Erdős-Ko-Rado sets of flags of finite sets

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Abstract

A flag of a finite set $S$ is a set $f$ of non-empty proper subsets of $S$ such that $A \subseteq B$ or $B \subseteq A$ for all $A, B \in f$. The set $\{|A| : A \in f\}$ is called the type of $f$. Two flags $f$ and $f'$ are in general position (with respect to $S$) when $A \cap B = \emptyset$ or $A \cup B = S$ for all $A \in f$ and $B \in f'$. We study sets of flags of a fixed type $T$ that are mutually not in general position and are interested in the largest cardinality of these sets. This is a generalization of the classical Erdős-Ko-Rado problem. We will give some basic facts and determine the largest cardinality in several non-trivial cases. For this we will define graphs whose vertices are flags and the problem is to determine the independence number of these graphs.

Keywords: Erdős-Ko-Rado sets, Kneser graphs, independence number

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1 Introduction

For integers $n \geq 0$ we define $[n] := \{i \in \mathbb{Z} \mid 1 \leq i \leq n\}$. Let $S$ be a non-empty finite set. We call two subsets of $S$ in general position (with respect to $S$) if their union is $S$ or their meet is empty, which means that their intersection is as small as possible. A flag of $S$ is a set $f$ of non-empty proper subsets of $S$ such that $X \subseteq Y$ or $Y \subseteq X$ for all $X, Y \in f$. The set $\{|X| : X \in f\}$ is the type of $f$. Two flags $f$ and $f'$ of $S$ are in general position, if $X$ and $X'$ are in general position for all $X \in f$ and all $X' \in f'$. For $T \subseteq [[S] - 1]$ the graph whose vertices are the flags of $S$ of type $T$ with two vertices being adjacent when the corresponding flags are in general position will be denoted by $\Gamma(S, T)$. If $S = [n]$, we use the notation $\Gamma(n, T)$. If $|T| = 1$, these graphs are called Kneser graphs.

The $q$-analog of these graphs are obtained from vector spaces $F_q^n$ of dimension $n \geq 2$ where $F_q$ is the Galois field of order $q$. A flag of $F_q^n$ is a set $f$ of subspaces such that $X \subseteq Y$ or $Y \subseteq X$ for all $X, Y \in f$, and the set $\{|\dim(X)| : X \in f\}$ is
the type of $f$. Two flags $f$ and $f'$ are in general position, if $X + X' = F_q^n$ or $X \cap X' = \{0\}$ for all $X \in f$ and $X' \in f'$. For $T \subseteq [n - 1]$, the graph whose vertices are the flags of type $T$ of $F_q^n$ with two vertices being adjacent when the corresponding flags are in general position will be denoted by $\Gamma_q(n, T)$.

In this paper we are interested in the independence number of the graphs $\Gamma(n, T)$ for $\emptyset \neq T \subseteq [n - 1]$. For $|T| = 1$ the problem reduces to the famous Erdős-Ko-Rado problem, which was solved in [2], but up to my knowledge the problem was not considered for $|T| > 1$. In the $q$-analogue situation, the independence number of the graphs $\Gamma_q(n, T)$ with $|T| = 1$ was determined by Frankl and Wilson [3], see also Hsieh [4] and Newman [5]. However, for the graphs $\Gamma_q(n, T)$ also several results for $|T| > 1$ are known, see [1].

For every non-empty subset $T$ of $[n - 1]$ the graph $\Gamma(n, T)$ is regular and non-empty, so its independence number is at most one half of the number of its vertices with equality if and only if the graph is bipartite. This is always the case when $T$ contains an element $i$ with $n - i \in T$. If $\max T \leq \frac{n}{2}$ or $\min(T) \geq \frac{n}{2}$, then the independence number can easily be derived form the above mentioned result of Erdős-Ko-Rado. Section 2 is devoted to these facts as well as to several other basic facts.

There are pairs $(n, T)$ with $\emptyset \neq T \subseteq [n - 1]$ for which it is to be a non-trivial problem to determine the independence number of $\Gamma(n, T)$. There are different techniques one might try, for example techniques that have been successful in the case $|T| = 1$. One of the techniques is to apply the Hoffman bound, but for the cases I have tested, this bound is too weak when $|T| > 1$ (except in some easy cases that are handled in Section 2 by different methods). One may consider the underlying coherent configuration and try to obtain a bound by semidefinite programming. But for the non-trivial cases I have tried this also does not give tight bounds. Another well-known technique is the cycle method and in fact a variant of this method will be used to show the first result of the paper. In order to state it, we denote the set of vertices of a graph $\Gamma$ by $V(\Gamma)$.

**Theorem 1.1.** For positive integers $b$ and $n$ and every non-empty subset $T$ of $[n]$ with $\max(T) + 3b \leq 2n < 4b$, the independence number of the graph $\Gamma(n, T \cup \{b\})$ is equal to $\binom{n-1}{b} \cdot |V(\Gamma(b, T))|$. An independent set of this size of $\Gamma(n, T \cup \{b\})$ is given by all flags of type $T \cup \{b\}$ whose sets of cardinality $b$ are contained in $[n - 1]$. We remark that for certain $T$, $b$ and $n$ there are non-equivalent independent sets of the same size. In fact, in Section 3 we construct several families of independent sets of $\Gamma(n, T)$ when $|T| = 2$ and determine which of these are the largest ones. We will see that there are either one, two or three non-equivalent independents sets of maximal size among these examples. In Section 4 we show that these examples are in fact largest independent sets when $T = \{a, b\}$ with $b > \frac{n}{2}$ and $a + 3b \leq 2n$. The above theorem is a corollary of this result. In Section 5 we show in one other case, namely $T = \{1, n - 2\}$ that the examples are best possible, which results in the following theorem.

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Theorem 1.2. For \( n \geq 5 \) the independence number of \( \Gamma(n, \{1, n-2\}) \) is \( \binom{n}{3} + 2 \) and there exist exactly three non-equivalent independent sets of this cardinality.

The proof is by induction on \( n \) but we prove the induction step \( n-1 \rightarrow n \) only for \( n \geq 10 \). The induction basis \( 5 \leq n \leq 9 \) is verified by computer. In fact our result is more general, in that it proves an induction step in a more general setting. However, the proof of the induction basis for the other cases seems to be out of range for computers. Details are given in Section 5.

The above theorems combined with the results presented in Section 2 determine the independence number of all graphs \( \Gamma(n, T) \) with \( n \leq 6 \). Several open problems will be given in Section 6.

2 Basic observations

Notation. 1. For every positive integer \( n \) we denote by \( \omega_n \) the permutation of \([n-1]\) with \( \omega_n(i) = n - i \) for all \( i \in [n-1] \). For subsets \( T \) of \([n-1]\), we define \( \omega_n(T) = \{\omega_n(i) \mid i \in T\} \).

2. For every non-empty finite set \( M \) and all sets \( S \subseteq T \subseteq [\lvert M \rvert - 1] \) we denote by \( \text{proj}_{T, S}^M \) the map from the set of all flags of \( M \) of type \( T \) to the set of all flags of \( M \) of type \( S \) that maps each flag \( f \) of type \( T \) to the flag \( \{A \in f : \lvert A \rvert \in S\} \). If \( M = [n] \), then we write \( \text{proj}_{T, S}^n \).

3. For each finite graph \( \Gamma \) we denote its independence number by \( \alpha(\Gamma) \).

Lemma 2.1. Let \( n \geq 2 \) be an integer and \( \emptyset \neq T \subseteq [n-1] \). Then we have.

(a) The graph \( \Gamma(n, T) \) is regular.

(b) \( \Gamma(n, T) \cong \Gamma(n, \omega_n(T)) \) (duality).

(c) \( \alpha(\Gamma(n, T)) \leq \frac{1}{2} \lvert V \Gamma(n, T) \rvert \) with equality if and only if \( \Gamma(n, T) \) is bipartite.

(d) If there exist \( i, j \in T \) with \( i + j = n \), then \( \Gamma(n, T) \) is bipartite and \( \alpha(\Gamma(n, T)) = \frac{1}{2} \lvert V \Gamma(n, T) \rvert \).

Proof. (a) Since the symmetric group on the set \([n]\) induces a vertex transitive automorphism group of \( \Gamma(n, T) \), this graph is regular.

(b) The map \( f \mapsto ([n] \setminus A \mid A \in f) \) for every flag \( f \) of \([n]\) induces a bijective map \( \iota \) from the vertex set of \( \Gamma(n, T) \) to the vertex set of \( \Gamma(n, \omega_n(T)) \). By the laws of DeMorgan two subsets of \([n]\) are in general position if and only if their complements in \([n]\) are in general position. Thus two flags of type \( T \) are in general position if and only if their images under \( \iota \) are in general position. Therefore \( \iota \) is an isomorphism between the graphs.

(c) This is true for every finite regular non-empty graph.
(d) We remark that we allow that $i = j$. Every flag $f$ of type $\{i,j\}$ is in general position to exactly one flag of type $\{i,j\}$, namely $\{[n] \setminus A \mid A \in f\}$. If we take from every such pair one flag, we obtain a set $X$ consisting of flags of type $\{i,j\}$ that are mutually not in general position. The set consisting of all flags $f$ of type $[n]$ that satisfy $proj_{T,(i,j)}^n(f) \in X$ has the property that no two of its flags are in general position and moreover this set contains $\frac{1}{2}VT(n,T)$ elements. By (c), the graph is bipartite. \hfill \Box

**Remark 2.2.** If there exists $i \in T$ with $j = n - i \in T$, then $\Gamma(n,T)$ is disconnected and each component is a complete regular bipartite graph.

**Lemma 2.3.** Let $S$ and $T$ be distinct non-empty subsets of $[n-1]$ with $S \subseteq T$.

(a) Let $f_1$ and $f_2$ be flags of $[n]$ of type $T$. If $f_1$ and $f_2$ are in general position, then so are $proj_{T,S}^n(f_1)$ and $proj_{T,S}^n(f_2)$. In other words, $proj_{T,S}^n$ is a graph homomorphism from $\Gamma(n,T)$ onto $\Gamma(n,S)$.

(b) We have

$$\alpha(\Gamma(n,T)) \geq \alpha(\Gamma(n,S)) \cdot \frac{|VT(n,T)|}{|VT(n,S)|}.$$  \hfill (1)

**Proof.** (a) This follows immediately from the definition of general position.

(b) Let $\mathcal{F}$ be an independent set of $\Gamma(n,S)$ with $|\mathcal{F}| = \alpha(\Gamma(n,S))$. By (a), the preimage $\mathcal{F}$ of $\mathcal{F}$ under the map $proj_{T,S}^n$ is an independent set of $\Gamma(n,T)$. The fraction $v$ in its is the common number of preimages of every vertex of $\Gamma(n,S)$ under the projection map $proj_{T,S}^n$. Hence $|\mathcal{F}| = v \cdot \alpha(\Gamma(n,S))$. \hfill \Box

**Remark 2.4.** In general we do not have equality in (1). For example for $n = 6$, $S = \{4\}$ and $T = \{1, 4\}$ we have $\alpha(\Gamma(n,S)) = \binom{5}{4} = 5$ and the fraction in (1) is $4$ whereas we shall see in Theorem 5.10 that $\alpha(\Gamma(n,T)) = 22$.

Now we present a situation where (1) holds with equality.

**Proposition 2.5.** Let $S$ and $T$ be distinct non-empty subsets of $[n-1]$ satisfying

$$S \subseteq T, \min(S) + \max(S) \leq n \quad \text{and} \quad \max(T \setminus S) < \min(S).$$

Then two flags of $\Gamma(n,T)$ are in general position if and only if their images under $proj_{T,S}^n$ are in general position. In particular

$$\alpha(\Gamma(n,T)) = \alpha(\Gamma(n,S)) \cdot |VT(\min(S),T \setminus S)|.$$  \hfill (2)

**Proof.** Let $f_1$ and $f_2$ be flags of $\Gamma(n,T)$ and $g_1$ and $g_2$ their images under $proj_{T,S}^n$. If $f_1$ and $f_2$ are in general position, then so are their images by the previous lemma. Now assume that $g_1$ and $g_2$ are in general position. Let $A_i, B_i \in g_i$ with $|A_i| = \min(S)$ and $|B_i| = \max(S) = \max(T)$. Since $\min(S) + \max(S) \leq n$ and since $g_1$ and $g_2$ are in general position we have $A_1 \cap B_2 = \emptyset$. If $C_i \in f_i \setminus g_i$,
In this section we study examples of independent sets of the graphs \( \Gamma(\mathbb{n}, T) \) with \( \mathbb{n} \leq 8 \). The results in this section determine the independence number of \( \Gamma(\mathbb{n}, T) \) in all cases in which the independence number for the graphs \( \Gamma(\mathbb{n}, T) \) is known.

3 Examples

In this section we study examples of independent sets of the graphs \( \Gamma(\mathbb{n}, \{a, b\}) \) with \( a < b \). Since the case \( \max\{a, b\} \leq \frac{n}{2} \) and the dual case \( \min\{a, b\} \geq \frac{n}{2} \) have been treated in Corollary 2.7, we assume that \( a < \frac{n}{2} < b \). The case \( a + b = n \) has been treated in Lemma 2.1 (d), so that we can assume \( a + b \neq n \). In view of Lemma 2.1 (b) we will restrict ourselves therefore to the situation when \( a + b < n \). For simplicity a flag \( \{A, B\} \) of type \( \{a, b\} \) will mostly be written as an ordered pair \( (A, B) \) with \( |A| = a \) and \( |B| = b \). Recall that \( [0] = \emptyset \).
Example 3.1. Let \( n, a \) and \( b \) be positive integers with \( a < \frac{b}{2} < b \) and \( a + b < n \). For every integer \( i \) with \( 0 \leq i \leq 2b - n + 1 \) we denote by \( \mathcal{F}_i(n, a, b) \) the set of all vertices \((A, B)\) of \( \Gamma(n, \{a, b\})\) that fulfill at least one of the following conditions.

(I) \( [i] \subseteq B \subseteq [n - 1] \),

(II) \( \min A \leq i \) and \( \min A \subseteq B \).

For \( i = 2b - n + 1 \), let \( \bar{\mathcal{F}}_i(n, a, b) \) be the set of all vertices \((A, B)\) of \( \Gamma(n, \{a, b\})\) that satisfy \( [i] \subseteq B \) or (II); hence \( \mathcal{F}_i(n, a, b) \subseteq \bar{\mathcal{F}}_i(n, a, b) \).

Lemma 3.2. Let \( n, a, b \) be positive integers with \( a + b < n \) and \( b > \frac{n}{2} \).

(a) For \( 0 \leq i \leq 2b - n \) the set \( \mathcal{F}_i(n, a, b) \) is a maximal independent set of \( \Gamma(n, \{a, b\}) \).

(b) For \( i = 2b - n + 1 \), the set \( \bar{\mathcal{F}}_i(n, a, b) \) is a maximal independent set of \( \Gamma(n, \{a, b\}) \) and it is the only maximal independent that contains \( \mathcal{F}_i(n, a, b) \). Moreover \( |\mathcal{F}_i \setminus \mathcal{F}_j| + \binom{2s-2}{a} \binom{s-1}{a} \) where \( s = n - b \).

Proof. We put \( \mathcal{F}_i = \mathcal{F}_i(n, a, b) \). It is easy to see that all sets in question are independent sets. To investigate maximality we consider the situation that there exists an integer \( i \) with \( 0 \leq i \leq 2b - n + 1 \) and a flag \((A, B)\) such that \((A, B) \notin \mathcal{F}_i(n, a, b) \) and \( \mathcal{F}_i \cup \{(A, B)\} \) is independent. We have to show that \( i = 2b - n + 1 \) and \((A, B) \in \mathcal{F}_i(n, a, b) \).

First we want to show that \( [i] \subseteq B \). Assume the contrary. Then there exists a positive integer \( j \leq i \) with \( [j - 1] \subseteq B \) and \( j \notin B \). Since \((A, B) \notin \mathcal{F}_i\), then \( \min(A) > j - 1 \) and hence \( \min(A) \geq j + 1 \). We have \( |B \setminus [j - 1]| \geq |B| - (i - 1) = b - 1 - i \geq n - b \). Hence there exists a subset \( T \) of cardinality \( n - b \) of \([n]\) with \( A \subseteq T \subseteq B \setminus [j - 1] \). Put \( B' := [n] \setminus T \). Then \( B \cap B' = [n] \) and \( |B' \setminus B| = n - b \geq a \).

Choose \( A' \subseteq B' \setminus B \) with \( |A'| = a \) and \( j \in A' \). Then \((A, B)\) and \((A', B')\) are in general position. Since \( \min(A') = j \) and \( j \subseteq B' \) we have \((A', B') \in \mathcal{F}_i\). But \( \mathcal{F}_i \cup \{(A, B)\} \) is independent, contradiction.

Therefore \( [i] \subseteq B \). Since \((A, B) \notin \mathcal{F}_i\), then \( \min(A) > i \) and \( n \in B \). We want to show that \( i = 2b - 1 - n \). Assume the contrary that \( i \leq 2b - n \). Then \( |B \setminus [i]| = b - i \geq n - b \). As \( a < n - b \), we thus find a subset \( T \) of \( B \setminus [i] \) with \( A \cup \{n\} \subseteq T \) and \( |T| = n - b \). Put \( B' := [n] \setminus T \). Then \( B \cap B' = [n] \) and \( |B' \setminus B| = n - b \geq a \).

Choose \( A' \subseteq B' \setminus B \) with \( |A'| = a \). Then \((A, B)\) and \((A', B')\) are in general position. Since \( [i] \subseteq B' \subseteq [n - 1] \) we have \((A', B') \in \mathcal{F}_i\). But \( \mathcal{F}_i \cup \{(A, B)\} \) is independent, contradiction.

Hence \( i = 2b - 1 - n \). This proves (a). We have moreover shown that every flag \((A, B)\) that is not in \( \mathcal{F}_i \) and such that \( \mathcal{F}_i \cup \{(A, B)\} \) is independent satisfies \( [i] \subseteq B \) and hence \((A, B) \in \bar{\mathcal{F}}_i(n, a, b) \). This is the first part of (b). Since the elements \((A, B)\) of \( \bar{\mathcal{F}}_i(n, a, b) \) are the flags with \( [i] \cup \{n\} \subseteq B \) and \( \min(A) \geq i + 1 \), we see that there exists \( \binom{n - i - 1}{b - i} \) such flags.
Lemma 3.3. Let \( n, a, b \) be positive integers with \( a + b < n \) and \( b > \frac{n}{2} \). For \( 0 \leq i \leq 2b - n + 1 \), the cardinality of \( F_i(n, a, b) \) is

\[
\binom{n-1-i}{b-i} \binom{b-i}{a} + \binom{n-b+a-1}{a-1} \left( \binom{n}{b-a} - \binom{n-i}{b-a-i} \right).
\]

Moreover, with

\[
i_0 := b - 1 - \frac{(n-b)(n-b-1)}{a}
\]

we have the following. If \( i_0 < 0 \), then the largest cardinality occurs only for \( i = 0 \). If \( i_0 \geq 0 \) is an integer, then the largest cardinality occurs for \( i \in \{i_0, i_0 + 1\} \). If \( i_0 \geq 0 \) but \( i_0 \) is not an integer, then the largest cardinality occurs only for \( i = \lfloor i_0 \rfloor \).

Proof. First notice that the condition \( a + b < n \) implies that \( i_0 \leq 2b - n - 1 \). The number of flags satisfying only the first condition of Example 3.1 is \( \binom{n-1-i}{b-i} \binom{b-i}{a} \). With the notation \( j := \min(A) \) in the second condition of Example 3.1 we find

\[
|F_i(n, a, b)| = \binom{n-1-i}{b-i} \binom{b-i}{a} + \sum_{j=1}^{i} \binom{n-j}{a-1} \binom{n-a-j+1}{b-a-j+1}
\]

\[
= \binom{n-1-i}{b-i} \binom{b-i}{a} + \sum_{j=1}^{i} \binom{n-b+a-1}{n-b} \binom{n-j}{n-b+a-1}
\]

\[
= \binom{n-1-i}{b-i} \binom{b-i}{a} + \binom{n-b+a-1}{a-1} \left( \binom{n}{b-a} - \binom{n-i}{b-a-i} \right)
\]

This proves the formula for \( F_i(n, a, b) \). We also find

\[
|F_i(n, a, b)| = \binom{n-b+a-1}{a-1} \binom{n}{b-a} + \frac{(n-b)^2 + a(i-b))(n-1-i)!}{(n-b+a)!n!(b-i-a)!(n-b+a)!(n-b+a)!}
\]

Using this, an easy calculation shows that \( |F_{i+1}| \geq |F_i| \) if and only if \( i \leq i_0 \) and that \( |F_{i+1}| = |F_i| \) if and only if \( i = i_0 \). This implies the second statement of the lemma. \( \square \)

Definition. We call two independent sets of a graph equivalent, if the are in the same orbit under the automorphism group of the graph.

Proposition 3.4. With \( a, b, n \) and \( i_0 \) as in Lemma 3.3 we have the following.

(a) If \( i_0 \) is an integer, then \( F_{i_0}(n, a, b) \) and \( F_{i_0+1}(n, a, b) \) are non-equivalent independent sets of \( \Gamma(n, \{a, b\}) \).

(b) If \( a + b = n - 1 \), then \( i_0 = 2b - n - 1 \) and the sets \( F_{i_0}(n, a, b) \), \( F_{i_0+1}(n, a, b) \) and \( F_{i_0+2}(n, a, b) \) are non-equivalent independent set of \( \Gamma(n, \{a, n-a-1\}) \) of size \( \binom{n}{2a+1} \binom{2a}{a} + \binom{2a}{a} \).
Proof. Put \( F_i = F_i(n, a, b) \) for \( i \in \{i_0, i_0 + 1\} \) and \( F_i = \tilde{F}_i(n, a, b) \) for \( i = i_0 + 2 \).

(a) **Part 1.** Consider \( i \in \{i_0, i_0 + 1\} \) and a vertex \((A, B)\) that does not lie in \( F_i \). In this part we determine the number of neighbors of \((A, B)\) in \( F_i \) or a lower bound for this number.

**Case 1.** \([i] \subseteq B\).

As \((A, B) \notin F_i(n, a, b)\), it follows that \( \min(A) \geq i + 1 \) and \( n \in B \). With
\[
z_1(i) := \binom{n-i-1}{n-b} \binom{b-i}{a} \quad \text{and} \quad z_2(i) := \binom{n-i-1}{n-b} \binom{b-i-1}{a}
\]
there exist exactly \( z_1(i) \) such vertices \((A, B)\) and exactly \( z_2(i) \) have \( n \notin A \). The neighbors of \((A, B)\) in \( F_i \) are the vertices \((A', B')\) satisfying \([i] \subseteq B' \subseteq [n-1]\) with \(|B \cap B'| = 2b - n\) and \( B' \cap A = \emptyset \) and \( A' \subseteq B' \setminus B \). It follows that the number of neighbors of \((A, B)\) in \( F_i \) is
\[
m_1(i) := \binom{b-i - a - 1}{n - b - a - 1} \binom{n-b}{a}, \quad \text{if} \quad n \notin A, \quad (5)
\]
\[
m_2(i) = \binom{b-i - a}{n - b - a} \binom{n-b}{a}, \quad \text{if} \quad n \in A. \quad (6)
\]

**Case 2.** \([i] \not\subseteq B\).

Then \([j - 1] \subseteq B \) and \( j \notin B \) for some integer \( j \in [i] \). As \((A, B) \notin F_i(n, a, b)\), then \( A \subseteq B \setminus [j - 1] \). Therefore \(|B \setminus (A \cup [j - 1])| = b - a + 1 - j\). Therefore the number of neighbors \((A', B')\) of \((A, B)\) in \( F_i(n, a, b)\) that satisfy \([j] \subseteq B'\) and \( j = \min A' \) is
\[
m(j) := \binom{b-a+1-j}{n-b-a} \binom{n-b-1}{a-1}.
\]

If \( n \in B \), then \((A, B)\) has neighbors \((A', B')\) in \( F_i \) that satisfy \([i] \subseteq B' \subseteq [n-1]\) and \( \min(A') > i \). Hence the number of neighbors of \((A, B)\) in \( F_i(n, a, b)\) is at least \( m_3(i) := m(i) \) and equality occurs if and only if \( j = i \) and \( n \notin B \). The number of such vertices \((A, B)\) is
\[
z_3(i) := \binom{n-i-1}{n-b-2} \binom{b+1-i}{a}.
\]

**Part 2.** Now we use this information to show that \( F_{i_0} \) and \( F_{i_0+1} \) are not equivalent. Recall the definition of \( i_0 \) in \( [9] \) and that \( a + b \leq n - 1 \). This implies that \( m_2(i_0) > m_1(i_0) \) and \( m_3(i_0) > m_1(i_0) \). Hence, every vertex outside \( F_{i_0} \) has at least \( m_1(i_0) \) neighbors in \( F_{i_0} \) and there are exactly \( z_2(i_0) \) vertices with \( m_1(i_0) \) neighbors in \( F_{i_0} \).

We have seen that there are vertices outside \( F_{i_0+1} \) with \( m_1(i_0 + 1) \) neighbors in \( F_{i_0+1} \). If \( a + b < n - 1 \), then \( m_1(i_0) > m_1(i_0 + 1) \) and hence \( F_{i_0} \) and \( F_{i_0+1} \) are not equivalent.
Now consider the case $a + b = n - 1$. Then $i_0 = 2b - n - 1$ and $m_1(i_0) = m_2(i_0 + 1)$. Also, using $a + b = n - 1$, we see that $m_3(i_0 + 1) \geq m_1(i_0 + 1)$ with equality if and only if $a = 1$. If $a > 1$, it follows that $z_1(i_0 + 1)$ vertices have exactly $m_1(i_0)$ neighbors in $F_{i_0+1}$. Since $z_1(i_0 + 1) < z_2(i_0)$, it follows in this case that $F_{i_0}$ and $F_{i_0+1}$ are not equivalent. Finally consider the case $a = 1$ and $b = n - 2$. Then $i_0 = n - 5$ and $m_1(i_0) = m_k(i_0 + 1)$ for $k = 1, 2, 3$. Also, there are exactly $z_1(i_0 + 1) + z_3(i_0 + 1) = 6 + 3 = 9$ vertices with $m_1(i_0)$ neighbors in $F_{i_0+1}$ and $z_2(i_0) = 12$ vertices with $m_1(i_0)$ neighbors in $F_{i_0}$. Thus, also in this situation, the two independent sets are not equivalent.

(b) Put $F_{10+2} = F_{i_0+2(n, a, b)}$. In this part we have $a + b = n - 1$, which implies that $i_0 = 2b - n - 1 = b - a - 2 \geq 0$, $m_1(i_0 + 1) = m_2(i_0 + 1) = a + 1$ and $m_3(i_0 + 1) = 2a$. From the proof of (a) we know the following:

- The vertices of $M_1 := \{(A, B) | [i_0 + 1] \subseteq B, \min(A) \geq i_0 + 2\}$ have $a + 1$ neighbors in $F_{i_0+1}$ and $|M_1| = z_1(i_0 + 1)$.
- The vertices of $M_2 := \{(A, B) | [i_0] \subseteq B, \ i_0 + 1, n \notin B, \min(A) > i_0 + 1\}$ have $2a$ neighbors in $F_{i_0+1}$ and $|M_2| = z_3(i_0 + 1)$.
- Every other vertex outside $F_{i_0+1}$ has more than $2a$ neighbors in $F_{i_0+1}$.

In a similar way one sees the following.

- The vertices of $M_1' := \{(A, B) | [i_0 + 1] \subseteq B, \ i_0 + 2 \notin B, \min(A) > i_0 + 2\}$ have $a + 1$ neighbors in $F_{i_0+2}$ and $|M_1'| = z_1(i_0 + 1)$.
- The vertices of $M_2' := \{(A, B) | [i_0] \subseteq B, \ i_0 + 1, n \notin B, \min(A) = i_0 + 2\}$ have $2a$ neighbors in $F_{i_0+2}$ and $|M_2'| = z_3(i_0 + 1)$.
- Every other vertex outside $F_{i_0+2}$ has more than $2a$ neighbors in $F_{i_0+2}$.

Hence the number of vertices with $a + 1$ neighbors in $F_{i_0+1}$ is equal to the number of vertices with $a + 1$ neighbors in $F_{i_0+2}$. We have seen in the proof of (a) that this number is different from the number of vertices that have $a + 1$ neighbors in $F_{i_0}$. Thus $F_{i_0}$ and $F_{i_0+2}$ are not equivalent. In order to see that $F_{i_0+1}$ and $F_{i_0+2}$ are not equivalent, we consider two cases. We remark that this case is slightly more complicated, since for each $j$, the number of vertices with exactly $j$ neighbors in $F_{i_0+1}$ is equal to the number of vertices with exactly $j$ neighbors in $F_{i_0+2}$.

Case 1. $a \geq 2$, that is $a + 1 < 2a$.

Then the set of vertices with $2a$ neighbors in $F_{i_0+1}$ or $F_{i_0+2}$ is $M_2$ respectively $M_2'$. The elements of $F_{i_0+1}$ that have at least one neighbor in $M_1$ are $(A', B')$ with $[i] \subseteq B'$ and $\min(A') = i$. Depending on $n \in A'$ or $n \notin A'$, the number of neighbors of $(A', B')$ in $M_1$ is different. The elements of $F_{i_0+2}$ that have a neighbor in $M_2'$ are $(A', B')$ with $\min(A') = i - 1, [i - 1] \subseteq B'$, $i \notin B'$.
By symmetry all these vertices have the same number of neighbors in $M_2$. This implies that $F_{i_0+1}$ and $F_{i_0+2}$ are not equivalent.

Case 2. $a = 1$, so $a + 1 = 2a$.

Then $M_1 \cup M_2$ consists of the vertices with $2a = a + 1$ neighbors in $F_{i_0+1}$, and $M'_1 \cup M'_2$ consists of the vertices with $2a = a + 1$ neighbors in $F_{i_0+2}$.

We have $|M_1| = 6$ and $|M_2| = 3$ where $M_1$ consists of $(A, B)$ with $A = \{c\}$, $B = [n-4] \cup \{c, n\}$, or $A = \{n\}$, $B = [n-4] \cup \{c, n\}$, and $M_2$ consists of the vertices $(A, B)$ with $A = \{c\}$ and $B = [n] \setminus \{n-4, n\}$, where $n - 3 \leq c \leq n - 1$.

We see that the first three vertices of $M_1$ have no neighbor in $M_1 \cup M_2$.

The set $M'_1$ consists of the six vertices $B = [n-4] \cup G$ and $A = \{c\} \subseteq G$ where $G$ is 2-subset of $\{n-2, n-1, n\}$, and $M_2$ consists of the three vertices with $B = [n-5] \cup G$ where $G = \{n-2, n-2, n-1\}$ and $A = \{c\}$ with $c \in G$. Hence every vertex of $M'_1 \cup M'_2$ has at least one neighbor in $M'_1 \cup M'_2$.

We have shown that graphs induced on $M_1 \cup M_2$ and on $M'_1 \cup M'_2$ are not isomorphic, and this implies that the sets $F_{i_0+1}$ and $F_{i_0+2}$ are not equivalent.

$\square$

**Notation.** For positive integers $n, a, b$ with $a + b < n$ and $a < \frac{n}{2} < b$ we denote by $f(n, a, b)$ the largest cardinality of the independent sets that have been constructed in Examples 3.1 that is

$$f(n, a, b) = \max\{|F_j(n, a, b)| : 0 \leq j \leq 2b - n + 1\}$$

$$= \binom{n - 1 - i}{b - i} \binom{b - i}{a} + \binom{n - b + a - 1}{a - 1} \left( \binom{n}{b} - \binom{n - i}{b - a - 1} \right)$$

where

$$i := \max\{0, \left\lfloor b - 1 - \frac{(n-b)(n-b-1)}{a} \right\rfloor \}.$$  (7)

4 The cycle method

In this section we consider the graphs $\Gamma(n, \{a, b\})$ with $a < \frac{n}{2} < b$ and $a + 3b \leq 2n$. The number $i_0$ defined in (3) is negative and hence $i$ defined in (4) is zero, so the results in Section 3 show that $F_0(n, a, b)$ is an independent set with

$$|F_0(n, a, b)| = f(n, a, b) = \binom{n-1}{b}.$$ 

We now show that there is in fact no larger independent set.

**Theorem 4.1.** *For positive integers $a, b$ and $n$ with $n < 2b$ and $a + 3b \leq 2n$ we have* 

$$\alpha(\Gamma(n, \{a, b\}) = \binom{n-1}{b}.$$
Proof. The above example shows that the independence number can not be smaller. To show that equality holds, we consider $M := \{0, 1, \ldots, n - 1\}$ and an independent set $\mathcal{F}$ of flags of $M$ of type $\{a, b\}$. Put $s := 2b - n$ and notice that $a < 2s + a \leq b$. Let $X$ be the set consisting of all pairs $(f, c)$ with vertices $f = \{A, B\} \in \mathcal{F}$ and $n$-cycles $c = (m_0, \ldots, m_{n-1})$ of the elements of $M$ such that there exists an index $i$ with $B = \{m_i, j \mid 1 \leq j \leq b\}$ and $A = \{m_{i+s+j} \mid 1 \leq j \leq a\}$, where indices are modulo $n$.

Each $f \in \mathcal{F}$ occurs in precisely $a!(b - a)!(n - b)!$ pairs of $X$. Therefore we have $|X| = |\mathcal{F}|a!(b - a)!(n - b)!$.

Now consider an $n$-cycle $c = (m_0, \ldots, m_{n-1})$ and let $t$ be the number of pairs of $X$ in which $c$ occurs. Let these pairs be $(f_i, c)$, $1 \leq i \leq t$, where $f_i = \{A_i, B_i\}$ with $|A_i| = a < b = |B_i|$. We want to show that $t < n - b$. For this we may assume that $t \geq 1$. Consider $1 \leq i, j \leq t$. As $f_i$ and $f_j$ are vertices that are not connected by an edge we have $|B_i \cap B_j| > 2b - n$ or $A_i \cap B_j \neq \emptyset$ or $A_j \cap B_i \neq \emptyset$.

Since $b \geq 2s + a$ and $(f_i, c), (f_j, c) \in X$, we see that each of $A_i \cap B_j \neq \emptyset$ and $A_j \cap B_i \neq \emptyset$ implies that $|B_i \cap B_j| \geq s + 1 > 2b - n$. Hence $|B_i \cap B_j| > 2b - n$ in any case. Thus any two of the sets $B_i := [n] \setminus B_i$ have non-empty intersection.

As in the classical proof of Katona, see Lemma 2.14.1 in [4], it follows that $t \leq n - b$. As there are $(n - 1)!$ $n$-cycles, it follows that $|X| \leq (n - 1)! \cdot (n - b)$.

Comparing the lower and upper bound for $|X|$ implies the desired result.  

Using Proposition 2.5 we find the following corollary, which is Theorem 1.1 from the introduction in a slightly different formulation.

**Corollary 4.2.** Let $a$, $b$ and $n$ be positive integers with $n < 2b$ and $a + 3b \leq 2n$. Then for every subset $T$ of $[a - 1]$ with we have

$$\alpha(\Gamma(n, T \cup \{a, b\}) = \binom{n-1}{b} \binom{b}{a} \cdot |V\Gamma(a, T)|.$$

**Remarks 4.3.**

1. The set $\mathcal{F}_0(n, a, b)$ is in general not the only largest independent set of $\Gamma(n, \{a, b\})$. For example for $n = 7$ and type $\{2, 4\}$ there are 14 non-equivalent independence sets of size 90 as can be verified by computer and $\mathcal{F}_0(7, 2, 4)$, $\mathcal{F}_1(7, 2, 4)$ and $\mathcal{F}_2(7, 2, 4)$ are among these.

2. For $(n, a, b) = (9, 1, 6)$ the condition $a + 3b \leq 2n$ is not fulfilled, but it can be verified by computer that $\alpha(\Gamma(n, \{a, b\}) = \binom{n-1}{b} \binom{b}{a}.$

5 **The chromatic number of $\Gamma(n, \{1, n - 2\})$**

In this section we classify the largest independent sets of $\Gamma(n, \{1, n - 2\})$ for $n \geq 5$ using induction on $n$. The induction step can be proved without significantly larger effort for more general graphs. We formulate this in Proposition 5.2 which is the main result of this section.

Recall that we have defined $f(n, a, b)$ for positive integers $n, a, b$ with $n > a + b$ and $a < \frac{n}{2} < b$ in the end of Section 3 and that $f(n, a, b)$ is the cardinality of
Lemma 5.1. For integers $n, a, b$ with $n \geq a + b + 1$ and $a < \frac{n}{2} < b$ we have

$$f(n, a, b) = f(n - 1, a, b - 1) + \binom{n - 1}{b - 1} \binom{b - 1}{a - 1}$$

and with $i$ as in (1) we have

$$n \cdot f(n - 1, a, b - 1) - (b - a) f(n, a, b) = \frac{[(n - b)^2 + a(i - b)(n - 1 - i)!}{a! (n - b)! (b - i - a)!}$$

Proof. This follows from straightforward calculations. For the second equation, one may apply (4) using that $f(n, a, b) = |\mathcal{F}(n, a, b)|$ and $f(n - 1, a, b - 1) = |\mathcal{F}_{i-1}(n - 1, a, b - 1)|$. \hfill \Box

Let $a, b$ and $n$ be positive integers with $n \geq a + b + 1$ and $a < \frac{n}{2} < b$. Let $M$ be a set with $n$ elements. In this section an independent set $\mathcal{F}$ of $\Gamma(M, \{a, b\})$ will be called an independent set of standard type if $|\mathcal{F}| = f(n, a, b)$ and if there exists a graph isomorphism $\mu$ of $\Gamma(M, \{a, b\})$ onto $\Gamma(n, \{a, b\})$ such that $\mu(\mathcal{F})$ is equal to one of the independent sets of $\Gamma(n, \{a, b\})$ that were constructed in Example 3.1.

Proposition 5.2. Let $a, b, n \geq 1$ be integers with $n \geq a + b + 1$ and $a < \frac{n}{2} < b$. Define $i$ as in (1) and assume that

$$n \geq \frac{[(n - b)^2 + a(i - b)(n - 1 - i)!}{a! (n - b)! (b - i - a)!} + 3a + 1. \quad (8)$$

Assume furthermore that $f(n - 1, a, b - 1)$ is the independence number of $\Gamma(n - 1, \{a, b - 1\})$ and that every independent set of $\Gamma(n - 1, \{a, b - 1\})$ is of standard type. Then $f(n, a, b)$ is the independence number of $\Gamma(n, \{a, b\})$ and every independent set of $\Gamma(n, \{a, b\})$ is of standard type.

For the proof of the proposition we define $M := [n]$ and consider an independent set $\mathcal{F}$ of $\Gamma(M, \{a, b\})$ with $|\mathcal{F}| = f(n, a, b)$. Recall that the vertices of $\Gamma(n, \{a, b\})$ are flags of type $\{a, b\}$, which we denote as ordered pairs $(A, B)$ where $A \subseteq B \subseteq M$, $a = |A|$ and $b = |B|$. In order to prove the proposition, we have to show that $\mathcal{F}$ is of standard type. This will be done in several lemmas in which we use for $c \in M$ the notation

$$\mathcal{F}_c := \{(A, B) \in \mathcal{F} | c \in B \setminus A\}$$

and

$$\tilde{\mathcal{F}}_c := \{(A, B \setminus \{c\}) | (A, B) \in \mathcal{F}_c\}.$$

Lemma 5.3. If the set $S$ has cardinality $n - 1$, then the graph $\Gamma(S, \{a, b - 1\})$ has independence number $f(n - 1, a, b - 1)$ and its largest independent set are of standard type.
Lemma 5.4. For every element \( c \in M \), the set \( \mathcal{F}_c \) is an independent set of the graph \( \Gamma(M \setminus \{ c \}, \{ a, b-1 \}) \) with \( |\mathcal{F}_c| \leq f(n-1, a, b-1) \).

Proof. Since \( \mathcal{F} \) is an independent of \( \Gamma(M, \{ a, b \}) \), then \( \mathcal{F}_c \) is an independent set of \( \Gamma(M \setminus \{ c \}, \{ a, b-1 \}) \). Apply Lemma 5.3.

Lemma 5.5. We have

\[
\sum_{c \in M} (f(n-1, a, b-1) - |\mathcal{F}_c|) = \frac{(n-b)^2 + a(i-b)(n-1-i)!}{a!(n-b)!(b-i-a)!}. (9)
\]

Proof. Every element of \( \mathcal{F} \) is a member of \( \mathcal{F}_c \) for \( b-a \) elements \( c \) of \( M \). Therefore \( |\mathcal{F}|(b-a) = \sum_{c \in M} |\mathcal{F}_c| \). Since \( |\mathcal{F}| = f(n, a, b) \), the claim follows from Lemma 5.1.

Lemma 5.6. If there exists an element \( x \in M \) such that \( x \in B \) for all elements \( (A, B) \) of \( \mathcal{F} \), then \( \mathcal{F} \) is of standard type.

Proof. Lemma 5.4 gives \( |\mathcal{F}_x| \leq f(n-1, a, b-1) \). Since \( x \in B \) for all \( (A, B) \) that \( \mathcal{F} \setminus \mathcal{F}_x \) of the type \( (A, B) \) of \( M \) of type \( (a, b) \) with \( x \in A \). This implies that \( |\mathcal{F} \setminus \mathcal{F}_x| \leq (n-1)(b-1)(a-1) \). Hence

\[
|\mathcal{F}| = |\mathcal{F}_x| + |\mathcal{F} \setminus \mathcal{F}_x| \leq f(n-1, a, b-1) + \binom{n-1}{b-1}\binom{b-1}{a-1} = f(n, a, b)
\]

where we have used Lemma 5.1 in the last step. Since \( |\mathcal{F}| = f(n, a, b) \), it follows that \( |\mathcal{F}_x| = f(n-1, a, b-1) \) and that \( \mathcal{F} \setminus \mathcal{F}_x \) consists of all flags \( (A, B) \) of \( M \) of type \( (a, b) \) with \( x \in A \). Since \( |\mathcal{F}_x| = |\mathcal{F}_x| \), Lemma 5.3 shows that the independent set \( \mathcal{F}_x \) of \( \Gamma(M \setminus \{ x \}, \{ a, b-1 \}) \) is of standard type. From the construction in Example 3.1 it follows that \( \mathcal{F} \) is of standard type.

Lemma 5.7. Suppose that \( c \) is an element of \( M \) with \( |\mathcal{F}_c| = f(n-1, a, b-1) \).

(a) There exists a unique index \( \hat{c} \neq c \) such that \( \hat{c} \in B \) for all \( (A, B) \in \mathcal{F}_c \).

(b) Every vertex \( (A, B) \) of \( \Gamma(M, \{ a, b \}) \) with \( \hat{c} \in A \) and \( c \in B \setminus A \) lies in \( \mathcal{F}_c \).

(c) If \( (A, B) \in \mathcal{F}_c \), then \( \hat{c} \in B \) or \( c \in A \).

Proof. Since \( |\mathcal{F}_c| = |\mathcal{F}_c| = f(n-1, a, b-1) \), Lemma 5.3 shows that \( \mathcal{F}_c \) is an independent set of \( \Gamma_c := \Gamma(M \setminus \{ c \}, \{ a, b-1 \}) \) of standard type. Inspection of the independent sets of standard type shows that there exists a unique element \( \hat{c} \in M \setminus \{ c \} \) such that \( \hat{c} \in B' \) for every element \( (A, B') \) of \( \mathcal{F}_c \). This proves (a). The inspection also shows that \( (A, B') \in \mathcal{F}_c \) for every vertex \( (A, B') \) of \( \Gamma_c \) with \( \hat{c} \in A \). This proves (b).
We prove (c) indirectly. Assume that \(F\) contains a vertex \((A, B)\) with \(\hat{c} \notin B\) and \(c \notin A\). Then (a) shows that \((A, B) \notin F_c\). Since \(c \notin A\), this implies that \(c \notin B\). Hence \(B \subseteq M \setminus \{c, \hat{c}\}\). Choose a set \(T\) with \(A \subseteq T \subseteq B\) and \(|T| = n - b\) and put \(B_0 := M \setminus T\). Then \(|B_0| = b\) and \(|B \setminus B_0| = |T| = n - b = a + 1\). Also \(c, \hat{c} \in B_0\). Since \(n - b \geq a + 1\), there exists a set \(A_0 \subseteq B_0\) of cardinality \(a\) satisfying \(\hat{c} \in A_0\) and \(c \notin A_0\). From (b) we see that \((A_0, B_0) \in F\). We have \(B \cup B_0 = M\) as well as \(A \cap B_0 = A_0 \cap B = \emptyset\). Hence \((A_0, B_0)\) and \((A, B)\) are adjacent vertices. As both vertices lie in \(F\) and since \(F\) is an independent set, this is the desired contradiction.

\(\Box\)

**Notation.** For every \(c \in M\) with \(|F_c| = f(n - 1, a, b - 1)\) we denote by \(\hat{c}\) the element provided by the previous lemma.

**Lemma 5.8.** Let \(C\) be the set of all elements \(c \in M\) with \(|F_c| = f(n-1, a, b-1)\).

(a) If there exist \(d \in M\) and \(T \subseteq C\) with \(|T| > a\) and \(\hat{c} = d\) for all \(c \in T\), then \(d \in B\) for all \((A, B) \in F\).

(b) If \(c_1, c_2 \in C\), then \(\hat{c}_1 = \hat{c}_2\) or \(\hat{c}_1 = c_2\) or \(\hat{c}_2 = c_1\).

**Proof.** (a) Assume on the contrary that \(F\) contains an element \((A, B)\) with \(d \notin B\). Then Lemma 5.7(c) shows \(T \subseteq A\). As \(|T| > a\), this is impossible.

(b) Assume on the contrary that \(c_1, \hat{c}_1, c_2, \hat{c}_2\) are four distinct elements. Since \(n - b \geq a + 1 \geq 2\) we see that \(M\) has subsets \(B_1\) and \(B_2\) of cardinality \(b\) with \(c_1, \hat{c}_1 \in B_1 \setminus B_2\) and \(c_1, \hat{c}_2 \in B_2 \setminus B_1\) and \(B_1 \cup B_2 = M\). Then \(|B_1 \setminus B_2| = n - b \geq a + 1\) and hence there exists a set \(A_1 \subseteq B_1 \setminus B_2\) with \(|A_1| = a\) and \(\hat{c}_1 \in A_1\) and \(c_1 \notin A_1\). The same argument shows that there exists \(A_2 \subseteq B_2\) with \(|A_2| = a\) and \(\hat{c}_2 \in A_2\) and \(c_2 \notin A_2\). Lemma 5.7(b) shows that \((A_1, B_1)\), \((A_2, B_2) \in F\). But these two vertices are adjacent, contradiction.

\(\Box\)

**Lemma 5.9.** Suppose that there exists a set \(T \subseteq M\) with \(|T| = 3a + 1\) and \(|F_c| = f(n - 1, a, b - 1)\) for all \(c \in T\). Then there exists an element \(x \in M\) with \(x \in B\) for all \((A, B) \in F\).

**Proof.** This follows from Lemma 5.8 if there are \(a + 1\) elements \(c \in T\) with the same \(\hat{c}\). We may thus assume that this does not occur. As \(|T| > 3a\), this implies that there exits \(c_1, c_2, c_3, c_4 \in T\) such that \(\hat{c}_1, \hat{c}_2, \hat{c}_3, \hat{c}_4\) are pairwise distinct. Lemma 5.8(b) shows that we may assume that \(\hat{c}_2 = c_1\). Since \(\hat{c}_3 \neq \hat{c}_4\), we may also assume that \(\hat{c}_3 \neq c_2\). Then \(c_2, c_3, \hat{c}_2, \hat{c}_3\) are mutually distinct. Lemma 5.8(b) gives a contradiction.

\(\Box\)

**Proof of Proposition 5.2** Using (8) we see from Lemma 5.4 and 5.5 that \(|F_c| = f(n - 1, a, b - 1)\) for at least \(3a + 1\) elements \(c \in M\). Then Lemma 5.9 shows that there exists an element \(x \in M\) with \(x \in B\) for all \((A, B) \in F\). Lemma 5.6 now proves the proposition.
REMARK. The proposition may be used as an induction step $n - 1 \rightarrow n$ to determine the independence number of $\Gamma(n, a, n - s)$ for given $a$ and $s$ (with $\frac{n}{2} > s \geq a + 1 \geq 2$) when $n$ satisfies (8) for $a$ and $b = n - s$, that is for

$$n \geq \frac{s^2 - a(1 + x)](s + x)!}{a! s!(1 + x - a)!} + 3a + 1. \tag{10}$$

where $x = \lfloor (s(s - 1)/a) \rfloor$. For $a = 1$ and $s = 2$, this is $n \geq 10$, and so the induction basis requires to show that the largest independent sets of $\Gamma(9, \{1, 7\})$ are of standard type. This can be done by computer. For all other values of $a$ and $s$, the integer $n$ must already be so large that I did not succeed to verify that induction hypothesis by computer. For example for $a = 2$ and $s = 3$, we already need $n \geq 37$, so the induction hypothesis requires to show that the independence number of $\Gamma(36, \{1, 33\})$, which is a graph with 235620 vertices, is 58947. So we obtain a theorem only in the case when $a = 1$ and $s = 2$, that is $b = n - 2$.

**Theorem 5.10.** For all integers $n \geq 5$ the graph $\Gamma(n, \{1, n - 2\})$ has independence number $\binom{n}{3} + 2$ and every independent set of $\Gamma(n, \{1, n - 2\})$ of cardinality $\binom{n}{3} + 2$ is equivalent to $F_{n-5}(n, 1, n-2)$ or $F_{n-4}(n, 1, n-2)$ or $\overline{F}_{n-3}(n, 1, n-2)$ as defined in Example 3.1.

**Proof.** Using induction on $n$ and Proposition 5.2, we only have to verify that the statement is true for $n \in \{5, 6, 7, 8, 9\}$. This can easily be done by computer for example using the package grape in GAP.

6 Open problems

A natural question is probably if not to determine the independence number of all graphs $\Gamma(n, M)$ to do this for the case $|M| = 2$. The only open case here for $n \leq 8$ is $n = 8$ and $M = \{2, 5\}$. We give some information on this case and some more special problems below.

**Lemma 6.1.** If $\alpha$ is the chromatic number of $\Gamma(8, \{2, 5\})$, then $230 \leq \alpha \leq 240$.

**Proof.** The set $F_1(8, 2, 5)$ is an independent set of $\Gamma(8, \{2, 5\})$ with 230 elements. This gives the lower bound. For the upper bound we use that $\Gamma(7, \{2, 4\})$ has independence number 90 by Theorem 4.1. Suppose $F$ is an independent set of $\Gamma(8, \{2, 5\})$. For each $c \in [8]$, let $F_c$ be the set of elements $(A, B)$ of $F$ with $c \in B \setminus A$. As in the proof of Lemma 5.5 we see that $\sum_{c=1}^{8} |F_c| = (5 - 2) \cdot |F|$. Each set $F_c$ is an independent set of the graph $\Gamma([8] \setminus \{c\}, \{2, 5\})$ and hence has at most 90 elements. It follows that $3|F| \leq 8 \cdot 90$, so $|F| \leq 240$. □

Remark. The smallest eigenvalue of the adjacency matrix $A$ of $\Gamma(8, \{2, 5\})$ is $-\frac{3}{2}(1 + \sqrt{17})$, so the Hoffman bound gives $\alpha \leq 257$ for the independence number.
If one considers the coherent configuration corresponding to $\Gamma(8, \{2, 5\})$, then one can apply semidefinite programming, but in this case it gives nothing better than the Hoffman bound. It can be checked that $A$ has the same number of positive and negative eigenvalues, so the inertia bound applied to $A$ does not give a better bound, but I did not check other matrices. Finally, since the independence number is larger than $560/3$, also the clique-coclique bound does not help to find a better bound for $\alpha$

Problem 1: Determine the chromatic number of $\Gamma(8, T)$ in the cases that $T$ is one of the sets $\{2, 5\}$, $\{1, 3, 5\}$ and $\{1, 4, 5\}$. These are all open problems for $n = 8$ up to duality.

Problem 2: Determine the chromatic number of $\Gamma(9, T)$ for $T = \{1, 6\}$ and for $T = \{2, 6\}$. These are the only open cases for $n = 9$ and $|T| = 2$.

Problem 3: It can be checked by computer that every independent set of $\Gamma(7, \{1, 4\})$ is equivalent to $\mathcal{F}_0(7, \{1, 4\})$. Find an computer-free argument for this fact.

Problem 4: Are there constructions of independent sets of $\Gamma(n, \{a, b\})$ with $a + b < n$ and $a < \frac{n}{2} < b$ that are larger than the ones constructed in Examples 3.1

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