A FAMILY OF ENTROPY-CONSERVATIVE FLUX FUNCTIONS FOR THE EULER EQUATIONS*

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Abstract. Entropy-conservative numerical flux functions can be used to construct high-order, entropy-stable discretizations of the Euler and Navier-Stokes equations. The purpose of this short communication is to present a novel family of such entropy-conservative flux functions. The proposed flux functions are solutions to quadratic optimization problems and admit closed-form, computationally affordable expressions. We establish the properties of the flux functions including their continuous differentiability, which is necessary for high-order discretizations.

Key words. stability, entropy, high-order, summation-by-parts

AMS subject classifications. 65M06, 65M60, 65M12

1. Introduction. Stability is a necessary property for practical discretizations of the time-dependent Euler and Navier-Stokes equations. While there are different ways of defining stability in the context of computational fluid dynamics, the focus here is on nonlinear entropy stability. Entropy stability is a valuable property to build into a discretization, because a bound on entropy implies an $L^2$ bound on the state variables [6, 18].

Although entropy stability has a long history [10, 19, 11, 20, 14, 1], the work of Fisher and Carpenter [7, 8], and their collaborators [9, 2], has generated renewed interest in the subject. Using summation-by-parts (SBP) finite-difference operators [8], Fisher and Carpenter constructed high-order entropy-stable semi-discretizations that have the potential to be both highly accurate and robust. While their initial work was with tensor-product finite-difference operators, the approach has since been generalized to include any operators with the SBP property and diagonal mass matrices [2, 4, 5].

Entropy-stable SBP discretizations rely on two-point entropy-conservative (EC) flux functions, and such flux functions are the focus of this work. The earliest EC flux function was presented by Tadmor [20]; however, because it relies on a path integral through phase space, it has been considered too computationally expensive for use in practice. The numerical flux of Ismail and Roe [12] was arguably the first practical EC flux. More recently, Chandrashekar [3] proposed an EC flux that also conserves kinetic energy (KE) in the sense of Jameson [13]. In [16], Ranocha presented a general procedure for constructing EC flux functions based on defining suitable differential mean values.

In this short communication, we present a family of affordable EC flux functions based on the solution of a quadratic minimization problem. Different flux functions are obtained by different choices of the target flux and the matrix that defines the norm. To the best of our knowledge, this construction is conceptually novel compared to existing EC fluxes.

The minimization problem and the resulting flux are described in Section 2, where

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we also show that the proposed flux is $C^1$ continuous. Some implementation details are discussed in Section 3.1, and numerical verifications are presented in Section 3.2.

2. Construction and analysis of the flux functions.

2.1. Preliminaries. Consider the time-dependent Euler equations:

\begin{equation}
\frac{\partial u}{\partial t} + \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} = 0,
\end{equation}

where $u = [\rho, \rho v_x, \rho v_y, \rho v_z, e]^T$ is the vector of conservative variables: density, $\rho$; momentum per unit volume, $[\rho v_x, \rho v_y, \rho v_z]^T$; and total energy per unit volume, $e$. The inviscid fluxes are denoted by $f_x, f_y, f_z$. The flux in the $x$ direction is given by

\[ f_x(u) = [\rho v_x, \rho v_x^2 + p, \rho v_y v_x, \rho v_z v_x, (e + p)v_x]^T, \]

where $p = (\gamma - 1)(e - \rho V^2/2)$ is the pressure, with $V^2 \equiv v_x^2 + v_y^2 + v_z^2$. The fluxes in the $y$ and $z$ coordinate directions are defined similarly.

We define the (mathematical) entropy as

\[ U(u) = -\frac{\rho S}{\gamma - 1}, \quad S(u) = \ln \left( \frac{p}{\rho^\gamma} \right), \]

where $S$ is the physical entropy and $\gamma$ is the ratio of specific heats. While other choices of entropy can be used to build entropy-stable discretizations of the Euler equations, the above definition is the only one that also symmetrizes the viscous terms in the Navier-Stokes equations.

The entropy variables are defined by the gradient of $U$ with respect to the conservative variables:

\[ w(u) \equiv (\nabla_u U)^T = \begin{bmatrix} \gamma - S & \rho v_x & \rho v_y & \rho v_z & -p \end{bmatrix}^T. \]

Above, and in the following, we adopt the convention that the gradient of a scalar is a row vector; otherwise, all vectors are assumed to be column vectors.

Taking the integral inner product between (2.1) and the entropy variables $w$ over a given volume $\Omega$ produces a conservation statement for $U$ (see [21] for the details):

\begin{equation}
\frac{d}{dt} \int_{\Omega} U \, d\Omega + \int_{\partial \Omega} (F_x n_x + F_y n_y + F_z n_z) \, d\Gamma = 0,
\end{equation}

where $[F_x, F_y, F_z]^T = [v_x U, v_y U, v_z U]^T$ are the entropy fluxes. When considering the unique weak solution to (2.1), the equality in (2.2) becomes an inequality implying that the total entropy is non-increasing in time. It is (2.2), or its inequality variant, that entropy-stable discretizations seek to mimic in order to bound the entropy.

We will make use of a couple relations involving the Euler fluxes, entropy variables, and entropy fluxes. The first relation is that [21]

\[ w(u)^T f_x(u) = F_x(u) + \psi_x(u), \]

where $\psi_x(u) = \rho v_x$ is the potential flux. The second relation is

\[ \nabla_u F_x = w^T \nabla_u f_x. \]
We conclude this section with the definition of an entropy-conservative (EC) flux function.

**Definition 2.1 (Entropy-conservative flux function).** The function \( \tilde{f}_x : \mathbb{R}^5 \to \mathbb{R}^5 \) is an entropy-conservative numerical flux function in the \( x \) direction if it satisfies the following three properties.

1. \( \tilde{f}_x \) is symmetric in its arguments: \( \tilde{f}_x(u^-, u^+) = \tilde{f}_x(u^-, u^+) \).
2. \( \tilde{f}_x \) is consistent: \( \tilde{f}_x(u, u) = f_x(u) \).
3. \( \tilde{f}_x \) satisfies the entropy-conservation condition

\[
(w^+ - w^-)^T \tilde{f}_x(u^-, u^+) = \psi_x^+ - \psi_x^-,
\]

where \( w^- = w(u^-), w^+ = w(u^+), \psi_x^- = \psi_x(u^-), \) and \( \psi_x^+ = \psi_x(u^+) \).

Similar definitions apply in the \( y \) and \( z \) directions.

**2.2. Optimization-based entropy-conservative flux.** The proposed EC flux, which we will denote by \( \tilde{f}_x \), is defined as the solution to the following quadratic optimization problem.

\[
\begin{align*}
\min_{\tilde{f}_x} \quad & \frac{1}{2} \| \tilde{f}_x - \bar{f}_x \|_{M^{-1}}^2, \\
\text{s.t.} \quad & \Delta w^T \tilde{f}_x = \Delta \psi_x,
\end{align*}
\]

where \( \Delta w \equiv w^+ - w^- \) and \( \Delta \psi_x = \psi_x^+ - \psi_x^- \). The function \( \tilde{f}_x = \tilde{f}_x(u^-, u^+) \) is any symmetric and consistent flux function, e.g. an average of the Euler flux.

We will refer to \( \tilde{f}_x \) as the target flux, since the optimization problem \((2.4)\) seeks the numerical flux \( \tilde{f}_x \) that is as “close” as possible to \( f_x \) while satisfying the entropy-conservation condition \((2.3)\). Here, “close” is defined by the norm

\[ \| y \|_{M^{-1}}^2 \equiv y^T M^{-1} y, \]

where \( M^{-1} \) is a symmetric positive-definite matrix.

**Remark 2.2.** The optimization statement \((2.4)\) defines a family of flux functions, because different norms, induced by \( M^{-1} \), can be used in the objective function \((2.4a)\), and any flux \( f_x(u^-, u^+) \) that is symmetric and consistent can be used as the target flux.

The optimization problem \((2.4)\) is a quadratic program with a strictly convex objective and linear constraint. To find its solution, we need to consider two cases based on whether or not \( \Delta w = w^+ - w^- = 0 \).

**Case \( \Delta w \neq 0 \):** The Jacobian of the constraint in \((2.4b)\) is simply \( \Delta w^T \). Thus, in the case under consideration, the constraint Jacobian has full-row rank and \((2.4)\) has a unique solution; see, for example, [15, Chap. 16]. Furthermore, when \( \Delta w \neq 0 \) problem \((2.4)\) has the following closed-form solution:

\[
\tilde{f}_x = \bar{f}_x - \left( \frac{\Delta w^T \bar{f}_x - \Delta \psi_x}{\Delta w^T M \Delta w} \right) M \Delta w.
\]

**Case \( \Delta w = 0 \):** Here, we must have \( \Delta \psi_x = 0 \) in order for the constraint to be satisfied. This condition will be satisfied provided \( \psi_x^- = \psi_x(u^-) \) and \( \psi_x^+ = \psi_x(u^+) \). In this case, the constraint becomes trivial and the solution to \((2.4)\) is \( \tilde{f}_x = \bar{f}_x \).
We have established the following result.

**Theorem 2.3.** If \( \psi^- = \psi_x(w^-) \) and \( \psi^+ = \psi_x(w^+) \), then the solution to (2.4) is given by

\[
\hat{f}_x = \begin{cases} 
  \bar{f}_x, & \text{if } \Delta w = 0, \\
  f_x - \left( \frac{\Delta w^T \bar{f}_x - \Delta \psi_x}{\Delta w^T M \Delta w} \right) M \Delta w, & \text{if } \Delta w \neq 0,
\end{cases}
\]

where \( \Delta w = w^+ - w^- \) and \( \Delta \psi_x = \psi^+_x - \psi^-_x \).

Next, we show that the numerical flux defined in Theorem 2.3 is an entropy-conservative flux if the left and right states determine the entropy variables.

**Theorem 2.4.** Let \( \hat{f}_x = \hat{f}_x(u^-, u^+) \) be a symmetric and consistent flux function. If the entropy variables are defined by their respective conservative variables — that is, \( w^- = w(u^-) \) and \( w^+ = w(u^+) \) — then the numerical flux given by Theorem 2.3 is symmetric, consistent, and entropy conservative.

**Proof.** First, since \( w^- = w(u^-) \) and \( w^+ = w(u^+) \), the assumption on the potential flux in Theorem 2.3 implies that \( \psi^+_x = \psi_x(w^+) = \psi_x(w(u^+)) \) and \( \psi^-_x = \psi_x(w^-) = \psi_x(w(u^-)) \). Furthermore, the difference in entropy variables can be expressed in terms of the conservative variables as \( \Delta w(u^-, u^+) = w(u^+) - w(u^-) \); similarly for \( \Delta \psi_x(u^-, u^+) \).

Symmetry of \( \hat{f}_x \) then follows from straightforward algebra using the symmetry of the target flux, \( \hat{f}_x(u^-, u^+) = \hat{f}_x(u^+, u^-) \), and the anti-symmetry of the differences \( \Delta w(u^-, u^+) = -\Delta w(u^+, u^-) \) and \( \Delta \psi_x(u^-, u^+) = -\Delta \psi_x(u^+, u^-) \).

When \( u^+ = u^- = u \) we have \( \Delta w(u^-, u^+) = 0 \) and \( \hat{f}_x = \bar{f}_x \); thus, consistency is satisfied because \( \bar{f}_x \) is consistent by assumption.

Finally, the numerical flux \( \hat{f}_x \) is the solution to (2.4), which requires that it satisfy the entropy-conservation condition (2.3). 

**2.3. \( C^1 \) continuity of the flux.** We have established that the flux \( \hat{f}_x \) is an entropy-conservative flux function according to Definition 2.1, but this is not sufficient for its use in high-order discretizations satisfying the summation-by-parts property. To achieve high-order accuracy, it was shown in [5] that the entropy-conservative flux must also be \( C^1 \) continuous. In order to prove this level of continuity, we will need the following two lemmas.

**Lemma 2.5.** Any symmetric, consistent flux function \( \bar{f}_x(u^-, u^+) \) that is differentiable satisfies

\[
\nabla \bar{f}_x(u, u) = \nabla u \bar{f}_x(u, u) = \frac{1}{2} \nabla_x f_x(u)
\]

where \( \nabla \bar{f}_x \) denotes the gradient with respect to the first argument of \( \bar{f}_x \) and \( \nabla_x \bar{f}_x \) denotes the gradient with respect to the second argument.

**Proof.** One can easily show that \( \nabla \bar{f}_x = \nabla_x \bar{f}_x \) along \( w^- = u^+ = u \) by using the definition of the derivative and the symmetry of the flux function. The right-hand side of (2.5) then follows by differentiating the consistency condition \( \bar{f}_x(u, u) = f_x(u) \).

**Lemma 2.6.** Suppose that \( \bar{f}_x(u^-, u^+) \) is a symmetric and consistent flux function that is \( C^2 \) continuous on its domain. Assume that pressure is positive in some neighborhood containing the states \( w^- \) and \( w^+ \), so that \( w^- \) and \( w^+ \) remain bounded. Define
illustrates why entropy-conservative flux functions are

\[ \Delta w^T \tilde{f}_x - \Delta \psi_x = O(\epsilon^3). \]

Proof. To simplify notation, let \( u^- = u \) so that \( u^+ = u + \epsilon v \). Under the assumptions, we can apply Taylor’s theorem to \( \Delta w, \tilde{f}_x, \) and \( \Delta \psi_x \). Doing so we find

\[
\Delta w^T \tilde{f}_x - \Delta \psi_x = \left[ \epsilon (\nabla_u w)v + \frac{\epsilon^2}{2} v^T (\nabla_u^2 w) v + O(\epsilon^2) \right]^T \left[ f_x + \frac{\epsilon}{2} (\nabla_u f_x) v + O(\epsilon^2) \right] \\
- \epsilon (\nabla_u \psi_x) v - \frac{\epsilon^2}{2} v^T (\nabla_u^2 \psi_x) v + O(\epsilon^3)
\]

Recall that the Euler flux satisfies \( w^T f_x = \psi_x + F_x \), and the entropy flux satisfies the differential relation \( \nabla_u F_x = w^T \nabla_u f_x \). Using these two relations we find that the order \( \epsilon \) term in the expansion of \( \Delta w^T \tilde{f}_x - \Delta \psi_x \) is zero:

\[
f_x^T \nabla_u w - \nabla_u \psi_x = f_x^T \nabla_u w - \nabla_u \psi_x + w^T \nabla_u f_x - \nabla_u F_x = 0
\]

The order \( \epsilon^2 \) term is also zero, because it is the derivative of the left-hand side of (2.6), which was just shown to be zero. That is,

\[
(\nabla_u^2 w)^T f_x = (\nabla_u w)^T \nabla_u f_x - \nabla_u^2 \psi_x = \nabla_u \left[ (\nabla_u w)^T f_x - (\nabla_u \psi_x)^T \right].
\]

Thus, \( \Delta w^T \tilde{f}_x - \Delta \psi_x = O(\epsilon^3) \), as desired.

Remark 2.7. Lemma 2.6 illustrates why entropy-conservative flux functions are not necessary for well-resolved smooth flows: under these conditions any symmetric, consistent flux function will satisfy the entropy-conservation condition to high accuracy. On the other hand, entropy-conservative flux functions are useful when \( u^+ - u^- \) is large, such as under-resolved flows or flows with shocks.

Theorem 2.8. If the assumptions of Lemma 2.6 hold, then the entropy-conservative flux \( \tilde{f}_x \), defined in Theorem 2.3, has continuous first partial derivatives.

Proof. The flux \( \tilde{f}_x \) is \( C^1 \) for \( u^+ \neq u^- \), because it consists of products and quotients of \( C^1 \) functions. We only need to show that it has continuous partial derivatives along \( u^+ = u^- \) where \( \Delta w = 0 \).

We begin by finding the directional derivative for \( u^+ = u^- \) in the arbitrary direction \( v \). As before, let \( u^- = u \) and \( u^+ = u + \epsilon v \) with \( v \) a unit vector. Then, using Taylor’s Theorem and Lemma 2.5, we have

\[
\tilde{f}_x(u, u + \epsilon v) = f_x(u) + \frac{\epsilon}{2} (\nabla_u f_x) v + O(\epsilon^2), \tag{2.7a}
\]

\[
\Delta w(u, u + \epsilon v) = \epsilon (\nabla_u w) v + O(\epsilon^2), \tag{2.7b}
\]

\[
\left( \Delta w^T M \Delta w \right)_{(u, u + \epsilon v)} = \epsilon^2 v^T (\nabla_u w)^T M (\nabla_u w) v + O(\epsilon^2). \tag{2.7c}
\]
Using these asymptotic expressions, together with Lemma 2.6, we can expand the difference \( f_x(u, u + \epsilon v) - f_x(u, u) \), which we will subsequently use to evaluate the directional derivative at \( u^- = u^+ = u \).

\[
\begin{align*}
\hat{f}_x(u, u + \epsilon v) - \hat{f}_x(u, u) &= \left[ f_x - \left( \frac{\Delta w^T f_x - \Delta \psi_x}{\Delta w^T M \Delta w} \right) M \Delta w \right] (u, u + \epsilon) - \hat{f}_x(u, u) \\
&= f_x(u) + \frac{\epsilon}{2} (\nabla u f_x) v + O(\epsilon^2) + \frac{O(\epsilon^3)}{O(\epsilon)} O(\epsilon) - f_x(u) \\
&= \frac{\epsilon}{2} (\nabla u f_x) v + O(\epsilon^2).
\end{align*}
\]

Thus, along \( u^- = u^+ = u \), the derivative of \( \hat{f}_x \) with respect to its second argument is \( \frac{\epsilon}{2} \nabla u f_x \), since the following (Fréchet) derivative definition is satisfied.

\[
\lim_{\epsilon \to 0} \frac{\| f_x(u, u + \epsilon v) - f_x(u, u) - \frac{\epsilon}{2} (\nabla u f_x) v \|}{\epsilon \| v \|} = 0,
\]

where the norm in the above limit is the usual 2-norm. Furthermore, it follows from symmetry that \( f_x(u, u + \epsilon v) = f_x(u + \epsilon v, u) \), so we also have \( \nabla f_x = \frac{1}{2} \nabla u f_x \) along \( u^- = u^+ = u \).

In order to prove that the derivative is continuous, we must show that \( \nabla \hat{f}_x \to \frac{1}{2} \nabla u f_x \) as \( u^+ \to u^- \), and similarly for \( \nabla f_x \). Differentiating \( f_x \) with respect to \( u^- \), and evaluating the result at \( u^- = u \) and \( u^+ = u + \epsilon v \), we obtain (note that \( \nabla \Delta w = -\nabla \Delta w^- \))

\[
\nabla \hat{f}_x = \nabla f_x + \left[ \frac{(\Delta w^T \dot{\bar{f}}_x - \Delta \psi_x)}{(\Delta w^T M \Delta w)^2} \right] M \Delta w \left( \frac{2 \Delta w^T M \nabla w^-}{O(\epsilon)} \right) + \left[ \frac{(\Delta w^T \dot{f}_x - \Delta \psi_x)}{(\Delta w^T M \Delta w)} \right] M \nabla w^- \left( \frac{\Delta w^T \Delta \psi_x}{O(\epsilon)} \right) + \left[ \frac{1}{\Delta w^T M \Delta w} \right] \frac{M \Delta w}{O(\epsilon)} + \left[ \frac{- (\nabla \Delta w^-)^T \bar{f}_x + \Delta w^T \nabla f_x + \nabla \psi_x}{O(\epsilon^2)} \right].
\]

To obtain the asymptotic expressions on the right, we used the expansions (2.7b) and (2.7c), and Lemma 2.6. In addition, the asymptotic expression for the last term on the last line above — i.e. the derivative \( \nabla \left( \Delta w^T \dot{f}_x - \Delta \psi_x \right) \) — also used (2.6) in the proof of Lemma 2.6.

To summarize, as \( \epsilon \to 0 \), we see that \( \nabla \hat{f}_x \to \nabla \bar{f}_x \to \frac{1}{2} \nabla u f_x \). A similar analysis shows that \( \nabla \tilde{f}_x \to \frac{1}{2} \nabla u f_x \) as \( \epsilon \to 0 \). Therefore, the derivative of \( \hat{f}_x \) is \( C^1 \) on its domain.

3. Implementation and verification.

3.1. Implementation details. The most efficient and accurate evaluation of \( \hat{f}_x \) will obviously depend on the choice of the target flux \( f_x \) and the matrix \( M \). Nevertheless, we can make a few general observations regarding the implementation of \( \hat{f}_x \) that should hold across a wide range of such choices.
For $\Delta w \neq 0$, recall that the flux in Theorem 2.3 is given by
\[
\hat{f}_x = \bar{f}_x - \left( \frac{\Delta w^T \hat{f}_x - \Delta \psi_x}{\Delta w^T M \Delta w} \right) M \Delta w.
\]
If $\Delta w$ is small, the above expression must be evaluated with care to avoid round-off errors. Lemma 2.6 provides some guidance on this matter. If we evaluate $p \Delta w^T \bar{f}_x - \Delta \psi_x \Delta \psi_x^T \Delta w$ first, and then divide by $\Delta w^T M \Delta w$, we will ensure that an $O(\epsilon^4)$ term is divided by an $O(\epsilon^2)$ term.

With finite-precision accuracy, we also need a threshold with which to distinguish between the cases $\Delta w = 0$ and $\Delta w \neq 0$ in the definition of $\hat{f}_x$. Suppose we are interested in using double precision accuracy. Based on the above arguments, when $\Delta w \neq 0$, the term added to $\bar{f}_x$ is $O(\epsilon^2)$. Consequently, if $\epsilon < 10^{-15}$, we can set $\hat{f}_x = \bar{f}_x$ with an error on the order $10^{-30}$. This should produce acceptable relative accuracy even when the flux components in $\bar{f}_x$ are small, e.g. near stagnation points.

We conclude this section with a suggestion to improve efficiency. In high-order entropy-stable SBP discretizations, the EC flux appears in products with a skew-symmetric matrix. For example, for the $i$th node on an element $\kappa$, the volume term in the residual evaluation is of the form
\[
\sum_{j=1}^{n_\kappa} S_{ij} \hat{f}_x(u_i, u_j),
\]
where $S_{ij}$ are the entries in a skew-symmetric matrix and $n_\kappa$ is the number of collocation nodes on $\kappa$. Given the symmetry of $\hat{f}_x(u_i, u_j)$ and skew-symmetry of $S_{ij}$, we can reuse the product $S_{ij} \hat{f}_x(u_i, u_j)$ on node $j$ by introducing a sign change. This is true for any EC flux and is well known. However, for the proposed family of EC flux functions, we can gain additional efficiency by avoiding the recomputation of the entropy variables. To do this, we express the numerical flux in the form $\hat{f}_x(u_i, u_j) = f_x(u_i, u_j, w_i, w_j)$; that is, the entropy variables are passed into the flux function as “independent” variables. In pseudocode, the above summation would then be implemented as follows ($r_i \in \mathbb{R}^5$ denotes the residual vector on node $i$):

```
for $i = 1 : n_\kappa$
do
    $w_i \leftarrow w(u_i)$
    for $j = (i + 1) : n_\kappa$ do
        $w_j \leftarrow w(u_j)$
        $f_{ij} \leftarrow \hat{f}_x(u_i, u_j, w_i, w_j)$
        $r_i \leftarrow r_i + S_{ij} f_{ij}$
        $r_j \leftarrow S_{ij} f_{ij}$
    end for
end for
```

Precomputing the entropy variables, as shown above, becomes increasingly important to reducing FLOPS as $n_\kappa$ increases.

3.2. Numerical verifications. It is beyond the scope of this short communication to provide an exhaustive comparison of the newly proposed family of entropy-conservative fluxes with existing fluxes. However, it is prudent that we verify the theory on at least one member of the family, and this is the objective of the present section. For the verification we consider the solution of the one-dimensional Euler equa-
tions on a periodic domain:
\[
\frac{\partial u}{\partial t} + \frac{\partial f_x}{\partial x} = 0, \quad \forall \; x \in [0,1],
\]
\[
u(t,1) = u(t,0), \quad \text{and} \quad u(0,x) = u_0(x).
\]
An exact solution to this problem is given by an entropy wave\(^1\) satisfying \(v_x = 1, p = 1,\) and
\[
\rho(t,x) = 1 + \frac{1}{2} \sin [2\pi (x-v_xt)].
\]
The exact solution at \(t = 0\) is used to define the initial condition \(u_0(x)\).

To construct the optimization-based EC flux, we choose \(M = 1,\) and, for the target flux, we select
\[
\bar{f}_x = \left[ \begin{array}{c}
\{\rho v_x \} \\
\{v_x \} (\{v_x \} + \{p \})
\end{array} \right],
\]
where \(\{ \cdot \} \) denotes the arithmetic mean of two variables, e.g. \(\{x\} \equiv \frac{1}{2} (x^+ + x^-)\). Based on these choices, and referring to Theorem 2.3, the EC numerical flux becomes
\[
\hat{f}_x = \begin{cases}
\bar{f}_x, & \text{if } \Delta w = 0, \\
\bar{f}_x - \left( \frac{\Delta w^T \bar{f}_x - \Delta \psi_x}{\Delta w^T \Delta w} \right) \Delta w, & \text{if } \Delta w \neq 0.
\end{cases}
\]
We will refer to \(\hat{f}_x^*\) as the EC-opt numerical flux.

The EC-opt flux is incorporated into an entropy-conservative scheme that uses collocation Legendre-Gauss (LG) operators with the summation-by-parts property; see, for example, [5] for the details of the discretization. We also consider the same discretization with the Ismail-Roe EC flux [12]. Finally, we compare the EC schemes with a conventional collocation discontinuous-Galerkin discretization of the Euler equations in which the Roe numerical flux [17] is used to couple the elements. All three semi-discretizations are discretized in time using the classical 4th-order Runge-Kutta scheme.

To assess the entropy conservation of the three schemes, we advanced the solutions for one period on coarse grids with 5 elements and degree \(p = 2\) operators. This coarse solution space was chosen to ensure that \(\Delta w^T \bar{f}_x - \Delta \psi_x\) is sufficiently large to cause a loss of entropy conservation in the conventional discretization that uses the Roe flux. In addition, to avoid the effects of temporal errors, the CFL number was set to 0.001.

Table 1 lists the change in total entropy for the three schemes. Here, the change in entropy is defined as
\[
\Delta U = 1^T H U_{t=1} - 1^T H U_{t=0},
\]
where \(U\) is a vector of entropy at the nodal degrees of freedom in the mesh, \(H\) is the diagonal mass matrix, and \(1\) is a vector of ones that is the same size as \(U\). The EC discretizations, using either EC-opt or the Ismail-Roe flux, are clearly entropy conservative, as expected. The Roe-based scheme is not entropy conservative, but this not surprising since it includes upwinding [21].

\(^1\)The authors wish to thank Mark Carpenter for suggesting this exact solution.
Table 1
Change in total (integrated) entropy after one period using different numerical flux functions applied to the entropy-wave problem.

| flux func. | Roe            | Ismail-Roe      | EC-opt          |
|------------|----------------|-----------------|-----------------|
| ∆ entropy  | $-2.24 \times 10^{-3}$ | $1.51 \times 10^{-14}$ | $-4.97 \times 10^{-14}$ |

We conclude by demonstrating that the accuracy of EC-opt is comparable to the other two discretizations. Accuracy also provides a means to falsify Theorem 2.8, since the $C^1$ continuity of EC-opt is necessary for high-order truncation errors in the entropy-conservative discretization.

To assess the accuracy of the schemes, we run the entropy-wave problem until $t = 0.1$. We consider Legendre-Gauss operators from $p = 1$ to $p = 4$ and a range of grid resolutions from $n = 10$ elements to $n = 80$ elements. The CFL number is again restricted to 0.001 to limit the impact of temporal errors. For this accuracy study, the discretizations using the EC fluxes include Lax-Friedrichs-type entropy-stable dissipation [5].

Tables 2 to 4 list the results of the accuracy study. We use the $L^2$ error in the density variable as the error metric. For this smooth problem, all three discretizations achieve similar density errors. We also observe that all three discretizations exhibit an even-odd effect in which even degree operators converge at a rate of $p$ and odd degree operators converge at a rate of $p + 1$. Similar even-odd effects have been reported elsewhere in the literature [4, 5].

4. Summary and Discussion. We have presented an optimization-based construction for entropy-conservative (EC) flux functions. The proposed approach seeks the flux that is the closest, in some norm, to a target flux function under the constraint that the entropy-conservation condition is met. The approach defines a family of EC flux functions, because different target fluxes and norms can be used in the optimization statement. Although an optimization problem is used to define the EC flux, the optimization problems are quadratic programs and admit closed-form (affordable) solutions. We showed that the EC fluxes are $C^1$ continuous, so they can be used in high-order entropy-stable discretizations.

Finally, while our verification demonstrated comparable accuracy with the Ismail-Roe numerical flux, we make no claim regarding the superiority of one or another EC flux. Such claims will require many studies. Furthermore, because the proposed approach defines an infinite family of EC fluxes, it is not immediately obvious which members of this family will perform best with respect to various metrics. This is the topic of future research.

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Table 2
Roe numerical flux, accuracy on entropy-wave problem.

| n  | $L^2$ error | rate | $L^2$ error | rate | $L^2$ error | rate | $L^2$ error | rate |
|----|-------------|------|-------------|------|-------------|------|-------------|------|
| 10 | 9.92e-03    | —    | 2.54e-03    | —    | 2.20e-04    | —    | 1.96e-05    | —    |
| 20 | 1.28e-03    | 2.95 | 5.50e-04    | 2.21 | 5.08e-06    | 5.43 | 1.11e-06    | 4.15 |
| 40 | 2.25e-04    | 2.51 | 1.09e-04    | 2.34 | 1.48e-07    | 5.11 | 5.14e-08    | 4.43 |
| 80 | 4.95e-05    | 2.18 | 2.05e-05    | 2.41 | 5.07e-09    | 4.86 | 2.13e-09    | 4.60 |

Table 3
Ismail-Roe numerical flux, accuracy on entropy-wave problem.

| n  | $L^2$ error | rate | $L^2$ error | rate | $L^2$ error | rate | $L^2$ error | rate |
|----|-------------|------|-------------|------|-------------|------|-------------|------|
| 10 | 9.70e-03    | —    | 2.49e-03    | —    | 2.11e-04    | —    | 1.87e-05    | —    |
| 20 | 1.27e-03    | 2.93 | 5.47e-04    | 2.19 | 5.15e-06    | 5.36 | 1.09e-06    | 4.10 |
| 40 | 2.25e-04    | 2.50 | 1.09e-04    | 2.33 | 1.49e-07    | 5.12 | 5.15e-08    | 4.41 |
| 80 | 4.96e-05    | 2.18 | 2.05e-05    | 2.41 | 5.07e-09    | 4.87 | 2.13e-09    | 4.59 |

Table 4
EC-opt numerical flux, accuracy on entropy-wave problem.

| n  | $L^2$ error | rate | $L^2$ error | rate | $L^2$ error | rate | $L^2$ error | rate |
|----|-------------|------|-------------|------|-------------|------|-------------|------|
| 10 | 9.71e-03    | —    | 2.48e-03    | —    | 2.11e-04    | —    | 1.88e-05    | —    |
| 20 | 1.27e-03    | 2.93 | 5.47e-04    | 2.18 | 5.16e-06    | 5.36 | 1.10e-06    | 4.10 |
| 40 | 2.24e-04    | 2.50 | 1.09e-04    | 2.33 | 1.49e-07    | 5.11 | 5.16e-08    | 4.41 |
| 80 | 4.94e-05    | 2.18 | 2.05e-05    | 2.41 | 5.09e-09    | 4.87 | 2.14e-09    | 4.59 |

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