Bounding the number of rational places using Weierstrass semigroups

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Abstract: Let $\Lambda$ be a numerical semigroup. Assume there exists an algebraic function field over $\mathbb{F}_q$ in one variable which possesses a rational place that has $\Lambda$ as its Weierstrass semigroup. We ask the question as to how many rational places such a function field can possibly have and we derive an upper bound in terms of the generators of $\Lambda$ and $q$. Our bound is an improvement to Lewittes’ bound in [6] which takes into account only the multiplicity of $\Lambda$ and $q$. From the new bound we derive significant improvements to Serre’s upper bound in the cases $q = 2, 3$ and $4$. We finally show that Lewittes’ bound has important implications to the theory of towers of function fields.

Keywords: Algebraic Function Field; Rational Place; Semigroup; Tower of Function Fields, Weierstrass Semigroup

MSC-class: 11G20; 14G15 (primary); 14H05; 14H25

1 Introduction

Throughout this paper by a function field we will always mean an algebraic function field of one variable unless otherwise stated. Given a function field $\mathbb{F}/\mathbb{F}_q$ we denote by $N(\mathbb{F})$ the number of rational places and we denote by $g(\mathbb{F})$ the genus. We will always assume that $\mathbb{F}_q$ is the full constant field of $\mathbb{F}$. For applications in coding theory it is desirable to have $N(\mathbb{F})/g(\mathbb{F})$
as high as possible as this allows for the construction of codes with good parameters. The above observation has led to an extensive research on the problem of deciding given a fixed constant field $\mathbb{F}_q$ and a fixed number $g$ what is the highest number $N_q(g)$ such that a function field $\mathbb{F} / \mathbb{F}_q$ exists with $N(\mathbb{F}) = N_q(g)$ and $g(\mathbb{F}) = g$.

Recall that, for any rational place the number of gaps in the corresponding Weierstrass semigroup $\Lambda$ equals the genus $g$ of the corresponding function field. That is, $\#(\mathbb{N}_0 \setminus \Lambda) = g$. This suggests that in some cases a Weierstrass semigroup $\Lambda$ for a rational place might hold more information about the number of rational places of the function field than does the genus alone. This theme was firstly explored by Lewittes in [6], though the bound by Stöhr and Voloch ([13, pp. 14-15]) induces a bound in terms of a Weierstrass semigroup under certain conditions. The smallest non zero element in a numerical semigroup $\Lambda$ is called the multiplicity of $\Lambda$ and we denote it by $\lambda_1$. Using function field theory Lewittes showed that if $\lambda_1$ is the multiplicity of a Weierstrass semigroup corresponding to a rational place of $\mathbb{F} / \mathbb{F}_q$ then $N(\mathbb{F}) \leq q\lambda_1 + 1$ holds. In the present paper we derive an improved upper bound on $N(\mathbb{F})$ as we take into account not only the multiplicity but also all the other elements in a generating set of $\Lambda$. Let

$$\Lambda = \langle \lambda_1, \ldots, \lambda_m \rangle = \{ a_1\lambda_1 + \cdots a_m\lambda_m \mid a_1, \ldots, a_m \in \mathbb{N}_0 \}$$

$0 < \lambda_1 < \cdots < \lambda_m$ be the Weierstrass semigroup of a rational place in a function field $\mathbb{F} / \mathbb{F}_q$. Our new finding is that $N(\mathbb{F}) \leq \#(\Lambda \setminus \cup_{i=1}^{m} (q\lambda_i + \Lambda)) + 1$ holds. Here, $\gamma + \Lambda$ means $\{ \gamma + \lambda \mid \lambda \in \Lambda \}$. Lewittes’ bound can be viewed as a corollary to the new bound as $\#(\Lambda \setminus (q\lambda_1 + \Lambda)) + 1 = q\lambda_1 + 1$ holds. The new bound is often better than Lewittes’ bound. For $q$ being 2, 3 and 4 we get as a corollary to the new bound a significant improvement to the Serre bound. As will be demonstrated in this paper Lewittes’ bound has implications to the theory of towers of function fields. We show that Lewittes’ bound holds information that describes rather accurately certain aspects of the second tower of Garcia and Stichtenoth. Finally, we show that one cannot hope to construct asymptotically good towers of function fields having telescopic Weierstrass semigroups.

The paper is organized as follows. In Section 2 we prove the bounds and state some corollaries. The rest of the paper is devoted to investigating how good are the bounds and how much does the new bound improve upon Lewittes’ bound. In Section 3 we first deal with a selection of semigroups of high genus and we then apply for $q$ being equal to 2, 3 and 4 the bounds to all semigroups of genus 8. In Section 4 we estimate $N_q(g)$ for $q$ being equal
to 2, 3 and 4. Finally, in Section 5 we are concerned with asymptotically good towers of function fields. Section 6 is the conclusion.

2 The bounds

Throughout this paper let \( \Lambda \) be a numerical semigroup with finitely many gaps (meaning that \( \mathbb{N}_0 \setminus \Lambda \) is finite) and let \( \{\lambda_1, \ldots, \lambda_m\} \) be a generating set for \( \Lambda \) with \( 0 < \lambda_1 < \cdots < \lambda_m \). The reader may think of \( \lambda_1, \ldots, \lambda_m \) as being a minimal generating set but being so or not is actually of no implication.

**Definition 1** Let \( \Lambda \) be a numerical semigroup with finitely many gaps. If there does not exist a function field over \( \mathbb{F}_q \) having a rational place which Weierstrass semigroup equals \( \Lambda \) then we write \( N_q(\Lambda) = 0 \). If such function fields exist we define

\[
N_q(\Lambda) = \max \{ N(F) | F \text{ is a function field over } \mathbb{F}_q \text{ having a rational place which Weierstrass semigroup equals } \Lambda \}.
\]

Our main result is:

**Theorem 1** Let \( \Lambda = \langle \lambda_1, \ldots, \lambda_m \rangle \) be a numerical semigroup with finitely many gaps, \( 0 < \lambda_1 < \cdots < \lambda_m \). We have

\[
N_q(\Lambda) \leq \#(\Lambda \setminus \bigcup_{i=1}^{m} (q\lambda_i + \Lambda)) + 1 \quad (1)
\]

which implies

\[
N_q(\Lambda) \leq \#(\Lambda \setminus (q\lambda_1 + \Lambda)) = q\lambda_1 + 1. \quad (2)
\]

Here, \( \gamma + \lambda \) means \( \{ \gamma + \lambda \mid \lambda \in \Lambda \} \).

**Proof:** Let \( F/F_q \) be a function field. Let its rational places be \( \mathcal{P}_1, \ldots, \mathcal{P}_{N-1}, \mathcal{P} \) and assume that the Weierstrass semigroup corresponding to \( \mathcal{P} \) is \( \Lambda \). Define \( R = \bigcup_{s=0}^{\infty} \mathcal{L}(s\mathcal{P}) \) and let \( R_{-1} = \{0\} \) and \( R_t = \bigcup_{s=0}^{t} \mathcal{L}(s\mathcal{P}) \) for \( t \in \mathbb{N}_0 \). It is well known that

\[
R_t = R_{t-1} \quad \text{if} \quad t \in \mathbb{N}_0 \setminus \Lambda
\]

\[
dim(R_t) = \dim(R_{t-1}) + 1 \quad \text{if} \quad t \in \Lambda. \quad (3)
\]

Here \( \dim \) denotes the dimension as a vector space over \( \mathbb{F}_q \). Let \( \varphi : R \to \mathbb{F}_q^{N-1} \) be the map \( \varphi(f) = (f(P_1), \ldots, f(P_{N-1})) \) and define \( E_t = \varphi(R_t) \) for \( t \in \mathbb{N}_0 \cup \{-1\} \). From \( \Box \) we observe that \( \dim(E_{-1}) = 0 \) and that
\[ \dim(E_t) = \dim(E_{t-1}) \text{ for all } t \in \mathbb{N}_0 \setminus \Lambda. \] For \( t \in \Lambda \) we can either have \( \dim(E_t) = \dim(E_{t-1}) \) or \( \dim(E_t) = \dim(E_{t-1}) + 1 \). The map \( \varphi \) is surjective meaning that for \( t \) large enough \( \dim(E_t) = N - 1 \). Hence, if we can give an upper bound on the number of \( t \in \Lambda \) for which \( \dim(E_t) = \dim(E_{t-1}) + 1 \) holds then this upper bound will also be an upper bound on the number \( N - 1 \). To prove \( \text{(1)} \) we therefore only need to show that \( \dim(E_t) = \dim(E_{t-1}) + 1 \) cannot happen when \( t \in q\lambda + \Lambda \) for some \( i \). For this purpose let for \( i = 1, \ldots, m, \ x_i \in \mathbb{R} \) be an element with \( -v_P(x_i) = \lambda_i \). Here \( v_P \) is the valuation corresponding to \( P \). Given \( t = q\lambda + \lambda \) with \( \lambda \in \Lambda \) choose \( f \in R\lambda \setminus R_{\lambda-1} \). We have \( x_i^t f \in R_t \setminus R_{t-1} \) and \( x_i f \in R_{t-1} \). Clearly, \( \varphi(x_i^t f) = \varphi(x_i f) \) and the proof of \( \text{(1)} \) is complete. Finally to see that \( \text{(2)} \) follows from \( \text{(1)} \) we note that \( \#(\Lambda \setminus (\cup_{i=1}^m q\lambda_i + \Lambda)) \leq \#(\Lambda \setminus q\lambda_1 + \Lambda) \) and we use the following lemma.

\[ \square \]

**Lemma 1** Let \( \Lambda \subseteq \mathbb{N}_0 \) be a semigroup with finitely many gaps. For \( \lambda \in \Lambda \) we have \( \#(\Lambda \setminus (\lambda + \Lambda)) = \lambda \).

**Proof:** See [5, Lem. 5.15]. \[ \square \]

**Example 1** In this example we apply Theorem 1 to the semigroup

\[ \Lambda = \langle \lambda_1 = 3, \lambda_2 = 5 \rangle \]

in the case \( q = 2 \). We have

\[ \Lambda \setminus (q\lambda_1 + \Lambda) = \{0, 3, 5, 8, 10, 13\}, \]

whereas

\[ \Lambda \setminus ((q\lambda_1 + \Lambda) \cup (q\lambda_2 + \Lambda)) = \{0, 3, 5, 8\}. \]

Lewittes’ bound [2] states \( N_2(\Lambda) \leq 7 \), whereas the new bound [1] gives \( N_2(\Lambda) \leq 5 \).

The following Proposition gives us some information on how good or bad the bound in \( \text{(1)} \) can possibly be.

**Proposition 1** We have

\[ q\lambda_1 + 1 - g \leq \#(\Lambda \setminus (\cup_{i=1}^m (q\lambda_i + \Lambda))) + 1 \leq \min\{q\lambda_1 + 1, q^m + 1\}. \]
Proof: To see the first inequality observe that there are at least 
$q\lambda_1 - g$
elements in $\Lambda$ that are smaller than $q\lambda_1$. Regarding the last inequality the 
upper bound $\lambda_1 q + 1$ comes from Theorem 1. To see the upper bound 
$q^m + 1$ we note that all $\lambda \in \Lambda$ can be written as $a_1\lambda_1 + \cdots + a_m\lambda_m$ for some 
$a_1, \ldots, a_m \in \mathbb{N}_0$. If $\lambda \in \Lambda \setminus (\cup_{i=1}^m (q\lambda_i + \Lambda))$ then necessarily $a_1, \ldots, a_m < q$
must hold.

Recall, that by $N_q(g)$ we denote the maximal number of rational places that 
a function field can possibly have if its full constant field is $\mathbb{F}_q$ and its genus 
is $g$. Serre’s bound

$$|N_q(g) - (g + 1)| \leq g\lfloor 2\sqrt{q} \rceil$$

implies that if $\Lambda$ is of genus $g$ then

$$N_q(\Lambda) \leq g\lfloor 2\sqrt{q} \rceil + q + 1$$

holds. Writing $r = \frac{\lambda_1 - 1}{g}$ we see that for

$$r \leq \frac{2\sqrt{q}}{q}$$

the right side of (2) (and therefore also the right side of (1)) is always better 
than (4). On the contrary Proposition 1 tells us that the new bound (1) can 
ot produce a number smaller than $q\lambda_1 + 1 - g$. We conclude that for

$$r > \frac{2\sqrt{q} + 1}{q}$$

the bound (1) can not compete with (5). Observe however, that as $r$ is at 
most 1 the condition in (5) is never satisfied for $q = 2, 3, 4, 5$.

We now present some corollaries to Theorem 1. The first corollary is a trivial 
restatement of Lewittes’ bound (2).

Corollary 1 If $\mathbb{F}/\mathbb{F}_q$ possesses $N$ rational places then for all of the corre-
sponding Weierstrass semigroups we have

$$\lambda_1 \geq (N - 1)/q.$$  (6)

Example 2 The norm-trace function field defined by 
$x^{q^r - 1/(q - 1)} - x^{q^r - 1} - y^{q^r - 2} - \cdots - y$ has $N = q^{2r - 1} + 1$ rational places (see [3]). All but one 
correspond to affine points of the curve. The last one is denoted by $P_\infty$. It is well-known that the Weierstrass semigroup of $P_\infty$ equals 
$(q^r - 1, (q^r - 1)/(q - 1))$. That is, $\lambda_1 = q^r - 1$. But also $(N - 1)/q^r$ equals $q^r - 1$ and 
the norm-trace function field therefore is an example where the bound in 
Corollary 2 is reached.
Corollary 2 Define
\[ t = \#\{\lambda \in \Lambda \mid \lambda \in [\lambda_1 + 1, \lambda_1 + [\lambda_1/q] - 1]\}. \]
We have
\[ N_q(\Lambda) \leq q\lambda_1 - t + 1. \] (7)

Proof: For \( \lambda \in \Lambda \) with \( \lambda \in [\lambda_1 + 1, \lambda_1 + [\lambda_1/q] - 1] \) we have \( q\lambda \neq q\lambda_1 + \eta \) for any \( \eta \in \Lambda \) as there are no non zero \( \eta \in \Lambda \) with \( \eta < \lambda_1 \). Therefore the number on the right side of (1) is at least \( t \) smaller than the number on the right side of (2). \( \square \)

Example 3 Consider the case \( \lambda_1 = g + 1 \). That is, the case \( \Lambda = \{0, g + 1, g + 2, \ldots\} \). The number \( t \) from Corollary 2 becomes equal to \( \lceil (g+1)/q \rceil - 1 \).
Hence
\[ N_q(\Lambda) \leq q(g + 1) + 2 - \lceil (g + 1)/q \rceil \] (8)
holds. Given \( \lambda > \lambda_1 \) we have \( q\lambda \notin q\lambda_1 + \Lambda \) if and only if \( \lambda \in [\lambda_1 + 1, \lambda_1 + [\lambda_1/q] - 1] \) and \( q\lambda + \eta \in q\lambda_1 + \Lambda \) holds for all \( \eta \in \Lambda \{0\} \). Hence, for the particular semigroup in the present example, we have
\[ \#(\Lambda \setminus (\cup_{i=1}^m (\lambda_i + \Lambda))) = q\lambda_1 - t + 1 = q(g + 1) + 2 - \lceil (g + 1)/q \rceil. \]
The following remark gives some criteria under which the bounds (1) and (2) are the same.

Remark 1 The conductor of a semigroup \( \Lambda \subseteq \mathbb{N}_0 \) with finitely many gaps is the smallest number \( c \) such that there are no gaps greater or equal to \( c \). The conductor is known to be smaller or equal to \( 2g \) (Prop. 5.7). If \( q\lambda_1 + c \leq q\lambda_2 \) then it is clear that the number on the right side of (1) is the same as the number on the right side of (2). In particular the numbers are the same if \( q\lambda_1 + 2g \leq q\lambda_2 \)

We conclude this section by mentioning that by using the theory of algebraic geometric codes one can sharpen (1) to
\[ N_q(\Lambda) \leq \# (\Lambda \cap \{\lambda \mid \lambda \leq N' + 2 - 2\}) \setminus (\cup_{i=1}^m (q\lambda_i + \Lambda)) + 1. \] (9)
Here, \( N' \) is some a priori known upper bound on \( N_q(\Lambda) \), e.g. \( N' = N_q(g) \). We should mention that in our experiments we have not been able to find any example where this improvement gives a number that is simultaneously smaller than both \( N' \) and the right side of (1).

In the following sections we investigate how good is the new bound (1) and in particular how good is it compared to Lewittes’ bound. We start by investigating a selection of concrete semigroups.
3 Examples

In this section we apply the bounds (1) and (2) to a number of concrete semigroups.

Example 4 In Table 1 we consider a collection of 7 semigroups. We apply the bounds to a number of fields of characteristics 2 and 3. Restricting to characteristics 2 and 3 allows us to get information on the number $N_q(g)$ from van der Geer and van der Vlugt’s table in [2]. An entry $x/y$ in the row named “bounds” indicates that Lewittes’ bound produces $x$ and that the new bound produces $y$. An interval in the row named $N_q(g)$ means that $N_q(g)$ is known to be in this interval. An * in the same row means that the table in [2] is empty. Table 1 illustrates that the new bound can be quite an improvement to Lewittes’ bound and that it can be much smaller than $N_q(g)$ also when Lewittes’ bound is not. It is clear that we get the most significant results for smallest $q$.

Example 5 From [10] we get all semigroups of genus 8. There are 66 of them. In Table 2 and Table 3 we then apply the bounds (1) and (2) to the cases of $q$ being 2, 3 and 4. As in the previous example an entry $x/y$ means that Lewittes’ bound produces $x$ whereas the new bound produces $y$. From [2] we know that $N_2(8) = 11$, $N_3(8) \in \{17, 18\}$ and $N_4(8) \in \{21, 22, 23, 24\}$. Lewittes’ bound tells us that in a function field over $\mathbb{F}_2$ of genus 8 and with $N_2(8) = 11$ rational places 13 semigroups are not allowed as Weierstrass semigroups of a rational place. The new bound gives us that 33 semigroups are not allowed. Assuming that $N_3(8) = 18$ Lewittes’ bound excludes 26 semigroups whereas the new bound excludes 31 semigroups. Assuming $N_4(8) = 24$ we get the exact same picture.

4 Bounds on $N_q(g)$

From Lewittes’ bound (2) we immediately get $N_q(g) \leq q(g + 1) + 1$ as the multiplicity of a semigroup with $g$ gaps can be at most $g + 1$. This fact is not stressed in [6] as the paper contains slightly better bounds on $N_q(g)$ namely $N_q(g) \leq qg + 2$ ([5, Th. 1, part (a)]) and $N_2(g) \leq 2g - 2$ ([10, Eq. (19)]). The latter bounds are slightly better than Serre’s upper bound in the case of $q$ being equal to 2, 3 and 4. We now investigate the implication of the new result (1) for establishing bounds on $N_q(g)$. We get the following proposition.
| $\Lambda$ | $g = 20$ | $\Lambda = (8, 9, 20)$ | $\Lambda = (13, 15, 17, 18, 20)$ | $\Lambda = (13, 15, 24, 31)$ | $\Lambda = (20, 22, 23, 24, 26)$ | $\Lambda = (13, 14, 20)$ | $\Lambda = (16, 17, 18, 19)$ | $\Lambda = (10, 11, 20, 22)$ |
|---|---|---|---|---|---|---|---|---|
| $q$ | 2 | 3 | 4 | 8 | 9 | 16 | 2 | 3 | 4 | 8 | 9 | 16 | 2 | 3 | 4 | 8 | 9 | 16 | 2 | 3 | 4 | 8 | 9 | 16 |
| $\Lambda = (8, 9, 20)$ | $g = 20$ | $\Lambda = (13, 15, 17, 18, 20)$ | $\Lambda = (13, 15, 24, 31)$ | $\Lambda = (20, 22, 23, 24, 26)$ | $\Lambda = (13, 14, 20)$ | $\Lambda = (16, 17, 18, 19)$ | $\Lambda = (10, 11, 20, 22)$ |
| $N_q(g)$ | 19 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 |
| $N_q(g)$ | 19 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 |
| $N_q(g)$ | 19 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 |
| $N_q(g)$ | 19 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 | 18 – 21 | 30 – 34 | 40 – 45 | 76 – 83 | 70 – 91 | 127 – 139 |

Table 1: Semigroups from Example 4.
| Semigroup    | $q = 2$ | $q = 3$ | $q = 4$ |
|-------------|---------|---------|---------|
| $\langle 2, 17 \rangle$ | 5/5     | 7/7     | 9/9     |
| $\langle 3, 10, 17 \rangle$ | 7/6     | 10/10   | 13/13   |
| $\langle 3, 11, 16 \rangle$ | 7/7     | 10/10   | 13/13   |
| $\langle 3, 13, 14 \rangle$ | 7/7     | 10/10   | 13/13   |
| $\langle 4, 6, 13 \rangle$ | 9/9     | 13/13   | 17/17   |
| $\langle 4, 6, 15, 17 \rangle$ | 9/9     | 13/13   | 17/17   |
| $\langle 4, 7, 17 \rangle$ | 9/6     | 13/11   | 17/17   |
| $\langle 4, 9, 10 \rangle$ | 9/9     | 13/12   | 17/17   |
| $\langle 4, 9, 11 \rangle$ | 9/7     | 13/13   | 17/17   |
| $\langle 4, 9, 14, 15 \rangle$ | 9/8     | 9/9     | 17/17   |
| $\langle 4, 10, 11, 17 \rangle$ | 9/9     | 13/13   | 17/17   |
| $\langle 4, 10, 13, 15 \rangle$ | 9/9     | 13/13   | 17/17   |
| $\langle 4, 11, 13, 14 \rangle$ | 9/9     | 13/13   | 17/17   |
| $\langle 5, 6, 13 \rangle$ | 11/7    | 16/12   | 21/18   |
| $\langle 5, 6, 14 \rangle$ | 11/7    | 16/12   | 21/19   |
| $\langle 5, 7, 9 \rangle$ | 11/7    | 16/13   | 21/19   |
| $\langle 5, 7, 11 \rangle$ | 11/9    | 16/14   | 21/19   |
| $\langle 5, 7, 13, 16 \rangle$ | 11/8    | 16/14   | 21/20   |
| $\langle 5, 8, 9 \rangle$ | 11/9    | 16/15   | 21/20   |
| $\langle 5, 8, 11, 12 \rangle$ | 11/9    | 16/14   | 21/21   |
| $\langle 5, 8, 11, 14, 17 \rangle$ | 11/9    | 16/15   | 21/20   |
| $\langle 5, 8, 12, 14 \rangle$ | 11/9    | 16/15   | 21/21   |
| $\langle 5, 9, 11, 12 \rangle$ | 11/9    | 16/16   | 21/21   |
| $\langle 5, 9, 11, 13, 17 \rangle$ | 11/9    | 16/15   | 21/21   |
| $\langle 5, 9, 12, 13, 16 \rangle$ | 11/10   | 16/16   | 21/21   |
| $\langle 5, 11, 12, 13, 14 \rangle$ | 11/11   | 16/16   | 21/21   |

Table 2: Semigroups from Example 5
| Semigroup          | $q = 2$ | $q = 3$ | $q = 4$ |
|-------------------|---------|---------|---------|
| $(6, 7, 8, 17)$   | $13/8$  | $19/15$ | $25/22$ |
| $(6, 7, 9, 17)$   | $13/10$ | $19/17$ | $25/22$ |
| $(6, 7, 10, 11)$  | $13/11$ | $19/16$ | $25/21$ |
| $(6, 7, 10, 15)$  | $13/10$ | $19/17$ | $25/23$ |
| $(6, 7, 11, 15, 16)$ | $13/9$ | $19/16$ | $25/23$ |
| $(6, 8, 11, 13, 15)$ | $13/11$ | $19/19$ | $25/25$ |
| $(6, 8, 10, 13, 15, 17)$ | $13/12$ | $19/19$ | $25/25$ |
| $(6, 8, 10, 11, 15)$ | $13/12$ | $19/19$ | $25/25$ |
| $(6, 8, 10, 11, 13)$ | $13/11$ | $19/18$ | $25/25$ |
| $(6, 8, 9, 10)$   | $13/10$ | $19/19$ | $25/25$ |
| $(6, 8, 9, 11)$   | $13/10$ | $19/19$ | $25/25$ |
| $(6, 8, 9, 13)$   | $13/11$ | $19/19$ | $25/25$ |
| $(6, 9, 10, 11, 14)$ | $13/12$ | $19/19$ | $25/25$ |
| $(6, 9, 10, 11, 13)$ | $13/11$ | $19/19$ | $25/25$ |
| $(6, 9, 10, 13, 14, 17)$ | $13/12$ | $19/19$ | $25/25$ |
| $(6, 9, 11, 13, 14, 16)$ | $13/12$ | $19/19$ | $25/25$ |
| $(6, 10, 11, 13, 14, 15)$ | $13/12$ | $19/19$ | $25/25$ |
| $(7, 8, 9, 10, 11)$ | $15/10$ | $22/18$ | $29/26$ |
| $(7, 8, 9, 10, 12)$ | $15/10$ | $22/18$ | $29/26$ |
| $(7, 8, 9, 10, 13)$ | $15/10$ | $22/18$ | $29/26$ |
| $(7, 8, 9, 11, 12)$ | $15/11$ | $22/18$ | $29/27$ |
| $(7, 8, 9, 11, 13)$ | $15/11$ | $22/18$ | $28/27$ |
| $(7, 8, 9, 12, 13)$ | $15/11$ | $22/18$ | $29/27$ |
| $(7, 8, 10, 12, 13)$ | $15/12$ | $22/19$ | $29/27$ |
| $(7, 8, 10, 11, 12)$ | $15/11$ | $22/19$ | $29/29$ |
| $(7, 8, 10, 11, 13)$ | $15/11$ | $22/19$ | $29/27$ |
| $(7, 8, 11, 12, 13, 17)$ | $15/12$ | $22/20$ | $29/28$ |
| $(7, 9, 10, 11, 12, 13)$ | $15/11$ | $22/20$ | $29/27$ |
| $(7, 9, 10, 11, 13, 15)$ | $15/11$ | $22/20$ | $29/28$ |
| $(7, 9, 10, 12, 13, 15)$ | $15/12$ | $22/21$ | $29/28$ |
| $(7, 9, 11, 12, 13, 15, 17)$ | $15/12$ | $22/21$ | $29/28$ |
| $(7, 10, 11, 12, 13, 15, 16)$ | $15/13$ | $22/21$ | $29/29$ |
| $(8, 9, 10, 11, 12, 13, 14)$ | $17/13$ | $25/22$ | $33/31$ |
| $(8, 9, 10, 11, 12, 13, 15)$ | $17/13$ | $25/22$ | $33/31$ |
| $(8, 9, 10, 11, 12, 14, 15)$ | $17/13$ | $25/22$ | $33/31$ |
| $(8, 9, 10, 11, 13, 14, 15)$ | $17/13$ | $25/22$ | $33/31$ |
| $(8, 9, 10, 12, 13, 14, 15)$ | $17/14$ | $25/22$ | $33/32$ |
| $(8, 9, 11, 12, 13, 14, 15)$ | $17/14$ | $25/24$ | $33/32$ |
| $(8, 10, 11, 12, 13, 14, 15, 17)$ | $17/15$ | $25/23$ | $33/33$ |
| $(9, 10, 11, 12, 13, 14, 15, 16, 17)$ | $19/15$ | $28/26$ | $37/35$ |

Table 3: Semigroups from Example 5
Proposition 2

\[ N_q(g) \leq (q - \frac{1}{q})g + q + 2 - \frac{1}{q}. \] \hspace{1cm} (10)

Proof: The proof uses Corollary 2. A reasonable estimate of the number \( t \) in Corollary 2 can be given in terms of \( \lambda_1 \) and \( g \) alone. We have

\[ t \geq \max\{\left\lceil \frac{\lambda_1}{q} \right\rceil - 1 - (g - (\lambda_1 - 1)), 0\} \] \hspace{1cm} (11)

\[ \geq \max\{\frac{\lambda_1}{q} + \lambda_1 - g - 2, 0\} \] \hspace{1cm} (12)

as there are \( g - (\lambda_1 - 1) \) gaps greater than \( \lambda_1 \). Observe that \( \frac{\lambda_1}{q} + \lambda_1 - g - 2 \leq 0 \) holds for \( \lambda_1 \leq \frac{q}{q+1}(g + 2) \). Hence, \( N_q(g) \leq \max\{K_1, K_2\} \) where

\[ K_1 := q(\frac{q}{q+1}(g + 2)) + 1 \]

\[ K_2 := \max\{q\lambda_1 - (\frac{\lambda_1}{q} + \lambda_1 - g - 2) + 1 \mid \frac{q}{q+1}(g + 2) \leq \lambda_1 \leq g + 1\}. \]

The maximal value of \( q\lambda_1 - (\frac{\lambda_1}{q} + \lambda_1 - g - 2) + 1 \) is attained for \( \lambda_1 = g + 1 \) and the proposition follows. \hfill \Box

Observe, that the bound (10) was obtained by showing that the semigroup considered in Example 3 is the worst case. That is, (10) is almost the same as (8). With the last part of Example 3 in mind we cannot hope to improve upon Proposition 2 using our method.

For \( q \) being equal to 2, 3 and 4 Proposition 2 is much better than Serre’s upper bound. We get

\[ N_2(g) \leq \frac{1}{2}g + 3 \cdot \frac{1}{2}, \]

\[ N_3(g) \leq \frac{2}{3}g + 4 \cdot \frac{2}{3}, \]

\[ N_4(g) \leq \frac{3}{4}g + 5 \cdot \frac{3}{4}, \]

whereas Serre’s upper bound states

\[ N_2(g) \leq 2g + 3, \]

\[ N_3(g) \leq 3g + 4, \]

\[ N_4(g) \leq 4g + 5. \]

For values of \( q \) greater than or equal to 5 Serre’s upper bound is much better than Proposition 2. Even though Proposition 2 is better than Serre’s
upper bound for three values of $q$ it does not provide information that is not already known. For $q$ being equal to 2, 3 and 4 and $g$ being not too small Ihara’s bound

$$N_q(g) \leq q + 1 + \lfloor(\sqrt{(8g+1)}g^2 + 4(q^2 - q)g - g)/2\rfloor.$$  \hspace{1cm} (13)

namely outperforms the bound in Proposition \[2\] For $q = 2$ of course the bound

$$N_2(g) \leq (0.83)g + 5.35,$$

which has been produced by the Oesterl-Serre method (see \[9\] Ex. 1.6.19) even more outperforms Proposition \[2\]

\section{5 Towers of function fields}

With the results in the previous sections in mind unsurprisingly Lewittes’ bound has implications for the theory of asymptotically good towers of function fields. Recall, that a sequence of function fields $(F^{(1)}/\mathbb{F}_q, F^{(2)}/\mathbb{F}_q, \ldots)$ is called a tower if $F^{(i)} \subseteq F^{(i+1)}$ holds for all $i \geq 1$. Given a tower of function fields we write $N^{(i)} = N(F^{(i)})$, $g^{(i)} = g(F^{(i)})$ and we say that the tower is asymptotically good if $g^{(i)} \to \infty$ for $i \to \infty$ and $\liminf_{i \to \infty} (N^{(i)}/g^{(i)}) = \kappa$ holds for some $\kappa > 0$. We mention that the interest in asymptotically good towers of function fields partly comes from the fact that they give rise to arbitrary long codes with good parameters (see \[12\] Sec. VII.2 for the details). We now present a corollary to Theorem \[1\] concerning asymptotically good towers.

\textbf{Corollary 3} Assume a tower of function fields is given with $g^{(i)} \to \infty$ for $i \to \infty$ and $\liminf_{i \to \infty} (N^{(i)}/g^{(i)}) = \kappa > 0$ (that is, the tower is asymptotically good). Let $(P^{(1)}, P^{(2)}, \ldots)$ be any sequence such that $P^{(i)}$ is a rational place of $F^{(i)}$ for $i = 1, 2, \ldots$. Let $\lambda_1^{(i)}$ be the multiplicity of the Weierstrass semigroup related to $P^{(i)}$ and let $m_i$ be the number of generators in some description of $\Lambda^{(i)}$. We have

\begin{align*}
\liminf_{i \to \infty} \frac{\lambda_1^{(i)}}{g^{(i)}} &\geq \frac{\kappa}{q} \quad \text{(14)} \\
m_i &\to \infty \quad \text{for} \quad i \to \infty \quad \text{(15)}
\end{align*}

\textbf{Proof:} From Lewittes’ bound \[2\] we know that $N^{(i)} \leq q\lambda_1^{(i)} + 1$. Applying the assumptions from the corollary we get

$$\liminf_{i \to \infty} \left(\frac{q\lambda_1^{(i)} + 1}{g^{(i)}}\right) \geq \kappa \Rightarrow \liminf_{i \to \infty} \frac{\lambda_1^{(i)}}{g^{(i)}} \geq \frac{\kappa}{q}.$$
To see the last part of the corollary observe that by Proposition 1 \( N(i) \leq q^{m_i} + 1 \) holds.

We next apply Corollary 3 to the case of the asymptotically good tower of function fields over \( \mathbb{F}_{q^2} \) which was introduced by Garcia and Stichtenoth in [1]. This tower is defined by \( F(1) = \mathbb{F}_{q^2}(x_1) \) and \( F(i+1) = F(i)(x_{i+1}) \) with

\[
x_{i+1}^q + x_{i+1} = \frac{x_i^q}{x_i^{q-1} + 1}.
\]

The tower is actually as good as a tower over \( \mathbb{F}_{q^2} \) can possibly be. We have

\[
g(i) = \begin{cases} 
(q^{i/2} - 1)^2 & \text{if } i \equiv 0 \mod 2 \\
(q^{(i+1)/2} - 1)(q^{(i-1)/2} - 1) & \text{if } i \equiv 1 \mod 2 
\end{cases}
\]

which of course implies \( g(i) \to \infty \) for \( i \to \infty \) and we have

\[
\lim_{i \to \infty} \left( \frac{N(i)}{g(i)} \right) = q - 1.
\]

The element \( x_1 \in F(1) \subseteq F(i) \) has in \( F(i) \) a unique pole which we denote by \( \mathcal{P}_{\infty}^{(i)} \). This place is known to be rational. Our interest in \( \mathcal{P}_{\infty}^{(i)} \) comes from the convenient fact that the Weierstrass semigroup \( \Lambda^{(i)} \) of \( \mathcal{P}_{\infty}^{(i)} \) was established in [11]. This allows us to apply Corollary 3 and Remark 1. For \( i \geq 1 \) define

\[
c(i) = \begin{cases} 
q^i - q^{i/2} & \text{if } i \equiv 0 \mod 2 \\
q^i - q^{(i+1)/2} & \text{if } i \equiv 1 \mod 2 
\end{cases}
\]

then \( \Lambda(1) = \mathbb{N}_0 \), and for \( i \geq 1 \) we have

\[
\Lambda(i+1) = q\Lambda(i) \cup \{ x \in \mathbb{N}_0 \mid x \geq c(i+1) \}.
\]

Assume for a moment that \( q > 2 \). The three smallest non gaps of \( \Lambda(i) \) are 0, \( q^{i-1} \), \( 2q^{i-1} \) (this can be verified by an induction proof). Hence, if we write \( \Lambda(i) = \langle \lambda_1^{(i)}, \ldots, \lambda_{m(i)}^{(i)} \rangle \) with \( 0 < \lambda_1^{(i)} < \cdots < \lambda_{m(i)}^{(i)} \) then we get \( \lambda_1^{(i)} = q^{i-1} \) and \( \lambda_2^{(i)} \geq 2q^{i-1} \). From (16) we see that \( g(i) < q^i \) holds. Therefore

\[
q^2 \lambda_1^{(i)} + 2g(i) \leq q^2 \lambda_2^{(i)},
\]

and by Remark 1 the bounds (1) and (2) from Theorem 1 will produce the same results when applied to \( \mathcal{P}_{\infty}^{(i)} \) in the cases \( q > 2 \). Hence, there is no point in trying to improve upon Corollary 3 for the case of Garcia and
Stichtenoth’s second tower by taking into account the new bound (1).

Whether or not \( q > 2 \) or \( q = 2 \) holds we have \( \lambda_{1}^{(i)} = q^{i-1} \), and therefore

\[
\lim_{i \to \infty} \left( \frac{\lambda_{1}^{(i)}}{g^{(i)}} \right) = 1/q.
\]

For comparison Corollary 3 reads

\[
\lim \inf_{i \to \infty} \left( \frac{\lambda_{1}^{(i)}}{g^{(i)}} \right) \geq \frac{(q - 1)/q^2}{q}.
\]

For large values of \( q \) Corollary 3 therefore gives a reasonable picture of the situation in the case of the tower considered.

We conclude this section by showing that one cannot hope to produce asymptotically good towers having telescopic Weierstrass semigroups. This fact may be known to some of the researchers of asymptotically good towers; but we have not been able to find any reference on it.

**Definition 2** Let \( \Lambda = \langle a_1, \ldots, a_k \rangle \in \mathbb{N}_0 \) be a semigroup for which we have \( \gcd(a_1, \ldots, a_k) = 1 \). For \( 1 \leq j \leq k \) define \( d_j := \gcd(a_1, \ldots, a_j) \) and \( \Lambda_j := \langle a_1/d_j, \ldots, a_j/d_j \rangle \). If \( a_j/d_j \in \Lambda_{j-1} \) for \( 2 \leq j \leq k \) then \( \Lambda \) is said to be telescopic.

We will need the following result corresponding to [5, Lem. 5.34].

**Lemma 2** If \( \Lambda = \langle a_1, \ldots, a_k \rangle \) describes a telescopic semigroup as in Definition 2 then for any \( \lambda \in \Lambda \) there exist (uniquely) determined non-negative integers \( x_1, \ldots, x_k \) such that \( 0 \leq x_j \leq d_{j-1}/d_j \) for \( 2 \leq j \leq k \) and 

\[
\lambda = \sum_{j=1}^{k} x_j a_j.
\]

**Proposition 3** Let \( (F^{(1)}/\mathbb{F}_q, F^{(2)}/\mathbb{F}_q, \ldots) \) be a tower of function fields such that for all \( i \) (or alternatively at least infinitely many \( i \)) the following holds: \( F^{(i)} \) possesses a rational place \( \mathcal{P}^{(i)} \) having a telescopic Weierstrass semigroup \( \Lambda^{(i)} \). Then the tower is asymptotically bad.

**Proof:** Only in the case that \( N^{(i)} \to \infty \) for \( i \to \infty \) we can hope to get an asymptotically good tower. We therefore assume that \( N^{(i)} \) satisfies this condition. As \( \Lambda^{(i)} \) is telescopic we know that there exists a description \( \Lambda^{(i)} = \langle \lambda_1^{(i)}, \ldots, \lambda_{m_i}^{(i)} \rangle \) satisfying the conditions in Definition 2. We will assume that we have chosen a description of this kind with \( m_i \) smallest possible. Following Definition 2 we have \( d_j^{(i)} := \gcd(\lambda_1^{(i)}, \ldots, \lambda_j^{(i)}) \) for \( 1 \leq j \leq m_j \). Clearly, \( d_j^{(i)} \mid d_{j-1}^{(i)} \) for \( j \geq 2 \) and by minimality of \( m_i \) Lemma 2
implies \( d_{j-1}^{(i)} \geq 2d_j^{(i)} \). The genus \( g^{(i)} \) is given by the following closed form expression (see \[5\] Pro. 5.35)

\[
g^{(i)} = \left(1 + \sum_{j=2}^{m_i} \left(\frac{d_j-1}{d_j} - 1\right)\lambda_j^{(i)}\right)/2,
\]

and therefore \( g^{(i)} \geq \frac{m_i-1}{2}\lambda_1^{(i)} \) holds. On the other hand Lewittes’ bound \(2\) states \( N^{(i)} \leq q\lambda_1^{(i)} + 1 \). Hence, \( \lim_{i \to \infty} \frac{N^{(i)}}{g^{(i)}} = 0 \) follows immediately from \([15]\).

\[\square\]

6 Concluding remarks and acknowledgements

We would like to mention that it is possible to give an alternative proof of Theorem \([1]\) by using results on order domain theory from \([4]\) in combination with the footprint bound from Gröbner basis theory. It would be nice if the methods from the present paper can to some extend be used to deal with algebraic function fields of more variables. We leave it for further research to decide if this is possible. The gonality of a curve is closely related to the multiplicities \(\lambda_1\) of the Weierstrass semigroups studied in the present paper. More generally the notion of the gonality sequences that one finds in the papers \([7]\), \([8]\) and \([14]\) on generalized Hamming Weights is closely related to the generators \(\lambda_1, \ldots, \lambda_m\) of the Weierstrass semigroups studied in the present paper. We leave it for further research to decide if the new bound \([1]\) has some implications for the theory of gonality sequences.

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