A COUNTEREXAMPLE TO QUESTION 1 OF
"A SURVEY ON THE TURAEV GENUS OF KNOTS"

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Abstract. In “A survey on the Turaev genus of knots,” Champanerkar and Kofman propose several open questions. The first asks whether the polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph obtained from a diagram with minimal Turaev genus is an invariant of the knot. We answer negatively by showing a counterexample obtained from the two diagrams of $8_{21}$ on the KnotAtlas and KnotScape.

1. Introduction

Champanerkar and Kofman offer a very complete “survey on the Turaev genus of knots” [CK14], and we defer the reader to this short survey rather than repeat most of the background here.

In an earlier work with Stoltzfus, they define a polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph $G$ obtained from a diagram with minimal Turaev genus. This comes from the Bollobás-Riordan-Tutte polynomial $C(G, X, Y, Z)$.

Proposition 1.1. [CKS07, Proposition 3.2] Let $q(G; t, Y) = C(G; 1, Y, tY^{-2})$. Then $q(G; t, Y)$ is a polynomial in $t$ and $Y$ such that

$$q(G; t) := q(G; t, 0) = \sum_j a_j t^j$$

where $a_j$ is the number of quasi-trees of genus $j$. Consequently, $q(G; 1)$ equals the number of quasi-trees of $G$.

Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus in [DFK+10, Theorem 3.2] show that the evaluation of this polynomial with $t = -1$ gives the determinant of the knot.

The recent survey paper asks whether the polynomial itself is an invariant when the Turaev genus $g_T$ of the diagram is equal to that of the knot, that is, when it is minimal.

Question 1.2. [CK14, Question 1] Let $G$ be the all-A ribbon graph for a diagram $D$ of a knot $K$. If $g_T(D) = g_T(K)$, is $q(G; t)$ an invariant of $K$?

We give a negative answer to this question by providing a counterexample.

Theorem 1.3. The polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph obtained from diagram with minimal Turaev genus is not an invariant of the knot.

We prove this Theorem 1.3 by considering the two diagrams of $8_{21}$ obtained from the Knot Atlas [BNMea] and KnotScape [HT99]. We address these cases in Examples 2.1 and 2.2, respectively.

We rely on an algorithm given by Armond, Druivenga, and Kindred in [ADK14] to obtain alternating diagrams on a surface with minimal Turaev genus.

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2. A counterexample: diagrams from the Knot Atlas and KnotScape

In Examples 2.1 and 2.2 below, we count the quasi-trees of the all-A ribbon graph obtained from diagrams coming from the Knot Atlas \[\text{BNMea}\] and KnotScape \[\text{HT99}\], respectively, as shown in Figure 1. We show that the polynomial \(q(G, t)\) is not invariant on the knot.

**Figure 1.** The knot 8\(21\) presented in diagrams given by the Knot Atlas \[\text{BNMea}\] and KnotScape \[\text{HT99}\].

**Example 2.1.** Consider first the knot diagram of 8\(21\) given by the Knot Atlas \[\text{BNMea}\], as shown in Figure 1. This diagram has Turaev genus 2. We perform a Reidemeister III move on the upper central three crossings to obtain a diagram of Turaev genus 1.

Armond, Druivenga, and Kindred \[\text{ADK14}\] give an algorithm to obtain an alternating diagram on a surface. We apply this to obtain a Heegaard diagram, where the dashed and dotted lines represent \(\alpha\) and \(\beta\) curves, respectively, as given on the left-hand side in Figure 2.

**Figure 2.** Alternating diagrams on the torus for 8\(21\) coming from the KnotAtlas and KnotScape, respectively, after applying the algorithm of \[\text{ADK14}\].

We checkerboard color this diagram on the torus to obtain the all-A ribbon graph given on the left-hand side in Figure 3. We proceed to count the number of quasi-trees.

First of all, any spanning tree of \(G\) must contain exactly two edges from the loop consisting of edges \(a, b,\) and \(c\). From here the spanning trees fall into two classes: those with one of the two edges \(g\) and \(h\) and those with neither \(g\) nor \(h\). Any spanning tree in the first class must contain one of the two edges \(d\) and \(e\) giving a total of \(3 \times 2 \times 2 = 12\) spanning trees in the first class. Any spanning tree in the second class must contain two of the three edges \(d, e,\) and \(f\) giving a total of \(3 \times 3 = 9\) spanning trees in the second class. Thus for this ribbon graph we get \(a_0 = 21\).

A quasi-tree of \(G\) with genus 1 must contain all of the edges \(a, b,\) and \(c\) as well as one of the two edges \(g\) and \(h\) and again these quasi-trees fall into two classes: those that contain the edges \(d\) and
e but not f and those that contain the edge f and exactly one of the edges d and e. This gives us \( a_1 = 4 + 2 = 6 \) for this ribbon graph.

Thus, we obtain \( q(G, t) = 6t + 21 \).

**Example 2.2.** Now consider the knot diagram of \( 8_{21} \) given by KnotScape [HT99] having Turaev genus 1 already and appearing on the right-hand side of Figure 1.

We apply the algorithm of [ADK14] to obtain an alternating diagram on a surface, which again is a Heegaard diagram, where the dashed and dotted lines represent \( \alpha \) and \( \beta \) curves, respectively, as given on the right-hand side in Figure 2.

We checkerboard color this diagram on the torus to obtain the all-A ribbon graph, given in Figure 3 from [DFK+10], which we include on the right-hand side in our Figure 3.

As observed in [DFK+10], this ribbon graph contains 9 spanning trees and 24 genus-1 quasi-trees, yielding
\[
q(G, t) = 24t + 9.
\]

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