On the $L_p$ Brunn-Minkowski theory and the $L_p$ Minkowski problem for $C$-coconvex sets *

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Abstract

Let $C$ be a pointed closed convex cone in $\mathbb{R}^n$ with vertex at the origin $o$ and having nonempty interior. The set $A \subset C$ is called a $C$-coconvex set if the volume of $A$ is finite and $A^* = C \setminus A$ is a closed convex set. For $0 < p < 1$, the $p$-co-sum of $C$-coconvex sets is introduced, and the corresponding $L_p$ Brunn-Minkowski inequality for $C$-coconvex sets is established. We also define the $L_p$ surface area measures, for $0 \neq p \in \mathbb{R}$, of certain $C$-coconvex sets, which are critical in deriving a variational formula of the volume of the Wulff shape associated with a family of functions obtained from the $p$-co-sum. This motivates the $L_p$ Minkowski problem aiming to characterize the $L_p$ surface area measures of $C$-coconvex sets. The existence of solutions to the $L_p$ Minkowski problem for all $0 \neq p \in \mathbb{R}$ is established. The $L_p$ Minkowski inequality for $0 < p < 1$ is proved and is used to obtain the uniqueness of the solutions to the $L_p$ Minkowski problem for $0 < p < 1$.

For $p = 0$, we introduce $(1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2$, the log-co-sum of two $C$-coconvex sets $A_1$ and $A_2$ with respect to $\tau \in (0, 1)$, and prove the log-Brunn-Minkowski inequality of $C$-coconvex sets. The log-Minkowski inequality is also obtained and is applied to prove the uniqueness of the solutions to the log-Minkowski problem that characterizes the cone-volume measures of $C$-coconvex sets. Our result solves an open problem raised by Schneider in [Schneider, Adv. Math., 332 (2018), pp. 199-219].

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1 Introduction and overview of the main results

Motivated by the elegant works by Khovanskii and Timorin [23] and Milman and Rotem [34], Schneider in his pioneer work [41] took the first step to build up the Brunn-Minkowski theory for $C$-coconvex sets. Let $C$ be a pointed closed convex cone in $\mathbb{R}^n$ with vertex at the origin $o$ and having nonempty interior. The set $A \subset C$ is called a $C$-coconvex set if the volume of $A$ is finite and $A^* = C \setminus A$ is closed and convex (the set $A^*$ will be called a $C$-close set). The set $A^*$ is said to be $C$-full when the $C$-coconvex set $A$ is bounded and nonempty. For two $C$-coconvex sets $A_1$ and $A_2$ with $A_1^* = C \setminus A_1$ and $A_2^* = C \setminus A_2$, the “co-sum” of $A_1$ and $A_2$, denoted by $A_1 \oplus A_2$, is defined as $A_1 \oplus A_2 = C \setminus (A_1^* + A_2^*)$, where “+” denotes the usual Minkowski addition. Note that $A_1^* + A_2^* \subset C$ is closed and convex. Moreover, $V_n(A_1 \oplus A_2)$, the volume of $A_1 \oplus A_2$, indeed satisfies the following inequality

\[ V_n(A_1 \oplus A_2)^{\frac{1}{n}} \leq V_n(A_1)^{\frac{1}{n}} + V_n(A_2)^{\frac{1}{n}}. \quad (1.1) \]

This shows that $V_n(A_1 \oplus A_2)$ is finite and hence $A_1 \oplus A_2$ defines a $C$-coconvex set. Inequality (1.1) is indeed equivalent to

\[ V_n((1 - \lambda)A_1 \oplus \lambda A_2)^{\frac{1}{n}} \leq (1 - \lambda)V_n(A_1)^{\frac{1}{n}} + \lambda V_n(A_2)^{\frac{1}{n}} \quad (1.2) \]

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for $\lambda \in (0, 1)$, where $V_n(\lambda A) = \lambda^n V_n(A)$ for $\lambda A = \{\lambda x : x \in A\}$. It has been also proved in [41] that equality holds for (1.2) if and only if $A_1 = \alpha A_2$ for some $\alpha > 0$. Inequality (1.2) (or equivalently (1.1)) is called the Brunn-Minkowski inequality for $C$-coconvex sets and plays essential roles in the development of the Brunn-Minkowski theory for $C$-coconvex sets in [41]. In particular, Schneider used inequality (1.2) to obtain the Minkowski inequality for $C$-coconvex sets:

$$\nabla_1(A_1, A_2)^n \leq V_n(A_1)^{n-1}V_n(A_2),$$

with equality if and only if $A_1 = \alpha A_2$ for some $\alpha > 0$ (see Section 2 for notations).

The Brunn-Minkowski inequality for $C$-coconvex sets reveals the dissimilarity to the celebrated Brunn-Minkowski inequality for convex bodies (convex compact sets in $\mathbb{R}^n$ with nonempty interiors). The Brunn-Minkowski inequality for convex bodies reads: for any $\lambda \in (0, 1)$ and for any two convex bodies $K_1$ and $K_2$, one has

$$V_n((1 - \lambda)K_1 + \lambda K_2)^{\frac{1}{n}} \geq (1 - \lambda)V_n(K_1)^{\frac{1}{n}} + \lambda V_n(K_2)^{\frac{1}{n}}.$$  (1.4)

Inequality (1.4) shows that $V_n(\cdot)$ is $\frac{1}{n}$-concave in terms of the Minkowski sum of convex bodies, while inequality (1.2) implies that $V_n(\cdot)$ is $\frac{1}{n}$-convex in terms of the co-sum of $C$-coconvex sets. In particular, inequalities (1.2) and (1.4) have opposite directions of inequalities; this has been carried over to their equivalent Minkowski type inequalities.

On the other hand, the Brunn-Minkowski theories for $C$-coconvex sets and for convex bodies also exhibit similarity. For instance, the surface area measures for $C$-coconvex sets can be defined in a way rather similar to the surface area measures of convex bodies. Let $C^o$ be the polar cone of $C$, i.e., $C^o = \{x \in \mathbb{R}^n : x \cdot y \leq 0 \text{ for all } y \in C\}$, where $x \cdot y$ means the usual inner product of $x, y \in \mathbb{R}^n$. Denote by $\partial E$ and $\text{int} E$ the boundary and the interior of $E \subset \mathbb{R}^n$, respectively. Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n$, $\Omega_C = S^{n-1} \cap \text{int} C^o$ and $\text{A}^\bullet$ be a $C$-close set. Thus the Gauss map $\nu_{\text{A}^\bullet} : \partial \text{A}^\bullet \cap \text{int} C \rightarrow S^{n-1}$ can be defined by the usual way; $\nu_{\text{A}^\bullet}(x)$ is an outer unit normal vector of $\partial \text{A}^\bullet$ at $x \in \partial \text{A}^\bullet \cap \text{int} C$. The surface area measures of $\text{A}^\bullet$ (see [41, p. 201]) and $\text{A}$ on $\Omega_C$, denoted by $\mathcal{S}_{n-1}(\text{A}^\bullet, \cdot)$ and $\mathcal{S}_{n-1}(\text{A}, \cdot)$ respectively, can then be defined by: for any Borel set $\eta \subset \Omega_C$,

$$\mathcal{S}_{n-1}(\text{A}, \eta) = \mathcal{S}_{n-1}(\text{A}^\bullet, \eta) = \mathcal{H}^{n-1}(\nu_{\text{A}^\bullet}^{-1}(\eta)),$$  (1.5)

where $\nu_{\text{A}^\bullet}^{-1}$ denotes the reverse of $\nu_{\text{A}^\bullet}$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Note that the surface area measure of $\text{A}^\bullet$ is uniquely determined, because the set of singular boundary points of $\text{A}^\bullet$ has $(n-1)$-dimensional Hausdorff measure zero. With the help of the surface area measures in (1.5), $V_n(\text{A}_1)$ and $\nabla_1(\text{A}_1, \text{A}_2)$ for $C$-coconvex sets $\text{A}_1$ and $\text{A}_2$ have the integral expressions (see (2.9) and (2.11), respectively) rather similar to those for the volume and the mixed volume for convex bodies (see e.g., [40]). Moreover, analogous to the classical Minkowski problem [35, 36], a Minkowski problem aiming to characterize the surface area measures of $C$-coconvex sets has been proposed by Schneider in [41]. A solution to this problem was obtained in [41, Theorem 3] when the given finite Borel measure has compact support in $\Omega_C$. This result was improved in [42, Theorem 1] where the given Borel measure is only assumed to be finite (i.e., removing the restriction on its support). The uniqueness of the solution was also established in [41, Theorem 2].

We would like to mention that the Minkowski problem to characterize the surface area measure for certain unbounded convex surfaces (or convex sets) has been done earlier by Chou and Wang [11], Pogorelov [38], and Urbas [48].

The classical Minkowski problem has a long history which can be traced back to [1, 14, 35, 36]. In his groundbreaking paper [26], Lutwak introduced the elegant $L_p$ Minkowski problem extending the classical Minkowski problem in a natural but nontrivial way. Since then, the $L_p$ Minkowski
problem has attracted great attention, see e.g., [10, 12, 19, 20, 21, 22, 25, 30, 47, 57, 58, 59] among others. In the case $p = 0$, the $L_p$ Minkowski problem for convex bodies reduces to the log-Minkowski problem, and amazing progress has been made to this challenging problem (see e.g., [2, 3, 6, 9, 18, 44, 45, 56]). No doubt that the $L_p$ Minkowski problem has nice applications (see e.g., [13, 17, 29, 53]) and plays a central role in the development of the $L_p$ Brunn-Minkowski theory of convex bodies, a theory founded by Lutwak [26, 27] based on the combination of the volume and the $L_p$ addition of convex bodies. Contributions to this $L_p$ theory include, e.g., [7, 15, 16, 28, 31, 33, 37, 43, 49, 51, 52, 54, 55].

A major goal in this paper is to develop an $L_p$ Brunn-Minkowski theory for $C$-coconvex sets, and provide the counterpart of the $L_p$ Brunn-Minkowski theory for convex bodies. Our first contribution is to define an $L_p$ addition of $C$-coconvex sets in Section 3, which is called the $p$-co-sum of $C$-coconvex sets. That is, for $0 < p < 1$ and two $C$-coconvex sets $A_1, A_2$, the $p$-co-sum of $A_1$ and $A_2$, denoted by $A_1 \oplus_p A_2$, is defined by

$$
(A_1 \oplus_p A_2)^* = C \cap \bigcap_{u \in \Omega_C} \left\{ x \in \mathbb{R}^n : x \cdot u \leq \overline{h}(A_1 \oplus_p A_2, u) \right\},
$$

where $\overline{h}(A, \cdot) : \Omega_C \rightarrow \mathbb{R}$ is the support function of a $C$-coconvex set $A$ on $\Omega_C$ (see (2.8) for the definition of $\overline{h}$), and for $0 < p < 1$,

$$
\overline{h}(A_1 \oplus_p A_2, u) = (\overline{h}(A_1, u)^p + \overline{h}(A_2, u)^p)^{1/p}, \quad \text{for} \quad u \in \Omega_C.
$$

The set $A_1 \oplus_p A_2$ turns out to be a $C$-coconvex set and this result will be presented in Theorem 3.2. Moreover, we also establish the $L_p$ Brunn-Minkowski inequality for $C$-coconvex sets in Theorem 3.1: Let $A_1, A_2$ be $C$-coconvex sets. If $0 < p < 1$, then

$$
V_n(A_1 \oplus_p A_2)^{\frac{n}{p}} \leq V_n(A_1)^{\frac{n}{p}} + V_n(A_2)^{\frac{n}{p}},
$$

with equality if and only if $A_1 = \alpha A_2$ for some $\alpha > 0$. The above $L_p$ Brunn-Minkowski inequality for $C$-coconvex sets reduces to inequality (1.1) if $p = 1$. It also shows that $V_n(\cdot)$ is $\frac{n}{p}$-concave in terms of the $p$-co-sum of $C$-coconvex sets, which clearly demonstrates its difference from the $L_p$ Brunn-Minkowski inequality for convex bodies (namely $V_n(\cdot)$ is $\frac{n}{p}$-concave in terms of the $L_p$ addition of convex bodies) [26]. The above $L_p$ Brunn-Minkowski inequality for $C$-coconvex sets provides precisely the “complementary” analogue of the $L_p$ Brunn-Minkowski inequality for convex bodies for $0 < p < 1$ conjectured by Böröczky, Lutwak, Yang, and Zhang in [5]. Note that this conjectured inequality is still quite open in general and important progress has been made recently by Chen, Huang, Li and Liu [8], and Kolesnikov and Milman [24].

Our second contribution is to develop the $L_p$ surface area measures of $C$-coconvex sets and study the related $L_p$ Minkowski problem for $0 \neq p \in \mathbb{R}$. Let $\omega \subset \Omega_C$ be a compact set. A closed convex set $A^* \subset C$ is $C$-determined by $\omega$ if

$$
A^* = C \cap \bigcap_{u \in \omega} H^-(u, h(A^*, u)),
$$

where $h(A^*, \cdot)$ denotes the support function of $A^*$ on $\Omega_C$ (see its definition in (2.6)). The collection of all closed convex sets that are $C$-determined by $\omega$ will be denoted by $\mathcal{H}(C, \omega)$. The $L_p$ surface area measure of a $C$-coconvex set $A$, with $A^* = C \setminus A \in \mathcal{H}(C, \omega)$, is denoted by $S_{n-1,p}(A, \cdot)$ and defined by

$$
S_{n-1,p}(A, \cdot) = \overline{h}(A, \cdot)^{1-p} S_{n-1}(A, \cdot) \quad \text{on} \quad \omega.
$$
The $L_p$ surface area measure of $A$ naturally appears in the variational formula derived from the volume of the Wulff shape associated with $(C, \omega, f_\tau)$, where $f_\tau$ is given by (4.25). This variational formula is presented and proved in Theorem 4.1. A fundamental question related to the $L_p$ surface area measure is the following $L_p$ Minkowski problem (i.e., Problem 4.1):

**The $L_p$ Minkowski problem.** Let $0 \neq p \in \mathbb{R}$ and $\omega \subset \Omega_C$ be a compact set. Under what necessary and/or sufficient conditions on a finite Borel measure $\mu$ on $\omega$ does there exist a set $A^* \in \mathcal{K}(C, \omega)$ with $A = C \setminus A^*$ such that $\mu = S_{n-1,p}(A, \cdot)$?

A solution to this $L_p$ Minkowski problem is established in Theorem 6.1, which reads: Let $\omega$ be a compact set of $\Omega_C$. Suppose that $\mu$ is a nonzero finite Borel measure on $\Omega_C$ whose support is concentrated on $\omega$. For $0 \neq p \in \mathbb{R}$, there exists a $C$-full set $A^*_0$ ($A_0 = C \setminus A^*_0$) such that

$$
\mu = c \cdot S_{n-1,p}(A_0, \cdot) \quad \text{with} \quad c = \frac{1}{nV_n(A_0)} \left( \int_\omega h(A_0, u)^p \, d\mu(u) \right).
$$

Section 5 aims to develop the $L_p$ Minkowski inequality for the $L_p$ mixed volume of $C$-coconvex sets (see Theorem 5.1). In particular, we prove that for $0 < p < 1$ and for two $C$-coconvex sets $A_1, A_2$ such that $A^*_1 \in \mathcal{K}(C, \omega)$ and $A^*_2 \in \mathcal{K}(C, \omega)$, one has

$$
\nabla_p(A_1, A_2) = \frac{1}{n} \int_\omega \bar{h}(A_2, u)^p \, dS_{n-1,p}(A_1, u) \leq V_n(A_1)^{\alpha_p} V_n(A_2)^{\beta_p},
$$

with equality if and only if $A_1 = \alpha A_2$ for some $\alpha > 0$. Again, this $L_p$ Minkowski inequality has its form similar to the $L_p$ Minkowski inequality for convex bodies [26]; but these two $L_p$ Minkowski inequalities have different directions of inequalities and work for different ranges of $p$. One important application of the $L_p$ Minkowski inequality for $C$-coconvex sets is to obtain the uniqueness of the solutions to the $L_p$ Minkowski problem, see Theorems 5.3 and 6.2 for more details.

We would like to emphasize that although the $L_p$ Minkowski problem for $C$-coconvex sets resembles the $L_p$ Minkowski problem for convex bodies in many ways (such as their formulations), these two problems are completely different, for instance, the former one is solvable for all $0 \neq p \in \mathbb{R}$, but the existence of solutions to the latter one is still unknown for many $p \in \mathbb{R}$ (in particular for $p < -n$).

We also make contributions to develop the $L_0$ (or log) Brunn-Minkowski theory for $C$-coconvex sets. In Section 7, we introduce the log-co-sum of two $C$-coconvex sets $A_1$ and $A_2$ with respect to $\tau \in (0, 1)$ (or, for simplicity, the log-co-sum of $A_1$ and $A_2$), denoted by $(1-\tau) \circ A_1 \oplus_0 \tau \circ A_2$, whose support function takes the following form:

$$
\bar{h}((1-\tau) \circ A_1 \oplus_0 \tau \circ A_2, u) = \bar{h}(A_1, u)^{1-\tau} \bar{h}(A_2, u)^{\tau}, \quad \text{for } u \in \Omega_C.
$$

See (7.51), (7.52) and (7.53) for more details. The log-Brunn-Minkowski inequality of $C$-coconvex sets is proved in Theorem 7.1, which asserts that $V_n(\cdot)$ is log-convex in terms of the log-co-sum; namely for $\tau \in (0, 1)$ and for two $C$-coconvex sets $A_1, A_2$,

$$
V_n((1-\tau) \circ A_1 \oplus_0 \tau \circ A_2) \leq V_n(A_1)^{1-\tau} V_n(A_2)^{\tau}.
$$

Equality characterization for the log-Brunn-Minkowski inequality is given in Theorem 7.2, where we also establish the log-Minkowski inequality (7.58): for two nonempty $C$-coconvex sets $A_1$ and $A_2$,

$$
\nabla_0(A_1, A_2) = \frac{1}{n} \int_{\Omega_C} \log \left( \frac{\bar{h}(A_2, u)}{\bar{h}(A_1, u)} \right) \bar{h}(A_1, u) \, dS_{n-1}(A_1, u) \leq \frac{V_n(A_1)}{n} \cdot \log \left( \frac{V_n(A_2)}{V_n(A_1)} \right),
$$
with equality if and only if $A_1 = \alpha A_2$ for some $\alpha > 0$. These inequalities provide precisely “complementary” analogues of the log-Brunn-Minkowski and log-Minkowski inequalities for convex bodies conjectured by Böröczky, Lutwak, Yang, and Zhang in [5]. Note that the log-Brunn-Minkowski and log-Minkowski inequalities for convex bodies are still quite open in general and have received a lot of attention, see e.g., [4, 32, 39, 46, 50]. The significance of our log-Minkowski inequality for $C$-coconvex sets (i.e., (7.58)) can also be seen from the fact that this inequality gives a positive answer to an open problem raised by Schneider in [41]. Indeed, Schneider in [41] proposed the log-Minkowski problem for $C$-coconvex sets aiming to characterize the cone-volume measures of $C$-close sets, and also provided solutions to the log-Minkowski problem in [41, Theorems 4 and 5]. Schneider raised an open problem regarding the uniqueness of the solutions to the log-Minkowski problem for $C$-coconvex sets [41, p. 203]. In Theorem 7.3, we use the the log-Minkowski inequality for $C$-coconvex sets (7.58) to confirm that solutions to the log-Minkowski problem for $C$-coconvex sets are indeed unique.

2 Background and preliminaries

The central objects of interest in this paper are the closed convex sets and related $C$-coconvex sets in the fixed pointed closed convex cone $C$ defined in the Euclidean space $\mathbb{R}^n$. By $V_n(\cdot)$ we mean the volume. The notations and definitions in this paper mainly follow those in [40, 41] for consistence.

The dot product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. A set $E \subset \mathbb{R}^n$ is said to be convex if $\lambda x + (1 - \lambda)y \in E$ for any $\lambda \in [0, 1]$ and $x, y \in E$. We say $C \subset \mathbb{R}^n$ a closed convex cone if $C$ is a closed and convex subset with nonempty interior such that $\lambda x \in C$ for any $x \in C$ and $\lambda \geq 0$. Note that, if $C$ is a closed convex cone, then $\lambda C = C$ for any $\lambda > 0$ and especially $C + C = C$. A closed convex cone $C$ is called a pointed cone if $-C \cap C = \{0\}$, where $0$ denotes the origin of $\mathbb{R}^n$ and $-C = \{-x : x \in C\}$.

Denote by $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ the unit sphere in $\mathbb{R}^n$. For $t \in \mathbb{R}$ and $u \in S^{n-1}$, let

$$H(u, t) = \{x \in \mathbb{R}^n : x \cdot u = t\}$$

and $H(u, t)$ is a hyperplane with normal vector $u \in S^{n-1}$. Define the upper and lower halfspaces $H^+(u, t)$ and $H^-(u, t)$, respectively, by the following formulas:

$$H^+(u, t) = \{x \in \mathbb{R}^n : x \cdot u \geq t\} \quad \text{and} \quad H^-(u, t) = \{x \in \mathbb{R}^n : x \cdot u \leq t\}.$$

Let $C \subset \mathbb{R}^n$ be a fixed pointed closed convex cone. Associated to $C$, there is a polar cone $C^o$ defined by

$$C^o = \{x \in \mathbb{R}^n : x \cdot y \leq 0 \text{ for all } y \in C\}.$$

Define $\Omega_C$, an open subset of $S^{n-1}$, by $\Omega_C = S^{n-1} \cap \text{int} C^o$, where $\text{int} C^o$ is the interior of $C^o$. It is easily checked that the set $H^+(u, t) \cap C$ is bounded with nonempty interior for any $u \in \Omega_C$ and $t < 0$. Moreover, $H^+(u, 0) \cap C = \{0\}$ for $u \in \Omega_C$. Throughout the paper, let $\zeta \in S^{n-1} \setminus (S^{n-1} \cap C^o)$ be fixed such that $x \cdot \zeta > 0$ for all $x \in C \setminus \{0\}$. The existence of such $\zeta$ is guaranteed by the fact that the closed convex cone $C$ is pointed. For simplicity, such a fixed $\zeta \in S^{n-1}$ will not appear in the following notations:

$$H_t = \{x \in \mathbb{R}^n : x \cdot \zeta = t\}, \quad H_t^- = \{x \in \mathbb{R}^n : x \cdot \zeta \leq t\},$$

for $t \geq 0$, $M_t = M \cap H_t^-$ for $M \subseteq C$ and for $t > 0$. Note that $M_t$ is always bounded for $t > 0$. When $M = C$, we use $C_t$ for the set $C \cap H_t^-$.
A set $L \subset \mathbb{R}^n$ is said to be a convex body if $L$ is a convex compact set with nonempty interior. Associated to a convex body $L$ is its support function $h(L, \cdot) : S^{n-1} \to \mathbb{R}$ which can be formulated by

$$h(L, u) = \max \{ x \cdot u : x \in L \}.$$ 

We say that a sequence of convex bodies $L_j, j \in \mathbb{N}$, converges to a convex body $L_0$ in terms of the Hausdorff metric if $h(L_j, \cdot) \to h(L_0, \cdot)$ uniformly on $S^{n-1}$ as $j \to \infty$. The support function can be used to uniquely determine a convex body. Similarly, the support function of a $C$-close set $A^\bullet$ can be defined in a similar manner. Recall that $A^\bullet \subset C$ is $C$-close if $A^\bullet$ is a closed convex subset and the volume of $C \setminus A^\bullet$ is positive and finite. A closed convex set $A^\bullet \subset C$ is said to be $C$-full if $C \setminus A^\bullet$ is bounded and nonempty.

If we let $h(A^\bullet, \cdot) : \text{int}C^\circ \to \mathbb{R}$ be the support function of a $C$-close set $A^\bullet$, then $h(A^\bullet, \cdot)$ can be formulated by

$$h(A^\bullet, x) = \sup \{ x \cdot y : y \in A^\bullet \}, \quad \text{for } x \in \text{int}C^\circ. \tag{2.6}$$

Properties for the support functions of $C$-close sets are similar to those for the support functions of convex bodies. For example, $h(A^\bullet, \cdot)$ for a $C$-close set $A^\bullet$ is a sublinear function in $\text{int}C^\circ$ and has positive homogeneity of degree 1. Again, the support function $h(A^\bullet, \cdot)$ and $A^\bullet$ can be (uniquely) determined by each other, in particular, we can have

$$A^\bullet = C \cap \bigcap_{u \in \Omega_C} H^-(u, h(A^\bullet, u)), \tag{2.7}$$

where $H(u, h(A^\bullet, u))$ supports $A^\bullet$ at some point $y \in \partial A^\bullet$ with outer normal vector $u$. It is easily checked that $-\infty < h(A^\bullet, \cdot) < 0$ on $\Omega_C$ and the maximum can be indeed obtained in (2.6).

The convergence of a sequence of $C$-close sets is given as follows, see e.g., [41]. Let $N_0 = \mathbb{N} \cup \{0\}$. 

**Definition 2.1.** Let $\{A^\bullet_j\}_{j \in N_0}$ be a sequence of $C$-close sets. If there exists $t_0 > 0$ such that $A^\bullet_j \cap C_{t_0} \neq \emptyset$ for all $j \in \mathbb{N}$, and for all $t \geq t_0$

$$\lim_{j \to \infty} (A^\bullet_j \cap C_t) = A^\bullet_0 \cap C_t,$$

in terms of the Hausdorff metric, then $\{A^\bullet_j\}_{j \in \mathbb{N}}$ is said to be convergent to $A^\bullet_0$, which is written by $A^\bullet_j \to A^\bullet_0$ as $j \to \infty$.

Note that $\partial A^\bullet \cap A^\circ$ if a $C$-coconvex set $A = C \setminus A^\bullet$ is nonempty. A nice thing is that $\partial A \setminus \partial C$ coincides with $\partial A^\bullet \setminus \partial C$. This nice property provides a good way to uniquely determine $A$ through $\overline{h}(A, \cdot) : \text{int}C^\circ \to \mathbb{R}$ formulated by

$$\overline{h}(A, x) = -h(A^\bullet, x), \quad \text{for } x \in \text{int}C^\circ. \tag{2.8}$$

Hence, $\overline{h}(A, x + y) \geq \overline{h}(A, x) + \overline{h}(A, y)$ and $\overline{h}(A, \lambda x) = \lambda \overline{h}(A, x)$ for $\lambda > 0$ and $x, y \in \text{int}C^\circ$. Clearly, $0 < \overline{h}(A, \cdot) < \infty$ in $\text{int}C^\circ$. Following [41], we call $\overline{h}(A, \cdot)$ the support function of $A$. In a similar manner, one can also define $S_{n-1}(A, \cdot)$, the surface area measure of $A$, as follows: for any Borel set $\eta \subset \Omega_C$,

$$S_{n-1}(A, \eta) = S_{n-1}(A^\bullet, \eta) = \mathcal{H}^{n-1}(\nu_{A^\bullet}^{-1}(\eta)),$$

where $S_{n-1}(A^\bullet, \cdot)$ is the surface area measure of $A^\bullet$ defined in (1.5). It has been proved in [41, Lemma 1] that

$$V_n(A) = \frac{1}{n} \int_{\Omega_C} \overline{h}(A, u) dS_{n-1}(A, u). \tag{2.9}$$
Again, $V_n(\lambda A) = \lambda^n V_n(A)$ for all $\lambda > 0$ and clearly
\[ V_n(A_1) \leq V_n(A_2) \] (2.10)
for any two $C$-coconvex sets $A_1$ and $A_2$ such that $A_1 \subseteq A_2$. The mixed volume of two $C$-coconvex sets $A_0, A_1$ [41, p. 219], denoted by $\overline{V}(A_0, \ldots, A_0, A_1)$ or by $\overline{V}_1(A_0, A_1)$ for short, is given by
\[ \overline{V}_1(A_0, A_1) = \overline{V}(A_0, \ldots, A_0, A_1) = \frac{1}{n} \int_{\Omega_C} \overline{h}(A_1, u) d\overline{S}_{n-1}(A_0, u). \] (2.11)
Consequently, one has the following Minkowski inequality for $\overline{V}_1(A_0, A_1)$ (see [41, (27)]):
\[ \overline{V}_1(A_0, A_1)^n \leq V_n(A_0)^{n-1} V_n(A_1), \]
with equality if and only if $A_0 = \alpha A_1$ for some $\alpha > 0$. Moreover, (2.11) implies $\overline{V}_1(A, A_2) \leq \overline{V}_1(A, A_2)$ for $C$-coconvex sets $A_1, A_2, A$ such that $A_1 \subseteq A_2$.

3 The $p$-co-sum of $C$-coconvex sets

In this section, the $p$-co-sum of two $C$-coconvex sets for $0 < p < 1$ will be introduced and related properties will be provided. In particular, we prove that the $p$-co-sum of $C$-coconvex sets for $0 < p < 1$ is still a $C$-coconvex set. For $\alpha > 0$ and a $C$-coconvex set $A$, let $\alpha \circ A = \alpha \overline{h}_A$ if there is no confusion on the value of $p$.

**Definition 3.1.** Let $A_1, A_2$ be two $C$-coconvex sets. For $p \in (0, 1)$ and $\alpha_1, \alpha_2 \geq 0$ (not both zero), define $\overline{h} : \text{int} C^0 \to \mathbb{R}$ as follows: for all $x \in \text{int} C^0$,
\[ \overline{h}(x) = (\alpha_1 \overline{h}(A_1, x)^p + \alpha_2 \overline{h}(A_2, x)^p)^{\frac{1}{p}}. \] (3.12)

We would like to mention that Definition 3.1 also works for $p = 1$. In this case, it goes back to the co-sum of two $C$-coconvex sets (see e.g., [41]):
\[ \overline{h}(\alpha_1 A_1 \oplus \alpha_2 A_2, x) = \alpha_1 \overline{h}(A_1, x) + \alpha_2 \overline{h}(A_2, x), \quad \text{for } x \in \text{int} C^0, \] (3.13)
where the addition “$\oplus$” is the co-sum given by $\alpha_1 A_1 \oplus \alpha_2 A_2 = C \setminus (\alpha_1 A_1^* + \alpha_2 A_2^*)$. The case $p = 0$ will be discussed in Section 7.

The following lemma asserts that $-\overline{h}$ is the support function of a closed convex set contained in $C$.

**Lemma 3.1.** Let $p \in (0, 1)$ and $\alpha_1, \alpha_2 \geq 0$ (not both zero). If $A_1$ and $A_2$ are nonempty $C$-coconvex sets, then $-\overline{h}$, where $\overline{h}$ is given in (3.12), is a sublinear function in $\text{int} C^0$ and hence uniquely determines a closed convex set contained in $C$.

**Proof.** First of all, as $A_1$ and $A_2$ are nonempty $C$-coconvex sets, then $\overline{h}(A_1, x) > 0$ and $\overline{h}(A_2, x) > 0$ for all $x \in \text{int} C^0$. As $\alpha_1, \alpha_2 \geq 0$ are not both zero, it is trivial to have $\overline{h}(x) > 0$ for all $x \in \text{int} C^0$.

Secondly, for any $x \in \text{int} C^0$ and $t > 0$,
\[ \overline{h}(tx) = (\alpha_1 \overline{h}(A_1, tx)^p + \alpha_2 \overline{h}(A_2, tx)^p)^{\frac{1}{p}} = t (\alpha_1 \overline{h}(A_1, x)^p + \alpha_2 \overline{h}(A_2, x)^p)^{\frac{1}{p}} = t \overline{h}(x). \]
That is, $\overline{h}$ and hence $-\overline{h}$ have positive homogeneity of degree 1 in $\text{int} C^0$. 

Thirdly, let \( T_p(a_1, a_2) = (a_1^p + a_2^p)^{\frac{1}{p}} \) for positive numbers \( a_1, a_2 \in \mathbb{R}^+ \) and \( p \in (0, 1) \). The inverse Minkowski’s inequality yields that, for any positive numbers \( a_i, b_i, c_i \in \mathbb{R}^+ \) such that \( a_i \geq b_i + c_i \) for \( i = 1, 2 \),
\[
T_p(a_1, a_2) \geq T_p(b_1, b_2) + T_p(c_1, c_2).
\]
(3.14)
Since \( A_1, A_2 \) are two \( C \)-coconvex sets, for \( \lambda \in [0, 1] \) and \( x, y \in \text{int}C^o \), one has
\[
h \left( C \setminus (\alpha_i^{1/p} A_i), \lambda x + (1 - \lambda)y \right) \leq h \left( C \setminus (\alpha_i^{1/p} A_i), \lambda x \right) + h \left( C \setminus (\alpha_i^{1/p} A_i), (1 - \lambda)y \right),
\]
for \( i = 1, 2, \) or equivalently
\[
\mathcal{H} \left( \alpha_i^{1/p} A_i, \lambda x + (1 - \lambda)y \right) \geq \mathcal{H} \left( \alpha_i^{1/p} A_i, \lambda x \right) + \mathcal{H} \left( \alpha_i^{1/p} A_i, (1 - \lambda)y \right).
\]
For convenience, for \( i = 1, 2 \), let
\[
a_i = \mathcal{H}(\alpha_i^{1/p} A_i, \lambda x + (1 - \lambda)y), \quad b_i = \mathcal{H}(\alpha_i^{1/p} A_i, \lambda x) \quad \text{and} \quad c_i = \mathcal{H}(\alpha_i^{1/p} A_i, (1 - \lambda)y).
\]
Employing (3.14), one has
\[
\mathcal{H}(\lambda x + (1 - \lambda)y) = T_p(a_1, a_2) \geq T_p(b_1, b_2) + T_p(c_1, c_2)
\]
\[
= \lambda(\alpha_1 \mathcal{H}(A_1, x)^p + \alpha_2 \mathcal{H}(A_2, x)^p)^{\frac{1}{p}} + (1 - \lambda)(\alpha_1 \mathcal{H}(A_1, y)^p + \alpha_2 \mathcal{H}(A_2, y)^p)^{\frac{1}{p}}
\]
\[
= \lambda \mathcal{H}(x) + (1 - \lambda)\mathcal{H}(y).
\]
Thus \( \mathcal{H} \) is concave and hence \(-\mathcal{H}\) is convex in \( \text{int}C^o \). Let
\[
(\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2)^* = C \cap \bigcap_{u \in \partial^c} H^- \left( u, -\mathcal{H}(u) \right).
\]
(3.15)
Clearly \( (\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2)^* \) is a closed convex set contained in \( C \). Moreover, \( \circ \notin (\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2)^* \) as \(-\mathcal{H}(x) < 0 \) for all \( x \in \text{int}C^o \).

Based on (3.15), we define
\[
\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2 = C \setminus (\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2)^*.
\]
(3.16)
For convenience, we (formally) let \( \mathcal{H}(\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2, \cdot) = \mathcal{H} \), and \( \mathcal{H} \) becomes the support function of a \( C \)-coconvex set once \( \alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2 \) is proved to be a \( C \)-coconvex set. To fulfill this goal, one needs to prove that \( V_n(\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2) \) is finite and positive. The following lemma is required.

**Lemma 3.2.** Let \( A_1, A_2 \) be \( C \)-coconvex sets, and \( p \in (0, 1) \). For \( 0 < \lambda < 1 \), one has
\[
(1 - \lambda) \circ A_1 \oplus_p \lambda \circ A_2 \subseteq (1 - \lambda)A_1 \oplus \lambda A_2,
\]
with equality if and only if \( A_1 = A_2 \).

**Proof.** First of all, the function \( t^p \) is strictly concave on \( t \in (0, +\infty) \) if \( 0 < p < 1 \). Thus,
\[
(1 - \lambda)\mathcal{H}(A_1, u) + \lambda\mathcal{H}(A_2, u))^p \geq (1 - \lambda)\mathcal{H}(A_1, u)^p + \lambda\mathcal{H}(A_2, u)^p,
\]
for all \( u \in C \).
for all \( u \in \Omega_C \). Equality holds if and only if \( \overline{h}(A_1, u) = \overline{h}(A_2, u) \) for all \( u \in \Omega_C \), namely, \( A_1 = A_2 \). This, together with (3.12) and (3.13), yields that

\[
\overline{h}((1 - \lambda)A_1 \oplus \lambda A_2, u) = (1 - \lambda)\overline{h}(A_1, u) + \lambda \overline{h}(A_2, u)
\]

\[
\geq ((1 - \lambda)\overline{h}(A_1, u)^p + \lambda \overline{h}(A_2, u)^p)^{\frac{1}{p}}
\]

\[
= \overline{h}((1 - \lambda) \circ A_1 \oplus_p \lambda \circ A_2, u).
\]

It follows from (3.15) that

\[
((1 - \lambda) \circ A_1 \oplus_p \lambda \circ A_2)^* = C \cap \bigcap_{u \in \Omega_C} H^- \left( u, -\overline{h}((1 - \lambda) \circ A_1 \oplus_p \lambda \circ A_2, u) \right)
\]

\[
\supseteq C \cap \bigcap_{u \in \Omega_C} H^- \left( u, -\overline{h}((1 - \lambda)A_1 \oplus \lambda A_2, u) \right)
\]

\[
= ((1 - \lambda)A_1 \oplus \lambda A_2)^*.
\]

Thus, the desired argument (3.17) follows immediately from (3.16):

\[
(1 - \lambda) \circ A_1 \oplus_p \lambda \circ A_2 = C \setminus ((1 - \lambda) \circ A_1 \oplus_p \lambda \circ A_2)^*
\]

\[
\subseteq C \setminus ((1 - \lambda)A_1 \oplus \lambda A_2)^*
\]

\[
= (1 - \lambda)A_1 \oplus \lambda A_2.
\]

Clearly, equality holds in (3.17) if and only if \( A_1 = A_2 \).

We now prove the \( L_p \) Brunn-Minkowski inequality for \( C \)-coconvex sets for \( 0 < p < 1 \). Note that the case \( p = 1 \) (i.e., inequality (1.1)) has been discussed in [41]. The case \( p = 0 \) will be discussed in Section 7.

**Theorem 3.1.** Let \( A_1, A_2 \) be \( C \)-coconvex sets. If \( 0 < p < 1 \), then

\[
V_n(A_1 \oplus_p A_2)^{\frac{p}{n}} \leq V_n(A_1)^{\frac{p}{n}} + V_n(A_2)^{\frac{p}{n}},
\]

(3.18)

with equality if and only if \( A_1 = \alpha A_2 \) for some \( \alpha > 0 \).

**Proof.** Let \( 0 < \lambda < 1 \). By (2.10), Lemma 3.2, and the Brunn-Minkowski inequality (1.2), we have

\[
V_n(A_1 \oplus_p A_2)^{\frac{1}{n}} = V_n \left( (1 - \lambda) \circ ( (1 - \lambda)^{-\frac{1}{p}} A_1 ) \oplus_p \lambda \circ ( \lambda^{-\frac{1}{p}} A_2 ) \right)^{\frac{1}{n}}
\]

\[
\leq V_n \left( (1 - \lambda)(1 - \lambda)^{-\frac{1}{p}} A_1 \oplus \lambda \lambda^{-\frac{1}{p}} A_2 \right)^{\frac{1}{n}}
\]

\[
\leq (1 - \lambda) V_n \left( (1 - \lambda)^{-\frac{1}{p}} A_1 \right)^{\frac{1}{n}} + \lambda V_n(\lambda^{-\frac{1}{p}} A_2)^{\frac{1}{n}}
\]

\[
= (1 - \lambda)^{1 - \frac{1}{p}} V_n(A_1)^{\frac{1}{n}} + \lambda^{1 - \frac{1}{p}} V_n(A_2)^{\frac{1}{n}}.
\]

Let \( \lambda = \frac{V_n(A_2)^{\frac{1}{n}}}{V_n(A_1)^{\frac{1}{n}} + V_n(A_2)^{\frac{1}{n}}} \). Then

\[
V_n(A_1 \oplus_p A_2)^{\frac{1}{n}} \leq \frac{V_n(A_1)^{\frac{p}{n}} + V_n(A_2)^{\frac{p}{n}}}{(V_n(A_1)^{\frac{p}{n}} + V_n(A_2)^{\frac{p}{n}})^{1 - \frac{1}{p}}} = \left( V_n(A_1)^{\frac{p}{n}} + V_n(A_2)^{\frac{p}{n}} \right)^{\frac{1}{p}}.
\]
After rearrangement, one gets (3.18) as desired. The equality condition, namely $A_1 = \alpha A_2$ for some $\alpha > 0$, follows from the equality conditions of (1.2) and (3.17). In particular,

$$V_n(A_1 \oplus_p A_2)^{\frac{p}{n}} = V_n((\alpha A_2) \oplus_p A_2)^{\frac{p}{n}} = (1 + \alpha^p) V_n(A_2)^{\frac{p}{n}} = V_n(A_1)^{\frac{p}{n}} + V_n(A_2)^{\frac{p}{n}}$$

if $A_1 = \alpha A_2$ for some $\alpha > 0$.

Indeed, it follows from Lemma 3.1 and Theorem 3.1 that if $0 < p < 1$ and $\alpha_1, \alpha_2 > 0$, then

$$V_n(\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2)^{\frac{p}{n}} = V_n\left(\alpha_1^{\frac{1}{p}} A_1 \oplus_p \alpha_2^{\frac{1}{p}} A_2\right)^{\frac{p}{n}} \leq \alpha_1 V_n(A_1)^{\frac{p}{n}} + \alpha_2 V_n(A_2)^{\frac{p}{n}} < \infty$$

for any two $C$-coconvex sets $A_1, A_2$. On the other hand, $\overline{h}(\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2, x) > 0$ for all $x \in \text{int} C^o$ as $\overline{h}(A_1, x) > 0$ and $\overline{h}(A_2, x) > 0$, one has $V_n(\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2) > 0$. This observation can be summarized as the following theorem.

**Theorem 3.2.** Let $A_1, A_2$ be two $C$-coconvex sets. For $p \in (0,1)$ and $\alpha_1, \alpha_2 \geq 0$ (not both zero), the set $\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2$ given in (3.16) does define a nonempty $C$-coconvex set, whose support function is $\overline{h}$ given by (3.12), namely, for all $x \in \text{int} C^o$,

$$\overline{h}(\alpha_1 \circ A_1 \oplus_p \alpha_2 \circ A_2, x) = \left(\alpha_1 \overline{h}(A_1, x)^p + \alpha_2 \overline{h}(A_2, x)^p\right)^{\frac{1}{p}}.$$

## 4 A variational formula of the volume related to the $p$-co-sum

In this section, we will prove a variational formula of the volume of the Wulff shape associated with a family of functions obtained from the $p$-co-sum. Motivated by this variational formula, the $L_p$ surface area measure can be introduced and the related $L_p$ Minkowski problem can be posed. Recall that for a $C$-close set $A^*$, the surface area measure $S_{n-1}(A^*, \cdot)$ defined on $\Omega_C$ could be infinite. In order to better deal with the surface area measure $S_{n-1}(A^*, \cdot)$ and related Minkowski problems, we will concentrate on a special class of $C$-close sets, namely the sets that are $C$-determined by $\omega$.

Throughout the rest of the paper, we always let $\omega \subset \Omega_C$ be a nonempty and compact set. As in the Brunn-Minkowski theory for convex bodies, Schneider in [41] introduced the $C$-coconvex analogue of the Wulff shape, which provides a powerful tool in establishing the variational formula regarding the co-sum and plays essential roles in finding solutions to the Minkowski problem that aims to characterize the surface area measures of $C$-coconvex sets. For $f : \omega \to \mathbb{R}$ a positive and continuous function on $\omega$, define the Wulff shape associated with $(C, \omega, f)$ to be a closed convex set of the following form:

$$A^*_f = C \cap \bigcap_{u \in \omega} H^-(u, -f(u)).$$

The Wulff shape $A^*_f$ is a $C$-full set and $C \setminus A^*_f$ is bounded and nonempty. Please see Figure 1 for a typical Wulff shape in $\mathbb{R}^2$.

We also would like to mention that the Wulff shape $A^*_f$ is $C$-determined by $\omega$. Hereafter, a closed convex set $A^* \subseteq C$ of the following form

$$A^* = C \cap \bigcap_{u \in \omega} H^-(u, h(A^*, u))$$

is called $C$-determined by $\omega$. The collection of all closed convex sets that are $C$-determined by $\omega$ is denoted by $\mathcal{K}(C, \omega)$. Clearly, $aA^* \in \mathcal{K}(C, \omega)$ for any $a > 0$ and for any $A^* \in \mathcal{K}(C, \omega)$. It was
Figure 1: A Wulff shape associated with \((C, \omega, f)\) for \(\omega = \{u_1, u_2, u_3, u_4\}\) and \(f : \omega \to (0, \infty)\) in \(\mathbb{R}^2\). The line \(H_t\) is to illustrate Lemma 4.2.

proved in [41] that if \(A^* \in \mathcal{K}(C, \omega)\), then \(A = C \setminus A^*\) is bounded and \(\overline{S}_{n-1}(A, \omega) = S_{n-1}(A^*, \omega)\) is finite.

We summarize some important properties of the Wulff shape into the following lemma for the easy future citation. Please read [41, pp. 220-221] for more details.

Lemma 4.1. Let \(f : \omega \to \mathbb{R}\) be a positive and continuous function on \(\omega\). Then

\[
h(A^*_j, u) \leq -f(u) \quad \text{for } u \in \omega, \tag{4.19}
\]

\[
h(A^*_j, u) = -f(u) \quad \text{almost everywhere with respect to the surface area measure } S_{n-1}(A^*_j, \cdot),
\]

\[
S_{n-1}(A^*_j, \Omega \setminus C) = 0 \quad \text{and}
\]

\[
V_n(f) := V_n(C \setminus A^*_j) = \frac{1}{n} \int_\omega f(u) \, dS_{n-1}(A^*_j, u). \tag{4.20}
\]

Note that if \(f = -h(A^*, \cdot)\) for some \(A^*\) which is \(C\)-full and \(C\)-determined by \(\omega\), then \(A^*_j = A^*\) due to (2.7); in this case, one has

\[
V_n(-h(A^*, \cdot)) = V_n(C \setminus A^*) = \frac{1}{n} \int_\omega -h(A^*, u) \, dS_{n-1}(A^*, u).
\]

It can be easily checked that \(V_n(f_j) \to V_n(f)\) if \(f_j \to f\) uniformly on \(\omega\). It has been proved in [41, Lemma 5] that the Wulff shape \(A^*_{j_0}\) associated with \((C, \omega, f_j)\) for \(j \in \mathbb{N}\) converges to the Wulff shape \(A^*_{j_0}\) associated with \((C, \omega, f_0)\) if the sequence of positive and continuous functions \(f_j : \omega \to \mathbb{R}\) converges uniformly to the positive and continuous function \(f_0 : \omega \to \mathbb{R}\). Moreover, [41, Lemma 6] asserts that if \(\{A^*_j\}_{j \geq 1}\) is a sequence in \(\mathcal{K}(C, \omega)\) such that \(A^*_j \to A^*_{j_0}\) for some \(C\)-full set \(A^*_{j_0}\), then \(A^*_{j_0} \in \mathcal{K}(C, \omega)\).

The following lemma is extracted from [41, Lemma 8]. See Figure 1 for an illustration.

Lemma 4.2. There is a constant \(t > 0\) with the following property: if \(A^* \in \mathcal{K}(C, \omega)\) and \(V_n(C \setminus A^*) = 1\), then \(C \cap H_t \subset A^*\).

The following lemma is the weak convergence of surface area measures defined on \(\omega\). Although its proof has already been given in the proof of [41, Lemma 7], we still provide the detailed proof here for completeness and for the convenience of future citation.
Lemma 4.3. Let \( \{A_i^\bullet\}_{i \geq 1} \subset \mathcal{K}(C, \omega) \) and \( A_i^\bullet \in \mathcal{K}(C, \omega) \). Let \( A_i = C \setminus A_i^\bullet \) for all \( i \in \mathbb{N}_0 \). If \( A_i \to A_0^\bullet \), then \( \mathcal{S}_{n-1}(A_i, \cdot) \to \mathcal{S}_{n-1}(A_0, \cdot) \) weakly on \( \omega \). That is, for any continuous function \( f : \omega \to \mathbb{R} \), one has,
\[
\int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_i, u) \to \int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_0, u).
\]
Moreover, if continuous functions \( f_i : \omega \to \mathbb{R} \) \( (i \in \mathbb{N}) \) satisfy that \( f_i \to f \) uniformly on \( \omega \), then
\[
\int_{\omega} f_i(u) \, d\mathcal{S}_{n-1}(A_i, u) \to \int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_0, u).
\]

Proof. Suppose that \( A_i^\bullet \to A_0^\bullet \). It follows from Definition 2.1 that for sufficiently large \( t > 0 \), one always has \( A_i^\bullet \cap C_t \to A_0 \cap C_t \) in terms of the Hausdorff metric. Note that for sufficiently large \( t > 0 \), \( A_i^\bullet \cap C_t \) for all \( i \in \mathbb{N}_0 \) is a convex body. Hence, for sufficiently large \( t > 0 \),
\[
S_{n-1}(A_i^\bullet \cap C_t, \cdot) \to S_{n-1}(A_0^\bullet \cap C_t, \cdot) \quad \text{weakly on} \quad S^{n-1}.
\]
That is, for any continuous function \( F : S^{n-1} \to \mathbb{R} \) and for sufficiently large \( t > 0 \),
\[
\int_{S^{n-1}} F(u) \, dS_{n-1}(A_i^\bullet \cap C_t, u) \to \int_{S^{n-1}} F(u) \, dS_{n-1}(A_0^\bullet \cap C_t, u).
\]
Tietze’s extension theorem implies that, for any continuous function \( f : \omega \to \mathbb{R} \), there is a continuous function \( F : S^{n-1} \to \mathbb{R} \) such that
\[
F(u) = \begin{cases} 
    f(u) & \text{on} \, \omega, \\
    0 & \text{on} \, S^{n-1} \setminus \Omega_C.
\end{cases}
\]
Note that \( S^{n-1} = \omega \cup (\Omega_C \setminus \omega) \cup (S^{n-1} \setminus \Omega_C) \) and \( S_{n-1}(A_i^\bullet, \Omega_C \setminus \omega) = 0 \) (see [41, (29)]). Consequently, for all \( i \in \mathbb{N}_0 \),
\[
\int_{S^{n-1}} F(u) \, dS_{n-1}(A_i^\bullet \cap C_t, u) = \int_{\omega} f(u) \, dS_{n-1}(A_i^\bullet \cap C_t, u) = \int_{\omega} f(u) \, dS_{n-1}(A_i^\bullet, u),
\]
where the equalities follow from \( F = 0 \) on \( S^{n-1} \setminus \Omega_C \) and the facts that, for sufficiently large \( t > 0 \), \( S_{n-1}(A_i^\bullet \cap C_t, \cdot) = S_{n-1}(A_i^\bullet, \cdot) \) on \( \omega \) and \( S_{n-1}(A_i^\bullet \cap C_t, \Omega_C \setminus \omega) = 0 \). This, together with (4.23) and the fact that \( S_{n-1}(A_i, \cdot) = S_{n-1}(A_i^\bullet, \cdot) \), yields that
\[
\lim_{i \to \infty} \int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_i, u) = \int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_0, u).
\]
This concludes the proof for the weak convergence of \( \mathcal{S}_{n-1}(A_i, \cdot) \to \mathcal{S}_{n-1}(A_0, \cdot) \) on \( \omega \). In particular,
\[
\lim_{i \to \infty} \mathcal{S}_{n-1}(A_i, \omega) = \mathcal{S}_{n-1}(A_0, \omega).
\]
Now let \( f_i \to f \) uniformly on the compact set \( \omega \subset \Omega_C \) with \( f \) and each \( f_i \) being continuous for \( i \in \mathbb{N} \). For any \( \varepsilon > 0 \), there exists an \( i_0 \in \mathbb{N} \), such that, for all \( i > i_0 \), one has \( |f_i(u) - f(u)| < \varepsilon \) for any \( u \in \omega \). Together with (4.21), (4.24), and the fact that \( \mathcal{S}_{n-1}(A_0, \omega) = \mathcal{S}_{n-1}(A_0^\bullet, \omega) \) is finite as \( A_0^\bullet \in \mathcal{K}(C, \omega) \), one gets,
\[
0 \leq \lim_{i \to \infty} \left| \int_{\omega} f_i(u) \, d\mathcal{S}_{n-1}(A_i, u) - \int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_0, u) \right|
\leq \lim_{i \to \infty} \left| \int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_i, u) - \int_{\omega} f(u) \, d\mathcal{S}_{n-1}(A_0, u) \right| + \lim_{i \to \infty} \int_{\omega} |f_i(u) - f(u)| \, d\mathcal{S}_{n-1}(A_i, u)
\leq \varepsilon \mathcal{S}_{n-1}(A_0, \omega).
\]
The desired convergence (4.22) holds after taking \( \varepsilon \to 0^+ \). \( \square \)
Let \(A^\bullet \in \mathcal{K}(C, \omega)\) be \(C\)-determined by \(\omega\) and \(f : \omega \to \mathbb{R}\) be a continuous function on the compact set \(\omega \subseteq \Omega_C\). Let \(A = C \setminus A^\bullet\). Define \(f_\tau : \omega \to \mathbb{R}\) for \(\tau \in (-\tau_0, \tau_0)\), where \(\tau_0 > 0\) is a constant small enough, by

\[
f_\tau (u) = \left( \tilde{h}(A, u)^p + \tau f(u) \right)^{1/p} \quad \text{for } u \in \omega. \tag{4.25}\]

Note that both \(\tilde{h}(A, \cdot)\) and \(f\) are continuous functions on the compact set \(\omega\). Hence one can let \(\tau_0\) be a constant such that

\[
0 < \tau_0 < \frac{\min_{u \in \omega} \tilde{h}(A, u)^p}{\max_{u \in \omega} |f(u)|}.
\]

Clearly, for each \(\tau \in (-\tau_0, \tau_0)\), \(f_\tau\) is also a positive and continuous function on \(\omega\). It is easy to verify that \(f_\tau \to \tilde{h}(A, \cdot)\) uniformly on \(\omega\) as \(\tau \to 0\). Hence the Wulff shape \(A^\bullet\) associated with \((C, \omega, f_\tau)\) converges to \(A^\bullet\), by [41, Lemma 5]. Moreover, it can be verified that

\[
\lim_{\tau \to 0} \frac{f_\tau (u) - \tilde{h}(A, u)}{\tau} = \frac{1}{p} f(u) \tilde{h}(A, u)^{1-p} \quad \text{uniformly on } \omega. \tag{4.26}
\]

We are now ready to state and prove the variational formula regarding \(V_n(f_\tau) = V_n(C \setminus A^\bullet)\). Although Theorem 4.1 can be obtained from [41, Lemma 7] by letting \(G(\tau, \cdot)\) in [41, Lemma 7] to be \(f_\tau\), we again provide a detailed proof in this paper for completeness. Note that \(f_0 = \tilde{h}(A, \cdot)\) and hence \(V_n(f_0) = V_n(A)\).

**Theorem 4.1.** Let \(A^\bullet \in \mathcal{K}(C, \omega)\) be \(C\)-determined by \(\omega\) and \(f : \omega \to \mathbb{R}\) be a continuous function on \(\omega\). Let \(A = C \setminus A^\bullet\) be defined by (4.25). For all \(0 \neq p \in \mathbb{R}\), one has

\[
\left. \frac{dV_n(f_\tau)}{d\tau} \right|_{\tau=0} = \lim_{\tau \to 0} \frac{V_n(f_\tau) - V_n(f_0)}{\tau} = \frac{1}{p} \int_{\omega} f(u) \tilde{h}(A, u)^{1-p} \, d\mathcal{S}_{n-1}(A, u). \tag{4.27}
\]

**Proof.** Note that the Wulff shape \(A^\bullet\) associated with \((C, \omega, f_\tau)\) is \(C\)-determined by \(\omega\), and hence \(S_{n-1}(A^\bullet, \Omega_C \setminus \omega) = 0\) (see [41, p. 220]). For convenience, let \(A_\tau = C \setminus A^\bullet\). Formula (2.11), Lemma 4.1 and the fact that \(\mathcal{S}_{n-1}(A_\tau, \cdot) = \mathcal{S}_{n-1}(A^\bullet, \cdot)\) imply that

\[
nV_n(f_\tau) = nV_n(A_\tau) = \int_{\omega} f_\tau (u) \, d\mathcal{S}_{n-1}(A_\tau, u), \tag{4.28}
\]

\[
nV_1(A_\tau, A) = \int_{\omega} \tilde{h}(A, u) \, d\mathcal{S}_{n-1}(A_\tau, u).
\]

Recall that \(A^\bullet \to A^\bullet\), and hence \(V_n(A_\tau) \to V_n(A)\) by (4.28). The uniform convergence of (4.26), together with Lemma 4.3, yields that

\[
\lim_{\tau \to 0^+} \frac{V_n(A_\tau) - V_1(A_\tau, A)}{\tau} = \frac{1}{n} \lim_{\tau \to 0^+} \int_{\omega} \frac{f_\tau (u) - \tilde{h}(A, u)}{\tau} \, d\mathcal{S}_{n-1}(A_\tau, u)
\]

\[
= \frac{1}{np} \int_{\omega} f(u) \tilde{h}(A, u)^{1-p} \, d\mathcal{S}_{n-1}(A, u). \tag{4.29}
\]

Similarly, Lemma 4.1 (in particular, (4.19)) yields that \(\tilde{h}(A_\tau, u) = -h(A^\bullet, u) \geq f_\tau (u)\) for all \(u \in \omega\), and hence

\[
\liminf_{\tau \to 0^+} \frac{V_1(A_\tau, A) - V_n(A)}{\tau} = \frac{1}{n} \liminf_{\tau \to 0^+} \int_{\omega} \frac{\tilde{h}(A_\tau, u) - \tilde{h}(A, u)}{\tau} \, d\mathcal{S}_{n-1}(A, u)
\]

\[
\geq \frac{1}{np} \int_{\omega} f(u) \tilde{h}(A, u)^{1-p} \, d\mathcal{S}_{n-1}(A, u).
\]
Together with the Minkowski inequality for $C$-coconvex sets (1.3), one has,
\[
\frac{1}{np} \int f(u) \ h(A, u)^{1-p} \ d\mathcal{S}_{n-1}(A, u) \leq \liminf_{\tau \to 0^+} \frac{V_1(A, A_\tau) - V_n(A)}{\tau} \leq V_n(A) \frac{n-1}{n} \liminf_{\tau \to 0^+} \frac{V_n(A_\tau)^{\frac{1}{n}}}{\tau} - V_n(A)^{\frac{1}{n}}. \tag{4.30}
\]
Similarly, one can also get, by (4.29),
\[
\frac{1}{np} \int f(u) \ h(A, u)^{1-p} \ d\mathcal{S}_{n-1}(A, u) = \limsup_{\tau \to 0^+} \frac{V_n(A_\tau) - V_1(A_\tau, A)}{\tau} \geq \limsup_{\tau \to 0^+} \frac{V_n(A_\tau) - V_n(A)^{\frac{a-1}{n}} V_n(A)^{\frac{1}{n}}}{\tau} = V_n(A) \frac{n-1}{n} \limsup_{\tau \to 0^+} \frac{V_n(A_\tau)^{\frac{1}{n}}}{\tau} - V_n(A)^{\frac{1}{n}}. \tag{4.31}
\]
By (4.30), (4.31) and the fact that $\liminf \leq \limsup$, we have
\[
\frac{d(V_n(A_\tau)^{\frac{1}{n}})}{d\tau} \bigg|_{\tau=0^+} = \lim_{\tau \to 0^+} \frac{V_n(A_\tau)^{\frac{1}{n}}}{\tau} - V_n(A)^{\frac{1}{n}} = \frac{V_n(A)^{\frac{a-1}{n}}}{np} \int_\omega f(u) \ h(A, u)^{1-p} \ d\mathcal{S}_{n-1}(A, u).
\]
Thus, by the L'Hospital rule and the fact that $V_n(f_\tau) = V_n(A_\tau)$, one gets
\[
\frac{dV_n(f_\tau)}{d\tau} \bigg|_{\tau=0^+} = \frac{d(V_n(A_\tau)^{\frac{1}{n}})}{d\tau} \bigg|_{\tau=0^+} \cdot nV_n(A)^{\frac{n-1}{n}} = \frac{1}{p} \int_\omega f(u) \ h(A, u)^{1-p} \ d\mathcal{S}_{n-1}(A, u).
\]
Following the same lines, one can also get
\[
\frac{dV_n(f_\tau)}{d\tau} \bigg|_{\tau=0^-} = \frac{1}{p} \int_\omega f(u) \ h(A, u)^{1-p} \ d\mathcal{S}_{n-1}(A, u)
\]
and hence the desired formula (4.27) follows. \hfill \square

Motivated by (4.27), one can define the $L_p$ surface area measure of $A^* \in \mathcal{K}(C, \omega)$ (or equivalently of $A = C \setminus A^*$) and the $L_p$ mixed volume of $A$ and a continuous function $g : \omega \to \mathbb{R}$ as follows.

**Definition 4.1.** Let $A^* \in \mathcal{K}(C, \omega)$ be $C$-determined by $\omega$ and $A = C \setminus A^*$. For $0 \neq p \in \mathbb{R}$, the $L_p$ surface area measure of $A^*$, denoted by $S_{n-1, p}(A^*, \cdot)$, on $\omega$ is absolutely continuous with respect to $S_{n-1}(A^*, \cdot)$ such that
\[
\frac{dS_{n-1, p}(A^*, u)}{dS_{n-1}(A^*, u)} = (-h(A^*, u))^{1-p}, \text{ for } u \in \omega.
\]
Equivalently, the $L_p$ surface area measure of $A$, denoted by $S_{n-1, p}(A, \cdot)$, is defined by $S_{n-1, p}(A, \cdot) = S_{n-1, p}(A^*, \cdot)$, and hence
\[
\frac{dS_{n-1, p}(A, u)}{dS_{n-1}(A, u)} = \overline{h}(A, u)^{1-p}, \text{ for } u \in \omega.
\]
Let $g : \omega \to \mathbb{R}$ be a positive and continuous function on $\omega$. The $L_p$ mixed volume of $A$ and $g$, denoted by $\nabla_p(A, g)$, is defined by

$$\nabla_p(A, g) = \frac{1}{n} \int_{\omega} g(u)^p \, \overline{h}(A, u)^{1-p} \, d\overline{S}_{n-1}(A, u) = \frac{1}{n} \int_{\omega} g(u)^p \, d\overline{S}_{n-1,p}(A, u). \quad (4.32)$$

A fundamental question related to the $L_p$ surface area measure is the following $L_p$ Minkowski problem. A solution to this $L_p$ Minkowski problem will be provided in Section 6.

**Problem 4.1 (The $L_p$ Minkowski problem).** Let $0 \neq p \in \mathbb{R}$ and $\omega \subset \Omega_C$ be a compact set. Under what necessary and/or sufficient conditions on a finite Borel measure $\mu$ on $\omega$ does there exist a $C$-close set $A^*$ with $A = C \setminus A^*$ such that $\mu = \overline{S}_{n-1,p}(A, \cdot)$?

## 5 The $L_p$ Minkowski Inequality and the Unique Determination of $C$-coconvex Sets for $0 < p < 1$

In this section, the $L_p$ Minkowski inequality related to the $L_p$ mixed volume for $C$-coconvex sets for $0 < p < 1$ is established. Such $L_p$ Minkowski inequality and the $L_p$ Brunn-Minkowski inequality (3.18) can be viewed as the fundamental elements in the $L_p$ Brunn-Minkowski theory for $C$-coconvex sets. Based on the $L_p$ Minkowski inequality, the unique determination for $C$-coconvex sets by the $L_p$ surface area measure for $0 < p < 1$ is provided.

Indeed, based on (4.27) and (4.32), if $A = C \setminus A^*$ with $A^* \in \mathcal{K}(C, \omega)$ and $f : \omega \to \mathbb{R}$ is a positive and continuous function on a compact set $\omega$, one has, for all $0 \neq p \in \mathbb{R}$,

$$\frac{p}{n} \lim_{\tau \to 0} \frac{V_n(f_{\tau}) - V_n(f_0)}{\tau} = \frac{1}{n} \int_{\omega} f(u) \overline{h}(A, u)^{1-p} \, d\overline{S}_{n-1}(A, u) = \nabla_p(A, f^{1/p}), \quad (5.33)$$

where $f_{\tau}$ is defined by (4.25).

**Definition 5.1.** Let $A = C \setminus A^*$ with $A^* \in \mathcal{K}(C, \omega)$ and $A_1 = C \setminus A_1^*$ with $A_1^* \in \mathcal{K}(C, \omega)$. Define the $L_p$ mixed volume of $A$ and $A_1$, denoted by $\nabla_p(A, A_1)$, for $0 \neq p \in \mathbb{R}$ as

$$\nabla_p(A, A_1) = \frac{1}{n} \int_{\omega} \overline{h}(A_1, u) \overline{h}(A, u)^{1-p} \, d\overline{S}_{n-1}(A, u) = \frac{1}{n} \int_{\omega} \overline{h}(A_1, u) \overline{h}(A, u)^{1-p} \, d\overline{S}_{n-1,p}(A, u). \quad (5.34)$$

From Definition 5.1, one can check that $\nabla_p(A, A) = V_n(A)$ and $\nabla_p(\alpha A, \beta A_1) = \alpha^{n-p} \beta^p \nabla_p(A, A_1)$ for any $\alpha, \beta > 0$ and two $C$-coconvex sets $A, A_1$ such that $A^*, A_1^* \in \mathcal{K}(C, \omega)$. We now prove the following $L_p$ Minkowski inequality.

**Theorem 5.1.** Let $A, A_1$ be two $C$-coconvex sets such that $A^* \in \mathcal{K}(C, \omega)$ and $A_1^* \in \mathcal{K}(C, \omega)$. For $0 < p < 1$, one has

$$\nabla_p(A, A_1)^n \leq V_n(A)^{n-p} V_n(A_1)^p, \quad (5.35)$$

with equality if and only if $A = \alpha A_1$ for some $\alpha > 0$.

**Proof.** By (5.34), if $p = 1$, one gets (see also (2.11)),

$$\nabla_1(A, A_1) = \frac{1}{n} \int_{\omega} \overline{h}(A_1, u) \, d\overline{S}_{n-1}(A, u).$$

Moreover, $\nabla_1(A, A) = V_n(A)$ (see also (2.9) and the fact that $S_{n-1}(A^*, \Omega_C \setminus \omega) = 0$ [41, (29)]), and hence, a probability measure on $\omega$ can be defined as follows:

$$d\nu = \frac{\overline{h}(A, \cdot)}{n V_n(A)} \, d\overline{S}_{n-1}(A, \cdot).$$

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It follows from the Hölder’s inequality that, for \(0 < p < 1\), one has
\[
\nabla_p(A, A_1) = \frac{1}{nV_n(A)} \int_\omega \bar{h}(A_1, u)^p \bar{h}(A, u)^{1-p} \, d\mathcal{S}_{n-1}(A, u)
= \int_\omega \left( \frac{\bar{h}(A_1, u)}{\bar{h}(A, u)} \right)^p d\nu(u)
\leq \left( \int_\omega \frac{\bar{h}(A_1, u)}{\bar{h}(A, u)} \, d\nu(u) \right)^p
= \left( \frac{\bar{V}_1(A, A_1)}{V_n(A)} \right)^p.
\tag{5.36}
\]

Employing the Minkowski inequality (1.3) to (5.36), one gets
\[
\nabla_p(A, A_1)^n \leq V_n(A)^n \left( \frac{\nabla_1(A, A_1)}{V_n(A)} \right)^{np} \leq V_n(A)^n \left( \frac{V_n(A_1)}{V_n(A)} \right)^p = V_n(A)^{n-p} V_n(A_1)^p.
\]

This is the desired inequality (5.35). The characterization of equality for inequality (5.35) is an easy consequence of the characterization of equality for inequality (1.3) and for the Hölder’s inequality (applied in (5.36)).

**Remark 5.1.** Although inequality (5.35) was proved for special \(C\)-coconvex sets, its proof indeed can be used to show that, for \(0 < p < 1\) and for any two \(C\)-coconvex sets \(A\) and \(A_1\),
\[
\nabla_p(A, A_1) = \frac{1}{n} \int_{\Omega_C} \bar{h}(A_1, u)^p \bar{h}(A, u)^{1-p} \, d\mathcal{S}_{n-1}(A, u) \leq V_n(A)^{\frac{n-p}{n}} V_n(A_1)^{\frac{p}{n}},
\]
with equality if and only if \(A = \alpha A_1\) for some \(\alpha > 0\).

**Theorem 5.2.** Let \(A_1, A_2\) be two \(C\)-coconvex sets such that \(A_1^*, A_2^* \in \mathcal{K}(C, \omega)\). For \(0 < p < 1\), let \(\bar{h} = (\bar{h}(A_1, \cdot)^p + \bar{h}(A_2, \cdot)^p)^{1/p}\) on \(\omega\) and \(\bar{A}^* \in \mathcal{K}(C, \omega)\) be the Wulff shape associated with \((C, \omega, \bar{h})\).

Then
\[
V_n(\bar{A})^\frac{p}{n} \leq V_n(A_1)^\frac{p}{n} + V_n(A_2)^\frac{p}{n}, \tag{5.37}
\]
where \(\bar{A} = C \setminus \bar{A}^*\), is equivalent to the \(L_p\) Minkowski inequality (5.35).

**Proof.** Let \(\tau > 0\) and \(\bar{A}^* \in \mathcal{K}(C, \omega)\) be the Wulff shape associated with \((C, \omega, \bar{h}_\tau)\), where \(\bar{h}_\tau = (\bar{h}(A_1, \cdot)^p + \tau \bar{h}(A_2, \cdot)^p)^{1/p}\) on \(\omega\). Let \(\bar{A}_\tau = C \setminus \bar{A}^*\). Inequality (5.37) for \(0 < p < 1\) implies that, for all \(\tau > 0\),
\[
g(\tau) = V_n(\bar{A}_\tau)^\frac{p}{n} - V_n(A_1)^\frac{p}{n} - \tau V_n(A_2)^\frac{p}{n} \leq 0.
\]

Taking use of Theorem 4.1 (or see (5.33)), one gets
\[
\lim_{\tau \to 0^+} \frac{g(\tau) - g(0)}{\tau} = \lim_{\tau \to 0^+} \frac{V_n(\bar{A}_\tau)^\frac{p}{n} - V_n(A_1)^\frac{p}{n} - \tau V_n(A_2)^\frac{p}{n}}{\tau}
= \frac{p}{n} V_n(A_1)^{\frac{p}{n} - 1} \lim_{\tau \to 0^+} \frac{V_n(\bar{A}_\tau) - V_n(A_1)}{\tau} - V_n(A_2)^\frac{p}{n}
= V_n(A_1)^{\frac{p}{n}} \nabla_p(A_1, A_2) - V_n(A_2)^\frac{p}{n}.
\]
The desired $L_p$ Minkowski inequality follows easily from $g(0) = 0$ and $g(\tau) \leq 0$ for $\tau > 0$.

Conversely, we assume that the $L_p$ Minkowski inequality holds. Note that $\bar{A} \in \mathcal{K}(C, \omega)$ and $\bar{h} = (\bar{h}(A_1, \cdot)^p + \bar{h}(A_2, \cdot)^p)^{1/p}$ on $\omega$. For $A^* \in \mathcal{K}(C, \omega)$, by (5.34), one has,

$$V_p (A, \bar{A}) = \frac{1}{n} \int_{\omega} \bar{h} (\bar{A}, u)^p \, d\mathcal{S}_{n-1,p} (A, u)$$

$$= \frac{1}{n} \int_{\omega} (\bar{h}(A_1, u)^p + \bar{h}(A_2, u)^p) \, d\mathcal{S}_{n-1,p} (A, u)$$

$$= \frac{1}{n} \int_{\omega} \bar{h}(A_1, u)^p \, d\mathcal{S}_{n-1,p} (A, u) + \frac{1}{n} \int_{\omega} \bar{h}(A_2, u)^p \, d\mathcal{S}_{n-1,p} (A, u)$$

$$= V_p (A, A_1) + V_p (A, A_2).$$

Employing the $L_p$ Minkowski inequality, one gets

$$V_n (\bar{A}) = V_p (\bar{A}, \bar{A}) = V_p (\bar{A}, A_1) + V_p (\bar{A}, A_2) \leq V_n (\bar{A})^{\frac{n-p}{n}} (V_n (A_1)^{\frac{p}{n}} + V_n (A_2)^{\frac{p}{n}}).$$

Simplifying this, one can get the desired inequality (5.37).

\[ \square \]

**Remark 5.2.** Let $A_1, A_2$ be two $C$-coconvex sets such that $A_1^*, A_2^* \in \mathcal{K}(C, \omega)$. For $0 < p < 1$, one cannot expect to have $\bar{A} = A_1 \oplus_p A_2$. Indeed, it is easily checked that $\bar{A} \subseteq A_1 \oplus_p A_2$ because $\bar{A}^* = C \setminus \bar{A}$ is the $C$-coconvex set generated by less halfspaces. However, counterexamples show that $\bar{A} \neq A_1 \oplus_p A_2$ can happen, and an example is provided in Figure 2. Moreover, as $\bar{A} \subseteq A_1 \oplus_p A_2$, it follows from Theorem 3.1 that

$$V_n (\bar{A}) \leq V_n (A_1 \oplus_p A_2)^{\frac{p}{n}} \leq V_n (A_1)^{\frac{p}{n}} + V_n (A_2)^{\frac{p}{n}}.$$

Clearly, if equality holds in (5.37), then equality holds in (3.18) as well, and this requires that $A_1 = \alpha A_2$ for some $\alpha > 0$. On the other hand, if $A_1 = \alpha A_2$ for some $\alpha > 0$, then

$$\bar{A} = A_1 \oplus_p A_2 = (\alpha A_2) \oplus_p A_2 = (1 + \alpha^p)^{1/p} A_2,$$

which clearly implies the equality in (5.37). This implies that the equality characterizations for inequalities (5.35) and (5.37) are indeed the same.

Figure 2: Let $\omega = \{u_1, u_2\}$ and $p = 1/2$. The support functions of $\bar{A}$ and $A_1 \oplus_p A_2$ at $u_3$ are

$$\frac{15 + 4\sqrt{3} + 2\sqrt{15}}{3\sqrt{2}}$$ (about 6.9943) and

$$\frac{5\sqrt{2} + 4\sqrt{3}}{2}$$ (about 6.9996), respectively.
The following result is for the unique determination of \( C \)-coconvex sets by the \( L_p \) surface area measure for \( 0 < p < 1 \). In particular, it can be applied to obtain the uniqueness of solutions to the \( L_p \) Minkowski problem (i.e., Problem 4.1), if its solution exists; see Theorem 6.2 for more details.

**Theorem 5.3.** Let \( A_1 = C \setminus A_1^* \) and \( A_2 = C \setminus A_2^* \) be two \( C \)-coconvex sets such that \( A_1^*, A_2^* \in \mathcal{K}(C, \omega) \). Then \( A_1 = A_2 \), if the following identity holds for \( 0 < p < 1 \):

\[
\mathcal{H}(A_1, \cdot)^{1-p} S_{n-1}(A_1, \cdot) = \mathcal{H}(A_2, \cdot)^{1-p} S_{n-1}(A_2, \cdot).
\]  

**Proof.** By (5.34) and (5.38), one gets

\[
\nabla_p(A_1, A_2) = \frac{1}{n} \int_\omega \mathcal{H}(A_2, u)^p \mathcal{H}(A_1, u)^{1-p} dS_{n-1}(A_1, u)
= \frac{1}{n} \int_\omega \mathcal{H}(A_2, u)^p \mathcal{H}(A_2, u)^{1-p} dS_{n-1}(A_2, u) = V_n(A_2).
\]

The \( L_p \) Minkowski inequality (5.35) yields

\[
\frac{V_n(A_2)}{V_n(A_1)} = \frac{\nabla_p(A_1, A_2)}{\nabla_p(A_1, A_1)} \leq \left( \frac{V_n(A_1)}{V_n(A_2)} \right)^{\frac{n-2}{n}} \leq \left( \frac{V_n(A_2)}{V_n(A_1)} \right)^{\frac{n}{2}}.
\]  

This implies \( V_n(A_2) \leq V_n(A_1) \). Similarly, we have \( V_n(A_2) \geq V_n(A_1) \). Thus \( V_n(A_2) = V_n(A_1) \) and inequality (5.39) becomes equality. This further leads to \( A_1 = A_2 \).

More results for the unique determination of \( C \)-coconvex sets can be obtained. For example, we have the following theorem.

**Theorem 5.4.** Let \( A_1, A_2 \) be two \( C \)-coconvex sets such that their companion sets \( A_1^*, A_2^* \in \mathcal{K}(C, \omega) \). Let \( V_n(A_1) \geq V_n(A_2) \) and \( 0 < p < 1 \).

(i) If \( V_n(A_1) \leq \nabla_p(A_1, A_2) \), then \( A_1 = A_2 \).

(ii) If \( V_n(A_1) \leq \nabla_p(A_2, A_1) \), then \( A_1 = A_2 \).

(iii) If \( \nabla_p(A_1, A_1) = \nabla_p(A, A_2) \) for any \( C \)-coconvex set \( A \) such that \( A \in \mathcal{K}(C, \omega) \), then \( A_1 = A_2 \).

**Proof.** Only the proof of the first case will be demonstrated and other cases can be proved along the same lines. By the \( L_p \) Minkowski inequality (5.35), under the assumption in Theorem 5.4, one has

\[
V_n(A_1)^n \leq \nabla_p(A_1, A_2)^n \leq V_n(A_1)^{n-p} V_n(A_2)^p,
\]

with equality in the second inequality if and only if \( A_1 = \alpha A_2 \) for some \( \alpha > 0 \). The above inequality yields that \( V_n(A_1) \leq V_n(A_2) \). Thus \( V_n(A_1) = V_n(A_2) \) by the condition, and then \( A_1 = A_2 \).

6 A solution to the \( L_p \) Minkowski problem

In this section, we will provide a solution to the \( L_p \) Minkowski problem (i.e., Problem 4.1) for all \( 0 \neq p \in \mathbb{R} \). Solutions to the \( L_p \) Minkowski problem for \( p = 1 \) and \( p = 0 \) have been provided in [41].

**Theorem 6.1.** Let \( \omega \) be a compact set of \( \Omega_C \). Suppose that \( \mu \) is a nonzero finite Borel measure on \( \Omega_C \) whose support is concentrated on \( \omega \). For \( 0 \neq p \in \mathbb{R} \), there exists a \( C \)-full set \( A_0^* \) \((A_0 = C \setminus A_0^*)\) such that

\[
\mu = c \cdot S_{n-1, p}(A_0, \cdot) \quad \text{with} \quad c = \frac{1}{nV_n(A_0)} \int_\omega \mathcal{H}(A_0, u)^p d\mu(u).
\]  

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Proof. Denote by $C^+(\omega)$ the set of all continuous and positive functions on $\omega$. For any $\alpha > 0$ and $f \in C^+(\omega)$, one can check that the Wulff shape $A_{\alpha f}^\bullet$ associated with $(C, \omega, \alpha f)$ is a dilation of the Wulff shape $A_f^\bullet$ associated with $(C, \omega, f)$, namely $A_{\alpha f}^\bullet = \alpha A_f^\bullet$. Recall (4.20) that $V_n(f) = V_n(C \setminus A_f^\bullet)$ and hence

$$V_n(\alpha f) = \alpha^n V_n(f). \tag{6.41}$$

For $p > 0$, consider the following optimization problem:

$$\sup \left\{ \mathcal{L}(f) : f \in C^+(\omega) \right\}, \tag{6.42}$$

where $\mathcal{L}(f)$ is the functional on $C^+(\omega)$ defined by

$$\mathcal{L}(f) = V_n(f)^{\frac{1}{p}} \int_\omega f(u)^p d\mu(u). \tag{6.43}$$

It follows from (6.41) that the optimization problem (6.42) has homogeneity of degree 0.

The key to solve the $L_p$ Minkowski problem is to find a $C$-full set $A_0^\bullet \in \mathcal{K}(C, \omega)$, such that $h(A_0, \cdot)$ with $A_0 = C \setminus A_0^\bullet$ is positive on $\omega$ and is a solution to the optimization problem (6.42). Indeed, if such $A_0^\bullet$ exists and $h(A_0, \cdot)$ solves (6.42), then for any continuous function $g : \omega \to \mathbb{R}$, one must have

$$\left. \frac{\partial \mathcal{L}((h(A_0, \cdot)p + tg)^{\frac{1}{p}})}{\partial t} \right|_{t=0} = 0,$$

due to the fact that the optimization problem (6.42) has homogeneity of degree 0. This, together with Theorem 4.1, further yields that, for any continuous function $g : \omega \to \mathbb{R}$,

$$\frac{1}{n} \left( \int_\omega h(A_0, u)^p d\mu(u) \right) \cdot \left( \int_\omega g(u)h(A_0, u)^{1-p} dS_{n-1}(A_0, u) \right) = V_n(A_0) \cdot \int_\omega g(u) d\mu(u). \tag{6.44}$$

As $g$ is arbitrary, one can get

$$\mu = \frac{1}{nV_n(A_0)} \left( \int_\omega h(A_0, u)^p d\mu(u) \right) \cdot h(A_0, \cdot)^{1-p} S_{n-1}(A_0, \cdot) = c \cdot S_{n-1, p}(A_0, \cdot).$$

Then $A_0$ satisfies (6.40) as desired.

Now let us claim that the optimization problem (6.42) has $h(A_0, \cdot)$ as one of its optimizers for some $A_0 = C \setminus A_0^\bullet$ such that $A_0^\bullet \in \mathcal{K}(C, \omega)$ is a $C$-full set. To this end, by Lemma 4.1, one gets for any $f \in C^+(\omega)$, $V_n(f) = V_n(C \setminus A_f^\bullet) = V_n(h(C \setminus A_f^\bullet, \cdot))$ and

$$h(C \setminus A_f^\bullet, u) = -h(A_f^\bullet, u) \leq f(u), \quad \text{for } u \in \omega.$$

It follows from (6.43) that $\mathcal{L}(h(C \setminus A_f^\bullet, \cdot)) \geq \mathcal{L}(f)$ for $p > 0$. Hence an optimizer for

$$\sup \left\{ \mathcal{L}(h(Q, \cdot)) : Q = C \setminus Q^\bullet \text{ such that } Q^\bullet \in \mathcal{K}(C, \omega) \right\}, \tag{6.45}$$

for $p > 0$ is also an optimizer of the optimization problem (6.42). Again the optimization problem (6.45) has homogeneity of degree 0, and hence it is equivalent to the following optimization problem: for $p > 0$,

$$\Theta = \sup \left\{ \int_\omega h(Q, u)^p d\mu(u) : Q = C \setminus Q^\bullet \text{ such that } V_n(Q) = 1 \text{ and } Q^\bullet \in \mathcal{K}(C, \omega) \right\}. \tag{6.46}$$
Clearly, the supremum is taken over a nonempty set:

\[ \mathcal{Q} = \left\{ Q = C \setminus Q^\bullet : V_n(Q) = 1 \text{ and } Q^\bullet \in \mathscr{K}(C, \omega) \right\} \neq \emptyset. \]

Recall that \( \zeta \in S^{n-1} \) is a special and fixed unit vector which will not appear in the following notations (see Section 2): for \( t \geq 0 \),

\[ H_t = \left\{ x \in \mathbb{R}^n : x \cdot \zeta = t \right\} \quad \text{and} \quad H_t^- = \left\{ x \in \mathbb{R}^n : x \cdot \zeta \leq t \right\}. \]

Lemma 4.2 implies that there exists a constant \( t_0 > 0 \), such that \( C \cap H_{t_0} \subset Q^\bullet \) for any \( Q \in \mathcal{Q} \).

Hence,

\[ \int_{\omega} \overline{h}(Q, u)^p \, d\mu(u) \leq \int_{\omega} (\inf (C \cap H_{t_0}, u))^p \, d\mu(u) := c_0. \]

Note that \( c_0 \) is a universal constant and is independent of the choice of \( Q \in \mathcal{Q} \). This shows that the optimization problem (6.46) is well-defined and the supremum \( \Theta \) is finite. Moreover \( \Theta > 0 \).

Take a sequence \( \{Q_i\}_{i \in \mathbb{N}} \subset \mathcal{Q} \) such that

\[ 0 < \Theta = \lim_{i \to \infty} \int_{\omega} \overline{h}(Q_i, u)^p \, d\mu(u). \]

Again, by Lemma 4.2, the constant \( t_0 > 0 \) as above satisfies that \( C \cap H_{t_0} \subset Q^\bullet_i \) for any \( i \in \mathbb{N} \). In particular, \( \emptyset \neq Q^\bullet_i \cap H_{t_0}^\circ \subset C \cap H_{t_0} \) for all \( i \in \mathbb{N} \). Therefore, the sequence of convex bodies \( \{Q^\bullet_i \cap H_{t_0}^\circ\}_{i \in \mathbb{N}} \) is bounded and the Blaschke selection theorem can be applied to get a subsequence, say \( \{Q^\bullet_{i_k} \cap H_{t_0}^\circ\}_{k \in \mathbb{N}} \), converging to some compact convex set, say \( K \). Due to the fact that \( C \cap H_{t_0} \subset Q^\bullet_i \) for any \( i \in \mathbb{N} \), there exists a closed convex set \( A^0 \subset C \) such that \( K = A^0 \cap H_{t_0} \) and \( Q^\bullet_{i_k} \to A^0 \). Moreover, \( V_n(A^0) = V_n(C \setminus A^0) = \lim_{k \to \infty} V_n(C \setminus Q^\bullet_{i_k}) = 1 \).

This further gives that \( A^0 \in \mathscr{K}(C, \omega) \) is a \( C \)-full set. It is also easily checked that \( \overline{h}(Q_{i_k}, \cdot) \) converges to \( \overline{h}(A^0, \cdot) \) uniformly on \( \omega \). We can also claim that \( \overline{h}(A^0, \cdot) \) is positive on \( \omega \) by contradiction. That is, if \( \overline{h}(A^0, u_0) = 0 \) for some \( u_0 \in \omega \), then

\[ 0 = \overline{h}(A^0, u_0) = h(A^0, u_0) = \max\{x \cdot u_0 : x \in A^0_{\bullet}\}. \]

Note that \( \omega \subset \Omega_C \) is a compact set, and \( x \cdot u_0 < 0 \) for all \( o \neq x \in C \). This further yields \( o \in A^0_{\bullet} \) and hence \( \overline{h}(A^0, u) = h(A^0_{\bullet}, u) = 0 \) for all \( u \in \omega \). In this case,

\[ A^0_{\bullet} = C \cap \bigcap_{u \in \omega} H^-(u, h(A^0_{\bullet}, u)) = C \cap \bigcap_{u \in \omega} H^-(u, 0) = C \]

and then \( A^0 = C \setminus A^0_{\bullet} = \emptyset \). This contradicts \( V_n(A^0) = 1 \) and hence \( \overline{h}(A^0, \cdot) \) is positive on \( \omega \). In conclusion, \( A^0 \in \mathcal{Q} \) and

\[ \Theta = \lim_{k \to \infty} \int_{\omega} \overline{h}(Q_{i_k}, u)^p \, d\mu(u) = \int_{\omega} \overline{h}(A^0, u)^p \, d\mu(u). \]

That is, \( A^0 \) solves the optimization problem (6.46) (and equivalently (6.45) and (6.42)). This completes the proof for the case \( p > 0 \).

Now let \( p < 0 \). In this case, we consider the following optimization problem:

\[ \inf \left\{ \mathcal{L}(f) : f \in C^+(\omega) \right\}. \]
Again the optimization problem (6.47) has homogeneity of degree 0. Moreover, by Lemma 4.1 and (6.43), one gets $\mathcal{L}(\overline{f}(C \setminus K_f, \cdot)) \leq \mathcal{L}(f)$. Hence an optimizer for

$$\inf \left\{ \int_{\omega} \overline{h}(Q, u)^p \, d\mu(u) : Q = C \setminus Q^\bullet \text{ such that } V_n(Q) = 1 \text{ and } Q^\bullet \in \mathcal{K}(C, \omega) \right\}$$  \hspace{1cm} (6.48)

for $p < 0$ is also an optimizer of the optimization problem (6.47). Clearly the optimization problem (6.48) is well defined. Repeating the arguments in the case $p > 0$, one can find $A_0^\bullet \in \mathcal{K}(C, \omega)$ which is also $C$-full, such that $A_0 \in \mathcal{Q}$ solves the optimization problem (6.48). Consequently, (6.44) holds for any continuous function $g : \omega \to \mathbb{R}$ and the desired formula (6.40) follows. \hfill \Box

**Theorem 6.2.** Let $\omega$ be a compact set of $\Omega_C$. Suppose that $\mu$ is a nonzero finite Borel measure on $\Omega_C$ whose support is concentrated on $\omega$. If $p \in \mathbb{R}$ and $p \neq 0, n$, then there exists a $C$-full set $A^\bullet$ ($A = C \setminus A^\bullet$) such that

$$\mu = \mathcal{S}_{n-1,p}(A, \cdot) = \overline{h}(A, \cdot)^{1-\tau} \mathcal{S}_{n-1}(A, \cdot).$$  \hspace{1cm} (6.49)

Moreover, if $0 < p < 1$, the solution to the $L_p$ Minkowski problem is unique.

**Proof.** By Theorem 6.1, there exists $A_0$ such that $A_0^\bullet = C \setminus A_0 \in \mathcal{K}(C, \omega)$ is a $C$-full set and (6.40) holds, namely $\mu = c \cdot \mathcal{S}_{n-1,p}(A_0, \cdot)$. The easily checked fact that $\mathcal{S}_{n-1,p}(A_0, \cdot)$ has homogeneity of degree $n - p$ implies that $A = c^{n-\tau} A_0$ satisfies (6.49), as desired. The uniqueness of the solutions to the $L_p$ Minkowski problem for $0 < p < 1$ follows immediately from Theorem 5.3. \hfill \Box

### 7 The log-Brunn-Minkowski and log-Minkowski inequalities

In this section, the log-co-sum of two $C$-coconvex sets will be introduced and related properties will be provided. In particular, we prove that the log-co-sum of $C$-coconvex sets is still a $C$-coconvex set. The log-Brunn-Minkowski and log-Minkowski inequalities will be established. The log-Minkowski inequality is applied to confirm that the solutions to the log-Minkowski problem (i.e., Problem 4.1 for $p = 0$), which aims to characterize the cone-volume measures of $C$-coconvex sets, are unique. Hence an open problem raised by Schneider in [41, p. 203] is solved.

Let us begin with the limit of (3.12) as $p \to 0^+$. Let $A_1$ and $A_2$ be two nonempty $C$-coconvex sets. For $\tau \in (0, 1)$ and $0 < p < 1$, then for all $x \in \text{int}C^\circ$,

$$\lim_{p \to 0^+} \left[ (1 - \tau) \overline{h}(A_1, x)^p + \tau \overline{h}(A_2, x)^p \right] = \overline{h}(A_1, x)^{1-\tau} \overline{h}(A_2, x)^\tau.$$  \hspace{1cm} (7.50)

For simplification, we write

$$\overline{h}_\tau(x) = \overline{h}(A_1, x)^{1-\tau} \overline{h}(A_2, x)^\tau.$$  \hspace{1cm} (7.51)

**Lemma 7.1.** Let $\tau \in (0, 1)$, and $A_1, A_2$ be nonempty $C$-coconvex sets. Then the function $\overline{h}_\tau : \text{int}C^\circ \to (0, \infty)$ is concave and has positive homogeneity of degree 1 in $\text{int}C^\circ$.

**Proof.** As $A_1$ and $A_2$ are nonempty $C$-coconvex sets, one sees that both $\overline{h}(A_1, x)$ and $\overline{h}(A_2, x)$ are positive for all $x \in \text{int}C^\circ$. Thus, $\overline{h}_\tau(x) > 0$ for all $x \in \text{int}C^\circ$. It is easy to verify that $\overline{h}_\tau(sx) = s \cdot \overline{h}_\tau(x)$ for any $x \in \text{int}C^\circ$ and $s > 0$. This shows that $\overline{h}_\tau(\cdot)$ has positive homogeneity of degree 1 in $\text{int}C^\circ$. Let $\lambda \in (0, 1)$ and $x, y \in \text{int}C^\circ$. It follows from (7.50), (7.51), and (the proof
of Lemma 3.1 (in particular, (3.14)) that
\[
\overline{h}_\tau(\lambda x + (1 - \lambda)y) = \lim_{p \to 0^+} \left[ (1 - \tau)\overline{h}(A_1, \lambda x + (1 - \lambda)y)^p + \tau\overline{h}(A_2, \lambda x + (1 - \lambda)y)^p \right]^{\frac{1}{p}} \\
\geq \lim_{p \to 0^+} \lambda \left[ (1 - \tau)\overline{h}(A_1, x)^p + \tau\overline{h}(A_2, x)^p \right]^{\frac{1}{p}} \\
+ \lim_{p \to 0^+} (1 - \lambda) \left[ (1 - \tau)\overline{h}(A_1, y)^p + \tau\overline{h}(A_2, y)^p \right]^{\frac{1}{p}} \\
= \lambda\overline{h}_\tau(x) + (1 - \lambda)\overline{h}_\tau(y).
\]

This shows that \(\overline{h}_\tau\) is concave in \(\text{int} C^\circ\) as desired.

By Lemma 7.1, \(\overline{h}_\tau\) is convex, strictly negative, and has positive homogeneity of degree 1 in \(\text{int} C^\circ\). Therefore, \(\overline{h}_\tau\) uniquely determines a closed convex set contained in \(C\). Indeed, one can restrict \(\overline{h}_\tau\) to \(\Omega_C = S^{n-1} \cap \text{int} C^\circ\) and let
\[
((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2)^\bullet = C \cap \bigcap_{u \in \Omega_C} H^-(u, -\overline{h}_\tau(u)). 
\tag{7.52}
\]

Consequently, for all \(u \in \Omega_C\)
\[
h\left(((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2)^\bullet, u\right) = -\overline{h}_\tau(u).
\]

Moreover, \(\circ \notin ((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2)^\bullet\) as \(\overline{h}_\tau > 0\) on \(\Omega_C\). Based on (7.52), let
\[
(1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2 = C \setminus ((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2)^\bullet 
\tag{7.53}
\]
and this set will be called the log-co-sum of \(C\)-coconvex sets \(A_1\) and \(A_2\) with respect to \(\tau\). Note that, for any \(u \in \Omega_C\),
\[
\overline{h}\left((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2, u\right) = \overline{h}(A_1, u)^{1-\tau} \overline{h}(A_2, u)^\tau. 
\tag{7.54}
\]

The following theorem states the log-Brunn-Minkowski inequality for two \(C\)-coconvex sets, and hence shows that the set \((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2\) is a nonempty \(C\)-coconvex set.

**Theorem 7.1.** Let \(\tau \in (0, 1)\). If \(A_1\) and \(A_2\) are nonempty \(C\)-coconvex sets, then
\[
V_n((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2) \leq V_n(A_1)^{1-\tau}V_n(A_2)^\tau. 
\tag{7.55}
\]

**Proof.** By the arithmetic-geometric inequality and (7.51), it follows that for all \(p \in (0, 1)\) and \(u \in \Omega_C\),
\[
\overline{h}\left((1 - \tau) \diamond A_1 \oplus_p \tau \diamond A_2, u\right) = \left[ (1 - \tau)\overline{h}(A_1, u)^p + \tau\overline{h}(A_2, u)^p \right]^{\frac{1}{p}} \\
\geq \overline{h}(A_1, u)^{1-\tau} \overline{h}(A_2, u)^\tau = \overline{h}_\tau(u).
\]

Thus, by (3.15), (7.52) and Theorem 3.2, one gets
\[
((1 - \tau) \diamond A_1 \oplus_0 \tau \diamond A_2)^\bullet = C \cap \bigcap_{u \in \Omega_C} H^-(u, -\overline{h}_\tau(u)) \\
\supseteq C \cap \bigcap_{u \in \Omega_C} H^-(u, -\overline{h}\left((1 - \tau) \diamond A_1 \oplus_p \tau \diamond A_2, u\right)) \\
= ((1 - \tau) \diamond A_1 \oplus_p \tau \diamond A_2)^\bullet.
\]

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Taking the complement with respect to $C$, one gets

$$(1 - \tau) \diamond A_1 \oplus_0 \tau \circ A_2 \subseteq (1 - \tau) \circ A_1 \oplus_p \tau \circ A_2.$$ 

This further yields, by the $L_p$ Brunn-Minkowski inequality (3.18), that

$$V_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2) \leq V_n((1 - \tau) \circ A_1 \oplus_p \tau \circ A_2)$$

$$\leq ((1 - \tau)V_n(A_1)^{\frac{p}{n}} + \tau V_n(A_2)^{\frac{p}{n}})^\frac{n}{p} < \infty. \quad (7.56)$$

Therefore, $(1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2$ has finite volume and hence is a $C$-coconvex set. To obtain inequality (7.55), one lets $p \to 0^+$ in (7.56), and then

$$V_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2) \leq \lim_{p \to 0^+} ((1 - \tau)V_n(A_1)^{\frac{p}{n}} + \tau V_n(A_2)^{\frac{p}{n}})^\frac{n}{p} = V_n(A_1)^{1 - \tau} V_n(A_2)^{\tau}.$$ 

This completes the proof of the theorem. \( \square \)

Unfortunately, the proof in Theorem 7.1 does not give the characterization of equality for (7.55). However, by the fact that the set $(1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2$ is a nonempty $C$-coconvex set proved in Theorem 7.1, the following theorem establishes the log-Minkowski inequality and the log-Brunn-Minkowski inequality (7.55) with characterization of equalities. Define the $L_0$ (or log) mixed volume of two nonempty $C$-coconvex sets $A_1$ and $A_2$ by

$$V_0(A_1, A_2) = \frac{1}{n} \int_{\Omega_C} \log \left( \frac{\overline{h}(A_2, u)}{\overline{h}(A_1, u)} \right) \overline{h}(A_1, u) \, dS_{n-1}(A_1, u), \quad (7.57)$$

provided the integral exists and is finite.

**Theorem 7.2.** Let $A_1$ and $A_2$ be two nonempty $C$-coconvex sets. The following log-Minkowski inequality holds with equality if and only if $A_1 = \alpha A_2$ for some $\alpha > 0$:

$$\frac{V_0(A_1, A_2)}{V_n(A_1)} \leq \frac{1}{n} \cdot \log \left( \frac{V_n(A_2)}{V_n(A_1)} \right). \quad (7.58)$$

Moreover, the log-Brunn-Minkowski inequality (7.55) holds with equality if and only if $A_1 = \alpha A_2$ for some $\alpha > 0$.

**Proof.** The proof follows along the lines for those in the proofs of Theorems 5.1 and 5.2. Indeed, a probability measure on $\Omega_C$ can be defined as follows:

$$d\nu = \overline{h}(A_1, \cdot) \frac{nV_n(A_1)}{\overline{h}(A_1, u)} \, dS_{n-1}(A_1, \cdot).$$

It follows from the Minkowski inequality (1.3) and Jensen’s inequality (as the logarithmic function is strictly concave) that

$$\frac{V_0(A_1, A_2)}{V_n(A_1)} = \int_{\Omega_C} \log \left( \frac{\overline{h}(A_2, u)}{\overline{h}(A_1, u)} \right) d\nu(u)$$

$$\leq \log \left( \int_{\Omega_C} \frac{\overline{h}(A_2, u)}{\overline{h}(A_1, u)} \, d\nu(u) \right)$$

$$= \log \left( \int_{\Omega_C} \frac{\overline{h}(A_2, u)}{nV_n(A_1)} \, dS_{n-1}(A_1, u) \right)$$

$$= \log \left( \frac{\overline{V}_1(A_1, A_2)}{V_n(A_1)} \right)$$

$$\leq \frac{1}{n} \cdot \log \left( \frac{V_n(A_2)}{V_n(A_1)} \right). \quad (7.59)$$

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This is the desired inequality (7.58). The characterization of equality for inequality (7.58) is an easy consequence of the characterization of equality for inequality (1.3) and for Jensen’s inequality (applied in (7.59)).

Although the log-Brunn-Minkowski inequality (7.55) has already been proved in Theorem 7.1, it can also be obtained by taking use of inequality (7.58) as follows: as \((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2\) is \(C\)-coconvex, one has

\[
\frac{\nabla_0((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2)}{\nabla_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2)} \leq \frac{1}{n} \log \left( \frac{\nabla_n(A_1)}{\nabla_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2)} \right),
\]

(7.60)

\[
\frac{\nabla_0((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2, A_1)}{\nabla_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2, A_2)} \leq \frac{1}{n} \log \left( \frac{\nabla_n(A_2)}{\nabla_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2)} \right).
\]

(7.61)

By (7.54) and (7.57), one can get

\[
0 = (1 - \tau)\nabla_0((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2, A_1) + \tau\nabla_0((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2, A_2).
\]

Together with (7.60) and (7.61), one gets

\[
0 \leq (1 - \tau) \log \left( \frac{\nabla_n(A_1)}{\nabla_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2)} \right) + \tau \log \left( \frac{\nabla_n(A_2)}{\nabla_n((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2)} \right),
\]

(7.62)

which is exactly the desired inequality (7.55) after simplification. To characterize the equality for (7.55), without loss of generality, we can assume that \(V_n(A_1) = V_n(A_2) = 1\) (due to the homogeneity of (7.55) for \(A_1\) and \(A_2\)). In this case, if equality holds in inequality (7.55), one must have equality in (7.62) and hence in (7.60) and (7.61). It follows from the characterization for equality in (7.58) that both \(A_1, A_2\) are dilations of \((1 - \tau) \circ A_1 \oplus_0 \tau \circ A_2\). This further implies that equality holds in (7.55) only if \(A_1 = \alpha A_2\) for some \(\alpha > 0\). Conversely, it is trivial to check that the equality holds in (7.55) if \(A_1 = \alpha A_2\) for some \(\alpha > 0\) and this completes the characterization of the equality of the log-Brunn-Minkowski inequality (7.55).

\[
\square
\]

In [41, Lemma 9], the cone-volume measure of a \(C\)-close set \(A^\bullet\), denoted by \(V_{A^\bullet}\), on \(\Omega_C\) has the following representation: for any Borel sets \(\eta \subset \Omega_C\),

\[
V_{A^\bullet} (\eta) = \frac{1}{n} \int h(A^\bullet, u) dS_{n-1}(A^\bullet, u),
\]

(7.63)

Again let \(V_{C \setminus A^\bullet} = V_{A^\bullet}\) be the cone-volume measure of \(C \setminus A^\bullet\). Clearly, \(V_{A^\bullet}(\Omega_C) = V_n(C \setminus A^\bullet) < \infty\) as \(C \setminus A^\bullet\) is a \(C\)-coconvex set. Hence the cone-volume measure \(V_{A^\bullet}\) is finite on \(\Omega_C\). The problem to characterize the cone-volume measure of \(C\)-close sets has been proposed by Schneider in [41]. This problem may be called the \(L_0\) (or log) Minkowski problem.

**Problem 7.1** (Log-Minkowski problem). Under what necessary and/or sufficient conditions on a nonzero finite Borel measure \(\mu\) on \(\Omega_C\) does there exist a \(C\)-close set \(A^\bullet\) with \(A = C \setminus A^\bullet\) such that \(\mu = \nabla_{A^\bullet}\)?

The existence of solutions to the log-Minkowski problem has been provided in [41, Theorems 4 and 5]. The following result confirms that the solutions to the log-Minkowski problem are indeed unique and hence solves the open problem raised by Schneider in [41, p. 203].

**Theorem 7.3.** Let \(A_1^\bullet, A_2^\bullet\) be two \(C\)-close sets such that \(A_1 = C \setminus A_1^\bullet\) and \(A_2 = C \setminus A_2^\bullet\) are two \(C\)-coconvex sets. Then \(A_1 = A_2\), if the following identity holds on \(\Omega_C\):

\[
\nabla_{A_1} = \nabla_{A_2}.
\]

(7.64)

In particular, the solutions to the log-Minkowski problem (i.e., Problem 7.1) are unique.
Proof. It follows from (7.57), (7.63) and (7.64) that \( V_n(A_1) = V_n(A_2) \) and

\[
\nabla_0(A_1, A_2) = \frac{1}{n} \int_{\Omega_C} \log \left( \frac{\bar{h}(A_2, u)}{\bar{h}(A_1, u)} \right) \bar{h}(A_1, u) \, dS_{n-1}(A_1, u) \\
= \int_{\Omega_C} \log \left( \frac{\bar{h}(A_2, u)}{\bar{h}(A_1, u)} \right) \, dV_{A_1}(u) \\
= \int_{\Omega_C} \log \left( \frac{\bar{h}(A_2, u)}{\bar{h}(A_1, u)} \right) \, dV_{A_2}(u) \\
= -\frac{1}{n} \int_{\Omega_C} \log \left( \frac{\bar{h}(A_1, u)}{\bar{h}(A_2, u)} \right) \bar{h}(A_2, u) \, dS_{n-1}(A_2, u) \\
= -\nabla_0(A_2, A_1). \tag{7.65}
\]

An application of the log-Minkowski inequality (7.58), together with the fact that \( V_n(A_1) = V_n(A_2) \), yields

\[
\frac{\nabla_0(A_1, A_2)}{V_n(A_1)} \leq \frac{1}{n} \cdot \log \left( \frac{V_n(A_2)}{V_n(A_1)} \right) = 0 \quad \text{and} \quad \frac{\nabla_0(A_2, A_1)}{V_n(A_2)} \leq \frac{1}{n} \cdot \log \left( \frac{V_n(A_1)}{V_n(A_2)} \right) = 0. \tag{7.66}
\]

It can be easily checked by (7.65) and (7.66) that

\[
\frac{\nabla_0(A_1, A_2)}{V_n(A_1)} = \frac{1}{n} \cdot \log \left( \frac{V_n(A_2)}{V_n(A_1)} \right) = 0.
\]

By the characterization of equality for the log-Minkowski inequality (7.58), one gets \( A_1 = \alpha A_2 \) for some \( \alpha > 0 \). Moreover, \( V_n(A_2) = V_n(A_1) = V_n(\alpha A_2) = \alpha^n V_n(A_2) \), which implies \( \alpha = 1 \). Hence \( A_1 = A_2 \) as desired.

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References

[1] A.D. Aleksandrov, On the theory of mixed volume. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies, Mat. Sb., Russian, 3 (1938): 27-46. An English translation is available in A.D. Aleksandrov, Selected Works Part I: Selected Scientific Papers. Edited by Y.G. Reshetnyak and S.S. Kutateladze, Gordon and Breach Publishers, Amsterdam 1996.

[2] K.J. Böröczky, P. Hegedűs and G. Zhu, On the discrete logarithmic Minkowski problem, Int. Math. Res. Not. IMRN, 2016 (2016): 1807-1838.

[3] K.J. Böröczky and M. Henk, Cone-volume measure of general centered convex bodies, Adv. Math., 286 (2016): 703-721.

[4] K.J. Böröczky and P. Kalantzopoulos, Log-Brunn-Minkowski inequality under symmetry, \texttt{arXiv:2002.12239}.
[5] K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang, *The log-Brunn-Minkowski inequality*, Adv. Math., 231 (2012): 1974-1997.

[6] K.J. Böröczky, E. Lutwak, D. Yang and G. Zhang, *The logarithmic Minkowski problem*, J. Amer. Math. Soc., 26 (2013): 831-852.

[7] S. Campi and P. Gronchi, *The $L_p$-Busemann-Petty centroid inequality*, Adv. Math., 167 (2002): 128-141.

[8] S. Chen, Y. Huang, Q. Li and J. Liu, *The $L_p$-Brunn-Minkowski inequality for $p < 1$*, Adv. Math., 368 (2020): 107166.

[9] S. Chen, Q. Li and G. Zhu, *The logarithmics Minkowski problem for non-symmetric measures*, Trans. Amer. Math. Soc., 371 (2019): 2623-2641.

[10] W. Chen, *$L_p$ Minkowski problem with not necessarily positive data*, Adv. Math., 201 (2006): 77-89.

[11] K. Chou and X. Wang, *Minkowski problems for complete noncompact convex hypersurfaces*, Topol. Methods Nonlinear Anal., 6 (1995): 151-162.

[12] K. Chou and X. Wang, *The $L_p$ Minkowski problem and the Minkowski problem in centroaffine geometry*, Adv. Math., 205 (2006): 33-83.

[13] A. Cianchi, E. Lutwak, D. Yang and G. Zhang, *Affine Moser-Trudinger and Morrey-Sobolev inequalities*, Calc. Var. Partial Differential Equations, 36 (2009): 419-436.

[14] W. Fenchel and B. Jessen, *Mengenfunktionen und konvexe Körper*, Danske Vid. Selskab. Mat.-fys. Medd., 16 (1938): 1-31.

[15] C. Haberl, *$L_p$ intersection bodies*, Adv. Math., 217 (2008): 2599-2624.

[16] C. Haberl and F.E. Schuster, *General $L_p$ affine isoperimetric inequalities*, J. Differential Geom., 83 (2009): 1-26.

[17] C. Haberl and F.E. Schuster, *Asymmetric affine $L_p$ Sobolev inequalities*, J. Funct. Anal., 257 (2009): 641-658.

[18] M. Henk and E. Linke, *Cone-volume measures of polytopes*, Adv. Math., 253 (2014): 50-62.

[19] Y. Huang, J. Liu and L. Xu, *On the uniqueness of $L_p$ Minkowski problems: The constant $p$-curvature case in $\mathbb{R}^3$*, Adv. Math., 281 (2015): 906-927.

[20] D. Hug, E. Lutwak, D. Yang and G. Zhang, *On the $L_p$-Minkowski problem for polytopes*, Discrete Comput. Geom., 33 (2005): 699-715.

[21] H. Jian, J. Lu and X. Wang, *Nonuniqueness of solutions to the $L_p$ Minkowski problem*, Adv. Math., 281 (2015): 845-856.

[22] H. Jian, J. Lu and G. Zhu, *Mirror symmetric solutions to the centro-affine Minkowski problem*, Calc. Var. Partial Differential Equations, 55 (2016), Art. 41, 22 pp.

[23] A. Khovanskiĭ and V. Timorin, *On the theory of coconvex bodies*, Discrete Comput. Geom., 52 (2014): 806-823.

[24] A.V. Kolesnikov and E. Milman, *Local $L_p$-Brunn-Minkowski inequalities for $p < 1$*, Mem. Amer. Math. Soc. (2020), in press. [arXiv:1711.01089](http://arxiv.org/abs/1711.01089).
[25] J. Lu and X. Wang, *Rotationally symmetric solutions to the $L_p$ Minkowski problem*, J. Differential Equations, 254 (2013): 983-1005.

[26] E. Lutwak, *The Brunn-Minkowski-Firey theory I: Mixed volume and the Minkowski problem*, J. Differential Geom., 38 (1993): 131-150.

[27] E. Lutwak, *The Brunn-Minkowski-Firey theory II: Affine and geominimal surface areas*, Adv. Math., 118 (1996): 244-294.

[28] E. Lutwak, D. Yang and G. Zhang, *$L_p$ affine isoperimetric inequalities*, J. Differential Geom., 56 (2000): 111-132.

[29] E. Lutwak, D. Yang and G. Zhang, *Sharp affine $L_p$ Sobolev inequalities*, J. Differential Geom., 62 (2002): 17-38.

[30] E. Lutwak, D. Yang and G. Zhang, *On the $L_p$-Minkowski problem*, Trans. Amer. Math. Soc., 356 (2004): 4359-4370.

[31] E. Lutwak and G. Zhang, *Blaschke-Santaló inequalities*, J. Differential Geom., 47 (1997): 1-16.

[32] L. Ma, *A new proof of the Log-Brunn-Minkowski inequality*, Geom. Dedicata, 177 (2015): 75-82.

[33] M. Meyer and E. Werner, *On the $p$-affine surface area*, Adv. Math., 152 (2000): 288-313.

[34] E. Milman and L. Rotem, *Complemented Brunn-Minkowski inequalities and isoperimetry for homogeneous and non-homogeneous measures*, Adv. Math., 262 (2014): 867-908. Corrigendum: Adv. Math., 307 (2017): 1378-1379.

[35] H. Minkowski, *Allgemeine Lehrsätze über die konvexen Polyeder*, Nach. Ges. Wiss. Göttingen, (1897): 198-219.

[36] H. Minkowski, *Volumen und Oberfläche*, Math. Ann., 57 (1903): 447-495.

[37] G. Paouris and E. Werner, *Relative entropy of cone measures and $L_p$ centroid bodies*, Proc. London Math. Soc. (3), 104 (2012): 253-286.

[38] A. Pogorelov, *An analogue of the Minkowski problem for infinite complete convex hypersurfaces*, Dokl. Akad. Nauk, 250 (1980): 553-556.

[39] C. Saroglou, *Remarks on the conjectured log-Brunn-Minkowski inequality*, Geom. Dedicata, 177 (2015): 353-365.

[40] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, 2nd edition, Encyclopedia Math. Appl., Cambridge Univ. Press, Cambridge, 2014.

[41] R. Schneider, *A Brunn-Minkowski theory for coconvex sets of finite volume*, Adv. Math., 332 (2018): 199-234.

[42] R. Schneider, *Minkowski type theorems for convex sets in cones*, Acta Math. Hungar., 164 (2021): 282-295.

[43] C. Schütt and E. Werner, *Surface bodies and p-affine surface area*, Adv. Math., 187 (2004): 98-145.

[44] A. Stancu, *The discrete planar $L_0$-Minkowski problem*, Adv. Math., 167 (2002): 160-174.

[45] A. Stancu, *The necessary condition for the discrete $L_0$-Minkowski problem in $\mathbb{R}^2$*, J. Geom., 88 (2008): 162-168.
A. Stancu, *The logarithmic Minkowski inequality for non-symmetric convex bodies*, Adv. Appl. Math., 73 (2016): 43-58.

V. Umanskiy, *On solvability of two-dimensional L_p-Minkowski problem*, Adv. Math., 180 (2003): 176-186.

J. Urbas, *The equation of prescribed Gauss curvature without boundary conditions*, J. Differential Geom., 20 (1984): 311-327.

E. Werner and D. Ye, *New L_p-affine isoperimetric inequalities*, Adv. Math., 218 (2008): 762-780.

Y. Yang and D. Zhang, *The log-Brunn-Minkowski inequality in \( \mathbb{R}^3 \)*, Proc. Amer. Math. Soc., 147 (2019): 4465-4475.

D. Ye, *L_p geominimal surface areas and their inequalities*, Int. Math. Res. Not. IMRN, 2015 (2015): 2465-2498.

D. Ye, B. Zhu and J. Zhou, *The mixed L_p geominimal surface area for multiple convex bodies*, Indiana Univ. Math. J., 64 (2015): 1513-1552.

G. Zhang, *The affine Sobolev inequality*, J. Differential Geom., 53 (1999): 183-202.

Y. Zhao, *On L_p-affine surface area and curvature measures*, Int. Math. Res. Not. IMRN, 2016 (2016): 1387-1423.

B. Zhu, J. Zhou and W. Xu, *L_p mixed geominimal surface area*, J. Math. Anal. Appl., 422 (2015): 1247-1263.

G. Zhu, *The logarithmic Minkowski problem for polytopes*, Adv. Math., 262 (2014): 909-931.

G. Zhu, *The L_p Minkowski problem for polytopes for 0 < p < 1*, J. Funct. Anal., 269 (2015): 1070-1094.

G. Zhu, *The centro-affine Minkowski problem for polytopes*, J. Differential Geom., 101 (2015): 159-174.

G. Zhu, *The L_p Minkowski problem for polytopes for p < 0*, Indiana Univ. Math. J., 66 (2017): 1333-1350.

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