Higher-order moment portfolio optimization via difference-of-convex programming and sums-of-squares

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Abstract
We are interested in developing DC (Difference-of-Convex) programming approach for solving higher-order moment (Mean-Variance-Skewness-Kurtosis) portfolio selection problem. The portfolio selection with higher moments can be formulated as a nonconvex quartic multivariate polynomial optimization. Based on the recent development in Difference-of-Convex-Sums-of-Squares (DCSOS) decomposition techniques for polynomial optimization, we can reformulate this problem as a DC program which can be solved by a well-known DC algorithm - DCA. We have also proposed an improved DC algorithm called Boosted-DCA (BDCA) based on an Armijo type line search to accelerate the convergence of DCA. We introduce this acceleration technique to both DC algorithm based on DCSOS decomposition proposed in this paper and the DC algorithm based on universal DC decomposition proposed in our previous paper. Results in numerical simulation show good performance of our proposed algorithms in portfolio optimization.

Keywords: Higher moment portfolio optimization, Difference-of-Convex programming, Difference-of-Convex-Sums-of-Squares decomposition, Boosted-DCA

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1. Introduction

The concepts of portfolio optimization and diversification are fundamental to understand the financial markets and financial decision making. The major breakthrough came in [Markowitz (1952)] with the publication of the mean-variance portfolio selection model (MV model) developed by Harry Markowitz (Nobel Laureate in Economics in 1990). This model provided an answer to the fundamental question: How should an investor allocate funds among the possible investment choices? Markowitz firstly quantified return and risk of a security, using the statistical measures of its expected return and variance. Then,
he suggested that investors should consider return and risk together, and determine the allocation of funds based on their return-risk trade-off. Before Markowitz’s seminal article, the finance literature had treated the interplay between return and risk in an ad hoc fashion. Based on MV model, the investors are going to find among the infinite number of portfolios that achieve a particular return objective with the smallest variance. The portfolio theory had a major impact on academic research and the financial industry as a whole which is often called “the first revolution of the Wall Street”. More discussions about MV model can be found in the review article Steinbach (2001).

However, It is often asserted that the application of MV model assumes normal (Gaussian) return distributions or quadratic utility functions which is not always true in real financial market. It was known that many return distributions in the market exhibit fat tails and asymmetry which will significantly affect portfolio performance Jobst & Zenios (2001). Therefore, many scholars suggested introducing higher moments such as skewness (3rd order moment) and kurtosis (4th order moment) into portfolio optimization model, since the presence of skewness and kurtosis are very important in asset pricing Arditti & Levy (1975); Jondeau & Rockinger (2003). E.g., Harvey & Siddique (2000) showed that in the presence of positive skewness, investors may be willing to accept a negative expected return. There is a rich literature that has attempted to model higher moments in the pricing of derivative securities, starting from the classic models of Merton (1976) (jump-diffusions) and Heston (1993) (stochastic volatility), see Bhandari & Das (2009) for more relevant works.

The first work attempted to extend the MV model to higher moments was proposed by Jean (1971). Some noteworthy works such as Arditti & Levy (1975) and Levy & Markowitz (1979) were mainly focused on the mean-variance-skewness model (MVS model). Later, more extensions of higher moment portfolio models adapted kurtosis had been investigated by several authors (e.g., De Athayde & Flores Jr (2004), Maringer & Parpas (2009), and Harvey et al. (2010) etc). In fact, higher order moment portfolio model can be seen as approximations to general expected utility function, where one considers a Taylor series expansion of the utility function and drops the higher order terms from the expansion. More information about different portfolio selection models were discussed in the survey article on the 60 years’ development in portfolio optimization Kolm et al. (2014). However, in practice, the higher moment portfolio models are seldom used. There are many reasons, typically, practitioners rely upon a utility function based on mean-variance approximations, which is mainly based on the statements in Levy & Markowitz (1979) that mean-variance approximations often perform well enough, and in Markowitz (2014) that “the persistence of the Great Confusion - that MV analysis is applicable in practice only when return distributions are Gaussian or utility functions quadratic - is as if geography textbooks of 1550 still described the Earth as flat.” Moreover, due to the limitation of the computational power, constructing and solving a higher moment portfolio is very hard which is often intractable even for a quartic polynomial approximation in a small sized assets. Despite the complexity of the higher moment portfolio models, fortunately, with the rapid development of CPU and GPU hardwares, as well as the adequate computer memory, the computational power available today is possible to deal with some higher moment portfolios (at least portfolios of moderate size).

In this paper, we will consider a general higher moment portfolio model which takes Mean, Variance, Skewness, Kurtosis into consideration, called MVS$\kappa$ model. This model consists of maximizing the mean and the skewness of the portfolio while minimizing the
variance and the kurtosis. The problem can be written as a nonconvex quartic polynomial optimization problem which is in general NP-hard (see e.g., Pham & Niu (2011); Ahmadi et al. (2013)). Thus we can’t expect to propose a polynomial time global optimization algorithm to solve this problem. In Maringer & Parpas (2009), the authors proposed using two stochastic algorithms: Differential Evolution (DE) and Stochastic Differential Equation (SDE) for higher moment portfolio optimization; Pham & Niu (2011) proposed a DC (Difference-of-convex) programming approach which reformulates the MVSK model as a DC program based on an universal DC decomposition of polynomial function over a compact convex set. Then using an efficient DC programming approach called DCA to solve the corresponding DC program. Recently, Ban et al. (2016) uses machine learning approaches (e.g., regularization and cross-validation) on higher moment portfolio optimization. Chen et al. (2017) presents a new class of nonnegative symmetric tensors to reformulate a kurtosis minimization higher moment portfolio model as a multilinear form optimization model which can be solved by the MBI or the BCD method. In our work, we will focus on constructing a new DC decomposition for MVSK model based on a recently developed Difference-of-convex-sums-of-squares (DCSOS) decomposition technique in Niu (2018), this kind of decomposition is expected to produce a better DC decomposition than universal DC decomposition for polynomial function. We will also investigate a Boosted-DCA (BDCA) based on an Armijo-type line search to accelerate the convergence of DCA.

The paper is organized as follows: Section 2 presents MVSK higher moment portfolio optimization model. DC programming formulation of MVSK model based on DCSOS decomposition will be developed in Section 3. DC programming algorithm for finding KKT solutions of MVSK model are investigated in Section 4. Then, we will discuss how to boost DCA based on Armijo-type line search in the next section. Applying the acceleration technique to universal DC decomposition will be introduced in Section 6. Numerical experimental results comparing with different DC algorithms with randomly generated portfolio dataset and real stock dataset are reported in Section 7. Conclusions and perspectives are discussed in the last section.

2. Higher moment portfolio optimization model

Consider a portfolio with \(n\) assets. In this section, we will describe the higher moment portfolio model (Mean-Variance-Skewness-Kurtosis, cf. MVSK).

2.1. Portfolio inputs

Let \(n\) be the number of assets, and \(T\) be the number of periods. Let us denote \(R_{i,t}\) the return rates on asset \(i \in \{1, \ldots, n\}\) in period \(t \in \{1, \ldots, T\}\). The return rate of the asset \(i\) is denoted by \(R_i\), and \(R = (R_i) \in \mathbb{R}^n\) stands for the return rate vector of the portfolio. Let us denote \(\mathbb{E}\) the expectation operator, the inputs of MVSK model consists of the mean, variance, skewness and kurtosis of the portfolio returns defined by:

1. Mean of return rates, denoted by \(\mu \in \mathbb{R}^n\), defined as:

\[
\mu_i := \mathbb{E}(R_i) = \frac{1}{T} \sum_{t=1}^{T} R_{i,t}.
\]
2. Covariance of return rates, denoted by \( \Sigma = (\sigma_{i,j}) \in \mathbb{R}^{n^2} \), defined as:

\[
\sigma_{ij} := \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)] = \frac{1}{T-1} \sum_{t=1}^{T} (R_{i,t} - \mu_i)(R_{j,t} - \mu_j).
\] (2)

3. Co-skewness of return rates, denoted by \( S = (S_{i,j,k}) \in \mathbb{R}^{n^3} \), defined as:

\[
S_{i,j,k} := \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)] = \frac{1}{T} \sum_{t=1}^{T} (R_{i,t} - \mu_i)(R_{j,t} - \mu_j)(R_{k,t} - \mu_k).
\] (3)

4. Co-kurtosis of return rates, denoted by \( K = (K_{i,j,k,l}) \in \mathbb{R}^{n^4} \), defined as:

\[
K_{i,j,k,l} := \mathbb{E}[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)] = \frac{1}{T} \sum_{t=1}^{T} (R_{i,t} - \mu_i)(R_{j,t} - \mu_j)(R_{k,t} - \mu_k)(R_{l,t} - \mu_l).
\] (4)

These inputs can be written as tensors and easily computed from data via formulations (1), (2), (3) and (4). Note that these tensors have perfect symmetry, e.g., \( \Sigma \) is a real symmetric positive semi-definite matrix, and the values of \( S_{i,j,k} \) (resp. \( K_{i,j,k,l} \)) with any permutation of the index \( (i,j,k) \) (resp. \( (i,j,k,l) \)) are equals. Therefore, we only need to compute \( \binom{n+1}{2}, \binom{n+2}{3} \) and \( \binom{n+3}{4} \) independent elements respectively. When dealing with these higher moments and co-moments, it is convenient to “slice” these tensors and create one big matrix out of the slices. In our previous work (Pham & Niu, 2011), we have discussed using Kronecker product \( \otimes \) to rewrite co-skewness (resp. co-kurtosis) tensor as \( n \times n^2 \) (resp. \( n \times n^3 \)) matrix by the formulations:

\[
\hat{S} = \mathbb{E}[(R - \mu)(R - \mu)^T \otimes (R - \mu)^T],
\]

\[
\hat{K} = \mathbb{E}[(R - \mu)(R - \mu)^T \otimes (R - \mu)^T \otimes (R - \mu)^T].
\]

Then converting \( \hat{S} \) and \( \hat{K} \) into sparse matrices by keeping only the independent elements based on symmetry. This computation technique will be very useful when dealing with large-scale cases with big \( n \).

2.2. Mean-Variance-Skewness-Kurtosis portfolio model

Let us denote the decision variable of a portfolio (called portfolio weights) as \( x \in \mathbb{R}^n \). We suppose no short-selling, i.e., \( x \geq 0 \) and sums up to one, thus \( x \) is restricted in a standard \((n - 1)\)-simplex \( \Omega := \{ x \in \mathbb{R}_+^n : e^T x = 1 \} \) where \( e \) denotes the vector of ones.

Given the portfolio inputs, the first four orders of portfolio moments are defined as:

1. First order moment (Mean of the portfolio):

\[
m_1(x) = \mu^T x.
\]
2. Second order moment (Variance of the portfolio):

\[ m_2(x) = x^\top \Sigma x. \]

3. Third order moment (Skewness of the portfolio):

\[ m_3(x) = \sum_{i,j,k=1}^n S_{i,j,k} x_i x_j x_k. \]

4. Fourth order moment (Kurtosis of the portfolio):

\[ m_4(x) = \sum_{i,j,k,l=1}^n K_{i,j,k,l} x_i x_j x_k x_l. \]

The MVSK portfolio optimization model consists of maximizing the expected return and the skewness while minimizing the variance and the kurtosis \cite{Fabozzi2006}. The positive skewness is desirable since it corresponds to higher returns albeit with low probability, while kurtosis is undesirable since it implies that the investor is exposed to more risk \cite{Parpas2006}.

Let us denote \( F: \mathbb{R}^n \rightarrow \mathbb{R}^4 \) defined by:

\[ F(x) := (-m_1(x), m_2(x), -m_3(x), m_4(x))^\top. \]

The MVSK model is described as a multi-objective optimization problem as:

\[
\min \{ F(x) : x \in \Omega \},
\]

which can be further reformulated as a weighted single-objective optimization:

\[
\min_{x \in \Omega} f(x) = c^\top F(x)
\]

where the parametric vector \( c \) denotes the investor’s preference verifying \( 0 \leq c \leq 1 \) and \( c^\top c = 1 \). For instance, \( c_1 = 1 \) means that the investor is risk-seeking, and \( c_2 = 1 \) for risk-aversing. The MVSK model provides more freedom in describing investor’s preference than the classical mean-variance framework. A rational investor’s preference is high odd moments, as this would decrease extreme values on the side of losses and increase them on the side of gains. Similarly, the investor prefers low even moments, as this implies decreased dispersion and therefore less uncertainty of returns \cite{Scott1980}.

3. DC programming formulation for MVSK model via sums-of-squares

The MVSK model as a nonconvex quartic polynomial optimization problem can be formulated as a DC (Difference-of-Convex) programming problem, since any polynomial function is \( \mathcal{C}^\infty \) which is indeed a DC function. However, constructing a DC decomposition for a high order polynomial, i.e., rewriting the original polynomial as difference of two convex polynomials, is not trivial for polynomial of degree higher than 2. In \cite{Pham2011}, we have discussed the construction of a DC decomposition for the objective...
function $f$ using an *universal DC decomposition* technique in form of $f(x) = \frac{\rho}{2} \|x\|^2 - \left(\frac{\rho}{2} \|x\|^2 - f(x)\right)$ with some suitable parameter $\rho$. Specifically, $\rho$ must be greater than an upper bound for the spectral radius of the Hessian matrices of $f$ over $\Omega$. The quality of this kind of decomposition is highly depending on the parameter $\rho$, and a small $\rho$ is always preferred than a big one. The reason is that when $\rho$ is too big, the DC components $\frac{\rho}{2} \|x\|^2$ and $\frac{\rho}{2} \|x\|^2 - f(x)$ are more convex. We have proved in (Niu, 2018; Niu et al., 2019) that a better DC decomposition must be an undominated DC decomposition whose DC components should be less convex as possible.

In this paper, we will investigate a different DC decomposition technique without estimating the parameter $\rho$, namely DCSOS (Difference-of-Convex-SOS) decomposition, which is based on Convex-Sums-of-Squares (CSOS) decomposition of polynomials introduced in (Niu, 2018). The basic idea of DCSOS decomposition is to represent any polynomial as difference of two Convex and Sums-Of-Squares (i.e., CSOS) polynomials, and we have proved that any polynomial can be rewritten as DCSOS in polynomial time by solving an SDP. The minimal degree for DC components equals to the degree of the polynomial if it is even, and equals to the degree of the polynomial plus one if it is odd.

In (MVSK) model, we are going to find DC decompositions for moment functions $m_i, i = 1, \ldots, 4$ based on DCSOS. The first two moments $m_1$ and $m_2$ are linear and quadratic convex respectively, so they are already convex. We will only focus on DC decompositions for $m_3$ and $m_4$ which will be discussed in next subsections.

### 3.1. DC decomposition for $m_3$

By symmetry of the co-skewness tensor $S$, we can rewrite $m_3$ as

$$m_3(x) = \sum_{i,j,k=1}^{n} S_{i,j,k} x_i x_j x_k$$

$$= \sum_{i=1}^{n} S_{i,i,i} x_i^3 \quad \text{three common indices}$$

$$+ \binom{3}{1} \sum_{i=1}^{n} \sum_{k \neq i} S_{i,i,k} x_i^2 x_k \quad \text{two common indices}$$

$$+ 3! \sum_{1 \leq i < j < k \leq n} S_{i,j,k} x_i x_j x_k \quad \text{no common index}$$

Let $\mathcal{N} = \{1, \ldots, n\}$, $\mathcal{P} = \{(i,k) : i \in \mathcal{N}, k \neq i\}$, and $\mathcal{Q} = \{(i,j,k) : 1 \leq i < j < k \leq n\}$, with the number of elements $|\mathcal{N}| = n$, $|\mathcal{P}| = n(n-1)$, and $|\mathcal{Q}| = \binom{n}{3}$, then the expression of $m_3$ is simplified as:

$$m_3(x) = \sum_{i \in \mathcal{N}} S_{i,i,i} x_i^3 + 3 \sum_{(i,j) \in \mathcal{P}} S_{i,i,k} x_i^2 x_k + 6 \sum_{(i,j,k) \in \mathcal{Q}} S_{i,j,k} x_i x_j x_k.$$ 

Thus, there are three types of monomials $x_i^3$, $x_i^2 x_k$ and $x_i x_j x_k$ in $m_3$ whose DC decompositions are given as follows:

- For $x_i^3, \forall i \in \mathcal{N}$: Since $x_i^3$ is locally convex on $\mathbb{R}_+^n \supset \Omega$, a DC decomposition for $x_i^3$
on $\mathbb{R}^n_+$ is explicitly given by

$$ x_i^3 = g_i(x) - h_i(x) $$

with

$$ g_i(x) = x_i^3; h_i(x) = 0 $$  \hspace{1cm} (5) $$

being both convex functions on $\mathbb{R}^n_+$. Their gradients are

$$ \nabla g_i(x) = 3x_i^2 e_i; \quad \nabla h_i(x) = 0_{\mathbb{R}^n} $$  \hspace{1cm} (6) $$

with $e_i$ being the $i$-th canonical vector of $\mathbb{R}^n$.

• For $x_i^2 x_k, \forall (i,k) \in \mathcal{P}$: a DCSOS formulation is

$$ x_i^2 x_k = \frac{1}{4}(x_i^2 - 0) ((x_k + 1)^2 - (x_k - 1)^2) $$

$$ = \frac{1}{8} \left[ (x_i^2 + (x_k + 1)^2)^2 + (x_k - 1)^4 \right] - \frac{1}{8} \left[ (x_k + 1)^4 + (x_k - 1)^2 \right]^2 $$

$$ = g_{i,k}(x) - h_{i,k}(x) $$

where

$$ g_{i,k}(x) = \frac{1}{8} \left[ (x_i^2 + (x_k + 1)^2)^2 + (x_k - 1)^4 \right] $$, \hspace{1cm} (7) $$

$$ h_{i,k}(x) = \frac{1}{8} \left[ (x_k + 1)^4 + (x_i^2 + (x_k - 1)^2)^2 \right] $$ \hspace{1cm} (8) $$

are both CSOS on $\mathbb{R}^n$. Their gradients are

$$ \nabla g_{i,k}(x) = \frac{1}{2} x_i \left( x_k^2 + 2x_k + x_i^2 + 1 \right) e_i + \frac{1}{2} \left( 2x_k^3 + x_i^2 x_k + 6x_k + x_i^2 \right) e_k $$, \hspace{1cm} (9) $$

$$ \nabla h_{i,k}(x) = \frac{1}{2} x_i \left( x_k^2 - 2x_k + x_i^2 + 1 \right) e_i + \frac{1}{2} \left( 2x_k^3 + x_i^2 x_k + 6x_k - x_i^2 \right) e_k $$ \hspace{1cm} (10) $$

• For $x_i x_j x_k, \forall (i,j,k) \in \mathcal{Q}$: a DCSOS decomposition is given in a similar fashion as

$$ x_i x_j x_k = (x_i x_j)(x_k) $$

$$ = \frac{1}{16} \left( (x_i + x_j)^2 - (x_i - x_j)^2 \right) ((x_k + 1)^2 - (x_k - 1)^2) $$

$$ = \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k + 1)^2)^2 + ((x_i - x_j)^2 + (x_k - 1)^2)^2 \right] $$

$$ - \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k - 1)^2)^2 + ((x_i - x_j)^2 + (x_k + 1)^2)^2 \right] $$

$$ = g_{i,j,k}(x) - h_{i,j,k}(x) $$
where
\[ g_{i,j,k}(x) = \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k + 1)^2)^2 + ((x_i - x_j)^2 + (x_k - 1)^2)^2 \right], \quad (11) \]
\[ h_{i,j,k}(x) = \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k - 1)^2)^2 + ((x_i - x_j)^2 + (x_k + 1)^2)^2 \right] \quad (12) \]
are both CSOS on \( \mathbb{R}^n \). Their gradients are
\[
\nabla g_{i,j,k}(x) = \frac{1}{4} \left( (x_i x_k^2 + 2 x_j x_k + 3 x_i x_j^2 + x_i^3 + x_i) e_i \right.
+ \frac{1}{4} (x_j x_k^2 + 2 x_i x_k + x_j^3 + 3 x_i^2 x_j + x_j) e_j
+ \frac{1}{4} (x_k^3 + x_j^2 x_k + x_i^2 x_k + 3 x_k + 2 x_j x_k) e_k,
\]
(13)
\[
\nabla h_{i,j,k}(x) = \frac{1}{4} \left( (x_i x_k^2 - 2 x_j x_k + 3 x_i x_j^2 + x_i^3 + x_i) e_i \right.
+ \frac{1}{4} (x_j x_k^2 - 2 x_i x_k + x_j^3 + 3 x_i^2 x_j + x_j) e_j
+ \frac{1}{4} (x_k^3 + x_j^2 x_k + x_i^2 x_k + 3 x_k - 2 x_i x_k) e_k. \quad (14)\]

The next proposition is an immediate consequence for DC decomposition of \( m_3 \):

**Proposition 1.** Let us define the index sets:

\[ \mathcal{I}^+(S) := \{ i \in \mathcal{N} : S_{i,i,i} > 0 \}; \mathcal{I}^-(S) := \{ i \in \mathcal{N} : S_{i,i,i} < 0 \}; \]
\[ \mathcal{J}^+(S) := \{ (i,k) \in \mathcal{P} : S_{i,i,k} > 0 \}; \mathcal{J}^-(S) := \{ (i,k) \in \mathcal{P} : S_{i,i,k} < 0 \}; \]
\[ \mathcal{K}^+(S) := \{ (i,j,k) \in \mathcal{Q} : S_{i,j,k} > 0 \}; \mathcal{K}^-(S) := \{ (i,j,k) \in \mathcal{Q} : S_{i,j,k} < 0 \}. \]

A DC decomposition for \( m_3 \) is

\[ m_3(x) = g_{m_3}(x) - h_{m_3}(x) \]

with
\[
g_{m_3}(x) = \sum_{i \in \mathcal{I}^+(S)} S_{i,i,i} g_i(x) + 3 \sum_{(i,j) \in \mathcal{J}^+(S)} S_{i,i,k} g_{i,k}(x) - 3 \sum_{(i,j) \in \mathcal{J}^-(S)} S_{i,i,k} h_{i,k}(x) + 6 \sum_{(i,j,k) \in \mathcal{K}^+(S)} S_{i,j,k} g_{i,j,k}(x) - 6 \sum_{(i,j,k) \in \mathcal{K}^-(S)} S_{i,j,k} h_{i,j,k}(x), \quad (15)\]
There are 5 types of monomials that can be rewritten as DC decomposition as follows:

\[ h_{m_3}(x) = - \sum_{i \in S} S_{i,i,i} g_i(x) + 3 \sum_{(i,j) \in J^+} S_{i,i,k} h_{i,k}(x) - 3 \sum_{(i,j) \in J^+} S_{i,i,k} g_{i,k}(x) + 6 \sum_{(i,j) \in K^+} S_{i,j,k} h_{i,j,k}(x) - 6 \sum_{(i,j) \in K^+} S_{i,j,k} g_{i,j,k}(x). \] (16)

being both convex functions on \( \mathbb{R}^n \), where \( g_i, g_{i,k}, h_{i,k}, g_{i,j,k} \) and \( h_{i,j,k} \) are given respectively by \([5, 7, 8, 11, 12]\). Their gradients are computed accordingly.

### 3.2. DC decomposition for \( m_4 \)

The DC decomposition for \( m_4 \) are constructed in a similar way as in \( m_3 \). Based on the symmetry of the co-kurtosis tensor \( K \), the function \( m_4 \) can be rewritten as

\[ m_4(x) = \sum_{i,j,k,l=1}^{n} K_{i,j,k,l} x_i x_j x_k x_l \]

\[ = \sum_{i=1}^{n} K_{i,i,i,i} x_i^4 \quad \text{four common indices} \]

\[ + \left( \frac{4}{1} \right) \sum_{i=1, k \neq i}^{n} K_{i,i,i,k} x_i^3 x_k \quad \text{three common indices} \]

\[ + \left( \frac{4}{2} \right) \sum_{i=1, k > i}^{n} K_{i,i,k,k} x_i^2 x_k^2 \quad \text{two common indices} \]

\[ + 4! \sum_{i < j < k < l} K_{i,j,k,l} x_i x_j x_k x_l \quad \text{no common index} \]

where \((j < k) \neq i\) means \(j < k\) and \(j \neq i, k \neq i\).

Let \( \mathcal{N} = \{1, \ldots, n\} \), \( \mathcal{P} = \{(i, k) : i \in \mathcal{N}, k \neq i\} \), \( \mathcal{P} = \{(i, k) : k > i\} \), \( \mathcal{Q} = \{(i, j, k) : i \in \mathcal{N}, (j < k) \neq i\} \), and \( \mathcal{R} = \{(i, j, k, l) : 1 \leq i < j < k < l \leq n\} \). The size of these sets are \(|\mathcal{N}| = n\), |\(\mathcal{P}\)| = \(n(n-1)\), |\(\mathcal{Q}\)| = \(n(n-2)\), and |\(\mathcal{R}\)| = \(n(n-2)(n-3)\). Then \( m_4 \) is simplified as

\[ m_4(x) = \sum_{i \in \mathcal{N}} K_{i,i,i,i} x_i^4 + 4 \sum_{(i, k) \in \mathcal{P}} K_{i,i,i,k} x_i^3 x_k + 6 \sum_{(i, k) \in \mathcal{Q}} K_{i,i,k,k} x_i^2 x_k^2 + 12 \sum_{(i, j, k) \in \mathcal{Q}} K_{i,j,k,k} x_i x_j x_k x_l + 24 \sum_{(i, j, k, l) \in \mathcal{R}} K_{i,j,k,l} x_i x_j x_k x_l. \]

There are 5 types of monomials \(x_i^4, x_i^3 x_k, x_i^2 x_k^2, x_i^2 x_j x_k, x_i x_j x_k x_l\). Each monomial can be rewritten as DC decomposition as follows:

- For \(x_i^4, \forall i \in \mathcal{N}\): it is already CSOS, thus a DCSOS decomposition is

\[ x_i^4 = x_i^4 - 0 = \tilde{g}_i(x) - \tilde{h}_i(x) \]

where

\[ \tilde{g}_i(x) = x_i^4; \tilde{h}_i(x) = 0 \] (17)
are both convex functions on $\mathbb{R}^n$. Their gradients are
\[
\nabla \tilde{g}_i(x) = 4x_i^3e_i; \quad \nabla \tilde{h}_i(x) = 0_{\mathbb{R}^n}.
\] (18)

• For $x^3_i x_k, \forall (i,k) \in P$: a DCSOS decomposition is
\[
x^3_i x_k = (x^2_i)(x_i x_k)
\]
\[
= \frac{1}{4}(x^2_i - 0)((x_i + x_k)^2 - (x_i - x_k)^2)
\]
\[
= \frac{1}{8} \left[ (x^2_i + (x_i + x_k)^2)^2 + (x_i - x_k)^4 \right] - \frac{1}{8} \left[ (x_i + x_k)^4 + (x_i^2 + (x_i - x_k)^2)^2 \right]
\]
\[
= \tilde{g}_{i,k}(x) - \tilde{h}_{i,k}(x)
\]
where
\[
\tilde{g}_{i,k}(x) = \frac{1}{8} \left[ (x_i^2 + (x_i + x_k)^2)^2 + (x_i - x_k)^4 \right]
\] (19)
\[
\tilde{h}_{i,k}(x) = \frac{1}{8} \left[ (x_i + x_k)^4 + (x_i^2 + (x_i - x_k)^2)^2 \right]
\] (20)
are both CSOS on $\mathbb{R}^n$. Their gradients are
\[
\nabla \tilde{g}_{i,k}(x) = \frac{1}{2} x_i \left( 7x_i^2 + 3x_i x_k + 5x_k^2 \right) e_i + \frac{1}{2} (2x_k^3 + 7x_i x_k e_i) e_k,
\] (21)
\[
\nabla \tilde{h}_{i,k}(x) = \frac{1}{2} x_i \left( 7x_i^2 - 3x_i x_k + 5x_k^2 \right) e_i + \frac{1}{2} (2x_k^3 + 7x_i x_k e_i) e_k.
\] (22)

• For $x^2_i x^2_k, \forall (i,k) \in \tilde{P}$: a DCSOS decomposition is
\[
x^2_i x^2_k = \frac{1}{2} (x^2_i + x^2_k)^2 - \frac{1}{2} (x^4_i + x^4_k) = \tilde{g}_{i,k}(x) - \tilde{h}_{i,k}(x)
\]
where
\[
\tilde{g}_{i,k}(x) = \frac{1}{2} (x^2_i + x^2_k)^2; \quad \tilde{h}_{i,k}(x) = \frac{1}{2} (x^4_i + x^4_k)
\] (23)
are both CSOS on $\mathbb{R}^n$. Their gradients are
\[
\nabla \tilde{g}_{i,k}(x) = 2 \left( x_k^2 + x_i^2 \right) (x_i e_i + x_k e_k),
\] (24)
\[
\nabla \tilde{h}_{i,k}(x) = 2 \left( x_i^3 e_i + x_k^3 e_k \right).
\] (25)
• For $x_i^2 x_j x_k$, $\forall (i, j, k) \in \mathcal{Q}$: a DCSOS decomposition is
\[
x_i^2 x_j x_k = (x_i^2)(x_j x_k)
\]
\[
= \frac{1}{4} (x_i^2 - 0) ((x_j + x_k)^2 - (x_j - x_k)^2)
\]
\[
= \frac{1}{8} \left[ (x_i^2 + (x_j + x_k)^2) + (x_j - x_k)^4 \right] - \frac{1}{8} \left[ (x_j + x_k)^4 + (x_i^2 + (x_j - x_k)^2)^2 \right]
\]
\[
= \tilde{g}_{i,j,k}(x) - \tilde{h}_{i,j,k}(x)
\]
where
\[
\tilde{g}_{i,j,k}(x) = \frac{1}{8} \left[ (x_i^2 + (x_j + x_k)^2) + (x_j - x_k)^4 \right],
\]
\[
\tilde{h}_{i,j,k}(x) = \frac{1}{8} \left[ (x_j + x_k)^4 + (x_i^2 + (x_j - x_k)^2)^2 \right]
\]
are both CSOS functions on $\mathbb{R}^n$. Their gradients are
\[
\nabla \tilde{g}_{i,j,k}(x) = \frac{1}{2} x_i \left( x_i^2 + 2x_j x_k + x_j^2 + x_i^2 \right) e_i + \frac{1}{2} (6 x_j x_k^2 + x_i^2 x_k + 2 x_j^3 + x_i^2 x_j) e_j
\]
\[
+ \frac{1}{2} (2 x_k^3 + 6 x_j^2 x_k + x_i^2 x_k + x_i^2 x_j) e_k,
\]
\[
\nabla \tilde{h}_{i,j,k}(x) = \frac{1}{2} x_i \left( x_i^2 - 2 x_j x_k + x_j^2 + x_i^2 \right) e_i + \frac{1}{2} (6 x_j x_k^2 - x_i^2 x_k + 2 x_j^3 + x_i^2 x_j) e_j
\]
\[
+ \frac{1}{2} (2 x_k^3 + 6 x_j^2 x_k + x_i^2 x_k - x_i^2 x_j) e_k.
\]

• For $x_i x_j x_k x_l$, $\forall (i, j, k, l) \in \mathcal{R}$: a DCSOS decomposition is
\[
x_i x_j x_k x_l = (x_i x_j)(x_k x_l)
\]
\[
= \frac{1}{16} ((x_i + x_j)^2 - (x_i - x_j)^2)((x_k + x_l)^2 - (x_k - x_l)^2)
\]
\[
= \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k + x_l)^2)^2 + ((x_i - x_j)^2 + (x_k - x_l)^2)^2 \right]
\]
\[
- \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k - x_l)^2)^2 + ((x_i - x_j)^2 + (x_k + x_l)^2)^2 \right]
\]
\[
= g_{i,j,k,l}(x) - h_{i,j,k,l}(x)
\]
where
\[
g_{i,j,k,l}(x) = \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k + x_l)^2)^2 + ((x_i - x_j)^2 + (x_k - x_l)^2)^2 \right],
\]
\[
h_{i,j,k,l}(x) = \frac{1}{32} \left[ ((x_i + x_j)^2 + (x_k - x_l)^2)^2 + ((x_i - x_j)^2 + (x_k + x_l)^2)^2 \right]
\]
are both CSOS on \( \mathbb{R}^n \). Their gradients are

\[
\nabla g_{i,j,k,l}(x) = \frac{1}{4} (x_i x_i^2 + 2x_j x_k x_l + x_i x_k^2 + 3x_i x_j^2 + x_i^3) e_i + \frac{1}{4} (x_j x_i^2 + 2x_i x_k x_l + x_j x_k^2 + x_j^3 + 3x_i^2 x_j) e_j + \frac{1}{4} (3x_k x_i^2 + 2x_i x_j x_l + x_k x_i^2 + x_k^2 x_j + x_i^2 x_k) e_k + \frac{1}{4} (x_l^3 + 3x_k^2 x_l + x_j^2 x_l + x_i^2 x_l + 2x_i x_j x_k) e_l.
\]

(32)

\[
\nabla h_{i,j,k,l}(x) = \frac{1}{4} (x_i x_i^2 - 2x_j x_k x_l + x_i x_k^2 + 3x_i x_j^2 + x_i^3) e_i + \frac{1}{4} (x_j x_i^2 - 2x_i x_k x_l + x_j x_k^2 + x_j^3 + 3x_i^2 x_j) e_j + \frac{1}{4} (3x_k x_i^2 - 2x_i x_j x_l + x_k x_i^2 + x_k^2 x_j + x_i^2 x_k) e_k + \frac{1}{4} (x_l^3 + 3x_k^2 x_l + x_j^2 x_l + x_i^2 x_l - 2x_i x_j x_k) e_l.
\]

(33)

**Proposition 2.** Let us define the index sets:

\[
\mathcal{I}^+(K) := \{i \in \mathcal{N} : K_{i,i,i,i} > 0\}; \mathcal{I}^-(K) := \{i \in \mathcal{N} : K_{i,i,i,i} < 0\};
\]

\[
\mathcal{J}^+(K) := \{(i, k) \in \mathcal{P} : K_{i,i,k,k} > 0\}; \mathcal{J}^-(K) := \{(i, k) \in \mathcal{P} : K_{i,i,k,k} < 0\};
\]

\[
\mathcal{J}^+(K) := \{(i, k) \in \mathcal{P} : K_{i,i,k,k} > 0\}; \mathcal{J}^-(K) := \{(i, k) \in \mathcal{P} : K_{i,i,k,k} < 0\};
\]

\[
\mathcal{K}^+(K) := \{(i, j, k) \in \mathcal{Q} : K_{i,j,k,k} > 0\}; \mathcal{K}^-(K) := \{(i, j, k) \in \mathcal{Q} : K_{i,j,k,k} < 0\};
\]

\[
\mathcal{L}^+(K) := \{(i, j, k, l) \in \mathcal{R} : K_{i,j,k,l} > 0\}; \mathcal{L}^-(K) := \{(i, j, k, l) \in \mathcal{R} : K_{i,j,k,l} < 0\}.
\]

A DC decomposition of \( m_4 \) is given by:

\[ m_4(x) = g_{m_4}(x) - h_{m_4}(x) \]

with

\[
g_{m_4}(x) = \sum_{i \in \mathcal{I}^+(K)} K_{i,i,i,i} \tilde{g}_i(x) + 4 \sum_{(i,k) \in \mathcal{J}^+(K)} K_{i,i,i,k} \tilde{g}_{i,k}(x) - 4 \sum_{(i,k) \in \mathcal{J}^-(K)} K_{i,i,i,k} \tilde{h}_{i,k}(x) + 6 \sum_{(i,k) \in \mathcal{J}^+(K)} K_{i,i,k,k} \tilde{g}_{i,k}(x) - 6 \sum_{(i,k) \in \mathcal{J}^-(K)} K_{i,i,k,k} \tilde{h}_{i,k}(x) + 12 \sum_{(i,j,k) \in \mathcal{K}^+(K)} K_{i,j,j,k} \tilde{g}_{i,j,k}(x) - 12 \sum_{(i,j,k) \in \mathcal{K}^-(K)} K_{i,j,j,k} \tilde{h}_{i,j,k}(x) + 24 \sum_{(i,j,k,l) \in \mathcal{L}^+(K)} K_{i,j,k,l} \tilde{g}_{i,j,k,l}(x) - 24 \sum_{(i,j,k,l) \in \mathcal{L}^-(K)} K_{i,j,k,l} \tilde{h}_{i,j,k,l}(x)
\]

(34)
3.3. DC programming formulation of (MVSK)

Now, a DC decomposition for the polynomial objective function of (MVSK) is

\[
h_{m_4}(x) = -\sum_{i \in I^+(K)} K_{i,i,i,i} \tilde{g}_i(x) + 4 \sum_{(i,k) \in J^+(K)} K_{i,i,i,k} \tilde{h}_{i,k}(x) - 4 \sum_{(i,k) \in J^+(K)} K_{i,i,k,k} \tilde{g}_{i,k}(x) + 6 \sum_{(i,k) \in J^+(K)} K_{i,i,k,k} \tilde{h}_{i,k}(x) - 6 \sum_{(i,k) \in J^+(K)} K_{i,i,k,k} \tilde{g}_{i,k}(x) + 12 \sum_{(i,j) \in L^+(K)} K_{i,i,j,k} \tilde{h}_{i,j,k}(x) - 12 \sum_{(i,j) \in L^+(K)} K_{i,i,j,k} \tilde{g}_{i,j,k}(x) + 24 \sum_{(i,j,k,l) \in L^+(K)} K_{i,i,j,k,l} \tilde{h}_{i,j,k,l}(x) - 24 \sum_{(i,j,k,l) \in L^+(K)} K_{i,j,k,l} \tilde{g}_{i,j,k,l}(x)
\]

being both convex functions on \(\mathbb{R}^n\), where \(\tilde{g}_i, \tilde{g}_{i,k}, \tilde{h}_{i,k}, \tilde{g}_{i,k}, \tilde{h}_{i,j,k}, \tilde{g}_{i,j,k}, \tilde{g}_{i,j,k}, \tilde{h}_{i,j,k}, \tilde{g}_{i,j,k,l}\) and \(h_{i,j,k,l}\) are given respectively in \([17, 19, 20, 23, 26, 27, 30]\) and \([31]\). Their gradients are computed accordingly.

4. DC programming algorithm for solving (DCP)

In this section, we will discuss how to use DCA (an efficient DC algorithm) for finding KKT solutions of (DCP). Firstly, we will give a short description about DC program and DCA, then we apply DCA to solve (DCP).

4.1. DC programming and DCA

Let us denote \(\Gamma_0(\mathbb{R}^n)\), the set of lower semi-continuous (l.s.c.) and proper convex functions defined on \(\mathbb{R}^n\) to \((-\infty, +\infty]\) under the convention that \(+\infty - (+\infty) = +\infty\). The standard DC program is given by

\[
\alpha = \min \{ f(x) := g(x) - h(x) : x \in \mathbb{R}^n \},
\]

where \(g\) and \(h\) are both convex functions in \(\Gamma_0(\mathbb{R}^n)\).
Let us denote by $\alpha$ the optimal value of the standard DC program, and suppose that $\alpha$ is finite (i.e., the DC program has an optimal solution), then the necessary optimality condition for $x^* \in \mathbb{R}^n$ being a stationary point of the standard DC program is
\[
\partial g(x^*) \cap \partial h(x^*) \neq \emptyset,
\]
where $\partial h(x^*)$ denotes the subdifferential of $h$ at $x^*$, defined by, e.g. [Rockafellar 1970],
\[
\partial h(x^*) := \{ y \in \mathbb{R}^n : h(x) \geq h(x^*) + \langle x - x^*, y \rangle, \forall x \in \mathbb{R}^n \}.
\]
The subdifferential generalizes the derivative in the sense that the convex function $h$ is differentiable at $x^*$ if and only if $\partial h(x^*)$ reduces to the singleton $\{ \nabla h(x^*) \}$. Thus, if $g$ and $h$ are both differentiable convex functions on $\mathbb{R}^n$, this optimality condition reduces to the classical first order optimality condition for unconstrained optimization as $\nabla f(x^*) = \nabla g(x^*) - \nabla h(x^*) = 0$.

Considering a convex constrained DC program:
\[
\min \{ g(x) - h(x) : x \in C \}
\]
where $C \subset \mathbb{R}^n$ is a nonempty closed convex set, then by introducing the indicator function $\chi_C$ of the convex set $C$:
\[
\chi_C(x) = \begin{cases} 
0 & \text{if } x \in C, \\
+\infty & \text{otherwise.}
\end{cases}
\]
The convex constrained DC program is equivalent to standard DC program as
\[
\text{argmin}\{g(x) - h(x) : x \in C\} \Leftrightarrow \text{argmin}\{(g + \chi_C)(x) - h(x) : x \in \mathbb{R}^n\}
\]
where $g + \chi_C$ and $h$ are both convex functions in $\Gamma_0(\mathbb{R}^n)$.

An efficient DC Algorithm for solving standard DC program, called DCA, was first introduced by Pham Dinh Tao in 1985 as an extension of subgradient methods, and extensively developed by Le Thi Hoai An and Pham Dinh Tao since 1994. DCA consists of solving the standard DC program by a sequence of convex optimization problems as
\[
x^{k+1} \in \arg\min\{g(x) - \langle x, y^k \rangle : x \in \mathbb{R}^n\}
\]
with $y^k \in \partial h(x^k)$.

The above convex optimization problems are in fact derived from convex overestimations of the DC objective function $f$ at current iterate $x^k$, denoted $f^k$, which is constructed by linearizing the DC component $h$ at $x^k$ and taking $y^k \in \partial h(x^k)$ as
\[
f(x) = g(x) - h(x) \leq g(x) - (h(x^k) + \langle x - x^k, y^k \rangle) = f^k(x), \forall x \in \mathbb{R}^n.
\]
Then $f^k(x)$ is simplified as $g(x) - \langle x, y^k \rangle$ by removing the constant term $-h(x^k) - \langle x^k, y^k \rangle$.

DCA applies to convex constrained DC program yields a similar scheme as:
\[
x^{k+1} \in \arg\min\{g(x) - \langle x, y^k \rangle : x \in C\}
\]
with $y^k \in \partial h(x^k)$. 

DCA will be terminated if one of the following conditions verified:

- ∥x^{k+1} - x^k(1 + ∥x^k∥) ≤ ε_1 (i.e., the sequence \{x^k\} converges).
- |f(x^{k+1}) - f(x^k)|/(1 + |f(x^k)|) ≤ ε_2 (i.e., the sequence \{f(x^k)\} converges).

A convergence theorem for DCA (see e.g., (Pham & Le Thi 1997)) states that DCA starting with an arbitrary initial point \(x^0 \in \mathbb{R}^n\) will generate a sequence \{x^k\} such that

- The sequence \{f(x^k)\} is decreasing and bounded below.
- Every limit point of the sequence \{x^k\} converges to a stationary point (i.e., general KKT point) of the standard DC program.

The reader is referred to (Pham & Le Thi 1997, 1998; Le Thi & Pham 2003; Pham & Le Thi 2005; Pham et al. 2016; Le Thi & Pham 2018) for more topics on DC programming and DCA.

4.2. DCA for solving (DCP)

Since \(G\) and \(H\) are convex polynomial functions given in (36) and (37), thus \(H\) is differentiable, we can compute \(\nabla H\) as:

\[
\nabla H(x) = c_3 \nabla g_{m_3}(x) + c_4 \nabla h_{m_4}(x)
\]

DCA requires solving a sequence of convex optimization problems as:

\[
x^{k+1} \in \text{argmin} \{ G(x) - \langle x, \nabla H(x^k) \rangle : x \in \Omega \}.
\]

The detailed DCA is described as in Algorithm 1.

**Algorithm 1 DCA for (DCP)**

**Input:** Initial point \(x^0 \in \mathbb{R}^n\); Tolerance for optimal value \(\varepsilon_1 > 0\); Tolerance for optimal solution \(\varepsilon_2 > 0\);

**Output:** Optimal solution \(x^*\); Optimal value \(f^*\);

1: \(k \leftarrow 0; \Delta f \leftarrow +\infty; \Delta x \leftarrow +\infty;\)
2: while \(\Delta f \leq \varepsilon_1\) or \(\Delta x \leq \varepsilon_2\) do
3: \(x^{k+1} \in \text{argmin}\{G(x) - \langle x, \nabla H(x^k) \rangle : x \in \Omega\};\)
4: \(f^* \leftarrow f(x^{k+1});\)
5: \(x^* \leftarrow x^{k+1};\)
6: \(\Delta f \leftarrow |f^* - f(x^k)|/(1 + |f^*|);\)
7: \(\Delta x \leftarrow \|x^* - x^k\|/(1 + \|x^*\|);\)
8: \(k \leftarrow k + 1;\)
9: end while

**Theorem 3** (Convergence theorem of DCA Algorithm 1). DCA Algorithm 1 will generate a sequence \(\{x^k\}\) such that

- The sequence \(\{f(x^k)\}\) is decreasing and bounded below.
- Any limit point of \(\{x^k\}\) is a stationary point of (DCP).
Proof. This theorem is an immediate consequence of the general convergence theorem of DCA. We just need to show that both the sequences \( \{ x^k \} \) and \( \{ f(x^k) \} \) are bounded. The sequence \( \{ x^k \} \) is bounded since \( x^k, k \in \mathbb{N} \) are included in \( \Omega \) which is a nonempty compact convex set. The boundness of the sequence \( \{ f(x^k) \} \) is obvious since any polynomial is bounded in a compact convex set.

Note that if \( G \) is a strictly convex function on \( \Omega \), then the sequence \( \{ x^k \} \) generated by DCA Algorithm 1 will converge to a stationary point of \( \text{(DCP)} \). Otherwise, we can terminate the algorithm by the convergence of \( \{ f(x^k) \} \).

5. Boosted-DCA

For a nonlinear DC function whose optimal solution lies in a flat region, i.e., the objective function becomes very flat near the optimal solution, it is often observed that the convergence of DCA becomes very slow. This is due to the fact that the convex overestimation \( f^k \) in a flat region is in general not flat since \( g \) is not flat, thus \( f^k \) will fit poorly the object function \( f \) in flat region which yields a small step to the next iterate by DCA. For example, in our MVSK model, the convex overestimation \( f^k \) of a quartic polynomial objective function \( f \) is still a convex quartic polynomial function. When \( f \) is very flat in a region, then \( f \) as difference of two convex quartic polynomials is more likely as a locally affine function, but \( f^k \) as \( g \) plus an affine function may be far from \( f \) in this region. Therefore, \( f^k \) will be a poor over estimation of \( f \) in a flat region, and particularly worse when the degree of \( g \) is high.

In order to improve the performance of DCA for higher degree polynomial optimization, we propose a Boosted-DCA (called BDCA) which consists of introducing a line search using Armijo-type rule to get an improved iterate from the one obtained by DCA. The idea to introduce such a line search to boost an algorithm is firstly introduced by Fukushima-Mine in [Fukushima & Mine, 1981; Mine & Fukushima, 1981], then applied to DC program without constraint under the assumptions of strong convexity in DC components and Lojasiewicz property in \( f \) [Aragón Artacho et al., 2018]. In our paper, we will extend this technique to convex constrained DC program, and prove that with a weaker condition, we have the next theorem for \( \text{(DCP)} \):

Theorem 4. For DC program \( \text{(DCP)} \), and for all \( x^k \) and \( x^{k+1} \) two consecutive iterates generated by DCA Algorithm 1, we have
\[
\langle \nabla f(x^{k+1}), x^{k+1} - x^k \rangle \leq 0.
\]

Proof. Since \( x^{k+1} \) solve the convex optimization (38), thus \( x^{k+1} \) is a KKT point of (38). Let us define its Lagrangian function:
\[
L(x, \mu, \lambda) = G(x) - \langle x, \nabla H(x^k) \rangle - \mu(e^T x - 1) - \lambda^T x
\]
with the Lagrangian multipliers \( \lambda \in \mathbb{R}^n_+ \) and \( \mu \in \mathbb{R} \). Then \( x^{k+1} \) satisfies the following
KKT system:
\[
\begin{align*}
\nabla G(x^{k+1}) - \nabla H(x^k) - \mu e - \lambda &= 0 \\
e^T x^{k+1} &= 1 \\
\lambda^T x^{k+1} &= 0 \\
x^{k+1} &\geq 0, \lambda \geq 0
\end{align*}
\]

In addition with \(x^k \in \Omega\), i.e., \(e^T x^k = 1, x^k \geq 0\), and with the well-known fact that for any differentiable convex function \(H\) (even without strong convexity), we have \(\nabla H\) is monotone, i.e.,
\[
\langle \nabla H(x^k) - \nabla H(x^{k+1}), x^k - x^{k+1} \rangle \geq 0,
\]

it follows that
\[
\langle \nabla f(x^{k+1}), x^{k+1} - x^k \rangle = \langle \nabla G(x^{k+1}) - \nabla H(x^{k+1}), x^{k+1} - x^k \rangle = \langle \nabla H(x^k) + \mu e + \lambda - \nabla H(x^{k+1}), x^{k+1} - x^k \rangle = \langle \nabla H(x^k) - \nabla H(x^{k+1}), x^{k+1} - x^k \rangle + \langle \mu e + \lambda, x^{k+1} - x^k \rangle \\
\leq 0 + \mu \langle e, x^{k+1} - x^k \rangle + \langle \lambda, x^{k+1} - x^k \rangle \\
= \mu \langle e, x^{k+1} \rangle - \mu \langle e, x^k \rangle + \langle \lambda, x^{k+1} \rangle - \langle \lambda, x^k \rangle \leq 0.
\]

\[\square\]

**Theorem 5.** Under assumption of strong convexity in \(G\) and \(H\), then for all \(x^k\) and \(x^{k+1}\) two consecutive iterates generated by DCA Algorithm \[\text{[2]}\], there exists a constant \(\rho > 0\) such that
\[
\langle \nabla f(x^{k+1}), x^{k+1} - x^k \rangle \leq -\rho \|x^{k+1} - x^k\|^2.
\]
i.e., \(d = x^{k+1} - x^k\) (if \(d \neq 0\)) is a descent direction for \(f\) at \(x^{k+1}\).

*Proof.* For strongly convex function \(H\), i.e., there exists a constant \(\rho > 0\) such that \(H - \frac{\rho}{2}\|\cdot\|^2\) is a convex function, then \(\nabla H\) is strongly monotone, i.e.,
\[
\langle \nabla H(x^k) - \nabla H(x^{k+1}), x^k - x^{k+1} \rangle \geq \rho \|x^k - x^{k+1}\|^2. \quad (39)
\]

By analogue as in Theorem \[\text{[4]}\] it follows that
\[
\langle \nabla f(x^{k+1}), x^{k+1} - x^k \rangle = \langle \nabla H(x^k) - \nabla H(x^{k+1}), x^{k+1} - x^k \rangle + \langle \mu e + \lambda, x^{k+1} - x^k \rangle \\
\leq -\rho \|x^k - x^{k+1}\|^2 + \mu \langle e, x^{k+1} - x^k \rangle + \langle \lambda, x^{k+1} - x^k \rangle \\
= -\rho \|x^k - x^{k+1}\|^2 + \langle \lambda, x^{k+1} \rangle - \langle \lambda, x^k \rangle \geq 0
\]

\[
\leq -\rho \|x^k - x^{k+1}\|^2.
\]

Noting \(d = x^{k+1} - x^k\), if \(d \neq 0\), then \(\langle \nabla f(x^{k+1}), x^{k+1} - x^k \rangle \leq -\rho \|d\|^2 < 0\), which implies that \(d\) is a descent direction for \(f\) at \(x^{k+1}\).

\[\square\]

The above Theorems \[\text{[4]}\] and \[\text{[5]}\] provide us a potentially descent direction as \(x^{k+1} - x^k\).
to proceed a line search for accelerating the convergence of DCA.

Concerning on step size for line search, we can use an Armijo-type rule. Let us denote $\alpha$ the step size moving from $x^{k+1}$ to a new candidate $\hat{x}^{k+1}$ in the direction $d$ as

$$\hat{x}^{k+1} = x^{k+1} + \alpha d. \tag{40}$$

The initial step size can be set as $\alpha = \sqrt{2} \|d\|$ since the distance between any two points of $\Omega$ is $\leq \sqrt{2}$, then it follows from (40) that $\alpha = \|\hat{x}^{k+1} - x^{k+1}\|/\|d\| \leq \sqrt{2} \|d\|$.

Now, let $\beta \in (0, 1)$ denote the contraction parameter to reduce the step size $\alpha$ as

$$\alpha = \beta \alpha.$$ 

We can choose $\beta = 0.5$ to reduce half of step size each time, or set $\beta = \frac{\sqrt{5} - 1}{2} \approx 0.618$ to reduce based on the golden ratio.

We will stop reducing the step size if $\hat{x}^{k+1} \in \Omega$ and verifying

$$f(\hat{x}^{k+1}) \leq f(x^{k+1}) - \alpha \|d\|^2.$$ 

When the step size is too small, e.g., if $\alpha \leq 10^{-8}/\|d\|$, then we do not need to reduce $\alpha$ any more since the distance between $\hat{x}^{k+1}$ and $x^{k+1}$ is also too small (say, $\leq 10^{-8}$). In this case, we will stop line search and return the initial point $x^{k+1}$ as $\hat{x}^{k+1}$. The proposed Armijo-type line search is described in Algorithm 2:

**Algorithm 2** Armijo line search

**Input:** Potentially descent direction $d = x^{k+1} - x^k$; current iterate $x^{k+1}$; contraction parameter $\beta \in (0, 1)$ (e.g., $\beta = 0.618$); initial step size $\alpha > 0$ (e.g., $\alpha = \sqrt{2} \|d\|$); stopping tolerance for line search $\epsilon > 0$.

**Output:** Potentially improved candidate $\hat{x}^{k+1}$.

1: while $\alpha > \epsilon/\|d\|$ do
2: $\hat{x}^{k+1} \leftarrow x^{k+1} + \alpha d$;
3: $\Delta \leftarrow f(x^{k+1}) - f(\hat{x}^{k+1}) - \alpha \|d\|^2$;
4: if $\Delta \geq 0$ and $\hat{x}^{k+1} \in \Omega$ then
5: return $\hat{x}^{k+1}$;
6: end if
7: $\alpha \leftarrow \beta \alpha$;
8: end while
9: $\hat{x}^{k+1} \leftarrow x^{k+1}$;
10: return $\hat{x}^{k+1}$.

Combing the Armijo line search Algorithm 2 with DCA Algorithm 1, we can establish a BDCA stated in Algorithm 3:

**Theorem 6** (Convergence theorem of BDCA Algorithm 3). BDCA Algorithm 3 will generate a sequence $\{x^k\}$ such that

- The sequence $\{f(x^k)\}$ is decreasing and bounded below.
- Any limit point of $\{x^k\}$ is a stationary point of (DCP).
Algorithm 3 BDCA for solving (DCP)

**Input:** Initial point \( x^0 \in \mathbb{R}_+^n \); Tolerance for optimal value \( \varepsilon_1 > 0 \); Tolerance for optimal solution \( \varepsilon_2 > 0 \);

**Output:** Optimal solution \( x^* \); Optimal value \( f^* \);

1: \( k \leftarrow 0 \); \( \Delta f \leftarrow +\infty \); \( \Delta x \leftarrow +\infty \);
2: \textbf{while} \( \Delta f \leq \varepsilon_1 \) or \( \Delta x \leq \varepsilon_2 \) \textbf{do}
3: \( x^{k+1} \leftarrow \arg\min \{ G(x) - \langle x, \nabla H(x^k) \rangle : x \in \Omega \} \);
4: \textbf{if} \( x^k \in \Omega \) and \( x^{k+1} \in \Omega \) \textbf{then}
5: \( d \leftarrow x^{k+1} - x^k \);
6: \( \text{Use Armijo Algorithm 2 to get an improved candidate} \ \hat{x}^{k+1} \text{ from} \ x^{k+1} \);
7: \( x^{k+1} \leftarrow \hat{x}^{k+1} \);
8: \textbf{end if}
9: \( f^* \leftarrow f(x^{k+1}) \);
10: \( x^* \leftarrow x^{k+1} \);
11: \( \Delta f \leftarrow |f^* - f(x^k)|/(1 + |f^*|) \);
12: \( \Delta x \leftarrow ||x^* - x^k||/(1 + ||x^*||) \);
13: \( k \leftarrow k + 1 \);
14: \textbf{end while}

**Proof.** This theorem is an immediate consequence of Theorem 3. The new sequence \( \{x^k\} \) is bounded on \( \Omega \), and

\[
-\infty < \min\{f(x) : x \in \Omega\} \leq f(\hat{x}^{k+1}) \leq f(x^{k+1}) \leq f(x^k)
\]

implies that \( \{f(x^k)\} \) is decreasing and bounded below. Moreover, every limit point of \( \{x^k\} \), denoted by \( x^* \), verifies that \( x^* \in \arg\min\{G(x) - \langle x, \nabla H(x^*) \rangle : x \in \Omega\} \), then \( d = 0 \), and BDCA will stop at \( x^* \). \( \square \)

![Figure 1](image.png)

Figure 1: BDCA boosts DCA at \( x^{k+1} \) in the direction \( d \) to get a better candidate \( \hat{x}^{k+1} \).

Figure illustrates how BDCA boost the convergence of DCA. We observe that using
DCA Algorithm 1, we construct a convex overestimation \( f^k \) of \( f \) at \( x^k \), whose minimum is \( x^{k+1} \). Then we proceed a line search in the decent direction \( d = x^{k+1} - x^k \) at \( x^{k+1} \) using Armijo line search Algorithm 2, we then find a better candidate \( \hat{x}^{k+1} \) in the direct direction verifying \( f(\hat{x}^{k+1}) < f(x^{k+1}) \). Thus BDCA boosts the convergence of DCA towards the minimum \( x^* \). This boost effect will be particularly useful when the curvature of the function \( f \) becomes flat around the optimal solution and the convex overestimation \( f^k \) is not good enough to fit \( f \) around the current iterate.

Note that the new sequence established by Algorithm 3 may converge to a different solution even the DC components are strongly convex. Figure 2 illustrates how it works.

![Figure 2: BDCA and DCA could converge to a different solution](image)

We can observe in Figure 2 that when \( \alpha \) is well chosen, it is possible to find from \( x^{k+1} \) in the descent direction \( d \) a better candidate \( \hat{x}^{k+1} \) located around a better local minimum \( y^* \). If we use DCA from \( x^k \) without line search, we will probably converge to the nearest and worse local minimum \( x^* \). This particular feature could be very useful as an heuristic to escape a neighborhood of local minimum. In practice, if we want to increase the probability of escape, we can set in Algorithm 2 both the parameters \( \alpha \) and \( \beta \) as some big values, since the bigger the \( \alpha \) and \( \beta \) are, the further the search point would be. However, this could also slow down the Armijo line search. Otherwise, if we want a faster line search, we can set some small values for \( \alpha \) and \( \beta \), but the probability to escape the nearest local minimum is reduced as well.

6. Boosted-DCA based on universal DC decomposition

We can also introduce the Boosted-DCA for universal DC decomposition of MVSK model proposed in (Pham & Niu, 2011; Niu, 2010). The universal DC decomposition yields a DC program defined as

\[
\min_{x} \quad f(x) = G(x) - H(x) \quad \text{s.t.} \quad x \in \Omega
\]

(41)
where $\bar{G}(x) = \frac{\rho}{2}\|x\|_2^2$ is a convex quadratic function on $\mathbb{R}^n$, and $\bar{H}(x) = \frac{\rho}{2}\|x\|_2^2 - f(x)$ is a differentiable convex function on $\Omega$. A suitable parameter $\rho > 0$ is chosen based on the Proposition 2 in (Pham & Niu, 2011) as:

$$\forall x \in \Omega, \rho(\nabla^2 f(x)) \leq 2c_2 \|\Sigma\|_\infty + 6c_3 \max_{1 \leq i \leq n} \left( \sum_{j,k=1}^{n} |S_{i,j,k}| \right) + 12c_4 \max_{1 \leq j,k \leq n} \left( \sum_{i,l=1}^{n} |K_{i,j,k,l}| \right) = \rho.$$ 

Applying DCA to the DC program (41), called Universal DCA (UDCA), the function $\bar{H}$ is differentiable whose gradient is computed by

$$\nabla \bar{H}(x) = \rho x + c_1 \mu - 2c_2 \Sigma x + c_3 \frac{\nabla^2 m_3(x)x}{2} - c_4 \frac{\nabla^2 m_4(x)x}{3},$$

where

$$\nabla^2 m_3(x) = \left( 6 \sum_{k=1}^{n} S_{i,j,k} x_k \right)_{(i,j) \in \mathcal{N}^2} \quad \text{and} \quad \nabla^2 m_4(x) = \left( 12 \sum_{k,l=1}^{n} K_{i,j,k,l} x_k x_l \right)_{(i,j) \in \mathcal{N}^2}.$$

The convex sub-problem in UDCA is a strictly convex quadratic program given by

$$x^{k+1} \in \operatorname{argmin} \left\{ \frac{\rho}{2} \|x\|_2^2 - \langle x, \nabla \bar{H}(x^k) \rangle : x \in \Omega \right\} \quad (42)$$

which leads to the projection of $\nabla \bar{H}(x^k)/\rho$ on the standard simplex $\Omega$. The later problem can be solved very efficiently by a strongly polynomial Block Pivot Principal Pivoting Algorithm (BPPPA) presented in (Judice & Pires, 1992; Pham & Niu, 2011), or by the direct projection method presented in (Gondran & Minoux, 1984; Held et al., 1974).

By introducing the Armijo line search in UDCA, we get a similar Boosted-UDCA, called UBDCA, described in Algorithm 4. The only difference between Algorithms 3 and 4 is in the line 3 where the convex polynomial optimization (38) is replaced by the convex quadratic optimization (42). So its convergence theorem will be the same.

Note that solving a convex quadratic optimization (42) will be faster than solving a convex quartic polynomial optimization (38) in practice. However, it is hopefully that the convex overestimation provided by DCSOS decomposition could fit better the high order polynomial objective function $f$ than the convex quadratic overestimation. Therefore, BDCA Algorithm 3 could require less number of iterations and get better computed results than BDCA Algorithm 4. It is worth noting that if the number of iterations required by the two boosted DC algorithms are almost the same for solving some problems, then Algorithm 4 will be faster than Algorithm 4.

7. Numerical simulation

7.1. Experimental setup

Our DCA and BDCA algorithms are implemented on MATLAB based on a self-developed continuously evolving MATLAB DC optimization toolbox which consists of a series of classes: a DC function class (called dcfunc), a DC programming problem class (called dcp), and a DC algorithm class (called dca). This toolbox provides a general
Algorithm 4 UBDCA for solving (41)

Input: Initial point $x^0 \in \mathbb{R}^n_+$; Tolerance for optimal value $\epsilon_1 > 0$; Tolerance for optimal solution $\epsilon_2 > 0$; 

Output: Optimal solution $x^*$; Optimal value $f^*$;

1: $k \leftarrow 0; \Delta f \leftarrow +\infty; \Delta x \leftarrow +\infty;$
2: while $\Delta f \leq \epsilon_1$ or $\Delta x \leq \epsilon_2$ do
3: Compute $x^{k+1}$ by solving the convex quadratic program (42);
4: if $x^k \in \Omega \& \& x^{k+1} \in \Omega$ then
5: $d \leftarrow x^{k+1} - x^k$;
6: Use Algorithm 2 to get an improved candidate $\hat{x}^{k+1}$ from $x^{k+1}$;
7: $x^{k+1} \leftarrow \hat{x}^{k+1}$;
8: end if
9: $f^* \leftarrow f(x^{k+1})$;
10: $x^* \leftarrow x^{k+1}$;
11: $\Delta f \leftarrow |f^* - f(x^k)|/(1 + |f^*|)$;
12: $\Delta x \leftarrow ||x^* - x^k||/(1 + ||x^*||)$;
13: $k \leftarrow k + 1$;
14: end while

modeling tool to build DC functions, DC programming problems and use a general DCA to solve DC programs conveniently. The version 1.0 of this toolbox will be published in our future paper and released for extensive tests on github soon.

Our DC algorithms proposed in this paper are tested on a Dell Workstation equipped with 4 Intel i7-6820HQ 2.70GHz CPU and 32 GB RAM. We use YALMIP (Lofberg, 2004) to build multivariate polynomials on MATLAB. The convex optimization sub-problems (38) required in Algorithm 1 is solved by nonlinear local optimization solver Knitro (Byrd et al., 2006) which is chosen based on our observations that it appears to be the fastest local solver using interior point method for solving convex polynomial optimization among other tested ones such as MATLAB fmincon, IPOPT (Wächter & Biegler, 2006) and CVX (Grant et al., 2008). The quadratic convex optimization problem (41) is solved by BPPPA algorithm which can be easily implemented on MATLAB (see e.g., (Pham & Niu, 2011) for more details about BPPPA).

7.2. Data description

We use two datasets in our experiments. The first dataset is randomly generated with the return rates $R_{i,t} \in [-0.1,0.1]$. This dataset is used to test the computational performance of different DC algorithms. The second dataset consists of 51 weekly returns of 1151 assets in Shanghai A shares ranged from January 2018 to December 2018 downloaded from CSMAR http://www.gtarsc.com/database. These data are used to analyze the optimal portfolios and plot efficient frontier on real stock market.

7.3. Higher moment computation

The input four moments (mean, covariance, co-skewness and co-kurtosis) are computed using the formulations (1), (2), (3) and (4). The “curse of dimensionality” is still a crucial question to the MVSK model. Two important issues need to be noted:
1. The sparsity issue of the moment data: As we have explained in our previous work (Pham & Niu, 2011) that the moments are often dense tensors which yields the MVSK model as a dense nonconvex quartic polynomial optimization. The construction of a dense higher degree multivariate polynomial in MATLAB is very time consuming using either MATLAB symbolic toolbox or YALMIP modeling tool. For instance, when the number of assets \( n = 25 \), constructing an MVSK model will take about 30 minutes, and most of computations are related on the construction of the quartic polynomial objective function. Figure 3 illustrates the increase of the number of assets \( n \) v.s. the model construction time. We can see that the construction time increases dramatically when \( n > 14 \).

![Figure 3: Number of assets \( n \) v.s. MVSK model construction time on MATLAB](image)

In our knowledge, there is still no more efficient way in the construction of a high degree polynomial function in MATLAB. The sparsity issue is the most important point to limit the size of the constructible MVSK model in practice.

2. The computer memory issue of the moment data: Based on the symmetric of the moment tensors, it is unnecessary to allocate full computer memories for saving all high order moments. E.g., due to the limitation of allowable MATLAB array size, the construction of an \( n^4 \) co-kurtosis tensor with \( n = 300 \) yields about 60.3 GB memory which is intractable in our 32 GB RAM equipment. In our previous work (Pham & Niu, 2011), we have tried using Kronecker product and MATLAB mex programming technique to compute co-skewness (resp. co-kurtosis) tensor as \( n \times n^2 \) (resp. \( n \times n^3 \)) sparse matrix by keeping only the independent elements. Even though, saving huge amounts of moment data in memory is still very space consuming. In order to overcome this difficulty, in this paper, we propose computing the co-skewness and the co-kurtosis entries just-in-time (called JIT technique) when they are used through the formulations (3) and (4) without saving them in memory at all. Since these moment coefficients need only to be computed once in the construction of the polynomial objective function with YALMIP, and the size of the resulting polynomial is not very large. Once the polynomial is constructed in YALMIP, the evaluation of the polynomial at any given point is very fast. Using
this approach, we will save tremendously the memory space and overcome the bottleneck of the computer memory issue.

7.4. Performance of DC algorithms

The numerical performance of our proposed algorithms DCA (Algorithm 1) and BDCA (Algorithm 2) are compared with our previously proposed DC algorithm based on universal DC decomposition UDCA in (Pham & Niu 2011), as well as UBDCA Algorithm 4 with Armijo line search. The convex optimization sub-problems (38) required in DCA and BDCA are solved by Knitro via interior-point-method, while the quadratic convex optimization sub-problems (42) required in UDCA and UBDCA are solved by BPPPA algorithm.

Note that the comparison of UDCA with many existing solvers such as Gloptipoly, LINGO and fmincon (SQP and Trust Region) have been reported in our previous work (Pham & Niu 2011), therefore, we will only focus on the comparison among different DC algorithms (DCA, BDCA, UDCA and UBDCA) in this paper.

We use the randomly generated dataset by varying the number of assets \( n \) from 4 to 20 by step increments of 2 and with \( T \) fixed to 30 periods. For each \( n \), we randomly generate 3 models in which the investor’s preference parameter \( c \) is randomly chosen with \( c^Tc = 1 \) and \( c \geq 0 \). The rate of returns \( R_{i,t} \) are randomly generated in \([-0.1, 0.1]\) for all \( i \in \{1, \ldots, n\}, t \in \{1, \ldots, T\} \) by using MATLAB function \texttt{rand}. Finally, we constructed 27 random MVSK models. The initial point \( x^0 \) is randomly chosen as an integer vector in \( \{0, 1\}^n \). The reason to use such an integer initial point is due to our observations that the optimal solution for MVSK model is often sparse, i.e., consists of many zero entries. In DCA, the tolerance for optimal value \( \varepsilon_1 = 10^{-5} \) and the tolerance for optimal solution \( \varepsilon_2 = \sqrt{\varepsilon_1} \). In Armijo line search, the contraction parameter \( \beta = 0.618 \), the initial step size \( \alpha = \frac{\sqrt{2}}{\beta^2} \), and the stopping tolerance for line search \( \varepsilon = 10^{-8} \).

Figure 4 illustrates the solution time v.s. the number of problems solved. We observe

![Figure 4: Number of models solved v.s. total solution time for DCA, BDCA, UDCA, and UBDCA](image)

that the fastest algorithm is UBDCA which solve all models within 15.8 seconds, then UDCA with 44.5 seconds, BDCA with 72.1 seconds, and DCA with 180.7 seconds. Thus, BDCA is 2.51 times faster than DCA, and UBDCA is 2.82 times faster than UDCA.
We conclude that the proposed boosted algorithms with Armijo-type line search indeed accelerate DC algorithms.

Table 1 summarizes the detailed tested models information including the number of variables (n), the number of period (T) and the number of monomials with nonzero coefficients in objective function (nnz), as well as the solution information for DC algorithms including the number of iterations (iter), the solution time (time), and the objective value (obj).

Among the four DC algorithms, BDCA seems always providing the best numerical solution by conserving the relatively small computing time. As we expected, we can observe that both DCA and BDCA get always better numerical results (with smaller objective values) than UDCA and UBDCA. It is not surprising to see that DCA and BDCA are slower than UDCA and UBDCA in average, this is because DCA and BDCA need to solve more difficult convex programming sub-problems than UDCA and UBDCA, meanwhile the difference in the average number of iterations is not too large (20 for DCA, 12 for BDCA, 48 for UDCA and 12 for UBDCA). But the quality of the numerical results provided by DCA and BDCA are often better, and the average iterations in DCA and BDCA are also smaller than UDCA and UBDCA, which indicates that DCSOS decomposition technique proposed in this paper provides indeed better DC decomposition than the universal DC decomposition.

Note that a fast convex optimization solver for problem (38) is extremely important to the performance of DCA and BDCA. It deserves more attention to develop faster convex optimization techniques for (38) in our future work.

7.5. Draw efficient frontier

We can generate the efficient frontier of the portfolio provided by the MVSK model. To do this, we randomly choose 10 potentially ‘good’ assets from the dataset of Shanghai A shares based on their positive average returns within 51 weeks, and fix the expected return of the portfolio \( m_1 \) in an interval \([a, b]\), e.g., given a set of expected returns \( \{r_k\} \) taken from 0.01 to 0.3 by step 0.005, then we add the linear constraint \( m_1(x) = r_k \) into the MVSK model which turns to:

\[
\min \{c_2m_2(x) - c_3m_3(x) + c_4m_4(x) : x \in \Omega, m_1(x) = r_k \}.
\]

Solving these problems by BDCA, we get a set of optimal solutions \( \{(x^*)^k\} \). These solutions are optimal portfolios located in the efficient frontier. Let us plot the Mean-Variance efficient frontier in Figure 6 (as the top portion of the blue line) by joining the set of points \( \{(m_2((x^*)^k), r_k)\} \). The horizontal axis is the Risk given by the variance of portfolio, i.e., \( m_2((x^*)^k) \), and the vertical axis is the expected return \( m_1((x^*)^k) = r_k \). We observe that the optimal portfolio with highest return is found at 0.9% with a highest risk 0.0084. The shape of the efficient frontier confirms exactly the modern portfolio theory as portion of hyperbola.

We can also plot higher moments efficient frontiers in Figure 6, the Mean-Variance-Skewness efficient frontier (on the left) and the Mean-Variance-Kurtosis efficient frontier (on the right). We observe that the Skewness and the Kurtosis of the optimal portfolios are both increasing when the Mean (return) and the Variance (risk) are big enough. This indicates that a high mean-variance portfolio choice could increase the probability of gains (high skewness) but with more uncertainty of returns (high kurtosis).
Table 1: Numerical performances of DCA, BDCA, UDCA and UBDCA
8. Conclusion and perspectives

In this paper, we have proposed several DC programming approaches for solving the higher moment (mean-variance-skewness-kurtosis) portfolio optimization problem which can be reformulated as a nonconvex quartic polynomial optimization problem. Our contributions are mainly focused on the construction of DC programming formulations based on DCSOS decomposition technique for MVK portfolio model, then we apply an efficient DC algorithm - DCA for finding its KKT solution. We have also investigated an acceleration technique based on Armijo-type line search to boost DC algorithms (DCA with DCSOS decomposition and UDCA with universal DC decomposition). Numerical simulations demonstrate that the boosted DC algorithms (BDCA and UBDCA) converge more rapidly to KKT solutions than DCA and UDCA. Moreover, our new DC programming formulation based on DCSOS decomposition provides a better DC decomposition.
than the universal DC decomposition since it often provides better local optimal solution and requires less number of iterations.

Concerning on future works, firstly, as forgoing mentioned, a fast convex optimization solver for problem (38) is extremely important to the performance of DCA based on DCSOS decomposition. It deserves more attention to develop faster solution method based on the specific structure of (38). For example, based on its DCSOS structure, we can introduce additional variables to replace the corresponding SOS as convex quadratic constraints, which will reformulate the high order convex optimization problem (38) as a convex quadratically constrained quadratic optimization problem. The later one can be solved much more efficiently using convex quadratic optimization solvers or second order cone programming solvers as CPLEX, Gurobi and Mosek. Moreover, since we do not really need to solve the convex optimization problem (38) in DCA, but only to find a better feasible point $x^{k+1} \in \Omega$ such that $f(x^{k+1}) < f(x^k)$. Therefore, it is possible to think about using descent algorithms such as Wolfe algorithm to search a better feasible point $x^{k+1}$ from the initial point $x^k$ without entirely solving the convex optimization (38). This technique is called partial solution strategy which is demonstrated as a useful technique to get better performance in many large-scale optimizations (see e.g., Niu & Pham [2014]).

Another important topic is to develop an efficient polynomial modeling toolbox in MATLAB to construct polynomial and its differentiation more efficiently. As we have observed, the MATLAB symbolic toolbox and YALMIP are inefficient for polynomial construction. Moreover, YALMIP can’t even save polynomial objects (sdpvar objects) into mat files, so we can’t save MVSK models that have been constructed in YALMIP over a long time to the hard disk so that they can be reloaded and tested in the future. To overcome this annoying problem, we have developed a multivariate polynomial class (called mpoly) in MATLAB which is used to convert the YALMIP sdpvar object as a savable format into mat file. We are going to extend mpoly class in future as a professional polynomial optimization modeling toolbox for constructing efficiently polynomial functions and polynomial optimization problems in MATLAB. Researches in these directions will be reported subsequently.

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