The \((\mathfrak{gl}_m, \mathfrak{gl}_n)\) Duality in the Quantum Toroidal Setting

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Abstract: On a Fock space constructed from \(mn\) free bosons and lattice \(\mathbb{Z}_{mn}\), we give a level \(n\) action of the quantum toroidal algebra \(\mathcal{E}_m\) associated to \(\mathfrak{gl}_m\), together with a level \(m\) action of the quantum toroidal algebra \(\mathcal{E}_n\) associated to \(\mathfrak{gl}_n\). We prove that the \(\mathcal{E}_m\) transfer matrices commute with the \(\mathcal{E}_n\) transfer matrices after an appropriate identification of parameters.

1. Introduction

Duality of integrable systems is a very interesting, deep, and somewhat mysterious phenomenon which is being intensively explored.

The simplest example is the \((\mathfrak{gl}_m, \mathfrak{gl}_n)\) duality of Gaudin systems. Recall that for any reductive Lie algebra \(\mathfrak{g}\), and any \(\mathfrak{g}\) weight \(\mathbf{p} \in \mathfrak{h}^*\), there exists a remarkable commutative subalgebra \(\mathcal{B}(\mathbf{p}) \subset U(\mathfrak{g}[x])\) called the algebra of higher Gaudin Hamiltonians. Here \(\mathfrak{g}[x] = \mathfrak{g} \otimes \mathbb{C}[x]\) is the current algebra, and \(\mathfrak{h} \subset \mathfrak{g}\) is the Cartan subalgebra. The algebra \(\mathcal{B}(\mathbf{p})\) can be obtained using structures of conformal field theory and the so-called “center on the critical level”, see [FFR].

Let \(V_k\) be the vector representation of \(\mathfrak{gl}_k\). Then, by the classical construction, we have commuting actions of \(\mathfrak{gl}_m\) and \(\mathfrak{gl}_n\) in the spaces \(S^*(V_m \otimes V_n)\) and \(\wedge^*(V_m \otimes V_n)\). These actions can be extended to the actions of \(\mathfrak{gl}_m[x]\) and \(\mathfrak{gl}_n[x]\) such that \(V_m \otimes V_n = \oplus_{i=1}^m V_m(u_i)\) as \(\mathfrak{gl}_m[x]\) module and \(V_m \otimes V_n = \oplus_{j=1}^m V_n(\tilde{u}_j)\) as \(\mathfrak{gl}_n[x]\) module. Here \(u_i, \tilde{u}_j \in \mathbb{C}\) are arbitrary parameters, and \(V_m(u_i), V_n(\tilde{u}_j)\) are \(\mathfrak{gl}_m[x]\) and \(\mathfrak{gl}_n[x]\) evaluation modules, respectively. The actions of \(\mathfrak{gl}_m[x]\) and \(\mathfrak{gl}_n[x]\) do not commute.

The parameters \(\{u_i\}\) canonically determine a \(\mathfrak{gl}_n\) weight \(\mathbf{p} \in (\mathfrak{h}_n)^*\), while the parameters \(\{\tilde{u}_j\}\) canonically determine a \(\mathfrak{gl}_m\) weight \(\mathbf{p} \in (\mathfrak{h}_m)^*\). It turns out that the two commutative subalgebras of higher Gaudin Hamiltonians \(\mathcal{B}(\mathbf{p}) \subset U(\mathfrak{gl}_m[x])\) and \(\mathcal{B}(\tilde{\mathbf{p}}) \subset U(\mathfrak{gl}_n[x])\) commute with each other, see [MTV]. Moreover, these subalgebras coincide and one can write explicitly \(\mathfrak{gl}_m\) higher Gaudin Hamiltonians in terms of \(\mathfrak{gl}_n\) higher Gaudin Hamiltonians and vice versa.
The spectrum of the Gaudin model is found by the Bethe ansatz method. The duality described above gives a correspondence between solutions of the Bethe ansatz equations for $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ models which can also be described via an appropriate Fourier transform related to the bispectral involution for rational solutions of the KP hierarchy, see [MTV1].

In [MTV2], a similar duality is described on the level of the Bethe ansatz equations between the trigonometric $\mathfrak{gl}_m$ Gaudin model and the XXX $\mathfrak{gl}_n$ model (related to Yangians). It is expected that the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality can be lifted to the duality of the XXZ models (related to quantum affine algebras).

Another example, which motivated this paper, is a duality between local and non-local integrals of motion in the quantum KdV system [BLZ, BLZ1, BLZ2]. Consider the free vertex operator algebra generated by one field $h(z) = \sum_{i \in \mathbb{Z}} h_i z^{-i}$, where $\{h_i\}$ are generators of the Heisenberg algebra satisfying $[h_i, h_j] = i\delta_{i+j}$. The local integrals of motion are given by integrals of the form $\int T_{2n}(z) dz$, $n \in \mathbb{Z}_{\geq 0}$, where $T_{2n}(z)$ are local currents. For example $T_2(z)$ is the Virasoro current $T_2(z) = \frac{1}{2} : h(z)^2 : = +\lambda h'(z)$, $\lambda \in \mathbb{C}$, $T_4(z) =: T_2(z)^2 :$, and so on.

Non-local integrals of motion are multiple integrals of vertex operators. For example, the first one has the form $\int \int S_1(z) S_2(w) dz dw$, where $S_1(z)$, $S_2(w)$ are vertex operators, the second non-local integral of motion is a four-fold integral, and so forth.

In the KdV case local and non-local integrals look very different, but they mutually commute. Understanding the situation with Bethe ansatz and the correspondence between the spectra of local and non-local integrals of motion is still far from complete.

There are many integrable systems of KdV type. Conjecturally, it is possible to find local and non-local integrals of motion in almost all $W$ algebras, in the universal enveloping algebras of affine Kac-Moody Lie algebras (including superalgebras), and in coset vertex algebras.

In the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality mentioned above, the two sets of integrals in duality are given by the same construction. In contrast, in the case of vertex operator algebras, the constructions of local and non-local integrals of motion are very different. The situation can be explained if we consider quantum versions.

In [FKSW, KS], “local” and “non-local” integrals of motion are constructed inside certain deformed $W$ algebras: the quantum toroidal $\mathfrak{gl}_1$ and $\mathfrak{gl}_n$ algebras, see [FJM]. Both are given by similar multiple integrals (hence they are both non-local). In the conformal limit the deformed $W$ algebras turn into vertex operator algebras, and the two sets of integrals of motion become different looking local and non-local integrals of motion. Moreover, the same deformed $W$ algebras admit alternative conformal limits where the situation is reversed: “local” integrals of motion become non-local and “non-local” ones become local.

In this paper we study a quantum affine analog of the $(\mathfrak{gl}_m, \mathfrak{gl}_n)$ duality.

Let $\mathcal{E}_k(\mathfrak{sl}_k(q_1, q_2, q_3))$ be the quantum toroidal algebra of type $\mathfrak{gl}_k$, where $k \in \mathbb{Z}_{\geq 1}$ and $q_1, q_2, q_3 \in \mathbb{C}$ are parameters satisfying $q_1 q_2 q_3 = 1$. For the definition, see Appendix A. The quantum toroidal algebra has a vertical subalgebra isomorphic to the quantum affine algebra $U_q \mathfrak{gl}_k \subset \mathcal{E}_k$ where $q^2 = q_2$. Inside a completion $\hat{\mathcal{E}}_k$ of $\mathcal{E}_k$ one has a commutative algebra of transfer matrices $\mathcal{B}_q(p)$, which we call the Bethe algebra of integrals of motion. The Bethe algebra of integrals of motion depends on an affine weight $p \in (\hat{\mathcal{H}}_k)^*$, see [FJMM] and Sect. 4.2 below.

We consider $mn$ free bosons and a lattice $\mathbb{Z}_{mn}$, where $m, n \in \mathbb{Z}_{\geq 1}$. We use the vertex operator construction of [Sa] to define actions of $\mathcal{E}_m(q_1, q_2, q_3)$ of level $n$ and of
\( \mathcal{E}_n(\hat{q}_1, q_2, \hat{q}_3) \) of level \( m \) on the total Fock space \( \mathbb{F}_{m,n} \). The actions depend on parameters 
\( u_1, \ldots, u_n \) and \( \hat{u}_1, \ldots, \hat{u}_m \), respectively.\(^1\)

Then the actions of vertical subalgebras \( U_q \hat{gl}_m \) and \( U_q \hat{gl}_n \) commute. The total Fock space \( \mathbb{F}_{m,n} \) as a \( U_q \hat{gl}_m \) module is a direct sum of various products of \( n \) integrable irreducible level one modules. As \( \mathcal{E}_m(q_1, q_2, q_3) \) modules those products also become tensor products of irreducible modules with spectral parameters, see Lemma 3.4. Similarly, as a \( U_q \hat{gl}_n \) module \( \mathbb{F}_{m,n} \) is a direct sum of various products of \( m \) integrable irreducible level one modules which are irreducible \( \mathcal{E}_n(\hat{q}_1, q_2, \hat{q}_3) \) modules with spectral parameters.

Next, we consider Bethe algebras. The new feature is that, while the parameters \( \{\hat{u}_j\} \) determine finite-dimensional part of the affine weight \( \hat{p} \in (\hat{h}_m)^* \), the parameter \( \hat{q}_1 \) determines the “loop direction” of the latter. Similarly the dual affine weight \( \check{p} \in (\check{h}_n)^* \) is determined from \( \{u_j\} \) and \( q_1 \). Then we prove that the corresponding Bethe algebras \( \mathfrak{B}_q(p) \subset \mathcal{E}_m \) and \( \mathfrak{B}_q(\check{p}) \subset \mathcal{E}_n \) mutually commute, see Theorem 4.2.

The action of integrals of motion in \( \mathbb{F}_{m,n} \) is given by multiple integrals described in [FJM], see also Proposition 4.1 below. The integrals of motion of [FKSW,KS] can be recovered in the case of \( m = 1 \).

The spectrum of Bethe algebra of integrals of motion is expected to be given by Bethe ansatz method. For the case of \( \mathcal{E}_1 \) it is proved in [FJMM1,FJMM2]. For the case of \( \mathcal{E}_2 \) the precise statement is conjectured in [FJM]. Therefore, the duality described in this paper suggests a correspondence of solutions of two different Bethe ansatz equations. It would be interesting to understand this correspondence.

The KdV nonlocal and local integrals of motion are expected to be related to coefficients of expansions of the same function at zero and infinity respectively, see formulas (55) and (66) in [BLZ]. It is an important problem to find and to prove a similar relation between integrals of motion corresponding to \( \mathcal{E}_n \) and \( \mathcal{E}_m \).

Also it would be interesting to study the various limits of the duality described in this paper.

From the technical point of view, our proof of duality is purely computational, following the lines of [FKSW,KS]. To give a more conceptual explanation to the duality phenomena is an issue left for the future.

The plan of the paper is as follows.

In Sect. 2 we collect preliminary materials concerning the bosons and the total Fock space \( \mathbb{F}_{m,n} \). In Sect. 3 we define actions of two quantum toroidal algebras \( \mathcal{E}_m = \mathcal{E}_m(q_1, q_2, q_3) \) and \( \hat{\mathcal{E}}_n = \hat{\mathcal{E}}_n(\hat{q}_1, q_2, \hat{q}_3) \) on \( \mathbb{F}_{m,n} \), and state the commutativity of the quantum affine subalgebras \( U_q \hat{gl}_m \) and \( U_q \hat{gl}_n \) (Theorem 3.6). In Sect. 4 we introduce integrals of motion associated with \( \mathcal{E}_m \) and \( \hat{\mathcal{E}}_n \). We then state the main theorem concerning their commutativity (Theorem 4.2).

The text is followed by four appendices. In Appendix A we give the presentation of the quantum toroidal algebra \( \mathcal{E}_m \). In Appendix B, contractions of various currents are summarized in tables. The commutativity of the quantum affine subalgebras \( U_q \hat{gl}_m \) and \( U_q \hat{gl}_n \) is proved in Appendix C. Proof of the duality is given in Appendix D.

**Notation.** Throughout the text we fix parameters \( q, \check{q}, \hat{d} \in \mathbb{C}^* \) and define

\[
q_1 = q^{-1}d, \quad q_2 = q^2, \quad q_3 = q^{-1}d^{-1}, \\
\hat{q}_1 = \hat{q}^{-1}\hat{d}, \quad \hat{q}_2 = \hat{q}^2, \quad \hat{q}_3 = q^{-1}\hat{d}^{-1}.
\]

\(^1\) In the main text we use notation \( u_0 = u_n \) and \( \hat{u}_0 = \hat{u}_m \).
so that \( q_2 = \tilde{q}_2, q_1 q_2 q_3 = \tilde{q}_1 \tilde{q}_2 \tilde{q}_3 = 1 \). We assume that \((q_1, q_2, q_3)\) are generic, in the sense that if \( q^i_1 q^j_2 q^k_3 = 1 \) for \( i, j, k \in \mathbb{Z} \), then \( i = j = k \). We assume the same for \((\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)\).

We use symbols for ordered products

\[
\prod_{1 \leq i \leq N} A_i = A_1 A_2 \cdots A_N, \quad \prod_{1 \leq i \leq N} A_i = A_N A_{N-1} \cdots A_1
\]

and infinite products

\[
(z_1, \ldots, z_r; p)_{\infty} = \prod_{i=1}^{r} \prod_{k=0}^{\infty} (1 - z_i p^k), \quad \Theta_p(z) = (z, pz^{-1}, p; p)_{\infty}.
\]

We set \( \theta(P) = 1 \) if statement \( P \) is true and \( \theta(P) = 0 \) otherwise. For a positive integer \( N \), we write \( \delta_{i,k}^{(N)} = \theta(i \equiv k \mod N) \).

2. Preliminaries

In this section we collect preliminary materials which will be used to construct representations of algebras \( \mathcal{E}_m \) and \( \tilde{\mathcal{E}}_n \) in Sect. 3.

2.1. Total Fock space. Let \( m, n \) be positive integers. We introduce a set of bosons \( \{a_{r}^{i,j} | r \in \mathbb{Z} \setminus \{0\}, 0 \leq i \leq m-1, 0 \leq j \leq n-1\} \) such that

\[
[a_{r}^{i,j}, a_{s}^{k,l}] = \delta_{i,k} \delta_{j,l} \delta_{r+s,0} \frac{[r]^2}{r},
\]

where \([x] = (q^x - q^{-x})/(q - q^{-1})\).

Consider a vector space with basis \( \{|m\}\) labeled by \( m = (m_s, r)_{0 \leq s \leq m-1, 0 \leq r \leq n-1} \in \mathbb{Z}^{mn} \). We define linear operators \( e^{\pm \xi_{i,j}}, \partial_{i,j} \) by setting

\[
e^{\pm \xi_{i,j}}|m\rangle = (-1)^{\sum_{s=0}^{j} \sum_{t=0}^{i} m_s t} e^{\sum_{s=0}^{j} \sum_{t=0}^{i} m_s t} |m \pm \delta_{i,j}\rangle,
\]

\[
\partial_{i,j}|m\rangle = m_{i,j}|m\rangle,
\]

where \( \delta_{i,j} = (\delta_{i,0} \delta_{j,1}) \). We have then

\[
e^{\epsilon_{i,j}} e^{\epsilon_{k,l}} = -e^{\epsilon_{k,l}} e^{\epsilon_{i,j}} \quad \text{if} \quad (i, j) \neq (k, l),
\]

\[
|m\rangle = \prod_{0 \leq i \leq m-1} \prod_{0 \leq j \leq n-1} e^{m_{i,j} \epsilon_{i,j}} |0\rangle = e^{m_{0,0} \epsilon_{0,0}} \cdots e^{m_{m-1,n-1} \epsilon_{m-1,n-1}} |0\rangle.
\]

Define the total Fock space by

\[
\mathbb{F}_{m,n} = \mathbb{C}[[a_{r}^{i,j} | r>0, 0 \leq i \leq m-1 ] \otimes \bigoplus_{m \in \mathbb{Z}^{mn}} \mathbb{C}|m\rangle.
\]
We write the operators $a_{i,j}^{i,j} \otimes \text{id}, \text{id} \otimes e^{i,j}, \text{id} \otimes \partial_{i,j}$ on $\mathbb{F}_{m,n}$ simply as $a_{i,j}^{i,j}, e^{i,j}, \partial_{i,j}$. We extend the range of the superfixes $i, j$ to $\mathbb{Z}$ by demanding periodicity $a_{i,j}^{i,j} = a_{i+\delta,j}^{i,j}$, and likewise for $e^{i,j}, \partial_{i,j}$.

The space $\mathbb{F}_{m,n}$ carries a $\mathbb{Z}$ grading with the degree assignment

$$\text{deg}(a_{i,j}^{i,j} \ldots a_{s,t}^{i,j} | m) = \sum_{l=1}^{N} n_l + \frac{1}{2} \sum_{s,t} m_{s,t}^2.$$ 

For a polynomial $X$ in the oscillators $a_{i,j}^{i,j}$ $(r \neq 0)$ and the ‘zero mode’ operators $e^{i,j}, \partial_{i,j}$, we shall use the normal ordering symbol $:X:$. The rule is as usual; $a_{i,j}^{i,j}$ with $r > 0$ (resp. $r < 0$) are placed to the right (resp. left), and $\partial_{i,j}$ are placed to the right of $e^{i,j}$. The ordering of the $e^{i,j}$’s is kept unchanged, so that $: e^{i,j} : e^{k,j} : = e^{i,j} e^{k,j}$.

### 2.2. Bosons $b_{i,j}^{i,j}$

We introduce auxiliary bosons $b_{i,j}^{i,j}, \tilde{b}_{i,j}^{i,j}$ by setting

$$b_{i,j}^{i,j} = q^r (q_3 a_{i,j}^{i,j} - a_{i,j}^{i,j}), \quad b_{i,j}^{i,j} = q_1 a_{i,j}^{i,j} - a_{i,j}^{i,j},$$

$$\tilde{b}_{i,j}^{i,j} = -q^r (q_3 a_{i,j}^{i,j} - a_{i,j}^{i,j}), \quad \tilde{b}_{i,j}^{i,j} = -(q_1 a_{i,j}^{i,j} - a_{i,j}^{i,j}),$$

for $r > 0$. They are subject to linear relations

$$b_{i,j}^{i,j} + b_{i,j}^{i,j} = \tilde{q}_3^r b_{i,j}^{i,j} + q_3 b_{i,j}^{i,j},$$

$$b_{i,j}^{i,j} + b_{i,j}^{i,j} = \tilde{q}_1^r b_{i,j}^{i,j} + q_1 b_{i,j}^{i,j}.$$ 

Commutation relations of these bosons are obtained from (2.1).

**Lemma 2.1.** The bosons (2.2)–(2.3) satisfy the commutation relations

$$[b_{i,j}^{i,j}, b_{i,j}^{k,l}] = \frac{[r]^2}{r} q^r \left((1 + q_2^{-r}) \delta_{i,k}^{(m)} - q_1^r \delta_{i,j+1,k} - q_3^r \delta_{j+1,1}^{(m)} \right) \delta_{j,l}^{(n)},$$

$$[\tilde{b}_{i,j}^{i,j}, \tilde{b}_{i,j}^{k,l}] = \frac{[r]^2}{r} q^r \delta_{i,k}^{(m)} \left((1 + q_2^{-r}) \delta_{j,l}^{(m)} - \tilde{q}_1^r \delta_{j,l+1}^{(m)} - \tilde{q}_3^r \delta_{j+1,l}^{(m)} \right),$$

$$[b_{i,j}^{i,j}, \tilde{b}_{i,j}^{k,l}] = \frac{[r]^2}{r} q^r \left(q_3^r \delta_{i,j+1,k} - \delta_{i,k}^{(m)} \right) \left(\tilde{q}_1^r \delta_{j,l+1}^{(m)} - \delta_{j,l}^{(m)} \right),$$

$$[\tilde{b}_{i,j}^{i,j}, b_{i,j}^{k,l}] = \frac{[r]^2}{r} q^r \left(q_1^r \delta_{i,j+1,k} - \delta_{i,k}^{(m)} \right) \left(\tilde{q}_3^r \delta_{j,l+1}^{(m)} - \delta_{j+1,l}^{(m)} \right).$$

\[\square\]

We shall also use currents $A_{i,j}^{i,j}(z), B_{i,j}^{i,j}(z), \tilde{A}_{i,j}^{i,j}(z), \tilde{B}_{i,j}^{i,j}(z)$. These are formal series of the form $X(z) = \sum_{r \neq 0} X_r z^{-r}$, whose Fourier coefficients $X_r$ are given in terms of $b_{i,j}^{i,j}, \tilde{b}_{i,j}^{i,j}$ as follows.

$$A_{i,j}^{i,j} = -\frac{1}{[r]} q^{-r} q^{(n-1)r} \tilde{q}_3^{- r} b_{i,j}^{i,j}, \quad A_{i,j}^{i,j} = \frac{1}{[r]} q^{(n-2)r} \left(\tilde{q}_3^{jr} b_{i,j}^{i,j} + (1 - q_3^r) \sum_{t=j+1}^{n-1} \tilde{q}_3^{tr} b_{i,j}^{i,t} \right),$$

(2.10)
\[ B_{r}^{i,j} = \frac{1}{[r]} q^r \left( \tilde{q}^{ir}_{1} b_{r}^{i,j} + (1 - q_{s}^{-2}) \sum_{t=0}^{j-1} \tilde{q}^{tr}_{1} b_{r}^{i,t} \right), \quad B_{-r}^{i,j} = -\frac{1}{[r]} q^{-ir} \tilde{b}_{-r}^{i,j}. \] (2.11)

\[ A_{r}^{i,j} = -\frac{1}{[r]} q^{-(m-1)r} q_{s}^{-ir} \tilde{b}_{r}^{i,j}, \quad A_{-r} = \frac{1}{[r]} q^{(m-2)r} \left( q_{s}^{3r} \tilde{b}_{r}^{i,j} + (1 - q_{s}^{-2}) \sum_{s=i+1}^{m-1} q_{s}^{3r} \tilde{b}_{-r}^{s,j} \right). \] (2.12)

\[ \tilde{B}_{r}^{i,j} = \frac{1}{[r]} q^r \left( q_{1}^{ir} \tilde{b}_{r}^{i,j} + (1 - q_{s}^{-2}) \sum_{s=0}^{j-1} q_{s}^{3r} \tilde{b}_{r}^{j,s} \right), \quad \tilde{B}_{-r}^{i,j} = -\frac{1}{[r]} q^{-ir} \tilde{b}_{-r}^{i,j}. \] (2.13)

Here \( 0 \leq i \leq m - 1, 0 \leq j \leq n - 1 \) and \( r > 0 \).

### 3. Fock Representation

We consider two quantum toroidal algebras in parallel: algebra \( \mathcal{E}_{m} = \mathcal{E}_{m}(q_{1}, q_{2}, q_{3}) \) with parameters \( q_{1}, q_{2}, q_{3} \), and algebra \( \tilde{\mathcal{E}}_{n} = \tilde{\mathcal{E}}_{n}(\tilde{q}_{1}, q_{2}, \tilde{q}_{3}) \) with parameters \( \tilde{q}_{1}, q_{2}, \tilde{q}_{3} \). Our convention about quantum toroidal algebras is summarized in Appendix A. We say that an \( \mathcal{E}_{m} \) module has level \( n \) if the central element \( C \) acts as a scalar \( q^{n} \). In this section, we introduce a level \( n \) action of \( \mathcal{E}_{m} \) and a level \( m \) action of \( \tilde{\mathcal{E}}_{n} \) on the same total Fock space \( \mathbb{F}_{m,n} \); see Propositions 3.3 and 3.5 below.

#### 3.1. Level one representations. Let \( m \geq 2 \). We begin with the case \( n = 1 \), considering a level one action of \( \mathcal{E}_{m} \) on \( \mathbb{F}_{m,1} \). Apart from minor modification, the following result is due to [Sa] (see also [STU]).

**Proposition 3.1.** Let \( u \in \mathbb{C}^{x} \). The following formulas give a level one representation of \( \mathcal{E}_{m} \) on \( \mathbb{F}_{m,1} \):

\[ q^{\partial_{i,1}}, \quad C = q, \quad D = q^{\text{deg}}, \quad H_{i,r} = b_{r}^{i,0}, \quad H_{i,-r} = b_{-r}^{i,0}, \]

\[ E_{i}(z) = u^{-\partial_{i,0}} : e^{A_{i,0}(z)} : U^{i,0}(z), \quad F_{i}(z) = u^{\partial_{i,0}} : e^{B_{i,0}(z)} : V^{i,0}(z). \]

Here \( 0 \leq i \leq m - 1, r > 0 \), the bosons \( b_{r}^{i,0} \) and the currents \( A_{i,0}(z), B_{i,0}(z) \) are those with \( n = 1 \), and

\[ U^{i,0}(z) = e^{-\epsilon_{i,0}} e^{\epsilon_{i-1,0}} z^{\partial_{i-1,0}-\partial_{i,0}+1} d^{(\partial_{i-1,0}+\partial_{i,0})/2}, \]

\[ V^{i,0}(z) = e^{-\epsilon_{i,0}} e^{\epsilon_{i-1,0}} z^{-\partial_{i-1,0}+\partial_{i,0}+1} d^{-(\partial_{i-1,0}+\partial_{i,0})/2}. \]

We write the level one \( \mathcal{E}_{m} \) module given above as \( \mathbb{F}_{m,1}(u) \).

For \( u \in \mathbb{C}^{x}, v \in \mathbb{Z}/m\mathbb{Z} \) and \( t, s \in \mathbb{Z} \), let \( \mathcal{F}_{m}^{(v)}(u; t, s) \) denote the level one irreducible \( \mathcal{E}_{m} \) module such that (i) it has highest weight \( (P_{0}(z), \ldots, P_{m-1}(z)) \) where

\[ P_{i}(z) = 1 \quad (i \neq v), \quad P_{v}(z) = q \frac{1 - q_{2}^{-1}u/z}{1 - u/z}, \]

(see Appendix A for the definition of highest weight), (ii) the highest weight vector has degree \( t \), and (iii) the central element \( q^{\sum_{i=0}^{m-1} \epsilon_{i}} \) acts as a scalar \( q^{s} \).
Lemma 3.2. We have a decomposition into submodules

$$\mathbb{F}_{m,1}(u) = \bigoplus_{s \in \mathbb{Z}} \mathcal{F}_m^{(v)}(u^{(s)}; t^{(s)}, s),$$

where $s = ml + v (l \in \mathbb{Z}, 0 \leq v \leq m - 1)$, and

$$u^{(s)} = (-)^m d^{-s-m/2}qu, \quad t^{(s)} = \frac{v}{2} (l + 1)^2 + \frac{m - v}{2} l^2.$$

The submodule $\mathcal{F}_m^{(v)}(u^{(s)}; t^{(s)}, s)$ is generated by a highest weight vector

$$v^{(s)} = \{l + 1, \ldots, l + 1, l, \ldots, l\}.$$

3.2. Higher level representations. For general $n \geq 1$, we identify the vector space $\mathbb{F}_{m,n}$ with the tensor product

$$\mathbb{F}_{m,1}(u_0) \otimes \cdots \otimes \mathbb{F}_{m,1}(u_{n-1}).$$

We give it an $\mathcal{E}_m$ module structure via the iterated coproduct $\Delta^{(n-1)}$ (see Appendix A for the definition of $\Delta$). For convenience we further gauge transform the action of $x \in \mathcal{E}_m$ as

$$\Delta^{(n-1)}x \rightarrow d^Z(\Delta^{(n-1)}x)d^{-Z}, \quad Z = -\frac{1}{2} \sum_{s=0}^{m-1} \sum_{0 \leq t \neq t' \leq n-1} (s + \frac{1}{2}) \partial_{s,t} \partial_{s,t'}.$$  (3.1)

We set further

$$e_i = \sum_{t=0}^{n-1} \partial_{i,t}, \quad \hat{e}_j = -\sum_{s=0}^{m-1} \partial_{s,j}.$$  (3.2)

Identifying $1 \otimes \cdots \otimes b_{\pm r}^i 0 \otimes \cdots \otimes 1$ with $d^jr b_{\pm r}^{i,j}$, we obtain formulas for the action of generators of $\mathcal{E}_m$, which we summarize below.

Proposition 3.3. Let $m \geq 2$, $n \geq 1$, $u_0, \ldots, u_{n-1} \in \mathbb{C}^\times$. The following formulas give a level $n$ action of $\mathcal{E}_m$ on $\mathbb{F}_{m,n}$:

$$q^{e_i} = q^{e_i}, \quad C = q^n, \quad D = q^{\text{deg}},$$  (3.3)

$$H_i, r = \sum_{j=0}^{n-1} q_j^{jr} b_{r}^{i,j}, \quad H_i, -r = \sum_{j=0}^{n-1} q_j^{(n-1)r} b_{-r}^{i,j},$$  (3.4)

$$E_i(z) = \sum_{j=0}^{n-1} u_j^{\delta_{i,0}} E_i^{i,j}(z), \quad E_i^{i,j}(z) =: e^{A^{i,j}(z)} : U_i^{i,j}(z),$$  (3.5)

$$F_i(z) = \sum_{j=0}^{n-1} u_j^{\delta_{i,0}} F_i^{i,j}(z), \quad F_i^{i,j}(z) =: e^{B^{i,j}(z)} : V_i^{i,j}(z).$$  (3.6)
where \(0 \leq i \leq m - 1, \ 0 \leq l \leq n - 1\) and \(r > 0\). We set for \(1 \leq i \leq m - 1\)

\[
U^{i,j}(z) = e^{-e_i,j} e^{e_{i-1,j}} (q^{n-1-j}z)^{\partial_{i-1,j} - \partial_{i,j} + 1}
\times d^{(1/2-i)} e^{e_{i-1,j} + (1/2+i)} e^{e_{i,j}} q^{\sum_{t=j+1}^{n-1} (\partial_{t-1,t} - \partial_{t,t})}.
\]

(3.6)

\[
V^{i,j}(z) = e^{-e_{i-1,j}} e^{e_{i,j}} (q^j z)^{-\partial_{i-1,j} + \partial_{i,j} + 1}
\times d^{-(1/2-i)} e^{e_{i-1,j} - (1/2+i)} e^{e_{i,j}} q^{\sum_{t=0}^{j-1} (\partial_{t-1,t} - \partial_{t,t})}.
\]

(3.7)

and for \(i = 0\)

\[
U^{0,j}(z) = e^{-e_{0,j}} e^{e_{m-1,j}} (q^{n-1-j}z)^{\partial_{m-1,j} - \partial_{0,j} + 1}
\times d^{(1/2-m)} e^{e_{m-1,j} + (1/2+i)} e^{e_{m-1,j}} q^{\sum_{t=j+1}^{n-1} (\partial_{m-1,t} - \partial_{0,t})}.
\]

(3.8)

\[
V^{0,j}(z) = e^{-e_{m-1,j}} e^{e_{0,j}} (q^j z)^{-\partial_{m-1,j} + \partial_{0,j} + 1}
\times d^{-(1/2-m)} e^{e_{m-1,j} - (1/2+i)} e^{e_{m-1,j}} q^{\sum_{t=0}^{j-1} (\partial_{m-1,t} - \partial_{0,t})}.
\]

(3.9)

□

Denote this module by \(\mathbb{F}_{m,n}(u_0, \ldots, u_{n-1})\). From Lemma 3.2, we obtain the following.

**Lemma 3.4.** We have a decomposition into submodules

\[
\mathbb{F}_{m,n}(u_0, \ldots, u_{n-1}) = \bigoplus_{s_0, \ldots, s_{n-1} \in \mathbb{Z}} \mathbb{F}^{(v_0)}(u_0^{(s_0)}; t^{(s_0)}, s_0) \otimes \cdots \otimes

\mathbb{F}^{(v_{n-1})}(u_{n-1}^{(s_{n-1})}; t^{(s_{n-1})}, s_{n-1}),
\]

where \(s_j = ml_j + v_j (l_j \in \mathbb{Z}, 0 \leq v_j \leq m - 1)\), and

\[
u^{(s_j)} = (-1)^m d^{-s_j - m/2} q u_j, \quad t^{(s_j)} = \frac{v_j}{2} (l_j + 1)^2 + \frac{m - v_j}{2} l_j^2.
\]

□

We remark that Proposition 3.3 holds true also for \(m = 1\) with the modification

\[
E_0(z) = a^{-1} \sum_{j=0}^{n-1} u_{j}^{(-1)} : e^{A_{0,j}(z)} : d^{\partial_{0,j}}, \quad F_0(z) = b^{-1} \sum_{j=0}^{n-1} u_{j} : e^{B_{0,j}(z)} : d^{-\partial_{0,j}},
\]

with \(ab = q(1 - q_1)(1 - q_3)\). In this paper we restrict ourselves to the case \(m \geq 2\).

### 3.3. Representations of \(\tilde{E}_n\)

In a similar manner we can write a representation of \(\tilde{E}_n\) on the same Fock space. We write the generators of \(\tilde{E}_n\) with checks, as \(\tilde{E}_i(z), \tilde{F}_i(z), \tilde{K}_i^\pm(z)\) and so forth.
**Proposition 3.5.** Let $n \geq 2$, $m \geq 1$, $u_0, \ldots, u_{m-1} \in \mathbb{C}^\times$. The following formulas give a level $m$ action of $\hat{\mathfrak{g}}_n$ on $\mathbb{F}_{m,n}$:

$$q^{\hat{e}_{ij}} = q^{\hat{\mathfrak{g}}_{ij}}, \quad \hat{C} = q^m, \quad \hat{D} = q^{\text{deg}}, \quad (3.10)$$

$$\hat{H}_{j,r} = \sum_{i=0}^{m-1} q_3^{-ir} \hat{h}_{r,ij}, \quad \hat{H}_{j,-r} = \sum_{i=0}^{m-1} q_1^{-(m-1)r} q_1^{-ir} \hat{h}_{r,ij}, \quad (3.11)$$

$$\hat{E}_j(z) = \sum_{i=0}^{m-1} u_i^{-\delta_j,0} \hat{e}_{i,j}(z), \quad \hat{E}_{i,j}(z) =: e^\hat{A}_{i,j}(z) : \hat{U}_{i,j}(z), \quad (3.12)$$

$$\hat{F}_j(z) = \sum_{i=0}^{m-1} u_i^{-\delta_j,0} \hat{f}_{i,j}(z), \quad \hat{F}_{i,j}(z) =: e^\hat{B}_{i,j}(z) : \hat{V}_{i,j}(z), \quad (3.13)$$

where $0 \leq i \leq m - 1$, $0 \leq j \leq n - 1$ and $r > 0$. We set for $1 \leq j \leq n - 1$

$$\hat{U}_{i,j}(z) = e^{\hat{F}_{i,j}} e^{-\hat{E}_{i,j}} (q^{m-1-i} z)^{-\hat{h}_{i,j-1} + \hat{h}_{j+i+1}} \times d^{(1/2-j)} \hat{E}_{j-1+(1/2+j)} \hat{E}_{j-1} (\hat{h}_{i,j-1} - \hat{h}_{j+1}) \sum_{s=1}^{m-1} (\hat{s}_{i,j-1} - \hat{s}_{j,s}), \quad (3.14)$$

$$\hat{V}_{i,j}(z) = e^{\hat{F}_{i,j}} e^{-\hat{E}_{i,j}} (q^{j} z)^{\hat{h}_{i,j-1} - \hat{h}_{j+1}} \times d^{-(1/2-j)} \hat{E}_{j-1-(1/2+j)} \hat{E}_{j-1} (\hat{h}_{i,j-1} - \hat{h}_{j+1}) q^{-\sum_{s=0}^{i-1} (\hat{s}_{i,j-1} - \hat{s}_{j,s})}, \quad (3.15)$$

and for $j = 0$

$$\hat{U}_{i,0}(z) = e^{\hat{F}_{i,0}} e^{-\hat{E}_{i,0}} (q^{m-1-i} z)^{-\hat{h}_{i,n-1} + \hat{h}_{i,0}+1} \times d^{(1/2-n)} \hat{E}_{n-1+(1/2)} \hat{E}_{n} (\hat{h}_{i,n-1} - \hat{h}_{0,0}+1) \sum_{s=1}^{m-1} (\hat{s}_{i,n-1} - \hat{s}_{0,s}), \quad (3.16)$$

$$\hat{V}_{i,0}(z) = e^{\hat{F}_{i,0}} e^{-\hat{E}_{i,0}} (q^{i} z)^{\hat{h}_{i,n-1} - \hat{h}_{0,0}+1} \times d^{-(1/2-n)} \hat{E}_{n-1-(1/2)} \hat{E}_{n} (\hat{h}_{i,n-1} - \hat{h}_{0,0}+1) q^{-\sum_{s=0}^{i-1} (\hat{s}_{i,n-1} - \hat{s}_{0,s})}. \quad (3.17)$$

$\square$

### 3.4. Quantum affine subalgebras

Recall that we have a vertical subalgebra $U_m \subset \mathfrak{e}_m$ isomorphic to the quantum affine algebra $U_q \mathfrak{g}_m$ generated by the currents $E_i(z)$, $F_i(z)$, $1 \leq i \leq m - 1$, and $D$. We also have the vertical subalgebra $\mathfrak{U}_n \subset \hat{\mathfrak{g}}_n$ generated by $\hat{E}_i(z), \hat{F}_i(z)$ with $1 \leq i \leq n - 1$, and $\hat{D}$.

**Theorem 3.6.** On the total Fock space $\mathbb{F}_{m,n}$, the action of the vertical subalgebras $U_m$ and $\mathfrak{U}_n$ mutually commute. $\square$

Theorem 3.6 will be proved in Appendix C. The picture is summarized in Fig. 1.

The Fock space $\mathcal{F}_m^{(v)}(u; t, s)$ can be constructed as a semi-infinite wedge product of level zero vector representations, see [FJMM3]. The usual $(\mathfrak{g}_m, \mathfrak{g}_n)$ duality uses either symmetric or skew-symmetric powers of vector representations. Therefore, it is natural to think that our construction is a quantum affine analog of the latter.
4. \((E_m, \check{E}_n)\) Duality

In this section, we present integrals of motion associated with quantum toroidal algebras \(E_m\) and \(\check{E}_n\) following [FJM]. We then state the duality.

4.1. Dressed currents. Recall that we deal with algebras \(E_m = E_m(q_1, q_2, q_3)\) and \(E_n^{\vee} = E_n^{\vee}(\check{q}_1, q_2, \check{q}_1)\) where \(q_1, q_2, \check{q}_1\) are independent parameters. We set

\[
p = \check{q}_1^{m}, \quad p^* = \check{q}_3^{-n}, \quad \check{p} = q_1^{n}, \quad \check{p}^* = q_3^{-m}.
\]  

(4.1)

We assume that \(|p|, |p^*|, |\check{p}|, |\check{p}^*| < 1\).

Introduce the dressed currents

\[
E_i(z) = K_i^-(z)^{-1}E_i(z), \quad K_i^- (z) = \prod_{l \geq 0} \check{K}_i^- (p^*^l z),
\]

(4.2)

\[
F_i(z) = F_i(z)K_i^+(z)^{-1}, \quad K_i^+ (z) = \prod_{l \geq 0} \check{K}_i^+ (p^{-l} z),
\]

(4.3)

where \(\check{K}_i^\pm (z) = (K_i)^\mp K_i^\pm (z)\), see (A.2). They are given respectively by formulas (3.4) and (3.5), replacing \(A^{i,j}(z), B^{i,j}(z)\) by the currents \(A^{i,j}(z), B^{i,j}(z)\) with Fourier coefficients

\[
A_{r}^{i,j} = \frac{-1}{[r]} q^{-(n-1)r} \check{q}_3^{-jr} b_r^{i,j},
\]

(4.4)

\[
A_{-r}^{i,j} = \frac{1}{[r]} q^{(n-2)r} \check{q}_3^{jr} \left( b_{-r}^{i,j} - \frac{1 - q_2^{-r}}{1 - p^{-r}} \sum_{t=0}^{n-1} \check{q}_3^{-tr} b_{-r}^{i,j-t} \right),
\]

(4.5)

\[
B_{r}^{i,j} = \frac{1}{[r]} q^r \check{q}_1^{jr} \left( b_r^{i,j} - \frac{1 - q_2^{-r}}{1 - p^r} \sum_{t=0}^{n-1} \check{q}_1^{tr} b_r^{i,j+t} \right), \quad B_{-r}^{i,j} = -\frac{1}{[r]} \check{q}_1^{-jr} b_{-r}^{i,j}.
\]

The dual counterparts are

\[
\check{E}_i(z) = K_i^- (\check{p})^l E_i(z), \quad \check{K}_i^- (z) = \prod_{l \geq 0} \check{K}_i^- (\check{p}^*^l z),
\]

(4.6)

\[
\check{F}_i(z) = F_i(z)K_i^+ (\check{p})^{-l}, \quad \check{K}_i^+ (z) = \prod_{l \geq 0} \check{K}_i^+ (p^{-l} z),
\]

(4.7)
with
\[
\tilde{A}^{i,j}_r = -\frac{1}{[r]} q^{-(m-1)r} q_3^{-ir} \tilde{b}^{i,j}_r,
\]
\[
\tilde{A}^{i,j}_{-r} = \frac{1}{[r]} q^{(m-2)r} q_3^{ir} \left( \tilde{b}^{i,j}_{-r} - \frac{1 - q_2}{1 - \tilde{p}^sr} \sum_{s=0}^{m-1} q_3^{-sr} \tilde{b}^{i-s,j}_r \right),
\]
\[
\tilde{B}^{i,j}_r = \frac{1}{[r]} q^{(m-2)r} q_3^{ir} \left( \tilde{b}^{i,j}_{-r} - \frac{1 - q_2}{1 - \tilde{p}^sr} \sum_{s=0}^{m-1} q_3^{-sr} \tilde{b}^{i+s,j}_r \right),
\]
\[
\tilde{B}^{i,j}_{-r} = -\frac{1}{[r]} q^{ir} \tilde{b}^{i,j}_{-r}.
\]

The currents \( A^{i,j}(z), B^{i,j}(z) \) are \( m \)-periodic in \( i \), whereas \( \tilde{A}^{i,j}(z), \tilde{B}^{i,j}(z) \) are \( n \)-periodic in \( j \). They are quasi-periodic with respect to the other index,

\[
A^{i,j-n}(z) = A^{i,j}(p^* z), \quad B^{i,j-n}(z) = B^{i,j}(pz),
\]
\[
\tilde{A}^{i-m,j}(z) = \tilde{A}^{i,j}(\bar{p}^* z), \quad \tilde{B}^{i-m,j}(z) = \tilde{B}^{i,j}(\bar{p} z).
\]

4.2. Integrals of motion. Elliptic deformation of integrals of motion in CFT were constructed in [FKSW, FKSW1, KS] as operators acting on Fock spaces. In [FJM] they were identified with Taylor coefficients of transfer matrices associated with quantum toroidal algebras. Let us briefly recall this.

We introduce parameters \( \bar{p}, \bar{p}_1, \ldots, \bar{p}_{m-1} \) and set

\[
p = \bar{p} q^{-n}, \quad p^* = \bar{p} q^n, \quad p_i = \bar{p}_i q^{-\varepsilon_i-1+\varepsilon_i}, \quad p^*_i = \bar{p}_i q^{\varepsilon_i-1-\varepsilon_i},
\]

where \( p, p^* \) are those in (4.1). Let \( R \) be the universal \( R \) matrix of \( E_m \) corresponding to the coproduct \( \Delta \) in Appendix A. Transfer matrices are weighted traces over \( \mathcal{T}^\mu(u) = \mathcal{T}^\mu_m(u; 0, 0) \),

\[
T_\mu(u; p) = \text{Tr}_{\mathcal{T}^\mu(u), 1} \left( (\bar{p}^d \prod_{i=1}^{m-1} \bar{p}_i^{-\tilde{A}_i})_1 R_{12} \right) q^d,
\]
\[
T^\mu(u; p^*) = \text{Tr}_{\mathcal{T}^{\mu*(u), 1}} \left( (\bar{p}^d \prod_{i=1}^{m-1} \bar{p}_i^{-\tilde{A}_i})_1 R_{21}^{-1} \right) q^{-d}.
\]

Here we set \( D = q^d, \tilde{A}_i = \varepsilon_1 + \cdots + \varepsilon_i \), the suffixes 1, 2 refer to the tensor components and trace is taken on the first component.

The transfer matrices (4.14), (4.15) are formal series in \( u^{\mp 1} \). Integrals of motion are defined as their coefficients (up to some normalization constants \( c_N, c^*_N \) which are irrelevant here, see [FJM] for the explicit formulas in cases of \( m = 1, 2 \)).

\[
T_\mu(u; p) = \sum_{M=0}^\infty u^{-M} c_M G_{\mu,M}(p), \quad T^\mu(u; p^*) = \sum_{M=0}^\infty u^M c^*_M \cdot G^*_M(p^*).
\]

We call \( G_{\mu,M}(p), G^*_M(p^*) \) integrals of motion of the first and the second kind, respectively. In order to write explicit formulas for them, we prepare some symbols.
Let $\vartheta_{\mu}^{(m)}$ denote the theta function associated with the root lattice $\tilde{Q}^{(m)}$ of $\mathfrak{sl}_m$:

$$\vartheta_{\mu}^{(m)}(z_1, \ldots, z_m; p) = \sum_{\beta \in \tilde{Q}^{(m)} + \Lambda_\mu} p^{(\beta, \beta)/2} \prod_{s=1}^{m-1} p_s^{-\langle \beta, \alpha_s \rangle} \prod_{i=1}^{m} z_i^{\langle \beta, \alpha_i \rangle}.$$ 

Define functions $h_{\mu, M}, h_{\mu, M}^*$ by

$$h_{\mu, M}(x_1, \ldots, x_M; x_{m, M}; p) = \prod_{i=1}^{m} \prod_{a < b} \Theta_p(x_{i, b}/x_{i, a} \cdot q_2 x_{i, b}/x_{i, a}) \prod_{a, b} \Theta_p(q_1 x_{1, b}/x_{m, a}) \prod_{i=1}^{m} x_{i, a}^{M-2a+1} \cdot \vartheta_{\mu}^{(m)} \left( \prod_{a=1}^{M} x_{1, a}, \ldots, \prod_{a=1}^{M} x_{m, a}; p \right).$$

$$h_{\mu, M}^*(x_1, \ldots, x_M; x_{m, M}; p^*) = \prod_{i=1}^{m} \prod_{a < b} \Theta_p^*(x_{i, b}/x_{i, a} \cdot q_2 x_{i, b}/x_{i, a}) \prod_{a, b} \Theta_p^*(q_1 x_{1, b}/x_{m, a}) \prod_{i=1}^{m} x_{i, a}^{M-2a+1} \cdot \vartheta_{\mu}^{(m)} \left( \prod_{a=1}^{M} x_{1, a}, \ldots, \prod_{a=1}^{M} x_{m, a}; p^* \right).$$

We have the quasi-periodicity property

$$h_{\mu, M}(x_1, \ldots, p x_{i, a}, \ldots, x_{m, M}; p) = p_i^{M-2a+1+M(\delta_{i, m} - \delta_{i, 1})},$$

$$h_{\mu, M}^*(x_1, \ldots, p^* x_{i, a}, \ldots, x_{m, M}; p^*) = (p_i^*)^{-1} q_2^{M-2a+1+M(\delta_{i, m} - \delta_{i, 1})},$$

where $p_m := (p_1 \cdots p_{m-1})^{-1}, p_m^* := (p_1^* \cdots p_{m-1}^*)^{-1}$.

**Proposition 4.1 [FJM].** The integrals of motion $G_{\mu, M}(p), G_{\mu, M}^*(p^*)$ are given by the following multiple integrals.

$$G_{\mu, M}(p) = \int \cdots \int \prod_{i=1}^{m} \prod_{a=1}^{M} \frac{dx_{i, a}}{2\pi \sqrt{-1} x_{i, a}} \prod_{1 \leq a \leq M} \mathcal{F}_1(x_{1, a}) \cdots \prod_{1 \leq a \leq N} \mathcal{F}_m(x_{m, a}) \times h_{\mu, M}(x_1, \ldots, x_M, \ldots, x_{m, M}; p),$$

$$G_{\mu, M}^*(p^*) = \int \cdots \int \prod_{i=1}^{m} \prod_{a=1}^{M} \frac{dx_{i, a}}{2\pi \sqrt{-1} x_{i, a}} \prod_{1 \leq a \leq M} \mathcal{E}_m(x_{m, a}) \cdots \prod_{1 \leq a \leq N} \mathcal{E}_1(x_{1, a}) \times h_{\mu, M}^*(x_1, \ldots, x_M, \ldots, x_{m, M}; p^*),$$

where $\mathcal{F}_m(z) = \mathcal{F}_0(z), \mathcal{E}_m(z) = \mathcal{E}_0(z)$. The integrals in $G_{\mu, M}(p)$ (resp. $G_{\mu, M}^*(p^*)$) are taken to be the unit circle $|x_{i, a}| = 1$ when $|q_1|, |q_3| < 1$ (resp. $|q_1|, |q_3| > 1$), and by analytic continuation in the general case. □
More details about the contours will be given in Sect. D.1.

Operators $G_{\mu, M}(p), G_{\mu, M}^*(p^*)$ act on each subspace of fixed ‘weight’ $m,$

$$\mathbb{C}[[a_{i, j}]_{r > 0, 0 \leq j \leq m - 1}] \otimes |m \rangle \subset F_{m, n}.$$ 

By construction they commute with each other,

$$[G_{\mu, M}(p), G_{\nu, N}(p)] = [G_{\mu, M}(p), G_{\nu, N}^*(p^*)] = [G_{\mu, M}(p^*), G_{\nu, N}(p^*)] = 0.$$

4.3. Duality. In a similar manner, we have the dual counterparts of integrals of motion

$$\tilde{G}_{\nu, N}(\tilde{p}) = \int \cdots \int \prod_{j=1}^{n} \prod_{a=1}^{N} \frac{dy_{j, a}}{2\sqrt{\gamma}} y_{j, a} \prod_{1 \leq a \leq N} \tilde{F}_1(y_{1, a}) \cdots \prod_{1 \leq a \leq N} \tilde{F}_n(y_{n, a})$$

$$\times \tilde{h}_{\nu, N}(y_1, \ldots, y_1, N, \ldots, y_1, 1, \ldots, y_N; \tilde{p}),$$

$$\tilde{G}_{\nu, N}^*(\tilde{p}^*) = \int \cdots \int \prod_{j=1}^{n} \prod_{a=1}^{N} \frac{dy_{j, a}}{2\sqrt{\gamma}} y_{j, a} \prod_{1 \leq a \leq N} \tilde{E}_1(y_{1, a}) \cdots \prod_{1 \leq a \leq N} \tilde{E}_n(y_{n, a})$$

$$\times \tilde{h}_{\nu, N}^*(y_1, \ldots, y_1, N, \ldots, y_1, 1, \ldots, y_N; \tilde{p}^*).$$

They depend on parameters $\tilde{p} = (\tilde{p}, \tilde{p}_1, \ldots, \tilde{p}_{n-1}),$ $\tilde{p}^* = (\tilde{p}^*, \tilde{p}_1^*, \ldots, \tilde{p}_{n-1}^*)$ related to $\tilde{p}, \tilde{p}_1, \ldots, \tilde{p}_{n-1}$ in the same way as in (4.12) (with $m$ and $n$ interchanged).

The following is the main result of this paper.

**Theorem 4.2.** Assume that the parameters $q, q_1, \tilde{q}_1, u_i, \tilde{u}_i$ are generic. Then we have

$$[G_{\mu, M}(p), \tilde{G}_{\nu, N}(\tilde{p})] = [G_{\mu, M}^*(p^*), \tilde{G}_{\nu, N}(\tilde{p})]$$

$$= [G_{\mu, M}(p), \tilde{G}_{\nu, N}^*(\tilde{p}^*)] = [G_{\mu, M}^*(p^*), \tilde{G}_{\nu, N}^*(\tilde{p}^*)] = 0$$

for all $\mu \in \mathbb{Z}/m\mathbb{Z}, \nu \in \mathbb{Z}/n\mathbb{Z}$ and $M, N \geq 1,$ provided

$$\tilde{p}_i = \frac{\tilde{u}_i}{\tilde{u}_{i-1}} \quad (1 \leq i \leq m - 1),$$

$$\tilde{p}_l = \frac{u_l}{u_{l-1}} \quad (1 \leq l \leq n - 1).$$

(4.18)

(4.19)

□

Theorem 4.2 will be proved in Appendix D.

**Remark 4.3.** When one of the components of $p$ is zero, namely, $\tilde{p} = 0,$ see (4.12), (4.13), the algebra of integrals of motion of $\tilde{\mathcal{E}}_k$ coincides with the Bethe algebra of the horizontal $U_q\hat{\mathfrak{gl}}_k$ algebra. The subspace of the top degree in the Fock module $\mathcal{F}_k^{(v)}(u; t, s)$ is preserved by the horizontal algebra and as $U_q\hat{\mathfrak{gl}}_k$ module is isomorphic to evaluation module associated to the $v$-th fundamental representation of $U_q\hat{\mathfrak{gl}}_k.$ Thus Theorem 4.2 implies the duality of $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ XXZ systems.
In this section we give the definition of the algebra $E_m$. Although we take $m \geq 2$ in the main text, we include the definition for $m = 1$ for completeness.

Let $P$ be a free $\mathbb{Z}$ module with basis $\varepsilon_i, i \in \mathbb{Z}/m\mathbb{Z}$, equipped with the inner product $( , ) : P \times P \to \mathbb{Z}$ such that $\varepsilon_i$ are orthonormal. We set $\bar{\alpha}_i = \varepsilon_{i-1} - \varepsilon_i$.

Define further functions $g_{i,j}(z,w)$ by

$$g_{i,j}(z,w) = \begin{cases} 
z - q_1 w & (i \equiv j - 1), \\
z - q_2 w & (i \equiv j), \\
z - q_3 w & (i \equiv j + 1), \\
z - w & (i \neq j, j \pm 1), 
\end{cases}$$

and set $d_{i,j} = d_i^{-1} (i \equiv j \equiv 1, m \geq 3) = -1 (i \neq j, m = 2), = 1$ (otherwise).

By definition, $E_m = E_m(q_1, q_2, q_3)$ is a unital associative algebra generated by $E_{i,k}, F_{i,k}, H_{i,r}$ and invertible elements $q^h, C, D$, where $i \in \mathbb{Z}/m\mathbb{Z}, k \in \mathbb{Z}, r \in \mathbb{Z}\setminus\{0\}, h \in P$. We write $K_i = q^{\bar{\alpha}_i}$. The defining relations are given in terms of generating series

$$E_i(z) = \sum_{k \in \mathbb{Z}} E_{i,k} z^{-k}, \quad F_i(z) = \sum_{k \in \mathbb{Z}} F_{i,k} z^{-k}, \quad (A.1)$$

$$K_i^\pm(z) = K_i^{\pm1} \exp(\pm (q - q^{-1}) \sum_{r > 0} H_{i,\pm r} z^{r}). \quad (A.2)$$

The relations are as follows.

$C, K$ relations

$C$ is central, $q^h q^{h'} = q^{h+h'}$ ($h, h' \in P$), $q^0 = 1$, $D q^h = q^h D$,

$q^h E_i(z) q^{-h} = q^{(h,\bar{\alpha}_i)} E_i(z), \quad q^h F_i(z) q^{-h} = q^{-(h,\bar{\alpha}_i)} F_i(z),$

$q^h K_i^\pm(z) = K_i^{\pm1} (z) q^h$ ($h \in P$),

$DE_i(z) D^{-1} = E_i(q z), \quad DF_i(z) D^{-1} = F_i(q z) \quad DK_i^\pm(z) D^{-1} = K_i^\pm(q z),$

$H-E$ and $H-F$ relations For $r \neq 0$,

$$[H_{i,r}, E_j(z)] = a_{i,j}(r) C^{-(r+|r|)/2} z^r E_j(z),$$

$$[H_{i,r}, F_j(z)] = -a_{i,j}(r) C^{-(r-|r|)/2} z^r F_j(z),$$
\[ [H_{i,r}, H_{j,s}] = \delta_{r+s,0} \cdot a_{i,j}(r) \frac{C^r - C^{-r}}{q - q^{-1}}, \]

where
\[ a_{i,j}(r) = \frac{|r|}{r} \left( (q^r + q^{-r})\delta_{i,j}^{(m)} - d^r \delta_{i,j-1}^{(m)} - d^{-r} \delta_{i,j+1}^{(m)} \right). \]

**E-F relations**
\[ [E_i(z), F_j(w)] = \frac{\delta_{i,j}}{q - q^{-1}} \left( \delta(C \frac{w}{z}) K_i^+(w) - \delta(C \frac{z}{w}) K_i^-(z) \right). \]

**E-E and F-F relations**
\[ [E_i(z), E_j(w)] = 0, \quad [F_i(z), F_j(w)] = 0 \quad (i \neq j, j \pm 1), \]
\[ d_{i,j} g_{i,j}(w) E_i(z) E_j(w) + g_{j,i}(w, z) E_j(w) E_i(z) = 0, \]
\[ d_{j,i} g_{j,i}(w, z) F_i(w) F_j(w) + g_{i,j}(z, w) F_j(w) F_i(z) = 0. \]

We omit the Serre relations which are not used in this paper.

The currents \( E_i(z), F_i(z), i \in \mathbb{Z}/m\mathbb{Z} \), together with \( D \), generate a subalgebra \( \mathfrak{U}_m \subset \mathcal{E}_m \) isomorphic to the quantum affine algebra \( U_q \mathfrak{g} \mathfrak{l}_m \). We call \( \mathfrak{U}_m \) the vertical subalgebra.

The elements \( E_{i,0}, F_{i,0}, q^h, i \in \mathbb{Z}/m\mathbb{Z}, h \in P \), also generate a subalgebra isomorphic to the quantum affine algebra \( U_q \hat{\mathfrak{g}} \mathfrak{l}_m \) which we call the horizontal subalgebra.

In particular, note that in terminology of \([\text{FJM,FJMM}]\) \( E_i(z), F_i(z), K_i(z) \) are perpendicular generators. Since we rarely use other generators in this paper we omit the sign \( \perp \) used in those papers from our notation.

We use the following coproduct, which is opposite to the one used in \([\text{FJM}]\):
\[ \Delta x = x \otimes x \quad (x = q^x, C, D), \]
\[ \Delta E_i(z) = E_i(C_2 z) \otimes K_i^-(z) + 1 \otimes E_i(z), \]
\[ \Delta F_i(z) = F_i(z) \otimes 1 + K_i^+(z) \otimes F_i(C_1 z), \]
\[ \Delta K_i^+(z) = K_i^+(z) \otimes K_i^+(C_1 z), \]
\[ \Delta K_i^-(z) = K_i^-(C_2 z) \otimes K_i^-(z). \]

Here \( C_1 = C \otimes 1 \) and \( C_2 = 1 \otimes C \).

Highest weight modules of \( \mathcal{E}_m \) are defined in terms of generators \( \theta^{-1}(E_i(z)), \theta^{-1}(F_i(z)) \), \( \theta^{-1}(K_i^\pm(z)) \) obtained by applying Miki’s automorphism \( \theta \) which interchanges the vertical and horizontal subalgebras \([\text{Mi}]\), see also \([\text{FJM,M}]\). Let \( P = (P_0(z), \ldots, P_{m-1}(z)) \in \mathbb{C}(z)^m \) be an \( m \)-tuple of rational functions which are regular at \( z^{\pm 1} = \infty \) and satisfy \( P_i(0)P_i(\infty) = 1 \). An \( \mathcal{E}_m \) module is a highest weight module of highest weight \( P \) if it is generated by a vector \( w \) satisfying
\[ \theta^{-1}(E_i(z))w = 0 \quad (i = 0, 1, \ldots, m - 1), \]
\[ \theta^{-1}(K_i^\pm(z))w = P_i(z)w \quad (i = 0, 1, \ldots, m - 1). \]

In the last line, \( P_i(z) \) stands for its expansion at \( z^{\pm 1} = \infty \).

The following formulas are used to calculate highest weights of level one modules:
\[ \theta^{-1}(H_{i,1}) = -(-d)^{-i} \left[ \cdots \cdot \left[ F_{0,0}, F_{m-1,0} \right]_q \cdots, F_{i+1,0} q, F_{i,0} \right]_q, \quad (1 \leq i \leq m - 1) \]
\[ F_{i-1,0} q, F_{i,0} q^2, \quad (0 \leq i \leq m - 1) \]
\[ \theta^{-1}(H_{0,1}) = -(-d)^{-m+1} \left[ \cdots \cdot \left[ F_{1,1}, F_{2,0} \right]_q, \cdots, F_{m-1,0} q, F_{0,-1} q^2 \right]_q, \]

where \( m \geq 2 \) and \([X, Y]_p = XY - pYX\).
Lemma B.1. Let $0 \leq i, k \leq m - 1, 0 \leq j, l \leq n - 1$. The contractions of the oscillator part of the non-dressed currents are given in Tables 1, 2, 3 and 4 below. In all other cases the contraction is 1. \(\square\)
Table 5. $E_{ij}^{(z)}(w)^{osc} E_{kl}^{(w)}(w)^{osc}$, $(z)_{\infty} = (z; p^*)_{\infty}$

|     | $j < l$                              | $j = l$                              | $j > l$                              |
|-----|--------------------------------------|--------------------------------------|--------------------------------------|
| $i \equiv k$ | $(q_2 w/z)_{\infty} q_2^{-1} w/z_{\infty}$ | $(1 - w/z) (q_2 w/z)_{\infty} (p^* q_2^{-1} w/z_{\infty})^{-1}$ | $(p^* q_2 w/z_{\infty}) (p^* q_2^{-1} w/z_{\infty})^{-1}$ |
| $i + 1 \equiv k$ | $(q_1 w/z)_{\infty} q_3^{-1} w/z_{\infty}$ | $(q_1 w/z)_{\infty} (p^* q_3^{-1} w/z_{\infty})^{-1}$ | $(p^* q_3 w/z_{\infty}) (p^* q_3^{-1} w/z_{\infty})^{-1}$ |
| $i - 1 \equiv k$ | $(q_3 w/z)_{\infty} q_1^{-1} w/z_{\infty}$ | $(q_1 w/z)_{\infty} (p^* q_3^{-1} w/z_{\infty})^{-1}$ | $(p^* q_3 w/z_{\infty}) (p^* q_1^{-1} w/z_{\infty})^{-1}$ |

Table 6. $F_{ij}^{(z)}(w)^{osc} F_{kl}^{(w)}(w)^{osc}$, $(z)_{\infty} = (z; p)_{\infty}$

|     | $j < l$                              | $j = l$                              | $j > l$                              |
|-----|--------------------------------------|--------------------------------------|--------------------------------------|
| $i \equiv k$ | $(q_2^{-1} w/z_{\infty}) (q_2 w/z_{\infty})^{-1}$ | $(1 - w/z) (q_2^{-1} w/z_{\infty}) (p q_2 w/z_{\infty})^{-1}$ | $(p q_2 w/z_{\infty}) (p q_2^{-1} w/z_{\infty})^{-1}$ |
| $i + 1 \equiv k$ | $(q_1^{-1} w/z_{\infty}) (q_1 w/z_{\infty})^{-1}$ | $(p q_3^{-1} w/z_{\infty}) (q_1 w/z_{\infty})^{-1}$ | $(p q_3 w/z_{\infty}) (p q_1^{-1} w/z_{\infty})^{-1}$ |
| $i - 1 \equiv k$ | $(q_3^{-1} w/z_{\infty}) (q_3 w/z_{\infty})^{-1}$ | $(p q_1^{-1} w/z_{\infty}) (q_3 w/z_{\infty})^{-1}$ | $(p q_3 w/z_{\infty}) (p q_1^{-1} w/z_{\infty})^{-1}$ |

Table 7. $E_{ij}^{(z)}(w)^{osc} F_{kl}^{(w)}(w)^{osc}$

|     | $j = l$                              | $j \neq l$                              |
|-----|--------------------------------------|--------------------------------------|
| $i \equiv k$ | $(1 - q^{-n+2j} w/z_{\infty})^{-1} (1 - q^{-n+2j+2} w/z_{\infty})^{-1}$ | 1 |
| $i + 1 \equiv k$ | $1 - q^{-n+2j} q_3^{-1} w/z_{\infty}$ | 1 |
| $i - 1 \equiv k$ | $1 - q^{-n+2j} q_1^{-1} w/z_{\infty}$ | 1 |

Table 8. $F_{kl}^{(w)}(w)^{osc} E_{ij}^{(z)}(w)^{osc}$

|     | $j = l$                              | $j \neq l$                              |
|-----|--------------------------------------|--------------------------------------|
| $i \equiv k$ | $(1 - q^{-n-2j} z/w_{\infty})^{-1} (1 - q^{-n-2j-2} z/w_{\infty})^{-1}$ | 1 |
| $i + 1 \equiv k$ | $1 - q^{-n-2j} q_3 z/w_{\infty}$ | 1 |
| $i - 1 \equiv k$ | $1 - q^{-n-2j} q_1 z/w_{\infty}$ | 1 |

**Lemma B.2.** Let $0 \leq i, k \leq m - 1, 0 \leq j, l \leq n - 1$. The contractions for the oscillator part of the dressed currents are given by Tables 5, 6, 7 and 8 below. In Table 5 we use $(z)_{\infty} = (z; p^*)_{\infty}$ while in Table 6 we use $(z)_{\infty} = (z; p)_{\infty}$. □

**Lemma B.3.** Set

$$
\begin{align*}
\bar{a}_{i,k}^{(m)} & = -\delta_{i-1,k}^{(m)} + 2 \delta_{i,k}^{(m)} - \delta_{i+1,k}^{(m)}, \\
\bar{p}_{i,k}^{(m)} & = i \bar{a}_{i,k}^{(m)} + m \delta_{i,0}^{(m)} (\delta_{k,0}^{(m)} - \delta_{k,-1}^{(m)}).
\end{align*}
$$
The contractions of the zero mode part are given as follows.

\[ U^{i,j}(z)U^{k,l}(w) =: U^{i,j}(z)U^{k,l}(w) : \]
\[ = (q^{n-j-1}z^{-\delta_{j+1,k}} - \delta_{j+1,k}) d^{-1/2} (1 - \theta(j,l) \tilde{a}^{(m)}_{i,k}) q^{-\theta(j,l) \tilde{a}^{(m)}_{i,k}}. \]  

(B.1)

\[ V^{i,j}(z)V^{k,l}(w) =: V^{i,j}(z)V^{k,l}(w) : \]
\[ = (q^{j-1}z^{\delta_{j,l}}) d^{-1/2} (1 - \theta(j,l) \tilde{a}^{(m)}_{k,l}) q^{-\theta(j,l) \tilde{a}^{(m)}_{k,l}}. \]  

(B.2)

\[ U^{i,j}(z)V^{k,l}(w) =: U^{i,j}(z)V^{k,l}(w) : \]
\[ = (q^{n-j-1}z^{\delta_{j+1,k}} - \delta_{j+1,k}) d^{-1/2} (1 + \theta(j,l) \tilde{a}^{(m)}_{k,l}) q^{\theta(j,l) \tilde{a}^{(m)}_{k,l}}. \]  

(B.3)

\[ V^{k,l}(w)U^{i,j}(z) =: V^{k,l}(w)U^{i,j}(z) : \]
\[ = (q^{l}w^{\delta_{j+l,k}}) d^{-1/2} (1 + \theta(j,l) \tilde{a}^{(m)}_{i,k}) q^{\theta(j,l) \tilde{a}^{(m)}_{i,k}}. \]  

(B.4)

B.2. Contractions between \( \mathcal{E}_m \) and \( \tilde{\mathcal{E}}_n \).

Lemma B.4. Let \( 0 \leq i, k \leq m - 1, 0 \leq j, l \leq n - 1 \). The contractions of the oscillator part of the dressed currents with the dual ones are given by Tables 9, 10, 11, 12, 13 and 14 below. In all other cases the contractions are 1. \( \square \)
Lemma B.5. Let $0 \leq i, k \leq m - 1$ and $0 \leq j, l \leq n - 1$, and set

$$D_{i,k}^{(m)} = \delta_{i,k}^{(m)} - \delta_{i-1,k}^{(m)}, \quad D_{j,l}^{(n)} = \delta_{j,l}^{(n)} - \delta_{j,l-1}^{(n)}. \quad (B.5)$$

The contractions of the zero mode part are given as follows.

\[
\begin{align*}
U^{i,j}(z) \tilde{U}^{k,l}(w) &=: U^{i,j}(z) \tilde{U}^{k,l}(w) : \\
&= \left( q^{n-1-j} z \right) D_{i,k}^{(m)} D_{j,l}^{(n)} \delta_{i,k}^{(m)} - \delta_{i-1,k}^{(m)} D_{j,l}^{(n)} q D_{i,k}^{(m)} \delta_{j,l-1}^{(n)} - \delta_{j,l}^{(n)}, \\
\tilde{U}^{k,l}(w) U^{i,j}(z) &=: \tilde{U}^{k,l}(w) U^{i,j}(z) :] \\
&= \left( q^{m-1-k} w \right) D_{i,k}^{(m)} D_{j,l}^{(n)} \delta_{i,k}^{(m)} - \delta_{i-1,k}^{(m)} D_{j,l}^{(n)} q D_{i,k}^{(m)} \delta_{j,l-1}^{(n)} - \delta_{j,l}^{(n)} D_{j,l}^{(n)}, \\
V^{i,j}(z) \tilde{V}^{k,l}(w) &=: V^{i,j}(z) \tilde{V}^{k,l}(w) : \\
&= \left( q^{j} w \right) D_{i,k}^{(m)} D_{j,l}^{(n)} \delta_{i,k}^{(m)} - \delta_{i-1,k}^{(m)} D_{j,l}^{(n)} q D_{i,k}^{(m)} \delta_{j,l-1}^{(n)} - \delta_{j,l}^{(n)} D_{j,l}^{(n)}, \\
\tilde{V}^{k,l}(w) V^{i,j}(z) &=: \tilde{V}^{k,l}(w) V^{i,j}(z) :] \\
&= \left( q^{k} w \right) D_{i,k}^{(m)} D_{j,l}^{(n)} \delta_{i,k}^{(m)} - \delta_{i-1,k}^{(m)} D_{j,l}^{(n)} q D_{i,k}^{(m)} \delta_{j,l-1}^{(n)} - \delta_{j,l}^{(n)} D_{j,l}^{(n)}, \\
U^{i,j}(z) \tilde{V}^{k,l}(w) &=: U^{i,j}(z) \tilde{V}^{k,l}(w) : \\
&= \left( q^{n-1-j} z \right) D_{i,k}^{(m)} D_{j,l}^{(n)} \delta_{i,k}^{(m)} - \delta_{i-1,k}^{(m)} D_{j,l}^{(n)} q D_{i,k}^{(m)} \delta_{j,l-1}^{(n)} - \delta_{j,l}^{(n)} D_{j,l}^{(n)}, \\
\tilde{V}^{k,l}(w) U^{i,j}(z) &=: \tilde{V}^{k,l}(w) U^{i,j}(z) :] \\
&= \left( q^{k} w \right) D_{i,k}^{(m)} D_{j,l}^{(n)} \delta_{i,k}^{(m)} - \delta_{i-1,k}^{(m)} D_{j,l}^{(n)} q D_{i,k}^{(m)} \delta_{j,l-1}^{(n)} - \delta_{j,l}^{(n)} D_{j,l}^{(n)}.
\end{align*}
\]
The following Lemmas will be used in Section D. They can be proved by a straightforward calculation using (2.4),(2.5) or the definition (3.6)–(3.9) and (3.14)–(3.17).

Lemma B.6. Assume that \( 1 \leq i \leq m - 1 \) and \( 1 \leq l \leq n - 1 \). Then we have

\[
\begin{align*}
E^{i,l}(z) \tilde{E}^{i,l}(w) : +q^{-2} : E^{i,l-1}(z) \tilde{E}^{i-1,l}(w) := 0 & \quad \text{if } w = q^{-m+n} q_{\tilde{z}} z, \\
F^{i,l}(z) \tilde{F}^{i,l}(w) : +q^{2} : F^{i,l-1}(z) \tilde{F}^{i-1,l}(w) := 0 & \quad \text{if } w = q_{1} q_{1}^{-l} z, \\
\hat{E}^{i,l}(z) \hat{F}^{i,l}(w) : +q^{-2} : \hat{E}^{i,l-1}(z) \hat{F}^{i-1,l}(w) := 0 & \quad \text{if } w = q^{n} q_{1}^{-l} \hat{z}. 
\end{align*}
\]  

(\text{B.6}) (\text{B.7}) (\text{B.8})

Lemma B.7. Define

\[
\begin{align*}
E^{i,-1}(z) &= E^{i,n-1}(p z) d_{q}^{-n} q^{-\tilde{e}_{-1} + \tilde{e}_{i}}, \\
\tilde{E}^{-1,l}(z) &= \tilde{E}^{m-1,l}(p z) d_{q}^{-m} q^{-\tilde{e}_{-1} + \tilde{e}_{i}}, \\
F^{i,-1}(z) &= F^{i,n-1}(p z) d_{q}^{-n} q^{-\tilde{e}_{-1} + \tilde{e}_{i}}, \\
\tilde{F}^{-1,l}(z) &= \tilde{F}^{m-1,l}(p z) d_{q}^{-m} q^{-\tilde{e}_{-1} + \tilde{e}_{i}}.
\end{align*}
\]

Then for \( 0 \leq i \leq m - 1, 0 \leq l \leq n - 1 \) we have

\[
\begin{align*}
E^{i,l}(z) \tilde{E}^{i,l}(w) : +q^{-2} : E^{i,l-1}(z) \tilde{E}^{i-1,l}(w) := 0 & \quad \text{if } w = q^{-m+n} q_{3}^{-l} \hat{z}, \\
F^{i,l}(z) \tilde{F}^{i,l}(w) : +q^{2} : F^{i,l-1}(z) \tilde{F}^{i-1,l}(w) := 0 & \quad \text{if } w = q_{1}^{-l} \tilde{z}, \\
\hat{E}^{i,l}(z) \hat{F}^{i,l}(w) : +q^{-2} : \hat{E}^{i,l-1}(z) \hat{F}^{i-1,l}(w) := 0 & \quad \text{if } w = q^{n} q_{1}^{-l} \hat{z}.
\end{align*}
\]  

(\text{B.9}) (\text{B.10}) (\text{B.11})

\[\square\]

Appendix C. Proof of Theorem 3.6

In this section we prove Theorem 3.6. We begin with

Lemma C.1. Assume that \( 1 \leq i \leq m - 1, 1 \leq l \leq n - 1 \) and \( r \neq 0 \). Then

\[
[H_{s,r}, \tilde{H}_{s,-r}] = 0, \quad [H_{s,r}, \tilde{b}^{s,l}_{-r}] = 0 \quad (0 \leq s, s' \leq m - 1, 0 \leq t, t' \leq n - 1).
\]

Proof. Suppose \( r > 0 \). Using (2.8) we compute

\[
[H_{s,r}, \tilde{b}^{s,l}_{-r}] = -\frac{[r]^{2}}{r} q^{r} (\delta^{(m)}_{s-1,s'} - \delta^{(m)}_{s,s'}) \sum_{j=0}^{n-1} (q_{1}^{(j+1)k} \delta^{(n)}_{j,l-1} - q_{1}^{-1} \delta^{(n)}_{j,l}) = 0,
\]

where we use \( 1 \leq l \leq n - 1 \). The other cases are similar. \(\square\)

Lemma C.2. Assume that \( 1 \leq i, k \leq m - 1, 1 \leq j, l \leq n - 1 \). Then formulas for the contractions in Tables 9, 10, 11, 12, 13 and 14 hold true if we replace \( E^{i,j}(z) \overset{\text{osc}}{\Rightarrow}, F^{i,j}(z) \overset{\text{osc}}{\Rightarrow}, \tilde{E}^{k,l}(w) \overset{\text{osc}}{\Rightarrow}, \tilde{F}^{k,l}(w) \overset{\text{osc}}{\Rightarrow} \) by \( E^{i,j}(z) \overset{\text{osc}}{\Rightarrow}, F^{i,j}(z) \overset{\text{osc}}{\Rightarrow}, \tilde{E}^{k,l}(w) \overset{\text{osc}}{\Rightarrow}, \tilde{F}^{k,l}(w) \overset{\text{osc}}{\Rightarrow} \), respectively.
Proof. By the definition we have

\[ A_r^{i,j} = A_r^{i,j}, \quad A_{-r}^{i,j} = A_{-r}^{i,j} - \frac{q - q^{-1}}{1 - p^* r} H_{i,-r}, \]

\[ B_r^{i,j} = B_r^{i,j} + \frac{q - q^{-1}}{1 - p^* r} H_{i,r}, \quad B_{-r}^{i,j} = B_{-r}^{i,j}, \]

and similarly for those with checks. Hence the assertion follows from Lemma C.1. \( \Box \)

From Lemma C.1 we obtain the commutativity

\[ [H_{i,r}, \tilde{H}_{i,r}] = 0, \]
\[ [H_{i,r}, \tilde{E}_l(w)] = [H_{i,r}, \tilde{F}_l(w)] = 0, \]
\[ [E_i(z), \tilde{H}_{i,r}] = [F_i(z), \tilde{H}_{i,r}] = 0 \]

provided \( i, l \neq 0 \). To show Theorem 3.6 it remains to check the following.

**Lemma C.3.** Assuming \( 1 \leq i \leq m - 1, 1 \leq l \leq n - 1 \) we have

\[ [E_i(z), \tilde{E}_l(w)] = 0, \quad [F_i(z), \tilde{F}_l(w)] = 0, \]
\[ [E_i(z), \tilde{F}_l(w)] = 0, \quad [\tilde{E}_i(z), F_l(w)] = 0. \]

*Proof.* Lemma B.1 and Lemma B.5 allow us to compute the commutators of \( E_i^{l,j}(z), \]
\( F_i^{l,j}(z) \) with \( \tilde{E}_k^{l,j}(w), \tilde{F}_k^{l,j}(w) \). In each case only two terms survive, with the result

\[ [E_i^{l,j}(z), \tilde{E}_k^{l,j}(w)] = \delta\left(q^{m-n} q_i^* q_i^* q^{1-w}_3 \frac{1}{z}\right) (q^{m-i-1} \tilde{d}^l w)^{-1} \]
\[ \times \left( \delta_{i,k} \delta_{j,l} : E_i^{l,j}(z) \tilde{E}_k^{l,j}(w) : + \delta_{i-k, l} \delta_{j,l-1} q^{-2} : E_i^{l-1,j}(z) \tilde{E}_k^{l-1,j}(w) : \right), \]  
(C.1)

\[ [F_i^{l,j}(z), \tilde{F}_k^{l,j}(w)] = \delta\left(q_1^* q_i^* q^{1-w}_3 \frac{1}{z}\right) (q^{-i-1} \tilde{d}^l w)^{-1} \]
\[ \times \left( q^{-2} \delta_{i, k} \delta_{j,l} : F_i^{l,j}(z) \tilde{F}_k^{l,j}(w) : + \delta_{i, k} \delta_{j,l-1} : F_i^{l-1,j}(z) \tilde{F}_k^{l-1,j}(w) : \right), \]  
(C.2)

\[ [E_i^{l,j}(z), \tilde{F}_k^{l,j}(w)] = \delta\left(q^{n} q_i^* q_i^* q^{1-w}_3 \frac{1}{z}\right) (q^{-i-1} \tilde{d}^l w)^{-1} \]
\[ \times \left( \delta_{i-k, l} \delta_{j,l} : E_i^{l,j}(z) \tilde{F}_k^{l-1,j}(w) : + q^{-2} \delta_{i, k} \delta_{j,l-1} : E_i^{l-1,j}(z) \tilde{F}_k^{l-1,j}(w) : \right). \]  
(C.3)

Using Lemma B.6 and summing over \( j, k \), we obtain the desired equalities. \( \Box \)

**Appendix D. Commutativity of \( G_{\mu,M}(p) \) and \( \tilde{G}_{\nu,N}(\tilde{p}) \)**

In this section we prove Theorem 4.2. Since the working is all very similar, we illustrate the proof for the simplest integrals of motion of the first kind, namely

\[ G_{\mu,1}(p) = \int \cdots \int_{C} \prod_{s=1}^{m} \frac{d x_s}{2 \pi \sqrt{-1} x_s} F_1(x_1) \cdots F_m(x_m) h_{\mu,1}(x_1, \ldots, x_m; p), \]
\[ \tilde{G}_{\nu,1}(\tilde{p}) = \int \cdots \int_{\tilde{C}} \prod_{t=1}^{n} \frac{d y_t}{2 \pi \sqrt{-1} y_t} \tilde{F}_1(y_1) \cdots \tilde{F}_n(y_n) \tilde{h}_{\nu,1}(y_1, \ldots, y_n; \tilde{p}). \]
Fig. 2. The contour in the $x_i$ plane.

D.1. Contours. The integration cycles $\mathcal{C}, \mathcal{C}$ are specified as follows.

For $h_{\mu,1}(p)$, the poles of the integrand come from the denominators of $h_{\mu,1}(x_1, \ldots, x_m; p)$ given in (4.16), as well as from the contractions given in Table 6. Altogether the integrand takes the form

$$ F_1(x_1) \cdots F_m(x_m) h_{\mu,1}(x_1, \ldots, x_m; p) = \sum_{j_1, \ldots, j_m: \mathcal{F}_1, j_1(x_1) \cdots \mathcal{F}_m, j_m(x_m): \prod_{j=1}^m (q_1 x_{j+1}/x_j, q_3 x_j/x_{j+1}; p) \infty} \varphi_{j_1, \ldots, j_m}(x_1, \ldots, x_m) $$

with some holomorphic function $\varphi_{j_1, \ldots, j_m}(x_1, \ldots, x_m).$ Here we set $x_0 = x_m$ and $x_{m+1} = x_1.$ For each $i$, the poles with respect to the variable $x_i$ consist of two groups,

- $p^k q_3 x_{i-1}, p^k q_1 x_{i+1}$ ($k \geq 0$),
- $p^{-k} q^{-1}_1 x_{i-1}, p^{-k} q^{-1}_3 x_{i+1}$, ($k \geq 0$).

The cycle $\mathcal{C}$ is such that $x_i$ encircles the first group separating it from the second, see Fig. 2 below:

Similarly, $\mathcal{C}$ is such that $y_l$ encircles $p^k q_3 y_{l-1}, p^k q_1 y_{l+1}$ keeping $p^{-k} q^{-1}_1 y_{l-1}, p^{-k} q^{-1}_3 y_{l+1}$ outside for all $k \geq 0$, where $y_0 = y_n$ and $y_{n+1} = y_1$.

D.2. Commutators. First we note the following relations of formal series, which follow from Lemmas B.4 and B.5.

Lemma D.1. (1) If $i \equiv k \mod m$ and $j \equiv l \mod n$, then

$$ E^{i,j}(z) \tilde{E}^{k,l}(w) - q^{\delta_{i,0} - \delta_{l,0}} q^{k,l}(w) E^{i,j}(z) =: E^{i,j}(z) \tilde{E}^{k,l}(w) :$$

$$ \times \delta \left( q^{m-n} q_3^{-k} z^{-j} x^{-1} \frac{w}{z} \right) \left( q^{n-1-j} z \right)^{-1} d^{-k} q^{-\delta_{l,0}}. $$
\[ F^{i,j}(z) F^{k,l}(w) - q^{-\delta_{i,0} + \delta_{j,0}} F^{k,l}(w) F^{i,j}(z) =: F^{i,j}(z) \tilde{F}^{k,l}(w) : \]
\[ \times \delta\left(q_{1}^{-k} q_{1}^{j} \frac{w}{z}\right) (q^{j} z)^{-1} d^{-k} q^{-1 + \delta_{0,1}}. \]

(2) If \( i \equiv l \pmod{m} \) and \( j \equiv l - 1 \pmod{n} \), then
\[ E^{i,j}(z) \tilde{F}^{k,l}(w) - q^{-\delta_{i,0} + \delta_{j,0}} \tilde{F}^{k,l}(w) E^{i,j}(z) =: E^{i,j}(z) \tilde{F}^{k,l}(w) : \]
\[ \times \delta\left(q_{n}^{-m-n} q_{3}^{-j-1} \frac{w}{z}\right) (q^{n-1-j} z)^{-1} d^{-k} q^{-1 + \delta_{0,1}}, \]
\[ F^{i,j}(z) \tilde{F}^{k,l}(w) - q^{\delta_{i,0} - \delta_{j,0}} \tilde{F}^{k,l}(w) F^{i,j}(z) =: F^{i,j}(z) \tilde{F}^{k,l}(w) : \]
\[ \times \delta\left(q_{1}^{-k-1} q_{1}^{j+1} \frac{w}{z}\right) (q^{j} z)^{-1} d^{-k-1} q^{\delta_{0,1}}. \]

(3) If \( i \equiv k \pmod{m} \) and \( j \equiv l - 1 \pmod{n} \), then
\[ E^{i,j}(z) \tilde{F}^{k,l}(w) - q^{\delta_{i,0} - \delta_{j,0}} \tilde{F}^{k,l}(w) E^{i,j}(z) =: E^{i,j}(z) \tilde{F}^{k,l}(w) : \]
\[ \times \delta\left(q_{n}^{-m-k} q_{3}^{-j-1} \frac{w}{z}\right) (q^{n-1-j} z)^{-1} d^{-k} q^{-1 + \delta_{0,1}}, \]
and if \( i \equiv 1 \pmod{m} \) and \( j \equiv l \pmod{n} \), then
\[ E^{i,j}(z) \tilde{F}^{k,l}(w) - q^{-\delta_{i,0} + \delta_{j,0}} \tilde{F}^{k,l}(w) E^{i,j}(z) =: E^{i,j}(z) \tilde{F}^{k,l}(w) : \]
\[ \times \delta\left(q_{n}^{-k+1} q_{3}^{-j+1} \frac{w}{z}\right) (q^{n-1-j} z)^{-1} d^{-k-1} q^{-\delta_{0,1}}. \]

(4) In all cases other than those given above, we have
\[ E^{i,j}(z) \tilde{F}^{k,l}(w) = \tilde{F}^{k,l}(w) E^{i,j}(z) \times q^{-D_{i,k} \delta_{0,j} + D_{i,j} \delta_{0,i}}, \]
\[ F^{i,j}(z) \tilde{F}^{k,l}(w) = \tilde{F}^{k,l}(w) F^{i,j}(z) \times q^{D_{i,k} \delta_{0,j} - D_{i,j} \delta_{0,i}}, \]
\[ E^{i,j}(z) \tilde{F}^{k,l}(w) = \tilde{F}^{k,l}(w) E^{i,j}(z) \times q^{D_{i,k} \delta_{0,j} + D_{i,j} \delta_{0,i}}. \]

We recall that \( D_{i,k}^{(m)}, D_{i,j}^{(n)} \) are given in (B.5).

Our task is to check the vanishing of the commutator
\[ [G_{\mu,1}(p), \tilde{G}_{\nu,1}(\tilde{p})] = \int \cdots \int \frac{dx_{s}}{2\pi \sqrt{-1} x_{s}} \int \cdots \int \frac{dy_{t}}{2\pi \sqrt{-1} y_{t}} \times [F_{1}(x_{1}) \cdots F_{m}(x_{m}), \tilde{F}_{1}(y_{1}) \cdots \tilde{F}_{n}(y_{n})] h_{\mu,1}(x_{1}, \ldots, x_{m}; p) \tilde{h}_{\nu,1}(y_{1}, \ldots, y_{n}; \tilde{p}) \]
in the sense of matrix elements.

We begin by rewriting
\[ [F_{1}(x_{1}) \cdots F_{m}(x_{m}), \tilde{F}_{1}(y_{1}) \cdots \tilde{F}_{n}(y_{n})] = \sum_{0 \leq j_{1}, \ldots, j_{m} \leq n-1} \sum_{0 \leq k_{1}, \ldots, k_{n} \leq m-1} u_{j_{1}} u_{k_{1}} [F_{1}^{j_{1}}(x_{1}) \cdots F_{m}^{j_{m}}(x_{m}), \tilde{F}_{1}^{k_{1}}(y_{1}) \cdots \tilde{F}_{n}^{k_{n}}(y_{n})]. \]
For \( j = (j_1, \ldots, j_m) \), \( k = (k_1, \ldots, k_n) \), let \( c_{a,b}(j, k) = D^{(m)}_{a,k_i} \delta_{0,b} - D^{(n)}_{j,a} \delta_{a,0} \). Pushing \( F_m^{j,m}(x_m) \) through to the right, we obtain

\[
F_m^{j,m}(x_m) \cdot \prod_{1 \leq I \leq n} \tilde{F}^j_t(y_I) = \sum_{l=1}^{n} q^{m-1} c_{m,b}(j,k) \prod_{1 \leq I \leq l-1} \tilde{F}^j_t(y_I) \cdot [F_m^{j,m}(x_m), \tilde{F}^j_t(y_I)]_{q^{m,l}(j,k)} \cdot \prod_{l+1 \leq I \leq n} \tilde{F}^j_t(y_I) + q^{m-k_n} \prod_{1 \leq I \leq n} \tilde{F}^j_t(y_I) \cdot F_m^{j,m}(x_m),
\]

where we use \( \sum_{b=1}^{m} c_{a,b}(j,k) = D^{(m)}_{a,k_n} \). Continuing the same way and noting that \( \sum_{a=1}^{m} D^{(m)}_{a,k_n} = 0 \), we arrive at

\[
[F_1(x_1) \cdots F_m(x_m), \tilde{F}^1_t(y_1) \cdots \tilde{F}^n_t(y_n)] = \sum_{0 \leq i_1, \ldots, i_m \leq m-1} \sum_{0 \leq k_1, \ldots, k_n \leq m-1} u_{i_m} \tilde{u}_{k_n} \sum_{i=1}^{m} q^{c_{i,i}(j,k)+\sum_{a=i+1}^{m} D^{(m)}_{a,k}} \times \prod_{1 \leq s \leq i-1} F^{s,i}_{j}(x_s) \prod_{1 \leq t \leq i-1} \tilde{F}^{t,i}_{j}(y_t) \cdot [F^{i,i}_{j}(x_i), \tilde{F}^{i,i}_{j}(y_i)]_{q^{i,i}(j,k)} \prod_{l+1 \leq I \leq n} \tilde{F}^{j}_t(y_I) \prod_{i+1 \leq s \leq m} F^{s,i}_{j}(x_s).
\]

By Lemma D.1, the factor \([F^{i,i}_{j}(x_i), \tilde{F}^{i,i}_{j}(y_i)]_{q^{i,i}(j,k)}\) is a sum of two terms containing delta functions. By the same argument as in the proof of Lemma C.3, these terms cancel out under the integral if \( 1 \leq i \leq m - 1 \) and \( 1 \leq l \leq n - 1 \). It remains to consider the three cases

(i) \( 1 \leq i \leq m - 1, l = n \)
(ii) \( i = m, 1 \leq l \leq n - 1 \)
(iii) \( i = m, l = n \).

D.3. Commutativity of \( G_{\nu,1} \) and \( \tilde{G}_{\nu,1} \). We now consider the case (i) in some detail. By Lemma C.3, non-trivial contributions arise only from terms with \((j_i, k_l) = (0, i), (n-1, i-1)\). Apart from a common factor, they give a sum \( I + II \) with

\[
I = \tilde{u}_i \delta(q_{j}^{i-1} y_n/x_i) x_i^{-1} : F^{i,0}_{j}(x_i) \tilde{F}^{i,0}_{j}(q_{j}^{i-1} x_i) : q^{i-1},
\]

\[
II = \tilde{u}_{i-1} \delta(q_{j}^{i-1} py_n/x_i) q^{n-1} x_i^{-1} : F^{i,n-1}_{j}(x_i) \tilde{F}^{i-1,0}_{j}(p^{-1} q_{j}^{i-1} x_i) : q^{i-1}.
\]

The powers \( q^{i-1}, q^{i-1} \) are due to \( \sum_{a=i+1}^{m} D^{(m)}_{a,k} \).

Let us examine the pole structure of the term \( I \). With respect to \( y_n \), the product \( F^{i,0}_{j}(x_i) \tilde{F}^{i,0}_{j}(y_n) \) has a pole at \( q_{j}^{i-1} x_i \) which corresponds to the delta function. In addition, there is another pole at \( y_n = p^{-1} q_{j}^{i-1} x_i \) coming from the contraction of \( \tilde{F}^{i,0}_{j}(y_n) F^{i+1,n-1}_{j}(x_{i+1}) \).
Upon taking the residue at $y_n = q_1^i x_i$, a new pole $p^{-1} q_1 x_i+1$ is produced with respect to $x_i$. This pole must be inside the contour of the $x_i$ integral because analytic continuation is made from the region $|y_n| \gg |x_i+1|$. The pole $x_i = q_3^{i-1} x_i+1$ comes only from the contraction of $F^{i,0}(x_i) F^{i+1,0}(x_i+1)$. On the other hand, the contraction of $\tilde{F}^{i,0}(y_n) F^{i+1,0}(x_i+1)$ gives $1 - q^2 q_1^{i+1} x_i+1 / y_n$, which cancels the pole $x_i = q_3^{i-1} x_i+1$ after the residue is taken in $y_n$. The situation is summarized in Fig. 3.

Consider similarly the product $F^{i,n-1}(x_i) \tilde{F}^{i-1,0}(y_n)$ corresponding to $II$. Poles in $y_n$ occur at $y_n = p^{-1} q_1^i x_i$, which corresponds to the delta function, and at $y_n = q_1^{i-1} x_i-1$ which comes from $F^{i-1,0}(x_i-1) \tilde{F}^{i-1,0}(y_n)$. Upon integration the latter produces a pole at $x_i = p q_1^{i-1} x_i-1$, which should be taken outside the contour of $x_i$ integral. The pole at $x_i = q_3 x_i-1$ comes only from $F^{i-1,n-1}(x_i-1) F^{i,n-1}(x_i)$. After $y_n$ integration it is canceled by the contraction factor $1 - q^2 q_1^{i-1} y_n / x_i-1$ coming from $F^{i-1,n-1}(x_i-1) \tilde{F}^{i,0}(y_n)$. See Fig. 4.
After taking the residue in $y_n$, we perform the shift $x_i \rightarrow px_i$ in the term $II$. From Figs. 3 and 4 the resulting contours become the same.

In view of the identity (B.10), the term $I$ becomes

$$-\tilde{u}_i x_i^{-1} \tilde{\gamma}^{-n} q : F^{i,n-1}(px_i) \tilde{F}^{i-1,0}(q_i x_i) : q^{-\epsilon_i-1+\epsilon_i}.$$

After the shift, the term $II$ becomes

$$\tilde{u}_{i-1} x_i^{-1} q^{-n+\delta_{i,1}} p^{-1} : F^{i,n-1}(px_i) \tilde{F}^{i-1,0}(q_i x_i) :.$$

In addition, the quasi-periodicity (4.10) of $h_{\mu,i}$ give rise to a factor $p_i q_2^{-\delta_{i,1}} = \tilde{p}_i q^{-2\delta_{i,1}}$ which we bring to the position next to $F^{i,n-1}(px_i) \tilde{F}^{i-1,0}(y_n)$ noting

$$\prod_{i+1 \leq s \leq m} F^{s,j_i}(x_s) \cdot p_i = q^{1+\delta_{i,1}} \times p_i \prod_{i+1 \leq s \leq m} F^{s,j_i}(x_s).$$

Putting all these together, we conclude that the two terms cancel provided $\tilde{p}_i \tilde{u}_{i-1} = \tilde{u}_i$.

In a similar manner, we can show that the vanishing holds in the case (ii) of Sect. D.2 provided $\tilde{p}_i u_{i-1} = u_i$. In the case (iii) the condition for vanishing becomes $\tilde{p}_m \tilde{u}_{m-1} = \tilde{u}_0$ and $\tilde{p}_n u_{n-1} = u_0$, which is consistent.

For general $M, N$ we repeat essentially the same computation (cf. [FKSW, KS]). We omit further details.

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