Hermite and poly-Bernoulli mixed-type polynomials

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Abstract
In this paper, we consider Hermite and poly-Bernoulli mixed-type polynomials and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give various identities associated with Stirling numbers, Bernoulli and Frobenius-Euler polynomials of higher order.

1 Introduction
For \( r \in \mathbb{Z}_{\geq 0} \), as is well known, the Bernoulli polynomials of order \( r \) are defined by the generating function to be

\[
\sum_{n=0}^{\infty} \frac{B_n^{(r)}(x)}{n!} t^n = \left( \frac{t}{e^t - 1} \right)^r (see [1–16]).
\]

For \( k \in \mathbb{Z} \), the polylogarithm is defined by

\[
\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}.
\]

Note that \( \text{Li}_1(x) = -\log(1 - x) \).

The poly-Bernoulli polynomials are defined by the generating function to be

\[
\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!} (see [5, 8]).
\]

When \( x = 0 \), \( B_n^{(k)} = B_n^{(k)}(0) \) are called the poly-Bernoulli numbers (of index \( k \)).

For \( \nu \neq 0 \) \( \in \mathbb{R} \), the Hermite polynomials of order \( \nu \) are given by the generating function to be

\[
e^{-\frac{x^2}{2t}} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\nu)}(x) \frac{t^n}{n!} (see [6, 12, 13]).
\]

When \( x = 0 \), \( H_n^{(\nu)} = H_n^{(\nu)}(0) \) are called the Hermite numbers of order \( \nu \).
In this paper, we consider the Hermite and poly-Bernoulli mixed-type polynomials $HB_{n}^{(v,k)}(x)$ which are defined by the generating function to be

$$e^{\frac{-u^2}{2}} \frac{\ln(1-e^{-t})}{1-e^{-t}} e^{xt} = \sum_{n=0}^{\infty} H_{n}^{(v,k)}(x) \frac{t^n}{n!},$$

(1.5)

where $k \in \mathbb{Z}$ and $v (\neq 0) \in \mathbb{R}$.

When $x = 0$, $H_{n}^{(v,k)} = H_{n}^{(v,k)}(0)$ are called the Hermite and poly-Bernoulli mixed-type numbers.

Let $\mathcal{F}$ be the set of all formal power series in the variable $t$ over $\mathbb{C}$ as follows:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k t^k \bigg| a_k \in \mathbb{C} \right\}.$$  

(1.6)

Let $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ denote the vector space of all linear functionals on $\mathbb{P}$.

$(L|p(x))$ denotes the action of the linear functional $L$ on the polynomial $p(x)$, and we recall that the vector space operations on $\mathbb{P}^*$ are defined by $(L + M|p(x)) = (L|p(x)) + (M|p(x))$, $(cL|p(x)) = c(L|p(x))$, where $c$ is a complex constant in $\mathbb{C}$. For $f(t) \in \mathcal{F}$, let us define the linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n \rangle = a_n \quad (n \geq 0).$$

(1.7)

Then, by (1.6) and (1.7), we get

$$\langle t^k|x^n \rangle = n! \delta_{n,k} \quad (n, k \geq 0),$$

(1.8)

where $\delta_{n,k}$ is the Kronecker symbol.

For $f_k(t) = \sum_{k=0}^{\infty} \frac{(f(t)|x^k)}{k!} t^k$, we have $(f_k(t)|x^n) = (L|x^n)$. That is, $L = f_k(t)$. The map $L \mapsto f_k(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in $t$ and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional.

We call $\mathcal{F}$ the umbral algebra and the umbral calculus is the study of umbral algebra. The order $O(f)$ of the power series $f(t) \neq 0$ is the smallest integer for which $a_k$ does not vanish. If $O(f) = 0$, then $f(t)$ is called an invertible series. If $O(f) = 1$, then $f(t)$ is called a delta series. For $f(t), g(t) \in \mathcal{F}$, we have

$$\langle f(t)g(t)|p(x) \rangle = \langle f(t)|g(t)p(x) \rangle = \langle g(t)|f(t)p(x) \rangle.$$  

(1.9)

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we have

$$f(t) = \sum_{k=0}^{\infty} \frac{(f(t)|x^k)}{k!} t^k, \quad p(x) = \sum_{k=0}^{\infty} \frac{(t^k|p(x))}{k!} x^k \quad (\text{see [8, 9, 11, 13, 14]}).$$

(1.10)

By (1.10), we get

$$p^{(k)}(0) = \langle t^k|p(x) \rangle = \langle 1|p^{(k)}(x) \rangle,$$

(1.11)

where $p^{(k)}(0) = \frac{d^k p(x)}{dx^k} \big|_{x=0}$. 


From (1.11), we have

\[ t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad (\text{see} \ [8, 9, 13]). \] (1.12)

By (1.12), we easily get

\[ e^{it} p(x) = p(x + y), \quad \{e^{it} \mid p(x)\} = p(y). \] (1.13)

For \( O(f(t)) = 1, O(g(t)) = 0 \), there exists a unique sequence \( s_n(x) \) of polynomials such that

\[ \langle g(t)f(t)^k \mid x^n \rangle = n! s_{n,k} \quad (n, k \geq 0). \]

The sequence \( s_n(x) \) is called the Sheffer sequence for \( (g(t), f(t)) \) which is denoted by \( s_n(x) \sim (g(t), f(t)) \).

Let \( p(x) \in P, f(t) \in F \). Then we see that

\[ \langle f(t)|xp(x)\rangle = \langle h(t)f(t)p(x) \rangle = \left( \frac{df(t)}{dt} \right) p(x), \]

where \( p(x) \in P, h(t) \in F, \)

\[ \frac{1}{g(f(t))} e^{\tilde{f}(t)} = \sum_{n=0}^{\infty} s_n(y) \frac{t^n}{n!}, \]

where \( \tilde{f}(t) \) is the compositional inverse for \( f(t) \) with \( f(\tilde{f}(t)) = t \),

\[ s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(y) p_{n-k}(x), \quad \text{where} \quad p_n(x) = g(t) s_n(x), \]

\[ f(t) s_n(x) = n s_{n-1}(x), \quad s_{n+1}(x) = \left( x - \frac{g(t)}{f(t)} \right) \frac{1}{f'(t)} s_n(x), \]

and the conjugate representation is given by

\[ s_n(x) = \sum_{j=0}^{n} \binom{x^n}{j} \langle g(\tilde{f}(t))^{-1} f'(t)^j \mid x^n \rangle x^j. \] (1.19)

For \( s_n(x) \sim (g(t), f(t)), r_n(x) \sim (h(t), l(t)) \), we have

\[ s_n(x) = \sum_{m=0}^{n} C_{n,m} r_m(x), \]

where

\[ C_{n,m} = \frac{1}{m!} \left( \frac{h(\tilde{f}(t))}{g(f(t))} \right) \frac{1}{f(t)^m} \left( \frac{g(t)}{l(t)} \right)^m \]

(see \([8, 9, 13]\)). (1.20)

\[ \text{(1.21)} \]
In this paper, we consider Hermite and poly-Bernoulli mixed-type polynomials and investigate the properties of those polynomials which are derived from umbral calculus. Finally, we give various identities associated with Bernoulli and Frobenius-Euler polynomials of higher order.

2 Hermite and poly-Bernoulli mixed-type polynomials

From (1.5) and (1.16), we note that
\[ HB^{(\nu,k)}_n(x) \sim \left( e^{\frac{\nu^2}{2}} \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})}, t \right), \]  
and, by (1.3), (1.4) and (1.16), we get
\[ B^{(k)}_n(x) \sim \left( \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})}, \right), \] \[ H^{(\nu)}_n(x) \sim \left( e^{\frac{\nu^2}{2}}, t \right), \text{ where } n \geq 0. \]

From (1.18), (2.1), (2.2) and (2.3), we have
\[ tB^{(k)}_n(x) = nB^{(k)}_{n-1}(x), \quad tH^{(\nu)}_n(x) = nH^{(\nu)}_{n-1}(x), \quad tHB^{(\nu,k)}_n(x) = nHB^{(\nu,k)}_{n-1}(x). \]  

By (1.5), (1.8) and (2.1), we get
\[ HB^{(\nu,k)}_n(x) = e^{\frac{\nu^2}{2}} \text{Li}_k(1 - e^{-t}) \sum_{m=0}^{\infty} \frac{1}{m!} \left( -\frac{\nu}{2} \right)^m B^{(k)}_{n-2m}(x) \]
\[ = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{2m}{m!} \right) \left( -\frac{\nu}{2} \right)^m B^{(k)}_{n-2m}(x). \]  

Therefore, by (2.5), we obtain the following proposition.

**Proposition 1** For \( n \geq 0 \), we have
\[ HB^{(\nu,k)}_n(x) = \sum_{m=0}^{\infty} \frac{n}{2m} \left( \frac{2m!}{m!} \right) \left( -\frac{\nu}{2} \right)^m B^{(k)}_{n-2m}(x). \]

From (1.5), we can also derive
\[ HB^{(\nu,k)}_n(x) = \text{Li}_k(1 - e^{-t}) e^{\frac{\nu^2}{2}} \sum_{m=0}^{\infty} \frac{1}{(m+1)^2} \frac{1 - e^{-t}}{m!} H^{(\nu)}_n(x) \]
\[ = \sum_{m=0}^{\infty} \frac{1}{(m+1)^2} \sum_{j=0}^{m} \binom{m}{j} (-1)^j H^{(\nu)}_n(x-j). \]  

(2.6)
Therefore, by (2.6), we obtain the following theorem.

**Theorem 2** For \( n \geq 0 \), we have

\[
HB_n^{(v,k)}(x) = \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{j=0}^{m} \binom{m}{j} (-1)^j H_n^{(v)}(x-j).
\]

By (1.5), we get

\[
HB_n^{(v,k)}(x) = e^{x^{1/2}} B_n^{(k)}(x) = \sum_{l=0}^\infty \frac{1}{l!} \left( -\frac{v}{2} \right)^l t^l B_n^{(k)}(x).
\]

\[
= \sum_{l=0}^\infty \frac{1}{l!} \sum_{m=0}^{n} \frac{1}{(m+1)^k} (-1)^j \binom{m}{j} t^l (x-j)^n.
\]

\[
= \sum_{l=0}^\infty \sum_{j=0}^{n} \sum_{m=j}^{n} \binom{n}{2l} \frac{(2l)!}{l!} \left( -\frac{v}{2} \right)^l \binom{n}{m} \binom{m}{j} (x-j)^{n-2l}.
\]

Therefore, by (2.7), we obtain the following theorem.

**Theorem 3** For \( n \geq 0 \), we have

\[
HB_n^{(v,k)}(x) = \sum_{l=0}^\infty \sum_{j=0}^{n} \binom{n}{2l} \frac{(2l)!}{l!} \left( -\frac{v}{2} \right)^l \binom{n}{m} \binom{m}{j} (x-j)^{n-2l}.
\]

By (2.6), we get

\[
HB_n^{(v,k)}(x) = \sum_{m=0}^{n} \frac{(1-e^{-t})^m}{(m+1)^k} H_n^{(v)}(x).
\]

\[
= \sum_{m=0}^{n} \frac{1}{(m+1)^k} \sum_{a=0}^{n-m} \binom{m}{a} (-1)^a S_2(a+m,m)(n)_{a+m} H_n^{(v)}(x).
\]

\[
= \sum_{m=0}^{n} \sum_{a=0}^{n-m} \left( -1 \right)^{n-a-m} \binom{n}{a} \binom{m}{n-a} S_2(n-a,m) H_n^{(v)}(x).
\]

\[
= (-1)^n \sum_{a=0}^{n} \sum_{m=0}^{n-a} \left( -1 \right)^{m+a} \binom{n}{a} S_2(n-a,m) H_n^{(v)}(x),
\]

where \( S_2(n,m) \) is the Stirling number of the second kind.

Therefore, by (2.8), we obtain the following theorem.

**Theorem 4** For \( n \geq 0 \), we have

\[
HB_n^{(v,k)}(x) = (-1)^n \sum_{a=0}^{n} \sum_{m=0}^{n-a} \left( -1 \right)^{m+a} \binom{n}{a} S_2(n-a,m) H_n^{(v)}(x).
\]
From (1.19) and (2.1), we have

\[ H_{n}^{(\nu, k)}(x) = \sum_{j=0}^{n} \binom{n}{j} \left( e^{-\frac{x^2}{2}} \frac{L_k(1-e^{-t})}{1-e^{-t}} \right)^{n-j} x^j \]

\[ = \sum_{j=0}^{n} \binom{n}{j} e^{-\frac{x^2}{2}} B_{n-j}^{(k)}(x) x^j \]

\[ = \sum_{j=0}^{n} \binom{n}{j} \sum_{l=0}^{n-j} \binom{n-j}{l} \left( -\frac{2}{l} \right)^l (n-j)_{2l} \left( e^{-t} \right)^{l} B_{n-j-2l}^{(k)} x^j \]

\[ = \sum_{j=0}^{n} \left\{ \sum_{l=0}^{n-j} \binom{n-j}{l} \left[ \frac{2l!}{l!} \left( -\frac{2}{l} \right)^l B_{n-j-2l}^{(k)} \right] \right\} x^j. \]  \tag{2.9}

Therefore, by (2.9), we obtain the following theorem.

**Theorem 5** For \( n \geq 0 \), we have

\[ H_{n}^{(\nu, k)}(x) = \sum_{j=0}^{n} \binom{n}{j} \sum_{l=0}^{n-j} \binom{n-j}{l} \left( -\frac{2}{l} \right)^l (n-j)_{2l} \left( e^{-t} \right)^{l} B_{n-j-2l}^{(k)} x^j. \]

**Remark** By (1.17) and (2.1), we easily get

\[ H_{n}^{(\nu, k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} H_{j}^{(\nu, k)}(x) y^{n-j}. \]  \tag{2.10}

We note that

\[ H_{n}^{(\nu, k)}(x) \sim \left( g(t) = e^{\frac{x^2}{2}} \frac{1-e^{-t}}{L_k(1-e^{-t})} f(t) = t \right). \]  \tag{2.11}

From (1.18) and (2.11), we have

\[ H_{n+1}^{(\nu, k)}(x) = \left( x - \frac{g'(t)}{g(t)} \right) H_{n}^{(\nu, k)}(x). \]  \tag{2.12}

Now, we observe that

\[ \frac{g'(t)}{g(t)} = (\log(g(t)))' \]

\[ = (\log e^{\frac{x^2}{2}} + \log(1-e^{-t}) - \log(L_k(1-e^{-t})))' \]

\[ = vt + e^{-t} \left( 1 - \frac{L_{k-1}(1-e^{-t})}{L_k(1-e^{-t})} \right). \]  \tag{2.13}
By (2.12) and (2.13), we get

\[
HB_{n+1}^{(v,k)}(x) = xHB_n^{(v,k)}(x) + \frac{t}{e^t - 1} \left( \frac{1}{m^2} - \frac{1}{m^{k+1}} \right) (1 - e^{-t})^{m-1}
\]

(2.14)

It is easy to show that

\[
\frac{\text{Li}_k(1 - e^{-t}) - \text{Li}_{k-1}(1 - e^{-t})}{1 - e^{-t}} = \sum_{m=2}^{\infty} \left( \frac{1}{m^2} - \frac{1}{m^{k+1}} \right) (1 - e^{-t})^{m-1}
\]

\[
= \left( \frac{1}{2^2} - \frac{1}{2^{k+1}} \right) t + \cdots .
\]

(2.15)

Thus, by (2.15), we get

\[
\frac{\text{Li}_k(1 - e^{-t}) - \text{Li}_{k-1}(1 - e^{-t}) x^n}{t(1 - e^{-t})} = \frac{\text{Li}_k(1 - e^{-t}) - \text{Li}_{k-1}(1 - e^{-t})}{1 - e^{-t}} \frac{x^{n+1}}{n+1}.
\]

(2.16)

From (2.16), we can derive

\[
e^{-\frac{\nu t^2}{2}} \frac{t}{e^t - 1} \frac{\text{Li}_k(1 - e^{-t}) - \text{Li}_{k-1}(1 - e^{-t})}{1 - e^{-t}} x^n
\]

\[
= \frac{1}{n+1} \left( \sum_{l=0}^{n+1} \frac{B_l}{l!} t^l \right) (HB_{n+1}^{(v,k)}(x) - HB_{n+1}^{(v,k-1)}(x))
\]

\[
= \frac{1}{n+1} \sum_{l=0}^{n+1} \frac{B_l}{l!} t^l (HB_{n+1}^{(v,k)}(x) - HB_{n+1}^{(v,k-1)}(x))
\]

\[
= \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l (HB_{n+1-l}^{(v,k)}(x) - HB_{n+1-l}^{(v,k-1)}(x)).
\]

(2.17)

Therefore, by (2.14) and (2.17), we obtain the following theorem.

**Theorem 6** For \( n \geq 0 \), we have

\[
HB_{n+1}^{(v,k)}(x) = xHB_n^{(v,k)}(x) - vnHB_{n-1}^{(v,k)}(x)
\]

\[
- \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} B_l (HB_{n+1-l}^{(v,k)}(x) - HB_{n+1-l}^{(v,k-1)}(x)).
\]

(2.18)

Let us take \( t \) on both sides of (2.18). Then we have

\[
(n+1)HB_n^{(v,k)}(x) = (xt + 1)HB_0^{(v,k)}(x) - vn(n-1)HB_{n-2}^{(v,k)}(x)
\]
\begin{align*}
&\quad -\frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} (n+1-l)B_l \{ HB_{n-1}^{(v,k)}(x) - HB_{n-l}^{(v,k-1)}(x) \} \\
&= nxHB_{n-1}^{(v,k)}(x) + HB_{n}^{(v,k)}(x) - vn(n-1)HB_{n-2}^{(v,k)}(x) \\
&\quad - \sum_{l=0}^{n} \binom{n}{l} B_l (HB_{n-1}^{(v,k)}(x) - HB_{n-l}^{(v,k-1)}(x)), \tag{2.19}
\end{align*}

where \( n \geq 3 \).

Thus, by (2.19), we obtain the following theorem.

**Theorem 7** For \( n \geq 3 \), we have

\[
\sum_{l=0}^{n} \binom{n}{l} B_l HB_{n-l}^{(v,k-1)}(x)
\]

\[
= (n+1)HB_{n}^{(v,k)}(x) - n \left( x + \frac{1}{2} \right) HB_{n-1}^{(v,k)}(x)
\]

\[
+ n(n-1) \left( v + \frac{1}{12} \right) HB_{n-2}^{(v,k)}(x)
\]

\[
+ \sum_{l=0}^{n-3} \binom{n}{l} B_{n-l} HB_{l}^{(v,k)}(x).
\]

By (1.5) and (1.8), we get

\[
HB_{n}^{(v,k)}(y)
\]

\[
= \left\{ e^{\frac{i \sqrt{2} }{\tau}} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right\} x^n
\]

\[
= \left\{ \partial_{\tau} e^{\frac{i \sqrt{2} }{\tau}} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right\} x^{n-1}
\]

\[
= \left\{ \partial_{\tau} e^{\frac{i \sqrt{2} }{\tau}} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right\} x^{n-1}
\]

\[
\quad + \left\{ e^{\frac{i \sqrt{2} }{\tau}} \left( \partial_{\tau} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right) x^{n-1} \right\}
\]

\[
= -v(n-1) \left\{ e^{\frac{i \sqrt{2} }{\tau}} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right\} x^{n-2}
\]

\[
\quad + y \left\{ e^{\frac{i \sqrt{2} }{\tau}} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right\} x^{n-1}
\]

\[
\quad + \left\{ e^{\frac{i \sqrt{2} }{\tau}} \left( \partial_{\tau} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right) x^{n-1} \right\}
\]

\[
= -v(n-1)HB_{n-2}^{(v,k)}(y) + yHB_{n-1}^{(v,k)}(y)
\]

\[
\quad + \left\{ e^{\frac{i \sqrt{2} }{\tau}} \left( \partial_{\tau} \frac{Li_k(1-e^{-\frac{\tau}{i}})}{1-e^{-\frac{\tau}{i}}} e^{\frac{\tau}{i}} \right) x^{n-1} \right\}. \tag{2.20}
\]
Now, we observe that
\[
\partial_t \left( \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} \right) = \frac{\text{Li}_{k-1}(1-e^{-t}) - \text{Li}_k(1-e^{-t})}{(1-e^{-t})^2} e^{-t}. \tag{2.21}
\]
From (2.21), we have
\[
\left( e^{-\frac{v^2 t^2}{2}} \left( \partial_t \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} \right) e^{\nu t} | x^{v-1} \right) = e^{-\frac{v^2 t^2}{2}} \left( \frac{\text{Li}_{k-1}(1-e^{-t}) - \text{Li}_k(1-e^{-t})}{(1-e^{-t})^2} \right) e^{\nu t} \left| \frac{1}{n} t x^n \right|
\]
\[
= \frac{1}{n} e^{-\frac{v^2 t^2}{2}} \frac{\text{Li}_{k-1}(1-e^{-t}) - \text{Li}_k(1-e^{-t})}{1-e^{-t}} e^{\nu t} t \left| \frac{1}{e^t - 1} x^n \right|
\]
\[
= \frac{1}{n} \sum_{l=0}^{n} \binom{n}{l} B_l \left( e^{-\frac{v^2 t^2}{2}} \frac{\text{Li}_{k-1}(1-e^{-t}) - \text{Li}_k(1-e^{-t})}{1-e^{-t}} e^{\nu t} \right) \left| \frac{1}{e^t - 1} x^n \right|
\]
\[
= \frac{1}{n} \sum_{l=0}^{n} \binom{n}{l} B_l \left\{ \text{HB}^{(v,k-1)}_{n-l}(y) - \text{HB}^{(v,k)}_{n-l}(y) \right\}, \tag{2.22}
\]
where \(B_n\) are the ordinary Bernoulli numbers which are defined by the generating function to be
\[
t \left( \frac{e^t}{e^t - 1} - 1 \right) = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.
\]
Therefore, by (2.20) and (2.22), we obtain the following theorem.

**Theorem 8** For \(n \geq 2\), we have
\[
\text{HB}^{(v,k)}_n(x) = -v(n-1)\text{HB}^{(v,k)}_{n-2}(x) + x\text{HB}^{(v,k)}_{n-1}(x)
+ \frac{1}{n} \sum_{l=0}^{n} \binom{n}{l} B_l \left( \text{HB}^{(v,k-1)}_{n-l}(x) - \text{HB}^{(v,k)}_{n-l}(x) \right).
\]

Now, we compute
\[
\left\langle e^{-\frac{v^2 t^2}{2}} \text{Li}_k(1-e^{-t}) | x^{v-1} \right\rangle
\]
in two different ways.

On the one hand,
\[
\left\langle e^{-\frac{v^2 t^2}{2}} \text{Li}_k(1-e^{-t}) | x^{v-1} \right\rangle
= \left\langle e^{-\frac{v^2 t^2}{2}} \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} (1-e^{-t}) | x^{v-1} \right\rangle
= \left\langle e^{-\frac{v^2 t^2}{2}} \frac{\text{Li}_k(1-e^{-t})}{1-e^{-t}} | (1-e^{-t})x^{v-1} \right\rangle
\]
\[ = \left\{ e^{-\frac{u^2}{2}} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} \right\} x^{n+1} - (x - 1)^{n+1} \]
\[ = \sum_{m=0}^{n} (-1)^{n-m} \binom{n+1}{m} e^{-\frac{u^2}{2}} \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} x^m \]
\[ = \sum_{m=0}^{n} (-1)^{n-m} \binom{n+1}{m} \text{HB}^{(\nu, k)}_m. \]  
(2.23)

On the other hand,
\[ \langle e^{-\frac{u^2}{2}} \text{Li}_k(1 - e^{-t}) | x^{n+1} \rangle \]
\[ = \{ \text{Li}_k(1 - e^{-t}) e^{-\frac{u^2}{2}} x^{n+1} \} \]
\[ = \left\{ \int_0^t (\text{Li}_k(1 - e^{-s}))' ds e^{-\frac{u^2}{2}} x^{n+1} \right\} \]
\[ = \left\{ \int_0^t e^{-\frac{u^2}{2}} \frac{\text{Li}_{k-1}(1 - e^{-s})}{1 - e^{-s}} ds e^{-\frac{u^2}{2}} x^{n+1} \right\} \]
\[ = \sum_{l=0}^{\infty} \left( \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} B^{(k-1)}_m \frac{l+1}{(l+1)!} \right) \left\{ H^{(\nu)}_m(x) \right\} \]
\[ = \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} B^{(k-1)}_m \frac{1}{(l+1)!} \left( t^{l+1} H^{(\nu)}_{m+l} \right) \]
\[ = \sum_{l=0}^{n} \sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B^{(k-1)}_m H^{(\nu)}_{n+l}. \]  
(2.24)

Therefore, by (2.23) and (2.24), we obtain the following theorem.

**Theorem 9** For \( n \geq 0 \), we have
\[ \sum_{m=0}^{n} (-1)^{n-m} \binom{n+1}{m} \text{HB}^{(\nu, k)}_m \]
\[ = \sum_{m=0}^{n} \sum_{l=0}^{n} (-1)^{l-m} \binom{l}{m} \binom{n+1}{l+1} B^{(k-1)}_m H^{(\nu)}_{n+l}. \]

Let us consider the following two Sheffer sequences:
\[ \text{HB}^{(\nu, k)}_n(x) \sim \left( e^{-\frac{u^2}{2}} \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})}, t \right) \]  
(2.25)

and
\[ B^{(r)}_n(x) \sim \left( \left( \frac{e^t - 1}{t} \right)^r, t \right) \quad (r \in \mathbb{Z}_{\geq 0}). \]  
(2.26)

Let us assume that
\[ \text{HB}^{(\nu, k)}_n(x) = \sum_{m=0}^{n} C_{n,m} B^{(r)}_m(x). \]  
(2.27)
Then, by (1.20) and (1.21), we get
\[
C_{n,m} = \frac{1}{m!}\left(\left(\frac{e^t - 1}{t}\right)^r t^m \left| e^{-\frac{x^2}{2}} \frac{\text{Li}_2(1 - e^{-t})}{1 - e^{-t}} x^m\right| \right)
\]
\[
= \frac{1}{m!}\left(\left(\frac{e^t - 1}{t}\right)^r t^m HB_n^{(v,k)}(x)\right) = \frac{1}{m!}(n_m)\left(\left(\frac{e^t - 1}{t}\right)^r HB_{n-m}^{(v,k)}(x)\right)
\]
\[
= \left(\frac{n}{m}\right)\sum_{l=0}^{\infty} \frac{r!}{(l+r)!} S_2(l + r, r)\left| t^l HB_n^{(v,k)}(x)\right|
\]
\[
= \left(\frac{n}{m}\right)\sum_{l=0}^{n-m} (n - m) \frac{r!}{(l+r)!} S_2(l + r, r)HB_{n-m-l}^{(v,k)}
\]
\[
= \left(\frac{n}{m}\right)\sum_{l=0}^{n-m} \frac{r!}{(l+r)!} S_2(l + r, r)HB_{n-m-l}^{(v,k)}
\] (2.8)

Therefore, by (2.27) and (2.28), we obtain the following theorem.

**Theorem 10** For \( n, r \in \mathbb{Z}_{\geq 0} \), we have
\[
HB_n^{(v,k)}(x) = \sum_{m=0}^{n} \left(\left(\frac{n}{m}\right)\sum_{l=0}^{\infty} \frac{r!}{(l+r)!} S_2(l + r, r)HB_{n-m-l}^{(v,k)}\right)|B_m^{(r)}(x)|.
\]

For \( \lambda \neq 1 \in \mathbb{C}, \, r \in \mathbb{Z}_{\geq 0} \), the Frobenius-Euler polynomials of order \( r \) are defined by the generating function to be
\[
\left(1 - \frac{\lambda}{e^t - \lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda) \frac{t^n}{n!} \quad \text{(see [1, 4, 7, 9, 10])}. \] (2.29)

From (1.16) and (2.29), we note that
\[
H_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^t - \lambda}{1 - \lambda}\right)^r, t\right).
\] (2.30)

Let us assume that
\[
HB_n^{(v,k)}(x) = \sum_{m=0}^{n} C_{n,m}H_m^{(r)}(x|\lambda).
\] (2.31)

By (1.21), we get
\[
C_{n,m} = \frac{1}{m!}\left(\left(\frac{e^t - \lambda}{1 - \lambda}\right)^r t^m \left| e^{-\frac{x^2}{2}} \frac{\text{Li}_2(1 - e^{-t})}{1 - e^{-t}} x^m\right| \right)
\]
\[
= \frac{(n)_m}{m!(1 - \lambda)^r} \left(\sum_{l=0}^{r} \binom{r}{l} (-\lambda)\frac{-t^l}{l!} HB_{n-m}^{(v,k)}(x)\right)
\]
\[
= \frac{(n)_m}{(1 - \lambda)^r} \sum_{l=0}^{r} \binom{r}{l} (-\lambda)^{-l-1} t^l HB_{n-m}^{(v,k)}(x)|B_m^{(r)}(x)|
\]
\[
= \frac{(n)_m}{(1 - \lambda)^r} \sum_{l=0}^{r} \binom{r}{l} (-\lambda)^{-l-1} HB_{n-m}^{(v,k)}(1).
\] (2.32)
Therefore, by (2.31) and (2.32), we obtain the following theorem.

**Theorem 11** For \( n, r \in \mathbb{Z}_{\geq 0} \), we have

\[
HB_n^{(r, k)}(x) = \frac{1}{(1-\lambda)^r} \sum_{m=0}^{n} \binom{n}{m} \left( \sum_{l=0}^{r} \binom{r}{l} (-\lambda)^{r-l} HB_{n-m}^{(l, k)}(t) \right) H_m^{(r)}(x|\lambda).
\]

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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