Trigonometrically fitted fifth-order explicit two-derivative Runge-Kutta method with FSAL property

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Abstract. A new trigonometrically fitted two-derivative explicit Runge-Kutta (TFTDRK) method of order five with FSAL property for solving system of first-order ordinary differential equations (ODEs) with oscillatory solutions are derived. The new method is derived using the property of First Same As Last (FSAL). This method has the advantageous to merge totally first-order ordinary differential systems which their solutions are linear composition of the set of functions \( \{ e^{i \omega t} ; e^{(-i \omega t)} \} \), or equivalently \( \{ \sin(\omega t) ; \cos(\omega t) \} \) when \( \omega > 0 \) is the dominant frequency of the problem. We analyzed the stability of our method. The numerical results are presented to illustrate the competence of TFTDRK method compared with some well-known TFRK methods.

Keywords: Explicit methods ; First-order ODEs; FSAL technique; TDRK methods.

1. Introduction

In this paper, we are interested in the effective numerical methods for solving ordinary differential equations of first order of the form

\[
y' = f(x, y), \quad y(x_0) = y_0, \tag{1}
\]

where \( f: \mathbb{R}^d \to \mathbb{R}^d \), and \( y \in \mathbb{R}^d \) whose solutions show a pronounced oscillatory behaviour. This kind of problem takes place in varies of applied science like mechanics, electronics, molecular dynamics, engineering and so on. Based on the oscillatory attribute of the problem (1), researchers have suggested trigonometrical and exponential fitting numerical methods with frequency coefficients (see [1,2,3,4,5]). A good theoretical basis of these techniques was given by Gautschi [6] and Lyche [7]. You et al. [8] proposed the trigonometrically fitted Scheifele methods for the perturbed oscillators problems. Vanden Berghe et al. [9,10] proposed an exponentially fitted explicit RK method for solving ordinary differential systems of first-order whose solutions are written as linear combinations of the set of functions \( \{ \exp(i \omega t) ; \exp(-i \omega t) \} \), or equivalently \( \{ \sin(\omega t) ; \cos(\omega t) \} \) when \( \omega > 0 \) is the dominant frequency of the problem. According to this idea, Franco [11] derived exponentially fitted Runge-Kutta Nyström (RKN) methods as well as the embedded RKN(4,3) pair method to solve second-order ODEs. Simos et al. [12] developed an exponentially fitted Runge-Kutta method for solving the system of first-order initial value problems with oscillatory solutions. Chan and Tsai [13], constructed an explicit Two Derivative Runge-Kutta (TDRK) methods of order up to seven. An feature of the TDRK methods is that they can use fewer function evaluations to reach higher order methods compared with the RK methods. Zhang et al. [14] constructed a new trigonometrically fitted fifth-order explicit two-derivative Runge-Kutta method based on Simos approach [12] for solving the Schrödinger equation and related
oscillatory problems. Fang et al. [15], derived exponentially two derivative Runge-Kutta (EFTDRK) methods of order four for solving Schrödinger Equation. Recently, Fang et al. [16], constructed trigonometrically fitted two-derivative Runge-Kutta (TFDRK) methods of orders four and five with two-stage and three stage respectively for solving second order ODEs which possess oscillatory solutions. More recently, Yang et al. [17], developed exponentially two derivative Runge-Kutta (EFTDRK) methods for solving first order ODEs by using the second derivative inside the scheme as well as by updating the local error of the internal stages.

Motivated by the previous studies, we are going here to construct a new trigonometrically fitting two-derivative Runge-Kutta (TFTDRK5F) method of order five with FSAL property for the numerical integration of the system of first-order ODEs. This paper is organized as follows. Section 2 presents the order conditions for special explicit TDRK methods up to order five as well as we derive a new fifth-order TDRK method with FSAL property. We construct an explicit trigonometrically fitted TDRK (TFTDRK5F) method of order five with FSAL property and analyze its linear stability in Section 3. In Section 4, Numerical results are presented to illustrate the efficiency of the new method in comparison with the existing TFRK methods in the literature. Conclusions are given in Section 5.

2. The Explicit TDRK Method

We consider the special form of explicit TDRK methods studied by Chan and Tsai [13]

\begin{equation}
Y_j = y_{n+1} + \sum_{k=1}^{s} a_j \ g(x_n + \epsilon_k h, Y_j), \quad j = 2, \ldots, s,
\end{equation}

where

\begin{align*}
Y_{n+1} &= y_n + h f(x_n, y_n) + h^2 \sum_{j=1}^{s} \hat{b}_j \ g(x_n + \epsilon_j h, Y_j),
\end{align*}

(2)

where \( y^n(x) = g(x, y) = f_x(x, y) + f_y(x, y) f(x, y) \). The special explicit TDRK method (2) includes only one function evaluation of \( f \) and \( s \) function evaluations of \( g \) per step. The coefficients of the special explicit TDRK method (2) can be expressed in Butcher tableau as follows:

\[
\begin{array}{cccc|c}
0 & 0 & 0 & 0 \\
c_2 & a_2 & 0 & \\
c_3 & a_3 & a_3 & 0 \\
\vdots & \vdots & \vdots & \ddots \\
c_s & a_{s+1} & a_{s+1} & \ldots & a_{s-1} & 0 \\
0 & b_1 & b_2 & \ldots & b_{s-1} & b_s \\
\end{array}
\]

According to [13], the order conditions for TDRK method up to five order are presented as follows:

order 2:

\[\sum_{j=1}^{s} \hat{b}_j = \frac{1}{2}\]

(3)

order 3:

\[\sum_{j=1}^{s} \hat{b}_j \epsilon_j = \frac{1}{6}\]

(4)

order 4:

\[\sum_{j=4}^{s} \hat{b}_j \epsilon_j ^2 = \frac{1}{12}\]

(5)
order 5:
\[
\sum_{j=2}^{5} b_j c_j^2 = \frac{1}{20}, \quad \sum_{j=3}^{5} \sum_{k=2}^{j-1} b_j a_j c_k = \frac{1}{120}.
\]

(6)

In practice, the following Nyström row assumption is helpful
\[
\sum_{k=1}^{j} a_j = \frac{1}{2} c_j^2, \quad j = 2, \ldots, s.
\]

(7)

2.1. A Fifth order TDRK with FSAL Property

We are going to derive a new TDRK method with “First Same As Last” (FSAL) property where
\[ b_i = a_s, \quad i = 1, \ldots, s - 1, \quad \text{and} \quad b_s = 0. \]

(8)

The feature of “First Same As Last” (FSAL) technique is that the fourth stage can be reused as the first stage of the next step. Therefore, the efficient number of function evaluations is three per step. In order to construct four-stage fifth-order TDRK method by using FSAL technique, we will satisfy the order conditions (3)–(6) together with the simplifying assumption (7).

Choosing \( \epsilon_2 = \frac{1}{3}, \epsilon_3 = \frac{8}{5} \) and solving equations (3)–(7) simultaneously yield a fifth order TDRK method with FSAL property denoted as TDRK5F, which is given in the following Butcher tableau:

1. The TDRK5F method with FSAL property

\[
\begin{array}{c|ccc}
0 & 0 \\
\hline
\frac{1}{3} & 1 & 0 \\
\frac{4}{5} & -2 & 42 & 0 \\
1 & 5 & 9 & 25 & 0 \\
\hline
\end{array}
\]

3. The Construction of explicit TFTDRK method

3.1. The Modified TDRK method

Observe that the solution of the problem (1) in most applications is oscillatory. This motivates us to consider the following modified TDRK method
\[
Y_j = y_n + \epsilon_j \frac{d}{h} f(x_n, y_n) + h^2 \sum_{k=1}^{j-1} a_j (v) g(x_{k} + c_k h, Y_k), \quad j = 2, \ldots, s,
\]

(9)
\[ y_{n+1} = y_n + h f(x_n, y_n) + h^2 \sum_{j=1}^{s} b_j(v) g(x_n + c_j h, y_j). \]  \hfill (9)

where \( v = uh, \ u \) is the dominant frequency of the solution to the problem and \( h \) is the step size. It can be expressed in Butcher tableau as follows:

| 0      | 1      | 0      |
|--------|--------|--------|
| \( c_2 \) | \( y_2(v) \) | \( a_2(v) \) | \( a_2(v) \) |
| \( c_3 \) | \( y_3(v) \) | \( a_3(v) \) | \( a_3(v) \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( c_s \) | \( y_s(v) \) | \( a_{s1}(v) \) | \( a_{s1}(v) \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( b_2(v) \) | \( b_2(v) \) | \( b_2(v) \) | \( b_2(v) \) |

Based on the approach which is given in [9,10,11], we require the method (9) to integrate exactly the family of functions \( \{ \exp(i\theta); \exp(i\theta) \} \), or equivalently \( \{ \sin(\varphi); \cos(\varphi) \} \) with \( \varphi > 0 \). As a result, we obtain the following equations

\[ \exp(\pm i\varphi) = 1 \pm i\varphi \; y_j(v) - \varphi \sum_{k=1}^{s} a_j(v) \exp(\pm i\varphi), \; j = 2, \ldots, s, \]  \hfill (10)

\[ \exp(\pm i\varphi) = 1 \pm i\varphi - \varphi \sum_{j=1}^{s} b_j(v) \exp(\pm i\varphi), \; v = uh. \]  \hfill (11)

The following trigonometrical fitting conditions are equivalent to the system (10)-(11)

\[ \sum_{k=1}^{s} a_j(v) \cos(c_k v) = \frac{1-c_j(v)}{v^2}, \]  \hfill (12)

\[ \sum_{k=1}^{s} a_j(v) \sin(c_k v) = \frac{c_j(v) y_j(v) - s [c_j(v)]}{v^2}, \]  \hfill (13)

\[ \sum_{j=1}^{s} b_j(v) \cos(c_j v) = \frac{1-c_j(v) v}{v^2}, \]  \hfill (14)

\[ \sum_{j=1}^{s} b_j(v) \sin(c_j v) = \frac{v-s [c_j(v)]}{v^2}. \]  \hfill (15)

If the modified TDRK5F method (9) satisfies the trigonometrically-fitted conditions (12)-(15), then the method is denoted as TFTDRK5F method.

### 3.2. A fifth-order TFTDRK method with FSAL property

We will construct fifth-order TFTDRK method with 4 stages with FSAL technique [18] (the last evaluation at any step is the same as the first evaluation at the next step) by using the order conditions (3)-(6) and the trigonometrical fitting conditions (12)-(15). In this case, the TFTDRK method require 3 function evaluations per step except for the first step it requires 4 function evaluations, and they are given by the tableau

| 0      | 1      | 0      |
|--------|--------|--------|
| \( c_2 \) | \( y_2(v) \) | \( a_2(v) \) | \( a_2(v) \) |
| \( c_3 \) | \( y_3(v) \) | \( a_3(v) \) | \( a_3(v) \) |
| 1      | 1      | \( b_3(v) \) | \( b_3(v) \) |

| \( b_2(v) \) | \( b_2(v) \) | \( b_2(v) \) | \( b_2(v) \) |

| \( b_3(v) \) | \( b_3(v) \) | \( b_3(v) \) | \( b_3(v) \) |

| \( b_4(v) \) | \( b_4(v) \) | \( b_4(v) \) | \( b_4(v) \) |

| \( b_5(v) \) | \( b_5(v) \) | \( b_5(v) \) | \( b_5(v) \) |
From the trigonometrical-fitting conditions (12)–(15) in addition to the second and third order conditions (3)–(4) and choosing \( \{ \varepsilon_2, \varepsilon_3, \alpha_3 \} = \left\{ \frac{1}{4}, \frac{3}{4}, -\frac{2}{3} \right\} \), we obtain

\[
\alpha_2 (v) = \frac{1-c (\frac{2}{3} v)}{v^4}, \quad \alpha_3 (v) = \frac{2(v^4+1)}{1} \frac{1-c (\frac{2}{3} v)}{v^4 c (\frac{2}{3} v)},
\]

\[
y_2 (v) = \frac{3}{c} \left( \frac{2}{3} v \right), \quad y_3 (v) = \frac{2(v^4+1)}{1} s (\frac{2}{3} v) + s \left( \frac{2}{3} v \right),
\]

\[
\ell_1 = \frac{u}{c}, \quad \ell_2 = \frac{u}{c}, \quad \ell_3 = \frac{u}{c},
\]

\[
H = 6 \sin \left( \frac{2}{3} v \right) + 6 \sin \left( \frac{2}{3} v \right) + \sin (v) - 6 v \cos \left( \frac{2}{3} v \right) - 7 v^2 \sin \left( \frac{2}{3} v \right) - 14 v \cos (v) + 10 v^2 \sin \left( \frac{2}{3} v \right) - 20 \sin \left( \frac{2}{3} v \right) - 20 \cos \left( \frac{2}{3} v \right) + 20 v \cos \left( \frac{2}{3} v \right),
\]

\[
C = 30 v^2 \sin \left( \frac{2}{3} v \right) + 10 v^2 \sin \left( \frac{2}{3} v \right) + 6 v^2 \sin \left( \frac{2}{3} v \right) + 14 v^2 \sin (v) - 24 v^2 \sin \left( \frac{2}{3} v \right) - 20 v^2 \sin \left( \frac{2}{3} v \right) + 5 v^2 \sin \left( \frac{2}{3} v \right) + 10 v^2 \sin \left( \frac{2}{3} v \right) - 20 v.
\]

The method will be denoted as TFTDRK5F and for small values of \( v \) the Taylor series expansions for the coefficients are given as follows:

\[
y_2 (v) = 1 - \frac{1}{5} v^2 + \frac{1}{9} v^4 - \frac{1}{3} v^6 + \frac{1}{2} v^8 + \ldots,
\]

\[
y_3 (v) = 1 + \frac{1}{9} v^2 + \frac{2}{3} v^4 + \frac{8}{15} v^6 + \frac{5}{15} v^8 + \ldots,
\]

\[
u_2 (v) = \frac{1}{1} - \frac{1}{1} v^2 + \frac{1}{5} v^4 - \frac{1}{2} v^6 + \frac{1}{2} v^8 + \ldots,
\]

\[
u_3 (v) = \frac{4}{1} + \frac{1}{6} v^2 + \frac{7}{2} v^4 + \frac{2}{15} v^6 + \frac{1}{1} v^8 + \ldots,
\]

\[
b_1 (v) = \frac{5}{4} + \frac{5}{1} v^2 + \frac{1}{11} v^4 + \frac{4}{2} v^6 - \frac{3}{6} v^8 + \ldots,
\]

\[
b_2 (v) = \frac{9}{2} - \frac{1}{2} v^2 - \frac{9}{3} v^4 - \frac{4}{5} v^6 + \frac{1}{6} v^8 + \ldots,
\]

\[
b_3 (v) = \frac{7}{3} + \frac{1}{4} v^2 + \frac{7}{2} v^4 + \frac{1}{5} v^6 + \frac{3}{1} v^8 + \ldots,
\]
3.3. **Stability of TDRK5F method**

In this section, we will discuss the stability of TDRK5F method. We consider the following test equation:

\[ y' = \lambda y \quad , \quad \lambda > 0, \]  

By applying TDRK method (2) to (16) produces the following difference equation

\[ y_{n+1} = M(\nu)y_n, \quad \nu = \lambda h, \]

where

\[ M(\nu) = (1 + \nu^2 b^T (I - \nu^2 A)^{-1} \epsilon) + (\nu + \nu^2 b^T (I - \nu^2 A)^{-1} \zeta) \]

where \( I \) is the identity matrix and \( A \) is the coefficient of the new TDRK5F with \( \epsilon = [0 \frac{1}{3} \frac{2}{5} 1]^T, \quad \zeta = [1 1 1 1], \quad b^T = [\frac{5}{4} \frac{9}{2} \frac{2}{3} 0]. \)

The stability function of TDRK5F method is as follows:

\[ M(\nu) = 1 + \nu + \frac{1}{2} \nu^2 + \frac{5}{6} \nu^3 + \frac{1}{2} \nu^4 + \frac{1}{1} \nu^5 + \frac{1}{7} \nu^6. \]

In Figure 1, we plot the stability regions of the TDRK5F method.

![Figure 1: The stability region of TDRK5F method](image)

**4. Numerical Results**

In this section, we will solve some test problems to show the efficiency of the new TDRK5F method as compared with some efficient RK methods which are selected from the scientific literature. The following methods are used in comparison:

- **TFTDRK5F**: The fifth-order TFTDRK5F method with FSAL property derived in this paper.
- **TFRK5A**: The TFRK5A method of order five derived by Anastassi and Simos in [21].
- **TFRK5A1**: The TFRK5A1 method of order five derived by Anastassi and Simos in [22].
- **TFRK5S**: The TFRK5S method of order five derived by Sakas and Simos in [23].

The accuracy criteria calculated by taking \( l_t \) of the maximum absolute error as follows:

\[ \text{The accuracy} = l_t (\max(|y(x_n) - y_n|)). \]

The problems are integrated in the interval [0,1000].
Problem 1: [18]

\[ y'' = -y, \quad y(0) = 0, \quad y'(0) = 1 \]

The analytic solution and frequency are as follows \( y(x) = \sin(x), \quad \nu = 1 \).

Problem 2: [20]

\[ y'' + \begin{pmatrix} 13 & -12 \\ -12 & 13 \end{pmatrix} y = \begin{pmatrix} 9 \cos(2x) - 12 \sin(2x) \\ -12 \cos(2x) + 9 \sin(2x) \end{pmatrix}, \quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y'(0) = \begin{pmatrix} -4 \\ 8 \end{pmatrix}. \]

The analytic solution is \( y(x) = \begin{pmatrix} \sin(x) - \sin(5x) + \cos(2x) \\ \sin(x) + \sin(5x) + \sin(2x) \end{pmatrix} \)

In this problem the frequency is \( \nu = 5 \).

Problem 3: [19]

\[ y''_1 + y'_1 = 0.001 \cos(x), \quad y(0) = 1, \quad y'(0) = 0, \quad y''_2 + y'_2 = 0.001 \sin(x), \quad y(0) = 0, \quad y'(0) = 0.9995. \]

The analytic solutions is \( y_1(x) = \cos(x) + 0.005x \sin(x), \quad y_2(x) = \sin(x) + 0.005x \cos(x), \)

and we choose \( \nu = 1 \).

Figure 2: The performance curves with \( h = 1.1 - 0.2, \quad i = 0, 1, 2, 3, 4 \).

for Problem 1.
Figure 3: The performance curves with $h = 1/2^i$, $i = 3, 4, 5, 6, 7$, for Problem 2

Figure 4: The performance curves with $h = 1/2^i$, $i = 1, 2, 3, 4, 5$, for Problem 3.

It can be observed from Figures 2–4, the robustness and efficiency of TFTDRK5F method as compared with TFRK methods chosen from the scientific literature.

5. Conclusion

A new trigonometrically fitted two-derivative Runge-Kutta method of algebraic order five with FSAL property is derived in this paper for problems with oscillatory solutions. We also analyzed the linear stability of the new TFTDRK5F method. Numerical results are implemented to show the robustness and competence of the new TFTDRK method compared with the adapted (exponentially or trigonometrically fitted) RK methods in the recent literature.
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