ROBUST ADAPTIVE SLIDING MODE CONTROL OF MARKOVIAN JUMP SYSTEMS WITH UNCERTAIN MODE-DEPENDENT TIME-VARYING DELAYS AND PARTLY UNKNOWN TRANSITION PROBABILITIES

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ABSTRACT

This paper deals with the problems of stochastic stability and sliding mode control for a class of continuous-time Markovian jump systems with mode-dependent time-varying delays and partly unknown transition probabilities. The design method is general enough to cover a wide spectrum of systems from those with completely known transition probability rates to those with completely unknown transition probability rates. Based on some mode-dependent Lyapunov-Krasovski functionals and making use of the free-connection weighting matrices, new delay-dependent conditions guaranteeing the existence of linear switching surfaces and the stochastic stability of sliding mode dynamics are derived in terms of linear matrix inequalities (LMIs). Then, a sliding mode controller is designed such that the resulted closed-loop system’s trajectories converge to predefined sliding surfaces in a finite time and remain there for all subsequent times. This paper also proposes an adaptive sliding mode controller design method which applies to cases in which mode-dependent time-varying delays are unknown. All the conditions obtained in this paper are in terms of LMI feasibility problems. Numerical examples are given to illustrate the effectiveness of the proposed methods.

Key Words: Linear matrix inequality (LMI), Markovian jump systems (MJSs), Mode-dependent time-varying delay, Partly unknown transition probabilities, Sliding mode control, Stochastic stability

I. Introduction

Markovian jump systems (MJSs), first introduced in [1], are a class of stochastic hybrid systems described by a set of classical differential equations along with a finite state Markov process representing the discrete state or jump. Transition probability rates are statistical values determining the behavior of system’s jumps. The complete knowledge of the transition probabilities simplifies the analysis and control of the MJSs to a large degree. Due to vast applications in various real world problems, including those in the networked control systems, aerospace systems, and
manufacturing systems (see [2][8], etc.), in the past few decades many have devoted their research to the study of Markovian jump systems and several results have been achieved. For example, see [9][12] and references therein. However, as a drawback, these results suffer from the assumption of fully-known transition probability rates.

Despite this common assumption, in most cases, all or parts of the elements in the transition probabilities matrix are not known a priori. The likelihood of a complete measurement regarding transition probabilities in practical cases is quite controversial, and it can also simultaneously be costly or time-consuming. Therefore, rather than gauging or estimating all the elements of transition probabilities matrix, it is a better choice to study more general MJSs with partly unknown transition probabilities. Recently, several interesting results on stability, stabilisation and filtering problems for MJSs with partly unknown transition probabilities have been addressed. For example, we refer readers to [13][17].

Meanwhile, time delays occur frequently in many practical control systems such as biological systems, heating systems and networked control systems. Particularly, time delays are well known as a source of instability and poor performance of a control system [18]. Accordingly, many results related to stability, stabilisation, and filtering of time delay Markovian jump systems have been obtained. See [18][22] and references therein for example. In terms of their stability conditions, these results are mainly classified into two categories: delay-dependent and delay-independent conditions. Applying the information regarding the size of delays, the delay-dependent criteria are considered to be less-conservative than the delay-independent ones, especially when the size of the delay is small. Recently, Markovian jump systems with mode-dependent time delays where the time delays depend on the system modes have been studied, and many topics such as stability, stabilisation and control of such systems have been investigated [23][26]. In this paper, the mode-dependent time-varying delayed Markovian jump system is considered, and new delay-dependent conditions are obtained in terms of less-conservative LMIs.

On the other hand, sliding mode control (SMC) is one of the most important robust control methods for uncertain or nonlinear systems. The main concept of SMC design is to utilize a discontinuous control law to drive the state trajectories of the closed-loop system to the predesigned sliding surface in a finite time and to maintain there for all subsequent times. The sliding surface is designed in advance with desired properties such as stability, regulation, disturbance rejection capability, tracking, etc. During the last decade, sliding mode control for MJSs has attracted a considerable interest. See [27][33] for example. However, in most of the mentioned works, it is assumed that all elements of transition probability rate matrix are known and accessible. This assumption drastically eases up the design process but at the same time sets limits on the generality of results especially in practical applications.

Authors in [34] investigate the problem of sliding mode control for Markovian jump systems with partly unknown transition probabilities, however they do not address the delay problem. Due to the significant effects of delays on the system’s performance, it is essential to consider potential delays in the study of control and stochastic stability. Thus, bringing the mode-dependent time-varying delay in the problem of MJS sliding mode design with partly known probability rates could be considered as one of the main contributions of this paper. As a result of taking the delay into account, the obtained stochastic stability conditions using the LMI framework would become much more complex than those presented in the earlier works in this field, such as in [34].

In most practical situations, the time delay functions are not exactly known though, in some cases, their bounds are available. Therefore, the desirable delay states cannot be employed in the SMC law in these cases. However, in some results in the literature such as [35][36], not only are the bounds of the time delays assumed to be known, but the time delay functions used in control law are also supposed to be known precisely. This is definitely a very restrictive condition. To overcome this problem, here we present a new adaptive sliding mode controller for MJSs with unknown mode-dependent time-varying delays.

Motivated by the above discussion, this paper considers the SMC design for delayed Markovian jump systems with partly unknown transition rates. The stochastic stability of sliding mode dynamics is assured based on a new stochastic Lyapunov-Krasovskii functional combining with Jensen’s inequality and usage of free-connection weighting matrices. The Lyapunov functional includes an upper bound, a lower bound, and a derivative bound of the mode-dependent time-varying delay, so less-conservative delay-dependent conditions are obtained in terms of LMIs, guaranteeing the existence of the desired linear sliding surface and the stochastic stability of sliding mode dynamics. Afterward, with the assistance of a mode-dependent Lyapunov function and free
weighting connection matrices, we design a sliding mode controller to ensure the reachability of closed-loop trajectories to the desired switching surface in a finite time. Finally, we propose a novel adaptive SMC law design method to handle unknown mode-dependent time-varying delays in the system under consideration. This is another major contribution of this paper which is presented in Theorem 3.

The remainder of this paper is organized as follows: Section II gives the problem statement and preliminary information. Section III first considers the problems of stochastic stability of sliding mode dynamics and the design procedure of a desired SMC law to ensure the stochastic stability of closed-loop system. Afterward, it generalizes the results by proposing an adaptive SMC law. Numerical examples and the conclusion are given in sections IV and V respectively.

II. Problem statement and preliminaries

Consider the following stochastic continuous-time Markovian jump system defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$:

$$
\dot{x}(t) = A(r_t)x(t) + A_d(r_t)x(t - \tau(t)) + B(r_t)[u(t) + F(r_t)w(t)]
$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is controller input, $w(t) \in \mathbb{R}^l$ is the disturbance, $\{r_t, t > 0\}$ is the continuous-time Markov process which takes value in a finite state space $\ell = \{1, 2, ..., N\}$ with generator $\lambda_{ij}$,

$$
\Pr(r_{t+h} = j \mid r_t = i) = \begin{cases} 
\lambda_{ij}h + o(h), & \text{if } j \neq i \\
1 + \lambda_{ii}h + o(h), & \text{if } j = i
\end{cases}
$$

(2)

where $\lambda_{ij} \geq 0$ ($i, j \in \ell, j \neq i$) represents the transition rate from mode $i$ at time $t$ to mode $j$ at time $t+h$ with $\lambda_{ii} = -\sum_{j=1, j\neq 1}^{N} \lambda_{ij}$ for each $i \in \ell$, and $h > 0, \lim_{h \to 0} (o(h)/h) = 0$. Besides, the Markov process transition probability rate matrix $A$ is defined by

$$
A = \begin{bmatrix}
\lambda_{11} & \lambda_{12} & \ldots & \lambda_{1N} \\
\lambda_{21} & \lambda_{22} & \ldots & \lambda_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{N1} & \lambda_{N2} & \ldots & \lambda_{NN}
\end{bmatrix}
$$

For convenience, for each possible value $r_t = i, i \in \ell$, we define $A(r_t) = A_i, A_d(r_t) = A_{di}, B(r_t) = B_i$ and $F(r_t) = F_i$. Then, system (1) can be described by

$$
\dot{x}(t) = A_i x(t) + A_{di} x(t - \tau_i(t)) + B_i [u(t) + F_i w(t)]
$$

(3)

where $A_i, A_{di}, B_i$ and $F_i$ are known constant matrices of appropriate dimensions. It is assumed that

$$
\|F_i w(t)\| \leq f_i, \quad i \in \ell
$$

(4)

with $f_i > 0$. Besides, $\tau_i(t)$ denotes mode-dependent time-varying delay (whether known or unknown), satisfying the following conditions:

$$
0 \leq h_1 \leq h_{11} \leq \tau_i(t) \leq h_{21} \leq h_2, \quad \tau_i(t) \leq \mu_i
$$

(5)

where $h_1 = \min_{i \in \ell} h_{11}$ and $h_2 = \max_{i \in \ell} h_{21}$. In this paper, the transition probability rates are considered to be partly unknown, i.e., some elements in matrix $\Lambda$ are unknown (They can be fully known or fully unknown as well). For distinctive notation, we define $\ell = \ell_{K} \cup \ell_{uK}$ by:

$$
\ell_K = \{j : \lambda_{ij} \text{ is known}\}
$$

$$
\ell_{uK} = \{j : \lambda_{ij} \text{ is unknown}\}
$$

(6)

and if $\ell_K \neq \emptyset$, it is also described as

$$
\ell_K = \{\kappa_1, \ldots, \kappa_q\}, \quad 1 \leq q \leq N
$$

(7)

where $\kappa_q \in \mathbb{N}^+$ stands for the $q^{th}$ known element with index $\kappa_q$ in the $i^{th}$ row of matrix $A$. Taking these definitions into account, we study a more general class of Markovian jump systems.

Before proceeding further, we will introduce the following definition and some lemmas which are indispensable in deriving the proposed stability criterion.

**Definition 1** [37] The Markovian jump system $\dot{x}(t) = A(r_t)x(t)$ is said to be stochastically stable (SS) if there exists a finite positive constant $T(x_0, r_0)$ such that the following holds for any initial condition $(x_0, r_0)$:

$$
\mathbb{E} \left\{ \int_0^\infty \|x(t)\|^2 dt \mid x_0, r_0 \right\} < T(x_0, r_0)
$$

**Lemma 1** [37] Let $A, D$ and $F$ be real matrices of appropriate dimensions with $F$ satisfying $F^T F < I$. Then for any scalar $\epsilon > 0$ and vectors $x, y \in \mathbb{R}^n$, the following statement holds:

$$
2x^T AFDy \leq \epsilon^{-1} x^T A A^T x + \epsilon y^T D^T D y.
$$

(8)
Lemma 2 Suppose $\gamma_1 \leq \gamma(t) \leq \gamma_2$, where $\gamma(.) : \mathbb{R}_+ \to \mathbb{R}_+$, and $\mathbb{R}_+$ (or $\mathbb{Z}_+$) $\to \mathbb{R}_+$ (or $\mathbb{Z}_+$). Then, for any constant matrices $\Xi_1, \Xi_2$ and $\Xi$ with proper dimensions, the following matrix inequality
\begin{equation}
\Xi + (\gamma(t) - \gamma_1) \Xi_1 + (\gamma_2 - \gamma(t)) \Xi_2 < 0
\end{equation}
holds, if and only if
\begin{equation}
\Xi + (\gamma_2 - \gamma_1) \Xi_1 < 0,
\Xi + (\gamma_2 - \gamma_1) \Xi_2 < 0.
\end{equation}

III. Main results

In order to obtain a regular form of system (3), we first choose a nonsingular matrix $T_i$ such that following equality holds (27):
\begin{equation}
T_i B_i = \begin{bmatrix} 0_{(n-m) \times m} \\ B_{2i} \end{bmatrix}
\end{equation}
in which $B_{2i} \in \mathbb{R}^{m \times m}$ is nonsingular. For convenience, we partition $T_i$ as follows:
\begin{equation}
T_i = \begin{bmatrix} U_{ti} \\ U_{2ti} \end{bmatrix}
\end{equation}
where $U_{ti} \in \mathbb{R}^{n \times m}$ and $U_{2ti} \in \mathbb{R}^{n \times (n-m)}$ are two sub-blocks of a unitary matrix resulting from the singular value decomposition of $B_i$, that is:
\begin{equation}
B_i = \begin{bmatrix} U_{ti} & U_{2ti} \end{bmatrix} \begin{bmatrix} \Sigma_i \\ 0_{(n-m) \times m} \end{bmatrix} J_i^T
\end{equation}
where $\Sigma_i \in \mathbb{R}^{m \times m}$ is a diagonal positive-definite matrix and $J_i \in \mathbb{R}^{m \times m}$ is a unitary matrix. Then the state transformation $z(t) = T_i x(t)$ is applied to system (3) to derive the following regular form:
\begin{equation}
\dot{z}(t) = \bar{A}_t z(t) + \bar{A}_{di} z(t - \tau(t)) + \begin{bmatrix} 0_{(n-m) \times m} \\ B_{2i} \end{bmatrix} u(t) + F_i w(t)
\end{equation}
in which, $\bar{A}_t = T_i A_t T_i^{-1}$ and $\bar{A}_{di} = T_i A_{di} T_i^{-1}$. System (11) can be written as follows:
\begin{equation}
\dot{z}_1(t) = \bar{A}_{1ti} z_1(t) + \bar{A}_{12ti} z_2(t) + \bar{A}_{d1ti} z_1(t - \tau(t)) + \bar{A}_{d21i} z_2(t - \tau(t))
\end{equation}
\begin{equation}
\dot{z}_2(t) = \bar{A}_{21i} z_1(t) + \bar{A}_{22i} z_2(t) + \bar{A}_{d21i} z_1(t - \tau(t)) + \bar{A}_{d22i} z_2(t - \tau(t)) + B_{2i} u(t) + F_i w(t)
\end{equation}
where $z_i(t) \in \mathbb{R}^{n-m}, z_2(t) \in \mathbb{R}^m$ and other parameters are obtained as follows:
\begin{equation}
\bar{A}_{1ti} = U_{2ti}^T A_t U_{2ti}, \quad \bar{A}_{12ti} = U_{2ti}^T A_{di} U_{2ti}, \quad \bar{A}_{d1ti} = U_{2ti}^T A_t U_{2ti}, \quad \bar{A}_{d21i} = U_{2ti}^T A_{di} U_{2ti},
\end{equation}
\begin{equation}
\bar{A}_{d22i} = U_{ti}^T A_{di} U_{ti}, \quad B_{2i} = \Sigma_i J_i^T.
\end{equation}

Based on sliding mode control theory (19) (40), it is known that (12) denotes the sliding mode dynamics. Therefore, we design the following linear sliding surface:
\begin{equation}
s(t) = \begin{bmatrix} C_{ti} \\ C_{2i} \end{bmatrix} z(t)
\end{equation}
where $C_{2i}$ is invertible for each $i \in \ell$. By defining $C_i = C_{gi}^{-1} C_{ti}$ and substituting $z_2(t) = -C_i z_1(t)$ and $z_2(t - \tau(t)) = -C_i z_1(t - \tau(t))$ to sliding dynamics (12), we have
\begin{equation}
\dot{z}_1(t) = \bar{A}_t z_1(t) + \bar{A}_{di} z_1(t - \tau(t)),
\end{equation}
\begin{equation}
\bar{A}_t = \bar{A}_{1ti} - \bar{A}_{12ti} C_i, \quad \bar{A}_{di} = \bar{A}_{d1ti} - \bar{A}_{d21i} C_i.
\end{equation}
By means of sliding mode control theory, when the state trajectories of the closed-loop system drive onto the sliding surface and maintain there for all subsequent times, we have $s(t) = 0$ and $\dot{s}(t) = 0$. Now, we are in the position to present main results of this paper. In the following, in Theorem 1 we design linear sliding surface parameter $C_i$ for the stochastic stability of sliding mode dynamics (15). Then, in Theorem 2 we construct a desired SMC law $u(t)$ which ensures that state trajectories of the closed-loop system enter the predefined sliding surface in finite time.

Theorem 1 The sliding mode dynamics (15) with mode-dependent time-varying delays $\tau_i(t)$ and partly unknown transition probabilities (6), is stochastically stable if there exist matrices $X_i > 0, \dot{Q}_{ti} > 0, \dot{Q}_{2ti} > 0, \dot{Q}_{3i} > 0, \dot{Q}_t > 0, \dot{Q}_2 > 0, \dot{Q}_3 > 0, \dot{R}_t > 0, \dot{R}_2 > 0, \dot{V}_t = V_t^T, \dot{W}_{ri} = W_{ri}^T$ with $r = 1, 2, 3, M_i, \dot{N}_i, \dot{S}_i$, and $Y_i$ such that the sets of LMIs (16)-(28) hold for each $i \in \ell$.

\begin{equation}
\begin{bmatrix} -V_i & X_i \\ X_i & -X_j \end{bmatrix} \leq 0, \quad i \in \ell_i^K, \quad j \in \ell_i^L
\end{equation}
\begin{equation}
X_j - V_i \geq 0, \quad i \in \ell_i^L, \quad j = i
\end{equation}
\[
\hat{\Delta}_{1i} = \begin{bmatrix}
\dot{\theta}_{bi} + \dot{\phi}_i + \dot{\phi}_i^T & h_2 \dot{A}_{im}^T & h_21 \dot{A}_{im}^T & h_1 \dot{M}_i & h_21 \dot{N}_i & \Gamma_i(X_i) \\
* & -h_2 \ddot{R}_1 & 0 & 0 & 0 & 0 \\
* & 0 & -h_2 \ddot{R}_2 & 0 & 0 & 0 \\
* & * & * & h_1(\ddot{R}_1 - 2X_1) & 0 & 0 \\
* & * & * & 0 & h_21(\ddot{R}_1 + \ddot{R}_2 - 4X_i) & -\Xi_i(X_i) \\
\end{bmatrix} < 0 \quad (16)
\]

\[
\hat{\Delta}_{2i} = \begin{bmatrix}
\dot{\theta}_{bi} + \dot{\phi}_i + \dot{\phi}_i^T & h_2 \dot{A}_{im}^T & h_21 \dot{A}_{im}^T & h_2 \dot{M}_i & h_21 \dot{S}_i & \Gamma_i(X_i) \\
* & -h_2 \ddot{R}_1 & 0 & 0 & 0 & 0 \\
* & 0 & -h_2 \ddot{R}_2 & 0 & 0 & 0 \\
* & * & * & h_2(\ddot{R}_1 - 2X_1) & 0 & 0 \\
* & * & * & 0 & h_21(\ddot{R}_2 - 2X_i) & -\Xi_i(X_i) \\
\end{bmatrix} < 0 \quad (17)
\]

where for \(j \in \ell^i K, i \in \ell^i K \) (\(b = 1\))

\[
\begin{bmatrix}
\lambda_{ij} \dot{Q}_{ij} - 2X_j + \alpha \dot{Q}_i & \lambda_{ij} \dot{X}_j \\
\lambda_{ij} \dot{X}_j & \lambda_{ij} \dot{W}_{ij} \\
\end{bmatrix} < 0, j \in \ell^i K, \ j \neq i \quad (20)
\]

\[
\dot{Q}_{ij} - 2X_j + \dot{W}_{ij} \leq 0, \ j \in \ell^i K, \ j \neq i \quad (21)
\]

\[
\dot{Q}_{ii} - 2X_i + \dot{W}_{ii} \geq 0, \ j = i \quad (22)
\]

and for \(j \in \ell^i K, i \in \ell^i K \) (\(b = 2\))

\[
\begin{bmatrix}
\lambda_{ij} \dot{Q}_{2j} - 2X_j + \alpha \dot{Q}_2 & \lambda_{ij} \dot{X}_j \\
\lambda_{ij} \dot{X}_j & \lambda_{ij} \dot{W}_{2j} \\
\end{bmatrix} < 0, j \in \ell^i K, \ j \neq i \quad (23)
\]

\[
\dot{Q}_{2j} - 2X_j + \dot{W}_{2j} \leq 0, \ j \in \ell^i K, \ j \neq i \quad (24)
\]

\[
\dot{Q}_{2i} - 2X_i + \dot{W}_{2i} \geq 0, \ j = i \quad (25)
\]

with

\[
\begin{align*}
\dot{\phi}_i &= [\dot{\hat{M}}_i - \hat{M}_i + \hat{N}_i - \hat{S}_i - \hat{\bar{S}}_i - \hat{\bar{N}}_i] \quad (31) \\
\hat{M}_i &= \begin{bmatrix} \hat{M}_{1i}^T & \hat{M}_{2i}^T & \hat{M}_{3i}^T & \hat{M}_{4i}^T \end{bmatrix}^T, \quad (32) \\
\hat{N}_i &= \begin{bmatrix} \hat{N}_{1i}^T & \hat{N}_{2i}^T & \hat{N}_{3i}^T & \hat{N}_{4i}^T \end{bmatrix}^T, \quad (33) \\
\hat{S}_i &= \begin{bmatrix} \hat{S}_{1i}^T & \hat{S}_{2i}^T & \hat{S}_{3i}^T & \hat{S}_{4i}^T \end{bmatrix}^T, \quad (34)
\end{align*}
\]
\[
\begin{align*}
\hat{A}_{im} &= \begin{bmatrix} A_{i11} X_i - \bar{A}_{i20} Y_i \end{bmatrix} - \begin{bmatrix} A_{i11} X_i - \bar{A}_{i20} Y_i \end{bmatrix} = 0 \quad (35) \\
\Gamma_i(X_i) &= \begin{bmatrix} h_1 X_i & h_2 X_i & h_2 X_i & \hat{f}_i(X_i) \end{bmatrix}, \quad (36) \\
\Xi_i(X_i) &= \text{diag} \left\{ h_1 Q_1, h_2 Q_2, h_2 Q_3, \hat{\Xi}_i(X_i) \right\}, \quad (37) \\
\hat{f}_i(X_i) &= \begin{bmatrix} \sqrt{\lambda_{1,i}} X_i, \ldots, \sqrt{\lambda_{i,i}} X_i, \\
\sqrt{\lambda_{1,i+1}} X_i, \ldots, \sqrt{\lambda_{i,i}} X_i \end{bmatrix}, \\
\hat{\Xi}_i(X_i) &= \text{diag} \left\{ X_{K1}^i, \ldots, X_{K1-i}^i, X_{K1+1}^i, \ldots, X_{K1}^i \right\} 
\end{align*}
\]

in which \( \alpha \) represents the number of known elements of transition probability matrix for \( j \neq i \), and \( \lambda_1, \ldots, \lambda_i \) are defined in (7). Moreover, if the LMIs (16) have a feasibility solution in terms of \( X_i \) and \( Y_i \), the parameter \( C_i \) can be computed by

\[
C_i = Y_i X_i^{-1} \quad (38)
\]

**Proof:** Choose the stochastic Lyapunov-Krasovskii functional candidate as follows:

\[
V(z_i(t), i) = V_1(z_i(t), i) + V_2(z_i(t), i) + V_3(z_i(t), i) + V_4(z_i(t), i)
\]

with

\[
V_1(z_i(t), i) = z_i^T(t) P_i z_i(t), \\
V_2(z_i(t), i) = \int_{t-h_1}^{t} z_i^T(s) Q_{i1} z_i(s) ds + \int_{t-h_1}^{t} z_i^T(s) Q_{2i} z_i(s) ds + \int_{t-h_1}^{t} \int_{t-\tau_i(t)}^{t} z_i^T(s) Q_{3i} z_i(s) ds ds, \\
V_3(z_i(t), i) = \int_{t-h_2}^{t} z_i^T(s) Q_{1i} z_i(s) ds + \int_{t-h_2}^{t} \int_{t-\tau_i(t)}^{t} z_i^T(s) Q_{2i} z_i(s) ds ds + \int_{t-h_2}^{t} \int_{t-\tau_i(t)}^{t} z_i^T(s) Q_{3i} z_i(s) ds ds, \\
V_4(z_i(t), i) = \int_{t-h_2}^{t} \int_{t-\tau_i(t)}^{t} z_i^T(s) R_1 z_i(s) ds ds + \int_{t-h_2}^{t} \int_{t-\tau_i(t)}^{t} z_i^T(s) R_2 z_i(s) ds ds
\]

By using the weak infinitesimal operator of the Lyapunov function \( \mathcal{L}V(z_i(t), i) \) [41], we have:

\[
\mathcal{L}V_1(z_i(t), i) = z_i^T(t) \left\{ \hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j \in \mathcal{L}} \lambda_{ij} P_j \right\} z_i(t) + 2 z_i^T(t) P_i \bar{A}_{i2i} z_i(t - \tau_i(t))
\]

\[
\mathcal{L}V_2(z_i(t), i) = z_i^T(t) \left[ Q_{1i} + Q_{2i} \right] z_i(t) + z_i^T(t - h_1)(Q_{2i} - Q_{1i}) z_i(t - h_1) + \int_{t-h_1}^{t} z_i^T(s) (\lambda_{ij} Q_{ij}) z_i(s) ds + \int_{t-h_1}^{t} z_i^T(s) (\lambda_{ij} Q_{ij}) z_i(s) ds + \int_{t-h_1}^{t} z_i^T(s) (\lambda_{ij} Q_{ij}) z_i(s) ds + \int_{t-h_1}^{t} z_i^T(s) (\lambda_{ij} Q_{ij}) z_i(s) ds
\]

\[
\mathcal{L}V_3(z_i(t), i) = z_i^T(t) \left( h_1 Q_1 + h_2 Q_2 + h_2 Q_3 \right) z_i(t) - \int_{t-h_1}^{t} z_i^T(s) Q_1 z_i(s) ds - \int_{t-h_1}^{t} z_i^T(s) Q_2 z_i(s) ds - \int_{t-h_1}^{t} z_i^T(s) Q_3 z_i(s) ds
\]

\[
\mathcal{L}V_4(z_i(t), i) = z_i^T(t) \left( h_1 Q_1 + h_2 Q_2 + h_2 Q_3 \right) z_i(t) - \int_{t-h_2}^{t} z_i^T(s) Q_1 z_i(s) ds - \int_{t-h_2}^{t} z_i^T(s) Q_2 z_i(s) ds - \int_{t-h_2}^{t} z_i^T(s) Q_3 z_i(s) ds
\]
where $h_{21} = h_2 - h_1$. By using Newton-Leibniz formula, for any matrices $M_i$, $N_i$ and $S_i$, we have:

$$
2\zeta_i^T(t) M_i \left[ z_i(t) - z_i(t - \tau_i(t)) \right] - \int_{t-\tau_i(t)}^t \dot{z}_i(s) ds = 0
$$

(44)

$$
2\zeta_i^T(t) N_i \left[ z_i(t - \tau_i(t)) - z_i(t - h_2) \right] - \int_{t-h_2}^{t-\tau_i(t)} \dot{z}_i(s) ds = 0
$$

(45)

$$
2\zeta_i^T(t) S_i \left[ z_i(t - h_1) - z_i(t - \tau_i(t)) \right] - \int_{t-h_1}^{t-\tau_i(t)} \dot{z}_i(s) ds = 0
$$

(46)

where

$$
\zeta_i(t) = \left[ z_i^T(t) z_i^T(t - \tau_i(t)) z_i^T(t - h_1) z_i^T(t - h_2) \right]^T
$$

Then, by adding left sides of (44), (45) and (46) to (43) and using Jensen’s inequality [42] and lemma 1 we have:

$$
\mathcal{L}V_i(z_i(t), i) \leq \left \langle \tilde{A}_i z_i(t) + \tilde{A}_d z_i(t - \tau_i(t)) \right \rangle^T \cdot
\begin{bmatrix}
(h_2 R_1 + h_{21} R_2) \\
\tilde{A}_i z_i(t) + \tilde{A}_d z_i(t - \tau_i(t)) \\
(\tau_i(t) - h_1) S_i R_2^{-1} S_i \\
[M_i - M_i 0 0] \\
[M_i - M_i 0 0] \\
[0 N_i 0 - N_i] \\
[0 N_i 0 - N_i] \\
[0 - S_i S_i 0] \\
[0 - S_i S_i 0]
\end{bmatrix} \zeta_i(t)
$$

(47)

Finally,

$$
\mathcal{L}V(z_i(t), i) = \mathcal{L}V_i(z_i(t), i) + \mathcal{L}V_2(z_i(t), i) + \mathcal{L}V_3(z_i(t), i) + \mathcal{L}V_4(z_i(t), i)
$$

(48)

If following conditions in (41) and (42) holds,

$$
\sum_{j \in \ell} \lambda_{ij} Q_{ij} \leq Q_1
$$

(49)

$$
\sum_{j \in \ell} \lambda_{ij} Q_{2ij} \leq Q_2
$$

(50)

$$
\sum_{j \in \ell} \lambda_{ij} Q_{3ij} \leq Q_3
$$

(51)

Then, we can rewrite $\mathcal{L}V(z_i(t), i)$ as follows:

$$
z_i^T(t) \left \langle \tilde{A}_i^T P_i + P_i \tilde{A}_i + \sum_{j \in \ell} \lambda_{ij} P_j + Q_{4i} \right \rangle z_i(t) + \mathcal{L}V_4(z_i(t), i)
$$

(52)
Now, by Substituting $\bar{A}_i$ and $\tilde{A}_{di}$ from (15) into (52), we have:

$$
\mathcal{L}V(z_i(t), i) \leq \zeta_i^T(t) \left\{ \theta_i + \phi_i + \phi_i^T h_2 R_i A_{im} + A_{im}^T h_2 R_i A_{im} + \tau_i(t) M_i R_i^{-1} M_i^T + (h_2 - \tau_i(t)) N_i (R_1 + R_2)^{-1} N_i^T + (\tau_i(t) - h_1) S_i R_2^{-1} S_i^T \right\} \zeta_i(t)
$$

with

$$
\theta_i = \begin{bmatrix}
\theta_{1ii} & \theta_{12i} \\
\ast & -(1 - \mu_i) Q_{2i} \\
0 & Q_{2i} - Q_{1i} \\
0 & 0
\end{bmatrix},
$$

$$
\theta_{12i} = (\bar{A}_{1ii} - \bar{A}_{12i} C_i)^T P_i + P_i (\bar{A}_{1ii} - \bar{A}_{12i} C_i) + \sum_{j \in \ell} \lambda_{j} P_j + Q_{1i} + Q_{3i} + h_1 Q_1 + h_2 Q_2 + h_2 Q_3,
$$

$$
\phi_i = [M_i - M_i + N_i - S_i, S_i - N_i],
$$

$$
A_{im} = [\tilde{A}_i \tilde{A}_{di} 0 0] = [(\bar{A}_{1ii} - \bar{A}_{12i} C_i) (\bar{A}_{d1ii} - \bar{A}_{d12i} C_i) 0 0]
$$

Obviously, from lemma [2] and Schur complement, we can see $\mathcal{L}V(z_i(t), i) < 0$ if following conditions hold:

$$
\Delta_{1i} = \begin{bmatrix}
\theta_i + \phi_i + \phi_i^T h_2 A_{im}^T h_21 A_{im}^T \\
\ast \ast \ast \ast
\end{bmatrix}
\begin{bmatrix}
\ast & h_21 A_{im}^T \\
\ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
$$

$$
\Delta_{2i} = \begin{bmatrix}
\theta_i + \phi_i + \phi_i^T h_2 A_{im}^T h_21 A_{im}^T \\
\ast \ast \ast \ast
\end{bmatrix}
\begin{bmatrix}
\ast & h_21 A_{im}^T \\
\ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
$$

By pre- and post-multiplying both sides of $\Delta_{1i}$ and $\Delta_{2i}$ with $\bar{X}_i = \text{diag} \{ \bar{X}_i, I, I, X_i, X_i \}$, respectively, where $\bar{X}_i = \text{diag} \{ X_i, X_i, X_i, X_i \}$ and $X_i = P_i^{-1}$, we can define:

$$
Y_i = C_i X_i, \quad \bar{R}_i = R_i^{-1}, \quad \bar{R}_2 = R_2^{-1},
$$

$$
Q_{1i} = X_i Q_{1i} X_i, \quad Q_{2i} = X_i Q_{2i} X_i, \quad Q_{3i} = X_i Q_{3i} X_i,
$$

$$
\bar{M}_i = X_i M_i X_i, \quad \bar{N}_i = X_i N_i X_i, \quad \bar{S}_i = X_i S_i X_i,
$$

$$
\phi_i = [\bar{M}_i - \bar{M}_i + \bar{N}_i - \bar{S}_i, \bar{S}_i - \bar{N}_i],
$$

$$
\tilde{A}_{im} = A_{im} X_i = \begin{bmatrix}
\tilde{A}_{11i} X_i - \bar{A}_{12i} Y_i \\
\tilde{A}_{d11i} X_i - \bar{A}_{d12i} Y_i
\end{bmatrix} 0 0
$$

Then, we have:

$$
\bar{\Delta}_{1i} = \begin{bmatrix}
\tilde{\theta}_i + \tilde{\phi}_i + \tilde{\phi}_i^T h_2 \tilde{A}_{im}^T h_21 \tilde{A}_{im}^T \\
\ast \ast \ast \ast
\end{bmatrix}
\begin{bmatrix}
\ast & h_21 \tilde{A}_{im}^T \\
\ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
$$

$$
\Delta_{2i} = \begin{bmatrix}
\theta_i + \phi_i + \phi_i^T h_2 A_{im}^T h_21 A_{im}^T \\
\ast \ast \ast \ast
\end{bmatrix}
\begin{bmatrix}
\ast & h_21 A_{im}^T \\
\ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
\begin{bmatrix}
\ast & \ast & \ast
\end{bmatrix}
$$

$$
\bar{\Delta}_{1i} < 0 \quad (55)
$$

$$
\Delta_{2i} < 0 \quad (56)
$$

$$
\bar{\Delta}_{2i} < 0 \quad (57)
$$
where transition probability rates is obtained. Matrices instead of fixed ones, less-conservative \( \tilde{\theta}_{1i} = \begin{bmatrix} \tilde{\theta}_{11i} & \tilde{\theta}_{12i} \\ \ast & -(1 - \mu_1) \tilde{Q}_{2i} \\ 0 & 0 \\ 0 & \hat{Q}_{2i} - \tilde{Q}_{1i} \\ 0 & 0 & -\hat{Q}_{2i} \end{bmatrix} \) (58)

\[
\tilde{\theta}_{1ii} = (\tilde{A}_{11i} X_i - \tilde{A}_{12i} Y_i) + (\tilde{A}_{11i} X_i - \tilde{A}_{12i} Y_i)^T \\
+ X_i \sum_{j \in \ell} \lambda_{ij} X_j^{-1} X_i + \hat{Q}_{1i} + \hat{Q}_{3i} \\
+ h_1 X_i Q_1 X_i + h_{21} X_i Q_2 X_i + h_2 X_i Q_3 X_i,
\]

\[
\tilde{\theta}_{12i} = (\tilde{A}_{d11i} X_i - \tilde{A}_{d12i} Y_i)
\]

Since \( \sum_{j \in \ell} \lambda_{ij} = 0 \), we can rewrite \( \tilde{\theta}_{1ii} \) as follows:

\[
\tilde{\theta}_{1ii} = (\tilde{A}_{11i} X_i - \tilde{A}_{12i} Y_i) + (\tilde{A}_{11i} X_i - \tilde{A}_{12i} Y_i)^T \\
+ \hat{Q}_{1i} + \hat{Q}_{3i} + \sum_{j \in \ell} \lambda_{ij} (X_i X_j^{-1} X_i) \\
- \sum_{j \in \ell} \lambda_{ij} V_i + h_1 X_i Q_1 X_i \\
+ h_{21} X_i Q_2 X_i + h_2 X_i Q_3 X_i,
\]

where \( V_i = V_i^T \) are free-connection weighting matrices. In order to solve the problem of MJSs with partly known transition probability rates, we separate the known and unknown elements of transition probabilities matrix by using (6):

\[
\tilde{\theta}_{1ii} = (\tilde{A}_{11i} X_i - \tilde{A}_{12i} Y_i) + (\tilde{A}_{11i} X_i - \tilde{A}_{12i} Y_i)^T \\
+ \hat{Q}_{1i} + \hat{Q}_{3i} + \sum_{j \in \ell_k} \lambda_{ij} (X_i X_j^{-1} X_i - V_i) \\
+ \sum_{j \in \ell_k} \lambda_{ij} (X_i X_j^{-1} X_i - V_i) \\
+ h_1 X_i Q_1 X_i + h_{21} X_i Q_2 X_i + h_2 X_i Q_3 X_i,
\]

In fact, with usage of free-connection weighting matrices instead of fixed ones, less-conservative stability criterion for MJSs with partly known transition probability rates is obtained.

Note that, for \( Q > 0 \), the following matrix inequality always holds:

\[
(X_i - Q^{-1}) Q (X_i - Q^{-1}) = X_i Q X_i - 2 X_i + Q^{-1} \geq 0
\]

and so we have:

\[
- X_i Q X_i \leq -2 X_i + Q^{-1}
\]

which is a useful tool for the development of our results. Now, by using Schur complement, it is obvious that \( \Delta_{1i} \) (16) and \( \Delta_{2i} \) (17) in Theorem 1 are obtained from \( \Delta_{1i} \) and \( \Delta_{2i} \), for each \( i \in \ell_k (b = 1) \), and \( i \in \ell_{uk} (b = 2) \) in the LMI framework. In which, \( \tilde{\theta}_{1i} \) for each \( i \in \ell_k (b = 1) \), and \( i \in \ell_{uk} (b = 2) \) are expressed in Theorem 1.

Besides, we consider unknown elements of probabilities matrix in (60) as follows:

\[
\sum_{j \in \ell_{uk}} \lambda_{ij} (X_i X_j^{-1} X_i - V_i) \leq 0
\]

Since \( \lambda_{ij} \geq 0 \) \((i, j \in \ell, j \neq i)\) and \( \lambda_{ii} < 0 \) \((\lambda_{ii} = -\sum_{j=1, j \neq i}^N \lambda_{ij})\), it is straightforward that (63) holds if the sets of LMIs (18) and (19) in Theorem 1 satisfy for \( i \in \ell_k \) and \( i \in \ell_{uk} \), respectively.

Furthermore, in order to consider unknown elements of probabilities matrix in (49–51), by using free connection weighting matrices \( W_{ri} = W_{ri}^T \) with \( r = 1, 2, 3 \), and separating the known and unknown elements of transition probabilities matrix by (6), we have:

\[
\sum_{j \in \ell_k} \lambda_{ij} (Q_{ij} - W_{ji}) + \sum_{j \in \ell_{uk}} \lambda_{ij} (Q_{ij} - W_{ji}) \leq Q_i
\]

\[
\sum_{j \in \ell_k} \lambda_{ij} (Q_{2ij} - W_{2ji}) + \sum_{j \in \ell_{uk}} \lambda_{ij} (Q_{2ij} - W_{2ji}) \leq Q_2
\]

\[
\sum_{j \in \ell_k} \lambda_{ij} (Q_{3ij} - W_{3ji}) + \sum_{j \in \ell_{uk}} \lambda_{ij} (Q_{3ij} - W_{3ji}) \leq Q_3
\]

At first we consider (64) and rewrite it as follows:

\[
\lambda_{ii} (Q_{ii} - W_{ii}) \\
+ \sum_{j \in \ell_{k,j \neq i}} \left( \lambda_{ij} (Q_{ij} - W_{ji}) - \frac{1}{\alpha} Q_i \right) \\
+ \sum_{j \in \ell_{uk,j \neq i}} \lambda_{ij} (Q_{ij} - W_{ji}) \leq 0
\]

where \( \alpha \) is the number of known elements of transition probabilities matrix for \( j \neq i \). Then, it is straightforward that (67) holds if the following set of LMIs satisfies

\[
\sum_{j \in \ell_{k,j \neq i}} \left( \lambda_{ij} (Q_{ij} - W_{ji}) - \frac{1}{\alpha} Q_i \right) < 0
\]

and

\[
\sum_{j \in \ell_{uk,j \neq i}} \lambda_{ij} (Q_{ij} - W_{ji}) \leq 0
\]
and
\[
\lambda_{ii} (Q_{ii} - W_{ii}) \leq 0
\] (70)

In (68), following LMI holds for each \( j \in \ell^i_X, j \neq i \):
\[
\lambda_{ij} (Q_{ij} - W_{ij} - \frac{1}{\alpha} Q_i) < 0
\] (71)

Pre- and post-multiplying both sides of (71) with \( X_j \), and using Schur complement and the inequality in (62) yields:
\[
\begin{bmatrix}
\lambda_{ij} \dot{Q}_{ij} - 2X_j + \alpha \dot{Q}_i & \lambda_{ij} X_j \\
\lambda_{ij} X_j & \lambda_{ij} \ddot{W}_{ii}
\end{bmatrix} < 0
\] (72)

In (69), the following LMI holds for each \( j \in \ell^i_uK, j \neq i \):
\[
(Q_{ij} - W_{ij}) \leq 0
\] (73)

By pre- and post-multiplying both sides of (73) with \( X_j \), and using the inequality in (62), we have:
\[
\dot{Q}_{ij} - 2X_j + \ddot{W}_{ii} \leq 0
\] (74)

Since \( \lambda_{ii} < 0 \), in (70), the following LMI holds for \( j = i \):
\[
Q_{ii} - W_{ii} \geq 0
\] (75)

Pre- and post-multiplying both sides of (75) with \( X_i \), and considering the inequality in (62) yields:
\[
\dot{Q}_{ii} - 2X_i + \ddot{W}_{ii} \geq 0
\] (76)

where \( \dot{Q}_i = Q_i^{-1}, \ddot{W}_{ii} = W_{ii}^{-1}, \dot{X}_i = X_i \dot{Q}_i \).

The same above calculation is applied to (65) and (66), respectively. Therefore, by considering (29)-(37), if the set of LMIs (16)-(28) is satisfied; i.e., \( C_i = Y_i X_i^{-1} \), then \( \dot{V}(x_i(t), i) < 0 \) and stochastic stability of sliding mode dynamics (15) is verified. This concludes the proof. \( \square \)

**Remark 1** In the Theorem 7, the sets of LMIs (16)-(28) guaranteeing stochastic stability of sliding mode dynamics (15) are delay-dependent. In other words, the obtained conditions include the information on the size of delay and its derivative; i.e., \( h_1, h_2 \) and \( \mu_i \). Note that by using these bounds, less-conservative stability criteria is obtained. To the best of our knowledge, in most of the studies such as [48][49][77], derivative of mode-dependent time-varying delay is assumed to be less than one. In this paper, we use new techniques to remove the derivative restriction and to obtain less-conservative criteria. To reach this purpose, we construct a novel stochastic Lyapunov-Krasovskii functional candidate including the information of delay size. Therefore, in our design method, the derivative of the mode-dependent time-varying delay may be larger than one; i.e., in this paper, the mode-dependent time-varying delay is more general.

**Remark 2** In the proof process, we encounter some nonlinear terms in the form of \( X_i Q X_i \) with \( Q > 0 \) that makes it hard to express stochastic stability criterion in terms of LMIs. In order to solve this problem, some assumptions have been presented by other researchers such as in [48][49] which may not be satisfied. In fact, the authors in [48][49] defined new variables such as \( R_i = X_i Q X_i \) or \( R = X_i Q X_i \), where \( X_i \) and \( R_i \) or \( R \) are design parameters. However, these equalities are impossible to fulfill for each \( i \in \ell \) because of the mode-dependent matrix \( Q \). In detail, finding a constant \( Q \) for all obtained design parameters \( X_i \) and \( R_i \) or \( R \) such that \( Q = X_i^{-T} R_i X_i^{-1} \) or \( Q = X_i^{-T} R X_i^{-1} \), for each \( i \in \ell \) are generally impossible [50]. In this paper, we use a new approach to deal with this constraint and finally derive stochastic stability conditions in terms of LMIs.

In the following theorem, a sliding mode controller \( u(t) \) is synthesized to guarantee the reachability of sliding surface \( s(t) = 0 \) for each \( i \) for Markovian jump systems with mode-dependent time-varying delays and partly unknown transition probability rates.

**Theorem 2** Consider the Markovian jump system (11) with mode-dependent time-varying delays \( \tau_i(t) \) and partly unknown transition probabilities (6). Suppose that the linear sliding surface is given by (14) and \( C_i \) is obtained in Theorem 7 and there exist matrices \( \Omega_i > 0 \) and \( \tilde{V}_i = \hat{V}_i^T \) such that the following sets of LMIs hold for each \( i \in \ell \):
\[
\Omega_j - \tilde{V}_i \leq 0, \quad j \in \ell^i\mu, j \neq i
\] (77)
\[
\Omega_j - \tilde{V}_i \geq 0, \quad j \in \ell^i\mu, j = i
\] (78)

Then, the state trajectories of system (11) can be reached the sliding surface \( s(t) = 0 \) in the finite time by the following SMC law:
\[
u(t) = - (C_{2i} B_{2i})^{-1} \left\{ \begin{bmatrix} C_{1i} & C_{2i} \end{bmatrix} \cdot (\tilde{A}_i z(t) + \tilde{A}_{di} z(t - \tau_i(t))) \right. \\
- (\epsilon_i + f_i) \text{sign} (B_{2i}^T C_{2i}^T \Omega_i s(t)) \\
- \frac{1}{2} (\Omega_i C_{2i} B_{2i})^{-1} \sum_{j \in \ell^i\mu} \lambda_{ij} (\Omega_j - \tilde{V}_j) s(t)
\] (79)

where \( \epsilon_i > 0 \) is a given small constant.
Proof: Choose the appropriate mode-dependent Lyapunov function candidate as

\[ V(z, t, i) = \frac{1}{2} s^T(t) \Omega_i s(t) \]  

According to (11) and (14), we have

\[ \dot{s}(t) = \left[ \begin{array}{cc} C_{t_i} & C_{2i} \end{array} \right] \left( \dot{A}_i z(t) + \dot{A}_{di} z(t - \tau_i(t)) \right) 
+ \left[ \begin{array}{c} 0 \\ B_{2i} \end{array} \right] \left[ \begin{array}{c} u(t) + F_i w(t) \end{array} \right] \]  

Applying the weak infinitesimal operator of the Lyapunov function \( \mathcal{L} V(z, t, i) \) and using (81), yields

\[ \mathcal{L} V(z, t, i) = s^T(t) \Omega_i 
\times \left\{ \left[ \begin{array}{cc} C_{t_i} & C_{2i} \end{array} \right] \left( \dot{A}_i z(t) + \dot{A}_{di} z(t - \tau_i(t)) \right) \right\} 
+ s^T(t) \Omega_i C_{2i} B_{2i} \left[ u(t) + F_i w(t) \right] 
+ \frac{1}{2} \sum_{j \in \ell} \lambda_{ij} s^T(t) \Omega_j s(t) \]  

From \( \sum_{j \in \ell} \lambda_{ij} = 0 \), it follows that following equation holds for arbitrary matrices \( \hat{V}_i = \hat{V}_i^T \)

\[ -\frac{1}{2} \sum_{j \in \ell} \lambda_{ij} s^T(t) \hat{V}_i s(t) = 0, \quad i \in \ell \]  

Adding the left side of (83) into (82) and separating the known and unknown elements of transition probabilities matrix by (6), yields:

\[ \mathcal{L} V(z, t, i) = s^T(t) \Omega_i \left\{ \left[ \begin{array}{cc} C_{t_i} & C_{2i} \end{array} \right] \right\} \left( \dot{A}_i z(t) + \dot{A}_{di} z(t - \tau_i(t)) \right) 
+ s^T(t) \Omega_i C_{2i} B_{2i} \left[ u(t) + F_i w(t) \right] 
+ \frac{1}{2} \sum_{j \in \ell} \lambda_{ij} s^T(t) \left( \Omega_j - \hat{V}_i \right) s(t) 
+ \frac{1}{2} \sum_{j \in \ell_{u\kappa}} \lambda_{ij} s^T(t) \left( \Omega_j - \hat{V}_i \right) s(t) \]  

By substituting SMC law (79) into (84), we have

\[ \mathcal{L} V(z, t, i) = s^T(t) \Omega_i C_{2i} B_{2i} \left\{ - (\epsilon_i + f_i) \cdot \right. 
\left. \text{sign} \left( B_{2i}^T C_{2i}^T \Omega_i s(t) + F_i w(t) \right) \right\} 
+ \frac{1}{2} \sum_{j \in \ell_{u\kappa}} \lambda_{ij} s^T(t) \left( \Omega_j - \hat{V}_i \right) s(t) \]  

Note that if the sets of LMIs (77) and (78) hold for \( i \in \ell_{\kappa} \) and \( i \in \ell_{u\kappa} \), then the following inequalities hold.

\[ \frac{1}{2} \sum_{j \in \ell_{u\kappa}} \lambda_{ij} s^T(t) \left( \Omega_j - \hat{V}_i \right) s(t) < 0 \]  

Thus, from (77) and (78), we have

\[ \mathcal{L} V(z, t, i) \leq - (\epsilon_i + f_i) \left\| B_{2i}^T C_{2i}^T \Omega_i s(t) \right\| \] 
\[ + f_i \left\| B_{2i}^T C_{2i}^T \Omega_i s(t) \right\| \] 
\[ \leq - \epsilon_i \left\| B_{2i}^T C_{2i}^T \Omega_i s(t) \right\| < 0 \]  

Note that

\[ \left\| \Omega_i s(t) \right\|^2 = \left( \Omega_i \hat{V}_i^T s(t) \right)^T \Omega_i \left( \Omega_i \hat{V}_i^T s(t) \right) \geq \lambda_{\min}(\Omega_i) \left\| \Omega_i \hat{V}_i^T s(t) \right\|^2 \]  

and so we have

\[ \mathcal{L} V(z, t, i) \leq - \varrho_i V^{\frac{1}{2}}(z, t, i) \]  

\[ \varrho_i = \sqrt{2} \epsilon_i \min_{i \in \ell} (\lambda_{\min}(\Omega_i))^\frac{1}{2} \times \] 
\[ \min_{i \in \ell} \left\{ \left( \lambda_{\min}(\Omega_i) \right)^{\frac{1}{2}} \right\} \]  

where \( \varrho_i > 0 \). Using Dynkin’s formula (51), this yields

\[ 2E[V(z(t), r(t))]^{\frac{1}{2}} \leq - \varrho_i t + 2V^{\frac{1}{2}}(z(0), r(0)) \]  

Thus, there exists an instant \( t^* = 2V^{\frac{1}{2}}(z_0, r_0)/\varrho_i \) such that \( V(z, t, i) = 0 \), and consequently \( s(t) = 0 \), for \( t > t^* \). Therefore, by applying the sliding control law (79), the state trajectories of closed-loop system can enter the desired sliding surface (14) in finite time. This completes the proof.

**Remark 3** Note that in some practical situations, the time delay functions \( \tau_i(t), i = 1, 2, \ldots, N \) are not explicitly known a priori, and consequently, the desired delay states \( z(t - \tau_i(t)) \) cannot be employed in the control law (79) in these cases. To cope with this kind
of practical problems, this paper also proposes another sliding mode controller in Theorem 3 for considered systems with unknown mode-dependent time-varying delays \(\tau_i(t)\).

Before proceeding, we give the following assumption:

**Assumption 1** According to the Razumikhin Theorem [52], there exists a constant \(r > 0\) such that the following inequality holds:

\[
\|z(t + \theta)\| \leq r \|z(t)\|, \quad \theta \in [-d, 0]
\]  

(91)

In (91), \(r\) is an unknown constant which should be first estimated by designing an adaptive law. If \(r(t)\) represents the estimate of \(r\), we have the following estimation error:

\[
\dot{\hat{r}}(t) = r(t) - r
\]  

(92)

Now we can present the following theorem to obtain an adaptive sliding mode control law for system (11) with unknown mode-dependent time-varying delays.

**Theorem 3** Consider the Markovian jump system (11) with unknown mode-dependent time-varying delays \(\tau_i(t)\) and partly unknown transition probabilities \(\bar{p}_i\). Suppose that the linear sliding surface is given by (13) and \(C_1\) is obtained in Theorem 1 and there exist matrices \(\Omega_j > 0\) and \(\hat{H}_i = \hat{H}_i^T\) such that the following sets of LMIs hold for each \(i \in \ell\):

\[
\Omega_j - \hat{H}_i \leq 0, \quad j \in \ell_{uK}, j \neq i
\]  

(93)

\[
\Omega_j - \hat{H}_i \geq 0, \quad j \in \ell_{uK}, j = i
\]  

(94)

Then, the state trajectories of system (11) can reach the sliding surface \(s(t) = 0\) in finite time by the SMC law (95) and the adaptive law (96).

\[
\dot{r}(t) = \frac{1}{\beta} \min_{i \in N} \left\{ \|s(t)\| \|\Omega_i\| \left[ \begin{array}{cc} C_{1i} & C_{2i} \end{array} \right] \|\bar{A}_{di}\| \|z(t)\| \right\}
\]  

(96)

where \(r(0) = 0\), \(\epsilon_i > 0\) is a given small constant, and \(\beta > 0\) is a given scalar.

**Proof:** Choose the appropriate mode-dependent Lyapunov function candidate as

\[
V(z, t, i) = \frac{1}{2} \left\{ s^T(t) \Omega_i s(t) + \beta \dot{r}^2(t) \right\}
\]  

(97)

Applying the weak infinitesimal operator of the Lyapunov function \(\mathbf{LV}(z, t, i)\) and using (81), yields

\[
\mathbf{LV}(z, t, i) = s^T(t) \Omega_i \left\{ \left[ \begin{array}{cc} C_{1i} & C_{2i} \end{array} \right] \left( \bar{A}_i z(t) + \bar{A}_{di} z(t - \tau_i(t)) \right) \right\}
\]  

\begin{align*}
&+ s^T(t) \Omega_i C_{2i} B_{2i} [u(t) + F_i w(t)] \\
&+ \frac{1}{2} \sum_{j \in \ell_{lK}} \lambda_{ij} s^T(t) \Omega_j s(t) + \beta \ddot{r}(t) \dot{r}(t)
\end{align*}

(98)

Considering arbitrary matrices \(\hat{H}_i = \hat{H}_i^T\) and separating the known and unknown elements of transition probabilities matrix by (9), yields:

\[
\mathbf{LV}(z, t, i) = s^T(t) \Omega_i \left\{ \left[ \begin{array}{cc} C_{1i} & C_{2i} \end{array} \right] \left( \bar{A}_i z(t) + \bar{A}_{di} z(t - \tau_i(t)) \right) \right\}
\]  

\begin{align*}
&+ s^T(t) \Omega_i C_{2i} B_{2i} [u(t) + F_i w(t)] \\
&+ \frac{1}{2} \sum_{j \in \ell_{lK}} \lambda_{ij} \left( s^T(t) \left( \Omega_j - \hat{H}_i \right) s(t) \right) + \beta \ddot{r}(t) \dot{r}(t)
\end{align*}

(99)

Notice that, the sets of LMIs (93) and (94) are equivalent to following inequality

\[
\frac{1}{2} \sum_{j \in \ell_{lK}} \lambda_{ij} \left( s^T(t) \left( \Omega_j - \hat{H}_i \right) s(t) \right) < 0
\]  

(100)

for \(i \in \ell_{lK}\) and \(i \in \ell_{uK}\), respectively.

By considering (91) in Assumption 1, we have

\[
\mathbf{LV}(z, t, i) \leq \|s(t)\| \cdot \|\Omega_i\| \left\{ \left[ \begin{array}{cc} C_{1i} & C_{2i} \end{array} \right] \right\} \|z(t)\|
\]  

\begin{align*}
&+ \|\bar{A}_i\| \cdot \|z(t)\| + r \|\bar{A}_{di}\| \cdot \|z(t)\|
\end{align*}

\begin{align*}
&+ s^T(t) \Omega_i C_{2i} B_{2i} u(t) \\
&+ \|B_{2i} C_{2i}^T \Omega_i s(t)\| \cdot \|F_i w(t)\| \\
&+ \frac{1}{2} \sum_{j \in \ell_{lK}} \lambda_{ij} \left( s^T(t) \left( \Omega_j - \hat{H}_i \right) s(t) \right)
\end{align*}

(101)

By substituting SMC law (95) into (101), we have

\[
\mathbf{LV}(z, t, i) \leq \frac{\dot{r}(t)\|s(t)\| \cdot \|\Omega_i\| \left\{ \left[ \begin{array}{cc} C_{1i} & C_{2i} \end{array} \right] \right\} \|\bar{A}_{di}\| \cdot \|z(t)\|}{\epsilon_i \|\Omega_i s(t)\| + \beta \ddot{r}(t) \dot{r}(t)}
\]  

(102)
where \( \tau \) is the mode-dependent time-varying delay.

Now, by substituting the adaptive law (96), we have

\[
\mathcal{L}V(z,t,i) \leq -\epsilon_i \|\Omega_i s(t)\| < 0
\]

where \( \epsilon_i > 0 \) is a given small constant. The rest of proof is similar to Theorem 2 and omitted here. The proof is completed. \( \square \)

IV. Numerical examples

In this section, we present numerical examples to illustrate the merits of the proposed approaches. Consider the sliding mode control for system (3) with partly unknown transition probabilities (6), three operating modes, i.e. \( N = 3 \) and the following system matrices and parameters:

\[
\begin{align*}
A_1 &= \begin{bmatrix}
-1 & 0 \\
2 & -2
\end{bmatrix}, & A_{d1} &= \begin{bmatrix}
-2 & 0.1 \\
0.5 & -1
\end{bmatrix}, & B_1 &= \begin{bmatrix}
1 \\
0
\end{bmatrix}, \\
A_2 &= \begin{bmatrix}
-0.15 & -0.49 \\
1.5 & -2.1
\end{bmatrix}, & A_{d2} &= \begin{bmatrix}
0 & -3 \\
0.1 & 0.5
\end{bmatrix}, \\
B_2 &= \begin{bmatrix}
2 \\
-1
\end{bmatrix}, & A_3 &= \begin{bmatrix}
-0.3 & -0.15 \\
1.5 & -1.8
\end{bmatrix}, \\
A_{d3} &= \begin{bmatrix}
-0.5 & 0.2 \\
0.1 & -0.3
\end{bmatrix}, & B_3 &= \begin{bmatrix}
1 \\
-1
\end{bmatrix}, \\
F_1 &= 1, & F_2 &= 1, & F_3 &= 1, & w(t) &= 0.1 \sin(t).
\end{align*}
\]

The mode-dependent time-varying delay \( \tau_i(t) \) satisfies (5) with \( h_1 = 0.3, \ h_2 = 0.5, \ \mu_1 = 0.6, \ \mu_2 = 0.4 \) and \( \mu_3 = 1.1 \). The transition probability rate matrix is described as

\[
A = \begin{bmatrix}
? & ? & 1.1 \\
0.2 & ? & ? \\
0.9 & 0.2 & -1.1
\end{bmatrix}
\]

Case I. By taking advantage of Matlab \(^\text{©} \text{LM} \) Toolbox to solve set of LMIs (16)-(28) in Theorem 1, we obtain a feasible solution as follows:

\[
X_1 = 0.8974, \quad X_2 = 0.9079, \quad X_3 = 0.9217, \\
Y_1 = -0.1495, \quad Y_2 = 0.2235, \quad Y_3 = 1.1734,
\]

and from (38), we have

\[
C_1 = -0.1666, \quad C_2 = 0.2462, \quad C_3 = 1.2731
\]

which give a stable sliding mode dynamics (15). Now, Solving LMIs (77)-(78) in Theorem 2 to design a SMC law of the form (79), yields

\[
\Omega_1 = 2.7392, \quad \Omega_2 = 1.4755, \quad \Omega_3 = 0.4918, \\
\tilde{V}_1 = 2.1074, \quad \tilde{V}_2 = 0.9837, \quad \tilde{V}_3 = 1.
\]

By choosing \( f_1 = f_2 = f_3 = 0.1 \) and \( \epsilon_1 = \epsilon_2 = \epsilon_3 = 0.2 \) and considering \( \tau_1(t) = 0.4 + 0.1 \sin(5t), \tau_2(t) = 0.45 + 0.05 \sin(6t) \) and \( \tau_3(t) = 0.42 + 0.07 \cos(11t) \), we have the following simulation results: Figure 1 shows the switching of the three operating modes. Figures 2 and 3 depict the state trajectories \( z_1(t) \) and \( z_2(t) \) of the closed loop system, respectively, for the initial values \( z(0) = [1 \ 1 \ 1]^T \). Moreover, the control input \( u(t) \) is given in Figure 4. Some slight discontinuities might appear in control signal, which are effects of random jumps in Markovian jump system. To make a firm conclusion, simulation of the closed-loop system with 10 different realizations of the stochastic process \( \tau_i \) is done and the state trajectories \( z_1(t) \) and \( z_2(t) \) are portrayed in Figures 5 and 6 respectively.

Case II. In other situations when delay functions \( \tau_i(t) \) are unknown, by solving LMIs (93)-(94) and applying the sliding mode controller (95)-(96) proposed in Theorem 3, the following simulation results are obtained: the states of the closed-loop system \( z_1(t) \) and
$z_2(t)$ are shown in Figure 7 with the initial values given by $z(0) = [1 \ 1]^T$. Moreover, the control input $u(t)$, and $r(t)$ with initial condition $r(0) = 0$ are portrayed in Figures 8 and 9, respectively. The adaptive law is given as:

$$\dot{r}(t) = 0.1454 \| s(t) \| \cdot \| z(t) \|$$

(104)

with $\beta = 2$.

The simulation results demonstrate that by applying the proposed SMC law, the state trajectories of the closed-loop system are driven onto the predefined sliding surface in finite time which verifies our main results.

V. Conclusion

A sliding mode control design for mode-dependent time varying delayed Markovian jump systems with partly unknown transition probabilities has been investigated. By using a new stochastic Lyapunov Krasovskii functional candidate combining with Jensen’s inequality and free-connection weighting matrix method, the sufficient delay-dependent conditions for stochastic stability of sliding mode dynamics has been presented in terms of LMIs. A SMC law has been synthesized to ensure the reachability of the closed-loop system’s state trajectories to the specified sliding surface in finite time. In our design approach, the information of delay size has been considered and derivative of mode-dependent time-varying delay may be larger than one. Therefore, less-conservative criteria have been derived.
In addition, an adaptive sliding mode controller has been designed to apply to cases, where mode-dependent time-varying delays are unknown. All of the conditions for the stability of sliding mode dynamics and SMC law design are expressed in terms of LMIs. Finally, numerical examples have been provided to demonstrate the validity of the main results.

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Fig. 9. Adaptive Estimate $r(t)$

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