A GENERALISED MONGE-AMPÈRE EQUATION

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Abstract. We consider a generalised complex Monge-Ampère equation on a compact Kähler manifold and treat it using the method of continuity. For complex surfaces, we prove an easy existence result. We also prove that (for three-folds and a related real PDE in a ball in $\mathbb{R}^3$), as long as the Hessian is bounded below by a pre-determined constant (while moving along the method of continuity path), a smooth solution exists. Finally, we prove existence for another real PDE in a 3-ball, which is a local, real version of a conjecture of X.X. Chen.

1. Introduction

Let $(X, \omega)$ be an $n$-dimensional, compact, Kähler manifold. Here, we consider a generalised complex Monge-Ampère PDE (to be solved for a smooth function $\phi$)

\begin{equation}
\alpha_0(\omega + dd^c \phi)^n + \alpha_1 \wedge (\omega + dd^c \phi)^{n-1} + \ldots + \alpha_{n-1} \wedge (\omega + dd^c \phi) = \eta
\end{equation}

where $\eta$, $\alpha_i$ are smooth, closed forms satisfying the obvious necessary condition $\int_X \eta = \int_X (\alpha_0 \omega^n + \alpha_1 \wedge \omega^{n-1} + \ldots)$. When $\eta > 0$, $\alpha_i = 0 \forall \ i \neq 0$ and $\alpha_0 = 1$, equation 1.1 is the one introduced by Calabi [3] and solved by Aubin [2] and Yau [4]. Equations of this type are ubiquitous in geometry. A version of this generalised one appeared in [15]. The geometric applications of this equation shall be explored elsewhere.

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2. Statements of results

We state a somewhat general theorem about uniqueness, openness and $C^0$ estimates. The proof is quite standard (adapted largely from [6] which is in turn based on [4]). Although the theorem is folklore, we haven’t found the precise statement (in this level of generality) in the literature on the subject. In what follows, positivity of $(p, p)$ forms is strong positivity. Let $B$ be the product of Banach submanifolds of forms wherein, an element of $B$ is of the form $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \phi)$ where $\alpha_i$ are $C^{1, \beta}$ $(i, i)$, closed forms and $\phi$ is a $C^{3, \beta}$ function satisfying $\int_M \phi = 0$, $n\alpha_0(\omega + dd^c \phi)^{n-1} + (n-1) \alpha_1 \wedge (\omega + dd^c \phi)^{n-2} + \ldots + \alpha_{n-1} > 0$ and, $\int_X (\sum_i \alpha_i \wedge \omega^{n-i}) \neq 0$. Also, let $\bar{B}$ be the Banach submanifold of $C^{1, \beta}$ top forms $\gamma$ with $\int_X \gamma = 1$ and $\gamma > 0$.

**Theorem 2.1.** If $\alpha_0 \omega^n + \alpha_1 \omega^{n-1} + \ldots > 0$, $\eta > 0$ and, $d\alpha_i = 0$, then, any smooth solution $\phi$ of 1.1 satisfying $\int_X \phi \omega^n = 0$ and, $\kappa \geq K \omega^n$ where $K > 0$ and $\sum_k (\alpha_k (\omega + dd^c \phi)^{n-k} - \alpha_k \omega^{n-k}) = \kappa \wedge dd^c \phi$, is bounded a priori: $\|\phi\|_{C^0} \leq C_{\eta}$. Also, if $\alpha_i > 0 \forall i$ and, if there exists a smooth solution $\phi$ such
that \( \omega + dd^c \phi > 0 \), it is unique (upto a constant) among all such solutions; In addition, the mixed derivatives of \( \phi \) are bounded a priori \( \| \phi \|_{C^{1,1}} \leq C \eta \).

The map \( T : B \to \tilde{B} \) defined by \( T(\alpha_0, \alpha_1, \ldots, \phi) = \sum_{i} \alpha_i / (\sum_{i} \alpha_i / \omega^{n-1}) \) is open and, so is the restriction of \( T \) to a subspace defined by fixing the \( \alpha_i \). Also, a level set of this map is locally a graph with \( \phi \) being a function of the \( \alpha_i \).

When \( n = 2 \), and, \( \alpha_0 = 1, \eta - \alpha_2 + \frac{\alpha^2}{2} > 0 \), there exists a unique, smooth solution to 1.1 satisfying \( \omega + dd^c \phi + \frac{\alpha^2}{4} > 0 \).

In particular, if \( \alpha_i = \omega_i \) for some \( i \) and all the other \( \alpha_j \) are small enough, then, by the solution of the \( k \)-Hessian equations \([9], [10]\) we have a smooth solution of equation 1.1.

One may formulate a version of the same problem locally as a Dirichlet problem on a pseudoconvex domain in \( \mathbb{C}^n \). In this context, we note that viscosity solutions to the Dirichlet problem exist by \([5]\) and \([7]\). We also have the following result for a real version of the PDE:

**Theorem 2.2.** The following Dirichlet problem on the ball \( B \) of radius 1 centred at the origin

\[
\begin{align*}
\det(D^2u) + \Delta u &= tf + (1-t)36 \\
u|_{\partial B} &= 0 \\
f &> 36
\end{align*}
\]

has a unique smooth solution at \( t = T \) if \( f \) is smooth and, for all \( t \in [0, T) \), smooth solutions \( u_t \) exist and satisfy \( D^2u_t > 3 \).

A similar result holds for complex three-folds.

**Theorem 2.3.** If \( \alpha > 0, \omega > 0 \) are smooth Kähler forms on a compact Kähler manifold \( (X, \omega_0) \), then, there exists a constant \( C > 0 \) depending only on \( \alpha \) and \( \omega_0 \) such that, the equation

\[
(\omega + dd^c u_t)^3 + \alpha^2(\omega + dd^c u_t) = \frac{e^{tf}(\omega^3 + \alpha^2 \omega)}{e^{tf}(\omega^3 + \alpha^2 \omega)}(\omega^3 + \alpha^2 \omega)
\]

has a unique smooth solution at \( t = T \), if for all \( t \in [0, T) \), smooth solutions exist and satisfy \( \omega + dd^c u > C \omega_0 \).

Finally, we present a local, real version of a conjecture of X.X. Chen (conjecture 4 in \([15]\) made in the compact complex manifold case). Some progress has been made in a few special cases \([16]\). However, in all these cases, the problem was reduced to an inverse Hessian equation. We prove existence in a special case here, using the method of continuity. Actually, a far more general result was proven in \([17]\), but, results on the Bellman equation were used (as opposed to a direct method of continuity). Such results may not carry over in an obvious way to the manifold case and hence our proof of this “toy model”.

**Theorem 2.4.** If \( f > 0 \) is a smooth function on \( \bar{B}(0,1) \) (the closed unit ball), then, the following Dirichlet problem has a unique, smooth, convex solution.

\[
\begin{align*}
\det(D^2u) - \Delta u &= f \\
u|_{\partial B} &= 0 \\
f &> 0
\end{align*}
\]
3. Standard results used

For the convenience of the reader, we have included statements of some standard results in the form that we use in the proofs.

Our principal tool to study fully nonlinear PDE like equation 1.1, is the method of continuity (It is like a flow technique. In fact this analogy was exploited more seriously, to great advantage, in [8]). To solve $Lu = f$ where $L$ is a nonlinear operator, one considers the family of equations $Lu_t = \gamma(t)$ where, $\gamma(1) = f$ and $\gamma(0) = g$ such that, at $t = 0$, one has a solution $Lu_0 = g_0$. Then, one proves that the set of $t \in [0,1]$ for which the equation has a solution is both, open and closed (and clearly non-empty). In order to prove openness, one considers $L$ to be a map between appropriate Banach spaces. Then, the implicit function theorem of Banach spaces proves openness. However, while dealing with equations like Monge-Ampère equations, one has to verify that certain conditions like ellipticity are preserved along the “continuity path”. This is crucial because, in order to solve the linearised equation and, to prove that indeed one has a solution in an appropriate Banach space, one needs ellipticity in these cases. In fact, in a few of the cases we shall consider, ellipticity is not preserved and hence, the best we can do is a “short-time” existence result. In order to prove closedness, one needs to prove uniform (i.e. independent of $t$) a priori estimates for $u$. In our case, we shall need these estimates in $C^{2,\alpha}$ in order to use the Arzela-Ascoli theorem to conclude closedness. These estimates are usually proved by improving on lower order estimates. Once, one produces a $C^{2,\alpha}$ solution, one “bootstraps” the regularity (at each $t \in [0,1]$) using the Schauder estimates. The Schauder estimates on a compact manifold (without boundary) are (they can be derived easily using similar interior and boundary ones in domains in $\mathbb{R}^n$ [1]).

**Theorem 3.1.** Schauder a priori estimates on a Riemannian manifold: If $Lu = f$, where $L$ is a second-order, uniformly elliptic operator with smooth coefficients, and, $u$ is a $C^{2,\alpha}$ and $f$ is a $C^{0,\alpha}$ function,

$$
\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{C^0} + \|f\|_{C^{0,\alpha}})
$$

In order to derive a priori estimates, we shall use standard techniques as in [4], [6] for the manifold case, and, [7] for the Euclidean case. The main blackbox is the Evans-Krylov-Safanov theory for proving $C^{2,\alpha}$ estimates from $C^2$ ones. This requires (apart from uniform ellipticity) concavity of the equation. There is a similar version for the complex case. The real version is:

**Theorem 3.2.** Let $u$ be a smooth function on the unit ball satisfying,

$$
F(D^2u, x, Du) = g
$$

on the unit ball in $\mathbb{R}^n$ centred at the origin $B(0,1)$ with $u = 0$ on the boundary of the ball. Here, $F$ is a smooth function defined on a convex open set of symmetric $n \times n$ matrices $\times \mathbb{R} \times \mathbb{R}^n$ which satisfies,

a) Uniform ellipticity on solutions : There exist positive constants $\lambda$ and $\Lambda$ so that $0 < \lambda |\xi|^2 \leq F_{ij}(D^2u, x, Du)\xi_i\xi_j \leq \Lambda |\xi|^2$ for all vectors $\xi$ and all $u$ satisfying the equation.

b) Concavity on a convex open set : $F$ is a concave function on a convex open set of symmetric matrices (containing $D^2u$ for all solutions $u$).

Then, $\|u\|_{C^{2,\alpha}(\bar{B}(0,1))}$ $\leq C$ where $C$ and $\alpha$ depend on the first and second derivatives of $F$, $\|u\|_{C^2(\bar{B})}$, $\|g\|_{C^2(\bar{B})}$, $n$, $\lambda$ and $\Lambda$.

The complex, interior version (that we need) is:

**Theorem 3.3.** Let $u$ be a $C^4$ function on the unit ball in $\mathbb{C}^n$ satisfying

$$
F(u_{i\bar{j}}, z, \bar{z}) = 0
$$
for a $C^{2,\beta}$ function $F(x,p,\bar{p})$ satisfying,

a) Uniform ellipticity on solutions: There exist positive constants $\lambda$ and $\Lambda$ so that $0 < \lambda |\xi|^2 \leq F_{ij}(dd^c u, z, \bar{z})\xi_i\xi_j \leq \Lambda |\xi|^2$ for all vectors $\xi$ and all $u$ satisfying the equation.

b) Concavity on a convex open set: $F$ is a concave function on a convex open set of hermitian matrices (containing $u_{ij}$ for all solutions $u$).

Then, $\|u\|_{C^{2,\alpha}(B(0,\frac{1}{2}))} \leq C$ where, $C$ and $\alpha$ depend on $\lambda$, $\Lambda$, $n$, and $\|u_{ij}\|_{C^0(B)}$ and uniform bounds on the first and second derivatives of $F$ evaluated at $u$.

The proofs are standard [1], [14], [13], [12]. Usually, one proves these estimates when the PDE is concave on all symmetric matrices. In Monge-Ampère equations, one needs a weaker requirement of being concave on a convex open set of symmetric matrices [7] (the proofs go through easily with this requirement).

To conclude, we add a few words about uniqueness. The usual technique for demonstrating uniqueness (due to Calabi) of $Lu = f$ is to assume two solutions $u_1$ and $u_2$, and, to write $0 = Lu_1 - Lu_2 = \int_0^1 \frac{dt}{dt} (tu_2 + (1-t)u_1) dt$. If the integrand is an elliptic operator, by the maximum principle, $u_1 = u_2$.

4. Proofs of the Theorems

4.1. Proof of theorem 2.1. This proof is similar to the one for the usual Monge-Ampère equation [6].

The $C^0$ estimate: As usual, without loss of generality, we may change the normalisation to $\sup \phi = -1$ i.e. we may add $-1 - \sup \phi$ to $\phi$. Indeed, if the new $\phi$ has a $C^0$ estimate, then, $\int_X \phi = 0$ yields the desired $C^0$ estimate. This means, we just have to find a lower bound on $\phi$. Certainly $\phi$ has an $L^1$ bound [6]. Let $\phi = -\phi_-$ (so that $\phi_- \geq 1$). Subtracting $\Theta = \sum k \alpha_k \wedge \omega^{n-k}$ and then, multiplying the equation by $\phi_-^p$ and integrating, we have (here $\eta = e^f \Theta$),

$$\begin{align*}
- \int \phi_-^p dd^c \phi_- \wedge \kappa &= \int \phi_-^p (e^f - 1) \Theta \\
\int \phi_-^p (e^f - 1) \Theta &\leq c \|\phi_-\|_{L^p}^p \\
- \int \phi_-^p dd^c \phi_- \wedge \kappa &= \int d(\phi_-^p) \wedge d^c \phi_- \wedge \kappa \\
&= c \int d(\phi_-^{\frac{p+1}{p+1}}) \wedge d^c (\phi_-^{\frac{p+1}{p+1}}) \wedge \kappa \\
&\geq C \|\nabla (\phi_-^{\frac{p+1}{p+1}})\|_{L^2}^2 \\
&\geq C_1 \left( \int \phi_-^{\frac{p+1}{p+1}} \right)^{\frac{n-1}{n}} - C_2 \int \phi_-^{p+1}
\end{align*}$$

where the last inequality follows from the Sobolev embedding theorem. Upon rearranging, we have

$$\|\phi_-\|_{L^{(p+1)(n)/(n-1)}} \leq (C(p + 1))^{\frac{1}{p+1}} \|\phi_-\|_{L^{p+1}}$$
The Moser-iteration procedure gives $\sup |\phi| \leq C\|\phi\|_{L^2}$. If we prove that the right hand side is controlled by the $L^1$ norm of $\phi$, we will be done. Indeed,

$$C\|\phi\|_{L^1} \geq \int \phi(1 - e^{f})\Theta$$

$$\geq C\|\nabla \phi\|_{L^2}^2$$

$$\geq C\|\phi - \langle \phi \rangle\|_{L^2}^2$$

$$\|\phi\|_{L^2} \leq C(\|\phi\|_{L^1} + 1)$$

where, we have used the Poincaré inequality. Hence, proved.

**Uniqueness**: If $\phi_1$ and $\phi_2$ are two solutions, upon subtraction we have,

$$\sum_i \alpha_i \wedge ((\omega + dd^c \phi_1)^{n-i} - (\omega + dd^c \phi_2)^{n-i}) = 0$$

$$\Rightarrow \int_0^1 \sum_k k\alpha_k \wedge (\omega + dd^c \phi_1 + tdd^c (\phi_2 - \phi_1))^{n-k-1} dt \wedge dd^c (\phi_2 - \phi_1) = 0$$

Thus, by the maximum principle, $\phi_2 - \phi_1$ is a constant.

**The mixed derivatives estimate**: When $\alpha_i > 0$,

$$\eta \geq \alpha_{n-1} \wedge (\omega + dd^c \phi) > C(\text{tr}(\omega + dd^c \phi))$$

where $C > 0$. Since $0 < \omega + dd^c \phi$, the eigenvalues of $dd^c \phi$ are bounded above. Thus, the mixed second derivatives of $\phi$ are bounded. Note that, by the Schauder estimate [11], the first derivatives are bounded as well.

**Openness**: The map $T$ is smooth. Its Gâteaux derivative is $DT(0, 0, \ldots, 0, \chi) = (n\alpha_0(\omega + dd^c \phi)^{n-1} + (n-1)\alpha_1 \wedge (\omega + dd^c \phi)^{n-2} + \ldots + \alpha_{n-1}) \wedge dd^c \chi$. It is clearly a bounded surjection (by the Schauder theory) onto its image if $n\alpha_0(\omega + dd^c \phi)^{n-1} + (n-1)\alpha_1 \wedge (\omega + dd^c \phi)^{n-2} + \ldots + \alpha_{n-1} > 0$. If $DT$ is restricted to vectors of the form $(0, 0, \ldots, 0, \chi)$, then, it is a Banach space isomorphism. Hence, by the implicit function theorem of Banach manifolds, openness is guaranteed. In fact, it also guarantees that, on a level set, $\phi$ can be solved for (locally), in terms of $\alpha_i$.

**The n=2 case**: The equation we have is equivalent to

$$(\omega + dd^c \phi + \frac{\alpha_1}{2})^2 = \eta - \alpha_2 + \frac{\alpha_1^2}{4}$$

This is just the usual Monge-Ampère equation and hence we are done.

### 4.2. Proof of theorem 2.2

**Uniqueness** is proved as before. We shall only prove existence. Let $Lu = det(D^2u) + \Delta u$. To this end, we use the method of continuity. Consider the equation

$$Lu_t = tf + (1 - t)L\phi$$

$$u_t|_{\partial B} = 0$$

$$\phi = \frac{3}{2} \sum x_i^2 - \frac{3}{2}$$

(4.1)

When $t = 0$, it has a smooth solution, namely, $\phi$.

**Openness**: Let $\Omega \subset C^{2,\alpha}_0(B)$ be the set of $u$ such that $D^2u > 3$ (where the subscript 0 indicates vanishing on the boundary). This is an open subset. Define $T : \Omega \to C^{0,\alpha}_0$ to be $T(u_t) = det(D^2u_t) +$
closedness. If \( u_\ast \) is a solution of \( 4.1 \), then, it is easy to see that \( DT_{u_\ast} \) is a linear isomorphism. Hence, by the inverse function theorem of Banach manifolds, we see that the set of \( t \) for which there is a solution is open.

**Closedness:** Suppose there is a sequence \( t_i \rightarrow t \) such that there are smooth solutions \( u_{t_i} \) satisfying \( D^2 u > 3 \). Then, we wish to prove that a subsequence of the \( u_{t_i} \) converges to a smooth solution \( u_t \) in the \( C^{2,\beta} \) topology. This requires \textit{apriori} estimates (the convergence following from the Arzela-Ascoli theorem). We shall prove the same for the equation \( 2.2 \). We just have to prove the \( C^{2,\alpha} \) estimate in order to ensure smoothness (by the Schauder theory).

\textit{C}^0 \textit{estimate:} Note that \( \Delta u \leq f \). Hence, for \( A >> 1 \), \( 0 > f - \Delta(A \sum x_i^2) = \Delta(u - A \sum x_i^2) \). The minimum principle implies that \( u \geq A \sum x_i^2 - A \). Since, \( \Delta u > 9 \), \( u \leq 0 \) by the maximum principle. Thus, \( \|u\|_{C^0} \leq C \).

\textit{C}^1 \textit{estimate:} Differentiating both sides using the operator \( D \), \( \text{tr}((\text{Hess}u)^{-1}D^2w) + \Delta w = Df \) where \( w = Du \). Just as before, by adding or subtracting a large multiple of \( \sum x_i^2 \) to \( w \) and using the maximum principle, we see that \( \|Du\|_{C^0} \) is controlled by its supremum on the boundary. The tangential boundary derivatives are 0. Since, \( A \sum x_i^2 - A \leq u \leq 0 \), \( \|\partial_x u\|_{C^0} \leq 2A \). Hence, \( \|u\|_{C^1} \leq C \).

\textit{C}^2 \textit{estimate:} Since \( \Delta u \leq Lu \leq f \) and \( \Delta u > 0 \), \( \|u_{ij}\|_{C^0} \leq C \). Hence, \( \|u\|_{C^2} \leq C \).

\textit{C}^{2,\alpha} \textit{estimate:} So far, we haven’t used anything about the sequence except that \( D^2 u_{t_i} > 0 \). This will change presently. For any function \( F : \mathbb{R} \rightarrow \mathbb{R} \), \( F(\det(D^2 u) + \Delta u) = F(f) \). If we choose the function appropriately, then the resulting equation will be a concave, uniformly elliptic Monge Ampère PDE to which we may apply the Evans-Krylov theory to extract a \( C^{2,\alpha} \) estimate.

We claim that, the function \( F(x) = \int_{36}^{x} e^{-\frac{x}{4}} dt \) is such that, \( g(\lambda_1, \lambda_2, \lambda_3) = F(\sum \lambda_i + \lambda_1 \lambda_2 \lambda_3) \) has a uniformly positive gradient and is concave if \( \lambda_i > 3 \). By using the \( C^2 \) estimate and theorem \( 3.2 \), we have the desired estimate.

We shall prove the aforementioned fact: Let \( x = \sum \lambda_i + \lambda_1 \lambda_2 \lambda_3 \). We see that \( \frac{\partial g}{\partial \lambda_i} \bigg|_{D^2 u} = e^{-\frac{x}{4}} (1 + \frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_i}) > e^{-\frac{x}{4}} \) and is less than \( 1 + 3f \) where we have evaluated the derivative at the eigenvalues of the Hessian of a solution of equation \( 2.2 \). Hence, it is uniformly elliptic.

If \( (v_1, v_2, v_3) \in \mathbb{R}^3 \), then \( -v_i v_j \frac{\partial^2 g}{\partial x_i \partial x_j} = e^{-\frac{x}{4}} (x(v_1(1 + \lambda_2 \lambda_3) + v_2(1 + \lambda_3 \lambda_1) + v_3(1 + \lambda_1 \lambda_2))) - 2e^{-\frac{x}{4}} (v_1 v_2 \lambda_3 + v_2 v_3 \lambda_1 + v_3 v_1 \lambda_2) \), which is in turn equal to \( e^{-\frac{x}{4}} (v_1^2 \alpha + \beta v_1 + \gamma) \geq 0 \iff \beta^2 - 4 \alpha \gamma \leq 0 \) and,

\[
\frac{\beta^2 - 4 \alpha \gamma}{4} = (v_2(x(1 + \lambda_2 \lambda_3)(1 + \lambda_1 \lambda_2 - \lambda_3) + v_3(x(1 + \lambda_1 \lambda_2)(1 + \lambda_2 \lambda_3) - \lambda_3))^2 - x(1 + \lambda_2 \lambda_3)^2(v_2^2 x(1 + \lambda_1 \lambda_3)^2 + v_3^2 x(1 + \lambda_1 \lambda_2)^2 + 2v_2 v_3(x(1 + \lambda_1 \lambda_3)(1 + \lambda_1 \lambda_2) - \lambda_1)) \leq \tilde{\alpha} v_2^2 + \tilde{\beta} v_2 + \tilde{\gamma} \leq 0
\]

with the last inequality holding if and only if \( \tilde{\alpha} \leq 0 \) and \( \tilde{\beta}^2 - 4 \tilde{\alpha} \tilde{\gamma} \leq 0 \). Let us assume (without loss of generality) that \( v_3 \neq 0 \) and that \( \lambda_1 < \lambda_2 < \lambda_3 \).

\[
\tilde{\alpha} = (x(1 + \lambda_2 \lambda_3)(1 + \lambda_1 \lambda_3) - \lambda_3)^2 - x^2(1 + \lambda_2 \lambda_3)^2(1 + \lambda_1 \lambda_3)^2 = \lambda_3^2 - 2 \lambda_3 x(1 + \lambda_2 \lambda_3)(1 + \lambda_1 \lambda_3) \leq -2x \lambda_3^2 \frac{2x}{3} \]
$$\frac{\tilde{\gamma}}{v_3^3} = (x(1 + \lambda_2 \lambda_1)(1 + \lambda_2 \lambda_3) - \lambda_2)^2 - x^2(1 + \lambda_2 \lambda_3)^2(1 + \lambda_1 \lambda_2)^2$$
$$\leq -\frac{4x^2\lambda_3^2}{3}$$

$$\frac{\tilde{\beta}}{2v_3^2} = (x(1 + \lambda_2 \lambda_3)(1 + \lambda_1 \lambda_2) - \lambda_2)(x(1 + \lambda_2 \lambda_3)(1 + \lambda_1 \lambda_3) - \lambda_3)$$
$$- x(1 + \lambda_2 \lambda_3)^2(x(1 + \lambda_1 \lambda_3)(1 + \lambda_1 \lambda_2) - \lambda_1)$$

$$\left(\frac{\tilde{\beta}}{2v_3^2}\right)^2 = (x(1 + \lambda_2 \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1 + \lambda_1 \lambda_2 \lambda_3) - \lambda_2 \lambda_3)^2$$
$$\leq x^4(1 + \lambda_2 \lambda_3)^2$$

$$\frac{\tilde{\beta}^2 - 4\tilde{\alpha}^2}{4v_3^2} \leq 0$$

Hence proved.

Remark: Writing equation 1.1 for $n = 3$ and $\alpha_0 = 1$ we have,

$$(\omega + dd^c \phi)^3 + \alpha_1(\omega + dd^c \phi)^2 + \alpha_2(\omega + dd^c \phi) = \eta$$

$$\Rightarrow (\omega + \frac{\alpha_1}{3} + dd^c \phi)^3 + (\alpha_2 - \frac{\alpha_1^2}{3})(\omega + \frac{\alpha_1}{3} + dd^c \phi) = \eta - \frac{2\alpha_1^3}{27} + \frac{\alpha_1 \alpha_2}{3}$$

A local, real version of a special case of the above is equation 2.2.

4.3. Proof of theorem 2.3. Once again, we apply the method of continuity. We shall impose several conditions on $C$ (as we go along). It should be large enough so that, whenever $\beta > C_0$, $\beta^3 \geq 3\lambda_2 \lambda_3$ (Indeed, if $K > 0$ and $B > 0$ are given, det($A$) > $K$tr($BA$) for sufficiently large $A > 0$). Obviously, at $t = 0, u = 0$ solves the equation. Openness and uniqueness, follow from theorem 2.1. As before, if $t_i \to t$ is a sequence such that there exist smooth solutions $u_i$ satisfying $\omega + dd^c u_i \to C_0$, then, we shall prove that a subsequence converges to a smooth solution $u$ in the $C^{2,h}$ topology. As usual, we need a priori estimates for this.

The $C^0$ and the mixed derivative estimates follow directly from theorem 2.1. We have to prove the $C^{2,\alpha}$ estimate (thus proving existence and smoothness as before). It suffices to prove a local (interior) estimate. We shall accomplish this via the complex version of the (interior) Evans-Krylov theory done in [13] and [14].

The local (in a ball) version of the equation is

$$\det(\phi_{ij}) + \text{tr}(B^{-1}[\phi_{ij}]) = f$$
$$\phi_{ij} > C > 1$$
$$f > C^3 + 9\|B^{-1}\|^2C$$

(4.2)

where $B^{-1}_{ij} = \det(\alpha)[\alpha]^{-1}_{ij}$. We claim that the function $g(A) = F(\det(A) + \text{tr}(B^{-1}A))$ from hermitian matrices satisfying $A > CID$ to $\mathbb{R}$ (where $F(x) = \int_c^x e^{-\frac{t^2}{2}}dt$) is concave and uniformly elliptic. Let the eigenvalues of $A$ be $\lambda_1, \lambda_2$ and $\lambda_3$. The uniform ellipticity is trivial (as in the proof of theorem 2.2). The concavity is also somewhat similar to theorem 2.2, but requires some modification. Indeed
(here \( V \) is an arbitrary hermitian matrix and \( x = \det(A) + \text{tr}(B^{-1}A) \),
\[
g''(V, V) = g''(x)(\det(A)\text{tr}(A^{-1}V) + \text{tr}(B^{-1}V))^2 \\
+ \ g'(x)(-\det(A)\text{tr}((A^{-1}V)^2) + \det(A)(\text{tr}(A^{-1}V))^2)
\]

We wish to prove that \( g''(V, V) < 0 \) for every hermitian \( V \). Let’s diagonalise the positive-definite form \( B^{-1} \), i.e. \( PB^{-1}P^\dagger = I \) for some matrix \( P \). Define \( \tilde{A} = (P^\dagger)^{-1}AP^{-1} \) and \( \tilde{V} = (P^\dagger)\text{tr}(P^{-1}V)P^{-1} \).

Now, using a unitary matrix \( U \), we may diagonalise \( \tilde{A} \) i.e. \( \tilde{A} = U\tilde{A}U^\dagger = \text{diag}(a_1, a_2, a_3) \) where \( a_1 \leq a_2 \leq a_3 \) and \( \tilde{V} = U\tilde{V}U^\dagger \). This implies that \( \det(\tilde{A}) \det(B) = \det(A) \text{and} \text{tr}(\tilde{A}) = \text{tr}(B^{-1}A) \).

Let \( \tilde{V}_{ii} = v_i \). Hence,
\[
g''(V, V) = -xe^{-\frac{x^2}{2}}(\det(B)a_1a_2a_3(\sum \frac{v_i}{a_i}) + \sum v_i)^2 \\
+ e^{-\frac{x^2}{2}}((\sum \frac{v_i}{a_i})^2 - (\sum \frac{v_i^2}{a_i^2} + 2(\frac{|v_{12}|^2}{a_1a_2} + |v_{23}|^2 + \frac{|v_{13}|^2}{a_1a_3}))) \\
\leq -xe^{-\frac{x^2}{2}}(\sum v_i(\det(B)a_1a_2a_3 + 1))^2 + 2\det(B)e^{-\frac{x^2}{2}}(v_1v_2a_3 + v_2v_3a_1 + v_3v_1a_2) \\
= e^{-\frac{x^2}{2}}(Pv_i^2 + Qv_1 + R)
\]

where,
\[
P = -x(\det(B)a_2a_3 + 1)^2 \leq 0 \\
Q = 2(\det(B)(v_2a_3 + v_3a_2) - x(\det(B)a_2a_3 + 1)(v_2(\det(B)a_1a_3 + 1) + v_3(\det(B)a_1a_2 + 1))) \\
R = 2\det(B)v_2v_3a_1 - x(v_2(\det(B)a_1a_3 + 1) + v_3(\det(B)a_1a_2 + 1))^2
\]
as before, we want \( Q^2 - 4PR < 0 \). Assume (without loss of generality) that \( v_3 = 1 \).
\[
\frac{Q^2 - 4PR}{4} = Jv_2^2 + Kv_2 + L
\]

where,
\[
J = \det(B)a_3(\det(B)a_3 - 2x(\det(B)a_2a_3 + 1)(\det(B)a_1a_3 + 1)) \\
< \det(B)^2a_3(1 - 2a_1a_2\det(B)) \\
< 0 \\
K = 2(a_2a_3(\det(B))^2 - x(\det(B)a_2a_3 + 1)((\det(B))^2a_1a_2a_3 + \det(B)(a_2 + a_3 - a_1))) \\
L = \det(B)a_2(\det(B)a_2 - 2x(\det(B)a_2a_3 + 1)(\det(B)a_1a_2 + 1))
\]

where the first inequality follows from the assumption that \( \det(A) = \det(B)a_1a_2a_3 > 3\sum a_i \). Hence,
\[
\frac{K^2}{4} = (a_2a_3(\det(B))^2 - x(\det(B)(\det(B)a_2a_3 + 1)(\det(B)a_1a_2a_3 + (a_2 + a_3 - a_1))))^2 \\
\leq x^4(\det(B)a_2a_3 + 1)^2(\det(B))^2 \\
J \leq -2x\det(B)a_3^2(\det(B)a_1a_2a_3 + a_1 + a_2) \\
\leq -2x\det(B)a_3^2\frac{2x}{3} \\
K \leq -2x\det(B)a_2^2\frac{2x}{3}
\]

Thus, \( K^2 - 4JL < 0 \) implying that \( g \) is concave. The \( C^{2,\alpha} \) estimate follows from theorem 3.3.
4.4. Proof of theorem 2.4. We use the method of continuity again. As before, openness follows easily using the Implicit function theorem on Banach spaces. Here, we prove only the a priori estimates. Smoothness follows by bootstrapping, as indicated earlier. Lastly, we shall also prove the uniqueness of convex solutions.

$C^0$ estimate: Since $u$ is convex, its maximum is attained on the boundary and hence, $u \leq 0$. Let $\phi = \frac{4}{7}r^2 - \frac{4}{7}$ where $\mu > 0$, and $\mu^3 - 3\mu \geq \max f$; then, subtracting $\det(D^2u) - \Delta u$ from $\det(D^2\phi) - \Delta \phi$, we have (assume that the eigenvalues of $D^2u$ are $\lambda_i$),

$$L(\phi - u) = \det(D^2\phi) - \det(D^2u) - \Delta(\phi - u) = (\mu - \lambda_1)(\frac{\mu^2 + \frac{4}{7}(\lambda_2 + \lambda_3) + \lambda_2\lambda_3}{3}) + (\mu - \lambda_2) \ldots$$

$$= \mu^3 - \mu - f > 0$$

We see that, since $\mu^2 > 3$, hence, $L$ is an elliptic operator acting on $\phi - u$ with $L(\phi - u) > 0$. So, by the maximum principle, $\phi < u$. This gives us a $C^0$ estimate on $u$.

$C^1$ estimate: Follows from ellipticity as before.

$C^2$ estimate: For future use, notice that, at least two of the eigenvalues of $D^2u$ are larger than 1. Taking derivatives of the equation we have (let $u_0$ be the minimum of $u$),

$$\det(D^2u)\text{tr}((D^2u)^{-1}D^2u_i) - \Delta u_i = f_i$$

$$\det(D^2u)\text{tr}((D^2u)^{-1}D^2\Delta u) - \Delta \Delta u = \Delta f + \sum_i \det(D^2u)\text{tr}(((D^2u)^{-1}D^2u_i)^2)$$

$$- \det(D^2u)\sum_i (\text{tr}((D^2u)^{-1}D^2u_i))^2$$

Let $A = \det(D^2u)(D^2u)^{-1} - I$. Consider $g = \Delta u + \mu(u - u_0) > 0$ (we shall choose the constant $\mu > 0$ later. It can depend on $\|u\|_{C^1}$ and other constants). Notice that, if $g$ is bounded, then, so is $\Delta u$ and thus, $D^2u$ is bounded. At the maximum of $g$ (if it occurs in the interior), $\Delta u_i = -\mu u_i$ and $\text{tr}(AD^2g) \leq 0$. This implies,

$$0 \geq \Delta f + \mu\text{tr}(AD^2u) - \frac{|-\mu \nabla u + \nabla f|^2}{\Delta u + f}$$

$$\geq C_1(\mu) - \frac{C_2(\mu)}{\Delta u + f} + (2\Delta u + 3f)\mu$$

Hence, $\Delta u$ is bounded at that point. Thus, $g$ is bounded at that point. This implies that $\Delta u$ is bounded everywhere. If the maximum of $g$ occurs on the boundary (call the max $g_0$), we shall have to analyse it separately. Let $\tilde{g} = g + g_0(1 - 2r^2)$. Clearly, the maximum of $\tilde{g}$ has to occur in the interior. There, $D\tilde{g} = 0$ and $\text{tr}(AD^2\tilde{g}) \leq 0$. Hence (here, we assume that, $\text{tr}(A) = \sum (\lambda_i\lambda_j - 1)$ and, that $g_0$ is sufficiently large compared to constants; If not, we are done),

$$0 \geq -C_1(\mu) + \mu\text{tr}(A)g_0 - C_2g_0 - (C_4g_0 + C_6(\mu))\text{tr}(A) - \frac{\rho}{\Delta u + f}$$

Choosing $\mu > E$, we see that, $g_0$ is bounded. Notice that, this also implies a lower bound on $D^2u$. This is because, $M\lambda_i > \lambda_1\lambda_2\lambda_3 > f$.

$C^{2,\alpha}$ estimate: Notice that, the set $Y$ of positive, symmetric matrices satisfying $\det(A) - \text{tr}(A) > 0$ is a convex open set (lemma 4.16 of [17]). Also, our equation may be written as $-1 = -\frac{\Delta u}{\det(D^2u)} -$
\[ \frac{f}{\det(D^2 u)} = F(D^2 u, x) \] which is certainly concave on \( Y \) by the same lemma in [17]. It is uniformly elliptic on solutions as long as the eigenvalues of the Hessian are bounded below and above (which they are, by the \( C^2 \) estimates). Theorem 3.2 yields the desired estimates.

**Uniqueness**: If \( u_1 \) and \( u_2 \) are two convex solutions of the equation \( F(u) = -1 \) (as above), then, upon subtraction, \( 0 = \int_0^1 \text{tr}\left( (-I + \det(D^2 u_t)(D^2 u_t)^{-1})D^2(u_2 - u_1) \right) dt = L(u_2 - u_1) \) where, \( L \) is elliptic. By the maximum principle, \( u_1 = u_2 \).

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