On Timelike Tubular Weingarten Surfaces in Minkowski 3-Space

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Abstract

In this paper, we study the timelike tubular Weingarten surfaces in 3-dimensional Minkowski space $\mathbb{IR}^3_1$. We have obtained some conditions for being $(K_{II}, H)$, $(K_{II}, K)$, timelike tubular Weingarten surfaces where $K$ is the second Gaussian curvature the Gaussian curvature and the mean curvature, respectively.

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1 Introduction

Let $f$ and $g$ be smooth functions on a surface $M$ in 3-dimensional Minkowski space $\mathbb{IR}^3_1$. The existence of a nontrivial functional relation $\Phi (f, g) = 0$ namely the Jacobian determinant of $f$ and $g$ functions $\Phi (f, g) = \det \begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix} = 0$, where $f_s = \frac{\partial f}{\partial s}$, $f_t = \frac{\partial f}{\partial t}$. A surface is called a Weingarten surface, if there is a nontrivial relation $\Phi (k_1, k_2) = 0$ between its principal curvatures $k_1$ and $k_2$, or, equivalently there is a nontrivial relation $\Phi (K, H) = 0$ between its Gaussian curvature $K$ and mean curvature $H$. Also we call a surface as a linear Weingarten surface such that the linear combination $aK + bH = c$ is constant along each ruling, where $a, b, c \in \mathbb{R}, (a, b, c) \neq (0, 0, 0)$. Several geometers have studied Weingarten and linear Weingarten surfaces. For non-developable surfaces Khnel studied $\{H, K_{II}\}$ and $\{K, K_{II}\}$ Weingarten surfaces \cite{9}. Then, G. Stamou extended this article and gave linear Weingarten surfaces which satisfy $aK_{II} + bH + cH_{II} = d$ is constant along rulings where $a^2 + b^2 + c^2 \neq 0$ \cite{13}. Also in 3-dimensional Minkowski space linear Weingarten surfaces which is foliated by pieces of circles and linear Weingarten helicoidal surfaces under cubic screw motion studied in \cite{4}, \cite{5}. F. Dillen and W. Sodsiri examined
Weingarten and linear Weingarten surfaces in 2005 for the 3-dimensional Minkowski space \([1], [2], [3]\).

A tubular Weingarten and linear Weingarten surface were studied by Ro and Yoon in 3-dimensional Euclidean space \(E^3\) \([11]\). Karacan and Bukcu constructed tubular surfaces with the assistance of an alternative moving frame \([7]\). Then geodesics and singular points of tubular surfaces are researched by using one parameter spatial motion along a curve in Minkowski 3-space \([6], [8]\).

In this paper, we give some theorems and conclusions related to time like tubular Weingarten and linear Weingarten surfaces in 3-dimensional Minkowski space \(I\mathbb{R}_3^1\).

2 Preliminaries

The Minkowski 3-space \(I\mathbb{R}_3^1\) is the Euclidean 3-space \(I\mathbb{R}^3\) provided with the indefinite inner product given by

\[
\langle \cdot, \cdot \rangle = -dy_1^2 + dy_2^2 + dy_3^2,
\]

where \((y_1, y_2, y_3)\) is natural coordinates of \(I\mathbb{R}_3^1\). Since \(\langle \cdot, \cdot \rangle\) is indefinite inner product, recall that a vector \(\beta \in I\mathbb{R}_3^1\) can have one of the three causal characters it can be spacelike if \(\langle \beta, \beta \rangle > 0\) or \(\beta = 0\), timelike if \(\langle \beta, \beta \rangle < 0\) and null (lightlike) if \(\langle \beta, \beta \rangle = 0\) and \(\beta \neq 0\). Similarly, an arbitrary curve \(\gamma = \gamma (t)\) in \(I\mathbb{R}_3^1\) can locally called as timelike, if its velocity vector \(\gamma' (t)\) is timelike. Recall that the norm of a vector is given by \(\| \beta \| = \sqrt{\langle \beta, \beta \rangle}\) and that the timelike \(\gamma (t)\) is said to be of unit speed if \(\langle \gamma ' (t), \gamma ' (t) \rangle = -1\). Moreover, the velocity of curve \(\gamma (t)\) is the function \(\nu (s) = \| \gamma ' (t) \|\). Denote by \(\{ t, n, b \}\) the moving Frenet frame along the curve \(\gamma (t)\) in the Minkowski space \(I\mathbb{R}_3^1\). Then Frenet formula of \(\gamma (t)\) in the space \(I\mathbb{R}_3^1\) is defined by \([10]\).

\[
t' = \kappa n
\]
\[
n' = \kappa t + \tau b
\]
\[
b' = -\tau n
\]

where the prime denotes the differentiation with respect to \(t\) and we denote by \(\kappa, \tau\) the curvature and the torsion of the curve \(\gamma\). Since \(\gamma\) is a timelike curve

\[
\langle t, t \rangle = -1, \langle b, b \rangle = \langle n, n \rangle = 1,
\]
\[
\langle t, n \rangle = \langle t, b \rangle = \langle b, n \rangle = 0.
\]

The vector product of the vectors \(\beta = (\beta_1, \beta_2, \beta_3)\) and \(\mu = (\mu_1, \mu_2, \mu_3)\) is defined by

\[
\beta \wedge \mu = (\beta_3 \mu_2 - \beta_2 \mu_3, \beta_3 \mu_1 - \beta_1 \mu_3, \beta_1 \mu_2 - \beta_2 \mu_1).
\]

We denote a timelike surface \(M\) in \(I\mathbb{R}_3^1\) by

\[
x (s, t) = (x_1 (s, t), x_2 (s, t), x_3 (s, t))
\]

Let \(U\) be the standard unit normal spacelike vector field on a surface \(M\) defined by \(U = \frac{x_s \wedge x_t}{\| x_s \wedge x_t \|}\), where \(x_s = \frac{\partial x(s, t)}{\partial s}\) and \(x_t = \frac{\partial x(s, t)}{\partial t}\). Then the first
fundamental form \( I \) and the second fundamental form \( II \) of a timelike surface \( M \) are defined by, respectively

\[
I = E ds^2 + 2F ds dt + G dt^2,
\]
\[
II = e ds^2 + 2f ds dt + g dt^2,
\]

where

\[
E = \langle x_s, x_s \rangle, \quad F = \langle x_s, x_t \rangle, \quad G = \langle x_t, x_t \rangle
\]
\[
e = \langle x_{ss}, U \rangle, \quad f = \langle x_{st}, U \rangle, \quad g = \langle x_{tt}, U \rangle.
\]

On the other hand, the Gaussian curvature \( K \) and the mean curvature \( H \) are given by, respectively

\[
K = \frac{eg - f^2}{2(EG - F^2)} (U, U)
\]
\[
H = \frac{eG - 2fF + gE}{2(EG - F^2)} (U, U).
\]

From Brioschi’s formula in a Minkowski 3-space, we can compute \( K_{II} \) of a surface by replacing the components of the first fundamental form \( E, F, G \) by the components of the second fundamental form \( e, f, g \) respectively in Brioschi’s formula. Consequently, the second Gaussian curvature \( K_{II} \) of a non-developable surface is defined by [12]

\[
K_{II} = \frac{1}{(eg - f^2)^2} \left\{ \begin{array}{ccc}
-\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\
\frac{1}{2}g_{tt} & e & f \\
\frac{1}{2}e_t & f & g
\end{array} \right. - \left[ \begin{array}{ccc}
0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\
e & f & g \end{array} \right].
\]

3 Timelike Tubular Surfaces of Weingarten Types

Let \( \gamma : (a, b) \to IR^3_1 \) be a smooth unit speed timelike curve of finite length which is topologically imbedded in \( IR^3_1 \). The total space \( N_\gamma \) of the normal bundle of \( \gamma((a, b)) \) in \( IR^3_1 \) is naturally diffeomorphic to the direct product \( (a, b) \times IR^3_1 \) via the translation along \( \gamma \) with respect to the induced normal connection. For sufficiently small \( r > 0 \), the tubular surface of radius \( r \) about the curve \( \gamma \) is the set:

\[
T_r(\gamma) = \{ \exp_{\gamma(t)} \vartheta \mid \vartheta \in N_\gamma, \|\vartheta\| = r, a < t < b \}.
\]

For a sufficiently small the tubular timelike surface \( T_r(\gamma) \) is a smooth surface in \( IR^3_1 \). Then the parametric equation of the timelike tubular surface \( T_r(\gamma) \) can be expressed as

\[
x(t, \theta) = \gamma(t) + r(\cos \theta n + \sin \theta b)
\]

(3.1)

Furthermore, we have the natural frame \( \{x_t, x_\theta\} \) is given by

\[
x_t = (1 + r\kappa \cos \theta) t + r\tau (\cos \theta b - \sin \theta n) = \alpha t + r\tau v, \quad x_\theta = r(\cos \theta b - \sin \theta n) = rv,
\]

(3.2)

where we put \( \alpha = 1 + r\kappa \cos \theta \) and \( v = \cos \theta b - \sin \theta n \). From which the components of the first fundamental form are

\[
E = -\alpha^2 + r^2 \tau^2, \quad F = r^2 \tau, \quad G = r^2.
\]

(3.3)
On the other hand, the unit normal spacelike vector field $U$ is obtained by

$$U = \frac{x_t \wedge x_\theta}{\|x_t \wedge x_\theta\|} = -\cos \theta n - \sin \theta b,$$

from this, the components of the second fundamental form of $x$ are given by

$$e = r \tau^2 - \kappa \alpha \cos \theta, \quad f = r \tau, \quad g = r.$$ 

If the second fundamental form is non-degenerate, $e g - f^2 \neq 0$, that is, $\kappa, \alpha$ and $\cos \theta$ are nowhere vanishing. In this case, we can define formally the second Gaussian curvature $K_{II}$ on $T_r(\gamma)$. On the other hand, the Gauss curvature $K$, the mean curvature $H$ and the second Gaussian curvature $K_{II}$ are given by,

respectively

$$K = \frac{2 \cos \theta}{r \alpha}, \quad (3.4)$$

$$H = -\frac{(1 + 2 r \kappa \cos \theta)}{2 r \alpha}, \quad (3.5)$$

$$K_{II} = \frac{1}{4 r^2 \alpha^2 \cos^2 \theta} (4 r^2 \kappa^2 \cos^4 \theta + 6 r \kappa \cos^3 \theta + \cos^2 \theta + 1). \quad (3.6)$$

Differentiating $K, H$ and $K_{II}$ with respect to $t$ and $\theta$, we get

$$K_t = \frac{\kappa' \cos \theta}{r \alpha^2}, \quad K_\theta = -\frac{\kappa \sin \theta}{r \alpha^2}, \quad (3.7)$$

$$H_t = -\frac{\kappa' \cos \theta}{2 \alpha^2}, \quad H_\theta = \frac{\kappa \sin \theta}{2 \alpha^2}, \quad (3.8)$$

$$\begin{align*}
(K_{II})_t &= \frac{1}{4 r^2 \alpha^2 \cos^2 \theta} (2 r^3 \kappa^2 \kappa' \cos^4 \theta + 6 r^2 \kappa \kappa' \cos^3 \theta + 4 r \cos^2 \theta - 2 r^2 \kappa \kappa' \cos \theta - 2 r \kappa'), \\
(K_{II})_\theta &= \frac{1}{4 r^2 \alpha^4 \cos^3 \theta} (-2 r^3 \kappa^3 \cos^6 \theta \sin \theta - 6 r^2 \kappa^2 \cos^5 \theta \sin \theta - 4 r \kappa \cos^4 \theta \sin \theta + 4 r^3 \kappa^2 \cos^3 \theta \sin \theta + 6 r \kappa \cos^2 \theta \sin \theta + 2 \cos \theta \sin \theta). 
\end{align*} \quad (3.9)$$

Now, we investigate a tubular timelike surface $T_r(\gamma)$ in $IR^3_1$ satisfying the Jacobi equation $\Phi (X, Y) = 0$. By using (3.7) and (3.8), $\Phi (X, Y) = 0$ satisfies identically the Jacobi equation $\Phi (K, H) = K_t H_\theta - K_\theta H_t = 0$. Therefore, $T_r(\gamma)$ is a Weingarten surface. We consider a timelike tubular $T_r(\gamma)$ with non-degenerate second fundamental form in $IR^3_1$ satisfying the Jacobi equation

$$\Phi (K, K_{II}) = K_t (K_{II})_\theta - K_\theta (K_{II})_t = 0 \quad (3.10)$$

with respect to the Gaussian curvature $K$ and the second Gaussian curvature $K_{II}$. Then, by (3.7) and (3.9) equation (3.10) becomes

$$r^2 \kappa^2 \kappa' \cos^2 \theta \sin \theta + 2 r \kappa \kappa' \cos \theta \sin \theta + \kappa' \sin \theta = 0$$

Since this polynomial is equal to zero for every $\theta$, all its coefficients must be zero. Therefore, we conclude that $\kappa' = 0$. We suppose that a timelike tubular $T_r(\gamma)$
with non-degenerate second fundamental form in $IR^3_1$ is $(H,K_{II})$-Weingarten surface. Then it satisfies the equation

$$H_t (K_{II})_\theta - H_\theta (K_{II})_t = 0,$$  \hfill (3.11)

which implies

$$r^2 \kappa^2 \kappa' \cos^2 \theta \sin \theta + 2r \kappa \kappa' \cos \theta \sin \theta + \kappa' \sin \theta = 0$$  \hfill (3.12)

from (3.12) we can obtain $\kappa' = 0$.

Consequently, we have the following theorems:

**Theorem 3.1** A timelike tubular surface in a Minkowski 3-space is a Weingarten surface.

**Theorem 3.2** Let $(X,Y) \in \{(K,K_{II}),(H,K_{II})\}$ and let $T_r(\gamma)$ be a timelike tubular surface in Minkowski 3-space with non-degenerate second fundamental form. If $T_r(\gamma)$ is a $(X,Y)$-Weingarten surface, then the curvature of $T_r(\gamma)$ is a non-zero constant.

Finally, we study a timelike tubular $T_r(\gamma)$ in $IR^3_1$ is a linear Weingarten surface, that is, it satisfies the equation

$$aK + bH = c.$$  \hfill (3.13)

Then, by (3.4) and (3.5) we have

$$(2ak - 2br\kappa - 2r^2ck) \cos \theta - b - 2rc = 0.$$  

Since $\cos \theta$ and 1 are linearly independent, we get

$$2ak - 2br\kappa - 2r^2ck = 0, \quad b = -2rc,$$

which imply

$$\kappa (a + cr^2) = 0.$$  

If $a + cr^2 \neq 0$, then $\kappa = 0$. Thus, $T_r(\gamma)$ is an open part of a circular cylinder. Next, suppose that a timelike tubular $T_r(\gamma)$ with non-degenerate second fundamental form in $IR^3_1$ satisfies the equation

$$aK + bK_{II} = c.$$  \hfill (3.14)

By (3.4) and (3.6), equation (3.14) becomes

$$(4ark^2 + 4br^2\kappa^2 - 4cr^2\kappa^2) \cos^4 \theta + (4ak + 6br\kappa - 8cr^2\kappa) \cos^3 \theta + (b - 4cr) \cos^2 \theta + b = 0.$$  

Since the identity holds for every $\theta$, all the coefficients must be zero. Therefore, we have

$$4ark^2 + 4br^2\kappa^2 - 4cr^2\kappa^2 = 0,$$

$$4ak + 6br\kappa - 8cr^2\kappa = 0,$$

$$b - 4cr = 0,$$

$$b = 0.$$
Thus, we get $b = 0, c = 0$ and $\kappa = 0$. In this case, the second fundamental form of $T_r(\gamma)$ is degenerate.

Suppose that a timelike tubular $T_r(\gamma)$ with non-degenerate second fundamental form in $\mathbb{IR}_1^3$ satisfies the equation

$$aH + bK_{II} = c.$$ 

By (3.5), (3.6) and (3.15), we have

$$(-4ar^2\kappa^2 + 4br^2\kappa^2 - 4cr^3\kappa^2) \cos^4 \theta + (-6a\kappa + 6b\kappa - 8cr^2\kappa) \cos^3 \theta + (-2a + b - 4cr) \cos^2 \theta + b = 0.$$

from which we can obtain $b = 0$ and $\kappa = 0$.

Consequently, we have the following theorems:

**Theorem 3.3** Let $T_r(\gamma)$ be a timelike tubular surface satisfying the linear equation $aK + bH = c$. If, $a + br \neq 0$, then it is an open part of a circular cylinder.

**Theorem 3.4** Let $(X, Y) \in \{(K, K_{II}), (H, K_{II})\}$. Then there are no $(X, Y)$-linear Weingarten tubular in Minkowski 3-space $\mathbb{IR}_1^3$.

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