Quantum tori, mirror symmetry and deformation theory

Yan Soibelman
Department of Mathematics,
Kansas State University,
Manhattan, KS 66506, USA
1 Introduction

1.1
Mathematical models of dualities in quantum physics is a very interesting and intriguing area of research. It became clear after the work of Kontsevich (see [Ko1]-[Ko3]) that the “right” framework for general duality theorems is the (yet non-existing) theory of moduli spaces of $A_\infty$-categories. Informally, an $A_\infty$-category (with extra conditions imposed) models a “projective non-commutative space” together with the formal moduli space of its deformations. Moduli space of such non-commutative spaces (whatever it is) consists of many “connected components”. The boundary of the compactification of a component may contain “cusps”. A non-commutative space can degenerate into a commutative one at a cusp. In some cases one can assign to a pair $(\text{component, cusp})$ a “dual” pair. The corresponding $A_\infty$-categories are equivalent. In particular, their Hochschild cohomologies (interpreted as tangent spaces to the moduli space of $A_\infty$-categories) are isomorphic. This idea of Kontsevich has received spectacular confirmation in homological mirror symmetry program (see [Ko1], [KoSo2]).

1.2
Even if the degeneration of a non-commutative space at a cusp is not commutative, one hopes to extend the dualities to the boundary stratum. The purpose of this paper is to explain without details the idea of non-commutative compactification in an example of abelian varieties. It will be discussed at length elsewhere (see [So]). The paper is an extended version of a series of talks I gave at Ecole Polytechnique, MSRI, Stanford University, Max-Planck Institut für Mathematik in Bonn, Oberwolfach workshop on non-commutative geometry and Moshe Flato Euroconference in Dijon in 1999-2000. In this paper we are going to treat many aspects informally, in order to explain main ideas.

Let us start with an example of what can be thought of as a non-commutative compactification.

Example 1 The universal covering of the moduli space of elliptic curves admits a “non-commutative” compactification with the boundary stratum con-
sisting of the universal covering of the moduli space of quantum (= non-commutative) tori. Moreover, the SL(2, Z)-symmetry extends from the upper-half plane to the boundary real line.

We understand a “space” as a category of certain sheaves on it. Thus the “moduli space” of “spaces” corresponds to the “moduli space” of categories. The latter “moduli space” $\mathcal{M}$ can be a usual manifold or orbifold (a compactification of $\mathcal{M}$ is often a compact manifold with corners). Let us consider a path $\gamma : [0, 1] \to \mathcal{M}$ between an interior point of the compactified moduli space and a point of the boundary. Then we have a 1-parameter family of categories $\mathcal{C}_t$ along the path. It can happen that $\mathcal{C}_t, t \in [0, 1)$ is a category of sheaves on a topological space, while $\mathcal{C}_1$ is not. We are going to treat $\mathcal{C}_1$ as a category of sheaves on a “non-commutative” space. Such non-commutative spaces constitute a “non-commutative stratum” of the compactification $\overline{\mathcal{M}}$.

In the example above, we think about an elliptic curve $E_q = \mathbb{C}^*/q\mathbb{Z}$, $q = e^{2\pi i \tau}$, $\text{Im}(\tau) > 0$ as about the bounded derived category $D^b(E_q)$ of coherent sheaves on it (we are going to discuss below reasons why derived categories appear in the story). Let us consider the universal covering of the moduli space of complex structures. Thus we have a family of categories $D^b(E_q)$ parametrized by the upper-half plane $\mathcal{H} = \{ \tau | \text{Im}(\tau) > 0 \}$. As $\text{Im}(\tau) = 0$ the elliptic curve does not exist as a “commutative” space. We will argue that the degenerate object is represented by the derived category of certain modules over the algebra of functions on a quantum torus. From this point of view elliptic curves and quantum tori belong to the same family of non-commutative spaces. Also, the $SL(2, \mathbb{Z})$-symmetry should be present for all $q \neq 0$. See Section 3 for more details. □

1.3

Relevance of derived categories (more technically, triangulated $A_\infty$-categories with finite-dimensional Hochschild cohomology) can be roughly justified by the following arguments:

\footnote{The idea to treat 2-dimensional quantum tori as limits of elliptic curves appeared in [CDS], but it was not made precise there. Motivated by the preliminary version of present paper Yu. Manin suggested in [M2] interesting ideas about quantum tori and abelian varieties defined over arbitrary complete normed fields.}
1) Deformation theory of an $A_\infty$-category is controlled by its Hochschild complex. Being defined properly (see [KoSo1]) such a category always appears together with the formal moduli space of its deformations.

2) Reconstruction theorems, mainly due to A. Bondal, D. Orlov and A. Polishchuk. Here are two examples.

**Theorem 1 ([BO]).** Let $X$ be a smooth irreducible variety with ample canonical or anticanonical sheaf, and $Y$ be a smooth algebraic variety. If the bounded derived categories of coherent sheaves $D^b(X)$ and $D^b(Y)$ are equivalent as triangulated categories, then $X$ is isomorphic to $Y$.

If $X$ is a Calabi-Yau manifold (for example, an elliptic curve) then the theorem is not true. Nevertheless, often one can recover from $D^b(X)$ some information about $X$.

**Theorem 2 ([O1]).** For each abelian variety $X$ there are finitely many non-isomorphic abelian varieties which have the bounded derived category of coherent sheaves equivalent to $D^b(X)$.

1.4

Physicists discovered quantum tori in various theories (see [CDS], [SW]). Morita equivalence of quantum tori was interpreted as a new duality (see [RS], [S1]). One hopes that the concept of non-commutative compactification will help to construct and investigate mathematical models for dualities in quantum physics.

As an example let us consider the Homological Mirror Conjecture (HMC) of Kontsevich (see [Ko1]). Homological Mirror Conjecture says that the category $D^b(E_q)$ is equivalent to the bounded derived category $D^b(F(T^2_q))$ of the Fukaya category $F(T^2_q)$ (see for ex. [Ko1] for the definition of the Fukaya category) of the symplectic torus $T^2_q = (T^2, 1/Re(\tau)dx_1 \wedge dx_2)$. Let us imagine, that as $q$ approaches to a point at the circle $|q| = 1$, both derived categories degenerate into the same $A_\infty$-category. Hence non-commutative degenerations become manifestly equivalent. This might give an insight to HMC. We will speculate about the category that can appear as such degeneration. Roughly speaking, it is the (derived) category of bundles with connections which are flat along a foliation. The foliations appear as degenerate complex
structures. Global sections of such bundles are modules over the algebra of functions on a quantum torus. Thus one gets a “non-commutative” description of the corresponding boundary stratum of the compactified universal covering of the moduli space of complex structures.

The foliations which appear in the story are not arbitrary. They carry affine structures on leaves. This makes the whole picture similar to the topological mirror symmetry of Strominger-Yau-Zaslow (see [SYZ]). In [SYZ] Calabi-Yau manifolds are foliated by special Lagrangian tori, hence fibers carry affine structures. On the other hand, the SYZ-picture is related to the “large” complex structure limit, while our approach seems to be of different nature. Hopefully, they both correspond to two different strata of the boundary of the compactified moduli space of $N = 2$ superconformal field theories. The commutative stratum was discussed in [KoSo2] in the case of abelian varieties (see also Section 5 below).

1.5

Two-dimensional quantum tori can be interpreted in many different ways. For example, they are Morita equivalent to algebras of foliations. Quantum tori can also be thought of as quantizations of tori equipped with constant symplectic structures (they appear in the open string theory in this way). Hence they fit into the framework of deformation theory. Deformation theory of a Poisson manifold $X$ gives rise to a formal family of categories $\hat{C}_h$ of modules over a quantized algebra $C^\infty(X)_h$ of smooth function on $X$, where $h$ is the formal parameter. The problem is to construct a “global” moduli space. The germ at $h = 0$ of this moduli space contains families of $\mathcal{A}_\infty$-categories $\mathcal{C}_h$, $h \in \mathbb{R}$ such that the formal completion of $\mathcal{C}_h$ at $h = 0$ gives $\hat{\mathcal{C}}_h$.

In general $\mathbb{R}$ can be replaced by some parameter space $\mathcal{M}_X$. If the global parameter space exists, one can speak about “dualities” for the “family of theories parametrized by $\mathcal{M}_X$”. In the case of quantum tori, the dualities can correspond to Morita equivalence of quantum tori, or to equivalence of some subcategories of the full categories of modules. The “duality group” should be $SL(2, \mathbb{Z})$ in order to agree with the interpretation of quantum tori as degenerate elliptic curves.

For general Poisson manifolds one can ask the following question.

**Question 1** For a given Poisson manifold $X$, is there a global parameter
space $\mathcal{M}_X$? If it exists, how to describe points corresponding to the equivalent categories? What is the "duality group" (and why it is a group)?

We are going to discuss later in the paper conjectures motivated by quantum groups. Interesting ideas in this direction can be found in [Fad].

1.6

Here is the content of the paper. In Section 2 we discuss analogies between quantum tori over $\mathbb{R}$ and abelian varieties over $\mathbb{C}$. In Section 3 we explain why quantum tori can be thought of as points of the boundary of the compactified Teichmüller space of an elliptic curve. Main idea here is to treat the Thurston boundary (given geometrically in terms of foliations) as a non-commutative space (i.e. as a category). Section 4 is devoted to mirror symmetry for quantum tori. We suggest to "compactify" the conventional homological mirror symmetry, extending it to the boundary of the universal covering of the moduli space of complex structures (Teichmüller space in the case of elliptic curves). Section 5 contains a comparison with the "commutative" picture. This material is borrowed from [KoSo2]. In Section 6 we discuss possible relations to theta functions and quasi-modular forms. We end this section with conjectures about dualities in quantum groups.

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2 Quantum tori and their representations

2.1 Generalities

We start with the definition of an algebraic quantum torus (see for example [M1]). Let $K$ be a complete normed field.
**Definition 1**  Quantum torus $T(L, \alpha)$ of rank $d$ (or dimension $d$) is defined by a free $\mathbb{Z}$-module $L$ of finite rank $d$ (lattice) and a bilinear form $\alpha : L \times L \rightarrow K$ such that

$$\alpha(m,n)\alpha(n,m) = 1, \alpha(m+n,l) = \alpha(m,l)\alpha(n,l).$$

More precisely, the coordinate ring $A(T(L, \alpha))$ of the quantum torus is a $K$-algebra with the unit generated by generators $e(n), n \in L$ subject to the relations

$$e(m)e(n) = \alpha(m,n)e(m+n).$$

Algebra of analytic functions $A^{an}(T(L, \alpha))$ is a completion of $A(T(L, \alpha))$ which consists of formal series $\sum_{n \in L} a_n e(n)$ with the coefficients $a_n$ decreasing as $|n| \rightarrow \infty$ faster than any power of $|n|$.

Assume that $K$ carries an involution $x \mapsto x^*$, such that $|x^*| = |x|^2$. Suppose that $\alpha(m,n)^{-1} = \alpha(m,n)$ for any $m, n \in L$. Then there is a natural involution of $A(T(L, \alpha))$ given by $e(m)^* = e(-m)$. It makes $A_{\alpha} = A(T(L, \alpha))$ into a $K$-algebra with involution. Clearly in this case $\alpha$ takes value in the subgroup $K_1 = \{ x \in K, |x| = 1 \}$. We will call such quantum tori unitary. We consider their coordinate rings as objects in the category of algebras with involutions.

**Definition 2**  Let $T(L, \alpha)$ be a unitary quantum torus. The algebra $B_{\alpha} := C^{\infty}(T(L, \alpha))$ of smooth functions on $T(L, \alpha)$ consists of series $f = \sum_{n \in L} a_n e(n)$ where the sequence $a_n$ decreases faster than any power of $|n|$, as $|n| \rightarrow \infty$.

This terminology is justified by the case $K = \mathbb{C}$, where one gets the algebra of smooth functions on a real torus. One has the natural embedding of algebras with involutions: $A_{\alpha} \subset B_{\alpha}$.

We are going to consider unitary quantum tori over the field $\mathbb{C}$ unless we say otherwise. The corresponding classical torus $L_{\mathbb{R}}/L$ will be denoted by $T(L)$ (here $L_{\mathbb{R}} = L \otimes \mathbb{R}$). Let $\Phi : L_{\mathbb{R}} \rightarrow L_{\mathbb{R}}^*$ be a linear map such that $\Phi(x)(y) + \Phi(y)(x) = 0$. We set $\varphi(x,y) = \Phi(x)(y)$. Thus we get a skew-symmetric bilinear form $\varphi : L_{\mathbb{R}} \times L_{\mathbb{R}} \rightarrow \mathbb{R}$. Taking $\alpha(x,y) = \exp(2\pi i \varphi(x,y))$ we obtain a unitary quantum torus, which will be denoted by $T(L, \varphi)$ or $T(L, \Phi)$. The corresponding algebras of functions (algebraic and smooth) will be denoted either by $A_\varphi, B_\varphi$ or by $A_\Phi, B_\Phi$. 

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Let $V_L = L_R \oplus L_R^*$. We equip this vector space with an odd symplectic structure. It is given by the canonical symmetric form $Q((x_1, l_1), (x_2, l_2)) = l_1(x_2) + l_2(x_1)$. Then the skew-symmetric map $\Phi$ defines a Lagrangian subspace $\text{graph}(\Phi) \subset V_L$.

One can define the Grassmannian $Gr_0(V_L)$ of Lagrangian subspaces in $V_L$. Then the map $\Phi \mapsto \text{graph}(\Phi)$ identifies quantum tori with an open subset of $Gr_0(V_L)$. Let $O(V_L, Q)$ be the group of linear automorphisms of $V_L$ preserving the form $Q$ (orthogonal group), and $SO(V_L, Q)$ be the corresponding special linear group. Then $O(V_L, Q)$ and $SO(V_L, Q)$ act transitively on $Gr_0(V_L)$. We will denote by $SO(L, L^\vee)$ the subgroup of $SO(V_L, Q)$ which preserves the lattice $L \oplus L^\vee$, where $L^\vee = \text{Hom}(L, \mathbb{Z})$.

### 2.2 Morita equivalence

The following theorem was proved in [RS] in the framework of $C^*$-algebras.

**Theorem 3** In the notation of the previous subsection, let $\text{graph}(\Phi)$ and $\text{graph}(\Phi')$ are conjugate by an element of the group $SO(L, L^\vee)$. Then the algebras $B_\Phi$ and $B_{\Phi'}$ are Morita equivalent.

Choosing a basis in $L$ one can identify the group $SO(L, L^\vee)$ with the group $SO(d, d, \mathbb{Z})$ of linear automorphisms of the vector space $\mathbb{R}^{2d}$ preserving the form $\sum_{1 \leq i \leq d} x_i x_{i+d}$ and the lattice $\mathbb{Z}^{2d}$.

Surprisingly, the same group appears in a different problem concerning derived categories of coherent sheaves on complex abelian varieties. We recall the following result.

Let $X$ and $Y$ be complex abelian varieties, $\hat{X}$ and $\hat{Y}$ are dual abelian varieties. We denote by $L_X$ and $L_Y$ the lattices of first homologies of $X$ and $Y$.

**Theorem 4** ([O1]). The derived category $D^b(X)$ is equivalent to $D^b(Y)$ iff there exists an isomorphism $X \times \hat{X} \to Y \times \hat{Y}$ which identifies $L_X \oplus L_X^\vee$ and $L_Y \oplus L_Y^\vee$ as odd symplectic lattices (both lattices are equipped with the canonical symmetric forms $Q_X$ and $Q_Y$ as above).
2.3 General algebraic scheme

We refer the reader to [KoSo2] for the background on $A_\infty$-categories. We are going to discuss here a general algebraic scheme which sheds some light on the similarity between the derived category of coherent sheaves on an abelian variety and the derived category of modules over the algebra of functions on a quantum torus. We will impose some restrictions on the objects. These restrictions can be relaxed. The reader should assume that “natural” conditions are imposed, so that “everything works”.

Let $C$ be an $A_\infty$-category over a field $k$ of characteristic zero, such that its Hochschild cohomology $HH^i(C)$ is finite-dimensional for all $i \geq 0$. In what follows we will assume that $k = \mathbb{C}$.

We assume that $HH^0(C)$ is a 1-dimensional vector space. It can be thought of as a Lie algebra of the group $Aut(Id_C)$ of automorphisms of the identity functor. The total cohomology space

$$\bigoplus_{i \geq 0} HH^i(C) = \bigoplus_{i,j \geq 0} Ext^i(Id_C, Id_C)$$

carries a graded Lie algebra structure (with the Gerstenhaber bracket on the Hochschild cohomology). Let $C = D^b(X)$, where $X$ is a smooth complex projective variety. Then $\bigoplus_{i \geq 0} HH^i(C) = \bigoplus_{i,j \geq 0} H^i(X, \wedge^j T_X)$ where $T_X = T_X^{1,0}$ is the holomorphic tangent bundle.

We assume that the Lie subalgebra $HH^1(C)$ is abelian. It admits the following interpretation. Let us consider the group $Aut(C)$ of automorphisms of the $A_\infty$-category $C$. It is the group of classes of isomorphisms $[F]$ of equivalence functors $F : C \to C$. We assume that it carries a structure of a Lie group. The Lie algebra $g_C$ of the connected component of the unit of $Aut^0(C) := G_C$ is isomorphic to $HH^1(C)$. Under our assumptions, the group $G_C$ is a finite-dimensional commutative Lie group over $\mathbb{C}$.

There exists a bundle $P$ over $G_C \times G_C$ with the fiber which is a $\mathbb{C}^*$-torsor. Let us describe it in detail. Let $F$ and $H$ be two functors, such that their isomorphism classes $[F], [H]$ belong to the group $G_C$. According to our assumption on $HH^0(C)$, the set of isomorphisms $Iso(F \circ H, H \circ F)$ is a $\mathbb{C}^*$-torsor. Suppose that $F_1$ and $H_1$ are another representatives of the classes $[F]$ and $[H]$ respectively. Then $F \sim F_1$ and $H \sim H_1$. This gives rise to an isomorphism of torsors $Iso(F \circ H, H \circ F) \simeq Iso(F_1 \circ H_1, H_1 \circ F_1)$.

The following lemma is easy to prove.
Lemma 1 This isomorphism of torsors depends on the equivalence classes \([F]\) and \([H]\) only. In other words, for different liftings of \([F]\) and \([H]\), we can canonically identify the torsors.

Thus we obtain a \(\mathbb{C}^\ast\)-torsor \(P = P_C\) over \(G_C \times G_C\) with the fibers \(P_{[F],[H]} = Iso(F \circ H, H \circ F)\).

The abelian group \(L_C := H_1(G_C, \mathbb{Z})\) carries a bilinear form \((,): L_C \times L_C \to \mathbb{Z}\), such that \((x,y) = c_1(P)(x \boxtimes y)\), where \(c_1(P)\) is the first Chern class of \(P\). We will keep the same notation for the \(\mathbb{C}\)-linear extension of the bilinear form to \(L_C \otimes \mathbb{C}\).

Since \(G_C\) is a Lie group, its fundamental group is commutative, and hence it is isomorphic to \(H_1(G_C, \mathbb{Z})\). Let \(\gamma\) be a loop based at the unit \(e\) of \(G_C\). We define a linear map \(p: L_C \otimes \mathbb{C} \to g_C\) by the formula \(p(\gamma) := \dot{\gamma}(e)\).

Then we have the following exact sequence of linear maps:

\[
0 \to \Lambda_C \to L_C \otimes \mathbb{C} \to g_C,
\]

where \(\Lambda_C\) is defined as the kernel of \(p\).

Conjecture 1 The symmetric bilinear form \((x,y)\) is non-degenerate, and the subspace \(\Lambda_C\) is maximal isotropic (i.e. Lagrangian) with respect to it.

It is not clear how to prove this conjecture for general \(A_\infty\)-categories. In two examples considered below (abelian varieties and quantum tori) the proofs are straightforward.

Assuming the conjecture, to the \(A_\infty\)-category \(\mathcal{C}\) we have assigned canonically a Lagrangian subspace \(\Lambda_C\) in the (odd) symplectic vector space \(L_C \otimes \mathbb{C}\). It is interesting to notice that the linear algebra data seem to contain certain information about “discrete” symmetries of the \(A_\infty\)-category. Apriori this was not obvious.

Remark 1 In the case of quantum tori we will need to consider non-Hausdorff Lie groups. Then the abelian Lie algebra \(L_C\) should be defined not as the first homology, but as the kernel of the exponential map \(\exp: g_C \to G_C\). Notice that \(g_C\) is defined canonically: it is the Lie algebra of infinitesimal symmetries of our \(A_\infty\)-category. The group \(G_C\) is not defined canonically. At the level of local Lie groups one can think about \(G_C\) as about quotient of the local Lie group corresponding to \(g_C\) by the discrete subgroup consisting of such \(\gamma \in g_C\) that \(\exp(\gamma)\) gives rise to an automorphism of the category \(\mathcal{C}\) isomorphic to the identity functor.
2.4 Example: abelian varieties and quantum tori

Let $\mathcal{C} = D^b(X)$ where $X$ is a complex abelian variety of dimension $n$. Then $G_{\mathcal{C}} \simeq \mathbb{C}^{2n}/\mathbb{Z}^{4n}$. The bundle $P$ is basically the tensor square of the Poincare line bundle on $X \times \hat{X}$ (this follows from the results of Polishchuk and Orlov, see [O1]). The odd symplectic form is the canonical symmetric form on $\mathbb{C}^{4n} = \mathbb{C}^{2n} \oplus (\mathbb{C}^{2n})^*$. It is easy to see that $HH^*(\mathcal{C}) \simeq \bigwedge^*(\mathbb{C}^{2n})$. The latter can be also interpreted as the algebra of functions on the odd Lagrangian subspace in $\mathbb{C}^{4n}$. Thus the formal deformation theory of $\mathcal{C}$ as an $A_\infty$-category is the same as the deformation of the corresponding odd Lagrangian subspace as a (non-linear) Lagrangian submanifold $\Lambda_{\mathcal{C}} \subset \mathbb{C}^{4n}$.

For a quantum torus $T(L, \Phi)$ of dimension $n$ one takes as $\mathcal{C}$ the derived category of modules of finite rank over the algebra $B_\Phi(L)$. It is expected that automorphisms of the category come from automorphisms of the algebra. Then $G_{\mathcal{C}} \simeq T^n/\mathbb{Z}^n$ is a non-Hausdorff Lie group (it is the group of automorphisms of the algebra $B_\Phi(L)$ modulo inner automorphisms). Following the general scheme outlined above, one obtains a Lagrangian subspace (it coincides with $\text{graph}(\Phi)$) in the odd symplectic vector space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus (\mathbb{R}^n)^*$. For the categories discussed in this subsection the cohomology $HH^0$ is 1-dimensional, and $HH^1$ is a commutative Lie algebra. Moreover, the Conjecture 1 holds in both cases.

3 Complex structures and foliations

3.1 Foliations and degenerate complex structures

Let $X$ be an even-dimensional smooth real manifold, which carries a complex structure. The latter is given by an integrable subbundle $T^{0,1}_X \subset T_X \otimes \mathbb{C}$ of anti-holomorphic directions, where $T_X$ is the real tangent bundle of $X$. Suppose that we have a family of complex structures degenerating into a real foliation of the rank equal to $\dim_\mathbb{C} X$. We will call such degenerations maximal. In fact one should consider subsheaves of the tangent sheaf $T_X$ because the foliation can be singular. Degeneration gives rise not just to a foliation $F$ of $X$, but also to an isomorphism $j$ of vector bundles $F$ and $T_X/F$. The isomorphism satisfies some integrability conditions, which give rise to an affine structure on the fibers of $F$. To be more precise, the space $\text{Hom}(F, T_X/F)$ can be interpreted as a tangent space $T_F(\mathcal{M})$ to $F$ in the
“moduli space” $\mathcal{M}$ of all foliations on $X$. When $T_{X}^{0,1}$ and the holomorphic subbundle $T_{X}^{1,0}$ get close to each other, one obtains a tangent vector in the space $T_{F}(\mathcal{M})$. Similarly, for any fiber $F_y, y \in X$ of the foliation $F$, the vector space $(T_{X}/F)_y$ can be identified with the tangent space to the leaf $F_y$ in the “moduli space” space of all leaves of the foliation $F$ (it does not depend on a point of the leaf). It is easy to check that the integrability condition implies the following result.

**Proposition 1** The isomorphism $j : F \rightarrow T_{X}/F$ gives rise to an isomorphism of the tangent space to a leaf $F_y$ in the space of leaves, with the space of commuting vector fields on the leaf. In particular it yields an affine structure on the leaf $F_y$.

Informally this result can be explained in the following way. Tangent space to a “degenerating” complex manifold is invariant with respect to the rotation by 90 degrees (multiplication by $i = \sqrt{-1}$). Let us move a point $x$ along some leaf $F_y$ of the limiting foliation $F$. Then the rotation by 90 degrees of vectors from $T_{x}X/F_x$ transforms them into $F_x$. On the other hand, the bundle $T_{X}/F$ carries a canonical flat connection (Bott connection). We can identify the spaces $T_{X,x}/F_x$ for close points $x$ using the connection. Hence a choice of commuting vectors at the space $T_{X,y}/F_y$ gives rise to commuting vector fields along the leaf $F_y$.

One can show that the space of pairs $(F, j)$ being factorized by the natural action of the multilicative group $\mathbb{R}^*$ (dilations of $j$) is generically a subspace of the real codimension 1 in the space of integrable subbundles in $T_{G}X$. The latter can be considered as a subvariety of the total space of the bundle of Grassmannians $\Gamma(X, Gr(T_{G}X))$. In this way one gets a compactification of the universal covering of the moduli space of complex structures on $X$. In the case of curves the compactification adds the Thurston boundary to the Teichmüller space.

One can try to “compactify” the category of coherent sheaves on a complex manifold. The category of sheaves equipped with flat connections along the foliation, which is the maximally degenerate complex structure, serves as a “point” of the “non-commutative” stratum of the boundary. In the case of elliptic curves the compactification is compatible with the natural $SL(2, \mathbb{Z})$-action. Indeed, in both cases (complex structure given by $\tau \in \mathcal{H}$ and affine foliation on $T^2$ given by $dt = \varphi dx$) if the values of parameters
are $SL(2, \mathbb{Z})$-conjugate, then there is an automorphism of $T^2$ (as a smooth manifold) which identifies the corresponding subbundles (in $T_C T^2$ and $T_R T^2$ respectively).

More generally, one considers the space of complex polarizations on a given compact complex manifold $X$. To every polarization $\tau$ one assigns the corresponding bounded derived category of coherent sheaves. The space of polarizations admits a natural compactification by real foliations, as we discussed above. The question is: what is the category which should be assigned to a foliation $F$ which is a point of the boundary? In the case of maximal degeneration we suggest to take the (derived) category of $F$-local systems (see next subsection).

**Remark 2** Compactification by pairs $(F, j)$ modulo the action of $R^*$ is not compatible with the action of the group of diffeomorphisms $Diff(X)$. In particular, it does not descends to a compactification of the moduli space of complex structures on $X$. In 1-dimensional case the action of the mapping class group extends to the Thurston boundary of the Teichmüller space.

### 3.2 Foliations and “small” modules over quantum tori

Let $X$ be a smooth manifold of dimension $2n$, and $F$ be a foliation of $X$ of rank $n$. Let $W$ be a sheaf of $C^\infty_X$-modules on $X$, where $C^\infty_X$ is the sheaf of smooth functions on $X$.

**Definition 3** We say that $W$ carries an $F$-connection, if we are given a morphism of sheaves $\nabla : W \to W \otimes F^*$ such that $\nabla_v(fs) = f\nabla(s) + v(f)s$ for any germs $f \in C^\infty_X$, $s \in W$, and $v \in F$ (here we identify the subbundle $F$ with the sheaf of sections), $F^* = \text{Hom}(F, C^\infty_X)$.

The category $\text{Sh}(X, F)$ of sheaves of finite rank, which carry an $F$-connection form a tensor category. The curvature of an $F$-connection is defined in the usual way. Sheaves which carry $F$-connections with zero curvature are called $F$-flat. If an $F$-flat sheaf is locally free (i.e. it corresponds to a smooth vector bundle on $X$), it is called an $F$-local system. We denote by $D^b(X, F)$ the bounded derived category of the category $\text{Loc}(X, F)$ of $F$-local systems on $X$. If $X$ carries a symplectic form $\omega$, and $F$ is a Lagrangian foliation then we will sometimes add $\omega$ to the notation. Foliations described
Let $F$ be an affine foliation of rank $n$ on the standard torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. This means that $F$ is defined as $(V \oplus \mathbb{R}^{2n})/\mathbb{Z}^{2n}$ for some $n$-dimensional vector subspace $V \subset \mathbb{R}^{2n}$. Let us choose the subspace $S$ such that $S \oplus V = \mathbb{R}^{2n}$, and $S$ defines a closed $n$-dimensional submanifold $Y$ in $T^{2n}$. There is a pull-back functor from the category $\text{Loc}(T^{2n}, F)$ to the category of vector bundles on $Y$. On the other hand, if $(\Lambda, \nabla)$ is an $F$-local system on $X$, then the holonomy of the connection $\nabla$ defines an action of the group $\mathbb{Z}^n$ on the restriction $\Lambda|_Y$. Since $\mathbb{Z}^n$ acts on $Y$, we obtain a structure of a $\mathbb{Z}^n \rtimes C^\infty(Y)$-module on the space of section $\Gamma(Y, \Lambda)$. We can complete the cross-product algebra thus getting the algebra of smooth functions on a quantum torus acting on $\Gamma(Y, \Lambda)$.

**Definition 4**

a) Let $B_\Phi(Y)$ be the algebra of smooth functions on the quantum torus described above. We call a $B_\Phi(Y)$-module small if it is projective as a module over the commutative subalgebra $C^\infty(Y)$.

b) More generally, let $B_\Phi$ be an algebra of smooth functions on a quantum torus $T(L, \Phi)$ such that $\text{rk}(L) = 2n$, and $\Phi$ defines a symplectic structure on $L_\mathbb{R} = L \otimes \mathbb{R}$. Let $L_0 \subset L$ be a Lagrangian sublattice (i.e. $L_0 \otimes \mathbb{R}$ is a Lagrangian subspace in $L_\mathbb{R}$). We say that a $B_\Phi$-module $M$ is small with respect to $L_0$, if it is projective with respect to the maximal commutative subalgebra of $B_\Phi$ spanned by $e(\lambda), \lambda \in L_0$.

Clearly small modules with respect to a given Lagrangian sublattice $L_0$ form a category $B_\Phi(L_0) - \text{mod}$. Let us consider an example of 2-dimensional tori. Then $\text{rk}(L) = 2$, $\text{rk}(L_0) = 1$. We will identify $L$ with $\mathbb{Z}^2$ and $L_0$ with $\mathbb{Z} \oplus 0 \subset \mathbb{Z}^2$. Let $\varphi \in \mathbb{R} \setminus \mathbb{Q}$, and $F$ be the affine foliation $dt = \varphi dx$ in the standard coordinates in $\mathbb{R}^2 = L \otimes \mathbb{R}$.

**Proposition 2** The category $B_\varphi(L_0) - \text{mod}$ is equivalent to the category of $F$-local systems on the torus $T(L) = L_\mathbb{R}/L$.

**Proof.** We have already constructed a functor from $F$-local systems to the category of small modules. Namely, every $F$-local system $V$ being restricted to the equator of $T^2$, gives rise to a projective module over the algebra $C^\infty(T^1)$. Since $\varphi$ is irrational, the foliation defines an action of the group $\mathbb{Z}$ on $T^1 = (x \text{ mod } \mathbb{Z}, 0)$. The holonomy of the flat connection defines the structure of a small module on $\Gamma(T^1, V_{T^1})$. 

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An inverse functor is constructed such as follows. Let $M$ be a small module over $B_\varphi$. We denote the standard generators of $B_\varphi$ by $e_1$ and $e_2$. Then $M$ is a projective module over the subalgebra $B_\varphi(e_2)$ generated by $e_2$. Thus we have a vector bundle $\hat{M}$ over $T^1 = \{(x \mod \mathbb{Z}, 0)\} \subset T^2$, such that $\Gamma(T^1, \hat{M}) = M$.

Let $V \subset \Gamma(T^1, \hat{M}) \otimes C^\infty(\mathbb{R})$ consists of elements $f$ such that $f(x+1, t) = f(x, t)$ and $f(x, t+1) = e_1(f(x-\varphi, t))$ (here we write formulas in coordinates $(x, t) \in \mathbb{R}^2$ rather than in coordinates $(x \mod \mathbb{Z}, t) \in T^1 \times \mathbb{R}$). It is easy to check that they are global sections of the vector bundle $V \to T^2$ such that for its pull-back to $\mathbb{R}^2$ we have: the fiber $V(x, t)$ is naturally isomorphic to $\hat{M}_x$.

We define an $F$-connection $\nabla_F$ on $V$ by identifying infinitesimally closed fibers: $g_\varepsilon : f(x, t) \mapsto f(x + \varepsilon \varphi, t + \varepsilon)$. Then $g_1 : f(x, t) \mapsto f(x + \varphi, t + 1) = e_1(f(x - \varphi + \varphi, t)) = e_1(f(x, t))$. The action of the holonomy of $\nabla_F$ (shift on the period $t = 1$) is equivalent to the action of the generator $e_1$ on the module $M$. Thus the action of $\mathbb{Z} \ltimes C^\infty(T^1)$ on $\{f(x, 0)\} = M$ given by the $F$-local system is the same as the structure of $B_\varphi$-module on $M$. The Proposition is proved.■

**Remark 3** 1) In order to identify the (derived) categories of coherent sheaves on the elliptic curves $E_{\tau_1}$ and $E_{\tau_2}$ we use a bimodule, which is a sheaf of regular functions on the graph of an automorphism $f : T^2 \to T^2$ identifying the complex structures. If $p_i : E_{\tau_i} \times E_{\tau_2} \to E_{\tau_i}$, $i = 1, 2$ are natural projections, then the equivalence functor is given by $M \mapsto p_2^* (p_1^* M \otimes \mathcal{O}_{\text{graph}(f)})$. In the case of tori with affine foliations we basically use the same description. Notice that it does not give a Morita equivalence of the corresponding quantum tori. It gives an equivalence of the categories of small modules.

2) One should notice that vector bundles with $F$-connections form a tensor category, while projective (or all finite) modules over quantum tori do not.

3) Small modules are similar to holonomic $D$-modules. Their algebraic version was studied from this point of view in [Sab].

Suppose we have a free abelian Lie group $G$ together with a dense embedding of $G$ into the group $\text{Aut}(T^n)$ of the affine automorphisms of the torus $T^n$. Then to have a vector bundle $V$ over $T^n$ together with a lifting of the action of $G$ to $F$, is the same as to have a module over the quantum torus defined as a completed cross-product of the group algebra of $G$ and $C^\infty(T^n)$. We used this simple observation in the case when the action of $G = \mathbb{Z}$ was
induced by an affine foliation of $T^2$. In general we have a functor \( \{ \text{Affine foliations} \} \to \{ \text{Small modules} \} \). Notice that for \( n > 1 \) there is no inverse to this functor. In other words we cannot recover a foliation from a small module over the algebra $B_\phi$. Indeed, we have less affine foliations than quantum tori. Quantum torus is given by a skew-symmetric form. Hence the dimension of the moduli space of quantum tori of the rank $2n$ is equal to $n(2n-1)$. At the same time the moduli space of affine foliations has the dimension $n^2$ (every such a foliation is given by the graph of a linear morphism $\mathbb{R}^n \to \mathbb{R}^n$). These two numbers coincide only if $n = 1$.

### 3.3 Coherent sheaves on elliptic curves and quantum tori

Here we recall a result from [BG] and use it to provide another link between the derived category of coherent sheaves on elliptic curves and the derived category of certain modules over quantum tori. Assume that we have fixed a non-zero complex number $q$ such that $|q| < 1$. Let us consider a complex algebra $\bar{A}_q$ which is generated by the field of Laurent formal power series $\mathbb{C}((z))$ and an invertible element $\xi$ such that $\xi f(z) = f(qz) \xi$ for any $f \in \mathbb{C}((z))$. One defines an abelian category $\mathcal{M}_q$ such as follows. Objects of $\mathcal{M}_q$ are $\bar{A}_q$-modules $M$, which are $\mathbb{C}((z))$-modules of finite rank. In addition, it is required that there exists a free $\mathbb{C}[[z]]$-submodule $M_0 \subset M$ of maximal rank, such that $M_0$ is invariant with respect to the subalgebra $\mathbb{C}[\xi, \xi^{-1}] \subset \bar{A}_q$. Clearly $\mathcal{M}_q$ is a $\mathbb{C}$-linear rigid tensor category. It is proved in [BG] that $\mathcal{M}_q$ is equivalent to the tensor category $\text{Vec}^{ss}_0(E_q)$ of degree zero semistable holomorphic vector bundles on the elliptic curve $E_q = \mathbb{C}^*/q\mathbb{Z}$. Let us recall the idea of the proof. Using the Fourier-Mukai transform, applied to the vector bundles in question, one gets the category of sheaves with finite support. The latter category admits a description in terms of linear algebra data (a vector space equipped with an endomorphism). The same data describe objects of $\mathcal{M}_q$.

If we forget about tensor structures, then the category $\text{Vec}^{ss}_0(E_q)$ generates a subcategory $\text{D}^b_{fin}(E_q)$ of $\text{D}^b(E_q)$. Here we understand the word “generate” in the following sense: one allows to take extensions, direct summands of objects and all shifts of an object. We will use the notation $\text{D}^b(\mathcal{M}_q)$ for the derived category generated by $\mathcal{M}_q$. Then the previous discussion implies
the following result.

**Proposition 3** The categories $D^b(\mathcal{M}_q)$ and $D^b_{fin}(E_q)$ are equivalent.

One can use this theorem as a definition of $D^b_{fin}(E_q)$ in the case when $|q| = 1$.

**Question 2** Let $|q| = 1$. Is it true that $D^b(\mathcal{M}_q)$ is equivalent to a subcategory of the derived category of small modules over the quantum torus $B_q$?

**Question 3** Let $|q| < 1$. How to describe the category $D^b(E_q)$ in terms of modules over $B_q$ (or some related algebra)?

In order to answer the last question, one can consider the algebra generated by holomorphic functions on $\mathbb{C}^*$ and shifts $z \mapsto qz$. The question is whether it is possible to replace the algebra of holomorphic functions on $\mathbb{C}^*$ by something “more algebraic”. Then one would have a uniform description of the derived category of coherent sheaves on an elliptic curve and its “non-commutative” degeneration.

Hopefully, the above considerations can be generalized to the case when $\mathbb{C}(((z)))$ is replaced by a complete normed field (for example, to the $p$-adic case). Then it can be used as a definition of the derived category of coherent sheaves on quantum tori defined over such fields (cf. [M2]).

## 4 Mirror symmetry and deformation quantization

### 4.1 Reminder on Homological Mirror Conjecture

Homological Mirror Conjecture (HMC) was formulated by Kontsevich in 1993. We are not going to recall all the details here (see [Ko1], [KoSo2]). HMC is a claim about equivalence of two triangulated $A_\infty$-categories $D^b_{\infty}(X)$ and $F(X^\vee)$ for given mirror dual complex Calabi-Yau manifolds $X$ and $X^\vee$. The category $F(X^\vee)$ is the Fukaya category of $X^\vee$. It can be defined for any symplectic manifold $(M, \omega)$. Objects of $F(M)$ are pairs $(N, L)$ where $N$ is a Lagrangian submanifold of $M$ and $L$ is a unitary local system on $N$. For
two objects \((N_1, L_1), (N_2, L_2)\) such that \(N_1\) and \(N_2\) intersect transversally, one defines the space of morphisms as
\[
\text{Hom}((N_1, L_1), (N_2, L_2)) = \bigoplus_{x \in N_1 \cap N_2} \text{Hom}(L_{1x}, L_{2x})
\]
where \(L_{ix}, i = 1, 2\) are fibers of the local systems.

The structure of \(A_\infty\)-category on \(F(M)\) is given in terms of the following data:

1) A structure of complex on all spaces \(\text{Hom}((N_1, L_1), (N_2, L_2))\). In particular they are \(\mathbb{Z}\)-graded complex vector spaces with the grading defined by means of the Maslov index.

2) Higher compositions, which are morphisms of complexes
\[
m_k : \bigotimes_{0 \leq i \leq k} \text{Hom}(X_i, X_{i+1}) \rightarrow \text{Hom}(X_0, X_{k+1})[2 - k],
\]
for given objects \(X_i = (N_i, L_i) \in F(M)\). The composition \(m_k\) is defined in terms of the moduli space of holomorphic maps of \((k + 1)\)-gons \(C_{k+1}\) to \(M\), such that the \(i\)th side of \(C_{k+1}\) belongs to \(N_i\). Thus, \(m_1\) is basically the differential in the Floer complex associated with the pair of Lagrangian submanifolds \(N_1\) and \(N_2\). Formulas for \(m_k, k \geq 2\) involve also the monodromies of flat connections along the sides of polygons as well as areas of the polygons computed with respect to the symplectic form on \(M\). There are compatibility conditions for the morphisms \(m_k\).

The category \(D^b_\infty(X)\) is an \(A_\infty\)-version of the derived category of coherent sheaves on \(X\). Its objects are bounded complexes of holomorphic vector bundles. Essentially the same \(A_\infty\)-category is given by the dg-category of dg-modules over the dg-algebra of Dolbeault forms \(\Omega^{0,*}(X)\).

### 4.2 The case of elliptic curves

In the case of elliptic curves the HMC was proved in [PZ] (see also [AP]). One starts with a symplectic 2-dimensional torus \((T^2, \omega)\), where \(\omega\) is a constant symplectic form. In fact one should also fix a real number \(B\) (more precisely, the cohomology class \([Bdx \wedge dy] \in H^2(T^2, \mathbb{R})/H^2(T^2, \mathbb{Z}) = \mathbb{R}/\mathbb{Z}\)). Then one defines the Fukaya category \(F(T^2)\) using the complexified symplectic form \(Bdx \wedge dy + i\omega\). All the definitions are standard, only the areas of polygons become complex numbers (see details in [PZ]). The symplectic form \(\omega\) determines a unique flat metric \(A(dx^2 + dy^2)\) on \(T^2\) such that \(A = \int_{T^2} \omega\).
It gives a unique complex structure on $T^2$ (because the size of the chamber of the lattice $\mathbb{Z}^2$ is fixed). In this way we obtain an elliptic curve (i.e. 1-dimensional Calabi-Yau manifold). Let $X = E_\tau = \mathbb{C}^*/q^\mathbb{Z}$ be this curve, $q = e^{\exp(2\pi i \tau)}, \text{Im}(\tau) > 0$. Then the dual Calabi-Yau manifold is the elliptic curve $E_\rho = \mathbb{C}^*/e^{2\pi i \rho^\mathbb{Z}}, \rho = B + iA$.

Mirror symmetry functor can be described explicitly. On the symplectic side of HMC one has closed 1-dimensional submanifolds in $T^2$ which carry unitary (quasi-unitary for the version of HMC considered in [PZ]) local systems. On the complex side of HMC one has complexes of holomorphic vector bundles (they generate the derived category of coherent sheaves). The dictionary between symplectic and complex sides translates standard $(m, n)$ geodesics equipped with trivial 1-dimensional local systems into holomorphic vector bundles of rank $n$ with the first Chern class $m$. For general abelian varieties a version of HMC was proved in [KoSo2] using methods of Morse theory and non-arithmetic analysis.

### 4.3 “Compactified” Homological Mirror Symmetry

Homological mirror conjecture implies that formal moduli spaces of deformations of $D^b_{\infty}(X)$ and $F(X^\vee)$ for dual Calabi-Yau manifolds are isomorphic. The tangent space to the moduli space of deformations of an $A_\infty$-category $\mathcal{C}$ is $\oplus_{k \geq 0} Ext^k(\text{Id}, \text{Id})$. Here one takes $Ext$ groups in the category of functors $\mathcal{C} \to \mathcal{C}$, and $\text{Id}$ is the identity functor. In other words, the tangent space is given by the Hochschild cohomology of the category. The Yoneda product of $Ext$ groups gives rise to the product on the tangent space.

In the case of $D^b_{\infty}(X)$ the tangent space is $\oplus_{p,q \geq 0} H^p(X, \Lambda^q T_X)$. In the case of $F(X^\vee)$ it is $H^*(X^\vee)$. The product in the latter case is the small quantum cohomology product defined in terms Gromov-Witten invariants (see [Ko1] for details). The formal moduli spaces are bases of semi-infinite variations of Hodge structures (see [B2]). The mirror symmetry functor should identify the corresponding Hodge filtrations for $D^b_{\infty}(X)$ and $F(X^\vee)$ (or, better, semi-infinite variations of Hodge structures, as suggested by Barannikov, see [B3]).

We mention here that non-commutative analog of the variations of Hodge structures with applications to mirror symmetry was introduced in [B2-B3].

In the case of elliptic curves (more generally, abelian varieties) there are no Gromov-Witten invariants, but the rest of the picture is present. Then
one can ask the following question.

**Question 4** What happens with both sides of HMC as $\text{Im}(\tau) \to 0$?

We have argued that the derived category of coherent sheaves on the elliptic curve $E_\tau$ degenerates into the derived category of $F$-local systems, where $F$ is a foliation on $T^2$ (degenerate complex structure). Equivalently, it is the derived category of the category of small modules over the algebra of functions on the corresponding quantum torus. Another way to treat this degeneration is to consider the derived category of “coherent sheaves on the non-commutative upper-half plane”. This means that we start with an algebra $B$ which is the algebra over $C[q,q^{-1}]$ consisting of series $f = \sum_{m,n} a_{m,n} x^m y^n$ where $xy = qyx$, the coefficients $a_{m,n}$ are rapidly decreasing, $m, n \in \mathbb{Z}$, and $n$ runs through a finite set. If $|q| \neq 1$ modules of finite rank over this algebra are the same as $\mathbb{Z}$-equivariant sheaves of finite rank on $C^*$, i.e. the same as coherent sheaves on $E_q = C^*/q\mathbb{Z}, q = e^{2\pi i \tau}$. If $|q| = 1$ then every such module admits a structure of module over the algebra $B_\tau$ (i.e. we allow infinite sums in $n$). Then the derived category of coherent sheaves over $B$ can be treated as a family of derived categories, which has as fibers elliptic curves and quantum tori for different values of $q$. Now the $SL(2,\mathbb{Z})$-invariance of categories is manifest. In fact one has two copies of $SL(2,\mathbb{Z})$ acting on the family of categories. One copy acts by fractional transformations of $\tau$. It gives rise to an isomorphism of elliptic curves if $\text{Im}(\tau) > 0$, and equivalence of the categories of small modules if $\text{Im}(\tau) = 0$. The equivalence in this case is given by a bimodule $H_{\tau,g}(\tau)$, where $g \in SL(2,\mathbb{Z})$. One can check that $H_{\tau,g1g2}(\tau) \simeq H_{g2(\tau),g1g2(\tau)} \otimes H_{\tau,g2(\tau)}$, and $H_{\tau,\tau}$ corresponds to the identity functor.

Another copy of $SL(2,\mathbb{Z})$ acts by automorphisms of the algebra $B_q$ for a fixed $q$ which is not a root of 1. Namely, $x \mapsto x^a y^b, y \mapsto x^c y^d$ is an algebra automorphism iff $ad - bc = 1$. Taking $a = d = 0, b = -c = 1$ one gets an analog of the Fourier-Mukai transform in the case of quantum tori.

We treat this picture as a non-commutative compactification of the universal covering of the moduli space of elliptic curves.

It is natural to expect that there is similar compactification of the “symplectic” side of HMC. In the next section we are going to present some arguments in favor of the idea, that the degenerate Fukaya category of the symplectic torus $(T^2, (1/\tau)\omega)$ is again the category of small modules over the
algebra $B_\varphi$, where $\varphi = \text{Im}(\tau)$, and $B_\varphi$ is interpreted as a quantized algebra of functions on the torus.

The “compactified” HMC should be a statement about an equivalence of families of $A_\infty$-categories, both parametrized by $\tau \in \mathcal{H} = \{z \in \mathbb{C} | \text{Im}(z) \geq 0\}$. For $\tau \in \mathcal{H}$ one has $A_\infty$-categories from the conventional HMC, and for $\tau \in \mathbb{R}$ one has two equivalent categories of modules for the same quantum torus, but described in two different ways:

a) as the category of $F$-local systems on $T^2$;

b) as the category of modules over a quantized symplectic torus, which correspond to Lagrangian submanifolds.

Although two descriptions a) and b) give rise to equivalent categories, the mirror symmetry functor produces a non-trivial identification of the Hodge filtrations on the cohomology of $X = T^2$. Hodge filtrations are filtrations on the periodic cyclic homology of the $A_\infty$-categories under considerations. Periodic cyclic homology of either $A_\infty$-category is isomorphic to the total cohomology of the underlying space. Mirror symmetry functor should interchange the Hodge filtrations on periodic cyclic homology.

In the case of the 2-dimensional quantum torus $T^2(\mathbb{Z}, \varphi)$ one has two filtrations on the periodic cyclic homology (the latter is isomorphic to the cohomology of the usual torus $T^2$). First filtration arises from the interpretation of the quantum torus as a quantized symplectic manifold. It has one non-trivial term corresponding to the line spanned by the vector $(1, [\omega])$, where $1 \in H^0(T^2)$ is the unit in cohomology, and $[\omega]$ is the cohomology class of the symplectic form $\omega = \frac{1}{\varphi} dx \wedge dt$. Second filtration arises from the interpretation of the quantum torus in terms of the foliation. The only non-trivial term of this filtration corresponds to the straight line spanned by the class of the foliation 1-form $dx - \varphi dt$. Now we have filtrations in even and odd cohomology respectively. The mirror symmetry functor interchanges odd and even cohomology, interchanging also the Hodge filtrations. This picture is a “limiting” one for the homological mirror symmetry in the case of elliptic curves. Therefore the mirror symmetry functor in the case of quantum tori identifies the (derived) category of modules over the algebra $B_\varphi$ of smooth functions on $T(L, \varphi)$ with itself. One can say that the quantum torus is mirror dual to itself. The mirror functor acts non-trivially interchanging odd and even cohomologies, and identifying the Hodge filtrations described above (cf. with the case of complex abelian varieties considered in [GLO]).
4.4 Modules over quantized algebras and Fukaya category

In order to define the Fukaya category one needs a symplectic manifold. There is a simpler abelian category depending on symplectic structure. Let \((M, \omega)\) be a symplectic manifold. Then it admits a deformation quantization (i.e. formal family of associative products with the prescribed first jet). The quantized algebra \(C_\infty(M)_\hbar\) is not defined canonically. On the other hand, the abelian category \(C_M\) of \(C_\infty(M)_\hbar\)-modules is defined canonically.

Let \(\Lambda \subset M\) be a Lagrangian submanifold. Identifying a neighborhood of \(\Lambda\) with a neighborhood of some manifold \(X\) in \(T^*X\), one can construct a \(C_\infty(M)_\hbar\)-module \(W_\Lambda\), which is canonically defined as an object of \(C_M\). For the modules \(W_\Lambda\) one expects the following theorem (cf. [Gi]).

**Theorem 5** Let \(\Lambda_i, i = 1, 2\) be two transversal Lagrangian submanifolds in a symplectic manifold \((M, \omega), \dim M = 2n\) (i.e. they are transversal at all intersection points). Then \(\text{Ext}^i_{C_M}(W_{\Lambda_1}, W_{\Lambda_2}) = 0\) for all \(i \neq n\), and \(\text{Ext}^n_{C_M}(W_{\Lambda_1}, W_{\Lambda_2}) = C|_{\Lambda_1 \cap \Lambda_2}\).

Let us consider the simplest case of 1-dimensional Lagrangian subspaces in the standard symplectic \(\mathbb{R}^2\). We will be considering algebraic quantization, not a smooth one. Let \(p, q\) be coordinates in \(\mathbb{R}^2\) such that \([p, q] = 1\), and \(A\) be the Weyl algebra (standard quantization of this symplectic manifold, so that \([p, q] = \hbar \cdot 1\)). Let \(\Lambda_1\) be the line \(q = 0\), and \(\Lambda_2\) be the line \(p = 0\). Then \(W_{\Lambda_1} = A/Aq\), and \(W_{\Lambda_2} = A/Ap\). Clearly we have the following free resolution of \(W_{\Lambda_1}\):

\[
E^* : A \rightarrow A \rightarrow W_{\Lambda_1} \rightarrow 0,
\]

where the first map is a multiplication by \(q\), and the second map is the natural projection. Then \(\text{Hom}(E^*, W_{\Lambda_2})\) gives the complex \(W_{\Lambda_2} \rightarrow W_{\Lambda_2}\), where the only map is given by the multiplication by \(q\). It is clear that there is an isomorphism of \(C[q]\)-modules: \(W_{\Lambda_2} \simeq C[q]\). Therefore the complex \(W_{\Lambda_2} \rightarrow W_{\Lambda_2}\) has trivial cohomology \(H^0\), and \(H^1\) is isomorphic to \(C[q]/qC[q] \simeq C\). Hence the only non-trivial Ext-group is \(\text{Ext}^1(W_{\Lambda_1}, W_{\Lambda_2}) = C\). This proves the theorem in the simplest case.

Similarly one can check that for a given \(\Lambda\) there is an isomorphism
⊕i≥0 Ext^i(W_Λ, W_Λ) \simeq H_{dR}(\Lambda). More generally, let W_{\Lambda_i, L_i}, i = 1, 2 be small modules, corresponding to closed Lagrangian transversal submanifolds \Lambda_i which carry local systems L_i. Assume that the Lagrangian submanifolds intersect transversally. Then

\text{Ext}^n(W_{\Lambda_1, L_1}, W_{\Lambda_2, L_2}) = \bigoplus_{x \in \Lambda_1 \cap \Lambda_2} \text{Hom}((L_1)_x, (L_2)_x), and all other \text{Ext}-groups are trivial.

In the case of 2-dimensional tori there are two different interpretations of the category of modules over the corresponding quantum torus:

a) as the category of small modules;

b) as the category of modules over a quantized algebra.

To a Lagrangian submanifold with a local system on it, one assigns (in both cases a) and b)) an object of the corresponding category of modules. It is natural to expect that these objects correspond to each other under the equivalence of categories a) and b).

Although these observations show certain similarity of the Fukaya category F(M) with the category C_M of modules over the quantized function algebra, there is a difference in their structures. Namely, Maslov index is not visible in C_M, and there is no non-trivial A_\infty-structure on the latter category. On the other hand, Maslov index appears in the definition of the structure of A_\infty-category on F(M).

**Question 5** Is there an A_\infty-extension of the category C_M which involves \textquotedblleft graded\textquotedblright objects (similarly to graded Lagrangian manifolds in the construction of F(M))? 

This question is interesting even in the linear case (i.e. when M is the standard symplectic \mathbb{R}^{2n}). If the answer to the question is affirmative, then for any three Lagrangian submanifolds intersecting transversally there is a generalized Yoneda composition Ext^*(W_{\Lambda_1}, W_{\Lambda_2}) \times Ext^*(W_{\Lambda_2}, W_{\Lambda_3}) \to Ext^*(W_{\Lambda_1}, W_{\Lambda_3}) which involves \textquotedblleft instanton corrections\textquotedblright, i.e. counting of holomorphic polygons with the edges belonging to \Lambda_i, i = 1, 2, 3.

### 4.5 Fukaya category for Lagrangian foliations and Moyal formula

Fukaya suggested in [Fu1] to construct a version of the category F(T^{2n}) for Lagrangian foliations. To a foliation he assigned the algebra of foliation, to
a pair of “good” foliations he assigned a bimodule over the corresponding algebras. This bimodule is a kind of a Floer complex. The bimodule is not projective over either of the algebras.

Fukaya proposed an $A_\infty$-structure on the category of Lagrangian foliations. Objects of the category are Lagrangian foliations together with transversal measures. Spaces $\text{Hom}(F_1, F_2)$ are analogs of Floer complexes constructed for “good” foliations $F_i$, $i = 1, 2$. The structure of $A_\infty$-category is given by the “higher compositions” $m_k, k \geq 1$. Composition map $m_2$ is defined by the formula which is similar to the formula for the Moyal $\ast$-product on a symplectic torus (see for example [We]):

$$(f \ast g)(x) = \left(\frac{1}{\pi \hbar}\right)^{2n} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} e^{-2\pi i \omega(x-y,x-z)/\hbar} f(y)g(z)dydz.$$

The formula contains the summation of the exponents of symplectic areas of triangles, which makes it similar to the formula for $m_2$. The latter contains the exponents of the symplectic areas of holomorphic polygons. Let us recall Fukaya’s formula in the case when Lagrangian foliations are obtained from real vector subspaces in the standard symplectic vector space $\mathbb{R}^{2n}$.

If one has three affine Lagrangian foliations: $F_i$, $i = 1, 2, 3$, such that the Maslov index of this triple is zero, then the formula looks such as follows:

$$m_2(\tau_2)(f \otimes g)(x, c, z) = \int_{F_2} \int_{F_2} e^{-Q(a, b, c, \omega)} f(x, a, y)g(y, b, z)d\tau_2(F_2).$$

The notation in the formula is explained in [Fu1]. Roughly speaking, $\tau_2$ is a transversal measure for $F_2$, so it determines a section of $|\Lambda^{\text{top}}T^{(2n)}/TF_2|$. Functions $f$ and $g$ live on the holonomy groupoids of the pairs of foliations. The most interesting datum is the function $Q$ which is defined as

$$Q(a, b, c, \omega) = \int_{\Delta_{a,b,c}} \omega,$$

where $\Delta_{a,b,c}$ is the geodesic triangle with vertices in $a, b, c \in \mathbb{C}^n$. This leads to the multi-dimensional theta-functions (see [Fu1]).

**Question 6** Can one explain the similarity of these two formulas from the point of view of degeneration of the Fukaya category?
Apparently, the Fukaya category and the category of certain modules over a quantized algebra of functions belong to the same connected component of the “moduli space of $A_\infty$-categories”.

5 The “large complex structure” limit

The content of this subsection is borrowed from [KoSo2].

Non-commutativity can appear as a result of degeneration of the complex structure into a foliation. The conventional approach to mirror symmetry suggests to consider the “large complex structure” limit, so that conjecturally Calabi-Yau manifolds become foliated by special Lagrangian tori (see [SYZ]). This gives rise to a stratum in the compactified “moduli space of conformal field theories. It is described in terms of “classical” theories. The formal neighborhood of a classical theory in the moduli space can be reconstructed from classical data by means of some formal rules.

In this subsection we will discuss an example of this kind. For more details and applications to mirror symmetry see [KoSo2]. In [GW] the conjecture similar to ours was formulated independently. It was verified in [GW] in the case of K3 surfaces.

Here is the idea. Suppose that we study degenerations of a family of $n$-dimensional Calabi-Yau manifolds $X_\varepsilon$ as $\varepsilon \to 0$. Every $X_\varepsilon$ carries the Calabi-Yau metric $g(\varepsilon) = g_{ij}(\varepsilon), 1 \leq i, j \leq N$. Let us rescale $g(\varepsilon)$ in such a way that the diameter of $X_\varepsilon$ will be $O(1)$ as $\varepsilon \to 0$. We consider the limiting space $X_0$, where the limit is taken in Gromov-Haudorff metric.

The following conjectural description of $X_0$ was suggested in [KoSo2]:

1) $X_0$ is a metric space. It contains an $n$-dimensional Riemannian manifold $X_0^{sm}$. The dimension of $X_0^{sm} = X_0 \setminus X_0^{sm}$ is less or equal than two.

2) The manifold $X_0^{sm}$ carries an affine structure (i.e. flat torsion-free connection on the tangent bundle $T_{X_0^{sm}}$).

3) There is a covariantly flat lattice $\Gamma \subset T_{X_0^{sm}}$. In affine local coordinates it is given by $\mathbb{Z}^n \subset \mathbb{R}^n$.

4) Let us identify locally $X_0^{sm}$ with $\mathbb{R}^n$. Then the Riemannian metric $g$ on $X_0^{sm}$ is Kähler-Einstein. This means that $g = \partial^\sharp H$ for some function $H$, such that $\det(g) = \text{const}$ (the Monge-Ampere equation).

These data give rise to a fibration of flat tori on $X_0^{sm}$.

This conjectural picture is related to the mirror symmetry in the following
way. Consider (locally) the graph $dH$ in $V = \mathbb{R}^n \oplus \mathbb{R}^n$. The latter space is a symplectic manifold equipped with two Lagrangian foliations arising from the coordinate spaces. The graph $dH$ is a Lagrangian submanifold in it. Since $H$ is defined up to the adding of an affine function, the graph itself is defined up to translations. Translated submanifolds are still Lagrangian in $V$, so we will not pay much attention to this ambiguity. Let $p_i, i = 1, 2$ be the canonical projections of $V$ to the coordinate subspaces. Then the Monge-Ampere equation corresponds to the condition $p_1^* (\text{vol}_{\mathbb{R}^n}) = p_2^* (\text{vol}_{\mathbb{R}^n})$ where $\text{vol}$ denotes the standard volume form.

Now we see that the whole picture is symmetric, so we can interchange dual affine structures. Then $H$ gets replaced by its Legendre transformation. It does not change $X_0^{sm}$, and the limiting metric $g$. It interchanges the dual affine structures and the dual lattices. Hence it interchanges the corresponding dual fibrations of the flat tori. This duality is geometric mirror symmetry. Conjecturally, degenerations of families of dual Calabi-Yau manifolds in the limit of “large complex structure”, lead to the two dual fibrations of flat tori over the same base $X_0^{sm}$.

From the point of view of $N = 2$ superconformal field theory, the whole picture is classical, not quantum. It is shown in [KoSo2] how it simplifies the counting. It turns out that the counting of pseudo-holomorphic discs can be reduced to the counting of certain binary trees in $X_0^{sm}$. It is also explained in [KoSo2] that the geometric degeneration is compatible with certain degenerations of $N = 2$ superconformal field theories.

**Example 2** Moduli space of conformal field theories with the central charge $c = 2$ is a product of two modular curves. These theories can be described as sigma-models with a target space, which is a 2-dimensional torus. Two separate “infinite” limits for each modular curve give rise to the two moduli spaces. Maximal degenerations give rise to bundles over a circle with the fibers which are circles themselves. The mirror symmetry relates two dual circle bundles over the same base.

5.1 Non-commutative stratum

As we have seen above, degenerations of the complex structure can be described in terms of either commutative or non-commutative geometry. It
would be interesting to understand what kind of non-commutative theories can be obtained in this way.

Let us consider the case of $K3$ surfaces elliptically fibered over $\mathbb{P}^1$. We can deform the complex structure on a surface in such a way, that it becomes a foliation on each non-degenerate fiber, and singular foliation on the degenerate fibers and on the base (which is a 2-sphere $S^2$). It is expected that to such geometric picture one can assign a family of “generically non-commutative” theories on quantum tori (foliated fibers).

**Question 7** *How to describe these theories?*

One also expects that similar non-commutative degenerations exist for arbitrary Calabi-Yau manifolds. Loosely speaking, one can make into non-commutative all special Lagrangian tori in [SYZ] picture, thus adding a new non-commutative stratum to the moduli space of $N = 2$ superconformal theories.

### 6 Other related topics

#### 6.1 Quantum theta functions

Let $T(L, \exp(2\pi i \varphi))$ be a quantum torus, $\varphi \in \mathbb{R}$. We fix a quadratic form $Q : L \times L \to \mathbb{C}$ with negative imaginary part and linear functional $l : L \to \mathbb{R}$. Let us also fix a symmetric bilinear form $(,)$ on $L_{\mathbb{R}}$. We write $Q(x) = (\Omega x, x)$ for some matrix $\Omega$ from the Siegel upper-half space. In particular $\Omega$ is symmetric with respect to the bilinear form.

To these data we associate a *quantum theta function* (see [M1]):

$$\theta(Q, l, \varphi) = \sum_{a \in L} e^{2\pi i (Q(a) + l(a))} e(a).$$

For any $t \in \text{Hom}(L, \mathbb{C}^*)$ we define an automorphism of $T(L, 2\pi i \varphi)$ by the formula:

$$t^*(e(a)) = t(a)e(a).$$

Every element $\xi \in L$ defines $t_{\xi} \in \text{Aut}(T(L, 2\pi i \varphi))$ such that $t_{\xi}(e(a)) = \exp(2\pi i (-a, \Omega \xi) + \varphi(a, \xi))e(a)$.

The following result is straightforward (see [M1]).
Proposition 4 For an arbitrary $\xi$ as above we have:

$$t^*_\xi(\theta) = e^{-2\pi i (Q(\xi)-l(\xi))} e(\xi)\theta.$$ 

Question 8 Let us write $Q(a) = (\Omega a, a)$, $\varphi(a, b) = (\Phi(a), b)$. Is there an analog of the functional equation for $\theta = \theta(\Omega, l, \Phi)$?

We plan to return to this question in the next paper. Notice that standard proofs of the functional equation for theta-functions which use the Poisson summation formula do not work in quantum case.

6.2 Quasi-modular forms

There is an approach to the mirror symmetry for elliptic curves due to Dijkgraaf (see [Di1]). It is the “elliptic” version of the counting of higher genus curves in the Calabi-Yau manifold. Main result of [Di1] is a statement that the generating function counting certain coverings of the elliptic curve is a quasi-modular form in the sense of Kaneko and Zagier (see [KZ]).

Dijkgraaf pointed out that for a fixed modular parameter $\tau$ and fixed Kähler class $t$ the related quantum field theory is based on a four-dimensional lattice of signature $(2, 2)$ in $\mathbb{C}^2$ (see [DVV]). Thus one has the group $O(2, 2, \mathbb{Z})$ as the duality group of the quantum theory. This group contains $\mathbb{Z}_2$ as a subgroup. The latter is responsible for the mirror symmetry. The theory depends on a flat metric on a 2-dimensional torus and a skew-symmetric bilinear form on the lattice defining the torus.

The partition function of the theory is quasi-modular. It can be modified, so that it enjoys modular properties, but becomes non-holomorphic.

More precisely, for any $g > 1$ Dijkgraaf defines a formal power series

$$F_g(q) = \sum_{d \geq 1} N_{g,d} q^d,$$

where $q = exp(2\pi i \rho)$. and $N_{g,d}$ counts the virtual number of ramified coverings of a complex elliptic curve $\mathcal{E}$ by smooth complex curves of genus $g$. There are $2g - 2$ ramification points of index 2, and coverings have degree $d$. One can interpret $\rho$ as the complexified area of the underlying torus.

The following result can be found in the cited paper by Dijkgraaf. It was rigorously proved in [KZ].
Proposition 5 $F_g \in \mathbb{Q}[E_2, E_4, E_6]$ and has weight $6g - 6$, where $E_k$ are Eisenstein series.

Since $E_2$ is not a modular function, same is true for $F_g$. But $E_2$ is quasi-modular in the sense of [KZ], so does $F_g$.

It is natural to ask about the non-commutative version of the Proposition. If it exists, then the corresponding counting function should be a kind of non-commutative limit of $F_g$. It is interesting to understand whether quantum tori can be related to certain degenerations of quasi-modular forms.

Informally, “boundary modular forms” should correspond to the limits of quasi-modular forms as modular parameter approaches to a real number. It is natural to ask how this limit should be understood. For example, one can consider limiting functions as distributions or hyperfunctions.

The question about such a limit might be related to a different question about “boundary” modular forms in the sense of Zagier. In Zagier’s work, rational points of the boundary line of the upper-half plane play a special role. More precisely, he constructs embeddings of various spaces of “honest” modular (or, hopefully, quasi-modular) forms to the space of “smooth function” on $\mathbb{Q}$, modulo “rational functions”. The image of this embedding consists of functions having modular properties.

Let us consider an example of a “boundary modular form” (I learned it from Don Zagier). Let $s(p, q)$ be a unique (after normalization) function defined for relatively prime integers $p, q$ in the following way:

a) $s(p, q) \in \mathbb{Q}$;
b) $s(p + nq, q) = s(p, q)$;
c) $s(-p, q) = -s(p, q)$;
d) $s(p, q) + s(q, p) = (q^2 + p^2 + 1 - 3pq)/12pq$.

Then as a function of $x = p/q$ it satisfies the following conditions:

1) $s(x + 1) = s(x)$;
2) $s(-1/x) = s(x) + P(x)$,
where $P(x)$ is the RHS of d).

Then one can give a precise meaning to the following statement:

the function $s(x)$ is modular as an element of the set of functions on $\mathbb{P}^1(\mathbb{Q})$, modulo “smooth” rational functions.

Question 9 What is the image of the Dijkgraaf’s partition function under the “boundary embedding”?
From the point of view of the classical theory of Eichler, Manin and Shimura, the boundary modular forms might be related to integrals of modular forms over geodesics between cusps in Lobachevsky plane. It is natural to ask about the meaning of other geodesics. One can expect that geodesics between arbitrary boundary points should be treated within the framework of the “non-commutative” geometry. Indeed, for non-cuspidal points the geodesics are dense in the corresponding modular curve.

6.3 On the Morita equivalence and deformation quantization

Hopefully Morita equivalence of quantum tori can be generalized to other quantized function algebras. It should be a statement that quantized algebras corresponding to different Poisson structures and different values of the quantization parameter produce Morita equivalent algebras. Notice, however, that the conventional deformation theory deals with formal series in a parameter $\hbar$. From this point of view the transformation $\hbar \mapsto -1/\hbar$ does not have sense. In order to allow such transformations (obviously we need them in order to treat Morita equivalent quantum tori) we need to have a “global” moduli space of quantizations, not just a formal scheme over $k[[\hbar]]$. The problem cannot be resolved at the level of Poisson structures. It is “quantum” problem. One can have a discrete group of symmetries of the quantum problem, which do not exists at the level of Poisson structures.

**Question 10** Are there examples of deformation quantization problems which model this phenomenon?

In quantum groups one often has series convergent in $\hbar$. Let $q = \exp(2\pi i \hbar)$. It is interesting to look at the quantized homogeneous spaces (for example quantum groups themselves) in the case $|q| = 1$.

For example, let us consider the categories of projective modules over the quantized coordinate rings $\mathbb{C}[G]_q$, where $q = e^{2\pi i / l}$, where $l$ is a positive integer number, and $G$ is a simple complex Lie group. One can see that the quantized coordinate ring becomes a projective module of finite rank $N = l^{\dim(G)}$ over the subalgebra of the center, which is isomorphic to the algebra $\mathbb{C}[G]$ of functions on the Poisson-Lie group $G$. 
Question 11 Is it true that for all primitive roots of 1 of the same order the quantized coordinate rings are Morita equivalent?

We would like to know the answer to a more general question. Namely, what is the “duality group” acting on the quantization parameter, such that if parameters $q$ and $q'$ belong to the same orbit, then the corresponding algebras of functions on the quantized homogeneous $G$-spaces $C[X]_q$ and $C[X]_{q'}$ are Morita equivalent? More generally, the duality group should act on Poisson structures as well.

In the case of quantized coordinate rings of simple Lie groups at roots of unity, the centers are non-isomorphic, so one cannot expect the group $SL(2, \mathbb{Z})$ be the “duality group” of the theory. If the answer to the above question is positive, then the “duality group” in question is the Galois group $\text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$, where $\mathbb{Q}^{ab}$ is the maximal abelian extension of the field of rational numbers.

This is a toy-model for the global dualities in quantum physics. Indeed, we have local quantizations given by perturbative series, and we search for a global non-perturbative theory.

Another interesting question is to find an analog of the mirror symmetry for the deformation quantization. In the case of quantum tori we discussed it in Section 4.

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Address: Department of Mathematics KSU, Manhattan, KS 66506, USA
e-mail: soibel@math.ksu.edu