Non-Orientable M(atrix) Theory

Nakwoo Kim and Soo-Jong Rey

Physics Department & Center for Theoretical Physics
Seoul National University, Seoul 151-742 KOREA

nakwoo@fire.snu.ac.kr, sjrey@gravity.snu.ac.kr

abstract

M(atrix) theory description is investigated for M-theory compactified on non-orientable manifolds. Relevant M(atrix) theory is obtained by Fourier transformation in a way consistent with T-duality. For nine-dimensional compactification on Klein bottle and Möbius strip, we show that M(atrix) theory is (2+1)-dimensional $\mathcal{N}=8$ supersymmetric $U(N)$ gauge theory defined on dual Klein bottle and dual Möbius strip parameter space respectively. The latter requires a twisted sector consisting of sixteen chiral fermions localized parallel to the boundary of dual Möbius strip and defines Narain moduli space of Chaudhuri-Hockney-Lykken (CHL) heterotic string. For six-dimensional CHL compactification $((S_1/\mathbb{Z}_2) \otimes T^4) / \Gamma_{\text{CHL}}$ we show that low-energy dynamics of M(atrix) theory is described by (5+1)-dimensional $\mathcal{N}=8$ supersymmetric $U(N) \times U(N)$ gauge theory defined on dual orbifold parameter space of $(\tilde{S}_1 \otimes \tilde{K}^3)/\mathbb{Z}_2$. Spacetime spectrum is deduced from BPS gauge field configurations consistent with respective involutions and is shown to agree with results from M-theory analysis.

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1 Introduction

At present, M(atrix) theory is the only available non-perturbative description of M-theory, the theory which unifies all known perturbative string theories. Based on light-front Hamiltonian formalism and holographic principle, it has been identified that the fundamental degrees of freedom of M-theory are Dirichlet zero-branes (D0-branes) of Type IIA string. Dynamics of N D0-partons is described by $\mathcal{N} = 16$ supersymmetric U(N) Yang-Mills quantum mechanics and it is the latter that defines the M(atrix) theory.

One of the most pressing issues in M(atrix) theory is to identify proper description of M-theory compactification. In the limit $d$-dimensional compactified space shrinks to a vanishing size, it has been found that Fourier transformation and T-duality yields the corresponding M(atrix) theory is $(d+1)$-dimensional quantum field theory which reduces to a supersymmetric Yang-Mills theory at low energy. Consider a compactified space $\mathcal{X}_d = \mathbb{R}_d/\Gamma$, a quotient of $d$-dimensional flat space by a symmetry group $\Gamma$. M(atrix) theory on $\mathcal{X}_d$ is obtained by taking into account of all image D0-partons located at orbits of $\Gamma$. Dynamical fields of the M(atrix) theory are gauge potential, transverse coordinates $X^\mu \equiv (A_0, X^i), \quad (i = 1, \cdots, 9)$, and super-partner spinor $\Theta^\alpha, \quad (\alpha = 1, \cdots, 16)$. As the volume of $\mathcal{X}_d$ is shrunken to zero, the configurations are most conveniently described in terms of T-dual picture. Under T-duality winding excitations are mapped into momentum excitations. As such, sum over winding string configurations is equivalent to Fourier transformation in dual space $\tilde{\mathcal{M}}$. For the simplest $S^1$ compactification, the transformation has been performed explicitly and the result turns out to be large-N limit of $(1+1)$-dimensional $\mathcal{N} = 16$ supersymmetric U(N) gauge theory with one adjoint matter multiplet.

Also studied are M(atrix) theory description of M-theory compactified on orbifolds. Among the more interesting ones are $S_1/\mathbb{Z}_2$, $T^5/\mathbb{Z}_2$ and $T^9/\mathbb{Z}_2$ orbifolds. Interesting feature common to all of them is that corresponding M(atrix) gauge theories are chiral, hence, entails nontrivial dynamics not encountered in maximally supersymmetric toroidal compactifications. Charge conservation, supersymmetry and gauge anomaly cancellations then introduce twisted sector uniquely to the M(atrix) theory. The twisted sectors all arise from fixed points of the orbifold, which can be reduced or eliminated completely if one take further quotient by freely acting involutions. The simplest quotient manifolds obtained by freely acting involution arise in two-dimensions: Klein bottle and Möbius strip. M-theory compactified on these manifolds have been discussed briefly. It has been identified that M-theory compactified on Klein bottle is a projection of Type IIA string, while the one on Möbius strip is Chaudhuri-Hockney-Lykken\textsuperscript{2} or light-cone.
(CHL) string \[\text{[CHL]}\] with \(E_8\) or \(SO(16)\) gauge groups.

In this paper, we construct M(atrix) theories for compactifications on non-orientable manifolds via straightforward application of Fourier transformation and T-duality. As will be shown in sections 3 and 4, corresponding M(atrix) theories are (2+1)-dimensional \(\mathcal{N} = 8\) supersymmetric gauge theories living on dual Klein bottle or M"obius strip parameter spaces. These so-called non-orientable M(atrix) theories are quotients of the M(atrix) theory compactified on \(T_2\) by appropriate freely acting involutions, and are obtained straightforwardly from Fourier and T-duality transformations. For both Klein bottle and M"obius strip, the M(atrix) gauge group is \(U(N) \times U(N)\) except that, at the boundary of M"obius strip, the gauge group is promoted to \(SO(2N)\). To ensure local charge conservation and cancel supersymmetry and gauge anomalies, a twisted sector consisting of sixteen (1+1)-dimensional chiral fermions needs to be introduced. Deformation of chiral fermions away from the orbifold boundary corresponds to turning on Wilson line in the CHL heterotic string. We also study BPS configurations of M(atrix) gauge theory and deduce spacetime spectrum from them. We find a complete agreement with the spectrum deduced earlier from string theory consideration. Non-orientable M(atrix) theories can be straightforwardly generalized to compactification on other higher-dimensional quotient spaces by freely acting involutions. In section 5, we illustrate this for toroidally compactified six-dimensional CHL string. We find that, at low-energy, the corresponding M(atrix) theory is supersymmetric gauge theory whose parameter space is an orbifold limit of \(S_1 \times K3\) modded out by \(\mathbb{Z}_2\) automorphism of K3. While this work is finished and is being written, we have received a work [20] that overlaps with parts of sections 3 and 4. We disagree, however, with part of the derivation and conclusion thereof.

2 M(atrix) theory on Torus

We begin with recapitulating M(atrix) theory description for M-theory compactified on a torus \(T_2\). It is intended to fix notations and essential aspects that will become relevant for later discussions. Consider M(atrix) theory for compactification on a rectangular torus \(T_2\) of size \((2\pi R_1) \times (2\pi R_2)\). The theory is defined in terms of \(N\) D0-partons living on \(T_2 \times M_8^+\). Large-\(N\) limit \(N \to \infty\) will be implicit throughout. Dynamics among D0-partons is described completely once all allowed configurations of open fundamental string connecting the partons are specified. They are encoded into \((N \times N)\) M(atrix) fields \(X^\mu \equiv (A_0, X^i)\), where \(A_0(t)\) denotes gauge potential and \(X^i(t)\) \((i = 1, \ldots, 9)\) transverse position matrices. To account for the parton dynamics correctly, it is necessary to take into consideration of string configurations that wind around one-cycles of \(T_2\). This is achieved most conveniently by arraying image D0-partons at
orbits of the symmetry group \( \Gamma = \mathbb{Z} \otimes \mathbb{Z} \) on the covering space \( \mathbb{R}_2 \), viz. \( \mathbb{T}_2 = \mathbb{R}_2/(\mathbb{Z} \otimes \mathbb{Z}) \).

Located on each fundamental cell, which is labelled by an indicial vector \( \mathbf{k} \equiv (k_1, k_2), \quad k_1, k_2 = \cdots, -2, -1, 0, 1, 2, \cdots \), are exactly \( N \) D0-partons. Introduce covering space M(atrix) fields \( X_{\mu, \mathbf{l}}^{\mathbf{k}, \mathbf{l}} \) denoting interactions along \( \mu \)-direction between \( N \) D0-partons in \( \mathbf{k} = (k_1, k_2) \)-th and \( N \) D0-partons in \( \mathbf{l} = (l_1, l_2) \)-th cells. The cell indices \( \mathbf{k}, \mathbf{l} \) are treated as generalized row and column indices. Thus \( X_{\mu, \mathbf{l}}^{\mathbf{k}, \mathbf{l}} \) for each \( \mathbf{k}, \mathbf{l} \) are \((N \times N)\) sub-matrices of infinite-dimensional matrices. Let also \( \mathbf{e}_1 \equiv (1, 0), \quad \mathbf{e}_2 \equiv (0, 1) \) denote action of \( \Gamma \) on the covering space along each direction and \( G_{ij} = \text{diag}(R_1, R_2) \) metric of \( \mathbb{T}_2 \). Modding out by the symmetry group \( \Gamma = \mathbb{Z} \times \mathbb{Z} \) imposes the following periodicity conditions:

\[
\begin{align*}
X_{1, \mathbf{k} + \mathbf{e}_1, \mathbf{k} + \mathbf{e}_1}^{1} &= X_{1, \mathbf{k}, \mathbf{k}}^{1} + \mathbf{n} \cdot (2\pi \mathbf{G}) \cdot \mathbf{e}_1 \mathbb{I}_{N \times N}, \\
X_{2, \mathbf{k} + \mathbf{e}_2, \mathbf{k} + \mathbf{e}_2}^{2} &= X_{2, \mathbf{k}, \mathbf{k}}^{2} + \mathbf{n} \cdot (2\pi \mathbf{G}) \cdot \mathbf{e}_2 \mathbb{I}_{N \times N}, \\
X_{\mu, \mathbf{l} + \mathbf{e}_1, \mathbf{l} + \mathbf{e}_1}^{\mu} &= X_{\mu, \mathbf{k} + \mathbf{e}_2, \mathbf{l} + \mathbf{e}_2}^{\mu} = X_{\mu, \mathbf{k}, \mathbf{l}}^{\mu}, \quad (\mathbf{k} \neq \mathbf{l})
\end{align*}
\]

where \( \mathbf{n} = (n_1, n_2) \) are integer-valued lattice vector. By successive operation of lattice shift, it is possible to bring the configuration at \( (\mathbf{k}, \mathbf{l}) \) to the one at \( (\mathbf{k} - \mathbf{l}, \mathbf{0}) \). Thus, we define \( X_{\mu}(\mathbf{k} - \mathbf{l}) = X_{\mu, \mathbf{k} - \mathbf{l}, 0}^{\mu} \) and express the periodicity condition Eq.(1) compactly in terms of Fourier transformed fields defined on two-dimensional parameter space \( \mathbf{y} \equiv (y_1, y_2): \)

\[
X_{\mu}(\mathbf{y}, t) = \sum_{\mathbf{k}} X_{\mu}(\mathbf{k}, t) \exp \left[ i\mathbf{k} \cdot \mathbf{G} \cdot \mathbf{y} \right], \quad (i = 1, \cdots, 9)
\]

where

\[
\begin{align*}
\tilde{G}_{ij} &= \text{diag}\left( R_1, R_2 \right), \\
R_{1,2} &= \ell_{11}/R_{11} R_{1,2}.
\end{align*}
\]

To emphasize distinctive role of compactified and non-compact spacetime coordinates, we introduce \((2+1)\)-dimensional fields \( A_\alpha = (A_0, A_1 \equiv X^1, A_2 \equiv X^2) \) and \( Y^i = X^i (i = 3, \cdots, 9) \). Then, the Chan-Paton condition Eq.(4) turns into boundary condition of the \((2+1)\)-dimensional fields:

\[
\begin{align*}
A_\alpha(\mathbf{y}, t) &= A_\alpha(\mathbf{y} + 2\pi \tilde{G} \cdot \mathbf{n}, t) \\
Y^i(\mathbf{y}, t) &= Y^i(\mathbf{y} + 2\pi \tilde{G} \cdot \mathbf{n}, t), \quad \mathbf{n} = (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z}.
\end{align*}
\]

After Fourier transformation is done explicitly, the M(atrix) quantum mechanics turns into \((2+1)\)-dimensional \( \mathcal{N} = 16 \) supersymmetric \( U(N) \) gauge theory with one adjoint matter multiplet, which is equivalent to dimensional reduction of \((9+1)\)-dimensional supersymmetric \( U(N) \) Yang-Mills theory to \((2+1)\)-dimensions. The transverse rotational invariance is realized as
\( \text{Spin}_R(7) \) R-symmetry. Lagrangian of the gauge theory is given by

\[
L = \frac{1}{2g_{YM}^2} \int_{T^2} d^2y \ Tr \left\{ F_{\alpha\beta} F^{\alpha\beta} + 2D_\alpha Y_i D^\alpha Y^i - [Y_i, Y_j][Y^i, Y^j] \\
-2\bar{\psi}_A \gamma^\alpha D_\alpha \psi_A + 2\bar{\psi}_A \gamma^i A_{AB} [Y_i, \psi_B] \right\}
\]

(5)

Denoting the M-direction radius \( R_{11} \), the gauge coupling is related to the parameters of the compactification torus \( T^2 \) as

\[
g_{YM}^2 = \frac{2R_{11}}{R_1 R_2}.
\]

(6)

Our notations are as follows. The SO(9,1) gamma matrices in Majorana-Weyl representation are decomposed into tensor product of SO(2,1) × SO(7) gamma matrices:

\[
\Gamma^\alpha = \begin{bmatrix} 0 & \gamma^\alpha \otimes \mathbb{I}_{8 \times 8} \\ \gamma^\alpha \otimes \mathbb{I}_{8 \times 8} & 0 \end{bmatrix} \quad \Gamma^i = \begin{bmatrix} 0 & -\mathbb{I}_{2 \times 2} \otimes \gamma^i \\ +\mathbb{I}_{2 \times 2} \otimes \gamma^i & 0 \end{bmatrix}
\]

(7)

such that

\[
\Gamma^{(11)} \equiv \Gamma^0 \ldots \Gamma^9 = \begin{bmatrix} +\mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{bmatrix}.
\]

(8)

All the spinors \( \psi_A \) satisfy \( \Gamma^{(11)} \psi_A = +\psi_A \). The (2+1)-dimensional gamma matrices are denoted by \( \gamma^\alpha \):

\[
\gamma^0 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & +i \\ +i & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix}
\]

(9)

Our notations are as follows. The thirty-two supersymmetries of M-theory can be decomposed as \( 16_+ \oplus 16_- \) of SO(9,1). Once the infinite momentum frame is chosen, the supersymmetries in the \( 16_- \) are broken and become non-linearly realized kinematical supersymmetries of M(atrix) theory. The other half, \( 16_+ \) supersymmetries are unbroken and defines the dynamical supersymmetries of M(atrix) theory. Then, the \( 16_\pm \) spinors associated with dynamical and kinematical supersymmetries are reduced to (2+1)-dimensional spinors \( \epsilon_A \) and \( \eta_A \) that are taken to be in inequivalent, opposite-sign representations of the (2+1)-dimensional Clifford algebra.

The M(atrix) gauge theory possesses thirty-two supercharges, among which half of them are dynamical supersymmetries in the light-front kinematics:

\[
\delta \epsilon A_\alpha = \frac{i}{2} \gamma_\alpha \psi_A \\
\delta \epsilon X^i = -\frac{1}{2} \gamma^i A_{AB} \psi_B \\
\delta \epsilon \psi_A = -\frac{i}{4} F_{\alpha\beta} \gamma^{\alpha\beta} \epsilon_A - \frac{i}{2} D_\alpha X_i \gamma^\alpha \gamma^i A_{AB} \epsilon_B - \frac{i}{4} [X_i, X_j] \gamma^{ij} A_{AB} \epsilon_B.
\]

(10)
The other half are kinematical supersymmetries

\[ \delta_\eta A_\alpha = \delta_\eta X_i = 0 \]
\[ \delta_\eta \psi_A = \eta_A \mathbb{1} \]

(11)

and acts only on the center-of-mass U(1)⊂U(N). It is evident that the M(atrix) theory has Spin(7)_R R-symmetry, which in fact encodes the rotational invariance on the transverse non-compact space.

3 M(atrix) Theory on Klein bottle

Consider nine-dimensional M-theory compactified on a Klein bottle \( K_2 \). Klein bottle \( K_2 \) of area \((\pi R_1) \times (2\pi R_2)\) is obtained as a quotient of torus \( T_2 \) of area \((2\pi R_1) \times (2\pi R_2)\) by symmetry group \( \Gamma_K \):

\[ \Gamma_K : \ x \to \hat{x} + \pi G \cdot e, \quad \hat{x} \equiv (-x^1, x^2) , \quad e \equiv (e^1, e^2), \]

(12)

viz. parity along 1-direction accompanied by half-period shifts in both directions. To define the corresponding M(atrix) theory, we put \( N \) D0-partons on \( K_2 \) and study their dynamics. From the covering space \( T_2 \) point of view, this is to place \( 2N \) D0-partons: original \( N \) D0-partons and image \( N \) D0-partons. They form a single \( \mathbb{Z}_2 \) orbit of the symmetry group \( \Gamma_K \). In what follows D0-parton dynamics on the \( K_2 \) will be referred as Klein bottle M(atrix) theory.

3.1 Chan-Paton Condition

In this section, we will find prescription of Klein bottle M(atrix) theory in two steps. We first study \( 2N \) D0-parton dynamics on a single fundamental cell of \( T_2 \). Subsequently, utilizing the result in section 2, we obtain the Klein bottle M(atrix) theory in terms of \( \mathbb{R}_2 \) covering space of \( T_2 \) with an appropriate projection, viz. \((2+1)\)-dimensional gauge theory on a dual parameter space (which will be determined in due course of foregoing construction).

On a single fundamental cell of \( T_2 \), infinitesimally short open string configurations among \( 2N \) D0-partons define parton dynamics and are denoted by \((2N \times 2N)\) matrix fields \( X^\mu \equiv (A_0, X^i) \quad (i = 1, \cdots , 9) \). Compactification on Klein bottle \( K_2 \) is then realized by imposing \( \Gamma_K \) projection on the D0-parton configurations, hence, on these matrix fields. In the previous work [3], we have found that the projection is given in terms of local Chan-Paton conditions on each fundamental cell of \( T_2 \):

\[ A_0 = -M \cdot A_0^T \cdot M^{-1} \]
\begin{align}
X^1 &= -M \cdot X^{1T} \cdot M^{-1} - (\pi \cdot G) \cdot e_1 I_{N \times N} \otimes \sigma^3 \\
X^2 &= +M \cdot X^{2T} \cdot M^{-1} - (\pi \cdot G) \cdot e_2 I_{N \times N} \otimes \sigma^3, \\
X^I &= +M \cdot X^{IT} \cdot M^{-1}, \quad (I = 3, \ldots, 9), \quad (13)
\end{align}
where \( G_{ij} = \text{diag} \). \((R_1, R_2)\) denotes the metric of covering space \( \mathbb{T}_2 \) and \( M = I \otimes \sigma^1, I \otimes \sigma^2 \) as was determined \(^3\) in the previous work [5]. We have taken transpose action between the original and the image D0-partons since, in the Type IIA string limit, the Klein bottle corresponds to orientifold and gives rise to unoriented membrane. Note that Chan-Paton condition should include a condition on M(atrix) gauge potential \( A_0 \) as well in order to maintain the mapping as a symmetry of M(atrix) quantum mechanics, by-now a well understood fact.

From M(atrix) gauge symmetry point of view, what does the inhomogenous term proportional to \( I_{N \times N} \otimes \sigma^3 \) in Eq. (13) signify? To understand this note that, for the above choices of \( M \),
\[ \pi R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \pi R \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{2} \pi R M \cdot \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \cdot M^{-1} \quad (15) \]
Hence, the Chan-Paton conditions for \( X^{1,2} \) can be re-expressed as
\begin{align}
(X^1 + \frac{1}{2} \pi R_1 I) &= -M \cdot (X^1 + \frac{1}{2} \pi R_2 I)^T \cdot M^{-1} \\
(X^2 + \frac{1}{2} \pi R_2 I) &= +M \cdot (X^2 + \frac{1}{2} \pi R_2 I)^T \cdot M^{-1}. \quad (16)
\end{align}
This shows that along 1,2-directions, there are nontrivial Wilson line backgrounds but the D0-parton coordinates measured relative to these Wilson line satisfy homogeneous Chan-Paton condition.

We now extend the Chan-Paton condition to other image cells in the covering space. Following exactly the same procedure as in Section 2 for \( \mathbb{T}_2 \), we introduce cell-indexed coordinate matrices \( X^\mu_{k,l} \), which are \((2N \times 2N)\) matrices for each \( k, l \). The matrices in Eq. (13) are diagonal sub-blocks \((k = l)\) of these infinite-dimensional matrices. We now extend the Chan-Paton condition Eq. (13) to all image cells in \( \mathbb{R}_2 \) covering space and rewrite them in terms of \( X^\mu_{k,l} \) matrices. In particular, the diagonal, \( 2N \)-dimensional sub-blocks of the extended condition should yield the same condition as Eq. (13). With such proviso, the extended Chan-Paton condition is given by
\[ A_{0k,l} = - \mathcal{M}_{k,k'} \left( M \cdot A_0^{T_{k,k'}} \cdot M^{-1} \right) \mathcal{M}^{-1}_{k',l} \]
\(^3\) Equivalently, the matrices along the Klein bottle can be parametrized as
\begin{align}
X^1 &= \begin{pmatrix} X & U \\ V & -X^T \end{pmatrix}, \quad X^2 = \begin{pmatrix} Y + \pi R_2 I & W \\ Z & +Y^T \end{pmatrix}. \quad (14)
\end{align}
\[ X_{1,k,l}^i = -M_{k',k} \left( M \cdot X_{1T}^{k',k} \cdot M^{-1} \right) M_{-1}^{k',k} \]
\[ X_{2,k,l}^i = +M_{k',k} \left( M \cdot X_{2T}^{k',k} \cdot M^{-1} \right) M_{-1}^{k',k} \]
\[ X_{4,k,l}^i = +M_{k',k} \left( M \cdot X_{4T}^{k',k} \cdot M^{-1} \right) M_{-1}^{k',k} \]

where \( M_{k,l} \) is a unitary matrix, whose choice should be determined from consistency with T-duality. In our notation, the Wilson line along 2-direction is suppressed, but its presence should be borne in mind throughout. In fact, this Wilson line breaks the U(2N) covering space gauge group into U(N) \times U(N).

In the previous work \cite{5}, we have classified all possible choices of \( M_{k,l} \) from consistency with area-preserving diffeomorphism gauge symmetry, and have found that \( M_{k,l} = (-)^{k,e} \delta(k - \hat{l}) \), where \( \hat{k} \equiv (-k_1, k_2) \). This choice is unique \cite{6} and, as we will see momentarily, turns out to be the choice also consistent with T-duality of M(atrix) theory. Using this result and the matrices \( X^\mu(k) \) introduced in section 2, the Chan-Paton condition now reads

\[ A_0(k, t) = -(-)^{k,e} M \cdot A_0^T(-\hat{k}, t) \cdot M^{-1} \]
\[ X^1(k, t) = -(-)^{k,e} M \cdot X_1^T(-\hat{k}, t) \cdot M^{-1} \]
\[ X^2(k, t) = +(-)^{k,e} M \cdot X_2^T(-\hat{k}, t) \cdot M^{-1} \]
\[ X^4(k, t) = +(-)^{k,e} M \cdot X_4^T(-\hat{k}, t) \cdot M^{-1} \].

The relative minus sign in the argument of right hand side is reflects the fact that cell indices in Eq. (17) are transposed so that the condition becomes compatible with single cell condition Eq. (13). Fourier transforming \( X^\mu(k) \) to (2+1)-dimensional fields,

\[ A_\alpha(y, t) = \sum_k A_\alpha(k, t) \exp \left[ i \mathbf{k} \cdot \vec{G} \cdot \mathbf{y} \right] \]
\[ Y^I(y, t) = \sum_k Y^I(k, t) \exp \left[ i \mathbf{k} \cdot \vec{G} \cdot \mathbf{y} \right] \]

where \( \alpha = 0, 1, 2 \), the above Chan-Paton conditions turn into

\[ A_0(y, t) = -M \cdot A_0^T(-\hat{y} + \pi \vec{G} \cdot \mathbf{e}, t) \cdot M^{-1} \]
\[ A_1(y, t) = -M \cdot A_1^T(-\hat{y} + \pi \vec{G} \cdot \mathbf{e}, t) \cdot M^{-1} \]
\[ A_2(y, t) = +M \cdot A_2^T(-\hat{y} + \pi \vec{G} \cdot \mathbf{e}, t) \cdot M^{-1} \]
\[ Y^I(y, t) = +M \cdot Y^I(-\hat{y} + \pi \vec{G} \cdot \mathbf{e}, t) \cdot M^{-1} \]
\[ \psi_A(y, t) = \Gamma \perp M \cdot \psi_A^T(-\hat{y} + \pi \vec{G} \cdot \mathbf{e}, t) \cdot M^{-1} \]

where again \( \hat{y} \equiv (-y_1, y_2), \; \mathbf{e} \equiv (e_1, e_2), \; \Gamma \perp = \gamma^0 \gamma^1, \; (\Gamma \perp)^2 = 1 \), and \( \pm \) signs are for two different choices of \( M \). With covering space Chan-Paton condition imposed, the M(atrix) \footnote{The other consistent choice is \( M_{k,l} = (-)^{k,e} \delta(k - \hat{l}) \), but the final form of Chan-Paton condition is unchanged.}
theory turns into (2+1)-dimensional gauge theory whose covering space gauge group U(2N) is broken to U(N)×U(N) by the presence of $A_2$ Wilson line. Furthermore, the parameter space on which the gauge theory lives is dual Klein bottle $\tilde{K}_2$ since, according to Eq. (24), the parameter space is a quotient of $\tilde{T}_2$ by dual symmetry group:

$$\Gamma_K : \ y \to -\hat{y} + \pi\tilde{G} \cdot e,$$

viz. $\tilde{K}_2 = \tilde{T}^2/\Gamma_K$ of volume $(2\pi R_1) \times (\pi R_2)$. Note that in $\tilde{K}_2$ the coordinate that parity operation acts is interchanged compared to the $K_2$. We conclude that M(atrix) theory description of M-theory compactified on Klein bottle $K_2$ is defined by (2+1)-dimensional $\mathcal{N} = 8$ supersymmetric U(N)×U(N) gauge theory living on dual Klein bottle $\tilde{K}_2$.

In fact, the Chan-Paton condition Eq. (24) can be seen to be consistent with T-duality of M(atrix) theory. Recall that T-duality of M(atrix) theory is defined in terms of T-duality of D0-parton themselves. Thus, consider Type IIA string compactified on Klein bottle $K_2 = T_2/\Gamma_K$. The action of $\Gamma_K$ is a product of $(\mathcal{P}_1\Omega) \cdot \mathcal{S}$, where $\mathcal{P}_{1,2}$ are parity inversion along 1, 2 directions, $\Omega$ worldsheet parity inversion and $\mathcal{S}$ half-period shift along both directions. Under T-duality along 1-direction, the theory turns into Type IIB string compactified on $(\tilde{S}_1 \times S_1)/(\mathcal{S} \cdot \Omega)$. Under another T-duality along 2 direction, the worldsheet parity $\Omega$ is mapped into $\mathcal{P}_2 \cdot \Omega$, and the theory is turned into $\tilde{I}A$ string compactified on $\tilde{T}_2/(\mathcal{P}_2 \cdot \Omega \cdot \mathcal{S})$. The compactification is again on Klein bottle but with inverted volume and parity transformation direction compared to the starting Klein bottle $K_2$. In fact, it is precisely the dual Klein bottle $\tilde{K}_2$ we have identified just above. The D0-partons are now T-dualized into D2-branes wrapped around the dual Klein bottle $\tilde{K}_2$. The large-N limit of (2+1)-dimensional world-volume gauge theory of $\tilde{I}A$ D2-branes on $\tilde{K}_2$ reduces to the Klein bottle M(atrix) theory we have deduced from the first-principle Chan-Paton conditions.

Having now understood the Chan-Paton conditions systematically, we can make a short-cut derivation of the Klein bottle M(atrix) theory. Begin with (2+1)-dimensional $\mathcal{N} = 16$ supersymmetric U(2N) gauge theory on $\tilde{T}_2$, relevant for M(atrix) theory on $T_2$. Consider the following set of transformations to the gauge theory. The first is orientation reversal or, equivalently, complex conjugation

$$\Omega : \ A_0(y,t) \to \Omega \cdot A_0(y,t) \cdot \Omega^{-1} = M \cdot A_0^T(y,t) \cdot M^{-1}$$
$$A_1(y,t) \to \Omega \cdot A_1(y,t) \cdot \Omega^{-1} = M \cdot A_1^T(y,t) \cdot M^{-1}$$
$$A_2(y,t) \to \Omega \cdot A_2(y,t) \cdot \Omega^{-1} = M \cdot A_2^T(y,t) \cdot M^{-1}$$
$$Y_I(y,t) \to \Omega \cdot Y_I(y,t) \cdot \Omega^{-1} = M \cdot Y_IT(y,t) \cdot M^1$$
$$\psi_A(y,t) \to \Omega \cdot \psi_A(y,t) \cdot \Omega^{-1} = M \cdot \psi_A^T(y,t) \cdot M^{-1}$$

(26)
where \( M \) is an arbitrary matrix subject to Hermiticity condition \( M^{-1} \cdot M^T = \pm I \),

\[
\mathcal{S} : \quad A_0(y, t) \rightarrow S \cdot A_0(S \cdot y, t) \cdot S^{-1} = A_0(y + \pi \tilde{G} \cdot e, t) \\
A_1(y, t) \rightarrow S \cdot A_1(S \cdot y, t) \cdot S^{-1} = A_1(y + \pi \tilde{G} \cdot e, t) + (\pi \tilde{G}) \cdot e_1 \\
A_2(y, t) \rightarrow S \cdot A_2(S \cdot y, t) \cdot S^{-1} = A_2(y + \pi \tilde{G} \cdot e, t) + (\pi \tilde{G}) \cdot e_2 \\
Y^I(y, t) \rightarrow S \cdot Y^I(S \cdot y, t) \cdot S^{-1} = Y^I(y + \pi \tilde{G} \cdot e, t) \\
\psi_A(y, t) \rightarrow S \cdot \psi_A(S \cdot y, t) \cdot S^{-1} = \psi_A(y + \pi \tilde{G} \cdot e, t),
\]

(27)

the half-period shift transformations along both directions of \( \tilde{T}_2 \) and gauge connections, and

\[
\mathcal{P} : \quad A_0(y, t) \rightarrow \mathcal{P} \cdot A_0(\mathcal{P} \cdot y, t) \cdot \mathcal{P}^{-1} = + A_0(\hat{y}, t) \\
A_1(y, t) \rightarrow \mathcal{P} \cdot A_1(\mathcal{P} \cdot y, t) \cdot \mathcal{P}^{-1} = + A_1(\hat{y}, t) \\
A_2(y, t) \rightarrow \mathcal{P} \cdot A_2(\mathcal{P} \cdot y, t) \cdot \mathcal{P}^{-1} = - A_2(\hat{y}, t) \\
Y^I(y, t) \rightarrow \mathcal{P} \cdot Y^I(\mathcal{P} \cdot y, t) \cdot \mathcal{P}^{-1} = - Y^I(\hat{y}, t) \\
\psi_A(y, t) \rightarrow \mathcal{P} \cdot \psi_A(\mathcal{P} \cdot y, t) \cdot \mathcal{P}^{-1} = \Gamma^\perp \psi_A(\hat{y}, t),
\]

(28)

the parity transformation in (2+1) dimensions, where we have used the fact that \( Y^I \)'s are pseudo-scalars.

It is straightforward to check that the starting (2+1) dimensional gauge theory is invariant under a simultaneous tranformation \( \Omega \cdot S \cdot \mathcal{P} \) of Eqs. (26 - 28). In fact, the \( \mathbb{Z}_2 \) action \( \Omega \cdot S \cdot \mathcal{P} \) is exactly identical to the covering space Chan-Paton condition Eq. (24). Hence, if we mod out the theory by this \( \mathbb{Z}_2 \) symmetry group, we obtain precisely the Klein bottle M(atrix) theory defined on dual Klein bottle \( \tilde{K}_2 \). Allowed choices of \( M \) in Eq. (26) are precisely the ones \( M = I \otimes \sigma^1 \) and \( I \otimes \sigma^2 \) permitted in Eq. (24). Moreover, it is to be noted that the action of \( S \) in Eq. (27) is accompanied by turning on constant \( A_{1,2} \) gauge field backgrounds. This background then breaks the starting U(2N) gauge group down to U(N)×U(N) and no further gauge symmetry enhancement is permitted. Intuitively this can be understood from the fact that the defining \( N \) D0-partons and their mirror partons never come close due to the fact that the \( \mathbb{Z}_2 \) involution is free.

To summarize, in M(atrix) theory, nine-dimensional M-theory compactified on Klein bottle is described by (2+1)-dimensional \( \mathcal{N} = 8 \) supersymmetric U(N)×U(N) gauge theory living on dual Klein bottle parameter space.
3.2 BPS Branes and Massless Spacetime Spectra

In M(atrix) theory various BPS branes arise as composite bound-states of D0-partons. By identifying BPS branes that are consistent with the orbifold projection $\Gamma$, as proposed and utilized in other orbifolds [3], it is possible to extract massless spacetime spectrum. We adopt the same strategy and deduce $d = 9$ spacetime spectrum now. To do so, we utilize two possible types of D0-parton bound-states. Two types of D0-parton bound-states are threshold bound-states representing M-theory graviton and Landau-level orbiting bound-states representing electric BPS states coupled to M-theory three-form potential. Threshold bound-states represent graviton. Projecting out odd components of the bound-state under $\Gamma_K$, hence, $\bar{\Gamma}_K$, one finds that the threshold bound-state of D0-parton give rise only to dimensionally reduced gravitons. The Landau-orbiting bound-state corresponds in M(atrix) gauge theory to a quantum of magnetic flux $B$, represented by half-integer valued first Chern class

$$\text{Tr} \int_{\bar{K}_2} \frac{d^2y}{2\pi} B = \frac{m}{2}, \quad m \in \mathbb{Z},$$

and gives rise to membrane configuration wrapped around the compactified Klein bottle. As is easily verified, this configuration is invariant under the $\Gamma_K$ action. In the limit the area of Möbius strip vanishes, infinite tower of wrapping states become massless densely and the Aspinwall-Schwarz dimension (9-th) dimension opens up.

The second is a quantum of electric flux $E_2$ on $\bar{K}_2$

$$\text{Tr} \int_0^{\pi \bar{R}_2} d\mathbf{y} \cdot E_2 = \frac{g_{YM}^2 \bar{R}_2}{4 \bar{R}_1} n, \quad n \in \mathbb{Z}.$$ 

The third is a photon propagating around the $x^1$ direction. Each of these three configurations in gauge theory represent M-theory BPS states that couple minimally to spacetime gauge fields. Thus, spacetime massless spectrum can be inferred from BPS spectrum of M(atrix) gauge theory.

Membrane located in $\mathbb{R}_9$ is described by

$$Y^I = \begin{pmatrix} P & 0 \\ 0 & +P^T \end{pmatrix}, \quad Y^J = \begin{pmatrix} Q & 0 \\ 0 & +Q^T \end{pmatrix}$$

so that

$$Z_2 = [Y^I, Y^J] = \frac{1}{N} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Total BPS charge vanishes and is not compatible with $\Gamma$ projection. Thus, there is no membrane propagating in $\mathbb{R}_9$ and we conclude that there is no massless three-form potential in the spacetime spectrum.
Membrane partially wrapped on $K_2$ is given by

$$A_1 = \begin{pmatrix} P & 0 \\ 0 & -P^T \end{pmatrix}, \quad Y^I = \begin{pmatrix} Q & 0 \\ 0 & +Q^T \end{pmatrix}$$

(33)

carries nontrivial BPS charge. Propagation of the state compatible with $\Gamma$ projection is in $R_9$. Therefore, we conclude that there is two-form potential in the spacetime spectrum. The commutator is $\mathbb{Z}_2$ even, hence, gives rise to a consistent configuration. On the other hand, membrane partially wrapped around the other direction of $K_2$ carries no BPS charge, hence, does not give rise to second two-form potential in the spectrum. Also, electric field excitation gives rise to BPS particle state, which couples minimally to the $B_{9\mu}$ component. Altogether, we have found that the spacetime spectrum includes graviton, single two-form tensor potential and dilaton. This is precisely the spectrum of Type IIB string upon compactification on $S_1/\Gamma_2$.

The parameter space of the M(atrix) gauge theory is Klein bottle, hence, do not have any orbifold fixed boundaries. Since the gauge theory in that case has no room for potential gauge nor supersymmetry anomalies, we conclude that there is no twisted sector in the M(atrix) theory. This in turn implies that there is no charged states, hence, no gauge symmetry group in spacetime spectrum.

### 4 M(atrix) Theory on Möbius strip

Next, consider another possible nine-dimensional M-theory compactification on a Möbius strip $M_2$. Möbius strip $M_2$ of area $(w\pi R_1) \times (2\pi R_2) / 2$ may be obtained as a quotient of torus $T_2$ of area $(2\pi R_1) \times (2\pi R_2)$ by symmetry group

$$\Gamma_M : \quad x \rightarrow \tilde{x},$$

(34)

where $\tilde{x} \equiv (x^2, x^1)$, and is encompassed by fundamental domain $0 \leq x^1 \leq (R_1/R_2) x^2, \quad 0 \leq x^2 \leq 2\pi R_2$. After rotation of the coordinates by $\pi/4$, the Möbius strip constructed as above is equivalent to $(T_2/\mathcal{P} \cdot \Omega) / S$, viz. $\mathbb{Z}_2$ quotient of cylinder. To define the corresponding Möbius M(atrix) theory, we put $N$ D0-branes on $M_2$ and study their dynamics on covering space $T_2$. On $T_2$, this amounts to putting $2N$ D0-partons. They form a single $\mathbb{Z}_2$ orbit of the symmetry group $\Gamma_M$ on $T_2$. M-theory compactification on Möbius strip has been proposed as a dual to strong coupling limit of CHL string [10, 11]. In this section, we study the corresponding M(atrix) theory and show that indeed the CHL string spectrum as well as Wilson line moduli space follow from the theory.
4.1 Chan-Paton Condition

Again we describe Möbius M(atrix) theory in terms of dynamics among 2N D0-partons on a single fundamental cell of covering space $T_2$. The Chan-Paton conditions to D0-partons on the single fundamental cell are given by

$$
A_0 = - M \cdot A_0^T \cdot M^{-1} \\
X^1 = + M \cdot X^{2T} \cdot M^{-1} \\
X^2 = + M \cdot X^{1T} \cdot M^{-1} \\
X^I = + M \cdot X^{IT} \cdot M^{-1} \quad (I = 3, \cdots, 9), \\
\psi_A = \Gamma_\perp M \cdot \psi_A^T \cdot M^{-1}, \quad \Gamma_\perp \equiv \frac{i}{2}(\gamma^1 - \gamma^2),
$$

where the spinor projection operator $(\Gamma_\perp)^2 = 1$. We now extend the Chan-Paton condition over the image $T_2$ cells in the covering space $R_2$. Following exactly the same procedure as Klein bottle compactification case, we find

$$
A_{0k,m} = - M_{k,k'} (M \cdot A_{0}^{T m',k'} \cdot M^{-1}) M^{-1 m',m} \\
X^1_{k,m} = + M_{k,k'} (M \cdot X^{2T m',k'} \cdot M^{-1}) M^{-1 m',m} \\
X^2_{k,m} = + M_{k,k'} (M \cdot X^{1T m',k'} \cdot M^{-1}) M^{-1 m',m} \\
X^I_{k,m} = + M_{k,k'} (M \cdot X^{IT m',k'} \cdot M^{-1}) M^{-1 m',m}; \\
\psi_{A_{k,m}} = \Gamma_\perp M_{k,k'} (M \cdot \psi_A^{T m',k'} \cdot M^{-1}) M^{-1 m',m}.
$$

(36)

In the previous work [5], we have also identified possible choices of $M_{k,1}$ for Möbius strip from the consistency of area-preserving diffeomorphism gauge symmetry. It was found there that a unique choice is $M_{k,m} = \delta(k - \tilde{m})$ where $\tilde{m} \equiv (m_2, m_1)$. The Chan-Paton conditions on the covering space is found to be

$$
A_0(k,t) = - M \cdot A_0^T (-\bar{k}, t) \cdot M^{-1} \\
X^1(k,t) = + M \cdot X^{2T} (-\bar{k}, t) \cdot M^{-1} \\
X^2(k,t) = + M \cdot X^{1T} (-\bar{k}, t) \cdot M^{-1} \\
X^I(k,t) = + M \cdot X^{IT} (-\bar{k}, t) \cdot M^{-1} \\
\psi_A(k,t) = \Gamma_\perp M \cdot \psi_A^{T} (-\bar{k}, t) \cdot M^{-1}.
$$

(37)

Again, the relative minus sign in the argument on the right hand side is a direct reflection of the transposition in the covering space Chan-Paton condition. In terms of Fourier transformed

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5 Alternative possible choice is $M_{k,m} = (-)^{k\cdot(e_1 + e_2)} \delta(k - \tilde{m})$, but again, this choice results in equivalent covering space Chan-Paton conditions.
fields, the Chan-Paton conditions become

\[
A_0(y,t) = -M \cdot A_0^T(-\tilde{y},t) \cdot M^{-1}
\]
\[
A_1(y,t) = +M \cdot A_2^T(-\tilde{y},t) \cdot M^{-1}
\]
\[
A_2(y,t) = +M \cdot A_1^T(-\tilde{y},t) \cdot M^{-1}
\]
\[
Y^I(y,t) = +M \cdot Y^{IT}(-\tilde{y},t) \cdot M^{-1}
\]
\[
\psi_A(y,t) = \Gamma \perp M \cdot \psi_A^T(-\tilde{y},t) \cdot M^{-1}.
\]

(38)

With these covering space Chan-Paton conditions imposed, the M(atrix) theory turns into (2+1)-dimensional gauge theory on dual Möbius strip \(\tilde{\mathcal{M}}_2\), since according to Eq. (38), the parameter space is a quotient of \(\tilde{T}_2\) by dual symmetry group:

\[
\tilde{\Gamma}_M : \ y \to -\tilde{y}.
\]

(39)

Hence, \(\tilde{\mathcal{M}}_2 = \tilde{T}_2/\tilde{\Gamma}_M\). Note that the boundary direction of original and dual Möbius strips is exchanged each other.

The above covering space Chan-Paton condition is again consistent with T-duality of M(atrix) theory. To exhibit this, recall that the Möbius strip \(\mathcal{M}_2\) is obtained as a quotient of cylinder \(C_2 = (S_1/\mathcal{P}_1 \cdot \Omega) \times S_1\) by \(\mathbb{Z}_2\) symmetry group \(\mathcal{S}\) that shifts both coordinates of covering space \(T_2\) by half periods. Thus, D0-parton dynamics on \(\mathcal{M}_2\) may be understood in terms of \(\mathcal{S}\) projection of Type IIA / heterotic string theories. Under T-duality along 1-direction, the theory is mapped into Type I string on \((\tilde{S}_1 \times S_1) / \mathcal{S}\). The D0-partons are mapped into D1-strings of Type I string theory. Upon another T-duality along 2-direction, the worldsheet parity inversion \(\Omega\) is mapped into \(\Omega \cdot \mathcal{P}_2\). The resulting theory is Type \(\tilde{I}A\) string compactified on \((\tilde{S}_1 \times (\tilde{S}_1/\mathcal{P}_2 \cdot \Omega)) / \mathcal{S}\). The resulting compactification space is again a Möbius strip but now with inverted volume and exchanged parity transformation direction.

We can proceed directly from (2+1)-dimensional \(\mathcal{N} = 16\) supersymmetric U(4N) gauge theory on \(\tilde{T}_2\) and obtain a short-cut derivation of the above Chan-Paton conditions. It is then straightforward to recognize that the starting gauge theory is invariant under a combined action of \(\Omega \cdot \mathcal{P}_-\), where \(\Omega\) is the orientation reversal or complex conjugation and \(\mathcal{P}_-\) is parity inversion along the diagonal:

\[
P_- : \quad A_0(y^\pm, t) \to \mathcal{P} \cdot A_0(\mathcal{P} \cdot y^\pm, t) \cdot \mathcal{P}^{-1} = +A_0(\mp y^\pm, t) \]
\[
A_-(y^\pm, t) \to \mathcal{P} \cdot A_-(\mathcal{P} \cdot y^\pm, t) \cdot \mathcal{P}^{-1} = +A_-(\mp y^\pm, t)
\]
\[
A_+(y^\pm, t) \to \mathcal{P} \cdot A_+(\mathcal{P} \cdot y^\pm, t) \cdot \mathcal{P}^{-1} = -A_+(\mp y^\pm, t)
\]
\[
Y^I(y^\pm, t) \to \mathcal{P} \cdot Y^{I}(\mathcal{P} \cdot y^\pm, t) \cdot \mathcal{P}^{-1} = -Y^I(\mp y^\pm, t)
\]
\[
\psi_A(y^\pm, t) \to \mathcal{P} \cdot \psi_A(\mathcal{P} \cdot y^\pm, t) \cdot \mathcal{P}^{-1} = i\gamma^- \psi_A(\mp y^\pm, t).
\]

(40)
where \( y^\pm \equiv y^1 \pm y^2, A^\pm = A_1 \pm A_2 \) and \( \gamma^\pm = \frac{1}{\sqrt{2}}(\gamma^1 \pm \gamma^2). \) Modding out the defining gauge theory by \( \Omega \cdot \mathcal{P}_- \), we obtain precisely the same conditions as the covering space Chan-Paton conditions Eq. (38).

To summarize, Möbius M(atrix) theory is described by (2+1)-dimensional \( \mathcal{N} = 8 \) super-symmetric \( U(N) \times U(N) \) gauge theory on the dual Klein bottle \( \tilde{K}_2 \):

\[
L_{\text{untwisted}} = -\frac{1}{g_{\text{YM}}^2} \int_{\mathcal{M}_2} d^2y \Tr \left[ F_{\alpha \beta} F^{\alpha \beta} + 2D_{\alpha} Y_{i} D^{\alpha} Y^i - [Y_i, Y_j][Y^i, Y^j] \right. \\
\left. - 2\bar{\psi}_A \gamma^\alpha D_\alpha \psi_A + 2i \bar{\psi}_A \gamma^i \gamma_{AB} [Y^i, \psi_B] \right].
\]

(45)

Note that an overall factor of two has been inserted since we write the action only on the fundamental domain of the Möbius strip. This (2+1)-dimensional bulk gauge theory will be called as **untwisted sector** of CHL M(atrix) theory.

### 4.2 Twisted Sector and Gauge Symmetry

The \( \mathbb{Z}_2 \) involution has a orientifold fixed line \( x = y \). The orientifold plane carries -8 units of D8-brane charge as can be seen, for example, from the D0-parton scattering off the orbifold fixed circle \( y^+ = 0 \). In fact, at the orbifold fixed circle, from the covering space Chan-Paton condition, the action of \( \Omega \cdot \mathcal{P}_- \) imposes boundary conditions on the fields:

\[
(A_0, A_-, \psi_{2A})(y^-) \quad \text{antisymmetric} \\
\partial_+(A_0, A_-, \psi_{2A})(y^-) \quad \text{symmetric} \\
(A_+, Y^i, \psi_{1A})(y^-) \quad \text{symmetric} \\
\partial_+(A_+, Y^i, \psi_{1A})(y^-) \quad \text{antisymmetric}
\]

(46)

These boundary conditions modify the M(atrix) gauge theory in several ways. First, they break half the supersymmetry. The dynamical supersymmetry parameters \( \epsilon_A \) that appear in Eq. (10) must be taken to be invariant under the \( \mathbb{Z}_2 \) projection onto Möbius strip, \( \epsilon_A = i\gamma^- \epsilon_A \).

This results in \( \mathcal{N} = 8 \) supersymmetry in 2+1 dimensions. Simultaneously, to respect the boundary conditions on the fermions, the kinematical supersymmetry parameters \( \eta_A \) should satisfy \( \eta_A = -i\gamma^- \eta_A \). The relative sign difference between the dynamical and the kinematical supersymmetry projections reflects the fact that \( \epsilon_A \) and \( \eta_A \) belong to inequivalent representations of the (2+1)-dimensional Clifford algebra.

At the orbifold fixed boundary, the gauge transformation \( A_\alpha \to U(-i\partial_\alpha + A_\alpha)U^{-1} \) respects the boundary conditions only if \( U^T \cdot U(y^+ = 0, y^-) = \pm \mathbb{1} \), viz. only gauge transformations in an O(2N) subgroup of U(2N) are permitted. Under the boundary O(N) gauge
group, the fields $A_0, A_1, \psi_{2A}$ transform as the adjoint representation, while $A_2, Y^i, \psi_{1A}$ transform as the symmetric representation. The fermions $\psi_A$ have normalizable modes which are independent of $y^+$. These modes behave as (1+1)-dimensional spinors. From (2+1)-dimensional Dirac equation, it is straightforward to see that $\psi_{1A}$ and $\psi_{2A}$ are in opposite chirality in the (1+1)-dimensional world. These modes therefore generate $O(2N)$ gauge anomaly

$$\frac{8I_2(adj.) - 8I_2(symm.)}{2} = -16I_2(fund.),$$

where the factor of 1/2 comes from the $\Gamma_2$ involution in defining the Möbius strip from the cylinder.

This gauge anomaly is cancelled by introducing a twisted sector consisting of sixteen left-moving Majorana-Weyl fermions $\chi(y^-)$ in the fundamental representation of $O(2N)$. Much in parallel to the case of heterotic M(atrix) theory on cylinder, it is expected that this is also the choice that cancels potential supersymmetry anomaly. In order to cancel the anomaly locally, the fermions should be localized at the boundary of Möbius strip.

The twisted sector of Möbius M(atrix) theory involves Majorana-Weyl fermions $\chi_M$. They are left-moving, hence, are (0,8) supersymmetry singlets. They also transform in the fundamental representations of the boundary M(atrix) gauge group $O(2N) \subset U(2N)$ as well as in the global ‘flavor’ symmetry group $SO(16)$ associated with the eith D8-branes and their images present at the boundary. Their Lagrangian is given by

$$L_{\text{twisted}} = \oint_{\partial M_2} dy^- i\chi_M \left( \delta_{MN} (D_0 + D_-) + i (B_0 + B_-)_{MN} \right)_{y^+ = 0} \chi_N. \quad (47)$$

It is tacitly assumed that (1+1)-dimensional boundary dynamics of the gauge fields $A_\pm$ is part of the untwisted sector Lagrangian. We also have included couplings to the background field $B_{MN}$ which are in the adjoint representation of $SO(16)$. This is in fact the spacetime gauge field of strongly coupled CHL heterotic string, hence, turning on $B_{MN}$ corresponds to turning on Wilson line moduli.

### 4.3 Chern-Simons Coupling and Wilson Line Moduli

An interesting and important question is to understand the Narain moduli space of strongly coupled CHL string in terms of M(atrix) theory. The reason why this is so is because the moduli spaces of CHL string contain points of not only simply-laced but also non-simply-laced enhanced symmetry, as well as higher Kac-Moody level realization of gauge symmetry. In the strongly coupled limit, how are these features realized? For example, in D=4 compactifications, it has been known that $Sp(2n)$ and $SO(2n+1)$ are interchanged under electric-magnetic duality.

In this section, as a first step toward a complete understanding of these issues, we address
how the Wilson line moduli space is realized in M(atrix) theory.

As in $E_8 \times E_8$ heterotic M(atrix) theory on $S_1$, the Wilson line is realized in terms of the positions of twisted sector Majorana-Weyl spinors on the dual Möbius strip. We have argued that these spinors are $y^+$-independent normalizable zero modes on a circle parallel to the boundary of Möbius strip. Geometric moduli of moving them away from the boundary and splitting among themselves then correspond to realization of Wilson line moduli of strongly coupled CHL string. In order for this deformation to be compatible with local cancellation of supersymmetry and gauge anomalies, it is necessary to include Chern-Simons term to the M(atrix) theory. Turning on Chern-Simons term is actually more involved, since it is directly a result of massive Type IIA supergravity background. Nevertheless, the procedure is essentially the same as that for heterotic M(atrix) theory. It is convenient to introduce new fields

$$Z^i = \left( z(y^+) \right)^{1/3} Y^i$$

The final Möbius M(atrix) theory Lagrangian is then given by:

$$L_{\text{Mobius}} = -\frac{1}{g_{YM}^2} \int d^2 y \text{Tr} \left\{ z(y^+) F_{\alpha \beta} F^{\alpha \beta} + 2 z^{1/3} (y^+) D_\alpha Z^i D^\alpha Z^i - z^{-1/3} (y^+) [Z_i, Z_j][Z^i, Z^j] 
- 2 i z^{1/3} (y^+) \bar{\psi}_A \gamma^\alpha D_\alpha \psi_A - \frac{dz^{1/3}}{dy^+} \bar{\psi}_A \psi_A + 2 i \bar{\psi}_A \gamma^i_{AB}[Z_i, \psi_B]
- \frac{4}{3} \frac{dz}{dy^+} \epsilon^{\alpha \beta \gamma} (A_\alpha \partial_\beta A_\gamma + i \frac{2}{3} A_\alpha A_\beta A_\gamma) \right\} 
+ i \sum_{M, N = 1}^{8} \oint dy^- \text{Tr} \left\{ \delta^{MN} (D_0 + D_-) + i (B_0 + B_-)^{MN} \right\} \chi_N .$$

5 M(atrix) Theory of $d = 6$ CHL Compactification

So far, we have studied M(atrix) theory on the simplest, two-dimensional non-orientable manifolds. In this section, we extend our study to higher dimensional Ricci flat non-orientable manifold. Consider CHL heterotic string compactified toroidally down to six-dimensions. In strong coupling limit, this theory is described as a quotient of M-theory compactified on $(S_1/\mathbb{Z}_2) \times T^4$ by the CHL projection group $\Gamma^{\text{CHL}}_2$, which acts as half-period shifts:

$$\Gamma^{\text{CHL}}_2 : \mathbf{x} \to \mathbf{x} + (\pi G) \cdot \mathbf{e},$$

of $\mathbf{x} = (x^1, x^2)$ coordinates along $S_1/\mathbb{Z}_2$ and one of the four coordinates of $T_4$. Such half-period shift has previously been used for an extensive construction of string dual pairs \cite{13, 14}. After rotating the coordinates by $\pi/4$, it is straightforward to recognize that the simultaneous action
of $\mathbb{Z}_2$ defining the cylinder $C_2$ and $\Gamma_{\text{CHL}}^2$ gives rise to the involution defining Möbius strip $M_2$ out of covering space torus $T_2$. It is always implicit that the quotient is accompanied by $\Omega$ that flips the sign of the three-form potential $C_{MNP}$.

What is the relevant M(atrix) theory description of this strong coupling CHL compactification? For $T_5$ and $T_5/\mathbb{Z}_2$ compactifications, it has been suggested that the relevant M(atrix) theory is six-dimensional little string theory with (2,0) or (1,0) chiral supersymmetries \cite{17}. Likewise, we expect that strong coupling CHL compactification on $M \times T_3$ is described by a six-dimensional little string theory with (1, 0) chiral supersymmetry living on orbifold limit parameter space of $(S_1 \otimes K3)/\mathbb{Z}_2$. Here, the $\mathbb{Z}_2$ involution of the parameter space acts simultaneously on $S_1$ as a half-period shift and on $K3$ as an involution $\sigma$. As a modest check of this conjecture, we show below that the low-energy dynamics of the M(atrix) theory, viz. (1,0) little string theory, is given by (5+1)-dimensional (1,0) supersymmetric $U(N) \times U(N)$ gauge theory living on dual parameter space $(\tilde{S}_1 \otimes \tilde{K}3)/\mathbb{Z}_2$. The low-energy effective description via supersymmetric gauge theory is obtained essentially by the same procedure as the two-dimensional compactifications. For vanishingly small size of $M_2 \otimes T_3$, we properly take into account of all possible winding string configurations between D0-partons via Fourier transformation. In terms of covering space description,

$$M_2 \otimes T_3 = \Gamma \backslash \mathbb{R}^5/\mathbb{Z}_2,$$

D0-parton dynamics on covering space is described by (1,0) supersymmetric $U(N)$ gauge theory on dual $\tilde{T}^5$ parameter space. It now remains to determine the action of an appropriate involution to this gauge theory.

In the covering space $T^5$, low energy effective dynamics of M(atrix) theory is described by six-dimensional gauge theory on dual parameter space $\tilde{T}^5$ with $\mathcal{N} = 16$ supersymmetry and gauge group $U(N)$. Let us denote the coordinates of parameter space $\tilde{T}^5$ as $y \equiv (y^1, y^2, \cdots, y^5)$. The orbifold under consideration is obtained by modding out by $\Gamma_{\text{CHL}}^2 = \mathcal{P} \cdot \Omega$. Acting on fields, they have the following action

$$\Omega : \begin{align*}
A_0(y, t) &\rightarrow \Omega \cdot A_0(y, t) \cdot \Omega^{-1} = -A_0^T(y, t) \\
A_1(y, t) &\rightarrow \Omega \cdot A_1(y, t) \cdot \Omega^{-1} = -A_1^T(y, t) \\
A_2(y, t) &\rightarrow \Omega \cdot A_2(y, t) \cdot \Omega^{-1} = -A_2^T(y, t) \\
Y^I(y, t) &\rightarrow \Omega \cdot Y^I(y, t) \cdot \Omega^{-1} = -Y^{IT}(y, t),
\end{align*}$$

(52)

where we have taken $M = \mathbb{I} \otimes \sigma^2$,

$$\mathcal{P} : \begin{align*}
A_0(y, t) &\rightarrow \mathcal{P} \cdot A_0(\mathcal{P} \cdot y, t) \cdot \mathcal{P}^{-1} = +A_0(-\hat{y}, t)
\end{align*}$$
\[
A_1(y, t) \rightarrow P \cdot A_1(P \cdot y, t) \cdot P^{-1} = +A_1(-\hat{y}, t)
\]
\[
A_2(y, t) \rightarrow P \cdot A_2(P \cdot y, t) \cdot P^{-1} = -A_2(-\hat{y}, t)
\]
\[
Y^I(y, t) \rightarrow P \cdot Y^I(P \cdot y, t) \cdot P^{-1} = -Y^I(-\hat{y}, t)
\]

(53)

where \( \hat{y} \equiv P \cdot y = (y^1, -y^2, y^3, \cdots, y^5) \). Finally, \( S \) is

\[
S : A_0(y, t) \rightarrow S \cdot A_0(S \cdot y, t) \cdot S^{-1} = +A_0(y + \pi \tilde{G} \cdot e, t)
\]
\[
A_1(y, t) \rightarrow S \cdot A_1(S \cdot y, t) \cdot S^{-1} = +A_2(y + \pi \tilde{G} \cdot e, t)
\]
\[
A_2(y, t) \rightarrow S \cdot A_2(S \cdot y, t) \cdot S^{-1} = -A_1(y + \pi \tilde{G} \cdot e, t)
\]
\[
Y^I(y, t) \rightarrow S \cdot Y^I(S \cdot y, t) \cdot S^{-1} = +Y^I(y + \pi \tilde{G} \cdot e, t)
\]

(54)

The combined action then acts on the covering space gauge theory fields as

\[
\Gamma_{2}^{CHL} : A_0(y, t) = -A_0^T(-\hat{y} + \pi \tilde{G} \cdot e, t)
\]
\[
A_1(y, t) = -A_1^T(-\hat{y} + \pi \tilde{G} \cdot e, t)
\]
\[
A_2(y, t) = -A_2^T(-\hat{y} + \pi \tilde{G} \cdot e, t)
\]
\[
Y^I(y, t) = +Y^I(-\hat{y} + \pi \tilde{G} \cdot e, t)
\]

(55)

It is then evident that the resulting gauge theory is defined on a parameter space

\[
\frac{\left(\tilde{S}_1 \times (\tilde{T}^4/\mathbb{Z}_2)\right)}{\tilde{\Gamma}_2^{CHL}}
\]

(56)

where

\[
\tilde{\Gamma}_2^{CHL} : y \rightarrow -\hat{y} + \left(\pi \tilde{G}\right) \cdot e.
\]

(57)

Thus the parameter space is \( \tilde{\Gamma}_2^{CHL} \) quotient of (orbifold limit of) \( \tilde{S}_1 \times K3 \). Note that the same parameter space has been identified as Type IIA compactification space dual to the CHL heterotic string [15]. We conclude that strongly coupled CHL heterotic string compactified toroidally to six dimensions is described in M(atrix) theory by six-dimensional \( \mathcal{N} = 8 \) supersymmetric \( U(N) \times U(N) \) gauge theory living on \( \tilde{S}_1 \times K3 \).

In fact, the resulting gauge theory can be understood via T-duality of M(atrix) theory as well. We begin with the D0-partons living on \( [(S_1/\mathbb{Z}_2) \times T_4]/\Gamma_2^{CHL} \). Rewriting the compactification as

\[
M_2 \times T_3 = [(S_1/\mathbb{Z}_2) \times T_4]/\Gamma_2^{CHL}
\]

(58)
we now make T-duality transformations successively. Let us first T-dualize the $S_1/Z_2$ direction. This maps the IIA D0-brane into IIB D1-string living on the space

$$\left(\tilde{S}_1 \times T_4\right)/\Gamma_{\text{CHL}}^2.$$  \hspace{1cm} (59)

The D8-branes and $\Omega_8$ orientifold plane that defines the twisted sector is mapped to D9-branes and $\Omega_9$ plane. Next, making T-duality transformations along the $T_4$ directions, the $\Omega_9$-plane and the D9-branes turn into $\Omega_5$-plane and D9-branes into D5-branes. The IIA D0-partons are now mapped into D5-branes wrapped around the compactified directions. After the complete T-duality transformation, we have M(atrix) theory described via Type IIB string compactified on

$$\left(\tilde{S}_1 \times (\tilde{T}_4/Z_2)\right)/\Gamma_2^\text{CHL}.$$  \hspace{1cm} (60)

which is nothing but the dual parameter space identified just above. The order-2 group $\Gamma_2^\text{CHL}$ acts the same as before, viz. half-period shift along dual circle direction and one of the dual four-tori. The dual space is nothing but orbifold limit of the dual IIA theory that has been identified previously \[15\].

Incidentally, the very same gauge theory is also obtained from the world-volume theory of Type IIB NS five-brane \[17, 18, 19\]. There, the gauge coupling constant is independent of string coupling parameter $\lambda_B$. In the limit $\lambda_B \to 0$, bulk dynamics decouples from the five-brane world-volume dynamics.

The theory consists of twisted sector as well. On the parameter space, the orbifold fixed points are located at eight fixed boundary circles. Located on each circle are (1+1)-dimensional chiral fermions. In the bulk of the parameter space, the gauge theory has a gauge group $U(N) \times U(N)$ and the matter fields are all in adjoint representation. As such, there is no six-dimensional gauge anomaly. What about at the boundary circles? At the boundary, there are zero-modes supported at the boundaries. These are cancelled precisely by the twisted sector fermions. Again, we find that the boundary (1+1)-dimensional gauge anomaly is cancelled. Incidentally, through the Wess-Zumino consistency condition, the supersymmetry anomaly is guaranteed to be absent if the gauge anomaly is cancelled. It is precisely these twisted sector chiral fermions that describe the Wilson line moduli deformation of CHL string compactification. The transverse positions of $S_1$ circles on which the twisted sector chiral fermions wave function is localized can be deformed continuously into the interior of the $\tilde{T}_4/Z_2$. 

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6 Discussion

In this paper, we have studied M(atrix) theory description of M-theory compactified on unoriented two-dimensional manifold: Klein bottle and Möbius strip. The corresponding M(atrix) theories are gauge theory of area-preserving diffeomorphism of Klein bottle and Möbius strip respectively. We have found that the M(atrix) theories are (2+1)-dimensional $\mathcal{N} = 8$ supersymmetric $U(N)$ gauge theory. The parameter space on which each M(atrix) gauge theory lives is *dual Klein bottle* or *dual Möbius strip* respectively.

To illustrate how the result can be extended to higher-dimensional non-orientable manifold, we have investigated strongly coupled limit of toroidally compactified six-dimensional CHL heterotic string. We have shown that low-energy limit of corresponding M(atrix) theory is six-dimensional gauge theory with gauge group $U(N) \times U(N)$ and with $(1,0)$ supersymmetry. We have shown that the parameter space on which the gauge theory lives is *dual $\tilde{S}_1 \otimes \tilde{K}^3/\mathbb{Z}_2$*, where the $\mathbb{Z}_2$ action acts as parity on $\tilde{S}_1$ and as a half-period shift on one of the coordinates of $\tilde{K}^3$. Seiberg has proposed that M(atrix) theory description of Type IIA on K3 is given by Neveu-Schwarz five-brane compactified on $\tilde{K}^3 \times S_1$. We expect that the description extends to the present situation, where K3 is modded out further by a freely acting involution $\mathbb{Z}_2$. The M(atrix) theory description should then be in terms of Neveu-Schwarz five-brane on dual manifold $\tilde{K}^3 \times S_1/\mathbb{Z}_2$. In fact, the gauge theory we have deduced corresponds to low-energy description of Neveu-Schwarz five-brane of Type IIB theory compactified on this dual space. Details of the investigation will be reported elsewhere.

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