Mathematical Theory of Powerful Tornadoes in the Atmosphere

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Abstract. We propose a mathematical theory of powerful tornadoes for which the angular velocity of air in the vortex ring of the mother cloud generating a tornado in the upper layers of the atmosphere is much greater than the angular velocity of air currents near the Earth’s surface. This theory can have practical applications, in particular, it can be used to determine, with any given accuracy, the air velocity within a tornado, as well as the funnel-shaped boundary of the tornado.

1. Introduction

The phenomenon of a tornado consists in the formation in the atmosphere near the surface of the Earth of a swirling upward flow of air, which has tremendous destructive power. Each year, more than a thousand tornadoes are recorded on Earth. It is currently believed that a tornado is generated by vortex movement in the mother cloud in the upper atmosphere.

Observations show [1, 2] that the vortex generating a tornado has a complex three-dimensional structure consisting of simultaneous air circulation in the poloidal and azimuthal directions and forming a curved torus (vortex ring). There is no clear understanding of the causes of the vortex ring and the mechanism by which it generates a tornado. A simple mathematical model proposed in [3, 4] allows one to investigate the causal relationship between the presence of vortex motion in the mother cloud and the appearance of a swirling upward air flow.

Although this model does not take into account a number of important features of the tornado, it seems to give some answer to two key questions, which ultimately is the goal of any study on the tornado problem: 1) where and when tornadoes can occur and what is their power? 2) what measures should be taken to prevent a possible tornado or to neutralize a tornado that has already occurred? To answer these questions on the basis of the simple model proposed in [3, 4], it would be necessary to examine more closely the data of monitoring vortex formations in the upper atmosphere and pressure distribution in the atmospheric air.

There are about a hundred relevant publications on the problem of tornadoes, of which the most important ones are briefly reviewed below.

In [5], an analytical model is proposed for a vortex non-viscous flow with a vertical symmetry axis and a motionless core in the form of a funnel. It is assumed that the air density within the core is smaller than that of the moving air whose rotation velocity is growing in the direction of the funnel border and is maximal near the ground. In [6], a model is proposed for the formation of atmospheric tornado-type vortices due to instability caused by the growth of the vertical component of the air velocity in the direction of the earth’s surface or an increase of...
concentration of suspended particles. Such conditions are typical for the initial stage of the development of a tornado and are realized in thunderclouds in the atmosphere. A class of analytical solutions of the Navier-Stokes equations for viscous incompressible fluids is obtained in [7]. These solutions can be used to predict the characteristics of certain vortex flows, in particular, tornadoes. An analytical study of ascending swirling air flow is carried out in [8]. According to this investigation, the Coriolis force plays a key role in the formation of tornadoes. It should be observed that a number of assumptions made in [8] contradict the generally accepted views on the formation and stability of tornadoes.

Numerical simulation of a tornado faces the problem of setting correct initial and boundary conditions. Moreover, when using turbulent models of air flows in a tornado, the problem is the calculation of empirical turbulent transport coefficients. Nevertheless, some important results have been obtained in this direction. In [9], one considers the process of tornado formation due to convective instability near the earth's ground. In [10–13], following [9], calculations of vortices are carried out and the determining role of the swirl ratio in the formation and stability of the vortex structure of a tornado is established. The process of tornado formation due to air rotation in a thundercloud is studied in [14], with the effects of turbulence of air currents, water or sand impurities, and compressibility taken into account. In [15, 16], developing the approaches of [14], one calculates three-dimensional velocity fields and air pressure in a tornado. An analysis of the dynamics of tornado-like flows on the basis of the large vortex method (LES method) was carried out in [17–21], where is was shown, in particular, that the compressibility of air has almost no effect on the vortex dynamics in a tornado. A two-fluid model of tornado is examined in [22, 23], the first (main) liquid being water vapor, which condenses upon a sharp change in pressure, the second liquid consisting of solid particles involved in the air flow by the vortex tornado flow.

A more detailed bibliography is given in [2].

The studies described in [5–23] deal with direct mathematical modeling of tornadoes based on the numerical solution of gas dynamics equations as applied to the Earth’s atmosphere, taking into account the three-dimensionality of motion, turbulence of air currents, the dependence of the dynamic viscosity of air on water or sand dust, etc. The main obstacle in this approach is the choice of the initial and boundary conditions, the extrapolation of which on the basis of the observational data inevitably leads to large errors that make useless the calculated tornado parameters.

The mathematical model of a tornado proposed in [3, 4] and the present paper does not take into account the above and other important features of the phenomenon and simplifies the problem of choosing boundary conditions, which makes it possible to give a theoretical solution to the problem of predicting and preventing tornadoes.

In the present study, the ideas of [3, 4] are developed in two directions. First, we construct a theory of powerful tornadoes (these are of the greatest practical interest) which gives a fairly complete analysis of the tornado phenomenon. In particular, this theory can be used to establish a relationship between the angular velocity of rotation of air masses in the mother cloud and the vertical velocity of air in a tornado at high altitudes, as well as to calculate the thickness of the boundary layer at the Earth’s surface, which is formed when air rotates in a tornado. Secondly, we propose a method to find the tornado boundary, which has a funnel-shaped profile.

2. Basic Concepts and Equations
The tornado phenomenon is caused by processes occurring in the atmospheric air. To describe these processes, we use the tools of continuum mechanics, more precisely, the Navier-Stokes equations for viscous incompressible air,

\[ \rho = \text{const}, \quad \text{div} \mathbf{U} = 0, \quad \rho \frac{\partial \mathbf{U}}{\partial t} + \rho \mathbf{U} \cdot \nabla \mathbf{U} + \nabla p = 2 \mu \text{Div} \mathbf{def} U + \rho g, \tag{1} \]
where $\rho$ is the air density, $p$ is its pressure, $U$ is its hydrodynamic velocity; $\mu = \text{const}$ is the dynamic viscosity coefficient; $g$ is the constant gravitational acceleration; $\text{def}U$ is the strain tensor of the vector field $U$.

Neglecting the motion of the mother cloud, we assume that the dynamics of air masses within the tornado has axial symmetry with respect to some instantaneous axis of rotation, which changes its location with the mother cloud. The motion of the latter is not considered here. Then, the hydrodynamic parameters of air, $U$ and $p$, near the tornado axis can be approximated in the cylindrical coordinates, with any given accuracy, by functions of the form

$$U_r = rA(t, z), \quad U_\phi = rB(t, z), \quad U_z = C(t, z), \quad p = \Phi(t, z) + r^2Q(t, z).$$

We make the following principal assumptions: 1) the tornado phenomenon is caused by the air flow in the near-axis region, the so-called “trunk” of a tornado; 2) the air flow in the near-axis region is described by a special solution of the Navier-Stokes system (1), namely, a solution of the form (2). Substituting the expressions (2) into system (1), we easily obtain the following result.

**Theorem 1.** Functions (2) satisfy equations (1) if and only if $\partial Q/\partial z \equiv 0$, and the complex-valued function $u = A + iB$ and the real-valued function $C$ satisfy the nonlinear system of equations

$$\frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} = -2Reu,$$

$$\frac{\partial C}{\partial z} = -2Reu,$$

where $\text{Re}$ denotes the real part of a complex number. Moreover, we have $Q = Q(t)$, and $\Phi$ is uniquely (to within an additive constant) determined from of the equation

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial z} \left( \frac{C^2}{2} + \frac{2\mu}{\rho}Reu + \frac{\Phi}{\rho} + gz \right) = 0$$

provided that the functions $u$ and $C$ have been found from (3).

The equations of system (1) are written in a non-inertial reference frame rigidly fixed to the surface of the Earth, with the axis $z$ directed upwards. Given the rotation of the Earth around its axis, it is necessary to take into account, in system (1), the inertial forces, namely, the Coriolis force and the centrifugal force. As shown in [3, 4], in the case of studying tornadoes, these additional forces are sufficiently small and can be neglected. The Coriolis force can be of some importance, at most, in setting the boundary condition for system (3) on the surface of the Earth, $z = 0$ (see below). Moreover, according to [3], the approximation of incompressible atmosphere is valid even for devastating tornadoes of class F4 on the Fujita scale, with air velocities about 100 m/s.

System (3) should be supplemented with an initial condition for the function $u$ at $t = 0$ and boundary conditions for $C$ at the point $z = 0$ and $u$ at $z = 0$ and $z = \infty$. The value of $C$ at $z = 0$ is determined by the physical condition of nonpenetration on the rigid surface of the Earth: $C(t, 0) \equiv 0$. Below, we will mainly consider stationary solutions of system (3), in which case $Q$ and $u(\infty)$ do not depend on $t$ and the boundary condition at infinity has the form $u(\infty) = i\sqrt{2Q/\rho}$.

**Definition.** A tornado (resp., antitornado) is a stationary solution of system (3) such that there is a finite limit $\lim_{z \to +\infty} C(z) > 0$ (resp., $\lim_{z \to -\infty} C(z) < 0$).

We seek the solutions of tornado (antitornado) type from the boundary value problem (3), (5),

$$u(0) = u_0, \quad u(\infty) = i\sqrt{2Q/\rho}, \quad C(0) = 0, \quad u_0 \in \mathbb{C},$$

(5)
by the stabilization method [3, 4]. First, let us rewrite problem (3), (5) in dimensionless form,

\[
\frac{\partial u}{\partial t} + C \frac{\partial u}{\partial z} - \frac{1}{R} \frac{\partial^2 u}{\partial z^2} + u^2 + \Gamma = 0, \quad \frac{\partial C}{\partial z} = -2Reu, \tag{6}
\]

\[
u(0) = u_0, \quad u(\infty) = i\sqrt{\Gamma}, \quad C(0) = 0, \tag{7}
\]

\[
\frac{\partial C}{\partial t} + \frac{\partial}{\partial z} \left( \frac{C^2}{2} + \frac{2}{R} Reu + M_0^2 \Phi + g_0 z \right) = 0, \tag{8}
\]

where \( R = \rho U_0 L_0 / \mu \) is the Reynolds number, \( M_0 = p_0^{1/2} (\rho U_0^2)^{-1/2} \) is the modified Mach number, \( \Gamma = 2Q_l^3 / \rho \), \( g_0 = gL_0 / U_0^2 \), \( t_0 = L_0 / U_0 \); and \( L_0, U_0, \) and \( p_0 \) are the characteristic scales of length, velocity, and pressure, respectively.

### 3. Some Calculation Results

The numerical approach to the investigation of problem (3), (5) is described in [3, 4].

First, consider problem (3), (5) with the boundary condition \( u(0) = i\omega \), where \( \omega = \omega_c \) is the angular velocity at which the Coriolis force swirls the radial air currents near the surface of the Earth. It is not difficult to show [2] that \( \omega_c = -\Omega E \sin \varphi \), where \( \Omega E = 7.3 \cdot 10^{-5} \text{s}^{-1} \) is the angular velocity of the Earth’s rotation around its axis, \( \varphi \) is the latitude of the area where the tornado occurs. In this way, the Coriolis force is taken into account in the boundary conditions.

On the other hand, the value of \( Q \) in (2) determines the radial pressure drop from the center of the tornado \( r = 0 \) to its periphery \( r = \infty \). It is intuitively clear that the swirling upward air currents (i.e., flows of the tornado type) are possible only if \( Q > 0 \). Therefore, we consider the boundary conditions of the form

\[
u(0) = i\omega, \quad u(\infty) = i\sqrt{2Q/\rho}, \quad C(0) = 0, \tag{9}
\]

where \( \omega > 0, Q \geq 0, \) and in (9) we take the nonnegative value of the square root (if \( \omega < 0 \), then we take its negative value in (9)). Setting \( t_0 = \omega^{-1} \), we rewrite condition (9) in dimensionless form as follows:

\[
u(0) = i, \quad u(\infty) = i\sqrt{\Gamma}, \quad C(0) = 0, \quad \Gamma = 2Q/(\rho \omega^2). \tag{10}
\]

A numerical analysis of the boundary value problem (6), (10) with \( R = 1 \) (Theorem 2 stated below implies that these results are valid for any Reynolds number \( R \)) allows us to claim that:

1) for \( \Gamma > 1 \), problem (6), (10) has a stationary solution of tornado type;

2) for \( 0 \leq \Gamma < 1 \), problem (6), (10) has a stationary solution of antitornado type;

3) for \( \Gamma = 1 \), problem (6), (10) has an explicit stationary solution: \( u(z) \equiv i, C(z) \equiv 0 \);

4) system (6) with the boundary conditions \( u(0) = u_0, u(\infty) = u_\infty, C(0) = 0, \) for \( u_\infty \neq i\sqrt{\Gamma} \) and any \( u_0 \), has no stationary solutions;

5) system (6) with the boundary conditions \( u(0) = -i, u(\infty) = i, C(0) = 0 \) has no stationary solutions, and it has a stationary solution if the boundary conditions are \( u(0) = i\omega, u(\infty) = i, C(0) = 0 \) with \( |\omega| < 10^{-2} \) (it is assumed that \( \Gamma = 1 \) in (6)).

The graphs demonstrating the statements 1)–5) are given in [3, 4]. From the statements 1)–5), one can make the following important conclusions, which we formulate for dimensional properties.

Suppose that \( u_0 = i\omega, \omega \geq 0, Q \geq 0 \) in (5) and the root value is nonnegative. Then, the statements 1)–3) imply that the boundary value problem (3), (5) has a stationary solution and there is a critical value, \( Q_{cr} = \rho \omega^2 / 2 \), such that for \( Q_{cr} < Q \), this solution is of tornado type, for \( 0 \leq Q < Q_{cr} \), it is of antitornado type, and for \( Q = Q_{cr} \), it reduces to the rotation of air around the axis \( z \) with constant angular velocity \( \omega \).
The appearance of the critical value of $Q$ can be physically explained by the balance of forces near the Earth’s surface acting in the radial direction on the volume element of air, namely, the centrifugal force $p r \omega^2$ and the radial component of the antigradient of pressure $(-\partial p/\partial r) = -2Qr$. If the centrifugal force is greater than the antigradient of pressure, $\rho \omega^2 > 2Q$, then the air near the Earth’s surface is moving away from the symmetry axis and there appears a downward flow of antitornado type, otherwise, for $\rho \omega^2 < 2Q$, the air is moving towards the symmetry axis and a flow of tornado type is formed.

Denoting by $\Omega$ the angular velocity of air in the vortex ring of the mother cloud, we can write the boundary condition at infinity in the form $u(\infty) = i\Omega$. Therefore, statement 4) for $Q \geq 0$ implies the following important result: if $\Omega \neq \sqrt{2Q/\rho}$, then, for any $u_0$, the boundary value problem (3), (5) has no stationary solutions; a fortiori, there are no solutions of tornado type. Thus, the relation $\Omega^2 = 2Q/\rho$ is a necessary condition for the existence of a tornado. This principal result can have practical applications with regard to tornado prediction [3, 4].

Statement 5) implies that the role of the Coriolis force is probably the opposite of that which is usually attributed to it [2]. The Coriolis force prevents the occurrence of tornadoes, especially if the direction of rotation of the air near the Earth’s surface due to this force is opposite to the direction of rotation of the air masses in the upper layers of the atmosphere and the angular velocities of both rotations are comparable. This result can be used for recommendations on the prevention of tornadoes [3, 4].

4. Theory of Powerful Tornadoes

We seek an approximate solution of problem (6), (7) in the form of a double power series with respect to two real parameters, namely, the real and the imaginary parts of the complex parameter $a = u_0 \Gamma^{-1/2} = a_1 + ia_2$, assuming that it is small, $|a| \ll 1$. It turns out (see below) that the leading term of this expansion does not depend on $u_0$, which simplifies the problem of choosing the boundary condition for $u(0)$ in (7). The physical meaning of the condition $|a| \ll 1$ can be easily understood if we take $u_0 = \omega_0$. Then, the dimensionless value $|u_0| \Gamma^{-1/2}$ is equal to $|\omega_0/\Omega|$ where $\Omega = (2Q/\rho)^{1/2}$ is the angular velocity of the air vortex at infinity and the condition $|a| \ll 1$ is equivalent to $|\Omega| \gg |\omega|$, which corresponds to a powerful tornado with the angular velocity of air masses in the mother cloud being much greater than the angular velocity of air rotation near the surface of the Earth. The desired expansion is constructed on the basis of the following simple result.

**Theorem 2.** Let $\Gamma \neq 0$ and

\[ u = |\Gamma|^{1/2} \tilde{u}, \quad C = |\Gamma|^{1/4} R^{-1/2} \tilde{C}, \quad z = x |\Gamma|^{-1/4} R^{-1/2}, \quad t = \tau |\Gamma|^{-1/2}. \tag{11} \]

Then the functions $u$, $C$ of $(t, z)$ satisfy system (6) if and only if the functions $\tilde{u}$, $\tilde{C}$ of $(\tau, x)$ satisfy the system

\[ \frac{\partial \tilde{u}}{\partial \tau} + C \frac{\partial \tilde{u}}{\partial x} - \frac{\partial^2 \tilde{u}}{\partial x^2} + \tilde{u}^2 \pm 1 = 0, \quad \frac{\partial \tilde{C}}{\partial x} = -2Re \tilde{u}, \tag{12} \]

where the sign “+” is taken for $\Gamma > 0$ and the sign “−” corresponds to $\Gamma < 0$.

Consider the case $\Gamma > 0$. If $u(z)$, $C(z)$ is a stationary solution of problem (6), (7), then Theorem 2 ensures that $\tilde{u}(x) = \Gamma^{-1/2} u(x \Gamma^{-1/4} R^{-1/2})$, $\tilde{C}(x) = \Gamma^{-1/4} R^{1/2} C(x \Gamma^{-1/4} R^{-1/2})$ is a stationary solution of system (12) with the sign “+” and the boundary conditions $\tilde{u}(0) = u_0 \Gamma^{-1/2}$, $\tilde{u}(\infty) = 1$, $\tilde{C}(0) = 0$. If $|u_0| \Gamma^{-1/2} \ll 1$, then, assuming analytic dependence of stationary solutions of system (12) with the above boundary conditions on the boundary value of $\tilde{u}$ at $x = 0$, we consider perturbations with respect to this boundary value and calculate, with any given accuracy, the functions $\tilde{u}(x)$, $\tilde{C}(x)$, and therefore, the functions $u(z)$, $C(z)$ with the help of (11). Note that the functions $\tilde{u}(x)$, $\tilde{C}(x)$ are perturbations of $u_0(x)$, $C_0(x)$, the latter
being stationary solutions of system (12) with the boundary conditions $u(0) = 0, u(\infty) = i, C(0) = 0$. The graphs of these functions in figure 1 show that we have a solution of tornado type.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig1.png}
\caption{Graphs of $C_0(x)$, $Re u_0(x)$, $Im u_0(x)$ with the boundary conditions $u(0) = 0$, $u(\infty) = i$, $C(0) = 0$.}
\end{figure}

Let us seek stationary solutions $u_\alpha(x), C_\alpha(x)$ of system (12) with “+” and the boundary conditions $u(0) = a, u(\infty) = i, C(0) = 0$ using the perturbations theory with respect to the parameter $a = (a_1, a_2) = a_1 + ia_2$, which is later taken equal to $a = u_0 \Gamma^{-1/2}$:

$$u_\alpha(x) = \sum_\alpha u_\alpha(x)a^\alpha, \quad C_\alpha(x) = \sum_\alpha C_\alpha(x)a^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\alpha_1 \geq 0, \alpha_2 \geq 0$ are integers, $a^\alpha = a_1^{\alpha_1}a_2^{\alpha_2}$, $|\alpha| = \alpha_1 + \alpha_2$, $u_\alpha(x)$ are complex-valued, $C_\alpha(x)$ are real-valued, and the sums in (13) are over all multi-indices $\alpha$. Substituting (13) into (12) with “+”, gathering similar terms, and equating the coefficients of the same powers $a^\alpha$, we find that $u_{(0,0)}(x) = u_0(x), C_{(0,0)}(x) = C_0(x)$, and $u_\alpha(x), C_\alpha(x)$, for $|\alpha| > 0$, are solutions of the following linear boundary value problems with stationary boundary conditions:

$$C_0 \frac{\partial u_\alpha}{\partial x} + C_\alpha \frac{\partial u_0}{\partial x} - \frac{\partial^2 u_\alpha}{\partial x^2} + 2u_0u_\alpha + \sum_{0 < \beta < \alpha} \left( u_\beta u_{\alpha-\beta} + C_\beta \frac{\partial u_{\alpha-\beta}}{\partial x} \right) = 0,$$

$$\frac{\partial C_\alpha}{\partial x} = -2Re u_\alpha, \quad u_\alpha(0) = u_\alpha(\infty) = 0, \quad C_\alpha(0) = 0, \quad |\alpha| > 1,$$

\begin{align*}
\begin{cases}
  u_{(1,0)}(0) = 1, & u_{(1,0)}(\infty) = 0, & C_{(1,0)}(0) = 0, \\
  u_{(0,1)}(0) = i, & u_{(0,1)}(\infty) = 0, & C_{(0,1)}(0) = 0,
\end{cases}
\end{align*}

where the inequality $\beta = (\beta_1, \beta_2) < \alpha = (\alpha_1, \alpha_2)$ for multi-indices means that $\beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2$ and $\alpha \neq \beta$, and $(0, 0) = 0$.

According to (14), the chain of the equations for the coefficients $u_\alpha, C_\alpha, |\alpha| > 0$, can be split (this allows us to calculate all these coefficients consecutively), and the differential operator in the equations for $u_\alpha, C_\alpha$ is linear and the same for all $\alpha$. In order to find the coefficients $u_\alpha, C_\alpha$, we should consecutively solve problems (14) by the stabilization method, and to that end, we should introduce the derivative $\partial u_\alpha/\partial r$ into the left-hand side of the equation for $u_\alpha$. The zero order approximation has the form

$$u(z) = \Gamma^{1/2} \left[ u_0 \left( z\Gamma^{1/4}R^{1/2} \right) + O \left( u_0 \Gamma^{-1/2} \right) \right], \quad u_0 \Gamma^{-1/2} \to 0,$$

$$C(z) = \Gamma^{1/4}R^{-1/2} \left[ C_0 \left( z\Gamma^{1/4}R^{1/2} \right) + O \left( u_0 \Gamma^{-1/2} \right) \right], \quad u_0 \Gamma^{-1/2} \to 0,$$

the graphs of the standard functions $u_0(x), C_0(x)$ are represented in figure 1. It should be emphasized that the leading term of the expansion (15) does not depend on $u_0$. Relations (15) imply some important practically applicable formulas, of which we select two.
First, we have the asymptotic (in the sense of mathematical analysis) relation
\[
C(\infty) \sim \Gamma^{1/4} R^{-1/2} C_0(\infty) = 1.35 \Gamma^{1/4} R^{-1/2}, \quad u_0 \Gamma^{-1/2} \to 0,
\]
where we have taken into account the results of calculations represented in figure 1, which yield
\( C(\infty) \approx 1.35 \). In particular, the growth of \( C(\infty) \) is proportional to \( \Gamma^{1/4} \) as \( \Gamma \to +\infty \) and \( u_0 \) is fixed.

Secondly, let \( x_\infty \) be the altitude at which \( u_0(x), C_0(x) \) stabilize to constants. According to the results represented in figure 1, we have \( x_\infty \approx 15 \). Then, the variation of \( u(z), C(z) \) is substantial only in the boundary layer \( 0 \leq z \leq x_\infty \Gamma^{-1/4} R^{-1/2} = 15 \Gamma^{-1/4} R^{-1/2} \) adjacent to the Earth’s surface, and outside that layer, these functions are approximately constant, \( u(z) \approx i\sqrt{\Gamma}, C(z) \approx C_0(\infty) \Gamma^{1/4} R^{-1/2} = 1.35 \Gamma^{1/4} R^{-1/2} \).

Formulas (15) yield the zero order approximation for the expansions (13). The first order approximation, with \( u_0 = b_1 + i b_2 \), has the form
\[
u(z) = \Gamma^{1/2} \left[ u_0(x) + (b_1 u_1(x) + b_2 u_2(x))\Gamma^{-1/2} + R_2 \right], \quad R_2 = O \left( |u_0|^2 \Gamma^{-1} \right),
\]
\[
C(z) = \Gamma^{1/4} R^{-1/2} \left[ C_0(x) + (b_1 C_1(x) + b_2 C_2(x))\Gamma^{-1/2} + S_2 \right], \quad S_2 = O \left( |u_0|^2 \Gamma^{-1} \right),
\]
\[
x = z\Gamma^{1/4} R^{-1/2}, \quad |u_0| \Gamma^{-1/2} \to 0,
\]
where \( u_i(x), C_i(x), i = 1, 2, \) are stationary solutions of the following boundary value problem on the half-line \( 0 \leq x < +\infty \):
\[
\frac{\partial u_i}{\partial \tau} + C_0 \frac{\partial u_i}{\partial x} + C_1 \frac{\partial u_0}{\partial x} - \frac{\partial^2 u_i}{\partial x^2} + 2u_0 u_i = 0, \quad \frac{\partial C_i}{\partial x} = -2Re u_i, \quad i = 1, 2,
\]
\[
u_1(0) = 1, \quad u_1(\infty) = 0, \quad C_1(0) = 0,
\]
\[
u_2(0) = i, \quad u_2(\infty) = 0, \quad C_2(0) = 0.
\]

Figure 2 shows the graphs of the functions \( u_i(x), C_i(x), i = 1, 2, \) obtained by the numerical solution of the stabilization problems (17) on the basis of the already known \( u_0(x), C_0(x) \).

The second order approximation has a more intricate form,
\[
u(z) = \Gamma^{1/2} \left[ u_0(x) + \frac{b_1 u_1(x) + b_2 u_2(x)}{\Gamma^{1/2}} + \frac{b_1^2 u_1(0) + b_1 b_2 u_11(x) + b_2^2 u_02(x)}{\Gamma} + R_3 \right],
\]
\[
C(z) = \frac{\Gamma^{1/4}}{R^{1/2}} \left[ C_0(x) + \frac{b_1 C_1(x) + b_2 C_2(x)}{\Gamma^{1/2}} + \frac{b_1^2 C_1(0) + b_1 b_2 C_11(x) + b_2^2 C_02(x)}{\Gamma} + S_3 \right],
\]
\[
x = z\Gamma^{1/4} R^{-1/2}, \quad u_0 = b_1 + i b_2, \quad R_3 = O \left( |u_0|^3 \Gamma^{-3/2} \right), \quad S_3 = O \left( |u_0|^3 \Gamma^{-3/2} \right),
\]
\[|u_0| \Gamma^{-1/2} \to 0,\]
where \( u_\alpha(x) \), \( C_\alpha(x) \) for \( \alpha = (2,0) \), \( \alpha = (1,1) \), \( \alpha = (0,2) \) are stationary solutions of the boundary value problem

\[
\frac{\partial u}{\partial \tau} + C \frac{\partial u}{\partial x} + C \frac{\partial u_0}{\partial x} - \frac{\partial^2 u}{\partial x^2} + 2u_0u_i + f_\alpha(x) = 0, \quad \frac{\partial C}{\partial x} = -2Reu,
\]

\( u(0) = u(\infty) = 0, \quad C(0) = 0, \)

where \( f_\alpha(x) = u_1^2 + C_1 \partial u_1/\partial x \) for \( \alpha = (2,0) \), \( f_\alpha(x) = u_2^2 + C_2 \partial u_2/\partial x \) for \( \alpha = (0,2) \), and \( f_\alpha(x) = 2u_1u_2 + C_3 \partial u_2/\partial x + C_2 \partial u_1/\partial x \) for \( \alpha = (1,1) \).

The functions \( u_\alpha(x), C_\alpha(x) \), \( |\alpha| = 2 \), are obtained by numerically solving problems (19) by the stabilization method on the basis of the known \( u_i(x), C_i(x), i = 0,1,2 \) (see figures 1, 2).

Their graphs are represented in figure 3. In a similar way, higher order approximations can be constructed, but their expressions become more and more cumbersome.

![Figure 3. Graphs of \( u_\alpha(x), C_\alpha(x) \) for \( 1 - \alpha = (2,0), 2 - \alpha = (0,2), 3 - \alpha = (1,1) \).](image)

Note that above we have dealt with formal power series. The convergence of the series (13) and the possibility of their term-by-term differentiation (used when obtaining boundary value problems (14)) are much more difficult questions and require special consideration. In the general case, for integer \( p \geq 1 \), we have

\[
u_\alpha(x) = \sum_{|\alpha|<p} u_\alpha(x)a^\alpha + R_p, \quad C_\alpha(x) = \sum_{|\alpha|<p} C_\alpha(x)a^\alpha + S_p,
\]

\[
R_p = \sum_{|\alpha|\geq p} u_\alpha(x)a^\alpha = O(|a|^p), \quad S_p = \sum_{|\alpha|\geq p} C_\alpha(x)a^\alpha = O(|a|^p),
\]

\( |\alpha| = |u_0| |\alpha|^{-1/2} \to 0. \)

For \( p = 1, 2, 3 \), the expansions (20) have been constructed above. It seems that their construction for \( p = 3 \) does not make sense. Indeed, let us estimate the order of magnitude of \( O(|a|^p) \) as \( a \to 0 \) for the boundary condition \( u_0 = i\omega_c \), where \( \omega_c \) is the angular velocity of air near the Earth’s surface due to the Coriolis force. In dimensionless form, we have \( a = \omega_c/\Omega \), where \( \Omega \) is the angular velocity in the vortex ring. For a typical air velocity, say \( U = 100 \text{ m/s} \), and the radius \( r = 10 \text{ km} \) of the vortex ring in the mother cloud, we obtain \( \Omega = U/r = 10^{-2} \text{ s}^{-1} \) and \( |\alpha| = 7.3 \times 10^{-3} \). Therefore, for \( p = 3 \), the residual terms in the expressions (20) are of the order of magnitude \( \sim 10^{-6} \) and should be neglected. Therefore, it makes no sense to construct further approximations, since their contribution to the approximation formulas (16), (18) would be negligibly small. Note that in the case of \( u_0 = i\omega_c \), the approximation formulas (16), (18), (20) become much simpler. For instance, (18) turns to

\[
u(z) = \Gamma_1^{1/2}\left[u_0(x) + \frac{\omega_c}{\Omega} u_2(x) + \left(\frac{\omega_c}{\Omega}\right)^2 u_{02}(x) + O\left(\frac{\omega_c}{\Omega}\right)^3\right],
\]

\[
C(z) = \Gamma_1^{1/4} R_1^{-1/2}\left[C_0(x) + \frac{\omega_c}{\Omega} C_2(x) + \left(\frac{\omega_c}{\Omega}\right)^2 C_{02}(x) + O\left(\frac{\omega_c}{\Omega}\right)^3\right],
\]

\[
x = z \Gamma_1^{-1/4} R_1^{1/2}, \quad \omega_c/\Omega \ll 1,
\]

where \( u_0(x) \), \( C_0(x) \) for \( \alpha = (2,0) \), \( \alpha = (1,1) \), \( \alpha = (0,2) \) are stationary solutions of the boundary value problem.
where $\Gamma^{1/2}$ and $R$ could be conveniently expressed through the Rossby number $Ro=(2\Omega z_0t_0)^{-1}$, which determines the characteristic time $t_0$. We have $\Gamma^{1/2}=(\Omega/\Omega_c)(2Ro)^{-1}$, $R=(\rho/\mu)L^2_0\Omega_cRo$. In tornado problems, it is usually assumed [2] that the value of $Ro$ varies from $10^3$ to $10^5$. Taking into account that $\mu/\rho = 0.15 \text{ cm}^2/\text{s}$, $L_0 = 1 \text{ cm}$, we find that the approximate range of $R$ is from 1 to 100.

The above theory of devastating tornadoes makes it possible to determine their parameters with any given accuracy by reducing all calculations to solving standard boundary value problems of the type (17), (19).

5. Finding the Tornado Boundary

Observations show [1, 2] that the shape of a tornado in the atmosphere is that of a funnel. The boundary of this funnel is a surface determined by the condition that at its points the air pressure on the tornado side should be equal to the air pressure in the atmosphere unperturbed by the air motion in the tornado. In the case of axial symmetry, the said boundary $r(z)$ (tornado profile) is found from the equation

$$Qr^2 + \Phi(z) = p(z) \Leftrightarrow r = ((p(z) - \Phi(z))/Q)^{1/2}, \quad (21)$$

where $p(z)$ is the air pressure at the altitude $z$ in the unperturbed region of the atmosphere. Assuming that the atmosphere is adiabatic, we have [24]

$$p(z) = p_0 \left(1 - zg\gamma s^{-2}\right)^{-\gamma/(\gamma-1)}, \quad (22)$$

where $c_s = (\gamma p_0/\rho_0)^{1/2}$ is the speed of sound, $\gamma > 1$ is the adiabatic exponent; $p_0$, $\rho_0$ are, respectively, air pressure and density near the Earth’s surface, $z = 0$. Formula (22) is valid for the altitudes $0 \leq z \leq c_s^2/g$, which, for $c_s = 330 \text{ m/s}$, $g = 9.8 \text{ m/s}^2$, yields the interval $0 \leq z \leq 11 \text{ km}$ sufficient for studying the tornado phenomenon. Equation (21), combined with (8) and the fact that $\partial C/\partial t = 0$, yields the dimensionless identities

$$\gamma \Gamma M^{-2}r^2/2 + \Phi(z) = p(z), \quad C^2/2 + 2R^{-1}Reu + M^2 \gamma^{-1}\Phi + g_0z = \text{const},$$

Hence, we finally obtain the following expression for the tornado profile:

$$r(z) = \left(\frac{2}{\Gamma}\right)^{1/2} \left\{\frac{C(z)^2}{2} + \frac{2Reu(z)}{R} + \frac{M^2}{\gamma} \left(1 - \frac{zg_0}{M\gamma}\right)^{-\gamma/(\gamma-1)} + g_0z + \text{const}\right\}^{1/2}, \quad (23)$$

where $M = c_s/U_0$ is the Mach number, $g_0$ is defined in (8). The functions $C(z)$, $u(z)$ are obtained as numerical solutions of the boundary value problem (6), (7), and therefore, depend on the boundary value $u_0$ at the point $z = 0$. The constant in (23) is chosen from the condition that the radicand is nonnegative. For powerful (in the sense of section 4) tornadoes, the expression for $r(z)$ is simplified, since (15) ensures that the sum of the first two terms of the radicand in (23) is of the order $\sim \Gamma^{-1/2}R^{-1}$, which is much less than the order $\sim M^2/\Gamma$ of the sum of the other two terms, as $u_0\Gamma^{-1/2} \rightarrow 0$. Indeed, the relation $M^2/\Gamma \gg \Gamma^{-1/2}R^{-1}$ is equivalent to $M^2R\Gamma^{-1/2} = (c_s^2/\Omega)(\rho/\mu) \gg 1$, where $\Omega = (2Q/\rho)^{1/2}$ is the angular velocity of air in the vortex ring of the mother cloud. The last condition, $M^2R\Gamma^{-1/2} \gg 1$, for $c_s = 3.3 \cdot 10^4 \text{ cm/s}$, $\mu/\rho = 0.15 \text{ cm}^2/\text{s}$, holds a fortiori. Therefore, for powerful tornadoes, the first two terms of the radicand in (23) can be neglected, and we obtain the following approximate formula for $r(z)$:

$$r(z) = c_s\Omega^{-1} \left\{2\gamma^{-1} \left(1 - zg\gamma s^{-2}\right)^{-\gamma/(\gamma-1)} + 2gzc_s^{-2} + \text{const}\right\}^{1/2}. \quad (24)$$
Our further results vary according to the value of the adiabatic exponent \( \gamma \), which depends on the assumptions about the structure of air. If air is considered to be a gas consisting of monoatomic particles with two or three degrees of freedom, then \( \gamma = 2 \) or \( \gamma = 5/3 \), respectively. If we take into account that air, in fact, consists mainly of diatomic molecules of nitrogen and oxygen, then \( \gamma = 9/7 \), provided that vibrations of atoms in the diatomic molecule have been taken into account, and \( \gamma = 7/5 \) otherwise. For details pertaining to the calculation of \( \gamma \), see [25]. The simplest expression for \( r(z) \) is obtained for \( \gamma = 2 \):

\[
r(z) = c_s \Omega^{-1} \left\{ 1 + z^2 g^2 c_s^{-4} + \text{const} \right\}^{1/2} = \left\{ r(0)^2 + g^2 c_s^{-2} \Omega^{-2} z^2 \right\}^{1/2}.
\]

(25)

The curve (25) corresponds to a funnel with its base of radius \( r_0 \) at the altitude \( z_0 \). Its graph is represented in figure 4. The boundary of the funnel-shaped tornado is obtained by rotating the graph around the axis \( z \), which qualitatively agrees with observation data. For other adiabatic exponents, \( 2 > \gamma > 1 \), the picture is similar, but the tornado profile \( r(z) \) calculated by (24) has a minimum at the point \( z_0 = (c_s^2/g)[1 - (\gamma - 1)^{-1}] \), and therefore, the funnel has a neck at the altitude \( z_0 \). The condition that the radicand in (24) is nonnegative is equivalent to it being nonnegative at the point of minimum \( z_0 \), which yields

\[
2 \left( 1 - \frac{(\gamma - 1)^{-1}}{\gamma} \right) + \text{const} \geq 0 \Rightarrow \left( \frac{r(0) \Omega}{c_s} \right)^2 \geq \frac{2(\gamma - 1)}{\gamma} \left[ (\gamma - 1)^{-2} - 1 \right].
\]

Hence we obtain the following constraint on \( r(0) \) and \( \Omega \):

\[
r(0) \Omega \geq h(\gamma) c_s.
\]

(26)

**Figure 4.** Graph of \( r(z) = (r(0)^2 + r_1^2(z))^{1/2} - 1 \); \( r_1(z) = (gc_s^{-1} \Omega^{-1}) z - 2 \).

For \( \gamma = 5/3 \), we have \( h(\gamma) = 0.34 \). For \( \gamma = 2 \), we have \( h(\gamma) = 0 \), condition (26) holds always and imposes no constraints on \( r(0), \Omega \). Let us use (26) to estimate the derivative \( r'(0) \) for \( 2 > \gamma > 1 \). From (24), it follows that

\[
r'(0) = -g(\Omega^2 r(0))^{-1} (2 - \gamma) (\gamma - 1)^{-1} \Rightarrow |r'(0)| \leq \frac{gr(0)(h^2(\gamma)c_s^2)^{-1}(2 - \gamma)(\gamma - 1)^{-1}}{\Omega^2}.
\]

Observations show [1, 2] that the funnel radius \( r(0) \) at the Earth’s surface varies from several meters to about one kilometer. Therefore, the above estimate for average values \( r(0) \sim 100 \text{ m} \) implies that \( |r'(0)| \ll 1 \), and therefore, the tornado boundary near the Earth’s surface is almost orthogonal to it. From the estimate (26), another important conclusion can be made: The smaller the radius of the tornado funnel at the Earth’s surface, the greater the angular velocity of the air masses of the vortex ring in the mother cloud that generates the tornado.

The expression (24) can be rewritten in a more convenient form:

\[
r(z) = \left\{ r(0)^2 + 2g^2 \Omega^{-2} z + c_s^2 \Omega^{-2} z^2 \left[ 1 - (1 - z g c_s^{-2})^{\gamma/(\gamma - 1)} \right] \right\}^{1/2}, \quad 0 \leq z \leq c_s^2 g^{-1},
\]

where \( r(0) \) and \( \Omega \) satisfy the inequality (26).

Thus, the tornado model considered in [3, 4] and here yields a qualitatively adequate prediction of the geometric shape of a powerful tornado.
6. Conclusion
We have considered a mathematical model of the tornado phenomenon based on some special exact solutions of the Navier-Stokes equations. In particular, solutions of this type describe upward air currents swirling around some axis. Such air flows are commonly called tornadoes. The reliability of the results obtained is the same as that of the equations of gas dynamics of air flows, the applicability of which to the analysis of the tornado phenomenon is generally accepted. The question whether the air dynamics equation should take into account the Coriolis force needs further discussion. Our investigation shows that the Coriolis force is not a determining factor in the formation of tornadoes, since tornado solutions exist even if this force is neglected in the dynamic equations. However, air viscosity is a determining factor in the formation of tornadoes.

We have constructed a theory of powerful tornadoes for which the angular velocity of air rotation near the Earth’s surface due to the Coriolis force is much smaller than the angular velocity of air rotation in the vortex ring of the mother cloud which generates the tornado. For powerful tornadoes, we obtain analytic expressions approximating (with any given accuracy) the air velocity in a tornado and its geometric boundary. These theoretical constructions can be used for the investigation of powerful anticyclones and reduce the problems of predicting and preventing tornadoes to a purely technical level.

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