Detecting the Beaming Effect of Gravitational Waves

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The standard model used in the detection of gravitational waves (GWs) presumes that the sources are not moving relative to the observer, but in reality they are. For light waves it is known that the relative motion changes the apparent brightness of the source, which is referred to as the “beaming effect”. Here we investigate such an effect for GW observations and calculate the “apparent amplitude” as a function of the relative velocity. We find that the observed amplitude deviates from the intrinsic one and that the difference can be larger than the calibration accuracy of LIGO even when the velocity is (0.1 – 1) % the speed of light. Moreover, the relative strength of the two GW polarizations also changes. Neglecting these effects would lead to an incorrect estimation of the distance and orbital inclination of a GW source, or induce a spurious signal that appears to be incompatible with general relativity. The magnitude of these effects is not a monotonic function of the relative velocity, which differs from the beaming effect for light and reveals a remarkable difference between gravitational and electromagnetic radiation.

Introduction.—The direct detection of gravitational waves (GWs) with the ground-based observatories LIGO and Virgo [1–5] has led in the last few years to the advent of data-based gravitational wave astronomy. A key ingredient in the detections has been the success of numerical relativity (NumRel) to reproduce the inspiral, merger, and ringdown of compact binaries (e.g. [6–8]), which in alliance with various inspiral modelling methods (e.g. [9–12]) can provide us with realistic waveform templates. However, the high cost of NumRel simulations imposes a severe restriction to the templates that can be developed [13, 14]. Therefore, the parameter space is usually limited to the mass ratio of the compact objects and their spins [15].

This restriction implicitly assumes there is no relative velocity between the center-of-mass (CoM) of the source and the observer. Even though, the assumption is plausible for many cases, the possibility of a non-vanishing relative velocity has been investigated for velocities as high as (40 – 70) % of the speed of light [16], as well as the mechanism that could lead to this situation [17].

In the case of electromagnetic waves, it is known that a relative motion induces a Doppler shift and the beaming effect [18]. For GWs, the first effect also applies and can be detected in various ways [16, 19–21]. However, the beaming effect, i.e. the dependence of the GW amplitude $h$ on the relative velocity $v$, is not as well studied. Inconsistent conclusions exist in the literatures, e.g., Ref. [19] asserts that $h$ does not depend on $v$ to linear order, but Ref. [20] suggests that the energy flux is beamed by the same factor as in the case of light. Since the estimation of the distance to the source is directly related to $h$ [22], and an erroneous retrieval of $h$ from data could be misinterpreted as a deviation from General Relativity [23], it is important to address the beaming effect for GWs. Throughout this Letter, we use $G = c = 1$.

The basic scenario.—We first revisit the textbook experiment: A pulse of light sent out and reflected by a mirror (e.g. [24]). This simple thought experiment has the advantage of laying bare the fundamentals of interferometric techniques as used in real detectors such as LIGO and LISA [25, 26]. However, the textbook formulae are derived assuming $v = 0$, which is not the case in our problem. Therefore, we tackle the problem from a different approach.

We choose a coordinate system $(t, x, y, z)$ at rest with respect to the CoM of the GW source. This allows us to adopt standard formulae to describe GW radiation [27, 28]. Furthermore, we (i) adopt the harmonic coordinate condition [29], (ii) expand the spacetime metric to linear order, and (iii) set the wave vector $k$ of the GW in the $z$-direction and the $+$-polarization in the direction of the $x, y$ coordinates. With these standard considerations, the spacetime metric far away from the GW source reduces to

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

(1)

where $\eta_{\mu\nu} := (1, -1, -1, -1)$ is the Minkowski metric and $h_{\mu\nu}$ represents the GW, with

$$h_{xx} = -h_{yy} = h_+^\omega(t, z) \quad \text{and} \quad h_{xy} = h_{yx} = h_{\times}^\omega(t, z)$$

(2)

being the only non-vanishing components.

To simplify the analysis, we focus on only one harmonic of the GW radiation, which has a frequency of $\omega$ and an
amplitude of $h$. Suppose the relative strength of the two polarizations are $\epsilon_+$ and $\epsilon_x$, we then can write

$$h^\omega_{+,-,x}(t,z) = \epsilon_{+,x} h e^{i (\omega t - kz)},$$

where $i$ is the imaginary unit and $\omega = k$ in natural units.

Now, suppose the electromagnetic pulse sets off at the time $t_s$ from the location of the emitter $p_E(t_s)$, bounces back at the time $t_e$ by the reflector located at $p_R(t_e)$, and finally returns to the emitter at $p_E(t_e)$ at the time $t_e$. We want to calculate the duration it takes, i.e., $t_e - t_s$, for the pulse to finish the round trip.

While in the textbook example the emitter and the reflector are both fixed in space, in the more general case, which is the focus of this work, they are moving relative to the GW source, i.e. $p_E$ and $p_R$ are both functions of the time. So in the following we provide a general framework of calculating $t_e - t_s$ without presumptions on the relative motion between the source and the detector (formed by the emitter and the reflector).

Since light travels along null geodesics, we use $\gamma_{\mu\nu} dx^\mu dx^\nu = 0$ to compute the light travel time. Expanding the line element to linear order of $h$, we derive

$$dt = \sqrt{dx^2 + dy^2 + dz^2} - \frac{1}{2} \frac{1}{\sqrt{dx^2 + dy^2 + dz^2}} \times [F_+(dx,dy) h^\omega_{+}(t,z) + F_x(dx,dy) h^\omega_{x}(t,z)],$$

where we have introduced the “polarization patterns” as

$$F_+(a,b) := (a^2 - b^2)$$

and $F_x(a,b) := 2ab$.

Integrating Eq. (4) using the boundaries $p_E(t_s)$, $p_R(t_e)$, and $p_E(t_e)$, we find

$$t_e - t_s = \int_{p_E(t_s)}^{p_E(t_e)} \left( \sqrt{dx^2 + dy^2 + dz^2} - \frac{1}{2} \frac{1}{\sqrt{dx^2 + dy^2 + dz^2}} h^\omega(t,z) \right) P(dx,dy) + \int_{p_R(t_e)}^{p_R(t_s)} \left( \sqrt{dx^2 + dy^2 + dz^2} - \frac{1}{2} \frac{1}{\sqrt{dx^2 + dy^2 + dz^2}} h^\omega(t,z) \right) P(dx,dy).$$

In the last equation we have adopted for compactness $h^\omega(t,z) := h e^{i (\omega t - kz)}$ and

$$P(a,b) := \epsilon_+ F_+(a,b) + \epsilon_x F_x(a,b).$$

To solve the above integral, we parameterize the four spacetime coordinates by an affine parameter along the geodesic of the light (see §1 in the supplemental material). Keeping all the terms up to the first order in $h$, we derive

$$t_e - t_s = \sqrt{(p_R(t_e) - p_E(t_s))^2} + \frac{1}{2} \left[ \frac{1}{\sqrt{(p_R(t_e) - p_E(t_s))^2}} \times \frac{h^\omega(t_e, z_E(t_e)) - h^\omega(t_s, z_E(t_s))}{\omega(t_e - t_s) - k(z_E(t_e) - z_E(t_s))} + \frac{h^\omega(t_e, z_R(t_e)) - h^\omega(t_s, z_R(t_s))}{\omega(t_e - t_s) - k(z_E(t_e) - z_R(t_s))} \right],$$

where we have taken the imaginary unit and $\omega = k$ in natural units.

One can verify the last equation by considering the textbook example in which the detector is at rest relative to the GW source. In this case we can write

$$p_E(t) = 0 \text{ and } p_R(t) = \hat{L} \hat{p},$$

where $L$ is the arm length of the detector and $\hat{p} = (\hat{x}, \hat{y}, \hat{z})$ is a unit vector pointing from the emitter to the reflector, both quantities defined in the rest frame of the source when there is no GW. To complete the calculation, we notice that (i) the travel time of the light for the outbound and inbound trip can be approximated by

$$t_r - t_s = t_1 [1 + O(h)]$$

and

$$t_e - t_r = t_2 [1 + O(h)],$$

where $t_1$ and $t_2$ denote the light travel times without GWs, (ii) the typical wavelength $\lambda$, that a detector like LIGO or LISA is sensitive to, satisfies the condition $L/\lambda \ll 1$ [25, 26], and (iii) when there is no relative motion the coordinate time and the proper time of the emitter are the same, $L$ equals the arm length in the rest frame of the detector $L_0$, and $t_1 = t_2 = L_0$. Under these circumstances, we can expand Eq. (8) to the first order of $h$ and $L/\lambda$. After some algebra and using the short hand notation $h^\omega_{+,-,x}(t) := h^\omega_{+,-,x}(t,0)$, we find that

$$\tau_e - \tau_s = 2L_0 \left( 1 - \frac{1}{2} \left[ F_+ (\hat{x}, \hat{y}) h^\omega_+ (\tau_s) + F_x (\hat{x}, \hat{y}) h^\omega_x (\tau_s) \right] \right),$$

where $\tau_s$ and $\tau_e$ are the proper times at the emitter when the pulse leaves and returns, respectively. Since the above proper time is measured by the same clock, it is coordinate independent. This equation is equivalent to that derived in text books (e.g. [24]).

The effect of relative motion.—Now we consider a relative motion between the detector and the GW source with a velocity $v = (v_x, v_y, v_z)$. We keep the assumption that the GW source is at rest relative to our coordinate system and hence $h_{\mu\nu}$ has the same components as in Eq. (2). The problem reduces to solving the light travel time between the two ends of a moving detector. In this case, the response of the detector, i.e. Eq. (11), will
change fundamentally because the spatial coordinates of the emitter and reflector are no longer fixed.

In particular, their motion do not only have a linear component, $vt$, but also a non-linear one caused by the distortion of GWs. We solve the free-fall geodesic equations for the emitter and reflector up to linear order of $\hbar$ (§2 in the supplemental material). The resulting spatial coordinates of the geodesics are

$$p_E(t) = vt + \frac{i\alpha h\bar{\omega}(t)}{\bar{\omega}(1-v_z)},$$  

(12a)

$$p_R(t) = L\hat{p} + vt + \frac{i\alpha h\bar{\omega}(t, L\hat{z})}{\bar{\omega}(1-v_z)},$$  

(12b)

where $\omega := \omega(1-v_z)$ and $\alpha := (\alpha_x, \alpha_y, \alpha_z)$ denotes

$$\alpha_x := \frac{1}{2}v_x P(v_x, v_y) - (1 - v_z)[\epsilon_x v_z + \epsilon_x v_y],$$  

(13a)

$$\alpha_y := \frac{1}{2}v_y P(v_x, v_y) - (1 - v_z)[\epsilon_x v_z - \epsilon_y v_y],$$  

(13b)

$$\alpha_z := -\frac{1}{2}(1 - v_z) P(v_x, v_y).$$  

(13c)

We note that $1 - v_z$ enters the equations because $\omega - kv_z$ is the rate at which the GW phase changes.

Eq. (12) indicates that the emitter and reflector wiggle as they advance along their geodesics. This wiggling can be understood from the fact that a four-velocity has always constant magnitude: due to the deformation of the space, i.e. the GW, the four-velocity has to rearrange to preserve its magnitude, which changes the direction of the trajectory. Moreover, Eq. (13) shows that when $v_x = v_y = 0$ the wiggling effect vanishes. This is because in our gauge the $t$- and $z$-components are not deformed by the GW.

Knowing $p_E(t)$ and $p_R(t)$, we can use them in Eq. (8) and derive the duration, in coordinate time, for the round trip of the light. The calculation is analogous to that without velocity but with two differences. (i) The length of the arm, $L$, and the light travel times, $t_1$ and $t_2$, differ from those in in the previous paragraph by a coordinate transformation. (ii) The wiggling of the emitter and reflector changes the length that the light travels. Effectively, the second point means the first-order terms in Eq. (10) contribute to the calculation of the first and third terms in Eq. (8), so we keep them. After some algebra, keeping only the linear terms in $h$ and $L/\lambda$, we find

$$t_e - t_s = 2\gamma L_0 \left(1 + \frac{1}{2} h\bar{\omega}(t_s) \left[\frac{P(v_x, v_y)}{1-v_z} - P\left(\hat{x} + \hat{y}, \hat{y} + \hat{y}\right) + \hat{z} v \cos(\theta) P\left(\hat{x}, \hat{y}\right)\right]\right)$$  

(14)

(§4 in the supplemental material), where $\gamma := (1-v^2)^{-1/2}$ is the Lorentz factor, $\hat{x}, \hat{y} := v_x, v_y/(1 - v_z)$, and $\theta$ is the angle spanned by the relative velocity and the arm of the detector, as seen in the rest frame of the source. The last equation is coordinate-dependent. To transform it into the coordinate-independent proper time of the emitter, we use

$$t_e - t_s = \gamma(t_e - t_s) + \gamma L_0 \frac{P(v_x, v_y)}{1-v_z} h\bar{\omega}(t_s)$$  

(15)

(§3 in the supplemental material). The term $\bar{\omega}$ in the last equation is the Doppler-shifted frequency, which the moving detector will perceive.

Finally, we find that the light-travel time which a clock fixed at the emitter would measure is

$$t_e - t_s = 2 L_0 \left(1 - \frac{1}{2} \left(\bar{F}_+ (\hat{p}, \hat{v}) h\bar{\omega}(t_s) + \bar{F}_x (\hat{p}, \hat{v}) h\bar{\omega}(t_s)\right)\right),$$  

(16)

where $\bar{F}_+$ and $\bar{F}_x$ are two new polarization patterns

$$\bar{F}_+, x (p, v) := F_+, x \left(\hat{x} + \hat{y}, \hat{y} + \hat{y}\right) - v \hat{z} \cos(\theta) F_+, x \left(\hat{x}, \hat{y}\right).$$  

(17)

These new patterns differ in many ways from the classic ones in Eq. (8). This difference results directly from the wiggling of the emitter and reflector relative to the GW source. Only in three special cases do these polarization patterns reduce to the classic ones. (i) When there is no relative motion. In this case the GW frequency is not shifted either so that we recover the classical light travel time in Eq. (11). (ii) There is a relative motion but only in the $z$-direction, i.e. $\hat{y}_x = \hat{y}_y = 0$. As we have discussed previously in this section, in this case the velocity four-vector is not affected by the GW so that the wiggling effect vanishes. (iii) The relative motion is in an arbitrary direction but the detector has a special orientation such that the arm is aligned with the wavefront, i.e. $\hat{z} = 0$. In this configuration the emitter and the reflector encounter the same GW phase and hence they wiggle also in phase. We note that this is the case considered in Ref. [13], where the authors claimed that to linear order there is no effect of relative motion on GW amplitude.

We emphasize that our approach allows us to find, in a unified way, two effects on GW observation due to a relative motion. First, we recover the well-known Doppler effect for GWs. Second, we find a new effect on the light travel time which affects the measurement of GW amplitude, as we show in the next section.

The beaming effect for GWs.—A detector infers the amplitude of a passing GW by, in practice, tracing the difference between the light travel times along two arms pointing in two directions $\hat{p}_1$ and $\hat{p}_2$. We now know that given the time difference $\delta \tau$ one should use our polarization patterns, Eq. (17), to solve for the intrinsic GW amplitude $h$. However, if one does not consider the relative motion and uses, instead, Eqs. (9) and (11) to infer
a GW amplitude $h'$, this amplitude will be different from the correct one.

To see the difference between $h$ and $h'$, we restrict ourselves to a problem in which the GW source is a circular binary and its orbital inclination and sky position are known [27]. In this case we can set up a coordinate system as described at the beginning of the letter and focus on retrieving the GW amplitude from the measurement of $\Delta \tau$. We consider a representative example in which the two arms have equal length, $L_0$, and are oriented in the directions $\mathbf{p}_1 = (\sqrt{0.5}, 0, 0\sqrt{0.5})$ and $\mathbf{p}_2 = (0, 0\sqrt{0.5}, 0\sqrt{0.5})$. We further assume a relative velocity in the $x$-direction with arbitrary magnitude, i.e., $\mathbf{v} = (v, 0, 0)$. We choose this configuration because it is straightforward to set up and, at the same time, different from any of the three aforementioned special cases.

In this example, the real relationship between the maximum time difference, $\Delta \tau$, and the GW amplitude is

$$h = \frac{\Delta \tau}{L_0 |\epsilon_\times(1 + v - |v|v^2/2) - \epsilon_\times v|}. \quad (18)$$

We see that (i) to obtain the correct GW amplitude one needs to consider the relative velocity and (ii) the two polarizations depend on the velocity in different ways. An observer not aware of the relative motion would derive $h' = \Delta \tau/(L_0\epsilon_\times)$ and, based on it, infer that the detector is blind to the $\times$-polarization.

Fig. 1 shows the difference between $h$ and $h'$ as a function of $v$ and different orbital inclinations of the binary. We find that the apparent GW amplitude, $h'$, depends on the relative velocity. Such an effect is well-known for light [18]. However, in the case of GWs, the effect is fundamentally different in three ways. First, the magnitude of this beaming effect does not increase monotonically with the velocity. Second, given a velocity the apparent amplitude can be amplified or reduced depending on the ratio of the two polarizations (i.e. the orbital inclination of the binary). Third, according to Eq. (18), the effect depends on the direction of the velocity even when the velocity vector is perpendicular to the line-of-sight, while in the case of light there is no such dependence.

Ignoring the beaming effect would introduce a systematic error in the amplitude $h$ measured by a laser interferometer such as LIGO or LISA. Fig. 2 shows this systematic error could be bigger than the previous calibration accuracy of LIGO, which is $1 - 5\%$ [30]. For example, when the binary has a high inclination angle of $\gtrsim 65^\circ$, a velocity of $1 - 3\%$ of the speed of light would already lead to a systematic error that exceeds the best calibration accuracy of the previous LIGO runs. Interestingly, recent stellar-dynamics models have shown that velocities as large as $10\%$ of the speed of light are possible for LIGO sources [16, 17, 31]. In the future, LIGO could further improve its calibration accuracy to $0.1 - 0.2\%$ [32, 33]. According to the same figure, this improvement would allow the detection of velocities below $1\%$ of the speed of light, for most of the inclinations.

An observer, without knowing the relative motion, not only would derive an incorrect GW amplitude but also interpret wrongly the relative contribution of the two polarizations. Fig. 3 shows the discrepancy between the intrinsic ratio, $\epsilon_\times/\epsilon_\times$, and the apparent one, $\epsilon_\times/\epsilon_\times$, which is calculated based on Eq. (11). When there is no velocity, our detector is blind to the $\times$-polarization, so the ratio $\epsilon_\times/\epsilon_\times$ is zero regardless of the intrinsic mixture of the two polarizations. However, the detector starts to perceive the $\times$-polarization when there is a relative motion and its contribution to the signal increases with the velocity. Such a dependence of the apparent polarization ratio on the relative velocity has important implications for GW observations, because observers will use this ratio to infer the orientation and inclination of the binary.

In conclusion, despite the seminal work of Isaacson [34] which shows that the propagation of GWs is, in many as-

![FIG. 1. Relative difference between the intrinsic GW amplitude $h$ and the inferred one $h'$, as a function of the velocity $v$. Different lines refer to different orbital inclinations for the binary.](image)

![FIG. 2. Systematic error, $\sigma_h := |h' - h|/h$, as a function of the relative velocity. The line styles are the same as in Fig. 1. The cyan shaded area shows the calibration accuracy of the previous LIGO observational runs. The pink one shows the improved calibration accuracy for future runs.](image)
 detectors and hence alters the apparent ratio of the two detectors and hence changes the apparent GW ampli-

pects, similar to that of light, we found that the beaming effect for GWs, however, differs fundamentally from that for light. First, it affects the light travel times in GW detectors and hence changes the apparent GW amplitude. Second, it affects the polarization patterns of the detectors and hence alters the apparent ratio of the two polarizations. Our results have important implications for future study of astrophysics and fundamental physics using GWs, because of the wide use of GW amplitude to determine the distance of the source [22] and the polarization ratio to infer the orbital inclination [35] or test alternative theories of gravitation [23].

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SUPPLEMENTAL MATERIAL

Integrate using affine parameterization.—In this section we show how the right side of Eq. (6) can be solved. Here we only explain the procedure for the first integral which we denote as $I_1$. The second integral can be solved analogously with the corresponding boundaries.

We start parameterizing the spatial coordinates $p := (x, y, z)$ by an affine parameter $\xi$, described by the two parameters $\beta := (\beta^1, \beta^2, \beta^3)$ and $\gamma := (\gamma^1, \gamma^2, \gamma^3)$:

$$p = \beta \xi + \gamma. \tag{19}$$

Assuming that the two boundaries of the integration cor-

respond to two values $\xi_a$ and $\xi_b$, we can write

$$p_E(t_s) = \beta \xi_a + \gamma$$

$$p_R(t_r) = \beta \xi_b + \gamma. \tag{20}$$

From these conditions the parameters $\beta$ and $\gamma$ can be determined as

$$\beta = \frac{p_R(t_r) - p_E(t_s)}{\xi_b - \xi_a}$$

$$\gamma = p_R(t_r) - \beta \xi_b. \tag{21}$$

Applying the parametrization in Eq. (19), the integral $I_1$ takes the form

$$I_1 = \int_{\xi_a}^{\xi_b} \left( \sqrt{\beta^2 - \frac{1}{2} \frac{P(\beta^1, \beta^2)}{\sqrt{\beta^2}} h(\theta, \beta^3 \xi + \gamma^3) \right) d\xi \tag{22}$$

In the above equation the time, $t$, is also a function of $\xi$. Up to linear order in $h$ the relation between $t$ and $\xi$ is linear, i.e.,

$$t = \beta^0 \xi + \gamma^0. \tag{23}$$

Considering that $I_1$ describes the out bounding light beam, we can see that the boundaries of the time related to $\xi_a$ and $\xi_b$ are $t_s$ and $t_r$, respectively. Therefore, we find for the parameters $\beta^0$ and $\gamma^0$:

$$\beta^0 = \frac{t_r - t_s}{\xi_b - \xi_a}, \quad \gamma^0 = t_r - \beta^0 \xi_b. \tag{24}$$

Using the parametrization of the time in Eq. (23) for the integral in Eq. (22) we find

$$I_1 = \int_{\xi_a}^{\xi_b} \left( \sqrt{\beta^2 - \frac{1}{2} \frac{P(\beta^1, \beta^2)}{\sqrt{\beta^2}}} \right. \times h(\beta^0 \xi + \gamma^0, \beta^3 \xi + \gamma^3) \right) d\xi. \tag{25}$$

Together with Eqs. (21) and (24), we find

$$I_1 = \sqrt{\beta^0 - \frac{P(\beta^1, \beta^2)}{\sqrt{\beta^2}}} \times \int \frac{h(\theta, \beta^3 \xi_a + \gamma^3)}{\omega \beta^3 - k^3} d\xi. \tag{26}$$

Finally, to eliminate the betas, we multiply the second term with $(\xi_b - \xi_a)^2 / (\xi_b - \xi_a)^2 = 1$ and use again Eqs. (21) and (24) to find the first part of Eq. (8).

The Geodesic of a moving particle.—In this section we solve the geodesic of a test particle under the influence of GWs. We say the particle is initially at the point $p_0 = (x_0, y_0, z_0)$ and moves with a velocity $v = (v_x, v_y, v_z)$, whereas the GW source is far away and at rest. We apply the coordinate system and the metric introduced in the first paragraph of the letter. In this harmonic coordinate system, the geodesic equation is Lorentz covariant [36–38].
The geodesic equation for a point particle in a force free space is \[ \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma - \frac{1}{c^2} \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma \dot{x}^\alpha = 0, \] (27)
where we define \( x^0 := t \), to write the geodesic equation in terms of the coordinate time, the dot denotes the coordinate time derivative, and \( \Gamma^\alpha_{\beta\gamma} \) are the Christoffel symbols. Moreover, the geodesic of a massive particle has to fulfill the constraint
\[ g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \left( \frac{dt}{d\tau} \right)^2 = 1, \] (28)
where \( \tau \) is the proper time along the geodesic (see next section). For the metric in Eq. (1), the Christoffel symbols have the form
\[ \Gamma^\alpha_{\beta\gamma} = -\frac{1}{2} h^{\alpha\nu} \left( \epsilon^\beta_{\delta} (\delta^\gamma_{\beta} - \delta^\gamma_{\delta}) + \epsilon^\gamma_{\delta} (\delta^\beta_{\delta} - \delta^\beta_{\gamma}) - \epsilon_{\beta\gamma} (\eta^{\alpha\nu} - \eta^{\alpha0}) \right), \] (29)
where \( \delta^\gamma_{\beta} \) is the Kronecker-Delta and \( \epsilon_{\mu\nu} := (h_{\mu\nu}|_{z=0})/h \).

We assume the velocity of the spatial coordinates \( p \) can be separated into the initial velocity, \( v \), and a function \( f := (f_x, f_y, f_z) \) of order 1 describing the effect of the GW:
\[ \dot{p} = v + f. \] (30)

Accordingly, \( z(t) = z_0 + v_z t + g \), where \( g \) is a function fulfilling \( \ddot{g} = f_z \). Up to linear order in \( h \) the geodesic equation is
\[ \dot{p} + i\omega \alpha h^\circ(t, z_0) = 0, \] (31)
where \( \alpha \) is introduced in Eqs. (13). The general solution to the differential equation is
\[ p(t) = p_0 + vt + \frac{i\omega x h^\circ(t, z_0)}{\bar{\omega}(1 - v_z)}, \] (32)
which also fulfills the constraint in Eq. (28).

Considering the special case where the initial velocity vanishes, \( \alpha \) is also zero and the geodesic reduces to
\[ p(t) = p_0, \] (33)
consistent with the classic notion that a particle at rest stays at rest in the presence of a GW.

Proper Time along the Geodesic.—In this section we calculate the proper time along the geodesic derived in the previous section. To linear order it is related to the coordinate time as
\[ d\tau = dt \sqrt{1 - v^2 + \left( \frac{2v \cdot \alpha}{1 - v_z} + P(v_x, v_y) \right) h^\circ(t, z_0)}, \] (34)
Expanding the term inside the square root to the linear order of \( h \) and noticing that
\[ v \cdot \alpha = -\frac{1}{2} (1 - v_z) P(v_x, v_y) - \frac{P(v_x, v_y)}{2\gamma^2}, \] (35)
we find
\[ d\tau = \left( \frac{1}{\gamma} - \frac{1}{2} \frac{P(v_x, v_y)}{\gamma(1 - v_z)} \right) dt. \] (36)

Integrating the left side from \( \tau_a \) to \( \tau_b \) and the right side from \( t_a \) to \( t_b \) we find
\[ \tau_b - \tau_a = \frac{1}{\gamma} (t_b - t_a) + \frac{1}{2} \frac{P(v_x, v_y)}{\gamma(1 - v_z)} \times i \left[ h^\circ(t_b, z_0) - h^\circ(t_a, z_0) \right]. \] (37)
For zero velocity, \( \tau_b - \tau_a \) equals \( t_b - t_a \), so that the proper time is the same as the coordinate time. For a non-vanishing velocity and neglecting the terms linear in \( h \), \( \tau_b - \tau_a \) is approximately \( (t_b - t_a)/\gamma \). Rearranging Eq. (37) we find
\[ t_b - t_a = \frac{1}{\gamma} (t_a - t_b) + \frac{1}{2} \frac{P(v_x, v_y)}{\gamma(1 - v_z)} \times i \left[ h^\circ(t_b, z_0) - h^\circ(t_a, z_0) \right], \] (38)
where we have used \( t = \gamma\tau \) to replace \( t \) in the \( h \)-functions. Traveling time of the light.—In this section we derive Eq. (14) by solving Eq. (8) to linear order. To calculate \( t_s, t_c, \) and \( t_e \) in Eq. (8), we use the relationship described in Eq. (10), where the light travel time without GWs is
\[ t_{1,2} = \gamma^2 L \left( \sqrt{1 - v^2 \sin^2(\theta)} \pm v \cos(\theta) \right). \] (39)
The length of the arm in the coordinate frame is
\[ L = \frac{L_0}{\gamma \sqrt{1 - v^2 \sin^2(\theta)}}. \] (40)

We consider contributions up to linear order in \( h \). Therefore, from Eq. (10), except for the \( 0^\text{th} \)-order coefficients, only the linear coefficients, say \( C_1 \) and \( C_2 \), respectively, enter our calculations. We expect the difference between them to be of the order of \( L/\lambda \). For realistic detectors and velocities (we require \( v \leq 0.99 \)), this difference is much smaller than one. Therefore, we can write \( C_1 = C_2 \).

We now consider the terms in Eq. (8) separately. First, for the roots in the first and third term we simplify them by performing four steps: (i) replace the times using the approximations specified above, (ii) expand the roots using \( h \ll 1 \), (iii) expand the \( h \)-functions around \( t_s \) up to linear order of \( L/\lambda \)
\[ h^\circ(t_s + t_1, L \hat{z}) \approx h^\circ(t_s) + i(\bar{\omega} t_1 - kL \hat{z}) h^\circ(t_s), \] (41a)
\[ h^\circ(t_s + t_1 + t_2) \approx h^\circ(t_s) + i\bar{\omega}(t_1 + t_2) h^\circ(t_s), \] (41b)
Because of the form of which in turn fulfills

\[
\sqrt{(p^R(t_r) - p^E(t_r))^2} + \sqrt{(p^E(t_e) - p^R(t_r))^2} = 2\gamma L_0
\times \left(1 + C_0 h v^2 - \frac{1}{\omega (1 - v_z)} \left( \omega v \cdot \alpha - \frac{\omega z}{\gamma^2} \hat{p} \cdot \alpha \right) \bar{h}\bar{\omega}(t_s) \right). \tag{42}
\]

Next, we consider the second and fourth terms in Eq. (8). For the first parts of these two terms we approximate the times as before and then use the following properties of $P$ to simplify the result:

\[
P(a + b, c + d) = P(a, c) + 2P(a, c, b, d) + P(b, d), \tag{43a}
\]

\[
P(\lambda a, \lambda b) = \lambda^2 P(a, b), \tag{43b}
\]

where

\[
P(a, b, c, d) := \epsilon_+(ac - bd) + \epsilon_-(ad + bc), \tag{44}
\]

which in turn fulfills

\[
P(\lambda a, \lambda b, c, d) = P(a, b, \lambda c, \lambda d) = \lambda P(a, b, c, d). \tag{45}
\]

Because of the $h$-functions in the later terms we only need to keep the $0^{th}$-order terms here. Therefore, we derive

\[
\frac{P(x_r(t_r) - x_c(t_s), y_c(t_r) - y_c(t_s))}{\sqrt{(p^R(t_r) - p^E(t_s))^2}} = \frac{L^2}{t_1} P(\hat{x}, \hat{y}) + 2LP(\hat{x}, \hat{y}, v_x, v_y) + t_1 P(v_x, v_y), \tag{46a}
\]

\[
\frac{P(x_c(t_r) - x_r(t_s), y_r(t_r) - y_r(t_s))}{\sqrt{(p^E(t_r) - p^R(t_s))^2}} = \frac{L^2}{t_2} P(\hat{x}, \hat{y}) - 2LP(\hat{x}, \hat{y}, v_x, v_y) + t_2P(v_x, v_y). \tag{46b}
\]

For the second parts of the second and forth term, we expand the $h$-functions as in Eqs. (44) and find that

\[
\frac{1}{\omega (t_r - t_s) - k(z_r(t_r) - z_e(t_s))} = \bar{h}\bar{\omega}(t_s), \tag{47a}
\]

\[
\frac{1}{\omega (t_e - t_r) - k(z_e(t_e) - z_r(t_r))} = \bar{h}\bar{\omega}(t_s). \tag{47b}
\]

Replacing the terms in Eq. (8) using eqs. (42), (46) and (47), further simplifying Eq. (42) using Eq. (35) and

\[
\hat{p} \cdot \alpha = \frac{v \cos(\theta) - \hat{z}}{2} P(v_x, v_y) - (1 - v_z) P(\hat{x}, \hat{y}, v_x, v_y), \tag{48}
\]

and contracting the result using Eqs. (43), we finally find

\[
t_e - t_s = 2\gamma L_0 \left(1 + C_1 h v^2 + \frac{\bar{h}\bar{\omega}(t_s)}{2\gamma^2} \frac{P(v_x, v_y)}{1 - v_z} - P(\hat{x} + \hat{r}, \hat{y} + \hat{v} \hat{z}) + \hat{z} v \cos(\theta) P(\hat{x}^2, \hat{y}^2) \right). \tag{49}
\]

Next, to get rid of $C_1$, we use Eq. (10) to replace the left-hand-side of the last equation, keeping the assumption that $C_1 = C_2$. Then we can solve for $C_1$ and finally find Eq. (14).

As a last step we transform the coordinate time into the proper time of the emitter. To do that, we substitute the left side of Eq. (14) using Eq. (15), which is derived from Eq. (38) noticing that (i) $z_0 = 0$, (ii) $t_a$ and $t_b$ are replaced by $t_s$ and $t_e$, respectively, and accordingly for the proper times, (iii) $h$-functions can be expanded as in Eqs. (41), and (iv) $t_s = \gamma t_s$. Finally, using $t_s = \gamma t_s$ again, we find Eq. (16).
