Lattice Study of the Massive Schwinger Model with a $\theta$ term under Lüscher’s ”Admissibility” condition*

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We present a numerical study of the massive two-flavor QED in two dimensions with the gauge action proposed by Lüscher, which allows only “admissible” gauge fields. We find that the admissibility condition does not allow any topology changes by the local updation in Hybrid Monte Carlo algorithm so that the configurations in each topological sector can be generated separately. By developing a new method to sum over different topological sectors, we investigate $\theta$ vacuum effects. Combining with domain-wall fermion action, we obtain the fermion mass dependence and $\theta$ dependence of the meson masses, which are consistent with the analytic results by mass perturbation in the continuum theory.

1. Introduction

The studies of chiral symmetries on the lattice have made much progress recently. There are two keys in these developments; (1) Ginsparg-Wilson relation \[1\]:

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D,$$

(1)

and (2) Lüscher’s “admissibility” condition [2,3]:

$$\| 1 - U_{\mu\nu}(x) \| \leq \epsilon \text{ for all } x, \mu, \nu.$$  

(2)

where $a$ denotes the lattice spacing, $U_{\mu\nu}(x)$ denotes a plaquette and $\epsilon$ is a fixed constant.

The first key, the Ginsparg-Wilson relation, gives a redefinition of chiral symmetries without fermion doublers at the classical level. The second key, the ”admissibility” condition Eq. (2), makes gauge fields smooth. Under this condition the gauge fields are classified by a topological charge which corresponds to that of the continuum theory, giving a well-defined chiral anomaly at the quantum level.

In two-dimensional QED, the topological charge is defined as

$$Q = \sum_{x,\mu,\nu} \left[ -\frac{1}{2} \epsilon_{\mu\nu} \ln U_{\mu\nu}^P(x) \right] \left( \text{integer} \right),$$

where $U_{\mu\nu}^P(x)$ denotes a plaquette and $\epsilon$ is a fixed constant.

$$\lim_{a \to 0} \frac{1}{2} \int_{T^2} d^2x \frac{1}{a} \epsilon_{\mu\nu} F_{\mu\nu}(x).$$  

(3)

While numerical studies on Ginsparg-Wilson fermion have been performed, no study of Lüscher’s admissible gauge fields has been done before.

We present a numerical study of two-dimensional QED with Lüscher’s gauge action which satisfies admissibility condition automatically [4]. We show that topological structure is realized on the lattice, $\theta$ dependence of observables can be evaluated by our new method. As a result we obtain the mass of isotriplet meson which is consistent with the continuum theory. This work may offer a new approach to investigate “topology” in the lattice gauge theories.

2. Lattice Simulations

We take Lüscher’s gauge action combined with the domain-wall fermion action:

$$S = \beta S_G + \sum_{\text{flavor}=1}^{2} S_F,$$  

(4)

$$S_G = \left\{ \begin{array}{ll} \sum_{x,\mu,\nu} \frac{(1 - \text{Re} U_{\mu\nu}^P(x))}{\epsilon} & \text{if admissible} \\
\infty & \text{otherwise} \end{array} \right\},$$  

(5)

$$S_F : \text{the domain-wall fermion action including Pauli-Villars fields},$$  

(6)
where $\beta = 1/e^2$. The Lüscher’s condition is automatically satisfied with this action.

We use the Hybrid Monte Carlo (HMC) method with pseudo-fermions and conjugate gradient (CG) algorithm for the inversion of the fermion matrix. 50 molecular dynamics steps with step-size $\Delta \tau = 0.02$ are performed for each trajectory and configurations are updated per 10 trajectories. We also check the admissibility at the Metropolis test. The simulations are carried out on a $16 \times 16 \times 6$ lattice with $\beta = 0.5$ and $\epsilon = 1.0$. Fermion mass $m$ is taken to be 0.1, 0.2, 0.3 and 0.4.

### 3. Topological Sectors and Reweighting

We set the initial link variables as follows,

$$U^{cl[N]}_1(x,y) = \exp\left\{\frac{2\pi N_i}{L} \delta_{x,y}\right\},$$

$$U^{cl[N]}_2(x,y) = \exp\left\{\frac{2\pi N_i}{L^2} x\right\},$$

where $L = 16$ is the lattice size. This configuration gives constant background electric fields over the torus and minimizes the gauge action in the sector with topological charge $N$. As Fig.1 shows, Lüscher’s action generates configurations without any topology changes.

Thus admissible gauge fields are separated into topological sectors and configurations are generated in each sector with Lüscher’s action. The next task is to sum up the expectation values in each sector with correct weights.

$$\langle O \rangle^\text{full}_{\beta,m} = \frac{\sum_{N=-\infty}^{+\infty} e^{i\theta N} \langle O \rangle^N_{\beta,m} R^N(\beta,m)}{\sum_{N=-\infty}^{+\infty} e^{i\theta N} R^N(\beta,m)},$$

where

$$R^N(\beta,m) = \frac{Z_N(\beta,m)}{Z_0(\beta,m)},$$

$$\langle O \rangle^N_{\beta,m} : \text{expectation value in each sector}$$

$$Z_N(\beta,m) : \text{generating functional in each sector.}$$

We developed a new method to calculate the reweighting factor $R^N(\beta,m)$ [4]. By separating $R^N(\beta,m)$ into three factors as

$$R^N(\beta,m) = \exp(-\beta S^{N}_{G_{\text{min}}}),$$

Figure 1. The comparison of the evolution of the topological charge with Wilson’s gauge action and Lüscher’s gauge action for the same lattice spacings determined from the string tension.

$$\times \left[ \frac{\int d\nu_1 d\nu_2 \det(D^N_{DW})^2}{\det(D^N_{PV})^2} \right] \times \left[ \frac{\int d\nu_1 d\nu_2 \det(D^N_{PW})^2}{\det(D^N_{PV})^2} \right] \times e\left[ \int_{\beta} d\beta \left( \langle S_G - S^{N}_{G_{\text{min}}} \rangle_{\beta,m} - (S^{N}_{G_{\beta},m}) \right) \right],$$

where $S^{N}_{G_{\text{min}}}$ in the first factor denotes the minimum of the gauge action in $N$ sector, the second factor is the contribution from free fermion determinants given by the integral of Polyakov loops, and the third one is a $\beta$ integral of the expectation value of the gauge action.

Once the reweighting factor is known, we can evaluate theta dependence of observables as in Eq. (8).
4. Meson Masses

We calculated the isotriplet meson propagators at various $\theta$ as

$$\sum_y \langle \pi(x, y)\pi(0,0) \rangle_{full} =$$

$$\sum_{N=-4}^{4} e^{iN\theta} \sum_y \langle \pi(x, y)\pi(0,0) \rangle_N^{\beta,m} R_N^{\beta,m}.$$  \hspace{1cm} (10)

Here we have ignored $|N| > 4$ sectors since they only give contributions less than 1.2 % of zero sector. Fitting them into hyperbolic cosine function, we evaluated the meson mass.

As Figs.2 and 3 show, the fermion mass and $\theta$ dependence of the pion mass are consistent with that of continuum theory at small $\theta$ [5–7];

$$m_\pi \propto \left( m \cos \frac{\theta}{2} \right)^{2/3}.$$  \hspace{1cm} (11)

Moreover, it is remarkable the residual mass in the chiral limit is consistent with zero within statistical error even in this strong coupling regime;

$$m_\pi(m \rightarrow 0) = -0.057 \pm 0.060.$$  \hspace{1cm} (12)

5. Summary

We performed a numerical simulation of 2 dimensional theory with Lüscher’s gauge action combined with domain-wall fermion action. We also proposed a new method for summing over different topological sectors.

Applying Lüscher’s condition to numerical simulations we find that the lattice gauge fields are separated into topological sectors. Our reweighting method enables the summation over topological sectors. We obtained the results which are qualitatively consistent with that in the continuum theory. It is also notable that the residual mass of the domain-wall fermion in the chiral limit is consistent with zero even in the strong coupling regime.

As future works, we plan to perform more precise and quantitative studies of 2 dimensional QED and comparison to the results in conventional lattice formalism [8]. Studying how to extend our approach to 4 dimensional QCD is also important.

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