TWIN PARADOX AND THE LOGICAL FOUNDATION OF
RELATIVITY THEORY

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ABSTRACT. We study the foundation of space-time theory in the framework of first-order
logic (FOL). Since the foundation of mathematics has been successfully carried through
(via set theory) in FOL, it is not entirely impossible to do the same for space-time theory (or
relativity). First we recall a simple and streamlined FOL-axiomatization Specrel of special
relativity from the literature. Specrel is complete with respect to questions about inertial
motion. Then we ask ourselves whether we can prove the usual relativistic properties
of accelerated motion (e.g., clocks in acceleration) in Specrel. As it turns out, this is
practically equivalent to asking whether Specrel is strong enough to “handle” (or treat)
accelerated observers. We show that there is a mathematical principle called induction
(IND) coming from real analysis which needs to be added to Specrel in order to handle
situations involving relativistic acceleration. We present an extended version AccRel of
Specrel which is strong enough to handle accelerated motion, in particular, accelerated
observers. Among others, we show that the Twin Paradox becomes provable in AccRel,
but it is not provable without IND.

Key words: twin paradox, relativity theory, accelerated observers, first-order logic, axiom-
atization, foundation of relativity theory

1. INTRODUCTION

The idea of elaborating the foundation of space-time (or foundation of relativity) in a
spirit analogous with the rather successful foundation of mathematics (FOM) was initiated
by several authors including, e.g., David Hilbert [12] or leading contemporary logician
Harvey Friedman [9, 10]. Foundation of mathematics has been carried through strictly
within the framework of first-order logic (FOL), for certain reasons. The same reasons
motivate the effort of keeping the foundation of space-time also inside FOL. One of the
reasons is that staying inside FOL helps us to avoid tacit assumptions, another reason
is that FOL has a complete inference system while higher-order logic cannot have one
by Gödel’s incompleteness theorem, see e.g., Väänänen [24, p.505] or [2, Appendix]. For
more motivation for staying inside FOL (as opposed to higher-order logic), cf. e.g., Ax [3],
Pambuccian [17, 2, Appendix 1: “Why exactly FOL”], [1], but the reasons in Väänänen
[24], Ferreirós [8], or Wołenski [26] also apply.

Following the above motivation, we begin at the beginning, namely first we recall a
streamlined FOL axiomatization Specrel of special relativity theory, from the literature. Specrel
is complete with respect to (w.r.t.) questions about inertial motion. Then we ask
ourselves whether we can prove the usual relativistic properties of accelerated motion (e.g.,
clocks in acceleration) in Specrel. As it turns out, this is practically equivalent to asking
whether Specrel is strong enough to “handle” (or treat) accelerated observers. We show
that there is a mathematical principle called induction (IND) coming from real analysis
which needs to be added to Specrel in order to handle situations involving relativistic
acceleration. We present an extended version AccRel of Specrel which is strong enough to
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We show that the so-called Twin Paradox becomes provable in \textit{AccRel}. It also becomes possible to introduce Einstein’s equivalence principle for treating gravity as acceleration and proving the gravitational time dilation, i.e. that gravity “causes time to run slow”.

What we are doing here is not unrelated to Field’s “Science without numbers” programme and to “reverse mathematics” in the sense of Harvey Friedman and Steven Simpson. Namely, we systematically ask ourselves which mathematical principles or assumptions (like, e.g., \textit{IND}) are really needed for proving certain observational predictions of relativity. (It was this striving for parsimony in axioms or assumptions which we alluded to when we mentioned, way above, that \textit{Specrel} was “streamlined”.)

The interplay between logic and relativity theory goes back to around 1920 and has been playing a non-negligible role in works of researchers like Reichenbach, Carnap, Suppes, Ax, Szekeres, Malament, Walker, and of many other contemporaries.\footnote{In passing we mention that Etesi-Németi \cite{EtesiNemeti}, Hogarth \cite{Hogarth} represent further kinds of connection between \textit{logic} and \textit{relativity} not discussed here.}

In Section 2 we recall the FOL axiomatization \textit{Specrel} complete w.r.t. questions concerning inertial motion. There we also introduce an extension \textit{AccRel} of \textit{Specrel} (still inside FOL) capable for handling accelerated clocks and also accelerated observers. In Section 3 we formalize the Twin Paradox in the language of FOL. We formulate Theorems 3.1, 3.2 stating that the Twin Paradox is provable from \textit{AccRel} and the same for related questions for accelerated clocks. Theorems 3.3, 3.4 state that \textit{Specrel} is not sufficient for this, more concretely that the induction axiom \textit{IND} in \textit{AccRel} is needed. In Sections 4, 5 we prove these theorems.

Motivation for the research direction reported here is nicely summarized in Ax \cite{Ax}, Suppes \cite{Suppes}; cf. also the introduction of \cite{Ax}. Harvey Friedman’s \cite{Friedman1, Friedman2} present a rather convincing general perspective (and motivation) for the kind of work reported here.

2. AXIOMATIZING SPECIAL RELATIVITY IN FOL

In this paper we deal with the kinematics of relativity only, i.e. we deal with motion of \textit{bodies} (or \textit{test-particles}). The motivation for our choice of vocabulary (for special relativity) is summarized as follows. We will represent motion as changing spatial location in time. To do so, we will have reference-frames for coordinatizing events and, for simplicity, we will associate reference-frames with special bodies which we will call \textit{observers}. We visualize an observer-as-a-body as “sitting” in the origin of the space part of its reference-frame, or equivalently, “living” on the time-axis of the reference-frame. We will distinguish \textit{inertial} observers from non-inertial (i.e. accelerated) ones. There will be another special kind of bodies which we will call \textit{photons}. For coordinatizing events we will use an arbitrary \textit{ordered field} in place of the field of the real numbers. Thus the elements of this field will be the “\textit{quantities}” which we will use for marking time and space. Allowing arbitrary ordered fields in place of the reals increases flexibility of our theory and minimizes the amount of our mathematical presuppositions. Cf. e.g., Ax \cite{Ax} for further motivation in this direction. Similar remarks apply to our flexibility oriented decisions below, e.g., keeping the number \textit{d} of space-time dimensions a variable. Using coordinate systems (or reference-frames) instead of a single observer independent space-time structure is only a matter of didactical convenience and visualization, furthermore it also helps us in weeding out unnecessary axioms from our theories. Motivated by the above, we now turn to fixing the FOL language of our axiom systems.

The first occurrences of concepts used in this work are set by boldface letters to make it easier to find them. Throughout this work, if-and-only-if is abbreviated to \textit{iff}.
Let us fix a natural number \( d \geq 2 \) for the dimension of the space-time that we are going to axiomatize. Our first-order language contains the following non-logical symbols:

- unary relation symbols \( B \) (for \textbf{Bodies}), \( \text{Ob} \) (for \textbf{Observers}), \( \text{IOb} \) (for \textbf{Inertial Observers}), \( \text{Ph} \) (for \textbf{Photons}) and \( F \) (for \textbf{quantities} which are going to be elements of a Field),
- binary function symbols \( +, \cdot \) and a binary relation symbol \( \leq \) (for the field operations and the ordering on \( F \)), and
- a \( 2 + d \)-ary relation symbol \( W \) (for \textbf{World-view relation}).

The bodies will play the role of the "main characters" of our space-time models and they will be "observed" (coordinatized using the quantities) by the observers. This observation will be coded by the world-view relation \( W \). Our bodies and observers are basically the same as the "test particles" and the "reference-frames", respectively, in some of the literature.

We read \( B(x) \), \( \text{Ob}(x) \), \( \text{IOb}(x) \), \( \text{Ph}(x) \), \( F(x) \) as "\( x \) is a body", "\( x \) is an observer", "\( x \) is an inertial observer", "\( x \) is a photon", "\( x \) is a field-element". We use the world-view relation \( W \) to talk about coordinatization, by reading \( W(x,y,z_1,\ldots,z_d) \) as "observer \( x \) observes (or sees) body \( y \) at coordinate point \( \langle z_1,\ldots,z_d \rangle \)". This kind of observation has no connection with seeing via photons, it simply means coordinatization.

\( B(x) \), \( \text{Ob}(x) \), \( \text{IOb}(x) \), \( \text{Ph}(x) \), \( F(x) \), \( W(x,y,z_1,\ldots,z_d) \), \( x = y, x \leq y \) are the so-called atomic formulas of our first-order language, where \( x, y, z_1,\ldots,z_d \) can be arbitrary variables or terms built up from variables by using the field-operations "+" and "\cdot". The formulas of our first-order language are built up from these atomic formulas by using the logical connectives \textit{not} (\( \neg \)), \textit{and} (\( \land \)), or (\( \lor \)), \textit{implies} (\( \Rightarrow \)), \textit{if-and-only-if} (\( \iff \)) and the quantifiers \textit{exists} \( x \) (\( \exists x \)) and for all \( x \) (\( \forall x \)) for every variable \( x \).

Usually we use the variables \( m, k, h \) to denote observers, \( b \) to denote bodies, \( ph \) to denote photons and \( p_1,\ldots,q_1,\ldots \) to denote quantities (i.e. field-elements). We write \( p \) and \( q \) in place of \( p_1,\ldots,p_d \) and \( q_1,\ldots,q_d \), e.g., we write \( W(m,b,p) \) in place of \( W(m,b,p_1,\ldots,p_d) \), and we write \( \forall p \) in place of \( \forall p_1,\ldots,p_d \) etc.

The models of this language are of the form

\[ \mathfrak{M} = \langle U; B, \text{Ob}, \text{IOb}, \text{Ph}, F, +, \cdot, \leq, W \rangle, \]

where \( U \) is a nonempty set, \( B, \text{Ob}, \text{IOb}, \text{Ph}, F \) are unary relations on \( U \), etc. A unary relation on \( U \) is just a subset of \( U \). Thus we use \( B, \text{Ob} \) etc. as sets as well, e.g., we write \( m \in \text{Ob} \) in place of \( \text{Ob}(m) \).

Having fixed our language, we now turn to formulating an axiom system for special relativity in this language. We will make special efforts to keep all our axioms inside the above specified first-order logic language of \( \mathfrak{M} \).

Throughout this work, \( i, j \) and \( n \) denote positive integers. \( F^n := F \times \ldots \times F \) \( (n\text{-times}) \) is the set of all \( n \)-tuples of elements of \( F \). If \( a \in F^n \), then we assume that \( a = \langle a_1,\ldots,a_n \rangle \), i.e. \( a_i \in F \) denotes the \( i \)-th component of the \( n \)-tuple \( a \).

The following axiom is always assumed and is part of every axiom system we propose.

**AxFrame:** \( \text{Ob} \cup \text{Ph} \subseteq B \), \( \text{IOb} \subseteq \text{Ob} \), \( U = B \cup F \), \( W \subseteq \text{Ob} \times B \times F^d \), \( + \) and \( \cdot \) are binary operations on \( F \), \( \leq \) is a binary relation on \( F \) and \( \langle F; +, \cdot, \leq \rangle \) is an Euclidean ordered field, i.e. a linearly ordered field in which positive elements have square roots.\(^2\)

\(^2\)For example, the ordered fields of the real numbers, the real algebraic numbers, and the hyper-real numbers are Euclidean but the ordered field of the rational numbers is not Euclidean and the field of the complex numbers cannot be ordered. For the definition of (linearly) ordered field, cf. e.g., Rudin [13] or Chang-Keisler [4].
In pure first-order logic, the above axiom would look like \( \forall x \left[ (\text{Ob}(x) \lor \text{Ph}(x)) \implies B(x) \right] \) etc. In the present section we will not write out the purely first-order logic translations of our axioms since they will be straightforward to obtain. The first-order logic translations of our next three axioms \( \text{AxSelf}^- \), \( \text{AxPh} \), \( \text{AxEv} \) can be found in the Appendix.

Let \( \mathcal{M} \) be a model in which \( \text{AxFrame} \) is true. Let \( \bar{\mathcal{M}} := \langle F; +, \cdot, \leq \rangle \) denote the ordered field reduct of \( \mathcal{M} \). Here we list the definitions and notation that we are going to use in formulating our axioms. Let \( 0, 1, -, /, \sqrt{\cdot} \) be the usual field operations which are definable from “+” and “.”. We use the vector-space structure of \( F^n \), i.e. if \( p, q \in F^n \) and \( \lambda \in F \), then \( p + q, -p, \lambda p \in F^n \); and \( o := \langle 0, \ldots, 0 \rangle \) denotes the origin. The Euclidean-length of \( a \in F^n \) is defined as \( |a| := \sqrt{a_1^2 + \ldots + a_n^2} \). The set of positive elements of \( F \) is denoted by \( F^+ := \{ x \in F : x > 0 \} \). Let \( p, q \in F^d \). We use the notation \( p_s := \langle p_2, \ldots, p_d \rangle \) for the space component of \( p \) and \( p_t := p_1 \) for the time component of \( p \). We define the line through \( p \) and \( q \) as \( pq := \{ q + \lambda(p - q) : \lambda \in F \} \). The set of lines is defined as \( \text{Lines} := \{ pq : p \neq q \land p, q \in F^d \} \). The slope of \( p \) is defined as \( \text{slope}(p) := |p_s|/|p_t| \) if \( p_t \neq 0 \) and is undefined otherwise; furthermore \( \text{slope}(pq) := \text{slope}(p - q) \) if \( p_t \neq q_t \) and is undefined otherwise. \( F^d \) is called the coordinate system and its elements are referred to as coordinate points. The event (the set of bodies) observed by observer \( m \) at coordinate point \( p \) is:

\[
ev_m(p) := \{ b \in B : W(m, b, p) \}.
\]

The mapping \( p \mapsto ev_m(p) \) is called the world-view (function) of \( m \). The coordinate domain of observer \( m \) is the set of coordinate points where \( m \) observes something:

\[
\text{Cd}(m) := \{ p \in F^d : ev_m(p) \neq \emptyset \}.
\]

The life-line (or trace) of body \( b \) as seen by observer \( m \) is defined as the set of coordinate points where \( b \) was observed by \( m \):

\[
tr_m(b) := \{ p \in F^d : W(m, b, p) \} = \{ p \in F^d : b \in ev_m(p) \}.
\]

The life-line \( tr_m(m) \) of observer \( m \) as seen by himself is called the self-line of \( m \). The time-axis is defined as \( \bar{t} := \{ \langle x, 0, \ldots, 0 \rangle : x \in F \} \).

![World-View Diagram](image)

**Figure 1.** For the basic definitions mainly for \( f^n_m \).

Now we are ready to build our space-time theories by formulating our axioms. We formulate each axiom on two levels. First we give an intuitive formulation, then we give a precise formalization using our notation.
The following natural axiom goes back to Galileo Galilei and even to the Norman-French Oresme of around 1350, cf. e.g., [1, p.23, §5]. It simply states that each observer thinks that he rests in the origin of the space part of his coordinate system.

\textbf{AxSelf}⁻: The self-line of any observer is the time-axis restricted to his coordinate domain:

\[ \forall m \in \text{Ob} \quad tr_m(m) = \tilde{t} \cap Cd(m). \]  

A FOL-formula expressing \textbf{AxSelf}⁻ can be found in the Appendix.

The next axiom is about the constancy of the speed of the photons, cf. e.g., [3, §2.6]. For convenience, we choose 1 for their speed.

\textbf{AxPh}: For every inertial observer, the lines of slope 1 are exactly the traces of the photons:

\[ \forall m \in \text{IOb} \quad \{tr_m(ph) : ph \in \text{Ph}\} = \{l \in \text{Lines} : \text{slope}(l) = 1\}. \]  

A FOL-formula expressing \textbf{AxPh} can be found in the Appendix.

We will also assume the following axiom:

\textbf{AxEv}: All inertial observers observe the same events:

\[ \forall m,k \in \text{IOb} \forall p \in F^d \exists q \in F^d \quad ev_m(p) = ev_k(q). \]  

A FOL-formula expressing \textbf{AxEv} can be found in the Appendix.

\[ \text{Specrel}_0 := \{\text{AxSelf}⁻, \text{AxPh}, \text{AxEv}, \text{AxFrame}\}. \]  

Since, in some sense, \textbf{AxFrame} is only an “auxiliary” (or book-keeping) axiom about the “mathematical frame” of our reasoning, the heart of \text{Specrel}_0 consists of three very natural axioms, \textbf{AxSelf}⁻, \textbf{AxPh}, \textbf{AxEv}. These are really intuitively convincing, natural and simple assumptions. From these three axioms one can already prove the most characteristic predictions of special relativity theory. What the average layperson usually knows about relativity is that “moving clocks slow down”, “moving spaceships shrink”, and “moving pairs of clocks get out of synchronism”. We call these the \textbf{paradigmatic effects} of special relativity. All these can be proven from the above three axioms, in some form, cf. Theorem 2.2. E.g., one can prove that “if \(m, k\) are any two observers not at rest relative to each other, then one of \(m, k\) will “see” or “think” that the clock of the other runs slow”. However, \text{Specrel}_0 does not imply yet the inertial approximation of the so-called Twin Paradox.\(^3\) In order to prove the inertial approximation of the Twin Paradox also, and to prove all the paradigmatic effects in their strongest form, it is enough to add one more axiom \textbf{AxSym} to \text{Specrel}_0. This is what we are going to do now.

We will find that studying the relationships between the world-views is more illuminating than studying the world-views in themselves. Therefore the following definition is fundamental. The \textbf{world-view transformation} between the world-views of observers \(k\) and \(m\) is the set of pairs of coordinate points \( \langle p, q \rangle \) such that \(m\) and \(k\) observe the same nonempty event in \(p\) and \(q\), respectively:

\[ f^k_m := \{(p, q) \in F^d \times F^d : ev_k(p) = ev_m(q) \neq \emptyset\}, \]  

cf. Figure 1. We note that although the world-view transformations are only binary relations, axiom \textbf{AxPh} turns them into functions, cf. (iii) of Proposition 5.1 way below.

\textbf{Convention} 2.1. Whenever we write “\( f^k_m(p) \)”, we mean that there is a unique \( q \in F^d \) such that \( \langle p, q \rangle \in f^k_m \), and \( f^k_m(p) \) denotes this unique \( q \). I.e. if we talk about the value \( f^k_m(p) \) of \( f^k_m \) at \( p \), we tacitly postulate that it exists and is unique (by the present convention).

\(^3\)This inertial approximation of the twin paradox is formulated as \textbf{AxTp}^n at the end of Section 3 below Theorem 3.7.
The following axiom is an instance (or special case) of the Principle of Special Relativity, according to which the “laws of nature” are the same for all inertial observers, in particular, there is no experiment which would decide whether you are in absolute motion, cf. e.g., Einstein [6] or [5] §2.5 or [15] §2.8. To explain the following formula, let \( p, q \in \mathbb{F}^d \). Then \( p_t - q_t \) is the time passed between the events \( ev_m(p) \) and \( ev_m(q) \) as seen by \( m \) and \( f_k^m(p)_t - f_k^m(q)_t \) is the time passed between the same two events as seen by \( k \). Hence \( |(f_k^m(p)_t - f_k^m(q)_t)/(p_t - q_t)| \) is the rate with which \( k \)’s clock runs slow as seen by \( m \). The same explanation applies when \( m \) and \( k \) are interchanged.

**AxSym:** Any two inertial observers see each other’s clocks go wrong in the same way:

\[
\forall m, k \in \text{IOb} \quad \forall p, q \in \mathbb{F}^d \quad |f_k^m(p)_t - f_m^k(q)_t| = |f_m^k(p)_t - f_k^m(q)_t|.
\]

(10)

All the axioms so far talked about inertial observers, and they in fact form an axiom system complete w.r.t. the inertial observers, cf. Theorem 2.2 below.

\[
\text{Specrel} := \{ \text{AxSelf}^-, \text{AxPh}, \text{AxEv}, \text{AxSym}, \text{AxFrame} \}.
\]

(11)

Let \( p, q \in \mathbb{F}^d \). Then

\[
\mu(p) := \begin{cases} 
\sqrt{|p_t^2 - |p_s|^2|} & \text{if } p_t^2 - |p_s|^2 \geq 0, \\
-\sqrt{|p_t^2 - |p_s|^2|} & \text{otherwise}
\end{cases}
\]

(12)

is the (signed) Minkowski-length of \( p \) and the Minkowski-distance between \( p \) and \( q \) is defined as follows:

\[
\mu(p, q) := \mu(p - q).
\]

(13)

A motivation for the “otherwise” part of the definition of \( \mu(p) \) is the following: \( \mu(p) \) codes two kinds of information, (i) the length of \( p \) and (ii) whether \( p \) is time-like (i.e. \( |p_t| > |p_s| \)) or space-like. Since the length is always non-negative, we can use the sign of \( \mu(p) \) to code (ii).

Let \( f : \mathbb{F}^d \to \mathbb{F}^d \) be a function. \( f \) is said to be a **Poincaré-transformation** if \( f \) is a bijection and it preserves the Minkowski-distance, i.e. \( \mu(f(p), f(q)) = \mu(p, q) \) for all \( p, q \in \mathbb{F}^d \). \( f \) is called a **dilation** if there is a positive \( \delta \in \mathbb{F} \) such that \( f(p) = \delta p \) for all \( p \in \mathbb{F}^d \) and \( f \) is called a **field-automorphism-induced** mapping if there is an automorphism \( \pi \) of the field \( \langle \mathbb{F}, +, \cdot \rangle \) such that \( f(p) = \langle \pi p_1, \ldots, \pi p_d \rangle \) for all \( p \in \mathbb{F}^d \). The following is proved in [2] 2.9.4, 2.9.5 and in [13] 2.9.4--2.9.7.

Let \( \Sigma \) be a set of formulas and \( \mathcal{M} \) be a model. \( \mathcal{M} \models \Sigma \) denotes that all formulas in \( \Sigma \) are true in model \( \mathcal{M} \). In this case we say that \( \mathcal{M} \) is a **model of** \( \Sigma \).

**Theorem 2.2.** Let \( d > 2 \), let \( \mathcal{M} \) be a model of our language and let \( m, k \) be inertial observers in \( \mathcal{M} \). Then \( f_m^k \) is a Poincaré-transformation whenever \( \mathcal{M} \models \text{Specrel} \). More generally, \( f_k^m \) is a Poincaré-transformation composed with a dilation and a field-automorphism-induced mapping whenever \( \mathcal{M} \models \text{Specrel}_0 \).

**Remark 2.3.** Assume \( d > 2 \). Theorem 2.2 is best possible in the sense that, e.g., for every Poincaré-transformation \( f \) over an arbitrary Euclidean ordered field there are a model \( \mathcal{M} \models \text{Specrel} \) and inertial observers \( m, k \) in \( \mathcal{M} \) such that the world-view transformation \( f_k^m \) between \( m \)'s and \( k \)'s world-views in \( \mathcal{M} \) is \( f \), see [2] 2.9.4(iii), 2.9.5(iii)]. Similarly for the second statement in Theorem 2.2. Hence, Theorem 2.2 can be refined to a pair of completeness theorems, cf. [2] 3.6.13, p.271]. Roughly, for every Euclidean ordered field, its Poincaré-transformations (can be expanded to) form a model of \( \text{Specrel} \). Similarly for the other case.
It follows from Theorem 2.2 that the paradigmatic effects of relativity hold in Specrel in their strongest form, e.g., if \( m \) and \( k \) are observers not at rest w.r.t. each other, then both will “think” that the clock of the other runs slow. Specrel also implies the “inertial approximation” of the Twin Paradox, see e.g., [2, 2.8.18], and [21]. It is necessary to add AxSym to Specrel to in order to be able to prove the inertial approximation of the Twin Paradox by Theorem 2.2, cf. e.g., [21].

We now begin to formulate axioms about non-inertial observers. The non-inertial observers are called accelerated observers. To connect the world-views of the accelerated and the inertial observers, we are going to formulate the statement that at each moment of his life, each accelerated observer sees the nearby world for a short while as an inertial observer does. To formalize this, first we introduce the relation of being a co-moving observer. The (open) ball with center \( c \in F^n \) and radius \( \varepsilon \in F^+ \) is:

\[
B_\varepsilon(c) := \{ a \in F^n : |a - c| < \varepsilon \}.
\] (14)

\( m \) is a co-moving observer of \( k \) at \( q \in F^d \), in symbols \( m \succ_q k \), if \( q \in Cd(k) \) and the following holds:

\[
\forall \varepsilon \in F^+ \exists \delta \in F^+ \forall p \in B_\delta(q) \cap Cd(k) \ [p - f^k_m(p)] | \leq \varepsilon |p - q|.
\] (15)

Behind the definition of the co-moving observers is the following intuitive image: as we zoom into smaller and smaller neighborhoods of the given coordinate point, the world-views of the two observers are more and more similar. Notice that \( f^k_m(q) = q \) if \( m \succ_q k \).

The following axiom gives the promised connection between the world-views of the inertial and the accelerated observers:

**AxAcc:** At any point on the self-line of any observer, there is a co-moving inertial observer:

\[
\forall k \in Ob \ \forall q \in tr_k(k) \ \exists m \in IOb \ m \succ_q k.
\] (16)

Let \( \text{AccRel}_0 \) be the collection of the axioms introduced so far:

\[
\text{AccRel}_0 := \{ \text{AxSelf}^-, \text{AxPh}, \text{AxEv}, \text{AxSym}, \text{AxAcc}, \text{AxFrame} \}.
\] (17)

Let \( \mathfrak{R} \) denote the ordered field of the real numbers. Accelerated clocks behave as expected in models of \( \text{AccRel}_0 \) when the ordered field reduct of the model is \( \mathfrak{R} \) (cf. Theorems 3.1, 3.2 and in more detail Prop 3.2, Rem 3.3); but not otherwise (cf. Theorems 3.3, 3.7). Thus to prove properties of accelerated clocks (and observers), we need more properties of the field reducts than their being Euclidean ordered fields. As it turns out, adding all FOL-formulas valid in the field of the reals does not suffice (cf. Corollary 3.6). The additional property of \( \mathfrak{R} \) we need is that in \( \mathfrak{R} \) every bounded non-empty set has a supremum, i.e. a least upper bound. This is a second-order logic property (because it concerns all subsets of \( \mathfrak{R} \)) which we cannot use in a FOL axiom system. Instead, we will use a kind of “induction” axiom schema. It will state that every non-empty, bounded subset of the ordered field reduct which can be defined by a FOL-formula using possibly the extra part of the model, e.g., using the world-view relation, has a supremum. We now begin to formulate our FOL induction axiom schema.

4This way of imitating a second-order formula by a FOL-formula schema comes from the methodology of approximating second-order theories by FOL ones, examples are Tarski’s replacement of Hilbert’s second-order geometry axiom by a FOL schema or Peano’s FOL induction schema replacing second-order logic induction.

To abbreviate formulas of FOL we often omit parentheses according to the following convention. Quantifiers bind as long as they can, and \( \land \) binds stronger than \( \implies \). E.g.,
∀x. φ ∧ ψ \implies \exists y. δ ∧ η means \( \forall x ((\phi \land \psi) \implies \exists y (\delta \land \eta)) \). Instead of curly brackets we sometimes write square brackets in formulas, e.g., we may write \( \forall x ((\phi \land \psi) \implies [\exists y \delta] \land \eta) \).

If \( \varphi \) is a formula and \( x \) is a variable, then we say that \( x \) is a free variable of \( \varphi \) iff \( x \) does not occur under the scope of either \( \exists x \) or \( \forall x \).

Let \( \varphi \) be a formula; and let \( x, y_1, \ldots, y_n \) be all the free variables of \( \varphi \). Let \( \mathfrak{M} = \langle U; B, \ldots \rangle \) be a model. Whether \( \varphi \) is true or false in \( \mathfrak{M} \) depends on how we associate elements of \( U \) to these free variables. When we associate \( d, a_1, \ldots, a_n \in U \) to \( x, y_1, \ldots, y_n \), respectively, then \( \varphi(d, a_1, \ldots, a_n) \) denotes this truth-value, thus \( \varphi(d, a_1, \ldots, a_n) \) is either true or false in \( \mathfrak{M} \). For example, if \( \varphi \) is \( x \leq y_1 + \ldots + y_n \), then \( \varphi(0, 1, \ldots, 1) \) is true in \( \mathfrak{R} \) while \( \varphi(1, 0, \ldots, 0) \) is false in \( \mathfrak{R} \). \( \varphi \) is said to be true in \( \mathfrak{M} \) if \( \varphi \) is true in \( \mathfrak{M} \) no matter how we associate elements to the free variables. We say that a subset \( H \) of \( F \) is definable by \( \varphi \) iff there are \( a_1, \ldots, a_n \in U \) such that \( H = \{ d \in F : \varphi(d, a_1, \ldots, a_n) \} \) is true in \( \mathfrak{M} \).

\textbf{AxSup}_\varphi: Every subset of \( F \) definable by \( \varphi \) has a supremum if it is non-empty and bounded:

\[
\forall y_1, \ldots, y_n \quad [\exists x \in F \varphi] \land [\exists b \in F (\forall x \in F \varphi \implies x \leq b)]
\implies (\exists s \in F \forall b \in F (\forall x \in F \varphi \implies x \leq b) \iff s \leq b). \quad (18)
\]

We say that a subset of \( F \) is definable iff it is definable by a FOL-formula. Our axiom scheme \textbf{IND} below says that every non-empty bounded and definable subset of \( F \) has a supremum.

\[
\textbf{IND} := \{ \textbf{AxSup}_\varphi : \varphi \text{ is a FOL-formula of our language } \}. \quad (19)
\]

Notice that \textbf{IND} is true in any model whose ordered field reduct is \( \mathfrak{R} \). Let us add \textbf{IND} to \textbf{AccRel}_0 and call it \textbf{AccRel}:

\[
\textbf{AccRel} := \textbf{AccRel}_0 \cup \textbf{IND}. \quad (20)
\]

\textbf{AccRel} is a countable set of FOL-formulas. We note that there are non-trivial models of \textbf{AccRel}, cf. e.g., Remark 3.3 way below. Furthermore, we note that the construction in Misner-Thorne-Wheeler [10, Chapter 6 entitled “The local coordinate system of an accelerated observer”, especially pp. 172-173 and Chapter 13.6 entitled “The proper reference frame of an accelerated observer” on pp. 327-332] can be used for constructing models of \textbf{AccRel}. Models of \textbf{AccRel} are discussed to some detail in [21]. Theorems 3.1 and 3.2 (and also Prop. 5.2, Rem. 5.3) below show that \textbf{AccRel} already implies properties of accelerated clocks, e.g., it implies the Twin Paradox.

\textbf{IND} implies all the FOL-formulas true in \( \mathfrak{R} \), but \textbf{IND} is stronger. Let \textbf{IND}^- denote the set of elements of \textbf{IND} that talk only about the ordered field reduct, i.e. let

\[
\textbf{IND}^- := \{ \textbf{AxSup}_\varphi : \varphi \text{ is a FOL-formula in the language of the ordered field reduct of our models} \}. \quad (21)
\]

Now, \textbf{IND}^- together with the axioms of linearly ordered fields is a complete axiomatization of the FOL-theory of \( \mathfrak{R} \), i.e. all FOL-formulas valid in \( \mathfrak{R} \) can be derived from them.\(^5\) However, \textbf{IND} is stronger than \textbf{IND}^-, since \( \textbf{AccRel}_0 \cup \textbf{IND}^- \not= \textbf{Tp} \) by Corollary 3.6 below, while \( \textbf{AccRel}_0 \cup \textbf{IND} \models \textbf{Tp} \) by Theorem 3.1. The strength of \textbf{IND} comes from the fact that the formulas in \textbf{IND} can “talk” about more “things” than just those in the language of \( \mathfrak{R} \) (namely they can talk about the world-view relation, too). For understanding how \textbf{IND} works, it is important to notice that \textbf{IND} does not speak about the field \( \mathfrak{F} \) itself, but instead, it speaks about connections between \( \mathfrak{F} \) and the rest of the model \( \mathfrak{M} = \langle \mathfrak{F}, \ldots, \mathfrak{W} \rangle \).

\(^5\)This follows from a theorem of Tarski, cf. Hodges [13, p.68 (b)] or Tarski [22, by Theorem 14.7 herein or 21, Proposition A.0.1].
Why do we call **IND** a kind of induction schema? The reason is the following. **IND** implies that if a formula \( \varphi \) becomes false sometime after 0 while being true at 0, then there is a “first” time-point where, so to speak, \( \varphi \) becomes false. This time-point is the supremum of the time-points until which \( \varphi \) remained true after 0. Now, \( \varphi \) may or may not be false at this supremum, but it is false arbitrarily “close” to it afterwards. If such a “point of change” for the truth of \( \varphi \) cannot exist, **IND** implies that \( \varphi \) has to be true always after 0 if it is true at 0. (Without **IND**, this may not be true.)

3. **MAIN RESULTS: ACCELERATED CLOCKS AND THE TWIN PARADOX IN OUR FOL AXIOMATIC SETTING**

Twin Paradox (TP) concerns two twin siblings whom we shall call Ann and Ian. (“A” and “I” stand for accelerated and for inertial, respectively). Ann travels in a spaceship to some distant star while Ian remains at home. TP states that when Ann returns home she will be **younger** than her twin brother Ian. We now formulate TP in our FOL language.

The **segment** between \( p \in F^d \) and \( q \in F^d \) is defined as:

\[
[pq] := \{\lambda p + (1 - \lambda)q : \lambda \in F \land 0 \leq \lambda \leq 1\}. \tag{22}
\]

We say that observer \( k \) is in **twin-paradox relation** with observer \( m \) iff whenever \( k \) leaves \( m \) between two meetings, \( k \) measures less time between the two meetings than \( m \):

\[
\forall p, q \in tr_k(k) \quad \forall p', q' \in tr_m(m) \quad \langle p, p' \rangle, \langle q, q' \rangle \in f_m^k \land [pq] \subset tr_k(k) \land [p'q'] \not\subset tr_m(m) \quad \Rightarrow \quad |q_t - p_t| < |q'_t - p'_t|, \tag{23}
\]

*cf. Figure 2.* In this case we write \( Tp(k < m) \). We note that, if two observers do not leave each other or they meet less than twice, then they are in twin-paradox relation by this definition. Thus two inertial observers are always in this relation.

![Figure 2](image)

**Tp**: Every observer is in twin-paradox relation with every inertial observer:

\[
\forall k \in Ob \quad \forall m \in IOb \quad Tp(k < m). \quad \tag{24}
\]
Let \( \varphi \) be a formula and \( \Sigma \) be a set of formulas. \( \Sigma \models \varphi \) denotes that \( \varphi \) is true in all models of \( \Sigma \). Gödel’s completeness theorem for FOL implies that whenever \( \Sigma \models \varphi \), there is a (syntactic) derivation of \( \varphi \) from \( \Sigma \) via the commonly used derivation rules of FOL. Hence the next theorem states that the formula \( Tp \) formulating the Twin Paradox is provable from the axiom system \( \text{AccRel} \).

**Theorem 3.1.** \( \text{AccRel} \models Tp \) if \( d > 2 \).

The proof of the theorem is in Section 5.

Now we turn to formulating a phenomenon which we call Duration Determining Property of Events.

**Ddpe:** If each of two observers observes the very same (non-empty) events in a segment of their self-lines, they measure the same time between the end points of these two segments:

\[
\forall k, m \in \text{Ob} \ \forall p, q \in \text{tr}_k(k) \ \forall p', q' \in \text{tr}_m(m) \quad \emptyset \not\subseteq \{ev_k(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\} \implies \quad \quad \quad \quad \quad \quad (25)
\]

\[|q_t - p_t| = |q'_t - p'_t|,\]

see the right hand side of Figure 2.

The next theorem states that \( \text{Ddpe} \) also can be proved from our FOL axiom system \( \text{AccRel} \).

**Theorem 3.2.** \( \text{AccRel} \models \text{Ddpe} \) if \( d > 2 \).

The proof of the theorem is in Section 5.

**Remark 3.3.** The assumption \( d > 2 \) cannot be omitted from Theorem 3.1. However, Theorems 3.1 and 3.2 remain true if we omit the assumption \( d > 2 \) and assume auxiliary axioms \( \text{AxIob} \) and \( \text{AxLine} \) below, i.e.

\[
\text{AccRel} \cup \{\text{AxIob}, \text{AxLine}\} \models Tp \land \text{Ddpe} \quad (26)
\]

holds for \( d = 2 \), too. A proof for the latter statement can be obtained from the proofs of Theorems 3.1 and 3.2 by [21] items 4.3.1, 4.2.4, 4.2.5] and [11] Theorem 1.4(ii)].

**AxIob:** In every inertial observer’s coordinate system, every line of slope less than 1 is the life-line of an inertial observer:

\[
\forall m \in \text{Iob} \quad \{\text{tr}_m(k) : k \in \text{Iob}\} \supseteq \{l \in \text{Lines} : \text{slope}(l) < 1\}. \quad (27)
\]

**AxLine:** Traces of inertial observers are lines as observed by inertial observers:

\[
\forall m, k \in \text{Iob} \quad \text{tr}_m(k) \in \text{Lines}. \quad (28)
\]

**Question 3.4.** Can the assumption \( d > 2 \) be omitted from Theorem 3.2 i.e. does \( \text{AccRel} \models \text{Ddpe} \) hold for \( d = 2 \)?

The following theorem says that Theorems 3.1 and 3.2 do not remain true if we omit the axiom scheme \( \text{IND} \) from \( \text{AccRel} \). If a formula \( \varphi \) is not true in a model \( \mathcal{M} \), we write \( \mathcal{M} \not\models \varphi \).

**Theorem 3.5.** For every Euclidean ordered field \( \mathcal{F} \) not isomorphic to \( \mathcal{R} \), there is a model \( \mathcal{M} \) of \( \text{AccRel}_0 \) such that \( \mathcal{M} \not\models Tp, \mathcal{M} \not\models \text{Ddpe} \) and the ordered field reduct of \( \mathcal{M} \) is \( \mathcal{F} \).
The proof of the theorem is in Section 5.

By Theorems 3.1 and 3.2, IND is not true in the model M mentioned in Theorem 3.5. This theorem has strong consequences, it implies that to prove the Twin Paradox, it does not suffice to add all the FOL-formulas valid in \( \mathcal{R} \) (to AccRel\(_0\)). Let \( Th(\mathcal{R}) \) denote the set of all FOL-formulas valid in \( \mathcal{R} \).

**Corollary 3.6.** \( Th(\mathcal{R}) \cup \text{AccRel}_0 \not\models Tp \) and \( Th(\mathcal{R}) \cup \text{AccRel}_0 \not\models Ddpe \).

The proof of the corollary is in Section 5.

An ordered field is called non-Archimedean if it has an element \( a \) such that, for every positive integer \( n \), \(-1 < \sum_{i=0}^{n-1} a^i < 1\). We call these elements infinitesimally small.

The following theorem says that, for countable or non-Archimedean Euclidean ordered fields, there are quite sophisticated models of AccRel\(_0\) in which \( Tp \) and \( Ddpe \) are false.

**Theorem 3.7.** For every Euclidean ordered field \( F \) which is countable or non-Archimedean, there is a model \( \mathcal{M} \) of AccRel\(_0\) such that \( \mathcal{M} \not\models Tp \), \( \mathcal{M} \not\models Ddpe \), the ordered field reduct of \( \mathcal{M} \) is \( \mathcal{F} \) and (i)–(iv) below also hold in \( \mathcal{M} \).

(i) Every observer uses the whole coordinate system for coordinate-domain:

\[
\forall m \in \text{Ob} \quad \text{Cd}(m) = F^d. \tag{29}
\]

(ii) At any point in \( F^d \), there is a co-moving inertial observer of any observer:

\[
\forall k \in \text{Ob} \quad \forall q \in F^d \exists m \in \text{IOb} \quad m \succ_q k. \tag{30}
\]

(iii) All observers observe the same set of events:

\[
\forall m, k \in \text{Ob} \quad \forall p, q \in F^d \exists q \in F^d \quad ev_m(p) = ev_k(q). \tag{31}
\]

(iv) Every observer observes every event only once:

\[
\forall m \in \text{Ob} \quad \forall p, q \in F^d \quad ev_m(p) = ev_m(q) \implies p = q. \tag{32}
\]

The proof of the theorem is in Section 5.

![Figure 3](image-url)
Finally we formulate a question. To this end we introduce the inertial version of the twin paradox and some auxiliary axioms. In the inertial version of the twin paradox, we use the common trick of the literature to talk about the twin paradox without talking about accelerated observers. We replace the accelerated twin with two inertial ones, a leaving and an approaching one.

We say that observers \( k_1 \) and \( k_2 \) are in inertial twin-paradox relation with observer \( m \) if the following holds:

\[
\forall p, q, r \in tr_{k_1}(k_1) \cap tr_{k_2}(k_2) \quad \forall p', q' \in tr_m(m) \quad \forall r' \in F^d \\
\langle p, p' \rangle, \langle r, r' \rangle \in f_{m}^{k_1} \wedge \langle r, r' \rangle, \langle q, q' \rangle \in f_{m}^{k_2} \wedge p'_t < r'_t < q'_t \wedge r'_t \notin [p'q'] \implies |q'_t - p'_t| > |q_t - r_t| + |r_t - p_t|,
\]

(33)

cf. Figure 3. In this case we write \( Tp(k_1k_2 < m) \).

\( AxTp^i \): Every three inertial observers are in inertial twin-paradox relation:

\[
\forall m, k_1, k_2 \in IOb \quad Tp(k_1k_2 < m).
\]

(34)

\( AxTrn \): To every inertial observer \( m \) and coordinate point \( p \) there is an inertial observer \( k \) such that the world-view transformation between \( m \) and \( k \) is the translation by vector \( p \):

\[
\forall m \in IOb \quad \forall p \in F^d \quad \exists k \in IOb \quad \forall q \in F^d \quad f^{m}_{k}(q) = q + p.
\]

(35)

\( AxLt \): The world-view transformation between inertial observers \( m \) and \( k \) is a linear transformation if \( f^{m}_{k}(o) = o \):

\[
\forall m, k, \exists q \in F^d \quad \forall \lambda \in F \quad f^{m}_{k}(o) = o \implies f^{m}_{k}(p + q) = f^{m}_{k}(p) + f^{m}_{k}(q) \wedge f^{m}_{k}(\lambda p) = \lambda f^{m}_{k}(p).
\]

(36)

**Question 3.8.** Does Theorem 3.1 remain true if we replace \( AxSym \) in \( AccRel \) with the inertial version \( AxTp^i \) of the twin paradox and we assume the auxiliary axioms \( AxLt \), \( AxIOb \) and \( AxTrn \)? Cf. Question 5.6. We note that \( AxTp^i \) and \( AxLt \) are true in the models of \( Specrel \) in case \( d > 2 \), cf. [11, Theorem 1.2], [15, Theorem 2.8.28] and [21, §3].

4. PIECES FROM NON-STANDARD ANALYSIS: SOME TOOLS FROM REAL ANALYSIS CAPTURED IN FOL

In this section we gather the statements (and proofs from \( AccRel \)) of the facts we will need from analysis. The point is in formulating these statements in FOL and for an arbitrary ordered field in place of using the second-order language of the ordered field \( \mathbb{R} \) of reals.

In the present section \( AxFrame \) is assumed without any further mentioning.

Let \( a, b, c \in F^d \). We say that \( b \) is between \( a \) and \( c \) iff \( a < b < c \) or \( a > b > c \). We use the following notation: \([a, b] := \{x \in F : a \leq x \leq b\}\) and \((a, b) := \{x \in F : a < x < b\}\).

**Convention 4.1.** Whenever we write \([a, b]\), we assume that \( a, b \in F \) and \( a \leq b \). We also use this convention for \((a, b)\).

Let \( H \subseteq F^n \). Then \( p \in F^n \) is said to be an accumulation point of \( H \) if for all \( \varepsilon \in F^+ \), \( B^+(p) \cap H \) has an element different from \( p \). \( H \) is called open if for all \( p \in H \), there is an \( \varepsilon \in F^+ \) such that \( B^+(p) \subseteq H \). Let \( R \subseteq A \times B \) and \( S \subseteq B \times C \) be binary relations. The composition of \( R \) and \( S \) is defined as: \( R \circ S := \{ \langle a, c \rangle \in A \times C : \exists b \in B \langle a, b \rangle \in R \wedge \langle b, c \rangle \in S \} \). The domain and the range of \( R \) are denoted by \( \text{Dom}(R) := \{a \in A : \exists b \in B \langle a, b \rangle \in R\} \) and \( \text{Rng}(R) := \{b \in B : \exists a \in A \langle a, b \rangle \in R\} \), respectively. \( R^{-1} \) denotes the inverse of \( R \), i.e. \( R^{-1} := \{\langle b, a \rangle \in B \times A : \langle a, b \rangle \in R\} \). We think of a function as a special binary relation. Notice that if \( f, g \) are functions, then
We call \( f \) composed by a translation. \( f : A \to B \) denotes that \( f \) is a function from \( A \) to \( B \), i.e. \( \text{Dom}(f) = A \) and \( \text{Rng}(f) \subseteq B \). Notation \( f : A \xrightarrow{\circ} B \) denotes that \( f \) is a partial function from \( A \) to \( B \); this means that \( f \) is a function, \( \text{Dom}(f) \subseteq A \) and \( \text{Rng}(f) \subseteq B \).

Let \( f : F \xrightarrow{\circ} F^n \). We call \( f \) continuous at \( x \) if \( x \in \text{Dom}(f) \), \( x \) is an accumulation point of \( \text{Dom}(f) \) and the usual formula of continuity holds for \( f \) and \( x \), i.e.

\[
\forall \varepsilon \in F^+ \exists \delta \in F^+ \forall y \in \text{Dom}(f) \quad |y - x| < \delta \implies |f(y) - f(x)| < \varepsilon.
\]

(37)

We call \( f \) differentiable at \( x \) if \( x \in \text{Dom}(f) \), \( x \) is an accumulation point of \( \text{Dom}(f) \) and there is an \( a \in F^n \) such that

\[
\forall \varepsilon \in F^+ \exists \delta \in F^+ \forall y \in \text{Dom}(f) \quad |y - x| < \delta \implies |f(y) - f(x) - (y - x)a| \leq \varepsilon |y - x|.
\]

(38)

This \( a \) is unique. We call this \( a \) the derivate of \( f \) at \( x \) and we denote it by \( f'(x) \). \( f \) is said to be continuous (differentiable) on \( H \subseteq F \) iff \( H \subseteq \text{Dom}(f) \) and \( f \) is continuous (differentiable) at every \( x \in H \). We note that the basic properties of the differentiability remain true since their proofs use only the ordered field properties of \( \mathfrak{R} \), cf. Propositions 4.2, 4.3 and 4.4 below.

Let \( f, g : F \xrightarrow{\circ} F^n \) and \( \lambda \in F \). Then \( \lambda f : F \xrightarrow{\circ} F^n \) and \( f + g : F \xrightarrow{\circ} F^n \) are defined as

\[
\lambda f := \{ (x, \lambda f(x)) : x \in \text{Dom}(f) \} \quad \text{and} \quad f + g := \{ (x, f(x) + g(x)) : x \in \text{Dom}(f) \cap \text{Dom}(g) \}.
\]

Let \( h : F \xrightarrow{\circ} F \). \( h \) is said to be increasing on \( H \) iff \( H \subseteq \text{Dom}(h) \) and for all \( x, y \in H \), 

\[
h(x) < h(y) \quad \text{if} \quad x < y,
\]

and \( h \) is said to be decreasing on \( H \) iff \( H \subseteq \text{Dom}(h) \) and for all \( x, y \in H \), 

\[
h(x) > h(y) \quad \text{if} \quad x < y.
\]

**Proposition 4.2.** Let \( f, g : F \xrightarrow{\circ} F^n \) and \( h : F \xrightarrow{\circ} F \). Then (i)–(v) below hold.

(i) If \( f \) is differentiable at \( x \) then it is also continuous at \( x \).

(ii) Let \( \lambda \in F \). If \( f \) is differentiable at \( x \), then \( \lambda f \) is also differentiable at \( x \) and 

\[
(\lambda f)'(x) = \lambda f'(x).
\]

(iii) If \( f \) and \( g \) are differentiable at \( x \) and \( x \) is an accumulation point of \( \text{Dom}(f) \cap \text{Dom}(g) \), then \( f + g \) is differentiable at \( x \) and 

\[
(f + g)'(x) = f'(x) + g'(x).
\]

(iv) If \( h \) is differentiable at \( x \), \( g \) is differentiable at \( h(x) \) and \( x \) is an accumulation point of \( \text{Dom}(h \circ g) \), then \( h \circ g \) is differentiable at \( x \) and 

\[
(h \circ g)'(x) = h'(x)g'(h(x)).
\]

(v) If \( h \) is increasing (or decreasing) on \( (a, b) \), differentiable at \( x \in (a, b) \) and \( h'(x) \neq 0 \), then \( h^{-1} \) is differentiable at \( h(x) \).

**on the proof.** Since the proofs of the statements are based on the same calculations and ideas as in real analysis, we omit the proof, cf. [18, Theorems 28.2, 28.3, 28.4 and 29.9].

Let \( i \leq n. \pi_i : F^n \to F \) denotes the \( i \)-th projection function, i.e. \( \pi_i : p \mapsto p_i \). Let \( f : F \xrightarrow{\circ} F^n \). We denote the \( i \)-th coordinate function of \( f \) by \( f_i \), i.e. \( f_i := f \circ \pi_i \). We also denote \( f_1 \) by \( f \). A function \( A : F^n \to F^j \) is said to be an affine map if it is a linear map composed by a translation.6

The following proposition says that the derivate of a function \( f \) composed by an affine map \( A \) at a point \( x \) is the image of the derivate \( f'(x) \) taken by the linear part of \( A \).

**Proposition 4.3.** Let \( f : F \xrightarrow{\circ} F^n \) be differentiable at \( x \) and let \( A : F^n \xrightarrow{\circ} F^j \) be an affine map. Then \( f \circ A \) is differentiable at \( x \) and 

\[
(f \circ A)'(x) = A(f'(x)) - A(o).\]

In particular, 

\[
f'_i(x) = f'_i(x), \quad i.e. \quad f'_i(x) = f'(x).
\]

**on the proof.** The statement is straightforward from the definitions.

---

6I.e. \( A \) is an affine map if there are \( L : F^n \to F^j \) and \( a \in F^j \) such that \( A(p) = L(p) + a, \quad L(p + q) = L(p) + L(q) \) and \( L(\lambda p) = \lambda L(p) \) for all \( p, q \in F^n \) and \( \lambda \in F \).
$f : F \rightarrow F$ is said to be **locally maximal** at $x$ iff $x \in \text{Dom}(f)$ and there is a $\delta \in F^+$ such that $f(y) \leq f(x)$ for all $y \in B_\delta(x) \cap \text{Dom}(f)$. The **local minimality** is defined analogously.

**Proposition 4.4.** If $f : F \rightarrow F$ is differentiable on $(a, b)$ and locally maximal or minimal at $x \in (a, b)$, then its derivate is 0 at $x$, i.e. $f'(x) = 0$.

**Proof.** The proof is the same as in real analysis, cf. e.g., [19, Theorem 5.8].

Let $\mathfrak{M} = \langle U; \ldots \rangle$ be a model. An $n$-ary relation $R \subseteq F^n$ is said to be **definable** iff there is a formula $\varphi$ with only free variables $x_1, \ldots, x_n, y_1, \ldots, y_i$ and there are $a_1, \ldots, a_i \in U$ such that

$$R = \{\langle p_1, \ldots, p_n \rangle \in F^n : \varphi(p_1, \ldots, p_n, a_1, \ldots, a_i)\} \text{ is true in } \mathfrak{M}.$$  \hspace{1cm} (39)

Recall that **IND** says that every non-empty, bounded and definable subset of $F$ has a supremum.

**Theorem 4.5** (Bolzano’s Theorem). Assume **IND**. Let $f : F \rightarrow F$ be definable and continuous on $[a, b]$. If $c$ is between $f(a)$ and $f(b)$, then there is an $s \in [a, b]$ such that $f(s) = c$.

**Proof.** Let $c$ be between $f(a)$ and $f(b)$. We can assume that $f(a) < f(b)$. Let $H := \{x \in [a, b] : f(x) < c\}$. Then $H$ is definable, bounded and non-empty. Thus, by **IND**, it has a supremum, say $s$. Both $\{x \in (a, b) : f(x) < c\}$ and $\{x \in (a, b) : f(x) > c\}$ are non-empty open sets since $f$ is continuous on $[a, b]$. Thus $f(s)$ cannot be less than $c$ since $s$ is an upper bound of $H$ and cannot be greater than $c$ since $s$ is the smallest upper bound. Hence $f(s) = c$ as desired.

**Theorem 4.6.** Assume **IND**. Let $f : F \rightarrow F$ be definable and continuous on $[a, b]$. Then the supremum $s$ of $\{f(x) : x \in [a, b]\}$ exists and there is an $y \in [a, b]$ such that $f(y) = s$.

**Proof.** The set $H := \{y \in [a, b] : \exists c \in F \forall x \in [a, y] \ f(x) < c\}$ has a supremum by **IND** since $H$ is definable, non-empty and bounded. This supremum has to be $b$ and $b \in H$ since $f$ is continuous on $[a, b]$. Thus $\text{Ran}(f) := \{f(x) : x \in [a, b]\}$ is bounded. Thus, by **IND**, it has a supremum, say $s$, since it is definable and non-empty. We can assume that $f(a) \neq s$. Let $A := \{y \in [a, b] : \exists c \in F \forall x \in [a, y] \ f(x) < c < s\}$. By **IND**, $A$ has a supremum. At this supremum, $f$ cannot be less than $s$ since $f$ is continuous on $[a, b]$ and $s$ is the supremum of $\text{Ran}(f)$.

Throughout this work $Id : F \rightarrow F$ denotes the identity function, i.e. $Id : x \mapsto x$.

**Theorem 4.7** (Mean Value Theorem). Assume **IND**. Let $f : F \rightarrow F$ be definable, differentiable on $[a, b]$. If $a \neq b$, then there is an $s \in (a, b)$ such that $f'(s) = \frac{f(b) - f(a)}{b - a}$.

**Proof.** Assume $a \neq b$. Let $h := (f(b) - f(a))Id - (b - a)f$. Then $h$ is differentiable on $[a, b]$ and $h'(x) = (f(b) - f(a)) - (b - a)f'(x)$ for all $x \in [a, b]$ by (ii) and (iii) of Proposition 4.2 since $Id$ is differentiable on $[a, b]$ and its derivate is 1 for all $x \in [a, b]$. If $h$ is constant on $[a, b]$,\footnote{i.e. if there is a $c \in F$ such that $h(x) = c$ for all $x \in [a, b]$,} then $h'(s) = 0$ for all $s \in (a, b)$. Otherwise, by Theorem 4.6 there is a maximum or minimum of $h$ different from $h(a) = f(b)a - bf(a) = h(b)$ at an $s \in (a, b)$. Hence $h'(s) = 0$ by Proposition 4.4. This completes the proof since $a \neq b$ and $h'(s) = f(b) - f(a) - (b - a)f'(s)$.

**Corollary 4.8** (Rolle’s Theorem). Assume **IND**. Let $f : F \rightarrow F$ be definable and differentiable on $[a, b]$. If $f(a) = f(b)$ and $a \neq b$, then there is an $s \in (a, b)$ such that $f'(s) = 0$.\footnote{i.e. if there is a $c \in F$ such that $h(x) = c$ for all $x \in [a, b]$,}
Proposition 4.9. Assume \textbf{IND}. Let \( f, g : F \rightarrow F \) be definable and differentiable on \((a, b)\). If \( f'(x) = g'(x) \) for all \( x \in (a, b) \), then there is a \( c \in F \) such that \( f(x) = g(x) + c \) for all \( x \in (a, b) \).

\textbf{proof.} Assume that \( f'(x) = g'(x) \) for all \( x \in (a, b) \). Let \( h := f - g \). Then \( h'(x) = f'(x) - g'(x) = 0 \) for all \( x \in (a, b) \) by (ii) and (iii) of Proposition 4.2. If there are \( y, z \in (a, b) \) such that \( h(y) \neq h(z) \) and \( y \neq z \), then, by the Mean Value Theorem, there is an \( x \) between \( y \) and \( z \) such that \( h'(x) = \frac{h(z) - h(y)}{z - y} \neq 0 \) and this contradicts \( h'(x) = 0 \). Thus \( h(y) = h(z) \) for all \( y, z \in (a, b) \). Hence there is a \( c \in F \) such that \( h(x) = c \) for all \( x \in (a, b) \). \( \blacksquare \)

5. PROOFS OF THE MAIN RESULTS

In the present section \textbf{AxFrame} is assumed without any further mentioning.

Let \( \sim : F \rightarrow F^d \) be the natural embedding defined as \( \sim : x \mapsto (x, 0, \ldots, 0) \). We define the \textbf{life-curve} of observer \( k \) as seen by observer \( m \) as \( Tr^k_m := \sim \circ f^k_m \). Throughout this work we denote \( \sim(x) \) by \( \hat{x} \), for \( x \in F \). Thus \( Tr^k_m(t) \) is the coordinate point where \( m \) observes the event “\( k \)’s wristwatch shows \( t \)”, i.e., \( Tr^k_m(t) = p \) iff \( ev_m(p) = ev_k((t, 0, \ldots, 0)) = ev_k(\hat{t}) \).

In the following proposition, we list several easy but useful consequences of some of our axioms.

Proposition 5.1. Let \( m \in \text{IOb} \) and \( k \in \text{Ob} \). Then (i)–(viii) below hold.

(i) Assume \textbf{AxPh}. Then \( Cd(m) = F^d \) and for all distinct \( p, q \in F^d \), \( ev_m(p) \neq ev_m(q) \).

(ii) Assume \textbf{AxPh} and \textbf{AxSelf}–. Then \( tr_m(m) = \hat{t} \).

(iii) Assume \textbf{AxPh}. Then \( f^k_m : F^d \rightarrow F^d \) and \( Tr^k_m : F \rightarrow F^d \).

(iv) Assume \textbf{AxPh} and \textbf{AxEv}. If \( k \in \text{IOb} \), then \( f^k_m : F^d \rightarrow F^d \) is a bijection and \( Tr^k_m : F \rightarrow F^d \) is an injection.

(v) Assume \textbf{AxEv}. Let \( h \in \text{IOb} \). Then \( f^k_h = f^k_m \circ f^m_h \) and \( Tr^k_h = Tr^k_m \circ f^m_h \).

(vi) Assume \textbf{AxAcc} and \textbf{AxEv}. Then \( tr_k(k) \subseteq \text{Dom}(f^k_m) \).

(vii) Assume \textbf{AxSelf}–. Then \( \{ \hat{x} \colon x \in \text{Dom}(Tr^k_m) \} \subseteq tr_k(k) \) and \( \text{Rng}(Tr^k_m) \subseteq tr_m(k) \).

(viii) Assume \textbf{AxAcc}, \textbf{AxEv} and \textbf{AxSelf}–. Then \( \{ \hat{x} \colon x \in \text{Dom}(Tr^k_m) \} = tr_k(k) \).

\textbf{proof.} To prove (i), let \( p, q \in F^d \) be distinct points. Then there is a line of slope 1 that contains \( p \) but does not contain \( q \). By \textbf{AxPh}, this line is the trace of a photon. For such a photon \( ph \), we have \( ph \in ev_m(p) \) and \( ph \notin ev_m(q) \). Hence \( ev_m(p) \neq ev_m(q) \) and \( ev_m(p) \neq \emptyset \). Thus (i) holds.

(ii) follows from (i) since \( tr_m(m) = Cd(m) \cap \hat{t} \) by \textbf{AxSelf}–.

(iii) and (iv) follow from (i) by the definitions of the world-view transformation and the life-curve.

To prove (v), let \( \langle p, q \rangle \in f^k_h \). Then \( ev_k(p) = ev_h(q) \neq \emptyset \). Since, by \textbf{AxEv}, \( h \) and \( m \) observe the same set of events, there is an \( r \in F^d \) such that \( ev_m(r) = ev_h(q) \). But then \( \langle p, r \rangle \in f^k_m \) and \( \langle r, q \rangle \in f^m_h \). Hence \( \langle p, q \rangle \in f^k_h \circ f^m_h \). Thus \( f^k_h \subseteq f^k_m \circ f^m_h \). The other inclusion follows from the definition of the world-view transformation. Thus \( f^k_h = f^k_m \circ f^m_h \) and \( Tr^k_h = \sim \circ f^k_h = \sim \circ f^k_m \circ f^m_h = Tr^k_m \circ f^m_h \).

To prove (vi), let \( q \in tr_k(k) \). By \textbf{AxAcc}, there is an \( h \in \text{IOb} \) such that \( h \) is a co-moving observer of \( k \) at \( q \). For such an \( h \), we have \( f^k_h(q) = q \) and, by (v), \( \text{Dom}(f^k_h) \subseteq \text{Dom}(f^k_m) \). Thus \( q \in \text{Dom}(f^k_m) \).

To prove (vii), let \( \langle x, q \rangle \in Tr^k_m \). Then \( \langle \hat{x}, q \rangle \in f^k_m \). But then \( ev_k(\hat{x}) = ev_m(q) \neq \emptyset \). Thus \( \hat{x} \in Cd(k) \cap \hat{t} \). By \textbf{AxSelf}–, \( \hat{x} \in tr_k(k) \); and this proves the first part of (vii). By \( \hat{x} \in tr_k(k) \), we have \( k \in ev_k(\hat{x}) = ev_m(q) \). Thus \( q \in tr_m(k) \) and this proves the second part of (vii).
The “⊆ part” of (viii) follows from (vii). To prove the other inclusion, let \( p \in tr_k(k) \). Then, by \textbf{AxSelf} and (vi), \( p \in \hat{t} \cap \text{Dom}(f^k_m) \). Thus there are \( x \in F \) and \( q \in F^d \) such that \( \hat{x} = p \) and \( \langle p, q \rangle \in f^k_m \). But then \( \langle x, q \rangle \in \hat{\circ} f^k_m = Tr^k_m \). Hence \( x \in \text{Dom}(Tr^k_m) \). \( \blacksquare \)

We say that \( f \) is well-parametrized iff \( f : F \to F^d \) and the following holds: if \( x \in \text{Dom}(f) \) is an accumulation point of \( \text{Dom}(f) \), then \( f \) is differentiable at \( x \) and its derivate at \( x \) is of Minkowski-length 1, i.e. \( \mu(f'(x)) = 1 \). Assume \( \mathfrak{S} = \mathfrak{R} \). Then the curve \( f \) is well-parametrized according to Minkowski-length, i.e. for all \( x, y \in F \), if \( \langle x, y \rangle \subseteq \text{Dom}(f) \), the Minkowski-length of \( f \) restricted to \( \langle x, y \rangle \) is \( y - x \). (By Minkowski-length of a curve we mean length according to Minkowski-metric, e.g., in the sense of Wald [25, p. 43, (3.3.7)]). \textbf{Proper time} or \textbf{wristwatch time} is defined as the Minkowski-length of a time-like curve, cf. e.g., Wald [25, p. 44, (3.3.8)], Taylor-Wheeler [23, 1-1-2] or d’Inverno [5, p. 112, (8.14)]. Thus a curve defined on a subset of \( \mathfrak{R} \) is well-parametrized iff it is parametrized according to proper time, or wristwatch-time. (Cf. e.g., [5, p. 112, (8.16)].)

The next proposition states that life-curves of accelerated observers in models of \textbf{AccRel} are well-parametrized. This implies that accelerated clocks behave as expected in models of \textbf{AccRel}. Remark 5.3 after the proposition will state a kind of “completeness theorem” for life-curves of accelerated observers, much in the spirit of Remark 2.3.

\textbf{Proposition 5.2}. Assume \textbf{AccRel} and \( d > 2 \). Let \( m \in \text{IOb} \) and \( k \in \text{Ob} \). Then \( Tr^k_m \) is well-parametrized and definable.

\textbf{proof}. Let \( m \in \text{IOb} \), \( k \in \text{Ob} \). Then \( Tr^k_m \) is definable by its definition. Furthermore, \( f^k_m : F^d \to F^d \) and \( Tr^k_m : F \to F^d \) by (iii) of Proposition 5.1. Let \( x \in \text{Dom}(Tr^k_m) \) be an accumulation point of \( \text{Dom}(Tr^k_m) \). We would like to prove that \( Tr^k_m \) is differentiable at \( x \) and its derivate at \( x \) is of Minkowski-length 1. \( \hat{x} \in tr_k(k) \) by (vii) of Proposition 5.1. Thus, by \textbf{AxAcc}, there is a co-moving inertial observer of \( k \) at \( \hat{x} \). By Proposition 4.3, we can assume that \( m \) is a co-moving inertial observer of \( k \) at \( \hat{x} \), i.e. \( m \vartriangleright_{\hat{x}} k \), because of the following three statements. By (v) of Proposition 5.1 for every \( h \in \text{IOb} \), either of \( Tr^k_m \) and \( Tr^k_h \) can be obtained from the other by composing the other by a world-view transformation between inertial observers. By Theorem 2.2, world-view transformations between inertial observers are Poincaré-transformations. Poincaré-transformations are affine and preserve the Minkowski-distance.

Now, assume that \( m \) is a co-moving inertial observer of \( k \) at \( \hat{x} \). Then \( f^k_m(\hat{x}) = \hat{x} \), \( z\hat{1} = \hat{z} \) and \( Tr^k_m(z) = f^k_m(z) \) for every \( z \in \text{Dom}(Tr^k_m) \). Therefore
\[
\forall y \in \text{Dom}(Tr^k_m) \quad |Tr^k_m(y) - Tr^k_m(x) - (y - x)\hat{1}| = |f^k_m(y) - y|.
\] (40)
Since \( \text{Dom}(f^k_m) \subseteq Cd(k) \) and \( \hat{y} \in \text{Dom}(f^k_m) \) if \( y \in \text{Dom}(Tr^k_m) \), we have that for all \( \delta \in F^+ \),
\[
\forall y \in \text{Dom}(Tr^k_m) \quad |y - x| < \delta \implies \hat{y} \in B_\delta(\hat{x}) \cap Cd(k).
\] (41)
Let \( \varepsilon \in F^+ \) be fixed. Since \( m \vartriangleright_{\hat{x}} k \) and \( f^k_m : F^d \to F^d \), there is a \( \delta \in F^+ \) such that
\[
\forall p \in B_\delta(\hat{x}) \cap Cd(k) \quad |p - f^k_m(p)| \leq \varepsilon|p - \hat{x}|.
\] (42)
Let such a \( \delta \) be fixed. By (41), (42) and the fact that \( |\hat{y} - \hat{x}| = |y - x| \), we have that
\[
\forall y \in \text{Dom}(Tr^k_m) \quad |y - x| < \delta \implies |\hat{y} - f^k_m(\hat{y})| \leq \varepsilon|y - x|.
\] (43)
By this and (40), we have
\[
\forall y \in \text{Dom}(Tr^k_m) \quad |y - x| < \delta \implies |Tr^k_m(y) - Tr^k_m(x) - (y - x)\hat{1}| \leq \varepsilon|y - x|.
\] (44)
Thus \( (Tr^k_m)'(x) = \hat{1} \). This completes the proof since \( \mu(\hat{1}) = 1 \). \( \blacksquare \)
Remark 5.3. Well parametrized curves are exactly the life-curves of accelerated observers, in models of $\text{AccRel}_0$, as follows. Let $\mathfrak{F}$ be an Euclidean ordered field and let $f : F \to F^d$ be well-parametrized. Then there are a model $\mathfrak{M}$ of $\text{AccRel}_0$, $m \in \text{Ob}$ and $k \in \text{Ob}$ such that $T^k_m = f$ and the ordered field reduct of $\mathfrak{M}$ is $\mathfrak{F}$. Recall that if $\mathfrak{F} = \mathfrak{R}$, then this $\mathfrak{M}$ is a model of $\text{AccRel}$. This is not difficult to prove by using the methods of the present paper.

We say that $p \in F^d$ is vertical iff $p \in \bar{t}$.

Lemma 5.4. Let $f : F \to F^d$ be well-parametrized. Then (i) and (ii) below hold.

(i) Let $x \in \text{Dom}(f)$ be an accumulation point of $\text{Dom}(f)$. Then $f_i$ is differentiable at $x$ and $|f'_i(x)| \geq 1$. Furthermore, $|f'_i(x)| = 1$ iff $f_i(x)$ is vertical.

(ii) Assume IND and that $f$ is definable. Let $[a, b] \subseteq \text{Dom}(f)$. Then $f_i$ is increasing or decreasing on $[a, b]$. If $f_i$ is increasing on $[a, b]$ and $a \neq b$, then $f'_i(x) \geq 1$ for all $x \in [a, b]$.

proof. Let $f$ be well-parametrized.

To prove (i), let $x \in \text{Dom}(f)$ be an accumulation point of $\text{Dom}(f)$. Then $f'(x)$ is of Minkowski-length 1. By Proposition 13, $f_i$ is differentiable at $x$ and $f'_i(x) = f'(x)_i$. Now, (i) follows from the fact that the absolute value of the time component of a vector of Minkowski-length 1 is always greater than 1 and it is 1 iff the vector is vertical.

To prove (ii), assume IND and that $f$ is definable. Let $[a, b] \subseteq \text{Dom}(f)$. From (i), we have $f'_i(x) \neq 0$ for all $x \in [a, b]$. Thus, by Rolle’s theorem, $f_i$ is injective on $[a, b]$. Thus, by Bolzano’s theorem, $f_i$ is increasing or decreasing on $[a, b]$ since $f_i$ is continuous and injective on $[a, b]$. Assume that $f_i$ is increasing on $[a, b]$ and $a \neq b$. Then $f'_i(x) \geq 0$ for all $x \in [a, b]$ by the definition of the derivate. Hence, by (i), $f'_i(x) \geq 1$ for all $x \in [a, b]$.

Theorem 5.5. Assume IND. Let $f : F \to F^d$ be definable, well-parametrized and $[a, b] \subseteq \text{Dom}(f)$. Then (i) and (ii) below hold.

(i) $b - a \leq |f_i(b) - f_i(a)|$.

(ii) If $(f(x)_s \neq f(a)_s$ for an $x \in [a, b]$, then $b - a < |f_i(b) - f_i(a)|$.

proof. Let $f : F \to F^d$ be definable, well-parametrized and $[a, b] \subseteq \text{Dom}(f)$. We can assume that $a \neq b$. For every $i \leq d$, $f_i$ is definable and differentiable on $[a, b]$ by Proposition 4.3. Then, by the Main Value Theorem, there is an $s \in (a, b)$ such that $f'_i(s) = \frac{f_i(b) - f_i(a)}{b-a}$. By (i) of Lemma 5.4, we have $1 \leq |f'_i(s)|$. But then, $b - a \leq |f_i(b) - f_i(a)|$. This completes the proof of (i).

To prove (ii), let $x \in [a, b]$ be such that $f(x)_s \neq f(a)_s$. Let $1 < i \leq d$ be such that $f_i(x) \neq f_i(a)$. Then, by the Main Value Theorem, there is an $y \in (a, b)$ such that $f'_i(y) = \frac{f_i(x) - f_i(a)}{x-a} \neq 0$. Thus $f_i(y)$ is not vertical. Therefore, by (i) of Lemma 5.4, we have $1 < |f'_i(y)|$. Thus, by the definition of the derivate, there is a $z \in (y, b)$ such that $1 < \frac{|f_i(z) - f_i(y)|}{z-y}$. Hence we have

$$z - y < |f_i(z) - f_i(y)|.$$ (45)

Let us note that $a < y < z < b$. By applying (i) to $[a, y]$ and $[z, b]$, respectively, we get

$$y - a \leq |f_i(y) - f_i(a)| \quad \text{and} \quad b - z \leq |f_i(b) - f_i(z)|.$$ (46)

$f_i$ is increasing or decreasing on $[a, b]$ by (ii) of Lemma 5.4. Thus $f_i(a) < f_i(y) < f_i(z) < f_i(b)$ or $f_i(a) > f_i(y) > f_i(z) > f_i(b)$. Now, by adding up the last three inequalities, we get

$$b - a < |f_i(b) - f_i(a)|.$$
Let \( a \in F^d \). For convenience, we introduce the following notation: \( a^+ := a \) if \( a_t \geq 0 \) and \( a^+ := -a \) if \( a_t < 0 \). A set \( H \subseteq F^d \) is called **twin-paradoxical** iff \( \bar{1} \in H \), \( o \not\in H \), slope\((p) < 1 \) if \( p \in H \), for all \( p \in F^d \) if slope\((p) < 1 \), then there is a \( \lambda \in F \) such that \( \lambda p \in H \), and for all distinct \( p, q, r \in H \) and for all \( \lambda, \mu \in F^+, \lambda^+ + \mu q^+ \) implies that \( \lambda + \mu < 1 \).

A positive answer to the following question would also provide a positive answer to Question 5.8, cf. [21] §3.

**Question 5.6.** Assume IND. Let \( f : F \to F^d \) be definable such that \( f \) is differentiable on \([a, b]\) and \( f(a), f(b) \in \bar{t} \). Furthermore, let the set \( \{ f'(x) : x \in [a, b] \} \) be a subset of a twin-paradoxical set. Are then (i) and (ii) below true?

(i) \( b - a \leq |f_t(b) - f_t(a)| \).

(ii) If \( f(x)_s \neq f(a)_s \) for an \( x \in [a, b] \), then \( b - a < |f_t(b) - f_t(a)| \).

**Theorem 5.7.** Assume IND. Let \( f, g : F \to F^d \) be definable and well-parametrized. Let \( [a, b] \subseteq Dom(f) \) and \([a', b'] \subseteq Dom(g)\) be such that \( \{ f(r) : r \in [a, b] \} = \{ g(r') : r' \in [a', b'] \} \). Then \( b - a = b' - a' \).

**proof.** By (ii) of Lemma 5.4, \( f_t \) is increasing or decreasing on \([a, b]\) and so is \( g_t \) on \([a', b']\).

We can assume that \( Dom(f) = [a, b] \), \( Dom(g) = [a', b'] \) and that \( f_t \) and \( g_t \) are increasing on \([a, b]\) and \([a', b']\), respectively.\(^8\) Then \( Rng(f) = Rng(g) \). Furthermore, \( f \) and \( g \) are injective since \( f_t \) and \( g_t \) are such. Since \( Rng(f) = Rng(g) \) and \( g_t \) is injective, \( f \circ g^{-1} = f_t \circ g_t^{-1} \).

Let \( h := f \circ g^{-1} = f_t \circ g_t^{-1} \). Since \( Rng(f_t) = Rng(g_t) \) and \( f_t \) and \( g_t \) are increasing, \( h \) is an increasing bijection between \([a, b]\) and \([a', b']\). Hence \( h(a) = a' \) and \( h(b) = b' \). We are going to prove that \( b - a = b' - a' \) by proving that there is a \( c \in F \) such that \( h(x) = x + c \) for all \( x \in [a, b] \). We can assume that \( a \neq b \) and \( a' \neq b' \). By Lemma 5.4, \( f_t \) and \( g_t \) are differentiable on \([a, b]\) and \([a', b']\), respectively, and \( f'_t(x) > 0 \) for all \( x \in [a, b] \) and \( g'_t(x') > 0 \) for all \( x' \in [a', b'] \). By (iv) and (v) of Proposition 4.2, \( h = f_t \circ g_t^{-1} \) is also differentiable on \((a, b)\). By \( h = f \circ g^{-1} \), we have \( h(x) = h'(x)g'(h(x)) \) for all \( x \in (a, b) \) by (iv) of Proposition 4.2. Since both \( f'(x) \) and \( g'(h(x)) \) are of Minkowski-length 1 and their time-components are positive\(^9\) for all \( x \in (a, b) \), we conclude that \( h'(x) = 1 \) for all \( x \in (a, b) \). By Proposition 4.9 we get that there is a \( c \in F \) such that \( h(x) = x + c \) for all \( x \in (a, b) \) and thus for all \( x \in [a, b] \) since \( h \) is an increasing bijection between \([a, b]\) and \([a', b']\).

**proof of Theorem 5.1.** Assume AccRel and \( d > 2 \). Let \( m \in IOB \) and \( k \in Ob \). Let \( p, q, p', q' \in tr_k(k) \) be such that \( \langle p, p' \rangle, \langle q, q' \rangle \in f^k_m \), \([pq] \subseteq tr_k(k) \) and \([pq'] \not\subseteq tr_m(k) \), cf. Figure 2. Let us abbreviate \( Tr^k_m \) by \( Tr \). We are going to prove that \( |q_t - p_t| < |q'_t - p'_t| \) by applying Theorem 5.5 to \( Tr \) and \([p_t, q_t] \). By Proposition 5.2

\[ Tr : F \to F^d \]

is well-parametrized and definable.\(^{47}\)

By \textbf{AxSelf}, \( p, q, p', q' \in \bar{t} \). By \( \bar{p}_t = p \), \( \bar{q}_t = q \), \( \langle p, p' \rangle \in f^k_m \), and by \( Tr = \sim \circ f^k_m \),

\[ Tr(p_t) = p' \quad \text{and} \quad Tr(q_t) = q'. \]

By \( p, q \in tr_k(k) \) and \( \langle p, p' \rangle, \langle q, q' \rangle \in f^k_m \), we have that \( p', q' \in tr_m(k) \). Thus, by \([pq'] \not\subseteq tr_m(k) \), we have that \( p' \neq q' \). Hence, by \( 48 \), \( p_t \neq q_t \). We can assume that \( p_t < q_t \). By (viii) of Proposition 5.1 \( \{ \bar{x} : x \in Dom(Tr) \} = tr_k(k) \). Since \([pq] \subseteq tr_k(k) \),

\[ [p_t, q_t] \subseteq Dom(Tr). \]

\(^8\)It can be assumed that \( f_t \) is increasing on \([a, b]\) because the assumptions of the theorem remain true when \( f \) and \([a, b]\) are replaced by \(-Id \circ f \) and \([-b, -a]\), respectively, and \( f_t \) is decreasing on \([a, b]\) iff \((-Id \circ f)_t \) is increasing on \([-b, -a]\).

\(^9\)i.e. \( f'_t(x) > 0 \) and \( g'_t(h(x)) > 0 \)
By (i) of Lemma 5.3 (17) and (19), we have that $Tr_t$ is differentiable on $[p_t, q_t]$, thus it is continuous on $[p_t, q_t]$. Let $x' \in [p'[q'] \subseteq \bar{t}$ be such that $x' \not\in tr_m(k)$. By Bolzano’s theorem and (18), there is an $x \in [p_t, q_t]$ such that $Tr_t(x) = x'$. Let such an $x$ be fixed. $Tr(x) \in tr_m(k)$ since $Rng(Tr) \subseteq tr_m(k)$ by (vii) of Proposition 5.1. But then $Tr(x) \neq x'$. Hence $Tr(x) \not\in \bar{t}$. Thus 

$$x \in [p_t, q_t] \text{  and  } Tr(x)_s \neq Tr(p_t)_s$$

Since $Tr(p_t) = p' \not\in \bar{t}$. Now, by (17)–(19) above, we can apply (ii) of Theorem 5.3 to $Tr$ and $[p_t, q_t]$, and we get that $|q_t - p_t| < |Tr_t(q_t) - Tr_t(p_t)| = |q'_t - p'_t|$. 

\section*{Proof of Theorem 5.7} Assume AccRel and $d > 2$. Let $k$ and $m$ be observers. Let $p, q \in tr_k(k), p', q' \in tr_m(m)$ be such that $\emptyset \not\in \{ev_{\emptyset}(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\}$, cf. the right hand side of Figure 2. Thus $[pq] \subseteq Cd(k)$ and $[p'q'] \subseteq Cd(m)$. By AxSelf-$t$, $tr_k(k) = Cd(k) \cap \bar{t}$ and $tr_m(m) = Cd(m) \cap t$. Therefore $[pq] \subseteq tr_k(k) \subseteq \bar{t}$ and $[p'q'] \subseteq tr_m(m) \subseteq \bar{t}$. We can assume that $p_t \leq q_t$ and $p'_t \leq q'_t$. Let $h \in IOb$. We are going to prove that $|q_t - p_t| = |q'_t - p'_t|$, by applying Theorem 5.7 as follows: let $[a, b] := [p_t, q_t], [a', b'] := [p'_t, q'_t], f := \text{Tr}_k$ and $g := \text{Tr}_m$. By (vii) of Proposition 5.1 by $[pq] \subseteq tr_k(k)$ and by $[p'q'] \subseteq tr_m(m)$, we conclude that $[a, b] \subseteq Dom(f)$ and $[a', b'] \subseteq Dom(g)$. By Proposition 5.2 $f$ and $g$ are well-parametrized and definable. We have $\{f(r) : r \in [a, b]\} = \{g(r') : r' \in [a', b']\}$ since $\{ev_k(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\}$. Thus, by Theorem 5.7 we conclude that $b - a = b' - a'$. Thus $|q_t - p_t| = |q'_t - p'_t|$ and this is what we wanted to prove.

\textbf{Proofs of Theorems 5.5 and 3.7} We will construct three models. Let $\mathcal{F} = (\mathbb{F}; +, \cdot, \leq)$ be an Euclidean ordered field different from $\mathfrak{R}$. For every $p \in F^d$, let $m_p : F^d \rightarrow F^d$ denote the translation by vector $p$, i.e. $m_p : q \mapsto q + p$. $f : F^d \rightarrow F^d$ is called translation-like iff for all $q \in F^d$, there is a $\delta \in F^+$ such that for all $p \in B_{\delta}(q)$, $f(p) = m_{f(q)} - q(p)$ and for all $p, q \in F^d, f(p) = f(q)$ and $p, q \in \emptyset$ imply that $q \in \emptyset$. Let $k : F^d \rightarrow F^d$ be translation-like. First we construct a model $\mathfrak{M}_{(\mathcal{F}, \mathcal{R})}$ of AccRel and (i) and (ii) of Theorem 3.7 for $\mathcal{F}$ and $k$, which will be a model of (iii) and (iv) of Theorem 3.7 if $k$ is a bijection. We will show that $Tp$ is false in $\mathfrak{M}_{(\mathcal{F}, \mathcal{R})}$. Then we will choose $\mathcal{F}$ and $k$ appropriately to get the desired models in which $Ddpe$ is false, too.

Let the ordered field reduct of $\mathfrak{M}_{(\mathcal{F}, \mathcal{R})}$ be $\mathcal{F}$. Let $\{I_1, I_2, I_3, I_4, I_5\}$ be a partition$^{10}$ of $F$ such that every $I_i$ is open, $x \in I_2 \iff x + 1 \in I_3 \iff x + 2 \in I_4$ and for all $y \in I_1$ and $z \in I_j, y \leq z \iff i \leq j$. Such a partition can easily be constructed.$^{11}$ Let

$$k'(p) := \begin{cases} p & \text{if } p_t \in I_1 \cup I_5, \\ p - \hat{1} & \text{if } p_t \in I_1, \\ p + \hat{1} & \text{if } p_t \in I_3, \\ p + \langle 0, 1, 0, \ldots, 0 \rangle & \text{if } p_t \in I_2 \end{cases}$$

for every $p \in F^d$, cf. Figure 4. It is easy to see that $k'$ is a translation-like bijection. Let $IOb := \{m_p : p \in F^d\}, Ob := IOb \cup \{k, k'\}, Ph := \{l \in \text{Lines} : \text{slope}(l) = 1\}$ and $B := Ob \cup Ph$. Recall that $o := \langle 0, \ldots, 0 \rangle$ is the origin. First we give the world-view of $m_o$ then we give the world-view of an arbitrary observer $h$ by giving the world-view transformation between $h$ and $m_o$. Let $tr_{m_o}(ph) := ph$ and $tr_{m_o}(h) := \{h(x) : x \in \bar{t}\}$ for all $ph \in Ph$ and $h \in Ob$. And let $ev_{m_o}(p) := \{b \in B : p \in tr_{m_o}(b)\}$ for all $p \in F^d$. Let $f_{m_o}^p := h$ for all $h \in Ob$. From these world-view transformations, we can obtain the world-view of each observer $h$ in the following way: $ev_h(p) := ev_{m_o}(h(p))$ for all $p \in F^d$.

$^{10}$I.e. $I_i$’s are disjoint and $F = I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5$.

$^{11}$Let $H \subseteq F$ be a non-empty bounded set that does not have a supremum. Let $I_1 := \{x \in F : \exists h \in H, \ x < h\}, I_2 := \{x + 1 \in F : x \in I_1\} \setminus I_1, I_5 := \{x + 1 \in F : x \in I_2\}, I_4 := \{x + 1 \in F : x \in I_3\}$ and $I_5 := F \setminus (I_1 \cup I_2 \cup I_3 \cup I_4)$. 
And from the world-views, we can obtain the W relation as follows: for all $h \in \text{Ob}$, $b \in B$ and $p \in F^d$, let $W(h, b, p)$ iff $b \in ev_h(p)$. Thus we are given the model $M_{(\mathfrak{g}, k)}$. We note that $f^h_m = m \circ h^{-1}$ and $m_{h(q) - q} \succ q$ for all $m, h \in \text{Ob}$ and $q \in F^d$. It is easy to check that the axioms of $\text{AccRel}_q$ and (i) and (ii) of Theorem 3.7 are true in $M_{(\mathfrak{g}, k)}$ and that if $k$ is a bijection, then (iii) and (iv) of Theorem 3.7 are also true in $M_{(\mathfrak{g}, k)}$. Let $p, q \in \tilde{t}$ be such that $p_t \in I_1$, $q_t \in I_4$; and let $p' := k'(p) = p$, $q' := k'(q) = q - \hat{1}$ and $m := m_o$. It is easy to check that $T_p$ is false in $M_{(\mathfrak{g}, k)}$ for $k'$, $m$, $p$, $q'$ and $q'$, i.e. $p, q \in tr_{k'}(k')$, $p', q' \in tr_m(m)$, $\langle p, p' \rangle, \langle q, q' \rangle \in f^e_{m'}$, $[pq] \subseteq tr_{k'}(k')$, $[p'q'] \not\subseteq tr_m(k')$ and $|q_t - p_t| \neq |q'_t - p'_t|$, cf. Figure 4.

To construct the first model, let $\mathfrak{g}$ be an arbitrary Euclidean ordered field different from $\mathfrak{R}$ and let $\{I_1, I_2\}$ be a partition of $F$ such that for all $x \in I_1$ and $y \in I_2$, $x < y$. Let

$$k(p) := \begin{cases} p \\ p - \hat{1} \end{cases} \text{ if } p_t \in I_1,$$

$$\text{if } p_t \in I_2 \tag{52}$$

for every $p \in F^d$, cf. Figure 5. It is easy to see that $k$ is translation-like. Let $p, q \in \tilde{t}$ be such that $p_t, p_t + 1 \in I_1$ and $q_t, q_t - 1 \in I_2$; and let $p' := k(p) = p$, $q' := k(q) = q - \hat{1}$ and $m := m_o$. It is also easy to check that $D\text{dpe}$ is false in $M_{(\mathfrak{g}, k)}$ for $k$, $m$, $p$, $q$, $p'$ and $q'$, i.e. $p, q \in tr_k(k)$, $p', q' \in tr_m(m)$, $\emptyset \not\subseteq \{ev_k(r) : r \in [pq]\} = \{ev_m(r') : r' \in [p'q']\}$ and $|q_t - p_t| \neq |q'_t - p'_t|$, cf. Figure 5. This completes the proof of Theorem 3.5.

To construct the second model, let $\mathfrak{g}$ be an arbitrary non-Archimedean Euclidean ordered field. Let $a \sim b$ if $a, b \in F$ and $a - b$ is infinitesimally small. It is easy to see that $\sim$ is an equivalence relation. Let us choose an element from every equivalence class of $\sim$ and let $\tilde{a}$ denote the chosen element equivalent with $a \in F$. Let $k(p) := \langle p_t + \tilde{p}_t, p_a \rangle$ for every $p \in F^d$, cf. Figure 5. It is easy to see that $k$ is a translation-like bijection. Let $p := o$, $q := \hat{1}$, $p' := k(p) = \langle \tilde{0}, 0, \ldots, 0 \rangle$, $q' := k(q) = \langle 1 + \tilde{1}, 0, \ldots, 0 \rangle$ and $m := m_o$. It is also easy to check that $D\text{dpe}$ is false in $M_{(\mathfrak{g}, k)}$ for $k$, $m$, $p$, $q$, $p'$ and $q'$, cf. Figure 5.

To construct the third model, let $\mathfrak{g}$ be an arbitrary countable Archimedean Euclidean ordered field and let $k(p) := \langle f(p_t), p_a \rangle$ for every $p \in F^d$ where $f : F \to F$ is constructed as follows, cf. Figures 5, 6. We can assume that $\mathfrak{g}$ is a subfield of $\mathfrak{R}$ by Theorem 1 in §VIII].

Let $a$ be a real number that is not an element of $F$. Let us enumerate the elements of $[a, a + 2] \cap F$ and denote the $i$-th element with $r_i$. First we cover $[a, a + 2] \cap F$ with
first model, $\mathcal{F} \neq \mathcal{R}$ is Euclidean, \textit{Ddpe} is false: $|p_t - q_t| \neq |p'_t - q'_t|$

second model, $\mathcal{F}$ is non-Archimedean, \textit{Ddpe} is false: $|p_t - q_t| \neq |p'_t - q'_t|$

third model, $\mathcal{F}$ is countable Archimedean, \textit{Ddpe} is false: $|p_t - q_t| \neq |p'_t - q'_t|$

Figure 5. for the proofs of Theorems 3.5 and 3.7.

infinitely many disjoint subintervals of $[a, a + 2]$ such that the sum of their length is 1, the length of each interval is in $F$ and the distance of the left endpoint of each interval from $a$ is also in $F$. We are going to construct this covering by recursion. In the $i$-th step, we will use only finitely many new intervals such that the sum of their length is $1/2^i$. In the first step, we cover $r_1$ with an interval of length $1/2$. Suppose that we have covered $r_i$ for each $i < n$. Since we have used only finitely many intervals yet, we can cover $r_n$ with an
interval that is not longer than $1/2^n$. Since $\sum_{i=1}^{\infty} 1/2^i < 1$, it is easy to see that we can choose finitely many other subintervals of $[a, a + 2]$ to be added to this interval such that the sum of their length is $1/2^n$. We are given the covering of $[a, a + 2]$. Let us enumerate these intervals. Let $I_i$ be the $i$-th interval, $d_i$ be the length of $I_i$, $d_0 := 0$ and $a_i \geq 0$ the distance of $a$ and the left endpoint of $I_i$. $\sum_{i=1}^{\infty} d_i = 1$ since $\sum_{i=1}^{\infty} 1/2^i = 1$. Let

$$f(x) := \begin{cases} 
  x & \text{if } x < a, \\
  x - 1 & \text{if } a + 2 \leq x, \\
  x - a_n + \sum_{i=0}^{n-1} d_i & \text{if } x \in I_n
\end{cases}$$

for all $x \in F$, cf. Figure 6. It is easy to see that $k$ is a translation-like bijection. Let $p, q \in F^d$ be such that $p_t < a$ and $a + 2 < q_t$; and let $p' := k(p) = p$, $q' := k(q) = q - \hat{1}$ and $m := m_o$. It is also easy to check that $\text{Ddpe}$ is false in $\mathfrak{M}_{(\mathfrak{F}, k)}$ for $k$, $m$, $p$, $q$, $p'$ and $q'$, cf. Figure 5.

proof of Corollary 3.6. Let $\mathfrak{F}$ be a field elementarily equivalent to $\mathfrak{R}$, i.e. such that all FOL-formulas valid in $\mathfrak{R}$ are valid in $\mathfrak{F}$, too. Assume that $\mathfrak{F}$ is not isomorphic to $\mathfrak{R}$. E.g. the field of the real algebraic numbers is such. Let $\mathfrak{M}$ be a model of $\text{AccRel}_0$ with field-reduct $\mathfrak{F}$ in which neither $\text{Tp}$ nor $\text{Ddpe}$ is true. Such an $\mathfrak{M}$ exists by Theorem 3.5. Since $\mathfrak{M} \models \text{Th}(\mathfrak{R})$ by assumption, this shows $\text{Th}(\mathfrak{R}) \cup \text{AccRel}_0 \models \text{Tp} \lor \text{Ddpe}$.

In a subsequent paper, we will discuss how the present methods and in particular $\text{AccRel}$ and $\text{IND}$ can be used for introducing gravity via Einstein’s equivalence principle and for proving that gravity “causes time run slow” (known as gravitational time dilation). In this connection we would like to point out that it is explained in Misner et al. [16, pp. 172-173, ...
327-332] that the theory of accelerated observers (in flat space-time!) is a rather useful first step in building up general relativity by using the methods of that book.

APPENDIX

A FOL-formula expressing $AxSelf^-$ is:
\[
\forall m \forall p \quad \text{Ob}(m) \land F(p_1) \land \ldots \land F(p_d) \implies \\
\left[ W(m, m, p) \iff \exists b \ B(b) \land W(m, b, p) \land p_2 = 0 \land \ldots \land p_d = 0 \right].
\] (54)

A FOL-formula expressing $AxPh$ is:
\[
\forall m \forall p \forall q \quad \text{IOb}(m) \land F(p_1) \land F(q_1) \land \ldots \land F(p_d) \land F(q_d) \implies \\
\left[ (p_1 - q_1)^2 = (p_2 - q_2)^2 + \ldots + (p_d - q_d)^2 \iff \\
\exists ph \ F(ph) \land W(m, ph, p) \land W(m, ph, q) \land \\
\forall ph \forall \lambda \ F(ph) \land F(\lambda) \land W(m, ph, p) \land W(m, ph, q) \implies W(m, ph, q + \lambda(p - q)) \right].
\] (55)

A FOL-formula expressing $AxEv$ is:
\[
\forall m \forall k \forall p \quad \text{IOb}(m) \land \text{IOb}(k) \land F(p_1) \land \ldots \land F(p_d) \implies \exists q \\
F(q_1) \land \ldots \land F(q_d) \land (\forall b \ B(b) \implies [W(m, b, p) \iff W(k, b, q)]). 
\] (56)

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