ORDINARY DIFFERENTIAL SYSTEMS IN DIMENSION THREE
WITH AFFINE WEYL GROUP SYMMETRY OF TYPES
$D_4^{(1)}, B_3^{(1)}, G_2^{(1)}, D_3^{(2)}$ AND $A_2^{(2)}$

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Abstract. We present a four-parameter family of ordinary differential systems in dimension three with affine Weyl group symmetry of type $D_4^{(1)}$. By obtaining its first integral, we can reduce this system to the second-order non-linear ordinary differential equations of Painlevé type. We also study this system restricted its parameters. Each system can be obtained by connecting some invariant divisors in the system of type $D_4^{(1)}$. Each system admits affine Weyl group symmetry of types $B_3^{(1)}, G_2^{(1)}, D_3^{(2)}$ and $A_2^{(2)}$, respectively. These symmetries, holomorphy conditions and invariant divisors are new.

1. Introduction

In this paper, we present a 4-parameter family of ordinary differential systems in dimension three with affine Weyl group symmetry of type $D_4^{(1)}$. By obtaining its first integral, we can reduce this system to the second-order non-linear ordinary differential equations of Painlevé type. This reduced system parametrizes the first-order ordinary differential equation:

$$\frac{dX}{dt} = \frac{b(t)}{2\eta}X(X + 1)(X + 1 - \eta)(X - \eta), \quad b(t) \in \mathbb{C}(t), \quad \eta \in \mathbb{C} - \{0\}.$$  

We also study this system restricted its parameters. Each system admits affine Weyl group symmetry of types $B_3^{(1)}, G_2^{(1)}, D_3^{(2)}$ and $A_2^{(2)}$, respectively.

Each system can be obtained by connecting some invariant divisors in the system of type $D_4^{(1)}$.

The Bäcklund transformations of each system satisfy

$$s_i(g) = g + \frac{\alpha_i}{f_i} \{f_i, g\} + \frac{1}{2!} \left(\frac{\alpha_i}{f_i}\right)^2 \{f_i; \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[x, y, z]),$$

where poisson bracket $\{,\}$ satisfies the relations:

$$\{z, x\} = \{z, y\} = 1, \quad \{x, y\} = 0.$$  

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

These symmetries, holomorphy conditions and invariant divisors are new.

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In this paper, we study the third-order ordinary differential system:

\[
\begin{align*}
\frac{2\eta}{b(t)} \frac{dx}{dt} &= -2x(y-1)yz(x-\eta)(2x-2y+1-\eta) - 2(2\alpha_0 + 2\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_4)x^3y \\
&\quad - 2(\alpha_0 + \alpha_1)xy^3 + (5\alpha_0 + 5\alpha_1 + 6\alpha_2 + 3\alpha_3 + \alpha_4)x^2y^2 \\
&\quad - \{5\alpha_0 + 5\alpha_1 + 6\alpha_2 + 2\alpha_3 - 3(2\alpha_0 + 2\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_4)\eta\}x^2y \\
&\quad + \{3(\alpha_0 + \alpha_1) - (4\alpha_0 + 6\alpha_1 + 6\alpha_2 + \alpha_3 + \alpha_4)\eta\}xy^2 + x^4 \\
&\quad + 2(\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 - \eta)x^3 + 2\alpha_1\eta y^3 \\
&\quad + \{(\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3)(\eta^2 - 3\eta + 1) + \alpha_4\eta^2\}x^2 + \alpha_1(\eta - 3)\eta y^2 \\
&\quad + \{-(\alpha_0 + \alpha_1) + 2(2\alpha_0 + 3\alpha_1 + 3\alpha_2 + \alpha_3)\eta - (2\alpha_0 + 2\alpha_1 + 4\alpha_2 + \alpha_3 + \alpha_4)\eta^2\}xy \\
&\quad + (\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3)(\eta - 1)\eta x - \alpha_1(\eta - 1)\eta y, \\
\frac{2\eta}{b(t)} \frac{dy}{dt} &= -2x(y-1)yz(x-\eta)(2x-2y+1-\eta) - 2(\alpha_3 + \alpha_4)x^3y \\
&\quad - 2x^3(\alpha_0 + \alpha_1 + 4\alpha_2 + 2\alpha_3 + 2\alpha_4) + x^2y^2(\alpha_0 + \alpha_1 + 6\alpha_2 + 5\alpha_3 + 5\alpha_4) \\
&\quad + x^2y\{-(\alpha_0 + \alpha_1 + 6\alpha_2 + 4\alpha_3 + 6\alpha_4) + 3(\alpha_3 + \alpha_4)\eta\} \\
&\quad + xy^2\{3(\alpha_0 + \alpha_1 + 4\alpha_2 + 2\alpha_3 + 2\alpha_4) - (2\alpha_0 + 6\alpha_2 + 5\alpha_3 + 5\alpha_4)\eta\} + y^4 + 2\alpha_4 x^3 \\
&\quad + 2y^3((\alpha_0 + 2\alpha_2 + \alpha_3 + \alpha_4)\eta - 1) - \alpha_4(3\eta - 1)x^2 \\
&\quad + y^2\{1 - 3(\alpha_0 + 2\alpha_2 + \alpha_3 + \alpha_4)\eta + (\alpha_0 + 2\alpha_2 + \alpha_3 + \alpha_4)\eta^2\} \\
&\quad + xy\{-(\alpha_0 + \alpha_1 + 4\alpha_2 + 2\alpha_3 + 2\alpha_4) + 2(\alpha_0 + 3\alpha_2 + 2\alpha_3 + 3\alpha_4)\eta - (\alpha_3 + \alpha_4)\eta^2\} \\
&\quad + \alpha_4(\eta - 1)\eta x - (\alpha_0 + 2\alpha_2 + \alpha_3 + \alpha_4)(\eta - 1)\eta y, \\
\frac{2\eta}{b(t)} \frac{dz}{dt} &= (2x - 2y + 1 - \eta)(2x^2y + 2x^2z - x^2 - y^2\eta - 2(x + 1)xy + \eta(x + y))z^2 \\
&\quad + z\{-2x^3(\alpha_3 + \alpha_4) + 2y^3(\alpha_0 + \alpha_1) + 2x^2y(\alpha_0 + \alpha_1 + 6\alpha_2 + 2\alpha_3 + 2\alpha_4) \\
&\quad - 2x^2y(2\alpha_0 + 2\alpha_1 + 6\alpha_2 + \alpha_3 + \alpha_4) + x^2\{-(\alpha_0 + \alpha_1 + 6\alpha_2 + 4\alpha_3 + 3(\alpha_3 + \alpha_4)\eta\} \\
&\quad + y^2(-3(\alpha_0 + \alpha_1) + (4\alpha_0 + 6\alpha_2 + \alpha_3 + \alpha_4)\eta) \\
&\quad - 4xy\{-(\alpha_0 + \alpha_1 + 3\alpha_2 + \alpha_3) + (\alpha_0 + 3\alpha_2 + \alpha_3 + \alpha_4)\eta\} \\
&\quad + x\{-(\alpha_0 + \alpha_1 + 4\alpha_2 + 2\alpha_3) + 2(\alpha_0 + 3\alpha_2 + 2\alpha_3)\eta - (\alpha_3 + \alpha_4)\eta^2\} \\
&\quad + y\{\alpha_0 + \alpha_1 - 2(2\alpha_0 + 3\alpha_2 + \alpha_3)\eta + (2\alpha_0 + 4\alpha_2 + \alpha_3 + \alpha_4)\eta^2\} \\
&\quad - (\alpha_0 + 2\alpha_2 + \alpha_3)(\eta - 1)\eta + \alpha_2[(\alpha_0 + \alpha_1 + 2\alpha_2 - \alpha_3 - \alpha_4)x^2 \\
&\quad + (\alpha_0 + \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4)y^2 - 2(\alpha_0 + \alpha_1 - \alpha_3 - \alpha_4)xy \\
&\quad + x\{\alpha_0 + \alpha_1 - 2\alpha_3 - (2\alpha_0 + 2\alpha_2 - \alpha_3 - \alpha_4)\eta\} \\
&\quad + y\{\alpha_0 - \alpha_1 + 2\alpha_2 + 2\alpha_3 + (2\alpha_0 - \alpha_3 - \alpha_4)\eta\} + (\eta - 1)(\alpha_2 + \alpha_3 + (\alpha_0 + \alpha_2)\eta)\}. 
\end{align*}
\]
Here \(x, y\) and \(z\) denote unknown complex variables, \(b(t) \in \mathbb{C}(t)\), \(\eta \in \mathbb{C} - \{0\}\), and \(\alpha_0, \alpha_1, \ldots, \alpha_4\) are complex parameters satisfying the relation:

\[
\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1.
\]

**Theorem 2.1.** Let us consider the following ordinary differential system in the polynomial class:

\[
\begin{align*}
\frac{dx}{dt} &= f_1(x, y, z), \\
\frac{dy}{dt} &= f_2(x, y, z), \\
\frac{dz}{dt} &= f_3(x, y, z).
\end{align*}
\]

We assume that

(A1) \(\text{deg}(f_i) = 6\) with respect to \(x, y, z\).

(A2) The right-hand side of this system becomes again a polynomial in each coordinate system \((x_i, y_i, z_i)\) \((i = 0, 1, \ldots, 4)\).

\[
\begin{align*}
0) & \quad x_0 = x - \eta, \quad y_0 = y, \quad z_0 = z - \frac{\alpha_0}{x - \eta}, \\
1) & \quad x_1 = x, \quad y_1 = y, \quad z_1 = z - \frac{\alpha_1}{x}, \\
2) & \quad x_2 = x + \frac{\alpha_2}{z}, \quad y_2 = y + \frac{\alpha_2}{z}, \quad z_2 = z, \\
3) & \quad x_3 = x, \quad y_3 = y - 1, \quad z_3 = z - \frac{\alpha_3}{y - 1}, \\
4) & \quad x_4 = x, \quad y_4 = y, \quad z_4 = z - \frac{\alpha_4}{y}.
\end{align*}
\]

Then such a system coincides with the system \([4]\).

These transition functions satisfy the condition:

\[
dx_i \wedge dy_i \wedge dz_i = dx \wedge dy \wedge dz \quad (i = 0, 1, \ldots, 4).
\]

**Theorem 2.2.** The system \([4]\) admits the affine Weyl group symmetry of type \(D_4^{(1)}\) as the group of its Bäcklund transformations, whose generators are explicitly given as follows:
with the notation \((\ast) := (x, y, z; \alpha_0, \alpha_1, \ldots, \alpha_4)\),

\[

text{\begin{align*}
    s_0 : (\ast) &\rightarrow \left(x, y, z - \frac{\alpha_0}{x}; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4\right), \\
    s_1 : (\ast) &\rightarrow \left(x, y, z - \frac{\alpha_1}{x}; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4\right), \\
    s_2 : (\ast) &\rightarrow \left(x + \frac{\alpha_2}{z}, y + \frac{\alpha_2}{z}; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2\right), \\
    s_3 : (\ast) &\rightarrow \left(x, y, z - \frac{\alpha_3}{y - 1}; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4\right), \\
    s_4 : (\ast) &\rightarrow \left(x, y, z - \frac{\alpha_4}{y}; \alpha_0, \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4\right).
\end{align*}}

We note that the generators \(s_0, s_1, \ldots, s_4\) are determined by the invariant divisors \((2.3)\) (see next proposition).

**Proposition 2.3.** The system \((4)\) has the following invariant divisors:

| parameter's relation | \(f_i\) |
|----------------------|---------|
| \(\alpha_0 = 0\)    | \(f_0 := x - \eta\) |
| \(\alpha_1 = 0\)    | \(f_1 := x\) |
| \(\alpha_2 = 0\)    | \(f_2 := z\) |
| \(\alpha_3 = 0\)    | \(f_3 := y - 1\) |
| \(\alpha_4 = 0\)    | \(f_4 := y\) |

We note that when \(\alpha_1 = 0\), we see that the system \((4)\) admits a particular solution \(x = 0\).

### 3. Reduction of the \(D_{4}^{(1)}\) system

In this section, we show that the system \((4)\) has its first integral. Thanks to this first integral, we can reduce the system \((4)\) to the second-order non-linear ordinary differential equations.

**Proposition 3.1.** The system \((4)\) has its first integral:

\[
(7) \quad \frac{d(x - y)}{dt} = \frac{b(t)}{2\eta}(x - y)(x - y + 1)(x - y + 1 - \eta)(x - y - \eta).
\]

Under the condition

\[
(8) \quad b(t) = \frac{2\eta}{t(t - 1)(t + \eta)(t + \eta - 1)},
\]

the equation \((7)\) admits a particular solution:

\[
(9) \quad x = y - t.
\]
THEOREM 3.2. Under the conditions

\begin{align}
  b(t) &= \frac{2\eta}{t(t-1)\{t^2 + (2\eta - 1)t + \eta(\eta - 1)\}}, \\
  x &= y - t,
\end{align}

for the system (4) we make the change of parameters and variables

\begin{align}
  \alpha_0 &= A_1, \quad \alpha_1 = A_0, \quad \alpha_2 = A_2, \quad \alpha_3 = A_3, \quad \alpha_4 = A_4, \quad X := y, \quad Y := z
\end{align}

from \(\alpha_0, \alpha_1, \ldots, \alpha_4, x, y, z\) to \(A_i, X, Y\). Then the system (4) can also be written in the new variables \(X, Y\) and parameters \(A_i\) as a Hamiltonian system. This new system tends to

\begin{align}
  \frac{dX}{dt} &= \frac{\partial H_{VI}}{\partial Y}, \quad \frac{dY}{dt} = -\frac{\partial H_{VI}}{\partial X}
\end{align}

with the polynomial Hamiltonian

\begin{align}
  H_{VI}(X, Y, t; A_1, A_0, A_2, A_3, A_4) &= \frac{1}{t(t-1)}[Y^2(X - t)(X - 1)X - \{(A_1 - 1)(X - 1)X + A_3(X - t)X
  
  + A_4(X - t)(X - 1)\}Y + A_2(A_0 + A_2)X]
\end{align}

as \(\eta \to \infty\).

This system is the Painlevé VI system.

We also see that the equation (7) admits a particular solution:

\begin{align}
  x &= y.
\end{align}

THEOREM 3.3. Under the condition

\begin{align}
  x &= y,
\end{align}

for the system (4) we make the change of variables

\begin{align}
  X := y, \quad Y := z
\end{align}

from \(x, y, z\) to \(X, Y\). Then the system (4) can also be written in the new variables \(X, Y\) as a Hamiltonian system. This new system tends to

\begin{align}
  \frac{dX}{dt} &= \frac{\partial H}{\partial Y}, \quad \frac{dY}{dt} = -\frac{\partial H}{\partial X}
\end{align}

with the polynomial Hamiltonian

\begin{align}
  H(X, Y, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) &= -\frac{\eta - 1}{2\eta}b(t)[-(X - 1)(X - \eta)X^2Y^2 - \{2\alpha_2(X - 1)(X - \eta) - \alpha_0\eta(X - 1) - \alpha_3(X - \eta)\}XY
  
  - \alpha_2^2X^2 + \alpha_2\{1 - (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_4) + (1 - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4))\eta\}X].
\end{align}
Elimination of $Y$ from the system (18) gives the second-order non-linear ordinary differential equation for the variable $X$:

$$
\frac{d^2 X}{dt^2} = \left\{ \frac{1}{2(X-1)} + \frac{1}{X} + \frac{1}{2(X-\eta)} \right\} \left( \frac{dX}{dt} \right)^2 + \frac{db(t)}{b(t)} \frac{dX}{dt} - \frac{b(t)^2}{8\eta^2(X-1)(X-\eta)} \left[ (\eta-1)^3 X^2 \{ (\alpha_0 \eta + \alpha_3) X - (1 - (\alpha_1 + 2\alpha_2 + \alpha_4)) \eta \} \right.
\times \left\{ (\alpha_0 \eta - \alpha_3) X + (1 - (2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_4)) \eta \} \right].
$$

(20)

**Proposition 3.4.** The system (18) admits the affine Weyl group symmetry of type $D_4^{(1)}$ as the group of its B"acklund transformations, whose generators are explicitly given as follows: with the notation $(*) := (X, Y; \alpha_0, \alpha_1, \ldots, \alpha_4)$,

$\begin{align*}
s_0 : (*) & \rightarrow \left( X, Y - \frac{\alpha_0}{X - \eta}; -\alpha_0, \alpha_1, \alpha_2 + \alpha_0, \alpha_3, \alpha_4 \right), \\
s_1 : (*) & \rightarrow \left( X, Y - \frac{\alpha_1}{X}; \alpha_0, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3, \alpha_4 \right), \\
s_2 : (*) & \rightarrow \left( X + \frac{\alpha_2}{Y}, Y; \alpha_0 + \alpha_2, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2, \alpha_4 + \alpha_2 \right), \\
s_3 : (*) & \rightarrow \left( X, Y - \frac{\alpha_3}{X - 1}; \alpha_0, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3, \alpha_4 \right), \\
s_4 : (*) & \rightarrow \left( X, Y - \frac{\alpha_4}{X}; \alpha_0, \alpha_1, \alpha_2 + \alpha_4, \alpha_3, -\alpha_4 \right).
\end{align*}$

4. The System of Type $B_3^{(1)}$

In this section, we present a 3-parameter family of ordinary differential systems in dimension three with affine Weyl group symmetry of type $B_3^{(1)}$. This system is equivalent to the system (4) restricted its parameters.

This system can be obtained by connecting the invariant divisors $x$ and $y$ in the system (4).

For this system, we can discuss some reductions to the second-order ordinary differential equations in the same way in previous section because this system is equivalent to the system (4) only restricted its parameters.

**Theorem 4.1.** Let us consider the following ordinary differential system in the polynomial class:

$$
\begin{align*}
\frac{dx}{dt} &= f_1(x, y, z), \\
\frac{dy}{dt} &= f_2(x, y, z), \\
\frac{dz}{dt} &= f_3(x, y, z).
\end{align*}
$$

We assume that

(A1) $\text{deg}(f_i) = 6$ with respect to $x, y, z$. 


(A2) The right-hand side of this system becomes again a polynomial in each coordinate system \((x_i, y_i, z_i)\) \((i = 0, 1, 2, 3)\).

\[
\begin{align*}
0) \ x_0 &= x - \eta, \quad y_0 = y, \quad z_0 = z - \frac{\beta_0}{x - \eta}, \\
1) \ x_1 &= x, \quad y_1 = y - 1, \quad z_1 = z - \frac{\beta_1}{y - 1}, \\
2) \ x_2 &= x + \frac{\beta_2}{z}, \quad y_2 = y + \frac{\beta_2}{z}, \quad z_2 = z, \\
3) \ x_3 &= x, \quad y_3 = y, \quad z_3 = z - \frac{\beta_3(x + y)}{xy}.
\end{align*}
\]

Then such a system coincides with the system (4) with the parameter’s relations:

\[
\begin{align*}
\alpha_4 &= \alpha_1, \quad \beta_0 := \alpha_0, \quad \beta_1 := \alpha_3, \quad \beta_2 := \alpha_2, \quad \beta_3 := \alpha_1.
\end{align*}
\]

Here, the complex parameters \(\beta_0, \beta_1, \beta_2, \beta_3\) satisfy the relation:

\[
\beta_0 + \beta_1 + 2\beta_2 + 2\beta_3 = 1.
\]

These transition functions satisfy the condition:

\[
dx_i \wedge dy_i \wedge dz_i = dx \wedge dy \wedge dz \quad (i = 0, 1, 2, 3).
\]

**Theorem 4.2.** This system admits the affine Weyl group symmetry of type \(B_3^{(1)}\) as the group of its Bäcklund transformations, whose generators are explicitly given as follows: with the notation \((\ast) := (x, y, z; \beta_0, \beta_1, \beta_2, \beta_3),\)

\[
\begin{align*}
s_0 : (\ast) &\rightarrow \left( x, y, z - \frac{\beta_0}{x - \eta}; -\beta_0, \beta_1, \beta_2 + \beta_0, \beta_3 \right), \\
s_1 : (\ast) &\rightarrow \left( x, y, z - \frac{\beta_1}{y - 1}; \beta_0, -\beta_1, \beta_2 + \beta_1, \beta_3 \right), \\
s_2 : (\ast) &\rightarrow \left( x + \frac{\beta_2}{z}, y + \frac{\beta_2}{z}; \beta_0 + \beta_2, \beta_1 + \beta_2, -\beta_2, \beta_3 + \beta_2 \right), \\
s_3 : (\ast) &\rightarrow \left( x, y, z - \frac{\beta_3(x + y)}{xy}; \beta_0, \beta_1, \beta_2 + 2\beta_3, -\beta_3 \right).
\end{align*}
\]

The Bäcklund transformations of each system satisfy

\[
s_i(g) = g + \frac{\alpha_i}{f_i} \{f_i, g\} + \frac{1}{2!} \left( \frac{\alpha_i}{f_i} \right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[x, y, z]),
\]

where poisson bracket \(\{,\}\) satisfies the relations:

\[
\{z, x\} = \{z, y\} = 1, \quad \{x, y\} = 0.
\]

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

We note that all generators \(s_0, s_1, s_2, s_3\) are determined by the invariant divisors (4.3) (see next proposition).
Proposition 4.3. This system has the following invariant divisors:

| parameter’s relation | \( f_i \) |
|----------------------|---------|
| \( \beta_0 = 0 \)    | \( f_0 := x - \eta \) |
| \( \beta_1 = 0 \)    | \( f_1 := y - 1 \) |
| \( \beta_2 = 0 \)    | \( f_2 := z \) |
| \( \beta_3 = 0 \)    | \( f_3 := xy \) |

We note that when \( \beta_3 = 0 \), after we make the birational transformations:

\[
(26) \quad x_3 = xy, \ y_3 = y, \ z_3 = z
\]

we see that in the coordinate system \((x_3, y_3, z_3)\) the system admits a particular solution \(x_3 = 0\).

5. The System of Type \(D_3^{(2)}\)

In this section, we present a 2-parameter family of ordinary differential systems in dimension three with affine Weyl group symmetry of type \(D_3^{(2)}\). This system is equivalent to the system (4) restricted its parameters.

Theorem 5.1. Let us consider the following ordinary differential system in the polynomial class:

\[
\begin{align*}
\frac{dx}{dt} &= f_1(x, y, z), \\
\frac{dy}{dt} &= f_2(x, y, z), \\
\frac{dz}{dt} &= f_3(x, y, z).
\end{align*}
\]

We assume that

(A1) \( \text{deg}(f_i) = 6 \) with respect to \( x, y, z \).

(A2) The right-hand side of this system becomes again a polynomial in each coordinate system \((x_i, y_i, z_i)\) \((i = 1, 2, 3)\).

1) \( x_1 = x - \eta, \ y_1 = y - 1, \ z_1 = z - \frac{\beta_0(x + y - \eta - 1)}{(x - \eta)(y - 1)}, \)

\[
(27) \quad x_2 = x + \frac{\beta_1}{z}, \ y_2 = y + \frac{\beta_1}{z}, \ z_2 = z,
\]

2) \( x_3 = x, \ y_3 = y, \ z_3 = z - \frac{\beta_2(x + y)}{xy}. \)

Then such a system coincides with the system (4) with the parameter’s relations:

\[
(28) \quad \alpha_4 = \alpha_1, \ \alpha_0 = \alpha_3, \ \beta_0 := \alpha_1, \ \beta_1 := \alpha_2, \ \beta_2 := \alpha_3.
\]

Here, the complex parameters \(\beta_0, \beta_1, \beta_2\) satisfy the relation:

\[
(29) \quad \beta_0 + \beta_1 + \beta_2 = \frac{1}{2}.
\]
These transition functions satisfy the condition:
\[ dx_i \wedge dy_i \wedge dz_i = dx \wedge dy \wedge dz \quad (i = 1, 2, 3). \]

**Theorem 5.2.** This system admits the affine Weyl group symmetry of type $D_3^{(2)}$ as the group of its Bäcklund transformations, whose generators are explicitly given as follows: with the notation $(\ast) := (x, y, z; \beta_0, \beta_1, \beta_2)$,

\[
\begin{align*}
s_0 : (\ast) &\to \left( x, y, z - \frac{\beta_0(x + y - \eta - 1)}{(x - \eta)(y - 1)}; -\beta_0 + 2\beta_0, \beta_2 \right), \\
s_1 : (\ast) &\to \left( x + \frac{\beta_1}{z}, y + \frac{\beta_1}{z}, z; \beta_0 + \beta_1, -\beta_1 + 2\beta_1 \right), \\
s_2 : (\ast) &\to \left( x, y, z - \frac{\beta_2(x + y)}{xy}; \beta_0, \beta_1 + 2\beta_2, -\beta_2 \right).
\end{align*}
\]

We note that the generators $s_0, s_1, s_2$ are determined by the invariant divisors (5.3) (see next proposition).

**Proposition 5.3.** This system has the following invariant divisors:

| parameter’s relation | $f_i$ |
|----------------------|-------|
| $\beta_0 = 0$        | $f_0 := (x - \eta)(y - 1)$ |
| $\beta_1 = 0$        | $f_1 := z$ |
| $\beta_2 = 0$        | $f_2 := xy$ |

6. **The system of type $G_2^{(1)}$**

In this section, we present a 2-parameter family of ordinary differential systems in dimension three with affine Weyl group symmetry of type $G_2^{(1)}$. This system is equivalent to the system (4) restricted its parameters

\[
\alpha_4 = \alpha_3 = \alpha_1, \quad \beta_0 := \alpha_0, \quad \beta_1 := \alpha_2, \quad \beta_2 := \alpha_3.
\]

Here, the complex parameters $\beta_0, \beta_1, \beta_2$ satisfy the relation:

\[
\beta_0 + 2\beta_1 + 3\beta_2 = 1.
\]

**Theorem 6.1.** This system admits the affine Weyl group symmetry of type $G_2^{(1)}$ as the group of its Bäcklund transformations, whose generators are explicitly given as follows: with the notation $(\ast) := (x, y, z; \beta_0, \beta_1, \beta_2)$,

\[
\begin{align*}
s_0 : (\ast) &\to \left( x, y, z - \frac{\beta_0}{x - \eta}; -\beta_0 + \beta_1 + \beta_0, \beta_2 \right), \\
s_1 : (\ast) &\to \left( x + \frac{\beta_1}{z}, y + \frac{\beta_1}{z}, z; \beta_0 + \beta_1, -\beta_1 + \beta_2 + \beta_1 \right), \\
s_2 : (\ast) &\to \left( x, y, z - \frac{\beta_2(y(y - 1) + x(y - 1) + xy)}{xy(y - 1)}; \beta_0, \beta_1 + 3\beta_2, -\beta_2 \right).
\end{align*}
\]
We note that the generators $s_0, s_1, s_2$ are determined by the invariant divisors (6.2) (see next proposition).

**Proposition 6.2.** This system has the following invariant divisors:

| parameter's relation | $f_i$          |
|----------------------|---------------|
| $\beta_0 = 0$        | $f_0 := x - \eta$ |
| $\beta_1 = 0$        | $f_1 := z$    |
| $\beta_2 = 0$        | $f_2 := xy(y - 1)$ |

7. **The system of type $A_2^{(2)}$**

In this section, we present a 1-parameter family of ordinary differential systems in dimension three with affine Weyl group symmetry of type $A_2^{(2)}$. This system is equivalent to the system (4) restricted its parameters

(32) $\alpha_4 = \alpha_3 = \alpha_0 = \alpha_1, \; \beta_0 := \alpha_2, \; \beta_1 := \alpha_1$.

Here, the complex parameters $\beta_0, \beta_1$ satisfy the relation:

(33) $\beta_0 + 2\beta_1 = \frac{1}{2}$.

**Theorem 7.1.** This system admits the affine Weyl group symmetry of type $A_2^{(2)}$ as the group of its Bäcklund transformations, whose generators are explicitly given as follows: with the notation $(\ast) := (x, y, z; \beta_0, \beta_1),

s_0 : (\ast) \rightarrow \left( x + \frac{\beta_0}{z}, y + \frac{\beta_0}{z}, -\beta_0, \beta_1 + \beta_0 \right),

s_1 : (\ast) \rightarrow (x, y, z - \beta_1 \left\{ y(x - \eta)(y - 1) + x(x - \eta)(y - 1) + xy(y - 1) + xy(x - \eta) \right\} xy(x - \eta)(y - 1) - \beta_0 + 4\beta_1, -\beta_1).

We note that the generators $s_0, s_1$ are determined by the invariant divisors (7.2) (see next proposition).

**Proposition 7.2.** This system has the following invariant divisors:

| parameter's relation | $f_i$          |
|----------------------|---------------|
| $\beta_0 = 0$        | $f_0 := z$    |
| $\beta_1 = 0$        | $f_1 := xy(x - \eta)(y - 1)$ |

8. **Appendix A**

It is well-known that the fifth Painlevé equation has symmetries under the affine Weyl group of type $A_3^{(1)}$. In this section, we present a 3-parameter family of the systems of the first-order ordinary differential equations.
Theorem 8.1. The fifth Painlevé equation is equivalent to a 3-parameter family of the systems of the first-order ordinary differential equations:

\[
\begin{align*}
\frac{df_0}{dt} &= -\frac{\varphi}{2}(-2f_0f_1f_2 + af_0f_2 + (\alpha_0 + \alpha_1 + \alpha_3)f_0 - \alpha_0f_2), \\
\frac{df_1}{dt} &= \frac{\varphi}{2}(-f_0f_1 - f_1^2f_2 + af_0f_1 + af_1f_2 - a\alpha_1 + (\alpha_1 + \alpha_3)f_1), \\
\frac{df_2}{dt} &= -\frac{\varphi}{2}(-2f_0f_1f_2 + af_0f_2 + (\alpha_1 + \alpha_2 + \alpha_3)f_2 - \alpha_2f_0).
\end{align*}
\]

Here, \(f_0, f_1, f_2\) denote unknown complex variables and \(\alpha_0, \alpha_1, \alpha_2, \alpha_3\) and \(a\) are constant complex parameters with \(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1\) and \(\varphi\) is a nonzero parameter which can be fixed arbitrarily.

We see that the system has its first integral:

\[
\frac{d(f_2 - f_0)}{dt} = f_2 - f_0.
\]

We can solve this equation by

\[
f_2 - f_0 = e^{(t+c)}.
\]

Here we set

\[
t + c = \log T, \ x := f_0, \ y := f_1,
\]

then we can obtain the fifth Painlevé system:

\[
\begin{align*}
\frac{dx}{dT} &= \frac{\partial H_V}{\partial y} = -\frac{2x^2y}{T} + \frac{ax^2}{T} - 2xy + \left(a + \frac{\alpha_1 + \alpha_3}{T}\right)x - \alpha_0, \\
\frac{dy}{dT} &= -\frac{\partial H_V}{\partial x} = \frac{2xy^2}{T} + y^2 - \frac{2axy}{T} - \left(a + \frac{\alpha_1 + \alpha_3}{T}\right)y + \frac{a\alpha_1}{T}
\end{align*}
\]

with the polynomial Hamiltonian

\[
H_V = -\frac{x^2y^2}{T} + \frac{ax^2y}{T} - xy^2 + \left(a + \frac{\alpha_1 + \alpha_3}{T}\right)xy - \alpha_0y - \frac{a\alpha_1}{T}x.
\]

Theorem 8.2. This system admits the affine Weyl group symmetry of type \(A_3^{(1)}\) as the group of its Bäcklund transformations, whose generators \(s_0, s_1, s_2, s_3, \pi\) are explicitly given.
as follows:

\begin{equation}
(40)
\end{equation}

\begin{align*}
s_0 : (f_0, f_1, f_2; \alpha_0, \alpha_1, \alpha_2, \alpha_3) & \rightarrow \left( f_0, f_1 + \frac{\alpha_0 f_0}{f_1}, f_2; -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3 + \alpha_0 \right), \\
s_1 : (f_0, f_1, f_2; \alpha_0, \alpha_1, \alpha_2, \alpha_3) & \rightarrow \left( f_0 - \frac{\alpha_1 f_0}{f_1}, f_1, f_2 - \frac{\alpha_1 f_0}{f_1}; \alpha_0 + \alpha_1, -\alpha_1, \alpha_2 + \alpha_1, \alpha_3 \right), \\
s_2 : (f_0, f_1, f_2; \alpha_0, \alpha_1, \alpha_2, \alpha_3) & \rightarrow \left( f_0, f_1 + \frac{\alpha_2 f_0}{f_2}, f_2; \alpha_0, \alpha_1 + \alpha_2, -\alpha_2, \alpha_3 + \alpha_2 \right), \\
s_3 : (f_0, f_1, f_2; \alpha_0, \alpha_1, \alpha_2, \alpha_3) & \rightarrow \left( f_0 - \frac{\alpha_3 f_0}{f_1 - a}, f_1, f_2 - \frac{\alpha_3 f_0}{f_1 - a}; \alpha_0 + \alpha_3, \alpha_1, \alpha_2 + \alpha_3, -\alpha_3 \right), \\
\pi : (f_0, f_1, f_2; \alpha_0, \alpha_1, \alpha_2, \alpha_3) & \rightarrow (f_2, f_1, f_0; \eta, \alpha_2, \alpha_1, \alpha_0, \alpha_3).
\end{align*}

The Bäcklund transformations of each system satisfy

\begin{equation}
(41) \quad s_i(g) = g + \frac{\alpha_i}{f_i} \{f_i, g\} + \frac{1}{2!} \left( \frac{\alpha_i}{f_i} \right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}[f_0, f_1, f_2]),
\end{equation}

where poisson bracket \{,\} satisfies the relations:

\begin{equation}
(42) \quad \{f_0, f_1\} = \{f_2, f_1\} = 1, \quad \{f_0, f_2\} = 0.
\end{equation}

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

9. Appendix B

It is well-known that the third Painlevé equation has symmetries under the affine Weyl group of type $C_2^{(1)}$. In this section, we present a new representation of the third Painlevé equation.

**Theorem 9.1.** The third Painlevé equation can be written in the following symmetric form:

\begin{equation}
(43) \quad \begin{cases}
\frac{df_0}{dt} = -2f_0 f_1 f_2 + (\alpha_0 + 2\alpha_1)f_0 - \alpha_0 f_2, \\
\frac{df_1}{dt} = (f_0 + f_2)f_1^2 - 2\alpha_1 f_1 + \eta, \\
\frac{df_2}{dt} = -2f_0 f_1 f_2 + (2\alpha_1 + \alpha_2)f_2 - \alpha_2 f_0.
\end{cases}
\end{equation}

Here, $f_0, f_1$ and $f_2$ denote unknown complex variables and $\alpha_0, \alpha_1$ and $\alpha_2$ are constant parameters with $\alpha_0 + 2\alpha_1 + \alpha_2 = 1$ and $\eta$ is a nonzero parameter which can be fixed arbitrarily.

We see that the system \[43\] has its first integral:

\begin{equation}
(44) \quad \frac{d(f_0 - f_2)}{dt} = f_0 - f_2.
\end{equation}
We can solve this equation by

\[(45) \quad f_0 - f_2 = e^{(t+c)}.\]

Here we set

\[(46) \quad t + c = \log T, \quad x := \frac{1}{f_1}, \quad y := -(f_1 f_2 + \alpha_2) f_1,\]

then we can obtain the third Painlevé system:

\[(47) \quad \begin{cases} \frac{dx}{dT} = \frac{\partial H_{III}}{\partial y} = \frac{2x^2 y}{T} - \frac{\eta x^2}{T} + \frac{2(\alpha_1 + \alpha_2) x}{T} - 1, \\ \frac{dy}{dT} = -\frac{\partial H_{III}}{\partial x} = -\frac{2xy^2}{T} + \frac{2\eta xy}{T} - \frac{2(\alpha_1 + \alpha_2) y}{T} + \frac{\eta \alpha_2}{T} \end{cases}\]

with the polynomial Hamiltonian

\[(48) \quad H_{III} = \frac{x^2 y^2 - \eta x^2 y + 2(\alpha_1 + \alpha_2) x y - T y - \eta \alpha_2 x}{T}.\]

**Theorem 9.2.** This system admits extended affine Weyl group symmetry of type $C_2^{(1)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, s_2, \pi$ are explicitly given as follows:

\[
s_0 : (f_0, f_1, f_2; \eta, \alpha_0, \alpha_1, \alpha_2) \rightarrow \left( f_0, f_1 + \frac{\alpha_0}{f_0}, f_2; \eta, -\alpha_0, \alpha_1 + \alpha_0, \alpha_2 \right),
\]

\[
s_1 : (f_0, f_1, f_2; \eta, \alpha_0, \alpha_1, \alpha_2) \rightarrow \left( f_0 - \frac{2\alpha_1}{f_1}, f_1, f_2 - \frac{2\alpha_1}{f_1} + \eta \right),
\]

\[
s_2 : (f_0, f_1, f_2; \eta, \alpha_0, \alpha_1, \alpha_2) \rightarrow \left( f_0, f_1 + \frac{\alpha_2}{f_2}, f_2; \eta, \alpha_0, \alpha_1 + \alpha_2, -\alpha_2 \right),
\]

\[
\pi : (f_0, f_1, f_2; \eta, \alpha_0, \alpha_1, \alpha_2) \rightarrow (f_2, f_1, f_0; \eta, \alpha_2, \alpha_1, \alpha_0).
\]

**Figure 1.** The transformations described in Theorem 9.2 satisfy the relations:

\[s_0^2 = s_1^2 = s_2^2 = \pi^2 = (s_0 s_2)^2 = (s_0 s_1)^4 = (s_1 s_2)^4 = 1, \quad \pi(s_0, s_1, s_2) = (s_2, s_1, s_0) \pi.\]
Theorem 9.3. For the system (34) of type $A_3^{(1)}$, we make the change of parameters

$$
\alpha_0 = \beta_0, \quad \alpha_1 = -\frac{\eta}{a}, \quad \alpha_2 = \beta_2, \quad \alpha_3 = 2\beta_1 + \frac{\eta}{a}, \quad \varphi = -2,
$$

from $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ to $\beta_0, \beta_1, \beta_2$. This new system tends to the system (43) of type $C_2^{(1)}$ as $a \to 0$.

We note that

$$
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \beta_0 + 2\beta_1 + \beta_2 = 1.
$$

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