On crown-free families of subsets

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Abstract

The crown $O_{2t}$ is a height-2 poset whose Hasse diagram is a cycle of length $2t$. A family $F$ of subsets of $[n] := \{1, 2, \ldots, n\}$ is $O_{2t}$-free if $O_{2t}$ is not a weak subposet of $(F, \subseteq)$. Let $La(n, O_{2t})$ be the largest size of $O_{2t}$-free families of subsets of $[n]$. De Bonis-Katona-Swanepoel proved $La(n, O_4) = \binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lceil \frac{n}{2} \rceil}$. Griggs and Lu proved that $La(n, O_{2t}) = (1 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all even $t \geq 4$. In this paper, we prove $La(n, O_{2t}) = (1 + o(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}$ for all odd $t \geq 7$.

1 Introduction

We are interested in estimating the maximum size of family of subsets of the $n$-set $[n] := \{1, \ldots, n\}$ avoiding a given (weak) subposet $P$. The starting point of this kind of problem is Sperner’s Theorem from 1928 [18], which determined that the maximum size of an antichain in the Boolean lattice $B_n := (2^n, \subseteq)$ is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

For posets $P = (P, \leq)$ and $P' = (P', \leq')$, we say $P'$ is a weak subposet of $P$ if there exists an injection $f: P' \to P$ that preserves the partial ordering, meaning that whenever $u \leq' v$ in $P'$, we have $f(u) \leq f(v)$ in $P$ (see [19]). Throughout the paper, when we say subposet, we mean weak subposet. The height $h(P)$ of poset $P$ is the maximum size of any chain in $P$.

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A family $F$ of subsets of $[n]$ can be viewed as a subposet of $B_n$. If $F$ contains no subposet $P$, we say $F$ is $P$-free. We are interested in determining the largest size of a $P$-free family of subsets of $[n]$, denoted $La(n, P)$.

In this notation, Sperner’s Theorem \cite{18} gives that $La(n, P_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, where $P_k$ denotes the path poset on $k$ points, usually called a chain of size $k$. Let $B(n, k)$ be the middle $k$ levels in the Boolean lattice $B_n$ and $\Sigma(n, k) := |B(n, k)|$. Erdős \cite{9} proved that $La(n, P_k) = \Sigma(n, k)$. Griggs-Li-Lu \cite{14} showed that the similar results hold for a wide class of posets including diamonds $D_k$ ($A < B_1, \ldots, B_k < C$, for $k = 3, 4, 7, 8, 9, 15, 16, \ldots$), harps $H(l_1, l_2, \ldots, l_k)$ (consisting of chains $P_1, \ldots, P_k$ with their top elements identified and their bottom elements identified, for $l_1 > l_2 > \cdots > l_k$).

For any poset $P$, we define $e(P)$ to be the maximum $m$ such that for all $n$, the union of the $m$ middle levels $B(n, m)$ does not contain $P$ as a subposet. For any $F \subset 2^{[n]}$, define its Lubell value $h_n(F) := \sum_{F \in F} 1/|F|$. Let $\lambda_n(P) = \max \{h_n(F) : F \subset 2^{[n]}, P\text{-free}\}$. A poset $P$ is called uniform-$L$-bounded if $\lambda_n(P) \leq e(P)$ for all $n$. Griggs-Li \cite{13} proved $La(n, P) = \Sigma(n, e(P))$ if $P$ is uniform-$L$-bounded. The uniform-$L$-bounded posets include $P_k$ (for any $k \geq 1$), diamonds $D_k$ (for $k \in [2^{m-1} - 1, 2^m - \lfloor \frac{m}{2} \rfloor - 1]$ where $m := \lfloor \log_2(k+2) \rfloor$), and harps $H(l_1, l_2, \ldots, l_k)$ (for $l_1 > l_2 > \cdots > l_k$), and other posets.

For any poset $P$, Griggs-Lu \cite{15} conjectured the limit $\pi(P) := \lim_{n \to \infty} \frac{La(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists and is an integer. This conjecture is based on various known cases. For example, an $r$-fork poset $V_r$, which has elements $A < B_1, \ldots, B_r$, $r \geq 2$. Katona and Tarján \cite{16} obtained bounds on $La(n, V_2)$ that he and DeBonis \cite{7} extended in 2007 to general $V_r$, $r \geq 2$, proving that

$$\left(1 + \frac{r-1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq La(n, V_r) \leq \left(1 + 2^{r-1} - \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

While the lower bound is strictly greater than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, we see that $La(n, V_r) \sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Earlier, Thanh \cite{20} had investigated the more general class of broom-like posets. Griggs and Lu \cite{15} studied the even more general class of baton posets. These are tree posets (meaning that their Hasse diagrams are trees.) Griggs and Lu \cite{15} proved that $\pi(T) = 1$ for any tree poset $T$ of height 2. Bukh \cite{4} proved that $\pi(T) = e(T)$ for any general tree poset $T$.

The most notable unsolved case is the diamond poset $D_2$. Griggs and Lu first observed $\pi(D_2) \in [2, 2.296]$. Axenovich, Manske, and Martin \cite{3} came
up with a new approach which improves the upper bound to 2.283. Griggs, Li, and Lu [14] further improves the upper bound to $2.27 + 3\cdot 11$. Very recently, Kramer-Martin-Young [17] recently proved $\pi(D_2) \leq 2.25$.

The crown $O_{2t}$ is another family of posets, which are neither trees nor uniform-L-bounded. For $k \geq 2$, the crown $O_{2t}$ is a height-2 poset whose Hasse diagram is a cycle of length $2t$. For $t = 2$, $O_4$ is also known as the butterfly poset; De Boinis-Katona-Swanepoel [8] proved $\Pi(D_2) = \Sigma(n, O_4)$.

Griggs and Lu [15] proved that $\La(n, O_{2t}) = (1 + o(1))(\frac{n}{2^t})$ for all even $t \geq 4$. For odd $t \geq 3$, Griggs and Lu showed that $\La(n, O_{2t}) = (1 + o(1))(\frac{n}{2^{t-1}})$ is asymptotically at most $1 + \frac{1}{\sqrt{2}}$, which is less than 2. In this paper, we determine all $\pi(O_{2t})$ except for $O_6$ and $O_{10}$.

**Theorem 1.1** For odd $t \geq 7$, we have $\La(n, O_{2t}) = (1 + o(1))(\frac{n}{2^{t-1}})$.

The proof of this theorem uses the concept of a $k$-partite representation, which was originally introduced by Conlon [6] to prove a similar Turán-type result on hypercubes. (Conlon’s result will be stated in Section 2.)

**Definition 1.2** A poset $P$ of height 2 has a $k$-partite representation if there exist two integers $k, l$, and a family $P \subseteq (\begin{array}{c} l \\ k \end{array}) \cup (\begin{array}{c} l \\ k \end{array})$ such that

- The poset $(P, \subseteq)$ contains $P$ as a subposet.
- And $G := G(P)$, a $k$-uniform hypergraph with $V(G) = [l]$ and $E(G) = P \cap \left(\begin{array}{c} l \\ k \end{array}\right)$ is $k$-partite.

Here is our main result.

**Theorem 1.3** Suppose that a poset $P$ of height 2 has a $k$-partite representation for some $k \geq 2$. Then $\La(n, P) = (1 + o(1))(\frac{n}{2^t})$.

Conlon [6] proved that for all crowns $O_{2t}$ except for $t = 2, 3, 5$ have $k$-partite representations for some $k$. For example, $O_{4t}$ (for $t \geq 2$) has a 2-partite representation $P$ such that $G(P)$ is the even-cycle $C_{2t}$. Similarly, $O_{2kt}$ (for $t \geq 2$) has a $k$-partite representation $P$ such that $G(P)$ is the tight $k$-uniform cycle $C_{kt}$. The first non-trivial case is $O_{14}$. The following 3-representation of $O_{14}$ is given by Conlon [6]:
Here $k = 3$, $l = 7$, and

\[ \mathcal{P} = \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{1, 2, 3\}, \{2, 3, 4\}, \{2, 4, 5\}, \{1, 2, 5\}, \{1, 5, 6\}, \{1, 6, 7\}, \{1, 2, 7\}\}. \]

It is easy to check that all the 3-edges $\{1, 2, 3\}, \{2, 3, 4\}, \{2, 4, 5\}, \{1, 2, 5\}, \{1, 5, 6\}, \{1, 6, 7\}, \{1, 2, 7\}$ form a 3-partite 3-uniform hypergraph. Thus, $\mathcal{P}$ is a 3-partite representation of $O_14$.

For $t \geq 4$ and $t \neq 5$, $O_{2t}$ has a $k$-partite representation for some $k$ (see [6]). It implies $La(n, O_{2t}) = (1 + o(1))\left(\frac{n}{\binom{k}{2}}\right)$. Theorem 1.1 is a corollary of Theorem 1.3. We also give an alternative proof for Griggs-Lu’s result $La(n, O_{4t}) = (1 + o(1))\left(\frac{n}{\binom{k}{2}}\right)$ for $t \geq 2$.

The rest of the paper is organized as follows. In section 2, we will first review Conlon’s theorem on Turán problems on hypercubes; then we will prove an interesting Tuán-Ramsey result for $k$-partite $k$-uniform hypergraphs. Finally Theorem 1.3 will be proved in section 3.

## 2 Turán problems on hypergraphs

### 2.1 Turán problem on hypercubes

The problem of determining $La(n, O_{2t})$ is closely related to the Turán problem on the hypercube $Q_n$, i.e., the Hasse diagram of the Boolean lattice $B_n$. Erdős [10] first posed the problem of determine the size of maximum subgraph of hypercube $Q_n$ forbidding a cycle $C_{2k}$. Let $\text{Ex}(H, Q_n)$ be the maximum size of a subgraph of $Q_n$ forbidding a given graph $H$. Let $\pi(H, Q_n) = \lim_{n \to \infty} \frac{\text{Ex}(H, Q_n)}{|E(Q_n)|}$. This limit always exists. Chung [3] proved that $\pi(C_{4k}, Q_n) = 0$ for all $k \geq 2$.

Alon et al. [1, 2] gave a characterization of all subgraphs $H$ of the hypercube which are Ramsey, that is, such that every $k$-edge-colouring of a sufficiently large $Q_n$ contains a monochromatic copy of $H$; in particular, $C_{4k+2}$ (for
$k \geq 2$) are Ramsey. Füredi and Özkahya\cite{Furedi1995, Ozkahya2006} showed that, for $t > 3$,\[ \pi(C_{4t+2}, Q_n) = 0. \] Conlon\cite{Conlon2013} proved the following theorem, which covers all known bipartite graphs $H$ with $\pi(H, Q_n) = 0$.

**Theorem 2.1 (Conlon’s Theorem\cite{Conlon2013})** Suppose that $H$ is the Hasse diagram of a height-2 poset, which admits a $k$-partite representation. Then $\pi(H, Q_n) = 0$.

In\cite{Conlon2013}, the $k$-partite representation is defined over bipartite graphs. His definition is equivalent to ours. Conlon\cite{Conlon2013} observed $C_{2t}$ (for $t \geq 4$ and $t \neq 5$) admits a $k$-partite representation for some $k$; thus, his result implies $\pi(C_{2t}, Q_n) = 0$ for all $t \geq 4$ except for $t = 5$.

### 2.2 A Lemma on $k$-partite $k$-uniform hypergraph

Conlon\cite{Conlon2013} used the following classical result of Erdős\cite{Erdos1960} regarding the extremal number of complete $k$-partite $k$-uniform hypergraphs.

**Lemma 2.2** Let $K^{(k)}_k(s_1, \ldots, s_k)$ be the complete $k$-partite $k$-uniform hypergraph with partite sets of size $s_1, \ldots, s_k$. Then any $K^{(k)}_k(s_1, \ldots, s_k)$-free $r$-uniform hypergraph can have at most $O(n^{k-\delta})$ edges, where $\delta = \left(\prod_{i=1}^{k-1} s_i\right)^{-1}$.

In the scenario of the Boolean lattice, for any poset $P$ having $k$-partite representation, we need prove that any family $\mathcal{F}$ of size $(1 + \epsilon)\left(\binom{n}{\frac{k-1}{2}}\right)$ contains $P$. Note that $\mathcal{F}$ is much sparser comparing to the full Boolean lattice $2^n$. Lemma 2.2 is not strong enough for our purpose. We need the following lemma for Ramsey-Turán problems on hypergraphs, which may have independent interest.

**Lemma 2.3** For any positive integers $k$, $s_1, \ldots, s_k$, and $r$, consider a collection $\mathcal{H} := \{H_i\}_{i \in I}$ (with an index set $I$) of $k$-uniform hypergraphs over a common vertex set $[n]$. Suppose that for each $i \in I$, $H_i$ does not contain $K^{(k)}_k(s_1, \ldots, s_k)$ as a sub-hypergraph, and for each $S \subset \binom{[n]}{k-1}$ there are at most $r$ hypergraphs $H_i$ having edges containing $S$. Then, the total number of edges in this family is at most $O(n^{k-\delta})$, where $\delta = \left(\prod_{i=1}^{k-1} s_i\right)^{-1}$. 

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Remark: Since every hypergraph $H_i$ contains no $K^{(k)}_k(s_1, \ldots, s_k)$, then $|E(H_i)| = O(n^{k-\delta})$ by Lemma 2.2. This lemma says if the family of hypergraphs cover each $(k-1)$-set at most $r$ times then the total number of edges is still $O(n^{k-\delta})$, where the hidden constant in $O(\cdot)$ depends on $k, s_1, \ldots, s_k,$ and $r$, but not on $n$.

Our proof extensively uses the following convexity inequality, (also see Lemma 2.3 of [15].) Suppose that $X$ is a random variable taking non-negative integer values. If for any positive integer $s$, $E(X) > s - 1$, then

$$E\left(\frac{X}{s}\right) \geq \left(\frac{EX}{s}\right).$$

Proof of Lemma 2.3: Let $H$ be the hypergraph on the vertex set $[n]$ with $E(H) = \bigcup_{i \in I} E(H_i)$. Observe that each edge in $H$ can appear in at most $r H_i$’s. Thus,

$$\sum_{i \in I} |E(H_i)| \leq r |E(H)|.$$

Since $r$ is a constant, it suffices to prove $|E(H)| = O(n^{k-\delta})$. Deleting overlapped edges will not affect the magnitude of $|E(H)|$. Without loss of generality, we can assume that edges of different $H_i$ are distinct. If an edge $F$ of $H$ is in $H_i$, then we say this edge has color $i$. By hypothesis, $H$ has no monochromatic copies of $K^{(k)}_k(s_1, \ldots, s_k)$.

Without loss of generality, we assume $n$ is divisible by $k$ and write $n = km$. Consider a random $k$-partition of $[n] = V_1 \cup V_2 \cup \cdots \cup V_k$ where each part has the equal size $m$. We say an edge $F$ is crossing (to this partition), if $F$ intersects every $V_i$ with exactly once. The probability of an edge $F$ being crossing is

$$\Pr(F \text{ is crossing}) = \left(\frac{n}{k}\right)^k \binom{\frac{n}{k}}{k} > \frac{k!}{k^k}.$$

There exists a partition so that the number of crossing edges in $H$ at least $\frac{k!}{k^k}|E(H)|$.

Now we fix this partition $[n] = V_1 \cup \cdots V_k$. Let $H'$ be the subgraph consisting of all crossing edges in $H$ and $H'_i$ be the subgraph consisting of all crossing edges in $H_i$ for $i \in I$. It is sufficient to show $|E(H')| = O(m^{k-\delta})$, since $n = km$ and $k$ is a constant.

Set $|E(H')| \approx C m^{k-\delta}$ (with a big constant $C$ chosen later). For $i \in \{1, s_1\}$ with $i = 1, 2, \ldots, k$, we would like to estimate the number of monochro-
matic (ordered) copies, denoted by \(f(t_1, t_2, \ldots, t_k)\), of \(K_k^{(k)}(t_1, \ldots, t_k)\) with the first \(t_1\) vertices in \(V_1\), the second \(t_2\) vertices in \(V_2\), and so on.

**Claim a:** For \(0 \leq l \leq k - 1\), we have

\[
f(s_1, \ldots, s_l, 1, 1, \ldots, 1) \geq (1 + o(1)) \frac{(\frac{c}{m^k})^{l} \prod_{j=1}^{l} s_j}{\prod_{j=1}^{l} (s_j!^{r_{s_j}-1}) \prod_{u=j+1}^{k} s_u} m^{k-l+\sum_{j=1}^{l} s_j}.
\]

We prove claim (a) by induction on \(l\). For the initial case \(l = 0\), the claim is trivial since \(f(1,1,\ldots,1) = |E(H')| \approx Cm^{k-\delta}\).

We assume Claim (a) holds for \(l\). Now consider the case \(l + 1\). For any \(S \in \binom{V_1}{s_1} \times \cdots \times \binom{V_l}{s_l} \times V_{l+2} \times \cdots \times V_k\), let \(d_S^i\) be the number of vertices \(v\) in \(V_{l+1}\) such that all edges in the induced subgraph of \(H'\) on \(S \times \{v\}\) have color \(i\). Let \(d_S = \sum_{i \in I} d_S^i\). We have

\[
f(s_1, \ldots, s_l, 1, 1, \ldots, 1) = \sum_{S} \sum_{i \in I} d_S^i; \tag{2}
\]

\[
f(s_1, \ldots, s_l, s_{l+1}, 1, \ldots, 1) = \sum_{S} \sum_{i \in I} \binom{d_S^i}{s_{l+1}}. \tag{3}
\]

Note that \(S\) contains at least \(k - 1\) vertices. By hypothesis, for a fixed \(S\), at most \(r\) of those \(d_S^i\) are non-zero; say \(d_S^{i_1}, \ldots, d_S^{i_r}\). Applying the convex inequality (1), we have

\[
\sum_{i \in I} \binom{d_S^i}{s_{l+1}} = \sum_{j=1}^{r} \binom{d_S^{i_j}}{s_{l+1}} \geq r \binom{d_S}{s_{l+1}},
\]

provided \(d_S > r(s_{l+1} - 1)\).

Let \(\bar{d}_l\) be the average of \(d_S\). By equation (2) and inductive hypothesis, we have

\[
\bar{d}_l \geq \frac{\sum_S \sum_{i \in I} d_S^i}{m^{k-l+\sum_{j=1}^{l} s_j}} \geq (1 + o(1)) \frac{m (\frac{c}{m^k})^{l} \prod_{j=1}^{l} s_j}{\prod_{j=1}^{l} (s_j!^{r_{s_j}-1}) \prod_{u=j+1}^{k} s_u}. \tag{4}
\]
Let $S$ be the set of $S$ satisfying $d_S > r(s_{l+1} - 1)$. Let $\bar{d}^*$ be the average of $d_S$ over $S \in S$. Clearly, $\bar{d}^* \geq \bar{d}$ since $\bar{d} \gg r(s_{l+1} - 1)$ Thus,

\[
f(s_1, \ldots, s_l, s_{l+1}, 1, \ldots, 1) \geq \sum_{s \in S} \sum_{i \in I} \left( \frac{d^*_S}{s_{l+1}} \right) \\
\geq \sum_{s \in S} r \left( \frac{d^*_S / r}{s_{l+1}} \right) \\
\geq r |S| \bar{d}^* \left( \frac{d^* / r}{s_{l+1}} \right) \\
= \frac{|S| \bar{d}^*}{s_{l+1}} \left( \frac{d^* / r}{s_{l+1} - 1} \right) \\
\geq \frac{(\bar{d} - r(s_{l+1} - 1))m^{k-1+\sum_{j=1}^{l}(s_j-1)}(\bar{d}/r)}{(s_{l+1}-1)} \\
= \left(1 + O \left( \frac{1}{\bar{d}} \right) \right) \frac{\bar{d}^{k-1+\sum_{j=1}^{l}(s_j-1)}}{s_{l+1}!r^s_{l+1-1}m^{k-1+\sum_{j=1}^{l}(s_j-1)}}. \tag{5}
\]

Combining with equation (4), we get

\[
f(s_1, \ldots, s_l, s_{l+1}, 1, \ldots, 1) \geq (1 + o(1)) \frac{(c/m^\gamma) \prod_{j=1}^{l+1} s_j}{\prod_{j=1}^{l+1} (s_j!r^{s_j-1}) \prod_{u=j+1}^{l+1} s_u} m^{k-1+\sum_{j=1}^{l+1} s_j}. \tag{6}
\]

The inductive proof is finished.

Applying Claim (a) with $l = k - 1$, we get

\[
f(s_1, s_2, \ldots, s_{k-1}, 1) \geq (1 + o(1)) \frac{(c/m^\gamma) \prod_{j=1}^{k-1} s_j}{\prod_{j=1}^{k-1} (s_j!r^{s_j-1}) \prod_{u=j+1}^{k-1} s_u} m^{1+\sum_{j=1}^{k-1} s_j} \\
= (1 + o(1)) \frac{C \prod_{j=1}^{k-1} s_j \sum_{j=1}^{k-1} s_j}{\prod_{j=1}^{k-1} (s_j!r^{s_j-1}) \prod_{u=j+1}^{k-1} s_u}. \tag{5}
\]

For any $S \in \binom{V_1}{s_1} \times \cdots \times \binom{V_k}{s_k}$, let $d_S$ be the number of vertices $v$ in $V_{l+1}$ such that the edges in the induced subgraph of $H'$ on $S \times \{v\}$ are monochromatic. Since $H'$ contains no monochromatic copy of $K_k^{(k)}(s_1, \ldots, s_k)$, we have $d_S \leq rs_k$. It implies

\[
f(s_1, s_2, \ldots, s_{k-1}, 1) = \sum_{S} d_S \leq rs_k m^{\sum_{j=1}^{k-1} s_j}. \tag{6}
\]
Choosing $C > 2(rs_k)^{\frac{1}{\sum_{a=1}^{k} u_a}} \cdot \prod_{j=1}^{k-1} (s_j)^{r-s_j-1} \cdot \frac{1}{\sum_{a=1}^{k} u_a}$, equations (5) and (6) contradict each other. Hence, $|E(H')| < Cm^{k-\delta}$. It implies $\sum_{i \in I} |E(H_i)| = O(m^{k-\delta}) = O(n^{k-\delta})$. The proof of the lemma is finished. □

3 Proof of main Theorem

We need the following two lemmas on binomial coefficients.

Lemma 3.1 (see Lemma 2.1 of [13]) For any positive integer $n$, we have

$$\frac{1}{2^n} \sum_{|i - \frac{n}{2}| > 2\sqrt{n \ln n}} \binom{n}{i} < \frac{2}{n^2}. \quad (7)$$

Lemma 3.2 For any $i, j \in (\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$, if $|i - j| = o \left(\frac{\sqrt{n}}{\sqrt{\ln n}}\right)$, then

$$\frac{n!}{i! j!} = 1 + o(1). \quad (8)$$

Proof: Without loss of generality, we can assume $j > i \geq \frac{n}{2}$. We have

$$\frac{n!}{i! j!} = \prod_{l=1}^{j-i} \frac{n!}{(i+l)!} = \prod_{l=1}^{j-i} \frac{i+l}{n-i-l+1} = \prod_{l=1}^{j-i} \left(1 + \frac{2(i+l) - n - 1}{n-i-l+1}\right).$$

Since $i + l \in (\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$, we have

$$\frac{|2(i+l) - n - 1|}{n-i-l+1} \leq \frac{4\sqrt{n \ln n} + 1}{\frac{n}{2} - 2\sqrt{n \ln n}} = (1 + o(1)) \frac{8\ln n}{\sqrt{n}}.$$

Thus, we get

$$\frac{n!}{i! j!} \leq \left(1 + (1 + o(1)) \frac{8\ln n}{\sqrt{n}}\right)^{j-i} = 1 + o(1).$$
The proof of the lemma is finished. □

To prove Theorem 1.3, we need to show that for any $\epsilon > 0$ any family $\mathcal{F} \subset 2^{[n]}$ of size $(1 + \epsilon)\binom{n}{\lfloor n/2 \rfloor}$ must contain the subposet $P$. Without loss of generality, we can assume that $\mathcal{F}$ only contains subsets of sizes in the interval $(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$. This is because the number of subsets of size not in $I$ (see Lemma 3.1) is at most

$$\sum_{|I - \frac{n}{2}| > 2\sqrt{n \ln n}} \binom{n}{l} \leq \frac{2^{n+1}}{n^2} = O\left(\frac{\binom{n}{\lfloor n/2 \rfloor}}{n^{3/2}}\right),$$

which is negligible compared to $\epsilon\binom{n}{\lfloor n/2 \rfloor}$.

Taking a random permutation $\sigma$ of the set $[n]$, a (random) full chain is the chain

$$\emptyset \subset \{\sigma(1)\} \subset \{\sigma(1), \sigma(2)\} \subset \cdots \subset [n].$$

Let $X$ be the number of subsets in both $\mathcal{F}$ and a random full chain. The expected value of $X$ is exactly the Lubell value of $\mathcal{F}$:

$$\mathbb{E}(X) = h_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}. \quad (*)$$

It is clear that

$$\mathbb{E}(X) \geq \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}} = 1 + \epsilon. \quad (10)$$

Combining equation $(10)$ and the convexity inequality $(\Pi)$ with $s = 2$, we have

$$\mathbb{E}\left(\binom{X}{2}\right) \geq \left(\frac{\mathbb{E}(X)}{2}\right)^2 \geq \frac{\epsilon}{2} \mathbb{E}(X). \quad (11)$$

For any two subsets $A \subset B$, the probability that a random full chain hits both $A$ and $B$ is $\frac{|A|!(|B| - |A|)!(n - |B|)!}{n!}$. By linearity, we get

$$\mathbb{E}\left(\binom{X}{2}\right) = \sum_{A,B \subset \mathcal{F}, A \subset B} \frac{|A|!(|B| - |A|)!(n - |B|)!}{n!} \quad (12)$$

The following Lemma was implicitly proved when Griggs and Lu [15] proved $\text{La}(T) = (1 + o(1))\binom{n}{\lfloor n/2 \rfloor}$ for any tree poset of height 2. The statement works for any poset of height 2, not just those having $k$-partite representation. We state it here as a lemma for the future references, and also provide a proof for completeness.
Lemma 3.3 Let $P$ be a finite poset of height 2 and $\mathcal{F}$ be a $P$-free $\mathcal{F}$ family of subsets of $[n]$ with the Lubell value $h_n(\mathcal{F}) \geq 1 + \epsilon$. Suppose that every subset in $\mathcal{F}$ has size in the interval $(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$. Then, we have

$$\sum_{A,B \in \mathcal{F}, |B| - |A| = 1} \frac{|A|(|B| - |A|)! (n - |B|)!}{n!} \geq (1 + o(1)) eh_n(\mathcal{F}).$$

(13)

Proof: Let $Y$ be the random variable counting a triple $(A, S, B)$ (on the random full chain) satisfying

$$A \subset S \subset B \quad A, B \in \mathcal{F}.$$

We have

$$E(Y) = \sum_{A, B \in \mathcal{F}, A \subset S \subset B} \frac{|A|(|S| - |A|)! (|B| - |S|)! (n - |B|)!}{n!}$$

$$= \sum_{A, B \in \mathcal{F}, A \subset B} \frac{|A|(|B| - |A|)! (n - |B|)!}{n!} \sum_{S: A \subset S \subset B} \frac{1}{(|B| - |A|) (|S| - |A|)}$$

$$= \sum_{A, B \in \mathcal{F}, A \subset B} \frac{|A|(|B| - |A|)! (n - |B|)!}{n!} (|B| - |A| - 1)$$

$$\geq \sum_{A, B \in \mathcal{F}, A \subset B, |B| - |A| > 1} \frac{|A|(|B| - |A|)! (n - |B|)!}{n!}.$$

(14)

Any poset $P$ of height 2 is a subposet of $K_{r,r}$ (the complete height-2-poset) for some $r$. Since $\mathcal{F}$ is $P$-free, there are no $2r$ subsets $A_1, A_2, \ldots, A_r, B_1, \ldots, B_r \in \mathcal{F}$ satisfying $A_i \subset S \subset B_j$ for $1 \leq i \leq r$ and $1 \leq j \leq r$.

For any fixed subset $S$, either “at most $r - 1$ subsets in $\mathcal{F}$ are supersets of $S$” or “at most $r - 1$ subsets in $\mathcal{F}$ are subsets of $S$”. Define

$$\mathcal{G}_1 = \{ S \mid |S| \in (\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n}), S \text{ has at most } r - 1 \text{ subsets in } \mathcal{F} \}.$$

$$\mathcal{G}_2 = \{ S \mid |S| \in (\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n}), S \text{ has at most } r - 1 \text{ supersets in } \mathcal{F} \}.$$
The union $G_1 \cup G_2$ covers all subsets with sizes in $(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$.

Rewrite $E(Y)$ as

$$E(Y) = \sum_{S: |S| - \frac{n}{2} < 2\sqrt{n \ln n}} \frac{1}{(\frac{n}{2})} \sum_{A \in F, A \subset S} 1 \sum_{B \in F, S \subset B} \frac{1}{(n - |S|)} \sum_{\substack{B \in F, S \subset B}} \frac{1}{(n - |B|)}.$$  \hspace{1cm} (15)

For $S \in G_1$, we have

$$\sum_{B \in F, S \subset B} \frac{1}{(n - |S|)} \leq \frac{r - 1}{n} = O(\frac{1}{n}).$$  \hspace{1cm} (16)

It implies

$$\sum_{S \in G_1} \frac{1}{(\frac{n}{2})} \sum_{A \in F, A \subset S} 1 \sum_{B \in F, S \subset B} \frac{1}{(n - |S|)} \leq \sum_{S \in G_1} \frac{1}{(\frac{n}{2})} \sum_{A \in F, A \subset S} 1 \sum_{B \in F, S \subset B} \frac{1}{(n - |S|)} = O\left(\frac{1}{n}\right).$$

Recall $E(X) = h_n(F) = \sum_{A \in F} \frac{1}{(|A|)}$ and $\sum_{S \in G_1} \frac{1}{(|S|)} \leq 4\sqrt{n \ln n}$. We have

$$\sum_{S \in G_1} \frac{1}{(|S|)} \sum_{A \in F, A \subset S} 1 \sum_{B \in F, S \subset B} \frac{1}{(n - |S|)} \leq O\left(\frac{\sqrt{\ln n}}{\sqrt{n}} E(X)\right).$$

Similarly, we have

$$\sum_{S \in G_2} \frac{1}{(|S|)} \sum_{A \in F, A \subset S} 1 \sum_{B \in F, S \subset B} \frac{1}{(n - |S|)} = O\left(\frac{\sqrt{\ln n}}{\sqrt{n}} E(X)\right).$$

Thus, we have

$$E(Y) = O\left(\frac{\sqrt{\ln n}}{\sqrt{n}} E(X)\right) = o(\epsilon E(X)).$$  \hspace{1cm} (17)

Combining inequalities (11), (14), (17), with equation (12), we have

$$\sum_{A, B \in F, |B| - |A| = 1} \frac{|A|!(|B| - |A|)! (n - |B|)!}{n!} = E\left(\frac{X}{2}\right) - E(Y) \geq (1 - o(1))\epsilon E(X).$$  \hspace{1cm} (18)
The proof of Lemma is finished. □

**Proof of Theorem 1.3** Now we assume that $P$ has a $k$-partite representation and $\mathcal{F}$ is a $P$-free $\mathcal{F}$ family of subsets of $[n]$ with the Lubell value $h_n(\mathcal{F}) = 1 + \epsilon$. We further assume that every subset in $\mathcal{F}$ has size in the interval $(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$. Let $X$ be the random variable counting the number of subsets of $\mathcal{F}$ hit by a random full chain. Note $E(X) = h_n(\mathcal{F})$. By Lemma 3.3, we have

$$\sum_{A, B \in \mathcal{F}, |B|-|A|=1} \frac{|A|!(|B|-|A|)!(n-|B|)!}{n!} \geq (1 - o(1))\epsilon E(X). \quad (19)$$

We define $N(B) = \{A \in \mathcal{F} \mid A \subset B, |A| = |B| - 1\}$ and $d(B) = |N(B)|$. We have

$$\sum_{B \in \mathcal{F}} \frac{1}{\binom{n}{|B|}} d(B) = \sum_{A, B \in \mathcal{F}, |B|=|A|} \frac{|A|!(|B|-|A|)!(n-|B|)!}{n!}. \quad (20)$$

Let $\bar{d} := \frac{1}{E(X)} \sum_{B \in \mathcal{F}} \frac{d(B)}{\binom{n}{|B|}}$ be the weighted average of $d(B)$. Since $|B| = (1 + o(1))\frac{n}{2}$ for any $B \in \mathcal{F}$, by equation (20) and inequality (19), we have

$$\bar{d} = \frac{1}{E(X)} \sum_{B \in \mathcal{F}} \frac{d(B)}{\binom{n}{|B|}}$$

$$= (1 + o(1)) \frac{n}{2E(X)} \sum_{B \in \mathcal{F}} \frac{d(B)}{|B|}$$

$$= (1 + o(1)) \frac{n}{2E(X)} \sum_{A, B \in \mathcal{F}, |B|=|A|} \frac{|A|!(|B|-|A|)!(n-|B|)!}{n!}$$

$$\geq (1 + o(1)) \frac{en}{2}.$$  

A pair of sets $(S, B)$ is said to form a $k$-configuration if

1. $S \subset B$, $|S| = |B| - k$, and $B \in \mathcal{F}$;
2. for any $x \in B \setminus S$, $B \setminus \{x\} \in \mathcal{F}$.

Since $|B| - \frac{n}{2} \leq 2\sqrt{n \ln n}$, $|S|$ belongs to the interval $J := (\frac{n}{2} - 2\sqrt{n \ln n} - k, \frac{n}{2} + 2\sqrt{n \ln n} - k)$. Set $\mathcal{S} := \cup_{s \in J} \binom{n}{s}$. For any $S \in \mathcal{S}$, let $L(S)$ be the
number of such configurations over a fixed set \( S \). We have

\[
\sum_{S \in S} \frac{L(S)}{\binom{n}{|S|}} = (1 + o(1)) \sum_{S \in S} \frac{L(S)}{\binom{n}{|S|+k}}
\]

\[
= (1 + o(1)) \sum_{B \in \mathcal{F}} \binom{n}{|B|} \left( \frac{d(B)}{k} \right)
\]

\[
\geq (1 + o(1)) \mathbb{E}(X) \left( \frac{d}{k} \right) \quad \text{by the convexity inequality (1)}
\]

\[
\geq (1 + o(1)) \frac{e^k}{2^k k!} n^k \mathbb{E}(X)
\]

\[
\geq \frac{e^k}{2^k k!} n^k.
\]

Partition \( J \) into small sub-intervals \( \{J_\lambda\}_{\lambda \in \Lambda} \) with equal length \( \sqrt{n \ln n} \). There are \( 4 \ln^{3/2} n \) of such sub-intervals. Setting \( S_\lambda := \bigcup_{s \in J_\lambda} \binom{n}{|s|} \), we have \( S = \bigcup_{\lambda \in \Lambda} S_\lambda \).

By an average argument, there is a \( \lambda_0 \in \Lambda \) so that

\[
\sum_{S \in S_\lambda_0} \frac{L(S)}{\binom{n}{|S|}} \geq \frac{1}{4 \ln^{3/2} n} \sum_{S \in S} \frac{L(S)}{\binom{n}{|S|}} \geq \frac{e^k}{2^{k+2} k! \ln^{3/2} n} \frac{n^k}{n}.
\] (21)

Suppose that \( \binom{n}{s} \) for \( s \in J_{\lambda_0} \) reaches the maximum at \( s = s_0 \). Note that \( |s - s_0| \leq \sqrt{n \ln n} \). By Lemma 3.2, we have

\[
\binom{n}{s} = (1 - o(1)) \binom{n}{s_0}.
\] (22)

Combining equations (21) and (22), we get

\[
\sum_{S \in S_{\lambda_0}} L(S) \geq (1 - o(1)) \frac{e^k}{2^{k+2} k! \ln^{3/2} n} \frac{n^k}{n} \binom{n}{s_0}.
\] (23)

Observe that there is a chain decomposition of \( S_{\lambda_0} \) into \( \binom{n}{s_0} \) chains. There exists one chain \( C \) satisfying

\[
\sum_{S \in C} L(S) \geq \frac{1}{\binom{n}{s_0}} \sum_{S \in S_{\lambda_0}} L(S) \geq (1 - o(1)) \frac{e^k}{2^{k+2} k! \ln^{3/2} n} \frac{n^k}{n}.
\] (24)
For any \( S \in \mathcal{C} \), we define a \( k \)-uniform hypergraph \( H_S \) on the vertex set \([n]\) as follows: a \( k \)-set \( F \) is an edge of \( H_S \) if \( S \cap F = \emptyset \) and \((S, S \cup F)\) forms a \( k \)-configuration.

Let \( \mathcal{P} \subset \binom{[l]}{k-1} \cup \binom{[l]}{k} \) be the \( k \)-representation of \( P \) and \( G(\mathcal{P}) \) be the \( k \)-uniform hypergraph associated with \( \mathcal{P} \). Since \( G(\mathcal{P}) \) is \( k \)-partite, there is a \( k \)-partition
\[
[l] = V_1 \cup V_2 \cdots \cup V_k
\]
such that all edges of \( G(\mathcal{P}) \) are crossing. For \( 1 \leq i \leq k \), set \( s_i := |V_i| \).

Clearly, we have \( G(\mathcal{P}) \subset K^{(k)}(s_1, \ldots, s_k) \).

**Claim b:** The hypergraph \( H_S \) contains no copies of \( K^{(k)}(s_1, \ldots, s_k) \) as a sub-hypergraph. Otherwise, \( H_S \) contains \( G(\mathcal{P}) \) as a subgraph. By the definition of \( H_S \), \( \{S \cup F\}_{F \in \mathcal{P}} \subset \mathcal{F} \). As a poset, \( \{S \cup F\}_{F \in \mathcal{P}} \subset \mathcal{F} \) is isomorphic to \( \mathcal{P} \). Thus, \( \mathcal{F} \) contains a subposet \( P \).

**Claim c:** For any \((k-1)\)-set \( T \), the number of edges of \( H_S \) (for \( S \in \mathcal{C} \)) containing \( T \) is at most \( r \). Otherwise, there exists a chain
\[
S_1 \subset S_2 \subset \cdots \subset S_r
\]
such that \( T \in E(H_{S_i}) \) for all \( 1 \leq i \leq r \). By the definition of \( H_{S_i} \), we have \( T \cup S_i \in \mathcal{F} \). Thus, \((S_1 \cup T), (S_2 \cup T), \ldots, (S_r \cup T)\) forms an \( r \)-chain in \( \mathcal{F} \). This chain contains the subposet \( P \). Contradiction.

By Claims (b) and (c), the collection \( \mathcal{H} := \{H_S\}_{S \in \mathcal{C}} \) satisfies the conditions of Lemma 2.3. Hence, the total number of edges in \( \mathcal{H} \) is \( O(n^{k-\delta}) \), where \( \delta = \left( \prod_{i=1}^{k-1} s_i \right)^{-1} \) is a positive constant. Note that an edge in \( H_S \) is 1-1 corresponding to a \( k \)-configuration \((S, B)\). Thus, we have
\[
\sum_{S \in \mathcal{C}} L(S) = O(n^{k-\delta}). \tag{25}
\]

Combining equations (24) with (25), we get
\[
\epsilon^k = O\left( \frac{\ln^{3/2} n}{n^\delta} \right) = o(1).
\]

This contradicts the assumption that \( \epsilon \) is a constant. The proof of the theorem is finished. \( \square \)
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