New minimal \((4, n)\)-regular matchstick graphs

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Abstract: A matchstick graph is a graph drawn with straight edges in the plane such that the edges have unit length, and non-adjacent edges do not intersect. We call a matchstick graph \((m, n)\)-regular if every vertex has only degree \(m\) or \(n\). In this article the authors present the latest known \((4, n)\)-regular matchstick graphs for \(4 \leq n \leq 11\) with a minimum number of vertices.

1 Introduction

A matchstick graph is a graph drawn with straight edges in the plane such that the edges have unit length, and non-adjacent edges do not intersect. We call a matchstick graph \((m, n)\)-regular if every vertex has only degree \(m\) or \(n\).

For \(m \leq n\) minimal \((4, n)\)-regular matchstick graphs with a minimum number of vertices only exist for \(4 \leq n \leq 11\). The smallest known \((4, n)\)-regular matchstick graph for \(n = 4\), also named 4-regular, is the so called Harborth graph (Fig. 2) consisting of 52 vertices and 104 edges. Since 1986 the second smallest known 4-regular matchstick graph consists of 60 vertices and 120 edges (Fig. 5). On July 3, 2016 the authors presented a new second smallest known example with 54 vertices and 108 edges (Fig. 3), which is based on the Harborth graph [10]. On April 15, 2016 Mike Winkler presented an example with 57 vertices and 114 edges with
a whole new geometry (Fig. 4) [11][12]. For each $n > 11$ only infinite graphs with an infinite number of vertices exists. Figures 21 - 23 show infinite graphs for $n = 12$ and $n = 13$.

It is an open problem how many different $(4, n)$-regular matchstick graphs with a minimum number of vertices for $4 \leq n \leq 11$ exist and which is the least minimal number. ”Our knowledge on matchstick graphs is still very limited. It seems to be hard to obtain rigid mathematical results about them. Matchstick problems constructing the minimal example can be quite challenging. But the really hard task is to rigidly prove that no smaller example can exist.” [4]

In this article the authors present the $(4, n)$-regular matchstick graphs for $5 \leq n \leq 11$ with the smallest currently known number of vertices as well as further new minimal examples for $n = 4, n = 5$ and $n = 6$ with 108, 114, 121 and 126 edges. The earlier versions of these graphs were presented on the Math Magic1 website of Erich Friedman [1], which can be seen on an older version of this site [7] from 2015.

The authors discovered the new minimal $(4, n)$-regular matchstick graphs in the days from March 14 - September 24, 2016 and presented them for the first time in the German mathematics internet forum Matroids Matheplanet2.

The rigidity of the graphs has been verified by Stefan Vogel. Details can be found in the thread of the graph theory forum [5]. The method which was used for the calculations he describes in a separate German article [9]. The rigidity of the graphs is also a proof for their geometry and their existenz.

2 Rigid subgraphs

The geometry of the new graphs, except for $n = 11$, is not complicated. Most of these graphs have a high degree of symmetry. There are two types of most used rigid subgraphs, which we call the kite (Fig. 1 a) and the triplet kite (Fig. 1 d). The kite is a $(4, 2)$-regular matchstick graph consisting of 12 vertices and 21 edges and has a vertical symmetry. Two kites can be connected to each other in two useful ways. We call these

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1http://www2.stetson.edu/efriedma/mathmagic/1205.html
2http://www.matheplanet.de
subgraphs the *double kite* (Fig. 1 b) and the *reverse double kite* (Fig. 1 c), both consisting of 22 vertices and 42 edges. The reverse double kite offers the possibility to connect two of the inner vertices with an additional unit length edge. This property has been used for $n = 5$ (Fig. 18). What makes the subgraphs (b) and (c) so useful is the fact that they have only two vertices of degree 2. Two of these subgraphs can be used to connect two vertices of degree 2 at different distances by using them like clasps. This property has been used for $n = 9$, $n = 10$ and $n = 11$ (Fig. 11 - 13).

![Kites](image)

**Figure 1: Rigid subgraphs**

The triplet kite is a $(4, 3, 2)$-regular matchstick graph consisting of 22 vertices and 41 edges and has a vertical symmetry. Three triplet kites can be connected together in a way so that the three vertices of the outer triangles become the same. The so formed 4-regular matchstick graph (Fig. 4) offers the possibility to connect each two of the three inner vertices with an additional unit length edge. This property has been used for $n = 5$ (Fig. 7) and $n = 6$ (Fig. 8). The graphs for $n = 9$ (Fig. 11) and $n = 11$ (Fig. 13) use additional subgraphs and are the only graphs which contain outer edges, which are not part of an equilateral triangle. The graph for $n = 11$, by far the largest graph in this article, has a more complicated geometry, which had to be calculated with a CAS by Stefan Vogel. The graphs for $n = 7$ (Fig. 9) are the only flexible graphs and do not consist...
of the kite-based subgraphs. The geometry of the graphs for \( n = 5 \) and \( n = 6 \) with 121 edges (Fig. 15 - 17) are based on the subgraph (d). These graphs are the previous versions of the smallest known follow-up graphs in Figure 7 and 8.

3 The first four smallest known 4-regular matchstick graphs

In 1986 Heiko Harborth [3] presented the smallest known example of a 4-regular matchstick graph. The rigidity and the existence of this graph could be proven first time in 2006 by Eberhard H.-A. Gerbracht [2].

![4-regular matchstick graph with 52 vertices and 104 edges.](image)

Figure 2: 4-regular matchstick graph with 52 vertices and 104 edges.
On July 3, 2016 the authors discovered a new second smallest known example of a 4-regular matchstick graph. This rigid graph has a vertical and a horizontal symmetry and is based on the Harborth graph. The history of this graph is a little bit intricate and begins on April 24, 2016 [10].

On April 15, 2016 Mike Winkler discovered a new third smallest known example of a 4-regular matchstick graph [11][12]. This rigid graph has a rotational symmetry of order 3 and consists of three overlapping triplet kites.

The graph in Figure 4 is also the basis for the smallest known \((4, n)\)-regular matchstick graphs for \(n = 5\) and \(n = 6\).
From 1986 to April 15, 2016 the second smallest known example of a 4-regular matchstick graph consisted of 60 vertices and 120 edges.

Figure 5: 4-regular matchstick graphs with 60 vertices and 120 edges.

The graphs in Figure 4 are flexible and each of them can be transformed into the other. Whereby the graph v2 shows only one possibility of transforming. The graph v1 has a rotational symmetry of order 12 and consists of 12 identical subgraphs consisting of 7 vertices and 10 edges (red). Each transformed version, as the graph v2, has a rotational symmetry of order 6 and consists of 6 identical subgraphs consisting of 12 vertices and 20 edges (red).

The transformation in the graph v2 has been chosen so, that the line segment between the vertices $a$ and $b$ measures exactly two unit lengths. This kind of transforming is exactly the one we need to build the triplet kite with (Fig. 6). This shows the close geometric relationship between the graphs in Figure 4 and Figure 5.

Figure 6: The geometry of the triplet kite.
The smallest known \((4, n)\)-regular matchstick graphs for \(5 \leq n \leq 11\)

Figure 7: \((4, 5)\)-regular matchstick graph with 57 vertices and 115 edges. Discovered April 15, 2016 by M. Winkler. This rigid graph has a vertical symmetry and contains three overlapped triplet kites.

Figure 8: \((4, 6)\)-regular matchstick graph with 57 vertices and 117 edges. Discovered April 15, 2016 by M. Winkler. This rigid graph has a rotational symmetry of order 3 and contains three overlapped triplet kites.
Discovered March 17, 2016 by M. Winkler, except for v3, which was discovered by P. Dinkelacker. The graph v3 is based on the graph v1 and has a slightly different internal geometry. These graphs are flexible. The graph v1 can be transformed into the graph v2, and the graph v3 can be transformed into the graph v4. The v1-based graphs always have a point symmetry. The graph v3 has a horizontal symmetry if the angles consist of 30 degrees and their multiples. For all other degrees the transformed graph is as asymmetric as the graph v4. The two vertices of degree 7 share an edge from which each graph gets its new minimality. The graphs v3 and v4 offer the possibility to rearrange two edges in the middle of the right side. It is unknown whether a rigid or kite-based \((4, 7)\)-regular matchstick graph with 159 edges or less exists.
Figure 10: $(4, 8)$-regular matchstick graph with 62 vertices and 126 edges. Discovered March 14, 2016 by P. Dinkelacker. This rigid graph has a horizontal symmetry and consists only of kites. Two reverse double kites and one double kite.

Figure 11: $(4, 9)$-regular matchstick graph with 135 vertices and 273 edges. Discovered May 18, 2016 by P. Dinkelacker. This rigid graph has a vertical symmetry and contains four double kites and four slightly modified single kites.
Discovered March 17, 2016 by P. Dinkelacker. This rigid graph is asymmetric and consists only of kites. Three double kites, two reverse double kites and one single kite. It remains an interesting question whether a symmetric (4, 10)-regular matchstick graph with 231 edges or less exists.
Figure 13: (4, 11)-regular matchstick graph with 403 vertices and 813 edges.

Discovered September 24, 2016 by S. Vogel and M. Winkler. This rigid graph is asymmetric and contains five double kites and five reverse double kites. There exists a few asymmetric variations of this graph with 813 edges, because the clasps can be varied. But the current design requires the least place in the plane.
The interesting and flexible part of this graph lies in the neighborhood of the vertices of degree 11, as the next detail image shows. There exists a very small rhombus. The long outside edges measure exactly two unit lengths.

Figure 14: Detail around the top vertex of degree 11 in Figure 13.

The degrees of the angles between the edges around the centered vertex. Counter-clockwise beginning with the angle between the red edges.

32.362519660072210, 40.49207000332465, 25.382433534610843, 34.890820876760450, 32.21894760945070, 34.514335947363630, 29.108515978283318, 36.31491131809427, 29.550687898877964, 35.065359484316880, 30.09939768884507.
5 Further new minimal and infinite \((4, n)\)-regular match-stick graphs

Before the authors had discovered the smallest known \((4, n)\)-regular match-stick graphs of chapter 4, they found further new minimal graphs for \(n = 5\) and \(n = 6\) consisting of 121 or 126 edges.

Figure 15: \((4, 5)\)-regular matchstick graphs with 60 vertices and 121 edges.

Discovered April 13, 2016 by M. Winkler. The rigid graphs v1, v2, v3 and v4 have a point symmetry.
Figure 16: $(4,5)$-regular matchstick graphs with 60 vertices and 121 edges.

Discovered April 13-14, 2016 by M. Winkler. The rigid graphs $v_5$, $v_6$ and $v_7$ are asymmetric. The rigid graph $v_8$ has a vertical symmetry.
Figure 17: \((4,6)\)-regular matchstick graphs with 60 vertices and 121 edges.

Discovered April 14, 2016 by M. Winkler, except for v3, which was discovered by P. Dinkelacker. The rigid graphs v1 and v2 are asymmetric. The rigid graphs v3 and v4 have a vertical symmetry. The vertical symmetry of these graphs was discovered by P. Dinkelacker and is based on the equivalent asymmetric and point symmetric graphs of this chapter by M. Winkler.
Figure 18: \((4, 5)\)-regular matchstick graph with 62 vertices and 126 edges.

Discovered March 14, 2016 by M. Winkler. This rigid graph has a point symmetry and consists of two slightly modified reverse double kites and two single kites.

Figure 19: \((4, 6)\)-regular matchstick graph with 62 vertices and 126 edges.

Discovered March 14, 2016 by P. Dinkelacker. This rigid graph has a vertical and a horizontal symmetry and consists only of kites. Two double kites and two single kites.
Figure 20: (4,6)-regular matchstick graph with 62 vertices and 126 edges.

Discovered March 14, 2016 by M. Winkler. This rigid graph has a point symmetry and consists only of kites. Two reverse double kites and two single kites. This graph needs less space in the plane than the (4, 6)-graph in Figure 18.

Figure 21: Infinite (4, 12)-regular and (4, 13)-regular matchstick graphs.

For each $n > 11$ only infinite (4, $n$)-regular matchstick graphs with an infinite number of vertices exist. The graphs of Figure 21 show only the simplest version of these kind of graphs with one vertex of degree $n$ in the center. But there exists infinite different constructions with up to infinite number of vertices of degree $n$. All these kinds of infinite graphs are high
flexible. The next four examples of infinite (4, 12)-regular matchstick graphs will illustrate this.

Figure 22: Infinite (4, 12)-regular matchstick graphs with 2 vertices of degree 12.
Figure 23: Infinite \((4, 12)\)-regular matchstick graphs flexibility examples.
6 References

1. E. Friedman, Math Magic, Problem of the Month (December 2005), Problem 4, Smallest Known m/n Matchstick Graphs. (weblink)
   (http://www2.stetson.edu/efriedma/mathmagic/1205.html)

2. E. H.–A. Gerbracht, Minimal Polynomials for the Coordinates of the Harborth Graph, 2006, arXiv:math/0609360[math.CO]. (weblink)
   (https://arxiv.org/pdf/math/0609360v3.pdf)

3. H. Harborth, Match Sticks in the Plane, The Lighter Side of Mathematics. Proceedings of the Eugéne Strens Memorial Conference of Recreational Mathematics & its History, Calgary, Canada, July 27 - August 2, 1986 (Washington) (R. K. Guy and R. E. Woodrow, eds.), Spectrum Series, The Mathematical Association of America, 1994, pp. 281-288.

4. S. Kurz and G. Mazzuoccolo, 3-regular matchstick graphs with given girth, Geombinatorics Quarterly Volume 19, Issue 4, April 2010, pp. 156-175. (weblink)
   (http://arxiv.org/pdf/1401.4360v1.pdf)

5. Matroids Matheplanet, Thread in the graph theory forum. (weblink)
   (http://www.matheplanet.de/matheplanet/nuke/html/viewtopic.php?topic=216644&start=0)
   (Nicknames used in the forum: haribo = Peter Dinkelacker, Slash = Mike Winkler)

6. Siemens PLM Software, Solid Edge 2D-Drafting ST8. (weblink)
   (http://www.plm.automation.siemens.com/de_de/)

7. Wayback Machine Internet Archive. (weblink)
   (https://archive.org)
   (https://web.archive.org/web/20151209031635/http://www2.stetson.edu/~efriedma/mathmagic/1205.html)[1]

8. Wikipedia, Matchstick graph. (weblink)
   (https://en.wikipedia.org/wiki/Matchstick_graph)
9. S. Vogel, Beweglichkeit eines Streichholzgraphen bestimmen, July 2016. (weblink)
(http://www.matheplanet.de/matheplanet/nuke/html/article.php?sid=1757&mode=&order=0)

10. M. Winkler, Der große Bruder des Harborth-Graphen, July 2016. (weblink)
(http://www.matheplanet.de/matheplanet/nuke/html/article.php?sid=1758&mode=&order=0)

11. M. Winkler, Ein 4-regulärer Streichholzgraph mit 114 Kanten, May 2016. (weblink)
(http://www.matheplanet.de/matheplanet/nuke/html/article.php?sid=1746&mode=&order=0)

12. M. Winkler, Ein neuer 4-regulärer Streichholzgraph, Mitteilungen der Deutschen Mathematiker-Vereinigung (DMV), Band 24, Heft 2, Seiten 7475, ISSN (Online) 0942-5977, ISSN (Print) 0947-4471, DOI: 10.1515/dmvm-2016-0031, July 2016. (weblink)
(http://www.degruyter.com/view/j/dmvm.2016.24.issue-2/dmvm-2016-0031/dmvm-2016-0031.xml)

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