REALIZING A HOMOLOGY CLASS OF A COMPACT MANIFOLD BY A HOMOLOGY CLASS OF AN EXPLICIT CLOSED SUBMANIFOLD—A NEW APPROACH TO THOM’S WORKS ON HOMOLOGY CLASSES OF SUBMANIFOLDS—

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Abstract. It is a classical important problem of differential topology by Thom; for a homology class of a compact manifold, can we realize this by a closed submanifold with no boundary? This is true if the degree of the class is smaller or equal to the half of the dimension of the outer manifold under the condition that the coefficient ring is \( \mathbb{Z}/2\mathbb{Z} \). If the degree of the class is smaller or equal to 6 or equal to \( k-2 \) or \( k-1 \) under the condition that the coefficient ring is \( \mathbb{Z} \) where \( k \) is the dimension of the manifold, then this is also true. As a specific study, for 4-dimensional closed manifolds, the topologies (genera) of closed and connected surfaces realizing given 2nd homology classes have been actively studied, for example.

In the present paper, we consider the following similar problem; can we realize a homology class of a compact manifold by a homology class of an explicit closed manifold embedded in the (interior of the) given compact manifold? This problem is considered as a variant of previous problems. We present an affirmative answer via important theory in the singularity theory of differentiable maps: lifting a given smooth map to an embedding or obtaining an embedding such that the composition of this with the canonical projection is the given map. Presenting this application of lifting smooth maps and related fundamental propositions is also a main purpose of the present paper.

1. Introduction and fundamental notation and terminologies.

The following problem, essentially launched by Thom, is a classical important problem in differential topology ([24]). We discuss the problem and related problems here in the smooth category. Let \( A \) be a module.

Problem 1. For a homology class \( c \in H_j(X; A) \) of a closed manifold \( X \), can we realize this by a closed submanifold \( Y \) with no boundary or can we represent \( c \) as \( i_\ast(\nu_Y) = c \) for a generator \( \nu_Y \in H_{\dim Y}(Y; A) \) of the module \( H_{\dim Y}(Y; A) \) and the inclusion \( i : Y \to X \)?

This is true if the degree of the class is smaller or equal to the half of the dimension of the outer manifold under the condition that the coefficient \( A \) is the ring \( \mathbb{Z}/2\mathbb{Z} \). If the degree of the class is smaller or equal to 6 or equal to \( \dim X - 2 \) or \( \dim X - 1 \) under the conditions that the outer manifold \( X \) is orientable and that the coefficient is the ring \( \mathbb{Z} \), then this is also true. [1] is on classes we cannot represent in this way.

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Problem 2. For a 2nd homology class of a 4-dimensional closed manifold, how about the orientability and the genus of a closed and connected surface realizing this?

This gives various, explicit, important and interesting problems. See [15] for example. In low dimensional differential topology, these kinds of problems are actively studied via technique on low dimensional topology, gauge theory and so on.

1.1. Problems studied in this paper related to these problems by Thom.
In this paper, related to the problems before, we consider the following problem.

Problem 3. For a homology class \( c \in H_j(X; A) \) of a compact manifold \( X \), can we realize the homology class by a homology class of a closed manifold \( Y \) satisfying \( \partial Y = \emptyset \) embedded in the (interior of the) manifold or can we represent \( c \) as \( i_\ast(c') = c \) for a class \( c' \in H_j(Y; A) \) of the module \( H_j(Y; A) \) and the inclusion \( i : Y \to X \)? Moreover, can we obtain \( Y \) in a constructive way?

Note that we consider manifolds of arbitrary dimensions in the present paper.

1.2. Fold maps and Reeb spaces.

1.2.1. Fold maps. We introduce terminologies on differentiable maps. A singular point of a differentiable map \( c : X \to Y \) is a point at which the rank of the differential of the map drops or the image of the differential is smaller than the dimension of \( Y \). A singular value of the map is a point realized as a value at a singular point of the map. The set \( S(c) \) of all singular points is the singular set of the map. The singular value set is the image \( c(S(c)) \) of the singular set. The regular value set of the map is the complementary set \( Y - c(S(c)) \) of the singular value set and a regular value is a point in the regular value set.

Manifolds are assumed to be smooth or differentiable and of \( C^\infty \) and so are maps between manifolds. A diffeomorphism on a manifold is always smooth and the diffeomorphism group of it is defined as the group of all diffeomorphisms on it.

Definition 1. Let \( m \geq n \geq 1 \) be integers. A smooth map from an \( m \)-dimensional smooth manifold with no boundary into an \( n \)-dimensional smooth manifold no boundary is said to be a fold map if at each singular point \( p \), the form is

\[
(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)
\]

for some coordinates and an integer \( 0 \leq i(p) \leq \frac{m-n+1}{2} \).

For a fold map in Definition 1, we can obtain the following two properties.

(1) For any singular point \( p \), the \( i(p) \) is unique \( (i(p) \) is called the index of \( p \)).
(2) The set consisting of all singular points of a fixed index of the map is a closed submanifold of dimension \( n-1 \) with no boundary of the domain and the restriction to the singular set is an immersion of codimension 1.

A fold map is a special generic map if \( i(p) = 0 \) for all singular points. A Morse function with exactly two singular points on a homotopy sphere and canonical projections of unit spheres are simplest examples of special generic maps. If \( m = n \), then a fold map is always special generic.
1.2.2. Reeb spaces. The Reeb space of a continuous map is defined as the space of all connected components of preimages of the map.

Definition 2. Let $X$ and $Y$ be topological spaces. For a continuous map $c : X \to Y$, we define a relation on $X$ as $p_1 \sim_c p_2$ if and only if $p_1$ and $p_2$ are in a same connected component of $c^{-1}(p)$ for some $p \in Y$. Thus $\sim_c$ is an equivalence relation on $X$ and we denote the quotient space $X/\sim_c$ by $W_c$ and call it the Reeb space of $c$.

We denote the induced quotient map from $X$ into $W_c$ by $q_c$ and we can define $\bar{c} : W_c \to Y$ uniquely by $c = \bar{c} \circ q_c$. See also [17] for example.

Proposition 1 ([23]). For a fold map, the Reeb space is a polyhedron whose dimension is equal to that of the target manifold.

For suitable classes of these maps, Reeb spaces inherit topological properties of the manifolds admitting the maps. See [8], [9], [10] and [21] for example.

1.3. Organization of the present paper. This paper concerns Problem 3 before and also presents a new answer to important problems on differentiable maps between manifolds, lifting of maps, which is a key ingredient in an explicit problem related to Problem 3. The organization of the paper is as the following. In the next section, we introduce a class of standard-spherical fold maps. It is defined as a fold map such that indices of all singular points are 0 or 1, that preimages of regular values are points or disjoint unions of standard spheres: if the codimension of the map $f : M \to N$ is $n - m = -1$, then assume also that the domain is an orientable manifold. A simple fold map $f$ is a fold map such that $q_f|_{S(f)}$ is injective. Special generic maps form a proper subclass of these classes.

In the third section, we present examples of simple standard-spherical fold maps. After that, we consider lifting these maps to embeddings or obtaining embeddings such that the compositions of the embeddings with the canonical projections are the given maps. This is a fundamental, important and interesting studies in the theory of singularity of differentiable maps and this gives strong tools in obtaining main results of the present paper. We also present a new answer for a kind of these problems as Theorem 1: this is also an important ingredient in main results. The last section is devoted to main results or explicit answers to Problem 3 via tools and theory in the third section.

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2. Special generic maps, standard-spherical fold maps and simple fold maps

Throughout this paper, let $m \geq n \geq 1$ be integers, $M$ be a closed and connected manifold of dimension $m$, $N$ be a manifold of dimension $n$ without boundary and $f : M \to N$ be a smooth map unless otherwise stated. In addition, the structure groups of bundles such that the fibers are manifolds are assumed to be (subgroups of) diffeomorphism groups unless otherwise stated or these bundles are assumed to be so-called smooth bundles: PL bundles are discussed as exceptional cases. A linear bundle is a smooth bundle whose fiber is a $(k+1)$-dimensional (closed) unit disc or the $k$-dimensional unit sphere in $\mathbb{R}^{k+1}$ and whose structure group is a
subgroup of the \((k+1)\)-dimensional one \(O(k+1)\) acting linearly in a canonical way. A PL bundle is a bundle whose fiber is a polyhedron and whose structure group is a group consisting of PL homeomorphisms on the fiber.

Definition 3. A fold map \(f : M \to N\) is said to be standard-spherical if the following properties hold (we can easily know the definitions of a crossing and a normal crossing of a smooth immersion and we omit the definition).

1. The restriction map \(f|_{S(f)}\) is an immersion whose crossings are normal.
2. Indices of singular points are 0 or 1.
3. If \(m - n = 1\) and there exists a singular point of index 1 of \(f\), then \(M\) is orientable.
4. Preimages of regular values are disjoint unions of points or standard spheres.

Example 1. A special generic map \(f\) is a map of this class. Under the condition \(m = n\) its Reeb space \(W_f\) is regarded as a manifold diffeomorphic to \(M\). Under the condition \(m > n\) its Reeb space \(W_f\) is an \(n\)-dimensional compact manifold we can smoothly immerse into \(N\): \(f\) is represented as the composition of \(q_f\) with the immersion and furthermore, \(q_f(S(f)) = \partial W_f\) and \(q_f|_{S(f)}\) is injective.

Definition 4. We say a fold map \(f\) is simple if the restriction of \(q_f\) to the singular set is injective.

Special generic maps are simple. Let \(m > n\). For a simple standard-spherical map \(f\), for the complementary set of the union of the regular neighborhoods, the restriction of \(q_f\) to the preimage gives a smooth bundle whose fiber is \(S^{m-n}\) and the fiber of the bundle \(N(C)\) is Y-shaped and the bundle \(q_f^{-1}(N(C))\) is a smooth bundle whose fiber is diffeomorphic to a manifold obtained by removing the interior of a disjoint union of three standard closed discs of dimension \(m - n\) smoothly and disjointly embedded into \(S^{m-n}\) for each connected component \(C\) consisting of singular points of index 1. For this, see also [18] for example. For each connected component \(C\) consisting of singular points of index 0, the fiber of the PL bundle is a closed interval and the smooth bundle is a linear bundle and this holds for general fold maps by [19].

Under the condition \(m > n\), generally, for a simple fold map \(f\), for each connected component \(C\) of \(q_f(S(f))\), its small regular neighborhood \(N(C)\) is represented as a PL bundle over \(C\) whose fiber is a closed interval or a Y-shaped 1-dimensional polyhedron and the composition of the restriction of the map \(q_f\) to the preimage \(q_f^{-1}(N(C))\) of the total space \(N(C)\) of the bundle with the canonical projection to \(C\) is a smooth bundle.

Under the condition \(m = n\), for each connected component \(C\) of the singular set, we can take a closed tubular neighborhood \(N(C) \subset M\) and the canonical projection to \(C\) gives a linear bundle whose fiber is a closed interval.

Remark 1. In the case of a simple standard-spherical fold map \(f\), we do not assume that the restriction map \(f|_{S(f)}\) is an immersion whose crossings are normal: we may assume this but the assumption is not essential.

3. S-trivial standard-spherical fold maps and lifting these maps to embeddings.

Definition 5. In section 2, if in the case \(m > n\) the PL bundle \(N(C)\) over \(C\) and the smooth bundle \(q_f^{-1}(N(C))\) over \(C\) are trivial for each \(C\), then the simple fold
map $f$ is said to be $S$-trivial. If in the case $m = n$, the linear bundle $N(C)$ over $C$ is trivial for each $C$, then the fold map $f$ is said to be $S$-trivial.

We present several examples of $S$-trivial standard-spherical fold maps.

**Example 2.**

1. A special generic map $f : M \to N$ between equidimensional manifolds such that the following properties hold.
   - (a) $M$ is orientable.
   - (b) $S(f)$ is orientable.

   For example, let $M$ be a homotopy sphere for example. [5] implies that for any homotopy class of continuous maps between $S^m$ and $S^n$ where $m = n$, we can find a fold map satisfying the properties before.

2. Projections of bundles whose fibers are standard spheres: they are also special generic maps having no singular point.

3. [3], [7] and [20] present $S$-trivial standard-spherical fold maps. Through them, we can know that 3-dimensional closed and orientable manifolds of a class (the class of graph manifolds) admit such maps into surfaces. For such a map, we can regard the Reeb space $W_f$ as the shadow of the manifold. Roughly speaking, a shadow is a polyhedron at each point of which there exists a small regular neighborhood PL homeomorphic to a regular neighborhood of a point of a Reeb space of a simple fold map into the plane as before. Moreover, the space of all points the regular neighborhoods of which are 2-dimensional discs containing the points in the interiors is a 2-dimensional manifold and an integer called a gleam is assigned to each connected component of this. For the definition, see [7] and see also [25] and [26]. See also Remark 2.

4. [12] and [14] present construction of infinite families of such maps and infinite families of closed and connected manifolds admitting them starting from fundamental examples including special generic maps and examples in [8], [9] and [11] for example via surgery operations to the maps and the manifolds.

**Remark 2.** In fact, in Example 2 (3) a shadow is a 2-dimensional polyhedron of a wider class in fact and an integer or a rational number of the form $\frac{k}{7}$ where $k$ is an integer is assigned to each connected component of the 2-dimensional manifold as before: the number is said to be a gleam. Similarly, from a fold map $f$ such that the restriction map $f|_{S(f)}$ is an immersion whose crossings are normal on a 3-dimensional closed and orientable manifold into a surface, which always exists, and the Reeb space $W_f$, we can obtain a shadow.

The following proposition is a fundamental and key proposition in the present paper. $\text{Emd}(X, Y)$ denotes the space of all the smooth embeddings of a manifold $X$ into a manifold $Y$ endowed with the so-called $C^\infty$ Whitney topology (see [6]).

**Fact 1 ([2]).** $\text{Emd}(X, Y)$ is $\max\{0, \min\{2 \dim Y - 3 \dim X - 4, \dim Y - \dim X - 2\}\}$-connected if $\dim Y \geq \dim X + 2 \geq 3$ is assumed.

See also [16] for Fact 1. The following is for example shown in Lemma 1 of [13].

**Lemma 1.** For a Morse function $\bar{f}$ on a $k$-dimensional compact manifold satisfying $k \geq 1$ and either of the following properties, we can represent this as a composition of an embedding into $\mathbb{R}^{k+1}$ with a canonical projection.
A Morse function is a function on a closed unit disc of dimension $k$ with exactly one singular point in the interior such that the preimage of one of the two extrema and the boundary coincide.

$k > 1$ and a Morse function is a function on a compact manifold obtained by removing the interior of the union of three disjointly smoothly embedded $k$-dimensional standard closed discs in $S^k$ with exactly one singular point in the interior such that the preimages of the two extrema are a connected component of the boundary and the disjoint union of the remaining two connected components of the boundary, respectively.

A sketch of the proof. The former case follows by the definition of a singular point of index 0 of a fold map. The latter case follows by the argument as the following.

(1) Decompose the $k$-dimensional manifold into two $k$-dimensional compact submanifolds with corners along $(k-1)$-dimensional submanifolds: one is a disjoint union of two copies of the total space of a product bundle over a closed interval whose fiber is a standard closed disc of dimension $k-1$ and the other is diffeomorphic to the product of a closed interval and a standard closed disc of dimension $k-1$, which is so-called a 1-handle of the Morse function.

(2) On each connected component of the disjoint union of the total spaces of the product bundles, the original function is regarded as a projection of the bundle by virtue of a kind of Ehresmann’s fibration theorem ([4]).

Proposition 2 ([16] and an extension of the result there.). For an S-trivial standard-spherical fold map $f : M \to N$, there exists an embedding $F$ such that $f = \pi_{N,k} \circ F$ where $\pi_{N,k} : N \times \mathbb{R}^k \to N$ is the canonical projection onto $N$ if $n + k \geq \max\{\frac{3m+3}{2}, m + n + 1\}$.

Proof. For each connected component $C$ of $q_f(S(f))$, consider a small regular neighborhood $N(C)$ and the local smooth map represented as the composition of $q_f|_{q_f^{-1}(N(C))}$ with $\bar{f}|_{N(C)}$, we can consider an embedding so that the composition with the composition of the canonical projection $N(C) \times \mathbb{R}^k \to N(C)$ with $\bar{f}|_{N(C)}$ is the original map: the assumptions that the map is S-trivial together with arguments related to so-called Thom’s isotopy theorem and Lemma 1 and that $n + k \geq m + n + 1$ enable us to do this. We extend this over the complementary set of the union of $N(C)$ in $W_f$. The essential assumption is that the space $\text{Emb}(S^{m-n}, \mathbb{R}^k)$ is $(n-1)$-connected together with the relation $n + k \geq m + n + 1$. We consider suitable cell decompositions of $W_f$ and the complementary set. On each 1-cell in the complementary set, the original map is regarded as the projection of a trivial smooth bundle and we can construct an embedding as before. We can extend this as projections over 2-cells. We can do this inductively over $k$-cells where $2 \leq k \leq n$. This completes the proof. 

We explain about the fact that $\text{Emb}(S^{m-n}, \mathbb{R}^k)$ is $(n-1)$-connected. In Fact 1, the inequality $2k - 3(m - n) - 4 \geq 2(\frac{3m+3}{2} - n) - 3(m - n) - 4 = n - 1$ completes the proof.

Nishioka studied only cases of special generic maps on closed, connected and orientable manifolds into Euclidean spaces satisfying the relation $m > n$. For the methods here, see also [22] for example. They are on a hot topic of the singularity
theory of differentiable maps: lifting a smooth map to a smooth map of a suitable
class into a higher dimensional space or finding a representation of the original map
by a composition of a map of the suitable class with a canonical projection. Note
also that in these problems, as the author know, the target manifolds of smooth
maps are Euclidean spaces.

The following theorem presents another new explicit answer for a kind of these
problems and important in a main result or Theorem 3 later.

**Theorem 1.** For an S-trivial standard-spherical fold map $f : M \to N$ from a
3-dimensional closed, connected and orientable manifold into a 2-dimensional man-
ifold $N$ with no boundary such that for the shadow defined from the map and the
Reeb space in Example 2 (3) and Remark 2, all gleams are even, there exists an
embedding $F : M \to N \times \mathbb{R}^k$ such that $f = \pi_{N,k} \circ F$ where $\pi_{N,k} : N \times \mathbb{R}^k \to N$ is
the canonical projection onto $N$ for $k \geq 3$.

**Proof.** Around each connected component $C$ of $q_f(S(f))$ and each connected com-
ponent of $W_f - \bigcup_C N(C)$ where we abuse notation in the proof of Proposition 2, we
can construct smooth map into $\mathbb{R}^{2+k}$ similarly. By gluing them together, we obtain
a desired embedding. The assumption that the gleams of the shadow are always
even guarantees this. We explain this shortly to complete the proof: to know more
precisely, we need to know about terminologies and notions on shadows more and
see the cited articles.

First we investigate the case where each connected component $O$ of $W_f - \bigcup_C N(C)$ is not a closed surface. The restriction of $f$ to the preimage $q_f^{-1}(N(C))$
is, for suitable coordinates, represented as the composition of a product map of
either of the following Morse functions and the identity map $id_C$ with a suitable
immersion into $N$.

1. A Morse function on a closed unit disc of dimension 2 with exactly one
   singular point in the interior such that the preimage of one of the two
   extrema and the boundary coincide.
2. A Morse function on a compact, connected and orientable surface whose
genus is 0 and whose boundary consists of exactly three connected compo-
nents with exactly one singular point in the interior such that the preimage
of the two extrema are a connected component of the boundary and the
disjoint union of the remaining two connected components of the boundary.

In the situation of the proof of Proposition 2, they are represented as the composi-
tions of embeddings into $I \times \mathbb{R}^3$ with the canonical projection to the first component
where $I$ is a closed interval, regarded as a fiber of a suitable trivial bundle over $C$.
We can consider functions obtained by considering the 1-dimensional higher ver-
sions of these two functions as the second Morse function in Lemma 1. They are also
represented as the compositions of embeddings into $I \times \mathbb{R}^3$ with the canonical pro-
jection to the first component where $I$ is a closed interval, regarded as a fiber of the
trivial bundle over $C$, in the same situation. Furthermore, the original functions are
obtained as restrictions of the corresponding functions on the 3-dimensional closed
unit disc or the 3-dimensional compact manifold obtained by removing the interior
of the union of three disjointly smoothly embedded 3-dimensional standard closed
discs in $S^3$. Moreover, the boundary of the domain of each function on the surface is
obtained as a closed submanifold with no boundary of the boundary of the domain.
of the Morse function on the 3-dimensional manifold so that each connected component, diffeomorphic to a circle, is an equator of each of the three 2-dimensional standard spheres in the boundary of the 3-dimensional manifold. Distinct circles are embedded as equators in distinct 2-dimensional spheres in the boundary of the 3-dimensional manifold.

On each connected component \( O \) of \( W_f - \bigcup_{C} N(C) \), the map \( f \) is regarded as the composition of the projection of a linear bundle whose fiber is a circle with a suitable immersion into \( N \). We can represent the projection as the composition of an embedding into \( O \times \mathbb{R}^3 \) with the canonical projection to \( O \). We can also construct a map regarded as the composition of the projection of a linear bundle whose fiber is a 2-dimensional standard sphere with a suitable immersion into \( N \) and we can represent this as the composition of the composition of an embedding into \( O \times \mathbb{R}^3 \) with the canonical projection to \( O \) with the last immersion. Furthermore, the original projection is obtained as the restriction of the corresponding projection of the total space of the linear bundle whose fiber is the 2-dimensional sphere by restricting each fiber to the equator of the sphere. In other words, the original bundle is regarded as a subbundle.

We can glue the local maps on 3-dimensional and 4-dimensional manifolds together to obtain global embeddings into \( N \times \mathbb{R}^{2+k} \) and a smooth map into \( N \). Note for example that we obtain the map on the 3-dimensional closed manifold \( M \) into \( N \) as the given map \( f \). Connected components of the boundaries where we glue the maps together are, regarded as total spaces of trivial linear bundles over circles whose fibers are \( S^1 \) or \( S^2 \) and the restrictions of the map \( q_f \) to the spaces give the projections. Note that these fibers are in a preimage of the projection \( \pi_{N,k} : N \times \mathbb{R}^k \rightarrow N \) and \( S^2 \) is embedded as a so-called unknot in \( \mathbb{R}^k \) in the smooth category. \( S^1 \) is, as explained, embedded in \( S^2 \) as an equator. The assumption on the gleams imply that the way we glue the maps yields global maps. Regard each copy of \( S^1 \times S^2 \) as a trivial linear bundle over \( S^1 \) equipped with a projection regarded as a canonical projection to the first component. The bundle isomorphisms between two of these bundles is regarded as a product map of diffeomorphisms between the base spaces, regarded as \( S^1 \times \{\ast_1\} \) and fibers, regarded as \( \{\ast_2\} \times S^2 \).

Last we investigate the case where there exists no singular point of the map \( f \) and this completes the proof. We can show the statement similarly by the assumption on the gleam or the discussion on the attachments of the local maps and spaces just before (there exists exactly one connected component a gleam is assigned to). Note also that this case accounts for the case which we cannot regard as the first case.

This completes the proof. \( \square \)

Example 3 presents infinitely many examples of maps and 3-dimensional closed, connected and orientable manifolds to which we can apply Theorem 1. We omit fundamental notions on general \( k \)-dimensional linear bundles such that oriented linear bundles and the Euler classes of them, defined as \( k \)-th integral cohomology classes.

Example 3. Projections of bundles whose fibers are circles over closed surfaces, which are also regarded as 2-dimensional linear bundles, such that the Euler classes are divisible by 2 (if it is oriented), are simplest examples satisfying the assumption of Theorem 1.
4. Main theorems

Definition 6. For a homology class of a compact manifold, if for the homology group of this, the coefficient is a commutative ring \( R \) having the unique identity element \( 1 \neq 0 \in R \) and a homology class \( c \neq 0 \) satisfies the following properties, then it is said to be a UFG.

1. If \( rc = 0 \) for \( r \in R \), then \( r = 0 \).
2. For any element \( r \in R \) which is not a unit and any homology class \( c' \), \( c \) is never represented as \( rc' \).

Definition 7. Let \( R \) be a commutative ring having the unique identity element \( 1 \neq 0 \in R \). For a UFG homology class \( c \in H_j(X; R) \) of a compact manifold \( X \), the class \( c^* \in H^j(X; R) \) satisfying the following properties is said to be the dual \( c \).

1. \( c^*(c) = 1 \)
2. For any submodule \( B \) of \( H_j(X; R) \) such that the internal direct sum of the submodule generated by \( c \) and \( B \) is \( H_j(X; R) \), \( c^*(B) = 0 \).

We denote the dual of a UFG homology class \( c \) as in Definition 7. Hereafter, we assume that \( R := \mathbb{Z}, \mathbb{Q}, \mathbb{Z}/k\mathbb{Z} \) where \( k > 1 \) is an integer. We denote the Poincaré dual to a homology class or cohomology class \( c \) by \( \text{PD}_R(c) \) where \( X \) is a closed, connected and orientable manifold. If \( R := \mathbb{Z}/2\mathbb{Z} \), then we do not assume the orientability of \( X \) or \( X \) may not be orientable.

Definition 8. For two closed, connected, oriented and equidimensional manifolds \( X \) and \( Y \), we denote by \( \nu_X \in H_{\dim X}(X; \mathbb{Z}) \) and \( \nu_Y \in H_{\dim Y}(Y; \mathbb{Z}) \) the fundamental classes of the manifolds or the generators compatible with the orientations: the groups are isomorphic to \( \mathbb{Z} \). The mapping degree of a continuous map \( c : X \to Y \) is the integer \( d(c) \) satisfying \( \nu_Y = d(c)\nu_X \). For two closed, connected and equidimensional manifolds \( X \) and \( Y \), we denote by \( \nu_X \in H_{\dim X}(X; \mathbb{Z}/2\mathbb{Z}) \) and \( \nu_Y \in H_{\dim Y}(Y; \mathbb{Z}/2\mathbb{Z}) \) the \( \mathbb{Z}/2\mathbb{Z} \) fundamental classes of the manifolds or the generators of the groups, isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). The \( \mathbb{Z}/2\mathbb{Z} \) mapping degree of a continuous map \( c : X \to Y \) is the integer \( d(c) = 0, 1 \) satisfying \( \nu_Y = d(c)\nu_X \).

Definition 9. Let \( X \) be a compact manifold. Let \( Y \) be a closed and connected manifold satisfying \( \dim Y < \dim X \). A class \( c_Y \in H_j(Y; R) \) is realized by a class \( c_X \in H_j(X; R) \) if for an embedding \( i_{Y,X} : Y \to X \) whose image is in \( \text{Int} X \), \( c_X = i_{Y,X}*(c_Y) \) holds.

Theorem 2. Let \( R := \mathbb{Z}, \mathbb{Q}, \mathbb{Z}/k\mathbb{Z} \). Let \( X \) be a compact manifold and let \( c \in H_n(X; R) \) be realized by a closed and connected manifold \( N \) of dimension \( n > 0 \). Let a closed and connected manifold \( M \) of dimension \( m \geq n \) admit a smooth map \( f : M \to N \) satisfying the following properties.

1. There exists a preimage \( F \) of a regular value representing a class \( c_F \in H_{m-n}(M; R) \).
2. There exists an integer \( k \) and a UFG \( c_{F,0} \in H_{m-n}(M; R) \) satisfying \( c_F = kc_{F,0} \) and \( H_{m-n}(M; R) \) is the internal direct sum of the submodule generated by the one element set \( \{c_{F,0}\} \) and a suitable submodule.
3. \( f \) is an S-trivial standard-spherical fold map.

We also assume the following conditions.

1. If \( R \) is not isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), then \( M \) and \( N \) are orientable and \( N \) is oriented so that the fundamental class is \( \nu_N \in H_n(N; R) \). We also use
\( \nu_N \in H_n(N; R) \) for the \( \mathbb{Z}/2\mathbb{Z} \) fundamental class where \( R \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \).

(2) We can take an embedding \( i_{N,X} : N \to X \) satisfying \( i_{N,X}(N) \subset \text{Int}X \) and \( c = i_{N,X,*}(\nu_N) \) so that the normal bundle of the image is trivial.

(3) \( \dim X \geq \max\{\frac{3m+3}{2}, m + n + 1\} \).

In this situation, \( kc \) is realized by the class \( k\text{PD}_R(c_{F,0}^*) \).

**Proof.** In the situation of Proposition 2, we consider a map \( i_{N,X} \circ f : M \to i_{N,X}(N) \) and take the total space of the normal bundle of the image, which is a trivial linear bundle over the image, instead of \( "N \times \mathbb{R}^k" \) in the situation of Proposition 2. By the definitions of \( c_F \) and the embedding \( i_{N,X} \), we have the result. \( \square \)

**Theorem 3.** Let \( R := \mathbb{Z}, \mathbb{Q}, \mathbb{Z}/k\mathbb{Z} \). Let \( X \) be a compact and spin manifold and let \( c \in H_2(X; R) \) be realized by a closed, connected and orientable manifold \( N_0 \) of dimension \( n = 2 \). Let \( M \) be a closed and connected manifold of dimension \( m \geq n = 2 \). Let \( N \) be a closed, connected and orientable manifold of dimension \( n = 2 \). If \( R \) is not isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), then \( M \) is assumed to be orientable and \( N \) and \( N_0 \) are oriented so that the fundamental classes are \( \nu_N \in H_2(N; R) \) and \( \nu_{N_0} \in H_2(N_0; R) \), respectively. In this case, we also assume that there exists a smooth map \( c_N : N \to N_0 \) of mapping degree \( k_2 \). We also use \( \nu_N \in H_2(N; R) \) and \( \nu_{N_0} \in H_2(N_0; R) \) for \( \mathbb{Z}/2\mathbb{Z} \) fundamental classes where \( R \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) and in this case we also assume that there exists a smooth map \( c_N : N \to N_0 \) of \( \mathbb{Z}/2\mathbb{Z} \) mapping degree \( k_2 \). Suppose that a smooth map \( f : M \to N \) satisfying the following properties exists.

(1) There exists a preimage \( F \) of a regular value representing a class \( c_F \in H_{m-2}(M; R) \).

(2) There exists an integer \( k \) and a UFG \( c_{F,0} \in H_{m-2}(M; R) \) satisfying \( c_F = kc_{F,0} \) and \( H_{m-2}(M; R) \) is the internal direct sum of the submodule generated by the one element set \( \{c_F\} \) and a suitable submodule.

(3) \( f \) is an S-trivial standard-spherical fold map.

Last we also assume the following conditions.

(1) We can take an embedding \( i_{N_0,X} : N_0 \to X \) satisfying \( i_{N_0,X}(N_0) \subset \text{Int}X \) and \( c = i_{N_0,X,*}(\nu_{N_0}) \), which is assumed in the beginning.

(2) \( \dim X \geq \max\{\frac{3m+3}{2}, m + 3\} \).

(3) If \( f \) is an S-trivial map as in Theorem 1 with \( (m, n) = (3, 2) \), then \( \dim X \geq 5 \).

In this situation, \( kk_2c \) is realized by the class \( kk_2\text{PD}_R(c_{F,0}^*) \).

**Proof.** First by the assumption, \( \dim X \geq 2n + 1 \) holds. The existence of \( c_N : N \to N_0 \) yields the existence of an embedding \( i_{N,X} : N \to X \) satisfying \( i_{N,X}(N) \subset \text{Int}X \) and \( k_2c = k_2i_{N,X,*}(\nu_N) \). The assumptions that \( X \) is spin and that \( N \) and \( N_0 \) are orientable guarantee that the normal bundle of the image is trivial together with fundamental arguments on characteristic classes of linear bundles over manifolds related to smooth embeddings of smooth manifolds. This follows by Theorem 2 (and the proof) together with Theorem 1 (and the proof). \( \square \)

**Theorem 4.** Let \( R := \mathbb{Z}, \mathbb{Q}, \mathbb{Z}/k\mathbb{Z} \). Let \( X \) be a compact and spin manifold and let \( c \in H_3(X; R) \) be realized by a closed, connected and orientable manifold \( N_0 \) of dimension \( n = 3 \). Let \( M \) be a closed and connected manifold of dimension
\( m \geq n = 3 \). Let \( N \) be a closed, connected and orientable manifold \( N \) of dimension \( n = 3 \). If \( R \) is not isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), then \( M, N \) and \( N_0 \) are orientable and \( N \) and \( N_0 \) are oriented so that the fundamental classes are \( \nu_N \in H_3(N; R) \) and \( \nu_{N_0} \in H_3(N_0; R) \), respectively. In this case, we also assume that there exists a smooth map \( c_N : N \to N_0 \) of mapping degree \( k_3 \). We also use \( \nu_N \in H_3(N; R) \) and \( \nu_{N_0} \in H_3(N_0; R) \) for \( \mathbb{Z}/2\mathbb{Z} \) fundamental classes where \( R \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) and in this case we also assume that there exists a smooth map \( c_N : N \to N_0 \) of \( \mathbb{Z}/2\mathbb{Z} \) mapping degree \( k_3 \). Suppose that a smooth map \( f : M \to N \) satisfying the following properties exists.

1. There exists a preimage \( F \) of a regular value representing a class \( c_F \in H_{m-3}(M; R) \).
2. There exists an integer \( k \) and a UFG \( c_{F,0} \in H_{m-3}(M; R) \) satisfying \( c_F = kc_{F,0} \) and \( H_{m-3}(M; R) \) is the internal direct sum of the submodule generated by the one element set \{\( c_{F,0} \)\} and a suitable submodule.
3. \( f \) is an S-trivial standard-spherical fold map.

We also assume the following conditions.

1. We can take an embedding \( i_{N_0,X} : N_0 \to X \) satisfying \( i_{N_0,X}(N_0) \subset \text{Int}X \) and \( c = i_{N_0,X}^*(\nu_{N_0}) \), which is assumed in the beginning.
2. \( \dim X \geq \max\{\frac{3m+3}{2}, m + 4\} \).

In this situation, \( kk_3c \) is realized by the class \( kk_3\text{PD}_R(c_{F,0}^*) \).

**Proof.** First by the assumption, \( \dim X \geq 2n + 1 \) holds. The existence of \( c_N : N \to N_0 \) yields the existence of an embedding \( i_{N,X} : N \to X \) satisfying \( i_{N,X}(N) \subset \text{Int}X \) and \( k_3c = k_3i_{N,X}^*(\nu_N) \). The assumptions that \( X \) is spin and that \( N \) and \( N_0 \) are orientable guarantee that the normal bundle of the image is trivial together with fundamental arguments on characteristic classes of linear bundles over manifolds related to smooth embeddings of smooth manifolds. Theorem 2 (and the proof) completes the proof. \( \square \)

Last we present important facts in constructing explicit cases explaining these theorems well.

1. For two closed, connected, oriented and equidimensional manifolds \( X \) and \( Y \) where \( Y \) is a standard sphere of dimension larger than 0 and an arbitrary integer \( k \), there exists a smooth map \( c : X \to Y \) whose mapping degree is \( k \). This is important as an example for \( c_N \) in Theorems 3 and 4.
2. For two closed, connected and equidimensional manifolds \( X \) and \( Y \) where \( Y \) is a standard sphere of dimension larger than 0 and an arbitrary element \( k \in \mathbb{Z}/2\mathbb{Z} \), there exists a smooth map \( c : X \to Y \) whose \( \mathbb{Z}/2\mathbb{Z} \) mapping degree is \( k \). This is important as an example for \( c_N \) in Theorems 3 and 4.
3. The projections of product bundles over closed and connected manifolds whose fibers are closed and connected manifolds present infinitely many examples for maps \( f \) in Theorems 2, 3 and 4.
4. The projections of linear bundles over standard spheres whose fibers are unit spheres present infinitely many examples for maps \( f \) in Theorems 2, 3 and 4 especially for the case \( R = \mathbb{Z}/k\mathbb{Z} \).

In addition, Example 2 is also important.
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