Comments on J. F. Ritt’s book “Integration in Finite Terms”

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Abstract

The First and Second Liouville’s Theorems provide correspondingly criterium for integrability of elementary functions “in finite terms” and criterium for solvability of second order linear differential equations by quadratures. The brilliant book of J.F. Ritt contains proofs of these theorems and many other interesting results. This paper was written as comments on the book but one can read it independently. The first part of the paper contains modern proofs of The First Theorem and of a generalization of the Second Theorem for linear differential equations of any order. In the second part of the paper we present an outline of topological Galois theory which provides an alternative approach to the problem of solvability of equations in finite terms. The first section of this part deals with a topological approach to representability of algebraic functions by radicals and to the 13-th Hilbert problem. This section is written with all proofs. Next sections contain only statements of results and comments on them (basically no proofs are presented there).

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1 Preface

I saw J.F. Ritt’s book [Rit48] for the first time in 1969 when I was an undergraduate student. I just started to work on topological obstructions to representability of algebraic functions by radicals and on an algebraic version of the 13th Hilbert problem on representability of algebraic functions of several complex variables by composition of algebraic functions of fewer number of variables. My beloved supervisor Vladimir Igorevich Arnold was very interested in these questions.

J.F. Ritt’s approach, which uses the theory of complex analytic functions and geometry, was very different from a formal algebraic approach. I was very intrigued and I started to read the book trying to get a feeling about the subject and avoiding all the details for the first reading.

My first impression was that the book was brilliant and the presented theory was ingenious. Simultaneously with the reading I obtained the very first results of topological Galois theory. Since then I have spent a few years developing it. I had hoped to return back to the book later, but I never made it (life is life!).

Even a brief reading turned out to be very useful. It helped me to formalize the definition of the Liouvillian classes of functions and the definition of the functional differential fields and their extensions. Later this experience helped me to find an appropriate definition of the class of Pfaffian functions playing the crucial role in a transcendental generalization of real algebraic geometry developed in the book [Kho91].

That is why I was really happy when Michael Singer invited me to write comments for a reprint of the book. I started to read it again after almost a half century break. I have to confess that it was hard for me to follow all the details. The reason is that J.F. Ritt uses an old mathematical language (the book was written about seventy years ago). Nevertheless I still think that that the book is brilliant and Liouville’s and Ritt’s ideas are ingenious.

In section 2 we present modern definitions of Liouvillian classes of functions and modern proofs of the First Liouville’s Theorem and of the Second Liouville’s Theorem. I hope that this modern presentation will help readers understand better the subject and J.F. Ritt’s book.

In the section 3 we present an outline of topological Galois theory which provides an alternative approach to the problem of solvability of equations in finite terms. We use the definition of classes of functions by the list of basic functions and the list of admissible operations presented in the section 2.2.

A few words about our proof of the First Liouville’s Theorem. All main ideas of the proof are presented in the book. I tried to clarify what is hidden behind the integration used by Liouville. I think that there are two statements which were not mentioned explicitly in the book: 1) a closed 1-form with elementary integral whose possible form was found by Liouville is locally invariant under the Galois group action, assuming that the Galois group is connected; 2) A class of closed 1-forms locally invariant under a connected Lie group action can be described explicitly. In fact all arguments needed for proving the first statement are presented in the book. Liouville used an explicit integration for description of closed 1-forms locally invariant under a natural action of the additive and the multiplicative groups of complex numbers.

A few words about our proof of the Second Liouville’s Theorem. In fact we prove its generalization applicable to linear homogeneous differential equation of any order. J.F. Ritt’s book contains basically all results needed for our version of the proof. Only some statements are missing there (but all arguments needed for their proofs are presented in some form in the book).

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2 Solvability of Equations in Finite Terms.

2.1 Introduction

Let \( K \) be a subfield of the field of meromorphic functions on a connected domain \( U \) of the complex line closed under the differentiation (i.e. if \( f \in K \) then \( f' \in K \)). Such field \( K \) with the operation of differentiation \( f \rightarrow f' \) provides an example of functional differential field.

Liouville’s First Theorem suggests conditions on a function \( f \) from a function differential field \( K \) which are necessary and sufficient for representability of an indefinite integral of \( f \) in generalized elementary functions over \( K \).

Liouville’s Second Theorem suggests conditions on second order homogeneous linear differential equation whose coefficients belong to a function differential field \( K \) which are necessary and sufficient for its solvability by generalized quadratures over \( K \).

The Liouville’s theory can be generalized to an abstract differential field \( K \), whose elements are not necessarily meromorphic functions (see [Kho14], [dPS03]). Abstract algebraic results are not directly applicable to integrals of elementary functions and to solutions of linear differential equations which could be multivalued, could have singularities and so on. For their applications some extra arguments are needed. Such arguments are presented in Chapter I of J.F. Ritt’s book.

This section has a following content

In section 2.2 we define functional differential fields, generalized elementary functions and generalized quadratures over such fields. Material of section 1 was inspired by Chapter I of the book [Rit48].

In section 2.3 we prove Liouville’s First Theorem making use of algebraic groups actions. Our proof can be considered as a modernization of the Liouville’s proof presented in Chapter II and Chapter III of the book [Rit48].

In section 2.4 we prove Liouville’s Second Theorem and its generalizations for homogeneous linear differential equations of any order. Our proof uses slightly modernized Liouville’s-Ritt’s arguments presented in Chapters V and VI of J.F. Ritt’s book.

2.2 Generalized Elementary Functions and Generalized Quadratures.

2.2.1 Introduction

The results presented in this section are inspired by the the material of Chapter 1 from Ritt’s book [Rit48]. We discuss here definitions and general statements related to functional and abstract differential fields and classes of their extensions including generalized elementary extensions and extensions by generalized quadratures. We follow mainly the presentation from the book [Kho14].

A natural definitions of generalized elementary function and of a function representable by generalized quadratures over \( K \) (see definitions 2.6, 2.7, 2.8, and 2.9 below) are hard to deal with. In particular they make use of a non algebraic operation of composition of functions. Algebraic definitions (see definitions 2.2 and 2.3 below) use solution of the simplest differential equations instead of composition of functions. We explain how the natural definitions can be reduced to the algebraic ones.
2.2.2 Differential fields and their extensions

Let us start with some pure algebraic definitions.

2.2.2.1 Abstract differential fields A field \( F \) is said to be a differential field if an additive map \( a \rightarrow a' \) is fixed that satisfies the Leibnitz rule \( (ab)' = a'b + ab' \). The element \( a' \) is called the derivative of \( a \). An element \( y \in F \) is called a constant if \( y' = 0 \). All constants in \( F \) form the field of constants. We add to the definition of differential field an extra condition that the field of constants is the field of complex numbers (for our purpose it is enough to consider fields satisfying this condition). An element \( y \in F \) is said to be: an exponential of \( a \) if \( y' = a'y \); an exponential of integral of \( a \) if \( y' = ay \); a logarithm of \( a \) if \( y' = a'/a \); an integral of \( a \) if \( y' = a \). In each of these cases, \( y \) is defined only up to an additive or a multiplicative complex constant.

Let \( K \subset F \) be a differential subfield in \( F \). An element \( y \) is said to be an integral over \( K \) if \( y' = a \in K \). An exponential of integral over \( K \), a logarithm over \( K \), and an integral over \( K \) are defined similarly.

Suppose that a differential field \( K \) and a set \( M \) lie in some differential field \( F \). The adjunction of the set \( M \) to the differential field \( K \) is the minimal differential field \( K\langle M \rangle \) containing both the field \( K \) and the set \( M \). We will refer to the transition from \( K \) to \( K\langle M \rangle \) as adjoining the set \( M \) to the field \( K \).

2.2.2.2 Generalized elementary extensions Let \( F \supset K \) be an extension of a differential field \( K \).

Definition 2.1. The differential field \( F \) is said to be a generalized elementary extension of the differential field \( K \) if \( K \subset F \) and there exists a chain of differential fields \( K = F_0 \subseteq \cdots \subseteq F_n \supset F \) such that \( F_{i+1} = F_i < y_i > \) for every \( i = 0, \ldots, n-1 \) where \( y_i \) is an exponential, a logarithm, or an algebraic element over \( F_i \).

An element \( a \in F \) is a generalized elementary element over \( K, K \subset F \), if it is contained in a certain generalized elementary extension of the field \( K \).

The following lemma is obvious.

Lemma 2.1. An extension \( K \subset F \) is a generalized elementary extension if and only if there exists a chain of differential fields \( K = F_0 \subseteq \cdots \subseteq F_n \supset F \) such that for every \( i = 0, \ldots, n-1 \), either \( F_{i+1} \) is a finite extension of \( F_i \), or \( F_{i+1} \) is a pure transcendental extension of \( F_i \) obtained by adjoining finitely many exponentials and logarithms over \( F_i \).

2.2.2.3 Extensions by generalized quadratures Let \( F \supset K \) be an extension of a differential fields \( K \).

Definition 2.2. The differential field \( F \) is said to be an extension of the differential field \( K \) by generalized quadratures if \( K \subset F \) and there exists a chain of differential fields \( K = F_0 \subseteq \cdots \subseteq F_n \supset F \) such that \( F_{i+1} = F_i < y_i > \) for every \( i = 0, \ldots, n-1 \) where \( y_i \) is an exponential of integral, an integral, or an algebraic element over \( F_i \). An element \( a \in F \) is representable by generalized quadratures over \( K, K \subset F \), if it is contained in a certain generalized extension of the field \( K \) by elementary generalized quadratures.

Definition 2.3. An extension \( F \) of a differential field \( K \) is said to be:

1) a generalized extension by integral if there are \( y \in F \) and \( f \in K \) such that \( y' = f \), \( y \) is transcendental over \( K \), and \( F \) is a finite extension of the field \( K(y) \),

2) a generalized extension by exponential of integral if there are \( y \in F, f \in K \) such that \( y' = fy \), \( y \) is transcendental over \( K \), and \( F \) is a finite extension of the field \( K(y) \).

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1These are also called liouvillian extensions
The following lemma is obvious.

**Lemma 2.2.** An extension $K \subset F$ is an extension by generalized quadratures if there is a chain $K = F_0 \subset \cdots \subset F_n$ such that $F \subset F_n$ and for every $i = 0, \ldots, n-1$ or $F_{i+1}$ is a finite extension of $F_i$, or $F_{i+1}$ is a generalized extension by integral of $F_i$, or $F_{i+1}$ is a generalized extension by exponential integral of $F_i$.

### 2.2.3 Functional differential fields and their extensions

Let $K$ be a subfield in the field $F$ of all meromorphic functions on a connected domain $U$ of the Riemann sphere $\mathbb{C}^1 \cup \infty$ with the fixed coordinate function $x$ on $\mathbb{C}^1$. Suppose that $K$ contains all complex constants and is stable under differentiation (i.e. if $f \in K$, then $f' = df/dx \in K$). Then $K$ provides an example of a functional differential field.

Let us now give a general definition.

**Definition 2.4.** Let $U, x$ be a pair consisting of a connected Riemann surface $U$ and a nonconstant meromorphic function $x$ on $U$. The map $f \to df/\pi^* dx$ defines the derivation in the field $F$ of all meromorphic functions on $U$ (the ratio of two meromorphic 1-forms is a well-defined meromorphic function). A functional differential field is any differential subfield of $F$ (containing all complex constants).

The following construction helps to extend functional differential fields. Let $K$ be a differential subfield of the field of meromorphic functions on a connected Riemann surface $U$ equipped with a meromorphic function $x$. Consider any connected Riemann surface $V$ together with a nonconstant analytic map $\pi : V \to U$. Fix the function $\pi^* x$ on $V$. The differential field $F$ of all meromorphic functions on $V$ with the differentiation $\varphi' = d\varphi/\pi^* dx$ contains the differential subfield $\pi^* K$ consisting of functions of the form $\pi^* f$, where $f \in K$. The differential field $\pi^* K$ is isomorphic to the differential field $K$, and it lies in the differential field $F$. For a suitable choice of the surface $V$, an extension of the field $\pi^* K$, which is isomorphic to $K$, can be done within the field $F$.

Suppose that we need to extend the field $K$, say, by an integral $y$ of some function $f \in K$. This can be done in the following way. Consider the covering of the Riemann surface $U$ by the Riemann surface $V$ of an indefinite integral $y$ of the form $f dx$ on the surface $U$. By the very definition of the Riemann surface $V$, there exists a natural projection $\pi : V \to U$, and the function $y$ is a single-valued meromorphic function on the surface $V$. The differential field $F$ of meromorphic functions on $V$ with the differentiation $\varphi' = d\varphi/\pi^* dx$ contains the element $y$ as well as the field $\pi^* K$ isomorphic to $K$. That is why the extension $\pi^* K(y)$ is well defined as a subfield of the differential field $F$. We mean this particular construction of the extension whenever we talk about extensions of functional differential fields. The same construction allows to adjoin a logarithm, an exponential, an integral or an exponential of integral of any function $f$ from a functional differential field $K$ to $K$. Similarly, for any functions $f_1, \ldots, f_n \in K$, one can adjoin a solution $y$ of an algebraic equation $y^n + f_1 y^{n-1} + \cdots + f_n = 0$ or all the solutions $y_1, \ldots, y_n$ of this equation to $K$ (the adjunction of all the solutions $y_1, \ldots, y_n$ can be implemented on the Riemann surface of the vector-function $y = y_1, \ldots, y_n$). In the same way, for any functions $f_1, \ldots, f_{n+1} \in K$, one can adjoin the $n$-dimensional $\mathbb{C}$-affine space of all solutions of the linear differential equation $y^{(n)} + f_1 y^{(n-1)} + \cdots + f_n y + f_{n+1} = 0$ to $K$. (Recall that a germ of any solution of this linear differential equation admits an analytic continuation along a path on the surface $U$ not passing through the poles of the functions $f_1, \ldots, f_{n+1}$.)

Thus, all above-mentioned extensions of functional differential fields can be implemented without leaving the class of functional differential fields. When talking about extensions of functional differential fields, we always mean this particular procedure.

The differential field of all complex constants and the differential field of all rational
functions of one variable can be regarded as differential fields of functions defined on the Riemann sphere.

### 2.2.4 Classes of functions and operations on multivalued functions

An indefinite integral of an elementary function is a function rather than an element of an abstract differential field. In functional spaces, for example, apart from differentiation and algebraic operations, an absolutely non-algebraic operation is defined, namely, the composition. Anyway, functional spaces provide more means for writing “explicit formulas” than abstract differential fields. Besides, we should take into account that functions can be multivalued, can have singularities and so on.

In functional spaces, it is not hard to formalize the problem of unsolvability of equations in explicit form. One can proceed as follows: fix a class of functions and say that an equation is solvable explicitly if its solution belongs to this class. Different classes of functions correspond to different notions of solvability.

#### 2.2.4.1 Defining classes of functions by the lists of data

A class of functions can be introduced by specifying a list of basic functions and a list of admissible operations. Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations. Below, we define the class of generalized elementary functions and the class of generalized elementary functions over a functional differential field $K$ in exactly this way.

Classes of functions, which appear in the problems of integrability in finite terms, contain multivalued functions. Thus the basic terminology should be made clear. We work with multivalued functions “globally”, which leads to a more general understanding of classes of functions defined by lists of basic functions and of admissible operations. A multivalued function is regarded as a single entity. Operations on multivalued functions can be defined. The result of such an operation is a set of multivalued functions; every element of this set is called a function obtained from the given functions by the given operation. A class of functions is defined as the set of all (multivalued) functions that can be obtained from the basic functions by repeated application of admissible operations.

#### 2.2.4.2 Operations on multivalued functions

Let us define, for example, the sum of two multivalued functions on a connected Riemann surface $U$.

**Definition 2.5.** Take an arbitrary point $a$ in $U$, any germ $f_a$ of an analytic function $f$ at the point $a$ and any germ $g_a$ of an analytic function $g$ at the same point $a$. We say that the multivalued function $\varphi$ on $U$ generated by the germ $\varphi_a = f_a + g_a$ is representable as the sum of the functions $f$ and $g$.

For example, it is easy to see that exactly two functions of one variable are representable in the form $\sqrt{x} + \sqrt{x}$, namely, $f_1 = 2\sqrt{x}$ and $f_2 \equiv 0$. Other operations on multivalued functions are defined in exactly the same way. For a class of multivalued functions, being stable under addition means that, together with any pair of its functions, this class contains all functions representable as their sum. The same applies to all other operations on multivalued functions understood in the same sense as above.

In the definition given above, not only the operation of addition plays a key role but also the operation of analytic continuation hidden in the notion of multivalued function. Indeed, consider the following example. Let $f_1$ be an analytic function defined on an open subset $V$ of the complex line $\mathbb{C}^1$ and admitting no analytic continuation outside of $V$, and let $f_2$ be an analytic function on $V$ given by the formula $f_2 = -f_1$. According to our definition, the zero function is representable in the form $f_1 + f_2$ on the entire complex line. By the commonly accepted viewpoint, the equality $f_1 + f_2 = 0$ holds inside the region $V$ but not outside.
Working with multivalued functions globally, we do not insist on the existence of a common region, were all necessary operations would be performed on single-valued branches of multivalued functions. A first operation can be performed in a first region, then a second operation can be performed in a second, different region on analytic continuations of functions obtained on the first step. In essence, this more general understanding of operations is equivalent to including analytic continuation to the list of admissible operations on the analytic germs.

2.2.5 Generalized elementary functions

In this section we define the generalized elementary functions of one complex variable and the generalized elementary functions over a functional differential field. We also discuss a relation of these notions with generalized elementary extensions of differential fields. First we’ll present needed lists of basic functions and of admissible operations.

List of basic elementary functions

1. All complex constants and an independent variable $x$.
2. The exponential, the logarithm, and the power $x^\alpha$ where $\alpha$ is any constant.
3. The trigonometric functions sine, cosine, tangent, cotangent.
4. The inverse trigonometric functions arcsine, arccosine, arctangent, arccotangent.

Lemma 2.3. Basic elementary functions can be expressed through the exponentials and the logarithms with the help of complex constants, arithmetic operations and compositions. Lemma 2.3 can be considered as a simple exercise. Its proof can be found in [Kho14].

List of some classical operations

1. The operation of composition takes functions $f,g$ to the function $f \circ g$.
2. The arithmetic operations take functions $f,g$ to the functions $f + g, f - g, fg, \text{ and } f/g$.
3. The operation of differentiation takes function $f$ to the function $f'$.
4. The operation of integration takes function $f$ to a solution of equation $y' = f$ (the function $y$ is defined up to an additive constant).
5. The operation of taking exponential of integral takes function $f$ to a solution of equation $y' = fy$ (the function $y$ is defined up to a multiplicative constant).
6. The operation of solving algebraic equations takes functions $f_1, \ldots, f_n$ to the function $y$ such that $y^n + f_1y^{n-1} + \cdots + f_n = 0$ (the function $y$ is not quite uniquely determined by functions $f_1, \ldots, f_n$ since an algebraic equation of degree $n$ can have $n$ solutions).

Definition 2.6. The class of generalized elementary functions of one variable is defined by the following data: List of basic functions: basic elementary functions. List of admissible operations: Compositions, Arithmetic operations, Differentiation, Operation of solving algebraic equations.

Theorem 2.1. A (possibly multivalued) function of one complex variable belongs to the class of generalized elementary functions if and only if it belongs to some generalized elementary extension of the differential field of all rational functions of one variable.
Theorem 2.1 follows from Lemma 2.3 (all needed arguments can be found in [Kho14]).

Let $K$ be a functional differential field consisting of meromorphic functions on a connected Riemann surface $U$ equipped with a meromorphic function $x$.

**Definition 2.7.** Class of generalized elementary functions over a functional differential field $K$ is defined by the following data.

- List of basic functions: all functions from the field $K$.
- List of admissible operations: Operation of composition with a generalized elementary function $\phi$ that takes $f$ to $\phi \circ f$, Arithmetic operations, Differentiation, Operation of solving algebraic equations.

**Theorem 2.2.** A (possibly multivalued) function on the Riemann surface $U$ belongs to the class of generalized elementary functions over a functional differential field $K$ if and only if it belongs to some generalize elementary extension of $K$.

Theorem 2.2 follows from Lemma 2.3 (all needed arguments can be found in [Kho14]).

### 2.2.6 Functions representable by generalized quadratures

Here we define functions of one complex variable representable by generalized quadratures and functions representable by generalized quadratures over a functional differential field. We also discuss a relation of these notions with extensions of functional differential fields by generalized quadratures. First we’ll present needed lists of basic functions and of admissible operations.

**Definition 2.8.** The class of functions of one complex variable representable by generalized quadratures is defined by the following data:

- List of basic functions: basic elementary functions.
- List of admissible operations: Compositions, Arithmetic operations, Differentiation, Integration, Operation of taking exponential of integral, Operation of solving algebraic equations.

**Theorem 2.3.** A (possibly multivalued) function of one complex variable belongs to the class of functions representable by generalized quadratures if and only if it belongs to some extension of the differential field of all constant functions of one variable by generalized quadratures.

Theorem 2.3 follows from Lemma 2.3 (All needed arguments can be found in [Kho14]).

Let $K$ be a functional differential field consisting of meromorphic functions on a connected Riemann surface $U$ equipped with a meromorphic function $x$.

**Definition 2.9.** The class of functions representable by generalized quadratures over the functional differential field $K$ is defined by the following data:

- List of basic functions: all functions from the field $K$.
- List of admissible operations: Operation of composition with a generalized elementary function $\phi$ that takes $f$ to $\phi \circ f$, Arithmetic operations, Differentiation, Integration, Operation of taking exponential of integral, Operation of solving algebraic equations.

**Theorem 2.4.** A (possibly multivalued) function on the Riemann surface $U$ belongs to the class of generalized quadratures over a functional differential field $K$ if and only if it belongs to some extension of $K$ by generalized quadratures.

Theorem 2.4 follows from Lemma 8 (all needed arguments can be found in [Kho14]).

### 2.3 Liouville’s First Theorem and Actions of Lie Groups

#### 2.3.1 Introduction

In 1833 Joseph Liouville proved the following fundamental result.
Liouville’s First Theorem: An integral \( y \) of a function \( f \) from a functional differential field \( K \) is a generalized elementary function over \( K \) if and only if \( y \) is representable in the form

\[
y(x) = \int_{x_0}^{x} f(t) \, dt = r_0(x) + \sum_{i=1}^{m} \lambda_i \ln r_i(x),
\]

where \( r_0, \ldots, r_m \in K \) and \( \lambda_1, \ldots, \lambda_m \) are complex constants.

For large classes of functions algorithms based on Liouville’s Theorem make it possible to either evaluate an integral or to prove that the integral cannot be “evaluated in finite terms”.

In this section we prove the Liouville’s First Theorem. We follow the presentation from the paper [Kho18a].

Let \( K(y) \supset K \) be an extension obtained by adjoining to a functional differential field \( K \) an integral \( y \) over \( K \). The differential Galois group \( G \) of this extension does not contain enough information to determine if the integral \( y \) belongs to a generalized elementary extension of \( K \) or not. Indeed, if the integral \( y \) does not belong to \( K \) then group \( G \) is always the same: it is isomorphic to the additive group of complex numbers. From this fact one can conclude that the Galois theory is not sensitive enough for proving Liouville’s First Theorem. Nevertheless, Liouville’s First Theorem can be proved using differential Galois groups.

The first step towards such a proof was suggested by Abel (see sections 2.3.3.1 and 2.3.3.2). This step is related to algebraic extensions and their finite Galois groups.

A second step (see section 2.3.4) deals with a pure transcendental extension \( F \) of a functional differential field \( K \), obtained by adjoining \( k + n \) logarithms and exponentials, algebraically independent over \( K \). The differential Galois group of the extension \( K \subset F \) is an \((k + n)\)-dimensional connected commutative algebraic group \( G \). It has a natural representation as a group of analytic automorphisms of an analytic variety \( X \). Thus \( G \) acts not only on the differential field \( F \) but also on other objects such as closed 1-forms on \( X \). This action plays a key role in our proof (see section 2.3.4.1).

### 2.3.2 Outline of an inductive proof

Let us outline an inductive proof of Liouville’s Theorem.

**Definition 2.10.** A function \( g \) is a generalized elementary function of complexity \( \leq k \) if there is a chain \( K = F_0 \subset F_1 \subset \ldots \subset F_k \) of functional differential fields such that \( g \in F_k \) and for any \( 0 \leq i < k \) either \( F_{i+1} \) is a finite extension of \( F_i \), or \( F_{i+1} \) is a pure transcendental extension of \( F_i \) obtained by adjoining finitely many exponentials, and logarithms over \( F_i \).

We will prove the following induction hypothesis \( I(m) \): Liouville’s Theorem is true for every integral \( y \) of complexity \( \leq m \) over any functional differential field \( K \). The statement \( I(0) \) is obvious: if \( y \in K \), then \( y = r_0 \in K \). Now let \( y' \in K \) and \( y \in F_k \). Since \( y' \in F_1 \), by induction \( y = R_0 + \sum_{i=1}^{q} \lambda_i \ln R_i \), where \( R_0, R_1, \ldots, R_q \in F_1 \). We need to show that \( y \) is representable in the form \( 1 \) with \( r_0, \ldots, r_m \in F_0 = K \) We have the following two cases to consider:

1. \( F_1 \) is a finite extension of \( F_0 = K \). The statement of induction hypothesis in that case was proved by Abel and is called the Abel’s Theorem. We will present its proof in section 2.3.3.
2. \( F_1 \) a pure transcendental extension of \( F_0 = K \) obtained by adjoining exponentials and logarithms over \( K \). We will deal with this case in section 2.3.4.
2.3.3  Algebraic case

In Subsection 2.3.3.1 we discuss finite extensions of differential fields. In subsection 2.3.3.2 we present a proof of the Abel’s Theorem.

2.3.3.1  An algebraic extension of a functional differential field  Let

\[ P(z) = z^n + a_1z^{n-1} + \cdots + a_n \quad (2) \]

be an irreducible polynomial over \( K, P \in K[z] \). Then \( P \) over the field \( \hat{K} \) contains \( K \) and a root \( z \) of \( P \).

Lemma 2.4. The field \( K(z) \) is stable under the differentiation.

Proof. Since \( P \) is irreducible over \( K \), the polynomial \( \frac{\partial P}{\partial x} \) has no common roots with \( P \) and is different from zero in the field \( K[z]/(P) \). Let \( M \) be a polynomial satisfying a congruence \( M \frac{\partial P}{\partial z} \equiv -\frac{\partial P}{\partial x} (mod P) \). Differentiating the identity \( P(z) = 0 \) in the field \( F \), we obtain that \( \frac{\partial P}{\partial x}(z) + \frac{\partial P}{\partial z}(z) = 0 \), which implies that \( z' = M(z) \). Thus the derivative of the element \( z \) coincides with the value \( z \) of a polynomial \( M \). Lemma 2.4 follows from this fact.

Let \( K \subset F \) and \( \hat{K} \subset \hat{F} \) be functional differential fields, and \( P, \hat{P} \) irreducible polynomials over \( K, \hat{K} \) correspondingly. Suppose that \( F, \hat{F} \) contain roots \( z, \hat{z} \) of \( P, \hat{P} \).

Theorem 2.5. Assume that there is an isomorphism \( \tau : K \to \hat{K} \) of differential fields \( K, \hat{K} \) which maps coefficients of the polynomial \( P \) to the corresponding coefficients of the polynomial \( \hat{P} \). Then \( \tau \) can be extended in a unique way to the differential isomorphism \( \rho : K(z) \to \hat{K}(\hat{z}) \).

Theorem 2.5 could be obtained by the arguments used in the proof of Lemma 2.4.

2.3.3.2  Induction hypothesis for an algebraic extension  Let \( z_1, \ldots, z_n \) be the roots of the polynomial \( P \) given by (2) and let \( F_1 = K(z_1) \). Assume that there is an element \( y_1 \in F_1 \), such that \( y_1' \in K, M_i \in K[z] \) and \( y_1' \) is representable in the form

\[ y_1' = \sum_{i=1}^{q} \lambda_i \frac{(M_i(z_1))'}{M_i(z_1)} + (M_0(z_1))'. \quad (4) \]

Abel’s Theorem Under the above assumptions the element \( y_1' \) is representable in the form (1) with polynomials \( M_i \) independent of \( z_1 \), i.e. with \( M_0, M_1, \ldots, M_q \in K \).

Proof. Let \( y_1 \) be equal to \( Q(z_1) \) where \( Q \in K[z] \). For any \( 1 \leq j \leq n \) let \( y_j \) be the element \( Q(z_j) \). According to Theorem 19 the identity (4) implies the identity

\[ y_j' = \sum_{i=1}^{q} \lambda_i \frac{(M_i(z_j))'}{M_i(z_j)} + (M_0(z_j))'. \quad (5) \]

Since \( y_j' \in K \) we obtain \( n \) equalities \( y_1' = \cdots = y_n' \). To complete the proof it is enough to take the arithmetic mean of \( n \) equalities (4). Indeed the elements \( \hat{M}_i = \prod_{1 \leq k \leq n} M_i(z_k) \) and \( \hat{M}_0 = \sum_{1 \leq k \leq n} M_0(z_k) \) are symmetric functions in the roots of the polynomial \( P \) thus \( \hat{M}_0, \ldots, \hat{M}_q \in K \).

Remark. The proof uses implicitly the Galois group \( G \) of the splitting field of the polynomial \( P \) over the field \( K \). The group \( G \) permutes the roots \( y_1, \ldots, y_n \) of \( P \). The element \( \hat{M}_i = \prod_{1 \leq k \leq n} M_i(z_k) \) and \( \hat{M}_0 = \sum_{1 \leq k \leq n} M_0(z_k) \) are invariant under the action of \( G \) thus they belong to the field \( K \).
2.3.4 Pure transcendental case

Here we prove the induction hypothesis in the pure transcendental case. First we will state the corresponding Theorem \ref{thm:2.6} and will outline its proof.

Let $F_1$ be a functional differential field obtained by extension of the functional differential field $K$ by adjoining algebraically independent over $K$ functions

$$ y_1 = \ln a_1, \ldots, y_k = \ln a_k, z_1 = \exp b_1, \ldots, z_n = \exp b_n $$

(6)

where $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ are some functions from $K$. We will assume that $F_1$ consists of meromorphic functions on a connected Riemann surface $U$ and the differentiation in $K$ using a meromorphic function $x$ on $U$. Let $X$ be the manifold $U \times G$ where $G = \mathbb{C}^k \times (\mathbb{C}^*)^n$.

Consider a map $\gamma: U \to \mathbb{C}^k \times (\mathbb{C}^*)^n$ given by formula

$$ \gamma(p) = y_1(p), \ldots, y_k(p), z_1(p), \ldots, z_n(p) $$

(7)

where the functions $y_i$, $z_j$ are defined by \ref{eq:2.11}.

Let $X$ be the product $U \times (\mathbb{C})^k \times (\mathbb{C}^*)^n$. Denote by $\Gamma \subset X$ the graph of the map $\gamma$. Consider a germ $\Phi$ of a complex valued function at the point $a \in X$.

**Definition 2.11.** We say that $\Phi$ is a logarithmic type germ if $\Phi$ is representable in the form $\Phi_a = R_0 + \sum_{i=1}^q \lambda_i \ln R_i$, where $R_i$ are germs at the point $a \in X$ of rational functions of $(y_1, \ldots, y_k, z_1, \ldots, z_n)$ with coefficients in $K$ and $\lambda_j$ are complex numbers.

**Theorem 2.6.** Let $\Phi$ be a logarithmic type germ at a point $a = (p_0, \gamma(p_0)) \in \Gamma$. Then the germ of the function $\Phi(p, \gamma(p))$ at the point $p_0 \in U$ is a germ of an integral over $K$ if and only if $\Phi$ is representable in the following form

$$ \Phi(p, y, z) = \Phi(p, \gamma(p_0)) + \sum_{i=1}^k c_i(y_i - y_i(p_0)) + \sum_{j=1}^n t_j \ln \frac{z_j}{z_j(p_0)} $$

(8)

where $r_0$ is a germ of a function from the field $K$ and $c_i, t_j$ are complex constants.

Theorem \ref{thm:2.6} proves induction hypothesis in the pure transcendental case. Indeed the germ $\Phi(p, \gamma(p_0))$ given by \ref{eq:2.11} is a germ of a function from the field $K$ and according to \ref{eq:2.10} the identities $c_i y_i = c_i \ln a_i$, $t_j \ln z_j = t_j b_j$ hold. We split the claim of Theorem 7 into two parts.

First we consider the natural action of the group $G = (\mathbb{C}^k) \times (\mathbb{C}^*)^n$ on $X = U \times G$ and we describe all germs of closed 1-forms locally invariant under this action. Corollary \ref{cor:2.2} claims that each such 1-form is a differential of a function representable in the form \ref{eq:2.11}.

Second we show that if the germ $\Phi$ satisfies the conditions of Theorem \ref{thm:2.6} then the germ $d\Phi$ is locally invariant under the action of the group $G$ (see Theorem \ref{thm:2.10}).

2.3.4.1 Locally invariant closed 1-forms Let $G$ be a connected Lie group acting by diffeomorphisms on a manifold $X$. Let $\pi: G \to \text{Diff}(X)$ be a corresponding homomorphism from $G$ to the group $\text{Diff}(X)$ of diffeomorphisms of $X$. For a vector $\xi$ from the Lie algebra $\mathfrak{g}$ of $G$ the action $\pi$ associates the vector field $V_\xi$ on $X$. The germ $\omega_{x_0}$ at a point $x_0 \in X$ of a differential form $\omega$ on $X$ is locally invariant under the action $\pi$ if for any $\xi \in \mathfrak{g}$ the Lie derivative $L_{V_\xi} \omega$ is equal to zero.

**Lemma 2.5.** The germ of the differential $d\varphi_{x_0} = \omega_{x_0}$ of a smooth function $\varphi$ is locally invariant under the action $\pi$ if and only if for each $\xi \in \mathfrak{g}$ the Lie derivative $L_{V_\xi} \varphi$ is a constant $M(\xi)$ (which depends on $\xi$).

**Proof.** Applying “Cartan’s magic formula” $L_{V_\xi} \omega = i_{V_\xi} d\omega + (i_{V_\xi} \omega)$ we obtain that $L_{V_\xi} \omega = 0$ if and only if $d(L_{V_\xi} \varphi) = 0$ which means that $L_{V_\xi} \varphi$ is constant. \qed
The following theorem characterizes locally invariant closed 1-forms more explicitly.

**Theorem 2.7.** The germ of the differential \( d\varphi_{x_0} = \omega_{x_0} \) of a smooth complex valued function \( \varphi \) is locally invariant under the action \( \pi \) if and only if there exists a local homomorphism \( \rho \) of \( G \) to the additive group \( \mathbb{C} \) of complex numbers such that for any \( g \in G \) in a neighborhood of the identity the following relation holds:

\[
\varphi(\pi(g)x_0) = \varphi(x_0) + \rho(g).
\]

**Proof.** For \( \xi \in G \) the Lie derivative \( L_{V_\xi} \varphi \) is constant \( M(\xi) \) by Lemma 8. Let us show that for \( \xi \in [G] \) where \([G]\) is the commutator of \( G \) the constant \( M(\xi) \) equals to zero. Indeed if \( \xi = [\tau, \rho] \) then

\[
L_{V_\xi} \varphi = L_{V_\tau} L_{V_\rho} \varphi - L_{V_\tau} L_{V_\rho} \varphi = L_{V_\tau} M(\tau) - L_{V_\rho} M(\rho) = 0.
\]

Thus the linear function \( M : G \to \mathbb{C} \) mapping \( \xi \) to \( M(\xi) \) provides a homomorphism of \( G \) to the Lie algebra of the additive group \( \mathbb{C} \) of complex numbers. Let \( \rho \) be the local homomorphism of \( G \) to \( \mathbb{C} \) corresponding to the homomorphism \( M \).

Consider a function \( \phi \) on a neighborhood of the identity in \( G \) defined by the following formula: \( \phi(g) = \varphi(x_0) + \rho(g) \). By definition on a neighborhood of identity the function \( \phi \) has the same differential as the function \( \varphi(\pi(g)x) \). Values of these functions at the identity are equal to \( \varphi(x_0) \). Thus these functions are equal. \( \square \)

Assume that \( X = U \times G \) where \( U \) is a manifold and an action \( \pi \) is given by the formula \( \pi(g)(x, g_1) = (x, g g_1) \). Applying Theorem 2.7 to this action we obtain the following corollary.

**Corollary 2.1.** If germ of differential \( d\varphi = \omega \) of a smooth complex valued function \( \varphi \) at a point \( (x_0, g_0) \in U \times G \) is locally invariant under the action \( \pi \) then in a neighborhood of the point \( (x_0, g_0) \) the following identity holds:

\[
\varphi(x, g) = \varphi(x, g_0) + \rho(g g_0^{-1}). \tag{9}
\]

where \( \rho \) is a local homomorphism of \( G \) to the additive group of complex numbers.

**Proof.** Follows from Theorem 9 since the element \( g g_0^{-1} \) maps the point \( (x, g_0) \) to the point \( (x, g) \). \( \square \)

Let \( G \) be the group \( \mathbb{C}^k \times (\mathbb{C}^*)^n \) where \( \mathbb{C} \) and \( \mathbb{C}^* \) are additive and duplicative group of complex numbers. We will consider the group \( \mathbb{C}^k \times (\mathbb{C}^*)^n \) with coordinate functions \( (y, z) = (y_1, \ldots, y_k, z_1, \ldots, z_n) \) assuming that \( z_1 \cdot \ldots \cdot z_n \neq 0 \).

**Corollary 2.2.** If in the assumptions of Corollary 2.1 for \( G = \mathbb{C}^k \times (\mathbb{C}^*)^n \) in a neighborhood of \( (x_0, y_0, z_0) \in U \times (\mathbb{C}^k \times (\mathbb{C}^*)^n) \) the following identity holds

\[
\varphi(x, y, z) = \varphi(x, y_0, z_0) + \sum_{1 \leq i \leq k} \lambda_i (y_i - (y_0)_i) + \sum_{1 \leq j \leq n} \mu_j \ln \frac{z_j}{(z_0)_j}
\]

where \( \lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_n \) are complex constants.

**Proof.** Follows from 9 since any local homomorphism \( \rho \) from the group \( \mathbb{C}^k \times (\mathbb{C}^*)^n \) to the additive group of complex numbers can be given by formula

\[
\rho(y_1, \ldots, y_k, z_1, \ldots, z_n) = \sum_{i \leq k} \lambda_i y_i + \sum_{j \leq n} \mu_j z_j
\]

where \( \lambda_i \) and \( \mu_j \) are complex constants. \( \square \)
2.3.4.2 Vector field associated to a logarithmic-exponential extension  We use the notation introduced in subsection 2.3.4. Let $G$ be the group $\mathbb{C}^k \times (\mathbb{C}^*)^n$ and let $X$ be the product $U \times G$ consider the map $\gamma : U \to \mathbb{C}^k \times (\mathbb{C}^*)^n$ given by the following formula:

$$y_1 = \ln a_1, \ldots, y_k = \ln a_k, z_1 = \exp b_1, \ldots, z_n = \exp b_n. \quad (10)$$

The map $\gamma$ satisfies the following differential relation:

$$d\gamma = da_1/a_1, \ldots, da_k/a_k, z_1 db_1, \ldots, z_n db_n$$

**Definition 2.12.** Let $V$ be a meromorphic vector field on $X$ defined by the following conditions. If $V_a$ is the value of $V$ at the point $a = (p, y_1, \ldots, y_k, z_1, \ldots, z_n) \in X$ then $\langle dx, V_a \rangle = 1$, $\langle dy_i, V_a \rangle = a_i'/a_i(p)$ for $1 \leq i \leq k$, $\langle dz_j, V_a \rangle = b_j'/b_j(p)$ for $1 \leq j \leq n$.

The vector field $V$ is regular on $U^0 \times G$ where $U^0$ is an open subset in $U$ which does not contain the zeroes and poles of the functions $a_1, \ldots, a_k$ and poles of functions $b_1, \ldots, b_n$ poles and zeros and poles of the 1-form $dx$. By construction the graph $\Gamma = (p, \gamma(p)) \subset X$ of the map $\gamma$ is an integral curve for the differential equation on $X$ defined by the vector field $V$.

The following lemmas are obvious.

**Lemma 2.6.** The vector field $V$ is invariant under the action $\pi$ on $X$. For each element $g \in G$ the curve $g\Gamma \subset X$ of the graph $\Gamma$ of $\gamma$ is an integral curve for $V$.

**Lemma 2.7.** The field $K(y, z)$ of rational functions in $y_1, \ldots, y_k, z_1, \ldots, z_n$ over the field $K$ is invariant under the action $\pi$ on $X$. For each vector $\xi \in G$ in the Lie algebra $G$ of $G$ the Lie derivative $LV_{\xi}R$ of $R \in K(y, z)$ belongs to $K(y, z)$.

2.3.4.3 Pure transcendental logarithmic exponential extension  We will assume below that the components $\{\gamma\}$ of $\gamma$ are algebraically independent over $K$.

**Liouville’s principal.** If a polynomial $P \in K[y_1, \ldots, y_k, z_1, \ldots, z_n]$ vanishes on the graph $\Gamma \subset X$ of the map $\gamma$ then $P$ is identically equal to zero.

**Proof.** If $P$ is not identically equal to zero then the components of $\gamma$ are algebraically dependent over the field $K$. \(\square\)

**Theorem 2.8.** The extension $K \subset F_1$ is isomorphic to the extension of $K$ by the field of rational functions $K(y, z)$ in $(y_1, \ldots, y_k, z_1, \ldots, z_n)$ over $K$ considered as the field of functions on $X$ equipped with the differentiation sending $f \in K(y, z)$ to the Lie derivative $LV_{\xi}f$ with respect to the vector field $V$ introduced in definition 8.

**Proof.** By assumption components $\{\gamma\}$ of the map $\gamma$ are algebraically independent over $K$ thus each function from the extension obtained by adjoining to $K$ these components is representable in the unique way as a rational function from $K(y, z)$. By definition the derivatives of the components $\{\gamma\}$ coincide with their Lie derivatives with respect to the vector field $V$. \(\square\)

The action $\pi$ of the group $G = \mathbb{C}^k \times (\mathbb{C}^*)^n$ on $X$ induces the action $\pi^*$ of $G$ on the space of functions on $X$ containing the field $K(y, z)$. The vector field $V$ is invariant under the action $\pi$. Thus $\pi^*$ acts on $K(y, z) \sim F_1$ by differential automorphisms. It is easy to see that a function $f \in K(y, z)$ is fixed under the action $\pi^*$ if and only if $f \in K$, i.e. the group $G$ is isomorphic to the differential Galois group of the extension $K \subset F_1$. We proved the following result.
Theorem 2.9. The differential Galois group of the extension $K \subset F_1$ is isomorphic to the group $G$. The Galois group is induced on the differential field $K(y, z)$ with the differentiation given by Lie derivative with respect to the field $V$ by the action of $G$ on the manifold $X = U \times \mathbb{C}^k \times (\mathbb{C}^*)^n$.

Now we are ready to complete inductive proof of Liouville’s First Theorem.

Theorem 2.10. Let $\Phi$ be a logarithmic type germ at a point $a = (p_0, \gamma(p_0)) \in \Gamma \subset X$. If the germ of the function $\Phi(p, \gamma(p))$ on $U$ at the point $p_0 \in U$ is a germ of an integral $f$ over $K$ then the germ of the differential $d\Phi$ at the point $a \in X$ is locally invariant under the action $\pi$.

Proof. By the assumption of Theorem the restriction of the function $(L_V \Phi - f)$ on $\Gamma$ is equal to zero. Since the function $(L_V \Phi - f)$ belongs to the field $K(y, z)$ the function $(L_V \Phi - f)$ by Liouville’s principle is equal to zero identically on $X$. In particular it is equal to zero on the integral curve $g\Gamma$ the vector field $V$, where $g$ is an element of the group $G$. Thus the restrictions of function $L_V \pi(g)^*(\Phi - f)$ to $\Gamma$ equals to zero. Since the function $f$ is invariant under the action $\pi^*$ we obtain that the restriction on $\Gamma$ of $L_V (\Phi - \pi^*(g)\Phi)$ is equal to zero. Differentiating this identity we obtain that for any $\xi \in G$ the restriction on $\Gamma$ of $L_V (L_V \xi \Phi)$ equals to zero. Thus on $\Gamma$ the function $L_V \xi \Phi$ is constant. Lemma 27 implies that the function $L_V \xi \Phi$ belongs to the field $K(x, y)$. Thus the function $L_V \xi \Phi$ is a constant on $X$ by Liouville’s principal. Thus the 1-form $d\Phi$ is locally invariant under the action $\pi$ by Lemma 22. Theorem 31 is proved.

Thus we complete proof of Theorem 21 and the inductive proof of Liouville’s First Theorem.

2.4 Liouville’s Second Theorem and its Generalizations

2.4.1 Introduction

In 1839 Joseph Liouville proved the following fundamental result.

Liouville’s Second Theorem A second order homogeneous linear differential equation

$$y'' + a_1y' + a_ny = 0$$

whose coefficients $a_1, a_2$ belong to a functional differential field $K$ is solvable by generalized quadratures over $K$ if and only if it has a solution of the form $y_1 = \exp z$ where $z'$ is algebraic over $K$.

Much later this theorem was generalized for homogeneous linear differential equations of any order. Consider an equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0$$

whose coefficients $a_i$ belong to $K$.

Theorem 2.11. If the equation (12) has a solution representable by generalized quadratures over $K$ then it necessarily has a solution of the form $y_1 = \exp z$ where $z'$ is algebraic over $K$.

The following lemma is obvious.

Lemma 2.8. Assume that the equation (12) has a solution $y_1$ representable by generalized quadratures over $K$. Then the equation (12) can be solved by generalized quadratures over $K$ if and only if the linear differential equation of order $(n - 1)$ over the differential field $K(y_1)$ obtained from (10) by the reduction of order using the solution $y_1$ is solvable by generalized quadratures over $K(y_1)$.
Indeed on one hand each solution of the equation obtained from (12) by the reduction of order using $y_1$ can be expressed in the form $(y/y_1)'$ where $y$ is a solution of (12). On the other hand any solution $y$ of the equation $(y/y_1)' = u$, where $u$ is represented by generalized quadratures over $K(y_1)$, is representable by generalized quadratures over $K$, assuming that $y_1$ representable by generalized quadratures over $K$.

Thus Theorem 2.11 provides the following criterium for solvability of the equation (12) by generalized quadratures.

**Theorem 2.12.** The equation (12) is solvable by generalized quadratures over $K$ if and only if the following conditions hold:

1) the equation (12) has a solution $y_1$ of the form $y_1 = \exp z$ where $z' = f$ is algebraic over $K$,

2) the linear differential equation of order $(n - 1)$ over $K(y_1)$ obtained from (12) by the reduction of order using the solution $y_1$ is solvable by generalized quadratures over $K(y_1)$.

For $n = 2$ Theorem 2.12 is equivalent to Liouville's Second Theorem because linear differential equations of first order are automatically solvable by quadratures.

The standard proof (E. Picard and E. Vessiot, 1910) of Theorem 2.11 uses the differential Galois theory and is rather involved (see [dPS03]).

In the case when the equation (12) is a Fuchsian differential equation and $K$ is the field of rational function of one complex variable Theorem 2.12 has a topological explanation (see section 3.3 and [Kho14]) which allows to prove much stronger version of this result). But in general case Theorem 2.12 does not have a similar visual explanation.

Maxwell Rosenlicht in 1973 proved [Ros73] the following theorem.

**Theorem 2.13.** Let $n$ be a positive integer, and let $Q$ be a polynomial in several variables with coefficients in a differential field $K$ and of total degree less than $n$. Then if the equation

$$u^n = Q(u, u', u'', \ldots)$$

(13)

has a solution representable by generalized quadratures over $K$, it has a solution algebraic over $K$.

The logarithmic derivative $u = y'/y$ of any solution of the equation (12) satisfies the generalized Riccati equation of order $n - 1$ associated with (12), which is a particular case of the equation (13). Rosenlicht showed that Theorem 2.11 easily follows from Theorem 2.12 applied to the corresponding generalized Riccati equation (see section 2.4.2). In modern differential algebra abstract fields equipped with an operation of differentiation are considered. The Rosenlicht’s proof of Theorem 2.12 is not elementary: it is applicable to abstract differential fields of characteristic zero and makes use of the valuation theory.

The logarithmic derivative $u = y'/y$ of any solution $y$ of the homogeneous linear differential equation of second order (11) satisfies the Riccati equation

$$u' + a_1 u + a_2 + u^2 = 0.$$  

(14)

To prove the Liouville’s Second Theorem Liouville and Ritt proved first Theorem 2.12 for the Riccati equation (14). To do that J.F. Ritt (in his simplification of the Liouville's proof) considered a special one parametric family of solutions of (14) and used an expansion of these solutions as functions of the parameter into converging Puiseux series (see Chapter V

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2 According to Michael Singer the valuation theory used in Rosenlicht’s proof is a fancy way of using power series methods (private communication).
in \cite{Rit48}. J.F.Ritt used a generalization of the following theorem based on ideas suggested by Newton.

Consider an algebraic function \( z(y) \) defined by an equation \( P(y, z) = 0 \) where \( P \) is a polynomial with coefficients in a subfield \( K \) of \( \mathbb{C} \). Then all branches of the algebraic function \( z(y) \) at the point \( y = \infty \) can be developed into converging Puiseux series whose coefficients belong to a finite extension of the field \( K \).

A **generalized Newton’s Theorem** claims that the similar result holds if instead of a numerical field of coefficients one takes a field \( K \) whose elements are meromorphic functions on a connected Riemann surface. In J.F.Ritt’s book \cite{Rit48} this result is proved in the same way as its classical version using the Newton’s polygon method.

J.F.Ritt’s proof is written in old mathematical language and does not fit into our presentation. Theorem 2.15 provides an exact statement of the generalized Newton’s Theorem. It is presented without proof: the main arguments proving it are well known and classical. One also can obtain a proof modifying J.F.Ritt’s exposition. Theorem 2.15 plays a crucial role in section 2.4.3. For the sake of completeness I will present its modern proof in a separate paper.

In this section we discuss a proof of Theorem 2.12 which does not rely on the valuation theory. It generalizes J.F.Ritt’s arguments (makes use of the Puiseux expansion via a generalized Newton’s Theorem) and provides an elementary proof of the classical Theorem 2.11.

We follow the presentation from the paper \cite{Kho18b}.

### 2.4.2 Generalized Riccati equation

Here we define the generalized Riccati equation and reduce Theorem 2.11 to Theorem 2.12. In this section we also generalize Theorem 2.11 for nonlinear homogeneous equations (this generalization will not be used in the next sections).

Assume that \( u \) is the logarithmic derivative of a non identically equal to zero meromorphic function \( y \), i.e the relation \( y' = uy \) holds.

**Definition 2.13.** Let \( D_n \) be a polynomial in \( u \) and in its derivatives \( u, u', \ldots, u^{(n-1)} \) up to order \( (n-1) \) defined by induction by the following conditions:

\[
D_0 = 1; \quad D_{k+1} = \frac{dD_k}{dx} + uD_k.
\]

**Lemma 2.9.** 1) The polynomial \( D_n \) has integral coefficients and \( \text{deg } D_n = n \). The degree \( n \) homogeneous part of \( D_n \) equals to \( u^n \) (i.e. \( D_n = u^n + \tilde{D}_n \) where \( \text{deg } \tilde{D}_n < n \)).

2) If \( y \) is a function whose logarithmic derivative equals to \( u \) (i.e. if \( y' = uy \)) then for any \( n \geq 0 \) the relation \( y^{(n)} = D_n(u)y \) holds.

Both claims of Lemma 2.9 can be easily checked by induction.

Consider a homogeneous linear differential equation (12) whose coefficients \( a_i \) belong to a differential field \( K \).

**Definition 2.14.** The equation

\[
D_n + a_1D_{n-1} + \cdots + a_nD_0 = 0
\]

of order \( n - 1 \) is called the **generalized Riccati equation** for the homogeneous linear differential equation (12).

**Lemma 2.10.** A non identically equal to zero function \( y \) satisfies the linear differential equation (12) if and only if its logarithmic derivative \( u = y'/y \) satisfies the generalized Riccati equation (13).
Proof. Let \( y \) be a nonzero solution of (12) and let \( u \) be its logarithmic derivative. Then dividing (12) by \( y \) and using the identity \( y^{(k)}/y = D_k(u) \) we obtain that \( u \) satisfies (15). If \( u \) is a solution of (15) then multiplying (5) by \( y \) and using the identity \( y^{(k)} = D_k(u)y \) we obtain that \( y \) is a non zero solution of (12).

Corollary 2.3. 1) The equation (12) has a non zero solution representable by generalized quadratures over \( K \) if and only if the equation (15) has a solution representable by generalized quadratures over \( K \).

2) The equation (12) has a solution \( y \) of the form \( y = \exp z \) where \( z' = f \) is an algebraic function over \( K \) if and only if the equation (15) has an algebraic solution over \( K \).

Proof. 1) A non zero function \( y \) is representable by generalized quadratures over \( K \) if and only if its logarithmic derivative \( u = y'/y \) is representable by generalized quadratures over \( K \).

2) A function \( y \) is equal to \( \exp z \) where \( z' = f \) if and only if its logarithmic derivative is equal to \( f \).

The generalized Riccati equation (15) satisfies the conditions of Theorem 2.12. Thus Theorem 2.11 follows from Theorem 2.12 and from Corollary 2.3.

Let us generalize the results of this section. Consider an order \( n \) homogeneous equation

\[
P(y, y', \ldots, y^{(n)}) = 0
\]

where \( P \) is a degree \( m \) homogeneous polynomial in \( n + 1 \) variables \( x_0, x_1, \ldots, x_n \) over a functional differential field \( K \).

Definition 2.15. The equation

\[
P(D_0, D_1, \ldots, D_n) = 0
\]

of order \( n - 1 \) is called the \textit{generalized Riccati equation} for the homogeneous equation (16).

Lemma 2.11. A non identically equal to zero function \( y \) satisfies the homogeneous equation (10) if and only if its logarithmic derivative \( u = y'/y \) satisfies the generalized Riccati equation (17).

Corollary 2.4. 1) The equation (10) has a non zero solution representable by generalized quadratures over \( K \) if and only if the equation (17) has a solution representable by generalized quadratures over \( K \).

2) The equation (10) has a solution \( y \) of the form \( y = \exp z \) where \( z' = f \) is an algebraic function over \( K \) if and only if the equation (17) has an algebraic solution over \( K \).

Lemma 2.11 and Corollary 2.4 can be proved exactly in the way as Lemma 2.10 and Corollary 2.3.

Let us defined the \( \xi \)-\textit{weighted degree} \( \deg_\xi x^p \) of the monomial \( x^p = x_0^{p_0} \cdots x_n^{p_n} \) by the following formula:

\[
\deg_\xi x^p = \sum_{i=0}^{i=n} ip_i.
\]

We will say that a polynomial \( P(x_0, \ldots, x_n) \) satisfies the \( \xi \)-\textit{weighted degree condition} if the sum of coefficients of all monomials in \( P \) having the biggest \( \xi \)-weighted degree is not equal to zero. A polynomial \( P \) having a unique monomial with the biggest \( \xi \)-weighted degree automatically satisfies this condition. For example a degree \( m \) polynomial \( P \) containing a term \( ax_m^m \) with \( a \neq 0 \) automatically satisfies \( \xi \)-weighted degree condition.

Theorem 2.14. Consider the homogeneous equation (10) with the polynomial \( P \) satisfying the \( \xi \)-weighted degree condition. If this equation has a solution representable by generalized quadratures over \( K \) then it necessarily has a solution of the form \( y_1 = \exp z \) where \( z' \) is algebraic over \( K \).
Proof. It is easy to check that if the polynomial $P$ satisfies the $\xi$-weighted degree condition then the generalized Riccati equation (17) satisfies the conditions of Theorem 2.12. Thus Theorem 2.14 follows from Theorem 2.12 and Corollary 2.3.

Remark. There exists a complete analog of Galois theory for linear homogeneous differential equations (see [IPS03]). Theorem 2.11 can be proved using this theory. The differential Galois group of a nonlinear homogeneous differential equation (16) could be very small and for such equation a complete analog of Galois theory does not exist. Thus Theorem 2.14 cannot be proved in a similar way.

2.4.3 Finite extensions of fields of rational functions

Here we discuss finite extensions of the field $K(y)$ of rational functions over a subfield $K$ of the field of meromorphic function on a connected Riemann surface $U$. We also state Theorem 2.14 (generalized Newton’s Theorem) which plays a crucial role for this chapter.

Let $F$ be an extension of $K(y)$ by a root $z$ of a degree $m$ polynomial $P(z) \in (K[y])[z]$ over the ring $K[y]$ irreducible over the field $K(y)$. Let $X$ be the product $U \times \mathbb{C}^1$ where $\mathbb{C}^1$ is the standard complex line with the coordinate function $y$. An element of the field $K(y)$ can be considered as a meromorphic function on $X$. One can associate with the element $z \in F$ a multivalued algebraic function on $X$ defined by the equation $P(z) = 0$. Let $D(y)$ be the discriminant of the polynomial $P$. Let $\Sigma \subset U \times \mathbb{C}^1 = X$ be the hypersurface defined by equation $p_m(y) \cdot D(y) = 0$ where $p_m(y)$ is the leading coefficient of the polynomial $P$.

Lemma 2.12. 1) About a point $x \in X \setminus \Sigma$ the equation $P(z) = 0$ defines $m$ germs $z_i$ of analytic functions whose values at $x$ are simple roots of polynomial $P$.

2) Let $x$ be the point $(a, y) \in U \times \mathbb{C}^1 \setminus \Sigma$. Then the field $F$ is isomorphic to the extension $K_a(z_i)$ of the field $K_a$ of germs at $a \in U$ of functions from the field $K$ (considered as germs at $x = (a, y)$) extended by the germ $z_i$ at $x$ satisfying the equation $P(z) = 0$.

Proof. The statement 1) follows from the Implicit Function Theorem. The statement 2) follows from 1).

Below we state Theorem 2.14 which is a generalization of Newton’s Theorem about the expansion of an algebraic functions as convergent Puiseux series. It is stated without proof (see comments in section 2.4.1).

We use notations introduced in the beginning of this section. Let $z$ be an element satisfying a polynomial equation $P(z) = 0$ over the ring $K[y]$, where $K$ is a subfield of the field of meromorphic functions on $U$. Then there exists a finite extension $K_P$ of the field $K$ associated with the element $z$ such that the following theorem holds.

Theorem 2.15. There is a finite covering $\pi : U_P \to U \setminus O_P$ where $O_P \subset U$ is a discrete subset, such that the following properties hold:

1) the extension $K_P$ can be realized by a subfield of the field of meromorphic functions on $U_P$ containing the field $\pi^* K$ isomorphic to $K$.

2) there is a continuous positive function $r : U \setminus O_P \to \mathbb{R}$ such that in the open domain $W \subset (U \setminus O_P) \times \mathbb{C}^1$ defined by the inequality $|y| > r(a)$ all $m$ germs of $z_i$ at a point $(a, y)$ can be developed into converging Puiseux series

$$z_i = z_{i_k} y^{\frac{1}{p}} + z_{i_{k-1}} y^{\frac{1}{p}} + \ldots$$

(18)

whose coefficients $z_{i_j}$ are germs of analytic functions at the point $a \in U_P$ having analytical continuation as regular functions on $U_P$ belonging to the field $K_P$. 

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For the special case when $K$ is a subfield of the field of complex numbers, it is natural to refer to Theorem 2.15 as Newton’s Theorem. One can consider $K$ as a field of constant functions on any connected Riemann surface $U$. One can chose $O_P$ to be the empty set, $U_P$ to be equal $U$, the projection $π : V_P → U$ to be the identity map, the function $r : U → ℝ$ to be a big enough constant. In this case Theorem 2.15 states that an algebraic function $z$ has a Puiseux expansion at infinity whose coefficients belong to a finite extension $K_P$ of the field $K$. This statement can be proved by Newton’s polygon method.

Let $F$ be an extension of $K(y)$ by a root $z$ of the polynomial $P$ and let $K_P$ be the finite extension of the field $K$ introduced in Theorem 2.13. The extension $F_P$ of the field $K_P(y)$ by $z$ is easy to deal with. Denote the product $U_P × C^1$ by $X_P$.

**Lemma 2.13.** Let $x ∈ X_P$ be the point $(a, y_0) ∈ U_P × C^1$. Then the field $F_P$ is isomorphic to the extension $K_{P,a}(z_i)$ of the field $K_{P,a}$ of germs at $a ∈ U_P$ of functions from the field $K_P$ (considered as germs at $x = (a, y_0)$ of functions independent of $y$) extended by the germ at $x$ of the function $z_i$ defined by $[18]$.

Lemma 2.13 follows from Theorem 2.15.

### 2.4.4 Extension by one transcendental element.

Let $U$ be a connected Riemann surface and let $K$ be a differential field of meromorphic functions on $U$. Let $C^1$ be the standard complex line with the coordinate function $y$. Elements of the field $K(y)$ of rational functions over $K$ could be considered as meromorphic functions on $X = U × C^1$.

In the field $K(y)$ there are two natural operations of differentiation. The first operation $R(y) → \frac{∂R}{∂x}(y)$ is defined as follows: the derivative $\frac{∂}{∂x}$ of the independent variable $y$ is equal to zero, and derivative $\frac{∂}{∂y}$ of an element $a ∈ K$ is equal to its derivative $a'$ in the field $K$. For the second operation $R(y) → \frac{∂R}{∂y}(y)$ the derivative of an element $a ∈ K$ is equal to zero and the derivative of the independent variable $y$ is equal to one.

Let $K ⊂ F$ be differential fields and let $θ ∈ F$ be a transcendental element over $K$. Assume that $θ' ∈ K(θ)$. Under this assumption the field $K(θ)$ has a following description.

**Lemma 2.14.** 1) The map $τ : K(θ) → K(y)$ such that $τ(θ) = y$ and $τ(a) = a$ for $a ∈ K$ provides an isomorphism between the field $K(θ)$ considered without the operation of differentiation and the field $K(y)$ of rational functions over $K$.

2) If $τ(θ') = w ∈ K(y)$ then for any $R ∈ K(y)$ and $z ∈ K(θ)$ such that $τ(z) = R$ the following identity holds

$$τ(ζ') = \frac{∂R}{∂x} + \frac{∂R}{∂y}w.$$ (19)

**Proof.** The first claim of the lemma is straightforward. The second claim follows from the chain rule.

Let $Θ ⊂ X = U × C^1$ be the graph of function $θ : U → C^1$. The following lemma is straightforward.

**Lemma 2.15.** The differential field $K(θ)$ is isomorphic to the field $K(y)|_Θ$ obtained by restriction to $Θ$ of functions from the field $K(y)$ equipped with the differentiation given by $[10]$. For any point $a ∈ Θ$ The differential field $K(θ)$ is isomorphic to the differential field of germs at $a ∈ Θ$ of functions from $K(y)|_Θ$.

### 2.4.5 An extension by integral

Here we consider extensions of transcendance degree one of a differential field $K$ containing an integral $y$ over $K$ which does not belong to $K$, $y ∉ K$. 


2.4.5.1 A pure transcendental extension by integral. Let \( \theta \) be an integral over \( K \), i.e. \( \theta' = f \in K \). Assume that \( \theta \) is a transcendental element over \( K \).

Lemma 2.16. 1) The field \( K(\theta) \) is isomorphic to the field \( K(y) \) of rational functions over \( K \) equipped with the following differentiation

\[
R' = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial y} f. \tag{20}
\]

2) For every complex number \( \rho \in \mathbb{C} \) the map \( \theta \rightarrow \theta + \rho \) can be extended to the unique isomorphism \( G_{\rho} : K(\theta) \rightarrow K(\theta) \) which fixes elements of the field \( K \).

3) Each isomorphism of \( K(\theta) \) over \( K \) is an isomorphism \( G_{\rho} \) for some \( \rho \in \mathbb{C} \). Thus the Galois group of \( K(\theta) \) over \( K \) is the additive group of complex numbers \( \mathbb{C} \).

Proof. The claim 1) follows from Lemma 2.15. For any \( \rho \in \mathbb{C} \) the element \( \theta_{\rho} = \theta + \rho \) is a transcendental element over \( K \) and \( \theta_{\rho}' \) equals to \( f \). Thus the claims 2) is correct. The claim 3) follows because if \( y' = f \) then \( y = \theta_{\rho} \) for some \( \rho \in \mathbb{C} \). \( \square \)

2.4.5.2 A generalized extension by an integral. According to Lemma 2.16 the differential field \( K(\theta) \) is isomorphic to the field \( K(y) \) with the differentiation given by (20). Let \( F \) be an extension of \( K(\theta) \) by an element \( z \in F \) which satisfies some equation \( P(z) = 0 \) where \( P \) is an irreducible polynomial over \( K(\theta) \). The isomorphism between \( K(\theta) \) and \( K(y) \) transforms the polynomial \( P \) into some polynomial \( P \) over \( K(y) \). Below we use notation from section 2.4.3 and deal we the multivalued algebroid function \( z \) on \( X \) defined by \( P(z) = 0 \).

Assume that at a point \( x \in X \) there are germs of analytic functions \( z_i \) satisfying the equation \( P(z_i) = 0 \). Let \( \theta_p \) be the function \( \theta : U_P \rightarrow \mathbb{C}^1 \) and let \( \Theta_p \subset X = U \times \mathbb{C}^1 \) be its graph. The point \( x = (p, q) \in U \times \mathbb{C}^1 \) belongs to the graph \( \Theta_p(x) \) for \( \rho(x) = q - \theta(p) \).

Let \( K(y)|_{\Theta_p(x)} \) be the differential field of germs at the point \( x \in \Theta_p(x) \) of restrictions on \( \Theta_p(x) \) of functions from the field \( K(y) \) equipped with the differentiation given by (20).

Lemma 2.17. The differential field \( F \) is isomorphic to the finite extension of the differential field \( K(y)|_{\Theta_p(x)} \) obtained by adjoining the germ at \( x \in \Theta_p(x) \) of the restriction to \( \Theta_p(x) \) of an analytic germ \( z_i \) satisfying \( P(z_i) = 0 \).

Proof. For the trivial extension \( F = K(\theta) \) Lemma 2.17 follows from Lemmas 2.15 and 2.16. Theorem 2.5 allows one to complete the proof for non trivial finite extensions \( F \) of \( K(\theta) \). \( \square \)

According to section 2.4.3 with the polynomial \( P \) over \( K(y) \) one can associate the finite extension \( K_P \) of the field \( K \) and the Riemann surface \( U_P \) such that Theorem 2.15 holds. Since \( K \) is functional differential field the field \( K_P \) has a natural structure of functional differential field. Below we will apply Lemma 2.17 taking instead of \( K \) the field \( K_P \) and considering the extension \( F_P \supset K_P(\theta) \) by the same algebraic element \( z \in F \). The use of \( K_P \) instead of \( K \) allows one to apply the expansion (14) for \( z_i \).

Theorem 2.16. Let \( x \in X_P = U_P \times \mathbb{C}^1 \) be a point \( (a, y_0) \) with \( |y| >> 0 \). The differential field \( F_P \) is isomorphic to the extension of the differential field of functions at the point \( a \in U_P \) of functions from the differential field \( K_P \) by the following germs: by the germ at \( a \) of the integral \( \theta_{\rho(x)} \) of the function \( f \in K \), and by a germ at \( a \) of the composition \( z_i(\theta_{\rho}) \) where \( z_i \) is a germ at \( x \) of a function given by a Puiseux series (14).

Proof. Theorem 2.16 follows from Lemma 2.17 and Theorem 2.15. \( \square \)

\( \text{3} \) It is easy to check that if an integral \( \theta \) over \( K \) does not belong to \( K \), then \( \theta \) is a transcendental element over \( K \) (see [Kho14]). We will not use this fact.
Here we discuss lemma 2.18 providing an important step for our proof of Theorem 2.12. We will use notations from sections 2.4.5.1 and 2.4.5.2.

Let \( T(u, u', \ldots, u^{(N)}) \) be a polynomial in independent function \( u \) and its derivatives with coefficients from the functional differential field \( K \). Consider the equation

\[
T(u, u', \ldots, u^{(N)}) = 0. \tag{21}
\]

In general the derivative of the highest order \( u^{(N)} \) cannot be represented as a function of other derivatives via the relation (21). Thus even existence of local solutions of (21) is problematic and we have no information about global behavior of its solutions.

Assume that (21) has a solution \( z \) in a generalized extension by integral \( F \supset K(\theta) \) of \( K \).

The solution \( z \) has a nice global property: it is a meromorphic function on a Riemann surface \( V \) with a projection \( \pi: V \to U \) which proves a locally trivial covering above \( U \setminus O \), where \( O \subset U \) is discrete subset.

Moreover the existence of a solution \( z \) implies the existence of a family \( z(\rho) \) of similar solutions depending on a parameter \( \rho \): one obtains such family of solutions by using an integral \( \theta + \rho \) instead of the integral \( \theta \) (see Lemma 2.17). If the parameter \( \rho \) has big absolute value \( |\rho| >> 0 \) for a point \( a \in U_P \) of the germ \( z(\rho) \) can be expanded in the Puiseux series in \( \theta_\rho^\rho \):

\[
z_{i}(\rho) = z_{i_{k}}\theta_{\rho}^{\frac{k}{p}} + z_{i_{k-1}}\theta_{\rho}^{\frac{k-1}{p}} + \ldots \tag{22}
\]

The series is converging and so it can be differentiate using the relation \( \theta_\rho' = f \).

**Lemma 2.18.** If \( z_{i_{k}}' \neq 0 \) then the leading term of the Puiseux series for \( z_{i}(\rho) \) is \( z_{i_{k}}'\theta_{\rho}^{\frac{k}{p}} \).

Otherwise the leading term has degree \( < \frac{k}{p} \). The leading term of the derivative of any order of \( z_{i_{k}} \) has degree \( \leq \frac{k}{p} \).

Let us plug into the differential polynomial \( T(u, u', \ldots, u^{(N)}) \) the germ (22) and develop the result into Puiseux series in \( \theta_\rho \). If the germ \( z(\rho) \) is a solution of the equation (21) then all terms of this Puiseux series are equal to zero. In particular the leading coefficient is zero. This observation is an important step for proving Theorem 2.12.

### 2.4.6 An extension by exponential of integral

Here we consider extensions of transcendental degree one of a differential field \( K \) containing an exponential integral \( y \) over \( K \) which is not algebraic over \( K \).

#### 3.6.1. A pure transcendental extension by an exponential integral

Let \( \theta \) be an an exponential integral over \( K \), i.e \( \theta' = f \theta \) where \( f \in K \). Assume that \( \theta \) is a transcendental element over \( K \).

**Lemma 2.19.** 1) The field \( K(\theta) \) is isomorphic to the field \( K(y) \) of rational functions over \( K \) equipped with the following differentiation

\[
R' = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial y}fy. \tag{23}
\]

2) For every complex number \( \mu \in \mathbb{C}^* \) not equal to zero the map \( \theta \to \mu \theta \) can be extended to the unique isomorphism \( G_\mu: K(\theta) \to K(\theta) \) which fixes elements of the field \( K \).

\footnote{It is easy to check that if an exponential of integral \( \theta \) over \( K \) is algebraic over \( K \), then \( \theta \) is a radical over \( K \), i.e. \( \theta^k \in K \) for some positive integral \( k \) (see [Kho14]). We will not use this fact.}
3) Each isomorphism of $K(\theta)$ over $K$ is an isomorphism $G_\mu$ for some $\mu \in \mathbb{C}^*$. Thus the Galois group of $K(\theta)$ over $K$ is the multiplicative group of complex numbers $\mathbb{C}^*$.

Proof. Statement 1) follows from Lemma 2.14. For any $\mu \in \mathbb{C}^*$ the element $\theta_\mu = \mu \theta$ is a transcendental element over $K$ and $\theta_\mu' = f \theta$. Thus 2) is correct. Statement 3) follows from 2) because if $y' = fy$ and $y \neq 0$ then $y = \theta_\mu$ for some $\mu \in \mathbb{C}^*$.

2.4.6.2 A generalized extension by exponential of integral

According to Lemma 2.19 the differential field $K(\theta)$ is isomorphic to the field $K(y)$ with the differentiation given by (23).

Let $F$ be an extension of $K(\theta)$ by an element $z \in F$ which satisfies some equation $\tilde{P}(z) = 0$ where $\tilde{P}$ is an irreducible polynomial over $K(\theta)$. The isomorphism between $K(\theta)$ and $K(y)$ transforms the polynomial $\tilde{P}$ into some polynomial $P$ over $K(y)$. Below we use notation from section 2.4.3 and we deal with the multivalued algebroid function $z$ on $X$ defined by $P(z) = 0$.

Assume that at a point $x \in X$ there are germs of analytic function $z_i$ satisfying the equation $P(z_i) = 0$. Let $\theta_\mu$ be the function $(\mu \theta) : U_\mu \rightarrow \mathbb{C}^1$ and let $\Theta_\mu \subset X = U \times \mathbb{C}^1$ be its graph. The point $x = (p, q) \in U \times \mathbb{C}^1$ where $q \neq 0$ belongs to the graph $\Theta_\mu(x)$ if $\mu(x) = q \cdot \theta(p)^{-1}$.

Let $K(y)|_{\Theta_\mu(x)}$ be the differential field of germs at the point $x \in \Theta_\mu(x)$ of restrictions on $\Theta_\mu(x)$ of functions from the field $K(y)$ equipped with the differentiation given by (23).

Lemma 2.20. The differential field $F$ is isomorphic to the finite extension of the differential field $K(y)|_{\Theta_\mu(x)}$ obtained by adjoining the germ at $x \in \Theta_\mu(x)$ of the restriction to $\Theta_\mu(x)$ of an analytic germ $z_i$ satisfying $P(z_i) = 0$.

Proof. For the trivial extension $F = K(\theta)$ Lemma 2.20 follows from Lemmas 2.15, 2.19. Theorem 2.16 allows one to complete the proof for non-trivial finite extensions $F$ of $K(\theta)$.

According to section 2.4.3 with the polynomial $P$ over $K(y)$ one can associate the finite extension $K_P$ of the field $K$ and the Riemann surface $U_P$ such that Theorem 2.16 holds. Since $K$ is a functional differential field the field $K_P$ has a natural structure of functional differential field. Below we will apply Lemma 2.20 taking instead of $K$ the field $K_P$ and considering the extension $F_P \supset K_P(\theta)$ by the same algebraic element $z \in F$. The use of $K_P$ instead of $K$ allows to apply the expansion (18) for $z_i$.

Theorem 2.17. Let $x \in X_P = U_P \times \mathbb{C}^1$ be a point $(a, y_0)$ with $|y| >> 0$. The differential field $F_P$ is isomorphic to the extension of the differential field of germs at the point $a \in U_P$ by an exponential of an integral $\theta_\mu(x)$, where $\phi_\mu(x) \equiv f \theta_\mu(x)$ for the function $f \in K$, and by a germ at $a$ of the composition $z_i(\theta_\mu)$ where $z_i$ is a germ at $x$ of a function given by a Puiseux series (18).

Proof. Theorem 2.17 follows from Lemma 2.20 and Theorem 2.15.

2.4.6.3 Solutions of equations in a generalized extension by exponential of integral

Here we discuss Lemma 2.21 providing an important step for our proof of Theorem 2.12. Assume that $z$ has a solution $z$ in a generalized extension by exponential of integral $F \supset K(\theta)$ of $K$. The solution $z$ has a nice global property: it is a meromorphic function on a Riemann surface $V$ with a projection $\pi : V \rightarrow U$ which proves a locally trivial covering above $U \setminus O$, where $O \subset U$ is a discrete subset.

Moreover the existence of a solution $z$ implies the existence of a family $z(\mu)$ of similar solutions depending on a parameter $\mu \in \mathbb{C}^*$: one obtains such family of solutions by using an exponential of integral $\mu \theta$ instead of the exponential of integral $\theta$ (see 2.20). If the parameter $\mu$ has big absolute value $|\mu| >> 0$ for a point $a \in U_P$ of the germ $z(\mu)$ can be expand in the Puiseux series in $\phi_\mu$.
The series is converging and so it can be differentiated using the relation \( \theta' = f\theta \).

**Lemma 2.21.** If \( z_i' + \frac{1}{p} z_i \neq 0 \) then the leading term of the Puiseux series for \( z_i(\mu)' \) is \((z_i' + \frac{1}{p} z_i)\theta^\frac{1}{p}_i\). Otherwise the leading term has degree \( < \frac{1}{p} \). The leading term of derivative of any order of \( z_i \) has degree \( \leq \frac{1}{p} \).

Let us plug the differential polynomial \( T(u, u', u'', \ldots) \) the germ (24) and develop the result as a Puiseux series in \( \theta \). If the germ \( z_i(\mu) \) is a solution of the equation (21) then all terms of this Puiseux series are equal to zero. In particular the leading coefficient is zero. This observation is an important step for proofing Theorem 2.12.

**2.4.7 Proof of Rosenlicht’s theorem**

Here we complete an elementary proof of Theorem 2.12 discovered by Maxwell Rosenlicht [Ros73]. We will proof first the simpler Theorems 2.18 and 2.19 of a similar nature.

**Theorem 2.18.** Assume that the equation (13) over a functional differential field \( K \) has a solution \( z \in F \) where \( F \) is a generalized extension by integral of \( K \). Then (13) has a solution in the algebraic extension \( K_P \) of \( K \) associated with the element \( z \in F \).

**Proof.** If the constant term of the differential polynomial \( T(u, u', u'', \ldots) = u^n - Q(u, u', u'', \ldots) \) is equal to zero, then (13) has solution \( u = 0 \) belonging to \( K \). In this case we have nothing to prove.

Below we will assume that the constant term \( T_0 \) of \( T \) is not equal to zero. Thus the differential polynomial \( T \) has two special terms: the term \( u^n \) which is the only term of highest degree \( n \) and the term \( T_0 \) which is the only term of smallest degree zero.

Assume that (13) has a solution \( z \) in a generalized extension by integral \( F \supset K(\theta) \) of \( K \). According to section 2.3.5.3 the existence of such a solution \( z \) implies the existence of family \( z(\rho) \) of germs of solutions depending on a parameter \( \rho \) such that when the absolute value \( |\rho| \) is big enough \( z(\rho) \) can be expanded in Puiseux series (22) in \( \theta \).

We will show that the degree \( \frac{1}{p} \) of the leading term in (22) is equal to zero and the leading coefficient \( z_{i_0} \in K_P \) satisfies (13). This will prove Theorem 2.18.

According to Lemma 2.18 the leading term of the derivative of any order of \( z_{i_0} \) has degree \( \leq \frac{1}{p} \). Thus the leading term of the Puiseux series obtained by plugging (22) instead of \( u \) into differential polynomial \( Q \) has degree \( < n\frac{1}{p} \). The leading term of the Puiseux series obtained by arising (22) to the \( n \)-th power is equal to \( n\frac{1}{p} \). If \( \frac{1}{p} > 0 \) this term can not be canceled after plugging (22) instead of \( u \) into differential polynomial \( T \). Thus the degree \( \frac{1}{p} \) cannot be positive.

Let us plug (22) into the differential polynomial \( T(u, u', \ldots) - T_0 \). We will obtain a Puiseux series of negative degree if \( \frac{1}{p} < 0 \). Thus the term \( T_0 \) in the sum \( (T - T_0) + T_0 \) can not be canceled. Thus \( \frac{1}{p} \) cannot be negative.

We proved that \( \frac{1}{p} = 0 \). If in this case we plug (22) into the differential polynomial \( T(u, u', \ldots) \) we obtain a Puiseux series having only one term of nonnegative degree which is equal to zero. From Lemma 2.18 it is easy to see that this term equals to \( T(z_{i_0}, z_{i_0}', \ldots)\theta^0 \). Thus \( z_{i_0} \in K_P \) is a solution of (13). Theorem 2.18 is proved.

**Theorem 2.19.** Assume that equation (13) over a functional differential field \( K \) has a solution \( z \in F \) where \( F \) is a generalized extension by an exponential of integral of \( K \). Then (13) has a solution in the algebraic extension \( K_P \) of \( K \) associated with the element \( z \in F \).
Proof. Theorem 2.19 can be proved exactly in the same way as Theorem 2.18. Instead of Lemma 2.18 one has to use Lemma 2.20. In the case when leading term of the Puiseux expansion of $z_\mu$ has degree zero, the leading coefficient of its derivative equals to $z_\mu'$ (see Lemma 2.20). That is why the case $\frac{z}{p} = 0$ in Theorem 2.10 can be treated exactly in the same way as in Theorem 2.18.

Now we ready to prove Theorem 2.12.

Proof. Proof of Theorem 35 By assumption the equation (13) has a solution $z \in F$ where $F$ is an extension of $K$ by generalized quadratures. By definition there is a chain $K = F_0 \subset \cdots \subset F_m$ such that $F \subset F_m$ and for every $i = 0, \ldots, m - 1$ or $F_{i+1}$ is a finite extension of $F_i$, or $F_{i+1}$ is a generalized extension by integral of $F_i$, or $F_{i+1}$ is a generalized extension by exponential integral of $F_i$. We prove Theorem 2.12 by induction in the length $m$ of the chain of extension. For $m = 1$ Theorem 2.12 follows from Theorem 2.18 or from Theorem 2.19.

Assume that Theorem 2.12 is true for $m = k$. A chain $F_0 \subset F_1 \subset \cdots \subset F_{k+1}$ provides the chain $F_1 \subset \cdots \subset F_{k+1}$ of extensions of of length $k$ for the field $F_1$. Thus (13) has an algebraic solution $z$ over the field $F_1$. The extension $F_0 \subset \tilde{F}_1$, where $\tilde{F}_1$ is the extension of $F_1$ by the element $z$, is either an algebraic extension, or extension by generalized integral or extension by generalized exponential of integral. Thus for the extension $F_0 \subset \tilde{F}_1$ Theorem 2.12 holds.

We completed the inductive proof of Theorem 2.12.

3 Topological Galois Theory

3.1 Introduction

In this section we present an outline of topological Galois theory based on the book [Kho14]. The theory studies topological obstructions to solvability of equations “in finite terms” i.e. to their solvability by radicals, by elementary functions, by quadratures and by functions belonging to other Liouvillian classes.

As was discovered by Camille Jordan the Galois group of an algebraic equation over the field of rational functions of several complex variables has a topological meaning: it is isomorphic to the monodromy group of the algebraic function defined by this algebraic equation. Therefore the monodromy group is responsible for the representability of an algebraic function by radicals.

In the section 3.2 we present a topological proof of the nonrepresentability of algebraic functions by radicals. This proof is based on my old paper [Kho71]. It contains a germ of topological Galois theory.

Not only algebraic functions have a monodromy group. It is defined for any solution of a linear differential equation whose coefficients are rational functions and for many more functions, for which the Galois group does not make sense.

It is thus natural to try using the monodromy group for these functions instead of the Galois group to prove that they do not belong to a certain Liouvillian class. This particular approach is implemented in topological Galois theory (see [Kho14]), which has a one-dimensional version and a multidimensional version.

In the sections 3.3 and 3.4 we present an outline of the one-dimensional version and an outline of the multidimensional version. These sections contain definitions, statements of results and comments to them. Basically no proofs are presented.
3.2 On Representability of Algebraic Functions by Radicals

3.2.1 Introduction

This section is dedicated to a self contained simple proof of the classical criteria for representability of algebraic functions of several complex variables by radicals. It also contains a criteria for representability of algebroidal functions by composition of single-valued analytic functions and radicals, and a result related to the 13-th Hilbert problem.

Consider an algebraic equation

\[ P_n y^n + P_{n-1} y^{n-1} + \cdots + P_0 = 0, \]

whose coefficients \( P_1, \ldots, P_n \) are polynomials of \( N \) complex variables \( x_1, \ldots, x_N \). Camille Jordan discovered that the Galois group of the equation \( \text{(25)} \) over the field \( \mathbb{R} \) of rational functions of \( x_1, \ldots, x_N \) has a topological meaning (see theorem 3.2 below): it is isomorphic to the monodromy group of the equation \( \text{(25)} \).

According to the Galois theory, equation \( \text{(25)} \) is solvable by radicals over the field \( \mathbb{R} \) if and only if its Galois group is solvable. If the equation \( \text{(24)} \) is irreducible it defines a multivalued algebraic function \( y(x) \). The Galois theory and Theorem 3.2 imply the following criteria for representability of an algebraic function by radicals, which consists of two statements:

1) If the monodromy group of an algebraic function \( y(x) \) is solvable, then \( y(x) \) is representable by radicals.

2) If the monodromy group of an algebraic function \( y(x) \) is not solvable, then \( y(x) \) is not representable by radicals.

We reduce the first statement to linear algebra (see Theorem 3.5 below) following the book [Kho14].

We prove the second statement topologically without using Galois theory. Vladimir Igorevich Arnold found the first topological proof of this statement [Ale04]. We use another topological approach (see Theorem 3.6 below) based on the paper [Kho71]. This paper contains the first result of topological Galois theory [Kho14] and it gave a hint for its further development.

3.2.2 Monodromy group and Galois group

Consider the equation \( \text{(25)} \). Let \( \Sigma \subset \mathbb{C}^N \) be the hypersurface defined by equation \( P_n J = 0 \), where \( P_n \) is the leading coefficient and \( J \) is the discriminant of the equation \( \text{(25)} \). The monodromy group of the equation \( \text{(25)} \) is the group of all permutations of its solutions which are induced by motions around the singular set \( \Sigma \) of the equation \( \text{(25)} \). Below we discuss this definition more precisely.

At a point \( x_0 \in \mathbb{C}^N \setminus \Sigma \) the set \( Y_{x_0} \) of all germs of analytic functions satisfying equation \( \text{(25)} \) contains exactly \( n \) elements, i.e. \( Y_{x_0} = \{ y_1, \ldots, y_n \} \). Indeed, if \( P_n(x_0) \neq 0 \) then for \( x = x_0 \) equation \( \text{(25)} \) has \( n \) roots counted with multiplicities. If in addition \( J(x_0) \neq 0 \) then all these roots are simple. By the implicit function theorem each simple root can be extended to a germ of a regular function satisfying the equation \( \text{(25)} \).

Consider a closed curve \( \gamma \) in \( \mathbb{C}^N \setminus \Sigma \) beginning and ending at the point \( x_0 \). Given a germ \( y \in Y_{x_0} \) we can continue it along the loop \( \gamma \) to obtain another germ \( y_\gamma \in Y_{x_0} \). Thus each such loop \( \gamma \) corresponds to a permutation \( S_\gamma : Y_{x_0} \to Y_{x_0} \) of the set \( Y_{x_0} \) that maps a germ \( y \in Y_{x_0} \) to the germ \( y_\gamma \in Y_{x_0} \). It is easy to see that the map \( \gamma \mapsto S_\gamma \) defines a homomorphism from the fundamental group \( \pi_1(\mathbb{C}^N \setminus \Sigma, x_0) \) of the domain \( \mathbb{C}^N \setminus \Sigma \) with the base point \( x_0 \) to the group \( S(Y_{x_0}) \) of permutations. The monodromy group of the equation \( \text{(25)} \) is the image of the fundamental group in the group \( S(Y_{x_0}) \) under this homomorphism.

Remark. Instead of the point \( x_0 \) one can choose any other point \( x_1 \in \mathbb{C}^N \setminus \Sigma \). Such a choice will not change the monodromy group up to an isomorphism. To fix this isomorphism one can
functions. Any single valued algebraic function is a rational function of the form 

\[ y = \frac{p(x)}{q(x)} \]

where \( p(x) \) and \( q(x) \) are polynomials in \( x \). The automorphism is unique because the extension is generated by the rational functions using the arithmetic operations and radicals.

The field of rational functions of \( x_1, \ldots, x_N \) is isomorphic to the field \( \mathbb{R} \) of germs of rational functions at the point \( x_0 = 0 \) in \( \mathbb{C}^N \). Consider the field extension \( \mathbb{R}(y_1, \ldots, y_n) \) of \( \mathbb{R} \) by the germs \( y_1, \ldots, y_n \) at \( x_0 \) satisfying the equation (25).

**Lemma 3.1.** Every permutation \( S \) from the monodromy group can be uniquely extended to an automorphism of the field \( \mathbb{R}(y_1, \ldots, y_n) \) over the field \( \mathbb{R} \).

**Proof.** Every element \( f \in \mathbb{R}(y_1, \ldots, y_n) \) is a rational function of \( x, y_1, \ldots, y_n \). It can be continued meromorphically along the curve \( y \in \pi_1(\mathbb{C}^n \setminus \Sigma, x_0) \) together with \( y_1, \ldots, y_n \). This continuation gives the required automorphism, because the continuation preserves the arithmetical operations and every rational function returns back to its original values (since it is a single-valued valued function). The automorphism is unique because the extension is generated by \( y_1, \ldots, y_n \).

By definition the **Galois group** of the equation (25) is the group of all automorphisms of the field \( \mathbb{R}(y_1, \ldots, y_n) \) over the field \( \mathbb{R} \). According to Lemma 3.1 the monodromy group of the equation (25) can be considered as a subgroup of its Galois group. Recall that by definition a multivalued function \( y(x) \) is algebraic if all its meromorphic germs satisfy the same algebraic equation over the field of rational functions.

**Theorem 3.1.** A germ \( f \in \mathbb{R}(y_1, \ldots, y_n) \) is fixed under the monodromy action if and only if \( f \in \mathbb{R} \).

**Proof.** A germ \( f \in \mathbb{R}(y_1, \ldots, y_n) \) is fixed under the monodromy action if and only if \( f \) is a germ of a single valued function. The field \( \mathbb{R}(y_1, \ldots, y_n) \) contains only germs of algebraic functions. Any single valued algebraic function is a rational function.

According to the Galois theory Theorem 3.1 can be formulated in the following way.

**Theorem 3.2.** The monodromy group of the equation (25) is isomorphic to the Galois group of the equation (25) over the field \( \mathbb{R} \).

Below we will not rely on Galois theory. Instead we will use Theorem 3.2 directly.

**Lemma 3.2.** The monodromy group acts on the set \( Y_{x_0} \) transitively if and only if the equation (25) is irreducible over the field of rational functions.

**Proof.** Assume that there is a proper subset \( \{y_1, y_2, \ldots, y_k\} \) of \( Y_{x_0} \) invariant under the monodromy action. Then the basic symmetric functions \( r_1 = y_1 + \cdots + y_k \), \( r_2 = \sum_{i<j} y_i y_j \), \( \ldots \), \( r_k = y_1 \cdots y_k \) belong to the field \( \mathbb{R} \). Thus \( y_1, y_2, \ldots, y_k \) are solutions of the degree \( k < n \) equation \( y^k - r_1 y^{k-1} + \cdots + (-1)^k r_k = 0 \). So equation (25) is reducible. On the other hand if the equation (25) can be represented as a product of two equations over \( \mathbb{R} \) then their roots belong to two complementary subsets of \( Y_{x_0} \) which are invariant under the monodromy action.

**Corollary 3.1.** An irreducible equation (25) defines a multivalued algebraic function \( y(x) \) whose set of germs at \( x_0 \in \mathbb{C}^N \setminus \Sigma \) is the set \( Y_{x_0} \) and whose monodromy group coincides with the monodromy group of the equation (25).

**Theorem 3.2** Corollary 3.1 and the Galois theory immediately imply the following result.

**Theorem 3.3.** An algebraic function whose monodromy group is solvable can be represented by rational functions using the arithmetic operations and radicals.
A stronger version of Theorem \ref{thm:main} can be proven using linear algebra (see Theorem 10 in the next section).

### 3.2.3 Action of solvable groups and representability by radicals

In this section, we prove that if a finite solvable group \( G \) acts on a \( \mathbb{C} \)-algebra \( V \) by automorphisms, then all elements of \( V \) can be expressed by the elements of the invariant subalgebra \( V_0 \) of \( G \) by taking radicals and adding.

This construction of a representation by radicals is based on linear algebra. More precisely we use the following well known statement: any finite abelian group of linear transformations of a finite-dimensional vector space over \( \mathbb{C} \) can be diagonalized in a suitable basis.

**Lemma 3.3.** Let \( G \) be a finite abelian group of order \( n \) acting by automorphisms on \( \mathbb{C} \)-algebra \( V \). Then every element of the algebra \( V \) is representable as a sum of elements \( x_i \in V \), such that \( x_i^n \) lies in the invariant subalgebra \( V_0 \) of \( G \), i.e., in the fixed-point set of the group \( G \).

**Proof.** Consider a finite-dimensional vector subspace \( L \) in the algebra \( V \) spanned by the \( G \)-orbit of an element \( x \). The space \( L \) splits into a direct sum \( L = L_1 + \cdots + L_k \) of eigenspaces for all operators from \( G \). Therefore, the vector \( x \) can be represented in the form \( x = x_1 + \cdots + x_k \), where \( x_1, \ldots, x_k \) are eigenvectors for all the operators from the group. The corresponding eigenvalues are \( n \)-th roots of unity. Therefore, the elements \( x_1^n, \ldots, x_k^n \) belong to the invariant subalgebra \( V_0 \).

**Definition 3.1.** We say that an element \( x \) of the algebra \( V \) is an \( n \)-th root of an element \( a \) if \( x^n = a \).

We can now restate Lemma \ref{lem:3.3} as follows: every element \( x \) of the algebra \( V \) is representable as a sum of \( n \)-th roots of some elements of the invariant subalgebra.

**Theorem 3.4.** Let \( G \) be a finite solvable group of order \( n \) acting by automorphisms on \( \mathbb{C} \)-algebra \( V \). Then every element \( x \) of the algebra \( V \) can be obtained from the elements of the invariant subalgebra \( V_0 \) by taking \( n \)-th roots and summing.

We first prove the following simple statement about an action of a group on a set. Suppose that a group \( G \) acts on a set \( X \), let \( H \) be a normal subgroup of \( G \), and denote by \( X_0 \) the subset of \( X \) consisting of all points fixed under the action of \( G \).

**Lemma 3.4.** The subset \( X_H \) of the set \( X \) consisting of the fixed points under the action of the normal subgroup \( H \) is invariant under the action of \( G \). There is a natural action of the quotient group \( G/H \) on the set \( X \) with the fixed-point set \( X_0 \).

**Proof.** Suppose that \( g \in G \), \( h \in H \). Then the element \( g^{-1}hg \) belongs to the normal subgroup \( H \). Let \( x \in X_H \). Then \( g^{-1}hx = x \), or \( h(g(x)) = g(x) \), which means that the element \( g(x) \in X \) is fixed under the action of the normal subgroup \( H \). Thus the set \( X_H \) is invariant under the action of the group \( G \). Under the action of \( G \) on \( X_H \), all elements of \( H \) correspond to the identity transformation. Hence the action of \( G \) on \( X_H \) reduces to an action of the quotient group \( G/H \).

We now proceed with the proof of Theorem \ref{thm:3.4}.

**Proof.** (of Theorem \ref{thm:3.4}) Since the group \( G \) is solvable, it has a chain of nested subgroups \( G = G_0 \supset G_1 \supset \cdots \supset G_m = e \) in which the group \( G_m \) consists of the identity element \( e \) only, and every group \( G_i \) is a normal subgroup of the group \( G_{i-1} \). Moreover, the quotient group \( G_{i-1}/G_i \) is abelian. Let \( V_0 \subset \cdots \subset V_m = V \) denote the chain of invariant subalgebras of the algebra \( V \) with respect to the action of the groups \( G_0, \ldots, G_m \). By Lemma 9 the abelian group \( G_{i-1}/G_i \) acts naturally on the invariant subalgebra \( V_i \), leaving the subalgebra \( V_{i-1} \)
pointwise fixed. The order $m_i$ of the quotient group $G_{i-1}/G_i$ divides the order of the group $G$. Therefore, Lemma 3.5 is applicable to this action. We conclude that every element of the algebra $V_i$ can be expressed with the help of summation and $n$-th root extraction by the elements of the algebra $V_{i-1}$. Repeating the same argument, we will be able to express every element of the algebra $V$ by the elements of the algebra $V_0$ using a chain of summations and $n$-th root extractions. \[ \square \]

Theorem 3.5. An algebraic function whose monodromy is solvable can be represented by rational functions by root extractions and summations.

Proof. One can prove Theorem 3.5 by applying Theorem 3.4 to the monodromy action by automorphisms on the extension $\mathcal{R}(y_1, \ldots, y_n)$ with the field of invariants $\mathcal{R}$. \[ \square \]

3.2.4 Topological obstruction to representation by radicals

Let us introduce some notation.

By $G_m$ we denote the $m$-th commutator subgroup of the group $G$. For any $m \geq 1$, the group $G_m$ is a normal subgroup in $G$.

By $F(D, x_0)$ we denote the fundamental group of the domain $U = \mathbb{C}^N \setminus D$ with the base point $x_0 \in U$, where $D$ is an algebraic hypersurface in $\mathbb{C}^N$.

Let $H(D, m)$ be the covering space of the domain $\mathbb{C}^N \setminus D$ corresponding to the subgroup $F^m(D, x_0)$ of the fundamental group $F(D, x_0)$.

We will say that an algebraic function is an $R$-function if it becomes a single-valued function on some covering $H(D, m)$.

Lemma 3.5. If $m_1 \geq m_2$ and $D_1 \supset D_2$ then there is a natural projection $\rho : H(D_1, m_1) \rightarrow H(D_2, m_2)$. Thus if a function $y$ becomes a single-valued function on $H(D_2, m_2)$ then it certainly becomes a single-valued function on $H(D_1, m_1)$.

Proof. Let $\rho : F(D_1, x_0) \rightarrow F(D_2, x_0)$ be the homomorphism induced by the embedding $\rho : \mathbb{C}^N \setminus D_1 \rightarrow \mathbb{C}^N \setminus D_2$. Lemma 3.5 follows from the following obvious inclusions: $\rho^{-1}(F^m(D_2, x_0)) \subset F^m(D_1, x_0)$ and $F^m(D_2, x_0) \subset F^m(D_1, x_0)$. \[ \square \]

Lemma 3.6. If $y_1$ and $y_2$ are $R$-function then $y_1 + y_2$, $y_1 - y_2$, $y_1 \cdot y_2$ and $y_1/y_2$ also are $R$-functions.

Proof. Assume that $R$-functions $y_1$ and $y_2$ become single-valued functions on the coverings $H(D_1, m_1)$ and $H(D_2, m_2)$. By Lemma 3.5, the functions $y_1,y_2$ become single-valued on the covering $H(D, m)$ where $D = D_1 \cup D_2$ and $m = \max(m_1,m_2)$. Thus the functions $y_1 + y_2$, $y_1 - y_2$, $y_1 \cdot y_2$ and $y_1/y_2$ also become single-valued on the covering $H(D, m)$. The proof is completed since $y_1 + y_2$, $y_1 - y_2$, $y_1 \cdot y_2$ and $y_1/y_2$ are algebraic functions. \[ \square \]

Lemma 3.7. Composition of an $R$-function with the degree $q$ radical is an $R$-function.

Proof. Assume that the function $y$ defined by (24) is $R$-function which becomes a single-valued function on the covering $H(D_1, m)$. Let $D_2 \subset \mathbb{C}^N$ be the hypersurface, defined by the equation $P_n P_0 = 0$, where $P_n$ and $P_0$ are the leading coefficient and the constant term of the equation (25). According to Lemma 3.5, the function $y$ becomes a single-valued function on the covering $H(D, m)$ where $D = D_1 \cup D_2$. Let $h_0 \in H(D, m)$ be a point whose image under the natural projection $\rho : H(D, m) \rightarrow \mathbb{C}^N \setminus D$ is the point $x_0$. One can identify the fundamental groups $\pi_1(H(D, m), h_0)$ and $F^m(D, x_0)$.

By definition of $D_2$ the function $y$ never equals to zero or to infinity on $H(D, m)$. Hence $y$ defines a map $y : H(D, m) \rightarrow \mathbb{C} \setminus \{0\}$. Let $y_* : \pi_1(H(D, m), h_0) \rightarrow \pi_1(\mathbb{C} \setminus \{0\}, y(h_0))$ be the induced homomorphism of the fundamental groups. The group $\pi_1(H(D, m), h_0)$ is
But becomes single-valued on the covering \( y \). Conversely, if \( m \) some natural number \( y \) of arbitrary small neighborhood of a point ramified germs of call In this section we describe a local version of Theorem 3.7.

3.2.6 Local representability

as it was used in the proof of Theorem 3.5.

\[ \text{Lemma 3.8. An algebraic function } y \text{ is an } R\text{-function if and only if its monodromy group is solvable.} \]

\[ \text{Proof. Assume that } y \text{ is defined by (25). Let } D \text{ be the hypersurface } P_nJ = 0 \text{ where } P_n \text{ is the leading coefficient and } J \text{ is the discriminant of (25). Let } M \text{ be the monodromy group of } y. \text{ Consider the natural homomorphism } p : F(D, x_0) \to M. \text{ If } M \text{ is solvable then for some natural number } m \text{ the } m\text{-th commutator of } M \text{ is the identity element } e. \text{ The function } y \text{ becomes single-valued on the covering } H(D, m) \text{ since } F^m(D, x_0) \subset p^{-1}(M^m) = p^{-1}(e). \]

Conversely, if \( y \) is a single-valued function on some covering \( H(D, m) \) then \( p(F^m(D, x_0)) = e. \) But \( p(F^m(D, x_0)) = M^m. \) Thus the monodrogy group \( M \) is solvable. \]

\[ \text{Theorem 3.6. If an algebraic function has unsolvable monodromy group then it cannot be represented by compositions of rational functions and radicals} \]

\[ \text{Proof. Lemma 3.7 and Lemma 3.8 show that the class of } R\text{-functions is closed under arithmetic operations and compositions with radicals. Lemma 3.8 shows that the monodromy group of any } R\text{-function is solvable.} \]

3.2.5 Compositions of analytic functions and radials

In this section we describe a class of multivalued functions in a domain \( U \subset \mathbb{C}^N \) representable by composition of single-valued analytic functions and radicals.

A multivalued function \( y \) in \( U \) is called an \textit{algebroidal function} in \( U \) if it satisfies an irreducible equation

\[ y^n + f_{n-1}y^{n-1} + \cdots + f_0 = 0 \]

whose coefficients \( f_{n-1}, \ldots, f_0 \) are analytic functions in \( U \). An algebroidal function could be considered as a continuous multivalued function in \( U \) which has finitely many values.

\[ \text{Theorem 3.7. (Kho70, Kho71) A multivalued function } y \text{ in the domain } U \text{ can be represented by composition of radicals and single valued analytic functions if and only } y \text{ is an algebroidal function in } U \text{ with solvable monodromy group.} \]

To prove the “only if” part one can repeat the proof of Theorem 3.6 replacing coverings over domains \( \mathbb{C}^N \setminus D \) by coverings over domains \( U \setminus \tilde{D} \) where \( \tilde{D} \) is an analytic hypersurface in \( U \).

To prove Theorem 3.7 in the opposite direction one can use Theorem 3.4 in the same way as it was used in the proof of Theorem 3.6.

3.2.6 Local representability

In this section we describe a a local version of Theorem 3.7.

Let \( y \) be an algebroidal function in \( U \) defined by (26). One can localize the equation (26) at any point \( p \in U \), i.e. one can replaced the coefficients \( f_i \) of the equation (26) by their germs at \( p \). After such a localization the equation (26) can became reducible, i.e. it can became representable as a product of irreducible equations. Thus an algebroidal functions \( y \) in arbitrary small neighborhood of a point \( p \) defines several algebroidal functions, which we will call \textit{ramified germs of } \( y \) \textit{at } \( p \). For a ramified germ of \( y \) at \( p \) the monodromy group is defined.
(as the monodromy group of an algebroidal function in an arbitrary small neighborhood of the point p).

A ramified germ of an algebroidal function y of one variable x in a neighborhood of a point \( p \in \mathbb{C}^1 \) has a simple structure: its monodromy group is a cyclic group \( \mathbb{Z}/m\mathbb{Z} \) and it can be represented as a composition of a radical and an analytic single-valued function: \( y(x) = f((x-p)^{1/m}) \) where m is the ramification order of y. The following corollary follows from Theorem 16.

**Corollary 3.2.** ([Kho70], [Kho71])

1) If a multivalued function y in the domain \( U \) can be represented by composition of an algebroidal functions of one variable and single valued analytic functions then the monodromy group of any ramified germ of y is solvable.

2) If the monodromy group of a ramification germ of y at \( p \) is solvable then in a small neighborhood of \( p \) it can be represented by composition of radicals and single valued analytic functions.

The local monodromy group of an algebroidal function y at a point \( p \in U \) is the monodromy group of the equation (26) in an arbitrary small neighborhood of the point p. The ramified germs of y at the point p correspond to the orbits of the local monodromy group actions. This statement can be proven in the same way as Lemma 3.2 was proved.

### 3.2.7 Application to the 13-th Hilbert problem

In 1957 A.N. Kolmogorov and V.I. Arnold proved the following totally unexpected theorem which gave a negative solution to the 13-th Hilbert problem.

**Theorem 3.8.** (Kolmogorov–Arnold) Any continuous function of \( n \) variables can be represented as the composition of functions of a single variable with the help of addition.

The 13-th Hilbert problem has the following algebraic version which still remains open:

*Is it possible to represent any algebraic function of \( n > 1 \) variables by algebraic functions of a smaller number of variables with the help of composition and arithmetic operations?*

An entire algebraic function y in \( \mathbb{C}^n \) is an algebraic function defined in \( U = \mathbb{C}^N \) by an equation (26) whose coefficient \( f_i \) are polynomials. An entire algebraic function could be considered as a continuous algebraic function.

It turns out that in Kolmogorov–Arnold Theorem one cannot replace continuous functions by entire algebraic functions.

**Theorem 3.9.** ([Kho70], [Kho71]) If an entire algebraic function can be represented as a composition of polynomials and entire algebraic functions of one variable, then its local monodromy group at each point is solvable.

**Proof.** Theorem 3.9 follows from from Corollary 3.2.

**Corollary 3.3.** A function \( y(a, b) \), defined by equation \( y^5 + ay + b = 0 \), cannot be expressed in terms of entire algebraic functions of a single variable by means of composition, addition and multiplication.

**Proof.** Indeed, it is easy to check that the local monodromy group of y at the origin is the unsolvable permutation group \( S_5 \) (see [Kho70], [Kho71]).

Division is not a continuous operation and it destroys the locality. One cannot add division to the operations used in Theorem 3.9. It is easy to see that the function \( y(a, b) \) from Corollary 3.2 can be expressed in terms of entire algebraic functions of a single variable by means of composition and arithmetic operations: \( y(a, b) = g(b/\sqrt[5]{a^5}) \sqrt[5]{a} \), where \( g(u) \) is defined by equation \( g^5 + g + u = 0 \).
The following particular case of the algebraic version of the 13-th Hilbert problem still remains open.

*Problem.* Show that there is an algebraic function of two variables which cannot be expressed in terms of algebraic functions of a single variable by means of composition and arithmetic operations.

### 3.3 One Dimensional Topological Galois Theory

#### 3.3.1 Introduction

In the one-dimensional version we consider functions from Liouvillian classes as multi-valued analytic functions of one complex variable. It turns out that there exist topological restrictions on the way the Riemann surface of a function from a certain Liouvillian class can be positioned over the complex plane. If a function does not satisfy these restrictions, then it cannot belong to the corresponding Liouvillian class.

Besides a geometric appeal, this approach has the following advantage. Topological obstructions relate to branching. It turns out that if a function does not belong to a certain Liouvillian class by topological reasons then it automatically does not belong to a much wider class of functions. This wider class can be obtained if we add to the Liouvillian class all single valued functions having at most a countable set of singularities and allow them to enter all formulas.

The composition of functions is not an algebraic operation. In differential algebra, this operation is replaced with a differential equation describing it. However, for example, the Euler $\Gamma$-function does not satisfy any algebraic differential equation. Hence it is pointless to look for an equation satisfied by, say, the function $\Gamma(\exp x)$ and one cannot describe it algebraically (but the function $y = \exp(\Gamma(x))$ satisfies the equation $y' = \Gamma'y$ over a differential field containing $\Gamma$ and it makes sense in the differential algebra). The only known results on non-representability of functions by quadratures and, say, the Euler $\Gamma$-functions are obtained by our method.

On the other hand, our method cannot be used to prove that a particular single valued meromorphic function does not belong to a certain Liouvillian class.

There are the following topological obstructions to representability of functions by generalized quadratures, $k$-quadratures and quadratures (see section 3.3.6).

Firstly, the functions representable by generalized quadratures and, in particular, the functions representable by $k$-quadratures and quadratures may have no more than countably many singular points in the complex plane (see section 3.3.4).

Secondly, the monodromy group of a function representable by quadratures is necessarily solvable (see section 3.3.6). There are similar restrictions for for a function representable by generalized quadratures and $k$-quadratures. However, these restrictions are more involved. To state them, the monodromy group should be regarded not as an abstract group but rather as a transitive subgroup in the permutation group. In other terms, these restrictions make use not only of the monodromy group but rather of the *monodromy pair* of the function consisting of the monodromy group and the stabilizer of some germ of the function (see section 3.3.7).

One can prove that the only reasons for unsolvability in finite terms of Fuchsian linear differential equations are topological (see section 3.3.12). In other words, if there are no topological obstructions to solvability of a Fuchsian equation by generalized quadratures (by $k$-quadratures, by quadratures), then this equation is solvable by generalized quadratures (by $k$-quadratures or by quadratures respectively). The proof is based on a linear-algebraic part of differential Galois theory (dealing with linear algebraic groups and their differential invariants).
3.3.2 Solvability in finite terms and Liouvillian classes of functions

An equation is solvable “in finite terms” (or is solvable “explicitly”) if its solutions belong to a certain class of functions. Different classes of functions correspond to different notions of solvability in finite terms.

A class of functions can be introduced by specifying a list of **basic functions** and a list of **admissible operations** (see section 2.2.4). Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations. (see section 2.2.3) Below, we define classes of functions in exactly this way.

Liouvillian classes of functions, which appear in the problems of integrability in finite terms, contain multivalued functions. Thus the operations on multivalued functions have to be defined. Such a definition can be found in section 2.2.4 (note that in the multidimensional case we use a slightly different, more restricted definition of the operations on multivalued functions).

We need the list of **basic elementary functions** (see section 2.2.5). In essence, this list contains functions that are studied in high-school and which are frequently used in pocket calculators.

We also use the classical operations on functions, such as the arithmetic operations, the operation of composition and so on. The list of such operations is presented in section 2.2.5.

We can now return to the definition of Liouvillian classes of single variable functions.

3.3.2.1 Functions representable by radicals

List of basic functions: all complex constants, an independent variable $x$. List of admissible operations: arithmetic operations and the operation of taking the $n$-th root $f^{\frac{1}{n}}$, $n = 2, 3, \ldots$, of a given function $f$.

3.3.2.2 Functions representable by $k$-radicals

List of basic functions: all complex constants, an independent variable $x$. List of admissible operations: arithmetic operations and the operation of taking the $n$-th root $f^{\frac{1}{n}}$, $n = 2, 3, \ldots$, of a given function $f$, the operation of solving algebraic equations of degree $\leq k$.

3.3.2.3 Elementary functions

List of basic functions: basic elementary functions. List of admissible operations: compositions, arithmetic operations, differentiation.

3.3.2.4 Generalized elementary functions

This class of functions is defined in the same way as the class of elementary functions. We only need to add the operation of solving algebraic equations to the list of admissible operations.

3.3.2.5 Functions representable by quadratures

List of basic functions: basic elementary functions. List of admissible operations: compositions, arithmetic operations, differentiation, integration.

3.3.2.6 Functions representable by $k$-quadratures

This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations of degree at most $k$ to the list of admissible operations.

3.3.2.7 Functions representable by generalized quadratures

This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations to the list of admissible operations.
3.3.3 Simple formulas with complicated topology

Developing topological Galois theory I followed the following plan:

I. To find a wide class of multivalued functions such that:
   a) it is closed under all classical operations;
   b) it contains all entire functions and all functions from each Liouvillian class;
   c) for functions from the class the monodromy group is well defined.

II. To use the monodromy group instead of the Galois group inside the class.

Let us discuss some difficulties that one need to overcome on this way.

Example 3.1. Consider an elementary function $f$ defined by the following formula:

$$f(z) = \ln \sum_{j=1}^{n} \lambda_j \ln(z - a_j)$$

where $a_j$ are different points in the complex line, and $\lambda_j \in \mathbb{C}$ are constants.

Let $\Lambda$ denote the additive subgroup of complex numbers generated by the constants $\lambda_1, \ldots, \lambda_n$. It is clear that if $n > 2$, then for almost every collection of constants $\lambda_1, \ldots, \lambda_n$, the group $\Lambda$ is everywhere dense in the complex line.

Lemma 3.9. If the group $\Lambda$ is dense in the complex line, then the elementary function $f$ has a dense set of logarithmic ramification points.

Proof. Let $g$ be the multivalued function defined by the formula

$$g(z) = \sum_{j=1}^{n} \lambda_j \ln(z - a_j).$$

Take a point $a \neq a_j, j = 1, \ldots, n$ and let $g_a$ be one of the germs of $g$ at $a$. A loop around the points $a_1, \ldots, a_n$ adds the number $2\pi i \lambda$ to the germ $g_a$, where $\lambda$ is an element of the group $\Lambda$. Conversely, every germ $g_a + 2\pi i \lambda$, where $\lambda \in \Lambda$, can be obtained from the germ $g_a$ by the analytic continuation along some loop. Let $U$ be a small neighborhood of the point $a$, such that the germ $g_a$ has a single-valued analytic continuation $G$ on $U$. The image $V$ of the domain $U$ under the map $G : U \to \mathbb{C}$ is open. Therefore, in the domain $V$, there is a point of the form $2\pi i \lambda$, where $\lambda \in \Lambda$. The function $G - 2\pi i \lambda$ is one of the branches of the function $g$ over the domain $U$, and the zero set of this branch in the domain $U$ is nonempty. Hence, one of the branches of the function $f = \ln g$ has a logarithmic ramification point in $U$. \qed

The set $\Sigma$ of singular points of the function $f$ is a countable set (see section 3.3.4). Under assumptions of Lemma 3.9 the set $\Sigma$ is everywhere dense.

It is not hard to verify that the monodromy group (see section 3.3.7) of the function $f$ has the cardinality of the continuum. This is not surprising: the fundamental group $\pi_1(\mathbb{C} \setminus \Sigma)$ has obviously the cardinality of the continuum provided that $\Sigma$ is a countable dense set.

One can also prove that the image of the fundamental group $\pi_1(\mathbb{C} \setminus \{\Sigma \cup b}\})$ of the complement of the set $\Sigma \cup b$, where $b \notin \Sigma$, in the permutation group is a proper subgroup of the monodromy group of $f$.

The fact that the removal of one extra point can change the monodromy group, makes all proofs more complicated.

Thus even simplest elementary functions can have dense singular sets and monodromy groups of cardinality of the continuum. In addition the removal of an extra point can change their monodromy groups.
3.3.4 Class of $S$-functions

In this section, we define a broad class of functions of one complex variable needed in the construction of topological Galois theory.

**Definition 3.2.** A multivalued analytic function of one complex variable is called a $S$-function, if the set of its singular points is at most countable.

Let us make this definition more precise. Two regular germs $f_a$ and $g_b$ defined at points $a$ and $b$ of the Riemann sphere $S^2$ are called equivalent if the germ $g_b$ is obtained from the germ $f_a$ by the analytic continuation along some path. Each germ $g_b$ equivalent to the germ $f_a$ is also called a regular germ of the multivalued analytic function $f$ generated by the germ $f_a$.

A point $b \in S^2$ is said to be a singular point for the germ $f_a$ if there exists a path $\gamma : [0,1] \to S^2$, $\gamma(0) = a$, $\gamma(1) = b$ such that the germ has no analytic continuation along this path, but for any $\tau$, $0 \leq \tau < 1$, it admits an analytic continuation along the truncated path $\gamma : [0, \tau] \to S^2$.

It is easy to see that equivalent germs have the same set of singular points. A regular germ is called a $S$-germ, if the set of its singular points is at most countable. A multivalued analytic function is called a $S$-function if each of its regular germ is a $S$-germ.

**Theorem 3.10.** (on stability of the class of $S$-functions) The class $S$ of all $S$-functions is stable under the following operations:

1. differentiation, i.e. if $f \in S$, then $f' \in S$;
2. integration, i.e. if $f \in S$ and $g' = f$, then $g \in S$;
3. composition, i.e. if $g$, $f \in S$, then $g \circ f \in S$;
4. meromorphic operations, i.e. if $f_i \in S$, $i = 1, \ldots, n$, the function $F(x_1, \ldots, x_n)$ is a meromorphic function of $n$ variables, and $f = F(f_1, \ldots, f_n)$, then $f \in S$;
5. solving algebraic equations, i.e. if $f_i \in S$, $i = 1, \ldots, n$, and $f^n + f_1 f^{n-1} + \cdots + f_n = 0$, then $f \in S$;
6. solving linear differential equations, i.e. if $f_i \in S$, $i = 1, \ldots, n$, and $f^{(n)} + f_1 f^{(n-1)} + \cdots + f_n f = 0$, then $f \in S$.

**Remark.** Arithmetic operations and the exponentiation are examples of meromorphic operations, hence the class of $S$-functions is stable under the arithmetic operations and the exponentiation.

**Corollary 3.4.** (see [Kho14]) If a multivalued function $f$ can be obtained from single valued $S$-functions by integration, differentiation, meromorphic operations, compositions, solutions of algebraic equations and linear differential equations, then the function $f$ has at most countable number of singular points.

**Corollary 3.5.** A function having uncountably many singular points cannot be represent by generalized quadratures. In particular it cannot be a generalized elementary function and it cannot be represented by $k$-quadratures or by quadratures.

**Example 3.2.** Consider a discrete group $\Gamma$ of fractional linear transformations of the open unit ball $U$ having a compact fundamental domain. Let $f$ be a nonconstant meromorphic function on $U$ invariant under the action of $\Gamma$. Each point on the boundary $\partial U$ belongs to the closure of the set of poles of $f$, thus the set $\Sigma$ of singular points of $f$ contains $\partial U$. So $\Sigma$ has the cardinality of the continuum and $f$ cannot be expressed by generalized quadratures.
3.3.5 Monodromy group of a $S$-function

The monodromy group of a $S$-function $f$ is the group of all permutations of the sheets of the Riemann surface of $f$ which are induced by motions around the singular set $\Sigma$ of the function $f$. Below we discuss this definition more precisely.

Let $F_{x_0}$ be the set of all germs of the $S$-function $f$ at point $x_0 \notin \Sigma$. Consider a closed curve $\gamma$ in $S^2 \setminus \Sigma$ beginning and ending at the point $x_0$. Given a germ $y \in F_{x_0}$ we can continue it along the loop $\gamma$ to obtain another germ $y_\gamma \in Y_{x_0}$. Thus each such loop $\gamma$ corresponds to a permutation $S_\gamma : F_{x_0} \to F_{x_0}$ of the set $F_{x_0}$ that maps a germ $y \in F_{x_0}$ to the germ $y_\gamma \in F_{x_0}$.

It is easy to see that the map $\gamma \to S_\gamma$ defines a homomorphism from the fundamental group $\pi_1(S^2 \setminus \Sigma, x_0)$ of the domain $S^2 \setminus \Sigma$ with the base point $x_0$ to the group $S(F_{x_0})$ of permutations. The monodromy group of the $S$-function $f$ is the image of the fundamental group in the group $S(F_{x_0})$ under this homomorphism.

Remark. Instead of the point $x_0$ one can choose any other point $x_1 \in S^2 \setminus \Sigma$. Such a choice will not change the monodromy group up to an isomorphism. To fix this isomorphism one can choose any curve $\gamma : I \to \mathbb{C}^N \setminus \Sigma$ where $I$ is the segment $0 \leq t \leq 1$ and $\gamma(0) = x_0$, $\gamma(1) = x_1$ and identify each germ $f_{x_0}$ of $f$ with its continuation $f_{x_1}$ along $\gamma$.

3.3.6 Strong non representability by quadratures

One can prove the following theorem.

Theorem 3.11. (see [Kho14]) The class of all $S$-functions, having a solvable monodromy group, is stable under composition, meromorphic operations, integration and differentiation.

Definition 3.3. A function $f$ is strongly nonrepresentable by quadratures if it does not belong to a class of functions defined by the following data. List of basic functions: basic elementary functions and all single valued $S$-function. List of admissible operations: compositions, meromorphic operations, differentiation and integration.

Theorem 3.11 implies the following corollary.

Corollary 3.6 (Result on quadratures). If the monodromy group of an $S$-function $f$ is not solvable, then $f$ is strongly non representable by quadratures.

Example 3.3. The monodromy group of an algebraic function $y(x)$ defined by an equation $y^5 + y - x = 0$ is the unsolvable group $S_5$. Thus $y(x)$ provides an example of a function with finite set of singular points, which is strongly non representable by quadratures.

The following Corollary contains a stronger result on non representability of algebraic functions by quadratures.

Corollary 3.7. If an algebraic function of one complex variable has unsolvable monodromy group then it is strongly non representable by quadratures.

For algebraic functions of several complex variables there is a result similar to Corollary 3.7.

3.3.7 The monodromy pair

The monodromy group of a function $f$ is not only an abstract group but is also a transitive group of permutations of germs of $f$ at a non singular point $x_0$.

Definition 3.4. The monodromy pair of an $S$-function $f$ is a pair of groups, consisting of the monodromy group of $f$ at $x_0$ and the stationary subgroup of a certain germ of $f$ at $x_0$.

The monodromy pair is well defined, i.e. this pair of groups, up to isomorphisms, does not depend on the choice of the non singular point and on the choice of the germ of $f$ at this point. The intersection of the stationary subgroups of all germs of $f$ at $x_0$ is the identity element since the monodromy group acts transitively on this set.
Definition 3.5. A pair of groups $[\Gamma, \Gamma_0]$ is an almost normal pair if there is a finite subset $A \subset \Gamma$ such that the intersection $\bigcap_{a \in A} a(\Gamma_0)a^{-1}$ is equal to the identity element.

Definition 3.6. The pair of groups $[\Gamma, \Gamma_0]$ is called an almost solvable pair of groups if there exists a sequence of subgroups

$$\Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m, \quad \Gamma_m \subset \Gamma_0,$$

such that for every $i, 1 \leq i \leq m - 1$ group $\Gamma_{i+1}$ is a normal divisor of group $\Gamma_i$ and the factor group $\Gamma_i/\Gamma_{i+1}$ is either a commutative group, or a finite group.

Definition 3.7. The pair of groups $[\Gamma, \Gamma_0]$ is called a $k$-solvable pair of groups if there exists a sequence of subgroups

$$\Gamma = \Gamma_1 \supseteq \cdots \supseteq \Gamma_m, \quad \Gamma_m \subset \Gamma_0,$$

such that for every $i, 1 \leq i \leq m - 1$ group $\Gamma_{i+1}$ is a normal divisor of group $\Gamma_i$ and the factor group $\Gamma_i/\Gamma_{i+1}$ is either a commutative group, or a subgroup of the group $S_k$ of permutations of $k$ elements.

We say that group $\Gamma$ is almost solvable or $k$-solvable if pair $[\Gamma, e]$, where $e$ is the group containing only the unit element, is almost solvable or $k$-solvable respectively.

It is easy to see that an almost normal pair of groups $[\Gamma, \Gamma_0]$ is almost solvable or $k$-solvable if and only if the group $\Gamma$ is almost solvable or $k$-solvable respectively.

3.3.8 Strong non-representability by $k$-quadratures

One can prove the following theorem.

Theorem 3.12. (see [Kho14]) The class of all $S$-functions, having a $k$-solvable monodromy pair, is stable under composition, meromorphic operations, integration, differentiation and solutions of algebraic equations of degree $\leq k$.

Definition 3.8. A function $f$ is strongly non representable by $k$-quadratures if it does not belong to a class of functions defined by the following data. List of basic functions: basic elementary functions and all single valued $S$-function. List of admissible operations: compositions, meromorphic operations, differentiation, integration and solutions of algebraic equations of degree $\leq k$.

Theorem 3.12 implies the following corollary.

Corollary 3.8 (Result on $k$-quadratures). If the monodromy pair of an $S$-function $f$ is not $k$-solvable, then $f$ is strongly non representable by $k$-quadratures.

Example 3.4. The monodromy group of an algebraic function $y(x)$ defined by an equation $y^n + y - x = 0$ is the permutation group group $S_n$. For $n \geq 5$ the group $S_n$ is not an $(n - 1)$-solvable group. Thus $y(x)$ provides an example of a function with finite set of singular points which is strongly non representable by $(n - 1)$-quadratures.

This example can be generalized.

Corollary 3.9. (see [Kho14]) If an algebraic function of one complex variable has non $k$-solvable monodromy group then it is strongly non representable by $k$-quadratures.

Theorem 3.13. (see [Kho14]) An algebraic function of one variable whose monodromy group is $k$-solvable, can be represented by $k$-radicals.

Results similar to Corollary 3.9 and Theorem 3.13 hold also for algebraic functions of several complex variables.

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3.3.9 Strong non-representability by generalized quadratures

One can prove the following theorem.

**Theorem 3.14.** (see [Kho14]) The class of all $S$-functions, having an almost solvable monodromy pair, is stable under composition, meromorphic operations, integration, differentiation and solutions of algebraic equations.

**Definition 3.9.** A function $f$ is **strongly non representable by generalized quadratures** if it does not belong to a class of functions defined by the following data. List of basic functions: basic elementary functions and all single valued $S$-function. List of admissible operations: compositions, meromorphic operations, differentiation, integration and solutions of algebraic equations.

Theorem 3.14 implies the following corollary.

**Corollary 3.10 (Result on generalized quadratures).** If the monodromy pair of an $S$-function $f$ is not almost solvable, then $f$ is strongly non representable by generalized quadratures.

Suppose that the Riemann surface of a function $f$ is a universal covering space over the Riemann sphere with $n$ punched points. If $n \geq 3$ then the function $f$ is strongly non representable by generalized quadratures. Indeed, the monodromy pair of $f$ consists of the free group with $n - 1$ generators, and its unit subgroup. It is easy to see that such a pair of groups is not almost solvable.

**Example 3.5.** Consider the function $z(x)$, which maps the upper half-plane onto a triangle with vanishing angles, bounded by three circular arcs. The Riemann surface of $z(x)$ is a universal covering space over the sphere with three punched points $^5$. Thus $z(x)$ is strongly non representable by generalized quadratures.

**Example 3.6.** Let $K_1$ and $K_2$ be the following elliptic integrals, considered as the functions of the parameter $x$:

$$K_1(x) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - t^2x^2)}} \quad \text{and} \quad K_2(x) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{(1 - t^2)(1 - t^2x^2)}}$$

The functions $z(x)$ can be obtained from $K_1(x)$ and from $K_2(x)$ by quadratures. Thus both functions $K_1(x)$ and $K_2(x)$ are strongly non representable by generalized quadratures.

In the next section we will list all polygons $G$ bounded by circular arcs for which the Riemann map of the upper half-plane onto $G$ is representable by generalized quadratures.

3.3.10 Maps of the upper half-plane onto a curved polygon

Consider a polygons $G$ on the complex plane bounded by circle arcs, and the function $f_G$ establishing the Riemann mapping of the upper half-plane onto the polygon $G$. The Riemann–Schwarz reflection principle allows one to describe the monodromy group $L_G$ of the function $f_G$ and to show that all singularities of $f_G$ are simple enough. This information together with Theorem 3.14 provide a complete classification of all polygons $G$ for which the function $f_G$ is representable in explicit form (see [Kho14]).

If a polygon $G$ is obtained from a polygon $G$ by a linear transformation $w : \mathbb{C} \to \mathbb{C}$ then $f_{\tilde{G}} = w(f_G)$. Thus it is enough to classify $G$ up to a linear transformation.

---

$^5$it is easy to see that the function $z(x)$ maps its Riemann surface to the open ball whose boundary contains the vertices of the triangle. These properties of the function $z(x)$ play the crucial role in Picard’s beautiful proof of his Little Picard Theorem.
3.3.10.1 The first case of integrability: the continuations of all sides of the polygon $G$ intersect at one point.

Mapping this point to infinity by a fractional linear transformation, we obtain a polygon $G$ bounded by straight line segments. All transformations in the group $L(G)$ have the form $z \rightarrow az+b$. All germs of the function $f = f_G$ at a non-singular point $c$ are obtained from a fixed germ $f_c$ by the action of the group $L(G)$ consisting of the affine transformations $z \rightarrow az+b$. The germ $R_c = (f''/f')_c$ is invariant under the action of the group $L(G)$. Therefore, the germ $R_c$ is a germ of a single valued function $R$. The singular points of $R$ can only be poles (see [Kho14]). Hence the function $R$ is rational. The equation $f''/f' = R$ is integrable by quadratures. This integrability case is well known. The function $f$ in this case is called the Christoffel–Schwarz integral.

3.3.10.2 The second case of integrability: there is a pair of points such that, for every side of the polygon $G$, these points are either symmetric with respect to this side or belong to the continuation of the side.

We can map these two points to zero and infinity by a fractional linear transformation. We obtain a polygon $G$ bounded by circle arcs centered at point 0 and intervals of straight rays emanating from 0 (see [Kho14]). All transformations in the group $L(G)$ have the form $z \rightarrow az, z \rightarrow b/z$. All germs of the function $f = f_G$ at a non-singular point $c$ are obtained from a fixed germ $f_c$ by the action of the group $L(G)$:

$$f_c \rightarrow af_c, f_c \rightarrow b/f_c.$$ 

The germ $R_c = (f''/f')_c$ is invariant under the action of the group $L(G)$. Therefore, the germ $R_c$ is a germ of a single valued function $R$. The singular points of $R$ can only be poles (see [Kho14]). Hence the function $R$ is rational. The equation $R = (f''/f')^2$ is integrable by quadratures.

3.3.10.3 The finite nets of circles. To describe the third case of integrability we need to define first the finite net of circles on the complex plane. The classification of finite groups, generated by reflections in the Euclidian space $\mathbb{R}^3$ is well known. Each such group is the symmetry group of the following bodies:

1. a regular $n$-gonal pyramid,
2. a regular $n$-gonal diheron, or the body formed by two equal regular $n$-gonal pyramidsharing the base,
3. a regular tetrahedron,
4. a regular cube or icosahedron,
5. a regular dodecahedron or icosahedron.

All these groups of isometries, except for the group of dodecahedron or icosahedron, are solvable.

The intersections of the unit sphere, whose center coincides with the barycenter of the body, with the mirrors, in which the body is symmetric, is a certain net of great circles. Stereographic projections of each of them is a net of great circles on complex plane defined up to a fractional linear transformation. The nets corresponding to the bodies listed above will be called the finite nets of circles.
3.3.10.4 The third case of integrability: every side side of a polygon $G$ belongs to some finite net of circles. In this case the function $f_G$ has finitely many branches. Since all singularities of the function $f_G$ are algebraic (see [Kho14]), the function $f_G$ is an algebraic function. For all finite nets but the net of dodecahedron or icosahedron, the algebraic function $f_G$ is representable by radicals and solutions of degree five algebraic equations (in other words $f_G$ is representable by $k$-radicals).

3.3.10.5 The strong non-representability. Our results imply the following:

Theorem 3.15. (see [Kho14]) If a polygon $G$ bounded by circles arcs does not belong to one of the three cases described above, then the function $f_G$ is strongly non representable by generalized quadratures.

3.3.11 Non-solvability of linear differential equations

Consider a homogeneous linear differential equation

$$y^{(n)} + r_1 y^{(n-1)} + \cdots + r_n y = 0,$$  \hspace{1cm} (27)

whose coefficients $r_i$’s are rational functions of the complex variable $x$. The set $\Sigma \subset \mathbb{C}$ of poles of $r_i$’s is called the set of singular points of the equation (27). At a point $x_0 \in \mathbb{C} \setminus \Sigma$ the germs of solutions of (27) form a $\mathbb{C}$-linear space $V_{x_0}$ of dimension $n$. The monodromy group $M$ of the equation (27) is the group of all linear transformations of the space $V_{x_0}$ which are induced by motions around the set $\Sigma$. Below we discuss this definition more precisely.

Consider a closed curve $\gamma$ in $\mathbb{C} \setminus \Sigma$ beginning and ending at the point $x_0$. Given a germ $y \in V_{x_0}$ we can continue it along the loop $\gamma$ to obtain another germ $y_\gamma \in V_{x_0}$. Thus each such loop $\gamma$ corresponds to a map $M_\gamma : V_{x_0} \to V_{x_0}$ of the space $V_{x_0}$ to itself that maps a germ $y \in V_{x_0}$ to the germ $y_\gamma \in V_{x_0}$. The map $M_\gamma$ is linear since an analytic continuation respects the arithmetic operations. It is easy to see that the map $\gamma \to M_\gamma$ defines a homomorphism of the fundamental group $\pi_1(\mathbb{C} \setminus \Sigma, x_0)$ of the domain $\mathbb{C} \setminus \Sigma$ with the base point $x_0$ to the group $GL(n)$ of invertible linear transformations of the space $V_{x_0}$.

The monodromy group $M$ of the equation (27) is the image of the fundamental group in the group $GL(n)$ under this homomorphism.

Remark. Instead of the point $x_0$ one can choose any other point $x_1 \in \mathbb{C} \setminus \Sigma$. Such a choice will not change the monodromy group up to an isomorphism. To fix this isomorphism one can choose any curve $\gamma : I \to \mathbb{C}^N \setminus \Sigma$ where $I$ is the segment $0 \leq t \leq 1$ and $\gamma(0) = x_0$, $\gamma(1) = x_1$ and identify each germ $y_{x_0}$ of solution of (27) with its continuation $y_{x_1}$ along $\gamma$.

Lemma 3.10. The stationary subgroup in the monodromy group $M$ of the germ $y \in V_{x_0}$ of almost every solution of the equation (27) is trivial (i.e. contains only the unit element $e \in M$).

Proof. The monodromy group $M$ contains countable many linear transformations $M_i$. The space $L_i \subset V_{x_0}$ of fixed points of a non identity transformation $M_i$, is a proper subspace of $V_{x_0}$. The union $L$ of all subspaces $L_i$ is a measure zero subset of $V_{x_0}$. The stationary subgroup in $M$ of $y \in V_{x_0} \setminus L$ is trivial. \hspace{1cm} $\square$

Theorem 3.16. (see [Kho14]) If the monodromy group of the equation (27) is not almost solvable (is not $k$-solvable, or is not solvable) then almost every solution of (27) is strongly non representable by generalized quadratures (correspondingly, is strongly non representable by $k$-quadratures, or is strongly non representable by quadratures).
Consider a homogeneous system of linear differential equations

\[ y' = Ay \]  

(28)

where \( y = (y_1, \ldots, y_n) \) is the unknown vector valued function and \( A = \{a_{i,j}(x)\} \) is a \( n \times n \) matrix, whose entries are rational functions of the complex variable \( x \). One can define the monodromy group of the equation (28) exactly in the same way as it was defined for the equation (27).

We will say that a vector valued function \( y = (y_1, \ldots, y_n) \) belongs to a certain class of functions if all of its components \( y_i \) belong to this class. For example the statement "a vector valued function \( y = (y_1, \ldots, y_n) \) is strongly non representable by generalized quadratures" means that at least one component \( y_i \) of \( y \) is strongly non representable by generalized quadratures.

**Theorem 3.17.** If the monodromy group of the system (28) is not almost solvable (is not \( k \)-solvable, or is not solvable) then almost every solution of (28) is strongly non representable by generalized quadratures (correspondingly, is strongly non representable by \( k \)-quadratures, or is strongly non representable by quadratures).

### 3.3.12 Solvability of Fuchsian equations

The differential field of rational functions of \( x \) is isomorphic to the differential field \( \mathcal{R} \) of germs of rational functions at the point \( x_0 \in C \setminus \Sigma \). Consider the differential field extension \( \mathcal{R}\langle y_1, \ldots, y_n \rangle \) of \( \mathcal{R} \) where the germs \( y_1, \ldots, y_n \) form a basis in the space \( \mathcal{V}_{x_0} \) of solutions of the equation (27) at \( x_0 \).

**Lemma 3.11.** Every linear map \( M_\gamma \) from the monodromy group of equation (27), can be uniquely extended to a differential automorphism of the differential field \( \mathcal{R}\langle y_1, \ldots, y_n \rangle \) over the field \( \mathcal{R} \).

**Proof.** Every element \( f \in \mathcal{R}\langle y_1, \ldots, y_n \rangle \) is a rational function of the independent variable \( x \), the germs of solutions \( y_1, \ldots, y_n \) and their derivatives. It can be continued meromorphically along the curve \( \gamma \in \pi_1(C \setminus \Sigma, x_0) \) together with \( y_1, \ldots, y_n \). This continuation gives the required differential automorphism, since the continuation preserves the arithmetical operations and differentiation, and every rational function of \( x \) returns back to its original values (since it is a single-valued valued function). The differential automorphism is unique because the extension is generated by \( y_1, \ldots, y_n \).

The differential Galois group (see [Kho14], [dPS03]) of the equation (27) over \( \mathcal{R} \) is the group of all differential automorphisms of the differential field \( \mathcal{R}\langle y_1, \ldots, y_n \rangle \) over the differential field \( \mathcal{R} \). According to Lemma 32 the monodromy group of the equation (27) can be considered as a subgroup of its differential Galois group over \( \mathcal{R} \).

The differential field of invariants of the monodromy group action is a subfield of \( \mathcal{R}\langle y_1, \ldots, y_n \rangle \), consisting of the single-valued functions. In contrast to the algebraic case, in the case of differential equations the field of invariants under the action of the monodromy group can be bigger than the field of rational functions. The reason is that the solutions of differential equations may grow exponentially in approaching the singular points or infinity.

**Example 3.7.** All solutions of the simplest differential equation \( y' = y \) are single-valued exponential functions \( y = C \exp x \), which are not rational.

For the wide class of Fuchsian linear differential equations all the solutions, while approaching the singular points and the point infinity, grow polynomially.

The following Frobenius theorem is an analog for Fuchsian equations of C.Jordan theorem (see [Kho14]) for algebraic equations.
Theorem 3.18 (Frobenius). For Fuchsian differential equations the subfield of the differential field \( \mathbb{R}(y_1, \ldots, y_n) \), consisting of single-valued functions, coincides with the field of rational functions.

A system of linear differential equations \( (28) \) is called a Fuchsian system if the matrix \( A \) has the following form:

\[
A(x) = \sum_{i=1}^{k} \frac{A_i}{x - a_i},
\]

(29)

where the \( A_i \)'s are constant matrices. Linear Fuchsian system of differential equations in many ways are similar to linear Fuchsian differential equations.

In construction of explicit solutions of linear differential equations the following theorem is needed.

Theorem 3.19 ((Lie–Kolchin)). Any connected solvable algebraic group acting by linear transformations on a finite-dimensional vector space over \( \mathbb{C} \) is triangularizable in a suitable basis.

Using the Frobenius Therem and Lie–Kolchin Theorem one can prove that the only reasons for unsolvability of Fuchsian linear differential equations and systems of linear differential equations are topological. In other words, if there are no topological obstructions to solvability then such equations and systems of equations are solvable. Indeed, the following theorems hold:

Theorem 3.20. (see [Kho14]) If the monodromy group of the linear Fuchsian differential equation \( (27) \) is almost solvable (is \( k \)-solvable, or is solvable) then every solution is representable by generalized quadratures (correspondingly, is representable by \( k \)-quadratures, or is representable by quadratures).

Theorem 3.21. (see [Kho14]) If the monodromy group of the linear Fuchsian system differential equations \( (28) \) is almost solvable (is \( k \)-solvable, or is solvable) then every solution is representable by generalized quadratures (correspondingly, is representable by \( k \)-quadratures, or is representable by quadratures).

3.3.13 Fuchsian systems with small coefficients

In general the monodromy group of a given Fuchsian equation is very hard to compute. It is known only for very special equations, including the famous hypergeometric equations. Thus Theorems 5.20 and 5.21 are not explicit.

If the matrix \( A(x) \) in the system \( (28) \) is triangular then one can easily solve the system by quadratures. It turns out that if the matrix \( A(x) \) has the form \( (29) \), where the matrices \( A_i \)'s are sufficiently small, then the system \( (28) \) with a non triangular matrix \( A(x) \) is unsolvable by generalized quadratures for a topological reason.

Theorem 3.22. (see [Kho14]) If the matrices \( A_i \)'s are sufficiently small, \( \|A_i\| < \varepsilon(a_1, \ldots, a_k, n) \), then the monodromy group of the system

\[
y' = \left( \sum_{i=1}^{k} \frac{A_i}{x - a_i} \right)y
\]

(30)
is almost solvable if and only if the matrices \( A_i \)'s are triangularizable in a suitable basis.

Corollary 3.11. If in the assumptions of Theorem 19 the matrices \( A_i \)'s are not triangularizable in a suitable basis then almost every solution of the system \( (6) \) is strongly non representable by generalized quadratures.
3.3.14 Polynomials invertible by radicals

In 1922 J.F.Ritt published (see [Rit22]) the following beautiful theorem which fits nicely into topological Galois theory.

**Theorem 3.23.** (J.F. Ritt) The inverse function of a polynomial with complex coefficients can be represented by radicals if and only if the polynomial is a composition of linear polynomials, the power polynomials $z \rightarrow z^n$, Chebyshev polynomials and polynomials of degree at most 4.

**Outline of proof (following [BK16])**

1) *Every polynomial is a composition of primitive ones:* Every polynomial is a composition of polynomials that are not themselves compositions of polynomials of degree $> 1$. Such polynomials are called *primitive*. Recall that a permutation group $G$ acting on a non-empty set $X$ is called *primitive* if $G$ acts transitively on $X$ and $G$ preserves no nontrivial partition of $X$. A polynomial is primitive if and only if the monodromy group of inverse of the polynomial acts primitively on its branches.

2) *Reduction to the case of primitive polynomials:* A composition of polynomials is invertible by radicals if and only if each polynomial in the composition is invertible by radicals. Indeed, if each of the polynomials in composition is invertible by radicals, then their composition also is. Conversely, if a polynomial $R$ appears in the presentation of a polynomial $P$ as $P = Q \circ R \circ S$ and $P^{-1}$ is representable by radicals, then $R^{-1} = S \circ P^{-1} \circ Q$ is also representable by radicals. Thus it is enough to classify only the primitive polynomials invertible by radicals.

3) *A result on solvable primitive permutation groups containing a full cycle:* A primitive polynomial is invertible by radicals if and only if the monodromy group of the inverse of the polynomial is solvable. Since it acts primitively on its branches and contains a full cycle (corresponding to a loop around the point at infinity on the Riemann sphere), the following group-theoretical result of Ritt is useful for the classification of polynomials invertible by radicals:

**Theorem 3.24.** (on primitive solvable groups with a cycle) Let $G$ be a primitive solvable group of permutations of a finite set $X$ which contains a full cycle. Then either $|X| = 4$, or $|X|$ is a prime number $p$ and $X$ can be identified with the elements of the field $F_p$ so that the action of $G$ gets identified with the action of the subgroup of the affine group $AGL_1(p) = \{x \rightarrow ax + b | a \in (F_p)^*, b \in F_p\}$ that contains all the shifts $x \rightarrow x + b$.

4) *Solvable monodromy groups of the inverse of primitive polynomials:* It can be shown by applying the Riemann–Hurwitz formula that among the groups in Theorem 3.24 on primitive solvable groups with a cycle, only the following groups can be realized as monodromy groups of inverse of primitive polynomials: 1. $G \subset S_4$, 2. Cyclic group $G = \{x \rightarrow x + b\} \subset AGL_1(p)$, 3. Dihedral group $G = \{x \rightarrow \pm x + b\} \subset AGL_1(p)$.

5) *Description of primitive polynomials invertible by radicals:* It can be easily shown (see for instance [see [Rit22], [BK16]]) that the following result holds:

**Theorem 3.25.** If the monodromy group of inverse of a primitive polynomial is a subgroup of the group $\{x \rightarrow \pm x + b\} \subset AGL_1(p)$, then up to a linear change of variables the polynomial is either a power polynomial or a Chebyshev polynomial.

Thus the polynomials whose inverse have monodromy groups 1-3 are respectively 1. Polynomials of degree four. 2. Power polynomials up to a linear change of variables. 3. Chebyshev polynomials up to a linear change of variables.

In each of these cases the fact that the polynomial is invertible by radicals follows from solvability of the corresponding monodromy group or from explicit formulas for its inverse (see for instance [BK16]).
3.3.15 Polynomials invertible by \( k \)-radicals

In this section we discuss the following generalization of J.F.Ritt’s Theorem.

\textbf{Theorem 3.26.} (see [BK16]) A polynomial invertible by radicals and solutions of equations of degree at most \( k \) is a composition of power polynomials, Chebyshev polynomials, polynomials of degree at most \( k \) and, if \( k \leq 14 \), certain primitive polynomials whose inverse have exceptional monodromy groups. A description of these exceptional polynomials can be given explicitly.

The proofs rely on classification of monodromy groups of inverse of primitive polynomials obtained by Müller based on group-theoretical results of Feit and on previous work on primitive polynomials whose inverse have exceptional monodromy groups by many authors. Besides the references to these highly involved and technical results an outline of the proof of Theorem 40 is not complicated and it resembles the outline of the proof of Ritt’s Theorem.

Let us start with some background on representability by \( k \)-radicals.

\textbf{Definition 3.10.} Let \( k \) be a natural number. A field extension \( L/K \) is \( k \)-radical if there exists a tower of extensions \( K = K_0 \subset K_1 \subset \ldots \subset K_n \) such that \( L \subset K_n \) and for each \( i \), \( K_{i+1} \) is obtained from \( K_i \) by adjoining an element \( a_i \), which is either a solution of an algebraic equation of degree at most \( k \) over \( K_i \), or satisfies \( a_i^m = b \) for some natural number \( m \) and \( b \in K_i \).

\textbf{Theorem 3.27.} (see [Kho14]) A Galois extension \( L/K \) of fields of characteristic zero is \( k \)-radical if and only if its Galois group is \( k \)-solvable.

An algebraic function \( z = z(x) \) of one or several complex variables is said to be representable by \( k \)-radicals if the corresponding extension of the field of rational functions is a \( k \)-radical extension.

\textbf{Theorem 2.15} and C. Jordan’s Theorem (see sections 3.2.1 and 3.2.2) imply the following corollary.

\textbf{Corollary 3.12.} An algebraic function is representable by \( k \)-radicals if and only if its monodromy group is \( k \)-solvable.

(Note that Theorem 3.13 above coincides with a part of Theorem 3.27).

Let us outline briefly the main steps in the proof of Theorem 3.26.

\textbf{Outline of proof of Theorem 3.26:}

1) Exactly as in Ritt’s theorem one can show that a composition of polynomials is invertible by \( k \)-radicals if and only if each polynomial in the composition is invertible by \( k \)-radicals. Thus one can reduce Theorem 40 to the case of primitive polynomials.

2) Feit and Jones totally classified all primitive permutation groups of \( n \) elements containing a full cycle.

3) Using this classification and Riemann–Hurwitz formula, Müller listed all groups of permutations of \( n \) elements which are monodromy groups of inverses of degree \( n \) primitive polynomials.

4) For each group from Müller’s list of groups of permutations of \( n \) elements one can determine the smallest \( k \) for which it is \( k \)-solvable and choose the exceptional groups for which \( k \) is smaller than \( n \).

5) For each such exceptional group one can explicitly describe polynomials whose inverse has the exceptional monodromy group.
3.4 Multidimensional Topological Galois Theory

3.4.1 Introduction

In this section we present an outline of the multidimensional version of topological Galois theory. The presentation is based on the book [Kho14]. It contains definitions, statements of results and comments on them. Basically no proofs are presented.

In topological Galois theory for functions of one variable (see section 2 and [Kho14]), it is proved that the way the Riemann surface of a function is positioned over the complex line can obstruct the representability of this function “in finite terms” (i.e. its representability by radicals, by quadratures, by generalized quadratures and so on). This not only explains why many algebraic and differential equations are not solvable in finite terms, but also gives the strongest known results on their unsolvability.

In the multidimensional version of topological Galois theory analogous results are proved. But in the multidimensional case all constructions and proofs are much more complicated and involved than in the one dimensional case (see [Kho14]).

3.4.2 Classes of functions

An equation is solvable “in finite terms” (or is solvable “explicitly”) if its solutions belong to a certain class of functions. Different classes of functions correspond to different notions of solvability in finite terms.

A class of functions can be introduced by specifying a list of basic functions and a list of admissible operations. Given the two lists, the class of functions is defined as the set of all functions that can be obtained from the basic functions by repeated application of admissible operations. Below, we define Liouvillian classes of functions in exactly this way.

Classes of functions, which appear in the problems of integrability in finite terms, contain multivalued functions. Thus the basic terminology should be made clear.

We understand operations on multivalued functions of several variables in a slightly more restrictive sense than operations on multivalued functions of single variable (the one dimensional case is discussed in section 3.3 and in [Kho14]).

Fix a class of basic functions and some set of admissible operations. Can a given function (which is obtained, say, by solving a certain algebraic or a differential equation) be expressed through the basic functions by means of admissible operations? We are interested in various single valued branches of multivalued functions over various domains. Every function, even if it is multivalued, will be considered as a collection of all its single valued branches. We will only apply admissible operations (such as arithmetic operations and composition) to single valued branches of the function over various domains. Since we deal with analytic functions, it suffices only to consider small neighborhoods of points as domains.

We can now rephrase the question in the following way: can a given function germ at a given point be expressed through the germs of basic functions with the help of admissible operations? Of course, the answer depends on the choice of a point and on the choice of a single valued germ at this point belonging to the given multivalued function. It turns out, however, that for the classes of functions interesting to us the desired expression is either impossible for every germ of a given multivalued function at every point or the “same” expression serves all germs of a given multivalued function at almost every point of the space.

For functions of one variable, we use a different, extended definition of operations on multivalued functions, in which the multivalued function is viewed as a single object. This definition is essentially equivalent to including the operation of analytic continuation in the list of admissible operations on analytic germs (all details can be found in [Kho14]). For functions of many variables, we need to adopt the more restrictive understanding of operations on multivalued functions, which is, however, no less (and perhaps even more) natural.
3.4.3 Specifics of the multidimensional case

I was always under impression that a full-fledged multidimensional version of topological Galois theory was impossible. The reason was that, to construct such a version for the case of many variables, one would need to have information on extendability of function germs not only outside their ramification sets but also along these sets. It seemed that there was nothing to extract such information from.

To illustrate the problem consider the following situation. Let \( f \) be a multivalued analytic function on \( \mathbb{C}^n \), whose set of singular points is an analytic set \( \Sigma_f \subset \mathbb{C}^n \). Let \( f_a \) be an analytic germ of \( f \) at a point \( a \in \mathbb{C}^n \). Let \( g : (\mathbb{C}^k, b) \to (\mathbb{C}^n, a) \) be an analytic map. Consider a germ \( \varphi_b \) at the point \( b \in \mathbb{C}^k \) of the composition \( f_a \circ g_b \). One can ask the following questions:

1) Is it true that \( \varphi_b \) is a germ of a multivalued function \( \varphi \) on \( \mathbb{C}^k \), whose set of singular points \( \Sigma_\varphi \) is contained in a proper analytic subset of \( \mathbb{C}^k \)?

2) Is it true that the monodromy group \( M_\varphi \) of \( \varphi \) corresponding to motions around the set \( \Sigma_\varphi \subset \mathbb{C}^k \) can be estimated in terms of the monodromy group \( M_f \) of \( f \) corresponding to motions around the set \( \Sigma_f \subset \mathbb{C}^n \)? For example, if \( M_f \) is a solvable group is it true that \( M_\varphi \) also is a solvable group?

If the image \( g(\mathbb{C}^k) \) is not contained in the singular set \( \Sigma_f \) then the answers to the both questions are positive: the set \( \Sigma_\varphi \) belongs to the analytic set \( g^{-1}(\Sigma_f) \) and the group \( M_\varphi \) is a subgroup of a certain factor group of \( M_f \). These statements are not complicated and can be proved by the same arguments as in the one dimensional topological Galois theory.

Assume that the multivalued function \( f \) has an analytic germ \( f_a \) at a point \( a \) belonging to the singular set \( \Sigma_f \) (some of the germs of the multivalued function \( f \) may appear to be nonsingular at singular points of this function). Assume now that the image \( g(\mathbb{C}^k) \) is contained in the singular set \( \Sigma_f \) and \( a = g(b) \). It turns out that for the germ \( \varphi_b = f_a \circ g_b \) the answers to the both questions also are positive. In this situation all the proofs are more involved. They use new arguments from multidimensional complex analysis and from group theory.

It turns out that function germs can sometimes be automatically extended along their ramification sets (see [Kho14]). That new statement from complex analysis suggests the positive answer to the first question.

To describe the connection between the monodromy group of the function \( f \) and the monodromy groups of the composition \( \varphi = f \circ g \), we introduce and develop the notion of pullback closure for groups (see [Kho14]). The use of this operation, in turn, forces us to reconsider all arguments we used in the one dimensional version of topological Galois theory. As a result we obtain a positive answer to the second question.

3.4.4 Liouvillian classes of multivariate functions

In this section we define Liouvillian classes of functions for the case of several variables. These classes are defined in the same way as the corresponding classes for functions of one variable (see section 3.2.2 and [Kho14]). The only difference is in the details.

We fix an ascending chain of standard coordinate subspaces of strictly increasing dimension: \( 0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n \subset \cdots \) with coordinate functions \( x_1, \ldots, x_n, \ldots \) (for every \( k > 0 \), the functions \( x_1, \ldots, x_k \) are coordinate functions on \( \mathbb{C}^k \)). Below, we define Liouvillian classes of functions for each of the standard coordinate subspaces \( \mathbb{C}^k \).

To define Liouvillian classes, we will need the list of basic elementary functions and the list of classical operations.

List of basic elementary functions.

1. All complex constants and all coordinate functions \( x_1, \ldots, x_n \) for every standard coordinate subspace \( \mathbb{C}^n \).
2. The exponential, the logarithm and the power $x^\alpha$, where $\alpha$ is any complex constant.

3. Trigonometric functions: sine, cosine, tangent, cotangent.

4. Inverse trigonometric functions: arcsine, arccosine, arctangent, arccotangent.

Let us now turn to the list of classical operations on functions.

**List of classical operations.**

1. **Operation of composition** that takes a function $f$ of $k$ variables and functions $g_1, \ldots, g_k$ of $n$ variables to the function $f(g_1, \ldots, g_k)$ of $n$ variables.

2. **Arithmetic operations** that take functions $f$ and $g$ to the functions $f + g$, $f - g$, $fg$ and $f/g$.

3. **Operations of partial differentiation with respect to independent variables.** For functions of $n$ variables, there are $n$ such operations: the $i$-th operation assigns the function $\frac{\partial f}{\partial x_i}$ to a function $f$ of the variables $x_1, \ldots, x_n$.

4. **Operation of integration** that takes $k$ functions $f_1, \ldots, f_k$ of the variables $x_1, \ldots, x_n$, for which the differential one-form $\alpha = f_1dx_1 + \cdots + f_kdx_k$ is closed, to the indefinite integral $y$ of the form $\alpha$ (i.e. to any function $y$ such that $dy = \alpha$). The function $y$ is determined by the functions $f_1, \ldots, f_k$ up to an additive constant.

5. **Operation of solving an algebraic equation** that takes functions $f_1, \ldots, f_n$ to the function $y$ such that $y^n + f_1y^{n-1} + \cdots + f_n = 0$. The function $y$ may not be quite uniquely determined by the functions $f_1, \ldots, f_n$, since an algebraic equation of degree $n$ can have $n$ solutions.

We now resume defining Liouvillian classes of functions.

### 3.4.4.1 Functions of $n$ variables representable by radicals.

List of basic functions: All complex constants and all coordinate functions. List of admissible operations: composition, arithmetic operations and the operation of taking the $m$-th root $f^{1/m}$, $m = 2, 3, \ldots$, of a given function $f$.

### 3.4.4.2 Functions of $n$ variables representable by $k$-radicals.

This class of functions is defined in the same way as the class of functions representable by radicals. We only need to add the operation of solving algebraic equations of degree $\leq k$ to the list of admissible operations.

### 3.4.4.3 Elementary functions of $n$ variables.

List of basic functions: basic elementary functions. List of admissible operations: composition, arithmetic operations, differentiation.

### 3.4.4.4 Generalized elementary functions of $n$ variables.

This class of functions is defined in the same way as the class of elementary functions. We only need to add the operation of solving algebraic equations to the list of admissible operations.

### 3.4.4.5 Functions of $n$ variables representable by quadratures.

List of basic functions: basic elementary functions. List of admissible operations: composition, arithmetic operations, differentiation, integration.
3.4.4.6 Functions of \( n \) variables representable by \( k \)-quadratures. This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations of degree at most \( k \) to the list of admissible operations.

3.4.4.7 Functions of \( n \) variables representable by generalized quadratures. This class of functions is defined in the same way as the class of functions representable by quadratures. We only need to add the operation of solving algebraic equations to the list of admissible operations.

3.4.5 Strong non representability in finite terms

Topological obstructions to the representability of functions in finite terms relate to branching. It turns out that if a function does not belong to a certain Liouvillian class by topological reasons then it automatically does not belong to a much wider \textit{extended Liouvillian class of functions}.

Such an extended Liouvillian class is defined as follows: its list of admissible operations is the same as in the original Liouvillian class and its list of basic functions is the list of basic function in the original class extended by all single valued functions of any number of variables having a proper analytic set of singular points.

Definition 3.11. A germ \( f \) is a germ of function belonging to the \textit{extended class of functions representable by quadratures} if it can be represented by germs of basic elementary functions and by germs of single valued functions, whose set of singular points is a proper analytic set, by means of composition, integration, arithmetic operations and differentiation.

Definition 3.12. A germ \( f \) is \textit{strongly non representable by quadratures} if it is not a germ of function from the extended class of functions representable by quadratures.

The definition of \textit{strong non representability of a germ \( f \) by radicals, by \( k \)-radical, by elementary functions, by generalized elementary functions, by \( k \)-quadratures and by generalized quadratures} is similar to the above definition.

3.4.6 Holonomic systems of linear differential equations

Consider a system of \( N \) linear differential equations

\[
L_j(y) = 0, \quad j = 1, \ldots, N,
\]

\[
L_j(y) = \sum a_{i_1,\ldots,i_n} \frac{\partial^{i_1+\cdots+i_n} y}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = 0,
\]

of an unknown function \( y \), whose coefficients \( a_{i_1,\ldots,i_n} \) are analytic functions in a domain \( U \subset \mathbb{C}^n \).

The system is \textit{holonomic} if at every point \( a \in U \) the \( \mathbb{C} \)-linear space \( V_a \) of germs \( y_a \) satisfying the system has finite dimension, \( \dim_{\mathbb{C}} V_a = d(a) < \infty \). Holonomic systems can be considered as a multidimensional generalization of linear differential equation on one unknown function of a single variable. Kolchin obtained a generalization of the Picard–Vessiot theory (Galois theory for linear differential equations) to the case of holonomic systems of differential equations [APS03].

The holonomic system has the following properties:

1) There exists an analytic \textit{singular hypersurface} \( \Sigma \subset U \) such that the dimension \( d(a) = \dim_{\mathbb{C}} V_a \) is constant \( d(a) = d \) on \( U \setminus \Sigma \).

2) Let \( \gamma : I \to U \setminus \Sigma \) be a continuous map, where \( I \) is the unit segment \( 0 \leq t \leq 1 \) and \( \gamma(0) = a, \gamma(1) = b \). Then the space \( V_a \) of solutions of \( \{31\} \) at the point \( a \) admits analytic
continuation along $\gamma$ and the space obtained by the continuation at the point $b$ is the space $V_b$ of solutions of (31) at the point $b$.

3) If all equations of the system (31) admit analytic continuation to some domain $W$, then the system obtained by such a continuation is a holonomic system in the domain $W$.

Let $a \notin \Sigma$ be a point not belonging to the hypersurface $\Sigma$. Take an arbitrary path $\gamma(t)$ in the domain $U$ originating and terminating at $a$ and avoiding the hypersurface $\Sigma$. Solutions of this system admit analytic continuations along the path $\gamma$, which are also solutions of the system. Therefore, every such path $\gamma$ gives rise to a linear map $M_\gamma$ of the solution space $V_a$ to itself. The collection of linear transformations $M_\gamma$ corresponding to all paths $\gamma$ form a group, which is called the monodromy group of the holonomic system.

3.4.7 $SC$-germs

There is a wide class of $S$-functions in one variable containing all Liouvillian functions and stable under classical operations, for which the monodromy group is defined. The class of $S$-functions plays an important role in the one dimensional version of topological Galois theory (see [Kho14, sec. 2]). Is there a sufficiently wide class of multivariate function germs with similar properties?

For a long time, I thought that the answer to this question was negative. In this section the class of $SC$-germs is defined. This provides an affirmative answer to this question.

A subset $A$ in a connected $k$-dimensional analytic manifold $Y$ is called meager if there exists a countable set of open domains $U_i \subset M$ and a countable collection of proper analytic subsets $A_i \subset U_i$ in these domains such that $A \subset \bigcup A_i$.

The following definition plays a key role in what follows.

Definition 3.13. A germ $f_a$ of an analytic function at a point $a \in \mathbb{C}^n$ is an $SC$-germ if the following condition is fulfilled. For every connected complex analytic manifold $Y$, every analytic map $G : Y \to \mathbb{C}^n$ and every preimage $b$ of the point $a$, $G(b) = a$, there exists a meager set $A \subset Y$ such that, for every path $\gamma : [0, 1] \to Y$ originating at the point $b$, $\gamma(0) = b$, and intersecting the set $A$ at most at the initial moment, $\gamma(t) \notin A$ for $t > 0$, the germ $f_a$ admits an analytic continuation along the path $G \circ \gamma : [0, 1] \to \mathbb{C}^n$.

The following lemma is obvious.

Lemma 3.12. The class of $SC$-germs contains all germs of analytic functions on $\mathbb{C}^N \setminus \Sigma$ where $\Sigma$ is an analytic subset in $\mathbb{C}^N$ where $N$ is a natural number. In particular the class contains all analytic germs of $S$-functions of one variable and all germs of meromorphic functions of many variables.

The proof of the following Theorem 3.28 uses the results on extendability of multivalued analytic functions along their singular point sets (see [Kho14]).

Theorem 3.28. (on stability of the class of $SC$-germs) The class of $SC$-germs on $\mathbb{C}^n$ is stable under the operation of taking the composition with $SC$-germs of $m$-variable functions, the operation of differentiation and integration. It is stable under solving algebraic equations whose coefficients are $SC$-germs and under solving holonomic systems of linear differential equations whose coefficients are $SC$-germs.

Theorem 3.28 implies the following corollary.

Corollary 3.13. If a germ $f$ is not an $SC$-germ then $f$ is strongly non representable by generalized quadratures. In particular it cannot be a germ of a function belonging to a certain Liouvillian class.
3.4.8 Monodromy group of a SC-germs

The monodromy group and the monodromy pair of a SC-germ $f_a$ can be defined in same way as for S-functions of one variable. By definition the set $\Sigma \subset \mathbb{C}^n$ of singular points of $f_a$ is a meager set. Take any point $x_0 \in \mathbb{C}^n \setminus \Sigma$ and consider the action of the fundamental group $\pi_1(\mathbb{C}^n \setminus \Sigma, x_0)$ on the set $F_{x_0}$ of all germs equivalent to the germ $f_a$. The monodromy group of $f_a$ is the image of the fundamental group under this action. The monodromy pair of $f_a$ is the pair $[\Gamma, \Gamma_0]$ where $\Gamma$ is the monodromy group and $\Gamma_0$ is the stationary subgroup of a germ $f \in F_{x_0}$. Up to an isomorphism the monodromy group and the monodromy pair are independent of a choice of the point $x_0$ and the germ $f$.

Remark. If a SC-germ $f_a$ is defined at a singular point $a \in \Sigma$ then the monodromy group of $f_a$ along $\Sigma$ is defined: one can consider continuations of $f_a$ along curves $\gamma$ belonging to $\Sigma$ and define a singular set $\Sigma_1 \subset \Sigma$ for $f_a$ along $\Sigma$. The monodromy group of $f_a$ along $\Sigma$ corresponds to the action the fundamental group of $\pi_1(\Sigma \setminus \Sigma_1, x_1)$ on the set of germs at $x_1 \in \Sigma \setminus \Sigma_1$ obtained by continuation of $f_a$ along $\Sigma$. If the point $a$ belongs to $\Sigma_1$ then one can define also a monodromy group of $f_a$ along $\Sigma_1$ and so on. Thus in the multidimensional case one can associate to an SC-germ an hierarchy of monodromy groups. All these monodromy groups (and corresponding monodromy pairs) appear in multidimensional topological Galois theory. But the monodromy group and the monodromy pair we discuss above are most important for our purposes.

3.4.9 Stability of certain classes of SC-germs

One can prove the following theorems.

Theorem 3.29. (see [Kho14]) The class of all SC-germs, having a solvable monodromy group is stable under composition, arithmetic operations, integration and differentiation. This class contains all germs of basic elementary functions and all germs of single valued functions whose set of singular points is a proper analytic set.

Theorem 3.30. (see [Kho14]) The class of all SC-germs, having a $k$-solvable monodromy pair (see [Kho14, sec. 2]) is stable under composition, arithmetic operations, integration, differentiation and solution of algebraic equations of degree at most $k$. This class contains all germs of basic elementary functions and all germs of single valued functions whose set of singular points is a proper analytic set.

Theorem 3.31. (see [Kho14]) The class of all SC-germs, having an almost solvable monodromy pair (see [Kho14, sec. 2]) is stable under composition, arithmetic operations, integration, differentiation and solution of algebraic equations. This class contains all germs of basic elementary functions and all germs of single valued functions whose set of singular points is a proper analytic set.

Theorems 3.29 – 3.31 imply the following corollaries.

Result on quadratures If the monodromy group of a SC-germ $f$ is not solvable, then $f$ is strongly non representable by quadratures.

Result on $k$-quadratures If the monodromy pair of a SC-germ $f$ is not $k$-solvable, then $f$ is strongly non representable by $k$-quadratures.

Result on generalized quadratures If the monodromy pair of a SC-germ $f$ is not almost solvable, then $f$ is strongly non representable by generalized quadratures.
3.4.10 Solvability and non solvability of algebraic equation

Consider an irreducible algebraic equation

\[ P_n y^n + P_{n-1} y^{n-1} + \cdots + P_0 = 0 \]  

(32)

whose coefficients \( P_n, \ldots, P_0 \) are polynomials of \( N \) complex variables \( x_1, \ldots, x_N \). Let \( \Sigma \subset \mathbb{C}^N \) be the singular set of the equation (32) defined by the equation \( P_n J = 0 \) where \( J \) is the discriminant of the polynomial (32).

Theorem 3.32. (see [Kho14, sec. 2], [Kho71]) Let \( y_{x_0} \) be a germ of analytic function at a point \( x_0 \in \mathbb{C}^N \) satisfying the equation (32). If the monodromy group of the equation (32) is solvable (is \( k \)-solvable) then the germ \( y_{x_0} \) is representable by radicals (is representable by \( k \)-radicals).

According to Camille Jordan’s theorem (see [Kho71]) the Galois group of the equation (32) over the field \( \mathcal{R} \) of rational functions of \( x_1, \ldots, x_N \) it is isomorphic to the monodromy group of this equation (32). Thus Theorem 3.32 follows from Galois theory (see [Kho14], [Kho71]).

Theorem 3.33. (see [Kho14]) Let \( y_{x_0} \) be a germ of analytic function at a point \( x_0 \in \mathbb{C}^N \) satisfying the equation (32). If the monodromy group of the equation is not solvable (is not \( k \)-solvable) then the germ \( y_{x_0} \) is strongly non representable by quadratures (is strongly non representable by \( k \)-quadratures).

Theorem 3.33 follows from the results on quadratures and on \( k \)-quadratures from the previous section.

Consider the universal degree \( n \) algebraic function \( y(a_n, \ldots, a_0) \) defined by the equation

\[ a_n y^n + \cdots + a_0 = 0. \]  

(33)

It is easy to see that the monodromy group of the equation (33) is isomorphic to the group \( S_n \) of all permutations of \( n \) element. For \( n \geq 5 \) the group \( S_n \) is unsolvable and it is not \( k \)-solvable group for \( k < n \). Thus Theorem 3.33 implies the following strongest known version of the Abel-Ruffini Theorem.

Theorem 3.34. (A version of the Abel–Ruffini Theorem) Let \( y_a \) be a germ of analytic function at a point \( a \) satisfying the universal degree \( n \geq 5 \) algebraic equation. If \( n \geq 5 \) then the germ \( y_a \) is strongly non representable by \((n - 1)\) quadratures. In particular the germ \( y_a \) is strongly non representable by quadratures.

3.4.11 Solvability and non solvability of holonomic systems of linear differential equations

Consider a system of \( N \) linear differential equations

\[ L_j(y) = 0, \quad j = 1, \ldots, N, \]

\[ L_j(y) = \sum a_{i_1, \ldots, i_n} \frac{\partial^{i_1 + \cdots + i_n} y}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} = 0, \]  

(34)

on an unknown function \( y \), whose coefficients \( a_{i_1, \ldots, i_n} \) are rational functions of \( n \) complex variables \( x_1, \ldots, x_n \). Assume that the system (34) is holonomic in \( \mathbb{C}^n \setminus \Sigma_1 \) where \( \Sigma_1 \) is the union of poles of the coefficients \( a_{i_1, \ldots, i_n} \). Let \( \Sigma_2 \subset \mathbb{C}^n \setminus \Sigma_1 \) be the singular hypersurface of a holonomic system (34).

Every germ \( y_a \) of a solution of the system at a point \( a \in \mathbb{C}^n \setminus \Sigma \) where \( \Sigma = \Sigma_1 \cup \Sigma_2 \) admits an analytic continuation along every path avoiding the hypersurface \( \Sigma \) so the monodromy group of the system (34) is well-defined.
Theorem 3.35. (see [Kho14]) If the monodromy group of the holonomic system (34) is not solvable (not $k$-solvable, not almost solvable), then a germ $y_a$ of almost every solution at a point $a \in \mathbb{C}^n \setminus \Sigma$ is strongly non representable by quadratures (is strongly non representable by $k$-quadratures, is strongly non representable by generalized quadratures).

Theorem 3.35 follows from the results on quadratures, on $k$-quadratures and on generalized quadratures from section 3.4.9.

A holonomic system is said to be regular, if near the singular set $\Sigma$ and near infinity the solutions of the system grow at most polynomially.

Theorem 3.36. (see [Kho14]) If the monodromy group of a regular holonomic system is solvable (is $k$-solvable, is almost solvable), then a germ $y_a$ of almost every solution at a point $a \in \mathbb{C}^n \setminus \Sigma$ is representable by quadratures (is representable by $k$-quadratures, is representable by generalized quadratures).

3.4.12 Completely integrable systems of linear differential equations with small coefficients

Consider a completely integrable system of linear differential equations of the following form

$$dy = Ay,$$  

(35)

where $y = y_1, \ldots, y_N$ is an unknown vector-function, and $A$ is a $(N \times N)$-matrix consisting of differential one-forms with rational coefficients on the space $\mathbb{C}^n$ satisfying the condition of complete integrability $dA + A \wedge A = 0$ and having the following form:

$$A = \sum_{i=1}^{k} A_i \frac{dl_i}{l_i},$$

where $A_i$ are constant matrices, and $l_i$ are linear (not necessarily homogeneous) functions on $\mathbb{C}^n$.

If the matrices $A_i$ can be simultaneously reduced to the triangular form, then system (35), as any completely integrable triangular system, is solvable by quadratures. Of course, there exist solvable systems that are not triangular. However, if the matrices $A_i$ are sufficiently small, then there are no such systems. Namely, the following theorem holds.

Theorem 3.37. (see [Kho14]) A system (35) that does not reduce to the triangular form and such that the matrices $A_i$ have sufficiently small norms is unsolvable by generalized quadratures in the following strong sense. At every point $a \in \mathbb{C}^n$ where the matrix $A$ is regular, and for almost any germ $y_a = (y_1, \ldots, y_N)_a$ of a vector-function satisfying the system (35), there is a component $(y_i)_a$ which is strongly non representable by generalized quadratures.

Multidimensional Theorem 3.37 is similar to the one dimensional Corollary 38. Their proofs (see [Kho14]) are also similar. We only need to replace the reference to the (one-dimensional) Lappo-Danilevsky theory with the reference to the multidimensional version of it from [Lek91].

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