int_I \rho^3 \kappa Q' \theta_{tt}^2 + \frac{d}{dt} \int_I \rho^2 |(\kappa \theta_x)_t|^2 \leq c \int_I |(\kappa \theta_x)_t|^2 + c ||\theta_t||_{L^\infty}^2 + c \int_I \theta_x^2 + c \int_I u_{xt}^2 + c.
\]

Integrating it over \((0, t)\), and using (A4), (A5), Lemma 3.7, Lemma 3.9, Corollary 3.4, and Corollary 3.5, we have

\[
\int_I \rho^2 |(\kappa \theta_x)_t|^2 + \int_0^t \int_I \rho^2 \theta_{tt}^2 \leq c \int_I \rho^2 |(\kappa \theta_x)_t|^2(0) + c. \tag{3.49}
\]

By (A5), (3.39), (3.42), and Corollary 3.2, we get

\[
\int_I \rho^2 |(\kappa \theta_x)_t|^2(0) \leq c \int_I \rho^2 \theta_{xt}^2(0) + c \int_I \rho \theta_t^2(0) \\
\leq c ||\theta_0||_{H^3}^2 + c ||u_0||_{H^2}^2 + c \int_I \rho_x^2 u_{xt}^2(0) + c \\
\leq c + c \int_I |(\sqrt{\rho})_x|^2 \rho u_{xt}^2(0) \\
\leq c. \tag{3.50}
\]

Substituting (3.50) into (3.49), we complete the proof.

\[\square\]

**Corollary 3.7** Under the conditions of Theorem 3.1, it holds for any \(0 \leq t \leq T\)

\[
\int_I (\theta_{xx}^2 + \rho \theta_{xt}^2) \leq c.
\]

**Proof.** A direct calculation gives

\[
\rho \kappa \theta_{xt} = \rho (\kappa \theta_x)_t - \rho \kappa' \theta_x,
\]

which implies

\[
\int_I \rho^2 \theta_{xt}^2 \leq c \int_I \rho^2 |(\kappa \theta_x)_t|^2 + c ||\theta_x||_{L^\infty}^2 \int_I \rho \theta_t^2 \\
\leq c. \tag{3.51}
\]
Here we have used $(A_5), \text{Lemma }[3.8], \text{Lemma }[3.9], \text{Lemma }[3.11] \text{ and Corollary } 3.6$.

From the second inequality of (3.43), we obtain
\[
\int_I \theta_{xx}^2 \leq c \int_I \rho \theta_{x}^2 + c \int_I \rho \theta_t^2 + c
\]
\[
\leq c \int_I (\sqrt{\rho})_x^2 \rho \theta_t^2 + c
\]
\[
\leq c,
\]

where we have used Lemma 3.9, (3.51) and Lemma 3.10.

The next lemma will be used to get $H^3-$estimates of $u$.

**Lemma 3.12** Under the conditions of Theorem 3.1, it holds for any $0 \leq t \leq T$
\[
\int_I \rho u_{xt}^2 + \int_{Q_T} \rho^3 u_{tt}^2 \leq c.
\]

**Proof.** Similarly to Lemma 3.11, multiplying (3.23) by $\rho^2 u_{tt}$, and integrating it over $I$, we have
\[
\int_I \rho^3 u_{tt}^2 + \frac{1}{2} \frac{d}{dt} \int_I \rho^2 u_{xt}^2
\]
\[
= \int_I \rho \rho_t u_{xt}^2 - \int_I \rho \rho x u_{xt} u_{tt} - \int_I \rho^2 u_{tt} (\rho u_t + \rho_t u^2 + \rho u_t u_x + \rho u_x u_{tt} + P_{xt})
\]
\[
\leq c \int_I u_{xt}^2 - 4 \int_I \rho^2 (\sqrt{\rho})_x u_{xt} u_{tt} + \frac{1}{4} \int_I \rho^3 u_{tt}^2 + c \int_I \rho u_{tt}^2 + c \int_I (\rho Q)_x u_{tt}^2 + c \int_I (P_c)_x u_{tt}^2 + c
\]
\[
\leq \frac{1}{2} \int_I \rho^3 u_{tt}^2 + c \int_I u_{xt}^2 + c \|\theta_t\|^2_{L^\infty} + c \int_I \theta_{xt}^2 + c.
\]

Here, we have used integration by parts, Lemma 3.7, Lemma 3.8, Lemma 3.10, Corollary 3.3, Corollary 3.6 and the Cauchy inequality.

The first term of the right side can be absorbed by the left. After that, we have
\[
\int_I \rho^3 u_{tt}^2 + \frac{d}{dt} \int_I \rho^2 u_{xt}^2 \leq c \int_I u_{xt}^2 + c \|\theta_t\|^2_{L^\infty} + c \int_I \theta_{xt}^2 + c.
\]

Integrating this inequality on both side over $(0, t)$, and using Lemma 3.7, Corollary 3.4 and Corollary 3.6, we have
\[
\int_0^t \int_I \rho^3 u_{tt}^2 + \int_I \rho^2 u_{xt}^2 \leq \int_I \rho^2 u_{xt}^2(0) + c.
\]

Similarly to (3.50), we use (3.26), (3.33), (A3) and (A4) to get
\[
\int_I \rho^2 u_{xt}^2(0) \leq c \|u_0\|^2_{H^3} + c \|\theta_0\|^2_{H^2} + c \|\rho_0\|^2_{H^2} + c \int_I \rho u_{tt}^2(0) + c
\]
\[
\leq c.
\]

Substituting (3.53) into (3.52), we complete the proof.

By (3.31), Lemma 3.7, Lemma 3.8, Lemma 3.10, Lemma 3.12 and Corollary 3.6, we get the following corollary.
Corollary 3.8 Under the conditions of Theorem 3.1, it holds for any $0 \leq t \leq T$

$$\int_t u^3_{xxx} \leq c.$$

From the above estimates, we get

$$\|\sqrt{\rho}\|_{L^\infty} + \|\sqrt{\rho_t}\|_{L^\infty} + \|\rho\|_{H^2} + \|\rho_t\|_{H^1} + \|u\|_{H^3} + \|\rho u_t\|_{H^1} + \|\sqrt{\rho u_t}\|_{L^2} + \|\theta\|_{H^3} + \|\sqrt{\theta}\|_{L^2} + \|\rho\theta_t\|_{H^1} + \int_{Q_T} (u^2_{xt} + \rho^2_{tt} + \theta^2_t + \theta^2_{tt} + \theta^2_{xxt} + \rho^2 u^2_{tt} + \rho^3 \theta^2_{tt}) \leq c.$$

(3.54)

Corollary 3.9 Under the conditions of Theorem 3.1, there exists a positive constant $c_\delta$ depending on $\delta$ such that for any $(x, t) \in Q_T$, it holds

$$\begin{cases}
\rho(x, t) \geq \frac{\delta}{c} > 0, \\
\theta(x, t) \geq c_\delta > 0.
\end{cases}$$

(3.55)

Proof. By (3.5), (A_3), (A_4), Lemma 3.8 and Corollary 3.6, we have for any $(x, t) \in Q_T$

$$\rho(x, t) \geq \frac{\delta}{c}.$$

This gets (3.55)_1. (3.55)_2 can be got by (3.55)_1, (3.54), (3.7) and the maximum principle for parabolic equation. \qed

From (3.51), (3.55), (3.23) and (3.37), we obtain

$$\|\rho\|_{H^2} + \|\rho_t\|_{H^1} + \|u\|_{H^3} + \|u_t\|_{H^1} + \|\theta\|_{H^3} + \|\theta_t\|_{H^1} + \int_{Q_T} (u^2_{xt} + u^2_{xxt} + \rho^2_{tt} + \theta^2_t + \theta^2_{tt} + \theta^2_{xxt} + \rho^2 u^2_{tt} + \rho^3 \theta^2_{tt}) \leq c.$$

This proves Theorem 3.1 \qed

Proof of Theorem 1.1 Consider (1.1)-(1.4) with initial data replaced by $(\rho_0^\delta, u_0^\delta, \theta_0^\delta)$, we obtain from Theorem 3.1 that there exists a unique solution $(\rho^\delta, u^\delta, \theta^\delta)$, such that (3.54) and (3.55) are valid when we replace $(\rho, u, \theta)$ by $(\rho^\delta, u^\delta, \theta^\delta)$. With the estimates uniform for $\delta$, we take $\delta \to 0^+$ (take subsequence if necessary) to get a solution to (1.1)-(1.4) still denoted by $(\rho, u, \theta)$ which satisfies (3.54) by the lower semi-continuity of the norms. This proves the existence of the solutions as in Theorem 1.1. The uniqueness of the solutions can be proved by the standard method like in [4], we omit it for brevity. The proof of Theorem 1.1 is complete. \qed

4 Proof of Theorem 1.2

In this section, we use the similar arguments as in Section 3 to prove Theorem 1.2. Throughout this section, we denote $c$ to be a generic constant depending on $\rho_0, u_0, \theta_0, T$ and some other known constants but independent of $\delta$ for any $\delta \in (0, 1)$. 
Denote \( \rho_0^\delta = \rho_0 + \delta, \theta_0^\delta = \theta_0 + \delta \) and \( P_0^\delta = P(\rho_0^\delta, \theta_0^\delta) \), where \( \rho_0 \) and \( \theta_0 \) satisfy the same conditions as those in Theorem 1.2. Note that \( \rho_0^\delta \in H^4(I), \rho_0^\delta \geq \delta > 0, \theta_0^\delta \in H^3(I), \partial_x \theta_0^\delta|_{x=0.1} = \partial_x \theta_0|_{x=0.1} = 0 \), and

\[
\begin{align*}
\|\rho_0^\delta\|_{H^4} &\leq c, \\
\|\sqrt{\rho_0^\delta}\|_{L^\infty} &\leq c, \\
\|\theta_0^\delta\|_{H^3} &\leq c. 
\end{align*}
\]

Different from Section 3, we need to mollify \( g_3 \). Denote \( g_3^\delta = J_\delta * \overline{g}_3 \), then \( g_3^\delta \in C^\infty(I) \), where

\[
\overline{g}_3(x) = \begin{cases} 
-g_3(-x), & x \in [-1, 0), \\
g_3(x), & x \in I, \\
-g_3(2-x), & x \in (1, 2],
\end{cases}
\]

and \( J\delta(\cdot) = \frac{1}{\sqrt{\delta}} J(\frac{\cdot}{\sqrt{\delta}}) \), and \( J \) is the usual mollifier such that \( J \in C_0^\infty(\mathbb{R}) \), \( \text{supp} J \in (-1, 1) \), and \( \int_{\mathbb{R}} J(x)dx = 1 \). Since \( g_3 \in H^1_0(I) \), we have \( \overline{g}_3 \in H^1_0([-1, 2]) \) and

\[
\partial_x \overline{g}_3(x) = \begin{cases} 
g_3^\delta(-x), & x \in [-1, 0), \\
g_3^\delta(x), & x \in I, \\
g_3^\delta(2-x), & x \in (1, 2].
\end{cases}
\]

Claim:

\[
\begin{align*}
g_3^\delta \rightarrow g_3 & \quad \text{in } H^1(I), \text{ as } \delta \rightarrow 0, \\
\|g_3^\delta\|_{H^1(I)} &\leq c\|\overline{g}_3\|_{H^1([-1,2])} \leq c\|g_3\|_{H^1(I)}, \text{ for any } \delta \in (0,1), \\
\|\sqrt{\rho_0^\delta} (g_3^\delta)_{xx}\|_{L^2(I)} &\leq c, \text{ for any } \delta \in (0,1).
\end{align*}
\]

In fact, the proof of (4.2)\(_1\) and (4.2)\(_2\) can be found in [7]. We are going to prove (4.2)\(_3\).

\[
\sqrt{\rho_0^\delta} (g_3^\delta)_{xx} = \left( \sqrt{\rho_0^\delta} - \sqrt{\rho_0} \right)(g_3^\delta)_{xx} + \sqrt{\rho_0} (g_3^\delta)_{xx} = \frac{\delta (g_3^\delta)_{xx}}{\sqrt{\rho_0^\delta} + \sqrt{\rho_0}} + \sqrt{\rho_0} (g_3^\delta)_{xx} = A_1 + A_2.
\]

Recall \( J\delta(\cdot) = \frac{1}{\sqrt{\delta}} J(\frac{\cdot}{\sqrt{\delta}}) \), we conclude

\[
\|A_1\|_{L^2(I)} \leq \sqrt{\delta}\|((g_3^\delta)_{xx})_{x}\|_{L^2(I)} \leq c\|\overline{g}_3\|_{L^2([-1,2])} \leq c\|g_3\|_{L^2(I)},
\]

A direct calculation combining \( (\sqrt{\rho_0} (g_3)_{xx}) x \in L^2(I) \) gives

\[
\int_I |A_2|^2 \leq c.
\]

By (4.3), (4.4) and (4.5), we get (4.2)\(_3\).
Let \( u_0^\delta \) be the solution to the following elliptic problem for each \( \delta \in (0, 1) \):
\[
\begin{aligned}
&u_{0xx}^\delta - (P_0^\delta)_x = \rho_0^\delta g_3^\delta, \\
&u_0^\delta|_{x=0,1} = 0.
\end{aligned}
\]  
(4.6)

Since \( \rho_0^\delta = \rho_0 + \delta \in H^4(I), \theta_0^\delta = \theta_0 + \delta \in H^3(I), \) and \( g_3^\delta \in C^\infty(I) \), we obtain from the elliptic theory (see [7], (1.1), (1.2) and (4.6) that \( u_0^\delta \in H^3(I) \cap H^3_0(I) \) with the following properties:
\[
\begin{aligned}
&u_0^\delta \rightarrow u_0 \text{ in } H^3(I), \text{ as } \delta \rightarrow 0, \\
&\|u_0^\delta\|_{H^3(I)} \leq c \text{ for any } \delta \in (0, 1).
\end{aligned}
\]  
(4.7)

**Theorem 4.1** Consider the same assumptions as in Theorem 1.2. Then for any \( T > 0 \) and \( \delta \in (0, 1) \) there exists a unique global solution \((\rho, u, \theta)\) to (1.1)-(1.4) with initial data replaced by \((\rho_0^\delta, u_0^\delta, \theta_0^\delta)\), such that
\[
\begin{aligned}
&\rho \in C([0, T]; H^4), \quad \rho_t \in C([0, T]; H^3), \quad \rho_{tt} \in C([0, T]; H^1) \cap L^2([0, T]; H^2), \\
&\rho_{ttt} \in L^2(Q_T), \quad \rho \geq \frac{\delta}{5} > 0, \quad u \in C([0, T]; H^4 \cap H^1_0) \cap L^2([0, T]; H^5), \\
&u_t \in C([0, T]; H^2) \cap L^2([0, T]; H^3), \quad u_{tt} \in C([0, T]; L^2) \cap L^2([0, T]; H^3), \\
&\theta \in C([0, T]; H^3) \cap L^2([0, T]; H^4), \quad \theta_t \in C([0, T]; H^1) \cap L^2([0, T]; H^2), \\
&\theta_{tt} \in L^2([0, T]; L^2), \quad \theta \geq c_\delta > 0,
\end{aligned}
\]
where \( c_\delta \) is a constant depending on \( \delta \), but independent of \( u \).

**Proof of Theorem 4.1:**

Similarly to the proof of Theorem 3.1 Theorem 4.1 can be proved by some \textit{a priori} estimates globally in time.

For any given \( T \in (0, +\infty) \), let \((\rho, u, \theta)\) be the solution to (1.1)-(1.4) as in Theorem 4.1. Then we have the following estimates.

**Lemma 4.1** Under the conditions of Theorem 4.1 it holds for any \( 0 \leq t \leq T \)
\[
\begin{aligned}
&\|\sqrt{\rho}_x\|_{L^\infty} + \|\sqrt{\rho}_t\|_{L^\infty} + \|\rho\|_{H^2} + \|\rho_t\|_{H^1} + \|u\|_{H^3} + \|\rho u_t\|_{H^1} + \|\sqrt{\rho} u_t\|_{L^2} + \|\theta\|_{H^3} \\
&+ \|\theta_t\|_{L^2} + \|\theta_{tt}\|_{H^1} + \int_{Q_T} (u_{x1}^2 + P_0^\delta + \theta_{t1}^2 + \theta_{t2}^2 + \rho \theta_{tt}) \leq c.
\end{aligned}
\]

**Proof.** Though the initial velocity in Theorem 4.1 (i.e. \( u_0^\delta \)) is different from that in Theorem 3.1 (i.e. \( u_0 \)), both of them are bounded in \( H^3 \). It suffices to check if (3.26) and (3.39) work here. If so, Lemma 4.1 will be obtained from (3.54).

By (3.11) and (4.6)
\[
|\sqrt{\rho} u_t(x, 0)| \leq \frac{|u_{0xx}^\delta - P(\rho_0^\delta, \theta_0^\delta)_x|}{\sqrt{\rho_0^\delta}} + \sqrt{\rho_0^\delta}|u_0 u_{0x}| \\
= \sqrt{\rho_0^\delta}|g_3^\delta| + \sqrt{\rho_0^\delta}|u_0 u_{0x}|.
\]
This gives
\[
\int P u_t^2(0) \leq c.
\]
Therefore, (3.26) is valid here.

Multiplying (3.7) by $\frac{1}{Q'(\theta)}\sqrt{\rho}$, taking $t \to 0^+$, and using (1.9), we have

$$
|\sqrt{\rho} \theta_t(x, 0)| \leq \frac{|(u_0^\delta)^2 + (\kappa(\theta_0^\delta)\theta_0x)|}{Q'(\theta_0^\delta)\sqrt{\rho_0^\delta}} + \left|\sqrt{\rho_0^\delta}u_0^\delta \theta_0x\right| + \left|\sqrt{\rho_0^\delta} \theta_0^\delta u_0x\right|
$$

$$
\leq \frac{u_0^2 + (\kappa(\theta_0)\theta_0x)}{Q'(\theta_0^\delta)\sqrt{\rho_0^\delta}} + c\left|u_0^\delta - u_0x\right| + \left|\frac{(\kappa(\theta_0^\delta)\theta_0x) - (\kappa(\theta_0)\theta_0x)}{Q'(\theta_0^\delta)\sqrt{\rho_0^\delta}} + c
\right|
$$

$$
\leq c|g_2| + \frac{c\delta}{\sqrt{\rho_0^\delta}}(1 + |\theta_0x|) + \frac{c|u_0^\delta - u_0x|}{\sqrt{\delta}}.
$$

Note that $\|u_0^\delta - u_0\|_{L^2(I)} \leq c\sqrt{\delta}$ by (1.9) and (1.6). This gives

$$
\int_I \rho \theta_t^2(0) \leq c \int_I g_2^2 + c \int_I \theta_0xx + c \leq c.
$$

Therefore, (3.39) is valid here. $\square$

**Lemma 4.2** Under the conditions of Theorem 4.1, it holds for any $0 \leq t \leq T$

$$
\int_I u_{xt}^2 + \int_{Q_T} \rho u_{tt}^2 \leq c.
$$

**Proof.** Multiplying (3.23) by $u_{tt}$, integrating it over $I$, and using integration by parts, Lemma 2.2, Lemma 4.1 and the Cauchy inequality, we have

$$
\int_I \rho u_{tt}^2 + \frac{d}{dt} \int_I u_{xt}^2
$$

$$
= -\frac{1}{2} \frac{d}{dt} \int_I \rho u_t^2 + \frac{1}{2} \int_I \rho u_{tt}^2 - \frac{d}{dt} \int_I \rho u_{xt} u_t + \int_I \rho u u_{xt} u_t + \int_I \rho u_t u_{xt} u_t + \int_I \rho u_t^2 u_t
$$

$$
+ \int_I \rho u_{xt} u_{xt} - \int_I \rho u_{tt} u_{xt} u_t - \int_I \rho u u_{xt} u_{tt} + \frac{d}{dt} \int_I P_t u_{xt} - \int_I P_t u_{xt}
$$

$$
\leq \frac{d}{dt} \int_I \left(P_t u_{xt} - \frac{1}{2} \rho u_t^2 - \rho u u_{xt} u_t\right) + c \int_I u_{xt}^2 \int_I \rho_{tt}^2
$$

$$
+c \int_I u_{xt}^2 + c \int_I \rho_{tt}^2 + \frac{1}{2} \int_I \rho u_{tt}^2 - \int_I P_t u_{xt} + c.
$$

This gives

$$
\int_I \rho u_{tt}^2 + \frac{d}{dt} \int_I u_{xt}^2 \leq \frac{d}{dt} \int_I (2P_t u_{xt} - \rho_t u_t^2 - 2\rho u u_{xt} u_t) + c \int_I u_{xt}^2 \int_I \rho_{tt}^2 + c \int_I u_{xt}^2
$$

$$
+c \int_I \rho_{tt}^2 - \frac{d}{dt} \int_I P_t^2 - 2 \int_I P_t (u_{xt} - P_t) + c. \tag{4.8}
$$

We are going to estimate the last term of the right side of (4.8). By (A2)-(A4), integration
by parts, Lemma 4.1 and the Cauchy inequality, we have

\[-2 \int_t^1 P_{tt}(u_{xt} - P_t)\]

\[= -2 \int_t^1 (\rho Q)_{tt}(u_{xt} - P_t) - 2 \int_t^1 (P_c)_{tt} [u_{xt} - (\rho Q)_t - (P_c)_t]\]

\[\leq -2 \int_t^1 [(\kappa \theta_x)_x + u_x^2 - (\rho P)_{xx} - \rho \theta P'_{xx}] (u_{xt} - P_t)
+ c \int_t^1 \rho \theta_t^2 + c \int_t^1 u_{xt}^2 + c \int_t^1 \rho \theta_t^2 + c.\]

This, combining (3.11), (A2)–(A4), Lemma 4.1 and the Cauchy inequality, concludes

\[-2 \int_t^1 P_{tt}(u_{xt} - P_t)\]

\[\leq c + 2 \int_t^1 (\kappa \theta_x)_t (\rho u_t + \rho u u_x)_t + c \int_t^1 u_{xt}^2 + c \int_t^1 \rho \theta_t^2
- 2 \int_t^1 (\rho Q)_t (\rho u_t + \rho u u_x)_t + 2 \int_t^1 (\rho \theta P'_{xx})_t (u_{xt} - P_t) + c \int_t^1 \rho \theta_t^2 + c.\]

Substituting (4.9) into (4.8), we get

\[\frac{1}{2} \int_t^1 \rho u_{tt}^2 + \frac{d}{dt} \int_t^1 u_{xt}^2 \leq \frac{d}{dt} \int_t^1 (2P_t u_{xt} - \rho u_t^2 - 2\rho u u_x u_t) + c \int_t^1 u_{xt}^2 \int_t^1 \rho \theta_t^2 + c \int_t^1 u_{xt}^2
+ c \int_t^1 \rho \theta_t^2 - \frac{d}{dt} \int_t^1 P_t^2 + c \int_t^1 (\kappa \theta_x)_t^2 + c \int_t^1 \rho \theta_t^2 + c.\]

Integrating (4.10) over (0, t), and using (1.1), integration by parts, (3.8), (3.11), (4.2), (4.6), and Lemma 4.1, we have

\[\frac{1}{2} \int_0^t \int_t^1 \rho u_{tt}^2 + \int_t^1 u_{xt}^2\]

\[\leq \int_t^1 (2P_t u_{xt} + (\rho u)_x u_t^2 - 2\rho u u_x u_t) + c \int_t^1 \int_t^1 u_{xt}^2 \int_t^1 \rho \theta_t^2 + c \int_t^1 \int_t^1 (\kappa \theta_x)_t^2 + c\]

\[= \int_t^1 (2P_t u_{xt} - 2\rho u u_x u_t - 2\rho u u_x u_t) + c \int_t^1 \int_t^1 u_{xt}^2 \int_t^1 \rho \theta_t^2 + c \int_t^1 \int_t^1 (\kappa \theta_x)_t^2 + c\]

\[\leq \frac{1}{2} \int_t^1 u_{xt}^2 + c \int_t^1 \rho \theta_t^2 + c \int_t^1 \rho \theta_t^2 u_{xt}^2 + c \int_t^1 \rho \theta_t^2 u_{xt}^2 + c \int_t^1 \int_t^1 u_{xt}^2 \int_t^1 \rho \theta_t^2 + c \int_t^1 \int_t^1 (\kappa \theta_x)_t^2 + c\]

which implies

\[\int_0^t \int_t^1 \rho u_{tt}^2 + \int_t^1 u_{xt}^2 \leq c \int_0^t \rho \theta_t^2 + c \int_0^t \int_t^1 \rho \theta_t^2 + c \int_0^t \int_t^1 (\kappa \theta_x)_t^2 + c.\]

Using the Gronwall inequality and Lemma 4.1, we complete the proof of the lemma.
Corollary 4.1  Under the conditions of Theorem 4.1, it holds for any \(0 \leq t \leq T\)
\[
\int_{Q_T} u_{xxt}^2 \leq c.
\]

Proof. It follows from (3.23), Lemma 4.1 and \((A_2)-(A_4)\) that
\[
\int_{Q_T} u_{xxt}^2 \leq c \int_{Q_T} \rho u_{tt}^2 + c \int_{Q_T} \rho^2 u_{xt}^2 + \int_{Q_T} \rho^2 u_{x}^2 + c \int_{Q_T} \rho^2 u_{xxt}^2 + c \int_{Q_T} \rho^2 u_{xxt}^2 + c \int_{Q_T} (\rho Q)_{xxt}^2 + c \int_{Q_T} (P_c)_{xxt}^2
\]
\[
\leq c + c \int_{0}^{T} \|u_{x}\|_{L^\infty}^2 + c \int_{Q_T} u_{x}^2 + c \int_{Q_T} \rho_{xxt}^2 + c \int_{Q_T} \theta_{xxt}^2 + c
\]
\[
\leq c.
\]
This proves Corollary 4.1. \(\square\)

Lemma 4.3  Under the conditions of Theorem 4.1, it holds for any \(0 \leq t \leq T\)
\[
\int_{I} \left( \rho_{xxx} + \rho_{xxt}^2 + \rho_{tt}^2 \right) + \int_{Q_T} \left( \rho_{xxt}^2 + u_{xxxx}^2 \right) \leq c.
\]

Proof. Differentiating (3.31) with respect to \(x\), we have
\[
\rho_{xxx} = -\rho_{xxxx} u - 4\rho_{xxxx} u_x - 6\rho_{x} u_{xxx} - 4\rho_{x} u_{xxx} - \rho_{xxxx}.
\]  (4.12)

Multiplying (4.12) by \(2\rho_{xxx}\), integrating the resulting equation over \(I\), and using integration by parts and the H"{o}lder inequality, we have
\[
\frac{d}{dt} \int_{I} \rho_{xxx}^2 = -7 \int_{I} \rho_{xxx}^2 u_x - 12 \int_{I} \rho_{xxx} \rho_{xxxx} u_{xx} - 8 \int_{I} \rho_{x} \rho_{xxx} u_{xxx} - 2 \int_{I} \rho \rho_{xxx} u_{xxxx}
\]
\[
\leq 7 \|u_x\|_{L^\infty} \int_{I} \rho_{xxx}^2 + 12 \|u_{xx}\|_{L^\infty} \|\rho_{xxx}\|_{L^2} \|\rho_{xxxx}\|_{L^2}
\]
\[
+ 8 \|\rho_{x}\|_{L^\infty} \|\rho_{xxx}\|_{L^2} \|u_{xxx}\|_{L^2} + 2 \|\rho\|_{L^\infty} \|\rho_{xxx}\|_{L^2} \|u_{xxxx}\|_{L^2}.
\]

By Lemma 4.1 and the Cauchy inequality, we get
\[
\frac{d}{dt} \int_{I} \rho_{xxx}^2 \leq c \int_{I} \rho_{xxx}^2 + c \int_{I} u_{xxxx}^2 + c. \tag{4.13}
\]

Differentiating (3.33) with respect to \(x\), we have
\[
u_{xxxx} = \rho_{xxx} u_t + 2\rho_{x} u_{xt} + \rho_{xxt} + (\rho_x u_{x})_x + (\rho u_{x})_x + (\rho u_{xx})_x
\]
\[
+ (P_c)_{xxx} + (\rho Q)_{xxx}. \tag{4.14}
\]

By (4.11), (A6) and Lemma 4.1, we have
\[
\int_{I} u_{xxxx}^2 \leq c \int_{I} \rho u_{xxt}^2 + c \int_{I} \rho_{xxx}^2 + c. \tag{4.15}
\]

By (4.13), (4.15), Corollary 4.1 and the Gronwall inequality, we get
\[
\int_{I} \rho_{xxx}^2 \leq c. \tag{4.16}
\]
It follows from (4.15), (4.16) and Corollary 4.1 that

$$\int_{Q_T} u_{xxx}^2 \leq c.$$ 

A direct calculation, combining (3.11), (3.31), (4.16), Lemma 4.1 Corollary 4.1 and Lemma 4.2 implies

$$\int_I (\rho_{xtt}^2 + \rho_{tt}^2) + \int_{Q_T} \rho_{tt}^2 \leq c.$$ 

The proof of Lemma 4.3 is complete.

The next lemma play the most important role in getting $H^4$ estimates of $u$.

**Lemma 4.4** Under the conditions of Theorem 4.1 it holds for any $0 \leq t \leq T$

$$\int_I \rho^3 u_{tt}^2 + \int_{Q_T} \rho^2 u_{xtt}^2 \leq c.$$ 

**Proof.** Differentiating (3.23) with respect to $t$, we have

$$(\rho u_{tt})_t + \rho u_{tt} + \rho_t u_{tt} + (\rho_t uu_x + \rho u u_{tx} + \rho uu_{xt})_t + P_{xtt} = u_{xttt}. \quad (4.17)$$

Multiplying (4.17) by $\rho^2 u_{tt}$ ($\gamma_2$ is to be decided later), and integrating the resulting equation over $I$, we have

$$\frac{1}{2} \frac{d}{dt} \int_I \rho^{\gamma_2} u_{tt}^2 + \int_I \rho^{\gamma_2} u_{xtt}^2$$

$$= \frac{\gamma_2 - 3}{2} \int_I \rho^{\gamma_2} \rho_t u_{tt}^2 - \int_I (\rho_{tt} u_t + \rho_t uu_x + 2 \rho_t u_{tx} + 2 \rho_{tt} u_{xt} + 2 \rho_{tt} u_{xt} + P_{xtt})(\rho^{\gamma_2} u_{tt})$$

$$- \int I \rho^{\gamma_2+1} u_{tt}^2 u_x - \int I \rho^{\gamma_2+1} uu_{tt} u_x - \gamma_2 \int I \rho^{\gamma_2-1} \rho_x u_{tt} u_x$$

$$\leq c \| (\sqrt{\rho})_t \|_{L^\infty} \int_I \rho^{\gamma_2+1} u_{tt}^2 + c \int_I \rho^{\gamma_2} u_{tt}^2 - \int I \rho^{\gamma_2} u_{tt} (\rho Q)_{x} - \int I \rho^{\gamma_2} u_{tt} (P_c)_{x}$$

$$+ c \int I \rho^{\gamma_2+1} u_{tt}^2 - \int I \rho^{\gamma_2+1} uu_{tt} u_x - 2 \gamma_2 \int I \rho^{\gamma_2-1/2} u_{ttx} u_t (\sqrt{\rho})_x + c$$

$$\leq c \int I \rho^{\gamma_2+1/2} u_{tt}^2 + c \int I \rho^{\gamma_2} u_{tt}^2 + \int I \rho^{\gamma_2} u_{tt} (\rho Q)_{tt} + \gamma_2 \int I \rho^{\gamma_2-1} \rho_x u_{tt} (\rho Q)_{tt}$$

$$+ c \| \rho_{tt} \|_{L^2}^2 + \frac{1}{4} \int I \rho^{\gamma_2} u_{tt}^2 + c \int I \rho^{\gamma_2} u_{tt}^2 + c \int I \rho^{\gamma_2-1} u_{tt}^2 + c$$

$$\leq c \int I \rho^{\gamma_2-1} u_{tt}^2 + c \int I \rho^{\gamma_2} u_{tt}^2 + \frac{1}{2} \int I \rho^{\gamma_2} u_{tt}^2 + c \int I \rho^{\gamma_2} (|\rho Q|_{tt})^2$$

$$+ c \int I \rho^{2 \gamma_2-2} u_{tt}^2 + c \int I \rho^2 |(\rho Q)|_{tt}^2 + c \| \rho_{ttx} \|_{L^2}^2 + c.$$ 

Here, we have used integration by parts, the Cauchy inequality, $A_2$, $A_3$, Lemma 2.2

Lemma 4.1 Lemma 4.2 and Lemma 4.3.

After the third term of the right side is absorbed by the left, we have

$$\frac{d}{dt} \int I \rho^{\gamma_2+1} u_{tt}^2 + \int I \rho^{\gamma_2} u_{xtt}^2 \leq c \int I \rho^{\gamma_2-1} u_{tt}^2 + \int I \rho^{\gamma_2} u_{tt}^2 + \int I \rho^{\gamma_2} (|\rho Q|_{tt})^2$$

$$+ c \int I \rho^{2 \gamma_2-2} u_{tt}^2 + c \int I \rho^2 |(\rho Q)|_{tt}^2 + c \| \rho_{ttx} \|_{L^2}^2 + c. \quad (4.18)$$
By Lemma 4.2, we know $\int_{Q_T} \rho u_t^2 \leq c$. This implies that the minimum of $\gamma_2$ we should take in (4.18) is 2. Substituting $\gamma_2 = 2$ into (4.18), we have

$$\frac{d}{dt} \int_I \rho^3 u_t^2 + \int_I \rho^2 u_{x tt}^2 \leq c \int_I \rho u_t^2 + c \int_I \rho |(\rho Q)_{tt}|^2 + c \int_I \rho^2 u_{xtt} + c. \quad (4.19)$$

We are going to estimate $\int_I \rho |(\rho Q)_{tt}|^2$. Using Lemma 4.1, Lemma 4.2 and Lemma 4.3, we have

$$\int_I \rho |(\rho Q)_{tt}|^2 = \int_I \rho |\rho u_t Q + 2\rho_t^2 \theta_t + \rho Q'' \theta_t^2 + \rho Q' \theta_{tt}|^2 \leq c + c\|\theta_t\|_{L^\infty}^2 + c \int_I \rho^2 \theta_t^2. \quad (4.20)$$

Substituting (4.20) into (4.19), integrating the resulting inequality over $(0, t)$, and using Lemma 4.1, Lemma 4.2 and Lemma 4.3, we get

$$\int_I \rho^3 u_{tt}^2 + \int_0^t \int_I \rho^2 u_{x tt}^2 \leq \int_I \rho^3 u_{tt}^2(x, 0) + c. \quad (4.21)$$

Using (4.21), (4.2), (4.6), (4.7), (3.8), (3.23) and (4.21), we have

$$\int_I \rho^3 u_{tt}^2(x, 0) \leq c,$$

which combining (4.21) completes the proof. \hfill \Box

**Lemma 4.5** Under the conditions of Theorem 4.3 it holds for any $0 \leq t \leq T$

$$\int_I \rho u_{xxt}^2 + \int_{Q_T} (u_{x x x t}^2 + \rho \theta^2_{x x t}) \leq c.$$

**Proof.** By (3.23), we have

$$\int_I \rho^2 u_{x x t}^2 \leq c \int_I \rho^3 u_{tt}^2 + c \int_I \rho \rho_t^2 u_t^2 + c \int_I \rho \rho_t^2 u^2_{x t} + c \int_I \rho^3 u_{tt}^2 u_{x t}^2 + c \int_I \rho |(\rho Q)_{x t}|^2 + c \int_I \rho |(\rho Q)_{x x t}|^2 \leq c,$$

where we have used (A2)-(A4), Lemma 2.2, Lemma 4.1, Lemma 4.2 and Lemma 4.3.

It follows from (3.37) and (A5) that

$$\int_{Q_T} \rho \theta^2_{x x t} \leq c \int_{Q_T} \rho^3 |\theta_t|^2 \theta_{x x t}^2 + c \int_{Q_T} \rho^3 |\theta_{x t}|^2 \theta_{x x t}^2 + c \int_{Q_T} \rho^3 |\theta_{x x t}|^2 \theta_{x t}^2 + c \int_{Q_T} \rho^3 |\theta^3_{x t}|^2 \theta_{x t}^2 + c \int_{Q_T} \rho^3 |\theta^3_{x x t}|^2 \theta_{x t}^2 + c \int_{Q_T} \rho^3 |\theta^3_{x x x t}|^2 \theta_{x t}^2,$$

which, combining (A4), (A5) and Lemma 4.1 gives

$$\int_{Q_T} \rho \theta^2_{x x t} \leq c \int_{Q_T} \theta_{x t}^2 + c \int_{Q_T} \rho \theta^2_{x t} + c \int_0^T \|\theta_t\|_{L^\infty}^2 + c \int_{Q_T} u_{x x t}^2 + c \quad (4.22)$$
Differentiating (3.33) with respect to $t$, we get

$$u_{xxxx} = 2(\sqrt{\rho})_x \sqrt{\rho} u_{tt} + \rho u_{xxt} + \rho_x u_{xt} + \rho_{xx} u_{tx} + \rho_{xxx} u_{tx} + \rho_{xxxx} u_x + 2 \rho u_{xxt} + \rho u_{txx} + \rho_{xx} u_{tx} + \rho u_{xxx} + \rho u_{xxxx} + (\rho Q)_{xxt} + (P c)_{xxt}.$$  

This, together with (4.22), (A6), Lemma 2.2, Lemma 4.1, Lemma 4.2, Lemma 4.3, Lemma 4.4 and Corollary 4.1, implies

$$\int_{Q_T} u_{xxxx}^2 \leq c + c \int_{Q_T} (\rho \theta_{xx}^2 + \theta_{xt}^2) + c \int_0^T \|\theta_t\|_{L^\infty}^2 + c \int_{Q_T} \rho_{xx}^2 \leq c.$$  

This completes the proof.  

From (4.15), Lemma 4.3 and Lemma 4.5, we get the following corollary immediately.

**Corollary 4.2** Under the conditions of Theorem 4.1, it holds for any $0 \leq t \leq T$

$$\int_I u_{xxxx}^2 \leq c.$$  

**Corollary 4.3** Under the conditions of Theorem 4.1, it holds

$$\int_{Q_T} \theta_{xxxx}^2 \leq c.$$  

**Proof.** Differentiating (3.42) with respect to $x$, we have

$$\kappa \theta_{xxxx} = -4 \kappa' \theta_{xxx} - 3 \kappa' \theta_{xx}^2 - 3 \kappa'' \theta_x^2 \theta_{xx} - (\kappa'' \theta_x^3)_x - 2 (u_x u_{xx})_x + (\rho Q' \theta_t)_x$$  

$$+ (\rho_x Q' \theta_t)_x + (\rho Q'' \theta_x \theta_t)_x + (\rho u Q' \theta_x)_x + (\rho \theta Q' u_x)_x.$$  

This, combining (A5), (A6), Lemma 3.3, Lemma 4.1 and Lemma 4.5, implies

$$\int_{Q_T} \theta_{xxxx}^2 \leq c + c \int_{Q_T} (\rho \theta_{xx}^2 + \theta_{xt}^2) + c \int_0^T \|\theta_t\|_{L^\infty}^2 + c \int_{Q_T} |\kappa''| \theta_x^8 \leq c.$$  

This proves Corollary 4.3.  

**Lemma 4.6** Under the conditions of Theorem 4.1, it holds for any $0 \leq t \leq T$

$$\int_I \rho_{xxxx}^2 + \int_{Q_T} u_{xxxx}^2 \leq c.$$  

**Proof.** Differentiating (4.12) with respect to $x$, multiplying the resulting equation by $2 \rho_{xxxx}$, integrating over $I$, and using integration by parts, Lemma 4.1, Lemma 4.3, Corollary 4.2 and the Cauchy inequality, we get

$$\frac{d}{dt} \int_I \rho_{xxxx}^2 = -9 \int_I \rho_{xxxx}^2 u_x - 20 \int_I \rho_{xxxx} \rho_{xxxx} u_{xx} - 20 \int_I \rho_{xxxx} \rho_{xxxx} u_{xxx}$$  

$$- 10 \int_I \rho_x \rho_{xxxx} u_{xxxx} - 2 \int_I \rho \rho_{xxxx} u_{xxxx}$$  

$$\leq c \int_I \rho_{xxxx}^2 + c \int_I u_{xxxx}^2 + c.$$  

(4.23)
Now we estimate the second term of the right-hand side of (4.23).

Differentiating (4.14) with respect to \(x\), we have

\[
\begin{align*}
    u_{xxxxx} &= \rho_{xxx} u_t + 3\rho_{xx} u_{xt} + 3\rho_{x} u_{xxt} + \rho_{xxxt} + (\rho_x u_u)_{xx} + (\rho u_x)_{xx} \\
    &\quad + (\rho u_{xx})_{xx} + (\rho Q)_{xxxx} + (P_c)_{xxxx}.
\end{align*}
\]

This, combining (A_6), Lemma 2.2, Lemma 4.1, Lemma 4.3 and Corollary 4.2 concludes

\[
\int_I u_{xxxxx}^2 \leq c \int_I u_{xxxx}^2 + c \int_I u_{xxxt}^2 + c \int_I \rho_{xxxx}^2 + c \int_I \theta_{xxxx}^2 + c.
\]  

(4.24)

Substituting (4.24) into (4.23), and using Corollary 4.1, Corollary 4.3, Lemma 4.5 and the Gronwall inequality, we obtain

\[
\int_I \rho_{xxxx}^2 \leq c.
\]  

(4.25)

It follows from (4.24), (4.25), Corollary 4.1, Corollary 4.3 and Lemma 4.5 that

\[
\int_{Q_T} u_{xxxxx}^2 \leq c.
\]

This completes the proof of Lemma 4.6.

\[\square\]

**Corollary 4.4** Under the conditions of Theorem 4.1, it holds for any \(0 \leq t \leq T\)

\[
\int_t (\rho_{xxt}^2 + \rho_{xxxt}^2) + \int_{Q_T} (\rho_{xxt}^2 + \rho_{xxxt}^2) \leq c.
\]

Here we have used the following inequality when we get the upper bound of \(\rho_{xxt}\):

\[
\rho_x^2 u^2_t = 2 \left( \sqrt{(\rho)} \sqrt{\rho_x} \right)^2 u^2_t \leq c \rho u^2_t.
\]

From the above estimates, we get

\[
\begin{align*}
    &\left\| \left( \sqrt{\rho} \right)_x \right\|_{L^\infty} + \left\| \left( \sqrt{\rho} \right)_t \right\|_{L^\infty} + \left\| \rho \right\|_{H^4} + \left\| \rho_t \right\|_{H^3} + \left\| \rho_{tt} \right\|_{H^1} + \left\| u \right\|_{H^4} \\
    &\left\| u_t \right\|_{H^1} + \left\| \rho_x^2 u_{tt} \right\|_{L^2} + \left\| \sqrt{\rho u_{xxt}} \right\|_{L^2} + \left\| \theta \right\|_{H^3} + \left\| \sqrt{\rho \theta}_t \right\|_{L^2} + \left\| \rho \theta_{x} \right\|_{L^2} \\
    &\quad + \int_{Q_T} \left( \rho^2 u_{xxt}^2 + \rho u_{xxt}^2 + u_{xxt}^2 + u_{xxxx}^2 \right) + \int_{Q_T} \left( \rho_{xxt}^2 + \rho_{xxxt}^2 + \theta_{xxxx}^2 + \theta_{xxt}^2 + \rho \theta_{xxt}^2 + \rho^3 \theta_{ht}^2 \right) \leq c.
\end{align*}
\]

(4.26)

From (4.26) and (3.55), we get

\[
\begin{align*}
    &\left\| \rho \right\|_{H^4} + \left\| \rho_t \right\|_{H^3} + \left\| \rho_{tt} \right\|_{H^1} + \left\| u \right\|_{H^4} + \left\| u_t \right\|_{H^2} + \left\| u_{tt} \right\|_{L^2} + \left\| \theta \right\|_{H^3} + \left\| \theta_t \right\|_{H^1} \\
    &\quad + \int_{Q_T} \left( u_{xxt}^2 + u_{xxxt}^2 + u_{xxxx}^2 \right) + \int_{Q_T} \left( \rho_{xxt}^2 + \rho_{xxxt}^2 + \theta_{xxxx}^2 + \theta_{xxt}^2 + \theta_{ht}^2 \right) \leq c(\delta),
\end{align*}
\]

where \(c(\delta)\) is a positive constant, and may depend on \(\delta\).

The proof of Theorem 4.1 is complete. \[\square\]

**Proof of Theorem 1.2:**
Consider (1.1)-(1.4) with initial data replaced by \((\rho_0^\delta, u_0^\delta, \theta_0^\delta)\), we obtain from Theorem 4.1 that there exists a unique solution \((\rho^\delta, u^\delta, \theta^\delta)\) such that (4.26) and (3.55) are valid when we replace \((\rho, u, \theta)\) by \((\rho^\delta, u^\delta, \theta^\delta)\). With this estimates uniform for \(\delta\), we take \(\delta \to 0^+\) (take subsequence if necessary) to get a solution to (1.1)-(1.4) still denoted by \((\rho, u, \theta)\). By the lower semi-continuity of the norms, we have

\[
\|(\sqrt{\rho})_x\|_{L^\infty} + \|(\sqrt{\rho})_t\|_{L^\infty} + \|\rho\|_{H^4} + \|\rho_t\|_{H^3} + \|\rho_{tt}\|_{H^1} + \|u\|_{H^4} + \|u_t\|_{H^1} + \|\sqrt{\rho}u_x\|_{L^2} + \|\theta\|_{H^3} + \|\sqrt{\rho}\theta_t\|_{L^2} + \|\rho\theta_{xt}\|_{L^2} + \int_{Q_T} (u_x^2 + u_{xxxt}^2 + u_{xxxxx}^2) \\
+ \int_{Q_T} (\rho_{tt}^2 + \rho_x^2 + \theta_{xx}^2 + \theta_t^2 + \theta_{xx}^2) \leq c,
\]

which proves the existence of the solutions as in Theorem 1.2. The uniqueness of the solutions can be proved by the standard method like in [4], we omit it for brevity. The proof of Theorem 1.2 is complete.

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A blow-up criterion of strong solution to a 3D viscous liquid-gas two-phase flow model with vacuum

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Abstract

In this paper, we get a unique local strong solution to a 3D viscous liquid-gas two-phase flow model in a smooth bounded domain. Besides, a blow-up criterion of the strong solution for $\frac{2}{3}\mu > \lambda$ is obtained. The method can be applied to study a blow-up criterion of the strong solution to Navier-Stokes equations for $\frac{2}{3}\mu > \lambda$, which improves the corresponding result about Navier-Stokes equations in [15] where $7\mu > \lambda$. Moreover, all the results permit the appearance of vacuum.

Keyword: Liquid-gas two-phase flow model, local strong solution, blow-up criterion, vacuum.

AMS Subject Classification (2000): 76T10, 76N10, 35L65.

1 Introduction

In this paper, we consider the following 3D viscous liquid-gas two-phase flow model

\begin{align}
\begin{cases}
 m_t + \text{div}(mu) = 0, \\
 n_t + \text{div}(nu) = 0, \\
 (mu)_t + \text{div}(mu \otimes u) + \nabla P(m, n) = \mu \Delta u + (\mu + \lambda) \nabla \text{div}u, & \text{in } \Omega \times (0, \infty),
\end{cases}
\end{align}

with the initial and boundary conditions

\begin{align}
(m, n, u)|_{t=0} = (m_0, n_0, u_0), & \text{ in } \overline{\Omega}, \\
 u(x, t) = 0, & \text{ on } \partial \Omega \times [0, \infty),
\end{align}

where $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain. Here $m = \alpha l\rho_l$ and $n = \alpha g\rho_g$ denote the liquid mass and gas mass, respectively; $\mu, \lambda$ are viscosity constants, satisfying

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0,$$
which implies $\mu + \lambda \geq \frac{1}{4} \mu > 0$.

The unknown variables $\alpha_l, \alpha_g \in [0, 1]$ denote respectively the liquid and gas volume fractions, satisfying the fundamental relation: $\alpha_l + \alpha_g = 1$. Furthermore, the other unknown variables $\rho_l$ and $\rho_g$ denote respectively the liquid and gas densities, satisfying equations of state: $\rho_l = \rho_{l,0} + \frac{P-P_{l,0}}{a_l^2}$, $\rho_g = \frac{P}{a_g^2}$, where $a_l, a_g$ are sonic speeds, respectively, in the liquid and gas, and $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants; $u$ denotes velocity of the liquid and gas; $P$ is the common pressure for both phases, which satisfies

$$P(m, n) = C^0 \left(-b(m, n) + \sqrt{b(m, n)^2 + c(m)}\right),$$

with $C^0 = \frac{1}{2}a_l^2$, $k_0 = \rho_{l,0} - \frac{\rho_{0}}{a_l^2} > 0$, $a_0 = \left(\frac{a_l}{a_g}\right)^2$ and

$$b(m, n) = k_0 - m - \left(\frac{a_g}{a_l}\right)^2 n = k_0 - m - a_0 n,$$

$$c(n) = 4k_0 \left(\frac{a_g}{a_l}\right)^2 n = 4k_0 a_0 n.$$

For more information about the above models, please refer to [11, 14, 20] and references therein.

The investigation of model (1.1) has been a topic during the last decade. There are many results about the numerical properties of this model or related model. However, there are few results providing insight into existence, uniqueness, regularity, asymptotic behavior and decay rate estimates concerning the two-phase liquid-gas models of the form (1.1). Let us review some previous works about the viscous liquid-gas two-phase flow model. For the model (1.1) in 1D, when the liquid is incompressible and the gas is polytropic, i.e., $P(m, n) = C\rho_l^{\gamma} \left(\frac{n}{\rho_l - m}\right)^{\gamma}$, Evje and Karlsen in [4] studied the existence and uniqueness of the global weak solution to the free boundary value problem with $\mu = \mu(m) = k_1 \frac{m^\beta}{(\rho_l - m)^{\beta+1}}$, $\beta \in (0, \frac{1}{2})$, when the fluids connected to vacuum state discontinuously.

Yao and Zhu extended the results in [4] to the case $\beta \in (0, 1]$, and also obtained the asymptotic behavior and regularity of the solution, see [18]. Evje, Flåtten and Friis in [2] also studied the model with $\mu = \mu(m, n) = k_2 \frac{n^\beta}{(\rho_l - m)^{\beta+1}}$ ($\beta \in (0, \frac{1}{2})$) in a free boundary setting when the fluids connected to vacuum state continuously, and obtained the global existence of the weak solution. Also, for the case of connecting to vacuum state continuously, Yao and Zhu investigated the free boundary problem to the model with constant viscosity coefficient, and obtained the existence and uniqueness of the global weak solution by the line method, where a new technique was introduced to get the key upper and lower bounds of gas and liquid masses $n$ and $m$, cf. [19]. Specifically, when both of the two fluids are compressible, one can consult the reference [3].

Recently, Yao, Zhang and Zhu obtained the existence of the global weak solution to the 2D model when the initial energy is small, see [20]. Furthermore, they proved a blow-up criterion in terms of the upper bound of the liquid mass for the strong solution to the 2D model in a smooth bounded domain, cf. [21]. Because of the complexity of the pressure $P(m, n)$, they in [21] can only deal with the case: there is no initial vacuum, i.e., $m_0 > 0$, $n_0 > 0$. Then, what will happen when the vacuum appears? In this paper, we prove the local existence of strong solution and give a blow-up criterion to the 3D viscous liquid-gas two-phase flow model in a smooth bounded domain with vacuum.

The main results are stated as follows.
**Theorem 1.1** (Local existence). Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^3$ and $q \in (3, 6]$. Assume that the initial data $m_0, n_0, u_0$ satisfy $m_0, n_0 \in W^{1,q}(\Omega)$, $u_0 \in H^1_0(\Omega) \cap H^2(\Omega)$, $0 \leq \overline{s}_0 m_0 \leq n_0 \leq \overline{s}_0 m_0$ in $\overline{\Omega}$, where $\overline{s}_0$ and $\overline{s}_0$ are positive constants. The following compatible condition is also valid:

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla P(m_0, n_0) = \sqrt{m_0^2 g}, \quad \text{for some } g \in L^2(\Omega). \quad (1.6)$$

Then, there exist a $T_0 > 0$ and a unique strong solution $(m, n, u)$ to the problem (1.1)-(1.5), such that

$$0 \leq \overline{s}_0 m \leq n \leq \overline{s}_0 m, \quad (m, n) \in C([0, T_0]; W^{1,q}(\Omega)), \quad (m_1, n_1) \in L^\infty(0, T_0; L^q(\Omega)),
$$

$$P \in L^\infty(0, T_0; W^{1,q}(\Omega)), \quad u \in L^\infty(0, T_0; H^1_0(\Omega) \cap H^2(\Omega)) \cap L^2(0, T_0; H^1_0(\Omega)),
$$

$$\sqrt{m_0} u_t \in L^\infty(0, T_0; L^2(\Omega)), \quad u_t \in L^2(0, T_0; H^1_0(\Omega)). \quad (1.7)$$

Furthermore, under the assumption

$$\lambda < \frac{25}{3} \mu, \quad (1.8)$$

we can establish a blow-up criterion of the strong solution:

**Theorem 1.2.** Under the assumptions of Theorem 1.1 if $T^* < \infty$ is the maximal existence time for the strong solution $(m, n, u)(x, t)$ to the problem (1.1)-(1.5) stated in Theorem 1.1 then

$$\lim_{T \to T^*} \sup \|m\|_{L^\infty(0, T; L^\infty(\Omega))} = \infty, \quad (1.9)$$

provided that (1.8) holds.

**Remark 1.1.** (i) For $\Omega = \mathbb{R}^3$, we can also get a unique strong solution to (1.1)-(1.5) and the blow-up criterion (1.9) by using the ideas of [1, 15] to modify the proofs of Theorem 1.1 and Theorem 1.2 slightly.

(ii) The proof of Theorem 1.1 and Theorem 1.2 implies that the following blow-up criterion would be obtained if the restriction (1.8) is removed:

$$\lim_{T \to T^*} \sup \left(\|m\|_{L^\infty(0, T; L^\infty(\Omega))} + \|\sqrt{m_0} u\|_{L^1(0, T; L^{s'}(\Omega))}\right) = \infty, \quad (1.10)$$

where $\frac{2}{s} + \frac{3}{s'} \leq 1$ and $3 < s' \leq \infty$. (1.10) is similar to [7].

(iii) Under the assumption (1.8), we can use our methods in Lemma 5.2 together with the estimates in [15] to get the following blow-up criterion of strong solution to Navier-Stokes equations:

$$\lim_{T \to T^*} \sup \|\rho(t)\|_{L^\infty(0, T; L^\infty(\Omega))} = \infty.$$

This relaxes the restriction $7\mu > \lambda$ in [15]. And our result can be viewed to be a generalization of [15].

We should mention that the methods introduced by Sun, Wang and Zhang in [15], Cho, Choe and Kim in [1] for Navier-Stokes equations will play a crucial role in our proof here. There are many results about blow-up criterion of the strong solution for the Navier-Stokes equations in addition to [7]. For the 2D compressible Navier-Stokes equations, Sun and Zhang in [16] obtained a blow-up criterion in terms of the upper bound of the density for the strong solution. For the 3D compressible Navier-Stokes equations, Sun, Wang and Zhang in [15] obtained a blow-up criterion in terms of the upper bound of the density for the strong solution, under the restriction $\lambda < 7\mu$. In both papers, the initial vacuum ($\rho_0 \geq 0$) was allowed and the domain included both the bounded smooth domain and $\mathbb{R}^N, N = 2, 3$. It also worths mentioning recent works [8, 9], under the assumptions

$$N = 2, \mu > 0, \mu + \lambda \geq 0, \Omega = T^2;$$
or

\[ N = 3, \, \lambda < 7\mu, \, \mu > 0, \, \text{and} \, 2\mu + 3\lambda \geq 0, \, \Omega \text{ is a smooth domain including } \mathbb{R}^3, \]

Huang and Xin proved the following blow-up criterion: if \( T^* < \infty \) is the maximal time of the existence of the strong solution, then

\[
\lim_{T \to T^*} \int_0^T \|\nabla u(t)\|_{L^p(\Omega)} dt = \infty.
\]  

(1.11)

Huang, Li and Xin in their recent paper [10] removed the restriction \( \lambda < 7\mu \) for \( N = 3 \), and got the blow-up criterion of strong solution:

\[
\lim_{T \to T^*} \int_0^T \|D(u)(t)\|_{L^\infty(\Omega)} dt = \infty,
\]

where \( D(u) = \frac{1}{2}(\nabla u + \nabla u') \). For the non-isentropic compressible Navier-Stokes equations, under the conditions: \( N = 2, \mu > 0, \mu + \lambda \geq 0, \Omega = T^2 \text{ or } [0, 1]^2; N = 3, \lambda < 7\mu, \mu > 0, \) and \( 2\mu + 3\lambda \geq 0, \Omega \) is a smooth bounded domain, please refer to [12, 5].

In Theorem 1.2, we give a blow-up criterion in terms of the upper bound of the liquid mass under the relaxed restriction (1.8), which improves the corresponding result about Navier-Stokes equations in [15] where \( 7\mu > \lambda \). Here, if the liquid mass is upper bounded, we can obtain a high integrability of the velocity, \( \sup_{0 \leq t \leq T} \int_\Omega \|m\|_p dx \leq C \), for some \( r \in (3, 4) \), see Lemma 5.2. Moreover, in order to overcome the singularity brought by the pressure \( P(m, n) \) when there is vacuum, we need the assumption: \( 0 \leq \Sigma_0 m_0 \leq n_0 \leq \Sigma_0 m_0 \), where \( \Sigma_0 \) and \( \Sigma_0 \) are positive constants.

2 Preliminaries

In this section, we give some useful lemmas which will be used in the next three sections, where \( N = 2, 3 \).

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^N \) be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function \( u \in W^{1,p}_0(\Omega) \) or \( u \in W^{1,p}(\Omega), \int_\Omega u dx = 0 \):

\[
\|u\|_{L^{p'}(\Omega)} \leq C_1 \|\nabla u\|_{L^p(\Omega)}\|u\|_{L^{p'}(\Omega)}^{1-\alpha},
\]

(2.1)

where \( \alpha = (1/r' - 1/p')(1/r' - 1/p + 1/N)^{-1} \); moreover, if \( p < N \), then \( p' \in [r', pN/(N - p)] \) for \( r' \leq pN/(N - p) \), and \( p' \in [pN/(N - p), r'] \) for \( r' > pN/(N - p) \). If \( p \geq N \), then \( p' \in [r', \infty) \) is arbitrary; moreover, if \( p > N \), then inequality (2.1) is also valid for \( \alpha = \infty \). The positive constant \( C_1 \) in inequality (2.1) depends on \( N, p, r', \alpha \) and the domain \( \Omega \) but independent of the function \( u \).

**Lemma 2.2.** Let \( \Omega \subset \mathbb{R}^N \) be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function \( u \in W^{1,p}(\Omega) \):

\[
\|u\|_{L^{p'}(\Omega)} \leq C_2 (\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)}\|u\|_{L^{p'}(\Omega)}^{1-\alpha}),
\]

(2.2)

where \( N, p, r', p' \) and \( \alpha \) are the same as those in Lemma 2.1. The positive constant \( C_2 \) in inequality (2.2) depends on \( N, p, r', \alpha \) and the domain \( \Omega \) but independent of the function \( u \).

The above two lemmas can be found in [13, 17] and the references therein.
Next, we give some $L^p$ ($p \in (1, \infty)$) regularity estimates for the solution of the following boundary problem:

\[
\begin{aligned}
LU := \mu \Delta U + (\mu + \lambda) \nabla \text{div} U &= F, \quad \text{in } \Omega, \\
U(x) &= 0, \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(2.3)

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $L$ is the Lamé operator, $U = (U_1, U_2, \cdots, U_N)$, $F = (F_1, F_2, \cdots, F_N)$. From (1.4), we know that (2.3) is a strong elliptic system. If $F \in W^{-1,2}(\Omega)$, then there exists an unique weak solution $U \in H^1_0(\Omega)$. In the subsequent context, we will use $L^{-1}F$ to denote the unique solution $U$ of the system (2.3) with $F$ belonging to some suitable space such as $W^{-1, p}(\Omega)$. Sun, Wang and Zhang in [15] [16] give the following estimates:

**Lemma 2.3.** Let $p \in (1, \infty)$, and $U$ be a solution of (2.3). Then there exists a constant $C$ depending only on $\mu$, $\lambda$, $p$, $N$ and $\Omega$ such that

1. if $F \in L^p(\Omega)$, then
\[
\|U\|_{W^{2,p}(\Omega)} \leq C\|F\|_{L^p(\Omega)};
\]  

(2.4)

2. if $F \in W^{-1, p}(\Omega)$ (i.e., $F = \text{div} f$ with $f = (f_{ij})_{N \times N}$, $f_{ij} \in L^p(\Omega)$), then
\[
\|U\|_{W^{1,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)};
\]  

(2.5)

3. if $F = \text{div} f$ with $f_{ij} = \partial_k h_{ij}^k$ and $h_{ij}^k \in W^{1, p}_0(\Omega)$ for $i, j, k = 1, 2, \cdots, N$, then
\[
\|U\|_{L^p(\Omega)} \leq C\|h\|_{L^p(\Omega)}.
\]  

(2.6)

**Lemma 2.4.** If $F = \text{div} f$ with $f = (f_{ij})_{N \times N}$, $f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega)$, then $\nabla U \in \text{BMO}(\Omega)$ and there exists a constant $C$ depending only on $\mu$, $\lambda$ and $\Omega$ such that

\[
\|\nabla U\|_{\text{BMO}(\Omega)} \leq C(\|f\|_{L^\infty(\Omega)} + \|f\|_{L^2(\Omega)}).
\]  

(2.7)

Here $\text{BMO}(\Omega)$ denotes the John-Nirenberg’s space of bounded mean oscillation whose norm is defined by

\[
\|f\|_{\text{BMO}(\Omega)} = \|f\|_{L^2(\Omega)} + [f]_{\text{BMO}(\Omega)},
\]

with the semi-norm

\[
[f]_{\text{BMO}(\Omega)} = \sup_{x \in \Omega, r \in (0, d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}(y)| dy,
\]

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center $x$ and radius $r$ and $d$ is the diameter of $\Omega$. For a measurable subset $E$ of $\mathbb{R}^N$, $|E|$ denotes its Lebesgue measure and

\[
 f_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y) dy.
\]

**Lemma 2.5.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ and $f \in W^{1, p}(\Omega)$ with $p \in (N, \infty)$. Then there exists a constant $C$ depending on $p$, $N$ and the Lipschitz property of the domain $\Omega$ such that

\[
\|f\|_{L^\infty(\Omega)} \leq C (1 + \|f\|_{\text{BMO}(\Omega)}) \ln(e + \|\nabla f\|_{L^p(\Omega)}).
\]  

(2.8)
3 Global existence for the linearized system

Consider
\[
\begin{align*}
  m_t + \text{div}(mv) &= 0, \\
  n_t + \text{div}(nv) &= 0, \\
  mu_t + mv \cdot \nabla u + \nabla P(m, n) &= Lu, \quad \text{in } \Omega \times (0, \infty),
\end{align*}
\]  
(3.1)

with the initial and boundary conditions
\[
(m, n, u)|_{t=0} = (m_0, n_0, u_0), \quad \text{in } \Omega.
\]  
(3.2)

\[
u(x, t) = 0, \quad \text{on } \partial \Omega \times [0, \infty),
\]  
(3.3)

where \( \Omega \subseteq \mathbb{R}^3 \) is a smooth bounded domain.

Throughout the rest of the paper, we denote \( W^{k,p} = W^{k,p}(\Omega) \) for \( k \geq 0 \) and \( 1 < p \leq \infty \) with the norm \( \| \cdot \|_{W^{k,p}} \). Particularly, \( H^2 = W^{2,2} \), and \( L^p = W^{0,p} \). Let \( \Omega \) be a smooth bounded domain.

**Theorem 3.1.** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^3 \) and \( q \in (3, 6] \). Assume \( m_0, n_0 \in W^{1,q} \), \( u_0 \in H^1 \cap H^2 \), \( m_0 \geq \delta > 0, n_0 \geq \delta > 0, v \in C([0, T]; H^1 \cap H^2) \cap L^2(0, T; W^{2,q}) \) and \( v_i \in L^2(0, T; H_0^1) \). Then there exists a unique strong solution \((m, n, u)\) to (3.1)-(3.3) such that
\[
(m, n) \in C([0, T]; W^{1,q}), \quad (m_t, n_t) \in C([0, T]; L^q),
\]
\[
P \in C([0, T]; W^{1,q}), \quad u \in C([0, T]; H^1 \cap H^2) \cap L^2(0, T; W^{2,q}),
\]
\[
u_i \in C([0, T]; L^2) \cap L^2(0, T; H_0^1), \quad m > 0, \quad n > 0 \text{ in } Q_T.
\]

**Proof.** By (1.1), (3.1) and (3.2), we get
\[
\begin{align*}
  m, n &\in C([0, T]; W^{1,q}); \quad m_t, n_t \in C([0, T]; L^q), \\
  \sup_{0 \leq t \leq T} \|m(t)\|_{W^{1,q}} &\leq \|m_0\|_{W^{1,q}} \exp \left\{ C \int_0^T \|\nabla v(s)\|_{W^{1,q}} ds \right\}, \\
  \sup_{0 \leq t \leq T} \|n(t)\|_{W^{1,q}} &\leq \|n_0\|_{W^{1,q}} \exp \left\{ C \int_0^T \|\nabla v(s)\|_{W^{1,q}} ds \right\}, \\
  0 &< \delta \exp \left\{ - \int_0^T \|\nabla v(s)\|_{L^\infty} ds \right\} \leq m \leq \|m_0\|_{L^\infty} \exp \left\{ \int_0^T \|\nabla v(s)\|_{L^\infty} ds \right\}, \\
  0 &< \delta \exp \left\{ - \int_0^T \|\nabla v(s)\|_{L^\infty} ds \right\} \leq n \leq \|n_0\|_{L^\infty} \exp \left\{ \int_0^T \|\nabla v(s)\|_{L^\infty} ds \right\}.
\end{align*}
\]  
(3.4)

This immediately gives
\[
P(m, n) \in C([0, T]; W^{1,q}), \quad P(m, n)_t \in C([0, T]; L^q).
\]  
(3.5)

It follows from (3.1), (3.4), (3.5) and (1.1) that
\[
u \in C([0, T]; H^1 \cap H^2) \cap L^2(0, T; W^{2,q}), \quad u_t \in C([0, T]; L^2) \cap L^2(0, T; H_0^1).
\]

\[\Box\]

4 Proof of Theorem 1.1

In this section, we get a unique local strong solution to (1.1) with \( m_0 \geq \delta > 0, n_0 \geq \delta > 0 \), and obtain some estimates uniformly for \( \delta \) (see Theorem 4.1). Theorem 1.1 will be obtained after constructing a sequence of approximate solutions \((m^\delta, n^\delta, u^\delta)\) by giving the initial data \((m_0, n_0)\) in Theorem 1.1 a lower bound \( \delta \), using the estimates in Theorem 4.1 and taking \( \delta \to 0 \) (taking subsequence if necessary).
Theorem 4.1. Under the conditions of Theorem \[\text{[1]}\] we assume \(m_0 \geq \delta > 0, n_0 \geq \delta > 0\). Then there exists a time \(T_0 > 0\) independent of \(\delta\) and a unique strong solution \((m, n, u)\) to \((1.1)-(1.5)\) such that

\[
(m, n) \in C([0, T_0]; W^{1,q}), \quad (m_t, n_t) \in C([0, T_0]; L^q),
\]

\[
P \in C([0, T_0]; W^{1,q}), \quad u \in C([0, T_0]; H^1_0 \cap H^2) \cap L^2(0, T_0; W^{2,q}),
\]

\[
u_t \in C([0, T_0]; L^2) \cap L^2(0, T_0; H^1_0), \quad m > 0, \quad n > 0 \quad \text{in} \quad Q_{T_0}.
\]

Moreover, we have the following estimates:

\[
\sup_{0 \leq t \leq T_0} \int_{\Omega} m|u_t|^2 + \int_0^{T_0} \int_{\Omega} |\nabla u_t|^2 \leq C,
\]

\[
\|u\|_{L^2(0,T_0;W^{2,q})} + \|u\|_{L^\infty(0,T_0;H^1_0 \cap H^2)} + \|m\|_{L^\infty(0,T_0;W^{1,q})} + \|n\|_{L^\infty(0,T_0;W^{1,q})} \leq C,
\]

\[
\frac{\delta_0}{T} \leq \delta_0 m \leq n \leq \delta_0 m, \quad \text{in} \quad Q_{T_0},
\]

\[
\|P\|_{L^\infty(Q_{T_0})} + \|P_m\|_{L^\infty(Q_{T_0})} + \|P_n\|_{L^\infty(Q_{T_0})} \leq C,
\]

where \(C\) is a positive constant, independent of \(\delta\).

To prove this theorem, we first construct a sequence of approximate solutions inductively as follows

(i) Define \(u^0 = 0\), and assume \(u^{k-1} \in C([0, T]; H^1_0 \cap H^2) \cap L^2(0, T; W^{2,q}) \cap H^1(0, T; H^1_0)\) was defined for \(k \geq 1\).

(ii) By Theorem \[\text{[5.1]}\] we can get \((m^k, n^k, u^k)\) with the regularities in Theorem \[\text{[5.1]}\] satisfying

\[
\begin{aligned}
m_t^k + \text{div}(m^k u^{k-1}) &= 0, \\
n_t^k + \text{div}(n^k u^{k-1}) &= 0, \\
m^k u_t^k + m^k u^{k-1} \cdot \nabla u^k + \nabla P^k &= Lu^k, \quad \text{in} \quad \Omega \times (0, T),
\end{aligned}
\]

where \(P^k = P(m^k, n^k)\). The initial and boundary conditions are stated as follows

\[
(m^k, n^k, u^k)_{l=0} = (m_0, n_0, u_0), \quad \text{in} \quad \overline{\Omega},
\]

\[
u^k(x, t) = 0, \quad \text{on} \quad \partial \Omega \times [0, T].
\]

Throughout this paper, we denote

\[
\Phi_K(t) = \max_{1 \leq k \leq K} \left(1 + \|m^k(t)\|_{L^\infty}\right), \quad \Psi_{K,r}(t) = \max_{1 \leq k \leq K} \left(1 + \int_{\Omega} m^k |u^{k-1}|^r \right),
\]

for \(r \in (3, 4)\) and \(K \in \mathbb{Z}_+\). The next step is to make some estimates for \((m^k, n^k, u^k)\) \((k \geq 1)\) independent of \(k\) and \(\delta\).

Lemma 4.1. Under the conditions of Theorem \[\text{[4.7]}\] we have for all \(k \geq 1\)

\[
0 < \delta_0 m^k \leq n^k \leq \delta_0 m^k, \quad \text{in} \quad Q_T.
\]

Proof. It follows from \[\text{[4.1]}_1\] and \[\text{[4.1]}_2\] that

\[
\left(\frac{n^k}{m^k}\right)_t + u^{k-1} \cdot \nabla \left(\frac{n^k}{m^k}\right) = 0.
\]

This implies

\[
\frac{d}{ds} \left(\frac{n^k}{m^k}\right)(X(s; x, t), s) = 0,
\]

(4.4)
where $X(s; x, t)$ is given by:

$$
\left\{ \begin{array}{l}
\frac{d}{ds} X(s; x, t) = u^{k-1}(X(s; x, t), s), \quad 0 \leq s < t, \\
X(t; x, t) = x.
\end{array} \right.
$$

Integrating (4.4) over $(0, t)$, and using the assumption $\delta_0 m_0 \leq n_0 \leq \delta_0 m_0$, we complete the proof of Lemma 4.1.

**Lemma 4.2.** Under the conditions of Theorem 4.1, we have for all $1 \leq k \leq K$

\begin{align}
0 < P^k \leq C \Phi_K(t) & \text{ in } Q_T, \quad (4.5) \\
0 < P^k_{m^k} \leq C & \text{ in } Q_T, \quad (4.6) \\
0 < P^k_{n^k} \leq C & \text{ in } Q_T, \quad (4.7)
\end{align}

where $C$ is a positive constant, independent of $K$, $\delta$ and $T$.

**Proof.** (4.5) can be obtained by Lemma 4.1 and (1.5). A direct calculation gives

\begin{align}
P^k_{m^k} &= C \left\{ 1 - \frac{b(m^k, n^k)}{\sqrt{b^2(m^k, n^k) + c(n^k)}} \right\} > 0, \quad (4.8) \\
P^k_{n^k} &= C \left\{ a_0 + \frac{a_0}{\sqrt{b^2(m^k, n^k) + c(n^k)}} (m^k + a_0 n^k + k_0) \right\} > 0. \quad (4.9)
\end{align}

Obviously, we get (4.6) by (4.8). To get (4.7), it suffices to prove

$$
\left( \frac{m^k + a_0 n^k + k_0}{\sqrt{b^2(m^k, n^k) + c(n^k)}} \right)^2 \leq C.
$$

In fact,

\begin{align*}
\left( \frac{m^k + a_0 n^k + k_0}{\sqrt{b^2(m^k, n^k) + c(n^k)}} \right)^2 &= \frac{(m^k)^2 + a_0^2 (n^k)^2 + k_0^2 + 2k_0 m^k + 2a_0 m^k n^k + 2a_0 k_0 n^k}{(m^k)^2 + a_0^2 (n^k)^2 + k_0^2 - 2k_0 m^k + 2a_0 m^k n^k + 2a_0 k_0 n^k} \\
&= 1 + \frac{4k_0 m^k}{(k_0 - m^k)^2 + a_0^2 (n^k)^2 + 2a_0 m^k n^k + 2a_0 k_0 n^k} \\
&\leq 1 + \frac{4k_0 m^k}{2a_0 k_0 n^k} \\
&\leq C,
\end{align*}

where we have used Lemma 4.1. This completes the proof of Lemma 4.2.

As in [15], we denote $w^k = u^k - h^k$, where $h^k$ is the unique solution to

\begin{align}
\begin{cases}
Lh^k = \nabla P^k, & \text{in } \Omega \times (0, T], \\
h^k|_{\partial \Omega} = 0.
\end{cases} \quad (4.10)
\end{align}

From Lemma 2.3 we get for any $p \in (1, \infty)$

\begin{align}
||h^k||_{W^{1,p}} \leq C||P^k||_{L^p}, \\
||h^k||_{W^{2,p}} \leq C||\nabla P^k||_{L^p}. \quad (4.11)
\end{align}
and 

$$\int$$

Now we estimate the term of the right side in (4.13) as follows:

$$\begin{align*}
\mathbf{C} &\leq C_{3} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\mathbf{m} &\leq C_{m} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\mathbf{w} &\leq C_{w} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\Omega &\leq C_{\Omega} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\Phi &\leq C_{\Phi} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\mu &\leq C_{\mu} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\lambda &\leq C_{\lambda} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\Phi &\leq C_{\Phi} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\mu &\leq C_{\mu} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right], \\
\lambda &\leq C_{\lambda} \left[ \int_{\Omega} \left| \nabla \Phi \right|^{2} + C \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right].
\end{align*}$$

where

$$F^{k} = -u^{k-1} \cdot \nabla u^{k} - L^{-1} \nabla P_{t}^{k} = -u^{k-1} \cdot \nabla u^{k} + L^{-1} \nabla (P_{t}^{k} u^{k-1}) + L^{-1} \nabla \left[ (m^{k} p^{k}_{m} + n^{k} p^{k}_{n} - P^{k}) \text{div} u^{k-1} \right].$$

**Lemma 4.3.** Under the conditions of Theorem 4.1, we have for all $1 \leq k \leq K$, $3 < r \leq 4$

$$\int_{0}^{T} \int_{\Omega} m^{k} |\nabla w_{t}^{k}|^{2} \leq C \sup_{0 \leq t \leq T} \Phi_{K}(t) \exp \left\{ C \int_{0}^{T} |\Phi_{K}(s)|^{2} |\Psi_{K,r}(s)|^{2} ds \right\},$$

and

$$\int_{0}^{T} \int_{\Omega} |\nabla^{2} w_{t}^{k}|^{2} \leq C \sup_{0 \leq t \leq T} \Phi_{K}(t) \exp \left\{ C \int_{0}^{T} |\Phi_{K}(s)|^{2} |\Psi_{K,r}(s)|^{2} ds \right\},$$

where $C$ is a positive constant, independent of $K$, $\delta$ and $T$.

**Proof.** Multiplying (4.12) by $w_{t}^{k}$, integrating over $\Omega$, and using integration by parts and Cauchy inequality, we have

$$\begin{align*}
\int_{\Omega} m^{k} |w_{t}^{k}|^{2} + \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mu| |\nabla w_{t}^{k}|^{2} + (\mu + \lambda) |\text{div} w_{t}^{k}|^{2} &
\leq \frac{1}{2} \int_{\Omega} m^{k} |w_{t}^{k}|^{2} + \frac{1}{2} \int_{\Omega} m^{k} |F^{k}|^{2},
\end{align*}$$

which implies

$$\int_{\Omega} m^{k} |w_{t}^{k}|^{2} + \frac{d}{dt} \int_{\Omega} |\mu| |\nabla w_{t}^{k}|^{2} + (\mu + \lambda) |\text{div} w_{t}^{k}|^{2} \leq \int_{\Omega} m^{k} |F^{k}|^{2}. \quad (4.13)$$

Now we estimate the term of the right side in (4.13) as follows:

$$\begin{align*}
\int_{\Omega} m^{k} |F^{k}|^{2} &
\leq C \int_{\Omega} (m^{k} |u^{k-1}|^{2} |\nabla u^{k}|^{2} + C \Phi_{K}(t) \int_{\Omega} |L^{-1} \nabla \text{div} (P_{t}^{k} u^{k-1})|^{2} + C \Phi_{K}(t) \int_{\Omega} (|m^{k}|^{2} + |P^{k}|^{2}) |\text{div} u^{k-1}|^{2} \\
&
\leq C \left( \int_{\Omega} (m^{k} |u^{k-1}|^{2})^{\frac{1}{2}} \right)^{2} \left[ \int_{\Omega} |\nabla u^{k}|^{2} \right]^{\frac{1}{2}} + C \Phi_{K}(t) \int_{\Omega} |P_{t}^{k} u^{k-1}|^{2} + C \Phi_{K}(t)^{3} \int_{\Omega} |\text{div} u^{k-1}|^{2} \\
&
\leq C \left( \int_{\Omega} (m^{k} |u^{k-1}|^{2})^{\frac{1}{2}} \right)^{2} \left[ \int_{\Omega} |\nabla u^{k}|^{2} \right]^{\frac{1}{2}} + C \Phi_{K}(t)^{3} \int_{\Omega} |\nabla u^{k}|^{2} + C \Phi_{K}(t)^{3} \int_{\Omega} |\text{div} u^{k-1}|^{2} \\
&
\leq C \Phi_{K}(t)^{\frac{1}{2}} \left[ \int_{\Omega} |\nabla w_{t}^{k}|^{2} \right]^{\frac{1}{2}} + C \Phi_{K}(t)^{3} \left( |\nabla w_{t}^{k}|^{2} + |\nabla v_{t}^{k}|^{2} \right) + C \Phi_{K}(t)^{3} \int_{\Omega} |\mu| |\nabla u^{k-1}|^{2} + (\mu + \lambda) |\text{div} u^{k-1}|^{2} \\
&
\leq C \Phi_{K}(t)^{3} \left( |\nabla w_{t}^{k}|^{2} + |\nabla v_{t}^{k}|^{2} + (\mu + \lambda) |\text{div} u^{k-1}|^{2} \right) + C \Phi_{K}(t)^{3} \int_{\Omega} |\mu| |\nabla u^{k-1}|^{2} + (\mu + \lambda) |\text{div} u^{k-1}|^{2} \\
&
\leq C \Phi_{K}(t)^{3} \left( |\nabla w_{t}^{k}|^{2} + |\nabla v_{t}^{k}|^{2} + (\mu + \lambda) |\text{div} u^{k-1}|^{2} \right) + C \Phi_{K}(t)^{3} \int_{\Omega} \left( |\mu| |\nabla u^{k-1}|^{2} + (\mu + \lambda) |\text{div} u^{k-1}|^{2} \right), \quad (4.14)
\end{align*}$$
where we have used Lemma 2.2, Lemma 2.3, Lemma 4.1, Lemma 4.2 and Young inequality: \( ab \leq \varepsilon a^p + (e^p - 1) b^q \) for any \( \varepsilon > 0, p, q > 0 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

By Lemma 2.3 and (4.12), we have

\[
\| \nabla^2 w^k \|_{L^2} \leq C \Phi_K(t) \int_\Omega m^k |w_r^{k^2} |^2 + C \Phi_K(t) \int_\Omega m^k |F^k| ^2. \tag{4.15}
\]

Substituting (4.15) into (4.14), we have

\[
\int_\Omega m^k |F^k| ^2 \leq C \varepsilon [\Phi_K(t)] \frac{2}{3m} [\Psi_{K,r}(t)] \left( \int_\Omega m^k |w_r^{k^2} |^2 + \int_\Omega m^k |F^k| ^2 \right)
\]

\[
+ C [\Phi_K(t)] \frac{2}{3m} [\Psi_{K,r}(t)] \left( e^{\frac{2}{3m}} + 1 \right) \int_\Omega \mu |\nabla w^k| ^2 + C [\Phi_K(t)] \frac{2}{3m} [\Psi_{K,r}(t)] \frac{2}{3m}
\]

\[
+ C [\Phi_K(t)] ^3 \int_\Omega [\mu |\nabla w^{k^2} |^2 + (\mu + \lambda) |\text{div} w^{k^2} |^2] + C [\Phi_K(t)] ^3 \int_\Omega [\mu |\nabla w^{k^2} |^2 + (\mu + \lambda) |\text{div} w^{k^2} |^2] + C [\Phi_K(t)] ^5,
\]

where we have used (4.11).

Take \( \varepsilon = \frac{1}{4m^2} [\Phi_K(t)] \frac{2}{3m} [\Psi_{K,r}(t)] \frac{2}{3m} \), we have

\[
\int_\Omega m^k |F^k| ^2 \leq \frac{1}{3} \int_\Omega m^k |w_r^{k^2} |^2 + C [\Phi_K(t)] \frac{2}{3m} [\Psi_{K,r}(t)] \frac{2}{3m} \int_\Omega \mu |\nabla w^k| ^2 + C [\Phi_K(t)] ^5 [\Psi_{K,r}(t)] \frac{2}{3m}
\]

\[
+ C [\Phi_K(t)] ^3 \int_\Omega [\mu |\nabla w^{k^2} |^2 + (\mu + \lambda) |\text{div} w^{k^2} |^2]. \tag{4.16}
\]

Combining (4.13) and (4.16), we get

\[
\frac{2}{3} \int_0^t \int_\Omega m^k |w_r^{k^2} |^2 + \frac{d}{dt} \int_\Omega \mu |\nabla w^k| ^2 + (\mu + \lambda) |\text{div} w^k |^2
\]

\[
\leq C [\Phi_K(t)] \frac{2}{3m} [\Psi_{K,r}(t)] \frac{2}{3m} \int_\Omega \mu |\nabla w^k| ^2 + C [\Phi_K(t)] ^5 [\Psi_{K,r}(t)] \frac{2}{3m}
\]

\[
+ C [\Phi_K(t)] ^3 \int_\Omega [\mu |\nabla w^{k^2} |^2 + (\mu + \lambda) |\text{div} w^{k^2} |^2].
\]

Integrating over \( (0, t) \), we have

\[
\frac{2}{3} \int_0^t \int_\Omega m^k |w_r^{k^2} |^2 + \int_\Omega \mu |\nabla w^k| ^2 + (\mu + \lambda) |\text{div} w^k |^2
\]

\[
\leq C \int_0^t [\Phi_K(s)] \frac{2}{3m} [\Psi_{K,r}(s)] \frac{2}{3m} \int_\Omega \mu |\nabla w^k| ^2 + C \int_0^t [\Phi_K(s)] ^5 [\Psi_{K,r}(s)] \frac{2}{3m}
\]

\[
+ C \int_0^t [\Phi_K(s)] ^3 \int_\Omega [\mu |\nabla w^{k^2} |^2 + (\mu + \lambda) |\text{div} w^{k^2} |^2] + C. \tag{4.17}
\]
Denote $A_K(t) = \sup_{1 \leq k \leq K} \int_{\Omega} [\mu|\nabla u^k|^2 + (\mu + \lambda)|\text{div} u^k|^2]$, we obtain from (4.17) and noticing $\frac{t^2}{2} > 3$ for $3 < r \leq 4$ and $\Phi_K \geq 1$, $\Psi_{K,r} \geq 1$

$$A_K(t) \leq C \int_0^t [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} A_K(s) + C \int_0^t [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} + C.$$  

By Gronwall inequality, we get

$$A_K(t) \leq C \exp \left( C \int_0^t [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} ds \right).$$

where we have used the inequality: $y \leq \exp(y)$ for $y \geq 0$. By (4.15), (4.16), (4.17) and (4.18), we complete the proof of Lemma 4.3.

From (4.11) and Lemma 4.3 we immediately give the following corollary.

**Corollary 4.1.** Under the conditions of Theorem 4.1, we have for all $1 \leq k \leq K$ and $3 < r \leq 4$

$$\|\nabla u^k\|_{L^\infty(0,T;L^2)} \leq C \sup_{0 \leq t \leq T} \Phi_K(t) \exp \left( C \int_0^t [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} ds \right),$$

and

$$\|\nabla u^k\|_{L^2(0,T;L^6)} \leq C \left[ \sup_{0 \leq t \leq T} \Phi_K(t) + \sqrt{T} \right] \exp \left( C \int_0^T [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} ds \right),$$

where $C$ is a positive constant, independent of $K$, $\delta$ and $T$.

Now we give higher order estimates for $u^k$.

**Lemma 4.4.** Under the conditions of Theorem 4.1, we have for all $1 \leq k \leq K$ and $3 < r \leq 4$

$$\int_{\Omega} m^k |u^k|^2 + \int_0^t \int_{\Omega} [\mu|\nabla u^k|^2 + (\mu + \lambda)|\text{div} u^k|^2] \leq C \sup_{0 \leq s \leq T} [\Phi_K(s)]^2 \exp \left( C \int_0^t [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} ds \right) + \int_{\Omega} |g|^2$$

$$+ C \sup_{0 \leq s \leq T} \Phi_K(s) \int_0^t \Phi_K(\tau) \left( 1 + \left\| \sqrt{m^k} u^k \right\|_{L^2} \right) \left\| \nabla u^k \right\|_{L^2} \exp \left( C \int_0^t [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} ds \right)$$

$$+ C \sup_{0 \leq s \leq T} \Phi_K(s) \int_0^t \Phi_K(\tau) \left( 1 + \left\| \sqrt{m^{k-1} u^{k-1}} \right\|_{L^2} \right) \left\| \nabla u^{k-1} \right\|_{L^6} \exp \left( C \int_0^t [\Phi(s)]^{\frac{t}{\tau^2}} [\Psi_{K,r}(s)]^{\frac{2}{\tau^2}} ds \right),$$

where $u^k = u^k_t + u^{k-1} \cdot \nabla u^k$, $0 \leq t \leq T$, and $C$ is a positive constant, independent of $K$, $\delta$ and $T$.

**Proof.** (4.1) can be rewritten as

$$m^k \dot{u}^k + \nabla P^k - Lu^k = 0.$$  

(4.19)

Differentiating (4.19) with respect to $t$, and using (4.1), we conclude

$$m^k \dot{u}^k_t + m^k u^{k-1} \cdot \nabla u^k + \nabla P^k + \text{div}(\nabla P^k \otimes u^{k-1})$$

$$= Lu^k - L(u^{k-1} \cdot \nabla u^k) + \text{div}(Lu^k \otimes u^{k-1}).$$  

(4.20)
Multiplying (4.20) by $\dot{u}^k$, integrating the resulting equation over $\Omega$, and using integration by parts, we obtain

\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} m^k |\dot{u}^k|^2 &+ \int_{\Omega} \left( \mu |\nabla \dot{u}^k|^2 + (\mu + \lambda) |\text{div}\dot{u}^k|^2 \right) \\
&= \int_{\Omega} \left( P^k \text{div}\dot{u}^k + (u^{k-1} \cdot \nabla \dot{u}^k) \cdot \nabla P^k \right) + \mu \int_{\Omega} \nabla (u^{k-1} \cdot \nabla u^k) : \nabla \dot{u}^k + (\mu + \lambda) \int_{\Omega} \text{div}(u^{k-1} \cdot \nabla u^k) \text{div}\dot{u}^k \\
&\quad - \int_{\Omega} (u^{k-1} \cdot \nabla u^k) \cdot (\mu \Delta u^k + (\mu + \lambda) \nabla \dot{u}^k) \\
&= \int_{\Omega} \left( P^k \text{div}\dot{u}^k + (u^{k-1} \cdot \nabla \dot{u}^k) \cdot \nabla P^k \right) + \mu \int_{\Omega} \left[ \nabla (u^{k-1} \cdot \nabla u^k) : \nabla \dot{u}^k - (u^{k-1} \cdot \nabla u^k) \cdot \Delta u^k \right] \\
&\quad + (\mu + \lambda) \int_{\Omega} \left[ \text{div}(u^{k-1} \cdot \nabla u^k) \text{div}\dot{u}^k - (u^{k-1} \cdot \nabla u^k) \cdot \nabla \text{div}\dot{u}^k \right] \\
&= I_1 + I_2 + I_3. \tag{4.21}
\end{align*}

Now we estimate $I_1$, $I_2$ and $I_3$ as follows:

\begin{align*}
I_1 &= \int_{\Omega} \left[ (P^k m^k + P^k m^k) \text{div}\dot{u}^k + (u^{k-1} \cdot \nabla \dot{u}^k) \cdot \nabla P^k \right] dx \\
&= \int_{\Omega} \left[ (\nabla m^k \cdot \nabla \dot{u}^k) \text{div}\dot{u}^k - u^{k-1} \cdot \nabla P^k \text{div}\dot{u}^k + (u^{k-1} \cdot \nabla u^k) \cdot \nabla P^k \right] \\
&= \int_{\Omega} \left[ (\nabla m^k \cdot \nabla \dot{u}^k) \text{div}\dot{u}^k + P^k \text{div}(u^{k-1} \cdot \nabla u^k) - P^k \text{div}(u^{k-1} \cdot \nabla \dot{u}^k) \right] \\
&\leq C \Phi_{K(t)} \|u^{k-1}\|_{L^2}^2 \|\nabla \dot{u}^k\|_{L^2}, \tag{4.22}
\end{align*}

where we have used integration by parts, (4.1), (4.1), Lemma 4.1, Lemma 4.2 and Hölder inequality.

\begin{align*}
I_2 &= \mu \int_{\Omega} \left[ \nabla (u^{k-1} \cdot \nabla u^k) : \nabla \dot{u}^k + \nabla (u^{k-1} \cdot \nabla \dot{u}^k) : \nabla \dot{u}^k \right] \\
&= \mu \int_{\Omega} \left[ \nabla (u^{k-1} \cdot \nabla \dot{u}^k) : \nabla \dot{u}^k + (u^{k-1} \cdot \nabla \dot{u}^k) : \nabla \dot{u}^k + \nabla \dot{u}^k : \nabla \dot{u}^k \right] \\
&= \mu \int_{\Omega} \left[ \nabla (u^{k-1} \cdot \nabla \dot{u}^k) : \nabla \dot{u}^k + (u^{k-1} \cdot \nabla \dot{u}^k) : \nabla \dot{u}^k - \nabla \dot{u}^k : \nabla \dot{u}^k \right] \\
&\leq C \|\nabla \dot{u}^k\|_{L^2} \|\nabla \dot{u}^k\|_{L^2} \|\nabla \dot{u}^k\|_{L^2}, \tag{4.23}
\end{align*}

where we have used integration by parts and Hölder inequality.

\begin{align*}
I_3 &= (\mu + \lambda) \int_{\Omega} \left[ \text{div}(u^{k-1} \cdot \nabla u^k) \text{div}\dot{u}^k + \nabla \cdot (u^{k-1} \cdot \nabla \dot{u}^k) \text{div}\dot{u}^k \right] \\
&= (\mu + \lambda) \int_{\Omega} \left[ \text{div}(u^{k-1} \cdot \nabla u^k) \text{div}\dot{u}^k + \text{div}\dot{u}^k (\nabla u^{k-1})' : \nabla \dot{u}^k + (u^{k-1} \cdot \nabla \dot{u}^k) \text{div}\dot{u}^k \right] \\
&= (\mu + \lambda) \int_{\Omega} \left[ \text{div}(u^{k-1} \cdot \nabla u^k) \text{div}\dot{u}^k + \text{div}\dot{u}^k (\nabla u^{k-1})' : \nabla \dot{u}^k - \text{div}\dot{u}^k \text{div}\dot{u}^k \text{div}\dot{u}^k + (u^{k-1} \cdot \nabla \dot{u}^k) \text{div}\dot{u}^k \right] \\
&= (\mu + \lambda) \int_{\Omega} \left[ (\nabla u^{k-1})' : \nabla \dot{u}^k \text{div}\dot{u}^k + \text{div}\dot{u}^k (\nabla u^{k-1})' : \nabla \dot{u}^k - \text{div}\dot{u}^k \text{div}\dot{u}^k \text{div}\dot{u}^k \right].
\end{align*}
By (4.26) and Lemma 2.3, we get
\[ s \text{satisfies} \]
In the following, we estimate the term \( \|\nabla u^k\|_{L^6}^4 \). From equation (4.1) and (4.10), we know that \( w^k \) satisfies
\[ \begin{cases} 
Lw^k = m^k u^k, & \text{in } \Omega, \\
w^k(x) = 0, & \text{on } \partial \Omega. 
\end{cases} \] 
By (4.26) and Lemma 2.3, we get
\[ \|w^k\|_{H^2} \leq C\|m^k u^k\|_{L^2} \leq C \sqrt{\Phi(t)} \|\sqrt{m^k u^k}\|_{L^2}, \]
which together with the interpolation inequality, Sobolev inequality, and (4.11) yields
\[ \|\nabla u^k\|_{L^6}^4 \leq C \|\nabla u^k\|_{L^2} \|\nabla u^k\|_{H^1}^2 \leq C \|\nabla u^k\|_{L^2} \|\nabla w^k\|_{L^2} + \|\nabla h^k\|_{L^6} \leq C \|\nabla u^k\|_{L^2} \|\nabla u^k\|_{L^6}^2 \Phi_K(t) \|\nabla w^k\|_{H^1} \leq C \|\nabla u^k\|_{L^2} \|\nabla u^k\|_{L^6}^2 \Phi_K(t) + \sqrt{\Phi(t)} \|\sqrt{m^k u^k}\|_{L^2} \leq C \Phi_K(t) \|\nabla u^k\|_{L^2} \|\nabla u^k\|_{L^6}^2 (1 + \sqrt{m^k u^k}\|_{L^2}). \]
This together with Corollary 4.1 gives
\[ \|\nabla u^k\|_{L^6}^4 \leq C(1 + \|\sqrt{m^k u^k}\|_{L^2}) \Phi_K(t) \|\nabla u^k\|_{L^6}^2 \sup_{0 \leq s \leq T} \Phi_K(s) \exp \left\{ C \int_0^t [\Phi_K(s)]^{\frac{1}{2}} [\Psi_{K, r}(s)]^{\frac{1}{2}} ds \right\}. \] 
Similarly, we have
\[ \|\nabla u^{k-1}\|_{L^6}^4 \leq C(1 + \|\sqrt{m^{k-1} u^{k-1}}\|_{L^2}) \Phi_K(t) \|\nabla u^{k-1}\|_{L^6}^2 \sup_{0 \leq s \leq T} \Phi_K(s) \exp \left\{ C \int_0^t [\Phi_K(s)]^{\frac{1}{2}} [\Psi_{K, r}(s)]^{\frac{1}{2}} ds \right\}. \] 
Substituting (4.27) and (4.28) into (4.25), we get
\[ \frac{d}{dt} \int_\Omega m^k |u^k|^2 + \int_\Omega \mu |\nabla u^k|^2 + (\mu + \lambda)|\text{div} \ u^k|^2 \]
\[ \leq C\Phi_K(t)(1 + \|\sqrt{m^k u^k}\|_{L^2}) \sup_{0 \leq s \leq T} \Phi_K(s) \exp \left\{ C \int_0^t [\Phi_K(s)]^{\frac{1}{2}} [\Psi_{K, r}(s)]^{\frac{1}{2}} ds \right\} + C\Phi_K(t)(1 + \|\sqrt{m^{k-1} u^{k-1}}\|_{L^2}) \|\nabla u^{k-1}\|_{L^6}^2 \sup_{0 \leq s \leq T} \Phi_K(s) \exp \left\{ C \int_0^t [\Phi_K(s)]^{\frac{1}{2}} [\Psi_{K, r}(s)]^{\frac{1}{2}} ds \right\} \]
\[ + C[\Phi_K(t)]^2 \sup_{0 \leq s \leq T} \Phi_K(s) \exp \left\{ C \int_0^t [\Phi_K(s)]^{\frac{1}{2}} [\Psi_{K, r}(s)]^{\frac{1}{2}} ds \right\}. \]
Integrating over (0, t) and using (1.6), we complete the proof of Lemma 4.4. \qed
Note that $T > 0$ and $r \in (3, 4]$ are arbitrary for all the above estimates which will be useful to get the blow-up criterion of the solution in the next section. To obtain the strong solutions, we have to take $T$ small enough. Therefore, we assume $T \in (0, 1)$. Moreover, we take $r = 4$ for simplicity.

Suppose for $1 \leq k \leq K$

$$\|u^{k-1}\|_{L^2(0,T;W^{2,q})} + \|u^{k-1}\|_{L^\infty(0,T;H^2)} + \sup_{0 \leq s \leq T} \int_\Omega m^{k-1} |\dot{u}^{k-1}|^2 \leq M_1, \quad (4.29)$$

for $M_1 > 1$ large enough.

Throughout the rest of the section, we denote by $C$ a generic positive constant which may be dependent of $\mu$, $\lambda$, $\Omega$, $m_0$, $n_0$, $u_0$ and other known constants but independent of $M_1$, $K$, $\delta$ and $T$.

**Lemma 4.5.** Under the conditions of Theorem 4.4, we have for any $k \geq 1$ and $T \leq M_1^{-8}$

$$\begin{align*}
\sup_{0 \leq s \leq T} \int_\Omega m^k |\dot{u}^k|^2 + \int_0^T \int_\Omega |\nabla \dot{u}^k|^2 & \leq CM_1^4, \\
\|u^k\|_{L^2(0,T;W^{2,q})} + \|u^k\|_{L^\infty(0,T;H^2)} & \leq M_1, \\
\|m^k\|_{L^2(0,T;W^{1,q})} + \|m^k\|_{L^\infty(0,T;H^0)} & \leq M_1, \\
\frac{1}{C} \leq \sum_{k=1}^M m^k \leq n^k & \leq \frac{1}{C} \text{ in } Q_T, \\
\|P_k^k\|_{L^\infty(Q_T)} + \|P_k^k\|_{L^\infty(Q_T)} + \|P_k^k\|_{L^\infty(Q_T)} & \leq C.
\end{align*}$$

**Proof.** By (4.29), (3.4), (4.1) and Sobolev inequality, we have

$$\sup_{0 \leq s \leq T} \|m^k(t)\|_{W^{1,q}} \leq C \exp(CM_1^4 T_k^4),$$

$$\sup_{0 \leq s \leq T} \|\dot{u}^k(t)\|_{W^{1,q}} \leq C \exp(CM_1^4 T_k^4).$$

Denote $T_1 = M_1^{-2}$, we get for $T \leq T_1$

$$\begin{align*}
\sup_{0 \leq s \leq T} \|m^k(t)\|_{W^{1,q}} & \leq C, \\
\sup_{0 \leq s \leq T} \|\dot{u}^k(t)\|_{W^{1,q}} & \leq C.
\end{align*} \quad (4.30)$$

(4.30) and Sobolev inequality give

$$\sup_{0 \leq s \leq T} \Phi_K(t) \leq C. \quad (4.31)$$

This together with (4.5) implies

$$\|P^k\|_{L^\infty(Q_T)} \leq C. \quad (4.32)$$

By (4.6), (4.7), (4.30) and (4.32), we have

$$\|P^k\|_{L^\infty(0,T;W^{1,q})} \leq C. \quad (4.33)$$

We obtain from (4.29), (4.31) and Sobolev inequality

$$\sup_{0 \leq s \leq T} \Psi_{K,1}(t) \leq CM_1^4. \quad (4.34)$$

It follows from Corollary 4.1, (4.31), and (4.34) that for $T \leq T_1$

$$\|\nabla \dot{u}^k\|_{L^2(0,T;L^2)} + \|\nabla \dot{u}^k\|_{L^\infty(0,T;L^2)} \leq C \exp\left(CM_1^8\right).$$
Take $T_2 = M_1^{-8}$, we have for $T \leq T_2$

$$
\|\nabla u^k\|_{L^2(0,T;L^2)} + \|\nabla u^k\|_{L^{\infty}(0,T;L^2)} \leq C. \quad (4.35)
$$

By Lemma\ref{4.4}, \ref{4.29}, \ref{4.31}, \ref{4.34}, \ref{4.35}, and Sobolev inequality, we get for $0 \leq t \leq T \leq T_2$

$$
\begin{align*}
\int_\Omega m^k|\dot{u}^k|^2 + \int_0^t \int_\Omega (\mu|\nabla u^k|^2 + (\mu + \lambda)\langle \text{div} u^k \rangle^2) \\
\leq C + \int_\Omega |g|^2 + C \int_0^t \int_\Omega |\nabla u^k|^2 + C \int_0^t \int_\Omega \left(1 + \sqrt{m^k |\dot{u}^k| L^2} \right) \left(1 + \sqrt{m^{k-1} |\dot{u}^{k-1}| L^2} \right) \|\nabla u^{k-1}\|_{H^1}^2 \\
\leq C + \int_\Omega |g|^2 + C \left(1 + \sup_{0 \leq s \leq T} \|\sqrt{m^k \dot{u}^k}\|_{L^2} \right) + C M_1^2 T.
\end{align*}
$$

Using Cauchy inequality, we have for $T \leq T_2$

$$
\sup_{0 \leq s \leq T} \int_\Omega m^k|\dot{u}^k|^2 + \int_0^T \int_\Omega |\nabla u^k|^2 \leq C + C \int_\Omega |g|^2 + C M_1^2 T. \quad (4.36)
$$

By \ref{4.19}, Lemma\ref{2.3}, \ref{4.31}, \ref{4.35} and \ref{4.36}, we get for $T \leq T_2$

$$
\|u^k\|_{L^2(0,T;W^{2,4})} + \|u^k\|_{L^{\infty}(0,T;H^2)} \leq C + C \int_\Omega |g|^2 + C M_1^2 T.
$$

Since $T \leq M_1^{-8}$, we have

$$
\sup_{0 \leq s \leq T} \int_\Omega m^k|\dot{u}^k|^2 + \int_0^T \int_\Omega |\nabla u^k|^2 + \|u^k\|_{L^2(0,T;W^{2,4})} + \|u^k\|_{L^{\infty}(0,T;H^2)} \leq C + C \int_\Omega |g|^2.
$$

Let $M_1 \geq C + C \int_\Omega |g|^2$, we obtain for $T \leq T_2$

$$
\sup_{0 \leq s \leq T} \int_\Omega m^k|\dot{u}^k|^2 + \int_0^T \int_\Omega |\nabla u^k|^2 + \|u^k\|_{L^2(0,T;W^{2,4})} + \|u^k\|_{L^{\infty}(0,T;H^2)} \leq M_1. \quad (4.37)
$$

By induction, \ref{4.37} is valid for any $k \in [1, K]$. Since $M_1$ is independent of $K$, and $K$ is arbitrary, we conclude that \ref{4.37} is actually valid for all $k \geq 1$.

From \ref{4.37}, we obtain for $T \leq T_2$

$$
\sup_{0 \leq s \leq T} \int_\Omega m^k|\dot{u}^k|^2 + \int_0^T \int_\Omega |\nabla u^k|^2 \leq C M_1^4.
$$

Summarily, we have for any $k \geq 1$ and $T \leq T_2$

$$
\begin{align*}
\sup_{0 \leq s \leq T} \int_\Omega m^k|\dot{u}^k|^2 + \int_0^T \int_\Omega |\nabla u^k|^2 & \leq C M_1^4, \\
\|u^k\|_{L^2(0,T;W^{2,4})} + \|u^k\|_{L^{\infty}(0,T;H^1)^{\cap H^2}} & \leq M_1, \\
\|m^k\|_{L^2(0,T;W^{1,4})} + \|m^k\|_{L^{\infty}(0,T;W^1)} & \leq C, \\
\frac{s_{0\delta}}{T} \leq s_T m^k \leq n_k \leq \frac{s_0 m^k}{T}, & \text{ in } Q_T, \\
\|P^k\|_{L^2(Q_T)} + \|P^k_m\|_{L^{\infty}(Q_T)} + \|P^k_{n_k}\|_{L^{\infty}(Q_T)} & \leq C. \quad (4.38)
\end{align*}
$$

The proof of Lemma 4.5 is completed. \hfill \Box
We will show that the full sequence \((m^k, n^k, u^k)\) converges to a solution of (1.1)-(1.5). To do this, we denote
\[
\overline{m}^{k+1} = m^{k+1} - m^k, \quad \overline{n}^{k+1} = n^{k+1} - n^k, \quad \overline{u}^{k+1} = u^{k+1} - u^k \quad \text{and} \quad \overline{P}^{k+1} = P^{k+1} - P^k.
\]
It follows from (4.1) that
\[
m^{k+1}u_i^{k+1} + m^k u_i \cdot \nabla u \cdot u^{k+1} - Lu^{k+1} + \nabla \overline{P}^{k+1} = \overline{m}^{k+1}(-u^k_i - \nabla u^k) - m^k u^k \cdot \nabla u^k.
\]
(4.39)

Multiplying (4.39) by \(\overline{u}^{k+1}\), and using (4.38) and Sobolev inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} m^{k+1} |\overline{u}^{k+1}|^2 + \int_{\Omega} |\mu| \overline{u}^{k+1}|^2 + (\mu + \lambda) (\text{div} \overline{u}^{k+1})^2
\]
\[
= \int_{\Omega} \overline{P}^{k+1} \text{div} \overline{u}^{k+1} + \int_{\Omega} \overline{m}^k (-u^k_i - \nabla u^k) \cdot \overline{u}^{k+1} - \int_{\Omega} m^k (\overline{u}^k \cdot \nabla u^k) \cdot \overline{u}^{k+1}
\]
\[
\leq ||\overline{P}^{k+1}||_{L^2} ||\text{div} \overline{u}^{k+1}||_{L^2} + ||\overline{m}^{k+1}||_{L^2} ||u^k||_{L^2} + ||\nabla u^k||_{L^2} ||\overline{u}^{k+1}||_{L^2} + ||\sqrt{m^k}||_{L^2} ||\nabla u^k||_{L^2} ||\overline{u}^{k+1}||_{L^2}
\]
\[
\leq ||\overline{P}^{k+1}||_{L^2} ||\text{div} \overline{u}^{k+1}||_{L^2} + C||\overline{m}^{k+1}||_{L^2} (||\nabla u^k||_{L^2}^2 + M_1^k) ||\nabla \overline{u}^{k+1}||_{L^2} + C||\sqrt{m^k}||_{L^2} ||\nabla \overline{u}^{k+1}||_{L^2} M_1
\]
\[
\leq \frac{1}{2} \int_{\Omega} |\mu| \overline{u}^{k+1}|^2 + (\mu + \lambda) (\text{div} \overline{u}^{k+1})^2 + C||\overline{m}^{k+1}||_{L^2}^2 + C||\overline{m}^{k+1}||_{L^2}^2 (||\nabla u^k||_{L^2}^2 + M_1^k) + CM_1^2 ||\sqrt{m^k}||_{L^2}^2.
\]

This gives
\[
\frac{d}{dt} \int_{\Omega} m^{k+1} |\overline{u}^{k+1}|^2 + \int_{\Omega} |\mu| \overline{u}^{k+1}|^2 + (\mu + \lambda) (\text{div} \overline{u}^{k+1})^2
\]
\[
\leq C||\overline{m}^{k+1}||_{L^2}^2 + C||\overline{m}^{k+1}||_{L^2}^2 (||\nabla u^k||_{L^2}^2 + M_1^k) + CM_1^2 ||\sqrt{m^k}||_{L^2}^2.
\]
(4.40)

From (4.1), we have
\[
m_i^{k+1} + \overline{m}^{k+1} \text{div} u^k + m^k \text{div} \overline{u}^k + u^k \cdot \nabla \overline{m}^{k+1} + \overline{u}^k \cdot \nabla m^k = 0.
\]
(4.41)

Multiplying (4.41) by \(\overline{m}^{k+1}\), integrating over \(\Omega\), and using integration by parts and (4.38), we have
\[
\frac{d}{dt} \int_{\Omega} ||\overline{m}^{k+1}||^2 = - \int_{\Omega} \overline{m}^{k+1} \cdot \text{div} u^k - 2 \int_{\Omega} (m^k \text{div} \overline{u}^k + \overline{u}^k \cdot \nabla m^k) \overline{m}^{k+1}
\]
\[
\leq C||u^k||_{W^{2,4}} ||\overline{m}^{k+1}||_{L^2}^2 + C||\overline{m}^{k+1}||_{L^2} (||\nabla \overline{u}^k||_{L^2} + ||\overline{u}^k \cdot \nabla m^k||_{L^2})
\]
\[
\leq C||u^k||_{W^{2,4}} ||\overline{m}^{k+1}||_{L^2}^2 + C||\overline{m}^{k+1}||_{L^2} (||\nabla \overline{u}^k||_{L^2} + ||\nabla m^k||_{L^2} ||\overline{u}^k||_{L^2})
\]
\[
\leq C||u^k||_{W^{2,4}} ||\overline{m}^{k+1}||_{L^2}^2 + C||\overline{m}^{k+1}||_{L^2} ||\nabla \overline{u}^k||_{L^2}.
\]
(4.42)

Similarly, we have from (4.1)2
\[
\frac{d}{dt} \int_{\Omega} ||\overline{u}^{k+1}||^2 \leq C||u^k||_{W^{2,4}} (||\overline{m}^{k+1}||_{L^2}^2 + ||\overline{m}^{k+1}||_{L^2}^2) + C(||\overline{m}^{k+1}||_{L^2} + ||\overline{m}^{k+1}||_{L^2}) ||\nabla \overline{u}^k||_{L^2}.
\]
(4.43)

By (4.42)-(4.43) and Cauchy inequality, we have
\[
\frac{d}{dt} \int_{\Omega} (||\overline{m}^{k+1}||^2 + ||\overline{u}^{k+1}||^2) \leq C||u^k||_{W^{2,4}} (||\overline{m}^{k+1}||_{L^2}^2 + ||\overline{m}^{k+1}||_{L^2}^2) + C(||\overline{m}^{k+1}||_{L^2} + ||\overline{m}^{k+1}||_{L^2}) ||\nabla \overline{u}^k||_{L^2}.
\]
(4.44)

By (4.40), (4.44) and Cauchy inequality, we get
\[
\frac{d}{dt} \int_{\Omega} (||\overline{m}^{k+1}||^2 + ||\overline{u}^{k+1}||^2) + m^k \text{div} \overline{u}^k \cdot \overline{u}^k + \int_{\Omega} |\mu| \overline{u}^{k+1}|^2 + (\mu + \lambda) (\text{div} \overline{u}^{k+1})^2
\]
This together with Gronwall inequality, (4.38) and 
where 
Recalling the notations of 
for any 
Denote 
By (4.45), we have 

for any \( \varepsilon > 0 \). Denote 

This together with Gronwall inequality, (4.38) and \( \varphi^{k+1}(0) = 0 \) implies 

where \( \overline{C} \) depends on \( M_1 \) and other known constants related to \( C \).

By (4.45), we have 

Take \( \varepsilon = \frac{1}{16} \exp(\overline{C} + C)(CM_1^2 + 1) \), and \( T_0 = \min\{T_2, \varepsilon\} \), we have for \( T = T_0 \)

This implies 

Recalling the notations of \( \varphi^{k+1}(t) \) and \( \psi^{k+1}(t) \), we get 

Denote \( m = \sum_{i=2}^{\infty} m_i + m^1 \), \( n = \sum_{i=2}^{\infty} n_i + n^1 \), \( u = \sum_{i=1}^{\infty} u_i \), we have 

\[ \|m^k - m\|_{L^\infty(0,T_0;L^2)} + \|n^k - n\|_{L^\infty(0,T_0;L^2)} + \|u^k - u\|_{L^2(0,T_0;H^1_0)} < \infty. \]
\[
\begin{align*}
&= \left\| \sum_{l=k+1}^{\infty} \eta_l \right\|_{L^\infty(0,T_0;L^2)} + \left\| \sum_{l=k+1}^{\infty} \eta_l \right\|_{L^\infty(0,T_0;L^2)} + \left\| \sum_{l=k+1}^{\infty} \eta_l \right\|_{L^2(0,T_0;H_0^1)} \\
&\leq \sum_{l=k+1}^{\infty} \left( \left\| \eta_l \right\|_{L^\infty(0,T_0;L^2)} + \left\| \eta_l \right\|_{L^\infty(0,T_0;L^2)} + \left\| \eta_l \right\|_{L^2(0,T_0;H_0^1)} \right) \to 0,
\end{align*}
\]
as \(k \to \infty\). Here we have used (4.46).

Therefore, we conclude the convergence of the full sequence \((m^k, n^k, u^k)\) as \(k \to \infty\)
\[
\begin{align*}
m^k &\to m, \quad \text{in} \quad L^\infty(0,T_0;L^2), \\
n^k &\to n, \quad \text{in} \quad L^\infty(0,T_0;L^2), \\
u^k &\to u, \quad \text{in} \quad L^2(0,T_0;H_0^1). (4.47)
\end{align*}
\]

It follows from Lemma 4.5 (4.47) and lower semi-continuity of norms, we conclude that \((m, n, u)\) is a solution to (1.1)-(1.5) under the conditions of Theorem 4.1 and that the following estimates uniform for \(\delta\) are obtained

\[
\begin{align*}
&\left\| u_0^{\delta} \right\|_{L^2(0,T_0;W^{2,q})} + \left\| u_0^{\delta} \right\|_{L^\infty(0,T_0;H_0^1 \cap H^2)} + \left\| m_0^{\delta} \right\|_{L^\infty(0,T_0;W^{1,q})} + \left\| n_0^{\delta} \right\|_{L^\infty(0,T_0;W^{1,q})} \leq \bar{C}, \\
&\frac{\delta}{\bar{C}} \leq \delta_0 \leq n \leq \bar{C}_0 m, \quad \text{in} \quad Q_{T_0}, \\
&\left\| P \right\|_{L^\infty(Q_{T_0})} + \left\| P_m \right\|_{L^\infty(Q_{T_0})} + \left\| P_n \right\|_{L^\infty(Q_{T_0})} \leq \bar{C}.
\end{align*}
\]
The uniqueness can be obtained similar to the proceeding of the convergence of full sequence. The proof of Theorem 4.1 is completed.

**Proof of Theorem 1.1**

Denote \(m_0^{\delta} = m_0 + \delta, n_0^{\delta} = n_0 + \delta, \) we have as \(\delta \to 0\)
\[
\begin{align*}
m_0^{\delta} &\to m_0, \quad \text{in} \quad W^{1,q}, \\
n_0^{\delta} &\to n_0, \quad \text{in} \quad W^{1,q}. (4.48)
\end{align*}
\]

Since \(0 \leq \delta_0 m_0 \leq n_0 \leq \bar{C}_0 m_0, \) in \(\bar{\Omega},\) without loss of generality, we assume \(0 < \delta_0 \leq 1 \leq \bar{C}_0, \) we have
\[
\delta_0 m_0^{\delta} \leq n_0^{\delta} \leq \bar{C}_0 m_0^{\delta}, \quad \text{in} \quad \bar{\Omega}. (4.49)
\]

By Lemma 2.3 we can find a \(u_0^{\delta} \in H_0^1 \cap H^2\) for each \(\delta > 0\) such that
\[
\begin{align*}
-Lu_0^{\delta} + \nabla P(m_0^{\delta}, n_0^{\delta}) &= \sqrt{m_0^{\delta}}, \quad \text{in} \quad \Omega, \\
u_0^{\delta} &\mid_{\partial \Omega} = 0. (4.50)
\end{align*}
\]

It follows from (1.6), (4.48), (4.50), and Lemma 2.3 that
\[
u_0^{\delta} \to u_0, \quad \text{in} \quad H^2,
\]
as \(\delta \to 0\).

Consider (1.1)-(1.5) with initial data \((m_0, n_0, u_0)\) replaced by \((m_0^{\delta}, n_0^{\delta}, u_0^{\delta})\), we obtain from Theorem 4.1 that there exists a \(T_0 > 0\) independent of \(\delta\) and a unique solution \((m^\delta, n^\delta, u^\delta)\) for each \(\delta > 0\) with the following estimates:
\[
\begin{align*}
&\left\| \sup_{0 \leq \tau \leq T_0} \int_\Omega m^\delta |u_\tau^\delta|^2 \right\|_{L^\infty(0,T_0;L^2)} + \left\| \int_0^T \int_\Omega |\nabla u_\tau^\delta|^2 \right\|_{L^\infty(0,T_0;L^2)} \leq C, \\
&\left\| u_\tau^\delta \right\|_{L^2(0,T_0;W^{2,q})} + \left\| u_\tau^\delta \right\|_{L^\infty(0,T_0;H_0^1 \cap H^2)} + \left\| m^\delta \right\|_{L^\infty(0,T_0;W^{1,q})} + \left\| n^\delta \right\|_{L^\infty(0,T_0;W^{1,q})} \leq C, \\
&\frac{\delta}{\bar{C}} \leq \delta_0 m^\delta \leq n^\delta \leq \bar{C}_0 m^\delta, \quad \text{in} \quad Q_{T_0}, \\
&\left\| P \right\|_{L^\infty(Q_{T_0})} + \left\| P_m \right\|_{L^\infty(Q_{T_0})} + \left\| P_n \right\|_{L^\infty(Q_{T_0})} \leq C,
\end{align*}
\]
where $C$ is a positive constant, independent of $\delta$.

Taking $\delta \to 0$ (take subsequence if necessary), and using the similar arguments in [1], under the conditions of Theorem 1.1 we get a solution $(m, n, u)$ to (1.1)-(1.5) with the regularities like in Theorem 1.1. The uniqueness can be proved by the similar arguments in the proof of Theorem 4.1. The proof of Theorem 1.1 is completed.

5 Proof of Theorem 1.2

Let $(m, n, u)$ be a strong solution to the problem (1.1)-(1.5) in $Q_T$ with the regularity stated in Theorem 1.1. We assume that the opposite holds, i.e.

$$\lim \sup_{T \to T^*} \|m\|_{L^\infty(0,T;L^\infty)} \leq M < \infty. \quad (5.1)$$

In this section, we denote by $C$ a generic positive constant which may depend on $\mu$, $\lambda$, $\Omega$, $m_0$, $n_0$, $u_0$, $M$, $T^*$, and the parameters in the expression of $P$ in (1.5).

Similar to Lemma 4.1, we get the first lemma:

**Lemma 5.1.** Under the conditions of Theorem 1.2, we have for all $0 \leq T < T^*$

$$\underline{s}m \leq n \leq \overline{s}m, \text{ in } Q_T.$$

**Lemma 5.2.** Under the conditions of Theorem 1.2 there exists some $r \in (3, 4]$ such that

$$\sup_{0 \leq t \leq T} \int_{\Omega} |u|^r \, dx \leq C, \quad 0 \leq T < T^*.$$

**Proof.** Multiplying (1.1) by $|u|^{r-2}u$, integrating the resulting equation over $\Omega$, and using integration by parts, we obtain

$$\frac{d}{dt} \int_{\Omega} |u|^r + \int_{\Omega} r|u|^{r-2} \left( \mu |\nabla u|^2 + (\lambda + \mu) |\text{div}u|^2 + \mu (r - 2) |\nabla |u||^2 \right)$$

$$= r \int_{\Omega} \text{div}(|u|^{r-2}u)P - r(r - 2)(\mu + \lambda) \int_{\Omega} \text{div}u |u|^{r-2}u \cdot \nabla |u|. \quad (5.2)$$

For $\varepsilon_1 \in (0, 1)$, define

$$\phi(\varepsilon_1, r) = \begin{cases} \frac{4\varepsilon_1 \mu (r-1)}{3(r-2)(\mu + \lambda) - 4\mu}, & \text{if } r > 2 + \frac{4\mu}{3(\mu + \lambda)} \\ 0, & \text{otherwise.} \end{cases}$$

**Case 1:** If

$$\int_{\Omega} |u|^r \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 \leq \phi(\varepsilon_1, r) \int_{\Omega} |u|^{r-2} |\nabla |u||^2. \quad (5.3)$$

A direct calculation gives

$$|\nabla u|^2 = |u|^2 \left| \nabla \left( \frac{u}{|u|} \right) \right|^2 + |\nabla |u||^2, \quad (5.4)$$

and

$$\text{div}u = |u| \text{div} \left( \frac{u}{|u|} \right) + \frac{u \cdot \nabla |u|}{|u|}. \quad (5.5)$$

By (5.2) and (5.5), we get

$$\frac{d}{dt} \int_{\Omega} |u|^r + \int_{\Omega} r|u|^{r-2} \left( \mu |\nabla u|^2 + (\lambda + \mu) |\text{div}u|^2 + \mu (r - 2) |\nabla |u||^2 \right)$$
\[
\begin{align*}
&= r \int_\Omega \text{div}(|u|^{r-2}u)P - r(r-2)(\mu + \lambda) \int_\Omega |u|^{r-2}u \cdot \nabla u \text{div} \left( \frac{u}{|u|} \right) - r(r-2)(\mu + \lambda) \int_\Omega |u|^{r-4}u \cdot \nabla |u|^2 \\
&\leq r \int_\Omega \text{div}(|u|^{r-2}u)P + \frac{r(r-2)(\mu + \lambda)}{4} \int_\Omega |u|^r |\text{div} \left( \frac{u}{|u|} \right)|^2 \\
&\leq r \int_\Omega \text{div}(|u|^{r-2}u)P + \frac{3r(r-2)(\mu + \lambda)}{4} \int_\Omega |u|^r |\nabla \left( \frac{u}{|u|} \right)|^2, \\
&\quad \text{(5.6)}
\end{align*}
\]
where we have used Cauchy inequality. By (5.4) and (5.6), we get
\[
\begin{align*}
&\frac{d}{dt} \int_\Omega m|u|^r + \mu r \int_\Omega |u|^{r-2} |\nabla |u||^2 \\
&\leq r \int_\Omega \text{div}(|u|^{r-2}u)P + \left( \frac{3r(r-2)(\mu + \lambda)}{4} - \mu r \right) \int_\Omega |u|^r |\nabla \left( \frac{u}{|u|} \right)|^2 \\
&\leq C \int_\Omega m \frac{\overline{\omega}^2}{r} |u|^{r-2} |\nabla u| + \phi(e_1, r) \left( \frac{3r(r-2)(\mu + \lambda)}{4} - \mu r \right) \int_\Omega |u|^{r-2} |\nabla |u||^2 \\
&\leq \varepsilon \int_\Omega |u|^{r-2} |\nabla u|^2 + \frac{C}{4\varepsilon} \left( \int_\Omega m|u|^r \right)^{\overline{\omega}^2} + \phi(e_1, r) \left( \frac{3r(r-2)(\mu + \lambda)}{4} - \mu r \right) \int_\Omega |u|^{r-2} |\nabla |u||^2, \\
&\quad \text{(5.7)}
\end{align*}
\]
where we have used \( P(m, n) \leq Cm^{\overline{\omega}} \), which can be obtained from (5.1), Lemma 5.1 and the expression of \( P \).

This together with (5.3) and (5.4) gives
\[
\begin{align*}
&\frac{d}{dt} \int_\Omega m|u|^r + r \left( \mu r - \phi(e_1, r) \left( \frac{3r(r-2)(\mu + \lambda)}{4} - \mu \right) \right) \int_\Omega |u|^{r-2} |\nabla |u||^2 \\
&\leq \varepsilon [1 + \phi(e_1, r)] \int_\Omega |u|^{r-2} |\nabla |u||^2 + \frac{C}{4\varepsilon} \left( \int_\Omega m|u|^r \right)^{\overline{\omega}^2}.
\end{align*}
\]
Taking \( \varepsilon = [1 + \phi(e_1, r)]^{-1} r \left( \mu r - \phi(e_1, r) \left( \frac{3r(r-2)(\mu + \lambda)}{4} - \mu \right) \right) \), we get
\[
\frac{d}{dt} \int_\Omega m|u|^r \leq \frac{C[1 + \phi(e_1, r)]}{4\mu r - 1 - \phi(e_1, r) [3r(r-2)(\mu + \lambda) - 4\mu r]} \left( \int_\Omega m|u|^r \right)^{\overline{\omega}^2}, \\
\quad \text{(5.7)}
\]
for any \( r \in (3, 4) \).

**Case 2:**
\[
\int_\Omega |u|^{r-2} |\nabla u| \left( \frac{u}{|u|} \right) |^2 \geq \phi(e_1, r) \int_\Omega |u|^{r-2} |\nabla |u||^2. \\
\quad \text{(5.8)}
\]
By (5.2), we get
\[
\begin{align*}
&\frac{d}{dt} \int_\Omega m|u|^r + \int_\Omega r|u|^{r-2} \left( \mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2 + \mu r - 2 |\nabla |u||^2 \right) \\
&\quad = r \int_\Omega \text{div}(|u|^{r-2}u)P - r(r-2)(\mu + \lambda) \int_\Omega \text{div}u|u|^{\overline{\omega}^2} |u|^{\overline{\omega}^2} u \cdot \nabla |u|.
\end{align*}
\]
It follows from (5.4), (5.9) and Cauchy inequality that for any $\epsilon_0 \in (0, 1)$

$$\frac{d}{dt} \int_{\Omega} m|u'| + \int_{\Omega} \mu r |u|^{r-2} |\nabla u|^2 + \mu (r-2) r \int_{\Omega} |u|^{r-2} |\nabla u|^2$$

$$\leq C \int_{\Omega} m^{\frac{2}{r-2}} |u|^{r-2} |\nabla u| + \frac{r(r-2)^2 (\mu + \lambda)}{4} \int_{\Omega} |u|^{r-2} |\nabla u|^2.$$  \hspace{1cm} (5.9)

It follows from (5.4), (5.9) and Cauchy inequality that for any $\epsilon_0 \in (0, 1)$

$$\frac{d}{dt} \int_{\Omega} m|u'| + \int_{\Omega} \mu r |u|^{r-2} |\nabla u|^2 + \mu (r-2) r \int_{\Omega} |u|^{r-2} |\nabla u|^2$$

$$\leq C \int_{\Omega} m^{\frac{2}{r-2}} |u|^{r-2} |\nabla u| + C \int_{\Omega} m^{\frac{2}{r-2}} |u|^{r-2} |\nabla u| + \frac{r(r-2)^2 (\mu + \lambda)}{4} \int_{\Omega} |u|^{r-2} |\nabla u|^2$$

$$\leq C \int_{\Omega} m^{\frac{2}{r-2}} |u|^{r-2} |\nabla u| + \mu \epsilon_0 \int_{\Omega} |u'| |\nabla \left( \frac{u}{|u|} \right)|^2 + \frac{C}{4 \mu \epsilon_0} \left( \int_{\Omega} m|u'| \right)^{\frac{r-2}{2}}$$

$$+ \frac{r(r-2)^2 (\mu + \lambda)}{4} \int_{\Omega} |u|^{r-2} |\nabla u|^2.$$  \hspace{1cm} (5.10)

Combining (5.8), we have

$$\frac{d}{dt} \int_{\Omega} m|u'| + r \left( \mu (1 - \epsilon_0) \phi(e_1, r) + \mu (r - 1) - \frac{(r-2)^2 (\mu + \lambda)}{4} \right) \int_{\Omega} |u|^{r-2} |\nabla u|^2$$

$$\leq C \int_{\Omega} m^{\frac{2}{r-2}} |u|^{r-2} |\nabla u| + \frac{C}{4 \mu \epsilon_0} \left( \int_{\Omega} m|u'| \right)^{\frac{r-2}{2}}.$$  \hspace{1cm} (5.10)

(Sub-Case 2): If $3 \in (2 + \frac{4 \mu}{3(\mu + \lambda)}, \infty)$, i.e., $\mu < 3 \lambda$, we have for $r \in [3, \infty)$

$$\phi(e_1, r) = \frac{4 \epsilon_1 \mu (r-1)}{3(r-2)(\mu + \lambda) - 4 \mu}.$$  \hspace{1cm} (5.11)

Define

$$f(\epsilon_0, \epsilon_1, r) = \mu (1 - \epsilon_0) \phi(e_1, r) + \mu (r - 1) - \frac{(r-2)^2 (\mu + \lambda)}{4}.$$  \hspace{1cm} (5.12)

By (5.11) and (5.12), we have

$$f(\epsilon_0, \epsilon_1, r) = \frac{4 \epsilon_1^2 (1 - \epsilon_0) \epsilon_1 (r-1)}{3(r-2)(\mu + \lambda) - 4 \mu} + \mu (r - 1) - \frac{(r-2)^2 (\mu + \lambda)}{4},$$  \hspace{1cm} (5.13)

for $r \in [3, \infty)$. Particularly,

$$f(0, 1, 3) = \frac{8 \mu^2}{3 \lambda - \mu} + \frac{7 \mu - \lambda}{4} > 0.$$

Here we have used $\frac{\mu}{3} < \lambda < \frac{25}{12} \mu$.

Since $f(\epsilon_0, \epsilon_1, r)$ is continuous w.r.t. $(\epsilon_0, \epsilon_1, r)$ in $(0, 1) \times (0, 1) \times [3, \infty)$, there exists $\epsilon_0, \epsilon_1 \in (0, 1)$ and $r_1 \in (3, 4]$ such that

$$f(\epsilon_0, \epsilon_1, r_1) > 0.$$
From (5.10), Cauchy inequality and Hölder inequality, we obtain
\[
\frac{d}{dt} \int_{\Omega} m|u|^{r_1} + r_1 f(\varepsilon_0, \varepsilon_1, r_1) \int_{\Omega} |u|^{r_1-2} |\nabla u|^{2} \\
\leq r_1 f(\varepsilon_0, \varepsilon_1, r_1) \int_{\Omega} |u|^{r_1-2} |\nabla u|^{2} + \frac{C}{4r_1 f(\varepsilon_0, \varepsilon_1, r_1)} \left( \int_{\Omega} m|u|^{r_1} \right)^{\frac{r_1-2}{r_1}} + \frac{C}{4\mu r_1 \varepsilon_0} \left( \int_{\Omega} m|u|^{r_1} \right)^{\frac{r_1-2}{r_1}}.
\]
Therefore,
\[
\frac{d}{dt} \int_{\Omega} m|u|^{r_1} \leq C\left( \frac{1}{f(\varepsilon_0, \varepsilon_1, r_1)} + \frac{1}{\mu \varepsilon_0} \right) \left( \int_{\Omega} m|u|^{r_1} \right)^{\frac{r_1-2}{r_1}}.
\]

(Sub-Case 2): If \(3 \not\in (2 + \frac{4\mu}{s(\mu+1)}, \infty)\), i.e. \(\mu \geq 3\lambda\).

In this case, it is easy to verify that the following inequality holds for any \(r \in (3, 4]\)
\[
r \left[ \mu(1 - \varepsilon_0)\phi(\varepsilon_1, r) + \mu(r - 1) - \frac{(r - 2)^2(\mu + \lambda)}{4} \right] > 2\mu.
\]

We obtain from (5.10), (5.15), Cauchy inequality and Hölder inequality
\[
\frac{d}{dt} \int_{\Omega} m|u|^{r} + 2\mu \int_{\Omega} |u|^{r-2} |\nabla u|^{2} \\
\leq C \int_{\Omega} m|u|^{r-2} |\nabla u|^{2} + \frac{C}{4\mu \varepsilon_0} \left( \int_{\Omega} m|u|^{r} \right)^{\frac{r-2}{r}} \\
\leq 2\mu \int_{\Omega} |u|^{r-2} |\nabla u|^{2} + \frac{C}{\mu \varepsilon_0} \left( \int_{\Omega} m|u|^{r} \right)^{\frac{r-2}{r}}.
\]
This implies
\[
\frac{d}{dt} \int_{\Omega} m|u|^{r} \leq \frac{C}{\mu \varepsilon_0} \left( \int_{\Omega} m|u|^{r} \right)^{\frac{r-2}{r}}.
\]

Particularly, (5.16) is also valid for \(r = r_1\).

Summarily, for Case 2, we obtain from (5.14) and (5.16)
\[
\frac{d}{dt} \int_{\Omega} m|u|^{r_1} \leq C \left( \int_{\Omega} m|u|^{r_1} \right)^{\frac{r_1-2}{r_1}},
\]
for some constant \(C\), if \(\lambda < \frac{2\mu}{3}\) and (5.8) is valid.

It follows from (5.7) and (5.17) that
\[
\frac{d}{dt} \int_{\Omega} m|u|^{r_1} \leq C \left( \int_{\Omega} m|u|^{r_1} \right)^{\frac{r_1-2}{r_1}},
\]
for some constant \(C\), if \(\lambda < \frac{2\mu}{3}\).

Since \(\frac{r_1-2}{r_1} \in (0, 1)\), we complete the proof of Lemma 5.2 after using young inequality and Gronwall inequality in (5.18) and still denoting \(r_1\) by \(r\). \(\square\)

Similar to Lemma 4.2 and Corollary 4.1, we get the next lemma.

**Lemma 5.3.** Under the conditions of Theorem 1.2 and (5.7), for \(0 \leq T < T^*\), we have
\[
\begin{align*}
\|P\|_{L^p(Q_T)} + \|P_m\|_{L^p(Q_T)} + \|P_n\|_{L^p(Q_T)} & \leq C, \\
\|\nabla u\|_{L^2(0,T;L^6)} + \|\nabla u\|_{L^\infty(0,T;L^2)} & \leq C.
\end{align*}
\]

(5.19)
Here we have used (5.1) and Lemma 5.2.

**Lemma 5.4.** Under the conditions of Theorem 1.2 and (5.1), for $0 \leq T < T^*$, we have
\[
\int_{\Omega} m|\dot{u}|^2 + \int_0^T \int_{\Omega} |\nabla \dot{u}|^2 \leq C,
\]
where $\dot{u} = u_t + u \cdot \nabla u$.

**Proof.** Similar to Lemma 4.4, we have
\[
\int_{\Omega} m|\dot{u}|^2 + \int_0^T \int_{\Omega} \left( \mu|\nabla \dot{u}|^2 + (\mu + \lambda)|\text{div} \dot{u}|^2 \right) \leq C + C \int_0^T \left( 1 + \|\sqrt{m} \dot{u}\|_{L^2} \|\nabla u\|_{L^6}^2 \right),
\]
where we have used (5.19) and Cauchy inequality. This together with Gronwall inequality and (5.19) completes the proof of Lemma 5.4.

Denote $w = u - h$, where $h$ is the solution to
\[
\begin{cases}
Lh = \nabla P(m, n), & \text{in } \Omega \times (0, T],

h|_{\partial \Omega} = 0.
\end{cases}
\]

Similar to (4.26), we have
\[
\begin{cases}
Lw = \dot{m}u, & \text{in } \Omega \times (0, T],

w|_{\partial \Omega} = 0.
\end{cases}
\]

Due to (5.1), (5.21), Lemma 2.3 and Lemma 5.4, we immediately give the following result.

**Corollary 5.1.** Under the conditions of Theorem 1.2 and (5.1), for $0 \leq T < T^*$, we have
\[
\|w\|_{L^2(0,T;W^{2,q})} \leq C.
\]

In the following, we give the estimates of the derivatives of $m$ and $n$.

**Lemma 5.5.** Under the conditions of Theorem 1.2 and (5.1), for $0 \leq T < T^*$, we have
\[
\sup_{t \in [0,T]} \| (\nabla m, \nabla n)(t) \|_{L^q} \leq C.
\]

**Proof.** Differentiating the equation (1.1) with respect to $x_i$, then multiplying both sides of the resulting equation by $q|\partial_i m|^{q-2} \partial_i m$, we get
\[
\partial_i |\partial_i m|^q + \text{div}(\partial_i m \partial_i u) + (q-1)|\partial_i m|^q \text{div} u + g|\partial_i m|^{q-2} \partial_i \partial_i m \text{div} u + q|\partial_i m|^{q-2} \partial_i m \partial_i u \cdot \nabla m = 0. \tag{5.22}
\]

Integrating (5.22) over $\Omega$, we obtain
\[
\frac{d}{dt} \int_{\Omega} |\nabla m|^q \leq C \int_{\Omega} |\nabla u||\nabla m|^q + q \int_{\Omega} m|\nabla u||\nabla m|^{q-1} \leq C \|\nabla u\|_{L^q} \|\nabla m\|_{L^q}^q + C \|\nabla^2 u\|_{L^q} \|\nabla m\|_{L^q}^{q-1}. \tag{5.23}
\]
Similarly,\[ \frac{d}{dt} \int_\Omega |\nabla n|^q \leq C\|
abla u\|_{L^\infty}^q \|
abla n\|_{L^q}^q + C\|
abla^2 u\|_{L^q} \|
abla n\|_{L^q}^{q-1}. \tag{5.24} \]

By (5.23) - (5.24), we have\[ \frac{d}{dt} (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \leq C(1 + \|\nabla w\|_{L^\infty} + \|\nabla h\|_{L^\infty})(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) + C\|\nabla^2 w\|_{L^q} + C\|\nabla^2 h\|_{L^q}. \tag{5.25} \]

Since \( P_m \) and \( P_n \) are bounded, we have from (5.20) and Lemma 2.3\[ \|\nabla^2 h\|_{L^q} \leq C(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}). \tag{5.26} \]

Applying (5.25) - (5.26) and Sobolev inequality, we get\[ \frac{d}{dt} (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \leq C(1 + \|\nabla w\|_{W^{2,q}} + \|\nabla h\|_{L^\infty})(\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) + C\|\nabla^2 w\|_{L^q}. \tag{5.27} \]

Using (5.19), (5.20), Lemmas 2.3 - 2.5, we have\[ \|\nabla h\|_{L^\infty} \leq C \left( 1 + \|\nabla h\|_{BMO(\Omega)} \ln(e + \|\nabla^2 h\|_{L^q}) \right) \leq C \left( 1 + \ln\|\nabla^2 h\|_{L^q} \right) \leq C \left( 1 + \ln(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \right). \tag{5.28} \]

From (5.27) - (5.28) and Cauchy inequality, we get\[ \frac{d}{dt} (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \leq C \left( 1 + \|\nabla w\|_{W^{2,q}} + \ln(e + \|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) \right) (\|\nabla m\|_{L^q} + \|\nabla n\|_{L^q}) + C\|\nabla^2 w\|_{L^q}. \tag{5.29} \]

Denote \( G(t) = e + \|\nabla m(t)\|_{L^q} + \|\nabla n(t)\|_{L^q} \), we have from (5.29)\[ \frac{d}{dt} G(t) \leq C\|\nabla^2 w\|_{L^q} + C(1 + \|\nabla w\|_{W^{2,q}})G(t) + CG(t) \ln G(t). \tag{5.30} \]

Multiplying (5.30) by \( \frac{1}{G(t)} \), and using \( G > 1 \), we have\[ \frac{d}{dt} \ln G(t) \leq C(1 + \|\nabla w\|_{W^{2,q}}) + C \ln G(t). \tag{5.31} \]

Using Gronwall inequality, Corollary 5.1 and (5.31), we complete the proof of Lemma 5.5. \( \square \)

**Lemma 5.6.** Under the conditions of Theorem 1.2 and (5.1), for \( 0 \leq T < T^* \), we have\[ \|u\|_{L^\infty(0,T;H^2)} + \|u\|_{L^2(0,T;W^{2,q})} \leq C. \]

**Proof.** Rewrite (1.1)3 as\[ Lu = m\dot{u} + \nabla P(m,n). \]

By (5.1), (5.19), Lemma 2.3, Lemma 5.4, and Lemma 5.5, we have\[ \|u\|_{H^2} \leq C(\|m\dot{u}\|_{L^2} + \|\nabla P\|_{L^2}). \]
\[ C(\|m\dot{u}\|_{L^2} + \|\nabla m\|_{L^2} + \|\nabla n\|_{L^2}) \leq C, \]

and

\[ \|u\|_{L^2(0,T;W^{2,q})} \leq C(\|\dot{u}\|_{L^2(0,T;L^q)} + \|\nabla u\|_{L^2(0,T;L^q)}) \leq C(\|\nabla \dot{u}\|_{L^2(0,T;L^2)} + \|\nabla m\|_{L^2(0,T;L^q)} + \|\nabla n\|_{L^2(0,T;L^q)}) \leq C. \]

□

By (5.1), Lemma 5.4, Lemma 5.6, and Sobolev inequality, we get the following result.

**Corollary 5.2.** Under the conditions of Theorem 1.2 and (5.1), for \(0 \leq T < T^*\), we have

\[ \int_{\Omega} m|u_t|^2 + \int_0^T \int_{\Omega} |\nabla u|^2 \leq C. \]

By (5.1), Lemma 5.3, Lemma 5.5, Lemma 5.6, and Corollary 5.2, we know that \(T^*\) is not the maximal existence time for the strong solution \((m,n,u)(x,t)\) to the problem (1.1)-(1.5). This is a contradiction with the definition of \(T^*\). Therefore, (5.1) is invalid, i.e.

\[ \lim_{T \to T^*} \sup_{T} \|m(t)\|_{L^\infty(0,T;L^\infty)} = \infty. \]

The proof of Theorem 1.2 is completed. □

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Global Classical Spherically Symmetric Solution to 
Compressible Navier-Stokes Equations with Large 
Initial Data and Vacuum

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Abstract

In this work, we prove the existence and uniqueness of the global classical spherically symmetric solution to the compressible isentropic Navier-Stokes equations with vacuum in a bounded domain or exterior domain Ω of \( \mathbb{R}^n (n \geq 2) \). This is an extensive work of [6], where the global classical solution of compressible Navier-Stokes equations in one dimension was obtained. As pointed out in [6], the regularity of velocity \( u \in H^1_{loc}([0, \infty); H^3) \) could not be improved to \( C^1([0, \infty); H^3) \), otherwise, it would blow up in finite time. Compared to [6], the regularities of velocity with respect to the space variables are improved. The analysis is based on some new mathematical techniques and some new useful estimates. This can be viewed to be the first result on global classical solution with large initial data and vacuum.

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Contents

1. Introduction ................................................................. 2
2. Proof of Theorem 1.1 ...................................................... 5
3. Proof of Theorem 1.2 ...................................................... 13
Acknowledgement .......................................................... 22
References ........................................................................ 22

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1 Introduction

In this paper, we consider the initial-boundary value problem of compressible isentropic Navier-Stokes equations in a bounded domain or exterior domain $\Omega$ of $\mathbb{R}^n (n \geq 2)$:

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0, \quad \rho \geq 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P(\rho) &= \mu \Delta u + (\mu + \lambda) \nabla (\nabla \cdot u) + \rho f,
\end{aligned}
$$

(1.1)

for $(x, t) \in \Omega \times (0, +\infty)$, where

$$
\Omega = \{x \mid a < |x| < b\}, \quad 0 < a < b \leq \infty, \quad f(x, t) = f(|x|, t) \frac{x}{|x|},
$$

$\rho$ and $u = (u_1, u_2, \cdots, u_n)$ denote the density and the velocity respectively; $P(\rho) = K\rho^\gamma$, for some constants $\gamma > 1$ and $K > 0$, is the pressure function; $f$ is the external force; the viscosity coefficients $\mu$ and $\lambda$ satisfy the natural physical restrictions: $\mu > 0$ and $2\mu + n\lambda \geq 0$.

We consider the initial condition:

$$(\rho, u)|_{t=0} = (\rho_0, u_0) \text{ in } \overline{\Omega},$$

(1.2)

and the boundary condition:

$$u(x, t) \to 0, \text{ as } |x| \to a \text{ or } b, \text{ for } t \geq 0,$$

(1.3)

where

$$\rho_0(x) = \rho_0(|x|), \quad u_0(x) = u_0(|x|) \frac{x}{|x|}.$$

We are looking for a classical spherically symmetric solution $(\rho, u)$:

$$\rho(x, t) = \rho(r, t), \quad u(x, t) = u(r, t) \frac{x}{r},$$

where $r = |x|$, and $(\rho, u)(r, t)$ satisfies

$$
\begin{aligned}
\rho_t + (\rho u)_r + m \frac{\rho u}{r} = 0, \quad \rho \geq 0, \\
(\rho u)_t + (\rho u^2)_r + m \frac{\rho u^2}{r} + P_r = \nu(u_r + m \frac{u}{r}) + \rho f,
\end{aligned}
$$

(1.4)

for $(r, t) \in (a, b) \times (0, \infty)$, with the initial condition:

$$(\rho(r, t), u(r, t))|_{t=0} = (\rho_0(r), u_0(r)) \text{ in } I,$$

(1.5)

and the boundary condition:

$$u(r, t) \to 0, \text{ as } r \to a \text{ or } b, \text{ for } t \geq 0,$$

(1.6)

where $m = n - 1$, $\nu = 2\mu + \lambda \geq \frac{2(n-1)}{n}\mu > 0$ and $I = [a, b]$.

Let’s review some previous work in this direction. When the viscosity coefficient $\mu$ is constant, the local classical solution of non-isentropic Navier-Stokes equations in Hölder spaces was obtained by Tani in [20] with $\rho_0$ bounded below away from zero. Using delicate energy
methods in Sobolev spaces, Matsumura and Nishida showed in [17, 18] that the existence of the global classical solution provided that the initial data was small in some sense and away from vacuum. There are also some results about the existence of global strong solution to the Navier-Stokes equations with constant viscosity coefficient when \( \rho_0 > 0 \), refer for instance to [11] for the isentropic flow. Jiang in [12] proved the global existence of spherically symmetric smooth solutions in Hölder spaces to the equations of a viscous polytropic ideal gas in the domain exterior to a ball in \( \mathbb{R}^n \) (\( n = 2 \) or 3) when \( \rho_0 > 0 \). For general initial data, Kawohl in [14] got the global classical solution with \( \rho_0 > 0 \) and the viscosity coefficient \( \mu = \mu(\rho) \) satisfying

\[
0 < \mu_0 \leq \mu(\rho) \leq \mu_1, \quad \text{for } \rho \geq 0,
\]

where \( \mu_0 \) and \( \mu_1 \) are constants. Indeed, such a condition includes the case \( \mu(\rho) \equiv \text{const.} \)

In the presence of vacuum, Lions in [16] used the weak convergence method to show the existence of global weak solution to the Navier-Stokes equations for isentropic flow with general initial data and \( \gamma \geq \frac{9}{5} \) in three dimensional space. Later, the restriction on \( \gamma \) was relaxed by Feireisl, et al [9] to \( \gamma > \frac{3}{2} \). Unfortunately, this assumption excludes for example the interesting case \( \gamma = 1 \) (air, et al). Jiang and Zhang relaxed the condition to \( \gamma > 1 \) in [13] when they considered the global spherically symmetric weak solution. It worths mentioning a result due to Hoff in [11], the existence of a weak solution for positive initial density has been proved when \( \gamma = 1 \).

There were few results about strong solution when the initial density may vanish until Salvi and Straškraba in [19], where \( \Omega \) is a bounded domain, \( P = P(\cdot) \in C^2(0, \infty) \), \( \rho_0 \in H^2 \), \( u_0 \in H^1_0 \cap H^2 \), and satisfied the compatibility condition:

\[
Lu_0(x) - \nabla P(\rho_0)(x) = \sqrt{\rho_0}g, \quad \text{for } g \in L^2.
\] (1.7)

Afterwards, Cho, Choe and Kim in [2, 3, 4] established some local and global existence results about strong solution in bounded or unbounded domain with initial data different from [19] still satisfying (1.7). Particularly, Choe and Kim in [4] showed the radially symmetric strong solution existed globally in time for \( \gamma \geq 2 \) in annular domain. As it is pointed out in [4] that the results have been proved only for annular domain and cannot be extended to a ball \( \Omega = B_R = \{x \in \mathbb{R}^n : |x| < R < \infty \} \), because of a counter-example of Weigant [21]. Precisely, for \( 1 < \gamma < 1 + \frac{1}{n-1} \), Weigant constructed a radially symmetric strong solution \( (\rho, u) \) in \( B_R \times (0, 1) \) such that \( \|\rho(\cdot, t)\|_{L^\infty(B_R)} \to \infty \) as \( t \to 1^- \). A recent paper [8] written by Fan, Jiang and Ni improved the result in [4] to the case \( \gamma \geq 1 \).

The local classical solution was obtained by Cho and Kim in [5] when the initial density may vanish and satisfying the following compatible conditions:

\[
Lu_0(x) - \nabla P(\rho_0)(x) = \rho_0|g_1(x) - f(x, 0)|,
\] (1.8)

for \( x \in \overline{\Omega} \), \( g_1 \in D^1_0 \) and \( \sqrt{\rho_0}g_1 \in L^2 \). Recently, we used some new estimates to get a unique globally classical solution \( \rho \in C^1([0, \infty); H^3) \) and \( u \in H^1_{\text{loc}}([0, \infty); H^3) \) to one dimensional compressible Navier-Stokes equations in a bounded domain when the initial density may
vanish, cf. [6]. Since Xin in [22] showed that the smooth solution \((\rho, u) \in C^1([0, \infty); H^3(\mathbb{R}))\) must blow up when the initial density is of nontrivial compact support, so the regularities of \(u\) with respect to time variable obtained in [6] could not be improved, refer to [6] for more details.

Since the system (1.4) have the one dimensional feature, the results in [6] are possible to obtain here. Besides, the regularities of \(u\) with respect to space variable could be improved though these of \(u\) with respect to time variable couldn’t be done. This causes some new challenges, which are handled by some new estimates.

This can be viewed to be the first result on global classical solution with large initial data and vacuum.

Notations:

1. \(Q_T = I \times [0, T], \bar{Q}_T = \Omega \times [0, T]\) for \(T > 0\).
2. For \(p \geq 1\), \(L^p = L^p(\Omega)\) denotes the \(L^p\) space with the norm \(\| \cdot \|_{L^p}\). For \(k \geq 1\) and \(p \geq 1\), \(W^{k,p} = W^{k,p}(\Omega)\) denotes the Sobolev space, whose norm is denoted as \(\| \cdot \|_{W^{k,p}}\); \(H^k = W^{k,2}(\Omega)\).
3. For an integer \(k \geq 0\) and \(0 < \alpha < 1\), let \(C^{k+\alpha}_\Omega\) denote the Schauder space of function on \(\Omega\), whose \(k\)th order derivative is Hölder continuous with exponent \(\alpha\), with the norm \(\| \cdot \|_{C^{k+\alpha}_\Omega}\).
4. For an integer \(k \geq 0\), denote \(H^k_r = H^k_r(I) = \left\{ u \mid \sum_{i=0}^{k} \int_I r^m |\partial_r^i u|^2 < \infty \right\}\), with the norm \(\| \cdot \|_{H^k_r} = \left( \sum_{i=0}^{k} \int_I r^m |\partial_r^i u|^2 \right)^{\frac{1}{2}}\).
5. \(L^2_r = H^0_r\).
6. \(D^{k,2} = D^{k,2}_0\) is the closure of \(C^\infty_0(\Omega)\) in \(D^{1,2}\).
7. \(L := \mu \Delta + (\mu + \lambda)\nabla \text{div}\) is the Lamé operator.

Our main results are stated as follows.

**Theorem 1.1** Assume that \(\rho_0 \geq 0\) satisfies \(\rho_0 \in L^1 \cap H^2, \rho^*_0 \in H^2, u_0 \in D^3 \cap D^1_0\) and \(f \in C([0, \infty); H^1), f_t \in L^2_{\text{loc}}([0, \infty); L^2),\) and the initial data \(\rho_0, u_0\) satisfy the compatible condition (1.5) with \(g_1(x) = g_1(r) \frac{x}{r}\). Then for any \(T > 0\), there exists a unique global classical solution \((\rho, u)\) to (1.3) satisfying

\[
(\rho, \rho^n) \in C([0, T]; H^2), \quad \rho \geq 0, \quad (\rho u)_t \in C([0, T]; H^1),
\]

\[
u \in C([0, T]; D^3 \cap D^1_0), \quad u_t \in L^\infty([0, T]; D^1_0) \cap L^2([0, T]; D^2), \quad \sqrt{\rho}u_{tt} \in L^2(\bar{Q}_T).
\]
Remark 1.1 (i) Note that if \( \Omega \) is bounded and locally Lipschitzian, then \( D^{k,p} = W^{k,p} \). See [10] for the proof.

(ii) By Sobolev embedding theorems, we have

\[
H^k(I) \hookrightarrow C^{k-\frac{1}{2}}(I), \quad \text{for } k = 1, 2, 3,
\]

this together with the regularities of \((\rho, u)\) give

\[
(\rho, \rho^\gamma) \in C([0,T];H^{1+\frac{1}{2}}(I)), \quad u \in C([0,T];C^{2+\frac{1}{2}}(I)).
\]

Since \((\rho(x, t), u(x, t)) = (\rho(r, t), u(r, t)\frac{\sqrt{r}}{2})\), we get

\[
(\rho, \rho^\gamma) \in C([0,T];H^{1+\frac{1}{2}}(\Omega)), \quad u \in C([0,T];C^{2+\frac{1}{2}}(\Omega)),
\]

which means \((\rho, u)\) is the classical solution to (1.1)-(1.3).

Theorem 1.2 Consider the same assumptions as in Theorem 1.1, and in addition assume that \( \rho_0 \in H^5, \rho_0^\gamma \in H^5, \nabla(\sqrt{\rho_0}) \in L^\infty, \sqrt{\rho_0}\nabla^2 g_1 \in L^2, \rho_0 \nabla^3 g_1 \in L^2, \ u_0 \in D^5, \ f \in C([0,\infty);H^3) \cap L^2_{loc}([0,\infty);H^1), \ f_t \in C([0,\infty);H^1) \cap L^2_{loc}([0,\infty);H^2) \) and \( f_{tt} \in L^2_{loc}([0,\infty);L^2) \).

Then the regularities of the solution obtained in Theorem 1.1 can be improved as follows:

\[
(\rho, \rho^\gamma) \in C([0,T];H^5), \quad \sqrt{\rho} \in W^{1,\infty}(\tilde{Q}_T), \quad (\rho u)_t \in L^\infty([0,T];H^3) \cap L^2([0,T];H^4),
\]

\[
 u \in L^\infty([0,T];D^1) \cap L^2([0,T];D^3), \quad u_t \in L^\infty([0,T];D^2) \cap L^2([0,T];D^4)
\]

\[
(\sqrt{\rho} \nabla^2 u_t, \rho \nabla^3 u_t) \in L^\infty([0,T];L^2), \quad \sqrt{\rho} \nabla^4 u_t \in L^2(\tilde{Q}_T), \quad \rho^2 u_{tt} \in L^2(\tilde{Q}_T), \quad \rho^2 u_{ttt} \in L^2(\tilde{Q}_T),
\]

\[
\rho^2 u_{tt} \in L^\infty([0,T];L^2) \cap L^2([0,T];H^1), \quad \rho^2 u_{tt} \in L^\infty([0,T];H^1) \cap L^2([0,T];D^2).
\]

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2 by giving some estimates similar to [6] and some new estimates, such as Lemma 3.6, Lemma 3.7 and Lemma 3.8.

The constants \( \nu \) and \( K \) play no role in the analysis, so we assume \( \nu = K = 1 \) without loss of generality.

2 Proof of Theorem 1.1

In this section, we get a unique global classical solution to (1.4)-(1.6) with initial density \( \rho_0 \geq \delta > 0 \) and \( b < \infty \) by some a priori estimates globally in time based on the local solution. Moreover, the estimates are independent of \( b \) and \( \delta \). Next, we construct a sequence of approximate solutions to (1.4)-(1.6) under the assumption \( \rho_0 \geq \delta > 0 \). We obtain the global classical solution to (1.4)-(1.6) for \( \rho_0 \geq 0 \) and \( b < \infty \) after taking the limits \( \delta \to 0 \). Based on the global classical solution for the case of \( b < \infty \), where the estimates are uniform for \( b \), then we can get the solution in the exterior domain by using the similar arguments as that in [3].

In the section, we denote by “\( c \)” the generic constant depending on \( a, \|\rho_0\|_{H^2}, \|\rho_0^\gamma\|_{H^2}, \|u_0\|_{D^3}, \|u_0\|_{D^4}, T \) and some other known constants but independent of \( \delta \) and \( b \).

Before proving Theorem 1.1, we give the following auxiliary theorem.
Theorem 2.1 Consider the same assumptions as in Theorem 1.7, and in addition assume that \( \rho_0 \geq \delta > 0 \) and \( b < \infty \). Then for any \( T > 0 \), there exists a unique global classical solution \((\rho, u)\) to \((1.4)-(1.6)\) satisfying

\[
\rho \in C([0,T]; H^2(I)), \quad \rho \geq \frac{\delta}{c}, \quad u_{tt} \in L^2([0,T]; L^2(I)),
\]

\[
u \in C([0,T]; H^3(I) \cap H^1_0(I)), \quad u \in C([0,T]; H^2_0(I) \cap L^2([0,T]; H^2(I)).
\]

The local solution in Theorem 2.1 can be obtained by the successive approximations like in \([3, 5]\), we omit it here for simplicity. The regularities guarantee the uniqueness (refer for instance to \([2, 3]\)). Based on it, Theorem 2.1 can be proved by some \textit{a priori} estimates globally in time.

For \( T \in (0, +\infty) \), let \((\rho, u)\) be the classical solution of \((1.4)-(1.6)\) as in Theorem 2.1. Then we have the following estimates (cf. \([4]\) and \([8]\)):

\textbf{Lemma 2.1} For any \( 0 \leq t \leq T \), it holds

\[
\|(\rho, \rho^\gamma)\|_{H^2} + \|(\rho_t, (\rho^\gamma)_t)\|_{H^1} + \int_0^T \|(\rho u_t, (\rho^\gamma) u_t)\|^2_{L^2} \leq c, \quad \rho \geq \frac{\delta}{c},
\]

and

\[
\int_I (r^m \rho u^2_t + r^m u^2_{rr} + r^m u^2_r + r^{m-2} u^2) + \int_{Q_T} (r^m u^2_t + r^{m-2} u^2_t + r^m u^2_{rrr}) \leq c,
\]

where \( \|h_1, h_2\|_X = \|h_1\|_X + \|h_2\|_X \), for some Banach space \( X \), and \( h_i \in X, i=1,2 \).

\textbf{Remark 2.1} (i) For the estimates about \( \rho^\gamma \), since \( \rho \) and \( \rho^\gamma \) satisfy the linear transport equations: \( \rho_t + u \cdot \nabla \rho + \rho \nabla \cdot u = 0 \) and \( (\rho^\gamma)_t + u \cdot \nabla (\rho^\gamma) + \gamma \rho^\gamma \nabla \cdot u = 0 \) respectively, then by using the similar arguments as that of Lemma 3.6 in \([4]\), we get

\[
\frac{d}{dt} \left\{ \int \nabla^2 \rho(x, t) dx + \int |\nabla^2 (\rho^\gamma)(x, t)|^2 dx \right\} 
\leq c \|\nabla u(\cdot, t)\|_{H^1} (\|\nabla \rho(\cdot, t)\|_{H^3} + \|\nabla (\rho^\gamma)(\cdot, t)\|_{H^1}) 
\leq c \|\nabla \rho(\cdot, t)\|_{H^1} + \|\nabla (\rho^\gamma)(\cdot, t)\|_{H^1} + \|G(\cdot, t)\|_{H^2}^2,
\]

where \( G = \nu \nabla \cdot u - \rho^\gamma \) is the effective viscous flux; we have used the estimates \( \|u(\cdot, t)\|_{D^1} \leq c \), \( \|u(\cdot, t)\|_{D^2} \leq c \) and \( \|\rho(\cdot, t)\|_{H^1} \leq c \) in \([4]\). Since \( \|G(\cdot, t)\|_{H^2} \in L^2(0, T) \) given in \([3]\), hence, an application of the Gronwall inequality gives

\[
\|(\rho, \rho^\gamma)(\cdot, t)\|_{H^2} \leq c,
\]

where we have used the following Sobolev inequalities for radially symmetric functions defined in \( \Omega' = \{x \in \mathbb{R}^n; x > a > 0\} \): \( \|\nabla \rho\|_{L^\infty} \leq c \|\nabla \rho\|_{H^1}, \|\nabla u\|_{L^\infty} \leq c \|\nabla u\|_{H^1}. \) From (\ast) and a direct calculation, we get

\[
\|(\rho, \rho^\gamma)\|_{H^2} \leq c.
\]

(ii) For the case \( a \geq 0 \) and \( \gamma \) different from that in \([21]\), it is interesting to investigate the existence of global strong or classical spherically symmetric solution. We leave it as a forthcoming paper.
From (1.4), we get

\[ \rho u_t + \rho uu_r + (\rho^\gamma)_r = \left( u_r + \frac{mu}{r} \right)_r + \rho f. \]  

(2.1)

Differentiating (2.1) with respect to \( t \), we have

\[ \rho u_{tt} + \rho_t u_t + \rho_t uu_r + \rho u_t u_r + \rho uu_{rt} + (\rho^\gamma)_{rt} = u_{rrt} + mr^{-2}(ru_r - u_t) + \rho_t f + \rho f_t. \]  

(2.2)

**Lemma 2.2** For any \( 0 \leq t \leq T \), it holds

\[ \int_I (r^m u_{rt}^2 + r^{m-2} u_t^2) + \int_{Q_T} r^m \rho u_{tt}^2 \leq c. \]

**Proof.** Multiplying (2.2) by \( r^m u_{tt} \), integrating over \( I \), and using integration by parts, Lemma 2.1 and Cauchy inequality, we have

\[
\int_I r^m \rho u_{tt}^2 + \frac{1}{2} \frac{d}{dt} \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) = \int_I r^m (\rho_t f + \rho f_t) u_{tt} - \int_I r^m (\rho_{tt} u_t + \rho_{tt} u_r + \rho_{tt} u_r + \rho uu_{rt} + (\rho^\gamma)_{rt}) u_{tt} \\
\leq - \int_I r^m \rho_t u_t u_{tt} - \int_I r^m (\rho_t u_r - \rho_t f) u_{tt} + c \int_I r^m \rho u_{tt}^2 + \frac{1}{2} \int_I r^m \rho u_{tt}^2 + c \int_I r^m u_{rt}^2 \\
+ \int_I r^m (\rho^\gamma)_{rt} u_{tt} + \int_I mr^{m-1}(\rho^\gamma)_{tt} + c \int_I r^m f_t^2,
\]

where we have used Lemma 2.1 and the following inequality found in [4]:

\[
\sup_{a \leq r \leq b} |u(r, t)| \leq c \left[ \int_I (r^m u_r^2 + mr^{m-2} u_r^2)(r, t) dr \right]^{\frac{1}{2}}, \text{ for } u(a, t) = 0,
\]

(2.3)

and the one-dimensional Sobolev inequality:

\[
\sup_{a \leq r \leq b} |\varphi(r)| \leq c||\varphi||_{H^1(I)}.
\]

(2.4)
Thus

\[
\begin{align*}
&\frac{1}{2} \int_I r^m \rho u_t^2 + \frac{1}{2} \frac{d}{dt} \int_I (r^m u_{tt}^2 + mr^m - 2 u_t^2) \\
\leq & - \int_I r^m \rho_t u_t u_{tt} - \int_I r^m (\rho_t uu_r - \rho_t f) u_{tt} + \int_I r^m (\rho \gamma) u_{tt} + \int_I mr^m - 1 (\rho \gamma) u_{tt} \\
&+ c \int_I r^m u_{tt}^2 + c \int_I r^m f_t^2 + c \\
= & - \frac{d}{dt} \int_I \left[ \frac{1}{2} r^m \rho_t u_t^2 + r^m (\rho_t uu_r - \rho_t f) u_t - r^m (\rho \gamma) u_{tt} - mr^m - 1 (\rho \gamma) u_t \right] \\
&+ \frac{1}{2} \int_I r^m \rho_t u_t^2 + \int_I r^m (\rho_t uu_r + \rho_t uu_r - \rho_t f - \rho_t f) u_t \\
&- \int_I r^m (\rho \gamma) u_{tt} + \int_I mr^m - 1 (\rho \gamma) u_t + c \int_I r^m u_{tt}^2 + c \int_I r^m f_t^2 + c \\
\leq & - \frac{d}{dt} \int_I \left[ \frac{1}{2} r^m \rho_t u_t^2 + r^m (\rho_t uu_r - \rho_t f) u_t - r^m (\rho \gamma) u_{tt} - mr^m - 1 (\rho \gamma) u_t \right] \\
&- \frac{1}{2} \int_I (r^m \rho u_t) u_{tt} + c \| u_t \|_{L^\infty} \int_I r^m (\rho_t^2 + u_t^2) + c \| u_t \|_{L^\infty}^2 \int_I r^m (\rho_t^2 + u_t^2) \\
&+ c \| u_t \|_{L^\infty} \left( \int_I r^m \rho_t^2 \right)^{\frac{1}{2}} \left( \int_I r^m u_{tt}^2 \right)^{\frac{1}{2}} + \| u_t \|_{L^\infty} \int_I r^m (\rho_t^2 + f + \rho_t^2 + f_t^2) \\
&+ c \int_I r^m (\rho \gamma) u_{tt}^2 + c \int_I (r^m u_{tt}^2 + mr^m - 2 u_t^2) + c \int_I r^m f_t^2 + c,
\end{align*}
\]

where we have used \([1.4], [2.3]\), Cauchy inequality, Hölder inequality and Lemma 2.1. This
together with (2.3), (2.4), Cauchy inequality, integration by parts and Lemma 2.1 implies

\[
\frac{1}{2} \int_I r^m \rho u_t^2 + \frac{1}{2} \frac{d}{dt} \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) \leq - \frac{d}{dt} \int_I \left[ \frac{1}{2} r^m \rho u_t^2 + r^m (\rho_t uu_r - \rho_t f) u_t - r^m (\rho^\gamma)_{rt} u_{rt} - mr^{m-1} (\rho^\gamma)_{rt} u_t \right] \\
+ \int_I r^m (\rho u_t + \rho_t u) u_t u_r + c \left[ \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) \right]^{\frac{1}{2}} \left[ \int_I r^m (\rho_{tt}^2 + f^2 + f_t^2) + 1 \right] \\
+ c \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) + c \int_I r^m (\rho^\gamma)_{tt} + c \int_I r^m f_t^2 + c \\
\leq - \frac{d}{dt} \int_I \left[ \frac{1}{2} r^m \rho u_t^2 + r^m (\rho_t uu_r - \rho_t f) u_t - r^m (\rho^\gamma)_{rt} u_{rt} - mr^{m-1} (\rho^\gamma)_{rt} u_t \right] \\
+ c\|u_t\|_L^2 \left( \int_I r^m \rho u_t^2 + \int_I r^m \rho_t^2 \right) + c \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) \left[ \int_I r^m (\rho_{tt}^2 + f^2 + f_t^2) + 1 \right] \\
+ c \int_I r^m (\rho^\gamma)_{tt} + \rho_{tt}^2 + f_t^2 + c \\
\leq - \frac{d}{dt} \int_I \left[ \frac{1}{2} r^m \rho u_t^2 + r^m (\rho_t uu_r - \rho_t f) u_t - r^m (\rho^\gamma)_{rt} u_{rt} - mr^{m-1} (\rho^\gamma)_{rt} u_t \right] \\
+ c \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) \left[ \int_I r^m (\rho_{tt}^2 + f^2 + f_t^2) + 1 \right] + c \int_I r^m (\rho^\gamma)_{tt} + \rho_{tt}^2 + f_t^2 + c.
\]

Integrating the above inequality over \((0, t)\), we have

\[
\frac{1}{2} \int_0^t \int_I r^m \rho u_t^2 + \frac{1}{2} \frac{d}{dt} \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) \leq - \int_I \left[ \frac{1}{2} r^m \rho u_t^2 + r^m (\rho_t uu_r - \rho_t f) u_t - r^m (\rho^\gamma)_{rt} u_{rt} - mr^{m-1} (\rho^\gamma)_{rt} u_t \right] \\
+ c \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) \left[ \int_I r^m (\rho_{tt}^2 + f^2 + f_t^2) + 1 \right] + c \\
\leq \int_I \left[ \frac{1}{2} r^m \rho u_t^2 + c\|u_t\|_L^2 \right] + c \left( \int_I r^m u_{rt}^2 \right) + c \left( \int_I mr^{m-2} u_t^2 \right) \\
+ c \int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2) \int_I r^m (\rho_{tt}^2 + f_t^2) + c,
\]

where we have used (1.3), (1.8), (2.3), Lemma 2.1 Hölder inequality and the following equalities:

\[
\int_I (r^m u_{rt}^2 + mr^{m-2} u_t^2)|_{t=0} = c \int_\Omega |\nabla u_t|^2 |_{t=0},
\]

\[
u_t|_{t=0} = \rho_0^{-1} (L u_0 + \rho_0 f (0) - \nabla P (\rho_0) - \rho_0 u_0 \cdot \nabla u_0) = g_1 - u_0 \cdot \nabla u_0 \in D_0^1.
\]
We apply integration by parts, (2.3), (2.4), Hölder inequality, Cauchy inequality and Lemma 2.1 to obtain

\[
\int_{0}^{t} \int_{I} r^m \rho u_{tt}^2 + \frac{1}{2} \int_{I} (r^m u_{rt}^2 + mr^m u_{t}^2) \\
\leq - \int_{I} r^m \rho u_t u_{rt} + c \left[ \int_{I} (r^m u_{rt}^2 + mr^m u_{t}^2) \right]^\frac{1}{2} + c \int_{0}^{t} \int_{I} (r^m u_{rt}^2 + mr^m - 2 u_{t}^2) \int_{I} r^m (\rho_{tt}^2 + f_t^2) + c \\
\leq c \left( \int_{I} r^m \rho u_t^2 \right)^\frac{1}{2} \left( \int_{I} r^m u_{rt}^2 \right)^\frac{1}{2} + c \left[ \int_{I} (r^m u_{rt}^2 + mr^m - 2 u_{t}^2) \int_{I} r^m (\rho_{tt}^2 + f_t^2) + c \right] \\
\leq \frac{1}{4} \int_{I} (r^m u_{rt}^2 + mr^m - 2 u_{t}^2) + c \int_{0}^{t} \int_{I} (r^m u_{rt}^2 + mr^m - 2 u_{t}^2) \int_{I} r^m (\rho_{tt}^2 + f_t^2) + c. 
\]

Therefore,

\[
\int_{0}^{t} \int_{I} r^m \rho u_t^2 + \frac{1}{2} \int_{I} (r^m u_{rt}^2 + mr^m - 2 u_{t}^2) \\
\leq c \int_{0}^{t} \int_{I} (r^m u_{rt}^2 + mr^m - 2 u_{t}^2) \int_{I} r^m (\rho_{tt}^2 + f_t^2) + c. 
\]

By Gronwall inequality, we get

\[
\int_{Q_T} r^m \rho u_t^2 + \int_{I} (r^m u_{rt}^2 + r^m - 2 u_{t}^2) \leq c. 
\]

The proof of Lemma 2.2 is completed.

**Lemma 2.3** For any \(0 \leq t \leq T\), it holds

\[
\int_{I} r^m u_{rrr}^2 \leq c. 
\]

**Proof.** Differentiating (2.1) with respect to \(r\), we get

\[
u_{rrr} = \rho_r u_t + \rho u_{rt} + \rho_r u_{t} + \rho u_{t} + \rho u_{t} + (\rho^2)_{rr} - \frac{m u_{rr} - u}{r} + \frac{2m(r u_r - u)}{r^3} - \rho_r f - \rho f_r. \tag{2.5}
\]

By (2.5), (2.3), (2.4) and Lemma 2.1, we obtain

\[
\int_{I} r^m u_{rrr}^2 \leq c \int_{I} (r^m u_{rt}^2 + mr^m - 2 u_{t}^2) + c \int_{I} r^m (\rho^2)_{rr}^2 + c \int_{I} r^m (f^2 + f_r^2) + c \leq c.
\]

The proof of Lemma 2.3 is completed.

**Lemma 2.4** For any \(0 \leq t \leq T\), it holds

\[
\int_{I} r^m (\rho_{tt}^2 + (\rho^2)_{tt}^2) + \int_{Q_T} r^m u_{rrt}^2 \leq c.
\]
Proof. By (2.2), (2.3), (2.4), Lemma 2.1 and Lemma 2.2 we get
\[
\int_{Q_T} r^m u_{rtt}^2 \leq c \int_{Q_T} r^m \rho^2 u_r^2 + c \int_{Q_T} r^m \rho^2 u_r^2 + c \int_{Q_T} r^m \rho^2 u_r^2 + c \int_{Q_T} r^m \rho^2 u_r^2 + c \int_{Q_T} r^m \rho^2 u_r^2 + c \int_{Q_T} r^m \rho^2 u_r^2 + c \int_{Q_T} r^m \rho^2 u_r^2 + c \int_{Q_T} r^m \rho^2 u_r^2 \leq c.
\]
Multiplying (1.4) by \(\gamma \rho^{-1}\), we get
\[
(r^\gamma)_t + \gamma \rho^\gamma u_r + (r^\gamma)_t u + m \gamma r^{-1} \rho^\gamma u = 0. \tag{2.6}
\]
(1.4), (2.3), (2.4), Lemma 2.1 and Lemma 2.2 imply
\[
\int_I r^m (\rho^2 u_t^2 + (r^\gamma)_t)^2 \leq c.
\]
This proves Lemma 2.3. \(\square\)

To sum up, we get
\[
\|(\rho, \rho^\gamma)\|_{H^2} + \|(\rho_t, (\rho^\gamma)_t)\|_{H^1} + \|(\rho u, (\rho^\gamma) u)\|_{L^2} \leq c, \quad \rho \geq \frac{\delta}{c}, \tag{2.7}
\]
and
\[
\int_I r^m(u_{rt}^2 + r^{-2} u_r^2 + u_{rr}^2 + u_t^2 + u_r^2 + r^{-2} u_r^2) + \int_{Q_T} r^m(\rho u_{tt}^2 + u_{rtt}^2) \leq c. \tag{2.8}
\]
By (2.7) and (2.8), we get
\[
\begin{align*}
\|\rho\|_{H^2(I)} + \|(\rho_t, u_t)\|_{H^1(I)} + \|\rho u\|_{L^2(I)} + \|u\|_{H^1(I)} + \int_0^T (\|u_t\|_{H^1(I)} + \|u_{tt}\|_{L^2(I)}) & \leq c(b, \delta), \\
\rho & \geq \frac{\delta}{c}.
\end{align*}
\]
This proves Theorem 2.1. \(\square\)

Proof of Theorem 1.1

Let \(b < \infty\), and denote \(\rho_0^\delta = \rho_0 + \delta\), for \(\delta \in (0, 1)\), we have
\[
\rho_0^\delta \to \rho_0, \quad \text{in } H^2(I), \tag{2.9}
\]
\[
(r_0^\delta)_t \to \rho_0^\gamma, \quad \text{in } H^2(I). \tag{2.10}
\]
Let \(u_0^\delta\) be the unique solution to the equation:
\[
\left(u_{0r} + \frac{mu_0^\delta}{r}\right)_r - [(\rho_0^\delta)_t]_r = \rho_0^\delta[g_1 - f(0)], \quad \text{in } I, \tag{2.11}
\]
for \(u_0^\delta|_{\partial I} = 0\). (1.4) implies
\[
(u_{0r} + \frac{mu_0}{r})_r - [(\rho_0^\gamma)_t]_r = \rho_0[g_1 - f(0)], \quad \text{in } I, \tag{2.12}
\]
for \( u_0|_{\partial I} = 0 \).

From (2.9)-(2.12) and the standard elliptic estimates, we obtain

\[
u^\delta_0 \to u_0, \quad \text{in } H^3(I), \quad \text{as } \delta \to 0.
\] (2.13)

Consider (1.4)-(1.6) with initial-boundary data replaced by

\[(\rho^\delta, u^\delta)|_{t=0} = (\rho^\delta_0, u^\delta_0), \quad \text{in } I,
\]

and

\[u^\delta|_{\partial I} = 0, \quad \text{for } t \geq 0.
\]

Then we get a unique solution \((\rho^\delta, u^\delta)\) for each \(\delta > 0\) by Theorem 2.1.

It follows from (2.7) and (2.8) that

\[
\|(\rho^\delta, (\rho^\delta)^\gamma)\|_{H^2(I)} + \|((\rho^\delta)^\gamma)_t\|_{H^1(I)} + \|((\rho^\delta)^\gamma)_tt\|_{L^2(I)} \leq c,
\] (2.14)

and

\[
\rho^\delta \geq \frac{\delta}{c}; \quad \|u^\delta_t\|_{H^1(I)} + \|u^\delta\|_{H^3(I)} + \int_{Q_T} (\rho^\delta|u^\delta_{tt}|^2 + |u^\delta_{rt}|^2) \leq c(b).
\] (2.15)

Based on the estimates in (2.11) and (2.13), we get a solution \((\rho, u)\) to (1.4)-(1.6) after taking the limit \(\delta \to 0\) (take the subsequence if necessary), satisfying

\[
\begin{cases}
(\rho, \rho^\gamma) \in L^\infty([0,T];H^2(I)), & (\rho_t, (\rho^\gamma)_t) \in L^\infty([0,T];H^1(I)), \\
(\rho_{tt}, (\rho^\gamma)_{tt}) \in L^\infty([0,T];L^2(I)), & \rho \geq 0, \quad \sqrt{\rho} u_{tt} \in L^2([0,T];L^2(I)), \\
u \in L^\infty([0,T];H^3(I) \cap H^1_0(I)), & u_t \in L^\infty([0,T];H^3(I)) \cap L^2([0,T];H^2(I)).
\end{cases}
\] (2.16)

Since \(u \in L^\infty([0,T];H^3(I))\) and \(u_t \in L^\infty([0,T];H^3_0(I))\), then we get \(u \in C([0,T];H^2(I))\) (refer to [7]). By (1.4), (2.6) and similar arguments as that in [4, 5], we get

\[
\rho \in C([0,T];H^2(I)), \quad \rho_t \in C([0,T];H^1(I)),
\] (2.17)

and

\[
\rho^\gamma \in C([0,T];H^2(I)), \quad (\rho^\gamma)_t \in C([0,T];H^1(I)).
\] (2.18)

Denote \(G = (u_r + \frac{mu}{r} - \rho^\gamma)_r + \rho f\). By (2.1) and (2.16), we have

\[G = \rho u_t + \rho uu_r \in L^2([0,T];H^2(I)),
\]

\[G_t = (\rho u_t + \rho uu_r)_t \in L^2([0,T];L^2(I)).
\]

By the embedding theorem (7), we have \(G \in C([0,T];H^1(I))\). Since \(\rho f \in C([0,T];H^1(I))\), we get

\[
\left(u_r + \frac{mu}{r} - \rho^\gamma\right)_r \in C([0,T];H^1(I)).
\]

This means

\[
u_r + \frac{mu}{r} - \rho^\gamma \in C([0,T];H^2(I)).
\] (2.19)
By (2.18) and (2.19), we get
\[ u_r + \frac{mu}{r} \in C([0, T]; H^2). \]
This together with \( u \in C([0, T]; H^2(I)) \) implies
\[ u \in C([0, T]; H^3(I)). \] 
(2.20)

(1.4), (2.17), (2.18) and (2.20) give
\[ (\rho u)_t \in C([0, T]; H^1(I)). \] 
(2.21)

Denote
\[ \rho(x, t) = \rho(r, t), \quad u(x, t) = u(r, t) \frac{x}{r}. \] 
(2.22)

It follows from (2.16)-(2.18) and (2.20)-(2.22), we complete the proof of Theorem 1.1 for \( b < \infty \). For \( b = \infty \), we can use the similar methods as that in [3] together with the estimates (2.7) and (2.8) uniform for \( b \) to get it. We omit details here for simplicity.

\[ \square \]

3 Proof of Theorem 1.2

Similarly to the proof of Theorem 1.1 we need the following auxiliary theorem.

**Theorem 3.1** Consider the same assumptions as in Theorem 1.2 and in addition assume that \( \rho_0 \geq \delta > 0 \) and \( b < \infty \). Then for any \( T > 0 \), there exists a unique global classical solution \( (\rho, u) \) to (1.4)-(1.6) satisfying
\[ \rho \in C([0, T]; H^5(I)), \quad \rho \geq \frac{\delta}{\gamma}, \quad u \in C([0, T]; H^5(I) \cap H^1_0(I)) \cap L^2([0, T]; H^6(I)), \]
\[ u_t \in C([0, T]; H^3(I) \cap H^1_0(I)) \cap L^2([0, T]; H^4(I)), \]
\[ u_{tt} \in C([0, T]; H^1_0(I)) \cap L^2([0, T]; H^2(I)), \quad u_{ttt} \in L^2([0, T]; L^2(I)). \]

Similarly to the proof of Theorem 2.1 Theorem 3.1 can be proved by some \textit{a priori} estimates globally in time. Since (2.7) and (2.8) are also valid here, we need some other \textit{a priori} estimates about higher order derivatives of \( (\rho, u) \). The generic positive constant \( c \) may depend on the initial data presented in Theorem 1.2 and other known constants but independent of \( \delta \) and \( b \).

**Lemma 3.1** For any \( 0 \leq t \leq T \), it holds
\[ \int_I r^n [\rho_{rr}^2 + \rho_{r}^2 + |(\rho^\gamma)_{rrr}|^2 + |(\rho^\gamma)_{rrt}|^2] + \int_{Q_T} r^n [\rho_{ttt}^2 + |(\rho^\gamma)_{ttt}|^2 + u_{rrrr}^2] \leq c. \]

**Proof.** Taking derivative of order three on both sides of (1.4) with respect to \( r \), we have
\[ \rho_{rrrr} = -mr^{-1}\rho_{u_{rr}} - 3mr^{-1}\rho_{u_{r}} - 3mr^{-1}\rho_{u_{rr}}u + \frac{3m\rho_{u_{rr}}}{r^2} - \frac{6m\rho_{u_{r}}}{r^3} - mr^{-1}\rho_{rrrr}u \\
+ \frac{3m\rho_{r}u}{r^2} + \frac{6m\rho_{r}u}{r^3} - \frac{6m\rho_{r}u}{r^4} + \frac{6m\rho_{u}}{r^4} - \rho_{rrrr}u - 4\rho_{u_{rr}}u_{r} - 6\rho_{u_{r}}u_{rr} \\
- 4\rho_{u_{rr}} - \rho_{u_{rrrr}}. \] 
(3.1)
Multiplying (3.1) by $r^m \rho_{rrr}$, integrating the resulting equation over $I$, and using integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_I r^m \rho_{rrr}^2 = \int_I r^m \rho_{rrr} \left[ -mr^{-1} \rho u_{rrr} - 3mr^{-1} \rho_r u_{rr} - 3mr^{-1} \rho_{rr} u_r + \frac{3m \rho_u}{r^2} 
\right]
\]

\[
- \frac{6m \rho u_r}{r^3} - mr^{-1} \rho_{rrr} u_r + \frac{3m \rho u_r}{r^2} + \frac{6m \rho_r u_r}{r^2} - \frac{6m \rho_r u_r}{r^3} + \frac{6m \rho u_r}{r^4} - 4 \rho_{rr} u_r - 6 \rho_{rr} u_r
\]

By Sobolev inequality, \(2.3\), \(2.4\), \(2.7\), \(2.8\) and Cauchy inequality, we get

\[
\frac{d}{dt} \int_I r^m \rho_{rrr}^2 \leq c \int_I r^m \rho_{rrr}^2 + c \int_I r^m u_{rrr}^2 + c.
\] (3.2)

Similarly to (3.2), we get from (2.6)

\[
\frac{d}{dt} \int_I r^m |(\rho r)_{rrr}|^2 \leq c \int_I r^m |(\rho r)_{rrr}|^2 + c \int_I r^m u_{rrr}^2 + c.
\] (3.3)

By (3.2) and (3.3), we have

\[
\frac{d}{dt} \int_I r^m (\rho_{rrr}^2 + |(\rho r)_{rrr}|^2) \leq c \int_I r^m (\rho_{rrr}^2 + |(\rho r)_{rrr}|^2) + c \int_I r^m u_{rrr}^2 + c.
\] (3.4)

Differentiating (2.5) with respect to $r$, we have

\[
u_{rrrr} = \rho_{rrr} u_t + 2 \rho_r u_r + \rho_{rr} u_t + (\rho_r u_r + \rho u_r^2 + \rho u_{rr}^2), r + (\rho r)_{rrr} - mr^{-1} u_{rrr}
\]

\[
+ 3mr^{-2} u_{rr} - 6mr^{-3} u_r + \frac{6m u}{r^4} - \rho_{rr} f - 2 \rho_r f_t - \rho f_{rrr}.
\] (3.5)

From (3.5), \(2.3\), \(2.4\), \(2.7\) and \(2.8\), we obtain

\[
\int_I r^m u_{rrr}^2 \leq c \int_I r^m |(\rho r)_{rrr}|^2 + c \int_I r^m \rho u_{rrr}^2 + c \int_I r^m f_{rr}^2 + c.
\] (3.6)

By (3.4), (3.6), \(2.4\) and \(2.7\), we get

\[
\frac{d}{dt} \int_I r^m (\rho_{rrr}^2 + |(\rho r)_{rrr}|^2) \leq c \int_I r^m (\rho_{rrr}^2 + |(\rho r)_{rrr}|^2) + c \int_I r^m u_{rrr}^2 + c \int_I r^m f_{rr}^2 + c.
\]

By Gronwall inequality and \(2.8\), we obtain

\[
\int_I r^m (\rho_{rrr}^2 + |(\rho r)_{rrr}|^2) \leq c.
\] (3.7)

It follows from \(1.4\), \(2.3\), \(2.4\), \(2.6\), \(2.7\), \(2.8\), \(3.6\), \(3.7\) and Lemma \(3.1\) that

\[
\int_I r^m \rho_{rrr}^2 + |(\rho r)_{rrr}|^2 + \int_{Q_T} r^m \rho_{rrr}^2 + |(\rho r)_{rrr}|^2 + u_{rrr}^2 \leq c.
\]

The proof of Lemma \(3.1\) is completed.

\(\square\)

**Lemma 3.2** For any $T > 0$, we have

\[
\| \sqrt{\rho} \|_{L^\infty(Q_T)} + \| (\sqrt{\rho})_t \|_{L^\infty(Q_T)} \leq c.
\]
Proof. Multiplying (1.4) by $\frac{1}{2\sqrt{\rho}}$, we have

$$
(\sqrt{\rho})_t + \frac{mr^{-1}}{2}\sqrt{\rho}u + \frac{1}{2}\sqrt{\rho}u_r + (\sqrt{\rho})_r u = 0. \tag{3.8}
$$

Differentiating (3.8) with respect to $r$, we get

$$
(\sqrt{\rho})_{rt} + \frac{mr^{-1}}{2}(\sqrt{\rho})_ru + \frac{mr^{-1}}{2}\sqrt{\rho}u_r - \frac{m\sqrt{\rho}u}{2r^2} + 3(\sqrt{\rho})_{uu} + \frac{1}{2}\sqrt{\rho}u_{rr} + (\sqrt{\rho})_{rru} u = 0.
$$

Denote $h = (\sqrt{\rho})_r$, we have

$$
h_t + h_r u + h\left(\frac{mr^{-1}}{2} u + \frac{3}{2} u_r\right) + \frac{mr^{-1}}{2}\sqrt{\rho}u_r - \frac{m\sqrt{\rho}u}{2r^2} + \frac{1}{2}\sqrt{\rho}u_{rr} = 0,
$$

which implies

$$
\frac{d}{dt}\left\{ h \exp\left[\int_0^t \left(\frac{mr^{-1}}{2} u + \frac{3}{2} u_r\right)(r(\tau, y), \tau)d\tau\right] \right\} = - \left(\frac{mr^{-1}}{2}\sqrt{\rho}u_r - \frac{m\sqrt{\rho}u}{2r^2} + \frac{\sqrt{\rho}u_{rr}}{2}\right) \times \exp\left[\int_0^t \left(\frac{mr^{-1}}{2} u + \frac{3}{2} u_r\right)(r(\tau, y), \tau)d\tau\right], \tag{3.9}
$$

where $r(t, y)$ satisfies

$$
\begin{cases}
\frac{dr(t, y)}{dt} = u(r(t, y), t), \quad 0 \leq t < s, \\
r(s, y) = y.
\end{cases}
$$

Integrating (3.9) over $(0, s)$, we get

$$
h(y, s) = \exp\left(- \int_0^s \left(\frac{mr^{-1}}{2} u + \frac{3}{2} u_r\right)(r(\tau, y), \tau)d\tau\right) h(r(0, y), 0)
- \int_0^s \left(\frac{mr^{-1}}{2}\sqrt{\rho}u_r - \frac{m\sqrt{\rho}u}{2r^2} + \frac{\sqrt{\rho}u_{rr}}{2}\right) \times \exp\left(\int_0^t \left(\frac{mr^{-1}}{2} u + \frac{3}{2} u_r\right)(r(\tau, y), \tau)d\tau\right) \right) dt.
$$

This together with (2.3), (2.4), (2.7) and (2.8) implies

$$
\| (\sqrt{\rho})_r \|_{L^\infty(Q_T)} \leq c. \tag{3.10}
$$

From (3.9), (3.10), (2.3), (2.4), (2.7) and (2.8), we get

$$
\| (\sqrt{\rho})_t \|_{L^\infty(Q_T)} \leq c.
$$

The proof of Lemma 3.2 is completed. \hfill \Box

Lemma 3.3 For any $0 \leq t \leq T$, it holds

$$
\int_I r^m(\rho^2 u_{tt}^2 + \rho u_{rr}^2 + u_{rrrr}^2) + \int_{Q_T} r^m(\rho^2 u_{rrt}^2 + u_{rrrrr}^2) \leq c.
$$
Proof. Differentiating (2.2) with respect to $t$, we get
\begin{align*}
\rho u_{ttt} + \rho u_t u_t + 2 \rho u_{tt} u_t + \rho u_{tt} u_t + 2 \rho u_{tt} u_t + 2 \rho u_t u_t + 2 \rho u_{tt} u_t + 2 \rho u_t u_t \\
+ \rho u_{tt} (\rho^\gamma)_{rt} = u_{r r t} + m r^{-2} (u_{r t t} - u_t) + \rho_t f + 2 \rho_t f_t + \rho_f t.
\end{align*}
(3.11)

Multiplying (3.11) by $r^m \rho^2 u_{tt}$, integrating the resulting equation over $I$, and using integration by parts and Cauchy inequality, we have
\begin{align*}
\frac{1}{2} \int_I r^m \rho^3 u_{tt}^2 + \int_I (r^m \rho^2 u_{rrt}^2 + m r^{m-2} \rho^2 u_{tt}^2) \\
= - \frac{1}{2} \int_I r^m \rho^2 \rho u_t u_t - \int_I r^m \rho^2 u_t \left[ \rho_t u_t + \rho u_t u_t + 2 \rho u_{tt} u_t + 2 \rho u_{tt} u_t + \rho u_{tt} u_t \right] \\
+ 2 \rho u_{tt} u_t + 2 \rho u_{tt} u_t + (\rho^\gamma)_{rt} \right] - 2 \int_I r^m \rho \rho_t u_{tt} u_t + \int_I r^m \rho^2 u_t (\rho_t f + 2 \rho_t f_t + \rho_f t) \\
\leq c \int_I r^m \rho u_{tt}^2 + \frac{1}{4} \int_I r^m \rho^2 u_{tt}^2 + c \int_I r^m |(\rho^\gamma)_{rt}|^2 - 4 \int_I r^m \rho u_{tt} \sqrt{\rho} u_t (\sqrt{\rho})_r \\
+ c \int_I (r^m \rho^2 f^2 + \rho^2 f_t^2 + \rho^2 f_{tt}^2) + c \\
\leq c \int_I r^m \rho u_{tt}^2 + \frac{1}{4} \int_I r^m \rho^2 u_{tt}^2 + c \int_I r^m |(\rho^\gamma)_{rt}|^2 + \frac{1}{4} \int_I r^m \rho^2 u_{tt}^2 + \int_I r^m f_{tt}^2 + c,
\end{align*}
where we have used (2.3), (2.4), (2.7), (2.8), Lemma 3.2 and Cauchy inequality.

Thus,
\begin{align*}
\frac{d}{dt} \int_I r^m \rho^3 u_{tt}^2 + \int_I (r^m \rho^2 u_{rrt}^2 + m r^{m-2} \rho^2 u_{tt}^2) \\
\leq c \int_I r^m \rho u_{tt}^2 + c \int_I r^m |(\rho^\gamma)_{rt}|^2 + \int_I r^m f_{tt}^2 + c.
\end{align*}
(3.12)

Integrating (3.12) over $(0, t)$, and using (2.8) and Lemma 3.1, we get
\begin{align*}
\int_I r^m \rho^3 u_{tt}^2(t) + \int_0^t \int_I (r^m \rho^2 u_{rrt}^2 + m r^{m-2} \rho^2 u_{tt}^2) \leq \int_I r^m \rho^3 u_{tt}^2(0) + c.
\end{align*}
(3.13)

By (2.1), (2.2), (1.8) and $\sqrt{\rho_0 \nabla^2 g_1} \in L^2$, we have
\begin{align*}
\int_I r^m \rho^3 u_{tt}^2(0) \leq c.
\end{align*}
(3.14)

(3.13) and (3.14) give
\begin{align*}
\int_I r^m \rho^3 u_{tt}^2 + \int_{Q_T} (r^m \rho^2 u_{rrt}^2 + r^{m-2} \rho^2 u_{tt}^2) \leq c.
\end{align*}
(3.15)

By (2.2), Cauchy inequality, (2.3), (2.4), (2.7), (2.8) and (3.15), we have
\begin{align*}
\int_I r^m \rho u_{rrt}^2 & \leq c \int_I r^m \rho u_{tt}^2 + c \int_I r^m \rho^2 u_{tt}^2 + c \int_I r^m \rho u_{tt}^2 u_t^2 + c \int_I r^m \rho u_{tt}^2 u_t^2 \\
& + c \int_I r^m \rho u_{tt}^2 u_t^2 + c \int_I r^m |(\rho^\gamma)_{rt}|^2 + c \int_I r^m \rho u_{tt}^2 u_t^2 + c \int_I r^m \rho^2 u_{tt}^2 \\
& + c \int_I r^m \rho^2 f^2 + c \int_I r^m \rho^2 f_{tt}^2 \\
& \leq c.
\end{align*}
(3.16)
Differentiating (2.2) with respect to \( r \), we get

\[
    u_{rrrt} = \rho_{rt}u_r + 2(\sqrt{\rho})_{r}\sqrt{\rho}u_{rr} + \rho_{r}u_{r} + \rho_{r}u_{rr} + \rho_{rr}u_{r} + \rho_{rr}u_{rr} + \rho_{rrr}u_{r} + (\rho^\gamma)_{rrr} - mr^{-1}u_{rr} + 2mr^{-2}u_{rr}
\]

\[
    - \frac{2m\rho}{r^3} - \rho_{rt}f_{r} - \rho_{r}f_{r} - \rho_{rr}f_{r}.
\]

By (3.15), (3.17), (3.17), (2.3), (2.4), (2.7), (2.8), Lemma 3.1 and Lemma 3.2, we have

\[
    \int_{Q_T} r^m u_{rr}^2 \leq c.
\]

By (3.19) and (3.20), we obtain

\[
    \int_{Q_T} r^m u_{rrr}^2 \leq c.
\]

The proof of Lemma 3.3 is completed.

\[ \square \]

**Lemma 3.4** For any \( 0 \leq t \leq T \), it holds

\[
    \int_{I} r^m [\rho_{rrr}^2 + |(\rho^\gamma)_{rrr}|^2] + \int_{Q_T} r^m u_{rrr}^2 \leq c.
\]

**Proof.** Differentiating (3.1) with respect to \( r \), we get

\[
    \rho_{rrr} + mr^{-1}\rho_{u_{rrr}} + 4mr^{-1}\rho_{r_{u_{rrr}}} + 6mr^{-1}\rho_{rr_{u_{rrr}}} - \frac{4m\rho_{rru_{rrr}}}{r^2} + \frac{12m\rho_{rrrru_{rr}}}{r^3} - \frac{18m\rho_{rrrru_{rr}}}{r^4}
\]

\[
    + 4mr^{-1}\rho_{r_{rrr}} + 6mr^{-1}\rho_{rr_{rrr}} + \frac{4m\rho_{rrrrr}}{r^2} - \frac{12m\rho_{rrrru_{rr}}}{r^2} - \frac{12m\rho_{rrrru_{rr}}}{r^2} + \frac{24m\rho_{rrrru_{rr}}}{r^3}
\]

\[
    + \frac{12m\rho_{rrr}u_{rrr}}{r^3} - \frac{6m\rho_{r_{rrr}}}{r^4} - \frac{24m\rho_{r_{rrr}}}{r^4} + \frac{24m\rho_{rrr}}{r^5} + \rho_{u_{rrr}} + 5\rho_{r_{rrr}u_{rrr}}
\]

\[
    + 10\rho_{rrr}u_{rrr} + 10\rho_{rrrr}u_{rrr} + 5\rho_{rrrr}u_{rrr} + 5\rho_{rrrr}u_{rrr} = 0
\]

(3.18)

Multiplying (3.18) by \( r^m \rho_{rrr} \), integrating over \( I \), and using integration by parts, (2.3), (2.4), (2.7), (2.8), Lemma 3.1 Lemma 3.3 we get

\[
    \frac{d}{dt} \int_{I} r^m \rho_{rrr}^2 \leq c \int_{I} r^m \rho_{rrr}^2 + c \int_{I} r^m u_{rrr}^2 + c.
\]

(3.19)

Similarly, we have

\[
    \frac{d}{dt} \int_{I} r^m |(\rho^\gamma)_{rrr}|^2 \leq c \int_{I} r^m |(\rho^\gamma)_{rrr}|^2 + c \int_{I} r^m u_{rrr}^2 + c.
\]

(3.20)

By (3.19) and (3.20), we obtain

\[
    \frac{d}{dt} \int_{I} r^m (\rho_{rrr}^2 + |(\rho^\gamma)_{rrr}|^2) \leq c \int_{I} r^m (\rho_{rrr}^2 + |(\rho^\gamma)_{rrr}|^2) + c \int_{I} r^m u_{rrr}^2 + c.
\]

(3.21)

Now we estimate \( \int_{I} r^m u_{rrr}^2 \). Differentiating (3.5) with respect to \( r \), we have

\[
    u_{rrr} = \rho_{rrr}u_r + 3\rho_{r_{rrr}}u_{rr} + 3\rho_{rr_{rrr}}u_r + (\rho_{rrr}u_{rrr} + \rho_{rrr}u_{rrr} + \rho_{rrr}u_{rrr} + \rho_{rrr}u_{rrr})_r + (\rho^\gamma)_{rrr}
\]

\[
    - mr^{-1}u_{rr} + 4mr^{-2}u_{rrr} - 12mr^{-3}u_{rr} + 24mr^{-4}u_{rr} - \frac{24m\rho}{r^5}
\]

\[
    - \rho_{rrr}f_{r} - 3\rho_{rrr}f_{rr} - 3\rho_{rrr}f_{rr} - \rho_{rrr}f_{rrr}.
\]

(3.22)
It follows from (3.22), (2.3), (2.4), (2.7), (2.8), Lemma 3.1 and Lemma 3.3 that
\[
\int_I r^m u^{2}_{rrrrr} \leq c \int_I r^m \rho^2 u_{rr}^2 + c \int_I r^m \rho^2 u_{rrt}^2 + c \int_I r^m |(\rho^\gamma)_{rrrrr}|^2 + c \int_I r^m \rho^2 (f_r^2 + f_t^2 + f_{rrrr}^2) + c. \tag{3.23}
\]

Since
\[
\int_I r^m \rho^2 u^2_{rrt} = 4 \int_I r^m \rho |(\sqrt{\rho})_r|^2 u_{rrt}^2.
\]
This together with (2.7), (2.8), Lemma 3.2 and Lemma 3.3 gives
\[
\int_I r^m u^2_{rrrrr} \leq c \int_I r^m \rho^2 u_{rrt}^2 + c \int_I r^m |(\rho^\gamma)_{rrrrr}|^2 + c \int_I r^m (f_r^2 + f_t^2 + f_{rrrr}^2) + c. \tag{3.24}
\]
Substituting (3.24) into (3.21), we obtain
\[
\frac{d}{dt} \int_I r^m (\rho^2_{rr} + |(\rho^\gamma)_{rrrrr}|^2)
\leq c \int_I r^m (\rho^2_{rr} + |(\rho^\gamma)_{rrrrr}|^2) + c \int_I r^m u^2_{rrrrr} + c \int_I r^m (f_r^2 + f_t^2 + f_{rrrr}^2) + c.
\]
Using Gronwall inequality and Lemma 3.3 we get
\[
\int_I r^m (\rho^2_{rr} + |(\rho^\gamma)_{rrrrr}|^2) \leq c. \tag{3.25}
\]
It follows from (3.24), (3.25) and Lemma 3.3 that
\[
\int_{Q_T} r^m u^2_{rrrrr} \leq c.
\]
This proves Lemma 3.4.

From (1.4), (2.3), (2.4), (2.6), (2.7), (2.8) and Lemmas 3.1, 3.4, we immediately get the following estimate.

**Lemma 3.5** For any $0 \leq t \leq T$, it holds
\[
\int_I r^m \left[ \rho^2_{rr} + |(\rho^\gamma)_{rr}|^2 + \rho^2_{rrt} + |(\rho^\gamma)_{rrrrr}|^2 \right] 
+ \int_{Q_T} r^m \left[ \rho^2_{tt} + |(\rho^\gamma)_{ttt}|^2 + \rho^2_{rttt} + |(\rho^\gamma)_{rrtt}|^2 \right] \leq c.
\]

**Lemma 3.6** For any $0 \leq t \leq T$, it holds
\[
\int_I (r^m \rho^4 u^2_{rrtt} + r^{m-2} \rho^4 u^2_{tt}) + \int_{Q_T} r^m \rho^5 u^2_{ttt} \leq c.
\]
Proof. Multiplying (3.11) by $r^m \rho^4 u_{tt}$, integrating the resulting equation over $I$, and using integration by parts, we have

$$
\int_I r^m \rho^5 u_{tt}^2 + \frac{1}{2} \frac{d}{dt} \int_I (r^m \rho^4 u_{tt}^2 + mr^m - 2 \rho^4 u_{tt}^2) = 2 \int_I r^m \rho^3 \rho u_{tt}^2 - 4 \int_I r^m \rho^3 \rho_r u_{tt} u_{rt} - \int_I r^m \rho^4 u_{tt} \left[ \rho u_{tt} + 2 \rho u_t + \rho_t uu_r + 2 \rho_t u_r + 2 \rho t u_r + \rho u_t u_r + 2 \rho t u_r + \rho u u_{rt} + (\rho')_{rt} \right] + 2 \int_I mr^m - 2 \rho^3 \rho u_{tt}^2
$$

$$
+ \int_I r^m \rho^4 u_{tt} \left( \rho t f + 2 \rho f_t + \rho f_{tt} \right) 
\leq c \int_I r^m \rho^2 u_{tt}^2 - 8 \int_I r^m \rho^2 u_{tt} \rho(\sqrt{\rho}) u_{tt} + \frac{1}{4} \int_I r^m \rho^5 u_{tt}^2 + c + c \int_I r^m (f^2 + f_t^2 + f_{tt}^2) \leq c \int_I r^m \rho^2 u_{tt}^2 + \frac{1}{2} \int_I r^m \rho^2 u_{tt}^2 + c \int_I r^m (f^2 + f_t^2 + f_{tt}^2) + c,
$$

where we have used (2.3), (2.4), (2.7), (2.8). Lemma 3.2, Lemma 3.3, Lemma 3.5 and Cauchy inequality.

Thus,

$$
\int_I r^m \rho^5 u_{tt}^2 + \frac{d}{dt} \int_I (r^m \rho^4 u_{tt}^2 + mr^m - 2 \rho^4 u_{tt}^2) \leq c \int_I r^m \rho^2 u_{tt}^2 + c \int_I r^m (f^2 + f_t^2 + f_{tt}^2) + c.
$$

By (1.3), (2.2) and $\rho_0 \nabla^3 g_1 \in L^2$, we have

$$
\int_{Q_T} r^m \rho^2 u_{tt}^2 + \int_{Q_T} (r^m \rho^4 u_{tt}^2 + mr^m - 2 \rho^4 u_{tt}^2) \leq c.
$$

The proof of Lemma 3.6 is completed. □

Lemma 3.7 For any $0 \leq t \leq T$, it holds

$$
\int_I r^m (\rho^2 u_{rrtt}^2 + |\partial_5^5 u|^2) + \int_{Q_T} r^m (\rho^3 u_{rrtt}^2 + \rho u_{rrr}^2) \leq c.
$$

Proof. From (3.17), (2.3), (2.4), (2.7), (2.8), Lemmas 3.1, 3.3 and Lemma 3.6 we get

$$
\int_I r^m \rho^2 u_{rrtt}^2 \leq c \int_I r^m (f_t^2 + f_r^2 + f_{rt}^2) + c \leq c.
$$

This combining (3.24) and Lemma 3.4 gives

$$
\int_I r^m |\partial_5^5 u|^2 \leq c.
$$

By (3.11), (2.3), (2.4), (2.7), (2.8), Lemma 3.1, Lemma 3.3 and Lemma 3.6 we get

$$
\int_{Q_T} r^m \rho^3 u_{rrtt}^2 \leq c \int_{Q_T} r^m (f_t^2 + f_{tt}^2) + c \leq c.
$$

(3.26)
Similarly, we get

\begin{align*}
&\text{By (3.26), (3.27), (2.3), (2.4), (2.7), (2.8), Lemmas 3.1-3.3 and Lemma 3.5 we obtain} \\
&\int_{Q_T} r^m \rho u^2_{rrrrt} \leq c \int_{Q_T} r^m \rho^2 |(\sqrt{\eta})_r|^2 u^2_{rt} + c \int_{Q_T} r^m \rho^3 u^2_{rrtt} \\
&\quad + c \int_{Q_T} r^m (f^2_{rr} + f^2_{rrt} + f^2_{rrrt}) + c \\
&\quad \leq c.
\end{align*}

The proof of Lemma 3.7 is completed. \qed

**Lemma 3.8** For any $0 \leq t \leq T$, it holds

\[ \int_I r^m (|\partial_t \rho|^2 + |\partial_t (\rho^2)|^2) + \int_{Q_T} r^m |\partial_t^6 u|^2 \leq c. \]

**Proof.** Differentiating (3.18) with respect to $r$, we obtain

\begin{align*}
&\partial_t^5 \rho_t + mr^{-1} \rho u_{rrrr} + mr^{-1} \rho u_{rrrr} - mr^{-2} \rho u_{rrrr} + \left(4mr^{-1} \rho u_{rrrr} + 6mr^{-1} \rho u_{rrr}\right) \\
&\quad - \frac{4mr \rho u_{rrr}}{r^2} + 12mr \rho u_{rr} - \frac{18mr \rho u_{rr}}{r^4} + 4mr^{-1} \rho u_{rrr} \right) + mr^{-1} \rho u_{rrrr} + mr^{-1} \rho u_{rrrr} \\
&\quad - \frac{5mr \rho u_{rrrr}}{r^2} - \frac{4mr \rho u_{rrr}}{r^2} + \frac{8mr \rho u_{rrr}}{r^4} + mr^{-1} \rho u_{rrrr} + mr^{-1} \rho u_{rrrr} \\
&\quad + \frac{12mr \rho u_{rrr}}{r^3} - \frac{6mr \rho u_{rr}}{r^4} + \frac{24mr \rho u_{rr}}{r^5} + \frac{24mr \rho u_{rr}}{r^5} \right) + 6mr \partial_t^5 u + 6 \partial_t^5 u + 15mr u_{rrrrrr} \\
&\quad + 20mr u_{rrrr} + 15mr u_{rrrrrr} + 6 \partial_t^5 u + u \partial_t^5 \rho = 0. \tag{3.28}
\end{align*}

Multiplying (3.28) by $r^m \partial_t^5 \rho$, and using (2.3), (2.4), (2.7), (2.8), Lemma 3.1 Lemma 3.3 Lemma 3.4 and Lemma 3.7 we have

\[ \frac{d}{dt} \int_I r^m |\partial_t^5 \rho|^2 \leq c \int_I r^m |\partial_t^5 \rho|^2 + c \int_I r^m |\partial_t^6 u|^2 + \int_I r^m (|\partial_t^5 \rho|^2) u + c \]

\[ \leq c \int_I r^m |\partial_t^5 \rho|^2 + c \int_I r^m |\partial_t^6 u|^2 + c. \tag{3.29} \]

Similarly, we get

\[ \frac{d}{dt} \int_I r^m |\partial_t^5 (\rho^2)|^2 \leq c \int_I r^m |\partial_t^5 (\rho^2)|^2 + c \int_I r^m |\partial_t^6 u|^2 + c. \tag{3.30} \]

Differentiating (3.22) with respect to $r$, we get

\begin{align*}
&\partial_t^5 u = \rho_{rrrrt} u_t + 4 \rho_{rrrr} u_t + 6 \rho_{rr} u_{rr} + 4 \rho_{rr} u_{rr} + \rho_{rrrr} + (\rho_{rr} u_{rr} + \rho_{rr} u_{rr}) \right) + \rho_{rrrr} \\
&\quad + \rho_{rrrr} + \rho_{rrrr} + \left(\rho_{rrrr} + 4 \rho_{rrrr} + \rho_{rrrr} + 4 \rho_{rrrr} + \rho_{rrrr} + \rho_{rrrr} + \frac{24mr \rho u}{r^5} \right) \\
&\quad - \rho_{rrrr} f - 4 \rho_{rrrr} f + 6 \rho_{rr} f_{rrr} - 4 \rho_{rr} f_{rrr} - \rho_{rrrr}. \tag{3.31}
\end{align*}
This proves Theorem 3.1.

By (3.34) and (3.35), we get
\[
\int_I r^m u_r^2 + c \int_I r^m \rho u_r^2 + c \int_I r^m u_{rr}^2 + c \int_I r^m |\partial_t \rho|^2 + c \int_I r^m \left( f_r^2 + f_{rr}^2 + f_{rrr}^2 \right) + c.
\]
(3.32)

It follows from (3.29), (3.30), (3.32), (2.8), Lemma 3.3 and Lemma 3.7 and Gronwall inequality that
\[
\int_I r^m |\partial_t \rho|^2 + |\partial_t \rho|^2 \leq c.
\]
(3.33)

From (3.32), (3.33), (2.8), Lemma 3.3 and Lemma 3.7, we obtain
\[
\int_{Q_T} r^m |\partial_t^2 u|^2 \leq c.
\]
The proof of Lemma 3.8 is completed. □

It follows from (1.411), (2.6), (2.7), (2.8) and Lemmas 3.1-3.8 that
\[
\| (\rho, \rho^\gamma) \|_{H^2_T} + \| (\rho_t, \rho^\gamma) \|_{H^2_T} + \| ((\sqrt{\rho})_r, (\sqrt{\rho})_t) \|_{L^\infty(Q_T)} \leq c, \quad \rho \geq \frac{\delta}{c},
\]
(3.34)

and
\[
\int_I r^m (\rho^3 u_{tt}^2 + \rho^4 u_{tt}^2 + r^{-2} \rho^4 u_{tt}^2 + \rho^2 u_{rr}^2 + u_r^2 + u_t^2 + r^{-2} u_t^2 + |\partial_t \rho|^2 + |\partial_t^2 u|^2
\]
\[
+ u_{rr}^2 + u_r^2 + u_t^2 + r^{-2} u_t^2) + \int_{Q_T} r^m (\rho u_{tt}^2 + \rho^2 u_{tt}^2 + \rho^5 u_{tt}^2 + \rho^3 u_{r}^2 + \rho u_{r}^2 + u_{r}^2 + \rho u_{rr}^2 + \rho u_{rr}^2 + u_{r}^2 + \rho u_{rr}^2 + |\partial_t^2 u|^2 \leq c.
\]
(3.35)

By (3.34) and (3.35), we get
\[
\| \rho \|_{H^2(I)} + \| \rho_t \|_{H^1(I)} + \| u \|_{H^2(I)} + \| u_t \|_{H^1(I)} + \| u_{tt} \|_{H^1(I)} + \int_0^T (\| u \|_{H^2(I)}^2 + \| u_t \|_{H^2(I)}^2
\]
\[
+ \| u_{tt} \|_{H^2(I)}^2 + \| u_{tt} \|_{L^2(I)}^2) \leq c(b, \delta).
\]

This proves Theorem 3.1 □

Proof of Theorem 1.2

Since (3.34) and (3.35) are uniform for \( b \) and \( \delta \), it suffices to prove Theorem 1.2 for the case \( b < \infty \). We follow the strategy as the proof of Theorem 1.1 and use Theorem 3.1. After taking \( \delta \to 0 \) (take subsequence if necessary), we get a solution \((\rho, u)\) of (1.4)-(1.6) satisfying
\[
\left\{
\begin{array}{l}
(\rho, \rho^\gamma) \in L^\infty([0, T]; H^5(I)), \quad ((\sqrt{\rho})_r, (\sqrt{\rho})_t) \in L^\infty(Q_T), \\
(\rho_t, \rho^\gamma) \in L^\infty([0, T]; H^4(I)), \quad u \in L^\infty([0, T]; H^5(I)) \cap L^2([0, T]; H^6(I)), \\
u_t \in L^\infty([0, T]; H^3(I)) \cap L^2([0, T]; H^4(I)), \\
(\sqrt{\rho} \rho^2 u_t, \rho \partial_t^2 u_t) \in L^\infty([0, T]; L^2(I)) \cap L^2([0, T]; H^1(I)), \\
\sqrt{\rho} \rho^2 u_t \in L^2(Q_T), \quad \rho^3 u_{ttt} \in L^2(Q_T), \quad \rho^3 u_{ttt} \in L^2(Q_T), \\
\rho^2 u_{tt} \in L^\infty([0, T]; H^1(I)) \cap L^2([0, T]; H^2(I)), \\
\rho^2 u_{tt} \in L^\infty([0, T]; H^1(I)) \cap L^2([0, T]; H^2(I)).
\end{array}
\right.
\]
(3.36)
It follows from $u \in L^\infty([0,T];H^5(I)) \cap L^2([0,T];H^6(I)), u_t \in L^2([0,T];H^3(I)), \quad (1.4)$, and the similar arguments as [3, 4] that
\[ \rho, \rho^\gamma \in C([0,T];H^5(I)). \]

Denote $\rho(x, t) = \rho(r, t)$, $u(x, t) = u(r, t) \frac{x}{r}$, then $(\rho, u)$ is the unique solution to (1.1)-(1.3) with the regularities in Theorem 1.2. The proof of Theorem 1.2 is completed. \hfill \Box

**Remark 3.1** By our method, it seems that the regularities of $u$ could not be improved to $L^\infty([0,T];D^6(\Omega))$ and $L^2([0,T];D^8(\Omega))$, even if the systems (1.1)-(1.3) have initial data, $f$ and $g_1$ smooth enough. More precisely, based on (3.34) and (3.35), the next two a priori estimates about $u$ for (1.4)-(1.6) are
\[ \int_I r^m \rho^\gamma u^2_{ttt} + \int_{Q_T} r^m \rho^6 (u_{r,ttt}^2 + r^{-2} u_{ttt}^2) \leq c, \quad (3.37) \]
and
\[ \int_I r^m \rho^8 (u_{r,ttt}^2 + r^{-2} u_{ttt}^2) + \int_{Q_T} r^m \rho^9 u^2_{tttt} \leq c. \quad (3.38) \]

(3.37) and (3.38) respectively implies
\[ \int_I r^m \rho |\partial_r^5 u|^2 + \int_{Q_T} r^m |\partial_r^7 u|^2 \leq c, \quad (3.39) \]
and
\[ \int_I r^m \rho^2 |\partial_r^5 u|^2 + \int_{Q_T} r^m \rho |\partial_r^7 u|^2 \leq c. \quad (3.40) \]

Just because of the appearance of the vacuum, we can’t obtain the regularities $u$ in $L^\infty(0,T;D^6(\Omega))$ and $L^2(0,T;D^8(\Omega))$.

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