Shell structure in the lightest nuclei emerges from an unnaturally small expansion parameter in nuclear effective field theory

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The strong interaction, i.e., quantum chromodynamics at the low energy nuclear regime, is notoriously known to be challenging for predictive modelling. Here, we use the simplest possible nuclear effective field theory (EFT), and show that in the case of the magnetic structure of nuclear systems with $A = 2$ and $A = 3$ nucleons it is highly precise, as well as predictive. The theoretical framework is the pionless EFT (πEFT), of point nucleons with contact interactions, expanded consistently up to next-to-leading order (NLO) in perturbation theory, i.e., including only eleven low-energy parameters, and augmented by novel bayesian analysis of theoretical uncertainties. The theory predicts accurately the shell structure reflected in the values of the magnetic moments and reactions of these nuclei, within $\approx 1\%$ calculated theoretical uncertainty. We show that this perfect prediction originates in implicit a-posteriori properties of the calculation, particularly an unexpectedly small expansion parameter, as well as vanishing contribution from the two-body isoscalar current in contrast with the naïve dimensional analysis of πEFT, indicating a misconception in identifying the breakdown scale to the theory. We discuss possible origins and consequences of these findings.

I. INTRODUCTION

Low-energy magnetic transitions and magnetic moments of nuclei are sensitive to the nuclear structure and, thus, are a window into the properties of the nuclear interaction. In addition, the electromagnetic interaction shares several similarities with the weak interaction, thus providing insight into weak reactions with nuclei, which are of importance in the theoretical modeling of stellar evolution and supernovae, and are sometimes out of reach experimentally. A direct calculation of these magnetic observables from the more fundamental theory, \textit{i.e.}, from Quantum Chromodynamics (QCD), is challenging due to the non-perturbative character of QCD at the nuclear regime.

In fact, lattice formulations of QCD have only recently been able to calculate such properties, though the calculations are still restricted to unphysical quark masses \[1, 2\]. The latter reproduce the empirical fact that in nature and for the unphysical quark masses considered by lattice QCD calculations, the nuclear magnetic moments of nuclei of masses $A = 2, 3$ follow a perturbative trend, a shell model trend, in the sense that the Deuteron ($d$ or $^2$H) magnetic moment, \(\langle \mu_d \rangle\), is found to be approximately the sum of the proton and neutron magnetic moments, \(\langle \mu_d \rangle \approx \mu_p + \mu_n\), and the Triton ($^3$H) and Helium-3 magnetic moments are approximately the magnetic moments of the proton and neutron, respectively, consistent with a shell model behavior of a valence nucleon weakly bound to a core.

This seemingly perturbative and linear behavior is in sharp contrast with the non-perturbative quantum character of the structure of these nuclei. The strong interaction between the nucleons leads in the $A = 2$ nuclear system to large nucleon-nucleon scattering lengths compared to the range of the interaction, or equivalently, as Bethe showed using an effective range expansion (ERE) \[3\], to a vanishing binding energy of the Deuteron, compared to the natural scale of QCD excitations. Moreover, in $A = 3$ nuclei, the large nucleon-nucleon scattering length leads to a realization of the Efimov effect, making the introduction of a three-body force an essential to stabilize these nuclei.

ERE can be formally phrased by the use of effective field theory (EFT) \[4, 5\]. EFT expansion relies on scale separation in the momentum-energy regime, and thus it is natural for these nuclei, since the binding momenta, $Q$, set by the scattering lengths, are well separated from the pion scale, related to the effective range. This is the “pionless” version of nuclear EFT (πEFT), applicable with the breakdown scale set by the pion mass $m_\pi \sim 140$ MeV \[6, 7\]. The momentum scale suggests that the viable degrees of freedom eminent in determining the structure are nucleons, which interact through contact interactions. πEFT assumes all particles, but the nucleons, are “integrated out”, and their properties dictate the size of the effective Lagrangian coefficients, called low-energy constants (LECs). The value of the LECs can be determined from experimental data \[8, 9\] or using Lattice QCD simulations \[10\].

Indeed, in the field theoretical formalization of πEFT, wave functions are a result of a loop integration over all the possible momenta with a cutoff $\Lambda$ \[4, 5, 11, 12\]. It can be analytically and numerically shown that the solution of the $A = 3$ integral equations reveals a strong dependence on the cutoff, resulting in the addition of a one-parameter 3-body force counter-term, $H(\Lambda)$, al-
ready at leading order (LO) as expected from the Efimov effect [4], to assure renormalization group (RG) invariance. A recent surge in the study of realistic nuclei using #EFT has shown that an additional counter-term is needed to ensure RG invariance at next-to-leading order (NLO), when the Coulombic repulsion between protons is considered [13]. In this work, we focus on the question whether the magnetic structure of $A = 2$ and $A = 3$ systems can be described successfully with such a simple theory, consistent with the differences between $A = 2$ and $A = 3$ systems, or should many-body counter term be introduced to assure RG invariance similar to those in the strong Hamiltonian. We make use of our recently developed renormalization group invariant method to calculate $A = 3$ matrix elements within #EFT [14] to study simultaneously $A = 2$ and $A = 3 M_1$ properties beyond LO. We introduce a novel method for assessing the theoretical uncertainty due to this EFT truncation, utilizing the perturbative property of this approach. Moreover, we study different ways to implement the perturbative expansion of #EFT to NLO, roughly analogous to choosing different expansion points inside the convergence radius of a Taylor expansion. We find that one of these, named the $Z$-parameterization, shows improved convergence pattern over the other. Finally, we show that the statistical analysis of the $A < 4 M_1$ observables leads to unexpected results, which might be connected, as numerically hinted, to the nature of the renormalization group flow of QCD to low-energies, through EFTs with higher breakdown scale, which are of interest in nuclear physics due to their relevance to the study of heavier nuclei.

To this end, we examine the four well-measured low-energy magnetic “$M_1$” reactions, in the the $A = 2, 3$ mass nuclear systems, i.e., the magnetic moments of the bound nuclei $\langle \mu_d \rangle$, $\langle \mu^{\text{He}} \rangle$ and $\langle \mu^{\text{H}} \rangle$, and the cross-section $\sigma_{n\rho}$ for the radiative capture $n + p \to d + \gamma$ for thermal neutrons [17]. Previous studies using phenomenological approach to nuclear physics (see a recent review in Ref. [18]) and using chiral EFT [19], however do not allow to study the reactions in a consistent perturbative approach, as well as from RG perspective. Within the #EFT framework, a large body of work has been done on the $A = 2$ aforementioned observables, with particular emphasis on $\sigma_{n\rho}$, due to its relevance to big-bang nucleosynthesis in energy regimes characterized by large experimental uncertainties [20]. Recently, exploratory studies of the $A = 3$ magnetic $M_1$ calculations, either to small cutoffs in a configuration space Schrödinger equation representation of #EFT [24], or without including the Coulomb interactions for $^3\text{He}$ [26]. As a result of the approximations, these studies could not approach the aforementioned questions.

II. SETTING UP #EFT TO NEXT-TO-LEADING ORDER

At LO, the two-body #EFT Lagrangian is minimally built with SU(2) nucleon fields to reproduce the bound spin-triplet channel ($t$, Deuteron) binding momentum, $\gamma_t \sim O(Q)$ and the unbound spin-singlet channel, $s$, scattering length, $a_s \sim O(1/Q)$. These values are unnatural compared to the QCD excitations (specifically, compared to the pion mass), and are the signature of the “strength” of the strong interaction, i.e., of the unnaturally large cross-section compared to the nuclear matter-radius. Details of the interaction enter at NLO, through its effective range in the triplet and singlet channels, $\rho_t$ and $\rho_s$, respectively, which scale as $O(1/\Lambda_b)$. Clearly, the nuclear system at low energies is characterized by the properties of two-body clusters, thus, it is convenient to use a Hubbard-Stratonovich (H-S) transformation to equivalently reformulate #EFT with dynamical dinucleon fields alongside with the nucleon field [27, 28]. The fields, $t$ and $s$, have quantum numbers of two coupled nucleons in an S-wave spin-triplet and -singlet state, respectively. This simplifies the calculation of a three-body amplitude by turning it into an effective two-body scattering problem of a dinucleon and a nucleon (see, for example, Refs. [27, 28]).

This explanation entails that the effective ranges vanish at LO, and receive a finite value at NLO. For the unbound spin-singlet state the natural choice is called effective-range (ER) parameterization in which $\rho_s = \rho_s^{\text{LO}} + \rho_s^{\text{NLO}}$, where $\rho_s^{\text{LO}}$ is the experimental value. However, for the bound triplet channel, there is another useful choice for its effective range value at NLO. Since it is bound, the long-range properties of the Deuteron wave function can be set by a quantity $Z_d$, defined through the Deuteron asymptotic S-state normalization, $A_S$, such that $A_S \equiv \sqrt{Z_d} \gamma_t$. $A_S$ roughly dictates the long range normalization of the deuteron wave function, $Z_d = \frac{1}{1 - \gamma_t \rho_t} \approx 1 + \gamma_t \rho_t$.

In the ER expansion, $\rho_t$ is fixed to its the physical value already at NLO, which is reflected in $\sim 17\%$ deviation of $Z_d$ from its physical value. The alternative arrangement of the #EFT is called $Z$-paramerization, and it fixes $Z_d$ to its experimental value at NLO, i.e., $Z_d^{\text{NLO}} = Z_d^{\text{exp}} \approx 1.690(3)$, while $\rho_t$ is subsequently derived to be, $\rho_t^2 = \frac{Z_d^{\text{exp}} - 1}{\gamma_t^{\text{LO}}} = 2.976(5)$ fm, deviating significantly by about 40% (!) from its physical value (see Tab. [1]). [29, 34]. It was shown that this choice recovers elastic scattering data better and converges faster in the EFT expansion [32, 35].

In the following, we use both NLO parameterizations, check that they are RG invariant and compare them in the context of $A < 4 M_1$ observables.

The three-nucleon scattering amplitude is a result of
the full solution of the coupled channel Faddeev integral equations. The different channels for $^3$H are the spin-triplet - $t$ (representing an “off-shell” Deuteron, $d$, dibaryon), and the spin-singlet - $s$ ($nn, np$). For $^3$He, the contributing channels are the spin-triplet - $t$, spin-singlet - $s$ ($np$) and $pp$ [11], where the latter is required because of the Coulomb force between the protons, which is fully considered. The nuclear amplitudes we use here are taken explicitly from Ref. [14], where they are benchmarked numerically, and validated using the binding energy difference between $^3$H-$^3$He.

III. M$_1$ OBSERVABLES IN THE A < 4 SYSTEMS

$M_1$ observables at vanishing momentum transfer are related to the electromagnetic nuclear current density $\hat{J}(\vec{q})$. Explicitly, the magnetic moment of a bound state is just the expectation value of the operator:

$$\hat{\mu} = -\frac{i}{2} \vec{\sigma} \times \hat{J}(\vec{q}) \big|_{q=0},$$  \hspace{1cm} (1)

while $\sigma_{np}$ is proportional to the transition matrix element of the same operator between the neutron and proton, $S = 0$ state, to the Deuteron, $S = 1$ state [24, 27, 40].

A magnetic photon interaction with a nucleus can be modeled effectively as interaction with ever-growing clusters of nucleons. In yEFT, LO includes a single nucleon interaction with a photon, while the interaction of a magnetic photon with two-body clusters appears for the first time at NLO [22, 24, 27].

The one-body electromagnetic Lagrangian is given by (see, for example, Ref. [23]):

$$\mathcal{L}^{1-B}_{\text{magnetic}} = \frac{e}{2M} N^\dagger \left( \kappa_0 + \kappa_1 T_3 \right) \vec{\sigma} \cdot \vec{B} N,$$  \hspace{1cm} (2)

where $\vec{B}$ is the magnetic field, $e$ is the electron charge, while $\kappa_0$ and $\kappa_1$ are the isoscalar and isovector magnetic moments of the nucleon.

The NLO interaction of a magnetic photon with a two-body nuclear field is given by the two-body electromagnetic Lagrangian in the form of two four-nucleon-one-magnetic-photon operators:

$$\mathcal{L}^{2-B\text{-magnetic}} = e \left[ L_1' \left( N^T P_s^A N \right)^\dagger \left( N^T P_t^A N \right) B_t \right. $$
$$- L_2' \left( N^T P_s^A N \right)^\dagger \left( N^T P_t^A N \right) B_k + h.c. \bigg].$$  \hspace{1cm} (3)

Applying the H-S transformation on eq. (3) leads to the interaction in terms of the dibaryon fields (see [14, 24]):

$$\mathcal{L}^{2-B\text{-magnetic}} = \frac{e}{2M} \left[ \kappa_1 L_1 (t^\dagger s + s^\dagger t) \cdot \vec{B} - i e \varepsilon^{ijk} \kappa_0 L_2 (t^\dagger t^i) \cdot B_k \right].$$  \hspace{1cm} (4)

The LECs $L_1$ and $L_2$ can be separated to a LO part, which is related to a consistent ERE, and NLO pure two-body contributions [24, 41],

$$L_1(\mu) = -\frac{\rho_s + \rho_t}{\sqrt{\rho_s \rho_t}} + \frac{4}{\gamma t \rho_t} \frac{I_{1s}^s(\mu)}{NLO},$$  \hspace{1cm} (5)

$$L_2(\mu) = -\frac{2}{\gamma t} + \frac{2}{\gamma t} \frac{I_{1s}^t(\mu)}{NLO},$$  \hspace{1cm} (6)

where $\rho_s$, $\rho_t$ are renormalization scale independent for $\mu \to \infty$. In this work, contrary to past studies on the electromagnetic properties of light nuclei, which arbitrarily took $\mu = m_\pi$, we check the full renormalizability to essentially infinite cutoffs, and use the value of the parameters at $\mu \to \infty$. We note that since the photon field $\vec{A}$ fulfills $\vec{B} = \vec{\nabla} \times \vec{A}(\vec{x})$, the scattering operator $\hat{\mu}$ is given by the prefactor of $\vec{B}$ in eqs. (2) and (4). Feynman rules are extracted trivially using this fact.

Given the above, the $A < 4 \ M_1$ observables are calculated consistently up to NLO in the following way [14]:

$$\langle \hat{\mu} \rangle = \langle \hat{\mu} \rangle^{1-B}_{LO} \times \begin{pmatrix} \frac{1}{LO} + \delta(\hat{\mu})^{1-B}_{ERE} + \delta(\hat{\mu})^{2-B}_{ERE} + \delta(\hat{\mu})^{2-B}_{NLO \text{ magn. op.}} - \delta(\hat{\mu})^{2-B}_{LO \text{ stor. inter.}} \end{pmatrix}.$$  \hspace{1cm} (7)

A. The $A < 4 \ M_1$ matrix elements

1. Two-nucleon electromagnetic matrix elements

The matrix element of $\hat{\mu}$ (eq. [11]) between two-nucleon states is represented diagrammatically in Fig. [1]. This matrix element is related to the calculation of $\sigma_{np} (\langle \hat{\mu}_{ab} \rangle)$ if the initial state is in a relative $S_0$ ($S_1$) state. The field $S_1$ state represents the deuteron.
From Fig. 1 one concludes that up to NLO, \( \langle \hat{\mu}_d \rangle \) is given by:

\[
\langle \hat{\mu}_d \rangle = \kappa_0 \left\{ 2Z_{NLO}^d + Z_{LO}^d [\gamma_I \rho_t L_2(\mu)] \right\} = 2\kappa_0 \left[ 1 + \frac{\gamma_I}{\gamma_I - \kappa_1} \frac{\gamma_I}{\gamma_I - \kappa_2} \right], \tag{8}
\]

The cross-section of \( n + p \rightarrow d + \gamma \) is related to the matrix element \( Y \) by:

\[
\sigma_{np} = 2\alpha_\pi \left( \gamma_I^2 + q^2/4 \right) \frac{a_s^2}{M^4 q^2} Y_{np}^2 = 2\alpha_\pi \gamma_I^2 a_s^2 \left( \frac{2\kappa_1}{M^4 q} \right)^2 (Y_{np}')^2, \tag{9}
\]

where \( Y_{np} \) is the sum over all the diagrams of Fig. 1 and \( q = 0.0069 \text{MeV}/c \) is the momentum transfer for thermal neutrons [10, 26, 42].

The normalized matrix element, \( Y_{np}' \), up to NLO is also obtained by Fig. 1 to yield:

\[
Y_{np}' = \left( 1 - \frac{1}{\gamma_I a_s} \right) \times \left[ 1 + \sqrt{Z_N^{NLO} - 1 - \frac{\gamma_I a_s}{\gamma_I a_s - 1} \left( \frac{\gamma_I}{\gamma_I - 4} \right)} \right]. \tag{10}
\]

The above expressions up to higher-order corrections can be found in the literature (see, e.g., [24, 43]).

2. Three-nucleon electromagnetic matrix elements

In Ref. [14], we presented a general perturbative diagrammatic approach for calculating one- and two-body matrix elements between initial and final three-nucleon bound-states up to NLO. In this work, we use this method to calculate the \( M_1 \) observables in the \( A = 3 \) system (see Appendix A). The \( A = 3 \) \( M_1 \) observables, can be separated to different contributions, similarly to Eq. [7] though calculated numerically (using the experimental input parameters shown in Tab. II), see Ref. [14].

IV. RESULTS AND ANALYSIS

The \( A < 4 \) \( M_1 \) observables depend upon the physical (RG invariant) values of the LECs, i.e., \( l_{1,2}' \sim \infty \). In past works, the experimental values of the \( A = 2 \) observables (\( \sigma_{np} \) and \( \langle \hat{\mu}_d \rangle \)) were used to fix these LECs [24, 41]. Here, we calculate consistently the \( A < 4 \) \( M_1 \) observables which depend on the same LECs, so we can extract these LECs from two observables and then use them to predict the remaining two observables. Therefore, we have six independent ways for calibrating the LECs. These calibrations are used to check whether this formalism can be consistently used to describe simultaneously \( A = 2 \) and \( A = 3 \) \( M_1 \) observables using \#EFT .

Tab. II (a) summarizes our predictions for \( l_{1,2}' \sim \infty \) and \( M_1 \) observables up to NLO in both \( Z \)- and \( ER \)- parameterization. For each row, the ‘+’ denotes the \( M_1 \) observables used to calibrate \( l_{1,2}' \sim \infty \). For example, the first row of Tab. II (a) shows the LECs fixed from \( A = 3 \) observables and our prediction of \( A = 2 \) magnetic observables, while the second row of Tab. II (a) shows the LECs fixed from \( A = 2 \) observables and the prediction of \( A = 3 \) magnetic observables. Note that for each \( M_1 \) observable we have three predictions.

The numerical results of \( l_{1}'(\Lambda) \) and \( l_{2}'(\Lambda) \) are shown in Fig. 2 and show that they become RG invariant at \( \mu \sim \) few \( A_0 \), if they are fixed in the \( A = 3 \) systems.
FIG. 2: Numerical results for LECs $l_1'$ ($\Lambda$) (left panel) and $l_2'$ ($\Lambda$) (right panel), calibrated from the $M_1 = 3$ observables as a function of the cutoff $\Lambda$. The long (short) dotted-dashed lines are the numerical results for the ER-($Z$-) parameterization.

| $l_1'^\infty / 10^{-2}$ | $l_2'^\infty / 10^{-2}$ | $\langle \hat{\mu}_H \rangle$ [NM] | $\langle \hat{\mu}_He \rangle$ [NM] | $\langle \hat{\mu}_d \rangle$ [NM] | $Y_n^\prime$ |
|-------------------------|-------------------------|----------------------------------|----------------------------------|----------------------------------|---------|
| LO                     | 0 (0)                   | 0 (0)                            | 2.76 (2.78)                      | 1.84 (1.84)                      | 0.88 (0.88) |
| NLO                    | 4.72 (14.2)             | -1.6 (4.1)                       | *                                | *                                | 0.87 (0.92) |
|                         | 4.66 (9.0)              | -2.6 (-2.6)                      | 2.978 (2.76)                     | 2.145 (1.89)                     | *        |
|                         | 4.66 (9.0)              | -2.4 (29)                        | *                                | 2.144 (1.66)                     | *        |
|                         | 4.66 (9.0)              | -0.13 (-31)                      | *                                | 0.88 (0.61)                      | *        |
|                         | 4.92 (15.2)             | -2.6 (-2.6)                      | *                                | 2.143 (2.23)                     | *        |
|                         | 4.60 (13.4)             | -2.6 (-2.6)                      | 2.967 (2.91)                     | *                                | 1.255 (1.32) |
| Mean                   | 4.73 (13.0)             | -1.7 (-0.04)                     | 2.98 (2.75)                      | 2.144 (1.93)                     | 0.87 (0.89) |
| std                    | 0.2 (2.8)               | 1.1 (25)                         | 0.015 (0.16)                     | 0.001 (0.28)                     | 0.01 (0.26) |
| Exp data               | 2.979 [44]              | 2.128 [44]                       | 0.857 [16]                       | 1.253 [17]                       |         |

(a) Numerical results for $l_1'^\infty$, $l_2'^\infty$ and $A = 2, 3$ $M_1$ observables. The nominal value is calculated using $Z$-parameterization, while the number in brackets is calculated using the ER-parameterization. “Mean” denotes the mean value of the $M_1$ observable based on its three (independent) predictions, while “std” denotes the standard deviation of these independent predictions.

TABLE II. (b) The order-by-order contributions of the $M_1$ matrix elements, based on their mean values given in Tab. II (a). The nominal value is calculated using $Z$-parameterization, while the number in brackets is calculated using ER-parameterization. The three nuclear magnetic moments are given in nuclear magnetons [NM].

In addition, in Tab. II (b) we present the three contributions to the different $M_1$ observables for both the ER- and $Z$-parameterizations.

This verifies that the πEFT calculations presented in this paper are purely perturbative, i.e., consistently organizing the expansion in a perturbative manner, and built theoretically without including any higher-order terms. Moreover, an order-by-order renormalization was obtained, as shown numerically, by the cutoff invariance (see Fig. 2) with a small expansion parameter of about $0.05 - 0.2$. In chiral effective field theory ($\chi$EFT), as well as in πEFT configuration space schemes [25, 45], a cutoff variation is frequently used to obtain an uncertainty estimate. Here we show that the main advantage of using the current formalism of πEFT is the cutoff invariance, which even for $A = 3$ calculations is obtained at a natural scale [46]. This cutoff independence not only removes questions regarding residual cutoff dependencies that might contribute to the total uncertainty [25, 45], but also allows giving physical meaning to the size of the NLO contribution.
A. Z-parameterization is better than the ER-parameterization at NLO

The comparison between the results of the Z- and ER-parameterizations reveals some interesting features. The ratios between the NLO and LO values are of the same order of magnitude, slightly smaller in the ER-parameterization. Naively, this can be interpreted as an indication of a better convergence pattern of the ER-parameterization. However, a closer look shows the contrary. In the ER-parameterization, one observes a large cancellation between the different contributions to the NLO, i.e., between the range corrections and the pure two-body contact contributions. Each of the NLO contributions is usually more than 10% of the LO, while the final NLO contribution is an order of magnitude smaller. The Z-parameterization shows a natural convergence pattern, where all the contributions are of the same order of magnitude. Second, and as a consequence of the former point, the resulting statistical standard deviations (std) between the predictions using the different LECs calibrations (see Tab. II (a)) for the four magnetic observables are of an order of magnitude bigger in the ER-parameterization than those of the Z-parameterization. Third, the statistical fluctuations in the sizes of the LECs, as can be seen in Tab. II (a), are much bigger in the ER-parameterization.

The large variations and fluctuations of the ER-parameterization raise questions about its relevance at NLO for predictions of electromagnetic observables. The advantage of Z-parameterization over ER-parameterization at NLO, as explicitly demonstrated here in the magnetic properties of the $A < 4$ systems, is consistent with the initial motivation for introducing Z-parameterization \cite{30, 32, 35}. In the next sub-sections, we examine the $\pi\pi$EFT NLO’s contributions and estimate its truncation error only for the Z-parameterization, which shows a more natural convergence pattern, and therefore expected to have better predictive power.

B. Isoscalar two-body coupling is consistent with zero

Interestingly, we find that while $l_2^{\infty}$ has minor dependence on the $M_1$ observables used for its calibration, i.e., $\Delta l_1^{\infty}/l_1 \approx 3\%$, the standard deviation of $l_2^{\infty}$ is of the same order of magnitude as $l_2^{\infty}$, i.e., $\Delta l_2^{\infty} / l_2^{\infty} \approx 70\%$, in the Z-parameterization. The differences are even more significant in the case of ER-parameterization, where $\Delta l_2^{\infty} / l_2^{\infty} \approx 21\%$, and $\Delta l_{2 ER}^{\infty}$ is two orders of magnitude larger than $l_{2 ER}^{\infty}$.

Moreover, the NLO contribution to the Deuteron magnetic moment is very small, in fact, it is much smaller than the NLO contributions of the other observables. The two-body contribution to $\hat{\mu}_d$, as seen in eq. \cite{8}, depends only on $l_2^{\infty} = (\pm 1.7 \pm 1.1) \cdot 10^{-2}$.

These two observations show that $l_2^{\infty}$ is consistent with zero. One interpretation, which we conjecture, is that the isoscalar pure two-body contribution, whose LEC is $l_2^{\infty}$, might be regarded as a higher order than NLO, in contrast to the naïve dimensional analysis of $\pi\pi$EFT, where the two-body isovector and isoscalar contributions are both treated as NLO \cite{27, 43}.

To check the consistency of our conjecture, we study the ramifications of vanishing $l_2^{\infty}$. Similarly to Tab. II (a), for each row in Tab. III the ‘$\times$’ denotes the $M_1$ observable used for $l_2^{\infty}$ calibration.

| $l_1^{\infty} / 10^{-2}$ | $\langle \mu_3(\pi_H) \rangle_{[NM]}$ | $\langle \langle \mu_3(\pi_H) \rangle_{[NM]} \rangle$ | $\% NLO / LO$ |
|--------------------------|---------------------------------|---------------------------------|------------------|
| 4.36                     | $\times$                        | -2.10                           | 6%               |
| 4.97                     | 3.00                            | $\times$                        | 13%              |
| 4.66                     | 2.99                            | -2.11                           | 13%              |
| Mean                     | 4.7                             | 2.99                            | 12.18            |
| Std                      | 0.6                             | 0.01                            | 0.006            |

Table III: Numerical results for the calibrated values of $l_1^{\infty}$ and the resulting predictions of $M_1$ observables up to NLO for the Z-parameterization for the case that $l_2^{\infty} = 0$.

Table II shows that setting $l_2^{\infty}$ to zero does not reduce the quality of $l_1^{\infty}$ and $M_1$ predictions, in terms of the size of the NLO contribution compared to the LO one, and the statistical accuracy of the predictions given different experimental constraints. This implies that there is no inconsistency in assuming that the isoscalar $l_2^{\infty}$ contribution to the $M_1$ matrix elements is suppressed compared to NLO contributions. This is a main result of this paper, and it will be discussed further in subsection IV C. One is also tempted to compare the predictions to the experimental values. However, for a complete comparison, one needs to estimate the theoretical uncertainty, which is the subject of the next subsection.

C. Estimating theoretical uncertainty

The aforementioned fact that EFT is a systematic expansion in some small parameter, $\delta$, is particularly helpful for estimating theoretical uncertainties in the calculation. A common approach is to study the residual cutoff dependence and to use it as a measure for the uncertainty (see, e.g., \cite{25, 45} in the context of $\pi\pi$EFT). The order-by-order renormalization group invariance achieved at a few times the physical breakdown scale in this work removes this source of uncertainty.

An additional approach to estimating the theoretical uncertainty is studying the truncation error in the systematic EFT expansion \cite{17}. In order to do so, let us write the $\pi\pi$EFT expansion for any $M_1$ observable as:

$$\langle M_1 \rangle = \langle M_1 \rangle_{LO} \cdot \left(1 + c_{M_1}^{NLO} \cdot \delta + \mathcal{O}(\delta^2)\right). \quad (11)$$

EFT suggests that $c_{M_1}^{NLO}$ is of natural size, and thus the
truncation error is dictated by $\delta$. In $\text{#EFT}$, the naïve expansion parameter is estimated from $\delta \approx \frac{m^2}{\Lambda^2} \approx \frac{4}{\pi}$. Using this expansion parameter, $\delta$, is usually the starting point for estimating theoretical uncertainties [45]. Here, for the first time, we estimate the expansion parameter, $\delta$, directly from the calculations. This is made possible by our ability to calibrate three-body observables. Firstly, see Tab. III, the ratios of the NLO to LO contribution are found to be in the range of $0.05 - 0.13$. Secondly, since $\hat{\mu}_d$ has a vanishing NLO contribution, its deviation from the experiment can be regarded as $\text{N}^2\text{LO}$, and assuming a natural convergence, we expect the ratio of this contribution to the LO contribution to be $(\text{N}^2\text{LO}/\text{LO}) \approx (\text{NLO}/\text{LO})^2$, or $\text{NLO}/\text{LO} \approx 0.1$. Lastly, the different calibration methods of $l_1^\infty$ (see Tab. III) lead to a variation in the predictions for the different $l_1^\infty$-dependent observables. This variation represents the contribution from higher orders. Thus, the ratio of the variation to the NLO contribution should be of the order of the expansion parameter. Using Tab. III this leads to $(\text{N}^2\text{LO}/\text{NLO}) \approx 0.04 - 0.1$.

If one assumes that the expansion parameter $\delta$ is common to all $M_1$ observables, then one can use the results to assess the value of $\delta$. In order to do this, let us take the log average of all the aforementioned estimates of the expansion parameter: 
$$\log \left[ a_{\text{N}^2\text{LO}} / a_{\text{N}^1\text{LO}} \right] = \log \delta + \log R.$$ 

The numbers $R$ are positive natural numbers and are not biased; they therefore should be distributed about “1”, and a sum over the logarithms of the different “R”’s should vanish. The log average of many such estimates should converge to $\log \delta$. The fact that this is a finite-size sample means that there remains a measure of uncertainty in determining $\delta$, represented as a distribution. We find that at a 95% degree of belief, the expansion parameter is within the range of $0.05 < \delta < 0.13$. The above suggests that the expansion converges faster, by more than a factor of 3, than the naïve $\text{#EFT}$ estimate.

In order to check the sensitivity of the expansion parameter to the number of observables, we calculate the Cumulative Density Functions (CDFs) of $\delta$, the expansion parameter, with all the $n = 7$ constraints: the NLO contributions of $\langle \mu_1 \rangle$, $\langle \mu_1 \text{He} \rangle$, $Y_{np}$, the $\text{N}^2\text{LO}$ contribution of $\langle \mu_d \rangle$, and the variation of $l_1^\infty$ stems from the three electromagnetic observables. Also, we calculate the CDF of $\delta$ only with the $n = 4$ first constraints stemming from the order of the calculation and not from the LEC variation. As shown in Fig. 3 with a 70% degree of belief, the effect of the change is rather small (a change of about 20% in the estimated truncation error). At higher degrees of belief, especially above 90%, the truncation error significantly depends upon the number of constraints, as can be expected due to the small number of such constraints.

The truncation error of an expansion, given a prior which represents the naturalness of the expansion, follows a posterior that was calculated in Refs. [47][49]. In the current case, since the expansion parameter is unknown, i.e., follows the prior distribution in Fig. 3, one should fold these two distributions to find the posterior distribution of these two distributions. The formalism is further explained in Appendix A.

![FIG. 3: Cumulative Density Functions (CDFs) of $\delta$, the expansion parameter. The blue curve represents a calculation that takes into account the constraints of the NLO contributions of $\langle \mu_1 \rangle$, $\langle \mu_1 \text{He} \rangle$, $Y_{np}$, the $\text{N}^2\text{LO}$ contribution of $\langle \mu_d \rangle$, and the variation of $l_1^\infty$. The orange curve takes into account only the first four constraints. The red lines limit the 10% – 90% probability range.](image)

![FIG. 4: Cumulative Density Functions (CDFs) for the different observables, as calculated using eq. (A-2). Horizontal lines are the 70% and 90% degrees of belief. We show CDFs relevant to expansion parameter priors with $n = 4$ (solid lines) and $n = 7$ (dashed lines) constraints, as explained in Fig. 3)](image)

D. Final results compared to experiment

Using $Z$-parameterization, our final predictions for the electromagnetic interactions are, to 70% degree of belief:
where the uncertainty for each $M_1$ observable is estimated from our calculation (see Fig. 3). These results are visually shown in Fig. 5, where for each observable, the bands correspond to the theoretical uncertainty in the Z-parameterization calculation, as calculated in the previous sub-section.

V. DISCUSSION

The results presented in the previous section indicate that #EFT has a predictive power. The prescription we presented, i.e., an NLO calculation in the Z-parameterization, augmented with an uncertainty estimate, reaches percent level accuracy and precision, showing an excellent agreement with the experimental results such as $\langle \mu_p \rangle$, $\langle \mu_3 \rangle$, and $\langle \mu_d \rangle$. The stars are the experimental values while the dotes are the shell model prediction.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The strength of $A = 2, 3 M_1$ observables with a 70% degree of belief. The bands correspond to the theoretical Z-parameterization theoretical uncertainty from the calculations shown Fig. 3. The stars are the experimental values while the dotes are the shell model prediction.}
\end{figure}

The other surprising fact is the unnaturally small expansion parameter $\delta = 5\% - 10\% \ll \frac{\Lambda}{Q} \approx \frac{1}{4}$. The fact that the NLO contribution is so small is traced to the shell-structure of the magnetic moments of the $A = 2, 3$ bound nuclei. A natural expansion parameter would have created $\approx 30\%$ deviation from the shell model prediction. We remind that the #EFT prediction of the quantum structure of the nuclei is inconsistent with shell model for these nuclei.

As the expansion parameter is reminiscent of the breakdown scale, a small expansion parameter may hint that this low energy EFT is actually representing different expansion physics.

For example, König et al. have shown that many features of the structure of nuclei emerge from a strictly perturbative expansion about the unitary limit, which is accompanied by a different expansion parameter: $x_0/Q \sim 10\% - 15\%$. Using the unitary limit, i.e. $1/|a_1| \ll Q \ll \Lambda_0$, they were able to calculate physical observables such as $^3$H and $^3$He binding energy up to NLO, which are very similar to those obtained using a physical value of $a_1$. The similar expansion parameter, which we empirically find in the current work, might hint towards a common physical origin.

A different explanation can be related to an expan-
sion about a leading Wigner-SU(4) symmetry, in which the nucleon is an SU(4) symmetry object, the isovector and spin SU(2) sub-groups. This has been suggested by Vanasse and Phillips [57], who examine $^3$H and $^3$He point charge radius in this limit. They indeed found that δ is smaller than the naïve #EFT estimate. Our result can be related to this suggestion, in particular since one notices that using the Z-parameterization is close to the SU(4) limit, as $ρ_1 ≈ ρ_s$.

Vanasse and Phillips further point-out that the emergence of SU(4) symmetry in #EFT calculations is surprising, as this symmetry is usually emergent at the QCD breakdown scale, i.e., $λ_{QCD} ≈ 1$ GeV, and thus it is not trivial that it will survive the Renormalization Group flow below the pion mass scale. Here we note that such a property would also explain the survival through Renormalization Group flow of the suppression of the isoscalar two-body coupling. Such a property of the flow might be related to the specific operator, which in the $M_1$ case can be traced back to SU(4) generators.

Nuclear EFT expansions, and their power counting, is still a matter of debate in the literature, it is of interest to further study these suggestions.

VI. SUMMARY

In this paper, we present a detailed study of $A = 2, 3 M_1$ observables using #EFT up to NLO, analytically built and numerically verified to be RG invariant by order-by-order consistent and controlled perturbative expansion, to describe the structure and dynamics of the reactions, making use of the low characteristic momentum of the reactions and the involved nuclei.

We check two different NLO arrangements, i.e., ER-parameterization, that uses at NLO the value of the $^3S_1$ effective range, and Z-parameterization, that fixes the Deuteron pole position exactly at NLO. In both cases, the next-to-leading order contribution amounts to less than 10% correction, which is smaller than the naïve expansion parameter of #EFT.

The four observables are used to fix, in six different ways, $δ_1^∞$ and $δ_2^∞$, two unknown NLO LECs representing two-nucleon electromagnetic isovector and isoscalar nuclear currents, respectively.

This process leads to an unusually small NLO contribution, which in the case of the magnetic moments, means that the deviation from shell model predictions is small. Thus, #EFT recovers the shell structure, only due to the unexpectedly small expansion parameter. Moreover, in both parameterizations, we find that the correction to their matrix element originating from the two-body isoscalar low-energy constant, $δ_2^∞$, is consistent with zero, again in contrast to the naïve dimensional analysis of pionless EFT. Together with the small expansion parameter, these are two surprising deviations from the naïve #EFT expectation, which hint towards properties of the renormalization group flow of QCD into #EFT low-energy regime. The small expansion parameter might be a result of SU(4) symmetry, which in principle exists in EFTs with breakdown scale of $∼ 1$ GeV, where the shell-model structure is a LO effect. A vanishing isoscalar two-body contribution at NLO is a property of chiral symmetry, due to the isovector character of the pion. The fact that these properties survive the renormalization group flow to low energies is probably an observable specific property which is of interest in the context of understanding power counting and nuclear EFTs, and so the $M_1$ observables can serve as a starting point for studies of power counting systematics in nuclear EFTs.

We further judge the validity of the NLO parameterizations, through the fluctuations in the values of the predicted observables, using the resulting spectrum of LEC values fixed by different choices of observables. The Z-parameterization is found to have a natural convergence pattern and very stable results. The ER-parameterization, however, is found to have large fluctuations in the predicted results. We, therefore, focused on the Z-parameterization.

We demonstrate that by using the Z-parameterization, the values of the short-range strengths are consistent in the $A = 2$ and $A = 3$ systems, showing no need in a three-body current. We develop a Bayesian approach to estimate the theoretical uncertainty due to the truncation of the EFT expansion. This is found to be about 1% for the calculated observables. The results reproduce high precision the experimental values of the $M_1$ observables, within a 70% degree of belief band.

We found that #EFT predictions for the electromagnetic observables have unprecedented precision and accuracy, comparable with χEFT calculations [54], which have about a factor of 3 more parameters and lack the robust uncertainty estimate and consistency we give here. These, as summarized in Fig. 5 verify and validate the way we applied #EFT at NLO.

Hence, this calculation opens a new path to model-independent calculations of low-energy electromagnetic and weak reactions, including reactions taking place in the interior of stars.

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APPENDIX A - A BAYESIAN APPROACH TO ESTIMATE THE CONVERGENCE RATE

In the following, we expand upon the approach we used to evaluate the truncation uncertainty and the size of the expansion parameter of a set of observables whose expansion is \( \langle M_1 \rangle = (M_1)_{LO} \cdot (1 + c_{M_1}^{NLO} \delta + c_{M_1}^{NLO} \delta^2 + O(\delta^3)) \).

A.I. The Bayesian probability distribution of the expansion parameter

We use information theory arguments to understand the expected behaviour of the expansion convergence rate. The ratio of the \( k^{th} \) and \( l^{th} \) expansion terms should be proportional to \( \delta^{k-l} \) (\( \delta \) is the expansion parameter), i.e.,

\[
r_{k-l}^{M_1} \delta^{k-l} = \frac{N_k^L}{N_l^L} \delta^{k-l}.
\]

\( \delta \)EFT formalism suggests that \( r_{k-l}^{M_1} \) should be a natural number. We interpret this as a statement regarding the nature of the distribution of these numbers, in layman’s terms, one would be surprised if these numbers deviate much from 1. In other words, these coefficients have some natural range of change \( \delta < r_{k-l}^{M_1} < \alpha \), where \( \alpha \) is a measure of naturalness. One can expect \( \alpha \) a factor of 2-3, while bigger variations are acceptable as long as they are rare. From a Bayesian point of view, \( r_{k-l}^{M_1} \) are independent and identically distributed random variables (i.i.d) with an average of about 1 and their logarithm has a (unknown) standard deviation of log \( \alpha \).

Information theory now states that the probability density function (pdf) \( f(r) \) should maximize the entropy \( S[f] = - \int dr f(r) \log f(r) \) subject to the constraints \( \log r = 0 \) and \( (\log r - \log r)^2 = \log \alpha \). Thus, the log-average of \( r_{k-l}^{M_1} \delta^{k-l} \) should be the expansion parameter \( (k-l)\log \delta \). These lead to a pdf \( f(r) \) that is a log-normal distribution.

One can now use a sample of the size \( n \) to estimate the expansion parameter, i.e.,

\[
\log \delta = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{k-l} \log \left( r_{k-l}^{M_1(i)} \delta^{k-l} \right).
\]

Then, by using Bayes theorem, the resulting distribution for the expansion parameter is Student’s \( t \)-distribution with \( n-1 \) degrees of freedom

\[
\frac{\log \delta - \log \delta}{\sigma^{2}/\sqrt{n}} \sim T(\sigma^2, n-1).
\]

A.II. The Bayesian probability distribution of the truncation error of an expansion, given a prior for the expansion parameter

A good estimate for the truncation error is the maximal coefficient in an expansion of order \( k \), multiplied by \( \delta^{k+1} \). In Refs. [47][49], a Bayesian probability distribution is calculated for the truncation error under the assumption of a natural expansion, albeit in the case where the expansion parameter \( \delta \) is known. In what follows, we combine their idea with the probability distribution for the expansion parameter found above, to find the Bayesian probability distribution that the NLO result will deviate by \( \Delta \) from the true value. Then,

\[
pr\left( \Delta \left| \left\{ a_{M_1}^{NLO} \right\}_{k=1}^{n} \right. \right) = \int d\delta pr\left( \Delta \left| \left\{ a_{M_1}^{NLO} \right\}_{k=1}^{n}, \delta \right. \right) \cdot pr\left( \delta \left| \left\{ a_{M_1}^{NLO} \right\}_{k=1}^{n} \right. \right).
\]

\( pr(\Delta) \left| \left\{ a_{M_1}^{NLO} \right\}_{k=1}^{n}, \delta \right. \) is calculated in Ref. [48], and at NLO, is roughly constant for \( |\Delta| \leq R_\xi(\delta) \), and decays as \( 1/|\Delta|^3 \) for \( |\Delta| \geq R_\xi(\delta) \), where \( R_\xi(\delta) = \max\left( 1, \left\{ a_{M_1}^{NLO} \right\}_{k=1}^{n} \right) \delta^2 \). The exact functional form depends on the prior assumption for \( \left\{ a_{M_1}^{NLO} \right\}_{k=1}^{n} \), but a common behavior of all the checked priors is that at a degree of belief of \( \frac{1}{\sqrt{\pi}} \) (translating to \( \approx 67\% \) for NLO calculations) the resulting truncation error is less than \( R_\xi(\delta) \delta^{k+1} \).

As the pdf for \( \delta \), we take the Student’s \( t \)-distribution found in the previous subsection.

APPENDIX B - CALCULATING THE \( A = 3 \) BOUND STATES

In this appendix, we present the general method for calculating an an \( A = 3 \) matrix element in \( \delta \)EFT and its application for calculating \( M_1 \) observables of an \( A = 3 \) system.

The \( A = 3 \) magnetic moments are defined as matrix elements between \( A = 3 \) bound state wave functions of \( \psi^{3H} \), \( \psi^{3He} \), using the general formalism introduced in Ref. [14].

\[
\langle O \rangle = a^J \langle S, S_z, I, I_z, E, q| O^I O^O | S, S_z, I, I_z, E \rangle,
\]

Where \( a^J \) originates from the reduction of the multipole operator (See [58]) and :

- \( \langle \frac{1}{2}, S_z, J, m_z | \frac{1}{2}, S_z' \rangle \neq 0 \)
- \( I_z^l = \left\{ \begin{array}{ll} -I_z & \text{O}^I = \tau^+ \\ I_z & \text{O}^I = \tau^0 \end{array} \right. \)

where \( \tau^\pm \) is isospin ladder operators. \( \psi_{k,j} \) is the initial (final) three-nucleon wave function as defined in Ref. [14].

1. \( A = 3 \), matrix element one-body operator

In Ref. [14], we showed that at LO, the three-nucleon normalization can be written as:

\[
1 = \sum_{\mu,\nu} \langle \psi_{\mu}^i | O^\text{norm}_{\mu,\nu} \langle E_i | \psi_{\nu}^i \rangle,
\]

\( A = 3 \)
where $O_{\mu\nu}^{\text{norm}}(E_i)$ is the normalization operator such that:

$$O_{\mu\nu}^{\text{norm}}(E_i) = \frac{\partial}{\partial E} \left[ \hat{I}_{\mu\nu}(E,p,p') - M_{ij} y_{ji} a_{\mu\nu}^i K^i_{\mu\nu}(p',p,E) \right] \bigg|_{E=E_i},$$

where:

$$K^i_{\mu\nu} = \begin{cases} K_0(p',p,E) & i = ^3\text{H} \\ K_0(p',p,E) + K^C_{\mu\nu}(p',p,E) & i = ^3\text{He} \end{cases}$$

and

$$a_{\mu\nu} = \begin{bmatrix} \nu \\ \mu \\ t \\ s \\ 1 \\ 3 \\ 3 \\ -1 \end{bmatrix},$$

$$a_{\mu\nu}' = \begin{bmatrix} \nu \\ \mu \\ t \\ n_p \\ 1 \\ 1 \\ -1 \\ 2 \\ -2 \\ 0 \end{bmatrix},$$

are a result of the different projection operators (see Ref. 59 for example) and we have defined the operation:

$$A(..., p) \otimes B(p, ...) = \int A(..., p) B(p, ...) \frac{p^2}{2\pi^2} dp .$$

$$\hat{I}_{\mu\nu}(p,p',E) = \frac{2\pi^2}{p^2} \delta(p-p') D_{\mu}(E,p)^{-1} \delta_{\mu\nu}$$

$$K_0(p,p',E) = \frac{1}{2pp'} Q_0 \left( \frac{p^2 + p'^2 - ME}{pp'} \right).$$

where $\delta_{\mu\nu}$ is the Kronecker delta and:

$$Q_0(a) = \frac{1}{2} \int_{-1}^{1} \frac{1}{x+a} dx .$$

$K^C_{\mu\nu}(p',p,E)$ is the $\mu,\nu$ index of the one-photon exchange matrix, $K^{C}(p',p,E)$ (see Ref. 14), $\mu, \nu = t, s$ are the different triton channels, $\mu = t, s, pp$ are the different $^3\text{He}$ channels, $y_{ji}, a_{\mu\nu}$ are the nucleon-dibaryon coupling constants for the different channels, $a_{\mu\nu}$ ($a_{\mu\nu}'$) are a result of the $n-d$ ($p-d$) doublet-channel projection (59) and $D_{\mu}(E,p)$ is the dibaryon propagator (14, 5, 14).

A general one-body operator, can be written as a generalization of a three-nucleon normalization operator for the case of both energy and momentum transfer, between initial (i) and final (f) $A = 3$ bound-state wave functions ($\psi_{i,f}$). The general operator $O_{j,i}$ is factorized into the following parts:

$$O_{j,i} = O^j O^T O_{j,i}(q_0, q),$$

where $O^j$, the spin part of the operator whose total spin is $J$, and $O^T$, the isospin part of the operator, that depend on the initial and final quantum numbers. The spatial part of the operator, $O_{j,i}(q_0, q)$, is a function of the three-nucleon wave function’s binding energies, $(E_i, E_j)$ and the energy and momentum transfer $(q_0, q)$, respectively.

In the case of a triton $\beta$-decay, the spin and isospin one-body operators are combinations of Pauli matrices, so their reduced matrix element ($\langle ||F||, ||GT|| \rangle$) can be easily calculated as a function of the three-nucleon quantum total spin and isospin numbers. In Ref. 14 we showed that the reduced matrix element of such an operator can be written as:

$$\langle ||O^1_{j,i}^{B}(q_0, q)|| \rangle = \left\langle \frac{1}{2} ||O^j|| \right\rangle \left\langle \frac{1}{2} \right\rangle \left( \frac{1}{2} \right)^2 \langle I^T \rangle I_z \langle \frac{1}{2} \rangle \times \sum_{\mu,\nu} \langle \psi_j^i \rangle \psi_j^f \left( d_{\mu\nu}^i \hat{I}(q_0, q) \right) + a_{\mu\nu}^i \left( \hat{K}(q_0, q) + \hat{K}^C_{\mu\nu}(q_0, q) \right) \rangle \langle \psi_f^s \rangle .$$

such that for $i = j$:

$$d_{\mu\nu}^i = \delta_{\mu,\nu}.$$

$$d_{\mu\nu}^i = \begin{cases} a_{\mu\nu} & i = j = ^3\text{H} \\ a_{\mu\nu}' & i = j = ^3\text{He} \end{cases} \ldots$$

The spatial parts of operator are are denoted by $\hat{I}(E,q_0, q), \hat{K}(q_0, q)$ and $\hat{K}^C_{\mu\nu}(E,q_0, q)$. The full analytical expressions for $\hat{I}(E,q_0, q)$ and $\hat{K}(E,q_0, q)$ are given in Ref. 14 while $\hat{K}^C_{\mu\nu}(q_0, q)$ are the diagrams that contain a one-photon interaction in addition to the energy and momentum transfer. A derivation of an analytical expression for these diagrams is too complex, so they were calculated numerically only. $a_{\mu\nu}$ and $d_{\mu\nu}^i$ are a result of the $N - d$ doublet-channel projection coupled to $O^j O^T$ (for more details, see Ref 14).

2. Two-body matrix element

In contrast to the normalization operator given in eq. (B-3), which contains only one-body interactions, a typical EFT electroweak interaction contains also the following two-body interactions up to NLO:

$$t^i t^i, s^i s^i, (s^i t + h.c) \ldots$$

under the assumption of energy and momentum conservation. The diagrammatic form of the different two-body interactions is given in Ref. 14.

B.I. magnetic $A = 3$ matrix element

Based on Ref. 14, the three-nucleon magnetic moment matrix element that contains one-body interac-
tions, \( \hat{\mu}^{(1-B)} \), can be written as:

\[
\langle \hat{\mu}^{(1-B)} \rangle = \left( \frac{1}{2} \left[ \sigma \right] \frac{1}{2} \right) \sqrt{\frac{6}{\pi}}
\times \sum_{\mu, \nu} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \psi_{\mu}^j(p') \psi_{\nu}^j(p) \gamma_0 + \gamma_0 \gamma_3 \gamma_3 \gamma_3 \gamma_2 \gamma_2 \gamma_2 \gamma_2
\times \left[ \tilde{T}_{\mu\nu}(0,0) + a_{\mu\nu}^{ij} \tilde{\mathcal{K}}(0,0) \right] \psi_i^j(p), \quad (B-16)
\]

where the full analytical expressions for the direct and exchange spatial operators, \( \tilde{T}(E, q_0, q) \) and \( \tilde{\mathcal{K}}(E, q_0, q) \), are given in Ref. [14].

For \(^3\text{H}\):

\[
d_{\mu\nu}^{nn} = \begin{vmatrix}
\mu & n & np \\
n & t & \nu \\
np & 0 & \mu_n \\
\end{vmatrix}
\]

\[
\hat{\nu}^{\mu} = \begin{vmatrix}
\nu & t & np \\
n & \nu & n \\
np & 0 & \nu \\
\end{vmatrix}
\]

and

\[
\hat{\nu}^{\mu} = \begin{vmatrix}
\nu & t & np \\
n & \mu & n \\
np & 0 & \mu \\
\end{vmatrix}
\]

is just the normalization condition of the bound state (see Ref. [14]).

At LO, Fig. B.1 (a) is given by:

\[
\mu_n y_t^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \psi_t^{3H}(p') \psi_t^{3H}(p) \times \pi \left[ \tilde{T}(p) + \tilde{T}(p') \right]
\]

\[
\times \left[ \tilde{T}(p) + \tilde{T}(p') \right] \delta(p - p') \psi_t^{3H}(p') \right] .
\]

(B-23)

The three-nucleon magnetic moment matrix element that contains two-body interactions, \( \hat{\mu}^{(2-B)} \), can be written as the sum of all two-body interactions (see Ref. [14]):

\[
\langle \hat{\mu}^{(2-B)} \rangle = \int \frac{d^3p}{(2\pi)^3} \left[ \frac{2}{3} L_2 \psi_t^{3H}(p) + L_1 a_{ls}^{(2)} \left[ \psi_t(p) \times \psi_{np}(p) \right] \right],
\]

where

\[
a_{ls}^{(2)} = \begin{vmatrix}
-\frac{2}{3} & 1 \\
1 & 3 \text{He} \\
\end{vmatrix}
\]

(B-26)

Fig. B.1 (c) is given by:

\[
L_2 \int \frac{d^3p}{(2\pi)^3} \left[ \psi_t^{3H}(p) \right]^2 .
\]

(B-27)

**FIG. B.1:** Three of the diagrams contributing to the triton magnetic moment. The one-body diagrams are coupled to \( \mu_n \cdot \sigma \) (a) and \( \mu_p \cdot \sigma \) (b), diagram (c) is coupled to the two-body LEC \( L_2 \). The double lines are the propagators of the dibaryon field \( D_1 \) (solid). The red lines represent the neutron propagator, while the blue lines represent the proton propagator.
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