CHERNOFF-LIKE COUNTEREXAMPLES RELATED TO UNBOUNDED OPERATORS

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Dedicated to the memory of Professor Paul R. Chernoff

Abstract. In this paper, we give an example of a closed unbounded operator whose square domain and adjoint’s square domain are equal and trivial. Then, we come up with an essentially self-adjoint whose square has a trivial domain.

1. Introduction

The striking example by Chernoff is well known to specialists. It states that there is a closed, unbounded, densely defined, symmetric and semi-bounded operator $A$ such that $D(A^2) = \{0\}$. This was obtained in [3] and came in to simplify a construction already obtained by Naimark in [7]. It is worth noting that Schmüdgen [8] obtained almost simultaneously (as Chernoff) that every unbounded self-adjoint $T$ has two closed and symmetric restrictions $A$ and $B$ such that

$$D(A) \cap D(B) = \{0\} \quad \text{and} \quad D(A^2) = D(B^2) = \{0\}.$$ 

This result by Schmüdgen (which was generalized later by Brasche and Neidhardt in [2]; see also [1]) is great but remains fairly theoretical. There seems to be no other simple Chernoff-like (whatever simplicity means) example around in the literature except the one by Chernoff. It is worth recalling that this type of operator cannot be self-adjoint nor can be normal. It cannot be invertible either.

So, what we will do here is to completely avoid Chernoff’s (or Naimark’s) construction and get a closed operator whose square has a trivial domain. As a bonus, its adjoint’s square domain is also trivial. The example is based on matrices of unbounded operators. So, we refer readers to [10] for properties of block operator matrices.

In our second example, we give an essentially self-adjoint bounded not-everywhere-defined operator whose square has also a trivial domain. Recall that Chernoff’s example cannot be essentially self-adjoint as it is already closed.

Finally, we assume that readers are familiar with the basic properties of bounded and unbounded operators. See [6] and [9] for further reading. We recall a few facts which may not be well known to some readers.

Recall that $C_0^\infty(\mathbb{R})$ denotes here the space of infinitely differentiable functions with compact support. The following result, whose proof relies upon the Paley–Wiener theorem, is well known.

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**Theorem 1.1.** If \( f \in C_0^{\infty}(\mathbb{R}) \) is such that \( \hat{f} \in C_0^{\infty}(\mathbb{R}) \), then \( f = 0 \).

One may wonder whether Theorem 1.1 remains valid for the so-called cosine Fourier transform? The answer is obviously no, as any non-zero odd function provides a counterexample. However, the same idea of proof of Theorem 1.1 works to establish the following (and we omit the proof).

**Theorem 1.2.** If \( f \in C_0^{\infty}(\mathbb{R}) \) is even and such that its cosine Fourier transform too is in \( C_0^{\infty}(\mathbb{R}) \), then \( f = 0 \).

### 2. Main counterexamples

**Lemma 2.1.** There are unbounded self-adjoint operators \( A \) and \( B \) such that

\[
D(A^{-1}B) = D(BA^{-1}) = \{0\}
\]

(where \( A^{-1} \) and \( B^{-1} \) are not bounded).

**Proof.** Let \( A \) and \( B \) be two unbounded self-adjoint operators such that

\[
D(A) \cap D(B) = D(A^{-1}) \cap D(B^{-1}) = \{0\},
\]

where \( A^{-1} \) and \( B^{-1} \) are not bounded. An explicit example of such a pair on \( L^2(\mathbb{R}) \) may be found in [4, Proposition 13, Section 5] (the idea is in fact due to Paul R. Chernoff). It reads: Let \( A = e^{-H} \) where \( H = \text{id}/dx \) and \( A \) is defined on its maximal domain, say. Then \( A \) is a non-singular (unbounded) positive self-adjoint operator. Now, set \( B = VAV \) where \( V \) is the multiplication operator by

\[
v(x) = \begin{cases} 
-1, & x < 0, \\
1, & x \geq 0.
\end{cases}
\]

Then \( V \) is a fundamental symmetry, that is, \( V \) is unitary and self-adjoint. Moreover, \( A \) and \( B \) obey

\[
D(A) \cap D(B) = D(A^{-1}) \cap D(B^{-1}) = \{0\}.
\]

This is not obvious and it is carried out in several steps.

For our purpose, we finally have

\[
D(A^{-1}B) = \{x \in D(B) : Bx \in D(A^{-1})\} = \{x \in D(B) : Bx = 0\} = \{0\},
\]

for \( B \) is one-to-one. Similarly, we may show that \( D(BA^{-1}) = \{0\} \).

**Theorem 2.2.** There is a densely defined unbounded and closed operator \( T \) on a Hilbert space such that

\[
D(T^2) = D(T^{*2}) = \{0\}.
\]

**Proof.** Let \( A \) and \( B \) be two unbounded self-adjoint operators such that

\[
D(A) \cap D(B) = D(A^{-1}) \cap D(B^{-1}) = \{0\},
\]

where \( A^{-1} \) and \( B^{-1} \) are not bounded. Now, define

\[
T = \begin{pmatrix} 0 & A^{-1} \\ B & 0 \end{pmatrix}
\]
on $D(T) := D(B) \oplus D(A^{-1}) \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Since $A^{-1}$ and $B$ are closed, we may show that $T$ is closed on $D(T)$. Moreover, $D(T)$ is dense in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Now,
\[
T^2 = \begin{pmatrix} 0 & A^{-1} \\ B & 0 \end{pmatrix} \begin{pmatrix} 0 & A^{-1} \\ B & 0 \end{pmatrix} = \begin{pmatrix} A^{-1}B & 0 \\ 0 & BA^{-1} \end{pmatrix}.
\]
By Lemma 2.1, we have
\[
D(T^2) = D(A^{-1}B) \oplus D(BA^{-1}) = \{0\} \oplus \{0\} = \{(0, 0)\},
\]
as needed.

Finally, we know (cf. [10] or [5]) that
\[
T^* = \begin{pmatrix} 0 & B \\ A^{-1} & 0 \end{pmatrix}
\]
because $A^{-1}$ and $B$ are both self-adjoint. As above,
\[
T^{*2} = \begin{pmatrix} BA^{-1} & 0 \\ 0 & A^{-1}B \end{pmatrix}
\]
on $D(T^{*2}) = D(BA^{-1}) \oplus D(A^{-1}B) = \{(0, 0)\}$, marking the end of the proof.

The second promised example is given next.

**Proposition 2.3.** There exists a densely defined essentially self-adjoint operator $A$ such that $D(A^2) = \{0\}$.

**Proof.** Define
\[
L^2_{\text{even}}(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : f(x) = f(-x) \text{ almost everywhere in } \mathbb{R}\}.
\]
Then $L^2_{\text{even}}(\mathbb{R})$ is closed in $L^2(\mathbb{R})$ and so it is in fact a Hilbert space with respect to the induced $L^2(\mathbb{R})$-inner product. Next define
\[
\text{even-}C_0^\infty(\mathbb{R}) := \{f \in C_0^\infty(\mathbb{R}) : f(x) = f(-x)\}.
\]

Let $\mathcal{F}$ be the usual $L^2$-Fourier transform and let $\mathcal{F}_0$ be its restriction to even-$C_0^\infty(\mathbb{R})$. Hence $\mathcal{F}_0 \subset \mathcal{F}$ and so $\mathcal{F}_0^* \subset \mathcal{F}^*$ whereby $\mathcal{F}^* = \mathcal{F}_0^*$. Therefore
\[
\mathcal{F}_0 = \mathcal{F} = \mathcal{F}.
\]

Set $A = \mathcal{F}_0 + \mathcal{F}_0^*$ (that is, $A$ is just the cosine Fourier transform) which is defined on even-$C_0^\infty(\mathbb{R})$ because $\mathcal{F}_0^*$ is defined on the whole of $L^2_{\text{even}}(\mathbb{R})$. Then $A$ is densely defined because even-$C_0^\infty(\mathbb{R})$ is dense in $L^2_{\text{even}}(\mathbb{R})$ (note also that $A$ is bounded but not everywhere defined). Moreover, $A$ is symmetric for
\[
A = \mathcal{F}_0 + \mathcal{F}_0^* \subset \overline{\mathcal{F}_0} + \mathcal{F}_0^* \subset (\mathcal{F}_0 + \mathcal{F}_0^*)^* = A^*.
\]
Moreover, $D(A^*) = L^2_{\text{even}}(\mathbb{R})$. As above, we obtain $\overline{A} = A^*$, that is, $A$ is essentially self-adjoint.
Now, by known properties of the distributive laws of not-everywhere-defined operators (see e.g. [9]), we may write

\[ A^2 = (F_0 + F_0^\ast)(F_0 + F_0^\ast) \]

\[ = F_0(F_0 + F_0^\ast) + F_0^\ast(F_0 + F_0^\ast) \]

\[ = F_0(F_0 + F_0^\ast) + F_0^\ast F_0 + F_0^2 \]  
(for \( F_0^\ast \) is defined everywhere).

But,

\[ D[F_0(F_0 + F_0^\ast)] = \{ f \in \text{even-}C_0^\infty(\mathbb{R}) : (F_0 + F_0^\ast)f \in \text{even-}C_0^\infty(\mathbb{R}) \} = \{ 0 \} \]

by Theorem 1.2. Accordingly,

\[ D(A^2) = D[F_0(F_0 + F_0^\ast)] \cap D(F_0^\ast F_0) \cap D(F_0^2) = \{ 0 \}, \]

as needed.

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