Quantum channels that preserve the commutativity

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We identify and characterize all the local quantum channels that preserves the set of classical states, i.e., does not create any quantum correlations. At first we show that the quantum correlations cannot be created from without if and only if the local quantum channel preserves the commutativity, i.e., the images of any two commuting states also commute. And then we provide an operational necessary and sufficient criterion for a known quantum channel to preserve the commutativity as well as a single observable to witness an arbitrary unknown commutativity-preserving channel. All the distance-based measures for quantum correlations, e.g., the geometric mean, are non-increasing while the quantum discord defined by von Neumann measurements can be increasing or decreasing under local commutativity-preserving channels.

Quantum correlations, including the entanglement and the quantum discord, are useful resources that play a fundamental role in various quantum informational processes. Quantum correlations beyond entanglement, i.e., quantum correlations found in separable states, e.g., as quantified by quantum discord [1, 2], are argued to be responsible for the speedup in certain quantum computational tasks such as deterministic quantum computation with one qubit [3, 4]. The operational interpretations via state merging [5] for the quantum discord establish firmly the status of quantum discord as a useful resource beside the entanglement. Also the quantum discord is related to the completely positive maps and quantum phase transitions in certain physical systems. As a resource it is therefore important to understand how the quantum correlations behave under local noises or operations.

As is well-known, the entanglement is non-increasing under local operations and classical communications (LOCC). Especially the entanglement cannot be created from a separable state using only LOCCs. This property of entanglement is characteristic for its various quantitative measures. It is natural to ask what kinds of local operations that play the role of LOCCs for quantum correlations and especially what kind of local operations that do not create the quantum correlations as quantified by quantum discord. Operations allowed by quantum mechanics are quantum channels, i.e., all possible dynamical processes described by trace-preserving completely positive (CP) maps. In the case of qubit channels this problem has been solved in [3]: qubit channels that do not create the quantum discord are either unital, i.e., mapping identity operator to identity operator, or semi-classical, i.e., nullifying quantum correlations in any state. For higher dimensional systems there are examples of unital channels that are able to create quantum correlations. The problem of characterizing all the quantum channels for higher dimensional system that do not create any quantum correlations is left open.

Here we shall resolve this problem in its full generosity. At first we show that a quantum channel does not create quantum correlation if and only if the given channel preserves the commutativity. And then we present an operational sufficient and necessary criterion for a known channel to be commutativity-preserving (CoP). Also a single observable is proposed to witness the commutativity-preserving property of an arbitrarily unknown quantum channel. As an example, we show that the mixing of a given CoP channel with the identical channel preserves still the commutativity if and only if the given channel is a cloning channel. Finally the behaviors of various quantitative measures of quantum correlations, especially the quantum discord, under CoP channels are discussed.

Although there are many different measures to quantify the quantum correlations beyond entanglement, bipartite classical states as well as quantum-classical or classical-quantum states are unambiguously defined. In what follows we shall use the languages of quantum discord. A bipartite state $\varrho_{AB}$ is said to be a classical-quantum state if and only if it has a vanishing $A$-discord, i.e., there exist an orthonormal basis $\{|k\rangle\}$ for subsystem $A$, a probability distribution $p_k$, and a set of density matrices of $\sigma^B_k$ of subsystem $B$ such that

$$\varrho_{AB} = \sum_k p_k |k\rangle \langle k| A \otimes \sigma^B_k.$$  

The classical-quantum state can be defined similarly, i.e., with a vanishing $B$-discord, and a classical state is both classical-quantum and quantum-classical, i.e., states with vanishing $A$-discord and $B$-discord. Operationally, a given bipartite state has zero $A$-discord if and only if $T_{B}(\varrho_{AB} \lambda^B_{\mu})$ are mutually commuting where $\{\lambda^B_{\mu}\}$ is an arbitrary operator basis for subsystem $B$ [7]. For an unknown state with four copies a single observable can witness the nonzero quantum discord [11, 12].

Our first result concerns what kinds of local quantum operations preserve the set of states with zero $A$-discord.

**Theorem 1** For a bipartite system $AB$ a local quantum channel $\Lambda$ acting on subsystem $A$ preserves the set of states with vanishing $A$-discord, i.e., the image of any zero $A$-discord state still has a zero $A$-discord, if and only if the channel $\Lambda$ preserves the commutativity, i.e., for any two density matrices $\varrho, \sigma$ it holds

$$[\varrho, \sigma] = 0 \Rightarrow [\Lambda(\varrho), \Lambda(\sigma)] = 0.$$  

**Proof** Consider a bipartite state $\varrho_{AB}$ with a zero $A$-discord as given in Eq. (1) and a local quantum channel $\Lambda$ acting only
on subsystem $A$ that preserves the commutativity. Since all projections $\{ |k\rangle \langle k| \}$ in an arbitrary basis commute their images $\Lambda(|k\rangle \langle k|)$ are also mutually commuting. As a result the image of the bipartite $\varrho_{AB}$ has also a zero $A$-discord according to the operational criterion [7]. If the local quantum channel $\Lambda$ does not preserve the commutativity then there exist two commuting states $\varrho, \sigma$ of subsystem $A$ such that $[\Lambda(\varrho), \Lambda(\sigma)] \neq 0$. While the state $\varrho_{AB} = \varrho_A \otimes |0\rangle \langle 0| + \sigma_A \otimes |1\rangle \langle 1|$ has a zero $A$-discord, its image $\Lambda \otimes \mathcal{I}(\varrho_{AB})$ has a nonzero $A$-discord, also thanks to the operational criterion [7]. That is to say the quantum channel $\Lambda$ that does not preserve the commutativity is able to create nonzero quantum discord from a state with a zero quantum discord.

Q.E.D.

In the following we shall characterize all the CoP channels for qudit, i.e., quantum system with $d$ energy levels. Let $\{ |k\rangle \}_{k=1}^d$ be the computational basis and $\{ \lambda_k |0\leq \mu \leq D := d^2 - 1 \}$ be a set of orthonormal Hermitian operator basis for qudit with $\lambda_0$ being identity and $\text{Tr}(\lambda_0 \lambda_{\mu}) = -d \delta_{\mu 0}$. The structure constants $f_{\mu \nu \tau} = -i \text{Tr}(\lambda_\mu \lambda_\nu \lambda_\tau)/d^2$ with $\mu, \nu, \tau = 0, 1, \ldots, D$ are totally antisymmetric with respect to three indices. A quantum channel $\Lambda$, i.e., a trace-preserving completely positive map, can be represented in three equivalent ways: the operator-sum representation, i.e., Kraus operators, the unitary representation on system plus environment, and a two-qudit state $R_{\Lambda} = \Lambda \otimes \mathcal{I}(\Phi)$ where $\Phi = |\Phi\rangle \langle \Phi|$ is the projection of the maximal entangled state $|\Phi\rangle = \sum_{n=1}^d |nn\rangle_{AB}$ (not normalized) in which qudit $A$ is called as the system qudit while the qudit $B$ is called as the reference qudit.

Consider four copies of the two-qudit state $R_{\Lambda} = \Lambda \otimes \mathcal{I}(\Phi)$ corresponding to a given channel $\Lambda$ and label four system qudits that the channels act on by $A_1$ and four reference qudits by $B_i$ with $i = 1, 2, 3, 4$. Let $V_{ij}$ be the swapping operator acting on the system (or reference) qudits $i$ and $j$ and denote by $X = V_{12}V_{23}V_{34}$ the cyclic permutation of four qudits. The quantum discord witness introduced in [12] is an collective observable on four copies

$$W = \frac{X_A + X_A^\dagger}{2} \otimes (V_{12}V_{34} - V_{13}V_{24}).$$

The 2-qudit state $R_{\Lambda}$ has a vanishing $A$-discord if and only if $\text{Tr}(R_{\Lambda} \otimes 4 W) = 0$ [12]. For an unknown 2-qudit state with four copies we have only to measure the quantum discord witness $W$ to see whether the state has a vanishing discord or not. Also we introduce another collective observable

$$Z = \frac{X_A + X_A^\dagger}{2} \otimes V_{14}V_{23}(V_{13} + V_{24} - V_{12} - V_{34})$$

and denote $L = -(I_A \otimes V_{14}V_{23}) Z$. We note that all these collective observables, especially $W$ and $Z$, can be measured by introducing some auxiliary qubits, some controlled swapping gates with qubits as sources, and qubit measurements. Now we are ready to formulate our main results:

**Theorem 2** The following statements are equivalent:

i) The qudit channel $\Lambda$ preserves the commutativity.

ii) Given four copies of $R_{\Lambda} = \Lambda \otimes \mathcal{I}(\Phi)$ it holds

$$d\text{Tr}(R_{\Lambda} \otimes 4 W) = \text{Tr}(R_{\Lambda} \otimes 4 Z).$$

iii) In a given orthonormal Hermitian operator basis $\{ \lambda_\mu \}$ with $\lambda_0$ being identity for arbitrary $\mu, \nu = 0, 1, \ldots, D$ it holds

$$\text{Tr}(\Lambda(\lambda_\mu), \Lambda(\lambda_\nu)) = \sum_{D}^\alpha \frac{1}{2} 
\sum_{\sigma, \tau, \nu = 1}^D f_{\mu \nu \tau} \text{Tr}[\Lambda(\lambda_\alpha), \Lambda(\lambda_\beta)].$$

Proof By definition a channel $\Lambda$ preserves the commutativity if and only if the images of two arbitrary commuting density matrices $|\varrho, \sigma\rangle = 0$ still commute, i.e., $[\Lambda(\varrho), \Lambda(\sigma)] = 0$. Since commuting density matrices have common eigenstates, it is equivalent to require $i[\Lambda(k_\nu), \Lambda(l_\mu)] = 0$ for arbitrary $l, k$ where $\{ |k\rangle \}$ is a given qudit basis and $|k_\nu\rangle = U(k)|k\rangle |k\rangle U^\dagger$. Since for a Hermitian matrix $H$ the condition $H = 0$ is equivalent to $\text{Tr}(H^\dagger H) = 0$, a qudit channel $\Lambda$ preserves the commutativity if and only if

$$0 = -\int dU \sum_{k,l=1}^d \text{Tr}[\Lambda(k_\nu), \Lambda(l_\mu)]^2
\text{Tr}(R_{\Lambda} \otimes 4 X_A + X_A^\dagger \otimes \int dU U^{\otimes 4} O^T U^{\otimes 4})
= \text{Tr}(R_{\Lambda} \otimes 4 ((d+1)L + dW - Z))
/(d(d+1)(d+2))$$

where the integral is over the unique Haar measure of unitary matrices and $O = \sum_{k,l=1}^d k \otimes (k \otimes l \otimes k \otimes l)$. To obtain the second equality above we have used the properties $\Lambda(\varrho) = \text{Tr}(R_{\Lambda} \otimes 4 \varrho_0)$ and $\text{Tr}(X g_1 \otimes g_2 \otimes g_3 \otimes g_4) = \text{Tr}(g_1 g_2 g_3 g_4)$. To calculate the integral $\int dU U^{\otimes 4} O^{O^{\otimes 4}}$ in the third line in the equation above we have used the fact that $O = \sum_{\sigma \in S_4} \varrho_0 V_{\sigma}$, since $O^{O^{\otimes 4}} = 0$ for arbitrary $U$, for some real numbers $O_{\sigma}$, where $V_{\sigma}$ is the 4-qudit operator representing the permutation $\sigma \in S_4$. To compute the coefficients $O_{\sigma}$ we can either use the Collins-Sniady formula [13] or solve the linear equation $\text{Tr}(O V_{\sigma}) = \text{Tr}(O V_{\sigma})$ with 24 variables $\{ O_{\sigma} \} \sigma \in S_4$. Taking into account $\text{Tr}(O V_{12}) = \text{Tr}(O V_{13}) = \text{Tr}(O V_{12} V_{23} V_{34}) = d(d-1)$ and $\text{Tr}(O V_{13}) = \text{Tr}(O V_{24}) = \text{Tr}(O V_{13} V_{24}) = -d(d-1)$ with $\text{Tr}(O V_{12}) = 0$ otherwise, we obtain

$$\int dU U^{\otimes 4} O^{O^{\otimes 4}} = \frac{(d+1)\tilde{L} + d(V_{12}V_{34} - V_{13}V_{24}) + V_{14}V_{23} \tilde{L}}{d(d+1)(d+2)},$$

where $\tilde{L} = V_{12} + V_{34} - V_{13} - V_{24}$. On the other hand it is rather straightforward to check the above equality by comparing the coefficients $\text{Tr}(O V_{\sigma})$ with $\text{Tr}(\bar{O} V_{\sigma})$ which should be identical.

In what follows the summation from 0 to $D$ over repeated indices is always assumed. In a local orthogonal observable basis $\{ \lambda_\mu \otimes \lambda_\nu^T \}$ for two qudits we have expansion $d\Phi =
sufficient condition for CoP becomes simply
that is defined by
concerned mainly in bipartite states and it is straightforward
we obtain immediately
In the above calculations the identity
which leads to
Now let us consider a nontrivial mixing
as a result of the identity channel
is obviously another example of
and thus the Hamiltonian channel preserves the commutativity if and only if
d = 3 or p = 1.
Now let us consider a nontrivial mixing
in which the equality holds true for arbitrary traceless qubit observable given by
we refer to the
Thus as long as H ≠ 0, p ≠ 1, and d ≥ 3 we have δλν > 0, i.e., the mixing λp of a Hamiltonian channel with the identity channel preserves the commutativity if and only if d = 2 or H = 0 or p = 1.
Furthermore we note that the condition
Thus to say that the channel is a semi-classical channel that is defined by
being a d-outcome POVM and \{ϕk\} an arbitrary basis.
In fact the semi-classical channel nullifies the quantum discord of all the states, i.e., it brings any bipartite state to a classical-quantum state. Thus in the case of qubits, all the CoP channels are either unital, i.e., \(Λ(I) = I\) or semi-classical, which reproduces the results for qubit channels in [6]. The cloning channel \(C(\rho) ∝ IT\rho + c\rho\) is obviously another example of CoP channel for all 0 ≤ c ≤ 1. One special case is the identity channel \(I = 0\).
As an example of non-CoP channel we consider the channel
where we have denoted by
To find out the eigenvalue of \(K\) let us look at its square
in order to find out the eigenvalue of \(K\) let us look at its square
with projectors \(P_0 = \delta_{\rho}\). In order to find out the eigenvalue of \(\rho\) let us look at its square

As a result the spectrum of \(K\) is {2, ±1, 0, ±2/d} and the eigenvalue 2 is non-degenerated with eigenstate \(|ψ\rangle\) and eigenvalue −1 is degenerated with \(D\) eigenstates |fτ⟩ in the case of \(d ≥ 3\). Thus \((2 - K)(1 + K) ≥ 0\) so that δλν = 0 if and only if \(|\Lambda⟩ = a|ψ⟩ + \sum_{\tau} h_{\tau}^{|f_\tau⟩}\) for some \(a, h_{\tau}, e_{\tau}\), i.e., the channel \(Λ\) is a linear combination of the identity channel and a Hamiltonian channel with \(H = \sum_\tau h_{\tau}^{|f_\tau⟩}/d\). In the case of
$d \geq 3$ the channel $\Lambda$ preserves the commutativity if and only if $H = 0$, i.e., the channel $\Lambda$ is exactly the cloning channel $C$.

**Corollary** In the case of $d \geq 3$ a mixing $p\mathcal{I} + \bar{p}\Lambda (1 > p > 0)$ of an arbitrary CoP channel $\Lambda$ with the identity channel $\mathcal{I}$ preserves the commutativity if and only if the channel $\Lambda$ is a cloning channel $C(\rho) \propto I\text{Tr}\rho + c\rho$.

The example of a unital non-CoP channel in the case of $d \geq 3$ given in [6] is a mixing of the identity channel with a semi-classical channel, which is definitely not a cloning channel. Another example of non-CoP channel is the mixing of the identical channel with a unitary channel in the case of $d \geq 3$ while it preserves the commutativity in the case of $d = 2$.

Finally, let us consider various quantitative measures for quantum correlations under the CoP channels. First of all since the CoP channels preserve the set of quantum classical states, all distance-based measures are non-increasing under CoP channels. In fact any such kind of measure is defined by the minimal distance to the set of classical states with some suitable distances such as the distance, geometric measure, or the relative entropy. There are two different definitions of quantum discord with one over orthogonal projective measurements and one over all possible measurements. It turns out that the quantum discord over von orthogonal projections does not have this desirable property. Let us consider the following 2-qubit state $\rho = \sum_{a,b=0}^{3} R_{ab}\sigma_{a} \otimes \sigma_{b}$ with

$$R = \frac{1}{4} \begin{pmatrix}
1 & 1/4 & -1/2 & 1/4 \\
1/4 & 2/5 & 0 & 0 \\
-1/6 & 0 & 1/5 & 0 \\
-1/20 & 0 & 0 & -1/5
\end{pmatrix}$$

(11)

Its $A$-discord over orthogonal projections can easily be found numerically 0.0314231 while for the state $\varrho_{a} = (\varrho + u_{A}u_{A}^{\dagger})/2$ where the single qubit unitary transformation $u_{A} = \sin \frac{\pi}{\sqrt{2}} + i\theta_{3} \sin \frac{\pi}{\sqrt{2}} - i\theta_{2} \cos \frac{\pi}{\sqrt{2}}$ acts on subsystem $A$ only, the $A$-discord reads 0.0325923 which is slightly larger. As to the case of quantum discord defined via minimization over all possible measurements no examples of increasing under local CoP channels is found so far. We conjecture that the quantum discord over general measurements, which can be different from that over von Neumann measurements even in the qubits cases [14], is non-increasing under local CoP channels.

Note added. On finishing our manuscript Hu et al. obtained Theorem 1 [14]. The CoP channel $\Lambda(\rho) \propto I\text{Tr}\rho + c\rho$ (with $|c| \leq 1$) provides a counterexample to their conjecture.

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