Some new applications of $\beta$-Laplace integral transform

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Abstract. The $\beta$-Laplace integral transform ($\beta$-LIT) has been introduced recently by Gaur et. al. which is a generalization of Laplace transform. This new $\beta$-LIT shows some special properties over Laplace integral transform. In the current paper, we investigate the applicability of the $\beta$-LIT technique to solve partial differential equation such as initial value problem of first order and second order PDE, wave equation, Diffusion equation, Potential and current equation, Heat conduction equation, Axisymmetric Heat conduction equation, Inhomogeneous partial differential equation, inhomogeneous wave equation, Stroke and Rayleigh problem in fluid dynamics and illustrate by some examples.

1. INTRODUCTION

It is obvious that many of physical science and engineering science processes are best described by Partial differential equation. Partial differential equation has many applications in the various fields such as electrical engineering, chemical science, physical science, life science, mechanics, control theory, economics, image processing, signal processing vibration and control, statistics, kinetic model, Riesz potential, geophysics, fluid dynamics, bio-medical engineering, chaos theory and many more [1,2,5,14,15,16,17].

Laplace integral transform has been effectively used for almost 150 years in mathematics, physics, and engineering science for solving numerous problems [7,8,9,18,20].

Recently, Gaur et al. introduced a generalization of Laplace transform (LT) and named $\beta$-Laplace integral transform ($\beta$-LIT) which shows some mathematical advantage over Laplace transform. This new $\beta$-LIT is very interesting transform and shows many special properties [11,12,13].

For any exponentially bounded and sectionally continuous function $\psi: [0, \infty) \rightarrow \mathbb{R}$, the $\beta$-LIT is defined by [12]

$$\mathcal{L}_\beta \{\psi\}(s) = \int_0^\infty e^{-s \xi} \psi(\xi) d\xi \ , \ \beta > 1, \Re(s) > \frac{\alpha}{\log \beta} . \quad (1)$$

Where $\alpha$ is exponential order of the function $\psi(x)$, and $s \in \mathbb{C}$.

In the current paper, we investigate the applicability of the $\beta$-Laplace integral transform technique to solve partial differential equation such as initial value problem of first order and second order PDE, wave equation, Diffusion equation, Potential and current equation, Heat conduction equation,
Axisymmetric Heat conduction equation, Inhomogeneous partial differential equation, inhomogeneous wave equation, Stroke and Rayleigh problem in fluid dynamics and illustrate by some example [3,4,6,10,19].

In our previous paper [11], we have seen that in case of ODE, the β-LIT gives rise to an algebraic equation for the transform which can be easily solved by the ordinary methods. In case of Partial Differential Equation (PDE), We shall show that the β-LIT gives rise to an ODE for the transform which can be easily solved by the ordinary methods.

2. MAIN RESULTS

Theorem 2.1. Let the function \( u(x,t) \) be defined for \( a \leq x \leq b \) and \( t > 0 \) and \( \mathcal{L}_\beta \{ u(x,t) \} = U_\beta(x,s) \), then

(i) \[ \mathcal{L}_\beta \left( \frac{\partial u}{\partial t} \right) (s) = s \log \beta U_\beta - u(x,0). \]

(ii) \[ \mathcal{L}_\beta \left( \frac{\partial^2 u}{\partial t^2} \right) (s) = (s \log \beta)^2 U_\beta - s \log \beta u(x,0) - \left( \frac{\partial u}{\partial t} \right)_{t=0}. \]

(iii) \[ \mathcal{L}_\beta \left( \frac{\partial u}{\partial x} \right) (s) = \frac{dU_\beta}{dx}. \]

(iv) \[ \mathcal{L}_\beta \left( \frac{\partial^2 u}{\partial x^2} \right) (s) = \frac{d^2U_\beta}{dx^2}. \]

Proof.
(i) By definition

\[
\mathcal{L}_\beta \left( \frac{\partial u}{\partial t} \right) (s) = \int_0^\infty \beta^{-st} \frac{\partial u}{\partial t} dt,
\]

\[
\mathcal{L}_\beta \left( \frac{\partial u}{\partial t} \right) (s) = \lim_{r \to \infty} \left( \beta^{-st} u(x,t) \right)_0^r + s \log \beta \int_0^r \beta^{-st} u(x,t) dt,
\]

\[
\mathcal{L}_\beta \left( \frac{\partial u}{\partial t} \right) (s) = s \log \beta U_\beta - u(x,0). \tag{2}
\]

(ii) Let \( \frac{\partial u}{\partial t} = v \), then

\[
\mathcal{L}_\beta \left( \frac{\partial^2 u}{\partial t^2} \right) (s) = s \log \beta \mathcal{L}_\beta \{ v \} - v(x,0),
\]

\[
\mathcal{L}_\beta \left( \frac{\partial^2 u}{\partial t^2} \right) (s) = s \log \beta \left[ s \log \beta \mathcal{L}_\beta \{ u \} - u(x,0) \right] - \left( \frac{\partial u}{\partial t} \right)_{t=0}, \tag{3}
\]

(iii) By definition

\[
\mathcal{L}_\beta \left( \frac{\partial u}{\partial x} \right) (s) = \int_0^\infty \beta^{-st} \frac{\partial u}{\partial x} dt,
\]

Applying Leibnitz rule [16]

\[
\mathcal{L}_\beta \left( \frac{\partial u}{\partial x} \right) (s) = \frac{d}{dx} \int_0^\infty \beta^{-st} u dt = \frac{dU_\beta}{dx}. \tag{4}
\]

(iv) Let \( \frac{\partial u}{\partial x} = v \), then
\[ L_\beta \left( \frac{\partial^2 u}{\partial x^2} \right)_s = L_\beta \left( \frac{\partial v}{\partial x} \right)_s = \frac{d}{dx} \left[ L_\beta \{ v \}_s \right] = \frac{d}{dx} \left[ L_\beta \{ u \}_s \right] \]
\[ = \frac{d}{dx} \left[ \frac{dU_\beta}{dx} \right] = \frac{d^2 U_\beta}{dx^2}. \]  

(5)

3. ILLUSTRATIVE EXAMPLE

3.1 Initial and Boundary Value Problem of First Order

Example 3.1.1. \( \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u. \)

Given \( u(x, 0) = 6e^{-3x}, \ x > 0, \ t > 0. \)

Solution. Taking \( \beta \)-Laplace Integral transform both the sides of the given PDE,

\[ L_\beta \left( \frac{\partial u}{\partial x} \right)_s = 2L_\beta \left( \frac{\partial u}{\partial t} \right)_s + L_\beta \{ u \}_s, \]

Using theorem (2.1),

\[ \frac{dU_\beta}{dx} = 2 \left[ \text{slog} \beta U_\beta - u(x, 0) \right] + U_\beta, \]

By the given boundary conditions

\[ \frac{dU_\beta}{dx} - (2\text{slog} \beta + 1)U_\beta = -12e^{-3x}, \]

Which is an ordinary linear differential equation, after solving, we obtain

\[ U_\beta = ce^{(2\text{slog} \beta + 1)x} + \frac{6}{\text{slog} \beta + 2} e^{-3x}, \]

Where \( c \) is the constant of integration.

Since, when \( x \to \infty, \ u(x, t) \) must be bounded,

Therefore, when \( x \to \infty, U_\beta(x, s) \) should also be bounded.

Hence \( c = 0. \)

From above equation, we get

\[ U_\beta(x, s) = \frac{6}{\text{slog} \beta + 2} e^{-3x}, \]

Now taking inverse \( \beta \)-LIT of both sides \( u(x, t) = 6e^{-2t-3x}. \)

Example 3.1.2. Consider the equation

\[ x \frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = x, \quad t > 0, \ x > 0, \]

With the conditions

\[ u(0,t) = 0 \quad \text{for} \ t > 0. \]
\[ u(x,0) = 0 \quad \text{for} \ x > 0, \]

Solution. Apply the \( \beta \)-LIT

\[ \text{slog} \beta U_\beta(x, s) + x \frac{dU_\beta}{dx} = \frac{x}{\text{slog} \beta}, \quad U_\beta(0, s) = 0. \]

after solving,

\[ U_\beta(x, s) = Cx^{-s} + \frac{x}{\text{slog} \beta (\text{slog} \beta + 1)}, \]

Where \( C \) is integral constant, Since \( U_\beta(0, s) = 0, \implies C = 0 \) for a bounded solution.

Therefore,


\[ U_\beta(x,s) = \frac{x}{s \log(\beta(s)) + 1} = x \left( \frac{1}{s \log(\beta)} - \frac{1}{s \log(\beta) + 1} \right), \]

taking Inverse \( \beta \)-LIT

\[ u(x, t) = x(1 - e^{-t}). \]

**Example 3.1.3.** Consider the equation

\[ \frac{\partial u(x,t)}{\partial x} + x \frac{\partial u(x,t)}{\partial t} = x, \quad t > 0, \quad x > 0, \]

With the conditions

\[ u(0, t) = 0 \quad \text{for} \quad t > 0. \]
\[ u(x, 0) = 0 \quad \text{for} \quad x > 0, \]

**Solution.** Apply the \( \beta \)-LIT

\[ x s \log(\beta) U_\beta(x,s) + \frac{du}{dx} = \frac{x}{s \log(\beta)} U_\beta(0, s) = 0. \]

after solving, obtain

\[ U_\beta(x,s) = \frac{1}{(s \log(\beta))^2} + C \beta^{-\frac{1}{2}x^2}, \]

Where, \( C \) is an integral constant, since solution is bounded. Therefore,

\[ U_\beta(0, s) = 0, \Rightarrow C = -\frac{1}{(s \log(\beta))^2}. \]

After putting the value of \( C \),

\[ U_\beta(x,s) = \frac{1}{(s \log(\beta))^2} \left[ 1 - \beta^{-\frac{1}{2}x^2} \right]. \]

taking Inverse \( \beta \)-LIT

\[ u(x, t) = t - \left( t - \frac{1}{2} x^2 \right) H \left( t - \frac{x^2}{2} \right). \]

Or, equivalently

\[ u(x, t) = \begin{cases} 
\frac{x^2}{2}, & \text{if} \quad t > \frac{x^2}{2} \\
t, & \text{if} \quad t < \frac{x^2}{2}.
\end{cases} \]

3.2 Initial-Boundary Value Problem of Second Order

**Example 3.2.1.** Let consider equation,
\[
\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = xt,
\]

Given
\[
u(x, 0) = 0, \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0.
\]

**Solution.** Taking \(\beta\)-LIT both the sides of the given PDE
\[
\mathcal{L}_\beta \left( \frac{\partial^2 u}{\partial x^2} \right) - \mathcal{L}_\beta \left( \frac{\partial^2 u}{\partial t^2} \right) = \mathcal{L}_\beta (xt),
\]

And we get
\[
\frac{d^2 U_\beta}{dx^2} - \left[ \left( \frac{\log \beta}{s} \right)^2 U_\beta - \frac{s \log \beta u(x, 0) - \left. \frac{\partial u}{\partial t} \right|_{t=0}}{s \log \beta} \right] = \frac{x}{(s \log \beta)^2}.
\]

Applying the boundary conditions,
\[
\frac{d^2 U_\beta}{dx^2} - \left( \frac{\log \beta}{s} \right)^2 U_\beta = \frac{x}{(s \log \beta)^2},
\]

after solving,
\[
U_\beta = c_1 \beta^s x + c_2 \beta^{-s} x - \frac{x}{(s \log \beta)^2}.
\]

From the boundary conditions, \(x \to \infty, U_\beta = 0 \Rightarrow c_1 = 0,\)

Again when \(x = 0, U_\beta = 0 \Rightarrow c_2 = 0,\)

After we get
\[
U_\beta = -\frac{x}{(s \log \beta)^2}.
\]

Now taking inverse \(\beta\)-LIT of both sides
\[
u(x, t) = -\frac{x^3}{6}.
\]

### 3.3 The Heat Equation

**Definition 3.3.1.** Heat Equation
Heat equation (one dimensional) is given by the PDE
\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.
\]

Where \(u(x, t)\) denote the temperature of the body.

\(k\) is Diffusivity and \(k = \frac{c^2}{c_p^2} K\) is Thermal Conductivity, \(c\) is specific heat, \(\rho\) is Density. All these have been assumed constants.

**Example 3.3.2.** The temperature \(u(x, t)\) of the body is given by the BVP
\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad x, t > 0.
\]

Boundary conditions:
\[
u(x, 0) = 0, \quad u(0, t) = U_0.
\]

**Solution.** Since the temperature of the body is finite.

therefore, \(\exists K(> 0), \forall x, t, \) such that
\[
|u(x, t)| < K.
\]

Taking \(\beta\)-LIT of both sides
\[
\mathcal{L}_\beta \left( \frac{\partial u}{\partial t} \right)_{(s)} = k \mathcal{L}_\beta \left( \frac{\partial^2 u}{\partial x^2} \right)_{(s)}',
\]

\[
\frac{\partial u_\beta}{\partial t} - u(x, 0) = k \frac{d^2 u_\beta}{dx^2}, \quad \text{where } \mathcal{L}_\beta \{u(x, t)\} = U_\beta(x, s)
\]

\[
\frac{d^2 U_\beta}{dx^2} - \frac{\log \beta}{k} U_\beta = 0
\]

After solving, we get
\[
U_\beta(x, s) = c_1 e^{\frac{s \log \beta}{k}} + c_2 e^{-\frac{s \log \beta}{k}},
\]

Where \(c_1\) and \(c_2\) are the constants of integration.

To determine \(c_1\) and \(c_2:\)
Since \( u(x, t) \) is finite as \( x \to \infty \)
Then \( U_\beta (x, s) \) is finite when \( x \to \infty \) otherwise the temperature will be infinity
\[ \therefore c_1 = 0 \]

Applying the \( \beta \)-LIT and the condition \( u(0, t) = U_0, \)
\[ L_\beta \{ u(0, t) \} = U_0 L_\beta \{ 1, \} \]
\[ U_\beta (0, s) = \frac{u_0}{s \log \beta} \]

Taking \( x = 0 \), gives \( c_2 = \frac{u_0}{s \log \beta} \).

Now using equations
\[ U_\beta (x, s) = \frac{u_0}{s \log \beta} e^{-\sqrt{s \log \beta} \frac{x}{\kappa}} \],

Now taking inverse \( \beta \)-LIT
\[ u(x, t) = U_0 \left\{ 1 - erf \left( \frac{x}{2 \sqrt{\kappa t}} \right) \right\} \],

### 3.4 Diffusion Equation in Finite Medium

#### Example 3.4.1.
Let consider the diffusion equation
\[ \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, \quad t > 0, \]
With conditions,
\[ u(x, 0) = 0, \quad 0 < x < a, \]
\[ u(0, t) = K, \quad t > 0, \]
\[ \left( \frac{\partial u(x, t)}{\partial x} \right)_{x=a} = 0, \quad t > 0, \]
Where, \( K \) is a constant.

**Solution.** Applying the \( \beta \)-LIT with respect to \( t \) and obtain
\[ \frac{d^2 U_\beta}{dx^2} - \frac{s \log \beta}{\kappa} U_\beta = 0, \quad 0 < x < a, \]
\[ U_\beta (0, s) = \frac{M}{s \log \beta}, \quad \left( \frac{d}{dx} (U_\beta (x, s)) \right)_{x=a} = 0. \]

The general solution of the differential equation is
\[ U_\beta (x, s) = A \cosh \left( x \sqrt{\frac{s \log \beta}{\kappa}} \right) + B \sinh \left( x \sqrt{\frac{s \log \beta}{\kappa}} \right). \]

After applying the given condition, we get the value of constants \( A \) and \( B \).
Now we have
\[ U_\beta (x, s) = \frac{K}{s \log \beta} \cdot \frac{\cosh \left( a - x \sqrt{\frac{s \log \beta}{\kappa}} \right)}{\cosh \left( a \sqrt{\frac{s \log \beta}{\kappa}} \right)}. \]
Finally, taking inverse \( \beta \)-LIT
The inverse β-LIT can be found by applying Cauchy Residue Theorem. Then we have

\[ u(x, t) = M \left[ 1 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \sin \left( \frac{(2n-1)\pi x}{2a} \right) e^{-\frac{(2n-1)^2 \pi^2 \kappa t}{2a^2}} \right]. \]

### 3.5 Wave Equation

**Definition 3.5.1.** Wave Equation (One dimensional equation)

Wave equation (one dimensional)

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0, \quad t > 0 \]

With the conditions

\[
\begin{align*}
    u(0, t) &= A f(t), \quad t \geq 0, \\
    u(x, t) &\to 0 \text{ as } x \to \infty, \quad t \geq 0, \\
    u(x, t) &= \frac{\partial u(x, t)}{\partial t}, \quad \text{at } t = 0 \text{ for } 0 < x < \infty.
\end{align*}
\]

**Solution.** Taking β-LIT

\[
\mathcal{L}_\beta \left\{ \frac{\partial^2 u}{\partial t^2} \right\}(s) = c^2 \mathcal{L}_\beta \left\{ \frac{\partial^2 u}{\partial x^2} \right\}(s),
\]

\[
(s \log \beta)^2 U_\beta(x, s) - s \log \beta u(x, 0) - \left( \frac{\partial u(x, t)}{\partial t} \right)_{t=0} = c^2 \frac{d^2 U_\beta}{dx^2},
\]

Where,

\[ U_\beta(x, s) = U_\beta(x, s). \]

By the conditions,

\[
\begin{align*}
    \frac{d^2 U_\beta}{dx^2} - \frac{(s \log \beta)^2}{c^2} U_\beta &= 0, \text{ for } 0 \leq x < \infty, \\
    U_\beta(x, s) &= A \mathcal{L}_\beta \{ f(t) \}(s) \text{ at } x = 0, \\
    U_\beta(x, s) &\to 0 \text{ as } x \to \infty.
\end{align*}
\]

After, solving the differential equation

\[ U_\beta(x, s) = A \mathcal{L}_\beta \{ f(t) \}(s) \beta^{\frac{cx}{x_c}}. \]

Taking inverse β-LIT

\[ u(x, t) = A f \left( t - \frac{x}{c} \right) H \left( t - \frac{x}{c} \right) \]
Solution can be re-written as

\[ u(x, t) = \begin{cases} 
0, & t < \frac{x}{c} \\
Af \left( t - \frac{x}{c} \right), & t > \frac{x}{c}
\end{cases} \]

### 3.6 Equation of Current and Potential in a Transmission Line

**Example 3.6.1.** Let consider potential \( V(x, t) \) and current \( I(x, t) \) satisfy the following system

\[
L \frac{\partial I}{\partial t} + RI = -\frac{\partial V}{\partial x},
\]
\[
C \frac{\partial V}{\partial t} + GV = -\frac{\partial I}{\partial x}.
\]

In a transmission line. Where, \( L, R, C, G \) denote the inductance, resistance, capacitance, leakage conductance of the transmission line.

**Solution.** After eliminating \( V \) or \( I \), we get the same equation which is in the following form

\[
\frac{1}{c^2} u_{tt} - u_{xx} + au_t + bu = 0.
\]

Where \( c^2 = (LC)^{-1}, \ a = LG + RC, \) and \( b = RG. \)

Above equation is known as Telegraph Equation.

To solve the equation, there may be some cases:

(a) Transmission line without loss \((R = 0 \text{ and } G = 0)\)

If \( R = 0 \) and \( G = 0 \) then classical wave equation

\[
u_{tt} = c^2 u_{xx}. \]

The solution of this equation can be carried out from example(?) with the following conditions in the potential \( V(x, t) \):

\[ V(x, t) = V_{0f}(t) \text{ at } x = 0, \ t > 0. \]

With conditions

\[ x = 0 \text{ for } t > 0, \]

and

\[ V(x, t) \rightarrow 0 \text{ as } x \rightarrow \infty \text{ for } t > 0. \]

Then the solution is given by

\[ V(x, t) = V_{0f} \left( t - \frac{x}{c} \right) H \left( t - \frac{x}{c} \right). \]

(b) Ideal submarine cable (The kelvin ideal cable)

For kelvin ideal cable, \( L = 0, \) and \( G = 0 \) equation reduces to the classical diffusion equation

\[ u_t = \kappa u_{xx}, \]

Where \( \kappa = a^{-1} = (RC)^{-1}. \)
Solve as same as Example. After using the given conditions, we get potential $V(x,t)$

$$V(x,t) = V_0 \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right)$$

(c) A Non-inductive leady cable

For a non-inductive leady cable, $L = 0$ and $G \neq 0$. We get

$$V_{xx} - aV_t - bV = 0,$$

With conditions

$$V(0,t) = H(t) \text{ and } V(x,t) \to 0 \text{ as } x \to \infty.$$ The β-Laplace integral transform of the problem is

$$\frac{d^2V_\beta}{dx^2} = (a \log \beta + b) V_\beta,$$

$$V_\beta(0,s) = \frac{1}{s \log \beta}, \quad V_\beta(x,s) \to 0 \text{ as } x \to \infty.$$ Then the solution is

$$V_\beta(x,s) = \frac{1}{s \log \beta} e^{\left[-x(s \log \beta a + b)\right]^\frac{1}{2}}.$$ Taking inverse β-Laplace integral transform

$$V(x,t) = \frac{1}{2} e^{x\sqrt{\beta}} \text{erfc} \left( \frac{x}{2} \sqrt{\frac{a}{t}} + \sqrt{\frac{bt}{a}} \right) + \frac{1}{2} e^{-x\sqrt{\beta}} \text{erfc} \left( \frac{x}{2} \sqrt{\frac{a}{t} - \frac{bt}{a}} \right).$$

(d) Heaviside distortion less cable

For Heaviside distortion less cable, $R = \frac{G}{L} = \kappa = \text{constant}$, the potential $V(x,t)$ and the current $I(x,t)$ satisfy the same equation

$$u_{tt} + 2\kappa u_t + \kappa^2 u = c^2 u_{xx}, \quad 0 \leq x < \infty, \quad t > 0.$$ Apply the β-Laplace Transform and we get

$$\frac{d^2V_\beta}{dx^2} = \left( \frac{s \log \beta + \kappa}{c} \right)^2 V_\beta,$$

Then the solution for $V_\beta(x,s)$ with the boundary conditions is

$$V_\beta(x,s) = V_0 L_\beta \{f(t)\}(s) e^{\left[-\left( \frac{s \log \beta + \kappa}{c} \right)x \right]}.$$ Taking inverse β-Laplace transform, and obtain
\[ V(x, t) = V_0 e^{-\frac{kx}{c} f \left( t - \frac{x}{c} \right)} H \left( t - \frac{x}{c} \right). \]

Where, wave propagating at a velocity \( c = \left(\frac{L}{C} \right)^{\frac{1}{2}}. \)

### 3.7 Axisymmetric Heat Conduction Equation

**Example 3.7.1.** Let consider axisymmetric heat conduction equation

\[ u_t = \kappa (u_{rr} + \frac{1}{r} u_r), \quad 0 \leq r < a, \quad t > 0, \]

With conditions

\[ u_t(r, 0) = 0 \quad \text{for} \quad 0 < r < a, \]
\[ u(r, t) = f(t) \quad \text{at} \quad r = a \quad \text{for} \quad t > 0, \]

Where \( \kappa \) and \( T_0 \) are constants.

**Solution.** Apply the \( \beta \)-Laplace integral transform to equation gives

\[ \frac{d^2 U_\beta}{dr^2} + \frac{1}{r} \frac{d U_\beta}{dr} - \frac{s \text{log} \beta}{\kappa} U_\beta = 0. \]

Or,

\[ r^2 \frac{d^2 U_\beta}{dr^2} + r \frac{d U_\beta}{dr} - r^2 \left( \frac{s \text{log} \beta}{\kappa} \right) U_\beta = 0. \]

Above equation is known as Standard Bessel Equation, and the solution is given by

\[ U_\beta(r, s) = A I_0 \left( r \sqrt{\frac{s \text{log} \beta}{\kappa}} \right) + B K_0 \left( r \sqrt{\frac{s \text{log} \beta}{\kappa}} \right) \]

Where \( I_0(x) \) and \( K_0(x) \) are modified Bessel function of zero order and \( A \) and \( B \) are integral constants.

Since \( K_0 \left( r \sqrt{\frac{s \text{log} \beta}{\kappa}} \right) \) is unbounded at \( r = 0 \), then we get \( B \equiv 0 \). After putting the value

\[ U_\beta(r, s) = A I_0 \left( r \sqrt{\frac{s \text{log} \beta}{\kappa}} \right) \]

Using transformed boundary condition \( U_\beta(a, s) = \mathcal{L}_\beta \{ f(t) \}_{(s)} \), we obtain

\[ U_\beta(r, s) = \mathcal{L}_\beta \{ f(t) \}_{(s)} I_0 \left( r \sqrt{\frac{s \text{log} \beta}{\kappa}} \right) = \mathcal{L}_\beta \{ f(t) \}_{(s)} \mathcal{L}_\beta \{ g(t) \}_{(s)} \]

Where,

\[ \mathcal{L}_\beta \{ g(t) \}_{(s)} = \frac{I_0 \left( r \sqrt{\frac{s \text{log} \beta}{\kappa}} \right)}{I_0 \left( a \sqrt{\frac{s \text{log} \beta}{\kappa}} \right)} \]

By Convolution Theorem, the solution takes the form
Where,

\[ g(t) = \frac{\log \beta}{2\pi i} \int_{c-i\infty}^{c+i\infty} \beta s^{t} \int_{0}^{\infty} r \left( \frac{\log \beta}{s} \right) ds. \]

This integral can be carried out by applying the theory of residue of complex analysis where the poles of the integrand are at the points \( s = \pm \omega_{n}, n = 1, 2, 3, \ldots \) and \( \omega_{n} \) are the roots of \( J_{0}(a\alpha) = 0 \). The residue at pole \( s = \omega_{n} \) is

\[ (2\omega a) \sum_{n=1}^{\infty} \frac{\alpha J_{0}(\alpha \omega_{n})}{J_{1}(\alpha \omega_{n})} e^{-\omega_{n}^{2} x} \]

So that

\[ g(t) = \left( \frac{2\omega}{a} \right) \sum_{n=1}^{\infty} \frac{\alpha J_{0}(\alpha \omega_{n})}{J_{1}(\alpha \omega_{n})} e^{-\omega_{n}^{2} x}. \]

The solution becomes

\[ u(x, t) = \left( \frac{2\omega}{a} \right) \sum_{n=1}^{\infty} \frac{\alpha J_{0}(\alpha \omega_{n})}{J_{1}(\alpha \omega_{n})} r(t - \tau) e^{-\omega_{n}^{2} x}. \]

### 3.8 Inhomogeneous Partial Differential Equation

**Example 3.8.1.** Let an inhomogeneous problem

\[ u_{x} = -\omega \sin \omega t, \quad t > 0 \]

With conditions

\[ u(x, 0) = x, \quad u(0, t) = 0. \]

**Solution.** Applying the \( \beta \)-Laplace integral transform which gives

\[ \frac{d U_{\beta}}{d x} = \frac{\log \beta}{(\log \beta)^{2} + \omega^{2}} \]

After, solving we get

\[ U_{\beta}(x, s) = \frac{\log \beta x}{(\log \beta)^{2} + \omega^{2}} + A \]

Where A is integral constant. Since \( U_{\beta}(0, s) = 0 \), \( \Rightarrow A = 0 \), and taking \( \beta \)-Laplace integral transform

\[ u(x, t) = x \cos \omega t. \]

### 3.9 Inhomogeneous Wave Equation

**Example 3.9.1.** Consider an inhomogeneous wave equation

\[ \frac{1}{c^{2}} u_{tt} - u_{xx} = k \sin \left( \frac{nx}{a} \right), \quad 0 < x < a, \quad t > 0, \]

\[ u(x, 0) = 0 = u_{t}(x, 0), \quad 0 < x < a, \]
\[ u(0, t) = 0 = u(a, t), \quad t > 0, \]

Where \( a, c \) and \( k \) are constants.

**Solution.** Applying the \( \beta \)-LIT which gives

\[
\frac{d^2 U_\beta}{dx^2} - \frac{(\text{slo}g \beta)^2}{c^2} U_\beta = -\frac{k}{s} \sin \left( \frac{\pi x}{a} \right),
\]

\[ U_\beta(0, s) = 0 = U_\beta(a, s). \]

After solving above differential equation, we get

\[ U_\beta(x, s) = A\beta \left( \frac{\pi x}{s} \right) + B\beta \left( \frac{-\pi x}{s} \right) + \frac{ks\sin \left( \frac{\pi x}{a} \right)}{a^2\text{slo}g \beta \left( (\text{slo}g \beta)^2 + \frac{\pi^2 x^2}{a^2} \right)}. \]

In view of equation \( A = B = 0 \), and hence, the solution becomes

\[ U_\beta(x, s) = \frac{ks\sin \left( \frac{\pi x}{a} \right)}{a^2\text{slo}g \beta \left( (\text{slo}g \beta)^2 + \frac{\pi^2 x^2}{a^2} \right)}. \]

Taking inverse \( \beta \)-LIT, then solution is,

\[ u(x, t) = \frac{k}{(\pi c)^2} \left[ 1 - \cos \left( \frac{\pi c t}{a} \right) \right] \sin \left( \frac{\pi x}{a} \right). \]

### 3.10 Rayleigh and the Stokes Problem in Fluid Dynamics

**Example 3.10.1.** Consider the Stokes problem,

\[ u_t = \nu u_{zz}, \quad t > 0, \quad z > 0, \]

With some following conditions

\[
\begin{align*}
  u(z, t) &= U_0 e^{i\omega t} \quad \text{on} \quad z = 0, \quad t > 0, \\
  u(z, t) &\to 0 \quad \text{as} \quad z \to \infty, \quad t > 0, \\
  u(z, t) &\to 0 \quad \text{at} \forall z > 0, \quad t \leq 0.
\end{align*}
\]

Where, \( U_0 \) is a constant, and \( u(z, t) \) represent the fluid velocity.

**Solution.** Applying the \( \beta \)-Laplace transform and given boundary conditions. Then we get

\[ U_\beta(z, s) = \frac{U_0}{\text{slo}g \beta - i\omega} e^{-\frac{z}{\sqrt{\nu}}} \int \frac{\text{slo}g \beta}{\nu} \]

Using the inverse \( \beta \)-Laplace integral transform, we obtain the solution

\[ u(z, t) = \frac{U_0}{2} e^{i\omega t} \left[ e^{-\frac{z}{2\sqrt{\nu}}} \text{erf} c \left( \frac{z}{2\sqrt{\nu}t} - \sqrt{i\omega t} \right) + e^{\frac{z}{2\sqrt{\nu}}} \text{erf} c \left( \frac{z}{2\sqrt{\nu}t} + \sqrt{i\omega t} \right) \right]. \]
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