Boussinesq-like problems in discrete media

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Abstract

Vertical loads acting on the surface of a half-space made of discrete and elastic particles are supported by a network of force chains that changes with the specific realization of the packing. These force chains can be transformed into equivalent stress fields, but the obtained values are usually different to those expected from the solution of the corresponding boundary value problem. In this research we explore the relationship between discrete and continuum approaches to Boussinesq-like problems in the light of classical statistical mechanics. We anticipate that some components of the extensive stress tensor (i.e. the product of the stress by the volume) follow exponential distributions whose parameters are given by the value obtained from a continuum approach. This has been validated through massive numerical simulation with the discrete element method.

1 Introduction

The estimation of stressess and strains in a homogenous isotropic linear elastic half space loaded on its surface is one of the most known problems in geotechnics. The solution for the case of a vertical point force was given by [4]. The two dimensional version (i.e. a vertical line load acting on the surface) was solved a few years later by [9]. When point forces are replaced by surface loads, solutions can be got from the superposition of infinitesimal loading states. We will refer to the situations in which there is a vertical load acting on the surface of a half-space as Boussinesq-like problems. From the point of view of continuum mechanics, the stress field in Boussinesq-like conditions is obtained by solving the equations governing the corresponding linear elastic boundary value problem. These equations include
three tensor partial differential equations for the balance of linear momentum and six infinitesimal strain-displacement relations. The system of differential equations is completed by a set of linear algebraic constitutive relations (Hooke's law). For example, in the case of 2D and a finite surface load, the line that connects all points below the ground surface objected to the same vertical pressure is a well-known stress bulb (Figure 1).

When the half-space not a continuum elastic medium but a dense packing of discrete elastic particles, the load acting on the surface is supported by a network interparticle forces. In a static equilibrium state, these forces organize in force chains in such a way that some particles are highly loaded while others are not (Figure 1). The volumetric average of the stress field within any particle is determined by its local interparticle forces and is generally different to that predicted at that location by classical solutions of Boussinesq-like problems. The specific distribution, value, orientation and ramification of force chains is determined by the features of the system, the boundary conditions and the history of the packing. Although the specific network of force chains resulting after a process cannot be anticipated, its statistics has attracted attention from the scientific community [12, 14, 13].

The purpose of this research article is to compare both continuum and discrete approaches and link them through statistics. For example, it is intuitively accepted (not yet proven to our knowledge) that the ensemble
average of the stress corresponds to the value given by the solution of the associated elastic boundary value problem (with equivalent values for all intervening parameters, namely, Young and Poisson moduli and density). However not only the mean value of the stress but also its statistical distribution could be of interest for some applications (e.g. reliability analyses). In this research work we have used statistical mechanics principles to anticipate these distributions. Then we have compared them to those measured from simulation. This has been done by numerically generating many packings of a system of discrete elastic particles that are objected to the same macroscopic boundary conditions.

The methodology has been applied to two different problems: the case of a granular half-space under its own weight (Case 1) and a Boussinesq-like problem that is equal to Case 1 plus a finite surface load (Case 2). The work is presented as follows:

(i) A theoretical approach to boundary value problems, stress homogenization techniques and statistical mechanics.

(ii) A description of the numerical method and the performed experiments.

(iii) A presentation of results followed with a discussion and an illustrative application.

(iv) A conclusion.

2 Methods

2.1 Continuum mechanics: the classical solutions

2.1.1 Case 1: 2D half-space under its own weight

In the absence of any load on the surface, this is just by the action of the gravity, the expected stress field can be determined from the weight of the overlying material:

\[ \sigma_{zz,0} = \gamma z, \]  

(1)

where \( \gamma \) is the unitary weight (in kN/m\(^3\)). \( \gamma = (1 - n) \rho_s g \), \( \rho_s \) is the density of the material of the particles, \( g \) is the gravitational acceleration and \( n \) is the porosity of the packing.

The horizontal stress also increases with depth, but it does at a rate given

\[ \sigma_{xz,0} = \gamma z. \]  

An ensemble is an idealization consisting of a large number of virtual copies of the system randomly generated with the same procedure.
the at-rest coefficient of lateral earth pressure, \( K_0 = \nu / (1 - \nu) \), being \( \nu \) the Poisson’s modulus: \( \sigma_{xx} = K_0 \gamma z \).

Therefore both the vertical and the horizontal stresses exclusively depend on the depth and all the points located at the same depth are symmetric.

### 2.1.2 Case 2: 2D Half-space with gravity and a finite surface load

For the case of surface loads acting on elastic half spaces, classical solutions can be revisited. A derivation of the solution to Boussinesq and Flamant problems is found in [16]. In both cases the stresses caused by the load decrease with depth but change with the horizontal distance to the applied load. In 2D, the vertical stress in the \( x-z \) plane caused by a surface load \( p \) is given by

\[
\sigma_{zz,p} = \frac{P}{\pi \left[ (\theta_1 - \theta_2) + \sin \theta_1 \cos \theta_1 - \sin \theta_1 \cos \theta_2 \right]},
\]

with \( \theta_1 = \arctan \left( \frac{x-X_1}{z} \right) \) and \( \theta_1 = \arctan \left( \frac{x-X_2}{z} \right) \) and \( X_1, X_2 \) the left and right limits of the surface load.

However numerical and laboratory experiments with no gravity are difficult. The bearing capacity of the half-space would be too low and the particles on the free surface could fly, so the hypothesis of small strains would not apply. For that reason, we have studied the case in which gravity and surface loads act at the same time. In such circumstances both solutions (Eqs. 1 and 2) can be superposed, because the behavior of the material is linear elastic and the equations of equilibrium and compatibility are linear too. Then, \( \sigma_{zz} = \sigma_{zz,g} + \sigma_{zz,p} \).

### 2.2 Stress homogenization: from discrete to continuum media

A heterogeneous continuum medium can be partitioned into domains. Every domain \( m \) in equilibrium has an inner stress field that matches the solution of the corresponding elastic problem. In the absence of body forces, the static equilibrium condition is \( \sigma_{ij,i} = 0 \). The boundary condition at a given point is \( \sigma_{ij} n_j = p_j \), with \( n_j, p_j \) representing the \( j \)-component of the normal and load vectors, respectively. Considering these two conditions and using the Gauss-Ostrogradsky theorem, the average stress field within the domain \( m \) is given by [1]:

\[
\langle \sigma_{ij}^m \rangle = \frac{1}{V_m} \int \sigma_{ij}^m dV_m = \frac{1}{V_m} \sum_l x_{lm}^m F_{jm}^{mn},
\]

\( 4 \)
where $V_m$ is the volume of the domain, $F_{jn}^m$ is the $j$-component of the interaction force between domains $m$ and $n$ and $x_{in}^m$ is the $i$-component of the application point. Equation 3 is independent of the origin of the coordinate system because particles are in static equilibrium. Therefore, the average stress field within a domain $m$, $\langle \sigma_{ij}^m \rangle$, can be obtained from the external forces and the positions where they are applied. The tensor product of these vectors is equal to the average stress field multiplied by the volume of the domain $\Sigma_{ij}^m = \langle \sigma_{ij}^m \rangle V_m$. We will refer to this extensive tensor quantity, $\Sigma_{ij}^m$, as extensive stress (also known as force-moment), which is expressed in energy units. A very important property of the extensive stress tensor is that it is additive: the extensive stress of a composite body (i.e. its average stress tensor multiplied by its volume) can be obtained either by adding the extensive stress of each separate component or by adding the tensor product of the external forces on that body by their position vectors.

2.3 Statistical mechanics approaches

Statistical mechanics is the branch of physics that deals with systems made of a large number of constituents. Although it was originally developed for thermal systems, it can be applied to granular media. Several approaches [8, 7, 6, 11, 10, 15] have been proposed since first Edward’s model in 1989. We have followed that based on the extensive stress [11, 10, 15] and have shown that it properly works for engineering practical purposes. The basic idea of statistical mechanics is [2] that among the solutions of a physical problem (e.g. the static arrangement of particles in mechanical equilibrium) there is a class that is compatible to the elastic boundary problem. This class still contains an enormous number of solutions and in the absence of further information there is not any a priori reason for favoring one of these more than any other (principle of equal a priori probabilities). This is not a mechanical, but a statistical assumption, because mechanics alone cannot solve the problem uniquely. Furthermore, in equilibrium theory the role of dynamics is trivial: the problem is essentially statistical one. A half-space made of densely packed particles can be partitioned into domains according to a Voronoi diagram. The volume of each Voronoi domain $V_m$ includes the volume of the elastic particle and an associated void space. The $M$ domains of the system have inner stress fields whose average value can be obtained from their interaction forces. Each component of the extensive stress of a subsystem $A$, made of $N$ particles (with $N < M$), is given by $\Sigma_{ij}^A = \sum_{m=1}^{N} \Sigma_{ij}^m$. The average stress in $A$ is by $\langle \sigma_{ij}^A \rangle = \Sigma_{ij}^A / V^A$, with
\[ V^A = \sum_{m=1}^{N} V^m. \]

We postulate that a domain \( m \) located at a position \((x, z)\) may take any allowed value for the extensive stress tensor, provided that the average value is externally controlled by the macroscopic boundary value problem. Other word said, if the packing is driven to make forces redistribute, the domain \( m \) will surely change its extensive stress, but the average (over either the ensemble or a set of symmetric domains of the same volume located at the same depth) must be equal to that expected from a continuum approach. The simplest approach is to consider that domains take independently the values of extensive stress. Although this is not completely true, the packing is so hyperstatic that the assumption could be acceptable for practical purposes.

Statistical samples can be generated by collecting \( N \) domains that would be objected to identical stress conditions in an equivalent boundary value problem. This is the case of either domains that are located around the same position in different packings objected to the same constrictions and generated with the same protocol (ensemble sample) or of domains of the same packing supposed to be in symmetric stress conditions (packing sample). The data can be classified into a discrete set of extensive stress values \( \{\Sigma_{ij,1}, \Sigma_{ij,2}, \ldots, \Sigma_{ij,r}\} \). The total number of domains with a extensive stress value of \( \Sigma_{ij,r} \) is denoted as \( N_{ij,r} \) and the number of permutations of the multiset formed by the extensive stress values \( \Sigma_{ij,r} \) is given by the multinomial coefficient:

\[
\Omega = \binom{N}{N_{ij,1}, \ldots, N_{ij,r}, \ldots} = \frac{N!}{N_{ij,1}! \cdots N_{ij,r}! \cdots},
\]

For horizontal and vertical components of the extensive stress tensor \((ij = xx \text{ or } ij = zz)\), the most probable distribution is that maximizing \( \Omega \) under the constraints \( \sum_r N_{ii,r} = N \) and \( \sum_r N_{ii,r} \Sigma_{ii,r} = \Sigma_{ii} = \mu_{ii} N \) with \( \Sigma_{ii,r} \geq 0 \). Negative values are not allowed because tensile interparticle forces do not exist in the interaction model. Then by using natural logarithms, Lagrange multipliers and the Stirling approximation, the celebrated Maxwell-Boltzmann statistics is found. This statistics states that the probability of finding a domain \( m \) with a given vertical extensive stress values

\[ \text{This is actually a simplification since an additional constraint is given by the total volume. This has not considered in this research because the particle size distribution is almost uniform and the variability of local volumes is quite small.} \]

\[ \text{There is actually a minimum value for the extensive stress that is given by the own weight of the domain, but this is often negligible when compared to the weight of the overlaying material.} \]
is:
\[ P(\Sigma_i = \Sigma_{i,r}) = \frac{N_r}{N} = \frac{e^{-\Sigma_{i,r}/\mu_{ii}}}{\sum_s e^{-\Sigma_{i,s}/\mu_{ii}}} . \] (5)

If possible values are not discrete but continuously distributed, then an exponential distribution is obtained:
\[ f(\Sigma_{ii}) = \frac{1}{\mu_{ii}} e^{-\Sigma_{ii}/\mu_{ii}} . \] (6)

Shear stresses can take any positive or negative value, provided that the average is null and that the maximum shear stress is limited by interparticle friction, so this approach does not apply to the expected distribution.

### 2.3.1 Case 1: 2D half-space under its own weight

In this case, the statistical distribution of vertical extensive stress is an exponential distribution of mean \( \mu_{zz,g}(z) = \gamma z V \) (Equation 1), where \( V \) is the volume of the domains. For horizontal stresses, the mean is \( \mu_{xx,g}(z) = K_0 \gamma z V \).

### 2.3.2 Case 2: 2D Half-space with gravity and a finite surface load

When a finite surface load acts on the half-space, the symmetry of the points located at the same depth is broken and packing samples are not possible. The statistical distribution generated in an ensemble sample is expected to be exponential and the corresponding mean is given by the horizontal distance to the load \( \mu_{zz,p(x,z)} = \sigma_{zz,p(x,z)} V \) (with \( \sigma_{zz,p(x,z)} \) obtained from equation 2).

When gravity and surface loads act a the same time, the vertical extensive stress at a given point is expected to follow an exponential distribution of mean \( \mu_{zz} = \mu_{zz,g} + \mu_{zz,p} \).

### 2.4 Numerical modeling

#### 2.4.1 The discrete element method

The discrete element method (DEM), implemented in YADE-DEM (\url{www.yade-dem.org}) has been used to generate packing corresponding to cases 1 and 2. The DEM computes the motion of the solid particles by considering particle-particle interactions. We followed a common frictional-Hookean DEM approach, in

\[ \text{www.yade-dem.org} \]
which normal interaction forces grow linearly with overlaps. The overlap is defined as $\delta_{ij} = (R_i + R_j) - r_{ij} = (R_i + R_j) - |r_j - r_i|$, where $R_{i,j}$ and $r_{i,j}$ are the radius and position vector of particles $i$ and $j$, respectively. The normal contact force per unit of length acting on particle $i$ due to particle $j$ is:

$$F_{n,ij} = -k_n \delta_{ij},$$

where the contact stiffness is $k_n = 2ER_iR_j/ (R_i + R_j)$ (in N.m$^{-1}$) and $E$ is the Young’s modulus (in Pa).

Tangential forces are produced in opposition to incremental lateral displacements. These forces are limited by the value of normal forces and friction coefficients ([3]). $F_{sij} = -\min (K_s u_{ij}, \tan \phi |F_{n,ij}|) u_{ij}/|u_{ij}|$, where $u_{ij}$ is the lateral displacement between the two particles previously in contact ($\delta_{ij} \geq 0$) and $\phi$ is the friction angle and $k_s$ is an elastic stiffness parameter.

### 2.4.2 Numerical experiments

Two sets of numerical experiments were performed. Each set included around one thousand numerical realizations. In these sets, the packings were generated by randomly pouring 5000 particles within a 1.0 m wide domain and waiting for an almost complete dissipation of kinetic energy. The sets differ from each other in the interparticle friction angle and in the presence or absence of a surface load. An additional wider packing (made of 50000 particles within a 10.0 m wide domain) was also generated to get a packing sample in Case 1. Gravity acted downwards with $g = 9.81 \text{ m/s}^2$. Surface loads were applied in Case 2 by gently and vertically (downwards) moving a rigid body of length $2a$ and centered at $x = 0.0$. Once the total vertical force on this rigid element was equal to $2ap$, the simulation was stopped.

In all the cases, a quasi uniform particle size distribution was used (i.e. all the diameters lying in the interval $D \pm \Delta D$).

Once a packing was in equilibrium in either Case 1 or Case 2, the statistical distributions of the extensive stress of the particles was measured. To do that, we selected a domain of volume $V_c = 0.0025 \text{ m}^3$ located at several control positions. The volume associated to the closest particle to the control point is always larger than the control volume. Then we assume that the average stress field within the control volume is equal to the average stress field within the volume associated to the particle, and correct data as $\Sigma_{ii}^{m,c} = \Sigma_{ii}^{m} V_c / V^m$.

In Case 1 all the particles whose center was located at a height $h_{O} \pm \Delta h_{O}$ from the bottom in the 10.0 m wide packing were considered for a packing
Table 1: Parameters used in the DEM numerical simulations to generate ensemble samples.

| Parameter                     | Value Case1 | Value Case2 | Units       |
|-------------------------------|-------------|-------------|-------------|
| Number of particles           | $N$         | 5000        | -           |
| Number of simulations         | #           | 1296        | 896         | -           |
| Simulation width              | $L$         | 1.0         | m           |
| Mean diameter                 | $D$         | 0.01        | m           |
| Diameter dispersion           | $\Delta D/D$| 0.05        | -           |
| Young’s modulus               | $E$         | $1.0 \cdot 10^{10}$ | Pa        |
| Material density              | $\rho_s$   | $2.6 \cdot 10^{3}$ | kg.m$^{-3}$ |
| Interparticle friction        | $\Phi$     | $\pi/6$     | 0 rad       |
| Loading width                 | $2a$        | -           | 0.045 m     |
| Surface load                  | $p$         | -           | $44.4 \cdot 10^{3}$ Pa |
| Control point O               | $x_O$       | 0.00        | 0.00 m      |
|                              | $h_O$       | 0.10        | 0.35 m      |
| Control point A               | $x_A$       | -           | 0.08 m      |
|                              | $h_A$       | -           | 0.35 m      |
| Control point B               | $x_B$       | -           | 0.15 m      |
|                              | $h_B$       | -           | 0.35 m      |

ensemble. In both Cases 1 and 2 ensemble samples were generated by collecting of values from particles that were located at the same control point in many 1.0 m wide packings. In Case 1, there was a single point O located right below the center of the box at a given depth. In Case 2, three control points were considered: point O right below the center of the surface load and points A and B located at the same height that O but that horizontally separate a given distance (leftwards and rightwards) from the center of the load. The simulation box was large enough to ensure that the stress field caused by the surface loading $p$ is below 0.05$p$ at the boundaries. The properties of the particles used in the simulations are shown in Table 1.

3 Results

3.0.1 Case 1: 2D half-space under its own weight

The obtained height of the half space after pouring the particles under gravity action and with interparticle friction angle $\phi = \pi/6$ was $H = 0.50 \pm 0.01$ m. The average porosity, $n = 0.21 \pm 0.01$.

In Fig 2 the statistical distribution of packing and ensemble samples are compared to the expected exponential distribution (Equation 6), of mean
Figure 2: Expected (exponential) and measured statistical distribution of vertical extensive stress in Case 1.

\[ \mu_{zz,g} = \gamma z V = (1 - n) (H - h_0) \rho_s g V. \]

Both packing and ensemble samples followed the same statistical distribution, which fits very well that predicted. It is worth mentioning that in DEM simulations particles do not achieve a fully equilibrium state but they remain vibrating around equilibrium positions. The smaller the timestep the lower the remaining kinetic energy. The theories presented here (section 2.2) are based on pure static equilibrium but our packing protocols and the chosen simulation parameters cannot fully achieve that state, affecting the obtained distribution. That is the reason why we think the measured distribution is shifted with respect to the theoretical one and this is particularly noticeable for the lowest values of extensive stress.

3.0.2 Case 2: 2D Half-space with gravity and a finite surface load

In this case particles were frictionless and they consequently packed more tightly. In effect the obtained height of the half space after dropping the particles by gravity was \( H = 0.46 \pm 0.01 \) m and its average porosity was reduced to \( n = 0.17 \pm 0.01 \).

In Figure 3 the three statistical distributions obtained at three control points through an ensemble of different realizations are compared to the expected exponential distributions whose means were given by equations 1 and 2.
In the three cases we observe that distributions fit those predicted by the theory (although there is again a small mismatching that is more noticeable for lowest values).

4 Discussion

The statistical distributions of extensive stress that were expected under certain hypotheses correspond to those measured in numerical experiments. The fact that these values follow exponential distributions could be interesting for geotechnics. In most common situation the number of intervening particles on the desired scale is extremely large. Then, because of the law of large numbers, the average stress matches that predicted from continuum approaches. But when this is not the case, continuum approaches fail. That could be the case of heterogeneous terrains, altered zones (fractures, faults, etc.), numerical or laboratory models with discrete media, etc.

As an illustrative example, let be the case of a rigid rectangular framework of width $L$ covered by a granular fill of height $H$ and made of particles of diameter $D$ (Figure 4). Following a continuum approach, the total pressure acting on the top of the framework should be equal to $p = \gamma H$. We have performed a direct sampling Monte Carlo simulation in which instead of a continuum fill, there was a finite number of particles $N = L/D$ inter-
acting with the framework whose equivalent vertical extensive stress value $\Sigma_{zz} = \sigma_{zz}D^2$ followed a exponential distribution. For each value of $N$ we run 10000 trials and then evaluated the variability of the average pressure with the coefficient:

$$C_{V}^{95} = \frac{p_{95} - \bar{p}}{\bar{p}} \sim 2 \frac{\sigma_p}{\bar{p}} = 2C_V,$$

where $p_{95}$ is the 95th-percentile, $\sigma_p$ is the standard deviation, $\bar{p} = \gamma z$ is the mean and $C_V$ is the coefficient of variation. The variability of resulting stress is more significant as $N$ decreases. Results are shown in Figure 4. For example with $N = 100$, the 95th-percentile occurs at $1.2\bar{p}$. It means that if there are 100 particles acting on the framework, in 5% of cases the total pressure was 20% higher than the expected value.

A different way to approach this illustrative example, is considering that the average stress is got from the contribution of particles whose extensive stress follows an exponential distribution. Then the average pressure on the top of the framework would follow an Erlang distribution of shape parameter $N$ and rate parameter $N/(\gamma H)$. The mean value would be $\gamma H$, and the variance $(\gamma H)^2/N$. Then, if the coefficient of variation is defined as the ratio of the standard deviation to the mean value, it would be given by $C_v = 1/\sqrt{N}$. This result agrees with Figure 4 and it means that with 100
particles, in 2.5% of cases, the stress would be 20% higher than the average and with 10000 particles this variation would drop to 2%.

5 Conclusion

We have used classical statistical mechanics to anticipate the statistical distribution of the vertical extensive stress (this is the average vertical stress field of a domain multiplied by its volume) in two cases: an elastic half-space made of elastic particles under its own weight and the same half-space under the action of the gravity and of a vertical surface load. Under certain hypotheses, the obtained statistical distributions are exponential and the corresponding means can be predicted from the solution of equivalent boundary value problem. In the former the expected vertical stress basically depends on the depth, the density of the material, the average porosity of the packing and the gravitational acceleration. In the latter the mean additonally changes with the applied surface load (value and area of application), the depth and the horizontal distance to the load. Although some strong hypotheses were done, the approach gives good results for practical purposes. Anticipating the statistical distributions of stresses can be useful in situations in which the size of the discrete particles (which could also represent heterogeneities, fragments, etc.) is comparable to the scale of the problem. For example, they can be used to estimate the probability of finding a total stress acting on a surface that is \( x \) times higher than the value obtained from the corresponding continuum approach.

This research helps to fill gaps between discrete and continuum geotechnical approaches (and thus between micro and macrogeotechnics) and opens a way to treat other seminal problems in geotechnics. The methodology leads to non-deterministic geotechnical models, which are useful for reliability analyses.

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