Classification of real three-dimensional Poisson–Lie groups

Ángel Ballesteros, Alfonso Blasco and Fabio Musso

Departamento de Física, Universidad de Burgos, 09001 Burgos, Spain
E-mail: angelb@ubu.es, ablasco@ubu.es and fimusso@ubu.es

Received 18 January 2012
Published 12 April 2012
Online at stacks.iop.org/JPhysA/45/175204

Abstract

All real three-dimensional Poisson–Lie (PL) groups are explicitly constructed and fully classified under group automorphisms by making use of their one-to-one correspondence with the complete classification of real three-dimensional Lie bialgebras given in Gomez (2000 J. Math. Phys. 41 4939). Many of these 3D PL groups are non-coboundary structures, whose Poisson brackets are given here for the first time. Casimir functions for all three-dimensional PL groups are given, and some features of several PL structures are commented.

PACS numbers: 02.20.Qs, 02.20.Uw, 02.30.Ik

1. Introduction

Poisson–Lie (PL) groups are Poisson structures on Lie groups for which the group multiplication is a Poisson map, and play an outstanding role in the theory of classical integrable models (see [2–9] and references therein). Indeed, they were initially introduced by Drinfel’d [2] to give a geometrical description of the Poisson algebra defined by the elements of the transition matrix for a large class of Hamiltonian systems that are integrable through the inverse scattering method. Later on, Semenov-Tian-Shansky [3] established the connection between PL groups and the group of dressing transformations of a completely integrable system (see also [9]). Moreover, quantum groups are just Hopf algebra quantizations of PL groups [10–14], quantum spaces are the quantizations of their corresponding Poisson homogeneous spaces [3, 15–17] and the Poisson analogues of quantum algebras can be understood as PL structures defined on the dual group [18, 19]. The relevance of certain PL groups and their Lie bialgebras in the theory of T-dual sigma models [20–22] and in (2+1) gravity [23–27] is also worth stressing. Moreover, being instances of Poisson–Hopf algebras, PL groups can be used to construct N-particle integrable systems through the so-called coalgebra symmetry method (see [28–31]) and some specific PL structures on solvable groups have recently been shown to arise as the Poisson manifolds underlying the integrability of a large class of Lotka–Volterra systems [32].

As established by Drinfel’d, there exists a one-to-one correspondence between the PL structures on a (connected and simply connected) Lie group $G$ and the Lie bialgebra structures
(g, δ) on its Lie algebra g. Therefore, the classification problem for PL structures on G is the same as the classification (under automorphisms) of the Lie bialgebra structures on g = Lie(G). Complementarily, we recall that if a given Lie bialgebra (g, δ) is a coboundary one, this means that the cocommutator δ is obtained from a classical r-matrix, and in this case the full PL group structure associated with (g, δ(r)) is obtained through the Sklyanin bracket given by r. For simple Lie algebras, this problem has been studied in [33, 34]; in this case, all Lie bialgebras are of the coboundary type and their classification reduces to obtain all constant solutions of the classical Yang–Baxter equation. However, for non-semisimple groups, non-coboundary Lie bialgebras can exist and for them no Sklyanin bracket is available. In this case, the PL group associated with a given non-coboundary Lie bialgebra has to be obtained by solving the compatibility conditions between the group product and the Poisson bracket and by imposing that the linearization of the latter corresponds to the Lie bialgebra under consideration (see [35–39]).

The aim of this paper is to complete the classification and the explicit construction of PL structures on all real 3D Lie groups by taking into account that the full classification of real 3D Lie bialgebra structures was performed in the remarkable paper [1]. It turns out that the vast majority of Lie bialgebra found in [1] for the non-semisimple cases were non-coboundaries and, to the best of our knowledge, the PL groups associated with many of these non-coboundaries have not been explicitly constructed so far. Throughout the paper, we will follow the notation from [40, 41], which classifies all real 3D Lie algebras into nine classes called A3,i (i = 1, . . . , 9), which we will appropriately connect with the classification used in [1] (see [42] for the translation into the original Bianchi classification).

Therefore, in this work we present the new classifications of PL groups corresponding to the solvable Lie algebras A3,2, A3,4 (the (1+1) Poincaré algebra), A3,5 and A3,7, together with the corresponding Casimir function for each of the PL brackets. We recall that all the 3D real coboundary PL groups have been explicitly constructed in [43] through their corresponding r-matrices, and all these results will be also recovered here in a more general framework. Also, PL structures on the Heisenberg–Galilei (1+1) group A3,1 were obtained in [44] and their classification in terms of their corresponding Lie bialgebra structures was performed in [45, 36]; the classification of PL structures on the Euclidean group A3,6 was given in [35] and the ‘book group’ A3,3 structures have recently been analysed in [19] through the same methodology as we will follow in the present paper and with the emphasis put on their applications in integrable PL dynamics. We emphasize that all these known results will be recovered from a common computational perspective and by adding the Casimir functions for all the PL structures.

The paper is organized as follows. In section 2, we tersely recall the relevant theory of PL groups needed for the scope of this paper and, simultaneously, we also describe the methodology that we have used in order to obtain the results that we will present in the following. In section 3, we give the complete classification for PL groups corresponding to each of the inequivalent real 3D Lie algebras. For each of these nine Lie groups, we find by direct computation the most generic (multiparametric) Poisson bracket that is compatible with the group multiplication. Afterwards, we will compare the (dual of the) linearization of such generic PL bracket with the classification of Lie bialgebra structures on the corresponding Lie algebra given in [1], thus obtaining the equivalence classes of PL structures. Among them, both the coboundary and non-coboundary cases are identified, and the Casimir functions for all of them are computed. In order to illustrate the method, the new classification for the 3D solvable group generated by the Lie algebra A3,2 is first explained in detail, and for the rest of the cases we present the results in a schematic way. In section 4, we give some concluding remarks on the results presented here and suggest some of their possible applications.
2. Method

2.1. PL structures on a Lie group G

A PL group is a Lie group $G$ together with a Poisson structure $\{,\}$ on $C^\infty(G)$, such that the multiplication $\mu : G \otimes G \to G$ is a Poisson map, namely

$$\{ f \circ \mu, g \circ \mu \} = \{ f, g \} \circ \mu(u,v), \quad u,v \in G, \quad f, g \in C^\infty(G). \quad (1)$$

In the language of Poisson–Hopf algebras [12], the pull-back of the multiplication $\mu$ on $G$ defines a coproduct map $\Delta : C^\infty(G) \to C^\infty(G \otimes G)$ through

$$\Delta(f)(u \otimes v) = f(\mu(u,v)) \quad u,v \in G, \quad f \in C^\infty(G). \quad (2)$$

In terms of the coproduct map $\Delta$, the homomorphism property (1) can be written in the form

$$\{ \Delta(f), \Delta(g) \} = \Delta(\{ f, g \}). \quad (3)$$

Our aim is to obtain explicitly and to classify all the (simply connected) real PL groups of dimension 3. It is well known that there exist nine non-isomorphic real 3D Lie algebras that cannot be decomposed as a direct sum of lower dimensional real Lie algebras, for which we will follow the structure constants and the basis $\{e_1, e_2, e_3\}$ given in [40]. For any of these Lie algebras $g$, we will select a faithful three-dimensional representation $\varrho$ and construct the matrix Lie group element as follows:

$$M = \exp(z \varrho(e_1)) \exp(y \varrho(e_2)) \exp(x \varrho(e_3)).$$

Then, we will introduce a set of coordinate functions on $M$ and we use equation (2) to define their coproduct. For instance, if $X$ is the coordinate function corresponding to the $i, k$ entry of $M$,

$$X(M) = M_{ik},$$

then from equation (2), it follows that the coproduct of $X$ will be given by

$$\Delta(X)(M \otimes M) = \sum_{j=1}^{3} M_{ij} \otimes M_{jk}. \quad (4)$$

Once the coproduct is defined for the three coordinate functions (denoted by $X, Y, Z$ and expressed in terms of the initial ones $(x, y, z)$), we will look for the most generic (multiparametric) Poisson bracket for which equation (3) holds. Note that if the coordinate functions correspond to linear combinations of the $M_{ij}$ entries, then the natural Ansatz for the Poisson bracket is a quadratic (and obviously antisymmetric) expression in the matrix group entries ($X^1 = X, X^2 = Y, X^3 = Z$) of the form

$$\{X^\alpha, X^\beta\}(M) = \sum_{i,j,k=1}^{3} c_{ijk}^\alpha \beta M_{ij} M_{jk}, \quad \alpha, \beta = 1, 2, 3, \quad (5)$$

where $c_{ijk}^\alpha \beta = -c_{ijk}^\beta \alpha$ are constant parameters to be determined. Plugging the coordinate functions into equation (3) and using Ansatz (5), we obtain a set of linear equations for the coefficients $c_{ijk}^\alpha \beta$ that can be easily solved by using a symbolic manipulation program. Afterwards, we impose the Jacobi identity

$$\{X, \{Y, Z\}\} + \{Z, \{X, Y\}\} + \{Y, \{Z, X\}\} = 0$$

and obtain a set of quadratic equations for the remaining coefficients $c_{ijk}^\alpha \beta$. Since condition (3) is quite restrictive, it turns out that for the nine groups it is always possible to find the general solution of these equations.
In this way, we obtain a multiparametric Poisson bracket on $C^\infty(G)$ that is both invariant under coproduct (4) and quadratic in the group matrix entries. Now we have to check whether it is the most general one, in the sense that it contains as particular cases all the inequivalent (under group automorphisms) PL structures on $G$, which have to be unambiguously identified. Indeed, since the hypothesis that the Poisson bracket is quadratic in the group matrix entries is restrictive, this could not be the case. To this aim, we use the one-to-one correspondence between PL groups on $G$ and Lie bialgebra structures on $g$ (see [12]) that we describe in the following.

As we will show, this Ansatz works for seven of the nine three-dimensional real Lie groups. As we will explicitly comment for the remaining two cases, in each case only one non-quadratic term has to be added to the quadratic Ansatz (5) on the PL bracket in order to get one ‘lost’ PL structure. Apart from that minor variation, the proposed methodology is exactly the same, works for the nine 3D cases and could be used in higher dimensional cases.

2.2. Classification through the correspondence with Lie bialgebras on $g$

If $G$ is a PL group with Lie algebra $g$, then it is always possible to define a canonical Lie algebra structure on $g^*$ through

\[ [\xi_1, \xi_2]^e = (d[f_1, f_2])^e, \]  

where $\xi_1, \xi_2 \in g^*$, $e$ is the identity element of $G$ and $f_1, f_2 \in C^\infty(G)$ are chosen in such a way that $(df_i)^e = \xi_i$. This same bracket can be written in the form

\[ [\xi_1, \xi_2]^e = \delta^*(\xi_1 \otimes \xi_2), \]  

where $\delta^*$ is the dual of a 1-cocyle of $g$ with values in $g \otimes g$ (the cocommutator),

\[ \delta([X, Y]) = [\delta(X), Y \otimes 1 + 1 \otimes Y] + [X \otimes 1 + 1 \otimes X, \delta(Y)]. \]  

The pair $(g, \delta)$ is called the tangent Lie bialgebra of the PL structure on $G$. We have the following key theorem, due to Drinfel’d [2].

**Theorem 1.** Let $G$ be a Lie group with Lie algebra $g$. If $G$ is a PL group, then $g$ has a natural Lie bialgebra structure, called the tangent Lie bialgebra of $G$. Conversely, if $G$ is connected and simply connected, every Lie bialgebra structure on $g$ is the tangent Lie algebra of a unique Poisson structure on $G$ which makes $G$ into a PL group.

So in order to identify all PL structures on $G$, we have to classify all the possible cocommutators $\delta$ on the Lie algebra $g$, i.e. all the tangent Lie bialgebras on $g$.

Again this implies the solution of a set of linear equations (the cocycle condition (8)) for a coantisymmetric map $\delta$ with arbitrary coefficients, together with a further quadratic equation on them (the Jacobi coidentity that ensures that $g^*$ is a Lie algebra). Once this problem has been solved, we can compare the Lie algebra structure on $g^*$ coming from cocommutator (7) with the one (6) coming from the linearization of our quadratic PL bracket. By theorem 1, if we find that the latter one is a particular case of the former one, then our PL bracket is not the most general one. This happens only for two solvable Lie algebras, namely for $A_{3,4}$ and $A_{3,6}$.

In both cases, we find that it suffices to add a single non-quadratic term in order to obtain a family of generic PL brackets on the group $G$. Finally, the corresponding Casimir functions for these generic brackets are also explicitly found.

Afterwards, in order to classify all the so-obtained PL structures in equivalence classes under group automorphisms, we first identify all the coboundary cases. This is done by solving
the modified classical Yang–Baxter equation on the Lie algebra $g$ associated with the Lie group $G$ and by using the components $r^{ij}$ of any of these solutions to construct the Sklyanin bracket

$$\{f, g\} = r^{ij} (X^i_L f X^j_L g - X^i_R f X^j_R g)$$

$f, g \in C^\infty(G)$,

where $X^i_L, X^i_R, i = 1, 2, 3$ are, respectively, the left- and right-invariant vector fields under the group action. By comparing this Sklyanin bracket with the generic one previously obtained, we can identify the particular values of the free parameters in our generic PL brackets that give rise to coboundary structures. Obviously, all the remaining ones will be non-coboundaries.

Finally, the equivalence classes of PL structures are identified by using Gomez’s classification [1] of all inequivalent bialgebra structures (under Lie algebra automorphisms) on the nine real three-dimensional algebras. Indeed, by theorem 1, all the Lie algebra structures induced on $g^*$ by the inequivalent cocommutators on $g$ (see equation (7)) found in [1] have to appear as particular cases (corresponding to particular values of the parameters coming from the full PL structures previously computed) of the linearization of our PL structures on $G$ (see equation (6)). Due to the one-to-one correspondence between PL structures on the Lie group $G$ and the Lie bialgebra structures on its Lie algebra $g$, a representative of each of the equivalence classes of PL structures on $G$ can be explicitly obtained, and the classification problem is solved.

3. Results

In this section, we present in a schematic way the full classification of 3D real PL groups. We present in more detail the computations for the solvable Lie algebra $A_{3,2}$ and for the rest of the cases, we list the relevant results in the same order and with the same notation.

3.1. PL structures on the solvable group generated by $A_{3,2}$

1) **Commutation relations:** we write (following [40]) the commutation relations of the Lie algebra under scrutiny. In particular, the solvable Lie algebra $A_{3,2}$ is defined by the commutation relations

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2, \quad [e_1, e_2] = 0.$$

2) **Representation:** we give a faithful three-dimensional representation $\varphi$, which in the case of $A_{3,2}$ is given by

$$\varphi(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi(e_3) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3) **Matrix group element:** the matrix group element

$$M = \exp(z \varphi(e_1)) \exp(y \varphi(e_2)) \exp(x \varphi(e_3))$$

is computed in terms of the local coordinates on the group given by $(z, y, x)$. When useful, we will define a change of variables by rewriting the $M$-entries using capital letters. The aim of this change of variables is to simplify the expressions for the coproduct and for the PL brackets (and also to simplify the identification of Poisson brackets that are quadratic in the matrix group element entries). The expression for the matrix group element is used to compute the group multiplication law and the corresponding coproduct.

Namely, the matrix group element for $A_{3,2}$ is given by

$$M = \begin{pmatrix} \exp(-x) & -x \exp(-x) & y + z \\ 0 & \exp(-x) & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X & \ln(X)X & Z \\ 0 & X & Y \\ 0 & 0 & 1 \end{pmatrix}.$$
Hereafter we will use $X, Y, Z$ as coordinates on the group; since $X = \exp(-x)$, the coordinate $X$ has the constraint $X > 0$.

(4) **Coproduct:** it comes from the group multiplication law, and in terms of the coordinate functions is obtained by solving the set of equations ($r = \dim(M)$)

$$\Delta(M_{ij}) = \sum_{k=1}^{r} M_{ik} \otimes M_{kj}, \quad i, j = 1, \ldots, r,$$

for $X, Y, Z$ (here $M_{ij}$ denotes the function that maps the group element $M$ onto its $i,j$ entry).

In the case of $A_{3,2}$, we realize that the $X$ coordinate can be obtained by taking the $M_{11}$ entry, the $Y$ coordinate is just $M_{23}$ and $Z$ is $M_{13}$. According to definition (2), the coproduct of $X, Y, Z$, evaluated on the two group elements $M_1$ and $M_2$, will be given, respectively, by the (11), (23) and (13) entries of the product $M_1 \cdot M_2$. Namely,

$$M_1 \cdot M_2 = \begin{pmatrix} X_1 & X_1 \ln(X_1) & Z_1 \\ 0 & X_1 & Y_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_2 & X_2 \ln(X_2) & Z_2 \\ 0 & X_2 & Y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} X_1X_2 & X_1X_2 \ln(X_1X_2) & X_1Z_2 + X_1 \ln(X_1)Y_2 + Z_1 \\ 0 & X_1X_2 & X_1Y_2 + Y_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now by the identification

$$X_1 = X \otimes 1, \quad X_2 = 1 \otimes X, \quad Y_1 = Y \otimes 1, \quad Y_2 = 1 \otimes Y, \quad Z_1 = Z \otimes 1, \quad Z_2 = 1 \otimes Z,$$

we obtain the coproducts for the coordinate functions

$$\Delta(X) = X \otimes X,$$

$$\Delta(Y) = X \otimes Y + Y \otimes 1,$$

$$\Delta(Z) = X \otimes Z + X \ln(X) \otimes Y + Z \otimes 1.$$

(5) **PL brackets:** the most general PL bracket on the group is obtained through Ansatz (5) by solving equations (3) for the coproducts of the coordinate functions. In the $A_{3,2}$ case, the most generic quadratic Poisson bracket for which (9) is a Poisson map is given by the three-parametric structure

$$\{X, Y\}_1 = 0,$$

$$\{X, Z\}_1 = -a_1X^2 + b_1XY + a_1X,$$

$$\{Y, Z\}_1 = c_1(1 - X^2) + \frac{b_1}{2}Y^2 + a_1Y$$

(10)

Note that since one of the group entries is the unity, the quadratic bracket contains linear and constant terms. The Jacobi identity for this bracket is automatically satisfied, and all the possible Lie bialgebra structures on $A_{3,2}$ can be obtained as the dual of the linearization of (10) (see equation (6)). The subindex in the Poisson bracket labels the different possible families of solutions for the PL bracket when compatibility (3) is imposed. In this case, there is only one of such families.

(6) **Casimir function:** the Casimir function for (10) is found to be

$$\mathcal{C} = \frac{2c_1(1 + X^2) + Y(-2a_1(-1 + X) + b_1Y)}{X},$$

which is real and well defined for all values of the parameters $(a_1, b_1, c_1)$. As we shall see in other groups, the Casimir function could be different depending on the values of the parameters appearing in the generic PL structure.
(7) **Coboundary cases:** the most general skewsymmetric candidate for the constant classical \( r \)-matrix on the Lie algebra \( A_{3,2} \) is

\[
    r = r^{12} e_1 \wedge e_2 + r^{13} e_1 \wedge e_3 + r^{23} e_2 \wedge e_3,
\]

where \( (r^{12}, r^{13}, r^{23}) \) are free real parameters. Now we have to impose that \( r \) is a solution of the mCYBE

\[
    [\xi \otimes 1 \otimes 1 + 1 \otimes \xi \otimes 1 + 1 \otimes 1 \otimes \xi, [[r, r]]] = 0, \quad \xi \in A_{3,2},
\]

and this condition leads to \( r^{23} = 0 \). Therefore, the coboundary PL structures for \( A_{3,2} \) will be generated by

\[
    r = r^{12} e_1 \wedge e_2 + r^{13} e_1 \wedge e_3.
\]

In order to identify these structures within (10), we compute the left- and right-invariant vector fields for the \( A_{3,2} \) group, which read

\[
    L_1 = L_x = e^{-x} \frac{\partial}{\partial x}, \quad R_1 = R_x = \frac{\partial}{\partial x},
\]

\[
    L_2 = L_y = -x e^{-x} \frac{\partial}{\partial x} + e^{-x} \frac{\partial}{\partial y}, \quad R_2 = R_y = \frac{\partial}{\partial y},
\]

\[
    L_3 = L_z = \frac{\partial}{\partial z}, \quad R_3 = R_z = -(y + z) \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.
\]

Using them we can compute the Sklyanin bracket

\[
    \{ f, g \} = r^{11} \left( L_x f L_y g - R_x f R_y g \right)
\]

that for the coordinate functions reads

\[
    \{ x, y \} = 0, \quad \{ x, z \} = r^{13} (1 - \exp(-x)), \quad \{ y, z \} = r^{12} (1 - \exp(-2x)) - r^{13} y.
\]

Passing to the \( X, Y, Z \) variables, we obtain

\[
    \{ X, Y \} = 0, \quad \{ X, Z \} = r^{13} X (X - 1), \quad \{ Y, Z \} = -r^{12} (X^2 - 1) - r^{13} Y.
\]

Therefore, we conclude that (10) is a coboundary structure if \( b_1 = 0 \), with \( a_1 = -r^{13} \) and \( c_1 = r^{12} \). Non-coboundary cases will appear whenever \( b_1 \neq 0 \).

(8) **Isomorphism with Lie algebras in [1]:** first of all we have to identify the change of bases in the Lie algebra that transforms our algebra into the generators \( e_0, e_1, e_2 \) used in [1].

In particular, the algebra \( A_{3,2} \) is isomorphic to the algebra \( \tau'_1(1) \) in Gomez classification through

\[
    e_1 = e_1 \quad e_2 = e_2 \quad e_3 = -e_0.
\]

(9) **Correspondence with the classification of Lie bialgebras [1]:** in the case of \( A_{3,2} \), the linearization of the PL bracket (10) gives

\[
    \{ x, y \} = 0, \quad \{ x, z \} = -a_1 x - b_1 y, \quad \{ y, z \} = 2 c_1 x + a_1 y,
\]

where \( x, y, z \) are the duals to \( e_3, e_2, e_1 \), respectively. The corresponding cocommutator (in the basis [1]) is given by

\[
    \delta(e_0) = -a_1 e_0 \wedge e_1 + 2 c_1 e_1 \wedge e_2, \quad \delta(e_1) = 0 \quad \delta(e_2) = -a_1 e_1 \wedge e_2 + b_1 e_0 \wedge e_1.
\]

By comparing cocommutator (11) with the four inequivalent classes of Lie bialgebra structures for \( \tau'_1(1) \) found by Gomez, we find which particular values of the parameters
In general, this correspondence will be summarized in the form of a table (table 1). In its first column, we write the number that identifies the type of Lie bialgebra (last column of table III in [1]), and the symbol (∗) is used to distinguish the coboundary Lie bialgebras. In the second one, we give the Poisson bracket {, }, we are considering (in the case of $A_{3,2}$ we have only one) and in the following columns, the particular values of the $(a_1, b_1, c_1)$ parameters for which the linearization of the PL bracket coincides with the Lie bialgebra parameters from [1]. We recall here the assumptions of [1] on the parameters $\alpha, \beta, \lambda, \omega$. All are assumed to be nonzero real numbers; moreover, $\alpha$ and $\beta$ can be rescaled (by an appropriate automorphism of the Lie algebra $g$) to arbitrary nonzero values, $\omega$ can be rescaled to any nonzero value of the same sign and $\lambda$ is an essential parameter. Also the parameters $\rho$ and $\mu$ are subject to the constraints $-1 \leq \rho \leq 1$ and $\mu \geq 0$.

In this way, each row in the table corresponds to one of the inequivalent PL structures on the group under consideration, which can be explicitly obtained by substituting the values of the $(a_1, b_1, c_1)$ parameters given in the table into the generic expression (10). In this way, for $A_{3,2}$, the following inequivalent PL structures are obtained.

| Lie bialgebra in [1] | {, } | $a_1$ | $b_1$ | $c_1$ |
|----------------------|------|------|------|------|
| 12 (∗)              | [, ]| 0    | 0    | $-\omega$ |
| (8) (∗)             | [, ]| 1    | 0    | 0    |
| 13                  | [, ]| 0    | $\lambda$ | 0 |
| 14                  | [, ]| 0    | $\lambda$ | $-\omega$ |

Remarks: note that there is not a one-to-one correspondence between the number of parameters in the generic PL bracket (10) and the number of inequivalent classes of PL structures. Also, we mention that there seems to be a misprint in [1], since in order to obtain correct expressions for the cocommutators of these Lie bialgebras, the generators $e_1$ and $e_2$ have to be interchanged.

3.2. PL structures on the Heisenberg group generated by $A_{3,1}$

(1) Commutation relations:

$[e_2, e_1] = e_1$, $[e_1, e_2] = 0$, $[e_1, e_3] = 0$.

Note that these are also the commutation rules of the massless (1+1) Galilei Lie algebra.

(2) Representation:

$\varrho(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\varrho(e_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\varrho(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(3) Matrix group element:

$M = \begin{pmatrix} 1 & y & xy+z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$.

(4) Coproduct:

$\Delta(X) = X \otimes 1 + 1 \otimes X$,

$\Delta(Y) = Y \otimes 1 + 1 \otimes Y$,

$\Delta(Z) = Z \otimes 1 + 1 \otimes Z + Y \otimes X$. 
(5) PL brackets: in this case we obtain three multiparametric families of Poisson structures that are compatible with the previous coproduct:

\[
\begin{align*}
[X, Y]_1 &= a_1 X + b_1 Y, \\
[X, Z]_1 &= \frac{1}{2} X^2 + c_1 X + d_1 Y + b_1 Z, \quad (b_1 \neq 0) \\
[Y, Z]_1 &= -\frac{a_1^2 d_1}{b_1^2} X - \frac{b_1}{2} Y^2 - \frac{2a_1 d_1 - b_1 c_1}{b_1} Y - a_1 Z; \\
[X, Y]_2 &= 0, \\
[X, Z]_2 &= a_2 X + b_2 Y, \\
[Y, Z]_2 &= c_2 X + d_2 Y; \\
[X, Y]_3 &= a_3 X, \\
[X, Z]_3 &= \frac{a_3}{2} X^2 + b_3 X, \quad (a_3 \neq 0) \\
[Y, Z]_3 &= -c_3 X + b_3 Y - a_3 Z.
\end{align*}
\]

(6) Casimir function: note that for the structures 2 and 3, it depends on the values of the corresponding parameters, which are indicated between parentheses:

\[
C_1 = \frac{2(b_1 c_1 - a_1 d_1) X + b_1^2 (2Z - XY) - 2d_1 (a_1 X + b_1 Y) \log (a_1 X + b_1 Y)}{(a_1 X + b_1 Y)}
\]

\[
C_2 = \left\{ \begin{array}{ll}
\left[\frac{(a + a_2 - d_2) Y - 2c_2 X}{(a_2 + d_2) Y + 2c_2 X}\right] & (a_2 + d_2) \arctan \left( \frac{2c_2 X - (a_2 - d_2) Y}{(a_2 + d_2) Y + 2c_2 X} \right) + \frac{1}{2} \alpha \ln \left[ \frac{\alpha^2}{4} Y^2 + \left( \frac{a_2 - d_2}{2} Y - c_2 X \right)^2 \right] & (a_2 + d_2) > 0 \\
\exp \left( \frac{2(a_2 + d_2) c_2 Y}{a_2 - d_2 + 2c_2 X} \right) & (a_2 - d_2) > 0 \end{array} \right.
\]

\[
C_3 = \left\{ \begin{array}{ll}
X \exp \left( -\frac{a_2 Y}{c_2 X} \right) & (d_2 = a_2, \quad b_2 = 0, \quad c_2 \neq 0) \\
Y \exp \left( -\frac{a_2 X}{b_2 Y} \right) & (a_2 = d_2, \quad c_2 = 0, \quad b_2 \neq 0) \\
X & (a_2 = d_2, \quad b_2 = c_2 = 0) \\
Y & (c_3 \neq 0)
\end{array} \right.
\]

\[
\begin{align*}
C_3 &= \left\{ \begin{array}{ll}
\frac{X \exp \left( \frac{2b_3 Y + a_3 (XY - 2Z)}{2b_3 Y + a_3 (XY - 2Z)} \right)}{X} & (c_3 = 0)
\end{array} \right.
\]
(7) Coboundary cases: the most generic classical $r$-matrix on this algebra is
\[ r = r^{12} e_1 \wedge e_2 + r^{13} e_1 \wedge e_3 + r^{23} e_2 \wedge e_3. \]

The Sklyanin bracket associated with this $r$ is just the PL bracket $\{ \cdot, \cdot \}_2$ provided that
\[ a_2 = d_2 = -r^{23}, \quad b_2 = c_2 = 0 \]
(the coefficients $r^{12}$ and $r^{13}$ do not play any role in the above Poisson bracket). Therefore, the PL brackets $\{ \cdot, \cdot \}_1$ and $\{ \cdot, \cdot \}_3$ are always non-coboundary ones.

(8) Isomorphism with the $\mathfrak{n}_3$ Lie algebra in [1]:
\[ e_1 = e_0, \quad e_2 = e_1, \quad e_3 = e_2. \]

(9) Correspondence with the classification of Lie bialgebras.

Table 2. Classification of PL structures on $A_{3,1}$.

| Lie bialgebra in [1] | $\{ \cdot, \cdot \}_1$ | $a_i$ | $b_i$ | $c_i$ | $d_i$ |
|---------------------|------------------------|-------|-------|-------|-------|
| (5–5′)              | $\{ \cdot, \cdot \}_2$| $-\rho$| 0     | 0     | $-1$  |
| (12)                | $\{ \cdot, \cdot \}_2$| $-1$  | 0     | 1     | $-1$  |
| (15)                | $\{ \cdot, \cdot \}_2$| $-\mu$| 1     | 1     | $-\mu$|
| 17                  | $\{ \cdot, \cdot \}_2$| 0     | 0     | 1     | 0     |
| (13)                | $\{ \cdot, \cdot \}_3$| $-1$  | 0     | $\lambda$| $-\mu$|
| (10)                | $\{ \cdot, \cdot \}_3$| $-1$  | 0     | 0     | $-\mu$|

(10) Remarks: a first study of the PL structures on the Heisenberg group and their quantizations was performed in [44] and Heisenberg Lie bialgebras together with their corresponding quantum algebras were presented in [45]. The classification and construction of Heisenberg PL groups given in [36] are included in table 2. The only coboundary structure corresponds to the ‘isolated point’ (5–5′) with $\rho = 1$. It can also be checked that the PL brackets $\{ \cdot, \cdot \}_1$ and $\{ \cdot, \cdot \}_3$ are isomorphic for generic values of the parameters, which explains the absence of $\{ \cdot, \cdot \}_1$ in table 2.

### 3.3. PL structures on the ‘book group’ generated by $A_{3,3}$

(1) Commutation relations:
\[ [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_2] = e_0. \]

(2) Representation:
\[ \varrho(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varrho(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varrho(e_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

(3) Matrix group element:
\[ M = \begin{pmatrix} \exp(-x) & 0 & z \\ 0 & \exp(-y) & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X & 0 & Z \\ 0 & X & Y \\ 0 & 0 & 1 \end{pmatrix}, \quad X > 0. \]

(4) Coproduct:
\[ \Delta(X) = X \otimes X, \]
\[ \Delta(Y) = X \otimes Y + Y \otimes 1, \]
\[ \Delta(Z) = X \otimes Z + Z \otimes 1. \]
(5) PL brackets:

\[ [X, Y]_1 = a_1(X^2 - X) - b_1XY - 2c_1XZ, \]
\[ [X, Z]_1 = d_1(X^2 - X) + 2e_1XY + b_1XZ, \]
\[ [Y, Z]_1 = f_1(1 - X^2) + e_1Y^2 + b_1YZ - d_1Y + c_1Z^2 + a_1Z. \]

(6) Casimir:

\[ C = \frac{1}{X} \left( f_1(1 + X^2) + d_1(-1 + X)Y + e_1Y^2 + a_1Z(1 - X) + Z(b_1Y + c_1Z) \right) . \]

(7) Coboundary cases: the most generic classical \( r \)-matrix is now

\[ r = r_{12}e_1 \wedge e_2 + r_{13}e_1 \wedge e_3 + r_{23}e_2 \wedge e_3, \]

which corresponds to the PL brackets given by

\[ a_1 = r_{23}^3, \quad d_1 = r_{13}^3, \quad f_1 = r_{12}^1, \quad b_1 = c_1 = e_1 = 0. \]

(8) Isomorphic to the algebra \( \tau_3(1) \) in [1] through the change of variables

\[ e_1 = e_1, \quad e_2 = e_2, \quad e_3 = -\epsilon_0. \]

(9) Correspondence with the classification of Lie bialgebras.

| Lie bialgebra in [1] | \[,\] | \( a \) | \( b \) | \( c \) | \( d \) | \( e \) | \( f \) |
|----------------------|------|------|------|------|------|------|------|
| 5 (\( \rho = 1 \)) (\( \ast \)) | \[,\] | 0 | 0 | 0 | 0 | 0 | -1 |
| 6 (\( \rho = 1, \chi = \epsilon_0 \wedge \epsilon_1 \)) (\( \ast \)) | \[,\] | 0 | 0 | 0 | -1 | 0 | 0 |
| 7 (\( \rho = 1 \)) | \[,\] | 0 | \( \lambda \) | 0 | 0 | 0 | 0 |
| (1) | \[,\] | 0 | \( \lambda \) | 0 | 0 | 0 | -\( \alpha \) |
| (2) | \[,\] | 0 | 0 | \( \lambda/2 \) | 0 | \( \lambda/2 \) | -\( \omega \) |
| 9 | \[,\] | 0 | 0 | \( \lambda/2 \) | 0 | \( \lambda/2 \) | 0 |
| 10 | \[,\] | 0 | 0 | -\( 1/2 \) | 0 | 0 | 0 |
| 11 | \[,\] | 0 | 0 | -\( 1/2 \) | 0 | 0 | -\( \omega \) |
| (3) | \[,\] | 0 | 0 | -\( 1/2 \) | -\( \alpha \) | 0 | -\( \omega \) |

(10) Remarks: note that in table 3 the parameter \( a \) is always 0. This is due to the fact that \( a \) is equivalent to \( d \) under the Lie group automorphism \( Y \leftrightarrow Z \). This classification of PL structures on the book group has recently been presented in [19], where it has been explicitly shown that many of these structures correspond—under suitable changes of local coordinates—to Poisson versions of 3D quantum algebras (see also [18], in which the PL group corresponding to (1) with \( b_1 = 1/f_1 \) was constructed). Also, the PL structure given by case (7) has recently been shown to underlie the integrability of a class of 3D Lotka–Volterra systems (see [19] and references therein).

### 3.4. PL structures on the (1+1) Poincaré group generated by \( A_{3,4} \)

(1) Commutation relations:

\[ [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2, \quad [e_1, e_2] = 0. \]

(2) Representation:

\[ \varphi(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi(e_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
(3) Matrix group element:

\[
M = \begin{pmatrix}
\exp(-x) & 0 & z \\
0 & \exp(x) & -y \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
X & 0 & Z \\
0 & X^{-1} & Y \\
0 & 0 & 1
\end{pmatrix}, \quad X > 0.
\]

(4) Coproduct:

\[
\Delta(X) = X \otimes X,
\]

\[
\Delta(Y) = X^{-1} \otimes Y + Y \otimes 1,
\]

\[
\Delta(Z) = X \otimes Z + Z \otimes 1.
\]

(5) PL brackets: we have two different families,

\[
\{X, Y\}_1 = -a_1 XY + b_1 (X - 1),
\]

\[
\{X, Z\}_1 = a_1 YZ - c_1 Y - b_1 Z;
\]

\[
\{X, Y\}_2 = a_2 (1 - X),
\]

\[
\{X, Z\}_2 = b_2 (X^2 - X),
\]

\[
\{Y, Z\}_2 = b_2 Y + a_2 Z + c_2 \ln(X).
\]

(6) Casimirs:

\[
C_1 = \begin{cases} 
  c_1 (X - 1) + a_1 Z & (a_1 \neq 0) \\
  b_1 (1 - X) + a_1 XY & (a_1 = 0)
\end{cases}
\]

\[
C_2 = \exp \left[ \frac{b_2 XY + a_2 Z}{c_2 (X - 1)} \right] \left( \frac{X Y}{X - 1} \right).
\]

(7) Coboundary cases: the generic classical r-matrix is

\[
r = r^{12} e_1 \wedge e_2 + r^{13} e_1 \wedge e_3 + r^{23} e_2 \wedge e_3,
\]

which means that the bracket \(\{\, , \}\_1\) is coboundary for the following values of the parameters:

\[
a_1 = 0, \quad b_1 = r^{23}, \quad c_1 = -r^{13}
\]

(the coefficient \(r^{12}\) does not play any role in the above Poisson bracket). The bracket \(\{\, , \}\_2\) is always a non-coboundary one.

(8) Isomorphic to the algebra \(\tau_3 (-1)\) in [1] through the change of variables

\[
e_1 = \varepsilon_1, \quad e_2 = \varepsilon_2, \quad e_3 = -\varepsilon_0.
\]

(9) Correspondence with the classification of Lie bialgebras.

**Table 4.** Classification of PL structures on \(A_{3,4}\).

| Lie bialgebra in [1] | \(\{\, , \}\) | \(a_i\) | \(b_i\) | \(c_i\) |
|---------------------|---------------|---------|---------|---------|
| 6 \((\rho = -1, X = \varepsilon_0 \wedge \varepsilon_1)\) \(\ast\) | \([1, 1]\) | 0 | 0 | 1 |
| 7 \((\rho = -1)\) | \([1, 1]\) | \(-\lambda\) | 0 | 0 |
| (11) \(\ast\) | \([1, 1]\) | 0 | \(-\alpha\beta\) | \(\alpha\) |
| 5 | \([1, 1]\) | 0 | 0 | 1 |
| 8 | \([1, 2]\) | \(-\alpha\) | 0 | 1 |
| (14) | \([1, 2]\) | \(\alpha\lambda\) | \(-\alpha\) | 1 |
Remarks: the \( A_{3,5} \) algebra is isomorphic to the (1+1) Poincaré algebra written in a ‘null-plane’ basis. Note that the Poisson bracket \([ , ]\) is not quadratic in the group matrix entries since it contains a term of the type \( c \ln(X) \), which we have been forced to include if we want to recover the non-coboundary PL structures 5, 8 and (14). If we write both families of Poisson brackets in the local coordinates \((x, y, z)\), we obtain
\[
\{x, y\}_1 = a_1 y + b_1 (1 - \exp(x))
\]
\[
\{x, z\}_1 = c_1 \exp(-x) - 1 + a_1 z
\]
\[
\{y, z\}_1 = y(a_1 z - c_1) + b_1 z
\]
\[
\{x, y\}_2 = a_2 \exp(x) - 1
\]
\[
\{x, z\}_2 = b_2 (1 - \exp(-x))
\]
\[
\{y, z\}_2 = c_2 x + b_2 y - a_2 z
\]
where the \( \{y, z\}_1 \) bracket would be the Poisson version for the non-commutative (null-plane) Minkowski spacetime associated with the corresponding quantum Poincaré group (see [46] and references therein).

### 3.5. PL structures on the solvable group generated by \( A_{3,5} \)

1. **Commutation relations:**
   \[
   [e_1, e_3] = e_1, \quad [e_2, e_3] = \rho e_2, \quad [e_1, e_2] = 0, \quad 0 < |\rho| < 1.
   \]
2. **Representation:**
   \[
   \varrho(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varrho(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & \rho & 0 \end{pmatrix}, \quad \varrho(e_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\rho & 0 \\ 0 & 0 & 0 \end{pmatrix}.
   \]
3. **Matrix group element:**
   \[
   M = \begin{pmatrix} \exp(-x) & 0 & z \\ 0 & \exp(-\rho x) & \rho y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} X & 0 & Z \\ 0 & X^\rho & Y \\ 0 & 0 & 1 \end{pmatrix}, \quad X > 0.
   \]
4. **Coproduct:**
   \[
   \Delta(X) = X \otimes X,
   \Delta(Y) = X^\rho \otimes Y + Y \otimes 1,
   \Delta(Z) = X \otimes Z + Z \otimes 1.
   \]
5. **PL brackets:** we have three different families:
   \[
   [X, Y]_1 = -a_1 X Y + b_1 X (X^\rho - 1),
   \]
   \[
   [X, Z]_1 = c_1 (X - X^2) + \frac{a_1}{\rho} X Z \quad (a_1 \neq 0),
   \]
   \[
   [Y, Z]_1 = \frac{\rho b_1 c_1}{a_1} (1 - X^{1+\rho}) + a_1 Y Z + \rho c_1 Y + b_1 Z;
   \]
   \[
   [X, Y]_2 = 0,
   \]
   \[
   [X, Z]_2 = a_2 (X - X^2),
   \]
   \[
   [Y, Z]_2 = b_2 (1 - X^{1+\rho}) + \rho a_2 Y;
   \]
   \[
   [X, Y]_3 = a_3 X (X^\rho - 1),
   \]
   \[
   [X, Z]_3 = 0 \quad (a_3 \neq 0),
   \]
   \[
   [Y, Z]_3 = b_3 (1 - X^{1+\rho}) + a_3 Z.
   \]
(6) Casimirs:

\[ C_1 = X^{-\rho} \left( b_1(1 - X^\rho) + a_1Y \right) \left( \rho c_1(X - 1) - a_1Z \right)^\rho, \]
\[ C_2 = \left( 1 - \frac{1}{X} \right)^\rho \left( b_2(1 - X^\rho) + \rho a_2Y \right), \]
\[ C_3 = (X^{-\rho} - 1) \left( b_3(X - 1) - a_3Z \right)^\rho. \]

(7) Coboundary cases: the bracket \{,\}_2 is coboundary for the following values of the parameters and \( r \)-matrix:

\[ a_2 = -r^{13}, \quad b_2 = \rho r^{12}, \quad r = r^{12}e_1 \wedge e_2 + r^{13}e_1 \wedge e_3. \]

The bracket \{,\}_3 is coboundary when

\[ a_3 = \rho r^{23}, \quad b_3 = \rho r^{12}, \quad r = r^{12}e_1 \wedge e_2 + r^{23}e_2 \wedge e_3. \]

The bracket \{,\}_1 is always a non-coboundary one.

(8) Isomorphic to the algebra \( \tau_3(\rho), 0 < |\rho| < 1 \), in [1] through the change of basis

\[ e_1 = e_1, \quad e_2 = e_2, \quad e_3 = -e_0. \]

(9) Correspondence with the classification of Lie bialgebras.

| Lie bialgebra in [1] \{,\}_i | a_i | b_i | c_i |
|-----------------------------|-----|-----|-----|
| 5 (\( * \)) \{,\}_2       | 0   | -\rho |   |
| 6 (\( \chi = e_0 \wedge e_1 \) \( * \)) \{,\}_2 | 1   | 0   | -   |
| 7 \{,\}_1 \lambda \rho | 0   | 0   |     |

(10) Remarks: to the best of our knowledge, this case has not been considered in the literature so far. Note that the non-coboundary PL group corresponding to the Lie bialgebra 7 is one of the Lotka–Volterra brackets [32] that generalize the PL bracket previously obtained on the book group \( A_{3,3} \).

3.6. PL structures on the 2D Euclidean group generated by \( A_{3,6} \)

(1) Commutation relations:

\[ [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1, \quad [e_1, e_2] = 0. \]

(2) Representation:

\[ \varrho(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varrho(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varrho(e_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

(3) Matrix group element:

\[ M = \begin{pmatrix} \cos(x) & -\sin(x) & y \\ \sin(x) & \cos(x) & -z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} C & -S & Y \\ S & C & Z \\ 0 & 0 & 1 \end{pmatrix}, \quad C^2 + S^2 = 1. \]

(4) Coproduct:

\[ \Delta(C) = C \otimes C - S \otimes S, \]
\[ \Delta(S) = S \otimes C + C \otimes S, \]
\[ \Delta(Y) = C \otimes Y - S \otimes Z + Y \otimes 1, \]
\[ \Delta(Z) = S \otimes Y + C \otimes Z + Z \otimes 1. \]
(5) PL brackets:

\[
\begin{align*}
[C, S]_1 &= 0, \\
[C, Y]_1 &= a_1 (1 - C^2) + b_1 S (1 - C) - c_1 SY, \\
[C, Z]_1 &= a_1 S (1 - C) + b_1 (C^2 - 1) - c_1 SZ, \\
[S, Y]_1 &= -a_1 CS + b_1 (C^3 - C) + c_1 CY, \\
[S, Z]_1 &= a_1 (C^2 - C) + b_1 CS + c_1 CZ, \\
[Y, Z]_1 &= a_1 Z + b_1 Y - \frac{c_1}{2} (Y^2 + Z^2); \\
[C, S]_2 &= 0, \\
[C, Y]_2 &= a_2 (1 - C^2) + b_2 S (1 - C), \\
[C, Z]_2 &= a_2 S (1 - C) + b_2 (C^2 - 1) \quad (c_2 \neq 0), \\
[S, Y]_2 &= -a_2 CS + b_2 (C^2 - C), \\
[S, Z]_2 &= a_2 (C^2 - C) + b_2 CS, \\
[Y, Z]_2 &= a_2 Z + b_2 Y + c_2 \arccos(C).
\end{align*}
\]

(6) Casimirs:

\[
C_1 = \begin{cases} 
\arctan \left( \frac{c_1 Z - a_1 (1 - C) + b_1 S}{b_1 (1 - C) + a_1 S - c_1 Y} \right) & (c_1 \neq 0), \\
\arctan \left( \frac{C}{S} \right) & (c_1 = 0), \\
a_1 Y - b_1 Z + \frac{S (a_1 Z + b_1 Y)}{C - 1} & (c_1 = 0), 
\end{cases}
\]

\[
C_2 = c_2 \ln (1 - C) + a_2 Y - b_2 Z + \frac{S (a_2 Z + b_2 Y + c_2 \arccos(C))}{C - 1}.
\]

(7) Coboundary cases: the generic classical \( r \)-matrix is given by

\[
r = r^{12} e_1 \wedge e_2 + r^{13} e_1 \wedge e_3 + r^{23} e_2 \wedge e_3.
\]

The bracket \( \{,\} \) is coboundary for the following values of the parameters:

\[
a_1 = r^{13}, \quad b_1 = -r^{23}, \quad c_1 = 0,
\]

(the coefficient \( r^{12} \) does not play any role in the above Poisson bracket), while the bracket \( \{,\}_2 \) is always a non-coboundary one.

(8) Isomorphic to the algebra \( s_3(0) \) in \([1]\) through the change of basis

\[
e_1 = e_1, \quad e_2 = e_2, \quad e_3 = -e_0.
\]

(9) Correspondence with the classification of Lie bialgebras.

| Lie bialgebra in [1] | \( \{,\} \) | \( a_1 \) | \( b_1 \) | \( c_1 \) |
|----------------------|-------------|---------|---------|---------|
| (9)                  | \( \{,\}_1 \) | 0       | 0       | \(-\lambda\) |
| 15'                  | \( \{,\}_2 \) | 0       | 0       | \(-\omega\) |
| (11') (*)           | \( \{,\}_1 \) | -1      | 0       | 0       |
| (14')               | \( \{,\}_2 \) | \(-\alpha\) | 0       | \(-\lambda\) |

(10) Remarks: in this case, the Poisson bracket \( \{,\}_2 \) is not quadratic in the group matrix entries since it contains a term \( \arccos(C) \), which has been allowed in order to obtain the PL structure corresponding to the Lie bialgebras \( 14' \) and \( 15' \). PL structures on the Euclidean group were first studied in [35], where the case \( 14' \) is lacking.
3.7. PL structures on the solvable Lie group generated by $A_{3,7}$

(1) Commutation relations:

\[ [e_1, e_3] = \mu e_1 - e_2, \quad [e_2, e_3] = e_1 + \mu e_2, \quad [e_1, e_2] = 0, \quad \mu > 0. \]

(2) Representation:

\[ \varphi(e_1) = \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi(e_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mu \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi(e_3) = \begin{pmatrix} -\mu & -1 & 0 \\ 1 & -\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

(3) Matrix group element:

\[
M = \begin{pmatrix}
\exp(-\mu \sin(x)) & -\exp(-\mu \sin(x)) & y + \mu z \\
\exp(-\mu \cos(x)) & \exp(-\mu \cos(x)) & \mu y - z \\
0 & 0 & 1
\end{pmatrix},
\]

\[
C^2 + S^2 - \exp\left(-2\mu \arctan\left(\frac{S}{C}\right)\right) = 0.
\]

(4) Coproduct:

\[
\Delta(C) = C \otimes C - S \otimes S, \\
\Delta(S) = S \otimes C + C \otimes S, \\
\Delta(Y) = C \otimes Y - S \otimes Z + Y \otimes 1, \\
\Delta(Z) = S \otimes Y + C \otimes Z + Z \otimes 1.
\]

(5) PL brackets:

\[
[C, S]_1 = 0, \\
[C, Y]_1 = \frac{1}{2a_1}(\mu C + S)\left[c_1^2(1 + \mu^2)(-Y + \mu Z) + b_1(-2a_1S + b_1(-Y + \mu Z))
+ 2c_1[a_1(-1 + C + \mu S) + b_1\mu (Y - \mu Z)]\right], \\
[C, Z]_1 = -\frac{1}{2a_1}(\mu C + S)\left[b_1^2(\mu Y + Z) - 2b_1[a_1(C - 1) + c_1\mu(\mu Y + Z)]
+ c_1\left[2a_1\mu(C - 1) - 2a_1S + c_1(1 + \mu^2)(\mu Y + Z)\right]\right], \\
[S, Y]_1 = \frac{1}{2a_1}(C - \mu S)\left[c_1^2(1 + \mu^2)(-Y + \mu Z) + b_1(-2a_1S + b_1(-Y + \mu Z))
+ 2c_1[a_1(-1 + C + \mu S) + b_1\mu (Y - \mu Z)]\right], \\
[S, Z]_1 = \frac{1}{2a_1}(C - \mu S)\left[b_1^2(\mu Y + Z) - 2b_1[a_1(C - 1) + c_1\mu(\mu Y + Z)]
+ c_1\left[2a_1\mu(C - 1) - 2a_1S + c_1(1 + \mu^2)(\mu Y + Z)\right]\right], \\
[Y, Z]_1 = -\frac{1}{4a_1}\left[-4a_1^2\left(C^2 + S^2 - 1\right) + 4a_1\left[c_1 + \mu b_1 - c_1\mu^2\right]Y + (b_1 - 2c_1\mu)Z
+ (1 + \mu^2)(c_1^2 + (b_1 - c_1\mu)^2)(Y^2 + Z^2)\right],
\]

(a_1 \neq 0);

\[
[C, S]_2 = 0, \\
[C, Y]_2 = \frac{a_2(Y - \mu Z)(\mu C + S)}{\mu}, \\
[C, Z]_2 = \frac{a_2(\mu Y + Z)(\mu C + S)}{\mu}.
\]
\( \{S, Y\}_2 = -\frac{a_2(Y - \mu Z)(C - \mu S)}{\mu} \),
\( \{S, Z\}_2 = -\frac{a_2(\mu Y + Z)(C - \mu S)}{\mu} \),
\( \{Y, Z\}_2 = -\frac{a_2(\mu^2 + 1)(Y^2 + Z^2)}{2\mu} \).

(6) Casimirs:
\[ C_1 = \arctan \left( \frac{S}{C} \right) + \frac{1}{i + \mu} \ln \left[ \frac{-2ia_1(-1 + C - iS) + (i + \mu)(ib_1 + c_1 - ic_1\mu)(Y - iZ)}{(i + \mu)(b_1 - c_1(i + \mu))} \right] \]
\[ + \frac{1}{(-i + \mu)} \ln \left[ \frac{2ia_1(-1 + C + iS) + (\mu - i)(ib_1 + c_1 + ic_1\mu)(Y + iZ)}{(\mu - i)(b_1 + c_1(i - \mu))} \right]. \]
\[ C_2 = \arctan \left( \frac{S}{C} \right) + \frac{2\mu}{1 + \mu^2} \left[ \arctan \left( \frac{Z}{Y} \right) - \frac{\mu}{2} \ln(Y^2 + Z^2) \right]. \]

(7) Coboundary cases: for this algebra the most general classical \( r \)-matrix is
\[ r = r_{12} e_1 \wedge e_2, \]
which corresponds to the bracket \( \{,\} \) for the following values of the parameters:
\[ a_1 = r_{12}(1 + \mu^2), \quad b_1 = c_1 = 0. \]
The bracket \( \{,\} \) is always a non-coboundary one.

(8) Isomorphic to the algebra \( s_3(\mu) \) in [1] through the change of basis
\[ e_1 = e_1, \quad e_2 = e_2, \quad e_3 = -e_0. \]

(9) Correspondence with the classification of Lie bialgebras.

| Table 7. Classification of PL structures on \( A_{3,7} \). |
|-----------------|-------|-------|-------|
| Lie bialgebra in [1] | \{,\} | \( a_1 \) | \( b_1 \) | \( c_1 \) |
| 15 (\( \ast \)) | \( [,] \) | -\( \omega \) | 0 | 0 |
| 16 | \( [,] \) | -\( \lambda \) | - | - |

(10) Remarks: the \( \mu \rightarrow 0 \) limit of the Lie algebra \( A_{3,7} \) is just the Euclidean Lie algebra \( A_{3,6} \).
Therefore, the former can be thought of as a deformation of the latter (see the \( A_{3,7} \) group element). Note that the PL brackets are always quadratic in the group matrix entries, provided that the constraint \( C^2 + S^2 - \exp(-2\alpha \arctan(\frac{Z}{Y})) = 0 \) is imposed. As has been stressed in [1], when \( \mu = 1 \) the non-coboundary Lie bialgebra 16 is self-dual.

3.8. PL structures on the \( SL(2, \mathbb{R}) \) group generated by \( A_{3,8} \)

(1) Commutation relations:
\[ [e_1, e_3] = -2e_2, \quad [e_1, e_2] = e_1, \quad [e_2, e_3] = e_3. \]

(2) Representation:
\[ \varphi(e_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \varphi(e_2) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varphi(e_3) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]
(3) Matrix group element:
\[
M = \begin{pmatrix}
  e^{2y} & xe^{2z} \\
  ze^{2y} & xe^{2z} + e^{-2x}
\end{pmatrix} = \begin{pmatrix}
  Y & X \\
  Z & W
\end{pmatrix}, \quad Y > 0, \quad YW - XZ = 1.
\]

(4) Coproduct:
\[
\Delta(X) = X \otimes W + Y \otimes X \\
\Delta(Y) = Y \otimes Y + X \otimes Z, \\
\Delta(Z) = Z \otimes Y + W \otimes Z, \\
\Delta(W) = Z \otimes X + W \otimes W.
\]

(5) PL bracket:
\[
\{X, Y\}_1 = -a_1 X^2 + b_1 XY + c_1 (1 - Y^2), \\
\{X, Z\}_1 = -(a_1 X + c_1 Z)(Y + W), \\
\{X, W\}_1 = -a_1 X^2 - b_1 XW + c_1 (1 - W^2), \\
\{Y, Z\}_1 = a_1 (1 - Y^2) - b_1 YZ - c_1 Z^2, \\
\{Y, W\}_1 = 2b_1 (1 - WY) + (c_1 Z - a_1 X)(Y - W), \\
\{Z, W\}_1 = a_1 (W^2 - 1) - b_1ZW + c_1 Z^2.
\]

(6) Casimir:
\[
C = \begin{cases}
  \frac{a_1 (W - Y) - b_1 Z}{a_1 X + c_1 Z} & (a_1 \neq 0 \text{ or } c_1 \neq 0) \\
  \frac{W - Y}{Z} & (a_1 = b_1 = 0) \\
  \frac{X}{Z} & (a_1 = c_1 = 0).
\end{cases}
\]

(7) Coboundary cases: since the Lie algebra \(A_{3,8}\) is simple, the PL tensor is always a coboundary one, with the following \(r\)-matrix:
\[
r = r_{12} e_1 \wedge e_2 + r_{13} e_1 \wedge e_3 + r_{23} e_2 \wedge e_3.
\]

By computing the Sklyanin bracket, we obtain the following identification for the parameters:
\[
a_1 = -r_{12}, \quad b_1 = -2r_{13}, \quad c_1 = r_{23}.
\]

(8) Isomorphic to the algebra \(sl(2, \mathbb{R})\) in [1] through the change of basis
\[
e_1 = \sqrt{2}e_1, \quad e_2 = -e_0, \quad e_3 = \sqrt{2}e_2.
\]

(9) Correspondence with the classification of Lie bialgebras.

### Table 8. Classification of PL structures on \(A_{3,8}\).

| Lie bialgebra in [1] | \{, \}_1 | a_i | b_i | c_i |
|----------------------|----------|-----|-----|-----|
| 1 (\*)              | \{, \}_1 | 0   | \lambda/2 | 0 |
| 2 (\*)              | \{, \}_1 | 0   | 0   | \sqrt{3}\lambda/4 |
| 3 (\*)              | \{, \}_1 | \sqrt{3}\lambda/4 | 0 | 0 |

(10) Remarks: the Lie bialgebra 1 corresponds to the PL structure underlying the standard quantum \(SL(2, \mathbb{R})\) group, case 2 corresponds to the standard quantum deformation of \(SO(2, 1)\) and case 3 corresponds to the non-standard quantum \(SL(2, \mathbb{R})\) group.
3.9. PL structures on the SO(3) group generated by $A_{3,9}$

1. Commutation relations:

\[ [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2. \]

2. Representation:

\[
\begin{align*}
\varphi(e_1) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\varphi(e_2) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\varphi(e_3) &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

3. Matrix group element:

\[
M = \begin{pmatrix} C_xC_y & -S_xC_y & S_y \\ C_xS_yS_z + S_xC_z & -S_xS_yS_z + C_xC_z & -C_xS_z \\ -C_xS_yS_z + S_xC_z & S_xS_yS_z + C_xC_z & C_xC_z \end{pmatrix},
\]

where we used the shorthand notation $C_\alpha = \cos(\alpha)$ and $S_\alpha = \sin(\alpha)$.

4. Coproduct: it would be immediate to write it for the matrix entries of $M$, and in terms of the local group coordinates, it can be formally written as

\[
\begin{align*}
\Delta(x) &= \pi - \arccos \left( \frac{-C_xC_y \otimes C_xC_y + S_xC_y \otimes S_xC_y + S_xC_x \otimes C_xS_x + C_xS_x \otimes C_xC_x - S_x \otimes S_x}{\sqrt{1 - \sin(\Delta(y))}} \right), \\
\Delta(y) &= \arcsin \left( C_xC_x \otimes S_y + S_xC_x \otimes C_xS_x + S_y \otimes C_yC_y \right), \\
\Delta(z) &= \pi - \arccos \left( \frac{C_yS_xC_z \otimes S_yC_yS_xS_z + S_yS_xC_yC_z + S_yS_xC_xC_z - C_yC_yC_z \otimes C_yC_yC_z}{\sqrt{1 - \sin(\Delta(y))}} \right).
\end{align*}
\]

5. PL bracket: the most general Poisson bracket compatible with the above coproduct maps is given by

\[
\begin{align*}
\{x, y\}_1 &= \frac{a_1 \sin(y) + b_1 \sin(x) \cos(y) + c_1 \cos(x) \cos(y) - c_1}{\cos(y)}, \\
\{x, z\}_1 &= \frac{a_1 \sin(z) + b_1 \cos(x) - b_1 \cos(z) - c_1 \sin(x)}{\cos(y)}, \\
\{y, z\}_1 &= \frac{-a_1 \cos(z) \cos(y) + b_1 \sin(z) \cos(y) - a_1 + c_1 \sin(y)}{\cos(y)}.
\end{align*}
\]

6. Casimir: in the case $b_1 = c_1 = 0$ (which is the only essential PL bracket, see below), it reads

\[
\mathcal{C} = \frac{x}{2} - \arctan \left[ \frac{\sin \left( \frac{\sqrt{a_1^2 + b_1^2}}{2} \right)}{\sin \left( \frac{\sqrt{a_1^2 + b_1^2}}{2} \right)} \right].
\]

7. Coboundary cases: again, the Lie algebra $A_{3,9}$ is simple, so its PL tensors are always coboundaries. The generic $r$-matrix is given by

\[
r = r^{12}e_1 \wedge e_2 + r^{13}e_1 \wedge e_3 + r^{23}e_2 \wedge e_3,
\]

which means that

\[
a = r^{12}, \quad b = r^{13}, \quad c = r^{23}.
\]
(8) Isomorphic to the algebra $so(3)$ in [1] through the change of basis
\[ e_1 = e_1, \quad e_2 = e_2, \quad e_3 = e_0. \]

(9) Correspondence with the classification of Lie bialgebras.

Table 9. Classification of PL structures on $A_{3,9}$.

| Lie bialgebra in [1]  | $\{ , \}$ | $a_1$ | $b$ | $c$ |
|----------------------|------------|------|----|----|
| 4                    | $\{ , \}_1$ | $\lambda$ | 0  | 0  |

(10) Remarks: the Lie bialgebra 4 is the one generated by the classical $r$-matrix $r = a_1 e_1 \wedge e_2$ (which is equivalent to the generic three-parametric $r$-matrix through the appropriate automorphism), whose Sklyanin bracket is given above and provides the semiclassical limit of the quantum $SO(3)$ group.

4. Conclusions

In this paper, we have constructed and classified all the possible PL structures for the nine real 3D Lie groups. For each of the PL brackets, we have given the explicit expressions for the coproduct map and for the corresponding Casimir function. Moreover, our results are fully consistent with the complete classification of Lie bialgebra structures given in [1], which we have used in order to identify all the inequivalent PL structures under generic group automorphisms.

For each Lie group, the PL structures are obtained by solving the cocycle condition through direct computation and by assuming initially a quadratic dependence of the PL bracket in terms of the group matrix entries (this assumption has to be relaxed in only two cases). The solutions so obtained are grouped into multiparametric families of Poisson brackets, which in many cases provide non-coboundary PL structures that have not been constructed so far. In particular, the PL structures for the groups corresponding to the solvable Lie algebras $A_{3,2}, A_{3,4}$ (the (1+1) Poincaré algebra), $A_{3,5}$ and $A_{3,7}$ are—to the best of our knowledge—presented here for the first time.

The approach presented here can be straightforwardly implemented in order to obtain and classify PL structures of non-semisimple Lie groups in higher dimensions, for which known results are scarce and essentially deal with coboundary structures [37, 47–50]. In particular, it is known that all the Lie bialgebra structures of groups built as semidirect products between (2+1) and (3+1) spacetime rotations (with arbitrary signature) and translations are coboundaries [51], although only the (2+1) [52] and the (3+1) Poincaré classification [51, 53, 54] were completed. However, a bunch of non-coboundary Lie bialgebras arise in the classification of the massless (3+1) Galilei PL groups [38] and similar results can be expected for the classification of the PL structures on the 4D and 5D real Lie groups whose Lie algebras are classified in [40].

Consequently, the results presented here can be useful in two different directions. On one hand, they provide a complete and closed chart of the semiclassical counterparts of all possible 3D real quantum groups, which in many non-semisimple (and non-coboundary) cases are yet unexplored. On the other hand, as in the case of Lotka–Volterra equations [32], it could happen that other relevant dynamical systems whose Hamiltonian structure is provided by quadratic Poisson algebras could find a group theoretical interpretation as PL structures on certain (possibly non-simple) Lie groups. Work on these two lines is in progress.
Acknowledgments

This work was partially supported by the Spanish MICINN under grant MTM2010-18556 and by INFN-MICINN (grant AIC-D-2011-0711).

References

[1] Gomez X 2000 J. Math. Phys. 41 4939
[2] Drinfel’d V G 1983 Sov. Math. Dokl. 27 68
[3] Semenov-Tian-Shansky M A 1985 Publ. Res. Inst. Math. Sci. 21 1237
[4] Kosmann-Schwarzbach Y 1987 Publ. IRMA (Lille) 5 No 12
[5] Kosmann-Schwarzbach Y and Magri F 1988 Ann. Inst. Henri Poincare 49 433
[6] Lu J H and Weinstein A 1990 J. Diff. Geom. 31 501
[7] Alekseev A Y and Malkin A Z 1994 Commun. Math. Phys. 162 147
[8] Alekseevsky D, Grabowski J, Marmo G and Michor P W 1998 J. Geom. Phys. 26 340
[9] Kosmann-Schwarzbach Y 2004 Lie bialgebras, Poisson Lie groups and dressing transformations Integrability of Nonlinear Systems (Lect. Notes in Phys. vol 638) (Berlin: Springer) pp 107–73
[10] Drinfel’d V G 1987 Quantum groups Proc. Int. Cong. Math. Berkeley 1986 ed A V Gleason (Providence, RI: American Mathematical Society) p 798
[11] Semenov-Tian-Shansiki M A 1992 Theor. Math. Phys. 93 1292
[12] Chari V and Pressley A 1994 A Guide to Quantum Groups (Cambridge: Cambridge University Press)
[13] Majid S 1995 Foundations of Quantum Group Theory (Cambridge: Cambridge University Press)
[14] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1995 J. Math. Phys. 36 631
[15] Grabowski J, Marmo G and Michor P W 1999 Mod. Phys. Lett. A 14 2109
[16] Ballesteros A, Blasco A and Musso F 2012 J. Phys. A: Math. Theor. 45 105205
[17] Klimcik C and Severa P 1995 Phys. Lett. B 351 455
[18] Klimcik C and Severa P 1996 Phys. Lett. B 372 65
[19] Lledó M A and Varadarajan V S V 1998 Lett. Math. Phys. 45 247
[20] Alekseev A Y and Malkin A Z 1995 Commun. Math. Phys. 169 99
[21] Fock V V and Rosly A A 1992 ITEP-92-72 (arXiv:math/9802054v2 [math.QA])
[22] Meusburger C and Schroers B J 2008 J. Math. Phys. 49 083510
[23] Ballesteros A, Blasco A and Musso F 2011 Phys. Lett. A 375 3370
[24] Belavin A A and Drinfel’d V G 1982 Funct. Anal. Appl. 16 159
[25] Stolin A A 1991 Math. Scand. 69 81
[26] Sobczyk J 1996 J. Phys. A: Math. Gen. 29 2887
[27] Kowalczyk E 1997 Acta Phys. Pol. B 28 1893
[28] Opanowicz A 1998 J. Phys. A: Math. Gen. 31 8387
[29] Brihaye Y, Kowalczyk E and Maslanka P 2001 Mod. Phys. Lett. A 16 321
[30] Vysoky J 2011 Poisson structures on Lie groups Diploma Thesis CTU, Prague
[31] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 J. Math. Phys. 17 986
[32] Mubarakzyanov G M 1963 Izv. Vyssh. Uchebn. Zaved. Mat. 1 114
[33] Mubarakzyanov G M 1963 Izv. Vyssh. Uchebn. Zaved. Mat. 3 99
[34] Smotl L and Hlavaty L 2002 Int. J. Mod. Phys. A 17 4043
[35] Rezaei-Aghdam A, Hemmati M and Raskar A 2005 J. Phys. A: Math. Gen. 38 3981
[36] Kupershmidt B 1993 J. Phys. A: Math. Gen. 26 L929
[37] Ballesteros A, Herranz F J and Parashar P 1997 J. Phys. A: Math. Gen. 30 1149
[38] Ballesteros A, Herranz F J, del Olmo M A, Pereña C M and Santander M 1995 J. Phys. A: Math. Gen. 28 7113

21
[47] Ballesteros A, Celeghini E and Herranz F J 2000 J. Phys. A: Math. Gen. 33 3431
[48] Ballesteros A and Herranz F J 1996 J. Phys. A: Math. Gen. 30 4307
[49] Ballesteros A, Herranz F J and Parashar P 2000 J. Phys. A: Math. Gen. 33 3445
[50] Kupershmidt B A 1994 J. Phys. A: Math. Gen. 27 L47
[51] Zakrzewski S 1995 Poisson Poincaré groups Quantum Groups, Formalism and Applications ed J Lukierski et al (Warsaw: Polish Scientific) p 433
[52] Stachura P 1998 J. Phys. A: Math. Gen. 31 4555
[53] Podlés P and Woronowicz S L 1996 Commun. Math. Phys. 178 61
[54] Zakrzewski S 1997 Commun. Math. Phys. 185 285