ASSessing Mediation Processes in Parallel Bilinear Spline Growth Curve Models in the Framework of Individual Measurement Occasions

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July 20, 2021

ABSTRACT

Multiple existing studies have developed multivariate growth models with nonlinear functional forms to explore joint development where two longitudinal records are associated over time. However, multiple repeated outcomes are not necessarily synchronous. Accordingly, it is of interest to investigate an association between two repeated variables on different occasions, for example, how a short-term change of one variable affects a long-term change of the other(s). One statistical tool for such analyses is longitudinal mediation models. In this study, we extend latent growth mediation models with linear trajectories (Cheong et al., 2003) and develop two models to evaluate mediational processes where the bilinear spline (i.e., the linear-linear piecewise) growth model is utilized to capture the change patterns. We define the mediational process as either the baseline covariate or the change of covariate influencing the change of the mediator, which, in turn, affects the change of the outcome. We present the proposed models by simulation studies. Our simulation studies demonstrate that the proposed mediational models can provide unbiased and accurate point estimates with target coverage probabilities with a 95% confidence interval. To illustrate modeling procedures, we analyze empirical longitudinal records of multiple disciplinary subjects, including reading, mathematics, and science test scores, from Grade K to Grade 5. The empirical analyses demonstrate that the proposed model can estimate covariates’ direct and indirect effects on the change of the outcome. Through the real-world data analyses, we also provide a set of feasible recommendations for empirical researchers. We also provide the corresponding code for the proposed models.

Keywords  Mediation Processes with Nonlinear Trajectories · Unknown Knot Locations · Individual Measurement Occasions · Simulation Studies

1 Introduction

In the studies with a longitudinal design, multiple variables of interest are collected together. For example, in clinical trials, the primary endpoint and multiple secondary endpoints are often recorded repeatedly. In observational studies, such as The Early Childhood Longitudinal Study, Kindergarten Class (ECLS-K), test scores of multiple disciplinary subjects are collected in each school year. Given that a longitudinal process exhibit a nonlinear relationship with respect to time $t$ if there are some periods where change is more rapid than in others, earlier studies, such as Blozis (2004); Blozis et al. (2008); Peralta et al. (2020); Liu and Perera (2021), have developed multivariate growth models (MGMs) (Grimm et al., 2016, Chapter 8) to explore joint nonlinear developmental processes, with a focus on an association between these processes by estimating the covariances of between-construct growth factors.

These MGMs have been shown useful to examine joint nonlinear development; however, the longitudinal processes are not necessarily synchronous in multiple domains. In the educational field, for example, reading is a crucial

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According to Baron and Kenny (1986), the most common situation in social science research is partial mediation, where we may be of interest to investigate why students’ reading ability positively affects their science ability at a certain causal effects are instantaneous and that the magnitude of a causal effect remains the same over time. On the other hand, occasions of the predictor, mediator, and outcome. However, the utilization of cross-sectional data assumes that the applying traditional mediation models to cross-sectional data. On the one hand, the causal relationships often take time to develop longitudinal models to investigate mediational processes to estimate regression coefficients between growth factors.

1.1 Cross-sectional Mediation Model

In this section, we first introduce the cross-sectional mediation model proposed by Baron and Kenny (1986), with which we may be of interest to investigate why students’ reading ability positively affects their science ability at a certain grade. One possible explanation of the relationship between reading and science ability could be that students with greater reading scores tend to perform better in mathematics tests, leading to higher science ability. Such a model proposes that mathematics ability partially explains the association between reading ability and science ability. Here, the role of mathematics skills in accounting for the connection between the other two abilities can be formally assessed by following Baron and Kenny’s four steps to test for mediation.

1. Step 1: Test whether the reading ability (predictor) and the science ability (outcome) are related directly. We specify the regression coefficient of the path between the predictor and outcome as $c$. Accordingly, the first assumption of mediation is that $c$ has to be different from 0.

2. Step 2: Test whether the reading ability (predictor) is related to the mathematics ability (mediator). We specify the regression coefficient of the path between the predictor and mediator as $a$, and therefore, the second assumption of mediation is that $a$ has to be different from 0.

3. Step 3: Test whether the mathematics ability (mediator) is related to the science ability (outcome) while controlling for the predictor. With specifying the path between the mediator and outcome as $b$, the third assumption of mediation is that $b$ should be different from 0.

4. Step 4: Test whether the relationship between the predictor and outcome changes when we include the mediator in the model. We specify the path coefficient between the predictor and outcome with the inclusion of the mediator as $c'$. We achieve complete mediation when the relationship between the predictor and the outcome disappears when controlling the mediator (i.e., $c' = 0$), whereas we have partial mediation when the relationship between these two variables reduces when introducing the mediator variable (i.e., $c' < c$).

According to Baron and Kenny (1986), the most common situation in social science research is partial mediation, where the mediator only explains part of the relationship between the predictor and the outcome. We then discuss how to test for partial mediation and decompose the total effect to obtain the mediated effect and direct effect. Following earlier studies, for example, MacKinnon et al. (2000), the total effect $c$ can be decomposed into the direct effect $c'$ and the indirect effect through the other two paths, $a$ and $b$; that is, that is, we can express the total effect $c = c' + a \times b$, where the product of $a$ and $b$ is a metric of the size of the mediated effect.

One challenge that lies in mediation analyses is obtaining standard errors of mediated effects, based on which we can conduct a hypothesis test for statistically significant partial mediation and construct a 95% confidence interval (CI) for the indirect effect. Generally, earlier studies have recommended employing the (multivariate) Delta method, for example, Sobel (1982, 1986); MacKinnon et al. (2002) to approximate the standard error of the product of $a$ and $b$, or utilizing bootstrap techniques to calculate more accurate estimates (Hayes, 2009; Shrout and Bolger, 2002). Moreover, Cheung and Lau (2008) has shown that bias-corrected bootstrap CIs performs best in tests for mediated effects compared to other methods. In the current study, we do not intend to develop a novel approach to obtaining standard errors of mediated effects or compare different approaches for constructing 95% CIs. Instead, we utilize the Delta method to obtain standard errors of mediated effects as more accurate standard errors are out of this study’s research interests. More importantly, this type of standard errors can be automatically obtained from multiple software of the structural equation modeling (SEM) framework such as Mplus 8 and OpenMx.

Earlier studies, for example, Gollob and Reichardt (1987) have described that multiple fundamental problems of applying traditional mediation models to cross-sectional data. On the one hand, the causal relationships often take time to unfold, implying that the magnitude of the causal effect usually depends on the elapsed time between measurement occasions of the predictor, mediator, and outcome. However, the utilization of cross-sectional data assumes that the causal effects are instantaneous and that the magnitude of a causal effect remains the same over time. On the other hand,
using cross-sectional data leaves out measurements at previous times that are potential predictors of the constructed model. Additionally, Cole and Maxwell (2003); Maxwell and Cole (2007) have demonstrated that cross-sectional mediation models typically yield substantially biased estimates for longitudinal parameters. Accordingly, longitudinal data are to be favored for testing mediation hypotheses.

### 1.2 Longitudinal Mediation Model

In this section, we first briefly review three modeling frameworks to investigate mediation in longitudinal data. Researchers have explored mediational processes in three frameworks: cross-lagged panel models (Cole and Maxwell, 2003), latent growth curve models (Cheong et al., 2003), and latent change score models (MacKinnon, 2008, Chapter 8). Cole and Maxwell (2003) and MacKinnon (2008, Chapter 8) have provided extensive overviews of the application of cross-lagged panel mediation models. We need at least three measurements of all three constructs in a fully-specified model, the predictor (for example, reading ability), the mediator (for example, mathematics ability), and the outcome (for example, science ability) to test how the reading ability at the first wave (i.e., $t_1$) affects mathematics ability at the second wave (i.e., $t_2$), and in turn, influences science ability at the third wave (i.e., $t_3$).

Cheong et al. (2003) describes how an intervention program (the predictor) affects the change of the outcome through the initial status of the mediator using a parallel latent growth curve model. MacKinnon (2008, Chapter 8) demonstrated how to extend the model to investigate three parallel longitudinal processes. Additionally, MacKinnon (2008, Chapter 8) and Selig and Preacher (2009) have described how to employ latent change score models (McArdle and Nesselroade, 1994) to assess mediation. In addition to the mediational effects that can be examined by the cross-lagged panel models, the latent change score modeling framework also allows for investigating how the change of the predictor during the first time interval (i.e., $t_2 - t_1$) influences the change of the mediator during the second time interval (i.e., $t_3 - t_2$), and in turn, affects the change of the outcome during the third time interval (i.e., $t_4 - t_3$). Selig and Preacher (2009) has shown that the mediation effects can also be any combination of measurements and change of constructs. Earlier studies, for example, MacKinnon (2008, Chapter 8) and Selig and Preacher (2009), have provided more detailed discussion in terms of the theory of change for three variables (i.e., the predictor, mediator, and outcome), the role of time, and the types of indirect effects of these three longitudinal mediation models.

The present study focuses on the latent growth mediation models. The existing latent growth mediation models developed in Cheong et al. (2003) and MacKinnon (2008, Chapter 8) assume that change patterns of each repeated variable take the linear functional form. However, it has been suggested to employ a two-stage piecewise parallel growth model instead of the parallel linear growth model to evaluate nonlinear trajectories (Cheong et al., 2003). Accordingly, in this present study, we developed two models to investigate mediational processes where the parallel bilinear spline growth model (PBLSGM) (Liu and Perera, 2021) is employed to examine joint longitudinal processes. The first model can be utilized to evaluate a mediational process with a baseline predictor and subsequently two variables that are repeatedly measured over time, while the second model can be employed to investigate a mediational process with three longitudinal variables. With the assumed linear-linear piecewise functional form, we have the initial status, short-term change, measurement at the knot (i.e., the change-point at which two linear segments join together), and long-term change for each longitudinal process.

In addition to the initial status and the change rate of each piece, the change point or knot where the change of the growth rate occurs must be determined when employing the linear-linear piecewise functional form to explore a longitudinal process. Earlier studies have demonstrated that the knot can either be pre-specified by domain theories (Dumenci et al., 2019; Flora, 2008; Riddle et al., 2015; Sterba, 2014), or be estimated as an unknown parameter (Cudeck and du Toit, 2003; Harring et al., 2006; Kwok et al., 2010; Kohli, 2011; Kohli et al., 2013; Kohli and Harring, 2013; Preacher and Hancock, 2015; Liu, 2019; Liu et al., 2019a,b; Dominicus et al., 2008; McArdle and Wang, 2008; Wang and McArdle, 2008; Muniz Terrera et al., 2011; Kohli et al., 2015; Lock et al., 2018).

In the two proposed models, we assume that the knot of each longitudinal process is unknown and to be estimated for two considerations. On the one hand, time is important in a longitudinal mediation model, and for each process, the knot is the transition time of the two stages. Accordingly, we want to avoid any unnecessary pre-specification of the knot. On the other hand, the knot measurement can be viewed as a developmental milestone or an event of a longitudinal process. For example, the development of mathematics ability (i.e., mediator process) that achieves the milestone may lead to the event or the long-term development of science ability (i.e., the outcome process). So ideally, the knot of the predictor process occurs first, followed by the knot of the mediator process and then the knot of the outcome process. We want to obtain data-driven evidence for this temporal order from the proposed models. Although multiple existing studies, for example, Preacher and Hancock (2015); Liu et al. (2019a); Peralta et al. (2020); Liu and Perera (2021) have demonstrated that the random effect of the knot can also be estimated, we assume that each process-specified knot is the same across all individuals since the knot variance is out of the research interests of the present study. Additionally, similar to Liu and Perera (2021), we construct the proposed models in the framework of
We then write the repeated outcome as \( \eta \) where \( y \).

The remainder of this article is organized as follows. In the method section, we start from a bilinear spline growth model to estimate a fixed knot for a univariate longitudinal process. We then extend it to longitudinal mediation models and introduce the model specification of the proposed models where we define the mediational process as either the baseline covariate or the change of covariate influencing the change of the mediator, which, in turn, affects the change of the outcome. Next, we describe the model estimation and model evaluation that is realized by the Monte Carlo simulation. We then present simulation results and evaluate the proposed models in terms of non-convergence rate and the performance measures, which include the relative bias, the empirical standard error (SE), the relative root-mean-squared error (RSME), and the empirical coverage probability for a nominal 95% confidence interval of each parameter. Next, in the application section, we demonstrate how to apply the proposed models to examine a data set of longitudinal records of reading, mathematics, and science abilities from Early Childhood Longitudinal Study, Kindergarten Class of 2010 – 11 (ECLS-K: 2011). Finally, discussions are framed regarding practical considerations, methodological considerations, and future directions.

2 Method

2.1 Bilinear Spline Growth Curve Model with a Fixed Knot

In this section, we briefly describe a latent growth curve (LGC) model with a linear-linear functional form to estimate a fixed knot in the framework of individual measurement occasions. As shown in Figure 1, we specify a separate linear function for each of the two phases of the developmental process and express the measurement for the \( i^{th} \) individual at the \( j^{th} \) time in the framework of individual measurement occasions as

\[
y_{ij} = \begin{cases} 
\eta_{0i}^{[y]} + \eta_{1i}^{[y]} t_{ij} + \epsilon_{ij}^{[y]} & t_{ij} \leq \gamma^{[y]} \\
\eta_{1i}^{[y]} + \eta_{2i}^{[y]} (t_{ij} - \gamma^{[y]}) + \epsilon_{ij}^{[y]} & t_{ij} > \gamma^{[y]}
\end{cases}
\]

(1)

where \( y_{ij} \) and \( t_{ij} \) are the measurement and measurement occasion of the \( i^{th} \) individual at wave \( j \). In Equation (1), \( \eta_{0i}^{[y]} \), \( \eta_{1i}^{[y]} \) and \( \eta_{2i}^{[y]} \) are the individual-level intercept, first slope and second slope, which are usually called ‘growth factors’; they three along with the fixed knot \( \gamma^{[y]} \) together determine the functional form of the growth curve of \( y_i \).

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Existing SEM software such as Mplus and the R package OpenMx does not allow for conditional statement of an unknown parameter such as \( \gamma^{[y]} \) in Equation (1), so the piecewise functional form above cannot be specified directly. In order to unify pre- and post-knot expressions, we have to reparameterize growth factors, which can be realized through multiple approaches. Harring et al. (2006) proposed to reparameterize three growth factors in Equation (1) (i.e., \( \eta_{0i}^{[y]} \), \( \eta_{1i}^{[y]} \) and \( \eta_{2i}^{[y]} \)) to the average of the two intercepts, the average of the two slopes, and the half difference between the two slopes. Alternatively, Grimm et al. (2016, Chapter 11) suggested reexpressing the three growth factors to the measurement at the knot and two slopes. Additionally, Liu et al. (2019a) reparameterized \( \eta_{0i}^{[y]} \), \( \eta_{1i}^{[y]} \) and \( \eta_{2i}^{[y]} \) as the measurement at the knot, the average of the two slopes, and the half difference between the two slopes.

The reparameterized growth factors proposed in Harring et al. (2006); Liu et al. (2019a) are no longer directly related to the underlying developmental process and therefore are less useful in the present study where we want to estimate regression coefficients between original growth factors. Accordingly, we follow the reparameterized approach in Grimm et al. (2016, Chapter 11), where all three reparameterized coefficients are still directly related to the growth patterns. We then write the repeated outcome as

\[
y_{ij} = (\eta_{0i}^{[y]} + \eta_{1i}^{[y]} \gamma^{[y]}) + \eta_{1i}^{[y]} \min(0, t_{ij} - \gamma^{[y]}) + \eta_{2i}^{[y]} \max(0, t_{ij} - \gamma^{[y]}) + \epsilon_{ij}^{[y]}
\]

where \( y_{ij}^{[y]} \) is the measurement at the knot of the \( i^{th} \) individual.
2.2 Model Specification of Mediation Model with Baseline Covariate and Longitudinal Mediator and Outcome

In this section, we extend the univariate linear-linear latent growth curve model to analyze a mediational process with baseline covariate as well as longitudinal mediator and outcome. Suppose both the mediator and the outcome take the linear-linear functional form with an unknown fixed knot. In such situations, we can construct a mediational process where the baseline covariate influences the change of the outcome directly and indirectly through its effect on the change of the mediator. The longitudinal mediation model can be specified as

\[
\begin{pmatrix}
  m_i \\
  y_i 
\end{pmatrix}
= \begin{pmatrix}
  \Lambda_i^m \\
  0 \\
  \Lambda_i^y 
\end{pmatrix}
\times
\begin{pmatrix}
  \eta_i^m \\
  \eta_i^y 
\end{pmatrix}
+ \begin{pmatrix}
  \epsilon_i^m \\
  \epsilon_i^y 
\end{pmatrix},
\]

where \( m_i \) and \( y_i \) are a \( J \times 1 \) vector of the repeated measurements of the mediator and outcome of the \( i \)th individual, respectively (in which \( J \) is the number of measurement occasions). With an assumption that the trajectories of mediator and outcome process take the linear-linear functional form with an unknown knot, \( \eta_i^u \) (\( u = m, y \)) is a \( 3 \times 1 \) vector of growth factors,

\[
\eta_i^u = \begin{pmatrix}
  \eta_i^{u_1} \\
  \eta_i^{u_2} \\
  \eta_i^{u_3}
\end{pmatrix}^T,
\]

representing the slope of the first stage, the measurement at the knot and the slope of the second stage. The corresponding factor loadings \( \Lambda_i^u \), a \( J \times 3 \) matrix, is expressed as

\[
\Lambda_i^u = \begin{pmatrix}
  \min(0, t_{ij} - \gamma^u) & \max(0, t_{ij} - \gamma^u) & 1 \\
  0 & 0 & 1 \\
  & & 1
\end{pmatrix} \quad (j = 1, \cdots, J).
\]

where the subscript \( i \) indicates that we build the model in the framework of individual measurement occasions. Additionally, \( \epsilon_i^u \) is a \( J \times 1 \) vector of residuals of the \( i \)th individual.

To define a mediational process with a baseline covariate as well as longitudinal mediator and outcome, we regress the growth factors of the mediator process on the predictor for the \( i \)th individual

\[
\eta_i^m = \alpha^m + B_{x \rightarrow m} x_i + \zeta_i^m,
\]

where \( x_i \), which is either continuous or binary, is the baseline covariate of the \( i \)th individual, \( \alpha^m \) is a \( 3 \times 1 \) vector of growth factor intercepts of the mediator process (which is the mean vector of growth factors of the mediator if the covariate is centered), and \( B_{x \rightarrow m} \) is a \( 3 \times 1 \) vector of regression coefficients from the predictor to the first slope, measurement at the knot and second slope of the mediator process, that is

\[
B_{x \rightarrow m} = \begin{pmatrix}
  \beta_{11}^x \\
  \beta_{12}^x \\
  \beta_{13}^x \\
  \beta_{21}^x \\
  \beta_{22}^x \\
  \beta_{23}^x \\
  \beta_{31}^x \\
  \beta_{32}^x \\
  \beta_{33}^x \\
\end{pmatrix}
\]

Additionally, we need to regress the growth factors of the outcome process on the predictor and the growth factors of the mediator process

\[
\eta_i^y = \alpha^y + B_{x \rightarrow y} x_i + B_{m \rightarrow y} \eta_i^m + \zeta_i^y,
\]

where \( \alpha^y \) is a \( 3 \times 1 \) vector of growth factor intercepts of the outcome process, \( B_{x \rightarrow y} \) is a \( 3 \times 1 \) vector of regression coefficients from the predictor to the first slope, measurement at the knot and second slope of the outcome process

\[
B_{x \rightarrow y} = \begin{pmatrix}
  \beta_{11}^x \\
  \beta_{12}^x \\
  \beta_{13}^x \\
  \beta_{21}^y \\
  \beta_{22}^y \\
  \beta_{23}^y \\
  \beta_{31}^y \\
  \beta_{32}^y \\
  \beta_{33}^y \\
\end{pmatrix}
\]

Additionally, \( B_{m \rightarrow y} \) is a \( 3 \times 3 \) matrix of regression coefficients from the growth factors of the mediator process to those of the outcome process. The coefficient matrix can be expressed as

\[
B_{m \rightarrow y} = \begin{pmatrix}
  \beta_{11}^{m \rightarrow y} & 0 & 0 \\
  \beta_{12}^{m \rightarrow y} & \beta_{12}^{m \rightarrow y} & 0 \\
  \beta_{13}^{m \rightarrow y} & \beta_{13}^{m \rightarrow y} & \beta_{13}^{m \rightarrow y}
\end{pmatrix}
\]

In Equations (4), (6), (7), the superscript and subscript of \( \beta \) together define the path of the corresponding coefficient. For example, \( \beta_{11}^{x \rightarrow m} \) is the path coefficient from the baseline covariate to the first slope of the mediator process. Similarly, \( \beta_{1r}^{m \rightarrow y} \) is the coefficient from the first slope of the mediator process to the knot measurement of the outcome process. Additionally, \( \zeta_i^y \) is a \( 3 \times 1 \) vector of deviations of the \( i \)th individual from the mean values of the variable-specific growth factors. We list six possible paths of indirect effects of the predictor on the outcome process in the specified model in Table 1.

Insert Table 1 about here
2.3 Model Specification of Mediation Model with Longitudinal Covariate, Mediator and Outcome

In this section, we extend the univariate linear-linear latent growth curve model to investigate a mediational process with a longitudinal covariate, mediator, and outcome. Suppose all three variables take the linear-linear functional form with an unknown fixed knot. In such situations, we can construct a longitudinal mediation model to explore how the change of covariate influences the change of the outcome directly and indirectly through its effect on the change of the mediator. The longitudinal mediation model can be specified as

\[
\begin{pmatrix}
\mathbf{x}_i \\
\mathbf{m}_i \\
\mathbf{y}_i
\end{pmatrix} = \begin{pmatrix}
\mathbf{A}_1^{[x]} & 0 & 0 \\
0 & \mathbf{A}_1^{[m]} & 0 \\
0 & 0 & \mathbf{A}_1^{[y]}
\end{pmatrix} \times \begin{pmatrix}
\mathbf{\eta}_i^{[x]} \\
\mathbf{\eta}_i^{[m]} \\
\mathbf{\eta}_i^{[y]}
\end{pmatrix} + \begin{pmatrix}
\mathbf{\epsilon}_i^{[x]} \\
\mathbf{\epsilon}_i^{[m]} \\
\mathbf{\epsilon}_i^{[y]}
\end{pmatrix},
\]

which also defines the covariate process in addition to the mediator and outcome processes specified in Equation (2). In Equation (8), \(\mathbf{x}_i\) is a \(J \times 1\) vector of the repeated measures of the \(i^{th}\) individual, \(\mathbf{A}_1^{[x]}\), \(\mathbf{\eta}_i^{[x]}\), and \(\mathbf{\epsilon}_i^{[x]}\) are its growth factors (a \(3 \times 1\) vector), the corresponding factor loadings (a \(J \times 3\) matrix), and the residuals (a \(J \times 1\) vector), respectively. We then write the growth factors of the covariate process as deviations from the corresponding growth factor means

\[
\mathbf{\eta}_i^{[x]} = \mu^{[x]}_i + \zeta^{[x]}_i,
\]

where \(\mu^{[x]}_i\) and \(\zeta^{[x]}_i\) are the mean vector of growth factors (a \(3 \times 1\) vector) and the deviations (a \(3 \times 1\) vector) of the covariate process. The growth factors of the mediator process are regressed on those of the covariate, as shown below

\[
\begin{pmatrix}
\mathbf{\eta}_i^{[m]} \\
\mathbf{\eta}_i^{[y]}
\end{pmatrix} = \begin{pmatrix}
\alpha^{[m]} + \mathbf{B}^{[x \rightarrow m]} \mathbf{\eta}_i^{[x]} + \mathbf{\zeta}^{[m]}_i \\
\alpha^{[y]} + \mathbf{B}^{[x \rightarrow y]} \mathbf{\eta}_i^{[x]} + \mathbf{\zeta}^{[y]}_i
\end{pmatrix},
\]

where \(\alpha^{[m]}\) is a \(3 \times 1\) vector of growth factor intercepts of the mediator, \(\mathbf{B}^{[x \rightarrow m]}\) is a \(3 \times 3\) matrix of regression coefficients from the growth factors of the covariate process to those of the mediator process

\[
\mathbf{B}^{[x \rightarrow m]} = \begin{pmatrix}
\beta^{[x \rightarrow m]}_{11} & 0 & 0 \\
\beta^{[x \rightarrow m]}_{12} & \beta^{[x \rightarrow m]}_{11} & 0 \\
\beta^{[x \rightarrow m]}_{13} & \beta^{[x \rightarrow m]}_{12} & \beta^{[x \rightarrow m]}_{11}
\end{pmatrix}.
\]

Similarly, the growth factors of the outcome process are regressed on those of the covariate and mediator

\[
\begin{pmatrix}
\mathbf{\eta}_i^{[y]}
\end{pmatrix} = \begin{pmatrix}
\alpha^{[y]} + \mathbf{B}^{[x \rightarrow y]} \mathbf{\eta}_i^{[x]} + \mathbf{\zeta}^{[y]}_i
\end{pmatrix},
\]

where \(\alpha^{[y]}\) is a \(3 \times 1\) vector of growth factor intercepts of the outcome, \(\mathbf{B}^{[x \rightarrow y]}\) (\(\mathbf{B}^{[m \rightarrow y]}\)) is a \(3 \times 3\) matrix of regression coefficients from the growth factors of the covariate (mediator) process to those of the outcome process. The coefficient matrix \(\mathbf{B}^{[x \rightarrow y]}\) is defined as

\[
\mathbf{B}^{[x \rightarrow y]} = \begin{pmatrix}
\beta^{[x \rightarrow y]}_{11} & 0 & 0 \\
\beta^{[x \rightarrow y]}_{12} & \beta^{[x \rightarrow y]}_{11} & 0 \\
\beta^{[x \rightarrow y]}_{13} & \beta^{[x \rightarrow y]}_{12} & \beta^{[x \rightarrow y]}_{11}
\end{pmatrix},
\]

and \(\mathbf{B}^{[m \rightarrow y]}\) has the same expression as such in Equation (7). Ten possible paths of indirect effects of the predictor process on the outcome process in the specified model are listed in Table 1.

2.4 Model Estimation

To simplify estimation, we make the following four assumptions. First, we assume that the covariate’s growth factors are normally distributed. Second, we assume that the mediator’s growth factors are normally distributed conditional on the baseline covariate (or growth factors of the covariate process). The third assumption is that the outcome’s growth factors are normally distributed conditional on the baseline covariate (or growth factors of the covariate process) and the growth factors of the mediator process. Accordingly, \(\mathbf{\zeta}^{[u]}_i \sim \text{MVN}(\mathbf{0}, \Psi^{[u]}_i)\) \((u = x, m, y)\), where \(\Psi^{[u]}_i\) is a \(3 \times 3\) (unexplained) variance-covariance matrix of growth factors of the variable \(u\). We also assume that the individual residuals \(\mathbf{\epsilon}^{[m]}_i\) are identical and independent normal distributions over time, and the covariances between residuals are homogeneous over time. We define \(\mathbf{I}\) is a \(J \times J\) identity matrix, and for the model specified in Section 2.2 and Section 2.3, the corresponding variance-covariance matrix is

\[
\begin{pmatrix}
\mathbf{\epsilon}^{[m]}_i \\
\mathbf{\epsilon}^{[y]}_i
\end{pmatrix} \sim \text{MVN} \left( \mathbf{0}, \begin{pmatrix}
\theta^{[m]}_e \mathbf{I} & \theta^{[m \rightarrow y]}_e \\
\theta^{[y \rightarrow m]}_e & \theta^{[y]}_e \mathbf{I}
\end{pmatrix} \right),
\]
We estimate \( mxAlgebra() \) we can specify the additional parameters in the function (outcome) process and those in Table 1; then

\[
\Theta_1 = \{ \mu_x, \Phi_x, \alpha^{[u]}, B^{[x \rightarrow m]}, B^{[y \rightarrow x]}, B^{[y \rightarrow m]}, \Psi^{[u]}, \theta^{[x]}, \theta^{[m]}, \theta^{[y]} \} (u = m, y)
\]

to list the parameters specified in the first proposed longitudinal mediation model, where \( B^{[x \rightarrow m]} \) and \( B^{[y \rightarrow x]} \) are those defined in Equations (4) and (6), respectively.

The parameters in the proposed model in Section 2.3 include the mean vector (\( \mu^{[x]} \)) and variance-covariance matrix (\( \Psi^{[s]} \)) of the growth factors of the mediator (outcome) process, the coefficients between growth factors of three longitudinal processes, the residual variances and covariances. We define \( \Theta_2 \) as

\[
\Theta_2 = \{ \mu^{[n]}, \Psi^{[n]}, \alpha^{[n]}, B^{[x \rightarrow m]}, B^{[y \rightarrow x]}, B^{[y \rightarrow m]}, \Psi^{[n]}, \theta^{[x]}, \theta^{[m]}, \theta^{[y]} \} (u = m, y)
\]

to list the parameters specified in the second longitudinal mediation model, where \( B^{[x \rightarrow m]} \) and \( B^{[y \rightarrow x]} \) are those defined in Equations (11) and (13), respectively.

We estimate \( \Theta_1 \) and \( \Theta_2 \) using full information maximum likelihood (FIML) to account for the individual measurement occasions and the potential heterogeneity of individual contributions to the likelihood function. In this work, we constructed the proposed longitudinal mediation models using the R package OpenMx with CSOLNP optimizer (Pritikin et al., 2015; Neale et al., 2016; Boker et al., 2020; Hunter, 2018).

In addition to the parameters that can be estimated directly, in practice, we are also interested in estimating the conditional mean vector and variance-covariance matrix of the growth factors of the mediator (outcome) process and indirect effects as well as the total effects listed in Table 1. When fitting the model using the R package OpenMx, we can specify the additional parameters in the function mxAlgebra() (Boker et al., 2020). We need to provide the algebraic expressions of the conditional mean vector and variance-covariance matrix of growth factors of the mediator (outcome) process and those in Table 1; then OpenMx is capable of computing the point estimates along with their standard errors of these additional parameters (that is realized by the Delta Method). We provide the expressions of the conditional mean vector and variance-covariance matrix of growth factors of the mediator (outcome) process in the Online Supplementary Document. Additionally, we provide OpenMx code in the online appendix (https://github.com/Veronica0206/Extension_projects) to demonstrate how to construct the proposed models and estimate these additional parameters.

3 Model Evaluation

In this section, we examine the proposed longitudinal mediation models using Monte Carlo simulation studies to examine the performance measures, including the relative bias, the empirical standard error (SE), the relative root-mean-square error (RMSE), and the empirical coverage probability for a nominal 95\% confidence interval of each parameter of the proposed models. Table 2 lists the definitions and estimates of these four performance measures. Specifically, the relative bias and empirical SE quantify whether the model, on average, targets the population values and whether estimates are precise. The relative RMSE integrates the bias and the precision metric into one measure. The coverage probability tells how well the interval estimate covers the corresponding population value.

We decided to replicate the simulation study \( S = 1,000 \) by an empirical approach following Morris et al. (2019). Specifically, we run a pilot simulation study, the standard error of bias of all parameters except the (unexplained) variance of knot measurement of each longitudinal process (i.e., \( \psi^{[u]}_{\gamma}, u = x, m, y \)) was less than 0.15. Therefore, we
We carried out the following general steps for the simulation study of the proposed mediation models:

We considered three levels of indirectly immediate effects and two levels of indirectly delayed effects. Similarly, we considered the knot of each longitudinal process is in the middle of the study duration (i.e., $\mu_{\gamma}^{u} = 2.5, u = x, m, y$). We considered the other condition, ten measurement occasions, for two reasons. On the one hand, we wanted to examine whether an increasing number of repeated measures would improve model performance. More importantly, this condition allowed us to evaluate the model performance under two scenarios: (1) the knot of each longitudinal process is in the middle of the study duration (i.e., $\mu_{\gamma}^{u} = 4.5, u = x, m, y$); (2) the transition time of the mediator (covariate) process occurs earlier than that of the outcome (mediator) process (i.e., $\mu_{\gamma}^{m} = 3.5$ and $\mu_{\gamma}^{y} = 5.5$ for the first model, while $\mu_{\gamma}^{x} = 3.5, \mu_{\gamma}^{m} = 3.5$ and $\mu_{\gamma}^{y} = 5.5$). In addition, to account for individual measurement occasions, we allowed a time window with width ($-0.25$, $+0.25$) around each wave, which is considered a ‘medium’ deviation as in Coulombe et al. (2015).

In the simulation studies, we also want to evaluate how the ratio between indirect effects to direct effects affects the proposed models. Accordingly, for the first mediation model, we fixed the coefficients of the covariate to the growth factors of the mediator process. For the elements in $B^{[m \rightarrow y]}$, we considered the diagonal coefficients and off-diagonal coefficients separately since the diagonal elements are related to the immediate effects, while the off-diagonal elements are related to the delayed effects in a mediational process. We considered three levels of indirectly immediate effects and two levels of indirectly delayed effects. Similarly, we fixed the coefficients in $B^{[x \rightarrow m]}$ and $B^{[x \rightarrow y]}$, but adjusted those in $B^{[m \rightarrow y]}$ to fix the direct effects and manipulate indirect effects of the second longitudinal mediation model. Additionally, we examined three common change patterns, two levels of residual variances ($\theta_{u}^{[m]} = 1$ or $2$) and set the residual correlation as 0.3. We also examined the models at two levels of sample size, $n = 200$ and $n = 500$. We provide the detailed simulation designs for the two longitudinal mediation models in the Online Supplementary Document.

3.1 Design of Simulation Study

The parameters of the most interest in the proposed model are the coefficients $B^{[x \rightarrow m]}$, $B^{[x \rightarrow y]}$ and $B^{[m \rightarrow y]}$, based on which we obtain direct, indirect, and total effects that the baseline covariate (or covariate process) has on the outcome process. The conditions hypothesized to influence the estimation of these coefficients, and other model parameters, included sample size, the number of repeated measurements, the knot locations, the ratio of indirect effects to direct effects, shapes of trajectories, and measurement precision. Accordingly, we fixed the factors, for example, the mean vectors and variance-covariance matrices of the growth factors of longitudinal processes, that presumably do not affect the model performance meaningfully.

In the simulation design, the most important factor is the number of repeated measures since the proposed model is employed to analyze a longitudinal data set. Intuitively, the model should perform better with more repeated measurements. We also realized that the transition time to the second stage of the covariate (mediator) process, ideally, should occur no later than that of the mediator (outcome) process in longitudinal mediation models. We then decided to test two different levels of the number of measurements: six and ten. We selected six as the minimum number of repeated measurements to ensure the proposed model was fully identified\(^3\). For the conditions with six repeated measures, we set the transition time to the second stage of the mediator and outcome process at halfway of study duration ($\mu_{\gamma}^{m} = 2.5, u = x, m, y$). We considered the other condition, ten measurement occasions, for two reasons. On the one hand, we wanted to examine whether an increasing number of repeated measures would improve model performance. More importantly, this condition allowed us to evaluate the model performance under two scenarios: (1) the knot of each longitudinal process is in the middle of the study duration (i.e., $\mu_{\gamma}^{u} = 4.5, u = x, m, y$); (2) the transition time of the mediator (covariate) process occurs earlier than that of the outcome (mediator) process (i.e., $\mu_{\gamma}^{m} = 3.5$ and $\mu_{\gamma}^{y} = 5.5$ for the first model, while $\mu_{\gamma}^{x} = 3.5, \mu_{\gamma}^{m} = 3.5$ and $\mu_{\gamma}^{y} = 5.5$). In addition, to account for individual measurement occasions, we allowed a time window with width ($-0.25$, $+0.25$) around each wave, which is considered a ‘medium’ deviation as in Coulombe et al. (2015).

In the simulation studies, we also want to evaluate how the ratio between indirect effects to direct effects affects the proposed models. Accordingly, for the first mediation model, we fixed the coefficients of the covariate to the growth factors of the outcome progress so that the covariate account for 13% variability of the growth factors (i.e., the medium level according to Cohen (1988, Chapter 9)) and manipulated the coefficients from the covariate to the growth factors of the mediator process and those between growth factors of the mediator and outcome process to adjust indirect effects. Specifically, we considered three levels of $B^{[x \rightarrow m]}$ so that the covariate explains zero, medium (i.e., 13%), and substantial (i.e., 26%) variability of the growth factors of the mediator process. For the elements in $B^{[m \rightarrow y]}$, we considered the diagonal coefficients and off-diagonal coefficients separately since the diagonal elements are related to the immediate effects, while the off-diagonal elements are related to the delayed effects in a mediational process. We considered three levels of indirectly immediate effects and two levels of indirectly delayed effects. Similarly, we fixed the coefficients in $B^{[x \rightarrow m]}$ and $B^{[x \rightarrow y]}$, but adjusted those in $B^{[m \rightarrow y]}$ to fix the direct effects and manipulate indirect effects of the second longitudinal mediation model. Additionally, we examined three common change patterns, two levels of residual variances ($\theta_{u}^{[m]} = 1$ or $2$) and set the residual correlation as 0.3. We also examined the models at two levels of sample size, $n = 200$ and $n = 500$. We provide the detailed simulation designs for the two longitudinal mediation models in the Online Supplementary Document.

3.2 Data Generation and Simulation Step

We carried out the following general steps for the simulation study of the proposed mediation models:

\(^2\)Monte Carlo SE(Bias) = $\sqrt{Var(\hat{\theta})}/S$ (Morris et al., 2019).

\(^3\)Bollen and Curran (2005) has shown that the latent growth model with linear-linear functional form can be identified with at least five waves with a knot at the midway of study duration, though no studies provided information of model identification for the model with an unknown knot.

\(^4\)The immediate effects could be, for example, the coefficient from the first slope of the mediator process to the first slope of the outcome process or that from the knot measurement of the mediator process to the knot measurement of the outcome process.

\(^5\)The delayed effects could be, for example, the coefficients from the first slope of the mediator process to the knot measurement of the outcome process.
1. Generate the baseline covariate (or growth factors of the covariate process), growth factors of the mediator process, and those of the outcome process simultaneously with the prespecified mean vectors and variance-covariance matrices using the R package MASS (Venables and Ripley, 2002).

2. Generate a scaled and equally-spaced time structure with \( J \) waves \( t_j \) and obtain individual measurement occasions: \( t_{ij} \sim U(t_j - 0.25, t_j + 0.25) \) by allowing a time-window with width \((-0.25, 0.25)\) around each wave.

3. Calculate factor loadings for each individual of the (covariate or) mediator or outcome process from individual measurement occasions and the process-specific knot location.

4. Calculate the values of the repeated measurements for each longitudinal process from corresponding growth factors, factor loadings, as well as residual variances and covariance.

5. Implement the proposed models on the generated data set, estimate the parameters, and construct corresponding 95% Wald CIs.

6. Repeat the steps as mentioned above until having 1,000 convergent solutions.

4 Results

4.1 Model Convergence

We first examined the convergence rate under each condition for each model before evaluating how the proposed longitudinal mediation models performed. The proposed model converged satisfactorily. For the first model, where we have baseline covariate and longitudinal mediator and outcome, 300 conditions out of a total of 324 conditions reported a 100% convergence rate. Only 24 condition(s) reported up to four non-convergence replications. For the second model, in which we have longitudinal covariate, mediator, and outcome, 104 conditions out of a total 108 conditions reported a 100% convergence rate. Only four condition(s) reported up to two non-convergence replications.

4.2 Performance Measures

We present the performance metrics, including the relative bias, empirical standard error (SE), relative root-mean-square error (RMSE), and empirical coverage probability (CP) for a nominal 95% confidence interval of each parameter for the proposed models. Both models are capable of estimating parameters unbiasedly, precisely and exhibiting the target confidence interval coverage in general. For each parameter of interest, given the size of the conditions and parameters, we first obtained its four performance measures over 1,000 replications under each condition in the simulation design. We then summarized these metrics across conditions as the corresponding median and range. We provide the summary of the four performance metrics for the two models in the Supplementary Document.

Both models yielded unbiased point estimates along with small empirical standard errors. Specifically, for both models, the magnitude of relative biases of the growth factor means was under 0.004, suggesting that we are able to have unbiased mean trajectories from both models. Additionally, the magnitude of relative biases of the coefficients from the covariate to the outcome growth factors was under 0.03 for the first model, while that of the coefficients from covariate growth factors to mediator (outcome) growth factors except the one from the first slope of the covariate to the second slope of the mediator (outcome) was under 0.05. The median values of relative biases of the indirect effects of the first and the second models were up to 0.016 and 0.030, respectively.

To further examine the relative bias pattern for the coefficients with some bias greater than 10%, including the coefficient from the first slope of the covariate to the second slope of the mediator (outcome) in the second mediation model, we plotted the relative bias under each condition for \( \beta_{12}^{[xu]} \) and \( \beta_{12}^{[xy]} \) in Figures 2a and 2b, respectively. From the figures, we noticed that the estimates of these two coefficients were satisfactory in general (i.e., less than 10%), and most factors that we considered in the simulation design, such as the sample size, the knot locations, the magnitude of indirect effects, and the trajectory shapes, did not affect the relative bias meaningfully. These biased estimates were generated under the conditions with the shorter study duration (i.e., six measurements) and larger residual variance (i.e., \( \theta^{[u]} = 2 \)).

\[ \text{Insert Figure 2 about here} \]

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\[ ^{6} \text{The convergence was defined as arriving at OpenMx status code 0 that indicates a successful optimization until up to 10 runs with different sets of initial values (Neale et al., 2016).} \]
Moreover, estimates obtained from the proposed models were precise: the magnitude of empirical standard errors of the slope- or knot-related parameters were under 0.25 although this value of the parameters related to the knot measurement could achieve 0.57. The relatively large empirical SEs of the parameters related to the knot measurement were due to the large scale of their population values (the population value of intercept means was around 100).

The proposed models are capable of estimating parameters accurately. For both models, the magnitude of relative RMSEs of growth factor means was under 0.09, and for knot was under 0.04. The relative RMSE magnitude of the coefficients, indirect effects, and direct effects was relatively large. In addition, both models performed well regarding empirical coverage since the coverage probabilities of all parameters were around 0.95. We noticed that the coverage probabilities of indirect effects could achieve 100% that is greater than the nominal coverage probability (95%).

As stated earlier, we constructed a 95% Wald confidence interval with the SE obtained by the Delta Method and generated by mla/gebra() automatically for each indirect effect, which should be very similar to the Sobel SE (Sobel, 1982, 1986; Baron and Kenny, 1986). Accordingly, the issue of overestimated SE is in line with an earlier study (Cheung, 2009) and within our expectation to see.

5 Application

In this section, we demonstrate how to employ the proposed longitudinal mediation models to analyze longitudinal records from the Early Childhood Longitudinal Study Kindergarten Cohort: 2010-2011 (ECLS-K: 2011). This section includes two examples. We illustrate how to apply the first model to investigate how the baseline attentional focus affects the development of reading ability and then the development of mathematics ability. We then demonstrate how to implement the second model to explore how the development of reading ability affects that of mathematics and science ability. We randomly extracted 400 students from ECLS-K: 2011 with complete records of repeated reading, mathematics, science item response theory (IRT) scaled scores, age at each wave, and baseline attentional focus.

ECLS-K: 2011 is a national longitudinal study of children registered in about 900 kindergarten programs starting from 2010 – 2011 school year in the United States. In ECLS-K: 2011, participants’ reading and mathematics ability were evaluated in nine waves: fall and spring of kindergarten (2010 – 2011), first (2011 – 2012) and second (2012 – 2013) grade, respectively, as well as spring of 3rd (2014), 4th (2015) and 5th (2016) grade, respectively. Only about 30% students were evaluated in the fall of 2011 and 2012 (Lê et al., 2011). Students’ science assessment started from the spring of kindergarten; accordingly, it was only examined in eight waves. We used children’s age (in months) instead of their grade-in-school to obtain individual measurement occasions in analyses. In the subsample, 47.75% and 52.25% of children were boys and girls. Additionally, the extracted sample was represented by 41.75% White, 9.00% Black, 38.00% Latinx, 6.00% Asian, and 5.25% others.

5.1 Univariate Development

Ideally, as stated earlier, the knot of the covariate process, indicating a milestone or an event of it, occurs first, followed by the knot of the mediator process and then that of the outcome process. Therefore, we can specify the paths from the covariate knot to the other two knots and the path from the mediator knot to the outcome knot if we observe this temporal order. Otherwise, these paths between the knots may not be reasonable since we usually do not expect a later event predict an earlier one. Accordingly, we first constructed a univariate latent growth curve model because we want to estimate the fixed knot for each longitudinal process and then decide possible paths for the proposed models in this section. The estimated knots of the development of reading, mathematics, and science ability were around 93, 100 and 100 months, respectively, suggesting that we can construct longitudinal models with the paths between these knots.

5.2 Baseline Attentional Focus and Longitudinal Reading and Mathematics Ability

In this section, we built the first longitudinal mediation model to assess how the baseline attentional focus affects the development of reading ability and then that of mathematics ability. As shown in Figure 3a, the estimates obtained from the first model lead to a model trajectory that sufficiently captures the smooth line of observed trajectories for each ability. Table 3 presents the estimates of parameters of interest for the first model. Post-knot development of both abilities slowed down substantially. The transition to the slower rate occurred earlier in reading ability (93 months) than in mathematics ability (99 months). Baseline attentional focus positively affects the pre-knot development as well as the knot measurement of reading ability and the knot measurement of mathematics ability. For example, the knot measurement of reading ability and mathematics ability increased 5.9 and 2.3 with one standardized unit increase in baseline attentional focus. Moreover, a child who developed more rapidly in reading ability tended to...
develop more rapidly in mathematics; a child who had higher scores at the knot of reading development tended to have higher mathematics scores at its knot. Additionally, pre-knot development of reading affects the knot measurement of mathematics development positively (estimate was 2.871 with the p-value of 0.0453).

In terms of mediate effects, the baseline attentional focus is associated with the pre-knot development, knot measurement, and post-knot development of the mathematics process indirectly through the corresponding counterpart of the reading process. In addition, we noticed that the effects on the post-knot development of both abilities of the baseline attentional focus were trivial.

5.3 Longitudinal Reading, Mathematics and Science Ability

In this section, we built the second longitudinal mediation model to evaluate how the development of reading ability affects the development of mathematics ability and then that of science ability. From Figure 3b, the estimates obtained from the second model also lead to a model-implied trajectory that sufficiently captures the smooth line of observed values for each disciplinary subject. We list the estimates of parameters of interest for the second model in Table 4, where we can see post-knot development of science ability slowed down slightly. The change to the slower rate occurred earliest in reading ability (93 months), followed by mathematics ability (99 months), and then science ability (99 months). In general, a child who developed more rapidly in reading ability tended to develop more rapidly in mathematics (science) at an early stage, a child who had higher scores at the knot of reading development tended to have higher mathematics (science) scores at the corresponding knot. We also noticed that pre-knot development of mathematics impacted the knot measurement of science IRT scores negatively, which seems counter-intuitive. We will provide possible explanations in the Discussion section.

In terms of mediate effects, the pre-knot development, knot measurement, and post-knot development of the reading ability positively affected the corresponding values of science ability through those of mathematics ability. Additionally, pre-knot development of reading ability influenced the knot measurement of mathematics scores and then that of science scores. The knot measurement of reading ability also impacted the post-knot development of mathematics skills and then that of science skills. In terms of total effects, all three growth factors of reading development affected the corresponding immediate and delayed growth factors of science development. For example, the total effects of the pre-knot development of reading ability on the pre-knot development, knot measurement, and post-knot development of science ability were 0.339, 7.347, and −0.330, respectively.

6 Discussion

In this article, we propose two longitudinal mediation models to evaluate how a covariate, either its baseline value or its process, affects a mediator process, and thus an outcome process. We employ the bilinear spline functional form to approximate the underlying change patterns of all longitudinal processes in the proposed models. This functional form allows for exploring how the change rate during the early stage or the value at a milestone (i.e., the measurement at a knot) of the covariate (mediator) process affects the change rate during the late stage of the mediator (outcome) process. We conducted extensive simulation studies to evaluate the proposed models in terms of convergence rate and performance metrics, including the relative bias, empirical standard error, relative RMSE, and coverage probability. We also illustrate the proposed models using a real-world data set from longitudinal records of reading, mathematics, and science IRT scores from Grade K to Grade 5. The results demonstrate the models’ valuable capabilities of capturing the underlying change patterns of nonlinear longitudinal processes, estimating direct and indirect effects on the development of the outcome of the covariate.

6.1 Practical Considerations

In this section, we provide a set of recommendations for empirical researchers who are interested in employing the proposed models. First, we recommend building a univariate growth curve model before constructing the proposed models as we did in the Application section. These univariate growth curve models allow us to assess whether the proposed functional form can capture the underlying change patterns of each longitudinal process. More importantly, we obtain the estimated knot of each longitudinal process, which allows for examining whether the temporal order of the covariate knot, mediator knot, and outcome knot is satisfied. If we have the temporal order as in the Application section, it is reasonable to add the paths between these knots. Otherwise, we may want to remove the paths to avoid a
logical fallacy. For example, if the outcome knot occurred earlier than the mediator knot, we may not want to have the path from the mediator knot to the outcome knot in longitudinal mediation models.

Second, we constructed full models in the simulation study to examine model performance in estimating each possible parameter within research interests. However, we suggest only use the models with all possible paths for exploratory purposes. We do not recommend including all paths to avoid the potential collinearity issue in a confirmatory model, as we encountered in the Application section. In the second example, we found that the direct effects on the knot measurement of science ability of the pre-knot development of reading and mathematics ability were 9.455 and −8.667, respectively. Although pre-knot development of mathematics ability might negatively affect the knot measurement of science ability, a more reasonable explanation is that the pre-knot development of reading and mathematics abilities were highly correlated. Therefore, the estimate of either individual effect was invalid, although the bundle of the pre-knot development of the two abilities can still predict the knot measurement of science ability. Accordingly, the paths included in the confirmatory model, where the individual effect is of interest, need to be carefully determined by specific research questions.

Third, the purpose of our examples was pedagogical; however, researchers should be aware of the multiple statistical tests conducted when fitting a longitudinal mediation model and should consider adjusting hypothesis tests to control the Type I error rate or false discovery rate. Furthermore, decisions about the importance of insights should not only be based on p-values but also consider effect sizes, prior evidence, and alternative explanations (Wasserstein et al., 2019). For instance, in the first example, the effects size of the baseline attentional focus on the post-knot development of reading ability was −0.047. Although it is statistically significant at the 0.05 level, this finding may not be meaningful in practice due to the trivial effect size. Additionally, one should carefully interpret the effect of the pre-knot slope of the covariate to the post-knot slope of the mediator (outcome), especially under conditions with fewer measurement occasions and less precise measurement where the estimates could exhibit some bias greater than 10% based on our simulation study.

6.2 Methodological Considerations and Future Directions

There are several directions to consider for future studies. First, the present study allows for employing the bilinear spline functional form to approximate nonlinear change patterns. This functional form is versatile and valuable for exploring longitudinal processes where different change rates correspond to different stages. More importantly, the estimated knot, which indicates a milestone or an event of a longitudinal process, is especially useful in a mediation model because it helps avoid a logical fallacy and ensure the consistency between theory and statistical method. Although the linear-linear piecewise functional form is sufficient in most cases, such as a developmental process with earlier and later stages in psychological phenomena or a recovery process with short-term and long-term recovery periods in biomedical fields, a functional form with more linear segments may also prove useful in practice. The proposed models can also be extended accordingly.

Second, we constructed the Wald confidence interval (CI) with the SE obtained by the Delta Method for the indirect effects and the total effects in the simulation studies and real-world data analyses. In the simulation studies, we noticed that the coverage effects could achieve 100% that is greater than the nominal coverage probability (95%). Another possible future direction is to construct other types of confidence intervals, such as the bootstrap CIs. Additionally, it is worth conducting simulation studies to evaluate these alternative methods for constructing CIs.

Third, we may not want to include all possible paths to construct a parsimonious model and avoid the potential collinearity issue. Although we recommend building the reduced model driven by specific research questions, it is still worth developing hypothesis testing for comparing the full model and reduced models, especially in an exploratory stage where even empirical researchers only have vague assumptions about causal relationships.

6.3 Concluding Remarks

To summarize, in this article, we propose two longitudinal mediation models to explore multiple nonlinear longitudinal processes. We can evaluate how the baseline covariate or the covariate process affects the outcome process through the mediator process with the proposed models. As discussed above, the proposed methods can be further extended in practice and further examined in methodology.

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**Figure 1:** Within-individual Change over Time with Bilinear Spline Functional Form
Figure 2: Summary of Relative Biases for Coefficients with Some Bias Greater than 10%

(a) Coefficient from Slope 1 of Covariate to Slope 2 Mediator
(b) Coefficient from Slope 1 of Covariate to Slope 2 Outcome

Figure 3: Model-Implied Trajectory and Smooth Line of Development of Academic Abilities

(a) First Longitudinal Mediation Model
(b) Second Longitudinal Mediation Model
Table 1: Possible Indirect Effects and Total Effects on Outcome Process of Covariate

| Baseline Covariate → Longitudinal Mediator → Longitudinal Outcome |
|---------------------------------------------------------------|
| **Putative Mediator** | **Indirect Effects** | **Estimates** |
| $\eta_1^{[m]}$ | $x \rightarrow \eta_1^{[m]} \rightarrow \eta_1^{[y]}$ | $\gamma_{12} \times \eta_1^{[m]} \rightarrow \eta_1^{[m]} \rightarrow \eta_1^{[y]}$ |
| $\eta_1^{[y]}$ | $x \rightarrow \eta_1^{[m]} \rightarrow \eta_1^{[y]}$ | $\beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_2^{[m]}$ | $x \rightarrow \eta_2^{[m]} \rightarrow \eta_2^{[y]}$ | $\beta_2^{[x \rightarrow m]} \times \beta_2^{[m \rightarrow y]}$ |

**Total Effects**

| **Indirect Effects** | **Estimates** |
|---------------------|----------------|
| $x \rightarrow \eta_1^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $x \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |

**Longitudinal Covariate → Longitudinal Mediator → Longitudinal Outcome**

| **Putative Mediator** | **Indirect Effects** | **Estimates** |
|-----------------------|---------------------|----------------|
| $\eta_1^{[m]}$ | $\eta_2^{[y]} \rightarrow \eta_2^{[m]} \rightarrow \eta_2^{[y]}$ | $\beta_2^{[x \rightarrow m]} \times \beta_2^{[m \rightarrow y]}$ |
| $\eta_1^{[y]}$ | $\eta_2^{[m]} \rightarrow \eta_2^{[y]}$ | $\beta_2^{[x \rightarrow m]} \times \beta_2^{[m \rightarrow y]}$ |
| $\eta_2^{[m]}$ | $x \rightarrow \eta_2^{[m]} \rightarrow \eta_2^{[y]}$ | $\beta_2^{[x \rightarrow m]} \times \beta_2^{[m \rightarrow y]}$ |

**Total Effects**

| **Indirect Effects** | **Estimates** |
|---------------------|----------------|
| $\eta_1^{[y]} \rightarrow \eta_1^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_1^{[y]} \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_1^{[y]} \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |

**Total Effects**

| **Indirect Effects** | **Estimates** |
|---------------------|----------------|
| $\eta_1^{[y]} \rightarrow \eta_1^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_1^{[y]} \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_1^{[y]} \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_1^{[y]} \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_1^{[y]} \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |
| $\eta_1^{[y]} \rightarrow \eta_2^{[y]}$ | $\beta_1^{[x \rightarrow y]} + \beta_1^{[x \rightarrow m]} \times \beta_1^{[m \rightarrow y]}$ |

**Total Effects**

| **Estimates** |
|----------------|
| $\beta_2^{[x \rightarrow m]} \times \beta_2^{[m \rightarrow y]}$ |

**Total Effects**

| **Estimates** |
|----------------|
| $\beta_2^{[x \rightarrow m]} \times \beta_2^{[m \rightarrow y]}$ |

**Total Effects**

| **Estimates** |
|----------------|
| $\beta_2^{[x \rightarrow m]} \times \beta_2^{[m \rightarrow y]}$ |
Table 2: Performance Metric: Definitions and Estimates

| Criteria         | Definition                                      | Estimate                                      |
|------------------|------------------------------------------------|-----------------------------------------------|
| Relative Bias    | $E_\hat{\theta}(\hat{\theta} - \theta)/\theta$ | $\sum_{s=1}^{S}(\hat{\theta}_s - \theta)/S\theta$ |
| Empirical SE     | $\sqrt{Var(\hat{\theta})}$                    | $\sqrt{\sum_{s=1}^{S}(\hat{\theta}_s - \bar{\theta})^2/(S-1)}$ |
| Relative RMSE    | $\sqrt{E_\hat{\theta}(\hat{\theta} - \theta)^2}/\theta$ | $\sqrt{\sum_{s=1}^{S}(\hat{\theta}_s - \theta)^2/S}/\theta$ |
| Coverage Probability | $Pr(\hat{\theta}_{\text{low}} \leq \theta \leq \hat{\theta}_{\text{upper}})$ | $\sum_{s=1}^{S} I(\hat{\theta}_{\text{low},s} \leq \theta \leq \hat{\theta}_{\text{upper},s})/S$ |

1 $\theta$: the population value of the parameter of interest
2 $\hat{\theta}$: the estimate of $\theta$
3 $S$: the number of replications and set as 1,000 in our simulation study
4 $s = 1, \ldots, S$: indexes the replications of the simulation
5 $\hat{\theta}_s$: the estimate of $\theta$ from the $s^{th}$ replication
6 $\bar{\theta}$: the mean of $\hat{\theta}_s$'s across replications
7 $I()$: an indicator function
Table 3: Estimates of Longitudinal Mediation Model for Reading and Mathematics Ability

| Para. | Reading Ability | Mathematics Ability |
|-------|----------------|---------------------|
|       | Est. (SE)       | P value             | Est. (SE)       | P value             |
| $\mu_{y_1}$ | 2.009 (0.028)  | $< 0.0001^*$        | 1.761 (0.020)  | $< 0.0001^*$        |
| $\mu_{y_2}$ | 108.955 (1.011) | $< 0.0001^*$        | 94.651 (1.024) | $< 0.0001^*$        |
| $\mu_{y_3}$ | 0.702 (0.016)  | $< 0.0001^*$        | 0.779 (0.018)  | $< 0.0001^*$        |
| $\gamma$ | 93.424 (0.317) | $< 0.0001^*$        | 99.127 (0.413) | $< 0.0001^*$        |

Growth Factor Means

| Para. | Covariate to Reading Ability | Covariate to Mathematics Ability |
|-------|-------------------------------|----------------------------------|
|       | Est. (SE) P value             | Est. (SE) P value                |
| $\psi_{11}$ | 0.183 (0.020)  | $< 0.0001^*$        | 0.048 (0.007)  | $< 0.0001^*$        |
| $\psi_{12}$ | 278.52 (20.765) | $< 0.0001^*$        | 83.196 (7.403) | $< 0.0001^*$        |
| $\psi_{22}$ | 0.042 (0.006)  | $< 0.0001^*$        | 0.029 (0.006)  | $< 0.0001^*$        |

Growth Factor (Unexplained) Variances

| Para. | Direct Effects | Covariate to Mathematics Ability |
|-------|----------------|----------------------------------|
|       | Est. (SE) P value | Est. (SE) P value                |
| $x \rightarrow \eta^{[1]}$ | 0.127 (0.026)  | $< 0.0001^*$        | 0.024 (0.016)  | 0.1336                |
| $x \rightarrow \eta^{[2]}$ | 5.897 (0.857)  | $< 0.0001^*$        | 2.277 (0.538)  | $< 0.0001^*$        |
| $x \rightarrow \eta^{[3]}$ | $-0.047 (0.014)$ | $0.008^*$         | $-0.017 (0.015)$ | 0.2571                |

| Para. | Indirect Effects | Total Effects |
|-------|-----------------|---------------|
|       | Est. (SE) P value | Est. (SE) P value |
| $x \rightarrow \eta^{[1]} \rightarrow \eta^{[2]}$ | 0.057 (0.013)  | $< 0.0001^*$        |
| $x \rightarrow \eta^{[1]} \rightarrow \eta^{[3]}$ | 0.366 (0.197)  | 0.0632                |
| $x \rightarrow \eta^{[2]} \rightarrow \eta^{[2]}$ | 0.002 (0.008)  | 0.8026                |
| $x \rightarrow \eta^{[2]} \rightarrow \eta^{[1]}$ | 4.055 (0.606)  | $< 0.0001^*$        |
| $x \rightarrow \eta^{[2]} \rightarrow \eta^{[3]}$ | 0.016 (0.012)  | 0.1824                |
| $x \rightarrow \eta^{[3]} \rightarrow \eta^{[2]}$ | $-0.031 (0.012)$ | 0.0098*                |

$^*$ indicates statistical significance at 0.05 level.
### Table 4: Estimates of Longitudinal Mediation Model for Reading, Mathematics and Science Ability

#### Growth Factor Means

| Para. | Reading Ability | Mathematics Ability | Science Ability |
|-------|-----------------|---------------------|-----------------|
|       | Est. (SE)       | P value             | Est. (SE)       | P value | Est. (SE) | P value |
| $\mu_\gamma$ | 2.006 (0.208) | $< 0.0001^*$ | 1.762 (0.202) | $< 0.0001^*$ | 0.814 (0.014) | $< 0.0001^*$ |
| $\mu_\psi$ | 108.886 (1.021) | $< 0.0001^*$ | 94.486 (1.022) | $< 0.0001^*$ | 53.609 (0.584) | $< 0.0001^*$ |
| $\mu_\phi$ | 0.703 (0.016) | $< 0.0001^*$ | 0.782 (0.018) | $< 0.0001^*$ | 0.574 (0.013) | $< 0.0001^*$ |
| $\gamma$ | 93.378 (0.317) | $< 0.0001^*$ | 99.006 (0.408) | $< 0.0001^*$ | 99.160 (0.252) | $< 0.0001^*$ |

#### Growth Factor (Unexplained) Variance

| Para. | Reading Ability | Mathematics Ability |
|-------|-----------------|---------------------|
|       | Est. (SE)       | P value             |
| $\psi_{11}$ | 0.190 (0.021) | $< 0.0001^*$ |
| $\psi_{1\gamma}$ | 319.694 (23.975) | $< 0.0001^*$ |
| $\psi_{22}$ | 0.045 (0.006) | $< 0.0001^*$ |

#### Direct Effects

| Para. | Growth Factors of Reading Ability to Those of Mathematics Ability |
|-------|------------------|
|       | Est. (SE)       | P value |
| $\eta_{1}^{[1]} \rightarrow \eta_{1}^{[m]}$ | 0.472 (0.043) | $< 0.0001^*$ |
| $\eta_{1}^{[2]} \rightarrow \eta_{1}^{[m]}$ | 4.569 (1.050) | 0.0056 |
| $\eta_{2}^{[1]} \rightarrow \eta_{2}^{[m]}$ | $-0.020 (0.082)$ | 0.8073 |
| $\eta_{2}^{[2]} \rightarrow \eta_{2}^{[m]}$ | 0.695 (0.028) | $< 0.0001^*$ |
| $\eta_{2}^{[3]} \rightarrow \eta_{2}^{[m]}$ | 0.003 (0.003) | 0.3173 |
| $\eta_{2}^{[4]} \rightarrow \eta_{2}^{[m]}$ | 0.775 (0.213) | 0.0003 * |

#### Indirect Effects

| Para. | Reading Ability | Mathematics Ability |
|-------|-----------------|---------------------|
|       | Est. (SE)       | P value             |
| $\eta_{1}^{[1]} \rightarrow \eta_{1}^{[m]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{1}^{[2]} \rightarrow \eta_{1}^{[m]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{1}^{[3]} \rightarrow \eta_{1}^{[m]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{1}^{[4]} \rightarrow \eta_{1}^{[m]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{2}^{[1]} \rightarrow \eta_{2}^{[m]} \rightarrow \eta_{2}^{[y]}$ | — | — |
| $\eta_{2}^{[2]} \rightarrow \eta_{2}^{[m]} \rightarrow \eta_{2}^{[y]}$ | — | — |
| $\eta_{2}^{[3]} \rightarrow \eta_{2}^{[m]} \rightarrow \eta_{2}^{[y]}$ | — | — |
| $\eta_{2}^{[4]} \rightarrow \eta_{2}^{[m]} \rightarrow \eta_{2}^{[y]}$ | — | — |

#### Total Effects

| Para. | Est. (SE) | P value |
|-------|-----------|---------|
| $\eta_{1}^{[1]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{1}^{[2]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{1}^{[3]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{1}^{[4]} \rightarrow \eta_{1}^{[y]}$ | — | — |
| $\eta_{2}^{[1]} \rightarrow \eta_{2}^{[y]}$ | — | — |
| $\eta_{2}^{[2]} \rightarrow \eta_{2}^{[y]}$ | — | — |
| $\eta_{2}^{[3]} \rightarrow \eta_{2}^{[y]}$ | — | — |
| $\eta_{2}^{[4]} \rightarrow \eta_{2}^{[y]}$ | — | — |

$^*$ indicates statistical significance at 0.05 level.