A family of symmetric second degree semiclassical forms of class $s = 2$

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1 Introduction and basic background

Second degree forms have been introduced since 1995 [13]. These forms are characterized by the fact that their formal Stieltjes function $S(u)$ satisfies a quadratic equation $BS^2(u) + CS(u) + D = 0$ where $B \neq 0$ and $C$ are polynomials and $D$ is a polynomial defined in terms of the previous ones. They have been studied in [7,16] and [17] in the framework of the orthogonality on several intervals. Later on, in [12] and [13] an algebraic approach to such second degree forms as an extension of the Tchebychev forms is given. Notice that every second degree form is semiclassical, i.e., there exist two polynomials $\Phi(x)$ and $\Psi(x)$, where $\Phi(x)$ is monic and $\deg \Psi > 0$, such that $(\Phi(x)u)' + \Psi(x)u = 0$ [11,13]. In [3], the authors determine all the classical forms (i.e., semiclassical of class $s = 0$) which are of second degree. Hermite, Laguerre and Bessel are not of second degree. Only Jacobi forms which satisfy a certain condition possess this property. Later on, in [2], Beghdadi determines all the symmetric second degree semiclassical forms of class $s = 1$.

The aim of this work is to approach the problem of determining all the symmetric semiclassical forms of class $s = 2$ which are of second degree when $\Phi(0) = 0$. The first section is devoted to the preliminary
results and notations used in the sequel. In the second section, we prove that a symmetric semiclassical form \( u \) is a second degree if and only if its odd part \( x\sigma u \) is also second degree form. Using this result, we give all the forms which we look for. Three canonical cases for the polynomial \( \Phi \) arise: \( \Phi(x) = x^2 \), \( \Phi(x) = x^4 \) and \( \Phi(x) = x^2(x^2 - 1) \). As it turned out, we obtained explicitly a family of nonsymmetric second degree semiclassical forms of class \( s = 1 \) which generalize the classical ones.

In the sequel, we will recall some basic definitions and results. The field of complex numbers is denoted by \( \mathbb{C} \). The vector space of polynomials with coefficients in \( \mathbb{C} \) is represented as \( \mathcal{P} \) and its dual space is represented as \( \mathcal{P}' \). We denote by \( \langle u, f \rangle \) the effect of \( u \in \mathcal{P}' \) on \( f \in \mathcal{P} \). In particular, we denote by \( (u)_n := (u, x^n), n \geq 0 \), the moments of \( u \). For any linear form \( u \), any polynomial \( h \), let \( Du = u' \) and \( hu \) be the forms defined by duality:

\[
\langle u', f \rangle := -\langle u, f' \rangle, \quad \langle hu, f \rangle := \langle u, hf \rangle, \quad f \in \mathcal{P}.
\] (1)

We recall the definition of right-multiplication of a form by a polynomial:

\[
(uf)(x) := \left(u, \frac{xh(x) - \xi h(\xi)}{x - \xi}\right), \quad u \in \mathcal{P}', \quad h \in \mathcal{P}.
\] (2)

By duality, we obtain the Cauchy’s product of two forms:

\[
\langle uv, f \rangle := \langle u, vf \rangle, \quad u, v \in \mathcal{P}', f \in \mathcal{P}.
\] (3)

Consequently,

\[
(uv)_n = \sum_{i+j=n} (u)_i(v)_j, \quad n \geq 0.
\] (4)

We define [14] the form \((x - c)^{-1}u, c \in \mathbb{C}, \) through

\[
\langle (x - c)^{-1}u, f \rangle := \langle u, \theta_c f \rangle
\] (5)

with

\[
(\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}, \quad u \in \mathcal{P}', f \in \mathcal{P}.
\] (6)

From the definitions, it results \((u\theta_0 f)(x) = (u, f(\xi) - f(0)) \), \( u \in \mathcal{P}', f \in \mathcal{P} \).

We introduce the operator \( \sigma : \mathcal{P} \rightarrow \mathcal{P} \) defined by \( (\sigma f)(x) = f(x^2) \) for all \( f \in \mathcal{P} \). By transposition, we define \( \sigma u \):

\[
\langle \sigma u, f \rangle = \langle u, \sigma f \rangle \quad u \in \mathcal{P}', f \in \mathcal{P}.
\] (7)

Consequently, \( (\sigma u)_n = (u)_{2n} \).

We will also use the so-called formal Stieltjes function associated with \( u \in \mathcal{P}' \) that is defined by

\[
S(u)(z) = -\sum_{n \geq 0} \frac{(u)_n}{z^{n+1}}.
\] (8)

The following auxiliary results will be used in the sequel [14, 15].

**Lemma 1.1** For any \( f \in \mathcal{P} \) and \( u, v \in \mathcal{P}' \)

\[
(fu)' = fu' + f'u,
\] (9)

\[
(u\theta_0 f)(x) = (\theta_0 (uf))(x),
\] (10)

\[
f(x)(\sigma u) = \sigma (f(x^2)u),
\] (11)

\[
\sigma u' = 2(\sigma (xu))',
\] (12)

\[
\sigma (uv) = (\sigma u)(\sigma v) + x^{-1}(\sigma (xu)\sigma (xv)),
\] (13)

\[
S(uv)(z) = -zS(u)(z)S(v)(z).
\] (14)
The form $u$ is called regular if there exists a polynomial sequence $\{B_n\}_{n \geq 0}$, $\deg B_n = n$, such that $(u, B_n B_m) = r_n \delta_{nm}, r_n \neq 0, n \geq 0$.

In this case $\{B_n\}_{n \geq 0}$ is said to be orthogonal with respect to $u$. It satisfies the recurrence relation (see, for instance, the monograph by Chihara [4])

$$
B_0(x) = 1, \quad B_1(x) = x - \beta_0, \\
B_{n+2}(x) = (x - \beta_{n+1}) B_{n+1}(x) - \gamma_{n+1} B_n(x), \quad n \geq 0. \tag{15}
$$

The regularity of $u$ means that we must have $\gamma_{n+1} \neq 0, n \geq 0$.

In this paper, we suppose that the forms are normalized (i.e., $(u)_0 = 1$).

**Definition 1.2** [13] The form $u$ is called a second degree form if it is regular and if there exist two polynomials $B$ and $C$ such that

$$
B(z) S^2(u)(z) + C(z) S(u)(z) + D(z) = 0, \tag{16}
$$

where $D$ is a polynomial depending on $B$, $C$, and $u$ given by

$$
D(z) = (u \theta_0 C)(z) - (u^2 \theta_0^2 B)(z). \tag{17}
$$

The regularity of $u$ means that we must have $B \neq 0; C^2 - 4BD \neq 0$ and $D \neq 0$.

The following expressions are equivalent to (16), [13]:

$$
B(x) u^2 = x C(x) u, \quad (u^2, \theta_0 B) = (u, C). \tag{18}
$$

In the sequel, we shall suppose $B$ to be monic. The polynomials $B$ and $C$, given in (16) or by (18), are not unique, because $B$ and $C$ can be multiplied by an arbitrary polynomial. If in (16) the polynomials $B, C$ and $D$ are coprime, then the pair $(B, C)$ is called a primitive pair. The primitive pair is unique.

Let us recall that a form $u$ is called semiclassical when it is regular and there exist two polynomials $\Phi$ and $\Psi$, where $\Phi(x)$ is monic and $\deg(\Psi) \geq 1$, such that

$$
(\Phi u)' + \Psi u = 0. \tag{19}
$$

The class of the semiclassical form $v$ is $s = \max(\deg \Psi - 1, \deg \Phi - 2)$ if and only if the following condition is satisfied

$$
\prod_c \left( |\Phi'(c) + \Psi(c)| + \|u, \theta_c \Psi + \theta_c^2 \Phi\| \right) > 0, \tag{20}
$$

where $c$ goes over the zeros set of $\Phi$ [14].

When $s = 0, u$ is called a classical form.

As a result, if $u$ is a semiclassical form of class $s$ satisfying (19), then the shifted form $\hat{u} = (h_{a-1} \circ \tau_{-b}) u, a \in \mathbb{C}^*, b \in \mathbb{C}$ is of class $s$ satisfying the equation

$$
(\hat{\Phi} \hat{u})' + \hat{\Psi} \hat{u} = 0 \tag{21}
$$

with

$$
\hat{\Phi}(x) = a^{-t} \Phi(ax + b), \quad \hat{\Psi}(x) = a^{1-t} \Psi(ax + b), \quad t = \deg(\Phi) \tag{22}
$$

where, for each polynomial $f$

$$
\langle \tau_b u, f \rangle := \langle u, \tau_{-b} f \rangle := \langle u, f(x + b) \rangle, \quad \langle h_a u, f \rangle := \langle u, h_a f \rangle := \langle u, f(ax) \rangle.
$$

A second degree form $u$ is a semiclassical form and satisfies (19), with [13]

$$
k \Phi(x) = B(x)(C^2(x) - 4B(x)D(x)) \tag{23}
$$

$$
k \Psi(x) = -\frac{3}{2} B(x)(C^2(x) - 4B(x)D(x))', \quad k \neq 0,
$$

where $k$ is a normalization factor.
The second degree character is kept by shifting. Indeed, if \( u \) is a second degree form satisfying (18), then \( \hat{u} \) is also second degree form [13]. It satisfies
\[
\hat{B}(x)\hat{u}^2 = x\hat{C}(x)\hat{u}, \quad \langle \hat{u}^2, \theta_0 \hat{B} \rangle = \langle \hat{u}, \hat{C} \rangle.
\]
with
\[
\hat{B}(x) = a^{-r}B(ax + b), \quad \hat{C}(x) = a^{1-r}C(ax + b), \quad r = \deg(B).
\]

**Lemma 1.3** [2] Let \( u \) be a second degree semiclassical form satisfying (19)–(20). The class of \( u \) is \( s = \deg \Phi - 2 = \deg \Psi - 1 \).

We finish this section by recalling this important result.

**Theorem 1.4** [3] Among the classical forms, only the Jacobi forms \( J(k - \frac{1}{2}, l - \frac{1}{2}) \) are second degree forms, provided \( k + l \geq 0, k, l \in \mathbb{Z} \) which satisfy
\[
\left((x^2 - 1)J\left(k - \frac{1}{2}, l - \frac{1}{2}\right)\right)' + \left(-(k + l + 1)x + k - l\right)J\left(k - \frac{1}{2}, l - \frac{1}{2}\right) = 0.
\]

# 2 Symmetric second degree semiclassical forms

## 2.1 Algebraic properties

We recall that a form \( u \) is called symmetric if \((u)_{2n+1} = 0, n \geq 0 \). The conditions \((u)_{2n+1} = 0, n \geq 0 \), are equivalent to the fact that the corresponding sequence of monic orthogonal polynomials (MOPS) \( \{B_n\}_{n \geq 0} \) satisfies the recurrence relation (15) with \( \beta_n = 0, n \geq 0 \) [4].

In addition, the sequence \( \{B_n\}_{n \geq 0} \) has the following quadratic decomposition
\[
B_{2n}(x) = P_n(x^2), \quad B_{2n+1}(x) = xR_n(x^2), \quad n \geq 0.
\]

The sequences \( \{P_n\}_{n \geq 0} \) and \( \{R_n\}_{n \geq 0} \) are respectively orthogonal with respect to \( \sigma u \) and \( x\sigma u \). We have for instance:
\[
P_{n+2}(x) = (x - \beta_{n+1}^P)P_{n+1}(x) - \gamma_{n+1}^P P_n(x), \quad n \geq 0,
\]
\[
P_1(x) = x - \beta_0^P, \quad P_0(x) = 1,
\]
with
\[
\beta_0^P = \gamma_1, \quad \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3}, \quad \gamma_{n+1}^P = \gamma_{2n+1} \gamma_{2n+2}, \quad n \geq 0.
\]

We have the following characterisations.

**Proposition 2.1** [2] The even part \( \sigma u \) of a symmetric second degree form \( u \) is also second degree form.

**Proposition 2.2** Let \( u \) be a regular and symmetric form. The following statements are equivalent:

(a) \( u \) is a second degree form

(b) The odd part \( x\sigma u \) of \( u \) is a second degree form.

**Proof** “(a) \( \implies \) (b)” According to Proposition 2.1 and the fact that the multiplication by a polynomial preserves the quadratic property.

“(b) \( \implies \) (a)” We denote by \( v \) the normalized form defined by \( \gamma_1 v = x\sigma u \). We suppose that \( x\sigma u \) is a second degree form. Then there exist two polynomials \( B_1 \) and \( C_1 \) such that
\[
B_1(z)S^2(v)(z) + C_1(z)S(v)(z) + D_1(z) = 0,
\]
where
\[
D_1(z) = (uv^0C_1)(z) - (v^2u_0^2B_1)(z).
\]
From (8) and the fact that $u$ is a symmetric form, we have

$$S(v)(z^2) = \gamma_1^{-1} z S(u)(z) + \gamma_1^{-1}. \quad (31)$$

Make a change of variable $z \longrightarrow z^2$ in (29), multiply by $\gamma_1^2$ and substitute (31) in the resulting equation, we get (16) with

$$\begin{aligned}
B(z) &= z^2 B_1(z^2), \\
C(z) &= 2z B_1(z^2) + \gamma_1 z C_1(z^2), \\
D(z) &= B_1(z^2) + \gamma_1 C_1(z^2) + \gamma_1^2 D_1(z^2).
\end{aligned} \quad (32)$$

From (6), we have $(u\theta_0(\xi C_1(\xi^2))) = (u C_1(\xi^2))$ (3)). Using (2), we obtain

$$\begin{aligned}
(u\theta_0(\xi C_1(\xi^2)))(z) &= \left\{ u, \frac{z C_1(z^2) - \xi C_1(\xi^2)}{z - \xi} \right\} \\
&= \left\{ u, z\xi(\theta z C_1)(\xi^2) + \frac{z^2 C_1(z^2) - \xi^2 C_1(\xi^2)}{z^2 - \xi^2} \right\}.
\end{aligned}$$

But $(u, z\xi(\theta z C_1)(\xi^2)) = 0$ since $u$ is a symmetric form, then

$$\begin{aligned}
(u\theta_0(\xi C_1(\xi^2)))(z) &= \left\{ u, \frac{z^2 C_1(z^2) - \xi^2 C_1(\xi^2)}{z^2 - \xi^2} \right\} \\
&= \left\{ \sigma u, \xi \frac{C_1(z^2) - C_1(\xi)}{z^2 - \xi} + C_1(z^2) \right\}.
\end{aligned}$$

by virtue of (7). Therefore,

$$(u\theta_0(\xi C_1(\xi^2)))(z) = \gamma_1 (v\theta_0 C_1)(z^2) + C_1(z^2). \quad (33)$$

Replacing $B_1$ by $C_1$ in (33), we get

$$\begin{aligned}
(u\theta_0(\xi B_1(\xi^2)))(z) &= \gamma_1 (v\theta_0 B_1)(z^2) + B_1(z^2). \quad (34)
\end{aligned}$$

From (6), we have $(u^2\theta_0^2(\xi^2 B(\xi^2)))(z) = (u^2 B(\xi^2))(z)$ and by (13), we have $\sigma u^2 = (\sigma u)^2$ because $u$ is a symmetric form. Then, using the same process described above with $(u^2, B_1)$ instead of $(u, C_1)$, we get

$$\begin{aligned}
(u^2\theta_0^2(\xi^2 B(\xi^2)))(z) &= B_1(z^2) + \left\{ \xi (\sigma u)^2, \frac{B_1(z^2) - B_1(\xi)}{z^2 - \xi} \right\}.
\end{aligned}$$

But, from (6), we have

$$\frac{B_1(z^2) - B_1(\xi)}{z^2 - \xi} = \frac{z^2 (\theta_0 B_1)(z^2) - \xi (\theta_0 B_1)(\xi)}{z^2 - \xi} = \frac{(\theta_0 B_1)(z^2) - \xi (\theta_0 B_1)(\xi)}{z^2 - \xi}.$$

Then, we get

$$\begin{aligned}
(u^2\theta_0^2(\xi^2 B(\xi^2)))(z) &= B_1(z^2) + 2\gamma_1 (\theta_0 B_1)(z^2) + \left\{ \xi^2 (\sigma u)^2, \frac{(\theta_0 B_1)(z^2) - (\theta_0 B_1)(\xi)}{z^2 - \xi} \right\}, \quad (35)
\end{aligned}$$

since $(\sigma u)^2, \xi = (u^2, \xi^2) = 2\gamma_1$, by (4) and (15).

Now, using (4) and taking into account $x\sigma u = \gamma_1 v$, we prove that

$$\xi^2 (\sigma u)^2 = (\xi (\sigma u)^2)^2 + 2\xi^2 \sigma u = \gamma_1^2 v^2 + 2\gamma_1 \xi v.$$

Then, (35) becomes

$$(u^2\theta_0^2(\xi^2 B(\xi^2)))(z) = B_1(z^2) + 2\gamma_1 (\theta_0 B_1)(z^2) + \gamma_1^2 (v^2 \theta_0^2 B_1)(z^2) + 2\gamma_1 \left( \frac{v (\theta_0 B_1)(z^2) - \xi (\theta_0 B_1)(\xi)}{z^2 - \xi} \right).$$
But,

$$\left\langle v, \frac{\xi(\theta_0 B_1)(z^2) - \xi(\theta_0 B_1)(\xi)}{\xi^2 - \xi} \right\rangle = -(\theta_0 B_1)(z^2) + (v\theta_0 B_1)(z^2).$$

Therefore, we deduce

$$\left( u^2 \theta_0^2 (\xi^2 B_1(\xi^2)) \right) (z) = B_1(z^2) + \gamma_1^2 \left( v^2 \theta_0 B_1 \right)(z^2) + 2 \gamma_1 (v \theta_0 B_1)(z^2). \quad (36)$$

Thus, on account of (30), (32)–(34) and (36), we conclude that the polynomials $B$, $C$ and $D$ given by (32) verify the relation (17).

Hence $u$ is also a second degree form. \hfill \square

Using Proposition 2.1, Bégdadi gives all the symmetric second degree semiclassical forms of class $s = 1$:

**Theorem 2.3** [2] Among the symmetric semiclassical forms of class $s = 1$, only the forms denoted by $I(k - \frac{1}{2}, l - \frac{1}{2})$ are second degree forms, provided $k + l \geq 0$, $l \neq 0$, $k, l \in \mathbb{Z}$ which satisfy

$$\left( x(x^2 - 1) I \left( k - \frac{1}{2}, l - \frac{1}{2} \right) \right)' + \left( -2(k + l + 1)x^2 + 2l + 1 \right) I \left( k - \frac{1}{2}, l - \frac{1}{2} \right) = 0.$$  \hfill (38)

The form $I = I(k - \frac{1}{2}, l - \frac{1}{2})$ possesses the following representation [2]:

$$\langle I, f \rangle = \frac{\Gamma(k + l + 1)}{\Gamma \left( k + \frac{1}{2} \right) \Gamma \left( l + \frac{1}{2} \right)} \int_{-1}^{1} \frac{x^2(1 - x^2)^k}{\sqrt{1 - x^2}} f(x) \, dx, \quad k \geq 0, l > 0.$$  \hfill (39)

**Remark** Unfortunately, we are not able to determine all the symmetric second degree semiclassical forms of class $s = 2$ by Proposition 2.1, especially because $\sigma u$ is among the second degree semiclassical forms of class $s = 1$ which are unknown.

2.2 Symmetric second degree semiclassical forms of class $s = 2$: case $\Phi(0) = 0$

Let us begin with an example $\mathcal{V}$ among the symmetric forms which is a second degree semiclassical form of class $s = 2$ satisfying (19) with $\Phi(0) = 0$. This example is given in [1]. The form $\mathcal{V}$ satisfies (16) with

$$B(z) = z^4(z^2 - 1), \quad C(z) = 2z^3(z^2 - 1), \quad D(z) = z^2(z^2 - 1) - \lambda^2, \quad (37)$$

and (19) with

$$\Phi(x) = x^2(x^2 - 1), \quad \Psi(x) = -x^3. \quad (38)$$

The corresponding MOPS of $\mathcal{V}$ satisfies (15) with

$$\gamma_1 = \lambda, \quad \gamma_{2n+2} = \frac{1 - 2(n + 1)\lambda}{1 - 2n\lambda}, \quad \gamma_{2n+3} = \frac{1}{4} \frac{1 - 2n\lambda}{1 - 2(n + 1)\lambda}, \quad n \geq 0. \quad (39)$$

Now, we state the following result which is essential for this work.

**Proposition 2.4** [2] Let $u$ be a symmetric semiclassical form of class $s$, satisfying (19). If $s$ is even then $\Phi$ is even and $\Psi$ is odd. If $s$ is odd then $\Phi$ is odd and $\Psi$ is even.
In the sequel, we suppose \( s = 2 \), \( u \) is symmetric, and \( \Phi(0) = 0 \). Then, according to the above proposition, \( u \) satisfies (19) with

\[
\Phi(x) = c_4x^4 + c_2x^2, \quad \Psi(x) = a_3x^3 + a_1x, \quad |c_4| + |a_3| \neq 0.
\]

Then, using the fact that \( \Phi \) is monic and the semiclassical character is kept by shifting, we distinguish three canonical cases for \( \Phi \): \( \Phi(x) = x^2 \), \( \Phi(x) = x^4 \), \( \Phi(x) = x^2(x^2 - 1) \).

**First case:** \( \Phi(x) = x^2 \)

According to Lemma 1.3, this case is excluded because \( s = 2 \neq \text{deg} \Phi - 2 \).

**Second case:** \( \Phi(x) = x^4 \)

Let \( \Psi(x) = a_3x^3 + a_1x \). After multiplying (19) by \( x \), applying the operator \( \sigma \) and using (11)–(12), we obtain

\[
(x^2(x\sigma u)') + \frac{1}{2}((a_3 - 1)x + a_1)(x\sigma u) = 0.
\]

Then \( x\sigma u = \gamma_1 B(\alpha) \) where \( B(\alpha) \) is the classical Bessel form with \( a_3 = -4\alpha + 1 \) and \( a_1 = -4 \). Recall that the form \( B(\alpha) \) satisfies (19) with

\[
\Phi(x) = x^2, \quad \Psi(x) = -2(\alpha x + 1), \quad \alpha \neq -\frac{n}{2}, \quad n \in \mathbb{N}.
\]

Since \( B(\alpha) \) is not a second degree form [3], according to Proposition 2.2, we conclude that \( u \) is not a second degree form.

**Third case:** \( \Phi(x) = x^2(x^2 - 1) \)

This case is mentioned in [6] and [18], when the authors gave all the symmetric semiclassical forms of class \( s = 2 \) with \( \Phi(x) = x^2(x^2 - 1) \). These forms satisfy

\[
(x^2(x^2 - 1)u') + ((-2\alpha - 2\beta - 3)x^3 + (2\beta + 1)x)u = 0, \quad \gamma_1(\alpha + \beta + 1) \neq \beta.
\] (40)

Taking into account [18], we have

\[
\gamma_1 = \lambda, \quad \gamma_{2n+2} = \frac{(n + \beta + 1)(n + \alpha + \beta + 1)d_{n+1}(\lambda)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)d_{n}(\lambda)}, \quad n \geq 0,
\]

\[
\gamma_{2n+3} = \frac{(n + 1)(n + \alpha + 1)d_{n}(\lambda)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 3)d_{n+1}(\lambda)}, \quad n \geq 0.
\] (41)

with

\[
d_{n}(\lambda) = \begin{cases} \frac{\Gamma(\beta + 1)\Gamma(\alpha + \beta + 1)\Gamma(n + 1)\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(n + \beta + 1)\Gamma(n + \alpha + \beta + 1)} + \frac{\beta}{\alpha + \beta + 1} - \lambda, \beta(\alpha + \beta + 1) \neq 0, n \geq 0, \\ 1 - \lambda \sum_{k=0}^{n-1} \frac{(2k + 1)\Gamma(\alpha + k + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + k + 2)}, \alpha + \beta = -1, n \geq 0, \\ \frac{1}{\alpha + 1} - \lambda \sum_{k=0}^{n-1} \frac{2k + \alpha + 2}{(k + 1)(k + \alpha + 1)}, \beta = 0, n \geq 0, \left( \sum_{0}^{1} = 0 \right). \end{cases}
\] (42)

The regularity condition is

\[
\alpha \neq -n - 1, \quad \beta \neq -n - 1, \quad \alpha + \beta \neq -n - 1, \lambda \neq 0, \quad d_{n}(\lambda) \neq 0 \quad n \in \mathbb{N}.
\]

In the sequel, we denote by \( \mathcal{L}(\alpha, \beta, \lambda) \) the form \( u \) which satisfies (40).

We have \( \mathcal{V} = \mathcal{L}(-\frac{1}{2}, -\frac{1}{2}, \lambda) \).

**Theorem 2.5** Among the symmetric semiclassical forms of class \( s = 2 \) satisfying (19) with \( \Phi(0) = 0 \), only the forms \( \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda) \) are second degree forms, provided \( p + q \geq 0, \lambda^{-1} \neq \frac{(p+q)}{2q-1}, \quad p, q \in \mathbb{Z} \).
Proof After multiplying (40) by \( x \), applying the operator \( \sigma \) and using (11)–(12), we obtain
\[
(x(x - 1)(xu))' + (-\alpha + \beta + 2)x + \beta + 1)(isu) = 0.
\]
Let us make the suitable shift for \((isu)\)
\[
\widehat{(isu)} = \left( h_{\left(-\frac{1}{2}\right)^-} \circ \tau_{\left(-\frac{1}{2}\right)} \right)(isu).
\]
Using (22), \((isu)\) satisfies (21) with
\[
\hat{\Phi}(x) = x^2 - 1, \quad \hat{\psi}(x) = -(\alpha + \beta + 2)x + \alpha - \beta.
\]
Therefore, we have \( \left( h_{\left(-\frac{1}{2}\right)^-} \circ \tau_{\left(-\frac{1}{2}\right)} \right)(isu \mathcal{L}(\alpha, \beta, \lambda)) = \lambda \mathcal{J}(\alpha, \beta) \)
where \( \mathcal{J}(a, b) \) is the classical Jacobi form with Pearson equation
\[
((x^2 - 1)\mathcal{J}(a, b))' + (-\alpha + b + 2)x + a - b)\mathcal{J}(a, b) = 0.
\]
According to Theorem 1.4, Proposition 2.2 and the fact that the shifted form of a second degree form is also
second degree form, we obtain: \( \mathcal{L}(\alpha, \beta, \lambda) \) is a second degree semiclassical form of class \( s = 2 \) if and only if
\( \alpha = p - \frac{1}{2}, \beta = q - \frac{1}{2}, \lambda^{-1} = \frac{2(p + q)}{2q - 1}, p + q \geq 0, p, q \in \mathbb{Z}. \)
\[\square\]

Let us now give the polynomial coefficients \( B, C \) and \( D \) of (16) corresponding to these forms. For this, we need
the following lemmas.

**Lemma 2.6** [3] Let \( u \) and \( v \) be two regular forms satisfying the following relation:
\[
M(x)u = N(x)v,
\]
where \( M(x) \) and \( N(x) \) are two polynomials.
If \( u \) is a second degree form satisfying (16), then \( v \) is also a second degree form and satisfies
\[
\bar{B}(z)S^2(v)(z) + \bar{C}(z)S(v)(z) + \bar{D}(z) = 0,
\]
with
\[
\begin{cases}
\bar{B}(z) = B(z)N^2(z), \\
\bar{C}(z) = N(z)[2B(z)((u\theta_0N(z)) - (u\theta_0M(z)) + M(z)C(z)], \\
\bar{D}(z) = B(z)((u\theta_0N(z)) - (u\theta_0M(z))^2 + M(z)C(z)((u\theta_0N(z)) - (u\theta_0M(z)) + M^2(z)D(z).
\end{cases}
\]

**Lemma 2.7** We have
\[
x^2 \mathcal{L}(\alpha, \beta, \lambda) = \mu \mathcal{L}(\alpha, \beta + 1, \lambda),
\]
\[
(x^2 - 1)\mathcal{L}(\alpha, \beta, \lambda) = \mu \mathcal{L}(\alpha + 1, \beta, \lambda),
\]
where \( \mu \) is the normalization factor.

Proof The form \( u = \mathcal{L}(\alpha, \beta, \lambda) \) satisfies (40). Multiplying by \( x^2 \), we obtain
\[
(x^2(x^2 - 1)(xu))' + (-2\alpha + 2\beta + 5)x^3 + (2\beta + 3)x)(x^2u) = 0.
\]
Hence (48). Multiplying (40) by \((x^2 - 1)\), we obtain
\[
(x^2(x^2 - 1)((x^2 - 1)u)' + (-2\alpha + 2\beta + 5)x^3 + (2\beta + 1)x)((x^2 - 1)u) = 0.
\]
Hence (49). \[\square\]

Using Lemma 2.6, Lemma 2.7, and the fact that \( \mathcal{V} = \mathcal{L}(-\frac{1}{2}, -\frac{1}{2}, \lambda) \) and satisfies (16) with (37), the
elements \( B, C \) and \( D \) in (16) are given here in every case:
Proposition 2.8 Let us consider \( u = L(p - \frac{1}{2}, q - \frac{1}{2}, \lambda) \), where \( p \) and \( q \) are integers provided \( p + q \geq 0 \) and \( \lambda^{-1} \neq \frac{2(p+q)}{2q-1} \). Then, we have the following:

1. For \( p \geq 0 \),
   i. if \( q \geq 0 \), then
   \[
   u = \mu(x^2 - 1)^p x^{2q} \mathcal{V},
   \]
   \[
   \begin{cases}
   B(z) = z^4(z^2 - 1), \\
   C(z) = -2\mu z^4(z^2 - 1)Y(z) + 2\mu z^3(z^2 - 1)^{p+1}z^{2q+3}, \\
   D(z) = \mu^2 z^4(z^2 - 1)Y^2(z) - 2\mu z^2(z^2 - 1)^{p+1}z^{2q+3}Y(z) + \mu^2(z^2 - 1)^2 z^{2q} (z^2(z^2 - 1) - \lambda^2),
   \end{cases}
   \]
   where
   \[
   Y(z) = (\mathcal{V} \theta_0((\xi^2 - 1)^p \xi^{2q})) (z), \quad \mu = \left(\frac{\mu}{\mathcal{V}, (x^2 - 1)^p x^{2q}}\right) \mathcal{V}.
   \]
   
   ii. if \( q \leq -1 \) and \( p + q \geq 0 \), then
   \[
   x^{-2q} u = \mu (x^2 - 1)^p \mathcal{V},
   \]
   \[
   \begin{cases}
   B(z) = (z^2 - 1)z^{-4q+4}, \\
   C(z) = z^{-2q}\left\{2z^4(z^2 - 1)Y(z) + 2\mu z^3(z^2 - 1)^{p+1}\right\}, \\
   D(z) = z^4(z^2 - 1)Y^2(z) + 2\mu z^2(z^2 - 1)^{p+1}Y(z) + \mu^2(z^2 - 1)^2 z^{2q} (z^2(z^2 - 1) - \lambda^2),
   \end{cases}
   \]
   where
   \[
   Z(z) = (u \theta_0((\xi^2 - 1)^{-p})) (z) - \mu (\mathcal{V} \xi^{2q-1}) (z), \quad \mu = \left(\frac{u, (x^2 - 1)^{-p}}{\mathcal{V}, x^{2q}}\right).
   \]

2. For \( p \leq -1 \) and \( q \geq 1 \) such that \( p + q \geq 0 \), we have
   \[
   (x^2 - 1)^{-p} u = \mu x^{2q} \mathcal{V},
   \]
   \[
   \begin{cases}
   B(z) = z^4(z^2 - 1)^{-2p+1}, \\
   C(z) = (z^2 - 1)^{-p}\left\{2z^4(z^2 - 1)Z(z) + 2\mu z^3(z^2 - 1)^{2q+3}\right\}, \\
   D(z) = z^4(z^2 - 1)Z^2(z) + 2\mu z^2(z^2 - 1)^{2q+3}Z(z) + \mu^2 z^{2q} (z^2(z^2 - 1) - \lambda^2),
   \end{cases}
   \]
   where
   \[
   Z(z) = (u \theta_0((\xi^2 - 1)^{-p})) (z) - \mu (\mathcal{V} \xi^{2q-1}) (z), \quad \mu = \left(\frac{u, (x^2 - 1)^{-p}}{\mathcal{V}, x^{2q}}\right).
   \]

Integral representation

The form \( u = L(\alpha, \beta, \lambda) \) has the following representation [6, 15] (for \( \Re(\alpha) > -1 \), \( \Re(\beta) > 0 \))

\[
\langle u, f \rangle = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{1} |x|^{2\beta-1}(1 - x^2)^{\alpha} f(x)dx + \left(1 - \frac{\lambda(\alpha + \beta + 1)}{\beta}\right) f(0).
\]

From Theorem 2.5, we deduce the following:

A symmetric semiclassical form of class \( s = 2 \) satisfying (19) with \( \Phi(0) = 0 \) is a second degree form and positive definite if the weight function has the following expression:

\[
\begin{aligned}
   w(x) &= \lambda \frac{\Gamma(p + q + 1)}{\Gamma(p + q + 1/2)\Gamma(q + 1/2)} \frac{x^{2q-2}(1 - x^2)^p Y(1 - x^2)}{\sqrt{1 - x^2}} + \left(1 - \frac{2\lambda(p + q)}{2q - 1}\right) \delta_0, \\
   p &\in \mathbb{N}, q &\in \mathbb{N}^*, \lambda &\in \left\{0, \frac{2q - 1}{2(p + q)}\right\}
\end{aligned}
\]

where \( Y \) is the characteristic function of \( \mathbb{R}^+ \).
The case $p = 0$ and $q = 0$ is $\mathcal{V}$. This form is not positive definite, and has the integral representation [1]

$$\mathcal{V} = \delta_0 + \lambda Pf \frac{Y(1-x^2)}{\pi x^2\sqrt{1-x^2}}$$

(see [1]), with the definition

$$\left\langle Pf \frac{Y(1-x^2)}{x^2\sqrt{1-x^2}}, f \right\rangle = \lim_{\epsilon \to 0} \left( \int_{-\epsilon}^{\epsilon} \frac{f(x)\sqrt{1-x^2}}{x^2} \, dx + \int_{-\epsilon}^{\epsilon} \frac{f(x)(1-x^2)}{x^2} \, dx \right).$$

**Particular cases:**

**1** If $p = q = 1$ and $\lambda = \frac{1}{8}$ then $u = \frac{1}{2} \delta_0 + \frac{1}{2} \mathcal{U}$ where $\mathcal{U}$ is a Tchebychev form of second kind. Let us recall that its sequence $\{B_n\}_{n \geq 0}$ satisfies (15) with

$$\beta_n = 0, \quad \gamma_{2n+1} = \frac{n + 1}{4(n + 2)}, \quad \gamma_{2n+2} = \frac{n + 3}{4(n + 2)}, \quad n \geq 0.$$

In a very interesting work [5], J. Charris, G. Salas and V. Silva studied this sequence of orthogonal polynomials. This sequence is a particular case of a more general sequence considered in Example 1 presented in [10]. According to Theorem 2.5 we deduce that it is a second degree form.

**2** If $\lambda^{-1} = \frac{2p + q}{2q - 1}$ then $u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2})$. This means that the second degree forms $u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$ generalize the symmetric second degree forms of class $s = 1$.

In fact, from (40), $u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$ satisfies (19) with

$$\Phi(x) = x^2(x^2 - 1), \quad \Psi(x) = (-2p - 2q - 1)x^3 + 2qx.$$  \hspace{1cm} (60)

We have $\Phi'(0) + \Psi(0) = 0$ and $(u, \theta_0 \Psi + \phi_0^2 \Phi) = -2\lambda(p + q) + 2q - 1$.

Then, if $-2\lambda(p + q) + 2q - 1 = 0$ we can simplify (19)–(60) by $x$ and we necessarily have $p + q \neq 0$ because $\lambda(2q - 1) \neq 0$. Therefore, $\gamma_1 = \lambda = \frac{2q - 1}{2p + q}$ and $u$ verifies (19) with

$$\Phi(x) = x(x^2 - 1), \quad \Psi(x) = -2(p + q)x^2 + (2q - 1).$$

Here, $\Phi'(0) + \Psi(0) + (u, \theta_0 \Psi + \phi_0^2 \Phi) = 2(q - 1)$.

Hence, for $(p, q) = (k, l + 1)$, we get the statement of Theorem 2.3.

### 2.3 The study of $\sigma \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$

In this part, the focus will be put on $\sigma u$: the even part of $u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda)$, provided $p + q \geq 0$, $p, q \in \mathbb{Z}$.

The linear form $u$ verifies the functional equation

$$(x^2(x^2 - 1)u)' + ((-2p - 2q - 1)x^3 + 2qx)u = 0.$$

Multiplication by $x$ gives

$$(x^3(x^2 - 1)u)' + ((-2p - 2q - 2)x^4 + (2q + 1)x^2)u = 0.$$ Applying the operator $\sigma$ in both hand sides of the above equation and using (11)–(12), we obtain

$$\left(\Phi^p(x)\sigma u\right)' + \Psi^p(x)\sigma u = 0$$  \hspace{1cm} (61)

where $\Phi^p(x) = x^2(x - 1), \quad \Psi^p(x) = -(p + q + 1)x^2 + (q + \frac{1}{2})x$.

We have $\Psi^p(0) + (\Phi^p)'(0) = 0$ and $(\sigma u, \theta_0 \Psi^p + \phi_0^2 \Phi^p) = -(p + q)\lambda + q + \frac{1}{2}$.

Then, from Proposition 2.1 and the standard criterion (20), we obtain the following cases:

(i) If $2(p + q)\lambda \neq 2q - 1$ then $\sigma u$ is a nonsymmetric second degree form of class $s = 1$. 

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(ii) If \( \lambda^{-1} = \frac{2(p + q)}{q + 1} \) then \((h_{- \frac{1}{2}} \circ \tau_{\frac{1}{2}}) \sigma u = \mathcal{J}(p - \frac{1}{2}, q - \frac{3}{2})\) the classical second degree forms. Indeed, in this case, we necessarily have \( p + q \neq 0 \). Then, for \((p, q) = (k, l + 1)\), we obtain the statement of Theorem 1.4.

From (27) and (28), the coefficients \( \{\beta_n^p, \gamma_{n+1}^p\}_{n \geq 0} \) of \( \{P_n\}_{n \geq 0} \) are

\[
\beta_0^p = \gamma_1, \quad \beta_{n+1}^p = \gamma_{2n+2} + \gamma_{2n+3}, \quad \gamma_{n+1}^p = \gamma_{2n+1}\gamma_{2n+2},
\]

where \( \gamma_n, n \geq 1 \) are given by (41) and \((\alpha, \beta) = (p - \frac{1}{2}, q - \frac{1}{2})\).

**Proposition 2.9** Let \( u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda) \), where \( p \) and \( q \) are integer numbers with \( p + q \geq 0 \). Then, the second degree form \( \sigma u \) satisfies

\[
\tilde{B}(z)S^2(\sigma u)(z) + \tilde{C}(z)S(\sigma u)(z) + \tilde{D}(z) = 0 \tag{62}
\]

with:

(1) For \( p \geq 0 \),

(i) if \( q \geq 0 \), then

\[
\begin{align*}
\tilde{B}(z) &= z^3(z - 1), \\
\tilde{C}(z) &= -2\mu z^3(z - 1)\tilde{X}(z) + 2\mu(z - 1)^{p+1}z^{q+2}, \\
\tilde{D}(z) &= \mu^2z^3(z - 1)\tilde{X}^2(z) - 2\mu^2(z - 1)^{p+1}z^{q+2}\tilde{X}(z) + \mu^2(z - 1)^{2p}z^{2q}(z(z - 1) - \lambda^2),
\end{align*}
\]

where

\[
\tilde{X}(z) = \left( (\sigma \mathcal{V}) \theta_0 \left( (\xi - 1)^p \xi^q \right) \right)(z), \quad \mu = \left( (\mathcal{V}, (x^2 - 1)^p x^{2q}) \right)^{-1}.
\]

(ii) if \( q \leq -1 \) and \( p + q \geq 0 \), then

\[
\begin{align*}
\tilde{B}(z) &= (z - 1)z^{-2q+3}, \\
\tilde{C}(z) &= z^{1-q}\left[ 2z^2(z - 1)\tilde{Y}(z) + 2\mu(z - 1)^{p+1} \right], \\
\tilde{D}(z) &= z^3(z - 1)\tilde{Y}^2(z) + 2\mu z^2(z - 1)^{p+1}\tilde{Y}(z) + \mu^2(z - 1)^{2p}(z(z - 1) - \lambda^2),
\end{align*}
\]

where

\[
\tilde{Y}(z) = \left( (\sigma \mathcal{U}) \xi^{-q-1} \right)(z) - \mu \left( (\sigma \mathcal{V}) \theta_0 \left( (\xi - 1)^p \right) \right)(z), \quad \mu = \left( \mathcal{U}, (x^2 - 1)^p \right).\]

(2) For \( p \leq -1 \) and \( q \geq 1 \) such that \( p + q \geq 0 \), we have

\[
\begin{align*}
\tilde{B}(z) &= z^3(z - 1)^{-2p+1}, \\
\tilde{C}(z) &= z(z - 1)^{-p}\left[ 2z^2(z - 1)\tilde{Z}(z) + 2\mu(z - 1)^{q+1} \right], \\
\tilde{D}(z) &= z^3(z - 1)\tilde{Z}^2(z) + 2\mu(z - 1)z^{q+2}\tilde{Z}(z) + \mu^2z^{2q}(z(z - 1) - \lambda^2),
\end{align*}
\]

where

\[
\tilde{Z}(z) = \left( (\sigma \mathcal{U}) \theta_0 \left( (\xi - 1)^{-p} \right) \right)(z) - \mu \left( (\sigma \mathcal{V}) \xi^{q+1} \right)(z), \quad \mu = \left( \mathcal{U}, (x^2 - 1)^{-p} \right).\]

**Proof** From Proposition 2.8, we notice that the polynomial coefficients of the second degree equation (16) satisfied by \( u = \mathcal{L}(p - \frac{1}{2}, q - \frac{1}{2}, \lambda) \) are such that \( B \) and \( D \) are even and \( C \) is odd. Then, there exist \( B^e, C^o \) and \( D^e \) such that

\[
B(z) = B^e(z^2), \quad C(z) = zC^o(z^2), \quad D(z) = D^e(z^2).
\]

From (8) and the fact that \( u \) is a symmetric form, we have

\[
S(u)(z) = zS(\sigma u)(z^2). \tag{67}
\]
Substituting (66) and (67) in (16) and making a change of variable \( z^2 \rightarrow z \), we get (62) with,

\[
\begin{align*}
\bar{B}(z) &= zB^*(z), \\
\bar{C}(z) &= zC^*(z), \\
\bar{D}(z) &= D^*(z).
\end{align*}
\] (68)

From (2), (6) and (11), we easily prove that for a symmetric form \( w \), we have

\begin{equation}
(w\theta_0 f(\xi^2))(z) = z((\sigma w)\theta_0 f)(z^2), \quad f \in \mathcal{P}.
\end{equation} (69)

Hence, the desired result is obtained by using (69) and the expressions of \( B, C \) and \( D \) given in the three different cases of Proposition 2.8.

\[ \square \]

**Integral representation**

From (58)–(59), we get

\[
\langle \sigma u, f(x) \rangle = \langle u, f(x^2) \rangle = \left( 1 - \frac{2\lambda(p+q)}{2q-1} \right) f(0) + 2\lambda \frac{\Gamma(p+q+1)}{\Gamma(p+\frac{1}{2})\Gamma(q+\frac{1}{2})} \int_0^1 \frac{x^{2q-2}(1-x^2)^p}{\sqrt{1-x^2}} f(x^2) dx.
\]

Then, we obtain after a change of variables

\begin{equation}
\langle \sigma u, f \rangle = \left( 1 - \frac{2\lambda(p+q)}{2q-1} \right) f(0) + \lambda \Gamma(p+q+1) \int_0^1 \frac{x^{q-1}(1-x)^p}{\sqrt{x}(1-x)} f(x) dx, \quad p, q \in \mathbb{N}, q \neq 0.
\end{equation} (70)

Notice that this form is a particular case of the so-called Koornwinder linear functionals (see [8]).

**Remark**  Thanks to Proposition 2.2, we carry out the complete description of the symmetric second degree semiclassical forms of class \( s = 2 \) when \( \Phi(0) = 0 \). Unfortunately, the case when \( \Phi(0) \neq 0 \) is not covered by this Proposition and the description of these forms remains open.

Notice that this last set is not empty. Indeed, let us define the normalized form \( \mathcal{W} \) by \( \mathcal{W} = \mathcal{U} + \lambda \delta_1 + \lambda \delta_{-1}, \lambda \in \mathbb{C} - \{0\} \) where \( \mathcal{U} \) is a Tchebychev form of second kind. This form is symmetric and semiclassical of class \( s = 2 \) satisfying (19) with \( \Phi(x) = (x^2 - 1)^2 \) and \( \Psi(x) = -5x(x^2 - 1) \). It is a particular case of the so-called Koornwinder linear functionals (see [6,8] and [9] for more information).

Moreover, it is well known that \( \mathcal{U} \) is a second degree form verifying the quadratic equation (see [11])

\begin{equation}
S^2(\mathcal{U})(z) + 4zS(\mathcal{U})(z) + 4 = 0.
\end{equation} (71)

From \((\mathcal{W})_{2n} = (\mathcal{U})_{2n} + 2\lambda, (\mathcal{W})_{2n+1} = 0, n \geq 0, \) we get \( S(\mathcal{U})(z) = S(\mathcal{W})(z) + \frac{2\lambda z}{z^2 - 1} \). Then, substituting in (71), we obtain after multiplying by \( (z^2 - 1)^2 \)

\[(z^2 - 1)^2S^2(\mathcal{W})(z) + 4z(z^2 - 1)(z^2 + \lambda - 1)S(\mathcal{W})(z) + 4(2\lambda + 1)z^4 + 4(\lambda^2 - 2\lambda - 2)z^2 + 4 = 0.\]

Hence, \( \mathcal{W} \) is a symmetric second degree semiclassical form of class \( s = 2 \) satisfying (19) with \( \Phi(0) \neq 0 \).

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