1 Introduction

Recently, a non-equilibrium open-dissipative artificial neural network consisting of optical oscillators and Bose-Einstein condensates has been studied as a novel computing method for hard optimization problems. In those novel computing machines, the cost functions such as Ising Hamiltonian and XY Hamiltonian are mapped to the loss landscape of the artificial neural network rather than the standard approach to map a target Hamiltonian to an energy landscape used in classical and quantum annealing. By providing an appropriate gain to such an open-dissipative artificial neural network with a slow enough speed, a lowest-loss ground state of a target Hamiltonian should be spontaneously selected as a single oscillation/condensation mode.

A unique advantage of using degenerate optical parametric oscillators (DOPOs) as neurons is its hybrid quantum and classical characters. At below threshold, the quantum noise correlation formed among DOPOs realizes a quantum parallel search to identify a ground state before sizable mean-fields build up in DOPOs, while the pitchfork bifurcation above threshold amplifies the amplitude of a selected ground state exponentially to form a deterministic (classical) computation result. This particular system is referred to as coherent Ising machines (CIMs) in this paper. CIM has serious drawbacks in its actual performance. Mapping of a cost function to loss landscape often fails due to unequal amplitudes of constituent DOPOs, which is particularly serious for a problem with frustrated spins. Recently, an error detection and correction feedback mechanism has been proposed to overcome this problem in classical context. There is another challenge for CIM, which is a universal problem for any combinatorial optimizers including classical and quantum annealing. An exponentially many local minima easily trap a optimizer and make it to report a wrong answer when a problem size increases and yet a computational time is finite.

In this paper, we extend the error detection and correction feedback technique discussed in ref. to quantum domain. We show that by modulating a mutual coupling field and external pump rate base on an error signal, the DOPO amplitude heterogeneity and trapping in local minima can be partially but simultaneously suppressed, which leads to improved performance compared to an open-loop CIM...
Noise-free deamplification of the canonical coordinate $\hat{X}$ provided by degenerate parametric amplifying element with negative pump parameter $p (< 0)$ plays an important role in destabilizing local minima. The DOPO quantum states stay close to minimum uncertainty states with amplitude squeezing rather than amplitude anti-squeezing during an entire search process.

2 Principle of the proposed machine

The proposed machine is based on the measurement-feedback coupling CIM shown in Fig. 1. At each round trip in a ring resonator, every DOPO pulse amplitude (canonical coordinate) $\hat{X}_i (i = 1, \ldots, N)$ is measured by an optical homodyne detector and a corresponding Ising spin is decided by the sign of a measured amplitude, i.e. $S_i = \hat{X}_i / |\hat{X}_i|$, where $\hat{X}_i$ is an inferred amplitude. Note that this is an indirect and weak measurement, in which a probe beam carries its own vacuum noise and only a small portion of the internal DOPO pulse is extracted for measurement. If a current Ising energy $E(t) = -\sum_{i<k} J_{ik} S_i S_k$, computed by $S_i (i = 1, \ldots, N)$, is lower than the best Ising energy $E_{opt}$ previously visited ($E(t) < E_{opt}$), we increase the DOPO pump rate ($p > 0$) and simultaneously increase the mutual coupling field ($e > 0$) among DOPOs based on the Ising Hamiltonian in order to decrease an energy continuously by flipping “wrong spins” and preserving “correct spins”. This mode of operation is similar to that of an open-loop CIM, in which the pump rate $p$ is monotonically increased from below to above threshold or the mutual coupling field $e$ is monotonically increased. This step is called a mode “A”. If a current Ising energy $E(t)$ is roughly equal to the best Ising energy $E_{opt}$, we assume the machine already visited a new local minimum. We then decrease the pump rate $p$ close to zero ($p \simeq 0$) and eliminate the central potential barrier of the effective potential landscape, $V(X) = 1/2(1-p)X^2 + (1/4)g^2X^4$, to let each Ising spin to switch freely by the mutual coupling field $e$. In this way, we can avoid the notorious problem of trapping in a local minimum. This mode of operation is called a mode “B”. On the other hand, if a current Ising energy is higher than the best Ising energy previously visited ($E(t) > E_{opt}$), we assume the machine already started to escape from a local minimum by climbing up a potential. We then decrease the DOPO pump rate $p$ to a negative value ($p < 0$) to destabilize the current spin configuration more strongly and simultaneously maintain the high level of mutual coupling field among DOPOs in order to identify which spins should be flipped and which other spins should be maintained to move away from the previously visited local minimum. This mode of operation is called a mode “C”.

As a consequence of such dynamical modulation of the pump rate $p$ and the mutual coupling field $e(t)$,
Figure 2: (a) A system trajectory of CIM with error correction feedback (solid line) and a simple gradient descent system (dashed line). (b) An open-loop CIM employs a parametric amplification to search for a ground state. A measurement error is on the order of \( \frac{w}{\sqrt{R_B}} \), where \( w \) is a vacuum fluctuation and \( R_B \) is the reflectivity of an out-coupling beam splitter. (c) A closed-loop CIM with error correction feedback employs a parametric deamplification to search for a ground state.

the system trajectory avoids trapping in a local minimum and migrates from one local minimum to another in search of a true ground state, as shown in Fig. 2(a). The above three operational modes (A, B, C) repeat when the machine migrates over many local minima. On the contrary, a simple gradient descent system relaxes to a particular local minimum determined by an initial condition as shown by the dashed lines in Fig. 2(a).

In an open-loop CIM with an increasing pump rate or mutual coupling field, a noise-free parametric amplification gain and gain saturation (two photon absorption) make the state of each DOPO pulse evolve from an anti-squeezed vacuum state to a coherent state, for which the signal-to-noise ratio including detector noise in reading out a spin configuration is improved because of an increased mean-field, reduced quantum noise and constant measurement error, as shown in Fig. 2(b). In a closed-loop CIM with error detection and correction feedback, on the other hand, the noise-free parametric deamplification is switched on \( (p < 0) \) during a mode “C”, so that the quantum states of both spin flipping and spin preserving DOPO pulses become amplitude squeezed states, in which the mutual coupling field \( e \) decides which spins are flipped and which other spins are preserved, as shown in Fig. 2(c). A negative pump rate \( p (< 0) \) exponentially decreases the absolute amplitude \( |\langle \hat{X} \rangle| \), which helps to switch the polarity from positive to negative, and vice versa quickly. The negative pump rate also squeezes the amplitude noise \( \langle \Delta \hat{X}^2 \rangle \) when a DOPO crosses a zero point, which makes switching DOPO pulses more sensitive to a mutual coupling field \( e \).

3 Gaussian quantum theory

The proposed machine is modeled by the quantum mechanical master equation with the Liouvillian coupling terms that describe the dissipative coupling among DOPO pulses aided by a measurement-feedback circuit, two photon absorption loss (gain saturation) in a degenerate parametric amplifying device and background linear loss, respectively.\[20,21,22\] By expanding the field density operator by the Wigner function, we can obtain the Fokker-Plank equation for a signal field variable after adiabatic elimination of a pump-field variable and appropriate truncation of higher-order derivative terms. Then, we obtain the truncated-Wigner stochastic differential equation (W-SDE) using the Ito rule.\[19\] Validation of this ap-
proach (W-SDE) was confirmed by comparing the entanglement and quantum discord computed by this model with those computed by more exact positive-P stochastic differential equation (P-SDE) for an optical delay-line coupling CIM.

In the case of a large linear loss and small nonlinear coupling coefficient, i.e. \( g^2 = \kappa^2 / 2\gamma_s \xi_p \ll 1 \), where \( \kappa \) is a parametric interaction constant, \( \gamma_s \) is a signal photon field decay rate and \( \xi_p \) is a pump photon field decay rate, we can separate the i-th DOPO pulse amplitude to the mean field \( \mu_i = \langle X_i \rangle / \sqrt{2} \) and the variance \( \sigma_i = \langle \Delta \hat{X}_i^2 \rangle \). The equation of motion for the mean field \( \mu_i \) is,

\[
\frac{d}{dt} \mu_i = -\left( 1 + j \right) + p - g^2 \mu_i^2 + e(t) \sum_k J_{ik} \left( j \mu_k + \sqrt{\frac{j}{4}} w_k \right) + \sqrt{j} \langle \Delta \hat{X}_i^2 : \rangle w_i. \tag{1}
\]

Here a time \( t \) is normalized so that the background linear loss (amplitude decay rate) \( \gamma_s \) is one in this time unit, as indicated in the first term of the R.H.S. of Eq. \( 1 \). Another loss parameter \( j \) represents the normalized out-coupling rate for optical homodyne measurement (see Fig. 1), where \( j = R_B / \Delta t_c \), \( R_B \) is the reflectivity of the out-coupling beam splitter and \( \Delta t_c \) is the round trip time of a ring cavity. We assume \( R_B = j \Delta t_c \ll 1 \). \( p = S / \gamma_s \) is a linear gain coefficient provided by the parametric device and \( S \) is a squeezing/anti-squeezing parameter. The term \( g^2 \mu_i^2 \) expresses two photon absorption (gain saturation). A solitary DOPO without mutual coupling \( (J_{ik} = 0) \) and out-coupling loss for measurement \( (j = 0) \) has an oscillation threshold \( p_{th} = 1 \) and an average photon number at above threshold \( \langle \hat{n}_i \rangle \approx \mu_i^2 = (p - 1) / g^2 \). \( w_k \) is a zero-mean and variance-one real number Gaussian random variable, which accounts for a finite measurement uncertainty in optical homodyne detection and is mainly determined by a vacuum field fluctuation incident upon the open port of the out-coupling beam splitter (see Fig. 1). An inferred mean-field amplitude, \( \hat{\mu}_k = \mu_k + \sqrt{\frac{1}{4}} w_k \), is deviated from the true mean value \( \mu_k \) by the finite measurement uncertainty \( w_k \). \( J_{ik} \) is the Ising coupling coefficient and \( e(t) \) is a dynamically modulated feedback mean-field strength. \( \langle \Delta \hat{X}_i^2 \rangle = \langle \Delta \hat{X}_i^2 \rangle_{SQL} = 1/2 \) is the excess amplitude noise above the standard quantum limit. The second and third terms of the R.H.S. of Eq. \( 1 \) represent the (noisy) measurement feedback coupling term and the measurement-induced shift of the mean-field \( \mu_i \), respectively.

The equation of motion for the variance \( \sigma_i \) is obtained as,

\[
\frac{d}{dt} \sigma_i = \frac{d}{dt} \langle \Delta \hat{X}_i^2 \rangle = 2 \left[ -\left( 1 + j \right) + p - 3g^2 \mu_i^2 \right] \sigma_i - 2j(\sigma_i - 1/2)^2 + \left( 1 + j \right) + 2g^2 \mu_i^2. \tag{2}
\]

The first term of the R.H.S. of Eq. \( 2 \) manifests that the variance \( \sigma_i \) is attenuated by linear loss, amplified by parametric gain and attenuated by two-photon absorption loss. The second term of the R.H.S. of Eq. \( 2 \) represents the measurement-induced reduction of the DOPO pulse quantum state. Note that there is no state reduction if the internal DOPO pulse is in a coherent state \( (\sigma_i = 1/2) \), for which there is no quantum correlation between the internal DOPO pulse and the out-coupled pulse for measurement so that there is no back action imposed on the internal DOPO pulse quantum state by the measurement. This statement is also true for the measurement-induced mean-field shift as shown by the last term of the R.H.S. of Eq. \( 1 \), which disappears when \( \sigma_i = 1/2 \). The third term of the R.H.S. of Eq. \( 2 \) shows the variance increase by the incident vacuum field fluctuation via linear loss and the pump noise coupled to the DOPO pulse via gain saturation, respectively. The equation of motion for the variance \( \eta_i = \langle \Delta \hat{P}_i^2 \rangle \) is also obtained as,

\[
\frac{d}{dt} \eta_i = \frac{d}{dt} \langle \Delta \hat{P}_i^2 \rangle = 2 \left[ -\left( 1 + j \right) + p - g^2 \mu_i^2 \right] \eta_i + \left( 1 + j \right) + 2g^2 \mu_i^2. \tag{3}
\]

Note that this equation is decoupled from both \( \mu \) and \( \sigma \) so that we do not need to solve it to search the solutions of the combinatorial optimization problems. However, it is worth computing in order to understand the operation of the proposed CIM as discussed in the following sections.

When there is no external pumping \( (p = 0) \), Eqs. \( 1 \) and \( 2 \) show that there exists no mean-field \( \mu_i = 0 \) but there is a finite variance \( \sigma_i = 1/2 \). This is the vacuum field noise injected constantly from a zero
temperature reservoir. When the pump rate is far above threshold \((p \gg (1 + j))\), the mean-field is \(\mu_i = \sqrt{p/g^2}\) and the variance is approaching to \(\sigma_i = 1/2\). This is the noise associated with a coherent state produced in a highly excited DOPO.

The dynamically modulated feedback mean-field strength \(e_i(t)\) obeys the following equation,

\[
\frac{d}{dt}e_i(t) = -\beta \left[ g^2 \tilde{\mu}_i^2 - a \right] e_i(t).
\] (4)

Here \(\beta\) is a positive constant and \(a\) is the target squared amplitude. Note that the feedback mean-field strength is exponentially increased if the normalized inferred squared amplitude \(g^2 \tilde{\mu}_i^2\) is smaller than the target squared amplitude \(a\), while it is exponentially decreased if the opposite is true, i.e. \(g^2 \tilde{\mu}_i^2 > a\). \(\beta\) is considered as a rate to establish a steady state condition \(g^2 \tilde{\mu}_i^2 = a\). Finally, the pump rate \(p\) and target squared amplitude \(a\) are determined by the difference between the current Ising energy \(E(t)\) and the best Ising energy \(E_{opt}\) visited before:

\[
p(t) = \pi(t) - \rho_p \tanh \left( \frac{E(t) - E_{opt}}{\Delta} \right)
\] (5)

\[
a(t) = \alpha + \rho_a \tanh \left( \frac{E(t) - E_{opt}}{\Delta} \right)
\] (6)

Here \(\pi, \alpha, \rho, \) and \(\Delta\) are positive constants, and \(\tanh(x)\) is a hyperbolic tangent function. If \(\pi < \rho\), the pump rate \(p\) becomes negative when \(E(t) - E_{opt} \gg \Delta\) (mode “C”), while \(p\) is positive when the opposite is true (mode “A”), as described already in the previous section.

Figure 3: Dynamical behavior of CIM with error correction feedback. (a) Ising energy, (b) Histogram of evolution time \(t_s\) finding a ground state for the first time, (c) Mean-field amplitude \(\mu(t)\), (d) Mutual coupling field \(e(t)\), (e) Pump rate \(p(t)\) and target amplitude squared \(a(t)\), (f) Variance in canonical coordinate \(\langle \Delta \hat{X}^2 \rangle\) and (g) Variance in canonical momentum \(\langle \Delta \hat{P}^2 \rangle\). The feedback parameters are \(\alpha = 1.0, \pi = 0.1, \rho_a = \rho_p = 1.0, \Delta = 1/5\) and the gain saturation parameter \(g^2 = 10^{-4}\).
4 Numerical Simulation

4.1 Dynamical behavior of the machine

We solve MAX-CUT problems with randomly chosen 21-level discrete weights $J_{ik} = (-1, -0.9, \ldots, 0.9, 1)$, for which an exact solution with a lowest energy is obtained by brute force search. Figure 3 shows (a) the dynamical behavior of an inferred Ising energy $E(t)$, (b) histogram of evolution time $t_s$ when one of exact solutions is obtained for the first time for 10,000 runs, (c) mean-amplitude $\mu(t)$, (d) feedback mean-field strength $e(t)$, (e) pump rate $p(t)$ (in blue) and target squared amplitude $a(t)$ (in red), (f) canonical coordinate variance $\langle \Delta \hat{X}^2 \rangle$ and (g) canonical momentum variance $\langle \Delta \hat{P}^2 \rangle$. An evolution time is normalized by a linear loss rate $t_s = \gamma_s T$, where $T$ is a wall clock time. The results shown in Fig. 3 is a single run trajectory of the machine for a particular problem instance, except for the histogram of evolution time $t_s$ shown in the second panel (Fig. 3(b)) which is the result of 10,000 trials. A problem size (number of spins) of the instance is $N = 16$. The feedback parameters are set to $\alpha = 1.0$, $\pi = 0.2$, $\rho_a = \rho_p = 1.0$, $\Delta = 1/5$ and $\beta = 0.05$. The saturation parameter is $g^2 = 10^{-4}$. We also assume that a signal field lifetime $\tau_{ph} = 1/\gamma_s$ is 40 times of a round trip time, $\gamma_s \Delta t_c = 0.025$, which corresponds to the reflectivity of the out-coupling beam splitter $R_B = 0.025$.

As shown in Fig. 3(a), the inferred Ising energy $E(t)$ fluctuates a lot between $t = 0$ and $t = 10$, but settles down near the optimum energy $E_{opt}$ by $t = 12$, when the pitchfork bifurcation is observed and the average photon number of each DOPO reaches one ($\mu_i^2 = 1$). For more than 50% of 10,000 trials, the machine finds an exact solution near this bifurcation point (first bunch of the histogram between $t = 0$ and $t = 12$ in Fig. 3(b)). This is the situation that the machine finds exact solutions only after visiting a few local minima. For the rest of 10,000 trials ($< 50\%$), the second bunch of histogram is observed between $t = 40$ and $t = 60$, in which the machine needs to visit and escape from many local minima until it finally finds an exact solution.

![Figure 4: Variances $\langle \Delta \hat{X}^2 \rangle$ and $\langle \Delta \hat{P}^2 \rangle$ for (a) the closed-loop CIM and (b) the open-loop CIM. Numerical parameters are same as in Fig. 3](image)

As shown in Fig. 3(d), the feedback mean-field $e(t)$ initially increases exponentially and then saturates when the actual DOPO squared amplitude $g^2 \mu_i^2$ becomes equal to the target squared amplitude $a(t)$. The pump rate $p(t)$ fluctuates between $p = 1.2$ and $p = -0.8$ initially (between $t = 0$ and $t = 10$), but after $t = 12$ the pump rate is set to a negative value ($p = -0.8$) for most of time as shown in Fig. 3(e). This result indicates the instantaneous energy $E(t)$ is larger than the best energy $E_{opt}$ previously visited most of time, so that the machine is in the mode “C”. Several spins are flipped simultaneously with some intervals. At specific times $t = 42, 66, 91$ (almost periodically), $p(t)$ approaches $p = \pi = 0.2$ and $a(t)$ approaches $a = \alpha = 1$, which means the instantaneous energy $E(t)$ becomes nearly equal to $E_{opt}$ (see Eqs.(5) and (6)). The machine is close to a local minimum at those times. The flipping of many spins is also observed at those times as shown in Fig. 3(c), which indicates that the machine already
tries to escape from this local minimum. Finally Figs. 3(f) and (g) show that the canonical coordinate \( \hat{X} \) is squeezed (\( \langle \Delta \hat{X}^2 \rangle < 1/2 \)) while the canonical momentum \( \hat{P} \) is anti-squeezed (\( \langle \Delta \hat{P}^2 \rangle > 1/2 \)) at most of time. This is in sharp contrast to a standard open-loop CIM, in which the amplitude anti-squeezing, \( \langle \Delta \hat{X}^2 \rangle > 1/2 \), and the phase squeezing, \( \langle \Delta \hat{P}^2 \rangle < 1/2 \), are maintained all the time. It is noted that the quantum states of all DOPO pulses satisfy the minimum uncertainty product, \( \langle \Delta \hat{X}^2 \rangle \langle \Delta \hat{P}^2 \rangle = 1/4 \), with a very small excess factor of less than 30% or less, as shown in Fig. 4. This result suggests that DOPO quantum states remain nearly pure states during entire computation time in spite of an open-dissipative nature of the machine.

Figure 5: Performance comparison of the closed-loop CIM with error correction feedback and the open-loop CIM. The feedback parameters of the closed-loop CIM are \( \beta = 0.05, \rho_a = \rho_p = 1.0, \alpha = 1.0, \pi = 0.2, \Delta = 1/5, \) and \( e(0) = 1.0. \) The parameters for the open-loop CIMs are \( \beta = \rho_a = \rho_p = 0, \pi(t) = 0.5(1 + t/100), \) and \( e(0) = 1.0. \) The ratio of a round trip time to a signal amplitude lifetime is \( \Delta t_c/\tau_{ph} = 0.025. \) (a) Number of instances with the success probability larger than 99% vs. problem size. (b) Number of instances with the success probability lower than 5% vs. problem size. (c) Histograms of success probabilities for the closed-loop and open-loop CIMs for the problems size of \( N = 20. \) The saturation parameter is set to \( g^2 = 10^{-4}. \)

4.2 Performance comparison against open-loop CIM

To understand how the performance of a closed-loop CIM is compared to that of an open-loop CIM, we solve MAX-CUT problems with 21-level randomly chosen \( J_{ik} \) and varying problem size \( N = 4 \sim 20. \) A total of 1000 instances were generated for each problem size. Each problem instance is solved 100 times to evaluate the success probability. The maximum normalized computation time is fixed to \( t_{max} = 100. \) If the machine finds an exact solution at a certain time within \( t_{max}, \) we count it as a successful trial and evaluate the success probability by the total counts over 100 trials. Figure 5 shows the performance of the closed-loop CIM with error correction feedback together with that of the open-loop CIM. The number of instances with a success probability higher than 99% and lower than 5% are plotted as a function
of the problem size in Fig. 5(a) and (b), respectively. The feedback parameters of the closed-loop CIM are \( \beta = 0.05, \rho = 1.0, \alpha = 1.0, \pi = 0.2, \Delta = 1/5, \) and \( e(0) = 1.0. \) The ratio of a round trip time to a signal field lifetime is \( \Delta t_c / \tau_{ph} = 0.025. \) We study the open-loop CIM with the same Gaussian quantum model. We set the feedback parameters of \( \beta = 0, e(0) = 1.0, \) and \( \rho = 0 \) for the open-loop CIMs to have the feedback mean-field strength \( e_i(t) = e_i(0) = 1.0 \) constant. The pump rate \( p \) is linearly increased from \( p = 0.5 \) at \( t = 0 \) to \( p = 1.0 \) at \( t = 100 \) (above threshold). If the pump rate is abruptly switched on at \( t = 0 \) from \( p = 0 \) to \( p = 1.0, \) the success probability is much worse.\[19\]

As shown in Fig. 5(a), the performance of the closed-loop CIM is superior to the open-loop CIM. The probability of finding the instances with a success probability higher than 99% shown in Fig. 5(a), decreases much slower in the closed-loop CIM than the open-loop CIM. The probability of finding the instances with a success probability lower than 5% increases dramatically in the case of the open-loop CIM but not in the closed-loop CIM, as shown in Fig. 5(b). The histogram of the success probabilities of the closed-loop CIM and the open-loop CIM are compared in Fig. 5(c), where the problem size is \( N = 20 \) and the saturation parameter is \( g^2 = 10^{-4}. \) For the open-loop CIM, the instances are clearly separated into either hard (success probability \( \approx 0\% \)) or easy (success probability \( \approx 100\% \)) ones. However, the closed-loop CIM can solve all instances with a high success probability. There is no instance with the success probability close to zero.

### 4.3 Random sampling in the closed-loop CIM

In the previous section, we have shown the performance comparison of the closed-loop and open-loop CIMs, where the probability of finding one of the ground states in a single trial is evaluated. Here, we investigate how the proposed closed-loop CIM samples ground states and low-energy excited states for a given problem instance. In order to see the fair sampling performance of the machine, the probability of finding not only specific energy states but also degenerate states in a same energy states are studied. We chose a particular problem instance that has the largest number of degenerate ground states from randomly generated 1000 instances.

![Figure 6: Sampling property of (a) closed-loop CIM and (b) uncoupled DOPOs as a function of the excess energy \( \mathcal{E}. \) (i) The probability of finding the specific energy states by the closed-loop CIM. (ii) The theoretical Boltzmann distribution at an effective temperature of \( T_{eff} = 1.51, \) and (iii) the density of state for a given problem instance. All probabilities are normalized to unity. Inset shows the \( D_{KL} \) as a function of the \( T_{eff}. \)](image-url)
The selected instance has a problem size of \( N = 16 \) and has eight degenerate ground states, six degenerate first-excited states, and four degenerate second-excited states. The energy difference between each energy state is 0.2, which corresponds to an energy difference of a single spin flip and a minimum weight of 0.1. We solve this problem instance 10^4 times to evaluate the sampling performance. The saturation parameter is \( g^2 = 10^{-4} \) and the feedback parameters of the closed-loop CIM are \( \alpha = \rho_a = 1.0 \times g^2, \beta = 0.05/\alpha, \pi = 0.2, \rho_p = 1.0, \Delta = 1/5, \) and \( e(0) = 1.0 \). The ratio of a round trip time to a signal field lifetime is set to be \( \Delta t_c/\tau_{ph} = 0.025 \). The time interval \( \Delta t \) in the numerical integration of Eqs. is identical to the round trip time of \( \Delta t_c = 0.025 \). The maximum computation time is \( t_{\text{max}} = 100 \), which indicates that there are \( 4 \times 10^3 \) sampling events in a single trial. For comparison, we simulate independent DOPOs with the same Gaussian quantum model, but set the feedback parameters of \( \beta = 0, e(0) = 0, \) and \( \rho_p = 0 \) to cut off both the pump modulation and the feedback mean-field modulation.

Note that there still exist the measurement-induced shift of the mean-field \( \mu_i \) and reduction reduction of the variance \( \sigma_i \). The pump rate \( p = \pi(t) \) is linearly increased from \( \pi(0) = 1.5 \) (below the threshold) to \( \pi(100) = 2.5 \) (above the threshold). Note that the solitary DOPO threshold pump rate is \( 1 + j = 2 \).

Figure 6(i) shows the probabilities \( P_{\text{CIM}}(\mathcal{E}) \) of sampling a specific energy state by the closed-loop CIM (Fig. 6(a)) and by the independent DOPOs (Fig. 6(b)) vs. excess energy of \( \mathcal{E} = \mathcal{E}_{\text{Ising}} - \mathcal{E}_{\text{ground}} \) measured from the ground state energy. The probability \( P_{\text{CIM}}(\mathcal{E}) \) is evaluated by averaging out the individual probability distribution of the \( 10^4 \) trials. Each distribution for a trial is obtained by normalizing the histogram that represents how many times the machine samples a specific energy state at \( \mathcal{E}_i \). The probability \( P_{\text{CIM}}(\mathcal{E}) \) is favorably compared with to the theoretical Boltzmann sampling result as shown in Fig. 6(b). The Boltzmann distribution at an effective temperature of \( T_{\text{eff}} \) is given by

\[
P_{\text{Boltzmann}}(\mathcal{E}_i) = D(\mathcal{E}_i) \times \frac{1}{Z} \exp \left( -\frac{\mathcal{E}_i}{T_{\text{eff}}} \right)
\]

\[
Z = \sum_i D(\mathcal{E}_i) \exp \left( -\frac{\mathcal{E}_i}{T_{\text{eff}}} \right) \Delta\mathcal{E}.
\]

Here \( D(\mathcal{E}_i) \) is the density of state for the given problem instance and obtained by a brute force search (see Fig. 6(iii)), \( \mathcal{E}_i \) represents the excess energy of the i-th bin of the histograms, \( \Delta\mathcal{E} \) is the energy width of the histogram bin. The bin width of \( \Delta\mathcal{E} \) is 0.2 in Fig. 6. The effective temperature \( T_{\text{eff}} \) is estimated by minimizing Kullback-Leibler (KL) divergence \( D_{KL} \) between the simulated CIM probability distribution \( P_{\text{CIM}}(\mathcal{E}) \) and the Boltzmann distribution \( P_{\text{Boltzmann}} \) (See the inset of Fig. 6(ii)). The KL divergence \( D_{KL} \) between two probability distributions \( \{P_n\} \) and \( \{Q_n\} \) is defined by

\[
D_{KL}(P||Q) = \sum_n P_n \log \left( \frac{P_n}{Q_n} \right)
\]

Here we choose the Boltzmann distribution as \( \{P_n\} \) and the simulated distribution as \( \{Q_n\} \). The observed probability distributions \( P_{\text{CIM}}(\mathcal{E}_i) \) of the closed-loop CIM is well matched with the Bolzmann distribution \( P_{\text{Boltzmann}}(\mathcal{E}_i) \) at the fitted effective temperature of \( T_{\text{eff}} = 1.51 \). On the other hand, the independent DOPOs behaves like a bunch of spins at the high-temperature \( (T_{\text{eff}} \approx 1000) \).

Figure 7 shows how the closed-loop CIM samples the degenerate states in the lowest three energy states. The sampling performance of the closed-loop CIM is shown in Fig. 7(a) while that of \( N = 16 \) independent DOPOs is shown in Fig. 7(b). All degenerate states are found for the closed-loop CIM with a higher probability than 60% except for two complimentary ground states with the spin configurations of 13465 and 52070. Even for these hard states to sample, the probabilities are higher than 35%, indicating that 3 trials would be enough to pick up all of the ground, first excited and second excited states. On the other hand, the corresponding probability for independent DOPOs is much lower, i.e., of about 0.8-0.9%. This value is about seven times lower than a simple estimate \( \approx 6\% \) of random guessing \( (4 \times 10^3 \) random sampling against \( 2^{16} \) states). This decrease in the probability for independent DOPOs is caused by the fact that the response time of each DOPO is longer than the sampling period, leading to a lower effective sampling rate. If the pump rate \( p = \pi(t) \) is linearly increased from \( \pi(0) = 1.9 \) (a little below the
threshold) to $\pi(100) = 2.5$ (a little above the threshold), the DOPOs slowly evolve and result in an even lower probability of 0.3-0.4%.

The simulation result of the closed-loop CIM shown in Fig. 6 and 7 are obtained for a target squared amplitude of $\alpha = \rho_a = 1.0 \times g^2$. In this parameter condition, the feedback mean-field $e_i(t)$ is modulated to stabilize the DOPO mean-field amplitude $\mu_i$ around 1, where the amplitude of the quantum fluctuation is comparable to the mean-field. If we set the feedback parameters of $\alpha = \rho_a = 1.0$, the DOPO mean-field amplitude is stabilized around $\mu_i = 1/g$, which is two orders magnitude larger than the amplitude of the quantum fluctuation. The effective temperature $T_{eff}$ for the feedback parameters of $\alpha = \rho_a = 1.0$ is decreased to $T_{eff} = 0.34$ and the sampling efficiency (probability) of finding the two ground states of 13465 and 52070 and the second-excited states are decreased. These results indicate that a closed-loop CIM can realize efficient random sampling of degenerate ground states and low-energy excited states by adjusting the effective temperature $T_{eff}$ of the machine through the feedback parameter of $\alpha$.

## 5 Conclusion

We have numerically studied the performance of the closed-loop CIM with error detection and correction feedback, in which amplitude squeezed states of DOPO pulses are repeatedly monitored by optical homodyne measurement and displaced by error correction feedback signal. A Gaussian quantum model, which is derived from the measurement-feedback CIM master equation using the Wigner representation
for the field density operator, is used to simulate the dynamical behavior of mean-field amplitudes and variances. This approximate model is valid as far as a saturation parameter is small ($g^2 \ll 1$) and a signal field lifetime is much longer than a cavity round trip time ($\tau_{ph}/\Delta t_c \gg 1$).

The closed-loop CIM is expected to partially overcome the two drawbacks of the previously studied open-loop CIM: (1) exponentially increasing local minima trap a machine state as a problem size increases, and (2) mapping of a target Hamiltonian to a loss landscape fails due to DOPO amplitude heterogeneity. The above expectation is confirmed by comparing the performance of a closed-loop CIM to that of an open-loop CIM. Moreover, it is shown that the hopping behavior of a closed-loop CIM realizes efficient sampling of degenerate ground states and low-energy excited states, which is useful for various applications including the lead optimization for drug discovery.

In a future publication, we will report on the performance of a closed-loop CIM in an opposite operating regime of a low-Q cavity ($\Delta t_c/t_{ph} \gtrsim 0.1$), which requires a new theoretical tool beyond the present Gaussian quantum model based on the Wigner stochastic differential equation.

**Conflict of Interest**
The authors have no conflict of interest, financial or otherwise.

**Acknowledgements**
The authors wish to thank the useful discussions with H. Mabuchi, S. Ganguli, Z. Troczkai, P. Drummond and M. Reid.

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