USING THE SWING LEMMA AND CZÉDLI DIAGRAMS FOR CONGRUENCES OF PLANAR SEMIMODULAR LATTICES

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Abstract. A planar semimodular lattice $K$ is slim if $M_3$ is not a sublattice of $K$. In a recent paper, G. Czédli found four new properties of congruences of slim, planar, semimodular lattices, including the No Child Property: Let $P$ be the ordered set of join-irreducible congruences of $K$. Let $x, y, z \in P$ and let $z$ be a maximal element of $P$. If $x \neq y$, $x, y \prec z$ in $P$, then there is no element $u$ of $P$ such that $u \prec x, y$ in $P$.

We are applying my Swing Lemma, 2015, and a type of standardized diagrams of Czédli’s, to verify Czédli’s four properties.

1. Introduction

Let $K$ be a planar semimodular lattice. We call the lattice $K$ slim if $M_3$ is not a sublattice of $K$. In the paper [4], I found a property of congruences of slim, planar, semimodular lattices. In the same paper (see also Problem 24.1 in G. Grätzer [6]), I proposed the following:

Problem 1. Characterize the congruence lattices of slim planar semimodular lattices.

G. Czédli [2, Corollaries 3.4, 3.5, Theorem 4.3] found four new properties of congruence lattices of slim, planar, semimodular lattices.

Czédli’s Theorem [2]. Let $K$ be a slim, planar, semimodular lattice with at least three elements and let $P$ be the ordered set of join-irreducible congruences of $K$.

(i) Partition Property: The set of maximal elements of $P$ can be represented as the disjoint union of two nonempty subsets such that no two distinct elements in the same subset have a common lower cover.

(ii) Maximal Cover Property: If $x \in P$ is covered by a maximal element $y$ of $P$, then $y$ is not the only cover.

(iii) Four-Crown Two-pendant Property: There is no cover-preserving embedding of the ordered set $R$ in Figure 7 into $P$ satisfying the property: any maximal element of $R$ is a maximal element of $P$.

(iv) No Child Property: Let $x \neq y \in P$ and let $z$ be a maximal element of $P$. Let us assume that both $x$ and $y$ are covered by $z$ in $P$. Then there is no element $u \in P$ such that $u$ is covered by $x$ and $y$.

In this paper, we will prove this theorem using the Swing Lemma and Czédli diagrams, see Sections 2.1 and 2.2. By G. Grätzer and E. Knapp [11], every slim,
planar, semimodular lattice $K$ has a congruence-preserving extension $\overline{K}$ to a slim rectangular lattice. Any of the properties (i)--(iv) holds for $K$ iff it holds for $\overline{K}$. Therefore, in the rest of this paper, we can assume that $K$ is a slim rectangular lattice, simplifying the discussion.

**Outline.** Section 2 provides the results we need: the Swing Lemma, Czédli’s $C_1$-diagrams (we call them Czédli diagrams), and forks. Section 3 proves the Partition Property, Section 4 does the Maximal Cover Property, while Section 5 verifies the No Child Property. Finally, The Four-Crown Two-pendant Property is proved in Section 6.

2. **Background**

Most basic concepts and notation not defined in this paper are available in Part I of the book

https://www.researchgate.net/publication/299594715

It is available to the reader. We will reference it, for instance, as [CFL2, page 52].

2.1. **Swing Lemma.** An SPS lattice $K$ is a slim, planar, and semimodular lattice, see [CFL2, Chapter 4]. For an edge (prime interval) $E$ of $K$, let $E = [0_E, 1_E]$ and define $\text{col}(E)$, the color of $E$, as $\text{con}(E)$, the (join-irreducible) congruence generated by collapsing $E$ (see [CFL2, Section 3.2]). We write $P$ for $J(\text{Con} K)$, the ordered set of join-irreducible congruences of $K$.

As in my paper [5], for the edges $U, V$ of an SPS lattice $K$, we define a binary relation: $U$ swings to $V$, in formula, $U \swings V$, if $1_U \geq 1_V$, the element $1_U = 1_V$ covers at least three elements, and $0_V$ is neither the left-most nor the right-most element covered by $1_U = 1_V$; if also $0_U$ is such, then the swing is interior.

**Swing Lemma [G. Grätzer [5]].** Let $K$ be an SPS lattice and let $U$ and $V$ be edges in $K$. Then $\text{col}(V) \leq \text{col}(U)$ iff there exists an edge $R$ such that $U$ is up-perspective to $R$ and there exists a sequence of edges and a sequence of binary relations

$$R = R_0 \rho_1 R_1 \rho_2 \cdots \rho_n R_n = V,$$

where each relation $\rho_i$ is $\downarrow$ (down-perspective) or $\swings$ (swing).

In addition, the sequence (1) also satisfies

$$1_{R_0} \geq 1_{R_1} \geq \cdots \geq 1_{R_n}.$$

The following statements are immediate consequences of the Swing Lemma, see my papers [5] and [7].
Corollary 1. We use the assumptions of the Swing Lemma.

(i) If \( \text{col}(U) = \text{col}(V) \), then either \( U \) and \( V \) are perspective or there exist edges \( S \) and \( T \) so that \( U \uparrow S \supset T \downarrow V \), where the swing is interior.

(ii) If \( v \prec u \) in \( P \), then there exist edges \( U, L \) of color \( u \) and \( R, V \) of color \( v \) such that \( U \uparrow L \supset R \downarrow V \), as in see the first diagram of Figure 2.

Note that the covering relation in \( P \) can always be represented in a covering \( N_7 \).

![Figure 2. Illustrating Corollaries 2 and 3](image)

Corollary 2. Let the edge \( U \) be on the upper edge of \( K \). Then \( \text{col}(U) \) is a maximal element of \( P \). Conversely, if \( u \) is a maximal element of \( P \), then there is an edge \( U \) on the upper edge of \( K \) so that \( \text{col}(U) = u \).

Corollary 3. Let \( v \prec u \) in \( P \) and \( u \) be a maximal element of \( P \). If \( U \) is an edge on the upper edge of \( K \) with \( \text{col}(U) = u \), then there exist an edge \( L \) such that \( U \downarrow L \supset V \), as in the second diagram of Figure 2.

2.2. Czédi diagrams. In the diagram of a planar lattice \( K \), a normal edge (line) has a slope of 45° or 135°. If it is the first, we call it a normal-up edge (line), otherwise, a normal-down edge (line). Any edge of slope strictly between 45° and 135° is steep. We use the symbols \( \searrow, \nearrow, \swarrow, \nwarrow, \uparrow, \downarrow \), for normal, normal-up, normal-down, and steep, respectively.

Definition 4 (G. Czédi [1]). A diagram of an SPS lattice \( L \) is a Czédi-diagram if the middle edge of any covering \( N_7 \) is steep and all other edges are normal.

Theorem 5 (G. Czédi [1]). Every slim, planar, semimodular lattice \( K \) has a Czédi diagram.

See the illustrations in this paper for examples of Czédi diagrams. G. Czédi [1] calls these \( \mathcal{C}_1 \)-diagrams. He also defines \( \mathcal{C}_0 \)- and \( \mathcal{C}_2 \)-diagrams. For an alternative proof for the existence of Czédi diagrams, see G. Grätzer [10].

In this paper, \( K \) denotes a slim rectangular lattice with a fixed Czédi diagram.

Let \( C \) and \( D \) be maximal chains in an interval \([a, b]\) of \( K \) such that \( C \cap D = \{a, b\} \). If there is no element of \( K \) between \( C \) and \( D \), then we call \( C \cup D \) a cell. A four-element cell is a 4-cell. Opposite edges of a 4-cell are called adjacent. Planar semimodular lattices are 4-cell lattices, that is, all of its cells are 4-cells, see G. Grätzer and E. Knapp [11] Lemmas 4, 5] and [CFL2, Section 4.1] for more detail.

The following statement illustrates the use of Czédi diagrams.
Lemma 6. Let \( K \) be a slim rectangular lattice \( K \) with a fixed Czédli diagram and let \( X \) be a \( \bowtie \)-edge of \( K \). Then \( X \) is up-perspective either to an edge in the upper-left boundary of \( K \) or to a \( \perp \)-edge.

Proof. If \( X \) is not in the upper-left boundary of \( K \), then there is a 4-cell \( C \) whose lower-right edge is \( X \). If the upper-left edge is \( l \) or it is in the upper-left boundary, then we are done. Otherwise, we proceed the same way until we reach the upper-left boundary. □

2.3. Trajectories. G. Czédli and E. T. Schmidt \[3\] introduced a trajectory in \( K \) as a maximal sequence of consecutive edges, see also \[CFL2, Section 4.1\]. The top edge \( T \) of a trajectory is either in the upper boundary of \( K \) or it is \( l \). For such an edge \( T \), we denote by \( \text{traj}(T) \) the trajectory with top edge \( T \). Since an element \( a \) in a slim semimodular lattice has at most three covers (G. Grätzer and E. Knapp \[9, Lemma 8\]), a trajectory has at most one top edge and at most one \( \perp \)-edge. So we conclude:

Lemma 7. Let \( K \) be a slim rectangular lattice \( K \) with a fixed Czédli diagram. Let \( X \) and \( Y \) be \( \perp \)-edges of \( K \). Then \( \text{traj}(X) \) and \( \text{traj}(Y) \) are disjoint.

3. The Partition Property

We start with a lemma.

Lemma 8. Let \( X \) and \( Y \) be distinct edges on the upper-left boundary of \( K \) of color \( x \) and \( y \), respectively. Then there is no edge \( Z \) of \( K \) of color \( z \) such that \( Z \prec x, y \).

Proof. By way of contradiction, let \( Z \) be an edge of color \( z \) such that \( Z \prec x, y \). \( \ell_Y \) the \( \perp \)-line through \( 0_Y \). By Corollary \[3\] there exist edges \( L_X, L_Y, Z_X, Z_Y \) such that \( X \uparrow Z_X \cap Z_Y \cap \text{traj}(Z_X) \cap \text{traj}(Z_Y) \). This implies that \( Z_X \uparrow Z_Y \), contradicting that \( 0_Z \) is meet-irreducible or that \( Z_X \uparrow Z_Y \), contradicting that \( 0_Z \) is meet-irreducible. □

By Corollary \[2\] the set of maximal elements of \( \mathcal{P} \) is the same as the set of colors of edges in the upper boundaries, which set we can partition into the set of colors of edges in the upper-left and upper-right boundaries. No two distinct elements in the same subset have a common lower cover by Lemma \[5\]. This verifies the Partition Property.

4. The Maximal Cover Property

Let \( x \in \mathcal{P} \) be covered by a maximal element \( y \) of \( \mathcal{P} \). By Corollary \[3\] there is a covering \( N_f \) such that \( \text{col}(M) = x \) and \( \text{col}(L) = y \), using the notation of Figure \[1\]. Moreover, \( \text{col}(L) \) and \( \text{col}(R) \) are the only covers of \( x \) in \( \mathcal{P} \). By Corollary \[2\] there is an edge \( U \) in the upper-left boundary, so that \( U \bowtie L \). Similarly, there is an edge \( V \) in the upper-right boundary, so that \( V \bowtie R \). So if the Maximal Cover Property fails, then \( \text{col}(L) = \text{col}(R) \). This contradicts Corollary \[1\](i).

5. The No Child Property

By way of contradiction, let us assume that there are elements \( a, b, c, d \in \mathcal{P} \) with \( b \neq c \prec a, d \prec b, c \) in \( \mathcal{P} \), where \( a \) is maximal in \( \mathcal{P} \). By Corollary \[2\] the element \( a \) colors an edge \( A \) in the upper boundary of \( K \), say, in the upper-left boundary.
By Corollary 3, we get a covering $N_7$, with middle edge $B$, upper-left edge $L$ satisfying that $\col(B) = b$ and $A \cong L$. Similarly, we get another covering $N_7$, with middle edge $C$ and upper-left edge $L'$ satisfying that $\col(B) = b$ and $A \cong L'$. By Lemma 4, $\text{traj}(B)$ and $\text{traj}(C)$ are disjoint, contradicting the existence of the element $d \prec b, c \in \mathcal{P}$, see Corollary 1(ii).

6. The Four-Crown Two-pendant Property

By way of contradiction, assume that the ordered set $\mathcal{R}$ of Figure 1 is a cover preserving ordered subset of $\mathcal{P}$, where $a, b, c, d$ are maximal elements of $\mathcal{P}$. By Corollary 2, there are edges $A, B, C, D$ on the upper boundary of $K$, so that $\col(A) = a$, $\col(B) = b$, $\col(C) = c$, $\col(D) = d$. By left-right symmetry, we can assume that the edge $A$ is on the upper-left boundary of $K$. Since $p \prec a, b$ in $\mathcal{P}$, it follows from Lemma 8 that the edge $B$ cannot be on the upper-left boundary of $K$. So $B$ is on the upper-right boundary of $K$, and so is $D$. Similarly, $C$ is on the upper-left boundary of $K$. So there are four cases, (i) $C$ is below $A$ and $B$ is below $D$; (ii) $C$ is below $A$ and $D$ is below $B$; and so on. The first two are illustrated in Figure 3.

![Figure 3. Illustrating the proof of The Four-Crown Two-pendant Property](image)

We consider the first case. By Corollary 3 (see Figure 2), there is a covering $N_7$ with middle edge $P$ as in the first diagram of Figure 3 so that $A$ and $B$ are down-perspective to the left-upper edge and the right-upper edge of the covering $N_7$, respectively. We define, similarly, the edge $Q$ for $C$ and $B$, the edge $S$ for $A$ and $D$, the edge $R$ for $C$ and $D$, and the edge $U$ for $R$ and $P$.

The ordered set $\mathcal{R}$ is a cover preserving subset of $\mathcal{P}$, so we get the covering $N_7$ with middle edge $U$, upper-left edge $U_l$ and upper-right edge $U_r$; the edge $U$ is collapsed by $\text{con}(P) \land \text{con}(R)$. Then $R \cong U_l$ and $P \cong U_r$, contradicting that $\text{traj}(P)$ and $\text{traj}(R)$ do not meet in a 4-cell since $\ell(P)$ and $\ell(R)$ are both $\nabla$ and so parallel. This concludes the proof of the Four-Crown Two-pendant Property and of Czédli’s Theorem.

Of course, the diagrams in Figure 3 are only illustrations. The grid could be much larger, the edges $A, C$ and $B, D$ may not be adjacent, and there may be lots of other elements in $K$. However, our argument only utilized what is true (for instance, that some edges are $\nabla$) whatever the configuration.
The second case is similar, except that we get the edge $V$ and cannot get the edge $U$. The third and fourth cases follow the same way.

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