Monotonicity of a Class of Integral Functionals

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Abstract
In this note we prove a condition of monotonicity for the integral functional
\[ F(g) = \int_a^b h(x) d[-g(x)] \]
with respect to \( g \), a function of bounded variation.

Keywords: monotonicity, integral functional, function of bounded variation, structured population model, net reproduction function.

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1 Introduction
In the article [1] (“Nontrivial Equilibria of a Quasilinear Population Model”, in progress), I study a functional \( R(u) \) \( (u \in L^1(0, \infty)) \), said generalized net reproduction rate, to prove existence of non–zero equilibria in a general structured population model.

The monotonicity of \( R(u) \) is used in a Corollary to prove the non-existence of a non–zero stationary population if \( R(0) < 1 \) (a sufficient condition of existence being \( R(0) > 1 \)).

The original proposition about monotonicity, not so immediate, will be reduced to the integration by parts of an improper Stieltjes integral:
\[
\int_a^\infty h(x) d[-g(x)] = h(a) g(a) - \lim_{b \to \infty} h(b) g(b) + \int_a^\infty g(x) dh(x)
\]

2 Monotonicity Propositions
Assume \( 0 < a < b \leq \infty \).

From now on we denote via \( G(b) \) the value of \( G(b) \) if \( b < \infty \) and \( \lim_{x \to \infty} G(x) \) if \( b = \infty \). I will denote respectively in the cases \([a, b]\) and \([a, \infty)\).

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Proposition 1 Let $H, G$ be two given functions on $I$.

Let $H$ be increasing (non-decreasing), bounded, non-negative. Let $G$ be continuous and of bounded variation.

Define

$$F(G) := \int_a^b H(x) d[−G(x)]. \quad (1)$$

If $G(b) = 0$, then $F$ is increasing (non-decreasing) with respect to $G$, i.e. let be $A := \{\phi | \phi \in C([a,b]) \cap BV[a,b], \phi(b) = 0\}$: if $G_1, G_2 \in A$ and $G_1 < G_2$, then $F(G_1) < F(G_2)$ (respectively $F(G_1) \leq F(G_2)$).

Proof. a) Consider first the case $b < \infty$. $F(G)$ is well-defined; integrating by parts we have:

$$F(G) = -H(b) G(b) + H(a) G(a) + \int_a^b G(x) dH(x) = H(a) G(a) + \int_a^b G(x) dH(x). \quad (2)$$

The conclusion is immediate.

b) Consider the case $b = \infty$. For $H$ bounded and $G(x)$ converging for $x \to \infty$ we obtain immediately the existence of the improper integral and extend the formula of case a).

If $H(x)$ is not strictly increasing but only non-decreasing, the functional $F$ is only non-decreasing with respect to $G$.

Corollary 2 Let $H, G$ given functions on $I$.

Let $H$ be decreasing (non-increasing), bounded, non-negative. Let $G$ be continuous and of bounded variation.

Define $F_0(G) := \int_a^b H(x) dG(x)$.

If $G(b) = 0$, then $F_0$ is increasing (non-decreasing) with respect to $G$.

Example 1. Consider the functional

$$I(f) = \int_0^\infty dx \, h(x) \, f(x) \, e^{-\int_0^x dy f(y)} \quad (3)$$

where $h$ is positive, increasing and bounded. If $f \in L^1_{loc}(0, \infty)$, $f \geq 0$ and $\int_0^\infty dy f(y) = \infty \,(f \notin L^1(0, \infty))$, then $I$ is decreasing with respect to $f$.

This is a particular case of Prop. \([\) where $g(x) = e^{-\int_0^x dy f(y)}$ and

$$I(f) = \int_0^\infty dx \, h(x) \, d[−e^{-\int_0^x dy f(y)}].$$

Corollary 3 Consider $u \in L^1(0, \infty)$ and the functional

$$R(u) = \int_0^\infty h(x, u(\cdot)) \, f(x, u(\cdot)) \, e^{-\int_0^y dy f(y, u(\cdot))} \quad (4)$$

where $h$ and $f$ are defined from $(0, \infty) \times L^1_+(0, \infty)$ in $[0, \infty)$, $h$ is positive and bounded, $x \mapsto f \in L^1_{loc}(0, \infty)$ and $\int_0^\infty dy f(y) = 0$, and
• let \( x \mapsto h(x,u) \) be non-decreasing (increasing) for fixed \( u \)
• \( u \mapsto h(x,u) \) decreasing (or non-increasing) for fixed \( x \)
• \( u \mapsto f(x,u) \) non-decreasing (or increasing) for fixed \( x \)

Then \( R(u) \) is decreasing with respect to \( u \).

Proof. Take \( u_1, u_2 \in L^1_+(0,\infty) \) with \( u_1 < u_2 \). For Proposition 1, the integral

\[
\int_0^\infty h(x,u_1) f(x,u) e^{-\int_0^x dy f(y,u)}
\]

is decreasing with respect to \( f \), that is non-decreasing in \( u \): therefore this integral is non-increasing in \( u \) and we have

\[
\int_0^\infty h(x,u_1) f(x,u_1) e^{-\int_0^x dy f(y,u_1)} \geq \int_0^\infty h(x,u_1) f(x,u_2) e^{-\int_0^x dy f(y,u_2)}. \tag{5}
\]

As \( f \) is decreasing with respect to \( u \), we have

\[
\int_0^\infty h(x,u_1) f(x,u_2) e^{-\int_0^x dy f(y,u_2)} > \int_0^\infty h(x,u_2) f(x,u_2) e^{-\int_0^x dy f(y,u_2)}, \tag{6}
\]

so that \( R(u_1) > R(u_2) \).

(The case of the alternative conditions, given by the parenthesis, is analogous).

Example 2. Corollary \[\text{[III]}\] is applied to a model of population dynamics: let \( u = u(t,x) \geq 0 \) be a population density with respect to age or size \( x \geq 0 \). Existence of stationary solutions (i.e. equilibria) \( u = u(x) \) is related to a functional \( R(u) \), the net reproduction rate. In a generalized model (see \[\text{[I]}\]) where \( g \) and \( \mu \) depend on \( u \) in an infinite-dimensional kind, \( R(u) \) is represented by

\[
R(u) = \int_0^\infty dx \beta(x,u(x)) e^{-\int_0^x dy \frac{\mu(y,u(x))}{g(y,u(x))}} \frac{g(y,u(x))}{g(x,u(x))} \tag{7}
\]

where \( \beta \) represents fertility, \( \mu \) mortality and \( g \) is a coefficient of growth (the detailed model is given and discussed in \[\text{[I]}\]).

The condition of existence of a nonzero steady solution (with suitable regularity conditions) is requiring that \( R(u) = 1 \); see \[\text{[2, 3]}\] and \[\text{[I]}\]. See also \[\text{[4, 5, 8]}\].

If \( R(0) < 1 \) and monotonicity conditions hold, the zero solution is the unique equilibrium.

I prove in \[\text{[I]}\] that \( R(0) > 1 \) is a sufficient condition for existence of nontrivial stationary solutions. If monotonicity conditions do not hold, then \( R(0) > 1 \) is sufficient but it is not necessary and it is simple to give a counterexample.
3 More about the Application

The model is a generalized version of the classic Lotka-MacKendrick population model: consider a population density \( u = u(t, x) \), where \( t \in [0, T] \) represents time, \( x \in (0, \infty) \) is age or size and the total population \( P(t) \) is

\[
P(t) = \int_0^\infty u(t, x) \, dx.
\]

Consider the following functions: growth/diffusion \( g = g(x, u) \), mortality \( \mu = \mu(t, u) \), fertility \( \beta = \beta(x, u) \), depending on \( x \) and infinite-dimensionally depending on the population density \( u(t, \cdot) \). The model is

\[
\begin{align*}
   u_t(t, x) + (g(x, u(t, \cdot)) u(t, x))_x + \mu(x, u(t, \cdot)) u(t, x) &= 0, \quad (8) \\
   g(0, u(t, \cdot)) u(t, 0) &= \int_0^\infty dx \beta(x, u(t, \cdot)) u(t, x). \quad (9)
\end{align*}
\]

In particular, Eq. (9) gives the newborns.

The generalized net reproduction rate is defined as

\[
R(u) = \int_0^\infty \beta(x, u) \Pi(x, u) \, dx,
\]

where \( \Pi(x, u) = 1 \frac{1}{g(x, u)} e^{-\int_0^x \frac{\mu(y, u)}{g(y, u)} \, dy} \) is an auxiliary function, said generalized survival probability and it represents a stationary solution of Eq. (8), i.e. the differential part of the model.

In general \( \beta \) and \( \Pi \) depend on \( u \) in a functional way: for instance in Calsina and Saldana [2, 3] the dependence is given through a weighted integral; in my paper [1] the dependence is infinite-dimensional in a more general way, to manage hierarchical models.

Some examples are populations where fertility or mortality are influenced only by the immediately superior size: for instance a population of trees in a forest, where the contended resource is the light, that is intercepted by immediately taller trees than trees of size \( x \) but not by the trees that are very taller than \( x \). (For a case of tree population model, see [7]).

A stationary solution \( u \) of (8)–(9) exists if and only if \( u \) satisfies the functional equation

\[
u = G(u) \Pi(u), \quad (11)
\]

where \( G(u(\cdot)) = \int_0^\infty \beta(x', u(\cdot)) u(x') \, dx' \).

Eq. (11) is related to the condition \( R(u) = 1 \) that is used to prove the existence of nontrivial stationary solution (that is, nonzero). Under suitable regularity conditions, we have that \( R(0) > 1 \) is a sufficient condition.

With additional conditions on monotonicity of \( \beta/g \) and \( \mu/g \), the reproduction rate \( R(u) \) is monotone decreasing and we exclude existence of nontrivial solution if \( R(0) < 1 \). This is a recurrent condition in dynamics of populations.
4 Other Recurrences of the Functional in Literature

Conditions on $H$ and $G$ in Prop. 1 are analogous to conditions given in [6], Teorema 2.1, b) Teorema [6] Let $-\infty < a < b \leq \infty$ and let $h$ and $g$ be positive functions on $(a, b)$, where $g$ is continuous on $(a, b)$.

Assume that $h$ is increasing on $(a, b)$ and $g$ is decreasing on $(a, b)$ where $g(b^-) = 0$. Then, for any $p \in (0, 1]$,

$$\int_a^b h(x)d[-g(x)] \leq \left(\int_a^b h^p(x)d[-g^p(x)]\right)^p$$

(1.2) (12)

If $1 \leq p < \infty$, then the inequality (1.2) holds in the reversed direction.

In [9], the theorem above extends from $t^p$ to concave and convex functions $\phi$, when they are positive and differentiable.

At the present I have no ideas if this fact would have any meaning for $R(u)$ or eventually estimates of it in the spaces $L^p$, however I think that the similarities of conditions is not a coincidence.

Heinig and Maligranda’s original paper [6] treats monotone functions and Hölder inequalities on Hardy spaces. A related field can be about Fredholm-Volterra equations.

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