Order of approximation in the central limit theorem for associated random variables and a moderate deviation result

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Abstract.
An estimate of the order of approximation in the central limit theorem for strictly stationary associated random variables with finite moments of order $q > 2$ is obtained. A moderate deviation result is also obtained. We have a refinement of recent results in Çağın et al. (2016). The order of approximation obtained here is an improvement over the corresponding result in Wood (1983).

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1. Introduction
A set of random variables (rvs) $\{X_1, X_2, \ldots, X_k\}$ is said to be associated if for each pair of coordinatewise nondecreasing functions $f, g : \mathbb{R}^k \to \mathbb{R}$
\[
\text{Cov}(f(X_1, X_2, \ldots, X_k), g(X_1, X_2, \ldots, X_k)) \geq 0
\]
whenever the covariance exists.
A sequence $\{X_n\}$ of rvs is associated if for every $n \in \mathbb{N}$ the family $X_1, X_2, \ldots, X_n$ is associated.

In this paper we consider a strictly stationary sequence of centered square integrable associated rvs $\{X_n\}$. Central limit theorem (CLT) for $\{X_n\}$ was proved by Newman (1980) and a Berry-Esseen type theorem giving an estimate of the order of approximation in the CLT was proved by Wood (1983). In the case of finite third absolute moment $E|X_1|^3$ Wood’s result gives an estimate of the order $O(n^{-1/5})$. Birkel (1988) obtained a rate of the order $O(n^{-1/2} \log^2 n)$ under the strong additional assumption that the Cox-Grimmett coefficients $u(n)$ decrease exponentially. Birkel also provided an interesting example to show the reasonableness of the assumptions to obtain the above order of approximation. In that example he showed that the above rate cannot be obtained if $u(n)$ decreases only as a power. Thus there is a huge gap between the results of Wood and Birkel. In a recent paper Çağın et al. (2016) obtained another estimate of the order of approximation

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in the CLT for associated rvs and also obtained a moderate deviation type
result. However their estimate in the case of finite third absolute moment
$E|X_1|^3$ is quite complicated.

We recall here that Wood (1983) mentioned some examples of stationary
associated random variables in models for ferromagnets in mathematical
physics and in particular Ising model. Further, large deviation probability
and moderate deviation probability investigations received much attention
due to their importance in statistical inference and applied probability. We
refer to monographs by Vardhan (1984), Dembo and Zeitouni (1998) and
Hollander (2000) and recent papers by Wang (2015) and Çağın et al. (2016)
for other references. These investigations are also useful in the construction
of certain counter examples (see, for example, Tikhomirov, 1979 and Birkel,
1988).

We give an estimate of the order of approximation in the CLT which is
a refined version of the result in Çağın et al. (2016) and also prove cor-
responding moderate deviation result. In the case of $X_n$ with finite third
absolute moment, when Cox-Grimmett coefficients $u(n)$ are of order $n^{-\delta}$,
the order of approximation in the CLT is proved to go to zero as $n^{-3/8}$ as
$\delta \to \infty$. The main steps in the proof are the classical decomposition of the
partial sum $S_n = \sum_{j=1}^{n} X_j$ into blocks ( of size $p_n = [n^{1-\alpha}], 0 < \alpha < 1$ )
, coupling them with blocks variables with the same distributions but in-
dependent and use the inequality due to Newman (1980). Our approach
is similar to that in Çağın et al. However the estimate of the order of
approximation we obtain does not depend on the value of $\alpha$ whereas the
same obtained by Çağın et al. depends on $\alpha$. The refinement is in terms
of the assumptions, bound and simplification of the steps. This helps us
to get moderate deviation type result too under assumptions milder than
those in Çağın et al. (2016) and also get an order of approximation in the
CLT which is an improvement over the corresponding result in Wood (1983).

The paper is organized as follows. In Section 2 we introduce notation and
give some lemmas. In Section 3 we shall have a set of propositions that will
be used in later sections. Order of approximation in the CLT is investigated
in Section 4. Finally a moderate deviation type result is discussed in Section
5.

2. Notation

Let $\{X_n\}$ be a strictly stationary sequence of centered square integrable
associated rvs. Set $E(X_1^2) = \sigma_1^2$, $c_j = Cov(X_1, X_{1+j})$, $S_n = \sum_{j=1}^{n} X_j$, $ES_n^2 = s_n^2$ and $\sigma^2 = \sigma_1^2 + 2 \sum_{j=1}^{\infty} c_j > 0$. 

We assume that $\sum_{j=1}^{\infty} c_j < \infty$. Then

$$\frac{S_n}{\sigma \sqrt{n}} \overset{D}{\to} Z_1 \sim N(0,1)$$

where $N(0,1)$ denotes the standard normal distribution. The standard proof of this result involves writing $S_n$ as the sum of blocks of fixed size, approximating the distribution of $S_n$ by the distribution of corresponding sum of coupling block rvs (to be defined shortly) and appealing to the CLT for the coupling block rvs. We need more notation to explain this. Define initial blocks

$$Y_{j,n} = \sum_{i=(j-1)p_n+1}^{j p_n} X_i, \quad j = 1, 2, \ldots, m_n$$

and

$$Y_{m_n+1,n} = \sum_{i=m_n p_n+1}^{n} X_i$$

where $m_n = \lceil n/p_n \rceil$, $p_n < n/2$ and $\lceil r \rceil$ denotes the largest integer $\leq r$.

Clearly

$$S_n = \sum_{j=1}^{m_n} Y_{j,n} + Y_{m_n+1,n}.$$  

We note that $Y_{j,n}, j = 1, 2, \ldots, m_n$ are identically distributed. Further $n - m_n p_n \leq p_n$. We next define independent coupling blocks $Y_{j,n}^*, j = 1, 2, \ldots, m_n$, where $Y_{j,n}^* \overset{D}{=} Y_{j,n}$. Note that since the $X_k$ are strictly stationary, the rvs $Y_{j,n}^*$ are independent and identically distributed.

Set $p_n = \lceil n^{1-\alpha} \rceil$ where $0 < \alpha < 1$.

In what follows limits are taken as $n \to \infty$ and statements hold for sufficiently large values of $n$. We make some of the following assumptions on the covariances $c_j$ and moments of $X_k$ in the remainder:

Assumption $A_1$: $E|X_k|^q < \infty$ for some $q > 2$.

Assumption $A_2$: $|\frac{s^2}{n \sigma^2} - 1| = O(n^{-\theta})$ for some $\theta > 0$ where $s^2_n = ES_n^2$.

Assumption $A_3$: $u(n) = \sum_{j=n}^{\infty} c_j < C_1 n^{-\delta}$, where $\delta > 0$.

**Remark 2.1.** (i) If $k_n \to \infty$ such that $k_n/n \to 0$ then the assumption $A_2$ implies $|\frac{s^2}{n \sigma^2} - \frac{s^2_n}{k_n \sigma^2}| = O(k_n^{-\theta})$.

(ii) By the assumption that $\sum_{i=1}^{\infty} c_i < \infty$ and the fact that $\sigma^2 - \frac{s^2}{n} = 2u(n) + \frac{2}{n} \sum_{i=1}^{n-1} j c_j$ it follows that if the assumption $A_2$ holds for some $\theta > 0$ then the assumption $A_3$ holds for $\delta = \theta$ and conversely.

(iii) Under the assumption $A_1$ there exist positive constants $A$ and $B$ such that for all the positive integers $n$, $A n^{1/2} < s_n < B n^{1/2}$ and $A n^{q/2} < E|S_n|^q < B n^{q/2}$. (see (2.16) in Birkel, 1988)
Here and elsewhere $C_1, C_2, \ldots$ are positive constants independent of $n$. Further $\eta_1, \eta_2, \ldots$ are constants with absolute values $\leq 1$. The following result is known.

**Lemma 2.2.** (Newman’s inequality, 1980) Suppose $U_1, U_2, \ldots, U_n$ are associated rvs with finite variances. Then for any real numbers $t_1, t_2, \ldots, t_n$

$$
\left| E \left( \exp^{i \sum_{j=1}^{n} t_j U_j} \right) - \prod_{j=1}^{n} E \left( \exp^{it_j U_j} \right) \right| \leq \sum_{i=1}^{n} |t_i| |t_j| \text{Cov}(U_i, U_j).
$$

**Remark 2.3.** If $\{X_n\}$ is a sequence of associated rvs, the block rvs $Y_{1,n}, Y_{2,n}, \ldots, Y_{m_{\alpha,n}}$ are associated. Further the characteristic functions satisfy

$$
E \left( \exp^{i \sum_{j=1}^{m_{\alpha,n}} t_j Y_{j,n}^*} \right) = \prod_{j=1}^{m_{\alpha,n}} E \left( \exp^{it_j Y_{j,n}^*} \right) = \prod_{j=1}^{m_{\alpha,n}} E \left( \exp^{it_j Y_{j,n}} \right) \quad (1)
$$

because $Y_{j,n}^*$ are independent and $Y_{j,n}^* \overset{D}{=} Y_{j,n}$.

Let $T_1 = a_n n^{\alpha/2}$ and $T_2 = b_n n^{\alpha/2}$ where $a_n = (\log n)^a$ and $b_n = (\log n)^b$ with $a < b < 0$.

### 3. Some preliminary results

In this section we discuss some preliminary results that will be used later and these are of independent interest too. The following result notes that while dealing with asymptotic properties of $S_n/(\sigma s_n)$ it is adequate to consider the sum $\sum_{j=1}^{m_{\alpha,n}} Y_{j,n}$.

**Proposition 3.1.** Suppose the assumption $A_1$ holds. Then for $\mu_n = n^{-3\alpha/8}$, $0 < \alpha < 1$

$$
P(|Y_{m_{\alpha,n}+1,n}| > \mu_n s_n) < C_2 n^{-q\alpha/8}.
$$

To see this note that because of stationarity of $\{X_n\}$

$$
P(|Y_{m_{\alpha,n}+1,n}| > \mu_n s_n) < \frac{E|S_n-m_{\alpha,n}p_n|^q}{\mu_n^q s_n^q}.
$$

The result follows now from Remark 2.1 and the assumption $A_1$.

**Remark 3.2.** This is an improvement of the result in Step 3 of the Theorem 3.1 in Çağin et al. (2016) as it does not put any restriction on $\alpha$ and $q$.

(ii) One can chose $\mu_n = n^{-\mu \alpha}$, $0 < \mu < 1/2$ but the calculations become too complicated. See Remark 4.2 below.

Next we approximate the distribution of the sum of the original rvs by that of the coupling blocks; i.e., the distribution of $\sum_{j=1}^{m_{\alpha,n}} Y_{j,n}$ by that of $\sum_{j=1}^{m_{\alpha,n}} Y_{j,n}^*$. The method of approximation is based on the celebrated Berry-Esseen inequality and Newman’s inequality for associated rvs.
Proposition 3.3. Suppose the assumptions $A_1$ and $A_2$ hold. Then
\[
\sup_{x \in R} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq xs_n \right) - P \left( \sum_{j=1}^{m_n} Y_{j,n}^{*} \leq xs_n \right) \right| < C_3 \frac{b_n^2}{n^{\theta - \alpha(1 + \theta)}} I \left( \frac{2\theta}{3} \leq \alpha < \frac{\theta}{1 + \theta} \right) + C_4 \frac{1}{b_n^{n\alpha/2}} I \left( \alpha < \frac{2\theta}{3 + 2\theta} \right).
\]

Proof By the Berry-Essèen inequality and (1) we have
\[
\sup_{x \in R} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq xs_n \right) - P \left( \sum_{j=1}^{m_n} Y_{j,n}^{*} \leq xs_n \right) \right| < C_5 \int_{-T_2}^{T_2} \frac{1}{|t|} \left| E \left( e^{i \frac{t}{s_n} \sum_{j=1}^{m_n} Y_{j,n}} \right) - \prod_{j=1}^{m_n} E \left( e^{i \frac{t}{s_n} Y_{j,n}} \right) \right| dt + \frac{C_6}{T_2}.
\]

By the Lemma 2.2 with $U_j = Y_{j,n}$, $j = 1, 2, \cdots, m_n$ we have
\[
\left| E \left( e^{i \frac{t}{s_n} \sum_{j=1}^{m_n} Y_{j,n}} \right) - \prod_{j=1}^{m_n} E \left( e^{i \frac{t}{s_n} Y_{j,n}} \right) \right| \leq \frac{t^2}{s_n^2} \sum_{j,k=1,j>k}^{m_n} Cov(Y_{j,n}, Y_{k,n})
\]

\[
= \frac{t^2 m_n p_n}{2s_n^2} \left| \frac{s_{m_n}^2 p_n}{m_n p_n} - \frac{s_n^2}{p_n} \right|.
\]

In view of the Remark 2.1 we then have from (2)
\[
\sup_{x \in R} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq xs_n \right) - P \left( \sum_{j=1}^{m_n} Y_{j,n}^{*} \leq xs_n \right) \right| < \frac{C_7}{p_n} \int_{-T_2}^{T_2} |t| dt + \frac{C_6}{T_2}.
\]

Recalling that $p_n = [n^{1-\alpha}]$, $T_2 = b_n n^{\alpha/2}$ we note that the right side above goes to zero only for $\alpha < \theta/(1 + \theta)$. Further $(1 - \alpha)\theta \leq 3\alpha/2$ if and only if $\alpha \geq 2\theta/(3 + 2\theta)$. Hence
\[
\sup_{x \in R} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq xs_n \right) - P \left( \sum_{j=1}^{m_n} Y_{j,n}^{*} \leq xs_n \right) \right| \leq \frac{C_6}{b_n n^{\alpha/2}} I \left( \alpha < \frac{2\theta}{3 + 2\theta} \right) + C_8 \frac{b_n^2}{n^{(1 - \alpha)\theta - \alpha}} I \left( \frac{2\theta}{3 + 2\theta} \leq \alpha < \frac{\theta}{1 + \theta} \right).
\]

This completes the proof of the Proposition 3.3.
\textbf{Remark 3.4.} In Çağin et al. (2016) the above bound was obtained separately for the odd numbered blocks and the even numbered blocks. Further, the bound obtained above goes to zero faster than their corresponding bound.

Our next result is concerned with the approximation of the characteristic function of the sum of coupling blocks by the characteristic function of an appropriate normal variable.

\textbf{Proposition 3.5.} Denote $\varphi_j(t) = E\left(e^{itY_{j,n}}\right)$. Then under the assumptions $A_1$ and $A_2$, for $|t| < T_2$

$$\left| \prod_{j=1}^{mn} \varphi_j(t) - e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}} \right| \leq C_9 \frac{m_n |t|^q p_n q/2}{s_n} e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}}.$$

\textbf{Proof} Let us first consider the case $2 < q < 3$. Note that since $Y_{j,n}^* \overset{D}{=} Y_{j,n}$

$$\varphi_j(t/s_n) = 1 - \frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_1 \frac{|t|^q}{q(q-1)s_n^q} E|Y_{j,n}|^q.$$

For $|t| < T_1 = a_n n^{a/2}$, with $a_n = (\log n)^a$, $a < 0$

$$\frac{t^2 s_{p_n}^2}{s_n^2} < C_{10} a_n^2 \to 0.$$

Further

$$\frac{|t|^q}{s_n^q} E|Y_{j,n}|^q < C_{11} a_n^2 \to 0.$$

Hence $|\varphi_j(t/s_n) - 1| \to 0$ and therefore $\varphi_j(t/s_n)$ is bounded away from 0 for $|t| < T_1$ so that we can take its logarithm. Then for each $j$

$$\log \varphi_j(t/s_n) = -\frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_1 \frac{|t|^q}{q(q-1)s_n^q} E|Y_{j,n}|^q + \eta_2 \left[ -\frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_1 \frac{|t|^q}{q(q-1)s_n^q} E|Y_{j,n}|^q \right]^2$$

$$= -\frac{t^2 s_{p_n}^2}{2s_n^2} + \eta_3 \frac{|t|^q}{q(q-1)s_n^q} E|Y_{j,n}|^q.$$

Then using the fact $|e^x - 1| < |x| e^{|x|}$ we get

$$\left| \prod_{j=1}^{mn} \varphi_j(t/s_n) - e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}} \right| \leq e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}} m_n |t|^q E|Y_{1,n}|^q e^{-\frac{m_n |t|^q E|Y_{1,n}|^q}{s_n^q}}.$$

Note that $\frac{|t|^q E|Y_{1,n}|^q}{s_n^q} < a_n^{q-2} \to 0$ so that $\frac{m_n |t|^q E|Y_{1,n}|^q}{s_n^q} < \frac{m_n t^2 s_{p_n}^2}{4s_n^2}$ and hence

$$\left| \prod_{j=1}^{mn} \varphi_j(t/s_n) - e^{-\frac{m_n t^2 s_{p_n}^2}{2s_n^2}} \right| < C_{12} \frac{m_n |t|^q E|Y_{1,n}|^q}{s_n^q} e^{-\frac{m_n t^2 s_{p_n}^2}{4s_n^2}} \quad (3)$$

for $|t| < T_1$. We shall prove that the relation $(3)$ holds for $T_1 \leq |t| < T_2$ also.
Let $W_j, j = 1, 2, \ldots, m_n$ be rvs such that for each $j$, $W_j$ is independent of $Y_{j,n}^*$ and distributed as $Y_{j,n}^*$. Then $E(W_j - Y_{j,n}^*) = 0$, $E(W_j - Y_{j,n}^*)^2 = 2s_n^2$ and $E|W_j - Y_{j,n}^*|^q \leq 2^q E|Y_{j,n}^*|^q$. Further

$$|\varphi_j(t/s_n)|^2 = E\left(e^{i\frac{t}{s_n}(W_j - Y_{j,n}^*)}\right) = 1 - \frac{t^2 s_n^2}{s_n^2} + \frac{2^q |t|^q E|Y_{j,n}^*|^q}{q(q-1)s_n^q}.$$ 

Note that for $|t| < T_2 = b_n n^{\alpha/2}$ by the Lemma 2.2

$$|\varphi_j(t/s_n)|^2 < 1 - \frac{t^2 s_n^2}{s_n^2}.$$ 

Since $\frac{t^2 s_n^2}{s_n^2} \to 0$, using the fact $1 - u < e^{-u}$ for $u > 0$ we have $|\varphi_j(t/s_n)|^2 < \exp\left(-\frac{t^2 s_n^2}{4s_n^2}\right)$. Thus for $|t| < T_2$

$$\left|\prod_{j=1}^{m_n} \varphi_j(t/s_n) - e^{\frac{m_n t^2 s_n^2}{2s_n^2}}\right| < 2e^{\frac{m_n t^2 s_n^2}{4s_n^2}}. \quad (4)$$

Now to complete the proof of the claim that $(3)$ holds for $T_1 \leq |t| < T_2$ also, consider

$$C_{12} \frac{m_n |t|^q E|Y_{1,n}^*|^q}{s_n^q} > C_{12} \frac{m_n T_1^q |E S_{p_n}^2|^{q/2}}{s_n^q} > C_{15} n^\alpha a_n^q \to \infty.$$ 

Hence for $n$ large

$$C_{12} \frac{m_n |t|^q E|Y_{1,n}^*|^q}{s_n^q} > 2,$$

and the claim that $(3)$ holds for $T_1 < |t| < T_2$ also follows from $(4)$

The result of the Proposition then follows from $(3)$ and the Remark 2.1 in the case $2 < q < 3$.

In the case $q \geq 3$ we can expand $\log \varphi_j(t/s_n)$ using the third moment also and similar calculations lead to the same bound as above and hence the Proposition holds true for $q \geq 3$.

**Remark 3.6.** The above proof is similar to that in the Theorem 4.1 in Çağın et al. (2016) but has greater clarity. Further the final bound is a bit different because we use different values of $T$s.
The final result of this section is to approximate the normal distribution with the characteristic function \( e^{-\frac{m_1^2 s_n^2}{2s_n^4}} \) by the standard normal distribution.

**Remark 3.8.** The proof here is essentially the same as that of the Theorem 4.1 in Çagın et al. (2016) but is included for completeness.

**Proposition 3.9.** Let \( G_n(x) \) be the distribution function with the characteristic function \( \exp(-\frac{m_1^2 s_n^2}{2s_n^4}) \) and \( \Phi \) be the standard normal distribution function. Then

\[
\sup_{x \in \mathbb{R}} |G_n(x) - \Phi(x)| < C_{17} \frac{1}{b_n} \frac{n^\alpha/2}{1 + 2\theta} I \left( \alpha \leq \frac{2\theta}{1 + 2\theta} \right) + C_{18} \frac{1}{n^{(1-\alpha)\theta}} I \left( \alpha > \frac{2\theta}{1 + 2\theta} \right).
\]

**Proof** By the Berry-Esséen inequality

\[
\sup_{x \in \mathbb{R}} |G(x) - \Phi(x)| \leq C_{19} \int_{-T_2}^{T_2} \frac{1}{|t|} \left| e^{-\frac{m_1^2 s_n^2}{2s_n^4}} - e^{-t^2/2} \right| dt + C_{20} \frac{1}{T_2}.
\]

Using again the fact that \(|e^a - 1| \leq |a|e^{|a|}\) and recalling that \(\frac{m_1 s_n^2}{s_n^4} \to 1\) we have for large \(n\)

\[
\frac{1}{|t|} \left| e^{-\frac{m_1^2 s_n^2}{2s_n^4}} - e^{-t^2/2} \right| \leq e^{-\frac{t^2}{2}} \left| \frac{m_1 s_n^2}{s_n^4} - 1 \right| e^{\frac{t^2}{2}} \frac{m_1 s_n^2}{s_n^4} - 1.
\]

Since the normal distribution has all the moments finite

\[
\sup_{x \in \mathbb{R}} |G(x) - \Phi(x)| \leq C_{19} \frac{m_1 s_n^2}{s_n^4} - 1 + C_{20} \frac{1}{T_2}.
\]

\[
< C_{21} \frac{1}{n^{(1-\alpha)\theta}} + C_{20} \frac{1}{n^{\alpha/2} b_n}.
\]

Note that \(n^{-(1-\alpha)\theta} \to 0\) faster than \(b_n^{-1} n^{-\alpha/2}\) for \(\alpha < \frac{2\theta}{1+2\theta}\) while for \(\alpha > \frac{2\theta}{1+2\theta}\) \(b_n^{-1} n^{-\alpha/2} \to 0\) faster than \(n^{-(1-\alpha)\theta}\) giving us the stated bound.
Remark 3.10. The bound obtained here is a better bound than the corresponding bound in Çağin et al. (2016).

4. ORDER OF APPROXIMATION IN THE CLT

We now obtain an estimate of the order of approximation in the CLT which is a refined version of the result in Çağin et al. (2016). The refinement is in terms of the assumptions, bound and simplification of the steps. It also provides a better bound than the bound of order \( n^{-1/5} \) obtained from Wood’s result under the assumption of finiteness of third absolute moments. See Corollary 4.14 in Oliveira (2012).

Theorem 4.1. Let the assumptions \( A_1 \) and \( A_2 \) hold. Then

\[
\sup_{x \in \mathbb{R}} |P(S_n \leq x) - \Phi(x)| \leq C_{22} \max\left\{ n^{-\frac{q}{q+29}} I(2 < q \leq 8/3), n^{-\frac{q}{39+8+8q}} I(8/3 \leq q < 3), n^{-\frac{q}{11+8+8q}} I(q \geq 3) \right\}.
\]

In particular when \( q = 3 \) the bound becomes \( C_{22} n^{-\frac{9}{11+8+8q}} \).

Proof Recall \( \mu_n = n^{-3a/8} \). Then by the Proposition 3.11 after making elementary adjustments, we get

\[
\sup_{x \in \mathbb{R}} |P(S_n \leq x) - \Phi(x)| \leq \sup_{x \in \mathbb{R}} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq x s_n \right) - \Phi(x) \right| + P(|Y_{m_n+1,n}| > \mu_n s_n) + \sup_{x \in \mathbb{R}} \left| \Phi(x + \mu_n) - \Phi(x) \right|
\]

\[
< \sup_{x \in \mathbb{R}} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq x s_n \right) - \Phi(x) \right| + C_{23} \frac{1}{n^{2a/8}} + C_{24} \frac{1}{n^{3a/8}}
\]

\[
< \sup_{x \in \mathbb{R}} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq x s_n \right) - \Phi(x) \right| + C_{23} \frac{1}{n^{2a/8}} + C_{24} \frac{1}{n^{3a/8}}. \tag{5}
\]

Further by the Berry - Esseen inequality

\[
\sup_{x \in \mathbb{R}} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq x s_n \right) - \Phi(x) \right| \leq \sup_{x \in \mathbb{R}} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq x s_n \right) - P \left( \sum_{j=1}^{m_n} Y_{j,n}^* \leq x s_n \right) \right| + \sup_{x \in \mathbb{R}} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n}^* \leq x s_n \right) - \Phi(x) \right|
\]

\[
< \sup_{x \in \mathbb{R}} \left| P \left( \sum_{j=1}^{m_n} Y_{j,n} \leq x s_n \right) - P \left( \sum_{j=1}^{m_n} Y_{j,n}^* \leq x s_n \right) \right| + C_{25} \int_{-T_2}^{T_2} \frac{1}{t} \left| E \left( e^{i s_n \sum_{j=1}^{m_n} Y_{j,n}^*} - e^{-i t^2 / 2} \right) \right| dt + C_{26} \frac{1}{T_2}
\]
Let us say

\[ n \leq \frac{1}{T_2}. \]

where

\[ I_1 = \int_{-T_2}^{T_2} \left| \prod_{j=1}^{m_n} \varphi_j \left( \frac{t}{s_n} \right) - e^{-\frac{m_n t^2}{2 s_n^2}} \right| dt \]

and

\[ I_2 = \int_{-T_2}^{T_2} \left| e^{-\frac{m_n t^2}{2 s_n^2}} - e^{-t^2/2} \right| dt. \]

The bounds for the expressions on the right side of (5) and (6) are obtained from the Propositions 3.1, 3.3, 3.9, Corollary 3.7 and the value of \( T_2 \).

To obtain the final bound we compare \( n^{-q \alpha/8} \), \( b_n^2 n^{\theta(1-\alpha)-\alpha} \), \( b_n^{-1} n^{-\alpha/2} \), \( n^{-3\alpha/8} \), \( n^{-\alpha(q-2)/2} \) and \( n^{(1-\alpha)\theta} \) for various values of \( \alpha \) and \( q \). We consider the cases

\[ 2 < q \leq 3 \]

and the choice of \( \alpha \) in the ranges given below

\[ 0 < \alpha < \frac{2\theta}{3+2\theta} \]

\[ \frac{2\theta}{3+2\theta} < \alpha < \frac{2\theta}{q+2\theta} \]

\[ \frac{2\theta}{q+2\theta} \leq \alpha \leq \frac{\theta}{1+\theta} \]

\[ \frac{\theta}{1+\theta} \leq \alpha < \frac{2\theta}{1+2\theta} \]

\[ \frac{2\theta}{1+2\theta} \leq \alpha < 1. \]

For \( 2 < q < 8/3 \), after a tedious but elementary analysis we get the bound

\[ C_{27} \frac{1}{n^{\alpha(q-2)/2}} I \left( \left( 0 < \alpha \leq \frac{2\theta}{q + 2\theta} \right) \cup \left( \frac{\theta}{1+\theta} < \alpha \leq \frac{2\theta}{q - 2 + 2\theta} \right) \right) \]

\[ + C_{28} n^{-\alpha(1+\theta)} I \left( \frac{2\theta}{q + 2\theta} \leq \alpha < \frac{\theta}{1+\theta} \right) + C_{29} \frac{1}{n^{\theta(1-\alpha)}} I \left( \frac{2\theta}{q - 2 + 2\theta} \leq \alpha < 1 \right) \]

This can be simplified further. Since \( n^{-\alpha(q-2)/2} \) decreases as \( \alpha \) increases the best rate contributed by the first term is for the maximum value of \( \alpha \).

So we compare for \( \alpha = \frac{2\theta}{q + 2\theta} \) and \( \frac{2\theta}{q - 2 + 2\theta} \) and get the best rate \( n^{-\theta(q-2)/q+2\theta} \).

On the other hand for the same value of \( q \), the second term gives the rate

\[ b_n^2 n^{-\theta(q-2)/q+2\theta} \]

which is dominated by the previously obtained rate because \( b_n^2 \to 0 \) while the third term gives the rate \( n^{-\theta(q-2)/q+2\theta} \) which too is dominated by \( n^{-\theta(q-2)/q+2\theta} \). Thus for \( 2 < q \leq 8/3 \) we get the rate \( n^{-\theta(q-2)/q+2\theta} \).

In the case \( 8/3 \leq q \leq 3 \) the bound for the expression on the right side of (5) turns out to be
Thus in the case $8/3 \leq q \leq 3$ the best rate is $n^{-\frac{q\theta}{8+q+8\theta}}$. In the case $q \geq 3$ the bound for the expression on the right side of (5) turns out to be

$$C_{32} \frac{1}{n^{3\alpha/8}} I \left( 0 < \alpha < \frac{8\theta}{11+8\theta} \cup \left( \frac{\theta}{1+\theta} \leq \alpha \leq \frac{8\theta}{3+8\theta} \right) \right).$$

Thus in the case $q \geq 3$ the best rate is $C_{33} n^{-\frac{3\theta}{11+8\theta}}$.

The best bound turns out to be

$$C_{22} n^{-\frac{3\theta}{11+8\theta}},$$

establishing the result.

This completes the proof of the Theorem 4.1.

**Remark 4.2.**

1. In the border case of $q = 8/3$ both the bounds $n^{-\frac{\theta(q-2)}{q+2\theta}}$ and $n^{-\frac{q\theta}{8+q+8\theta}}$ coincide.

2. The bound in the Theorem 4.1 is independent of $\alpha$. However in Çağın et al. (2016) the bound depends on $\alpha$.

3. For $q = 3$ the rate is $n^{-\frac{3\theta}{11+8\theta}}$, which as $\theta \to \infty$, goes to $n^{-3/8}$ and this is far better rate than the rate $n^{-1/5}$ given in Oliveira’s book (2012).

4. As is to be expected the rate of convergence in the CLT improves as $q$ increases in the interval $(2, 3)$. Further as in the case of independent and identically distributed rvs the rate remains the same with finiteness of the moments of order $\geq 3$.

5. If $\mu_n$ is chosen as $e^{-\mu\alpha}$, $0 < \mu < 1/2$ instead of the above choice, the calculations become more complicated and we have to consider three cases; viz., $2 < q < \frac{1}{\mu}$, $\frac{1}{\mu} < q < \frac{2\mu}{1-2\mu}$ and $\frac{2\mu}{1-2\mu} < q \leq 3$ instead of $2 < q < 8/3$ and $8/3 \leq q < 3$ when $q < 3$. The best rate turns out to be $n^{-\frac{\mu\theta}{\mu+1+\theta}}$ for any choice of $q \in \left[\frac{2\mu}{1-2\mu}, 3\right]$. Interestingly the above interval collapses to the single point set consisting of 3 when $\mu = 3/8$.

5. **Moderate deviation result**

Çağın et al. (2016) recently obtained a moderate deviation result for associated rvs under strong conditions. Before we state and prove the moderate
deviation result, we shall recall a result of Frolov (2005) and apply it to coupling block rvs introduced earlier.

**Theorem 5.1.** (Theorem 1.1 in Frolov, 2005) Let \( \{Y_{k,n}, k = 1, 2, \ldots, k_n, n = 1, 2, \ldots, \} \) be an array of column-wise independent centered rvs with \( EY_{k,n}^2 = \sigma_{k,n}^2 < \infty \). Denote \( T_n = \sum_{k=1}^{k_n} Y_{k,n} \) and \( B_n = \sum_{k=1}^{k_n} \sigma_{k,n}^2 \). Assume for some \( q > 2 \), \( E[Y_{k,n}^q I(Y_{k,n} > 0)] = \beta_{k,n} < \infty \), \( B_n \rightarrow \infty \) and set

\[
M_n = \sum_{k=1}^{k_n} \beta_{k,n}, \quad L_n = \frac{M_n}{B_n^{3/2}},
\]

\[
\Lambda_n(t, s, \delta) = \frac{t}{B_n} \sum_{k=1}^{k_n} E(Y_{k,n}^2 I(-\infty < Y_{k,n} < -\delta \sqrt{B_n/s})).
\]

Assume that \( L_n \rightarrow 0 \), and that for each \( \delta > 0 \), \( \Lambda_n(x^4, x^5, \delta) \rightarrow 0 \). If \( x_n \rightarrow \infty \) such that

\[
x_n^2 - 2 \log(1/L_n) - (q-1) \log \log(1/L_n) \rightarrow -\infty, \tag{7}
\]

then

\[
P(T_n \leq x_n s_n) = (1 - \Phi(x_n))(1 + o(1)).
\]

Let the assumption \( A_1 \) hold for the original rvs \( X_n \). Recall that the block rvs \( Y_{k,n}^*, k = 1, 2, \ldots, m_n \) are independent and identically distributed for each \( n \) with \( E|Y_{k,n}|^q < \infty \) where \( q > 2 \). With \( Y_{k,n} = Y_{k,n}^*, k_n = m_n \),

\[
B_n = m_n \sigma_{p_n}^2 \sim n \sigma^2, \quad M_n \leq m_n E|Y_{k,n}^*|^q \sim n^{\alpha + (1-\alpha)/2}, \quad L_n \sim n^{\alpha(2-q)/2} \rightarrow 0
\]
as \( n \rightarrow \infty \). Further

\[
\Lambda_n(x^4, x^5, \delta) \leq \frac{x^4}{\sigma^2 n^{1-\alpha}} E(Y_{k,n}^2 I(D_n)). \tag{8}
\]

where \( D_n \) is the event \( |Y_{k,n}| > \delta \sqrt{n} \sigma / x^6 \) since \( Y_{k,n}^* \overset{D}{=} Y_{k,n} \). By the Hölder’s inequality and finiteness of moment of order \( q \) for \( Y_{k,n} \), we get

\[
E(Y_{k,n}^2 I(D_n)) \leq \left( E(Y_{k,n}^2 I(D_n))^q/2 \right)^{q/2} \left( E(I(D_n))^q/(q-2) \right)^{(q-2)/q} \\
\leq (E|Y_{1,n}|^q)^{2/q} (P(D_n))^{(q-2)/2} \leq p_n \left( \frac{E|Y_{1,n}|^q x_{5q}}{\delta^q q^{q/2} \sigma^q} \right)^{(q-2)/q},
\]

which results in the following bound from \( \delta \)

\[
\Lambda_n(x^4, x^5, \delta) \leq x^4 \left( \frac{p_n^{q/2} x_{5q}}{\delta^q q^{q/2} \sigma^q} \right)^{(q-2)/q} \\
\leq C_3 x_{5q-6} x_{\alpha(q-2)/2}.
\]

If \( x = x_n \sim (\log n)^\kappa \) and \( \kappa > 0 \) we then have \( \Lambda_n(x^4, x^5, \delta) \rightarrow 0 \) as \( n \rightarrow \infty \), so that all the conditions of the Theorem 5.1 hold and we then get the following moderate deviation result for the coupling block rvs \( Y_{k,n}^* \).
Theorem 5.2. If \( \{X_n\} \) is a sequence of centered associate rvs satisfying the assumption A1 then for the coupling block rvs \( Y_{j,n}^* \)

\[
P \left( \sum_{j=1}^{m_n} Y_{j,n}^* > x_n s_n \right) = (1 - \Phi(x_n))(1 + o(1))
\]

whenever \( x_n \) satisfies

\[
\limsup_{n \to \infty} \frac{x_n^2}{\log n} = \lambda < \alpha(q - 2).
\]

Remark 5.3. In the Theorem 4.2 of Çağin et al. (2016) the Assumption (B2) states the condition differently but a close look at the proof reveals that they indeed use \( \limsup_{n \to \infty} \frac{x_n^2}{\log n} < 1 \) which is similar to our assumption.

Corollary 5.4. Recall \( \mu_n = n^{-3\alpha/8} \). If \( x_n \) satisfies the relation (7) then so will \( x_n \pm \mu_n \) and we have

\[
P \left( \sum_{j=1}^{m_n} Y_{j,n}^* > (x_n \pm \mu_n) s_n \right) = (1 - \Phi(x_n))(1 + o(1))
\]
because \( \mu_n = o(1 - \Phi(x_n)) \). Here we use the fact \(|\Phi(x + \epsilon) - \Phi(x)| < \epsilon \).

Now we state and prove the moderate deviation result for \( S_n \).

Theorem 5.5. Let \( \{X_n\} \) be a sequence of centered stationary associated rvs satisfying the assumptions A1 and A2. Assume further

(i) \( \limsup_{n \to \infty} \frac{x_n^2}{\log n} = \lambda < \frac{q - 2}{2} \),

(ii) \( \theta \) in the assumption A2 is such that

\[
\theta > 1 + \lambda.
\]

Then

\[
P(S_n > x_n s_n) = (1 - \Phi(x_n))(1 + o(1)).
\]

Proof Choose \( \alpha \) in the definition of \( p_n \) such that

\[
\frac{1}{2} < \alpha < \frac{2\theta - \lambda}{2\theta + 2}.
\]

This is possible because of the assumption at (9). Let \( \epsilon_n = n^{-\epsilon} \) where

\[
0 < \epsilon < \frac{qa - \lambda}{2q}.
\]

This is possible because \( \lambda < (q - 2)/2 \) and \( \alpha > 1/2 \). The stated result follows from the Corollary 5.4 and the assumption (i) above if we prove

(a) \[ |P(S_n > x_n s_n) - P \left( \sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n) s_n \right) | = o(1 - \Phi(x_n)) \]
and

\[(b) \left| \left( \sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n) s_n \right) - \left( \sum_{j=1}^{m_n} Y_{j,n}^* > (x_n \pm \epsilon_n) s_n \right) \right| = o(1 - \Phi(x_n)).\]

To prove (a) recall from the Proposition 3.1

\[
P(S_n > x_n s_n) - P \left( \sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n) s_n \right) \leq P \left( |Y_{m_n+1,n}| > \epsilon_n s_n \right) \]

\[
< C_{35} \frac{p_n^{q/2}}{\epsilon_n^q n^{q/2}} < C_{36} n^{-q(\alpha-2\epsilon)/2} \quad (13)
\]

We get the result (a) if

\[
\frac{\sqrt{\log n}}{n^{(q(\alpha-2\epsilon) - \lambda)/2}} \to 0
\]

which follows from (11).

Next to prove (b) recall from the Proposition 3.3

\[
\left| \left( \sum_{j=1}^{m_n} Y_{j,n} > (x_n \pm \epsilon_n) s_n \right) - \left( \sum_{j=1}^{m_n} Y_{j,n}^* > (x_n \pm \epsilon_n) s_n \right) \right| \]

\[
< C_3 \frac{b_n^2}{n^{\theta-\alpha(1+\theta)}} I \left( \frac{2\theta}{3+2\theta} \leq \alpha < \frac{\theta}{1+\theta} \right) + C_4 \frac{1}{b_n n^{\alpha/2}} I \left( \alpha < \frac{2\theta}{3+2\theta} \right). \quad (14)
\]

The first term on the right side above is \(o(1 - \Phi(x_n))\) because (11) implies \(\theta - \alpha(1+\theta) > \lambda/2\). The second term on the right side of (14) is \(o(1 - \Phi(x_n))\) because \(\lambda < \alpha\). This completes the proof of the Theorem.

**Remark 5.6.** Çağin et al. (2016) proved the Theorem 5.1 making complicated assumptions of the type A2 as well as A3 with the conditions that \(\theta > 4\) and \(q > 3\). Further our proof does not require dealing with odd numbered and even numbered blocks separately nor does it need introduction of Gaussian centered variables similar to odd and even block sums.

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