On the Capability Of SuperKamiokande Detector To Define the Primary Parameters Of Muon And Electron Events

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Abstract. We develop a new discrimination procedure for separating electron neutrinos from muon neutrinos, based on detailed simulations carried out with GEANT 3.21 and with newly derived mean angular distribution functions for the charged particles concerned (muons and electrons/positrons), as well as the corresponding functions for the relative fluctuations. These angular distribution functions are constructed introducing a “moving point” approximation. Using our type definition procedure we are able to discriminate muons from electrons in Fully Contained Events with a probability of error of less than several %. At the same time, our geometrical reconstruction procedure, considering only the ring-like structure of the Cherenkov image, gives an unsatisfactory resolution for 1 GeV $e$ and $\mu$, with a mean vertex position error, $\delta r$, of 5–10 m and a mean directional error, $\delta \theta$, of about 6°–20°. In contrast, a geometrical reconstruction procedure utilizing the full image and using a detailed approximation of the event angular distribution works much better: for a 1 GeV $e$, $\delta r \sim 2$ m and $\delta \theta \sim 3^\circ$; for a 1 GeV $\mu$, $\delta r \sim 3$ m and $\delta \theta \sim 5^\circ$. At 5 GeV, the corresponding values are $\sim 1.4$ m and $\sim 2^\circ$ for $e$ and $\sim 2.9$ m and $\sim 4.3^\circ$ for $\mu$. The numerical values depend on a single PMT contribution threshold. The values quoted above are the minima with respect to this threshold. Even the methodologically correct approach we have adopted, based on detailed simulations using closer approximations than those adopted in the SK analysis, cannot reproduce the accuracies for particle discrimination, momentum resolution, interaction vertex location, and angular resolution obtained by the SK simulations, suggesting the assumptions in these may be inadequate.

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1. Introduction

The possible existence of neutrino oscillations is one of the most important issues in particle astrophysics as well as elementary particle physics at the present time. Among the positive and negative results reported for neutrino oscillation, experimental results for atmospheric neutrino by Super Kamiokande (hereafter, we abbreviate simply SK) has special position in the experiments concerned, because it is said that they have given the decisive and clear evidence for the existence of neutrino oscillation. The reasons as follows:

(1) They carried out the calibration experiments for the discrimination between muon and electron by electron accelerator beam whose energies are well known and established the clear discrimination between muon and electron for SK energy region concerned [1].

(2) Based on the well established discrimination procedure between muon and electron, they have analyzed Fully Contained Events and Partially Contained Events, whose energies covered from several hundreds MeV to several GeV. As the results of them, they have found significantly different zenith angle distribution for muon and electron, namely, muon deficit, and attributed such discrepancy to the neutrino oscillation between $\nu_\mu$ and $\nu_\tau$. As the most new one, they give $\sin^2 2\theta > 0.92$ and $1.5 \times 10^{-3} \text{eV}^2 < \Delta m^2 < 3.4 \times 10^{-3} \text{eV}^2$ at 90% confidence level [2].

(3) Also, they have analyzed Upward Through Going Particle Events and Stopping Particle Events. Most physical events under such category could be regarded as exclusively the muon (neutrino) induced events, not electron (neutrino) induced events, because the effective volume for muon is much larger than that for electron due to longer range of muon irrespective of the discrimination procedure between muon and electron which is indispensable for the analysis for Fully Contained Events and Partially Contained Events. Also, in this case, they have given the same parameters for neutrino oscillation which are obtained in the analysis of Fully Contained Events and Partially Contained Events [2].

Through three different kinds of the experiment performed by SK, all of which are constructed upon the well established procedure, it is said that SK has given clear and definite evidence for existence for the neutrino oscillation.

The analysis of Fully Contained Events and Partially Contained Events is closely and inevitably related to the discrimination procedure between electron and muon. The frequency of muon events with some energy occurred inside the detector is nearly the same as that of electron events unless oscillation exists and, therefore, the precise discrimination procedure between electron and muon is absolutely necessary.

Considering the great impact of SK experiment over other experiments concerned and theoretical physics, we feel we should examine the validities of the experimental results performed by SK, because nobody has examined them in the most comprehensive way, solely due to character of huge experiment, although the partial aspect of SK had been examined in fragmental way [3].
However, Mitsui et al have examined the validity of the discrimination procedure by SK and have pointed out the necessity of fluctuation effect into the discrimination procedure between muon and electron by SK [4].

We have examined validities of all the SK experiment, adopting quite different approach from the SK procedure.

2. Algorithm For Processing Cherenkov Light Images In SuperKamiokande Experiments

As can be inferred from [5, 6], the image processing technique at SuperKamiokande is based on events simulated with the aid of the GEANT3.21 code [7].

But, in reality, a small part of the simulation results is used by SK to construct models of $e, \mu$-event images, namely, the average angular distribution (more precisely, that which is averaged both over shower particles and over the ensemble of showers) of light emitted from the electromagnetic shower initiated by an electron. In the SuperKamiokande studies, the spatial distribution of the light source (that is, the shower) is not taken into account either in the transverse direction or along the shower axis — in other words, the shower is taken in the pointlike form. This may lead to significant distortions of the pattern in procedures for event-type recognition and event-geometry reconstruction, since the mean longitudinal length of a shower initiated by particles of energy 1 GeV is about 4 m, while the dimensions of the sensitive volume of the water tank do not exceed 40 m, the events being uniformly distributed over the whole tank volume. Further, a muon track is represented by a straight-line segment, the distribution of light emitted from it taking the form of a delta function (that is, the photons fly along the Cherenkov cone generatrices). This means that the effect of multiple scattering is neglected in [5, 6].

The disregard of the information about fluctuations of the light spatial and angular distributions that is contained in simulated events is yet another significant simplification that is not well justified in our opinion. The scale of relative fluctuations of the light angular distribution for events initiated by particles of energy about 1 GeV is about hundreds of percent, and it is this circumstance that must restrict substantially the potential for the reconstruction of the event type and geometry.

Mitsui et al. [4] also indicated that the event models chosen by the SuperKamiokande Collaboration are inadequate. They reproduced the SuperKamiokande procedure for events obtained from a Monte Carlo simulation with allowance for all possible fluctuations (including fluctuations of photoelectrons) and found errors in event-type identification that are much greater than those reported by the SuperKamiokande Collaboration (about 20% for events initiated by particles of energy below 1 GeV versus several percent).
3. Statement Of the Problem Of Estimating the Upper Limits For the Parameter Resolutions

Here, we did not aim at developing new algorithms for processing SuperKamiokande data, since this would require considerable resources and detailed knowledge of the experimental setup; instead, we just tried to set absolute limits on the potential of SuperKamiokande telescope that are associated with detector geometry and physical processes of light generation and propagation.

The problem of determining primary parameters of events can be simplified by breaking it down into three separate problems:

(i) that of determining the primary-particle momentum (energy) under the assumption that the particle type, the injection point (particle-production vertex), and the direction of particle motion are known;

(ii) that of identifying the primary-particle type under the assumption that the particle momentum (energy), the injection point, and the direction of particle motion are known;

(iii) that of determining injection point for a primary particle and the direction of its motion under the assumption that its type and momentum (energy) are known.

For all of the parameters, this approach is generally expected to give resolutions that are higher than those in the case of solving the total problem of determining all parameters simultaneously. Thus, our results must set limits on the resolutions of the SuperKamiokande telescope.

In the case where a muon track or an electron shower lies completely within the sensitive volume of the detector (so-called Fully Contained Events), a precise description of event geometry, especially in the SuperKamiokande detector, where the light absorption length is about 100 m, is not required for the first problem because the total amount of recorded Cherenkov light depends strongly on the primary particle momentum.

In this paper we limit our consideration to the analysis of Fully Contained Events.

As for the second and third problems, it is necessary to describe in detail the features of Cherenkov light emitted by a muon and an electron — at least in three dimensions — since, in order to determine the event type and event geometry, one needs the image pattern to compare with the actual experimental image. An oversimplified description of images can impair considerably the resolution in the event type and event geometry.

The process of optical-photon transformation into photoelectrons was not considered, because its analysis would require detailed information about photomultiplier tubes. This simplification does not change the initial purpose of estimating limits on resolutions, since the elimination of one source of fluctuations may only improve the respective estimates.

We decided to neglect the lateral distribution of particles in electron and muon events (its scale was on the order of a few tenths of a meter); that is, we assumed that Cherenkov light is emitted exclusively from the event axis. In view of this, the present
approach is inapplicable to events associated with particles moving overly close to the tank walls and nearly parallel to them, but it is quite suitable for estimating resolutions.

4. Monte-Carlo Simulation

In simulating images, we assumed that events occurred within a cylindrical water volume 16.9 m in radius and 36.2 m in height. Further, 11 408 photomultiplier tubes, the diameter of their photocathodes being 50 cm, were distributed uniformly over the walls
of the cylinder and over its bottom and top bases. This corresponds roughly to the Super-Kamiokande detector. The scattering of light and its reflection from the walls were disregarded, but its absorption was taken into account.

A modified GEANT3.21 code was used to obtain simulated Cherenkov light images for events in the above described analog of the Super-Kamiokande telescope and to develop an adequate model of muon and electron events. In our modification of the code, we abandoned the standard algorithms for tracking optical photons. Instead, the product photon was tracked along a straight line until it hit the wall and was considered to be recorded with the weight equal to the transmission coefficient for the traversed water layer if the trajectory of this photon intersected a circle imitating a photomultiplier tube.

The water refraction index was taken to be 1.34 for the whole wavelength range 300-600 nm considered here. Electrons were tracked up to the kinetic energy of 0.25 MeV, while muons were tracked up to their decay. As a result of event simulation one gets detailed Cherenkov images of \( e^- \)-showers/\( \mu^- \)-tracks in SK telescope.

For the construction of event parameter definition procedures one needs the models of event images that are close to real ones. Event parameters are defined through a comparison of these model images with different parameter values with experimental images.

Instead of the simple mean models used by SK (point-like one for \( e^- \)-events and straight line for \( \mu^- \)-events) we introduce 'moving-point' approximation models for both \( e^- \) and \( \mu^- \)-events. In this approximation \( e^- \)-shower/\( \mu^- \)-track is assumed to emit Cherenkov photons from a straight line following the primary direction but the angular distribution of light changes along this line i.e. both mean angular distribution and its fluctuation evolve with water layer depth.

The procedure for simulating electron and muon light angular distribution involved the following steps:

(i) The mean longitudinal length of a muon track (electron shower) was fixed to be the length from which 99.5% of the mean total number of Cherenkov photons are emitted.

(ii) This length was broken down into equal segments, their length and number depending on the particle type and energy and on the required accuracy in the image pattern.

(iii) The mean angular distribution of Cherenkov light, \( F_{\theta_i}^{e^\pm}(\theta) \), and its relative fluctuation \( \delta_{\theta_i}^{e^\pm}(\theta) \) were calculated for each segment \( i \).

Thus, the simulation provides an approximation of the mean angular distributions of light

\[
F_{e^\pm,\mu}(\theta_i, E_0, k) = \frac{\langle N_{e^\pm,\mu}(\theta_i, E_0, k) \rangle}{\Delta \Omega_i},
\]

and their relative fluctuations
\[
\delta_{e,\mu}(\theta_i, E_0, k) = \sqrt{\langle N_{e,\mu}^2(\theta_i, E_0, k) \rangle - \langle N_{e,\mu}(\theta_i, E_0, k) \rangle^2} / \langle N_{e,\mu}(\theta_i, E_0, k) \rangle.
\]

versus the light emission angle \(\theta\) and the water-layer thickness \(t\). Here \(\langle \ldots \rangle\) denotes the average over a large event sample, \(N_{e,\mu}(\theta_i, E_0, k)\) is a number of Cherenkov photons emitted from segment \(k\), \(\theta_i\) is the center of mass of the \(i\)-th histogram bin, and \(\Delta \Omega_i\) is the solid angle of the \(i\)-th bin. While calculating \(F_{e,\mu}(\theta_i, E_0, k)\) and \(\delta_{e,\mu}(\theta_i, E_0, k)\) we considered samples of 10 000 to 20 000 events and did not track Cherenkov photons, but we included their contributions in the histograms in \(\theta\) of bin width 1.875° immediately after light generation, irrespective of the azimuthal emission angle.

The number of the segments was varied from 7 to 24 in the calculations; segments of length 40 cm and 100 cm were used for events generated by particles of energy below 1 GeV and equal to 5 GeV, respectively.

In order to approximate the mean angular distribution of light within each individual segment of a muon track (electron shower), we took the model functions

\[
F_{\mu}(\theta; A, B, C, B1, B2) = 10 \left\{ A \exp \left[-B (\theta-C)^2 \right] \right\} + 10 \left[ B1/(1+B2 \theta^4) \right],
\]

\[
F_e(\theta; A, B, C, B1, B2, B3, B4) = 10 \left\{ \frac{A+B3/\left(1+B4+B1 \theta^4\right)}{1+B2 \theta} \right\}.
\]

The approximations of the mean angular distributions were obtained as the best least squares fits of the model functions to the histograms.

The shapes of the relative fluctuations are very complicated. However, we used linear interpolations with 7 to 8 nodes to describe the fluctuations, because a high accuracy was not necessary in that case.

Figure 1 shows examples of approximations of the mean angular distributions of light and their relative fluctuations in events initiated by electrons (a, b) and muons (c, d) of energy 500 MeV.

It should be noted that, for a given type and a given energy of the primary particle, the above approximations of the angular distributions of Cherenkov light are quite universal in the sense that they can be used to calculate the patterns of mean images and their variations for any possible geometry of events in any water detector.

5. Procedures For Reconstructing Primary Parameters Of Events

5.1. Reconstruction of Primary Energy (Momentum)

As was mentioned above, the energy for fully contained events of a specific type can be estimated on the basis of the total number of photons recorded by photomultiplier
Figure 2. Distribution in the total number of detected Cherenkov photons from muon and electron events initiated by 300 MeV/c particles. The injection point is in the center of the tank. Sample volume: 1000 events.

Table 1. Energy (momentum) determination errors

| momentum, MeV/c | 300 | 1000 |
|-----------------|-----|------|
| event type      | μ   | e    | μ   | e   |
| present work    | 10.6| 2.5  | 1.9 | 1.3 |
| [6]             | 3.0 | 5.3  | 2.4 | 3.2 |

Therefore, the energy (momentum) resolution can be estimated from the width of the distribution of the total number of recorded Cherenkov photons. For muons and electrons of momentum 300 MeV/c emitted approximately from the tank center, Fig. 2 shows the distributions of the total number of recorded photons. The relative fluctuations obtained in this study for events initiated by electrons and muons having two different momenta are presented in Table 1, along with the estimates of the momentum resolution from [6].

That fluctuations in muon events are much larger than those in events initiated by electrons is quite understandable: muons of energy about 1 GeV lose energy only by ionization; in a muon event, one particle carries the bulk of the energy, and the fluctuations of the total number of recorded photons reflect a wide diversity of possible muon-propagation histories. In an electromagnetic shower, the energy is distributed among many particles, with the result that fluctuations of its features are less pronounced.

As the muon momentum decreases from 1 GeV/c to 300 MeV/c, fluctuations of
the total number of emitted photons increase considerably, which is due to an increase in the relative contribution to Cherenkov radiation from that portion of muon events which experience the greatest fluctuations: the total number of photons depends on the location of the decay vertex along the track and on the energy of the decay electron, which also emits light.

The collection of light by detectors introduces additional uncertainties since only about one-third of the tank-wall area is covered with photomultiplier tubes and since the distribution of photons over this area is governed by the fluctuating spatial and angular distribution of light in the source.

It can be stated that the resolutions in electron events are in reasonable agreement (if we disregard fluctuations of photon transformation into photoelectrons) — that is, the uncertainties obtained in our study are smaller than those in [6]. At the same time, the resolution values for muons disagree, at least for momenta below 1 GeV/c.

There is a characteristic relationship between the total amount of light generated in muon and electron events at identical momenta of primary particles: the number of photons generated by a muon is smaller by $2.8 \times 10^4$ than the number of photons generated by an electromagnetic shower, this being due to the difference in the Cherenkov thresholds. It is necessary to take this fact into account in addressing the problem of event-type identification. It is logical to formulate this problem for events involving identical numbers of recorded Cherenkov photons. For the sub-GeV and GeV energy ranges considered here, this means that one should consider muon events of energy about 200 MeV higher than the energy of electron events.

5.2. Event-Type Identification

In the problems of event-type identification and the reconstruction of event geometry, each event is treated as some random vector $Q = Q_j$ whose components are the contributions (numbers of Cherenkov photons) to all photomultiplier tubes of the setup. Here, $j$ is the photomultiplier index $(j = 1, 2, \ldots, N)$ and $N$ is the number of photomultiplier tubes. In the procedure of identifying the event type, one considers two classes of events: $\omega_1 = e$ (electron) and $\omega_2 = \mu$ (muon). A Monte Carlo simulation of event optical images makes it possible to study the properties of images belonging to both classes — that is, to obtain the image distributions $F(Q_1, Q_2, \ldots, Q_N; \omega_i, E_0, r_0, \theta_0)$, which are the joint distributions of the light contributions $Q_j$ to photomultiplier tubes for the case where the particle type $\omega_i$, the particle energy $E_0$, the particle injection point $r_0$, and the quantity $\theta_0$ specifying the direction of particle motion are preset.

However, it is hardly possible to deal with such functions in practice, because one has to simulate a great event sample in order to obtain a distribution function that involves this extent of differentiation for each set $\omega_i, E_0, r_0, \theta_0$.

In order to construct a more realistic solution to this problem, it would be more appropriate to choose an adequate model of the distribution of the number of Cherenkov photons in an individual detector and to specify such a distribution in each individual
Photomultiplier tube in terms of only the first few of its moments. In the case being considered, these distributions are close to a normal distribution at rather large mean numbers of photons (Fig. 3).

Therefore, we can characterize their classes by the mean vector $\tilde{Q}_j(\omega, E_0, r_0, \theta_0)$ and the covariance matrix $\Sigma_Q(\omega, E_0, r_0, \theta_0) = cov(Q_j, Q_m)$ and treat the joint distributions of the contributions $Q_j$ as multidimensional normal distributions:

$$ p(Q; \omega, E_0, r_0, \theta_0) = (2\pi)^{-N/2} \left| \det \Sigma^{-1}_\omega \right| \cdot \exp \left\{ -\left( Q - \tilde{Q}_\omega \right)^T \Sigma^{-1}_\omega \left( Q - \tilde{Q}_\omega \right) \right\}$$

Here $\tilde{Q}_\omega = \tilde{Q}(\omega, E_0, r_0, \theta_0)$, $\Sigma_\omega = \Sigma_Q(\omega, E_0, r_0, \theta_0)$.

Yet, some event sample is required for calculating $\tilde{Q}$ and $\Sigma_Q$ in this case, but its size can be several orders of magnitude smaller than that in the general case, about some tens of events. Within our approach, it is assumed that the mean vector of an image, $\tilde{Q}$, and the vector of fluctuations, $\delta Q$, can be calculated with the aid of approximations

† In real cases, the distributions of features used for classification bear most often much less resemblance to normal distributions than the distributions in Fig. 3, but this does not prevent their approximation by a normal distribution for multidimensional classification. As a matter of fact, it is necessary that the distribution density for the features being considered have one maximum, not overly large asymmetry, and two first momenta. With increasing dimensionality of the feature vector, higher order moments become progressively less significant.
of the mean values and fluctuations of the angular distributions of Cherenkov light. In
electron (muon) events, typical values of the correlation coefficients change from 0.6
(0.8) for neighboring cells to 0.1 (0.1) for distant ones. As a result, the covariance
matrix proves to be close to a diagonal matrix (more precisely, to a sparse matrix):
from the outset, we treat the correlation coefficients of about 0.1 as vanishing ones,
whereupon each photomultiplier tube has only four neighbors having nonzero correlation
coefficients. We performed test calculations with such a covariance matrix, as well as
with a diagonal matrix. The results demonstrated an insignificant difference in the
quality of event-type identification for these two forms of matrices. Therefore, we can
neglect any correlations between $Q_j$ to simplify the form of the probability density

corresponding to a multidimensional normal distribution.

$$p(\bar{Q}; \omega_i, E_0, r_0, \theta_0) = (2\pi)^{-N/2} \cdot \left( \prod_{j=1}^{N} \delta Q_{j}^{\omega_i} \right)^{1/2} \cdot \exp \left\{ -\sum_{j=1}^{N} \frac{(Q_j - Q_j^{\omega_i})^2}{\delta Q_{j}^{\omega_i}} \right\} , \quad (2)$$

For given event geometry specified by the injection point $r_0$ and particle direction
$\theta_0$ and given energy $E_0$, we calculated the image patterns for the mean value,
$Q_j^{\omega_i} = Q_j^{e,\mu}(E_0, r_0, \theta_0)$, and for fluctuations,
$\delta Q_j^{\omega_i} = \delta Q_j^{e,\mu}(E_0, r_0, \theta_0)$, by the formulae

$$Q_j^{e,\mu}(E_0, r_0, \theta_0) = \sum_{k=1}^{n} \frac{S}{D_j^{2}} \cdot \cos \chi_{j,k} \cdot \exp \left( -\frac{D_{j,k}}{\lambda_{abs}} \right) \cdot F_{k}^{e,\mu}(\theta_{j,k}) , \quad (3)$$

$$\delta Q_j^{e,\mu}(E_0, r_0, \theta_0) = \sum_{k=1}^{n} \left[ \frac{S}{D_j^{2}} \cdot \cos \chi_{j,k} \cdot \exp \left( -\frac{D_{j,k}}{\lambda_{abs}} \right) \right]^2 \cdot \left[ F_{k}^{e,\mu}(\theta_{j,k}) \cdot \delta_{k}^{e,\mu}(\theta_{j,k}) \right]^2 , \quad (4)$$

where $k$ is the segment index; $n$ is the number of segments in the track (shower);
$S$ is the area of the circle representing a photomultiplier; $D_{j,k}$ is the distance from
the segment center to the photomultiplier center; $\cos \chi_{j,k}$ is the cosine of the angle between
the vector $\mathbf{D}_{j,k}$ and the photomultiplier axis, which is normal to the tank surface; $\theta_{j,k}$ is
the emission angle [that is, the angle between the track (shower) axis and the vector $\mathbf{D}_{j,k}$; and $\lambda_{abs}$ is the light absorption length in water. This formula for fluctuations suggests
the absence of correlations between the contributions of individual segments, this being
close to the actual situation since, in the samples of simulated electron events, the
absolute values of the correlation coefficients do not exceed 0.4 for neighboring segments
and are about 0.1 for more distant segments. In the muon events, typical absolute values
of the correlation coefficients are still lower: they are about 0.1 even for neighboring
segments; this may not be so only for the last segments of the track: radiation from
them is 100 times less intense than from other segments, but it is correlated because of
muon decay.

Once the typical features of the classes have been determined, we can formulate
a statistical test for identifying the event type. We use the Bayes decision rule, which
minimizes the decision error \[8\]. Under the assumption that the a priori probabilities for the electron and muon arrival are equal to each other (this corresponds approximately to the expected relation between the fluxes of these events), the ratio of the conditional probabilities for the electron and muon arrival in the case of recording the image \(Q\) can be represented in the form

\[
r = \frac{P(e/Q)}{P(\mu/Q)} = \frac{p(Q/e)}{p(Q/\mu)} = \left(\prod_{j=1}^{N} \frac{\delta Q^e_j}{\delta Q_j^e}\right)^{1/2} \cdot \exp \left\{ -\sum_{j=1}^{N} \left(\frac{Q_j^e - Q_j^e}{\delta Q_j^e}\right)^2 \right\},
\]

where \(p(Q/e)\) and \(p(Q/\mu)\) come from Eq.(2).

The simplest criterion used to identify the event type is

\[
q = 2ln r = q_\mu - q_{el} + C,
\]

where

\[
q_\mu = \sum_{j=1}^{N} \left(\frac{Q_j^{exp} - Q_j^\mu}{\delta Q_j^\mu}\right)^2,
\]

\[
q_{el} = \sum_{j=1}^{N} \left(\frac{Q_j^{exp} - Q_j^e}{\delta Q_j^e}\right)^2,
\]

\[
C = \frac{1}{2} ln \left(\frac{\prod_{j=1}^{N} \delta Q_j^e}{\prod_{j=1}^{N} \delta Q_j^\mu}\right),
\]

where \(Q_j^{exp}\) is the light contribution to the \(j\)th photomultiplier in the experimental image being considered. The event is assumed to be of the \(e\) or \(\mu\) type if \(q > 0\) or \(q < 0\), respectively; in the case of \(q = 0\), the event is rejected.

A somewhat more general form of the criterion can reduce the decision error in event-type identification. It can be taken in the form

\[
q = q_\mu - Aq_{el} + B,
\]

where \(A\) and \(B\) are tuned to minimize the identification errors. The application of this statistical test is similar to the application of the test used above.

Figure 4 shows typical distributions of \(q_\mu - q_e\) for simulated electron and muon events characterized by energies of 300 MeV and 500 MeV, respectively. The optimum boundary between the electron and muon samples for the first statistical test is indicated by the vertical line (that is, \(C\) is determined in this case by minimizing the identification error rather than by using the respective formula). The error associated with misidentifying an electron as a muon, as well as the error associated with misidentifying a muon as an electron, is about 10%. An analysis of events in the long tail of the muon distribution revealed that \(q_\mu - q_e\) values for muons that are
similar to those in $e$ events are due to decay electrons, which generate an isotropic mean distribution of light. The error of event-type identification can be reduced by using the light angular distribution in the range $0^\circ - 180^\circ$ rather than in the range $0^\circ - 90^\circ$ to calculate the pattern image. Figure 5 shows the $q_\mu - q_e$ distributions for the same event samples as in Fig. 4, but, in calculating the pattern image, the mean angular distributions and fluctuations of light emitted into the backward hemisphere are assumed to be constant and equal to their values at $\theta = 90^\circ$ rather than ignored, as was done in the first case. It can be seen that the distributions of the two event classes are separated much better in this case: after optimization, the error associated with misidentifying an electron as a muon becomes as small as a few tenths of a percent, while the error associated with misidentifying a muon as an electron proves to be 2 to 3%.

The application of the more general criterion permits us to improve even this result (Fig. 6). The criterion (7) optimized throughout the sub-GeV range, which embraces $e$ events at 300 MeV ($\mu$ events at 500 MeV), $e$ events at 500 MeV ($\mu$ events at 700 MeV), and $e$ events at 700 MeV ($\mu$ events at 900 MeV) makes it possible to reduce both identification errors to fractions of a percent.

Thus, the upper limits obtained for the errors in event-type identification are in reasonable agreement with the errors estimated by the SuperKamiokande Collaboration for sub-GeV energies [6]: 0.5% and 1.0% for, respectively, electron and muon events.

The above results illustrate the application of the statistical test to events characterized by the lowest of the energies considered here, this case being the most complicated for classification. The difference in the Cherenkov images for muons and electrons becomes more pronounced with increasing energy, this facilitating event-type
Figure 5. Typical distributions in $q_{\mu} - q_e$ criterion applied to artificial 300 MeV electron events and 500 MeV muon events. Cherenkov light angular distribution approximated in the range $0^\circ - 180^\circ$.

Figure 6. Correlation plot $q_{\mu} - q_e$ for artificial 300 MeV electron events (diamonds) and 500 MeV muon events (squares). Cherenkov light angular distribution approximated in the range $0^\circ - 180^\circ$. The green line represents a simple criterion shown in Fig 5, while the black one shows the best criterion.
identification.

6. Reconstruction of Event Geometry

6.1. TDC procedure

The SK analysis introduces a “TDC procedure” to determine the vertex position. The principle of the TDC procedure is to find the position where the time residuals, \( t_i \), for the PMTs being fitted are minimized. The time residual \( t_i \) of the \( i \)-th PMT is defined as

\[
\begin{align*}
  t_i &= t_i^0 - \frac{n}{c} \times \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}, \\
  &\quad \text{(8)}
\end{align*}
\]

where \( t_i^0 \) is the hit time of the \( i \)-th PMT, \((x_i, y_i, z_i)\) is the position of the \( i \)-th PMT, \((x, y, z)\) is the effective emitting point position and \( c/n \) is the velocity of the Cherenkov light in water. That all the light is emitted from the same effective point in space and comes to \( j \)-th PMT exactly at the moment corresponding to the mean time \( \bar{t}_j \) of the \( j \)-th PMT Cherenkov pulse is not really true. A simple equation for time residual used in a \( \chi^2 \)-like sum (the system of linear equations for the effective point coordinates \((x^*, y^*, z^*)\) is overdefined!) can give effective point estimate after the sum’s minimization (with respect to \((x^*, y^*, z^*)\)) even if the original assumption is not valid. The effective point thus deduced does not coincide with the center-of mass of the light emitting system (e-shower or \( \mu \)-track) because of specific mechanism of Cherenkov pulse formation and will usually differ from the event starting (injection) point.

The TDC procedure based only on \( \bar{t}_j \) cannot estimate the event direction because to define a direction one needs at least two points. Direction estimates could be obtained as a result of Cherenkov pulse shape analysis for each sufficiently illuminated optical module if the PMT and electronics are fast enough for such analysis.

Sakai shows the time residual distribution of typical event (a 1 GeV/c, electron) which is distributed over 50 nanoseconds (Sakai, p.38 [5]), assuming a point-like source. We simulate the Cherenkov light in the cascade shower using GEANT 3.21 and the tools we have developed. In Figure 7 we give one example for the time residual distribution for a 1 GeV primary electron based on Eq.(8) with the use of the detailed simulation of the cascade shower. In our calculation, we simulate shower particles and the accompanying Cherenkov light due to shower particle concerned. Then, we know the starting point of the primary electron. Shifting the starting point from the real point to as range of artificial ones, we can obtain the time residual distribution for each position, and examples are given in Figure 7.

Among the five different starting points, which includes the true one, the smallest standard deviation is obtained in the case of \( Z' = Z + 100 \) (cm), where \( Z' \) and \( Z \) denote the assumed vertex point and the real vertex point, respectively. Thus, the apparently most probable vertex point is not real one, but is offset by 100 cm from the real one. Of course, this is only one example and not average behavior. However we examined
many individual cases and confirm that this is usual character which should hold even for the average behavior.

The comparison of our simulations with the experimental data from the SK experiment (Sakai) reveals a large difference. The width of our time residuals distribution is within one nanosecond, while the width for SK is $\sim 50$ ns. The reasons are that we have not considered the PMT and electronics response functions, and that we have neglected light scattering.

We calculate the time residuals for electrons of 500 MeV, 1 GeV, 3 GeV and 5 GeV, assuming that all Cherenkov light comes from certain points of a shower/track. From these calculations one can see that the smallest time residuals do not give the vertex position but yield points shifted from the vertex point along the direction of the cascade shower, namely, 50 to 100 cm for 500 MeV electrons, 100 to 150 cm for 1 GeV (Figure 7) and 3 GeV electrons, and 150 to 200 cm for 5 GeV electrons. Such a tendency is quite understandable if we consider the size of the shower/track: the effective point should not be too close to the starting or ending points. The error of effective point location by minimizing the width of the distribution amounts to about 50 cm.

From the much larger width of the SK time residual distribution it is clear that in experimental conditions the effective point location error should be a few times greater because the minimum of the width as a function of effective point position would be much less pronounced.

For reasons mentioned above, we conclude that the SK TDC procedure is not
suitable for the determination of an accurate vertex position for electron events. The situation for the muon events is essentially the same but must be worse than in the case of electron events as muon events have a longer extent than the corresponding electron events.

6.2. ADC procedure

6.2.1. Procedure for Geometry Reconstruction Within the procedure used to reconstruct event geometry, it is assumed that the type of an event and its energy are known. Thus, one simultaneously seeks only the injection point \( r_0 \) and the direction \( \theta_0 \) of primary particle.

At the first step of the procedure, ring-shaped (arc-shaped) structures are sought in the optical image. In the case of a muon event, two ring-shaped or spot-shaped structures are observed since an electron produced in muon decay also emits Cherenkov light. We did not consider images from muons of momentum below 500 MeV/c and therefore selected the more intense of the two structures. Two-dimensional (or column-by-column) scanning in order to find the maximum above some threshold in the number of Cherenkov photons.

At the second step, the first \( r_1 \) and \( \theta_1 \) approximations to the geometric event parameters were determined by approximating the ring-shaped structure by the following simple conelike model of the event. The entire amount of light \( Q_{\text{tot}} \) emitted by a \( \mu \) track (e shower) is assumed to originate from a single point \( W \) on the track (shower axis) and to have an angular distribution of the form

\[
F(\theta) = \begin{cases} 
0, & \theta < \bar{\theta} - \Delta \theta \\
\alpha, & |\theta - \bar{\theta}| \leq \Delta \theta \\
0, & \theta > \bar{\theta} + \Delta \theta
\end{cases}
, \quad \alpha : \frac{2\pi}{\pi} \int_0^\pi F(\theta) \sin \theta d\theta = Q_{\text{tot}} \quad (9)
\]

In this case, a zero-order approximation can be arbitrary since a ring-shaped structure is usually quite distinct, while the conelike model of the Cherenkov light distribution has sharp edges. The approximation is performed by means of a numerical minimization of the following function with respect to the variables \( r \) and \( \theta \):

\[
G(r, \theta) = \sum_{l=1}^M \left( \frac{Q_{\text{expt}}(r_0, \theta_0) - Q_l(r, \theta)}{Q_{\text{expt}}(r_0, \theta_0)} \right)^2, \quad (10)
\]

Here, \( l \) is the photomultiplier index within the ring-shaped structure; \( M \) is the number of photomultipliers in the ring-shaped structure; \( Q_{\text{expt}}^l \) is the light contribution to the \( l \)th photomultiplier tube from the event being considered; and \( Q_l(r, \theta) \) is the estimate of this contribution according to the calculation within the above cone-like model \( F(\theta) \) of the light angular distribution,

\[
Q_l(r, \theta) = \frac{S}{D_{l,W}^2} \cdot \cos \chi_{l,W} \cdot \exp \left( -\frac{D_{l,W}}{\lambda_{\text{abs}}} \right) \cdot F(\theta_{l,W}), \quad (11)
\]
Figure 8. Error distribution for the vertex position for 300 MeV electrons injected at the scaled WUS point. The sample volume is 100.

Figure 9. Error distribution for the vertex position for 500 MeV muons injected at the scaled WUS point. The sample volume is 100.
The third (last) step of the procedure consists in improving the estimates of the geometric parameters by approximating the whole image (including only the contributions above some threshold $Q_{thr}$) by the pattern image calculated for the corresponding event class, $Q_{j}^{e,\mu}(E_0, r, \theta)$, within a detailed model of the Cherenkov light angular distribution as was described in the preceding section. The first approximation obtained at the second step of the procedure is used as the zero-order approximation for the approximation being considered. Specifically, one performs a numerical minimization of the following function with respect to $r$ and $\theta$:

$$H(r, \theta) = \sum_{j: Q_j^{expt} \geq Q_{thr}} \left( \frac{(Q_j^{expt}(E_0, r_0, \theta_0) - Q_j(E_0, r, \theta))^2}{\delta Q_j(E_0, r, \theta)} \right)$$  \hspace{1cm} (12)

An optimum threshold for the Cherenkov contribution to photomultiplier, $Q_{thr, op}$, can be chosen in such a way as to minimize the uncertainty in determining the geometric parameters (the geometric resolution accordingly being maximal in this case). The optimum threshold grows with increasing primary energy. This can be used to improve the resolution in an actual experiment since the energy can be estimated on the basis of the total amount of recorded light.

6.2.2. Results of our Analysis
By using the developed technique we analyze simulated events to determine the error distributions for the vertex position and for the particle direction. Here, we examine the error distributions for 300 MeV electrons and 500 MeV muons, which yield roughly the same amount of Cherenkov light.

In Figure 8 we give the error distribution for the vertex position for 300 MeV electrons for different Cherenkov threshold quantities. “Ring only” denotes that only the information from PMTs whose Cherenkov photons contribute to the Cherenkov ring are used for the estimation of the error. “Full proc., thr=1ph” denotes that information from “ring only” PMTs and also those exceeding 1 Cherenkov photon are utilized for the estimation on the error. For “full proc., thr=5” and “full proc., thr=10”, this latter threshold is raised to 5 and 10 photons, respectively.

In Figure 9 the error distribution for the vertex position for 500 MeV muons are given. It is clear that a wider error distribution is obtained for a “ring only” analysis. A narrower error distribution results from the “full proc, thr=10 ph” algorithm. This is the same as in the case of the electron. However, muons generally have wider error distributions than electrons: the mean error for 500 MeV muons for the vertex determination in the full analysis is 2.9 m while for 300 MeV electrons it amounts to 2 m.

In Figure 10, the error distribution for the direction of the 300 MeV electron is given. As expected, “ring only” gives the largest uncertainty distribution, while “full proc., thr=10ph” has the narrowest error distribution, with a mean error of about 3.7°.

In Figure 11, we give the corresponding distributions for muons. The same trend is seen as for electrons, though the muons have a wider uncertainty distribution. The mean
Figure 10. Error distribution for the direction for 300 MeV electrons injected at the scaled WUS point. The sample volume is 100.

Figure 11. Error distribution for the direction for 500 MeV muons injected at the scaled WUS point. The sample volume is 100.
direction uncertainty in the best case is $4.9^\circ$.

Now, we compare 1 GeV electrons with 1 GeV muons, both of which yield roughly the same quantity of Cherenkov light. In Figure 12, we give the error distribution for

**Figure 12.** Error distribution for the vertex position for 1 GeV electrons injected at the scaled WUS point. The sample volume is 100.

**Figure 13.** Error distribution for the vertex position 1 GeV muons injected at the scaled WUS point. The sample volume is 100.
Figure 14. Error distribution for the direction for 1 GeV electrons injected at the scaled WUS point. The sample volume is 100.

Figure 15. Error distribution for the direction 1 GeV muons injected at the scaled WUS point. The sample volume is 100.
the vertex position for 1 GeV electrons in the case of “full proc,thr=10”. The mean error is 1.9 m. In Figure 13, the corresponding quantities for the muon are plotted. The average error for the vertex position is 3.2 m. Again, the error of the vertex point for muons is larger than for electrons.

In Figure 14, the error distribution for the direction for 1 GeV electron is shown. The average direction error is 3.0° for “full proc,thr=10”. The corresponding quantities for the muon are plotted in Figure 15, where the mean error is 5.3°. Once again, the directional error for muons is larger than that for electrons.

In Table 3, we summarize the error distributions for the vertex points and the direction for both electrons and muons. Both mean errors and root mean square errors are given. Errors are also given for the different criteria, namely, different Cherenkov light threshold. From Figures 8 to 11 and Figures 12 to 15 and Table 3, it should be noticed the following:

(i) Of the different criteria considered, the “ring only” procedure results in the largest error. The reasons are as follows: The concept of the ring structure is essentially fuzzy, both in our procedure and the SK procedure, and information from ring structure is only part of the total information available for the pattern recognition. It is, therefore, natural that the vertex position and directional errors are largest in the “ring only” analysis. The standard SK analysis uses ring structure only, and their errors are amplified by that fact that the analysis ignores fluctuation effects.

(ii) Muons events have larger uncertainties than electron events. For both electrons and muons, the sources for the Cherenkov light are not point-like and have some extent in both cases. Significant errors come from the point-like approximation for electron events.

(iii) The optimal Cherenkov threshold for the third step of geometry reconstruction procedure depends on the primary energy of the particle concerned. For energies less than 1 GeV, third step with 10 photon threshold gives the best results for both muons and electrons among the alternatives considered. For 5 GeV electrons and muons, 20 photon threshold seems to be optimal for the third step.

(iv) The fact that the uncertainties for the determination of the vertex point and direction are rather large comes from the effect of fluctuations, namely the nature of the stochastic process concerned (an electron cascade shower or sequence of muon interactions with the medium). The utility of model developed in this paper, the moving point approximation model, is guaranteed, because it gives mean values and relative fluctuations precisely and takes all necessary geometrical considerations into account correctly. Even if additional errors exist, they should be negligible compared to the uncertainty caused by fluctuations. The rather large errors for the vertex point and the direction obtained by our model could not be reduced substantially, reflecting the essential nature of the physical processes concerned.
Table 2. Mean and standard deviation of the error in the vertex position and the direction due to primary electrons and primary muons. These are given for different criteria for the Cherenkov threshold. Ring proc. denotes errors estimated using the Cherenkov ring only. [1], [5], [10], [20] denote errors estimated by the combination of ring proc. with a Cherenkov photon threshold of 1, 5, 10 and 20 photons, respectively. Alm denotes the mean direction error in degrees. Als denotes the standard deviation for the corresponding mean values. Rm denotes the mean position error in metres. Rs denotes the standard deviation for the corresponding mean values.

| Q_{thr}, threshold | Alm (deg.) | Als (deg.) | Rm (m) | Rs (m) | Alm (deg.) | Als (deg.) | Rm (m) | Rs (m) | Alm (deg.) | Als (deg.) | Rm (m) | Rs (m) | Alm (deg.) | Als (deg.) | Rm (m) | Rs (m) |
|------------------|------------|------------|--------|--------|------------|------------|--------|--------|------------|------------|--------|--------|------------|------------|--------|--------|
| 300 MeV electron |            |            |        |        | 7.2        | 4.2        | 3.88   | 2.91   | 7.8        | 4.7        | 5.71   | 2.57   | 11.7       | 14.4       | 8.21   | 2.57   |
| 500 MeV muon     |            |            |        |        | 4.4        | 2.4        | 2.50   | 1.26   | 9.3        | 3.7        | 5.69   | 2.01   | 3.1        | 3.5        | 2.19   | 2.01   |
| 1 GeV electron   |            |            |        |        | 21.3       | 11.0       | 15.07  | 6.65   | 8.8        | 11.3       | 5.98   | 6.76   | 3.0        | 3.7        | 8.39   | 6.76   |
| 1 GeV muon       |            |            |        |        | 8.5        | 6.7        | 4.68   | 2.52   | 7.3        | 11.1       | 4.25   | 4.92   | 5.3        | 11.1       | 4.92   | 4.92   |
| 5 GeV electron   |            |            |        |        | 2.0        | 5.1        | 15.07  | 6.65   | 2.0        | 5.1        | 1.43   | 2.76   | 2.0        | 5.1        | 2.76   | 2.76   |
| 5 GeV muon       |            |            |        |        | 4.3        | 4.0        | 4.68   | 2.52   | 3.8        | 3.8        | 2.89   | 2.55   | 3.8        | 3.8        | 2.89   | 2.89   |
(v) The SK analyses, according to all published accounts, completely neglect fluctuations and also use point-like approximations for the electron cascade. Moreover, they neglect the scattering effects on muon track geometry. As we showed earlier, their simple approximations distort the mean values in certain parameter domains. The most probable reason for their low error estimates is the fact that they completely neglect fluctuations in the event development. Our results contradict clearly the fine positional resolution of 23 to 56 cm claimed by for the SK analysis (Kibayashi, p.73[9]).

(vi) It should be noticed that errors derived by us are lower limits. As already mentioned, we do not consider the production of photoelectrons in PMTs, and only consider direct Cherenkov photons in our discrimination procedures neglecting the diffusion of Cherenkov photons. If we include these factors in our procedure, then, the actual errors for the vertex position and the direction should be larger than that given here.

7. Summary and Conclusion

(1) Type definition procedure
SK procedure for event type definition is based on oversimplified models of events and is unlikely to give the type definition errors declared by SK. Our procedure, based on much more accurate event models, is potentially capable of enabling the error of less than 1% in type definition.

(2) The SK TDC procedure
The TDC procedure assumes that the Cherenkov light originates from a point, and thus does not determine the vertex position accurately, because the sources for the Cherenkov light have a non-negligible extent. In order to utilize the TDC meaningfully, we should take into account the extent of the source for the Cherenkov light in space and time. Further, ideally we should utilize not only arrival time of the Cherenkov light but also shape of the pulse in the PMTs.

(3) Errors for the vertex point and the direction
Our estimation (Figures 15 to 18 and Figures 19 to 22 and Table 3) shows non-negligible and inevitable errors for the vertex position and direction. It should be, particularly, noticed that the fluctuations in error in both vertex position and direction are rather big.

Kibayashi (p.73[9]) concludes that the uncertainty for vertex point is from 23 cm to 56 cm and the uncertainty for the direction from 0.9° to 3.0° using both the estimator for particle identification and the TDC adopted by the SK. As we have demonstrated, these appear to severely underestimate the error distributions, which is too far from reality.

(4) In the present paper, we do not take into consideration photoelectrons produced
by the Cherenkov light in the PMT, and we neglect scattering of the Cherenkov light. Therefore, our results on discrimination of electron events from muon events, only yield lower limits to the realistically achievable experimental errors.

On the basis of a statistical simulation of electron and muon events in a water Cherenkov telescope close in parameters to the SuperKamiokande telescope, we have constructed realistic models of actual events. Relying on these models, we have developed algorithms for event type identification and event geometry reconstruction. Although we did not aim at developing elaborate procedures for this telescope precisely, our calculations allowed us to obtain upper limits on the resolutions in energy, event type, and geometric parameters for telescopes of this class. The observed discrepancies — first and foremost, the fact that the geometric resolution obtained by the SuperKamiokande Collaboration is higher than our estimates — call for future investigations.

Construction of optimal experimental algorithm for SK data is beyond the scope of this paper and is better undertaken by those with a detailed understanding of the specifics of the detector.

A part of the results of the present paper are found in [10].

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