MULTI-OBJECTIVE CONVEX POLYNOMIAL OPTIMIZATION AND SEMIDEFINITE PROGRAMMING RELAXATIONS

LIGUO JIAO, JAE HYOUNG LEE†, AND NITHIRAT SISARAT

Abstract. This paper aims to find efficient solutions to a multi-objective programming problem with convex polynomial data. To this end, a hybrid method, which allows us to transform into a family of scalar convex polynomial optimization problems and does not destroy the properties of convexity, is considered. Then, we show an existence result of efficient solutions to under some mild assumption. Apart from this, we also show that the proposed scalar convex polynomial optimization problem generically possesses a unique optimal solution. In addition, we establish two kinds of representations of non-negativity of convex polynomials over convex semi-algebraic sets, and propose two kinds of finite convergence results of the Lasserre-type hierarchy of semidefinite programming relaxations for the (scalar) convex polynomial optimization problem under suitable assumptions. Finally, we show that finding efficient solutions to can be achieved successfully by solving hierarchies of semidefinite programming relaxations and checking a flat truncation condition.

1. Introduction

The multi-objective programming is used to denote a type of optimization problems, where two or more objectives are to be minimized over certain constraints. In this paper, we are interested in a multi-objective programming problem with convex polynomial data, i.e.,

\[
\text{(MP)} \quad \min_{\mathbb{R}_+^p} f(x) \quad \text{subject to} \quad g_i(x) \leq 0, \; i = 1, \ldots, m,
\]

where \( f(x) := (f_1(x), \ldots, f_p(x)) \), \( f_j : \mathbb{R}^n \to \mathbb{R}, \; j = 1, \ldots, p \), and \( g_i : \mathbb{R}^n \to \mathbb{R}, \; i = 1, \ldots, m \), are convex polynomials, and \( \mathbb{R}_+^p \) stands for the non-negative orthant of \( \mathbb{R}^p \). We denote the...
feasible set of (MP) by
\[ K := \{ x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, \ldots, m \}, \]
which is assumed to be nonempty (not necessarily compact) throughout this paper.

It is worth noting that, in general, there is no single optimal solution that simultaneously 
minimizes all the objective functions. In these cases, one aims to look for the “best preferred” 
solution, in contrast to an optimal solution. This leads to the concept of the so-called efficient 
(or Pareto-optimal) solutions in multi-objective programming. In fact, the efficient solutions 
are the ones that cannot be improved in one objective function without deteriorating their 
performance in at least one of the rest. Now, we state the concept of an efficient solution to 
(MP).

**Definition 1.1.** A point \( \bar{x} \in K \) is said to be an **efficient solution** to (MP) if
\[ f(x) - f(\bar{x}) \notin -\mathbb{R}^p_+ \setminus \{0\}, \ \forall x \in K. \]
In addition, if \( \bar{x} \) is an efficient solution to (MP), \( f(\bar{x}) \in \mathbb{R}^p \) is called a **non-dominated point**.

1.1. **Scalarization methods.** Usually, the problem (MP) can be solved (i.e., its efficient 
solutions be found) by solving related single objective programming problems. We call such 
a method by **scalarization approach**, and there are many types of scalarization approaches, 
for example, the weighted sum method, the \( \epsilon \)-constraint method, and the hybrid method 
(see [5, 6]). Below, we first describe the weighted sum method and the \( \epsilon \)-constraint method, 
and show their powerful aspects for solving some special cases of problem (MP). Then, we 
minutely discuss the hybrid method, which will be used in the paper to solve problem (MP).

**Weighted sum method.** The idea of this method is to associate each objective function with 
a weighting non-negative coefficient and then minimize the weighted sum of the objectives.
This method is a simple way to generate different efficient solutions, however, there are 
parameters as many as the number of objective functions, and this may be not easy to 
control with the parameters if the proposed multi-objective programming problems have a 
great number of objective functions. Further, this method is effective to deal with convex 
cases, but for the nonconvex cases, it may go awry. Besides, for a given non-dominated 
point, it is usually not easy to find a corresponding desired parameter. In other words, it 
may be not easy to set parameters to obtain a non-dominated point from a desired region 
of the image space (i.e., efficient solutions in the feasible set).

Nevertheless, the weighted sum method has been shown to be effective to solve some 
special case of problem (MP), e.g., the involving functions are linear (see Blanco et al. [4]); 
in this case, we call problem (MP) by **multi-objective linear programming** (MLP, for short). 
An MLP has become a very important area of activity within the optimization field since
the 1960s for its relevance in practice and as a mathematical topic in its own right; see [21].
The development of MLPs has come in parallel to the scalar counterpart and its theory
and algorithms have been developed continuously over the years. Among them, Blanco
et al. [4] presented a new method to solve MLPs, and this new method is based on a
transformation of any MLPs into a unique lifted semidefinite programming (SDP); however,
we would emphasize here that before their new method works, the weighted sum method is
used to transform the MLPs into a scalar linear programming problem.

**ε-constraint method.** This method is that one of the objective functions is minimized and
all the other objective functions are transformed into constraints by setting an upper bound
to each of them. Notwithstanding the fact that, in order to find efficient solutions by this
method, each transformed single objective optimization problem, as many as the number
of the objective functions, shall be solved; or the optimal solution of a single objective
optimization problem should be unique. The ε-constraint method was still proved to be
useful to solve some special case of problem (MP), e.g., the involving functions are SOS-
convex polynomial, see [19, 20]. More precisely, in [19, 20], the authors employed the
ε-constraint method to transform this special case of problem (MP) into a class of scalar
ones. Moreover, since the ε-constraint method did not destroy the SOS-convex properties,
along with these facts, an exact SDP approach was used to find the optimal solutions to the
corresponding scalar problems, then efficient solutions to the special case of problem (MP)
can also be found.

**Hybrid method.** In this paper, we are interested in the study of finding efficient solutions
to (MP) with convex polynomial data, and we do this by transforming (MP) to a scalar
one based on the hybrid method. Mathematically speaking, in connection with the problem
(MP), we consider the following (scalar) convex polynomial optimization problem [6, 7]:

\[
(P_z) \quad \bar{f}_z := \min_{x \in \mathbb{R}^n} \quad \lambda^T f(x) := \sum_{j=1}^{p} \lambda_j f_j(x) \\
\text{s.t.} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m, \\
\quad f_j(x) \leq f_j(z), \quad j = 1, \ldots, p,
\]

where \( \lambda \in \text{int} \mathbb{R}_+^p \) is fixed and the parameter \( z \in \mathbb{R}^n \). Note that \( \text{int} \mathbb{R}_+^p \) stands for the interior of \( \mathbb{R}_+^p \). Let

\[
K_z := \{ x \in \mathbb{R}^n : g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad f_j(x) \leq f_j(z), \quad j = 1, \ldots, p \}
\]

\footnote{A polynomial \( f \) is called *SOS-convex* if there exists a matrix polynomial \( F(x) \) such that the Hessian \( \nabla^2 f(x) \) factors as \( F(x)F(x)^T \); see [1, 2, 9].}
be the (possibly non-compact) feasible set of $(P_z)$. It is worth noting that $\lambda$ here does not play the role of a parameter but be fixed in $(P_z)$. It is worth also mentioning that the feasible set $K_z$ is nonempty whenever the parameter $z$ is selected in the feasible set $K$ of $(MP)$.

Actually, the hybrid method can be regarded as the one combining the weighted sum method with the $\epsilon$-constraint method; see, for example [7]. In other words, it does not require solving several problems, consider about uniqueness of an optimal solution to single objective optimization problems, or control weighting non-negative parameters. Even though in the view of computation, this method needs more process and time than the weighted method (since the number of constraints increases), one still may find non-dominated point (i.e., efficient solution(s) in the feasible set) from a desired region of the image space, which can be controlled by $\epsilon$-constraint method, in contrast to the weighted sum method.

1.2. Convex polynomial optimization. As mentioned in the above, the hybrid method is substantially one of the scalarization approaches, thus it is essential to recall some celebrated results in scalar (as opposed to multi-objective) polynomial optimization.

Indeed, if one restricts oneself to polynomial optimization (not necessarily convex), then one may approximate the optimal value and an optimal solution to a polynomial optimization problem as closely as desired, and sometimes solve such a polynomial optimization problem exactly; see [14, 15, 23]. Moreover, polynomial optimization problems have attracted a lot of attention in theoretical and applied mathematics over the years; see, for example, the related monographs [8, 16, 17]. Real algebraic geometry and semi-algebraic geometry are sub-fields in algebra that are strongly related to polynomial optimization problems; see [17]. Since these problems are, in general, very difficult, it is a natural choice to look for tractable relaxations. These relaxations are often based on some variant of a “Positivstellensatz” for given semi-algebraic sets [24, 25, 30]. Many researchers have proposed hierarchies of such relaxations that are based on moment and sum-of-squares approximations of the original problem, and give semidefinite programming (SDP) problems. For instance, under certain conditions, Lasserre [14] proposed a hierarchy of SDP problems that converge with their optimal values to the optimal value of the original polynomial optimization problem, see also [29].

The reasons why we restrict us to convex polynomial data are (i) under convexity, the hierarchy of semidefinite programming relaxations for polynomial optimization simplifies and has finite convergence, a highly desirable feature as convex problems are in principle easier to solve; see Lasserre [15] for more details; (ii) the Lasserre hierarchy of SDP relaxations with a slightly extended quadratic module for convex polynomial optimization problems always converges asymptotically even in the case of non-compact semi-algebraic feasible sets; see, [12]. Moreover, as Jeyakumar et al. [12] pointed out, the positive definiteness of
the Hessian of the associated Lagrangian at a saddle-point guarantees the finite convergence of the hierarchy.

1.3. Our contributions. In this paper, we make the following contributions to a multi-objective programming problem with convex polynomial data.

- First, we establish an existence result for efficient solutions to (MP) under some mild assumption. Apart from this, we show that for each \( \lambda \in \mathbb{R}^p_+ \), the problem \( (P_z) \) generically admits a unique optimal solution.
- Second, we give two kinds of representations of non-negativity of convex polynomials over convex semi-algebraic sets.
- Third, we formulate two kinds of Lasserre-type hierarchies of SDP relaxations for \( (P_z) \) and give finite convergence results for the hierarchy of SDP relaxations.
- Last, under the flat truncation condition, we show how to find efficient solutions to the problem (MP).

The outline of this paper is arranged as follows. In Sect. 2, we recall some notations and celebrated results that will be widely used throughout the paper. We establish an existence result for efficient solutions to (MP) under some mild assumption, and also show that for each \( \lambda \in \mathbb{R}^p_+ \), the problem \( (P_z) \) generically admits a unique optimal solution in Section 3. Section 4 shows two kinds of representations of non-negativity of convex polynomials over convex semi-algebraic sets; moreover, we formulate two kinds of Lasserre-type hierarchies of SDP relaxations for \( (P_z) \) and give their finite convergence results. Under the flat truncation condition, Section 5 provides a way to find efficient solutions to the problem (MP). Finally, the conclusion remarks is stated in Sect. 6.

2. Preliminaries

We begin this section by fixing some notations and preliminaries. We suppose \( 1 \leq n \in \mathbb{N} \) (\( \mathbb{N} \) is the set of non-negative integers) and abbreviate \((x_1, x_2, \ldots, x_n)\) by \( x \). The Euclidean space \( \mathbb{R}^n \) is equipped with the usual Euclidean norm \( \| \cdot \| \). The non-negative orthant of \( \mathbb{R}^n \) is denoted by \( \mathbb{R}^n_+ \).

The space of all real polynomials on \( \mathbb{R}^n \) is denoted by \( \mathbb{R}[x] \). Moreover, the space of all real polynomials on \( \mathbb{R}^n \) with degree at most \( d \) is denoted by \( \mathbb{R}[x]_d \). The degree of a polynomial \( f \) is denoted by \( \deg f \). We say that a real polynomial \( f \) is sum-of-squares if there exist real polynomials \( q_l, l = 1, \ldots, r \), such that \( f = \sum_{l=1}^r q_l^2 \). The set consisting of all sum-of-squares real polynomials is denoted by \( \Sigma[x] \). In addition, the set consisting of all sum-of-squares real polynomials with degree at most \( d \) is denoted by \( \Sigma[x]_d \). Given polynomials \( \{g_1, \ldots, g_m\} \subset \mathbb{R}[x] \), a quadratic module generated by the tuple \( g := (g_1, \ldots, g_m) \) is defined
by
\[ Q(g) := \left\{ \sigma_0 + \sum_{i=1}^{m} \sigma_i g_i : \sigma_i \in \Sigma[x], \ i = 0, 1, \ldots, m \right\}. \]

The set \( Q(g) \) is Archimedean if there exists \( p \in Q(g) \) such that the set \( \{ x \in \mathbb{R}^n : p(x) \geq 0 \} \) is compact.

The following result is a celebrated and important representation of positive polynomials over a semi-algebraic set when the quadratic module \( Q(-g) \) is Archimedean.

**Lemma 2.1** (Putinar’s Positivstellensatz). \[24\] Let \( f \) and \( g_i, i = 1, \ldots, m, \) be real polynomials on \( \mathbb{R}^n. \) Suppose that \( Q(-g) \) is Archimedean. If \( f \) is strictly positive on \( K := \{ x \in \mathbb{R}^n : g_i(x) \leq 0, \ i = 1, \ldots, m \} \neq \emptyset, \) then \( f \in Q(-g). \)

For a multi-index \( \alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, \) let us denote \( |\alpha| := \sum_{i=1}^{n} \alpha_i \) and \( \mathbb{N}^n_d := \{ \alpha \in \mathbb{N}^n : |\alpha| \leq d \}. \) The notation \( x^\alpha \) denotes the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \) The canonical basis of \( \mathbb{R}[x]_d \) is denoted by
\[(1) \quad v_d(x) := (x^\alpha)_{\alpha \in \mathbb{N}^n_d} = (1, x_1, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^2, \ldots, x_1^d, \ldots, x_n^d)^T,
\]
which has dimension \( s(d) := \binom{n+d}{d}. \)

Given an \( s(2d) \)-vector \( y := (y_\alpha)_{\alpha \in \mathbb{N}^n_{2d}} \) with \( y_0 = 1, \) let \( M_d(y) \) be the moment matrix of dimension \( s(d), \) with rows and columns labeled by \( (1) \) in the sense that
\[ M_d(y)(\alpha, \beta) = y_{\alpha+\beta}, \ \forall \alpha, \beta \in \mathbb{N}^n_d. \]

For a polynomial \( x \mapsto p(x) := \sum_{\gamma \in \mathbb{N}^n_w} p_\gamma x^\gamma, \) \( M_d(py) \) is the so-called localization matrix defined by
\[ M_d(py)(\alpha, \beta) = \sum_{\gamma \in \mathbb{N}^n_w} p_\gamma y_{\gamma+\alpha+\beta}, \ \forall \alpha, \beta \in \mathbb{N}^n_d. \]

For a symmetric matrix \( X, X \succeq 0 \) (resp., \( X \succ 0 \)) means that \( X \) is positive semidefinite (resp., positive definite). The gradient and the Hessian of a polynomial \( f \in \mathbb{R}[x] \) at a point \( \bar{x} \) are denoted by \( \nabla f(\bar{x}) \) and \( \nabla^2 f(\bar{x}), \) respectively.

### 3. Existence result and genericity result

In this section, we establish necessary and sufficient conditions for the existence of an efficient solution to the problem \( \text{(MP)} \) and show that, for each \( \lambda \in \mathbb{R}_+^p, \) the problem \( \text{(P)}_\lambda \) generically admits a unique optimal solution.
Recall the following (scalar) convex polynomial optimization problem \([6, 7]\) introduced in Section 1, which is transformed from \((\text{MP})\) by the hybrid method:

\[
\begin{aligned}
\bar{f}_z := \min_{x \in \mathbb{R}^n} \quad & \lambda^T f(x) := \sum_{j=1}^{p} \lambda_j f_j(x) \\
\text{s.t.} \\
& g_i(x) \leq 0, \quad i = 1, \ldots, m, \\
& f_j(x) \leq f_j(z), \quad j = 1, \ldots, p,
\end{aligned}
\]

where \(\lambda \in \text{int} \mathbb{R}_+^p\) is fixed and the parameter \(z \in \mathbb{R}^n\).

The following proposition suggests a way to obtain an efficient solution to problem \((\text{MP})\) by solving the problem \((P_z)\).

**Proposition 3.1.** [6, Proposition 12] Let \(z_0 \in K\). If \(\bar{x}\) is an optimal solution to \((P_{z_0})\), then \(\bar{x}\) is also an optimal solution to \((P_{\bar{x}})\), and so is an efficient solution to \((\text{MP})\).

Now, we recall a known lemma that shows an important existence result of solutions to (scalar) convex polynomial optimization problems.

**Lemma 3.1.** [3, Theorem 3] Let \(f\) and \(g_i, \quad i = 1, \ldots, m\), be convex polynomials on \(\mathbb{R}^n\). Let \(K := \{x \in \mathbb{R}^n : g_i(x) \leq 0, \quad i = 1, \ldots, m\}\). Suppose that \(\inf_{x \in K} f(x) > -\infty\). Then, \(\text{argmin}_{x \in K} f(x) \neq \emptyset\).

As a consequence of Lemma 3.1, the next theorem provides necessary and sufficient conditions for the existence of an efficient solution to \((\text{MP})\).

**Theorem 3.1** (Existence of Efficient Solutions). The following statements are equivalent:

(i) Problem \((\text{MP})\) admits an efficient solution.

(ii) There exists \(z_0 \in \mathbb{R}^n\) such that \(f(K) \cap (f(z_0) - \mathbb{R}_+^p)\) is a nonempty and bounded set.

**Proof.** We first prove (i) \(\Rightarrow\) (ii). To show this, let \(\bar{x} \in K\) be an efficient solution to \((\text{MP})\). Then we have \(f(K) \cap (f(\bar{x}) - \mathbb{R}_+^p) = \{f(\bar{x})\}\), and so, the assertion (ii) holds.

Conversely, we first note that \(f(K_{z_0}) = f(K) \cap (f(z_0) - \mathbb{R}_+^p)\). From the assertion (ii), the image \(f(K_{z_0})\) is nonempty and bounded. So, there exists a positive real number \(N\) such that \(\|f(x)\| \leq N\) for all \(x \in K_{z_0}\). It then follows from the Cauchy–Schwarz inequality that for all \(x \in K_{z_0}\),

\[
\left| \sum_{j=1}^{p} \lambda_j f_j(x) \right| = |\langle \lambda, f(x) \rangle| \leq ||\lambda|| ||f(x)|| \leq ||\lambda|| N,
\]

and so, \((P_{z_0})\) has a finite optimal value. Hence, in view of Lemma 3.1, there exists at least one optimal solution to \((P_{z_0})\), and the conclusion follows by applying Proposition 3.1. \(\square\)
It is worth mentioning that necessary and sufficient conditions for the existence of efficient solutions to multi-objective programming problems, in which the involving functions are locally Lipschitz, were given in \cite{13}.

Below, we give an example, which shows that the existence result of efficient solutions in the preceding theorem may go awry if the involving functions are not convex polynomials.

**Example 3.1.** Consider 2-dimensional multi-objective optimization problem

$$\min_{(x_1,x_2)\in K} \left(f_1(x_1, x_2), f_2(x_1, x_2)\right),$$

where \(f_1(x_1, x_2) = f_2(x_1, x_2) = (x_1 x_2 - 1)^2 + x_2^2\) are (non-convex) polynomials and let \(K := \mathbb{R}^2\). Note that the image of \(K\) under \(f = (f_1, f_2)\) is \(f(K) = \{ (w_1, w_2) \in \mathbb{R}^2 : w_1 = w_2 > 0 \}\).

So, we see that for any \((z_1, z_2) \in \mathbb{R}^n\), the section

$$f(K) \cap \left(f(z_1, z_2) - \mathbb{R}_+^2\right) = \{ (w_1, w_2) \in \mathbb{R}^2 : 0 < w_1 = w_2 \leq (z_1 z_2 - 1)^2 + z_2^2 \}$$

is nonempty and bounded, however, it is clear that there is no efficient solution to this problem.

We now show that for each \(\lambda \in \mathbb{R}_+^p\), the problem \((P_{z_0})\) generically admits a unique optimal solution. Before that, we recall some notions of semi-algebraic geometry; see \cite{8}.

**Definition 3.1** (Semi-algebraic Sets and Semi-algebraic Functions).

(i) A subset of \(\mathbb{R}^n\) is called semi-algebraic, if it is a finite union of sets of the form

$$\{ x \in \mathbb{R}^n : h_i(x) = 0, \ i = 1, \ldots, k, \ h_i(x) > 0, \ i = k + 1, \ldots, p \},$$

where all \(h_i\) are polynomials.

(ii) Let \(A \subset \mathbb{R}^n\) be a semi-algebraic set. A function \(\Phi : A \to \mathbb{R}\) is said to be semi-algebraic, if its graph is a semi-algebraic subset in \(\mathbb{R}^n \times \mathbb{R}\).

**Theorem 3.2.** Assume that there exists \(z_0 \in \mathbb{R}^n\) such that \(f(K) \cap (f(z_0) - \mathbb{R}_+^p)\) is a nonempty and bounded set. Then there exists an open and dense semi-algebraic subset \(U\) of \(\mathbb{R}_+^p\) such that for each \(\lambda \in U\), the problem \((P_{z_0})\) has a unique optimal solution.

**Proof.** Define the function \(\Phi : \mathbb{R}_+^p \to \mathbb{R}\) by

$$\Phi(\lambda) := \min_{x \in K_{z_0}} \lambda^T f(x).$$

Then \(\Phi(\cdot)\) is well-defined since, for every \(\lambda \in \mathbb{R}_+^p\), the problem \((P_{z_0})\) has optimal solutions (by the proof of Theorem 3.1). On the other hand, the graph of \(\Phi\) is

$$\{ (\lambda, t) \in \mathbb{R}_+^p \times \mathbb{R} : \forall x \in K_{z_0}, \ t \leq \lambda^T f(x) \text{ and } \forall \epsilon > 0, \ \exists x \in K_{z_0} \text{ s.t. } t > \lambda^T f(x) - \epsilon \}.$$
that Φ is a continuously differentiable function on \( U \). Hence, the function Φ is semi-algebraic. Then we can see that there exists an open and dense semi-algebraic subset \( U \) of \( \mathbb{R}_+^p \) such that Φ is a continuously differentiable function on \( U \) (see, e.g., [18, Theorem 6.1]). Observe that

\[
-\Phi(\lambda) = -\min_{x \in K_{z_0}} \lambda^T f(x) = \max_{x \in K_{z_0}} -\lambda^T f(x).
\]

Besides, we easily see that \(-\Phi\) is a convex function. Note that, for every \( \lambda \in U \),

\[
(2) \quad \{\nabla \Phi(\lambda)\} = \partial^L \Phi(\lambda) = -\partial(-\Phi)(\lambda),
\]

where the notation \( \partial^L \Phi(\lambda) \) and \( \partial(-\Phi)(\lambda) \) stand for the limiting subdifferential and the (classical) subdifferential (for convex functions) of Φ and \(-\Phi\) at \( \lambda \), respectively (for the definitions of the limiting subdifferential and the classical subdifferential, see, [22, Definition 1.77] and [26, Chapter 23], resp.).

Now, for simplicity, we define the function \( \phi: \mathbb{R}_+^p \times \mathbb{R}^n \to \mathbb{R} \) by \( \phi(\lambda, x) := -\lambda^T f(x) \). Then, by contradiction, we assume that for fixed \( \lambda^* \in U \), \((P_{z_0})\) has two distinguishing optimal solutions, say \( x^1_{\lambda^*}, x^2_{\lambda^*} \). Without loss of generality, we may assume that \( f(x^1_{\lambda^*}) \neq f(x^2_{\lambda^*}) \). Then we have

\[
-\Phi(\lambda^*) = \phi(\lambda^*, x^1_{\lambda^*}) = \phi(\lambda^*, x^2_{\lambda^*}), \quad \text{and}
\]

\[
\partial_\lambda \phi(\lambda^*, x^1_{\lambda^*}) = \{-f(x^1_{\lambda^*})\} \text{ and } \partial_\lambda \phi(\lambda^*, x^2_{\lambda^*}) = \{-f(x^2_{\lambda^*})\},
\]

where \( \partial_\lambda \phi \) denotes the (classical) subdifferential of \( \phi \) with respect to \( \lambda \). So, along with the definition of the subdifferential, we have

\[
\phi(\lambda, x^1_{\lambda^*}) - \phi(\lambda^*, x^1_{\lambda^*}) \geq \langle -f(x^1_{\lambda^*}), \lambda - \lambda^* \rangle, \quad \forall \lambda \in \mathbb{R}_+^p.
\]

Since \(-\Phi(\lambda^*) = \phi(\lambda^*, x^1_{\lambda^*})\), the above inequality leads to

\[
-\Phi(\lambda) - (-\Phi(\lambda^*)) = \left( \max_{x \in K_{z_0}} \phi(\lambda, x) \right) - \phi(\lambda^*, x^1_{\lambda^*}) \geq \langle -f(x^1_{\lambda^*}), \lambda - \lambda^* \rangle, \quad \forall \lambda \in \mathbb{R}_+^p.
\]

Thus, we have \(-f(x^1_{\lambda^*}) \in \partial(-\Phi)(\lambda^*)\). Similarly, we obtain \(-f(x^2_{\lambda^*}) \in \partial(-\Phi)(\lambda^*)\). Note that the subdifferential \( \partial(-\Phi)(\lambda^*) \) is convex. It follows from (2) that

\[
\{\nabla \Phi(\lambda^*)\} = -\partial(-\Phi)(\lambda^*) \supset \{(1-t)f(x^1_{\lambda^*}) + tf(x^2_{\lambda^*}) : t \in [0, 1]\},
\]

which is a contradiction. Consequently, the desired result follows.

\[\Box\]

4. Representation and finite convergence

In this section, we give two kinds of representations of non-negativity of convex polynomials over convex semi-algebraic sets. In addition, we formulate two kinds of Lasserre-type hierarchies of SDP relaxations for \((P_1)\) and establish their finite convergence results.
4.1. Representations of non-negativity of convex polynomials over convex semi-algebraic sets. Let $z \in K$ be given. Then, we define the quadratic module $Q$ generated by the tuples $-g := (-g_1, \ldots, -g_m)$ and $-f_z := (- (f_1 - f_1(z)), \ldots, -(f_p - f_p(z)))$ as

$$Q(-g, -f_z) := \left\{ \sigma_0 - \sum_{i=1}^{m} \sigma_i g_i - \sum_{j=1}^{p} \bar{\sigma}_j (f_j - f_j(z)) : \sigma_0, \sigma_1, \ldots, \sigma_m, \bar{\sigma}_1, \ldots, \bar{\sigma}_p \in \Sigma[x] \right\}.$$ 

Similarly, we define the following special quadratic module generated by the tuples $-g, -f_z$ and an additional polynomial $-\lambda^T f_z := -(\lambda^T f - \lambda^T f(z))$ as

$$\mathcal{M}(-g, -f_z, -\lambda^T f_z) := \left\{ \sigma_0 - \sum_{i=1}^{m} \mu_i g_i - \sum_{j=1}^{p} \nu_j (f_j - f_j(z)) - \sigma (\lambda^T f_z) : \sigma, \sigma_0 \in \Sigma[x], \mu \in \mathbb{R}_+^m, \nu \in \mathbb{R}_+^p \right\}.$$ 

Clearly, the module $\mathcal{M}(-g, -f_z, -\lambda^T f_z)$ is a subset of the quadratic module $Q(-g, -f_z)$. In connection with problem $\{P_z\}$, we define the Lagrangian-type function $L_z: \mathbb{R}^n \times \mathbb{R}_+^m \times \mathbb{R}_+^p \rightarrow \mathbb{R}$ as follows:

$$(3) \quad L_z(x, \mu, \nu) = \lambda^T f(x) + \sum_{i=1}^{m} \mu_i g_i(x) + \sum_{j=1}^{p} \nu_j (f_j(x) - f_j(z)).$$

**Definition 4.1.** We say that the triplet $(\bar{x}, \bar{\mu}, \bar{\nu}) \in K_z \times \mathbb{R}_+^m \times \mathbb{R}_+^p$ is a saddle point of (3) if the following inequality holds:

$$L_z(x, \bar{\mu}, \bar{\nu}) \geq L_z(z, \bar{x}, \bar{\nu}) \geq L_z(x, \mu, \nu), \forall x \in \mathbb{R}^n, \mu \in \mathbb{R}_+^m, \nu \in \mathbb{R}_+^p.$$ 

The following lemma, which plays a key role in deriving the desired results, shows that a convex polynomial with positive definiteness of its Hessian at some point is strictly convex and coercive.

**Lemma 4.1.** [12, Lemma 3.1] Let $f$ be a convex polynomial on $\mathbb{R}^n$. If $\nabla^2 f(x_0) \succ 0$ at some point $x_0 \in \mathbb{R}^n$, then $f$ is coercive and strictly convex on $\mathbb{R}^n$.

In what follows, we give a representation result for non-negativity of convex polynomials over convex semi-algebraic sets. Note that the result can be obtained by modifying the proof of [12, Theorem 3.1]; apart from this, we also need the following assumption.

**Assumption 4.1.** Let $z_0 \in K$. There exists a saddle point $(\bar{x}, \bar{\mu}, \bar{\nu}) \in K_{z_0} \times \mathbb{R}_+^m \times \mathbb{R}_+^p$ of the Lagrangian-type function $L_{z_0}$ such that $\nabla^2_{xx} L_{z_0}(\bar{x}, \bar{\mu}, \bar{\nu}) \succ 0$.

**Theorem 4.1.** (cf. [12, Theorem 3.1]) Consider problem $\{P_z\}$ at $z = z_0 \in K$. If Assumption 4.1 holds, then

$$\lambda^T f - \bar{f}_{z_0} \in Q(-g, -f_{z_0}).$$
Proof. Since \((\bar{x}, \bar{\mu}, \bar{\nu})\) is a saddle-point of the Lagrangian-type function \(L_{z_0}\), it follows that
\[
L_{z_0}(x, \bar{\mu}, \bar{\nu}) \geq L_{z_0}(\bar{x}, \bar{\mu}, \bar{\nu}) = \lambda^T f(\bar{x}), \quad \forall x \in \mathbb{R}^n
\]
and \(\bar{x}\) is an optimal solution to \((P_{z_0})\). Define a function \(h : \mathbb{R}^n \to \mathbb{R}\) by
\[
h(x) := L_{z_0}(x, \bar{\mu}, \bar{\nu}) - \lambda^T f(\bar{x})
= \lambda^T f(x) + \sum_{i=1}^{m} \bar{\mu}_i g_i(x) + \sum_{j=1}^{p} \bar{\nu}_j (f_j(x) - f_j(z_0)) - \lambda^T f(\bar{x}).
\]
Then \(h\) is a convex polynomial and \(h(x) \geq 0\) for all \(x \in \mathbb{R}^n\). Moreover, we easily see that \(h(\bar{x}) = 0 = \inf_{x \in \mathbb{R}^n} h(x)\). Since \(\nabla^2_{xx} L_{z_0}(\bar{x}, \bar{\mu}, \bar{\nu}) \succ 0\), we also see that the Hessian \(\nabla^2 h(\bar{x})\) is positive definite. It follows from Lemma 14 that the polynomial \(h\) is strictly convex and coercive. Furthermore, this implies that \(\bar{x}\) is the unique optimal solution to \(h\) over \(\mathbb{R}^n\). Now consider the set
\[
S := \{x \in \mathbb{R}^n : c - h(x) + \lambda^T f(z_0) - \lambda^T f(\bar{x}) \geq 0\},
\]
where \(c\) is some positive constant. Since \(z_0 \in K_{z_0}\), we see that \(\bar{x} \in S\), and so, the set \(S\) is nonempty and compact (since the polynomial \(h\) is coercive). Moreover, since
\[
c - h(x) + \lambda^T f(z_0) - \lambda^T f(\bar{x})
= c - \sum_{j=1}^{p} \lambda_j (f_j(x) - f_j(z_0)) - \sum_{i=1}^{m} \bar{\mu}_i g_i(x) - \sum_{j=1}^{p} \bar{\nu}_j (f_j(x) - f_j(z_0))
= c - \sum_{i=1}^{m} \bar{\mu}_i g_i(x) - \sum_{j=1}^{p} (\lambda_j + \bar{\nu}_j) (f_j(x) - f_j(z_0)) \in \mathcal{Q}(-g, -f_{z_0})
\]
and \(S = \{x \in \mathbb{R}^n : c - h(x) + \lambda^T f(z_0) - \lambda^T f(\bar{x}) \geq 0\}\) is compact, the quadratic module \(\mathcal{Q}(-g, -f_{z_0})\) is Archimedean. It follows from [28, Corollary 3.6] (see also [27, Example 3.18]) that there exist sum-of-squares polynomials \(\sigma_0, \sigma_1 \in \Sigma[x]\) such that, for each \(x \in \mathbb{R}^n\),
\[
h(x) = \sigma_0(x) + \sigma_1(x)(c - h(x) + \lambda^T f(z_0) - \lambda^T f(\bar{x})).
\]
Thus, we have
\[
\lambda^T f - \tilde{f}_{z_0} = \sigma_0 + c \sigma_1 - \sum_{i=1}^{m} (\bar{\mu}_i + \bar{\mu}_i \sigma_1) g_i - \sum_{j=1}^{p} (\bar{\nu}_j + (\lambda_j + \bar{\nu}_j) \sigma_1) (f_j - f_j(z_0)),
\]
thereby establishing the desired result. \(\square\)

Assumption 4.2. For a given point \(z_0 \in K\), the following two statements hold:

(i) the Slater-type condition holds for \((P_{z_0})\), that is, there exists \(\hat{x} \in \mathbb{R}^n\) such that \(g_i(\hat{x}) < 0\), for \(i = 1, \ldots, m\), and \(f_j(\hat{x}) < f_j(z_0), j = 1, \ldots, p\);

(ii) \(\sum_{j=1}^{p} \lambda_j \nabla^2 f_j(\bar{x}) > 0\), where \(\bar{x} \in \text{argmin}_{x \in K_{z_0}} \lambda^T f(x)\).
Slightly modifying [12, Theorem 3.2], we obtain the following representation of a convex polynomial, which is sharper than the result of Theorem 4.1.

**Theorem 4.2.** (cf. [12, Theorem 3.2]) Consider problem $\{P_z\}$ at $z = z_0 \in K$. Suppose that Assumption 4.2 holds. Then

$$\lambda^T f - \bar{f}_{z_0} \in \mathcal{M}(-g, -f_{z_0}, -\lambda^T f_{z_0}).$$

*Proof.* Let $\bar{x} \in \text{argmin}_{x \in K_{z_0}} \lambda^T f(x)$. Since the Slater-type condition holds for $(P_{z_0})$, by the KKT optimality conditions for convex optimization problems, there exist the Lagrangian multipliers $\bar{\mu} \in \mathbb{R}^n_+$ and $\bar{\nu} \in \mathbb{R}^p_+$ such that

$$0 = \sum_{j=1}^p \lambda_j \nabla f_j(\bar{x}) + \sum_{i=1}^m \bar{\mu}_i \nabla g_i(\bar{x}) + \sum_{j=1}^p \bar{\nu}_j \nabla f_j(\bar{x}),$$

$$0 = \bar{\mu}_i g_i(\bar{x}), \ i = 1, \ldots, m;$$

$$0 = \bar{\nu}_j (f_j(\bar{x}) - f_j(z_0)), \ j = 1, \ldots, p.$$

By defining a convex polynomial $h: \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$h(x) := \lambda^T f(x) + \sum_{i=1}^m \bar{\mu}_i g_i(x) + \sum_{j=1}^p \bar{\nu}_j (f_j(x) - f_j(z_0)) - \bar{f}_{z_0}.$$

It is easily verified that $h(x) \geq 0$ for all $x \in \mathbb{R}^n$, particularly, $h(\bar{x}) = 0$. On the other hand, since $\sum_{j=1}^p \lambda_j \nabla^2 f_j(\bar{x}) \succ 0$, it follows from Lemma 4.1 that $\lambda^T f$ is strictly convex and coercive on $\mathbb{R}^n$. Consequently, the set $F := \{x \in \mathbb{R}^n : c + \lambda^T f(z_0) - \lambda^T f(x) \geq 0\}$ is nonempty and compact, where $c$ is some positive constant. Moreover, the quadratic module $Q(c - \lambda^T f_{z_0})$ is Archimedean along with the fact that $c - \lambda^T f_{z_0} \in Q(c - \lambda^T f_{z_0})$ and $F$ is compact. In addition, as $h(x) \geq 0$ for all $x \in F$, $h(\bar{x}) = 0$ and $\nabla^2 h(\bar{x}) \succ 0$, $\bar{x}$ is a unique optimal solution to the problem $\min_{x \in F} h(x)$. Thanks to [28, Corollary 3.6] (see also [27, Example 3.18]), there exist $\sigma, \sigma_0 \in \Sigma[x]$ such that $h = \sigma_0 + \sigma(c - \lambda^T f_{z_0})$, and hence,

$$\lambda^T f - \bar{f}_{z_0} = \sigma_0 + c \sigma - \sum_{i=1}^m \bar{\mu}_i g_i - \sum_{j=1}^p \bar{\nu}_j (f_j - f_j(z_0)) - \sigma(\lambda^T f_{z_0}),$$

which is the desired result. \qed

**Remark 4.1.** It is worth noting that, in Theorem 4.2, the condition $\sum_{j=1}^p \lambda_j \nabla^2 f_j(\bar{x}) \succ 0$ guarantees the compactness of the feasible set $K_{z_0}$ for the problem $(P_{z_0})$. Indeed, let $\{x_k\} \subset K_{z_0}$ be an arbitrary sequence. Then, for each $k \in \mathbb{N}$, $x_k \in K_\lambda := \{x \in K : \lambda^T f(x) \leq \lambda^T f(z_0)\}$, which is compact as $\lambda^T f$ is coercive on $K$. So, there exists a subsequence $\{x_{k_l}\} \subset K_\lambda$ such that $x_{k_l} \to x^* \in K_\lambda$ as $l \to +\infty$. Since $x_{k_l} \in K_{z_0}$ for all $l \in \mathbb{N}$, by the continuity of each $f_j$, we have $x^* \in K_{z_0}$, and so, the set $K_{z_0}$ is nonempty and compact. This
yields that the efficient solution set of (MP) is nonempty. Note also that the problem (MP) admits an efficient solution if there exists \( x \in \mathbb{R}^n \) such that \( \sum_{j=1}^{p} \lambda_j \nabla^2 f_j(x) \succ 0 \).

4.2. Finite convergence for the Lasserre-type hierarchies of semidefinite programming relaxations. Let \( z \in K \) be given. With \( g_0 = 1 \), let \( r_i := \lceil \deg g_i / 2 \rceil \), \( i = 0, 1, \ldots, m \), and let \( d_j := \lceil \deg f_j / 2 \rceil \), \( j = 1, \ldots, p \), where the notation \( \lceil a \rceil \) stands for the smallest integer greater than or equal to \( a \). Now, for \( k \geq k_0 := \max \{ \max_i r_i, \max_j d_j \} \), consider the following semidefinite programming problem:

\[
\rho^k_z := \inf_y \sum_{j=1}^{p} \sum_{\alpha \in \mathbb{N}^n} \lambda_j (f_j)_\alpha y_\alpha \\
\text{s.t. } M_{k - r_i} (-g_i y) \succeq 0, \ i = 0, 1, \ldots, m, \\
\quad M_{k - d_j} ((f_j(z) - f_j) y) \succeq 0, \ j = 1, \ldots, p.
\]

It is worth noting that \((Q^k_z)\) is a Lasserre-type hierarchy of SDP relaxation of problem \((P_z)\) i.e., \( \rho^k_z \leq \rho^{k+1}_z \leq \cdots \leq \hat{\rho}_z \) for all \( k \geq k_0 \) (see, e.g., [17]).

Now, consider the following programming problem:

\[
\hat{\rho}^k_z := \sup_{\gamma, \sigma_i, \sigma_j} \gamma \\
\text{s.t. } \lambda^T f - \gamma = \sigma_0 - \sum_{i=1}^{m} \sigma_i g_i \wedge \sum_{j=1}^{p} \sigma_j (f_j - f_j(z)), \\
\quad \sigma_i \in \Sigma[x]_{k - r_i}, \ i = 0, 1, \ldots, m, \ \sigma_j \in \Sigma[x]_{k - d_j}, \ j = 1, \ldots, p.
\]

It is worth mentioning that \((Q^k_z)\) is the dual problem of \((Q^k_z)\) (see, [15, 17]). Note that, for a given \( z \in K \), the set \( K_z \) is nonempty. This implies that the feasible set of \((Q^k_z)\) is nonempty. So, if the feasible set of \((Q^k_z)\) is nonempty, then we see that \( \rho^k_z \geq \hat{\rho}^k_z \) for all \( k \geq k_0 \) by weak duality. Moreover, \((Q^k_z)\) has an asymptotic convergence in the sense that \( \hat{\rho}^k_z \uparrow \hat{f}_z \) as \( k \to \infty \) without any regularity conditions (see, e.g., [12, Theorem 2.1]).

Now, with the help of Theorem 4.1, we show the finite convergence for the hierarchy of SDP relaxations of \((P_z)\) in the next Theorem.

**Theorem 4.3.** Consider problem \((P_z)\) at \( z = z_0 \in K \). If Assumption 4.1 is satisfied, then there exists an integer \( \bar{k} \) such that \( \rho^k_{z_0} = \hat{\rho}^k_{z_0} = \hat{f}_{z_0} \) for all \( k \geq \bar{k} \), and both \((Q^k_{z_0})\) and \((Q^k_z)\) attain their optimal solutions.
Proof. By Theorem 4.1, there exist sum-of-squares polynomials \( \sigma_i, i = 0, 1, \ldots, m \), and \( \sigma_j, j = 1, \ldots, p \), such that

\[
\lambda^T f - \bar{f}_{z_0} = \sigma_0 - \sum_{i=1}^m \sigma_ig_i - \sum_{j=1}^p \sigma_j (f_j - f_j(z_0)).
\]

Let \( \bar{k} \geq \max \{ \deg \sigma_0, \max_i \{ \deg(\sigma_ig_i) \}, \max_j \{ \deg(\sigma_jf_j) \} \} \). Then, \( (\bar{f}_{z_0}, \sigma_0, (\sigma_i), (\sigma_j)) \) is a feasible solution of \( (\bar{Q}^k_{z_0}) \) for \( k = \bar{k} \), and so, we have \( \bar{f}_{z_0} \leq \bar{\rho}^k_{z_0} \) for \( k \geq \bar{k} \). Also, we can easily see that the sequence \( \{\bar{\rho}^k_{z_0}\} \) is monotonically increasing and bounded from above by \( \bar{f}_{z_0} \); in particular, \( \bar{\rho}^k_{z_0} \leq \bar{f}_{z_0} \) for all \( k \geq \bar{k} \). Thus, \( \bar{\rho}^k_{z_0} = \bar{f}_{z_0} \) for all \( k \geq \bar{k} \). In fact, \( (\bar{f}_{z_0}, \sigma_0, (\sigma_i), (\sigma_j)) \) is an optimal solution of \( (\bar{Q}^k_{z_0}) \) for all \( k \geq \bar{k} \).

On the other hand, by the weak duality between \( (Q^k_z) \) and \( (Q^k_{z_0}) \), we have \( \bar{\rho}^k_{z_0} \leq \rho^k_{z_0} \) for all \( k \geq \bar{k} \). Thus, we conclude that \( \bar{\rho}^k_{z_0} = \rho^k_{z_0} = \bar{f}_{z_0} \) for all \( k \geq \bar{k} \). In particular, it is clear that, for all \( k \geq \bar{k} \), \( \bar{y} = v_{2k}(\bar{x}) \) is an optimal solution to \( (Q^k_{z_0}) \), which completes the proof. \( \square \)

4.3. Finite convergence for the Lasserre-type hierarchy of sharp semidefinite programming relaxations. Let \( z \in K \) be given. Consider the following semidefinite programming problem:

\[
(P_z^k) \quad f_z^k := \inf_{y} \sum_{j=1}^{p} \sum_{a \in \mathbb{N}^2_{2k}} \lambda_j f_j a y_a
\]

s.t. \( M_k(y) \succeq 0 \),

\[
\sum_{a \in \mathbb{N}^2_{2k}} (g_i) a y_a \leq 0, \quad i = 1, \ldots, m,
\]

\[
\sum_{a \in \mathbb{N}^2_{2k}} (f_j) a y_a \leq f_j(z), \quad j = 1, \ldots, p,
\]

\[
M_{k-d}(\lambda^T f_z) y \succeq 0,
\]

where \( k \geq k_0 = \max \{ \max_j d_j, \max_i r_i \} \) and \( d_f := \max_j d_j \). It is worth noting that \( (P_z^k) \) is also Lasserre-type hierarchy of SDP relaxation for \( (P_z) \). Indeed, letting \( x \in K_z \) and \( y := v_{2k}(x) \), we see that \( y \) is a feasible solution of \( (P_z^k) \) with its value \( f(x) \). So, we have \( f_z^k \leq \bar{f}_z \) for all \( k \geq k_0 \). In addition, we see that \( f_z^k \leq f_z^{k+1} \) for all \( k \geq k_0 \) since \( (P_z^{k+1}) \) is more constrained than \( (P_z^k) \).
Below, we consider the dual problem \( (\overline{P}^k_z) \) of \( (P^k_z) \) as follows:

\[
(\overline{P}^k_z) \quad \overline{f}_z^k := \sup_{\gamma, \sigma_0, \mu, \nu} \gamma \\
\text{s.t.} \quad \lambda^T f - \gamma = \sigma_0 - \sum_{i=1}^{m} \mu_i g_i - \sum_{j=1}^{p} \nu_j (f_j - f_j(z)) - \sigma(\lambda^T f_z), \\
\sigma \in \Sigma[x]_k, \sigma_0 \in \Sigma[x]_{k-d}, \mu_i \geq 0, i = 1, \ldots, m, \nu_j \geq 0, j = 1, \ldots, p.
\]

Note that weak duality holds between \( (P^k_z) \) and \( (\overline{P}^k_z) \), i.e., \( \overline{f}_z^k \leq f_z^k \) for all \( k \geq k_0 \). Moreover, it is easily to verify that \( \overline{f}_z^k \leq \overline{f}_z^{k+1} \) for all \( k \geq k_0 \).

We now establish an asymptotic convergence result for the SDP relaxations \( (\overline{P}^k_z) \) under the positive definiteness of the Hessian of the objective function of problem \( (P^k_z) \) at some point, and the proof of this result can be obtained by slightly modifying the proof of [12, Theorem 2.1].

**Theorem 4.4.** Let \( z_0 \in K \) be given. If there exists \( \bar{x} \in \mathbb{R}^n \) such that \( \nabla^2(\lambda^T f)(\bar{x}) \succ 0 \), then \( \overline{f}_z^k \uparrow \overline{f}_{z_0} \) as \( k \to \infty \).

**Proof.** Let \( \epsilon > 0 \). We first claim that there exist \( \mu \in \mathbb{R}^m_+ \) and \( \nu \in \mathbb{R}^p_+ \) such that

\[
\lambda^T f(x) - \overline{f}_{z_0} + \sum_{i=1}^{m} \mu_i g_i(x) + \sum_{j=1}^{p} \nu_j (f_j(x) - f_j(z_0)) + \epsilon > 0, \quad \forall x \in \mathbb{R}^n.
\]

Since \( \lambda^T f(x) - \overline{f}_{z_0} \geq 0 \) for all \( x \in K_{z_0} \), observe that \( \lambda^T f - \overline{f}_{z_0} + \epsilon > 0 \) on \( K_{z_0} \). Then there exists \( \delta > 0 \) such that \( \lambda^T f(x) - \overline{f}_{z_0} + \epsilon > 0 \) for all \( x \in K_{z_0, \delta} \), where

\[
K_{z_0, \delta} := \{ x \in \mathbb{R}^n : g_i(x) \leq \delta, i = 1, \ldots, m, f_j(x) - f_j(z_0) \leq \delta, j = 1, \ldots, p \}.
\]

Otherwise, suppose that there exist sequences \( \{\delta_k\} \subset \mathbb{R}_+, \delta_k \to 0 \) and \( \{x_k\} \subset \mathbb{R}^n \) such that \( g_i(x_k) \leq \delta_k, i = 1, \ldots, m, f_j(x_k) - f_j(z_0) \leq \delta_k, j = 1, \ldots, p \), and \( \lambda^T f(x_k) - \overline{f}_{z_0} + \epsilon \leq 0 \). Then,

\[
0 \leq \inf_{x, w} \left\{ \sum_{i=1}^{m+p} w_i^2 : \lambda^T f(x) - \overline{f}_{z_0} + \epsilon \leq 0, g_i(x) \leq w_i, i = 1, \ldots, m, f_j(x) - f_j(z_0) \leq w_{m+j}, j = 1, \ldots, p \right\}
\]

\[
\leq \sum_{i=1}^{m+p} \delta_k^2 = (m+p)\delta_k^2 \to 0 \text{ as } k \to \infty.
\]

It follows from Lemma 3.1 that there exist \( x^* \in \mathbb{R}^n \) and \( w^* \in \mathbb{R}^{m+p} \) such that \( \lambda^T f(x^*) - \overline{f}_{z_0} + \epsilon \leq 0, g_i(x^*) \leq w_i^*, i = 1, \ldots, m, f_j(x^*) - f_j(z_0) \leq w_{j+m}^*, j = 1, \ldots, p \), and \( \sum_{i=1}^{m+p} (w_i^*)^2 = 0 \), i.e., \( \lambda^T f(x^*) - \overline{f}_{z_0} + \epsilon \leq 0, g_i(x^*) \leq 0, i = 1, \ldots, m, f_j(x^*) - f_j(z_0) \leq 0, j = 1, \ldots, p \), which contradicts the fact that \( \lambda^T f - \overline{f}_{z_0} + \epsilon \) is positive over \( K_{z_0} \).
Now, define $h: \mathbb{R}^n \to \mathbb{R}$ by

$$h(x) := \lambda^T f(x) - \bar{f}_{z_0} + \sum_{i=1}^m \mu_i g_i(x) + \sum_{j=1}^p \nu_j (f_j(x) - f_j(z_0)) + \epsilon, \forall x \in \mathbb{R}^n.$$ 

Along with (1), it is clear that $h(x) > 0$ for all $x \in \mathbb{R}^n$.

On the other hand, since $\sum_{j=1}^p \lambda_j \nabla^2 f_j(\bar{x}) > 0$ for some $\bar{x} \in \mathbb{R}^n$, it follows from Lemma 4.1 that $\lambda^T f$ is strictly convex and coercive on $\mathbb{R}^n$. Let $S := \{x \in \mathbb{R}^n : -\lambda^T f_{z_0}(x) \geq 0\}$. Then the set $S$ is nonempty and compact. Note that $h$ is positive on $S$. Moreover, since $-\lambda^T f_{z_0} \in \mathcal{Q}(-\lambda^T f_{z_0})$ and $S$ is compact, the quadratic module $\mathcal{Q}(-\lambda^T f_{z_0})$ is Archimedean. Thanks to Lemma 2.1 (Putinar’s Positivstellensatz), there exist $\sigma, \sigma_0 \in \Sigma[x]$ such that $h = \sigma_0 - \sigma(\lambda^T f_{z_0})$, i.e., for each $x \in \mathbb{R}^n$,

$$\lambda^T f(x) - \bar{f}_{z_0} + \epsilon = \sigma_0 - \sum_{i=1}^m \mu_i g_i(x) - \sum_{j=1}^p \nu_j (f_j(x) - f_j(z_0)) - \sigma(\lambda^T f_{z_0}).$$

So, $(\bar{f}_{z_0} - \epsilon, \sigma, \sigma_0, \mu, \nu)$ is a feasible solution of $(\bar{P}^k_{z_0})$ as soon as $k$ is large enough. Hence we have $\bar{f}_{z_0} - \epsilon \leq \bar{f}_k$. Finally, by weak duality between $(P_{z_0}^k)$ and $(\bar{P}^k_{z_0})$, we have $\bar{f}_k \leq f_{z_0}^k$ for all $k \geq k_0$. Besides, as shown before that $f_{z_0}^k \leq \bar{f}_{z_0}$, we thus conclude that $f_{z_0} - \epsilon \leq f_{z_0}^k \leq f_{z_0}$. As $\epsilon > 0$ is arbitrary, the desired result follows. 

We close this section by giving the next finite convergence result for the hierarchy of SDP relaxations of $(P^k_z)$, which is sharper than the one of Theorem 4.3.

**Theorem 4.5.** Consider problem $(P^k_z)$ at $z = z_0 \in K$. If Assumption 4.3 holds, then there exists an integer $\bar{k}$ such that $f_{z_0}^k = f_{z_0}^\bar{k} = \bar{f}_{z_0}$ for all $k \geq \bar{k}$. In addition, both $(P_{z_0}^k)$ and $(\bar{P}_{z_0}^k)$ attain their optimal solutions.

**Proof.** The proof is similar to the one of Theorem 4.3. It follows from Theorem 4.2 that there exist $\sigma, \sigma_0 \in \Sigma[x], \bar{\mu} \in \mathbb{R}^m_+$, and $\nu \in \mathbb{R}^p_+$ such that

$$\lambda^T f - \bar{f}_{z_0} = \sigma_0 - \sum_{i=1}^m \bar{\mu}_i g_i - \sum_{j=1}^p \nu_j (f_j - f_j(z_0)) - \sigma(\lambda^T f_{z_0}).$$

Let $\bar{k} \geq \max\{\deg \sigma_0, \deg \sigma + d_f, \max_i \{\deg g_i\}\}$. Then, $(\bar{f}_{z_0}, \sigma, \sigma_0, \bar{\mu}, \nu)$ is a feasible solution of $(\bar{P}^k_{z_0})$ for $k = \bar{k}$, and so, we have $\bar{f}_{z_0} \leq \bar{f}_k$. Moreover, we have already seen that $\bar{f}_k \leq f_{z_0}$ for all $k \geq \bar{k}$. Thus, we have $f_{z_0}^k = \bar{f}_{z_0}$ for all $k \geq \bar{k}$. In fact, $(\bar{f}_{z_0}, \sigma, \sigma_0, \bar{\mu}, \nu)$ is an optimal solution to $(\bar{P}^k_{z_0})$ for all $k \geq \bar{k}$.

On the other hand, by the weak duality between $(P_{z_0}^k)$ and $(\bar{P}_{z_0}^k)$, $f_{z_0}^k \leq \bar{f}_k$ for all $k \geq k_0$. Thus, we conclude that $f_{z_0}^k = f_{z_0}^k = \bar{f}_{z_0}$ for all $k \geq \bar{k}$. In particular, it is clear that, for all $k \geq \bar{k}$, $\bar{y} = \nu_{2k}(\bar{x})$ is an optimal solution to $(P_{z_0}^k)$, which completes the proof. \qed
5. Finding efficient solutions

Let \( z \in K \) be given, and let \( \bar{y} \) be an optimal solution to \((Q_k^z)\) (or \((P_k^z)\)). If the flat extension condition holds, that is,

\[
\text{rank } M_k(\bar{y}) = \text{rank } M_{k-k_0}(\bar{y}),
\]

then there exist at least \( \text{rank } M_k(\bar{y}) \) optimal solutions to \((P_z)\) (see, e.g., [17, Theorem 6.6]), and they can be efficiently extracted by a suitable algorithm (see, e.g., [17, Algorithm 6.9 in Section 6.1], [10, Section 2]). In addition, the flat extension condition (5) guarantees a finite convergence of Lasserre’s hierarchy, but the converse may not be true [23, Example 1.1].

Recently, a weak condition of the flat extension condition (5) was proposed by Nie [23]. That is, there exists an integer \( t \in [k_0, k] \) such that

\[
\text{rank } M_t(\bar{y}) = \text{rank } M_{t-k_0}(\bar{y}).
\]

Also, we say that \( \bar{y} \) has a flat truncation if the condition (6) holds for some \( t \in [k_0, k] \). Note that if \( \bar{y} \) has a flat truncation, then we can find at least \( \text{rank } M_t(\bar{y}) \) optimal solutions to \((P_z)\).

Assumption 5.1 (cf. Assumption 2.1 in [23]). Let \( \lambda \in \text{int } \mathbb{R}^p_+ \) be fixed. For a given \( z \in K \), there exists \( \bar{\rho} \in Q(-g, -f_z) \) such that for every \( I \subseteq \{1, \ldots, m\}, J \subseteq \{1, \ldots, p\} \), and

\[
V_{z,I,J} := \{ x \in \mathbb{R}^n : \text{there exist } \mu_i \geq 0, \ i \in I, \ and \ \nu_j \geq 0, \ j \in J, \ such \ that \\
\sum_{j=1}^p \lambda_j \nabla f_j(x) + \sum_{i \in I} \mu_i \nabla g_i(x) + \sum_{j \in J} \nu_j \nabla f_j(x) = 0, \\
g_i(x) = 0 \ (\forall i \in I), \ f_j(x) - f_j(z) = 0 \ (\forall j \in J)\},
\]

the intersection \( V_{z,I,J} \cap S_z \cap \mathcal{P} \) is finite, where \( S_z := \{ x \in \mathbb{R}^n : \lambda^T f(x) = \bar{f}_z \} \), and \( \mathcal{P} := \{ x \in \mathbb{R}^n : \bar{\rho}(x) \geq 0 \} \).

Let \( z_0 \in K \) be given. It is worth mentioning that Assumption 5.1 implies that \((P_{z_0})\) has finite optimal solutions (see [23]). This fact, together with convexity, implies that the problem \((P_{z_0})\) has a unique optimal solution. This consequence seems to be restrictive, however, as shown in Theorem 3.2, problem \((P_{z_0})\) generically admits a unique optimal solution, hence Assumption 5.1 is not very restrictive. Furthermore, Assumption 5.1 guarantees that the flat truncation (5) is not only a sufficient condition, but also a necessary condition for the finite convergence of the Lasserre hierarchy [23, Theorem 2.2].

Along with these facts, we propose the following result.

**Theorem 5.1.** Consider problem \((P_z)\) at \( z = z_0 \in K \). If Assumptions 4.1 and 5.1 hold, then \( \bar{x} := (\bar{y}_a)_{|a|=1} \) is an efficient solution to \((MP)\), where \( \bar{y} \) is an optimal solution to \((Q^{k}_{z_0})\) for some sufficiently large \( k \).
Proof. By Theorem 4.3 for sufficiently large $k$, $\tilde{\rho}_k = \rho_k = \tilde{f}_{z_0}$ and the optimal value of $(\tilde{Q}_z^k)$ is achievable. It follows from [23, Theorem 2.2] that every optimal solution to $(Q_z^k)$ has a flat truncation for some sufficiently large $k$, i.e., there exists an integer $t \in [k_0, k]$ such that

$$\text{rank } M_t(\bar{y}) = \text{rank } M_{t-k_0}(\bar{y}),$$

where $\bar{y}$ is an optimal solution to $(Q_z^k)$, and so, the problem $(P_z)$ has at least rank $M_t(\bar{y})$ optimal solutions.

On the other hand, Assumption 5.1 implies that $(P_z)$ has a unique solution, thus rank $M_t(\bar{y})$ and rank $M_{t-k_0}(\bar{y})$ should be equal to 1, and so, necessarily, $M_t(\bar{y}) = v_t(\bar{x})v_t(\bar{x})^T$ for some $\bar{x} \in \mathbb{R}^n$. Moreover, since $\bar{y}$ is a feasible solution of $(Q_z^k)$, we can easily see that $\bar{x}$ is also a feasible solution of $(P_z)$. It means that $\bar{y}$ is the vector of moments up to order $2t$ of the Dirac measure $\delta_{\bar{x}}$ at $\bar{x} \in K_z$, i.e., $\bar{y} = v_{2t}(\bar{x})$. This yields that $\bar{x}$ is an optimal solution to $(P_z)$. In particular, $\bar{x} = (\bar{y}_0)_{|\alpha| = 1}$. It follows from Proposition 3.1 that $(\bar{y}_0)_{|\alpha| = 1}$ is an efficient solution to $(MIP)$. \hfill \Box

The following lemma shows that a weak condition of Assumption 4.2 (ii) implies the validity of Assumption 5.1.

Lemma 5.1. Let $z_0 \in K$ be given. Assume that there exists $\bar{x} \in \mathbb{R}^n$ such that the Hessian $\nabla^2(\lambda^T f)(\bar{x})$ is positive definite. Then Assumption 5.1 holds.

Proof. Let $z_0 \in K$ be fixed. Since the Hessian $\sum_{j=1}^p \lambda_j \nabla^2 f_j(\bar{x})$ is positive definite, it follows from Lemma 4.1 that the polynomial $\lambda^T f$ is coercive and strictly convex. This implies that there is a unique optimal solution $\bar{x}$ to the problem $(P_z)$.

Let us denote the set of active constraints by

$$I(\bar{x}) \cup J(\bar{x}) := \{i : g_i(\bar{x}) = 0\} \cup \{j : f_j(\bar{x}) - f_j(z_0) = 0\}.$$

Then, without loss of generality, we can assume that the set $I(\bar{x}) \cup J(\bar{x})$ is nonempty; otherwise, we have $\nabla (\lambda^T f) (\bar{x}) = 0$. This implies that $S_{z_0} = \{x \in \mathbb{R}^n : \lambda^T f(x) = f_{z_0}\} = \{\bar{x}\}$, and so, in this case, Assumption 5.1 obviously holds.

Now, let $I \subset \{1, \ldots, m\}$ and $J \subset \{1, \ldots, p\}$ be any fixed. To finish the proof of this lemma, it suffers to show that the following two statements hold:

(i) If $\bar{x} \in V_{z_0, I, J} \cap S_{z_0}$, then $V_{z_0, I, J} \cap S_{z_0} = \{\bar{x}\}$;
(ii) otherwise, $V_{z_0, I, J} \cap S_{z_0} = \emptyset$.

We first prove that the assertion (i) is true. Assume to the contrary that there exists $\hat{x} \in \mathbb{R}^n$ such that $\hat{x} \neq \bar{x}$ and $\hat{x} \in V_{z_0, I, J} \cap S_{z_0}$. Then, there exist $\hat{\mu}_i \geq 0$, $i \in I$, and $\hat{\nu}_j \geq 0$, $j \in J$, such that
such that
\[ \sum_{j=1}^{p} \lambda_j \nabla f_j(\hat{x}) + \sum_{i \in I} \hat{\mu}_i \nabla g_i(\hat{x}) + \sum_{j \in J} \hat{\nu}_j \nabla f_j(\hat{x}) = 0, \]
(7)
\[ g_i(\hat{x}) = 0, \quad i \in I, \]
\[ f_j(\hat{x}) - f_j(z_0) = 0, \quad j \in J. \]

Now, consider the convex optimization problem
\[
\min_{x \in \mathbb{R}^n} \lambda^T f(x)
\]
\[
s.t. \quad g_i(x) \leq 0, \quad i \in I,
\]
\[
\quad f_j(x) \leq f_j(z_0), \quad j \in J.
\]
Then \( \hat{x} \) is a feasible solution of problem \( \hat{P}_{z_0} \). It follows from (7) that \( \hat{x} \) is indeed an optimal solution to problem \( \hat{P}_{z_0} \) with the optimal value \( \lambda^T f(\hat{x}) = \bar{f}_{z_0} \). On the other hand, since the set \( K_{z_0} \) is clearly a subset of the feasible set of problem \( \hat{P}_{z_0} \), \( \bar{x} \) is also an optimal solution of problem \( \hat{P}_{z_0} \), which contradicts to the fact that the problem \( \hat{P}_{z_0} \) has a unique optimal solution (due to the strictly convexity of \( \lambda^T f \)). Thus, the statement (i) holds.

(ii) Suppose to the contrary that the set \( V_{z_0,I,J} \cap S_{z_0} \) is nonempty. For simplicity, let \( \hat{x} \in V_{z_0,I,J} \cap S_{z_0} \). Then similar to the proof of assertion (i), we see that \( \hat{x} \) is an optimal solution to problem \( \hat{P}_{z_0} \), and so, \( \bar{x} \) is also an optimal solution of problem \( \hat{P}_{z_0} \), which contradicts to the fact that the problem \( \hat{P}_{z_0} \) has a unique optimal solution (due to the strictly convexity of \( \lambda^T f \)). Thus, the statement (i) holds.

The following example shows that Lemma 5.1 may fail if the Hessian \( \nabla^2(\lambda^T f)(x) \) is not positive definite for all \( x \).

**Example 5.1.** For simplicity, let us consider the following 2-dimensional scalar convex polynomial optimization problem:
\[
\min_{(x_1, x_2) \in \mathbb{R}^2} f_1(x_1, x_2)
\]
\[
s.t. \quad g_1(x_1, x_2) \leq 0,
\]
\[
\quad f_1(x_1, x_2) \leq f_1(z_1, z_2),
\]
where \( f_1(x_1, x_2) := (x_1 - x_2)^2 \) and \( g_1(x_1, x_2) = x_1 - x_2 + 1 \). we first note that a simple calculation shows that the Hessian \( \nabla^2 f_1(x_1, x_2) \) is not positive definite for all \( (x_1, x_2) \in \mathbb{R}^2 \).
Let $z_0 = (0, 1) \in \{(x_1, x_2) \in \mathbb{R}^2 : g_1(x_1, x_2) \leq 0\}$. Then it is easily verified that $S_{z_0} = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - x_2)^2 = 1\}$. Moreover, for $I = \{1\}$ and $J = \emptyset$, we have

$$V_{z_0, \{1\}, \emptyset} = \{(x_1, x_2) \in \mathbb{R}^2 : \text{there exists } \mu_1 \geq 0 \text{ such that } x_1 - x_2 + 1 = 0, \left(\begin{array}{c} x_1 - x_2 \\ -x_1 - x_2 \end{array}\right) + \mu_1 \left(\begin{array}{c} 1 \\ -1 \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\},$$

and so, we get $V_{z_0, \{1\}, \emptyset} \cap S_{z_0} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 - x_2 + 1 = 0\}$.

We are now ready to provide our final result which shows that finding efficient solutions to (MP) can be done via solving the Lasserre-type hierarchy of SDP relaxations.

**Theorem 5.2.** Consider problem $(P_{z_0})$ at $z = z_0 \in K$. If Assumption 4.2 holds, then $\bar{x} := (\bar{y}_i)_{|i|=1}$ is an efficient solution to (MP), where $\bar{y}$ is an optimal solution of $(P_{z_0})$ for some sufficiently large $k$.

**Proof.** It follows from Theorem 4.5 that for sufficiently large $k$, $\bar{f}_{z_0}^k = f_{z_0}^k = \bar{f}_{z_0}$ and the optimal value of $(\hat{P}_{z_0}^k)$ is achievable. Also, by Lemma 5.1 Assumption 5.1 holds, and the rest of the proof of this theorem can be constructed by using similar arguments as in the proof of Theorem 5.1. □

**Remark 5.1.** By employing the hybrid method to (MP), Algorithm 1 below shows that finding efficient solutions to (MP) can be done via solving hierarchies of SDP relaxations. It is worth mentioning that since the purpose of this paper is to find efficient solutions to (MP), the assumption of the positive definiteness of the Hessian of the associated Lagrangian (resp., the objective function) at a saddle-point (resp., an optimal solution) may be theoretical rather than practical. On the other hand, if the weighted sum polynomial $\lambda^T f$ is strongly convex, then, by Lemma 3.1 the problem (P) has an optimal solution (in fact, it is unique). In addition, since the Hessian $\nabla^2(\lambda^T f)$ of the weighted sum polynomial $\lambda^T f$ is positive definite on $\mathbb{R}^n$, we see that Assumption 4.2 (ii) holds. For simplicity, we illustrate our results by an example which satisfies all of the assumptions described above (see, Example 5.2).

We close the section by designing the following example, which illustrates how to find efficient solutions to (MP) with convex polynomial data via Algorithms 1.
Algorithm 1 Finding Efficient Solutions to \((\text{MP})\)

**Input**: Fix \(\lambda \in \text{int } \mathbb{R}^p\).

**Step 0.** Set \(k = k_0\).

**Step 1.** Pick \(z \in \mathbb{R}^n\) arbitrarily.

**Step 2.** If \(K_z = \emptyset\), then return to Step 1; otherwise, go to Step 3.

**Step 3.** Solve \((Q_k z)\) (or \((P_k z)\)) and obtain its optimal solution \(\bar{y}\).

**Step 4.** If the flat truncation condition (6) is satisfied, go to Step 5; otherwise, set \(k = k + 1\) and go back to Step 3.

**Step 5.** Extract a unique optimal solution \(\bar{x}\) to problem \((P_z)\) from \(\bar{y}\).

**Output**: Efficient solution \(\bar{x}\) (by Proposition 3.1).

**Example 5.2.** Consider the following 2-dimensional multi-objective convex polynomial optimization problem:

\[
(\text{MP})_1 \quad \min_{(x_1, x_2) \in \mathbb{R}^2} (f_1(x_1, x_2), f_2(x_1, x_2), f_3(x_1, x_2)) \\
\text{s.t.} \quad g_1(x_1, x_2) \leq 0, \\
\quad g_2(x_1, x_2) \leq 0,
\]

where \(f_1(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 2)^2\), \(f_2(x_1, x_2) = x_1 + x_2\), \(f_3(x_1, x_2) = x_1 + 2x_2\), \(g_1(x_1, x_2) = -x_1\), and \(g_2(x_1, x_2) = -x_2\). Let \(K_1 = \{(x_1, x_2) \in \mathbb{R}^2 : -x_1 \leq 0, -x_2 \leq 0\} = \mathbb{R}^2_+\) be the feasible set of \((\text{MP})_1\).

It is worth noting that the best known set of efficient solutions to \((\text{MP})_1\) is as follows:

\[
\{(x_1, x_2) \in \mathbb{R}^2 : \text{ either } 0 \leq x_1 \leq 0, x_2 = 0 \text{ or } x = t_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad t_1 + t_2 + t_3 = 1, \ t_i \geq 0, \ i = 1, 2, 3\}
\]

(see, e.g., [5, Example 2 in Chapter 6]).

Now, consider the following (scalar) optimization problem with \(\lambda = (\lambda_1, \lambda_2, \lambda_3) := (1, 1, 1)\),

\[
(\text{P}_z)_1 \quad \min_{(x_1, x_2) \in \mathbb{R}^2} (x_1 - 3)^2 + (x_2 - 2)^2 + 2x_1 + 3x_2 \\
\text{s.t.} \quad -x_1 \leq 0, -x_2 \leq 0, \\
\quad (x_1 - 3)^2 + (x_2 - 2)^2 \leq (z_1 - 3)^2 + (z_2 - 2)^2, \\
\quad x_1 + x_2 \leq z_1 + z_2, \\
\quad x_1 + 2x_2 \leq z_1 + 2z_2.
\]
Let \((z_1, z_2) \in K_1\) be any given. Then we see that the Slater-type condition for \((P_z)_1\) holds except \((z_1, z_2) = (3, 2)\). On the other hand, if we choose \((z_1, z_2) = (3, 2)\), then the feasible set of \((P_z)_1\) is \(\{(3, 2)\}\). So, in this case, we have \((3, 2)\) is an optimal solution to \((P_z)_1\), and so is an efficient solution to \((MP)_1\). Moreover, a simple computation yields that the Hessian \(\sum_{j=1}^{3} \nabla^2 f_j\) is positive definite on \(\mathbb{R}^2\), and hence, for all \(z := (z_1, z_2) \in K_1 \setminus \{(3, 2)\}\), all of the assumptions of Theorem 5.2 are satisfied.

On the other hand, for \(k \geq 1\), the hierarchy semidefinite programming problem, related with \((P_z)_1\), reads as follows

\[
(P_z^k)_1 \quad \inf_{y \in \mathbb{R}^{s(2k)}} \sum_{\alpha \in \mathbb{N}_{2k}^2} \sum_{j=1}^{3} (f_j)_{\alpha} y_{\alpha} \\
\text{s.t.} \quad M_k(y) \succeq 0, \\
\sum_{\alpha \in \mathbb{N}_{2k}^2} (g_i)_{\alpha} y_{\alpha} \leq 0, \quad i = 1, 2, \\
\sum_{\alpha \in \mathbb{N}_{2k}^2} (f_j)_{\alpha} y_{\alpha} \leq f_j(z), \quad j = 1, 2, 3, \\
M_{k-1}((-\lambda^T f_z)y) \succeq 0.
\]

Now, let us pick \(z = (1, 1) \in K_1\). Then we consider the problem \((P_z^k)_1\) with \(k = 1\)

\[
(P_z^1)_1 \quad \inf_{y \in \mathbb{R}^6} 13 - 4y_{(1,0)} - y_{(0,1)} + y_{(2,0)} + y_{(0,2)} \\
\text{s.t.} \quad M_1(y) = \begin{pmatrix} 1 & y_{(1,0)} & y_{(0,1)} \\ y_{(1,0)} & y_{(2,0)} & y_{(1,1)} \\ y_{(0,1)} & y_{(1,1)} & y_{(0,2)} \end{pmatrix} \succeq 0, \\
y_{(1,0)} \leq 0, \quad -y_{(0,1)} \leq 0, \\
13 - 6y_{(1,0)} - 4y_{(0,1)} + y_{(2,0)} + y_{(0,2)} \leq 5, \\
y_{(1,0)} + y_{(0,1)} \leq 2, \quad y_{(1,0)} + 2y_{(0,1)} \leq 3, \\
M_0((9 - \lambda^T f)y) = -4 + 4y_{(1,0)} + y_{(0,1)} - y_{(2,0)} - y_{(0,2)} \geq 0.
\]

Solving \((P_z^1)_1\) using GloptiPoly 3 \[11\] yields an optimal value \(8.875\) and an optimal solution

\[
\bar{y} = (1, 1.7500, 0.2500, 3.0625, 0.4375, 0.0625).
\]

Then, we easily check that \(\text{rank } M_1(\bar{y}) = 1 = \text{rank } M_0(\bar{y})\), and so, \(x = (\bar{y}_a)_{|a|=1} = (1.75, 0.25)\) is an optimal solution to \((P_z)_1\). It follows from Proposition 3.1 that \(x = (1.75, 0.25)\) is an efficient solution to \((MP)_1\).

In order to find more efficient solutions to \((MP)_1\), we need to parametrically move \(z \in K\). So, we give 1000 points \(z \in K\) arbitrarily to find more efficient solutions and Figure \[1\] shows
that, for the given 1000 points $z$, all of obtained efficient solutions to (MP)$_1$ belongs to the known set of efficient solutions to (MP)$_1$.

6. Conclusions

In this paper, we mainly investigated the issue that how to find efficient solutions of a multi-objective programming problem with convex polynomial data by using the well-known hybrid method and the SDP relaxation approach. To this end, an existence result for efficient solutions to (MP) under some mild assumption was firstly established; moreover, we proved that for each $\lambda \in \mathbb{R}^p$, the problem (P$_z$) generically possesses a unique optimal solution. Then, two kinds of representations of non-negativity of convex polynomials over convex semi-algebraic sets were formulated, and two kinds of Lasserre-type hierarchies of SDP relaxations for problem (P$_z$) with their finite convergence results were also discussed. Finally, we showed how to find efficient solutions to the problem (MP) by solving hierarchies of semidefinite programming relaxations and checking a flat truncation condition.

Acknowledgments. The authors would like to express their sincere thanks to Prof. Tiến-Sơn Phạm of University of Dalat for his valuable suggestions and warm helps.

References

[1] Ahmadi, A.A., Parrilo, P.A.: A convex polynomial that is not SOS-convex. Math. Program. 135(1-2), 275–292 (2012)
[2] Ahmadi, A.A., Parrilo, P.A.: A complete characterization of the gap between convexity and SOS-convexity. SIAM J. Optim. 23(2), 811–833 (2013)
[3] Belousov, E.G., Klatte, D.: A Frank–Wolfe type theorem for convex polynomial programs. Comput. Optim. Appl. 22(1), 37–48 (2002)

[4] Blanco, V., Puerto, J., Ali, S.E.H.B.: A semidefinite programming approach for solving multiobjective linear programming. J. Glob. Optim. 58, 465–480 (2014)

[5] Chankong, V., Haimes, Y.Y.: Multiobjective Decision Making: Theory and Methodology. Amsterdam: North-Holland (1983)

[6] Ehrgott, M.: Multicriteria Optimization (2nd ed.), Springer, Berlin (2005)

[7] Giannessi, F., Mastroeni, G., Pellegrini, L.: On the theory of vector optimization and variational inequalities. Image space analysis and separation. In: F. Giannessi (ed.): Vector Variational Inequalities and Vector Equilibria: Mathematical Theories. Nonconvex Optimization and Its Applications, vol. 38, pp. 141–215. Kluwer (2000)

[8] Hà, H.-V., Phâm, T.S.: Genericity in Polynomial Optimization. World Scientific Publishing (2017)

[9] Helton, J.W., Nie, J.W.: Semidefinite representation of convex sets. Math. Program. 122(1), 21–64 (2010)

[10] Henrion, D., Lasserre, J.: Detecting global optimality and extracting solutions in GloptiPoly. In: Positive Polynomials in Control. Lecture Notes in Control and Information Science vol. 312, pp. 293–310. Springer, Berlin (2005)

[11] Henrion, D., Lasserre, J.B., Loefberg, J.: GloptiPoly 3: moments, optimization and semidefinite programming. Optim. Methods Softw. 24, 761–779 (2009)

[12] Jeyakumar, V., Phảm, T.S., Li, G.: Convergence of the Lasserre hierarchy of SDP relaxations for convex polynomial programs without compactness. Oper. Res. Lett. 42, 34–40 (2014)

[13] Kim, D.S., Mordukhovich, B.S., Phảm, T.S., Tuyen, N. V.: Existence of efficient and properly efficient solutions to problems of constrained vector optimization. Optimization and Control (math.OC) (2018) arXiv:1805.00298 [math.OC]

[14] Lasserre, J.B.: Global optimization with polynomials and the problem of moments. SIAM J. Optim. 11(3), 796–817 (2001)

[15] Lasserre, J.B.: Convexity in semialgebraic geometry and polynomial optimization. SIAM J. Optim. 19(4), 1995–2014 (2009)

[16] Lasserre, J.B.: Moments, Positive Polynomials and their Applications, vol. 1. Imperial College Press, London (2010)

[17] Lasserre, J.B.: An Introduction to Polynomial and Semi-Algebraic Optimization. Cambridge University Press (2015)

[18] Lee, G.M., Phảm, T.S.: Stability and genericity for semi-algebraic compact programs. J. Optim. Theory Appl. 169(2), 473–495 (2016)
[19] Lee, J.H., Jiao, L.G.: Solving fractional multicriteria optimization problems with sum of squares convex polynomial data. J. Optim. Theory Appl. 176(2), 428–455 (2018)
[20] Lee, J.H., Jiao, L.G.: Finding efficient solutions for multicriteria optimization problems with sos-convex polynomials. Taiwanese J. Math (2019). doi:10.11650/tjm/190101
[21] Luc, D.T.: Multiobjective Linear Programming: An Introduction. Springer International Publishing, Switzerland (2016)
[22] Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications. Springer-Verlag, Berlin (2006)
[23] Nie, J.: Certifying convergence of Lasserre’s hierarchy via flat truncation. Math. Program. 142, 485–510 (2013)
[24] Putinar, M.: Positive polynomials on compact semi-algebraic sets. Indiana Univ. Math. J. 42(3), 969–984 (1993)
[25] Putinar, M., Vasilescu, F.-H.: Positive polynomials on semi-algebraic sets. C. R. Acad. Sci. Paris Sér. I Math. 328(7), 585–589 (1999)
[26] R. T. Rockafellar, Convex Analysis. Princeton Univ. Press, Princeton, New Jersey (1970)
[27] Scheiderer, C.: Sums of squares on real algebraic curves. Math. Z. 245, 725–760 (2003)
[28] Scheiderer, C.: Distinguished representations of non-negative polynomials. J. Algebra 289(2), 558–573 (2005)
[29] Schweighofer, M.: Optimization of polynomials on compact semialgebraic sets. SIAM J. Optim. 15(3), 805–825 (2005)
[30] Schm"udgen, K.: The $K$-moment problem for compact semi-algebraic sets. Math. Ann. 289(2), 203–206 (1991)

(Liguo Jiao) SCHOOL OF MATHEMATICAL SCIENCES, SOOCOW UNIVERSITY, SUZHOU 215006, JIANGSU PROVINCE, CHINA
E-mail address: hanchezi@163.com

(Jae Hyoung Lee) DEPARTMENT OF APPLIED MATHEMATICS, PUKYONG NATIONAL UNIVERSITY, BUSAN, 48513, KOREA
E-mail address: mc7558@naver.com

(Nithirat Sisarat) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, NARESUAN UNIVERSITY, PHITSANULOK 65000, THAILAND
E-mail address: nithirats@hotmail.com