FRAMED MOTIVIC DONALDSON–THOMAS INVARIANTS
OF SMALL CREPANT RESOLUTIONS

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ABSTRACT. For an arbitrary integer \( r \geq 1 \), we compute \( r \)-framed motivic PT and DT invariants of small crepant resolutions of toric Calabi–Yau 3-folds, establishing a “higher rank” version of the motivic DT/PT wall-crossing formula. This generalises the work of Morrison and Nagao.

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0. INTRODUCTION

Morrison and Nagao computed in [13] motivic Donaldson–Thomas (DT in short) and Pandharipande–Thomas (PT in short) invariants of small crepant resolutions \( Y_\sigma \) of the affine toric Calabi–Yau 3-fold

\[
X = \text{Spec } \mathbb{C}[x, y, z, w]/(x y - z^{N_1} w^{N_2}) \subset \mathbb{A}^4,
\]

generalising previous results on the resolved conifold [12]. Such resolutions \( Y_\sigma \to X \) are indexed by partitions \( \sigma \) of a polygon \( \Gamma_{N_1, N_2} \) naturally attached to \( X \) (more details in §2). The 1-framed quiver associated to \( \sigma \) gives rise to generating functions \( \text{PT}_1 \) and \( \text{DT}_1 \) of motivic invariants. We compute the generating functions \( \text{PT}_r \) and \( \text{DT}_r \) corresponding to an arbitrary number \( r \) of framings on the same quiver. The result can be expressed as an \( r \)-fold twisted product of the \( r = 1 \) case. Moreover, we establish an \( r \)-framed version of the motivic DT/PT correspondence for \( Y_\sigma \).

Our main result, proved in §4.2, is the following.

Theorem A. Let \( Y_\sigma \) be the crepant resolution of \( X \) corresponding to \( \sigma \). There are factorisations

\[
\text{PT}_r(Y_\sigma; s, T) = \prod_{i=1}^r \text{PT}_1(Y_\sigma; (-1)^{r+i} L^{N_1 s_i} s, T),
\]

\[
\text{DT}_r(Y_\sigma; s, T) = \prod_{i=1}^r \text{DT}_1(Y_\sigma; (-1)^{r+i} L^{N_1 s_i} s, T).
\]

Furthermore, the \( r \)-framed motivic DT/PT correspondence holds: there is an identity

\[
\text{DT}_r(Y_\sigma; s, T) = \text{DT}_r^\text{points}(Y_\sigma, s) \cdot \text{PT}_r(Y_\sigma; s, T),
\]

where \( \text{DT}_r^\text{points}(Y_\sigma, s) \) is the virtual motivic partition function of the Quot scheme of points on \( Y_\sigma \).

The series \( \text{DT}_r^\text{points}(\mathbb{A}^3, s) = \sum_n (\text{Quot}_{\mathbb{A}^3}(O^{\sigma r}, n))_{\text{vir}} \cdot s^n \), originating from the critical locus structure on \( \text{Quot}_{\mathbb{A}^3}(O^{\sigma r}, n) \), is studied in detail in [4, 18, 5]. The series \( \text{DT}_r^\text{points}(Y, s) \) was introduced and computed for all 3-folds \( Y \) in [22, §4], generalising the \( r = 1 \) case corresponding to \( \text{Hilb}^n Y \) [2]. See §3 for more details — for instance, an explicit formula for \( \text{DT}_r^\text{points}(Y_\sigma, s) \) will be given in Equation (3.3).

A first instance of Formulae (0.1) was computed in [4, Chap. 3] for the case of the resolved conifold and the resolution of a line of \( A_2 \) singularities.
The same factorisation of generating functions of “rank $r$ objects” into $r$ copies of generating functions of rank 1 objects, shifted precisely as in Formulae (0.1), has recently been observed in the context of higher rank K-theoretic DT invariants [8] and in string theory [17].

Even though the geometric meaning of the moduli spaces of quiver representations giving rise to the $r$-framed invariants (0.1), for arbitrary $r$, is not as clear as in the $r = 1$ case, we do believe that such moduli spaces have a sensable geometric interpretation as suitable “higher rank” analogues of the Hilbert scheme of curves in $Y_r$ (DT side) and the moduli space of stable pairs on $Y_r$ (PT side). We come back to this in Remark 4.9, where we discuss a geometric interpretation of the framed moduli spaces in the PT chamber for the case of the conifold and $\tilde{A}_2$ quivers.

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1. Background material

1.1. Rings of motives. In this subsection we recall the definitions of various rings where the motivic invariants we want to study live.

As in [12, 13], we let $\mathcal{M}_C$ be the Grothendieck ring of the category of effective Chow motives over $\mathbb{C}$ with rational coefficients [11], extended with $L^{-1/2}$. A lambda-ring structure on $\mathcal{M}_C$ is obtained by setting $\sigma_n(X) = [\text{Sym}^n X]$ and $\sigma_n(L^{1/2}) = L^{n/2}$ to define the lambda operations. In particular, there is a well defined notion of power structure and plethystic exponential on $\mathcal{M}_C$ (see e.g. [2, §2.5] or [6, §1.5.1] for their formal properties). We consider the dimensional completion [3]

$$\widehat{\mathcal{M}}_C = \mathcal{M}_C[1],$$

which is also a lambda-ring, and in which the motives $[\text{GL}_k]$ of all general linear groups are invertible.

1.1.1. The virtual motive of a critical locus. Let $U$ be a smooth $d$-dimensional $\mathbb{C}$-scheme, $f : U \to \mathbb{A}^1$ a regular function. The virtual motive of the critical locus $\text{crit} f = Z(df) \subset U$, depending on the pair $(U, f)$, is defined in [12, 13] as the motivic class

$$[\text{crit} f]_{\text{vir}} = (\mathbb{L}^{-\mu})^{-d} \cdot [\phi_f] \in M^d_{\text{vir}},$$

where $[\phi_f] \in \mathbb{K}_0^{\mu} (\text{Var}_{\mathbb{C}})$ is the (absolute) motivic vanishing cycle class defined by Denef and Loeser [7] and the ‘$\mu$’ decoration refers to $\mu$-equivariant motives, where $\mu$ is the group of all roots of unity. However, all the motivic invariants studied here will live in the subring $\mathcal{M}_C \subset \mathcal{M}^C_{\text{vir}}$ of classes carrying the trivial $\mu$-action, so we will not be concerned with the subtle structure of this larger ring.

Example 1.1. Set $f = 0$. Then crit $f = U$, $[\phi_f] = [U]$ and hence $[U]_{\text{vir}} = (\mathbb{L}^{-\mu})^{-\dim U} \cdot [U]$. For instance, $[\text{GL}_k]_{\text{vir}} = (\mathbb{L}^{-\mu})^{-k} \cdot [\text{GL}_k]$.

Remark 1.2. Our definition of $[\text{crit} f]_{\text{vir}}$ differs from the original one [2, §2.8], which is also the one used in [6, 5]. We decided to adopt the conventions in [12, 13] to keep close to the original formulae. In practice, the difference amounts to the substitution $L^{1/2} \leftrightarrow -L^{1/2}$. In particular, the Euler number specialisation with our conventions is $L^{1/2} \to 1$, instead of $L^{1/2} \to -1$.

1.2. Quivers: framings, and motivic quantum torus. A quiver $Q$ is a finite directed graph, determined by its sets $Q_0$ and $Q_1$ of vertices and edges, respectively, along with the maps $h$, $t : Q_1 \to Q_0$ specifying where an edge starts or ends. We use the notation

$$t(a) \bullet \xrightarrow{a} \bullet h(a)$$

to denote the tail and the head of an edge $a \in Q_1$.

All quivers in this paper will be assumed connected. The path algebra $\mathbb{C}Q$ of a quiver $Q$ is defined, as a $\mathbb{C}$-vector space, by using as a $\mathbb{C}$-basis the set of all paths in the quiver, including a trivial path $e_i$ for each $i \in Q_0$. The product is defined by concatenation of paths whenever the operation is possible, and 0 otherwise. The identity element is $\sum_{i \in Q_0} e_i \in \mathbb{C}Q$. 

On a quiver $Q$ one can define the Euler–Ringel form $\chi(-, -) : \mathbb{Z}^Q \times \mathbb{Z}^Q \to \mathbb{Z}$ by

$$\chi(a, b) = \sum_{i \in Q_0} a_i b_i - \sum_{a \in Q_1} a_{i(a)} b_{h(a)},$$

as well as the skew-symmetric form

$$\langle a, b \rangle = \chi(a, b) - \chi(b, a).$$

The following construction will be central in our paper.

**Definition 1.3** ($r$-framing). Let $Q$ be a quiver with a distinguished vertex $0 \in Q_0$, and let $r$ be a positive integer. We define the quiver $\tilde{Q}$ by adding one vertex, labelled $\infty$, to the original vertices in $Q_0$, and $r$ edges $\infty \to 0$. We refer to $Q$ as the $r$-framed quiver obtained out of $(Q, 0)$.

The $r$-framing construction was applied to the 3-loop quiver (on the left in Figure 1) in [4, 18, 1, 5], following the $r=1$ case studied by Behrend–Bryan–Szendrő [2], and to the conifold quiver (on the right in Figure 1) in [4]. In this paper, it will be applied more generally to the quivers arising in the work of Morrison–Nagao [13], which we briefly discuss in §2. The case $r=1$ was covered in [12, 13].

![Figure 1. The 3-loop quiver $L_3$ and the conifold quiver $Q_{\text{con}}$.](image)

Let $Q$ be a quiver. Define its motivic quantum torus (or twisted motivic algebra) as

$$\mathcal{T}_Q = \prod_{a \in \mathbb{N}^Q} \mathcal{M}_C \cdot y^a$$

with product rule

$$y^a \cdot y^b = (-1)^{\delta(a, b)} y^{a+b}.$$

If $\tilde{Q}$ is the $r$-framed quiver associated to $(Q, 0)$ via Definition 1.3, one has a decomposition

$$\mathcal{T}_{\tilde{Q}} = \mathcal{T}_Q \oplus \prod_{d \geq 0} \mathcal{M}_C \cdot y_{\infty}^d,$$

where we have set $y_{\infty} = y^{(0,1)}$. Similarly, a generator $y^a \in \mathcal{T}_Q$ will be identified with its image $y^{(a,0)} \in \mathcal{T}_{\tilde{Q}}$.

1.3. **Quiver representations and their stability.** Let $Q$ be a quiver. A representation $\rho$ of $Q$ is the datum of a finite dimensional $C$-vector space $\rho_i$ for every vertex $i \in Q_0$, and a linear map $\rho(a) : \rho_i \to \rho_j$ for every edge $a : i \to j$ in $Q_1$. The dimension vector of $\rho$ is the vector $\dim (\rho) = (\dim C \rho_i)_{i \in \mathbb{N}^Q}$, where $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

**Convention 1.** Let $Q$ be a quiver, $\tilde{Q}$ the associated $r$-framed quiver. The dimension vector of a representation $\tilde{\rho}$ of $\tilde{Q}$ will be denoted $(a, d)$, where $a \in \mathbb{N}^Q$ and $\dim C \tilde{\rho}_{\infty} = d \in \mathbb{N}$.

Representations of a quiver $Q$ form an abelian category, which is equivalent to the category of left modules over the path algebra $\mathbb{C}Q$ of the quiver. The space of all representations of $Q$, with a fixed dimension vector $a \in \mathbb{N}^Q$, is the affine space

$$\text{R}(Q, a) = \prod_{a \in Q_1} \text{Hom}_C(C^{a_{i(a)}}, C^{a_{h(a)}}).$$

The gauge group $\text{GL}_a = \prod_{a \in Q_0} \text{GL}_{a_i}$ acts on $\text{R}(Q, a)$ by $(g_i)_{i \in Q_0} \cdot (\rho)_{a \in Q_1} = (g_{h(a)} \circ \rho \circ g_{i(a)}^{-1})_{a \in Q_1}$. The quotient stack

$$\mathcal{M}(Q, a) = \text{[R}(Q, a)/\text{GL}_a]$$

parametrises isomorphism classes of representations of $Q$ with dimension vector $a$.

Following [12, 13], we recall the notion of (semi)stability of a representation.
Definition 1.4. A central charge is a group homomorphism \( Z : \mathbb{Z}^Q \to \mathbb{C} \) such that the image of \( \mathbb{N}^Q \setminus 0 \) lies inside \( \mathbb{H}_+ = \{ t e^{\sqrt{-1} \pi t} \mid t > 0, 0 < \varphi \leq 1 \} \). For every \( \alpha \in \mathbb{N}^Q \setminus 0 \), we denote by \( \varphi(\alpha) \) the real number \( \varphi \) such that \( \mathbb{Z}^{\alpha} = t e^{\sqrt{-1} \pi \varphi} \). It is called the phase of \( \alpha \) with respect to \( Z \).

Note that every vector \( \zeta \in \mathbb{R}^Q \) induces a central charge \( Z_\zeta \) if we set \( Z_\zeta(\alpha) = -\zeta \cdot \alpha + |\alpha| \sqrt{-1} \), where \( |\alpha| = \sum_{i \in Q_0} \alpha_i \). We denote by \( \varphi_\zeta \) the induced phase function, and we set \( \varphi_\zeta(\rho) = \varphi_\zeta(\dim \rho) \) for every representation \( \rho \) of \( Q \). The slope function attached to \( Z_\zeta \) assigns to \( \alpha \in \mathbb{N}^Q \setminus 0 \) the real number \( \mu_\zeta(\alpha) = \zeta \cdot \alpha / |\alpha| \). Note that \( \varphi_\zeta(\alpha) < \varphi(\beta) \) if and only if \( \mu_\zeta(\alpha) < \mu_\zeta(\beta) \) (cf. [13, Rem. 3.5]).

Definition 1.5. Fix \( \zeta \in \mathbb{R}^Q \). A representation \( \rho \) of \( Q \) is called \( \zeta \)-semistable if

\[
\varphi_\zeta(\rho') \leq \varphi_\zeta(\rho)
\]

for every nonzero proper subrepresentation \( 0 \neq \rho' \subsetneq \rho \). When strict inequality holds, we say that \( \rho \) is \( \zeta \)-stable. Vectors \( \zeta \in \mathbb{R}^Q \) are referred to as stability parameters.

For a fixed \( \zeta \), every representation \( \rho \) admits a unique filtration

\[
\text{HN}_\zeta(\rho) : \quad 0 = \rho_0 \subsetneq \rho_1 \subsetneq \cdots \subsetneq \rho_s = \rho,
\]

called the Harder–Narasimhan filtration, such that \( \rho_i / \rho_{i-1} \) is \( \zeta \)-semistable for \( 1 \leq i \leq s \), and there are strict inequalities \( \varphi_\zeta(\rho_1 / \rho_0) > \varphi_\zeta(\rho_2 / \rho_1) > \cdots > \varphi_\zeta(\rho / \rho_{s-1}) \).

Definition 1.6 ([12, § 1.3]). Let \( \alpha \in \mathbb{N}^Q \) be a dimension vector. A stability parameter \( \zeta \) is called \( \alpha \)-generic if for any \( 0 < \beta < \alpha \) one has \( \varphi_\zeta(\beta) \neq \varphi_\zeta(\alpha) \).

The sets of \( \zeta \)-stable and \( \zeta \)-semistable representations with given dimension vector \( \alpha \) form a chain of open subsets

\[
\mathbb{R}^{\zeta-\text{st}}(Q, \alpha) \subseteq \mathbb{R}^{\zeta-\text{ss}}(Q, \alpha) \subseteq \mathbb{R}(Q, \alpha).
\]

If \( \zeta \) is \( \alpha \)-generic, one has \( \mathbb{R}^{\zeta-\text{st}}(Q, \alpha) = \mathbb{R}^{\zeta-\text{ss}}(Q, \alpha) \).

1.4. Quivers with potential. Let \( Q \) be a quiver. Consider the quotient \( \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q] \) of the path algebra by the commutator ideal. An element \( W \in \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q] \), which is a finite linear combination of cyclic paths, is called a potential. Given a cyclic path \( w \) and an arrow \( a \in Q_1 \), one defines the noncommutative derivative

\[
\frac{\partial w}{\partial a} = \sum_{c' \in \text{paths in } Q} \sum_{c : w = cac' \in Q} c' c \in \mathbb{C}Q.
\]

This rule extends to an operator \( \partial / \partial a \) acting on every potential. Thus every potential \( W \) gives rise to a (two-sided) ideal \( I_W \subset \mathbb{C}Q \) generated by the paths \( \partial W / \partial a \) for all \( a \in Q_1 \). The quotient \( J = J(Q, W) = \mathbb{C}Q / I_W \) is called the Jacobi algebra of the quiver with potential \( \langle Q, W \rangle \). For every \( \alpha \in \mathbb{N}^Q \), a potential \( W = \sum_a \alpha_a c \) determines a regular function

\[
f_a : \mathbb{R}(Q, \alpha) \to \mathbb{A}^1, \quad \rho \mapsto \sum_{c \text{ cycle in } Q} \alpha_c \text{ Tr}(\rho(c)).
\]

The points in the critical locus \( \text{crit } f_a \subset \mathbb{R}(Q, \alpha) \) correspond to \( J \)-modules with dimension vector \( \alpha \). Fix an \( \alpha \)-generic stability parameter \( \zeta \in \mathbb{R}^Q \). If \( f_{a,\alpha} : \mathbb{R}^{\zeta-\text{ss}}(Q, \alpha) \to \mathbb{A}^1 \) is the restriction of \( f_a \), then

\[
\mathcal{M}(J, \alpha) = \frac{\text{crit } f_a / \mathbb{G}_a}{\mathbb{M}(J, \alpha) = \frac{\text{crit } f_{a,\alpha} / \mathbb{G}_a}{\text{vir}}}
\]

are, by definition, the stacks of \( \alpha \)-dimensional \( J \)-modules and \( \zeta \)-stable \( J \)-modules.

Definition 1.7. A quiver with potential \( \langle Q, W \rangle \) admits a cut if there is a subset \( I \subset Q_1 \) such that every cyclic monomial appearing in \( W \) contains exactly one edge in \( I \).

From now on we assume \( \langle Q, W \rangle \) admits a cut. This condition ensures that the motive \( \mathcal{M}(J, \alpha) \mid_{\mathbb{G}_a} \) introduced in the next definition is monodromy-free, i.e. it lives in \( \mathcal{M}_C \). See [12, § 1.4] for more details. All quivers considered in this paper admit a cut [13, § 4].
Definition 1.8 ([12]). We define motivic Donaldson–Thomas invariants
\[
\{\mathcal{M}(J, \alpha)\}_{\text{vir}} = \frac{\text{crit } f_{\tilde{u}}_{\text{vir}}}{[\text{GL}_u]_{\text{vir}}}
\]
(1.2)
\[
\{\mathcal{M}_{\zeta}(J, \alpha)\}_{\text{vir}} = (-1)^{\chi(J, \alpha)} \frac{[f_{\tilde{u}}_{\text{vir}}^{-1}(0)] - [f_{\tilde{u}}_{\text{vir}}^{-1}(1)]}{[\text{GL}_u]},
\]
in \(\overline{\mathcal{M}}_C\), where \([\text{GL}_u]_{\text{vir}}\) is taken as in Example 1.1. The generating function
(1.3)
\[
A_U = \sum_{\alpha \in \mathbb{N}_0} \{\mathcal{M}(J, \alpha)\}_{\text{vir}} \cdot y^\alpha \in T_Q
\]
is called the universal series attached to \((Q, W)\).

Definition 1.9 ([12, §2.4]). A stability parameter \(\zeta \in \mathbb{R}_0^+\) is called generic if \(\zeta \cdot \text{dim} \rho \neq 0\) for every nontrivial \(\zeta\)-stable \(J\)-module \(\rho\).

1.5. Framed motivic DT invariants. Let \(r \geq 1\) be an integer, \(Q\) a quiver, \(\tilde{Q}\) its \(r\)-framing with respect to a vertex \(0 \in Q_0\) (Definition 1.3). A representation \(\rho\) of \(\tilde{Q}\) can be uniquely written as a pair \((\rho, u)\), where \(\rho\) is a representation of \(Q\) and \(u = (u_1, \ldots, u_r)\) is an \(r\)-tuple of linear maps \(u_j : \rho_0 \rightarrow \rho_0\).

From now on, we assume all \(r\)-framed representations to satisfy \(\text{dim}_C \rho_\infty = 1\), so that by Convention 1 one has \(\text{dim} \tilde{\rho} = (\text{dim} \rho, 1)\).

Definition 1.10 ([16] and [12, Def. 3.1]). Let \(\zeta \in \mathbb{R}_0^+\) be a stability parameter. A representation \((\rho, u)\) of \(\tilde{Q}\) (or a \(J\)-module) with \(\text{dim}_C \rho_\infty = 1\) is said to be \(\zeta\)-\((\text{semi})\)stable if it is \((\zeta, \zeta_\infty)\)-\((\text{semi})\)stable in the sense of Definition 1.5, where \(\zeta_\infty = -\zeta \cdot \text{dim} \rho\).

Now fix a potential \(W\) on \(Q\). We define motivic DT invariants for moduli stacks of \(r\)-framed \(J\)-modules on \(Q\). Let \(\tilde{J}\) be the Jacobi algebra \(J_{\tilde{Q}, W}\), where \(W\) is viewed as a potential on \(\tilde{Q}\) in the obvious way. For a generic stability parameter \(\zeta \in \mathbb{R}_0^+\), and a dimension vector \(\alpha \in \mathbb{N}_0^d\), set
\(\tilde{\alpha} = \alpha \cdot \zeta, \quad \tilde{\zeta} = (\zeta, \zeta_\infty)\), \(\tilde{\alpha} = (\alpha, 1)\).

As in § 1.4, consider the functions
\[
R^{\tilde{u}}(\tilde{Q}, \tilde{\alpha}) \xrightarrow{f_{\tilde{u}}} R(\tilde{Q}, \tilde{\alpha}) \xrightarrow{\tilde{f}_\alpha} \mathbb{A}^d
\]
associated to the potential \(W\). Define the moduli stacks
\[
\mathcal{M}(\tilde{J}, \alpha) = \left[\text{crit } f_{\tilde{u}}_{\text{vir}} / [\text{GL}_u]_{\text{vir}}\right], \quad \mathcal{M}_{\zeta}(\tilde{J}, \alpha) = \left[\text{crit } f_{\tilde{u}}_{\text{vir}} / [\text{GL}_u]_{\text{vir}}\right].
\]

Definition 1.11. We define \(r\)-framed motivic Donaldson–Thomas invariants
\[
\{\mathcal{M}(\tilde{J}, \alpha)\}_{\text{vir}} = \frac{\text{crit } f_{\tilde{u}}_{\text{vir}}}{[\text{GL}_u]_{\text{vir}}}
\]
\[
\{\mathcal{M}_{\zeta}(\tilde{J}, \alpha)\}_{\text{vir}} = \frac{\text{crit } f_{\tilde{u}}_{\text{vir}}}{[\text{GL}_u]_{\text{vir}}}
\]
in \(\overline{\mathcal{M}}_C\), and the associated motivic generating functions
\[
A_U = \sum_{\alpha \in \mathbb{N}_0^d} \{\mathcal{M}(\tilde{J}, \alpha)\}_{\text{vir}} \cdot y^{\tilde{\alpha}} \in T_Q
\]
\[
A_{\zeta} = \sum_{\alpha \in \mathbb{N}_0^d} \{\mathcal{M}_{\zeta}(\tilde{J}, \alpha)\}_{\text{vir}} \cdot y^{\tilde{\alpha}} \in T_Q
\]
\[
Z_{\zeta} = \sum_{\alpha \in \mathbb{N}_0^d} \{\mathcal{M}_{\zeta}(\tilde{J}, \alpha)\}_{\text{vir}} \cdot y^{\alpha} \in T_Q.
\]

The fact that the \(r\)-framed invariants live in \(\overline{\mathcal{M}}_C\) (i.e. have no monodromy) follows from [12, Lemma 1.10]. The reason is that the dimension vector \(\tilde{\alpha} = (\alpha, 1)\) contains ‘1’ as a component.

Our main goal is to give a formula for \(Z_{\zeta}\), where \(\zeta\) is chosen in a PT (resp. DT) chamber.
2. Non-commutative crepant resolutions

Fix integers $N_0 > 0$ and $0 \leq N_1 \leq N_0$, and set $N = N_0 + N_1$. The cone realising the singular Calabi–Yau 3-fold $X = \text{Spec} C[x, y, z, w]/(x y - z^{N_0} w^{N_1})$ as a toric variety is the cone over a quadrilateral $\Gamma_{N_0 N_1}$ (or a triangle, if $N_1 = \Delta$). A partition $\sigma$ of $\Gamma_{N_0 N_1}$ is, roughly speaking, a subdivision of the polygon $\Gamma_{N_0 N_1}$ into $N$ triangles $\{\sigma_i\}_{0 \leq i \leq N-1}$ of area 1/2. We refer the reader to [15, §1.1] for the precise definition. We denote by $\Gamma_\sigma$ the resulting object — see Figure 2 for an example with $N_0 = 4, N_1 = 2$. Each internal edge $\sigma_{i, i+1}$ corresponds to a component $C_i$ of the exceptional curve in the resolution $Y_\sigma$ attached to $\Gamma_\sigma$, and $C_i$ is a $(-1, -1)$-curve (resp. a $(-2, 0)$-curve) if $\sigma_i \cup \sigma_{i+1}$ is a quadrilateral (resp. a triangle).

![Figure 2. A partition $\Gamma_\sigma$ of $\Gamma_{4,2}$.](image)

As explained in [15, 13], any partition $\sigma$ gives rise to a small crepant resolution $Y_\sigma \to X$ by taking the fan of $\Gamma_\sigma$, and any two such resolutions are related by a sequence of mutations. On the other hand, Nagao [15] explains how to associate to $\sigma$ a bipartite tiling of the plane. The general construction in [10] then produces a quiver with potential $(Q_\sigma, \omega_\sigma)$. Its Jacobi algebra $J_\sigma$ is derived equivalent to $Y_\sigma$ [15, §1].

The quiver $Q_\sigma$ has vertex set $\hat{\Gamma} = \{0, 1, \ldots, N-1\}$, identified with the cyclic group $\mathbb{Z}/N\mathbb{Z}$. Each vertex has an edge in and out of the next vertex. The partition prescribes which vertices carry a loop, as we now explain using the specific example of Figure 2. In that case, the partition $\sigma = \{\sigma_i\}_{0 \leq i \leq 5}$ can be identified with the ordered set of half-points

\[
\sigma = \{\left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, 1\right), \left(\frac{3}{2}, 0\right), \left(\frac{5}{2}, 0\right), \left(\frac{3}{2}, 1\right), \left(\frac{7}{2}, 0\right)\},
\]

where the $i$th element corresponds to the mid-point of the base of the $i$th triangle $\sigma_i$. A vertex $k \in \hat{\Gamma}$ will carry a loop if and only if $\sigma_k$ and $\sigma_{k+1}$ have the same $y$-coordinate. Thus, by cyclicity, in our case we get two vertices ($k = 0, 2$) carrying a loop. The resulting quiver is drawn in Figure 3.

![Figure 3. The quiver $Q_\sigma$ associated to the partition (2.1).](image)

For the definition of the potential $\omega_\sigma$, we refer the reader to [15, §1.2] or [13, §2.A]. It is proved in [13, §4] that $(Q_\sigma, \omega_\sigma)$ has a cut for all $\sigma$.

**Remark 2.1.** The quiver $Q_\sigma$ is symmetric. This implies that its motivic quantum torus $T_{Q_\sigma}$ is commutative.

From the point of view of root systems, a choice of partition $\sigma$ corresponds to the choice of a set of simple roots $\alpha_0, \ldots, \alpha_{N-1}$ of type $A_N$, that one can take as a basis of $\mathbb{Z}^\hat{\Gamma}$. Following the notation in [13], we denote by $\Delta^+_{\sigma^+}$, $\Delta^+_{\sigma^{-}}$ and $\Delta^+_{\sigma^{\alpha}}$ the sets of positive, positive real and positive imaginary roots, respectively. As in [13, §1], we set $a_{[a,b]} = \sum_{a \leq i \leq b} a_i$ for all $1 \leq a \leq b \leq N - 1$, and

\[
\begin{align*}
\Delta^+_{\sigma^+} &= \{a_{[a,b]} \mid 1 \leq a \leq b \leq N - 1\} \\
\Delta^+_{\sigma^{-}} &= \{a_{[a,b]} + n \cdot \delta \mid n \in \mathbb{Z}_{\geq 0}\} \\
\Delta^+_{\sigma^{\alpha}} &= \{n \cdot \delta \mid n \in \mathbb{Z}_{\geq 0}\},
\end{align*}
\]

where $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{N-1}$ is the positive minimal imaginary root.
3. Higher rank motivic DT theory of points

Let $F$ be a locally free sheaf of rank $r$ on a smooth 3-fold $Y$. Building on the case of $Y = \mathbb{A}^3$, settled in [4, 18, 5], a virtual motive for the Quot scheme $\text{Quot}_Y(F, n)$ was defined in [22, Def. 4.10] via power structures, along the same lines of the rank 1 case [2, § 4.1].

The generating function

$$\mathcal{DT}^\text{points}_r(Y, (-1)^r s) = \sum_{n=0}^N [\text{Quot}_Y(F, n)]_{\text{str}} \cdot ((-1)^r s)^n$$

was computed in [22, Thm. 4.11] as a plethystic exponential. Just as in the case of the naive motives [21], the generating function does not depend on $F$ but only on $r$ and on the motive of $Y$.

Consider the singular affine toric Calabi–Yau 3-fold $X = \text{Spec} \mathbb{C}[x, y, z, w]/(x y - z^{N_0} w^{N_1}) \subset \mathbb{A}^4$, and fix a partition $\sigma$ associated to the polygon $\Gamma_{N_0, N_1}$.

Lemma 3.1. Let $Y_\sigma$ be the crepant resolution of $X$ corresponding to $\sigma$. Then

$$[Y_\sigma] = L^3 + (N-1)L^2 \in K_0(\text{Var}_C).$$

Proof. The toric polygon of $Y_\sigma$ consists of $N = N_0 + N_1$ triangles $\{\sigma_i\}$ intersecting pairwise along the edges $\{\sigma_{i+1}\}$. The toric resolution $Y_\sigma$ is constructed by gluing the toric charts $U_{\sigma_i}$ along the open affine subvarieties $U_{\sigma_{i+1}}$. Thus, the class $[Y_\sigma]$ can be computed using the cut-and-paste relations, after noticing that $U_{\sigma_i} \simeq \mathbb{A}^3$ and $U_{\sigma_{i+1}} \simeq \mathbb{A}^2 \times \mathbb{C}^*$. The result is

$$[Y_\sigma] = \sum_{i=1}^N L^3 - \sum_{i=1}^{N-1} L^2 (L-1) = L^3 + (N-1)L^2.$$ 

By [5, Thm. A] (but see also [4, 18] for different proofs), after rephrasing the result using the conventions adopted in this paper (cf. Remark 1.2), one has

$$\mathcal{DT}^\text{points}_r(\mathbb{A}^3, (-1)^r s) = \prod_{m \geq 1} \prod_{k=0}^{r-1} (1 - L^{k+2} \frac{s^m}{s^m})^{-1} = \prod_{i=1}^r \mathcal{DT}^\text{points}_1(\mathbb{A}^3, \mathbb{L} \frac{s^{r+1}}{s^{r+1}}).$$

An easy power structure argument shows that the same decomposition into $r$ rank 1 pieces holds for every smooth 3-fold $Y$. In a little more detail (we refer the reader to [9] or to [2, 6] for the formal properties of the power structure on $\mathcal{M}_C$), we have

$$\mathcal{DT}^\text{points}_r(Y, (-1)^r s) = \mathcal{DT}^\text{points}_r(\mathbb{A}^3, (-1)^r s)^{L^3[Y]}$$

$$= \prod_{i=1}^r \mathcal{DT}^\text{points}_1(Y_i, -L \frac{s^{r+1}}{s^{r+1}})^{L^3[Y]}$$

$$= \prod_{i=1}^r \mathcal{DT}^\text{points}_1(Y_i, -L \frac{s^{r+1}}{s^{r+1}}).$$

Therefore, for any smooth 3-fold $Y$, we can write

$$\mathcal{DT}^\text{points}_r(Y, s) = \prod_{i=1}^r \mathcal{DT}^\text{points}_1(Y_i, (-1)^{r+1} L \frac{s^{r+1}}{s^{r+1}}).$$

By Lemma 3.1, the motivic partition of the Hilbert scheme of points on $Y_\sigma$ is

$$\mathcal{DT}^\text{points}_1(Y_\sigma, s) = \prod_{m \geq 1} \prod_{k=0}^{m-1} (1 - L^{k+1} \frac{s^m}{s^m} (-s)^{m+1})^{-1} (1 - L^{k+2} \frac{s^m}{s^m} (-s)^m)^{-1}$$

and this determines $\mathcal{DT}^\text{points}_r(Y_\sigma, s)$ via Equation (3.1). The result is

$$\mathcal{DT}^\text{points}_r(Y_\sigma, s) = \prod_{m \geq 1} \prod_{k=0}^{m-1} (1 - L^{k+1} \frac{s^m}{s^m} (-1)^r s^m)^{-1} (1 - L^{k+2} \frac{s^m}{s^m} (-1)^r s^m)^{-1}. $$
4. Motivic invariants of non-commutative crepant resolutions

4.1. Relations among motivic partition functions. Fix integers $N_0 > 0$ and $0 \leq N_1 \leq N_0$, and set $N = N_0 + N_1$. We consider the affine singular toric Calabi–Yau 3-fold

$$X_{N_0,N_1} = \text{Spec} \mathbb{C}[x, y, z, w]/(xy - z^N w^N) \subset \mathbb{A}^4.$$ 

Fix a partition $\sigma$ of the polygon $I_{N_0,N_1}$, and set $(Q, W, J) = (Q_\sigma, \omega_\sigma, I_\sigma)$ to ease notation, where $f_\sigma$ is the Jacobi algebra of the quiver with potential $(Q_\sigma, \omega_\sigma)$ whose construction we sketched in §2. The universal series

$$A^\sigma_d(y) = A^\sigma_d(y_0, \ldots, y_{N-1}) = \sum_{a \in \mathbb{N}^{N_0}} \left[ \mathcal{M}(J_\sigma, a) \right]_{\text{vir}} \cdot y^a \in T_Q,$$

defined in Equation (1.3), is the main object of study in the work of Morrison and Nagao [13].

Fix a generic stability parameter $\zeta$ (cf. Definition 1.9) on the unframed quiver $Q$. Consider the stacks $\mathfrak{M}^\zeta_d(J, \alpha)$ of $J$-modules all of whose Harder–Narasimhan factors have positive (resp. negative) slope with respect to $\zeta$. These stacks are defined as follows. Restrict the function $f_\sigma: R(Q, \alpha) \to \mathbb{A}^1$, defined by taking the trace of $\omega_\sigma$, to the open subschemes $R^\zeta(J, \alpha) \subset R(Q, \alpha)$ of representations satisfying the above properties. This yields two regular functions $f^\pm_\sigma: R^\zeta(J, \alpha) \to \mathbb{A}^1$, and we set $\mathfrak{M}^\zeta_d(J, \alpha) = (\zeta \pm f^\pm_\sigma)$. We define the virtual motives $[\mathfrak{M}^\zeta_d(J, \alpha)]_{\text{vir}}$, as in the second identity in Equation (1.2), and the associated motivic generating functions (depending on $\sigma$ via $J = I_\sigma$)

$$A^\sigma_d = \sum_{a \in \mathbb{N}^{N_0}} \left[ \mathfrak{M}^\zeta_d(J, \alpha) \right]_{\text{vir}} \cdot y^a \in T_Q.$$

The vertices of $Q$ are labeled from 0 up to $N - 1$. Let $\tilde{Q}$ be the r-framed quiver associated to $(Q, 0)$ (Definition 1.3). We let $\tilde{J} = J_{\tilde{Q}, W}$ be the Jacobi algebra of $(\tilde{Q}, W) = (\tilde{Q}_\sigma, \omega_\sigma)$. Now recall the motivic generating functions

$$\tilde{A}_\zeta, \tilde{A}_\zeta, \tilde{Z}_\zeta$$

introduced in Definition 1.11. We have to extend the relations between framed and unframed generating functions (in the same spirit of Mozgovoy’s work [14]) to general $r$. By the following lemma, the arguments are going to be essentially formal.

Lemma 4.1. In $T_{\tilde{Q}}$ there are identities

$$y_{\infty} \cdot y^{(a, 0)} = (-L^d)^{-ra_0} \cdot \tilde{y}^a, \quad y^{(a, 0)} \cdot y_{\infty} = (-L^d)^{ra_0} \cdot \tilde{y}^a.$$

Proof. Since $\infty \in \tilde{Q}_0$ has edges only reaching 0, and no vertex of $Q$ reaches $\infty$, we have $\chi((a, 0),(0, 1)) = 0$, and $\chi((0, 1),(a, 0)) = -ra_0$. The result follows by the product rule (1.1).

Corollary 4.2. In $T_{\tilde{Q}}$, there are identities

\begin{align*}
\tilde{A}_\zeta \cdot y_{\infty} & = y_{\infty} \cdot \tilde{A}_\zeta \cdot (L^d y_0, y_1, \ldots, y_{N-1}) \\
A^\sigma_d \cdot y_{\infty} & = y_{\infty} \cdot A^\sigma_d \cdot (L^d y_0, y_1, \ldots, y_{N-1}).
\end{align*}

Proof. We have

$$y_{\infty} \cdot \tilde{A}_\zeta \cdot (L^d y_0, y_1, \ldots, y_{N-1}) = \sum_{a \in \mathbb{N}^{N_0}} \left[ \mathfrak{M}_\zeta(J_\sigma, a) \right]_{\text{vir}} \cdot (L^d y_0)^a \cdot y^a \cdot y_{N-1}^a = \sum_{a \in \mathbb{N}^{N_0}} \left[ \mathfrak{M}_\zeta(J_\sigma, a) \right]_{\text{vir}} \cdot (L^d y_0 \cdot y^a) = \sum_{a \in \mathbb{N}^{N_0}} \left[ \mathfrak{M}_\zeta(J_\sigma, a) \right]_{\text{vir}} \cdot \tilde{y}^a = \tilde{A}_\zeta,$$

where we have applied Lemma 4.1 in the last step. The identity (4.2) follows by an identical argument.

Lemma 4.3 ([12, Proposition 3.5]). Let $Q$ be a quiver, $\zeta \in \mathbb{R}^Q$ a generic stability parameter, $\tilde{\rho}$ a representation (resp. $\tilde{J}$-module) of the $r$-framed quiver $\tilde{Q}$ with $\dim C \tilde{\rho}_\infty = 1$. Then there is a unique filtration

$$0 = \tilde{\rho}^0 \subset \tilde{\rho}^1 \subset \tilde{\rho}^2 \subset \tilde{\rho}^3 = \tilde{\rho}$$

such that the quotients $\tilde{\rho}^i = \tilde{\rho}^i / \tilde{\rho}^{i-1}$ satisfy:

1. $\tilde{\rho}^1_\infty = 0$, and $\tilde{\rho}^i \in R^i(Q, \dim \tilde{\rho}^1)$,
2. $\dim C \tilde{\rho}^2_\infty = 1$ and $\tilde{\rho}^2$ is $\zeta$-stable,
Lemma 4.4. Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_Q$, there are factorisations

\begin{align}
\tilde{A}_U &= A^+_U \cdot A^-_U, \\
A_U &= A^U_0 \cdot y_{\infty}.
\end{align}

Proof. Equation (4.3) is a direct consequence of the existence of the filtration of Lemma 4.3. Equation (4.4) follows directly from the following observation: given a framed representation $(\rho, u)$ with $\dim \rho = 1$, one can view $\rho$ as a sub-module $\rho \subset \tilde{\rho}$ of dimension $(\dim \rho, 0)$, and the quotient $\tilde{\rho}/\rho$ is the unique simple module of dimension $(0, 1)$, based at the framing vertex.

Following [13, §0], we define, for $\alpha \in \Delta_{\sigma,+}$, the infinite products

\begin{equation}
A_\alpha(y) = \begin{cases}
\prod_{j=0} (1 - L^{-j} y^a) & \text{if } \alpha \in \Delta^o_{\sigma,+} \text{ and } \sum_{k \neq j} \alpha_k \text{ is odd} \\
\prod_{j=0} (1 - L^{-j} y^a)^{-1} & \text{if } \alpha \in \Delta^o_{\sigma,+} \text{ and } \sum_{k \neq j} \alpha_k \text{ is even} \\
\prod_{j=0} (1 - L^{-j} y^a)^{-1-N}(1 - L^{-j+1} y^a)^{-1} & \text{if } \alpha \in \Delta^m_{\sigma,+}
\end{cases}
\end{equation}

where $\bar{I}_U \subset \bar{I} = (Q_0)_0$ denotes $^1$ the set of vertices carrying a loop, and $\alpha_k \in \mathbb{N}$ is the component of $\alpha$ corresponding to a vertex $k$.

Lemma 4.5 ([12, Lemma 2.6]). Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_Q$, there are identities

\begin{equation}
A^+_\zeta(y) = \prod_{\alpha \in \Delta_{\sigma,+}} A_\alpha(y).
\end{equation}

Lemma 4.6. Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_Q$, there is an identity

\begin{equation}
A^-_U(y) = \prod_{\alpha \in \Delta_{\sigma,+}} A_\alpha(y).
\end{equation}

Proof. By [13, Thm. 0.1] there is a factorisation

\begin{equation}
A^-_U(y) = \prod_{\alpha \in \Delta_{\sigma,+}} A_\alpha(y).
\end{equation}

Since $\zeta$ is generic, $\zeta \cdot \alpha \neq 0$ for all $\alpha \in \Delta_{\sigma,+}$. The result then follows by combining this factorisation with Equation (4.6).

Theorem 4.7. Let $\zeta \in \mathbb{R}^{Q_0}$ be a generic stability parameter. In $\mathcal{T}_Q$, there is an identity

\begin{equation}
Z_\zeta(y) = \frac{A^+_\zeta((-L^+)^r y_0, y_1, \ldots, y_{N-1})}{A^-_\zeta((-L^-)^r y_0, y_1, \ldots, y_{N-1})}
\end{equation}

Proof. Since $Q = Q_{\sigma}$ is symmetric (Remark 2.1), the algebra $\mathcal{T}_Q$ is commutative, therefore a power series $F \in \mathcal{T}_Q$ starting with the invertible element $1 \in \mathcal{M}_C$ will be invertible. For instance $A^+_\zeta$ and $A^-_\zeta$ are invertible. Therefore we can write

\begin{align}
y_{\infty} \cdot Z_\zeta((-L^+)^r y_0, y_1, \ldots, y_{N-1}) &= \tilde{A}_\zeta \\
&= (A^+_\zeta)^{-1} \cdot \tilde{A}_U \cdot (A^-_\zeta)^{-1} \quad \text{by } (4.1) \\
&= (A^+_\zeta)^{-1} \cdot (A^U_0 \cdot y_{\infty}) \cdot (A^-_\zeta)^{-1} \quad \text{by } (4.3) \\
&= (A^+_\zeta)^{-1} \cdot (A^+_U \cdot y_{\infty}) \cdot (A^-_U)^{-1} \quad \text{by } (4.4) \\
&= (A^-_U)^{-1} \cdot (A^+_U \cdot y_{\infty}) \cdot (A^-_U)^{-1} \quad \text{by } (4.7) \\
&= y_{\infty} \cdot A^-_\zeta (L^r y_0, y_1, \ldots, y_{N-1}) \cdot (A^-_U)^{-1} \quad \text{by } (4.2)
\end{align}

\footnote{The set $\bar{I}_U$ is denoted $\bar{I}_U$ in [13]. We changed the notation to avoid conflict with the number $r$ of framings.}
It follows that

\[ Z_c((-L^\frac{1}{2})' y_0, y_1, \ldots, y_{N-1}) = \frac{A_c((-L^\frac{1}{2})' y_0, y_1, \ldots, y_{N-1})}{A_c(y_0, y_1, \ldots, y_{N-1})}, \]

so the change of variable \( y_0 \to (-L^\frac{1}{2})' y_0 \) yields the result. \( \square \)

### 4.2. Computing invariants in the DT and PT chambers.

In this subsection we prove Theorem A. Define, for \( \alpha \in \Delta_{\sigma, r} \), the fraction

\[ Z^{(i)}_\alpha(y_0, y_1, \ldots, y_{N-1}) = \frac{A_\alpha((-L^\frac{1}{2})' y_0, y_1, \ldots, y_{N-1})}{A_\alpha((-L^\frac{1}{2})' y_0, y_1, \ldots, y_{N-1})}, \]

where \( A_\alpha \) is defined case by case in (4.5). Then one deduces the following explicit formulae:

\[ Z^{(i)}_\alpha((-1)^i y_0, y_1, \ldots, y_{N-1}) = \begin{dcases} \prod_{k=0}^{r_{\alpha, r-1}} (1 - L^{\frac{1}{2} - \frac{a}{m}} y^a) & \text{if } \alpha \in \Delta_{\sigma, r}^{re} \text{ and } \sum_{k \in I_0} a_k \text{ is odd} \\ \prod_{k=0}^{r_{\alpha, r-1}} (1 - L^{\frac{1}{2} - \frac{a}{m}} y^a)^{-1} & \text{if } \alpha \in \Delta_{\sigma, r}^{re} \text{ and } \sum_{k \in I_0} a_k \text{ is even} \end{dcases} \]

These identities can be easily rewritten uniformly in terms of the ‘rank 1’ generating functions:

\[ Z^{(i)}_\alpha((-1)^i y_0, y_1, \ldots, y_{N-1}) = \prod_{i=1}^{r} Z^{(i)}_\alpha(-L^\frac{1}{2} + i y_0, y_1, \ldots, y_{N-1}). \]

As in [13, §0], let us set

\[ s = y_0 y_1 \cdots y_{N-1}, \quad T_i = y_i, \quad T = (T_1, \ldots, T_{N-1}). \]

For \( 1 \leq a \leq b \leq N - 1 \), we let \( T_{[a, b]} = T_a \cdots T_b \) be the monomial corresponding to the homology class \( C_{[a, b]} = [C_a] + \cdots + [C_b] \in H_2(Y_{\sigma}, \mathbb{Z}) \), where \( C_i \subset Y_{\sigma} \) is a component of the exceptional curve. Let \( c(a, b) \) be the number of \((-1, -1)\)-curves in \( \{ C_i \mid a \leq i \leq b \} \). Then we set

\[ Z_{(a, b)}(s, T_{[a, b]}) = \begin{dcases} \prod_{m=1}^{m-1} \prod_{j=0}^{m-1} (1 - L^{\frac{j}{2} - \frac{a}{m}} (-s)^m T_{[a, b]}) & \text{if } c(a, b) \text{ is odd} \\ \prod_{m=1}^{m-1} \prod_{j=0}^{m-1} (1 - L^{\frac{j}{2} - \frac{a}{m}} (-s)^m T_{[a, b]})^{-1} & \text{if } c(a, b) \text{ is even} \end{dcases} \]

and

\[ Z_{im}(s) = \prod_{m=1}^{m-1} \prod_{j=0}^{m-1} (1 - L^{\frac{j}{2} - \frac{a}{m}} (-s)^m)^{1-N} (1 - L^{\frac{j}{2} - \frac{a}{m}} (-s)^m)^{-1}. \]

Fix, as in [13, §6.C], stability parameters

\[ \zeta_{PT} = (1 - N + \epsilon, 1, \ldots, 1), \quad \zeta_{DT} = (1 - N - \epsilon, 1, \ldots, 1), \]

with \( 0 < \epsilon \ll 1 \) chosen so that they are generic. We want to compute

\[ P T_r(Y_{\sigma}, s; T) = Z_{\zeta_{PT}}(s, T_1, \ldots, T_{N-1}), \quad D T_r(Y_{\sigma}; s, T) = Z_{\zeta_{DT}}(s, T_1, \ldots, T_{N-1}). \]

For \( r = 1 \), these are the generating functions computed in [13, Cor. 0.3]. We know by Equation (3.2) (see also [13, Cor. 0.3 (2)]) that

\[ Z_{im}(s) = DT_1^{points}(Y_{\sigma}, s), \]
and Morrison–Nagao proved that
\[
\PT_1(Y_{\alpha}; s, T) = \prod_{1 \leq a \leq b \leq N-1} Z_{(a,b)}(s, T_{(a,b)})
\]
(4.13)
\[
\DT_1(Y_{\alpha}; s, T) = Z_{\im}(s) \cdot \PT_1(Y_{\alpha}; s, T).
\]

As in [13, §6.C], we have
\[
\{ \alpha \in \Delta_{\alpha, +} \mid \zeta_{PT} \cdot \alpha < 0 \} = \Delta_{\alpha}^{rec+}
\]
(4.14)
\[
\{ \alpha \in \Delta_{\alpha, +} \mid \zeta_{DT} \cdot \alpha < 0 \} = \Delta_{\alpha}^{rec+} \cap \Delta_{\alpha}^{im},
\]
where the definition of the sets in the right hand sides was recalled in Equation (2.2). For the PT stability condition, we thus obtain
\[
\PT_r(Y_{\alpha}; s, T) = \frac{A_{PT}^r((-L^+)^r s, T_1, \ldots, T_{N-1})}{A_{PT}^r((-L^-)^r s, T_1, \ldots, T_{N-1})} = \prod_{\alpha \in \Delta_{\alpha}^{rec+}} \frac{A_{\alpha r}((-L^+)^r s, T_1, \ldots, T_{N-1})}{A_{\alpha r}((-L^-)^r s, T_1, \ldots, T_{N-1})} \quad \text{by (4.6) and (4.14)}
\]
\[
= \prod_{\alpha \in \Delta_{\alpha}^{rec+}} Z_{\alpha}^{(r)}(s, T_1, \ldots, T_{N-1}) \quad \text{by (4.9)}
\]
\[
= \prod_{\alpha \in \Delta_{\alpha}^{rec+}} \left( (-1)^{r+1} L^{-\frac{i}{r}+i} s, T_1, \ldots, T_{N-1} \right) \quad \text{by (4.11)}
\]
\[
= \prod_{\alpha \in \Delta_{\alpha}^{rec+}} \left( (-1)^{r+1} L^{-\frac{i}{r}+i} s, T_{(a,b)} \right) \quad \text{by (2.2)}
\]
\[
= \prod_{\alpha \in \Delta_{\alpha}^{rec+}} \left( (-1)^{r+1} L^{-\frac{i}{r}+i} s, T \right), \quad \text{by (4.13)}
\]

which proves the first identity in Theorem A.

Similarly,
\[
\prod_{\alpha \in \Delta_{\alpha}^{rec+}} A_{\alpha r}((-L^+)^r s, T_1, \ldots, T_{N-1}) = \prod_{\alpha \in \Delta_{\alpha}^{rec+}} Z_{\alpha}^{(r)}(s, T_1, \ldots, T_{N-1}) \quad \text{by (4.9)}
\]
\[
= \prod_{\alpha \in \Delta_{\alpha}^{rec+}} \left( (-1)^{r+1} L^{-\frac{i}{r}+i} s \right) \quad \text{by (4.11)}
\]
\[
= \prod_{\alpha \in \Delta_{\alpha}^{rec+}} \left( (-1)^{r+1} L^{-\frac{i}{r}+i} s, T_{(a,b)} \right) \quad \text{by (4.12)}
\]
\[
= \prod_{\alpha \in \Delta_{\alpha}^{rec+}} \left( (-1)^{r+1} L^{-\frac{i}{r}+i} s, T \right). \quad \text{by (3.1)}
\]

In particular, thanks to (4.14), the motivic DT/PT correspondence
\[
\DT_r(Y_{\alpha}; s, T) = \DT^\text{points}_r(Y_{\alpha}, s) \cdot \PT_r(Y_{\alpha}; s, T)
\]
holds. Note that, thanks to Equation (3.3), the right hand side is entirely explicit. Finally, the relation
\[
\DT_r(Y_{\alpha}; s, T) = \prod_{i=1}^{r} \left( Y_{\alpha}((-1)^{r+1} s L^{-\frac{i}{r}+i}, T) \right)
\]
follows from the factorisations of $\PT_r$ and $\DT^\text{points}_r$ as products of (equally shifted) $r = 1$ pieces, combined with the rank 1 DT/PT correspondence (4.13). The proof of Theorem A is complete.

**Remark 4.8.** A motivic DT/PT correspondence was obtained in [6] in the rank 1 case for the motivic contribution of a smooth curve in a 3-fold, refining the corresponding enumerative calculations [20, 19].

**Remark 4.9.** In the case when $Y_{\alpha}$ is the crepant resolution of the conifold singularity, corresponding to $N_0 = N_1 = 1$, the moduli space of framed quiver representation has a clear geometric interpretation for a choice of PT stability condition. Consider the moduli space $\mathcal{P}_\alpha(Y_{\alpha})$ parametrising Shesmani's highly
frozen stable triples [23], whose geometric points consist of framed multi-sections \( \mathcal{O}^r_{Y_{\sigma}} \to F \) with 0-dimensional cokernel, where \( F \) is a pure 1-dimensional sheaf \( F \) satisfying \( ch_2(F) = (a_0 - a_1)\mathbb{P}^1 \) and \( \chi(F) = a_0 \). In [4, Chap. 3] a scheme theoretic isomorphism \( \mathfrak{M}^r (J_{\sigma}, \alpha) \simeq \mathcal{P}_r (Y_{\sigma}) \) is constructed, and it is used to compute a first instance of Formula (0.1). A completely analogous result holds when \( Y_{\sigma} \) is the resolution of a line of \( A_2 \) singularities, corresponding to the case \( N_0 = 2, N_1 = 0 \) [4, Appendix 3.A]. We leave to future work a full geometric interpretation of the more general moduli spaces of framed quiver representations that we studied in this paper.

REFERENCES

1. Sjoerd Beentjes and Andrea T. Ricolfi, Virtual counts on Quot schemes and the higher rank local DT/PT correspondence, To appear in Math. Res. Lett., 2018.
2. Kai Behrend, Jim Bryan, and Balázs Szendrői, Motivic degree zero Donaldson–Thomas invariants, Invent. Math. 192 (2013), no. 1, 111–160.
3. Kai Behrend and Ajeet Dhillon, On the motivic class of the stack of bundles, Adv. Math. 212 (2007), no. 2, 617–644.
4. Alberto Cazzaniga, On some computations of refined Donaldson–Thomas invariants, PhD Thesis, University of Oxford, 2015.
5. Alberto Cazzaniga, Dimbinaina Ralaivaosaona, and Andrea T. Ricolfi, Higher rank motivic Donaldson–Thomas invariants of \( A^2 \) via wallcrossing, and asymptotics, https://arxiv.org/abs/2004.07020.
6. Ben Davison and Andrea T. Ricolfi, The local motivic DT/PT correspondence, https://arxiv.org/abs/1905.12458, 2019.
7. Jan Denef and François Loeser, Geometry on arc spaces of algebraic varieties, 3rd European congress of mathematicians (ECM), Barcelona, Spain, July 10–14, 2000. Volume I, Basel: Birkhäuser, 2001, pp. 327–348.
8. Nadir Fasola, Sergej Monavari, and Andrea T. Ricolfi, Higher rank K-theoretic Donaldson–Thomas theory of points, https://arxiv.org/abs/2003.13565, 2020.
9. Sabir M. Gusein-Zade, Igancio Luengo, and Alejandro Melle-Hernández, A power structure over the Grothendieck ring of varieties, Math. Res. Lett. 11 (2004), no. 1, 49–57.
10. Amihay Hanany and David Vegh, Quivers, tilings, branes and rhombi, J. High Energy Phys. (2007), no. 10, 029, 35.
11. Ju. I. Manin, Correspondences, motifs and monoidal transformations, Mat. Sb. (N.S.) 77 (119) (1968), 475–507.
12. Andrew Morrison, Sergey Mozgovoy, Kentaro Nagao, and Balázs Szendrői, Motivic Donaldson–Thomas invariants of the conifold and the refined topological vertex, Adv. Math. 230 (2012), no. 4-6, 2065–2093.
13. Andrew Morrison and Kentaro Nagao, Motivic Donaldson–Thomas invariants of small crepant resolutions, Algebra Number Theory 9 (2015), no. 4, 767–813.
14. Sergey Mozgovoy, Wall-crossing formulas for framed objects, Q. J. Math. 64 (2013), no. 2, 489–513.
15. Kentaro Nagao, Derived categories of small toric Calabi-Yau 3-folds and curve counting invariants, Q. J. Math. 63 (2012), no. 4, 965–1007.
16. Kentaro Nagao and Hiraku Nakajima, Counting invariants of perverse coherent sheaves and its wall-crossing, Int. Math. Res. Not. 2011 (2011), no. 17, 3885–3938.
17. Nikita Nekrasov and Nicolò Piazzalunga, Magnificent four with colors, Comm. Math. Phys. 372 (2019), no. 2, 573–597.
18. Andrea T. Ricolfi, Local Donaldson–Thomas invariants and their refinements, Ph.D. thesis, University of Stavanger, 2017.
19. Andrea T. Ricolfi, The DT/PT correspondence for smooth curves, Math. Z. 290 (2018), no. 1-2, 699–710.
20. Andrea T. Ricolfi, Local contributions to Donaldson–Thomas invariants, Int. Math. Res. Not. 2018 (2018), no. 19, 5995–6025.
21. Andrea T. Ricolfi, On the motive of the Quot scheme of finite quotients of a locally free sheaf, To appear in J. Math. Pures Appl., 2019.
22. Andrea T. Ricolfi, Virtual classes and virtual motives of Quot schemes on threefolds, https://arxiv.org/abs/1906.02557, 2019.
23. Artan Sheshmani, Higher rank stable pairs and virtual localization, Comm. Anal. Geom. 24 (2016), no. 1, 139–193.

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