Quantum Brownian Motion in a Periodic Potential and the Multi Channel Kondo Problem

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We study the motion of a particle in a periodic potential with Ohmic dissipation. In $D = 1$ dimension it is well known that there are two phases depending on the dissipation: a localized phase with zero temperature mobility $\mu = 0$ and a fully coherent phase with $\mu$ unaffected by the periodic potential. For $D > 1$, we find that this is also the case for a Bravais lattice. However, for non symmetric lattices, such as the honeycomb lattice and its $D$ dimensional generalization, there is a new intermediate phase with a universal mobility $\mu^*$. We study this intermediate fixed point in perturbatively accessible regimes. In addition, we relate this model to the Toulouse limit of the $D+1$ channel Kondo problem. This mapping allows us to compute $\mu^*$ exactly using results known from conformal field theory. Experimental implications are discussed for resonant tunneling in strongly coupled Coulomb blockade structures and for multi channel Luttinger liquids.

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The quantum mechanics of a particle in a periodic potential coupled to a dissipative environment is a fundamental problem in condensed matter physics [1]. A simple theory based on the Caldeira-Leggett model of Ohmic dissipation was proposed in the mid 1980’s as a possible description of the motion of a heavy charged particle in a metal [2]. In a one dimensional periodic potential it was shown that there are two zero temperature ($T = 0$) phases. For weak friction, the particle diffuses freely as if the periodic potential were absent. When the friction exceeds a critical value, however, the particle is localized in one of the minima of the potential.

Recently there has been renewed interest in this quantum Brownian motion (QBM) model in connection with quantum impurity problems [3] and boundary conformal field theory [4]. It is isomorphic to the problem of tunneling through a barrier in a Luttinger liquid, which is relevant to experiments in quantum wires [5] and tunneling in quantum Hall edge states [6]. Here the “coordinate” of the “particle” is the number of electrons that tunnel past the barrier. The periodic potential arises from the discreteness of the electron’s charge. The Luttinger liquid’s modes play the role of the dissipative bath. The particle’s mobility corresponds to the electrical conductance.

There are often multiple electron channels, due to spin and transverse degrees of freedom. The impurity problem then maps to a multi-dimensional periodic potential. In addition to the extended and localized phases, in two dimensions it has been shown that there can be additional non trivial phases [7,8], which may be accessed by tuning to a resonance. Using a similar analysis, Furusaki and Matveev recently found a similar intermediate phase in a model of resonant tunneling through a Coulomb blockade structure [9]. They argued that the resonance fixed point is that of the multi channel Kondo problem.

In this paper we consider the general problem of QBM on periodic lattices. We show that the lattice symmetry plays a crucial role in determining the $T = 0$ phases. For the honeycomb lattice and its $D$ dimensional generalization, there is a $T = 0$ phase described by an intermediate fixed point, which we relate to the $D + 1$ channel Kondo fixed point. Exploiting the mapping onto the Kondo problem, we compute exactly the fixed point mobility and critical exponents by borrowing results from conformal field theory. Integrating out the bath degrees of freedom, the QBM model is described by the Euclidean action

$$S = S_0[\mathbf{r}(\tau)] - \int_{\tau_c} d\tau \sum_{\mathbf{G}} v_{\mathbf{G}} e^{i2\pi\mathbf{G} \cdot \mathbf{r}(\tau)},$$

where $\mathbf{r}$ is the coordinate of the particle and $v_{\mathbf{G}}$ are dimensionless Fourier components of the periodic potential, defined at the reciprocal lattice vectors $\mathbf{G}$. (G is defined so that $\mathbf{G} \cdot \mathbf{R}$ is an integer for any lattice vector $\mathbf{R}$.) The coupling to the dissipative bath is described by

$$S_0[\mathbf{r}(\tau)] = \frac{1}{2} \int d\omega |\omega|^2 |\mathbf{r}(\omega)|^2,$$

where $\tau_c$ is a short time cutoff. The friction is proportional to the coefficient of this term. However, by rescaling $\mathbf{r}$ and $\mathbf{G}$ this coefficient may be fixed. The lattice constant thus controls the strength of the friction. In accordance with Ref. [9], we define the dimensionless parameter $g = |\mathbf{R}_{\text{min}}|^2$, where $|\mathbf{R}_{\text{min}}|$ is the Bravais lattice constant. $g$ is inversely proportional to the friction. A 1+1 dimensional version of this theory has recently been analyzed by Kondev and Henley [10].

Our system may be characterized by the mobility, which describes the average velocity of the particle in response to a uniform applied force. We define the dimensionless mobility $\mu$ as the ratio of the mobility to the “perfect” mobility obtained in the absence of the periodic potential. When $v_G = 0$, $\mu = 1$. $\mu$ may be computed from linear response theory,

$$\mu = \frac{\langle 2\pi/D \rangle}{\omega \rightarrow 0} \langle |\omega|^2 |\mathbf{r}(\omega)|^2 \rangle.$$
The effect of the periodic potential may be analyzed perturbatively in either of two limits. A weak potential may be studied by considering the renormalization group (RG) flows to leading order in \(v_G\),

\[
dv_G/d\ell = (1 - |G|^2)v_G.
\]  

Clearly, if the shortest reciprocal lattice vector satisfies \(|G_{\text{min}}| > 1\), then all \(v_G\) are irrelevant. The “small barrier” limit in which the particle diffuses freely is thus perturbatively stable. On the other hand, if \(|G_{\text{min}}| < 1\), then the system flows to a different strong coupling phase.

When the barriers are large, the particle is localized in one of the minima of the potential with a small probability for tunneling to another. It is then more natural to consider a dual representation in which the partition function is expanded in powers of the “fugacity” of these tunneling events \([8, 9]\). For a Bravais lattice, this may be generated by expanding the dual action,

\[
S = S_0[k(\tau)] - \int \frac{d\tau}{\tau_c} \sum \tau_G e^{2\tau R \cdot k(\tau)}. \tag{5}
\]

\(\tau_G\) may be interpreted as the matrix element for the particle to tunnel between minima connected by a lattice vector \(R\). Equivalently, \(k(\tau)\) describes the particle’s trajectory in momentum space in a potential with the symmetry of the reciprocal lattice. The RG flows to leading order in \(\tau_G\) are then

\[
d\tau_G/d\ell = (1 - |R|^2)\tau_G. \tag{6}
\]

The “large barrier” phase is thus perturbatively stable provided the shortest lattice vector satisfies \(|R_{\text{min}}| > 1\).

For a one dimensional lattice, \(|R_{\text{min}}|^2 = 1/g\) and \(|G_{\text{min}}|^2 = g\). Thus either the small or the large barrier limit is stable, but not both. There are two phases: for \(g < 1\) the system is localized and for \(g > 1\) the system has perfect mobility. Clearly, this is also the case in higher dimensions for a lattice with cubic symmetry.

In contrast, for a triangular lattice, \(|R_{\text{min}}|^2 = 1/g\), but \(|G_{\text{min}}|^2 = 4g/3\). It follows that for \(3/4 < g < 1\), both the small and large barrier limits are stable. There must therefore be an unstable fixed point separating the two phases, as indicated in Fig. 1a. A similar intermediate fixed point occurs in the single barrier problem of a spin 1/2 Luttinger liquid \([7]\).

A perturbative analysis of this fixed point is possible for small \(v\) in the vicinity of \(g = 3/4\). Specifically, consider a model with \(v_G = v\) for the six “nearest neighbor” reciprocal lattice vectors. For \(v > 0\), this produces a potential with minima forming a triangular lattice. For \(g = 3/4(1 + \epsilon)\), the RG flow to second order in \(v\) is

\[
dv/d\ell = -ev + 2v^2. \tag{7}
\]

Provided \(v > 0\) and \(\epsilon > 0\), there is an unstable fixed point \(v^* = \epsilon/2\), with RG eigenvalue \(\epsilon\). The dimensionless mobility at this fixed point is universal, \(\mu^* = 1 - (3\pi^2/2)^2\).

A similar analysis is possible in the dual theory for small \(t\). Since the dual lattice is also triangular, the results are identical, given the substitutions \(r \rightarrow k\), \(G \rightarrow R\), \(g \rightarrow 3/(4g)\) and \(v \rightarrow t\). \(\mu\) is mapped to \(1 - \mu\). For \(g = 1 - \epsilon\), the fixed point is at \(t^* = \epsilon/2\), with an exponent \(\epsilon\) and mobility \(\mu^* = (3\pi^2/2)^2\). For \(g = \sqrt{3}/2\), the theory is self dual, which implies that the fixed point mobility is \(\mu^* = 1/2\). Piecing these results together, we obtain the flow diagram in Fig. 1a.

When \(v < 0\), the minima of the potential described above form a honeycomb lattice. The honeycomb lattice is equivalent to the triangular lattice described above with a two site basis. If the triangular lattice constant is \(1/\sqrt{3}\), then we again have \(|G_{\text{min}}|^2 = 4g/3\). However \(|R_{\text{min}}|^2 = 1/(3g)\) is now shorter. Thus, for \(1/3 < g < 3/4\) both the large and small barrier limits are unstable, so that there must be a stable fixed point describing a new intermediate coupling phase. With \(v\) negative, Eq. (1) may also be viewed as the tight binding representation of a triangular lattice with \(\pi\) flux per plaquette, whose reciprocal is the honeycomb lattice. Our analysis applies to this dual theory as well. Callan et. al. \([8]\) have recently found similar intermediate phases on a square lattice with magnetic flux.

A perturbative analysis is again possible in the large and small barrier limits. For small barriers the fixed point of Eq. (7) is stable for \(v < 0\) and \(\epsilon < 0\). The RG eigenvalue and mobility at the fixed point are the same as above. In the large barrier theory we must keep track of the two site basis of the honeycomb lattice. There are three nearest neighbors \(R\) for each site on the A sublattice. For the B sublattice the nearest neighbors are \(-R\). The tunneling must alternate between the sublattices. This can be incorporated in the dual theory by introducing a spin 1/2 degree of freedom. For nearest neighbor hopping the dual action is then

\[
S = S_0[k] - \int \frac{d\tau}{\tau_c} \sum \frac{1}{R} \left[ \tau R e^{2\tau R \cdot k} + \tau^* e^{-i2\pi R \cdot k} \right], \tag{8}
\]

where \(R\) are among the 3 nearest neighbor lattice vectors.
of sublattice A and $\tau^\pm$ are spin 1/2 operators, $\sigma^\pm/2$. The intermediate fixed point may now be accessed perturbatively for $g = (1/3)(1 + \epsilon)$. We have computed the RG flow equation to order $t^3$,

$$\frac{dt}{d\ell} = ct - 3t^3.$$  \hspace{1cm} (9)

For $\epsilon > 0$ there is a stable fixed point at $t^* = \sqrt{\epsilon/3}$, with RG eigenvalue $2\epsilon$. The fixed point mobility is $\mu = \pi^2 \epsilon$.

The flow diagram for the honeycomb lattice as a function of $g$ is summarized in Fig. 1b. Unlike the cubic and triangular Bravais lattices, the $T = 0$ mobility does not exhibit a discontinuous jump from 0 to 1 as $g$ is increased. Rather, the mobility interpolates smoothly between the two limits in the intermediate phase for $1/3 < g < 3/4$.

Below we show that for $g = 1/2$, the intermediate fixed point is that of the 3 channel Kondo problem. Exploiting this mapping, the mobility for $g = 1/2$ may be computed exactly. The generalization of the above analysis to other lattices and higher dimensions is straightforward and will be given elsewhere. In general, the existence of a stable intermediate phase requires a non symmorphic lattice symmetry, with a vector connecting equivalent sites that is shorter than any lattice translation.

We now relate the stable intermediate fixed point to the multi channel Kondo problem by identifying the lattice symmetry in the Kondo problem. The Hamiltonian of the anisotropic $N$ channel Kondo model is [11].

$$\mathcal{H} = iv_F \sum_{a,s,i} \int dxv_{as}^i \partial_x \psi_{as} + 2\pi v_F \sum_{i,a} J_i S_{imp}^i s_a^i(0),$$

where $a, s, i$ are channel, spin, and space indices, $S_{imp}^i$ is the impurity spin, and $s_a^i(0) = \psi_{as}^i(0) \langle \sigma_{ss}^i/2 \rangle \psi_{as}^i(0)$ is the electronic spin in channel $a$ at $x = 0$. We consider an anisotropic model, characterized by dimensionless couplings $J_z$ and $J_x = J_y = J_{\perp}$. Our analysis closely parallels that of Emery and Kivelson for the two channel Kondo problem [12]. We first bosonize the theory, and then do a rotation in spin space which transforms the $J_z$ interaction. Upon integrating out the degrees of freedom away from $x = 0$, we obtain a theory in terms of the boson fields at the impurity which closely resembles the lattice models studied in this paper. The details of this mapping will be presented in a longer article, however its essence may be understood quite simply.

When $J_{\perp} = 0$ the states of the system may be characterized by the total spin $S_a^z$ in each of the $N$ channels. The possible values of $S_a^z$ form a $N$ dimensional cubic lattice. $J_\perp$ "hops" the system between sites on this lattice. Since $\mathcal{H}$ conserves the total spin of the electrons plus the impurity, the system is constrained to lie on one of two lattice planes with constant $S_{imp}^i + \sum_a S_a^i$ where $S_{imp}^i = \pm 1/2$. For $N = 3$ each lattice plane forms a triangular lattice, as sketched in Fig. 2. Viewed from the $(111)$ direction, the two lattice planes form a "corrugated" honeycomb lattice in which the two triangular sublattices are displaced in the perpendicular direction. For general $N$, the "lattice planes" consist of two interpenetrating $N - 1$ dimensional close packed lattices. For $N = 4$, they form a corrugated diamond lattice.

Now consider QBM on such a lattice described by

$$S = S_0|k| - \int \frac{dt}{\tau_c} \sum_{R_1} \left[ \tau^+ e^{i 2\pi (R_1k_1 + R_2k_2)} + h.c. \right].$$

For $N = 3$, $k$ is a 3 dimensional vector with components $k_1$ and $k_2$ parallel and perpendicular to the lattice plane. $R_i$ are chosen from the 3 nearest neighbor lattice vectors for the honeycomb lattice, and $R_2$ is the perpendicular displacement between the two sublattices. $\tau^\pm$ guarantee that the hopping alternates between the two lattice planes. This model is identical to the multi channel Kondo problem, with $t = J_{\perp} / 2$, provided the lattice constants are chosen to give the appropriate scaling for $J_{\perp}$. For $J_x = 0$, the dimension of $S_a^z(0)$ is 1, so the cubic lattice constant in Fig. 2 is unity. It follows that $|R_i|^2 = 1 - 1/N$. If the lattice constant of the close packed Bravais lattice is $1/\sqrt{3g}$, then $g = 1/2$. Finite $J_z$ may be treated nonperturbatively using bosonization [13], and affects the dimension of $S_a^z(0)$. This leads to a distortion the lattice in the perpendicular direction, $R_2^2 = (1 - NJ_x/2)^2/N$. Note that $R_2 = 0$ for $J_x = 2/N$, so that the perpendicular direction decouples. This is the $N$ channel generalization of the Toulouse limit [12–14]. A central point of this paper is that this limit of the $N$ channel Kondo model is identical to the $g = 1/2$ QBM model on a $N - 1$ dimensional "honeycomb" lattice.

Note that the motion perpendicular to the planes alternates, whereas the motion parallel to the planes does not. This gives rise to a renormalization of $R_2$ (or equivalently $J_z$) but not $R_1$. This may be seen from a RG analysis similar to that of Anderson, Yuval, and Hamann [13,10]. Expressing the flow equations in terms of $J_z$ and $J_{\perp}$, we find to order $J_{\perp}^2$,

$$dJ_z/d\ell = J_{\perp}^2 \left[ 1 - (N/2)J_z \right],$$

$$dJ_{\perp}/d\ell = J_{\perp} J_z \left[ 1 - (N/4)J_z \right] - (N/4)J_{\perp}^2.$$  \hspace{1cm} (10)

The RG flows are shown in Fig. 3. $J_z$ flows towards $2/N$, the Toulouse limit, shown by the dashed line. For $N = 3$ this is the same as the dashed line in Fig. 1b. The intermediate fixed point for the honeycomb lattice with $g = 1/2$ is the same as the fixed point of the 3 channel Kondo problem. Varying $g$ adiabatically connects the multi channel Kondo fixed point to the strong and weak barrier limits described perturbatively in (7) and (9). For
large $N$, the fixed point at $J_2 = J_z = 2/N$ approaches the strong barrier limit, and is perturbatively accessible. Perturbations which break the symmetry between the two sublattices are relevant and act like a magnetic field in the Kondo problem. The system then flows to a fixed point of the lower symmetry lattice.

Conformal field theory allows for an exact description of the multi channel Kondo fixed point. We can thus identify the critical exponents and mobility for the $N - 1$ dimensional generalized honeycomb lattice model for $g = 1/2$. The RG eigenvalue of the leading irrelevant operator at the fixed point is $-2/(N + 2)$. The mobility is computed by identifying the appropriate correlation function in the Kondo model. The analogue of $r$ in the Kondo model is the spin in each channel, $S^z_a$, projected onto the $N - 1$ dimensional lattice plane with $\sum_a S^z_a$ constant. This corresponds to the operator $\hat{O} = \psi_\alpha^+ \sigma^z \psi_\beta^+ \sigma^z T^A \psi_\alpha \sigma^z \psi_\beta$, where $T^A$ is one of the $N - 1$ diagonal generators of $SU(N)$. The “current” $\mathbf{r}$ corresponds to a flow of spin between the different channels. Ludwig and Affleck have computed all correlation functions of $\hat{O}$ exactly. Borrowing their results, we obtain for the fixed point mobility,

$$\mu^* = 2 \sin^2 \frac{\pi}{N + 2} \quad (12)$$

For $N = 2$, the Kondo fixed point has $\mu^* = 1$, and is at the “small barrier” limit. For $N = 3$, this value is plotted in Fig. 1b. For large $N$, the mobility and RG eigenvalue may be found perturbatively in a manner analogous to the $e$ expansion following Eq. (9). We have checked that they agree with the exact result to leading and sub leading order in $1/N$.

Recently, Furusaki and Matveev have studied Coulomb blockade resonances in a spin degenerate quantum dot with quantum point contact leads. For $\Delta \ll T \ll E_C$, ($\Delta$ is the dot’s level spacing and $E_C$ is the Coulomb charging energy), they argue that resonances are controlled by the 4 channel Kondo fixed point. This mapping may be understood in terms of the lattice of allowed charge states for the dot and the four lead/spin channels: a symmetric dot on resonance has the symmetry of a diamond lattice with $g = 1/2$. Our analysis allows us to identify the universal on resonance conductance. A voltage between the leads corresponds to a force $F = eV$, and the resulting current is $I = e\mathbf{r}$. From (12), with $N = 4$, $\mu^* = 1/2$, leading to $G^* = (1/2)e^2/h$. Our analysis also applies to resonant tunneling through a single resonant state (i.e. $T < \Delta$), for a multi channel Luttinger liquid with repulsive interactions. For two channels, a tunneling barrier maps to a two dimensional lattice with $g < 1$, which, when tuned to a resonance, can in principle have a non symmorphic distorted honeycomb lattice symmetry. Such resonances would be an adiabatic cousin of the 3 channel Kondo effect. Analogous resonances have been observed in a single channel Luttinger liquid.

In summary, we have presented a general theory of quantum Brownian motion on periodic lattices. For the honeycomb lattice and its $D$ dimensional generalization, there is a non trivial intermediate phase, which we have identified with the multi channel Kondo problem. Presumably, other non symmorphic lattices also display such phases, and it would be interesting to classify them using conformal field theory.

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