On complete Lie algebras

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Abstract
In this paper, the author gives two methods to construct complete Lie algebras. Both methods show that the derivation algebras of some Lie algebras are complete.

1 Introduction

A finite dimensional Lie algebra $\mathfrak{g}$ is called a complete Lie algebra if its center $C(\mathfrak{g})$ is trivial and all of its derivations are inner, i.e. $\text{Der}(\mathfrak{g}) = \text{ad}(\mathfrak{g})$.

There are many beautiful results on complete Lie algebras. A famous result is E. V. Stitzinger’s derivations towers theorem, which says, for any finite dimensional Lie algebra $\mathfrak{h}$ with trivial center, there is a positive integer $n$ such that $\text{Der}^n(\mathfrak{h})$ is a complete Lie algebra. In section 2, we construct some Lie algebras whose derivation algebras are complete. We show that if $\mathfrak{g}$ is a Lie algebra with $\mathfrak{b}$ a torus on it and $\tau$ the commutator of $\mathfrak{b}$ in $\text{Der}(\mathfrak{g})$ such that $(\mathfrak{g}, \mathfrak{b})$ is a non-degenerate pair, then $\tau \times_t \mathfrak{g}$ is isomorphic to the derivation algebra of $\mathfrak{b} \times_t \mathfrak{g}$ and is complete. Especially, if $\mathfrak{b}$ is a maximal torus, then $\mathfrak{b} \times_t \mathfrak{g}$ is complete. There are many unsolved questions on complete Lie algebras. For example, one may ask when a nilpotent Lie algebra is the nilpotent radical of some solvable complete Lie algebra. We give part answer to this question. In fact we show a nilpotent Lie algebra of non-degenerate type is the nilpotent radical of some solvable complete Lie algebra.

In section 3, at first, we give a method to calculate the derivation algebras of the full graphs of Lie algebras of certain type. Then we apply it to Heisenberg algebra to construct Lie algebras which are not complete but whose derivation algebras are complete.

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2 Complete Lie algebras associated to pairs of non-degenerate type

2.1 Notations and concepts

Let $g$ be a Lie algebra, and let $\text{Der}(g)$ denote its derivation algebra. A Lie subalgebra $b$ of $\text{Der}(g)$ is called a (maximal) torus on $g$ if $b$ is a (maximal) commutative subalgebra whose elements are all semisimple.

For a given torus $b$ on $g$ and any $\alpha \in b^*$, put

$$g_\alpha = \{ g \in g : D(g) = \alpha(D)g, \forall D \in b \}.$$  

Then $g$ has a decomposition

$$g = \bigoplus_{\alpha \in \Delta} g_\alpha. \quad (1)$$

A pair $(g, b)$ is called a non-degenerate pair if all $\alpha$ in formula (1) are nonzero.

If $g$ is nilpotent and $b_1, b_2$ are two maximal toruses on $g$, then there is some $\theta \in \text{Aut}g$ such that $\theta(b_1) = b_2$. Hence $(g, b_1)$ is a non-degenerate pair if and only if $(g, b_2)$ is. Then we say that $g$ is a nilpotent Lie algebra of non-degenerate type or a non-degenerate nilpotent Lie algebra.

If there is a homomorphism $\varphi$ from Lie algebra $s$ to $\text{Der}(g)$, we can define a Lie algebra $s \times_{\varphi} g$ via

$$[(s_1, g_1), (s_2, g_2)] = ([s_1, s_2], s_1(g_2) - s_2(g_1) + [g_1, g_2]). \quad (2)$$

If $s = \text{Der}(g)$, $s \times_{\varphi} g$ is called the full graph of $g$ and is denoted by $f(g)$. If $g$ is center free, then both $\text{Der}(g)$ and $f(g)$ are center free.

2.2 Complete Lie algebras associated to pairs of non-degenerate type

In this subsection, we assume $b$ is a torus on $g$ such that $(g, b)$ is a non-degenerate pair. Let $\tau$ be the commutator of $b$ in $\text{Der}(g)$. Let $h_1$ and $h$ denote $b \times_t g$ and $\tau \times g$, respectively. Then $h_1$ and $h$ have decompositions

$$h_1 = b \bigoplus \sum_{\alpha \in \Delta} g_\alpha$$

and

$$h = \tau \bigoplus \sum_{\alpha \in \Delta} g_\alpha,$$

respectively.

We identify $b$ with $\{(b, 0) : b \in b\}$ in $h_1$, $\tau$ with $\{(t, 0) : t \in \tau\}$ in $h$, and $g$ with $\{(0, g) : g \in g\}$ in $h_1$ and $h$ respectively.

**Lemma 1.** Let $g$, $b$, $\tau$ be as above.

(i) If $D$ is a derivation of $h$ such that $D(b) \subseteq \tau$, then $D = \text{ad}(t)$ for some $t \in \tau.$
(ii) If \( D \) is a derivation of \( h_1 \) such that \( D(b) \subseteq b \), then \( D|b = 0 \) and \( D|g = t \) for some \( t \in \tau \).

**Proof.** We need only to prove (i), since (ii) can be proved similarly.

We notice that each \( g_\alpha \) is stable under \( \tau \), since \( \tau \) commutes with \( b \).

Assume \( D e_\alpha = t_1 + \sum_{\beta \in \Delta} x_\beta \) \((t_1 \in \tau, x_\beta \in g_\beta)\) for a nonzero element \( e_\alpha \) in \( g_\alpha \) with \( \alpha \) in \( \Delta \). We have

\[
\alpha(b)(t_1 + \sum_{\beta \in \Delta} x_\beta) = [D, e_\alpha] + \sum_{\beta \in \Delta} \beta(b)x_\beta \quad (b \in b)
\]

since \( D[e_\alpha] = [D, e_\alpha] + [b, De_\alpha] \). Therefore, we obtain \( t_1 = 0 \), \( x_\beta = 0 \) \((\beta \neq \alpha)\) and

\[
[D, e_\alpha] = 0 \quad \forall \alpha \in \Delta, \ b \in b.
\]

Then we get \( Db = 0 \) for any \( b \) in \( b \) and \( D(g_\alpha) \subseteq g_\alpha \) for any \( \alpha \) in \( \Delta \). Hence, \( D|g \) commutates with \( \text{ad}(b)|g \) and so there exists a \( t \) in \( \tau \) such that

\[
D|g = \text{ad}(t)|g.
\]

Let \( D_1 \) be \( D - \text{ad}(t) \), then \( D_1|b \oplus g = 0 \).

For any \( t \) in \( \tau \) and any \( g \) in \( g \), we have

\[
[D_1 t, g] = D_1[t, g] - [t, D_1 g] = 0.
\]

Therefore, we obtain \( D_1 t = 0 \) as desired. \( \square \)

**Lemma 2.** Let \( g, b, \tau, h_1 \) and \( h \) be as above.

(i). For any derivation \( D \) of \( h \), there exist \( x_\alpha \in g_\alpha \) \((\alpha \in \Delta)\) such that

\[
Db = b' + \sum_{\alpha \in \Delta} \alpha(b)x_\alpha \quad \forall b \in b.
\]

with some \( b' \in \tau \).

(ii). For any derivation \( D \) of \( h_1 \), there exist \( x_\alpha \in g_\alpha \) \((\alpha \in \Delta)\) such that

\[
Db = b' + \sum_{\alpha \in \Delta} \alpha(b)x_\alpha \forall b \in b
\]

with some \( b' \in b \).

**Proof.** As in the proof of Lemma 1, we need only to prove (i), since (ii) can be proved similarly.

For any \( b \) in \( b \) assume that

\[
Db = b' + \sum_{\alpha \in \Delta} y_\alpha(b)
\]

with \( b' \in \tau \) and \( y_\alpha(b) \in g_\alpha \). Then, for \( b_1 \) in \( b \), we have

\[
-\sum_{\alpha \in \Delta} \alpha(b_1)y_\alpha(b) + \sum_{\alpha \in \Delta} \alpha(b)y_\alpha(b_1) = 0.
\]
\[ \alpha(b_1)y\alpha(b) = \alpha(b)y\alpha(b_1) \forall \alpha \in \Delta, \ b, b_1 \in b. \quad (9) \]

If \( \alpha(b) = 0 \), we find a \( b_1 \) in \( b \) with \( \alpha(b_1) \neq 0 \), then from the above formula we obtain \( y\alpha(b) = 0 \). If \( y\alpha(b) = 0 \) for all \( b \in b \), let \( x_\alpha = 0 \). If there exists a \( b \in b \) with \( y\alpha(b) \neq 0 \), then \( \alpha(b) \neq 0 \) and let \( x_\alpha = \frac{1}{\alpha(b)}y\alpha(b) \). Then from formula (9) we get
\[ y\alpha(b_1) = \frac{\alpha(b_1)}{\alpha(b)}y\alpha(b) = \alpha(b_1)x_\alpha \quad (10) \]
as desired.

We can now prove the following theorem.

**Theorem 1.** Let \( g \) be a Lie algebra, and \( b \) be a torus on \( g \) such that \((g, b)\) is a non-degenerate pair. Let \( \tau \) be the commutator of \( b \) in \( \text{Der}(g) \). Let \( h \) and \( h_1 \) denote \( \tau \times_t g \) and \( b \times_t g \) respectively. Then \( h \) is equal to \( \text{Der}(h_1) \) and is a complete Lie algebra.

**Proof.** (i). At first we prove \( h = \text{Der}(h_1) \).

Let \( D \) be a derivation of \( h_1 \). From lemma 2 (ii) we obtain
\[ Db = b' + \sum_{\alpha \in \Delta} \alpha(b)x_\alpha \forall b \in b \]
with some \( b' \in b \). Therefore \( D + \text{ad}(\sum x_\alpha) \) is a derivation of \( h_1 \) such that
\[ (D + \text{ad}(\sum x_\alpha))b = b' \in b \forall b \in b. \quad (11) \]

Lemma 1(ii) shows
\[ D + \text{ad}(\sum x_\alpha)|b = 0 \quad (12) \]
and
\[ D + \text{ad}(\sum x_\alpha)|g = t \quad (13) \]
for some \( t \in \tau \).

For any \((t_0, g_0)\) in \( h \), we define a derivation \( D_{(t_0, g_0)} \) of \( h_1 \) via
\[ D_{(t_0, g_0)}((b, g)) = (0, t_0(g) - b(g_0) + [g_0, g]) \forall (b, g) \in h_1. \quad (14) \]

Then we get a homomorphism from \( h \) to \( \text{Der}(h_1) \). It is easy to see \( D_{(t_0, g_0)} = 0 \) if and only if \((t_0, g_0) = 0 \). Thus we can regard \( h \) as a Lie subalgebra of \( \text{Der}(h_1) \) via (13). From formula (13), we see that every derivation \( D \) of \( h_1 \) lies in \( h \).

Therefore, \( h \) is equal to \( \text{Der}(h_1) \).

(ii). We are going to prove \( h \) is complete.

It is obvious that \( C(h) = 0 \). So it is sufficient to prove \( \text{Der}(h) = \text{ad}(h) \).

Let \( D \) be a derivation of \( h \). From lemma 2(i), we obtain
\[ Db = b' + \sum_{\alpha \in \Delta} \alpha(b)x_\alpha \forall b \in b \quad (15) \]
with some $b' \in \tau$. Therefore, we have

$$(D + \text{ad}(\sum_{\alpha \in \Delta} x_\alpha))b = b' \in \tau \forall b \in b.$$  \hspace{1cm} (16)

Then lemma 1(i) tells us that there exists some $t \in \tau$ such that

$$D + \text{ad}(\sum_{\alpha \in \Delta} x_\alpha) = \text{ad}(t)$$

So $D$ is in $\text{ad}(h)$. \hfill \Box

Now, we apply the above theorem to solvable complete Lie algebra. A nilpotent Lie algebra is called completable if it is the nilpotent radical of some solvable complete Lie algebra. We have the following proposition.

**Proposition 1.** Let $n$ be a nilpotent Lie algebra of non-degenerate type. Then $n$ is completable.

**Proof.** Let $b$ be a maximal torus over $n$. Then its commutator $\tau$ in $\text{Der}(n)$ is just $b$ itself. Therefore, $b \times n$ is a solvable complete Lie algebra. \hfill \Box

We use proposition 1 to construct some examples of completable nilpotent Lie algebras. Let $g$ be a Lie algebra and let $n$ be a positive integer. We will define a Lie algebra $g \oplus n = \{g = (g(1),...,g(n))| g(1),...,g(n) \text{ are elements of } g\}$ via

$$[g,g'](k) = \sum_{i+j=k} [g(i),g'(j)], \forall g,g' \in g \oplus n, 1 \leq i,j,k \leq n.$$  \hspace{1cm} (17)

We show $g \oplus n$ is a non-degenerate nilpotent Lie algebra. Let $D$ be a derivation of $g$ such that $D(g)(i) = ig(i)$ for any $g \in g \oplus n$, then $\{tD|t \in \mathbb{R}\}, g \oplus n$ is a non-degenerate pair. Hence, $g \oplus n$ is non-degenerate and so it is completable.

3 Some relation between complete Lie algebra and Heisenberg algebra

3.1 A general result

In this section, we are going to study full graphs of Lie algebras. Let $g$ be a Lie algebra and $S$ be a complete Lie algebra with a homomorphism $\varphi$ from $S$ to $\text{Der}(g)$. Let $h$ denote $S \times \varphi g$.

We define $Z_s(g)$ via

$$Z_s(g) = \{g \in C(g) : s(g) = 0 \forall s \in S\}. \hspace{1cm} (18)$$

By a simple calculate we obtain the following lemma.

**Lemma 3.** The center of $h$ is $\{0,g : g \in Z_s(g)\}$.

Let $F_s(g)$ be the set $\{D \in \text{Der}(g) : [D,D_1] \in \text{ad}(g), \forall D_1 \in \varphi(S)\}$ which is a Lie subalgebra of $\text{Der}(g)$. It is obvious that $\text{ad}(g)$ is its ideal. We have a natural induced homomorphism $\hat{\varphi}$ from $S$ to $\text{Der}(\text{Der}(g))$. If $S$ is just $\text{Der}(g)$, let $F(g)$ denote $F_s(g)$. Let $\hat{h}$ denote $S \times \hat{\varphi} F_s(g)$.
Lemma 4. If $C(\mathfrak{g}) = 0$, then $\text{Der}(\mathfrak{h}) \supseteq \tilde{\mathfrak{h}}$.
Furthermore, if one of the following conditions (a), (b) and (c) is satisfied, then $\text{Der}(\mathfrak{h}) = \tilde{\mathfrak{h}}$.
(a). $\text{Der}(\mathfrak{g}) \supseteq \mathcal{S} \supseteq \text{ad}(\mathfrak{g})$,
(b). $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$,
(c). $\mathfrak{g} = \mathfrak{b} \times \mathfrak{n}$ is a solvable Lie algebra with $(\mathfrak{n}, \mathfrak{b})$ a non-degenerate pair, and $\mathcal{S}$ annihilates $\mathfrak{b}$.

Proof. For any inner derivation $D = \text{ad}(g)$ of $\mathfrak{g}$, let $I(D)$ denote $g$. For any $(s, D)$ in $\tilde{\mathfrak{h}}$, a derivation $D(s, D)$ on $\mathfrak{h}$ is defined via

$$D(s, D)(s_1, g) = ([s, s_1], s(g) + D(g) + I([D, \varphi(s_1)]) \quad \forall (s_1, g) \in \mathfrak{h}. \quad (18)$$

By a simple calculate, we get the following assertions

(i). $D(s, D)$ is a derivation,
(ii). $[D(s_1, D_1), D(s_2, D_2)] = D([s_1, D_1]s_2, D_2),$

and

(iii). $D(s, D) = 0$ if and only if $s = 0$ and $D = 0$. Therefore, $\tilde{\mathfrak{h}}$ is a Lie subalgebra of $\text{Der}(\mathfrak{h})$.

Now we begin to prove $\text{Der}(\mathfrak{h}) = \tilde{\mathfrak{h}}$ if at least one of the conditions (a), (b) and (c) is satisfied. Let $D_1$ be a derivation of $\mathfrak{h}$. Write $D_1 = (T_1, T_2)$, where $T_1$ takes values in $\mathcal{S}$ and $T_2$ takes values in $\mathfrak{g}$. It is obvious that $T_1|\mathcal{S}$ is a derivation on $\mathcal{S}$. Since $\mathcal{S}$ is complete, we see $T_1|\mathcal{S} = \text{ad}(s)|\mathcal{S}$ with some $s \in \mathcal{S}$. Let $D = D_1 - D(s, 0)$, then $D = (T_1, T_2)$ with $T_1|\mathcal{S} = 0$. We are going to show $T_1 = 0$. By

$$D([0, g_1](s_1, g_2)) = [D(0, g_1), (s_1, g_2)] + [(0, g_1), D(s_1, g_2)],$$

we obtain

$$-T_1(s(g_1)) + T_1([g_1, g_2]) = [T_1(g_1), s] \quad \forall s \in \mathcal{S}, g_1, g_2 \in \mathfrak{g}. \quad (19)$$

We see

$$T_1|[\mathfrak{g}, \mathfrak{g}] = 0, \quad (20)$$

and

$$-T_1(s(g)) = [T_1(g), s]. \quad (21)$$

If (b) holds, it is obvious $T_1 = 0$. Next we assume (a) holds. From formulas (20) and (21), we obtain

$$[T_1(g), s] = 0 \quad \forall s \in \text{ad}(\mathfrak{g}), g \in \mathfrak{g}, \quad (22)$$

since $\text{ad}(\mathfrak{g})$ is contained in $\mathcal{S}$. Hence, $T_1$ annihilates $\mathfrak{g}$. Now we assume (c) holds. Condition (c) implies that $\mathfrak{g} = \mathfrak{b} + [\mathfrak{g}, \mathfrak{g}]$. From formula (21), we get

$$[T_1(b), s] = 0 \quad \forall b \in \mathfrak{b}, s \in \mathcal{S}. \quad (23)$$

Since $C(\mathcal{S}) = 0$, we see that $T_1$ annihilates $\mathfrak{b}$. Since $T_1|[\mathfrak{g}, \mathfrak{g}] = 0$, the proof of $T_1 = 0$ is completed.
We have already obtained $D = (0, T_2)$. The following equalities hold

$$T_2([(s_1, 0), (s_2, 0)]) = [(s_1, 0), T_2(s_2, 0)] + [T_2(s_1, 0), (s_2, 0)]$$  \tag{24}

$$T_2([(s, 0), (0, x)]) = [(s, 0), T_2(0, x)] + [T_2(s, 0), (0, x)]$$  \tag{25}

$$T_2([(0, x_1), (0, x_2)]) = [(0, x_1), T_2(0, x_2)] + [T_2(0, x_1), (0, x_2)].$$  \tag{26}

From formula (26), we see $T_2 \mid g \in \text{Der}(\mathfrak{g})$ and we may assume $T_2 \mid g = D$. From (25) we obtain that $\text{ad}(T_2(s)) = [D, \varphi(s)] \forall s \in S$.  \tag{27}

Therefore $D$ belongs to $F_s(\mathfrak{g})$ as desired.  \hfill \Box

From lemma 3 and lemma 4, it is easy to obtain the following theorem.

**Theorem 2.** Let $\mathfrak{g}$ be a Lie algebra with trivial center, and assume that $\text{Der}(\mathfrak{g})$ is a complete Lie algebra. Then the derivation algebra of the full graph $f(\mathfrak{g}) \times \varphi F(\mathfrak{g})$. Therefore, $f(\mathfrak{g})$ is a complete Lie algebra if and only if $F(\mathfrak{g}) = \mathfrak{g}$.

For any integer $n$, let $f^n(\mathfrak{g})$ denote $f(f^{n-1}(\mathfrak{g}))$ by induction.

The following corollary can be easily proved by induction.

**Corollary.** Let $\mathfrak{g}$ be a non-complete Lie algebra with trivial center, and assume that $\text{Der}(\mathfrak{g})$ is a complete Lie algebra such that $[\text{Der}(\mathfrak{g}), \text{Der}(\mathfrak{g})] \subseteq \text{ad}(\mathfrak{g})$. Then we have the following assertions

(i). $\text{Der}(f^n(\mathfrak{g})) = f^n(\text{Der}(\mathfrak{g}))$ is complete but $f^n(\mathfrak{g})$ is not,

(ii). The center of $f^n(\mathfrak{g})$ is trivial,

(iii). $[\text{Der}(f^n(\mathfrak{g})), \text{Der}(f^n(\mathfrak{g}))] \subseteq \text{ad}(f^n(\mathfrak{g}))$,

(iv). $\text{dim}([\text{Der}(f^n(\mathfrak{g})), \text{Der}(f^n(\mathfrak{g}))]) = \text{dim}(\text{Der}(f^n(\mathfrak{g}))) - \text{dim}([\text{Der}(\mathfrak{g}), \text{Der}(\mathfrak{g})] - \text{dim}(\mathfrak{g})$.

### 3.2 Apply the general result to Heisenberg algebras

In this section, let $\mathfrak{g}$ be a Heisenberg algebra such that $[\mathfrak{g}, \mathfrak{g}] = C(\mathfrak{g})$ and $\text{dim}(C(\mathfrak{g})) = 1$.

Let $c$ be a nonzero element of $C(\mathfrak{g})$. Then there is an anti-symmetric quadratic form $\psi$ on $\mathfrak{g}$ of rank $\text{dim}(\mathfrak{g}) - 1$ such that for any $s, y$ in $\mathfrak{g}$

$$[x, y] = \psi(x, y)c.$$  \tag{28}

Then dimension of $(\mathfrak{g})$ is equal to $2N + 1$ for some positive integer $N$. We may choose a basis $\{x_1, ..., x_N; y_1, ..., y_N; c\}$ such that for any $1 \leq i, j \leq N$,

$$\psi(x_i, x_j) = \psi(y_i, y_j) = \psi(c, x) = 0,$$  \tag{29}

and

$$\psi(x_i, y_j) = \delta_{ij}.$$  \tag{30}

In [5], the following three propositions are stated.
Proposition 2. Der(\(g\)) is simple complete Lie algebra, and has the following decomposition

\[
\text{Der}(g) = S \times \tau (b \times n),
\]

where, \(S \cong \text{sp}(2n, \mathbb{C})\), \(n = \text{ad}(g) \cong g/\mathbb{C}c\) is an irreducible \(S\)-module and \(\dim(b) = 1\).

Proposition 3. The full graph \(f(g)\) of \(g\) is not a complete Lie algebra, but its center is trivial.

Proposition 4. The derivation algebra \(\text{Der}(f(g))\) of \(f(g)\) is a complete Lie algebra with \(\dim(\text{Der}(f(g))) = \dim(f(g)) + 1\).

From proposition 3 and proposition 4, we obtain

\[
[\text{Der}(f(g)), \text{Der}(f(g))] \subseteq \text{ad}(f(g)).
\]

By corollary of theorem 2, we get

Theorem 3. Let \(g\) be a Heisenberg algebra, then for any positive integer \(n\),

(i). The center of \(f^n(g)\) is trivial,
(ii). \(\text{Der}(f^n(g)) = f^{n-1}(\text{Der}(f(g)))\) is complete but \(f^n(g)\) is not complete,
(iii). \([\text{Der}(f^n(g)), \text{Der}(f^n(g))] \subseteq \text{ad}(f^n(g))\),
(iv). \(\dim(\text{Der}(f^n(g))) - \dim(f^n(g)) = 1\).

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