ABOUT THE CHARACTERISTIC FUNCTION OF A SET

Prof. Mihály Bencze, Department of Mathematics, University of Braşov, Romania

Prof. Florentin Smarandache, Chair of Department of Math & Sciences, University of New Mexico, 200 College Road, Gallup, NM 87301, USA, E-mail: smarand@unm.edu

Abstract:
In this paper we give a method, based on the characteristic function of a set, to solve some difficult problems of set theory found in undergraduate studies.

Definition: Let’s consider $A \subset \neq \emptyset$ (a universal set), then $f_A : E \to \{0, 1\}$, where the function

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

is called the characteristic function of the set $A$.

Theorem 1: Let’s consider $A, B \subset E$. In this case $f_A = f_B$ if and only if $A = B$.

Proof.

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A = B \\ 0, & \text{if } x \notin A = B \end{cases} = f_B(x)$$

Reciprocally: For any $x \in A$, $f_A(x) = 1$, but $f_A = f_B$, therefore $f_B(x) = 1$, namely $x \in B$ from where $A \subset B$. The same way we prove that $B \subset A$, namely $A = B$.

Theorem 2: $f_A = 1 - f_A$, $\tilde{A} = C_E A$.

Proof.

$$f_A(x) = \begin{cases} 1, & \text{if } x \in \tilde{A} \\ 0, & \text{if } x \notin \tilde{A} \end{cases} = \begin{cases} 1 - 0, & \text{if } x \notin A \\ 0 - 1, & \text{if } x \in A \end{cases} = \begin{cases} 1 - \{0, & \text{if } x \notin A \\ 1 - \{1, & \text{if } x \in A \end{cases} = 1 - f_A(x)$$

Theorem 3: $f_{A \cap B} = f_A * f_B$. 
Proof.

\[ f_{A \cap B}(x) = \begin{cases} 
1, & \text{if } x \in A \cap B \\
0, & \text{if } x \notin A \cap B
\end{cases} \]

The theorem can be generalized by induction:

**Theorem 4:** \( f_{A_1 \cap A_2 \cap \ldots \cap A_n} = \prod_{k=1}^{n} f_{A_k} \)

**Consequence.** For any \( n \in \mathbb{N}^* \), \( f_M^n = f_M \).

**Proof.** In the previous theorem we chose \( A_1 = A_2 = \ldots = A_n = M \).

**Theorem 5:** \( f_{A \cup B} = f_A + f_B - f_A f_B \).

**Proof.**

\[ f_{A \cup B} = f_{A \cap B} = f_A f_B = 1 - f_A f_B = 1 - \left( 1 - f_A \right) \left( 1 - f_B \right) = f_A + f_B - f_A f_B \]

It can be generalized by induction:

**Theorem 6:** \( f_{\bigcup_{k=1}^{n} A_k} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} (-1)^{k-1} f_{A_{i_1} \cap \ldots \cap A_{i_k}} \)

**Theorem 7:** \( f_{A \cap B} = f_A \left( 1 - f_B \right) \)

**Proof.**

\[ f_{A \cap B} = f_{A \cup B} = f_A f_B = f_A \left( 1 - f_B \right) \]

It can be generalized by induction:

**Theorem 8:** \( f_{A_k \cap A_{k-1} \cap \ldots \cap A_1} = \sum_{k=1}^{n} (-1)^{k-1} f_{A_{i_1} \cap \ldots \cap A_{i_k}} \)

**Theorem 9:** \( f_{A \Delta B} = f_A + f_B - 2 f_A f_B \)

**Proof.**

\[ f_{A \Delta B} = f_{A \cup B} - f_{A \cap B} = f_A f_B = f_A + f_B - 2 f_A f_B \]

It can be generalized by induction:

**Theorem 10:** \( F_{A_1 \cap A_2 \cap \ldots \cap A_n} = \sum_{k=1}^{n} (-2)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{A_{i_1} \cap \ldots \cap A_{i_k}} \)

**Theorem 11:** \( f_{A \Delta B}(x, y) = f_A(x) f_B(y) \).
Proof. If \((x, y) \in A \times B\), then \(f_{A \times B}(x, y) = 1\) and \(x \in A\), namely \(f_A(x) = 1\) and \(y \in B\), namely \(f_B(y) = 1\), therefore \(f_A(x)f_B(y) = 1\). If \((x, y) \notin A \times B\), then \(f_{A \times B}(x, y) = 0\) and \(x \notin A\), namely \(f_A(x) = 0\) or \(y \notin B\), namely \(f_B(y) = 0\), therefore \(f_A(x)f_B(y) = 0\).

This theorem can be generalized by induction.

**Theorem 12:** \(f_{\times_{k=1}^n A_k}(x_1, x_2, \ldots, x_n) = \prod_{k=1}^n f_{A_k}(x_k)\).

**Theorem 13:** (De Morgan) \(\bigcup_{k=1}^n A_k = \bigcap_{k=1}^n A_k\)

**Proof.**

\[
f_{\bigcup_{k=1}^n A_k} = 1 - f_{\bigcap_{k=1}^n A_k} = 1 - \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{A_{i_1}} \cdots f_{A_{i_k}} f_M = \prod_{k=1}^n (1 - f_{A_k}) = \prod_{k=1}^n f_{A_k} = f_{\bigcap_{k=1}^n A_k}.
\]

We prove in the same way the following theorem:

**Theorem 14:** (De Morgan) \(\bigcap_{k=1}^n A_k = \bigcup_{k=1}^n \overline{A_k}\).

**Theorem 15:** \(\left(\bigcup_{k=1}^n A_k\right) \cap M = \bigcup_{k=1}^n (A_k \cap M)\).

**Proof.**

\[
f_{\left(\bigcup_{k=1}^n A_k\right) \cap M} = f_{\bigcup_{k=1}^n A_k} f_M = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{A_{i_1}} \cdots f_{A_{i_k}} f_M = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{A_{i_1}} \cdots f_{A_{i_k}} f_M = \sum_{k=1}^n (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{A_{i_1}} \cdots f_{A_{i_k}} f_M = f_{\bigcup_{k=1}^n (A_k \cap M)}.
\]

In the same way we prove that:

**Theorem 16:** \(\left(\bigcap_{k=1}^n A_k\right) \cup M = \bigcap_{k=1}^n (A_k \cup M)\).

**Theorem 17:** \(\left(\Delta_{k=1}^n A_k\right) \cap M = \Delta_{k=1}^n (A_k \cap M)\)

**Application.**

\(\left(\Delta_{k=1}^n A_k\right) \cup M = \Delta_{k=1}^n (A_k \cup M)\) if and only if \(M = \emptyset\).

**Theorem 18:** \(M \times \left(\bigcup_{k=1}^n A_k\right) = \bigcup_{k=1}^n (M \times A_k)\)

**Proof.**
\[
 f_{M \in \bigcup_{i=1}^{n} A_i} (x, y) = f_M(y) f_{\bigcup_{i=1}^{n} A_i} (x) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{A_{i_1}} (x) f_{A_{i_2}} (x) \ldots f_{A_{i_k}} (x) f_M (y) = \\
 = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{A_{i_1}} (x) f_{A_{i_2}} (x) \ldots f_{A_{i_k}} (x) f^k (y) = \\
 = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{A_{i_1} \times M (x, y)} f_{A_{i_k} (x, y)} = f_{\bigcup_{i=1}^{n} (M \times A_i)}
\]

In the same way we prove that:

**Theorem 19:** \( M \times \bigcap_{k=1}^{n} A_k = \bigcap_{k=1}^{n} (M \times A_k) \).

**Theorem 20:** \( M \times (A_1 - A_2 - \ldots - A_n) = (M \times A_1) - (M \times A_2) - \ldots - (M \times A_n) \).

**Theorem 21:** \( (A_1 - A_2) \cup (A_2 - A_3) \cup \ldots \cup (A_{n-1} - A_n) \cup (A_n - A_1) = \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k \)

**Proof 1.**

\[
 f_{(A_k - A_{k+1}) \cup (A_{k+1} - A_k)} = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{A_{i_1} - A_{i_2}} \ldots f_{A_{i_k} - A_{i_{k-1}}} = \\
 = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} (f_{A_{i_1}} - f_{A_{i_2}}) \ldots (f_{A_{i_{k-1}}} - f_{A_{i_k}}) = \\
 = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{A_{i_1}} \ldots f_{A_{i_k}} (1 - \prod_{p=1}^{n} f_{A_{i_p}}) = f_{\bigcup_{i=1}^{n} A_i} \left( 1 - \prod_{i=1}^{n} f_{A_i} \right) = f_{\bigcup_{i=1}^{n} A_i - \bigcap_{i=1}^{n} A_i}.
\]

**Proof 2.** Let’s consider \( x \in \bigcup_{i=1}^{n} (A_i - A_{i+1}) \), (where \( A_{n+1} = A_1 \)), then there exists \( k \) such that \( x \in (A_k - A_{k+1}) \), namely \( x \notin (A_k \cap A_{k+1}) = A_k \cap A_{k+1} \). Now we prove the inverse statement:

Let’s consider \( x \in \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k \), we show that there exists \( k \) such that \( x \in A_k \) and \( x \notin A_{k+1} \). On the contrary, it would result that for any \( k \in \{1, 2, \ldots, n\} \), \( x \notin A_k \) and \( x \in A_{k+1} \) namely \( x \notin \bigcup_{k=1}^{n} A_k \), it results that there exists \( p \) such that \( x \in A_p \), but from the previous reasoning it results that \( x \in A_{p+1} \), and using this we consequently obtain that \( x \in A_k \) for \( k = p, n \). But from \( x \notin A_k \), we obtain that \( x \in A_1 \), therefore, it results that \( x \in A_k \), \( k = 1, p \), from where \( x \in A_k \), \( k = 1, n \), namely \( x \in A_1 \cap \ldots \cap A_k \), that is a contradiction. Thus there exists \( r \) such that \( x \in A_r \) and \( x \notin A_{r+1} \), namely \( x \in (A_r - A_{r+1}) \) and therefore \( x \in \bigcup_{k=1}^{n} (A_k - A_{k+1}) \).
In the same way we prove the following theorem:

**Theorem 22:** \((A_i \Delta A_j) \cup (A_j \Delta A_i) \cup \ldots \cup (A_{n-1} \Delta A_n) = \bigcup_{k=1}^{n} A_k - \bigcap_{k=1}^{n} A_k\).

**Theorem 23:**
\[
(A_1 \times A_2 \times \ldots \times A_k) \cap \big( (A_{k+1} \times A_{k+2} \times \ldots \times A_{2k}) \cap \big( (A_n \times A_1 \times \ldots \times A_{k-1}) = (A_1 \cap A_2 \cap \ldots \cap A_n)^k. \]

**Proof.**
\[
\begin{align*}
\hat{f}_{(A_1 \times \ldots \times A_k) \cap \big( (A_{k+1} \times A_{k+2} \times \ldots \times A_{2k}) \cap \big( (A_n \times A_1 \times \ldots \times A_{k-1}) &= \hat{f}_{A_1 \times \ldots \times A_k} (x_1, \ldots, x_n) = \\
&= \hat{f}_{A_1 \times \ldots \times A_k} (x_1, \ldots, x_n) \cdots \hat{f}_{A_k \times \ldots \times A_{k-1}} (x_1, \ldots, x_n) = \\
&= (\hat{f}_{A_1} (x_1) \cdots \hat{f}_{A_k} (x_k)) \cdots (\hat{f}_{A_n} (x_1) \cdots \hat{f}_{A_{k-1}} (x_{k-1})) = \\
&= \hat{f}_{A_1}^k (x_1) \cdots \hat{f}_{A_k}^k (x_k) = \hat{f}_{A_1 \cap \ldots \cap A_k}^k (x_1, \ldots, x_n) = \\
&= \hat{f}_{(A_1 \cap \ldots \cap A_k) \times \ldots \times A_k}^k (x_1, \ldots, x_n). \\
\end{align*}
\]

**Theorem 24.** \((P(E), \cup)\) is a commutative monoid.

**Proof.** For any \(A, B \in P(E)\); \(A \cup B \in P(E)\), namely the intern operation. Because \((A \cup B) \cup C = A \cup (B \cup C)\) is associative, \(A \cup B = B \cup A\) commutative, and because \(A \cup \emptyset = A\) then \(\emptyset\) is the neutral element.

**Theorem 25:** \((P(E), \cap)\) is a commutative monoid.

**Proof.** For any \(A, B \in P(E)\); \(A \cap B \in P(E)\) namely intern operation. \((A \cap B) \cap C = A \cap (B \cap C)\) associative, \(A \cap B = B \cap A\), commutative \(A \cap E = A\), \(E\) is the neutral element.

**Theorem 26:** \((P(E), \Delta)\) is an abelian group.

**Proof.** For any \(A, B \in P(E)\); \(A \Delta B \in P(E)\), namely the intern operation. \(A \Delta B = B \Delta A\) commutative. The proof of associativity is in the XII\textsuperscript{th} grade manual as a problem. We’ll prove it using the characteristic function of the set.

\[
\hat{f}_{(A \Delta B) \cap C} = 4f_A f_B f_C - 2f_A f_B + f_B f_C + f_C f_A + f_A + f_B + f_C = \hat{f}_{A \Delta (B \cap C)}. \\
\]

Because \(A \Delta \emptyset = A\), \(\emptyset\) is the neutral element and because \(A \Delta A = \emptyset\); the symmetric element of \(A\) is \(A\) itself.

**Theorem 27:** \((P(E), \Delta, \cap)\) is a commutative Boole ring with a divisor of zero.

**Proof.** Because the previous theorem satisfies the commutative ring axioms, the first part of the theorem is proved. Now we prove that it has a divisor of zero. If \(A \neq \emptyset\) and \(B \neq \emptyset\) are two disjoint sets, then \(A \cap B = \emptyset\), thus it has divisor of zero. From Theorem 17 we get that it is distributive for \(n = 2\). Because for any \(A \in P(E)\); \(A \cap A = A\) and \(A \Delta A = \emptyset\) it also satisfies the Boole-type axioms.
Theorem 28: Let’s consider $H = \{ f \mid f : E \to \{0,1\} \}$, then $(H, \oplus)$ is an abelian group, where $f_A \oplus f_B = f_A + f_B - 2f_A f_B$ and $(P(E), \Delta) \cong (H, \oplus)$.

Proof. Let’s consider $F : P(E) \to H$, where $f(A) = f_A$, then, from the previous theorem we get that it is bijective and because $F(A \Delta B) = f_{A \Delta B} = F(A) \oplus F(B)$ it is compatible.

Theorem 29: $\text{card}(A_1 \Delta A_n) \leq \text{card}(A_1 \Delta A_2) + \text{card}(A_2 \Delta A_3) + \ldots + \text{card}(A_{n-1} \Delta A_n)$.

Proof. By induction. If $n = 2$, then it is true, we show that for $n = 3$ it is also true. Because $(A \cap A_2) \cup (A_2 \cap A_3) \subseteq A_2 \cup (A_1 \cap A_3)$;

$$\begin{align*}
\text{card}(M \cup N) &= \text{card}M + \text{card}N - \text{card}(M \cap N), \text{ and thus} \\
\text{card}A_2 + \text{card}(A_1 \cap A_3) - \text{card}(A_2 \cap A_3) - \text{card}(A_1 \cap A_2) \geq 0, \text{ can be written as} \\
\text{card}A_4 + \text{card}A_3 - 2\text{card}(A_1 \cap A_3) \leq & \\
\leq (\text{card}A_4 + \text{card}A_3 - 2\text{card}(A_1 \cap A_2)) + (\text{card}A_2 + \text{card}A_3 - 2\text{card}(A_2 \cap A_3)).
\end{align*}$$

But because of $(M \Delta N) = \text{card}M + \text{card}N - 2\text{card}(M \cap N)$

then $\text{card}(A_1 \Delta A_3) \leq \text{card}(A_1 \Delta A_2) + \text{card}(A_2 \Delta A_3)$. The proof of this step of the induction relies on the above method.

Theorem 30: $(P^2(E), \text{card}(A \Delta B))$ is a metric space.

Proof. Let $d(A, B) = \text{card}(A \Delta B) : P(E) \times P(E) \to \mathbb{R}$

1. $d(A, B) = 0 \Leftrightarrow \text{card}(A \Delta B) = 0 \Leftrightarrow \text{card}((A - B) \cup (B - A)) = 0$ but because $(A - B) \cap (B - A) = \emptyset$ we obtain $(A - B) + \text{card}(B - A) = 0$ and because $(A - B) = 0$ and $\text{card}(B - A) = 0$, then $A - B = \emptyset$, $B - A = \emptyset$, and $A = B$.

2. $d(A, B) = d(B, A)$ results from $A \Delta B = B \Delta A$.

3. As a consequence of the previous theorem $d(A, C) \leq d(A, B) + d(B, C)$.

As a result of the above three properties it is a metric space.

PROBLEMS

Problem 1.
Let’s consider $A = B \cup C$ and $f : P(A) \to P(A) \times P(A)$, where $f(x) = (X \cup B, X \cup C)$. Prove that $f$ is injective if and only if $B \cap C = \emptyset$.

Solution 1. If $f$ is injective. Then

$$f(\emptyset) = (\emptyset \cup B, \emptyset \cup C) = (B, C) = ((B \cap C) \cup B, (B \cap C) \cup C) = f(B \cap C)$$

from which we obtain $B \cap C = \emptyset$. Now reciprocally: Let’s consider $B \cap C = \emptyset$, then $f(X) = f(Y)$; it results that $X \cup B = Y \cup B$ and $X \cup C = Y \cup C$ or
\[ X = X \cup \emptyset = X \cup (B \cap C) = (X \cup B) \cap (X \cup C) = (Y \cup B) \cap (Y \cup C) = Y \cup (B \cap C) = Y \cup \emptyset = Y \]

namely it is injective.

**Solution 2.** Let’s consider \( B \cap C = \emptyset \) passing over the set function \( f(X) = f(Y) \) if and only if \( X \cup B = Y \cup B \) and \( X \cup C = Y \cup C \), namely \( f_{X \cup B} = f_{Y \cup B} \) and \( f_{X \cup C} = f_{Y \cup C} \) or \( f_X + f_B - f_X f_B = f_Y + f_B - f_Y f_B \) and \( f_X + f_C - f_X f_C = f_Y + f_C - f_Y f_C \) from which we obtain \( (f_X - f_Y)(f_B - f_C) = 0 \).

Because \( A = B \cup C \) and \( B \cap C = \emptyset \), we have

\[
(f_B - f_C)(u) = \begin{cases} 1, & \text{if } u \in B \\ -1, & \text{if } u \in C \end{cases} 
eq 0
\]

therefore \( f_X - f_Y = 0 \), namely \( X = Y \) and thus it is injective.

**Generalization.** Let \( M = \bigcup_{k=1}^{n} A_k \) and \( f : P(A) \to P''(A) \), where

\[
f(X) = (X \cup A_1, X \cup A_2, \ldots, X \cup A_n).
\]

Prove that \( f \) is injective if and only if \( A_1 \cap A_2 \cap \ldots \cap A_n = \emptyset \).

**Problem 2.** Let \( E \neq \emptyset \), \( A \in P(E) \), and \( f : P(E) \to P(E) \times P(E) \), where

\[
f(X) = (X \cap A, X \cup A).
\]

a. Prove that \( f \) is injective
b. Prove that \( \{f(x), x \in P(E)\} = \{(M, N) | M \subset A \subset N \subset E\} = K \).

c. Let \( g : P(E) \to K \), where \( g(X) = f(X) \). Prove that \( g \) is bijective and compute its inverse.

**Solution.**

a. \( f(X) = f(Y) \), namely \( (X \cap A, X \cup A) = (Y \cap A, Y \cup A) \) and then \( X \cap A = Y \cap A \), \( X \cup A = Y \cup A \), from where \( X \Delta A = Y \Delta A \) or \( (X \Delta A) \Delta A = (Y \Delta A) \Delta A \), \( X \Delta (A \Delta A) = Y \Delta (A \Delta A) \), \( X \Delta \emptyset = Y \Delta \emptyset \) and thus \( X = Y \), namely \( f \) is injective.

b. \( \{f(X), X \in P(E)\} = f(P(E)) \). We’ll show that \( f(P(E)) \subset K \). For any \( (M, N) \in f(P(E)) \), \( \exists X \in P(E) : f(X) = (M, N) \); \( (X \cap A, X \cup A) = (M, N) \).

From here \( X \cap A = M \), \( X \cup A = N \), namely \( M \subset A \) and \( A \subset N \) thus \( M \subset A \subset N \), and, therefore \( (M, N) \in X \).

Now, we’ll show that \( K \subset f(P(E)) \), for any \( (M, N) \in K \), \( \exists X \in P(E) \) such that \( f(X) = (M, N) \). \( f(X) = (M, N) \), namely \( (X \cap A, X \cup A) = (M, N) \) from where \( X \cap A = M \) and \( X \cup A = N \), namely \( X \Delta A = N - M \), \( (X \Delta A) \Delta A = (N - M) \Delta A \), \( X \Delta \emptyset = (N - M) \Delta A \), \( X = (N - M) \Delta A \), \( X = (N \cap \overline{M}) \Delta A \),

\[
X = ((N \cap \overline{M}) - A) \cup (A - (N \cap \overline{M})) = (N \cap \overline{M}) \cap A \cup (A \cap (N \cap \overline{M})) =
\]

\[
= (N \cap \overline{M} \cap A) \cup (A \cap (N \cap \overline{M})) = (N \cap \overline{A}) \cup ((A \cap N) \cup (A \cap M)) =
\]


\[ (N \cap \overline{A}) \cup (\emptyset \cup M) = (N - A) \cup M. \]

From here we get the unique solution: \( X = (N - A) \cup M. \)

We test \( f((N - A) \cup M) = \left( (N - A) \cup M \right) \cap A, (N - A) \cup M \cup A \)

but
\[ ((N - A) \cup M) \cap A = (N \cap \overline{A}) \cup M = (N \cap \overline{A}) \cup (M \cap A) = \]
\[ = (N \cap (\overline{A} \cap A)) \cup M = (N \cap \emptyset) \cup M = \emptyset \cup M = M \]

and
\[ ((N - A) \cup M) \cup A = (N - A) \cup (M \cup A) = (N - A) \cup A = (N \cap \overline{A}) \cup A = \]
\[ = (N \cup A) \cap (\overline{A} \cup A) = N \cap E = N, \quad f((N - A) \cup M) = (M, N). \]

Thus \( f(P(E)) = K. \)

c. From point a. we have that \( g \) is injective, from point b. we have that \( g \)
surjective, thus \( g \) is bijective. The inverse function is:

\[ g^{-1}(M, N) = (N - A) \cup M. \]

**Problem 3.** Let \( E \neq \emptyset, \quad A, B \in P(E) \) and \( f : P(E) \to P(E) \times P(E), \) where \( f(X) = (X \cap A, X \cap B). \)

a. Give the necessary and sufficient condition such that \( f \) is injective.

b. Give the necessary and sufficient condition such that \( f \) is surjective.

c. Supposing that \( f \) is bijective, compute its inverse.

**Solution.**

a. Suppose that \( f \) is injective. Then:

\[ f(A \cup B) = ((A \cup B) \cap A, (A \cup B) \cap B) = (A, B) = (E \cap A, E \cap B) = f(E), \]

from where \( A \cup B = E. \)

Now we suppose that \( A \cup B = E \), it results that:

\[ X = X \cap E = X \cap (A \cup B) = (X \cap A) \cup (X \cap B) = (Y \cap A) \cup (Y \cap B) = Y \cap (A \cup B) = Y \cap E = Y \]

namely from \( f(X) = f(Y) \) we obtain that \( X = Y \), namely \( f \) is injective.

b. Suppose that \( f \) is surjective, for any \( M, N \in P(A) \times P(B) \), there exists

\( X \in P(E), f(X) = (M, N), \ (X \cap A, X \cap B) = (M, N), \ X \cap A = M, \ X \cap B = N. \)

In special cases \( (M, N) = (A, \emptyset) \), there exists \( X \in P(E) \), from

\[ X \supset A, \emptyset = X \cap B \supset A \cap B, \ A \cap B = \emptyset. \]

Now we suppose that \( A \cap B = \emptyset \) and show that it is surjective.

Let \( (M, N) \in P(A) \times P(B) \), then \( M \subset A, \ N \subset B, \ \ M \cap B \subset A \cap B = \emptyset, \) and

\[ N \cap A \subset B \cap A = \emptyset, \] namely \( M \cap B = \emptyset, \ N \cap A = \emptyset \) and

\[ f(M \cup N) = ((M \cup N) \cap A, (M \cup N) \cap B) = \]
\[ = ((M \cap A) \cup (N \cap A), (M \cap B) \cup (N \cap B)) = (M \cup \emptyset, \emptyset \cup N) = (M, N), \]

8
for any \((M, N)\) there exists \(X = M \cup N\) such that \(f(X) = (M, N)\), namely \(f\) is surjective.

\[\text{c. We'll show that } f^{-1}((M, N)) = M \cup N.\]

**Remark.** In the previous two problems we can use the characteristic function of the set as in the first problem. We leave this method for the readers.

**Application.** Let \(E \neq \emptyset\), \(A_k \in P(E)\) \((k = 1, \ldots, n)\) and \(f : P(E) \to P^n(E)\), where \(f(X) = (X \cap A_1, X \cap A_2, \ldots, X \cap A_n)\).

Prove that \(f\) is injective if and only if \(\bigcup_{k=1}^n A_k = E\).

**Application.** Let \(E \neq \emptyset\), \(A_k \in P(E)\), \((k = 1, \ldots, n)\) and \(f : P(E) \to P^n(E)\), where \(f(X) = (X \cap A_1, X \cap A_2, \ldots, X \cap A_n)\).

Prove that \(f\) is surjective if and only if \(\bigcap_{k=1}^n \overline{A_k} = \emptyset\).

**Problem 4.** We name the set \(M\) convex if for any \(x, y \in M\) \(tx + (1 - t)y \in M\), for any \(t \in [0, 1]\).

Prove that if \(A_k\), \((k = 1, \ldots, n)\) are convex sets, then \(\bigcap_{k=1}^n A_k\) is also convex.

**Problem 5.** If \(A_k\), \((k = 1, \ldots, n)\) are convex sets, then \(\bigcap_{k=1}^n A_k\) is also convex.

**Problem 6.** Give the necessary and sufficient condition such that if \(A, B\) are convex/concave sets, then \(A \cup B\) is also convex/concave. Generalization for the \(\mathbb{N}\) set.

**Problem 7.** Give the necessary and sufficient condition such that if \(A, B\) are convex/concave sets then \(A \Delta B\) is also convex/concave. Generalization for the \(\mathbb{N}\) set.

**Problem 8.** Let \(f, g : P(E) \to P(E)\), where \(f(x) = A - X\), and \(g(x) = A \Delta X, A \in P(E)\).

Prove that \(f, g\) are bijective and compute their inverse functions.

**Problem 9.** Let \(A \circ B = \{(x, y) \in \square \times \square \mid \exists z \in \square : (x, z) \in A \text{ and } (z, y) \in B\}\). In a particular case let \(A = \{(x, \{x\}) \mid x \in \square\}\) and \(B = \{\{y\}, y\} \mid y \in \square\}.

Represent the \(A \circ A\), \(B \circ A\), \(B \circ B\) cases.

**Problem 10.**

i. If \(A \cup B \cup C = D, A \cup B \cup D = C, A \cup C \cup D = B, B \cup C \cup D = A\), then \(A = B = C = D\).
ii. Are there different $A$, $B$, $C$, $D$ sets such that

$A \cup B \cup C = A \cup B \cup D = A \cup C \cup D = B \cup C \cup D$?

**Problem 11.** Prove that $A \Delta B = A \cup B$ if and only if $A \cap B = \emptyset$.

**Problem 12.** Prove the following identity.

$$\bigcap_{i,j=1, i<j}^n A_i \cup A_j = \bigcup_{i=1}^n \left( \bigcap_{j=1, j>i}^n A_j \right)$$

**Problem 13.** Prove the following identities.

$$(A \cup B) - (B \cap C) = (A - (B \cap C)) \cup (B - C) = (A - B) \cup (A - C) \cup (B - C)$$

and

$$A - [(A \cap C) - (A \cap B)] = (A - B) \cup (A - C).$$

**Problem 14.** Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \cap B$ if and only if $A \subseteq B$ and $A \subseteq C$.

**Problem 15.** Prove the following identities:

$$(A - B) - C = (A - B) - (C - B),$$

$$(A \cup B) - (A \cup C) = B - (A \cap C),$$

$$(A \cap B) - (A \cap C) = (A \cap B) - C.$$

**Problem 16.** Solve the following system of equations:

$$\begin{cases}
A \cup X \cup Y = (A \cup X) \cap (A \cup Y) \\
A \cap X \cap Y = (A \cap X) \cup (A \cap Y)
\end{cases}$$

**Problem 17.** Solve the following system of equations:

$$\begin{cases}
A A \Delta X \Delta B = A \\
A \Delta Y \Delta B = B
\end{cases}$$

**Problem 18.** Let $X$, $Y$, $Z \subseteq A$. Prove that:

$Z = (X \cap \bar{Z}) \cup (Y \cap \bar{Z}) \cup (X \cap \bar{Z} \cap \bar{Y})$ if and only if $X = Y = \emptyset$.

**Problem 19.** Prove the following identity:

$$\bigcup_{k=1}^n [A_k \cup (B_k - C)] = \left( \bigcup_{k=1}^n A_k \right) \cup \left( \bigcup_{k=1}^n A_k \right) - C.$$

**Problem 20.** Prove that: $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

**Problem 21.** Prove that:
\[(A \Delta B) \Delta C = (A \cap B \cap C) \cup (\overline{A} \cap B \cap C) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap \overline{B} \cap C) \cup (A \cap B \cap C)\].

REFERENCES:

[1] Mihály Bencze, F. Popovici – Permutaciok - Matematikai Lapok, Kolozsvar, pp. 7-8, 1991.
[2] Pellegrini Miklós – Egy ujabb kiserlet, a retegezett halmaz. – M.L., Kolozsvar, 6, 1978.
[3] Halmazokra vonatkozo egyenletekrol – Matematikai Lapok, Kolozsvar, 6, 1970.
[4] Alkalmazasok a halmazokkal kapcsolatban - Matematikai Lapok, Kolozsvar, 3, 1970.
[5] Ion Savu – Produsul elementelor într-un grup finit comutativ – Gazeta Matematică Perf., 1, 1989.
[6] Nicolae Negoescu – Principiul includerii-excluderii – RMT 2, 1987.
[7] F. C. Gheorghe, T. Spiru – Teorema de prelungire a unei probabilități, dedusă din teorema de completare metrică – Gazeta Matematică, Seria A, 2, 1974.
[8] C. P. Popovici – Funcții Boolene – Gazeta Matematică, Seria A, 1, 1973.
[9] Algebra tankonyv IX oszt., Romania.
[10] Năstăescu stb. – Exerciții și probleme de algebră pentru clasele IX-XII – Romania.

[Published in Octogon, Vol. 6, No. 2, pp. 86-96, 1998.]