SELF-SHRINKERS WITH SECOND FUNDAMENTAL FORM OF CONSTANT LENGTH

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Abstract. In this note, we give a new and simple proof of a result in [DX11] which states that any smooth complete self-shrinker in $\mathbb{R}^3$ with second fundamental form of constant length must be a generalized cylinder $S^k \times \mathbb{R}^{2-k}$ for some $k \leq 2$. Moreover, we prove a gap theorem for smooth self-shrinkers in all dimensions.

0. Introduction

A one-parameter family of hypersurfaces $M_t \subset \mathbb{R}^{n+1}$ flows by mean curvature if

\begin{equation}
\partial_t x = -Hn,
\end{equation}

where $H$ is the mean curvature, $n$ is the outward pointing unit normal and $x$ is the position vector.

We call a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ a self-shrinker if it satisfies

\begin{equation}
H = \frac{\langle x, n \rangle}{2}.
\end{equation}

Under the mean curvature flow ("MCF"), $\Sigma$ is just moving by rescalings, i.e., $\Sigma_t \equiv \sqrt{-t} \Sigma$ gives a MCF.

Self-shrinkers play a key role in the study of MCF. By Huisken’s monotonicity formula [Hui90] and an argument of Ilmanen [Ilm97] and White, self-shrinkers provide all singularity models of the MCF. Although there are infinitely many of them, we only know few embedded complete examples, see [Ang92], [Cho94], [KM10], [Møl11] and [Ngu09]. Moreover, numerical results showed that it is impossible to give a complete classification of self-shrinkers in higher dimensions. However, under certain conditions, there are many classification results of self-shrinkers. In [CM12], Colding and Minicozzi proved that the only smooth complete embedded self-shrinkers with polynomial volume growth and $H \geq 0$ in $\mathbb{R}^{n+1}$ are generalized cylinders $S^k \times \mathbb{R}^{n-k}$ (where the $S^k$ has radius $\sqrt{2k}$). A natural question is under what other conditions can we conclude that a self-shrinker is a generalized cylinder. We are interested in those conditions involving the squared norm of the second fundamental form, i.e., $|A|^2$.

First, we consider the following question.

Conjecture 0.1. $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth complete embedded self-shrinker with polynomial volume growth. If $|A|^2 = \text{constant}$, then $\Sigma$ is a generalized cylinder.
The case \( n = 1 \) follows from a more general result by Abresch and Langer [AL86] stating that the only smooth complete and embedded self-shrinkers in \( \mathbb{R}^2 \) are the lines and a round circle. In \( \mathbb{R}^3 \), i.e., \( n = 2 \), the above conjecture was proved by Ding and Xin [DX11] using the following identity

\[
\frac{1}{2} L |\nabla A|^2 = |\nabla^2 A|^2 + (1 - |A|^2)|\nabla A|^2 - 3 \Xi - \frac{3}{2} |\nabla |A|^2|^2
\]

where the operator \( L = \Delta - \langle \frac{x}{2}, \nabla \cdot \rangle \), \( h_{ij} \) is the second fundamental form and \( \Xi = \sum_{i,j,k,l,m} h_{i,j,k} h_{i,j,l} h_{k,m} h_{m,l} - 2 \sum_{i,j,k,l,m} h_{i,j,k} h_{k,m} h_{i,m} h_{j,l} \).

In this note, we give a new and simple proof of the above result without heavy computation, more precisely, we prove the following theorem.

**Theorem 0.2.** Let \( \Sigma^2 \subset \mathbb{R}^3 \) be a smooth complete embedded self-shrinker with polynomial volume growth. If the second fundamental form of \( \Sigma^2 \) is of constant length, i.e., \( |A|^2 = \text{constant} \), then \( \Sigma^2 \) is a generalized cylinder \( S^k \times \mathbb{R}^{2-k} \) for \( k \leq 2 \).

The key idea in the proof is to analyze the point where \( |x| \) achieves its minimum. We mention that our method does not apply to higher dimensions to prove the Conjecture 0.1.

For self-shrinkers, there exists some gap phenomenon for the squared norm of the second fundamental form. Cao and Li [CL13] proved that any smooth complete self-shrinker with polynomial volume growth and \( |A|^2 \leq \frac{1}{2} \) in arbitrary codimension is a generalized cylinder. Colding, Ilmanen and Minicozzi [CIM13] showed that generalized cylinders are rigid in a strong sense that any self-shrinker which is sufficiently close to one of generalized cylinders on a large and compact set must itself be a generalized cylinder. Using this result we prove that any self-shrinker with \( |A|^2 \) sufficiently close to \( \frac{1}{2} \) must also be a generalized cylinder.

**Theorem 0.3.** Given \( n \) and \( \lambda_0 \), there exists \( \delta = \delta(n, \lambda_0) > 0 \) so that if \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a smooth embedded self-shrinker with entropy \( \lambda(\Sigma) \leq \lambda_0 \) and

- \( |A|^2 \leq \frac{1}{2} + \delta \),

then \( \Sigma^n \) is a generalized cylinder \( S^k \times \mathbb{R}^{n-k} \) for some \( k \leq n \).

**Remark 0.4.** It is expected that we could remove the entropy bound in Theorem 0.3 in dimensional two case. In other words, the bound for \( |A|^2 \) may imply the entropy bound in \( \mathbb{R}^3 \). Note that in \( \mathbb{R}^3 \), if \( \Sigma^2 \) is a closed self-shrinker with \( |A|^2 \leq C \), where \( C \) is a constant less than 1. Then by Gauss-Bonnet Formula, one can easily get that the genus of \( \Sigma^2 \) is 0 and therefore obtain an entropy bound for \( \Sigma^2 \).

1. **Background and Preliminaries**

   In this section, we recall some background and preliminaries for self-shrinkers from [CM12]. Throughout this note, we always assume self-shrinkers to be smooth complete embedded, without boundary and with polynomial volume growth.
Let $\Sigma \subset \mathbb{R}^{n+1}$ be a hypersurface, then $\Delta$, $\text{div}$, and $\nabla$ are the Laplacian, divergence, and gradient, respectively, on $\Sigma$. $\mathbf{n}$ is the outward unit normal, $H=\text{div}_\Sigma \mathbf{n}$ is the mean curvature, $A$ is the second fundamental form, and $x$ is the position vector. With this convention, the mean curvature $H$ is $n/r$ on the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ of radius $r$.

First, recall the operators $\mathcal{L}$ and $L$ defined by

\begin{align}
\mathcal{L} &= \Delta - \frac{1}{2} \langle x, \nabla \cdot \rangle, \\
L &= \Delta - \frac{1}{2} \langle x, \nabla \cdot \rangle + |A|^2 + \frac{1}{2}.
\end{align}

**Lemma 1.1** \cite{CM12}. If $\Sigma \subset \mathbb{R}^{n+1}$ is a smooth self-shrinker, then

\begin{align}
\mathcal{L}|x|^2 &= 2n - |x|^2, \\
L H^2 &= 2(\frac{1}{2} - |A|^2)H^2 + 2|\nabla H|^2, \\
L |A|^2 &= 2(\frac{1}{2} - |A|^2)|A|^2 + 2|\nabla A|^2.
\end{align}

A direct consequence of Lemma 1.1 is the following corollary.

**Corollary 1.2.** Let $\Sigma \subset \mathbb{R}^{n+1}$ be a smooth self-shrinker. If $|A|^2 \leq \frac{1}{2}$, then $\Sigma$ is a generalized cylinder $\mathbb{S}^k \times \mathbb{R}^n - k$ for some $k \leq n$. Moreover, if $|A|^2 < \frac{1}{2}$, then $\Sigma$ is a hyperplane.

Next, we introduce the concepts of the $F$-functional and the entropy of a hypersurface.

**Definition 1.3.** For $t_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$, the $F$-functional $F_{x_0,t_0}$ of a hypersurface $M \subset \mathbb{R}^{n+1}$ is defined as

\begin{equation}
F_{x_0,t_0}(M) = (4\pi t_0)^{-\frac{n}{2}} \int_M e^{-\frac{|x-x_0|^2}{4t_0}},
\end{equation}

and the entropy of $M$ is given by

\begin{equation}
\lambda(M) = \sup_{x_0,t_0} F_{x_0,t_0}(M),
\end{equation}

here the supremum is taking over all $t_0 > 0$ and $x_0 \in \mathbb{R}^{n+1}$.

2. **Proof of Theorem 0.2**

**Proof.** By Corollary 1.2 if $|A|^2 < \frac{1}{2}$, then $\Sigma$ is a hyperplane in $\mathbb{R}^3$. Therefore, in the following we only consider the case when $|A|^2 \geq \frac{1}{2}$.

For any point $p \in \Sigma$, we can choose a local orthonormal frame $\{e_1, e_2\}$ such that the coefficients of the second fundamental form $h_{ij} = \lambda_i \delta_{ij}$ for $i, j = 1, 2$, then

\begin{equation}
|\nabla A|^2 = h_{111}^2 + h_{222}^2 + 3h_{112}^2 + 3h_{221}^2 = |A|^2(|A|^2 - \frac{1}{2}),
\end{equation}
\( \lvert \nabla H \rvert^2 = (h_{111} + h_{221})^2 + (h_{112} + h_{222})^2. \)  

Since \( \lvert A \rvert^2 \) is constant, we have

\[ h_{11}h_{111} + h_{22}h_{221} = h_{11}h_{112} + h_{22}h_{222} = 0. \] (2.3)

First, we prove \( \lvert x \rvert > 0 \) on \( \Sigma \). We argue by contradiction. Suppose \( \Sigma \) goes through the origin, then at the origin, we have \( H = \lvert \nabla H \rvert = 0 \), therefore

\[ h_{11} + h_{22} = h_{111} + h_{221} = h_{112} + h_{222} = 0. \]

Combining this with (2.3), we get

\[ h_{111} = h_{222} = h_{112} = h_{221} = 0. \]

This implies that \( \lvert \nabla A \rvert^2 = \lvert A \rvert^2 (\lvert A \rvert^2 - \frac{1}{2}) = 0 \), i.e., \( \lvert A \rvert^2 = \frac{1}{2} \). By Corollary 1.2, we conclude that \( \Sigma \) is \( S^2 \) or \( S^1 \times \mathbb{R} \). However, this contradicts the assumption that \( \Sigma \) goes through the origin.

Note that \( \Sigma \) has polynomial volume growth implies that \( \Sigma \) is proper (see Theorem 4.1 in [CZ13]) and by the maximum principle \( \Sigma \) intersects \( S^2(2) \), so there exists a point \( p \in \Sigma \) minimize \( \lvert x \rvert \).

Now, at point \( p \), we have \( \lvert x \rvert > 0 \) and \( x^T = 0 \), where \( x^T \) is the tangential projection of \( x \). This implies that \( 4H^2(p) = \lvert x \rvert^2(p), \nabla H(p) = 0 \), and thus

\[ h_{111} + h_{221} = h_{112} + h_{222} = 0, \]

By (2.3), we get

\[ h_{111}(h_{11} - h_{22}) = h_{222}(h_{11} - h_{22}) = 0. \] (2.4)

If \( h_{111} = h_{222} = 0 \), then \( \lvert \nabla A \rvert^2 = 0 \) and therefore \( \lvert A \rvert^2 = \frac{1}{2} \). By Corollary 1.2, we conclude that \( \Sigma \) is a generalized cylinder.

If \( h_{11} = h_{22} \), then we have

\[ \lvert A \rvert^2 = 2h_{11}^2 = \frac{H^2(p)}{2} = \frac{\lvert x \rvert^2(p)}{8}. \] (2.5)

Since every smooth complete self-shrinker must intersect the sphere \( S^2(2) \), we conclude that \( \lvert x \rvert(p) \leq 2 \). By (2.3), this gives

\[ \lvert A \rvert^2 \leq \frac{1}{2}, \]

then the theorem follows immediately from Corollary 1.2. \qed
3. Proof of Theorem 0.3

First, we state two key ingredients from [CIM13]. The first one is the rigidity theorem for the generalized cylinders and the second one is the compactness theorem for self-shrinkers.

**Theorem 3.1 ([CIM13])**. Given \( n, \lambda_0 \) and \( C \), there exists \( R = R(n, \lambda_0, C) \) so that if \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a self-shrinker with entropy \( \lambda(\Sigma) \leq \lambda_0 \) satisfying

- \( \Sigma \) is smooth in \( B_R \) with \( H \geq 0 \) and \( |A| \leq C \) on \( B_R \cap \Sigma \),

then \( \Sigma \) is a generalized cylinder \( S^k \times \mathbb{R}^{n-k} \) for some \( k \leq n \).

**Lemma 3.2 ([CIM13])**. Let \( \Sigma_i \subset \mathbb{R}^{n+1} \) be a sequence of \( F \)-stationary varifolds with \( \lambda(\Sigma_i) \leq \lambda_0 \) and

\[
(3.1) \quad B_{R_i} \cap \Sigma_i \text{ is smooth with } |A| \leq C,
\]

where \( R_i \to \infty \). Then there exists a subsequence \( \Sigma'_i \) that converges smoothly and with multiplicity one to a complete embedded self-shrinker \( \Sigma \) with \( |A| \leq C \) and

\[
(3.2) \quad \lim_{i \to \infty} \lambda(\Sigma'_i) = \lambda(\Sigma).
\]

**Remark 3.3.** In the above lemma, the entropy bound is used to guarantee that the convergence is with finite multiplicity. Moreover, if the multiplicity is greater than one, then the limit is \( L \)-stable. Using the fact that there are no complete \( L \)-stable self-shrinkers with polynomial volume growth gives the multiplicity of the convergence must be one.

Now we are ready to give the proof of Theorem 0.3.

**Proof of Theorem 0.3.** We will argue by contradiction, so suppose there is a sequence of smooth embedded self-shrinkers \( \Sigma_i \neq S^k \times \mathbb{R}^{n-k} \) \( (k \leq n) \) with \( \lambda(\Sigma_i) \leq \lambda_0 \) and

\[
(3.3) \quad |A|^2 \leq \frac{1}{2} + \frac{1}{i}.
\]

By Lemma 3.2 there exists a subsequence \( \Sigma_i \) (still denoted by \( \Sigma_i \)) that converges smoothly and with multiplicity one to a complete embedded self-shrinker \( \Sigma \).

By (3.3), we can conclude that \( \Sigma \) satisfies \( |A|^2 \leq \frac{1}{2} \), and thus, \( \Sigma \) is a generalized cylinder. Now we choose the \( R \) in Theorem 3.1 and for \( N \) sufficiently large, \( \Sigma_m \) is very close to \( \Sigma \) on \( B_{2R} \cap \Sigma_m \) for \( m \geq N \), i.e.,

\[
\Sigma_m \text{ satisfies } H \geq 0 \text{ and } |A| \leq 1 \text{ on } B_R \cap \Sigma_m,
\]

then by the rigidity theorem of self-shrinkers, Theorem 3.1, \( \Sigma_m \) is a generalized cylinder. However, this contradicts our assumption that \( \Sigma_i \) is not a generalized cylinder, completing the proof. \( \Box \)
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