Floquet topological phase transition in $\alpha$-$T_3$ lattice

Bashab Dey and Tarun Kanti Ghosh
Department of Physics, Indian Institute of Technology-Kanpur, Kanpur-208 016, India

We investigate topological characteristics of the photon-dressed band structure of $\alpha$-$T_3$ lattice on being driven by off-resonant circularly polarized radiation. We obtain exact analytical expressions of the quasienergy bands over the first Brillouin zone. The broken time-reversal symmetry caused by the circularly polarized light lifts the triple point degeneracy completely at both the Dirac points. The gaps become unequal at $K$ and $K'$ (except at $\alpha = 0$ and 1), which reveals the absence of inversion symmetry in the system. At $\alpha = 1/\sqrt{2}$, the gap between flat and valence bands closes at $K$, while that between conduction and flat bands closes at $K'$, thereby restoring a semimetallic phase. At the gap closing point ($\alpha = 1/\sqrt{2}$) which is independent of the radiation amplitude, there is a reappearance of low-energy Dirac cones around $K$ and $K'$ points. Under the influence of the circularly polarized radiation, the $\alpha$-$T_3$ lattice is transformed from semimetal to a Haldane-like Chern insulator characterized by non-zero Chern number. The system undergoes a topological phase transition from $C = 1\,(-1)$ to $C = 2\,(2)$ at $\alpha = 1/\sqrt{2}$, where $C$ is the Chern number of the valence (conduction) band. This sets an example of a multiband system having larger Chern number. These results are supported by the appearance of chiral edge states in irradiated $\alpha$-$T_3$ nanoribbon.

I. INTRODUCTION

Non-trivial topological phases in electronic and photonic systems have drawn enormous interest since the discovery of quantum Hall effect.\cite{1} Topological insulators (TIs)\cite{2,3,4} are distinctive states of matter characterized by an insulating bulk gap and gapless chiral or helical edge/surface modes that are topologically protected.\cite{5,6,7,8} There are several classes of TIs, each of which is represented by a topological index. Chern insulators (CIs), also known as anomalous quantum Hall insulators (AQHIs) belong to a class of TIs characterized by a topological invariant called Chern number $C$ associated with each band. A band with non-zero $C$ gives rise to quantized Hall conductance even in the absence of a net magnetic flux. This feature was first predicted in an exotic 2D lattice model with broken time-reversal (TRS) symmetry, popularly known as the Haldane model\cite{9} and was later verified experimentally.\cite{10} These materials host chiral edge states (unidirectional propagating modes along an edge) that are robust against backscattering. The edge states are guaranteed by a non-zero Chern number of the bulk band through bulk-edge correspondence.\cite{11} On the other hand, $Z_2$ TIs, also called quantum spin Hall insulators (QSHIs), constitute another class of TIs in which the edge states are protected by time-reversal symmetry.\cite{12,13} $Z_2$ phases have been studied in large number of systems including two-dimensional (2D) quantum materials, strong spin-orbit coupled quantum wells and exotic lattice models. The gapless edge states in $Z_2$ TIs are helical i.e. they form pairs of counter-propagating modes with opposite spins along an edge, that are time-reversed copies of each other. The topology of these edge states is described by another topological invariant called the $Z_2$ index. The topological phases exist in static\cite{15} as well as in time-periodic systems.\cite{16,17,18} Such a periodic drive can also transform a topologically trivial insulator to a CI.\cite{19,20} Topological properties of periodically driven systems can be analyzed using Floquet theory.

The Chern number, in principle, can have any integral value. Most of the theoretically and experimentally studied CIs have Chern number $C = 1$. Therefore, it would be interesting to have a system with Chern number $C \geq 2$. Recently, large Chern numbers have been predicted\cite{21} and experimentally realized in photonic 2D square and hexagonal crystals.

FIG. 1: Schematic diagram of the $\alpha$-$T_3$ lattice illuminated by off-resonant circularly polarized light.

The $\alpha$-$T_3$ lattice, as shown in Fig. 1, is the extension of a honeycomb lattice. This is a conventional honeycomb lattice with two lattice points (A,B) and an additional lattice point (C) at the centre of each honeycomb cell. The quasiparticle can hop from C sites to the alternate vertices (say, B) of the same honeycomb lattice. The hopping amplitude between A and B sites is $t \cos \phi$ and that between the B and C sites is $t \sin \phi$, where the angle $\phi$ parameterizes the hopping amplitude. It is convenient to express the angle $\phi$ by another parameter $\alpha$ such that $\alpha = \tan \phi$. For $\phi = 0$ ($\alpha = 0$), the C site is decoupled from the honeycomb lattice and it resembles to the monolayer graphene. The upper-left $2 \times 2$ matrix
block in Eq. (1) describes the quasiparticle dynamics of monolayer graphene. For \( \alpha = 1 \) \((\phi = \pi/4)\), the \( \alpha-T_3 \) model becomes conventional dice or \( T_3 \) lattice having pseudospin-1/2. The \( \alpha-T_3 \) lattice with non-zero \( \alpha \) has three energy bands since it has three sublattices consisting of a hub site (B) connected to six rim sites (A, C). The dice lattice can naturally be built by growing trilayers of cubic lattices \((\text{e.g. SrTiO}_3/\text{SrIrO}_3/\text{SrTiO}_3)\) in (111) direction. It has been proposed theoretically that a dice lattice can be generated by interfering three counter-propagating pairs of identical laser beams \([22]\) on a plane with wavelength \( \lambda = 3a/2 \), with \( a \) being the lattice constant. Later, it was shown that the \( \alpha-T_3 \) optical lattice can be achieved by dephasing one of the pairs of the laser beams while keeping other parameters unaltered \([22]\). Therefore, a continuous change of \( \alpha \), through tuning phase of one of the three pairs of the laser beams, will allow us to study the changes in the topological properties of the system. The Hamiltonian of \( \text{Hg}_1-x\text{Cd}_x\text{Te} \) quantum well can be mapped to that of low-energy \( \alpha-T_3 \) model with effective \( \alpha = 1/\sqrt{3} \) on appropriate doping \([32]\).

In recent years, there have been several studies \([35,37,39]\) on various properties of the \( \alpha-T_3 \) lattice. The role of variable Berry phase in orbital susceptibility \([35]\) magnetotransport coefficients \([35]\) (quantized Hall conductivity and SdH oscillation) and optical conductivity \([37]\) of the \( \alpha-T_3 \) lattice has been established. Very recently, Floquet states and the variable Berry phase dependent photoinduced gap in \( \alpha-T_3 \) lattice irradiated by the circularly polarized light have been studied in detail \([22]\). It has been shown that an off-resonant radiation induces a gap in graphene, on the surface states of \( \text{Tl}_3\text{Si}_2\text{Sn} \), \( \text{MoS}_2 \) \([30]\) semi-Dirac system \([8]\), \( \text{MoS}_2 \) \([30]\) etc and transforms them to Chern insulating states.

Since the proposal of Haldane’s Chern insulator on a honeycomb lattice \([2] \), several multiband CIs such as kagomé \([21,22]\), dice \([21,22]\) and Lieb \([22]\) lattices with tunable parameters controlling the band topology have been studied. The \( \alpha-T_3 \) lattice is another example of a multiband system having trivial topology. In this work, we will show that an application of circularly polarized radiation on the \( \alpha-T_3 \) lattice makes it Haldane-type CI having non-zero Chern number and tuning the parameter \( \alpha \) leads to a topological phase transitions at \( \alpha = 1/\sqrt{2} \) by changing the Chern number of the valence (conduction) band from \( \mathcal{C} = 1(-1) \) to larger Chern number \( \mathcal{C} = 2(-2) \). This phase transition results in doubling of the number of chiral edge modes from one to two in irradiated \( \alpha-T_3 \) nanoribbon. First we derive Floquet-Magnus Hamiltonian of the \( \alpha-T_3 \) lattice for the entire Brillouin zone. We get exact analytical expressions of quasiequilibrium band structure over the full Brillouin zone. The triple-point degeneracy at the Dirac points is completely removed by breaking time-reversal symmetry due to time-periodic circularly polarized light. An intriguing state, independent of the radiation amplitude, appears at \( \alpha = 1/\sqrt{2} \), where the gap between flat and valence bands closes at \( K \), while that between conduction and flat bands closes at \( K' \). The low-energy bands around both \( K \) and \( K' \) points display a Dirac-like dispersion with the reduced slope, as compared to monolayer graphene, at the gap closing points.

This paper is arranged as follows. In Sec. II, we provide topological band structure of the \( \alpha-T_3 \) lattice irradiated by the circularly polarized light. In Sec. III, we present the analytical calculations of the Chern number and show that the system undergoes a topological phase transition at \( \alpha = 1/\sqrt{2} \). We present results of chiral edge states of irradiated \( \alpha-T_3 \) nanoribbon in Sec. IV. In Sec. V, a summary and conclusion of our results are presented.

II. TOPOLOGICAL BAND STRUCTURE OF \( \alpha-T_3 \) LATTICE IRRADIATED BY CIRCULARLY POLARIZED LIGHT

Considering only the nearest-neighbour (NN) hopping integrals, the rescaled tight-binding Hamiltonian for the \( \alpha-T_3 \) lattice is given by

\[
H_0(k) = \begin{pmatrix}
0 & h(k) \cos \phi & 0 \\
h^*(k) \cos \phi & 0 & h(k) \sin \phi \\
0 & h^*(k) \sin \phi & 0
\end{pmatrix}, \tag{1}
\]

where \( k = (k_x, k_y) \) and \( h(k) = \tau (e^{i k \cdot \delta_3} + e^{i \delta_2} + e^{i k \cdot \delta_3}) \).

Also, the three nearest neighbor position vectors with respect to the rim site B are \( \delta_1 = a(\sqrt{3}/2,-1/2), \delta_2 = a(-\sqrt{3}/2,-1/2) \) and \( \delta_3 = a(0,1) \), with \( a \) is the lattice constant of graphene. The energy-wavevector dispersion, independent of \( \alpha \), over the full Brillouin zone consists of three bands: two dispersive bands \( E_{\pm}(k) = \pm |h(k)| \) having electron-hole symmetry and a zero energy non-dispersive band \( E_0(k) = 0 \). The dispersion \( E_{\pm}(k) \) is identical to that of graphene. The corresponding normalized eigenvectors over the full Brillouin zone are given by

\[
\psi_{k,\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos \phi e^{-i \psi(k)} \\
\pm 1 \\
\sin \phi e^{i \psi(k)}
\end{pmatrix}, \quad \psi_{k,0} = \begin{pmatrix}
\sin \phi e^{-i \psi(k)} \\
0 \\
- \cos \phi e^{i \psi(k)}
\end{pmatrix},
\]

where \( h(k) = |h(k)| e^{-i \psi(k)} \). The low-energy quasiparticles in the \( \alpha-T_3 \) lattice are described by two-dimensional (2D) Dirac-Weyl equation. It is a semimetallic system in absence of any external fields/perturbations. It will behave like a TI if a Haldane-type energy gap is induced at the Dirac points by external means. Next we show that circularly polarized off-resonant radiation opens up gaps and induces topological states in \( \alpha-T_3 \) lattice.

An \( \alpha-T_3 \) lattice is irradiated with circularly polarized radiation falling normal to the lattice plane. The corresponding vector potential is \( A(t) = A_0(\cos \omega t, \sin \omega t) \), where \( A_0 = E_0/\omega \) with \( E_0 \) and \( \omega \) being the electric field amplitude and frequency of the radiation, respectively. By Pierl's substitution \( k \to (k+eA(t)/\hbar) \) in Eq. (1), we
obtain
\[ H(k, t) = \begin{pmatrix} 0 & h(k, t) \cos \phi & 0 \\ h^*(k, t) \cos \phi & 0 & h(k, t) \sin \phi \\ 0 & h^*(k, t) \sin \phi & 0 \end{pmatrix} \] (2)

where \( h(k, t) = \tau \sum_{j=1}^{3} e^{i(k+\epsilon A(t)/\delta) \cdot \delta_j} \). The Hamiltonian \( H(k, t) \) is periodic in time since \( A(t+T) = A(t) \) with the periodicity \( T = 2\pi/\omega \). Using the NN vectors \( \delta_j \) and the Jacobi-Anger expansion \( e^{iz \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\theta} \) with \( J_n(z) \) being the \( n \)-th order cylindrical Bessel function, we get
\[ h(k, t) = \tau \sum_{n=-\infty}^{\infty} J_n(\eta) \left[ e^{in\omega t} e^{ik \cdot \delta_3} + e^{-in(\pi/3+\omega t)} e^{ik \cdot \delta_2} + e^{in(\pi/3-\omega t)} e^{ik \cdot \delta_1} \right]. \]

Here \( \eta = e A_0 / \hbar \) is a dimensionless parameter characterizing the light intensity (can be expressed as square root of intensity and fine structure constant). Typically, \( \eta \ll 1 \) for the intensity of lasers and pulses available in the THz frequency domain. The off-resonant condition can be achieved when the photon energy is much larger than the band width of the undriven system i. e. \( \hbar \omega > 6\gamma \). When the light frequency satisfies the off-resonant condition, the band structure modifies by the second-order virtual photon absorption-emission processes.

The effective time-independent Hamiltonian valid under off-resonant condition is
\[ H_{\text{eff}}(k) = H_0(k) + [H_{-}(k), H_{+}(k)]/\hbar \omega + O(1/\omega^2), \] (3)

where
\[ H_{\pm}(k) = \frac{1}{T} \int_0^T dt \, e^{\mp i\omega t} H(k, t) \]
is the Fourier component of the Hamiltonian \( H(k, t) \). By Fourier transform, we obtain
\[ [H_{-}(k), H_{+}(k)] = \frac{\Delta(k)}{2} S_\pm(\alpha) \] (4)

where \( S_\pm(\alpha) \) is defined as
\[ S_\pm(\alpha) = 2 \begin{pmatrix} \cos^2 \phi & 0 & 0 \\ 0 & -\cos 2\phi & 0 \\ 0 & 0 & -\sin^2 \phi \end{pmatrix} \] (5)

and \( \Delta(k) = |g(k)|^2 - |f(k)|^2 = \hbar \omega \gamma(k) \) with
\[ g(k) = \tau J_1(\eta) \left[ e^{ik \cdot \delta_1} e^{i\pi/3} + e^{ik \cdot \delta_2} e^{-i\pi/3} - e^{ik \cdot \delta_3} \right] \] (6)
\[ f(k) = \tau J_1(\eta) \left[ -e^{ik \cdot \delta_1} e^{i\pi/3} - e^{ik \cdot \delta_2} e^{-i\pi/3} + e^{ik \cdot \delta_3} \right] \]

Hence, the light-matter coupling results in a mass term of the form \( \gamma(k) S_\pm(\alpha)/2 \) which lifts the three-fold degeneracy at the Dirac points. It can be shown that the mass term reduces to \( \mu^2 \beta^2 \hbar \omega S_\pm(\alpha)/2 \) in the linearized low energy limit where \( \beta = 3\gamma/(2\hbar \omega) \) and \( \mu = 1(-1) \) corresponds to \( K(K') \) valleys. On time-reversal operation, \( \gamma(k) \) changes sign, which implies the breaking of TRS in the system. Similarly, the term also changes sign on switching from right to left circular polarization. So, the mass term is trivially zero in case of linearly polarized light since it is a linear combination of both the polarizations with equal weights.

Interestingly, the mass term is Haldane-type which has opposite signs in the two valleys. In Haldane model, the NN hoppings do not accumulate Aharonov-Bohm (AB) phases since the net magnetic flux through a unit cell is zero. In this model, the magnetic flux is locally zero everywhere on the lattice plane at a given time. But the time-varying vector potential is spatially constant over the lattice, due to which NN hoppings acquire time-dependent AB phases.

The effective Hamiltonian \( H_{\text{eff}}(k) \) can now be written explicitly as
\[ H_{\text{eff}}(k) = \begin{pmatrix} \gamma(k) \cos^2 \phi & h(k) \cos \phi & 0 \\ h^*(k) \cos \phi & -\gamma(k) \cos 2\phi & h(k) \sin \phi \\ 0 & h^*(k) \sin \phi & -\gamma(k) \sin^2 \phi \end{pmatrix}. \] (8)

The Hamiltonian \( H_{\text{eff}}(k) \) satisfies the following anti-commutation relations for \( \alpha = 0 \) and \( \alpha = 1 \):
\[ \{H_{\text{eff}}^G(k), \mathcal{P}_G\} = 0, \quad \{H_{\text{eff}}^D(k), \mathcal{P}_D\} = 0. \] (9)

Here \( \mathcal{P}_G \) and \( \mathcal{P}_D \) are operators defined for graphene and dice respectively as follows:
\[ \mathcal{P}_G = \mathcal{K} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{P}_D = \mathcal{K} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \] (10)

with \( \mathcal{K} \) being the complex-conjugation operator. The relations imply that a band with energy \( \epsilon(k) \) will have a partner band with energy \( -\epsilon(k) \). This symmetry confirms the presence of a zero energy band in the three-band system. Hence, the flat band in dice lattice will not be perturbed by radiation. Also, the Hamiltonian is traceless, implying the sum of energies of the bands will be zero for all values of \( \alpha \).

The eigen values \( \epsilon_m(k) \) of \( H_{\text{eff}}(k) \) represent off-resonant quasienergy band structure. The characteristic equation for the eigen value problem turns out to be a depressed cubic equation: \( \lambda^3 + p \lambda + q = 0 \), where
\[ p = -\left[ |h(k)|^2 + \gamma(k)^2 \left( \cos^2 2\phi + \frac{\sin^2 2\phi}{4} \right) \right] \] (11)
\[ q = -\frac{\gamma(k)^3}{4} \sin^2 2\phi \cos 2\phi. \] (12)

The eigen values are of the form
\[ \epsilon_m(k) = 2\sqrt{\frac{p}{3}} \cos \left[ \frac{1}{3} \cos^{-1} \left( \frac{3q}{2p} \sqrt{\frac{3}{p}} \right) - \frac{2\pi m}{3} \right] \] (13)
with \( m = 0, 1 \) and 2 correspond to the quasienergies of the conduction, flat and valence bands, respectively. The band structure of the undriven \( \alpha-F_3 \) lattice is strongly modified by the off-resonant radiation and becomes \( \alpha \) as well as \( \eta \) dependent. It exhibits interesting features as we tune \( \alpha \), which will be discussed in detail. The components of the normalized eigen vectors \( \Psi_m(k) \) = \((a_m(k), b_m(k), c_m(k))^T \) can be written as

\[
a_m(k) = \frac{d(k) \sin \theta \cos \phi - \epsilon_m(k) b_m(k)}{\epsilon_m(k) - d(k) \cos \theta \cos^2 \phi} b_m(k) \tag{14}
\]
\[
c_m(k) = \frac{d(k) \sin \theta \sin \phi + \epsilon_m(k) b_m(k)}{\epsilon_m(k) + d(k) \cos \theta \sin^2 \phi} b_m(k) \tag{15}
\]

Now we shall analyze the topological band structures as we vary \( \alpha \) continuously. Figure 2 shows the low-energy topological bands for different values of \( \alpha \) in the first Brillouin zone. For three-band systems, there can be two distinct band gaps at the Dirac points: gap between i) conduction and flat bands (\( \Delta^F_{K/K'} \)); and ii) flat and valence bands (\( \Delta^K_{F/V} \)). At \( K/K' \) points. In presence of TRS breaking circularly polarized light, the triple point degeneracy at both the Dirac points is completely lifted (i.e. \( \Delta^F_{K/K'} \neq 0 \) and \( \Delta^K_{F/V} \neq 0 \)) except at \( \alpha = 1/\sqrt{2} \).

The photoinduced gaps at \( \alpha = 0 \) and 1 are \( \Delta^F_{K/K'} = \Delta^K_{F/V} = \beta^2 \hbar \omega \) and \( \Delta^K_{K/K'} = \Delta^K_{F/V} = \beta^2 \hbar \omega / 2 \), respectively. It is interesting to note that \( \Delta^F_{K/K'} = 0 \) but \( \Delta^K_{K/K'} \neq 0 \) and \( \Delta^K_{F/V} \neq 0 \) but \( \Delta^F_{F/V} = 0 \) at \( \alpha = 1/\sqrt{2} \). It implies that the band gaps at the Dirac points do not open completely at \( \alpha = 1/\sqrt{2} \). Note that this result is independent of the radiation amplitude \( \eta \) (as long as off-resonant approximation is valid). The partial closing of the band gap at \( \alpha = 1/\sqrt{2} \) can be deduced by obtaining the eigen values at a Dirac point (say \( \mathbf{K} \)) viz. \( \epsilon_0 = \cos^2 \phi, \epsilon_1 = -\cos 2\phi, \epsilon_2 = -\sin^2 \phi \). Equating \( \epsilon_0 \) with \( \epsilon_1 \), we find that the band touching occurs at \( \alpha = 1/\sqrt{2} \).

We present plots of \( \Delta^F_{K/K'} \) and \( \Delta^K_{F/V} \) vs \( \alpha \) in Fig. 3. The system exhibits an interesting property of \( \alpha \rightarrow 1/\alpha \) duality. The measurable quantities of the system will be same for \( \alpha \) and \( 1/\alpha \). Hence, similar gap-closing is also seen at \( \alpha = \sqrt{2} \). However, the duality exchanges the Dirac points i.e. \( \Delta^F_{K/K'}(\alpha) = \Delta^K_{F/V}(1/\alpha) \) and \( \Delta^K_{K/K'}(\alpha) = \Delta^K_{F/V}(1/\alpha) \).

The band gaps (in units of \( \beta^2 \hbar \omega \)) at the Dirac point \( \mathbf{K} \) are tabulated in Table I. For \( \mathbf{K}' \) point, one can easily check that \( \Delta^F_{K/K'} = \Delta^K_{F/V} \) and \( \Delta^K_{K/K'} = \Delta^K_{F/V} \) for given \( \alpha \).

\[ b_m(k) = \left[ 1 + \frac{d(k) \sin \theta \cos \phi - \epsilon_m(k) b_m(k)}{\epsilon_m(k) - d(k) \cos \theta \cos^2 \phi} \right]^2 + \left[ \frac{d(k) \sin \theta \sin \phi + \epsilon_m(k) b_m(k)}{\epsilon_m(k) + d(k) \cos \theta \sin^2 \phi} \right]^2 \right]^{-1/2}, \tag{16} \]

where we have parameterized \( \gamma(k) \) and \( h(k) \) as \( \gamma(k) = d(k) \cos \theta, h(k) = d(k) \sin \theta \epsilon^{-\imath \phi(k)} \), with \( d(k) = \sqrt{\gamma(k)^2 + h(k)^2} \).
TABLE I: Photoinduced gaps at the Dirac point $K$ as a function of $\alpha$.

| $\alpha$ | $\Delta K$ | $\Delta K_{\text{cf}}$ |
|---------|-----------|-----------------|
| $0 \leq \alpha \leq 1/\sqrt{2}$ | $1$ | $\cos^2 \phi + \cos 2\phi$ |
| $1/\sqrt{2} < \alpha < \sqrt{2}$ | $\cos 2\phi + \cos^2 \phi$ | $- (\cos 2\phi + \cos^2 \phi)$ |
| $\alpha \geq \sqrt{2}$ | $1$ | $\sin^2 \phi + \cos 2\phi$ |

FIG. 3: Plots of the photoinduced gaps at the Dirac point $K$ as a function of $\alpha$.

Substituting $k = K + q$ with $q \to 0$ in Eq. (5), we get low-energy Hamiltonian around $K$. Interestingly, the touching bands i.e. flat and conduction bands exhibit Dirac cones in the low-energy limit as

$$\epsilon_{0,1}(q) = \frac{\beta^2 \hbar \omega}{3} \pm \frac{\hbar v_f}{\sqrt{3}} |q|.$$ 

It is to be noted that in the field free case, $\epsilon_m(k) = \epsilon_m(-k)$ for all values of $\alpha$. But for the irradiated model, we have $\epsilon_m(k) = \epsilon_m(-k)$ for $\alpha = 0, 1$, and $\epsilon_m(-k) \neq \epsilon_m(k)$ for $\alpha \neq 0, 1$. This can be explained as follows

In the radiation-free case, the Kramer’s degeneracy ensured by TRS guarantees $\epsilon_m(k) = \epsilon_m(-k)$ irrespective of the value of $\alpha$ or other symmetries. On application of TRS-breaking circularly polarized light, the Kramer’s degeneracy is lifted. Now, the band structure will be an even function in $k$ only if the lattice has inversion symmetry (IS). Since graphene and dice lattice have IS, the band structure remains an even function in quasimomentum despite a broken TRS.

The topology of the band structure remains same if the energy gap in the band structure does not close and reopen while tuning the parameter continuously. Here we have seen that one of the gaps closes at $\alpha = 1/\sqrt{2}$ and opens when $1/\sqrt{2} < \alpha < \sqrt{2}$. Hence, we expect a transition in band topology at $\alpha = 1/\sqrt{2}$. In 2D, a change in the Chern number or TKNN integer can be used to identify whether the system undergoes a topological transition or not. In the next section, we show that there is indeed a topological phase transition at $\alpha = 1/\sqrt{2}$ (equivalently at $\alpha = \sqrt{2}$) by evaluating the Chern number explicitly as a function of $\alpha$.

III. CALCULATIONS OF CHERN NUMBER AND TOPOLOGICAL PHASE TRANSITIONS

We need to analyze the behaviour of the Berry connection and Berry curvature in order to calculate the Chern number of each band analytically\cite{57-59} as well as numerically\cite{60}. The Berry connection for the band $\epsilon_m(k)$ can be written as $A_m(k) = i(\Psi_m(k)|\nabla_k|\Psi_m(k))$. Under the gauge used in (14) and (15), the Berry connection reduces to

$$A_m(k) = s_m(k)|\nabla_k|\Psi_m(k),$$

with $s_m(k) = ||a_m(k)||^2 - |c_m(k)|^2$. The Berry curvature of the $m$-th band is defined as

$$\Omega_m(k) = \tilde{z} \cdot [\nabla_k \times A_m(k)].$$

Figure 4 shows the plots of $\Omega_2(k)$ around the two val-

![FIG. 4: Density-contour plots of the Berry curvature ($\Omega(k)$) around the two Dirac points $K$ and $K'$ for different values of $\alpha$. Top panel: $\alpha = 0$, Middle panel: $\alpha = 0.5$, Bottom panel: $\alpha = 1$. Here $k_x$ and $k_y$ are in units of $a^{-1}$.](image-url)
presence of inversion (IS) symmetry in the lattice, but a broken TRS. This also holds true for $\alpha = 1$ i.e. dice lattice [Fig. 4(e),(f)]. However, for $\alpha = 0.5$, $\Omega_2(\mathbf{k})$ is largely different in the two valleys [Fig. 4(d), (e)]. This is a signature of the absence of IS and a broken TRS.

The Berry connection depends on the gauge and may have singularities within the first Brillouin zone (FBZ). The Berry curvature of the $m$-th band is well defined when the quasienergy $\epsilon_m(\mathbf{k})$ is non-degenerate (i.e. $m$-th band does not touch any other bands) for all values of $\mathbf{k}$ within the FBZ. Contour plots of the Berry curvature near the two Dirac points $\mathbf{K}$ and $\mathbf{K'}$ for different values of $\alpha$ are shown in Fig. 4. The surface integral of Berry curvature $\Omega_m(\mathbf{k})$ over the FBZ gives $2\pi c_m$, where $c_m$ is an integer called the Chern number or TKNN index for the $m$-th band:

$$c_m = \frac{1}{2\pi} \int_{\text{FBZ}} \hat{z} \cdot [\nabla_\mathbf{k} \times A_m(\mathbf{k})] \, d^2\mathbf{k}. \quad (19)$$

It is important to mention here that any contributions due to gauge-dependent singularities in $A_m(\mathbf{k})$ must be excluded from the above equation. If a global gauge transformation removes all the singularities, then Chern number of the band will be trivially zero by Stokes theorem. Note that $c_m \neq 0$ implies the absence of a global gauge under which the Berry connection has no singularities in the FBZ. The Berry connection $A_m(\mathbf{k})$ given in Eq. (17) is proportional to $\nabla_\mathbf{k} \psi(\mathbf{k})$. The gauge-dependent singularities in the Berry connection $A_m(\mathbf{k})$ arise at the $\mathbf{k}$ points where the phase $\psi(\mathbf{k})$ of the off-diagonal matrix elements $h(\mathbf{k})$ is ill defined. It occurs if the function $h(\mathbf{k}) = 0$ for certain values of $\mathbf{k}$. In this band structure, since $h(\mathbf{K}) = h(\mathbf{K'}) = 0$, $\psi(\mathbf{K})$ and $\psi(\mathbf{K'})$ are not defined and hence there may be singularity in $A_m(\mathbf{k})$ at the Dirac points if $s_m(\mathbf{K}) \neq 0$ or $s_m(\mathbf{K'}) \neq 0$. Thus, we expect a non-zero Chern number in this case. First we calculate the Chern number for the valence band.

Now we calculate the Chern number for the valence band corresponding to $m = 2$. The variation $s_2(\mathbf{K})$ and $s_2(\mathbf{K'})$ with $\alpha$ is displayed in Fig. 5. Note that $s_2(\mathbf{K})$ and $s_2(\mathbf{K'})$ are evaluated very close to the Dirac points since they are not defined exactly at the Dirac points. These two functions can be written mathematically as

$$s_2(\mathbf{K}) = -\Theta(\alpha - 1/\sqrt{2}), \quad s_2(\mathbf{K'}) = \Theta(\sqrt{2} - \alpha), \quad (20)$$

where $\Theta(x)$ is the usual unit step function.

Now we calculate the Chern number of the valence band for $\alpha < 1/\sqrt{2}$. Since $s_2(\mathbf{K'}) = 1$ and $s_2(\mathbf{K}) = 0$, we have $A_2(\mathbf{K'})$ not defined and $A_2(\mathbf{K}) = 0$. For convenience, we remove the subscript $2$ from $A_2$, as we will stick to the quantities related to the valence band only. The Berry connection for valence band under the chosen gauge (say $A_1(\mathbf{k})$) has a singularity at $\mathbf{K'}$ point. Hence, $A_1(\mathbf{k})$ is not smoothly defined across the FBZ. We make a gauge transformation $A_1(\mathbf{k}) \rightarrow A_1(\mathbf{k}) - \nabla_\mathbf{k} \psi(\mathbf{k}) = [s_1(\mathbf{k}) - 1] \nabla_\mathbf{k} \psi(\mathbf{k})$, which gives $A_{11}(\mathbf{K'}) = 0$ and $A_{11}(\mathbf{K})$ not defined. In this gauge, $\mathbf{K}$ is the singular point. As long as there is a singularity under a given gauge, integral of the Berry curvature will not be defined if the gauge is chosen globally across the FBZ. So, we divide the FBZ, as depicted in Fig. 6, into two regions $R_I$ and $R_{II}$ surrounding $\mathbf{K}$ and $\mathbf{K'}$, respectively. We assign gauge-related Berry connections $A_{1I}(\mathbf{k})$ and $A_{1II}(\mathbf{k})$ in $R_I$ and $R_{II}$, respectively, so that the Berry curvature ($\Omega(\mathbf{k})$) obtained from them is well-defined in each region. $\Gamma_I$ and $\Gamma_{II}$ are contours enclosing $R_I$ and $R_{II}$ respectively. The two regions share a common boundary coinciding with $\Gamma_I$. Now, the Chern number can be written as

$$C_2 = \frac{1}{2\pi} \left[ \int_{R_I} \nabla_\mathbf{k} \times A_1(\mathbf{k}) + \int_{R_{II}} \nabla_\mathbf{k} \times A_{1II}(\mathbf{k}) \right] \cdot \hat{z} d^2\mathbf{k}$$

$$= \frac{1}{2\pi} \left[ \int_{\Gamma_I} \nabla_\mathbf{k} \psi(\mathbf{k}) \cdot d\mathbf{k} \right]$$

(21)

where we have used the fact that integral along outer
boundary of $\Gamma_{II}$ vanishes due to periodicity in $A(k)$ across the FBZ. The region $R_1$ can be chosen as an infinitesimally small circle around the Dirac point $K$. Then, the term within the brackets is the vorticity $\nu_K$ around $K$ point. Since $\nu_K = 2\pi$, $C_2 = 1$. The valence band is degenerate with flat band at $\alpha = 1/\sqrt{2}$. Hence, the Chern number of the valence band at $\alpha = 1/\sqrt{2}$ is not defined.

We have already seen that the three bands are non-degenerate again when $\alpha$ lying between $1/\sqrt{2}$ and $\sqrt{2}$ i.e. $1/\sqrt{2} < \alpha < \sqrt{2}$. In this case, $s_1(K') = 1$ and $s_1(K) = -1$. Hence, both $A_1(K)$ and $A_1(K')$ are not defined. Since, we need at least one non-singular point, this gauge choice is redundant. However, the gauge transformed $A_{II}(K') = [s_1(K') - 1]\nabla_k \psi(K') = s_1(K') \nabla_k \psi(K') = 0$ and $A_{II}(K) = [s_1(K) - 1]\nabla_k \psi(K) = s_1(K) \nabla_k \psi(K) = -2 \nabla_k \psi(K)$. Thus, we have $s_{II}(K') = 0$ and $s_{II}(K) = -2$, i.e. $A(k)$ is singular at $K$ and is non-singular at $K'$. On making a gauge transformation $A_{III}(k) = \frac{A_{II}(k) + 2\nabla_k \psi(k)}{s_1(k)} = \frac{s_1(k) + 2\nabla_k \psi(k)}{s_1(k)} \nabla_k \psi(k)$. Now, we have $s_{III}(K') = 2$ and $s_{III}(K) = 0$, i.e. $K$ is non-singular.

Again, we divide the BZ, similar to Fig. 3 into two regions $R_{II}$ and $R_{III}$ surrounding $K'$ and $K$, respectively. We assign gauge-related Berry connections $A_{II}(k)$ and $A_{III}(k)$ in $R_{II}$ and $R_{III}$, respectively, such that that $\Omega(k)$ is well-defined in each region. So, the Chern number is given by

$$C_2 = \frac{1}{2\pi} \left[ \int_{R_{II}} \nabla_k \times A_{II}(k) + \int_{R_{III}} \nabla_k \times A_{III}(k) \right] \cdot \hat{z} d^2k = \frac{1}{2\pi} \left[ 2 \oint_{\Gamma_1} \nabla_k \psi(k) \cdot dk \right] = 2,$$

where we have taken the infinitesimal loop around $K$ point to have the positive sense of rotation.

The Chern number for the valence band for all $\alpha$ as shown in Fig. 3 can be expressed as

$$C_2(\alpha) = \Theta(1/\sqrt{2} - \alpha) + 2\Theta(\alpha - 1/\sqrt{2}) - \Theta(\alpha - \sqrt{2}).$$

The Chern number for the non-degenerate flat band turns out to be zero for all values of $\alpha$ i.e. $C_1(\alpha) = 0$. Therefore, the Chern number for the conduction band corresponding to $m = 0$ is $C_0(\alpha) = -C_2(\alpha)$. Using Eqs. (18) and (19), we have also calculated the Berry curvature and Chern numbers $C_m$ numerically. Our numerical results support the exact analytical results. Figure 3 displays that the system undergoes a topological phase transition at $\alpha = 1/\sqrt{2}$ (also at $\alpha = \sqrt{2}$ due to the duality) since there is a change in the Chern number.

The anomalous Hall conductivity is directly related to the Chern number. When the Fermi energy is located in a band gap, the Hall conductivity can be expressed in terms of the Chern number as $\sigma_H = c^2/\hbar \sum_m C_m$, where $m$ is restricted to the filled bands below the Fermi energy. By complete filling of the valence band or both the valence and flat bands, the system becomes a QHI with the Hall conductivity $\sigma_H = c^2/\hbar$ for $1/\sqrt{2} < \alpha < \sqrt{2}$.

V. SUMMARY AND CONCLUSION

We have considered $\alpha$-$T_3$ lattice illuminated by intense circularly polarized radiation. Using the off-resonant approximation, we have derived exact analytical expressions of three quasienergy bands over the full Brillouin zone. It is observed that the triple point degeneracy is completely lifted due to the broken TRS symmetry.
FIG. 7: Radiation-dressed band structure of $\alpha$-$T_3$ nanoribbon with armchair edges for (a) $\alpha = 0$, (b) $\alpha = 0.5$, (c) $\alpha = 0.8$ and (d) $\alpha = 1.0$. The red/blue curves represent edge states propagating on top/bottom edges, while the black curves represent the bulk bands.

caused by the circularly polarized light. It leads to unequal photoinduced gaps at $K$ and $K'$ except for monolayer graphene and dice lattice, due to the lack of inversion symmetry. At $\alpha = 1/\sqrt{2}$, the semimetallic phase is restored by closing the gap between flat and valence bands at $K$, while the gap between conduction and flat bands at $K'$ still opens. The low-energy Dirac cones at $K$ and $K'$ points resurface at the gap closing point which is insensitive to the radiation amplitude. The $\alpha$-$T_3$ lattice illuminated by the circularly polarized radiation is transformed to a Haldane-like Chern insulator. We find that there is a topological phase transition from the Chern number $C = 1(0, -1)$ to a Chern number $C = 2(0, -2)$ at the band closing point, where $C$ is the Chern number of the valence (flat, conduction) band. This is an example of a three-band system having larger Chern number. The effect of non-trivial topology of the system should get reflected in the transport properties through the chiral edge channels as shown for the armchair configuration.

ACKNOWLEDGEMENT

We would like to thank Firoz Islam and Sonu Verma for useful discussions.

1 K. von Klitzing, Rev. Mod. Phys. 58, 519 (1986)
2 M. Z. Hasan and C. Kane, Rev. Mod. Phys. 82, 3045 (2010)
3 X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011)
4 L. Lu, J. D. Joannopoulos, and M. Soljacic, Nat. Photon. 8, 821 (2014)
5 C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005)
6 C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005)
7 B. Bernevig, T. L. Hughes, and S. C. Zhang, Science 314, 1757 (2006)
8 F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988)
9 G Jotzu, M Messer, R Desbuquois, M Lebrat, T Uehlinger, D Greif, and T Esslinger, Nature 515, 237 (2014)
10 X. L. Qi, Y. S. Wu, and S. C. Zhang, Phys. Rev. B 74, 085308 (2006)
11 E. Tang, J. W. Mei, and X. G. Wen, Phys. Rev. Lett. 106, 236802 (2011)
12 K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett. 106, 236803 (2011)
13 T. Neupart, L. Santos, C. Chamon, and C. Mudry, K. Sun, Z. Gu, H. Katsura, and S. Das Sarma, Phys. Rev. Lett. 106, 236804 (2011)
14 Y. Hatsugai, Phys. Rev. Lett. 71, 3697 (1993)
15 A. P. Schnyder, S. Ryu, A. Furuski, and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008)
16 W. Yao, A. H. Macdonald, and Q. Niu, Phys. Rev. Lett. 99, 047401 (2007)
17 T. Oka and H. Aoki, Phys. Rev. B 79, 081406(R) (2009)
18 J. I. Inoue and A. Tanaka, Phys. Rev. Lett. 105, 017401 (2010)
19 T. Kitagawa, T. Oka, A. Brataas, L. Fu, and E. Demler, Phys. Rev. B 84, 235108 (2011)
20 N. Lindner, G. Refael, and V. Galitski, Nat. Phys. 7, 490 (2011)
21 B. Dora, J. Cayssol, F. Simon, and R. Mossner, Phys. Rev. Lett. 108, 056602 (2012)
22 G. Usaj, P. M. Perez-Piskunow, L. E. F. Fora Torres, and C. A. Balseiro, Phys. Rev. B 90, 115423 (2014)
23 J W McIver, B Schulte, F.-U. Stein, T Matsuyama, G Jotzu, G Meier, A Cavalleri, arxiv: 1811.03512
24 S. A. Skirlo, L. Lu, and M. Soljacic, Phys. Rev. Lett. 113, 113904 (2014)
25 S. A. Skirlo, L. Lu, Y. Igarashi, Q. Yan, J. Joannopoulos, and M. Soljacic, Phys. Rev. Lett. 115, 253901 (2015)
26 B. Sutherland, Phys. Rev. B 34, 5208 (1986)
27 J. Vidal, R. Mosseri, and B. Doucot, Phys. Rev. Lett. 81, 5888 (1998)
28 J. I. Inoue and A. Tanaka, Phys. Rev. B 81, 235108 (2011)
29 S. A. Skirlo, L. Lu, Y. Igarashi, Q. Yan, J. Joannopoulos, and M. Soljacic, Phys. Rev. Lett. 115, 253901 (2015)
30 B. Sutherland, Phys. Rev. B 34, 5208 (1986)
31 J. Vidal, R. Mosseri, and B. Doucot, Phys. Rev. Lett. 81, 5888 (1998)
32 S. E. Korshinov, Phys. Rev. B 63, 134503 (2001)
33 M. Rizzi, V. Cataudella, and R. Fazio, Phys. Rev. B 73, 144511 (2006)
D. Bercioux, D. F. Urban, H. Grabert, and W. Hausler, Phys. Rev. A 80, 063603 (2009)

D. F. Urban, D. Bercioux, M. Wimmer, and W. Hausler, Phys. Rev. B 84, 115136 (2011)

J. D. Malcolm and E. J. Nicol, Phys. Rev. B 92, 035118 (2015)

J. D. Malcolm and E. J. Nicol, Phys. Rev. B 93, 165433 (2016)

M. Vigh, L. Oroszlány, S. Vajna, P. San-Jose, G. Dávid, J. Cserti, and Balázs Dora, Phys. Rev. B 88, 161413(R), (2013)

A. Raoux, M. Morigi, J. N. Fuchs, F. Piechon, and G. Montambaux, Phys. Rev. Lett. 112, 026402 (2014)

F. Wang and Y. Ran, Phys. Rev. B 84, 241103 (2011)

E. Illes, J. P. Carbotte, and E. J. Nicol, Phys. Rev. B 92, 245110 (2015)

T. Biswas and T. K. Ghosh, J. Phys.: Condens. Matter 28, 495302 (2016)

A. D. Kovacs, G. David, B. Dora, and J. Cserti, Phys. Rev. B 95, 035414 (2017)

SK Firoz Islam and P. Dutta, Phys. Rev. B 96, 045418 (2017)

T. Biswas and T. K. Ghosh, J. Phys.: Condens. Matter 30, 075301 (2018)

B. Dey and T. K. Ghosh, Phys. Rev. B 98, 075422 (2018)

A. Iurov, G. Gumbs, and D. Huang, arxiv: 1806.09172v2

D. O. Oriekhov, E. V. Gorbar, and P. Gusynin, J. Low Temp. Phys. 44, 1313 (2018)

D. O. Oriekhov, E. V. Gorbar, P. Gusynin, and D. O. Oriekhov, arxiv: 1812.10979v1

Yan-Ru Chen, Y. Xu, J. Wang, Jun-Feng Liu, and Z. Ma, Phys. Rev. B 99, 045420 (2019)

SK Firoz Islam, arxiv: 1901.07943v1

M. Ezawa, Phys. Rev. Lett. 110, 026603 (2013)

K. Saha, Phys. Rev. B 94, 081103(R) (2016)

M. Tahir, A. Macchon, and U. Schwingenschlogl, Phys. Rev. B 90, 125438 (2014)

E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. 106, 236802 (2011)

R. Liu, W.-C. Chen, Y.-F. Wang, and C.-D. Gong, J. of Phys.: Condens. Matter 24, 305602 (2012)

T. Andrijauskas, E. Anisimovas, M. Raciunas, A. Mekys, V. Kudriasov, I. B. Spielman, and G. Juzeliunas, Phys. Rev. A 92, 033617 (2015)

B. Jaworowski, A. Manolescu, and P. Potasz, Phys. Rev. B 92, 245119 (2015)

T. Dittrich, P. Hanggi, G. L. Ingold, B. Kramer, G. Schon, and W. Zweger (eds), Quantum Transport and Dissipation (Wiley-WCH, Weinheim, 1998)

M. Kohmoto, Ann. Phys. (N.Y.) 160, 343 (1985)

D. Xiao, M.-C. Chang, and Q. Niu, Rev. Mod. Phys. 82, 1959 (2010)

N. Goldman, G. Juzeliunas, P. Ohberg, and I. B. Spielman, Rep. Prog. Phys. 77, 126401 (2014)

T. Fukui, Y. Hatsugai, and H. Suzuki, J. Phys. Soc. Jpn. 74, 1674 (2005)

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. 49, 405 (1982)