Solitary wave solutions in plasma physics and acoustic gravity waves of some nonlinear evolution equations through enhanced MSE method

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Abstract
In this work, we probe the Gardner equation and the modified Benjamin-Bona-Mahony (mBBM) equation associated with plasma physics, acoustic-gravity waves in fluid mechanics, hydro-magnetic waves in cold plasma etc Exact wave solutions to the mentioned equations are studied analytically by the enhanced modified simple equation (EMSE) method. The solitary wave solutions are determined including free parameters. Setting definite values of the unknown parameters yield useful and stable solitary wave solutions. We have depicted some 3D and 2D graphs of the obtained solutions to comprehend the physical structure of the tangible events.

1. Introduction
The combination of the KdV and mKdV equation is the Gardner equation. In many areas of applied mathematics and physics, videlicet, plasma physics, nonlinear physics, fluid dynamics, the Gardner equation interprets the reality. Gardner equation is the most imperative model which is in the form [1].

\[ u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0. \]  

(1)

Many models can be extracted from this model by selecting different values of \( \alpha \), \( \beta \) and \( \gamma \). The KdV equation attain from the Gardner equation when we set \( \beta = 0 \) and \( \gamma = 1 \) in equation (1) which is important in the plasma physics and shallow water wave. Again, if we put \( \alpha = 0 \) and \( \gamma = 1 \), model (1) turns into the mKdV equation which is significant in the study of the multi-component plasma and electric circuits [2]. Therefore, numerous physical models, which intimately connected for studying physics the Gardner equation widely used. Thus, many methods have been executed to extract exact solutions, for instance the restrictive Taylor approximation for the Gardner and KdV equation [3], an analytic N-solitary wave solution by the Hirota bilinear method [4], the sub-ODE method [5], Wazwaz [6, 7] studied this equation and attain new solutions and kink solutions through the expansion method [8], the bilinear transformation method [9] and extended homoclinic test approach [10].

The modified Benjamin-Bona-Mahony (mBBM) equation [11] is in form

\[ u_t + u_x - u u_x + u_{xxx} = 0. \]  

(2)

This equation describes the acoustic-gravity waves in compressible fluids, the surface long waves in nonlinear dispersive media, acoustic waves in a harmonic crystals and hydro-magnetic waves in cold plasma. The mBBM model is most important in the study of shock and solitary waves. This equation is much studied in nonlinear physics. Different approaches are put in use, explicitly Manafianheiro is obtained the exact solutions by using the generalized \((G'/G)\)-expansion method [11], Bekir applied the \((G'/G)\)-expansion method [12], the Exp-function method [13], the tanh and the sine-cosine methods [14] to construct exact solutions of the mBBM equation. Therefore, to find the exclusive solutions to the mBBM equation [15] Benjamin implemented the first
integral method [16] and characterizes the approximation for surface long waves in nonlinear dispersive media [17] and Layeni used the new hyperbolic auxiliary function method [18] and fully discrete Galerkin approximations [19].

In this article, we interpret the EMSE method to extract exact solution to the Gardner equation and mBBM equation. There are many complex phenomena appeared in different field of mathematical physics and engineering that can be expressed by NLEEs. To examine the NLEEs there are many methods, such as the \((G'/G)\)-expansion method [20], the F-expansion method [21–23], the MSE method [24, 25] etc. The EMSE method provides large-number of solutions to the nonlinear evolution equations than other methods, namely the tanh method, the sine-cosine method etc.

The rest of the article has been arranged as, in section 2, outline of the EMSE method has been arranged. In section 3, solutions have been determined by means of the introduced method to the Gardner and mBBM equations. In section 4, we present graphical representations of the attained solutions. Finally in section 5, we have drawn the conclusion of this article.

2. Outline of the enhanced modified simple equation method

In this part, we discuss the EMSE method in order to derive exact soliton solutions to NLEEs arise in mathematical physics and engineering problems. Let us consider a NLEE

\[ E(u, u_t, u_x, u_{xx}, \cdots) = 0, \]

where \(u(x, t)\) is an unidentified function and \(E\) is a function of \(u(x, t)\) and its partial derivatives in which maximal order derivatives and nonlinear terms are attached. The steps of the method are as follows:

**Step 1:** We introduce the generalized wave transformation

\[ u(x, t) = u(\xi), \quad \xi = p(t)x \pm q(t), \]

where \(p(t)\) and \(q(t)\) are differentiable functions of \(t\). Making use of wave transformation (4), equation (3) converts into

\[ F(u, (\dot{p}x + \dot{q})u')p(t)u', \cdots) = 0, \]

where \(\equiv d/dt\) and \(\dot{\cdot} \equiv d/d\xi\).

**Step 2:** As per the EMSE method the solution structure of equation (5) is of the form

\[ u(\xi) = \sum_{j=0}^{n} c_j(t) \left( \frac{\varphi'(\xi)}{\varphi(\xi)} \right)^j; \quad c_n \neq 0, \]

where \(c_j(t)\) is arbitrary function of \(t\) such that \(c_n(t) \neq 0\) and \(\varphi(\xi)\) is an unrevealed function.

**Step 3:** In this step, we are devoted to evaluate the highest power of the series solution equation (6). To do that, we balance homogeneously among the nonlinear terms and the maximum order derivatives in equation (5).

**Step 4:** We compute the essential derivatives \(u', u'', \cdots\) of the considered solution \(u(\xi)\) and then inserting the derivatives into (5), we attained to an equation with the function \(\phi(\xi)\) and its derivatives as well as \(c_j(t)\). Setting the coefficients of \(x^j\varphi^{-1}, \varphi^{-1}; j = 0, 1, \cdots n + 1\) equal to zero, yield a system of equations. Solving the set of equations for the polynomial of \(\phi(\xi)\) and \(c_j(t)\), we attain to the require solution of equation (2).

3. Formulation of the solutions through the EMSE method

Now we will pay attention to put in use the EMSE method to extract the exact soliton solutions to the mBBM equation and the Gardner equation.

3.1. The gardner equation

According to step 3, we consider the subsequent traveling wave variable

\[ u(x, t) = u(\xi); \quad \xi = p(t)x + q(t), \]

Differentiating equation (7) with respect to \(\xi\), yields

\[ u_t = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} = (\dot{p}x + \dot{q})u'; \quad u_x = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} = p' u'. \]
Similarly, the higher derivatives are:

\[ u_{xx} = p^2 u'' \]

and \( u_{xxx} = p^3 u''' \).

Now, using the derivatives, equation (1) turns out into

\[ p\alpha (u^3)'u_s + p\beta (u^3)' + p^3\gamma u''' - (p\xi + q) u' = 0. \]  
(8)

Integrating equation (8), we attain

\[ p^3\gamma u''' + p\alpha u^2 + p\beta u^3 - (p\xi + q) u = 0. \]  
(9)

Using the balance between the principle of the highest order derivative \( u''' \) and the nonlinear term \( u^3 \), gives \( n = 1 \).

Therefore, solution of equation (9) takes the shape

\[ u = c_0(t) + c_1(t) \frac{\varphi'(\xi)}{\varphi(\xi)}, \quad c_1 \neq 0, \]  
(10)

Differentiating twice equation (10) with respect to \( \xi \), yields

\[ u' = c_1(t) \left\{ \frac{\varphi''}{\varphi} - \frac{\varphi'''}{\varphi^2} + 2\frac{\varphi'^3}{\varphi^3} \right\}, \]

\[ u'' = c_1(t) \left\{ \frac{\varphi''}{\varphi} - 3\frac{\varphi''\varphi'}{\varphi^2} + 2\frac{\varphi'^3}{\varphi^3} \right\}. \]  
(11)

Putting (10) and (11) into equation (9), we accomplish

\[ p^3\gamma c_1 \left\{ \frac{\varphi'''}{\varphi} - 3\frac{\varphi''\varphi'}{\varphi^2} + 2\frac{\varphi'^3}{\varphi^3} \right\} + \frac{p\alpha}{2} c_0(t) + c_1(t) \frac{\varphi'(\xi)}{\varphi(\xi)}^2 + \frac{p\beta}{3} \left( c_0(t) + c_1(t) \varphi'(\xi) \right)^3 + (p\xi + q) \left( c_0(t) + c_1(t) \frac{\varphi'(\xi)}{\varphi(\xi)} \right) = 0. \]

The above equation can be simplified as

\[ -p\xi c_0 - p\xi c_1 \frac{\varphi'}{\varphi} - \varphi c_0 + \frac{p\alpha}{2} c_0^2 + \frac{p\beta}{3} c_0^3 + \]

\[ \frac{1}{\varphi^2} \left( \alpha p^3 \gamma \varphi''' + p\alpha c_0 c_1 \varphi' + p\beta c_0 c_1 \varphi' - \varphi c_0 \varphi' \right) + \frac{1}{\varphi} \left( -3c_0 p^3 \gamma \varphi'' \varphi' + \frac{p\alpha}{2} c_1^2 \varphi'^2 + \frac{p\beta c_0}{3} \varphi^2 \right) + \frac{\varphi'^3}{\varphi^3} \left( 2p^3 \gamma c_1 + \frac{p\beta}{3} c_0 \right) = 0. \]

Equating the coefficient of \( \varphi^0 \), \( \varphi^2 \), \( \varphi^{-1} \), \( \varphi^{-2} \) and \( \varphi^{-3} \) to zero, we attain the subsequent set of algebraic and differential equations:

\[ -\varphi c_0 + \frac{p\alpha}{2} c_0^2 + \frac{p\beta}{3} c_0^3 = 0, \]  
(12)

\[ -\varphi c_1(t) = 0, \]  
(13)

\[ -\varphi c_1(t) = 0, \]  
(14)

\[ c_1 p^3 \gamma \varphi''' + p\alpha c_0 c_1 \varphi' + p\beta c_0 c_1 \varphi' - \varphi c_0 \varphi' = 0, \]  
(15)

\[ -3c_1 p^3 \gamma \varphi'' \varphi' + \frac{p\alpha}{2} c_1^2 \varphi'^2 + \frac{p\beta c_0}{3} c_1^2 \varphi'^2 = 0, \]  
(16)

\[ \left( 2p^3 \gamma c_1 + \frac{p\beta}{3} c_0 \right) \varphi^3 = 0. \]  
(17)

Since \( c_1 \neq 0 \) and \( \varphi' = 0 \), then equations (13) and (14) gives the following result

\[ p = 0, i.e. \ p = H \ (const) \]

From equation (12), we obtain

\[ c_0 = 0, \ \pm \alpha H + \frac{\sqrt{\alpha^2 H^2 + 8H\beta q}}{2H\beta} \]
From equation (17), we attain the value of $c_1$

$$c_1 = \pm \sqrt{-\frac{6H^2\gamma}{\beta}}$$, since $c_1 \neq 0$

Thus equation (15) can be rewritten as

$$\varphi' = \frac{6H^2\gamma}{c_1 \alpha + 2\beta c_0 c_1} \varphi''.$$ \hspace{1cm} (18)

Substituting the value of $\varphi'$ from (18) into equation (16), we achieve

$$\frac{\varphi''}{\varphi''} = 6\left(\frac{q - p\beta c_0 - p\alpha c_0}{H(c_1 \alpha + 2\beta c_0 c_1)}\right).$$ \hspace{1cm} (19)

Integrating equation (19), gives

$$\varphi'' = a_1 \exp\left\{\frac{6\left(q - p\beta c_0 - p\alpha c_0\right)}{H(c_1 \alpha + 2\beta c_0 c_1)} \xi\right\},$$ \hspace{1cm} (20)

where $\xi = p(t)x + q(t)$.

Inserting the value $\varphi''$ from equations (20) into (18), we achieve

$$\varphi' = \frac{6H^2\gamma}{c_1 \alpha + 2\beta c_0 c_1} \exp\left\{\frac{6\left(q - H\beta c_0 - H\alpha c_0\right)}{H(c_1 \alpha + 2\beta c_0 c_1)} \xi\right\}.$$ \hspace{1cm} (21)

Integrating with respect to $\xi$, we get

$$\varphi = \frac{H^3\gamma}{(q - H\beta c_0 - H\alpha c_0)} \exp\left\{\frac{6\left(q - H\beta c_0 - H\alpha c_0\right)}{H(c_1 \alpha + 2\beta c_0 c_1)} \xi\right\} + a_2(t),$$ \hspace{1cm} (22)

where $a_1(t)$ and $a_2(t)$ are arbitrary functions of $t$.

Putting the value of $\varphi$ and $\varphi'$ into solution (10), we obtain the following solution

$$u = c_0(t) + c_1(t) - \frac{6H^2\gamma}{c_1 \alpha + 2\beta c_0 c_1} \exp\left\{\frac{6\left(q - H\beta c_0 - H\alpha c_0\right)}{H(c_1 \alpha + 2\beta c_0 c_1)} \xi\right\} + a_2(t).$$ \hspace{1cm} (23)

Equation (22) is the exact solitary wave solution of the Burger Huxley equation.

**Case 1.** When $c_0 = 0$ and $c_1 = \sqrt{\frac{-6H^2\gamma}{\beta}}$, then equation (22) reduce to

$$u_1(\xi) = \frac{a_1 H \sqrt{-6\gamma \beta}}{\alpha} \exp\left\{\frac{q}{H^2\alpha} \sqrt{\frac{-6\gamma}{\beta}} \xi\right\},$$ \hspace{1cm} (24)

where $\xi = Hx + q(t)$.

If we set $a_1 = \pm 1$ and $a_1 = \frac{q}{H^2\alpha}$ into solution (22), we accomplish

$$u_{1,1} = \frac{6q}{aH} \left[1 + \tanh\left(\frac{6q}{H\alpha} \xi\right)\right].$$

$$u_{1,2} = \frac{6q}{aH} \left[1 + \coth\left(\frac{6q}{H\alpha} \xi\right)\right].$$

**Case 2.** If $c_0 = 0$ and $c_1 = -\sqrt{\frac{-6H^2\gamma}{\beta}}$, solution (22) reduce to
\[ u_2(\xi) = \sqrt{\frac{-6H^2\gamma}{2H\beta}} \frac{a_1H^{-\gamma}b^{-\gamma}}{\gamma} \exp\left\{ -\frac{q}{H^\alpha} \sqrt{\frac{6\beta}{\gamma}} \xi \right\} + a_2(t) \]  
\[ \xi = Hx + q(t). \]

On the other hand, if we set \( a_2 = \pm 1 \) and \( a_1 = \frac{q}{H^\gamma} \) into solution (22)

Then we have

\[ u_{2,1} = \frac{6q}{aH} \left[ 1 + \tanh\left( \frac{q}{2H^\gamma} \sqrt{\frac{6\beta}{\gamma}} \xi \right) \right] \]

\[ u_{2,2} = \frac{6q}{aH} \left[ 1 + \coth\left( \frac{q}{2H^\gamma} \sqrt{\frac{6\beta}{\gamma}} \xi \right) \right]. \]

**Case 03.** If \( c_0 = \frac{aH + \sqrt{a^2H^2 + 8H\beta q}}{2H\beta} \) and \( c_1 = \sqrt{\frac{-6H^2\gamma}{2H\beta}} \), the solution (22) becomes

\[ u_3(\xi) = \frac{aH + \sqrt{a^2H^2 + 8H\beta q}}{2H\beta} + \sqrt{\frac{-6H^2\gamma}{2H\beta}} \frac{6H^2\gamma a_1}{\beta a_1 + 2\beta a_1} \exp\left\{ \frac{6(q - p\beta c_0 - p\alpha c_0)}{H(c_1 + 2\beta a_1)} \xi \right\} + a_2(t) \]

**3.2. The modified Benjamin-Bona-Mahony equation**

In this subsection, the EMSE method is used in order to establish exact soliton solutions to the mBBM equation. We consider the mBBM equation [11]:

\[ u_t + uu_x - u^2u_x - u_{xxx} = 0, \]

According to step 3, we consider the following traveling wave variable

\[ u(x, t) = u(\xi); \quad \xi = pt + qt, \]

Making use of the wave variable (27), the mBBM equation converts into

\[ p^3u''' + pv (u^3)' - (\dot{p}x + \dot{q})u' - pu' = 0. \]

Integrating equation (28) with respect to \( \xi \), yields

\[ p^3u'' + pv (u^3') - (\dot{p}x + \dot{q})u - Pu = 0. \]

Using the homogeneous balance between the terms \( u^3 \) and \( u'' \), we attain \( n = 1 \).

Thus, solution shape is of the form

\[ u = c_0(t) + \frac{c_1(t)}{\varphi(\xi)}; \quad c_1 \neq 0. \]

Differentiating twice solution (30) with respect to \( \xi \), gives

\[ u' = c_1(t) \left\{ \frac{\varphi''}{\varphi} - \frac{\varphi'^2}{\varphi^2} \right\}; \]

\[ u'' = c_1(t) \left\{ \frac{\varphi'^3}{\varphi} - \frac{3\varphi''\varphi'}{\varphi^2} + 2\varphi^3 \right\}. \]

Inserting the solution (30) along with the relations presented in (31) into the equation (29), we attain

\[ -c_0\dot{p}x - \dot{q}c_1 + c_1 \frac{c_1}{\varphi} + \frac{1}{\varphi} \left( p^3c_1 \varphi'' + 3pv c_0 c_1 \varphi' - \dot{q}c_1 \varphi' - p c_1 \varphi' \right) \]

\[ + \frac{1}{\varphi^2} \left( 3pv c_1 c_1 \varphi'^2 - 3p^3 c_1 \varphi'^2 \right) + \frac{1}{\varphi^3} \left( pv c_1 \varphi'^3 + 2p^3 c_1 \varphi'^3 \right), \]

\[ -\dot{q}c_0 - p c_0 + c_0^3pv = 0. \]

Equating the coefficient of \( \varphi^0, x, \varphi^{-1}, \varphi^{-1}, \varphi^{-2} \) and \( \varphi^{-3} \), we derive the following equations:

\[ -\dot{q}c_0 - p c_0 + c_0^3pv = 0, \]
\(-c_0 \dot{p} x = 0,\) \hspace{1cm} (33)
\(-\dot{p} x \alpha \varphi' = 0,\) \hspace{1cm} (34)
\(p^4 \alpha \varphi'' + 3p\nu c_0^2 \alpha \varphi' - \dot{q} \alpha \varphi' - p \alpha \varphi' = 0,\) \hspace{1cm} (35)
\(3p\nu \alpha^2 \alpha_1 \varphi'' - 3p^3 \alpha \varphi' = 0,\) \hspace{1cm} (36)
\(p\nu \alpha_3 \varphi'' + 2p^3 \alpha \varphi' = 0.\) \hspace{1cm} (37)

Since \(c_1 \neq 0\) and \(\varphi' \neq 0,\) then (33) and (34) give the following result
\(p = H (\text{const}).\)

And from equation (32), we get
\(c_0 = 0, \quad \sqrt{\frac{H - \dot{q}}{\nu \alpha_0}}\)

From equation (37), we attain the value of \(c_1\)
\(c_1 = H \sqrt{-\frac{2}{\nu}} \text{ since } c_1 \neq 0.\)

We can rewrite the equation (36) as follows:
\(\varphi' = \frac{H^2 \varphi''}{\nu \alpha_0 c_0}\) \hspace{1cm} (38)

Now substituting the value of \(\varphi'\) from (38) into equation (35), provides
\(\frac{\varphi''}{\varphi'} = \left(\frac{q + p - 3p\nu c_0^2}{H\nu_\alpha c_0}\right).\) \hspace{1cm} (39)

Integrating equation (39), we attain
\(\varphi'' = a_1 \exp\left\{\frac{(q + H - 3H\nu c_0^2)}{H\nu_\alpha c_0}\xi\right\},\) \hspace{1cm} (40)
where \(\xi = Hx + q(t)\).

Substituting the value of \(\varphi''\) into equation (39), we acquire
\(\varphi' = \frac{H^2 a_1(t)}{\nu \alpha_0 c_0} \exp\left\{\frac{(q + H - 3H\nu c_0^2)}{H\nu_\alpha c_0}\xi\right\},\)

Integrating with respect to \(\xi\), we get
\(\varphi = \frac{H^2 a_1(t)}{q + H - 3H\nu c_0^2} \exp\left\{\frac{(q + H - 3H\nu c_0^2)}{H\nu_\alpha c_0}\xi\right\} + a_2(t)\) \hspace{1cm} (41)
where \(a_1(t)\) and \(a_2(t)\) are arbitrary functions of \(t\).

Putting the value of \(\varphi\) and \(\varphi'\) into the equation (30), we attain
\(u = c_0(t) + \frac{1}{\nu \alpha_0 c_0} \exp\left\{\frac{(q + p - 3p\nu c_0^2)}{H\nu_\alpha c_0}\xi\right\},\)

\(\frac{H}{q + H - 3H\nu c_0^2} \exp\left\{\frac{(q + H - 3H\nu c_0^2)}{H\nu_\alpha c_0}\xi\right\} + a_2(t)\)

Solution (42) is the general solitary wave solution of the mBBM equation.

**Case 1.** If \(c_0 = 0\) and \(c_1 = H \sqrt{\frac{2}{\nu}}\), the solution equation (42) is unbounded. So this case is rejected.

**Case 2.** Again, if \(c_0 = \sqrt{\frac{H + \dot{q}}{H\nu}}\) and \(c_1 = H \sqrt{\frac{2}{\nu}}\), the solution (42) reduce to
\(u_2(\xi) = \sqrt{\frac{H + \dot{q}}{H\nu}} + H^2 \sqrt{\frac{H}{v(H + \dot{q})}} a_1 \exp\left\{\frac{-H \sqrt{\frac{2(q + H)}{H}}}{2} \xi\right\} + a_2(t) - \frac{H^3 a_1}{2(q + H)} \exp\left\{\frac{-H \sqrt{\frac{2(q + H)}{H}}}{2} \xi\right\},\)
where \(\xi = Hx + q(t)\).
If we set \( a_1 = \pm 1 \) and \( a_t = \frac{2(q + H)}{H} \), the solution (43) becomes

\[
\begin{align*}
    u_{2,1} &= \sqrt{\frac{H + q}{Hv}} + \sqrt{\frac{H + q}{Hv}} \left[ 1 - \tanh \left( -H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \right], \\
    u_{2,2} &= \sqrt{\frac{H + q}{Hv}} + \sqrt{\frac{H + q}{Hv}} \left[ 1 - \coth \left( -H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \right].
\end{align*}
\]

**Case 3.** When \( c_0 = -\sqrt{\frac{H + q}{Hv}} \) and \( c_1 = H \sqrt{\frac{q}{v}} \), solution (42) reduce to

\[
\begin{align*}
    u_3(\xi) &= \sqrt{\frac{H + q}{Hv}} - H^2 \sqrt{\frac{H}{v(H + q)}} a_2(t) - \frac{H^2 a_1}{2(q + H)} \exp \left( H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \left[ \right. \\
    & \left. \quad - \frac{2(q + H)}{H} \left( \frac{1}{H} \right) \right].
\end{align*}
\]

where \( \xi = Hx + q(t) \).

If we set \( a_2 = \pm 1 \) and \( a_t = \frac{2(q + H)}{H} \) into the equation (44) becomes

\[
\begin{align*}
    u_{3,1} &= \sqrt{\frac{H + q}{Hv}} - \sqrt{\frac{H + q}{Hv}} \left[ 1 - \tanh \left( -H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \right], \\
    u_{3,2} &= \sqrt{\frac{H + q}{Hv}} - \sqrt{\frac{H + q}{Hv}} \left[ 1 - \coth \left( -H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \right].
\end{align*}
\]

**Case 4.** When \( c_0 = \sqrt{\frac{H + q}{Hv}} \) and \( c_1 = -H \sqrt{\frac{q}{v}} \) then equation (42) reduce to

\[
\begin{align*}
    u_4(\xi) &= \sqrt{\frac{H + q}{Hv}} - H^2 \sqrt{\frac{H}{v(H + q)}} a_2(t) - \frac{H^2 a_1}{2(q + H)} \exp \left( H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \left[ \right. \\
    & \left. \quad - \frac{2(q + H)}{H} \left( \frac{1}{H} \right) \right].
\end{align*}
\]

where \( \xi = Hx + q(t) \).

If we set \( a_2 = \pm 1 \) and \( a_t = \frac{2(q + H)}{H} \) into the solution (45), then it becomes

\[
\begin{align*}
    u_{4,1} &= \sqrt{\frac{H + q}{Hv}} - \sqrt{\frac{H + q}{Hv}} \left[ 1 - \coth \left( -H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \right], \\
    u_{4,2} &= \sqrt{\frac{H + q}{Hv}} - \sqrt{\frac{H + q}{Hv}} \left[ 1 - \tanh \left( -H \sqrt{-\frac{2(q + H)}{H}} \xi \right) \right].
\end{align*}
\]

### 4. Graphical representations

In this section, we explain the graphical demonstration of the obtained solutions to the Gardner equation and the mBBM equation. By implementing EMSE method, we obtained further exact solution of the Gardner equation and mBBM equation through free parameters. We have depicted the 3D and 2D figure of some attained solitary wave solutions through the computer algebraic software MAPLE.
4.1. The gardner equation
In this subsection, we have discussed about the nature and attitude of the obtaining traveling wave solution to the Gardner equation by implement EMSE method. The 3D and 2D figures of the solutions (23) and (24) are provided in the underneath:

4.2. The modified BBM equation
In this section, we will depict the 3D and 2D graphs of the determined solitary wave solutions to the mBBM equation. These exact solutions have different types of shape. With the help of MAPLE software, we have depicted the graphical view of some attained exact solutions.
Figure 2. (a): Periodic soliton depicted from the solution (43) for $a_1 = H = v = 1$, $a_2 = -2$ and $q(t) = t$. (b): Periodic soliton depicted from the solution (43) for $a_1 = H = v = 1$, $a_2 = -2$ and $q(t) = t$. (c): Kink type soliton drawn from solution (43) for $a_1 = 1$, $a_2 = -2$, $H = 1$, $v = 1$ and $q(t) = -2t$. (d): Kink type soliton drawn from solution (43) for $a_1 = 1$, $a_2 = -2$, $H = 1$, $v = 1$ and $q(t) = -2t$. (e): Bellshape solitary drawn from solution (43) for $a_1 = 1$, $a_2 = H = 1$, $v = 2$ and $q(t) = t$. (f): Bell shape soliton drawn from solution (43) for $a_1 = a_2 = H = 1$, $v = 2$ and $q(t) = t$. (g): Soliton drawn from solution (43) for $a_1 = v = 1$, $a_2 = H = -1$, and $q(t) = \text{sech}(t)$. (h): Soliton drawn from solution (43) for $a_1 = v = 1$, $a_2 = H = -1$, and $q(t) = \text{sech}(t)$. (i): Singular soliton drawn from solution (43) for $a_1 = v = a_2 = 1$, $H = -2$, and $q(t) = t$. (j): Singular soliton drawn from solution (43) for $a_1 = v = 1$, $a_2 = -1$ and $q(t) = t$. (k): Singular periodic soliton drawn from solution (43) for $a_1 = v = 1$, $H = 2$, $a_2 = -1$ and $q(t) = t$. (l): Singular periodic soliton drawn from solution (43) for $a_1 = v = 1$, $H = 1$, $a_2 = 1$ and $q(t) = t$. 


5. Comparison of the solutions

Lin [26] examined the Gardner equation through the tanh-coth method. The tanh-coth method is also an important analytical method to examine closed form solitary wave solutions. Therefore, we have compared the attained solutions with those obtained by the tanh-coth method (Lin [26]) and shown in table 1.

It is observed from the table 1 that the obtained solution \( u_{1,2} \) is similar to the solutions \( u_2 \) found by Lin [26]. It is also observable that, we have attained seven solutions. On the other hand, Lin [26] received only three solutions wherein the attained solution \( u_{1,2} \) is identical to the solution \( u_2 \) for specific values of the parameters which validates the introduced method. Besides, the attained remaining six solutions are different than the other two solutions obtained by Lin [26]. Therefore, we might assert the EMSE method is optimal than the tanh-coth method for the Gardner equation.

Yusufoglu [14] studied the mBBM equation through the extended tanh method. The extended tanh method is also a powerful method for searching solitary wave solutions. Therefore, we have compared the obtained solutions with those found by the extended tanh method (Yusufoglu [14]) and shown in table 2.

It is noteworthy to notice from table 2 that the obtained solution \( u_{1,1} \) is parallel to the solutions \( u_2 \) found by Yusufoglu [14]. It is also noticeable that, we have obtained nine solutions; on the other hand Yusufoglu [14] found only four solutions of which the solution \( u_2 \) is similar with the attained one solution \( u_{1,1} \). Besides, the attained remaining eight solutions have not been established by Yusufoglu [14]. Setting specific values of the parameters, we will get many new solutions from these general solutions. That is the specialty of this method.
6. Conclusion

In this article, we have established different type of soliton solutions to the Gardner equation and the mBBM equation through the EMSE method. The solutions include the bell shaped soliton, kink, singular kink, periodic, soliton etc. The method is compatible, useful and functional to substantiate new and further general closed form exact solutions to the NLEEs. We have depicted the 3D and 2D graphs to visualize the inner structure of the complicated nonlinear phenomena. This study shows that the EMSE method useful, reliable and provide adequate general solutions to NLEEs.

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Competing interests

There is no competing interest among the authors.

Author’s contributions

The study has been conducted in collaboration among the authors. All of the authors have worked effectively to develop the study. Manuscript has been scrutinized by all of the authors.

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Table 2. Comparison of the obtained solutions with the solutions found by Yusufoglu [14].

| Solutions found in Yusufoglu [14] | Solutions found in this article |
|----------------------------------|--------------------------------|
| \( u_1(x, t) = \frac{ab}{1 + (at + 1)^2} \tan \left( \frac{x + \frac{1}{2at} - 1}{t} \right) \) | \( u_{1,1}(x, t) = \tanh (x + t) \) |
| Setting \( a = -6, b = 1 \), the above solution turns into |
| \( u_2(x, t) = \tanh (x + t) \). | \( u_{1,1}(x, t) = \tanh (x + t) \) where \( q = -2, H = 1, v = -1, p(t) = \frac{1}{2q}, q(t) = \frac{1}{2q} \).
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