A priori bounds for degenerate and singular evolutionary partial integro-differential equations

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Abstract

We study quasilinear evolutionary partial integro-differential equations of second order which include time fractional p-Laplace equations of time order less than one. By means of suitable energy estimates and De Giorgi’s iteration technique we establish results asserting the global boundedness of appropriately defined weak solutions of these problems. We also show that a maximum principle is valid for such equations.

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1 Introduction and main result

Let $T > 0$, and $Ω$ be a bounded domain in $\mathbb{R}^N$. In this paper we are concerned with global a priori bounds for weak solutions of quasilinear problems of the form

\begin{equation}
\partial_t \left( k \ast (u - u_0) \right) - \text{div} \, a(t, x, u, Du) = b(t, x, u, Du), \quad t \in (0, T), \, x \in Ω,
\end{equation}

where $Du$ stands for the gradient of $u$ w.r.t. the spatial variables, $k \in L_{1, \text{loc}}(\mathbb{R}_+) \;$ is a singular kernel, and $k \ast v$ denotes the convolution on the positive halfline w.r.t. the time variable, that is

$$(k \ast v)(t) = \int_0^t k(t - \tau) v(\tau) \, d\tau, \quad t \geq 0.$$ 

We will assume that the kernel $k$ satisfies the following conditions.

(K1) $k$ is of type $\mathcal{PC}$, that is (cf. [24], [20]) $k \in L_{1, \text{loc}}(\mathbb{R}_+)$ is nonnegative and nonincreasing, and there exists a kernel $l \in L_{1, \text{loc}}(\mathbb{R}_+)$ such that $k \ast l = 1$ in $(0, \infty)$.

(K2) $l \in L_q([0, T])$ for some $q > 1$.

An important example is given by

\begin{equation}
k(t) = g_{1-\alpha}(t)e^{-\mu t} \quad \text{and} \quad l(t) = g_\alpha(t)e^{-\mu t} + \mu(1 \ast [g_\alpha(\cdot)e^{-\mu \cdot}](t)), \quad t > 0,
\end{equation}

with $\alpha \in (0, 1)$ and $\mu \geq 0$, see also [24], [20]. Here $g_\beta$ denotes the Riemann-Liouville kernel

\begin{equation}
g_\beta(t) = \frac{t^{\beta - 1}}{\Gamma(\beta)}, \quad t > 0, \; \beta > 0.
\end{equation}

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In this case, (1) amounts to a time fractional equation of order \( \alpha \in (0,1) \). Recall that for a (sufficiently smooth) function \( v \) on \( \mathbb{R}_+ \), the Riemann-Liouville fractional derivative \( D^\alpha_t v \) of order \( \alpha \in (0,1) \) is defined by \( D^\alpha_t v = \frac{d}{dt} (g_{1-\alpha} * v) \).

Letting \( p > 1 \) and \( \Omega_T = (0, T) \times \Omega \), we will further assume that the functions \( a : \Omega_T \times \mathbb{R}^{N+1} \to \mathbb{R} \) and \( b : \Omega_T \times \mathbb{R}^{N+1} \to \mathbb{R} \) are measurable and that they satisfy the structure conditions

(Q1) \[ (a(t,x,\xi,\eta)|\eta) \geq C_0|\eta|^p - c_0|\xi|^\gamma - \varphi_0(t,x), \]

(Q2) \[ |a(t,x,\xi,\eta)| \leq C_1|\eta|^{p-1} + c_1|\xi|^\frac{p-1}{q} + \varphi_1(t,x), \]

(Q3) \[ |b(t,x,\xi,\eta)| \leq C_2|\eta|^{q-1} + c_2|\xi|^{1-q} + \varphi_2(t,x), \]

for a.a. \( (t,x) \in \Omega_T \), and all \( \xi \in \mathbb{R} \), \( \eta \in \mathbb{R}^N \). Here \( C_i, c_i, i = 0,1,2 \), are positive constants, and

(Q4) The parameter \( \gamma \) lies in the range

\[ 1 < \gamma < \frac{1 - \frac{1}{q}}{p(1 - \frac{1}{q})} + \frac{2}{Nq} =: r. \]

(Q5) The functions \( \varphi_i, i = 0,1,2 \), defined on \( \Omega_T \) are nonnegative, \( \varphi_i^{\frac{p}{q}} \in L_1(\Omega_T) \), and \( \varphi_0, \varphi_2 \in L_{s}(\Omega_T) \), where

\[ s > \frac{1}{p} \left(1 - \frac{1}{q}\right) + \frac{1}{q} \left(1 - \frac{1}{q}\right) = \frac{N}{p} + q'. \]

Here, as usual, \( q \) and \( q' \) denote conjugate exponents, i.e. \( \frac{1}{q} + \frac{1}{q'} = 1 \).

The function \( u_0 = u_0(x) \) is a given data and plays the role of the initial data for the function \( u \). We will assume that \( u_0 \in L_2(\Omega) \).

Throughout the paper we will further assume that \( \partial \Omega \) satisfies the property of positive density, see Section 2.

Before describing the main results we give some comments on applications. Problems of the form (1) arise for example in mathematical physics when describing dynamic processes in materials with memory, e.g. in the theory of heat conduction with memory, see \cite{18} and the references therein. Time fractional diffusion equations which are obtained by taking \( k = g_{1-\alpha} \) in (1) are also used to model anomalous diffusion, see e.g. \cite{10}. In this context, these equations are termed subdiffusion equations (the time order \( \alpha \) lies in \((0,1))\); in the case \( \alpha \in (1,2) \), which is not considered here, one speaks of superdiffusion equations. We point out that our general setting also includes models which describe nonlinear diffusion phenomena. An important special case of (1) is the class of time fractional \( p \)-Laplace equations like e.g. (9) below. Let us further mention that time fractional diffusion equations of time order \( \alpha \in (0,1) \) are closely related to a class of Montroll-Weiss continuous time random walk models where the waiting time density behaves as \( t^{-\alpha-1} \) for \( t \to \infty \), see e.g. \cite{12}, \cite{13}, \cite{10}.

We say that a function \( u \) is a weak solution (subsolution, supersolution) of (1) in \( \Omega_T \), if \( u \) belongs to the space

\[ \tilde{V}_{q,p} := \{ v \in L_{2q}([0,T];L_2(\Omega)) \cap L_p([0,T];H^1_q(\Omega)) \text{ such that } \}

\[ k * v \in C([0,T];L_2(\Omega)), \text{ and } (k * v)|_{t=0} = 0 \}, \]
a(t, x, u, Du) and b(t, x, u, Du) are measurable, and for any nonnegative test function

\[ \eta \in \tilde{H}^{-1}_{p, s}(\Omega_T) := \tilde{H}^1_{p, s}(0, T; L_2(\Omega)) \cap L_p(0, T; \tilde{H}^1_{p, s}(\Omega)) \]

with \( \eta|_{t=T} = 0 \) there holds

\[ \int_0^T \int_\Omega \left( -\eta[k \ast (u - u_0)] + (a(t, x, u, Du)| Du) - b(t, x, u, Du)\eta \right) dx \, dt = \langle \leq, \geq \rangle 0. \]  

This definition makes sense, since under conditions (Q1)-(Q5) the integral in (5) is finite, by Hölder’s inequality and the parabolic embedding \( \tilde{V}_{q,p} \hookrightarrow L_r(\Omega_T) \) (see Proposition 2.1 below).

We point out that (H) is considered without any boundary conditions, in this sense weak solutions of (H) as defined above are local ones w.r.t. space. We further remark that weak solutions of (H) in the class \( \tilde{V}_{q,p} \) have been constructed in [24] in the linear case with \( p = 2 \). In view of the basic energy estimate (see below) and the known results in the case \( p = 2 \) the space \( \tilde{V}_{q,p} \) is the natural choice for weak solutions in the general case \( p \in (1, \infty) \). We strongly believe that under stronger assumptions on the nonlinearities \( a \) and \( b \) it is possible to prove the existence of weak solutions of (H) in the class \( \tilde{V}_{q,p} \) by means of the theory of monotone operators and the techniques developed in [24]. Notice also that the initial condition \( u|_{t=0} = u_0 \) has to be understood in a weak sense. One can show (24) that in case of sufficiently smooth functions \( u \) and \( k \ast (u - u_0) \), the condition \( (k \ast u)|_{t=0} = 0 \) implies \( u|_{t=0} = u_0 \).

To state our main results we set \( \Gamma_T = (0, T) \times \partial \Omega \) and \( y_+ := \max\{y, 0\} \). By \( |A| \) we denote the Lebesgue measure of a measurable set \( A \subset \mathbb{R}^N \). When we say that a function \( u \in \tilde{V}_{q,p} \) satisfies \( u \leq K \) a.e. on \( \Gamma_T \) for some number \( K \in \mathbb{R} \) we mean that \( (u - K)_+ \in L_p((0, T]; \tilde{H}^1_{p, s}(\Omega)) \), likewise for lower bounds on \( \Gamma_T \). This convention allows to formulate our results without extra smoothness assumptions on the boundary \( \partial \Omega \). Our main result reads as follows.

**Theorem 1.1** Let \( p > 1, T > 0 \), and \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Let the assumptions (K1),(K2),(Q1)-(Q5) be satisfied.

(i) (Subsolutions) Suppose \( u_0 \in L_2(\Omega) \) and that \( K \geq 0 \) is such that \( u_0 \leq K \) a.e. in \( \Omega \). Then there exists a constant

\[ C = C(N, p, q, C_0, c_0, C_2, c_2, \gamma, s, l|L_q((0, T]), |\varphi_0 + \varphi_1|L_q(\Omega_T), T, |\Omega|) \]  

such that for any weak subsolution \( u \in \tilde{V}_{q,p} \) of (7) in \( \Omega_T \) satisfying \( u \leq K \) a.e. on \( \Gamma_T \) there holds

\[ \text{ess sup}_{\Omega_T} u \leq 2 \left( K + \max \left\{ 1, C \left( \int_0^T \int_\Omega u_+^\gamma \, dx \, dt \right)^{\frac{s}{s-1}} \right\} \right), \]  

where \( r \) is defined in (Q4) and

\[ \theta = \frac{1}{N} (1 - \frac{1}{q}) - \frac{1}{p} \frac{(1 - \frac{1}{q}) + \frac{1}{\gamma}}{1 - \frac{1}{q}}. \]

(ii) (Supersolutions) Suppose \( u_0 \in L_2(\Omega) \) and that \( K \geq 0 \) is such that \( u_0 \geq -K \) a.e. in \( \Omega \). Then there exists a constant \( C \) like in (7) such that for any weak supersolution \( u \in \tilde{V}_{q,p} \) of (7) in \( \Omega_T \) satisfying \( u \geq -K \) a.e. on \( \Gamma_T \) there holds

\[ \text{ess inf}_{\Omega_T} u \geq -2 \left( K + \max \left\{ 1, C \left( \int_0^T \int_\Omega (-u)_+^\gamma \, dx \, dt \right)^{\frac{s}{s-1}} \right\} \right), \]  

\[ (8) \]
Note that $c_1$, $C_1$, and $\varphi_1$, which appear in (Q2), do not play any role in determining the constant in (7) and (8), respectively.

An important special case of (1) is the equation
\[
\partial_\alpha t (u - u_0) - \text{div} \left( \frac{|Du|^{p-2}Du}{p-2} \right) = f \quad \text{in } \Omega_T,
\]
with $\alpha \in (0, 1)$ and $p > 1$. We have the following result.

**Theorem 1.2** Let $\alpha \in (0, 1)$, $p > 1$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $u_0 \in L_\infty(\Omega)$ and that $f \in L_s(\Omega_T)$ with $s > \frac{N}{p} + \frac{1}{\alpha}$. Let further $q > 1$ be a fixed number satisfying
\[
s > \frac{N}{p} + \frac{q'}{N} > \frac{N}{p} + \frac{1}{\alpha}. \tag{10}
\]
Then for any weak solution $u \in \tilde{V}_{q,p}$ of (9) in $\Omega_T$ which is essentially bounded on $\Gamma_T$ there holds
\[
|u|_{L_\infty(\Omega_T)} \leq C(N, p, \alpha, s, |f|_{L_s(\Omega_T)}, T, |\Omega|, \max\{|u_0|_{L_\infty(\Omega)}, \text{ess sup}_{\Gamma_T} |u|\}). \tag{11}
\]

Here the condition on $f$ is sharp, at least in the cases $p = 2$, $\alpha \in (0, 1)$ and $p > 2$, $\alpha \in (p'/N, 1)$, as we will show in Section 5. We also remark that the estimate (11) is stable for $\alpha \to 1$, that is, the constants in the proof remain bounded as $\alpha \to 1$. Hence in this sense we recover well-known results for equations like the classical parabolic $p$-Laplace equation, which can be found in the monograph [7].

In this paper we further prove that in case of so-called homogenou s structures (see Section 6) the weak maximum principle for weak solutions takes the same form as in the classical parabolic case. This applies e.g. to equation (9) with $f = 0$.

In the literature not much seems to be known concerning a regularity theory for weak solutions to (1) in the general setting considered in this paper. To our knowledge, the only paper in this direction is [20], where the global boundedness of weak solutions was proved in the case $p = 2$ under similar assumptions on the kernel $k$ and the nonlinearities $a$ and $b$. On the other hand there exists a rather well developed regularity theory for degenerate ($p > 2$) and singular ($1 < p < 2$) parabolic equations of the form (1) with $\partial_t(k \ast (u - u_0))$ replaced by $\partial_t u$, see the monograph [2] and the references given therein as well as the recent work [8]. This theory includes besides local and global $L_\infty$-bounds also much deeper results such as Harnack and Hölder estimates for weak solutions. For the case $p = 2$ we also refer to [14] and [15]. In the time fractional case the situation is much harder due to the nonlocal nature of $\partial_\alpha t$. Recently, a weak Harnack inequality was proved for nonnegative weak supersolutions of (1) in a special case where $p = 2$ and $k = g_{1-\alpha}$, see [23]. Concerning results in stronger settings for (1) as well as abstract variants of it (mostly with $p = 2$) we refer to [1], [3], [6], [9], [10], [18], [21], [22].

Our proofs of the global $L_\infty$-bounds use De Giorgi’s iteration technique and are based on suitable truncated energy estimates for weak solutions of (1). These estimates are derived by combining the techniques from [20] and [7]. A key ingredient is the basic inequality (14) (see below) for nonnegative nonincreasing kernels. We further adopt the method of time regularization of the equation which goes back to [20] in the weak setting (see also [17]) and uses the Yosida approximations of the operator $B$ defined by $Bv = \partial_t(k \ast v)$, see Section 2.

The paper is organized as follows. In Section 2 we collect some preliminary results such as the basic inequality (14) and we explain the time regularization method in more detail. The
main result is proved in Sections 3 and 4. Section 3 is devoted to the truncated energy estimates and in Section 4 we carry out the iteration process. Section 5 gives the proof of Theorem 1.2. In Section 6 we establish the maximum principle for homogeneous structures, while Section 7 is concerned with the case of natural growth conditions.

2 Preliminaries

We first discuss an important method of regularizing kernels of type $PC$. Let $k,l \in L_{1,loc}(\mathbb{R}_+)$ be as in assumption (K1). For $1 \leq p < \infty$, $T > 0$, and a real Banach space $X$ we consider the operator $B$ defined by

$$ Bu = \frac{d}{dt} (k*u), \quad D(B) = \{ u \in L_p([0,T]; X) : k*u \in \mathcal{O}H^1_p([0,T]; X) \}, $$

where the zero means vanishing at $t = 0$. It is known that this operator is $m$-accretive in $L_p([0,T]; X)$, cf. [2], [5], [10]. Its Yosida approximations $B_n$, defined by $B_n = nB(n+B)^{-1}$, $n \in \mathbb{N}$, enjoy the property that for any $u \in D(B)$, one has $B_n u \to Bu$ in $L_p([0,T]; X)$ as $n \to \infty$. It has been shown in [20] that

$$ B_n u = \frac{d}{dt} (k_n*u), \quad u \in L_p([0,T]; X), \quad n \in \mathbb{N}, $$

where the kernel $k_n$ has the representation

$$ k_n = k + h_n, \quad n \in \mathbb{N}. \quad (12) $$

Here $h_n \in L_{1,loc}(\mathbb{R}_+)$ denotes the resolvent kernel associated with $nl$, that is

$$ h_n(t) + n(h_n*l)(t) = nl(t), \quad t > 0, \quad n \in \mathbb{N}. $$

It is further known that the kernels $k_n$, $n \in \mathbb{N}$, are also nonnegative and nonincreasing, and that in addition they belong to $H^1_1([0,T])$, see e.g. [19], [20], [24].

We also remark that (K1) implies that $l$ is completely positive, see e.g. Theorem 2.2 in [4]. Consequently, $l$ and $h_n$ are nonnegative for all $n \in \mathbb{N}$.

Note further that for any function $f \in L_p([0,T]; X)$, $1 \leq r < \infty$, there holds $h_n*f \to f$ in $L_p([0,T]; X)$ as $n \to \infty$. In fact, setting $u = l*f$, we have $u \in D(B)$, and

$$ B_n u = \frac{d}{dt} (k_n*u) = \frac{d}{dt} (k*l*h_n*f) = h_n*f \to Bu = f \quad \text{in } L_p([0,T]; X) $$

as $n \to \infty$. In particular, $k_n \to k$ in $L_1([0,T])$ as $n \to \infty$.

We next recall a fundamental identity for integro-differential operators of the form $\frac{d}{dt}(k*u)$. Suppose $k \in H^1_1([0,T])$ and $H \in C^4(\mathbb{R})$. Then for any sufficiently smooth function $u$ on $(0,T)$ one has for a.a. $t \in (0,T)$,

$$ H'(u(t)) \frac{d}{dt} (k*u)(t) = \frac{d}{dt} (k*H(u))(t) + \left( -H(u(t)) + H'(u(t))u(t) \right) k(t) $$

$$ + \int_0^t \left( H(u(t-s)) - H(u(t)) - H'(u(t))[u(t-s) - u(t)] \right)[-\dot{k}(s)] ds. \quad (13) $$

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This follows from a straightforward computation, see also [20, 23]. An integrated version of (13) can be found in [11, Lemma 18.4.1].

Equation (13) is highly important for deriving a priori estimates for problems of the form (1). In this paper (see also [20]) we will apply it to the functions

\[ H_+(y) = \frac{1}{2}(y_+)^2 \quad \text{and} \quad H_-(y) = \frac{1}{2}(y_-)^2 \]

defined for \( y \in \mathbb{R} \). Here \( y_- := \min\{y, 0\} \). Evidently, \( H_\pm \in C^1(\mathbb{R}) \) with derivative \( H'_\pm(y) = y_\pm, y \in \mathbb{R} \).

If the kernel \( k \) belongs to \( H^1_1([0,T]) \) and is nonnegative and nonincreasing, then it follows from (13) and the convexity of \( H_\pm \) that for any function \( u \in L^2([0,T]) \),

\[ u(t) \pm \frac{d}{dt} (k \ast u)(t) \geq \frac{1}{2} \frac{d}{dt} \left( k \ast (u_\pm)^2 \right)(t), \quad \text{a.a. } t \in (0,T). \] (14)

The following two lemmas concerning the geometric convergence of sequences of numbers will be needed for the De Giorgi iteration arguments below. The first one is contained, e.g., in [14, Chapter II, Lemma 5.6], see also [7, Chapter I, Lemma 4.1]. Its proof is by induction.

**Lemma 2.1** Let \( \{Y_n\}, n = 0, 1, 2, \ldots \), be a sequence of positive numbers, satisfying the recursion inequality

\[ Y_{n+1} \leq Cb^n Y_n^{1+\alpha}, \quad n = 0, 1, 2, \ldots, \]

where \( C, b > 1 \) and \( \alpha > 0 \) are given numbers. If

\[ Y_0 \leq C^{-1/\alpha} b^{-1/\alpha^2}, \]

then

\[ Y_n \leq C^{-1/\alpha} b^{-1/\alpha^2} b^{-n/\alpha}, \quad n \in \mathbb{N}, \]

in particular \( Y_n \to 0 \) as \( n \to \infty \).

**Lemma 2.2** Let \( \{Y_n\}, n = 0, 1, 2, \ldots \), be a sequence of positive numbers, satisfying the recursion inequality

\[ Y_{n+1} \leq Cb^n \left( Y_n^{1+\alpha} + Y_n^{1+\delta} \right), \quad n = 0, 1, 2, \ldots, \]

where \( C, b > 1 \) and \( \delta \geq \alpha > 0 \) are given numbers. If

\[ Y_0 \leq (2C)^{-1/\alpha} b^{-1/\alpha^2}, \]

then

\[ Y_n \leq (2C)^{-1/\alpha} b^{-1/\alpha^2} b^{-n/\alpha}, \quad n \in \mathbb{N}, \]

and thus \( Y_n \to 0 \) as \( n \to \infty \).

**Proof.** The assertion follows directly from the proof of the previous lemma and the trivial estimate

\[ Y_{n+1} \leq 2Cb^n Y_n^{1+\alpha}, \]

which holds whenever \( Y_n \leq 1 \), due to the assumption \( \delta \geq \alpha \).

We conclude this preliminary part with an interpolation result which will be frequently used in this paper.
Let \( T > 0 \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). For \( p, q \geq 1 \) we define the spaces
\[
V_{q,p} := V_{q,p}([0,T] \times \Omega) = L_{2,q}([0,T]; L_2(\Omega)) \cap L_p([0,T]; H^1_p(\Omega)),
\]
and
\[
V_{q,p}^0 := V_{q,p}^0([0,T] \times \Omega) = L_{2,q}([0,T]; L_2(\Omega)) \cap L_p([0,T]; \dot{H}^1_p(\Omega)),
\]
both equipped with the norm
\[
\|u\|_{V_{q,p}([0,T] \times \Omega)} := \|u\|_{L_{2,q}([0,T]; L_2(\Omega))} + \|Du\|_{L_p([0,T]; H^1_p(\Omega))}.
\]
We will assume that \( \partial \Omega \) satisfies the property of positive density, i.e. there exist \( \delta \in (0,1) \) and \( \rho_0 > 0 \) such that for any \( x_0 \in \Gamma \), any ball \( B(x_0, \rho) \) with \( \rho \leq \rho_0 \) we have that \( |\Omega \cap B(x_0, \rho)| \leq \delta |B(x_0, \rho)| \), cf. e.g. [7, Section I.1].

**Proposition 2.1** There exists a constant \( \tilde{C} = \tilde{C}(N, p, q) \) such that for every \( u \in V_{q,p}^0([0,T] \times \Omega) \) there holds
\[
\int_0^T \int_\Omega |u(t,x)|^r \, dx \, dt 
\leq \tilde{C}^r \left( \int_0^T \int_\Omega |Du(t,x)|^p \, dx \, dt \right)^{\frac{\beta \hat{r}}{p}} \left( \int_0^T \left( \int_\Omega |u(t,x)|^2 \, dx \right)^q \, dt \right)^{\frac{(1-\beta)\hat{r}}{pq}},
\]
where
\[
r = \frac{1 - \frac{1}{q} + \frac{2}{N}}{\frac{1}{p} \left( 1 - \frac{1}{q} \right) + \frac{2}{Nq}} \quad \text{and} \quad \beta = \frac{1 - \frac{1}{q} + \frac{2}{N}}{1 - \frac{1}{q} + \frac{2}{N}} \in [0,1].
\]

**Proof.** We proceed similar as in [7, Chapter I]). We first consider the case where \( p > 2N/(N+2) \). By the Gagliardo-Nirenberg inequality (see e.g. [7, Theorem 2.1]), we have for a.a. \( t \in (0,T) \)
\[
|u(t,\cdot)|_{L_r(\Omega)} \leq C_1(N,p)|Du(t,\cdot)|^\beta_{L_p(\Omega)} |u(t,\cdot)|^{1-\beta}_{L_2(\Omega)},
\]
where \( \beta \) and \( r \) are given by (18). In fact, a short computation shows that (18) implies that
\[
\beta = \frac{\frac{1}{2} - \frac{1}{q} + \frac{2}{N}}{\frac{1}{2} - \frac{1}{q} + \frac{2}{N}},
\]
which corresponds to condition (2.2) in [7, Theorem 2.1]. Taking the \( r \)th power in (19), integrating over \((0,T)\), and using Hölder’s inequality yields
\[
\|u\|_{L_r([\Omega \times (0,T)])} \leq C_1^r \|Du\|_{L_p(\Omega)}^{\beta \hat{r}} \|u\|_{L_r([\Omega \times (0,T) ; L_2(\Omega)])}^{(1-\beta)\hat{r}}, \quad \hat{r} = \frac{(1-\beta)rp}{p-\beta r},
\]
One verifies that \( \hat{r} = 2q \), and so (17) is valid.
If \( p \leq 2N/(N+2) \), then we have in particular \( p < N \), and thus by [7, Corollary 2.1]
\[
|u(t,\cdot)|_{L_{N/p}(\Omega)} \leq C_2(N,p)|Du(t,\cdot)|_{L_p(\Omega)}, \quad \text{a.a.} \ t \in (0,T).
\]
Using this and the fact that
\[ r(1 - \beta) \frac{Np}{Np - (N - p)r\beta} = 2 \]
it follows by means of Hölder’s inequality that
\[ \int_0^T \int_\Omega |u|^r \, dx \, dt = \int_0^T \int_\Omega |u|^{2r} |u|^{(1 - \beta)r} \, dx \, dt \]
\[ \leq \int_0^T \left( \int_\Omega |u|^{\frac{Np}{1 - \beta}} \, dx \right)^\frac{2r}{r} \left( \int_\Omega |u|^2 \, dx \right)^{\frac{1 - \beta}{r}} \, dt \]
\[ \leq C_{p,1}^2 \int_0^T \left( \int_\Omega |Du|^p \, dx \right)^\frac{2r}{p} \left( \int_\Omega |u|^2 \, dx \right)^{\frac{1 - \beta}{r}} \, dt. \]

As in the first case we may now apply Hölder’s inequality once more thereby proving (17). □

We remark that \( r \geq 2 \) if and only if \( p \geq \frac{2N}{N+2} \).

3 Energy estimates

The following lemma will be the starting point for all of the a priori estimates derived in this paper. It provides an equivalent weak formulation of (1) where the kernel \( k \) is replaced with the more regular kernel \( k_n \) (\( n \in \mathbb{N} \)) defined in (12). In what follows the kernels \( h_n \), \( n \in \mathbb{N} \), are as in Section 2.

**Lemma 3.1** Let \( p > 1 \), \( T > 0 \) and \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Let the assumptions (K1),(K2),(Q1)-(Q5) be satisfied and assume that \( u_0 \in L^2(\Omega) \). Then \( u \in \bar{V}_{q,p} \) is a weak solution (subsolution, supersolution) of (1) if and only if for every nonnegative function \( \psi \in H^1_0(\Omega) \) one has

\[ \int_\Omega \left( \psi \partial_t [k_n \ast (u - u_0)] + (h_n \ast a(\cdot, x, u, Du)[D\psi] - [h_n \ast b(\cdot, x, u, Du)]\psi \right) \, dx \]
\[ = (\leq, \geq) 0 \quad a.a. \ t \in (0, T), \ n \in \mathbb{N}. \quad (20) \]

**Proof.** The proof is analogous to the proof of Lemma 3.1 in [20]. For the reader’s convenience we repeat it here.

We may restrict ourselves to the subsolution case as the remaining cases can be treated analogously.

The ’if’ part can be seen as follows. Given an arbitrary nonnegative \( \eta \in H^{1,1}_2(\Omega_T) \) satisfying \( \eta|_{t=T} = 0 \), we take in (20) \( \psi(x) = \eta(t, x) \) for any fixed \( t \in (0, T) \), integrate from \( t = 0 \) to \( t = T \), and integrate by parts w.r.t. the time variable. Sending then \( n \to \infty \) yields (17); here we use the approximating properties of the kernels \( h_n \) described in Section 2.

To prove the ‘only–if’ part, we take the test function

\[ \eta(t, x) = \int_t^T h_n(\sigma - t) \varphi(\sigma, x) \, d\sigma = \int_0^{T-t} h_n(\sigma) \varphi(\sigma + t, x) \, d\sigma, \quad t \in (0, T), \ x \in \Omega, \quad (21) \]
with arbitrary \( n \in \mathbb{N} \) and nonnegative \( \varphi \in \dot{H}^{1,1}_2(\Omega_T) \) satisfying \( \varphi|_{t=T} = 0; \eta \) is nonnegative since \( \varphi \) and \( h_n \) are so (see Section 2). Then we have

\[
\eta(t,x) = \int_t^T h_n(t - s)\varphi(s,x) \, ds, \quad \text{a.a. } (t,x) \in \Omega_T.
\]

By Fubini’s theorem, we have

\[
\int_0^T \left( \int_t^T h_n(t-s)\psi_1(s) \, ds \right) \psi_2(t) \, dt = \int_0^T \psi_1(t) \left( \int_t^T h_n(t-s)\psi_2(s) \, ds \right) \, dt,
\]

for all \( \psi_1, \psi_2 \in L_2([0,T]) \). So it follows from \([5]\) and \( k_n = h_n * k \) (c.p. \([12]\)) that

\[
\int_0^T \int_\Omega \left( - \varphi_t[k_n * (u - u_0)] + (h_n * a(\cdot,x,u,Du))D\varphi - [h_n * b(\cdot,x,u,Du)]\varphi \right) \, dx \, dt \leq 0,
\]

for all \( n \in \mathbb{N} \). Observe that \( k_n * (u - u_0) \in \dot{H}_2^1([0,T]; L_2(\Omega)) \). Therefore, integrating by parts and using \( \varphi|_{t=T} = 0 \) yields

\[
\int_0^T \int_\Omega \left( \varphi \partial_t[k_n * (u - u_0)] + (h_n * a(\cdot,x,u,Du))D\varphi - [h_n * b(\cdot,x,u,Du)]\varphi \right) \, dx \, dt \leq 0, \tag{22}
\]

for all \( n \in \mathbb{N} \) and \( \varphi \in \dot{H}_2^1(\Omega_T) \) with \( \varphi|_{t=T} = 0 \). By means of a simple approximation argument, we then see that \((22)\) is valid for any \( \varphi \) of the form \( \varphi(t,x) = \chi_{(t_1,t_2)}(t)\psi(x) \), where \( \chi_{(t_1,t_2)} \) denotes the characteristic function of the time-interval \((t_1,t_2)\), \( 0 < t_1 < t_2 < T \), and \( \psi \in \dot{H}_2^1(\Omega) \) is nonnegative. Relation \((20)\) follows now from the Lebesgue differentiation theorem. \( \square \)

Our proof of the sup-bounds for subsolutions stated above relies on the subsequent truncated energy estimates.

**Proposition 3.1** Let \( p > 1 \), \( T > 0 \) and \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Let the assumptions \((K1),(K2),(Q1)-(Q5)\) be satisfied. Suppose that \( u_0 \in L_2(\Omega) \) is essentially bounded above in \( \Omega \). Then for any weak subsolution \( u \in V_{q,p} \) of \([1]\) with \( \text{ess sup}_{\Gamma_T} u < \infty \) and any \( \kappa \) satisfying the condition

\[
\kappa \geq \tilde{\kappa} := \max\{0, \text{ess sup}_{\Omega} u_0, \text{ess sup}_{\Gamma_T} u\}, \tag{23}
\]

there holds

\[
|u - \kappa|_{L_{2q}([0,T]; L_2(\Omega))} + |D(u - \kappa)|_{L_p([0,T] \times \Omega)} \\
\leq C \left( \int_0^T \int_{A_{\kappa}(t)} u^{\gamma} \, dx \, dt + \left( \int_0^T |A_{\kappa}(t)| \, dt \right)^{\frac{1}{\gamma}} \right)
\]

where

\[
A_{\kappa}(t) = \{ x \in \Omega : u(t,x) > \kappa \}, \quad t \in (0,T),
\]

and the constant \( C = C(N,p,q,C_0,c_0,C_2,c_2,\gamma,s,|l|_{L_\infty([0,T])},|\varphi_0 + \varphi_2|_{L_\infty(\Omega_T)},T,|\Omega|) \).
Proof. Let \( u \in \tilde{V}_{q,p} \) be a weak subsolution of (1) in \( \Omega_T \). Then (20) holds with the '≤' sign for any nonnegative function \( \psi \in H^1_p(\Omega) \). For \( t \in (0,T) \) we choose in (20) the test function \( \psi = u^+_\kappa := (u_\kappa)_+ \), where we set \( u_\kappa := u - \kappa \), and \( \kappa \in \mathbb{R} \) satisfies (23). The resulting inequality can be written as
\[
\int_{\Omega} \left( u^+_\kappa \partial_t (k_n \ast u_\kappa) + (h_n \ast a(\cdot, x, u, Du)|Du^+_\kappa) dx
\leq \int_{\Omega} \left( [h_n \ast b(\cdot, x, u, Du)|u^+_\kappa + u^+_\kappa (u_0 - \kappa)k_n) dx, \quad \text{a.a. } t \in (0,T). \tag{24}\]

By positivity of \( k_n \) and (23),
\[
\int_{\Omega} u^+_\kappa (u_0 - \kappa)k_n dx \leq 0, \quad \text{a.a. } t \in (0,T),
\]
Thanks to (14) we further have
\[
u^+_\kappa \partial_t (k_n \ast u_\kappa) \geq \frac{1}{2} \partial_t \left( k_n \ast (u^+_\kappa)^2 \right), \quad \text{a.a. } (t,x) \in \Omega_T. \tag{25}\]

Using these relations we infer from (24) that for a.a. \( t \in (0,T) \)
\[
\int_{\Omega} \left( \frac{1}{2} \partial_t (k_n \ast (u^+_\kappa)^2) + (h_n \ast a(\cdot, x, u, Du)|Du^+_\kappa) dx \leq \int_{\Omega} \left( [h_n \ast b(\cdot, x, u, Du)|u^+_\kappa dx. \tag{26}\]

We next convolve (20) with the nonnegative kernel \( l \) from assumption (K1), and observe that in view of
\[
k_n \ast (u^+_\kappa)^2 \in \omega H^1_1([0,T]; L_1(\Omega))
\]
and \( k_n = k \ast h_n \) we have
\[
l \ast \partial_t \left( k_n \ast (u^+_\kappa)^2 \right) = \partial_t \left( l \ast k_n \ast (u^+_\kappa)^2 \right) = h_n \ast (u^+_\kappa)^2.
\]
Sending then \( n \to \infty \), and selecting an appropriate subsequence, if necessary, we thus obtain
\[
\frac{1}{2} \int_{\Omega} (u^+_\kappa)^2 dx + l \ast \int_{\Omega} (a(\cdot, x, u, Du)|Du^+_\kappa) dx \leq l \ast \int_{\Omega} b(\cdot, x, u, Du)|u^+_\kappa dx \tag{27}\]
for a.a. \( t \in (0,T) \).

By the structure condition (Q1) we have
\[
\int_{\Omega} \left( a(t, x, u, Du)|Du^+_\kappa) dx = \int_{A_\kappa(t)} (a(t, x, u, Du)|Du) dx \geq \int_{A_\kappa(t)} \left( C_0 |Du|^p - c_0 |u|^{-1} - \tilde{\varphi}_0 \right) dx. \tag{28}\]

Employing (Q3) and Young's inequality we may further estimate
\[
\int_{\Omega} |b(t, x, u, Du)u^+_\kappa| dx \leq \int_{\Omega} \left( C_2 |Du|^{p-1} u^+_\kappa + c_2 |u|^{-1} u^+_\kappa + \tilde{\varphi}_2 u^+_\kappa \right) dx \leq \int_{A_\kappa(t)} \left( \frac{C_0}{2} |Du|^p + C_3 |u|^\gamma + \tilde{\varphi}_2 u^+_\kappa \right) dx, \tag{29}\]

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where the constant $C_3 > 0$ depends only on $C_0, C_2, c_2$, and $\gamma$. From (27), (28), and (29) we infer that for a.a. $t \in (0, T)$

$$\int_\Omega (u_\kappa^+)^2 \, dx + C_0 t \ast \int_\Omega |Du_\kappa^+|^p \, dx \leq 2l \ast F,$$

(30)

where

$$F(t) = \int_{A_\kappa(t)} ((C_3 + c_0)|u|^{\gamma} + \varphi_0 + \varphi_2 u_\kappa^+) \, dx.$$

Dropping the second term in (30), which is nonnegative, and applying Young’s inequality for convolutions yields

$$|u_\kappa^+|_{L_2([0,T];L_2(\Omega))} = |(u_\kappa^+)|_{L_q([0,T];L_1(\Omega))} \leq 2|t|_{L_q([0,T])} |F|_{L_1([0,T])}.$$  

(31)

On the other hand, we may also drop the first term in (30), convolve the resulting inequality with $k$, and use that $k \ast l = 1$, thereby obtaining

$$C_0 |Du_\kappa^+|_{L_p([0,T];L_p(\Omega))} \leq 2|F|_{L_1([0,T])}.$$  

(32)

By Hölder’s inequality and assumption (Q5) we have

$$\int_0^T \int_{A_\kappa(t)} \varphi_0 \, dx \, dt \leq |\varphi_0|_{L_{\eta p}(\Omega_T)} \left( \int_0^T |A_\kappa(t)| \, dt \right)^{\frac{1}{p}}.$$  

(33)

Next, set

$$\eta := \left(\frac{\beta}{p} + \frac{(1 - \beta)}{2}\right)^{-1} = \frac{1 - \frac{1}{p} + \frac{\beta}{p}}{\frac{1}{\beta} \left(1 - \frac{1}{p}\right) + \frac{1}{q}} > 1.$$  

(34)

Then the term involving $\varphi_2$ can be estimated as follows, where we use (Q5), Hölder’s and Young’s inequality, as well as Proposition (2.1)

$$\int_0^T \int_{A_\kappa(t)} \varphi_2 u_\kappa^+ \, dx \, dt \leq |\varphi_2|_{L_\gamma(\Omega_T)} |u_\kappa^+|_{L_{\gamma}(\Omega_T)} \left( \int_0^T |A_\kappa(t)| \, dt \right)^{\frac{\gamma'}{\gamma - \gamma'}} \leq \tilde{C}|\varphi_2|_{L_\gamma(\Omega_T)} |Du_\kappa^+|_{L_\beta(\Omega_T)} |u_\kappa^+|_{L_{\gamma}(\Omega_T)} \left( \int_0^T |A_\kappa(t)| \, dt \right)^{\frac{\gamma'}{\gamma - \gamma'}}$$

$$\leq \varepsilon^\eta |Du_\kappa^+|^\beta_{L_\beta(\Omega_T)} |u_\kappa^+|_{L_{\gamma}(\Omega_T)}^{\eta(1 - \beta)} + \varepsilon^{-\eta''} \tilde{C} \eta'^\eta |\varphi_2|_{L_\gamma(\Omega_T)} \left( \int_0^T |A_\kappa(t)| \, dt \right)^{\frac{\eta''(p - p')}{p'}} \leq \varepsilon^\eta (|u_\kappa^+|^2_{L_2([0,T];L_2(\Omega))} + |Du_\kappa^+|^p_{L_p(\Omega_T)}) + \varepsilon^{-\eta''} \tilde{C} \eta'^\eta |\varphi_2|_{L_\gamma(\Omega_T)} \left( \int_0^T |A_\kappa(t)| \, dt \right)^{\frac{\eta''(p - p')}{p'}}$$

(35)

for every $\varepsilon > 0$.

Combining (31), (33) and choosing $\varepsilon$ such that

$$2\varepsilon^\eta \left( |l|_{L_q([0,T])} + \frac{1}{C_0} \right) = \frac{1}{2}$$

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Proof. The proof is analogous to the previous one. (20) now holds with the ‘and the constant $C$’ where $u$ is like in Proposition 3.1. Replacing $u$ by $-u$ and $A_\kappa(t)$ by $A_\kappa(t)$ the same line of arguments as above yields the asserted estimate. \hfill \Box

The corresponding result for supersolutions reads as follows.

**Proposition 3.2** Let $p > 1$, $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let the assumptions $(K1),(K2),(Q1)-(Q5)$ be satisfied. Suppose that $u_0 \in L^2(\Omega)$ is essentially bounded below in $\Omega$. Then for any weak supersolution $u \in \hat{V}_{q,p}$ of $\{u\}$ with ess inf$_{\Gamma_T} u > -\infty$ and any $\kappa$ satisfying the condition

$$\kappa \geq \hat{\kappa} := -\min\{0, \text{ess inf}_\Omega u_0, \text{ess inf}_\Gamma u\},$$

we have

$$\|u + \kappa\|_{L^2((0,T];L^2(\Omega))}^2 + \|D(u + \kappa)\|_{L^p((0,T];L^p(\Omega))}^p \leq C \left( \int_0^T \int_{A_\kappa(t)} (-u)^\gamma dx dt + \left( \int_0^T |\hat{A}_\kappa(t)| dt \right)^{\frac{1}{\gamma}} \right)$$

where $\hat{A}_\kappa(t) = \{x \in \Omega: -u(t,x) > \kappa\}$, $t \in (0,T)$, and the constant $C$ is like in Proposition 3.1. \hfill \Box

**Proof.** The proof is analogous to the previous one. (20) now holds with the ‘$\geq$’ sign and we take $\psi = -(u + \kappa)_- \geq 0$. Replacing $u$ by $-u$ and $A_\kappa(t)$ by $\hat{A}_\kappa(t)$ the same line of arguments as above yields the asserted estimate. \hfill \Box

## 4 Iterative inequalities

Let $u \in \hat{V}_{q,p}$ be a weak subsolution of $\{u\}$ in $\Omega_T$. Set

$$\kappa_n = \kappa \left( 2 - \frac{1}{2^n} \right), \quad n = 0, 1, 2, \ldots,$$
where $\kappa \geq \max\{\kappa, 1\}$ will be chosen later. We further put

$$Y_n = \int_0^T \int_{A_{\kappa_n}(t)} (u - \kappa_n)^\gamma \, dx \, dt, \quad n = 0, 1, 2, \ldots$$

By Proposition 3.1 we have for all $n = 0, 1, 2, \ldots$

$$|(u - \kappa_{n+1})|_{L^2(0,T;L^2(\Omega))}^2 + |D(u - \kappa_{n+1})|_{L^p([0,T] \times \Omega)}^p \leq C \left( \int_0^T \int_{A_{\kappa_{n+1}}(t)} u^\gamma \, dx \, dt + \left( \int_0^T |A_{\kappa_{n+1}}(t)| \, dt \right)^{\frac{1}{\gamma^*}} \right). \tag{40}$$

To estimate the right-hand side, note first that

$$\int_0^T \left| A_{\kappa_{n+1}}(t) \right| \, dt \leq \int_0^T \int_{A_{\kappa_{n+1}}(t)} \left( \frac{u - \kappa_n}{\kappa_{n+1} - \kappa_n} \right)^\gamma \, dx \, dt \leq \frac{1}{(\kappa_{n+1} - \kappa_n)^\gamma} \int_0^T \int_{A_{\kappa_n}(t)} (u - \kappa_n)^\gamma \, dx \, dt = \frac{2^{\gamma(n+1)}}{\kappa_1} Y_n. \tag{41}$$

Further,

$$Y_n \geq \int_0^T \int_{A_{\kappa_{n+1}}(t)} (u - \kappa_n)^\gamma \, dx \, dt \geq \int_0^T \int_{A_{\kappa_{n+1}}(t)} u^\gamma \left( 1 - \frac{\kappa_n}{\kappa_{n+1}} \right)^\gamma \, dx \, dt \geq \frac{1}{2^{\gamma(n+2)}} \int_{A_{\kappa_{n+1}}(t)} u^\gamma \, dx \, dt.$$

Hence (41) implies that

$$|(u - \kappa_{n+1})|_{L^2(0,T;L^2(\Omega))}^2 + |D(u - \kappa_{n+1})|_{L^p([0,T] \times \Omega)}^p \leq C 2^{\gamma(n+2)} Y_n + C \left( \frac{2^{\gamma(n+1)}}{\kappa_1} \right)^{\frac{1}{\gamma^*}} Y_n^{\gamma^*}. \tag{42}$$

On the other hand, we have by Hölder’s inequality and Proposition 2.1

$$Y_{n+1} = \int_0^T \int_{A_{\kappa_{n+1}}(t)} (u - \kappa_{n+1})^\gamma \, dx \, dt \leq \left( \int_0^T \int_{A_{\kappa_{n+1}}(t)} (u - \kappa_{n+1})^\gamma \, dx \, dt \right)^\frac{\gamma}{\gamma^*} \left( \int_0^T |A_{\kappa_{n+1}}(t)| \, dt \right)^{1-\frac{\gamma}{\gamma^*}} \leq \tilde{C}^\gamma \left( |D(u - \kappa_{n+1})|_{L^p([0,T] \times \Omega)} \right)^{\beta\gamma} \left( |(u - \kappa_{n+1})|_{L^2(0,T;L^2(\Omega))} \right)^{(1-\beta)\gamma} \times \left( \int_0^T |A_{\kappa_{n+1}}(t)| \, dt \right)^{1-\frac{\gamma}{\gamma^*}},$$

where $\tilde{C} = \tilde{C}(N, p, q)$ and $r$ and $\beta$ are given by (18). Recall that

$$\frac{1}{\eta} = \frac{\beta}{p} + \frac{(1-\beta)}{2} = \frac{1}{2} \left( 1 - \frac{1}{q} \right) + \frac{1}{N} \frac{1}{1 - \frac{1}{q} + \frac{1}{r}} < 1.$$
Using (41) and (42) it then follows that

\[ Y_{n+1} \leq \tilde{C} \left[ 2^{\gamma(n+2)}Y_n + C \left( \frac{2^{\gamma(n+1)}}{\kappa^\gamma} \right)^{\frac{1}{\gamma}} Y_n^{\frac{1}{\gamma}} \right]^{1 - \frac{1}{\gamma}} \]

\[ \leq \tilde{C} (2C) \frac{1}{\kappa} Y_n^{1 - \gamma(1 - \frac{1}{\gamma})} 2^{\gamma(n+2)(\frac{1}{\gamma}+1)} \left( \frac{Y_n^{\frac{1}{\gamma}} + Y_n^{\frac{1}{\gamma} + 1}}{Y_n^{\frac{1}{\gamma}} + Y_n^{\frac{1}{\gamma} + 1}} \right) \]

\[ \leq \left( C\tilde{C} 4^{\gamma+2} \right)^{\gamma(1 - \gamma(1 - \gamma))} \left( 2^{\gamma(\gamma+1)} Y_n^{\frac{1}{\gamma} + 1 - \gamma} + Y_n^{\frac{1}{\gamma} + 1 - \gamma} \right). \]

Note that (see also (37))

\[ \gamma \frac{(\frac{1}{\eta} - \frac{1}{r})}{r} > 1 \iff \frac{1}{\eta} > \frac{1}{r} \iff s > \frac{r - \eta}{r - \eta} = \frac{\frac{1}{p}(1 - \frac{1}{q})}{\frac{1}{q} - \frac{1}{q}}. \]

The last condition is satisfied thanks to (Q5). Hence

\[ \alpha := \gamma \left( \frac{1}{\eta} - \frac{1}{r} \right) > 0 \]

as well as

\[ \delta := \gamma \left( \frac{1}{\eta} - \frac{1}{r} \right) \geq \alpha > 0. \]

It follows from Lemma 2.2 that \( Y_n \to 0 \) as \( n \to \infty \), provided that

\[ Y_0 = \int_0^T \int_{\Omega} (u - \kappa)_{+} dx dt \leq \left( \frac{2C \tilde{C} 4^{\gamma(\gamma+2)}}{\kappa^{\gamma(1 - \gamma)}} \right)^{-\frac{1}{\gamma}} \left( 2^{\gamma(\gamma+1)} \right)^{-\frac{1}{\gamma}}, \]

which in turn is certainly satisfied if

\[ \int_0^T \int_{\Omega} u_{+}^{\gamma} dx dt \leq \tilde{C}^{-1} \kappa^{\frac{1}{\gamma}} (1 - \frac{1}{\gamma}), \]

where

\[ \tilde{C} = \left( 2C \tilde{C} 4^{\gamma(\gamma+2)} \right)^{\frac{1}{\gamma}} \left( 2^{\gamma(\gamma+1)} \right)^{\frac{1}{\gamma}}. \]

The conditions (43) and \( \kappa \geq \max\{\tilde{\kappa}, 1\} \) are fulfilled when we set

\[ \kappa := \tilde{\kappa} + \max \left\{ 1, \left( \tilde{C} \int_0^T \int_{\Omega} u_{+}^\gamma dx dt \right)^{\frac{1}{\gamma(1 - \gamma)}} \right\}. \]

Since \( \kappa_n \to 2\kappa \) as \( n \to \infty \) we thus obtain

\[ \text{ess sup}_{\Omega_\tau} u \leq 2\kappa = 2 \left( \tilde{\kappa} + \max \left\{ 1, \left( \tilde{C} \int_0^T \int_{\Omega} u_{+}^\gamma dx dt \right)^{\frac{1}{\gamma(1 - \gamma)}} \right\} \right). \]

Finally a short computation shows that

\[ \frac{\alpha r}{\gamma} = \frac{\frac{1}{\gamma} \left( 1 - \frac{1}{q} \right) - \frac{1}{\gamma} \left[ \frac{1}{p} \left( 1 - \frac{1}{q} \right) + \frac{1}{q} \right]}{\frac{1}{p} \left( 1 - \frac{1}{q} \right) + \frac{1}{q}}. \]

The first part of the theorem is proved.

The second part is proved analogously replacing \( u \) by \( -u \) and \( A_\kappa(t) \) by \( \tilde{A}_\kappa(t) \) in the previous arguments and employing Proposition 3.2. \( \square \)
5 Proof of Theorem 1.2

To prove Theorem 1.2 note first that the kernel \( k = g_{1-\alpha} \) satisfies (K1) and (K2) with \( l = g_\alpha \) and any \( q \in (1, \frac{1}{1-\alpha}) \). In particular, any \( q > 1 \) satisfying condition (10) is admissible.

Let now \( q > 1 \) be fixed such that (10) holds and suppose that \( u \in \tilde{V}_{q,p} \) is a weak solution of (9) in \( \Omega_T \). Fix \( \gamma \in (1, \eta) \), where \( \eta \) is given by (34). Note that \( \eta < r \). Hence we may apply Theorem 1.1 which yields boundedness of \( u \) in \( \Omega_T \) and the estimate

\[
|u|_{L^\infty(\Omega_T)} \leq 2 \left( \max\{|u_0|_{L^\infty(\Omega)}, \sup_{\Gamma_T} |u|\} + \max \left\{ 1, C \left( \int_0^T \int_{\Omega} |u|^\gamma \, dx \, dt \right)^\frac{q}{\gamma} \right\} \right),
\]

where the constant \( C \) depends only on the data.

It remains to derive an a priori bound for the integral term involving \( |u| \) in terms of the data and the quantity \( \kappa := \max\{|u_0|_{L^\infty(\Omega)}, \sup_{\Gamma_T} |u|\} \). To this purpose we write \( |u| = u_+ + (-u)_+ \) and estimate \( |u_+|_{L^\infty(\Omega_T)} \) and \( |(-u)_+|_{L^\gamma(\Omega_T)} \) separately by means of the energy estimates from Propositions 3.1 and 3.2 respectively.

We have

\[
\int_0^T \int_{\Omega} u_+^\gamma \, dx \, dt \leq 2\gamma \left( \int_0^T \int_{\Omega} (u - \kappa)_+^\gamma \, dx \, dt + \kappa \gamma T|\Omega| \right),
\]

and by Propositions 2.1 and 3.1

\[
\int_0^T \int_{\Gamma_T} (u - \kappa)_+^\gamma \, dx \, dt \leq \|u - \kappa\|_{L^\infty(\Omega_T)}\gamma T|\Omega|\left(\frac{\gamma}{\gamma - \gamma' + 1}\right)\left(\frac{\gamma'}{\gamma - \gamma' + 1}\right) \\
\leq \varepsilon \tilde{C}^\gamma \|u - \kappa\|_{L^\infty(\Omega_T)}\gamma T|\Omega|\left(\frac{\gamma}{\gamma - \gamma' + 1}\right)\left(\frac{\gamma'}{\gamma - \gamma' + 1}\right) \\
\leq \varepsilon \tilde{C}^\gamma C \left( \int_0^T \int_{\Omega} u_+^\gamma \, dx \, dt + (T|\Omega|)^\frac{\gamma}{\gamma'} \right) + \varepsilon^{-\frac{\gamma}{\gamma - \gamma'} - 1} (T|\Omega|)^\frac{\gamma}{\gamma - \gamma'},
\]

for all \( \varepsilon > 0 \). Choosing \( \varepsilon \) sufficiently small we get a bound

\[
\int_0^T \int_{\Omega} u_+^\gamma \, dx \, dt \leq C = C(N, p, q, s, |f|_{L^\infty(\Omega_T)}, T, |\Omega|, \kappa).
\]

The bound for \( |(-u)_+|_{L^\gamma(\Omega_T)} \) is obtained analogously. Combining these estimates and (44) proves the assertion of Theorem 1.2.

The condition on \( f \) is sharp, at least in the cases \( p = 2, \alpha \in (0, 1) \), and \( p > 2, \alpha \in (p'/N, 1) \), as we will show in the following.

Suppose \( u \) is a solution of equation (9) with smooth data \( u_0, \) e.g. \( u_0 = 0 \). Then the optimal regularity for \( u \) is determined from the conditions

\[
\partial_t^\alpha (u - u_0) \in L_s(\Omega_T) \quad \text{and} \quad \text{div} \; (|Du|^{p-2} Du) \in L_s(\Omega_T).
\]

In the linear case \( p = 2 \) these lead to the maximal regularity class

\[
u \in Z := H^\alpha_s([0,T];L_s(\Omega)) \cap L_s([0,T];H^2_s(\Omega)),
\]
where \( H^α_s(J;X) \) denotes the vector-valued Bessel potential space of \( X \)-valued functions on the interval \( J \). In fact, the first space is a consequence of the first condition, by well-known properties of the fractional derivation operator, see e.g. [21].

The question is now for which \( s > 1 \) we have \( Z \hookrightarrow \mathcal{L}_∞(\Omega_T) \). This can be determined by means of cross interpolation and Sobolev embeddings. By the mixed derivative theorem (see [17]) we have for all \( \vartheta \in [0,1] \)

\[
Z \hookrightarrow H^{(1-\vartheta)α}_s([0,T];H^{2\vartheta}_s(\Omega)).
\]

Thus \( Z \hookrightarrow \mathcal{L}_∞(\Omega_T) \) if there exists \( \vartheta \in (0,1) \) such that

\[
(1-\vartheta)α > \frac{1}{s} \quad \text{and} \quad 2\vartheta > \frac{N}{s}.
\]

This is equivalent to

\[
\frac{N}{2s} < \vartheta < 1 - \frac{1}{sα}.
\tag{46}
\]

There exists \( \vartheta \in [0,1] \) satisfying (46) if and only if \( s > \frac{N}{2} + \frac{1}{s} \), which is exactly the condition for \( s \) in Theorem 1.2 in the case \( p = 2 \).

We now discuss a nonlinear case, namely let us assume that \( p > 2 \) and \( α \in (p'/N,1) \). Here we are led to the maximal regularity class

\[
u \in Z := H^α_s([0,T];L_s(\Omega)) \cap L_{s(p-1)}([0,T];H^2_α(\Omega)) \quad \text{with} \quad \frac{p-1}{q} = \frac{1}{s} + \frac{p-2}{N}.
\]

To see the second space suppose that \( u \in L^q([0,T];H^2_α(\Omega)) \) satisfies the second condition in (45). Assuming \( \bar{r} < N \) we have \( D_iD_ju \in L^q([0,T];L_r(\Omega)) \) and

\[
D_iu \in L^q([0,T];L^2_α(\Omega)) \hookrightarrow L^{\bar{q}}([0,T];L^{\bar{r}}(\Omega)),
\]

for all \( i,j = 1, \ldots, n \), and so the structure of the \( p \)-Laplace leads to the condition

\[
L_{\frac{q}{p-2}}([0,T];L_{\frac{q}{p-2}rN}(\Omega)) \times L^{\bar{q}}([0,T];L^{\bar{r}}(\Omega)) \subset L_{s}([0,T];L_s(\Omega)).
\]

This means we need

\[
\frac{p-2}{q} + \frac{1}{q} = \frac{1}{s} \quad \text{and} \quad \frac{(N-\bar{r})(p-2)}{\bar{r}N} + \frac{1}{\bar{r}} = \frac{1}{s},
\]

which in turn implies \( \bar{q} = s(p-1) \) and \( \bar{r} = \frac{N}{p} \).

As before we have to determine those \( s > 1 \) for which we have \( Z \hookrightarrow \mathcal{L}_∞(\Omega_T) \). Note first that the conditions \( α > \frac{1}{s} \) and \( 2 > \frac{N}{s} \) are necessary. Assuming this it follows that \( s > \max\{\frac{1}{s}, \frac{N}{s}\} \), in particular we have \( s \geq \frac{N}{p} \) due to \( p > 2 \). By the mixed derivative theorem we then have for all \( \vartheta \in [0,1] \)

\[
Z \hookrightarrow H^{(1-\vartheta)(\frac{1}{s}+\frac{N}{s})}_s([0,T];L_s(\Omega)) \cap L_{s(p-1)}([0,T];H^2_α(\Omega)) \hookrightarrow H^{(1-\vartheta)(\frac{1}{s}+\frac{N}{s})}_s([0,T];H^{\alpha_s}_s(\Omega)).
\]
with sharp embeddings. Hence $Z \hookrightarrow L_\infty(\Omega_T)$ if there exists $\vartheta \in (0,1)$ such that

\[(1-\vartheta)(\alpha - \frac{1}{s} + \frac{1}{s(p-1)}) > \frac{1}{s(p-1)} \quad \text{and} \quad \vartheta\left(2 - \frac{N}{p} + \frac{N_s}{s}\right) > \frac{N_s}{s},\]

which is equivalent to

\[0 < \omega_1 := \frac{\frac{N}{s}}{2 - \frac{N}{p} + \frac{N_s}{s}} < \vartheta < \frac{\alpha - \frac{1}{s} + \frac{1}{s(p-1)}}{\alpha - \frac{1}{s} + \frac{1}{s(p-1)}} =: \omega_2 < 1.\]

Thus it boils down to the condition $\omega_1 < \omega_2$. A short computation shows that this condition is in fact equivalent to $s > \frac{N_p}{p} + \frac{1}{\alpha}$.

Recall that we assumed that $\bar{r} < N$, that is $\hat{r} = \bar{r} < N$. The condition $\hat{r} < N$ in turn is equivalent to $s < N$. Thus $s$ has to satisfy the condition

\[\frac{N}{p} + \frac{1}{\alpha} < s < N,\]

which is possible, by the assumption $\alpha \in (p'/N,1)$. Hence in this case the condition on $f$ is optimal as well.

## 6 Homogenous structures

In this section we consider the special case of homogenous structures. By this we mean equations of the type

\[
\partial_t\left(k \ast (u - u_0)\right) - \text{div} a(t,x,u,Du) = 0, \quad t \in (0,T), \quad x \in \Omega,
\]

where

(HS) \[ (a(t,x,\xi,\eta)|\eta|) \geq C_0|\eta|^p, \quad |a(t,x,\xi,\eta)| \leq C_1|\eta|^{p-1}, \]

for a.a. $(t,x) \in \Omega_T$, and all $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^N$. Here $C_0$ and $C_1$ are positive constants.

In this situation the weak maximum principle takes the same form as in the classical parabolic case. Moreover the assumption (K2) can be dropped, $q = 1$ is here admissible.

**Theorem 6.1** Let $p > 1$, $T > 0$, and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume (K1) and (HS), and let $u_0 \in L_2(\Omega)$. Then for any weak subsolution (supersolution) $u \in \tilde{V}_{1,p}$ of (47), we have for a.a. $(t,x) \in \Omega_T$

\[ u(t,x) \leq \max \left\{ 0, \text{ess sup}_{\Omega} u_0, \text{ess sup}_{\Gamma_T} u \right\} \quad \left( u(t,x) \geq \min \left\{ 0, \text{ess inf}_{\Omega} u_0, \text{ess inf}_{\Gamma_T} u \right\} \right), \]

provided this maximum (minimum) is finite.

**Proof.** Note first that Lemma 3.1 also holds under the assumptions of Theorem 6.1. It suffices to consider the subsolution case. We take

\[ \kappa = \bar{\kappa} = \max \left\{ 0, \text{ess sup}_{\Omega} u_0, \text{ess sup}_{\Gamma_T} u \right\}, \]

(48)
assuming that this quantity is finite, and proceed as in the proof of Proposition 3.1. This yields
\[ |(u - \tilde{\kappa})|^2_{L^2(\Omega_T)} + |D(u - \tilde{\kappa})|^p_{L^p(\Omega_T)} \leq 0, \]
which immediately implies the assertion.

Theorem 6.1 shows in particular that the maximum principle holds in the usual form for weak solutions of the time fractional p-Laplace equation (9) with \( f = 0 \).

We remark that the case \( q = 1 \) can occur. In [20, Section 3] an example is given for a kernel \( k \) satisfying (K1) with \( l \notin L_q([0,T]) \) for all \( q > 1 \) and \( T > 0 \).

7 Natural growth conditions

Finally we consider the case of ‘natural’ or Hadamard growth conditions with respect to \( |Du| \).

For the sake of simplicity we suppose that

\[(Q) \quad (a(t,x,\xi,\eta)\eta) \geq C_0|\eta|^p, \quad |a(t,x,\xi,\eta)| \leq C_1|\eta|^{p-1}, \quad |b(t,x,\xi,\eta)| \leq C_2|\eta|^p, \]

for a.a. \((t,x) \in \Omega_T\), and all \( \xi \in \mathbb{R}, \eta \in \mathbb{R}^N \), where \( C_i, i = 0, 1, 2 \) are positive constants. In the classical parabolic case it is known that weak solutions of the corresponding problem under the conditions (Q) are in general not bounded. However there are results (also in a more general situation) which provide \( L_\infty \) bounds in terms of the data assuming in addition that the weak solution is bounded, see e.g. [7]. It turns out that corresponding results can be obtained for (1). Here we only prove such a result in the case where (Q) holds. It generalizes Theorem 4.3 in [20], where \( p = 2 \) is required. As in the previous section we may drop assumption (K2).

**Theorem 7.1** Let \( p > 1, T > 0, \) and \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Let (K1) and (Q) be satisfied, and suppose that \( u_0 \in L_\infty(\Omega) \). Then for any bounded weak solution \( u \in \bar{V}_{1,p} \) of (1),

\[ |u|_{L_\infty(\Omega_T)} \leq \max \left\{ |u_0|_{L_\infty(\Omega)}, \text{ess sup}_{\Gamma_T} |u| \right\}. \]

**Proof.** We proceed as in the proof of [7, Theorem 17.1], see also [20, Theorem 4.3]. Assume that \( K := \text{ess sup}_{\Omega_T} u > \tilde{\kappa} \), where \( \tilde{\kappa} \) is as in (15). Let \( \varepsilon > 0 \) be such that \( \kappa := K - \varepsilon \geq \tilde{\kappa} \). We then choose the test functions \( u_\kappa^+ = (u - \kappa)_+ \) and estimate similarly as above, using the conditions (Q). This yields

\[ |u_\kappa^+|^2_{L^2(\Omega_T)} + |Du_\kappa^+|^p_{L^p(\Omega_T)} \leq C(C_0,C_2)|Du_\kappa^+|^p_{L^p(\Omega_T)} \]

\[ \leq \varepsilon C(C_0,C_2)||Du_\kappa^+|^p_{L^p(\Omega_T)}. \]

Choosing \( \varepsilon \) sufficiently small, it follows that \( |u_\kappa^+|^2_{L^2(\Omega_T)} \leq 0 \), that is \( u \leq \kappa < K \) a.e. in \( \Omega_T \), a contradiction. Hence \( u \leq \tilde{\kappa} \) a.e. in \( \Omega_T \). The lower bound is obtained analogously.

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