DIFFERENTIALS OVER DIFFERENTIAL FIELDS

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Abstract. Given an algebra $A$ over a differential field $K$, we study derivations on $A$ that are compatible with the derivation on $K$. There is a universal object, which is a twisted version of the usual module of differentials, and we establish some of its basic properties. In the context of differential algebraic geometry, one gets a sheaf of these $\tau$-differentials which can be interpreted as certain natural functions on the prolongation of a variety, as studied by Buium. This sheaf corresponds to the Kodaira-Spencer class of the variety.

1. Introduction

In this paper, we study derivations of algebras over differential fields and the associated module of differentials. A main idea is to develop a theory analogous to that of Kähler differentials for differential (commutative) algebra. In the usual case, given a ring $A$, an $A$-algebra $B$, and a $B$-module, an $A$-derivation from $B$ to $M$ is a derivation whose kernel contains $A$. Here, we suppose that $A$ is in addition a differential field, and define a $\tau$-derivation from $B$ to $M$ to be a derivation that is compatible with the derivation on $A$, in a sense to be explained below. We call the universal object the module of $\tau$-differentials of $B$ over $A$, and establish some basic results about it.

In the second part of the paper, we describe the geometric meaning of the module of $\tau$-differentials, in the context of Buium’s differential algebraic geometry [Bui93, Bui94], that is, algebraic geometry over a differential field. Buium introduced the fundamental notion of the prolongation of a variety, which is a torsor of the tangent variety, and hence an affine bundle. Whereas a differential form on a variety $X$ can be viewed as a regular function on the tangent variety $TX$ that is linear on each fiber, a $\tau$-differential form on $X$ is a regular function on the prolongation $X^{(1)}$ that is affine on each fiber. And whereas $\text{Spec} \text{Sym}(\Omega_X)$ is $TX$, for smooth $X$, $\text{Spec} \text{Sym}(\Omega^\tau_X)$ is a $(\dim X + 1)$-vector bundle over $X$, which we call the prolongation cone of $X$, into which both $TX$ and $X^{(1)}$ naturally embed.

Many of the results of this paper hold in the more general context of algebras over a differential ring. Nonetheless, we have chosen to work over a differential field as this suffices for the applications we have considered. For a somewhat different, more geometric approach to some of this material, see also [Ros05], which was motivated by work of Hrushovski and Itai [HI03] on the model theory of differential fields.

2. Differentials over differential fields

In this section, we introduce and develop the theory of $\tau$-differentials, in analogy to the usual theory of differentials (see [Eis95, Mat89]). Throughout, $(K, \delta)$ will be a differential field, and all algebras will be $K$-algebras. We will also assume that $K$ contains an element $e$ with $\delta(e) = 1$, which is necessary for some of our main results. For example, under this assumption, for any $K$-algebra $R$, there is a canonical embedding of $R$ into the module $\Omega^\tau_{R/K}$ of $\tau$-differentials. But this is not true if the derivation on $K$ is trivial or, more generally, if the derivation of no element is a unit. (An example of a differential ring with this property is the fraction field of a polynomial ring $L[x]$, with $\delta(x) = x$ and $\delta(a) = 0$, for all $a \in L$.)

We then recall the definition of the prolongation of a $K$-algebra (see [Joh85, Bui93, Gil02]) and explain some connections with $\tau$-differentials.

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Definition 2.1. Let \((K, \delta)\) be a differential ring, \(R\) a \(K\)-algebra, and \(M\) an \(R\)-module. A map \(t : R \to M\) is a \(\tau\)-derivation (over \(K\)) if it is a derivation and, for all \(a, b \in K\), \(\delta(a)t(b) = \delta(b)t(a)\). We often write \(tr\) instead of \(t(r)\).

Let \(\text{Der}_K^\tau(R, M)\) denote the set of such \(\tau\)-derivations, which is an \(R\)-module.

Note that any \(K\)-linear derivation is also a \(\tau\)-derivation.

Definition 2.2. Let \(R\) be a \(K\)-algebra. The module of \(\tau\)-differentials of \(R\), denoted \(\Omega^\tau_R\), is the \(R\)-module generated by the set \(\{\tau(r) \mid r \in R\}\), with the relations
\begin{align*}
(1) & \quad \tau(r + s) = \tau(r) + \tau(s); \\
(2) & \quad \tau(rs) = r\tau(s) + s\tau(r); \\
(3) & \quad \delta(a)\tau(b) = \delta(b)\tau(a);
\end{align*}
for all \(r, s \in R\), and for all \(a, b \in K\). We often write \(\tau r\) instead of \(\tau(r)\). The map \(\tau : R \to \Omega^\tau_R\), taking \(r\) to \(\tau r\) is a \(\tau\)-derivation, called the universal \(\tau\)-derivation.

When the differential ring \((K, \delta)\) is understood, we will usually write \(\Omega^\tau_R\) instead of \(\Omega^\tau_R\).

Remarks 2.3. (1) From the definition, for any \(a, b \in K\), with \(\delta(b) \neq 0\), one gets that \(\tau a = \frac{\delta(a)}{\delta(b)} \tau b\). In particular, \(\tau a = \tau b\) if and only if \(\delta(a) = \delta(b)\) and \(\tau a = 0\) if and only if \(a\) is a constant, i.e., \(\delta(a) = 0\).

For \(e \in K\) such that \(\delta(e) = 1\), the universal derivation \(\tau\) maps \(K\) into the submodule \(Q\) of \(\Omega^\tau_R\) generated by \(\tau e\). There is a natural map \(\iota : R \to \Omega^\tau_R\), taking \(r\) to \(r\tau e\) mapping \(R\) onto \(Q\). This map does not depend on the choice of \(e\). Below, Lemma 2.10, we give a general condition under which this map is injective.

(2) There is also a natural surjective map from \(\lambda : \Omega^\tau_R \to \Omega_R\) taking \(\tau r\) to \(dr\). Below, Lemma 2.10 again, we show that the kernel of this map is \(i(R)\).

Lemma 2.4. Given a \(K\)-algebra \(R\), \(\Omega^\tau_R\) represents the functor from the category of \(R\)-modules to the category of Sets that sends \(M \mapsto \text{Der}_K^\tau(R, M)\). In other words, there is a natural bijection
\[\text{Hom}_{R\text{-mod}}(\Omega^\tau_R, M) \equiv \text{Der}_K^\tau(R, M).\]

Proof. As in the usual case. \(
\square
\)

Lemma 2.5. The \(R\)-module \(\Omega^\tau_R\) is isomorphic to the pushout \(P\) of the following diagram.
\[
\begin{array}{ccc}
R \otimes \Omega_K & \xrightarrow{\alpha} & \Omega_R \\
\downarrow{\delta} & & \downarrow{\beta} \\
R & \xrightarrow{f} & P
\end{array}
\]

where \(\alpha\) takes \(r \otimes da\) to \(rda\) and \(\delta\) is the \(R\)-module map taking \(r \otimes da\) to \(r\delta(a)\), for all \(r \in R, a \in K\).

Proof. By definition, \(P\) is isomorphic to the module \((R \oplus \Omega_R)/N\), where \(N\) is the submodule of \(R \oplus \Omega_R\) generated by \(\{\delta(a) \oplus 0 - 0 \oplus da \mid a \in K\}\). There is a natural surjection \(\Omega_R \to \Omega^\tau_R\), taking \(dr\) to \(\tau r\), whose kernel \(M\) is generated by the set \(\{\delta(a)db - \delta(b)da \mid a, b \in K\}\). Thus, to give a homomorphism \(f\) from \(\Omega^\tau_R\) to \(P\), it suffices to give a homomorphism \(F\) from \(\Omega_R\) to \(P\) whose kernel contains \(M\). Let \(F\) be the map that sends \(dr\) to \(0 \oplus dr\). We must then show that for all \(a, b \in K\), \(F(\delta(a)db - \delta(b)da) = 0\) in \(P\).

\[
F(\delta(a)db - \delta(b)da) = 0 \oplus \delta(a)db - 0 \oplus \delta(b)da = \delta(a)\delta(b)\oplus 0 - \delta(b)\delta(a)\oplus 0 = 0
\]

Note that \(f\) takes \(\tau r\) to \(0 \oplus dr\).

To prove that \(f\) is an isomorphism, we construct the inverse \(g : P \to \Omega^\tau_R\), which we also derive from a homomorphism \(G : R \oplus \Omega_R \to \Omega^\tau_R\). Choose again \(e \in K\) such that \(\delta(e) = 1\). For all \(r \in R\), let \(G(r \oplus 0) = r\tau e\) and \(G(0 \oplus dr) = \tau r\). To show that \(G\) determines a homomorphism \(g : P \to \Omega^\tau_R\), it suffices to show that for all \(a \in K\), \(G(\delta(a) \oplus 0 - 0 \oplus da) = 0\).

\[
G(\delta(a) \oplus 0 - 0 \oplus da) = \delta(a)\tau e + \tau a = \delta(a)\tau e + \delta(e)\tau a = 0
\]
as desired.
Note that for all \( r \in R \), in \( P \) we have \( r \oplus 0 = r\delta(e) \oplus 0 = 0 \oplus rde \), so every element of \( P \) can be written as a sum of elements of the form \( 0 \oplus rds \), with \( r, s \in R \). Thus \( g \) takes \( 0 \oplus rds \) to \( r\tau s \in \Omega^r_R \). Finally, it is clear that \( g \) is the inverse of \( f \), so \( f \) is indeed an isomorphism.

\( \square \)

**Remark 2.6.** Identifying \( \Omega^r_R \) with \( P \) via the above isomorphism \( f \), we see that \( \iota : R \to P \) takes \( r \) to \( 0 \oplus rde = r\delta(e) \oplus 0 = r \oplus 0 \). Thus \( \iota(R) \) is the submodule \( R \oplus 0 \subseteq P \). Below, we give a condition under which this submodule is free. (As far as we know, it is possible that it is always free. This would be true if, for example, the assumption in Lemma 2.8 that \( R \) is an integral domain is unnecessary.)

Let \( L \) be an extension field of a field \( F \). Recall that a set \( B \subseteq L \) is a differential basis of \( L \) over \( F \) if \( \{dx \mid x \in B\} \) is a basis of the \( L \)-vector space \( \Omega_{L/F} \). In characteristic 0, a differential basis is the same thing as a transcendence basis ([Mat89], p. 202).

**Lemma 2.7.** The kernel of the \( R \)-module map \( \delta : R \otimes_K \Omega_K \to R \), that takes \( r \otimes da \) to \( r\delta(a) \), is equal to \( R \otimes_K M \), for \( M \subseteq \Omega_K \) the \( K \)-vector space generated by \( \{\delta(a)db - \delta(b)da \mid a, b \in K\} \).

**Proof.** We first consider the case \( R = K \) is a field of finite transcendence degree \( n \). Then \( \ker(\delta) \) is an \((n-1)\)-dimensional vector space. It is clear that \( M \subseteq \ker(\delta) \), so it suffices to show that there are \( n - 1 \) linearly independent elements in \( M \). By assumption, there is an element \( e \in K \) such that \( \delta(e) = 1 \), which must be transcendental. Let \( \{e_1, \ldots, e_n\}, e_1 = e \), be a transcendence basis for \( K \), so \( \{de_1, \ldots, de_n\} \) is a differential basis of \( \Omega_K \). For \( i = 2, \ldots, n \), let \( v_i \in M \) equal \( \delta(e_i)de_1 - \delta(e_1)de_i = \delta(e_i)de_1 - de_i \). We claim that the \( v_i \) are linearly independent. Suppose that \( \sum_{i=2}^{n} c_i v_i = 0 \). Rearranging terms, one gets \( \sum_i c_i v_i = (\sum_i c_i \delta(e_i))de_1 + (\sum_i c_i de_i) \). By the linear independence of the \( de_i \), each \( c_i \) must be 0, proving the claim.

Next, let \( R = K \) be an arbitrary field. Suppose that \( \sum_i c_i dm_i \in \ker(\delta) \). Let \( L \) be the finitely generated subfield of \( K \) generated by the \( c_i \) and the \( m_i \). By the previous argument, \( \sum_i c_i dm_i \) is contained in the \( L \)-vector space generated by \( \{\delta(a)db - \delta(b)da \mid a, b \in L\} \), as desired.

Finally, let \( R \) be an arbitrary \( K \)-algebra. By above, we have an exact sequence of \( K \)-vector spaces, \( 0 \to M \to \Omega_K \to K \to 0 \). Tensoring with \( R \), one gets the desired exact sequence of free \( R \)-modules.

\( \square \)

**Lemma 2.8.** Let \( R \) be a \( K \)-algebra and an integral domain. The map \( \alpha : R \otimes_K \Omega_K \to \Omega_R \), taking \( r \otimes da \) to \( rda \), \( r \in R, a \in K \), is an injection.

**Proof.** Let \( L \) be the fraction field of \( R \). The map \( \alpha' : R \otimes_K \Omega_K \to \Omega_L \), taking \( r \otimes da \) to \( rda \), factors through \( \alpha \), so it suffices to show that \( \alpha' \) is injective.

Let \( B_K \) be a differential basis of \( K \), and let \( B_L \) be a differential basis of \( L \) such that \( B_K \subseteq B_L \). As \( R \otimes_K \Omega_K \) is a free \( R \)-module with basis \( \{db \mid b \in B_K\} \) and \( \alpha' \) preserves the linear independence of the \( db \), it is clear that \( \alpha' \) is injective, as desired.

\( \square \)

**Remark 2.9.** We do not know whether the assumption that \( R \) is an integral domain is necessary. In other words, is it true that for every \( K \)-algebra \( R \), the natural map \( R \otimes_K \Omega_K \to \Omega_R \) is injective?

**Lemma 2.10.** For any \( K \)-algebra \( R \), there is an exact sequence,

\[ R \overset{r}{\to} \Omega^r_R \overset{\lambda}{\to} \Omega_{R/K} \to 0. \]

Suppose that \( R \) is an integral domain. Then there is an exact sequence,

\[ 0 \to R \overset{r}{\to} \Omega^r_R \overset{\lambda}{\to} \Omega_{R/K} \to 0. \]

**Proof.** The first sequence is just the pushout of the first fundamental exact sequence along the map \( \delta \) defined in Lemma 2.6.

\[ \begin{array}{ccc}
R \otimes_K \Omega_K & \overset{\alpha}{\to} & \Omega_R \\
\delta & \downarrow & \beta \\
R & \overset{\iota}{\to} & \Omega^r_R \overset{\lambda}{\to} \Omega_{R/K} \\
\end{array} \]

Suppose now that \( R \) is an integral domain. By the previous lemma, \( \alpha : R \otimes_K \Omega_K \to \Omega_R \) is injective, and the second claim now follows.

\( \square \)
**Corollary 2.11.** Let $R$ be a smooth $K$-algebra. Then there is a split short exact sequence, $0 \to R \to \Omega^*_R \to \Omega^*_R \to 0$.

*Proof.* By the previous lemma and the fact that the first fundamental exact sequence extends to a split short exact sequence if $R$ is smooth over $K$. (See [Mat89, p. 193], also for the definition of smooth.) □

**Corollary 2.12.** Let $R$ be a finitely generated smooth $K$-algebra. Then $\Omega^*_R$ is a projective module.

*Proof.* Since $R$ is smooth, $\Omega^*_R$ is locally free and thus projective. Thus, by the previous corollary, $\Omega^*_R$ is the direct sum of two projective modules, and thus projective. (For the connection between locally free and projective modules, see [Eis95, Theorem A3.2].) □

The following, technical lemma will be useful in the proofs, below, of the $\tau$-versions of the first and second fundamental exact sequences. (See [Mat89, p. 193–4].)

**Lemma 2.13.** Let $R \to S$ be a map between $K$-algebras. Then $\Omega^*_S$ is isomorphic to the pushout $P$ of the following diagram, where $g,h$ are the natural maps.

\[
\begin{array}{ccc}
S \otimes_R \Omega^*_R & \xrightarrow{g} & \Omega^*_S \\
\downarrow h & & \downarrow \eta \\
S \otimes_R \Omega^*_R & \xrightarrow{\beta} & P \\
\end{array}
\]

*Proof.* Same idea as the proof of Lemma 2.5. □

The next two lemmas are $\tau$-versions of basic results about usual differentials. They can be proved directly, as in Matsumura, but we give different proofs, obtaining the $\tau$-sequences as pushouts of the usual ones.

**Lemma 2.14.** (First $\tau$-fundamental exact sequence) Let $R \to S$ be a map between $K$-algebras. There is an exact sequence of $S$-modules,

\[
\begin{array}{ccc}
S \otimes_R \Omega^*_R & \xrightarrow{\alpha} & \Omega^*_S \\
\downarrow \beta & & \downarrow \eta \\
S \otimes_R \Omega^*_R & \xrightarrow{\beta} & \Omega^*_S/R \\
\end{array}
\]

where $\alpha(s \otimes \tau r) = s \tau r$ and $\beta(\tau s) = ds$.

In addition, if $S$ is smooth over $R$, then there is a short exact sequence of $S$-modules.

\[
0 \to S \otimes_R \Omega^*_R \xrightarrow{\alpha} \Omega^*_S \xrightarrow{\beta} \Omega^*_S/R \to 0
\]

*Proof.* By Lemma 2.13 we get the desired sequence as the pushout of the first fundamental exact sequence.

\[
\begin{array}{ccc}
S \otimes_R \Omega^*_R & \xrightarrow{\alpha} & \Omega^*_S \\
\downarrow \beta & & \downarrow \eta \\
S \otimes_R \Omega^*_R & \xrightarrow{\beta} & \Omega^*_S/R \\
\end{array}
\]

The second claim follows as in Corollary 2.11. □

**Lemma 2.15.** (Second $\tau$-fundamental exact sequence) Let $R \to S$ be a surjective map of $K$-algebras with $\ker(f) = I$. Then there is an exact sequence of $S$-modules,

\[
I/I^2 \to S \otimes_R \Omega^*_R \xrightarrow{\alpha} \Omega^*_S \to 0
\]

where $\gamma(r) = 1 \otimes \tau r$ and $\alpha(s \otimes \tau r) = s \tau r$.

*Proof.* By Lemma 2.13 again, one gets the following diagram.

\[
\begin{array}{ccc}
I/I^2 & \xrightarrow{\mu} & S \otimes_R \Omega^*_R \\
\downarrow \eta & & \downarrow \zeta \\
I/I^2 & \xrightarrow{\gamma} & S \otimes_R \Omega^*_R \\
\end{array}
\]

□
The next lemma characterizes when the map $\gamma$ in the second $\tau$-fundamental exact sequence is injective. (As mentioned in Remark 2.6 the assumption that $R, S$ are integral domains is perhaps unnecessary.)

**Lemma 2.16.** Let $R \xrightarrow{\tau} S$ be a surjective map of $K$-algebras, which are integral domains, with $\ker(\tau) = I$. Then the map $\gamma$ in the previous diagram is injective if and only if $\mu$ is injective.

**Proof.** Clearly, if $\gamma$ is injective, then so is $\mu$. In the other direction, suppose that $\mu$ is injective, and let $N = (I/I^2)/\ker(\gamma)$. Letting $M = \langle \delta(a)db - \delta(b)da \mid a, b \in K \rangle$, $\ker(\tau) = S \otimes_K M$. By the right exactness of the tensor product, there is an exact sequence, $S \otimes_R (R \otimes_K M) \to S \otimes_R \Omega_R \xrightarrow{\tau} S \otimes_R \Omega_R \to 0$, so $\ker(\mu) = J$ is a homomorphic image of $S \otimes_R (R \otimes_K M)$, which is isomorphic to $S \otimes_K M$. By the Snake Lemma, we get the following diagram, with each horizontal sequence exact.

$$
\begin{array}{ccccccc}
0 & \to & \ker(\gamma) & \to & J & \to & S \otimes_K M & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I/I^2 & \xrightarrow{\mu} & S \otimes_R \Omega_R & \xrightarrow{\nu} & \Omega_S & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & N & \xrightarrow{\tau} & S \otimes_R \Omega_R & \xrightarrow{\alpha} & \Omega_S & \to & 0 \\
\end{array}
$$

Since $J$ is a homomorphic image of $S \otimes_K M$, $\zeta$ is an isomorphism, and $\ker(\gamma) = 0$, as desired. \hfill \Box

Given a ring $R$ and a multiplicative subset $U \subseteq R$, then $\Omega_R[U^{-1}] = R[U^{-1}] \otimes_R \Omega_R$. The next lemma establishes the analogous result for $\tau$-differentials.

**Lemma 2.17.** (Localization) Let $R$ be a $K$-algebra, $U$ a multiplicative subset of $R$. Then $\Omega_R[U^{-1}] = R[U^{-1}] \otimes_R \Omega_R$.

**Proof.** By Lemma 2.13 $\Omega_R[U^{-1}]$ is the pushout in the following diagram.

$$
\begin{array}{ccc}
R[U^{-1}] \otimes_R \Omega_R & \xrightarrow{g} & \Omega_R[U^{-1}] \\
\downarrow{h} & & \downarrow{\gamma} \\
R[U^{-1}] \otimes_R \Omega_R & \xrightarrow{j} & \Omega_R[U^{-1}] \\
\end{array}
$$

Since $g$ is an isomorphism, so is $j$. \hfill \Box

**Lemma 2.18.** (Base Change) Let $R$ be a $K$-algebra, and let $(K', \delta)$ be a differential field extension of $K$. Then $\Omega_{K' \otimes_K R/K'} \cong K' \otimes_K \Omega_R/K$.

**Proof.** Fix $e \in K$ such that $\delta(e) = 1$. Let $T : K' \otimes_K R \to K' \otimes_K \Omega_{R/K}$ be the $\tau$-derivation (over $K'$) that maps $(a \otimes r)$ to $(a \otimes r) + (\delta(a) \otimes r)$. By the universal property of $\tau$-differentials, this determines a $K' \otimes_K R$-module homomorphism $f : \Omega_{(K' \otimes_K R)/R/K} \to K' \otimes_K \Omega_{R/K}$ that maps $\tau(a \otimes r)$ to $(a \otimes r) + (\delta(a) \otimes r)$. Note also that in $\Omega_{(K' \otimes_K R)/R/K}$, $\tau(a \otimes r) = r\delta(a)\tau(\varepsilon \otimes 1) + a\tau(1 \otimes r)$.

Let $U : R \to \Omega_{(K' \otimes_K R)/R/K}$ be the $\tau$-derivation mapping $r$ to $\tau(1 \otimes r)$. As above, this determines an $R$-module homomorphism $g_0 : \Omega_R \to \Omega_{(K' \otimes_K R)/R/K}$ mapping $\tau r$ to $\tau(1 \otimes r)$. Since $\Omega_{(K' \otimes_K R)/R/K}$ is a $K' \otimes_K R$-module, $g_0$ determines a $(K' \otimes_K R)$-module homomorphism $g : K' \otimes_K \Omega_R \to \Omega_{(K' \otimes_K R)/R/K}$ taking $(a \otimes r)$ to $a\tau(1 \otimes r)$. It is easy to see that $g$ is the inverse of $f$, so $f$ is an isomorphism. \hfill \Box

The following result is a $\tau$-version of a standard fact.

**Lemma 2.19.** Let $R$ be a $K$-algebra, $f, g$ $\tau$-derivations from $R$ to $R$. Then the commutator, $[f, g] = fg - gf$, is also a $\tau$-derivation.
Proof. It is well-known that \([f,g]\) is a derivation, so it suffices to show that for all \(a, b \in K\), \(\delta(a)[f,g](b) = \delta(b)[f,g](a)\). Let \(e \in K\) be such that \(\delta(e) = 1\), so that for all \(a \in K\), \(f(a) = \delta(a)f(e)\) and \(g(a) = \delta(a)g(e)\). In fact, it is enough to show that for all \(a, b \in K\), \(\delta(a)[f,g](b) = \delta(a)\delta(b)[f,g](e)\), since this implies that for all \(a, b \in K\), \(\delta(a)f,g(b) = \delta(b)[f,g](a)\).

\[
[f, g](a) = fg(a) - gf(a) = f(\delta(a)g(e)) - g(\delta(a)f(e)) = (\delta(a)fg(e) + f(\delta(a))g(e)) - (\delta(a)gf(e) + g(\delta(a))f(e)) = \delta(a)[f,g](a) + g(\delta(a))f(e) - \delta(a)g(\delta(a))f(e) = \delta(a)[f,g](a)
\]

as desired.\qed

**Examples.**

**Lemma 2.20.** For any \(n\), let \(\epsilon : K[x_1, \ldots, x_n] \rightarrow K[x_1, \ldots, x_n]\) be the map that takes any polynomial \(f\) to \(f^\delta\), which is obtained from \(f\) by taking the derivative of each coefficient. Then \(\epsilon\) is a derivation on \(K[x_1, \ldots, x_n]\). Further, \(\epsilon\) commutes with each derivation \(\frac{\partial}{\partial x}\).

From now on, we will write \(\epsilon\) as \(\frac{\partial}{\partial x}\), and write \(\frac{\partial f}{\partial x}\) for \(\frac{\partial}{\partial x}(f)\), which equals \(f^\delta\).

**Proof.** The proofs are straightforward calculations. To simplify notation, we assume \(n = 1\). Clearly, \(\epsilon\) is additive, so it suffices to show that for any \(f, g, \epsilon(fg) = \epsilon(f)g + \epsilon(g)\). Let \(f = \sum_m a_m x^m\) and \(g = \sum_n b_n x^n\).

Then

\[
\epsilon \left( \left( \sum_m a_m x^m \right) \left( \sum_n b_n x^n \right) \right) = \epsilon \left( \sum_l \left( \sum_{m=0}^l a_m b_{l-m} x^l \right) \right) = \sum_l \left( \sum_{m=0}^l (a_m \delta(b_{l-m}) + \delta(a_m)b_{l-m}) \right) x^l = f \epsilon(g) + g \epsilon(f)
\]

as desired.

To prove that the derivations commute, it suffices to note that

\[
\frac{d}{dx} \sum_m a_m x^m = \sum_m \left( \frac{d}{dx} \delta(a_m) x^m \right) = \sum_m m \delta(a_m) x^{m-1} = \frac{d}{dx} \sum_m a_m x^m.
\]

The following lemma is immediate.

**Lemma 2.21.** Let \(R = K[x_1, \ldots, x_n]\), and choose \(\epsilon \in K\) such that \(\delta(\epsilon) = 1\). Then \(\Omega^\tau_R\) is a free module of rank \(n + 1\), with generators \(\langle \tau x_1, \ldots, \tau x_n, \tau \epsilon \rangle\). The universal \(\tau\)-derivation from \(R\) to \(\Omega^\tau_R\) is given by

\[
\tau f = \sum_m \frac{df}{dx} \tau x_i + \frac{df}{dx} \tau \epsilon.
\]

**Lemma 2.22.** Let \(R = K[x_1, \ldots, x_n]\), with \(\epsilon \in K\) such that \(\delta(\epsilon) = 1\). Then \(\Omega^\tau_R = R \epsilon \oplus (\oplus_{i=1}^n R \tau x_i)\), a free module of rank \(n + 1\).

**Proof.** One can adapt the proof that \(\Omega_{R/K}\) is a rank \(n\) free module, as in [Eis95]. We define \(R\)-module homomorphisms \(F : R^{n+1} \rightarrow \Omega^\tau_R\) and \(G : \Omega^\tau_R \rightarrow R^{n+1}\) such that \(F\) is surjective, and \(G\) is the inverse of \(F\). Let \(F\) take \((a_0, \ldots, a_n) \in R^{n+1}\) to \(a_0 \tau + \sum_{i=1}^n a_i \tau x_i \in \Omega^\tau_R\). Since \(\Omega^\tau_R\) is clearly generated by \(\{\tau \epsilon, \tau x_1, \ldots, \tau x_n\}\), \(F\) is surjective.

To define \(G\), note that by Lemma 2.23, there is a natural bijection between homomorphisms from \(\Omega^\tau_R\) to \(R\) and \(\tau\)-derivations from \(R\) to itself. Given such a \(\tau\)-derivation \(\tau r\), let \(T\) be the corresponding homomorphism. Thus, an \((n + 1)\)-tuple of \(\tau\)-derivations determines a homomorphism from \(\Omega^\tau_R\) to \(R^{n+1}\) in an obvious way.

Let \(\delta^\tau_0 : R \rightarrow R\) be the derivation from \(R\) to \(R\) that extends \(\delta\) on \(K\) and such that \(\delta^\tau_0(x_i) = 0\), for all \(i\). Clearly, \(\delta^\tau_0\) is also a \(\tau\)-derivation. For \(m, 1 \leq m \leq n\), let \(\delta_m^\tau\) be the usual partial derivative \(\partial / \partial x_m\), which
is also a τ-derivation. Now define $G : \Omega_R^n \to R^{n+1}$ by $G(\omega^r) = (T_0(\omega^r), \ldots, T_n(\omega^r))$. It is easy to see that $G \circ F = 1_{R^{n+1}}$, the identity map on $R^{n+1}$, as desired.

**Proposition 2.23.** Let $L$ be a field extending the differential field $K$, and let $e \in K$ be an element such that $\delta(e) = 1$. Given a set $B \subseteq L$, \{\tau b \mid b \in B\} \cup \{\tau e\}$ is a basis of the $L$-vector space $\Omega^*_L$ if and only if $B$ is a transcendence basis for $L$ over $K$.

**Proof.** By Lemma 2.10 there is an exact sequence $0 \to L \xrightarrow{\iota} \Omega^*_L \xrightarrow{\lambda} \Omega^*_L \to 0$, with $\iota(a) = a\tau e$ and $\lambda(\tau a) = da$. Let $B \subseteq L$. In one direction, suppose that $\{\tau b \mid b \in B\} \cup \{\tau e\}$ is a basis of $\Omega^*_L$. Then $\lambda(\{\tau b \mid b \in B\}) = \{db \mid b \in B\}$ is a basis of $\Omega^*_L$. Therefore, $B$ is a differential basis of $L$ over $K$, and thus a transcendence basis. In the other direction, if $B$ is a transcendence basis of $L$ over $K$, then it is a differential basis, so $\lambda$ maps the set $\{\tau b \mid b \in B\} \subseteq \Omega^*_L$ bijectively onto a basis of $\Omega^*_L$. Thus $\{\tau b \mid b \in B\} \cup \{\tau e\}$ is a basis of $\Omega^*_L$.

**Corollary 2.24.** Let $L$ be a field extension of $K$ of transcendence degree $= n$. Then $\dim \Omega^*_L = n + 1$.

**Prolongations.** Prolongations were introduced by Johnson [Joh85] in the context of what might be called differential commutative algebra. Buium [Bui93] incorporated this work into his differential algebraic geometry, and developed the notion of the prolongation of a variety. Here we briefly describe the algebraic version. In the next section, we use this to define the prolongation of a variety in the language of schemes.

A kernel is a ring homomorphism $f : A \to B$ together with a derivation $\delta$ from $A$ into $B$. A prolongation is a pair of kernels, $(f, \delta) : A \to B$ and $(g, \delta') : B \to C$, such that $\delta' \circ f = g \circ \delta$. More generally, one can define a prolongation sequence in the obvious way, which is how one gets, e.g., the higher prolongations of a variety. There are also natural notions of morphisms of kernels, and of prolongations, each of which gives a category.

The basic example of a kernel is a $K$-algebra $R$ over a differential field $(K, \delta)$. A prolongation of $K \to R$ is a kernel $(g, \delta_S) : R \to S$ such that $\delta_S = g \circ \delta$. A morphism between two such kernels $(g, \delta_S) : R \to S$ and $(h, \delta_T) : R \to T$ is an $R$-algebra morphism $j : S \to T$ such that $j \circ \delta_S = \delta_T$. There is a universal object in this category, which we simply call the prolongation of $R$, and denote $R^{(1)}$. Given any prolongation $(g, \delta_S) : R \to S$ in the category, there is a unique morphism from $R^{(1)}$ to $S$.

**Definition 2.25.** Let $R$ be a $K$-algebra. The (first) prolongation of $R$, denoted $R^{(1)}$, is $\text{Sym}(\Omega_R)/I$, where $\text{Sym}(\Omega_R)$ denotes the symmetric algebra of $\Omega_R$, and $I$ is the ideal generated by $\{da - \delta(a) \mid a \in K\}$. $R \to R^{(1)}$ is a prolongation with the natural derivation $\delta^{(1)} : R \to R^{(1)}$, with $\delta^{(1)}(r) = dr$, for all $r \in R$.

$\delta^{(1)} : R \to R^{(1)}$ is also a $\tau$-derivation, so there is a unique $R$-module homomorphism $t : \Omega^{(1)}_R \to R^{(1)}$ such that $t \circ \tau = \delta^{(1)}$. Below, we show that when $R$ is smooth over $K$, this homomorphism is an embedding. The proof uses the following known fact, whose geometric meaning is that the first prolongation of a smooth affine variety is isomorphic to the tangent variety.

**Proposition 2.26.** Let $R$ be a smooth $K$-algebra. Then $R^{(1)} \cong \text{Sym}(\Omega_{R/K})$.

**Proof.** Since $R$ is smooth, the first fundamental exact sequence, $0 \to R \otimes_K \Omega_K \to \Omega_R \xrightarrow{\psi} \Omega_{R/K} \to 0$ splits, so we can choose a splitting homomorphism $\eta : \Omega_{R/K} \to \Omega_R$. Let $\phi : \Omega_R \to R \otimes_K \Omega_K$ be the map $\phi(x) = x - \eta(x)$, so we have a (non-canonical) isomorphism $f : \Omega_R \to (R \otimes_K \Omega_K) \otimes \Omega_{R/K}$ given by $f(x) = \phi(x) \otimes 0 + 0 \otimes \psi(x)$. This map determines an isomorphism from $\text{Sym}(\Omega_R)$ to $\text{Sym}((R \otimes_K \Omega_K) \otimes \Omega_{R/K}) \cong \text{Sym}(R \otimes_K \Omega_K) \otimes R \text{Sym}(\Omega_{R/K})$, which we also call $f$.

Let $J$ be the ideal of $\text{Sym}(R \otimes_K \Omega_K)$ generated by the set $\{da - \delta(a) \mid a \in K\}$. The quotient $\text{Sym}(R \otimes_K \Omega_K)/J$ is naturally isomorphic to $R$, under the map $\delta$ that sends each $r \in R$ to itself and sends $da$ to $\delta(a)$ for each $a \in K$. Tensoring the exact sequence $0 \to J \to \text{Sym}(R \otimes_K \Omega_K) \to R \to 0$ by $\text{Sym}(\Omega_{R/K})$, one gets the exact sequence $0 \to J \otimes_R \text{Sym}(\Omega_{R/K}) \to \text{Sym}(R \otimes_K \Omega_K) \otimes_R \text{Sym}(\Omega_{R/K}) \to R \otimes_R \text{Sym}(\Omega_{R/K}) + 0$ since $\text{Sym}(\Omega_{R/K})$ is flat over $R$. Under the isomorphism $f : \text{Sym}(R \otimes_K \Omega_K) \otimes_R \text{Sym}(\Omega_{R/K}) \cong \text{Sym}(\Omega_R)$, the ideal $J \otimes_R \text{Sym}(\Omega_{R/K})$ corresponds to the ideal $I$ of $\text{Sym}(\Omega_R)$ generated by $\{da - \delta(a) \mid a \in K\}$, so $R \otimes_R \text{Sym}(\Omega_{R/K}) = \text{Sym}(\Omega_{R/K})$ is isomorphic to $R^{(1)}$, as desired. □
Proposition 2.27. Let $R$ be a smooth $K$-algebra. The (unique) $R$-module homomorphism $t : \Omega^r_R \to R^{(1)}$ such that $t \circ \tau = \delta^{(1)}$ is injective. Thus, there is a canonical embedding of $\Omega^r_R$ into $R^{(1)}$.

Proof. The homomorphism $t : \Omega^r_R \to R^{(1)} = \text{Sym}(\Omega_R)/I$ maps $\tau r$ to $dr + I$. Let $t_0$ be the canonical homomorphism from $\Omega_R$ to $R^{(1)}$, taking $dr$ to $dr + I$. Given the natural map $\beta : \Omega_R \to \Omega^r_R$, we have $t_0 = \beta \circ t$. Thus, to show that $t$ is injective, it suffices to show that $\ker(t_0) = \ker(\beta)$.

Recall from the proof of Lemma 2.10 that we have the following commutative diagram.

$$
\begin{array}{c}
0 & \to & R \otimes_K \Omega_K & \xrightarrow{\alpha} & \Omega_R & \xrightarrow{\beta} & \Omega_R/K & \to & 0 \\
& & \downarrow{\delta} & & \downarrow{\beta} & & \downarrow{=} & & \\
0 & \to & R & \xrightarrow{\iota} & \Omega_R^\lambda & \xrightarrow{\lambda} & \Omega_R/K & \to & 0 \\
\end{array}
$$

By the Snake Lemma, $\ker(\beta) = \ker(\delta)$, so we will show that $\ker(t_0) = \ker(\delta)$. (By Lemma 2.7, $\ker(\delta)$ is $R \otimes_K M$, $M \subseteq \Omega_K$ the $K$-vector space. In other words, given a field $a, b \in K$, though we do not use this here.)

We now calculate $\ker(t_0)$. Let $h : R^{(1)} \to \text{Sym}(\Omega_R/K)$ be the isomorphism from the previous proposition, and define $t_0 = h \circ t_0$, so $\ker(t_0) = \ker(\delta)$. The map $\tilde{t}_0$ is the composite of the maps

$$
\Omega_R \to \text{Sym}(\Omega_R) \xrightarrow{\tilde{f}} \text{Sym}(R \otimes_K \Omega_K) \otimes_R \text{Sym}(\Omega_R/K) \to R \otimes_R \text{Sym}(\Omega_R/K) \to \text{Sym}(\Omega_R/K).
$$

Explicitly, $m \in \Omega_R$ is sent to $(\tilde{\delta}(\phi(m)) \otimes 1) + (1 \otimes \psi(m)) \in R \otimes_R \text{Sym}(\Omega_R/K)$, with $\tilde{\delta}(\phi(m)) \in R$ and $\psi(m) \in \Omega_R/K$.

Thus, $t_0(m) = 0$ if and only if $\tilde{\psi}(m) = 0$ and $\tilde{\delta}(\phi(m)) = 0$. Here, $\psi(m) = 0$ if and only if $m \in R \otimes_K \Omega_K$ and, in case $\psi(m) = 0$, then $\phi(m) = m$. Thus, $m \in \ker(\tilde{t})$ if and only if $m \in R \otimes_K \Omega_K$ and $m \in \ker(\tilde{\delta})$. This completes the proof. 

3. Varieties, prolongations, and $\tau$-differential forms

In this section, we introduce the sheaf of $\tau$-differential forms on a variety over a differential field, and describe the connection to the prolongation of the variety, introduced by Buium (see [Bui93]). First we describe the construction of the prolongation, which is a torsor under the tangent bundle, and thus an affine bundle.

We adopt the following conventions. $(K, \delta)$ is an algebraically closed differential field with an element $e \in K$ such that $\delta(e) = 1$. A variety is an integral, separated $K$-scheme of finite type. We will only consider smooth, i.e., nonsingular, varieties (see [Har77], p. 268).

Affine bundles. Recall that an affine space is a principal homogeneous space of (the additive group of) a vector space. In other words, given a field $K$, a $K$-affine space is a triple $(A, V, \alpha)$, where $A$ is a set, $V$ a $K$-vector space, and $\alpha$ a regular action of $V$ on $A$, though we generally omit explicit mention of the function $\alpha$. We will say that the dimension of $A$ is just the dimension of $V$.

An affine map between $K$-affine spaces $(A, V)$ and $(B, W)$ is a function $f : A \to B$ such that there is a linear map $\lambda f : V \to W$ such that for all $a \in A, v \in V$, $f(v \cdot a) = (\lambda f(v)) \cdot f(e)$. There is also a natural ‘linearization’ functor $\lambda$ from the category of affine spaces to vector spaces, with $\lambda(A, V) = V$ and behaving on morphisms as described above.

Given a $K$-affine space $(A, V)$, there is an associated ‘dual’ vector space $(A, V)^\vee$ of affine maps from $A$ to $K$, of dimension $\dim A + 1$.

An affine bundle over a variety can then be defined in analogy to the definition of a vector bundle (e.g., see [Har77], p. 128).

Definition 3.1. Let $Y$ be a variety. A (geometric) affine bundle of rank $n$ over $Y$ is a variety $X$ with a morphism $f : X \to Y$, together with the data of an open covering $\{U_i\}_{i \in I}$ of $Y$ and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{A}^n_{U_i}$ such that for all $i, j \in I$ and open affine $V = \text{Spec} B \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{A}^n_{U_i}$ is given by an affine automorphism $\theta$ of $B[x_1, \ldots, x_n]$, i.e., $\theta(b) = b$ for $b \in B$ and $\theta(x_j) = c_j + \sum_k b_{jk}x_k$, for $c_j \in B$ and suitable $b_{jk} \in B$. 

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Remark 3.2. Given a rank $n$ affine bundle over a variety $Y$, and a point $p \in \text{Spec } B \subseteq Y$, the fiber $Y_p$ has the structure of an $n$-dimensional affine space over $\kappa(p) = B_p/p_p$.

Definition 3.3. A morphism of affine bundles, $f : X \to Y$ and $g : W \to Z$, is a pair of morphisms $s : X \to W, t : Y \to Z$ so that $t \circ f = g \circ s$ and, for any $a \in Y, b = t(a) \in Z$, there are affine neighborhoods $U = \text{Spec } A$ of $a, V = \text{Spec } B$ of $b$, so that

$$f^{-1}(U) \cong \text{Spec } A[x_1, \ldots, x_n]$$

$$g^{-1}(U) \cong \text{Spec } B[y_1, \ldots, y_m]$$

and the induced map $s : f^{-1}(U) \to g^{-1}(V)$ is given by an affine map $h : B[y_1, \ldots, y_m] \to A[x_1, \ldots, x_n]$ such that (i) $h$ maps $B$ to $A$ and $h|_B = t$ and (ii) $h(y_i) = a_i + \sum_j c_{ij}x_j$.

Equivalently, one can define an affine bundle as a torsor of a vector bundle.

Kernels and prolongations. We now describe Buium’s ‘globalization to the frame of schemes’ of Johnson’s work.

Definition 3.4. Let $X$ be a scheme, $F$ a sheaf of modules on $X$. A derivation $\delta$ from $O_X$ to $F$ is a set of derivations $\delta_U : O_{X|U} \to F|_U$, for $U \subseteq X$ open, compatible with the restriction maps. Let $\text{Der}(O_X, F)$ denote the set of such derivations. Likewise, let $\text{Der}^\tau(O_X, F)$ denote the set of $\tau$-derivations from $O_X$ to $F$, defined in the obvious way.

Given a morphism of schemes $g : Y \to X$, a derivation from $X$ to $Y$, also written $\delta : X \to Y$, is a derivation $\delta^g \in \text{Der}(O_X, g_*O_Y)$. Similarly for $\tau$-derivations.

Remark 3.5. For any $K$-variety $X$, the basic example of a $(K$-linear) derivation is the differential map $d : O_X \to T_X$. The map $d$ determines an embedding of $\Omega_{X/K}$ into $p_*O_{TX}$, $p : TX \to X$, as $O_X$-modules. One can also consider $\Omega_{X/K}$ as a sheaf of abelian groups on $TX$, which is not, however, an $O_{TX}$-module.

Below, we will see that the map $\delta(1)$ from $O_X$ to $O_{X^{(1)}}$, is a $\tau$-derivation, and closely related to $d$. In particular, over a field $K$ with a trivial derivation, $X^{(1)} = TX$, and $\delta(1) = d$.

The following lemma is clear.

Lemma 3.6. Let $R$ be a $K$-algebra and $M$ an $R$-module, so $X = \text{Spec } R$ is an affine scheme and $F = (M)^\sim$ is an $O_X$-module. There are natural bijections between $\text{Der}(R, M)$ and $\text{Der}(O_X, F)$, and also between $\text{Der}^\tau(R, M)$ and $\text{Der}^\tau(O_X, F)$.

Lemma 3.7. Let $X$ be a variety, $F$ a quasi-coherent sheaf on $X$, and $t$ a map from $O_X$ to $F$. To prove that $t$ is a $(\tau$-) derivation, it suffices to check that for any affine $Y \subseteq X, Y = \text{Spec } R, F|_Y = M^{\sim}$, $M$ an $R$-module, the map $t|_Y : R \to M$ is a $(\tau$-) derivation.

Proof. Straightforward. \qed

We can now give Buium’s definitions of kernel and prolongation for varieties. For affine varieties, these are obviously equivalent to Johnson’s.

Definition 3.8. A kernel from a variety $Z$ to a variety $X$ is a pair $(f, \delta), f$ a morphism from $Z$ to $X$ and $\delta \in \text{Der}(O_X, f_*O_Z)$.

There is also a ‘relative’ notion of a kernel for $K$-varieties, where $\delta$ must be compatible with the derivation on $K$. This makes $\delta$ into a $\tau$-derivation. For our purposes, the relative version is more important, and the only one that we will consider. Given any $K$-variety $X$, there is a natural kernel $X \to K$, as well as the fundamental example of the kernel from $X^{(1)}$ to $X$, defined below, which prolongs the former.

We first define the prolongation of an affine variety, and then show how to globalize.

Definition 3.9. Let $X$ be an affine $K$-variety, $X = \text{Spec } R, R$ a $K$-algebra. The first prolongation $X^{(1)}$ is $\text{Spec } \text{Sym}(\Omega_R)/I$, where $I$ is the ideal generated by $\{da - \delta(a) \mid a \in K\}$.

The projection from $X^{(1)}$ to $X$ is determined by the natural embedding of $R$ into $\text{Sym}(\Omega_R)/I$. By Lemma 3.6 the $\tau$-derivation from $\delta^{(1)} : R \to R^{(1)}$ corresponds to a $\tau$-derivation $\delta^{(1)} : X \to X^{(1)}$, making $X^{(1)} \to X$ into a kernel.
To see that $X^{(1)}$ is a torsor under the tangent bundle, it is useful to give an equivalent definition, in terms of representable functors (see, e.g., [EH00], sections I.4 and VI.1). This approach also globalizes more easily, that is, without explicit patching, and provides some insight into the connection between prolongations and $\tau$-derivations.

**Definition 3.10.** Let $X$ be an affine $K$-variety, $X = \text{Spec} R$, $R$ a $K$-algebra. By smoothness, there is an exact sequence of $R$-modules, $0 \to R \otimes_K \Omega_K \to \Omega_R \to \Omega_{R/K} \to 0$. Let $\delta : R \otimes_K \Omega_K \to R$ be the $R$-module homomorphism given by $\delta(r \otimes da) = r \cdot \delta(a)$. Let $F$ be the functor from the category of $R$-algebras to the category of Sets that associates to any $K$-algebra $S$, the set of pairs $(g, w)$, $g : R \to S$ a $K$-algebra map and $w : \Omega_R \to S$ a $K$-module homomorphism making the following diagram commute.

$$
\begin{array}{c}
0 \\
\downarrow \downarrow \\
R \\
\downarrow g \\
S
\end{array}
\quad
\begin{array}{c}
R \otimes_K \Omega_K \\
\downarrow \delta \\
\Omega_R \\
\downarrow \beta \\
\Omega_{R/K}
\end{array}
\quad
\begin{array}{c}
0
\end{array}
$$

Let $R^{(1)}$ be the $K$-algebra representing this functor, and let $X^{(1)} = \text{Spec} R^{(1)}$.

**Remark 3.11.** It is easy to check that $R^{(1)}$ is isomorphic to $\text{Sym}(\Omega_R)/I$, as in Definition 3.9.

We now give the general definition of the prolongation of a $K$-variety $X$ and show that it is a $TX$-torsor. (See [Bui93], p. 1392–93.)

**Definition 3.12.** Let $X$ be a $K$-variety, $f : X \to K$ the structure map. There is an exact sequence

$$
0 \to f^*\Omega_K \overset{\alpha}{\to} \Omega_X \overset{\beta}{\to} \Omega_{X/K} \to 0
$$

of $\mathcal{O}_X$-modules. Let $\tilde{\delta} : f^*\Omega_K \to \mathcal{O}_X$ be the map determined by $\tilde{\delta}$ from Definition 3.10. Let $F$ be the functor from $K$-schemes to Sets that takes a scheme $Z$ to the set of pairs $(g, w)$, $g : Z \to X$ a morphism of schemes, $w : \Omega_X \to g^*\mathcal{O}_X$ a map of $\mathcal{O}_X$-modules such that the following diagram commutes.

$$
\begin{array}{c}
\mathcal{O}_X \\
\downarrow g \\
g^*\mathcal{O}_Z
\end{array}
\quad
\begin{array}{c}
\Omega_X \\
\downarrow \delta \\
f^*\Omega_K
\end{array}
\quad
\begin{array}{c}
g^*\Omega_X \\
\downarrow \beta \\
g^*\mathcal{O}_Z
\end{array}
$$

The prolongation of $X$, written $X^{(1)}$, is the scheme representing $F$.

Explicitly, $X^{(1)}$ is the scheme $\text{Spec} \text{Sym}(\Omega_X)/I$, where $I$ is the ideal sheaf in the symmetric algebra $\text{Sym}(\Omega_X)$ generated by all elements of the form $\delta(x) - x, x$ a local section of $f^*\Omega_K$ (as in Definition 3.9).

Let $g^{(1)} : X^{(1)} \to X$ be the natural morphism of schemes, and $w^{(1)} : \Omega_X \to (g^{(1)})^*\mathcal{O}_X$ the natural map of $\mathcal{O}_X$-modules. $w^{(1)}$ determines a $\tau$-derivation in $\text{Der}^r(\mathcal{O}_X, g^{(1)}_*\mathcal{O}_X)$, which we call $\tau^{(1)}$, though Buium, who introduced it, called it $\tilde{\delta}$ ([Bui93], p. 1396). This makes $X^{(1)}$ into a kernel $(g^{(1)}, \tau^{(1)}) : X^{(1)} \to X$, which prolongs the kernel $X \to K$.

**Definition 3.13.** With the notation of the previous definition, let $G$ be the functor from $K$-schemes to Sets that takes a scheme $Z$ to the set of pairs $(g, v)$, $g : Z \to X$ a morphism of schemes and $v : \Omega_{X/K} \to g^*\mathcal{O}_Z$ a map of $\mathcal{O}_X$-modules. $G$ is represented by the tangent variety $TX$ (which can also be constructed as $\text{Spec} \text{Sym}(\Omega_{X/K})$).

Considering schemes as the functors they represent, we have functorial transformations $X^{(1)} \to X$ and $TX \to X$, and an action $TX \times_X X^{(1)} \to X^{(1)}$ given by

$$
((g, v), (g, w)) \mapsto (g, w + v \circ \beta), (g, v) \in G(Z), (g, w) \in F(Z)
$$

This makes $X^{(1)}$ into a $TX$-torsor.

Let $X, Y$ be varieties, and $f : X \to Y$ a morphism between them. We recall how the lifting map $f^{(1)} : X^{(1)} \to Y^{(1)}$ is defined, and that it is compatible with the torsor structure. (Compare [Bui93], p. 1435, where Buium writes that $f^{(1)} : X^{(1)} \to Y^{(1)}$ is ‘equivariant’ with respect to $df : TX \to TY$.) Given, $f : X \to Y$ the lifting $f^{(1)} : X^{(1)} \to Y^{(1)}$ determines a morphism in the category of kernels. In
Lemma 3.14. Given a variety $X$, the map $\tau: O_X \to p_* O_{X(1)}$ is a $\tau$-derivation, that is, $\tau \in \text{Der}^\tau(O_X, p_* O_{X(1)})$. 

$\tau$-differentials on schemes. The universal derivation from a $K$-algebra $R$ to $\Omega_{R/K}$ corresponds, geometrically, for a $K$-variety $X$, to a derivation in $\text{Der}(\mathcal{O}_X, \Omega_{X/K})$. Alternatively, it can be considered as belonging to $\text{Der}(\mathcal{O}_X, p_* \mathcal{O}_{TX})$, $p: TX \to X$, but this is really equivalent, as there is a canonical $\mathcal{O}_X$-module embedding of $\Omega_{X/K}$ in $p_* \mathcal{O}_X$. Thus $\Omega_{X/K}$ is also naturally a sheaf of abelian groups on $TX$. In differential algebraic geometry, there is a twisted version of this picture, described below.

Let $f: X \to Y$ be a morphism of varieties, and $f^{(1)}: X^{(1)} \to Y^{(1)}$ the lifting morphism on their prolongations. For $Y = A$, the affine line, one can modify $f^{(1)}$ to give element of $\mathcal{O}_{X^{(1)}}(X^{(1)})$, as we now describe. Thus we will get a map $\tau: \mathcal{O}_X(X) \mathcal{O}_{X^{(1)}}(X^{(1)})$. The natural bijection between $\text{Mor}(X, A)$ and $\mathcal{O}_X(X)$ takes $f, f^\#: K[x] \to \mathcal{O}_X(X)$ a ring homomorphism, to $f^\#(x) \in \mathcal{O}_X(X)$. Let $\tau_0: \mathcal{O}_X(X) \to \text{Mor}(\mathcal{O}_{X^{(1)}}, A^{(1)})$ be the map $f \mapsto f^{(1)}$. We have $A^{(1)} \cong T_A = \text{Spec} K[x, \delta x]$, so let $q: A^{(1)} \to A$ be the projection onto the fiber (not the base), given by the ring embedding $K[\delta x] \to K[x, \delta x]$. Thus $q$ induces a map from $\text{Mor}(\mathcal{O}_{X^{(1)}}, A^{(1)})$ to $\text{Mor}(\mathcal{O}_{X(1)}, A) \cong \mathcal{O}_{X(1)}(X^{(1)})$. Putting the pieces together, $\tau := q \circ \tau_0$ is the desired map from $\mathcal{O}_X(X)$ to $\mathcal{O}_{X^{(1)}}(X^{(1)})$.

One can also consider rational maps on $X$, that is, morphisms from open subschemes of $X$ to $A$. For each such $Y \subseteq X$, one gets a map $\tau_Y$ from $\mathcal{O}_Y(Y)$ to $\mathcal{O}_{Y^{(1)}}(Y^{(1)})$, as above. Equivalently, the $\tau_Y$’s determine a map, which we also call $\tau$, between the $\mathcal{O}_X$-modules $\mathcal{O}_X$ and $p_* \mathcal{O}_{X^{(1)}}$, for $p$ the canonical projection from $X^{(1)}$ to $X$.

**Lemma 3.14.** Given a variety $X$, the map $\tau: O_X \to p_* O_{X(1)}$ is a $\tau$-derivation, that is, $\tau \in \text{Der}^\tau(O_X, p_* O_{X(1)})$. 

Note that we also get that $\delta_T \circ f = f^{(1)} \circ \delta_R$, which is precisely the condition for having a morphism of kernels.

Next one wants to show that $f^{(1)}$ is compatible with the torsor structure, that is, that the following diagram is commutative.

$$
\begin{array}{llll}
X^{(1)} \times_X T X & \xrightarrow{f^{(1)} \times T f} & Y^{(1)} \times_Y T Y \\
m_X & \quad & m_Y \\
X^{(1)} & \xrightarrow{f^{(1)}} & Y^{(1)}
\end{array}
$$

This diagram corresponds to:

$$
\begin{array}{llll}
\text{Sym}(\Omega_T/I_T) \otimes \text{Sym}(\Omega_{T/K}) & \xrightarrow{f^{(1)} \otimes df} & \text{Sym}(\Omega_R/I_R) \otimes \text{Sym}(\Omega_{R/K}) \\
m_T & \quad & m_R \\
\text{Sym}(\Omega_T/I_T) & \xleftarrow{f^{(1)}} & \text{Sym}(\Omega_R/I_R)
\end{array}
$$

It now suffices to observe that the following diagram is commutative.

$$
\begin{array}{llll}
df(a) \otimes 1 & \xrightarrow{1 \otimes 1} & da \otimes 1 \\
\quad & \quad & \quad & \\
df(a) & \xleftarrow{1} & da
\end{array}
$$
Proof. By Lemma 3.7 it suffices to prove that for all affine subschemes $Y \subseteq X$, $Y \cong \text{Spec} T$, then $\tau_Y : T \to T^{(1)}$ is a $\tau$-derivation.

Passing to the category of $K$-algebras, and letting $R = K[x]$, $R^{(1)} = K[x, \delta x]$ one gets the following diagram.

$$
\begin{array}{cccc}
T^{(1)} & \xrightarrow{f^{(1)}} & R^{(1)} & \xrightarrow{\tau} & K[\delta x] \\
\downarrow{\langle p_T, \delta T \rangle} & & \downarrow{\langle p_R, \delta R \rangle} & & \\
T & \xrightarrow{f} & R & & 
\end{array}
$$

Given $f \in T = \mathcal{O}_X(X)$, $\tau(f) = f^{(1)} \circ \pi(\delta x)$, so we get $\tau(f) = \delta_T(f) = df \in \text{Sym}(\Omega_T)/I_T$, which is easily seen to be a $\tau$-derivation. □

**Proposition 3.15.** Let $X$ be a variety. The $\tau$-differential map $\tau : \mathcal{O}_X \to p_* \mathcal{O}_{X^{(1)}}$ is the same as Buium’s map, $\tau^{(1)} : \mathcal{O}_X \to p_* \mathcal{O}_{X^{(1)}}$.

Proof. Immediate from the proof of the previous lemma and Definition 3.12. □

We now define the coherent sheaf of $\tau$-differentials on a variety $X$, written $\Omega^\tau_X$, which will be locally free of rank $\dim(X) + 1$. As a subsheaf of $p_* \mathcal{O}_{X^{(1)}}$, these can be viewed as rational functions on $X^{(1)}$, which are affine maps on each fiber of $X^{(1)} \to X$.

For $X = \text{Spec} T$ affine, let $\hat{\Omega}^\tau_T$ be the submodule of $T^{(1)}$ generated by $\delta_T(T)$ (which is isomorphic to $\Omega^\tau_T$, by Proposition 2.27). Then $\Omega^\tau_X$ is the $\mathcal{O}_X$-sheaf $(\hat{\Omega}^\tau_T)^\wedge$, which naturally embeds in the $\mathcal{O}_X$-sheaf $(T^{(1)})^\wedge$. The following result globalizes this to varieties.

**Lemma 3.16.** Let $X$ be a $K$-variety, $\pi : X \to K$. There is the following diagram of sheaves on $X$, with each row exact. In particular, $\Omega^\tau_X$ is locally free of rank $\dim X + 1$.

$$
\begin{array}{cccccccc}
0 & \xrightarrow{} & \pi^* \Omega_K & \xrightarrow{} & \Omega_X & \xrightarrow{} & \Omega_{X/K} & \xrightarrow{} & 0 \\
\downarrow{\delta} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} & \\
0 & \xrightarrow{} & \mathcal{O}_X & \xrightarrow{} & \Omega^\tau_X & \xrightarrow{\Lambda} & \Omega_{X/K} & \xrightarrow{} & 0
\end{array}
$$

Proof. From the proof of Lemma 2.10 and Corollary 2.11. □

As an extension of $\Omega_X$ by $\mathcal{O}_X$, $\Omega^\tau_X$ corresponds to an element of the cohomology group $\text{Ext}^1(\Omega_X, \mathcal{O}_X)$, which is naturally isomorphic to $H^1(X, \Theta_{X/K})$, the dual sheaf of $\Omega_{X/K}$. In fact, $\Omega^\tau_X$ corresponds to the Kodaira-Spencer class of $X$, as defined in [Bui93], p. 1396, as can be easily seen by comparing the diagram in the preceding lemma with Buium’s construction. (See also [Ros05]).

**Proposition 3.17.** Given a variety $X$, the sheaf $\Omega^\tau_X$ corresponds to the Kodaira-Spencer class of $X$.

**Lemma 3.18.** Let $f : X \to Y$ be a morphism of $K$-varieties. Then there is an exact sequence of sheaves on $X$,

$$
f^* \Omega^\tau_Y \longrightarrow \Omega^\tau_X \longrightarrow \Omega_{X/Y} \longrightarrow 0
$$

Proof. From Lemma 2.16. □

**Remark 3.19.** Everything is functorial so, for example, given a morphism of $K$-varieties, $f : X \to Y$, the following diagram of $\mathcal{O}_X$-modules commutes.

$$
\begin{array}{cccccccc}
\Omega^\tau_X & \xrightarrow{f^* \Omega^\tau_Y} & \Omega^\tau_X & \xrightarrow{f^* \Omega^\tau_Y} & \Omega_{X/K} & \xrightarrow{f^* \Omega^\tau_Y} & \Omega_{X/Y} & \xrightarrow{f^* \Omega^\tau_Y} & 0 \\
\downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} & & \\
\Omega_{X/K} & \xrightarrow{f^* \Omega^\tau_Y} & \Omega_{X/Y} & \xrightarrow{f^* \Omega^\tau_Y} & 0
\end{array}
$$
4. The prolongation cone

We introduce a new construction, the prolongation cone of a variety. If the variety $X$ is smooth, then it will be the smallest vector bundle over $X$ into which both $TX$ and $X^{(1)}$ can be embedded.

**Definition 4.1.** Let $X$ be a $K$-variety and $\Omega^\ast_X$ the sheaf of $\tau$-differentials. The **prolongation cone** of $X$, written $CX$, is $\text{Spec} \text{Sym}(\Omega^\ast_X)$.

For $X$ smooth, $\Omega^\ast_X$ is a locally free sheaf, and $CX$ is the geometric vector bundle associated to it.

To prove that there is a closed embedding of $X^{(1)}$ in $CX$ over $X$, it suffices to prove this on affine subvarieties of $X$. Reformulated in terms of $K$-algebras, this is equivalent to showing that, for any $K$-algebra $R$, there is a natural surjective $R$-algebra homomorphism from $\text{Sym}(\Omega^\ast_R)$ onto $R^{(1)}$.

**Proposition 4.2.** Let $R$ be a $K$-algebra, and $\bar{R} := \text{Sym}(\Omega^\ast_R)$. There is a natural surjective $R$-algebra homomorphism $\bar{R} \to R^{(1)}$.

**Proof.** Let $S$ be the $R$-algebra $S := \text{Sym}(\Omega_R)$. Recall that the prolongation of $R$ is $R^{(1)} := S/J$, $J$ the ideal generated by $\langle da - \delta a \mid a \in K \rangle$. Let $\bar{R} := \text{Sym}(\Omega^\ast_R)$.

Let $f : S \to R^{(1)}$ be the natural quotient map, with kernel $J$. Since there is a natural surjection of $R$-modules from $\Omega_R$ to $\Omega^\ast_R$, there will also be a natural surjection $\bar{J} \to \Omega^\ast_R$ from $S$ onto $\bar{R}$, with some kernel $I$. Thus, to give the desired surjection from $\bar{R}$ onto $R^{(1)}$, it suffices to show that $I \subseteq J$. Then $f$ will factor through $g$, so that the desired surjection $h : \bar{R} \to R^{(1)}$, $f = h \circ g$, has kernel $\cong J/I$.

The kernel $N$ of the natural $R$-module homomorphism from $\Omega_R$ to $\Omega^\ast_R$ is generated by $\langle \delta(a)db - \delta(b)da \mid a,b \in K \rangle$ (which is immediate from the definition of $\Omega^\ast_R$, but see also Lemma 5.7 and the proof of Proposition 5.25). Thus the kernel $I$ of $g : S \to \bar{R}$ is the ideal $I := 0 \oplus N \oplus N^2 \oplus \ldots$.

Finally, we observe that $I \subseteq J$. The ideal $I$ is generated by elements of the form $\delta(a)db - \delta(b)da$, $a,b \in K$, which can be written as $da(db - \delta(b)) - db(da - \delta(a))$, and are thus in $J$.

**Corollary 4.3.** For any variety $X$, there is a natural closed embedding of $X^{(1)}$ into $CX$.

For any $K$-variety $X$, let $TX$ denote $\text{Spec} \text{Sym}(\Omega^\ast_{X/K})$, which equals the usual tangent variety of $X$ when $X$ is smooth. To prove that there is a closed embedding of $TX$ in $CX$, one can again, as above, reduce it to an assertion about $K$-algebras.

**Proposition 4.4.** Let $R$ be a $K$-algebra. Then there is an $R$-algebra homomorphism from $\text{Sym}(\Omega^\ast_R)$ onto $\text{Sym}(\Omega^\ast_{R/K})$.

**Proof.** The $R$-module homomorphism from $\Omega^\ast_R$ onto $\Omega^\ast_{R/K}$ determines such a map. Alternatively, one can argue as in the previous proposition.

**Corollary 4.5.** For any variety $X$, there is a natural closed embedding of $TX$ into $CX$.

Let $X$ be an affine variety, together with a closed embedding $X \to \mathbb{A}^n$. We now describe $CX$ and the embeddings of $X^{(1)}$ and $TX$ in $CX$ in local coordinates.

Above we defined an affine space as a principal homogeneous space of a vector space. For the next proposition, it will be helpful to recall an alternative characterization of an affine space as a coset $A$ of a linear subspace of a vector space $V$. In this case, say that $A$ is a proper affine space if $0 \notin A$. Given a proper $A$, let $A' := \{ca \mid c \in K, a \in A\}$, the smallest linear space containing $A$. If $A$ is proper, there is a surjective homomorphism $V^\vee \to \text{Aff}(A)$. If, additionally, $A$ has codimension 1, then this is an isomorphism. Note that, given an affine space $A \subseteq V$, there is an associated, isomorphic, affine space $A' \subseteq V' := V \times K$, $A' := \{a \times 1 \mid a \in A\}$, that is clearly proper. In particular, if $A = V$, then $A' \subseteq V'$ has codimension 1 and $V^\vee \cong \text{Aff}(A)$.

In the same way, one can also define an affine bundle over a variety $X$ to be a closed subvariety of a vector bundle $Y$ over $X$ with the obvious properties.

**Proposition 4.6.** Let $X = \text{Spec} B \subseteq \mathbb{A}^n$ be an affine variety. Let $A := K[x_1, \ldots, x_n]$ and $B = A/I$, $I$ an ideal of $A$. Let $C = K[x_1, \ldots, x_n, \tau x_1, \ldots, \tau x_n, \tau e]$. Then $CX = \text{Spec} C/J$, where $J$ is the ideal generated by $I$ and $\{ \tau f \mid f \in I \}$.
Corollary 4.7. Using notation from the previous proposition, the prolongation as desired. □

One can easily show that these embeddings, in local coordinates, are the same as the ones

Remark 4.8. CX to the intersection of TX is naturally isomorphic to the intersection of CX

In particular, for each f ∈ A, τf is an affine bundle map on X(1). For each K-valued point b ∈ X, τf(b) is an affine map on the affine space X(1).b.

We now consider the general case, X = Spec B, B = A/I. By Lemma 2.15 we have an exact sequence of B-modules,

\[ I/I^2 \xrightarrow{\alpha} B \otimes_A \Omega^*_A \xrightarrow{\beta} \Omega^*_B \to 0 \]

where \( \alpha(f) = 1 \otimes \tau f \), for \( f \in I \). The map \( \alpha \) is not necessarily injective, but there is always the related short exact sequence,

\[ 0 \to \alpha(I/I^2) \to B \otimes_A \Omega^*_A \xrightarrow{\beta} \Omega^*_B \to 0. \]

where \( \alpha(I/I^2) = \{ 1 \otimes \tau f \mid f \in I \} \subseteq B \otimes_A \Omega^*_A \).

Since \( \Omega^*_A \) is a free rank \( n + 1 \) \( A \)-module, \( B \otimes_A \Omega^*_A \) is a free rank \( n + 1 \) \( B \)-module, and \( \Omega^*_B \cong (B \otimes_A \Omega^*_A) / \alpha(I/I^2) \). For each K-valued point \( b \in X \), \( (X_b(1)) \circ \subseteq (\tau X_b)^\circ \) is the linear subspace equal to \( \cap_{f \in I} \Ker(\tau f(b)) \), as desired. □

The following corollary follows easily, by the standard embeddings of \( X(1) \) and \( TX \) in affine space.

Corollary 4.7. Using notation from the previous proposition, the prolongation \( X(1) \) is naturally isomorphic to the intersection of \( CX \) with the hyperplane of \( \mathbb{A}^{n+1} \) defined by \( \tau e - 1 = 0 \). Likewise, the tangent space \( TX \) is naturally isomorphic to the intersection of \( CX \) with the hyperplane of \( \mathbb{A}^{n+1} \) defined by \( \tau e = 0 \).

With these natural embeddings of \( X(1) \) and \( TX \) in \( CX \), \( TX \) is a vector subbundle of \( CX \), and \( X(1) \) is an affine subbundle of \( CX \). Further, \( X(1) \) is a \( TX \)-torsor under the action given by vector space addition in \( CX \).

Finally, \( TX \) and \( X(1) \) are (disjoint) principal divisors of \( CX \).

Remark 4.8. One can easily show that these embeddings, in local coordinates, are the same as the ones described above in terms of surjective \( R \)-algebra homomorphisms.

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