Information Theoretic Exemplification of the Impact of Transmitter-Receiver Cognition on the Channel Capacity

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Abstract—In this paper, we study, information theoretically, the impact of transmitter and/or receiver cognition on the channel capacity. The cognition can be described by state information, dependent on the channel noise and/or input. Specifically, as a new idea, we consider the receiver cognition as a state information dependent on the noise and we derive a capacity theorem based on the Gaussian version of the Cover-Chiang capacity theorem for two-sided state information channel. As intuitively expected, the receiver cognition increases the channel capacity and our theorem shows this increase quantitatively. Also, our capacity theorem includes the famous Costa theorem as its special cases.

Index Terms—transmitter-receiver cognition, Gaussian channel capacity, correlated side information.

I. INTRODUCTION

Information theoretic study of the impact of transmitter and/or receiver cognition on the channel capacity is a new idea and an important research issue. For example, one channel from view points of two receivers with different cognition and information on the channel, may have different capacities. The cognition at the transmitter or receiver can be described by the usual concept of information theory i.e., side information.

Side information channels have been extensively studied since the initiation by Shannon [1] and the subsequent study by Kusnetsov-Tsybakov [2]. The capacity of channel with side information (CSI) known causally only at the transmitter and only at the receiver has been determined by Gel’fand-Pinsker (GP) [3] and Heegard-El Gamal [4] respectively. Considering the GP theorem for the Gaussian channel, Costa [5] obtained an interesting result, i.e., the channel capacity in the presence of interference known at the transmitter is the same as the case without interference. Having extended the above results, Cover-Chiang [6] established a general capacity theorem for the channel with two-sided state information. We have many other important researches in the literature, e.g. [7]–[9]. The results obtained for side information point channel have been extended, at least at special cases, to multiuser channels [10]–[14].

As mentioned above, our motivation was the fact that cognition of the transmitter and receiver can affect the channel capacity. In order to quantify this effect, we illustrate the cognition as state information dependent on the channel noise and/or input. Then we derive a capacity theorem and prove that, as expected, the receiver cognition increases the channel capacity and our theorem shows this increase quantitatively. Our capacity theorem, while revealing the importance of Costa theorem, is a more general theorem and includes the Costa theorem as special cases.

In the remainder of this section, we briefly review the Cover-Chiang, the Gel’fand-Pinsker and the Costa theorems.

Cover-Chiang Theorem: Fig. 1 shows a channel with side information available non-causally at the transmitter and at the receiver.

Fig. 1. Channel with side information available non-causally at the transmitter and at the receiver.

\[
C = \max_{p(u,x|s_1)} [I(U; S_2, Y) - I(U; S_1)] \tag{1}
\]

where the maximum is over all distributions:

\[
p(y, x, u, s_1, s_2) = p(y|x, s_1, s_2) p(u|x) p(s_1) p(s_2) \tag{2}
\]

and \(U\) is an auxiliary random variable for conveying the information of the known \(S_1^n\) into \(X^n\).

It is important to note that the Markov chain:

\[
S_2 \rightarrow S_1 \rightarrow UX \tag{3}
\]

is satisfied for all above distributions.

Gel’fand-Pinsker Theorem: The situation \(S_2 = \phi\) (no side information at the receiver) leads to the Gel’fand-Pinsker theorem [3]: The memoryless channel with transition probability \(p(y|x, s_1)\) and the side information sequence \(S_1^n\) (which is
Encoder \( X^n \) \( p(y|x,s_1) \) Decoder \( Y^n \) \( \hat{W} \)

Fig. 2. Channel with side information known at the transmitter.

\[ W \rightarrow X^n \rightarrow Y^n \rightarrow \hat{W} \]

\[ S_1^n \]

\[ Z^n \]

Fig. 3. Gaussian channel with additive interference known at the transmitter.

\[ W \rightarrow X^n \rightarrow Y^n = X^n + S_1^n + Z^n \rightarrow \hat{W} \]

**Definition of the Channel**

Consider the Gaussian channel depicted in Fig. 2. Our channel is defined with properties D.1-D.3 below:

**D.1:** \( (S_1^n, S_2^n) \) are i.i.d. sequences with zero mean and jointly Gaussian distributions.

**D.2:** Random variables \( (X, S_1, S_2) \) have the covariance matrix \( K \):

\[
K = \begin{bmatrix}
\sigma_X^2 & \sigma_X \sigma_{S_1} \rho_{X S_1} & \sigma_X \sigma_{S_2} \rho_{X S_2} \\
\sigma_{X S_1} \rho_{X S_1} & \sigma_{S_1}^2 & \sigma_{S_1} \sigma_{S_2} \rho_{S_1 S_2} \\
\sigma_{X S_2} \rho_{X S_2} & \sigma_{S_1} \sigma_{S_2} \rho_{S_1 S_2} & \sigma_{S_2}^2
\end{bmatrix}.
\]

We suppose that \( S_2 \) is independent of \( X \) and \( S_1 \), so we have \( \rho_{X S_2} = \rho_{S_2 S_1} = 0 \). Moreover \( X^n \) is assumed to have the constraint \( E \{ X^2 \} = \sigma_X^2 \leq P \). All values in \( K \) except \( \sigma_X \), are fixed and must be considered as the definition of the channel.

**D.3:** The output sequence \( Y^n = X^n + S_1^n + S_2^n + Z^n \), where \( Z^n \) is the sequence of white Gaussian noise with zero mean and power \( \sigma_Z^2 = N \) \( (Z \sim N(0,N)) \) and independent of \( (X,S_1) \) and dependent on \( S_2 \) with \( \rho_{S_2 Z} \). For simplicity, we define:

\[
L_2 \triangleq E \{ S_2 Z \} = \sigma_{S_2} \sigma_Z \rho_{S_2 Z}.
\]

**D.4:** \( (X, U, S_1, S_2) \) form the Markov Chain \( S_2 \rightarrow S_1 \rightarrow U \). (We note that as mentioned earlier, this Markov chain [5] must be satisfied by all distributions \( p(y, x, u, s_1, s_2) \) in Cover-Chiang capacity theorem and is physically acceptable.) It is readily seen that all distributions \( p(y, x, u, s_1, s_2) \) specified with D.1-D.4 are in the form of [6] and hence we can use the extended version of Cover-Chiang theorem to random variables with continuous alphabets about the capacity of this channel.

**II. A Capacity Theorem for Analyzing the Impact of Transmitter-Receiver Cognition on Channel Capacity**

In this section we define and investigate a Gaussian channel in presence of two-sided information known non-causally at the transmitter and at the receiver. The side information at the transmitter and at the receiver is considered as additive interference at the receiver (Fig. 3). In comparison with Costa channel, our channel has two major modifications: 1) In Costa channel there is no condition for the correlation between the channel input \( X \) and the side information \( S_1 \). So \( \frac{1}{2} \log (1 + \frac{P}{N}) \) is the capacity of a channel in which the side information \( S_1 \) can be freely correlated to the channel input \( X \); so this capacity can not be used for a channel with a specific correlation between \( X \) and \( S_1 \). The correlation coefficient \( \rho_{X S_1} \) between \( X \) and \( S_1 \) is specified in our channel. 2) We suppose that the Gaussian side information \( S_2 \) known at the receiver, exists and is correlated to the channel noise \( Z \).

It is important to note that assuming the input random variable \( X \) and \( S_1 \) correlated to each other with a specific correlation coefficient, does not impose any restriction on \( X \)’s own distribution and the distribution of \( X \) is still free to choose.
Comparing our channel (defined with D.1-D.4) with Costa channel, a question may arise: (if we ignore $S_2$) what is the relationship between capacities of these channels? To answer this question let us consider a subset of all distributions (channels) $p(y, x, u, s_1, s_2)$ that satisfy D.1-D.4 and are similar but with different $\rho_{X,S_1}$. Since Costa channel imposes no restriction on $\rho_{X,S_1}$, these channels differ from the corresponding Costa channel on the restricted $\rho_{X,S_1}$. It is clear that searching for the capacity of the Costa channel is led to the maximum capacity in this subset. So if $C_D$ be the capacity of the channel defined with D.1-D.4, and $C$ be the capacity of the Costa channel, we can write:

$$C = \max_{\rho_{X,S_1}, \rho_{S_2|Z}} C_D.$$  \hspace{1cm} (9)

We will show that the situation that $(X, S_1, S_2)$ are jointly Gaussian and the auxiliary random variable $U$ is designed as linear combination of $X$ and $S_1$, is optimum and maximizes the transmitting rate. So we consider an important subset of the distributions $p(y, x, u, s_1, s_2)$ defined in D.1-D.4, as the set of all $p^*(y, x, u, s_1, s_2)$ that have the properties D.5 and D.6 below, in addition to D.1-D.4 (although the channel is defined only with D.1-D.4):

D.5: Random variables $(X, S_1, S_2)$ are jointly Gaussian distributed. $X$ is with zero mean and has the maximum power of $P$ (so $X \sim \mathcal{N}(0, P)$). Naming the covariance matrix in this special case as $K^*$, for simplicity, by defining $A_1 \triangleq E \{XS_1\}$, $Q_1 \triangleq \sigma_{S_1}^2$, and $Q_2 \triangleq \sigma_{S_2}^2$, we rewrite:

$$K^* = \begin{pmatrix} P & A_1 & 0 \\ A_1 & Q_1 & 0 \\ 0 & 0 & Q_2 \end{pmatrix}.$$ \hspace{1cm} (10)

D.6: Following Costa, we consider $U$ in the form of linear combination of $X$ and $S_1$ as $U = \alpha S_1 + X$.

For summarizing expressions, we define two following symbols:

$$d_{Q_1} \triangleq PQ_1 - A_1^2 = \sigma_{S_1}^2 \sigma_{S_2}^2 (1 - \rho_{X,S_1}^2),$$
$$d_{PQ_1} \triangleq Q_2 N - L_2^2 = \sigma_{S_2}^2 \sigma_{Z}^2 (1 - \rho_{S_2,Z}^2).$$ \hspace{1cm} (11) \hspace{1cm} (12)

**Capacity of the Channel**

**Theorem 1:** The Gaussian channel defined with properties D.1-D.4 has the capacity

$$C_D = \frac{1}{2} \log \left( 1 + \frac{P \left( 1 - \rho_{X,S_1}^2 \right)}{N \left( 1 - \rho_{S_2,Z}^2 \right)} \right).$$ \hspace{1cm} (13)

**Corollary 1:** As mentioned earlier, by (13) we can obtain Costa capacity by assuming $\rho_{S_2,Z} = 0$ and maximizing $C_D$ with $\rho_{X,S_1} = 0$.

**Corollary 2:** It is seen that if the side information $S_2$ is independent of the channel noise $Z$ (and so $\rho_{S_2,Z} = 0$), the capacity of the channel is equal to the capacity when there is no interference $S_2$. In other words, in this case the receiver can subtract the known $S_2^2$ from the received $Y^n$ without losing any worthy information. But when the state information $S_2$ is correlated with additive noise $Z, S_2$ is containing worthy information that increases the capacity, and hence subtracting $S_2$ is a wrong decoding strategy.

**Corollary 3:** It is seen that while, as intuitively expected, correlation between $S_2$ and $Z$ increases the capacity, the correlation between $X$ and $S_1$ decreases it.

**Proof of Theorem 1:** To prove the theorem, we first show that $C_D$ (13) is a lower bound for the capacity of the channel, then we show that $C_D$ is an upper bound for the capacity too, so $C_D$ is the capacity of the channel.

**Achievability part of the proof:** we use the extended version of Cover-Chiang capacity (1) to obtain a lower bound for the capacity of the channel: For all distributions $p(y, x, u, s_1, s_2)$ (with properties D.1-D.4) and its subset $p^*(y, x, u, s_1, s_2)$ (defined with properties D.1-D.6), we can write:

$$C_D = \max_{p(u,x|s_1)} \left[ I(U;Y, S_2) - I(U; S_1) \right] \geq \max_{p^*(u,x|s_1)p^*(x|s_1)} \left[ I(U;Y, S_2) - I(U; S_1) \right] = \max_{\alpha} \left[ I(U;Y, S_2) - I(U; S_1) \right] \triangleq \max_{\alpha} R_D(\alpha) = R_D^*(\alpha^*) \hspace{1cm} (14)$$

So $R_D^*(\alpha^*)$ is a lower bound for the capacity of the channel. To compute $R_D(\alpha)$ we write (details of computations are omitted for the brevity):

$$I(U;Y, S_2) = H(U) + H(Y, S_2) - H(U, Y, S_2),$$
$$I(U; S_1) = H(U) + H(S_1) - H(U, S_1),$$ \hspace{1cm} (15) \hspace{1cm} (16)

where:

$$H(Y, S_2) = \frac{1}{2} \log \left( (2\pi e)^2 \det (\text{cov}(Y, S_2)) \right),$$
$$H(U, Y, S_2) = \frac{1}{2} \log \left( (2\pi e)^2 \left( Q_2(P + Q_1 + 2A_1) + d_{PQ_1} \right) + (\alpha - 1)^2 Q_2 d_{Q_2} \right),$$
$$H(S_1) = \frac{1}{2} \log ((2\pi e) Q_1),$$
$$H(U, S_1) = \frac{1}{2} \log (2\pi e d_{Q_2}).$$ \hspace{1cm} (17) \hspace{1cm} (18) \hspace{1cm} (19) \hspace{1cm} (20)

Substituting (17)-(20) in (18) and (19), we obtain:

$$R_D^*(\alpha^*) = \frac{d_{Q_2} d_{Q_2} - A_1 d_{PQ_1}}{Q_2 d_{Q_2} - Q_1 d_{PQ_1}},$$ \hspace{1cm} (21)

and after maximizing it over $\alpha$, we conclude:

$$\alpha^* = \frac{Q_2 d_{Q_2} - A_1 d_{PQ_1}}{Q_2 d_{Q_2} - Q_1 d_{PQ_1}}.$$ \hspace{1cm} (22)

Now, if we compute $R_D^*(\alpha^*)$ by putting (21) into (24) and then rewrite the resulted expression in terms of $\sigma_X, \sigma_{S_1}, \sigma_{S_2}, \rho_{X,S_1}, \rho_{S_2,Z}$ by (8) and (10)-(12), we finally conclude:

$$R_D(\alpha^*) = \frac{1}{2} \log \left( 1 + \frac{P \left( 1 - \rho_{X,S_1}^2 \right)}{N \left( 1 - \rho_{S_2,Z}^2 \right)} \right).$$ \hspace{1cm} (23)
Converse part of the proof: For all distributions \( p(y, x, u, s_1, s_2) \) defined with properties D.1-D.4, we have:

\[
I(U; Y, S_2) - I(U; S_1) = -H(U | Y, S_2) + H(U | S_1) \\
\leq I(X; Y | S_1, S_2)
\]

where (27) follows from Markov chains \( S_2 \rightarrow S_1 \rightarrow U \) and \( U \rightarrow XS_1S_2 \rightarrow Y \), which are true for all distributions defined with properties D.1-D.4. Now from [1] and (27) we can write:

\[
C = \max_{p(u, x \mid S_1)} [I(U; Y, S_2) - I(U; S_1)] \leq \max_{p(x \mid S_1)} [I(X; Y | S_1, S_2)] = \mathcal{I}^*(X; Y | S_1, S_2),
\]

(28)

(29)

hence \( \mathcal{I}^*(X; Y | S_1, S_2) \) is an upper bound for the capacity of the channel. For computing it we write:

\[
I(X; Y | S_1, S_2) = H((X + Z), S_1, S_2) - H(S_1, S_2) - H(Z | S_2).
\]

(30)

So when (29) reaches to its maximum, \( (X, S_1, S_2) \) are jointly Gaussian and \( X \) has its maximum power of \( P \) and it means that \( \mathcal{I}^*(X; Y | S_1, S_2) \) is the value of (30) which is computed for distributions \( p^*(y, x, s_1, s_2) \) defined with properties D.1-D.6. After computing we have:

\[
H((X + Z), S_1, S_2) = \frac{1}{2} \log \left( 2\pi e^3 (Q_2 d_{Q_2} + Q_1 d_{PQ_1}) \right)
\]

(31)

\[
H(S_1, S_2) = \frac{1}{2} \log \left( 2\pi e^2 Q_2 \right)
\]

(32)

\[
H(Z, S_2) = \frac{1}{2} \log \left( 2\pi e^2 d_{PQ_1} \right)
\]

(33)

so we obtain from (30)-(33):

\[
\mathcal{I}^*(X; Y | S_1, S_2) = \frac{1}{2} \log \left( Q_2 d_{Q_2} + Q_1 d_{PQ_1} \right)
\]

(34)

\[
= \frac{1}{2} \log \left( 1 + \frac{P(1 - \rho_{X,S_1}^2)}{N(1 - \rho_{S_2,Z}^2)} \right)
\]

(35)

where (35) follows by rewriting (34) in terms of \( \sigma_X, \sigma_{S_1}, \sigma_{S_2}, \rho_{X,S_1}, \rho_{S_2,Z} \) by (8) and (10)-(12).

From (26) and (35), we conclude that \( C_D \) (13) is the capacity of the channel.

\[\square\]

III. Numerical Results

Fig. 5 illustrates the impact of the correlation of \( S_2 \) and the channel noise \( Z \) on the channel capacity. Figure plotted for independent \( X \) and \( S_1 \) (so \( \rho_{X,S_1} = 0 \)). It is seen that the more \( S_2 \) depends on the noise \( Z \), the larger capacity of channel is. On the condition of full dependency \( \rho_{S_2,Z} = \pm 1 \) the capacity of channel is infinite.

IV. Conclusion

We investigated the Gaussian channel in the presence of two-sided state information with dependency on the input and the channel noise. Having established a capacity theorem for the channel, we illustrated the impact of the receiver cognition (the correlation between the channel noise and state information known at the receiver) and the correlation between the input and the state information known at the transmitter, on the capacity of the channel.

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