Holographic cusped Wilson loops in $q$-deformed $AdS_5 \times S^5$ spacetime

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Abstract: In this paper, a minimal surface in $q$-deformed $AdS_5 \times S^5$ with a cusp boundary is studied in detail. This minimal surface is dual to a cusped Wilson loop in dual field theory. We find that the area of the minimal surface has both logarithmic squared divergence and logarithmic divergence. The logarithmic squared divergence cannot be removed by either Legendre transformation or the usual geometric subtraction. We further make an analytic continuation to the Minkowski signature, taking the limit such that the two edges of the cusp become light-like, and extract the anomalous dimension from the coefficient of the logarithmic divergence. This anomalous dimension goes back smoothly to the results in the undeformed case when we take the limit that the deformation parameter goes to zero.

Key words: AdS/CFT, Wilson loops

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1 Introduction

Integrability [1] and localization [2] now allow us to compute some important quantities in $\mathcal{N}=4$ super Yang-Mills theory as non-trivial functions of 't Hooft coupling $\lambda$ and the rank of the gauge group $N$4). These computations lead to results in strong coupling and give a non-trivial test of the famous $AdS/CFT$ correspondence [3–5]. The cusp anomalous dimension $f(\lambda)$ is among these interesting quantities and its value at finite $\lambda$ in the planar limit can be computed using this powerful integrability method [6, 7]. This function appears as a cusp anomaly of a light-like Wilson loop [8, 9]. It also appears as the coefficient in front of $\log S$ of the anomalous dimension of large spin twist-two operators (here $S$ is the spin of the operator) [10, 11]. The fact that these two approaches give the same function $f(\lambda)$ was proved in perturbative gauge theory in Refs. [12–14].

Both approaches for the cusp anomalous dimension have dual descriptions in the gravity side of gauge/string duality. The twist-two operator is dual to folded spinning strings in $AdS_5$ found by Gubser–Klebanov–Polyakov (GKP) [15]. The anomalous dimension of the operator can be obtained from the energy of the semi-classical string. The Wilson loop is dual to an open F-string in $AdS_5$, and the contour of the Wilson loop is just the boundary of the F-string worldsheet [16, 17]. The holographical dual of cusped light-like Wilson loops was studied in detail in Ref. [18] (see also Ref. [19]) by performing a nontrivial analytic continuation of the F-string solution dual to cusped Wilson loops in Euclidean space in Ref. [20]. The cusp anomalous dimension obtained from the open F-string solution coincides with the results from the closed string solution obtained in Ref. [15]. In Ref. [18], this was taken as evidence that [15] made the correct identification for string theory dual of the twist-two operators. In Ref. [21], the scaling limits of the above closed string solution and open string solution was shown to be equivalent through an analytic continuation and an $AdS_5$ isometry rotation. This explained, on the gravity side, why these two approaches give the same results for the anomalous dimension. This can be thought of as a kind of open-closed duality in the $AdS$ background.

It is obviously of great value to search for integrable structures in $AdS/CFT$ correspondence with fewer sup-
There are already several interesting features for the clas-
sical strings, which are different from the case without
the deformation parameter goes to zero, and the energy
part in the undeformed case. We take the limit that the deformation parameter tends
to zero.

The rest of this paper is structured as follows. In
the next section, we will find the F-string solution in q-
deformed $AdS_5 \times S^5$ dual to cusped Wilson loops. The
cusp anomalous dimension will be extracted from the
cusped Wilson loops in Section 3. The final section is
devoted to discussion and conclusions.

2 F-string solution dual to cusped Wilson

2.1 q-deformed $AdS_5 \times S^5$

In Ref. [41], an integrable deformation of type IIB
superstring theory on $AdS_5 \times S^5$ was constructed. From
this, the string frame metric and B-field for this string
background was given in Ref. [42]. Later a new coordinate
system was introduced in Ref. [52] which was inspired by studies of GKP (Gubser–Klebanov–Polyakov)
strings [15] in q-deformed $AdS_5$. A related Poincare-like
coordinate system for q-deformed $AdS_5$ was introduced
in Ref. [55]. This will be our starting point. We now list
the results of metric and B-field in these Poincare-like
coordinates. The metric for the q-deformed $AdS_5$ part in
Poincare coordinates is:

$$
\text{d}s^2 = \sqrt{1 + C^2} R^2 \left[ \frac{dy^2 + dr^2}{y^2 + C^2 (y^2 + r^2)} + \frac{C^2 (y dy + r dr)^2}{y^2 (y^2 + C^2 (y^2 + r^2))} \right.
$$

$$
+ \frac{(y^2 + C^2 (y^2 + r^2))^2}{(y^2 + C^2 (y^2 + r^2))^2 + C^2 r^2 \sin^2 \zeta} (\text{d} \zeta + \cos \zeta \text{d} \phi^2) + r^2 \sin^2 \zeta \text{d} \phi^2 \left. \right] .
$$

1) Some people choose the name $q$-deformation.
The metric for the $q$-deformed $S^5$ part is:
\[
\begin{align*}
\text{ds}^2 &= \sqrt{1+C^2R^2} \left[ \cos^2 \gamma d\theta^2 + \frac{d\gamma^2}{1+C^2\cos^2 \gamma} \\
&+ \frac{(1+C^2 \cos^2 \gamma) \sin^2 \gamma}{(1+C^2 \cos^2 \gamma)^2 + C^2 \sin^2 \gamma \sin^2 \xi} (d\xi^2 + \cos^2 \xi d\phi^2) \\
&+ \frac{\sin^2 \gamma \sin^2 \xi d\phi^2}{1+C^2 \cos^2 \gamma} \right].
\end{align*}
\]
(2)

The action of the Wess-Zumino term for the deformed AdS part is
\[
\mathcal{L}_{WZ}^{(1)} = \frac{C \sqrt{1+C^2 R^2}}{4\pi \alpha'} e^{\mu \nu} r^4 \sin 2 \zeta \partial_{\mu} \phi \partial_{\nu} \zeta
\times \left[ (C^2 r^2 + (1+C^2) z^2)^2 + C^4 r^4 \sin^2 \zeta \right],
\]
(3)

and for the deformed $S^5$ part

\[
\mathcal{L}_{WZ}^{(2)} = -\frac{C \sqrt{1+C^2 R^2}}{4\pi \alpha'} e^{\mu \nu} \sin^4 \gamma \sin 2 \xi \partial_{\mu} \phi \partial_{\nu} \zeta
\times \left[ (1+C^2 \cos^2 \gamma)^2 + C^2 \sin^4 \gamma \sin^2 \xi \right].
\]
(4)

It is easy to see that $C$ plays the role of deformation parameter and when we take the limit $C \to 0$, we will go back to the undeformed case.

### 2.2 Cusped Wilson loop

#### 2.2.1 Loops without a jump in deformed $S^5$

We now begin our computation of a minimal surface with a cusped loop boundary. This minimal surface is the worldsheet of an F-string in deformed $AdS_5 \times S^5$ dual to a cusped Wilson loop in the dual field theory. First we study the case with trivial dependence on the coordinates of deformed $S^5$, that is to say that the coordinates of deformed $S^5$ take constant values on the worldsheet.

At the boundary the Wilson loop is put in two lines:

\[
r \in [0, \infty), \quad \psi = 0, \quad \zeta = \frac{\pi}{2}, \quad \phi = 0,
\]
(5)

and

\[
r \in [0, \infty), \quad \psi = \Omega, \quad \zeta = \frac{\pi}{2}, \quad \phi = 0.
\]
(6)

The string worldsheet will extend to the bulk of deformed $AdS_5$. Let us choose $r$ and $\psi$ to be the coordinates of string worldsheet and start with the following ansatz:

\[
y = y(r, \psi), \quad \zeta = \frac{\pi}{2}, \quad \phi = 0,
\]
(7)

with the boundary condition

\[
y(r, 0) = y(r, \Omega) = 0.
\]
(8)

Taking into account the invariance of the metric in Eq. (1) under the scaling transformation

\[
y \to \lambda y, \quad r \to \lambda r,
\]

we expect the solution for $y(r, \psi)$ to take the form

\[
y(r, \psi) = \frac{r}{f(\psi)}
\]
(10)

The boundary condition now gives

\[
\lim_{\psi \to 0} f(\psi) = \lim_{\psi \to \Omega} f(\psi) = \infty.
\]
(11)

One can also check that the Wess-Zumino term in the worldsheet action will not affect the equation of motion for the ansatz chosen above.$^1$

Substituting the ansatz back into the target space metric in Eq. (1), we obtain the induced metric on the worldsheet,

\[
ds_{\text{ind}}^2 = R^2 \sqrt{1+C^2} \left[ \frac{1+f_0^2}{r^2} dr^2 + \frac{f_0^2}{r^2} d\psi^2 + \frac{1}{1+C^2(1+f_0^2)} d\psi^2 \right].
\]
(12)

Then the area of the surface is

\[
A = \sqrt{1+C^2} \int dr d\psi \frac{1}{1+C^2(1+f_0^2)} f^2 + f_0^2.
\]
(13)

So the Nambu–Goto action of the string is

\[
S_{NG} = \frac{1}{2\pi \alpha'} A = \frac{1}{2\pi \alpha'} \int dr d\psi \frac{1}{1+C^2(1+f_0^2)} f^2 + f_0^2.
\]
(14)

Therefore, finding the minimum surface in the bulk in this case reduces to a one dimensional variational problem with the Lagrangian,

\[
\mathcal{L} = \int d\psi \sqrt{1+C^2(1+f_0^2)} f^2 + f_0^2.
\]
(15)

We can solve this extreme value problem by making use of the translation invariance in $\psi$ ($L$ does not depend on $\psi$ explicitly), and the corresponding conserved charge is:

\[
E = \frac{1}{\sqrt{1+C^2(1+f_0^2)} f^2 + f_0^2}.
\]
(16)

Due to the symmetry of the system, $f$ will achieve its minimal value $f_0$ at $\psi = \Omega/2$, then we have $\partial_\psi f |_{\psi = \Omega/2} = 0$. Thus we can also express $E$ in terms of $f_0$,

\[
E = \frac{f_0 \sqrt{1+f_0^2}}{\sqrt{1+C^2(1+f_0^2)}}.
\]
(17)

$^1$ Things will be different if we choose $\zeta = 0, \psi = 0$ and a worldsheet along the $y, r, \phi$ directions. In this case, though the WZ term will not contribute to the worldsheet action, it does affect the string equation of motion.
We can work out $f'$ by equating these two expressions of $E$,
\[
\left( \frac{df}{dx} \right)^2 = (f^4 + f^2)(f^2 - f_0^2) \times \frac{1 + f_0^2 + f^2 + C(1 + f_0^2 + f^2 + f_0^2)}{f_0^2(1 + f_0^2)(1 + C^2 + C^2 f^2)}.
\] (18)

From this, we get the relation between $f_0$ and $\Omega$ as
\[
\frac{\Omega}{2} = f_0 \sqrt{1 + f_0^2} \int_{f_0}^{\infty} \frac{df}{f} \sqrt{\frac{1 + C^2 + C^2 f^2}{(f^2 - f_0^2)(1 + f^2 + f_0^2 + C(1 + f_0^2)(1 + f^2))}}.
\] (19)

By making the transformation $f = \sqrt{z^2 + f_0^2}$, we get
\[
\frac{\Omega}{2} = f_0 \sqrt{1 + f_0^2} \int_{f_0}^{\infty} \frac{dz}{(z^2 + f_0^2) \sqrt{(1 + z^2 + f_0^2)(1 + f_0^2 + z^2 + C^2(1 + f_0^2)(1 + f_0^2 + z^2))}}.
\] (20)

Substituting Eq. (18) back into the Lagrangian, we obtain:
\[
\mathcal{L} = 2 \sqrt{1 + C^2 R^2} \int_{f_0}^{\infty} \frac{df}{f^2} \sqrt{(1 + C^2 + C^2 f^2)(1 + f^2)}(1 + f_0^2 + z^2 + C^2(1 + f_0^2)(1 + f_0^2 + z^2 + C^2))}
\] (21)

where we have imposed an infrared cutoff for $y$, $y > \epsilon$ or $f < r/\epsilon$. We can further make the transformation $f = \sqrt{z^2 + f_0^2}$ as above and the integral becomes
\[
\mathcal{L} = \mathcal{L}(r, \epsilon) = \frac{2 \sqrt{1 + C^2 R^2}}{C} \int_{f_0}^{\infty} \frac{dt}{r t} \sqrt{\frac{1 + a t^2}{(1 + b t^2)(1 + C^2 t^2)}}.
\] (22)

with $a, b, c$ listed below:
\[
a = 1 + f_0^2, 
\] (23)
\[
b = \frac{1 + C^2 + C^2 f_0^2}{C^2}, 
\] (24)
\[
c = \frac{1 + 2 f_0^2 + C^2(1 + 2 f_0^2 + f_0^4)}{1 + C^2 + f_0^2 C^2}/(1 + f_0^2 C^2).
\] (25)

Here we will give an approximate analysis since a special function solution requires additional constraints for the parameters $f_0$ and $C$ and will not make the result clearer. By making a change of variable $z = 1/t$ and noticing that
\[
\sqrt{\frac{r^2}{\epsilon^2} - f_0^2} \approx \frac{r}{\epsilon},
\] (26)

for small $\epsilon$, we have
\[
\mathcal{L}(r, \epsilon) \approx \frac{2 \sqrt{1 + C^2 R^2}}{C} \int_{f_0/r}^{\infty} \frac{dt}{r t} \sqrt{\frac{1 + a t^2}{(1 + b t^2)(1 + C^2 t^2)}}.
\] (27)

To extract the divergent part of the integral, we expand the integrand around $t = \epsilon/r$,
\[
\mathcal{L}(r, \epsilon) \approx \frac{2 \sqrt{1 + C^2 R^2}}{C} \log \frac{r}{\epsilon} + L_{\text{finite}}.
\] (28)

Hence, the area can be evaluated as,
\[
S_{\text{NC}} \approx \frac{\sqrt{1 + C^2 R^2}}{2 \pi \alpha C} \log \frac{L}{\epsilon} - \frac{1}{2 \pi} F(\Omega, C) \log \frac{L}{\epsilon},
\] (29)
The area becomes:

$$A = \sqrt{1+C^2R^2} \int \frac{dr}{r} \int_0^{\theta_0} d\psi \sqrt{\frac{f^4+f^2+f^4}{1+C^2+C^2f^2} + (1+f^2)\theta^2}.$$  (34)

We focus on the Lagrangian density,

$$\mathcal{L} = \sqrt{\frac{f^4+f^2+f^4}{1+C^2+C^2f^2} + (1+f^2)\theta^2}.\quad (35)$$

It is then easy to find the two conserved charges of the system, the energy and the canonical momentum conjugate to $\theta$:

$$E = \frac{1}{\sqrt{1+C^2+C^2f^2}} \times \sqrt{\frac{f^4+f^2}{f^2+f^2+f^4+(1+C^2+C^2f^2)(1+f^2)\theta^2}}.\quad (36)$$

$$J = \frac{1+f^2}{\mathcal{L}} \theta'.\quad (37)$$

For the convenience of calculation, we introduce two new conserved quantities which are the combinations of $J$ and $E$,

$$p = \frac{1}{E},\quad q = \frac{J}{E} = \frac{1+C^2+C^2f^2}{f^2} \theta'.\quad (38)$$

We find immediately $\theta' = qf^2/(1+C^2+C^2f^2)$ and by substituting it into $p'$, $f'$ is easily obtained:

$$\left(\frac{df}{df}\right)^2 = \frac{1}{f^2(1+f^2)^2} \frac{1+C^2+C^2f^2}{f^2f^2((1+f^2)p^2-q^2-C^2)-C^2-1}.\quad (40)$$

The extreme value $f_0$ is determined from the condition $\partial_f f|_{f_0} = 0$ as follows:

$$\frac{f_0^2(1+f_0^2)p^2-f_0^2q^2}{1+C^2+C^2f_0^2} = 1.\quad (41)$$

The relation between $\Omega$ and $f_0$ is

$$\frac{\Omega}{2} = \int_{f_0}^{\infty} df \sqrt{1+C^2+C^2f^2} \frac{df}{\sqrt{1+C^2(1+f^2)(1+f^2)(f^2((1+f^2)p^2-q^2-C^2)-C^2-1)}}.\quad (42)$$

As in the previous subsection, we make the transformation $f = \sqrt{f_0^2+z^2}$ and obtain

$$\Omega = 2 \int_0^{\infty} dz \sqrt{1+C^2(1+f_0^2+z^2)} \frac{d\psi}{(f_0^2+z^2) \sqrt{((1+f_0^2)p^2-q^2-C^2)-C^2-1}}.\quad (43)$$

The relation between $\Theta$ and $f_0$ is

$$\Theta = \int_0^{\theta_0} \frac{qf^2}{1+C^2(1+f^2)} d\psi = 2 \int_0^{\infty} dz \sqrt{1+C^2(1+f_0^2+z^2)} \frac{d\psi}{(1+f_0^2+p^2(1+2f_0^2)-q^2-C^2)} = 2qf^2 \int_0^{\infty} dz \frac{d\psi}{\sqrt{1+C^2(1+f_0^2+z^2)}(1+C^2(1+f_0^2+z^2))(p^2z^2+p^2(1+2f_0^2)-q^2-C^2)}.\quad (44)$$

The area becomes:

$$A = 2 \sqrt{1+C^2R^2} \int \frac{dr}{r} \int_0^{\theta_0} d\psi \mathcal{L} = \sqrt{1+C^2R^2} \int \frac{dr}{r} \int_0^{\theta_0} d\psi \frac{pf^2(1+f^2)}{1+C^2+C^2f^2}$$

$$= 2 \sqrt{1+C^2R^2} \int \frac{dr}{r} \int_0^{\theta_0} d\psi \frac{pf\sqrt{1+C^2+C^2f^2}}{\sqrt{f^2((1+f^2)p^2-q^2-C^2)-C^2-1}} \frac{1}{\sqrt{1+C^2+C^2f^2}}$$

$$= 2 \sqrt{1+C^2R^2} \int \frac{dr}{r} \int_0^{\theta_0} \sqrt{\frac{1}{\sqrt{1+C^2+C^2f^2}}} \frac{dz}{\sqrt{z^2+k_1z+k_2}} \frac{\sqrt{z^2+k_1z+k_2}}{\sqrt{z^2+k_1z+k_2}}.$$

where

$$k_1 = f_0^2+1,\quad (48)$$

$$k_2 = \frac{1+C^2+C^2f_0^2}{C^2},\quad (49)$$

$$k_3 = \frac{p^2(2f_0^2+1)-q^2-C^2}{p^2}.\quad (50)$$

We can analyze the integral approximately by using a new variable $t=1/z$,

$$\int_0^{\infty} dz \frac{1}{\sqrt{1+C^2R^2}} \frac{1}{\sqrt{1+k_3^2}} \approx \int_0^{\infty} dt \frac{1}{t} \sqrt{1+k_3t^2} \approx \log t + \text{finite terms}.\quad (51)$$
Therefore the area is
\[ A = \frac{\sqrt{1+C^2} R^2}{C} \log \frac{L}{\epsilon} - \frac{1}{2\pi} F(\Omega, \Theta, C) \log \frac{L}{\epsilon}. \] (52)
We find that the structure of the divergences is the same as the no-jump case.

2.3 Renormalization of the area

Let us first recall the story in the undeformed case. When the contour of the Wilson loops is smooth, the bare area of the F-string worldsheet diverges universally as \( L/\epsilon \) [62] where \( L \) is the length of the loop and \( \epsilon \) is the cut-off as introduced in this paper. This divergence can be removed either via a Legendre transformation [20] or by a geometric subtraction [17], and these two methods are equivalent to each other. For the case with a cusp, beside this, this divergence term growing as \( \log(L/\epsilon) \). The leading divergence can be removed by either of the two methods, and the subleading \( \log(L/\epsilon) \) term will remain. This is consistent with the perturbative computations from the field theory side [20].

2.3.1 Legendre transformation

Firstly, we consider the loop with no dependence on the deformed \( S^5 \), where the only coordinate that needs to be replaced by its conjugate momentum is the radial coordinate \( y \). From the Nambu–Goto action (29), it can be easily obtained as
\[ P_y = \frac{\sqrt{1+C^2} R^2}{2\pi \alpha' \epsilon^2} \sqrt{1+C^2+C^2 f^2} \frac{1}{f'' + f^2}. \] (53)
Near the boundary \( y = \epsilon \) or \( f = r/\epsilon \), we can evaluate \( f' \) from (18),
\[ \left( \frac{df}{dy} \right)^2 \approx r^6 1+C^2(1+f_0^2) \] (54)
which indicates \( f'' > 0 > f' \). Thus, we obtain from Eq. (53),
\[ P_y \approx \frac{\sqrt{1+C^2} R^2}{2\pi \alpha' C \epsilon}. \] (55)
So the boundary term is
\[ -2 \int_{r}^{\epsilon} dr (P_y y) |_{y=\epsilon} \approx -\frac{\sqrt{1+C^2} R^2}{\pi \alpha' C} \log \frac{L}{\epsilon}. \] (56)
Notice this cannot be used to cancel the leading \( \log^2 \) divergence found in the previous section. The computation of the Legendre transformation for the case with a jump in deformed \( S^5 \) is similar and we arrive at the same conclusion.

2.3.2 Geometric subtraction

We may consider a geometric subtraction scheme which is performed by discarding two ‘flat’ planes in the deformed \( AdS \) space with the metric
\[ ds^2 = R^2 \sqrt{1+C^2} \left[ \frac{dy^2 + dr^2}{y^2 + C^2 (y^2 + r^2)} \right. \]
\[ + \frac{C^2 (y^2 dy^2 + r^2 dr^2 + 2 y d y d r)}{y^2 (y^2 + C^2 (y^2 + r^2))} \]. (57)
So the area to be subtracted is:
\[ A_s = 2 \int dy dr \sqrt{G_{yy} G_{rr}} - G_{yy} \]
\[ = 2R^2 \sqrt{1+C^2} \int_{y_1}^{y_2} dy \int_{r_1}^{r_2} dr \frac{1}{y \sqrt{y^2 + C^2 (y^2 + r^2)}} \]
\[ \approx 2R^2 \sqrt{1+C^2} \left( \log \frac{2C}{\sqrt{1+C^2}} \log \frac{y_2}{y_1} \right. \]
\[ \left. + \log r_2 \log \frac{y_2}{y_1} - \frac{1}{2} \log^2 y_2 + \frac{1}{2} \log^2 y_1 \right) \]. (58)
where \( y_1, r_1 = \epsilon \) and \( y_2, r_2 = L \) are the IR and UV cutoffs respectively. In other words, we have
\[ A_s \approx 2R^2 \sqrt{1+C^2} \left( \log \frac{2C}{\sqrt{1+C^2}} \log \frac{L}{\epsilon} + \log L \log \frac{L}{\epsilon} \right. \]
\[ \left. - \frac{1}{2} \log^2 L + \frac{1}{2} \log^2 \epsilon \right). \] (59)

From this result, one can see that the leading \( \log^2 \) divergence cannot be canceled using this geometric subtraction. One can also see that the Legendre transformation is not equivalent to the geometric subtraction, as we indicated earlier.

3 Anomalous dimension from cusped Wilson loop

The anomalous dimension can be obtained by the vacuum expectation value of a light-like Wilson loop with a cusp [18]. We will only consider the case without a jump in deformed \( S^5 \) at the cusp. The light-like system can be reached from the solution we have found by analytically continuing \( f_0 \rightarrow -i f_0 \) and taking \( f_0 \) to a fixed value which will be given later. Thus, the cusp angle \( \Omega \) becomes \( \pi + i \gamma \), with \(^1\)

\[ \gamma = P.P. \int_{-\infty}^{\infty} dz \frac{f_0 \sqrt{1-f_0^2} \sqrt{1+C^2-C^2 f_0^2+C^2 z^2}}{(z^2-f_0^2) \sqrt{1-f_0^2+z^2} \sqrt{z^2-2 f_0^2+1+C^2(1-f_0^2)(1-f_0^2+z^2)}}. \] (60)

\(^1\) The real part of \( \Omega \), which equals \( \pi \), comes from the residual at \( z = f_0 \).
The (renormalized) area $A$ now becomes
\[ A = \sqrt{(1+C^2)(1+C^2-f_0^2 C^2)} R^2 \log \frac{L}{\epsilon} \int_{-\infty}^{+\infty} dz \times \left\{ \frac{\sqrt{1+z^2-f_0^2}}{\sqrt{1+C^2-C^2 f_0^2+C^2 z^2}} \frac{1}{z C \sqrt{1+C^2-C^2 f_0^2+z^2}} \right\}. \] (61)

As discussed in the previous section, neither Legendre transformation nor geometric subtraction can cancel the leading $\log^2$ divergence, and to extract the anomalous dimension which comes from the coefficient of the logarithmic divergence, we subtract the leading divergence by hand in the above expression. In order to make the above two integrals real when $z \to 0$, we can choose $f_0^2$ to satisfy
\[ f_0^2 \leq \frac{C^2+1-\sqrt{C^2+1}}{C^2}. \] (62)

We then make the transformation
\[ f_0^2 = \frac{C^2+1-\sqrt{C^2+1+C^2 \delta}}{C^2}, \] (63)
which gives
\[ \delta = 1 - 2 f_0^2 + C^2(1-f_0^2)^2, \] (64)
and the integral can be expressed in terms of $\delta$ as
\[ \gamma = \int_{-\infty}^{+\infty} dz \frac{\sqrt{C^2+1-\sqrt{C^2+1+C^2 \delta}}}{\sqrt{C^2 z^2-2Cz^2+1+C^2 z^2+\delta}} \frac{\sqrt{1+C^2+C^2 \delta-1}}{\sqrt{1+C^2+C^2 \delta+C^2 z^2}} \frac{\sqrt{1+C^2+C^2 \delta+C^2 z^2+\delta}}{\sqrt{1+C^2+C^2 \delta+C^2 z^2+\delta}}. \] (65)

In order for the two edges of the cusped Wilson loops to be light-like, we need to take a limit such that $\gamma \to \infty$. This limit is given by $\delta \to 0$ (which obviously corresponds to $f_0^2 = \frac{C^2+1-\sqrt{C^2+1}}{C^2}$), and one can see the largest contribution stems from the term $\sqrt{C^2+1+C^2 \delta z^2+\delta}$ around $z \approx 0$, i.e. $z \in (-\epsilon, \epsilon)$. When $\delta \ll \epsilon \ll 1$, we get
\[ \gamma \approx \frac{C}{\sqrt{C^2+1-\sqrt{C^2+1}}} \log \delta. \] (66)

The same method can be applied to compute the area, which gives
\[ A \approx \frac{R^2 (C^2+1)^{1/4} \sqrt{C^2+1-1} \log \delta \log \frac{L}{\epsilon}}{C}. \] (67)

So the cusp anomaly is
\[ \tilde{\Gamma}_{cusp} = -\frac{A}{2\pi \alpha'} |\gamma| \log \frac{L}{\epsilon} = \frac{R^2 (1+C^2-\sqrt{1+C^2})}{2\pi \alpha' C^2}. \] (68)

In the $C \to 0$ limit, we have
\[ \tilde{\Gamma}_{cusp} = -\frac{R^2}{4\pi \alpha'}. \] (69)

By using the relation $R^2 = \alpha' \sqrt{\lambda}$ in the undeformed case with $\lambda$ the 't Hooft of the $\mathcal{N}=4$ super Yang–Mills theory,
regularization [55]. For the Wilson loops with a cusp, the bare area goes like
\[ \sqrt{1+C^2} R^2 \log \frac{L}{\epsilon} - \frac{1}{2\pi} F(\Omega, \Theta, C) \log \frac{L}{\epsilon}. \]

Now, the leading divergence scales as \( \log \frac{L}{\epsilon} \) which is not made finite due to the deformation, however it is less divergent than \( L/\epsilon \). In other words, \( q \)-deformation softens the divergence although it does not soften it into a finite term. This divergence can be removed neither by the Legendre transformation nor by geometric subtraction, and these two methods are no longer equivalent to each other. In general, \( q \)-deformation will spoil the asymptotic AdS geometry of the spacetime and the dual field theory will probably be a non-local one. Therefore the usual subtraction schemes suitable for the undeformed case may not be appropriate here. The results of our calculations strongly support the above analysis.

Another feature of our solution is that the cusped anomalous dimension obtained from the solution without a jump in deformed \( S^3 \) can be smoothly connected with the result in the undeformed case when we take the limit that the deformation parameter \( C \) tends to zero\(^1\). This is quite different from the case for the spinning folded GKP-like string [52, 53]. For the GKP string, the relation \( E-S \sim f(\lambda) \log S \) for large \( S \) was destroyed by the deformation. This makes us unable to extract the anomalous dimension from the GKP string side and compare it with the results given here from the cusped Wilson loops. The equivalent of these two approaches for the undeformed case is broken down by the deformation, partly because the background has a much smaller isometry group after deformation.

For the undeformed case, the solution dual to the cusped Wilson loop with two light-like edges in Ref. [18] was found to actually have four cusps using the embedding coordinations [61]. This observation made this solution play a key role in the holographic computations of four-gluon planar amplitudes at strong coupling. It should be interesting to try to embed the deformed \( AdS_5 \) into a higher-dimensional spacetime and study the geometry of the minimal surface dual to the cusped Wilson loop from this point of view.

It should be interesting to compute the holographic entanglement entropy [63–65] from this background to investigate whether the area law [66, 67] of the entanglement entropy is lost or not, since the dual field theory is probably a non-local one. However, to perform this computation, we need to know the metric in the Einstein frame. Since the metric in the string frame is known, we need to know the dilaton field. Some progress has been made in Refs. [54, 57], but a complete solution is still unknown. It should be valuable to find a consistent solution, including dilaton and Ramond–Ramond fields, and compute the holographic entanglement entropy. We hope to work on this point in the near future.

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\(^1\) However, the bare area of the solution is not smooth in the \( C \to 0 \) limit.

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