Feynman diagrams to three loops in three dimensional field theory

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Abstract

The two point integrals contributing to the self energy of a particle in a three dimensional quantum field theory are calculated to two loop order in perturbation theory as well as the vacuum ones contributing to the effective potential to three loop order. For almost every integral an expression in terms of elementary and dilogarithm functions is obtained. For two integrals, the master integral and the Mercedes integral, a one dimensional integral representation is obtained with an integrand consisting only of elementary functions. The results are applied to a scalar $\lambda \phi^4$ theory.
1 Introduction

When physical phenomena are described by quantum field theories, all the observable quantities are expressed in terms of functional integrals. Since these integrals can be evaluated exactly only in very special cases, one has to use lattice simulations or analytical approximation methods. The most common method is perturbation theory. One expands the desired quantity as a series of integrals represented by Feynman diagrams.

Although the field theories of particle physics are four dimensional, the importance of three dimensional theories has grown recently. The main reason is the method of dimensional reduction of a four dimensional finite temperature field theory to a three dimensional zero temperature effective theory [1–6]. This technique has been applied to the electroweak phase transition of the early universe [7–9]. Three dimensional field theories are also important in the theory of critical phenomena.

The nature of the phase transition is an important question in both main applications of three dimensional field theories. Thus the effective potential which gives the true ground state of the system has an essential significance. Unfortunately, in dimensionally reduced effective theory, perturbative calculations are applicable neither in the symmetric phase nor in the immediate vicinity of the phase transition. However, the perturbative results obtained deep in the broken phase can give new insight into the problem and the lattice results [8, 9].

At one loop level the perturbative calculations are fairly easy. When one needs higher corrections, the integrals get more complicated. In four dimensions these integrals have been studied recently by many authors [10–19] but a three dimensional discussion has been missing.

The purpose of this paper is to evaluate all the integrals needed for the self energy of a particle in a three dimensional scalar theory to two loop order and the ones needed for the effective potential to three loop order. As shown by Weiglein et al. [20], the self energy of a particle in a gauge field theory can be expressed in terms of these scalar integrals. Most of the integrals are calculated in a straightforward way, but with two integrals a different route must be chosen.

The paper is organized as follows. In Sect. 2 the simpler integrals are calculated explicitly. Sect. 3 is devoted to the two more intricate integrals. In Sect. 4 the integrals are applied to the calculation of the self energy and
the effective potential of scalar $\lambda \phi^4$ theory.

## 2 Evaluation of scalar integrals

### 2.1 Classification of integrals

In order to calculate the self energy of a specific system one needs to consider all possible one particle irreducible two point diagrams. Let us assume that the Lagrangian consists only of terms at most quartic in the fields. Then the diagrams can be composed from three and four leg vertices using Feynman rules.

Depending on the number of fields and the specific form of the Lagrangian the number of possible diagrams may vary. However, there is only a restricted set of different topologies these diagrams may have. At one loop level there are only two possible topologies. These two are the diagrams $\alpha$ and $\beta$ shown in Fig. 1. At two loop level the number of different topologies is eight. These are given in Fig. 2.

For the effective potential one needs the vacuum diagrams [21]. At one loop level there is of course only one possible topology, and two at the two loop level. At the three loop level the number is six. These diagrams are shown in Fig. 3.

Some of the integrals corresponding to these diagrams do not converge, but when they do, there is a relation between vacuum and two point integrals.

**Lemma 1** Let $f_2(p)$ be a two point integral with an external momentum of $p$ and $F_0(m)$ the vacuum diagram obtained by connecting the outer legs of $f_2$ with a particle of mass $m$. Then the following relation holds:

\[
f_2(p) = \frac{2\pi i}{p} \left( F_0(ip) - F_0(-ip) \right). \tag{1}
\]
Figure 2: Topologies of two loop two point diagrams
The lemma can easily be proven by taking a Fourier transform of both sides of Eq. (1).

One should also notice that in case there are more than one propagator with the same momentum, they can be separated to two integrals by partial fractioning or be written as a derivative:

\[
\frac{1}{(p^2 + m_1^2)(p^2 + m_2^2)} = \frac{1}{m_2^2 - m_1^2} \left( \frac{1}{p^2 + m_1^2} - \frac{1}{p^2 + m_2^2} \right),
\]

\[
\frac{1}{(p^2 + m^2)^2} = -\frac{1}{2m} \frac{\partial}{\partial m} \frac{1}{p^2 + m^2}.
\]

Some of the integrals also factorize into separate parts. Using these relations the number of integrals to be calculated can be reduced to six: \( \alpha, \beta, g, A, C \) and \( E \). Of these, diagrams \( \alpha \) and \( \beta \) are easily evaluated. Diagram \( \gamma \) is only a special case of \( g \). Diagrams \( a, b \) and \( f \) are related to diagrams \( A, B \) and \( C \), respectively, by Lemma \( \Pi \). Finally, all the diagrams \( c, d, e, h, D \) and \( F \) factorize to products of simpler diagrams. The results for all the integrals are given in explicit form in appendix A.
2.2 Diagram g

Consider now the diagram g called the sunset. Diagram $\gamma$ is a special case of this with vanishing external momentum. Let the masses of the particles be $m_1$, $m_2$ and $m_3$. In this integral the coordinate space method \cite{22,23} will be used. In $3 - 2\varepsilon$ dimensions the Fourier transform of the propagator is

$$V_i(x) = (\pi\mu^2)^{\varepsilon/2} \frac{1}{(2\pi)^{\frac{3}{2} - \varepsilon}} \left(\frac{m_i}{x}\right)^{\frac{1}{2} - \varepsilon} K_{\frac{1}{2} - \varepsilon}(m_i x).$$ \hspace{1cm} \hspace{1cm} (4)

Here $K_{\nu}(x)$ is the modified Bessel function. After the transform the integral reads

$$I_g(k; m_1, m_2, m_3) = \mu^{2\varepsilon} \int d^{3-2\varepsilon} R \text{Re} e^{i\vec{k} \cdot \vec{R}} \prod_i V_i(R).$$ \hspace{1cm} \hspace{1cm} (5)

The divergence occurs only on the limit $R \to 0$. Thus the integration can be split at $R = r$:

$$I_g(k) = \left(\frac{e^\gamma \bar{\mu}^2}{2k}\right)^{-\varepsilon} \int_0^r dRR^{\frac{3}{2} - \varepsilon} J_{\frac{1}{2} - \varepsilon}(kR) \prod_i V_i(R)$$

$$+ \frac{4\pi}{k} \int_r^\infty dRR \sin(kR) \prod_i V_i(R)$$

$$\equiv I_{g(a)}(k) + I_{g(b)}(k),$$ \hspace{1cm} \hspace{1cm} (6)

where $J_{\nu}(x)$ is the Bessel function and $\bar{\mu}$ is the \overline{MS} scale parameter $\bar{\mu}^2 = e^{-\gamma_e} 4\pi \mu^2$.

Since $I_{g(b)}$ converges, the usual three dimensional Fourier transform of the propagator can be used. In $I_{g(a)}$, that is when $R < r$, one can approximate the Bessel functions by the lowest order terms in their Laurent series:

$$J_{\frac{1}{2} - \varepsilon}(kR) = \frac{1}{\Gamma\left(\frac{3}{2} - \varepsilon\right)} \left(\frac{1}{2} kR\right)^{\frac{1}{2} - \varepsilon} + \mathcal{O}\left((kR)^{\frac{3}{2}}\right),$$ \hspace{1cm} \hspace{1cm} (7)

$$V_i(R) = \left(\frac{e^\gamma \bar{\mu}^2}{4}\right)^\varepsilon \frac{\Gamma\left(\frac{1}{2} - \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{4\pi} R^{-1+2\varepsilon}$$

$$- \left(\frac{e^\gamma \bar{\mu}^2}{4}\right)^\varepsilon \frac{\Gamma\left(-\frac{1}{2} + \varepsilon\right)}{\Gamma\left(-\frac{1}{2}\right)} \frac{1}{4\pi} m_i^{1-2\varepsilon} + \mathcal{O}(R).$$ \hspace{1cm} \hspace{1cm} (8)

The error vanishes as the limit $r \to 0$ is taken. For the present integral only the $\mathcal{O}(R^{-1})$ term of $V_i$ is needed. Now one is left with only a straightforward
task of integrating over powers of $R$. All the terms that are singular at the lower limit $R = 0$ are treated by analytical continuation to sufficiently great values of $\varepsilon$ so that they vanish.

The result is

$$\mathcal{I}_g^{(a)}(k) = \frac{1}{(4\pi)^2} \left( \frac{1}{4\varepsilon} + \log \bar{\mu} r + \frac{1}{2} + \gamma \right).$$

(9)

$\mathcal{I}_g^{(b)}$ is a normal three dimensional integral and it can be calculated using normal methods of multidimensional integration:

$$\mathcal{I}_g^{(b)}(k) = \frac{1}{(4\pi)^2} \left( \frac{1}{4\varepsilon} - \frac{m_1 + m_2 + m_3}{k} \arctan \frac{k}{m_1 + m_2 + m_3} \right. $$

$$ \left. - \gamma - \log r - \frac{1}{2} \log \left( (m_1 + m_2 + m_3)^2 + k^2 \right) \right).$$

(10)

Now one can write down the result:

$$\mathcal{I}_g(k) = \frac{1}{(4\pi)^2} \left( \frac{1}{4\varepsilon} - \frac{m_1 + m_2 + m_3}{k} \arctan \frac{k}{m_1 + m_2 + m_3} \right.$$

$$ \left. + \frac{1}{2} \log\frac{\bar{\mu}^2}{(m_1 + m_2 + m_3)^2 + k^2} + \frac{3}{2} \right).$$

(11)

### 2.3 Reducible diagrams

The integral $g$ together with the well known results for the integrals $\alpha$ and $\beta$ make it possible to calculate all the reducible integrals using Lemma 1 and Eq. (2). As a simple example, consider the diagram c:

$$\mathcal{I}_c = \int \frac{d^3-2\varepsilon k}{(2\pi)^3-2\varepsilon} \frac{d^3-2\varepsilon q}{(2\pi)^3-2\varepsilon} \frac{1}{(k^2 + m_1^2)(q^2 + m_2^2)(k^2 + m_3^2)((p - k)^2 + m_4^2)}.$$  

(12)

The $q$-integration is nothing but $\mathcal{I}_\alpha$. One then uses Eq. (2) to obtain

$$\mathcal{I}_c = -\frac{m_2}{4\pi} \frac{1}{m_1^2 - m_2^2} \int \frac{d^3-2\varepsilon k}{(2\pi)^3-2\varepsilon} \left( \frac{1}{k^2 + m_1^2} - \frac{1}{k^2 + m_3^2} \right) \frac{1}{(p - k)^2 + m_4^2}$$

$$= \frac{m_2}{4\pi} \frac{1}{m_1^2 - m_3^2} (\mathcal{I}_\beta(m_1, m_4) - \mathcal{I}_\beta(m_3, m_4))$$

$$= \frac{m_2}{(4\pi)^2 p(m_1^2 - m_3^2)} \left( \arctan \frac{p}{m_1 + m_4} - \arctan \frac{p}{m_3 + m_4} \right).$$

(13)
In a similar way the diagram D can be factorized:

\[ I_D = I_\alpha(m_5) \frac{1}{m_4^2 - m_3^2} (I_\gamma(m_1, m_2, m_3) - I_\gamma(m_1, m_2, m_4)) \]
\[ = \frac{m_5}{(4\pi)^3(m_3^2 - m_4^2)} \log \frac{m_1 + m_2 + m_4}{m_1 + m_2 + m_3}. \] (14)

The results of the integrals are convergent, but the integrals themselves are actually not, since they both contain the simple loop \( \alpha \), which is divergent. In dimensional regularization this divergence vanishes, but Lemma 1 not proven to hold with dimensional regularization. However, this loop is in both cases factorized as a separate integral and the remaining other factor is convergent. Therefore Lemma 1 should hold for these diagrams:

\[ I_c(p; m_1, m_2, m_3, m_4) = \frac{2\pi i}{p} (I_D(ip, m_4, m_1, m_3, m_2) - I_D(-ip, m_4, m_1, m_3, m_2)). \] (15)

Substitute the previous result to the right hand side to obtain

\[ I_c = \frac{m_2}{(4\pi)^2 p(m_1^2 - m_3^2)} \left( \arctan \frac{p}{m_1 + m_4} - \arctan \frac{p}{m_3 + m_4} \right), \] (16)

which is the same as Eq. (13).

2.4 Diagram E

Since the integrals E and g do not converge, Lemma 1 does not hold. Therefore one must start the calculation of E from the beginning. The coordinate space integral is

\[ I_E(m_1, m_2, m_3, m_4) = \mu^{2\varepsilon} \int d^{3-2\varepsilon} R \prod_{i=1}^{4} V_i(\vec{R}). \] (17)

Just like before, this is separated to two parts, only one of which is divergent:

\[ I_E = \left( \frac{e^\gamma \mu^2}{4} \right)^{-\varepsilon} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - \varepsilon\right)} 4\pi \int_0^r dRR^{2-2\varepsilon} \prod_i V_i(R) \]
\[ + \frac{1}{(4\pi)^3} \int_r^\infty dRR^2 \frac{1}{R^4} e^{-\sum_i m_i R}. \] (18)
In the expansion of $V_i$ one now has to take into account also the term of order $\mathcal{O}(1)$. Then the first integral reads

$$I_E^{(a)} = \frac{1}{(4\pi)^3} \left( \frac{e^\gamma \bar{\mu}^2}{4} \right)^{\varepsilon} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - \varepsilon\right)} \int_0^\infty dR R^{2-2\varepsilon}$$

$$\left[ \left( \frac{e^\gamma \bar{\mu}^2}{4} \right)^{4\varepsilon} \left( \frac{\Gamma\left(\frac{1}{2} - \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^4 R^{4+8\varepsilon} \right.$$

$$\left. - \left( \frac{e^\gamma \bar{\mu}^2}{4} \right)^{3\varepsilon} \left( \frac{\Gamma\left(\frac{1}{2} - \varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)} \right)^3 \left( \frac{e^\gamma \bar{\mu}^2}{4} \right)^{3\varepsilon} \frac{\Gamma\left(-\frac{1}{2} + \varepsilon\right)}{\Gamma\left(-\frac{1}{2}\right)} R^{-3+6\varepsilon} \sum_i m_i^{1-2\varepsilon} \right]$$

(19)

This simple integral gives

$$I_E^{(a)} = \frac{1}{(4\pi)^3} \left[ -\frac{1}{r} - \frac{1}{4} \sum_i m_i \left( \frac{1}{\varepsilon} + 4 + 4\gamma + 2 \log \frac{r^2 \bar{\mu}^3}{2m_i^3} \right) \right].$$

(20)

$I_E^{(b)}$ is also easily evaluated and one can write the result for the whole integral:

$$I_E = \frac{1}{(4\pi)^3} \sum_{i=1}^4 m_i \left( -\frac{1}{4\varepsilon} + 2 + \frac{1}{2} \log \frac{2m_i^2}{\mu^2} + \log \sum_j m_j^3 \right).$$

(21)

### 2.5 Diagram C

Consider now the diagram C. This integral converges and Fourier transform can be used directly in three dimensional space. The masses are as shown in Fig. 3.

$$I_C = \int_{p,q,k} \frac{1}{(p^2 + m_1^2)((k-p)^2 + m_2^2)((k-q)^2 + m_3^2)}$$

$$\frac{1}{(q^2 + m_3^2)(k^2 + m_2^2)}.$$ 

(22)

Taking Fourier transform of this gives

$$I_C = \frac{1}{(4\pi)^5} \int d^3 x_1 \frac{e^{-(m_1+m_2)x_1}}{x_1^2} \int d^3 x_2 \frac{e^{-m_3|x_1+x_2|-(m_3+m_4)x_2}}{|x_1 + x_2|x_2^2|}.$$ 

(23)
Concentrate now on the latter integration

\[
\int d^3x_2 \frac{e^{-m_5|x_1+x_2|-(m_3+m_4)x_2}}{|x_1+x_2|^2}
= \frac{2\pi}{m_5x_1} \left[ e^{-m_5x_1} \int_0^{x_1} \frac{dx}{x} (e^{-(m_3+m_4-m_5)x} - e^{-(m_3+m_4+m_5)x}) 
+ (e^{m_5x_1} - e^{-m_5x_1}) \int_{x_1}^{\infty} \frac{dx}{x} e^{-(m_3+m_4+m_5)x} \right]
= \frac{2\pi}{m_5x_1} \left[ e^{-m_5x_1} \left( \log \frac{m_3+m_4+m_5}{m_3+m_4-m_5} + \text{Ei}((m_3+m_4-m_5)x_1) \right) 
- e^{m_5x_1} \text{Ei}((m_3+m_4+m_5)x_1) \right],
\]

where \( \text{Ei}(x) \) is the exponential integral.

Substituting this into Eq. (23) and integrating gives the result

\[
\mathcal{I}_C = \lim_{r \to 0} \frac{1}{(4\pi)^3 2m_5} \left[ \log \frac{m_3+m_4+m_5}{m_3+m_4-m_5} (-\gamma - \log(m_1+m_2+m_5)r) 
- \frac{1}{2} \left( \zeta(2) + (\gamma + \log(m_3+m_4-m_5)r)^2 \right) - \text{Li}_2 \left( \frac{m_1+m_2+m_5}{m_3+m_4-m_5} \right) 
+ \frac{1}{2} \left( \zeta(2) + (\gamma + \log(m_3+m_4+m_5)r)^2 \right) + \text{Li}_2 \left( \frac{m_1+m_2-m_5}{m_3+m_4+m_5} \right) \right]
= \frac{1}{(4\pi)^3 2m_5} \left[ \log \frac{m_3+m_4+m_5}{m_3+m_4-m_5} \log \frac{(m_3+m_4)^2-m_5^2}{m_1+m_2+m_5} 
+ \text{Li}_2 \left( \frac{m_1+m_2-m_5}{m_3+m_4+m_5} \right) - \text{Li}_2 \left( \frac{m_1+m_2+m_5}{m_3+m_4-m_5} \right) \right].
\]

\( \text{Li}_2(x) \) is the Euler dilogarithm function and \( \zeta(x) \) is the Riemann zeta function.

### 3 Mercedes and the master integral

#### 3.1 Mercedes integral

The last diagram to be considered is A. Many different techniques have been proposed for the calculation [10–19], most often in four dimensions. The
coordinate space method of the preceding subsection does not work well even though the integral converges and three dimensional transform can be used. The transformed integral is namely as complicated as the original one. Other approaches are based on some series expansion of the integral but then the resummation of the series is usually not possible. Many other methods, like Mellin-Barnes transformation and Gegenbauer polynomials, work well only in four dimensional space. In four dimensions one has been able to reduce the general combination of masses to a one-dimensional integral.

The approach to be used here is similar to that of Kotikov [24]. A differential equation which the integral must satisfy is constructed and then this equation is solved. For propagators a shorthand notation will be used:

\[
\Delta^i_p = \frac{1}{p^2 + m_i^2}.
\]  

(26)

With the masses and the momenta defined as in Fig. 4, the integral reads

\[
\mathcal{I}_A(m_1, m_2, m_3, m_4, m_5, m_6) = \int \Delta^1_q \Delta^2_{q-k} \Delta^3_{p-q} \Delta^4_{k-p} \Delta^5_{p} \Delta^6_{k} \\
\equiv \int \Delta^{123456}.
\]  

(27)

Here integration over all momenta is assumed.

The first step is to integrate this by parts. The boundary terms vanish.

\[
\mathcal{I}_A = -\frac{1}{3} \int q_i \frac{\partial}{\partial q_i} \Delta^{123456}
\]
\[
I_A = \mathcal{I}_A - \mathcal{I}_C \equiv -\frac{1}{2m_i} \frac{\partial}{\partial m_i} \mathcal{I}_A(m_1, m_2, m_3, m_4, m_5, m_6). 
\] (30)

\( J^j_i \) can be obtained from the following expression by substituting the corresponding masses

\[
\mathcal{I}_C(m_1, m_2, m_3, m_4, m_5) \equiv -\frac{1}{2m_1} \frac{\partial}{\partial m_1} \mathcal{I}_C(m_1, m_2, m_3, m_4, m_5) \]

\[
= \frac{1}{(4\pi)^3 2m_1 ((m_1 + m_2)^2 - m_5^2)} \log \frac{m_1 + m_2 + m_3 + m_4}{m_3 + m_4 + m_5}. 
\] (31)

In appendix A the explicit expressions are given for necessary \( J^j_i \). Equation (29) can be transformed to a more suitable form by using the symmetry properties of the diagram. Obviously, the diagram is invariant in the following permutations of masses:

\[
\left(\begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}\right) \Rightarrow \left(\begin{array}{cccc}
2 & 3 & 1 \\
5 & 6 & 4
\end{array}\right) \Rightarrow \left(\begin{array}{cccc}
3 & 1 & 2 \\
6 & 4 & 5
\end{array}\right), 
\] (32)

because they are equivalent to rotations of the diagram.

Using these symmetries, Eq. (29) can be written in matrix form

\[
\mathcal{M}(m_1, m_2, m_3, m_4, m_5, m_6) K \equiv 
\begin{pmatrix}
2m_1^2 & A & B \\
A & 2m_2^2 & C \\
B & C & 2m_3^2
\end{pmatrix}
\begin{pmatrix}
K_1 \\
K_2 \\
K_3
\end{pmatrix}
= \begin{pmatrix}
\mathcal{I}_A + J_1 \\
\mathcal{I}_A + J_2 \\
\mathcal{I}_A + J_3
\end{pmatrix} \equiv \mathcal{I}_A I + J, 
\] (33)
where $I = (1, 1, 1)^T$ and $J = (J_1, J_2, J_3)^T$,

\[
A = m_1^2 + m_2^2 - m_6^2 \\
B = m_1^2 + m_3^2 - m_5^2 \\
C = m_2^2 + m_3^2 - m_4^2
\]

and

\[
J_1 = J_2^1 - J_6^1 + J_3^1 - J_5^5 \\
J_2 = J_3^2 - J_4^2 + J_2^1 - J_6^6 \\
J_3 = J_1^3 - J_5^1 + J_3^3 - J_4^4.
\]

(34)

However, since in the following all masses except $m_1$ will be fixed and it occurs in the matrix only in quadratic form, the following notation will be used for simplicity:

\[
\mathcal{M}(m_1^2) \equiv \mathcal{M}(m_1, m_2, m_3, m_4, m_5, m_6).
\]

(36)

### 3.2 Degenerate case

Consider first the degenerate case. If $|\mathcal{M}(m_1^2)| = 0$ the matrix has an eigenvalue of zero. Let $U$ be the corresponding eigenvector. Then

\[
0 = U^T \mathcal{M}(m_1^2) K = U^T I A + U^T J.
\]

(37)

Let $D_i$ be the minor of the determinant of matrix $\mathcal{M}(m_1^2)$ with one of the rows and column $i$ removed. Then a suitable eigenvector is

\[
U = \begin{pmatrix}
D_1 \\
-D_2 \\
D_3
\end{pmatrix}.
\]

(38)

Hence, the result for $I_A$ can be written down:

\[
I_A = -\frac{U^T J}{U^T I} = - \frac{|\mathcal{M}_J|}{|\mathcal{M}_I|},
\]

(39)

where $\mathcal{M}_X$ denotes matrix $\mathcal{M}(m_1^2)$ with one of the rows or columns replaced by vector $X$. Not every choice of row or column is always possible, but one
can use the symmetries of the diagram to transform the matrices to a form in which the chosen row is the first one. Therefore the first row will be used here.

As an example let us calculate the special case $m_1 = 0, m_5 = m_3, m_6 = m_2$. Then the matrix $\mathcal{M}$ reads

$$
\mathcal{M} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 2m_2^2 & m_2^2 + m_3^2 - m_4^2 \\
0 & m_2^2 + m_3^2 - m_4^2 & 2m_3^2
\end{pmatrix}.
$$

Then

$$
\mathcal{I}_A = -\left| \begin{array}{ccc}
J_1 & J_2 & J_3 \\
0 & 2m_2^2 & m_2^2 + m_3^2 - m_4^2 \\
0 & m_2^2 + m_3^2 - m_4^2 & 2m_3^2
\end{array} \right| = -J_1
$$

$$
= -J_2^4 + J_2^6 - J_3^4 + J_3^5
$$

$$
= -\tilde{\mathcal{I}}_C(m_2, m_3, m_2, m_3, m_4) + \tilde{\mathcal{I}}_C(m_2, m_4, 0, m_3, m_3) - \tilde{\mathcal{I}}_C(m_3, m_2, m_3, m_2, m_4) + \tilde{\mathcal{I}}_C(m_3, m_4, 0, m_2, m_2).
$$

Substituting the expression (31) one obtains

$$
\mathcal{I}_A(0, m_2, m_3, m_4, m_3, m_3) =
\frac{1}{(4\pi)^3} \left[ \frac{1}{2} \log \frac{2m_2}{m_2 + m_3 + m_4} + \log \frac{2m_2}{m_2 + m_3 + m_4} \right]
+ \left( \frac{1}{m_2} + \frac{1}{m_3} \right) \log \frac{2(m_2 + m_4)}{m_2 + m_3 + m_4}.
$$

In the case $m_2 = m_3 = m, m_4 = 0$ this result simplifies to

$$
\mathcal{I}_A(0, m, m, 0, m, m) =
\frac{1}{(4\pi)^3} \frac{1}{4m^3} (1 - \log 2),
$$

which is the result of [25].
3.3 Non-degenerate case

Consider now the non-degenerate case $|\mathcal{M}| \neq 0$. Now the matrix is invertible. The equation can then be solved for $K_1$. This leads to a first order linear differential equation:

$$\frac{\partial}{\partial m^2_1}\mathcal{I}_A = -K_1 = -\frac{|\mathcal{M}_I|\mathcal{I}_A + |\mathcal{M}_J|}{|\mathcal{M}(m^2_1)|}, \quad (44)$$

where in $\mathcal{M}_X$ substitution of $X$ to the first row is assumed. From now on for $\mathcal{I}_A$ a notation similar to (36) will be used:

$$\mathcal{I}_A(m^2_1) \equiv \mathcal{I}_A(m_1, m_2, m_3, m_4, m_5, m_6). \quad (45)$$

In the domain where the coefficients on the right hand side are continuous (44) now has a unique solution. If $\mathcal{I}_A$ is this solution, it has a form

$$\mathcal{I}_A(m^2_1) = -\exp\left(-\int^{m^2_1} |\mathcal{M}_I(t)| dt\right) \left[\int^{m^2_1} \exp\left(-\int^{s} |\mathcal{M}_I(t)| dt\right) \frac{|\mathcal{M}_J(s)|}{|\mathcal{M}(s)|} ds + C\right]. \quad (46)$$

The value of the integration constant $C$ is to be determined. Let the lower limit of the integrations be $\tilde{m}^2$, a point such that the coefficients are continuous. Now let $m^2_1$ approach $\tilde{m}^2$ to see that in this case $C = \mathcal{I}_A(\tilde{m}^2)$, since the exponential factor approaches unity and the integral in the brackets vanishes. Now let $m_0$ be such that $|\mathcal{M}(m^2_0)| = 0$. Since $\mathcal{I}_A(m^2_1)$ coincides with $\mathcal{I}_A(m^2_1)$ when $m^2_1 < m^2_0$ and $\mathcal{I}_A$ is continuous also when $m^2_1 = m^2_0$, they must coincide also at the point $m^2_1 = m^2_0$ for $\mathcal{I}_A$ to be continuous. Thus, when $m^2_1 \leq m^2_0$,

$$\mathcal{I}_A(m^2_1) = \mathcal{I}_A(m^2_0). \quad (47)$$

However, when $m^2_1 \to m^2_0$, the exponential factor in (46) diverges. Therefore the expression inside the brackets must vanish and one obtains

$$\mathcal{I}_A(m^2_1) = -\int_{m^2_0}^{m^2_1} \exp\left(\int_{m^2_1}^{s} \frac{|\mathcal{M}_I(t)|}{|\mathcal{M}(t)|} dt\right) \frac{|\mathcal{M}_J(s)|}{|\mathcal{M}(s)|} ds. \quad (48)$$

Now the following relation is true:

$$|\mathcal{M}_I(m^2_1)| = \frac{1}{2} \frac{\partial}{\partial m^2_1} |\mathcal{M}(m^2_1)|. \quad (49)$$
This can be seen as follows. Consider the determinant expanded as a sum of
minors:
\[
|M(m^2)| = 2m_1^2 \left| \begin{array}{ccc} 2m_2^2 & C & -A \end{array} \right| + \left| \begin{array}{ccc} B & C^2 & 2m_3^2 \end{array} \right|.
\]
(50)

The first minor is constant in \(m_1^2\) and \(\partial A/\partial m_1^2 = \partial B/\partial m_1^2 = 1\). Therefore
\[
\frac{\partial}{\partial m_1^2} |M(m_1^2)| = 2 \left| \begin{array}{ccc} 2m_2^2 & C & -A \end{array} \right| + \left| \begin{array}{ccc} B & C^2 & 2m_3^2 \end{array} \right|
\]
\[
= \left| \begin{array}{ccc} 1 & 1 & 1 \\ A & 2m_2^2 & C \\ B & C & 2m_3^2 \end{array} \right| + \left| \begin{array}{ccc} 1 & 1 & A \\ B & C & 2m_3^2 \end{array} \right| + \left| \begin{array}{ccc} 1 & 2m_2^2 & 1 \\ A & B & C^2 \\ B & C & 2m_3^2 \end{array} \right|
\]
\[
= 2 \left| \begin{array}{ccc} 1 & 1 & 1 \\ A & 2m_2^2 & C \\ B & C & 2m_3^2 \end{array} \right| = 2|M_I(m_1^2)|.
\]
(51)

Thus, the integral in the exponential can be calculated and one obtains
\[
\mathcal{I}_A(m_1^2) = -\int_{m_0^2}^{m_1^2} \frac{|M(s)|}{\sqrt{|M(m_1^2)|}} \frac{|M_I(s)|}{|M(s)|} ds
\]
\[
= -\frac{1}{\sqrt{|M(m_1^2)|}} \int_{m_0^2}^{m_1^2} \frac{|M_I(s)|}{|M(s)|} ds
\]
\[
= -\frac{2}{\sqrt{|M(m_1^2)|}} \int_{m_0}^{m_1} \frac{|M_I(x^2)|}{\sqrt{|M(x^2)|}} dx.
\]
(52)

This result holds only if the integrand has no singularities between \(m\) and
\(m_0\). However, \(M_I\) diverges only if \(m_1 = 0\), \(m_2 = 0\) or \(m_3 = 0\), i.e. there is
no closed massive loop in the diagram. That case must be treated separately
and leads to infrared divergences. An appropriate choice of \(m_0\) ensures that
the denominator \(\sqrt{|M(x^2)|}\) has no zeros on the domain of integration.

When evaluating the integral (52) one needs the following integral
\[
\int_0^1 \frac{\log(1 + \alpha t) dt}{\sqrt{1 - \beta t^2 + \gamma t^4(1 + \delta t)}}.
\]
(53)
Unfortunately this integral is not expressible in terms of usual special functions. Therefore one needs numerical evaluation of the last expression. Since the integrand consists of only elementary functions, this can easily be done.

As an example consider the case \(m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = m\). Now \(|M(x^2)| = 6m^4x^2 - 2m^2x^4\) and

\[
|M_f(x^2)| = 3m^4J_1 - x^2m^2(J_2 + J_3)
= \frac{m}{(4\pi)^3} \left[ \log \frac{4}{3} - \log \frac{3m + x}{2m + x} + \frac{x}{2m + x} \left( \frac{x}{2m - x} \log \frac{4m}{2m + x} - \log \frac{3m + x}{3m} \right) \right]. \tag{54}
\]

Since \(m_0\) can be chosen to be zero, the result is

\[
I_A(m, m, m, m, m, m) = \frac{1}{(4\pi)^3m^3} \left[ \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{3 - x^2}} \left( \log \frac{3}{4} + \log \frac{3 + x}{2 + x} + \frac{x}{4 - x^2} \log \frac{4}{2 + x} + \frac{x}{2 + x} \log \frac{3 + x}{3} \right) \right]. \tag{55}
\]

Numerical evaluation of this integral gives

\[
I_A(m, m, m, m, m, m) \approx 0.0217376 \frac{0.0217376}{(4\pi)^3m^3}. \tag{56}
\]

### 3.4 Master integral

The integral \(I_a\) is called the master integral, since all the other two loop two point integrals can be obtained from it by removing some of the propagators from the integrand. Since \(I_a\) is the two point counterpart of \(I_A\) and it converges, it can be calculated using Lemma 1:

\[
I_a(p; m_1, m_2, m_3, m_4, m_5) = \frac{2\pi i}{p} (I_A(m_3, m_1, m_2, ip, m_5, m_4) - I_A(m_3, m_1, m_2, -ip, m_5, m_4)). \tag{57}
\]

In case of two point diagrams it is more convenient to use the following notation

\[
M(p; m_1, m_2, m_3, m_4, m_5) \equiv M(m_3, m_1, m_2, ip, m_5, m_4)
\]
\[
\begin{pmatrix}
2m_3^2 & m_1^2 + m_2^2 - m_4^2 & m_2^2 + m_3^2 - m_5^2 \\
m_1^2 + m_3^2 - m_4^2 & 2m_1^2 & m_1^2 + m_3^2 + p^2 \\
m_2^2 + m_3^2 - m_5^2 & m_1^2 + m_2^2 + p^2 & 2m_2^2
\end{pmatrix}
\]

(58)

and again, for simplicity

\[
\mathcal{M}(p; m^2_3) \equiv \mathcal{M}(p; m_1, m_2, m_3, m_4, m_5).
\]

(59)

In the matrix \(\mathcal{M}\) only the squares of masses are present and therefore those are equal for both terms. One obtains

\[
\mathcal{I}_a(m^2_3) = -\frac{2}{\sqrt{|\mathcal{M}(p; m^2_3)|}} \int_{m_0}^{m_3} dx \frac{1}{\sqrt{|\mathcal{M}(p; x^2)|}} 
\left( |\mathcal{M}_J(x, m_1, m_2, ip, m_5, m_4)| - |\mathcal{M}_J(x, m_1, m_2, -ip, m_5, m_4)| \right).
\]

(60)

Now the matrices \(\mathcal{M}_J\) differ only in the first row. Hence

\[
\frac{2\pi i}{p} (|\mathcal{M}_J(\ldots, ip, \ldots)| - |\mathcal{M}_J(\ldots, -ip, \ldots)|) = |\mathcal{M}_H(\ldots, ip, \ldots)|,
\]

(61)

where

\[
H(\ldots, ip, \ldots) = \frac{2\pi i}{p} (J(\ldots, ip, \ldots) - J(\ldots, -ip, \ldots)).
\]

(62)

Since vector \(J\) consists of convergent vacuum integrals, one can use Lemma \(\square\) to notice that \(H\) consists of corresponding two point integrals:

\[
\begin{align*}
H_1(p; m_1, \ldots, m_5) &= H_1^4 - H_1^3 + H_2^3 - H_2^5 \\
H_2(p; m_1, \ldots, m_5) &= H_2^3 + H_3^1 - H_3^4 \\
H_3(p; m_1, \ldots, m_5) &= H_3^2 - H_3^5 + H_1^2,
\end{align*}
\]

(63)

where \(H_i^j\) is, similarly to \(J_i^j\), integral \(\mathcal{I}_f\) or \(\mathcal{I}_h\) which has been constructed by removing the particle \(j\) from integral \(\mathcal{I}_a\) and taking square of the propagator of particle \(i\). The explicit expressions of these functions are given in appendix \(\square\).

If one writes

\[
\mathcal{M}_H(p; m_1, m_2, m_3, m_4, m_5) \equiv \mathcal{M}_H(m_3, m_1, m_2, ip, m_5, m_4).
\]

(64)
Eq. (60) can be written in the form

\[
I_a(p;m_1, m_2, m_3, m_4, m_5) = -\frac{2}{\sqrt{|M(p;m_3^2)|}} \int_{m_0}^{m_3} \frac{|M_H(p;m_1, m_2, x, m_4, m_5)|}{\sqrt{|M(p;x^2)|}} x dx,
\]

where \(m_0\) is again a root of the equation

\[
|M(p;m_0^2)| = 0.
\]

As an example, consider now the case \(m_1 = \cdots = m_5 = m\). Then the matrix \(M(p;x^2) = M(p;m, m, x, m, m)\) is

\[
M(p;x^2) = \begin{pmatrix}
2x^2 & x^2 & x^2 \\
2x^2 & 2m^2 & 2m^2 + p^2 \\
2m^2 + p^2 & 2m^2 & 2m^2
\end{pmatrix},
\]

and the functions \(H_i\) are

\[
H_1 = \frac{1}{(4\pi)^2 m_p^2(p^2 + 4m^2)} \left( p \left( \arctan \frac{p}{2m} - \arctan \frac{p}{2m + x} \right) - m \log \frac{p^2 + (2m + x)^2}{(2m + x)^2} \right),
\]

\[
H_2 = H_3 = \frac{1}{(4\pi)^2 2m_p x(2m + x)} \left( \arctan \frac{p}{2m} - \arctan \frac{p}{2m + x} \right). \quad (68)
\]

Substituting these to Eq. (65) gives the result

\[
I_a(p;m, m, m, m, m) = \frac{1}{(4\pi)^2 m_p^2 \sqrt{p^2 + 3m^2}} \int_0^m \frac{dx}{\sqrt{(p^2 + 4m^2) - x^2}}
\]

\[
\left[ -\frac{2p}{2m + x} \left( \arctan \frac{p}{2m + x} - \arctan \frac{p}{2m} \right) + \log \frac{p^2 + (2m + x)^2}{(2m + x)^2} \right]. \quad (69)
\]

4 Effective potential and self energy in scalar theory
4.1 Lagrangian

As an application of the integrals calculated in the previous section scalar \( \lambda \phi^4 \) theory will now be discussed. This is the simplest possible nontrivial quantum field theory. However, it has also physical significance. In theory of critical phenomena it is in the same universality class as the Ising model. The Lagrangian of the theory is

\[
L = \frac{1}{2} (\partial_i \phi)^2 + \frac{1}{2} m_\phi^2 \phi^2 + \frac{1}{4} \lambda \phi^4. \tag{70}
\]

Suppose now that \( m_\phi^2 < 0 \). Then the minimum of the Lagrangian is not anymore in the origin. Now make a shift \( \phi \rightarrow \phi_0 + \phi \) to get a new broken Lagrangian. The terms linear in fields are discarded, since they cancel the tadpole terms in the true minimum when calculating the self energy. When calculating the effective potential they are also discarded, so that a presentation in terms of vacuum diagrams can be obtained [21]. The Lagrangian now reads

\[
L = \frac{1}{2} m_\phi^2 \phi_0^2 + \frac{1}{4} \lambda \phi_0^4 + \frac{1}{2} (\partial_i \phi)^2 + \frac{1}{2} m^2 \phi^2 + \lambda \phi_0 \phi^3 + \frac{1}{4} \lambda \phi^4, \tag{71}
\]

where

\[
m^2 = m_\phi^2 + 3 \lambda \phi_0^2. \tag{72}
\]

The Feynman rules of the theory are shown in Fig. 5.

4.2 Self energy and renormalization

Let us start analyzing the system by calculating the self energy. In Fig. 6 it is expanded to two loop order. The number of different Wick contractions corresponding to each diagram is given in table [2].
In dimensional regularization only diagram g diverges. To remove this divergence a mass counterterm $\delta m^2$ must be introduced. Using the given symmetry factors and integrals, the value of the divergent diagram is

$$\frac{6\lambda^2}{(4\pi)^2} \left( \frac{1}{4\varepsilon} + \frac{3}{2} - \frac{3m}{p} \arctan \frac{p}{3m} + \frac{1}{2} \log \frac{\bar{\mu}^2}{9m^2 + p^2} \right).$$  \hfill (73)

The correct value of the mass counterterm is then

$$\delta m^2 = \frac{\lambda^2}{(4\pi)^2} \frac{3}{2\varepsilon}. \hfill (74)$$

This gives rise to a running mass

$$m_0^2(\bar{\mu}) = \frac{6\lambda^2}{(4\pi)^2} \log \frac{\bar{\mu}}{\Lambda_m}, \hfill (75)$$

where $\Lambda_m$ is a dimensional parameter such that $m_0^2(\Lambda_m) = 0$. Thus, the mass $m^2$ is a function of both the renormalization point and the vacuum expectation value of the field:

$$m^2 = m^2(\bar{\mu}, \phi_0). \hfill (76)$$

Since no other divergences are present, the coupling constant $\lambda$ does not run.

The self energy can now be calculated by collecting all the relevant integrals and the two loop part of the result is

$$\Pi^{(2)} =$$
\[
\frac{1}{(4\pi)^2} \left\{ \lambda^2 \left[ \frac{9}{2} - 18 \frac{m}{p} \arctan \frac{p}{3m} + 3 \log \frac{\bar{\mu}^2}{9m^2 + p^2} \right]
+ \lambda^3 \phi_0^2 \left[ \frac{54}{p^2 + 4m^2} - \frac{9}{m^2} - \frac{54}{k^2} \left( \arctan \frac{p}{2m} \right)^2 \right]
- \frac{54}{pm} \left[ 2 \log 3 \arctan \frac{p}{2m} + i \left( \text{Li}_2 \left( -\frac{ip}{3m} \right) + \text{Li}_2 \left( -\frac{2m - ip}{m} \right) \right) \right]
+ \lambda^4 \phi_0^4 \left[ \frac{27}{m^3 p^2 (p^2 + 4m^2)} \left( 4p(2p^2 + 11m^2) \arctan \frac{p}{3m} \right) \right]
+ (6 \log 3 - 8)p(p^2 + 4m^2) \arctan \frac{p}{2m} - 6m(p^2 + 2m^2) \log \left( 1 + \frac{p^2}{9m^2} \right)
+ 3ip(p^2 + 4m^2) \left( \text{Li}_2 \left( -\frac{ip}{3m} \right) - \text{Li}_2 \left( \frac{ip}{3m} \right) \right)
+ \text{Li}_2 \left( -2 + \frac{ip}{m} \right) - \text{Li}_2 \left( -2 - \frac{ip}{m} \right) \right]
+ \frac{648}{mp^2 \sqrt{p^2 + 3m^2}} \int_0^m \frac{dx}{\sqrt{(p^2 + 4m^2) - x^2}}
\left( \frac{2p}{2m + x} \left( \arctan \frac{p}{2m + x} - \arctan \frac{p}{2m} \right) + \log \frac{p^2 + (2m+x)^2}{(2m+x)^2} \right) \right\}. (77)
\]

### 4.3 Effective potential

The calculation of the effective potential is very similar to that of the self energy. The diagrams to three loop order are given in Fig. [1] and the numbers of contractions in table [1].

The result can be written down at once

\[
V(\phi_0) = \frac{1}{2} m_0^2 \phi_0^2 + \frac{1}{4} \lambda \phi_0^4 - \frac{1}{12\pi} m^3
+ \frac{1}{(4\pi)^2} \left[ \frac{3}{4} \lambda m^2 - 3 \lambda^2 \phi_0^2 \left( \log \frac{\bar{\mu}}{3m} + \frac{1}{2} \right) \right]
+ \frac{1}{(4\pi)^3} \left\{ m\lambda^2 \left( 3 \log \frac{\bar{\mu}}{4m} + \frac{27}{8} \right) \right\}
\]
\[ \Pi(k) = \]
\[
-\left( \frac{1}{A_1} \right)^{-1} + \frac{1}{2} + \frac{1}{24} - \frac{1}{2} + \frac{1}{24} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2}
\]

Figure 6: Loop expansion of the scalar self energy

\[
\begin{align*}
+ \frac{\lambda^3 \phi_0^2}{m} & \left[ -\frac{9}{2} + \frac{9}{4} \pi^2 - \frac{27}{2} \left( \log \frac{4}{3} \right)^2 - 27 \text{Li}_2 \left( \frac{1}{4} \right) \right] \\
+ \frac{\lambda^4 \phi_0^4}{m^3} & \left[ -\frac{27}{8} \pi^2 + \frac{81}{4} \left( \log \frac{4}{3} \right)^2 + 54 \log \frac{4}{3} + \frac{81}{2} \text{Li}_2 \left( \frac{1}{4} \right) \right] \\
-54 & \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{3-x^2}} \left( \log \frac{3}{4} + \log \frac{3+x}{2+x} - \frac{x^2}{4-x^2} \log \frac{4}{2+x} + \frac{x}{2+x} \log \frac{3+x}{3} \right) \\
\end{align*}
\]  

This result agrees perfectly with that evaluated numerically in [26].

5 Conclusions

In this paper, the integrals necessary for the self energy to two loop level in a three dimensional scalar field theory have been evaluated explicitly as well as the ones necessary for the effective potential to three loop level. In almost every case the result can be expressed with elementary functions and dilogarithms.

A large part of the paper has been devoted to the evaluation of the two
most difficult integrals, the master integral and the Mercedes integral. Those are expressed in terms of a one dimensional integral representation with an integrand consisting only of elementary functions. This form makes numerical evaluation easy.

The results have been applied to $\lambda \phi^4$ scalar theory. It will be very interesting to extend them to gauge theories with scalars, like the U(1)+Higgs or SU(2)+Higgs models. This will lead to a large number of new diagrams. However, all the two loop integrals contributing to the self energy in a gauge field theory can be decomposed to a sum of scalar integrals. This is a laborious task for which a computer algebra system is needed. The results will help to deepen our understanding of the phase transitions in gauge theories, for example the electroweak phase transition of the early universe.

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### A Integrals

This is a complete list of the integrals corresponding to the diagrams of Figs. 1, 2 and 3.

\[ \alpha \) = \frac{-m}{4\pi} \left[ 1 + \varepsilon \left( 2 + 2 \log \frac{\bar{\mu}}{2m} \right) \right]. \quad (79) \\
\beta \) = \frac{1}{4\pi p} \arctan \frac{p}{m_1 + m_2}. \quad (80) \\
\gamma \) = \frac{1}{(4\pi)^2} \left( \frac{1}{4\varepsilon} + \frac{1}{2} + \log \frac{\bar{\mu}}{m_1 + m_2 + m_3} \right). \quad (81) \\
\delta \) = \frac{2}{\sqrt{|\mathcal{M}(p; m_1, m_2, m_3, m_4, m_5)|}} \int_{m_0}^{m_3} \left| \mathcal{M}_H(p; m_1, m_2, x, m_4, m_5) \right| \sqrt{|\mathcal{M}(p; m_1, m_2, x, m_4, m_5)|} x dx, \quad (82) \\

where

\[ \mathcal{M}(p; m_1, m_2, m_3, m_4, m_5) = \begin{pmatrix} 2m_2^2 & m_1^2 + m_3^2 - m_4^2 & m_2^2 + m_3^2 - m_5^2 \\ m_1^2 + m_3^2 - m_4^2 & 2m_1^2 & m_1^2 + m_4^2 + p^2 \\ m_2^2 + m_3^2 - m_5^2 & m_1^2 + m_2^2 + p^2 & 2m_2^2 \end{pmatrix}, \quad (83) \]

and

\[ \mathcal{M}_H(p; m_1, m_2, m_3, m_4, m_5) = \begin{pmatrix} H_1 & H_2 & H_3 \\ H_1 & H_4 & H_5 \\ H_2 & H_5 & H_1 \end{pmatrix}, \quad (84) \]

The functions \( H_i(p; m_1, m_2, m_3, m_4, m_5) \) are

\[ H_1(p; m_1, \ldots, m_5) = H_1^3 - H_1^1 + H_2^3 - H_2^2 \]
\[ H_2(p; m_1, \ldots, m_5) = H_2^1 + H_3^1 - H_3^4 \]
\[ H_3(p; m_1, \ldots, m_5) = H_3^3 - H_3^5 + H_1^2, \quad (85) \]

where

\[ H_1^3 = H_X(m_1, m_2, m_4, m_5) \]
\begin{align*}
H_1^4 &= H_Y(m_1, m_3, m_5, m_2) \\
H_2^3 &= H_X(m_2, m_1, m_5, m_4) \\
H_2^5 &= H_Y(m_2, m_3, m_4, m_1) \\
H_2^1 &= H_Z(m_4, m_2, m_3, m_5) \\
H_3^1 &= H_Z(m_4, m_3, m_2, m_5) \\
H_4^1 &= H_Z(m_1, m_3, m_5, m_2) \\
H_3^2 &= H_Z(m_5, m_3, m_1, m_4) \\
H_5^2 &= H_Z(m_2, m_3, m_4, m_1) \\
H_1^2 &= H_Z(m_5, m_1, m_3, m_4),
\end{align*}

and

\begin{align*}
H_X(m_1, m_2, m_3, m_4) &= \frac{1}{(4\pi)^2 2m_1 p ((m_1 + m_2)^2 + p^2)^2} \arctan \frac{p}{m_3 + m_4} \\
H_Y(m_1, m_2, m_3, m_4) &= \frac{1}{(4\pi)^2 2m_1 p ((p^2 + m_1^2 + m_4^2)^2 - 4m_1^2 m_2^2)} \\
&\quad \left( (p^2 + m_3^2 - m_1^2) \arctan \frac{p}{m_1 + m_2 + m_3} \\
&\quad + m_1 p \log \frac{p^2 + (m_1 + m_2 + m_3)^2}{(m_2 + m_3 + m_4)^2} \right) \\
H_Z(m_1, m_2, m_3, m_4) &= \frac{1}{(4\pi)^2 2m_2 p (m_4^2 - (m_2 + m_3)^2)} \\
&\quad \left( \arctan \frac{p}{m_1 + m_2 + m_3} - \arctan \frac{p}{m_1 + m_4} \right). \tag{87}
\end{align*}

\begin{align*}
b &= \frac{1}{(4\pi)^2 4pm_1 m_4 (m_4^2 - m_1^2)} \left( 2m_1 \log \frac{m_2 + m_3 + m_4}{m_2 + m_3 - m_4} \arctan \frac{p}{m_4 + m_5} \right. \\
&\quad + 2m_4 \log \frac{m_1 + m_2 + m_3}{m_2 + m_3 - m_1} \arctan \frac{p}{m_1 + m_5} \\
&\quad + \frac{m_4}{i} \left[ \text{Li}_2 \left( -\frac{m_5 + m_1 - ip}{m_2 + m_3 - m_1} \right) + \text{Li}_2 \left( -\frac{m_5 + m_1 + ip}{m_2 + m_3 - m_1} \right) \right].
\end{align*}
where the matrices $\mathcal{M}$ and $\mathcal{M}_f$ are

$$
\mathcal{M}(m_1, m_2, m_3, m_4, m_5, m_6) = 
\begin{pmatrix}
2m_1^2 & m_1^2 + m_3^2 - m_6^2 & m_1^2 + m_4^2 - m_5^2 \\
m_2^2 + m_3^2 - m_6^2 & 2m_2^2 & m_2^2 + m_5^2 - m_4^2 \\
m_1^2 + m_3^2 - m_5^2 & m_2^2 + m_3^2 - m_4^2 & 2m_3^2
\end{pmatrix},
$$
and

\[ M_f(m_1, m_2, m_3, m_4, m_5, m_6) = \begin{pmatrix}
J_1 & J_2 & J_3 \\
(m_1^2 + m_2^2 - m_6^2) & 2m_2^2 & m_2^2 + m_5^2 - m_4^2 \\
(m_1^2 + m_3^2 - m_5^2) & m_2^2 + m_3^2 - m_1^2 & 2m_2^2 \\
\end{pmatrix}. \tag{97}

The functions \( J_i(m_1, m_2, m_3, m_4, m_5, m_6) \) are

\[
\begin{align*}
J_1 &= J_2^1 - J_2^6 + J_3^1 - J_3^5 \\
J_2 &= J_3^2 - J_3^4 + J_1^2 - J_1^6 \\
J_3 &= J_1^3 - J_1^1 + J_3^3 - J_3^1,
\end{align*}
\tag{98}
\]

where

\[
\begin{align*}
J_2^1 &= \tilde{I}_C(m_2, m_3, m_5, m_6, m_4) \\
J_2^6 &= \tilde{I}_C(m_2, m_4, m_1, m_5, m_3) \\
J_3^1 &= \tilde{I}_C(m_3, m_2, m_5, m_6, m_4) \\
J_3^5 &= \tilde{I}_C(m_3, m_4, m_1, m_6, m_2) \\
J_3^2 &= \tilde{I}_C(m_3, m_4, m_1, m_6, m_5) \\
J_3^4 &= \tilde{I}_C(m_3, m_5, m_2, m_6, m_1) \\
J_1^2 &= \tilde{I}_C(m_1, m_6, m_3, m_4, m_5) \\
J_1^6 &= \tilde{I}_C(m_1, m_5, m_2, m_4, m_3) \\
J_1^3 &= \tilde{I}_C(m_1, m_2, m_4, m_5, m_6) \\
J_1^5 &= \tilde{I}_C(m_1, m_6, m_3, m_4, m_2) \\
J_2^3 &= \tilde{I}_C(m_2, m_1, m_4, m_5, m_6) \\
J_2^4 &= \tilde{I}_C(m_2, m_6, m_3, m_4, m_1),
\end{align*}
\tag{99}
\]

and \( \tilde{I}_C \) is the derivative of \( I_C \):

\[
\tilde{I}_C(m_1, m_2, m_3, m_4, m_5) = \frac{1}{(4\pi)^3 2m_1((m_1 + m_2)^2 - m_6^2)} \log \frac{m_1 + m_2 + m_3 + m_4}{m_3 + m_4 + m_5}. \tag{100}
\]

\[
B) = \frac{1}{(4\pi)^3 4m_5 m_6 (m_6^2 - m_5^2)}
\]

27
\[
\begin{align*}
C' &= \frac{1}{(4\pi)^3 2 m_5} \left[ \log \frac{m_3 + m_4 + m_5}{m_3 + m_4 - m_5} \log \frac{(m_3 + m_4)^2 - m_5^2}{(m_1 + m_2 + m_5)^2} 
+ 2 \text{Li}_2 \left( -\frac{m_1 + m_2 - m_5}{m_3 + m_4 + m_5} \right) - 2 \text{Li}_2 \left( -\frac{m_1 + m_2 + m_5}{m_3 + m_4 - m_5} \right) \right], \\
D' &= \frac{m_5}{(4\pi)^3 (m_3^2 - m_4^2)} \log \frac{m_1 + m_2 + m_4}{m_1 + m_2 + m_3}, \\
E' &= \frac{1}{(4\pi)^3} \sum_{i=1}^4 m_i \left( -\frac{1}{4\varepsilon} + 2 + \frac{1}{2} \log \frac{2m_i}{\bar{\mu}} + \log \frac{\sum_i m_j}{\bar{\mu}} \right), \\
F' &= \frac{1}{(4\pi)^3} \frac{m_1 m_4}{m_2 + m_3}.
\end{align*}
\]

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Erratum to “Feynman diagrams to three loops in three-dimensional field theory” 
[Nucl. Phys. B 480 (1997) 729]∗

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The sign of the constant term $-2$ was incorrect in Eqs. (21) and (A.26):

\[
\frac{1}{(4\pi)^3} \sum_{i=1}^{4} m_i \left( -\frac{1}{4\varepsilon} - 2 + \frac{1}{2} \log \frac{2m_i}{\mu} + \log \frac{\sum_j m_j}{\mu} \right). \tag{1}
\]

One $m_2$ should be $m_4$ in Eq. (A.9):

\[
H_Y(m_1, m_2, m_3, m_4) = \frac{1}{(4\pi)^2} \frac{1}{2m_1p} \left( (p^2 + m_1^2 + m_2^2) - 4m_1^2m_2^2 \right) \times \left( p^2 + (m_1 + m_2 + m_3)^2 \right) \arctan \frac{p}{m_1 + m_2 + m_3} + m_1 p \log \frac{p^2 + (m_1 + m_2 + m_3)^2}{(m_2 + m_3 + m_4)^2}. \tag{2}
\]

Five signs were incorrect in Eq. (A.10):

\[
(b) = \frac{1}{(4\pi)^2 4p m_1 m_4 (m_4^2 - m_1^2)} \left\{ 2m_4 \log \frac{m_1 + m_2 + m_3}{m_2 + m_3 - m_1} \arctan \frac{p}{m_1 + m_5} - 2m_1 \log \frac{m_2 + m_3 + m_4}{m_2 + m_3 - m_4} \arctan \frac{p}{m_4 + m_5} + i m_4 \left[ \text{Li}_2 \left( -\frac{m_5 + m_1 - ip}{m_2 + m_3 - m_1} \right) - \text{Li}_2 \left( -\frac{m_5 + m_1 + ip}{m_2 + m_3 - m_1} \right) \right] \right\}.
\]

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\[- \text{Li}_2 \left( \frac{-m_5 - m_1 - \text{ip}}{m_1 + m_2 + m_3} \right) + \text{Li}_2 \left( \frac{-m_5 - m_1 + \text{ip}}{m_1 + m_2 + m_3} \right) \]
\[ - \text{im}_1 \left[ \text{Li}_2 \left( \frac{-m_5 + m_4 - \text{ip}}{m_2 + m_3 - m_4} \right) - \text{Li}_2 \left( \frac{-m_5 + m_4 + \text{ip}}{m_2 + m_3 - m_4} \right) \right] \]
\[ - \text{Li}_2 \left( \frac{-m_5 - m_4 - \text{ip}}{m_2 + m_3 + m_4} \right) + \text{Li}_2 \left( \frac{-m_5 - m_4 + \text{ip}}{m_2 + m_3 + m_4} \right) \right\}. \quad (3)\]

In three of the quantities \( J^i \) listed in Eq. (A.21), one or two mass indices were incorrect:

\[ J^2_3 = \tilde{\mathcal{I}}_C(m_3, m_1, m_4, m_6, m_5), \]
\[ J^1_2 = \tilde{\mathcal{I}}_C(m_1, m_3, m_4, m_6, m_5), \]
\[ J^4_2 = \tilde{\mathcal{I}}_C(m_2, m_6, m_3, m_5, m_1). \quad (4)\]

In Eq. (A.23) the \( \pi^2/12 \) terms were missing:

\[ (B) = \frac{1}{(4\pi)^3(m_6^2 - m_5^2)} \]
\[ \left\{ \frac{1}{m_5} \left[ \frac{\pi^2}{12} + \frac{1}{4} \left( \log \frac{m_5 + m_1 + m_2}{m_5 + m_3 + m_4} \right)^2 \right] \right. \]
\[ + \frac{1}{2} \text{Li}_2 \left( \frac{m_5 - m_1 - m_2}{m_5 + m_3 + m_4} \right) + \frac{1}{2} \text{Li}_2 \left( \frac{m_5 - m_3 - m_4}{m_5 + m_1 + m_2} \right) \]
\[ \left. - \frac{1}{m_6} \left[ \frac{\pi^2}{12} + \frac{1}{4} \left( \log \frac{m_6 + m_1 + m_2}{m_6 + m_3 + m_4} \right)^2 \right] \right. \]
\[ + \frac{1}{2} \text{Li}_2 \left( \frac{m_6 - m_1 - m_2}{m_6 + m_3 + m_4} \right) + \frac{1}{2} \text{Li}_2 \left( \frac{m_6 - m_3 - m_4}{m_6 + m_1 + m_2} \right) \right\}. \quad (5)\]

On p. 740 the sign of the integration constant \( C \) should be the opposite. On the fourth line of p. 742 there should be \( \sqrt{|M(x^2)|} \) and on p. 743, below Eq. (59), \( |M(p; m_6^2)| = 0 \).

These errors change neither the results in Eqs. (77) and (78) nor any of the conclusions.

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