Weak cosmic censorship, dyonic Kerr–Newman black holes and Dirac fields

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Abstract

It was investigated recently, with the aim of testing the weak cosmic censorship conjecture, whether an extremal Kerr black hole can be converted into a naked singularity by interaction with a massless classical Dirac test field, and it was found that this is possible. We generalize this result to electrically and magnetically charged rotating extremal black holes (i.e. extremal dyonic Kerr–Newman black holes) and massive Dirac test fields, allowing magnetically or electrically uncharged or nonrotating black holes and the massless Dirac field as special cases.

We show that the possibility of the conversion is a direct consequence of the fact that the Einstein–Hilbert energy-momentum tensor of the classical Dirac field does not satisfy the null energy condition, and is therefore not in contradiction with the weak cosmic censorship conjecture. We give a derivation of the absence of superradiance of the Dirac field without making use of the complete separability of the Dirac equation in the dyonic Kerr–Newman background, and we determine the range of superradiant frequencies of the scalar field. The range of frequencies of the Dirac field that can be used to convert a black hole into a naked singularity partially coincides with the superradiant range of the scalar field. We apply horizon-penetrating coordinates, as our arguments involve calculating quantities at the event horizon. We describe the separation of variables for the Dirac equation in these coordinates, although we mostly avoid using it.

Keywords: Dirac field, weak cosmic censorship, dyonic Kerr–Newman black hole

1. Introduction

The well-known weak cosmic censorship conjecture (WCCC), stated originally by Penrose [1], asserts that naked singularities (i.e. gravitational singularities not hidden behind an event
horizon) generically cannot be produced in a physical process from regular initial conditions, if the matter involved in the process has reasonable properties. Although there is significant evidence in favour of the validity of this conjecture, finding a general proof remains one of the major unsolved problems of classical general relativity. (For a more detailed and precise description of the WCCC and for reviews on results regarding its validity see [2–7].)

As long as a complete proof is not available, it is interesting to test the WCCC in various special cases. One possible such test is a thought experiment in which a small particle is thrown at a Kerr–Newman black hole to assess if an overextremal Kerr–Newman spacetime, which contains a naked singularity, can arise after the particle has been absorbed by the black hole. This thought experiment was considered first in [8], where it was shown that an extremal Kerr–Newman black hole cannot be overcharged or overspun by throwing a pointlike test particle with electric charge into it. In particular, it was shown that if a particle has a charge or angular momentum that would make the black hole overextremal if it absorbed the particle, then the particle will not fall into the black hole. A simpler derivation of this result was given in [9]. In [10, 11] the result of [8] was extended to dyonic Kerr–Newman black holes, which are rotating black holes with both electric and magnetic charge. More recently another version of the thought experiment, in which various test fields (scalar, electromagnetic and Dirac) are used instead of point particles, was also considered [12–16]. It was found that the weak cosmic censorship is not violated in these cases either, with the exception of the case when the test field is a Dirac field [15]. Such a result is not surprising, since the WCCC is expected to be valid only for matter that has ‘reasonable’ properties, among which a suitable energy condition is included (see e.g. [2, 3]), and the Dirac field is well known not to satisfy the weak energy condition [17], in contrast with the scalar and electromagnetic fields. Studying the case of Dirac test fields is interesting, nevertheless, because fermionic matter has an important role in physics.

In the present paper we extend the result of [15], which applies to Kerr black holes and massless neutral Dirac fields, to charged rotating black holes and charged massive Dirac fields. For the sake of generality we allow the black hole to have magnetic charge as well, i.e. we consider dyonic Kerr–Newman black holes, but we stress that the cases of Kerr–Newman, Reissner–Nordström and Kerr black holes and neutral or massless Dirac fields can be obtained from the general case by suitable special choice of the parameters.

The arguments in this paper are technically different from [15] in a few aspects. First, we make little use of the complete separability of the Dirac equation in the dyonic Kerr–Newman background; we mainly use only Fourier expansion in the time and azimuthal angle variables, along with simple properties of the Dirac field. Second, we apply horizon-penetrating coordinates, since these are well suited for calculating fluxes at the event horizon. Third, instead of the Newman–Penrose formalism we use orthonormal tetrads and four-component Dirac spinor formalism. This is done to keep the formalism close to the usual Minkowski spacetime formulation of Dirac fields (see e.g. [81]). Fourth, we construct the energy and angular momentum currents using Noether’s theorem rather than the Einstein–Hilbert energy-momentum tensor, because the latter method is not suitable in the presence of external electromagnetic fields.

The Dirac field has another remarkable feature in which it differs from the scalar and electromagnetic fields, namely it does not exhibit superradiance in black hole spacetimes. After discussing the thought experiment we present a derivation of this result as well, because it requires arguments similar to those used for the thought experiment, and because the derivations that can be found in the literature usually apply the complete separability of the Dirac equation (see e.g. [17–23]), but we would like to emphasize that this is not necessary. Our derivation is similar to the one that is outlined in [2, 24]. Moreover, the non-superradiant
nature of the Dirac field is also related to its property that it does not satisfy the weak energy condition (see e.g. [2, 17, 18]). We determine the superradiant frequency range of the scalar field as well, because it has relevance for the thought experiment. The superradiance of the scalar field is discussed in several articles (see e.g. [2, 18, 33]), but usually at zero magnetic charge, and often in a way that relies on the complete separability of the field equation.

The paper is organized as follows. In section 2 the Dirac field is introduced and its conservation laws relevant for the thought experiment are discussed. This is done in a general setting, i.e. the discussion is not specialized to black hole spacetimes. In section 3 the relevant properties of dyonic Kerr–Newman black holes are recalled. In section 4 the thought experiment is described and the derivation of the main result, which indicates the possibility of the formation of a naked singularity as a result of the interaction of a black hole and a classical Dirac field, is presented. A discussion of the relevance of backreaction effects is also included. In section 5 the absence of superradiance of Dirac fields around dyonic Kerr–Newman black holes is derived and the superradiant frequency range of the scalar field is determined. Conclusions are given in section 6. In appendix A a part of the formalism of spinor fields in curved spacetime is recalled for completeness and to fix notation. In appendix B the separation of variables for the Dirac equation, pertaining to the horizon-penetrating coordinates and to the tetrad used in this paper, is described. The asymptotic behaviour of the radial functions at the event horizon is also determined.

The signature of metric tensors will be (+, −, −, −).

2. The Dirac field

The Lagrangian density of the Dirac field \( \Psi \) in fixed gravitational and electromagnetic fields is

\[
\mathcal{L} = \frac{1}{2} g^{\mu \nu} \left[ \bar{\Psi} i \gamma_\mu (\nabla_\nu + ie A_\nu) \Psi - (\nabla_\nu - ie A_\nu) \bar{\Psi} i \gamma_\mu \Psi \right] - m \bar{\Psi} \Psi,
\]

where \( A_\mu \) is the vector potential of the electromagnetic field, \( m \) is the mass parameter of the Dirac field and \( e \) is the electromagnetic coupling constant. For the definition of \( \nabla_\mu \), \( \gamma_\mu \) and \( \bar{\Psi} \) see appendix A. The Euler–Lagrange equation corresponding to \( \mathcal{L} \) is the Dirac equation, \( i \gamma^\mu (\nabla_\mu + ie A_\mu) \Psi = m \Psi \).

2.1. Conserved currents

The electric current of the Dirac field is

\[
j^\mu_{\text{em}} = -e \bar{\Psi} \gamma^\mu \Psi = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)}.
\]

The closely related current \( j^\mu = \bar{\Psi} \gamma^\mu \Psi \) is often called the particle number density current. The vector \( j^\mu \) has the important and well-known property that it is real, future directed and time-like or null for any Dirac spinor \( \Psi \), regardless of the equation of motion. Furthermore, one can also verify that \( (\bar{\Psi} \gamma_\mu \Psi)(\bar{\Psi} \gamma^\mu \Psi) = 4w^2 w \), where \( w = \bar{\Psi} \gamma^3 \Psi + \bar{\Psi} \gamma^2 \Psi \), thus \( j^\mu \) is null if and only if \( w = 0 \). These properties of \( j^\mu \) imply that the electric charge of a classical–Dirac field has a definite sign, which is the same as the sign of \(-e\).

Regarding conserved currents associated with Killing fields, a standard way in general relativity to construct such currents is to take \( T^{\mu \nu} K_\nu \), where \( K^\mu \) is the relevant Killing vector field and \( T^{\mu \nu} \) is the Einstein–Hilbert energy-momentum tensor obtained by the variation of the matter action with respect to the metric. The conservation of \( T^{\mu \nu} K_\nu \) follows from \( \nabla_\mu T^{\mu \nu} = 0 \) and from the Killing equation. Although the Lagrangian density (1) depends
explicitly (i.e. not only through the metric) on the tetrad field, the definition of the Einstein–Hilbert energy-momentum tensor can be extended to such cases (see e.g. [80]). However, as is well known, in the presence of external fields (in particular in the presence of an external electromagnetic field) generally \( \nabla_i T^{\mu\nu} = 0 \) and \( T^{\mu\nu} K_{\mu} \) is not conserved, thus one has to find some other way to construct a suitable conserved current. If the matter action is invariant under the diffeomorphisms generated by the Killing field, then Noether’s theorem is still available for this purpose. In the following we discuss the Noether currents of the Dirac field associated with Killing fields, and compare them with the currents \( T^{\mu\nu} K_{\mu} \).

Let us assume that coordinates are chosen so that there is one coordinate function, which we denote by \( t \), for which \( K^\mu = (\partial_t)^\mu \). In these coordinates \( K^\mu \) generates translations of \( t \). Let us also assume that the tetrad (and thus also \( g^m_n \)) is chosen so that it is invariant under \( t \)-translations. In addition, the vector potential of the external electromagnetic field is also assumed to be invariant under \( t \)-translations. In this case the action of the Dirac field is invariant under \( t \)-translations, and the straightforward application of Noether’s theorem gives the conserved current

\[
\mathcal{E}^\mu = \frac{\partial}{\partial \Psi} \partial_{\Psi} \Psi + \frac{\partial}{\partial \bar{\Psi}} \partial_{\bar{\Psi}} \bar{\Psi} - \delta^\mu \mathcal{L} = \frac{1}{2} (i \bar{\Psi} \gamma^\mu \partial_{\Psi} \Psi - i \partial_{\bar{\Psi}} \bar{\Psi} \gamma^\mu \Psi).
\]

On the right hand side the term \( \delta^\mu \mathcal{L} \) is omitted because \( \mathcal{L} = 0 \) if \( \Psi \) satisfies the Dirac equation. It is worth noting that \( \mathcal{E}^\mu \) is real, and if \( \Psi \) has the \( t \)-dependence \( \Psi = e^{-i\omega t} \), then \( \mathcal{E}^\mu = \omega j^\mu \).

If the electromagnetic field or \( e \) is zero, then \( T^\mu_\nu \) is also conserved, thus it is natural to ask what the relation between \( T^\mu_\nu \) and \( \mathcal{E}^\mu \) is in this case. In the following we show that the answer to this question is that the difference between these two currents is a current of the form \( \nabla_i f^{\mu\nu} \), where \( f^{\mu\nu} \) is antisymmetric, therefore \( T^\mu_\nu \) and \( \mathcal{E}^\mu \) can be considered to be equivalent. In fact we derive a more general result, equation (8), which holds also in the presence of an electromagnetic field. (8) will be useful in section 4.

The Einstein–Hilbert energy-momentum tensor of the Dirac field is

\[
T^{\mu\nu} = \frac{1}{4} (\bar{\Psi} \gamma^\mu (\nabla^\nu + ieA^\nu) \Psi + \bar{\Psi} i \gamma^\nu (\nabla^\mu + ieA^\mu) \Psi - (\nabla^\mu - ieA^\mu) \bar{\Psi} i \gamma^\nu \Psi - (\nabla^\nu - ieA^\nu) \bar{\Psi} i \gamma^\mu \Psi).
\]

We also introduce the similar tensor

\[
\hat{T}^{\mu\nu} = \frac{1}{2} (\bar{\Psi} \gamma^\mu (\partial_{\nu} + ieA_{\nu}) \Psi - (\partial_{\mu} - ieA_{\mu}) \Psi \gamma^\nu \psi),
\]

which will appear in section 4 as well, and we define \( f^{\mu\nu} \) as

\[
f^{\mu\nu} = -\frac{1}{8} i \bar{\Psi} \gamma^\nu \gamma^\mu \Psi - \gamma^{\nu \mu} \gamma^\nu \Psi.
\]

By evaluating \( \nabla_i f^{\mu\nu} \) one finds that if \( \Psi \) satisfies the Dirac equation, then

\[
\nabla_i f^{\mu\nu} = \frac{1}{4} [i \bar{\Psi} \gamma^\mu \nabla_i \Psi - \bar{\Psi} \gamma^\mu \nabla_i \Psi - \bar{\Psi} \gamma^\mu \Psi + \bar{\Psi} i \nabla_i \gamma^\mu \Psi] - \frac{1}{2} [i \bar{\Psi} \gamma^\mu (\partial_{\nu} - \partial_{\nu}) \Psi - i (\nabla_{\nu} - \partial_{\nu}) \bar{\Psi} \gamma^\mu \Psi] + \frac{1}{2} eA^\mu \bar{\Psi} \gamma^n \Psi - \frac{1}{2} eA^\nu \bar{\Psi} \gamma^\mu \Psi.
\]
From this result and from (4) and (5), it can be seen immediately that

\[ \hat{T}^\mu_i - T^\mu_i = \nabla_i f^{\mu\nu}. \]  

(8)

The current on the right hand side is conserved for arbitrary \( \Psi \), because \( f^{\mu\nu} \) is by definition antisymmetric.

By applying Stokes’s theorem it is easy to show, and is well known, that if a current has the form \( \nabla_i f^{\mu\nu} \), where \( f^{\mu\nu} \) is antisymmetric, then any corresponding charge associated with some hypersurface (which does not need to be space-like) is zero if the surface integral arising in the application of Stokes’s theorem vanishes. Therefore, in view of (8) \( \hat{T}^\mu_i \) and \( T^\mu_i \) can be considered to be equivalent.

In the absence of an electromagnetic field \( \hat{\mathcal{E}}^\mu = \mathcal{E}^\mu \), thus in this case (8) shows that \( \mathcal{E}^\mu \) and \( T^\mu_i \) are equivalent.

We note that a similar but more special result on the equivalence of \( \mathcal{E}^\mu \) and \( T^\mu_i \) can be found in [31].

3. The dyonic Kerr–Newman black holes

A dyonic Kerr–Newman black hole can be characterized by four parameters: the mass \( M \), the angular momentum per unit mass \( a \), the electric charge \( Q_e \) and the magnetic charge \( Q_m \). The angular momentum of the black hole is \( J = aM \), and \( Q_m = 0 \) corresponds to a usual Kerr–Newman black hole. The metric of the dyonic Kerr–Newman black hole spacetime with parameters \( (M, a, Q_e, Q_m) \) is the same as the Kerr–Newman metric with parameters \( (M, a, q) \), \( q^2 = Q_e^2 + Q_m^2 \), where \( q \) denotes the electric charge parameter of the Kerr–Newman metric. The parameters have to satisfy the inequality

\[ \eta = M^2 - Q_e^2 - Q_m^2 - a^2 \geq 0, \]  

(9)

otherwise the spacetime contains a naked singularity. The black hole is called extremal if \( \eta = 0 \). Under certain conditions, the dyonic Kerr–Newman black holes are the only static and asymptotically flat black hole solutions of the Einstein–Maxwell equations [34, 35].

The vector potential of the electromagnetic field of a dyonic Kerr–Newman black hole is

\[ A = Q_e A_e + Q_m A_m, \]  

(10)

where

\[ A_e = - \frac{r}{\Sigma} dt + \frac{ar \sin^2 \theta}{\Sigma} d\phi \]  

(11)

\[ A_m = \frac{a \cos \theta}{\Sigma} dt + \left[ \dot{\mathcal{C}} - \frac{r^2 + a^2 \cos^2 \theta}{\Sigma} \right] d\phi, \]  

(12)

\[ \Sigma = r^2 + a^2 \cos^2 \theta. \]  

(13)

These formulas are written in Boyer–Lindquist coordinates \( (t, r, \theta, \phi) \). The electromagnetic field derived from \( A_m \) is dual to the electromagnetic field derived from \( A_e \). The electromagnetic field does not depend on the constant \( \dot{\mathcal{C}} \), which can be used, by setting \( \dot{\mathcal{C}} = 1 \) or \( \dot{\mathcal{C}} = -1 \), to eliminate the Dirac string singularity of \( A_m \) along the positive or negative \( z \) axis (\( \theta = 0 \) and \( \theta = \pi \)), respectively. We set \( \dot{\mathcal{C}} \) to zero for a reason that is explained below.
3.1. Horizon-penetrating coordinates

In the following sections various quantities will be considered at the future event horizon. Since the Boyer–Lindquist coordinates do not cover the future event horizon, Eddington–Finkelstein-type ingoing horizon-penetrating coordinates, denoted by \((t, r, \theta, \varphi)\), will be used. These coordinates can be introduced by the transformation

\[
\tau = t - r + \int dr \frac{r^2 + a^2}{\Delta}, \quad \varphi = \phi + \int dr \frac{a}{\Delta},
\]

where \(\Delta = r^2 + a^2 + Q_e^2 + Q_m^2 - 2Mr\). The future event horizon is located in these coordinates at the constant value \(r = M + \sqrt{M^2 - (a^2 + Q_e^2 + Q_m^2)}\) of \(r\), and the metric is non-singular in these points. The inner horizon is located at \(r = M - \sqrt{M^2 - (a^2 + Q_e^2 + Q_m^2)}\). In the extremal case \(r = r_+ = M\). The \((\tau + r, \theta, \varphi)\) = constant lines are ingoing null geodesics, and there exists an \(r_0 < r_+\) such that the \(\tau = constant\) hypersurfaces are space-like in the domain \(r_0 < r\).

The \(r\) component \((A_e)_r\) of \(A_e\) with respect to the coordinates \((\tau, r, \theta, \varphi)\) is singular at the event horizon, but this singularity can be eliminated by the gauge transformation \(A_e \rightarrow A_e - \frac{\alpha}{\Delta} dr\). After this gauge transformation

\[
A_e = -\frac{r}{\Sigma} d\tau + \frac{a r \sin \theta d\theta}{\Sigma} - \frac{r}{\Sigma} dr.
\]

The \(r\) component of \(A_m\) with respect to the coordinates \((\tau, r, \theta, \varphi)\) is also singular if \(\tilde{C} \neq 0\), therefore we set \(\tilde{C} = 0\). Nevertheless, in order to treat the Dirac string singularity of \(A_m\), we introduce an explicit gauge parameter into it by adding \(C d\varphi\), where \(C\) is a real constant. Thus

\[
A_m = a \cos \theta d\tau + \left[ C - \frac{r^2 + a^2 \cos \theta}{\Sigma} \right] d\varphi + \frac{a \cos \theta}{\Sigma} dr.
\]

Generally \(A_m\) has a string singularity along the \(z\) axis (which corresponds to \(\theta = 0\) and \(\theta = \pi\)) because \(d\varphi\) is singular here, and its coefficient \((A_m)_\varphi\) does not cancel this singularity. However, in the special cases \(C = 1\) and \(C = -1\) the singularity is cancelled along the positive \(z\) axis \((\theta = 0)\) or along the negative \(z\) axis \((\theta = \pi)\), respectively. The string singularity can therefore be avoided by using two domains that cover the whole spacetime region of interest in such a way that one of the domains contains the entire positive \(z\) axis but is well separated from the negative \(z\) axis and the other one contains the entire negative \(z\) axis but is separated from the positive \(z\) axis. In the first domain the \(C = 1\) gauge is used then, and in the second domain the \(C = -1\) gauge. Suitable domains are given by the relations \(r_0 < r, 0 \leq \theta < \pi/2 + \epsilon\) and \(r_0 < r, \pi/2 - \epsilon < \theta \leq \pi\), where \(\epsilon\) is some small number. These domains will be denoted by \(\mathcal{D}_+\) and \(\mathcal{D}_-\). It should be kept in mind that the transition between the two domains involves a gauge transformation. This approach to treating the string singularity of \(A_m\) was proposed in [36] and was taken also in [11, 12, 16].

In the following sections and in appendix B, except in section 3.3, we use only the coordinates \((\tau, r, \theta, \varphi)\), and we also use the notation \(\zeta\) for the one-form \(dr\) (the exterior derivative of the coordinate function \(r\)), i.e.

\[
\zeta^\mu = (dr)^\mu.
\]

\(A_e, A_m\) and \(A\) will denote (15), (16) and \(A = Q_e A_e + Q_m A_m\), respectively.
3.2. Various important properties

In this section further important properties of dyonic Kerr–Newman black holes, which will be used in the subsequent sections, are collected. \( \partial_t \) and \( \partial_j \) are Killing fields; \( \partial_t \) is the generator of time translations and \( \partial_j \) is the generator of rotations around the axis of the black hole. \( A_e \) and \( A_m \) are also invariant under these symmetries. The Killing field

\[
\chi = \partial_r + \Omega_H \partial_r, \quad \Omega_H = \frac{a}{r_t^2 + a^2}
\]

is null at the event horizon. In the subsequent sections it will also be important that at the event horizon

\[
(A_e)_r \chi^\mu = -\frac{r_t}{r_t^2 + a^2}, \quad (A_m)_r \chi^\mu = C \Omega_H,
\]

and \( \zeta^\mu \) is parallel to \( \chi^\mu \). The relation between \( \zeta^\mu \) and \( \chi^\mu \) at the event horizon is

\[
\zeta^\mu = -\frac{r_t^2 + a^2}{r_t^2 + a^2 \cos^2 \theta} \chi^\mu,
\]

thus \( \zeta^\mu \) is past directed (in [16] \( \zeta^\mu \) was denoted by \( \omega^\mu \) and it was future directed because of the opposite signature of the metric there). (20) shows that \( \zeta^\mu \) is null at the event horizon, but it should be stressed that this property of \( \zeta^\mu \) follows directly from the facts that the event horizon is a null surface and is a level surface of the function \( r \).

It is useful to introduce the quantity \( \Phi_H \) as

\[
\Phi_H = \frac{r_t Q}{r_t^2 + a^2}.
\]

In the case of Kerr–Newman black holes, \( \Phi_H \) is known as the electrostatic potential of the horizon.

3.3. Tetrad

In order to define a suitable tetrad for the Kerr–Newman metric one can start with the Kinnersley-type tetrad (see also [37])

\[
V_\mu^0 = \frac{1}{\sqrt{2}} \left( \left( 1 + \frac{\Delta}{2 \Sigma} \right) dr + \left( \frac{1}{2} - \frac{\Sigma}{\Delta} \right) d\tau - \left( 1 + \frac{\Delta}{2 \Sigma} \right) a \sin^2 \theta \, d\phi \right)
\]

(22)

\[
V_\mu^1 = -\frac{a^2 \cos \theta \sin \theta}{\Sigma} dr + r \, d\theta + \frac{a(a^2 + r^2) \cos \theta \sin \theta}{\Sigma} d\phi
\]

(23)

\[
V_\mu^2 = \frac{ar \sin \theta}{\Sigma} d\tau + a \cos \theta \, d\theta - \frac{r(a^2 + r^2) \sin \theta}{\Sigma} d\phi
\]

(24)

\[
V_\mu^3 = \frac{1}{\sqrt{2}} \left( \left( -1 + \frac{\Delta}{2 \Sigma} \right) dr + \left( \frac{1}{2} + \frac{\Sigma}{\Delta} \right) d\tau + \left( 1 - \frac{\Delta}{2 \Sigma} \right) a \sin^2 \theta \, d\phi \right)
\]

(25)

given in Boyer–Lindquist coordinates. This can be transformed into the ingoing horizon-penetrating coordinates, but one finds that it is singular at the event horizon. This singularity can nevertheless be removed by a suitable local Lorentz transformation, similarly as for example in [32]. Thus in the present paper we use the Lorentz transformed non-singular tetrad.
\[ \mathcal{V}_\mu^0 \text{ related to } V_\mu^0 \text{ as} \]

\[ \mathcal{V}_\mu^0 = \frac{r^2}{\Delta} (V_\mu^0 + V_\mu^3) + \frac{\Delta}{r^2} (V_\mu^0 - V_\mu^3) \]

\[ \mathcal{V}_\mu^3 = \frac{r^2}{\Delta} (V_\mu^0 + V_\mu^3) - \frac{\Delta}{r^2} (V_\mu^0 - V_\mu^3) \]

\[ \mathcal{V}_\mu^1 = V_\mu^1, \quad \mathcal{V}_\mu^2 = V_\mu^2. \]

\( \mathcal{V}_\mu^0 \) and \( V_\mu^0 \) are invariant under time translations and under rotations around the axis of the black hole, and \( \mathcal{V}_\mu^0 \) tends to \( V_\mu^0 \) if \( r \to \infty \). It should also be mentioned that \( \mathcal{V}_\mu^0 \) and \( V_\mu^0 \) are not null tetrads, rather \( \mathcal{V}_\mu^0 V^\mu_\nu = \mathcal{V}_\mu^\nu \mathcal{V}_\nu^\mu = g^{\mu\nu} \), where \( g^{\mu\nu} = \text{diag}(1, -1, -1, -1) \). We note finally that another useful choice for \( V_\mu^0 \) would be the ‘canonical’ tetrad of Carter [37, 38].

4. The thought experiment

The thought experiment for testing the WCCC is assumed to proceed in the following way. Initially one has an extremal dyonic Kerr–Newman black hole, then a small amount of matter represented by a wave packet is thrown at it from a great distance. A certain part of the matter is absorbed by the black hole, the remaining part is scattered back to infinity, and finally the system settles down in another dyonic Kerr–Newman state with slightly different parameters.

Under an infinitesimally small change \((dM, dJ, dQ_\text{e}, dQ_\text{m})\) of the parameters \((M, J, Q_\text{e}, Q_\text{m})\) of a dyonic Kerr–Newman configuration, the change of \( \eta \) (which was introduced in (9)) is

\[ d\eta = 2 \left( \frac{M^2}{M^2 + a^2} dJ - \frac{a}{M^2 + a^2} dQ_\text{e} - \frac{Q_\text{e} M}{M^2 + a^2} dQ_\text{m} - \frac{Q_\text{m} M}{M^2 + a^2} dQ_\text{m} \right). \]

If one calculates the change \((dM, dJ, dQ_\text{e}, dQ_\text{m})\) of the parameters in the process described above, one should find \( d\eta > 0 \), if the final state is a dyonic Kerr–Newman black hole and cosmic censorship is not violated, whereas a result \( d\eta < 0 \) indicates the formation of a naked singularity, and thus a violation of the WCCC. Of course, in the case \( d\eta < 0 \) the last conclusion that the WCCC is violated can be drawn only if the matter used in the thought experiment does have the properties required in the WCCC.

In the calculation of \((dM, dJ, dQ_\text{e}, dQ_\text{m})\) the test matter approximation is used, i.e. the metric and the electromagnetic field are considered fixed and backreaction effects are neglected. The reason for taking the initial black hole state to be extremal is that the quantities \((dM, dJ, dQ_\text{e}, dQ_\text{m})\) are very small, in accordance with the test matter approximation.

There are several articles, e.g. [39–71], in which other versions or aspects of the thought experiment are studied. For instance, backreaction effects and subextremal initial black holes are considered in several papers. Furthermore, besides the thought experiment it is interesting to study the possibilities of observing naked singularities that may form if the WCCC is violated; see e.g. [74–79].

We turn now to the calculation of \( d\eta \). In the following the black hole is not restricted to be extremal unless explicitly stated. Applying (3) to the Killing fields \((\delta_\nu)^\mu\) and \((\partial_\nu)^\mu\) one obtains that the energy and angular momentum currents are given by the equations
\[ \mathcal{E}^{\mu} = \hat{T}^{\mu}_{\tau} + eA_{\tau}j^{\mu}, \]  

\[ \mathcal{J}^{\mu} = \hat{T}^{\mu}_{\varphi} + eA_{\varphi}j^{\mu}, \]

where \( \hat{T}^{\mu}_{\nu} \) is given by (5).

\( \hat{T}^{\mu}_{\nu} \) and \( j^{\mu} \) are gauge invariant and \( A_{\nu} \) does not depend on the gauge parameter \( C \), therefore \( \mathcal{E}^{\mu} \) is also independent of \( C \). \( A_{\varphi} \) does depend on \( C \), however, thus \( \mathcal{J}^{\mu} \) also depends on it. For this reason we take (as in [16]) the modified definition

\[ \mathcal{J}^{\mu} = \hat{T}^{\mu}_{\varphi} + e(A_{\varphi} - Q_{m}C)j^{\mu} \]

for \( \mathcal{J}^{\mu} \), which eliminates its dependence on \( C \). The conservation of \( \mathcal{J}^{\mu} \) is not affected by this modification, because \( j^{\mu} \) is conserved. The independence of \( \mathcal{E}^{\mu} \) and \( \mathcal{J}^{\mu} \) of \( C \) is important because the value of \( C \) is different in the domains \( \mathcal{D}_{a} \) and \( \mathcal{D}_{b} \).

The electric charge flux through the event horizon into the black hole is

\[ \frac{dQ}{d\tau} = \int_{H} \sqrt{-g} \, eJ_{\mu} \, d\theta d\varphi, \]

where \( H \) denotes the two-dimensional surface of the black hole (which is the relevant time slice of the event horizon), and the energy and angular momentum fluxes are

\[ \frac{dE}{d\tau} = -\int_{H} \sqrt{-g} \, (\hat{T}^{\tau}_{\tau} + eA_{\tau}j^{\tau}) \, d\theta d\varphi, \]

\[ \frac{dL}{d\tau} = \int_{H} \sqrt{-g} \, (\hat{T}^{\varphi}_{\varphi} + e(A_{\varphi} - Q_{m}C)j^{\varphi}) \, d\theta d\varphi, \]

where the quantities in the brackets are \( \mathcal{E}^{\tau} \) and \( \mathcal{J}^{\tau} \), respectively. The total energy, angular momentum and electric charge that fall through the event horizon are \( \int_{-\infty}^{\infty} \frac{dE}{d\tau} \), \( \int_{-\infty}^{\infty} \frac{dL}{d\tau} \) and \( \int_{-\infty}^{\infty} \frac{dQ}{d\tau} \), respectively. The metric and the electromagnetic field are taken to be fixed, therefore these quantities can be identified with \( dM \), \( dJ \) and \( dQ_{e} \), i.e. with the change of the mass, angular momentum and electric charge of the black hole. \( dQ_{m} = 0 \), since the Dirac field does not have magnetic charge.

From equations (33–35) above and from (18), (19) and (21) it follows immediately that

\[ -\int_{H} \sqrt{-g} \, \hat{T}^{\mu}_{\nu} \zeta^{\mu} \lambda^{\nu} \, d\theta d\varphi = \frac{dE}{d\tau} - \Omega_H \frac{dL}{d\tau} - \Phi_H \frac{dQ}{d\tau}. \]

Taking into account the relations \( dM = \int_{-\infty}^{\infty} \frac{dE}{d\tau} d\tau, dJ = \int_{-\infty}^{\infty} \frac{dL}{d\tau} d\tau \) and \( dQ_{e} = \int_{-\infty}^{\infty} \frac{dQ}{d\tau} d\tau \),

\[ -\int_{-\infty}^{\infty} d\tau \int_{H} \sqrt{-g} \, \hat{T}^{\mu}_{\nu} \zeta^{\mu} \lambda^{\nu} \, d\theta d\varphi = dM - \Omega_H dJ - \Phi_H dQ_{e} \]

is obtained from (36). It is easy to see that in the extremal case the right hand side in (37) is \( \frac{M}{2(M+\alpha^{2})} d\eta \), thus the sign of \( d\eta \) depends, in the extremal case, on the sign of \( \int_{-\infty}^{\infty} d\tau \int_{H} \sqrt{-g} \, \hat{T}^{\mu}_{\nu} \zeta^{\mu} \lambda^{\nu} \, d\theta d\varphi \).

In order to examine \( \int_{-\infty}^{\infty} d\tau \int_{H} \sqrt{-g} \, \hat{T}^{\mu}_{\nu} \zeta^{\mu} \lambda^{\nu} \, d\theta d\varphi \) it is useful to consider the Fourier expansion

\[ \Psi = \sum_{n} \int d\omega \, e^{-i\omega r} e^{i(n-CeQ_{a})\varphi} \psi_{\omega,n}^{\pm}(r, \theta) \]

of \( \Psi \), where \( e^{-i\omega r} e^{i(n-CeQ_{a})\varphi} \psi_{\omega,n}^{\pm}(r, \theta) \) are solutions of the Dirac equation. The term \(-CeQ_{m}\) in the factor \( e^{i(n-CeQ_{a})\varphi} \) is included because of the gauge transformation done in equation (16) (see also [37]). For \( e^{-i\omega r} e^{i(n-CeQ_{a})\varphi} \psi_{\omega,0}^{\pm}(r, \theta) \) to be single valued for both \( C = 1 \) and
\[ C = -1, \text{ both } n - eQ_m \text{ and } n + eQ_m \text{ have to be integer, implying that } n \text{ and } eQ_m \text{ are either integer or half-integer. The summation in (38) should therefore be done over } \mathbb{Z} \text{ if } eQ_m \text{ is integer and over } \frac{1}{2} + \mathbb{Z} \text{ if } eQ_m \text{ is half-integer. Far from the black hole, only modes with } |\omega| > m \text{ propagate waves. Using (38) one finds that}
\]
\[
\int_{-\infty}^{\infty} d\tau \int_{\mathcal{H}} \sqrt{-g} \, \hat{T}_{\mu\nu} \zeta^{\mu} \chi^{\nu} \, d\Omega d\phi = (2\pi)^2 \sum_n \int d\omega \int d\theta \, \sqrt{-g} \, \zeta^{\mu} \hat{\psi}_{\mu,\omega,n} \gamma^{\nu} \psi_{\nu,\omega,n} \left( \omega - n\Omega_H + \frac{eQ_m r_n}{r_n^2 + a^2} \right).
\]

(39)

In the derivation of (39) the only derivatives of \( \Psi \) that appear are \( \partial_r \Psi \) and \( \partial_\tau \Psi \), which are easy to evaluate, and formulas (18) and (19) can also be applied.

As was mentioned in section 3.2, \( \zeta^{\mu} \) is a past-directed null vector at the event horizon, and in section 2.1 it was also noted that \( \hat{\psi}_{\mu,\omega,n} \gamma^{\nu} \psi_{\nu,\omega,n} \) is a real future-directed null or time-like vector, therefore \( \zeta^{\mu} \hat{\psi}_{\mu,\omega,n} \gamma^{\nu} \psi_{\nu,\omega,n} \leq 0 \). The integrand on the right hand side of (39) is thus positive if
\[
\tilde{\omega} = \omega - n\Omega_H + \frac{eQ_m r_n}{r_n^2 + a^2} < 0
\]
and \( \zeta^{\mu} \hat{\psi}_{\mu,\omega,n} \gamma^{\nu} \psi_{\nu,\omega,n} \neq 0 \) at the event horizon. (Here the notation \( \tilde{\omega} \) has been introduced.) If \( \zeta^{\mu} \hat{\psi}_{\mu,\omega,n} \gamma^{\nu} \psi_{\nu,\omega,n} \tilde{\omega} \) is large at the event horizon mainly for those values of \( \omega \) and \( n \) for which \( \tilde{\omega} < 0 \), then it is possible for the whole integral (39) to be positive. In this case \( dM - \Omega_H dJ - \Phi_H dQ_e < 0 \), in particular in the extremal case \( d\eta < 0 \), indicating a possible violation of the WCCC.

Clearly \( \tilde{\omega} \) is negative if \( \omega \) has a sufficiently large negative value, but, more interestingly, \( \tilde{\omega} < 0 \) is possible even for \( \omega > 0 \), if
\[
n\Omega_H - \frac{eQ_m r_n}{r_n^2 + a^2} > 0.
\]

(41)

It is also interesting to note that the frequency range where \( \tilde{\omega} < 0 \) partially coincides with the range where the scalar field exhibits superradiance (see section 5.2).

In the special case when the charges \( Q_e \) and \( Q_m \) of the black hole are zero, one can use instead of (30) and (31) the energy and angular momentum currents obtained from the Einstein–Hilbert energy-momentum tensor, as is usually done in the literature (see, for example, [15, 17]). In view of the arguments in the last part of section 2.1, this would give the same result (namely \( -1 \times \) the right hand side of (39)) for \( dM - \Omega_H dJ - \Phi_H dQ_e \).

A tensor similar to \( \hat{T}^{\mu\nu} \) appears also in that version of the thought experiment in which the test field is a scalar field (see section 4.1 of [16] and section 5.2). In that case \( \hat{T}^{\mu\nu} \) satisfies the null energy condition \( \hat{T}^{\mu\nu} \chi_\mu \chi_\nu \geq 0 \) at the event horizon, and this implies that the WCCC is not violated. If this null energy condition held in the case of the Dirac test field, then it could be used in the same way as in the case of the scalar test field to show that
\[
dM - \Omega_H dJ - \Phi_H dQ_e \geq 0 \text{ and the WCCC is not violated.}
\]

In the case of the scalar field \( \hat{T}^{\mu\nu} \) is the Einstein–Hilbert energy-momentum tensor, and it was shown in the last part of section 2.1 that \( \hat{T}^{\mu\nu} \) is related to the Einstein–Hilbert energy-momentum tensor also in the case of the Dirac field. Moreover, \( \hat{T}^{\mu\nu} \zeta^{\mu} \chi^{\nu} = \hat{T}_\tau + \Omega_H \hat{T}_\tau \), therefore (8) and Stokes’s theorem imply that the left hand side of (37) can be written also as
\[
- \int_{-\infty}^{\infty} d\tau \int_{\mathcal{H}} \frac{\sqrt{-g}}{N} \, T_{\mu\nu} \zeta^{\mu} \chi^{\nu} \, d\Omega d\phi,
\]
where \( T_{\mu\nu} \) is the Einstein–Hilbert energy-momentum tensor of the Dirac field given by (4). Thus the result (37) for \( dM - \Omega_H dJ - \Phi_H dQ_e \) is
completely analogous to the result obtained in the case of the scalar field in [16], and one can say that the conversion of a black hole into a naked singularity by a Dirac field is possible because the Einstein–Hilbert energy-momentum tensor of the Dirac field does not satisfy the null energy condition \( T^\mu_\nu \chi_\mu \chi_\nu \geq 0 \).

The case of combined scalar and electromagnetic test matter was also considered in section 4.2 of [16], and also in that case it was found that \( dM = \Omega_d dJ - \varphi_H dQ_e = -\int_{-\infty}^{\infty} dr \int_H \sqrt{-g} \, T^\mu_{\nu} \zeta^\mu \chi^\nu \, d\theta d\varphi \), where \( T^\mu_{\nu} \) is the relevant Einstein–Hilbert energy-momentum tensor. This \( T^\mu_{\nu} \) satisfies the null energy condition, implying \( \zeta^\mu \chi^\nu \).

If the scalar field vanishes, then this case reduces to the case of a purely electromagnetic test field.

The integrand on the right hand side of (39) can be expressed in a more explicit form. At the event horizon

\[
\zeta^{\mu} \dot{\zeta}_{\mu} = \gamma \zeta^0 \frac{-r_e^2}{\sqrt{2} (r_e^2 + a^2 \cos^2 \theta)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]  

(42)

thus

\[
\zeta^\mu \bar{\Psi}_{\mu} \Psi = \bar{\Psi}_{\mu} \Psi = \frac{-r_e^2}{\sqrt{2} (r_e^2 + a^2 \cos^2 \theta)} (|\Psi_1|^2 + |\Psi_4|^2),
\]

where \( \Psi_1 \) and \( \Psi_4 \) denote the first and fourth components of \( \Psi \). Furthermore,

\[
\sqrt{-g} \zeta^\mu \bar{\Psi}_{\mu} \Psi = \frac{-r_e^2}{\sqrt{2} \sin \theta} (|\Psi_1|^2 + |\Psi_4|^2).
\]

These formulas hold for any spinor \( \Psi \), regardless of the Dirac equation, thus they hold also for \( \psi_{\omega,n} \). (39) can be rewritten therefore as

\[
\int_{-\infty}^{\infty} dr \int_H \sqrt{-g} \bar{\hat{T}}_{\mu \nu} \zeta^\mu \chi^\nu \, d\theta d\varphi

\]

\[
= (2\pi)^2 \sum_n d\omega \int_H d\theta \frac{-r_e^2}{\sqrt{2} \sin \theta} (|(\psi_{\omega,n})_1|^2 + |(\psi_{\omega,n})_4|^2) \bar{\omega}.
\]

Finally, for \( d\eta \leq 0 \) it is necessary that \( (\psi_{\omega,n})_1 \) or \( (\psi_{\omega,n})_4 \) be nonzero at the event horizon at least for some values of \( \omega \) and \( n \) for which \( \bar{\omega} < 0 \), therefore in principle it should be investigated if there is anything that could force \( (\psi_{\omega,n})_1 \) or \( (\psi_{\omega,n})_4 \) to be zero at the event horizon. If, invoking the separability of the Dirac equation (see appendix B), it is assumed that \( \psi_{\omega,n} \) is a linear combination of terms satisfying the ansatz ((B.2), (B.11) and (B.12)), then \( (\psi_{\omega,n})_1 \) or \( (\psi_{\omega,n})_4 \) is nonzero at the event horizon if \( R_e(r_e) \neq 0 \) in these terms ((B.2), (B.11) and (B.12) show that \( R_e \) does not enter \( (\psi_{\omega,n})_1 \) and \( (\psi_{\omega,n})_4 \)). As explained in more detail in appendix B.1, \( R_e(r_e) \) is not zero, therefore generally \( (\psi_{\omega,n})_1 \) and \( (\psi_{\omega,n})_4 \) do not have to be zero at the event horizon.

4.1. On possible backreaction effects

Regarding the question whether backreaction effects can be expected to prevent the formation of a naked singularity, it should be noted first that a result in a rigorous test field
approximation, which can be considered as a lowest-order approximation, indicating the formation of a naked singularity is more conclusive than a result which indicates that a naked singularity is not formed (as in the cases of scalar and electromagnetic fields), because the latter type of result does not exclude the possibility of the formation of a naked singularity outside the domain of validity of the test field approximation, whereas the first type of result implies that the formation of a naked singularity may be avoided only if the perturbation is sufficiently large so that higher-order effects can dominate. This also shows that considering higher-order effects is more important when naked singularity formation is excluded at the lowest order.

In the literature it has been emphasized that backreaction effects have to be taken into account properly, and that this usually restores the cosmic censor in scenarios in which it seems to be violated \[52, 61, 62, 68, 69\]. In these cases, in contrast with the case of the Dirac field, cosmic censorship is respected at the lowest order, i.e. in the rigorous test matter approximation, and the apparent violation of the WCCC arises because effects beyond the lowest order are included in some way, but only partially. The restoration of cosmic censorship is achieved by properly taking into account all relevant effects. For example, in \[52\] the overspinning of a near extremal Reissner–Nordström black hole by waves carrying angular momentum was considered. Such a setting immediately implies the inclusion of effects beyond the lowest order, because \(\eta\) depends on \(J\) through \(J^2\), thus in the lowest order the \(\eta\) parameter of a Reissner–Nordström black hole cannot be changed by changing its angular momentum. In \[52\] it was shown that although an apparent violation of the WCCC can be found if the change of \(\eta\) due to the change of \(J\) is not neglected but the waves are assumed to propagate on a fixed Reissner–Nordström background, cosmic censorship is restored if the effect of the waves on the background during the interaction process is also taken into account, as required by the consistency of the approximation applied. In \[41\] the overspinning of a slightly subextremal Kerr black hole with a test body was considered, and also in this study some higher-order quantities were not neglected, while radiative and self-force effects were not taken into account. Later in \[61, 62, 68, 69\] it was argued that self-force effects are not negligible in this scenario and they might be the main effect preventing the violation of the WCCC.

In \[54\] a further interesting effect is described: the formation of another horizon outside a Reissner–Nordström black hole when a charged shell that would be expected to destroy it is adiabatically lowered towards its event horizon. This scenario is hard to compare with the case of the Dirac field, but even if a similar effect can show up also in the latter case, it is a higher-order effect, therefore it cannot be expected to completely override the lowest-order result. Regarding the adiabatic lowering of charged objects, it should also be noted that it is not necessary to assume that the particle comes from infinity in the derivations in \[8, 9, 16\] of the result that in rigorous test particle approximation the cosmic censorship principle is respected.

5. Superradiance

The setting in which the phenomenon of black hole superradiance occurs is similar to that of the thought experiment, with the difference that the initial black hole is not necessarily extremal and the quantity of interest is the total energy \(dE\) that flows through the event horizon, instead of \(d\phi\). Superradiance occurs if \(dE\) has a sign that corresponds to an amplification of the energy of the field outside the event horizon. In addition, the angular
momentum and the electric charge of the field can also be considered in the study of superradiance.

In the literature it is usual to describe superradiance in terms of the amplitude of suitable radial functions which arise when the complete separation of the variables is carried out (see e.g. [17, 18, 33]), but in this section we do not use these amplitudes, in accordance with our aim to avoid the use of the complete separation of variables as much as possible.

The superradiance of individual energy and angular momentum modes can also be defined. In this case the quantity that determines if a certain mode is superradiant is the sign of the rate \( \frac{dE}{dt} \).

5.1. Absence of superradiance of Dirac fields

If some matter described by the Dirac field is thrown into a black hole, then the total electric charge absorbed by the black hole is

\[
\int_{-\infty}^{\infty} \int_{H} |e_j| \text{d}t \text{d}\varphi = e Q H \text{d}t \text{d}\varphi.
\]

By definition \( j^r = \zeta^j j^i \), and one can argue, in the same way as in section 4, that at the event horizon \( \zeta^i \) is past directed and null, and \( j^\mu \) is always future directed and null or time-like, thus \( j^r \leq 0 \), and so \( e Q H \text{d}t \text{d}\varphi < 0 \). This means that the total electric charge falling through the event horizon always has the same sign as the charge of the Dirac field, thus the charge outside the event horizon does not increase. In other words, the Dirac field does not show superradiance in relation to electric charge.

Considering energy and angular momentum, using the Fourier expansion (38) one finds that the total energy and angular momentum absorbed by the black hole are

\[
dE = - \int_{-\infty}^{\infty} \int_{H} |e_j| \text{d}t \text{d}\varphi \int_{H} \sqrt{-g} \ E^r \text{d}\theta \text{d}\varphi
\]

\[
= (2\pi)^2 \sum_{n} \int \text{d}\omega \int \text{d}\theta \sqrt{-g} \ (\omega \bar{\psi}_{\omega,n} \gamma^{\rho} \psi_{\omega,n})
\]

\[
dL = - \int_{-\infty}^{\infty} \int_{H} |e_j| \text{d}t \text{d}\varphi \int_{H} \sqrt{-g} \ J^r \text{d}\theta \text{d}\varphi
\]

\[
= (2\pi)^2 \sum_{n} \int \text{d}\omega \int \text{d}\theta \sqrt{-g} \ (-n \bar{\psi}_{\omega,n} \gamma^{\rho} \psi_{\omega,n}).
\]

In the derivation of (48) and (49) it is useful to write (30) and (32) as

\[
E^r = \frac{1}{2} (\bar{\Psi} \gamma^{\rho} \partial_r \Psi - \partial_r \bar{\Psi} \gamma^{\rho} \Psi) \quad \text{and} \quad J^r = \frac{1}{2} (\bar{\Psi} \gamma^{\rho} \partial_r \Psi - \partial_r \bar{\Psi} \gamma^{\rho} \Psi) - e Q H \text{d}t \text{d}\varphi,
\]

because the vector potential does not appear explicitly in the latter expressions.

Since, as explained in section 4, \( \bar{\psi}_{\omega,n} \gamma^{\rho} \psi_{\omega,n} \leq 0 \) at the event horizon, the integrands in (48) and (49) have the same signs as \( \omega \) and \( n \), respectively. Consequently, if the Fourier expansion of \( \Psi \) contains only positive frequency modes, then the energy falling through the event horizon is also positive and the energy outside the event horizon does not increase, and analogous statements can be made for negative frequency modes and for angular momentum.

This means that the Dirac field is not superradiant in relation to energy and angular momentum either.

Instead of considering solutions of the form (38), one can consider waves consisting of a single mode, \( e^{-\omega t} e^{ic \theta} e^{i\omega \tau} \bar{\psi}_{\omega,n}(r, \theta) \). \( dQ \), \( dE \) and \( dL \) are not meaningful for such waves, but one can study the rates \( \frac{dQ}{dt} = \int_{H} \sqrt{-g} \ e_j \text{d}t \text{d}\varphi \), \( \frac{dE}{dt} = - \int_{H} \sqrt{-g} \ E^r \text{d}t \text{d}\varphi \) and \( \frac{dL}{dt} = \int_{H} \sqrt{-g} \ J^r \text{d}t \text{d}\varphi \). These rates can be expressed as
\[
\frac{dQ}{d\tau} = 2\pi \int_H d\theta \sqrt{-g} \left( e \psi^{\mu, n} \gamma^\mu \psi_{\nu, m} \right) 
\]
(50)

\[
\frac{dE}{d\tau} = 2\pi \int_H d\theta \sqrt{-g} \left( -\omega \psi^{\mu, n} \gamma^\mu \psi_{\nu, m} \right) 
\]
(51)

\[
\frac{dL}{d\tau} = 2\pi \int_H d\theta \sqrt{-g} \left( -n \psi^{\mu, n} \gamma^\mu \psi_{\nu, m} \right) .
\]
(52)

Using these expressions one can argue in the same way as above that the Dirac field does not have superradiant modes.

5.2. Superradiant frequency range of scalar fields

The massive complex scalar field has the Lagrangian density

\[ \mathcal{L} = g^{\mu\nu} (\partial_{\mu} - i e A_{\mu}) \Phi^* (\partial_{\nu} + i e A_{\nu}) \Phi - m^2 \Phi^* \Phi \]
(53)

and the corresponding field equation \((\nabla^\mu + i e A^\mu)(\nabla_{\nu} + i e A_{\nu}) \Phi = -m^2 \Phi \). The electric current of the scalar field is

\[ j^\mu = \frac{\partial \mathcal{L}}{\partial A_{\mu}} = -i e [\Phi^* (\partial^\mu + i e A^\mu) \Phi - \Phi(\partial^\mu - i e A^\mu) \Phi^*]. \]
(54)

In [16] we found the energy and angular momentum current densities \(\mathcal{E}^\mu\) and \(\mathcal{J}^\mu\) by applying Noether’s theorem. The result for \(\mathcal{J}^\mu\) had to be modified in the same way as in section 4 to eliminate its dependence on the gauge parameter C. \(\mathcal{E}^\mu\) and \(\mathcal{J}^\mu\) are given by the expressions

\[ \mathcal{E}^\mu = \mathcal{T}_{\tau}^{\mu} = A_{\tau} j^\mu, \quad \mathcal{J}^\mu = \mathcal{T}_{\nu}^{\mu} = (A_{\nu} - Q_m C) j^\mu, \]
(55)

where

\[ \mathcal{T}_{\mu\nu} = (\partial_{\mu} - i e A_{\mu}) \Phi^* (\partial_{\nu} + i e A_{\nu}) \Phi + (\partial_{\nu} + i e A_{\nu}) \Phi (\partial_{\mu} - i e A_{\mu}) \Phi^* - g_{\mu\nu} \mathcal{L}. \]
(56)

Using the Fourier expansion

\[ \Phi = \sum_n \int d\omega \ e^{-i\omega t} e^{i(n - C Q_m) r} \phi_{\omega, n}^\pm(r, \theta) \]
(57)
of \(\Phi\), where \(e^{-i\omega t} e^{i(n - C Q_m) r} \phi_{\omega, n}^\pm(r, \theta)\) are solutions of the field equation, one obtains the following results for \(dQ\), \(dE\) and \(dL\):

\[ dQ = -\int_{-\infty}^{\infty} d\tau \int_H \sqrt{-g} \mathcal{J}^\mu d\theta d\varphi \]

\[ = -(2\pi)^2 \int d\omega \int_H d\theta \ 2(a^2 + r_0^2) \sin \theta \ \phi_{\omega, n}^\pm \phi_{\omega, n}^\pm e\omega \]
(58)

\[ dE = -\int_{-\infty}^{\infty} d\tau \int_H \sqrt{-g} \mathcal{E}^\mu d\theta d\varphi \]

\[ = (2\pi)^2 \int d\omega \int_H d\theta \ 2(a^2 + r_0^2) \sin \theta \ \phi_{\omega, n}^\pm \phi_{\omega, n}^\pm \omega e\omega \]
(59)

\[ dL = \int_{-\infty}^{\infty} d\tau \int_H \sqrt{-g} \mathcal{J}^\nu d\theta d\varphi \]

\[ = (2\pi)^2 \int d\omega \int_H d\theta \ 2(a^2 + r_0^2) \sin \theta \ \phi_{\omega, n}^\pm \phi_{\omega, n}^\pm n e\omega, \]
(60)
where $\tilde{\omega}$ is defined as in (40). In the derivation of (58)–(60) only the derivatives $\partial_r \Phi$ and $\partial_\theta \Phi$ of $\Phi$ appear, which can be evaluated easily, and one can also use (18)–(20) and (45). Due to the factor $g_{\mu\nu}$, the $-g_{\mu\nu}L$ term appearing in (56) does not make any contribution to $dE$ and $dL$, as $\delta_r^\mu = \delta_\theta^\mu = 0$. For $e^{-i\omega r}e^{i(n-eQ_m)\theta}\phi_{\omega,e}(r, \theta)$ to be single valued for both $C = 1$ and $C = -1$, both $n - eQ_m$ and $n + eQ_m$ have to be integer, implying that $n$ and $eQ_m$ are either integer or half-integer, thus in the summations $n$ should take integer values if $eQ_m$ is integer and half-integer values if $eQ_m$ is half-integer [73]. It should also be noted that far from the black hole, only modes with $|\omega| > m$ describe propagating waves.

Since $\phi_{\omega,e}^* \phi_{\omega,e}$ is positive unless $\phi_{\omega,e} = 0$, (59) shows that $dE$ is negative if the main contribution to the integral comes from the frequency range where

$$\omega \tilde{\omega} < 0.$$  \hspace{1cm} (61)

In this case the scalar field exhibits superradiance in the sense that the total energy of the field outside the event horizon increases. The sign of the integrands in (58) and (60) is also determined by $e\omega$ and $n\omega$, respectively, instead of $e\omega$ and $n\omega$.

For individual modes $e^{-i\omega r}e^{i(n-eQ_m)\theta}\phi_{\omega,e}(r, \theta)$, the rates $\frac{dQ}{d\tau}$, $\frac{dE}{d\tau}$ and $\frac{dL}{d\tau}$ can be expressed as

$$\frac{dQ}{d\tau} = -2\pi \int_H d\theta \ 2(a^2 + r_t^2) \sin \theta \ \phi_{\omega,e}^* \phi_{\omega,e} e\omega$$  \hspace{1cm} (62)

$$\frac{dE}{d\tau} = 2\pi \int_H d\theta \ 2(a^2 + r_t^2) \sin \theta \ \phi_{\omega,e}^* \phi_{\omega,e} \omega \tilde{\omega}$$  \hspace{1cm} (63)

$$\frac{dL}{d\tau} = 2\pi \int_H d\theta \ 2(a^2 + r_t^2) \sin \theta \ \phi_{\omega,e}^* \phi_{\omega,e} n\omega.$$  \hspace{1cm} (64)

Of course, the superradiant frequencies following from (63) are the same as those that follow from (59).

6. Conclusion

We showed that a dyonic Kerr–Newman black hole can be converted into a naked singularity by interaction with massive charged classical Dirac fields, generalizing a recent result [15] which applies to massless Dirac fields and Kerr black holes. We found that for this conversion the spectrum of the Dirac field has to be dominated by modes with temporal and angular frequencies $\omega$ and $n$ satisfying the inequality (40). We also showed that the creation of a naked singularity is possible because the null energy condition is not satisfied by the Einstein–Hilbert energy-momentum tensor of the Dirac field. This means that the classical Dirac field does not satisfy the criteria under which the WCCC is expected to hold, thus in a strict sense the possibility found in [15] and in the present paper does not contradict the WCCC. These features of the Dirac field are complementary to those of the scalar and the electromagnetic fields, which satisfy the null energy condition and as a consequence cannot convert a black hole into a naked singularity.

We gave a derivation of the absence of superradiance of the Dirac field around dyonic Kerr–Newman black holes, and we determined the temporal and angular frequencies for which the scalar field is superradiant. We found that the superradiant modes are those that satisfy the condition (61). The frequencies satisfying (40) partially agree with those satisfying (61).
Although the well-known separability of the scalar wave equation and of the Dirac equation around black holes is very useful for several purposes and is also often used in discussion of testing the WCCC or the superradiance phenomenon, its use could be largely avoided in this paper. Nevertheless, as we used it in an argument at the end of section 4, we described it in an appendix in horizon-penetrating coordinates.

In the derivations the test field approximation was applied, but we argued that the destruction of black holes cannot be completely prevented by backreaction effects. It was also assumed that the final state that arises after the interaction of the black hole and the Dirac field is again a dyonic Kerr–Newman configuration, without any Dirac hair. This assumption is in accordance with the no-hair conjecture, but its validity would nevertheless be interesting to investigate.

It is natural to hope that the formation of a naked singularity can be ruled out in quantum theory, since problems related to negative energies are absent in quantum field theory in Minkowski spacetime. Results on quantum effects have already appeared in the literature [31, 47–49, 72], and comments can be found also in [15]. We leave further investigations of the quantum Dirac field for future work.

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Appendix A. Dirac spinor fields in curved spacetime

In this paper we apply the tetrad formalism to incorporate Dirac spinor fields into general relativity, as described e.g. in [80].

We denote internal Lorentz vector indices by letters with an overbar. In contrast with Lorentz and spacetime vector indices, the normal position of Dirac spinor indices is taken to be the lower one, and hence cospinor indices are in the upper position. Dirac spinor indices are often not written out explicitly.

Lorentz vector and spinor indices are scalar indices with respect to spacetime diffeomorphisms; in particular, tensors having only Lorentz vector and spinor indices are scalars under spacetime diffeomorphisms. On the other hand, local Lorentz transformations do not act on the spacetime manifold and spacetime vector indices are scalar indices with respect to them.

We use orthonormal tetrad fields $V_{\mu}^\nu$, i.e. $g_{\rho\sigma} = V_{\mu}^\rho V_{\nu}^\sigma$, $g_{\rho\sigma} = \text{diag}(1, -1, -1, -1)$.

The Levi–Civita covariant differentiation is extended to tensors with Lorentz vector indices in the following way:

$$\nabla^\nu \nu^\rho = \partial_\mu \nu^\rho + S_{\lambda\mu}^{\sigma} \nu^\lambda, \quad \mathcal{S}_{\lambda\mu}^{\rho} = -V_\eta \nabla_{\mu}^{\eta} V_{\nu}^{\lambda}, \quad (A.1)$$

where $S_{\lambda\mu}^{\rho}$ is an analogue of the Christoffel symbols. On the right hand side the superscript $LC$ indicates that the Levi–Civita covariant differentiation should be used. With this definition the covariant derivative of the Lorentz metric tensor and of the tetrad field is zero: $\nabla_\mu g_{\nu\lambda} = V_{\mu}^{\eta} V_{\nu}^{\lambda} = 0$.

The Dirac gamma matrices $\gamma^\mu$ are defined as

$$\gamma^\mu = V_{\mu}^{\nu} \gamma^\nu, \quad (A.2)$$
where \( \gamma^\mu \) are the standard Minkowski space gamma matrices. We use the Weyl representation for them,

\[
\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \tag{A.3}
\]

where \( I \) denotes the \( 2 \times 2 \) identity matrix and \( \sigma^i \) are the Pauli sigma matrices (see [81]). \( \gamma^\mu \) has the property \( \{ \gamma^\mu, \gamma^\nu \} = 2\eta^{\mu\nu} \), where \( \{ , \} \) denotes the anticommutator \( \{ A, B \} = AB + BA \).

The Levi–Civita connection can be extended to tensors having Dirac spinor indices in the following way:

\[
\nabla_\mu \psi^\alpha = \partial_\mu \psi^\alpha + S_{\alpha\beta\mu} \psi^\beta, \tag{A.4}
\]

where

\[
S_{\alpha\beta\mu} = \frac{1}{4} \sigma^{\alpha\beta} S_{\nu\lambda\mu}, \quad \sigma^{\alpha\beta} = \frac{1}{2} \{ \gamma^\alpha, \gamma^\beta \}. \tag{A.5}
\]

With this definition of the covariant differentiation of spinors the covariant derivative of the gamma tensor is zero: \( \nabla_\nu \gamma^\mu = 0 \).

In the Weyl representation the Dirac conjugation for Dirac spinors with a lower spinor index is customarily expressed as \( \bar{\psi} = \psi^T \gamma^0 \), where the \( T \) denotes transposition, the \( * \) denotes componentwise complex conjugation and as usual the Dirac conjugation is denoted by an overbar.

Finally we note that we do not introduce any raising and lowering convention for spinor indices.

**Appendix B. Separation of variables for the Dirac equation**

In this appendix the separation of variables for the Dirac equation in the dyonic Kerr–Newman background is described. This is done in the horizon-penetrating coordinates \((\tau, r, \theta, \varphi)\), using the Kinnersley-type tetrad \( V^\mu \), introduced in section 3.3 and the Weyl representation for the Dirac gamma matrices described in appendix A. The separability of the Dirac equation in the dyonic Kerr–Newman background was shown first in [37], in Boyer–Lindquist coordinates. The reader is referred to this article and to [18, 19] for further references to earlier results. The recent article [30] focuses on the separability of the Dirac equation in horizon-penetrating coordinates, but only in Kerr geometry.

The asymptotic behaviour at the event horizon of the solutions of the radial equations that arise after the separation of variables is also discussed in a subsection.

The application of the method of separation of variables to finding solutions of the Dirac equation in the dyonic Kerr–Newman background begins with assuming that \( \Psi \) depends on \( \tau \) and \( \phi \) harmonically as

\[
\Psi(\tau, r, \theta, \varphi) = e^{-i\omega \gamma} e^{i(n - CeQ_m)\tau} \psi(r, \theta), \tag{B.1}
\]

where \( \omega \) and \( n \) denote the temporal and angular frequency, respectively. The term \(-CeQ_m\) in the factor \( e^{i(n - CeQ_m)\tau} \) is included because of the gauge transformation done in equation (16) in section 3.1 (see also [37]). After introducing the new field variables \( f_i, i = 1, \ldots, 4 \), as

\[
f_1 = \psi_1, \quad f_2 = \frac{1}{r} (r - ia \cos \theta) \psi_2, \quad f_3 = \frac{1}{r} (r + ia \cos \theta) \psi_3, \quad f_4 = \psi_4, \tag{B.2}
\]

where \( \psi_i, i = 1, \ldots, 4 \), denote the components of \( \psi \), the Dirac equation can be written in the form
\[ D_{f_3} + L_{f_4} = -im(r + ia \cos \theta)f_4 \] (B.3)
\[ D_{f_4} + L_{f_3} = -im(r + ia \cos \theta)f_2 \] (B.4)
\[ D_{f_3} - L_{f_2} = -im(r - ia \cos \theta)f_3 \] (B.5)
\[ D_{f_2} - L_{f_1} = -im(r - ia \cos \theta)f_4 \] (B.6)

where
\[ D_+ = \frac{\sqrt{2}}{r^2} \left[ -Mr + r^2 + i(2amr - 2\epsilon Q_e r^2 - 2a^2 \rho - 2r^3 \omega + r \Delta \omega) + r \Delta \psi \right] \] (B.7)
\[ D_- = \frac{1}{\sqrt{2}} (1 + ir \rho + r \Delta \psi) \] (B.8)
\[ L_+ = \frac{1}{2 \sin \theta} [ -2n + (1 + 2eQ_m) \cos \theta + a\omega(1 - \cos 2\theta) ] + \partial_\theta \] (B.9)
\[ L_- = \frac{1}{2 \sin \theta} [ 2n + (1 - 2eQ_m) \cos \theta - a\omega(1 - \cos 2\theta) ] + \partial_\theta. \] (B.10)

We note that in (B.2) the factors in front of \( \psi_2 \) and \( \psi_3 \) are chosen so that in the \( r \to \infty \) limit \( f_2 \to \psi_2 \) and \( f_3 \to \psi_3 \).

By taking the ansatz
\[ f_1(r, \theta) = R_+(r)S_+(\theta) \quad f_2(r, \theta) = R_-(r)S_-(\theta) \] (B.11)
\[ f_3(r, \theta) = R_+(r)S_-(\theta) \quad f_4(r, \theta) = R_-(r)S_+(\theta) \] (B.12)
the \( r \) and \( \theta \) parts of equations (B.3)-(B.6) become separated and one finds the ordinary differential equations
\[ D_R R_- - (\lambda - imr)R_+ = 0 \] (B.13)
\[ D_R R_+ + (\lambda + imr)R_- = 0 \] (B.14)
\[ L_S S_- + (\lambda - ma \cos \theta)S_+ = 0 \] (B.15)
\[ L_S S_+ - (\lambda + ma \cos \theta)S_- = 0 \] (B.16)
for \( R_+, R_-, S_+ \) and \( S_- \), where \( \lambda \) is the separation constant. Initially one introduces different separation constants in each equation (B.3)-(B.6), but then one sees that they have to be related.

By eliminating \( R_+ \) or \( R_- \) and \( S_+ \) or \( S_- \) one gets the second-order decoupled equations
\[ D_D D_R R_- - \frac{imr}{\sqrt{2}(\lambda - imr)}D_R R_+ + (\lambda^2 + m^2 r^2)R_- = 0 \] (B.17)
\[ D_D D_R R_+ - \frac{i\sqrt{2} m \Delta}{r(\lambda + imr)}D_R R_- + (\lambda^2 + m^2 r^2)R_+ = 0 \] (B.18)
and
\[ L_D L_S S_- - \frac{ma \sin \theta}{\lambda - ma \cos \theta}L_S S_+ + (\lambda^2 - m^2 a^2 \cos^2 \theta)S_- = 0 \] (B.19)
\[ L_D L_S S_+ + \frac{ma \sin \theta}{\lambda + ma \cos \theta}L_S S_- + (\lambda^2 - m^2 a^2 \cos^2 \theta)S_+ = 0. \] (B.20)
It is useful to write out the explicit form of \( L_{+}, L_{-}, D_{+}, D_{-} \):

\[
L_{+} S_{+} = -\frac{1}{8 \sin^2 \theta} \left[ 3 + 8n^2 + 8eQ_m + 4e^2Q_m^2 - 8an\omega + 3a^2\omega^2 \\
- 2(n(4 + 8eQ_m) + a(1 - 2eQ_m)\omega) \cos \theta \\
+ (-1 + 4e^2Q_m^2 + 8an\omega - 4a^2\omega^2) \cos 2\theta \\
+ 2(2a\omega - 4aeQ_m\omega) \cos 3\theta + a^2\omega^2 \cos 4\theta \right] S_+ \\

+ \frac{\cos \theta}{\sin \theta} \partial_\theta S_+ + \partial_\theta^2 S_+, \tag{B.21}
\]

\[
L_{-} S_{-} = -\frac{1}{8 \sin^2 \theta} \left[ 3 + 8n^2 - 8eQ_m + 4e^2Q_m^2 - 8an\omega + 3a^2\omega^2 \\
+ 2(n(4 - 8eQ_m) + a(1 + 2eQ_m)\omega) \cos \theta \\
+ (-1 + 4e^2Q_m^2 + 8an\omega - 4a^2\omega^2) \cos 2\theta \\
- (2a\omega + 4aeQ_m\omega) \cos 3\theta + a^2\omega^2 \cos 4\theta \right] S_- \\

+ \frac{\cos \theta}{\sin \theta} \partial_\theta S_- + \partial_\theta^2 S_-, \tag{B.22}
\]

and

\[
D_{+} D_{-} = \left[ -1 + 2ieQ_e + \frac{M - 2Ian}{r} + \omega(iM + ir + 2an - 2eQ_e r) \\
+ \frac{2i\omega(a^2 - \Delta)}{r} + \omega^2(\Delta - 2a^2 - 2r^2) \right] R_+ \\
+ \left[ M - r + \frac{2\Delta}{r} + 2i(-an + eQ_e r) + 2i\omega(a^2 + r^2 - \Delta) \right] \partial_\theta R_+ \\
- \Delta \partial_\theta^2 R_+, \tag{B.23}
\]

\[
D_{-} D_{-} = \left[ -1 + 2ieQ_e + \omega(3iM + ir + 2an - 2eQ_e r) \\
+ \omega^2(\Delta - 2a^2 - 2r^2) \right] R_- \\
+ \left[ 3(M - r) + 2i(-an + eQ_e r) + 2i\omega(a^2 + r^2 - \Delta) \right] \partial_\theta R_- \\
- \Delta \partial_\theta^2 R_- \tag{B.24}
\]

If the mass of the Dirac field is zero, then after multiplying by the integrating factor \( \sin \theta \) the angular equations \( B.19 \) and \( B.20 \) take a Sturm–Liouville form. The weight factor appearing in the scalar product for the solutions is also \( \sin \theta \). Although the case \( m \neq 0 \) is more complicated, it was studied e.g. in [25–29] at \( Q_m = 0 \).

In the \( r \to \infty \) limit the radial equations \( B.17 \) and \( B.18 \) become

\[
-\omega^2 R_\pm - \partial_\theta^2 R_\pm + m^2 R_\pm = 0. \tag{B.25}
\]

**B.1. Asymptotic behaviour of the radial functions at the event horizon**

The second-order radial equations \( B.17 \) and \( B.18 \) are singular at the event horizon; the singularity is regular if the black hole is not extremal and irregular if the black hole is extremal. By analysing the asymptotic behaviour of the solutions near the event horizon one finds two kinds of asymptotic behaviour. One of them is such that \( R_+ \) (or \( R_- \)) approaches a
finite nonzero value as \( r \to r_+ \); the other one is such that \( R_+ \) goes to zero and \( |R_-| \) to infinity as \( r \to r_- \). Solutions with the latter behaviour can be considered unphysical.

More specifically, in the non-extremal case \( R_+ \) takes the form

\[
c_1 (r - r_+)^s_1 y_1(r - r_+) + c_2 (r - r_+)^s_2 y_2(r - r_+),
\]

where \( c_1 \) and \( c_2 \) are integration constants, \( y_1 \) and \( y_2 \) are functions that are regular and nonzero at 0, and the characteristic exponents \( s_1 \) and \( s_2 \) are solutions of the indicial equation

\[
s(s - 1) - \frac{1}{r_+ - r_-} [M - r_+ + 2i(a^2 + r_+^2)\bar{\omega}]s = 0.
\]

The solutions of this equation are

\[
s_1 = 0, \quad s_2 = -\frac{1}{2} + \frac{2(a^2 + r_+^2)\bar{\omega}}{r_+ - r_-},
\]

thus the solution corresponding to \( c_2 = 0 \) is regular and nonzero at the event horizon, whereas the solution corresponding to \( c_1 = 0 \) is not regular but is vanishing at \( r_+ \). The latter solution can also be written as

\[
e^{\bar{\omega} \log(r - r_+)}y_2(r - r_+),
\]

showing that near the event horizon the absolute value of this solution behaves like \( \sim (r - r_+)^{1/2} \) and the oscillation frequency of its phase increases to infinity as \( r \to r_+ \).

In the extremal case the asymptotic behaviour of the \( R_+ \to 0 \) type solution of (B.18) at the event horizon is found to be

\[
\sim \exp\left[ -\frac{2i(a^2 + r_+^2)\bar{\omega}}{r_+ - r_-} + (1 + 2iQ_e + 4i\omega r_+)\log(r - r_+) \right],
\]

showing that the absolute value of this solution behaves like \( \sim (r - r_+) \) and the oscillation frequency of its phase increases to infinity as \( r \to r_+ \).

The characteristic exponents for \( R_- \) in the non-extremal case are

\[
s_1 = 0, \quad s_2 = -\frac{1}{2} + i \frac{2(a^2 + r_+^2)\bar{\omega}}{r_+ - r_-},
\]

as can be expected from (B.14) and (B.28). Similarly, in the extremal case the asymptotic behaviour of the \( |R_-| \to \infty \) type solution of (B.17) at the event horizon is

\[
\sim \exp\left[ -\frac{2i(a^2 + r_+^2)\bar{\omega}}{r - r_+} + (-1 + 2iQ_e + 4i\omega r_+)\log(r - r_+) \right].
\]

References

[1] Penrose R 1969 Gravitational collapse: the role of general relativity Riv. Nuovo Cimento 1 252 special number
[2] Wald R M 1984 General Relativity (Chicago: University of Chicago Press)
[3] Wald R M 1997 Gravitational collapse and cosmic censorship arXiv:gr-qc/9710068
[4] Joshi P S 2002 Cosmic censorship: a current perspective Mod. Phys. Lett. A 17 1067
[5] Clarke C J S 1994 A title of cosmic censorship Class. Quantum Grav. 11 1375
[6] Singh T P 1999 Gravitational collapse, black holes and naked singularities J. Astrophys. Astron. 20 221
[7] Krolak A 1999 Nature of singularities in gravitational collapse Prog. Theor. Phys. Suppl. 45 136
[8] Wald R M 1974 Gedanken experiments to destroy a black hole Ann. Phys. NY 83 548
[9] Needham T 1980 Cosmic censorship and test particles Phys. Rev. D 22 791
[10] Hiscock W A 1981 Magnetic charge, black holes and cosmic censorship Ann. Phys. 131 245
[11] Semiz I 1990 Dyon black holes do not violate cosmic censorship Class. Quantum Grav. 7 353
[12] Semiz I 2011 Dyonic Kerr–Newman black holes, complex scalar field and cosmic censorship Gen. Rel. Grav. 43 833
[13] Duztas K 2014 Electromagnetic field and cosmic censorship Gen. Rel. Grav. 46 1709
[14] Duztas K and Semiz I 2013 Cosmic censorship, black holes and integer-spin test fields Phys. Rev. D 88 064043
[15] Duztas K 2015 Stability of event horizons against neutrino flux: the classical picture Class. Quantum Grav. 32 075003
[16] Tóth G Z 2012 Test of the weak cosmic censorship conjecture with a charged scalar field and dyonic Kerr–Newman black holes Gen. Rel. Grav. 44 2019
[17] Chandrasekhar S 1983 The Mathematical Theory of Black Holes (New York: Oxford University Press)
[18] Brito R, Cardoso V and Pani P 2015 Superradiance: Energy Extraction, Black-Hole Bombs and Implications for Astrophysics and Particle Physics (Lecture Notes in Physics vol 906) (Berlin: Springer) 906
[19] Dolan S R and Dempsey D 2015 Bound states of the Dirac equation on Kerr spacetime Class. Quantum Grav. 32 184001 18
[20] Lee C H 1977 Massive spin-½ wave around a Kerr–Newman black hole Phys. Lett. B 68 152
[21] Unruh W 1973 Separrability of the neutrino equations in a Kerr background Phys. Rev. Lett. 31 1265 20
[22] Gueven R 1977 Wave mechanics of electrons in Kerr geometry Phys. Rev. D 16 1706
[23] Casals M, Dolan S R, Nolan B C, Ottewill A C and Winstanley E 2013 Quantization of fermions on Kerr space-time Phys. Rev. D 87 064027 6
[24] Arderucio B Superradiance: classical, relativistic and quantum aspects (arXiv:1404.3421 [gr-qc] section 3.4)
[25] Suffern K G, Fackerell E D and Cosgrove C M 1983 Eigenvalues of the Chandrasekhar–Page angular functions J. Math. Phys. 24 1350
[26] Kalnins E G and Miller W Jr 1992 Series solutions for the Dirac equation in Kerr–Newman spacetime J. Math. Phys. 33 286
[27] Finster F, Kamran N, Smoller J and Yau S T 2000 Nonexistence of time periodic solutions of the Dirac equation in an axisymmetric black hole geometry Commun. Pure Appl. Math. 53 902
[28] Finster F, Kamran N, Smoller J and Yau S T 2003 The long time dynamics of Dirac particles in the Kerr–Newman black hole geometry Adv. Theor. Math. Phys. 7 25
[29] Dolan S and Gair J 2009 The massive Dirac field on a rotating black hole spacetime: angular solutions Class. Quantum Grav. 26 175020
[30] Roken C The massive Dirac equation in Kerr geometry: separability in Eddington–Finkelstein-type coordinates and asymptotics (arXiv:1506.08038 [gr-qc])
[31] Unruh W G 1974 Second quantization in the Kerr metric Phys. Rev. D 10 3194
[32] Teukolsky S A 1973 Perturbations of a rotating black hole I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations Astrophys. J. 185 635
[33] Richartz M, Weinfurtner S, Penner A J and Unruh W G 2009 Generalised superradiant scattering Phys. Rev. D 80 124016
[34] Mazur P O 1982 Proof of uniqueness of the Kerr–Newman black hole solution J. Math. Phys. 15 3173
[35] Bunting G L 1983 Proof of the uniqueness conjecture for black holes PhD Thesis University of New England, Armidale, Australia
[36] Wu T T and Yang C N 1976 Dirac monopole without strings: monopole harmonics Nucl. Phys. B 107 365
[37] Semiz I 1992 The Dirac equation is separable on the dyon black hole metric Phys. Rev. D 46 5414
[38] Carter B 1968 Hamilton–Jacobi and Schrodinger separable solutions of Einstein’s equations Commun. Math. Phys. 10 280
[39] Hubeny V E 1999 Overcharging a black hole and cosmic censorship Phys. Rev. D 59 064013
[40] de Felice F and Yu Y 2001 Turning a black hole into a naked singularity Class. Quantum Grav. 18 1235
[41] Jacobson T and Sotiriou T P 2009 Over-spinning a black hole with a test body Phys. Rev. Lett. 103 141101
Jacobson T and Sotiriou T P 2009 Phys. Rev. Lett. 103 209903 (Erratum)
[42] Jensen B 1995 Stability of black hole event horizons Phys. Rev. D 51 5511
[43] Cohen J M and Gautreau R 1979 Naked singularities event horizons and charged particles Phys. Rev. D 19 2273
[44] Bekenstein J D and Rosenzweig C 1994 Stability of the black hole horizon and the Landau ghost Phys. Rev. D 50 7239
[45] Matsas G E A and da Silva A R R 2007 Overspinning a nearly extreme charged black hole via a quantum tunneling process Phys. Rev. Lett. 99 181301
[46] Matsas G E A, Richartz M, Saa A, da Silva A R R and Vanzella D A T 2009 Can quantum mechanics fool the cosmic censor? Phys. Rev. D 79 101502(R)
[47] Richartz M and Saa A 2011 Challenging the weak cosmic censorship conjecture with charged quantum particles Phys. Rev. D 84 104021
[48] Richartz M and Saa A 2008 Overspinning a nearly extreme black hole and the weak cosmic censorship conjecture Phys. Rev. D 78 081503
[49] Hod S 2008 Return of the quantum cosmic censor Phys. Lett. B 668 346
[50] Hod S and Piran T 2000 Cosmic censorship: the role of quantum gravity Gen. Rel. Grav. 32 2333
[51] Hod S 2002 Cosmic censorship, area theorem, and self-energy of particles Phys. Rev. D 66 024016
[52] Hod S 2008 Weak cosmic censorship: as strong as ever Phys. Rev. Lett. 100 121101
[53] Hod S 1999 Black hole polarization and cosmic censorship Phys. Rev. D 60 104031
[54] Hod S 2013 Cosmic censorship: formation of a shielding horizon around a fragile horizon Phys. Rev. D 87 024037 2
[55] Gao S and Lemos J P S 2008 Collapsing and static thin massive charged dust shells in a Reissner–Nordström black hole background in higher dimensions Int. J. Mod. Phys. A 23 2943
[56] Rocha J V and Cardoso V 2011 Gravitational perturbation of the BTZ black hole induced by test particles and weak cosmic censorship in AdS spacetime Phys. Rev. D 83 104037
[57] Rocha J V and Santarelli R 2014 Flowing along the edge: spinning up black holes in AdS spacetimes with test particles Phys. Rev. D 89 064065 6
[58] Delsate T, Rocha J V and Santarelli R 2014 Collapsing thin shells with rotation Phys. Rev. D 89 121501
[59] Rocha J V 2015 Gravitational collapse with rotating thin shells and cosmic censorship Int. J. Mod. Phys. D 24 1542002 09
[60] Bouhmadi-Lopez M, Cardoso V, Nerozzi A and Rocha J V 2010 Black holes die hard: can one spin up a black hole past extremality? Phys. Rev. D 81 084051
[61] Barausse E, Cardoso V and Khanna G 2010 Test bodies and naked singularities: is the self-force the cosmic censor? Phys. Rev. Lett. 105 261102
[62] Barausse E, Cardoso V and Khanna G 2011 Testing the cosmic censorship conjecture with point particles: the effect of radiation reaction and the self-force Phys. Rev. D 84 104006
[63] Isoyama S, Sago N and Tanaka T 2011 Cosmic censorship in overcharging a Reissner–Nordström black hole via charged particle absorption Phys. Rev. D 84 124024
[64] Saa A and Santarelli R 2011 Destroying a near-extremal Kerr–Newman black hole Phys. Rev. D 84 027501
[65] Zhang Y and Gao S S 2014 Testing cosmic censorship conjecture near extremal black holes with cosmological constants Int. J. Mod. Phys. D 23 1450044
[66] Li Z and Bambi C 2013 Destroying the event horizon of regular black holes Phys. Rev. D 87 124022 12
[67] Gao S and Zhang Y 2013 Destroying extremal Kerr–Newman black holes with test particles Phys. Rev. D 87 044028 4
[68] Colleoni M and Barack L 2015 Overspinning a Kerr black hole: the effect of self-force Phys. Rev. D 91 104024
[69] Colleoni M, Barack L, Shah A G and van de Meent M 2015 Self-force as a cosmic censor in the Kerr overspinning problem Phys. Rev. D 92 084044 8
[70] Semiz I and Duztas K 2015 Weak cosmic censorship, superradiance and quantum particle creation Phys. Rev. D 92 104021 10
[71] Duztas K and Semiz I Black hole evaporation as a cosmic censor (arXiv:1508.06685 [gr-qc])
[72] Duztas K Absorption probability of neutrino fields and Hawking radiation (arXiv:1503.05061 [gr-qc])

22
[73] Semiz I 1992 Klein–Gordon equation is separable on dyon black hole metric Phys. Rev. D 45 532
[74] Pugliese D, Quevedo H and Ruffini R 2013 Equatorial circular orbits of neutral test particles in the Kerr–Newman spacetime Phys. Rev. D 88 024042 2
[75] Kong L, Malafarina D and Bambi C 2014 Can we observationally test the weak cosmic censorship conjecture? Eur. Phys. J. C 74 2983
[76] Virbhadra K S and Ellis G F R 2002 Gravitational lensing by naked singularities Phys. Rev. D 65 103004
[77] Virbhadra K S and Keeton C R 2008 Time delay and magnification centroid due to gravitational lensing by black holes and naked singularities Phys. Rev. D 77 124014
[78] Joshi P S, Malafarina D and Narayan R 2014 Distinguishing black holes from naked singularities through their accretion disc properties Class. Quantum Grav. 31 015002
[79] Ortiz N, Sarbach O and Zannias T 2015 The shadow of a naked singularity Phys. Rev. D 92 044035 4
[80] Weinberg S 1972 Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York: Wiley) Ch 12 section 5
[81] Peskin M E and Schroeder D V 1995 An Introduction to Quantum Field Theory (Reading: Addison-Wesley)