Whitham Prepotential and Superpotential

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Abstract

$\mathcal{N} = 2$ supersymmetric $U(N)$ Yang-Mills theory softly broken to $\mathcal{N} = 1$ by the superpotential of the adjoint scalar fields is discussed from the viewpoint of the Whitham deformation theory for prepotential. With proper identification of the superpotential we derive the matrix model curve from the condition that the mixed second derivatives of the Whitham prepotential have a nontrivial kernel.

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I. Introduction

Gauge theory with $\mathcal{N} = 2$ supersymmetry has a successful exact description of low energy effective action in terms of curve, the meromorphic differential, the period integrals and above all the prepotential [1, 2]. This structure is shared by the corresponding classical integrable system of particles, which may be regarded as representing universality class of the field theories of this type [3]. In particular, the prepotential, which extracts all available information on low energy phenomena from the curve and the periods, can be extended to include infinite number of couplings as time variables [4, 5, 6] and becomes a generating functional (or $\tau$ function) of some kind from the point of view of the integrable hierarchy. It is therefore natural to investigate what role is played by the prepotential when the original theory is finitely as opposed to infinitesimally driven by a specific set of operators. The framework which fits this line of thoughts is deformation theory of prepotential called Whitham deformation. See [7] for reviews.

In more physical terms, this amounts to asking whether the notion of the prepotential be effective when the $\mathcal{N} = 2$ theory becomes softly broken to $\mathcal{N} = 1$ [8] or $\mathcal{N} = 0$. In fact there have been several such developments till now, a few of which are listed below. In [6], the Whitham deformation based on the formalism of [4] and the computation of [5] has been applied to $\mathcal{N} = 0$ through the spurion formalism of [9]. More recently, the gluino condensate prepotential has appeared through the determination of the effective superpotential for $\mathcal{N} = 1$ super Yang-Mills with the adjoint chiral superfield [10]. This prepotential has been identified as the matrix model free energy [11]. In the subsequent development of [12], the prepotential appeared via a fermionic shift symmetry \(^1\) associated with the system of three anomalous chiral Ward-Takahashi identities [15]. See a recent review [16] for more on this last development, and [17] for recent discussions on the relationship of the superpotential with the Lax matrix.

In this paper, we will add one more step to this list of developments in prepotential theory: we will find a more direct route and rationale leading to the appearance of the prepotential for $\mathcal{N} = 1$ theories. We begin with a few preliminaries. In the next section, we first give the basic ingredients of the Seiberg-Witten curve for the $\mathcal{N} = 2$ $SU(N)$ pure-super Yang-Mills theory (or the spectral curve of the $N$ site periodic Toda chain.) Next, we briefly review Whitham deformation theory for the prepotential. It is a generic framework and we need to find a proper condition that deforms the theory and that breaks $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. At the same time, the superpotential responsible for this deformation must be identified. In section three, we carry out this task. We examine the mixed second derivatives.

\(^1\)The developments concerning with this particular point include the work of [13, 14].
A main thrust of this paper is to observe that, in prepotential theory, the condition for a curve to degenerate or factorize is given by that the kernel of the matrix made of the mixed second derivatives of the Whitham prepotential be nontrivial:

$$\text{ker} \frac{\partial^2 \mathcal{F}}{\partial a^i T_\ell} \neq 0 \ .$$

(1.1)

We derive the matrix model curve, identifying the tree-level super potential with the vector belonging to this nontrivial kernel of the matrix above. In the classical limit where the scale parameter vanishes, expressions get simplified largely. We discuss this case separately in section four and show that the identification made in section three is indeed correct. We make several comments on the gluino condensate prepotential in the final section.

II. Some Preliminaries

A. The Seiberg-Witten curve

Let us first collect some basic ingredients of the Seiberg-Witten curve for the $\mathcal{N} = 2$ SU($N$) pure-super Yang-Mills theory (or the spectral curve of the $N$ site periodic Toda chain) and the meromorphic differential $dS_{SW}$ defined on this Riemann surface of genus $g = N - 1$. The curve can be represented by a hyperelliptic form:

$$Y^2 = P_N(x)^2 - 4 \Lambda^{2N} \ ,$$

$$P_N(x) \equiv \langle \det (x1 - \Phi) \rangle = \prod_{i=1}^{N}(x - p_i) = x^N - \sum_{k=2}^{N} u_k x^{N-k}$$

$$= \sum_{k=0}^{N} s_k(h_\ell)x^{N-k} \ ,$$

(2.2)

where $s_k(h_\ell)$ are the fundamental Schur polynomials of

$$h_\ell = \frac{1}{\ell} \langle tr_N \Phi^\ell \rangle = \frac{1}{\ell} \sum_{i=1}^{N} p_i^\ell , \ \ell = 2, \ldots , N \ .$$

(2.3)

The overall $U(1)$ can be decoupled from our consideration at this moment. In the Toda chain representation, this curve is parameterized by the spectral parameter $z$ by

$$P_N(x) = z + \frac{\Lambda^{2N}}{z} \ ,$$

$$Y = z - \frac{\Lambda^{2N}}{z} \ .$$

(2.4)
The general variations of eq. (2.4) give us the following equations:

\[ P'_N \delta x + \delta_u P_N = Y \delta \log z - N(Y - P_N) \delta \log \Lambda, \]
\[ \delta Y = P_N \delta \log z + N(Y - P_N) \delta \log \Lambda. \quad (2.5) \]

Here \( \delta_u \) denotes a generic variation with respect to \( u_k \). The distinguished meromorphic differential for the prepotential theory of the \( \mathcal{N} = 2 \) \( SU(N) \) pure-super Yang-Mills is

\[ dS_{SW} = xd \log z = xt(x)dx. \quad (2.6) \]
\[ t(x) = \frac{P'_N}{\sqrt{P^2_N - 4\Lambda^2}}. \quad (2.7) \]

This differential possesses a double pole at \( x = \infty_{\pm} \). Moduli derivatives with either \( z \) or \( x \) and \( \Lambda \) fixed generate the bases of the holomorphic differentials:

\[ \frac{\partial}{\partial u_k} dS_{SW} \mid_{z, \Lambda} = \frac{x^{N-k}}{Y} dx \equiv dv_k, \quad \frac{\partial}{\partial u_k} dS_{SW} \mid_{x, \Lambda} = dv_k - d \left( \frac{x^{N-k+1}}{Y} \right), \quad k = 2, \ldots N. \quad (2.8) \]

The differentials \( dv_k \) are related by \( u_k \) dependent numerical factors to the canonical holomorphic differentials \( d\omega_i \) which are normalized as \( f_{ij} d\omega_j = \delta^i_j \). Moduli derivatives are in general coordinate dependent and in this paper, we take them, keeping \( z \) fixed.

\section*{B. Whitham Prepotential}

It is possible to deform consistently both moduli of the Riemann surface and the meromorphic differential discussed in the last subsection without losing their defining properties. Namely,

\[ dS_{SW} \rightarrow dS, \quad \frac{\partial}{\partial u_k} dS \mid_{z, \Lambda} = \text{holomorphic}. \quad (2.9) \]

This is called Whitham deformation (of Toda integrable hierarchy). It has an effect of adding higher order poles to the original double poles of the meromorphic differential \( dS_{SW} \). The deformation is characterized by a set of the punctures and local coordinates in their neighborhood, which we denote generically by \( \xi \). In order to accomplish this deformation, we introduce a set of meromorphic differentials \( d\Omega_\ell \) which satisfy

\[ d\Omega_\ell = \xi^{-\ell-1} d\xi + \text{nonsingular part}, \quad \ell = 1, 2 \cdots. \quad (2.10) \]

This still leaves us with the possibility to add any linear combination of the holomorphic differentials \( d\omega_i \) to the right hand side. To remove this ambiguity, we require the condition

\[ \int_{A_i} d\Omega_\ell = 0. \quad (2.11) \]
Construction of the Whitham prepotential begins with introducing time variables via
\[
\frac{\partial dS}{\partial T_\ell} = d\Omega_\ell .
\] (2.12)

Let
\[
a^i \equiv \oint_A^i dS ,
\] (2.13)
be the local coordinates of the moduli space. After some reasonings [4], one concludes
\[
dS = \sum_{i=1}^g a^i d\omega_i + \sum_\ell T_\ell d\Omega_\ell ,
\] (2.14)
and
\[
T_\ell = \text{res}_{\xi=0} \xi^\ell dS .
\] (2.15)
Both \(a^i\) and \(T_\ell\) are regarded as independent variables. In \(dS_{SW}\), the first time alone is turned on. Invariant moduli \(h_k\) are expressible as
\[
h_k = h_k(a^i, T_\ell) .
\] (2.16)

The prepotential \(F(a^i, T_\ell)\) is introduced via
\[
\frac{\partial F}{\partial a^i} = \oint_{B^i} dS , \quad 1 \leq i \leq N-1
\] (2.17)
\[
\frac{\partial F}{\partial T_\ell} = \frac{1}{2\pi i \ell} \text{res}_{\xi=0} \xi^{-\ell} dS \equiv H_\ell(h_k) . \quad \ell = 1, 2, \ldots
\] (2.18)

The consistency of eq. (2.17) and that of eq. (2.18) are ensured respectively by the property that the period matrix is symmetric as well as by the Riemann identity. The right hand side of eq. (2.18), which we have denoted by \(H_\ell(h_k)\), is some polynomial [5] of the invariant moduli. Below, the local coordinates in the neighborhood of the punctures at \(x = \infty \pm\) are taken as
\[
\xi = z^{\pm\frac{i}{\pi}} .
\] (2.19)

III. Mixed Second Prepotential Derivatives: Derivation of the Matrix Model Curve

Now we come to the point of our paper. Differentiating eqs. (2.17) and (2.18) once again, we obtain two distinct formulas for the mixed second derivatives:
\[
\frac{\partial^2 F}{\partial a^i \partial T_\ell} = \oint_{B^i} d\Omega_\ell = \frac{1}{2\pi i \ell} \text{res}_{\xi=0} \xi^{-\ell} d\omega_j ,
\] (3.1)
where \( j = 1 \sim N - 1 \). As for \( \ell \), it is any positive integer. Note that eq. (3.1) has been given with no reference to \( dS \). The discussion in what follows holds, therefore, for the Whitham prepotential deformed by the arbitrary values of the time variables. As is stated in the introduction, we impose the condition that the kernel of this rectangular matrix be nontrivial:

\[
\ker \frac{\partial^2 F}{\partial a^i T_\ell} \neq 0 .
\]  

(3.2)

Equivalently

\[
\text{rank} \frac{\partial^2 F}{\partial a^i T_\ell} \leq N - 2 .
\]  

(3.3)

In the case in which the time variables are truncated to \( T_1, T_2 \cdots T_{N-1} \), the condition is of course stated as that of the vanishing determinant:

\[
\det \frac{\partial^2 F}{\partial a^i T_\ell} = 0 .
\]  

(3.4)

Eq. (3.2) tells us the existence of at least one nonvanishing column vector

\[
\begin{pmatrix}
  c^1 \\
  \vdots \\
  c^{N-1}
\end{pmatrix}
\]  

(3.5)

belonging to the kernel;

\[
0 = \sum_\ell \frac{\partial^2 F}{\partial a^i T_\ell} c_\ell = \sum_\ell \oint_{B^i} c_\ell d\Omega_\ell
\]  

(3.6)

\[
= \frac{1}{2\pi i} \text{res}_{\xi=0} \left( \sum_\ell \frac{c_\ell}{\ell} \xi^{-\ell} d\omega_i \right) .
\]  

(3.7)

Eq. (3.6) implies the existence of the meromorphic one-form

\[
d\tilde{\Omega} \equiv \sum_\ell c_\ell d\Omega_\ell .
\]  

(3.8)

whose period integral over any of the \( A_i \) and \( B^i \) cycles vanishes. Let us argue implications of this last statement. Obviously, once this property holds, we can integrate \( d\tilde{\Omega} \) along any path ending with a point \( z \) to define a function holomorphic except at punctures: \( f(z) = \int z d\tilde{\Omega} \). As for the order of the poles at the punctures, it is generically arbitrary according to the construction. This is, however, contradictory to the Weierstrass gap theorem [18], which says that, for \( g = N - 1 \) integers satisfying \( 1 = n_1 < n_2 < \cdots < n_g < 2g \), such function with
a pole of order \(n_j\) does not exist. This theorem is derived from the Riemann-Roch theorem. In order to avoid the contradiction, the degeneration of the surface must take place.

Eq. (3.2) also implies the existence of at least one non-vanishing co-vector (row vector)

\[
(\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_{N-1})
\]

such that

\[
0 = \sum_{i=1}^{N-1} \tilde{c}_i \frac{\partial^2 F}{\partial a^i T_\ell} = \sum_{i=1}^{N-1} \tilde{c}_i \frac{\partial H_{\ell+1}}{\partial a^i}.
\]

We see from eq. (2.18) that this is satisfied provided

\[
\sum_{i=1}^{N-1} \tilde{c}_i \frac{\partial h_k}{\partial a^i} = 0, \quad k = 2 \sim N.
\]

Eq. (3.11) tells us the degeneration of the curve from the point of view of the moduli space: the invariant moduli \(h_k, k = 2 \sim N\) become functions which actually depend on less than \(N - 1\) arguments. This equation is regarded as the counterpart of the statement of the vanishing discriminant [19].

We conclude that the original curve must get degenerated under the condition eq. (3.2) and the factorization of eq. (2.1) by the following kind [10] takes place:

\[
Y^2(x) = H_{N-n}^2(x) F_{2n}(x).
\]

\[
P'_N(x) = H_{N-n}(x) R_{n-1}(x)
\]

\[
t(x) = \frac{R_{n-1}(x)}{\sqrt{F_{2n}(x)}}.
\]

The polynomial \(R_{n-1}\) does not depend on \(N\) [20]. As is well-known, \(N - n\) non-intersecting cycles vanish once this condition is satisfied and the moduli space of the resulting vacua is codimension \(N - n\) subspace of the original \(\mathcal{N} = 2\) Coulomb branch. To smoothen out the singularities developed, let us imagine blowing up the surface and multiplying \(dv_k\) in eq. (2.8) by a factor which cancels the poles developed and which behaves as \(x\) at the infinities. Upon partial fractions, \(n\) independent differentials emerge from \(dv_k, k = 2, \cdots N\):

\[
\frac{x^{j-1}dx}{\sqrt{F_{2n}}} , \quad j = 1 \sim n.
\]

The differentials for \(j = 1 \sim n - 1\) are the holomorphic differentials on this reduced surface and the last one \(j = n\) has a pole at the infinities. This last one has been included due to the blow-up process and physically implies that the overall \(U(1)\) fails to decouple through this process.
Finally let us examine eq. (3.7). Let
\[
\sum_{\ell} \left( \frac{c_\ell}{\ell} \xi^{-\ell} \right)_+ \equiv W'_{K+1}(x) ,
\]  
where \((\cdots)_+\) denotes the part consisting of the non-negative powers of \(\cdots\) in the Laurent expansion in \(x\). We consider the case in which \(W'_{K+1}(x)\) is a polynomial of degree \(K (K \geq n)\) in \(x\) parameterized as
\[
W'_{K+1}(x) \equiv \prod_{j=1}^{K} (x - \alpha_j) .
\]  
We substitute eq. (3.15) for \(d\omega_i\) in eq. (3.7) to obtain
\[
0 = \text{res}_{x=\infty} \left( W'_{K+1}(x) \frac{x^{j-1}}{\sqrt{F_{2n}}} \right) , \quad j = 1 \sim n .
\]  
Hence
\[
\frac{W'_{K+1}(x)}{\sqrt{F_{2n}(x)}} = Q_{K-n}(x) + \sum_{\ell > n} \frac{\beta_{\ell}}{x^{\ell}} ,
\]  
where \(Q_{K-n}(x)\) is a polynomial of degree \(K - n\). We conclude
\[
F_{2n}(x) Q^2_{K-n}(x) = W'^2_{K+1}(x) + f_{K-1}(x) ,
\]  
where \(f_{K-1}(x)\) is a polynomial of degree \(K - 1\). It still remains to be seen that the function \(W_{K+1}(x)\) introduced above is in fact a tree-level superpotential. This is done in the next section. Let us also note that our discussion suggests a family of superpotentials continuously connected when the kernel is more than one-dimensional.

IV. Classical Limit

In the classical limit, \(\Lambda = 0\) and the curve gets simplified largely:
\[
Y = z = \prod_{\ell=1}^{N} (x - p_\ell) .
\]  
The original Seiberg-Witten differential becomes
\[
dS^{(\text{class})}_{SW} = xd\log z = \sum_{i=1}^{N} \frac{x}{x - p_i} dx .
\]  
The period integrals over the \(A_i\) cycles just pick up the poles at \(p_i\):
\[
a^{(\text{class})}_i = p_i .
\]
The canonical holomorphic differentials take the following form:

\[ d\omega_i^{(\text{class})} = \frac{\partial}{\partial p_i} dS_{SW}^{(\text{class})} \big|_{z = \frac{dx}{x - p_i}} . \]  

Eq. (2.14) is simply

\[ dS_{SW}^{(\text{class})} = \sum_{i=1}^{N} p_i d\omega_i + N dx . \]  

Let us suppose that

\[ z = \prod_{j=1}^{n} (x - \beta_j)^{N_j} , \quad \sum_{j=1}^{n} N_j = N , \]  

which means that \( N_j \) poles of eq. (4.2) (or \( N_j \) eigenvalues of the vev of the adjoint Higgs) coalesce to one point \( \beta_j \) for \( j = 1 \sim n \) and \((N - n)\) non-intersecting \( A_i \) cycles vanish. The canonical holomorphic differentials on this degenerate curve are

\[ d\omega_j^{(\text{class},\text{red})} = \frac{dx}{x - \beta_j} , \quad j = 1 \sim n , \]  

so that

\[ a_j^{(\text{red})} = \beta_j . \]  

Let us now examine the condition eq. (3.18). In the classical limit, eq. (3.18) becomes

\[ 0 = \text{res}_{x = \infty} \left( W'_{K+1}(x) d\omega_j^{(\text{class},\text{red})} \right) , \quad j = 1 \sim n . \]  

The residue is originally evaluated by the contour around infinity but by a contour deformation it becomes a residue at \( x = \beta_j \). We see that \( \beta_j \) must coincide with one of the roots \( \alpha_j \) of \( W'_{K+1} \). The vacuum values of the adjoint scalars are constrained to the extremum of \( W_{K+1} \). We conclude that the function \( W_{K+1} \) is the tree-level superpotential as is promised.

V. Discussion

Once the condition (3.2) is imposed upon, we have a reduced curve of genus \( g = n - 1 \). Let us set \( K = n \) for simplicity. In the context of the present paper, the coefficients (denoted by \( b_\ell \)) of the polynomial \( f_{K-1} \) are determined by the parameters \( \alpha_j \) of the superpotential. In [10, 21, 22], these \( b_\ell \) as opposed to \( u_k \) are treated as moduli. The gluino condensate prepotential has been given by the matrix model differential whose \( b_\ell \) moduli derivatives are almost holomorphic and these moduli derivatives are taken, keeping the coordinate \( x \) fixed as opposed to \( z \) fixed. In this setting the Whitham times have been identified with the symmetric polynomials made of \( \alpha_j \). Note that the \( S_i \) moduli (the \( A_i \) cycle integral of the matrix model differential) as opposed to \( a_i \) moduli have a quantum mechanical origin: they vanish as \( \Lambda \to 0 \). Also \( S \equiv \sum_{i=1}^{n} S_i \) is non-vanishing and the cutoff must be introduced at the infinities of the curve.
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