A LOWER BOUND ON THE ENTRIES OF THE PRINCIPAL EIGENVECTOR OF A GRAPH

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Abstract. We obtain a lower bound on each entry of the principal eigenvector of a non-regular connected graph.

1. Introduction

The theory of graph spectra, in whose earliest annals we find such illustrious names as Hoffman, Bose, Seidel, and Fiedler, has by now attained a fairly mature stage. Recent expositions of the theory may be found in the books [3, 6, 7, 9].

On the other hand, while the theory of graph eigenvectors may be already out of its infancy, it is still very much in a state of toddlerhood. The purpose of the present note is to make a modest contribution to one of the basic problems of this theory - the description of the entries of the principal eigenvector of a non-regular graph.

2. The problem

Let \( G \) be a connected graph on \( n \) vertices with adjacency matrix \( A \in \mathbb{R}^{n \times n} \). The following facts are widely known (and may be found in each of the references mentioned above):

- \( A \) is an irreducible nonnegative matrix.
- The spectral radius \( \rho(G) \) of \( A \) is a simple eigenvalue.
- The eigenvector \( x \in \mathbb{R}^n \) corresponding to \( \rho \) is positive entrywise.

We shall refer to \( \rho(G) \) as the spectral radius of \( G \) and when the context is clear, denote simply \( \rho = \rho(G) \). The vector \( x \) will be referred to as the principal eigenvector of \( G \). An alternative name, which we shall not use here, would be the Perron vector.

It is also very well known that:

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If $G$ is regular, then all the entries of $x$ are equal.

Papendieck and Recht in [11] were the first to study the problem of estimating the entries of $x$ in the case that $G$ is non-regular. Before presenting their result, we make an assumption which will be sustained throughout the rest of the note:

**Assumption 1.** The vector $x$ is normalized so that $\sum_{i=1}^{n} x_i^2 = 1$.

**Theorem 2.** [11] Let $G$ be a connected graph with principal eigenvector $x$. Let $x_{\text{max}}$ be the largest entry of $x$. Then

$$x_{\text{max}} \leq \frac{1}{\sqrt{2}}$$

Equality is attained if and only if $G = K_{1,n-1}$ is the star on $n$ vertices.

In fact, Papendieck and Recht’s full result is more general, holding for every $p$-norm ($p \in [1,\infty]$) and depending also on $\rho$. However, in the case of interest to us, $p = 2$, it reduces to $\frac{1}{\sqrt{2}}$.

### 3. Known bounds on $x$

Let us introduce some more notation: denote the degree of the $i$th vertex of $G$ by $d_i$. The subgraph of $G$ obtained by deleting the $i$th vertex (and all edges incident on it) will be denoted as $G_{(i)}$. The spectral radius of $G_{(i)}$ will be denoted by $\rho_i$. Note that since $G$ is connected, we have by [2, Corollary 2.1.5(b)]:

$$\rho > \rho_i.$$  

Cioabă and Gregory [5] have generalized Theorem 2 to give upper bounds on every entry of $x$.

**Theorem 3.** [5, Theorem 3.2] Let $G$ be a connected graph with principal eigenvector $x$. Then for every $1 \leq i \leq n$:

$$x_i \leq \frac{1}{\sqrt{1 + \frac{\rho^2}{d_i}}}.$$  

Equality is attained if and only if $x_i = x_{\text{max}}$, $d_i = n - 1$, and $G_{(i)}$ is regular.

A natural counterpart to Theorem 3 is given by Li, Wang, and Van Mieghem [8]:

**Theorem 4.** [8] Let $G$ be a connected graph with principal eigenvector $x$. Then for every $1 \leq i \leq n$:

$$x_i \geq \sqrt{\frac{\rho - \rho_i}{2\rho}}.$$
We remark that additional bounds for $x_{\text{max}}$ and $x_{\text{min}}$ can be found in [5, 10]. There are also in the literature results of a different kind where $\sum_{i \in S} x_i^2$ is estimated from above for subsets $S \subseteq V(G)$ which induce either empty [4, 8] or, more generally, regular subgraphs [1]. When $S$ is a singleton set such bounds reduce to an analogue of Theorem 3.

4. A NEW LOWER BOUND

Our new result is another lower bound on $x_i$, which is often, but not always, better than Theorem 4.

**Theorem 5.** Let $G$ be a connected graph with principal eigenvector $x$. Then for every $1 \leq i \leq n$:

$$x_i \geq \frac{1}{\sqrt{1 + \frac{d_i}{(\rho - \rho_1)^2}}}.$$

For the proof we need a lemma:

**Lemma 6.** [12, p. 148] Let the Hermitian matrix $A$ be partitioned as

$$A = \begin{bmatrix} a & b^T \\ b & B \end{bmatrix}$$

and let $x$ be a unit eigenvector of $A$ corresponding to the eigenvalue $\lambda$. If $\lambda$ is not an eigenvalue of $B$, then

$$|x_1|^2 = \frac{1}{1 + ||(\lambda I - B)^{-1}b||^2}.$$

**Proof of Theorem 5.** Without loss of generality, let $i = 1$ and suppose that $A$ is partitioned as in (1). Then $B$ is the adjacency matrix of $G_{(1)}$. As observed before: $\rho > \rho_1$. This means that $\rho$ is not an eigenvalue of $B$ and the hypothesis of Lemma 6 is satisfied. Thus we have

$$|x_1|^2 = \frac{1}{1 + ||(\rho I - B)^{-1}b||^2} \geq \frac{1}{1 + ||(\rho I - B)^{-1}||^2||b||^2},$$

where $||(\rho I - B)^{-1}||$ is the 2-norm, which is known to be equal to

$$\lambda_{\max}((\rho I - B)^{-1}) = \frac{1}{\lambda_{\min}(\rho I - B)} = \frac{1}{\rho - \lambda_{\max}(B)} = \frac{1}{\rho - \rho_1}.$$

Thus, since $||b||^2 = d_1$ we obtain

$$|x_1|^2 \geq \frac{1}{1 + \frac{d_1}{(\rho - \rho_1)^2}}.$$

\[\square\]
5. An example

Consider the following graph:

In the table we list the actual values of the principal eigenvector $x$ and the bounds given by all three theorems discussed.

| vertex name | vertex degree | Theorem 4   | Theorem 5   | $x_i$     | Theorem 3 |
|-------------|---------------|-------------|-------------|-----------|-----------|
| b           | 6             | 0.39725     | 0.45901     | 0.49917   | 0.5213    |
| c           | 6             | 0.374       | 0.41636     | 0.48264   | 0.5213    |
| g           | 4             | 0.29584     | 0.33114     | 0.39818   | 0.44634   |
| a           | 3             | 0.18076     | 0.14959     | 0.26109   | 0.39654   |
| e           | 3             | 0.25233     | 0.28276     | 0.34415   | 0.39654   |
| i           | 3             | 0.18904     | 0.16325     | 0.27064   | 0.39654   |
| d           | 2             | 0.17415     | 0.16949     | 0.24485   | 0.33261   |
| f           | 2             | 0.13045     | 0.096049    | 0.18786   | 0.33261   |
| h           | 1             | 0.044799    | 0.016093    | 0.065114  | 0.24198   |

As the table makes clear, Theorems 5 and Theorem 4 are, generally speaking, incomparable. Nevertheless, a rule of thumb may be discerned as to when is one better than the other: Theorem 5 works better for vertices of higher degree and Theorem 4 for vertices of low degree. As vertices $a, e, i$ show, however, this rule of thumb is not perfect.
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