SWAP OPERATORS AND ELECTRIC CHARGES OF FERMIONS

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Abstract

Electric Charges operator (ECO) in phase space formulation, proposed by Zenczykowski, is expressed in terms of swap operator $2 \otimes 2$, in some expressions for possible physical interpretations. An expression of an ECO in terms of a swap operator makes sense to the eigenvalues of the swap operator. An ECO including all the fermions of the standard model (SM) is constructed.

Keywords: Tensor commutation matrix, leptons, quarks

1 Introduction

Swap operator or tensor commutation matrix $n \otimes n$ is the matrix $U_{n \otimes n}$ which has the following properties: for any unicolumns and $n$ rows matrices $\alpha \in \mathcal{M}_{n \times 1}(\mathbb{C})$, $\beta \in \mathcal{M}_{n \times 1}(\mathbb{C})$

$$U_{n \otimes n} \cdot (\alpha \otimes \beta) = \beta \otimes \alpha$$ (1)
and for any two \( n \times n \)-matrices, \( A, B \in M_{n \times n}(\mathbb{C}) \)

\[
U_{n \otimes n} \cdot (A \otimes B) = (B \otimes A) \cdot U_{n \otimes n}
\]

(2)

In quantum information theory (Cf. for example [1]) the swap operator \( 2 \otimes 2 \)

\[
U_{2 \otimes 2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(3)

is written as linear combination of the tensor products of the Pauli matrices

\[
\sigma_1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

and the unit \( 2 \times 2 \)-matrix

\[
\sigma_0 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
U_{2 \otimes 2} = \frac{1}{2} \sigma_0 \otimes \sigma_0 + \frac{1}{2} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i
\]

(4)

Since \( U_{2 \otimes 2} \) is an unitary matrix, it gives a representation of the Dirac equation. [2] thinks to have given physical interpretation of the transformed

\[
U_{2 \otimes 2} \cdot \left( \sqrt{\frac{E + mc^2}{2E}} \frac{1}{\sqrt{2(1 + n_3)}} e^{\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - E t)} \left( \frac{1}{E + mc^2} \right) \otimes \left( \frac{1 + n_3}{n_1 + in_2} \right) \right)
\]

\[
= \sqrt{\frac{E + mc^2}{2E}} \frac{1}{\sqrt{2(1 + n_3)}} e^{\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - E t)} \left( \frac{1 + n_3}{n_1 + in_2} \right) \otimes \left( \frac{1}{E + mc^2} \right)
\]

(5)

by \( U_{2 \otimes 2} \) of a solution of the Dirac equation.

Fermions have quantum numbers \( I_3 \), the isospin and \( Y \), the hypercharge. The electric charge \( Q \) of a fermion is given by the Gell-Mann-Nishijima relation

\[
Q = I_3 + \frac{Y}{2}
\]

(6)

For the fermions of the SM these quantum numbers are given by the following table.
Neutral leptons  $\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$  $Q$  $I_3$  $Y$

Charged leptons  $e_L, \mu_L, \tau_L$  $e_R, \mu_R, \tau_R$  $-1$  $-1/2$  $-1$

Quarks $u, c, t$  $u_R^u, u_R^c, u_R^t, c_R^u, c_R^c, c_R^t, t_R^u, t_R^c, t_R^t$  $2/3$  $1/2$  $1/3$

Quarks $d, s, b$  $d_R^d, d_R^s, d_R^b, s_R^d, s_R^s, s_R^b, b_R^d, b_R^s, b_R^b$  $-1/3$  $-1/2$  $1/3$

A matrix Gell-Mann-Nishijima relation for eight leptons and quarks of the SM of the same generation is proposed by [3], in phase space formulation. According to the formula (4) it is easy to notice that this matrix Gell-Mann-Nishijima relation can be expressed in terms of $U_{2\otimes 2}$. In this paper, we will write this relation in some forms where physical interpretation of the action of the swap operator $U_{2\otimes 2}$ is possible. Then the eigenvalues and eigenvectors of $U_{2\otimes 2}$ will take physical sense. In the section 4, for including more fermions of the SM we will write a matrix formula giving the electric charges in terms of the swap operator $3 \otimes 3$,\[
U_{3\otimes 3} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{pmatrix}
\]

Then the physical sense given to the eigenvalues of the swap operators are maintained. In section 5, for including all the fermions of the SM we will write the matrix of the electric charges in terms of the swap operator $U_{4\otimes 4}$. For the calculus we have used SCILAB, a free mathematical software for numerical analysis.

2 Gell-Mann-Nishijima relation in phase space formulation

We re-write here the Gell-Mann-Nishijima relation in phase space approach [3], which yields the ECO of eight fermions, two leptons and two three-colored
quarks for a single SM generation, for example $e_L, \nu_e, u^L, d^L, d^u, u^L, d^L, \nu_e$. 

$$Q = I_3 + \frac{Y}{2}$$  \hspace{1cm} (7)$$ where  

$$I_3 = \frac{1}{2} \sigma_0 \otimes \sigma_0 \otimes \sigma_3$$  \hspace{1cm} (8)$$ the weak isospin, 

$$Y = \left( \frac{1}{3} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i \right) \otimes \sigma_0$$  \hspace{1cm} (9)$$ the weak hypercharge.

We remark that the operators $I_3$ and $Y$ act independantly on the vector field because in the expression of $I_3$, $\sigma_3$ is at the right side of the tensor product and in the expression of $Y$, $\left( \frac{1}{3} \sum_{i=1}^{3} \sigma_i \otimes \sigma_i \right)$ is at the left side.

3 Electric charges operator in terms of the swap Operator

According to the formula (11) we can introduce the operator $U_{2\otimes 2}$ into the expression of the operator weak hypercharge $Y$, and then into the expression of the ECO $Q$.

$$Y = \frac{2}{3} U_{2\otimes 2} \otimes \sigma_0 - \frac{1}{3} \sigma_0 \otimes \sigma_0 \otimes \sigma_0$$  \hspace{1cm} (10)$$ Hence  

$$Q = \frac{1}{2} \sigma_0 \otimes \sigma_0 \otimes \sigma_3 + \frac{1}{3} \left( U_{2\otimes 2} \otimes \sigma_0 - \frac{1}{2} \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \right)$$  \hspace{1cm} (11)$$ Or  

$$Q = \sigma_0 \otimes \sigma_0 \otimes \left( \frac{\sigma_3}{2} + \alpha \frac{\sigma_0}{6} \right) + \frac{1}{3} \left( U_{2\otimes 2} - \frac{1 + \alpha}{2} \sigma_0 \otimes \sigma_0 \right) \otimes \sigma_0$$  \hspace{1cm} (12)$$ with $\alpha$ a real parameter. 
If $\alpha = 0$, we have the relation (11), that is (7).

If $\alpha = -3$,  

$$Q = \sigma_0 \otimes \sigma_0 \otimes Q_L + \frac{1}{3} \left( U_{2\otimes 2} + \sigma_0 \otimes \sigma_0 \right) \otimes \sigma_0$$  \hspace{1cm} (13)$$
with $Q_L = \frac{\alpha_3}{2} - \frac{\alpha_0}{2}$ the Gell-Mann Nishijima relation for leptons in space-time approach. (Cf. for example [II])

If $\alpha = 1$,
$$Q = \sigma_0 \otimes \sigma_0 \otimes Q_Q + \frac{1}{3} (U_{2\otimes 2} - \sigma_0 \otimes \sigma_0) \otimes \sigma_0$$
(14)

with $Q_Q = \frac{\alpha_3}{2} + \frac{\alpha_0}{6}$ the Gell-Mann Nishijima relation for quarks in space-time approach.

If $\alpha = -1$,
$$Q = \sigma_0 \otimes \sigma_0 \otimes \frac{Q_L + Q_Q}{2} + \frac{1}{3} U_{2\otimes 2} \otimes \sigma_0$$
(15)

The eigenvalues of $U_{2\otimes 2}$ are once $-1$ and three times +1.

$$\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$
the eigenvector of $U_{2\otimes 2}$ associated to the eigenvalue $-1$, which is, according to (13), associated to leptons.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
are the eigenvectors associated to $+1$, which are, according to (14), associated to three-colored quarks.

The diagonal of $Q_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ are formed by the charges of $e_L$ and $\nu_{eL}$.

The diagonal of $Q_Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$ are formed by the charges of a up quark $u$ and a down quark $d$. The number of the eigenvalue $-1$ of $U_{2\otimes 2}$ are the number of the generation of leptons. +1 three times the eigenvalue of $U_{2\otimes 2}$, that is the number of the colors. Hence the eigenvalues of $Q$ are the charges of the eight fermions mentioned above.
4 Electric Charges Operator in Terms of Swap Operator $3 \otimes 3$

$$Q_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad Q_Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$$

are respectively the leptons and quarks electric charges operators in spacetime formulation.

$$Q_Q - Q_L = \frac{2}{3} \lambda_0$$

where $\lambda_0$ is the $3 \times 3$-unit matrix. Hence,

$$\lambda_0 \otimes \lambda_0 \otimes Q_Q + \frac{1}{3} (U_{3\otimes 3} - \lambda_0 \otimes \lambda_0) \otimes \lambda_0 = \lambda_0 \otimes \lambda_0 \otimes Q_L + \frac{1}{3} (U_{3\otimes 3} + \lambda_0 \otimes \lambda_0) \otimes \lambda_0$$

We denote it $Q$ like electric charge operator. The eigenvalues of the swap operator $U_{3\otimes 3}$ are $-1$ three times and $+1$ six times. From the above equation the eigenvalues $-1$ are associated to leptons whereas the eigenvalues $+1$ are associated to quarks. Following the result of [3], the three eigenvalues $-1$ are associated to the three generations of leptons whereas the three eigenvalues $+1$ are associated to three colors of left handed quarks and the three ones are associated to three colors of right handed quarks. The diagonal of $Q_L$ are formed by the charges of the leptons of the SM in a same generation, for example $\nu_{eL}$, $\nu_{\mu L}$ and $\nu_{\tau L}$. The diagonal of $Q_Q$ are formed by the charges of a up quark $u$, a down quark $d$ and a strange quark $s$. Hence the twenty seven eigenvalues of $Q$, $-1$ six times, $0$ three times, $-1/3$ twelve times and $+2/3$ six times can be the charges of following SM fermions $\nu_{eL}$, $\nu_{\mu L}$, $\nu_{\tau L}$, $e_L$, $\mu_L$, $\tau_L$, $e_R$, $\mu_R$, $\tau_R$, $u^c_L$, $u^b_L$, $u^g_L$, $u^c_R$, $u^b_R$, $u^g_R$, $d^c_L$, $d^b_L$, $d^g_L$, $d^c_R$, $d^b_R$, $d^g_R$, $s^c_L$, $s^b_L$, $s^g_L$, $s^c_R$, $s^b_R$, $s^g_R$.

$$Q_Q = I_{3Q} + \frac{Y_Q}{2}$$

where $I_{3Q} = \frac{1}{2} \lambda_3$ and $Y_Q = \frac{1}{\sqrt{3}} \lambda_8$, with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{pmatrix},$$
\[
\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\]
\[
\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\]
are the Gell-Mann matrices.

The swap operator \( U_{3 \otimes 3} \) can be written in terms of the Gell-Mann matrices under the following way [4]
\[
U_{3 \otimes 3} = \frac{1}{3} \lambda_0 \otimes \lambda_0 + \frac{1}{2} \sum_{i=1}^{8} \lambda_i \otimes \lambda_i
\]

So,
\[
Q = \lambda_0 \otimes \lambda_0 \otimes \left( -\frac{2}{9} \lambda_0 + \frac{1}{2} \lambda_3 + \frac{1}{2\sqrt{3}} \lambda_8 \right) + \frac{1}{6} \left( \sum_{i=1}^{8} \lambda_i \otimes \lambda_i \right) \otimes \lambda_0
\]

Q can be written under the form of the relation [7], where
\[
I_3 = \frac{1}{2} (\lambda_0 \otimes \lambda_0 \otimes \lambda_3 - \tau_1 \otimes \tau_1 \otimes \tau_1 - \tau_2 \otimes \tau_2 \otimes \tau_1 - \tau_3 \otimes \tau_3 \otimes \tau_1)
\]
\[
Y = \tau_1 \otimes \tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2 \otimes \tau_1 + \tau_3 \otimes \tau_3 \otimes \tau_1 + \frac{1}{\sqrt{3}} \lambda_0 \otimes \lambda_0 \otimes \lambda_8 + \frac{2}{3} (U_{3 \otimes 3} - \lambda_0 \otimes \lambda_0) \otimes \lambda_0
\]
or
\[
Y = \tau_1 \otimes \tau_1 \otimes \tau_1 + \tau_2 \otimes \tau_2 \otimes \tau_1 + \tau_3 \otimes \tau_3 \otimes \tau_1 + \lambda_0 \otimes \lambda_0 \otimes \left( -\frac{4}{9} \lambda_0 + \frac{1}{\sqrt{3}} \lambda_8 \right) + \frac{1}{3} \left( \sum_{i=1}^{8} \lambda_i \otimes \lambda_i \right) \otimes \lambda_0
\]

with \( \tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), \( \tau_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) and \( \tau_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \).

From [2], \( I_3 \) and \( Y \) commute, so they are simultaneously diagonalizable.

If we take \( Q_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) and \( Q_Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \) the above relations between \( Q_L \) and \( Q_Q \) will hold. The twenty seven eigenvalues of the
ECO $Q$ are $-1$ three times, $0$ six times, $-1/3$ six times and $+2/3$ twelve times. These eigenvalues can be the charges of the following fermions $\nu_{eL}$, $\nu_{\mu L}$, $\nu_{\tau L}$, $\nu_{eR}$, $\nu_{\mu R}$, $\nu_{\tau R}$, $e_L$, $\mu_L$, $\tau_L$, $c_L^b$, $c_L^g$, $c_R^b$, $c_R^g$, $\nu_{eL}$, $v_{eL}$, $v_{\mu L}$, $v_{\tau L}$, $v_{eR}$, $v_{\mu R}$, $v_{\tau R}$, $t_R^b$, $b_R^b$, $b_R^g$, $b_R^\tau$, $d_R^s$, $s_R^c$, $c_R^s$. The right handed neutrinos $\nu_{eR}$, $\nu_{\mu R}$, $\nu_{\tau R}$ whose charge is $0$ are not SM fermions.

5 Including all the fermions of the standard Model

For including all the fermions of the SM we are going to build an ECO in terms of the swap operator $U_{4\otimes 4}$.

The eigenvalues of the swap operator $U_{4\otimes 4}$ are $-1$ six times and $+1$ ten times. So, the ECO

$$Q = \Lambda_0 \otimes \Lambda_0 \otimes Q_Q + \frac{1}{3} (U_{4\otimes 4} - \Lambda_0 \otimes \Lambda_0) \otimes \Lambda_0 = \Lambda_0 \otimes \Lambda_0 \otimes Q_Q + \frac{1}{3} (U_{4\otimes 4} + \Lambda_0 \otimes \Lambda_0) \otimes \Lambda_0$$

where $\Lambda_0$ is the $4 \times 4$-unit matrix and $Q_Q = \begin{pmatrix} -1/3 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 2/3 \end{pmatrix}$.

$Q_L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are respectively the electric charges operators of quarks and leptons in spacetime formulation, has sixty four eigenvalues.

The diagonal of $Q_L$ are formed by the charges of four leptons of the same generation, for example $e_L$, $\nu_{eL}$, $e_R$ and $\nu_{eR}$, whereas the diagonal of $Q_Q$ are formed by the charges of a up quark $u$, a down quark $d$, a strange (or a bottom) quark $s$ and a charme (or a top) quark $c$. For the swap operator $U_{4\otimes 4}$, the four eigenvalues $-1$ are associated to four generations of leptons, the three eigenvalues $+1$ and the one eigenvalue $-1$, are respectively associated to the three colors of left handed quarks and the lepton which form the left handed leptoquark of the Pati-Salam model [5].

$$\begin{pmatrix} u^L_L & u^L_R & u^R_L & \nu_{eL} \\ d^L_L & d^L_R & d^R_L & e_L \end{pmatrix} = \begin{pmatrix} u^L_L & u^L_R & u^L_L & u^L_L \\ d^L_L & d^L_R & d^L_L & d^L_L \end{pmatrix}, \begin{pmatrix} s^L_L & s^L_R & s^L_L & \mu_L \\ c^L_L & c^L_R & c^L_L & \mu_L \end{pmatrix} = \begin{pmatrix} s^L_L & s^L_R & s^L_L & s^L_L \\ c^L_L & c^L_R & c^L_L & c^L_L \end{pmatrix}$$

where the other three eigenvalues $1$ and the one eigenvalue $-1$, are respectively associated to the three colors of right handed quarks and the lepton which form the right handed leptoquark of the Pati-Salam model.
Finally, the last four eigenvalues +1 are associated to four colors of quarks of color white. So, according to [6] we have considered the leptons in the leptoquarks as trinos included.

\[
\begin{pmatrix}
v_R^c & u_R^b & d_R^c & e_R^c \\
v_R^a & u_R^a & d_R^a & e_R^a \\
v_R^g & u_R^g & d_R^g & e_R^g \\
\end{pmatrix} = \begin{pmatrix}
v_R^c & u_R^b & d_R^c & e_R^c \\
v_R^a & u_R^a & d_R^a & e_R^a \\
v_R^g & u_R^g & d_R^g & e_R^g \\
\end{pmatrix}, \quad \begin{pmatrix}
s_R^c & s_R^b & s_R^g & \nu_R^c \\
s_R^a & s_R^a & s_R^a & \nu_R^a \\
s_R^g & s_R^g & s_R^g & \nu_R^g \\
\end{pmatrix} = \begin{pmatrix}
s_R^c & s_R^b & s_R^g & \nu_R^c \\
s_R^a & s_R^a & s_R^a & \nu_R^a \\
s_R^g & s_R^g & s_R^g & \nu_R^g \\
\end{pmatrix}
\]

and that ends the list of sixty four fundamental fermions, right handed neutrinos included.

\[
\begin{align*}
\Lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_2 &= \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \\
\Lambda_5 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_6 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_9 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{11} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{12} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
\Lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\Lambda_{14} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix},
\end{align*}
\]
\[ \Lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \]

are the 4 \times 4-Gell-Mann matrices.

The formula \[ U_{4\otimes4} = \frac{1}{4} \Lambda_0 \otimes \Lambda_0 + \frac{1}{2} \sum_{i=1}^{15} \Lambda_i \otimes \Lambda_i \]

yields

\[ Q = \Lambda_0 \otimes \Lambda_0 \otimes \left( -\frac{1}{12} \Lambda_0 - \frac{1}{36} \Lambda_3 + \frac{\sqrt{3}}{2} \Lambda_8 - \frac{1}{6\sqrt{6}} \Lambda_{15} \right) + \frac{1}{6} \left( \sum_{i=1}^{15} \Lambda_i \otimes \Lambda_i \right) \otimes \Lambda_0 \]

Conclusion

Thanks to [3], we have an ECO for two leptons and six colored quarks of the SM in one generation. This ECO can be expressed in terms of swap operator \(2 \otimes 2\). An ECO for more fermions of the SM in three generations have been obtained in terms of swap operator \(3 \otimes 3\).

These expressions allow to say that the eigenvalues \(-1\) of the swap operator are associated to leptons whereas the eigenvalues \(+1\) are associated to quarks.

According to the sense taken by an eigenvalue of a swap operator, for obtaining an ECO for all the fermions of the SM, in terms of the swap operator \(4 \otimes 4\), we have to introduce fourth generation of leptons, the model of leptoquark of Pati-Salam and quarks of color yellow.

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