Unification of Inflation with Dark Energy in $f(R)$ Gravity and Axion Dark Matter

S.D. Odintsov,$^{1,2,3}$ V.K. Oikonomou,$^{4,5,3}$

1) ICREA, Passeig Lluís Companys, 23, 08010 Barcelona, Spain
2) Institute of Space Sciences (IEEC-CSIC) C. Can Magrans s/n, 08193 Barcelona, Spain
3) Tomsk State Pedagogical University, 634061 Tomsk, Russia
4) Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
5) Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics, 634050 Tomsk, Russia (TUSUR)

In this work we introduce an effective model of $f(R)$ gravity containing a non-minimal coupling to the axion scalar field. The axion field is described by the misalignment model, in which the primordial $U(1)$ Peccei-Quinn symmetry is broken during inflation and the $f(R)$ gravity is described by the $R^2$ model, and in addition, the non-minimal coupling has the form $\sim h(\phi)R^2$, with $0 < \gamma < 0.75$. By appropriately constraining the non-minimal coupling at early times, the axion field remains frozen in its primordial vacuum expectation value, and the $R^2$ gravity dominates the inflationary era. As the Universe expands, when $H$ equals the axion mass $m_a$ and for cosmic times for which $m_a \gg H$, the axion field oscillates. By assuming a slowly varying evolution of the axion field, the axion energy density scales as $\rho_a \sim a^{-3}$, where $a$ is the scale factor, regardless of the background Hubble rate, thus behaving as cold dark matter. At late times, the axion still evolves as $\rho_a \sim a^{-3}$, however the Hubble rate of the expansion and thus the dynamical evolution of the Universe is controlled by terms containing the higher derivatives of $\sim R^2$, which are related to the non-minimal coupling, and as we demonstrate, the resulting solution of the Friedman equation at late times is an approximate de Sitter evolution. The late-time de Sitter Hubble rate scales as $H \sim \Lambda^{1/2}$, where $\Lambda$ is an integration constant of the theory, which has its allowed values very close to the current value of the cosmological constant. Finally, the theory has a prediction for the existence of a pre-inflationary primordial stiff era, in which the energy density of the axion scales as $\rho_a \sim a^{-6}$.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k, 98.80.Cq,11.25.-w

I. INTRODUCTION

For decades fundamental scalar fields were an important element of quantum field theory, however these remained undetected until the Higgs particle was discovered in 2012 [1]. This experimental verification of the theory showed that the spin zero elementary particles play some important role in model building, and it is possible that scalar fields may manifest themselves in other physical phenomena and processes. In cosmology, scalar fields have a prominent role, since quite many inflationary theories are based on the slow-rolling of the inflaton field [2,3]. On the other hand alternative theoretical models of modified gravity can also describe the inflationary era, see for example Ref. [4] for the pioneer Starobinsky model. Modified gravity theory is also capable to describe dark energy [5,6] and in addition, it can also successfully provide a unified description of early-time acceleration with late-time acceleration, see for example [7], for the first proposal towards this direction in the context of $f(R)$ gravity. Several viable $f(R)$ gravities unifying inflation with dark energy were also developed in Refs. [8–10], and also the possibility of incorporating the radiation and matter domination eras, was addressed in Ref. [11]. For some recent updates in the field of modified gravity, the reader is referred to the reviews [12–14]. Nevertheless, the nature of dark energy and especially of dark matter as well as their interaction still remain unclear. Dark matter has been believed for many years that consists of weakly interacting massive particles, and there are many observational motivations towards believing in the particle nature of dark matter, like the collision of galaxies observed in the Bullet Cluster. Up to date many theoretical proposals were introduced that could potentially make the observation of dark matter particles possible, having a wide range of masses, see for example Ref. [21] which focuses on the direct detection of dark matter candidates. However, many experiments up to date have not achieved in detecting any dark matter particle. Axion physics is a timely theoretical framework [21–27], due to the related experiments which have gained a lot of attention and many of the well motivated proposed experiments may reveal whether the axion exists [28–35], see also [36] for an insightful approach. Some of the proposals indicate that the axion mass could be extremely low, of the order $m_a \sim O(10^{-12})$eV,
and it is well known from the axion physics literature that axions with such a low mass could be perfect candidates for dark matter, at least some of it in the Universe [21]. In fact, the most promising model for axion is the so-called misalignment axion model, which is based on a broken Peccei-Quinn $U(1)$ symmetry during inflation [37], in contrast to the QCD axion models which have unbroken Peccei-Quinn $U(1)$ symmetry during inflation. Most of the misalignment models can be string theory motivated, since several string theory models predict such low-mass axion particles [21]. The detection of the axion particle is quite easy to realize and it is based on the fact that the axion can interact with photons in the presence of magnetic fields [38–40], thus leaving many possibilities of detection. To our opinion, the last forefront of dark matter, before the whole context collapses, is low mass particles like axions or even neutrinos, unless supersymmetry is confirmed experimentally by detecting some supersymmetric partner in the Large Hadron Collider.

It is interesting to note that among different dark matter models, one can mention also an $f(R)$ gravity related dark matter model presented in Ref. [18]. Furthermore, a theoretical attempt to unify inflation with dark energy and dark matter, using scalar Lagrange multipliers, can be found in Ref. [19].

In view of the dark matter and dark energy problem, in this paper we shall introduce an effective $f(R)$ gravity theory in the presence of a misalignment axion scalar field. The theory we shall present has a non-minimal coupling between the axion scalar field and a curvature dependent function $G(R)$, of primordial origin. The exact form of the $f(R)$ gravity will contain the $R^2$ model and this extra non-minimal coupling, with the function $G(R)$ being chosen as $G(R) \sim R^2$. $0 < \gamma < 0.75$. The misalignment axion model predicts that the axion is frozen at early times to its vacuum expectation value obtained by the breaking of the primordial $U(1)$ Peccei-Quinn symmetry, thus it is dominated dynamically by the $f(R)$ gravity at early times. Thus during inflation, the leading order terms of the $f(R)$ gravity dominate the evolution, and drive inflation. As the Universe expands, when $m_a \sim H$, with $H$ being the Hubble rate, the axion starts to oscillate and when $m_a \gg H$, the axion oscillates in a slow-varying way. Focusing on later and late times, where the curvature of the Universe is too low, the slow-varying assumption leads to the fact that its energy density scales as $\rho_a \sim a^{-3}$, where $a$ is the scale factor, regardless of the cosmological background in terms of the Hubble rate. This is a crucial observation, and as we demonstrate at late times, the background cosmological evolution in terms of the Hubble rate, is controlled by the non-minimally coupling term containing $G(R) \sim R^\gamma$. By solving the resulting Friedman equation at leading order, we show that a de Sitter expansion is achieved at late times of the form,

$$H(t_0) \simeq \frac{\sqrt{2} \sqrt{1 - \gamma \sqrt{\Lambda}}}{\sqrt{3 - 4\gamma}}, \quad (1)$$

which clearly describes and accelerating expansion, thus the model successfully generates a dark energy era. Intriguingly enough, for the integration constant $\Lambda$ which emerges from the theory, we have $\Lambda \sim H_0^2$. In addition, for the allowed values of the parameter $\gamma$, which are $0 < \gamma < 0.75$, and particularly when $0 < \gamma < 0.74$, and for $H_0$ taking the current observed value of the Hubble rate $H_0 \sim 10^{-17}eV$, the allowed values of the integration constant $\Lambda$ are $1.5 \times 10^{-66}eV^2 < \Lambda < 7.69231 \times 10^{-68}eV^2$, which are very close to the current allowed value of the cosmological constant $\Lambda_0 \sim 10^{-66}eV^2$. However, as $\gamma$ approaches the value 0.75, then the constant $\Lambda$ takes smaller values, for example when $\gamma = 0.749999999999$, then $\Lambda = 8 \times 10^{-77}eV^2$.

Thus with the present effective model we aim to demonstrate that an $f(R)$ gravity in the presence of a misalignment axion field, can describe the inflationary era, a dark matter component which scales at intermediate and late times as $a^{-3}$, and can also generate the late-time acceleration in the Universe. In addition, in our model, the primordial non-minimal axion coupling to the curvature can affect drastically only the late-time dynamics, since in all previous eras it had no particular effect. Finally, we shall show that the present model predicts a stiff matter era preexisting the inflationary era.

In all the following considerations, the background metric will assumed to be that of a flat Friedmann-Robertson-Walker (FRW), with line element,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \quad (2)$$

where $a(t)$ is as usual the scale factor.

**II. $f(R)$ GRAVITY WITH AXION DARK MATTER**

In this work we shall consider a vacuum $f(R)$ gravity theory in the presence of an axion dark matter scalar field, which is a canonical scalar field, and we shall assume a non-minimal coupling between the scalar field and higher
powers of the Ricci scalar. The gravitational action is the following,

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} f(R) + \frac{1}{2\kappa^2} h(\phi)G(R) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right], \]

(3)

where \( \kappa^2 = \frac{1}{8\pi G} \), with \( G \) Newton’s gravitational constant. For notational simplicity in deriving the gravitational equations of motion, we set,

\[ \mathcal{F}(R, \phi) = \frac{1}{\kappa^2} f(R) + \frac{1}{\kappa^2} h(\phi)G(R). \]

(4)

We shall use the metric formalism, so upon variation of the action (3) with respect to the metric tensor, for the FRW metric of Eq. (2), we get the following gravitational equations,

\[ 3H^2 F = \frac{1}{2} \dot{\phi}^2 + \frac{RF - \mathcal{F} + 2V}{2} - 3H \dot{F}, \]

\[ -3FH^2 - 2\dot{H}F = \frac{1}{2} \dot{\phi}^2 - \frac{RF - \mathcal{F} + 2V}{2} + \ddot{F} + 2H \dot{F}, \]

(5)

where \( F = \frac{\dot{\mathcal{F}}}{\mathcal{F}} \). Upon varying the gravitational action (3) with respect to the scalar field, we obtain the following equation of motion for the scalar field,

\[ \ddot{\phi} + 3H \dot{\phi} + \frac{1}{2} \left( -\mathcal{F}'(R, \phi) + 2V'(\phi) \right) = 0, \]

(6)

where the “prime” denotes differentiation with respect to the scalar field \( \phi \).

The choice of the \( f(R) \) gravity is the following,

\[ f(R) = R + \frac{1}{36H_i^2} R^2, \]

(7)

so it is basically chosen to be the well-known \( R^2 \) model of inflation, with the \( H_i \) units being eV. As for the non-minimally coupled term \( \sim h(\phi)G(R) \), the exact choice will depend on the phenomenology of the axion field and on the late-time behavior. The \( h(\phi) \) will be assumed to be inverse proportional to the scalar field,

\[ h(\phi) \sim \frac{1}{\phi^\delta}, \]

(8)

with \( \delta > 0 \), or similar and the reasoning for this choice will be explained in the next section. Also the \( G(R) \) function will be assumed to have the following form,

\[ G(R) \sim R^\gamma, \]

(9)

where \( \gamma \) takes values in the interval \( 0 < \gamma < 0.75 \).

III. UNIFIED DESCRIPTION OF INFLATION AND DARK ENERGY

A. The Evolution of the Axion Field

The evolution of the axion dark matter scalar field is essential for the theory, and everything is based on two intrinsic mechanisms of the axion scalar field, the primordial broken \( U(1) \) Peccei-Quinn like symmetry, and the axion field slow-varying reheating. Basically, the broken Peccei-Quinn symmetry refers to an era where non-perturbative sting theory phenomena take place and control the dynamics of the axion. A recent informative review on the axion field dynamics in the context of various theories can be found in Ref. [21]. We shall adopt many of the conventions and notation of Ref. [21]. One of the most sound phenomenological models for the axion field, that can mimic dark matter is the misalignment model, which eventually can describe some of the dark matter present in our Universe at present. One additional assumption crucial for the dynamical evolution of the theory we develop in this paper, is that the term containing the non-minimal coupling of the axion to the curvature scalar, namely \( \sim h(\phi)G(R) \), satisfies the following condition during the inflationary era and for all the cosmological eras that follow,

\[ 2V'(\phi) \gg \mathcal{F}'(R, \phi), \]

(10)
or equivalently,
\[ 2V'(\phi) \gg \frac{1}{\kappa^2} h'(\phi)G(R). \]  
(11)

By integrating Eq. (11) with respect to the scalar field \( \phi \), we obtain,
\[ 2V(\phi) \gg \frac{1}{\kappa^2} h(\phi)G(R), \]  
(12)

which is very important for the considerations that will follow on the dynamical evolution of the theory. The condition (10) is the only assumption of the theory on the dynamics of the scalar, and clearly restricts severely the functional form of the non-minimal coupling \( h(\phi) \). However, the behavior of \( h(\phi) \) and \( G(R) \) made in Eqs. (8) and (9) can satisfy the constraint (10). Essentially we need a small contribution of the term \( \sim h(\phi)G(R) \) during and after the inflationary era. However there will be a pre-inflationary era for which \( 2V(\phi) \sim \frac{\kappa^2}{2} h(\phi)G(R) \). During this pre-inflationary era, the axion field will not behave as dark matter, and we discuss this issue more concretely in a later subsection.

In view of the condition (10), the equation of motion for the axion scalar field (6) is mainly affected by the axion scalar potential, during and after the inflationary era. Thus the phenomenology of the misalignment model applies, in the context of which the primordial \( U(1) \) Peccei-Quinn symmetry is broken during the pre-inflationary era and during all the subsequent cosmological eras, leaving the axion field having a vacuum expectation value during the inflationary era. During the inflationary era, the axion mass \( m_a \) is constant, and this holds true for all cosmic times for which \( H \gg m_a \), where \( H \) is the Hubble rate. The axion field potential has the following form,
\[ V(\phi(t)) \simeq \frac{1}{2} m_a^2 \phi_i^2(t), \]  
(13)

where \( \phi_i \) is the value of the axion during the corresponding cosmic era. The axion field is overdamped during inflation, and the following initial conditions hold true for the values of the axion and its derivatives during inflation,
\[ \dot{\phi}(t_i) = \zeta < 1, \quad \phi(t_i) = f_a \theta_a, \]  
(14)

where \( t_i \) is the cosmic time during the inflationary period, \( f_a \) stands for the axion decay constant and \( \theta_a \) is the so-called initial misalignment angle. Due to the initial conditions, the axion is frozen during inflation and essentially contributes a small cosmological constant term during inflation, as we also show shortly. Basically, the effective equation of state (EoS) parameter for the axion \( w_a \) is approximately \( w_a \approx -1 \), so it contributes an effective cosmological term, and this is due to the fact that it is overdamped. As we show shortly in the following subsection, the cosmological dynamics or equivalently the Hubble rate \( H(t) \), will be strictly determined by the \( f(R) \) gravity during inflation. Due to the overdamping at early-times during inflation, we have \( \dot{\phi} = \zeta \ll 1 \) and also \( \dot{\phi} = \lambda \ll 1 \), and this behavior continues until \( H \approx m_a \), at which point the axion starts to oscillate. Before we continue with the description of the axion dynamics during the era \( H \approx m_a \), let us first fix the values of the quantities that appear in the above equations. Firstly, we shall assume that the inflationary scale will be \( H_I = \mathcal{O}(10^{13})\text{GeV} \), so the low-scale inflationary scenario is realized. Also for phenomenological reasons, the most plausible values for \( f_a, \theta_a \) and \( m_a \) are [21],
\[ f_a \sim \mathcal{O}(10^{11})\text{GeV}, \quad \theta_a \sim \mathcal{O}(1), \]  
(15)
\[ m_a \sim \mathcal{O}(10^{-12})\text{eV}. \]  
(16)

In order for the condition (12) to hold true during the inflationary era, the coupling function \( h(\phi) \) and the function \( G(R) \) must have appropriate forms, so let us have an idea on how these behave during inflation for the above choices of the parameters, and compare these to the scalar potential. For the axion field parameter values (15) and (16), the potential term \( 2\kappa^2 V(\phi_i) \) during inflation is of the order \( \mathcal{O}(3 \times 10^{-38})\text{eV} \), while for \( \delta = \mathcal{O}(3) \) and for \( 0 < \gamma < 0.75 \), the term \( \sim h(\phi)G(R) \) during inflation, is of the order \( \mathcal{O}(2 \times 10^{-88})\text{eV} \), so for this simple qualitative choice, the constraint (12) holds true during inflation.

As the Universe evolves and the Hubble rate values drop, after \( m_a \sim H \), and for cosmic times for which \( m_a \gg H \), the axion field has already started oscillating. Actually the oscillations start when \( m_a \sim H \), and this can be viewed as an inherent reheating mechanism in the theory. In view of the assumption (10), the scalar field equation of motion when becomes approximately,
\[ \ddot{\phi} + 3H \dot{\phi} + m_a^2 \phi = 0. \]  
(17)
One can seek slowly varying oscillating solutions of the form \[ \phi(t) = A(t) \cos(m_a t), \] where \( A(t) \) is a slowly-varying function of the cosmic time, which satisfies the constraints,

\[ \frac{\dot{A}}{m_a} \sim \frac{H}{m_a} \sim \epsilon \ll 1. \]  

Hence, plugging Eq. (18) in Eq. (17), and working at leading order in \( \epsilon \) we obtain the following equation,

\[ -\frac{2\dot{A}(t) \sin(m_a t)}{m_a} - \frac{3A(t)H(t) \sin(m_a t)}{m_a} = 0, \]  

so by expressing \( H = \frac{\dot{a}}{a} \), we get,

\[ \frac{dA}{A} = -\frac{3da}{2a}, \]  

which when solved yields,

\[ A \sim a^{-3/2}. \]  

The axion energy density is,

\[ \rho_a \sim \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m_a^2 \phi^2, \]  

where we omitted the non-minimal coupling contribution due to the condition (12). So in view of the solution (18) we get,

\[ \rho_a = m_a^2 A(t)^2 + \frac{1}{2} \dot{a} A(t)^2 \cos^2(m_a t) - m_a A(t) \dot{A}(t) \sin(m_a t) \cos(m_a t), \]  

so at leading order we have (21),

\[ \rho_a \sim A^2, \]  

or in view of the solution (22) we have equivalently,

\[ \rho_a \sim a^{-3}. \]  

The above result is of crucial importance since it shows that the axion field dynamics after the frozen state during inflation, is that of cold dark matter, regardless of the Hubble rate background. Thus the actual cosmological evolution \( H(t) \) is not affected by the scalar field, and in our model the \( f(R) \) gravity will determine the inflationary and late-time dynamics of the spacetime, with the axion field evolving as dark matter. Actually, for all the subsequent eras after inflation, for which \( H \ll m_a \), which corresponds to cosmic times \( t \gg 1/m_a \), the averaged EoS parameter for the axion scalar field is \( \langle w_{eff} \rangle \sim 0 \) due to the sinusoidal time dependence (21), and this holds true regardless of the background evolution, thus for any Hubble rate. This is important and this result holds true for any cosmic time for which \( m_a \gg H \), so well after the inflationary era.

In conclusion, the axion field during the inflationary era is frozen to its string originating vacuum expectation value acquired by the primordial pre-inflationary breaking of the Peccei-Quinn symmetry, and well after inflation it oscillates in a way so that its energy density evolves as \( \rho_a \sim a^{-3} \), so it evolves as cold dark matter, regardless the background evolution. Thus the background evolution is controlled by the \( f(R) \) gravity, both during inflation and at late-times, as we demonstrate in a quantitative way in the following two subsections.

**B. Background Evolution During the Inflationary Era: The Role of the \( f(R) \) Gravity**

In this section we shall discuss the essential features of the inflationary era for the \( f(R) \) axion model. As we already mentioned in the previous subsection, the axion field is frozen during the inflationary era, so it does not affect the cosmic evolution at all, and hence does not affect the solution \( H(t) \) of the Friedman equation. Here we shall quantify
this claim and prove that this behavior is indeed what happens. Let us consider the first Friedman equation \[\text{(5)}\], for the \(R^2 f(R)\) gravity and with the axion scalar field potential satisfying the condition \[\text{(12)}\], which holds true for all the cosmological eras during and after inflation. In this case, the first Friedman equation of Eq. \[\text{(5)}\] becomes,

\[
3H^2 \left( 1 + \frac{1}{18H_i^2} R + h(\phi)G'(R) \right) \simeq \frac{1}{2} \dot{\phi}^2 + \frac{1}{72H_i^2} R^2 + 2V\kappa^2 - \frac{1}{6H_i^2} H\dot{R} - 3H\dot{R} G'(R) h(\phi). \tag{27}
\]

During inflation, the curvature is of the order \(R \sim 12H_i^2 \sim \mathcal{O}(1.2 \times 10^{45})\text{eV}\), and thus the term containing \(\sim G'(R)\) in the left hand side of Eq. \[\text{(27)}\] is \(\sim R^{-1}\) so it is extremely suppressed during inflation. The same applies for the term \(\sim G''(R)\) in the right hand side, so these two terms can be omitted. As for the term \(\frac{1}{2}\kappa^2 \dot{\phi}^2\), in view of the initial conditions \[\text{(14)}\], by taking \(\zeta \sim \mathcal{O}(10^{-10})\) for example, this is of the order \(\frac{1}{2}\kappa^2 \dot{\phi}^2 \sim \mathcal{O}(8 \times 10^{-74})\text{eV}^2\), and also the axion potential term is of the order \(2V\kappa^2 \sim \mathcal{O}(3 \times 10^{-38})\text{eV}^2\). In contrast, the \(\sim R^2\) term is of the order, \(\frac{1}{72H_i^2} R^2 \sim \mathcal{O}(2 \times 10^{62})\text{eV}^2\), where we took \(H_i \sim \mathcal{O}(10^{13})\text{eV}\) for phenomenological reasons \[\text{[41]}\]. Therefore, comparing the scalar field and the \(f(R)\) gravity terms during inflation, it is obvious that the inflationary era is overwhelmed by the Starobinsky model, so the Hubble rate is determined by the \(f(R)\) gravity.

The model thus results to the well-know spectral index of primordial curvature perturbations \(n_s\) and the tensor-to-scalar ratio,

\[
n_s \sim 1 - \frac{2}{N}, \quad r \sim \frac{12}{N^2}, \tag{31}
\]

where \(N\) is the \(\epsilon\)-foldings number. The phenomenology of the Starobinsky model is the most successful among \(f(R)\) gravity models, so we do not discuss the inflationary era further. The purpose of this section was to evince that the axion field does not affect the cosmological evolution at all, during the inflationary era, because it is frozen in its vacuum expectation value, and its dynamics make it extremely overdamped, allowing the \(f(R)\) gravity to utterly control the the dynamical evolution of the Universe. Thus the background geometry is controlled solely by the \(f(R)\) gravity during inflation.

### C. The Dark Energy Era

In the previous subsection we showed that the \(f(R)\) gravity will dominate the Universe’s evolution during the inflationary era, so long as the axion field remains frozen around its vacuum expectation value. However, as the Universe expands, when \(H \sim m_a\), the axion field will start oscillating, thus its dynamics will change, since it is not frozen anymore, and following the considerations of section III-A, its dynamical evolution is given in Eq. \[\text{(15)}\] and also by assuming a slow-varying evolution, the axion field energy density will behave as \(\rho_a \sim a^{-3}\). Effectively, this indicates that the axion field for cosmic times corresponding to an era for which \(m_a \gg H\), will behave as dark matter. The background evolution for the eras between inflation and the dark energy era will be controlled possibly by both the \(f(R)\) gravity, the non-minimal coupling \(h(\phi)G(R)\) and the axion field, however our interest in this subsection is on the late-time era, where the curvature is significantly small. For clarity let us quote here the Friedman equation and we discuss the significance of the various terms appearing in it. The Friedman equation reads,

\[
3H^2 \left( 1 + \frac{1}{18H_i^2} R + h(\phi)G'(R) \right) \simeq \frac{1}{2} \dot{\phi}^2 + \frac{1}{72H_i^2} R^2 + 2V\kappa^2 - \frac{1}{6H_i^2} H\dot{R} - 3H\dot{R} G'(R) h(\phi), \tag{32}
\]

and it is clear that the dominant term in the left hand side of the above differential equation is \(\sim h(\phi)G'(R)\) because it contains the term \(G'(R) \sim R^{-1}\) and since \(0 < \gamma < 0.75\) it contains negative powers of the curvature. On the
right hand side, the dominant term is solely $-3H\dot{R}G''(R)h(\phi)$ for the reason that it contains $G''(R) \sim R^{\gamma - 2}$ which is strongly dominant for small curvatures. The potential term $2V\kappa^2$ is proportional to $\sim A^2$ and the kinetic term $\frac{1}{2}\kappa^2\dot{\phi}^2$ is depends on powers of $A$ and $A$, and since $A \sim a^{-3/2}$ these terms are subdominant to the term $\sim G''(R)$. The same applies to the term $\sim R^2$. To have a strong idea on how large is the term $\sim R^{\gamma - 2}$ recall that the Hubble rate today is approximately $H_0 \approx 10^{-33}\text{eV}$ so by choosing for example $\gamma = 0.74$, the term $G''(R)$ becomes $G''(R) \sim R^{\gamma - 2} \sim O(6.3 \times 10^{31})/(\text{eV})^{4-2\gamma}$, while the $R^2$ term is of the order $\frac{1}{2\pi^2^2}R^2 \sim O(2 \times 10^{-178})\text{eV}^2$, which is extremely small. Thus, the Friedman equation at late-times becomes,

$$3H^2h(\phi)G'(R) \simeq -3H\dot{R}G''(R)h(\phi),$$

which by using $G(R) \sim R^\gamma$, can be written as follows,

$$RH \simeq (1 - \gamma) \dot{H},$$

so by using $R = 12H^2 + 6\dot{H}$ for the FRW Universe and by omitting the term $\sim H^3$ which is subdominant at late times, the solution is,

$$H(t) \simeq \frac{\sqrt{2}\sqrt{1 - \gamma}\sqrt{\Lambda}}{\sqrt{3 - 4\gamma}} \tanh\left(\frac{1}{2}\left(\frac{\sqrt{2}\sqrt{3 - 4\gamma}\sqrt{\Lambda}}{\sqrt{1 - \gamma}} + \frac{\sqrt{2}\sqrt{3 - 4\gamma}\sqrt{\Lambda}t}{\sqrt{1 - \gamma}}\right)\right),$$

where $\Lambda$ and $\Theta$ are integration constants. Also from the above solution, the parameter $\gamma$ is constrained to take values in the interval $0 < \gamma < 0.75$. Effectively, at late times which correspond to large cosmic times, say at $t = t_0$, the Hubble rate is approximately constant,

$$H(t_0) \simeq \frac{\sqrt{2}\sqrt{1 - \gamma}\sqrt{\Lambda}}{\sqrt{3 - 4\gamma}}.$$

Therefore the late-time evolution is a de Sitter evolution, that the Universe evolves in an accelerating way. This can also be seen by calculating the deceleration parameter $q = -1 - \frac{\ddot{H}}{H^2}$, which for the Hubble rate (35) is equal to,

$$q = -1 - (3 - 4\gamma)\text{csch}^2\left(\frac{1}{2}\left(\frac{\sqrt{2}\sqrt{3 - 4\gamma}\sqrt{\Lambda}}{\sqrt{1 - \gamma}} + \frac{\sqrt{2}\sqrt{3 - 4\gamma}\sqrt{\Lambda}t}{\sqrt{1 - \gamma}}\right)\right),$$

which is clearly negative during the late-time era, thus an accelerating evolution occurs. Therefore, the axion coupling $h(\phi)G(R)$ affects the late-time era in a dominant way, since it controls the evolution making the Universe to accelerate. In effect, in some sense the microparticles primordial coupling $h(\phi)G(R)$ which was subdominant at the high curvature era and during inflation, becomes dominant at late times, causing the Universe to accelerate and thus providing a dark energy era. Also the effective equation of state of the Universe at late-times is,

$$w_{eff} = -1 - (3 - 4\gamma)\text{csch}^2\left(\frac{1}{2}\left(\frac{\sqrt{2}\sqrt{3 - 4\gamma}\sqrt{\Lambda}}{\sqrt{1 - \gamma}} + \frac{\sqrt{2}\sqrt{3 - 4\gamma}\sqrt{\Lambda}t}{\sqrt{1 - \gamma}}\right)\right),$$

so this becomes approximately $w_{eff} \sim -1$, due to the exponential decay of the hyperbolic cosecant function for large cosmic times.

Let us here quote an interesting phenomenological observation. By taking into account that today $H_0 \approx 10^{-33}\text{eV}$, for $\gamma = 0.74$ we get the approximate value of the integration constant $\Lambda = 7.69231 \times 10^{-66}\text{eV}^2$. Also for $\gamma = 0.2$ we have $\Lambda = 1.375 \times 10^{-66}\text{eV}^2$ and it can be shown that for $\gamma$ belonging in the interval $0 < \gamma < 0.75$, so when $0 < \gamma \leq 0.74$, the parameter $\Lambda$ takes values in the interval $1.5 \times 10^{-66}\text{eV}^2 < \Lambda < 7.69231 \times 10^{-66}\text{eV}^2$. The value of the actual cosmological constant today is approximately $\Lambda_0 \sim 10^{-66}\text{eV}^2$, so it is quite intriguing that for the allowed values of the dimensionless parameter $\gamma$, the integration constant $\Lambda$ takes values quite close to the actual value of the cosmological constant $\Lambda_0$ today, and also it has the same dimensions $\text{eV}^2$. In addition, as the parameter $\gamma$ approaches the value 0.75, then the constant parameter $\Lambda$ takes smaller values, for example, as we also noted in the introduction, when $\gamma = 0.74999999999$, then $\Lambda = 8 \times 10^{-77}\text{eV}^2$. So the question is, is there any possibility that the relation between the current Hubble rate and the cosmological constant is given by Eq. (38)? It is certainly known that for the present day cosmological constant it holds true that $\Lambda_0 \sim H_0^2$ so this is an intriguing result. However, as $\gamma$ approaches the limiting value $\gamma = 0.75$, the parameter $\Lambda$ takes much more smaller values in comparison to the cosmological constant today.
Before closing, let us note that a similar phenomenology can be obtained by using a pure $f(R)$ gravity model in addition to the axion scalar field, without the axion non-minimal coupling. In this case the gravitational action would be of the form,

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \left( R + \frac{1}{36H_i^2}R^2 - \delta R^3 \right) - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right]. \quad (39)$$

It can also be shown in this case that the $R^2$ gravity dominates inflation, and also at late times, the term $R^3$ dominates the evolution. However as we show in the next section, the non-minimally coupled axion model predicts also a stiff matter era preceding the inflationary era, which is not predicted from the model (39).

Before closing, some interesting questions must be discussed. Firstly, the models (3) and (39) produce qualitatively similar results for the dark energy era, and specifically for the late-time de Sitter solution, but we did not quote these for brevity. However, for the model (39) the pre-inflationary stiff era for the axion would absent. Secondly and important question is what are the predictions of the non-minimally coupled model (3) at a perturbative level, since all the study was performed at a background level, so what would be the effects of the non-minimal coupling at a perturbative level, and in effect, what would be the effects of this axion dark matter coupling to dark energy for structure formation? This question is quite interesting, though non-trivial to address directly in this paper. The perturbations for the combined theory for eras corresponding to cosmic times after inflation and before the dark energy era, mainly correspond to an $f(\phi, R)$ theory and were calculated in detail in [42], so in principle one can have a concrete picture on how the perturbations at intermediate eras evolve exactly. However, since during inflation the $R^2$ gravity dominates, the perturbative modes during the horizon exit, evolve in a similar way as in the pure $R^2$ model. An exact calculation in the context of $f(\phi, R)$ performed in [12], reveal the exact behavior of the perturbations evolution. Moreover, after inflation, the reheating picture changes drastically, so one must know the exact behavior of the $f(\phi, R)$ theory, in order to correctly study the reheating era and the subsequent matter formation era, this is not easy trivial to do analytically though. There is also another feature to be pointed out. The axion may be overdamped during inflation, but it actually acquires perturbations during inflation, so it may directly affect the primordial curvature perturbation modes, and hence affect their evolution, after horizon exit, as we already mentioned above. Although the effect are moderate, in an exact theory, these should be take into account. The theory we studied though is an effective theory, and we did not take these effects into account. Also after inflation, the axion energy density behaves as $\rho_\phi \sim a^{-3}$, but this is an average behavior for the axion field, and the average value may vary randomly within the observable Universe, as the assumed location of the observable Universe is altered in a comoving volume. Thus, locally the axion physics may alter, when for example intergalactic scales are considered. The physics of this discussion is highly non-trivial to even discuss in the context of the effective model we discussed, however these are noteworthy questions that should be addressed appropriately in future works.

D. A Prediction of the Theory: Stiff Pre-inflation Era

Up to now we assumed that the condition of Eq. (11) or equivalently Eq. (12) was holding true for cosmic times during and after the inflationary era. It is though conceivable that at some cosmic time before inflation, the potential could take smaller values in comparison to the value it takes when the axion field takes its vacuum expectation value corresponding to the broken Peccei-Quinn symmetry. In effect, at some pre-inflationary primordial era we could have $2V(\phi) \sim \frac{1}{\kappa^2} h(\phi) G(R)$. This can be indeed true if the $U(1)$ breaking occurs during a second order phase transition in the axion field, so that the potential continuously deforms to the value $\sim \frac{1}{2} m^2_\phi \phi^2$. When the condition $2V(\phi) \sim \frac{1}{\kappa^2} h(\phi) G(R)$ holds true, the differential equation (6) that governs the evolution of the scalar field becomes,

$$\ddot{\phi} + 3H \dot{\phi} = 0, \quad (40)$$

so by solving this we get,

$$\dot{\phi} \sim a^{-3}, \quad (41)$$

where $a$ is the scale factor, and in effect we have $\dot{\phi}^2 \sim a^{-6}$. Due to the condition $2V(\phi) \sim \frac{1}{\kappa^2} h(\phi) G(R)$, the energy density of the axion scalar field at this primordial era is approximately $\rho_\phi \sim \dot{\phi}^2$, so in view of the solution (41) we found, we have $\rho_\phi \sim a^{-6}$. This behavior is characteristic of a stiff matter fluid and thus the model predicts a stiff matter era before the inflationary era. It is intriguing to note that Zel’dovich had introduced the hypothesis that a stiff matter era occurred in the primordial Universe [43], however in the Zel’dovich the matter fluid consisted of a stiff cold baryon gas, whereas in our case the stiff matter fluid is the axion itself. Hence, the presence of the non-minimal coupling apart from affecting the late-time era, also affects the pre-inflationary primordial era, allowing for a stiff matter era to be realized.
IV. CONCLUDING REMARKS

The possibility of detecting the axion field in the next decades apart from exciting, is perhaps the last resort of dark matter particle physics. This is due to the fact that the experiments searching for large mass weakly interacting massive particles seem not to produce any hint for the existence of a dark matter particle, so unless some evidence for supersymmetry is found, one has a last possibility for detecting dark matter particles having extremely low masses. The misalignment axion model seems to be an appealing candidate, and in this paper we provided an effective theory that may unify the inflationary era with the dark energy era, and finally describe a dark matter dominated era. Particularly, we assumed that the $f(R)$ gravity is described by the $R^2$ model at early times and also contains a non-minimal coupling to the axion scalar field of the form $h(\phi)R^2$. Due to the fact that the axion field remains frozen in its primordial vacuum expectation value, the Starobinsky model controls the early-time dynamics, providing a viable quasi-de Sitter evolution. As the Universe expands, the axion field starts to oscillate, approximately when $H \sim m_a$. By assuming a slowly-varying evolution, the axion field energy density scales as $\rho_a \sim a^{-3}$, which describes a dark matter effective fluid, regardless what the actual Hubble rate of the background is. At late times, the non-minimal coupling dominates the evolution, and the resulting Hubble rate at late times is nearly a de Sitter one, thus describing an accelerating Universe. Also the theory predicts the existence of a stiff matter era for the axion field at a primordial pre-inflationary era, in which case the energy density scales as $\rho_a \sim a^{-6}$.

It is also known that in the context of the singular inflation scenario, one can unify the early-time acceleration with late-time acceleration in $f(R)$ gravity \cite{44}. Hence, it is natural to discuss the unification of singular-inflation with dark energy, including above the axion dark matter and non-minimal coupling terms. In the same way one can take into account logarithmic quantum gravity corrections in order to unify inflation with dark energy, as it was done without the axion dark matter field in Ref. \cite{43}. The inclusion of the axion dark matter field may strengthen the viability of these theories, providing a dark matter era and a smooth transition from the dark matter era to the dark energy era.

An interesting issue which we did not discuss, is the radiation domination era and specifically the reheating issue in the dawn of this era. Particularly, the axion oscillations may amplify the curvature oscillations caused by the $R^2$ gravity, thus altering the reheating temperature. Work is in progress towards this research line.

Acknowledgments

This work is supported by MINECO (Spain), FIS2016-76363-P, by project 2017 SGR247 (AGAUR, Catalonia) (S.D.O) and by Russian Ministry of Science and High Education, project No. 3.1386.2017.

\begin{thebibliography}{99}
\bibitem{} G. Aad \textit{et al.} [ATLAS Collaboration], Phys. Lett. B \textbf{716} (2012) 1 doi:10.1016/j.physletb.2012.08.020 \texttt{arXiv:1207.7213 [hep-ex]].}
\bibitem{} A. H. Guth, Phys. Rev. D \textbf{23} (1981) 347. doi:10.1103/PhysRevD.23.347
\bibitem{} A. D. Linde, Phys. Lett. \textbf{129B} (1983) 177. doi:10.1016/0370-2693(83)90837-7
\bibitem{} A. A. Starobinsky, Phys. Lett. \textbf{91B} (1980) 99. doi:10.1016/0370-2693(80)90670-X
\bibitem{} S. Capozziello, Int. J. Mod. Phys. D \textbf{11} (2002) 483 doi:10.1142/S0218271802002025 \texttt{[gr-qc/0210033].}
\bibitem{} S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D \textbf{70} (2004) 043528 doi:10.1103/PhysRevD.70.043528 \texttt{[astro-ph/0306438].}
\bibitem{} S. Nojiri and S. D. Odintsov, Phys. Rev. D \textbf{68} (2003) 123512 doi:10.1103/PhysRevD.68.123512 \texttt{[hep-th/0307288].}
\bibitem{} S. Nojiri and S. D. Odintsov, Phys. Lett. B \textbf{657} (2007) 238 doi:10.1016/j.physletb.2007.10.027 \texttt{[arXiv:0707.1941 [hep-th]].}
\bibitem{} S. Nojiri and S. D. Odintsov, Phys. Rev. D \textbf{77} (2008) 026007 doi:10.1103/PhysRevD.77.026007 \texttt{[arXiv:0710.1738 [hep-th]].}
\bibitem{} G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D \textbf{77} (2008) 046009 doi:10.1103/PhysRevD.77.046009 \texttt{[arXiv:0712.4017 [hep-th]].}
\bibitem{} S. Nojiri and S. D. Odintsov, Phys. Rev. D \textbf{74} (2006) 086005 doi:10.1103/PhysRevD.74.086005 \texttt{[hep-th/0608008].}
\bibitem{} S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rept. \textbf{592} (2017) 1 doi:10.1016/j.physrep.2017.06.001 \texttt{[arXiv:1705.11098 [gr-qc]].}
\bibitem{} S. Nojiri, S. D. Odintsov, Phys. Rept. \textbf{505}, 59 (2011);
\bibitem{} S. Nojiri, S. D. Odintsov, eConf \textbf{C0602061}, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007)].
\bibitem{} S. Capozziello, M. De Laurentis, Phys. Rept. \textbf{509}, 167 (2011); V. Faraoni and S. Capozziello, Fundam. Theor. Phys. \textbf{170} (2010). doi:10.1007/978-94-007-0165-6
\bibitem{} A. de la Cruz-Dombriz and D. Saez-Gomez, Entropy \textbf{14} (2012) 1717 doi:10.3390/e14091717 \texttt{[arXiv:1207.2633 [gr-qc]].}
\bibitem{} G. J. Olmo, Int. J. Mod. Phys. D \textbf{20} (2011) 413 doi:10.1142/S0218271811018925 \texttt{[arXiv:1101.3884 [gr-qc]].}
\end{thebibliography}
