STRICT STABILITY OF EXTENSION TYPES

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Abstract. We show that the extension types occurring in Riehl–Shulman’s work on synthetic (∞, 1)-categories can be interpreted in the intended semantics in a way so that they are strictly stable under substitution. The splitting method used here is due to Voevodsky in 2009. It was later generalized by Lumsdaine–Warren to the method of local universes.

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1. Introduction

In this note, we carry out the splitting method described by Voevodsky in [Voe09, Section 4], [Voe17], and recalled by Streicher [Str14b, Appendix C] to show that the so-called extension type in Riehl–Shulman’s simplicial homotopy type theory [RS17] can be interpreted in a way that makes them strictly stable under substitution.

1.1. Synthetic (∞, 1)-category theory. In [RS17], Riehl–Shulman introduce simplicial homotopy type theory (sHoTT) as a domain-specific language (DSL) to reason synthetically about (∞, 1)-categories. As a standard model one can consider the Reedy model structure on the simplicial presheaf category $[\Delta^{op}, \text{sSet}_{\text{Kan}}]$, where $\text{sSet}_{\text{Kan}}$ denotes the category of simplicial sets $\text{sSet} = [\Delta^{op}, \text{Set}]$ endowed with the Kan model structure. Accordingly, types get interpreted as simplicial spaces. The syntax makes it possible to internally single out those types that satisfy the Segal condition (i.e., weak composition of 1-arrows) and Rezk completeness (i.e. local univalence) which semantically correspond to the complete Segal aka Rezk spaces, known to implement (∞, 1)-categories [Rez01, JT07, Ber18, RV22]. In general, for a (strict) shape $\Phi$ [RS17, Section 2] the type $A^\Phi$ denotes the type of functions from $\Phi$ to $A$, thought of as $\Phi$-shaped diagrams in $A$. The Segal condition (in its internal form, after Joyal) is then expressed by demanding the induced map $A^{\Delta^2} \to A^{\Lambda^2_1}$ to be a weak equivalence [RS17, Sections 5 and 10].

1.2. Extension types. The theory of Segal and Rezk types makes crucial use of strict simplicial shapes and extension types [RS17, Section 3], the latter going
back to unpublished work of Lumsdaine–Shulman. The formation is as follows. Given a shape inclusion \( \Phi \hookrightarrow \Psi \), a type family \( A : \Psi \rightarrow \mathcal{U} \), and a partial section \( a : \prod_{t : \Phi} A(t) \). The extension type \( \langle \prod_{t : \Psi} A(t) \vbar a \rangle \) consists of all strict totalizations of \( a \), i.e., sections \( b : \prod_{t : \Psi} A(t) \) such that \( t : \Phi \vdash a(t) \equiv b(t) \), as indicated in the following diagram:

One reason why these extension types are important is that situations familiar from (simplicial and categorical) homotopy theory can be conveniently and homotopy-invariantly re-expressed in terms of them, such as right orthogonality up to homotopy. Namely, let \( \iota : \Phi \rightarrow \Psi \) be a shape inclusion and \( \pi : E \rightarrow B \) be a map between two types. In HoTT, we can assume \( \pi \) to be given as the projection \( \pi \equiv \text{pr}_1 : \left( \sum_{b : B} P(b) \right) \rightarrow B \) for a family \( P : B \rightarrow \mathcal{U} \) of \( \mathcal{U} \)-small types. Then, the shape inclusion \( \iota \) is right orthogonal to \( \pi \), i.e., the space of all fillers of any diagram

is contractible if and only if the proposition

holds.

To be able to literally make such and similar formulations inside our type theory we reflect the tope into the type layer. This has been described in [BW21, Section 2.4] and will be justified by our intended semantics, as we will see next.

Indeed, continuing the program of [RS17], some parts of synthetic \((\infty, 1)\)-category theory have recently in simplicial HoTT been developed in [CRS18, BW21, Wei22, Mar22], and in [WL20] within a (bi-)cubical variant of the theory. Many of these developments are crucially inspired by results from Riehl–Verity’s \( \infty \)-cosmos theory, a very general and powerful account to model-independent \((\infty, n)\)-category theory itself not formulated within type theory. Hence, several constructions and results in sHoTT can be seen as a type-theoretic translation of \( \infty \)-cosmological setting (at the moment essentially confined to \((\infty, 1)\)-categorical reasoning, cf. [RS17, Corollary 8.19]).

1.3. Internal \((\infty, 1)\)-categories. In fact, by Shulman’s far-reaching results [Shu19] about strictification of univalent universes we can present any \((\infty, 1)\)-topos\(^2\) by a suitably strict 1-categorical structure, a type-theoretic model topos (TTMT). Hence, simplicial HoTT has models in all TTMTs of the form \( \mathcal{E} \Delta^{\text{op}} \) for an arbitrary given TTMT \( \mathcal{E} \), where types get interpreted as Reedy fibrant simplicial diagrams, and

\(^1\)an instance of which is also to be found as path types in cubical type theory [CCHM18, OP16, Swa18]

\(^2\)in the sense of Grothendieck–Rezk–Lurie [Rez10, TV05, Lur09], not necessarily in the sense of the more recent, more general notion of elementary \((\infty, 1)\)-topos [Shu17a, Ras22]
1.4. **Strict stability of extension types.** An axiomatic notion of model of simplicial type theory is given in [RS17, Appendix A]. They prove that a wide class of model structures give rise to models of simplicial homotopy type theory with pseudo-stable extension types. It is pointed out that one can use the splitting methods of Voevodsky [Voe09, Voe17] and Lumsdaine–Warren [LW15] to rectify this to obtain strictly stable extension types in the model, i.e., the type formers commute with substitutions on the nose.

In general, the basic idea behind the splitting is as follows. Given a lex category \( \mathcal{C} \) and a universe \( \pi: \widetilde{\mathcal{U}} \to \mathcal{U} \) in \( \mathcal{C} \), one chooses for every morphism \( A: \Gamma \to \mathcal{U} \) a pullback:

\[
\begin{array}{c}
\Gamma \downarrow^A \downarrow^\pi \\
\Delta^A \downarrow^A \\
\Gamma \downarrow^A \\
\end{array}
\]

This yields a splitting of the full subfibration of the fundamental fibration \( \mathcal{C}[1] \to \mathcal{C} \) that corresponds to the universe \( \pi \), cf. [Voe09, Section 4, Definition 0.1], [LW15, Section 1], [Awo18, Subsection 1.1], [Str14b, Appendix C], [Rie22, Definition 2.4.2, Digression 6.13], and Section 3.2.

1.5. **Contribution.** Our contribution consists in applying the splitting mentioned in Section 1.4 to the extension type formers in simplicial HoTT so as to make them strictly stable under pullback. We are working w.r.t. to just one fixed universe. Furthermore, the shapes are taken to be fibrant as well. This simplifies the setting on a technical level while retaining its applicability to the class of models of the form \( s\mathcal{E} := \mathcal{E}\Delta^{op} \) for a TTMT \( \mathcal{E} \).

The work presented here is also to be found as [Wei22, Chapter 6] of the author’s doctoral dissertation under the supervision of Thomas Streicher at TU Darmstadt, Germany.

2. **Models of simplicial homotopy type theory**

2.1. **Shulman’s type-theoretic model toposes.** In [Shu19], Shulman provided the final missing part to establish the far-reaching conjecture that any Grothendieck (\( \infty,1 \))-topos gives rise to a model of homotopy type theory. Indeed, he proved that any such higher topos can be presented by a model structure in which provides enough univalent universes, validating propositional resizing, and which are strictly closed under all desired type formers.\(^3\) This had previously been established for certain classes of special cases. [Shu15b, Shu15a, Shu17b, Cis14] More generally, work connected to “internal languages” of higher categories is found in [AW09, GK17, KL18, KS19, Kap17]. An excellent exposition of Shulman’s article is given in Riehl’s mini-course notes [Rie22].

Given a type-theoretic model topos \( \mathcal{E} \), the internal presheaf category \( s\mathcal{E} := [\Delta^{op}, \mathcal{E}] \) of simplicial objects in \( \mathcal{E} \) is again a type-theoretic model topos.\(^4\) These are the “standard models” for simplicial homotopy type theory. Inside any such

\(^3\)with more general kinds of higher inductive types being work in progress

\(^4\)with the injective model structure [Shu19, Corollary 8.29]
We find a copy of the TTMT of spaces \( S \), which is embedded fibrantly and spatially-discretely via the constant diagram functor

\[
\begin{align*}
\text{Set} & \xrightarrow{\Delta_E} \mathcal{E} \\
& \xrightarrow{\Gamma_E} S \\
\end{align*}
\]

\[\xrightarrow{\text{Set}} \xrightarrow{\Delta_E} \mathcal{E} \xrightarrow{\text{spatially-discretely}} S \xrightarrow{\Gamma_E} \mathcal{E} \cong \mathcal{E}^\text{op} \]

\[\Delta_{sE} : S \hookrightarrow \mathcal{E}, \quad X \mapsto (n \mapsto \coprod_{X_n} 1).
\]

We adapt the coherence construction used in \([KL21, LW15, Awo18, Str14b, Str14a, LS20]\), based on \([Voe09, Voe17]\), so that in the model extension types à la \([RS17]\), going back to earlier work by Lumsdaine and Shulman, can be chosen in a way that is strictly stable under pullback.

Our presentation is inspired by \([KL21, Awo18]\) and also \([LW15]\) in method and style. We define the type formers and the data required from the rules by performing the corresponding constructions in suitable generic contexts. Substitution then corresponds to precomposition with the reindexing map, and applying type formers corresponds to postcomposition with an ensuing map between universes. This implies strict substitution stability on the nose, in particular avoiding choices of pullbacks (and dealing with the coherences that would ensue).

This is done with respect to just a single universe as e.g. in \([Awo18, Str14a, Str14b]\). Because extension types are similar to II-types, the splitting is also analogous to the method for II-types from the aforementioned sources.\(^5\)

We briefly recall Shulman’s setting and results that we are building on, although we will rarely need to be explicit about the machinery under the hood from here on.

A type-theoretic model topos (TTMT) \( \mathcal{E} \) ([Shu19, Definition 6.1], [Rie22, Definition 6.1.1]) is given by a Grothendieck 1-topos \( \mathcal{E} \), together with the structure of a right proper simplicial Cisinski model category, which is in addition simplicially cartesian closed (i.e., reindexing preserves simplicial copowers) and is also equipped with an appropriate notion of “type-theoretic” fibration.\(^6\) An immediate example is given by the prime model of homotopy type theory, the type-theoretic model topos of spaces \( S \), but there are many more examples. In particular, the class of TTMTs is closed under typical operations, such as slicing and small products, as well as passage to diagram categories and internal localizations. Moreover, any Grothendieck–Rezk–Lurie–(\(\infty, 1\))-topos can be presented by a type-theoretic model topos. As a consequence, the study of Rezk types in simplicial HoTT can be understood as a synthetic version of internal \(\infty\)-category theory in an arbitrary given \((\infty, 1)\)-topos.

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\(^5\)Since the same ideas have successfully generalized from global to local universes \([LW15, Shu19]\), we believe the same generalization would also work for strict extension types (which we do not consider in the present text, however). In particular, a formulation in (an appropriate extension of) Shulman’s setting \([Shu19, Appendix A]\) should be possible.

\(^6\)In Shulman’s terminology, a locally presentable and relatively acyclic notion of fibration structure, satisfying in addition a “fullness” condition.
Crucially, any TTMT hosts enough\(^7\) strict fibrant univalent universes [Shu19, Theorem 5.22, Theorem 11.2].\(^8\) Abstractly from the given fibration structure, the universes can be constructed using a version of the small object argument [Shu19, Theorem 5.9]. Then a splitting method due to Voevodsky [Voe09, KL21] akin to Giraud’s left adjoint splitting [Gir71, LW15, Str14b] is applied to obtain strict type formers for these universes.

3. Strict stability of extension types

3.1. General ideas. The rules for the extension types from [RS17] are recalled in Appendix A.

We adapt the coherence construction presented in [KL21, LW15, Awo18], based on [Voe09, Voe17], so that in the model extension types à la [RS17] can be chosen in a way that is strictly stable under pullback.

Our presentation is close to [KL21, Awo18] in style. We define the type formers and the data required from the rules by performing the corresponding constructions in suitable generic contexts. Substitution then corresponds to precomposition with the reindexing map. Applying type formers/operations corresponds to postcomposition with an ensuing map between universes. This implies strict substitution stability on the nose.

We work w.r.t. just a single universe, as is also done in [Awo18]. The extension types of [RS17, Figure 4] are both syntactically and semantically similar to \(\Pi\)-types: Intuitively, the extension type \(\langle \Pi_{\psi} A \mid \phi \rangle\) consists of sections \(b : \Pi_{\psi} A\) that on the subshape \(\varphi \rightarrow \psi\) coincide with the given section \(a\) in the sense of judgmental equality. Hence, they can be seen as a kind of “\(\Pi\)-types with side conditions” (for functions whose domain is a tope, \textit{a priori} a pre-type).

Since in the case of ordinary \(\Pi\)-types the splitting method has been generalized from global to local universes [LW15, Shu19], we believe the same should also work for strict extension types (which we do not consider in the present text, however). This has also been claimed in [RS17, Appendix A.2].

In particular, a formulation in (an appropriate extension of) Shulman’s setting [Shu19, Appendix A] should then be possible.

In contrast to most of [RS17, Appendix A] we will work completely internally to the topos, disregarding the extra “non-fibrant” structure made explicit in a comprehension category with shapes [RS17, Definition A.5].

This is justified, because shapes are reflected into types by a newly added rule, cf. [RS17, Section 2.4]:

\[
\begin{align*}
I \text{ cube} & \\
t : I \vdash \varphi \text{ tope} & \\
(t : I \mid \varphi) \text{ type}
\end{align*}
\]

In the model, this is validated because the simplicial shapes turn up as (spatially-discrete) fibrant objects, cf. Section 2.1.

In sum, this allows us to focus the presentation on the splitting of the extension types, without having to deal with also splitting the extra layers, introducing universes for “cofibrations” or similar.\(^9\)

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\(^7\) bounded from below by a cardinal
\(^8\) as usual, in a strong enough meta-theory such as ZFC with inaccessibles
\(^9\) Even though all of this should be possible if desired, cf. also the remarks about strict stability in [RS17, Appendix A.2].
We will work in a classical meta-theory with global choice and universes. Sometimes we use the internal extensional type theory (ETT) of the topos for an easy-to-parse denotation of the objects presenting generic contexts. This is along the lines of the presentation in [Awo18] and also [KL21, LW15].

As in [RS17, Theorem A.16], in the (injective) model structure of the TTMT presenting $s\mathcal{E}$ we interpret the extension types as ordinary 1-categorical pullbacks of Leibniz cotensors. Since the model structure at hand is type-theoretic [Shu15b, Definition 2.12] the Leibniz cotensors are fibrations, so are all their reindexings, which in particular includes the (dependent) extension types.\footnote{In the case of non-fibrant shapes, the Leibniz cotensor is still a fibration because the model structure is cartesian monoidal, cf. [RS17, Section 1, Section A.2].} In particular, the ensuing pullbacks are homotopy pullbacks, cf. also [Lur09, Proposition A.2.4.4 and Remark A.2.4.5].

In the non-dependent case the extension type $\langle \psi \to A \mid \phi \rangle_\alpha$ is thus given simply by \(\text{ordinary, 1-categorical pullback}\): For the dependent case the construction will have to be relativized as we will see later on.

We want to show that the interpretation of the extension types can be chosen in a strictly pullback-stable way. In particular, after [RS17, Definition A.10, Theorem A.16] given a type family $A : \Gamma \times \psi \to \mathcal{U}$ and a partial section $a : \Gamma \times \varphi \to \Gamma \times \psi$ a substitution of the extension type\footnote{Here and in the following, notation such as $\Pi(A, B)$ or $\langle \Pi[\varphi][A] \rangle_\alpha$ instead of $\prod_A B$, $\langle \prod_\varphi A \rangle_\alpha$ designates the (strictly stable) type formers (to be) defined in the interpretation, in line with [RS17, KL21, KL21, Awo18].} $\langle \Pi[\varphi][A] \rangle_\alpha$ is only considered along (type) context morphisms $\sigma : \Delta \to \Gamma$, leaving the shape inclusion $j : \varphi \to \psi$ fixed. Hence, we will understand the type former of the extension type as a family of type formers $\text{Ext@}_j$, given by

\[ \text{Ext@}_j(A,a) := \langle \Pi[\psi][A] \rangle_\alpha \]

indexed by the shape inclusions $j$.\footnote{Note that, \textit{a posteriori} this also yields stability w.r.t. to reindexing along cube context morphisms $\tau : J \to I$ pulling back shape inclusions $j : \varphi \to I \psi$: By their defining universal property, which only involves a condition on reindexings of the type context $\Gamma$, the (pseudo-stable) extension types from [RS17, Definition A.10] are determined uniquely up to isomorphism. The splitting then yields uniqueness up to equality. Then, considering reindexings along shape maps does not lead out of this class, so the choice remains strictly stable.}

The strict stability result can then be phrased as the following theorem:

**Theorem 3.1** (Strict stability of extension types). Let $\mathcal{E}$ be a type-theoretic model topos, and consider the interpretation of simplicial homotopy type theory à la [RS17, Shu19] in the type-theoretic model topos $s\mathcal{E}$. Then, a fixed splitting of the standard type formers induced by the universal fibration $\pi : \tilde{U} \to U$ for small fibrations yields an interpretation of the extension types $\text{Ext@}_j$ (where $j : \varphi \to I \psi$ is a fixed shape inclusion relative to some cube $I$) that is strictly stable under pullback. This means, for families $A : \Gamma \times \psi \to \mathcal{U}$ and partial sections $a : \Gamma \times \varphi \to \Gamma \times \psi$ $A$ we have:

\[
\begin{align*}
\sigma^* (\text{Ext@}_j(A,a)) &= \text{Ext@}_j(\sigma^* A, \sigma^* a) \\
\sigma^* (\lambda@_{j,A,a}(b)) &= \lambda@_{j,\sigma^* A, \sigma^* a}(\sigma^* b) \\
\sigma^* (\text{app@}_{j,A,a}(f,s)) &= \text{app@}_{j,\sigma^* A, \sigma^* a}(\sigma^* f, \sigma^* s)
\end{align*}
\]
In what we follows, we establish the proof of this theorem by splitting the extension type formers, assuming all the necessary and pre-established logical and model-categorical structure in the background.

3.2. **Global universe splitting.** Depending on the precise setup—in particular with extra shape layers—one might have to use e.g. the local universe method [LW15] to strictify the pseudo-stable tope logic beforehand [RS17, Definition A.7, Remark A.8]. But we will neglect this here, as mentioned before, since we are staying inside the given model structure on the topos, so it is enough just to consider a splitting of this.

A family in simplicial type theory is modeled by a composition

\[ A \to \Gamma \to \Phi \hookrightarrow \Xi, \]

where \( \Phi \hookrightarrow \Xi \) plays the role of a shape inclusion. Let \( \pi : \tilde{\mathcal{U}} \to \mathcal{U} \) be the type-theoretic universe in \( s\mathcal{E} \) (strictly à la Tarski, closed under all standard type formers) which classifies small Reedy fibrations and which exists by [Shu19, Corollary 8.29]. This means in particular we are considering the injective model structure on internal presheaves, which in the case of the base \( \Delta \) coincides with the Reedy model structure, cf. also [Cis14, Shu15a].

We just have to split w.r.t. the universe \( \pi : \tilde{\mathcal{U}} \to \mathcal{U} \). This is done after Streicher [Str14b, Appendix C] and Voevodsky [Voe09, Voe17] in the following way:

Using meta-theoretic global choice, for each family \( A : \Gamma \to \mathcal{U} \), we select a distinguished square

\[
\begin{array}{c}
\Gamma.A \\
\downarrow \ \ \ \ \ \downarrow q_A \\
\mathcal{U} \\
\end{array}
\]

A map from \( B : \Delta \to \mathcal{U} \) and \( A : \Gamma \to \mathcal{U} \) is given by a square:

\[
\begin{array}{c}
\Delta.B \\
\downarrow \sigma \\
\Gamma \\
\end{array}
\]

which is split cartesian if and only if \( B = A \circ \sigma \) and \( q_B = q_A \circ q_\sigma \):

\[
\begin{array}{c}

\end{array}
\]

This then necessarily implies that the left-hand side is a pullback as well. Since the identities involved are strict equalities between objects and morphisms involving distinguished squares, this models substitution up to equality on the nose by defining \( \sigma^* A := A[\sigma] := A \circ \sigma \).

When defining the generic contexts à la [KL21], we often define them in the internal extensional type theory (ETT) of the topos, as done in [Awo18, LW15].
particular, given a map \( f : B \to A \) the endofunctor defined via the adjoint triple
\[
\sum_f \dashv f^* \dashv \prod_f
\]
as
\[
P_f(X) = \sum_{A \to 1} \left( \prod_f(B^*(X)) \right)
\]
in the internal language ETT reads as
\[
P_f(X) = \sum_{a : A} X^{B_a},
\]
where \( B_a = f^* a \).

Moreover, for \( A : \Gamma \to \mathcal{U} \) we allow ourselves to abbreviate \( A := \Gamma.A = \tilde{U}_A = \pi^* A \), as is commonly done.

For the universal fibration \( \pi : \tilde{U} \to \mathcal{U} \), the total object denotes the generic context
\[
\tilde{U} = \sum_{A : U} A = [\text{the context } A : \mathcal{U}, a : A].
\]

We show how to define a strictly stable choice of extension types à la [Awodey 2018, Proposition 2.4]. To illustrate the analogy, we give a brief recollection how to define the generic contexts for the formation and introduction rule.

### 3.3. Recollection: Strictly stable \( \Pi \)-types.

The generic context for the \( \Pi \)-formation rule is given, in the internal language, by the type
\[
\mathcal{U}^{\Pi} := \sum_{A : \mathcal{U}} A = [\text{the context } A : \mathcal{U}, B : A \to \mathcal{U}],
\]
and the generic context for the introduction rule is given by
\[
\mathcal{U}^{\lambda \Pi} := \sum_{A : \mathcal{U}} \left( \sum_{B : A \to \mathcal{U}} \prod a : A B(a) = [\text{the context } A : \mathcal{U}, B : A \to \mathcal{U}, t : (a : A) \to B(a)] \right),
\]
which captures the generic dependent term.

The idea is that a map \( \langle A, B \rangle : \Gamma \to \mathcal{U}^{\Pi} \) precisely captures the input data for the \( \Pi \)-type former.

The universe then admits \( \Pi \)-types (cf. \( \Pi \)-structure, cf. [Kapulkin & Lumsdaine 2021, Definition 1.4.2, Theorem 1.4.15]) if there exist maps \( \Pi : \mathcal{U}^{\Pi} \to \mathcal{U}, \lambda : \mathcal{U}^{\lambda \Pi} \to \mathcal{U} \) making the square
\[
\begin{array}{ccc}
\mathcal{U}^{\lambda} & \xrightarrow{\lambda} & \tilde{U} \\
\downarrow & & \downarrow \\
\mathcal{U}^{\Pi} & \xrightarrow{\Pi} & \mathcal{U}
\end{array}
\]
commute, and moreover rendering as a pullback. These maps implement the formation and introduction rule, hence we take the maps
\[
\Pi : \mathcal{U}^{\Pi} \to \mathcal{U}, \Pi(A, B) := \prod_A B, \quad \lambda : \mathcal{U}^{\Pi} \to \mathcal{U}, \lambda(A, B, t) := \left\langle \prod_A (B), t \right\rangle
\]
induced from the structure of the ambient category as a type-theoretic model category.\(^{13}\)

\(^{13}\)Concretely, for fibrations \( p_A : \Gamma.A \to \Gamma, p_B : \Gamma.A.B \to \Gamma.A \) we define \( \Pi(A, B) := \prod_{p_A} (p_B) \) via the pushforward functor \( \prod_{p_A} \) which in this setting preserves fibrations.
In particular, for the formation rule, strict stability under pullback follows from strict commutation of the diagram

$$
\begin{array}{ccc}
\Delta & \xrightarrow{\pi} & \Pi \sigma^* \sigma^* B \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{(A,B)} & U^\Pi \\
\downarrow & & \downarrow \\
\Pi A B & \rightarrow & U
\end{array}
$$

as elaborated in [Awo18].

We now treat the extension types from [RS17] in a similar fashion.

3.4. **Strictly stable extension types: Generic contexts.** Recall the rules from [RS17, Figure 4].

To form an extension type, we start with a context fibered-over-shapes

$$\Gamma \rightarrow \Phi \rightarrow \Xi$$

and a separate\(^{14}\) shape inclusion

$$\psi \Rightarrow \varphi \Rightarrow I.$$  

The extension type is then formed for a pair \(\langle A, a \rangle\) of a type and a partial section

$$A : \Gamma \times \psi \rightarrow \Phi \times \psi \rightarrow \Xi \times I, \quad a : \Gamma \times \varphi \rightarrow A \psi A$$

In fact, the defining universal property for the extension types [RS17, Definition A.10] asks for substitution (pseudo-)stability of \(A\) and \(a\) along morphisms \(\sigma : \Delta \rightarrow \Gamma\). Hence, we in fact consider a family of type formers \((\text{Ext} @ j)_{j : \varphi \rightarrow I \psi}\) indexed externally by the shape inclusions \(j : \varphi \rightarrow \psi\), so that

$$(\text{Ext} @ j)(A, a) = (\Pi[\psi][A]^j_\psi).$$

Thus, the input data to form the extension type \(\text{Ext} @ j\) should be represented using a suitable generic context \(U^{\text{Ext} @ j}\) as a morphism

$$\langle A, a \rangle : \Gamma \rightarrow U^{\text{Ext} @ j}$$

with \(\langle A, a \rangle\) as in (1).\(^{15}\) In analogy to the case of \(\Pi\)-types, we form the generic contexts by

\begin{align*}
U^{\text{Ext} @ j} & := \sum_{A \in U^\varphi} j^* A, \\
U^{\lambda @ j} & := \sum_{A \in U^\varphi} \sum_{a : A^j} a^*(A^\psi).
\end{align*}

\(^{14}\)The semantic reason for this is explained right before [RS17, Theorem A.17]. Cf. also the explanation about the rules at the beginning of [RS17, Section 2.2].

\(^{15}\)Note that we are suppress the further structure \(\Gamma \rightarrow \Phi \rightarrow \Xi\) here, in line with [RS17, Definition A.10].
3.5. **Strictly stable extension types: Formation and introduction.** The formation and introduction rule are, top to bottom, Rule 1 and 2, resp., in [RS17, Figure 4] (or cf. Figure 1).

Then, at stage $\Gamma$ the object $U^{\lambda @ j}$ consists of triples $(A, a, b)$ as in

\[
\begin{array}{c}
\Gamma \times \varphi \\
\downarrow \quad a \\
\Gamma \times j \\
\downarrow \quad b \\
\Gamma \times \psi
\end{array}
\]

i.e., strictly $a = b \circ (\Gamma \times j)$. Considering the projection $U^{\lambda @ j} \to U^{\text{Ext} @ j}$ the fiber at an instance $(A, a)$ is exactly the semantic extension type

\[
\text{Ext} @ j(A, a) := \langle [\Pi [\psi] [A]]_{a}^{\psi} \rangle
\]

defined by the *split*\(^{16}\) cartesian square, after [RS17, Theorem A.16]:

\[
\begin{array}{ccc}
\langle [\Pi [\psi] [A]]_{a}^{\psi} \rangle & \rightarrow & A^{\psi} \\
\downarrow \quad \gamma & & \downarrow \quad j \\
\Gamma & \rightarrow & A^{\varphi \times (\Gamma \times \psi)^{\psi}} (\Gamma \times \psi)^{\psi}
\end{array}
\]

In particular, the right vertical map is a fibration,\(^{17}\) so the object $\text{Ext} @ j(A, a)$ is fibrant.

Note that in the extensional type theory of the ambient topos one could describe the extension type also as the $\Sigma$-type

\[
\sum_{b : \psi \to A} (b \circ j \equiv a),
\]

where $(-1) \equiv (-2)$ stands for the extensional identity type.

Now, in analogy to [KV18, Theorem 1.4.15], we define the map $\text{Ext} @ j : U^{\lambda @ j} \to \bar{U}$ by the universal property of the extension type. The generic context $U^{\lambda @ j}$ consists of triples $(A, a, b)$ s.t.

\[
\begin{array}{c}
\Gamma \\
\downarrow \quad \langle a, \eta \rangle \\
A^{\varphi \times (\Gamma \times \psi)^{\psi}} (\Gamma \times \psi)^{\psi}
\end{array}
\]

where $\eta : \Gamma \to (\Gamma \times \psi)^{\psi}$ denotes the transpose of the identity of $\Gamma \times \psi$. Now, we define the generic $\lambda$-term of the extension type as the gap map $(\lambda @ j)(b) := \hat{b}$ of the

\(^{16}\)i.e., $([\Pi [\psi] [A]]_{a}^{\psi})$ and its projection to $\Gamma$ have been chosen

\(^{17}\)If shapes are taken to be fibrant, this already follows from type-theoretic-ness of the model structure. Otherwise one would have to use that the model structure is cartesian monoidal as in [RS17, Lemma A.4].
pullback:

Then, the square

being a pullback precisely captures the universal property as discussed in [RS17, Definition A.10, Theorem A.16]. In the terminology of [KL21, Theorem 2.3.4] this means that the projection $U^\lambda_{@j} \to U^{\text{Ext}@j}$ is the *universal dependent extension type* over $U^{\text{Ext}@j}$. In particular, in presence of the splitting it is a *chosen* small fibration. Thus, as illustrated e.g. in [Awo18, Remark 2.6] we obtain strict pullback stability: everything in

commutes strictly on the nose because we have split the model structure from the get-go, yielding as desired

Similarly, and using that the generic lifts $b$ are given by gap maps of (strict) pullbacks we obtain

\[ \lambda @ j (A, a, b) \circ \sigma = \text{Ext}@j(A\sigma, a\sigma, b\sigma). \]

3.6. Strictly stable extension types: Elimination and computation. The elimination and computation rule are Rule 3 and 4, resp., in [RS17, Figure 4] (or cf. Figure 2).

To interpret the elimination rule, note first that the exponential $A^\psi$ comes with a (chosen) evaluation map:

\[ A^\psi \times_{\Gamma} (\Gamma \times \psi) \overset{\text{ev}}{\longrightarrow} A \]

\[ \Gamma \]
By pulling this back along the (chosen) map from the extension type, we obtain a map
\[ ev : \text{Ext}_{\mathcal{A}}(A,a) \times_{\Gamma} (\Gamma \times \psi) \to \Gamma \times A, \]
fibered over \( \Gamma \). Now, we want to define application \( \text{app}_{\mathcal{A}}(f,s) \) of a function \( f : \Gamma \to \Gamma \times \text{Ext}_{\mathcal{A}}(A,a) \) to a section \( s : \Gamma \to \Gamma \times \psi \). Analogously to [KL21, Theorem 1.4.15], we take the composition:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{(f,s)} & \text{Ext}_{\mathcal{A}}(A,a) \times_{\Gamma} (\Gamma \times \psi) \\
& & \downarrow \text{ev} \\
\Gamma & \xrightarrow{\text{app}_{\mathcal{A}}(f,s)} & A
\end{array}
\]

This validates the rule, since \( \text{app}_{\mathcal{A}}(f,s) : \Gamma \to \Gamma \times A \), as desired. For substitution along \( \sigma : \Delta \to \Gamma \), consider the following diagram involving chosen fibrations and split cartesian squares:

\[
\begin{array}{ccc}
\text{Ext}_{\mathcal{A}}(\sigma^* A,\sigma^* a) \times_{\Delta} (\Delta \times \psi) & \longrightarrow & \text{Ext}_{\mathcal{A}}(A,a) \times_{\Gamma} (\Gamma \times \psi) \\
\text{app}_{\mathcal{A}}(\sigma^* f,\sigma^* s) & \downarrow & \text{app}_{\mathcal{A}}(f,s) \\
\sigma^* \Delta & \xrightarrow{\langle \sigma^* f,\sigma^* a \rangle} & \Delta \\
\end{array}
\]

This uniquely defines the map \( \text{app}_{\mathcal{A}}(\sigma^* f,\sigma^* s) \), and by construction
\[ \sigma^* \text{app}_{\mathcal{A}}(f,s) = \text{app}_{\mathcal{A}}(\sigma^* f,\sigma^* s). \]

Furthermore the desired computation rule holds, saying that for a term \( s \) in the smaller tope \( \varphi \), one judgmentally has \( \text{app}_{\mathcal{A}}(f,s) \equiv a[s] \). This is established by the commutation of the diagram:

\[
\begin{array}{ccc}
\text{app}_{\mathcal{A}}(f,s) & \downarrow & \Gamma \times \varphi \\
\end{array}
\]

The \( \beta \)- and \( \eta \)-rule (cf. [RS17, Figure 4]) can be proven to hold similarly (Rule 5 and 6, resp., in [RS17, Figure 4], or cf. Figure 3).

4. Outlook

We have used Voevodsky’s splitting method from [Voe00, Section 4, Definition 0.1] to split the extension type formers from Riehl–Shulman’s synthetic \((\infty,1)\)-category theory [RS17].

Doing so, we have only worked using a single universe. We believe there to be no essential problems generalizing this line of reasoning to the case of a hierarchy of
universes, cf. [LW15, Shu19, RS17]. We also worked in a simplified technical setting with fibrant shapes, blurring the lines to the pre-type layers which are treated quite explicitly in [RS17, Appendix A]. While this can be justified semantically, one could still try to adapt the splitting to a setting more faithful to the original theory [RS17] and its fibered structure/multi-part contexts. Already in [RS17] it is suggested that it should be possible to do so along the lines of [LW15].

Yet another line of investigation could consist of trying to capture extension types in the recent 2-categorical framework of Coraglia–Di Liberti [CDL21].

APPENDIX A. Rules for extension types

The rules for the extension types of sHoTT have been taken from [RS17, Figure 4]. In simplicial HoTT, a general typing judgment has the form

\[ \Xi \mid \Phi \mid \Gamma \vdash A \text{ type} \]

where \( \Xi \) denotes a cube context, \( \Phi \) a shape context, and \( \Gamma \) a usual type context.

\[
\begin{array}{c}
\{t : I \mid \phi\} \text{ shape} \\
\{t : I \mid \psi\} \text{ shape} \\
\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash A \text{ type} \quad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash a : A \\
\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash \text{ctx} \\
\Xi \mid \Phi \mid \Gamma \vdash \prod_{t : I \mid \psi} A^{\phi}_{a} \text{ type} \\
\{t : I \mid \phi\} \text{ shape} \\
\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash A \text{ type} \quad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash b : A \\
\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash \lambda t \mid \psi. b : \prod_{t : I \mid \psi} A^{\phi}_{a} \\
\Xi \mid \Phi \mid \Gamma \vdash b \equiv a \\
\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash s : I \\
\Xi \mid \Phi \mid \Gamma \vdash s/t \\
\Xi \mid \Phi \mid \Gamma \vdash f(s) : A \\
\Xi \mid \Phi \mid \Gamma \vdash f(s) \equiv a[s/t] \\
\end{array}
\]

Figure 1. Extension types: formation and introduction

\[
\begin{array}{c}
\{t : I \mid \phi\} \text{ shape} \\
\{t : I \mid \psi\} \text{ shape} \\
\Xi, t : I \mid \Phi, \psi \mid \Gamma \vdash A \text{ type} \quad \Xi, t : I \mid \Phi, \phi \mid \Gamma \vdash s : I \\
\Xi \mid \Phi \mid \Gamma \vdash \psi[s/t] \\
\Xi \mid \Phi \mid \Gamma \vdash f(s) : A \\
\Xi \mid \Phi \mid \Gamma \vdash f(s) \equiv a[s/t] \\
\end{array}
\]

Figure 2. Extension types: elimination and computation
\[
\begin{aligned}
\{ t: I \mid \phi \} \text{ shape} & \quad \{ t: I \mid \psi \} \text{ shape} & \quad t: I \mid \phi \vdash \psi \\
\Xi \mid \Phi \vdash \Gamma \text{ ctx} & \quad \Xi, t: I \mid \Phi, \psi \mid \Gamma \vdash A \text{ type} & \quad \Xi, t: I \mid \Phi, \phi \mid \Gamma \vdash a: A \\
\Xi, t: I \mid \Phi, \phi \mid \Gamma \vdash b: A & \quad \Xi, s: I \mid \Phi \mid \psi[s/t] \quad \Xi \mid \Phi \vdash (\lambda t^I. b)(s) \equiv b[s/t]
\end{aligned}
\]

(Ext-\(\beta\))

\[
\begin{aligned}
\{ t: I \mid \phi \} \text{ shape} & \quad \{ t: I \mid \psi \} \text{ shape} & \quad t: I \mid \phi \vdash \psi \\
\Xi \mid \Phi \mid \Gamma \vdash f : \langle \prod_{t: I} A \mid \phi \rangle & \quad \Xi \mid \Phi \mid \Gamma \vdash f \equiv \lambda t^I. f(t)
\end{aligned}
\]

(Ext-\(\eta\))

\textbf{Figure 3.} Extension types: \(\beta\)-conversion and \(\eta\)-abstraction

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