Cosmic Microwave Background Polarization Signals from Tangled Magnetic Fields

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Tangled, primordial cosmic magnetic fields create small rotational velocity perturbations on the last scattering surface (LS) of the cosmic microwave background radiation (CMBR). For fields which redshift to a present value of $B_0 = 3 \times 10^{-9}$ Gauss, these vector modes are shown to generate polarization anisotropies of order $0.1 \mu K - 4 \mu K$ on small angular scales ($500 < l < 2000$), assuming delta function or a power law spectra with $n = -1$. About 200 times larger signals result for $n = 2$ spectra. Unlike inflation generated, scalar modes, these signals are dominated by the odd parity, B-type polarization, which could help in their detection.

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Magnetic fields in astronomical objects, like galaxies, could grow by the amplification of small seed fields by turbulent dynamo action [1]. However, the need to produce magnetic helicity in galaxies seems to severely constrain the efficiency of such dynamo action [2]. Alternatively, the galactic field could be of primordial origin [3], although, there is no compelling mechanism for producing the required field [4]. A primordial field that redshifted to a present value of $\sim 10^{-9}$ Gauss, tangled on galactic scales, could also significantly affect galaxy formation [4]. It is of considerable interest, therefore, to find ways of constraining or detecting such fields [4]. Observations of anisotropies in the CMBR, provide a potentially powerful constraint. Indeed, the CMBR temperature isotropy can be used to place limits, of order several nano-Gauss, on both the uniform [5] and tangled components of the magnetic fields [5]. However, temperature anisotropies in inflationary models of structure formation, will be dominated by "non-magnetic", scalar modes. So the detection of a magnetic field induced signal is likely to be difficult, except possibly on scales smaller than the Silk damping scale $\delta$. Here we point out the advantage of using alternatively, the polarization anisotropy. Note that scalar perturbations only produce, what is known as E-type polarization anisotropy. However, as we show here, tangled magnetic fields which drive significant vector perturbations, also lead to a distinctive, significant and potentially detectable B-type polarization anisotropy of the CMBR. This could help in separating their contribution from scalar contributions, to detect/constrain such tangled, primordial fields.

Polarization of the CMBR arises from the Thomson scattering of radiation from free electrons, and is sourced by the quadrupole component of the CMBR anisotropy. The evolution equations of temperature and polarization anisotropy for vector perturbations have been derived in detail in Ref. [1], in the total angular momentum representation. We use their results extensively below. The anisotropy in the temperature and polarization is expanded in terms of tensor spherical harmonics. This enables one to write evolution equations, for the moments, $\Theta_2^{(m)}$, $E_1^{(m)}$ and $B_1^{(m)}$, of the temperature anisotropy ($\Delta T/T$), the electric (E-) type and the odd parity, magnetic (B-) type polarization anisotropies, respectively. Here $l$ stands for the multipole number and $m = 0, \pm 1, \pm 2$, respectively, for scalar, vector and tensor perturbations. For vector perturbations ($m = \pm 1$), the magnetic type contribution dominates the polarization anisotropy $\Theta_2^{(1)}$. Its evolution is given by (Eqn. (77), (62) and (18) of [1]),

$$\frac{B_1^{(m)}(\tau_0, k)}{2l + 1} = -\sqrt{6} \int_0^{\tau_0} d\tau g(\tau, \tau_0) P^{(m)}(\beta_1^{(m)}(k(\tau_0 - \tau))$$

where $P^{(m)}(k, \tau) = [\Theta_2^{(m)} - \sqrt{6} E_2^{(m)}]/10$ and $\beta_1^{(1)}(x) = \sqrt{l(l+1)/2}/(l+2) j_l(x)/2x$, with $j_l(x)$ the spherical Bessel function of order $l$. The ‘visibility function’, $g(\tau_0, \tau) = k(\tau) \exp[-\int_0^{\tau_0} \kappa(\tau')d\tau']$, determines the probability that a photon reaches us at the conformal time $\tau_0$ if it was last scattered at the epoch $\tau$. Here $\kappa(\tau) = n_e(\tau)\sigma_T a(\tau)$, $n_e$ the electron number density, $\sigma_T$ the Thomson cross section, and $a(\tau)$ the cosmic scale factor normalised to unity at the present. We assume a flat universe throughout.

For standard recombination, $g$ is sharply peaked about the time of recombination. We therefore need to calculate $P^{(1)}$, and hence the quadrupole anisotropies around this epoch of last scattering. These can be analytically estimated using the tight-coupling approximation, $k/\kappa = kL\gamma \ll 1$. Here $L_{(l)}(\gamma) = (\kappa)^{-1}$ is the co-moving, photon mean free path. First, to leading order in this approximation, we have zero quadrupoles, and a dipole $\Theta_2^{(1)} = v_B^{(1)}$, where $v_B^{(1)}$ is the magnitude of the (vector or rotational component of) baryon fluid velocity field, in Fourier space. However, to the next order the quadrupole is not zero. It is generated from the dipole at the ‘last but one’ scattering of the CMBR. Using the moments of the Boltzmann equations for the temperature and polarization anisotropies (Eq. (60), (63) and (64) of [1]), one gets $\Theta_2^{(1)} = -4 E_2^{(1)}/\sqrt{6} = 4kL_{v_B^{(1)}/(3\sqrt{3})$ and hence $P^{(1)} = \Theta_2^{(1)}/4 = kL_{v_B^{(1)}/(3\sqrt{3})$. Using this in Eq. (1) gives an estimate of $B_1^{(1)}$, and the angular power spectra $C_{1_{BB}^{(1)}}$ due to B-type polarization anisotropy. We use Eq. (56) of [1] to relate $C_{1_{BB}^{(1)}}$ in terms of $B_1^{(1)}$ and get
Here we have included an extra factor of 2, since we have to sum over the power in both $m = +1$ and $m = -1$ contributions. The above expression for $C^{BB}_I$ is very closely related to Eq. (1) of Ref. [1] (henceforth, Paper I), for the temperature power spectrum $C_l$, due to tangled magnetic fields. One can make the same approximations as made there, to obtain an analytic estimate of $C^{BB}_I$.

Firstly, it suffices to approximate the visibility function as a Gaussian: $g(\tau_0, \tau) = (2\pi \sigma^2)^{-1/2} \exp[-(\tau - \tau_0)^2/(2 \sigma^2)]$, where $\tau_0$ is the conformal epoch of “last scattering” and $\sigma$ measures the width of the LSS. Using the expressions given in Ref. [2], we estimate $\tau_0 \sim 176.2 h^{-1} Mpc$ and $\sigma = 11.6 h^{-1} Mpc$. $(h$ is the Hubble constant in units of 100 km s$^{-1}$ Mpc$^{-1}$). We take $h = 0.75$ (henceforth, a baryonic density parameter $\Omega_0 = 0.02 h^{-2}$.) The dominant contributions to the integral over $\tau$ in Eq. (3) then come from a range $\sigma$ around the epoch $\tau = \tau_0$. Further, $j_l(k(\tau_0 - \tau))$ picks out $(k, \tau)$ values in the integrand which have $k(\tau_0 - \tau) \sim l$.

Let us also assume that $k L_v^{(1)}(k, \tau)$ varies slowly with $\tau$ (around $\tau \sim \tau_0$), and slowly with $k$ around $k \sim 1/R_i$, the regions which contribute dominantly to the integrals in Eq. (3). (Here $R_i = \tau_0 - \tau_0$). Then following exactly the arguments detailed in paper I, we get for $k \sigma \ll 1$, the analytical estimate,

$$\frac{l(l + 1)C^{BB}_I}{2\pi} \approx \left( \frac{k L_v^{(1)}(\tau_0)}{3} \right)^2 \frac{\pi}{4} \Delta^2_{\nu}(k, \tau_0) |_{k=1/R_i}. \quad (3)$$

Here, $\Delta^2_{\nu} = k^3 < [v_B^{(1)}(k, \tau_0)]^2 > + [v_B^{(1)}(k, \tau_0)]^2 > / (2\pi^2)$ is the power per unit logarithmic interval of $k$, residing in the net rotational velocity perturbation. And in the other limit, $k \sigma >> 1$, we get

$$\frac{l(l + 1)C^{BB}_I}{2\pi} \approx \left( \frac{k L_v^{(1)}(\tau_0)}{3} \right)^2 \frac{\sqrt{\pi}}{4} \Delta^2_{\nu}(k, \tau_0) / k \sigma |_{k=1/R_i}. \quad (4)$$

For small wavelengths, $C^{BB}_I$ is suppressed by a $1/k \sigma$ factor due to the finite thickness of the LSS. Note that in both cases, the polarization anisotropy, $\Delta T^{BB}_p(l) \approx (k L_v^{(1)}(\tau_0)/3) \times \Delta T(l)$, where, $\Delta T(l)$ is the temperature anisotropy computed in Paper I.

To evaluate $C^{BB}_I$, one needs to estimate $v_B^{(1)}$, for a general spectrum of magnetic inhomogeneities. We assume the magnetic field to be initially a Gaussian random field. On galactic scales and above, the induced velocity is generally so small, it does not lead to any appreciable distortion of the initial field [3]. So, to a very good approximation, the magnetic field simply redshifts as, $B(x, t) = b_0(x)/a^2$. The Lorentz force associated with the tangled field is then $F_L = (\nabla \times b_0) \times b_0/(4\pi a^3)$, which pushes the fluid, creating rotational velocity perturbations. Further, the magnetic stresses, say $\Pi_B$, can lead to metric perturbations. We focus on scales larger than the photon mean-free-path at decoupling, and describe the viscous effect due to photons, in the diffusion approximation. The Fourier transform of the linearised Euler equation for $v_B^{(1)}$, is given by $\square$, 

\[
\left( \frac{4}{3} \rho_\gamma + \rho_\delta \right) \frac{\partial (v_B^{(1)} - V)}{\partial t} + \rho_\delta \frac{d a}{dt} v_B^{(1)} - V + k^2 \eta v_B^{(1)} = F_{B_1}^{(1)} / 4\pi a^5. \quad (5)
\]

Here $V(k, t)$ is the vector component of the metric perturbation (t is comoving proper time), $\rho_\gamma$ the photon density, $\rho_\delta$ the baryon density, $\eta = (4/15)\rho_\gamma l_\|$, the shear viscosity coefficient associated with the damping due to photons, whose mean-free-path is $l_\| = L_\odot a(t)$. $F_{B_1}^{(1)}$ is the $m = 1$ component (defined as in ref. [1]) of $P_{ij}F_j$, the rotational part of the Lorentz force. We have defined the Fourier transforms of the magnetic field as, $b_0(x) = \sum_k b(k) \exp(ik \cdot x)$. Since the Lorentz force is non-linear in $b_0(x)$, this leads to the mode-coupling term $\mathbf{F}(k) = \sum_p [b(k + p) \cdot [b(k) - b(k + p)]$ projects $\mathbf{F}$ onto its transverse components (perpendicular to $k$).

The comoving Silk scale at recombination, $L_S = k^{-1}_S \sim (l_\|(t_n) a(t_n))^{-1/2} / a(t_n) \sim 6.8 h Mpc$, separates scales for which the damping term in $F_{BB}^{(1)}$, is important ($k L_S >> 1$) from those for which it is negligible ($k L_S << 1$) [3]. We can solve Eq.(4) analytically, in these two limits. For $k L_s << 1$, and when the fluid starts from rest ($v_B^{(1)}(\tau_0) = 0$), the damping due to the photon viscosity can be neglected compared to the Lorentz force. Integrating Eq.(6) gives $v_B^{(1)} = V + G_B^{(1)}(\tau - \tau_0)/(1 + S_\tau)$, where we have defined $G_B^{(1)} = 3F_{B_1}^{(1)}/[16\pi\rho_\delta]$, with $\rho_\delta$ the redshifted present day value of $\rho_\gamma$, and $S_\tau = (3\rho_\delta / 4\rho_\gamma)(\tau_0) \sim 0.73 (\Omega_0 / 0.02 h^{-2})$. The metric perturbation term $V$ is also smaller than the Lorentz force driven contribution to $v_B^{(1)}$, for large $l$ by a factor $\sim (l/(24)^{-1/2}) h^{-2}$ (see [3]); and so makes a negligible contribution to $C^{BB}_I$, for the small angular scales ($l > 400$) considered here. In the other limit, with $k L_s >> 1$, we can use the terminal-velocity approximation, neglecting the inertial terms in the Euler equation, and balance the Lorentz force by friction. This gives $v_B^{(1)} = (G_B^{(1)}/k L_S)/(5)^{-1}$, independent of $V$. We also need to specify the spectrum of the tangled magnetic field, $M(k)$. We define, $< b_i(k)b_j(k') > = \delta_{k,q} \delta_{p,p'} M(k)M(k')$, where $\delta_{k,q}$ is the Kronecker delta which is non-zero only for $k = q$. This gives $< b_0^2 > = 2 \int (dk/k) \Delta^2_{\delta}(k)$, where $\Delta^2_{\delta}(k) = k^3 M(k)/(2\pi^2)$ is the power per logarithmic interval in $k$ space residing in magnetic tangles, and we replace the summation over $k$ space by an integration. The ensemble average $< |v_B^{(1)}|^2 >$, and hence the $C^{BB}_I$, can be computed in terms of the
magnetic spectrum $M(k)$. It is convenient to define a dimensionless spectrum, $h(k) \equiv \Delta_k^2(k)/B_0^2$, where $B_0$ is a fiducial constant magnetic field. The Alfvén velocity, $V_A$, for this fiducial field is

$$V_A = \frac{B_0}{(16\pi\rho_0/3)^{1/2}} \approx 3.8 \times 10^{-4} B_{-9}.$$  

where $B_{-9} = (B_0/10^{-9} \text{Gauss})$. Also, as a measure of the B-type CMBR polarization anisotropy induced by the tangled magnetic field, we define the quantity

$$\Delta T^B_B(l) = \int_0^\infty \frac{d\mu}{(l+1)C^B_B(2\mu)^{1/2}T_0}.$$  

Note that when $k\sigma < 1$ (or $l < 500$), one also generally has $kL_S < 1$. The resulting $\Delta T^B_B$ can be estimated using Eq. (5), and ignoring viscous damping. A lengthy calculation gives, for such scales, $\Delta T^B_B(l) = T_0(\pi/32)^{1/2} I(k)[k^2V_A^2 \tau L^2(\tau_*)/3(1 + S_*)]$, where, $l = kR_*$. For scales with $kL_S > 1$ ($l > 1150$), we can use Eq. (5), and $\nu^2_{B1} = (G_B/k)(kL_0/5)^{-1}$. A similar calculation to that above gives, $\Delta T^B_B(l) = T_0(5\pi/12)\sqrt{2} I(k) V_A^2 (k_0/\tau_*)^{1/2}$. The function $I^2(k)$ is a dimensionless mode-coupling integral given by

$$I^2(k) = \int_0^\infty \frac{d\mu}{q} \int_1^0 d\mu' \frac{h(\mu)[(k + q\mu)]k^3}{(k^2 + q^2 + 2k\mu\mu')^{3/2}} \times (1 - \mu^2) \left[ 1 + \frac{(k + 2\mu)(k + q\mu)}{(k^2 + q^2 + 2k\mu\mu')^{2}} \right]$$

where $|k + q\mu| = (k^2 + q^2 + 2k\mu\mu')^{1/2}$. Putting in numerical values we estimate for $l < 500$ and $l > 1150$ respectively,

$$\Delta T^B_B(l) \approx 0.2\mu K \frac{l}{R_*} \left( \frac{B_{-9}}{3} \right)^2 \left( \frac{l}{400} \right)^2 \approx 0.3\mu K \frac{l}{R_*} \left( \frac{B_{-9}}{3} \right)^2 \left( \frac{l}{1500} \right)^{-1/2}$$

Further, for intermediate scales, $500 < l < 1150$, an estimate using Eq. (5), but ignoring viscous damping, gives $\Delta T^B_B(l) \approx 0.5\mu K(l/800)^{3/2} I(l/R_*) (B_{-9}/3)^2$. If the magnetic spectrum has a single scale, with $k(k + \mu) = k\delta_D(k - \mu)$, where $\delta_D(x)$ is the Dirac delta function, $<b_0^2> = B_0^2$ and the mode-coupling integral can be evaluated exactly. We find $I(k) = (k\mu_0)^2(1 - (k/2k_0)^2)^{-1/2}$ for $k < 2k_0$, and $I(k) = 0$ for larger $k$. So $I(k) \sim 1$ for $k \sim k_0$, with a maximum $I(\sqrt{2}k_0) = 1$. For $B_{-9} \sim 3$, one then expects a RMS $B^B_B \sim 0.2\mu K - 1\mu K$, depending on $k_0$ and $l$. We can also consider power law spectra, $M(k) = Ak^n$ cut-off at $k = k_c$, where $k_c$ is the Alfvén-wave damping length-scale. We fix $A$ by demanding that the field smoothed over a "galactic" scale $G_k = 16\pi\rho_0^{-1}$ (using a sharp $k$-space filter) is $B_0$, giving $h(k) = (n + 3)(k/k_0)^{3+n}$ ($n > -3$). We then find for $k << k_c$, $k_0$ (as is relevant for $l < 2000$), and $n > -3/2$, $T^B_B(l) = (28/15)(n + 3)/3^2/(3 + 2n)(k_0/kG)^{3+n}$. Using this in Eq. (6), we find $\Delta T^B_B \sim 0.16\mu K(l/400)^{-2/3}$ for $l < 500$ and $\Delta T^B_B \sim 5\mu K(l/1500)$ for $l > 1150$, for $n = -1$ and $B_{-9} \sim 3$. Much larger signals result for larger $n = 2$, say, and same $B_0$, with $\Delta T^B_B \sim 9.6\mu K(l/400)^{-2/3}$ for $l < 500$ and $\Delta T^B_B \sim 305.8\mu K(l/1500)$ for $l > 1150$. To complement the analytis, we have also computed $\Delta T^B_B$ for the above spectra, by evaluating the $\tau$ and $k$ integrals in Eq. (2) numerically (but using analytical approximations to $\nu^2_{B1}$). The results are shown in Figure 1. We see that for $B_0 \sim 3 \times 10^{-9} G$, one expects a RMS B-type CMBR polarization anisotropy of order $0.1\mu K - 4\mu K$ for $500 < l < 2000$, for delta function or a power law spectra with $n = -1$, and $\sim 200$ times larger signals for $n = 2$ spectra. These values compare reasonably with the analytical estimate from Eq. (5) (see also [13]). Further, both the tensor contribution of typical inflationary models (from CMBFAST [14]), and the foreground contribution (dominated by galactic dust polarized emission for high frequencies cf. [19]) are sub-dominant to the magnetic field induced signal, for large $l > 400$. Clearly with the sub-micro Kelvin sensitivities expected from Planck [20], these signals can be detected. Earlier work also emphasised the possibility of Faraday rotation and the depolarization of the CMBR due to differential Faraday rotation in a tangled magnetic field [4]. Note that the average Faraday rotation (in radians) between Thomson scatterings is given by $F = 3B_0/(2\pi\mu_0^2) \approx 0.23(B_0/3 \times 10^{-9} G)(\nu_0/300 GHz)^{-2}$, where $\nu_0$ is the observed frequency. The CMB could then become significantly de-polarised due to this effect, for $\nu_0 < 16.4 GHz(B_0/3 \times 10^{-9} G)^{1/2}$, but Faraday rotation effects will be negligible for say $\nu_0 > 40 GHz$, or the higher frequency instruments of the Planck Satellite. Note that scalar modes can also be induced by tangled...
magnetic fields and generate a purely E-type polarization. These are however of smaller amplitude than vector modes, due to the larger restoring force contributed by the radiation-baryon fluid pressure $\frac{\rho}{3}$. They are also strongly damped on scales smaller than the Silk scale, $L_s$, while the vector mode, survives on much smaller scales $> V_A L_s \frac{\rho}{3}$, or larger $l$ (see Figure 1). The tensor mode can also contribute to $\Delta T^{BB}$, but one can show that their effect is only important at large angular scales. Further, $\Delta T^{BB}$ could also be generated, after the universe gets re-ionized, though from the recent detection of multiple doppler peaks, the optical depth for scattering at re-ionization is probably very small $\frac{\rho}{3}$. A detailed computation of these effects will be presented elsewhere.

In conclusion, we have identified here, a new physical effect of tangled magnetic fields; that they can produce distinctive and potentially detectable B-type polarization anisotropy on arc minute scales. From Eq. (8) and Figure 1, we see that a tangled field with $B_0 \approx 3 \times 10^{-6} G$, induces a RMS B-type CMBR polarization anisotropy of order 0.1 $\mu K - 900 \mu K$ or larger, depending on $M(k)$ and $l$. The anisotropy in hot/cold spots could be several times larger, because the non-linear dependence of $C_l^{BB}$ on $M(k)$ will imply a non-Gaussian statistics for the anisotropies (see Paper I). Further in standard models all the $C_l$s have a sharp cut-off for $l > R_s/L_s$, due to Silk damping but strong damping of Alfvénic modes is expected only on scales smaller than $V_A L_s \frac{\rho}{3}$. Finally, since tangled magnetic fields produce predominantly B-type polarization, which are also dominant at large $l$, they can be distinguished from those produced by inflationary scalar and tensor perturbations. Satellite borne experiments like Planck \cite{22}, with the sub-micro Kelvin sensitivities should be able to detect and isolate the effects of magnetic fields, using CMBR polarization, if such fields indeed play a role in structure formation.

As this letter was prepared for (re)submission, preprints \cite{22} appeared which have some overlap with our work. We thank H. M. Antia and S. Sethi for help.

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\bibitem{15} Note that $V$ satisfies the vector Einstein equation, $\dot{V} + 2(\dot{a}/a)V = -8\pi G a^2 \Pi^{(1)} / k$, where ‘dot’ denotes a conformal derivative (see Eq (70) of Ref. (\cite{11})), and the vector part of the stress, $\Pi^{(1)} = F^{(1)} / k$. This integrates to give $V = -8\pi G F^{(1)} (\tau - \tau_i)/(k^2 a^2)$. The ratio of $V$ to the Lorentz force term at last scattering is then $-8\pi G(4\rho_0/3)(1 + S_\epsilon)/(k_\epsilon^2 a^2)$. So we can neglect the $V$ term contribution for large $l > 100$.
\bibitem{16} Alfvén modes survive damping on scales larger than $k_\epsilon^{-1} \approx V^{rel}_A L_s \frac{\rho}{3}$. For computing the effective Alfvén velocity, $V^{rel}_A$, we assume that the field smoothed over scales $2k^{-1}$ acts as an effective large-scale field for the cut-off scale perturbations. For $B_0 = 3$, we get $k_\epsilon = 14 Mpc^{-1}$ for $n = -1$, and $k_\epsilon = 5.4 Mpc^{-1}$ for $n = 2$.
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\bibitem{18} The numerically computed $l$ dependence, also agrees reasonably with the analytical prediction for $l > 500$, but differs somewhat at lower $l$; $\Delta T^{BB} \propto l^{2.5}$, compared to the expected $l^{4.5}$ dependence for $l < 500$. This is probably because the numerical integration treats more accurately the more rapid $k$ variation of $P^{(1)}$, for small $k$, and the effects of the finite thickness of the LSS.
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