A REVIEW OF THE LARGE $N$ LIMIT OF TENSOR MODELS

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Random matrix models encode a theory of random two dimensional surfaces with applications to string theory, conformal field theory, statistical physics in random geometry and quantum gravity in two dimensions. The key to their success lies in the $1/N$ expansion introduced by 't Hooft. Random tensor models generalize random matrices to theories of random higher dimensional spaces. For a long time, no viable $1/N$ expansion for tensors was known and their success was limited.

A series of recent results has changed this situation and the extension of the $1/N$ expansion to tensors has been achieved. We review these results in this paper.

Keywords: Random Tensors; $1/N$ expansion; Critical behavior.

1. Introduction

In many theories space appears as a fixed background, a manifold with a fixed metric. Length scales are defined with respect to this background. The scales encode the causality: fundamental physics at short distances determines the effective phenomena at large distances. It is however clear that the length scale are too crude to be applied in all instances: not only general relativity promotes the metric to a dynamical variable but also quantum field theory suggests that at the fundamental level space should become random, quantum. A fundamental theory of space is for now out of reach. Nevertheless we know that, no matter what this theory is, it must answer three major questions: how to define a statistical theory of random geometries, how to define an appropriate notion of scale and how to recover the usual space time as an effective phenomenon.

In this paper we present a framework to deal with random geometries in arbitrary dimensions. In this framework we answer precisely each of the three question formulated above.

Invariant probability distributions for random $N \times N$ matrices encode the most successful (but restricted to two dimensions) theory of random geometry we have so far. The moments and partition functions of such a probability distribution evaluate in terms of ribbon Feynman graphs. Ribbon graphs are in one to one correspon-

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*Random matrices were introduced by Wishart in statistics and used for the first time in physics by Wigner for the study of the spectra of heavy atoms.
dence with triangulated surfaces providing the bridge between random matrices and random geometries.

In his seminal work, 't Hooft realized that matrix models have a built-in notion of scale: the size of the matrix, \( N \). The perturbative expansion can be reorganized in powers of \( 1/N \) (indexed by the genus) and in the large \( N \) limit only planar graphs contribute. Matrix models undergo a phase transition to continuum infinitely refined surfaces because the planar graphs form a summable family (i.e., a power series with a finite radius of convergence). Random matrices provide the quantization of two-dimensional gravity coupled to conformal matter and through the KPZ correspondence they relate to conformal field theory in fixed geometry. In the double scaling limit matrix models provide a quantization of the string world sheet in string theory.

The resounding success of matrix models has inspired their generalization in higher dimensions to random tensor models. Until recently however tensor models have failed to provide an analytically controlled theory as, for a long time, no generalization of the \( 1/N \) expansion to tensors has been found.

The discovery of colored tensor models has drastically changed this situation. The colored models support a \( 1/N \) expansion indexed by the degree. The leading order (melon) graphs, triangulate the sphere in any dimension. They map to tree hence are a summable family (different from the planar family). Colored tensor models undergo a phase transition to a theory of continuous infinitely refined random spaces.

These results generalize to all invariant models for a random complex tensor. The colors arise as a canonical bookkeeping device labeling the indices of the tensor. The resulting universal theory of random tensors is the generalization of invariant matrix models to higher dimensions. The \( 1/N \) expansion can be realized dynamically as a renormalization group flow in tensor group field theories which are, in at least two simple cases, asymptotically free.

2. Tensor Models

Let us consider a covariant complex tensor of rank \( D \) transforming under the external tensor product of \( D \) fundamental representations of the unitary group

\[
T'_{a_1\ldots a_D} = \sum_{n_1,\ldots,n_D} U_{a_1 n_1} \cdots V_{a_D n_D} T_{n_1\ldots n_D}
\]

\[
\bar{T}'_{\bar{a}_1\ldots \bar{a}_D} = \sum_{n_1,\ldots,n_D} \bar{U}_{\bar{a}_D n_D} \cdots \bar{V}_{\bar{a}_1 n_1} \bar{T}_{n_1\ldots n_D} .
\]

We stress that each unitary group \( U(N_i) \) acts independently on its corresponding index. The tensor \( T \) has no symmetry properties under permutation of its indices. The dimensions \( N_1, \ldots, N_D \) can be different but for simplicity we set \( N_i = N, \forall i \). We denote by \( \bar{n}_i \) the indices of the complex conjugated tensor \( \bar{T} \) (which is contravariant...
of rank $D$), and by $\bar{n}$ the $D$-uple of integers $(n_1, \ldots, n_D)$. We define the color of an index as its position: $n_1$ (and $\bar{n}_1$) has color 1, $n_2$ (and $\bar{n}_2$) has color 2 and so on.

**Invariants and action.** An invariant tensor model is a probability measure defined by an invariant polynomial $S(T, \bar{T})$, which we call the “action”,

$$
d\nu = \frac{1}{Z} e^{-N^{D-1} S(T, \bar{T})} \left( \prod dT_{n_1 \ldots n_D} d\bar{T}_{n_1 \ldots n_D} \right).$$

(2)

By the fundamental theorem of classical invariants of the unitary group (see [53] for a modern proof) any invariant polynomial is a linear combination of trace invariants obtained by contracting pairs of covariant and contravariant indices in a product of tensor entries

$$
\text{Tr}(T, \bar{T}) = \sum \prod \delta_{n_1, \bar{n}_1} T_{n_1 \ldots} \bar{T}_{\bar{n}_1 \ldots},
$$

(3)

where all indices are saturated. Note that, in order to obtain an invariant, we always contract the first index $n_1$ of a $T$ with the first index $\bar{n}_1$ on some $\bar{T}$, the second index $n_2$ of $T$ with the second index $\bar{n}_2$ on some $\bar{T}$ and so on. That is we only contract indices of the same color.

The trace invariants can be represented graphically as $D$-colored graphs. We represent every $T_{n_1 \ldots n_L}$ by a white vertex $v$ and every $\bar{T}_{\bar{n}_1 \ldots \bar{n}_L}$ by a black vertex $\bar{v}$. The contraction of the two indices $n_i$ and $\bar{n}_i$, $\delta_{n_i \bar{n}_i}$, is represented by a line $l^i = (v, \bar{v})$ of color $i$ connecting the two vertices (see figure 1). For example the invariant

$$
\sum \delta_{a_1 p_1} \delta_{a_2 q_2} \delta_{a_3 r_3} \delta_{b_1 r_1} \delta_{b_2 p_2} \delta_{b_3 q_3} \delta_{c_1 q_1} \delta_{c_2 r_2} \delta_{c_3 p_3} T_{a_1 a_2 a_3} T_{b_1 b_2 b_3} T_{c_1 c_2 c_3} \bar{T}_{p_1 p_2 p_3} \bar{T}_{q_1 q_2 q_3} \bar{T}_{r_1 r_2 r_3},
$$

(4)

is represented by the leftmost graph in figure 1 (the vertex $a$ in the drawing represents $T_{a_1 a_2 a_3}$ and so on). The trace invariant associated to the graph $\mathcal{B}$ writes

$$
\text{Tr}_B(T, \bar{T}) = \sum \left( \prod_{\{v, \bar{v}\}, v, \bar{v} \in B} \delta_{n_v, \bar{n}_{\bar{v}}} \right) \prod_{v, \bar{v} \in B} T_{v \bar{v}} \bar{T}_{\bar{v} v},
$$

(5)

where $v$ and $\bar{v}$ run over the white and black vertices of $\mathcal{B}$ and $l^i$ runs over the lines of color $i$ of $\mathcal{B}$. The (unique) graph with two vertices connected by $D$ lines is called the $D$-dipole and is denoted $\mathcal{B}_1$. It represents the unique quadratic invariant

$$
\text{Tr}_{\mathcal{B}_1}(T, \bar{T}) = \sum_{\bar{n}, \bar{\bar{n}}} T_{\bar{n} \bar{\bar{n}}} \prod_{i=1}^D \delta_{n_i, \bar{n}_i},
$$

(6)
The most general “single trace” invariant tensor model is defined by the action

$$S(T, \bar{T}) = \text{Tr}_B(T, \bar{T}) + \sum_B t_B N^{-\omega(B)} \text{Tr}_B(T, \bar{T}),$$

where the sum over invariants includes only connected graphs $B$. The parameters $t_B$ are the coupling constants of the model. We have added in eq. (7) a scaling $N^{-\omega(B)}$ for each invariant as it simplifies some formulae. Up to trivial modifications all the results presented below hold in its absence.

**Feynman graphs.** We will discuss in the sequel the partition function

$$Z(t_B) = \int \left( \prod_{\vec{n}=\vec{n}} dT_{\vec{n}} d\bar{T}_{\vec{n}} \right) e^{-N^{D-1} S(T, \bar{T})},$$

as the expectations of the invariant observables (which are also trace invariants represented by $D$-colored graphs) are treated similarly.

By Taylor expanding with respect to $t_B$ and evaluating the Gaussian integral in terms of Wick contractions, $Z(t_B)$ becomes a sum over Feynman graphs. The graphs are made of effective vertices connected by effective propagators. The effective vertices are the invariants $\text{Tr}_B(T, \bar{T})$, that is they are themselves graphs $B$ with colors $1, \ldots, D$. The effective propagators correspond to pairings (Wick contractions) of $T_{a_1 \ldots a_D}$’s and $\bar{T}_{\bar{p}_1 \ldots \bar{p}_D}$’s. A Wick contraction with the quadratic part eq. (6) amounts to replacing the two tensors by $\prod_{i=1}^D \prod_{v=1}^d \delta_{n_v^n, \bar{n}_{\bar{v}}}$. We represent the effective propagators by dashed lines to which we assign a new color, 0. The Feynman graphs, denoted from now on $G$, are then $D + 1$ colored graphs (see figure 2).

We label $B_{(\rho)}$, $\rho = 1, \ldots, |\rho|$ the effective vertices (subgraphs with colors $1, \ldots, D$) of $G$ and denote $l^0$ the effective propagators (lines of color 0) of $G$. The free energy is a sum over connected Feynman graphs $G$

$$F(t_B) = -\ln Z(t_B) = \sum_G \frac{(-1)^{|\rho|}}{s(G)} \left( \prod_{\rho=1}^{|\rho|} t_{B_{(\rho)}} \right) A(G),$$

where $s(G)$ is a symmetry factor and $A(G)$ is the amplitude of $G$

$$A(G) = \sum_{(\vec{n}, \vec{\bar{n}})} \left[ \prod_{\rho} N^{D-1-\frac{2\omega(B_{(\rho)})}{D}} \prod_{i=1}^D \prod_{(v, \bar{v}) \in B_{(\rho)}} \delta_{n_v^n, \bar{n}_{\bar{v}}} \right] \left[ \prod_{l^0=(v, \bar{v})} \frac{1}{N^{D-1}} \prod_i \delta_{n_v^n, \bar{n}_{\bar{v}}} \right].$$

**Colored Graphs and triangulated spaces.** Matrix models generate ribbon graphs which represent triangulated surfaces. The colored graphs of tensor models represent triangulated spaces.

**Theorem 2.1.** A closed connected $D + 1$ colored graph is a $D$ dimensional normal simplicial pseudo manifold.
A pseudo manifold is a generalization of a manifolds having a finite number of conical singularities. This pseudo manifold can be built by gluing simplices. Consider $D = 3$, thus the $D + 1$ colored graphs have lines of colors 0, 1, 2 and 3.

Every four valent white (or black) vertex is dual to a positive (or negative) oriented tetrahedron. The lines emanating from a vertex are dual to the triangles bounding the tetrahedron (see figure 3(a)). The four triangles inherit the color of the lines 0, 1, 2 and 3. All the lower dimensional simplices are then canonically colored: the edge common to the triangles 1 and 2 is colored by the couple of colors 12, and the apex common to the triangles 1, 2 and 3 is colored by the triple of colors 123. A line in the graph represents the unique gluing of two tetrahedra which respects all the colorings.

Take the example presented in fig. 3(b). The line of color 3 represents the gluing of the two tetrahedra along triangles of color 3 such that the edges 13, 23 and 03 as seen from the positive tetrahedron are glued on the edges 13, 23 and 03 as seen from the negative tetrahedron (and similarly for apices).

The cellular structure of the resulting gluing of tetrahedra is encoded in the colors: an edge ($D - 2$ simplex for $D + 1$ colored graphs) 13 corresponds to a subgraph with colors 13 (see fig. 3(c)). Such subgraphs are called the faces of $G$.

A classical result guarantees that tensor models generate all manifolds:

**Theorem 2.2 (Pezzana’s Existence Theorem).** Any piecewise linear $D$ dimensional manifold admits a representation as a $D + 1$ colored graph.

An initial trace invariant (subgraph with colors 1, ..., $D$) represents a “chunk” of a $D + 1$ dimensional space. Set $D = 3$. As a graph with 3 colors (see fig. 4) an invariant represents a surface. When seen as a subgraph in a 3 + 1 colored Feynman graph, the invariant is decorated by lines of color 0. This amounts to taking the topological cone over the surface and obtain the 3 + 1 dimensional “chunk” of...
Fig. 4. A trace invariant and associated chunk.

space. Note that, if the surface represented by the invariant is non planar, taking the topological cone leads to a conical singularity in the $3+1$ dimensional space.

The $1/N$ expansion. The amplitude of a closed connected ribbon graph of a matrix model is $A(\mathcal{G}) = N^{2-2g(\mathcal{G})}$, where $g(\mathcal{G})$ is the genus of the graph. What replaces the genus in higher dimensions? We answer this question below.

The genus arises in matrix models because the numbers of vertices, lines and faces in a ribbon graph are not independent. A closed connected ribbon graph with $V = 2p$ trivalent vertices has $L = 3p$ lines and $F = p + 2 - 2g(\mathcal{G})$ faces, where $g(\mathcal{G})$ is the genus of $\mathcal{G}$. This generalizes in higher dimensions for $D$ colored graphs. Consider a closed connected $D$ colored graph $\mathcal{B}$ with $2p$ black and white vertices (and $Dp$ lines). The number of faces (subgraphs with two colors) of $\mathcal{B}$, is

$$F = \frac{(D-1)(D-2)}{2}p + (D-1) - \frac{2}{(D-2)!} \omega(\mathcal{B}),$$

where the degree of $\mathcal{B}$, $\omega(\mathcal{B})$, is a non negative integer. Of course a similar relation holds by shifting $D$ to $D + 1$ for $D + 1$ colored graphs $\mathcal{G}$.

Although, like the genus, the degree is a non negative integer it is not a topological invariant. It is an intrinsic number one can compute starting from the graph which mixes information about the topology and the cellular structure of $\mathcal{B}$. Some examples of $D + 1$ colored graphs and their degrees are presented in figure 5.

$$0 \quad 1 \quad 2$$

(a)

$$2 \quad 1 \quad 2$$

(b)

$$2 \quad 1 \quad 2$$

(c)

Fig. 5. $3+1$ colored graphs of degree (a) $\omega(\mathcal{G}) = 0$. (b) $\omega(\mathcal{G}) = 4$. (c) $\omega(\mathcal{G}) = 10$.

The idea is that, embedded in the graph $\mathcal{B}$, one can identify some special ribbon graphs $\mathcal{J}$, called jackets. One then counts the number of faces of each jacket in terms of its genus $g(\mathcal{J})$. Summing over the jackets one obtains the total number of faces of the colored graph in terms of the sum of these genera, the degree of the colored graph $\omega(\mathcal{B}) = \sum_{\mathcal{J}} g(\mathcal{J})$.

We chose the scaling of the invariants in the action eq. (7), $\omega(\mathcal{B})$, to be precisely
the degree of $B$. To evaluate the amplitude of a Feynman graph $G$ (eq. (10)), one needs to count the number of independent sums over indices. Each solid line (of colors 1, ..., $D$) represents the identification of one index $\delta_{n_\bar{n}}$. The dashed lines (of color 0) represent the identifications of $D$ indices $\prod_{i=1}^{D} \delta_{a_i\bar{p}_i}$. An index $n_i$ is identified first along a line of color $i$, then along a line of color 0, then along a line of color $i$ and so on until the cycle of colors 0 and $i$ closes. We obtain a free sum over an index (hence a factor $N$) for every face (subgraph with two colors) of colors 0 and $i$.

Combining this with the explicit scalings in eq. (10) we obtain the $1/N$ expansion in tensor models in arbitrary dimension:

**Theorem 2.3.** The amplitude of a closed connected Feynman ($D+1$ colored) graph generated by the action (7) is

$$A(G) = N^{D-\frac{2(D-1)}{D}} \omega(G).$$

For $D = 2$ the $2 + 1$ colored graphs are ribbon graphs, the degree is the genus and the $1/N$ expansion of tensor models reduces to the one of matrices.

**The leading order graphs.** In matrix models the planar graphs (of genus zero) dominate the $1/N$ expansion. They represent spherical surfaces and form a summable family. What generalizes planar graphs to arbitrary dimensions? We answer this question below.

The leading order graphs are the $D + 1$ colored graphs of degree zero. The structure of such graphs is quite different for $D = 2$ (matrices) and $D \geq 3$ (tensors). However, for all $D$, the graphs of degree zero represent spherical topologies and form a summable family.

A first example of a graph of degree zero is the $D + 1$ dipole (see fig. 5(a)). It has 2 vertices and $\frac{D(D-1)}{2}$ faces (subgraphs with two colors $ij$) hence degree 0. For $D \geq 3$, the $D + 1$ colored graphs of degree 0 with 2$p + 2$ vertices are obtained by inserting two vertices connected by $D$ lines arbitrarily on any line of a $D + 1$ colored graph of degree zero with 2$p$ vertices. This clearly fails for $D = 2$. We call these graphs melons (see figure 6).

The $D + 1$ dipole represents a sphere in $D$ dimensions obtained by identifying coherently two $D$ simplices along their boundary. Two $D + 1$ valent vertices connected by $D$-lines represent a $D$ dimensional ball. As the insertion of balls into spheres preserves the topology we have:

**Theorem 2.4.** For any $D$, the graphs of degree 0 have spherical topology.
The melonic graphs are generated by iterative insertions which can be mapped onto abstract (colored, D + 1-ary) trees. Melonic graphs thus form a summable family with a finite radius of convergence. They are weighted by the coupling constants $t_B$ of the model. When tuning the coupling constants to criticality, tensor models undergo a phase transition to infinitely refined continuous random spaces.

**The algebra of constraints.** The observables are graphs with D colors. At leading order only observables corresponding to melonic graphs $B$ contribute. To each observable we associate a Schwinger Dyson equation\[^{29,30}\] and at leading order only the equations corresponding to $D$ melons survive. Each equation translates into a constraint satisfied by the partition function $L_T Z = 0$ for an explicit partial differential operator $L_T$. The operators form a Lie algebra. As the melons are indexed by $D$ colored trees, the Lie algebra is indexed by colored rooted $D$-ary trees.\[^{19}\]

A colored rooted $D$-ary tree $T$ is a tree such that all its vertices (including the root) have at most $D$ descendants and all its lines have a color index, $1, \ldots, D$. Furthermore the direct descendants of any vertex are connected by lines with different colors.

Two trees $T_1$ and $T_2$ can be joined at a vertex $V \in T_1$. The joined tree $T_1 \star_V T_2$ is obtained as follows (see fig. 7, where $V$ is the bold vertex). We first cut the successors of $V$ in $T_1$ (and get a collection of branches, represented as dashed in figure 7(b)). We then glue the tree $T_2$, with its root on top of $V$. Finally we reattach the branches at the end of the branches of the appropriate color starting at $V$ (see fig. 7(c)).

All the vertices of $T_1$ except $V$ and all the vertices in $T_2$ except its root, which we denote ( ), are mapped into an unique vertex in $T_1 \star_V T_2$, while $V$ and ( ) are both mapped into $V$ in $T_1 \star_V T_2$. The constraint operators form the Lie algebra with Lie bracket

$$[L_{T_2}, L_{T_1}] = \sum_{V \in T_2} L_{T_2 \star_V T_1} - \sum_{V \in T_1} L_{T_1 \star_V T_2}. \quad (13)$$

The fact that the bracket is antisymmetric is clear. What is less obvious is that it respects the Jacobi identity. All the more so because the composition of trees is *not* associative. Indeed, $(T_1 \star_V T_2) \star_W T_3 = T_1 \star_{(V)} (T_2 \star_W T_3)$ only if $W \in T_2$ (or $W = V$). If $W \in T_1 \setminus V$ the right hand side is not defined. However the Jacobi
identity holds due to the presence of the sums. To see this we evaluate

\[
\left[ \mathcal{L}_{T_1}, [\mathcal{L}_{T_2}, \mathcal{L}_{T_3}] \right] = \left[ \mathcal{L}_{T_1}, \sum_{V \in T_2} \mathcal{L}_{(T_2 + V)T_3} - \sum_{V \in T_3} \mathcal{L}_{(T_3 + V)T_2} \right] \\
= \sum_{V \in T_2} \left( \sum_{V' \in T_1} \mathcal{L}_{(T_2 + V')T_3} - \sum_{V' \in T_2 \setminus V} \mathcal{L}_{(T_2 + V')T_3} \right) \\
- \sum_{V \in T_3} \left( \sum_{V' \in T_1} \mathcal{L}_{(T_3 + V')T_2} - \sum_{V' \in T_3 \setminus V} \mathcal{L}_{(T_3 + V')T_2} \right) \\
- \sum_{V \in T_3} \left( \sum_{V' \in T_1} \mathcal{L}_{(T_3 + V')T_2} - \sum_{V' \in T_3 \setminus V} \mathcal{L}_{(T_3 + V')T_2} \right),
\]

hence \( \left[ \mathcal{L}_{T_1}, [\mathcal{L}_{T_2}, \mathcal{L}_{T_3}] \right] + \left[ \mathcal{L}_{T_2}, [\mathcal{L}_{T_3}, \mathcal{L}_{T_1}] \right] + \left[ \mathcal{L}_{T_3}, [\mathcal{L}_{T_1}, \mathcal{L}_{T_2}] \right] \) writes

\[
\sum_{V \in T_2} \sum_{V' \in T_1} \mathcal{L}_{(T_2 + V')T_3} - \sum_{V \in T_2} \sum_{V' \in T_2 \setminus V} \mathcal{L}_{(T_2 + V')T_3} \\
- \sum_{V \in T_3} \sum_{V' \in T_1} \mathcal{L}_{(T_3 + V')T_2} - \sum_{V \in T_3} \sum_{V' \in T_3 \setminus V} \mathcal{L}_{(T_3 + V')T_2} \\
+ \sum_{V \in T_3} \sum_{V' \in T_1} \mathcal{L}_{(T_3 + V')T_2} + \sum_{V \in T_3} \sum_{V' \in T_3 \setminus V} \mathcal{L}_{(T_3 + V')T_2},
\]

which cancels due to identities like

\[
- \sum_{V \in T_2} \sum_{V' \in T_2 \setminus V} \mathcal{L}_{(T_2 + V')T_3} - \sum_{V \in T_2} \sum_{V' \in T_3 \setminus V} \mathcal{L}_{(T_2 + V')T_3} = 0,
\]
\[
\sum_{V \in T_2} \sum_{V' \in T_1} L_{T_1 \ast V} (T_2 \ast V \ast T_3) - \sum_{V \in T_1} \sum_{V' \in T_2 \setminus \{\}} L_{(T_1 \ast V \ast T_2) \ast V} = 0,
\]
\[
- \sum_{V \in T_1} \sum_{V' \in T_2} L_{T_1 \ast V'} (T_3 \ast V \ast T_2) + \sum_{V \in T_1} \sum_{V' \in T_3} L_{(T_1 \ast V_3) \ast V} = 0. \tag{16}
\]

This algebra generalizes to all orders to an algebra indexed by \(D\)-colored graphs. As it contains the dilation operator, in order to connect with conformal field theories in arbitrary dimensions it remains to properly identify the rotation operators. This requires the completion of this Lie algebra by the appropriate generalization of the negative part of the Virasoro (Witt) algebra.

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