Harmonic synchronization under all three types of coupling: position, velocity, and acceleration

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Abstract

Synchronization of identical harmonic oscillators interconnected via position, velocity, and acceleration couplings is studied. How to construct a complex Laplacian matrix representing the overall coupling is presented. It is shown that the oscillators asymptotically synchronize if and only if this matrix has a single eigenvalue on the imaginary axis. This result generalizes some of the known spectral tests for synchronization. Some simpler Laplacian constructions are also proved to work provided that certain structural conditions are satisfied by the coupling graphs.

1 Introduction

If a group of identical harmonic oscillators \( m_0 \ddot{x}_i + k_0 x_i = 0 \) (where \( m_0, k_0 > 0 \) and \( x_1, x_2, \ldots, x_q \in \mathbb{R} \)) are coupled through their relative velocities to form a network

\[
m_0 \ddot{x}_i + k_0 x_i + \sum_{j=1}^{q} b_{ij} (\dot{x}_i - \dot{x}_j) = 0
\]

(where \( b_{ji} = b_{ij} \geq 0 \) and \( b_{ii} = 0 \)) they sometimes display a remarkable behavior: synchronization, i.e., \( |x_i(t) - x_j(t)| \to 0 \) for all \( (i, j) \) as \( t \to \infty \). When they shall synchronize (or fail to do so) is now well known. All one has to do is check whether the graph \( B \) that the coupling \( (b_{ij})_{i,j=1}^{q} \) gives rise \( \text{to} \) is connected\(^2\) (or not). There is also an equivalent, yet more technical, test to determine synchronization. It employs the graph Laplacian

\[
B = \begin{bmatrix}
\sum_{j} b_{1j} & -b_{12} & \cdots & -b_{1q} \\
-b_{21} & \sum_{j} b_{2j} & \cdots & -b_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
-b_{q1} & -b_{q2} & \cdots & \sum_{j} b_{qj}
\end{bmatrix} =: \text{lap}(b_{ij})_{i,j=1}^{q}
\]

and makes a special case of \(^4\) Thm. 3.1:

**Test 1.** The oscillators \( \| \) synchronize if and only if \( \lambda_2(B) > 0 \)

There are instances in the physical world where the position coupling also plays a role in shaping the overall interconnection among the oscillators \( \| \). This has motivated the extension of the model \( \| \) to

\[
m_0 \ddot{x}_i + k_0 x_i + \sum_{j=1}^{q} b_{ij} (\dot{x}_i - \dot{x}_j) + \sum_{j=1}^{q} k_{ij} (x_i - x_j) = 0
\]

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1The graph \( B \) has \( q \) nodes and there is an edge between \( i \)th and \( j \)th nodes if \( b_{ij} > 0 \).

2See, e.g., \(^2\) for the definition of connected graph.

3\( \lambda_i(A) \) denotes the \( i \)th eigenvalue of \( A \in \mathbb{C}^{q \times q} \) with respect to the ordering \( \Re \lambda_1(A) \leq \Re \lambda_2(A) \leq \cdots \leq \Re \lambda_q(A) \).
In particular, for the synchronization of the oscillators (2), neither it is necessary that both nor it is sufficient that their union $\mathcal{B} \cup \mathcal{K}$ is. Even though the connectivity condition does not yield a straightforward extension, it turns out that Test 1 does. In a recent work [11, Cor. 6] it has been shown that

**Test 2.** The oscillators (2) synchronize if and only if $\text{Re } \lambda_2(B + jK) > 0$.

If one continues to walk in the direction of generalization that took us from the solely velocity-coupled network (1) to both position- and velocity-coupled one (2), the obvious next stop is the setup where the **acceleration** coupling is also present. Namely,

$$m_0\ddot{x}_i + k_0x_i + \sum_{j=1}^{q} m_{ij}(\ddot{x}_i - \ddot{x}_j) + \sum_{j=1}^{q} b_{ij}(\dot{x}_i - \dot{x}_j) + \sum_{j=1}^{q} k_{ij}(x_i - x_j) = 0$$

(3)

where $m_{ji} = m_{ij} \geq 0$ and $m_0 = 0$. In accordance with our previous notation we introduce the Laplacian

$$M = \text{lap}(m_{ij})_{i,j=1}^q$$

whose graph is denoted by $\mathcal{M}$. The motivation for studying this general coupling scheme is not purely theoretical; certain electrical oscillator networks under RLC-type coupling indeed obey the dynamics (3). Consider, for instance, the linear time-invariant (LTI) network of $q = 3$ coupled LC-tanks shown in Fig. 1; where $c_0$ and $\ell_0$ are, respectively, the capacitance and the inductance of the individual oscillators, $\ell_{12}$ is the inductance of the inductor connecting the nodes $\mathcal{G}$ and $\mathcal{S}$, $r_{23}$ is the resistance of the resistor connecting the nodes $\mathcal{S}$ and $\mathcal{R}$, and $c_{31}$ is the capacitance of the capacitor connecting the nodes $\mathcal{R}$ and $\mathcal{G}$. Letting $x_i$ be the $i$th node voltage, the dynamics of this simple example circuitry read

$$c_0\ddot{x}_1 + \ell_0^{-1}x_1 + c_{31}(\ddot{x}_1 - \ddot{x}_3) + \ell_{12}^{-1}(x_1 - x_2) = 0$$

$$c_0\ddot{x}_2 + \ell_0^{-1}x_2 + r_{23}^{-1}(\ddot{x}_2 - \ddot{x}_3) + \ell_{12}^{-1}(x_2 - x_1) = 0$$

$$c_0\ddot{x}_3 + \ell_0^{-1}x_3 + c_{31}(\ddot{x}_3 - \ddot{x}_1) + r_{23}^{-1}(\ddot{x}_3 - \ddot{x}_2) = 0.$$
believe that the problem we study here is novel. To the best of our knowledge, the collective behavior of harmonic oscillators connected via these three separate coupling graphs has not been studied before, despite the fact that there are real-world networks (e.g., Fig. 1) that would benefit from such an analysis. In particular, the possible effects of relative acceleration coupling on the evolution of simple oscillator networks (being a fertile subject of investigation notwithstanding) are yet unbeknownst to the lively literature on harmonic synchronization, which we review next.

The literature on synchronization of coupled harmonic oscillators has reached a certain maturity in the last decade. Most of the initial results concerned pure velocity coupling; works on position coupling appearing only later. One of the first comprehensive analyses of harmonic oscillators within the synchronization framework can be found in [4], where Ren considers time-varying oscillator dynamics under time-varying and asymmetrical velocity coupling. This work later enjoyed certain variations and generalizations. For instance, a type of coupling that becomes inactive when the distance between oscillators exceeds a threshold is studied in [8]. Nonlinearly-coupled harmonic oscillators are analyzed in [1], where an averaging technique is employed to establish synchronization. Among many other articles studying velocity-coupled harmonic oscillators are [10], where the information exchange between units takes place in an impulsive fashion; [14, 9], where sampled-data approaches are proposed to study synchronization; and [7], where both delayed measurements and negative coupling weights are allowed. The early investigations on the effect and utility of position coupling seem to go as far back as the work [15], where position coupling is considered together with velocity coupling, but not independently, in the sense that they share the same single Laplacian matrix. The works that succeeded [15] can be classified into two groups. One group removed velocity coupling from the picture altogether and allowed relative position measurements only, while the other group allowed in their setup both position and velocity couplings, where each has its own separate Laplacian matrix. To the first group belong, for instance, [6, 13], where synchronization is established via sampled-data strategies. A generalization to heterogeneous harmonic oscillator networks is later presented in [5]. Also related to the first group is [8], where bipartite consensus problem is considered under sampled position data. The second group contains the work [10], where an observability-like condition for synchronization is presented in terms of the pair of Laplacians describing the overall interconnection; and [12], where practical stochastic synchronization is studied under position and velocity couplings.

The remainder of the paper is organized as follows. In Section 2 we present a complex Laplacian matrix construction out of the network parameters \((M, B, K, m_0, k_0)\) and an associated eigenvalue test (that generalizes Tests 1 and 2) to determine whether the array of oscillators synchronize. We also provide a numerical example to emphasize the fact that synchronous behavior (or its absence) does depend on the individual oscillator parameters \((m_0, k_0)\); a peculiarity that the simpler networks [11] and [2] do not suffer from. Then, in Section 3 we bring forth some structural conditions on the coupling graphs \(M, B, K\) under which the eigenvalue test presented in Section 2 takes much simpler forms.

2 The second eigenvalue

Consider the network of \(q\) coupled harmonic oscillators \((3)\). When these units will eventually oscillate in unison is what we aim to find out here. For our purpose, we focus on the implications of the spectral properties of a \(q\)-by-\(q\) complex Laplacian matrix (yet to be constructed) on synchronization.

**Definition 1** The oscillators \((3)\) are said to synchronize if the solutions satisfy \(|x_i(t) - x_j(t)| \to 0\) as \(t \to \infty\) for all \((i, j)\) and all initial conditions.

The identity matrix is denoted by \(I \in \mathbb{R}^{q \times q}\) and the vector of all ones by \(1_q \in \mathbb{R}^q\). By construction the (previously defined) Laplacian matrices \(M, B, K \in \mathbb{R}^{q \times q}\) are all symmetric positive semidefinite and the null space of each contains the vector \(1_q\). By letting \(x = [x_1, x_2, \ldots, x_q]^T \in \mathbb{R}^q\) we can rewrite the dynamics \((3)\) as

\[
(M + m_0 I)\ddot{x} + B\dot{x} + (K + k_0 I)x = 0.
\]

We observe that every solution \(x(t)\) of \((3)\) is bounded. This fact can be established using the function

\[
V = \frac{1}{2} x^T(K + k_0 I)x + \frac{1}{2}\dot{x}^T(M + m_0 I)\dot{x}
\]

which
which is nonnegative because both \((K + k_0I) =: K_a\) and \((M + m_0I) =: M_a\) are positive definite matrices. Combining (1) and (5) yields \(V = -\dot{x}^T B \dot{x}\). Since \(B\) is positive semidefinite we have \(V \leq 0\). Therefore \(V(t) \leq V(0)\) for all \(t \geq 0\), which at once implies the boundedness of \(x(t)\). Now, being produced by an LTI system, \(x(t)\) can be written as a finite sum

\[x(t) = \sum_k \text{Re}(e^{\lambda_k t} p_k(t))\]

where \(\lambda_k \in \mathbb{C}\) are distinct and \(p_k(t)\) are polynomials with vector coefficients. In the light of boundedness we can then assert that \(\text{Re}\lambda_k \leq 0\) for all \(k\) and if \(\text{Re}\lambda_k = 0\) for some \(k\) then the corresponding polynomial \(p_k(t)\) must necessarily be of degree zero, i.e., a constant vector. Suppose now the oscillators (3) fail to synchronize. This implies that there exists a solution (4) where the sum contains an index \(k\) for which \(\lambda_k = j\omega\) and \(p_k(t) \equiv \xi\) with \(\omega \in \mathbb{R}_{>0}\) and \(\xi \in \mathbb{C}^q \setminus \text{span}\{1_q\}\). Because the function \(t \mapsto e^{j\omega t}\) has to satisfy (1), we have

\[(K_a - \omega^2 M_a)\xi + j\omega B\xi = 0.\]

(7)

Multiplying the above equation from left by \(\xi^*\) yields

\[\xi^* K_a \xi - \omega^2 \xi^* M_a \xi + j\omega \xi^* B \xi = 0.\]

(8)

Note that the terms \(\xi^* K_a \xi, \xi^* M_a \xi, \xi^* B \xi\) are all real because \(K_a, M_a, B \geq 0\). Therefore (8) implies \(\xi^* B \xi = 0\). Since \(B\) is symmetric positive semidefinite this means \(B \xi = 0\), whence \((K_a - \omega^2 M_a)\xi = 0\) by (7). To summarize, if the oscillators (3) do not synchronize then there exist a real number \(\omega > 0\) and a vector \(\xi \notin \text{span}\{1_q\}\) such that

\[
\begin{bmatrix}
(K + k_0 I) - \omega^2 (M + m_0 I)

B
\end{bmatrix} \xi = 0.

(9)

It is not difficult to see that the steps we have taken are reversible. That is, if we can find a real number \(\omega > 0\) and a vector \(\xi \notin \text{span}\{1_q\}\) satisfying (9) then we can construct the function \(t \mapsto \text{Re}(e^{j\omega t})\) which solves (1) thanks to (9). And this cannot a synchronous solution because \(\xi \notin \text{span}\{1_q\}\). We have therefore established:

**Lemma 1** The following are equivalent.

1. The oscillators (3) do not synchronize.

2. There exist \(\omega \in \mathbb{R}_{>0}\) and \(\xi \in \mathbb{C}^q \setminus \text{span}\{1_q\}\) satisfying (9).

The above lemma can be a useful test for synchronization, but it is worthwhile to search for a simpler way to determine when the oscillators synchronize. We now present the following alternative.

**Theorem 1** The oscillators (3) synchronize if and only if \(\text{Re}\lambda_2(A) > 0\) where

\[A := (M + m_0 I)^{-1/2} (B + jK) (M + m_0 I)^{-1/2} - j \frac{k_0}{m_0} I\]

(10)

is the complex Laplacian representing the network.

Note that when there is no inertial coupling (i.e., \(M = 0\)) we have \(\Lambda = m_0^{-1} (B + j K)\) and the condition for synchronization presented in Theorem 1 can be written as \(\text{Re}\lambda_2(m_0^{-1} (B + j K)) > 0\) which clearly is equivalent to \(\text{Re}\lambda_2(B + j K) > 0\). Furthermore, if there is only dissipative coupling (i.e., both \(M = 0\) and \(K = 0\)) the condition further reduces to \(\text{Re}\lambda_2(m_0^{-1} B) > 0\) which is equivalent to \(\lambda_2(B) > 0\) since \(B\) is real and symmetric. Therefore Theorem 1 generalizes the Tests 1 and 2 mentioned earlier in the paper. Note however that this generalization has one qualitative aspect which its corollaries do not manifest: it appears to depend not only on the coupling \((M, B, K)\) but also on the individual oscillator parameters \((m_0, k_0)\). Is this a superficial dependence? If not, there should exist a coupling \((M, B, K)\) for which one can find two pairs \((m_0', k_0')\) and \((m_0'', k_0'')\) such that the array of oscillators (4) described by \((M, B, K, m_0', k_0')\) is not synchronized but...
synchronize whereas the other set of parameters \((M, B, K, m_0, k_0)\) produces asynchronous solutions. It turns out that such couplings are not difficult to come by. (Hence the answer to our question is no.) We provide an example below.  

Consider the network of six coupled LC-tanks shown in Fig. 2. Letting \(x = [x_1, x_2, \cdots, x_6]^T\) denote the node voltage vector and setting \(m_0 = c_0\) and \(k_0 = \ell_0^{-1}\), the dynamics of this network obey (4) by the following coupling matrices

\[
M = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3}{8} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad K = \begin{bmatrix}
2 & -2 & 0 & 0 & 0 & 0 \\
-2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & -2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{2} & -\frac{3}{2} \\
0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{2}
\end{bmatrix}.
\]

Let us first study this circuit under the oscillator parameters \(c_0 = 2\) and \(\ell_0 = \frac{1}{2}\), yielding \((m_0, k_0) = (2, 2)\) and \(\omega_0 = 1\text{rad/sec}\) (where \(\omega_0 = \sqrt{k_0/m_0}\) is the frequency of uncoupled oscillations). The eigenvalues of the associated Laplacian (10) can be computed as \(\lambda_1 = 0, \lambda_2 = 0.0078 - j0.1409, \lambda_3 = 0.0088 + j1.5747, \lambda_4 = 0.0434 + j1.9338, \lambda_5 = 0.4452 + j0.1386, \lambda_6 = 0.4947 + j1.4484\). Since the second eigenvalue is on the open right half-plane the oscillators synchronize by Theorem 1. Consider once again the array in Fig. 2 this time with \(c_0 = 1\) and \(\ell_0 = 1\), yielding \((m_0, k_0) = (1, 1)\) and \(\omega_0 = 1\text{rad/sec}\). Note that in this second case even though we changed the oscillator parameters, the frequency of uncoupled oscillations is still the same. Despite this sameness however the eigenvalues of the Laplacian \((\lambda_1 = 0, \lambda_2 = 0.0107 - j0.2436, \lambda_3 = 0.0996 + j3.8647, \lambda_4 = 0.8666 + j0.2996, \lambda_5 = 1.0230 + j2.7936)\) tell us that the oscillators will fail to synchronize because the condition \(\text{Re}\lambda_2(\Lambda) > 0\) no longer holds. We end this section by the proof of Theorem 1.

**Proof of Theorem 1.** Recall the shorthand notation \(M_A = M + m_0I, K_A = K + k_0I,\) and \(\omega_0^2 = k_0/m_0\) we introduced earlier. We first establish some properties of the Laplacian (10). Since the real matrices \(M, K\) are symmetric positive semidefinite, the augmented matrices \(M_A, K_A\) are symmetric positive definite. Furthermore, for an eigenvalue \(\alpha \in \mathbb{R}\) and the corresponding eigenvector \(y \in \mathbb{C}^q\) satisfying \(My = \alpha y\) it is clear that we can write \(M_A^\alpha y = (\alpha + m_0)^\alpha y\) for any power \(\alpha \in \mathbb{R}\). Likewise, \(Ky = \alpha y\) implies \(K_A^\alpha y = (\alpha + k_0)^\alpha y\). This at once yields \(\Lambda 1_q = 0\) since \(M 1_q = B 1_q = K 1_q = 0\). Let \(D := M_A^{-1/2}BM_A^{-1/2}\) and \(R := M_A^{-1/2}K_AM_A^{-1/2}\). Note that \(D, R \geq 0\). We now show that \(\Lambda\) can have no eigenvalue on the open left half-plane. Let \(\lambda \in \mathbb{C}\) be an eigenvalue of \(\Lambda\) and \(u \in \mathbb{C}^q\) be the corresponding unit eigenvector. That is, \(\Lambda u = \lambda u\) and \(u^*u = 1\). We can write

\[\lambda = u^*\Lambda u = u^*(D + jR - j\omega_0^2I)u = u^*Du + j(u^*Ru - \omega_0^2u^*u)\]

whence follows that \(\text{Re}\lambda = u^*Du \geq 0\). Combining this with the fact that \(\Lambda\) has an eigenvalue at the origin (recall \(\Lambda 1_q = 0\)) allows us to write, without loss of generality, \(\lambda_1(\Lambda) = 0\).
Suppose now $\text{Re} \lambda_2(\Lambda) \leq 0$. Since $\text{Re} \lambda_i(\Lambda) \geq 0$ for all $i$ we have to have $\lambda_2(\Lambda) = j\mu$ for some $\mu \in \mathbb{R}$. This implies the existence of an eigenvector $\eta \notin \text{span}\{1_q\}$ satisfying $\Lambda\eta = j\mu\eta$. This is obvious if $\mu \neq 0$. It is still true when $\mu = 0$, i.e., when the eigenvalue at the origin is repeated. To see that suppose otherwise, i.e., $1_q$ were the sole eigenvector corresponding to the eigenvalue at the origin. Since the eigenvalue at the origin is repeated we then would have to have a generalized eigenvector $\eta_3$ satisfying $\Lambda\eta_3 = 1_q$. This however would produce the contradiction

$$q = 1_q^T 1_q = 1_q^T \Lambda\eta_3 = (\Lambda 1_q)^T \eta_3 = 0$$

due to the symmetry $\Lambda^T = \Lambda$. Now, without loss of generality let $\eta^* = 1$. We can write

$$j\mu = \eta^* \Lambda\eta = \eta^* D\eta + j(\eta^* R\eta - \omega_0^2)$$

which tells us $\eta^* D\eta = 0$. Consequently, since the real matrix $D$ is symmetric positive semidefinite, we have $D\eta = 0$. Recalling $D = M_a^{-1/2} B M_a^{-1/2}$ and defining $\xi := M_a^{-1/2} \eta$ we can then assert

$$B\xi = 0$$

(11)

because $M_a^{-1/2}$ is nonsingular. Observe that $\xi \notin \text{span}\{1_q\}$. This follows from the fact that $1_q$ is an eigenvector of $M_a$ and, consequently, of $M_a^{1/2}$. That is, $M_a^{1/2} 1_q \in \text{span}\{1_q\}$. Hence, if $\xi$ did belong to span $\{1_q\}$ then we would have $\eta = M_a^{1/2} \xi \in \text{span}\{1_q\}$. But this contradicts $\eta \notin \text{span}\{1_q\}$. Combining $D\eta = 0$ and $\Lambda\eta = j\mu\eta$ we obtain $R\eta - \omega_0^2 \eta = \mu\eta$ yielding $R\eta = (\omega_0^2 + \mu)\eta$. Since $R$ is symmetric positive definite all its eigenvalues are real and positive. This means $\omega = \sqrt{\omega_0^2 + \mu} > 0$ satisfies

$$0 = (R - \omega^2 I)\eta = (M_a^{-1/2} K a M_a^{-1/2} - \omega^2 I)\eta = M_a^{-1/2} (K_a - \omega^2 M_a) M_a^{-1/2} \eta$$

which lets us see

$$((K + k_0 I) - \omega^2 (M + m_0 I))\xi = 0.$$  

(12)

Combining (11), (12), and Lemma 1 we finally establish that the oscillators (3) do not synchronize.

We now show the other direction. Suppose the oscillators (3) do not synchronize. Then by Lemma 1 there exist $\omega > 0$ and $\xi \notin \text{span}\{1_q\}$ satisfying (11) and (12). Let $\mu = \omega^2 - \omega_0^2$ and $\eta = M_a^{1/2} \xi$. By retracing the steps we have taken in the first part of the proof we can easily reach $\Lambda\eta = j\mu\eta$ as well as establishing $\eta \notin \text{span}\{1_q\}$. This means (because $\Lambda 1_q = 0$) that the Laplacian $\Lambda$ has at least two eigenvalues on the imaginary axis. Hence we conclude that $\text{Re} \lambda_2(\Lambda) = 0$ since all the eigenvalues of $\Lambda$ are on the closed right half-plane.  

\section{3 Simpler characterizations under structural conditions}

In the previous section we have seen that for a given coupling $(M, B, K)$ whether the oscillators (3) synchronize or not depends in general on the individual oscillator parameters $(m_0, k_0)$ as well. In this section we investigate structural conditions on the coupling under which synchronization depends solely on the triple $(M, B, K)$. To this end, we need some notation first. A graph $G$ is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_q\}$ is the set of vertices (nodes) and the set $\mathcal{E}$ contains some (unordered) pairs $(v_i, v_j)$ with $i \neq j$, called edges. The set of vertices incident to an edge is denoted by $\text{ver}\mathcal{E} \subset \mathcal{V}$. That is, $\text{ver}\mathcal{E} = \{v_i : (v_i, v_j) \in \mathcal{E}\}$. We define the edge set describing the inertial coupling as $\mathcal{E}_m = \{(v_i, v_j) : m_{ij} > 0\}$. The sets $\mathcal{E}_b$ and $\mathcal{E}_k$ are defined, mutatis mutandis, for the dissipative and restorative couplings, respectively. Therefore the three graphs (introduced earlier) describing the coupling in the network (3) can be written as $\mathcal{M} = (\mathcal{V}, \mathcal{E}_m)$, $\mathcal{B} = (\mathcal{V}, \mathcal{E}_b)$, and $\mathcal{K} = (\mathcal{V}, \mathcal{E}_k)$.

\textbf{Theorem 2} The oscillators (3) synchronize if the graph $\mathcal{B}$ is connected.

\textbf{Proof.} That $\mathcal{B}$ is connected means its Laplacian $\mathcal{B}$ satisfies $\text{null} \mathcal{B} = \text{span}\{1_q\}$; see, for instance, [4] Lem. 3.1. The result then follows by Lemma 1. \hfill \blacksquare
For many applications, connectedness of the dissipative coupling graph could be too conservative an assumption. We now attempt to relax this requirement utilizing isolation, by which we mean the following. Two graphs (defined over the same vertex set) are isolated when their edges do not touch one another. More formally:

**Definition 2** The graphs \( M = (\mathcal{V}, \mathcal{E}_m) \) and \( K = (\mathcal{V}, \mathcal{E}_k) \) are said to be edge-isolated if \( \text{ver} \mathcal{E}_m \cap \text{ver} \mathcal{E}_k = \emptyset \).

**Theorem 3** Suppose the graphs \( M \) and \( K \) are edge-isolated. Then the oscillators \( (\mathcal{V}, \mathcal{E}_m, \mathcal{E}_k) \) synchronize if and only if \( \text{Re} \lambda_2(B + j(K - M)) > 0 \).

An immediate implication of Theorem 3 concerning the type of electrical networks we considered earlier in the paper is the following. If the coupling network is such that there is not a single node where the terminals of a capacitive connector and an inductive connector meet then whether the oscillators synchronize or not does not depend on the individual oscillator parameters \((c_0, f_0)\). Note also that Test 2 follows from Theorem 3 as a special case. In addition, Theorem 3 produces the following sister test.

**Corollary 1** The coupled oscillators

\[
m_0 \ddot{x}_i + k_0 x_i + \sum_{j=1}^{q} m_{ij}(\ddot{x}_i - \ddot{x}_j) + \sum_{j=1}^{q} b_{ij}(\dot{x}_i - \dot{x}_j) = 0
\]

synchronize if and only if \( \text{Re} \lambda_2(B - jM) > 0 \).

We need the following result for the proof of the theorem.

**Lemma 2** Let \( P, Q \in \mathbb{R}^{q \times q} \) be symmetric positive semidefinite matrices satisfying \( PQ = 0 \). Let \( \mu \in \mathbb{R} \) and the nonzero vector \( \eta \in \mathbb{C}^q \) satisfy

\[
(P - Q) \eta = \mu \eta. \tag{13}
\]

The following hold.

1. If \( \mu > 0 \) then \( P \eta = \mu \eta \) and \( Q \eta = 0 \).
2. If \( \mu < 0 \) then \( P \eta = 0 \) and \( Q \eta = -\mu \eta \).
3. If \( \mu = 0 \) then \( P \eta = 0 \) and \( Q \eta = 0 \).

**Proof.** Case 1: \( \mu > 0 \). Note that \( PQ = 0 \) implies \( QP = 0 \) because the matrices \( P, Q \) are symmetric. Multiplying both sides of (13) by \(-Q\) we obtain \(-\mu Q \eta = -Q P \eta + Q^2 \eta = Q(Q \eta)\) which tells us that the vector \( Q \eta \) if nonzero must be an eigenvector of \( Q \) with the negative eigenvalue \(-\mu \). But since \( Q \succeq 0 \) all its eigenvalues must be nonnegative. Hence \( Q \eta = 0 \). Then (13) gives us \( P \eta = \mu \eta \). Case 2: \( \mu < 0 \). Negating (13) we can write \((P - Q) \eta = (-\mu) \eta \). The result then follows from the previous case. Case 3: \( \mu = 0 \). This time (13) implies \( P \eta = Q \eta \). Multiplying both sides with \( P \) yields \( P^2 \eta = PQ \eta = 0 \) thanks to \( PQ = 0 \). Then we can proceed as follows \( 0 = \eta^* P^2 \eta = \|P \eta\|^2 \) because \( P \) is symmetric positive semidefinite. And \( \|P \eta\| = 0 \) means \( P \eta = 0 \). Then \( Q \eta = 0 \) follows by \( Q \eta = P \eta \).

**Proof of Theorem 3** Let \( M \) and \( K \) be edge-isolated. This implies that the product of their Laplacians vanish, i.e., \( MK = 0 \). This is obvious if either \( \text{ver} \mathcal{E}_m \) or \( \text{ver} \mathcal{E}_k \) is empty because an empty edge set means a zero Laplacian matrix. As for the case that both edge sets are nonempty we can always label the vertices such that \( \text{ver} \mathcal{E}_m = \{v_1, v_2, \ldots, v_r\} \) and \( \text{ver} \mathcal{E}_k = \{v_s, v_{s+1}, \ldots, v_q\} \) for some indices \( 2 \leq r < s \leq q - 1 \). The corresponding \((q \times q)\) Laplacians then enjoy the block diagonal form

\[
M = \begin{bmatrix} M_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 0 \\ 0 & K_2 \end{bmatrix}
\]

with \( M_1 \in \mathbb{R}^{r \times r} \) and \( K_2 \in \mathbb{R}^{(q-s+1) \times (q-s+1)} \) which makes it clear that \( MK = KM = 0 \). Let us introduce the shorthand notation \( \Gamma = B + j(K - M) \). The matrix \( \Gamma \) comes with the properties \( \Gamma 1_\eta = 0 \)
and \( \Re \lambda_i(\Gamma) \geq 0 \) for all \( i \). (The demonstration of these properties is very similar to the demonstration of the same properties satisfied by the matrix \( \Lambda \); see the proof of Theorem 1.) Hence, without loss of generality, we let \( \lambda_1(\Gamma) = 0 \).

Suppose \( \lambda_2(\Gamma) \leq 0 \). This means \( \lambda_2(\Gamma) = j \mu \) for some \( \mu \in \mathbb{R} \) because \( \Re \lambda_i(\Gamma) \geq 0 \) for all \( i \). Then we can find an eigenvector \( \xi \notin \text{span} \{1_q\} \) satisfying \( \Gamma \xi = j \mu \xi \) (see the proof of Theorem 1). Without loss of generality let \( \xi \) be a unit vector. By writing

\[
j \mu = \xi^* \Gamma \xi = \xi^* B \xi + j(\xi^* K \xi - \xi^* M \xi)
\]

we see at once that \( \xi^* B \xi = 0 \) because \( M, B, K \) are symmetric positive semidefinite matrices. Then follows

\[
B \xi = 0
\]

under which \( \Gamma \xi = j \mu \xi \) reduces to

\[
(K - M) \xi = \mu \xi .
\]

Let us now study (15) under all three possibilities. Case 1: \( \mu > 0 \). By Lemma 2 we have \( K \xi = \mu \xi \) and \( M \xi = 0 \). Choosing \( \omega = \sqrt{(\mu + k_0)/m_0} \) we can therefore write

\[
((K + k_0 I) - \omega^2 (M + m_0 I)) \xi = 0 .
\]

Case 2: \( \mu < 0 \). By Lemma 2 we have \( K \xi = 0 \) and \( M \xi = -\mu \xi \). This time choosing \( \omega = \sqrt{k_0/(-\mu + m_0)} \) we can establish (16). Case 3: \( \mu = 0 \). By Lemma 2 we have \( K \xi = M \xi = 0 \) and (16) holds with \( \omega = \sqrt{k_0/m_0} \). Hence for all cases (14) and (16) simultaneously hold. Lemma 1 then tells us that the oscillators (3) do not synchronize.

To show the other direction suppose the oscillators (3) do not synchronize. By Lemma 1 there exist \( \omega > 0 \) and \( \xi \notin \text{span} \{1_q\} \) such that (14) and (16) hold. Let us rewrite (16) as

\[
(K - \omega^2 M) \xi = (\omega^2 m_0 - k_0) \xi .
\]

There are three possibilities concerning (17). Case 1: \( \omega^2 m_0 - k_0 > 0 \). Lemma 2 allows us write \( K \xi = \mu \xi \) and \( M \xi = 0 \) with \( \mu = \omega^2 m_0 - k_0 \). Combining this with (14) we obtain

\[
\Gamma \xi = j \mu \xi .
\]

Case 2: \( \omega^2 m_0 - k_0 < 0 \). By Lemma 2 this time we have \( K \xi = 0 \) and \( M \xi = -\mu \xi \) with \( \mu = m_0 - k_0/\omega^2 \) and again (18) follows. Case 3: \( \omega^2 m_0 - k_0 = 0 \). In this final case we have to have \( K \xi = 0 \) and \( M \xi = 0 \) according to Lemma 2. Then (18) holds with \( \mu = 0 \). Now, in the light of \( \xi \notin \text{span} \{1_q\} \) and \( \Gamma 1_q = 0 \) we can deduce from (18) that \( \Gamma \) has at least two eigenvalues on the imaginary axis. Combining this with the fact that all the eigenvalues of \( \Gamma \) are on the closed right half-plane we reach the conclusion \( \Re \lambda_2(\Gamma) = 0 \).

Hence the result.

\[\blacksquare\]

4 Conclusion

In this paper we studied the collective behavior of harmonic oscillators that are communicating through inertial, dissipative, and restorative connectors. The coupling considered was fixed and symmetric. We showed that whether the oscillators tend to synchronize or not can be determined through the spectrum of a single complex Laplacian matrix, which is constructed from the three individual Laplacians, each representing a different type of coupling. We also provided certain structural conditions (on the coupling graphs) which render the Laplacian construction much simpler. The theorems presented here generalize some earlier results.
References

[1] C. Cai and S.E. Tuna. Synchronization of nonlinearly coupled harmonic oscillators. In Proc. of the American Control Conference, pages 1767–1771, 2010.

[2] R. Diestel. Graph Theory. Springer-Verlag, 1997.

[3] J. Liu, H. Li, and J. Luo. Impulse bipartite consensus control for coupled harmonic oscillators under a competitive network topology using only position states. IEEE Access, 7:20316–20324, 2019.

[4] W. Ren. Synchronization of coupled harmonic oscillators with local interaction. Automatica, 44:3195–3200, 2008.

[5] Q. Song, F. Liu, J. Cao, A.V. Vasilakos, and Y. Tang. Leader-following synchronization of coupled homogeneous and heterogeneous harmonic oscillators based on relative position measurements. IEEE Transactions on Control of Network Systems, 6:13–23, 2019.

[6] Q. Song, F. Liu, G. Wen, J. Cao, and Y. Tang. Synchronization of coupled harmonic oscillators via sampled position data control. IEEE Transactions on Circuits and Systems I: Regular Papers, 63:1079–1088, 2016.

[7] Q. Song, G. Lu, G. Wen, J. Cao, and F. Liu. Bipartite synchronization and convergence analysis for network of harmonic oscillator systems with signed graph and time delay. IEEE Transactions on Circuits and Systems I: Regular Papers, 66:2723–2734, 2019.

[8] H. Su, X. Wang, and Z. Lin. Synchronization of coupled harmonic oscillators in a dynamic proximity network. Automatica, 45:2286–2291, 2009.

[9] W. Sun, J. Lu, S. Chen, and X.Yu. Synchronisation of directed coupled harmonic oscillators with sampled-data. IET Control Theory and Applications, 8:937–947, 2014.

[10] S.E. Tuna. Synchronization of harmonic oscillators under restorative coupling with applications in electrical networks. Automatica, 75:236–243, 2017.

[11] S.E. Tuna. Synchronization of small oscillations. Automatica, 107:154–161, 2019.

[12] G. Wang, J. Ji, and J. Zhou. Practical stochastic synchronisation of coupled harmonic oscillators subjected to heterogeneous noises and its applications to electrical systems. IET Control Theory & Applications, 13:96–105, 2019.

[13] H. Zhang and J. Ji. Group synchronization of coupled harmonic oscillators without velocity measurements. Nonlinear Dynamics, 91:2773–2788, 2018.

[14] H. Zhang and J. Zhou. Synchronization of sampled-data coupled harmonic oscillators with control inputs missing. Systems & Control Letters, 61:1277–1285, 2012.

[15] Y. Zhang, Y. Yang, and Y. Zhao. Finite-time consensus tracking for harmonic oscillators using both state feedback control and output feedback control. International Journal of Robust and Nonlinear Control, 23:878–893, 2013.

[16] J. Zhou, H. Zhang, L. Xiang, and Q. Wu. Synchronization of coupled harmonic oscillators with local instantaneous interaction. Automatica, 48:1715–1721, 2012.