QUANTUM GROUP ACTIONS ON RINGS AND 
EQUIVARIANT K-THEORY

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Abstract. Let $U_q(g)$ be a quantum group. Regarding a (noncommutative) space 
with $U_q(g)$-symmetry as a $U_q(g)$-module algebra $A$, we may think of equivariant 
vector bundles on $A$ as projective $A$-modules with compatible $U_q(g)$-action. We 
construct an equivariant $K$-theory of such quantum vector bundles using Quillen’s 
equivalents categories, and provide means for its computation. The equivariant $K$-groups 
of quantum homogeneous spaces and quantum symmetric algebras of classical type 
are computed.

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1. INTRODUCTION

In recent years there has been much work exploring various types of noncommutative 
geometries with possible applications in physics. The best developed theory 
is Connes’ noncommutative differential geometry \cite{Connes} formulated within the framework of $C^*$-algebras, which incorporates $K$-theory and cyclic cohomology and has

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yielded new index theorems. Various aspects of noncommutative algebraic geometry have also been developed (see, e.g., [3, 29]).

Noncommutative generalisations of classical geometries are based on the strategy of regarding a space as defined by an algebra of functions, which is commutative in the classical case. In noncommutative geometry [10] one replaces this commutative algebra by noncommutative algebra; in analogy with the classical case, vector bundles are regarded as finitely generated projective modules over this algebra. One may then investigate problems with geometric origins by means of these algebraic structures. This permits cross fertilisation of algebraic and geometric ideas, and is expected to lead to mathematical advances in both areas. An important motivation from physics for studying noncommutative geometry is the notion that spacetime at the Planck scale becomes noncommutative [13]; thus noncommutative geometry may be a necessary ingredient for a consistent theory of quantum gravity.

Much of classical algebraic and differential geometry concerns algebraic varieties and manifolds with algebraic or Lie group actions. Correspondingly, in noncommutative geometry one studies noncommutative algebras with Hopf algebra actions. Natural examples of such noncommutative geometries are quantum analogues [15, 33, 34, 19] of homogeneous spaces and homogeneous vector bundles [7], which have proved useful for formulating the quantum group version [11] of the Bott-Borel-Weil theorem [7] into a noncommutative geometric setting [15, 33]. Some quantum homogeneous spaces such as quantum spheres have been particular objects of attention [11, 20, 19] because of potential physical applications. We note that interesting examples of Hopf algebras acting on noncommutative algebras are noncocommutative, and thus do not correspond to groups.

In this paper we study noncommutative geometries with quantum group symmetries. In particular, we shall study an equivariant algebraic K-theory of such spaces which is a generalisation of the equivariant algebraic K-theory of reductive group actions investigated by Bass and Haboush in [4, 5]. The equivariant topological K-theory of Lie group actions and algebraic group scheme actions have been developed in the celebrated papers [28, 12, 30]. Following the work of Bass and Haboush, several authors have addressed geometric themes in the context of the equivariant K-theory of algebraic vector bundles [6, 21, 22]. A recent treatment of the K-theory of compact Lie group actions in relation to representation theory may be found in [17, §12].

The need for an equivariant K-theory in the noncommutative setting was already clear in the classification of quantum homogeneous bundles [34]. Very recently, Nest and Voigt have extended the notion of Poincare duality in K-theory to the setting of compact quantum group actions [24] within the framework of $C^*$-algebras.

In our situation, the two crucial notions which are needed are those of an ‘equivariant noncommutative space’, which we shall take to mean a module algebra $A$ over a Hopf algebra [23], and an ‘equivariant vector bundle on $A$’ which we shall take to mean an equivariant module over the module algebra $A$. Specifically, a module algebra is an associative algebra that is also a module for the Hopf algebra, whose algebraic structure is preserved by the action; equivariant modules over module algebras were introduced in [34] in the context of quantum homogeneous spaces, and we generalise it here to arbitrary module algebras.
Let $U = U_q(g)$ be a quantum group defined over the field $\mathbb{k} = \mathbb{C}(q)$, and let $A$ be a module algebra over $U$. We introduce the category $\mathcal{M}(A, U)$ of $U$-equivariant $A$-modules which are finitely $A$-generated and locally $U$-finite. The full subcategory $\mathcal{P}(A, U)$ of $\mathcal{M}(A, U)$ consisting of finitely $A$-generated, locally $U$-finite, projective equivariant modules is an exact category in a natural way. Thus Quillen’s $K$-theory of exact categories applies to $\mathcal{P}(A, U)$, giving rise to an algebraic $K$-theory of module algebras which is equivariant under the action of the quantum group $U$ (Definition 2.4).

Properties of equivariant $K$-theory are developed for module algebras with filtrations which are stable under quantum group actions. Under a regularity assumption for the module algebras, we establish a relationship between the $K$-groups of the filtered algebras and the degree zero subalgebras of the corresponding graded algebras (see Theorem 4.3 for details). This may be regarded as an equivariant analogue of [27, §6, Theorem 7] on the higher algebraic $K$-theory of filtered rings.

We apply Theorem 4.3 to compute the equivariant $K$-groups for a class of module algebras over quantum groups, which we call quantum symmetric algebras. The results are summarised in Theorem 5.3. In establishing the regularity of the left Noetherian quantum symmetric algebras, some elements of the theory of Koszul algebras [26] are used, which are discussed in Section 5.2.

One of the main motivations of [5] is to prove the results [5, Theorems 1.1, 2.3, Cor. 2.4], which would be easy consequences of the Serre conjecture if one assumed that all reductive group actions on affine space are linearisable. Thus a natural question which arises from our work is whether there is a non-commutative version of the Serre conjecture for the quantum symmetric algebras we consider. In [2], the case of the natural representation of $U_q(gl_n)$ is discussed.

We also study the equivariant $K$-theory of quantum homogeneous spaces in detail. Given a quantum homogeneous space of a quantum group $U_q(g)$ which corresponds to a reductive quantum subgroup $U_q(l)$, we show that the equivariant $K$-groups are isomorphic to the $K$-groups of the exact category whose objects are the $U_q(l)$-modules (see Corollary 6.5 for the precise statement).

Properties of the categories $\mathcal{M}(A, U)$ and $\mathcal{P}(A, U)$, analogous to those of their classical analogues are established, and used in the study of equivariant $K$-theory. We prove a splitting lemma (Proposition 3.2), which enables us to characterise the finitely $A$ generated, locally $U$-finite, projective $U$-equivariant $A$-modules (Corollary 3.4). A similar result in the commutative setting was proved by Bass and Haboush [4, 5] for reductive algebraic group actions.

The equivariant algebraic $K$-theory constructed here generalises, in a completely straightforward manner, to Hopf algebras whose locally finite modules are semi-simple, e.g., universal enveloping algebras of finite dimensional semi-simple Lie algebras. In fact, the case of such universal enveloping algebras essentially covers the Bass-Haboush theory for semi-simple algebraic group actions when the module algebras are commutative. We also point out that the equivariant algebraic $K$-theory of a $U$-module algebra $A$ developed here is different from the usual algebraic $K$-theory of the smash product algebra $R := A \# U$, see Remark 2.6.

The organisation of the paper is as follows. In Section 2, we introduce various categories of equivariant modules for module algebras over quantum groups, and define the equivariant algebraic $K$-theory of quantum group actions. In Section 3,
the theory of equivariant modules is developed, and is used to study quantum group equivariant K-theory. In Section 4 we develop the equivariant K-theory of filtered module algebras, and in the remaining two sections we study concrete examples. In Section 5 we compute the equivariant K-groups of the quantum symmetric algebras, and in Section 6 we investigate in detail the equivariant K-groups of quantum homogeneous spaces.

2. EQUIVARIANT K-THEORY OF QUANTUM GROUP ACTIONS

The purpose of this section is to introduce an equivariant algebraic K-theory of quantum group actions. This theory generalises, in a straightforward way to arbitrary Hopf algebras.

2.1. Module algebras and equivariant modules. For any finite dimensional simple complex Lie algebra \( g \), denote by \( U := U_q(g) \) the quantum group defined over the field \( \mathbb{C}(q) \) of rational functions in \( q \); \( U \) has a standard presentation with generators \( \{ e_i, f_i, k_i^{\pm 1} \mid i = 1, \ldots, r \} \) (\( r = \text{rank}(g) \)), and relations which may be found e.g., in \([1]\). If \( g \) is a semi-simple Lie algebra, \( U \) will denote the tensor product of the quantum groups in the above sense, associated with the simple factors.

It is well known that \( U \) has the structure of a Hopf algebra; denote its co-multiplication by \( \Delta \), co-unit by \( \epsilon \) and antipode by \( S \). We shall use Sweedler’s notation for co-multiplication: given any \( x \in U \), write \( \Delta(x) = \sum (x) x(1) \otimes x(2) \). The following relations are among those satisfied by any Hopf algebra:

\[
\sum_{(x)} \epsilon(x(1)) x(2) = \sum_{(x)} x(1) \epsilon(x(2)) = x, \\
\sum_{(x)} S(x(1)) x(2) = \sum_{(x)} x(1) S(x(2)) = \epsilon(x).
\]

Let \( \Delta' \) be the opposite co-multiplication, defined by \( \Delta'(x) = \sum_{(x)} x(2) \otimes x(1) \) for any \( x \in U \).

Denote by \( \text{U-mod} \) the category of finite dimensional left \( U \)-modules of type-(1, \ldots, 1). Then \( \text{U-mod} \) is a semi-simple braided tensor category. A (left) \( U \)-module \( V \) is called locally finite if for any \( v \in V \), the cyclic submodule \( Uv \) generated by \( v \) is finite dimensional. We shall make use of the important fact that locally finite modules are semi-simple. We shall say that a locally finite \( U \)-module is type-(1, \ldots, 1) if all its finite dimensional submodules are type-(1, \ldots, 1).

An associative algebra \( A \) with identity 1 is a (left) module algebra over \( U \) if it is a left \( U \)-module, and the multiplication \( A \otimes_k A \to A \) and unit map \( k \to A \) are \( U \)-module homomorphisms. Explicitly, if we write the \( U \)-action on \( A \) as \( U \otimes_k A \to A \), \( x \otimes a \mapsto x \cdot a \), for all \( a \in A \) and \( x \in U \), then

\[
x \cdot (ab) = \sum_{(x)} (x(1) \cdot a)(x(2) \cdot b), \quad x \cdot 1 = \epsilon(x) 1.
\]

We call a \( U \)-module algebra \( A \) locally finite if it is locally finite as a \( U \)-module. If all its submodules are in \( \text{U-mod} \), we say that the locally finite \( U \)-module algebra is type-(1, \ldots, 1).
An element $a \in A$ is $U$-invariant if $x \cdot a = \epsilon(x)a$ for all $x \in U$. We denote by $A^U$ the submodule of $U$-invariants of $A$, that is,

$$A^U := \{ a \in A | x \cdot a = \epsilon(x)a, \quad \forall x \in U \}.$$

The fact that $U$ is a Hopf algebra implies that this is a subalgebra of $A$. Indeed, for all $a, b \in A^U$, we have

$$x \cdot (ab) = \sum_{(x)} (x_1) \cdot a (x_2) \cdot b = \sum_{(x)} \epsilon(x_1) \epsilon(x_2) ab = \epsilon(x)ab.$$

Hence $ab \in A^U$. We shall refer to $A^U$ as the subalgebra of $U$-invariants of $A$.

Let $M$ be a left $A$-module with structure map $\phi : A \otimes M \to M$, and also a locally finite left $U$-module with structure map $\mu : U \otimes M \to M$. Then $A \otimes_k M$ has a natural $U$-module structure

$$\mu' : U \otimes (A \otimes M) \to A \otimes M,$$

$$x \otimes (a \otimes m) \mapsto \sum_{(x)} x_1 \cdot a \otimes x_2 m.$$

The $A$-module and $U$-module structures of $M$ are said to be compatible if the following diagram commutes

$$\begin{array}{ccc}
U \otimes (A \otimes M) & \xrightarrow{id \otimes \phi} & U \otimes M \\
\mu' \downarrow & & \mu \downarrow \\
A \otimes M & \xrightarrow{\phi} & M.
\end{array}$$

In this case, $M$ is called a $U$-equivariant left $A$-module, or $A$-$U$-module for simplicity. A morphism between two $A$-$U$-modules is an $A$-module map which is at the same time also $U$-linear. We denote by $\text{Hom}_{A-U}(M, N)$ the space of $A$-$U$-morphisms.

Denote by $A$-$U$-$\text{mod}$ the category of locally $U$-finite $A$-$U$-modules (i.e., locally $U$-finite $U$-equivariant left $A$-modules), which as $U$-modules are of type-$(1, \ldots, 1)$. It is clear that $A$-$U$-$\text{mod}$ is an abelian category. Let $\mathcal{M}(A, U)$ be the full subcategory of $A$-$U$-modules consisting of finitely $A$-generated objects, and denote by $\mathcal{P}(A, U)$ the full subcategory of $\mathcal{M}(A, U)$ whose objects are the projective objects in $A$-$U$-$\text{mod}$.

**Remark 2.1.**

1. In this work, $U$ shall generally denote $U = U_q(\mathfrak{g})$, where $\mathfrak{g}$ is a reductive Lie algebra. In particular, it may happen that the root lattice has smaller rank than the weight lattice. We shall consider locally finite $U$-modules and locally finite $U$-module algebras which are of type-$(1, \ldots, 1)$ for the chosen dominant weights. In particular, the categories of finite dimensional modules considered are semisimple.

2. The categories just introduced are quantum analogues of those occurring in [5, Theorem 2.3].

We also define a $U$-equivariant right $A$-module $M$, as a left $U$-module which is also a right $A$-module, such that the module structures are compatible in the sense that

$$x(ma) = \sum_{(x)} (x_1)m (x_2) \cdot a$$
for all \( x \in U, a \in A \) and \( m \in M \). Similarly, we also define a \( U \)-equivariant \( A \)-\( bimodule \( M \), as a left \( U \)-module which is also an \( A \)-\( bimodule \), such that

\[
x(amb) = \sum_{(x)} (x(1) \cdot a)(x(2)m(x(3)) \cdot b)
\]

for all \( x \in U, a, b \in A \) and \( m \in M \).

Let \( R \) be a \( U \)-equivariant right \( A \)-module, and let \( B \) be a \( U \)-equivariant \( A \)-\( bimodule \). For any \( U \)-equivariant left \( A \)-module \( M \), \( R \otimes A M \) has the structure of left \( U \)-module, and \( B \otimes A M \) has the structure of \( U \)-equivariant left \( A \)-module, with the module structures define in the following way. For all \( r \in R, b \in B, a \in A \) and \( m \in M \),

\[
x(r \otimes_A m) = \sum_{(x)} x(1)r \otimes_A x(1)m,
\]

\[
x(b \otimes_A m) = \sum_{(x)} x(1)b \otimes_A x(1)m,
\]

\[
a(b \otimes_A m) = ab \otimes_A m.
\]

The following result is now clear.

**Lemma 2.2.** Let \( A \) be a locally finite \( U \)-module algebra. Let \( R \) be a \( U \)-equivariant right \( A \)-module, and let \( B \) be a \( U \)-equivariant \( A \)-\( bimodule \). Assume that both \( R \) and \( B \) are locally \( U \)-finite, then we have covariant functors

\[
R \otimes_A - : A-\text{U-mod} \to \text{U-Mod}_{l.f.},
\]

\[
B \otimes_A - : A-\text{U-mod} \to A-\text{U-mod},
\]

where \( \text{U-Mod}_{l.f.} \) is the category of locally finite \( U \)-modules. Furthermore, if \( B \) is also finitely \( A \)-generated as a left \( A \)-module, then we have the covariant functor

\[
B \otimes_A - : \mathcal{M}(A, U) \to \mathcal{M}(A, U).
\]

In this work, the term ‘module’ will mean left module unless otherwise stated.

### 2.2. Equivariant K-theory of quantum group actions

Recall that an exact category \( P \) is an additive category with a class \( E \) of short exact sequences which satisfies a series of axioms, see [27, p.99] or [16, Appendix A]. For our purposes, we may think of an exact category \( P \) as a full (additive) subcategory of an abelian category \( \mathcal{A} \) which is closed under extensions in \( \mathcal{A} \), that is, for any short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) in \( \mathcal{A} \), if \( M' \) and \( M'' \) are in \( P \), then \( M \) also belongs to \( P \). Typical examples of exact categories are

1. any abelian category with exact structure given by all short exact sequences, and
2. the full subcategory of finitely generated projects (left) modules of the category of (left) modules over a ring.

For any exact category \( P \) in which the isomorphism classes of objects form a set, one may define the Quillen category \( 
\text{Q}P \). Quillen’s algebraic K-groups [27] of the exact category \( P \) are defined to be the homotopy groups of the classifying space \( B(\text{Q}P) \) of \( \text{Q}P \):

\[
K_i(P) = \pi_{i+1}(B(\text{Q}P)), \quad i = 0, 1, \ldots.
\]
If $F : \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is an exact functor between exact categories, it induces a functor $QF : Q\mathcal{P}_1 \rightarrow Q\mathcal{P}_2$ between the corresponding Quillen categories. This functor then induces a cellular map $BQF : B(Q\mathcal{P}_1) \rightarrow B(Q\mathcal{P}_2)$, which in turn leads to the homomorphisms

$$F_* : K_i(\mathcal{P}_1) \rightarrow K_i(\mathcal{P}_2), \quad \text{for all } i.$$ 

We now turn to the definition of an equivariant algebraic K-theory of quantum group actions. The following fact is immediate from the definition of $\mathcal{P}(A, U)$.

**Theorem 2.3.** Let $A$ be a locally finite module algebra over the quantum group $U$. Then the category $\mathcal{P}(A, U)$ of finitely $A$-generated, locally $U$-finite, projective $U$-equivariant $A$-modules is an exact category.

The Quillen K-groups $K_i(\mathcal{P}(A, U))$ are therefore defined for $\mathcal{P}(A, U)$, and the following definition makes sense.

**Definition 2.4.** Let $A$ be a locally finite module algebra over the quantum group $U$. The $U$-equivariant algebraic K-groups of $A$ are defined by

$$K^U_i(A) := K_i(\mathcal{P}(A, U)), \quad i = 0, 1, \ldots.$$ 

It follows from standard facts [27, Theorem 1, p.102] that the fundamental group of $B(\mathcal{P}(A, U))$ is isomorphic to the Grothendieck group of $\mathcal{P}(A, U)$. Hence the $U$-equivariant K-group $K^U_0(A)$ is isomorphic to the Grothendieck group of $\mathcal{P}(A, U)$.

**Remark 2.5.** The $U$-equivariant algebraic K-groups $K^U_i(A)$ of $A$ are a generalisation to the quantum group setting of the equivariant algebraic K-groups of reductive algebraic group actions studied in [4, 5].

**Remark 2.6.** One may also consider the usual algebraic K-theory of the smash product $A\#U$ (see Appendix A for the definition of a smash product). This, however, is completely different from the equivariant K-theory of the $U$-module algebra $A$ introduced here. See Appendix A for more details.

### 3. Categories of equivariant modules

To study the equivariant K-groups introduced in the last section, we require some properties of various categories of equivariant modules. Fix a quantum group $U$ and a locally finite module algebra $A$ over $U$. As we have already declared in Remark 2.1 all locally finite $U$-modules and locally finite $U$-module algebras considered are assumed to be type-$(1, \ldots, 1)$.

**Lemma 3.1.** Let $M$ and $N$ be $A$-$U$-modules. Then there is a natural $U$-action on $\text{Hom}_A(M, N)$ defined for any $x \in U$ and $f \in \text{Hom}_A(M, N)$ by

$$ (xf)(m) = \sum_{(x)} x(2) f(S^{-1}(x(1))m), \quad \forall m \in M. $$ (3.1)
Proof. We first show that (3.1) defines a $U$-module structure on $\text{Hom}_k(M, N)$. For any $x, y \in U$ and $f \in \text{Hom}_k(M, N)$, we have

$$(y(xf))(m) = \sum_{(y)} y(2)(xf)(S^{-1}(y(1))m)$$

$$= \sum_{(y), (x)} y(2)x(2)f(S^{-1}(x(1))S^{-1}(y(1))m)$$

for all $m \in M$. Using the facts that for all $x, y \in U$, $S(yx) = S(x)S(y)$ and $\Delta(yx) = \sum_{(x), (y)} y(2)x(2) \otimes y(1)x(1)$, we can cast the far right hand side into

$$\sum_{(yx)} (yx)(2)f(S^{-1}((yx)(1))m) = ((yx)f)(m)$$

Thus $\text{Hom}_k(M, N)$ is a $U$-module.

Next we show that $\text{Hom}_A(M, N)$ is a $U$-submodule of $\text{Hom}_k(M, N)$. Let $f \in \text{Hom}_A(M, N)$, $a \in A$ and $x \in U$. Then for all $m \in M$, we have

$$(xf)(am) = \sum_{(x)} x(2)f(S^{-1}(x(1))(am))$$

$$= \sum_{(x)} x(3)f((S^{-1}(x(2)) \cdot a)S^{-1}(x(1))m)$$

$$= \sum_{(x)} x(3)((S^{-1}(x(2)) \cdot a)f(S^{-1}(x(1))m),$$

where the last step used the fact that $f$ is $A$-linear. The far right hand side can be rewritten as $\sum_{(x)} (x(3) \cdot (S^{-1}(x(2)) \cdot a))x(4)f(S^{-1}(x(1))m)$. By using the defining property of the antipode, we can further simplify it to obtain

$$\sum_{(x)} \epsilon(x(2))a(x(3)f(S^{-1}(x(1))m))$$

$$= \sum_{(x)} a(x(2)f(S^{-1}(x(1))m))$$

$$= a(xf)(m).$$

Hence $xf \in \text{Hom}_A(M, N)$, as required. \hfill \Box

For any $M, N$ in $\mathcal{M}(A, U)$, the $U$-action on $\text{Hom}_A(M, N)$ defined in Lemma 3.1 is semi-simple, and $\text{Hom}_A(M, N)$ belongs to $\mathcal{M}(A, U)$. To see this, we note that there exists a finite dimensional $U$-module $V$ which generates $M$ over $A$. Thus $\text{Hom}_A(M, N)$ is isomorphic to a submodule of $V^* \otimes_k N$ as a $U$-module, which is obviously locally finite and thus semi-simple over $U$.

**Proposition 3.2.** (Splitting lemma) Consider a short exact sequence

$$0 \rightarrow M' \rightarrow M \xrightarrow{p} M'' \rightarrow 0$$

in $A$-$U$-$\text{mod}$, where $M''$ is an object of $\mathcal{M}(A, U)$. If the exact sequence is $A$-split, then it is also split as an exact sequence of $A$-$U$-modules.
Proof. Since the above sequence is $A$-split, there is an $A$-module isomorphism
\[ M \overset{\sim}{\longrightarrow} M' \oplus M''. \]
Therefore $\text{Hom}_A(M'', M) \overset{p_0}{\longrightarrow} \text{Hom}_A(M'', M'') \longrightarrow 0$ is exact. This is an exact sequence of $U$-modules as the map $p \circ -$ is clearly $U$-linear. Since $M''$ is an object of $\mathcal{M}(A, U)$ and $M$ is locally $U$-finite, both hom-spaces are semi-simple $U$-modules. Hence we have the exact sequence of $U$-invariants $\text{Hom}_A(M'', M) U \overset{p_0}{\longrightarrow} \text{Hom}_A(M'', M'') U \longrightarrow 0$. Note that for any $A$-$U$-modules $N$ and $N'$, an element $f$ of $\text{Hom}_A(N, N')$ belongs to $\text{Hom}_{A-U}(N, N')$ if and only if $xf = \epsilon(x)f$ for all $x \in U$. Therefore, we have the exact sequence
\[ \text{Hom}_{A-U}(M'', M) \overset{p_0}{\longrightarrow} \text{Hom}_{A-U}(M'', M'') \longrightarrow 0. \]
Now any element of the pre-image of $\text{id}_{M''}$ splits the exact sequence of $A$-$U$-modules. $\square$

Given a finite dimensional $U$-module $V$, we define the free $A$-module $V_A = A \otimes_k V$ with the obvious $A$-action (given by left multiplication). We also define a $U$-action on it by
\[ x(a \otimes v) = \sum_{(x)} x_{(1)} \cdot a \otimes x_{(2)} v \]
for all $a \in A$, $v \in V$ and $x \in U$. These two actions are easily be shown to be compatible. Call $V_A$ a free $A$-$U$-module of finite rank. Since the module algebra $A$ is locally $U$-finite, $V_A$ is also locally $U$-finite, and hence belongs to $\mathcal{M}(A, U)$.

Lemma 3.3. For each object $M$ of $\mathcal{M}(A, U)$, there exists an exact sequence $V_A \longrightarrow M \longrightarrow 0$ in $\mathcal{M}(A, U)$, where $V_A$ is a free $A$-$U$-module of finite rank.

Proof. Given an object $M$ of $\mathcal{M}(A, U)$, we may choose any finite set of generators for it as an $A$-module, and consider the $U$-module $V$ generated by this set. Then $V$ is finite dimensional because of the local $U$-finiteness of $M$. We have the obvious surjective $A$-$U$-module map $V_A \longrightarrow M$, $a \otimes v \mapsto av$. Since $A$ is locally $U$-finite, a free $A$-$U$-module of finite rank is locally $U$-finite, thus the exact sequence $V_A \longrightarrow M \longrightarrow 0$ is in $\mathcal{M}(A, U)$. $\square$

As a corollary of Proposition 3.2, we have the following result.

Corollary 3.4. Let $A$ be a locally finite $U$-module algebra. For any object $P$ of $\mathcal{M}(A, U)$, the following conditions are equivalent:

1. $P$ is projective as an $A$-module;
2. $P$ is a projective object of $A$-$U$-$\text{mod}$;
3. $P$ is a direct summand of some free $A$-$U$-module $V_A = A \otimes_k V$ of finite rank.

Proof. Assume that $P$ is $A$-projective. Then given any exact sequence $M \rightarrow N \rightarrow 0$ in $A$-$U$-$\text{mod}$, we have the exact sequence
\[ \text{Hom}_A(P, M) \rightarrow \text{Hom}_A(P, N) \rightarrow 0 \]
of $U$-modules. Since both $\text{Hom}_A(P, M)$ and $\text{Hom}_A(P, N)$ are semi-simple as $U$-modules, this leads to the following exact sequence of $k$-vector spaces
\[ \text{Hom}_{A-U}(P, M) \rightarrow \text{Hom}_{A-U}(P, N) \rightarrow 0. \]
This proves that (1) implies (2).

Now assume that (2) holds. By Lemma 3.3, there exists a free $A$-$U$-module $V_A = A \otimes_k V$ of finite rank and an exact sequence $V_A \xrightarrow{p} P \rightarrow 0$ of $A$-$U$-modules. It follows from (2) that the identity map of $P$ factors through $p$. Hence the exact sequence splits in $\mathcal{M}(A, U)$. This proves that (2) implies (3).

It is evident that (3) implies (1), and the proposition follows. □

Recall that an algebra $A$ is called left regular if it is left Noetherian and every finitely generated left $A$-module has a finite resolution by finitely generated projective left $A$-modules.

**Proposition 3.5.** Let $A$ be a locally finite module algebra over $U$. Assume that $A$ is left regular. Then every object $M$ in $\mathcal{M}(A, U)$ admits a finite $P(A, U)$-resolution.

**Proof.** Let $M$ be an object in $\mathcal{M}(A, U)$. By Lemma 3.3, there exists an exact sequence $V_0, A \rightarrow M \rightarrow 0$ in $\mathcal{M}(A, U)$, where $V_0, A = A \otimes V_0$ is a free $A$-$U$-module of finite rank. Since $Ker(p_0)$ also belongs to $\mathcal{M}(A, U)$ because $A$ is left Noetherian, we may apply the same considerations to it, and inductively we obtain an $A$-free resolution $\cdots \rightarrow V_{d-1, A} \rightarrow V_{d-2, A} \rightarrow M \rightarrow 0$ in $\mathcal{M}(A, U)$ for $M$. Let $d$ be the projective dimension of $M$, which is finite because $A$ is regular. It follows from standard facts in homological algebra (see, e.g., [32, Lemma 4.1.6]) that the kernel $P$ of the map $V_{d-1, A} \rightarrow V_{d-2, A}$ is $A$-projective, hence belongs to $P(A, U)$ by Corollary 3.4. Thus we arrive at the $P(A, U)$-resolution

$$0 \rightarrow P \rightarrow V_{d-1, A} \rightarrow \cdots \rightarrow V_{1, A} \rightarrow V_{0, A} \rightarrow M \rightarrow 0.$$ 

This completes the proof of the proposition. □

For left regular module algebra, we have the following result, which is an analogue of [5, Theorem 2.3].

**Proposition 3.6.** Assume that the locally finite $U$-module algebra $A$ is left regular. Then there exist the isomorphisms

$$K_i^U(A) \cong K_i(\mathcal{M}(A, U)), \quad i = 0, 1, 2, \ldots .$$

**Proof.** Since $A$ is left regular, it must be left Noetherian. Thus $\mathcal{M}(A, U)$ is an abelian category, which has the natural exact structure consisting of all the short exact sequences. In view of Proposition 3.5, the embedding $P(A, U) \subset \mathcal{M}(A, U)$ satisfies the conditions of Quillen’s Resolution Theorem [31, Theorem 4.6]. The statement follows. □

4. **Equivariant K-theory of filtered module algebras**

In this section we develop properties of the equivariant $K$-theory of filtered module algebras over quantum groups. The main results here are Theorem 4.2 and Theorem 4.3 which are quantum analogues of [5, Theorem 3.2, Theorem 4.1]. The proofs of these theorems are adapted from [5, §3, §4] and [27, §6].
4.1. Graded module algebras. Let $S = \oplus_{n=0}^{\infty} S_n$ be a graded, locally finite $U$-module algebra. We assume that the $U$-action preserves the grading of $S$, that is, each $S_n$ is stable under the $U$-action. Then $A := S_0$, is a subalgebra of $S$. Set $S_+ = \oplus_{n>0} S_n$. Then $A$ may be identified with $S/S_+$. We shall consider positively graded $U$-equivariant $S$-modules, in which the $U$-action preserves the grading. We continue to assume that all modules are locally $U$-finite.

For (such) a graded $S$-$U$-module $N = \oplus_{i=0}^{\infty} N_i$, we set $T_i(N) = \text{Tor}_i^S(A, N), \ i \geq 0$.

These spaces have a natural $S$-$U$-module structure, which may be seen as follows. Take a sequence of graded $U$-equivariant $S$-modules which form a graded $S$-projective resolution for $N$; this may be done by taking, e.g., the usual normalised bar resolution for $N$, which is a sequence of graded $U$-equivariant $S$-modules as can be easily seen by examining the explicit definition of the differential. By Lemma 2.2 the Tor-groups $T_i(N)$ computed from such a resolution are $S$-$U$-modules.

In particular, we have $T_0(N) = N/S_0$ with the natural $S$-$U$-module structure.

Let us introduce an increasing filtration $0 = F_{-1}N \subset F_0N \subset F_1N \subset \ldots$ of $N$ by graded submodules, by taking $F_pN = \sum_{i \leq p} S_{N_i}$. Then $T_0(F_pN)_n = 0$ if $n > p$ and $T_0(F_pN)_n = T_0(N)_n$ if $n \leq p$. There is also a natural $S$-$U$-module surjection

\[(4.1) \quad S(-p) \otimes_A T_0(N)_p \longrightarrow F_pN/F_{p-1}N,\]

where $S(-p)$ is $S$ with the grading shifted by $p$, that is, $S(-p)_n = S_{n-p}$.

Remark 4.1. By [27] Lemma 1, p.117, if $T_i(N) = 0$ and $\text{Tor}_i^A(S, T_0(N)) = 0$ for all $i > 0$, then the maps (4.1) are isomorphisms.

Let $\mathcal{M}_{gr}(S, U)$ be the additive category of finitely $S$-generated, positively graded, and locally $U$-finite $S$-$U$-modules. If we assume that $S$ is left Noetherian, then $\mathcal{M}_{gr}(S, U)$ is abelian, and hence is an exact category. Its $K$-groups are naturally $\mathbb{Z}[t]$-modules with $t$ acting as the translation functor $N \rightarrow N(-1)$.

If $S$ is $A$-flat, then every $A$-projective resolution $P_* \rightarrow V$ of $V$ in $\mathcal{M}(A, U)$ gives rise to an $S$-projective resolution $S \otimes_A P_* \rightarrow S \otimes_A V$. Hence we have an exact functor $(S \otimes_A -) : \mathcal{M}(A, U) \rightarrow \mathcal{M}_{gr}(S, U)$, which induces homomorphisms

\[(4.2) \quad (S \otimes_A -)_* : K_i(\mathcal{M}(A, U)) \longrightarrow K_i(\mathcal{M}_{gr}(S, U))\]

of $K$-groups.

Theorem 4.2. Assume that $S$ is left Noetherian and $A$-flat. If $A = S/S_+$ has finite projective dimension as an $S$-module, then (4.2) extends to a $\mathbb{Z}[t]$-module isomorphism

\[(4.3) \quad \mathbb{Z}[t] \otimes_{\mathbb{Z}} K_i(\mathcal{M}(A, U)) \longrightarrow K_i(\mathcal{M}_{gr}(S, U)), \ \text{for each} \ i.\]

Proof. We adapt the proofs of [27] Theorem 5] and [3] Theorem 3.2] to the present setting. Let $\mathcal{M}_p$ denote the full subcategory of $\mathcal{M}_{gr}(S, U)$ with objects $N$ such that $T_i(N) = 0$ for all $i > p$. If the projective dimension of the $S$-module $A$ is $d$, then $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_d = \mathcal{M}_{gr}(S, U)$. For $N$ in $\mathcal{M}_p$, Lemma 3.3 gives a surjective $S$-$U$-map $V_S = S \otimes_k V \twoheadrightarrow N$, $s \otimes v \mapsto sv$, where $V$ is a finite dimensional $U$-submodule of $N$ which generates $N$ itself over $S$. Then the kernel $N'$ of the surjection belongs to $\mathcal{M}_{gr}(S, U)$ since $S$ is left Noetherian. Because $V_S$ is a free $S$-module, $T_i(V_S) = 0$.

for all $i > 0$. Hence the long exact sequence of Tor groups arising from the short exact sequence $0 \to N' \to V_S \to N \to 0$ yields $T_i(N) \cong T_{i-1}(N')$ for all $i > 0$. This then implies that $N'$ belongs to $\mathcal{M}_{p-1}$.

Therefore, the inclusion $\mathcal{M}_p \subset \mathcal{M}_{p+1}$ for every $p \geq 0$ satisfies the conditions of the Resolution Theorem \cite[Theorem 4.6]{31}, hence is a homotopy equivalence. This leads to the homotopy equivalence $\mathcal{M}_0 \subset \mathcal{M}_d = \mathcal{M}_{gr}(S, U)$, which induces the isomorphisms

\begin{equation}
K_i(\mathcal{M}_0) \cong K_i(\mathcal{M}_{gr}(S, U)), \quad \text{for all } i = 0, 1, \ldots.
\end{equation}

Let $V$ be an object in $\mathcal{M}(A, U)$, and let $P_\bullet \to V$ be an $A$-projective resolution. Since $S$ is $A$-flat, $S \otimes_A P_\bullet \to S \otimes_A V$ is an $S$-projective resolution, and hence $T_i(S \otimes_A V) = 0$, for all $i > 0$. Thus for any $V$ in $\mathcal{M}(A, U)$, $S \otimes_A V$ belongs to $\mathcal{M}_0$. Therefore, if $\mathcal{M}_{0,n}$ be the full subcategory of $\mathcal{M}_0$ whose objects are modules $M$ such that $M = F_nM$, then we have an exact functor

$$b : \mathcal{M}(A, U)^{n+1} \longrightarrow \mathcal{M}_{0,n},$$

$$(V_0, V_1, \ldots, V_n) \longmapsto \oplus_{p=0}^n S(-p) \otimes_A V_p.$$ 

This induces homomorphisms

$$b_* : K_i(\mathcal{M}(A, U))^{n+1} \longrightarrow K_i(\mathcal{M}_{0,n}).$$

Since $T_0$ is exact on $\mathcal{M}_0$, we also have an exact functor $c : \mathcal{M}_{0,n} \longrightarrow \mathcal{M}(A, U)^{n+1}$,

$$M \longmapsto (T_0(M)_0, T_0(M)_1, \ldots, T_0(M)_n),$$

and homomorphisms $c_* : K_i(\mathcal{M}_{0,n}) \longrightarrow K_i(\mathcal{M}(A, U))^{n+1}$.

Note that $c \circ b$ is equivalent to the identity functor, thus $c_* \circ b_* = \text{id}$.

On the other hand, any $M$ in $\mathcal{M}_{0,n}$ has a filtration

$$0 = F_{-1}M \subset F_0M \subset F_1M \subset \cdots \subset F_nM = M.$$ 

Because of the $A$-flat nature of $S$, Remark \ref{flat} applies and we have $F_pM = S(-p) \otimes_A T_0(M)_p$. Clearly each functor $\frac{F_pM}{F_{p-1}M}$ is exact. It follows from the additivity of characteristic filtrations \cite[Corollary 2, p.107]{31} \cite[Corollary 4.4]{31} that

$$\sum_{p=0}^n \left( \frac{F_p}{F_{p-1}} \right)_* = (F_n)_* = \text{id}.$$ 

Now observe that $(b \circ c)_* = \sum_{p=0}^n \left( \frac{F_p}{F_{p-1}} \right)_*$, hence $b_* \circ c_* = \text{id}$.

Passing to the limit $n \to \infty$ we have the following isomorphism for each $i$:

$$\mathbb{Z}[t] \otimes_{\mathbb{Z}} K_i(\mathcal{M}(A, U)) \longrightarrow K_i(\mathcal{M}_0).$$

Using the isomorphisms \eqref{isomorphism}, we arrive at the desired result. \hfill $\square$

4.2. Filtered module algebras. Let $S$ be a locally finite $U$-module algebra. Assume that $S$ has an ascending filtration $0 \subset F_{-1}S \subset F_0S \subset F_1S \subset \cdots$ such that $1 \in F_0S$, $\cup_p F_pS = S$ and $F_pSF_qS \subset F_{p+q}S$. We assume that the filtration is preserved by the $U$-action. Let

$$\overline{S} = \text{gr}(S) := \oplus_{p \geq 0} \overline{S}_p \quad \text{with} \quad \overline{S}_p := F_pS/F_{p-1}S,$$

and set $A = F_0S$ and $\overline{S}_+ = \oplus_{p > 0} \overline{S}_p$.\hfill $\square$
Theorem 4.3. Assume that $\overline{S}$ is left Noetherian and $A$-flat. If $A (= S/S_\ast)$ has a finite projective $\overline{S}$-resolution, then for $i = 0, 1, 2, \ldots$ there exist isomorphisms

$$K_i(M(A, U)) \sim K_i(M(S, U)).$$

If furthermore $A$ is regular, then $S$ is regular and there exist isomorphisms

$$K_i^U(A) \sim K_i^U(S).$$

Remark 4.4. If $g = 0$ and $U$ is generated by the identity, Theorem 4.3 reduces to a slightly weaker version of [27, Theorem 7]. See also the question raised in [27, p.118] (immediately below [27, Theorem 6]).

In order to prove the theorem, we need some preliminaries. Let $z$ be an indeterminate, and consider the graded algebra $S' = \oplus_{p \geq 0} (F_p S) z^p$, where $z$ is central in $S'$ and has degree 1. This is a subalgebra of $S[z]$. We endow $S'$ with a $U$-action by specifying that $z$ is $U$-invariant. This turns $S'$ into a $U$-module algebra. Let $S'_+ = \oplus_{p > 0} (F_p S) z^p$, then $A = S'/S'_+$. Note also that $\overline{S} = S'/zS'$.

The next result does not involve the $U$-action.

Lemma 4.5. Assume that $\overline{S}$ is left Noetherian and $A$-flat.

1. Then $S'$ is left Noetherian and $A$-flat.
2. If it is further assumed that $A (= S/S_\ast)$ has a finite projective $\overline{S}$-resolution, then $A (= S'/S'_\ast)$ also has a finite projective $S'$-resolution.

Proof. Filter $S'$ by letting $F_p S'$ consist of polynomials in $z$ with coefficients in $F_p S$. Then the associated graded algebra of $S'$ is given by

$$\text{gr}(S') = \oplus_{p \geq 0} F_p S' = \oplus_{p \geq 0} \overline{S} z^p.$$

Since $\overline{S}$ is left Noetherian, so also is $\overline{S}[z]$. This then implies that $S'$ is left Noetherian (see, e.g., [27, Lemma 3.(i), p.119]).

Given that $\overline{S}$ is $A$-flat, so also is every graded component $\overline{S}_p$, and in particular, $F_0 S = \overline{S}_0$. Corresponding to any short exact sequence $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ of $A$-modules, we have the commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & F_{p-1} S \otimes_A V' & \rightarrow & F_{p-1} S \otimes_A V & \rightarrow & F_{p-1} S \otimes_A V'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & F_p S \otimes_A V' & \rightarrow & F_p S \otimes_A V & \rightarrow & F_p S \otimes_A V'' & \rightarrow & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & \overline{S}_p \otimes_A V' & \rightarrow & \overline{S}_p \otimes_A V & \rightarrow & \overline{S}_p \otimes_A V'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

where the columns are obviously exact. Since the composition of the maps in the middle row is zero, exactness of the top and bottom rows will imply the exactness of the middle one. Thus by induction on $p$, we can show that $F_p S$ is $A$-flat for all $p$. This then immediately leads to the $A$-flatness of $S'$. 

To prove the second part of the lemma, we note that $\overline{S} = S'/zS'$ leads to $pd_{S}(\overline{S}) = 1$, where $pd_{R}(M)$ denotes the projective dimension of the left $R$-module $M$. Thus by the Change Rings Theorem [32, Theorem 4.3.1], $pd_{S'}(A) \leq pd_{\overline{S}}(A) + pd_{S}(\overline{S}) < \infty$. This completes the proof of the lemma.

Since $S = S'/(1 - z)S'$, it follows from Lemma 4.5(1) that $S$ is left Noetherian. Hence $K_{1}(\mathcal{M}(S, U))$ are defined.

The proof of the following lemma requires both the Localisation Theorem [27, Theorem 5, p.105] and Devissage Theorem [27, Theorem 4, p.104].

**Lemma 4.6.** There exists the following long exact sequence of $K$-groups:

$$
\cdots \longrightarrow K_{1}(\mathcal{M}_{gr}(\overline{S}, U)) \longrightarrow K_{1}(\mathcal{M}_{gr}(S', U)) \longrightarrow K_{1}(\mathcal{M}(S, U))
\longrightarrow K_{0}(\mathcal{M}_{gr}(\overline{S}, U)) \longrightarrow K_{0}(\mathcal{M}_{gr}(S', U)) \longrightarrow K_{0}(\mathcal{M}(S, U)) \longrightarrow 0.
$$

**Proof.** Recall the crucial facts that $z$ is $U$-invariant and is also central in $S'$. Let $N$ be the full subcategory of $\mathcal{M}_{gr}(S', U)$ consisting of modules killed by some power of $z$. This is a Serre subcategory, so we may define the quotient category $\mathcal{M}_{gr}(S', U)/N$. A more concrete way to view this construction is as follows. Let $S'[z^{-1}]$ be the localisation of $S'$ at $z^{-1}$. Then the localisation functor $\mathcal{M}_{gr}(S', U) \longrightarrow \mathcal{M}_{gr}(S'[z^{-1}], U)$ annihilates precisely the modules in $N$, and $\mathcal{M}_{gr}(S', U)/N$ is equivalent to $\mathcal{M}_{gr}(S'[z^{-1}], U)$.

For any object $M$ in $\mathcal{M}_{gr}(S'[z^{-1}], U)$, $z^{-1}$ acts as an isomorphism. Hence $M$ is uniquely determined by its degree 0 component. This leads to an equivalence of categories $\mathcal{M}_{gr}(S'[z^{-1}], U) \cong \mathcal{M}(S, U)$. Denote by $j : \mathcal{M}_{gr}(S', U) \longrightarrow \mathcal{M}(S, U)$ the composition of this equivalence with the localisation functor. Then $j : M \mapsto M/(1 - z)M$ for all $M$ in $\mathcal{M}_{gr}(S', U)$.

Now using Quillen’s Localisation Theorem [27, Theorem 5, p.105], we obtain (cf. op. cit. p. 115) a long exact sequence of $K$-groups:

$$
\cdots \longrightarrow K_{i}(\mathcal{N}) \longrightarrow K_{i}(\mathcal{M}_{gr}(S', U)) \longrightarrow K_{i}(\mathcal{M}(S, U)) \longrightarrow K_{i-1}(\mathcal{N}) \longrightarrow \cdots
\longrightarrow K_{0}(\mathcal{M}(S, U)) \longrightarrow 0.
$$

Since $\overline{S} = S'/zS'$, we see that $\mathcal{M}_{gr}(\overline{S}, U)$ is a full subcategory of $\mathcal{M}_{gr}(S', U)$ with objects the modules annihilated by $z$. Hence we have an inclusion $\mathcal{M}_{gr}(\overline{S}, U) \subset \mathcal{N}$. As any object $N$ of $\mathcal{N}$ is annihilated by $z^{k}$ for some $k$, we have the filtration $0 = z^{k}N \subset z^{k-1}N \subset \cdots \subset zN \subset N$ with $z^{i}N$ obviously annihilated by $z$ for each $i$. Hence the Devissage Theorem [27, Theorem 4, p.112] applies to this situation, inducing isomorphisms of $K$-groups $K_{i}(\mathcal{N}) \cong K_{i}(\mathcal{M}_{gr}(\overline{S}, U))$ for all $i \geq 0$. The lemma follows.

**Remark 4.7.** Let $i : \mathcal{M}_{gr}(\overline{S}, U) \longrightarrow \mathcal{M}_{gr}(S', U)$ denote the composition of the inclusions $\mathcal{M}_{gr}(\overline{S}, U) \subset \mathcal{N} \subset \mathcal{M}_{gr}(S', U)$. From the proof of the lemma we see that maps beside the connecting homomorphism in the long exact sequence are $i_{*}$ and $j_{*}$ as indicated below

$$
\cdots \longrightarrow K_{i}(\mathcal{M}_{gr}(\overline{S}, U)) \xrightarrow{i_{*}} K_{i}(\mathcal{M}_{gr}(S', U)) \xrightarrow{j_{*}} K_{i}(\mathcal{M}(S, U)) \longrightarrow \cdots
\.$$
Proof of Theorem 4.3. Since $\overline{S}$ satisfies the conditions of Theorem 4.2, by Lemma 4.5 $S'$ also satisfies the conditions of the theorem. Therefore, we have the $\mathbb{Z}[t]$-module isomorphisms
\begin{align}
(4.7) \quad \mathbb{Z}[t] \otimes_{\mathbb{Z}} K_i(\mathcal{M}(A, U)) & \longrightarrow K_i(\mathcal{M}_{gr}(\overline{S}, U)), \quad 1 \otimes g \mapsto (\overline{S} \otimes_A -)_* g, \\
& \mathbb{Z}[t] \otimes_{\mathbb{Z}} K_i(\mathcal{M}(A, U)) \longrightarrow K_i(\mathcal{M}_{gr}(S', U)), \quad 1 \otimes g \mapsto (S' \otimes_A -)_* g.
\end{align}

Let us now describe the map $i_*$ in Remark 4.7 more explicitly by finding the map $\delta$ which renders the following diagram commutative:
\[
\begin{array}{ccc}
K_i(\mathcal{M}_{gr}(\overline{S}, U)) & \xrightarrow{i_*} & K_i(\mathcal{M}_{gr}(S', U)) \\
\mathbb{Z}[t] \otimes_{\mathbb{Z}} K_i(\mathcal{M}(A, U)) & \xrightarrow{\delta} & \mathbb{Z}[t] \otimes_{\mathbb{Z}} K_i(\mathcal{M}(A, U)).
\end{array}
\]

Here the vertical isomorphisms are given by (4.7). For every $M$ in $\mathcal{M}(A, U)$, we have an exact sequence
\[
0 \longrightarrow S'(-1) \otimes_A M \xrightarrow{\sim} S' \otimes_A M \longrightarrow \overline{S} \otimes_A M \longrightarrow 0
\]
in $\mathcal{M}_{gr}(S', U)$, where $\overline{S} \otimes_A M$ is in $\mathcal{M}_{gr}(\overline{S}, U)$ but is regarded as an object of $\mathcal{M}_{gr}(S', U)$ via the inclusion $i$. Therefore the composition $i \circ (\overline{S} \otimes_A -)$ of functors $\mathcal{M}(A, U) \xrightarrow{\sim} \mathcal{M}(\overline{S}, U) \xrightarrow{i} \mathcal{M}_{gr}(S', U)$ fits into an exact sequence of functors
\[
0 \longrightarrow S'(-1) \otimes_A - \longrightarrow S' \otimes_A - \longrightarrow i \circ (\overline{S} \otimes_A -) \longrightarrow 0
\]
from $\mathcal{M}(A, U)$ to $\mathcal{M}_{gr}(S', U)$. By Corollary 1 of Theorem 2 in [27, p.106], we have
\[
i_* \circ (\overline{S} \otimes_A -)_* = (S' \otimes_A -)_* - (S'(-1) \otimes_A -)_* = (1 - t)(S' \otimes_A -)_*.
\]

From this formula it is evident that the map $\delta$ is multiplication by $1 - t$, which is injective with cokernel $K_i(\mathcal{M}(A, U))$.

Therefore $i_*$ is injective with cokernel isomorphic to $K_i(\mathcal{M}(A, U))$. Using this information in the long exact sequence of Lemma 4.6 we deduce that the connecting morphism is zero, and $K_i(\mathcal{M}(S, U))$ is isomorphic to the cokernel of $i_*$. Hence the composition of functors $\mathcal{M}(A, U) \longrightarrow \mathcal{M}_{gr}(S', U) \xrightarrow{j_*} \mathcal{M}(S, U)$, $M \mapsto S' \otimes_A M \mapsto S \otimes_A M$ (where $S = S'/(1 - z)S'$) induces an isomorphism
\[
K_i(\mathcal{M}(A, U)) \longrightarrow K_i(\mathcal{M}_{gr}(S', U)) \xrightarrow{j_*} K_i(\mathcal{M}(S, U)).
\]

This proves the first assertion of the theorem.

Given the conditions that $\overline{S}$ is left Noetherian and $A$ has finite projective dimension as a left $\overline{S}$-module, [27, Lemma 4, p.120] applies and hence the regularity of $A$ implies the regularity of $S$. Thus it follows from Proposition 3.6 that $K_i(\mathcal{M}(A, U)) = K_i^U(A)$ and $K_i(\mathcal{M}(S, U)) = K_i^U(S)$ for all $i$. Now the second part of the theorem immediately follows from the first part. $\square$
5. Quantum symmetric algebras

We now apply results from Section 4 to compute the equivariant K-groups of a class of module algebras over quantum groups. We shall refer to these module algebras as quantum symmetric algebras; these are quadratic algebras of Koszul type naturally arising from the representation theory of quantum groups. The quantised coordinate ring of affine $n$-space is an example. See [19] [33] [8] for other examples.

5.1. Equivariant K-theory of quantum symmetric algebras. Let $V$ be a finite dimensional vector space over a field $k$, and denote by $T(V)$ the tensor algebra over $V$. Given a subset $I$ of $V \otimes_k V$, we denote by $\langle I \rangle$ the two-sided ideal of $T(V)$ generated by $I$. Define the quadratic algebra

$$A := T(V)/\langle I \rangle.$$

We shall also use the notation $k\{V, I\}$ for $A$ to indicate the generating vector space $V$ and the defining relations of the algebra explicitly. The algebra $A$ is naturally $\mathbb{Z}_+$-graded since $\langle I \rangle$ is. We have $A = \bigoplus_{i=0}^{\infty} A_i$, with $A_0 = k$ and $A_1 = V$.

We shall say that a quadratic algebra $A = k\{V, I\}$ is of PBW type if there exists a basis $\{v_i \mid i = 1, 2, \ldots, d\}$ of $V$ such that the elements $v^a := v_1^{a_1}v_2^{a_2}\cdots v_d^{a_d}$, with $a := (a_1, a_2, \ldots, a_d) \in \mathbb{Z}_+^d$, form a basis (called the PBW basis) of $A$.

Let $k$ be the field $\mathbb{C}(q)$.

Lemma 5.1. Let $V$ be a finite dimensional module of type $(1, \ldots, 1)$ over a quantum group $U$. Let $I \subset V \otimes_k V$ be a $U$-submodule. Then the quadratic algebra $k\{V, I\} = T(V)/\langle I \rangle$ is a $U$-module algebra.

Proof. The tensor algebra $T(V)$ has a natural $U$-module algebra structure, with the $U$-action defined by using the co-multiplication. Since $I$ is a $U$-submodule of $V \otimes V$, so also is the two-sided ideal $\langle I \rangle$. Hence $A = T(V)/\langle I \rangle$ is a $U$-module algebra.

Definition 5.2. We call a $U$-module algebra $A = k\{V, I\}$ of the type defined in Lemma 5.1 a quantum symmetric algebra of the finite dimensional left $U$-modules of type $(1, \ldots, 1)$ if it admits a PBW basis.

If $A = k\{V, I\}$ admits a PBW basis, it is sometimes referred to as ‘flat’.

The next theorem is our main result concerning the equivariant K-theory of quantum symmetric algebras.

Theorem 5.3. Let $A = k\{V, I\}$ be a quantum symmetric algebra of a finite dimensional module $V$ over the quantum group $U$. Assume that $A$ is left Noetherian, then

$$K^U_i(A) = K_i(U\text{-mod}),$$

for all $i = 0, 1, \ldots$, where $U\text{-mod}$ is the category of finite dimensional left $U$-modules of type $(1, \ldots, 1)$.

We shall prove this result using Theorem 4.3 of Section 5.3. In order to do that, we need some results from the theory of Koszul algebras, which we now discuss.

5.2. Quadratic algebras and Koszul complexes. In this subsection, $k$ may be any field. Let $V$ be a finite dimensional vector space. Given a subspace $I$ of $V \otimes_k V$, we have the corresponding quadratic algebra $A = k\{V, I\}$. Let $V^*$ be the dual vector space of $V$, and define

$$I^\perp := \{\alpha \in V^* \otimes_k V^* \mid \alpha(w) = 0 \text{ for all } w \in I\}.$$
This definition implicitly uses the canonical isomorphism \((V \otimes_k V)^* \cong V^* \otimes_k V^*\). Henceforth \(\otimes\) will denote \(\otimes_k\). Let \(\langle I^\perp \rangle\) be the two-sided ideal of the tensor algebra \(T(V^*)\) over \(V^*\) generated by \(I^\perp\). We may define the quadratic algebra

\[ A^1 := T(V^*)/\langle I^\perp \rangle, \]

which is referred to as the dual quadratic algebra of \(A\). We endow \(T(V^*)\) with a \(\mathbb{Z}_- := -\mathbb{Z}_+\) grading with \(V^*\) having degree \(-1\). Then \(\langle I^\perp \rangle\) is a two-sided graded ideal, and \(A^1\) is \(\mathbb{Z}_-\)-graded with \(V^*\) having degree \(-1\).

Let \(A = \mathbb{k}\{V, I\}\) and \(A^i = \mathbb{k}\{V^i, I^\perp\}\) be dual quadratic algebras. Then \(A^i \otimes A\) has a natural algebra structure such that the subalgebras \(A^i\) and \(A^i\) commute. Let \(z\) be the image of the identity element of \(\text{Hom}_\mathbb{k}(V, V)\) in \(V^* \otimes V\) under the natural isomorphism, and let \(e_A\) be its image in \(A^i \otimes A\). It is easily verified that \(e_A^2 = 0\).

We regard \(A\) as a right \(A\)-module, and \(A^i\) as a left \(A^i\)-module. Then the graded dual \(A^* = \bigoplus_{i \in \mathbb{Z}_+} A^i\) of \(A\) with \(A^i = (A^i)^*\) has a natural right \(A\)-module structure. Hence \(A^* \otimes A\) is a right \(A^i \otimes A\)-module. The action of \(e_A\) defines a differential on \(A^* \otimes A\), yielding the Koszul complex of \(A\):

\[
\cdots \rightarrow A^2 \otimes A \rightarrow A^1 \otimes A \rightarrow A.
\]

**Remark 5.4.** We may also regard \(A\) as a left \(A\)-module and \(A^i\) as a right \(A^i\)-module. Then we have a Koszul complex of left \(A\)-modules:

\[
\cdots \rightarrow A \otimes A^2 \rightarrow A \otimes A^1 \rightarrow A,
\]

where the differential is \(\bar{e}_A = \sum_{i=1}^d v_i \otimes \bar{v}_i\).

For any pair of graded left \(A\)-modules \(M\) and \(N\), one may compute the extension spaces \(\text{Ext}^*\!(M, N) := \bigoplus_i \text{Ext}^i_A(M, N)\) defined as the right derived functor of the graded homomorphism \(\text{Hom}_A(M, N)\). Under the Yoneda product, \(\text{Ext}^*\!(\mathbb{k}, \mathbb{k})\) forms a graded algebra. A quadratic algebra \(A = \mathbb{k}\{V, I\}\) is called Koszul if \(\text{Ext}^*\!(\mathbb{k}, \mathbb{k}) \cong A^1\) as graded algebras. If \(A\) is Koszul, so is also \(A^i\).

A key property of a Koszul algebra is that the Koszul complex (5.1) of \(A\) is a graded free resolution of the base field \(\mathbb{k}\) regarded as a right \(A\)-module; that is, the complex

\[
\cdots \rightarrow A^2 \otimes A \rightarrow A^1 \otimes A \rightarrow A \rightarrow \mathbb{k} \rightarrow 0
\]

is exact. The map \(\epsilon\) is the augmentation \(A \rightarrow A/A_+\), where \(A_+ = \bigoplus_{i>0} A_i\). For the proof of this fact, see e.g., [20, Corollary II.3.2]. Similarly, (5.2) leads to a graded free resolution for the base field \(\mathbb{k}\) regarded as a left \(A\)-module in this case.

For quadratic algebras of PBW type, we have the following result.

**Theorem 5.5.** Let \(A = \mathbb{k}\{V, I\}\) be a quadratic algebra of PBW type, and denote by \(A^i\) its dual quadratic algebra. Then:

1. The algebra \(A\) is Koszul.
2. Let \(d = \dim_\mathbb{k} V\), then \(\dim_\mathbb{k} A_i = \binom{d + i - 1}{i}\) and \(\dim_\mathbb{k} A^i = \binom{d}{i}\) for all \(i \geq 0\).
3. The Koszul complexes (5.1) and (5.2) of \(A\) are graded free resolutions of length \(\dim_\mathbb{k} V\) of the base field \(\mathbb{k}\).
Proof. The Koszul nature of quadratic algebras of PBW type is a well-known fact, which was originally established in [25, Theorem 5.3].

The dimension of $A_i$ can be easily computed from the PBW basis. Let $h_A(z) = \sum_{i=0}^{\infty} z^i \dim_k A_i$ and $h_A! (z) = \sum_{i=0}^{\infty} z^i \dim_k A_i!$. Then $h_A(z) = 1/(1-z)^d$. It follows from some general facts on the Hilbert series of $\text{Ext}^\bullet(k,k)$ that $h_A(z)h_A!(-z) = 1$. By part (1), $A$ is Koszul, thus $A! \cong \text{Ext}^\bullet(k,k)$. Hence $h_A! (z) = (1+z)^d$, which implies the claimed dimension formula for $A_i! - i$.

The Koszul complexes are free resolutions since $A$ is Koszul by part (1). As $A_i! = 0$ for all $i > d$, the length of the resolutions is $\dim_k V$. □

Remark 5.6. Theorem 5.5.(3) will suffice for the purpose of proving Theorem 5.12 and Theorem 5.3 on $K$-groups. The results below give a direct proof that (left) Noetherian quantum symmetric algebras are regular.

Let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded module for a quadratic algebra $A = k\{V, I\}$. If $M$ is finitely generated, there exists some integer $r$ such that $M_i = 0$ for all $i < r$. That is, a finitely generated graded module must be bounded below. Let

$$\overline{M} := A_{-i} \otimes_A M \cong k \otimes_A M.$$  

The following result is a special case of [18, Theorem 4.6].

Theorem 5.7. Let $P$ be a finitely generated graded projective module over a quadratic algebra $A = k\{V, I\}$. Then $P$ is obtained from $\overline{P}$ by extension of scalars. That is, $P \cong A \otimes_k \overline{P}$. Therefore, all finitely generated graded projective modules over a quadratic algebra are free.

Recall that a complex of graded left $A$-modules

$$\cdots \rightarrow C_n \xrightarrow{\phi_n} C_{n-1} \xrightarrow{\phi_{n-1}} C_{n-2} \rightarrow \cdots$$

is called minimal if $\phi_n(C_n) \subseteq A_+ C_{n-1}$ for each $n$. A minimal resolution is defined similarly.

Theorem 5.8. Every finitely generated graded left module over a quadratic algebra $A = k\{V, I\}$ of PBW type has a minimal free resolution of length at most $\dim V$.

Proof. By [26, Proposition §1.4.2.], every finitely generated graded left $A$-module $M$ has a minimal free resolution

$$\mathcal{F} : \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0.$$  

(5.4)

Thus it suffices to show that this resolution has finite length. Let us compute $\text{Tor}_i^A(k, M)$ from the complex $k \otimes_A \mathcal{F}$. By using the minimality of $\mathcal{F}$, we obtain

$$\text{Tor}_i^A(k, M) = k \otimes_A F_i \quad \text{for all } i.$$  

In particular, $\text{Tor}_i^A(k, M) = 0$ if and only if $F_i = 0$.

On the other hand, we may also compute $\text{Tor}_i^A(k, M)$ from the complex obtained by tensoring (over $A$) the Koszul resolution of the base field $k$ (as a right $A$-module) with $M$. By part (3) of Theorem 5.5, $\text{Tor}_i^A(k, M) = 0$ for all $i > \dim V$. This leads to $F_i = 0$ for all $i > \dim V$ in the minimal free resolution for $M$. □

The following result is a consequence of Theorem 5.8.
Theorem 5.9. Every finitely generated left module over a quadratic algebra \( A = k\{V, I\} \) of PBW type has a free resolution of finite length.

Proof. Let \( A^n \xrightarrow{\phi} A^m \longrightarrow M \longrightarrow 0 \) be an exact sequence of left \( A \)-modules. We think of \( A^n \) and \( A^m \) as consisting of rows with entries from \( A \). Then \( \phi \) can be represented as an \( n \times m \)-matrix \((\phi_{ij})\) with entries \( \phi_{ij} \in A \), and acts on \( A^v \) by matrix multiplication from the right.

Now consider the algebra \( T = A[x] \), which consists of polynomials in \( x \) with coefficients in \( A \). We stipulate that \( x \) commutes with all elements of \( A \). Then \( A = T/(1-x)T \). It is easy to see that \( T \) is a quadratic algebra which is also Koszul.

Let \( r \) be the smallest integer such that every entry \( \phi_{ij} \) of the matrix of \( \phi \) is contained in \( A_0 \oplus A_1 \oplus \cdots \oplus A_r \). Then we can write \( \phi_{ij} = \phi_{ij}[0] + \phi_{ij}[1] + \cdots + \phi_{ij}[r] \) with \( \phi_{ij}[k] \in A_k \). Upon replacing the entries \( \phi_{ij} \) of the matrix by \( \phi_{ij}(x) = x^r \phi_{ij}[0] + x^{r-1} \phi_{ij}[1] + \cdots + \phi_{ij}[r] \), we obtain a rectangular matrix \( \hat{\phi} = (\hat{\phi}_{ij}) \) with entries which are homogeneous elements of \( T \) of degree \( r \).

Regard \( \hat{\phi} \) as a left \( T \)-module homomorphism \( T^n \longrightarrow T^m \), and let \( \hat{M} = \text{coker} \hat{\phi} \). Then \( M \) is a graded \( T \)-module, and \( A \otimes_T M = M \). By Theorem 5.9 we have a free resolution \( \mathcal{F} \) of \( \hat{M} \) with finite length. The homology of the complex \( A \otimes_T \mathcal{F} \) is \( \text{Tor}_T^*(A, M) \).

To compute \( \text{Tor}_T^*(A, \hat{M}) \), we tensor with \( \hat{M} \) the free resolution \( 0 \longrightarrow T \xrightarrow{1-x} T \longrightarrow A \longrightarrow 0 \) of the right \( T \)-module \( A \), obtaining \( 0 \longrightarrow \hat{M} \xrightarrow{1-x} \hat{M} \). Since the action of \( 1-x \) on any graded \( T \)-module is injective, the above complex is exact. This shows that

\[
\text{Tor}_T^*(A, \hat{M}) = 0, \quad \text{for all } i \geq 1. \tag{5.5}
\]

Note that \( A \otimes_T \mathcal{F} \) is also a complex of left \( A \)-modules, which are all free except for \( A \otimes_T \hat{M} = M \). From equation (5.5) we see that this complex is exact, thus is a free resolution of finite length for the \( A \)-module \( M \).

Recall that a left module \( E \) over a ring \( R \) is called stably free if there exists a free left \( A \)-module \( F \) of finite rank such that \( E \oplus F \cong R^n \) for some finite \( n \). Clearly a stably free module is finitely generated and projective.

It is easy to see that if a projective module admits a free resolution of finite length, it is stably free. Conversely, a projective \( R \)-module \( E \) with a free resolution \( 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0 \) of finite length \( n \) can be shown to be stably free by a simple induction on \( n \). Indeed, if \( n = 0 \), the claim is obviously true. Let \( E' = \text{Ker}(F_0 \longrightarrow E) \), then we have the following free resolution for \( E' \):

\[
0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow E' \longrightarrow 0.
\]

Since the length of the resolution is \( n - 1 \), by the induction hypothesis, \( E' \) is stably free. Hence \( E \) is stably free since \( F_0 \cong E \oplus E' \).

The next statement is an immediate consequence of Theorem 5.9.
Corollary 5.10. Every finitely generated projective module over a quadratic algebra \( A = \mathbb{k}\{V, I\} \) of PBW type is stably free.

Every quadratic algebra \( A = \mathbb{k}\{V, I\} \) of PBW type is a \( \mathbb{Z}_+ \)-graded algebra with degree 0 subalgebra \( \mathbb{k} \). If the algebra is assumed to be left Noetherian, then Theorem 5.9 implies that it is regular.

Lemma 5.11. A quadratic algebra of PBW type is left regular if it is left Noetherian. In particular, every left Noetherian quantum symmetric algebra is left regular.

We may now use [27, Theorem 7] to compute the usual algebraic \( K \)-groups \( K_i(A) \) of \( A \). The result is as follows.

Theorem 5.12. Let \( A \) be a quadratic algebra of PBW type, and assume that \( A \) is left Noetherian. Then \( K_i(A) = K_i(\mathbb{k}) \), \( i = 0, 1, \ldots \).

In particular, \( K_0(\mathbb{k}) = \mathbb{Z} \). This is consistent with Corollary 5.10.

5.3. Proof of Theorem 5.3. By Lemma 5.11 and Proposition 3.6, \( K_i^U(A) = K_i(\mathcal{M}(A, U)) \). Now \( A = A_0 + A_1 + \ldots \) is \( \mathbb{Z}_+ \)-graded with \( A_0 = \mathbb{k} \). Thus we may apply Theorem 4.2 to compute its U-equivariant \( K \)-groups. We have \( K_i^U(A) = K_i^U(\mathbb{k}) = K_i(\mathcal{P}(\mathbb{k}, U)) \) for all \( i = 0, 1, \ldots \).

Note that \( \mathcal{M}(\mathbb{k}, U) \) is the category of finite dimensional left \( U \)-modules of type \((1, 1, \ldots) \). As is well-known, \( \mathcal{M}(\mathbb{k}, U) \) is semi-simple, thus \( \mathcal{P}(\mathbb{k}, U) = \mathcal{M}(\mathbb{k}, U) = U\text{-mod} \).

In particular \( K_0^U(A) \) is the Grothendieck group of \( U\text{-mod} \).

5.4. Examples. In this section, we consider examples of quantum symmetric algebras arising from natural modules for the quantum groups associated with the classical series of Lie algebras. These quantum symmetric algebras also feature prominently in the study of the invariant theory of quantum groups [19].

Example 5.13. Coordinate algebra of a quantum matrix. A familiar example of quantum symmetric algebras is \( \mathcal{O}(M(m, n)) \), the coordinate algebra of a quantum \( m \times n \) matrix. It is generated by \( x_{ij} \) \((1 \leq i \leq m, 1 \leq j \leq n)\) subject to the following relations

\[
\begin{align*}
  x_{ij}x_{ik} &= q^{-1}x_{ik}x_{ij}, & j < k, \\
  x_{ij}x_{kj} &= q^{-1}x_{kj}x_{ij}, & i < k, \\
  x_{ij}x_{kl} &= x_{kl}x_{ij}, & i < k, j > l, \\
  x_{ij}x_{kl} &= x_{kl}x_{ij} - (q - q^{-1})x_{il}x_{kj}, & i < k, j < l.
\end{align*}
\]

(5.6)

It is well known that this is a module algebra over \( U_q(sl_n) \) with a PBW basis consisting of ordered monomials of the elements \( x_{ij} \). The \( U_q(sl_n) \)-action on \( \mathcal{O}(M(m, n)) \) can be described as follows. For each \( i \), the subspace \( \oplus_{j=1}^n \mathbb{k}x_{ij} \) is isomorphic to the natural module for \( U_q(sl_n) \). Thus \( \mathcal{O}(M(m, n)) \) is a quadratic algebra of the \( U_q(sl_n) \)-module \( V \) which is the direct sum of \( m \) copies of the natural module.

In particular, when \( m = 1 \), all relations but the first of (5.6) are vacuous, and we obtain the quantised coordinate algebra of affine \( n \)-space.
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By [9. §1], \(\mathcal{O}(M(m,n))\) is left Noetherian for all \(m\) and \(n\). By Theorem 5.12, the ordinary algebraic K-groups of \(\mathcal{O}(M(m,n))\) are given by \(K_i(\mathcal{O}(M(m,n))) = K_i(\mathbb{k})\) for all \(i\). Theorem 5.5 also applies, and we have

\[ K_i^{\mathbb{U}_q(sl_n)}(\mathcal{O}(M(m,n))) \cong K_i(\mathbb{U}_q(sl_n)-\text{mod}), \text{ for all } i. \]

**Example 5.14.** Quantum symmetric algebras associated with the natural modules for \(\mathbb{U}_q(so_m)\) and \(\mathbb{U}_q(sp_{2n})\). We first briefly recall the construction given in [19].

An important structural property of the quantum group \(\mathbb{U} = U_q(\mathfrak{g})\) associated with a simple Lie algebra \(\mathfrak{g}\) is the braiding of its module category provided by a universal \(R\)-matrix [14]. We may think of this as an invertible element in some completion of \(\mathbb{U} \otimes_{\mathbb{k}} \mathbb{U}\), which satisfies the following relations

\[
\begin{align*}
R\Delta(x) &= \Delta'(x)R, \quad \forall x \in \mathbb{U}, \\
(\Delta \otimes \text{id})R &= R_{13}R_{23}, \quad (\text{id} \otimes \Delta)R = R_{13}R_{12}, \\
(\epsilon \otimes \text{id})R &= (\text{id} \otimes \epsilon)R = 1 \otimes 1,
\end{align*}
\]

where \(\epsilon\) is the co-unit and \(\Delta'\) is the opposite co-multiplication. Here the subscripts of \(R_{13}\) etc. have the usual meaning as in [14]. It follows from the second line of (5.7) that \(R\) satisfies the celebrated Yang-Baxter equation 

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

Given a finite dimensional \(U\)-module \(V\), let \(R_{V,V}\) denote the automorphism of \(V \otimes V\) defined by the universal \(R\)-matrix of \(U\). Let \(P : V \otimes V \to V \otimes V\), \(v \otimes w \mapsto w \otimes v\), be the permutation, and define \(\tilde{R} = PR_{V,V}\). Then \(\tilde{R} \in \text{End}_U(V \otimes V)\) by (5.7), and \(\tilde{R}\) has characteristic polynomial of the form

\[
\prod_{i=1}^{k_+} (x - q^{\chi_i^{(+)}}) \prod_{j=1}^{k_-} (x + q^{\chi_j^{(-)})}),
\]

where \(\chi_i^{(+)\text{ and } \chi_i^{(-)}\) are integers, and \(k_+\) and \(k_-\) positive integers. Consider the \(U\)-submodule \(V_-\) of \(V \otimes V\) defined by

\[
V_- = \prod_{i=1}^{k_+} (\tilde{R} - q^{\chi_i^{(+)}}) (V \otimes V).
\]

We may now form the \(U\)-module algebra \(\mathbb{k}\{V, V_-\}\) associated to \(V\) and \(V_-\). There is a classification in [8, 35] of those irreducible \(U\)-modules \(V\) satisfying the condition that the corresponding quadratic algebras \(\mathbb{k}\{V, V_-\}\) admit PBW bases. In particular, the natural modules of the quantum groups associated with the classical Lie algebras all have this property [35].

Recall from [19] that if \(A\) and \(B\) are locally finite \(U\)-module algebras, \(A \otimes_{\mathbb{k}} B\) becomes a \(U\)-module algebra if its multiplication is twisted by the universal \(R\)-matrix. Explicitly, if we write \(R = \sum_t \alpha_t \otimes \beta_t\), then for all \(a, a' \in A\) and \(b, b' \in B\),

\[
(a \otimes b)(a' \otimes b') = \sum_t a(\beta_t \cdot a') \otimes (\alpha_t \cdot b)b'.
\]

If \(C\) is a third locally finite \(U\)-module algebra, the \(U\)-module algebras \((A \otimes B) \otimes C\) and \(A \otimes (B \otimes C)\) are canonically isomorphic [19]. Therefore, given \(\mathbb{k}\{V, V_-\}\) associated with an irreducible finite dimensional \(U\)-module, we have locally finite \(U\)-module algebras \(\mathbb{k}\{V, V_-\}^{\otimes m}\) for each positive integer \(m\).

For any vector space \(W\) we use the notation \(W^n = \oplus^n W\).
**Theorem 5.15.** Let $V$ be the natural module of $U_q(so_{2n})$ or $U_q(sp_{2n})$, and let $I_-$ be the $U$-submodule of $V \otimes V$ defined by (5.8). Then $S_q(V^m) := \mathbb{k}\{V, I_-\}^\otimes m$ is a Noetherian quantum symmetric algebra for every $m$.

Here by Noetherian we mean that the algebra is both left and right Noetherian.

To prove the theorem, we require some preliminaries. The following result from [9] will be of crucial importance.

**Lemma 5.16.** [9 Proposition I.8.17] Let $A$ be an associative algebra over $\mathbb{k}$. Let $u_1, u_2, \ldots, u_N$ be a finite sequence of elements which generate $A$. Assume that there exist scalars $q_{ij} \in \mathbb{k}^\times$, $\alpha_{ij}^t, \beta_{ij}^t \in \mathbb{k}$ such that for all $i < j$,

$$ u_j u_i = q_{ij} u_i u_j + \sum_{s=1}^{i-1} \sum_{t=1}^{N} \left( \alpha_{ij}^t u_s u_t + \beta_{ij}^t u_t u_s \right), $$

then $A$ is Noetherian.

Observe in particular that the algebra $\bar{A}$ presented in terms of the generators $u_1, \ldots, u_N$ and the relations (5.9) is Noetherian. Thus any algebra defined by the same generators subject to (5.9) and extra relations is a quotient of $\bar{A}$, and hence is also Noetherian.

Let $V$ be the natural $U_q(so_{2n})$-module, and let $\{v_a \mid a = 1, \ldots, 2n\}$ be a basis of weight vectors of $V$, with weights decreasing as $a$ increases. Order this basis in the natural way: $v_1, v_2, \ldots, v_{2n}$. Then $\mathbb{k}\{V, I_-\}$ is generated by $v_a$ ($1 \leq a \leq 2n$) with the following relations

$$ v_b v_a = q_{ab} v_a v_b, \quad a < b \leq 2n, \quad a + b \neq 2n + 1, $$

$$ v_{n+1} v_n = v_n v_{n+1}, $$

$$ v_{2n-i} v_{i+1} = q^2 v_{i+1} v_{2n-i} - q v_i v_{2n+1-i} + q v_{2n+1-i} v_i, \quad i \leq n - 1. $$

(5.10)

If we ignore the relation arising from the $i = n - 1$ case of the third equation in (5.10), the remaining relations define an algebra which satisfies the conditions of Lemma 5.16. Therefore, $\mathbb{k}\{V, I_-\}$ is Noetherian. It is known [8, 35] that $\mathbb{k}\{V, I_-\}$ admits a PBW basis, and hence is a Noetherian quantum symmetric algebra.

We now turn to Theorem 5.15.

**Proof of Theorem 5.15.** It was shown in [10] that $S_q(V^m)$ admits a PBW basis for every $m$, whence it suffices to prove that this algebra is Noetherian.

If $V$ is the natural $U_q(so_{2n})$-module, we may regard $S_q(V^m)$ as generated by $X_{ia}$ with $1 \leq i \leq m$ and $1 \leq a \leq 2n$, subject to the relations R(1) and R(2) below:

R(1): For any $i$,

$$ X_{ib} X_{ia} = q X_{ia} X_{ib}, \quad a < b \leq 2n, \quad a + b \neq 2n + 1, $$

$$ X_{i,n+1} X_{in} = X_{in} X_{i,n+1}, $$

$$ X_{i,2n-s} X_{i,s+1} = q^2 X_{i,s+1} X_{i,2n-s} - q X_{ia} X_{i,2n+1-s} + q X_{i,2n+1-s} X_{ia}, \quad s \leq n - 1. $$

R(2): For $i < j$ and all $a, b$,

$$ X_{jb} X_{ia} = q_{ab}^{-1} X_{ia} X_{jb} + \sum_t (\beta_{ij}^t \cdot X_{ia}) (\alpha_{ij}^t \cdot X_{jb}), $$

where $q_{ab}$ is $q^{-1}$ if $a = b$, is $q$ if $a + b = 2n + 1$, and is 1 otherwise.
Here $\sum a_i \mathbb{k}X_{ia} \cong V$ as $U$-module for each $i$, and we use the form $R = K + \sum_i \alpha_i' \otimes \beta_i'$ for the universal $R$-matrix, where $K$ acts by $K(X_{jb} \otimes X_{ia}) = q_{ab}^{-1}X_{jb} \otimes X_{ia}$. Actions of $\alpha_i'$ (resp. $\beta_i'$) increase (resp. decrease) weights. We have $\beta_i' \cdot X_{ia} = \zeta_{at}X_{ia}$ and $\alpha_i' \cdot X_{jb} = \eta_{tb}X_{jb}$ for some $\alpha > a$ and $b < b$, where $\zeta_{ta}$ and $\eta_{tb}$ are scalars such that $\zeta_{ta}\eta_{tb} \neq 0$ only for finitely many $t$.

Order the elements $X_{ia}$ as follows:

$$X_{m1}, X_{m2}, \ldots, X_{m,2n}; X_{m-1,1}, X_{m-1,2}, \ldots, X_{m-1,2n}; \ldots; X_1, X_2, \ldots, X_{1,2n}.$$ 

Note that relations $R(1)$ are the same as (5.10) with $v$ replaced by $X_{ia}$, and the order of the elements agrees with that of the $v$. The relations $R(2)$ may be re-written as

$$X_{ia}X_{jb} = q_{ab}X_{jb}X_{ia} - q_{ab} \sum_t (\beta_i' \cdot X_{ia})(\alpha_j' \cdot X_{jb})$$

$$= q_{ab}X_{jb}X_{ia} - q_{ab} \sum_t \zeta_{at}\eta_{tb}X_{ia}X_{jb}$$

for $i < j$ and all $a, b$. These relations are in the form of (5.9). Therefore $S_q(V^m)$ meets the conditions of Lemma 5.16 and hence is Noetherian. This completes the proof for the case of $U_q(so_{2n})$ when $V$ is the natural $U_q(so_{2n+1})$-module or natural $U_q(sp_{2n})$-module, there are defining relations of $S_q(V^m)$ analogous to those in the $U_q(so_{2n})$ case [19], and similar reasoning shows that $S_q(V^m)$ is also Noetherian. We leave the details of the proof in these cases to the reader. 

In view of Theorem 5.13 we may now apply Theorem 5.12 and Theorem 5.3 to compute the ordinary and equivariant $K$-groups of $S_q(V^m)$. We have

$$K_i(S_q(V^m)) \cong K_i(\mathbb{k}), \quad K_i^{U_q(g)}(S_q(V^m)) \cong K_i(U_q(g)-\text{mod}), \quad \text{for all } i,$$

where $U_q(g)$ is $U_q(so_m)$ or $U_q(sp_{2n})$.

**Example 5.17.** We consider the Weyl algebra $W_q$ of degree 1 over $\mathbb{k} = \mathbb{C}(q)$. It is generated by $x, y$ subject to the relation

$$xy - q^{-1}yx = 1.$$ 

This is a module algebra over $U_q(sl_2)$ if we identify $\{x, y\}$ with the standard basis of the natural $U_q(sl_2)$-module $\mathbb{k}^2$.

Let $F_tW_q$ be the span of the elements $x^{j-t}y^t$ with $i \geq j \geq t \geq 0$. Then we have a complete ascending filtration $0 \subset F_0W_q \subset F_1W_q \subset F_2W_q \subset \ldots$, which is stable under the $U_q(sl_2)$-action. The associated graded algebra $gr(W_q)$ is the $U_q(sl_2)$-module algebra generated by $x, y$ subject to the relation $xy = q^{-1}yx$. By Lemma 5.11 this algebra is left regular.

Therefore Theorem 5.12 and Theorem 5.3 apply to $W_q$, we obtain $K_i(W_q) = K_i(\mathbb{k})$ and $K_i^{U_q(sl_2)}(W_q) = K_i(\mathcal{P}(\mathbb{k}, U_q(sl_2)))$ ($i \geq 0$) for the Quillen $K$-groups and equivariant $K$-groups respectively. In the present case, $\mathcal{M}(\mathbb{k}, U_q(sl_2)) = \mathcal{P}(\mathbb{k}, U_q(sl_2)) = U_q(sl_2)-\text{mod}$ is the category of finite dimensional $U_q(sl_2)$-modules of type-$(1, \ldots, 1)$. Hence $K_i^{U_q(sl_2)}(W_q) = K_i(U_q(sl_2)-\text{mod})$ for all $i \geq 0$. 


Quantum homogeneous spaces [15] are a class of noncommutative geometries with quantum group symmetries, which have been widely studied (see, e.g., [11, 20] and the references therein). We shall develop their equivariant $K$-theory in this section. We mention that the equivariant $K_0$-groups of quantum homogeneous spaces have been determined in [34].

6.1. Quantum homogeneous spaces. We recall from [15, 34] some background material on quantum homogeneous spaces, which will be needed later. Let $V$ be an object in $U$-$mod$, and let $\pi : U \rightarrow \text{End}_k(V)$ be the corresponding matrix representation of $U$ relative to some basis of $V$. Then there exist elements $t_{ij}$ $(i, j = 1, 2, \ldots, \dim V)$ in the dual $U^*$ of $U$ such that for any $x \in U$, we have $\pi(x)_{ij} = \langle t_{ij}, x \rangle$ for all $i, j$. The $t_{ij}$ will be referred to as the coordinate functions of the finite dimensional representation $\pi$. It follows from standard facts in Hopf algebra theory [23] that the coordinate functions of all the various $U$-representations associated with the $U$-modules in $U$-$mod$ span a Hopf algebra $O_q(U)$, which is a Hopf subalgebra of the finite dual of $U$ (see [23] for this notion). We shall denote the co-multiplication and the antipode of $O_q(U)$ by $\Delta$ and $S$ respectively, but the co-unit will be denoted by $\epsilon_{0}$. Note that the co-unit (resp. unit) of $U$ becomes the unit (resp. co-unit) of $O_q(U)$.

We shall denote $O_q(U_q(\mathfrak{g}))$ by $A_{\mathfrak{g}}$ for brevity. There exist two natural actions $R$ and $L$ of $U$ on $A_{\mathfrak{g}}$ [15], which correspond to left and right translation in the context of Lie groups. These actions are respectively defined by

$$R_x f = \sum_{(f)} f_{(1)} \langle f_{(2)}, x \rangle, \quad L_x f = \sum_{(f)} \langle f_{(1)}, S(x) \rangle f_{(2)}$$

for all $x \in U$ and $f \in A_{\mathfrak{g}}$, where as $f(x) = \langle f, x \rangle$ for any $x \in U$ and $f$ in the finite dual of $U$. These two actions clearly commute by the coassociativity of the Hopf structure on $A_{\mathfrak{g}}$.

It can be shown that $A_{\mathfrak{g}}$ forms a $U$-module algebra under both actions. However, some care needs to be exercised in the case of $L$, as for any $f, g \in A_{\mathfrak{g}}$ and $x \in U$, we have

$$L_x (fg) = \sum_{(f), (g)} \langle f_{(1)} g_{(1)}, S(x) \rangle f_{(2)} g_{(2)}$$

$$= \sum_{(f), (g), (x)} \langle f_{(1)}, S(x_{(2)}) \rangle \langle g_{(1)}, S(x_{(1)}) \rangle f_{(2)} g_{(2)}$$

$$= \sum_{(x)} L_{x(2)}(f) L_{x(1)}(g).$$

This shows that under the action $L$, $A_{\mathfrak{g}}$ forms a module algebra over $U'$, which is $U$, with the opposite co-multiplication $\Delta'$.

Let $\Theta$ be a subset of $\{1, 2, \ldots, r\}$, where $r$ is the rank of $\mathfrak{g}$. We denote by $U_q(\mathfrak{l})$ the Hopf subalgebra of $U$ generated by the elements of $\{k_i^\pm \mid 1 \leq i \leq r\} \cup \{e_j, f_j \mid j \in \Theta\}$. We denote by $U_q(\mathfrak{l})$-$mod$ the category of finite dimensional left $U_q(\mathfrak{l})$-modules of type-$(1, \ldots, 1)$. This category is semisimple.
Following [15], we define

\[(6.1) \quad A = \{ f \in A_{\mathfrak{g}} \mid L_x(f) = \epsilon(x) f, \ \forall x \in U_q(\mathfrak{l}) \} = A_{\mathfrak{g}}^{U_q(\mathfrak{l})}.\]

This is the submodule of $U_q(\mathfrak{l})$-invariants of $A_{\mathfrak{g}}$.

The following result is fairly straightforward (cf. [15, 34]).

**Theorem 6.1.** The subspace $A$ forms a locally finite $U$-module algebra under the action $R$. Furthermore, $A$ is (both left and right) Noetherian.

Indeed, since $U_q(\mathfrak{l})$ is a Hopf subalgebra of $U$, it follows from the definition that $A$ is a subalgebra of $A_{\mathfrak{g}}$. Since left and right translations commute, the $U$-module algebra structure of $A_{\mathfrak{g}}$ under $R$ descends to $A$. Being a subalgebra of $A_{\mathfrak{g}}$ which is contained in the finite dual of $U$, $A$ must be locally finite under the $U$-action $R$. The fact that $A$ is Noetherian is proved in [34].

It was shown in [15] that the algebra $A$ is the natural quantum analogue of the algebra of complex valued (smooth) functions on the real manifold underlying a compact homogeneous space. A complex structure on the quantum homogeneous space was discussed and used in establishing Borel-Weil type theorems in [15].

**Remark 6.2.** A quantum homogeneous space defined this way is a quantisation of the real manifold underlying a compact homogeneous space. A complex structure (in a generalised sense) on the quantum homogeneous space was discussed and used in [15].

6.2. **Equivariant $K$-theory of quantum homogeneous spaces.** For any object $\Xi$ of $U_q(\mathfrak{l})$-$\text{mod}$, we define the induced $U$-module as follows:

\[(6.2) \quad S(\Xi) := \left\{ \zeta \in \Xi \otimes A_{\mathfrak{g}} \mid \sum_{(x)} (x_{(1)} \otimes L_{x_{(2)}})\zeta = \epsilon(x) \zeta, \ \forall x \in U_q(\mathfrak{l}) \right\} = (\Xi \otimes_k A_{\mathfrak{g}})^{U_q(\mathfrak{l})},\]

where $U_q(\mathfrak{l})$ acts on the tensor product via $(\text{id} \otimes L) \circ \Delta$.

Then $S(\Xi)$ is both a left $A$-module and left $U$-module with $A$ and $U$ actions defined, for $b \in A$, $x \in U$ and $\zeta = \sum v_i \otimes a_i \in S(\Xi)$, by

\[(6.3) \quad b\zeta = \sum v_i \otimes ba_i,\]

\[(6.4) \quad x\zeta = (\text{id}_\Xi \otimes R_x) \zeta = \sum v_i \otimes R_x(a_i).\]

We have

\[x(b\zeta) = \sum v_i \otimes R_x(ba_i) = \sum_{(x)} R_{x_{(1)}}(b)(x_{(2)})\zeta, \quad \text{for} \ b \in A.\]

Thus $S(\Xi)$ indeed forms a $U$-equivariant $A$-module.

The following results were proved in [15, 34].

**Theorem 6.3.**

1. Let $V$ be the restriction of a finite dimensional $U$-module to a $U_q(\mathfrak{l})$-module. Then $S(V) \cong V \otimes_k A$ in $\mathcal{M}(A, U)$.
2. For any object $\Xi$ in $U_q(\mathfrak{l})$-$\text{mod}$, $S(\Xi)$ is an object of $\mathcal{P}(A, U)$.
Recall from [15] that part (2) of the theorem follows from part (1). Indeed, \( \Xi \) can always be embedded in the restriction of some finite dimensional \( U \)-module as a direct summand. That is, there exist a finite dimensional \( U \)-module \( W \) and a \( U_q(1) \)-module \( \Xi^+ \) such that \( W \cong \Xi \oplus \Xi^+ \) as \( U_q(1) \)-module. It follows from part (1) of Theorem 6.3 that \( S(\Xi) \oplus S(\Xi^+) \cong W \otimes A \). By Corollary 6.4, \( S(\Xi) \) is in \( \mathcal{P}(A,U) \).

We extend (6.2) to a covariant functor

\[
S : U_q(1)\text{-mod} \longrightarrow \mathcal{P}(A,U),
\]

which acts on objects of \( U_q(1)\text{-mod} \) according to (6.2) and sends a morphism \( f \) to \( f \otimes \text{id}_A \). Since \( U_q(1)\text{-mod} \) is semi-simple and \( S(V \oplus W) = S(V) \oplus S(W) \) for any direct sum \( V \oplus W \) of objects in \( U_q(1)\text{-mod} \), the functor \( S \) is exact.

Let \( I = \{ f \in A | f(1) = 0 \} \); this is a maximal two-sided ideal of \( A \). We have \( A/I \cong \mathbb{k} \). For any \( x \in U_q(1) \) and \( a \in I \), \( (R_x(a),1) = \epsilon(x)a(1) = 0 \), thus \( I \) forms a \( U_q(1) \)-algebra under the restriction of the action \( R \). This implies that for any \( U \)-equivariant \( A \)-module \( M \), \( IM \) is a \( U_q(1) \)-equivariant \( A \)-submodule of \( M \). This can be seen from the following calculation: for any \( a \in I \) and \( m \in M \), we have \( x(am) = \sum_x R_{x(1)}(a)x(2)m \in IM \) for all \( x \in U_q(1) \).

Given a \( U \)-equivariant \( A \)-module \( M \), let \( M_0 = M/IM \); this is a \( U_q(1) \)-equivariant \( A \)-module in which \( a \in A \) acts as \( a(1) \in \mathbb{k} \). Denote the natural surjection by

\[
p : M \longrightarrow M_0.
\]

This is an \( A-U_q(1) \)-linear map.

For any object \( M \) in \( \mathcal{M}(A,U) \), we can find a finite dimensional \( U \)-submodule \( W \) which generates \( M \). By Lemma 6.3, the \( A \)-map \( A \otimes_k W \longrightarrow M \), \( a \otimes w \mapsto aw \), is surjective. Therefore \( M_0 = p(W) \). Note that \( W \) is semi-simple as \( U \)-module and hence also as \( U_q(1) \)-module. Thus \( M_0 \) belongs to \( U_q(1)\text{-mod} \).

We therefore have a covariant functor

\[
\mathcal{E} : \mathcal{M}(A,U) \longrightarrow U_q(1)\text{-mod},
\]

which sends an object \( M \) in \( \mathcal{M}(A,U) \) to \( M_0 \), and is defined on morphisms in the obvious way. We may restrict this functor to the full subcategory \( \mathcal{P}(A,U) \) to obtain a covariant functor

\[
\mathcal{E}_P : \mathcal{P}(A,U) \longrightarrow U_q(1)\text{-mod}.
\]

**Theorem 6.4.** The functors \( S : U_q(1)\text{-mod} \longrightarrow \mathcal{P}(A,U) \) and \( \mathcal{E}_P : \mathcal{P}(A,U) \longrightarrow U_q(1)\text{-mod} \) respectively defined by (6.5) and (6.7) are mutually inverse equivalences of categories.

We will prove the theorem in Section 6.3. It has the following consequence.

**Corollary 6.5.** There is an isomorphism of abelian groups

\[
K^U_i(A) \cong K_i(U_q(1)\text{-mod})
\]

for each \( i \geq 0 \), where \( U_q(1)\text{-mod} \) is the category of finite dimensional left \( U_q(1) \)-modules of type-\((1,\ldots,1) \) (which is semi-simple).

This implies in particular that \( K^U_i(A) \) is isomorphic to the Grothendieck group of \( U_q(1)\text{-mod} \), a result proved in [15].
6.3. **Proof of Theorem 6.4.** We now prove Theorem 6.4 by means of a series of lemmas.

For any \( V \) in \( U_q(\mathfrak{l})\)-mod, \( S(V) \) is the subspace of \( U_q(\mathfrak{l}) \)-invariants in \( V \otimes_k A_\mathfrak{g} \) with respect to the action \( \text{id}_V \otimes L_{U(q)} \). The linear map \( V \otimes_k A_\mathfrak{g} \to V \) given by

\[
\zeta = \sum_i v_i \otimes f_i \mapsto \zeta(1) = \sum_i f_i(1)v_i,
\]
induces a linear map (‘evaluation’)

\[
ev : S(V) \to V, \quad \zeta \mapsto \zeta(1).
\]

Note that \( S(V) \) is an \( A\)-\( U_q(\mathfrak{l}) \)-module with the standard actions defined by \( (6.3) \) and the restriction of \( (6.4) \). We may also define an \( A \)-module structure on \( V \) in which each \( a \in A \) acts as scalar multiplication by \( a(1) \). This makes \( V \) into an \( A\)-\( U_q(\mathfrak{l}) \)-module.

**Lemma 6.6.** Any \( V \) in \( U_q(\mathfrak{l})\)-mod may be regarded as an \( A\)-\( U_q(\mathfrak{l}) \)-module as above. The map \( (6.8) \) is \( A\)-\( U_q(\mathfrak{l}) \)-linear.

**Proof.** Given its importance, we give a brief proof of this lemma. For any \( \zeta \in S(V) \) and \( u \in U_q(\mathfrak{l}) \), we have

\[
\sum_{(u)}(u(1) \otimes L_{u(2)})\zeta \otimes u(3) = \zeta \otimes u
\]
by the \( U_q(\mathfrak{l}) \)-invariance of \( S(V) \). Using \( \sum_{(a)}(u(1) \otimes L_{u(2)})\zeta(u(3)) = u \cdot ev(\zeta) \) and \( \zeta(u) = ev((\text{id}_V \otimes R_u)\zeta) \), we obtain \( u \cdot ev(\zeta) = ev((\text{id}_V \otimes R_u)\zeta) \). Finally, for any \( a \in A \), we have \( ev(a\zeta) = a(1)ev(\zeta) \). This completes the proof.

In view of the lemma and the fact that the map \( (6.6) \) is \( U_q(\mathfrak{l}) \)-linear, we have the following \( U_q(\mathfrak{l}) \)-map for each \( V \):

\[
(6.9) \quad \epsilon_V : E \circ S(V) \to V, \quad p(\zeta) \mapsto ev(\zeta) = \zeta(1).
\]

**Proposition 6.7.** The map \( \epsilon_V \) defined by \( (6.9) \) for each object \( V \) in \( U_q(\mathfrak{l})\)-mod gives rise to a natural transformation

\[
(6.10) \quad \epsilon : E \circ S \to \text{id}_{U_q(\mathfrak{l})\text{-mod}};
\]

which is in fact a natural isomorphism.

**Proof.** For any map \( \alpha : V \to V' \) in \( U_q(\mathfrak{l})\)-mod, \( E \circ S(\alpha) \) is given by

\[
E \circ S(\alpha)(p(\zeta)) = p((\alpha \otimes \text{id}_A)\zeta), \quad \forall p(\zeta) \in E \circ S(V).
\]

Now \( \epsilon_V \circ p((\alpha \otimes \text{id}_A)\zeta) = ((\alpha \otimes \text{id}_A)\zeta)(1) = \alpha(\zeta(1)) \). This proves the commutativity of the following diagram

\[
\begin{array}{ccc}
E \circ S(V) & \xrightarrow{\epsilon_V} & V \\
\downarrow & & \downarrow \alpha \\
E \circ S(V') & \xrightarrow{\epsilon_V'} & V'
\end{array}
\]
where the left vertical map is \( E \circ S(\alpha) \). Hence \( \epsilon \) is a natural transformation between the functors \( E \circ S \) and \( \text{id}_{U_q(\mathfrak{l})\text{-mod}} \).

We have already noted that any module \( V \) in \( U_q(\mathfrak{l})\)-mod may be embedded in the restriction of a finite dimensional \( U \)-module \( W \) as a direct summand, that is,
by Theorem 6.3(2). Using this isomorphism, we obtain that $S \circ E \cong W$, $ev(S(W)) = W$, and that $\epsilon_W : E \circ S(W) \rightarrow W$ is an isomorphism. Since $S(V) \oplus S(V^\perp) \cong W \otimes_k A$, we have $\epsilon_W = \epsilon_V \oplus \epsilon_{V^\perp}$. As $\epsilon_W$ is an isomorphism, both $\epsilon_V$ and $\epsilon_{V^\perp}$ are isomorphisms.

It follows from Appendix A that there is a one to one correspondence between locally finite left $U$-modules (of type-(1, 1, . . . , 1)) and right $A_q$-comodules. For any locally finite left $U$-module $M$, denote the corresponding right $A_q$-comodule structure map by

$$\delta_M : M \rightarrow M \otimes_k A_q, \quad w \mapsto \delta_M(w) = \sum_{(w)} w_{(1)} \otimes w_{(2)}.$$ 

Let $M$ be and object of $\mathcal{M}(A, U)$, and consider the composition $M \xrightarrow{\delta_M} M \otimes_k A_q \xrightarrow{p \otimes \text{id}} p(M) \otimes_k A_q$. Note that for any $m \in M$ and $u \in U_q(1)$, we have

$$\sum p(u_{(1)}m_{(1)}) \otimes L_{u_{(2)}}(m_{(2)}) = \sum p(m_{(1)}) \otimes \langle m_{(2)}, u_{(1)} \rangle \langle m_{(3)}, S(u_{(2)}) \rangle m_{(4)} = \epsilon(u) \sum p(m_{(1)}) \otimes m_{(2)}.$$ 

Hence the image of this map is contained in $S(\mathcal{E}(M))$, and we obtain a map

$$\hat{\delta}_M = (p \otimes \text{id}) \circ \delta_M : M \rightarrow S(\mathcal{E}(M)),$$

which is clearly $A$-linear. Moreover for any $m \in M$ and $x \in U$, we have

$$\hat{\delta}_M(xm) = \sum p(m_{(1)}) \otimes R_xm_{(2)} = x\hat{\delta}_M(m).$$

Thus the map is also $U$-linear, and is therefore a homomorphism of $A - U$-modules. This leads to the following result, which was proved in [34, Proposition 3.8] in a slightly different form, and was used there to show that the Grothendieck groups of $\mathcal{P}(A, U)$ and $U_q(1)-\text{mod}$ are isomorphic.

**Lemma 6.8.** If $M$ is in $\mathcal{P}(A, U)$, then $\hat{\delta}_M : M \rightarrow S \circ \mathcal{E}(M)$ is an isomorphism of $A - U$-modules.

**Proposition 6.9.** The maps $\hat{\delta}_M$, for $M$ in $\mathcal{M}(A, U)$ define a natural transformation $\text{id}_{\mathcal{M}(A, U)} \rightarrow S \circ \mathcal{E}$.

**Proof.** Let $\beta : M \rightarrow N$ be a morphism in $\mathcal{M}(A, U)$. Then

$$S \circ \mathcal{E}(\beta) \delta_M = (p_N \otimes \text{id})(\beta \otimes \text{id})\delta_M,$$

where $p_N$ is the map [6.6] for $N$. Using the fact that $(\beta \otimes \text{id})\delta_M = \delta_N\beta$, we obtain $S \circ \mathcal{E}(\beta) \delta_M = \hat{\delta}_N \beta$. That is, the following diagram commutes

$$\begin{array}{cc}
M & \xrightarrow{\delta_M} & S \circ \mathcal{E}(M) \\
\beta \downarrow & & \downarrow \\
N & \xrightarrow{\hat{\delta}_N} & S \circ \mathcal{E}(N)
\end{array}$$

where the right vertical map is $S \circ \mathcal{E}(\beta)$. \qed

The next statement is an immediate consequence of Lemma 6.9 and Lemma 6.8.
Proposition 6.10. There is a natural isomorphism \( \hat{\delta} : \text{id}_{\mathcal{P}(A, U)} \longrightarrow \mathcal{S} \circ \mathcal{E}_\mathcal{P} \).

Proof of Theorem 6.4. It follows from Proposition 6.7 and Proposition 6.10 that the categories \( U_q(\mathfrak{l})\text{-mod} \) and \( \mathcal{P}(A, U) \) are equivalent. \( \square \)

Appendix A. Comodules and smash products

The notions of module algebras and equivariant modules can also be formulated in terms of co-algebras and comodules, which we discuss briefly here. Some of this material is used in Section 6.3. We shall also discuss the relationship between locally finite equivariant modules over a \( U \)-module algebra \( A \), and finitely generated modules over the smash product of \( A \) and \( U \).

Recall from Section 6.3 the coordinate functions of the \( U \)-representations associated with the \( U \)-modules in \( U\text{-mod} \) span a Hopf algebra \( \mathcal{O}_q(U) \).

A comodule over \( \mathcal{O}_q(U) \) is a vector space \( M \) with a \( k \)-linear map \( \delta : M \longrightarrow M \otimes_k \mathcal{O}_q(U) \) satisfying the following conditions:

\[
(\delta \otimes \text{id}_{\mathcal{O}_q(U)}) \circ \delta = (\text{id}_M \otimes \Delta) \delta, \quad (\text{id}_M \otimes \varepsilon) \delta = \text{id}_M.
\]

We use Sweedler’s notation \( \delta(v) = \sum_{(v)} v_1 \otimes v_2 \) for the co-action on \( v \in M \), so that \( v_1 \in M \), while \( v_2 \in \mathcal{O}_q(U) \).

A locally finite left \( U \)-module \( M \) of type-\((1, \ldots, 1)\) is naturally a right comodule over \( \mathcal{O}_q(U) \), and vice versa. The comodule structure map \( \delta : M \longrightarrow M \otimes \mathcal{O}_q(U) \) and the module structure map \( \phi : U \otimes M \longrightarrow M \) are related to each other by

\[
\delta(v)(x) = \phi(x \otimes v), \quad \text{for all} \quad x \in U, \; v \in M.
\]

In this definition, local \( U \)-finiteness of \( M \) is needed in order for \( \delta(v) \) to lie in \( M \otimes \mathcal{O}_q(U) \) for all \( v \in M \).

If a \( U \)-module is not locally finite, it does not correspond to any comodule over \( \mathcal{O}_q(U) \). The local finiteness condition is built into the definition of comodules because \( \mathcal{O}_q(U) \) is the Hopf algebra spanned by the coordinate functions of \( U \)-representations corresponding to objects in \( U\text{-mod} \).

Now a locally finite \( U \)-module algebra \( A \) of type-\((1, \ldots, 1)\) is nothing but an associative algebra which is a right comodule over \( \mathcal{O}_q(U) \) satisfying the conditions

\[
\delta_A(1_A) = 1_A \otimes \varepsilon, \quad \delta_A(ab) = \sum_{(a), (b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}, \quad \forall a, b \in A,
\]

where \( \delta_A \) is the comodule structure map of \( A \).

Let \( A \) be a locally finite \( U \)-module algebra of type-\((1, \ldots, 1)\), and as above, denote the co-action of \( \mathcal{O}_q(U) \) on any \( a \in A \) by \( a \mapsto \sum_{(a)} a_{(1)} \otimes a_{(2)} \).

A locally \( U \)-finite, type-\((1, \ldots, 1)\), \( U \)-equivariant \( A \)-module \( M \) is a left \( A \)-module, which is also a right \( \mathcal{O}_q(U) \)-comodule, such that the \( A \)-module structure and \( \mathcal{O}_q(U) \)-comodule structure \( \delta_M : M \longrightarrow M \otimes \mathcal{O}_q(U) \) are compatible in the sense that for all \( a \in A \) and \( v \in M \),

\[
\delta_M(av) = \sum_{(a), (v)} a_{(1)} v_{(1)} \otimes a_{(2)} v_{(2)}.
\]

A notion closely related to equivariant modules is the smash product [23, Definition 4.1.3]. Given a \( U \)-module algebra \( A \), one may construct the smash product \( R := \)}
$A \# U$, which is an associative algebra with underlying vector space $A \otimes_k U$ and multiplication defined by

$$(a \otimes u)(b \otimes v) = \sum_{(u)} a(u_{(1)} \cdot b) \otimes u_{(2)} v$$

for all $a, b \in A$ and $u, v \in U$.

It is easy to show that a left $U$-equivariant $A$-module is in fact a left $R$-module. However, a finitely generated $R$-module need not be locally $U$-finite, and so is not generally in $\mathcal{M}(A, U)$. Therefore, $\mathcal{P}(A, U)$ is different from the category of finitely generated projective $R$-modules.

A case in point is the following example. Take $A = \mathbb{C}(q)$ be equipped with trivial $U$-action (through the co-unit). In this case, both $\mathcal{M}(A, U)$ and $\mathcal{P}(A, U)$ coincide with $U\text{-mod}$. On the other hand, the smash product $R = A \# U$ is $U$ itself. The category of finitely generated projective $U$-modules is totally different from the category $U\text{-mod}$.

This shows that the equivariant K-theory of the $U$-module algebra $A$ introduced in Section 2.2 is quite different from the usual K-theory of the smash product $R := A \# U$, a fact which we have already pointed out in Remark 2.6.

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