Interference Management in a One-dimensional Space: Interference Dissolution

Mohaned Chraiti, Ali Ghrayeb and Chadi Assi

Abstract

This paper proposes a novel technique for dealing with interference in a one-dimensional space over time-invariant channels. The key idea of the proposed technique, which we refer to as Interference Dissolution (ID), is that it precodes signals in such a way that interfering signals are dissolved in the intended ones. Consequently, the receiver will only see the intended signals, thereby allowing it to separate the intended signals from the interfering ones. The proposed technique offers several advantages over its counterparts that have been presented in the literature, including lower computational complexity, more flexibility, higher achievable rate and lower symbol error rate. We consider a point-to-point channel with a single-antenna receiver. We show that the proposed ID technique can achieve transmission of two (uncoded) symbols per channel use with non-zero degree-of-freedom (DoF), i.e., symbols are perfectly separable at the receiver. These results are obtained in the asymptotic sense, i.e., at high signal-to-noise ratio (SNR). To characterize the achievable rate of ID for all range of SNR, we show that, assuming a random Gaussian input, ID achieves a rate within one bit per channel use from capacity. These results are confirmed numerically when the input signals belong to discrete constellations.

Index Terms

Degrees-of-freedom, interference alignment, interference dissolution, time-invariant channels.

I. INTRODUCTION

Interference is an inherent phenomenon in wireless networks, resulting from the concurrent transmissions of many signals over the same communication channel. Interference can be avoided
by transmitting different independent signals over orthogonal channels in time, frequency and/or space domains which fall short of achieving high throughput as the number of users grows rapidly. Efficient interference management is thus crucial for an efficient use of resources and for achieving high throughput. Traditionally, interference is considered as either noise or decoded along the intended signal [1]. Interference may thus cause severe performance degradation especially when the powers of the interference and the intended signals are of similar strength.

Much work has been done to deal with interference, including the so-called interference alignment [2], which has opened the possibility of providing significantly better performance than what has been traditionally thought possible. The basic idea of interference alignment is to confine multiple interfering signals to a specific sub-space, while saving the other sub-spaces for communicating the intended signals. Such approach makes the impact of the interference less severe at the receiver and keeps intended signals completely free from interference. This technique is especially relevant for interference channels with more than two users. Indeed, for the case where only two users share the same medium, the interference at each receiver is generated by only one user and the maximal degree-of-freedom offered by the channel is equal to one [3]- [4]. Simple transmission schemes such as alternating access to the medium, in time, between the two users achieves the maximum DoF. However, such simple scheme is not efficient when the number of users is large.

The potential benefit of interference alignment was first highlighted by Cadambe and Jafar [2], by applying it to the fully connected single-antenna $K$-user interference channel. The authors showed that for this channel with time-varying or frequency selective channel coefficients, $K^2$ total DoF (i.e., full DoF) are achievable, i.e., every user gets $\frac{1}{2}$ DoF. These results are encouraging in the sense that they imply that the single-antenna $K$-user Gaussian interference channel is not inherently interference limited for the case of time/frequency-varying channel. However, the above result requires unbounded channel diversity and it is unclear what the implication is for real systems with finite channel diversity [5]. The feasibility of interference alignment for more practical cases such as the time-invariant multiple-input multiple-output (MIMO) interference channel is studied in [5], where only a finite space diversity is available (no time or frequency diversity). The authors showed that the total achievable DoF, for a $K$-user interference channel, is at most two for an arbitrary number of users. This is in sharp contrast to the $\frac{K^2}{2}$ DoF result shown to be possible in [2] considering unbounded time or frequency diversity. Interference
management is far more challenging for time-invariant channels with finite diversity, when there is no time or frequency diversity.

Until recently, the proposed interference techniques, for multiple-user channels, manage interference by keeping some sub-spaces free from interference in conjunction with using successive decoding. Since it is believed that non free-interference sub-spaces cannot secure non zero DoF for communicating intended signals, the achievable DoF is roughly limited to the achievable DoF over the free-interference sub-spaces. These are due to the fact that these techniques are unable to efficiently separate the intended signals from the interfering ones in a one-dimensional space. Therefore, they are still far from achieving, for several time-invariant interference channels, the maximal performance offered by these channels, which motivates the work in this paper.

It is commonly believed that time-invariant channels are restrictive, when only a one-dimensional space is available. This perception stems from the fact that such channels are unable to incorporate vectorial interference management. To this end, some efforts have been made to show that interference alignment in a one-dimensional space is possible when signals belong to integer lattices (lattice constellations) [6]. In [7] and [8], the interfering signals from several users are aligned in such a way that they are confined to one lattice. However, these techniques essentially rely on a judicious choice of power allocation such that high or low signal to interference ratio conditions must hold, in conjunction with successive decoding.

The applicability of interference management in a one-dimensional space was first presented in [9] and [10] where no conditions on the interference power must be satisfied. They showed that intended and interfering signals can be jointly decoded if they are transmitted over rationally independent channels when the transmitted signals belong to a discrete constellation. They used the Diophantine approximation in number theory to prove the possibility of communicating intended signals, in the presence of interference, with non-zero DoF. Specifically, it was shown in [10] that if at each receiver, the channel gains corresponding to the interferers are rational, whereas the direct channel gains corresponding to the intended signal are irrational, \( \frac{K}{2} \) DoFs are achievable. In [11]- [12] more general results about this technique, referred to as lattice interference alignment, are presented. In [11], the authors showed that it is possible to jointly decode intended and interfering signals for most channel realizations and hence they proved that the interfering signals are naturally aligned by the channel. However, the DoF results in [11] and [13] are derived in the asymptotic sense, i.e., at high signal-to-noise ratio (SNR), which
naturally raises a concern about the performance of lattice interference alignment at finite SNR. In [13], the authors consider a $K$-user interference channel, where all interference (cross) channel gains are integers, and derived the achievable rate for finite SNR. Lattice interference alignment was extended in [13] in the context of secrecy communication over a $K$-user Gaussian wiretap channel in the presence of $N$ eavesdroppers. The authors showed that $\frac{1}{2} - \frac{1}{2K}$ secure DoFs are achieved for almost all transmitter-receiver pairs.

The lattice interference alignment techniques allowed for important progress towards the feasibility of interference management in a one-dimensional space for time-invariant channels. When very strong or weak interference conditions are not satisfied, lattice interference alignment techniques offer to the decoder the possibility of jointly decoding the interfering and intended information symbols. However, the applicability of these techniques is very sensitive to the ratio of each intended signal channel gain and the remainder signals channel gains. Two signals with the same associated channel gains are still inseparable at the receiver. Moreover, joint decoding implies that the receiver must simultaneously decode all signals that have different associated channel gains, including interfering signals. This leads to an exponential increase in the size of the codebook, which in turn increases the decoding computational complexity. To elaborate, consider the case in which the receiver observes a linear combination of $K$ signals belonging to the same constellation points $\mathcal{M}$. When the channel gains are different from each other, the size of the used codebook is thus equal to $|\mathcal{M}|^K$. Furthermore, the receiver requires knowledge of all channel gains and constellations of the interfering signals.

In this paper, we introduce a novel interference management approach in a one-dimensional space, which we refer to as interference dissolution (ID). The key idea is to precode signals in such a way that the interfering signals are forced to yield their transmit power to the intended signals, i.e., the interfering signals dissolve in the intended ones. As such, the receiver sees only the intended signals. The proposed approach essentially transforms the interference problem to an unknown channel problem. Using ID, the receiver can extract the intended signals without any need to extract the interfering (extra) signals, i.e., a joint decoding is not needed. In contrast with the schemes presented in [9]–[11], ID allows for the possibility of transmitting several signals with the same channel gains while keeping them separable at the receiver. Furthermore, ID offers lower computational complexity in comparison with the schemes presented in [9]–[11] by reducing the codebook size to $|\mathcal{M}|^2$. Another advantage of ID is that the receiver does not
require any information about the extra signals to decode the intended ones.

Similar to the previous interference management techniques in a one-dimensional space, the proposed ID technique has several applications including the X channel, the many-to-one channel, and the wiretap channel, among others. However, for the sake of clarity in presenting the proposed technique, we adopt in this paper a simple time-invariant channel with a one-dimensional space. In particular, we consider a real time-invariant point-to-point channel with a single-antenna receiver, implying that there is one DoF available. The transmitter wishes to communicate several independent information symbols per channel use (symbol period/time slot), while keeping them separable at the receiver. We stress here that the number of channel uses (or time slots) is strictly less than the number of independent symbols to be decoded, implying that the signal combinations are not enough to linearly resolve all the information symbols. The only restriction that ID has, however, is that the channel gains associated to two (intended or interfering) signals must be different.

In light of the above discussion, we may summarize the contributions of the paper as follows:

– We introduce the interference dissolution concept as a more flexible alternative to managing interference in a one-dimensional space over time-invariant channels.

– We show the capability of ID to extract the intended signals without decoding the interfering ones.

– We show that ID allows for the transmission of $\frac{K}{\left|\frac{K}{2}\right|+1} \rightarrow \frac{1}{2}$ independent symbols per channel use while keeping them separable at the receiver, where $K$ is the total number of transmitted signals.

– We derive the (asymptotic) optimal decoder for the proposed ID approach.

– We derive the achievable DoF and show that each signal achieves $\frac{\left|\frac{K}{2}\right|}{2\left|\frac{K}{2}\right|+1} \rightarrow \frac{1}{2}$ DoF suggesting that the two signals are separable at the receiver, i.e., ID manages interference in a one-dimensional space.

– We characterize the achievable rate of ID and show that it achieves near capacity performance.

The rest of the paper is organized as follows. Section II introduces the proposed ID technique and the corresponding optimal decoder. In Section III, the achievable DoF is analyzed and the achievable rate is derived. Simulation results are presented in Section VI. Concluding remarks are given in Section V.
Throughout the rest of the paper, we will use $\| \cdot \|$, $(\cdot)^T$ and $\langle \cdot, \cdot \rangle$ to denote the 2-norm, the transpose operators and the inner product two vectors.

II. INTERFERENCE DISSOLUTION (ID)

In this section, we illustrate the idea of the proposed ID considering a real time-invariant point-to-point channel with a single-antenna receiver. We consider signals belonging to real discrete constellations. We first present ID including its precoding and decoding. We then prove the optimality of the proposed decoder.

A. ID Precoding and Decoding

Let us assume that the transmitter wishes to communicate several independent signals per channel use with the receiver. The power of the transmitted signals are assumed to be the same. In the first channel use, the transmitter sends two intended signals $(s_1, s_2)$ and $K-2$ extra signals $(e_1, e_2, \ldots, e_{K-2})$. It should be noted that the extra signals are also intended by the receiver in the event that the receiver also wishes to decode them. The receiver receives a linear combination of the $K$ signals, that is,

$$y_1 = h_1 s_1 + h_2 s_2 + \sum_{k=1}^{K-2} g_k e_k + n_1,$$

where $\{h_1, h_2, g_1, \ldots, g_{K-2}\}$ are real channel gains, and $n_1$ is an additive white Gaussian noise (AWGN) with zero mean and variance $\sigma^2$.

Signals are drawn from a real discrete constellations with equal probability. Especially, the signals sets are constructed by pulse amplitude modulation (PAM) [14]. The intended and the extra signals may belong to two different real discrete constellations denoted by $\mathcal{M}_s = A_s\mathbb{Z}_{Q_s}^* = \{-A_s Q_s, \ldots, A_s(Q_s-1), A_s Q_s\}$ and $\mathcal{M}_e = A_e\mathbb{Z}_{Q_e}^* = \{-A_e Q_e, \ldots, A_e(Q_e-1), A_e Q_e\}$, respectively. $A_s$ and $A_e$ are two real constants controlling the minimum distance between the signals belonging to the same constellation points. Each transmitted signal is subject to a power constraint: $\forall i \in \{s, e\}, \sum_{l=-Q_i}^{Q_i} A_i l^2 \leq P$.

We first assume that the receiver is interested in decoding the intended signals only. As such, the extra signals are seen as interference. We show the capability of the proposed ID technique to extract the intended signals in the presence of interference. At the end of this subsection, we extend the proposed technique to extract all $K$ signals.
Signals Precoding

We assume that the channel state information (CSI) is available at the transmitter. However, the receiver only requires knowledge of the CSI pertaining the signals that will be decoded. Let us now assume that the receiver wishes to decode the intended signals. If the transmitter send two other linearly independent combinations of $s_1$, $s_2$ and $\sum_{k=1}^{K-2} g_k e_k$ to the receiver, it will be able to linearly solve the intended information symbols $s_1$ and $s_2$. However, when the transmitter sends only one more combination, linearly solving both intended signals is not possible. To deal with this, an interference management technique in a one-dimensional space is needed, otherwise successive decoding maybe used but this offer poor performance. Using the proposed ID technique, just an additional channel use is needed to perfectly extract the two desired signals. The idea of ID is to construct signals such that the extra signals dissolve into one of the intended signals $s_1$ or $s_2$ and the receiver sees just the desired signals. The transmitter reproduces the noiseless part of $y_1$ and then it calculates a dissolution factor $\beta$ by solving:

$$h_1 s_1 + \beta h_2 s_2 = h_1 s_1 + h_2 s_2 + \sum_{k=1}^{K-2} g_k e_k,$$

then $\beta$ is equal to

$$\beta = 1 + \frac{\sum_{k=1}^{K-2} g_k e_k}{h_2 s_2}.$$  

The signal received in the first channel use can be written as

$$y_1 = h_1 s_1 + \beta h_2 s_2 + n_1$$

which is equivalent to receiving $s_2$ via $\beta h_2$ which is unknown to the receiver. In the second channel use, the transmitter sends $-\beta s_1$ via $h_1$ and $s_2$ via $h_2$. The received signals vector can be written as

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_1 s_1 \\ h_2 s_2 \end{pmatrix} + \beta \begin{pmatrix} h_2 s_2 \\ -h_1 s_1 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

$$= \mathbf{v}(s_1, s_2) + \beta \mathbf{v}^\perp(s_1, s_2) + \mathbf{n},$$

where $n_2$ is AWGN with zero mean and variance $\sigma^2$. $\mathbf{n}$ is the AWGN vector $(n_1, n_2)^T$. $\mathbf{v}(s_1, s_2)$ and $\mathbf{v}^\perp(s_1, s_2)$ denote the vector $(h_1 s_1, h_2 s_2)^T$ and its orthogonal vector $(h_2 s_2, -h_1 s_1)^T$. The received signals vector is equivalent to transmitting $(s_1, s_2)$ via $(h_1, h_2)$ and $(s_2, -s_1)$ via $(\beta h_2, \beta h_1)$ which is unknown to the receiver. Therefore, the interference problem is transformed to an unknown channel problem.
**Intended Signals Decoding**

The receiver starts by building the set of all possible pairs of intended signals, namely,

$$S = \left\{ \mathbf{v}(\tilde{s}_1, \tilde{s}_2) = \begin{pmatrix} h_1\tilde{s}_1 \\ h_2\tilde{s}_2 \end{pmatrix} : (\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_s^2 \right\}.$$  

The codebook size is thus equal to $|\mathcal{M}_s|^2$ which is independent from the number of interfering signals, unlike the case in the schemes of [9]–[11]. Then, it provides, for each vector $\mathbf{v}(\tilde{s}_1, \tilde{s}_2) \in S$, the decision weight component:

$$w(\tilde{s}_1, \tilde{s}_2) = \frac{|\langle \mathbf{y} - \mathbf{v}(\tilde{s}_1, \tilde{s}_2), \mathbf{v}(\tilde{s}_1, \tilde{s}_2) \rangle|}{\|\mathbf{v}(\tilde{s}_1, \tilde{s}_2)\|}.$$  

(5)

It is clear that the noiseless part of the weight component $w(\tilde{s}_1, \tilde{s}_2)$ is equal to zero when $(\tilde{s}_1, \tilde{s}_2) = (s_1, s_2)$

$$w(s_1, s_2) = \frac{|\langle \beta \mathbf{v}^+(s_1, s_2), \mathbf{v}(\tilde{s}_1, \tilde{s}_2) \rangle|}{\|\mathbf{v}(\tilde{s}_1, \tilde{s}_2)\|} = 0.$$  

(6)

Otherwise, it takes a non zero value for almost all the channel realizations (see Lemma 1 in Subsection III-B and its proof in Appendix A). The decision rule consists therefore of choosing the signals vector $(\tilde{s}_1, \tilde{s}_2)$ that minimizes the weight component

$$(\tilde{s}_1, \tilde{s}_2) = \arg\min_{(\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_s^2} w(\tilde{s}_1, \tilde{s}_2).$$  

(7)

The optimality of this decision rule is proved in Subsection II-B. It should be noted that the receiver is able to decode the intended information symbols without decoding the extra ones and without any knowledge about them or their associated channel.

**Extra Signals Decoding**

In this part, we consider the case when the receiver also wishes to decode the extra signals. Obviously, CSI of the signals to be decoded is known to the receiver. The receiver considers $(e_1, e_2)$ as intended and the rest of the signals as interference. It proceeds as explained in the
previous part. It uses again the noiseless part of $y_1$, but this time to dissolve $h_1s_1 + h_2s_2 + \sum_{k=3}^{K} g_k e_k$ in $g_2 e_2$ by calculating the dissolution factor

$$\beta_2 = 1 + \frac{h_1s_1 + h_2s_2 + \sum_{k=3}^{K-2} g_k e_k}{g_2 e_2}.$$ 

In the third channel use, the transmitter sends the precoded signals. The receiver uses the signals vector $(y_1, y_3)$

$$
\begin{pmatrix}
  y_1 \\
  y_3
\end{pmatrix}
= \begin{pmatrix}
  g_1 e_1 \\
  g_2 e_2
\end{pmatrix} + \beta_2 \begin{pmatrix}
  g_2 e_2 \\
  -g_1 e_1
\end{pmatrix} + \begin{pmatrix}
  n_1 \\
  n_3
\end{pmatrix}$$

$$= v(e_1, e_2) + \beta_1 v^\perp(e_1, e_2) + n_2,$$

to decode the second signals pair, i.e., the first extra signals pair. $n_2$ is the AWGN vector $(n_1, n_3)$.

In the $(m + 2)^{th}$ channel use, the $(m)^{th}$ extra signals pair, i.e., $(m + 1)^{th}$ signals pair, $(1 \leq m \leq \lceil \frac{K}{2} \rceil - 1)$ is precoded then sent. The dissolution factor is $\beta_m$. The receiver uses the signals vector,

$$
\begin{pmatrix}
  y_1 \\
  y_{m+2}
\end{pmatrix}
= \begin{pmatrix}
  g_{2m-1} e_{2m-1} \\
  g_{2m} e_{2m}
\end{pmatrix} + \beta_m \begin{pmatrix}
  g_{2m} e_{2m} \\
  -g_{2m-1} e_{2m-1}
\end{pmatrix} + \begin{pmatrix}
  n_1 \\
  n_{m+2}
\end{pmatrix},$$

(9)

to decode the $(m - 1)^{th}$ extra signals pair.

After decoding the $\lceil \frac{K}{2} \rceil - 1$ signals pairs, in $\lceil \frac{K}{2} \rceil$ channel uses, it still needs to decode two extra signals when $K$ is even. If $K$ is odd, an extra signal is chosen to form a pair with the $K^{th}$ extra signal. Otherwise, the interference dissolution is applied for the $(\frac{K}{2})^{th}$ signals pair. Interference dissolution allows thus the transmission of $\frac{K}{\lceil \frac{K}{2} \rceil + 1} \to 2$ independent information symbols per channel use while keeping them separable at the receiver.

B. Interference dissolution decision rule analysis

The intended signals are assumed equally likely over the discrete constellation $\mathcal{M}_s$. Hence, the optimal detection rule is the Maximum Likelihood (ML). That is,

$$\hat{(s_1, s_2)} = \arg\max_{(\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_s^2} p(y|\tilde{s}_1, \tilde{s}_2)$$

$$= \arg\min_{(\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_s^2} \sigma^2(y - E[y])^T C_{\tilde{s}_1, \tilde{s}_2}^{-1}(y - E[y]),$$

where $C_{\tilde{s}_1, \tilde{s}_2}$ is the covariance matrix of the received signal $y$ given that $(\tilde{s}_1, \tilde{s}_2)$ was transmitted. Next, we use $\eta^2$ to denote the variance of the variable $\beta$ given $(\tilde{s}_1, \tilde{s}_2)$ was transmitted.
\[ \eta^2 = E[\beta^2] - (E[\beta])^2 \]
\[ = E \left[ \left( 1 + \frac{\sum_{k=1}^{K} g_k e_k}{h_2 s_2} \right)^2 \right] - \left( E \left[ 1 + \frac{\sum_{k=1}^{K} g_k e_k}{h_2 s_2} \right] \right)^2 \]  
\[ = P \frac{\sum_{k=1}^{K} (g_k)^2}{(h_2 s_2)^2}. \]  
(11)

The inverse of the covariance matrix is written as
\[ C^{-1}_{\tilde{s}_1, \tilde{s}_2} = \left( E \left[ \left( y - E[y] \right) \left( y - E[y] \right)^T \right] \right)^{-1} \]
\[ = \begin{pmatrix}
\eta^2 h_2^2 \tilde{s}_2^2 + \sigma^2 & -\eta^2 h_1 h_2 \tilde{s}_1 \tilde{s}_2 \\
-\eta^2 h_1 h_2 \tilde{s}_1 \tilde{s}_2 & \eta^2 h_1^2 \tilde{s}_1^2 + \sigma^2
\end{pmatrix}^{-1} \]
\[ = \begin{pmatrix}
\eta^2 h_2^2 \tilde{s}_1^2 + \sigma^2 & \eta^2 h_1 h_2 \tilde{s}_1 \tilde{s}_2 \\
\eta^2 h_1 h_2 \tilde{s}_1 \tilde{s}_2 & \eta^2 h_2^2 \tilde{s}_2^2 + \sigma^2
\end{pmatrix} \]
\[ = \frac{\sigma^2 (\eta^2 \| v(\tilde{s}_1, \tilde{s}_2) \|^2 + \sigma^2)}{\sigma^2 (\eta^2 \| v(\tilde{s}_1, \tilde{s}_2) \|^2 + \sigma^2)}. \]  
(12)

Let \((d_1, d_2)^T\) denote the vector \((y - v(\tilde{s}_1, \tilde{s}_2))\). In the following, we derive the explicit formula of the ML decision function in (10) to show that it is equal to the proposed weight component \(w^2(\tilde{s}_1, \tilde{s}_2)\) and hence prove the optimality of the decoder. Considering the formula of the covariance matrix, the ML decision function can be written as
\[
\frac{\eta^2 \langle y - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2}{\eta^2 \|v(\tilde{s}_1, \tilde{s}_2)\|^2 + \sigma^2} = \frac{\eta^2 (d_1 h_1 \tilde{s}_1 + d_2 h_2 \tilde{s}_2)^2 + \sigma^2 (d_1^2 + d_2^2)}{\eta^2 \|v(\tilde{s}_1, \tilde{s}_2)\|^2 + \sigma^2}
\]

\[
= \frac{\eta^2 \left( \|v(\tilde{s}_1, \tilde{s}_2)\|^2 (d_1^2 + d_2^2) - (d_1 h_2 \tilde{s}_2 - d_2 h_1 \tilde{s}_1)^2 \right) + \sigma^2 (d_1^2 + d_2^2)}{\eta^2 \|v(\tilde{s}_1, \tilde{s}_2)\|^2 + \sigma^2}
\]

\[
= (d_1^2 + d_2^2) \left( \|v(\tilde{s}_1, \tilde{s}_2)\|^2 + \sigma^2 \right) - \eta^2 (d_1 h_2 \tilde{s}_2 - d_2 h_1 \tilde{s}_1)^2
\]

\[
\frac{\eta^2 \|v(\tilde{s}_1, \tilde{s}_2)\|^2}{\eta^2 \|v(\tilde{s}_1, \tilde{s}_2)\|^2 + \sigma^2}
\]

\[
= d_1^2 + d_2^2 - \frac{\eta^2 (d_1 h_2 \tilde{s}_2 - d_2 h_1 \tilde{s}_1)^2}{\eta^2 \|v(\tilde{s}_1, \tilde{s}_2)\|^2 + \sigma^2}
\]

\[
\approx d_1^2 + d_2^2 - \frac{\eta^2 (d_1 h_2 \tilde{s}_2 - d_2 h_1 \tilde{s}_1)^2}{\eta^2 \|v(\tilde{s}_1, \tilde{s}_2)\|^2}
\]

\[
= \frac{(d_1^2 + d_2^2) \|V(\tilde{s}_1, \tilde{s}_2)\|^2 - (d_1 h_2 \tilde{s}_2 - d_2 h_1 \tilde{s}_1)^2}{\|V(\tilde{s}_1, \tilde{s}_2)\|^2}
\]

\[
= \frac{(d_1 h_1 \tilde{s}_1 + d_2 h_2 \tilde{s}_2)^2}{\|v(\tilde{s}_1, \tilde{s}_2)\|^2}
\]

\[
= \frac{\langle y - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2}{\||v(\tilde{s}_1, \tilde{s}_2)\|^2}
\]

The equality (a) comes from the assumption of the high SNR \( \eta^2 v(\tilde{s}_1, \tilde{s}_2)^2 + \sigma^2 \approx \eta^2 v(\tilde{s}_1, \tilde{s}_2)^2 \). The ML detector in (10) hence becomes

\[
(\tilde{s}_1, \tilde{s}_2) = \arg\min_{(\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_d^2} \frac{\langle y - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2}{\|v(\tilde{s}_1, \tilde{s}_2)\|^2}
\]

\[
= \arg\min_{(\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_d^2} w^2(\tilde{s}_1, \tilde{s}_2)
\]

\[
= \arg\min_{(\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_d^2} w(\tilde{s}_1, \tilde{s}_2).
\]

This proves the optimality of the proposed decoder, defined in (7), at high SNR.

### III. ON THE PERFORMANCE OF ID

Although the ID technique does not require asymptotic conditions, as shown in the previous section, we first analyze the performance of this technique in terms of DoF, when the signals belong to real discrete constellations. We prove that ID allows the transmission of two signals simultaneously with non zero DoF and hence the proof of the capability of ID to keep signals...
separable at the receiver, i.e., it manages interference in a one-dimensional space. We provide an explicit formula for a lower bound on the error probability. Moreover, we provide a lower bound on the DoF associated with each signal. However, the explicit lower bound on the ID achievable rate is intractable at finite SNR. To this end, we derive the explicit ID achievable rate, for the entire range of SNR, assuming a continuous Gaussian input. This provides a benchmark on the ID achievable rate.

A. Error Probability Performance

We proved above the optimality of the decoder proposed in (7) and showed that the decoder chooses the signal vector minimizing the decision weight component

$$w(\tilde{s}_1, \tilde{s}_2) = \frac{\langle y - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2}{\|v(\tilde{s}_1, \tilde{s}_2)\|^2}. $$

The receiver correctly decodes the intended signal vector $v(s_1, s_2)$ only if

$$w(s_1, s_2) < w(\tilde{s}_1, \tilde{s}_2) \quad \forall (\tilde{s}_1, \tilde{s}_2) \in \{M^2 : (\tilde{s}_1, \tilde{s}_2) \neq (s_1, s_2)\}.$$ For notational convenience, we use $y(\beta, s_1, s_2) = v(s_1, s_2) + \beta v^\perp(s_1, s_2)$ to refer to the noiseless part of the signal $y$ in (4). We have

$$w(s_1, s_2) = \frac{\langle y - v(s_1, s_2), v(s_1, s_2) \rangle^2}{\|v(s_1, s_2)\|^2} = \frac{\langle n, v(s_1, s_2) \rangle^2}{\|v(s_1, s_2)\|^2},$$

and

$$w(\tilde{s}_1, \tilde{s}_2) = \frac{\langle y - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2}{\|v(\tilde{s}_1, \tilde{s}_2)\|^2} = \frac{\langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2 + \langle n, v(\tilde{s}_1, \tilde{s}_2) \rangle^2 + 2\langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle \langle n, v(\tilde{s}_1, \tilde{s}_2) \rangle}{\|v(\tilde{s}_1, \tilde{s}_2)\|^2}. $$

At high SNR, the term

$$\frac{\langle n, v(\tilde{s}_1, \tilde{s}_2) \rangle^2}{\|v(\tilde{s}_1, \tilde{s}_2)\|^2} - \frac{\langle n, v(s_1, s_2) \rangle^2}{\|v(s_1, s_2)\|^2}$$

is negligible [15, 6.1.7] compared to the term

$$2\langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle \langle n, v(\tilde{s}_1, \tilde{s}_2) \rangle \|v(\tilde{s}_1, \tilde{s}_2)\|^2.$$
The inequality \( w(s_1, s_2) < w(\tilde{s}_1, \tilde{s}_2) \) then gives
\[
0 < 2\langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle / \|v(\tilde{s}_1, \tilde{s}_2)\|^2 + \langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2 / \|v(\tilde{s}_1, \tilde{s}_2)\|^2
\]
\[
\Rightarrow 2\langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle / \|v(\tilde{s}_1, \tilde{s}_2)\|^2 > -\langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle^2.
\] (17)

Given that \( n \) is an AWGN vector, we can obtain a lower bound on the probability of correct decision as follows
\[
P_e \geq Q\left( -d_{\text{min}} / 2\sigma \right) \right),
\]
where \( Q(\cdot) \) denotes the Q-function and
\[
d_{\text{min}} = \min_{(\tilde{s}_1, \tilde{s}_2) \neq (s_1, s_2)} \|\langle y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2), v(\tilde{s}_1, \tilde{s}_2) \rangle / \|v(\tilde{s}_1, \tilde{s}_2)\| \|.
\]

We now use the lower bound in (18) to provide an upper bound on the error probability, \( P_e \)
\[
P_e = 1 - P_c
\]
\[
\leq 1 - Q\left( -d_{\text{min}} / 2\sigma \right)
\]
\[
= Q\left( d_{\text{min}} / 2\sigma \right)
\]
\[
\leq \exp\left( -d_{\text{min}}^2 / 8\sigma^2 \right).
\] (19)

B. The Achievable DoF

We provide here a lower bound on the achievable DoF associated with the intended signals per channel use. The achievable DoF represents the rate of growth of the achievable rate with \( \frac{1}{2} \log_2(P) \) (\( \log_2(P) \) for the complex case) when \( P \) tends to infinity. The \( \frac{1}{2} \log_2(P) \) follows from the well known formula of the capacity for the point-to-point AWGN, namely \( \frac{1}{2} \log_2(1 + \text{SNR}) \), when the SNR tends to infinity. For the rest of this paper, without loss of generality, we assume that the intended signals are transmitted over the same channel \( h = h_1 = h_2 \) in order to simplify the analysis. The maximum achievable rate for \( s_1 \) and \( s_2 \) is written as
\[ R(s_1, s_2) = I(s_1, s_2; y) \]
\[ = H(s_1, s_2) - H(s_1, s_2 | y) \]
\[ = H(s_1) + H(s_2) - H(s_1, s_2 | y) \]
\[ \geq (1 - P_e) 2 \log_2 (2Q_s) - H(P_e), \]

where \( I(.) \) and \( H(.) \) denote respectively the mutual information and the entropy. The inequality (b) follows from the Fano’s inequality
\[ H(s_1, s_2 | y) \leq H(P_e) + P_e H(s_1, s_2). \]

In (20), we derived a lower bound on the achievable rate as a function of \( P_e \) which is given as function of \( d_{\text{min}}^2 \) in (19). It remains to provide an explicit lower bound on \( d_{\text{min}}^2 \) in order to derive an explicit lower bound on the achievable rate. In the following Lemma, we provide an explicit expression for the lower bound on \( d_{\text{min}}^2 \).

**Lemma 1.** For almost all channel realizations, there exists a real constant \( K \in \mathbb{R}^{+} \) such that
\[ d_{\text{min}}^2 \geq \frac{\kappa^2 A_s^2 K^2}{Q_s^2}. \]

**Proof:** See Appendix A. □

The minimum value of the noiseless part of \( w(\tilde{s}_1, \tilde{s}_2) \), over \( \{ (\tilde{s}_1, \tilde{s}_2) \in \mathcal{M}_s^2 : (\tilde{s}_1, \tilde{s}_2) \neq (s_1, s_2) \} \) equals to \( d_{\text{min}} \) which is strictly greater than zero for almost all the channel realizations. This proves that the noiseless part of \( w(\tilde{s}_1, \tilde{s}_2) \) gives zero only if \( (\tilde{s}_1, \tilde{s}_2) = (s_1, s_2) \).

Each transmitted signal from \( \{ s_1, s_2 \} \) is subject to a power constraint \( P \). When \( P \) is relatively large, it can be expressed as
Lemma 1, (19) and (22) give an upper bound on the error probability as follows.

\[ P_e \leq \exp \left( -\frac{3h^2K^2P}{8\sigma^2Q_s^4} \right). \]  

(23)

Let \( \epsilon \) be an arbitrarily small constant such that \( \epsilon > 0 \). In order to get small error probability as desired at high transmit power, the transmit constellation size may be selected as follows.

\[ Q_s^4 = P^{1-\epsilon} \Rightarrow Q = P^{1-\epsilon}. \]  

(24)

Now, we use equations (20), (23) and (24) to give a lower bound on the achievable data rate. That is,

\[ R_{s_1, s_2} \geq \left( 1 - \exp \left( -\frac{3h^2K^2P^\epsilon}{8\sigma^2} \right) \right) \times 2 \log_2 \left( 2P^{\frac{1+\epsilon}{4}} \right) - H \left( \exp \left( -\frac{3h^2K^2P^\epsilon}{8\sigma^2} \right) \right). \]  

(25)

Recall that the achievable DoF represents the rate of growth of the achievable rate with \( \frac{1}{2} \log_2(P) \) (log_2(P)) for the complex case when \( P \) tends to infinity. Therefore, the DoF associated with \( s_1 \) and \( s_2 \), in two channel uses, can be provided by using (25) as follows.

\[ \lim_{P \to \infty} \frac{R(s_1, s_2)}{\frac{1}{2} \log_2(P)} \geq \lim_{P \to \infty} \frac{2 \log_2 \left( 2P^{\frac{1+\epsilon}{4}} \right)}{\frac{1}{2} \log_2(P)} + \frac{H(0)}{\frac{1}{2} \log_2(P)} \]

\[ = \lim_{P \to \infty} \frac{4(1 - \epsilon) \log_2(P)}{4 \log_2(P)} = 1 - \epsilon. \]  

(26)

Since \( \epsilon \) is arbitrarily small, we can conclude that the intended signals get at least one DoF in two channel uses. The same holds true for the remaining \( K - 2 \) (extra) signals. Since the \( K \) signals are decoded in \( \left\lfloor \frac{K}{2} \right\rfloor + 1 \) channel uses, the proposed ID achieves \( \left\lfloor \frac{K}{2} \right\rfloor + 1 \) DoF, i.e., full DoF.
when $K$ is large. Hence, ID is capacity achieving at high SNR. In Section II, we showed that two independent signals can be transmitted per channel use when $K$ is large. This suggests that the proposed ID technique allows for transmitting up to two independent signals per channel use and each signal gets $\frac{1}{2}$ (i.e., non zero) DoF. This proves the capability of ID to keep signals separable at the receiver and hence its ability of achieving interference-free transmission in a one-dimensional space.

C. Achievable Rate

As shown above, the proposed ID technique allows for the transmission of two symbols per channel use, i.e., it manages interference in a one-dimensional space, where the transmitted signal belong to discrete constellations, suggesting that ID is asymptotically capacity achieving. It is of interest to study the achievable rate of the proposed ID technique for the entire range of SNR. For the discrete input, the explicit formula of the lower bound of the achievable rate, using the Fano’s inequality, is intractable at finite SNR. Therefore, in this subsection, we assume a real Gaussian input and derive an explicit expression for the achievable rate accordingly. Even if the achievable rate considering real Gaussian input provides an upper bound on the one considering the discrete input, it gives a benchmark for the performance of ID at finite SNR. The upper bound and lower bound, given by simulations in Section IV together characterize the achievable rate at finite SNR for the discrete constellation case.

We first start with the first two signals of interest, namely, $s_1$ and $s_2$ defined by (4). We then generalize the result to the remaining $K-2$ signals. For a given channel realization, the achievable rate for the intended signals is written as

$$R(s_1, s_2) = I(s_1, s_2; y_1, y_2)$$
$$= H(y_1, y_2) - H(y_1, y_2|s_1, s_2)$$
$$= \frac{1}{2}E \left[ \log_2 \left( \det(C(y_1, y_2)) \right) \right] - \frac{1}{2}E \left[ \log_2 \left( \det(C(y_1, y_2|s_1, s_2)) \right) \right],$$

where $C(y_1, y_2)$ the covariance of $(y_1, y_2)$. $C(y_1, y_2|s_1, s_2)$ denotes the covariance of $(y_1, y_2)$ given $(s_1, s_2)$. The explicit formulas are written as follows:
Consequently, the achievable rate in (27) becomes

\[ C(y_1, y_2) = \begin{pmatrix} E[y_1^2] & E[y_1 y_2] \\ E[y_1 y_2] & E[y_2^2] \end{pmatrix} \]

\[ = P \begin{pmatrix} 2|h|^2 + \sum_{k=1}^{K-2} |g_k|^2 & 0 \\ 0 & 2|h|^2 + \sum_{k=1}^{K-2} |g_k|^2 \end{pmatrix} + \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}. \tag{28} \]

\[ C(y_1, y_2|s_1, s_2) = \begin{pmatrix} E[y_1^2] & E[y_1 y_2] \\ E[y_1 y_2] & E[y_2^2] \end{pmatrix} \]

\[ = P \begin{pmatrix} \sum_{k=1}^{K-2} |g_k|^2 & -\sum_{k=1}^{K-2} |g_k|^2 s_1 s_2 \\ -\sum_{k=1}^{K-2} |g_k|^2 s_1 s_2 & \sum_{k=1}^{K-2} |g_k|^2 s_2^2 \end{pmatrix} + \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}. \tag{29} \]

Using (28) and (29), we can expand the two terms in (27), respectively, as follows.

\[ \log_2 (\det(C(y_1, y_2))) = \log_2 \left( 2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2 \right)^2 \]

\[ = 2 \log_2 \left( 2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2 \right). \tag{30} \]

\[ E \left[ \log_2 (\det(C(y_1, y_2|s_1, s_2))) \right] = E \left[ \log_2 \left( \left( P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2 \right)^{\sum_{k=1}^{K-2} |g_k|^2 s_1 s_2^2 + \sigma^2} \right) \right. \]

\[ - \left( \sum_{k=1}^{K-2} |g_k|^2 \right)^{2 s_1^2 s_2^2} \]

\[ = \log_2 \left( \sigma^2 \left( 2P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2 \right) \right) \]

\[ = \log_2 (\sigma^2) + \log_2 \left( 2P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2 \right). \tag{31} \]

Consequently, the achievable rate in (27) becomes

\[ R(s_1, s_2) = \frac{1}{2} \log_2 \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right) + \frac{1}{2} \log_2 \left( \frac{\sigma^2 + 2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{2P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2} \right). \tag{32} \]
Similar steps can be followed to find a similar expression for the remaining signal pairs. Specifically, the achievable rate for the $m^{th}$ signal pair is given as

$$R(e_{2m-1}, e_{2m}) = I(e_{2m-1}, e_{2m}; y_1, y_{m+2}) = \frac{1}{2} \log_2 \left[ \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right) \left( \frac{\sigma^2 + 2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{4P|h|^2 + 2P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2} \right) \right].$$

(33)

Armed with the above results, we now find the overall achievable rate per channel use, $R$. Note that $\lfloor \frac{K}{2} \rfloor + 1$ channel uses are used to transmit the $K$ signals. As such, $R$ can be expressed as shown in (34).

$$R = \frac{1}{\lfloor \frac{K}{2} \rfloor + 1} \left( R(s_1, s_2) + \sum_{m=1}^{\lfloor \frac{K-2}{2} \rfloor} R(e_{2m-1}, e_{2m}) \right)$$

$$= \frac{1}{2 \left( \lfloor \frac{K}{2} \rfloor + 1 \right)} \left( \left( \frac{K}{2} \right) \log_2 \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right) \right)$$

$$+ \sum_{m=0}^{\lfloor \frac{K-2}{2} \rfloor} \log_2 \left( \frac{\sigma^2 + 2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\min(1, m)4P|h|^2 + 2P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2} \right).$$

(34)

In the following, we provide a characterization of the achievable rate to within one bit from the capacity by providing a lower bound on the second term in (34). We have the following upper bound on the denominator of the second term in (34), that is

$$\min(1, m)4P|h|^2 + 2P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2 < 4P|h|^2 + 2P \sum_{k=1}^{K-2} |g_k|^2 + 2\sigma^2,$$

(35)

where $m \geq 0$. This upper bound and (34) together give a lower bound on the achievable rate.
as expressed in (36).

\[
R \geq \frac{[K/2]}{2 \left(\left\lfloor \frac{K}{2} \right\rfloor + 1 \right)} \left( \log_2 \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right) \right)
+ \sum_{m=0}^{[K/2]-1} \log_2 \left( \frac{\sigma^2 + 2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{4P|h|^2 + 2P \sum_{k=1}^{K-2} |g_k|^2 + \sigma^2 + \sigma^2} \right)
= \frac{[K/2]}{2 \left(\left\lfloor \frac{K}{2} \right\rfloor + 1 \right)} \left( \log_2 \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right) + \log_2 \left( \frac{1}{2} \right) \right)
= \frac{1}{2} \left( \log_2 \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right) - 1 \right).
\]

(36)

We can conclude that the achievable rate, using ID, is at most 1bit/s/Hz away from the channel capacity for the entire SNR space considering a Gaussian input.

It is known that the continuous Gaussian distributed input is capacity-achieving on the Gaussian channel. Therefore, the achievable DoF considering the Gaussian input is an upper bound on the achievable DoF when the input is constrained (i.e., belongs to a discrete constellation). The total DoF associated with \(s_1\) and \(s_2\), in two channel uses is equal to

\[
\lim_{P \to \infty} \frac{R(s_1, s_2)}{\frac{1}{2} \log_2(P)} = \lim_{P \to \infty} \frac{\frac{1}{2} \log_2 \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right)}{\frac{1}{2} \log_2(P)}
= 1.
\]

(37)

The result in (37) matches well the lower bound on the DoF given by (26), which proves the accuracy of the asymptotic analytical results given for the discrete constellation case. When \(K\) is large, the total achievable DoF per channel use is equal to

\[
\lim_{P \to \infty} \frac{R}{\frac{1}{2} \log_2(P)} \approx \lim_{P \to \infty} \left( \frac{\log_2 \left( 1 + \frac{2P|h|^2 + P \sum_{k=1}^{K-2} |g_k|^2}{\sigma^2} \right)}{\frac{1}{2} \log_2(P)} - 1 \right)
= 1.
\]

(38)

IV. SIMULATION RESULTS

We performed Monte Carlo simulations to evaluate the performance of ID in terms of the achievable rate and symbol error rate when the input signals belong to discrete constellations.
We recall that the previous theoretical results on the ID performances are provided for a given channel realization. In this section, we consider that the channel coefficients follow Rayleigh distribution with a variance equal to $\gamma = 1$. The ID symbol error rate (SER) and achievable rate are averaged over a large number of channel realizations. Thus, the simulation results represent the average SER and the ergodic achievable rate which were refer to here after by SER and achievable rate, respectively. We consider an AWGN with zero mean and variance $\sigma^2 = 1$. Moreover, we consider the more realistic case of complex input and complex channel gains.

A. Symbol Error rate

![Symbol error rate versus $\zeta$(dB) for 4 – QAM constellation points](image)

**Fig. 1:** Symbol error rate versus $\zeta$(dB) for 4 – QAM constellation points

In this subsection, we provide the SER as a function the transmit power. We consider the 4 – QAM for the signal constellations. Moreover, we set the total transmission power to $2P$ per channel use. Given that ID allows the transmission of two signals per channel use, in fairness, we compare the performance of ID to the performance of other techniques where the transmission rate is two signals per channel use. We show first the impact of interference at the receiver considering existing interference management technique, namely successive decoding [17] and lattice interference alignment [11]. The performance of our proposed technique is also compared to the $2 \times 2$ MIMO point-to-point channel where the transmitter sends two independent signals.
per channel use. Since, the receiver receives, in the case of $2 \times 2$ MIMO, two independent linear combinations of two signals per channel use, it is able to resolve the two signals. The transmit power per symbol is assumed constant and hence transmitting two non-precoded independent signals per channel use is capacity achieving for the case of $2 \times 2$ MIMO.

Fig. 1 shows the SER versus the transmit SNR, denoted by $\zeta$, which is defined by the ratio between the average per transmitted symbol to the noise power. As can be observed from Fig. 1, the receiver realizes a very high SER, using successive decoding, which stays fixed for different transmit powers which proves the inefficiency of this technique when the received signal are of similar strength. The figure also shows that the interference dissolution outperforms the Lattice interference alignment and provides close performance the $2 \times 2$ point-to-point channel.

**B. Normalized Achievable Rate**

![Normalized Rate Chart](image)

**Fig. 2: Normalized rate as function as $\zeta$(dB)**

In this subsection, we provide by simulations the ID achievable rate lower bounds based on the Fano’s inequality given in (20). In this paper we considered a point-to-point channel. Therefore, the obtained lower bound will be first compared to the rate given by the Fano’s inequality where just one signal is transmitted per channel use. Then we compare it to the point-to-point channel capacity $C = \log_2(1 + \text{SNR})$ and to the lattice interference alignment achievable rate lower bounds given by the Fano’s inequality. Since the transmitter sends uncoded signals, we hence...
expect that the lower bound given by the Fano’s inequality is much lower than the rate considering Gaussian inputs. The simulation results are given in terms of a normalized rate defined as the ratio of an achievable rate to the point-to-point channel capacity $C = \log_2(1 + \text{SNR})$.

Fig. 2 depicts the normalized rate of the Fano’s inequality lower bound on the ID achievable rate ($r_{ID}$) as a function of the $\zeta$. This result is compared to the normalized rate given by Fano’s inequality where just one signal is transmitted per channel use ($r_{FI}$). We also compare this result to the lattice interference alignment normalized rate ($r_{LIA}$). As expected, Fig. 2 shows that $r_{FI}$ is much lower than the unity which can be explained by the fact that we consider uncoded transmissions. It also demonstrates that $r_{ID}$ outperforms $r_{FI}$. Indeed, ID allows the transmission of two symbols per channel use. For a given bit rate, ID uses lower-level constellation compared to the case when the transmitter sends one signal per channel use. Hence, for the same energy per symbol, the distance between constellation points in the case of ID is larger than the one in the case when just one signal is transmitted by channel use. Since $r_{ID}$ outperforms $r_{FI}$ and its is well known that $r_{FI}$ approaches unity by using coded signals, $r_{ID}$ will also approach unity using coded signals. Hence, we can conclude that ID achieves near capacity rate also at finite SNR.

The figure also suggests that ID outperforms lattice interference alignment at low to medium SNR. This validates the results given in Fig. 1 which show that ID provides lower error probability than lattice interference alignment. We remark that, at high SNR, all those technique achieve full DoF, i.e., are capacity achieving.

V. CONCLUSION

In this paper, we introduce a new interference management scheme over tim-invariant channels with a one-dimensional space. We analyzed the performance of the proposed scheme, i.e., ID, in terms of achievable rate, DoF and symbol error rate performances. We showed that ID is able to achieve two (uncoded) symbols per channel use, while each symbols gets a non-zero DoF. We compared the performance of ID to its counterparts including successive decoding and lattice interference alignment. We showed the superiority of ID in all measures. We believe that similar favorable results may be obtained for other scenarios, including the X channel, the many-to-one channel, and wiretap channel among others.

Through this paper, we shown that the interference dissolution offers several advantages over
their counterparts presented in the literature. It provides lower computational complexity, lower SER and higher achievable rate. We showed that it provides closed SER to the $2 \times 2$ MIMO point-to-point channel. We believe that interference dissolution introduces a new ways to treat interference by transforming the interference problem to an unknown CSI problem which is much easier to deal with. Interference dissolution can be extended to be applied for multiple-user channels.

APPENDIX A

In this section, we use \( \begin{pmatrix} \bar{d}_1 \\ \bar{d}_2 \end{pmatrix} \) to denote the vector \( y(\beta, s_1, s_2) - v(\tilde{s}_1, \tilde{s}_2) \). \( d_{\min}^2 \) can then be written as

\[
\begin{align*}
\min_{(\tilde{s}_1, \tilde{s}_2) \in M_2} & \left( \frac{\langle \bar{d}_1, \bar{d}_2 \rangle, v(\tilde{s}_1, \tilde{s}_2) \rangle^2}{\| v(\tilde{s}_1, \tilde{s}_2) \|^2} \right) \\
= & \min_{(\tilde{s}_1, \tilde{s}_2) \in M_2} \left( \frac{(\bar{d}_1 h \tilde{s}_1 + \bar{d}_2 h \tilde{s}_2)^2}{(h \tilde{s}_1)^2 + (h \tilde{s}_2)^2} \right) \\
= & \min_{(\tilde{s}_1, \tilde{s}_2) \in M_2} \left( \frac{(\bar{d}_1 + \bar{d}_2 \frac{\tilde{s}_2}{\tilde{s}_1})^2}{1 + \left(\frac{\tilde{s}_2}{\tilde{s}_1}\right)^2} \right). 
\end{align*}
\]

(39)

Without loss of generality, we consider the case when \( |\tilde{s}_1| \geq |\tilde{s}_2| \) to derive a lower bound on \( d_{\min}^2 \). For the case \( |\tilde{s}_1| < |\tilde{s}_2| \), we proceed similarly to provide the same formula. At high transmit power i.e., high value of \( Q_s \), three cases are possible

\[
\lim_{Q_s \to \infty} \frac{\tilde{s}_2}{\tilde{s}_1} = \begin{cases} 
\pm 1 & \text{if } |\tilde{s}_1| \approx |\tilde{s}_2| \\
0 & \text{if } |\tilde{s}_1| \gg |\tilde{s}_2| \\
c & \text{otherwise}
\end{cases}
\]

(40)

where \( c \) denotes a constant (1 > |c| > \( \frac{1}{Q} \approx 0 \)). Next, we differentiate between the three cases and we derive the desired results for each case separately. In the following, we use Khintchine-Groshev theorem. From \( \llbracket \llbracket \), for almost all m-tuple reals number \( \{x_1, x_2, \ldots, x_m\} \) there exists a real constant \( K > 0 \) such that

\[
|p + x_1 q_1 + x_2 q_2 + \ldots + x_m q_m| > \frac{K}{(\max|q_i|)^m},
\]

(41)

for all \( p \in \mathbb{Z} \) and \( \{q_1, q_2, \ldots, q_m\} \in (\mathbb{Z}^*)^m \). So, it is important to note that \( \frac{1}{A_q}(s_1, s_2, \tilde{s}_1, \tilde{s}_2) \) is a set of integer in \((\mathbb{Z}_Q^*)^4 \). In section \( \llbracket \llbracket \) we assumed that at least two signals have two different
associated channel gains. This condition is important to guarantee that $\beta$ be a real number. Indeed, assuming that all signals are transmitted over a channel $h$ only the $m^{th}$ extra signal is transmitted over a channel $g_m$. In this case, $\beta = 1 + \sum_{k=1}^{K-2} \frac{q_k}{s_2} + \frac{g_m}{s_2}$. Now, it is clear that $\beta$ is a real number given that $g_m$ and $h$ are two independent real channel gains [11].

a) Case I ($\frac{\tilde{\alpha}}{s_1} \simeq \pm 1$): In this case, (39) becomes

$$d_{min}^2 \simeq \min_{\langle \tilde{s}_1, \tilde{s}_2 \rangle \in M^2_2} \left(\frac{d_1 \pm d_2}{2}\right)^2$$

$$= \min_{\langle \tilde{s}_1, \tilde{s}_2 \rangle \in M^2_2} \frac{h^2 A_s^2}{2} \left(\frac{s_1}{A_s} + \frac{s_2}{A_s} \pm \left(\tilde{s}_1 + \tilde{s}_2\right) + \beta \left(\frac{s_2}{A_s} - \frac{s_1}{A_s}\right)^2\right)$$

$$\geq \min_{\langle \tilde{s}_1, \tilde{s}_2 \rangle \in M^2_2} \frac{h^2 A_s^2}{2} \left(K_1 \right)^2$$

$$= \frac{h^2 A_s^2 K_1^2}{2Q_s^2}.$$

where $K_1$ is a constant in $\mathbb{R}^*+$. The inequality (c) is backed up by the Khintchine-Groshev theorem.

b) Case II ($\frac{\tilde{\alpha}}{s_1} \simeq 0$): In this case, (39) becomes

$$d_{min}^2 \simeq \min_{\langle \tilde{s}_1, \tilde{s}_2 \rangle \in M^2_2} d_1^2$$

$$= \min_{\langle \tilde{s}_1, \tilde{s}_2 \rangle \in M^2_2} \frac{h^2 A_s^2}{2} \left(\frac{s_1}{A_s} - \frac{s_2}{A_s} + \beta \left(\frac{s_2}{A_s}\right)^2\right)$$

$$\geq \min_{\langle \tilde{s}_1, \tilde{s}_2 \rangle \in M^2_2} \frac{h^2 A_s^2}{2} \left(K_2 \right)^2$$

$$= \frac{h^2 A_s^2 K_2^2}{Q_s^2},$$

where $K_2 \in \mathbb{R}^*+$. We also used the Khintchine-Groshev theorem to provide the lower bound.

c) Case III ($\frac{\tilde{\alpha}}{s_2} = c : 0 < \lim_{Q_s \to \infty} |c| < 1$): From (18), we have

$$d_{min}^2 = \min_{\langle \tilde{s}_1, \tilde{s}_2 \rangle \in M^2_2} \frac{|\langle \tilde{d}_1, \tilde{d}_2 \rangle \cdot (h\tilde{s}_1, h\tilde{s}_2)\rangle^2}{h^2 s_1^2 + h^2 s_2^2}.$$

(44)

Given that $\langle \tilde{s}_1, \tilde{s}_2, d_1, d_2 \rangle \neq 0$, the minimum is achieved when the vector $(\tilde{d}_1, \tilde{d}_2)$ is as orthogonal as possible to $(h\tilde{s}_1, h\tilde{s}_2)$, i.e., $(\tilde{d}_1, \tilde{d}_2)$ is the closet as possible to $(h\tilde{s}_2, -h\tilde{s}_1)$. From the kintchine-
Groshev theorem, there are two constants $\mathcal{K}_3^{(1)}, \mathcal{K}_3^{(2)}$ in $(\mathbb{R}^+)^2$ such that

\[
|\overline{d}_1 - h\tilde{s}_2| = |h|A_s \left| \frac{s_1}{A_s} - \frac{\tilde{s}_2}{A_s} + \beta \left( \frac{s_2}{A_s} \right) \right| > |h|A_s \frac{\mathcal{K}_3^{(1)}}{Q_s},
\]

\[
|\overline{d}_2 - (-h\tilde{s}_1)| = |h|A_s \left| \frac{\tilde{s}_2}{A_s} + \frac{s_1}{A_s} - \beta \left( \frac{s_1}{A_s} \right) \right| > |h|A_s \frac{\mathcal{K}_3^{(2)}}{Q_s}.
\]

Next, $\mathcal{K}_3$ refers to the $\min(\mathcal{K}_3^{(1)}, \mathcal{K}_3^{(2)})$. Considering (45) and (46) in (39) yields

\[
d_{\min}^2 = \min_{(\tilde{s}_1, \tilde{s}_2) \in \{M_2\}} \left( \frac{(\tilde{s}_1 + \tilde{s}_2)^2}{\tilde{s}_1^2 + \tilde{s}_2^2} \right)
\]

\[
\geq \min_{(\tilde{s}_1, \tilde{s}_2) \in \{M_2\}} \left( \frac{(h\tilde{s}_2 \pm hA_s \frac{\mathcal{K}_3^{(1)}}{Q_s}) \tilde{s}_1 + (-h\tilde{s}_1 \pm hA_s \frac{\mathcal{K}_3^{(2)}}{Q_s}) \tilde{s}_2}{\tilde{s}_1^2 + \tilde{s}_2^2} \right)^2
\]

\[
= \frac{h^2 A_s^2 \mathcal{K}_3^2}{Q_s^2} \min_{(\tilde{s}_1, \tilde{s}_2) \in \{M_2\}} \left( \frac{1 \pm c}{1 + c^2} \right)
\]

\[
= \frac{h^2 A_s^2 \mathcal{K}_3^2}{Q_s^2} c_{\min}^2.
\]

$c_{\min}$ is a constant independent of $Q_s$.

Considering the inequality (42), (43) and (47), we obtain a lower bound of $d_{\min}^2$ as follows

\[
d_{\min}^2 \geq \frac{h^2 A_s^2}{Q_s^2} \min \left( \frac{\mathcal{K}_1^2}{\mathcal{K}_3^2}, \frac{\mathcal{K}_2^2}{\mathcal{K}_3^2}, \frac{\mathcal{K}_3^2}{\mathcal{K}_3^2} c_{\min}^2 \right) = \frac{h^2 A_s^2}{Q_s^2} \mathcal{K}^2,
\]

where $\mathcal{K} = \min \left( \frac{\mathcal{K}_1^2}{\mathcal{K}_3^2}, \frac{\mathcal{K}_2^2}{\mathcal{K}_3^2}, \frac{\mathcal{K}_3^2}{\mathcal{K}_3^2} c_{\min}^2 \right)$.

\section*{References}

[1] X. Shang, G. Kramer, and B. Chen, “A new outer bound and the noisy-interference sum-rate capacity for gaussian interference channels,” \textit{IEEE Trans. Inf. Theory}, vol. 55, no. 2, pp. 689–699, Feb. 2009.

[2] V. Cadambe and S. Jafar, “Interference alignment and degrees of freedom of the k -user interference channel,” \textit{IEEE Trans. Inf. Theory}, vol. 54, no. 8, pp. 3425–3441, Aug. 2008.

[3] R. Etkin, D. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” \textit{IEEE Trans. Inf. Theory}, vol. 54, no. 12, pp. 5534–5562, Dec. 2008.

[4] T. Han and K. Kobayashi, “A new achievable rate region for the interference channel,” \textit{IEEE Trans. Inf. Theory}, vol. 27, no. 1, pp. 49–60, Jan. 1981.

[5] G. Bresler, D. Cartwright, and D. Tse, “Feasibility of interference alignment for the mimo interference channel,” \textit{IEEE Trans. Inf. Theory}, vol. 60, no. 9, pp. 5573–5586, Sep. 2014.
[6] R. Zamir, "Lattice Coding for Signals and Networks". Cambridge University Press, 2014.

[7] S. Sridharan, A. Jafarian, S. Vishwanath, and S. Jafar, “Capacity of symmetric K-user gaussian very strong interference channels,” in Proc. IEEE GLOBECOM, Nov. 2008.

[8] S. Sridharan, A. Jafarian, S. Vishwanath, S. Jafar, and S. Shamai, “A layered lattice coding scheme for a class of three user gaussian interference channels,” in Proc. Allerton Conference, Sept. 2008.

[9] A. Motahari, A. Khandani, and S. Gharam, “On the degrees of freedom of the 3-user gaussian interference channel: The symmetric case,” in Proc. IEEE ISIT, June 2009.

[10] R. Etkin and E. Ordentlich, “On the degrees-of-freedom of the K-user gaussian interference channel,” in Proc. IEEE ISIT, June 2009.

[11] A. Motahari, S. Oveis-Gharan, M.-A. Maddah-Ali, and A. Khandani, “Real interference alignment: Exploiting the potential of single antenna systems,” IEEE Trans. Inf. Theory, vol. 60, no. 8, pp. 4799–4810, Aug. 2014.

[12] O. Ordentlich and U. Erez, “On the robustness of lattice interference alignment,” IEEE Trans. Inf. Theory, vol. 59, no. 5, pp. 2735–2759, May 2013.

[13] X. Jianwei and S. Ulukus, “Real interference alignment for the K-user gaussian interference compound wiretap channel,” in Proc. Allerton Conference, 2010.

[14] D. A. Guimaraes, “Digital Transmission : A Simulation-Aided Introduction with VisSim/Comm”. Springer, 2009.

[15] A. Goldsmith, ”Wireless Communications”. Cambridge University Press, 2005.

[16] T. A. Schonhoff and A. A. Giordano, “Detection and Estimation Theory and Its Applications”. PearsonPrentice Hall, 2006.

[17] K. Marvin and M. Alouini, “Digital Communication Over Fading Channels”. John Wiley and Sons, 2000.