UPPER BOUNDS FOR THE TIGHTNESS OF THE $G_\delta$-TOPOLOGY

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Abstract. We prove that if $X$ is a regular space with no uncountable free sequences, then the tightness of its $G_\delta$ topology is at most continuum and if $X$ is in addition Lindelöf then its $G_\delta$ topology contains no free sequences of length larger than the continuum. We also show that the higher cardinal generalization of our theorem does not hold, by constructing a regular space with no free sequences of length larger than $\omega_1$, but whose $G_\delta$ topology can have arbitrarily large tightness.

1. Introduction

Given a space $X$, the $G_\delta$-modification of $X$ (or $G_\delta$-topology on $X$), $X_\delta$ is defined as the topology on $X$ which is generated by the $G_\delta$-subsets of $X$. The problem of bounding the cardinal invariants of $X_\delta$ in terms of those of $X$ is a well-studied one in set-theoretic topology. For example if $c$, $s$, $L$, $t$ denote respectively the cellularity, the spread, the Lindelöf degree and the tightness of $X$, then $c(X_\delta) \leq 2^{c(X)}$ for every compact space $X$ (see [7]), $s(X_\delta) \leq 2^{s(X)}$ for every Hausdorff space $X$ (see [1]) and $L(X_\delta) \leq 2^{L(X):t(X)}$ for every Hausdorff space $X$ (see [10]). This is nothing but a small sample of bounds for the $G_\delta$ topology that have been proved in the past; for more results and applications of the $G_\delta$ topology to homogeneous compacta we refer the reader to our paper [1] and its bibliography.

Note that we have not mentioned a bound for the tightness of the $G_\delta$ topology yet, and indeed finding such a bound seems to be particularly tricky. Answering a question posed in [1], Dow, Juhász, Soukup, Szentmiklóssy and Weiss [5] proved that the inequality $t(X_\delta) \leq 2^{t(X)}$ holds within the realm of regular Lindelöf spaces. The Lindelöf property is essential in their argument, and in fact the authors were able to construct a consistent example of a regular countably tight space $X$ such that $t(X_\delta)$ can be as big as desired. They left open whether a countably
tight space $X$ such that $t(X_\delta) > 2^{\aleph_0}$ can be found in ZFC. This question was later solved in the positive by Usuba \cite{12}, who also found a bound on the tightness of the $G_\delta$-modification of every countably tight space, modulo the consistency of a certain very large cardinal. More precisely, Usuba proved that if $\kappa$ is an $\omega_1$-strongly compact cardinal then $t(X_\delta) \leq \kappa$, for every countably tight space $X$. Chen and Szeptycki \cite{3} managed to prove a very tight consistent bound for the special class of Fréchet $\alpha_1$-spaces, namely $t(X_\delta) \leq \aleph_1$ if the Proper Forcing Axiom holds.

Exploiting the notion of a free sequence, we will give another bound on the tightness of the $G_\delta$ topology.

A free sequence is a special kind of discrete set that was introduced by Arhangel’skii and is one of the essential tools in his celebrated solution of the Alexandroff-Urysohn problem on the cardinality of first-countable compacta. Recall that the set $\{x_\alpha : \alpha < \kappa\} \subseteq X$ is a free sequence provided that $\{x_\beta : \beta < \alpha\} \cap \{x_\beta : \alpha \leq \beta < \kappa\} = \emptyset$ for each $\alpha < \kappa$. We define $F(X)$ to be the supremum of cardinalities of free-sequences in $X$. The cardinal functions $F(X)$ and $t(X)$ are intimately related. Indeed, $F(X) \leq L(X)t(X)$ for every space $X$ and $t(X) = F(X)$, for every compact Hausdorff space $X$. However, the gap between $F(X)$ and $t(X)$ can be arbitrarily large even for a Lindelöf space $X$, as observed by Okunev \cite{9}.

We will prove a result about the tightness of the $G_\delta$ modification which has the Dow, Juhász, Soukup, Szentmiklóssy and Weiss bound as a consequence and also implies the following new bound: if $X$ is a regular space such that $F(X) = \omega$, then $t(X_\delta) \leq 2^{\aleph_0}$. The higher cardinal generalization of this is not true, as we will construct, for every cardinal $\kappa$, a regular space $X$ such that $F(X) = \omega_1 < \kappa = t(X_\delta)$. As a byproduct of our bound we will obtain that if $X$ is a Lindelöf regular space such that $F(X) = \omega$ then $F(X_\delta) \leq 2^{\aleph_0}$.

Given a set $S$, we denote by $\mathcal{P}(S)$ the powerset of $S$ and by $[S]^{\leq \kappa}$ the set of all subsets of $S$ which have cardinality at most $\kappa$. For undefined notions see \cite{6}, but our notation regarding cardinal functions follows \cite{8}.

2. The tightness of the $G_\delta$-modification

Let $X$ be a space, let $W$ be a subset of $X$ and let $\kappa$ be an infinite cardinal. We say that a collection $\mathcal{U}$ of subsets of $X$ is a $\text{Cl}_\kappa$-cover of $W$ provided that for any $C \in [W]^{\leq \kappa}$ there is $U_C \in \mathcal{U}$ such that $\overline{C} \subseteq U_C$. 
We say that a space $X$ is $Cl_\kappa$-Lindelöf if whenever $W$ is a subset of $X$ and $\mathcal{U}$ is an open $Cl_\kappa$-cover of $W$, then $W$ is covered by countably many elements of $\mathcal{U}$.

**Lemma 1.** Every Lindelöf space $X$ is $Cl_{t(X)}$-Lindelöf.

*Proof.* It suffices to observe that every open $Cl_{t(X)}$-cover of a set $W \subseteq X$ is actually a cover of $W$. \qed

**Lemma 2.** Every space $X$ satisfying $F(X) = \omega$ is $Cl_\omega$-Lindelöf.

*Proof.* Let $W$ be a subset of $X$ and $\mathcal{U}$ be an open $Cl_\omega$-cover of $W$. Assume by contradiction that no countable subfamily of $\mathcal{U}$ covers $W$. We will then construct a free sequence of cardinality $\omega_1$ inside $W$.

Suppose that, for some $\beta < \omega_1$, we have chosen points $\{x_\tau : \tau < \beta\} \subset W$ and elements $U_\tau \in \mathcal{U}$ for every $\tau < \beta$ with the property that $\{x_\alpha : \alpha < \beta\} \subseteq U_\beta$. By our assumption, the family $\{U_\tau : \tau \leq \beta\}$ does not cover $W$, and therefore we can fix a point $x_\beta \in W \setminus \bigcup\{U_\tau : \tau \leq \beta\}$.

Eventually, $\{x_\tau : \tau < \omega_1\}$ is a free sequence of cardinality $\omega_1$ in $X$, which is a contradiction. \qed

**Theorem 3.** Let $X$ be a regular space and let $\kappa$ be an infinite cardinal. If $X$ is $Cl_\kappa$-Lindelöf, then $t(X) \leq 2^\kappa$.

*Proof.* Let $A$ be any subset of $X$ and fix a point $p$ in the $G_\delta$-closure of $A$.

Let $\mathcal{N}_\kappa(X) = \{C \in [X]^{\leq \kappa} : p \notin \overline{C}\}$. By the regularity of $X$, for every $C \in \mathcal{N}_\kappa(X)$, we can find disjoint open sets $U_C$ and $V_C$ such that $\overline{C} \subset U_C$ and $p \in V_C$.

Let $\phi$ be a choice function on $\mathcal{P}(X)$. We will build by induction an increasing family $\{W_\alpha : \alpha < \kappa^+\} \subset [A]^{2^\kappa}$.

Let $W_0$ be any subset of $A$ of cardinality $\leq 2^\kappa$ and assume we have already defined $\{W_\beta : \beta < \alpha\}$. If $\alpha$ is a limit ordinal then put $W_\alpha = \bigcup\{W_\beta : \beta < \alpha\}$. If $\alpha = \gamma + 1$ then let:

$$W_\alpha = W_\gamma \cup \{\phi(A \cap \bigcap\{V_C : C \in \mathcal{C}\}) : \mathcal{C} \in [\mathcal{N}_\kappa(W_\gamma)]^{\leq \omega}\}$$

Note that $|W_\alpha| \leq 2^\kappa$.

Finally, let $W = \bigcup\{W_\alpha : \alpha < \kappa^+\}$. Since $|W| \leq 2^\kappa$, it suffices to show that $p$ is in the $G_\delta$-closure of $W$.

Indeed, let $\{O_n : n < \omega\}$ be a family of open neighbourhoods of $p$.

**Claim.** There is a countable family $\mathcal{C}_n \subset \mathcal{N}_\kappa(W \setminus O_n)$ such that $W \setminus O_n \subset \bigcup\{U_C : C \in \mathcal{C}_n\}$. 


Proof of Claim. Since $O_n$ is an open neighbourhood of $p$, we have that $N_\kappa(W \setminus O_n) = [W \setminus O_n]^{\le \kappa}$ and therefore $U_C$ is defined for every $C \in [W \setminus O_n]^{\le \kappa}$ and $\overline{C} \subset U_C$. In particular, $\mathcal{U} = \{U_C : C \in N_\kappa(W \setminus O_n)\}$ is a $Cl_\kappa$-open cover of $W \setminus O_n$. Now, the statement of the claim follows from the fact that $X$ is a $Cl_\kappa$-Lindelöf space.

For every $n < \omega$, fix a family $\mathcal{C}_n$ satisfying the Claim and let $S = \bigcup \{\mathcal{C}_n : n < \omega\}$ and $S = \bigcup S$.

Since the set $S$ has cardinality at most $\kappa$, there is an ordinal $\delta < \kappa^+$ such that $S \subset W_\delta$. It follows then that the point $q = \phi(\bigcap_{C \in S} V_C \cap A)$ belongs to $W_{\delta+1} \subset W$.

Note that, for every $n < \omega$, $q \in \bigcap\{V_C : C \in \mathcal{C}_n\} \cap W \subset W \setminus \bigcup\{U_C : C \in \mathcal{C}_n\} \subset O_n \cap W$. Therefore $q \in \bigcap\{O_n : n < \omega\} \cap W$ and we are done. \qed

Corollary 4. (Dow, Juhász, Soukup, Szentmiklóssy and Weiss [5]) If $X$ is a Lindelöf regular space, then $t(X_\delta) \le 2^{t(X)}$.

Corollary 5. If $X$ is a regular space and $F(X) = \omega$, then $t(X_\delta) \le 2^\omega$.

It’s natural to ask whether the higher cardinal version of Corollary 5 holds true. The following theorem shows that this is not the case.

Let $\theta$ be a regular uncountable cardinal. Recall that an elementary submodel $M$ of $H(\theta)$ is said to be $\omega$-covering if for every countable subset $A$ of $M$ there is a countable set $B \in M$ such that $A \subset B$. The union of any elementary chain of elementary submodels of length $\omega_1$ is an $\omega$-covering elementary submodel, so $\omega$-covering submodels of cardinality $\omega_1$ exist in ZFC (see [4]).

Theorem 6. For every uncountable cardinal $\kappa$, there is a space $Y$ such that $F(Y) = \omega_1 < \kappa = t(Y_\delta)$.

Proof. Let $X = \Sigma(2^\kappa) = \{x \in 2^\kappa : |x^{-1}(1)| \le \aleph_0\}$ and let $p \in 2^\kappa$ be the point defined by $p(\alpha) = 1$, for every $\alpha < \kappa$. We will prove that $Y = X \cup \{p\}$ with the topology inherited from $2^\kappa$ is the required example.

The following Claim was proved by the second author in [11] for the case $\kappa = \omega_2$, but the argument works for any uncountable cardinal $\kappa$ without any modifications. We include it for the reader’s convenience.

Claim. $L(X) = \aleph_1$.

Proof of Claim. Let $\mathcal{U}$ be an open cover of $X$. Without loss of generality we can assume that for every $U \in \mathcal{U}$, there is a finite partial function $\sigma : \kappa \rightarrow 2$ such that $U = \{x \in 2^\kappa : \sigma \subset x\}$. The domain of $\sigma$ will then be called the support of $U$ and we will write $\text{supp}(U) = \text{dom}(\sigma)$.
Let \( \theta \) be a large enough regular cardinal and \( M \) be an \( \omega \)-covering elementary submodel of \( H(\theta) \) such that \( X, U, \kappa \in M \) and \( |M| = \aleph_1 \).

We claim that \( U \cap M \) covers \( X \). Indeed, let \( x \in X \) be any point and let \( A \in M \) be a countable set such that \( x^{-1}(1) \cap M \subset A \).

Let \( Z = \{ y \in X : (\forall \alpha \in \kappa \setminus A)(y(\alpha) = 0) \} \). Then \( Z \in M \) and \( Z \) is a compact subspace of \( X \). So there is a finite subfamily \( V \in M \) of \( U \) such that \( Z \subset \bigcup V \). Since \( V \) is finite, we have \( V \subset M \). It then follows that \( U \cap M \) covers \( Z \).

Let \( a \) be the point such that \( a(\alpha) = x(\alpha) \) for all \( \alpha \in M \cap \kappa \) and \( a(\alpha) = 0 \) for all \( \alpha \in \kappa \setminus M \). The fact that \( x^{-1}(1) \cap M \subset A \) implies that \( a \in Z \) and hence there is \( U \in U \cap M \) such that \( a \in U \). Note that \( \text{supp}(U) \) is a finite element of \( M \) and hence \( \text{supp}(U) \subset M \). But since \( x \) and \( a \) coincide on \( M \) we then have that \( x \in U \) as well, as we wanted.

This proves \( L(X) \leq \aleph_1 \), but we can’t have \( L(X) = \aleph_0 \) because \( X \) is countably compact non-compact. Hence \( L(X) = \aleph_1 \).

\[ \triangle \]

It is well known that \( X \) is Fréchet-Urysohn and hence \( X \) has countable tightness. Since \( F(X) \leq L(X) \cdot t(X) \) we have \( F(X) \leq \omega_1 \), but then also \( F(Y) \leq \omega_1 \). It’s easy to see that \( t(p, Y_\delta) = \kappa \).

\[ \square \]

In [2] Carlson, Porter and Ridderbos proved the following improvement of the Pytkeev inequality \( L(X_\delta) \leq 2^{L(X) \cdot t(X)} \) mentioned in the introduction.

Theorem 7. [2] (Theorem 2.7) If \( X \) is a Hausdorff space, then \( L(X_\delta) \leq 2^{L(X)F(X)} \).

Putting together Corollary 5 and the above theorem we obtain:

Corollary 8. Let \( X \) be a regular Lindelöf space such that \( F(X) = \omega \). Then \( F(X_\delta) \leq 2^{\aleph_0} \).

We don’t know whether the Lindelöf property can be removed from the above corollary.

Question 9. Let \( X \) be a regular space satisfying \( F(X) = \omega \). Is it true that \( F(X_\delta) \leq 2^\omega \) ?

It’s reasonable to conjecture that the higher cardinal version of Corollary 5 holds, at least for Lindelöf spaces.

Question 10. Let \( X \) be a regular (Lindelöf) space. Is it true that \( F(X_\delta) \leq 2^{F(X)} \) ?
Note that neither the consistent example from [5] of a regular countably tight space \(X\) such that \(t(X_\delta)\) can be arbitrarily large nor the example from Theorem 6 work for the above question since \(F(X) = |X|\) for the former and \(F(X_\delta) \leq 2^{\aleph_0}\) for the latter.

We finish with two easy bounds for the tightness of the \(G_\delta\) topology, by making using of the weight and the spread.

**Proposition 11.** Let \(X\) be a regular space. Then:

1. \(t(X_\delta) \leq 2^{d(X)}\).
2. \(t(X_\delta) \leq 2^{s(X)}\).

**Proof.** To prove (1) recall that \(w(X) \leq 2^{d(X)}\) for every regular space \(X\) (see [8]). Now \(t(X_\delta) \leq w(X_\delta) \leq w(X)^\omega \leq 2^{d(X)^\omega} = 2^{d(X)}\).

To prove (2) recall that \(nw(X) \leq 2^{s(X)}\) for every regular space \(X\) (see [8]) and proceed as before. \(\square\)

Proposition 11 (1) is not true for Hausdorff spaces, as the following example shows.

**Example 12.** There is a separable Hausdorff space \(X\) such that \(t(X_\delta) > 2^{\aleph_0}\).

**Proof.** Let \(Y\) be the Katětov extension of the integer. That is, if \(U\) is the set of all non-principal ultrafilters on \(\omega\) then \(Y = \omega \cup U\), every point of \(\omega\) is isolated and a basic neighbourhood of \(p \in U\) is a set of the form \(\{p\} \cup A \setminus F\), where \(A \in p\) and \(F\) is finite.

Let \(X = Y \cup \{\infty\}\), where \(\infty \notin Y\) and declare \(V \subset X\) to be a neighbourhood of \(\infty\) if and only if \(|X \setminus V| \leq 2^{\aleph_0}\). It is easy to see that \(X\) is a separable Hausdorff space and \(t(X_\delta) > 2^{\aleph_0}\). \(\square\)

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