AN INVITATION TO THE LOCAL STRUCTURES OF MODULI OF GENUS ONE STABLE MAPS

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Abstract. This informal note provides some elementary examples to motivate the local structural results of [1] on the moduli space of genus one stable maps to projective space. The hope is that these examples will be helpful for graduate students to learn this important subject.

1. Introduction

The moduli space $\overline{M}_g$ of stable curves of genus $g$ has been an important subject of study in algebraic geometry. It is a smooth Deligne-Mumford stack (orbifold). The moduli space $\overline{M}_g(\mathbb{P}^n, d)$ of stable maps of degree $d$ from genus-$g$ curves to the projective space $\mathbb{P}^n$ is a natural generalization of $\overline{M}_g$, but this generalization leads us from a smooth space to a space that can contain singularities as bad as possible (Vakil’s Murphy’s Law). For the purposes of some programs, it is important to obtain the (local) structures of the moduli space $\overline{M}_g(\mathbb{P}^n, d)$.

A point $[u, C]$ of $\overline{M}_g(\mathbb{P}^n, d)$ is (the isomorphism class of) a map

$$u : C \rightarrow \mathbb{P}^n$$

where $C$ is an algebraic curve of arithmetic genus $g$ with at worse nodal singularities such that the automorphism group of the map $[u, C]$ is finite. Here a nodal singularity locally is given by $(xy = 0)$; the point corresponding to the origin is called a node. An automorphism of $[u, C]$ means a morphism $\phi : C \rightarrow C$ such that $u \circ \phi = u$. The automorphism group of the map $[u, C]$ is finite if and only if any genus-0 irreducible component of $C$ contains at least three nodes whenever it is contracted by the map and any genus-1 irreducible component of $C$ contains at least one node whenever it is contracted by the map.

One may add $m$ marked points to the domain curve $C$ away from the nodes and obtain the moduli space of stable maps with $m$ markings, denoted $\overline{M}_{g,m}(\mathbb{P}^n, d)$. But, as far as singularity is concerned, $\overline{M}_{g,m}(\mathbb{P}^n, d)$
and $\overline{M}_g(\mathbb{P}^n, d)$ have exactly the same local singularity types simply because each marked point moves in a smooth local domain. Hence, as far as singularity types are concerned, we may only consider moduli spaces without markings.

When $g = 0$, $\overline{M}_{0,m}(\mathbb{P}^n, d)$ is smooth (as a stack or orbifold); when $g \geq 1$, $\overline{M}_g(\mathbb{P}^n, d)$ is singular (as a stack or orbifold). Indeed, if allowing arbitrary $g$ and $d$, Ravi Vakil showed that $\overline{M}_g(\mathbb{P}^n, d)$ can contain all possible singularity types over $\mathbb{Z}$. This seems to be a piece of bad news, but thinking positively, it also makes the spaces $\overline{M}_g(\mathbb{P}^n, d)$ ultimately rich as modular singularity models to investigate.

Historically though, $\overline{M}_g(\mathbb{P}^n, d)$ was introduced for the (sole) purpose of defining the Gromov-Witten invariants. An important standing problem in this area is to enumerate the (virtual) number of curves with fixed genus and degree in a smooth Calabi-Yau threefold in $\mathbb{P}^4$, or more generally, in a complete intersection $X$ in $\mathbb{P}^n$. In principle, the curve-counting business on $X$ can be done on the ambient space $\mathbb{P}^n$, using the defining equations of $X$. The moduli space $\overline{M}_g(X, d)$ of stable maps of degree $d$ from curves of genus $g$ to $X$ is naturally a submoduli space of $\overline{M}_g(\mathbb{P}^n, d)$. For example, let us assume that $X$ is a smooth hypersurface defined by $X = s^{-1}(0)$ for some section $s \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ with some positive integer $k$. If we let

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathbb{P}^n \\
\pi \downarrow & & \\
\overline{M}_g(\mathbb{P}^n, d)
\end{array}$$

be the universal family over the moduli space $\overline{M}_g(\mathbb{P}^n, d)$ with the universal map $f$ and let

$$\sigma = \pi_* f^* s \in \Gamma(\overline{M}_g(\mathbb{P}^n, d), \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)),$$

then we have

$$\overline{M}_g(X, d) = \sigma^{-1}(0).$$

If $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$ were locally free, then it would have a natural Euler class, and this Euler class would bridge the intersection theory on $\overline{M}_g(X, d)$ to the intersection theory on $\overline{M}_g(\mathbb{P}^n, d)$. Thus, resolving the non-locally free locus of the direct image sheaf $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$ is essential in GW theory of complete intersections in $\mathbb{P}^n$. Here, by resolving the sheaf, we mean...
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a diagram

\[
\begin{array}{ccc}
\tilde{f} : \tilde{X} & \longrightarrow & X \\
\tilde{\pi} & \downarrow & \pi \\
\tilde{M}_g(\mathbb{P}, d) & \longrightarrow & \overline{M}_g(\mathbb{P}^n, d),
\end{array}
\]

where \(\tilde{M}_g(\mathbb{P}, d) \longrightarrow \overline{M}_g(\mathbb{P}^n, d)\) is a blowup and \(\tilde{X} = X \times \overline{M}_g(\mathbb{P}^n, d)\) such that the direct image sheaf \(\tilde{\pi}_*\tilde{f}_*\mathcal{O}_{\mathbb{P}^n}(k)\) is locally free.

Thus, one sees that on the one hand, it is important for GW theory to resolve the sheaf \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(k)\); on the other hand, it is important for singularity theory to resolve \(\overline{M}_g(\mathbb{P}^n, d)\). The two problems are related. The point of [1] is that it is more natural to resolve the sheaves \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(k)\) first, and at least when the genus is low, resolving the sheaves will also ensure a resolution of the moduli space.

To see this point, observe that a stable map \([u] \in \overline{M}_g(\mathbb{P}^n, d)\), as a morphism, is given by the data

\[
u = [u_0, \cdots, u_n] : C \longrightarrow \mathbb{P}^n, \quad u_i \in H^0(u^*\mathcal{O}_{\mathbb{P}^n}(1));
\]

its deformation is determined by the combined deformation of the curve \(C\) and the sections \(\{u_i\}\). Since the deformation of the curve is unobstructed, the irregularity of \(\overline{M}_g(\mathbb{P}^n, d)\) is closely related to the non-local-freeness of the direct image sheaf \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(1)\) This alludes that desingularizations of \(\overline{M}_g(\mathbb{P}^n, d)\) should be governed by desingularizations of \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(1)\). For genus-1, this is true in the simplest form, a desingularization of \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(1)\) implies a desingularization of \(\overline{M}_1(\mathbb{P}^n, d)\).

Vakil and Zinger first discovered a desingularization of the main component of the moduli space \(\overline{M}_1(\mathbb{P}^n, d)\) in [2]. Their method is analytic in nature. They found a natural sequence of blowups that resolve singularities of \(\overline{M}_1(\mathbb{P}^n, d)\) and then showed that the same blowups also resolve the sheaves \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(k)\). As hinted in the last paragraph, from the algebro-geometric approach, it is more natural to resolve the sheaves first. In [1], we first obtain local structures of the sheaf \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(k)\). The structures of \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(1)\) allow us to derive local defining equations of the moduli space \(\overline{M}_1(\mathbb{P}^n, d)\). Having obtained these local equations, it is rather clear what loci one should blow up, how the resulting space turns out to be smooth, and why the resulting direct image sheaves become locally free.

This note, through some concrete examples, is solely devoted to reveal the structures of the sheaf \(\pi_*f_*\mathcal{O}_{\mathbb{P}^n}(k)\). By working out these examples, we hope the student will familiarize himself/herself with the
aspects of families of elliptic curves that are useful for obtaining the local structures of the moduli space of stable maps.

During the summer school, I gave four lectures on tropical curves and their applications to plane enumerative geometry. When the school was over, the organizers asked every speaker to write up his lecture notes for the proceeding. However, as there have been already several excellent expository articles on tropical curves that the students can easily find on arXiv and I did not feel that I could make any meaningful improvement, instead, I thought that an introductory note on elliptic stable maps should help students to learn this important subject. The idea is that through some examples the students will gain the intuition about the approach to the local structures of the stable map moduli $[1]$. I hope that the material will be a more converging addition to the proceeding, and it will be more useful for graduate students as well as for researchers. Most of the note should be accessible to any graduate student with some backgrounds on algebraic geometry.

This note recollects the toy examples that Jun Li and I calculated in the summer of 07 as the warm-up as well as the guide for our approach toward the general theory $[1]$, but, needless to say, all the mistakes in this detailed presentation must be due to my own oversight. I thank Jun Li, from whom I have learned a lot, for the collaboration. I also thank CMS of Zhejiang University and the organizers of the summer school, especially Lizhen Ji, for the excellent environment and for supporting the idea to include this note in the proceeding.

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2. The Structures of the Direct Image Sheaf

We begin with motivating the setups of our examples.

2.1. Motivation: reduction to local family.

2.1. Let \( \pi : \mathcal{X} \to \overline{M}_1(\mathbb{P}^n, d) \) be the universal family with the universal map \( f : \mathcal{X} \to \mathbb{P}^n \). As for any sheaf, the question on the structures of the direct image sheaf \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k) \) is (étale) local. Given any point \([u, C] = [u : C \to \mathbb{P}^n] \in \overline{M}_1(\mathbb{P}^n, d)\), we choose a small étale neighborhood \( U \ni [u, C] \subset \overline{M}_1(\mathbb{P}^n, d) \). By choosing \( k \) general hyperplanes \( H_1, \ldots, H_k \) of \( \mathbb{P}^n \), we can assume that

\[
 f^*(H_1 + \cdots + H_k) \cap C
\]

is a simple divisor \( \sum_{i=1}^m s_i \) of degree \( m = dk \) (\( s_1, \ldots, s_m \) are disjoint). Let \( S = f^*(H_1 + \cdots + H_k) \), then

\[
 f^* \mathcal{O}_{\mathbb{P}^n}(k) = \mathcal{O}_\mathcal{X}(S).
\]

By an étale base change, we may assume that \( S = \sum_{i=1}^m S_i \) where each \( S_i \) is a section of \( \pi : \mathcal{X} \to \mathcal{V} \) such that \( s_i = S_i \cap C \). Hence, the local structures of \( \pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k) \) is reflected in the structure of \( \pi_* \mathcal{O}_\mathcal{X}(S) \).

2.2. This motivates us to consider some examples of flat families of elliptic curves, \( \pi : Z \to B \) with a section \( S \), and study the associated direct image sheaf \( \pi_* \mathcal{O}_Z(mS) \).

Definition 2.3. The core \( C_e \) of a connected genus-one curve \( C \) is the unique smallest (by inclusion) subcurve of arithmetic genus one.

Figure 1. An elliptic curve \( C_e \) attached with 3 rational tails
2.4. Given any connected genus-one curve $C$, upon removing the core of $C$, the rest of irreducible components are all rational curves (i.e., genus zero curves), we denote their union by $C'$. We will call each connected component of $C'$ a tail, it is a tree of rational curves. If there are $r$ such connected components, we will say the curve $C$ has $r$ (rational) tails. See Figure 1 for an example of elliptic curve with three rational tails.

2.5. By Riemann-Roch, we can check that $R^1\pi_*\mathcal{O}_Z(mS) = 0$ at point $b \in B$ where the section $S$ meets the core of $Z_b$ but $R^1\pi_*\mathcal{O}_Z(mS)$ does not vanish otherwise. In particular, $\pi_*\mathcal{O}_Z(mS)$ is locally free at point $b \in B$ where the section meets the core of $Z_b$, but it is not locally free otherwise.

2.6. So we will study examples of families $\pi : Z \rightarrow B$ together with a section $S$ having a point $0 \in B$ such that

- for $b \neq 0$, the fiber $Z_b$ is smooth;
- the fiber $Z_0$ is an elliptic curve with $r$ tails;
- $S$ misses the core of $Z_0$ but meets the tails.

![Figure 2. The family $Z \rightarrow \mathbb{P}^1$](image)
We will begin with three toy examples in order of generality: a warm-up 1-tail case, an $r$-tail case ($r > 1$), and a case of more general type. Throughout the note, unless otherwise stated, we can work over any fixed algebraically closed base field $k$.

### 2.2. A 1-tail case.

We now begin to construct a one-parameter family of genus-1 curves whose central fiber is a smooth elliptic curve attached with a smooth rational curve. One may consult Figure 2 for picture of such a construction. The details are in the next two paragraphs.

#### 2.9. Let $b : Z \to \mathbb{P}^1 \times E$ be the blow up of $\mathbb{P}^1 \times E$ at $(p, e_0)$ where $p$ a fixed point on $\mathbb{P}^1$ and $e_0$ is a fixed point on $E$, respectively. Let $\pi : \mathbb{P}^1 \times E \to \mathbb{P}^1$ be the projection to the first factor and

$$\pi = \pi_1 \circ b : Z \to \mathbb{P}^1 \times E \to \mathbb{P}^1.$$ 

Note that this provides a one-parameter smoothing of an elliptic curve with one rational tail. We will denote the fiber $\pi^{-1}(t)$ by $Z_t$, $t \in \mathbb{P}^1$.

#### 2.10. Choose a generic point $e \in E$. Let $S, D$ be the proper transform of $\mathbb{P}^1 \times e_0$ and $\mathbb{P}^1 \times e$, respectively. Note that if we write the central fiber of $\pi$ as $C_o + C_a$ where $C_o$ is elliptic and $C_a$ is rational, then $\mathcal{O}_Z(D) = \mathcal{O}_Z(S + C_a)$. We will consider the direct image sheaf $L_m = \pi_* \mathcal{O}_Z(mS)$.

#### 2.11. We have a short exact sequence

$$0 \to \mathcal{O}_Z(mS) \to \mathcal{O}_Z(mS + D) \to \mathcal{O}_Z(mS + D)|_D \to 0$$

and a long exact sequence

$$0 \to \pi_* \mathcal{O}_Z(mS) \xrightarrow{\alpha_m} \pi_* \mathcal{O}_Z(mS + D) \xrightarrow{\beta_m} \pi_* \mathcal{O}_Z(mS + D)|_D \xrightarrow{\gamma_m} R^1 \pi_* \mathcal{O}_Z(mS) \to 0.$$ 

Here observe that $\pi_* \mathcal{O}_Z(mS + D)$ is locally free because one calculates that

$$\dim H^0(Z_t, \mathcal{O}_Z(mS + D)|_{Z_t}) = m + 1, \quad \text{for all } t \in \mathbb{P}^1$$

and

$$R^1 \pi_* \mathcal{O}_Z(mS + D) = 0.$$ 

Since

$$\mathcal{O}(mS + D)|_D = \mathcal{O}(D)|_D = N_{D \mid Z} \cong \mathcal{O}_D$$

and $\pi|_D : D \to \mathbb{P}^1$ is an isomorphism, we see that $\pi_* \mathcal{O}_Z(mS + D)|_D = \mathcal{O}_{\mathbb{P}^1}$.
2.12. Our goal is to describe explicitly the entire sequence (2.1).

2.13. The case $m = 0$ is somewhat special, we isolate it below.

$$0 \rightarrow \pi_* \mathcal{O}_Z \rightarrow \pi_* \mathcal{O}_Z(D) \rightarrow \pi_* \mathcal{O}_Z|_D \rightarrow R^1 \pi_* \mathcal{O}_Z \rightarrow 0.$$  

It is easy to see that this is

(2.2)  

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1} \rightarrow 0.$$  

2.14. Toward the general case of (2.1), we first consider another short exact sequence:

$$0 \rightarrow \mathcal{O}_Z(mS + D) \rightarrow \mathcal{O}_Z((m + 1)S + D) \rightarrow \mathcal{O}_Z((m + 1)S + D)|_S \rightarrow 0.$$  

Noting that

$$\mathcal{O}_Z((m + 1)S + D)|_S = N_{S/Z} \otimes^m \mathcal{O}_S(-m - 1)$$

and

$$R^1 \pi_* \mathcal{O}_Z(mS + D) = 0,$$

we obtain a short exact sequence of locally free sheaves

(2.3)  

$$0 \rightarrow \pi_* \mathcal{O}_Z(mS + D) \rightarrow \pi_* \mathcal{O}_Z((m + 1)S + D) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-m - 1) \rightarrow 0.$$  

We claim

Lemma 2.15. The sequence (2.3) splits and consequently

$$\pi_* \mathcal{O}_Z(mS + D) = \bigoplus_{i=0}^m \mathcal{O}_{\mathbb{P}^1}(-i).$$

Proof. We prove it by induction. For $m = 0$, it is clear that

$$\pi_* \mathcal{O}_Z(D) = \mathcal{O}_{\mathbb{P}^1}.$$  

Assume that the lemma is true for the case of $m$. Then by (2.3), we have

$$0 \rightarrow \bigoplus_{i=0}^m \mathcal{O}_{\mathbb{P}^1}(-i) \rightarrow \pi_* \mathcal{O}_Z((m + 1)S + D) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-m - 1) \rightarrow 0.$$  

Using Serre duality, one calculates that

$$\text{Ext}_{\mathbb{P}^1} \left( \mathcal{O}_{\mathbb{P}^1}(-m - 1), \bigoplus_{i=0}^m \mathcal{O}_{\mathbb{P}^1}(-i) \right) = 0,$$

hence the exact sequence above must be trivial. \qed
2.16. Let \([t, s] \) be the homogeneous coordinates of \( \mathbb{P}^1 \). We may assume that \( p = [0, 1] \). Then we have a canonical map \( \beta \),

\[
\begin{array}{c}
\mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{x_t} \mathcal{O}_{\mathbb{P}^1}
\end{array}
\]

which induces a canonical exact sequence

\[
\begin{array}{c}
0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{x_t} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\gamma} \mathcal{k}(p) \longrightarrow 0
\end{array}
\]

where \( \mathcal{k}(p) \) is the one-dimensional skyscraper sheaf supported at \( p \) and \( \gamma \) is the evaluation at \( p \). We make an observation here that any map \( \beta' : \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \) such that \( \gamma \circ \beta' = 0 \) is a scalar multiple of \( \beta \). This is because \( \dim \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}) = 2 \) and \( \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}) \) has a basis \( \{x_t, x_s\} \). In particular, any such nontrivial \( \beta' \) determine the same cokernel, namely, \( \mathcal{k}(p) \).

**Proposition 2.17.** Assume \( m > 0 \). Up to isomorphism, we have

1. \( \pi_* \mathcal{O}_Z(mS) \cong \bigoplus_{i=0}^{m} \mathcal{O}_{\mathbb{P}^1}(-i) \).
2. The map \( \alpha_m : \pi_* \mathcal{O}_Z(mS) \longrightarrow \pi_* \mathcal{O}_Z(mS + D) \cong \bigoplus_{i=0}^{m} \mathcal{O}_{\mathbb{P}^1}(-i) \) is the natural inclusion to the corresponding factors.
3. The map \( \beta_m \) is given by \( \beta : \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \).
4. The map \( \gamma_m \) is the evaluation at \( p \). In particular, all \( R^1 \pi_* \mathcal{O}_Z(mS) \) are isomorphic to the one-dimensional skyscraper sheaf \( \mathcal{k}(p) \) supported at \( p \).

**Proof.** To start, observe that for any nonnegative integer \( k \), a map from \( \mathcal{O}_{\mathbb{P}^1}(-k) \) to \( \mathcal{O}_{\mathbb{P}^1} \) is equivalent to a map from \( \mathcal{O}_{\mathbb{P}^1} \) to \( \mathcal{O}_{\mathbb{P}^1}(k) \), thus the space of all such maps is \( H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \).

Now consider the case \( m = 1 \) first. We have

\[
\begin{array}{c}
0 \longrightarrow \pi_* \mathcal{O}_Z(S) \xrightarrow{\alpha_1} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\beta_1} \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\gamma_1} R^1 \pi_* \mathcal{O}_Z(S) \longrightarrow 0
\end{array}
\]

By the observation in the start, we can express \( \beta_1 \) as

\[
(h_0, h_1) \mapsto c_0 h_0 + c_1 h_1
\]

where \( h_0, h_1 \in \mathcal{O}_{\mathbb{P}^1} \) and \( c_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \) \( (i = 0, 1) \) are two fixed sections. Since the cokernel of \( \beta_1 \) supports at \( p \), which follows from

\[
R^1 \pi_* \mathcal{O}_Z(mS) \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{k}(p) = H^1(\mathcal{O}_{C_p \cup C_1}(mS)) = \mathcal{k}
\]

by the base change property, hence \( c_0 = 0 \) and \( c_1 \) is a non-zero constant multiple of \( t \). In particular this also implies that \( R^1 \pi_* \mathcal{O}_Z(S) \) is the one-dimensional skyscraper sheaf \( \mathcal{k}(p) \) supported at \( p \), and \( \gamma_1 \) is the evaluation at the point \( p \). This proves the case \( m = 1 \).
For $m \geq 1$, consider the following natural commutative diagram of long exact sequence,

\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_*\mathcal{O}_Z(mS) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \pi_*\mathcal{O}_Z((m+1)S) \\
\beta_m & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^1} & \longrightarrow & R^1\pi_*\mathcal{O}_Z(mS) \longrightarrow 0 \\
\end{array}
\]

Note that the third downward arrow is an isomorphism. The diagram gives rise to the following one which we will use soon

\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_*\mathcal{O}_Z(S) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \pi_*\mathcal{O}_Z(mS) \\
\beta_1 & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathbb{P}^1} & \longrightarrow & R^1\pi_*\mathcal{O}_Z(S) \longrightarrow 0 \\
\end{array}
\]

We take splits of all $\pi_*\mathcal{O}_Z(kS + D)$ $1 \leq k \leq m$ so that all the inclusions $\pi_*\mathcal{O}_Z(kS + D) \hookrightarrow \mathcal{O}_Z(mS + D)$ are given by the natural factor inclusions. By the second square of the last diagram, we see that the $\beta_m$ restricted to the factor $\mathcal{O}_{\mathbb{P}^1}(-1)$ is nontrivial. So, up to a non-zero scalar, we can assume that it is given by multiplication by $t$.

As in the case of $m = 1$, we can express $\beta_m$ as

\[
(h_0, h_1, h_2, \ldots, h_m) \mapsto c_0h_0 + th_1 + c_2h_2 + \cdots + c_mh_m
\]

for some fixed $c_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i))$, where $h_0, \ldots, h_m \in \mathcal{O}_{\mathbb{P}^1}$. Using the base change property, we see that $R^1\pi_*\mathcal{O}_Z(mS)$ is supported at $p$, thus $c_0 = 0$ and $t \mid c_i$ for $i \geq 2$. So, we can write $c_i = ta_i$ with $a_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i - 1))(i \geq 2)$.

Hence we obtain that $\ker \beta_m$ is given by

\[
\{(h_0, h_1, h_2, \ldots, h_m)| - h_1 = a_2h_2 + \cdots + a_mh_m, h_0, h_2, \ldots, h_m \in \mathcal{O}_{\mathbb{P}^1}\}.
\]

This clearly is isomorphic to $\bigoplus_{1 \neq i = 0}^m \mathcal{O}_{\mathbb{P}^1}(-i)$. Therefore, by expressing an arbitrary element $(h_0, h_1, h_2, \ldots, h_m)$ as

\[
(h_0, -(a_2h_2 + \cdots + a_mh_m), h_2, \ldots, h_{m+1})
\]
we conclude that after a (possibly) new split of 
\[ \mathcal{O}_Z(mS + D) = \bigoplus_{i=0}^{m} \mathcal{O}_{\mathbb{P}^1}(-i), \]
we can identify \( \ker \beta_m \) with the direct summand \( \bigoplus_{i \neq 0}^{m} \mathcal{O}_{\mathbb{P}^1}(-i) \) and the map \( \beta_m \) is given by the restriction to the summand \( \mathcal{O}_{\mathbb{P}^1}(-1) \) which, in turn, is given by multiplying by \( t \).

All the rest of the statements follow immediately. \( \square \)

2.3. Many tails that are smoothed in one direction.

2.18. We can generalize the above to the case where the central fiber is an elliptic curve with \( k \) many tails. To do this, again on \( \mathbb{P}^1 \times E \), we pick up \( k \) points \( (p, e_1), \ldots, (p, e_k) \), and blow up \( \mathbb{P}^1 \times E \) at \( (p, e_1), \ldots, (p, e_k) \). This way, we obtain a smooth surface \( Z \) whose projection to \( \mathbb{P}^1 \) provides a family \( \pi : Z \to \mathbb{P}^1 \times E \) with the central fiber an elliptic curve with \( k \) rational tails, elsewhere the fiber is isomorphic to \( E \). In the moduli space, this represents a general direction along which all the nodes are smoothed simultaneously. See Figure 3.
2.19. Let $S_i$ be the proper transform of $\mathbb{P}^1 \times e_i$, $1 \leq i \leq k$ and $D$ be the proper transform of $\mathbb{P}^1 \times e$ where $e$ is a general point on $E$. Let $m = (m_1, \cdots, m_k)$ and $S = (S_1, \cdots, S_k)$. Let $m = \sum_i m_i$ and $mS = \sum_i m_i S_i$. We consider the pushforward sheaf $\pi_* \mathcal{O}_Z(mS + D)$.

2.20. Then all the previous results extend almost word by word to this case with $m$, $S$ replaced by $m$, $S$. This suggests that the direct image sheaf is sensitive only to the smoothing direction, but not to the number of tails. The point is that after carefully treating the case of one tail, it is almost routine to treat the case of multiple tails.

2.21. One can consult §2.4 below to see how to formulate similar statements and arguments. For example, in the inductive proof in this case, to go from $m$ to $m+1$, one goes from $\sum_i m_i S_i$ to $(\sum_i m_i S_i) + S_j$. Since the results of this subsection will not be used elsewhere in this work, we omit the routine details.

2.4. An $r$-tail case.

2.22. We will now construct a family $Z \rightarrow B$ such that the central fiber is a smooth elliptic curve attached with a connected chain of rational curves.

2.23. Consider the space $(\mathbb{P}^1)^r \times E$. Let $p = [0, 1] \in \mathbb{P}^1$. Pick $r$ distinct points $e_1, \cdots, e_r$ in $E$. For any $1 \leq i \leq r$, Set

$$W_i = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \times p \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \subset (\mathbb{P}^1)^r$$
where $p$ occurs in the $i$-th factor. Set $W'_i = W_i \times e_i \subset (\mathbb{P}^1)^r \times E$, $1 \leq i \leq r$. Let $b : Z \rightarrow (\mathbb{P}^1)^r \times E$ be the blowup of $\mathbb{P}^1)^r \times E$ along the disjoint union $\coprod_i W'_i$. Let $S_i$ be the proper transform of $(\mathbb{P}^1)^r \times e_i$ and $D$ be the proper transform of $\mathbb{P}^1)^r \times e$ where $e \in E$ is a point distinct from all $e_1, \cdots, e_r$.

**2.24.** Let $\pi$ be the projection $b : Z \rightarrow (\mathbb{P}^1)^r \times E$ followed by the projection to $(\mathbb{P}^1)^r$. Then this provides a family of nodal elliptic curves: the fiber over $\bar{p} := (p, \cdots, p)$ is a smooth elliptic curve with $r$ many rational tails; moving to a general direction represented by coordinates indexed by $i_1, \cdots, i_j$ ($1 \leq j \leq r$), the nodes created by blowing up $W'_{i_1}, \cdots, W'_{i_j}$ will be smoothed.

**2.25.** Figure 4 shows the case of $r = 2$. Here, the central fiber over $(p, p)$ is $C_e \cup C_1 \cup C_2$ where $C_e \cong E$ and $C_1 \cong C_2 \cong \mathbb{P}^1$; the section $D$ passes $C_e$; the sections $S_i$ pass $C_i$ ($i = 1, 2$).

**2.26.** Let $C_j$ be the exceptional divisor of $Z$ corresponding to $W'_j$. Observe that $\mathcal{O}_Z(S_j + C_j) = \mathcal{O}(D)$. Let $m = (m_1, \cdots, m_r)$ and $S = (S_1, \cdots, S_r)$. Let $m = \sum_i m_i$ and $mS = \sum_i m_i S_i$. We will study the local freeness of the direct image sheaf $\pi_* \mathcal{O}_Z(mS)$.

**2.27.** We have a short exact sequence
\begin{equation}
0 \rightarrow \mathcal{O}_Z(mS) \rightarrow \mathcal{O}_Z(mS + D) \rightarrow \mathcal{O}_Z(mS + D)|_D \rightarrow 0
\end{equation}
and a long exact sequence
\begin{equation}
\begin{array}{cccc}
0 & \rightarrow & \pi_* \mathcal{O}_Z(mS) & \rightarrow \pi_* \mathcal{O}_Z(mS + D) & \rightarrow \pi_* \mathcal{O}_Z(mS + D)|_D & \rightarrow 0 \\
\rightarrow & & \beta_m & & \gamma_m & \\
\end{array}
\end{equation}
\begin{equation}
\pi_* \mathcal{O}_Z(mS + D)|_D \rightarrow R^1 \pi_* \mathcal{O}_Z(mS) \rightarrow 0.
\end{equation}
As in **2.22** $\pi_* \mathcal{O}_Z(mS + D)$ is locally free and $R^1 \pi_* \mathcal{O}_Z(mS + D) = 0$.

**2.28.** Since $\mathcal{O}_Z(mS + D)|_D = \mathcal{O}_Z(D)|_D = N_{D/Z} \cong \mathcal{O}_D$ and $\pi|_D : D \rightarrow (\mathbb{P}^1)^r$ is an isomorphism, we see that $\pi_* \mathcal{O}_Z(mS + D)|_D = \mathcal{O}(\mathbb{P}^1)^r$.

**2.29.** The case of **2.27** when $m = 0$ is special, we isolate it below.
\begin{equation}
0 \rightarrow \pi_* \mathcal{O}_Z \rightarrow \pi_* \mathcal{O}_Z(D) \rightarrow \pi_* \mathcal{O}_Z(D)|_D \rightarrow R^1 \pi_* \mathcal{O}_Z \rightarrow 0.
\end{equation}
It is easy to see that this is
\begin{equation}
0 \rightarrow \mathcal{O}(\mathbb{P}^1)^r \rightarrow \mathcal{O}(\mathbb{P}^1)^r \rightarrow \mathcal{O}(\mathbb{P}^1)^r \rightarrow 0.
\end{equation}

**2.30.** To treat the general case, similar to **2.22** of the 1-tail case, we will first prove a formula for $\pi_* \mathcal{O}_Z(mS + D)$.
We begin with a lemma.
Lemma 2.31. $\pi_*N_{S\setminus Z}$ is isomorphic to $\mathcal{O}_{(\mathbb{P}^1)^r}(-W_j)$. And both are isomorphic to $\pi_j^*\mathcal{O}_{\mathbb{P}^1}(-1)$ where $\pi_j : (\mathbb{P}^1)^r \rightarrow \mathbb{P}^1$ is the projection to the $j$-th factor.

Proof. Any bundle on $(\mathbb{P}^1)^r$ is of the form $\bigotimes_{j=1}^r \pi_j^*\mathcal{O}_{\mathbb{P}^1}(a_j)$. We will determine the integer $a_j$ for each of $\pi_*N_{S\setminus Z}$ and $\mathcal{O}_{(\mathbb{P}^1)^r}(-W_j)$.

First, it is easy to determine that $\mathcal{O}_{(\mathbb{P}^1)^r}(W_j) = \pi_j^*\mathcal{O}_{\mathbb{P}^1}(1)$ by looking at the intersection numbers of $W_j$ with the coordinate lines. Hence $\mathcal{O}_{(\mathbb{P}^1)^r}(-W_j) = \pi_j^*\mathcal{O}_{\mathbb{P}^1}(-1)$, as desired.

To show the rest, again we can argue by computing intersection numbers. Let $l_i$ be the pre-image of the $i$-th coordinate line by the isomorphism $\pi|_{S_j} : S_j \xrightarrow{\pi} (\mathbb{P}^1)^r$. Then we have $S_j \cdot l_i = 0$ when $i \neq j$ because $l_i$ can be moved to become a section of $C_j \cong W_j \times \mathbb{P}^1 \rightarrow W_j$, disjoint from $S_j$. When $i = j$, we have 

$$(S_j + C_j) \cdot l_j = D \cdot l_j = 0.$$ 

Hence $S_j \cdot l_j = -C_j \cdot l_j$. But $C_j \cdot l_j = 1$. Hence $\pi_*N_{S\setminus Z} = \pi_j^*\mathcal{O}_{\mathbb{P}^1}(-1)$ because $N_{S\setminus Z} = \mathcal{O}_Z(S_j)|_{S_j}$.

We may also argue without calculating the intersection numbers. Since $S_j$ is the proper transform of the constant section $(\mathbb{P}^1)^r \times e_j$ in $(\mathbb{P}^1)^r \times E$, we have that the normal bundle $N_{S\setminus Z}$ is isomorphic to

$\mathcal{O}_{S_j}(-\text{exceptional divisor in } S_j)$

where the exceptional divisor in $S_j$ is the pre-image of $W_j$ in $S_j$. This implies that $N_{S\setminus Z} \cong \mathcal{O}_{(\mathbb{P}^1)^r}(-W_j)$.\hfill \Box

2.32. Now, consider the short exact sequence:

$$0 \rightarrow \mathcal{O}_Z(mS+D) \rightarrow \mathcal{O}_Z(mS+S_j+D) \rightarrow \mathcal{O}_Z(mS+S_j+D)|_S \rightarrow 0.$$ 

Noting that we have

$$\mathcal{O}_Z(mS+S_j+D)|_{S_j} = N_{S\setminus Z}^{\otimes (m_j+1)},$$

$$\pi_*N_{S\setminus Z}^{\otimes (m_j+1)} = \mathcal{O}_{(\mathbb{P}^1)^r}(-(m_j+1)W_j) \quad \text{(Lemma 2.31)},$$

and

$$R^1\pi_*\mathcal{O}_Z(mS+D) = 0,$$

hence we obtain a short exact sequence of locally free sheaves

$$(2.9) \quad 0 \rightarrow \pi_*\mathcal{O}_Z(mS+D) \rightarrow \pi_*\mathcal{O}_Z(mS+S_j+D) \rightarrow \mathcal{O}_{(\mathbb{P}^1)^r}(-(m_j+1)W_j) \rightarrow 0.$$
Lemma 2.33. The sequence (2.9) splits and consequently
\[ \pi_* \mathcal{O}_Z(mS + D) = \mathcal{O}_{(\mathbb{P}^1)^r} \oplus \bigoplus_{i=1}^{r} \bigoplus_{k=1}^{m_i} \mathcal{O}_{(\mathbb{P}^1)^r}(-kW_j). \]

Proof. We proceed by induction on \( \sum m_i = m \). When \( m = 0 \), we have
\[ \pi_* \mathcal{O}_Z(D) = \mathcal{O}_{(\mathbb{P}^1)^r}. \]
Hence the lemma holds in this case.
Assume that when \( \sum m_i = m \), the lemma holds. Then for the case of \( m + 1 \), by (2.9), we have
\[ 0 \to \mathcal{O}_{(\mathbb{P}^1)^r} \oplus \bigoplus_{i=1}^{r} \bigoplus_{k=1}^{m_i} \mathcal{O}_{(\mathbb{P}^1)^r}(-kW_j) \to \pi_* \mathcal{O}_Z(mS + S_j + D) \to \mathcal{O}_{(\mathbb{P}^1)^r}(-(m_j + 1)W_j) \to 0. \]
Now applying Serre’s duality and Lemma, we obtain
\[ \text{Ext}^1(\mathcal{O}_{(\mathbb{P}^1)^r}(-(m_j + 1)W_j), \mathcal{O}_{(\mathbb{P}^1)^r} \oplus \bigoplus_{i=1}^{r} \bigoplus_{k=1}^{m_i} \mathcal{O}_{(\mathbb{P}^1)^r}(-kW_j)) = 0, \]
hence the sequence is the trivial one. \( \square \)

2.34. Our aim is to explicitly describe the sheaf \( \pi_* \mathcal{O}_Z(mS) \). By (2.8), we may assume \( m > 0 \). Let \( [t_i, s_i] \) be the homogeneous coordinates of the i-th factor \( \mathbb{P}^1 \) in the product \( (\mathbb{P}^1)^r \). Then
\[ W_i = \{ t_i = 0 \}. \]
Note that \( \bigcap_i W_i = \bar{p} \). Let
\[ \mathcal{V}_1 = \bigoplus_{i=1}^{r} \mathcal{O}_{(\mathbb{P}^1)^r}(-W_i) = \bigoplus_{i} \pi_i^* \mathcal{O}_{\mathbb{P}^1}(-1). \]
Then we have a canonical map \( \beta \),
\[ (h_1, \ldots, h_r) \to \sum_i t_i h_i \]
which induces a canonical exact sequence
\[ 0 \to \ker \beta \to \mathcal{V}_1 \to \mathcal{O}_{(\mathbb{P}^1)^r} \to \mathcal{O}_{(\mathbb{P}^1)^r}|_W = \mathcal{O}_W = k(\bar{p}) \to 0 \]
where \( k(\bar{p}) \) is the structure sheaf of \( \bar{p} \). This sequence is a higher dimensional generalization of (2.3) and plays similar role in the proof.
Set
\[ \mathcal{V}_{m,0} = \mathcal{O}_{(\mathbb{P}^1)^r} \oplus \bigoplus_{j=1}^{r} \bigoplus_{k=2}^{m_j} \mathcal{O}_{(\mathbb{P}^1)^r}(-kW_j). \]
Proposition 2.35. Assume that $m_j \neq 0$ for all $1 \leq j \leq r$. Then up to isomorphism, we have a commutative diagram

$$
0 \longrightarrow \ker \beta \longrightarrow \mathcal{V}_1 \longrightarrow 0
$$

$$
0 \longrightarrow \pi_* \mathcal{O}_Z(m \mathbb{S}) \overset{\alpha_m}{\longrightarrow} \pi_* \mathcal{O}_Z(m \mathbb{S} + D) \overset{\beta}{\longrightarrow} \mathcal{O}(\mathbb{P}_1)^r \overset{\gamma}{\longrightarrow} \mathcal{K}(\tilde{p}) \longrightarrow 0
$$

where the first two downward arrows are inclusions, and the last two are isomorphisms. Moreover, we have

1. $\pi_* \mathcal{O}_Z(m \mathbb{S} + D) = \mathcal{V}_{m,0} \oplus \mathcal{V}_1$.
2. $\pi_* \mathcal{O}_Z(m \mathbb{S}) \cong \ker \beta_m = \mathcal{V}_{m,0} \oplus \ker \beta$.
3. $\beta_m \mid \mathcal{V}_1 = \beta$.

Proof. The proof is totally analogous to the proof given in Proposition 2.17 with main exception that $\ker \beta$ needs not to be trivial in this higher dimensional case. Below, we will provide enough details.

(1) is just a rephrase of Lemma 2.33. We will prove the rest all together.

The initial case is when all $m_j = 1, 1 \leq j \leq r$. Denote $(1, \cdots, 1) \in \mathbb{Z}^r$ by $1$. We have

$$
\pi_* \mathcal{O}_Z(1 \cdot \mathbb{S} + D) = \mathcal{O}(\mathbb{P}_1)^r \oplus \mathcal{V}_1.
$$

Hence

$$
0 \longrightarrow \pi_* \mathcal{O}_Z(1 \cdot \mathbb{S}) \overset{\alpha_1}{\longrightarrow} \mathcal{O}(\mathbb{P}_1)^r \oplus \mathcal{V}_1 \overset{\beta_1}{\longrightarrow} \mathcal{O}(\mathbb{P}_1)^r \overset{\gamma_1}{\longrightarrow} R^1 \pi_* \mathcal{O}_Z(1 \cdot \mathbb{S}) \longrightarrow 0
$$

So, we can express $\beta_1$ as

$$(h_0, h_1, \cdots, h_r) \mapsto c_0 h_0 + \sum_i c_i h_i$$

where $h_0, h_1 \in \mathcal{O}(\mathbb{P}_1)^r$, $c_0$ is a fixed scalar, and $c_i(t_i, s_i) \in H^0(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(1))$ are fixed sections. The same argument as in the case of $r = 1$ implies that $c_0 = 0$ and $t_i \mid c_i$. By changing coordinates if necessary, we can write $c_i = t_i$. Then $(h_0, h_1, \cdots, h_r) \in \ker \beta_1$ if and only if $\sum_i t_i h_i = 0$.

\footnote{When some $m_j = 0$, we can ignore $W_j, S_j$, and the family $C_j$ from the consideration, this tail will not affect the sheaf $\pi_* \mathcal{O}_Z(m \mathbb{S})$, and the problem is reduced to the previous $(r - 1)$-tail case. This is analogous to the situation of §2.3.}
That is, \( \ker \beta_1 = \ker \beta \oplus \mathcal{O}(p_1)_Y \). Hence the case \( m = 1 \) of the lemma follows.

When \( m \geq 1 \), meaning all \( m_j \geq 1 \), we have

\[
\begin{array}{ccc}
0 & \longrightarrow & \pi_\ast \mathcal{O}_Z(\sum_i S_i) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \pi_\ast \mathcal{O}_Z(m\mathcal{S})
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
& & \\
& & \\
\downarrow & & \downarrow \\
& & \\
& & \\
\beta_m & \longrightarrow & \mathcal{O}_{\mathbb{P}^1} \\
& & \\
& & \\
& & \\
\gamma_m & \longrightarrow & R^1\pi_\ast \mathcal{O}_Z(\sum_i S_i) \longrightarrow 0
\end{array}
\]

Hence we can assume that \( \beta_m|_{V_1} = \beta \). Then the map \( \beta_m \) is given by

\[
(h_0, (h_1, \ldots, h_r), h_{1,2}, \ldots, h_{1m_1}, \ldots, h_{r,2}, \ldots, h_{r,m_r})
\]

As in Proposition 2.37, \( c_0 = 0 \) and we may write \( c_{jk} = t_j a_{jk} \) (1 \( \leq j \leq r, 2 \leq k \leq m_j \)). Hence \((h_0, h_1, \ldots, h_r, h_{1,2}, \ldots, h_{1m_1}, \ldots, h_{r,2}, \ldots, h_{r,m_r}) \in \ker \beta_m \) if and only if

\[
\sum_{j=1}^r t_j h_j + \sum_{j=1}^r \sum_{k=2}^{m_j} c_{jk} h_{jk} = 0.
\]

We can split an arbitrary point

\[
(h_0, (h_1, \ldots, h_r), h_{1,2}, \ldots, h_{1m_1}, \ldots, h_{r,2}, \ldots, h_{r,m_r})
\]

as the sum of

\[
(h_0, (\ldots, - \sum_{k=2}^{m_j} a_{jk} h_{jk}, \ldots), h_{1,2}, \ldots, h_{1m_1}, \ldots, h_{r,2}, \ldots, h_{r,m_r})
\]

and

\[
(0, (\ldots, h_j + \sum_{k=2}^{m_j} a_{jk} h_{jk}, \ldots), 0, \ldots, 0).
\]

This gives a splitting of \( \pi_\ast \mathcal{O}_Z(m\mathcal{S} + D) \) as the direct sum of \( \mathcal{V}_{m,0} \) and \( \mathcal{V}_1 \). Under this direct sum, we see that \( \ker \beta_m = \mathcal{V}_{m,0} \oplus \ker \beta \). The rest follows straightforwardly.

2.36. Clearly, \( \ker \beta \), independent of \( m \), is the sole cause for non-local freeness of the sheaf \( \pi_\ast \mathcal{O}_Z(m\mathcal{S}) \). Blowup is to make \( \ker \beta \) locally free, thus resolves the sheaf \( \pi_\ast \mathcal{O}_Z(m\mathcal{S}) \).
Remark 2.37.  

(1) Note here that the essence of the arguments on the sheaf is independent of the genus of the family.

(2) Also, the decomposition $\mathcal{V}_1 = \bigoplus \pi_i^*\mathcal{O}_{\mathbb{P}^1}(-1)$ seems to suggest a relation with either restricting the family to the coordinate directions or projections to the coordinate directions.

2.38. Let $\varphi : (\mathbb{P}^1)^r \to (\mathbb{P}^1)^r$ be the blowup at the point $\bar{p} = (p, \cdots, p)$. We have the square

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\psi} & Z \\
\pi \downarrow & & \downarrow \pi \\
(\mathbb{P}^1)^r & \xrightarrow{\varphi} & (\mathbb{P}^1)^r.
\end{array}
\]

Proposition 2.39. The sheaf $\tilde{\pi}_*\psi^*\mathcal{O}_Z(mS)$ is locally free.

Proof. This can be checked directly. (See [1] for a general proof. The reader is encouraged to do this example.) \(\square\)

2.40. A further example.

To gain further intuition around deeper strata of the moduli space $\mathcal{M}_1(\mathbb{P}^n, d)$, let us look at the stratum of the type $o[a[b, c]]$. The interested reader is referred to [1] for further explanation of this notational scheme; he is also encouraged to construct a family for the second curve in Figure 5. Here, any curve in the stratum consists of a smooth elliptic curve $C_o$, a rational curve $C_a$ attached to $C_o$, and two more rational curves $C_b$ and $C_c$ attached to $C_a$. A line bundle over the family of the interest is trivial on both $C_o$ and $C_a$, and positive on $C_b$ and $C_c$.

We will construct explicitly such a family $Z$ over a three dimensional base $B$. We work over $\mathbb{C}$ in this subsection.

2.41. Let $Z_a$ be the family of nodal elliptic curve constructed in §2.2. That is, $Z_a$ is the blowup of $\mathbb{P}^1 \times E$ at $(p, e_0)$. We would like to have two disjoint sections that go through the rational curve of the central fiber. So, we take a small analytic disc $\Delta$ of the base center at $p$. We denote the family restricted to $\Delta$ by $Z_a|_{\Delta}$. So, let $S_{a,i}$ be two disjoint sections of $Z_a|_{\Delta}$ that go through the rational curve of the central fiber, $i = b, c$. Let $D_a$ be the proper transform of $\mathbb{P}^1 \times e$ in $Z_a$, $e \neq e_0$.

2.42. Consider $Z_a|_{\Delta} \times \mathbb{P}^1 \times \mathbb{P}^1$. Let $W_b = S_{a,b} \times p \times \mathbb{P}^1$ and $W_c = S_{a,c} \times \mathbb{P}^1 \times p$. Let $Z$ be the blowup of $Z_a|_{\Delta} \times \mathbb{P}^1 \times \mathbb{P}^1$ along $W_b \coprod W_c$.

Then we obtain a family of nodal elliptic curve over $B = \Delta \times \mathbb{P}^1 \times \mathbb{P}^1$.

\[Z \to B\]
such that: moving along the first coordinate direction, the node $a$ is smoothed, but the elliptic curve comes with two solid tails; moving along the second direction, the node $b$ is smoothed; moving along the last direction, the node $c$ is smoothed.

Figure 5. Two elliptic curves and two trees

2.43. Let $D$ be the proper transform of $D_a|_{\Delta \times \mathbb{P}^1 \times \mathbb{P}^1}$ in $Z$, $S_b$ the proper transform of $S_{a,b} \times \mathbb{P}^1 \times \mathbb{P}^1$, and $S_c$ the proper transform of $S_{a,c} \times \mathbb{P}^1 \times \mathbb{P}^1$.

Let $V_a = p \times \mathbb{P}^1 \times \mathbb{P}^1$, $V_b = p \times \mathbb{P}^1 \times \mathbb{P}^1$, $V_c = \mathbb{P}^1 \times p \times \mathbb{P}^1$.

Let $\pi_i$ be the projection of $\Delta \times \mathbb{P}^1 \times \mathbb{P}^1$ to the $i$-th factor, $i = a, b, c$.

A main new feature in this example is

Lemma 2.44.

$$\pi_* \mathcal{O}_Z(S_b)|_{S_b} = \mathcal{O}_B(-V_b - V_a) = \pi^*_a \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \pi^*_b \mathcal{O}_{\mathbb{P}^1}(-1).$$

$$\pi_* \mathcal{O}_Z(S_c)|_{S_c} = \mathcal{O}_B(-V_c - V_a) = \pi^*_a \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \pi^*_c \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. Both can be checked directly by calculating the degree on each coordinate direction.
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Let $\bar{p} = (p, e_0, p, p) \in B$. Then, almost identical arguments as in the sections §2.2 and 2.4 will lead to a canonical homomorphism

$$O_B(-V_b - V_a) \oplus O_B(-V_c - V_a) \overset{\beta}{\longrightarrow} O_B$$

$$(h_a, h_b, h_c) \longrightarrow (t_a t_b) h_b + (t_a t_c) h_c = t_a(t_b h_b + t_c h_c).$$

Hence we have,

**Proposition 2.45.** Up to isomorphism, we have a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \ker \beta \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \pi_* O_Z(S_b + S_c) \\
\beta & \longrightarrow & \pi_* O_Z(S_b + S_c + D) \\
\downarrow & & \downarrow \\
\beta_S \cdot O_B & \longrightarrow & R^1 \pi_* O_Z(S_b + S_c) \\
\end{array}
$$

where the first two downward arrows are inclusions, and the last two are isomorphisms. Moreover, we have

1. $\pi_* O_Z(S_b + S_c + D) = O_B \oplus O_B(-V_b - V_a) \oplus O_B(-V_c - V_a).$
2. $\pi_* O_Z(S_b + S_c) \cong \ker \beta_S = O_B \oplus \ker \beta.$
3. $\beta_S|_{O_B(-V_b - V_a) \oplus O_B(-V_c - V_a)} = \beta.$

**Proof.** Parallel to to the proofs in §2.2 and 2.4. □

**2.46.** In the next two subsections §2.6 and 2.7 we will give interpretations of the main results of [1] on the direct image sheaves in the spirit of the examples that the reader has already been through. We refer the proofs to [1].

2.6. The direct image sheaf on the moduli space: simple case.

**2.47.** Let $\pi : X \longrightarrow V$ be a flat family of nodal genus 1 curves. Let $0 \in V$ be a fixed point with the fiber

$$X_0 = \bigcup_{i=0}^r C_i$$

such that $C_0$ is a nodal elliptic curve, $C_i$ are smooth rational curves attached to $C_0$ at distinct smooth points.
2.48. We will first make sense of the subscheme $V_i \subset V$ that are locus of the $i$-th tail not smoothed ($1 \leq i \leq r$). We let $p : \mathcal{X} \to \mathcal{Y}$ be the stabilization\footnote{In this case, this means that the morphism $p$ contracts any rational components that has fewer than three nodes.}, let $\mathcal{E} \subset \mathcal{X}$ be the exceptional divisors, let $\mathcal{E}_i$ be the irreducible component of $\mathcal{E}$ that contains $C_i \subset \mathcal{X}_0$. We define

$$V_i = \pi(\mathcal{E}_i).$$

Lemma 2.49. The subset $V_i \subset V$ is canonically a closed subscheme whose ideal sheaf $\mathcal{I}_{V_i} \subset V$ is generated by a regular function $t_i \in \Gamma(\mathcal{O}_V)$.

So we can write $V_i = \{ t_i = 0 \}$. For every $1 \leq i \leq r$, let $\mathcal{X}_i$ be the restriction of the total family to $V_i$. Choose a section $S_i$ over $V$ that meets $\mathcal{E}_i$ ($1 \leq i \leq r$), and a generic section $D$ that misses all $\mathcal{E}_i$. All these can be done by shrinking $V$ if necessary.

2.50. Again we will consider sheaves of the form

$$\pi_* \mathcal{O}_X(\mathcal{mS}), \quad \mathcal{m} \geq 1.$$ 

We have a short exact sequence

$$0 \to \mathcal{O}_X(\mathcal{mS}) \to \mathcal{O}_X(\mathcal{mS} + D) \to \mathcal{O}_X(\mathcal{mS} + D)|_D \to 0$$

and a long exact sequence

$$0 \to \pi_* \mathcal{O}_X(\mathcal{mS}) \to \pi_* \mathcal{O}_C(\mathcal{mS} + D) \to$$

$$\pi_* \mathcal{O}_X(\mathcal{mS} + D)|_D \to R^1 \pi_* \mathcal{O}_X(\mathcal{mS}) \to 0. \tag{2.14}$$

2.51. Since $V$ is affine, we have an isomorphism $\pi_* \mathcal{O}_V(\mathcal{mS} + D)|_D(V) \cong \mathcal{O}_V$. We fix such an isomorphism. As in §2.4, consider the short exact sequence:

$$0 \to \mathcal{O}_X(\mathcal{mS} + D) \to \mathcal{O}_X(\mathcal{mS} + S_j + D) \to \mathcal{O}_X(\mathcal{mS} + S_j + D)|_S \to 0.$$

Because $\mathcal{O}_X(\mathcal{mS} + S_j + D)|_{S_j} = \mathcal{O}_X(S_j)^{m_j + 1}|_{S_j}$, we have $\pi_* \mathcal{O}_X(\mathcal{mS} + S_j + D)|_{S_j} = \mathcal{O}_V(-(m_j + 1)V_j)$, hence we obtain a short exact sequence of locally free sheaves

$$0 \to \pi_* \mathcal{O}_X(\mathcal{mS} + D) \to \pi_* \mathcal{O}_X(\mathcal{mS} + S_j + D) \to \mathcal{O}_V(-(m_j + 1)V_j) \to 0 \tag{2.15}$$

because $R^1 \pi_* \mathcal{O}_Z(\mathcal{mS} + D) = 0$. Then by arguing that $\text{Ext}^1 = 0$, we obtain that the sequence splits; then by induction, we get

Lemma 2.52. The sequence (2.15) splits and consequently

$$\pi_* \mathcal{O}_X(\mathcal{mS} + D) = \mathcal{O}_V \oplus \bigoplus_{i=1}^r \bigoplus_{k=1}^{m_i} \mathcal{O}_V(-kV_j).$$
2.53. Let
\[ Y_1 = \bigoplus_{i=1}^{r} \mathcal{O}_V(-V_i). \]

Then we have a canonical map \( \beta \),
\[
\begin{array}{ccc}
Y_1 & \longrightarrow & \mathcal{O}_V \\
(h_1, \ldots, h_r) & \longmapsto & \sum_i t_i h_i 
\end{array}
\]
which induces a canonical exact sequence
\[
0 \longrightarrow \ker \beta \longrightarrow Y_1 \xrightarrow{\beta} \mathcal{O}_V \xrightarrow{\gamma} \mathcal{O}_V|_W \longrightarrow 0
\]
where \( W = \cap V_i \).

Set
\[
Y_{m,0} = \mathcal{O}_V \oplus \bigoplus_{j=1}^{r} \bigoplus_{k=2}^{m_j} \mathcal{O}_V(-kV_j).
\]

We have

**Proposition 2.54.** Assume that \( m_j \neq 0 \) for all \( 1 \leq j \leq r \). Then up to isomorphism, we have a commutative diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & \ker \beta \\
& \searrow & \downarrow \\
& & Y_1 \\
0 & \longrightarrow & \pi_* \mathcal{O}_X(mS) \xrightarrow{\alpha_m} \pi_* \mathcal{O}_X(mS + D) \\
& \xrightarrow{\beta} & 0 \\
& \downarrow & \downarrow \\
& & \pi_* \mathcal{O}_X(mS) \\
& \xrightarrow{\beta_m} & \pi_* \mathcal{O}_X(mS + D) \\
& \downarrow & \downarrow \\
& & R^1 \pi_* \mathcal{O}_X(mS) \\
& \xrightarrow{\gamma_m} & 0 \\
\end{array}
\]
where the first two downward arrows are inclusions, and the last two are isomorphisms. Moreover, we have
\[
\begin{align*}
(1) & \quad \pi_* \mathcal{O}_X(mS + D) = Y_{m,0} \oplus Y_1. \\
(2) & \quad \pi_* \mathcal{O}_X(mS) \cong \ker \beta_m = Y_{m,0} \oplus \ker \beta. \\
(3) & \quad \beta_m|_{Y_1} = \beta.
\end{align*}
\]

**Proof.** It follows from the general proof in [1]. \( \square \)

\[\footnote{It would be nice to find an elementary proof of this proposition.}\]
2.7. The direct image sheaf around deeper strata: general case.

2.55. We let $\mathcal{X} \to V$ be a flat family of nodal curves of arithmetic genus one. We let $0 \in V$ be a fixed point with the fiber

$$\mathcal{X}_0 = \bigcup_{i=0}^r C_i$$

such that $C_0$ is a nodal elliptic curve, $C_i$ are connected trees of rational curves attached to $C_0$ at distinct smooth points.

2.56. For each $1 \leq i \leq r$, let $C_i^0$ be the irreducible component of $C_i$ that meets $C_0$. Note that as a nodal elliptic curve, $C_0$ contains either a smooth elliptic curve or a unique loop of rational curves. This is the core $C_e$ of $\mathcal{X}_0$.

2.57. We let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$ whose restriction to $C_0$ is trivial and whose restriction to $C_i^0$ are effective and with positive degree ($1 \leq i \leq r$). Thus, $C_0$ is the largest subcurve of $\mathcal{X}_0$ containing the core $C_e$ such that $\mathcal{L}|_{C_0}$ is trivial.

2.58. As in §2.6 by shrinking $V$ if necessary, we can choose sections $S_i$, $1 \leq i \leq r$, and $D$ such that $\mathcal{L}$ is of the form $\mathcal{O}_{\mathcal{X}}(mS)$ for some $m \geq 1$.

2.59. For each $C_i$, $1 \leq i \leq r$, let

$$C_{i_1}, \ldots, C_{i_{m_i}}$$

be the list of all irreducible (ghost) rational curves through which $C_i$ is attached to $C_e$. Observe that some (ghost) rational curves $C_{ih}$ are allowed to repeat as $C_{jk}$ for some $j \neq i$.

2.60. We have local variables $t_i$, $t_{ij}$, $1 \leq i \leq r$, $1 \leq j \leq n_i$, each of which is the smoothing parameter of the corresponding node. Let $V_i = \{t_i = 0\}$ and $V_{ij} = \{t_{ij} = 0\}$, $1 \leq i \leq r$, $1 \leq j \leq n_i$.

Let $\mathcal{V}_1 := \mathcal{V}_{\gamma,1} = \bigoplus_{i=1}^r \mathcal{O}_{\mathcal{V}}(-V_i - \sum_{j=1}^{n_i} V_{ij})$.

2.61. The reason that $\mathcal{O}_{\mathcal{V}}(-V_i - \sum_{j=1}^{n_i} V_{ij})$ occurs is because of the sequence

$$0 \to \mathcal{O}_{\mathcal{X}}(mS+D) \to \mathcal{O}_{\mathcal{X}}(mS+S_j+D) \to \mathcal{O}_{\mathcal{X}}(mS+S_j+D)|_S \to 0$$

and $\pi_* \mathcal{O}_{\mathcal{Z}}(mS + S_j + D)|_{S_i} = \mathcal{O}_{\mathcal{Z}}(S_i)|_{S_i} = \mathcal{O}_{\mathcal{V}}(-V_i - \sum_{j=1}^{n_i} V_{ij})$. 
Then we have a canonical map $\beta$,
\[
V_1 \longrightarrow \mathcal{O}_V
\]
which induces a canonical exact sequence
\[
0 \longrightarrow \ker \beta \longrightarrow V_1 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V|_W \longrightarrow 0
\]
where $W = \bigcap_{i=1}^r V_i$.

Set
\[
V_{m,0} := V_{\gamma, m, 0} = \mathcal{O}_V \oplus \bigoplus_{i=1}^r \bigoplus_{k=2}^{m_i} \mathcal{O}_V(-k(V_i + \sum_{j=1}^{n_i} V_{ij})).
\]

**Proposition 2.64.** Assume that $m_j \neq 0$ for all $1 \leq j \leq r$. Then up to isomorphism, we have a commutative diagram
\[
\begin{array}{cccccc}
0 & \longrightarrow & \ker \beta & \longrightarrow & \mathcal{V}_1 & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \pi_* \mathcal{O}_X(mS) & \longrightarrow & \mathcal{O}_V & \longrightarrow \mathcal{O}_V|_W \longrightarrow 0 \\
\downarrow \beta_m & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathcal{O}_V & \longrightarrow & R^1\pi_* \mathcal{O}_X(mS) \longrightarrow 0
\end{array}
\]
where the first two downward arrows are inclusions, and the last two are isomorphisms. Moreover, we have

1. $\pi_* \mathcal{O}_X(mS + D) = \mathcal{V}_{m,0} \oplus \mathcal{V}_1$.
2. $\pi_* \mathcal{O}_X(mS) \cong \ker \beta_m = \mathcal{V}_{m,0} \oplus \ker \beta$.
3. $\beta_m|_{\mathcal{V}_1} = \beta$.

**Proof.** Let $V' = V_i + \sum_{j=1}^{n_i} V_{ij}$. Then this reduces to the case of $r$-tails (Proposition 2.54). See also [1] for the general proof.

3. Extensions of sections on the central fiber

### 3.1. Motivation.

Consider the flat family of nodal elliptic curves $\pi : Z \longrightarrow B$. In this section, we look at sections in $H^0(Z_b, \mathcal{O}_{Z_b}(mS))$ over the central fiber and study their extensions by deformation theory. This is useful because of the following. Set theoretically, our deformation space is the union
\[
\bigcup_{b \in B} H^0(Z_b, \mathcal{O}_Z(mS)|_{Z_b})^\oplus n.
\]
The deformation space is singular at 0 if the core curve of $Z_0$ is ghost due to the nonvanishing of the first cohomology. Naively, this is caused by the existence of some sections in $H^0(Z_0, \mathcal{O}_{Z_0}(mS))$ that do not extend to neighboring fibers. Thus, it is instrumental to determine, at least for some examples, exactly what sections can be extended and what sections can not. (The singularities occurring in the genus one case is relatively simple comparing to the more general case. For more general deformation of sections of invertible sheaves, one may consult the book “Deformations of Algebraic Schemes” by Edoardo Sernesi.)

3.2. We will deal exclusively with the example of the 1-tail case as in §2.2. However, both the results and arguments work for arbitrary flat families of nodal elliptic curves.

3.2. Sections on the central fiber.

3.3. Let $b : Z \to \mathbb{A}^1 \times E$ be the blowup of $\mathbb{A}^1 \times E$ at $(0, e_0)$, where $e_0$ is a fixed point on $E$. Let $\pi : \mathbb{A}^1 \times E \to \mathbb{A}^1$ be the projection to the first factor and $\pi = \pi_1 \circ b$.

3.4. Choose a generic point $e \in E$. Let $S, D$ be the proper transform of $\mathbb{A}^1 \times e_0$ and $\mathbb{A}^1 \times e$, respectively. We will consider the direct image sheaf $L_m = \pi_{\ast} \mathcal{O}_{Z(mS)}$.

3.5. We write the central fiber $Z_0$ of $\pi$ as

$$Z_0 = C \cup C_a$$

where $C$ is elliptic and $C_a$ is rational. Choose homogeneous coordinates $[u_0, u_1]$ of $C_a$ such that $C \cap C_a = [1, 0]$ and $S \cap C_a = [0, 1]$. Then the space of sections of $\mathcal{O}_Z(mS)$ restricted to $C_a$ has a basis

$$u_1^m, u_0^{m-1}u_1, \ldots, u_1^m.$$ 

Since $\mathcal{O}_Z(mS)$ restricted to $C$ is trivial, sections on $C$ are constant, hence these constants are uniquely determined by the values of sections on $C_a$ at the point $C \cap C_a = [1, 0]$. This shows that $\mathcal{O}_Z(mS)$ restricted to $Z_0$ has $m + 1$ dimensional global sections spanned by

$$u_0^m, u_0^{m-1}u_1, \ldots, u_0u_1^{m-1}, u_1^m.$$ 

3.6. We first observe that $u_0^m$ extends to $Z$. Indeed, the tautological inclusion $\mathcal{O}_Z \to \mathcal{O}_Z(mS)$ induces a section $1 \in \Gamma(Z, \mathcal{O}_Z(mS))$ that vanishes along $S$ of order $m$. Thus this section restricts $Z_0$ must be the $u_0^m$ mentioned.

3.7. When $m = 1$, we claim that $u_1$ is the section that does not extend to the nearby fiber. For this, note that $u_1$ does not vanish on $S \cap C_a =
Consider

\[ \bigcup_{t \neq 0} \mathcal{O}_{Z_t}(S_t) \]

where \( S_t = S \cap Z_t \), then a section at a nearby fiber must vanish at the point \( S_t \). As \( t \) specializes to 0, the limit section on the central fiber must vanish at the point \( S \cap Z_0 = S \cap C_a = [0, 1] \). Combined, since \( u_0 \) extends, \( au_0 + bu_1 \) extends if and only if \( b = 0 \).

### 3.3. The first order extensions.

#### 3.8. We need to investigate how sections

\[ u_0^m, u_0^{m-1}u_1, \cdots, u_1^m \]

in \( H^0(Z_0, \mathcal{O}_Z(mS)) \) can be extended to

\[ H^0(kZ_0, \mathcal{O}_Z(mS)) \text{ for } k \geq 2. \]

To achieve this, we use Gieseker’s idea and instead look at

\[ H^0(kC + (k-1)C_a, \mathcal{O}_Z(mS)). \]

#### 3.9. We begin with studying the sheaf \( \mathcal{O}_Z(mS)|_{2C+C_a} \) and its global sections \( H^0(2C+C_a, \mathcal{O}_Z(mS)) \), which will be determined by the long exact sequence of cohomology of the exact sequence

\[ 0 \rightarrow \mathcal{O}_Z(mS - C)|_C \rightarrow \mathcal{O}_Z(mS)|_{2C} \rightarrow \mathcal{O}_Z(mS)|_C \rightarrow 0. \]

Alternatively, this is just

\[ 0 \rightarrow I/I^2 \rightarrow \mathcal{O}_Z/I^2 \rightarrow \mathcal{O}_Z/I \rightarrow 0 \]

where \( I \) is the ideal sheaf of \( C \). In particular, \( \mathcal{O}_Z(mS - C)|_C = N^\vee_{C\setminus Z} \).

By adjunction formula,

\[ N^\vee_{C\setminus Z} = K_Z|_C = \mathcal{O}_Z(E)|_C = \mathcal{O}_C(C \cap C_a) = \mathcal{O}_C(q) \]

where \( q = C \cap C_a \). As a consequence \( H^0(N^\vee_{C\setminus Z}) \cong k \) and \( H^1(N^\vee_{C\setminus Z}) = 0 \).

#### 3.10. From the exact sequence (3.1), we have

\[ 0 \rightarrow H^0(N^\vee_{C\setminus Z}) \rightarrow H^0(\mathcal{O}_Z(mS)|_{2C}) \rightarrow H^0(\mathcal{O}_Z(mS)|_C) \rightarrow 0. \]

Via the inclusion \( \mathcal{O}_{2C}(-C) \subset \mathcal{O}_{2C} \), \( H^0(N^\vee_{C\setminus Z}) \) has the standard section \( t \) which restricts to zero in \( \mathcal{O}_{2C\cap C_a} \).

#### 3.11. Since \( H^0(\mathcal{O}_Z(mS)|_C) \) consists of constant sections, by lifting constant sections over \( C \) to constant sections over \( 2C \), we obtain a canonical split

\[ H^0(\mathcal{O}_Z(mS)|_{2C}) = H^0(\mathcal{O}_Z(mS)|_C) \oplus H^0(N^\vee_{C\setminus Z}). \]
Thus, an arbitrary element of $H^0(\mathcal{O}_Z(mS)|_{2C})$ can be (canonically) expressed as $a + bt$ for scalars $a$ and $b$.

**3.12.** Then consider the exact sequence

$$
\begin{align*}
0 \rightarrow & \quad \mathcal{O}_Z(mS)|_{2C+C_a} \rightarrow \mathcal{O}_Z(mS)|_{2C} \oplus \mathcal{O}_Z(mS)|_{C_a} \\
\xrightarrow{(\phi_1,\phi_2)} & \quad \mathcal{O}_{2C\cap C_a} \rightarrow 0.
\end{align*}
$$

First, we have

$$
\mathcal{O}_{2C\cap C_a} = k[u_1]/(u_1^2).
$$

We already know that

$$
H^0(\mathcal{O}_Z(mS)|_{C_a}) = \text{span of } u_0^m, u_0^{m-1}u_1, \ldots, u_1^m.
$$

Let $\phi_i$ be as indicated in the sequence (3.2), then sections of $H^0(2C + C_a, \mathcal{O}_Z(mS))$ are pairs

$$(w_1, w_2) \in H^0(\mathcal{O}_Z(mS)|_{2C}) \times H^0(\mathcal{O}_Z(mS)|_{C_a})$$

such that

$$\phi_1(w_1) = \phi_2(w_2).$$

We can write $w_1 = a + bt$ and $w_2 = \sum_i a_i u_0^{m-i}u_1^i$, by solving

$$\phi_1(w_1) = a = \phi_2(w_2) = a_0 + a_1 u_1,$$

we see that $a = a_0$ and $a_1 = 0$. Hence we conclude

$$H^0(2C + C_a, \mathcal{O}_Z(mS)) = \text{span of the extensions of } u_0^m, u_0^{m-2}u_2, \ldots, u_1^m.
$$

**3.4. Further extensions.** We now let $W_n = (n + 1)C + nC_a$, viewed as a subscheme of $Z$.

**3.13.** In the previous subsection, we have already determined the space $H^0(W_1, \mathcal{O}_Z(mS))$. To find sections of $H^0(W_k, \mathcal{O}_Z(mS))$, we shall use induction based on the exact sequence

$$
\begin{align*}
0 \rightarrow & \quad \mathcal{O}_Z(mS - W_k)|_{Z_0} \rightarrow \mathcal{O}_Z(mS)|_{W_{k+1}} \rightarrow \mathcal{O}_Z(mS)|_{W_k} \rightarrow 0.
\end{align*}
$$

Taking global sections, we obtain exact sequence

$$
\begin{align*}
0 \rightarrow & \quad H^0(\mathcal{O}_Z(mS - W_k)|_{Z_0}) \rightarrow H^0(\mathcal{O}_Z(mS)|_{W_{k+1}}) \xrightarrow{\alpha} H^0(\mathcal{O}_Z(mS)|_{W_k}) \\
\rightarrow & \quad H^1(\mathcal{O}_Z(mS - W_k)|_{Z_0})
\end{align*}
$$

Because $W_k = kZ_0 + C$ and $\mathcal{O}_Z(Z_0) \cong \mathcal{O}_Z$,

$$
\mathcal{O}_Z(mS - W_k)|_{Z_0} \cong \mathcal{O}_Z(mS - C)|_{Z_0}.
$$

**3.14. Claim:** $H^1(\mathcal{O}_Z(mS - W_k)|_{Z_0}) = H^1(\mathcal{O}_Z(mS - C)|_{Z_0}) = 0.$
3.15. To prove the claim, we use an exact sequence similar to (3.2)
\[ 0 \rightarrow \mathcal{O}_Z(mS - C)|_{Z_0} \rightarrow \mathcal{O}_Z(mS - C)|_C \oplus \mathcal{O}_Z(mS - C)|_{C_a} \rightarrow \mathcal{O}_{C\cap C_a} \rightarrow 0. \]

Then taking cohomology, we have
\[ 0 \rightarrow H^0(\mathcal{O}_Z(mS - C)|_{Z_0}) \rightarrow H^0(\mathcal{O}_Z(mS - C)|_C) \oplus H^0(\mathcal{O}_Z(mS - C)|_{C_a}) \rightarrow \mathcal{O}_{C\cap C_a} \rightarrow H^1(\mathcal{O}_Z(mS - C)|_{Z_0}) \rightarrow H^1(\mathcal{O}_Z(mS - C)|_C) \oplus H^1(\mathcal{O}_Z(mS - C)|_{C_a}) \rightarrow 0. \]

\( (\phi_1, \phi_2) \) is easily seen to be surjective. Also, we have
\[ H^1(\mathcal{O}_Z(mS - C)|_C) = H^1(\mathcal{O}_C(q)) = 0, \]
\[ H^1(\mathcal{O}_Z(mS - C)|_{C_a}) = H^1(\mathcal{O}_{C_a}(m - 1)) = 0. \]

Thus the claim is proved.

3.16. From the claim, and by induction on \( k \), all the sections in \( H^0(\mathcal{O}_Z(mS)|_{W_k}) \) (\( k \geq 1 \)) extend to \( H^0(\mathcal{O}_Z(mS)|_{W_{k+1}}) \).

3.5. The other sheaves.

3.17. Let \( D_0 \) and \( D_1 \) be two sections of \( Z/\mathbb{P}_1 \) passing through two general points of \( C \). We now look at the sheaf
\[ \mathcal{M} = \mathcal{O}_Z(mS + D_0 - D_1). \]

3.18. First observe that \( H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{C_a}) = H^0(\mathcal{O}_{C_a}(m)) \) has the space of sections generated by
\[ u_0^m, u_0^{m-1}u_1, \ldots, u_1^m. \]

Next, using the sequence
\[ 0 \rightarrow \mathcal{O}_Z(mS + D_0 - D_1)|_{Z_0} \rightarrow \mathcal{O}_Z(mS + D_0 - D_1)|_C \oplus \mathcal{O}_Z(mS + D_0 - D_1)|_{C_a} \rightarrow \mathcal{O}_{C\cap C_a} \rightarrow 0, \]
and the fact that
\[ H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_C) = H^0(\mathcal{O}_C(q_0 - q_1)) = 0 \]
where \( q_i = D_i \cap C, i = 0, 1 \), we see that each \( u_0^i u_1^{m-i}, i > 0 \), extends by zero to \( \mathcal{O}_Z(mS + D_0 - D_1) \). Thus
\[ H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{Z_0}) = \text{span of the extensions of } u_0^{m-1}u_1, \ldots, u_1^m. \]
To proceed further, we need to determine $H^0(\mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1))$. For this, we look at
\begin{equation}
0 \to \mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1) \to \mathcal{O}_Z(mS + D_0 - D_1)|_{2\mathbb{C}} \to 0.
\end{equation}

Note that since $\mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1)$ is annihilated by $\mathcal{O}_{C_{2\mathbb{C}}}$, it is naturally a sheaf of $\mathcal{O}_C$-modules. And as such, $\mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1) \cong \mathcal{O}_C(q + q_0 - q_1)$ (recall that $q = C \cap C_a$). Taking the long exact sequence, we then get
\begin{equation}
0 \to H^0(\mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1)) \to H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{2\mathbb{C}}).
\end{equation}

Because $\mathcal{O}_Z(mS + D_0 - D_1)|_{C} = \mathcal{O}_C(q_0 - q_1)$, we conclude that the last term is zero. Hence we obtain
\begin{align*}
H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{2\mathbb{C}}) &= H^0(\mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1)) \\
&= H^0(\mathcal{O}_C(q + q_0 - q_1)) = k
\end{align*}
where the last isomorphism holds by Riemann-Roch.

We claim that the evaluation homomorphism
\begin{equation}
\phi_1 : H^0(\mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1)) \to \mathcal{O}_{2\mathbb{C} \cap C_a}
\end{equation}
has image $k \cdot u_1 \subset k[u_1]/(u_1^2) = \mathcal{O}_{2\mathbb{C} \cap C_a}$. Indeed, by evaluating sections of $\mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1)$, we obtain an exact sequence
\begin{equation}
0 \to \mathcal{I}_{Z_a \cap 2\mathbb{C}C_{2\mathbb{C}}}(mS + D_0 - D_1) \to \mathcal{I}_{C_{2\mathbb{C}}}(mS + D_0 - D_1) \to k \cdot u_1 \to 0.
\end{equation}

Because as $\mathcal{O}_C$-modules,
\begin{equation}
\mathcal{I}_{Z_a \cap 2\mathbb{C}C_{2\mathbb{C}}}(mS + D_0 - D_1) \cong \mathcal{O}_C(q + q_0 - q_1 - q);
\end{equation}
thus its first cohomology group vanishes. This proves the claim.

Now consider the sequence
\begin{equation}
0 \to H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{2\mathbb{C} + C_a}) \to H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{2\mathbb{C}}) \oplus H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{C_a}) \xrightarrow{\phi_1 \oplus \phi_2} H^0(\mathcal{O}_{2\mathbb{C} \cap C_a}).
\end{equation}

By the above, we may choose a generator $s$ of $H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{2\mathbb{C}})$ so that its image under $\phi_1$ is $u_1$. Then it is easy to see that $\sum_{i \geq 0} a_i u_0^{m-1-i} u_1^i$ extends to $H^0(\mathcal{O}_Z(mS + D_0 - D_1)|_{2\mathbb{C} + C_a})$ if and only if $a_0 = 0$. So, we find that
\begin{equation}
H^0(W_1, \mathcal{M}) = \text{span of extensions of } u_0^{m-1} u_1, \ldots, u_1^m.
This is the same as $H^0(Z_0, \mathcal{M})$.

3.22. Applying the parallel method as in the previous discussion, one can find that

$$H^1(\mathcal{O}_Z(mS + D_0 - D_1 - W_k)|_{Z_0}) = H^1(\mathcal{O}_Z(mS + D_0 - D_1 - C)|_{Z_0}) = 0$$

and then conclude that there are no obstructions to extend all sections from $H^0(W_1, \mathcal{M})$ to $H^0(W_k, \mathcal{M})$.

3.6. The trivialization of $\mathcal{O}_Z(mS)$.

3.23. We fix an analytic coordinate $z$ of the elliptic curve $E$ over $e_0 \in U_0 \subset E$ so $e_0 = (z = 0)$; we let $t$ be the standard coordinate of $\mathbb{A}^1$.

3.24. We let $E \subset Z$ be the exceptional divisor; we let $U \subset Z$ be the analytic neighborhood $\pi_2^{-1}(U_0)$ of $E \subset Z$. We let $[u_0, u_1]$ be the homogeneous coordinate of $E$ so that over $U$:

$$u_0z = u_1t.$$ 

We let $V_0 = U \cap (u_1 = 1)$, $V_1 = U \cap (u_0 = 1)$ and $V_2 = (C - e_0) \times \mathbb{A}^1$.

Then we have

1. the transition from $V_0$ to $V_2$: $(u_0, z) \mapsto (z, t) = (z, u_0z)$;
2. the transition from $V_1$ to $V_2$: $(u_1, t) \mapsto (z, t) = (u_1t, t)$;
3. the transition from $V_0$ to $V_1$: $(u_0, z) \mapsto (u_1, t) = (1/u_0, u_0^2z)$.

3.25. This way, we cover $Z$ by the open subsets: $V_0$, $V_1$ and $V_2$.

3.26. Since $S \cap V_2 = \emptyset$, $\mathcal{O}_Z(mS)|_{V_2}$ has an obvious trivialization. Also, we assume that $S \cap V_1 = \emptyset$, hence $\mathcal{O}_Z(mS)|_{V_1}$ also an obvious trivialization. We assume that $S$ intersects $E$ at $([0, 1], 0) \in V_0$.

3.27. We use the following local trivializations of $\mathcal{O}_Z(mS)$:

$$(u_0, z, \eta_0) \in \mathcal{O}_Z(mS)|_{V_0} \cong V_0 \times \mathbb{C},$$
$$(u_1, t, \eta_1) \in \mathcal{O}_Z(mS)|_{V_1} \cong V_1 \times \mathbb{C},$$
$$(z, t, \eta_2) \in \mathcal{O}_Z(mS)|_{V_2} \cong V_2 \times \mathbb{C}.$$ 

From $V_0$ to $V_1$, the transition function is the same as the transition function for $\mathcal{O}_{\mathbb{P}^1}(m)$, hence we have

$$u_0^m \eta_1 = \eta_0 u_1^m.$$ 

Since we use 1 to trivialize both $V_1$ and $V_2$, the transition from $V_1$ and $V_2$ is just identity

$$\eta_2 = \eta_1.$$ 

The two transitions patch all the local trivializations.
References

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