I. INTRODUCTION

The standard formulation of gravity in four space-time dimensions, General Relativity (GR), is well understood as a theory where the dynamical degrees of freedom are directly given by the spacetime metric. However, it is also possible to work on alternative formulations where the degrees of freedom are not explicitly the metric components, but other fields in terms of which the metric can be recovered. A remarkable and well known situation where this approach has been fruitful is the development of Loop Quantum Gravity (LQG), which was impulsed by the introduction of a real triad and a SL(2, C) connection 1-form that replaced the metric as the dynamical field of the theory. It was later realized that these variables, now known as Ashtekar variables, arise in the ADM decomposition of the self-dual Palatini Lagrangian (see, e.g., [24]). In addition, it was noted in [15] that the connection used by Ashtekar can be generalized in terms of a one parameter family of canonical transformations applied to the variables of a SO(3)-ADM phase space (see also [18, 19]). These new connections, known as Ashtekar-Barbero connections, avoid the necessity of imposing reality conditions a-posteriori at the expense of working with a more complicated Hamiltonian constraint. The parameter controlling the canonical transformations to get these variables is dubbed the \textit{(Barbero)-Immirzi} parameter. Although this parameter does not play a role in physical predictions at classical energy scales, it appears at the foundations of LQG. The Ashtekar-Barbero connection, together with the triad, serve as canonical variables for the study of the Holst action [14], which is a generalization of the Palatini action to include the Holst topological term weighted by the Immirzi parameter. When the Immirzi parameter is equal to the imaginary unit, i, the self-dual Palatini action is recovered.

Another example comes from Extended Theories of Gravity [7], these formulations try to address the observations that led to the dark energy/dark matter paradigm from a different perspective, often via the addition of higher order curvature invariants to the gravitational action. With the exception of Lanczos-Lovelock theories [18, 19], these models generically have equations of motion for the metric with derivatives of order higher than two. However, in a formulation à la Palatini where the metric and the curvature are considered as independent variables, the equations of motion are of second order. The metric and Palatini formulations coincide for the Einstein-Hilbert Lagrangian, but for more general Lagrangians the solutions to the Palatini formulation are a subset of the solutions to the metric equations [7].

The alternative theory of gravity that is at the basis of this work is known as MacDowell-Mansouri (MM) gravity [10] (see [3] for a review). This theory is an attempt to recast gravity as an ordinary Yang-Mills gauge theory, constructed purely from the field strength of the gauge potential on either the de Sitter (dS) group SO(4,1) for positive cosmological constant, or anti-de Sitter (AdS) SO(3,2) for negative cosmological constant. This gauge potential acts as an internal (A)dS connection whose principal feature is that it unifies the tetrad and the spin connection used in the Palatini formulation into a single object. This is done by associating the translational part of the gauge connection to the tetrad and the Lorentz part to the spin connection. For a geometrical interpretation of MM gravity see [14]. By explicitly breaking the original SO(4,1) or SO(3,2) gauge symmetry to its Lorentz subgroup, SO(3,1), one obtains the action for MM gravity, which turns out to be equivalent to Einstein-Hilbert gravity with cosmological constant, supplemented with the Euler topological term. The MM action is thus an elegant mathematical construct with deep connections to the infra-red and ultra-violet physics of the spacetime: it naturally includes a cosmological constant and signals the way to the inclusion of topological terms that modify the quantum predictions of the theory.
with respect to those of pure GR \[16\].

In this work we investigate the relation between an extension of MM gravity and the Immirzi parameter. Some years ago \[12\], it was suggested that the Immirzi ambiguity has similarities with the $\theta$-ambiguity of Yang-Mills theories. Pursuing this idea in the context of MM gravity and supergravity allows for an interpretation of the Immirzi parameter along the same lines as a $\theta$-parameter \[21, 22\]. Here we complement this picture by obtaining the contributions from the $\theta$-term derived in \[22\] directly from the explicit breaking of gauge symmetry.

An extension of the MM theory was proposed in \[22\] by constructing the action with the self-dual part of the curvature. As a result of this extension, the $SO(3,1)$ structure of the new action is equal to the MM action plus the Pontryagin topological term and the Holst term, which is part of the Nieh-Yan topological term and does not affect the classical equations of motion in the absence of torsion.

We introduce a further generalization in the same lines as \[22\], we formulate a linear combination of two MM-like actions, one for the self-dual part of the curvature and the other for the anti-self-dual part. Our motivation to explore this generalization is not to extend the theory but to gain freedom in the resulting coefficient for the Holst term, which is then identified as the Barbero-Immirzi parameter. Finally, using the self-dual formulation for CS gravity, we conjecture that the Immirzi ambiguity is related to the SL(2,$\mathbb{Z}$) invariance of the CS partition function.

The structure of the paper is as follows: in section II we present the Lagrangians that are admissible in a first order formulation of gravity, and we review how a subset of those terms is recovered from the original MM action. In section III we review the self-dual formulation and we introduce the contributions from the anti-self-dual curvature. A further step is presented in section IV where we obtain the (2+1)-dimensional version of our action and explore its relation to the two classically equivalent actions for gravity in this number of dimensions. Section V is devoted to conclusions.

II. FOUR DIMENSIONAL GRAVITY AND TOPOLOGICAL TERMS

In this section we expand on the details of some topics mentioned in the introduction, this is done in order to set up our framework and to fix notation. Given a theory with a certain set of symmetries and variables, one can write down an action by adding all the terms that can be constructed from those variables in compatibility with the given symmetries. For four dimensional gravity in a first order formalism we are then talking about the tetrad $e^\mu_\nu$, the $SO(3,1)$ connection 1-form $\omega^{ab}_\mu$ and the groups of diffeomorphisms and local Lorentz invariance, where Greek and Latin letters stand for space-time and internal $SO(3,1)$ indices respectively, both running from zero to three. Under these restrictions the most general first order action for gravity contains the Einstein-Cartan, Holst and cosmological constant terms, and it can be supplemented with the only topological invariants that can be constructed in four spacetime dimensions, namely the Euler, Pontryagin and Nieh-Yan terms (see, e.g. \[28\]).

All of those terms are respectively defined below (mod. global constant coefficients):

\[
\mathcal{L}_{EC} = -\epsilon^{\mu \nu \rho \sigma} e_{ab} e^c_{\mu} e^d_{\nu} R^{cd}_{\rho \sigma},
\]

\[
\mathcal{L}_H = -\epsilon^{\mu \nu \rho \sigma} e^a_{\mu} e^b_{\nu} R^a_{\rho \sigma},
\]

\[
\mathcal{L}_{CC} = \epsilon^{\mu \nu \rho \sigma} e_{a b} e^c_{\mu} e^d_{\nu} T^a_{\rho \sigma},
\]

\[
\mathcal{L}_E = \epsilon^{\mu \nu \rho \sigma} e_{a b} e^c_{\mu} e^d_{\nu} T^{c d}_{\rho \sigma},
\]

\[
\mathcal{L}_{P} = \epsilon^{\mu \nu \rho \sigma} R_{a b \rho \sigma} R^{a b} - \epsilon^{\mu \nu \rho \sigma} e_{a b} e^c_{\mu} e^d_{\nu} R^c_{\rho \sigma} R^d_{\rho \sigma},
\]

where $\epsilon$ is the Levi-Civita tensor density and the curvature is

\[
R_{a b} = \partial_{[a} \omega_{b]} - \partial_{b} \omega_{[a} + \omega_{[c} \omega_{b]} - \omega_{[b} \omega_{c]} - \omega_{b}^{a c} e^a_{c}.
\]

The Euler term is equivalent to the Gaussian-Bonnet term $R^2 - 4R_{a b} R^{a b} + R_{\rho \sigma \mu \nu} R^{\rho \sigma \mu \nu}$, this can be seen explicitly by expanding the product of Levi-Civita densities in terms of Kronecker deltas. The Pontryagin term is related to the more familiar Chern-Simons term which only exists in odd number of dimensions. The Nieh-Yan term reduces to the Holst term when the torsion vanishes. An action containing the Lagrangian densities \[1a\], \[1f\] and \[1d\] can be obtained from the Macdowell-Mansouri proposal,

\[
S_{MM} = \int d^4 x \epsilon^{\mu \nu \alpha \beta} \epsilon_{a b c d} R_{a b}^{\mu \nu} R^{c d}_{\alpha \beta}.
\]

The curvature $R_{a b}^{\mu \nu}$ is the four dimensional part of the curvature associated to the five dimensional internal group and it contains the usual curvature, eq. \[2\], plus some other terms related to the tetrad, i.e., it unifies the variables of a formulation a la Palatini in a single object. Let us write this in detail.

We take a connection $\omega^{AB}_\mu$, where as before Greek indices run from zero to three and they label a base four-dimensional Lorentzian spacetime $\mathcal{M}$, while capital Latin letters run from zero to four and they are associated to a $SO(4,1)$ or $SO(3,2)$ fiber bundle attached to the base manifold. The curvature of this internal (A)dS connection is

\[
R^{AB}_{\mu \nu} = \partial_{\mu} \omega^{AB}_\nu - \partial_{\nu} \omega^{AB}_\mu + \frac{1}{2} f^{[AB]}_{(CD)[EF]} \omega^{CD}_{\mu} \omega^{EF}_\nu,
\]

where

\[
f^{[AB]}_{(CD)[EF]} = \eta_{AC} \zeta_{B}^{[E} \partial^{F]} - \eta_{AD} \zeta_{B}^{[E} \partial^{F]} + \eta_{BD} \zeta_{A}^{[E} \partial^{F]} - \eta_{BC} \zeta_{A}^{[E} \partial^{F]}
\]
are the structure constants of the gauge group. The components of \( f^{[AB]}_{\{CD\}[EF]} \) can be separated according to the sector of the internal space to which they belong as \( f^{[ab]}_{\{cd\}[ef]} \). Here we skip the splitting of action (6b). This way to break the (A)dS group automatically keeps the torsion out of the action.

In the next section we present the anti-self-dual extension of (3), and we introduce the necessary ingredients to split it into the Lagrangian densities shown in eqs. (14-16). Here we skip the splitting of action (3) since it is along the same lines that we follow below for the splitting of the anti-self-dual extension.

### III. (ANTI-)SELF-DUAL GAUGE THEORY OF GRAVITY

In this section we present an action that includes the Einstein-Cartan, cosmological constant, Euler, Holst and Pontryagin terms. We show that when the action combines the self-dual and anti-self-dual parts of the curvature, the coefficient of the Holst term is free and can therefore be interpreted as the Immirzi parameter. We closely follow [22]. As we explained in the previous section, the idea is to consider a five dimensional (a)dS internal group, whose gauge field defines a curvature. With the four dimensional internal part of this curvature we formulate a quadratic action, and we decompose this action in terms of the standard 4d curvature and the tetrad. In [22], the MM action is generalized by considering the self-dual part of the connection, given by the upper sign in

\[
\pm \omega_{ab} = \frac{1}{2} \left( \omega_{ab} + i \frac{e_{ab}}{2} \right),
\]

(8)
to construct the corresponding quadratic action

\[
S_{SD} = \int d^4x \epsilon_{\mu\nu\alpha\beta} \epsilon_{\alpha\beta\gamma\delta} + R_{\nu\rho\mu} + R_{\alpha\beta}\]

\[
= \int d^4x \epsilon_{\mu\nu\alpha\beta} \epsilon_{\alpha\beta\gamma\delta} + R_{\nu\rho\mu} + R_{\alpha\beta}.
\]

(9)

where \( R_{\nu\rho\mu} \) is the self-dual part of (6b), given by the upper sign in

\[
\pm R_{\nu\rho\mu} = \frac{1}{2} \left( R_{\nu\rho\mu} + i \frac{e_{ab}}{2} \right).
\]

(10)

In [8] and [10] we already defined the anti-self-dual connection \( -\omega_{ab} \) and curvature \( -R_{\nu\rho\mu} \) for future use. To write the forthcoming expressions in a more compact way it is convenient to introduce the quantities \( \Sigma_{\mu\nu} = e_{\mu} e_{\nu} - e_{\nu} e_{\mu} \) whose (anti)-self-duals are

\[
\pm \Sigma_{\mu\nu} = \frac{1}{2} \left( \Sigma_{\mu\nu} + i \frac{e_{ab}}{2} \right),
\]

(11)

The (anti)-self-dual curvature is then

\[
\pm R_{\nu\rho\mu} = \frac{1}{2} \left( R_{\nu\rho\mu} + i \frac{e_{ab}}{2} \right),
\]

(12)

and from this expression it is easy to show

\[
\epsilon_{\mu\nu\rho\sigma} \pm R_{\nu\rho\mu} = \epsilon_{\mu\nu\rho\sigma} \pm \epsilon_{ab} \Sigma_{cd}.
\]

(13)

Using eqs. (11-16) to rewrite (9) in terms of \( R_{\nu\rho\mu} \) and \( \Sigma_{\mu\nu} \) we can see that it is equivalent to the MM action plus the Holst and Pontryagin terms.

Now let us propose a further generalization by considering an action that combines the self-dual and the anti-self-dual parts of the connection:

\[
S = \mathcal{G}_M \int d^4x \epsilon_{\mu\nu\rho\sigma} \epsilon_{abcd} (\tau + R_{\nu\mu} + R_{\rho\sigma} - \tau - R_{\nu\mu} - R_{\rho\sigma}),
\]

(15)

where \( \tau \), \( -\tau \) and \( \mathcal{G}_M \) are arbitrary parameters that will at the end be related to the Immirzi parameter, effective cosmological constant and gravitational constant. Rewriting this action in terms of the four dimensional curvature and tetrad we obtain

\[
S = \mathcal{G}_M \int d^4x \epsilon_{\mu\nu\rho\sigma} \epsilon_{abcd} \left( \tau + R_{\nu\mu} + R_{\rho\sigma} - \tau - R_{\nu\mu} - R_{\rho\sigma} \right)
\]

\[
- 2\mathcal{G}_M \lambda (\tau + \Sigma_{\mu\nu} + R_{\rho\sigma} + R_{\rho\sigma} - \tau - \Sigma_{\mu\nu} - R_{\rho\sigma})
\]

\[
+ 2\mathcal{G}_M \lambda^2 (\tau - R_{\nu\rho\mu} + R_{\alpha\beta} - R_{\alpha\beta}),
\]

(16)
In the last term we see the Lagrangian density $L_{cc}$, and we can already read off the effective value of the cosmological constant as

$$\lambda_{eff} = G_N \lambda^2 (\tau - \bar{\tau}).$$

(17)

As a reference, note that setting $\tau = 0$ and $\bar{\tau} = 1$, and dropping all the $^\pm$ superscripts what is left is the decomposition of the MM action: it includes – in addition to $L_{cc}$ – the Einstein-Cartan and Euler terms.

Using (11) and (13) we can expand the (anti)-self-dual curvatures to write (10) in terms of $R_{\mu\nu}^{ab}$ and $e^a_\mu$. After a straightforward calculation we get

$$S = G_N \int d^4 x \left[ (\tau - \bar{\tau}) \eta^{\mu \nu \alpha \beta} \epsilon_{abcd} \left( \frac{1}{2} R_{\mu
u}^{ab} R_{\alpha \beta}^{cd} - \lambda \Sigma_{\mu \nu}^{ab} R_{\alpha \beta}^{cd} + 2 \lambda^2 \epsilon^{a}_{\mu} \epsilon^{b}_{\nu} \epsilon^{c}_{\alpha} \epsilon^{d}_{\beta} \right) + (\tau - \bar{\tau}) \eta_{\alpha \beta} R_{\mu \nu}^{ab} R_{\alpha \beta}^{cd} - 2 \lambda \Sigma_{\mu \nu}^{ab} R_{\alpha \beta}^{cd} \right].$$

(18)

Setting $\tau = 0$ this reduces to the same terms that are obtained from the self-dual Lagrangian [9]. A key difference when $\tau \neq 0$ is that we are free to identify three independent parameters with physical significance:

$$G_N = G_N (\tau - \bar{\tau}),$$

(19a)

$$\lambda_{eff} = G_N \lambda^2 (\tau - \bar{\tau}),$$

(19b)

$$\beta^{-1} = G_N (\tau + \bar{\tau}),$$

(19c)

that correspond to the gravitational constant, the effective cosmological constant and the Immirzi parameter. When $\tau = 0$ the product $G_N \tau$ becomes a global coefficient of the action and we cannot identify a free Immirzi parameter anymore; indeed $G_N \tau$ would be the coefficient of the Einstein-Cartan Lagrangian, thus the Immirzi parameter would be degenerated with the gravitational constant.

The second line in (19) does not contribute to the classical equations of motion, this implies that the self-dual and anti-self-dual actions, and any linear combination of them, are classically equivalent – mod. global coefficients that can be reabsorbed in the bare values of $\lambda$ and $G_N$. An action similar to (19) was obtained in [21] (see also [23]) from different arguments that led to a modification of the MM action by the introduction of a $\theta$-term – in analogy to a Yang-Mills theory – whose coefficient is related to our $\tau$.

Up to now we have reviewed the MM action as well as a generalization of it which gives raise to the most general first order gravitational action that one can have in four dimensions (for vanishing torsion). In addition, we proposed a further generalization which allows us to have enough freedom to account for the cosmological constant, the gravitational constant and the Immirzi parameter. Now we want to exploit the fact that all these terms are obtained from an action that can be reduced to (2+1)-dimensions to identify the equivalent of the Immirzi parameter in three dimensional gravity.

IV. REDUCTION TO (2+1)-DIM

In this section we derive an action for three dimensional gravity that consists of a linear combination of two classically equivalent actions, known as the standard and exotic actions, introduced by Witten in [27]. The main feature of our derivation is that it allows for a straightforward identification of the three dimensional equivalent of the Immirzi parameter. There are a few proposals in the literature for what the Immirzi parameter might be in three dimensions, it is usually introduced as a classical ambiguity in analogy to the role that it plays in four dimensions [24][6]. Such an intuitive interpretation coincides with the result that we present below.

Using the (anti)-self-dual properties of the curvature,

$$\epsilon^{ab}_{\mu \nu} \pm R_{\mu \nu}^{cd} \mu = \pm 2i \pm R_{\mu \nu}^{ab},$$

(20)

we can rewrite (19) as a total derivative. To ease notation we briefly turn our formalism to differential forms. Then, action (19) becomes

$$S = \int d^4 x \epsilon^{\mu \nu \alpha \beta} \epsilon_{abcd} \left( \tau R^{ab}_{\mu \nu} R^{cd}_{\alpha \beta} - \bar{\tau} R^{ab}_{\mu \nu} R^{cd}_{\alpha \beta} \right) = 2i \int \tau R + \bar{\tau} R + 2i \int \tau R \wedge R = 2i \int (d^+ A + A \wedge A) \wedge (d^+ A + A \wedge A) + 2i \int (d^+ A + A \wedge A) \wedge (d^+ A + A \wedge A),$$

(21)

where we set $G_N = 1$ and we introduced the connections \pm A defined by

$$\pm R = d \pm A + \pm A \wedge \pm A.$$

(22)

It is not hard to see that the integrands are just $d(\pm A \wedge \pm A \wedge \pm A)$, so in (2+1) dimensions the action (19) is seen as a combination of the Chern-Simons actions for the (anti)-self-dual curvatures,

$$S_{CS} = 2i \tau \int (d^+ A + A \wedge A) \wedge (d^+ A + A \wedge A) + 2i \int (d^+ A + A \wedge A) \wedge (d^+ A + A \wedge A).$$

(23)

This action is topological and consistent with the fact that 2+1 gravity has no dynamical degrees of freedom. Turning back to the usual notation we have:

$$\frac{S_{CS}}{2i} = \tau \int d^3 x \epsilon^{\mu \nu \alpha} \left[ A^{ab}_{\mu} \partial_{\nu} A_{\alpha \mu} + A^{ab}_{\mu} + A^{ad}_{\mu} + A_{\alpha \mu} \right] + (+ \rightarrow -),$$

(24)
where \( a, b, c, d = 0, 1, 2, 3, \) \( \eta_{ab} = \text{diag}(-1, 1, 1, 1, 1), \) and the complex (anti-)self-dual connections are

\[
\pm A^{ab}_\mu = \frac{1}{2} \left( A^{ab}_\mu + \frac{i}{2} \varepsilon^{ac}_{\mu} A^{cd}_\mu \right). \tag{25}
\]

Let \( S^+_\text{CS} \) and \( S^-_\text{CS} \) be the first and second integrals in \([21]\), a straightforward calculation shows that

\[
S^\pm_\text{CS} = \int d^3x \frac{1}{2} \varepsilon^{\mu \nu \alpha} \left( A^{ab}_\mu \partial_\mu e_{\alpha A} + \frac{2}{3} A^a_{\mu a} A^{bc}_\mu A^c_{\alpha c} \right) + \frac{i}{4} \varepsilon^{\mu \nu \alpha} \varepsilon^{abcd} \left( A_{\mu ab} \partial_\mu e_{\alpha cd} + \frac{2}{3} A^a_{\mu a} A^{bc}_\mu A_{\alpha cd} \right). \tag{26}
\]

Then, each one of these actions contains the Chern-Simons action and the “theta term” for the internal gauge group. The total action \([21]\) will be a linear combination of these two terms with coefficient \( +\tau - \tau \) for the C-S part and \( +\tau + \tau \) for the theta term.

The previous actions are similar to the MM action in the sense that we have an internal group with one dimension more than the spacetime. Following the same idea, we use this extra dimension of the internal space to recover the information about the cosmological constant. We impose that the \( SO(3,1) \) connection can be decomposed as \( A^a_\mu = (A^A_\mu , A^A_\mu ) = (\omega^A_\mu , \sqrt{\lambda} e^A_\mu ) \) and \( \omega^A_\mu = \epsilon^{ABC} \omega^B_\mu C, \) where now \( A, B = 0, 1, 2, \) In this way we obtain

\[
S^\pm_\text{CS} = \int_X \frac{1}{2} \varepsilon^{\mu \nu \alpha} \left( \omega^A_\mu (\partial_\mu \omega_{\alpha A} - \partial_{\alpha} \omega_{\mu A}) + \frac{2}{3} \epsilon^{ABC} \omega^A_\mu \omega^B_\nu \omega^C_\alpha \right) + \lambda e^A_\mu (\partial_\mu e_{\alpha A} - \partial_{\alpha} e_{\mu A}) + 2 \lambda \epsilon^{ABC} e^A_\mu e^B_\nu \omega^C_\alpha \right) \pm \frac{i}{4} \sqrt{\lambda} \varepsilon^{\mu \nu \alpha} \left( e^A_\mu (\partial_\mu \omega_{\alpha A} - \partial_{\alpha} \omega_{\mu A}) \right) + \epsilon^{ABC} e^A_\mu e^B_\nu \omega^C_\alpha + \frac{1}{\lambda} \epsilon^{ABC} \epsilon^{EFG} \omega^E_\mu \omega^F_\nu \omega^G_\alpha. \tag{27}
\]

The actions in the first and second rows of the last expression are respectively the exotic and standard actions \( I \) and \( I \) \([27]\). The full action is then

\[
S_{\text{CS}} = i(\tau + \tau)\tilde{I} - 2\sqrt{\lambda}(\tau - \tau)I. \tag{28}
\]

As we saw in the previous section, the combination \( +\tau + \tau \) is related to the Immirzi parameter, and the exact term corresponding to it is in the exotic action. This is consistent with the fact that the sum of the standard and exotic actions is a classical ambiguity in 2 + 1-dimensional gravity. This explicit identification of the Immirzi parameter in terms of the ambiguity for the classical description of three dimensional gravity is a novel result of this work. As already suggested in \([12, 21, 23]\), the “Immirzi parameter” is analogous to the “\( \theta \)-term” in Yang-Mills theories. Furthermore in \([13]\) the transformation of the Chern-Simons gravity partition function was studied and it was concluded that eq. \([28]\) has the form of actions transforming under modular transformations, and that the partition function transforms as

\[
Z(\frac{at + b}{ct + d}) = (ct + d)^n (c\tau + d)^v Z(\tau), \tag{29}
\]

under modular transformations, relating actions with different coupling constants. From this we conjecture that the Immirzi ambiguity in \( 2 + 1 \) dimensions can be related to modular invariance of the partition function.

Before concluding we want to comment further on the relation between the derivation of the standard and exotic actions by Witten and the way – introduced in \([12]\) – in which they emerge here.

### A. Relation to Witten’s derivation of the standard and exotic actions

For concreteness we do not review Witten’s construction \([27]\) here, instead we just state that it is based on a Pontryagin topological invariant for the group \( SO(3,1) \) if the cosmological constant is positive (dS) or \( SO(2,2) \) if it is negative (AdS), and the posterior reduction to three dimensions of this topological invariant. The relevant field strength is

\[
F_{\mu \nu} = P_A (\partial_\mu e^A_{\nu} - \partial_\nu e^A_{\mu} + \epsilon^{ABC} (\omega^B_{\mu} e^C_{\nu} + e^B_{\nu} \omega^C_{\mu})), \tag{30}
\]

where \( P \) and \( J \) are the generators of translations and Lorentz transformations respectively, and the triad \( e^A_\mu \) and connection \( \omega^A_\mu \) are unified in a connection for (a splitting of) the groups mentioned above as \( A_\mu = e^A_\mu P_A + \omega^A_\mu J_A. \) By inspection one can suspect that equations \([32a], [32b] \) and \([31]\) are related in some way. Of course the internal groups are different: here \( A, B = 0, 1, 2, 3 \). However we can show that the three dimensional projection of the internal group in \([31]\) and \([27]\) performed in the same way as was done to obtain \([30]\) reproduces exactly the components of \( F_{\mu \nu} \). To see this, we first write \( R^{ab}_\mu \) in terms of its associated connection \( A^{ab}_\mu \) used at the beginning of this section, obtaining

\[
R^{ab}_\mu = \partial_\mu A^{ab} - \partial_b A^{a}_\mu + A^{ca}_\mu A^{b}_\nu - A^{ca}_\nu A^{a}_\mu b, \tag{31}
\]

and then we split it by means of \( A^{ab}_\mu = (A^{AB}_\mu , A^{A}_\mu ) = (\omega^{AB}_\mu , \sqrt{\lambda} e^A_\mu ). \) We find the contribution to 3d as

\[
R^{AB}_\mu = \partial_\mu A^{AB}_\nu - \partial_\nu A^{AB}_\mu + A^{CA}_\nu A^{B}_\mu C - A^{CA}_\mu A^{B}_\nu C = \partial_\mu \omega^{AB}_\nu - \partial_\nu \omega^{AB}_\mu + \omega^{CA}_\mu \omega^{CB}_\nu + \lambda e^A_\mu e^B_\nu e^C_\alpha. \tag{32a}
\]

\[
R^{A}_\mu = \partial_\mu e^A_\nu - \partial_\nu e^A_\mu + (e^B_\nu \omega^{AB}_\mu - \omega^{AB}_\mu e^B_\nu ). \tag{32b}
\]
The last line follows straightforwardly from \((6a)\) since \(e^3\) does not exist. To compare directly with \((31)\), we need to write both connections in the same notation (remember that before we introduced \(\omega^{AB}_{\mu} = \epsilon^{ABC} \omega_{\mu C}\)). It is easier to do this starting from the field strength \((31)\) and reinserting the definition \(J^A = \frac{1}{2} F^{ABC} J_{BC}\) [27] in order to form the appropriate combinations of the Levi-Civita tensor density and the spin connection:

\[
F_{\mu \nu} = P_A \left( \partial_\nu e^A_\mu - \partial_\mu e^A_\nu + \epsilon^{APQ} (\omega_{\mu P} e_{\nu Q} + \epsilon_{\mu P} \omega_{\nu Q}) \right) + J_A \left( \partial_\mu \omega^A_\nu - \partial_\nu \omega^A_\mu + \epsilon^{APQ} (\omega_{\mu P} \omega_{\nu Q} + \lambda e_{\mu P} e_{\nu Q}) \right) 
\]

\[
= P_A \left( \partial_\nu e^A_\mu - \partial_\mu e^A_\nu + \epsilon^{APQ} (\omega_{\mu P} e_{\nu Q} + \epsilon_{\mu P} \omega_{\nu Q}) \right) + \frac{1}{2} \epsilon^{ABC} J_{BC} \left( \partial_\mu \omega^A_\nu - \partial_\nu \omega^A_\mu + \epsilon_{APQ} (\omega^P_{\mu} \omega^Q_{\nu} + \lambda \epsilon^P_{\mu} \epsilon^Q_{\nu}) \right) 
\]

\[
= P_A \left( \partial_\nu e^A_\mu - \partial_\mu e^A_\nu - \omega^Q_{\mu e_{\nu Q} e_{\mu P}} \right) \left( \omega_{\mu P} e_{\nu Q} - \epsilon_{\mu P} \omega_{\nu Q} \right) + \frac{1}{2} J_{BC} \left( \partial_\mu \omega^B_{\nu} - \partial_\nu \omega^B_{\mu} + \omega^D_{\mu} \omega^C_{\nu} - \omega^D_{\nu} \omega^C_{\mu} + \lambda (\epsilon^B_{\mu} \epsilon^C_{\nu} - \epsilon^C_{\mu} \epsilon^B_{\nu}) \right), 
\]

where we have used

\[
\omega^D_{\mu B \omega_{\nu DC} - \omega^D_{\nu B \omega_{\mu DC} = \omega^D_{\mu B \omega_{\nu C} - \omega^D_{\nu B \omega_{\mu C}}} \quad (34) 
\]

In the last expression we see that the spacetime components of \(F_{\mu \nu}\) are a linear combination of the components of \((32a)\) and \((32b)\). This clarifies the relation between the field strength used by Witten and the curvature that before we introduced \(\omega_{\mu C} \omega_{\nu B}\).

V. CONCLUSIONS

In this paper we proposed a first order action for four dimensional gravity constructed with the self and anti-self-dual parts of the field strength of an (A)dS internal connection. After explicit symmetry breaking, the action reduces to a combination of the Einstein-Cartan, Holst, cosmological constant and Euler terms. The geometrical content of our model is the same as in [22], where only the self-dual part of the gauge connection is considered.

The advantage of adding the anti-self-dual terms is that now we are able to identify the Immirzi parameter as a combination of the free parameters of the action. This combination is not degenerated with the gravitational and cosmological constants, and it does not appear in the classical equations of motion. The decomposed form of the action, [18], can be also interpreted as coming from an usual MM formulation supplemented with a \(\theta\)-term, as argued in [21, 22].

Due to the (anti)-self-duality of the fields that we use, we can write their quadratic actions as Pontryagin terms. In consequence we can obtain their three-dimensional versions as Chern-Simons terms. In doing so, there is the underlying assumption that the spacetime has a well-defined boundary so that we can apply Stoke’s theorem. Evidently, this would not be the case for AdS or flat spacetimes. Nonetheless, we can start with a four dimensional manifold that has a well-defined tree dimensional boundary where the combination \(\tau - \tau + \tau\) is related to the four dimensional Immirzi parameter. We remark that our claim is not that we find a three dimensional version of the same physical system described by \([18]\), but rather our claim is as follows: in four spacetime dimensions we first identify a combination of parameters that corresponds to the Immirzi parameter, and then we take advantage of the structure of the action to find a situation where we can integrate out one spacetime dimension, and investigate the role of such combination of parameters in the resulting lower dimensional setting. By a convenient splitting of the connections, the Chern-Simons form gives raise to the standard and exotic actions for three dimensional gravity; these actions are classically equivalent in the sense that upon variation they lead to the same equations of motion. Thus, what we obtain from \([15]\) in three dimensions is a linear combination of the standard and exotic terms. The combination of \(\tau - \tau\) and \(\tau - \tau\) associated to the Immirzi parameter in four dimensions translates to the coefficient of the exotic action in the three dimensional action. This is satisfying from an heuristic point of view: both in three and four dimensional space-time the Immirzi parameter plays the role of a classical ambiguity. This identification has been proposed and used in other works [2, 4, 6] but without a formal derivation from a four dimensional action.

Finally, we conjecture that the Immirzi ambiguity can be understood from the modular invariance transformation of the \((2+1)\) partition function. Self-dual Chern-Simons theories in three dimensions have been widely studied in the literature for different reasons, in the context of gravity this is generically in the search of guiding principles to construct four dimensional theories (see for example [8, 10, 17] for specific applications or [29] for a general review). The analysis of what those developments perspective for this work.

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[30] We do not consider boundary terms. For a study of those terms see [3].
[31] Here A is a SO(3, 1) connection associated to the $R^a_{\mu
u}$, which is the SO(3, 1) projection of the SO(3, 2) (or SO(4, 1), depending on the sign of the cosmological constant) curvature $R^a_{\mu
u}$. 