Self-organization, ergodicity breaking, phase transition
and synchronization in two-dimensional traffic-flow
model

Yu Shi*

Technology (World Laboratory),
P. O. Box 8730, Beijing 100080, People’s Republic of China

and

Fudan-T. D. Lee Physics Laboratory and Department of Physics,
Fudan University, Shanghai 200433, People’s Republic of China†

October 21, 2021

* electronic address: yushi@fudan.ihep.ac.cn
† mailing address
Abstract

Analytical investigation is made on the two-dimensional traffic-flow model with alternative movement and exclude-volume effect between right and up arrows [Phys. Rev. A 46 R6124 (1992)]. Several exact results are obtained, including the upper critical density above which there are only jamming configurations, and the lower critical density below which there are only moving configurations. The observed jamming transition takes place at another critical density $p_c(N)$, which is in the intermediate region between the lower and upper critical densities. It is derived that $p_c(N) = CN^\alpha$, where $C$ and $\alpha$ are determined to be respectively 0.76 and −0.14 from previous numerical simulation. This transition is suggested to be a second-order phase transition, the order parameter is found. The nature of self-organization, ergodicity breaking and synchronization are discussed. Comparison with the sandpile model is made.

PACS numbers: 05.70.Ln, 64.60.Cn, 64.60.Ak, 89.40.+k
I. INTRODUCTION

Much attention has been paid on cellular automaton models to investigate complex systems. These models can be viewed as statistical models with dynamics. Recently some models approaching traffic-flow problems have been studied. There are mainly two classes of models. The model of one class, introduced by Nagel and Schrekenberg [1-2], is defined on a one-dimensional array or simple network, where each site can be in one of several states representing emptiness or occupied by a car with one of the possible velocities. The iterative rules include acceleration, deceleration, randomization and movement. Indications were found near the point of maximum throughout for a phase transition separating low-density lameller flow from high-density jammed behavior. Completely different from this model, the model of another class, introduced by Biham, Middleton and Levin (BML) [3], is defined on a square lattice with periodic boundary condition. Each site contains either an arrow pointing upwards, an arrow pointing to the right, or is empty. The dynamics is controlled by the traffic light, such that the right arrows move only on even time steps and the up arrows move on odd time steps. On even time steps, each right arrows moves one lattice constant to the right unless the site on its righhand side is occupied, either up or right. If it is blocked by another arrow it does not move, even if during the same time step the blocking arrow moves out of that site. Similar rules apply to the up arrows, which move upwards. The velocity \( v \) of an arrow is defined to be the number of success moves it makes within a time interval of its turn \( \tau \), which is just the number of odd or even time steps within a “real” time interval, be divided by \( \tau \). It has maximal value \( v = 1 \), indicating that the arrow was never blocked, while \( v = 0 \) means that the arrow was stopped for the entire duration. The average velocity \( \bar{v} \) for the system is obtained by averaging \( v \) over all the arrows in the system.

The BML model is fully deterministic. It is called to be self-organized because whatever
the initial configuration of the system is, in the asymptotic configurations, all the arrows move freely in their turn hence the velocity averaged over all the arrows is $\bar{v} = 1$, or they are all stucked. These two kinds of configurations are referred to as moving and jamming ones, respectively. In the language of dynamics, there are two and only two kinds of attractors. Arriving finally which of the two depends on both the density of arrows and the initial configuration. This essence escaped the attention in previous analytical attempts [6]. In order to show that the critical density above which all arrows are stuck is less than 1, Chau, Hui and Woo [7] find certain jamming configurations with density less than 1. But it was not proved that all the configurations of the same density are jamming. In fact, as has been indicated in Ref. [8], the density for a configuration to be jamming can be as small as $2/N$, where $N$ is lattice size, but this case is atypical. In a more complicated model [4], more realistic rules and results are adopted. We think that the most important factor leading to jamming transition are also the exclude-volume effect between cars (arrows) of different type, as in the BML model.

We may ask, is there any configuration in which some arrows are moving while others are blocked? The answer is yes. Consider a column whose two neighboring columns are all full of up arrows, this column is occupied by up arrows less than $N/2$. Then asymptotically the arrows in this column are moving forever independent of other arrows, which may be all blocked. But such configurations are very few, i.e., its volume in the phase space is negligible compared with that of moving or jamming configurations, especially when we consider asymptotic configurations after self-organization. Therefore one can only consider moving and jamming configurations.

Here we give the complete picture for asymptotic configurations, neglecting those with both moving and blocked arrows. For very low density, there are only moving configurations. For very high density, there are only jamming configurations. Therefore there are a
lower critical density and an upper critical density. Between these two critical densities, the asymptotic configuration can be moving or jamming, dependent on the initial configurations. As will be discussed, there is another critical density above which the asymptotic configurations are typically jamming with moving one negligible. In fact this is just the jamming transition uncovered in simulation in Ref. [3]. The contents of this article is as follows. For convenience of discussions we introduce some notations in Sec. II. In Secs. III and IV, some exact results are given. The necessary conditions for the system to be moving or jamming are discussed, thus the upper and lower critical densities are determined exactly. In Sec. V, by considering the typical pattern formation of the jamming cluster, we obtain the third critical density above which the observed jamming transition takes place, the dependence on the lattice size is determined. This transition is suggested by sound reason to be a second-order phase transition, the order parameter is found. Some discussions concerning the nature of self-organization, ergodicity breaking and synchronization are made in Sec. VI, where we also compare the BML model with the sandpile model.

II. NOTATIONS

For convenience of discussions, some notations are used throughout this article. The lattice is $N \times N$, the density of up (right) arrows is $p_\uparrow = n_\uparrow/N^2$ ($p_\rightarrow = n_\rightarrow/N^2$), where $n_\uparrow$ ($n_\rightarrow$) is the number of up (right) arrows. The total density of arrows is $p = p_\uparrow + p_\rightarrow$. The number of empty lattice points is denoted as $n_0$. The empty sites can be treated as occupied by holes, which are able to move. Each lattice point is given a coordinate $(i,j)$, where $i$ and $j$ are $x$ and $y$ coordinates, respectively. The lower-left corner is $(1,1)$. The periodic boundary condition can be expressed as

$$(i + N, j) = (i, j + N) = (i + N, j + N) = (i, j).$$

(1)
This periodicity makes every lattice point equivalent (when there is no arrow), similar to (but different from) that on a sphere. As will be used in later Sections, the appearance of the lattice can be transformed without any real change, for example, so that it can be seen that the lattice can also be viewed as a parallelogram also with the periodic boundary condition. This parallelogram is made up of $N$ lines parallel to the left-falling diagonal of the original square, on each of these lines there are also $N$ lattice points. Hence we say that the lattice is composed of $N$ left-falling diagonals. The lattice points on the line linking $(1, i)$ and $(i, 1)$ and on the line linking $(i + 1, N)$ and $(N, i + 1)$ belong to a same left-falling diagonal. Similarly, the lattice can also be viewed as being made up of right-falling diagonals. Since the arrows are right or up, the former viewpoint will be used in studying the moving configurations.

To avoid confusion, the word “state” is used for the lattice points, while “configuration” is for the whole system. The state of $(i, j)$ is denoted as $|i, j >$. $|i, j > = \uparrow$, $\rightarrow$ or 0 if $(i, j)$ is occupied by an up arrow, a right arrow or is empty, respectively. $|i, j >$ is, of course, dependent on time, so it can be written as $|i, j > (t)$ when needed, here $t$ is the corresponding one for the given arrow. It is obvious that $|i, j > (t) = |i + \delta, j > (t + \delta)$ if $|i, j > (t) = \rightarrow$, $|i, j > (t) = |i, j + \delta > (t + \delta)$ if $|i, j > (t) = \uparrow$. The case for $|i, j > = 0$ will be discussed in the next Section.

III. EXACT RESULTS ON MOVING CONFIGURATION

First we point out that not only jamming configuration, but also moving configuration is stationary, since all the same kind of arrows move simultaneously thus form a rigid body. Considering the sequential arrangement for different kinds of arrows, we may think that the
whole system moves freely as a rigid body. In fact if we make a Galilean transformation to let one kind of arrow static, the system of the other kind of arrow hops between two positions all the time. In the phase space, the system hops between two fixed points. The exact results are stated in the form of lemma and theorem.

**Lemma 1.-**In a moving configuration, at any time, if there is an up (right) arrow on a left-falling diagonal, then any other lattice points on this left-falling diagonal are also occupied by an up (right) arrow or is empty.

*Proof.* Assume $|i, j > = \uparrow$, while $|i - \delta, j + \delta > = \rightarrow$, where $\delta$ is a positive integer. If it is on an odd (even) time step, then after $\delta$ time steps in the turns for up (right) arrow, the right (up) arrow will be forbidden to move by the up (right) arrow. Because of periodic boundary condition, every lattice point other than $(i, j)$ on the same left-falling diagonal is $(i - \delta, j + \delta)$ with $\delta > 0$. Therefore there can not be both up arrows and right arrows on a same left-falling diagonal. Q.E.D

From this lemma, we know that all of the left-falling diagonals with the same kind of arrows and holes, form a rigid body moving freely. So for $|i, j > (t) = 0$, $|i, j > (t) = |i, j + \delta > (t + \delta)$ if it belongs to a left-falling diagonal with up arrows, while $|i, j > (t) = |i + \delta, j > (t + \delta)$ if it belongs to a left-falling diagonal with right arrows.

**Lemma 2.-**In a moving configuration, at any time, there is at least one left-falling diagonal without any arrows, i.e., only made up of holes.

*Proof.* Without lose of generality, consider an odd time step. For a left-falling diagonal of up arrows, its upside left-falling diagonal cannot be that of right arrows even if these right arrows are not in the same columns of those up arrows, or there will be a left-falling diagonal where there are both up and right arrows, which is forbidden by lemma 1.

There must be at least one left-falling diagonal of up arrows whose upside left-falling diagonal is not that of up arrows, or there are only up arrows on the whole lattice. This
upside left-falling diagonal should be empty.

On an odd time step, for a left-falling diagonal of right arrows, it is unnecessary for its righthand side left-falling diagonal, which is just the upside one, to be empty. If the righthand side left-falling diagonal is that of up arrows, it will leave an empty left-falling diagonal after its movement. Therefore the least number of empty left-falling diagonal is 1. Q.E.D.

Theorem 1.-For \( N > 2 \), the necessary condition of formation of any possible moving configuration is \( p \leq \frac{1}{2} + \frac{(N - 4)}{2N^2} \) if \( N \) is even, \( p \leq \frac{1}{2} - \frac{1}{N^2} \) if \( N \) is odd. This is also sufficient for that moving configuration can form.

Proof.-It is obvious that there cannot be any moving configuration with both kinds of arrows for \( N = 2 \), hence \( N > 2 \) is assumed in the following. In the connected left-falling diagonals with the same kind of arrows, any up (right) arrow on the upside (righthand side) left-falling diagonal should be on the upside (righthand side) of an empty site, or it is not moving configuration. For even number of connected left-falling diagonals with the same kind of arrows, the number of arrows is at most equal to the number of empty sites. For odd number of connected left-falling diagonals with the same kind of arrows, there can be at most \((N - 1) - 1\) more than empty sites in addition. To make the number of empty left-falling diagonals be the least value 1, all the left-falling diagonals with the same kind of arrows must be connected. So if \( N \) is even, the number of the left-falling diagonals with up arrows and that of left-falling diagonals with right arrows can be all odd, hence

\[
n_\rightarrow \leq n_0^{(1)} + (N - 2),
\]

while

\[
n_\rightarrow \leq n_0^{(2)} + (N - 2),
\]

where \( n_0^{(1)} \) \((n_0^{(2)})\) is the number of empty sites in left-falling diagonals with up (right) arrows. If \( N \) is odd, one of the numbers of the left-falling diagonals with the same kind of arrows is
even while another is odd. Hence either Eq. (2) is valid while

\[ n_\rightarrow \leq n_0^{(2)}, \quad (4) \]

or Eq. (3) is valid while

\[ n_\uparrow \leq n_0^{(1)}. \quad (5) \]

Combining with

\[ n_0 \geq n_0^{(1)} + n_0^{(2)} + N, \quad (6) \]

we obtain

\[ p \leq \frac{1}{2} + \frac{N - 4}{2N^2} \quad (7) \]

if \( N \) is even, and

\[ p \leq \frac{1}{2} - \frac{1}{N^2} \quad (8) \]

if \( N \) is odd. Q.E.D.

This theorem shows that the argument that the largest possible \( p \) for moving configuration is 2/3 is false.

**IV. EXACT RESULTS ON JAMMING CONFIGURATION**

Since an up arrow can only be blocked by the upside arrow, which can be up or right one, while a right arrow can only be blocked by the right side arrow, these arrows thus form a directed path in a jamming configuration. The directions of all the directed path here are rightward or upward. Considering the periodic boundary condition, it is easy to know the following lemma 3.

*Lemma 3.*-In the jamming configuration, starting from an arbitrary arrow, one can obtain a directed path which return to either this starting arrow or an arrow in this path.
Such a path can be called a closed path. If it return to the starting arrow, it is referred to as a circular path. Of course, whenever there is a closed path, there is a circular path, which is a part of the former.

**Lemma 4.** In the jamming configuration, there must be at least one circular path.

This is just the necessary condition of for a configuration to be jamming.

**Lemma 5.** The length of a circular path is $N$ if it is made up of only one kind of arrows, while is $2N$ if made up of both kinds. Here the length is defined as the number of lattice points.

**Proof.** The first half statement is obvious since the circular path made up of one kind of arrows is parallel to the edge of the square. If the circular path is made up of both kinds of arrows, because it is directed, generally it appears as two part in the square lattice, for example, one part is a directed path connecting $(1, J)$ and $(I, N)$, another is a directed path connecting $(I+1, 1)$ and $(N, J)$. Please note that the former two points should be the nearest neighbors of the latter two, respectively, while the end points of a diagonal are in fact next-nearest neighbor. The length of every directed path connecting two end points of a diagonal is $2N - 1$. The length of a circular path is $2N$. Q.E.D.

**Theorem 2.** The necessary condition of formation of any possible jamming configuration is $p \geq (1 + p_s/p_l)/N$, where $p_s$ and $p_l$ are respectively the smaller and larger one of $p_{\uparrow}$ and $p_{\rightarrow}$. This is also sufficient for that jamming configuration can form.

**Proof.** Suppose there is a circular path made up of only the arrows with larger density, and there are no other arrows of this kind. The other kind of arrows will surely be blocked by this circular path. Therefore $N_i = N$, $N_s = (p_s/p_l)N$. Thus $(1 + p_s/p_l)/N$ is the smallest density for the jamming configuration where the circular path is made up of one kind of arrows.

For a circular path made up of both kinds of arrows, the density is $2/N$, which is not
smaller than the value obtained from circular path made up of only the arrows with smaller density. They are equal when $p_\uparrow = p_\rightarrow$. Q.E.D.

From Secs. III and IV, we obtain the upper and lower critical densities. When the density is smaller than $(1 + p_s/p_l)/N$, there can only be moving configurations. There can only be jamming configurations when the density is larger than $1/2 + (N - 4)/2N^2$ if $N$ is even, or $1/2 - 1/N^2$ if $N$ is odd. For a density between these two critical values, both moving and jamming configurations are possible depending on the initial configurations. Whenever the initial configuration is given, the final stationary configuration is determined.

V. TYPICAL FORMATION OF JAMMING CONFIGURATION

Although we have obtained the two critical densities, it was found from the simulation that a jamming transition occurs at a third critical density below the upper critical density. This is because that the number of moving configurations with density larger than the third critical density are negligible compared with the jamming configurations with the same density, especially when asymptotic configurations are studied. Considering the typical case for the formation of a jamming configuration, we determine this third critical density, denoted as $p_c(N)$. This definition of $p_c(N)$ is different from and more reasonable than that in Ref. [3], where it is defined to be at the center of the region where moving and jamming configurations are both non-negligible.

The jamming transition occurs soon after a circular path forms. The circular path, of course, typically consists of both up and right arrows. By “typically”, we mean that it has the possibility near to 1 while the other possibility is very small. The circular path blocks the neighboring right arrows on its lefthand side, or say, upside, while blocks the neighboring
up arrows on its downside, or say, righthand side. These blocked arrows block other arrows further. Consequently, a global cluster with directed branching structure emerge. Here branching means that there are both an up and a right arrows connected to a same arrow, which belongs to the the higher branching level. The highest level is the circular path.

In the final jamming cluster, there are some end-arrows, which are the end-points of the paths, which are connected to the circular path therefore are the rests of the closed paths other than the circular path. For an end-arrow, if it is an up (right) arrow, there must be no right arrow on its lefthand side and there is no up arrow on its downside, while its upside (righthand side) must be occupied. So the density of end-arrows are

$$\rho_e(p) = p^2(1 - p\rightarrow)(1 - p\uparrow) = p^2 - p^3 + p^2p\rightarrow p\uparrow \approx p^2. \quad (9)$$

On the other hand, the average number of end-arrows connected through paths to a arrow on the circular path is just the average branching levels starting from that arrow, as a function of $N$, it is denoted as $b(N)$. Therefore,

$$\rho_e(p = p_c)N^2 = 2Nb(N). \quad (10)$$

It is expected that

$$b(N) \sim N^{1+2\alpha}, \quad (11)$$

from which one obtain

$$p_c(N) = CN^\alpha, \quad (12)$$

where $C$ is a coefficient, while $\alpha$ is the exponent.

The simulation results can be used to test the above result and determine $\alpha$ and $c$. With the approximate values of $p_c(N)$ for $N = 16, 32, 64, 128, 512$ observed from the values of $p$ at which the ensemble average of velocity defined there begin to be almost 0 on Fig. 3 in Ref. 8, we obtain a good fit to Eq. (12) with $\alpha = -0.14$ and $C = 0.76$. 
Eq. (12) suggests that the jamming cluster at $p_c$ is fractal with dimensionality $2 + \alpha = 1.86$, which is near to $91/48$, the fractal dimension of the infinite cluster of two-dimensional percolation [14]. This is reasonable since the jamming cluster forms soon after the circular path forms, which is simailar to percolation in this respect. But surely the jamming cluster is fractal only when the density is near to $p_c$. Therefore we suggest that the jamming transition at $p_c$ might be a second-order phase transition. From lemma 5, we know that the order parameter is the probability that an arbitrarily chosen arrow belongs to a closed path. This is similar to that the order parameter of percolation is the probability that an arbitrarily chosen occupied site or bond belongs to an infinite cluster. Extensive studies concerning this issue is anticipated.

**VI. DISCUSSIONS**

An intuitive analogy of the dynamics of this system is the motion of a ball on a structure with two valleys of different depth in gravitational field. This structure is smooth enough for the ball to roll but there is friction. When it is put initially low enough on this structure, the ball can only roll to the lower valley. The structure can be made so that for high enough initial position, the ball can only roll to the higher valley. For intermediate height, it will roll to either of the valleys depending on the initial position.

The two-dimensional traffic-flow model we have studied is in fact a closed system, almost so is the traffic within a city. Therefore the so-called “self-organization” is mainly the tendency to equilibrium, with the special pattern determined by the dynamical rules. We have seen that the two kinds of attractors are both stationary states. By a proper definition of entropy or free energy, the dynamical processes might be formalized to processes minimizing the free energy, the landscape of which has two minima.
For an intermediate value of density between upper and lower critical values, the phase space decomposes into unconnected components, since one attractor is reached given the initial configuration. For a certain density, the trajectory in each component is unidirectional and irreversible, ending in an attractor. The attractor is either a fixed point or two fixed points, between which the system keeps hopping. Concerning the evolutionary process, the dynamics is of ergodicity breaking\[9\]. So the ensemble average of velocity in Ref. \[8\] has no meaning as a velocity, it is nothing but the percentage of moving configurations among all possible configurations. Ensemble average of velocity cannot be done over all configurations. The velocity in the asymptotic configuration is either 1 or 0, belonging to disconnected components of the phase space, any attempt in giving an ensemble average velocity between 0 and 1 is meaningless.

We may observe some resemblance between this model and directed sandpile model \[10-12\]. This traffic model can be viewed as a sandpile model with the addition that there are two kinds of particles (arrows) with different toppling direction. The critical slop is 1 so that the arrow hops to the next only if there is no arrow there. Similar to the original sandpile model \[13\], there is nonlinear interaction of the Goldstone modes due to gradient-dependence, i.e. $k$-dependence in momentum representation. The leading term of the interaction can be expected to be proportional to $k_x + k_y$. But there is a great difference that it is a closed system, while the original sandpile model is an open system. In the traffic-flow model, the total number of arrows and even the number of up (right) arrows in each column (row) are conserved all the time from beginning. There is no exchanges with environment. In the original sandpile model, there are continuous exchanges with the environment, the conservation is kept only after the critical state is arrived. This critical state is metastable, while the traffic-flow reaches one of the stable configurations, which are separated by an infinite barrier. The fractal structure exhibits only in the jamming cluster formed near $p_c$. This jamming
transition at $p_c$ has been suggested to be a second-order phase transition. The criticality is not “self-organized” as in sandpile model, the tuning parameter is the density $p$, the order parameter is the probability that an arbitrarily chosen arrow belong to a closed path. For larger density the cluster must be compact. This can also be understood considering the interactions of Goldstone modes. For larger $p$, the interaction drastically reduces. Since in the final asymptotic configuration, all the arrows are moving or stucked, this model can also be viewed as providing an interesting mechanism for synchronization.

**ACKNOWLEDGEMENT**

This work was initiated in reading Refs. [2,4,7] provided by the authors. Hoi Fung Chau and Ruibao Tao are acknowledged for stimulating discussions. Chau is particularly thanked for critically reading the manuscript. This work is supported by China Postdoctoral Science Foundation.

**References**

[1] K. Nagel and M. Schreckenberg, J. Phys. (France) I 2, 2221 (1992).

[2] K. Nagel and S. Rasmussen, in Proceedings of Alife 4, edited by R. Brooks and P. Maes (MIT Press, Cambridge, MA, 1994).

[3] O. Biham, A. A. Middleton and D. Levine, Phys. Rev. A 46, 3290 (1992).

[4] J. Freund and T. Pöschel, Physica A (to appear); LANL e-print adap-org

[5] The definition of $\tau$ here is different from that in Ref.[3], where $\tau$ is the “real” time interval, the sum of the alternative odd and even time steps.
[6] T. Nagatani, J. Phys. Soc. Jpn, 62, 2656 (1993); Y. Ishibashi and M. Fukui, ibid, 63, 2882 (1994). There are also other defects in these papers.

[7] H. F. Chau, P. M. Hui and Y. F. Woo, preprint IASSNS-HEP-95/07; LANL e-print hep-th/9502002.

[8] There are issues to be clarified in Sec. VI concerning this quantity.

[9] There is nonrigoroussness here since usually ergodicity breaking is for components separated by finite free energy barrier hence can be recovered, while it is impossible here. However, we feel that “ergodicity breaking” is more suitable than “symmetry breaking”.

[10] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. 59, 381 (1987).

[11] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. 63 1659 (1989).

[12] Y. Shi and C. Gong, Comm. Theor. Phys. 19 157 (1993).

[13] S. P. Obukhov, Phys. Rev. Lett. 65 1395 (1990).

[14] D. Stauffer, Introduction to Percolation Theory (Taylor & France, London, 1985); J. Feder, Fractals (Plenum Press, New York and London, 1988).
FIG. 1. Equivalent transformation of the appearance of the lattice. The square $PORQ$ can be changed to the parallelogram $PRQS$. The coordinates of the marked points are $O(1, 1)$, $R(N, 1)$, $Q(N, N)$, $P(1, N)$, $S(1, N-1)$, $T(N-1, 1)$, $O'(1, N+1)$, $T'(N-1, N+1)$, $S'(1, 2N-1)$. $O$ and $O'$, $T$ and $T'$, $S$ and $S'$ are respectively the same points due to the periodic boundary condition.
FIG. 2. Log-log plot of $p_c(N)$, the critical density at which typical jamming configuration begins to form while the possibility of moving configuration is negligible, with lattice size $N$. The circles are the results observed from Ref. [3]. The straight line is the least square fit yielding $p_c(N) \approx CN^\alpha$, with $C = 0.76$, $\alpha = -0.14$. 