Entropy production and information fluctuations along quantum trajectories

B. Leggio, A. Napoli, A. Messina, and H.-P. Breuer

Dipartimento di Fisica e Chimica, Università di Palermo, Via Archirafi 36, 90123 Palermo, Italy
Physikalisches Institut, Universität Freiburg, Hermann-Herder-Straße 3, D-79104 Freiburg, Germany
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Employing the stochastic wave function method, we study quantum features of stochastic entropy production in nonequilibrium processes of open systems. It is demonstrated that continuous measurements on the environment introduce an additional, non-thermal contribution to the entropy flux, which is shown to be a direct consequence of quantum fluctuations. These features lead to a quantum definition of single trajectory entropy contributions, which accounts for the difference between classical and quantum trajectories and results in a quantum correction to the standard form of the integral fluctuation theorem.

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Fluctuation theorems (FTs) for nonequilibrium processes [1, 2] are a set of general laws describing the intrinsic fluctuating nature of thermodynamical quantities for systems far from their equilibrium states. They describe the probability distribution of measurement outcomes for some quantities characterizing a system, such as energy or entropy. The validity of these laws for classical systems has been theoretically predicted [3, 7] and experimentally verified [8, 11] under various conditions, and a classical formulation of FTs has been satisfactory given and can nowadays be considered an (almost) settled problem. On the other hand, many efforts have been made to provide quantum versions of these laws [12–18], but a full picture is not yet available and the quantum counterpart to FTs has not yet been fully understood. A theoretical description of stochastic entropy production has been given for classical [9, 19] and quantum trajectories [20], and employed for classical as well as quantum FTs [5, 9, 10, 14, 15, 21]. These approaches have never taken into account the full quantum features of stochastic dynamics: When considering quantum systems and the probability distribution of measurement outcomes, the role of the external observer can not be neglected and a full quantum description of single nonequilibrium processes can not be given without incorporating the backaction of measurements [22, 23]. Following a quantum trajectory amounts to a continuous measurement process, and it is reasonable to expect that it introduces a non-negligible term in entropy production. Moreover, because any measurement fundamentally affects the evolution of the monitored system, the previous proposals of quantum trajectories based on measures on the open quantum system [16, 20, 24] can not fully reproduce its quantum features, as these projective measurements partly hide the quantumness of the process. In the framework of open quantum dynamics, however, the time evolution of an open system can be monitored by continuous measurements on its environment [25]. In this work we apply the quantum stochastic wave function method [26, 27] within the Markovian approximation to a generic quantum system interacting with a bath and, in general, externally driven through a fixed protocol, to obtain an expression for its entropy production. In this framework we consider information (entropy) contributions as extracted by measurements on the environment. The ensemble Markovian dynamics of the open quantum system is described by the master equation

$$\dot{\rho} = -i[H_S(t), \rho] + \sum_i \gamma_i(t)\left(A_i(t)\rho \hat{A}_i^\dagger(t) - \frac{1}{2}\{A_i^\dagger(t)A_i(t), \rho\}\right),$$

where $H_S(t)$ is the free Hamiltonian of the open quantum system and the non-Hermitian operators $\{A_i\}$ are known as Lindblad operators. The time dependence of $H_S(t)$, $\gamma_i(t)$ and $A_i(t)$ originates, in general, from the open system-environment interaction and from the external protocol driving the open system out of a stationary state. In this context, a forward quantum trajectory of a driven open quantum system is described by a piecewise deterministic process (PDP) [28] for its wave function $|\psi\rangle$, characterized by jumps (or transitions) described by the action of the Lindblad operators $A_i$ and by a nonunitary deterministic time evolution given by the operator $U(t_f, t_s) = \mathcal{T}\exp\left\{-i \int_{t_s}^{t_f} H_{\text{eff}}(t)dt\right\}$, where $H_{\text{eff}}(t) = H_S(t) - \frac{i}{2} \sum_i \gamma_i(t) A_i^\dagger A_i$. The jumps occur at certain times $t_k$ with corresponding rates $\gamma_i$ and Lindblad jump operators $A_i$ (see Appendix for further details). As the only way to extract information about the open system is, in this context, a measurement on the environment, one only detects transitions of the open system. Therefore, the backward trajectory is fixed by the requirement that the open system performs transitions at the same time instants as the forward one with rates $\gamma_i$ and jump operators $B_i = A_i^\dagger$. The reason is that the Lindblad operators in the Markovian master equation [1] and in the weak coupling limit satisfy the condition $[H_S, A_i^\dagger] = \pm \epsilon_i A_i^\pm$, where $A_i^\pm = (A_i^-)^\dagger$ and they thus describe jumps in which an energy quantum $\epsilon_i$ is absorbed ($A_i$) or emitted ($A_i^\dagger$) by the open system. Analy-
ogously we assume the backward drift parts to be generated by the operator $U(t_f, t_s) = U^\dagger(t_s, t_f)$. In this work we denote the normalized state of the forward (backward) process right before the jump at time $t_k$ as $|\chi^{(b)}_k\rangle$ and the normalized state of the forward (backward) process right after the jump at time $t_k$ as $|\psi^{(b)}_k\rangle$. A generic example of such trajectories is schematically depicted in Fig. 1 where the forward process starts in the state $|\psi_0\rangle$ and ends in $|\psi_f\rangle$, while final state of the backward one is $|\psi^b_f\rangle$. Note that the forward and backward trajectories are connected by the requirement that the initial state of the backward process coincides with the final state of the forward one but, in general, $|\psi_0\rangle \neq |\psi^b_f\rangle$ due to the fact that $A_i^b A_i \neq I$ and $U(t_N, T)U(T, t_N) \neq I$. Applying the prescriptions of quantum unraveling methods, the ratio of forward ($P_f$) and backward ($P_b$) probabilities can be written as

$$P_f = \frac{P[\psi_0, t_0]}{P[\psi^b_f, T]} \prod_{k=1}^N \gamma_{i_k} |A_{i_k}| |\chi^{(f)}_{k-1}\rangle|^2 \times \prod_{k=1}^{N+1} \frac{|U(t_k, t_{k-1})| \langle \psi^b_{k-1}|^2}{|U^\dagger(t_k, t_{k-1})| \langle \psi^{(f)}_k|^2}.$$  

The goal of this work is to derive an integral FT for entropy production along these nonequilibrium processes. Therefore we aim at giving explicit expressions for entropy contributions along PDFs.

**FIG. 1.** (Color online) Pictorial representation of a forward quantum trajectory (dark blue states) and its backward counterpart (light green states) consisting of $N = 3$ jumps (vertical red arrows). The backward process is characterized by jumps through the channels $\{\gamma^b_{i_k}, B_{i_k} = A^b_{i_k}\}$ and by deterministic evolution periods according to $U(t_k, t_{k+1}) = U^\dagger(t_{k+1}, t_k)$.

**Entropy.** A single quantum trajectory (either forward or backward), being a nonequilibrium process, is characterized by a nonzero entropy production. In the ensemble picture, the entropy of a system is given by its von Neumann entropy $S_{VN} = -\text{Tr} (\rho \ln \rho)$. This definition can not however be applied in the framework of stochastic wave function methods, since $S_{VN}$ vanishes on pure states. In quantum unraveling contexts, however, the entropy of an open system along a single trajectory can be defined as $S[\psi] = -\ln P[\psi, t]$. In what follows, we will refer to such a quantity as quantum entropy. Note that its mathematical definition is formally analogous to the one employed for entropy in classical stochastic processes [19]. On ensemble level the time derivative of the open system quantum entropy is given by

$$\dot{S} = -\int d\psi \dot{P}[\psi, t] \ln P[\psi, t].$$  

Such a definition is the natural quantum extension of the one employed in many previous works on entropy FTs [2, 14, 19], but it has no classical analogue as it does not reduce to the usual form of entropy in the classical limit. It describes the knowledge about the open quantum system, extracted by an external observer measuring the environment and, as single realizations of quantum dynamics are fundamentally different from their classical counterpart, the information thus extracted can not in general be given any classical interpretation. Exploiting the explicit form of the master equation for $P[\psi, t]$ [30], it is possible to show that the single trajectory contribution to Eq. (3) can be written as $\dot{S}[\psi] = \dot{S}_j[\psi] + S_d[\psi]$, i.e. as the sum of two terms, one arising from the drift part of the PDP (describing the conditioned no-jump evolution of the open system) and one due to the open system jumps. Since both quantum jumps and drifts are detected by measurements, each of these terms describes a change in knowledge of the external observer about the open system. In addition, we show in what follows that both the jump and the drift entropic term contribute to entropy production along the nonequilibrium process.

**Entropy production.** As $\int \dot{S}_j[\psi] dt$ and $\int \dot{S}_d[\psi] dt$ only take into account the difference of entropy between initial and final states, but not the features of the transition connecting them, these terms then do not fully describe the information content of unraveling measurements: There are indeed two corresponding terms describing information about transitions, detected in the environment, and which correspond then to entropy flowing from the open quantum system to its bath. Since the full system is out of equilibrium the changes in open quantum system entropy and the flux to the bath are not the same in absolute value. Their difference is interpreted as a net total entropy production along a trajectory, and it is written as

$$\sigma = \Delta S[\psi] - \Delta S^e[\psi],$$  

$\Delta S^e[\psi]$ being the total entropy flux to the bath and $\Delta S[\psi] = \int_0^T dt \dot{S}[\psi]$. In addition, we define a single trajectory jump entropy production and a single trajectory
drift entropy production as, respectively,
\[ \sigma_j = \int_{t_0}^T dt \dot{S}_j[\psi] - \Delta S^j_0[\psi], \]
\[ \sigma_d = \int_{t_0}^T dt \dot{S}_d[\psi] - \Delta S^d_0[\psi]. \]  
In the rest of this work, we aim at giving an expression for the entropy production along a generic quantum trajectory, being thus valid both for what we defined as forward trajectory and for its backward counterpart.

**Jump entropy production.** Along a generic trajectory, a transition \( |\chi_k\rangle \to |\psi_k\rangle = \frac{A_{ik} |\chi_k\rangle}{||A_{ik} |\chi_k\rangle||} \) is characterized by a rate
\[ R_{ik}^D[\chi_k] = \gamma_{ik} ||A_{ik} |\chi_k\rangle||^2. \]  
In what follows, we refer to such a transition as direct jump. A direct jump is nothing but the transition experimentally detected within an unraveling approach while following a particular nonequilibrium process (which can, in turn, either be a forward or a backward trajectory). In contrast to a direct jump, we define also a reversed jump as \( |\psi_k\rangle \to |\xi_k\rangle = \frac{A^*_i |\psi_k\rangle}{||A^*_i |\psi_k\rangle||} \), which represents the reversed transition associated to the \( k \)-th direct jump, and which is a fictitious transition as it is not detected in the trajectory: It represents a tool which allows us to introduce a “direction” of a single jump and thus its entropy production, and as such it is intrinsically different from the backward process previously introduced. A reversed jump is associated with the rate
\[ R_{ik}^R[\chi_k] = \gamma_{ik} \frac{\langle \chi_k | (A^*_i A_{ik})^2 |\chi_k\rangle}{||A_{ik} |\chi_k\rangle||^2}. \]  
In analogy with classical systems [31], we define the jump entropy flux along a single full quantum trajectory as
\[ \Delta S^j_0[\psi] = - \sum_{k=1}^N R_{ik}^D[\chi_k] \ln \frac{R_{ik}^D[\chi_k]}{P[\psi_k, t_k]} . \]  
Note that this definition, despite being formally analogous to the one usually employed in FT contexts when considering pure jump processes [14, 15, 21], differs from it because of the structure of transition rates in Eqs. (7) and (8). The total change of the open system entropy along the process, due to jumps only, is \( \Delta S_j[\psi] = - \sum_{k=1}^N \ln \frac{R_{ik}^D[\chi_k]}{P[\psi_k, t_k]} \). As a consequence, we obtain the total jump entropy production along a full quantum trajectory consisting of \( N \) jumps as
\[ \sigma_j = \ln \prod_{k=1}^N \frac{R_{ik}^D[\chi_k] P[\chi_k, t_k]}{R_{ik}^D[\chi_k] P[\psi_k, t_k]} . \]  

**Drift entropy production.** In order to obtain a time local entropy balance equation for the drift contribution we subdivide each finite drift interval \( [t_{k-1}, t_k) \) into many small steps of size \( \delta t \). In each of these small time intervals the monitoring of the environment yields the result that no jump with any of the Lindblad operators \( A_i \) occurs. Conditioned on these events the state vector undergoes small changes which lead in the limit \( \delta t \to 0 \) to a smooth time evolution describing the drift process. The formulation is thus analogous to the one given for jump entropy contributions, provided one uses the correct expression for the drift probabilities. The latter are given by
\[ \Gamma^{D[R]}_m[\psi] = 1 - \Gamma^{D[R]}_m[\psi] \delta t, \]  
\[ \Gamma^{D[R]}_m[\psi] = \sum_i R^D_i[\psi] \] is the total direct (reversed) jump rate for the state \( |\psi\rangle \). The bath entropy contribution of each of these no-jump events is thus \( \delta S^d_0[\psi] \) = \( - \ln \frac{1}{\Gamma^{D[R]}_m[\psi]} \). In this formulation, \( \delta t \) is the time interval between two subsequent measurements on the environment. Moreover, since unraveling approaches correspond to continuous measuring processes, it is justified to assume such a time interval to be very small (usually lower bounded only by the resolution time of the measuring apparatus), such that \( \Gamma^{D[R]} \delta t \ll 1 \). Under this approximation, then, \( \Delta S^d_0[\psi] \sim \int_{t_0}^T dt \left( \Gamma^{D}(t) - \Gamma^{R}(t) \right) \). Exploiting Eqs. (7) and (8), one can easily prove that
\[ \int_{t_{k-1}}^{t_k} dt \Gamma^{D}(t) = - \ln ||U(t_k, t_{k-1})|\psi_{k-1}\rangle||^2 \]  
and
\[ \int_{t_{k-1}}^{t_k} dt \Gamma^{R}(t) = - \ln ||U(t_{k-1}, t_k)|\psi_k\rangle||^2, \]  
so that
\[ \Delta S^d_0[\psi] = - \ln \prod_{k=1}^{N+1} \frac{||U(t_{k-1}, t_k)|\psi_{k-1}\rangle||^2}{||U(t_k, t_{k+1})|\psi_k\rangle||^2}. \]  
The total drift-induced change of open quantum system entropy is
\[ \Delta S_d[\psi] = - \ln \prod_{k=1}^{N+1} \frac{||U(t_k, t_{k+1})|\psi_{k+1}\rangle||^2}{||U(t_{k-1}, t_k)|\psi_{k}\rangle||^2}. \]  
is the single trajectory drift entropy production. With the use of Eqs. (4), (9) and (10) it is straightforward to show that
\[ \sigma = \ln \left( \frac{P[\psi_0, t_0]}{P[\Xi, t_N]} \times \prod_{k=1}^{N} \frac{R_{ik}^D[\chi_k]}{R_{ik}^D[\chi_k]} \frac{P[\psi_{k+1}, t_{k+1}]}{P[\psi_{k}, t_k]} \right). \]  
Such an equation describes the total entropy production along a single quantum trajectory: In particular, since a quantum trajectory is followed by measuring the environment, \( \sigma \) is the total information the external observer acquires about the system through the knowledge of initial and final states of the process (\( \Delta S \)) minus the information extracted by measurements of all intermediate steps connecting them (\( \Delta S_d \)), detected in the bath.

**Entropy flux and quantum fluctuations.** To fully understand the physics described by the entropy flux terms introduced in above, let us analyze for instance its jump contribution in Eq. (9). In the case of a jump \( |\chi_k\rangle \to |\psi_k\rangle \)
the entropy flowing into the environment is given by
\[ \Delta S_{jk}^{\sigma} = \ln \frac{\gamma_b^{k}}{\gamma_i^{k}} + \ln \frac{\gamma_i^{k} R^R_k}{\gamma_b^{k} R^D_k}. \]  
(12)

On average the process has a preferred direction if the two rates are not equal. Since, in a weak coupling Markovian master equation with a thermal environment, \( \frac{\gamma_i^{k}}{\gamma_i^{k}} = e^{-\beta \epsilon^{k}} \) \( (\epsilon^{k} \) being the energy \( Q_E \) exchanged between system and environment during the transition \( A_{ik} \)), the first term on the r.h.s. of Eq. \( \ref{eq:12} \) is a standard thermodynamic entropic flux of the form \( -\frac{Q_E}{kT} \). The second term on the r.h.s. of Eq. \( \ref{eq:12} \) describes, on the other hand, how much information is produced by the system jumping through the particular decay channel \( A_{ik} \). We refer to such an additional term as nonthermal entropy flux \( \Delta S^{\text{nt}} \). We can characterize such a nonthermal flux by introducing the parameter \( \eta_k = 1 - \frac{\gamma_i^{k} R^R_k}{\gamma_b^{k} R^D_k} \).

According to its definition, \( \eta_k = 0 \) if the bias of the associated direct transition to the corresponding reversed one is only due to the direction of heat flux. Introducing the operator \( A_{ik} = \Lambda_i^{\dagger} A_i \) and exploiting the explicit expression of \( R^R \) and \( R^D \) one obtains
\[ \eta_k = -\frac{\text{Var}_1[\chi_k](\Lambda_{ik})}{||A_{ik}[\chi_k]||^2}, \]  
(13)

where \( \text{Var}_1(Q) = \int d\psi P(\psi) \left( \left< |\langle \psi | Q | \psi \rangle|^2 - \langle |\langle \psi | Q | \psi \rangle|^2 \right> \right) \), introduced in \( \ref{eq:12} \), is known to measure the average intrinsic quantum fluctuations of an operator \( Q \) during a dynamic process, and \( \text{Var}_1[\chi_k] \) is its single trajectory contribution due to the \( k \)-th jump. From the structure of \( \eta_k \) we infer that during the jump \( |\chi_k \rangle \rightarrow |\psi_k \rangle \), the exchange of information between system and environment goes beyond the standard thermodynamic form if and only if the operator \( A_{ik} \) has nonzero purely quantum fluctuations on the source state of the direct jump: The nonthermal entropic contribution has indeed the form \( \Delta S^{\text{nt}}_{jk} = \ln(1-\eta_k) \). The additional, nonthermal contribution to the jump entropy flux is directly linked to the quantum fluctuation of the operators \( A_{ik} \), which shows the nonclassical character of our results. Note that, thanks to the same formal structure of the jumps and the drifts transition rates, these results hold true also for the drift parts of a quantum trajectory. In particular, during a drift there is no standard thermodynamic entropy flux as the heat flux vanishes. However, thanks to the purely quantum fluctuations of the operator \( \Omega_k = U^\dagger(t_k,t_{k-1})U(t_k,t_{k-1}) \) on the state \( |\psi_{k-1} \rangle \), the generic \( k \)-th drift part of the full process is also associated to a purely quantum information flux between system and environment.

**Integral Fluctuation Theorems.** We investigate the statistical properties of \( \sigma_f \) in Eq. \( \ref{eq:14} \) along a forward process. To simplify the notation, in what follows we introduce the symbols \( D^R_k[\psi_k] = ||U(t_k,t_{k-1})|\psi_{k-1}\rangle||^2 \) and \( D^D_k[\psi_k] = ||U(t_{k-1},t_k)|\psi_{k}\rangle||^2 \). Moreover, rates along forward or backward trajectories will be denoted by specifying the trajectory directly in the functional dependence of the rates on the wave function, so that for example \( R_k^{R(D)}[\psi_{k}^{(f(b))}] \) is the reversed (direct) \( k \)-th jump rate of the forward (backward) trajectory. With these notations, the mean value of \( e^{-\sigma_f} \) (commonly considered in FTs contexts) can be evaluated as
\[ \int d\sigma_f P(\sigma_f) e^{-\sigma_f} = \left< \prod_{k=1}^{N} \frac{R^R_k[|\chi_k\rangle]}{R^D_k[|\chi_k\rangle]} \frac{D^R_k[\psi_k]}{D^D_k[\psi_k]} \right> = 1 + \zeta_f, \]  
(14)

where \( \langle \cdot \rangle \) stands for an average over all possible realizations of a nonequilibrium process. Eq. \( \ref{eq:14} \) shows that, in the case of quantum trajectories, \( \langle e^{-\sigma_f} \rangle \) is not a universal constant: the r.h.s. is indeed, in general, different from 1 and depends on the set of Lindblad operators characterizing the unraveling scheme. This results in a quantum correction \( \zeta_f \) to the classical result. We expect such a correction to be positive: Since, as commented previously, \( \sigma_f \) is the difference between the information extracted only by measuring initial and final states of a trajectory and the information available by following the full quantum trajectory, it is reasonable to expect that the latter is greater than the former. Therefore, on average we have \( \langle \sigma_f \rangle < 0 \) which leads to \( \zeta_f > 0 \). This is illustrated in the Appendix, where we study the predictions of Eq. \( \ref{eq:14} \) numerically for several model system. In particular, such a correction originates from the fundamental difference between a "backward process" (which is a real dissipative process) and "reversed" processes (which is the collection of all reversed processes and, as such, is fictitious). Such a distance is nothing but the consequence of the measuring scheme employed to characterize trajectories: information acquired about the system by the external observer is not symmetric under time reversal, and such a broken symmetry of knowledge produces different states in forward and backward transitions. This physically results in the presence of the nonthermal quantum entropic flux \( \ref{eq:12} \), which does not obey a standard FT. Indeed it has recently been shown \( \ref{eq:14} \) that, if only thermal energy exchanges during jumps are taken into account along quantum trajectories of an open two-level system, the standard universal form of FT holds. As a matter of fact, in a "standard"–like limit the nonthermal entropy flux vanishes both for drifts and jumps due to the fact that the operators \( \Omega_k \) and \( \Lambda_{ik} \) have vanishing quantum fluctuations (see Appendix), and in this case \( \zeta_f = 0 \) recovering the universal standard form of FT.

**Conclusions.** We have obtained an expression for stochastic entropy production along a purely quantum trajectory of a driven open system, defined through continuous measurements on the environment only. The quantum entropy thus defined, which is fundamentally different from the commonly employed von Neumann entropy, describes the observer gain/loss of information.
about the open system along single realization on quantum dynamics. We showed that the flux of such an entropy is not only associated to energy flux from/to the bath, defying common classical thermodynamic expectations. The additional information term results from purely quantum fluctuations of the transition operators along a trajectory. Due to this additional term, the quantum entropy of a stochastic trajectory does not obey the usual form of integral fluctuation theorem: the quantum correction $\zeta_f$ in Eq. (14) depends on the set of jump operators employed to unravel the master equation, and ultimately describes the difference between the physical backward trajectory and the fictitious reversed processes. In other words, such a correction is due to the lack of symmetry between forward and backward processes, which in turn originates from the existence of an external observer performing measurements on the bath to detect transitions. It is worth stressing that, contrarily to previous approaches to quantum FTs, our formulation does not require any knowledge about the full ensemble dynamics of the open quantum system, given by the solution of the master equation [1].

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APPENDIX A. STOCHASTIC WAVE FUNCTION METHOD AND QUANTUM TRAJECTORIES

The stochastic wave function approach (see e.g. [30]) for a detailed description of the method), for a quantum system whose ensemble evolution is given by Eq (1) of the main manuscript, describes single realizations of a dissipative quantum process as trajectories composed of subsequent drift parts interrupted by jumps, each of which happens at random times and along a randomly chosen channel \{γ_i \k_i \} with rate \γ_i \|A_i \|ψ\|^2\].

On the other hand, the probability that, after jumping at time \(t_k\), the system performs no further transitions up to time \(t_{k+1}\) is \(|U(t_{k+1},t_k)\|ψ\|^2\]. Clearly the probability of a trajectory is given by the product of probabilities for each drift and jump as \(P[ψ,t] = P[ψ_0,t_0] \prod_{i=1}^N \gamma_i \|A_i \|ψ\|^2 dt \prod_{i=1}^{N+1} |U(t_{k+1},t_k)\|ψ\|^2\].

Fixing a particular trajectory from time \(t_0\) to time \(T\) amounts then to specifying a number of jumps \(N\) and a set of time instants \(\{t_k\} (k = 1, \ldots , N)\) such that \(t_0 < t_1 < \cdots < t_N < T = t_{N+1}\), at which the wave function jumps along the channels \{γ_i \k_i \}. As commented in the main manuscript, the backward trajectory is characterized by jumps through channels \{γ_i^\dagger \k_i^\dagger \} at the same time instants as the direct one. On the other hand, the action of the nonunitary operator \(U(t_f,t_s)\) on a state during the drift interval \([t_s,t_f]\) reduces its norm in time, describing the decrease of probability of the no-jump event. Therefore, as the backward process itself is a physical dissipative process detected by measurements, its associated drift operator \(U(t_f,t_s)\) has to describe such a decrease of probability along the backward drifts, taking into account that a backward drift propagates the state of the system from time \(t_f\) to time \(t_s\) such that \(t_f > t_s\). Therefore, the hermitian part of the operator generating backward evolutions has to be unchanged, but its nonhermitian part has to be sign-reversed: This is achieved if one defines \(U(t_f,t_s) = U^\dagger(t_s,t_f)\). This means, however, that the final state of the backward process may be different from the initial state of the forward one, as in general \(A_i^\dagger A_i \neq I\) and \(U(t_N,T)U(T,t_N) \neq I\). The system state along a trajectory is always described by a wave function. As moreover the trajectory is a PDP, the wave function \(|ψ\rangle\) of the system is by all means a measurable variable, with associated probability distribution \(P[ψ,t]\).

The forward trajectory starts from a fixed state \(|ψ_0\rangle\) with probability \(P[ψ_0,t_0]\) and ends after a time \(τ\) in a random state \(|ψ_\tau(T)\rangle\), where \(T = t_0 + τ\). Analogously, the backward one starts from \(|ψ_\tau(T)\rangle\) with probability \(P[ψ_\tau(T),T]\) and ends in \(|ψ_\tau^\dagger(t_0)\rangle\). Clearly, the forward and backward trajectories are connected by the requirement that \(|ψ_\tau^\dagger(t_0)\rangle = |ψ_\tau^\dagger(T)\rangle\), i.e. the initial state of the backward process coincides with the final state of the forward one, but in general \(|ψ_\tau^\dagger(t_0)\rangle \neq |ψ_\tau^\dagger(T)\rangle\) due to the fact that \(A_i^\dagger A_i \neq I\) and \(U(t_N,T)U(T,t_N) \neq I\).

APPENDIX B. THE STANDARD CASE: JUMPS BETWEEN FREE HAMILTONIAN EIGENSTATES

As an example of the standard limit of our results, in the main manuscript we mentioned an open system without driving, whose Lindblad operators and decay rates remain constant in time. In the Markovian and weak coupling limit, its Lindblad operators satisfy \(\{H_S, A_i^\dagger A_i \} = ±ε_k A_i^\dagger A_i\). If now we assume the free Hamiltonian \(H_S\) to have nondegenerate energy gaps in its spectrum, the emission of an energy quantum \(ε_k\) is in a one-to-one correspondence with a transition between well defined energy levels \(|n\rangle\) and \(|m\rangle\) such that \(ω_n - ω_m = ε_k\), \(ω_i\) being the energy associated to the eigenstate \(|i\rangle\) of \(H_S\).

Assuming the spectrum of \(H_S\) is composed of \(N\) discrete levels (|1\rangle being the ground state) of increasing energy, the natural choice for the set of Lindblad operators is then

\[
A_{N(i-1)+j} = |i\rangle\langle j| \quad \text{for} \quad 1 \leq i < j \leq N, \quad (15)
\]

\[
A_{N(i-1)+j}^\dagger = |j\rangle\langle i| \quad \text{for} \quad 1 \leq i < j \leq N. \quad (16)
\]

Note that, thanks to the assumption of nondegenerate gaps in \(H_S\) and the form of the operators in the set \{\(A_k\}\}, it is not necessary for the system to start its trajectory in an eigenstate of \(H_S\) since after the first jump any wave function \(|ψ\rangle\) is projected to a well defined energy eigenstate. We can therefore assume, without loss of generality, that the system starts its trajectory from a generic yet fixed energy eigenstate \(|n\rangle\). The action of a jump operator \(|m\rangle\langle n|\) on such a state is then nothing but the transition \(|n\rangle \rightarrow |m\rangle\). The system performs then jumps only between eigenstates of its free Hamiltonian. Exploiting Eqs. (15) and (16), one notices that \(A_i^\dagger A_j = |i\rangle\langle j|\), so that the drift non-Hermitian Hamiltonian becomes \(H_{eff} = H_S - \frac{i}{2} \sum_{i} \bar{γ}_i |i\rangle\langle i|\), with \(\bar{γ}_i = \sum_{j} γ_{N(i-1)+j} \gamma_{N(i-1)+j}^\dagger\) the total relaxation rate associated with the energy level \(|i\rangle\). The drift operator \(U(t_k,t_{k-1})\) is then diagonal in the eigenbasis of \(H_S\) and introduces nothing but a phase factor to any evolving energy eigenstates. Any trajectory of this kind is equivalent to a pure jump process between energy eigenstates. We note two things: On the one hand, since the emission or absorption of an energy quantum always connects the same two states, and since drifts have no effects on the trajectory, backward and reversed processes are the same and the backward trajectory connects the same states as the forward one, but in reversed order. This in turn means that the quantum correction \(ζ_f\) in Eq. (14) of the main manuscript vanishes, and one recovers the standard form of fluctuation theorems. On the other hand, as expected, this
is due to the fact that nonthermal entropic fluxes are zero, since it can be straightforwardly shown that neither the operators $A_N^{j(i-1)+j-i(i+z)} = |j\rangle \langle j|$ nor the operators $U(t_k, t_{k-1})U(t_k, t_{k-1}) = \sum^N e^{-\gamma_i(t_k-t_{k-1})}|i\rangle \langle i|$ have purely quantum fluctuations on any energy eigenstates, as they are diagonal in such a basis. The process is thus, in this respect, fully classical.

APPENDIX C. DRIVEN TWO-LEVEL ATOM

As a simple, more interesting example of our results, we unravel the dynamics of a driven two-level atom $|e\rangle$ and $|g\rangle$ being, respectively, its excited and ground state under two different unraveling schemes somehow analogous to, respectively, the one describing a direct photodetection of emitted light and the so-called homodyne photodetection. We assume that the atom interacts with a reservoir of field modes which is not at zero Kelvin. The atom master equation, in the form of Eq. (1) of the main manuscript, is given by

$$\dot{\rho}(t) = -i\frac{\omega(t)}{2}[\sigma_z, \rho(t)] + \gamma_1(t)\left(\sigma_-\rho(t)\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho(t)\}\right) + \gamma_2(t)\left(\sigma_+\rho(t)\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho(t)\}\right),$$

where $\omega(t)$ accounts for the applied external driving, $\sigma_- = \sigma^\dagger_+ = |g\rangle\langle e|$ is the lowering operator of the atom and the rates $\gamma_1(t)$ and $\gamma_2(t)$ depend on the atom-field coupling parameter, on the structure of the state of the field and on its spectrum. Note that, as long as $\gamma_1(t), \gamma_2(t) \geq 0 \forall t$, Eq. (17) always implements a time-dependent Markovian dynamics. The “direct photodetection-like” unraveling yields two jump operators of the form

$$A_1 = \sigma_-,$$
$$A_2 = \sigma_+ = A_1^\dagger,$$

describing respectively emission and absorption of a quantum of light by the atom, with relaxation rates $\gamma_1(t)$ (emission) and $\gamma_2(t)$ (absorption).

On the other hand another suitable set of Lindblad operators, similar to the ones describing the “homodyne” photodetection process, is given by

$$A_1^\dagger (\beta) = \sigma_- - i\beta,$$
$$A_2^\dagger (\beta) = \sigma_+ + i\beta,$$
$$A_1^\dagger (\beta) = \sigma_- - i\beta^* = A_1^\dagger (\beta)^\dagger,$$
$$A_2^\dagger (\beta) = \sigma_+ + i\beta^* = A_1^\dagger (\beta)^\dagger,$$

for any $\beta \in \mathbb{C}$. The associated relaxation rates are $\gamma_1^\pm (t) = \frac{\gamma(t)}{2}$ and $\gamma_2^\pm (t) = \frac{\gamma(t)}{2}$. Note that the transformation of Lindblad operators leading to the set (20)-(23) produces no changes in the Hamiltonian part thanks to the fact that $A_1^\dagger (\beta) + A_1^\dagger (\beta) = 2A_1$ and $A_2^\dagger (\beta) + A_2^\dagger (\beta) = 2A_2$. It is easy to check that the master equation obtained using the four operators (20)-(23) reduces, for any $\beta$, to Eq. (17), therefore describing the same physical process on the ensemble level. Fixing $\beta$ one fixes a particular measuring scheme and, therefore, a particular set of Lindblad operators. In this way we are able, just by switching between the two sets (18),(19) and (20)-(23) and/or by tuning $\beta$, to investigate the dependence of $\zeta_f$ (Eq. (12) of the main manuscript) on the unraveling scheme employed.

We have performed simulations for the “direct photodetection-like” scheme, and for the “homodyne-like” scheme with different values of $\beta$, with fixed measurement step $\delta t$ and total time duration $T$, choosing $\omega(t) = \omega_0(1 - e^{-\frac{t}{T}})$, $\gamma_1(t) = g_1 e^{-\frac{t}{T}}$ and $\gamma_2(t) = g_2(1 - e^{-\frac{t}{T}})$. The parameters have been fixed such that $\delta t_1 = 1.3 * 10^{-3}$, $\delta t_2 = 10^{-3}$, $\delta t_3 = 2.7 * 10^{-3}$ and $\delta t_4 = 8 * 10^{-4}$. The initial atomic wave function is of the form $|\psi(t)\rangle = c_e|e\rangle + c_g|g\rangle$ and, for each trajectory, the complex values for $c_e$ and $c_g$ have been chosen randomly out of a uniform distribution of real values in $[0,1]$ for their modulus and of a uniform distribution of real angles in $[0,2\pi]$ for their relative phase. The results of these simulations are shown in Figs. 2 and 3, where $\langle e^{-\sigma t} \rangle$, evaluated as an average on $10^4$ quantum trajectories, is shown for, respectively, 10 different sets of values of rates and driving such that $\delta t = 0.8 * 10^{-4}$, $\delta t = 8 * 10^{-4}$, $\delta t = 2.7 * 10^{-3}$ and $\delta t = 8 * 10^{-4}$. The results of these simulations are shown in Figs. 2 and 3 where $\langle e^{-\sigma t} \rangle$, evaluated as an average on $10^4$ quantum trajectories, is shown for, respectively, 10 different sets of values of rates and driving such that $\delta t = 0.8 * 10^{-4}$, $\delta t = 8 * 10^{-4}$, $\delta t = 2.7 * 10^{-3}$ and $\delta t = 8 * 10^{-4}$.

Finally, we analyze the more familiar case in which the values of decay rates are determined by environmental properties only, i.e. the case of a thermal bath weakly interacting with the system: Fig. 4 shows results for time independent relaxation rates $\gamma_1 \propto \langle N \rangle + 1$ and $\gamma_2 \propto \langle N \rangle$, $\langle N \rangle$ being the average photon number in the field state. This further run of simulations, consisting of $3 \times 10^4$ for each point in the plot, has been performed analogously to the one reported in Fig. 2, keeping all the parameters fixed at the same value characterizing Fig. 2 with the only exception of $\omega_0$ which has been fixed such that $\omega_0 \delta t = 8 * 10^{-4}$ and, of course, the rates $\gamma_1$ and $\gamma_2$. The simulations have been performed for 8 different values of $\langle N \rangle$ such that $\langle N \rangle = 0.2 + 0.3k$, $k = 0, \ldots, 7$. Note that constant relaxation rates obeying $\gamma_1 = \gamma_2 + 1$ are obtained for a reservoir of modes in thermal equilibrium at fixed temperature, in which case $\gamma_2$ is proportional.
Two interesting features emerge from these simulations: first of all, the mean value $\langle e^{-\sigma_f} \rangle$ can be strongly different from 1 both for “direct photodetection-like” and “homodyne-like” schemes, resulting in a nonzero quantum correction $\zeta_f$. Therefore, even for such a simple system the difference between backward trajectory and reversed processes becomes nonneglegible. Secondly, $\langle e^{-\sigma_f} \rangle$ shows a clear dependence on the set $\{\omega_0, g_1, g_2\}$, on the bath average photon number $\langle N \rangle$ and on $|\beta|$, i.e. on the driving and decaying strengths, on the bath temperature and on the unraveling scheme employed. In particular, in the “direct photodetection-like” scheme $\langle e^{-\sigma_f} \rangle$ is very close to 1 in the case of a weakly decaying and driven system ($k = 1$) and increases smoothly with $k$ with a power-law like shape. Also in the case of the homodyne-like scheme a clear increasing trend is detected which suggests a monotonic increase of $\langle e^{-\sigma_f} \rangle$ with $|\beta|$, properly described by a quadratic function of $|\beta|^2$. Finally, it is interesting to note that, in the case of a thermal bath, $\langle e^{-\sigma_f} \rangle$ increases quadratically with the average photon number $\langle N \rangle$ but does not tend to 1 for $\langle N \rangle \to 0$, since also in the case of a zero temperature bath the system can perform quantum jumps and undergoes nontrivial drifts, resulting in a nonvanishing nonthermal entropy flux. These features may reasonably be employed to properly engineer a class of nonequilibrium processes with particular stochastic properties of entropy production.

to the average number of photons in the thermal state of the field. Tuning $\langle N \rangle$ in our simulations, therefore, corresponds to tuning the temperature of the field with which the two-level atom interacts (provided its spectrum stays constant).