We discuss the late-time behaviour of a dynamically perturbed Kerr black hole. We present analytic results for near extreme Kerr black holes that show that the large number of virtually undamped quasinormal modes that exist for nonzero values of the azimuthal eigenvalue $m$ combine in such a way that the field oscillates with an amplitude that decays as $1/t$ at late times. This prediction is verified using numerical time-evolutions of the Teukolsky equation. We argue that the observed behaviour may be relevant for astrophysical black holes, and that it can be understood in terms of the presence of a “superradiance resonance cavity” immediately outside the black hole.

A brief background. — Our understanding of the generic response of a black hole to dynamic perturbations is based on seminal work from 30 years ago. Exponentially damped quasinormal-mode (QNM) ringing was first observed in numerical experiments by Vishveshwar [1], and the subsequent late-time power-law fall-off (that all perturbative fields decay as $t^{-2l-3}$ in the Schwarzschild geometry) was discovered by Price [2]. A considerable body of work has since established the importance of these two phenomena for black-hole physics. We now know that most black-hole signals are dominated by the slowest damped QNMs, and many reliable methods for investigating these modes have been developed [3]. The nature of the late-time tail has also been studied in great detail. In particular it has been established that it is a generic effect independent of the presence of an event horizon: The tail arises from backscattering off of the weak gravitational potential in far zone [4]. However, the fact that our understanding has reached a mature level does not mean that no problems remain in this field. A few years ago, the quasinormal modes had been calculated also for Kerr black holes [5], but there were no actual calculations demonstrating the presence of power-law tails. Neither were there any dynamical studies of rotating black holes. Several recent developments have served to change this situation and improve our understanding of dynamical rotating black holes. Of particular relevance has been an effort to develop a reliable framework for perturbative time-evolutions of Kerr black holes [6]. There has also been recent efforts to analytically approximate the late-time power-law tails for Kerr black holes [7]. Furthermore, numerical relativity is reaching a stage where fully nonlinear studies of spinning black holes are feasible.

Kerr black-hole spectroscopy. — With the likely advent of gravitational-wave astronomy only a few years away the onus is on theorists to provide detailed predictions of the many scenarios that may lead to detectable gravitational waves. In this context, the question whether we can realistically hope to do “black-hole spectroscopy” by detecting QNM signals and inverting them to infer the black holes mass and angular momentum is highly relevant [10]. For slowly rotating black holes this presents a serious challenge. Using standard results one can readily estimate that the effective gravitational-wave amplitude for QNMs is (cf. similar estimates for pulsating stars [11])

$$h_{\text{eff}} \approx 4.2 \times 10^{-24} \left( \frac{\delta}{10^{-6}} \right)^{1/2} \left( \frac{M}{M_\odot} \right) \left( \frac{15 \text{Mpc}}{r} \right)$$

where $\delta$ is the radiated energy as a fraction of the black-hole mass $M$. The frequency of the radiation depends on the black-hole mass as $f \approx 12(M_\odot/M)$kHz. Given these relations, and recalling the estimated sensitivity of the generation of detectors that is under construction, the detection of QNM signals from slowly rotating solar-mass black holes seems rather unlikely. The situation will be rather different for low-frequency signals from supramassive black holes in galactic nuclei and detection with LISA, the space-based interferometric gravitational-wave antenna. Also, there is recent evidence that “middle weight” black holes, in the range $100 - 1000 M_\odot$ may exist [12]. For such black holes the most important QNMs would radiate at frequencies where the new generation of ground based detectors reach their peak sensitivity. If there are indeed such black holes out there we may hope to take their fingerprints in the future.

It has been suggested that QNM signals from rapidly rotating black holes would be easier to detect. This belief is based on the fact that some QNMs become very long lived as $a \to M$. In fact, mode calculations predict the existence of an infinite set of essentially undamped modes in the extreme Kerr limit [5]. The available investigations into the detectability of QNM signals have focused on the slow damping of these modes [10]. It has been shown that the decreased damping of the mode may increase the detectability considerably. However, these results have to be interpreted with some caution. What has been shown is the (anticipated) effect that a slower damped mode is easier to detect than a short-lived one, provided that the
modes are excited to a comparable amplitude. This is a rather subtle issue that pertains to the question whether it is easier to excite a slowly damped QNM than a short-lived one. Intuitively, one might expect this not to be the case. In similar physical situations the build-up of energy in a long-lived resonant mode takes place on a time-scale similar to the eventual mode damping. Thus it ought to be very difficult to excite a QNM that has characteristic damping several times longer than the dynamical timescale of the excitation process. This argument suggests that the amplitude of each long-lived mode ought to vanish in the limit \( a \to M \) when the e-folding time of the mode increases dramatically \[1\]. In view of this it would seem rather dubious to conclude that the detectability of a QNM signal actually improves as \( a \to M \). All may not be lost, however, because even if each individual QNM has an infinitesimal amplitude for rapidly spinning black holes a large number of modes approach the same limiting frequency as \( a \to M \). These modes may combine to give a considerable signal \[1\].

A surprising analytic result. — We want to assess the change in “detectability” of the QNMs as \( a \to M \), i.e. as we approach the extreme Kerr black hole case. As a suitable model problem, we consider a massless scalar field.

As is well known, the equation that governs such a field (which follows immediately from \( \Box \Phi = 0 \)) is similar to the master equation for both electromagnetic and gravitational perturbations of a rotating black hole that was first derived by Teukolsky \[13\]. In the following we briefly outline our calculation and discuss the main results. A more exhaustive discussion will be presented elsewhere. We use standard Boyer-Lindquist coordinates, and approach the QNM problem in the frequency domain (obtained via the integral transform used in \[10\]). Furthermore, we use the symmetry of the problem to separate the dependence on the azimuthal angle \( \varphi \). In essence, we are using a decomposition:

\[
\Phi = \frac{e^{i\varphi}}{2\pi} \sum_{l=0}^{\infty} \frac{R_{lm}(\omega, r)}{\sqrt{r^2 + a^2}} S_{lm}(\omega, \theta) e^{-i\omega t} d\omega
\]  

(2)

It should be noted that the rotation of the black hole couples the various multipoles through the (frequency dependent) spheroidal angular functions \( S_{lm} \).

In direct analogy with the Schwarzschild case \[10\] the initial value problem for the scalar field can be solved using a Green’s function constructed from solutions to the homogeneous radial equation for \( R_{lm}(\omega, r) \). One of the required solutions, that satisfies the causal condition at the event horizon \( r_+ = M + \sqrt{M^2 - a^2} \), has asymptotic behaviour

\[
R_{lm}^{\text{in}} \sim \begin{cases} 
    e^{-ikr_+} & \text{as } r \to r_+ , \\
    A_{\text{out}} e^{i\omega r_+} + A_{\text{in}} e^{-i\omega r_+} & \text{as } r \to +\infty .
\end{cases}
\]  

(3)

Here

\[
k = \omega - \frac{ma}{2M^2} = \omega - m\omega_+ ,
\]  

(4)

where \( \omega_+ \) is the angular velocity of the event horizon, and \( r_+ \) is the tortoise coordinate. It is useful to recall that a monochromatic wave is superradiant if it has frequency in the range \( 0 < \omega < m\omega_+ \) \[5\].

A QNM is defined as a frequency \( \omega_n \) at which \( A_{\text{in}} = 0 \). Assuming that \( A_{\text{in}} \approx (\omega - \omega_n)\alpha_n \) close to \( \omega = \omega_n \) we can deduce (via the residue theorem) that the contribution from each such mode to the evolution of the scalar field is

\[
\Phi_n(t, r, \theta) = \frac{A_{\text{out}}}{2\omega_n\alpha_n} e^{-i\omega_n(t-r_+)} \sum_{l=0}^{\infty} S_{lm}(\omega_n, \theta) I_{lm}
\]  

(5)

where \( I_{lm}(\omega_n, r) \) is a complicated expression that depends on the details of the initial data (here assumed to have support only far away from the black hole), cf. \[10\].

Let us now focus on the case of nearly extreme Kerr black holes, i.e. on the case \( a \approx M \). Then we can benefit from an approximation due to Teukolsky and Press \[13\], that suggests that there will exist an infinite set of QNMs that can be approximated by \[5,14\].

\[
\omega_n M \approx \frac{m}{2} - \frac{1}{4m} e^{(\theta - 2\pi)/\delta} (\cos \varphi - i \sin \varphi)
\]  

(6)

where \( \delta, \theta \) and \( \varphi \) are positive constants (not to be confused with the coordinates), and \( n \) is an integer labelling the modes. It is easy to see that as \( n \to \infty \) the modes become virtually undamped, and that they are located close to the upper limit of the superradiant frequency interval. That such a set of long-lived QNMs will exist agrees with other mode-calculations \[10\]. Given the location of the QNMs we can extend the calculation to deduce also the form of the asymptotic amplitudes \( A_{\text{out}} \) and \( A_{\text{in}} \) (or rather, the coefficient \( \alpha_n \)) for each \( \omega_n \). This enables us to approximate the contribution of each long-lived QNM to the field via \[\tilde{R}_{lm}\]. Doing this we find that the longest lived modes have exponentially small amplitudes. Thus we predict that the individual QNM will not in general be excited to a large amplitude, in agreement with the intuitive expectation. Expressing this result in terms of the effective amplitude of a corresponding gravitational-wave QNM, we would have

\[
h_{\text{eff}} \sim \sqrt{\text{Re} \omega_n A_{\text{out}}} \sim e^{-n\pi/2\delta} \text{Im} \omega_n \alpha_n \sim e^{-n\pi/2\delta}
\]  

(7)

In other words, the assumption that the long-lived modes may be easier to detect than (say) their short-lived counterparts for slowly rotating black holes is cast in serious doubt. A recent, more detailed, calculation of the QNM excitation coefficients for \( a \leq M \) supports this conclusion.

This does not, however, mean that the long-lived QNMs are without relevance. On the contrary, the fact...
that there is a large number of such modes has a very interesting consequence. After combining all the long-lived modes we find

$$\sum_{n=0}^{\infty} \frac{A_{\text{out}}}{\alpha_n} e^{-i\omega_n(t-r_c)} \sim \frac{e^{-i m \omega_{\pm} t}}{t} \quad \text{as } t \to \infty \quad (8)$$

This is an unexpected result: It suggests that, when summed, the contribution from the slowly damped QNMs of a near extreme Kerr black hole corresponds to a oscillating signal whose magnitude falls of with time as a power-law. Furthermore, the decay of this signal is considerable slower than the standard power-law tail. The decay of $1/t$ should be compared to the tail-results of, for example, Ori and Barack [3] that suggests that $\Phi \sim t^{-l-|m|-3-q}$ where $q = 0$ for even $l+m$ and $1$ for odd $l+m$ (derived only for non-extreme black holes). Hence, we predict that the oscillating QNM-tail will dominate the late-time behaviour of a perturbed near extreme Kerr black hole.

**Numerical confirmation.** — Our analytic result is obviously surprising. However, in view of the many approximations involved in the derivation of (8) considerable caution is warranted, and an alternative confirmation of the analytic prediction is desirable. Fortunately, the recent effort to develop a framework for doing perturbative time-evolutions for Kerr black holes [7] provides the means for testing our result. Hence, we have performed a set of evolutions (for various values of $m$) using the same scalar field code that was used to study superradiance in a dynamical context [8]. As initial data we have chosen a generic Gaussian pulse originally located far away from the black hole.

Our numerical evolution results can be succinctly summarised as follows (further details will be discussed elsewhere):

i) For extreme Kerr black holes ($a = M$) the numerical evolutions show the predicted oscillating $1/t$ behaviour for all $m \neq 0$, cf. Figure [1].

ii) For $a < M$ we find a similar behaviour, i.e. that the field is well approximated oscillations whose amplitude decays as $t^{-\mu}$, with $\mu$ rapidly increasing from 1 as $a$ departs from $M$, at late times.

iii) For axisymmetric perturbations ($m = 0$) the numerical evolution recovers the standard power-law tail. For our particular choice of initial data (that contains the $l = 0$ multipole) the tail falls off as $t^{-3}$.

Our interpretation of these results are: Firstly, the numerical evolutions confirm the analytic prediction for extreme Kerr black holes, i.e. that the field will oscillate with an amplitude that decays as $1/t$ at very late times. Secondly, and more important physically, the numerical data suggests that the late-time behaviour is qualitatively similar also in cases when $a$ is significantly smaller than $M$. The late-time behaviour was found to be consistent with an oscillating tail in all cases we have considered so far (essentially $a \geq 0.9M$). We have also verified that the observed late-time behaviour cannot be accounted for by a single slowly damped QNM when $a < M$. It is important to emphasise that the result for $a < M$ was not predicted by the analytical work (since the approximate modes we used are relevant only for $a \approx M$). In other words, the numerical experiments indicate that the effect could be of relevance for all rapidly rotating black holes and may well dominate the standard power-law tail for a large range of astrophysical black hole parameters. Intuitively one would expect there to exist a critical value of the rotation parameter $a$ above which the new effect becomes relevant. More detailed numerical work is needed to establish this, and investigate the role of the new effect further.

**FIG. 1.** A numerical evolution showing the late-time behaviour of a scalar field in the geometry of a rapidly rotating black hole. We show the field as viewed by an observer situated well away from the black hole for $a = M$. At late times the field falls off according to an oscillating power-law with the amplitude decaying as $1/t$.

**A physical interpretation.** — Given both the analytic prediction for extreme Kerr black holes and the numerical evolution results for $a \leq M$, an intriguing picture emerges. The results seem to suggest the existence of a new phenomenon in black-hole physics, with relevance at late times. We recall that the QNMs are typically interpreted, in analogy with scattering resonances in quantum physics, as originating from waves that are temporarily trapped close to the peak of the curvature potential (corresponding to the unstable photon orbit at $r = 3M$ in the Schwarzschild spacetime), and that the late-time power-law tail arises because of backscattering off of the weak potential in the far zone. Can the present results be interpreted in a similar intuitive vein? We think they can, and propose the following explanation: Consider the fate of an essentially monochromatic wave that falls onto the black hole. Provided that the frequency is in the interval $0 < \omega < m\omega_+$ the wave will be superradiant. In effect, this means that a distant observer will see waves “emerging from the horizon”, cf. [8], even though a local observer sees the waves crossing the event horizon (at $r_+$) [9]. This results in the scattered wave being amplified. In addition to this, one can establish that the effective potential has a peak outside the black hole (which is not immediately obvious since the potential is frequency depen-
dent in the Kerr case) for a range of frequencies including the superradiant interval. Now, the combination of the causal boundary condition at the horizon effectively corresponding to waves “coming out of the black hole” (according to a distant observer) and the presence of a potential peak leads to waves potentially being trapped in the region close to the horizon. In effect, there is a “superradiance resonance cavity” outside the black hole. Again according to a distant observer, waves can only escape from this cavity by leakage through the potential barrier to infinity. Presumably it is this leakage that then leads to the observed $1/t$ decay. Furthermore, it should be noted that superradiant amplification is strongest for frequencies close to $m\omega_+$. Thus superradiance effects a form of parametric amplification on waves in the cavity. At very late times, the dominant oscillation frequency ought to be that which experiences the strongest amplification, i.e. $m\omega_+$. This is, of course, exactly the result of our analytic calculation. The above argument is illustrated schematically in Figure 2.

At very late times, the dominant oscillation frequency is $\omega_+$, which is the frequency of the superradiant barrier to infinity. Presumably it is this leakage that then leads to the observed $1/t$ decay. Furthermore, it should be noted that superradiant amplification is strongest for frequencies close to $m\omega_+$. Thus superradiance effects a form of parametric amplification on waves in the cavity. At very late times, the dominant oscillation frequency ought to be that which experiences the strongest amplification, i.e. $m\omega_+$. This is, of course, exactly the result of our analytic calculation. The above argument is illustrated schematically in Figure 2.

**Concluding remarks.** We have presented the results of an investigation into the late-time behaviour of a perturbed Kerr black hole. An analytic calculation for the near extreme Kerr black hole case led to two important results. Firstly, we deduced that even though some QNMs become very slowly damped as $a \to M$ these modes will not be easier to detect with a gravitational-wave detector. Secondly, we arrived at the rather surprising prediction that the large number of virtually undamped QNMs that exist for each value of $m \neq 0$ combine in such a way that the field oscillates with an amplitude that decays as $1/t$ at late times. This decay is considerably slower than the standard power-law tail. The analytic prediction was then verified using numerical time-evolutions of the Teukolsky equation. These evolutions, performed for a larger range of the black-hole rotation parameter, suggest that the observed behaviour may well be present also for astrophysical black holes (which we recall must have $a \leq 0.998M$ [17]). Finally, we proposed an intuitive explanation of the observed phenomenon: That waves of certain frequencies are effectively trapped in a “superradiance resonance cavity” immediately outside the black hole. In conclusion, we find these results tremendously exciting: They indicate the presence of a new phenomenon in black-hole physics that may well be of astrophysical relevance.

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![Figure 2](image-url)