Approximation of conformal mappings by circle patterns

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Abstract

A circle pattern is a configuration of circles in the plane whose combinatorics is given by a planar graph $G$ such that to each vertex of $G$ there corresponds a circle. If two vertices are connected by an edge in $G$ then the corresponding circles intersect with an intersection angle in $(0, \pi)$ and these intersection points can be associated to the dual graph $G^*$.

Two sequences of circle patterns are employed to approximate a given conformal map $g$ and its first derivative. For the domain of $g$ we use embedded circle patterns where all circles have the same radius $\varepsilon_n > 0$ for a sequence $\varepsilon_n \to 0$ and where the intersection angles are uniformly bounded. The image circle patterns have the same combinatorics and intersection angles and are determined from boundary conditions (radii or angles) according to the values of $g$ ($|g'|$ or $\arg g'$). The error is of order $1/\sqrt{-\log \varepsilon_n}$. For quasicrystalline circle patterns the convergence result is strengthened to $C^\infty$-convergence on compact subsets and an error of order $\varepsilon_n$.

1 Introduction

Conformal mappings constitute an important class in the field of complex analysis. They may be characterized by the fact that infinitesimal circles are mapped to infinitesimal circles. Suitable discrete analogs are of actual interest in the area of discrete differential geometry and its applications, see [BSSZ08].

Bill Thurston first introduced in his talk [Thu85] the idea to use finite circles, in particular circle packings, to define a discrete conformal mapping. Remember that an embedded planar circle packing is a configuration of closed disks with disjoint interiors in the plane $\mathbb{C}$. Connecting the centers of touching disks by straight lines yields the tangency graph. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two circle packings whose tangency graphs are combinatorially the same. Then there is a mapping $g_\varepsilon : \mathcal{C}_1 \to \mathcal{C}_2$ which maps the centers of circles of $\mathcal{C}_1$ to the corresponding centers of circles of $\mathcal{C}_2$ and is an affine map on each triangular region corresponding to three mutually tangent circles. Various connections between circle packings and classical complex analysis have already been studied. A beautiful introduction and survey is presented by Stephenson in [Ste04]. In particular, several results concerning convergence, i.e. quantitative approximation of conformal mappings by $g_\varepsilon$, have been obtained, see [RS87, CR92, HS96, HS98].

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The class of circle patterns generalizes circle packings as for each circle packing there is an associated orthogonal circle pattern. Simply add a circle for each triangular face which passes through the three touching points. To define a circle pattern we use a planar graph as combinatorial data. The circles correspond to vertices and the edges specify which circles should intersect. The intersection angles are given using a labelling on the edges. Thus an edge corresponds to a kite of two intersecting circles as in Figure 1 (right). Moreover, for interior vertices the kites corresponding to the incident edges have disjoint interiors and their union is homeomorphic to a closed disk. Similarly as for circle packings, there are results on existence, rigidity, and construction of special circle patterns. See for example [Riv94, Sch97, AB00, BS04, BH03], where some of the references use a generalized notion of circle patterns.

Given two circle patterns $C_1$ and $C_2$ with the same combinatorics and intersection angles, define a mapping $g_{C_1} : C_1 \rightarrow C_2$ similarly as for circle packings. Namely, take $g_{C_1}$ to map the centers of circles and the intersection points of $C_1$ corresponding to vertices and faces of $G$ to the corresponding centers of circles and intersection points of $C_2$ and extend it to an affine map on each kite.

For a given conformal map $g$ we use an analytic approach and specify suitable boundary values for the radius or the angle function according to $|g'|$ or to $\arg g'$ respectively in order to define the (approximating) mappings $g_{C_1}$. Generalizing ideas of Schramm’s convergence proof in [Sch97] we obtain convergence in $C^1$ on compact sets if we take for $C_1^{(n)}$ a sequence of isoradial circle patterns (i.e. all radii are equal) with decreasing radii $\varepsilon_n \rightarrow 0$ which approximate the domain of $g$. Furthermore, we assume that the intersection angles are uniformly bounded away from 0 and $\pi$. Note in particular that the combinatorics of the circle pattern $C_1^{(n)}$ may be irregular or change within the sequence. Thus our convergence results applies to a considerably broader class of circle patterns as the known results of Schramm [Sch97], Matthes [Mat05], or Lan and Dai [LD07] for orthogonal circle patterns with square grid combinatorics.

The main idea of the proof is to consider a “nonlinear discrete Laplace equation” for the radius function. This equation turns out to be a (good) approximation of a known linear Laplace equation and can be used in the case of isoradial circle patterns to compare discrete and smooth solutions of the corresponding elliptic problems, that is the logarithm of the radii of $C_2^{(n)}$ and $\log |g'| = \text{Re}(\log g')$. Also, we obtain an a priori estimation of the approximation error of order $1/\sqrt{\log(1/\varepsilon_n)}$.

If the circle patterns $C_1^{(n)}$ additionally have only a uniformly bounded number of different edge directions, then the corresponding kite patterns are quasicrystalline rhombic embeddings and the circle patterns $\mathcal{C}_1^{(n)}$ are called quasicrystalline [BMS05]. For such embeddings we generalize an asymptotic development given by Kenyon in [Ken02] of a discrete Green’s function. Also, using similar ideas as Duffin in [Duf53], we generalize theorems of discrete potential theory concerning the regularity of solutions of a discrete Laplace equation. We then use these results together with the $C^1$-convergence for isoradial circle patterns and prove $C^\infty$-convergence on compact sets for a class of quasicrystalline circle patterns. In this case the approximation error is of order $\varepsilon_n$ (or $\varepsilon_n^2$ for square grid or hexagonal combinatorics and regular intersection angles). The proof generalizes a method used by He and Schramm in [HS98].

The article is organized as follows: First we introduce and remind the ter-
minology and some results on circle patterns in Section 2, focusing in particular on the radius and the angle function. In Section 3 we formulate and prove the theorems on $C^1$-convergence for isoradial circle patterns. After a brief review on quasicrystallic circle patterns in Section 4 we state and prove in Section 5 the theorem on $C^\infty$-convergence. The necessary results on discrete potential theory are presented in Appendix A. An extended and more details version of the results can be found in [Buc07].

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2 Circle patterns

To define circle patterns we use combinatorial data and intersection angles.

The combinatorics are specified by a $b$-quad-graph $D$, that is a strongly regular cell decomposition of a domain in $\mathbb{C}$ possibly with boundary such that all 2-cells (faces) are embedded and counterclockwise oriented. Furthermore all faces of $D$ are quadrilaterals, that is there are exactly four edges incident to each face, and the 1-skeleton of $D$ is a bipartite graph. We always assume that the vertices of $D$ are colored white and black. To these two sets of vertices we associate two planar graphs $G$ and $G^*$ as follows. The vertices $V(G)$ are all white vertices of $V(D)$. The edges $E(G)$ correspond to faces of $D$, that is two vertices of $G$ are connected by an edge if and only if they are incident to the same face of $D$. The dual graph $G^*$ is constructed analogously by taking for $V(G^*)$ all black vertices of $V(D)$. $D$ is called simply connected if it is the cell decomposition of a simply connected domain of $\mathbb{C}$ and if every closed chain of faces is null homotopic in $D$.

For the intersection angles, we use a labelling $\alpha : F(D) \to (0, \pi)$ of the faces of $D$. By abuse of notation, $\alpha$ can also be understood as a function defined on $E(G)$ or on $E(G^*)$. The labelling $\alpha$ is called admissible if it satisfies the following condition at all interior black vertices $v \in V_{int}(G^*)$:

$$\sum_{f \text{ incident to } v} \alpha(f) = 2\pi.$$  \hspace{1cm} (1)

Figure 1: Left: An example of a $b$-quad-graph $D$ (black edges and bicolored vertices) and its associated graph $G$ (dashed edges and white vertices). Right: The exterior intersection angle $\alpha$ of two intersecting circles and the associated kite built from centers and intersection points.
A regular square grid circle pattern.  
(b) A regular hexagonal circle pattern.  
(c) A rhombic embedding (a part of a Penrose tiling) and a corresponding isoradial circle pattern.

Figure 2: Examples of isoradial circle patterns.

**Definition 2.1.** Let $\mathcal{D}$ be a b-quad-graph and let $\alpha : E(G) \to (0,\pi)$ be an admissible labelling. An (immersed planar) circle pattern for $\mathcal{D}$ (or $G$) and $\alpha$ are an indexed collection $\mathcal{C} = \{C_z : z \in V(G)\}$ of circles in $\mathbb{C}$ and an indexed collection $\mathcal{K} = \{K_e : e \in E(G)\} = \{K_f : f \in F(\mathcal{D})\}$ of closed kites, which all carry the same orientation, such that the following conditions hold.

1. If $z_1, z_2 \in V(G)$ are incident vertices in $G$, the corresponding circles $C_{z_1}, C_{z_2}$ intersect with exterior intersection angle $\alpha([z_1, z_2])$. Furthermore, the kite $K_{[z_1, z_2]}$ is bounded by the centers of the circles $C_{z_1}, C_{z_2}$, the two intersection points, and the corresponding edges, as in Figure 1 (right). The intersection points are associated to black vertices of $V(D)$ or to vertices of $V(G^*)$.

2. If two faces are incident in $\mathcal{D}$, then the corresponding kites have one edge in common.

3. Let $f_1, \ldots, f_n \in F(\mathcal{D})$ be the faces incident to an interior vertex $v \in V_{int}(\mathcal{D})$. Then the kites $K_{f_1}, \ldots, K_{f_n}$ have mutually disjoint interiors. Their union $K_{f_1} \cup \cdots \cup K_{f_n}$ is homeomorphic to a closed disc and contains the point $p(v)$ corresponding to $v$ in its interior.

The circle pattern is called *embedded* if all kites of $\mathcal{K}$ have mutually disjoint interiors. It is called *isoradial* if all circles of $\mathcal{C}$ have the same radius.

Note that we associate a circle pattern $\mathcal{C}$ to an immersion of the kite pattern $\mathcal{K}$ corresponding to $\mathcal{D}$ where the edges incident to the same white vertex are of equal length. The kites can also be reconstructed from the set of circles using the combinatorics of $G$. Note further, that there are in general additional intersection points of circles which are not associated to black vertices of $V(\mathcal{D})$.

Some examples are shown in Figure 2.

There are also other definitions for circle patterns, for example associated to a Delaunay decomposition of a domain in $\mathbb{C}$. This is a cell decomposition such that the boundary of each face is a polygon with straight edges which is inscribed in a circular disk, and these disks have no vertices in their interior. The corresponding circle pattern can be associated to the graph $G^*$. The Poincaré-dual decomposition of a Delaunay decomposition with the centers of the circles as vertices and straight edges is a Dirichlet decomposition (or Voronoi diagram) and corresponds to the graph $G$.

Furthermore the definition of circle patterns can be extended allowing cone-like singularities in the vertices; see [BS04] and the references therein.
2.1 The radius function

Our study of a planar circle pattern $\mathcal{C}$ is based on characterizations and properties of its radius function $r_\mathcal{C} = r$ which assigns to every vertex $z \in V(G)$ the radius $r_\mathcal{C}(z) = r(z)$ of the corresponding circle $C_z$. The index $\mathcal{C}$ will be dropped whenever there is no confusion likely.

The following proposition specifies a necessary and sufficient condition for a radius function to originate from a planar circle pattern, see [BS04] for a proof. For the special case of orthogonal circle patterns with the combinatorics of the square grid, there are also other characterizations, see for example [Sch97].

**Proposition 2.2.** Let $G$ be a graph constructed from a b-quad-graph $\mathcal{D}$ and let $\alpha$ be an admissible labelling.

Suppose that $\mathcal{C}$ is a planar circle pattern for $\mathcal{D}$ and $\alpha$ with radius function $r = r_\mathcal{C}$. Then for every interior vertex $z_0 \in V_{\text{int}}(G)$ we have

$$\left( \sum_{[z,z_0] \in E(G)} f_\alpha([z,z_0])(\log r(z) - \log r(z_0)) \right) - \pi = 0, \tag{2}$$

where

$$f_\theta(x) := \frac{1}{2i} \log \frac{1 - e^{x-i\theta}}{1 - e^{x+i\theta}},$$

and the branch of the logarithm is chosen such that $0 < f_\theta(x) < \pi$.

Conversely, suppose that $\mathcal{D}$ is simply connected and that $r : V(G) \to (0, \infty)$ satisfies (2) for every $z \in V_{\text{int}}(G)$. Then there is a planar circle pattern for $G$ and $\alpha$ whose radius function coincides with $r$. This pattern is unique up to isometries of $\mathcal{C}$.

Note that $2f_\alpha([z,z_0])(\log r(z) - \log r(z_0))$ is the angle at $z_0$ of the kite with edge lengths $r(z)$ and $r(z_0)$ and angle $\alpha([z,z_0])$, as in Figure 1 (right). Equation (2) is the closing condition for the chain of kites corresponding to the edges incident to $z_0$ which is condition (3) of Definition 2.1.

For further use we mention some properties of $f_\theta$, see for example [Spr03].

**Lemma 2.3.** (1) The derivative of $f_\theta$ is $f'_\theta(x) = \frac{\sin \theta}{2(cosh x - cos \theta)} > 0$. So $f_\theta$ is strictly increasing.

(2) The function $f_\theta$ satisfies the functional equation $f_\theta(x) + f_\theta(-x) = \pi - \theta$.

(3) For $0 < y < \pi - \theta$ the inverse function of $f_\theta$ is $f_\theta^{-1}(y) = \log \frac{\sin y}{\sin(y+\theta)}$.

**Remark 2.4.** Equation (2) can be interpreted as a nonlinear Laplace equation for the radius function and is related to a linear discrete Laplacian which is common in the linear theory of discrete holomorphic functions; see for example [Duf68, Mer01, BMS05] for more details. This can be seen as follows.

Let $G$ be a planar graph and let $\alpha$ be an admissible labelling. Assume there is a smooth one parameter family of planar circle patterns $\mathcal{C}_\varepsilon$ for $G$ and $\alpha$ with radius function $r_\varepsilon$ for $\varepsilon \in (-1, 1)$. Then for every interior vertex $z_0$ with incident vertices $z_1, \ldots, z_m$ and all $\varepsilon \in (-1, 1)$ Proposition 2.2 implies that

$$\sum_{j=1}^m 2f_\alpha([z_j,z_0])(\log r_\varepsilon(z_j) - \log r_\varepsilon(z_0)) = 2\pi.$$
Differentiating this equation with respect to \( \varepsilon \) at \( \varepsilon = 0 \), we obtain

\[
\sum_{j=1}^{m} 2f'_{\alpha([z,j,z_0])}(\log r_0(z_j) - \log r_0(z_0))(v(z_j) - v(z_0)) = 0, \tag{3}
\]

where \( v(z) = \frac{d}{d\varepsilon} \log r_\varepsilon(z)\big|_{\varepsilon=0} \). Thus \( v \) satisfies a linear discrete Laplace equation with positive weights. Lemma 2.3 and a simple calculation show that

\[
2f'_{\alpha([z_1,z_2])}(\log r_0(z_1) - \log r_0(z_2)) = \left|\frac{p(v_1) - p(v_2)}{p(z_1) - p(z_2)}\right|. \tag{4}
\]

Here \( z_1, z_2 \in V(G) \) are two incident vertices which correspond to the centers of circles \( p(z_1), p(z_2) \) of the circle pattern \( \mathcal{C}_G \). The two other corner points of the same kite are denoted by \( p(v_1), p(v_2) \).

In analogy to smooth harmonic functions, the radius function of a planar circle pattern satisfies a maximum principle and a Dirichlet principle.

**Lemma 2.5** (Maximum Principle). Let \( G \) be a finite graph associated to a \( b \)-quad-graph as above with some admissible labelling \( \alpha \). Suppose \( \mathcal{C} \) and \( \mathcal{C}^* \) are two planar circle patterns for \( G \) and \( \alpha \) with radius functions \( r_\mathcal{C}, r_\mathcal{C}^* : V(G) \to (0,\infty) \). Then the maximum and minimum of the quotient \( r_\mathcal{C}/r_\mathcal{C}^* \) is attained at the boundary.

A proof can be found in [He99, Lemma 2.1]. If there exists an isoradial planar circle pattern for \( G \) and \( \alpha \), the usual maximum principle for the radius function follows by taking \( r_\mathcal{C}^* \equiv 1 \).

**Theorem 2.6** (Dirichlet Principle). Let \( \mathcal{D} \) be a finite simply connected \( b \)-quad-graph with associated graph \( G \) and let \( \alpha \) be an admissible labelling.

Let \( r : V_\partial(G) \to (0,\infty) \) be some positive function on the boundary vertices of \( G \). Then \( r \) can be extended to \( V(G) \) in such a way that equation (2) holds at every interior vertex \( z \in V_{int}(G) \) if and only if there exists any circle pattern for \( G \) and \( \alpha \). If it exists, the extension is unique.

By lack of a good reference we include a proof.

**Proof.** The only if part follows directly from the second part of Proposition 2.2.

To show the if part, assume that there exists a circle pattern for \( G \) and \( \alpha \) with radius function \( R : V(G) \to (0,\infty) \). A function \( \kappa : V(G) \to (0,\infty) \) which satisfies the inequality

\[
\left( \sum_{[z,z_0] \in E(G)} f_{\alpha([z,z_0])}(\log \kappa(z) - \log \kappa(z_0)) \right) - \pi \geq 0 \tag{5}
\]

at every interior vertex \( z \in V_{int}(G) \) will be called subharmonic in \( G \). Let \( b \) be the minimum of the quotient \( r/R \) on \( V_\partial(G) \) and let \( \kappa_1 \) be equal to \( r \) on \( V_\partial(G) \) and to \( bR \) on \( V_{int}(G) \). Then \( \kappa_1 \) is clearly subharmonic. The maximum of \( \kappa_1/R \) is attained at the boundary, which is a simple generalization of the Maximum Principle 2.3. Let \( r^* \) be the supremum of all subharmonic functions on \( G \) that coincide with \( r \) on \( V_\partial(G) \). Thus \( r^* \) is bounded from above by the maximum of \( r/R \) on \( V_\partial(G) \) which is finite. One easily checks that \( r^* \) satisfies condition (2).

The uniqueness claim follows directly from the Maximum Principle 2.3. \( \square \)
2.2 The angle function and relations to the radius function

Similarly as for polar coordinates of the complex plane, a suitably defined angle function can be interpreted as a “dual” to the radius function. We focus on connections between these functions and on a characterization for angle functions of circle patterns similar to Proposition 2.2.

Let \( G \) be a finite graph associated to a \( b \)-quad-graph \( \mathcal{D} \) and let \( \alpha \) be an admissible labelling. Denote by \( \hat{E}(\mathcal{D}) \) the set of oriented edges, where each edge of \( E(\mathcal{D}) \) is replaced by two oriented edges of opposite orientation. Let \( \mathcal{C} \) be a planar circle pattern for \( \mathcal{D} \) and \( \alpha \). Define an angle function \( \phi_\alpha = \phi \) on \( \hat{E}(\mathcal{D}) \) as follows. Denote by \( p(w) \) the point of \( \mathcal{C} \) corresponding to the vertex \( w \in V(\mathcal{D}) \). For \( \vec{e} = \frac{z}{\hat{E}(\mathcal{D})} \) set \( \phi_\alpha(\vec{e}) = \arg(p(v) - p(z)) \) to be the argument of \( p(v) - p(z) \), that is the angle between the positively oriented real axis and the vector \( p(v) - p(z) \). As this argument is only unique up to addition of multiples of \( 2\pi \), we will mostly consider \( \phi \in \mathbb{R}/(2\pi \mathbb{Z}) \). But note that \( e^{i\phi(\vec{e})} \) is well defined.

Furthermore
\[
\phi(\vec{e}) - \phi(-\vec{e}) = \pi \quad (\text{mod } 2\pi). \tag{6}
\]

Our choice of the angle function leads to the following connections between radius function \( r \) and angle function \( \phi \) for a planar circle pattern for \( \mathcal{D} \) and \( \alpha \).

Let \( f \in F(\mathcal{D}) \) be a face of \( \mathcal{D} \). Without loss of generality, we assume that the notation for the vertices and edges of \( f \) is taken from Figure 3 (left). More precisely, the white vertices \( z_- \) and \( z_+ \) of \( f \) and the black vertices \( v_- \) and \( v_+ \) are labelled such that the points \( z_-, v_, z_+, v_+ \) appear in this cyclical order using the counterclockwise orientation of \( f \). Furthermore the edges are labelled such that \( \vec{e}_1 = \hat{e}_z v_+ \), \( \vec{e}_2 = \hat{e}_z v_- \), \( \vec{e}_3 = \hat{e}_v z_+ \), \( \vec{e}_4 = \hat{e}_v z_- \). Then the following equations hold.

\[
\begin{align*}
\phi(\vec{e}_1) - \phi(-\vec{e}_3) &= \alpha(f) - \pi + 2f_{\alpha(f)}(\log r(z_+) - \log r(z_-)) \quad (\text{mod } 2\pi) \tag{7} \\
\phi(-\vec{e}_4) - \phi(\vec{e}_2) &= \alpha(f) - \pi + 2f_{\alpha(f)}(\log r(z_+) - \log r(z_-)) \quad (\text{mod } 2\pi) \tag{8} \\
\phi(\vec{e}_1) - \phi(\vec{e}_2) &= 2f_{\alpha(f)}(\log r(z_+) - \log r(z_-)) \quad (\text{mod } 2\pi) \tag{9} \\
\phi(-\vec{e}_4) - \phi(-\vec{e}_3) &= -2f_{\alpha(f)}(\log r(z_-) - \log r(z_+)) \quad (\text{mod } 2\pi) \tag{10} \\
\phi(-\vec{e}_3) - \phi(\vec{e}_2) &= \pi - \alpha(f) \quad (\text{mod } 2\pi) \tag{11} \\
\phi(-\vec{e}_4) - \phi(\vec{e}_1) &= \alpha(f) - \pi \quad (\text{mod } 2\pi) \tag{12}
\end{align*}
\]

Additionally, the angle function \( \phi \) satisfies the following

Lemma 2.7 (Monotonicity condition). Let \( z \in V(G) \) be a white vertex and let \( e_1, \ldots, e_n \) be the sequence of all incident edges in \( E(\mathcal{D}) \) which are cyclically ordered respecting the counterclockwise orientation of the circle \( C_z \). Then the values of \( \phi \in \mathbb{R} \) can be changed by suitably adding multiples of \( 2\pi \) such that \( \phi(e_j) \) is an increasing function of the index \( j \) and if \( z \) is an interior vertex, then also \( \phi(e_j) - \phi(e_1) < 2\pi \) for all \( j = 1, \ldots, n \).

The following theorem is useful to compare two circle patterns with the same combinatorics and intersection angles.

Theorem 2.8. Let \( \mathcal{D} \) be a \( b \)-quad-graph and let \( \alpha \) be an admissible labelling.

Let \( \mathcal{C} \) and \( \hat{\mathcal{C}} \) be two planar circle patterns for \( \mathcal{D} \) and \( \alpha \) with radius functions \( r_\mathcal{C} = r \) and \( r_{\hat{\mathcal{C}}} = \hat{r} \) and angle functions \( \phi_\mathcal{C} = \phi \) and \( \phi_{\hat{\mathcal{C}}} = \hat{\phi} \) respectively.
Then the difference $\hat{\phi} - \varphi$ gives rise to a function $\delta : V(G^*) \to \mathbb{R}$ such that the following condition holds on every face $f \in F(\mathcal{D})$.

\[
2f_{\alpha(f)} \left( \log \left( \frac{w(z_+)}{w(z_-)} \right) + \log \left( \frac{r(z_+)}{r(z_-)} \right) \right) - 2f_{\alpha(f)} \left( \log \left( \frac{r(z_+)}{r(z_-)} \right) \right) = \delta(v_+) - \delta(v_-)
\]

(13)

Here we have defined $w : V(G) \to \mathbb{R}$, $w(z) = r(z)/r(z)$ and the notation is taken from Figure 3 (left) as above.

Conversely, assume that $\mathcal{C}$ is a planar circle pattern for $\mathcal{D}$ and $\alpha$ with radius functions $r_\mathcal{C} = r$ and angle function $\varphi_\mathcal{C} = \varphi$. Let $\delta : V(G^*) \to \mathbb{R}$ and $w : V(G) \to \mathbb{R}_+$ be two functions which satisfy equation (13) for every face $f \in F(\mathcal{D})$. Then $(rw)$ and $(\varphi + \delta)$ are the radius and angle function of a planar circle pattern for $\mathcal{D}$ and $\alpha$. This circle pattern is unique up to translation.

Proof. If $\mathcal{C}$ and $\hat{\mathcal{C}}$ are two planar circle patterns for $\mathcal{D}$ and $\alpha$, equations (6), (11), and (12) imply that the difference $\hat{\phi} - \varphi$ is constant for all edges incident to any fixed black vertex $v \in V(G^*)$. Therefore $\delta \mod 2\pi$ is well defined on the intersection points and encodes the relative rotation of the star of edges at $v$. Also equation (13) holds modulo $2\pi$.

To obtain a function $\delta$ with values in $\mathbb{R}$, fix $\delta(v_0) \in [0, 2\pi)$ for one arbitrary vertex $v_0 \in V(G^*)$ (for each connected component of $G^*$). Define the values of $\delta$ for all incident vertices in $G^*$ by equation (13). Continue this construction until a value has been assigned to all vertices. This procedure leads to a well-defined function, as by Proposition 2.2 the sum of the left hand side of equation (13) is zero for simple closed paths in $E(G^*)$ around a white vertex.

To prove the converse claim, observe that if $r$ is a radius function, equation (13) implies that $(wr)$ fulfills equation (2). By Proposition 2.2 there is a circle pattern with radius function $(wr)$. Adjust the rotational freedom at one edge according to $(\varphi + \delta)$. Equation (13) implies that $(\varphi + \delta)$ is indeed the angle function of this circle pattern. $\square$

The preceding theorem motivates the definition of a comparison function for two circle patterns with the same combinatorics and intersection angles.

Let $G$ be a graph associated to a $b$-quad-graph $\mathcal{D}$ and let $\alpha$ be an admissible labelling. Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are planar circle patterns for $\mathcal{D}$ and $\alpha$ with radius functions $r_{\mathcal{C}_1}$ and $r_{\mathcal{C}_2}$ and angle functions $\varphi_{\mathcal{C}_1}$ and $\varphi_{\mathcal{C}_2}$ respectively. Let $\delta : V(G^*) \to \mathbb{R}$ be a function corresponding to $\varphi_{\mathcal{C}_2} - \varphi_{\mathcal{C}_1}$ as in Lemma 2.8.
Define a comparison function \( w : V(\mathcal{D}) \to \mathbb{C} \) by

\[
\begin{cases}
  w(y) = r_{\mathcal{E}_1}(y)/r_{\mathcal{E}_2}(y) & \text{for } y \in V(G), \\
  w(x) = e^{\delta(x)} \in S^1 & \text{for } x \in V(G^*).
\end{cases}
\] (14)

Note that \( w(y) \) is the scaling factor of the circle corresponding to \( y \in V(G) \) when changing from the circle pattern \( \mathcal{E}_1 \) to \( \mathcal{E}_2 \). \( w(x) \) gives the rotation of the edge-star at \( x \in V(G^*) \). Furthermore, \( w \) satisfies the following Hirota Equation for all faces \( f \in F(\mathcal{D}) \).

\[
w(x_0)w(y_0)a_0 - w(x_1)w(y_0)a_1 - w(x_1)w(y_1)a_0 + w(x_0)w(y_1)a_1 = 0 \quad (15)
\]

Here \( x_0, x_1 \in V(G^*) \) and \( y_0, y_1 \in V(G) \) are the black and white vertices incident to \( f \) and \( a_0 = x_0 - y_0 \) and \( a_1 = x_1 - y_1 \) are the directed edges. Equation (14) is the closing condition for the kite of \( \mathcal{E}_2 \) which corresponds to the face \( f \).

Angle functions associated to planar circle patterns can be characterized in a similar way as radius functions are qualified in Proposition 2.2.

**Proposition 2.9.** Let \( \mathcal{D} \) be a b-quad-graph with associated graphs \( G \) and \( G^* \) and let \( \alpha \) be an admissible labelling.

Suppose \( \mathcal{C} \) is a planar circle pattern for \( \mathcal{D} \) and \( \alpha \) with angle function \( \varphi = \varphi_\mathcal{C} \). Then \( \varphi \) satisfies equations (9), (11), (12), the Monotonicity condition 2.7 at every white vertex of \( \mathcal{D} \), and the following two conditions.

(i) Let \( f \) be a face of \( \mathcal{D} \) and let \( e_1 \) and \( e_2 \) be two edges incident to \( f \) and to the same white vertex and assume that \( e_1 \) and \( e_2 \) are enumerated in clockwise order as in Figure 4 (left). Define an angle \( \beta \in (0, \pi) \) by

\[
2\beta = \varphi(e_1) - \varphi(e_2) \quad (\text{mod } 2\pi),
\]

where the orientation of the edges is chosen such that the vectors point from a white vertex to a black vertex, as in Figure 4 (left). Then

\[
\beta + \alpha(f) < \pi. \quad (16)
\]

(ii) For an interior black vertex \( v \in V_{\text{int}}(G^*) \), denote by \( e_1, \ldots, e_n, e_{n+1} = e_1 \) all incident edges of \( \mathcal{D} \) in counterclockwise order and by \( f_j \) the face of \( \mathcal{D} \) incident to \( e_j \) and \( e_{j+1} \). Denote by \( e_j^* \) (\( j = 1, \ldots, n \)) the edge incident to \( e_j \) and \( f_j \) which is not incident to \( v \). For \( j = 1, \ldots, n \) define as above \( \beta_j \in (0, \pi) \) by \( 2\beta_j = \varphi(e_j^*) - \varphi(e_j^{*+}) \) (mod \( 2\pi \)), where we choose the same orientation of the edges from white to black vertices as above. Then

\[
\sum_{j=1}^{n} f_{\alpha(f_j)}(\beta_j) = 0. \quad (17)
\]

Conversely, suppose that \( \mathcal{D} \) is simply connected and that \( \varphi : \bar{E}(\mathcal{D}) \to \mathbb{R}/(2\pi\mathbb{Z}) \) satisfies equations (9), (11), (12), the Monotonicity condition 2.7, condition (10) at every white vertex of \( \mathcal{D} \), and condition (17) at every interior black vertex. Then there is a planar circle pattern for \( \mathcal{D} \) and \( \alpha \) with angle function \( \varphi \). This pattern is unique up to scaling and translation.
Proof. For a given planar circle pattern equations (1) and Lemma 2.7 hold. To show (16), consider a kite corresponding to a face of \( \mathcal{D} \) as in Figure 3 (left). Note that \( \beta_- = 2\beta \) by equation (9), \( \beta_- + \beta_+ + 2\alpha = 2\pi \), and \( \beta_-, \beta_+, \alpha > 0 \). Using notation of Figure 3 (right) we also deduce that \( f^{-1}_\alpha(\beta_j) = \log r_{j+1} - \log r_j \) for \( j = 1, \ldots, n \), where we identify \( r_1 = r_{n+1} \). Now (17) follows immediately.

In order to prove the converse claim, we construct a radius function \( r : V(G) \to (0, \infty) \) and build a circle pattern corresponding to \( r \) and \( \varphi \).

Let \( z \in V(G) \) be an interior white vertex. Set \( r(z) = 1 \). Consider a neighboring white vertex \( z' \in V(G) \) and the face \( f \in F(\mathcal{D}) \) incident to \( z \) and \( z' \). Denote the black vertices of \( \mathcal{D} \) incident to \( z \) and \( f \) by \( v_1, v_2 \) such that \( v_1, z, v_2, z' \) appear in counterclockwise order along the boundary of \( f \). Define \( \psi_\infty \in (0, 2\pi) \) by \( \psi_\infty = \varphi(zv_1) - \varphi(zv_2) \) (mod \( 2\pi \)). Condition (10) implies that \( r(z') := r(z)\exp(f^{-1}_\alpha(\psi_\infty/2)) \) is well defined and positive. We proceed in this way until a radius has been assigned to all white vertices. Condition (17) guarantees that these assignments do not lead to different values when turning around a black vertex (see Figure 3 (right) with \( \psi = 2\beta \)). Thus \( r \) is uniquely determined up to the choice of the initial radius, which corresponds to a global scaling. For each face \( f \in F(\mathcal{D}) \) construct a kite with lengths \( r(z_+), r(z_-) \) of the edges incident to the white vertices \( z_+, z_- \) of \( f \) respectively and angle \( \alpha = \alpha(f) \). Lay out one kite fixing the rotational freedom according to \( \varphi \). Successively add all other kites, respecting the combinatorics of \( \mathcal{D} \). By construction and assumptions, at every interior vertex the angles of the kites having this vertex in common add up to \( 2\pi \). Thus we obtain a circle pattern with angle function \( \varphi \).

\[
\beta_+ + \beta_+ + 2\alpha = 2\pi, \quad \beta_-, \beta_+, \alpha > 0.
\]

Now (17) follows immediately.

\section{C1-convergence with Dirichlet or Neumann boundary conditions}

In this section we state and prove our main results on convergence for isoradial circle patterns. We begin with Dirichlet boundary conditions, that is we first focus on the radius function with given boundary values.

**Theorem 3.1.** Let \( D \subset \mathbb{C} \) be a simply connected bounded domain, and let \( W \subset \mathbb{C} \) be open such that \( W \) contains the closure \( \overline{D} \) of \( D \). Let \( g : W \to \mathbb{C} \) be a locally injective holomorphic function. Assume, for convenience, that \( 0 \in D \).

For \( n \in \mathbb{N} \) let \( \mathcal{D}_n \) be a b-quad-graph with associated graphs \( G_n \) and \( G_n^* \) and let \( \alpha_n \) be an admissible labelling. We assume that \( \mathcal{D}_n \) is simply connected and that \( \alpha_n \) is uniformly bounded such that for all \( n \in \mathbb{N} \) and all faces \( f \in F(\mathcal{D}_n) \)

\[
|\alpha_n(f) - \pi/2| < C \quad (18)
\]

with some constant \( 0 < C < \pi/2 \) independent of \( n \).

Let \( \varepsilon_n \in (0, \infty) \) be a sequence of positive numbers such that \( \varepsilon_n \to 0 \) for \( n \to \infty \). For each \( n \in \mathbb{N} \), assume that there is an isoradial circle pattern for \( G_n \) and \( \alpha_n \), where all circles have the same radius \( \varepsilon_n \). Assume further that all centers of circles lie in the domain \( D \) and that any point \( x \in \overline{D} \) which is not contained in any of the disks bounded by the circles of the pattern has a distance less than \( C \varepsilon_n \) to the nearest center of a circle and to the boundary \( \partial D \), where \( C > 0 \) is some constant independent of \( n \). Denote by \( R_n \equiv \varepsilon_n \) and \( \phi_n \) the radius and the angle function of the above circle pattern for \( G_n \) and \( \alpha_n \). By abuse of
notation, we do not distinguish between the realization of the circle pattern, that is the centers of circles $z_n$, the intersection points $v_n$, and the edges connecting corresponding points in $\mathcal{G}_n$ or $G_n$, and the abstract b-quad-graph $\mathcal{P}_n$ and the graphs $G_n$ and $G_n^*$. Also, the index $n$ will be dropped from the notation of the vertices and the edges.

Define another radius function on $G_n$ as follows. At boundary vertices $z \in V_0(G_n)$ set

$$r_n(z) = R_n(z) |g'(z)|. \quad (19)$$

Using Theorem 2.6 extend $r_n$ to a solution of the Dirichlet problem on $G_n$. Let $z_0 \in V(G_n)$ be such that the disk bounded by the circle $C_{z_0}$ contains 0 and let $e = [z_0, v_0] \in E(\mathcal{G}_n)$ be one of the edges incident to $z_0$ such that $\phi_n(\vec{e}) \in [0, 2\pi)$ is minimal.

Let $\varphi_n$ be the angle function corresponding to $r_n$ that satisfies

$$\varphi_n(\vec{e}) = \arg(g'(v_0)) + \phi_n(\vec{e}). \quad (20)$$

Let $\mathcal{C}_n$ be the planar circle pattern with radius function $r_n$ and angle function $\varphi_n$. Suppose that $\mathcal{C}_n$ is normalized by a translation such that

$$p_n(v_0) = g(v_0), \quad (21)$$

where $p_n(v)$ denotes the intersection point corresponding to $v \in V(G_n^*)$. For $z \in D$ set

$$g_n(z) = p_n(w) \quad \text{and} \quad q_n(z) = \frac{r_n(v)}{R_n(v)} e^{i(\varphi_n(\vec{w}) - \phi_n(\vec{w}))},$$

where $w$ is a vertex of $V(G_n^*)$ closest to $z$ and $v$ is a vertex of $V(G_n)$ closest to $z$ such that $|v, w| \in E(\mathcal{G}_n)$.

Then $q_n \rightarrow g'$ and $g_n \rightarrow g$ uniformly on compact subsets in $D$ as $n \rightarrow \infty$.

**Remark 3.2.** The proof of Theorem 3.1 actually shows the following a priori estimations for the approximating functions $q_n$ and $g_n$.

$$\|g - g'\|_{V(G_n) \cap K} \leq C_1(- \log_2 \varepsilon_n)^{-\frac{1}{2}} \quad \text{and} \quad \|g_n - g\|_{V(G_n) \cap K} \leq C_2(- \log_2 \varepsilon_n)^{-\frac{1}{2}}$$

for all compact sets $K \subset D$, where the constants $C_1, C_2$ depend on $K, g, D$, and on the constants of Theorem 3.1.

We begin with an a priori estimation for the quotients of the radius functions.

**Lemma 3.3.** For $z \in V(G_n)$ set

$$h_n(z) = \log |g'(z)|, \quad t_n(z) = \log(r_n(z)/R_n(z)) = \log(r_n(z)/\varepsilon_n).$$

Then

$$h_n(z) - t_n(z) = O(\varepsilon_n).$$

Here and below the notation $s_1 = O(s_2)$ means that there is a constant $C$ which may depend on $W, D, g$, but not on $n$ and $z$, such that $|s_1| \leq Cs_2$ wherever $s_1$ is defined. A direct consequence of Lemma 3.3 is

$$r_n(z) = R_n(z) |g'(z)| + O(\varepsilon_n^2). \quad (22)$$

Our proof uses ideas of Schramm’s proof of the corresponding Lemma in [Sch97].
Proof. Consider the function

\[ p(z) = t_n(z) - h_n(z) + \beta|z|^2, \]

where \( \beta \in (0, 1) \) is some function of \( \varepsilon_n \). We want to choose \( \beta \) such that \( p \) will have no maximum in \( V_{\text{int}}(G_n) \).

Suppose that \( p \) has a maximum at \( z \in V_{\text{int}}(G_n) \). Denote by \( z_1, \ldots, z_m \) the incident vertices of \( z \) in \( G_n \) in counterclockwise order. Then we have

\[ t_n(z_j) - t_n(z) \leq x_j \quad (23) \]

for \( j = 1, \ldots, m \) where

\[ x_j = h_n(z_j) - h_n(z) - \beta|z_j|^2 + \beta|z|^2. \quad (24) \]

Since \( z \in V_{\text{int}}(G_n) \), we have \( |z| = O(1) \) and by assumption \( z - z_j = O(\varepsilon_n) \). With \( \beta \in (0, 1) \) this leads to \( \beta|z_j|^2 - \beta|z|^2 = O(\varepsilon_n) \). Using this estimate and the smoothness of \( \text{Re}(\log g') \), we get \( x_j = O(\varepsilon_n) \).

From (23), the definition of \( t_n(z) = \log(r_n(z)/R_n(z)) = \log r_n(z) - \log \varepsilon_n \) and the monotonicity of the sum in equation (2) (see Lemma 2.3 (i)), we get

\[ 0 = \left( \sum_{j=1}^{m} f_{\alpha(z,z_j)}(\log r_n(z_j) - \log r_n(z)) \right)_{=\, t_n(z_j) - t_n(z)} - \pi \leq \left( \sum_{j=1}^{m} f_{\alpha(z,z_j)}(x_j) \right) - \pi. \quad (25) \]

Remembering \( x_j = O(\varepsilon_n) \), we can consider a Taylor expansion of the right hand side of inequality (24) about 0 in order to make an \( O(\varepsilon_n^4) \)-analysis.

Consider the chain of faces \( f_j \) of \( \mathcal{P}_n \) (\( j = 1, \ldots, m \)) which are incident to \( z \) and \( z_j \). The enumeration of the vertices \( z_j \) (and hence of the faces \( f_j \)) and of the black vertices \( v_1, \ldots, v_m \) incident to these faces can be chosen such that \( f_j \) is incident to \( v_j \) and \( v_{j-1} \) for \( j = 1, \ldots, m \), where \( v_0 = v_m \). Furthermore, using this enumeration \( i(z_j - z) \) and \( v_{j-1} - v_j \) are parallel, see Figure 4.

Figure 4: A rhombic face of \( \mathcal{P}_n \) with oriented edges.

As each face \( f_j \) of an isoradial circle pattern is a rhombus we can write, using the notation of Figure 4

\[ z_j - z = a_{j-1} + a_j \quad \text{and} \quad v_j - v_{j-1} = a_j - a_{j-1}. \quad (26) \]

Denoting \( \alpha_j = \alpha([z,z_j]) \), \( l_j = |z_j - z| = 2\varepsilon_n \sin(\alpha_j/2) \), and \( \tilde{l}_j = |v_j - v_{j-1}| = 2\varepsilon_n \cos(\alpha_j/2) \), we easily obtain by simple calculations that

\[ f_{\alpha_j}(0) = (\pi - \alpha_j)/2, \quad f'_{\alpha_j}(0) = \tilde{l}_j/(2l_j), \quad f''_{\alpha_j}(0) = 0. \]

Taking into account that equation (2) holds with \( R_n \equiv \varepsilon_n \) and using the uniform boundedness \( \text{[15]} \) of the labelling \( \alpha \), inequality (24) yields

\[ 0 \leq \sum_{j=1}^{m} f'_{\alpha_j}(0) x_j + O(\varepsilon_n^4). \quad (27) \]
To evaluate this sum, expand
\[ h_n(z_j) - h_n(z) = \text{Re}(\log g'(z_j) - \log g'(z)) = \text{Re}(a(z_j - z) + b(z_j - z)^2) + O(\varepsilon_n^3) \]
and
\[ x_j = h_n(z_j) - h_n(z) - \beta |z_j|^2 + \beta |z|^2 = \text{Re}(a(z_j - z) + b(z_j - z)^2 - 2\beta z(z_j - z)) - \beta t_j^2 + O(\varepsilon_n^3). \]

Noting that \( f'_{\alpha_j}(0)(z_j - z) = (v_j - v_{j-1})/(2i) \) we get
\[ \sum_{j=1}^{m} f'_{\alpha_j}(0)x_j = \text{Re}\left( \frac{a - 2\beta z}{2i} \sum_{j=1}^{m} (v_j - v_{j-1}) + \frac{b}{2i} \sum_{j=1}^{m} (v_j - v_{j-1})(z_j - z) \right) 
- \beta \sum_{j=1}^{m} |l_j| \frac{l_j}{2} + O(\varepsilon_n^3). \]

Thus from inequality (28), remembering \( l_j = 2\varepsilon_n \sin(\alpha_j/2) \) and \( \tilde{l}_j = 2\varepsilon_n \cos(\alpha_j/2) \), we arrive at
\[ 0 \leq -\beta \varepsilon_n^2 \sum_{j=1}^{m} \sin(\alpha_j/2) \cos(\alpha_j/2) + O(\varepsilon_n^3) \quad \iff \quad \beta \sum_{j=1}^{m} \sin(\alpha_j) \leq O(\varepsilon_n). \]

Note that \( \varepsilon_n^2 \sum_{j=1}^{m} \sin(\alpha_j) > \pi \varepsilon_n^2 \) is the area of the rhombic faces incident to the vertex \( z \). Thus we conclude that \( \beta = O(\varepsilon_n) \). This means, that if we choose \( \beta = C \varepsilon_n \) with \( C > 0 \) a sufficiently large constant and if \( \varepsilon_n \) is small enough such that \( C \varepsilon_n < 1 \), then \( p \) will have no maximum in \( V_{int}(G_n) \). In that case, as we have \( p(z) = \beta |z|^2 = O(\varepsilon_n) \) on \( V_2(G_n) \), we deduce that \( p(z) \leq O(\varepsilon_n) \) in \( V(G_n) \) and thus
\[ t_n(z) - h_n(z) \leq O(\varepsilon_n) \quad \text{for } z \in V(G_n). \quad (28) \]

The proof for the reverse inequality is almost the same. The only modifications needed are reversing the sign of \( \beta \) and a few inequalities.

**Remark 3.4.** The statement of Lemma 3.3 can be improved to
\[ h_n(z) - t_n(z) = O(\varepsilon_n^2) \]
in the case of a 'very regular' isoradial circle pattern. These are isoradial circle patterns such that for every oriented edge \( e_{j1} = z_{j1} - z \in \hat{E}(G) \) incident to an interior vertex \( z \in V_{int}(G) \) there is another parallel edge \( e_{j2} = z_{j2} - z \in \hat{E}(G) \) with opposite direction incident to \( z \), that is \( e_{j2} = -e_{j1} \). Furthermore, the corresponding intersection angles agree: \( \alpha([z, z_{j1}]) = \alpha([z, z_{j2}]) \). This additional regularity property holds for example for an orthogonal circle pattern with the combinatorics of a part of the square grid, see Figure 2(a).

The proof of estimation (29) follows the same reasonings as above, but makes an \( O(\varepsilon_n^4) \)-analysis. The additional regularity implies that all terms of order \( \varepsilon_n^3 \) vanish.

**Definition 3.5.** For a function \( \eta : V(G) \to \mathbb{R} \) define a discrete Laplacian by
\[ \Delta \eta(z) = \sum_{[z,z_j] \in E(G)} 2f'_{\alpha([z,z_j])}(0)(\eta(z_j) - \eta(z)). \quad (30) \]
As \( f'_n(z_1,z_2)(0) > 0 \) one immediately has the following

**Lemma 3.6** (Maximum Principle). If \( \Delta \eta \geq 0 \) on \( V_{int}(G) \) then the maximum of \( \eta \) is attained at the boundary \( V_\partial(G) \).

The proof of Lemma 3.6 actually shows that \( t_n - h_n \) is almost harmonic. More precisely, we have \( \Delta(t_n - h_n) = \mathcal{O}(\varepsilon_n^2) \). Adding a suitable subharmonic function \( \beta |z|^2 \) with \( \beta > 0 \) big enough, we deduce that the resulting function \( \bar{p} \) is subharmonic, that is \( \Delta \bar{p} \geq 0 \), such that \( \bar{p} \) attains its maximum at the boundary. This is also important for our proof of the following lemma.

**Lemma 3.7.** Let \( t_n \) and \( h_n \) be defined as in Lemma 3.3. Let \( K \subset D \) be a compact subset in \( D \). Then the following estimation holds for every interior vertex \( z \in V_{int}(G_n) \cap K \) such that all its incident vertices \( z_1, \ldots, z_l \) are also in \( V_{int}(G_n) \cap K \):

\[
t_n(z_j) - h_n(z_j) - (t_n(z) - h_n(z)) = \mathcal{O}(\varepsilon_n(- \log \varepsilon_n)^{\frac{1}{2}}).
\]

(31) for \( j = 1, \ldots, l \). The constant in the \( \mathcal{O} \)-notation may depend on \( K \), but not on \( n \) or \( z \).

The proof of Lemma 3.7 uses the following estimation for superharmonic functions, which is a version of Corollary 3.1 of [SC97]; see also [SC97, Remark 3.2 and Lemma 2.1].

**Proposition 3.8** ([SC97]). Let \( G \) be an undirected connected graph without loops and let \( c : E(G) \to \mathbb{R}^+ \) be a positive weight function on the edges. Denote \( c(e) = c(x,y) \) for an edge \( e = [x,y] \in E(G) \) and assume that

\[
m = \max_{[x,y] \in E(G)} \sum_{[x,z] \in E(G)} \frac{c(x,z)}{c(x,y)} < \infty.
\]

Denote by \( d(x,y) \) the combinatorial distance between two vertices \( x, y \in V(G) \) in the graph \( G \). Let \( B_x(\varrho) = \{ y \in V(G) : d(x,y) \leq \varrho \} \) be the combinatorial ball of radius \( \varrho > 0 \) around the vertex \( x \in V(G) \). Fix \( x \in V(G) \) and \( R \geq 4 \) and set

\[
A_R = \sup_{1 \leq \varrho \leq R} \varrho^{-2} W_x(\varrho), \quad \text{where} \quad W_x(\varrho) = \sum_{z \in B_x(\varrho), y \in V(G) \atop d(x,z) < d(x,y)} c(z,y).
\]

Let \( u \) be a positive superharmonic function in \( B_x(R+1) \), that is

\[
\sum_{[z,w] \in E(G)} c(z,w)(u(z) - u(w)) \leq 0
\]

for all \( w \in B_x(R+1) \). Let \( y \) be incident to \( x \) in \( G \). Then

\[
\left| \frac{u(x)}{u(y)} - 1 \right| \leq \frac{4m^2 \sqrt{A_R}}{\sqrt{c(x,y) \log_2 R}}.
\]

**Proof of Lemma 3.7.** Lemma 3.3 implies that \( t_n(z_j) - t_n(z) = \mathcal{O}(\varepsilon_n) \) for all incident vertices \( z, z_j \in V(G_n) \) since \( h_n = \log |g'| \) is a \( C^\infty \)-function. Consider a Taylor expansion about 0 of

\[
0 = \left( \sum_{j=1}^{m} f_n(z_j)(t_n(z_j) - t_n(z)) \right) - \pi.
\]
Similar reasonings as in the proof of Lemma 3.3 imply that $\Delta t_n(z) = O(\varepsilon_n^2)$. Let $p = t_n - h_n + \beta |z|^2$ with $\beta \in (0, 1)$. Choosing $\beta = C\varepsilon_n$ with a sufficiently large constant $C > 0$ and $\varepsilon_n$ small enough we deduce similarly as in the proof of Lemma 3.3 that $\Delta p(z) \geq 0$ for all interior vertices $z \in V_{int}(G_n)$. Now define the positive function $\hat{p} = \varepsilon_n + \|p\| - p$. Then $\Delta \hat{p}(z) \leq 0$ for all $z \in V_{int}(G_n)$. The proof of Lemma 3.3 shows that there is a constant $C_1$, depending only on $p$, $D_n$, and the labelling $\alpha$, such that $\|p\| \leq C_1 \varepsilon_n$. Thus $\|\hat{p}\| \leq C_2 \varepsilon_n$ with $C_2 = 2C_1 + 1$.

To finish the proof, we apply Proposition 3.8 to the superharmonic function $\hat{p}$. Remember that $G_n$ is a connected graph without loops and $c(\varepsilon) := 2f_n(\varepsilon)(0) > 0$ defines a positive weight function on the edges. The bound (15) on the labelling $\alpha$ implies that

$$m = \max_{x,y \in E(G_n)} \frac{c(x,z)}{c(x,y)} \leq \frac{2\pi}{\cot(\pi/4 - C/2)} \frac{\cot(\pi/4 + C/2)}{\pi/2 - C} =: C_3 < \infty.$$ 

Let $x \in V_{int}(G_n)$. Note that

$$W_x(\varepsilon) = \sum_{z \in B_x(\varepsilon), y \in V(G_n) \atop d(x,z) < d(x,y)} c(z,y) \leq \max_{\varepsilon \in E(G_n)} c(\varepsilon) \left| F_w(x,\varepsilon) \right|,$$

where $F_w(x,\varepsilon)$ is the set of all faces of $\mathcal{G}_n$ with one white vertex $z \in B_x(\varepsilon)$ and $|F_w(x,\varepsilon)|$ denotes the number of faces of $F_w(x,\varepsilon)$. Now, $\max_{\varepsilon \in E(G_n)} c(\varepsilon) < \cot(\pi/4 - C/2)/2$ and

$$F_w(x,\varepsilon) \subseteq D_x((\varepsilon + 1)2\varepsilon) = \{w \in C : |w - x| \leq (\varepsilon + 1)2\varepsilon\},$$

as the edge lengths in $G_n$ are smaller than $2\varepsilon_n$. Remember that $F(f) = \varepsilon_n^2 \sin \alpha(f) > \varepsilon_n^2 \sin(\pi/2 - C)$ is the area of the face $f \in F(\mathcal{G}_n)$. Thus

$$|F_w(x,\varepsilon)| \leq \frac{\pi((\varepsilon + 1)2\varepsilon)^2}{\varepsilon_n^2 \sin(\pi/2 - C)} \leq \frac{16\pi \varepsilon^2}{\sin(\pi/2 - C)} =: g^2 C_4$$

for all $g \geq 1$. Therefore we obtain $A_R = \sup_{1 \leq \varepsilon \leq R} g^{-2}W_x(\varepsilon) < C_4$, where the upper bound $C_4$ is independent of $R \geq 4$ and $n \in \mathbb{N}$.

Let $K \subseteq D$ be compact. Denote by $d$ the Euclidean distance (between a point and a compact set or between closed sets of $\mathbb{R}^2 \cong \mathbb{C}$). Let $z \in V(G_n) \cap K$ and set $(R + 1) = d(z, V\partial(G_n))$ to be the combinatorial distance from $z$ to the boundary of $G_n$. Let $z_j \in V(G_n)$ be incident to $z$. As the labelling $\alpha$ is bounded, $\varepsilon_n \to 0$, and $\mathcal{G}_n$ approximates $D$, we deduce that $R \geq \varepsilon_n^{-1}C_5 \geq 4$ if $n \geq n_0$ is large enough. Thus for all $n \geq n_0$ and all $z \in V(G_n) \cap K$

$$1/\sqrt{\log_2 R} \leq 1/\sqrt{\log_2 C_5 - \log_2 \varepsilon_n} \leq \sqrt{2}/\sqrt{-\log_2 \varepsilon_n}$$

holds by our assumptions. Proposition 3.8 implies that

$$\frac{1}{\hat{p}(z_j)} - 1 \leq \frac{4C_3^2 \sqrt{C_4 \sqrt{2}}}{\sqrt{-c(z,z_j)} \log_2 \varepsilon_n}$$

for all incident vertices $z, z_j \in V(G_n) \cap K$ and $n \geq n_0$. As $c(z,z_j) \leq \cot(\pi/4 - C/2)/2$ and $\|\hat{p}\| \leq C_2 \varepsilon_n$ we finally arrive at the desired estimation

$$|t_n(z_j) - h_n(z_j) - (t_n(z) - h_n(z))| = |\hat{p}(z) - \hat{p}(z_j)| \leq 4C_3 \varepsilon_n (-\log_2 \varepsilon_n)^{-1/2}$$
where a \in V(G_n) \cap K and n \geq n_0, where the constant \(C_5\) depends on \(C_2, \ldots, C_5\), that is only on \(g, D, C, \hat{C}, \) and \(K\).

**Lemma 3.9.** Let \( \bar{e} = \overrightarrow{uv} \in \overline{E}(\mathcal{D}_n) \) be a directed edge with \( u \in V(G_n) \) and \( v \in V(G_n^*) \). Denote by \( \delta_n(e) \) the combinatorial distance in \( \mathcal{D}_n \) from \( e = [u, v] \) to \([z_0, v_0]\), that is the least integer \( k \) such that there is a sequence of edges 
\[
\{ [z_0, v_0] = e_1, e_2, \ldots, e_k = e \} \subseteq E(\mathcal{D}_n) \text{ such that the edges } e_{m+1} \text{ and } e_m \text{ are incident to the same face in } \mathcal{D}_n \text{ for } m = 1, \ldots, k - 1. \]
Then
\[
\varphi_n(\bar{e}) = \arg g'(v) + \phi_n(\bar{e}) + \delta_n(e)O(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}).
\] (32)

The constant in the notation \( O(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}) \) may depend on the distance of \( v \) to the boundary \( \partial D \).

Note that if \( \partial D \) is smooth, then \( \delta_n(e) = O(\varepsilon_n^{-1}) \). In general we have \( \delta_n(e) = O(\varepsilon_n^{-1}) \) on compact subsets \( K \subset D \), where the constant in the notation \( O(\varepsilon_n^{-1}) \) may depend on \( K \). In any case, on compact subsets of \( D \) we have
\[
\varphi_n(\bar{e}) = \arg g'(v) + \phi_n(\bar{e}) + O((-\log_2 \varepsilon_n)^{-\frac{1}{2}}). \tag{33}
\]

**Proof.** Using the notation of Figure 3 (left), equation (31) implies
\[
2f_\alpha(0)(\log |g'(z_+)| - \log |g'(z_-)|) = \frac{\nu_+ - \nu_-}{i(z_+ - z_-)} \operatorname{Re}(a(z_+ - z_-)) + O(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}})
\]
\[
= \frac{\nu_+ - \nu_-}{i(z_+ - z_-)} \operatorname{Re}(a(z_+ - z_-)) + O(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}})
\]
\[
= \frac{\nu_+ - \nu_-}{i(z_+ - z_-)} \operatorname{Im}(a(z_+ - z_-)) + O(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}})
\]
\[
= \varphi_n(\bar{e}) + g'(v_+) - g'(v_-) + O(\varepsilon_n(-\log_2 \varepsilon_n)^{-\frac{1}{2}}),
\]

where \( a = \frac{g''((z_+ + z_-)/2)}{g'((z_+ + z_-)/2)} = \frac{g''((v_+ + v_-)/2)}{g'((v_+ + v_-)/2)} \).

By Lemma 2.7 we can choose the angle functions \( \phi_n \) and \( \varphi_n \) on any minimal sequence of edges \( \{ [z_0, v_0] = e_1, e_2, \ldots, e_k = e \} \subseteq E(\mathcal{D}_n) \) such that equations (31) to (34) are satisfied without the \( \text{(mod 2\pi)} \)-term. Using the above considerations of \( 2f_\alpha(\log r_n(z_+) - \log r_n(z_-)) \) and the normalization of \( \varphi_n \), we arrive at equation (32).

**Proof of Theorem 3.1.** Consider a compact subset \( K \) of \( D \). Let \( z \in V(G_n) \cap K \) and \( v \in V(G_n^*) \cap K \) be vertices which are incident in \( \mathcal{D}_n \), that is \([z, v] \in E(\mathcal{D}_n)\). Then Lemmas 3.3 and 3.9 imply that
\[
\log g'(z) = \log |g'(z)| + i \varphi_n(\bar{e}v) = \log(r_n(z)/R_n(z)) + i(\varphi_n(\bar{e}v) - \phi_n(\bar{e}v)) + O((-\log_2 \varepsilon_n)^{-\frac{1}{2}}).
\]
As \( g' \) and thus the quotient \( r_n / R_n \) is uniformly bounded, we obtain
\[
g'(z) = \frac{r_n(z)}{R_n(z)} e^{i(\varphi_n(z) - \varphi_n(w))} + O((- \log_2 \varepsilon_n)^{-\frac{1}{2}}). \tag{34}
\]
This implies the uniform convergence on compact subsets of \( D \) of \( q_n \) to \( g' \).

Convergence of \( q_n \) is now proven by using suitable integrations of \( g' \) and \( q_n \).

Let \( w \in V(G_n^*) \) and consider a shortest path \( \gamma \) in \( G_n^* \) from \( v_0 \) to \( w \) with vertices \( \{v_0 = w_1, w_2, \ldots, w_k = w\} \subset V(G_n^*) \). Then
\[
g(w) = g(v_0) + \int_{\gamma} g'(\zeta) d\zeta = g(v_0) + \sum_{j=1}^{k-1} g'(w_{j+1})(w_{j+1} - w_j) + O(\varepsilon_n)
\]
= \( g(v_0) + \sum_{j=1}^{k-1} q_n(w_{j+1})(w_{j+1} - w_j) + O((- \log_2 \varepsilon_n)^{-\frac{1}{2}}) \),
because \( g'(w_j) - q_n(w_j) = O((- \log_2 \varepsilon_n)^{-\frac{1}{2}}) \), \( w_{j+1} - w_j = O(\varepsilon_n) \) and \( k = O(\varepsilon_n^{-1}) \) on compact sets. Thus it only remains to show that
\[
p_n(w) = g(v_0) + \sum_{j=1}^{k-1} q_n(w_{j+1})(w_{j+1} - w_j) + O((- \log_2 \varepsilon_n)^{-\frac{1}{2}}). \tag{35}
\]

Remembering
\[
q_n(w_+) = \frac{r_n(z_+)}{R_n(z_+)} e^{i(\varphi_n(w_{j+1}, z_+) - \varphi_n(w_{j+1}, z_-))} + O((- \log_2 \varepsilon_n)^{-\frac{1}{2}}),
\]
\[
(w_{j+1} - w_j) = 2R_n(z_+ \cos(\alpha([w_{j+1}, w_j])/2) e^{i(\varphi_n(w_{j+1}, z_+))/2}(\pi/2 - \alpha([w_{j+1}, w_j])/2)),
\]
we can conclude that
\[
q_n(w_{j+1})(w_{j+1} - w_j) = p_n(w_{j+1}) - p_n(w_j) + O(\varepsilon_n(- \log_2 \varepsilon_n)^{-\frac{1}{2}}),
\]
where \( z_-, z_+ \in V(G) \) are incident to \( w_{j+1} \) and \( w_j \) and we have used the notations as in Figure 3 (left) with \( w_j = v_\alpha \) and \( w_{j+1} = v_\beta \). As we have normalized \( g(v_0) = p(v_0) \), this proves equation \( 35 \) and therefore the uniform convergence of \( p_n \) to \( g \) on compact subsets of \( D \).

**Remark 3.10.** Theorem 3.1 may easily be generalized in the following ways. First, we may consider ‘nearly isoradial’ circle patterns which satisfy \( R_n(z) = O(\varepsilon_n) \) for all vertices \( z \in V(G_n) \) and \( R_n(z_1)/R_n(z_2) = 1 + O(\varepsilon_n^3) \) for all edges \( [z_1, z_2] \in E(G_n) \).

Second, we may omit the assumption that the whole domain \( D \) is approximated by the rhombic embeddings \( D_n \). Then the convergence claims remain true for compact subsets of any open domain \( D' \subset D \) which is covered or approximated by the rhombic embeddings and contains \( v_0 \).

Using the angle function instead of the radius function, we obtain the following analog of Theorem 3.1 for Neumann boundary conditions.
Theorem 3.11. Under the same assumptions as in Theorem 3.1 and with the same notation, assume further that $\varepsilon_n$ is sufficiently small such that for all $n \in \mathbb{N}$

$$\sup_{v \in D} \max_{\theta \in [0, 2\pi]} |\arg g'(v + 2\varepsilon_n e^{i\theta}) - \arg g'(v)| < \frac{\pi}{2} - C < \min_{e \in E(G_n)} (\pi - \alpha(e)).$$  (36)

Define an angle function on the oriented boundary edges by

$$\varphi_n(\vec{e}) = \phi_n(\vec{e}) + \arg g'(v),$$

where $\vec{e} = \vec{z}v \in \vec{E}(D_n)$ and $v \in V(G_n^*)$. Then there is a circle pattern $\mathcal{C}_n$ for $G_n$ and $\alpha_n$ with radius function $r_n$ and angle function $\varphi_n$ with these boundary values.

Suppose that this circle pattern is normalized such that

$$r_n(z_0) = R_n(z_0)|g'(z_0)|,$$

where $z_0 \in V(G_n)$ is chosen such that the disk bounded by the circle $C_{z_0}$ contains 0. Suppose further that $\mathcal{C}_n$ is normalized by a translation such that

$$p_n(v_0) = g(v_0),$$  (37)

where $p_n(v)$ denotes the intersection point corresponding to $v \in V(G_n^*)$.

Then $q_n \to g$ and $g_n \to g$ uniformly on compact subsets in $D$ as $n \to \infty$.

Proof. The existence claim for the circle pattern with Neumann boundary conditions follows from [BS04, Theorem 3] using the assumption (36).

Theorem 2.8 shows that the difference $\varphi_n - \phi_n$ gives rise to a function $\delta_n : V(G_n^*) \to \mathbb{R}$ with boundary values given by $\arg g'$. The proof of the convergence claim is similar to the proof of Theorem 3.1. The roles of $\delta_n = \varphi_n - \phi_n$ and $\log(r_n/R_n)$ have to be interchanged in Lemmas 3.3, 3.7, and 3.9 and similarly $\arg g' = \text{Im}(\log g')$ has to be considered instead of $\log |g'| = \text{Re}(\log g')$. The role of equation (2) is substituted by equation (17).

4 Quasicrystallic circle patterns

The order of convergence in Theorems 3.1 and 3.11 can be improved for a special class of isoradial circle patterns with a uniformly bounded number of different edge directions and a local deformation property. In the following, we introduce suitable terminology and some useful results.

As the kites of an isoradial circle pattern are in fact rhombi, an embedded isoradial circle pattern leads to a rhombic embedding in $\mathbb{C}$ of the corresponding b-quad-graph $\mathcal{D}$. Conversely, adding circles with centers in the white vertices of a rhombic embedding and radius equal to the edge length results in an embedded isoradial circle pattern.

Given a rhombic embedding of a b-quad-graph $\mathcal{D}$, consider for each directed edge $\vec{e} \in \vec{E}(\mathcal{D})$ the vector of its embedding as a complex number with length one. Half of the number of different values of these directions is called the dimension $d$ of the rhombic embedding. If $d$ is finite, the rhombic embedding is called quasicrystallic. A circle pattern for a b-quad-graph $\mathcal{D}$ is called a quasicrystallic circle pattern if there exists a quasicrystallic rhombic embedding of $\mathcal{D}$ and if the
intersection angles are taken from this rhombic embedding. The comparison
function of the isoradial circle pattern \( C_1 \) for \( D \) and the quasicrystallic circle
pattern \( C_2 \) will also be called comparison function for \( C_2 \).

Quasicrystallic circle patterns were introduced in \[BMS05\]. Certainly, this
property only makes sense for infinite graphs or infinite sequences of graphs
with growing number of vertices and edges.

In the following we will identify the b-quad-graph \( D \) with a rhombic embed-
ing of \( D \).

4.1 Quasicrystallic rhombic embeddings and \( Z^d \)

Any rhombic embedding of a connected
b-quad-graph \( D \) can be seen as a sort of pro-
jection of a certain two-dimensional subcom-
plex (combinatorial surface) \( \Omega_\mathcal{D} \) of the multi-
dimensional lattice \( Z^d \) (or of a multi-dimen-
sional lattice \( \mathcal{L} \) which is isomorphic to \( Z^d \)).

An illustrating example is given in Figure 5.

The combinatorial surface \( \Omega_\mathcal{D} \) in \( Z^d \) can
be constructed in the following way. Denote
the edges of \( \{\pm e_1, \ldots, \pm e_d\} \) at the origin which correspond to the edges of
\( \{\pm a_1, \ldots, \pm a_d\} \) incident to \( x_0 \) in \( D \), together with their endpoints. Successively
continue the construction at the new endpoints. Also, add two-dimensional
facets (faces) of \( Z^d \) corresponding to faces of \( D \), spanned by incident edges.

A combinatorial surface \( \Omega_\mathcal{D} \) in \( Z^d \) corresponding to a quasicrystallic rhom-

bic embedding can be characterized using the following monotonicity property,
see \[BMS05\], Section 6 for a proof.

**Lemma 4.1** (Monotonicity criterium). Any two points of \( \Omega_\mathcal{D} \) can be connected
by a path in \( \Omega_\mathcal{D} \) with all directed edges lying in one \( d \)-dimensional octant,
that is all directed edges of this path are elements of one of the \( 2^d \) subsets of
\( \{\pm e_1, \ldots, \pm e_d\} \) containing \( d \) linearly independent vectors.

An important class of examples of rhombic embeddings of b-quadgraphs can be
constructed using ideas of the grid projection method for quasiperiodic tilings
of the plane; see for example \[DKS85\], \[GR86\], \[Sen95\].

**Example 4.2** (Quasicrystallic rhombic embedding obtained from a plane). Let
\( E \) be a two-dimensional plane in \( R^d \) and \( t \in E \). Let \( e_1, \ldots, e_d \) be the standard orthono-
mal basis of \( R^d \). Fix a white vertex \( x_0 \in V(D) \) and the origin of \( R^d \). Add
the edges of \( \{\pm e_1, \ldots, \pm e_d\} \) at the origin which correspond to the edges of
\( \{\pm a_1, \ldots, \pm a_d\} \) incident to \( x_0 \) in \( D \), together with their endpoints. Successively
continue the construction at the new endpoints. Also, add two-dimensional
facets (faces) of \( Z^d \) corresponding to faces of \( D \), spanned by incident edges.

A combinatorial surface \( \Omega_\mathcal{D} \) in \( Z^d \) corresponding to a quasicrystallic rhom-

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\( E \) be a two-dimensional plane in \( R^d \) and \( t \in E \). Let \( e_1, \ldots, e_d \) be the standard orthono-
mal basis of \( R^d \). We assume that \( E \) does not contain any of the segments
\( s_j = \{t + \lambda e_j : \lambda \in [0, 1]\} \) for \( j = 1, \ldots, d \). Then we can choose positive
numbers \( c_1, \ldots, c_d \) such that the orthogonal projections \( P_E(c_j e_j) \) have length
1. (If \( E \) contains two different segments \( s_{j1} \) and \( s_{j2} \), the following construc-
tion only leads to the standard square grid pattern \( Z^2 \). If \( E \) contains exactly
one such segment \( s_j \), then the construction may be adapted for the remaining
Consider around each vertex $p$ of the lattice $L = c_1 \mathbb{Z} \times \cdots \times c_d \mathbb{Z}$ the hypercuboid $V = [-c_1/2, c_1/2] \times \cdots \times [-c_d/2, c_d/2]$, that is the Voronoi cell $p + V$. These translations of $V$ cover $\mathbb{R}^d$. We build an infinite monotone two-dimensional surface $\Omega^E(L)$ in $L$ by the following construction. The basic idea is illustrated in Figure 6 (left).

If $E$ intersects the interior of the Voronoi cell of a lattice point (i.e. $(p + V) \cap E \neq \emptyset$ for $p \in L$), then this point belongs to $\Omega^E(L)$. Undirected edges correspond to intersections of $E$ with the interior of a $(d-1)$-dimensional facet bounding two Voronoi cells. Thus we get a connected graph in $L$. An intersection of $E$ with the interior of a translated $(d-2)$-dimensional facet of $V$ corresponds to a rectangular two-dimensional face of the lattice. By construction, the orthogonal projection of this graph onto $E$ results in a planar connected graph whose faces are all of even degree (= number of incident edges or of incident vertices). A face of degree bigger than 4 corresponds to an intersection of $E$ with the translation of a $(d-k)$-dimensional facet of $V$ for some $k \geq 3$. Consider the vertices and edges of such a face and the corresponding points and edges in the lattice $L$. These points lie on a combinatorial $k$-dimensional hypercuboid contained in $L$. By construction, it is easy to see that there are two points of the $k$-dimensional hypercuboid which are each incident to $k$ of the given vertices. We choose a point with least distance from $E$ and add it to the surface. Adding edges to neighboring vertices splits the face of degree $2k$ into $k$ faces of degree 4. Thus we obtain an infinite monotone two-dimensional combinatorial surface $\Omega^E(L)$ which projects to an infinite rhombic embedding covering the whole plane $E$.

**Example 4.3** (Modification of the construction in Example 4.2). The method used in the preceding example can be modified to result in similar but different rhombic embeddings. The basic idea is illustrated in Figure 6 (right).

Let $E$ be a two-dimensional plane in $\mathbb{R}^d$ satisfying the same assumptions as in Example 4.2. Let $N \neq 0$ be a vector orthogonal to $E$. Let $\Delta$ be an equilateral triangle in $E$ with vertices $t_1, t_2, t_3$ and let $s$ be the intersection point of the bisecting lines of the angles. Consider the two-dimensional facets of the three-
dimensional tetrahedron $T(\Delta, N)$ spanned by the four vertices $t_1, t_2, t_3, s + N$. Exactly one of these facets is completely contained in $E$ (this is the triangle $\Delta$). We remove the triangle from $E$ and add instead the remaining facets of $T(\Delta, N)$. Let $E_\Delta$ be the resulting two-dimensional surface. Note that $E_\Delta$ is orientable like $E$. We remove the triangle from $E$ and add instead the remaining facets of $T(\Delta, N)$. Let $E_\Delta$ be the resulting two-dimensional surface. Note that $E_\Delta$ is orientable like $E$. Define $\gamma \in (0, \pi/2)$ by $\gamma = \arctan(2\sqrt{3}\|N\|/\|t_1 - t_2\|)$, where $\| \cdot \|$ denotes the Euclidean norm of vectors in $\mathbb{R}^d$. $\gamma$ is the acute angle between $E$ and the two-dimensional facets of $T(\Delta, N)$ not contained in $E$.

If $\gamma$ is small enough, then our assumptions imply that we can apply the same construction algorithm to $E_\Delta$ as for the plane $E$ in the previous example and obtain a monotone surface $\Omega^L(E_\Delta)$. The orthogonal projection of $\Omega^L(E_\Delta)$ onto $E$ is a rhombic embedding which coincides with the rhombic embedding from $\Omega^L(E)$ except for a finite part.

The quasicrystallic rhombic embedding of Example 4.3 may also obtained using the following general concept of local changes of rhombic embeddings.

**Definition 4.4.** Let $\mathcal{D}$ be a rhombic embedding of a finite simply connected b-quad-graph with corresponding combinatorial surface $\Omega_\mathcal{D}$ in $\mathbb{Z}^d$.

Let $\mathbf{z} \in V_{\text{int}}(\Omega_\mathcal{D})$ be an interior vertex with exactly three incident two-dimensional facets of $\Omega_\mathcal{D}$. Consider the three-dimensional cube with these boundary facets. Replace the three given facets with the three other two-dimensional facets of this cube. This procedure is called a flip; see Figure 7 for an illustration.

A vertex $z \in \mathbb{Z}^d$ can be reached with flips from $\Omega_\mathcal{D}$ if $z$ is contained in a combinatorial surface obtained from $\Omega_\mathcal{D}$ by a suitable sequence of flips. The set of all vertices which can be reached with flips (including $V(\Omega_\mathcal{D})$) will be denoted by $\mathcal{F}(\Omega_\mathcal{D})$.

For further use we enlarge the set $\mathcal{F}(\Omega_\mathcal{D})$ of vertices which can be reached by flips from $\Omega_\mathcal{D}$ in the following way. For a set of vertices $W \subset V(\mathbb{Z}^d)$ denote by $W^{[1]}$ the set $W$ together with all vertices incident to a two-dimensional facet of $\mathbb{Z}^d$ where three of its four vertices belong to $W$. Define $W^{[k+1]} = (W^{[k]})^{[1]}$ inductively for all $k \in \mathbb{N}$. In particular, we denote for some arbitrary, but fixed $\kappa \in \mathbb{N}$

$$\mathcal{F}_\kappa(\Omega_\mathcal{D}) = (\mathcal{F}(\Omega_\mathcal{D}))^{[\kappa]}.$$  

4.2 Quasicrystalline circle patterns and integrability

Let $\mathcal{D}$ be a quasicrystalline rhombic embedding of a b-quad-graph. The combinatorial surface $\Omega_\mathcal{D}$ in $\mathbb{Z}^d$ is important by its connection with integrability. See also [BS08] for a more detailed presentation and a deepened study of integrability and consistency.

In particular, a function defined on the vertices of $\Omega_\mathcal{D}$ which satisfies some 3D-consistent equation on all faces of $\Omega_\mathcal{D}$ can uniquely be extended to the brick

$$\Pi(\Omega_\mathcal{D}) := \{ n = (n_1, \ldots, n_d) \in \mathbb{Z}^d : a_k(\Omega_\mathcal{D}) \leq n_k \leq b_k(\Omega_\mathcal{D}), \ k = 1, \ldots, d \},$$
where \( a_k(\Omega_\mathcal{D}) = \min_{n \in V(\Omega_\mathcal{D})} n_k \) and \( b_k(\Omega_\mathcal{D}) = \max_{n \in V(\Omega_\mathcal{D})} n_k \). Note that \( \Pi(\Omega_\mathcal{D}) \) is the hull of \( \Omega_\mathcal{D} \). A proof may be found in [BMS05, Section 6]. This extension of a function using a 3D-consistent equation will now be applied for the comparison function \( w \) defined in (14) of two circle patterns. In particular, we take for \( \mathcal{C}_1 \) the isoradial circle pattern which corresponds to the quasicrystalline rhombic embedding \( \mathcal{D} \). Given another circle pattern \( \mathcal{C}_2 \) for \( \mathcal{D} \) with the same intersection angles, let \( w \) be the comparison function for \( \mathcal{C}_2 \). Note that the Hirota Equation (15) is 3D-consistent; see Sections 10 and 11 of [BMS05] for more details. Thus \( w \) considered as a function on \( V(\Omega_\mathcal{D}) \) can uniquely be extended to the brick \( \Pi(\Omega_\mathcal{D}) \) such that equation (15) holds on all two-dimensional facets. Additionally, \( w \) and its extension are real valued on white points \( V_w(\Omega_\mathcal{D}) \) and has value in \( S^1 \) for black points \( V_b(\Omega_\mathcal{D}) \). This can easily be deduced from the Hirota Equation (15).

The extension of \( w \) can be used to define a radius function for any rhombic embedding with the same boundary faces as \( \mathcal{D} \).

**Lemma 4.5.** Let \( \mathcal{D} \) and \( \mathcal{D}' \) be two simply connected finite rhombic embeddings of b-quad-graphs with the same edge directions. Assume that \( \mathcal{D} \) and \( \mathcal{D}' \) agree on all boundary facets. Let \( \mathcal{C} \) be an (embedded) planar circle pattern for \( \mathcal{D} \) and the labelling given by the rhombic embedding. Then there is an (embedded) planar circle pattern \( \mathcal{C}' \) for \( \mathcal{D}' \) which agrees with \( \mathcal{C} \) for all boundary circles.

**Proof.** Consider the monotone combinatorial surfaces \( \Omega_\mathcal{D} \) and \( \Omega'_\mathcal{D} \). Without loss of generality, we can assume that \( \Omega_\mathcal{D} \) and \( \Omega'_\mathcal{D} \) have the same boundary faces in \( \mathbb{Z}^d \). Thus they both define the same brick \( \Pi(\Omega_\mathcal{D}) = \Pi(\Omega'_\mathcal{D}) =: \Pi \). Given the circle pattern \( \mathcal{C} \), define the function \( w \) on \( V(\Omega_\mathcal{D}) \) by (14). Extend \( w \) to the brick \( \Pi \) such that condition (15) holds for all two-dimensional facets. Consider \( w \) on \( \Omega'_\mathcal{D} \) and build the corresponding pattern \( \mathcal{C}' \), such that the points on the boundary agree with those of the given circle pattern \( \mathcal{C} \). Equation (15) guarantees that all rhombi of \( \Omega'_\mathcal{D} \) are mapped to closed kites. Due to the combinatorics, the chain of kites is closed around each vertex. Since the boundary kites of \( \mathcal{C}' \) are given by \( \mathcal{C} \) which is an immersed circle pattern, at every interior white point the angles of the kites sum up to \( 2\pi \). Thus \( \mathcal{C}' \) is an immersed circle pattern.

Furthermore, \( \mathcal{C}' \) is embedded if \( \mathcal{C} \) is, because \( \mathcal{C}' \) is an immersed circle pattern and \( \mathcal{C}' \) and \( \mathcal{C} \) agree for all boundary kites.

## 5 \( C^\infty \)-convergence for quasicrystallic circle patterns

In order to improve the order of convergence in Theorem 3.1 we study partial derivatives of the extended radius function using the integrability of the Hirota equation (15) and a Regularity Lemma 5.6.

The following constants are useful to estimate the possible orders of partial derivatives for a function defined on \( F_\kappa(\Omega_\mathcal{D}) \). Note that \( F_\kappa(\Omega_\mathcal{D}) \subset \Pi(\Omega_\mathcal{D}) \).

**Definition 5.1.** Let \( \mathcal{D} \) be a rhombic embedding of a finite simply connected b-quad-graph with corresponding combinatorial surface \( \Omega_\mathcal{D} \) in \( \mathbb{Z}^d \). Let \( J \subset \{1, \ldots, d\} \) contain at least two different indices.
For $B \geq 0$ define a combinatorial ball of radius $B$ about $z \in V(\mathbb{Z}^d)$ using the directions $\{e_j : j \in J\}$ by

$$U_J(z, B) = \{\zeta = z + \sum_{j \in J} n_j e_j : \sum_{j \in J} |n_j| \leq B\}. \quad (38)$$

The radius of the largest ball about $z$ using these directions which is contained in $\mathcal{F}_\kappa(\Omega_\varphi)$ is denoted by $B_J(z, \mathcal{F}_\kappa(\Omega_\varphi)) = \max\{B \in \mathbb{N} : U_J(z, B) \subset \mathcal{F}_\kappa(\Omega_\varphi)\}$.

Denote by $d(\hat{z}, \partial \Omega_\varphi)$ the combinatorial distance of $\hat{z} \in V(\Omega_\varphi)$ to the boundary $\partial \Omega_\varphi$, that is the smallest integer $K$ such that there is a connected path with $K$ edges contained in $\Omega_\varphi$ from $\hat{z}$ to a boundary vertex of $\partial \Omega_\varphi$. For further use, we define the constant

$$C_J(\mathcal{F}_\kappa(\Omega_\varphi)) = \min \left\{ \frac{B_J(\hat{z}, \mathcal{F}_\kappa(\Omega_\varphi)) + 1}{d(\hat{z}, \partial \Omega_\varphi)} : \hat{z} \in V_{\text{int}}(\Omega_\varphi) \right\} > 0. \quad (39)$$

Note as an immediate consequence that for all $\hat{z} \in V_{\text{int}}(\Omega_\varphi)$

$$U_J(z, \lceil C_J(\mathcal{F}_\kappa(\Omega_\varphi))d(\hat{z}, \partial \Omega_\varphi) - 1 \rceil) \subset \mathcal{F}_\kappa(\Omega_\varphi),$$

where $\lceil s \rceil$ denotes the smallest integer bigger than $s \in \mathbb{R}$.

The following theorem is an improved version of Theorem 3.1 for a specified class of quasicrystallic circle patterns.

**Theorem 5.2.** Under the assumptions of Theorem 3.1 and with the same notation, let $d \in \mathbb{N}$ with $d \geq 2$ be a constant and assume further that $\mathcal{D}_n$ is a quasicrystallic rhombic embedding in $D$ with edge lengths $\varepsilon_n$ and dimension $d_n \leq d$. The directions of the edges are elements of the set $\{\pm a_1^{(n)}, \ldots, \pm a_{d_n}^{(n)}\} \subset S^1$ such that any two of the vectors of $\{a_1^{(n)}, \ldots, a_{d_n}^{(n)}\}$ are linearly independent. The possible angles are uniformly bounded, that is for all $n \in \mathbb{N}$ the scalar product is strictly bounded away from 1,

$$|\langle a_i^{(n)}, a_j^{(n)} \rangle| \leq \cos(\pi/2 + C) < 1, \quad (40)$$

for all $1 \leq i < j \leq d_n$ and some constant $0 < C < \pi/2$. Consequently, the intersection angles $\alpha_n$ are uniformly bounded in the sense that for all $n \in \mathbb{N}$ and all faces $f \in F(\mathcal{D}_n)$ there holds

$$|\alpha_n(f) - \pi/2| \leq C. \quad (41)$$

Let $\kappa \in \mathbb{N}$, let $J_0 \subset \{1, \ldots, d\}$ contain at least two indices, and let $B, C_{J_0} > 0$ be real constants. Suppose that $J_0 \subset \{1, \ldots, d_n\}$ for all $n \in \mathbb{N}$ and

$$C_{J_0}(\mathcal{F}_\kappa(\Omega_{\mathcal{D}_n})) \geq C_{J_0} > 0.$$

Then we have with the same definitions of $r_n, \phi_n, q_n$, and $p_n$ as in Theorem 3.1 that $q_n \to q'$ and $g_n \to g$ in $C^\infty(D)$ as $n \to \infty$, that is discrete partial derivatives of all orders of $q_n$ and $g_n$ converge uniformly on compact subsets to their smooth counterparts.

Simple examples of sequences of quasicrystallic circle patterns for this theorem are subgraphs of the suitably scaled infinite regular square grid or hexagonal circle patterns or subgraphs of suitably scaled infinite rhombic embeddings.
constructed in Examples 4.2 and 4.3 (see Figure 2). Simply connected parts of these rhombic embeddings which are large enough satisfy the conditions $C_J(\mathcal{F}_n(\Omega_{\mathcal{D}_n})) \geq C_0 > 0$ for all subsets $J \subset \{1, \ldots, d\}$, where the constant $C_0$ only depends on the construction parameters. This is a consequence of the simple combinatorics or of the modified construction in Example 4.3.

Remark 5.3. Similarly as for Theorem 3.1, the proof of Theorem 5.2 shows Remark 5.3.

Remark 5.4. There is an analogous version of Theorem 5.2 of $\mathcal{C}^\infty$-convergence for quasicrystallic circle patterns with Neumann boundary conditions.

For the proof of Theorem 5.2 we first define the comparison function $w_n$ for the circle pattern $\mathcal{C}_n$ according to (13) and extend it to $\mathcal{F}_n(\Omega_{\mathcal{D}_n})$. This extension is again denoted by $w_n$. The restriction of $w_n$ to white vertices is called extended radius function and denoted by $r_n$ as the original radius function for $\mathcal{C}_n$. We also extend $|g'|$ to $\mathcal{F}_n(\Omega_{\mathcal{D}_n})$ by defining $|g'(z)|$ to be the value $|g'(z)|$ at the projection $z$ of $z \in \mathbb{Z}^d$ onto the plane of the rhombic embedding $\mathcal{D}_n$. If $n$ is big enough, which will be assumed in the following, then $z \in D$ or $z \in W \setminus D$ and the distance of $z$ to $D$ is bounded independently of $n$.

Denote $t_n = \log(r_n/R_n) = \log(r_n/\varepsilon_n)$ and $h_n = \log|g'|$ as in Section 5. Using the extensions of $r_n$ and $|g'|$, these functions are defined on all white vertices of $\mathcal{F}_n(\Omega_{\mathcal{D}_n})$. Furthermore we have the following extension of Lemma 3.3.

Lemma 5.5. The estimation

$$h_n(z) - t_n(z) = O(\varepsilon_n).$$

holds for all white vertices $z \in V(\mathcal{F}_n(\Omega_{\mathcal{D}_n}))$. The constant in the $O$-notation may depend on $\kappa$ and on the dimension $d_n$ of $\mathcal{D}_n$.

Proof. Let $z \in V(\mathcal{F}(\Omega_{\mathcal{D}_n}))$ be a white vertex. By definition of $\mathcal{F}(\Omega_{\mathcal{D}_n})$ there is a combinatorial surface $\Omega(z)$ containing $z$ with the same boundary curve as $\Omega_{\mathcal{D}_n}$. Lemma 1.3 implies that we can define an embedded circle pattern $\mathcal{C}'$ using the values of $w_n$ on $\Omega_{\mathcal{D}_n}$. Now the claim follows from Lemma 3.3.

For $z \in V(\mathcal{F}_n(\Omega_{\mathcal{D}_n})) \setminus V(\mathcal{F}(\Omega_{\mathcal{D}_n}))$ observe that equation (13) may be used to extend the estimation of $h_n - t_n$. Each steps adds an error of order $\varepsilon_n$, therefore the final constant depends on $\kappa$ and on $d_n$.

Our main aim is to estimate the partial derivatives of the extended radius function in $\mathbb{Z}^{d_n}$. Such partial derivatives can generally be considered in direction of the vectors $v = \pm e_{j_1} \pm e_{j_2}$ for $0 \leq j_1, j_2 \leq d_n$ such that $e_{j_1}$ and $e_{j_2}$ are not collinear. Let $v_1, \ldots, v_{2d_n(d_n - 1)}$ be an enumeration of these vectors and set $V_n = \{v_1, \ldots, v_{2d_n(d_n - 1)}\}$. The corresponding enumeration of the directions $v = \pm a_{j_1}^{(n)} \pm a_{j_2}^{(n)}$ in $\mathcal{D}_n$ for $0 \leq j_1, j_2 \leq d$ is denoted by $v_1, \ldots, v_{2d_n(d_n - 1)}$. 
For any function \( h \) on white vertices \( z \in \mathbb{Z}^d \) and/or in \( z \in \mathbb{C} \) define \textit{discrete partial derivatives} in direction \( v_i \) or \( v_i \) by

\[
\partial_{v_i} h(z) = \frac{h(z + v_i) - h(z)}{\varepsilon_n |v_i|} \quad \text{and} \quad \partial_{v_i} h(z) = \frac{h(z + \varepsilon_n v_i) - h(z)}{\varepsilon_n |v_i|}
\]

respectively. Furthermore, we call a direction \( v_i \) or \( v_i \) to be \textit{contained in} \( \Omega_{\mathcal{D}_n} \) or \( \mathcal{D}_n \) at a vertex \( \hat{z} \) or \( z \) respectively if there is a two-dimensional facet of \( \Omega_{\mathcal{D}_n} \) incident to \( \hat{z} \) whose diagonal incident to \( \hat{z} \) is parallel to \( v_i \), that is \( \{ \hat{z} + \lambda v_i : \lambda \in [0,1] \} \subset \Omega_{\mathcal{D}_n} \).

Corresponding to these partial derivatives we use a scaled version of the Laplacian in (30). For a function \( \eta \) and an interior vertex \( z \) with incident vertices \( z_1, \ldots, z_L \), define the \textit{discrete Laplacian} by

\[
\Delta^{\varepsilon_n} \eta(z) := \Delta^{\varepsilon_n}_{v_1, \ldots, v_L} \eta(z) := \frac{1}{\varepsilon_n^L} \sum_{j=1}^{L} 2f_{\alpha_n(z_j)}(0)(\eta(z_j) - \eta(z)). \tag{42}
\]

Here \( v_{\alpha_j} = v([z, z_j]) = (z_j - z)/\varepsilon_n \) \( (j = 1, \ldots, L) \) and the notation \( \Delta^{\varepsilon_n}_{v_1, \ldots, v_L} \) emphasizes the dependence of the Laplacian on the directions \( v([z, z_j]) \) of the edges \( [z, z_j] \in E(G_n) \).

Let \( z_0 \in V_{int}(G_n) \) be an interior vertex and let \( v_{\mu_1}, \ldots, v_{\mu_L} \in V_n \) be the directions which correspond to the directions of the edges of \( G_n \) incident to \( z_0 \). Let \( z_1 \in F_n(\Omega_{\mathcal{D}_n}) \) be a vertex such that \( z_1 + v_{\mu_i} \in F_n(\Omega_{\mathcal{D}_n}) \) for all \( i = 1, \ldots, L \). Our next aim is to study \( \Delta^{\varepsilon_n}_{v_{\mu_1}, \ldots, v_{\mu_L}} t_n(z_1) \). For this purpose we assume that we have translated to \( z_1 \) the facets of \( \Omega_{\mathcal{D}_n} \) incident to \( z_0 \), that is we consider the (very small) monotone surface consisting of the two-dimensional facets incident to \( z_1 \) which contain a diagonal \( \{ z_1 + \lambda v_{\mu_i} : \lambda \in [0,1] \} \) for \( i = 1, \ldots, L \). Using the extension of the comparison function \( w_\varepsilon \), the closed chain of these two-dimensional facets incident to \( z_1 \) is mapped to a closed chain of kites. Thus we have

\[
\sum_{i=1}^{L} 2f_{\alpha_n}(t_n(z_1 + v_{\mu_i}) - t_n(z_1)) \in 2\pi \mathbb{N},
\]

where \( \alpha_{\mu_i} \) denotes the labelling of the two-dimensional facet containing \( z_1 \) and \( z_1 + v_{\mu_i} \). By assumption \( \sum_{i=1}^{L} 2f_{\alpha_n}(0) = 2\pi \) and we know that \( t_n(z_1 + v_{\mu_i}) - t_n(z_1) = O(\varepsilon_n) \) by Lemma 5.5. Since the intersection angles and thus the maximum number of neighbors are uniformly bounded, we deduce that

\[
\left( \sum_{i=1}^{L} 2f_{\alpha_n}(t_n(z_1 + v_{\mu_i}) - t_n(z_1)) \right) - 2\pi = 0
\]

if \( \varepsilon_n \) is small enough. Using a Taylor expansion about 0, we obtain

\[
\Delta^{\varepsilon_n}_{v_{\mu_1}, \ldots, v_{\mu_L}} t_n(z_1) = \frac{2}{\varepsilon_n^2} \left( -\sum_{i=1}^{L} \sum_{m=3}^{\infty} \frac{f_{\alpha_n}^{(m)}(0)}{m!} (t_n(z_1 + v_{\mu_i}) - t_n(z))^m \right)
\]

\[
= \varepsilon_n \left( -2 \sum_{i=1}^{L} \sum_{m=3}^{\infty} \frac{f_{\alpha_n}^{(m)}(0)}{m!} |v_{\mu_i}| m (\partial_{v_{\mu_i}} t_n(z_1))^m \varepsilon_n^{m-3} \right)
\]

\[
= \varepsilon_n F_{v_{\mu_1}, \ldots, v_{\mu_L}} (\varepsilon_n, \partial_{v_{\mu_1}} t_n, \ldots, \partial_{v_{\mu_L}} t_n; z_1), \tag{43}
\]
Note that $F_{x_1,\ldots,x_{n_k}}$ is a $C^\infty$-function in the variables $\varepsilon_n, \partial_{x_{n_1}} t_n, \ldots, \partial_{x_{n_k}} t_n$. This fact will be important for the proof of Lemma 5.7 below.

For further use we introduce the following notation. Let $K$ be a compact subset of $D$ and let $M > 0$. Denote by $\Omega_{\mathcal{D}_n}^{K,M}$ the part of $\Omega_{\mathcal{D}_n}$ with vertices of combinatorial distance bigger than $M$ to the boundary and whose corresponding vertices $z \in V(\mathcal{D}_n)$ lie in $K$. Let $J \subset \{1, \ldots, d\}$ contain at least two different indices. In order to consider partial derivatives within $K$ in the directions $\pm e_{j_1} \pm e_{j_2}$, where $j_1, j_2 \in J$ and $j_1 \neq j_2$, we attach a ball $U_J(\hat{z}, M)$ at each of these points:

$$U_J(K, M, \Omega_{\mathcal{D}_n}) = \bigcup_{\hat{z} \in V(\Omega_{\mathcal{D}_n}^{K,M})} U_J(\hat{z}, M). \quad (44)$$

Note that if $M \leq B_J(\hat{z})$, for example if $M \leq C_J(\mathcal{F}_n(\Omega_{\mathcal{D}_n}))d(\hat{z}, \partial \Omega_{\mathcal{D}_n}) - 1$ for all $\hat{z} \in V(\Omega_{\mathcal{D}_n}^{K,M})$, then $U_J(K, M, \Omega_{\mathcal{D}_n}) \subset \mathcal{F}_n(\Omega_{\mathcal{D}_n})$.

Furthermore we define $K + d_0$ to be the compact $d_0$-neighborhood of the compact set $K \subset \mathcal{C}$, that is

$$K + d_0 = \{ z \in \mathcal{C} : \varepsilon(K, z) \leq d_0 \},$$

where $\varepsilon$ denotes the Euclidean distance between a point and a compact set or between two compact subsets of $\mathcal{C}$.

The following lemma is important for our argumentation, as it gives an estimation of a partial derivative using estimations of the function and its Laplacian.

**Lemma 5.6** (Regularity Lemma). Let $\mathcal{D}$ be a quasicrystallic rhombic embedding with associated graph $G$ and labelling $\alpha$. Let $W \subset V(G)$ and let $u : W \rightarrow \mathbb{R}$ be any function. Let $M(u) = \max_{v \in W_{int}} |\Delta u(v)/(4F^*(v))|$, where

$$F^*(v) = \frac{1}{4} \sum_{[z,v] \in E(G)} c([z,v])|z - v|^2 = \frac{1}{2} \sum_{[z,v] \in E(G)} \sin \alpha([z,v])$$

is the area of the face of the dual graph $G^*$ corresponding to the vertex $v \in V_{int}(G)$. There are constants $C_5, C_6 > 0$, independent of $W$ and $u$, such that

$$|u(x_0) - u(x_1)| \rho \leq C_5 \|u\|_W + \rho^2 C_6 M(u) \quad (45)$$

for all vertices $x_1 \in W$ incident to $x_0$, where $\rho$ is the Euclidean distance of $x_0$ to the boundary $\partial \mathcal{D}$.

A proof is given in the appendix, see Lemma A.8. As a direct application we get

**Lemma 5.7.** Let $K \subset D$ be a compact set, $n_0 \in \mathbb{N}$ and $0 < d_0 < \varepsilon(K, \partial \mathcal{D}_n)$ for all $n \geq n_0$.

Let $k \in \mathbb{N}_0$ and let $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k} \in \bigcap_{n \geq n_0} V_n$ be $k$ not necessarily different directions. Let $J \subset \{1, \ldots, d\}$ be a minimal subset of indices such that $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}\} \subset \text{span}\{e_j : j \in J\}$. Let $B_0, C_0 \geq 0$ be some constants. Assume that all discrete partial derivatives using at most $k$ of the directions $\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_k}$ exist on $U_0 = U_J(K + d_0, B_0, \Omega_{\mathcal{D}_n})$ and are bounded on $U_0$ by $C_0 \varepsilon_n$ for all $n \geq n_0$.

Let $n \geq n_0$ and let $\Omega_{\mathcal{D}_n}$ be a two-dimensional monotone combinatorial surface. Let $\hat{\mathcal{D}}_n$ be the corresponding rhombic embedding with edge lengths $\varepsilon_n$ and such that

$$\{ \hat{z} \in V(\Omega_{\mathcal{D}_n}) : z \in V(\hat{\mathcal{D}}_n) \cap (K + d_0) \} \subset U_0.$$
Let \( z_0 \in V_n(\tilde{\mathcal{G}}_n) \cap (K + d_0/2) \) be a white vertex such that \( \phi(z, \partial \tilde{\mathcal{G}}_n) \geq d_0/2 \). Let \( v_{ik+1} \in V \) be a direction contained in \( \Omega_{\tilde{\mathcal{G}}_n} \) at \( z_0 \). Then there is a constant \( C_1 \), depending on \( K, D, g, \) and on the constants \( d_0, B_0, C_0, \kappa, C \), but not on \( z_0, v_{ik+1}, \) and \( n \), such that

\[
|\partial_{v_{ik+1}} \partial_{v_{ik}} \cdots \partial_{v_{i1}} (t_n - h_n)(z_0)| \leq C_1 \varepsilon_n.
\]

Proof. The proof is an application of the Regularity Lemma \[5.6\] for the function \( u = \partial_{v_{ik}} \cdots \partial_{v_{i1}} (t_n - h_n) \) and the part of the rhombic embedding \( \tilde{\mathcal{G}}_n \) contained in \( K + d_0 \). Note that the edge lengths of \( \tilde{\mathcal{G}}_n \) are \( \varepsilon_n \) and we suppose that \( \phi(z, \partial \tilde{\mathcal{G}}_n) \geq d_0/2 \).

By assumption we have \( \|u\|_{V(\tilde{\mathcal{G}}_n) \cap (K + d_0)} \leq C_0 \varepsilon_n \). As \( h_n \) is a \( C^\infty \)-function, this implies that \( \partial_{v_{ik}} \partial_{v_{ik}} \cdots \partial_{v_{i1}} t_n = \mathcal{O}(1) \) on \( U_0 \) for all possible directions \( v_i \in V \) whenever this partial derivative is defined in \( F_n(\Omega_{\tilde{\mathcal{G}}_n}) \).

Let \( z_1 \in V_{\text{int}}(\tilde{\mathcal{G}}_n) \cap (K + d_0) \) be an interior white vertex of \( \tilde{\mathcal{G}}_n \) and let \( v_{\mu_1}, \ldots, v_{\mu_L} \in V_n \) be the directions corresponding to the directions of the edges of \( \mathcal{G}_n \) incident to \( z_1 \). Note that

\[
\Delta_{v_{\mu_1} \ldots v_{\mu_L}} u = \Delta_{v_{\mu_1} \ldots v_{\mu_L}} \partial_{v_{\mu_1}} \cdots \partial_{v_{i1}} (t_n - h_n) = \partial_{v_{ik}} \cdots \partial_{v_{i1}} \Delta_{v_{\mu_1} \ldots v_{\mu_L}} (t_n - h_n).
\]

From the above consideration in \[43\] we use that \( \Delta_{v_{\mu_1} \ldots v_{\mu_L}} t_n = \varepsilon_n F_{v_{\mu_1} \ldots v_{\mu_L}} \), where \( F_{v_{\mu_1} \ldots v_{\mu_L}} \) is a \( C^\infty \)-function in the variables \( \varepsilon_n, \partial_{v_{\mu_1}} t_n, \ldots, \partial_{v_{\mu_L}} t_n \). From our assumptions we know that all partial derivatives \( \partial_{v_{\mu_1}} \partial_{v_{\mu_2}} \cdots \partial_{v_{i1}} t_n(z_1) \) and those containing less than \( k+1 \) of the derivatives \( \partial_{v_{\mu_1}}, \partial_{v_{\mu_2}}, \ldots, \partial_{v_{i1}} \) are defined and uniformly bounded by a constant independent of \( z_1 \) and \( \varepsilon_n \). Thus

\[
|\partial_{v_{ik}} \cdots \partial_{v_{i1}} (t_n - h_n)(z_1)| \leq C_{v_{\mu_1} \ldots v_{\mu_L}} \varepsilon_n,
\]

where the constant \( C_{v_{\mu_1} \ldots v_{\mu_L}} \) does not depend on \( z_1 \) and \( \varepsilon_n \). As \( h_n \) is a harmonic \( C^\infty \)-function, we obtain by similar reasonings as in the proof of Lemma \[33\] that

\[
\partial_{v_{ik}} \cdots \partial_{v_{i1}} \Delta_{v_{\mu_1} \ldots v_{\mu_L}} h_n = \mathcal{O}(\varepsilon_n).
\]

As \( V_n \) is a finite set, we deduce that \( \Delta^\varepsilon u \) is uniformly bounded on \( V_{\text{int}}(\tilde{\mathcal{G}}_n) \cap (K + d_0) \) by \( C_2 \varepsilon_n \) for some constant \( C_2 \) independent of \( \varepsilon_n \). Now the Regularity Lemma \[5.6\] gives the claim.

The following corollary of the preceding lemma constitutes the crucial step in our proof of Theorem \[5.2\]

**Lemma 5.8.** Let \( K \subset D \) be a compact set and let \( 0 < d_1 < \phi(K, \partial D) \). Let \( n_0 \in \mathbb{N} \) be such that \( K + d_1 \) is covered by the rhombi of \( \mathcal{G}_n \) for all \( n \geq n_0 \). Then there is a constant \( C_1 = C_1(K, C_0(J, \mathcal{F}_n(\Omega_{\tilde{\mathcal{G}}_n}))) > 0 \) such that for all \( z \in K + d_1 \)

\[
C_1 \varepsilon_n^{-1} \leq C_0(J, \mathcal{F}_n(\Omega_{\tilde{\mathcal{G}}_n})) - d(z, \partial \mathcal{G}_n) - 1.
\]

Furthermore, let \( k \in \mathbb{N}_0 \) and let \( v_{i_1}, \ldots, v_{i_k} \in V \) be \( k \) (not necessarily different) directions such that \( v_{i_l} \in \text{span}\{e_j : j \in J_0\} \) for all \( l = 1, \ldots, k \). Then there are constants \( n_0 \leq n_1(k, K) \in \mathbb{N} \) and \( C(k, K) > 0 \) which may depend on \( k, K, d_1, D, g, \kappa, C_0(J, \mathcal{F}_n(\Omega_{\tilde{\mathcal{G}}_n})), \) but not on \( \varepsilon_n \), such that for all \( n \geq n_1(k, K) \) we have

\[
\|\partial_{v_{i_k}} \cdots \partial_{v_{i_1}} (t_n - h_n)\|_{U_k} \leq C(k, K) \varepsilon_n,
\]

where \( U_k = U_{J,0}(K + 2^{-k} d_1, 2^{-k} C_1 \varepsilon_n^{-1}, \Omega_{\tilde{\mathcal{G}}_n}) \).
Proof. The existence of the constant $C_1$ follows from the fact that $d(z, \partial \mathcal{D}_n)/\varepsilon_n$ is bounded from below for $z \in (K + d_1)$ since the distance $\varepsilon(K + d_1, \partial \mathcal{D}_n) > 0$ is positive and the angles of the rhombi are uniformly bounded.

The proof of estimation (46) uses induction on the number of partial derivatives $k$. For $k = 0$ the claim has been shown in Lemma 5.5.

Let $k \in \mathbb{N}_0$ and assume that the claim is true for all $\nu \leq k$. Let $v_{ik+1} \in \mathbf{V}$ be a direction with $v_{ik+1} \in \{\pm e_j, \pm e_{j2}\} \subset \text{span}\{e_j : j \in J_0\}$. Using the induction hypothesis, we can apply Lemma 5.7 for $U_0 = U_k$, $d_0 = 2^{-k}d_1$, $\Omega_{\hat{\mathcal{D}}_n} = U\{j_1, j_2\}(K + 2^{-k}d_1, 2^{-k}C_1\varepsilon_n^{-1}, \Omega_{\partial n}) \subset U_k$ and the corresponding rhombic embedding $\hat{\mathcal{D}}_n$ obtained by projection, and $z_0 \in \hat{\mathcal{D}}_n \cap (K + 2^{-k-1}d_1)$. This completes the induction step and the proof.

Proof of Theorem 5.2 Identify $C$ with $\mathbb{R}^2$ in the standard way and fix two orthogonal unit vectors $e_1, e_2$. Define discrete partial derivatives $\partial_{e_1}, \partial_{e_2}$ in these directions using the discrete partial derivatives in two orthogonal directions $v_{i_1} = a_{j_1}^{(n)} + a_{j_2}^{(n)}$ and $v_{i_2} = a_{j_1}^{(n)} - a_{j_2}^{(n)}$ for $j_1, j_2 \in J_0$. This definition depends on the choice of $a_{j_1}^{(n)}, a_{j_2}^{(n)}$, which may be different for each $n$, but this does not affect the proof. As the possible intersection angles are bounded and as $h_n$ is a $C^\infty$-function, we deduce that

$$\|\partial_{e_{j_1}} \cdots \partial_{e_{j_1}} h_n - \partial_{j_1} \cdots \partial_{j_1} h_n\|_K \leq C_1(k, K)\varepsilon_n$$

on every compact set $K$ for $j_k, \ldots, j_1 \in \{1, 2\}$. Here $\partial_1, \partial_2$ denote the standard partial derivatives associated to $e_1, e_2$ for smooth functions and $C_1(k, K)$ is a constant which depends only on $K$, $k$, and $g$. Lemma 5.3 implies that

$$\|\partial_{e_{j_1}} \cdots \partial_{e_{j_1}} t_n - \partial_{e_{j_1}} \cdots \partial_{e_{j_1}} h_n\|_{U_{2}(K + 2^{-k}d_1, 2^{-k}C_1\varepsilon_n^{-1}, \Omega_{\partial n})} \leq C_2(k, K)\varepsilon_n$$

if $n$ is big enough. Using a version of Lemma 3.9 with error of order $O(\varepsilon_n)$, we deduce that $t_n + i(\varphi_n - \phi_n)$ converges to $\log g^t$ in $C^\infty(D)$. Now the convergence of $q_n$ and $g_n$ follows by similar arguments as in the proof of Theorem 5.1.

A Appendix: Properties of discrete Green’s function and regularity of solutions of discrete elliptic equations

In order to prove the Regularity Lemma 5.6 (see Lemma A.8), we present some results in discrete potential theory on quasicrystallic rhombic embeddings which are derived from a suitable asymptotic expansion of a discrete Green’s function.

Throughout this appendix, we assume that $\mathcal{G}$ is a (possibly infinite) simply connected quasicrystallic rhombic embedding of a b-quad-graph with edge directions $A = \{\pm a_1, \ldots, \pm a_d\}$. Also, the edge lengths of $\mathcal{G}$ are supposed to be normalized to one. Let $G$ be the associated graph built from white vertices.

Fix some interior vertex $x_0 \in V_{\text{int}}(G)$. Following Kenyon [Ken02] and Bobenko, Mercat, and Suris [BMS05], we define the discrete Green’s function $G(x_0, \cdot) : V(G) \to \mathbb{R}$ by

$$G(x_0, x) = -\frac{1}{4\pi^2} \int_\Gamma \frac{\log(\lambda)}{2\lambda} e(x; \lambda)d\lambda$$  (47)
for all $x \in V(G)$. Here $e(x; z) = \prod_{k=1}^{d} \left( \frac{z+a_k}{z-a_k} \right)^{n_k}$ is the discrete exponential function, where $n = (n_1, \ldots, n_d) = \bar{x} - \bar{x}_0 \in \mathbb{Z}^d$ and $\bar{x}, \bar{x}_0 \in V(\mathcal{D})$ correspond to $x, x_0 \in V(\mathcal{D})$ respectively. The integration path $\Gamma$ is a collection of $2d$ small loops, each one running counterclockwise around one of the points $\pm a_k$ for $k = 1, \ldots, d$. The branch of $\log(\lambda)$ depends on $x$ and is chosen as follows.

Without loss of generality, we assume that the circular order of the points of $\mathcal{A}$ on the positively oriented unit circle $\mathbb{S}^1$ is $a_1, \ldots, a_d, -a_1, \ldots, -a_d$. Set $a_{k+d} = -a_k$ for $k = 1, \ldots, d$ and define $a_m$ for all $m \in \mathbb{Z}$ by $2d$-periodicity. To each $a_m = e^{i \theta_m} \in \mathbb{S}^1$ we assign a certain value of the argument $\theta_m \in \mathbb{R}$: choose $\theta_1$ arbitrarily and then use the rule

$$\theta_{m+1} - \theta_m \in (0, \pi) \quad \text{for all } m \in \mathbb{Z}.$$ 

Clearly we then have $\theta_{m+d} = \theta_m + \pi$. The points $a_m$ supplied with the arguments $\theta_m$ can be considered as belonging to the Riemann surface of the logarithmic function (i.e. a branched covering of the complex $\lambda$-plane). Since $\Omega_\mathcal{D}$ is a monotone surface, there is an $m \in \{1, \ldots, 2d\}$ and a directed path from $x_0$ to $x$ in $\mathcal{D}$ such that the directed edges of this path are contained in $\{a_m, \ldots, a_{m+d-1}\}$, see [BMS05, Lemma 18]. Now, the branch of $\log(\lambda)$ in (17) is chosen such that

$$\log(a_l) = [i \theta_m, i \theta_{m+d-1}], \quad l = m, \ldots, m + d - 1.$$

Remember Definition 3.3 of the discrete Laplacian and the representation of its weights $c(|z_1, z_2|) = 2 f'_a(|z_1, z_2|)(0)$ in (1). Then there holds

Lemma A.1 ([Ken02 Theorems 7.1 and 7.3]). The discrete Green’s function $G(x_0, \cdot)$ defined in equation (17) has the following properties.

(i) $\Delta G(x_0, v) = -\delta_{x_0}(v)$, where the Laplacian is taken with respect to the second variable.

(ii) $G(x_0, x_0) = 0$.

(iii) $G(x_0, v) = O(\log(|v - x_0|))$.

Note that $G(x_0, \cdot)$ may also be defined by these three conditions.

A.1 Asymptotics for discrete Green’s function

Kenyon derived in [Ken02] an asymptotic development for the discrete Green’s function using standard methods of complex analysis. His result can be slightly strengthened to an error of order $O(1/|v - x_0|^2)$. Note, that there is the summand $-\log 2/(2\pi)$ missing in Kenyon’s formula (but not in his proof).

Theorem A.2 (cf. [Ken02 Theorem 7.3]). For $v \in V(G)$ there holds

$$G(x_0, v) = -\frac{1}{2\pi} \log(2|v - x_0|) - \frac{\gamma_{\text{Euler}}}{2\pi} + O \left( \frac{1}{|v - x_0|^2} \right). \quad (48)$$

Here $\gamma_{\text{Euler}}$ denotes the Euler $\gamma$ constant.
Proof. Consider a directed path $x_0 = w_0, \ldots, w_k = v$ in $\mathcal{D}$ from $x_0$ to $v$ such that the directed edges of this path are contained in $\{a_m, \ldots, a_{m+d-1}\}$ for some $1 \leq m \leq 2d$ as above. Note that $k$ is even since $x_0$ and $v$ are both white vertices of $\mathcal{D}$. The integration path $\Gamma$ in (17) can be deformed into a connected contour lying on a single leaf of the Riemann surface of the logarithm, in particular to a simple closed curve $\Gamma_1$ which surrounds the set $\{a_m, \ldots, a_{m+d-1}\}$ in a counterclockwise sense and has the origin and a ray $\mathcal{R}$ exterior. We also assume that $\Gamma$ is contained in the sector $\{z = re^{i\varphi} : r > 0, \varphi \in [\theta + \frac{\pi}{2} + \eta, \theta + \frac{3\pi}{2} - \eta]\}$ for some $\eta > 0$ independent of $v$, $x_0$, and $m$. This is possible due to the fact that $\theta_{m+d-1} - \theta_m < \pi - \delta$ for some $\delta > 0$ independent of $v$, $x_0$, and $m$.

Let $N = |v - x_0|$. Take $0 < \varrho_1 \ll 1/N^3$ and $\varrho_2 \gg N^3$, but not exponentially smaller than $1/N$ or bigger than $N$ respectively. The curve $\Gamma_1$ is again homotopic to a curve $\Gamma_2$ which runs counterclockwise around the circle of radius $\varrho_2$ about the origin from the angle $\theta$ to $\theta + 2\pi$, then along the ray $\mathcal{R}$ from $\varrho_2 e^{i\beta}$ to $\varrho_1 e^{i\beta}$, then clockwise around the circle of radius $\varrho_1$ about the origin from the angle $\theta + 2\pi$ to $\theta$, and finally back along the ray $\mathcal{R}$ from $\varrho_1 e^{i\beta}$ to $\varrho_2 e^{i\beta}$. Without loss of generality, we assume that $\mathcal{R}$ is the negative real axis.

Kenyon showed in [Ken02] that the integrals along the circles of radius $\varrho_1$ and $\varrho_2$ give

$$(-1)^k \frac{\log \varrho_1}{4\pi} (1 + O(N\varrho_1)) - \frac{\log \varrho_2}{4\pi} (1 + O(N/\varrho_2)).$$

The difference between the value of $\log z$ above and below the negative real axis is $2\pi i$ Thus the integrals along the negative real axis can be combined into

$$-\frac{1}{4\pi} \int_{-\varrho_2}^{-\varrho_1} \frac{1}{z} \prod_{j=0}^{k-1} \frac{z + b_j}{z - b_j} \, dz,$$

where $b_j = w_{j+1} - w_j \in \mathcal{A}$ is the directed edge from $w_j$ to $w_{j+1}$ and $k = O(N)$ is the number of edges of the path. This integral can be split into three parts: from $-\varrho_2$ to $-\sqrt{N}$, from $-\sqrt{N}$ to $-1/\sqrt{N}$, and from $-1/\sqrt{N}$ to $-\varrho_1$.

The integral is negligible for the intermediate range because

$$\left| \frac{t + e^{i\beta}}{t - e^{i\beta}} \right| \leq e^{2t \cos \beta/(t-1)^2}$$

for negative $t < 0$ and due to our assumptions.

For small $|t|$ we have

$$\prod_{j=0}^{k-1} \frac{t + b_j}{t - b_j} = \prod_{j=0}^{k-1} \frac{tb_j + 1}{tb_j - 1} = (-1)^k e^{2\sum_{j=1}^{k-1} b_j t (1 + O(kt^3))},$$

using the Neumann series and a Taylor expansion. Thus the integral near the origin is

$$-\frac{(-1)^k}{4\pi} \left( \int_{-\sqrt{N}}^{-\varrho_1} \frac{e^{2(\vartheta - \bar{\vartheta}_0)t}}{t} \, dt + \int_{-1/\sqrt{N}}^{-\varrho_1} O(kt^3) \frac{e^{2(\vartheta - \bar{\vartheta}_0)t}}{t} \, dt \right).$$
Applying similar reasonings and estimations as in Kenyon’s proof, we obtain

\[-\frac{(-1)^k}{4\pi} \left( \log(2\varrho_1(\bar{v} - \bar{x}_0)) + \gamma_{\text{Euler}} \right) + O\left(\frac{1}{N^2}\right).\]

Here \(\gamma_{\text{Euler}}\) denotes the Euler \(\gamma\) constant.

For large \(|t|\) the estimations are very similar. Since

\[
\prod_{j=0}^{k-1} \frac{t + b_j}{t - b_j} = \prod_{j=0}^{k-1} \frac{1 + \frac{b_j t^{-1} - 1}{1 - b_j t^{-1}}}{1} = e^{2\sum_{j=0}^{k-1} b_j t^{-1}} (1 + O(kt^{-3})),
\]

we get

\[
-\frac{(-1)^k}{4\pi} \left( \int_{-\varrho_2}^{-\sqrt{N}} \frac{e^{2(v-x_0)t^{-1}}}{t} dt + \int_{-\sqrt{N}}^{-\varrho_2} O(kt^{-3}) \frac{e^{2(v-x_0)t^{-1}}}{t} dt \right) = -\frac{1}{4\pi} \left( -\log \left( \frac{\varrho_2}{2(v-x_0)} \right) + \gamma_{\text{Euler}} \right) + O\left(\frac{1}{N^2}\right).
\]

Since \(k\) is even, the sum of all the above integral parts is therefore given by the right hand side of (18).

For a bounded domain we also define a discrete Green’s function with vanishing boundary values. Let \(W \subset V_{\text{int}}(G)\) be a finite subset of vertices. Denote by \(W_\partial \subset W\) the set of boundary vertices which are incident to at least one vertex in \(V(G) \setminus W\). Set \(W_{\text{int}} = W \setminus W_\partial\) the interior vertices of \(W\). Let \(x_0 \in W_{\text{int}}\) be an interior vertex. The discrete Green’s function \(G_W(x_0, \cdot)\) is uniquely defined by the following properties.

(i) \(\Delta G_W(x_0, v) = -\delta_{x_0}(v)\) for all \(v \in W_{\text{int}}\), where the Laplacian is taken with respect to the second variable.

(ii) \(G_W(x_0, v) = 0\) for all \(v \in W_\partial\).

In the following, we choose \(W\) to be a special disk-like set. Let \(x_0 \in V(G)\) be a vertex and let \(\rho > 2\). Denote the closed disk with center \(x_0\) and radius \(\rho\) by \(B_\rho(x_0) \subset \mathbb{C}\). Suppose that this disk is entirely covered by the rhombic embedding \(\mathcal{F}\). Denote by \(V(x_0, \rho) \subset V(G)\) the set of white vertices lying within \(B_\rho(x_0)\). For \(x_1 \in V_{\text{int}}(x_0, \rho)\) we denote

\[G_{x_0, \rho}(x_1, \cdot) = G_V(x_0, \cdot)(x_1, \cdot).\]

The asymptotics of the discrete Green’s function \(G\) from Theorem 3.2 can be used to derive the following estimations for \(G_{x_0, \rho}\).

**Proposition A.3.** There is a constant \(C_1\), independent of \(\rho\) and \(x_0\), such that

\[|G_{x_0, \rho}(x_0, v)| \leq C_1 / \rho\]

for all vertices \(v \in V_{\text{int}}(x_0, \rho)\) which are incident to a boundary vertex.

Furthermore, there is a constant \(C_2\), independent of \(\rho\) and \(x_0\), such that for all interior vertices \(x_1 \in V_{\text{int}}(x_0, \rho)\) incident to \(x_0\) and all \(v \in V(x_0, \rho)\) there holds

\[|G_{x_0, \rho}(x_0, v) - G_{x_0, \rho}(x_1, v)| \leq C_2/(|v - x_0| + 1).\]
Proof. Consider the function $h_\rho(x_0, \cdot) : V(x_0, \rho) \to \mathbb{R}$ defined by

$$h_\rho(x_0, v) = \mathcal{G}_{x_0, \rho}(x_0, v) - \mathcal{G}(x_0, v) - \frac{1}{2\pi}(\log(2\rho) + \gamma_{\text{Euler}}).$$

Then $h_\rho(x_0, \cdot)$ is harmonic on $V_{\text{int}}(x_0, \rho)$. For boundary vertices $v \in V_{\partial}(x_0, \rho)$ Theorem A.2 implies that $h_\rho(x_0, v) = \mathcal{O}(1/\rho)$. The Maximum Principle 3.6 yields $|h_\rho(x_0, v)| \leq C/\rho$ for all $v \in V(x_0, \rho)$ and some constant $C$ independent of $\rho$ and $v$. This shows the first estimation.

To prove the second claim, we also consider the harmonic function

$$h_\rho(x_1, v) = \mathcal{G}_{x_0, \rho}(x_1, v) - \mathcal{G}(x_1, v) - \frac{1}{2\pi}(\log(2\rho) + \gamma_{\text{Euler}})$$

for a fixed interior vertex $x_1 \in V_{\text{int}}(x_0, \rho)$ incident to $x_0$. By similar reasonings as for $h_\rho(x_0, \cdot)$, we deduce that $|h_\rho(x_1, v)| \leq \tilde{C}/\rho$ for all $v \in V(x_0, \rho)$ and some constant $\tilde{C}$ independent of $\rho$ and $v$. Theorem A.2 implies the desired estimation. \qed

### A.2 Regularity of discrete solutions of elliptic equations

In the following, we generalize and adapt some results of discrete potential theory for discrete harmonic functions obtained by Duffin in [Duf53, see in particular Lemma 1 and Theorem 3–5]. The proofs are very similar or use ideas of the corresponding proofs in [Duf53]. Our aim is to obtain the Regularity Lemma A.8.

#### Lemma A.4 (Green’s Identity)

Let $W \subset V(G)$ be a finite subset of vertices. Let $u, v : W \to \mathbb{R}$ be two functions. Then

$$\sum_{x \in W_{\text{int}}} (v(x) \Delta u(x) - u(x) \Delta v(x)) = \sum_{[p, q] \in E_0(W)} c([p, q]) (v(p)u(q) - u(p)v(q)),$$

where $E_0(W) = \{[p, q] \in E(W) : p \in W_{\text{int}}, q \in W_{\partial}\}$. (49)

#### Corollary A.5 (Representation of harmonic functions)

Let $u$ be a real valued harmonic function defined on $V(x_0, \rho)$. Then

$$u(x_0) = \sum_{q \in V_{\partial}(x_0, \rho)} c(q)u(q),$$

where

$$c(q) = \sum_{p \in V_{\text{int}}(x_0, \rho) \text{ and } [p, q] \in E(G)} c([p, q])\mathcal{G}_{x_0, \rho}(x_0, p) = \mathcal{O}(1/\rho).$$

The estimation in (50) is a consequence of Proposition A.3 and of the boundedness of the weights $c(e)$.

#### Theorem A.6

Let $u : V(x_0, \rho) \to \mathbb{R}$ be a non-negative harmonic function. There is a constant $C_3$ independent of $\rho$ and $u$ such that

$$\left|\frac{u(x_0)}{\pi \rho} - \frac{1}{\pi \rho} \sum_{v \in V_{\text{int}}(x_0, \rho)} F^*(v)u(v)\right| \leq \frac{C_3 u(x_0)}{\rho},$$

where $F^*(v) = \frac{1}{4} \sum_{[z, v] \in E(G)} c([z, v])|z - v|^2$ is the area of the face of the dual graph $G^*$ corresponding to the vertex $v \in V_{\text{int}}(G)$. (51)
Proof. Consider the function

$$p(z) = G(x_0, z) + \frac{1}{2\pi} \log(2\rho) + \gamma_{\text{Euler}} + \frac{|z - x_0|^2 - \rho^2}{4\pi \rho^2}.$$ 

Let $z \in V_{\text{int}}(G)$ be an interior vertex. Consider the chain of faces $f_1, \ldots, f_m$ of $\mathcal{D}$ which are incident to $z$ in counterclockwise order. The enumeration of the faces $f_j$ and of the black vertices $v_1, \ldots, v_m$ and the white vertices $z_1, \ldots, z_m$ incident to these faces can be chosen such that $f_j$ is incident to $v_{j-1}, v_j,$ and $z_j$ for $j = 1, \ldots, m,$ where $v_0 = v_m.$ Furthermore, using this enumeration we have

$$\frac{z_j - z}{|z_j - z|} = \frac{v_j - v_{j-1}}{|v_j - v_{j-1}|},$$

see Figure 4 with $z_\pm = z, z_+ = z_j, v_\pm = v_{j-1}, v_+ = v_j.$ As $|z_j - x_0|^2 - |z - x_0|^2 = -2\text{Re}((z - x_0)(z_j - z)) + |z_j - z|^2$ we deduce by very similar calculations as in the proof of Lemma 11.3 that

$$\Delta p(z) = \Delta G(x_0, v) + \frac{1}{4\pi \rho^2} \sum_{j=1}^m c([z, z_j]) \left( |z_j - x_0|^2 - |z - x_0|^2 \right)$$

$$= -\delta_{x_0}(z) + F^*(z)/(\pi \rho^2).$$

Let $v$ be incident to a vertex of $V_\partial(x_0, \rho).$ Theorem A.2 implies that

$$p(v) = -\frac{1}{2\pi} \log \frac{|v - x_0|}{\rho} + \frac{|v - x_0|^2 - \rho^2}{4\pi \rho^2} + O\left(\frac{1}{|v - x_0|^2}\right) = O(1/\rho^2).$$

Thus there is a constant $B_1,$ independent of $\rho$ and $v,$ such that $p_1(v) := p(v) + B_1/\rho^2 \geq 0$ and $|p_1(v)| \leq 2B_1/\rho^2$ for all vertices $v \in V_{\text{int}}(x_0, \rho)$ incident to a vertex of $V_\partial(x_0, \rho).$ Applying Green’s Identity A.4 to $p_1$ and the non-negative harmonic function $u,$ we obtain

$$u(x_0) - \frac{1}{\pi \rho^2} \sum_{v \in V_{\text{int}}(x_0, \rho)} F^*(v)u(v) = \sum_{x \in V_{\text{int}}(x_0, \rho)} (p_1(x)\Delta u(x) - u(x)\Delta p_1(x))$$

$$= \sum_{[z, q] \in E_\rho} c([z, q]) (p_1(z)u(q) - u(z)p_1(q)), \geq 0$$

$$\leq \frac{2B_1}{\rho^2} \frac{4\pi}{\rho} \sum_{q \in V_\partial(x_0, \rho)} u(q) \leq \frac{8\pi B_1 B_2}{\rho^2} u(x_0),$$

Here $E_\rho = \{[x, y] \in E(G) : x \in V_{\text{int}}(x_0, \rho), y \in V_\partial(x_0, \rho)\}$ and we have used the estimation

$$\sum_{[x, y] \in E(G)} c([x, y]) \leq \sum_{[x, y] \in E(G)} c([x, y])|x - y|^2 = 4F^*(x) < 4\pi$$

for all fixed vertices $x \in V_{\text{int}}(G).$ Furthermore $\sum_{y \in V_\partial(x_0, \rho)} u(y) \leq B_2 \rho u(x_0)$ for some constant $B_2 > 0$ as a consequence of Corollary A.5.

For the reverse inequality, note that there is also a constant $B_3$ independent of $\rho$ and $v$ such that $p_1(v) := p(v) - B_3/\rho^2 \leq 0$ and $|p_1(v)| \leq 2B_3/\rho^2$ for all vertices $v \in V_{\text{int}}(x_0, \rho)$ incident to a vertex in $V_\partial(x_0, \rho).$ Combining both estimation proves the claim. \[\square\]
Theorem [A.6] can be interpreted as an analog to the Theorem of Gauss in potential theory. Furthermore, we can deduce a discrete version of Hölder’s Inequality for non-negative harmonic function.

**Theorem A.7** (Hölder’s Inequality). Let \( u : V(x_0, \rho) \to \mathbb{R} \) be a non-negative harmonic function. There is a constant \( C_4 \), independent of \( \rho \) and \( u \), such that

\[
|u(x_0) - u(x_1)| \leq C_4 u(x_0)/\rho
\]  

(52)

for all vertices \( x_1 \in V(x_0, \rho) \) incident to \( x_0 \).

As a corollary of Hölder’s Inequality and of Proposition [A.3] we obtain the following result on the regularity of discrete solutions to elliptic equations.

**Lemma A.8** (Regularity Lemma). Let \( W \subset V(G) \) and let \( u : W \to \mathbb{R} \) be any function. Set \( M(u) = \max_{x \in W_{\text{int}}} |\Delta u(v)/(4F^*(v))| \), where \( F^*(v) \) is the area of the face dual to \( v \) as in Theorem [A.6]. Define \( \|\eta\|_W := \max\{|\eta(z)| : z \in W\} \). There are constants \( C_5, C_6 > 0 \), independent of \( W \) and \( u \), such that

\[
|u(x_0) - u(x_1)|\rho \leq C_5 \|u\|_W + \rho^2 C_6 M(u)
\]

(53)

for all vertices \( x_1 \in W \) incident to \( x_0 \in W_{\text{int}} \), where \( \rho \) is the Euclidean distance of \( x_0 \) to the boundary \( W_{\partial} \).

**Proof.** Let \( x_1 \in W \) be a fixed vertex incident to \( x_0 \).

First we suppose that \( \rho \geq 4 \). Consider the auxiliary function \( f(z) = M(u)|z - x_0|^2 \). Since \( |x_1 - x_0| < 2 \), we obviously have

\[
|f(x_0) - f(x_1)| = M(u)|x_1 - x_0|^2 \leq 4M(u).
\]

Let \( h : V(x_0, \rho) \to \mathbb{R} \) be the unique harmonic function with boundary values \( h(v) = u(v) + f(v) \) for \( v \in V_{\partial}(x_0, \rho) \). Hölder’s Inequality (52) and the Maximum Principle 3.6 imply that

\[
|h(x_0) - h(x_1)|\rho \leq B_1 \|h\|_{V(x_0, \rho)} \leq B_1 (\|u\|_W + M(u)\rho^2)
\]

for some constant \( B_1 \) independent of \( h, \rho, x_0, x_1 \).

Next consider \( s = u + f - h \) on \( V(x_0, \rho) \). Then

\[
\begin{align*}
\Delta s &= \Delta u + 4F^* M(u) \geq 0 \quad \text{on} \ V_{\text{int}}(x_0, \rho), \\
\gamma(s) &= 0 \quad \text{for} \ v \in V_{\partial}(x_0, \rho).
\end{align*}
\]

The Maximum Principle 3.6 implies \( s \leq 0 \). Green’s Identity 3.3 gives

\[
s(x_0) + \sum_{v \in V_{\text{int}}(x_0, \rho)} \mathcal{G}_{x_0, \rho}(x_0, v) \Delta s(v)
\]

\[
= \sum_{v \in V_{\text{int}}(x_0, \rho)} (\mathcal{G}_{x_0, \rho}(x_0, v) \Delta s(v) - s(v) \Delta \mathcal{G}_{x_0, \rho}(x_0, v))
\]

\[
= \sum_{[p, q] \in E_{\rho}} c([p, q])(\mathcal{G}_{x_0, \rho}(x_0, p)s(q) - s(p)\mathcal{G}_{x_0, \rho}(x_0, q)) = 0,
\]
where \( E_\rho = \{ [p, q] \in E(G) : p \in V_{\text{int}}(x_0, \rho), q \in V_\partial(x_0, \rho) \} \). Analogously,

\[
s(x_1) + \sum_{v \in V_{\text{int}}(x_0, \rho)} \mathcal{G}_{x_0, \rho}(x_1, v) \Delta s(v) = 0.
\]

Using the estimation \( \Delta s(v) \leq 8F^*(v)M(u) \) we deduce that

\[
|s(x_0) - s(x_1)| \leq \sum_{v \in V_{\text{int}}(x_0, \rho)} |\mathcal{G}_{x_0, \rho}(x_0, v) - \mathcal{G}_{x_0, \rho}(x_1, v)| 8F^*(v)M(u).
\]

Now Proposition A.3 implies that

\[
|s(x_0) - s(x_1)| \leq 8B_2M(u) \sum_{v \in V_{\text{int}}(x_0, \rho)} \frac{F^*(v)}{|v - x_0| + 1} \leq 8B_2M(u)B_3\rho,
\]

where \( B_2 \) and \( B_3 \) are constants independent of \( s, \rho, x_0, x_1 \).

Combining the above estimations for \( f, h, \) and \( s \), we finally obtain

\[
|u(x_0) - u(x_1)| \rho \leq |s(x_0) - s(x_1)| - (f(x_0) - f(x_1)) + h(x_0) - h(x_1) | \rho \\
\leq B_1|u||W + \rho^2(4 + B_1 + 8B_2B_3)M(u). \]

This implies the claim for \( \rho \geq 4 \). For \( \rho < 4 \) inequality (53) can be deduced from

\[
-4F^*(x_0)M(u) \leq \Delta u(x_0) = \sum_{[x_0, v] \in E(G)} c([x_0, v])(u(v) - u(x_0)) \leq 4F^*(x_0)M(u).
\]

using \( F^*(x_0) \leq \pi, \sum_{[x_0, v] \in E(G)} c([x_0, v]) \leq 4\pi, \) and the uniform boundedness of the weights \( c(e) \).

\[
\]

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