A MODEL FOR THE DENSITY DISTRIBUTION OF VIRIALIZED COLD DARK MATTER HALOS

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Received 1999 January 7; accepted 1999 February 19; published 1999 March 12

ABSTRACT

An analytic collapse model for the formation and density distribution of virialized cold dark matter halos is proposed. Hierarchical structure formation is taken into account explicitly. Monte Carlo methods are used to generate samples of mass histories of virialized halos. The mean density distribution found from the collapse model is in good agreement with numerical results in the mass range from \(10^{11}\) to \(10^{15}\) \(M_\odot\) and in the radial range from 0.05\(r_{200}\) to \(r_{200}\).

Subject headings: cosmology: theory — dark matter — galaxies: halos — methods: analytical

1. INTRODUCTION

The current standard paradigm of cosmological structure formation is summarized by three major points: (1) the universe on large scales is dominated by collisionless dark matter, (2) the seeds of collapsed systems are small-amplitude Gaussian fluctuations of the uniform primordial density field, and (3) the evolution toward virialized systems occurs by gravitational instability. In hierarchical models of structure formation, the amplitude of the primordial density fluctuations increases with decreasing scale. This leads to a picture in which small objects form first and, by subsequent merging, form larger objects later on. Numerical simulations (e.g., Lacey & Cole 1994) have confirmed this picture to a large extent.

A striking outcome of numerical simulations is the “universal” shape of the profile of virialized density distributions, e.g., of the form (Navarro, Frenk, & White 1996, hereafter NFW96)

\[
\rho(r) = \frac{\rho_0}{(r/r_0)[1 + (r/r_0)]^2}, \tag{1}
\]

where \(r_0\) is a characteristic length scale and \(\rho_0\) is a characteristic density. Recent numerical simulations (e.g., Moore et al. 1998) indicate a somewhat steeper innermost slope \(\rho \propto r^{-1.4}\). However, in the radial range \([0.05r_{200}, r_{200}]\), a broad agreement is found.

Early analytical studies of the density distribution of collapsed objects are based on spherically symmetric systems (Gunn & Gott 1973; Gunn 1977) forming around sharp density peaks. Gaussian initial conditions have been taken into account by Hoffman & Shcham (1985). These studies suggest a virialized density distribution of the form \(\rho \propto r^{-\gamma}\) with \(\gamma \sim 2\). From the point of view of the current standard paradigm of structure formation, these models suffer from one main defect: they assume continuous and self-similar infall. There have been several attempts to develop more consistent models. Most of these studies seek to explain the form of the density distribution given in equation (1). For example, it has been argued (Syer & White 1998) that it arises from repeated mergers. Semi-analytical (Avila-Reese, Firmiani, & Hernandez 1998) and combined analytical and numerical approaches have been conducted as well. By including dynamical friction and tidal stripping, Nusser & Sheth (1998), e.g., find that the final density profile mainly depends on the rate of infall. The scaling of the density distribution (eq. [1]) with formation redshift has been investigated by Raig et al. (1998).

One of the key issues regarding hierarchical structure formation is how well the formation history is remembered by a collapsed system. The aim of this Letter is to investigate this question. As a working hypothesis, we assume that each step of the formation history is remembered by the conservation of the corresponding total energy of infalling matter. Under this assumption, a simple spherical symmetric collapse model is proposed. The model is developed along the line of earlier analytical studies (e.g., Gunn 1977; Hoffman & Shcham 1985). However, the condition of self-similarity is abandoned and the hierarchical formation history is taken into account explicitly.

2. THE MODEL

The model is based on formation histories and a collapse model describing how infalling matter virializes. The cosmological context is implicitly present in the formation histories.

2.1. Formation History

The excursion set formalism (Bond et al. 1991; Lacey & Cole 1993, 1994) provides an analytical model of the merger history of a collapsed object. Given the mass \(M_0\) of an object at redshift \(z_0\), it predicts the probability distribution of the mass \(M_{k+1}\) of the progenitor at an earlier epoch \(z_{k+1}\). Monte Carlo methods can be used to generate particular merger histories. In doing so, we follow here the method of Eisenstein & Loeb (1996). Starting from a halo mass \(M_0\) at \(z_0 = 0\), the merger history is traced by stepping backward in time until a typical mass marking the seeding of the object is reached. A particular mass history then is described by a set \(\{(m_k, z_k), k = 0, 1, 2, \ldots, K\}\), where \(m_k = M_0 - M_{k+1}\) is the mass falling in at redshift \(z_k\) and \(K\) denotes the total number of formation steps.

2.2. Total Energy of Infalling Matter

In the spherical symmetric case, the total energy \(E_i\), the infalling mass \(m_i\), and its turnaround radius \(R_m\) are related by

\[
E_i = -\frac{GM_i M_{k+1}}{R_m}, \tag{2}
\]

where \(M_{k+1}\) is the mass interior of \(R_m\), i.e., the mass collapsed
at \( z > z_* \). For \( m_* \ll M_{k+1} \), \( R_m \) is approximately
\[
R_m = \left( \frac{3M_{k+1}}{4\pi \rho_m} \right)^{1/3},
\]
where \( \rho_m \) is the mean density of the region of mass \( M_{k+1} \) at maximum expansion. For an Einstein–de Sitter cosmology (\( \Omega = 1, \Lambda = 0 \)) considered here (e.g., Peebles 1980),
\[
\rho_m = \frac{9\pi \gamma^2 \rho_0 (1 + z_{k+1})^3}{16}.
\]

Here, \( \rho_0 \) is the background density at the current epoch \( z = 0 \) and \( z_{k+1} \) is the redshift at collapse. From equations (2) and (4), the total energy \( E_k \) of matter of mass \( m_k \) falling in at formation step \( k \) is
\[
E_k = -(3/4)^{1/3} \pi G m_1 M_{k+1} \rho_0^2 (1 + z_{k+1}).
\]

How does this approximation of \( E_k \) fit the general picture of hierarchical structure formation? The basic assumption that has been made is that the density distribution of collapsed matter is approximately spherically symmetric, an approximation that is reasonable for virialized systems. No assumption about the distribution of the infalling matter has been made, however. The infalling matter can be distributed continuously, or it can be concentrated in one or several spots at \( r = R_m \). In this sense, the approximation for \( E_k \) accounts for a hierarchical structure formation process in which larger objects form by the collapse of already collapsed smaller objects.

### 2.3. Collapse Model

The working hypothesis for the proposed collapse model is that a virialized halo has a memory of its formation history in the sense that at each formation step the total energy of infalling matter is conserved. To determine the virialized structure, two additional simplifying assumptions have to be made. We suppose that the dynamical timescale of the preexisting structure is short compared to the timescale of the infall (i.e., the preexisting structure is virialized) and that the structure already present reacts adiabatically to the infall. In detail, the three assumptions about infall of matter at formation step \( k \) are as follows.

1. **Total energy.**—The total energy of the infalling mass \( m_k \) is conserved. There is no energy dissipation, and
\[
E_k = \text{const}.
\]

2. **Virialization.**—The dynamical timescale of the preexisting structure is short compared to the timescale of the infall. The infalling mass \( m_k \) virializes in the gravitational potential originating from the preexisting, virialized structure. The corresponding virial theorem is (e.g., Binney & Tremaine 1987)
\[
2K_k + W_k + V_k = 0,
\]
where \( K_k \) is the kinetic energy, \( W_k \) is the potential energy, and \( V_k \) is related to the external gravitational potential.

3. **Adiabatic contraction.**—The preexisting structure reacts adiabatically to the infalling matter deposed within its extent.

This assumption is based on the fact that for periodic orbits, \( \int pdq \) (where \( p \) and \( q \) are then conjugate momentum and coordinate, respectively) is an adiabatic invariant (Blumenthal et al. 1986; Flores et al. 1993).

To simplify calculations, the density distribution resulting from virialization of the mass \( m_k \) is approximated by
\[
\tilde{\rho}_k(r) = \begin{cases} 
3m_k/4\pi R_k^3, & r \leq R_k; \\
0, & r > R_k;
\end{cases}
\]
where \( R_k \) is the characteristic (virial) radius determined by equation (7).

Before discussing assumptions 1, 2, and 3 in more detail, consider the sequence of formation steps. The formation of the virialized halo starts with the collapse of the mass \( m_k \) at redshift \( z_k \). There is no preexisting structure, and the collapse according to assumptions 1, 2, and 3 is equivalent to the familiar top-hat collapse. The resulting object (structure \( K \)) is characterized by the density distribution \( \tilde{\rho}_K \) (eq. [8]), where \( R_k \) is determined from equation (7) with \( V_k = 0 \).

The mass \( m_{k-1} \) collapses onto structure \( K \) at redshift \( z_{k-1} \). Its virialized density distribution \( \tilde{\rho}_{K-1} \) is determined by equations (7) and (8) depending on \( \tilde{\rho}_K \). The mass enhancement within \( R_k \) due to the infall of \( m_{k-1} \) leads to an adiabatic contraction of the innermost structure \( K \) (assumption 3). It is estimated here by considering the quantity \( R_k m_k \) as adiabatic invariant. In principle, this implies a self-similar mass distribution. Since we do not attempt to spatially resolve \( \tilde{\rho}_K \) within \( R_k \), this is a reasonable approximation here. The contraction of structure \( K \) due to the infall of \( m_{k-1} \), then is described by
\[
[m_{k-1}(R_k')] + m_k R_k' = m_k R_k,
\]
where \( m_{k-1}(R_k') \) is the fraction of the mass \( m_{k-1} \) which is inside the “contracted” radius \( R_k' \).

In formation step \( K - 2 \), the density distribution \( \tilde{\rho}_{K-2} \) is determined by equations (7) and (8) depending now on \( \tilde{\rho}_k \) and \( \tilde{\rho}_{K-1} \). The infall of mass \( m_{k-2} \) leads to a contraction of structure \( K - 1 \), and the structure \( K \) contracts because of \( m_{k-2} \) and the contraction of structure \( K - 1 \) (which contributes to a mass enhancement within \( R_k \) as well).

The formation of the collapsed object is completed at \( z = 0 \) after the formation steps \( K - 3, K - 4, \ldots, 0 \). The structure of the virialized halo is described as a superposition of concentric spheres of different but uniform density and defined by the set \( \{(m_k, R_k), k = 0, 1, 2, \ldots, K\} \). The total density profile is
\[
\rho(r) = \sum_{k=0}^{K} \tilde{\rho}_k(r).
\]
similarity and it includes the formation history. With respect to these differences, the presented model clearly adopts a more realistic view of hierarchical structure formation.

But how realistic are assumptions 1–3 in the context of hierarchical structure formation? A complete answer to this question lies beyond the scope of this Letter. The key issue probably is assumption 1. An attempt to semianalytically model heating of the preexisting structure by infalling matter has recently been made by Nusser & Sheth (1998). Their study seems to indicate that including energy dissipation (e.g., by drag forces; Ostriker & Turner 1979) does not drastically alter density profiles but leads to a slightly steeper slope in the innermost region of halos. Regarding assumption 2, the timescale for virialization that is comparable to the dynamical timescale $\tau_{\text{dyn}} \sim (4\pi G \rho_{\text{max}})^{-1/2}$ is of importance. As the virialized structure forms and the cuspy inner region emerges, $\tau_{\text{dyn}}$ rapidly gets smaller. From the outcome of the model calculation (see § 3), one finds that for time step $\tau_{\text{step}} = \Delta \tau \sim 0.05$, and after the first few formation steps, $\tau_{\text{step}} \leq \tau_{\text{dyn}}$. This indicates that it is reasonable to assume infalling matter to collapse onto an at least pre-virialized structure. Finally, it should be stressed that the model uses energy considerations to relate mass histories and virialized density distributions. These considerations depend on the gravitational potential, which itself is much smoother and more symmetric than the underlying density distribution (see also Hoffman 1988).

3. RESULTS

The results presented here are based on 50 Monte Carlo realizations of the formation history of halos of mass $M_0 = 10^{11}, 10^{13},$ and $10^{15} M_\odot$ in the context of the standard cold dark matter (SCDM) cosmology ($\Omega = 1, h = 0.5, \sigma_8 = 0.67$). A time step $\Delta \tau = 0.05$ has been chosen, and the mass limit below which the object is considered to be a seeding object is defined as $M_1 \leq 0.05 M_0$. While the time step $\Delta \tau$ affects the mean mass falling in at each formation step, the resulting virialized density distributions do not change significantly in the tested range $\Delta \tau = 0.01–0.2$.

3.1. Virialized Density Distribution

Figure 1 shows a superposition of the virialized density distributions. Data points mark the total density at various radii $R_i$. The sharply dropping density at the outer edge of the distributions originates from the missing infall of mass in excess of the defined halo mass. The radius at which this drop occurs roughly corresponds to the radius $r_{200}$, at which the mean overdensity drops below $\delta = 200$. The radius $r_{200}$ is known as the characteristic size of a virialized object. This implies that for $r > r_{200}$, the assumptions of the model break down. As a consequence, the density distribution at $r > r_{200}$ will not be considered hereafter. Table 1 presents the mean of $r_{200}$ and the corresponding mass $M_{200}$. The scatter of the innermost data points is due to the different formation masses $M_0$ and redshifts $z$, leading to different initial values of $R_i$ and $\rho_i$, respectively. For radii $r < R$, the structure of the density distributions are not resolved. Thus, the meaningful innermost radius for an individual density distribution is $r_{\text{vir}} = R_i$. The radial range of the model thus is $[r_{\text{vir}}, r_{200}]$.

Figure 2 shows the mean virialized density profile. Data points mark the mean of the 50 individual density profiles for the $10^{11}, 10^{13},$ and $10^{15} M_\odot$ halos (from left to right). Dashed lines indicate the rms scatter. The data points mark the radial range $[r_{\text{vir}}, r_{200}]$. The best-fit NFW96 profile (eq. [1]) is plotted as a solid line. Table 1 lists the concentration parameter $r_s/r_{200}$. Higher halo masses lead to less centrally concentrated halos. The comparison of Figures 2 and 3 with Figures 3 and 4 of NFW96 and corresponding concentration parameters illustrates the agreement with $N$-body simulations.

3.2. Circular Velocities

Mean circular velocities and the corresponding scatter are calculated from $v_c(r) = (GM(r)/r)^{1/2}$. The maximum values for the scatter are found roughly at $r = r_{\text{vir}}$. It is $\pm 13\%$, $\pm 10\%$, and $\pm 6\%$ for the $10^{11}, 10^{13},$ and $10^{15} M_\odot$ halos, respectively. These values are in good agreement with Eisenstein & Loeb (1996; see Fig. 2 in their paper). For the outer regions of the halos, the model predicts a significantly decreasing scatter that reaches values well below $10\%$ at $r = r_{200}$ (see Table 1). If confirmed by $N$-body simulations, this could be of importance for the interpretation of the Tully-Fisher, e.g., in terms of its

| $M_0$  | $M_{200}$ | $r_{200}$ | $V_{200}$ | $\rho_s$ | $r_s$ | $r_s/r_{200}$ |
|-------|-----------|-----------|-----------|----------|------|--------------|
| $10^{11}$ | 0.94 ± 0.05 | 0.117 ± 0.002 | 58.7 ± 0.97 | 3.25 × 10^{16} | 0.0044 | 0.038 |
| $10^{13}$ | 0.92 ± 0.05 | 0.541 ± 0.009 | 270 ± 4.6 | 7.33 × 10^{13} | 0.0373 | 0.069 |
| $10^{15}$ | 0.85 ± 0.05 | 2.44 ± 0.05 | 1222 ± 25 | 1.37 × 10^{13} | 0.3367 | 0.138 |

Fig. 1.—Superposition of the virialized density distributions resulting from the 50 mass histories and the collapse model discussed in § 2. The halo masses are $10^{11}, 10^{13},$ and $10^{15} M_\odot$ (from left to right).
relation to cosmological initial conditions (see Eisenstein & Loeb 1996). As a word of caution, it should be noted that the poor radial density resolution at \( r \leq r_{200} \) affects the circular rotation profiles. Compared to the NFW96 profile, the present model suggests a higher central density leading to flatter circular velocity profiles.

4. SUMMARY AND CONCLUSIONS

A simple analytic collapse model has been proposed for the hierarchical formation scenario of virialized cold dark matter halos. As a working hypothesis, it has been assumed that dark matter halos possess a memory of their formation history by conserving the total energy of infalling matter in each formation step. For the tested SCDM cosmology, the virialized mean density profiles are in good agreement with numerical results. This suggests that energy dissipation does not play a major role in the determination of the mean shape of virialized density profiles. As a consequence, the (nonlinear) virialized density distribution is tightly related to the (linear) density field of the infall region by the total energy of infalling matter.

An improved model could include more realistic density distributions \( \rho_0 \) and probably should differentiate between minor and major mergers (e.g., Salvador-Solé, Solanes, & Manrique 1998). While these modifications could allow for resolving the radial range \( r < 0.05 r_{200} \), it is believed that they will not drastically alter the results for \( r > 0.05 r_{200} \).

Stimulating discussions with L. Kofman and J. R. Bond are gratefully acknowledged. This work was supported by grant 81AN-052101 from the Swiss National Foundation.

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