Behavior of Totally Positive Differential Systems Near a Periodic Solution

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Abstract—A time-varying nonlinear dynamical system is called a totally positive differential system (TPDS) if its Jacobi admits a special sign pattern: it is tri-diagonal with positive entries on the super- and sub-diagonals. If the vector field of a TPDS is $T$-periodic then every bounded trajectory converges to a $T$-periodic solution. In particular, when the vector field is time-invariant every bounded trajectory of a TPDS converges to an equilibrium. Here, we use the spectral theory of oscillatory matrices to analyze the behavior near a periodic solution of a TPDS. This yields information on the perturbation directions that lead to the fastest and slowest convergence to or divergence from the periodic solution. We demonstrate the theoretical results using a model from systems biology called the ribosome flow model.

I. INTRODUCTION

Consider the time-varying nonlinear system:

\[ \dot{x}(t) = f(t, x(t)), \]

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ is continuously differentiable and $\Omega$ is a convex subset of $\mathbb{R}^n$. For $t \geq t_0 \geq 0$ and $a \in \Omega$, let $x(t, t_0, x_0)$ denote the solution of (1) at time $t$ with the initial condition $x(t_0) = x_0$. From here on we always take $t_0 = 0$ and write $x(t, x_0)$ for $x(t, 0, x_0)$. We assume throughout that for any $x_0 \in \Omega$ the solution $x(t, x_0)$ exists for all $t \geq 0$.

Let $J(t, x) := \frac{\partial f(t, x)}{\partial x}$ denote the Jacobian of the vector field $f$. Let $M^+ \subset \mathbb{R}^{n \times n}$ denote the set of $n \times n$ tri-diagonal matrices with positive entries of the super- and sub-diagonals, i.e. the subset of Jacobi matrices. System (1) is called a totally positive differential system (TPDS) if $J(t, x) \in M^+$ for all $(t, x)$. In particular, $J(t, x)$ is a Metzler matrix (i.e. a matrix whose off-diagonal entries are all non-negative), and also an irreducible matrix. This implies that (1) is strongly cooperative [27], i.e. if $a, b \in \Omega$, with $a \neq b$, and

\[ a - b \in \mathbb{R}_+^n := \{ x \in \mathbb{R}^n : x_i \geq 0, \ i = 1, \ldots, n \} \]

then

\[ x(t, a) - x(t, b) \in \mathbb{R}_+^n := \{ x \in \mathbb{R}^n : x_i > 0, \ i = 1, \ldots, n \} \]

for all $t > 0$. Cooperative systems play an important role in models from biology and neuroscience, where it is known that the state-variables represent quantities that can never attain negative values, e.g., concentration of molecules, average number of spikes in a neuron, etc. Cooperative systems enjoy a well-ordered behavior. By Hirsch’s quasi-convergence theorem [27], almost every bounded trajectory of a time-invariant (i.e. $f(t, x) = f(x)$) cooperative system converges to the set of equilibria.

TPDSs require a stronger condition on the Jacobian and consequently enjoy stronger properties. Smillie [25] proved that if the vector field is time-invariant then every bounded trajectory converges to an equilibrium. Smith [26] generalized this result and showed that if the vector field $f$ of a TPDS is $T$-periodic, that is,

\[ f(t, z) = f(t + T, z) \]

then every bounded trajectory converges to a $T$-periodic solution of the system. These results found many applications in fields such as systems biology and neuroscience [13], [4], [6], as well as several generalizations (see, e.g. [9], [28]).

The proofs of Smillie and Smith are based on using the number of sign changes in the vector of derivatives $\dot{x}(t)$ as an integer-valued Lyapunov function. Recently, it was shown that these results are closely related to the sign-variation diminishing property of totally positive matrices [12]. Recall that a matrix $A \in \mathbb{R}^{n \times n}$ is called totally non-negative (TN) if all its minors are non-negative, and totally positive (TP) if they are all positive. Such matrices have a rich and beautiful structure [18], [8]. For example, if $A \in \mathbb{R}^{n \times n}$ is TP and $v \in \mathbb{R}^n \setminus \{0\}$, then the number of sign variations in $Av$ is smaller or equal to the number of sign variations in $v$.

Schwarz [23] introduced and analyzed linear TPDS. He showed in particular that the following two conditions are equivalent: 1) $A \in M^+$; 2) the transition matrix $\exp(At)$ of the linear time-invariant (LTI) system $\dot{x} = Ax$ is TP for any $t > 0$. Unfortunately, his results were almost forgotten.

In many dynamical systems it is important to understand how a small perturbation affects the behavior near a periodic solution (and, in particular, near an equilibrium). In general, this is a difficult problem, as there is little explicit information on the periodic solution. Here, we use the spectral theory of TP matrices to analyze trajectories of the $T$-periodic TPDS (1) in the vicinity of a periodic solution (and in the special case of a time-invariant vector field, near an equilibrium). Our main result characterizes the sign pattern of the “most stable” and “most unstable” perturbation directions near a periodic solution. We also provide an intuitive explanation for the structure of these sign patterns. We demonstrate an application of these theoretical results to an important model from systems biology called the ribosome flow model (RFM) [32].

We note that when $n = 2$ and (1) is a strictly cooperative system then it is also a TPDS (as a $2 \times 2$ matrix is always

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tri-diagonal), so our results hold in this special case as well.

The remainder of this note is organized as follows. The next section briefly reviews several known definitions and results that are used later on. Section II presents our main results, and describes an application of these results to the RFM. The final section concludes. We use standard notation. Vectors [matrices] are denoted by small [capital] letters. For $A \in \mathbb{R}^{n \times m}$, $A^T$ is the transpose of $A$.

II. Preliminaries

We begin by briefly reviewing known results that will be used later on.

A. Sign variation diminishing property

An important property of TP matrices is that multiplying a vector by a TP matrix can never increase the number of sign variations. To explain this, we recall two definitions for the number of sign changes in a vector. For $z \in \mathbb{R}^n$, let $s^-(z)$ denote the number of sign changes in the vector $z$, after deleting all the zero entries (with $s^-(0)$ defined as zero). For example, $s^-(\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \end{bmatrix}^T) = 2$. For $z \in \mathbb{R}^n$, let $s^+(z)$ denote the maximal possible number of sign changes in the vector $z$, after replacing every zero entry by either plus one or minus one. For example, $s^+(\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \end{bmatrix}^T) = 4$. Note that these definitions imply that

$$0 \leq s^-(z) \leq s^+(z) \leq n - 1, \text{ for any } z \in \mathbb{R}^n. \quad (2)$$

The next result describes the sign variation diminishing property of TP matrices.

Proposition 1: [8] Let $A \in \mathbb{R}^{n \times n}$ be TP. Then $s^+(Ax) \leq s^-(x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

B. Oscillatory matrices

A matrix $A \in \mathbb{R}^{n \times n}$ is called oscillatory if it is TN and there exists an integer $k > 0$ such that $A^k$ is TP [10]. The smallest such $k$ is called the exponent of the oscillatory matrix $A$ [30]. A TN matrix is oscillatory if and only if (iff) it is non-singular and irreducible [10]. Oscillatory matrices have special spectral properties: their eigenvalues are all real, simple, and positive, and the corresponding eigenvectors have a special sign pattern [18]. The next result summarizes this spectral structure.

Proposition 2: [18] If $A \in \mathbb{R}^{n \times n}$ is oscillatory then all its eigenvalues are real, positive, and simple. Let $\lambda_i \in \mathbb{R}$ be an eigenvalue corresponding to $v_i$. For any $1 \leq i \leq n$, let $V^{ij} := \text{span}\{v_i, v_{i+1}, \ldots, v_j\}$. Then

$$i - 1 \leq s^-(z) \leq s^+(z) \leq j - 1, \text{ for any } z \in V^{ij} \setminus \{0\}. \quad (3)$$

Remark 3: Note that this implies in particular that

$$s^-(v_i) = s^+(v_i) = i - 1, \text{ for any } i \in \{1, \ldots, n\}. \quad (3)$$

In general, the eigenvectors may include zero entries. However, the zeros must be located such that (3) holds. In particular, $v^1$ cannot include any zero entries because $s^+(v^1) = 0$, and $v^n$ cannot include any zero entries because $s^-(v^n) = n - 1$.

Example 1: Let $A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 3 & 1 \\ 0.1 & 1 & 4 \end{bmatrix}$. All the $1 \times 1$ minors of $A$ (i.e., its entries) are non-negative. The $2 \times 2$ minors are 7, 3, 2, 2.8, 12, 8, 0.7, 3.9, 11, and the single $3 \times 3$ minor is $\text{det}(A) = 25.2$. Thus, $A$ is TN. Calculating all the minors of $A^2$ shows that they are all positive, so $A^2$ is TP. Thus, $A$ is oscillatory. Its eigenvalues are $\lambda_1 = 5.03851$, $\lambda_2 = 3.55435$, $\lambda_3 = 1.40714$, (all numerical values in this paper are to 5-digit accuracy) with corresponding eigenvectors: $v^1 = [0.55898, 0.569742, 0.602442]^T$, $v^2 = [0.746782, 0.206989, -0.632038]^T$, and $v^3 = [0.765516, -0.609679, 0.205614]^T$. Note that $s^-(v^i) = s^+(v^i) = i - 1$, $i \in \{1, 2, 3\}$.

For our purposes, it is important to know when a tri-diagonal matrix is an oscillatory matrix.

C. Tri-diagonal oscillatory matrices

Consider the $n \times n$ tri-diagonal matrix

$$T(a_i, b_i, c_i) := \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & c_{n-1} & a_n \end{bmatrix}.$$

(4)

Assume that $b_i, c_i \geq 0$, and that the entries satisfy the dominance condition:

$$a_i \geq b_i + c_{i-1}, \quad i = 1, \ldots, n,$$

(5)

with $b_n := 0$, and $c_0 := 0$. Then all the minors of $T$ are non-negative, i.e. $T$ is TN [8, Chapter 0]. If, furthermore, $T$ is non-singular and irreducible then it is oscillatory [10].

D. Linear TPDSs

Schwarz [23] considered the system

$$x(t) = A(t)x(t),$$

(6)

with $t \to A(t)$ continuous. The system is called a linear TPDS if the corresponding transition matrix $\Phi(t, t_0)$ (i.e., the matrix such that $x(t) = \Phi(t, t_0)x(t_0)$ for all $t \geq t_0 \geq 0$ and $x(t_0) \in \mathbb{R}^n$) is TP for any $t > t_0 \geq 0$. Schwarz showed that this holds iff: 1) $a_{ij}(t) = 0$ for all $|i-j| > 1$, 2) $a_{ij}(t) \geq 0$ for all $|i-j| = 1$, and 3) any of the functions in 2) does not vanish identically on any interval of positive length. In particular, if $A(t) \equiv A$ is constant then the system is a TPDS iff $A \in \mathbb{R}^{n \times n}$. One important property of a linear TPDS, that follows from Prop. II is that for any non-trivial solution $x(t)$,

$$s^+(x(t_2)) \leq s^-(x(t_1)), \text{ for all } t_2 > t_1 \geq 0.$$

(7)

Thus, the number of sign variations along a solution of the system is non-increasing. Furthermore, it can be shown that $s^-(x(t)) = s^+(x(t))$ except perhaps at up to $n - 1$ isolated points [12].
One implication of this is that if $\gamma(t)$ is a $T$-periodic solution of a linear TPDS (that is not the trivial solution $\gamma(t) \equiv 0$) then

$$s^-(\gamma(t)) = s^+(\gamma(t)) = s^- (\gamma(0))$$

for all $t \geq 0$.

The next section describes our main results. It is useful to begin with case of a linear TPDS, and then consider the nonlinear case.

### III. MAIN RESULTS

#### A. $T$-periodic linear TPDS

Consider the linear-time varying (LTV) system $\dot{x} = A(t)x$ with $A(t)$ a continuous and $T$-periodic matrix. We also assume that $A(t) \in \mathbb{M}^+$ for all $t \in [0, T)$, so (6) is a linear TPDS. Then any trajectory with an initial condition in $\Omega$ converges to a $T$-periodic solution of (6) (that is not necessarily unique).

The transition matrix satisfies $\Phi(0) = I$ and by Floquet theory [3], $\Phi(t + T) = \Phi(t)\Phi(T)$, for all $t \geq 0$. Let $B := \Phi(T)$. The eigenvalues of $B$ are called the characteristic multipliers of (6). Since $A(t) \in \mathbb{M}^+$, $\Phi(t)$ is TP for any $t > 0$ and in particular $B$ is TP. Let

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0,$$

(8)
denote the eigenvalues of $B$, and let $v^i \in \mathbb{R}^n$ denote the eigenvector corresponding to $\lambda_i$.

Let $\gamma(t)$ denote a $T$-periodic solution of (6) with $\gamma(0) \neq 0$. Then $\gamma(T) = B\gamma(0)$ and since $\gamma(T) = \gamma(0)$, this implies that there exists $p \in \{1, \ldots, n\}$ such that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p = 1 > \cdots > \lambda_n > 0,$$

(9)

Fix a “perturbation direction” $w \in \mathbb{R}^n$ and consider the difference

$$z(t) := x(t, \gamma(0) + w) - x(t, \gamma(0)).$$

(10)

Then for any integer $k > 0$,

$$z(kT) = \Phi(kT)z(0) = B^kz(0).$$

Let $c_i \in \mathbb{R}$ be such that $w = \sum_{i=1}^n c_i v^i$. Then

$$z(kT) = \sum_{i=1}^n c_i \lambda_i^k v^i.$$  

(11)

Combining this with (9) implies the following result.

**Proposition 4:** Consider the linear TPDS (6), and assume that it admits a $T$-periodic solution $\gamma(t)$ with $\gamma(0) \neq 0$. There are two possible cases.

Case 1. If $\lambda_1 > 1$ then the periodic solution is not orbitally stable, $v^1$ is the “worst” perturbation direction in the sense that this perturbation takes the state away from $\gamma$ as quickly as possible. Also, $\lambda_n \leq 1$, and if $\lambda_n < 1$ then $v^n$ is the “best” perturbation direction in the sense that this perturbation takes the state back to $\gamma$ as quickly as possible.

Case 2. If $\lambda_1 = 1$ then (9) implies that the periodic solution is stable and orbitally asymptotically stable, and that $v^2 [v^n]$ is the “worst” (“best”) perturbation direction in the sense that this perturbation takes the state back to $\gamma$ as slowly [quickly] as possible.

Note that for a TPDS, even if we do not have an explicit expression for $\gamma(t)$, we always know that the eigenvalues $\lambda_i$ of $B = \Phi(T)$ are real, positive, and distinct, and also the special sign pattern of each of the eigenvectors $v^i$.

The next simple example demonstrates Prop. 4.

**Example 2:** Consider (6) with $n = 2$ and

$$A(t) = \begin{bmatrix} -2 & 2 + \sin(t) \\ 2 + \sin(t) & -2 \end{bmatrix}.$$  

Note that $A(t) \in \mathbb{M}^+$ for all $t \geq 0$, and that $A(t)$ is $T$-periodic for $T = 2\pi$. The transition matrix is

$$\Phi(t) = \exp(-2t) \begin{bmatrix} \cosh(c(t)) & \sinh(c(t)) \\ \sinh(c(t)) & \cosh(c(t)) \end{bmatrix},$$

with $c(t) := 2t - \cos(t) + 1$. Note that every entry of $\Phi(t)$ is positive and that $\det(\Phi(t)) > 0$ (i.e. $\Phi(t)$ is TP) for all $t > 0$. Here,

$$B = \Phi(2\pi) = \exp(-4\pi) \begin{bmatrix} \cosh(4\pi) & \sinh(4\pi) \\ \sinh(4\pi) & \cosh(4\pi) \end{bmatrix}.$$  

The eigenvalues of $B$ are

$$\lambda_1 = 1, \quad \lambda_2 = \exp(-8\pi),$$

with corresponding eigenvectors

$$v^1 = [1 \ 1]^T, \quad v^2 = [-1 \ 1]^T.$$  

Hence, a $2\pi$-periodic solution is

$$\gamma(t) := x(t, v^1) = \Phi(t)v^1$$

$$= \exp(-2t)(\cosh(c(t)) + \sinh(c(t)))v^1$$

$$= \exp(1 - \cos(t))v^1.$$

As expected, $v^1 [v^2]$ has zero [one] sign changes. To demonstrate (11) in this case, define $z$ as in (10). Any $w \in \mathbb{R}^2$ can be written as $w = \frac{w_1 + w_2}{2}v^1 + \frac{w_2 - w_1}{2}v^2$, so

$$z(kT) = B^k w$$

$$= \frac{w_1 + w_2}{2}v^1 + \frac{w_2 - w_1}{2} \exp(-8\pi k)v^2.$$  

### B. $T$-periodic nonlinear TPDS

Consider the nonlinear system (1) with a $T$-periodic vector field. Assume that its trajectories evolve on the state-space $\Omega \subseteq \mathbb{R}^n$, and that

$$J(t, z) \in \mathbb{M}^+$$

(12)

for all $t \in [0, T), z \in \Omega$, that is, (1) is a TPDS.

We first consider the case where $f$ is time-invariant (and hence $T$-periodic for any $T$). The next result describes the special spectral structure of $J(z)$ at any point $z \in \Omega$.

**Proposition 5:** Fix $z \in \Omega$. The eigenvalues of $J(z)$ are
real and simple. Denote these eigenvalues by
\[ \alpha_1(z) > \alpha_2(z) > \cdots > \alpha_n(z), \]
and let \( v^k(z) \in \mathbb{R}^n \) denote the eigenvector corresponding to \( \alpha_k(z) \). For any \( 1 \leq i \leq j \leq n \), let \( V^{ij}(z) := \text{span}\{v^i(z), v^{i+1}(z), \ldots, v^j(z)\} \). Then
\[ i-1 \leq s^-(y) \leq s^+(y) \leq j-1, \quad \text{for any } y \in V^{ij}(z) \setminus \{0\}. \]
(13)

\[ \text{Proof:} \quad \text{Fix } z \in \Omega. \text{ Then } J(z) \text{ is a Jacobi matrix.} \]
Pick \( s > 0 \) large enough so that \( sI + J(z) \) satisfies the domination condition \( (5) \), and is nonsingular. Then \( sI + J(z) \) is TN, nonsingular and irreducible, so it is oscillatory. Applying Prop. \( \text{2} \) completes the proof.

In particular, if \( e \) is an equilibrium then the set of eigenvalues and eigenvectors of \( J(e) \) provides a rather complete picture of the dynamical behavior near \( e \). Indeed, the Hartman–Grobman theorem [3] asserts that the phase portrait near \( e \) (assuming that \( J(e) \) has no eigenvalues with a zero real part) is the same as the phase portrait of the LTI system \( \dot{y} = J(e)y \), up to a continuous change of coordinates \( x = p(y) \), with \( p : \mathbb{R}^n \to \mathbb{R}^n \) satisfying \( p(0) = e \).

Suppose for example that all the eigenvalues are negative. Then the eigenvector \( v^1(e) [v^n(e)] \) corresponds to the “slowest” “fastest” eigenvalue \( \alpha_1(e) [\alpha_n(e)] \). A perturbation of the steady-state in the form \( e \to p(\varepsilon v^1(e)) [e \to p(\varepsilon v^n(e))] \), with \( \varepsilon \) small, will be “compensated” at the slowest [fastest] possible rate.

\[ \text{Example 3: Consider the system:} \]
\[ \dot{x}_1 = -2x_1 + \tanh(x_1) + 2 \tanh(x_2), \]
\[ \dot{x}_2 = -x_2 + (1/2) \tanh(x_1) + \tanh(x_2). \]
(14)

Such systems appear in models of neural networks with \( \tanh(\cdot) \) as the activation function. It is straightforward to verify that there are three equilibrium points in \( \mathbb{R}^2 \):
\[ e^1 = [0 \ 0]^T, \quad e^2 = [1.28784 \ 1.28784]^T, \quad \text{and } e^3 = -e^2. \]

The Jacobian of \( (14) \) is
\[ J(x) = \begin{bmatrix} -2 + \cosh^{-2}(x_1) & 2 \cosh^{-2}(x_2) \\ (1/2) \cosh^{-2}(x_1) & -1 + \cosh^{-2}(x_2) \end{bmatrix}, \]
(15)
so \( J(x) \in M^+ \) for all \( x \in \mathbb{R}^2 \). The eigenvalues of \( J(e^1) \) are
\[ (\sqrt{5} - 1)/2, \quad (-\sqrt{5} - 1)/2, \]
with corresponding eigenvectors
\[ \begin{bmatrix} \sqrt{5} - 1 \\ 1 \end{bmatrix}^T, \quad \begin{bmatrix} -\sqrt{5} - 1 \\ 1 \end{bmatrix}^T. \]
The eigenvalues of \( J(e^2) \) (and of \( J(e^3) \)) are
\[ -0.672232, \quad -1.80202, \]
with corresponding eigenvectors
\[ \begin{bmatrix} 0.442698 \ 0.896671 \end{bmatrix}^T, \quad \begin{bmatrix} 0.992469 \ -0.122499 \end{bmatrix}^T. \]
As expected, the eigenvalues are real and distinct, and the eigenvectors have the specified sign pattern.

We now turn to consider the case where the nonlinear system is time-varying, and \( T \)-periodic with a minimal period \( T > 0 \). For \( x_0 \in \Omega \), let
\[ H(t, x_0) := \frac{\partial}{\partial x_0} x(t, x_0), \]
that is, \( H \) maps a change in the initial condition at time 0 to the change in the solution at time \( t \). Then \( H(0, x_0) = I \), and
\[ \dot{H}(t, x_0) := \frac{d}{dt} H(t, x_0) \]
\[ = \frac{\partial}{\partial x_0} f(t, x(t, x_0)) \]
\[ = J(t, x(t, x_0)) H(t, x_0). \]
(16)
Since \( J \in M^+ \), this is a linear time-varying TPDS. Let \( a := \gamma'(0) \), where \( \gamma(t) \) is a \( T \)-periodic solution of \( (1) \). Then for \( x_0 = a \) the linear TPDS \( (16) \) is also \( T \)-periodic, so \( \Phi(t) := H(t, a) \) satisfies \( \Phi(t + T) = \Phi(t) B \), with \( B := \Phi(T) \).

We conclude that
\[ \frac{\partial}{\partial x_0} x(kT, a) = B^k. \]

For \( \varepsilon > 0 \) and \( \omega \in \mathbb{R}^n \), let \( z(t) := x(t, a + \varepsilon \omega) - x(t, a) \).
Then for a fixed time \( t \geq 0 \), \( z(t) = \varepsilon \Phi(t) \omega + o(\varepsilon) \), so for a fixed \( k \),
\[ z(kT) = \varepsilon \Phi(kT) \omega + o(\varepsilon) = \varepsilon B^k \omega + o(\varepsilon). \]

Thus, as in the cases described above, \( B \) always has 1 as an eigenvalue and a corresponding eigenvector \( a \). The eigenvalues and eigenvectors of the TP matrix \( B \) determine the response to a small perturbation \( a \to a + \varepsilon \omega \).

The special sign pattern of the eigenvectors \( v^1 \) and \( v^n \) of \( B \) can be explained intuitively using the cooperative and tridiagonal structure of the Jacobian of a TPDS. We demonstrate this using a model from systems biology called the ribosome flow model (RFM).

\[ \text{C. Application: the RFM} \]

The RFM is a phenomenological model for the flow of “material” along a 1D chain composed of \( n \) consecutive sites. The RFM is the dynamic mean field approximation of a fundamental model from statistical mechanics called the totally asymmetric simple exclusion process (TASEP) [2, 1]. This model describes the stochastic motion of particles hopping along a 1D chain of sites while restricted by the simple exclusion principle, namely, two particles cannot occupy the same site at the same time.

The RFM, and more generally networks of interconnected RFMs, has been extensively used to model the flow of ribosomes along the mRNA during translation [19, 14, 13, 20, 34, 21, 17, 31, 33, 15], and also other important processes such as the transfer of the phosphoryl group from the sensor kinases to the ultimate target during phosphorelay [1].

The RFM is a set of \( n \) ODEs:
\[ \dot{x}_i(t) = \lambda_{i-1} x_{i-1}(t)(1-x_i(t)) - \lambda_i x_i(t)(1-x_{i+1}(t)), \]
(17)
with \( i = 1, \ldots, n \), \( x_0(t) \equiv 1 \), and \( x_{n+1}(t) \equiv 0 \). The state-variable \( x_i(t) \) represents the density at site \( i \) at time \( t \), normalized such that \( x_i(t) = 0 \) \( [x_i(t) = 1] \) corresponds to site \( i \) being empty [full] at time \( t \). The positive parameter \( \lambda_i \) represents the transition rate from site \( i \) to site \( i + 1 \). In particular, \( \lambda_0 \) \( [\lambda_n] \) is called the initiation [exit] rate, and the other \( \lambda_i \)'s are the elongation rates.

To explain (17), consider the equation for \( \dot{x}_2 \), namely,

\[
\dot{x}_2 = \lambda_1 x_1 (1 - x_2) - \lambda_2 x_2 (1 - x_3).
\]

This asserts that the change in the density at site 2 is the flow from site 1 to site 2 minus the flow from site 2 to site 3. The flow from site 1 to site 2, given by \( \lambda_1 x_1 (1 - x_2) \), is proportional to the density at site 1, \( 1 \) the amount of “free space” \( (1 - x_2) \) at site 2, and the transition rate \( \lambda_1 \) from site 1 to site 2. In particular, as the density in site 2 increases, the flow from site 1 to site 2 decreases. This is a “soft version” of the simple exclusion principle in TASEP.

An important property of the RFM, “inherited” from TASEP, is that allows to study the generation of “traffic jams” along the chain. Indeed, if some \( \lambda_i \) is very small w.r.t. the other rates then the exit rate from site \( i \) will be small. Then the density at site \( i \) will increase, and consequently, the transition rate from site \( i - 1 \) will decrease. In this way, a traffic jam of high density sites will evolve “behind” site \( i \). The dynamic evolution and implications of traffic jams of “biological particles” like ants, ribosomes, and molecular motors are attracting considerable interest (see, e.g., [22], [5], [24], [7]).

The output flow from the last site is \( R(t) := \lambda_n x_n(t) \). This represents the rate at which ribosomes exit the mRNA, i.e. the protein production rate.

The state space of RFM is the unit cube \( [0, 1]^n \). For \( a \in [0, 1]^n \), let \( x(t, a) \) denote the solution of the RFM at time \( t \) with \( x(0) = a \). The flow possesses a unique globally stable (GAS) equilibrium \( e \in (0, 1)^n \) \([13], [16] \), that is, \( \lim_{t \to \infty} x(t, a) = e \), for all \( a \in [0, 1]^n \). This steady-state represents a set of densities \( e_1, \ldots, e_n \) for which the flow into each site is equal to the flow out of this site. In physics, this is sometimes referred to as a nonequilibrium stationary state.

Let \( R := \lambda_n e_n \). This is the flow of ribosomes out of the chain, and thus the protein production rate, at equilibrium. It follows from (17) that

\[
\lambda_i e_i (1 - e_{i+1}) = R, \quad i = 0, \ldots, n, \tag{18}
\]

where \( e_0 := 1 \) and \( e_{n+1} := 0 \).

Let \( J(x) \) denote the Jacobian of the vector field in the RFM. Then \( J(x) \) admits the tridiagonal structure \( J(x) = T(a_i(x), b_i(x), c_i(x)) \) in \([4]\) with \( a_i(x) = -\lambda_i x_{i-1} - \lambda_i (1 - x_{i+1}) \), \( b_i(x) = \lambda_i x_i \), and \( c_i(x) = \lambda_i (1 - x_{i+1}) \), where \( x_0 := 1 \), and \( x_{n+1} := 0 \). Thus, the RFM is a TPDS on the invariant set \((0, 1)^n\), and Theorem \([3]\) implies that the eigenvalues of \( J(e) \) are real and simple and the corresponding eigenvectors satisfy the sign pattern \((13)\). Furthermore, since the RFM is contractive \([16]\), the eigenvalues of \( J(e) \) are all real, simple, and negative.

The sign pattern of the eigenvectors corresponding to the slowest [fastest] eigenvalue \( \alpha_1(e) \) \( \alpha_n(e) \) can be explained as follows. Recall that \( v^1(e) \) has zero sign variations. We may assume that \( v_i^1(e) > 0 \) for all \( i \). Then \( e \to e + \varepsilon v_i^1(e) \) is difficult to compensate because it increases all the densities in the chain. This corresponds a to very congested situation, and returning to the steady-state \( e \) takes more time.

On the other-hand, a perturbation in the form \( e \to e + \varepsilon v_n(e) \) is easier to compensate because \( v_n(e) \) has an alternating sign pattern, that is, we may assume that \( v_n(e) > 0 \) for \( i \) odd, and \( v_n(e) < 0 \) for \( i \) even. This represents a positive addition to \( e_1 \), a negative addition to \( e_2 \), a positive addition to \( e_3 \), and so on. The RFM dynamics compensates for this variation quite naturally: the additional density in an odd site generates an increased flow to the consecutive site and in this way also “automatically” takes care of the lower densities in the even sites.

The next example considers specific values for the rates for which \( e \) (and thus \( J(e) \)) is known explicitly for any \( n \).

**Example 4:** Consider the RFM with \( \lambda_0 = \lambda_n = 1/2 \), and \( \lambda_i = 1 \) for \( i = 1, \ldots, n - 1 \). It can be shown using the spectral representation of \( e \) \([19]\) that in this case \( e_i = 1/2 \), \( i = 1, \ldots, n \). This implies that \( J(e) = T(a_i(e), b_i(e), c_i(e)) \) is an \( n \times n \) Toeplitz tridiagonal matrix with \(-1 \) on the main diagonal, and \( 1/2 \) on the sub- and super-diagonals. It is well-known \([29]\) that the eigenvalues of such a matrix are

\[
\alpha_k = -1 + \cos(k\pi/(n + 1)), \quad k = 1, \ldots, n,
\]

and the eigenvector corresponding to \( \alpha_k \) is \( v^k = [\sin(\pi n^{-1}) \sin(2\pi n^{-1}) \ldots \sin(n\pi n^{-1})]^T \). In particular,

\[
v^1 := [\sin(\pi n^{-1}) \sin(2\pi n^{-1}) \ldots \sin(n\pi n^{-1})]^T.
\]

As expected, all the eigenvalues are real, negative and simple, all the entries of \( v^1 \) are positive, and the entries of \( v^n \) have the sign pattern \((+,-, +, \ldots)\). The maximal eigenvalue is \( \alpha_1 = -1 + \cos(\pi n/(n + 1)) \), Thus,

\[
\lim_{n \to \infty} \frac{\log(-\alpha_1(n))}{\log(n)} = \lim_{n \to \infty} \frac{\log(\frac{2n^2}{(n+1)^2} - \frac{x^2}{2(n+1)^2} + \ldots)}{\log(n)} = -2.
\]

Thus, the relaxation time of the system behaves like \( n^2 \) for large \( n \).

The minimal eigenvalue is \( \alpha_n = -1 + \cos(\pi n/(n + 1)) \), so

\[
\lim_{n \to \infty} \alpha_n = -2.
\]

Thus, the fastest convergence rate to \( e \), corresponding to a perturbation in the form \( e + \varepsilon v^n \), is independent of \( n \) for large \( n \).

**Remark 6:** The asymptotic behavior described in \([19]\) seems to hold for other rates as well. For example, Fig. \([4]\).
depicts \( \log(-\alpha_1(n)) \) as a function of \( \log(n) \) for the case where \( \lambda_i = 1 \) for \( i \in \{0, \ldots, n\} \). It may be seen that for large values of \( n \) the graph behaves like a line with slope \(-2\).

IV. CONCLUSION

Any bounded solution of a \( T \)-periodic TPDS converges to a \( T \)-periodic solution of the TPDS. If the \( T \)-periodic vector field represents a \( T \)-periodic excitation then this implies that the dynamical system entrains to the excitation. In particular, any bounded solution of a time-invariant TPDS converges to an equilibrium [12].

It is important to understand how a perturbation from a periodic solution affects the dynamics and, in particular, what are the convergence or divergence rates associated with different perturbation directions. In general, Floquet theory does not provide explicit answers to these questions. We used the spectral theory of TP matrices to analyze this question for TPDSs and showed that the “best” and “worst” perturbation directions have special sign patterns that admit an intuitive interpretation.

An interesting topic for future research is to use this information to design appropriate control algorithms for TPDSs.

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