Geometric Engineering in Toric F-Theory and GUTs with $U(1)$ Gauge Factors

Volker Braun,\textsuperscript{1} Thomas W. Grimm,\textsuperscript{2} and Jan Keitel\textsuperscript{2}

\textsuperscript{1}Mathematical Institute, University of Oxford
24-29 St Giles’, Oxford, OX1 3LB, United Kingdom

\textsuperscript{2}Max-Planck-Institut für Physik,
Föhringer Ring 6, 80805 Munich, Germany

ABSTRACT

An algorithm to systematically construct all Calabi-Yau elliptic fibrations realized as hypersurfaces in a toric ambient space for a given base and gauge group is described. This general method is applied to the particular question of constructing $SU(5)$ GUTs with multiple $U(1)$ gauge factors. The basic data consists of a top over each toric divisor in the base together with compactification data giving the embedding into a reflexive polytope. The allowed choices of compactification data are integral points in an auxiliary polytope. In order to ensure the existence of a low-energy gauge theory, the elliptic fibration must be flat, which is reformulated into conditions on the top and its embedding. In particular, flatness of $SU(5)$ fourfolds imposes additional linear constraints on the auxiliary polytope of compactifications, and is therefore non-generic. Abelian gauge symmetries arising in toric F-theory compactifications are studied systematically. Associated to each top, the toric Mordell-Weil group determining the minimal number of $U(1)$ factors is computed. Furthermore, all $SU(5)$-tops and their splitting types are determined and used to infer the pattern of $U(1)$ matter charges.

June 3, 2013

volker.braun@maths.ox.ac.uk; grimm, jkeitel@mpp.mpg.de
Contents

1 Introduction 2
2 Classifying Toric Sections 4
3 All $SU(5)$ Tops 10
4 Determining $U(1)$ Charges 11
   4.1 Different Splits and the Fundamental Representation . . . . . . . . 11
   4.2 A No-Go Theorem for the Antisymmetric Fields . . . . . . . . . . . 17
5 The Polytope of Compactifications 19
6 Flatness of the Fibration 20
   6.1 Codimension Two Fibers . . . . . . . . . . . . . . . . . . . . . . . . 22
   6.2 Flatness Criterion . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22
   6.3 Codimension Three Fibers . . . . . . . . . . . . . . . . . . . . . . . 23
   6.4 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 24
   6.5 Flattening Base Change . . . . . . . . . . . . . . . . . . . . . . . . . 25
7 Conclusions 27
A The Group Law on a Cubic 28

List of Figures

1 Toric F-theory flowchart . . . . . . . . . . . . . . . . . . . . . . . . . 3
2 The 16 reflexive polygons . . . . . . . . . . . . . . . . . . . . . . . . 7
3 All $SU(5)$ tops . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
4 The different splits of the $SU(5)$ . . . . . . . . . . . . . . . . . . . 14
5 The $\tau_{5,3}$ top . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15
6 Triangulations and $I_5 \rightarrow I_1^*$ degenerations . . . . . . . . . . . 18
7 $SU(5)$ models with top $\tau_{10,1}$ and $\tau_{3,6}$ . . . . . . . . . . . . . . . . 26
1 Introduction

Four-dimensional F-theory compactifications on elliptically fibered Calabi-Yau fourfolds allow for the exciting possibility to geometrically engineer interesting Grand Unified Theories (GUTs) [1, 2]. While the original models were local, and thus decoupled from gravity, vast progress has been made in the search for global realizations [3, 4, 5, 6, 7, 8, 9, 10]. To systematically approach this problem, one has to construct fully resolvable Calabi-Yau fourfolds, e.g. with $SU(5)$ and $SO(10)$ gauge group singularities [4, 6, 7, 8], but also needs to globally understand the physics of Abelian gauge symmetries. Abelian gauge symmetries are not localized on the degenerate fibers and, hence, depend on the global properties of the Calabi-Yau manifold. Therefore, a systematic study of allowed models requires us to control both the local singularity resolution for the non-Abelian groups and the global properties of $U(1)$ gauge symmetries. Studying global $U(1)$ symmetries is a topic of intense current research [11, 12, 13, 14, 15, 16, 17, 18, 19]. In this paper, we use toric geometry to develop an algorithm to construct compact Calabi-Yau fourfolds with a certain non-Abelian and Abelian gauge group for a specified base. The various steps are schematically pictured in Figure 1 and we will discuss them in the following.

In F-theory, the physics of space-time filling seven-branes is encoded in the degeneration of the elliptic fiber and the sections of the fibration. First, one has to specify a zero-section that embeds the base used as a Type IIB string background into the Calabi-Yau manifold. In this work we will only consider elliptic fibers that are hypersurfaces inside a toric ambient space. This leaves 16 choices for the fiber ambient space corresponding to the 16 inequivalent reflexive polygons in two dimensions. For a given fiber ambient spaces, one can then classify all toric sections in the corresponding elliptically fibered manifold. These form the toric Mordell-Weil group and determine
the minimal number of $U(1)$ factors in the four-dimensional effective theory. Because the fiber ambient space is the same for almost all fibers, the number of toric sections is independent of the chosen base and non-Abelian gauge group. As a first step, we will classify all such toric sections, their dependencies, and the toric Mordell-Weil group they generate. While this gives us a first understanding of allowed $U(1)$ symmetries, a complete model requires one to also specify the non-Abelian gauge symmetry and a base of the fibration, possibly giving rise to further non-toric sections.

Next, to generate a non-Abelian gauge group, one modifies the toric ambient space of the elliptic fiber over the seven-brane divisor in the base. The Calabi-Yau hypersurfaces is then the resolution of the non-Abelian singularity. The toric data is encoded in a so-called top \[20, 21\]. For each gauge group there is a finite number of tops \[20, 22\]. To actually classify all tops for a given gauge group requires some care to mod out by the remaining discrete symmetries of the top. As an example, we classify all 37 distinct $SU(5)$ tops for all 16 fiber types. Recently \[17, 18\], one of the fiber types together with its 5 distinct $SU(5)$ tops was considered in detail as a model for a rank-two Mordell-Weil group.

Fixing a particular top to obtain a desired gauge group determines the toric $U(1)$ charges and constrains the $U(1)$ charges of the matter fields. We demonstrate this for the example of $SU(5)$ with matter in $1$, $5$ and $10$ representations. For each section of the elliptic fibration, one can introduce a split of the affine Dynkin diagram obtained in the resolution of the non-Abelian gauge group.\footnote{In case of a single $U(1)$, our notion of splits also captures the situations occurring in the split-spectral cover analysis performed in \[3, 5, 23, 24\].} This split fixes the $U(1)$ charges of the $5$, $10$ matter modulo 5. Moreover, a detailed analysis of the $10$ matter fields shows that the fiber degeneration is fixed by the ambient space fan. This forces them...
to all have the same $U(1)$ charge. Our no-go result depends crucially on the fact that the fiber is a hypersurface of codimension-one in the ambient space fiber, and can be avoided using complete intersections [25].

Finally, we present an algorithm to construct all reflexive polytopes with a given top and base manifold. In the construction, we introduce an auxiliary polytope whose lattice points label the resulting inequivalent reflexive polytopes. Having found a method to construct appropriate compactification manifolds, we further discuss flatness which is tied to the compactification data. For phenomenological reasons, one is generally interested in low-energy effective gauge theories. Non-flat F-theory compactifications have tensionless strings, yielding an infinite towers of massless fields in the resulting effective theory. Therefore, the Calabi-Yau manifold must be a flat fibration, that is, have constant fiber dimension over all points in the base. Ensuring flatness in various codimensions in the base imposes additional constraints on the reflexive polytope specifying the Calabi-Yau manifold and we formulate these in terms of geometric conditions on the toric data. Importantly, flatness is non-generic in the sense that it corresponds to equations that points of the auxiliary polytope have to satisfy. In fact, requiring the fibration to be flat can rule out certain combinations of tops and base manifold as well as entire tops independent of the base manifold. To illustrate this point, we give examples for both cases.

The paper is organized as follows. In Section 2, we first list the toric sections for all 16 two-dimensional reflexive polygons in which the elliptic fiber can be embedded. We then study the toric subgroup of the Mordell-Weil group generated by them. In Section 3, all $SU(5)$ tops based on these 16 fiber types are listed and the number of induced toric $U(1)$ factors is determined. These constrain the $U(1)$ matter charges for $SU(5)$ matter and singlets, which is discussed in Section 4. In particular, splits of the tops induced by the $U(1)$ factors are introduced and a no-go theorem for 10-matter with different $U(1)$ charges in hypersurfaces is presented. In Section 5, we discuss the compactification data necessary in addition to the choice of a tops together with a base. Particular care has to be taken to ensure flatness of the fibration, as we discuss in Section 6. We introduce a toric flatness criterion and provide interesting compact $SU(5)$ GUT examples in this final section.

2 Classifying Toric Sections

In order to classify the gauge theory content arising from F-theory compactifications on Calabi-Yau manifolds, one might hope that information about the gauge group is already specified by local properties of the geometry, as their classification could be a more feasible task than trying to construct all possible Calabi-Yau manifolds in various dimensions. For non-Abelian gauge groups, this does turn out to be the case and the non-Abelian gauge group can be read off from the local singularity structure
Fiber polygon | $F_1$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ | $F_6$ | $F_7$ | $F_8$ | $F_9$ | $F_{10}$ | $F_{11}$ | $F_{12}$ | $F_{14}$
---|---|---|---|---|---|---|---|---|---|---|---|---|---
$r_{\text{min}}$ | 0 | 1 | 1 | 0 | 2 | 1 | 3 | 1 | 2 | 0 | 1 | 2 | 1
$r_{\text{max}}$ | 4 | 3 | 4 | 2 | 4 | 4 | 3 | 2 | 3 | 0 | 1 | 2 | 1

Table 1: Abelian $U(1)^r$ factors for all Calabi-Yau elliptic fibrations $X$ with $SU(5)$ tops and fixed base manifold $B = \mathbb{P}^3$. The minimal number $r_{\text{min}} = \text{rank } MW_T(X)$ is the number of toric $U(1)$ factors. All values $r_{\text{min}} \leq r \leq r_{\text{max}}$ are realized by a fourfold.

of the compactification manifold \cite{26, 27, 20, 28, 29}.

In this section, we contrast this with the Abelian gauge group factors arising from F-theory compactifications on elliptically fibered Calabi-Yau manifolds. For the purposes of this paper, we only consider hypersurfaces in toric varieties such that the ambient toric variety is itself fibered. In this case the toric ambient fiber is two-dimensional and generically again a toric variety. In terms of toric data, the entire toric ambient space is always specified by a reflexive polytope and the fibration amounts to a projection onto the base fan. The preimage of the origin must be a reflexive polygon, and this is the polygon determining the generic fiber of the fibration. In Figure 2 we list the 16 inequivalent reflexive polygons, all of which can arise as the fiber polygon. If the classification of $U(1)$ symmetries were to proceed analogously to that of non-Abelian gauge symmetries, then one would expect the fiber to fix the number of $U(1)$ factors uniquely. However, this is not the case. The easiest way to illustrate this fact is to find concrete counter-examples, and, indeed, these can easily be found. Using the general methods discussed in later sections of this paper, we constructed all different Calabi-Yau fourfolds with non-Abelian gauge group $SU(5)$ and base manifold $B = \mathbb{P}^3$. In Table 1 we list the different Abelian gauge groups we found for each of the 16 fiber polygons. As they are not all the same, the total number of $U(1)$ gauge factors is only captured by global properties of the Calabi-Yau geometry.\footnote{Over the discriminant, the fiber will not be a toric variety but a reducible union of two-dimensional toric varieties.} Nevertheless, there are some properties that are fixed by the ambient space fiber alone and we now focus on these.

A toric section of an elliptically fibered Calabi-Yau hypersurface is one that is obtained by setting one of the fiber coordinates equal to zero. The conditions for a toric section were first derived in \cite{30, 31}, but for completeness we restate them here. If we denote the fiber coordinates by $f_i$, then the divisor $V(f_i)$ defines a section if it cuts out a single point in the fiber, that is, the intersection product between $V(f_i)$

\footnote{Note that, apart from non-toric $U(1)$s, there can also be additional non-toric non-Abelian gauge groups. That is, gauge groups localized at discriminant components that are not defined by the vanishing of a single homogeneous coordinate. In particular, all $SU(5)$ models with fiber $F_{14}$ and base $\mathbb{P}^3$ turn out to have such additional non-Abelian gauge factors.}
and the elliptic fiber $E$ must be

$$[V(f_i)] \cap [E] = 1. \quad (1)$$

The genus-one curve $E$ can be thought of as the one-dimensional Calabi-Yau hypersurface in the fiber toric surface $F$, that is, the anti-canonical hypersurface

$$[E] = -c_1(F) = \sum_i [V(f_i)]. \quad (2)$$

We can always assume that the fiber fan is smooth. In this case, we have $[V(f_i)] \cap [V(f_j)] = \delta_{i,j-1} + \delta_{i,j+1}$ for $i \neq j$ and, therefore, eq. (1) implies that a toric section $V(f_i)$ must satisfy

$$[V(f_i)] \cap [V(f_i)] = -1. \quad (3)$$

To translate this into the geometry of the fan, let us denote the ray corresponding to the toric coordinate $f_i$ by $v_i$. Then (3) is satisfied if the lattice spanned by the edges connecting $v_i$ with its neighboring rays,

$$N_i = \text{span } (v_i - v_{i-1}, v_i - v_{i+1}) \quad (4)$$

is the same as the fan lattice $N$, i.e.

$$V(f_i) \text{ is a section } \iff N_i = N. \quad (5)$$

In particular, only vertices of a fiber polygon can give rise to toric sections. Given this simple geometric prescription, one can easily read off the toric sections of a given fiber polygon. In Figure 2, the vertices of the fiber polygon corresponding to sections are shown in red.

Having defined toric sections, we now define the toric Mordell-Weil group $\text{MW}_T$ if there is at least one toric section. First, recall that the sections of an elliptic fibration form an Abelian group using the group law on each elliptic curve fiber. This group is the Mordell-Weil group of the compactification manifold and plays an important role in F-theory compactifications \[32, 12, 13\]. The rank of the Mordell-Weil group, measuring the free part, equals the number of independent $U(1)$ factors. The torsion subgroup are discrete symmetries in F-theory that might be important for proton stability. Now, the toric sections form a finite subset of all sections, but that subset is usually not closed under the group law. However, if there is at least one toric section to use as the zero-section, then they generate a subgroup which we call the toric Mordell-Weil group

$$\text{MW}_T = \langle f_i - f_j \rangle \subset \text{MW}. \quad (6)$$

Hence, once the toric sections of a certain fiber polygon have been found one has to determine the relations among them. Clearly, the relations only need to be computed
Figure 2: The 16 reflexive polygons. $F_i$ and $F_{17-i}$ are dual for $i = 0, \ldots, 6$, and self-dual for $i = 7, \ldots, 10$. The corresponding toric surfaces are also known as $F_1 = \mathbb{P}^2$, $F_2 = \mathbb{P}^1 \times \mathbb{P}^1$, $F_3 = dP_1$, $F_4 = \mathbb{P}^2[1, 1, 2]$, $F_5 = dP_2$, $F_7 = dP_3$, $F_{10} = \mathbb{P}^2[2, 2, 3]$, where $dP_n$ are the del Pezzo surfaces obtained by blowing up $\mathbb{P}^2$ at $n$ points. Vertices defining toric sections are colored red. First derived in Figure 1 of [37].
| Fiber polygon | Toric sections | Relations | MW<sub>T</sub> |
|---------------|---------------|-----------|---------------|
| $F_3$         | $(1, 1) \simeq f_0$ |           | 0             |
| $F_5$         | $(0, -1) \simeq f_0$ |           | $\mathbb{Z} \oplus \mathbb{Z}$ |
|               | $(-1, -1) \simeq f_1$ |           |               |
|               | $(-1, 0) \simeq f_2$ |           |               |
| $F_6$         | $(0, 1) \simeq f_0$ |           | $\mathbb{Z}$ |
|               | $(-1, 1) \simeq f_1$ |           |               |
| $F_7$         | $(-1, -1) \simeq f_0$ | $\sigma_1 = \sigma_3 + \sigma_4$ | $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ |
|               | $(0, -1) \simeq f_1$ |           |               |
|               | $(1, 0) \simeq f_2$ |           |               |
|               | $(1, 1) \simeq f_3$ |           |               |
|               | $(0, 1) \simeq f_4$ | $\sigma_5 = \sigma_2 + \sigma_3$ |               |
|               | $(-1, 0) \simeq f_5$ |           |               |
| $F_8$         | $(-1, 1) \simeq f_0$ |           | $\mathbb{Z}$ |
|               | $(1, 1) \simeq f_1$ |           |               |
| $F_9$         | $(-1, -1) \simeq f_0$ | $\sigma_1 = \sigma_2 + \sigma_3$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
|               | $(0, -1) \simeq f_1$ |           |               |
|               | $(0, 1) \simeq f_2$ |           |               |
|               | $(1, 1) \simeq f_3$ |           |               |
| $F_{10}$      | $(-3, -2) \simeq f_0$ |           | 0             |
| $F_{11}$      | $(-1, -1) \simeq f_0$ | $\sigma_1 = 2\sigma_2$ | $\mathbb{Z}$ |
|               | $(0, -1) \simeq f_1$ |           |               |
|               | $(-1, 2) \simeq f_2$ |           |               |
| $F_{12}$      | $(-1, -1) \simeq f_0$ | $\sigma_1 = \sigma_3 + \sigma_4$ | $\mathbb{Z} \oplus \mathbb{Z}$ |
|               | $(1, -1) \simeq f_1$ |           |               |
|               | $(1, 0) \simeq f_2$ |           |               |
|               | $(0, 1) \simeq f_3$ | $\sigma_4 = \sigma_1 + \sigma_2$ |               |
|               | $(-1, 1) \simeq f_4$ |           |               |
| $F_{13}$      | $(-1, -2) \simeq f_0$ | $2\sigma_1 = 0$ | $\mathbb{Z}_2$ |
|               | $(-1, 2) \simeq f_1$ |           |               |
| $F_{14}$      | $(-1, 0) \simeq f_0$ | $\sigma_1 = 2\sigma_2$ | $\mathbb{Z}$ |
|               | $(-1, 2) \simeq f_1$ |           |               |
|               | $(2, -1) \simeq f_2$ | $\sigma_3 = \sigma_1 + \sigma_2$ |               |
|               | $(0, -1) \simeq f_3$ |           |               |
| $F_{15}$      | $(-1, -1) \simeq f_0$ | $2\sigma_2 = 0$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
|               | $(1, -1) \simeq f_1$ |           |               |
|               | $(1, 1) \simeq f_2$ | $\sigma_3 = \sigma_1 + \sigma_2$ |               |
|               | $(-1, 1) \simeq f_3$ |           |               |
| $F_{16}$      | $(-1, -1) \simeq f_0$ | $3\sigma_1 = 0$ | $\mathbb{Z}_3$ |
|               | $(2, -1) \simeq f_1$ |           |               |
|               | $(-1, 2) \simeq f_2$ | $\sigma_2 = 2\sigma_1$ |               |

**Table 2:** Toric sections corresponding to the reflexive polytopes and the toric subgroup of the Mordell-Weil group generated by them.
at a single sufficiently generic fiber. As we have just reviewed, a toric section $f_i = 0$ 

cuts out precisely one point on a generic elliptic curve contained in the fiber variety. Moreover, this point is rational if we choose the hypersurface to have rational coefficients and a rational point on the base. Note that an elliptic fibration is just an elliptic curve over the function field in the base, so the notions of sections and group law equally apply to the set of rational points $E(\mathbb{Q})$ on an elliptic curve $E$. The analogue of a toric section is a toric rational point, that is, a point where one of the homogeneous variables is equal to zero. We review the geometric group law in Appendix A. The generic Mordell-Weil group $E(\mathbb{Q})$ obtained in this way only depends on the fiber polygon, but not on the details of the embedding in a higher-dimensional reflexive polytope. However, note that special fibers, that is, over special points in the base or the restriction of non-generic hypersurface equations will in general have a different $E(\mathbb{Q})$.

We now apply this to the 16 toric surfaces that can arise as the ambient space of the generic fiber. We first pick a toric section, say $f_0$ in the notation of Table 2, as the zero section. The other sections, let us call them $\sigma_i = f_i - f_0$, $i > 0$, (7) 

then satisfy relations among them which determine the toric Mordell-Weil group $\text{MW}_T(X) \subseteq \text{MW}(X)$. All fiber polygons with sections contain the $\mathbb{P}^2$ polygon $F_1$, so all of their Calabi-Yau hypersurface equations are specializations of the cubic in $\mathbb{P}^2$. By picking a sufficiently generic fiber $E$ and applying the $9:1$ map from the cubic to the Weierstrass equation, we map the $f_i$ to specific rational points on the Weierstrass model of $E$. There, the group structure on $E(\mathbb{Q})$ can easily be determined. Comparing the generators for $E(\mathbb{Q})$ with the restrictions $\sigma_i|_E$, we obtain relations between the $\sigma_i|_E$. Up to 3-torsion, these must also hold in the toric Mordell-Weil group $\text{MW}_T(X)$ of the Calabi-Yau manifold $X$. However, there can be extra factors of $\mathbb{Z}_3$ owing to the fact that the map to the Weierstrass form includes a $3^2 : 1$ isogeny. Using the geometric group law in the actual fiber $E$, we have verified which relations hold directly and which hold only modulo three-torsion. The result are the actual relations between the toric sections listed in Table 2. Incidentally, and somewhat disappointingly for phenomenology purposes, the three polygons whose toric Mordell-Weil groups contain non-trivial torsion subgroups turn out to be precisely those that do not support an $SU(5)$ top.

Finally, let us stress that from a physical point of view, the presence of non-toric sections poses no problems. In practice, a vast number of torically constructed Calabi-Yau manifolds has such sections and while their defining equations may not be as easily found as in the toric case, for practical purposes it often suffices to know their homology classes [16].

5For example, the three toric sections in $F_{16}$ all map to the same section of the Weierstrass model.
3 All SU(5) Tops

By definition, a top \cite{20, 21} is the preimage of a base ray in a toric morphism where the Calabi-Yau hypersurface (in the total space) is a genus-1 fibration. This preimage is a lattice polytope with one facet, namely the kernel of the projection, passing through the origin. This special facet passing through the origin is the fiber polygon. It is again reflexive if the total space polytope is, and hence must be one of the 16 polygons shown in Figure 2. For example, consider an elliptically fibered K3 constructed as a hypersurface in a three-dimensional fibered toric variety. Then the three-dimensional reflexive polytope is split into two tops by the projection to the \( \mathbb{P}^1 \)-fan. The top determines the structure of the toric variety over the base divisor specified by the base ray. In particular, the top often enforces a particular Kodaira fiber in the hypersurface.

In the following, we will be particularly interested in elliptic fibrations with I\(_5\) Kodaira fibers, leading to SU(5) GUTs. In general, the SU(\(n\)) tops are spanned by two parallel facets lying in two lattice planes of distance one. For convenience, let us pick coordinates \((x, y, z)\) such that the fiber polygon vertices are at \(z = 0\) and all remaining vertices at height \(z = 1\). The circumference \(n\) of the facet at \(z = 1\) equals \(n\). Hence, an SU(5) top is defined by the fiber polygon together with a polygon with circumference 5 \cite{33, 34}. The defining data for all 37 SU(5)-tops is shown in Figure 3.\(^7\) Note that the \(GL(3, \mathbb{Z})\)-subgroup generated by \((x, y, z) \mapsto (x + \alpha z, y + \beta z, z)\) still acts on the tops after fixing the fiber polygon and therefore the \(x, y\) coordinates of the facet at \(z = 1\) can be shifted arbitrarily. This is why in the origin is only marked for the fiber polygons in Figure 3 but not for the polygons at height one.

There is another useful notation for tops. The dual polyhedron \(\tau^*\) of a top \(\tau\) is a non-compact semi-infinite prism whose cross-section is the dual fiber polygon \(F^*\). One of the dual vertices is \((x^*, y^*, z^*) = (0, 0, -1)\), the dual of the facet at \(z = 1\). The \(z^*\) value at which the semi-infinite prism is cut off defines a function on the remaining lattice points \(F^* \cap \mathbb{Z}^2\) of the dual fiber polygon. This function \(z^*: \partial F^* \cap \mathbb{Z}^2 \rightarrow \mathbb{Q}\) defines the top as

\[
\tau^* = \text{conv} \left\{ (p_x, p_y, z^*(p)) \mid p \in \partial F^* \cap \mathbb{Z}^2 \right\} + \mathbb{R}_+ \cdot (0, 0, 1), \tag{8}
\]

which is an equivalent notation to specify a top. Again, there is a residual \(GL(3, \mathbb{Z})\) symmetry action on the coordinate choices, since two functions \(z^*\) that differ only by a linear function define the same top. Shifting by a linear function, one can always bring \(z^*\) in a form where \(z^*(p) \geq -1\) for all boundary points and this convention has \(^6\)That is, the number of lattice points on the boundary.

\(^7\)Note that symmetries of the tops are divided out. For example, the \(F_5\) fiber polygon is symmetric under a \(\mathbb{Z}_2\) automorphism. It supports 5 distinct tops, of which one is symmetric under the \(\mathbb{Z}_2\) and four are not. If one were to disregard the symmetry, one would arrive at the 9 tops of \cite{15}.
been used in Figure 3.

In general, one can use a top to construct a local Kodaira fiber. This is the most general definition of a top, and has been used in [22]. Of course, not every local Kodaira fiber can appear in a compact elliptic fibration. However, all of the $SU(5)$ tops actually appear in elliptic K3 fibrations, that is, the tops occur in the list of 1052 half-K3 polytopes [20].

4 Determining $U(1)$ Charges

Having identified all toric Mordell-Weil generators for the 16 two-dimensional reflexive polytopes, an obvious question is whether one can make any general statements about matter representations charged under the $U(1)$ gauge groups corresponding to the Mordell-Weil generators. Similar to Section 2, it turns out that information about toric $U(1)$s is already contained in the top alone [16, 17]. For illustration, let us restrict ourselves to gauge groups such that the non-Abelian factor is $SU(5)$. However, nothing really depends on this choice and our reasoning can readily be extended to other gauge groups. Generic F-theory compactifications only give rise to two distinct non-trivial $SU(5)$ representations, 5 and 10. In particular, this is the case for generic hypersurfaces in toric varieties. This is because, generically, the rank 4 gauge group will only be enhanced to rank 5, that is, $SU(6)$ and $SO(10)$. The matter fields then arise from decomposing their adjoint representations. We begin by studying the constraints on the $U(1)$ charges of the 5 representation before proceeding with the latter representation.

4.1 Different Splits and the Fundamental Representation

Let us now examine the possible $U(1)$ charges of matter in the 5 representation in a systematic manner. In many ways, this is a straightforward generalization of the example analyzed in [16] for the case of a single $U(1)$ and we therefore reiterate the arguments made in the above paper in the next paragraphs.

By definition, a section cuts out a single point in the fiber of our elliptically fibered Calabi-Yau manifold over a dense subset of the base manifold $B$. Over generic points of the GUT divisor, however, the torus fiber degenerates. In the case of an $SU(5)$ gauge group, it splits into an $I_5$ Kodaira fiber consisting of 5 $\mathbb{P}^1$ components intersecting as the affine Dynkin diagram $\tilde{A}_4$. There is no reason for the sections of the compactification manifold to mark points on the same $\mathbb{P}^1$, and, in general, they

---

*If one requires the section to be holomorphic, then this is in fact true for all of $B$. Rational sections, however, may wrap entire fiber components over certain lower-dimensional loci of the base, see [13, 15, 16, 19] for related discussions.
**Figure 3:** The SU(5) tops based on the 16 reflexive polygons. Numbers next to boundary points of the facet in the z = 1 plane indicate which toric sections intersect the associated exceptional divisor.
Figure 3: (continued) The $SU(5)$ tops $\tau_{i,j}$ based on the 16 reflexive polygons. For each reflexive polygon (the fiber polygon at $z = 0$), the admissible facets at $z = 1$ are listed. Below each the values of $z_*$ on the vertices of the dual polygon (in clockwise order, starting at the “y”-axis) are given, which also an equivalent way of specifying the top. See discussion at the beginning of Section 3.
will intersect different \( \mathbb{P}^1 \) irreducible components. One is always free to designate any one of the sections to be the \textit{zero section}, that is, one takes the point marked by this section to be the identity element on the elliptic curve. Physically, the closed two-form corresponding to the zero section gives rise to the Kaluza-Klein vector \( A_\mu^0 \) in the expansion of the RR three-form. By choosing a zero section, one implicitly fixes the affine node of \( A_4 \) as the irreducible component of the \( I_5 \) Kodaira fiber that the zero section intersects.

Having designated a zero section, one then identifies the remaining \( \mathbb{P}^1 \) components with the simple roots of \( \mathfrak{su}(5) \) such that their intersections reproduce the inner products of the associated coroots. Note that identifying the \( \mathbb{P}^1 \) components with simple roots of \( \mathfrak{su}(5) \) is unique only up to a \( \mathbb{Z}_2 \) ambiguity corresponding to the outer automorphism \( D_i \leftrightarrow D_{5-i} \). After eliminating said ambiguity, one is left with only three possible intersections patterns in the case of a single \( U(1) \) generator, namely the first three of Figure 4. The physics behind these three different intersection patterns is that they constrain the charges of the 5 representation up to multiples\(^9\) of 5 in the non-trivial cases, that is, except for the 0-5 split. Concretely, the relation is

\[
U(1)\text{-generator with } i-(5-i)\text{-split } \Leftrightarrow \quad Q_{U(1)}(5) \equiv i \mod 5. 
\] (9)

\(^9\)More precisely, the charges are determined up to a multiple of 5 after rescaling the naive \( U(1) \) generator, obtained by applying the Shioda map [35, 36] to the Mordell-Weil generator, by a factor of 5 to make all charges integral.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The different splits for the case in which \( \sigma_0 \) denotes the zero section and \( \sigma_1 \) is one of possibly more independent Mordell-Weil generators. In the case of a single \( U(1) \), the \( i-(5-i)\)-split and the \( (5-i)-i\)-split are equivalent under to the \( \mathbb{Z}_2 \) outer automorphism of \( \mathfrak{su}(5) \).}
\end{figure}
This interpretation of the intersection pattern generalizes easily to the case of multiple Abelian gauge group factors. Having fixed a zero section and one of the two choices for assigning simple roots of $\mathfrak{su}(5)$ to the irreducible fiber components, one can compute the intersection numbers between the fiber $\mathbb{P}^1$'s and any given Mordell-Weil generator $\sigma_m$. For sections $\sigma_j$ with $j = 1, \ldots, n_{U(1)}$ intersecting the $i_j$-th fiber component we then find that all our matter representations have $U(1)$ charges satisfying

$$Q_{U(1)}(5) \equiv i_j \mod 5.$$  \hfill (10)

The intersection structure between toric sections and the irreducible fiber components is already fixed by the top alone. In fact, one can easily read off the intersection numbers from the geometry of the top. As reviewed in Section 3, a top is a three-dimensional polyhedron such that the origin is interior to a facet. In the case of $SU(n)$ gauge groups, it consists of two parallel polygons. One of these, the fiber polygon, is the facet containing the origin. It is a reflexive sub-polytope. The other polygon lies at height 1 and its integral boundary points reproduce the affine Dynkin diagram $\tilde{A}_n$. Now, let $v$ be a vertex of the fiber polygon corresponding to a toric section $\sigma$. Furthermore, let us denote the lattice points corresponding to the $i$-th fiber component by $w_i$. Then $\sigma$ intersects the $i$-th fiber component if and only if $v$ and $w_i$ share an edge. Using this prescription, we have determined the intersection numbers for all $SU(5)$ tops in Figure 3 by listing the sections intersecting a certain

**Figure 5:** Two different 3D visualizations of the entire top $\tau_{5,3}$. Fiber vertices corresponding to sections and lattice points associated to exceptional divisors intersecting them are colored red, as are the edges connecting them.
exceptional divisor next to the corresponding lattice point of the $z = 1$ facet of the top. To give an example, consider the top $\tau_{5,3}$. From Table 2, we see that the toric sections generate a subgroup $\text{MW}_T \cong \mathbb{Z} \oplus \mathbb{Z} \subseteq \text{MW}$ of the entire Mordell-Weil group. Now, pick $f_0$ as the zero section and assign simple roots $\alpha_i$ in clockwise order to the boundary points of $\tau_{5,3}$. Taking $f_1$ and $f_2$ as sections generating $U(1)_1$ and $U(1)_2$, we find that the charges of the $5$ representations must satisfy

$$Q_{U(1)_1}(5) \equiv 1 \mod 5 \quad \text{and} \quad Q_{U(1)_2}(5) \equiv 3 \mod 5.$$  \hfill (11)

Using the algorithm described in the next section, one can complete this top to a five-dimensional reflexive polyhedron. For simplicity, we choose the base manifold to be $\mathbb{P}^3$ and find that there are 30 inequivalent embeddings, one of which we list in Table 3. Independent of the chosen triangulation, one finds that a generic hypersurface inside this space possesses three distinct curves in the base over which the $SU(5)$ singularity is enhanced to $SU(6)$, giving rise to matter in the 5 representation of $SU(5)$. Going through similar calculations as for example in \cite{15,16,18}, one finds that the $U(1)$ charges of these three curves with respect to the generators associated with $(-1, -1, 0)$ and $(-1, 0, 0)$ are

$$5_{-4,-2}, \ 5_{1,-2}, \text{ and } 5_{1,3},$$  \hfill (12)

in agreement with eq. (11).

Finally, let us point out that the above notion of splits agrees with the cases that have been analyzed with the split spectral cover constructions \cite{3,5,23,24} only in the case of a single $U(1)$. As soon as there are multiple Abelian gauge symmetries,
our notation describes the “split” between the section generating the particular $U(1)$ symmetry and the zero section, whereas the split spectral cover constructions denote by split the factorization pattern of the spectral cover. Hence, when there are multiple $U(1)$s we determine a split with respect to each one of them.

4.2 A No-Go Theorem for the Antisymmetric Fields

We now turn to the $10$ matter fields of the $SU(5)$ GUT. Somewhat surprisingly, their geometric origin is rather different from the $5$ matter fields. Recall that the $5$ matter fields come from individual $\mathbb{P}^1$ in the $I_5$ Kodaira fiber degenerating into two irreducible components. This kind of degeneration will never yield a codimension-two $I_1$ Kodaira fiber where the $10$ matter field is localized: Splitting nodes of the $I_5$ Kodaira fiber will never eliminate the fundamental group $\pi_1(I_5) = \mathbb{Z}$ of the Kodaira fiber, but $\pi_1(I_1) = 0$. The only way to obtain a simply connected fiber is to have the hypersurface equation vanish identically on a toric curve of the top. That is, along the intersection of the irreducible components of the toric surfaces in the fiber of the ambient toric variety. Note that the irreducible components of the two-dimensional ambient space fiber correspond to the vertices of the top that are not interior to a facet and not part of the fiber polygon. They intersect in a toric curve $\simeq \mathbb{P}^1$ whenever the triangulation induced by the fan joins two vertices.

As we will see in Subsection 6.1 if a $SU(5)$-top contains a point interior to a facet then the fibration is not flat. This means that the low-energy theory is not an ordinary gauge theory and we only have to focus on tops without facet interior points. For an $SU(5)$ top this means that the facet at height $z = 1$ is a degenerate lattice pentagon with one of the lattice points at a midpoint of an edge. Up to isomorphism, there is only a single such lattice pentagon, see Figure 3. There are two fine triangulations $T_1$ and $T_2$ of this boundary facet and they are shown on the left hand side of Figure 6. Regardless of the triangulation, the degenerate toric ambient space fiber consists of 5 irreducible surfaces $V(d_1), \ldots, V(d_5)$. These always intersect cyclically in toric curves, that is, $V(d_i) \cap V(d_{i+1}) \simeq \mathbb{P}^1$. Depending on the triangulation, they additionally intersect as the internal 1-simplices in the triangulation, that is,

- Triangulation $T_1$: $V(d_1) \cap V(d_4) \simeq \mathbb{P}^1$ and $V(d_1) \cap V(d_3) \simeq \mathbb{P}^1$,
- Triangulation $T_2$: $V(d_1) \cap V(d_4) \simeq \mathbb{P}^1$ and $V(d_2) \cap V(d_4) \simeq \mathbb{P}^1$.

The Calabi-Yau hypersurface generically intersects the toric $\mathbb{P}^1$’s corresponding to the boundary 1-simplices in a point, and is a non-zero constant on the toric $\mathbb{P}^1$ corresponding to the internal 1-simplices. As argued in the beginning of this section, the $10$ matter is localized when the whole toric $\mathbb{P}^1$ is contained in the hypersurface, that
Figure 6: Left: The two possible fine triangulations of the lattice polygon at height $z = 1$ in the $SU(5)$-top. Right: The corresponding degeneration of the $I_5 \rightarrow I_1^*$ Kodaira fiber.

is, where the above constant happens to be zero. Since the internal 1-simplices are internal to the same facet of the top, the hypersurface always vanishes simultaneously on both toric curves. These two toric $P^1$s intersect in a toric point, the containing 2-simplex. Hence they form two nodes joined by an edge in the dual fiber diagram, which will turn out to be the middle two nodes of the $D_5$ extended Dynkin diagram.

Intersecting the hypersurface $Y$ with the ambient space irreducible surface components of a fiber yields additional curve components for the degenerate elliptic fiber. These necessarily contain the toric curves of the adjacent internal 1-simplices as irreducible components. For example, in triangulation $T_1$ the intersection $Y \cap V(d_1)$ contains both toric surfaces $V(d_1) \cap V(d_4)$ and $V(d_1) \cap V(d_3)$ as irreducible components. Likewise, $Y \cap V(d_5)$ contains none of the toric $P^1$ since the vertex $d_5$ is not adjacent to an interior 1-simplex. This fixes the degeneration of the $I_5$ Kodaira fiber, that is the 5 curves $Y \cap V(d_i)$ away from the matter curve, to be the one shown on the right hand side of Figure 6.

This is the key observation: The triangulation of the top fixes the degeneration of the codimension-one Kodaira fiber at the codimension-two $10$ matter curves of a toric hypersurface. Since the triangulation is fixed for a given manifold, the degeneration is the same for all $10$. Importantly, this behavior is different from that of the $5$ matter curves, where different degenerations occur over different codimension-two fibers. As a corollary, the $U(1)$ charges of all $10$ matter representations are equal. In other words, if one wants to construct F-theory GUTs such that the $10$ fields carry different $U(1)$ charges then one needs to consider complete intersections such that

---

10 This is at a codimension-one curve of the discriminant, that is, it is of codimension two in the base.
the fiber is at least codimension-two in the ambient space fiber \cite{37, 25}. Although we will not attempt such a construction in this paper, let us note that the toric machinery applies equally. The only difference is that one would construct a fibration such that the toric ambient space fiber is higher dimensional, for example, a reflexive 3-dimensional polytope instead of a reflexive polygon. Such a complete intersection top is then a 4-dimensional polytope with one facet being the 3-dimensional fiber polytope. The complete intersection tops can be combined into reflexive polytopes as described in \textit{Section 5}.

\section{The Polytope of Compactifications}

By definition, the top describes the degeneration of the elliptic fiber over a toric divisor in the base. The base divisor is one of the rays in the base fan. The obvious question is how this data can be completed to a compact Calabi-Yau manifold, that is, how to complete the top and the choice of base fan to a reflexive polytope. In fact, this has a nice answer: The remaining choices for a lattice polytope after fixing the tops and the base again amount to the integral points of a polytope. This just follows from convexity, and one needs to verify reflexivity and flatness of the fibration by hand.

In particular, we will be interested in the case of a single $SU(5)$ top together with trivial tops over the remaining rays of the base fan. For the purposes of this section, we will only consider the case where the base fan equals $\mathbb{P}^n$, whose rays are generated by the unit vectors $e_1, \ldots, e_n$ together with $-\sum e_i$. Then

- The fixed $SU(5)$ top can be chosen to project to $[0, e_1]$.
- The single point generating the trivial top over each of $e_2, \ldots, e_n$ can be chosen to have fiber coordinates $(0, 0)$ by a $GL(n, \mathbb{Z})$ rotation fixing all previous tops.
- The final point, generating the trivial top over $-\sum e_i$, has coordinates $(p_1, p_2) \in \mathbb{Z}^2$ with no remaining freedom of coordinate redefinition.

This parametrizes the choices of completion to a polytope by a pair of integers $(p_1, p_2)$. These are constrained by convexity: Having fixed the height-one points of the other tops, there is only a finite range of $(p_1, p_2)$ such that the fiber (preimage of the origin in the base) of the convex hull does not exceed the chosen fiber polygon. These are linear constraints, turning the allowed region for $(p_1, p_2)$ into a polygon (with not necessarily integral vertices).

It turns out that all lattice polytopes for a single $SU(5)$ top over $\mathbb{P}^n$ that one constructs just by demanding convexity, as above, are automatically reflexive. Their \footnote{This polytope is not necessarily integral, that is, its vertices are in general rational.}
total number for small values of $n$ is listed in Table 4. We included also the cases $\mathbb{P}^4$ and $\mathbb{P}^5$ that, when used as base, would not lead to gauge theories in four dimensions. However, the construction can be supplemented by additional polynomials specifying the actual base as hypersurface in $\mathbb{P}^4$ or complete intersection in $\mathbb{P}^5$. For example, the Fano threefold obtained by a quartic constraint in $\mathbb{P}^4$ is a viable choice for the base. Note that realizing the base itself as hypersurface or complete intersection can be also phenomenologically motivated. Such realizations allow for more exhaustive choices of fluxes on the GUT brane as demonstrated in the models of [4, 6]. This applies in particular to hypercharge flux [38, 39, 40] that is non-trivial on the GUT brane but trivial on the entire base manifold. Our construction thus extends straightforwardly to these more involved Calabi-Yau fourfold examples.

6 Flatness of the Fibration

Not all compactifications of F-theory give rise to ordinary gauge theories, as they may contain tensionless strings yielding an infinite tower of massless fields in the low-energy effective action. While there is nothing wrong with that, these theories have to be excluded when one looks for phenomenologically viable theories. Alternatively, one can try to lift all but finitely many of these massless fields through fluxes, but we will leave this for future work. The geometric origin of these massless strings [41, 42, 32, 43] are three-branes wrapping a curve inside a surface of vanishing volume in the F-theory limit. Such a surface must necessarily sit over a point in the discriminant locus, that is, in a fiber of the elliptic fibration that is at least two-dimensional. Clearly, this cannot happen if all degenerate fibers are of Kodaira type. Hence, any $K3$ hypersurface in a toric variety constructed by gluing two tops along the fiber polygon has all fibers one-dimensional. A fibration with the property that all fibers are of the same dimension is called flat.\footnote{Flat in the sense of homological algebra, that is, the functions in a neighborhood of each fiber are a flat module over the function ring of the base.}

For the case of hypersurfaces in toric varieties, there are two possible sources for non-flat fibers:

- The ambient toric fiber can jump in dimension. That is, the toric fibration of the ambient space can already fail to be flat [44]. This happens in particular if one places two non-Abelian tops on neighboring base rays such that the intersection is not a Miranda model [45].

- Even if the ambient toric fibration is flat, the hypersurface equation can vanish identically in the fiber direction for certain fibers. Then the fiber of the elliptic fibration becomes two-dimensional.
| Fiber | Top  | $N_{p^n}^{SU(5)}$ | $N_{p^2}^{SU(5)}$ | $N_{p^3}^{SU(5)}$ | $N_{p^4}^{SU(5)}$ | $N_{p^5}^{SU(5)}$ |
|-------|------|-------------------|-------------------|-------------------|-------------------|-------------------|
| $F_1$ | $\tau_{1,1}$ | 1 | 5 | 12 | 22 | 35 |
| $F_1$ | $\tau_{1,2}$ | 1 | 5 | 12 | 22 | 35 |
| $F_1$ | $\tau_{1,3}$ | 1 | 4 | 8 | 14 | 21 |
| $F_1$ | $\tau_{1,4}$ | 1 | 5 | 12 | 22 | 35 |
| $F_1$ | $\tau_{1,5}$ | 1 | 5 | 11 | 18 | 27 |
| $F_2$ | $\tau_{2,1}$ | 2 | 9 | 20 | 30 | 42 |
| $F_2$ | $\tau_{2,2}$ | 3 | 10 | 21 | 36 | 55 |
| $F_2$ | $\tau_{2,3}$ | 3 | 8 | 15 | 24 | 35 |
| $F_3$ | $\tau_{3,1}$ | 2 | 9 | 20 | 35 | 54 |
| $F_3$ | $\tau_{3,2}$ | 3 | 10 | 21 | 36 | 55 |
| $F_3$ | $\tau_{3,3}$ | 3 | 10 | 21 | 36 | 55 |
| $F_3$ | $\tau_{3,4}$ | 3 | 10 | 21 | 36 | 55 |
| $F_3$ | $\tau_{3,5}$ | 3 | 10 | 21 | 36 | 55 |
| $F_3$ | $\tau_{3,6}$ | 3 | 10 | 21 | 36 | 55 |
| $F_3$ | $\tau_{3,7}$ | 3 | 10 | 21 | 36 | 55 |
| $F_4$ | $\tau_{4,1}$ | 3 | 10 | 21 | 36 | 55 |
| $F_4$ | $\tau_{4,2}$ | 3 | 10 | 21 | 36 | 55 |
| $F_5$ | $\tau_{5,1}$ | 6 | 12 | 20 | 31 | 44 |
| $F_5$ | $\tau_{5,2}$ | 5 | 15 | 30 | 50 | 75 |
| $F_5$ | $\tau_{5,3}$ | 5 | 15 | 30 | 50 | 75 |
| $F_5$ | $\tau_{5,4}$ | 6 | 16 | 31 | 51 | 76 |
| $F_5$ | $\tau_{5,5}$ | 5 | 15 | 30 | 50 | 75 |
| $F_6$ | $\tau_{6,1}$ | 6 | 16 | 31 | 51 | 76 |
| $F_6$ | $\tau_{6,2}$ | 6 | 16 | 31 | 51 | 76 |
| $F_6$ | $\tau_{6,3}$ | 5 | 15 | 30 | 50 | 75 |
| $F_6$ | $\tau_{6,4}$ | 5 | 15 | 30 | 50 | 75 |
| $F_6$ | $\tau_{6,5}$ | 6 | 16 | 31 | 51 | 76 |
| $F_7$ | $\tau_{7,1}$ | 8 | 18 | 30 | 45 | 63 |
| $F_8$ | $\tau_{8,1}$ | 8 | 21 | 40 | 65 | 96 |
| $F_8$ | $\tau_{8,2}$ | 8 | 21 | 40 | 65 | 96 |
| $F_9$ | $\tau_{9,1}$ | 8 | 21 | 40 | 65 | 96 |
| $F_9$ | $\tau_{9,2}$ | 8 | 21 | 40 | 65 | 96 |
| $F_{10}$ | $\tau_{10,1}$ | 8 | 21 | 40 | 65 | 96 |
| $F_{11}$ | $\tau_{11,1}$ | 11 | 27 | 50 | 80 | 117 |
| $F_{12}$ | $\tau_{12,1}$ | 11 | 27 | 50 | 80 | 117 |
| $F_{14}$ | $\tau_{14,1}$ | 14 | 23 | 38 | 57 | 80 |

**Table 4:** Number $N_{p^n}^{SU(5)}$ of reflexive polytopes fibered over $\mathbb{P}^n$ with one $SU(5)$-top and $n$ trivial tops, modulo fiber-preserving automorphisms.
The flatness of the ambient toric fibration can easily be checked \cite{46, 47, 44} using toric methods. In particular, this is always the case when only a single non-trivial top is used. Hence, we will focus in the remainder of this paper on the second source for non-flat fibers. In this case, the non-flat fibers do not generally lie over toric fixed points.

6.1 Codimension Two Fibers

While elliptic $K3s$ are always flat fibrations, a toric Calabi-Yau threefold hypersurface can be non-flat even if the ambient toric fibration is. These codimension-two (over the base, codimension one inside the discriminant) non-flat fibers come from lattice points interior to the $z = 1$ facets. This is because a point interior to a facet corresponds to a toric divisor such that the hypersurface equation restricts to a (generically non-zero) constant. However, a point interior to a facet of the top is usually not interior to a facet of the entire 4-d polytope. This means that the hypersurface equation is not constant on the corresponding divisor in the ambient space, but only in the fiber direction. In fact, this fiber-wise constant is a section of a nef line bundle over the (toric) discriminant component, and therefore has a zero somewhere. This is the location of the non-flat fiber.

There is one loophole in the argument: If the base ray, over which the non-trivial top is placed, is itself a point interior to a facet of the base polytope, then a point interior to a facet of the top is also interior to a facet of the 4-d polytope. Geometrically, this means that the discriminant component is a curve of self-intersection $-2$ and the hypersurface again avoids the corresponding toric divisor entirely. However, this is not a physically desirable situation: The hypersurface equation restricted to this discriminant component is now independent of the point along the discriminant. Therefore, there are no codimension-two degenerations at all, and in particular no matter curves. Hence we will not consider this case in the following, and only allow tops with no points interior to facets.

For example, consider the del Pezzo surface of degree 7, that is $F_5$ in Figure 3, as the fiber polygon. Then one of the tops, namely $\tau_{5,1}$, will have non-flat fibers and the remaining 4 tops $\tau_{5,2}, \ldots, \tau_{5,5}$ yield flat fibrations in codimension two.

6.2 Flatness Criterion

Having described the flatness criterion for codimension-two fibers, we now proceed to generalize it to arbitrary codimension. As an example, we then apply it to the physically relevant case of codimension-three fibers in elliptically fibered fourfolds.

By an analogous argument as in the previous section, a Calabi-Yau hypersurface in
an ambient flat fibration will be flat itself if the hypersurface equation never vanishes identically in the fiber direction. For simplicity, consider the case where there is only a single non-trivial top. To understand the hypersurface equation we collect the monomials of the hypersurface equation \( p = 0 \) by their dependence on the top homogeneous coordinates \( z_\tau = \{z_{\tau,1}, \ldots, z_{\tau,k}\} \) as

\[
p(z_\tau, z) = \sum_{\vec{\nu}=\{i_1, \ldots, i_k\}\in I} z_\tau^\vec{\nu} \ p_\tau(z) = \sum_{\vec{\nu}\in I} z_\tau^\vec{\nu} \left( \sum_{\vec{\mu}\in J_\tau} a_{\tau,\vec{\mu}} z_\tau^\vec{\mu} \right), \quad a_{\tau,\vec{\mu}} \in \mathbb{C}.
\]  

(13)

The irreducible components of the degenerate fiber induced by the top are the toric divisors \( z_{\tau,\ell} = 0 \) corresponding to the integral points of the top that are not in the fiber polygon. One needs to check for every irreducible component that the fibration is flat. The irreducible fiber component \( z_{\tau,\ell} = 0 \) projects to one discriminant component \( D_\tau \), and the local coordinates on \( D_\tau \) are the rays in the star of \( \pi(\tau) \) in the base. Each of the polynomials \( p_\tau(z) \) only depends on the base coordinates and therefore defines a divisor \( V_\tau(p_\tau) \subset D_\tau \) on the discriminant. Then a generic hypersurface is a flat fibration over \( D_\tau \) if these divisors do not meet, that is,

\[
D_\tau \ni \bigcap_{\vec{\mu}\in J_\tau} V_\tau(p_\tau) = \bigcap_{\vec{\nu}=(i_1, \ldots, i_k)\in I} V_\tau(p_\tau) = \emptyset.
\]  

(14)

The summation range \( I \) is over all fiber monomials, that is, integral points of the dual of the top polytope intersected with the projection of the dual polytope of the ambient toric variety. The summation range \( J_\tau \) is the fiber of the projection of the dual polytope, that is, over all integral points of the dual polytope whose monomial is divisible by \( z_\tau^\vec{\nu} \).

Phrasing Subsection 6.1 in this language, if \( z_{\tau,\ell} \) corresponds to the ray generated by an integral point interior to a facet of the top then \( I = \{\vec{\nu}\} \) consists of only a single element. The corresponding divisor \( V_\tau(p_\tau) \subset D_\tau \) will generically be non-empty and, therefore, the fibration non-flat. The only loophole is if the divisor is empty, that is, \( J_\tau = \{\vec{\mu}\} \) consists of a single point which then must be a vertex of the dual polytope. But this means that the point was not just interior to a facet of the top, but interior to a facet (dual to \( \vec{\mu} \)) of the ambient toric variety.

### 6.3 Codimension Three Fibers

We now apply the flatness criterion to Calabi-Yau fourfold hypersurfaces. As we will see, flatness is a non-generic property in the sense that it imposes additional equations on the polytope of compactifications (as defined in Section 5). Hence, the flat fourfold fibrations are identified with integral points in a strictly smaller-dimensional polytope than the set of all convex lattice polytopes with the specified top and base.
The new source for non-flat fibers are irreducible fiber components such that there are only two distinct fiber monomials. These are integral points of the top such that their dual face in the dual top contains exactly two points, that is, such that the dual face is an interval. In other words, the corresponding integral point of the top is along an edge of the top such that it is contained in only two 2-faces. Note that this is the case for every SU(5)-top that is not already non-flat in codimension 2 due to an integral point interior to a facet. This is because the polygon at height $z = 1$ of the SU(5)-top, see Figure 3, is a lattice polygon with circumference 5 in lattice units. But such a lattice polygon has either an interior point or is degenerate, see also Figure 3. Therefore, each SU(5) top that is flat in codimension-two yields a non-trivial flatness condition in codimension-three associated to the integral point on the edge.

For simplicity, let us assume that the discriminant component $D_\tau$ of the SU(5)-top is a toric surface where any two effective divisors intersect. This will always be the case in the examples below, where we will be using $D_\tau = \mathbb{P}^2$. Consider now the toric divisor $z_\tau,\ell = 0$ corresponding to the integral point interior to an edge. The index set $I = \{\bar{i}^{(0)}, \bar{i}^{(1)}\}$ consists of two elements, corresponding to the two facets $F_{\tau,2}^{(0)}, F_{\tau,2}^{(1)}$ of the top adjacent to the edge. The fibration is then flat if and only if one of the divisors is trivial, say, $V_\tau(p_{\bar{i}^{(1)}}) \subset D_\tau$. This is the case if $J_{\bar{i}^{(1)}}$ contains a single element, which then must be a vertex of the dual ambient space polytope. Hence, the facet $F_{\tau,2}^{\bar{i}^{(1)}}$ of the top is contained in only a single facet of the ambient space polytope. As discussed in Subsection 6.1, this is not a particularly interesting case and we will ignore it in the following. Therefore, the facet $F_{\tau,2}^{\bar{i}^{(1)}}$ of interest is the one that contains at least one point of the fiber polygon.

### 6.4 Examples

The constraints from flatness of the fibration can rule out a fixed combination of top and base polytope. To see this explicitly, we will look at two examples in this section, namely the top $\tau_{10,1}$ and $\tau_{3,6}$, respectively, to construct an elliptic fibration over $\mathbb{P}^3$. Note that $\tau_{10,1}$ is the unique SU(5)-top in Weierstrass form, that is, with ambient space fiber $\mathbb{P}^2[1,2,3]$. The $\tau_{3,6}$ top was used in [16] and we will follow the same coordinate choice, which is different from Figure 3 but of course $GL(2,\mathbb{Z})$-equivalent to it. As described in Section 5, we can choose coordinates such that everything except the fiber coordinates of a single point are fixed. These are shown in Table 5.

Imposing convexity of the 5-d polytopes amounts to the inequalities

\[
\nabla_{10,1}(p_1, p_2) : \quad p_1 + p_2 \leq 4, \quad -p_1 + p_2 \leq 3, \quad p_1 - 2p_2 \leq 3
\]

\[
\nabla_{3,6}(p_1, p_2) : \quad p_1 + p_2 \leq 4, \quad -p_1 \leq 2, \quad p_1 - 2p_2 \leq 2, \quad -p_1 + p_2 \leq 3.
\]

"
The interior point of an edge in the $\tau_{10,1}$-top is $(-2, -1, 1, 0, 0)$. The relevant 2-face of the top for the flatness criterion is

$$F_{\tau_{10,1}, 2}^{(1)} = \langle (-3, -2, 1, 0, 0), (-1, 0, 1, 0, 0), (-3, -2, 0, 0, 0), (0, 1, 0, 0, 0) \rangle$$

This facet is contained in a the $(p_1, p_2)$-independent supporting hyperplane of the total polytope $\nabla_{10,1}(p_1, p_2)$ defined by

$$(1, -1, 0, -1, -1) \cdot \bar{x} + 1 = 0. \quad (17)$$

It is contained in further facets of $\nabla_{10,1}$ unless the final point $(p_1, p_2, -1, -1, -1)$ is also on this hyperplane, and therefore cannot span an independent facet. This is a linear equation for $(p_1, p_2)$. Together with the result for the second example, this equation is

$$\nabla_{10,1}(p_1, p_2): \quad p_1 - p_2 = -3$$
$$\nabla_{3,6}(p_1, p_2): \quad p_2 = 3 \quad (18)$$

The constraints coming from convexity and flatness are shown in Figure 7. We observe that there are many flat elliptic fibrations using the $\tau_{10,1}$ top, but none with the $\tau_{3,6}$ top.

### Table 5

Parametrization $(p_1, p_2) \in \mathbb{Z}^2$ of all polytopes with base $\mathbb{P}^3$ and top $\tau_{10,1}$ (left) and $\tau_{3,6}$ (right), respectively. The fibration is the projection on the last 3 coordinates.

| $\nabla_{10,1}(p_1, p_2)$ | fiber | base |
|-------------------------|-------|------|
| fiber                   | 1 0   | 0 0 0 |
|                         | 0 1   | 0 0 0 |
|                         | -3    | -2   | 0 0 0 |
| $\tau_{10,1}$           | 0 0   | 1 0 0 |
|                         | -1 0  | 1 0 0 |
|                         | -2   | -1  | 1 0 0 |
|                         | -3  | -2  | 1 0 0 |
|                         | -1  | -1  | 1 0 0 |
| trivial top             | 0 0   | 0 1 0 |
| trivial top             | 0 0   | 0 0 1 |
| trivial top             | $p_1$ | $p_2$ | -1 -1 -1 |

| $\nabla_{3,6}(p_1, p_2)$ | fiber | base |
|-------------------------|-------|------|
| fiber                   | 1 0   | 0 0 0 |
|                         | 0 1   | 0 0 0 |
|                         | -1 0  | 0 0 0 |
|                         | -1  | -1  | 0 0 0 |
| $\tau_{3,6}$            | -2   | -1  | 1 0 0 |
|                         | -1  | -1  | 1 0 0 |
|                         | -1  | 0  | 1 0 0 |
|                         | 0  | -1  | 1 0 0 |
|                         | 0 0  | 1 0 0 |
| trivial top             | 0 0   | 0 1 0 |
| trivial top             | 0 0   | 0 0 1 |
| trivial top             | $p_1$ | $p_2$ | -1 -1 -1 |

6.5 Flattening Base Change

It is perhaps unexpected that for $\mathbb{P}^3$, the simplest choice of base for a Calabi-Yau fourfold, the top $\tau_{3,6}$ cannot be used to construct a flat elliptic fibration. One might
speculate that this has something to do with the non-standard matter $U(1)$-charges that were found in [16]. However, this is not the case and one can easily build flat elliptic fibrations over other bases, leading to 4-d $SU(5)$ GUT gauge theories with matter charges among $5_{-2}$, $5_3$, $5_7$, $5_8$, $10_1$, and conjugates. In fact, already the base $P^1 \times P^2$ works for this purpose. Since it is instructional to consider a different base than just $P^n$, we will give some of the details of the possible reflexive polytopes for this base manifold.

First of all, not all divisors of the base are equivalent any more. For definiteness, we put the GUT divisor at $D_{\tau} = \{\text{pt.}\} \times P^2 \subset P^1 \times P^2$. Up to coordinate changes, there are now four integers parametrizing the possible embeddings in a 5-d polytope, see Table 6. The polytope of compactifications is now 4-dimensional, and contains 75 integral points. These are the 75 solutions to the convexity constraints. Again, it turns out that for this choice of base all polytopes that are allowed by convexity are actually reflexive. All have $h^{1,1} = 8$, corresponding to a single $U(1)$. Out of these, 3 polytopes yield a flat fibration. These are

$$(p_1, p_2; p_3, p_4) \in \{(0, 0; -3, -3), (0, 1; -3, -3), (1, 1; -3, -3)\}$$

Using the methods of [16], one readily verifies that the non-Abelian matter content once again consists of fields in the representations $5_{-2}$, $5_3$, $5_7$, $5_8$, $10_1$ of $SU(5) \times U(1)$. We hence conclude that it is easy to build 4-d $SU(5)$-GUT models with the $\tau_{3,6}$-top as long as the base is not $P^3$. 

**Figure 7:** The black points are the solution set $(p_1, p_2)$ for $SU(5)$ models with top $\tau_{10,1}$ (left) and $\tau_{3,6}$ (right) fibered over $\mathbb{P}^3$. The green polygon is the convexity constraint from eq. (15). The red line is the condition of flatness of the fibration, see eq. (18).
We introduced an algorithm to construct compact elliptically fibered Calabi-Yau fourfolds with specified gauge group and base space using toric geometry. This provides a systematic approach to geometrically engineering four-dimensional $\mathcal{N} = 1$ gauge theories coupled to gravity. While our approach is otherwise general, we restricted ourselves to situations in which the elliptic fiber is realized as a hypersurface inside a toric ambient space. To exemplify the steps involved, we constructed GUTs with non-Abelian gauge group $SU(5)$ while leaving the Abelian gauge sector unspecified. We described which parts of the construction can be performed independently of the base space and classified all fibers and $SU(5)$ tops. It is crucial to point out that a complete analysis depends on the entire Calabi-Yau space. First, the total number of $U(1)$ symmetries depends on the base and its embedding in the fibration, and we introduced an auxiliary polytope labeling all inequivalent choices for the latter. Second, we showed that ensuring flatness of the fibration imposes further non-trivial constraints that have to be checked for each base manifold individually. We formulated the flatness constraints as constraints on the toric data specifying the Calabi-Yau manifold. With these techniques at hand, we constructed numerous interesting fourfold examples with $SU(5) \times U(1)^n$ gauge group.

Using the methods from this paper, one is now in a position to perform a systematic scan over all elliptically fibered Calabi-Yau fourfolds embedded in toric ambient spaces over a fixed base. Only restricting the search to reflexive polytopes giving rise to a fixed
gauge group might turn this into a tractable problem with today’s computer power. One can then perform a survey of the different Abelian gauge factors. While we made progress towards the combinatorial geometry determining the matter spectrum and its $U(1)$ charges, there still remains work to be done. In particular, it would be desirable to determine all occurring matter representations from toric data alone, without needing to specify a particular hypersurface equation. Furthermore, it would be interesting to systematically study elliptic fibrations with fibers realized as complete intersections in higher-dimensional ambient spaces. This is in particular motivated by phenomenology, since we presented a no-go theorem ruling out different $U(1)$ charges for the antisymmetric representation of $SU(5)$. We therefore hope to return to this case in the future [25].

Having developed a formalism to extract the matter representations from a compact geometry, one has to include $G$-fluxes to obtain a chiral four-dimensional matter spectrum next. This can be done systematically by evaluating the three-dimensional Chern-Simons terms as in [48, 14, 16]. Doing so requires a detailed knowledge of allowed triangulations of the toric ambient spaces and, hence, of the phase structure in the three-dimensional gauge theory [48, 49]. Specifying the $G$-flux data allows one to completely determine the matter content, which is localized at codimension two in the base. At codimension three, a systematic analysis will be even more involved. Nevertheless, there is hope that model building using the billions of known Calabi-Yau fourfolds may soon become a tractable combinatorial problem, which would allow to extract generic features or general constraints of gauge theories coupled to gravity in F-theory.

Acknowledgments

We would like to thank Philip Candelas, Denis Klevers, Eran Palti, Raffaele Savelli, Sakura Schäfer-Nameki, and Timo Weigand for interesting discussions. The work of T.G. and J.K. is supported by a grant of the Max Planck Society.

A The Group Law on a Cubic

In this short appendix let us briefly review the geometric origin of the group law on the set of rational points of an elliptic curve. For a more exhaustive treatment of the extensive theory of elliptic curves we refer to the literature, e.g. [50]. We begin by discussing the group law in the case of the Weierstrass cubic. This is the affine curve

\[ E : \quad y^2 = x^3 + fx + g, \quad (20) \]
Figure 8: Example of the group law on the cubic $y^2 = x^3 + 2x + 3$; The point $P = (3 : 6 : 1)$ is, up to sign, the single generator of the Mordell-Weil group $E(\mathbb{Q}) \simeq \mathbb{Z}$. The graphics shows how to compute $P + P$.

or, by homogenizing, the cubic curve in $\mathbb{P}^2$ given by

$$E : y^2z = x^3 + fx^2z + gz^3. \quad (21)$$

Here, $(x : y : z)$ are homogeneous coordinates of $\mathbb{P}^2$ and the point at infinity is $O = (0 : 1 : 0)$. Given two rational points $P = (p_x : p_y : p_z)$ and $Q = (q_x : q_y : q_z)$, one can now define a group action as follows. If $P = Q$, we take $L$ to be the line tangent to $P$, otherwise we choose it to intersect both points. By Bézout’s theorem [51], $L$, being a curve of degree one, intersects $E$ exactly three times. Therefore, their intersection defines a third point $R$, which can be shown to be rational again. This naive map $(P, Q) \mapsto R$ does not form a group law yet, for example it lacks an identity element. However, this can be remedied. Take $L'$ to be another line through $R$ and the point at infinity. Then the third intersection point between $L'$ and $E$ defines the sum $P + Q$, which is now a group action on the set $E(\mathbb{Q})$ of rational points on $E$. In particular, $O$ is the identity element with respect to this action. Figure 8 illustrates the group law for the single generator of a rank 1 elliptic curve.

Finally, let us remark that choosing $L'$ to intersect $R$ and $O$ was an arbitrary choice. Replacing $O$ by any other rational point $O'$ still gives a valid group law with $O'$ now acting as the identity element with respect to this new group action.
References

[1] R. Donagi and M. Wijnholt, “Model Building with F-Theory,”
Adv.Theor.Math.Phys. 15 (2011) 1237–1318, 0802.2969

[2] C. Beasley, J. J. Heckman, and C. Vafa, “GUTs and Exceptional Branes in F-theory - I,”
JHEP 0901 (2009) 058, 0802.3391

[3] J. Marsano, N. Saulina, and S. Schafer-Nameki, “Monodromies, Fluxes, and Compact Three-Generation F-theory GUTs,”
JHEP 0908 (2009) 046, 0906.4672

[4] R. Blumenhagen, T. W. Grimm, B. Jurke, and T. Weigand, “Global F-theory GUTs,”
Nucl.Phys. B829 (2010) 325–369, 0908.1784

[5] J. Marsano, N. Saulina, and S. Schafer-Nameki, “Compact F-theory GUTs with U(1) (PQ),”
JHEP 1004 (2010) 095, 0912.0272

[6] T. W. Grimm, S. Krause, and T. Weigand, “F-Theory GUT Vacua on Compact Calabi-Yau Fourfolds,”
JHEP 1007 (2010) 037, 0912.3524

[7] C.-M. Chen, J. Knapp, M. Kreuzer, and C. Mayrhofer, “Global SO(10) F-theory GUTs,”
JHEP 1010 (2010) 057, 1005.5735

[8] J. Knapp, M. Kreuzer, C. Mayrhofer, and N.-O. Walliser, “Toric Construction of Global F-Theory GUTs,”
JHEP 1103 (2011) 138, 1101.4908

[9] T. Weigand, “Lectures on F-theory compactifications and model building,”
Class.Quant.Grav. 27 (2010) 214004, 1009.3497

[10] A. Maharana and E. Palti, “Models of Particle Physics from Type IIB String Theory and F-theory: A Review,”
1212.0555

[11] T. W. Grimm and T. Weigand, “On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs,”
Phys.Rev. D82 (2010) 086009, 1006.0226

[12] D. S. Park, “Anomaly Equations and Intersection Theory,”
JHEP 1201 (2012) 093, 1111.2351

[13] D. R. Morrison and D. S. Park, “F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds,”
JHEP 1210 (2012) 128, 1208.2695

[14] M. Cvetic, T. W. Grimm, and D. Klevers, “Anomaly Cancellation And Abelian Gauge Symmetries In F-theory,”
JHEP 1302 (2013) 101, 1210.6034
[15] C. Mayrhofer, E. Palti, and T. Weigand, “U(1) symmetries in F-theory GUTs with multiple sections,” JHEP 1303 (2013) 098, 1211.6742
[16] V. Braun, T. W. Grimm, and J. Keitel, “New Global F-theory GUTs with U(1) symmetries,” 1302.1854
[17] J. Borchmann, C. Mayrhofer, E. Palti, and T. Weigand, “Elliptic fibrations for SU(5) x U(1) x U(1) F-theory vacua,” 1303.5054
[18] M. Cvetic, D. Klevers, and H. Piragua, “F-Theory Compactifications with Multiple U(1)-Factors: Constructing Elliptic Fibrations with Rational Sections,” 1303.6970
[19] T. W. Grimm, A. Kapfer, and J. Keitel, “Effective action of 6D F-Theory with U(1) factors: Rational sections make Chern-Simons terms jump,” 1305.1929
[20] P. Candelas and A. Font, “Duality between the webs of heterotic and type II vacua,” Nucl.Phys. B511 (1998) 295–325, hep-th/9603170
[21] P. Candelas, A. Constantin, and H. Skarke, “An Abundance of K3 Fibrations from Polyhedra with Interchangeable Parts,” 1207.4792
[22] V. Bouchard and H. Skarke, “Affine Kac-Moody algebras, CHL strings and the classification of tops,” Adv.Theor.Math.Phys. 7 (2003) 205–232, hep-th/0303218
[23] E. Dudas and E. Palti, “On hypercharge flux and exotics in F-theory GUTs,” JHEP 1009 (2010) 013, 1007.1297
[24] M. J. Dolan, J. Marsano, N. Saulina, and S. Schafer-Nameki, “F-theory GUTs with U(1) Symmetries: Generalities and Survey,” Phys.Rev. D84 (2011) 066008, 1102.0290
[25] V. Braun, T. W. Grimm, and J. Keitel, “Work in progress.”
[26] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, et al., “Geometric singularities and enhanced gauge symmetries,” Nucl.Phys. B481 (1996) 215–252, hep-th/9605200
[27] S. H. Katz and C. Vafa, “Geometric engineering of N=1 quantum field theories,” Nucl.Phys. B497 (1997) 196–204, hep-th/9611090
[28] P. Candelas, E. Perevalov, and G. Rajesh, “Toric geometry and enhanced gauge symmetry of F theory / heterotic vacua,” Nucl.Phys. B507 (1997) 445–474, hep-th/9704097.
[29] S. Katz, D. R. Morrison, S. Schafer-Nameki, and J. Sully, “Tate’s algorithm and F-theory,” *JHEP* **1108** (2011) 094, [1106.3854](https://arxiv.org/abs/1106.3854) 2

[30] F. Rohsiepe, “Fibration structures in toric Calabi-Yau fourfolds,” [hep-th/0502138](https://arxiv.org/abs/hep-th/0502138) 2

[31] A. Grassi and V. Perduca, “Weierstrass models of elliptic toric K3 hypersurfaces and symplectic cuts,” *ArXiv e-prints* (Jan., 2012) [1201.0930](https://arxiv.org/abs/1201.0930) 2 2

[32] D. R. Morrison and C. Vafa, “Compactifications of F theory on Calabi-Yau threefolds. 2.,” *Nucl.Phys.* **B476** (1996) 437–469, [hep-th/9603161](https://arxiv.org/abs/hep-th/9603161) 2 6

[33] E. Perevalov and H. Skarke, “Enhanced gauged symmetry in type II and F theory compactifications: Dynkin diagrams from polyhedra,” *Nucl.Phys.* **B505** (1997) 679–700, [hep-th/9704129](https://arxiv.org/abs/hep-th/9704129) 3

[34] P. Candelas and H. Skarke, “F theory, SO(32) and toric geometry,” *Phys.Lett.* **B413** (1997) 63–69, [hep-th/9706226](https://arxiv.org/abs/hep-th/9706226) 3

[35] T. Shioda, “Mordell-Weil lattices and Galois representation, I,” *Proceedings of the Japan Academy, Series A, Mathematical Sciences* **65** (1989), no. 7, 268–271.

[36] T. Shioda, “On the Mordell-Weil lattices,” *Comment. Math. Univ. St. Paul* **39** (1990), no. 2, 211–240.

[37] M. Esole, J. Fullwood, and S.-T. Yau, “$D_5$ elliptic fibrations: non-Kodaira fibers and new orientifold limits of F-theory,” [1110.6177](https://arxiv.org/abs/1110.6177) 4.2

[38] C. Beasley, J. J. Heckman, and C. Vafa, “GUTs and Exceptional Branes in F-theory - II: Experimental Predictions,” *JHEP* **0901** (2009) 059, [0806.0102](https://arxiv.org/abs/0806.0102) 5

[39] R. Donagi and M. Wijnholt, “Breaking GUT Groups in F-Theory,” *Adv.Theor.Math.Phys.* **15** (2011) 1523–1604, [0808.2223](https://arxiv.org/abs/0808.2223) 5

[40] E. Palti, “A Note on Hypercharge Flux, Anomalies, and U(1)s in F-theory GUTs,” [1209.4421](https://arxiv.org/abs/1209.4421) 5

[41] N. Seiberg and E. Witten, “Comments on string dynamics in six-dimensions,” *Nucl.Phys.* **B471** (1996) 121–134, [hep-th/9603003](https://arxiv.org/abs/hep-th/9603003) 6

[42] D. R. Morrison and C. Vafa, “Compactifications of F theory on Calabi-Yau threefolds. 1,” *Nucl.Phys.* **B473** (1996) 74–92, [hep-th/9602114](https://arxiv.org/abs/hep-th/9602114) 6
[43] P. Candelas, D.-E. Diaconescu, B. Florea, D. R. Morrison, and G. Rajesh, “Codimension three bundle singularities in F theory,” *JHEP* **0206** (2002) 014, [hep-th/0009228](http://arxiv.org/abs/hep-th/0009228) [6]

[44] V. Braun, “Toric Elliptic Fibrations and F-Theory Compactifications,” *JHEP* **1301** (2013) 016, [1110.4883](http://arxiv.org/abs/1110.4883) [6]

[45] R. Miranda, “Smooth models for elliptic threefolds,” in *The birational geometry of degenerations (Cambridge, Mass., 1981)*, vol. 29 of *Progr. Math.* Birkhäuser Boston, Mass., 1983. [6]

[46] Y. Hu, C.-H. Liu, and S.-T. Yau, “Toric morphisms and fibrations of toric Calabi-Yau hypersurfaces,” *ArXiv Mathematics e-prints* (Oct., 2000) [arXiv:math/0010082](http://arxiv.org/abs/math/0010082) [6]

[47] V. Braun and A. Y. Novoseltsev, *Toric varieties framework for Sage*. The Sage Development Team, 2013. [http://sagemath.org/doc/reference/schemes/sage/schemes/toric/variety.html](http://sagemath.org/doc/reference/schemes/sage/schemes/toric/variety.html) [6]

[48] T. W. Grimm and H. Hayashi, “F-theory fluxes, Chirality and Chern-Simons theories,” *JHEP* **1203** (2012) 027, [1111.1232](http://arxiv.org/abs/1111.1232) [7]

[49] H. Hayashi, C. Lawrie, and S. Schafer-Nameki, “Phases, Flops and F-theory: SU(5) Gauge Theories,” [1304.1678](http://arxiv.org/abs/1304.1678) [7]

[50] P. Deligne, “Courbes elliptiques: formulaire,” *Modular Functions of One Variable IV* (1975) 53–73. [A]

[51] P. Griffiths and J. Harris, *Principles of algebraic geometry*, vol. 52. Wiley-interscience, 2011. [A]