Rankin-Cohen brackets for orthogonal Lie algebras and bilinear conformally equivariant differential operators

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Dedicated to Professor Toshi Kobayashi on the occasion of his birthday, with admiration.

Abstract
Based on the Lie theoretical methods of algebraic Fourier transformation, we classify in the case of generic values of inducing parameters the scalar singular vectors corresponding to the diagonal branching rules for scalar generalized Verma modules in the case of orthogonal Lie algebra and its conformal parabolic subalgebra with commutative nilradical, thereby realizing the diagonal branching rules in an explicit way. The complicated combinatorial structure of singular vectors is conveniently determined in terms of recursion relations for the generalized hypergeometric function $\genfrac{3}{2}{2}$. As a geometrical application, we classify bilinear conformally equivariant differential operators acting on homogeneous line bundles on the flag manifold given by conformal sphere $S^n$.

Key words: Generalized Verma modules, Diagonal branching rules, Rankin-Cohen brackets, Bilinear conformally equivariant differential operators.
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1 Introduction and Motivation
The subject of our article has its motivation in the Lie theory for finite dimensional simple Lie algebras applied to the problem of branching rules and composition structure of generalized Verma modules, and dually in the geometrical situation related to the construction of equivariant bilinear differential operators or Rankin-Cohen-like brackets associated to orthogonal groups.

The classical Rankin-Cohen brackets realized by holomorphic $SL(2, \mathbb{R})$-equivariant bilinear differential operators on the upper half plane $\mathbb{H}$ are devised, originally in a number theoretic context, to produce from a given
pair of modular forms another modular form. They turn out to be intertwining operators responsible for the ring structure on $SL(2, \mathbb{R})$ holomorphic discrete series representations, and can be analytically continued to the full range of inducing characters. Consequently, such operators were constructed by different techniques in several specific situations of interest related to Jacobi forms, Siegel modular forms, holomorphic discrete series of causal symmetric spaces of Cayley type, real symmetric pairs of split rank one, etc., cf. [8], [11], [21], [14].

The main result of the present article is the classification of Rankin-Cohen conformally equivariant bilinear differential (bidifferential) operators acting on sections of homogeneous line bundles over the conformal sphere $S^n$. These operators can be regarded as projectors onto irreducible summands in the decomposition of the tensor product of two particular representations of $SO_0(n + 1, 1, \mathbb{R})$. Our approach to this geometrical problem for finding equivariant bilinear differential operators is based on its conversion to a Lie algebraic problem of the characterization of homomorphisms of generalized Verma modules. Based on the techniques of algebraic Fourier transform for generalized Verma modules (F-method) recently initiated in [12], [13], [14], we solve this problem in the case of orthogonal Lie algebras and their conformal (commutative) parabolic subalgebras. The finer questions related to the description of composition series for special values of infinitesimal characters or the complete classification of all solutions for the system of PDEs via F-method are postponed to a future research.

An abstract description of the underlying multiplicity-free branching problem in question is known, cf. [15]. Our intention is the computation of precise positions of submodules for the branching subgroup in the whole representation space, whose explicit knowledge is required in many applications like geometric analysis on manifolds, number theory, etc.

There are various approaches to the questions discussed in our short article. For example, there is an analytic approach consisting of meromorphic continuation of invariant distributions given by a multilinear form on the principal series representations. A class of $SO_0(n + 1, 1, \mathbb{R})$ (i.e., conformally)-covariant linear and bilinear differential operators was realized in residues of meromorphically continued invariant trilinear form on principal series representations induced from characters, see e.g., [2]. In the geometrical context of generalization towards curved manifolds with parabolic structure, a classification of first order equivariant bilinear differential operators for parabolic subalgebras with commutative nilradicals (AHS structures) was completed in [19]. As for a geometrical application of conformally covariant bilinear differential operators, see [7]. We remark that the appearance of a higher hypergeometric functions in our main Theorem 4.1 parallels the results in [14] on Rankin-Cohen bracket for real split rank one symmetric spaces, e.g. the case of $(U(n, 1) \times U(n, 1), U(n, 1))$ corresponding to the diagonal embedding of complex projective space $\mathbb{CP}^n \hookrightarrow \mathbb{CP}^n \times \mathbb{CP}^n$ where the differential operators in concern are given by substituting differential operators into appropriate hypergeometric function $_2F_1$ (Jacobi and Gegenbauer polynomials.)

The structure of our article goes as follows. In Section 2, we reformulate the question of existence for equivariant bilinear differential operators
(the Rankin-Cohen brackets) in terms of purely abstract Lie theoretic classification scheme for diagonal branching rules of generalized Verma modules, and apply it to the case of real orthogonal Lie algebras \( so(n+1,1,\mathbb{R}) \) and their conformal parabolic Lie subalgebras \( \mathfrak{p} \). We review some necessary technical background, and discuss the abstract (or, qualitative) part of the branching problem taking its value in the Grothendieck group \( K(\mathcal{O}^p) \) of the Bernstein-Gelfand-Gelfand parabolic category \( \mathcal{O}^p \). We also review the procedure of algebraic Fourier transformation applied to generalized Verma modules, which will be used in Section 3. The main theme of our article is the quantitative part of the branching problem, see Section 3, which consists of the construction of a class of (scalar valued) singular vectors. Our approach to analyze singular vectors is based on the procedure of F-method applied to generalized Verma modules, where the action of positive nilradical of \( so(n+1,1,\mathbb{R}) \) reduces to a four term functional equation for the coefficients of singular vectors. To solve it is technically the most difficult part, including both analytic and combinatorial aspects of generalized hypergeometric functions \( _3F_2 \). The results of Sections 2, 3 are applied in Section 4 to produce classification and explicit formulas of bilinear conformally covariant differential operators corresponding to (scalar valued) singular vectors.

There are several ways allowing to produce the results equivalent to ours. For example, [20] is based on the construction of transvectants. [4] and [5] develop the (meromorphic continuation of) conformally invariant trilinear forms, or even the construction of (explicit formulas are of low order only) curved analogues on a Riemannian manifold via conformally invariant ambient metric construction in [3]. It follows from our construction via F-method the completeness of the constructed set, a property which is not automatic in the other approaches. In addition, our results do agree with those appearing in the above mentioned references. Moreover, an additional effort can be used to construct the lifts to homomorphisms of semi-holonomic Verma modules in conformal geometry, which then give the curved version of our construction parallel to the results in [3].

Throughout the article we denote by \( \mathbb{N} \) the set of natural numbers including zero.

## 2 F-method and diagonal branching problem for generalized Verma modules

In the present section we briefly review basic notations and results initiated and developed in [16], [14], [12], [13], [15], allowing in an explicit way to realize the diagonal branching problem for real orthogonal Lie algebras and their conformal parabolic subalgebras.

We denote by \( G_\mathbb{R} \) a connected real reductive Lie group with real Lie algebra \( \mathfrak{g}_\mathbb{R} \). \( P_\mathbb{R} \subset G_\mathbb{R} \) a parabolic subgroup and \( \mathfrak{p}_\mathbb{R} \) its Lie algebra, \( \mathfrak{p}_\mathbb{R} = \mathfrak{l}_\mathbb{R} \oplus \mathfrak{n}_\mathbb{R} \) the Levi decomposition of \( \mathfrak{p}_\mathbb{R} \) and \( \mathfrak{n}_\mathbb{R}^{-} \) its opposite nilradical, \( \mathfrak{g}_\mathbb{R} = \mathfrak{n}_\mathbb{R}^{-} \oplus \mathfrak{p}_\mathbb{R} \). Given a complex finite dimensional \( P_\mathbb{R} \)-module \( V \), we consider the induced representation \( (\pi, \text{Ind}_{P_\mathbb{R}}^{G_\mathbb{R}}(V)) \) of \( G_\mathbb{R} \) on the space of smooth complex valued sections of the homogeneous vector bundle \( G_\mathbb{R} \times_{P_\mathbb{R}} V \).
We denote the complexification of a given real Lie algebra or group by
\[ V \] with
\[ V = \{ f \in C^\infty(G, \mathbb{R}) \mid f(\mathbf{g} \cdot p) = p^{-1} \cdot f(\mathbf{g}), \mathbf{g} \in G, p \in P_k \}. \tag{1} \]

We denote the universal enveloping algebra of \( \mathfrak{g} \) as \( \mathcal{U}(\mathfrak{g}) \), where the space of \( \mathcal{U}(\mathfrak{g}) \)-invariant natural pairing
\[ \mathfrak{g} \times \mathcal{U}(\mathfrak{g}) \to \mathbb{C}, \tag{3} \]
where the space of \( \mathcal{U}(\mathfrak{g}) \)-equivariant differential operators \( \text{Ind}^{G_k}_{P_k}(V) \to \text{Ind}^{G_k}_{P_k}(W) \) is bijective to the space of \( (\mathfrak{g}, P_k) \)-homomorphisms \( \mathcal{M}_p(V) \to \mathcal{M}_p(W) \).

Recall that homomorphisms of generalized Verma modules are determined by their singular vectors.

A generalization of the previous framework is based on two compatible pairs of real Lie groups \((G_k, P_k)\) and \((G_k', P_k')\), where \( G_k' \subset G_k \) is a real reductive subgroup of \( G_k \) and \( P_k' = P_k \cap G_k' \) is compatible parabolic subgroup of \( G_k \). The Lie algebras of \( G_k, G_k' \) are denoted by \( \mathfrak{g}_k, \mathfrak{g}_k' \), \( \mathfrak{n}_k := \mathfrak{n} \cap \mathfrak{g}_k \) is the nilradical of \( \mathfrak{p}_k \), and \( L_k = L_k' \cap G_k' \) is the Levi subgroup of \( P_k' \). As in the previous paragraph, omitting the subscript \( \mathbb{R} \) denotes the complexification of real Lie algebra or group. Therefore, an irreducible \( L_k' \)-submodule \( W \) of \( \mathcal{M}_p(V) \) of
\[ \mathcal{M}_p(V) \nabla := \{ v \in \mathcal{M}_p(V) \mid Z \cdot v = 0 \text{ for all } Z \in \mathfrak{n}' \} \tag{4} \]
with the standard left action of \( Z \) on the generalized Verma module, gives a \( \mathcal{U}(\mathfrak{g}') \)-homomorphism \( \mathcal{M}_p'(W) \to \mathcal{M}_p(V) \).

The whole procedure of the F-method to find explicit singular vectors may be divided into the following three main steps:

**Step 1.** Computation of the infinitesimal action \( \mathfrak{d}_\pi(X) \) for \( X \in \mathfrak{n}_k \) on a chosen principal series representation \((\pi, \text{Ind}^{G_k}_{P_k}(V))\) of \( G_k \), realized in the non-compact picture. The induced representation \( \mathfrak{d}_\pi \) defines, by its restriction, the representation of \( \mathfrak{g} \) on the space \( C^\infty(N_{\mathfrak{g}_k}, V) \).

**Step 2.** Computation of the dual infinitesimal action \( \mathfrak{d}_\pi^\vee(X) \) for \( X \in \mathfrak{n}_k \) on the dual space \( \mathcal{D}_0'(N_{\mathfrak{g}_k}, V) \) of distributions on \( N_{\mathfrak{g}_k} \) with values in \( V \) and supported in the unit \( P_k \)-coset \([o]\). Here we recall that we realize \( N_{\mathfrak{g}_k} \) as the open Bruhat cell \( N_{\mathfrak{g}_k} = N_{\mathfrak{g}_k} \cdot P_k \) of the flag manifold \( G_k/P_k \). The Lie algebra \( \mathfrak{g} \) acts on the space of vector valued distributions by the dual action \( \mathfrak{d}_\pi^\vee \):

\[ \mathfrak{d}_\pi^\vee(X)(T)(f) = -T(\mathfrak{d}_\pi(X)(f)), \quad X \in \mathfrak{g}, f \in C^\infty(N_{\mathfrak{g}_k}, V). \]

This space is isomorphic with \( \mathcal{M}_p'(W) \) as \( \mathfrak{g} \)-modules by linear map
\[ \phi : \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V \to \mathcal{D}_0'(N_{\mathfrak{g}_k}, V) \]
determined by
\[ \phi(u \otimes v^\vee) : f \mapsto \langle v^\vee, (d\pi(u^\omega)f)(o) \rangle \]
with the map \( u \mapsto u^\omega \) given by the antiautomorphism of \( U(\mathfrak{g}) \) acting as \( X \mapsto -X \) on \( \mathfrak{g} \).

**Step 3.** Let us identify \( \mathfrak{n}_{\mathbb{R}-} \) with \( N_{\mathbb{R}-} \) by the exponential map. Based on the convention introduced in [12], the Fourier transform gives an isomorphism
\[ \mathcal{F} : D'[0](\mathfrak{n}_{\mathbb{R}-}) \overset{\sim}{\longrightarrow} \text{Pol}[n], \quad T \mapsto \mathcal{F}(T) \]  
defined by
\[ \mathcal{F}(T)(\xi) = T_x(e^{i\langle x, \xi \rangle}), \quad \text{for } x \in \mathfrak{n}_{\mathbb{R}-}, \xi \in \mathfrak{n}, \]
where \( \langle x, \xi \rangle \) is given by the Killing form on \( \mathfrak{g} \) and \( i \in \mathbb{C} \) denotes the complex unit. The isomorphism (5) can be extended to distributions with values in \( V^\vee \) by
\[ \mathcal{F} \otimes \text{Id}_{V^\vee} : D'[0](\mathfrak{n}_{\mathbb{R}-}, V^\vee) \rightarrow \text{Pol}[n] \otimes V^\vee. \]
The Fourier transform \( \mathcal{F} \otimes \text{Id}_{V^\vee} \) can then be used to define the action \( \tilde{d}\pi^\vee \) of \( \mathfrak{g} \) on the space \( \text{Pol}[n] \otimes V^\vee \) by
\[ \tilde{d}\pi^\vee = (\mathcal{F} \otimes \text{Id}_{V^\vee}) \circ d\pi^\vee \circ (\mathcal{F}^{-1} \otimes \text{Id}_{V^\vee}), \]
where the elements of \( \mathfrak{n} \) act by differential operators of second order as \( \mathfrak{n} \) is commutative. We set \( \varphi := \phi^{-1} \circ (\mathcal{F}^{-1} \otimes \text{Id}_{V^\vee}) \). Then \( \varphi \) gives a bijection
\[ \varphi : \text{Pol}[n] \otimes V^\vee \overset{\sim}{\longrightarrow} U(\mathfrak{g}) \otimes_{U(p)} V^\vee. \]  

To summarize, the algebraic Fourier transform (the F-method) for generalized Verma modules allows to convert the explicit form of the action \( (d\pi^\vee)(Z) \) to a system of partial differential equations \( (\tilde{d}\pi^\vee)(Z) \), i.e., it transforms the initial algebraic and combinatorial problem of computation of singular vectors in generalized Verma modules into analytic problem of solving a system of PDEs. In particular, we introduce
\[ \text{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee) := \{ f \in \text{Pol}[n] \otimes V^\vee : \tilde{d}\pi^\vee(Z)f = 0 \text{ for any } Z \in \mathfrak{n}' \}, \]  
so the inverse Fourier transform gives an \( L'_K \)-isomorphism
\[ \text{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee) \overset{\sim}{\longrightarrow} M^0_\phi(V^\vee)^{\mathfrak{n}'}. \]  
An explicit form of the action \( \tilde{d}\pi^\vee(Z) \) leads to a system of differential equations for elements in \( \text{Sol} \), allowing to describe its structure completely in some particular cases of interest (cf. [16] [14] [12].)

In the dual language of differential operators acting on principal series representations, the set of \( G'_\mathbb{R} \)-equivariant differential operators \( \text{Ind}^{G'_\mathbb{R}}_{G_{\mathbb{R}-}^0}(V) \rightarrow \text{Ind}^{G_{\mathbb{R}-}^0}_{G'_{\mathbb{R}-}^0}(V') \) is in bijective correspondence with the space of all \( (\mathfrak{g}', P'_{\mathbb{K}}) \)-homomorphisms \( M^0_\phi(V^\vee)^{\mathfrak{n}'} \rightarrow M^0_{\phi'}(V^\vee) \).

Now we describe the explicit choice of Lie algebras and representations to which we apply the previous general procedure, and we also discuss the
qualitative results on the underlying branching rule. Let $n \in \mathbb{N}$ be such that $n \geq 3$. In the rest of the article $\mathfrak{g}$ denotes complexification of the real Lie algebra $so(n + 1, 1, \mathbb{R})$ of the connected simple real Lie group $G_{\mathbb{R}} = SO_{\mathbb{R}}(n + 1, 1, \mathbb{R})$, and $\mathfrak{p}$ its maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{I} \ltimes \mathfrak{n}$, in the Dynkin diagrammatic notation for parabolic subalgebras given by omitting the first simple root of $\mathfrak{g}$. The Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ is isomorphic to complexification of $so(n, \mathbb{R}) \times \mathbb{R}$ and the commutative nilradical $\mathfrak{n}$ (resp., the opposite nilradical $\mathfrak{n}_-$) is isomorphic to $\mathbb{C}^n \simeq \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C}_{\mathbb{R}}$.

As for the matrix realization of this decomposition, we recall the Langlands decomposition $P_{\mathbb{R}} = L_{\mathbb{R}} N_{\mathbb{R}} = M_{\mathbb{R}} A_{\mathbb{R}} N_{\mathbb{R}}$. The group $M_{\mathbb{R}}$ is isomorphic to $SO_{\mathbb{R}}(n, \mathbb{R})$, and acts on $n_{\mathbb{R}} \simeq \mathbb{R}^n$ by its fundamental vector representation preserving the quadratic form $\sum_{i=1}^n x_i^2$. The group $A_{\mathbb{R}}$ is given by

$$A_{\mathbb{R}} = \{ \begin{pmatrix} a & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} | a \in \mathbb{R}^+ \} \simeq \mathbb{R}^+.$$  \hspace{1cm} (9)

Then the elements $p \in P_{\mathbb{R}}$ are given by block triangular matrices

$$p = \begin{pmatrix} a & * & * \\ 0 & m & * \\ 0 & 0 & a^{-1} \end{pmatrix}$$  \hspace{1cm} (10)

with $a \in \mathbb{R}^+$, $m \in SO(n, \mathbb{R})$. Let $\{E_j\}_{j=1, \ldots, n}$ and $\{E_{-j}\}_{j=1, \ldots, n}$ be the standard basis of root vectors in $n_{\mathbb{R}}$ and $n_{\mathbb{R}}^-$, respectively, given by

$$E_j = \begin{pmatrix} 0 & e_j & 0 \\ 0 & 0 & -e_j \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-j} = \begin{pmatrix} 0 & 0 & 0 \\ e_j & 0 & 0 \\ 0 & -e_j & 0 \end{pmatrix}, \quad 1 \leq j \leq n.$$  \hspace{1cm} (11)

We identify $n_{\mathbb{R}}^-$ with $\mathbb{R}^n$ and likewise $n_{\mathbb{R}}$ with $\mathbb{R}^n$ by using coordinates:

$$n_{\mathbb{R}} \simeq \{ Z : Z = (z_1, \ldots, z_n) \}, \quad n_{\mathbb{R}}^- \simeq \{ X : X^t = (x_1, \ldots, x_n) \}.$$  

Then we have for elements in $N_{\mathbb{R}}$ and $N_{\mathbb{R}}^-$

$$n = \exp Z = \begin{pmatrix} 1 & Z & -\frac{1}{2}Z^2 \\ 0 & \text{Id} & -Z^t \\ 0 & 0 & 1 \end{pmatrix} \in N_{\mathbb{R}},$$  \hspace{1cm} (12)

$$x = \exp X = \begin{pmatrix} 1 & 0 & 0 \\ X & \text{Id} & 0 \\ -\frac{1}{2}X^2 & -X^t & 1 \end{pmatrix} \in N_{\mathbb{R}^-},$$

where we set $|X|^2 := X^t X$ and $|Z|^2 := ZZ^t$. By slight abuse of notation, we write $p = man$ for (10).

Then the case of our interest is given by pairs of complexified Lie algebras

$$(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{p} \oplus \mathfrak{p}), \quad \text{diag}(\mathfrak{g}, \mathfrak{p}) = (\text{diag}(\mathfrak{g}), \text{diag}(\mathfrak{p})),$$  \hspace{1cm} (13)

where diag denotes the diagonal embedding, and the main task of the present article concerns the branching problem for the family of scalar...
generalized Verma $\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g})$-modules induced from characters of $\mathfrak{p} \oplus \mathfrak{p}$. In particular, we shall classify (out of a discrete subset of inducing parameters) the scalar-valued singular vectors in scalar generalized Verma modules for orthogonal Lie algebras, i.e., the singular vectors transforming in the trivial representation of the simple part of Levi factor $\mathfrak{l}$.

Let $\mathbb{C}_{\chi}(\mathbb{C}_{\mu})$ denote the one-dimensional representation of $P_{\mathfrak{p}}$ given by $p = man \mapsto a^\chi$ ($p = man \mapsto a^\mu$), and $\chi_{\lambda} : p_{\mathfrak{p}} \to \mathbb{C}_{\lambda}$ ($\chi_{\mu} : p_{\mathfrak{p}} \to \mathbb{C}_{\mu}$) its differential given by multiplication by $\lambda$ ($\mu$, respectively). An inducing character $\chi_{\lambda,\mu}$ of $\mathfrak{p} \oplus \mathfrak{p}$ is determined by two complex characters $\chi_{\mu}, \chi_{\lambda}$, $\lambda, \mu \in \mathbb{C}$,

$$\chi_{\lambda,\mu} \equiv (\chi_{\lambda} : \mathfrak{p} \oplus \mathfrak{p} \to \mathbb{C}, (p_1, p_2) \mapsto \chi_{\lambda}(p_1) \otimes \chi_{\mu}(p_2) \in \text{End}(\mathbb{C}_{\lambda} \otimes \mathbb{C}_{\mu}),$$

and the generalized Verma $\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g})$-module induced from character $(\chi_{\lambda}, \chi_{\mu})$ is

$$\mathcal{M}_{p \oplus p}^{\mathfrak{g}}(\mathbb{C}_{\lambda} \otimes \mathbb{C}_{\mu}) \equiv \mathcal{M}_{\lambda,\mu}(\mathfrak{g} \oplus \mathfrak{g}, p \oplus p) \equiv \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \otimes_{\mathcal{U}(p \oplus p)} (\mathbb{C}_{\lambda} \otimes \mathbb{C}_{\mu}),$$

where $\mathbb{C}_{\lambda} \otimes \mathbb{C}_{\mu}$ is a 1-dimensional representation $(\chi_{\lambda}, \chi_{\mu})$ of $\mathfrak{p} \oplus \mathfrak{p}$. As a vector space, $\mathcal{M}_{\lambda,\mu}(\mathfrak{g} \oplus \mathfrak{g}, p \oplus p)$ is isomorphic to the symmetric algebra $S(n_n \oplus n_n)$, where $n_n \oplus n_n$ is the complement of $p \oplus p$ in $\mathfrak{g} \oplus \mathfrak{g}$. Notice that we have an isomorphism

$$(n_n \oplus n_n)/(n_n \oplus n_n \cap \text{diag}(\mathfrak{g})) \simeq n_n$$
of diag(1)-quotient modules.

The symmetric algebra $S((n_n \oplus n_n)/(n_n \oplus n_n \cap \text{diag}(\mathfrak{g})))$ decomposes as diag(1)-module on irreducible submodules, with higher multiplicities in general. In particular, each diag(1)-module realized in homogeneity $k$ polynomials also appears in polynomials of homogeneity $(k + 2)$, $k \in \mathbb{N}$. As we have already explained, we focus on the case of 1-dimensional representations $\mathcal{V}_{\lambda,\mu} \simeq \mathbb{C}_{\lambda} \otimes \mathbb{C}_{\mu}$ regarded as $(\mathfrak{p} \oplus \mathfrak{p})$-modules with the trivial action of the simple part of Levi subalgebra and the nilradical and $\mathcal{V}_{\lambda,\mu} \simeq \mathbb{C}_{\mu}$ as diag(1)-modules $(\lambda, \mu, \nu \in \mathbb{C})$, and it is a result in classical invariant theory (see [10], [15]) that for each even homogeneity there is just one 1-dimensional module. Because $n_n$ is as $(\text{diag}(1)/\text{diag}(1), \text{diag}(1))$-module isomorphic to the character $\mathbb{C}_{-1}$, the following holds true in the Grothendieck group of $\mathcal{O}^n$, $\mathfrak{p} \simeq \text{diag}(\mathfrak{p})$. As a consequence of [15], Theorem 3.9, we have

**Corollary 2.1** Let

$$\mathfrak{g} \oplus \mathfrak{g} = so(n + 2, \mathbb{C}) \oplus so(n + 2, \mathbb{C}), \quad \text{diag}(\mathfrak{g}) \simeq so(n + 2, \mathbb{C})$$

with standard maximal parabolic subalgebras $\mathfrak{p} \oplus \mathfrak{p}$, diag($\mathfrak{p}$) given by omitting the first simple root in the corresponding Dynkin diagram.

Then the multiplicity $m(\nu, (\lambda, \mu))$ of $\mathcal{M}_{\nu}(\mathbb{C}_{\nu}) \equiv \mathcal{M}_{\lambda,\mu}(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{p} \oplus \mathfrak{p})$ is equal to one for $\nu = \lambda + \mu - 2j$, $j \in \mathbb{N}$, and zero otherwise. In the Grothendieck group $K(\mathcal{O}^n)$ of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^n$ holds

$$\mathcal{M}_{\lambda,\mu}(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{p} \oplus \mathfrak{p})|_{\text{diag}(\mathfrak{g})} \simeq \bigoplus_{j \in \mathbb{N}} \mathcal{M}_{\lambda + \mu - 2j}(\mathfrak{g}, \mathfrak{p}).$$

(16)
Although we work in one specific signature \((n+1,1)\), the results are easily extended to any real form of arbitrary signature.

3 The construction of singular vectors for diagonal branching rules applied to scalar generalized Verma modules for \(so(n+1, 1, \mathbb{R})\)

The rest of the article is devoted to the construction of a class of scalar valued singular vectors, whose abstract existence was concluded in Section 2, Corollary 2.1, using the tool of algebraic Fourier transform (F-method) for generalized Verma modules reviewed in Section 2. This can be regarded as a quantitative part of our diagonal branching problem.

3.1 Description of the representation

In this subsection we describe the representation of \(g \oplus g\) on scalar generalized Verma modules

\[
\mathcal{M}_{\lambda,\mu}(g \oplus g, p \oplus p) = \mathcal{U}(g \oplus g) \otimes \mathcal{U}(p \oplus p) \mathcal{C}_{\lambda,\mu} \simeq \mathcal{M}_\lambda(g, p) \otimes \mathcal{M}_\mu(g, p) \quad (17)
\]

for \(\mathcal{C}_{\lambda,\mu} = \mathcal{C}_\lambda \otimes \mathcal{C}_\mu\) in its Fourier image, i.e., we apply the F-method explained in Section 2. The first goal is to describe the action by elements in the nilradical \(\text{diag}(n)\) of \(\text{diag}(p)\) in terms of differential operators acting on the Fourier image of \(\mathcal{M}_{\lambda,\mu}(g \oplus g, p \oplus p)\). It can be derived from the explicit form of the action on the induced representation realized in the non-compact picture, and it follows from (17) that the problem can be reduced to the question on each component of the tensor product separately. Let us consider the complex representation \(\pi_\lambda\) of \(G_\mathbb{R} = \text{SO}_o(n+1, 1, \mathbb{R})\) on \(\text{Ind}_{G_\mathbb{R}}^{G_\mathbb{R}}(\mathcal{C}_\lambda)\), \(\lambda \in \mathbb{C}\), induced from the character \(p = \text{man} \mapsto a^\lambda, p \in P_\mathbb{R}\), on one dimensional representation space \(\mathbb{C}_\lambda \simeq \mathbb{C}\).

Here \(a \in A_\mathbb{R}\) is the abelian subgroup in the Langlands decomposition \(P_\mathbb{R} = M_\mathbb{R}A_\mathbb{R}N_\mathbb{R}\), \(M_\mathbb{R} \simeq \text{SO}(n, \mathbb{R})\), \(N_\mathbb{R} \simeq \mathbb{R}^n\). The character of \(P_\mathbb{R}\) is trivial on \(M_\mathbb{R}\) and \(N_\mathbb{R}\), and its value on \(a \in A_\mathbb{R} \simeq \mathbb{R}^+\) is \(a^\lambda\) (i.e., it is a complex character of \(A_\mathbb{R}\) on \(\mathbb{C}_\lambda\)).

Let \(x_j, j = 1, \ldots, n\), be the coordinates with respect to the standard basis \(\{E_i\}_{j=1,\ldots,n}\) of root vectors on \(n_-\), and \(\xi_j, j = 1, \ldots, n\), the coordinates on the Fourier transform of \(n_-\). We consider the family of differential operators

\[
Q_j(\lambda) = -\frac{1}{2}|x|^2 \partial_j + x_j(-\lambda + \sum_k x_k \partial_k), \quad j = 1, \ldots, n, \quad (18)
\]

\[
P^E_j(\lambda) = i \left(\frac{1}{2} \xi_j \Delta^E + (\lambda - \bar{\xi}^E) \partial_{\xi_j}\right), \quad j = 1, \ldots, n, \quad (19)
\]

where \(|x|^2 = x_1^2 + \cdots + x_n^2\),

\[
\Delta^E = \partial_{\xi_1}^2 + \cdots + \partial_{\xi_n}^2
\]
is the Laplace operator of positive signature, \( \partial_j = \frac{\partial}{\partial x_j} \) and \( \nabla^\xi = \sum_k \xi_k \partial \xi_k \) is the Euler homogeneity operator (\( i \in \mathbb{C} \) denotes the complex unit.) Via the exponential map, the non-compact picture of the induced representation \( \text{Ind}_{G^R}^G(C, \lambda) \) is given by

\[ C^\infty(n, \mathbb{C}) \cong C^\infty(n, \mathbb{C}) \otimes C_\lambda. \]

The following result is a routine computation, cf. [12]:

**Lemma 3.1** Let us denote by \( E_j \in n \) the standard basis elements, \( j = 1, \ldots, n \). Then

\[ d\pi_\lambda(E_j)(s \otimes v) = Q_j(\lambda)(s) \otimes v, \quad s \in C^\infty(n, \mathbb{C}), \quad v \in C_\lambda, \quad (20) \]

and the action of \( d\pi^\lambda_\vee \) on \( \text{Pol}[\xi_1, \ldots, \xi_n] \otimes C_\lambda^\vee \) is given by

\[ d\pi^\lambda_\vee(E_j)(f \otimes u) = P^j(\lambda)(f) \otimes u, \quad f \in \text{Pol}[\xi_1, \ldots, \xi_n], \quad u \in C_\lambda^\vee. \quad (21) \]

As for the action of remaining basis elements of \( g \) in the Fourier image of the induced representation, the action of \( n_- \) is given by multiplication by coordinate functions, the standard basis elements of the simple part of the Levi factor \( l' = [l, l] \cong \text{so}(n, \mathbb{C}) \) realized by matrices

\[ M_{i,j} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_{ir} \delta_{js} - \delta_{is} \delta_{jr} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i, j, r, s = 1, \ldots, n, \quad i < j, \]

act by first order differential operators

\[ M^\xi_{i,j} = (\xi_i \partial_{\xi_j} - \xi_j \partial_{\xi_i}), \quad i, j = 1, \ldots, n \]

and the basis element of the Lie algebra of \( A_R \) given by the matrix

\[ E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

acts as the homogeneity operator, \( \nabla^\xi = \sum_{i=1}^n \xi_i \partial_{\xi_i} \).

We introduce the diagonal embedding of the Lie algebra \( g_R \) as the differential of the diagonal map for the Lie group \( G_R \)

\[ G_R \hookrightarrow G_R \times G_R, \quad g \mapsto (g, g), \]

given by

\[ \text{diag}(g_R) : X \mapsto X \otimes 1 + 1 \otimes X, \quad \text{for all} \quad X \in g_R \]

and termed diagonaly embedded Lie algebra of \( g_R \). This notion restricts to any Lie subalgebra of \( g_R \). Then the action of \( \text{diag}(g_R) \) in the representation \( d\pi^\lambda_\vee \otimes d\pi^\mu_\vee \) on \( \text{Pol}[\xi_1, \ldots, \xi_n] \otimes C_\lambda^\vee \otimes \text{Pol}[\nu_1, \ldots, \nu_n] \otimes C_\mu^\vee \), the algebra of complex valued polynomials on \( n \oplus n \) with coordinates \( \xi \) resp. \( \nu \) on the first resp. second copy of \( n \) in \( n \oplus n \), by particular generators of the diagonal subalgebra \( \text{diag}(g_R) \) is given by

9
1. The multiplication by

$$(\xi_j \otimes 1) + (1 \otimes \nu_j), \quad j = 1, \ldots, n, \quad (22)$$

for the elements of $\text{diag}(n_{\mathfrak{s} -})$.

2. The action by first order differential operators with linear coefficients

$$M_{ij}^{\xi, \nu} = (M_{ij}^\xi \otimes 1) + (1 \otimes M_{ij}^\nu) = (\xi_i \partial_{\xi_j} - \xi_j \partial_{\xi_i}) \otimes 1 + 1 \otimes (\nu_j \partial_{\nu_i} - \nu_i \partial_{\nu_j}), \quad (23)$$

$i, j = 1, \ldots, n$, for the elements of the simple part of $\text{diag}(n_{\mathfrak{s}})$ and

$$E^\xi \otimes 1 + 1 \otimes E^\nu = \sum_{i=1}^{n} (\xi_i \partial_{\xi_i} \otimes 1 + 1 \otimes \nu_i \partial_{\nu_i}) \quad (24)$$

for the generator of $\text{diag}(n_{\mathfrak{s}})/[\text{diag}(n_{\mathfrak{s}}), \text{diag}(n_{\mathfrak{s}})]$.

3. The action by second order differential operators with at most linear coefficients

$$P_{ij}^{\xi, \nu}(\lambda, \mu) = (P_{ij}^\xi(\lambda) \otimes 1) + (1 \otimes P_{ij}^\nu(\mu))$$

$$= i(\frac{1}{2} \xi_j \Delta^\xi + (\lambda - E^\xi) \partial_{\xi_j}) \otimes 1$$

$$+ 1 \otimes (\frac{1}{2} \nu_j \Delta^\nu + (\mu - E^\nu) \partial_{\nu_j}), \quad (25)$$

$j = 1, \ldots, n$ for the elements $\text{diag}(n_{\mathfrak{s}})$.

### 3.2 Reduction to scalar differential equation in two variables

It follows from the discussion in Section 2 that $\text{diag}(l)$-modules for the diagonal branching rules, inducing singular vectors for generalized scalar Verma modules, are one dimensional. This means that they are annihilated by $\text{diag}(l^*) \simeq \text{so}(n, \mathbb{C})$, the simple part of the diagonal Levi subalgebra $\text{diag}(l) \simeq \text{so}(n, \mathbb{C}) \times \mathbb{C}$. It follows that the singular vectors are invariants of $\text{diag}(l^*)$ acting diagonally on the algebra of polynomials on $n \oplus n$ regarded as a $l \oplus l$-module. The following result is a special case of the first fundamental theorem in classical invariant theory, see e.g. [10], [13].

**Lemma 3.2** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional complex vector space with bilinear form $\langle \cdot, \cdot \rangle$ and $SO(V)$ the Lie group of automorphisms of $(V, \langle \cdot, \cdot \rangle)$. Then the subalgebra $\text{Pol}[V \oplus V]^{SO(V)}$ of $SO(V)$-invariants in the complex polynomial algebra $\text{Pol}[V \oplus V]$ (with $SO(V)$ acting diagonally on $V \oplus V$) is polynomial algebra generated by $\langle \xi, \xi \rangle$, $\langle \xi, \nu \rangle$ and $\langle \nu, \nu \rangle$. Here we use the convention that $\xi$ is a vector in the first summand $V$ of the direct sum $V \oplus V$ and $\nu$ in the second summand. Therefore, there is an algebra isomorphism

$$\text{Pol}[V \oplus V]^{SO(V)} \simeq \text{Pol}[\langle \xi, \xi \rangle, \langle \xi, \nu \rangle, \langle \nu, \nu \rangle].$$
In our case, the complex polynomial algebra is $Pol[\xi_1, \ldots, \xi_n, \nu_1, \ldots, \nu_n]$ and we use the notation $Pol[r, s, t]$ for the subalgebra of invariants generated by

$$r := \langle \xi, \nu \rangle = \sum_{i=1}^{n} \xi_i \nu_i,$$

$$s := \langle \xi, \xi \rangle = \sum_{i=1}^{n} \xi_i^2,$$

$$t := \langle \nu, \nu \rangle = \sum_{i=1}^{n} \nu_i^2.$$  \hfill (26)

The task of the present subsection is to rewrite the operators $P^e_{\xi, \nu} (\lambda, \mu)$ in the variables $r, s, t$, i.e., to reduce the action of $P^e_{\xi, \nu} (\lambda, \mu)$ from the polynomial ring to the algebra of $\text{diag}(\Gamma^*)$-invariants on $n \oplus n$.

We compute

$$\partial_{\nu_i} r = \xi_i, \partial_{\xi_i} r = \nu_i, \partial_{\nu_i} s = 0, \partial_{\xi_i} s = 2 \xi_i, \partial_{\nu_i} t = 2 \nu_i, \partial_{\xi_i} t = 0,$$

and

$$\partial_{\xi_i} = \nu_i \partial_r + 2 \xi_i \partial_s, \Delta^r = t \partial_s^2 + 4 r \partial_s \partial_t + 2 n \partial_t + 4 s \partial_r^2, \text{ for } i = 1, \ldots, n.$$  \hfill (27)

Note that analogous formulas for $\partial_{\nu_i}, \Delta^\nu$ can be obtained from those for $\xi$ by applying the change of variables

$$\xi_i \longleftrightarrow \nu_i, \; s \longleftrightarrow t, \; r \longleftrightarrow r.$$  \hfill (29)

We also have for all $i = 1, \ldots, n$

$$E^r \partial_{\xi_i} = \nu_i (E^r + 2 E^s) \partial_r + \xi_i (2 E^r + 4 E^s + 2) \partial_s,$$

and so taking all together we arrive at the operators

$$P_i^{e, r, t} (\lambda, \mu) = \xi_i (\frac{1}{2} s \partial_r^2 + (n + 2 \lambda - 2 - 2 E^s) \partial_t + (E^r + 2 E^s) - \mu) \partial_r$$

$$+ \nu_i \frac{1}{2} s \partial_r^2 + (n + 2 \mu - 2 - 2 E^r) \partial_t - (E^r + 2 E^s - \lambda) \partial_s.$$  \hfill (30)

acting on complex polynomial algebra $Pol[r, s, t], i = 1, \ldots, n$. A suitable linear combination of this vector-valued system of equations ($i = 1, \ldots, n$) leads to operators

$$P_{\xi}^{e, r, t} (\lambda, \mu) := \sum_{i=1}^{n} \xi_i P_i^{e, r, t} (\lambda, \mu) = s (\frac{1}{2} t \partial_r^2 + (n + 2 \lambda - 2 - 2 E^s) \partial_t, - (E^r + 2 E^s - \mu) \partial_r)$$

$$- (E^r + 2 E^s - \mu) \partial_r \big) + r \frac{1}{2} s \partial_r^2 + (n + 2 \mu - 2 - 2 E^r) \partial_t - (E^r + 2 E^s - \lambda) \partial_r),$$

$$P_{\nu}^{e, r, t} (\lambda, \mu) := \sum_{i=1}^{n} \nu_i P_i^{e, r, t} (\lambda, \mu) = r (\frac{1}{2} t \partial_r^2 + (n + 2 \lambda - 2 - 2 E^s) \partial_t, - (E^r + 2 E^s - \lambda) \partial_r)$$

$$- (E^r + 2 E^s - \mu) \partial_r \big) + t \frac{1}{2} s \partial_r^2 + (n + 2 \mu - 2 - 2 E^r) \partial_t - (E^r + 2 E^s - \lambda) \partial_r).$$  \hfill (32)
Notice that the second equation follows from the first one by the action of involution

\[ \lambda \leftrightarrow \mu, \ s \leftrightarrow t, \ r \leftrightarrow r. \]

In what follows we construct a set of homogeneous polynomial solutions of \( P_{r,s,t}^\xi(\lambda, \mu) \), \( P_{r,s,t}^\nu(\lambda, \mu) \) solving the system \( \{ P_{r,s,t}^\xi(\lambda, \mu) \}_{i=1}^n \). The uniqueness of the solution for the generic values of the inducing parameters implies the unique solution of the initial system of PDEs \( \text{(31)} \). Notice that \( \text{(32)} \) is the system of differential operators preserving the space of homogeneous polynomials in the variables \( r, s, t \), i.e. \( P_{r,s,t}^\eta(\lambda, \mu) \) commute with \( E_{r,s,t} := E_r + E_s + E_t \).

### 3.3 Polynomial solutions of the differential equation in two variables produced by the F-method

We start with a couple of simple examples.

**Example 3.3** Let us consider the polynomial of homogeneity one,

\[ p(r, s, t) = Ar + Bs + Ct, \ A, B, C \in \mathbb{C}. \]

The application of \( P_{r,s,t}^\xi(\lambda, \mu) \) yields

\[ P_{r,s,t}^\xi(\lambda, \mu)(Ar + Bs + Ct) = \xi_i(B(n + 2\lambda - 2)A\mu + \nu_i(C(n + 2\mu - 2) + A\lambda)) \quad (33) \]

for all \( i = 1, \ldots, n \). When \( A \) is normalized to be equal to 1, we get

\[ C = -\frac{\lambda}{n + 2\mu - 2}, \ B = -\frac{\mu}{n + 2\lambda - 2}. \]

If \( n + 2\mu - 2 \neq 0 \) and \( n + 2\lambda - 2 \neq 0 \), there is a unique homogeneous solution of \( P_{r,s,t}^\xi(\lambda, \mu)p(r, s, t) = 0 \) for all \( i = 1, \ldots, n \) given by

\[ p(r, s, t) = (n + 2\lambda - 2)(n + 2\mu - 2)r - \mu(n + 2\mu - 2)s - \lambda(n + 2\lambda - 2)t. \quad (34) \]

**Example 3.4** Let

\[ p(r, s, t) = Ar^2 + Bs^2 + Ct^2 + Drs + Est + Frt, \ A, B, C, D, E, F \in \mathbb{C} \]

be a general polynomial of homogeneity two. We have

\[ P_{r,s,t}^\xi(\lambda, \mu)p(r, s, t) = \xi_i[r(D(n + 2\lambda - 2)A\mu - 1) + s(B2(n + 2\lambda - 4) + D\mu) + t(A + E(n + 2\lambda - 2) + F(\mu - 2))], \]

\[ + \nu_i[r(F(n + 2\mu - 2) + A2(\lambda - 1)) + s(A + E(n + 2\mu - 2) + D(\lambda - 2)) + t(C2(n + 2\mu - 4) + F\lambda)], \quad (35) \]

for all \( i = 1, \ldots, n \). The equations

\[ \sum_i \xi_i P_{r,s,t}^\xi(\lambda, \mu) = 0, \ \sum_i \nu_i P_{r,s,t}^\nu(\lambda, \mu) = 0 \]
are equivalent to two systems of linear equations:

\[
\begin{align*}
D(n + 2\lambda - 2) + A2(\mu - 1) + A + E(n + 2\mu - 2) + D(\lambda - 2) &= 0, \\
F(n + 2\mu - 2) + A2(\lambda - 1) &= 0, \\
B2(n + 2\mu - 4) + D\mu &= 0, \\
A + E(n + 2\lambda - 2) + F(\mu - 2) &= 0, \\
C2(n + 2\mu - 4) + F\lambda &= 0,
\end{align*}
\]

resp.

\[
\begin{align*}
D(n + 2\lambda - 2) + A2(\mu - 1) &= 0, \\
B2(n + 2\mu - 4) + D\mu &= 0, \\
A + E(n + 2\lambda - 2) + F(\mu - 2) + F(n + 2\mu - 2) + A2(\lambda - 1) &= 0, \\
A + E(n + 2\mu - 2) + D(\lambda - 2) &= 0, \\
C2(n + 2\mu - 4) + F\lambda &= 0.
\end{align*}
\]

Both systems are equivalent under the involution $A \leftrightarrow A$, $E \leftrightarrow E$, $D \leftrightarrow F$, $B \leftrightarrow C$, $\lambda \leftrightarrow \mu$, and for $n + 2\mu - 2 \neq 0, n + 2\lambda - 2 \neq 0, n + 2\mu - 4 \neq 0$ and $n + 2\lambda - 4 \neq 0$ there is a unique solution invariant under this involution

\[
\begin{align*}
A &= 1, \quad F = \frac{-2(\lambda - 1)}{n + 2\mu - 2}, \quad C = \frac{\lambda(\lambda - 1)}{(n + 2\mu - 2)(n + 2\lambda - 4)}, \\
E &= 2\left(\frac{(\lambda - 2)(\mu - 2)- (1 + \frac{n}{2})}{(n + 2\mu - 2)(n + 2\lambda - 2)}\right), \quad D = \frac{-2(\mu - 1)}{(n + 2\lambda - 2)}, \\
B &= \frac{\mu(\mu - 1)}{(n + 2\lambda - 2)(n + 2\mu - 4)}.
\end{align*}
\]

To conclude, in the case when $n + 2\mu - 2 \neq 0, n + 2\lambda - 2 \neq 0, n + 2\mu - 4 \neq 0$ and $n + 2\lambda - 4 \neq 0$, the polynomial

\[
p(r, s, t) = (n + 2\lambda - 2)(n + 2\mu - 2)(n + 2\mu - 4)\omega^2 + 2(\lambda - 2)(\mu - 2) - (1 + \frac{n}{2})(n + 2\lambda - 2)(n + 2\mu - 4)r
\]

\[
+ 2(\mu - 1)(n + 2\lambda - 4)(n + 2\mu - 4)s
\]

\[
- 2(\mu - 1)(n + 2\lambda - 4)(n + 2\mu - 4)t
\]

\[
- 2(\lambda - 1)(n + 2\lambda - 2)(n + 2\mu - 4)rt
\]

is the unique solution of $P_i^{r,s,t}(\lambda, \mu)p(r, s, t) = 0$ of homogeneity two.

We now return back to the case of general homogeneity. Let

\[
p = p(r, s, t) = \sum_{0 \leq i, j, k \leq N} A_{i, j} s^i t^j r^k
\]

be a homogeneous polynomial of degree $N$, $deg(p) = N$, and write

\[
p = r^N p(1, \frac{s}{r}, \frac{t}{r}) = r^N \tilde{p}(u, v), ~ u := \frac{s}{r}, v := \frac{t}{r},
\]

\[
\tilde{p}(u, v) = \sum_{0 \leq i, j \leq N} A_{i, j} u^i v^j,
\]

(41)
where \( \tilde{p} \) is a polynomial of degree \( N \). The dehomogenesation \( (r, s, t) \rightarrow (r, u, v) \) is governed by coordinate change

\[
u := \frac{s}{r}, \quad \lambda := \frac{t}{r}, \quad r := r,
\]

such that

\[
\partial_s \rightarrow \frac{1}{r} \partial_u, \quad \partial_t \rightarrow \frac{1}{r} \partial_v, \quad \partial_r \rightarrow -\frac{1}{r^2} u \partial_u - \frac{1}{r^2} v \partial_v + \partial_r,
\]

\[
E^u \rightarrow E^\nu, \quad E^t \rightarrow E^\lambda, \quad E^r \rightarrow -E^u - E^v + E^r.
\]

Then the particular terms in \( P^{r,s,t}_{\xi}(\lambda, \mu) \), see (42), transform as

\[
\frac{1}{2} s \partial_s^2 \rightarrow \frac{1}{2} uv(E^u + E^v - E^r + 1)(E^u + E^v - E^r),
\]

\[
-r(E^v + 2E^s - \lambda) \partial_r \rightarrow (E^u + E^v - E^r)(E^u - E^v + E^r - \lambda - 1),
\]

\[
r(n + 2\mu - 2 - 2E^v) \partial_r \rightarrow (n + 2\mu - 2 - 2E^v) \partial_v,
\]

\[
\frac{1}{2} n s \partial_s^2 \rightarrow \frac{1}{2} uv(E^u + E^v - E^r + 1)(E^u + E^v - E^r),
\]

\[
-s(E^v + 2E^s - \mu) \partial_r \rightarrow u(E^u + E^v - E^r)(-E^u + E^v + E^r - \mu - 1),
\]

\[
s(n + 2\lambda - 2 - 2E^u) \partial_u \rightarrow (n + 2\lambda - 2 - 2E^u) E_u,
\]

and when acting on a polynomial of homogeneity \( N \), \( p(r, s, t) = r^N \tilde{p}(u, v) \) for a polynomial \( \tilde{p}(u, v) \) of degree \( N \) in \( u, v, E^v = N \) and we get

\[
P^{r,s,t}_{\xi}(\lambda, \mu) = \frac{1}{2} uv(E^u + E^v - N + 1)(E^u + E^v - N)
\]

\[
- (E^u)^2 + E^u (n + \lambda - 1) + (E^v - N)(-E^u + N - \lambda - 1)
\]

\[
+ (n + 2\mu - 2 - 2E^v) \partial_v
\]

\[
+ \frac{1}{2} u(E^u + E^v - N)(-E^u + 3E^v + N - 2\mu - 1).
\]

Similarly, we have

\[
P^{r,s,t}_{\nu}(\lambda, \mu) = \frac{1}{2} uv(E^u + E^v - N + 1)(E^u + E^v - N)
\]

\[
- (E^v)^2 + E^v (n + \mu - 1) + (E^u - N)(-E^v + N - \mu - 1)
\]

\[
+ (n + 2\lambda - 2 - 2E^u) \partial_u
\]

\[
+ \frac{1}{2} v(E^v + E^u - N)(-E^v + 3E^u + N - 2\lambda - 1).
\]

Let us denote \( A_{i,j}(\lambda, \mu) \) the coefficient by monomial \( u^i v^j \) in the polynomial \( \tilde{p}(u, v) \).

The assumption \( A_{i,j}(\lambda, \mu) = A_{j,i}(\mu, \lambda) \), combined with the symmetry between \( P^{r,s,t}_{\xi}(\lambda, \mu) \) and \( P^{r,s,t}_{\nu}(\lambda, \mu) \), allows to restrict to the action of \( P^{r,s,t}_{\nu}(\lambda, \mu) \) on a polynomial of degree \( N \) of the form

\[
\tilde{p}(u, v) = \sum_{\xi,j : 0 \leq |i+j| \leq N} A_{i,j}(\lambda, \mu) u^i v^j, \quad A_{i,j}(\lambda, \mu) = A_{j,i}(\mu, \lambda).
\]
Consequently, we convert the differential equation (43) into the four-term functional relation
\[
\begin{align*}
\left[\frac{1}{2}(i + j - N - 1)(i + j - N - 2)\right]A_{i-1,j-1}(\lambda, \mu) \\
+\left[-i^2 + i(n + \lambda - 1) + (j - N)(-j + N - \lambda - 1)\right]A_{i,j}(\lambda, \mu) \\
+\left[(j + 1)(n + 2\mu - 2 - 2j)\right]A_{i,j+1}(\lambda, \mu) \\
+\left[\frac{1}{2}(i + j - N - 1)(-i + 3j + N - 2\mu)\right]A_{i-1,j}(\lambda, \mu) = 0 \quad (48)
\end{align*}
\]
for \(i, j = 1, \ldots, N\) and \(j \geq i\), which recursively computes \(A_{i,j+1}(\lambda, \mu)\) in terms of \(A_{i-1,j-1}(\lambda, \mu), A_{i-1,j}(\lambda, \mu)\) and \(A_{i,j}(\lambda, \mu)\).

As for the normalization of \(A_{i,j}(\lambda, \mu)\), a singular vector can be normalized by multiplication by common denominator resulting in the coefficients valued in \(\text{Pol}[\lambda, \mu]\) rather than its quotient field \(\mathbb{C}(\lambda, \mu)\).

As we shall prove in the next Theorem, a consequence of (48) is the uniqueness of solution in the range \(\lambda, \mu \in \mathbb{C} \setminus \{m - \frac{n}{2} \mid m \in \mathbb{N}\}\). We observe that the uniqueness of solution fails for \(\lambda, \mu \in \{m - \frac{n}{2} \mid m \in \mathbb{N}\}\), which indicates the appearance of a non-trivial composition structure in the branching problem for generalized Verma modules. In the following Theorem we construct a set of singular vectors, which will be the representatives realizing abstract character formulas of the diagonal branching problem in Corollary 2.3.

**Theorem 3.5** Let us assume \(\lambda, \mu \in \mathbb{C} \setminus \{m - \frac{n}{2} \mid m \in \mathbb{N}\}, N \in \mathbb{N}\), and introduce the Pochhammer symbol \((x)_l = x(x + 1) \cdots (x + l - 1), l \in \mathbb{N}\), for \(x \in \mathbb{C}\). The four-term functional equation (48) for the coefficients \(\{A_{i,j}(\lambda, \mu)\}_{i,j \in \{1, \ldots, N\}}\) of (scalar valued) singular vectors fulfilling
\[
A_{j,i}(\lambda, \mu) = A_{i,j}(\mu, \lambda), \quad j \geq i,
\]
has the unique non-trivial solution given by the formula
\[
A_{i,j}(\lambda, \mu) = \frac{\Gamma(i + j - N)\Gamma(1 - \frac{n}{2} - \mu)\Gamma(1 - i + j - N + \lambda)\Gamma(\lambda + \frac{n}{2} - i)}{2^{2i+j}(1)_{i+j}j!\Gamma(-N)\Gamma(1 - N + \lambda)\Gamma(1 + j - \frac{n}{2} - \mu)\Gamma(\lambda + \frac{n}{2})}
\sum_{k=0}^{i}(-1)^k \binom{i}{k} (j - i + 1 + k)_{i-k}(\lambda + \frac{n}{2} - i)_{i-k}(\mu - N + 1)_k(\lambda - N + 1 - k)_k.
\]

**Proof:**

Let us first discuss the uniqueness of the solution. The knowledge of \(A_{i,j}(\lambda, \mu)\) for \(i + j \leq k_0\) allows to compute the coefficient \(A_{i,j+1}(\lambda, \mu)\) with \(i + j = k_0 + 1\) from the recursive functional equation, because of assumption \(\lambda, \mu \notin \{m - \frac{n}{2} \mid m \in \mathbb{N}\}\). The symmetry condition for \(A_{i,j}(\lambda, \mu)\) gives \(A_{j,i}(\mu, \lambda) = A_{i,j}(\lambda, \mu)\) and the induction proceeds by passing to the computation of \(A_{i,j+2}(\lambda, \mu)\). Note that all coefficients are proportional to \(A_{0,0}(\lambda, \mu)\) and its choice affects their explicit form.

The proof of the explicit form for \(A_{i,j}(\lambda, \mu)\) is based on the verification of the recursion functional equation (48). To prove that the left-hand side of (48) is trivial is equivalent to the following check: up to a product of
linear factors coming from $\Gamma$-functions, the left hand side is the sum of four polynomials in $\lambda, \mu$. A simple criterion for the triviality of a polynomial of degree $d$ we use is that it has $d$ roots (counted with multiplicity) and the leading monomial in a corresponding variable has coefficient zero.

It is straightforward but tedious to check that the left hand side of (43) has, as a polynomial in $\lambda$, the roots $\lambda = k - \frac{n}{2}$ for $k = 1, \ldots, i$ and its leading coefficient is zero. Let us first consider $\lambda = i - \frac{n}{2}$, so get after substitution

$$A_{i,j}(i - \frac{n}{2}, \mu) = \frac{(-1)^i(i + j - N - 1)\ldots(-N)}{2^{i+j}j!}, \quad (50)$$

and hence

$$A_{i,j+1}(i - \frac{n}{2}, \mu) = A_{i,j}(i - \frac{n}{2}, \mu) \cdot \frac{(-1)(i + j - N)(j - N - \frac{n}{2} + 1)}{2(j + 1)(j - \frac{n}{2} + \mu + 1)}. \quad (51)$$

Taken together, there remain just two contributions on the left hand side of (43) given by $A_{i,j}(i - \frac{n}{2}, \mu), A_{i,j+1}(i - \frac{n}{2}, \mu)$. Up to a common rational factor, their sum is proportional to

$$i\left(\frac{n}{2} - 1\right) + (j - N)(-j + N - i + \frac{n}{2} - 1) +$$

$$(j + 1)(n + 2\mu - 2 - 2j)(-1)\frac{(i + j - N)(j - N - \frac{n}{2} + 1)}{2(j + 1)(j - \frac{n}{2} + \mu + 1)} = 0,$$

which proves the claim. The proof of triviality of the left hand side at special values $\lambda = i - 1 - \frac{n}{2}, \ldots, 1 - \frac{n}{2}$ is completely analogous.

Note that there are some other equally convenient choices for $\lambda, \mu$ allowing the triviality check for (43), for example based on the choice $\lambda = k + N - 1, k = 1, \ldots, i$ or $\mu = N - k, k = 1, \ldots, i$.

The remaining task is to find the leading coefficient on the left hand side of (43) as a polynomial in $\lambda$. Because

$$(\lambda + \frac{n}{2} - i)_{i-k}^{\lambda \rightarrow \infty} \lambda^{-k},$$

$$(\lambda - N + 1 - k)_{k}^{\lambda \rightarrow \infty} \lambda^{-k}, \quad (52)$$

the polynomial is of degree $\lambda^{j-i} \frac{\lambda}{N} = \lambda^{j-i}, \ j \geq i$. The leading coefficient of $A_{i,j}(\lambda, \mu)$ is

$$\lim_{\lambda \rightarrow \infty} \frac{A_{i,j}(\lambda, \mu)}{\lambda^{j-i}} = \left(\sum_{k=0}^{i} (-1)^{k} \binom{i}{k} (j - i + 1 + k)(i - k)(\mu + N + 1)\right) \cdot$$

$$\frac{(-1)^{i+j}(i + j - N - 1)\ldots(-N)}{2^{i+j}j! (j - \frac{n}{2} + \mu) \ldots (1 - \frac{n}{2} + \mu)}. \quad (53)$$
There are three contributions to (48):

\[
\begin{align*}
(N - j + i) & \lim_{\lambda \to \infty} \frac{A_{i,j}(\lambda, \mu)}{\lambda^{j - i}}, \\
(j + 1)(n + 2\mu - 2 - 2j) & \lim_{\lambda \to \infty} \frac{A_{i,j+1}(\lambda, \mu)}{\lambda^{j+1-i}}, \\
\frac{1}{2}(i + j - N - 1)(-i + 3j + N - 2\mu) & \lim_{\lambda \to \infty} \frac{A_{i-1,j}(\lambda, \mu)}{\lambda^{j+1-i}},
\end{align*}
\]

whose sum is a polynomial in \(\mu\) multiplied by common product of linear polynomial. In order to prove triviality of this polynomial, it suffices as in the first part of the proof to find sufficient amount of its roots and to prove the triviality of its leading coefficient. For example in the case \(\mu = N - 1\), we get from (54) that the coefficients of this polynomial are proportional to the sum

\[
(N - j + i) + \frac{(j + 1)(i + j - N)}{(j - i + 1)} - \frac{i(-i + 3j + N - 2(N - 1))}{(j - i + 1)},
\]

which equals to zero. The verification of the required property for \(\mu = N - k, k = 2, \ldots, i\) is completely analogous. This completes the proof. \(\Box\)

Based on the notation (7), this completes the description of

\[
\text{Sol}(\mathfrak{g} \oplus \mathfrak{g}, \text{diag}(\mathfrak{g}); \mathbb{C}_{\lambda, \mu}),
\]

the space of scalar valued singular vectors in the Fourier image of generalized Verma modules characterizing solution space of a diagonal branching problem for \(\text{so}(n + 1, 1, \mathbb{R})\).

It is an interesting observation that the four term functional equation (48) for \(A_{i,j}(\lambda, \mu)\) can be simplified using the generalized hypergeometric function \(\text{$_3F_2$}\),

\[
\text{$_3F_2$}(a_1, a_2, a_3; b_1, b_2; z) := \sum_{m=0}^{\infty} \frac{(a_1)_m(a_2)_m(a_3)_m z^m}{(b_1)_m(b_2)_m m!},
\]

where \(a_1, a_2, a_3 \in \mathbb{C}, b_1, b_2 \in \mathbb{C} \setminus \{-N\}\) and \((x)_m = x(x+1)\ldots(x+m-1)\). In particular, it can be converted into the four term functional equation

\[
\frac{(n + 2\lambda)\Gamma(i + j - N)\Gamma(-\frac{n}{2} - \lambda)\Gamma(1 - i + j - N + \lambda)\Gamma(1 - \frac{n}{2} - \mu)}{2^{i+j}\Gamma(1 + i)\Gamma(-N)\Gamma(1 - N + \lambda)\Gamma(1 + j - N - \mu)} \cdot
\]

\[
(i(-2j + n + 2\mu)\text{$_3F_2$}(1 - i, N - \lambda, 1 - N + \mu; 1 - i + j, 1 - \frac{n}{2} - \lambda; 1) +
\]

\[
(i^2 - i(-1 + n + \lambda) + (j - N)(1 + j - N + \lambda)) \cdot
\]

\[
\text{$_3F_2$}(1 - i, N - \lambda, 1 - N + \mu; 2 - i + j, 1 - \frac{n}{2} - \lambda; 1) +
\]

\[
(1 + j)(i + j - N)(1 + i - j + N - \lambda) \cdot
\]

\[
\text{$_3F_2$}(-i, N - \lambda, 1 - N + \mu; 2 - i + j, 1 - \frac{n}{2} - \lambda; 1) = 0.
\]

(56)
Proof:
In particular, the diagonal coefficients $A_{i,i}(\lambda, \mu)$ can be written as

$$A_{i,i}(\lambda, \mu) = \frac{\Gamma(2i - N)}{\Gamma(-N)^2 \Gamma(1 + i - \frac{n}{2} - \mu) \Gamma(1 + i - \frac{n}{2} - \lambda)} \cdot$$

$$\Gamma(1 + i - \frac{n}{2} - \lambda) \Gamma(1 - \frac{n}{2} - \mu) 3\m F_2(-i, 1 - N + \lambda, N - \mu, 1; 1 - \frac{n}{2} - \mu; 1)$$

As for the second claim, it follows from the definition of $3\m F_2$ that

$$\frac{1}{\Gamma(1 + i - \frac{n}{2} - \mu) \Gamma(1 + i - \frac{n}{2} - \lambda)} \cdot$$

$$\Gamma(1 + i - \frac{n}{2} - \lambda) \Gamma(1 - \frac{n}{2} - \mu) 3\m F_2(-i, 1 - N + \lambda, N - \mu, 1; 1 - \frac{n}{2} - \mu; 1)$$

$$= \sum_{m=0}^{i} \frac{(-i)_m(N - \lambda)_m(1 - N + \mu)_m}{(1)_m(1 - \frac{n}{2} - \mu)_m(1 - \frac{n}{2} - \lambda)_m} \frac{1}{m!}.$$

Using basic properties of the Pochhammer symbol, e.g. $(x)_m = (-1)^m(-x + m - 1)_m$, an elementary manipulation yields the result.

\[\square\]

Let us mention that the diagonal coefficients $A_{i,i}(\lambda, \mu) = A_{i,i}(\mu, \lambda)$ are, up to a rational multiple coming from the ratio of the product of $\Gamma$-functions, symmetric with respect to $\lambda \leftrightarrow \mu$. As a consequence, these polynomials belong to the algebra of $\mathbb{Z}_2$-invariants:

$$\mathbb{C}[\lambda, \mu]^{\mathbb{Z}_2} \cong \mathbb{C}[\lambda \mu, \lambda + \mu].$$

Example 3.7 As an example, in the case of $i = 1$ we have

$$A_{1,1}(\lambda, \mu) = \frac{N(N - 1)(\lambda \mu - N(\lambda + \mu) + (1 - \frac{n}{2} + N(N - 1)))}{(2\lambda + n - 2)(2\mu + n - 2)}. \quad (59)$$
and
\[
A_{1,j}(\lambda, \mu) = \frac{\Gamma(1 + j - N)\Gamma(j - N + \lambda)\Gamma(1 - \frac{n}{2} - \mu)}{2^{j+1}(-1)^{j+1}\Gamma(1 + j)\Gamma(-N)\Gamma(1 - N + \lambda)\Gamma(1 + j - \frac{n}{2} - \mu)} \cdot \frac{j(-2 + n + 2\lambda) + 2(N - \lambda)(1 - N + \mu)}{(n + 2\lambda - 2)}
\]
for all \(j \in \{1, \ldots, N\}\).

Let us also remark that for special values \(\lambda, \mu \in \{m - \frac{n}{2} | m \in \mathbb{N}\}\), the formula \(A_{1,j}(\lambda, \mu)\) simplifies due to the factorization of the underlying polynomial. This factorization indicates so called factorization identity, when a homomorphism of generalized Verma modules quotients through a homomorphism of generalized Verma modules of one of its summands (in the source) or a target homomorphism of generalized Verma modules. This naturally leads to the question of full composition structure of the branching problem, which goes beyond the formulation in terms of the Grothendieck group of the Bernstein-Gelfand-Gelfand parabolic category \(\mathcal{O}^p\).

Let us summarize our results.

**Theorem 3.8** Let \(\mathfrak{g}_R = \text{so}(n+1,1, \mathbb{R})\) and \(\mathfrak{p}_R\) its conformal parabolic subalgebra with commutative nilradical. Then the diagonal branching problem for scalar generalized Verma \(\mathcal{U}(\mathfrak{g} \oplus \mathfrak{g})\)-modules induced from characters \(\chi_{\lambda,\mu}\) is determined, in the Grothendieck group \(K(\mathcal{O}^p)\) of Bernstein-Gelfand-Gelfand parabolic category \(\mathcal{O}^p\), by \(\mathcal{U}(\mathfrak{g})\)-isomorphism in Corollary 2.1, equation (16).

Assuming that \(\lambda, \mu \in \mathbb{C}\backslash\{m - \frac{n}{2} | m \in \mathbb{N}\}\), the summand \(\mathcal{M}_{\lambda+\mu-2N}(\mathfrak{g}, \mathfrak{p})\) in (12) is generated by scalar valued singular vector of homogeneity \(2N\) of the form (40):

\[
p(r,s,t) = \sum_{0 \leq i,j,k \leq N, i+j+k=N} A_{i,j}(\lambda, \mu)s^it^jrk,
\]

where the coefficients \(A_{i,j}(\lambda, \mu)\) are given by equation (49).

In particular, these singular vectors are non-zero, linearly independent and of expected weight (induced by the homogeneity), and the cardinality of the set of singular vectors is as predicted by Corollary 2.1.

We also remark that our results for the coefficients \(A_{i,j}(\lambda, \mu)\) of conformally invariant bilinear differential operators can be directly compared to the coefficients \(c_{r,s,t}\) derived in [20]. For example, the substitution for \(\lambda\) and \(\mu\), respectively, into our formulas the expression \(-n\lambda\) and \(-n\mu\), respectively, identifies our linear and quadratic solutions in Examples 3.3 and 3.4 with those given in [20] up to the multiple \(-1\). In fact, after a tedious but straightforward computation there is analogous comparison result for all coefficients, cf. [20], page 26, (4.4). The reason for different normalizations comes from exploiting different initial approaches to the same problem.
4 Application - the classification of bilinear conformally equivariant differential operators on line bundles

Let $M$ be a smooth (complex) manifold equipped with a filtration of its tangent bundle

$$0 \subset T^1M \subset \cdots \subset T^nM = TM,$$

$\mathcal{V} \rightarrow M$ a smooth (holomorphic) vector bundle on $M$ and $J^k\mathcal{V} \rightarrow M$ the weighted jet bundle over $M$ defined by

$$J^k\mathcal{V} = \bigcup_{x \in M} J_x^k\mathcal{V}, \quad J_x^k\mathcal{V} \xrightarrow{\sim} \oplus_{l=1}^k \text{Hom}(U_l(T_xM)), \quad (61)$$

where $U_l(T_xM)$ is the subspace of homogeneity at most $l$ elements in the universal enveloping algebra of the associated graded algebra $gr(T_xM)$. A bilinear differential pairing between sections of the bundle $\mathcal{V}$ and sections of the bundle $\mathcal{W}$ to sections of the bundle $\mathcal{Y}$ is a vector bundle homomorphism

$$B : J^k\mathcal{V} \times J^l\mathcal{W} \rightarrow \mathcal{Y}. \quad (62)$$

In the case when $M = G_\mathbb{R}/P_\mathbb{R}$ is a generalized flag manifold, a differential pairing is called equivariant if it commutes with the action of $G_{\mathbb{R}}$ on sections of homogeneous vector bundles $\mathcal{V}, \mathcal{W}, \mathcal{Y}$. Denoting $\mathcal{V}, \mathcal{W}, \mathcal{Y}$ the inducing complex $P_\mathbb{R}$-representations of $\mathcal{V}, \mathcal{W}, \mathcal{Y}$, the space of $G_{\mathbb{R}}$-equivariant differential pairings is in bijection with

$$((\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})) \otimes_{\mathcal{U}(\mathfrak{g})} \mathcal{U}(\mathfrak{g})) \text{Hom}(\mathcal{V} \otimes \mathcal{W}, \mathcal{Y}))^{P_\mathbb{R}} \cong \text{Hom}_{\mathcal{U}(\mathfrak{g})}(\mathcal{M}(\mathfrak{g}, p, \mathcal{V}^\vee), \mathcal{M}(\mathfrak{g} \oplus \mathfrak{p} \oplus \mathfrak{p}, (\mathcal{V}^\vee \otimes \mathcal{W}^\vee)), \quad (63)$$

where the superscript denotes the space of $P_\mathbb{R}$-invariant elements and $\mathcal{V}^\vee, \mathcal{W}^\vee$ denote the complex dual representations.

**Theorem 4.1** Let $G_\mathbb{R} = SO_\mathbb{R}(n+1,1, \mathbb{R})$ and $P_\mathbb{R}$ its conformal parabolic subgroup, $\lambda, \mu \in \mathbb{C} \setminus \{ m - \frac{n}{2} \mid m \in \mathbb{N} \}$ and $N \in \mathbb{N}$. Let us denote by $\mathcal{L}_\lambda$ the homogeneous line bundle on $n$-dimensional conformal sphere $G_\mathbb{R}/P_\mathbb{R} \cong S^n$ induced from the complex character $\chi_\lambda$ of $P_\mathbb{R}$. We denote by $\iota : G_\mathbb{R}/P_\mathbb{R} \hookrightarrow G_\mathbb{R}/P_\mathbb{R} \times G_\mathbb{R}/P_\mathbb{R}$ the diagonal embedding, and by $\iota^*$ the induced pull-back of sections of vector bundles given by restriction on the diagonal. Then there exists a set of bilinear conformally equivariant operators

$$B_N : C^\infty(G_\mathbb{R}/P_\mathbb{R}, \mathcal{L}_\lambda) \times C^\infty(G_\mathbb{R}/P_\mathbb{R}, \mathcal{L}_\mu) \rightarrow C^\infty(G_\mathbb{R}/P_\mathbb{R}, \mathcal{L}_{\lambda+\mu-2N}) \quad (64)$$

of the form

$$B_N = \sum_{0 \leq i, j, k \leq N} A_{i,j}(-\lambda, -\mu) s^i \tilde{r}^j \tilde{r}^k, \quad (65)$$

where the coefficients $A_{i,j}(\lambda, \mu)$ are given by (49) and

$$\tilde{s} = \sum_{i=1}^n \partial^2_{x_i} = \Delta_x, \quad \tilde{t} = \sum_{i=1}^n \partial^2_{y_i} = \Delta_y, \quad \tilde{r} = \sum_{i=1}^n \partial_{x_i} \partial_{y_i}. \quad (66)$$

The set $\{B_N\}_{N \in \mathbb{N}}$ determines uniquely the set of all scalar valued conformally equivariant bilinear differential operators.
Proof:

The proof is a direct consequence of Theorem 3.8 and duality (63), together with the application of inverse Fourier transform

\[ x_j \leftrightarrow -i\partial_{\xi_j}, \partial_{x_j} \leftrightarrow -i\xi_j \]

with \( i \in \mathbb{C} \) the imaginary unit.

\[ \square \]

In many applications, it is perhaps more convenient to express the bilinear differential operators in terms of tangent and normal coordinates

\[ t_i = \frac{1}{2}(\xi_i + \nu_i) \text{ resp. } n_i = \frac{1}{2}(\xi_i - \nu_i), \; i = 1, \ldots, n \]

to the diagonal submanifold \( \mathfrak{u}(G_k/P_k) \subset G_k/P_k \times G_k/P_k \), where

\[ r = \frac{1}{4} \left( \sum_{i=1}^{n} t_i^2 - \sum_{i=1}^{n} n_i^2 \right), \]
\[ s = \frac{1}{4} \left( \sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} n_i^2 + 2 \sum_{i=1}^{n} t_i n_i \right), \]
\[ t = \frac{1}{4} \left( \sum_{i=1}^{n} t_i^2 + \sum_{i=1}^{n} n_i^2 - 2 \sum_{i=1}^{n} t_i n_i \right). \]  

(67)

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