A SCHWARTZ-TYPE BOUNDARY VALUE PROBLEM IN
A BIHARMONIC PLANE

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Abstract. A commutative algebra \( \mathbb{B} \) over the field of complex numbers with the bases \( \{e_1, e_2\} \) satisfying the conditions \((e_1^2 + e_2^2)^2 = 0, e_1^2 + e_2^2 \neq 0\) is considered. The algebra \( \mathbb{B} \) is associated with the biharmonic equation. Consider a Schwartz-type boundary value problem on finding a monogenic function of the type \( \Phi(xe_1 + ye_2) = U_1(x, y)e_1 + U_2(x, y)ie_1 + U_3(x, y)e_2 + U_4(x, y)ie_2, (x, y) \in D\), when values of two components \( U_1, U_4 \) are given on the boundary of a domain \( D \) lying in the Cartesian plane \( xOy \). We develop a method of its solving which is based on expressions of monogenic functions via corresponding analytic functions of the complex variable. For a half-plane and for a disk, solutions are obtained in explicit forms by means of Schwartz-type integrals.

1. MONOGENIC FUNCTIONS IN A BIHARMONIC ALGEBRA

Definition 1. An associative commutative two-dimensional algebra \( \mathbb{B} \) with the unit \( 1 \) over the field of complex numbers \( \mathbb{C} \) is called biharmonic (see [1,2]) if in \( \mathbb{B} \) there exists a basis \( \{e_1, e_2\} \) satisfying the conditions

\[
(e_1^2 + e_2^2)^2 = 0, \quad e_1^2 + e_2^2 \neq 0.
\]

Such a basis \( \{e_1, e_2\} \) is also called biharmonic.

In the paper [2] I. P. Mel’nichenko proved that there exists the unique biharmonic algebra \( \mathbb{B} \), and he constructed all biharmonic bases in \( \mathbb{B} \). Note that the algebra \( \mathbb{B} \) is isomorphic to four-dimensional over the field of real numbers \( \mathbb{R} \) algebras considered by A. Douglis [3] and L. Sobrero [4].

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In what follows, we consider a biharmonic basis \(\{e_1, e_2\}\) with the following multiplication table (see [1]):

\[
e_1 = 1, \quad e_2^2 = e_1 + 2ie_2,
\]

where \(i\) is the imaginary complex unit. We consider also a basis \(\{1, \rho\}\) (see [2]), where a nilpotent element

\[
\rho = 2e_1 + 2ie_2
\]

satisfies the equality \(\rho^2 = 0\).

We use the euclidian norm \(\|a\| := \sqrt{|z_1|^2 + |z_2|^2}\) in the algebra \(\mathbb{B}\), where \(a = z_1 e_1 + z_2 e_2\) and \(z_1, z_2 \in \mathbb{C}\).

Consider a biharmonic plane \(\mu := \{\zeta = xe_1 + ye_2 : x, y \in \mathbb{R}\}\) which is a linear span of the elements \(e_1, e_2\) of the biharmonic basis (I) over the field of real numbers \(\mathbb{R}\). With a domain \(D\) of the Cartesian plane \(xOy\) we associate the congruent domain \(D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\}\) in the biharmonic plane \(\mu\) and the congruent domain \(D_z := \{z = x + iy : (x, y) \in D\}\) in the complex plane \(\mathbb{C}\). Its boundaries are denoted by \(\partial D, \partial D_\zeta\) and \(\partial D_z\), respectively. Let \(\overline{D_\zeta}\) (or \(\overline{D_z}\)) be the closure of domain \(D_\zeta\) (or \(D_z\)). In what follows, \(\zeta = xe_1 + ye_2, z = x + iy\) and \(x, y \in \mathbb{R}\).

Any function \(\Phi : D_\zeta \rightarrow \mathbb{B}\) has an expansion of the type

\[
\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) ie_1 + U_3(x, y) e_2 + U_4(x, y) ie_2,
\]

where \(U_l : D \rightarrow \mathbb{R}, l = 1, 2, 3, 4\), are real-valued component-functions. We shall use the following notation: \(U_l[\Phi] := U_l, l = 1, 2, 3, 4\).

**Definition 2.** A function \(\Phi : D_\zeta \rightarrow \mathbb{B}\) is monogenic in a domain \(D_\zeta\) if it has the classical derivative \(\Phi'(\zeta)\) at every point \(\zeta \in D_\zeta\):

\[
\Phi'(\zeta) := \lim_{h \to 0, h \in \mu} \left(\Phi(\zeta + h) - \Phi(\zeta)\right) h^{-1}.
\]

It is proved in [1] that a function \(\Phi : D_\zeta \rightarrow \mathbb{B}\) is monogenic in \(D_\zeta\) if and only if its each real-valued component-function in (3) is real differentiable in \(D\) and the following analog of the Cauchy – Riemann condition is fulfilled:

\[
\frac{\partial \Phi(\zeta)}{\partial y} e_1 = \frac{\partial \Phi(\zeta)}{\partial x} e_2 \quad \forall \zeta \in D_\zeta.
\]

All component-functions \(U_l, l = 1, 2, 3, 4\), in the expansion (3) of any monogenic function \(\Phi : D_\zeta \rightarrow \mathbb{B}\) are biharmonic functions (cf., e.g., [5, 6]), i.e., satisfy the biharmonic equation in \(D\):

\[
\Delta^2 U(x, y) \equiv \frac{\partial^4 U(x, y)}{\partial x^4} + 2 \frac{\partial^4 U(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 U(x, y)}{\partial y^4} = 0.
\]

Every monogenic function \(\Phi : D_\zeta \rightarrow \mathbb{B}\) is expressed via two corresponding analytic functions \(F : D_z \rightarrow \mathbb{C}, F_0 : D_z \rightarrow \mathbb{C}\) of the complex variable \(z\) in
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the form (cf., e.g., [5, 6]):

\[ \Phi(\zeta) = F(z)e_1 - \left( \frac{iy}{2} F'(z) - F_0(z) \right) \rho \quad \forall \zeta \in D_{\zeta}. \]

The equality (4) establishes one-to-one correspondence between monogenic functions \( \Phi \) in the domain \( D_{\zeta} \) and pairs of complex-valued analytic functions \( F, F_0 \) in the domain \( D_z \).

Using the equality (2), we rewrite the expansion (4) for all \( \zeta \in D_{\zeta} \) in the basis \( \{e_1, e_2\} \):

\[ \Phi(\zeta) = \left( F(z) - iyF'(z) + 2F_0(z) \right)e_1 + i \left( 2F_0(z) - iyF'(z) \right)e_2. \]

2. (\( k-m \))-PROBLEM FOR MONOGENIC FUNCTIONS

V. F. Kovalev [7] considered the following boundary value problem: to find a continuous function \( \Phi : \overline{D_{\zeta}} \rightarrow \mathbb{B} \) which is monogenic in a domain \( D_{\zeta} \) when values of two component-functions in (3) are given on the boundary \( \partial D_{\zeta} \), i.e., the following boundary conditions are satisfied:

\[ U_k(x, y) = u_k(\zeta), \quad U_m(x, y) = u_m(\zeta) \quad \forall \zeta \in \partial D_{\zeta} \]

for \( 1 \leq k < m \leq 4 \), where \( u_k \) and \( u_m \) are given functions.

We assume additionally that the sought-for function \( \Phi \) has the limit

\[ \lim_{\|\zeta\| \to \infty, \zeta \in D_{\zeta}} \Phi(\zeta) =: \Phi(\infty) \in \mathbb{B} \]

in the case where the domain \( D_{\zeta} \) is unbounded as well as every given function \( u_l, \ l \in \{k, m\} \), has a finite limit

\[ u_l(\infty) := \lim_{\|\zeta\| \to \infty, \zeta \in \partial D_{\zeta}} u_l(\zeta) \]

if \( \partial D_{\zeta} \) is unbounded.

We shall call such a problem by the \( (k-m) \)-problem. V. F. Kovalev [7] called it by a biharmonic Schwartz problem owing to its analogy with the classical Schwartz problem on finding an analytic function of the complex variable when values of its real part are given on the boundary of domain.

It was established in [7] that all biharmonic Schwartz problems are reduced to the main three problems: the (1-2)-problem or the (1-3)-problem or the (1-4)-problem.

It is shown (see [7] [8] [9]) that the main biharmonic problem is reduced to the (1-3)-problem. In [8], we investigated the (1-3)-problem for cases where \( D_{\zeta} \) is either a half-plane or a unit disk in the biharmonic plane. Its solutions were found in explicit forms with using of some integrals analogous to the classic Schwarz integral. In [9], using a hypercomplex analog of the Cauchy type integral, we reduced the (1-3)-problem to a system of integral equations and established sufficient conditions under which this system has the Fredholm property. It was made for the case where the boundary of domain belongs to a
class being wider than the class of Lyapunov curves that was usually required in the plane elasticity theory (cf., e.g., [10, 11, 12, 13, 14]).

In [15], there is considered a relation between (1-4)-problem and boundary value problems of the plane elasticity theory. Namely, there is considered a problem on finding an elastic equilibrium for isotropic body occupying \( D \) with given limiting values of partial derivatives \( \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \) for displacements \( u = u(x,y), v = v(x,y) \) on the boundary \( \partial D \). In particular, it is shown in [15] that such a problem is reduced to (1-4)-problem.

In this paper we develop methods for solving the (1-4)-problem. Obtained results are mostly analogous to appropriate results in [8, 9] dealing with the (1-3)-problem, but in contradistinction to the (1-3)-problem, which is solvable in a general case if and only if a certain natural condition is satisfied, the (1-4)-problem is solvable unconditionally.

Note that hypercomplex methods and suitable "analytic" functions are used for investigation of elliptic partial differential equations in the papers [3, 4, 16, 17, 18, 19, 20, 21, 22, 23].

3. Solving process of (1-4)-problem by means analytic functions of the complex variable

Consider a method for solving the (1-4)-problem that is based on the expression (4) of monogenic function by means of appropriate analytic functions of the complex variable.

For a continuous function \( u : \partial D_\zeta \longrightarrow \mathbb{R} \), by \( \hat{u} \) we denote the function defined on \( \partial D_z \) by the equality \( \hat{u}(z) = u(\zeta) \) for all \( z \in \partial D_z \).

In what follows, we assume that the domain \( D_z \) is simply connected (bounded or unbounded), and in this case we shall say that the domain \( D_\zeta \) is also simply connected.

**Lemma 1.** Let \( u_l : \partial D_\zeta \longrightarrow \mathbb{R}, l \in \{1, 4\}, \) be continuous functions and \( \Phi : \overline{D_\zeta} \longrightarrow \mathbb{B} \) be a solution of the (1-4)-problem. Then the function \( F \) in (4) is a solution of the Schwartz problem on finding a continuous function \( F : \overline{D_z} \longrightarrow \mathbb{C} \) which is analytic in \( D_z \) and satisfies the boundary condition:

\[
\text{Re} \ F(t) = \hat{u}_1(t) - \hat{u}_4(t) \quad \forall t \in \partial D_z.
\]

**Proof.** Consider the linear continuous multiplicative functional \( f : \mathbb{B} \longrightarrow \mathbb{C} \) such that \( f(\rho) = 0 \) and \( f(1) = 1 \). Then, it follows from the equality (2) that \( f(e_2) = i \).

Thus, acting by the functional \( f \) on the equalities (3), (4), we obtain the relations

\[
F(z) = f(\Phi(\zeta)) = U_1(x,y) - U_4(x,y) + i \left( U_2(x,y) + U_3(x,y) \right) \quad \forall z \in D_z.
\]

Inasmuch as the functional \( f \) is continuous and \( \Phi \) is a solution of the (1-4)-problem, the analytic function \( F \) permits a continuous extendibility to the boundary \( \partial D_z \) and satisfies the boundary condition (7). ∎
Lemma 2. Let the conditions of Lemma 1 be satisfied and, furthermore, the function \( yF'(z) \) permit a continuous extendibility to the boundary \( \partial D_z \), where \( F \) is a solution of the Schwartz problem with boundary condition (7). Then the function \( F_0 \) in (4) is a solution of the Schwartz problem on finding a continuous function \( F_0 : D_z \to \mathbb{C} \), which is analytic in \( D_z \) and satisfies the boundary condition

\[
\Re F_0(t) = \frac{1}{2} \left( \hat{u}_4(t) - \Im t \lim_{z \to t, z \in D_z} F'(z) \right) \forall t \in \partial D_z.
\]

Proof. We obtain the equality

\[
U_4[\Phi(\zeta)] = 2 \Re F_0(z) + y \Im F'(z) \forall \zeta \in D_\zeta
\]
as a corollary of the equality (5).

Under the conditions of Lemma 1, the Schwartz problem with boundary condition (7) is solvable. Taking into account that a function \( yF'(z) \) permits a continuous extendibility to the boundary \( \partial D_z \), we conclude that the function \( F_0 \) is a solution of the Schwartz problem with boundary condition (8).

By virtue of Lemmas 1 and 2 we can assert that a solving of the (1-4)-problem is reduced to successive solving processes of two Schwartz problems with boundary conditions (7), (8), respectively. Therefore, we get the following theorem.

Theorem 1. Let \( u_l : \partial D_\zeta \rightarrow \mathbb{R}, \ l \in \{1, 4\} \), be continuous functions and, furthermore, the function \( yF'(z) \) permit a continuous extendibility to the boundary \( \partial D_z \), where \( F \) is a solution of the Schwartz problem with boundary condition (7). Then a solution of the (1-4)-problem is expressed by the formula (4) or, the same, by the formula (5), where the function \( F_0 \) is a solution of the Schwartz problem with boundary condition (8).

A particular case of Theorem 1 is the following theorem, where all solutions of the homogeneous (1-4)-problem, i.e., with \( u_1 = u_4 \equiv 0 \), are described for an arbitrary (bounded or unbounded) simply connected domain \( D_\zeta \).

Theorem 2. The general solution of the homogeneous (1-4)-problem for an arbitrary simply connected domain \( D_\zeta \) is expressed by the formula

\[
\Phi(\zeta) = a_1 i e_1 + a_2 e_2,
\]

where \( a_1, a_2 \) are any real constants.

Proof. By Theorem 1 a solving process of the homogeneous (1-4)-problem consists of consecutive finding of solutions of two homogeneous Schwartz problems, viz.:

a) to find an analytic in \( D_z \) function \( F \) satisfying the boundary condition \( \Re F(t) = 0 \) for all \( t \in \partial D_z \). As a result, we have \( F(z) = ai \), where \( a \) is an arbitrary real constant;
b) to find similarly an analytic in $D_z$ function $F_0$ satisfying the boundary condition $\text{Re } F_0(t) = 0$ for all $t \in \partial D_z$.

Consequently, getting a general solution of the homogeneous (1-4)-problem in the form (5), we can rewrite it in the form (9).

Having an intention to develop a method for solving the inhomogeneous (1-4)-problem without an essential assumption that the function $y F'(z)$ permits a continuous extendibility to the boundary $\partial D_z$, we consider primarily questions about finding solutions of the inhomogeneous (1-4)-problem for some canonical domains, namely: a half-plane and a disk.

4. (1-4)-problem for a half-plane.

Consider the (1-4)-problem in the case where the domain $D_\zeta$ is the half-plane $\Pi^+ := \{\zeta = xe_1 + ye_2 : y > 0\}$.

Our aim is to find an explicit formula of solution of the (1-4)-problem for the half-plane $\Pi^+$ under the assumption that for every given function $u_l : \mathbb{R} \to \mathbb{R}$, $l \in \{1, 4\}$, its modulus of continuity

$$\omega_R(u_l, \varepsilon) = \sup_{\tau_1, \tau_2 \in \mathbb{R} : |\tau_1 - \tau_2| \leq \varepsilon} |u_l(\tau_1) - u_l(\tau_2)|$$

and the local centered (with respect to the infinitely remote point) modulus of continuity

$$\omega_{R, \infty}(u_l, \varepsilon) = \sup_{\tau \in \mathbb{R} : |\tau| \geq 1/\varepsilon} |u_l(\tau) - u_l(\infty)|$$

satisfy the Dini conditions

$$\int_0^1 \frac{\omega_R(u_l, \eta)}{\eta} d\eta < \infty,$$  

$$\int_0^1 \frac{\omega_{R, \infty}(u_l, \eta)}{\eta} d\eta < \infty.$$  

In the following theorem all integrals along the real axis are understood in the sense of their Cauchy principal values.

**Theorem 3.** Let every function $u_l : \mathbb{R} \to \mathbb{R}$, $l \in \{1, 4\}$, have a finite limit of the type (6) and the conditions (10), (11) be satisfied. Then the general solution of the (1-4)-problem for the half-plane $\Pi^+$ is expressed by the formula

$$\Phi(\zeta) = S_{\Pi^+}[u_1](\zeta) e_1 + S_{\Pi^+}[u_4](\zeta) ie_2 + a_1 ie_1 + a_2 e_2,$$

where

$$S_{\Pi^+}[u_l](\zeta) := \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{u_l(t)(1 + t\zeta)}{(t^2 + 1)} (t - \zeta)^{-1} dt \quad \forall \zeta \in \Pi^+, \quad l \in \{1, 4\},$$

and $a_1, a_2$ are any real constants.
Proof. For each function \( u_l, l \in \{1, 4\} \), the following equalities are fulfilled

\[
\lim_{\zeta \to \xi, \zeta \in \Pi^+} S_{\Pi^+}[u_l](\zeta) = u_l(\xi) + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u_l(t)}{t^2 + 1} \frac{1 + t\xi}{t - \xi} dt \quad \forall \xi \in \mathbb{R} ,
\]

\[
\lim_{\|\zeta\| \to \infty, \zeta \in \Pi^+} S_{\Pi^+}[u_l](\zeta) = u_l(\infty) - \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{t}{t^2 + 1} dt ,
\]

which are proved in Theorem 1 of the paper [8]. It follows from these equalities that the function

\[
(13) \quad \Phi(\zeta) = S_{\Pi^+}[u_1](\zeta) e_1 + S_{\Pi^+}[u_4](\zeta) ie_2
\]

is a solution of the (1-4)-problem for the half-plane \( \Pi^+ \).

The general solution of the (1-4)-problem in the form (12) is obtained by summarizing the particular solution (13) of the inhomogeneous (1-4)-problem and the general solution (9) of the homogeneous (1-4)-problem.

Note that in [7] the (1-4)-problem for the half-plane is solved under complementary assumptions, viz.: the function \( u_l \) belongs to a H"older space and \( u_l(\infty) = 0 \) for \( l \in \{1, 4\} \), that imply the conditions (10), (11).

5. (1-4)-Problem for a Disk

Now, let \( D_\zeta := \{ \zeta = xe_1 + ye_2 : \|\zeta\| \leq 1 \} \) be the unit disk in the biharmonic plane \( \mu \) and \( D_z := \{ z = x + iy : |z| \leq 1 \} \) be the unit disk in the complex plane \( \mathbb{C} \).

In order to construct a solution of the (1-4)-problem for the disk \( D_\zeta \), as well as in the case of (1-3)-problem in the paper [8], we use the integral

\[
(14) \quad S_{D_\zeta}[u](\zeta) := \frac{1}{2\pi i} \int_{\partial D_\zeta} u(\tau)(\tau + \zeta)(\tau - \zeta)^{-1} d\tau^{-1} d\tau \quad \forall \zeta \in D_\zeta
\]

being an analog of the complex Schwartz-type integral.

It is proved in [8] that if the modulus of continuity

\[
\omega(u, \varepsilon) := \sup_{\zeta_1, \zeta_2 \in \partial D_\zeta : \|\zeta_1 - \zeta_2\| \leq \varepsilon} ||u(\zeta_1) - u(\zeta_2)||
\]

of the function \( u : \partial D_\zeta \to \mathbb{R} \) satisfies the Dini condition

\[
(15) \quad \int_{0}^{1} \frac{\omega(u, \eta)}{\eta} d\eta < \infty ,
\]

then the integral (14) has limiting values on \( \partial D_\zeta \). Here we rewrite a formula for the mentioned limiting values (cf. the formulas (25), (26) in [8]) in the following form

\[
\lim_{\xi \to \zeta, \xi \in D_\zeta} S_{D_\zeta}[u](\xi) = u(\zeta) e_1 + S_0[\bar{u}](z)e_1 +
\]
Theorem 4. Let functions \( u_l : \partial D_\zeta \to \mathbb{R} \), \( l \in \{1, 4\} \), satisfy conditions of the type (15). Then the general solution of (1-4)-problem is expressed in the form
\[
\Phi(\zeta) = S_{D_\zeta}[u_1](\zeta) e_1 + S_{D_\zeta}[u_4](\zeta) ie_2 + \]
\[
+ \left( (b_1 + ib_2)\zeta + b \right)(e_1 + ie_2) + a_1 ie_1 + a_2 e_2 ,
\]
where
\[
b_1 := -\frac{1}{2\pi} \text{Im} \int_{\partial D_\zeta} \frac{\tilde{u}_1(t) - \tilde{u}_4(t)}{t^2} \, dt , \quad b_2 := -\frac{1}{2\pi} \text{Re} \int_{\partial D_\zeta} \frac{\tilde{u}_1(t) - \tilde{u}_4(t)}{t^2} \, dt ,
\]
and \( a_1, a_2 \) are any real constants.

Proof. Let us prove that there exists a particular solution of the (1-4)-problem in the form
\[
\Phi(\zeta) = S_{D_\zeta}[u_1](\zeta) e_1 + S_{D_\zeta}[u_4](\zeta) ie_2 + \]
\[
+ \left( b_1 e_1 + b_2 ie_1 + b_3 e_2 + b_4 ie_2 \right)\zeta + c_1 e_1 + c_2 ie_2 ,
\]
where unknown coefficients \( b_1, b_2, b_3, b_4, c_1, c_2 \) are need to be found.

In order to single out components \( U_l[\Phi] \), \( l \in \{1, 4\} \), of limiting values of the function (18) on \( \partial D_\zeta \), we use the equality (16) and get
\[
U_1[\Phi(\zeta)] = u_1(\zeta) + (B_1 - B_4 + b_1)x - (A_1 - A_4 - b_3)y + D_1 - D_4 + c_1 ,
\]
\[
U_4[\Phi(\zeta)] = u_4(\zeta) + (B_1 - B_4 + b_4)x - (A_1 - A_4 - b_2 - 2b_3)y + D_1 - D_4 + c_2
\]
for \( \zeta \in \partial D_\zeta \), where
\[
A_l := \frac{1}{2\pi} \text{Re} \int_{\partial D_\zeta} \frac{\tilde{u}_l(t)}{t^2} \, dt , \quad B_l := \frac{1}{2\pi} \text{Im} \int_{\partial D_\zeta} \frac{\tilde{u}_l(t)}{t^2} \, dt ,
\]
\[
D_l := \frac{1}{2\pi} \text{Im} \int_{\partial D_\zeta} \frac{\tilde{u}_l(t)}{t^3} \, dt , \quad l \in \{1, 4\}.
\]

Now, it is clear that the identities \( U_l[\Phi(\zeta)] \equiv u_l(\zeta) \), \( l \in \{1, 4\} \), hold on \( \partial D_\zeta \) if \( b_4 = b_1 = -(B_1 - B_4) \), \( b_3 = -b_2 = A_1 - A_4 \), \( c_1 = c_2 = -(D_1 - D_4) \).
Finally, substituting the found values for the coefficients $b_1$, $b_2$, $b_3$, $b_4$, $c_1$, $c_2$ to the partial solution (18) of the inhomogeneous (1-4)-problem and adding the general solution (9) of the homogeneous (1-4)-problem, after evident identical transformations we obtain the formula (17) for the general solution of the (1-4)-problem for the unit disk. □

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