Constraining Parameters of Generalized Cosmic Chaplygin Gas in Loop Quantum Cosmology

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ABSTRACT

We have assumed the FRW universe in loop quantum cosmology (LQC) model filled with the dark matter and the Generalized Cosmic Chaplygin gas (GCCG) type dark energy where dark matter follows the linear equation of state. We present the Hubble parameter in terms of the observable parameters $\Omega_{m0}$ and $H_0$ with the redshift $z$ and the other parameters like $A$, $B$, $w_m$, $\omega$ and $\alpha$ which coming from our model. From Stern data set (12 points)& SNe Type Ia 292 data (from Riess et al. (2004, 2007); Astier et al. (2006)) we have obtained the bounds of the arbitrary parameters by minimizing the $\chi^2$ test. The best-fit values of the parameters are obtained by 66%, 90% and 99% confidence levels. Next due to joint analysis with Stern+BAO and Stern+BAO+CMB observations, we have also obtained the bounds of the parameters $(A, B)$ by fixing some other parameters $\alpha, w_m$ and $\omega$. From the best fit values of the parameters, we have obtained the distance modulus $\mu(z)$ for our theoretical GCCG model in LQC and from Supernovae Type Ia (union2 sample 552 data from Amanullah et al. (2010) & Riess 292 data from Riess et al. (2004, 2007); Astier et al. (2006) ), we have concluded that our model is in agreement with the Supernovae Type Ia sample data. In addition, we have investigated in details about the various types of Future Singularities that may be formed in this model and it is notable that our model is completely free from any types of future singularities.

1. Introduction

It is known from recent observational study that our universe is expanding with an acceleration and that supported by different observations of the SNeIa (Perlmutter et al. (1998, 1999); Riess et al. (1998, 2004)], baryon acoustic oscillations (BAO) [Eisenstein et al. (2005)], large scale redshift surveys [Bachall et al. (1999); Tedmark et al. (2004)], the measurements of the cosmic microwave background (CMB) [Miller et al. (1999); Bennet et al. (2000)], WMAP
[Briddle et al. (2003); Spergel et al. (2003, 2007)] and effects of weak lensing [Jain et al. (2003)]. The recent trend among the researchers is that to find the methodology that triggers late inflation and for that researchers are mainly divided into two groups, one considering a modification in the geometry by adjusting the form of original general theory of relativity and other invoking any mysterious fluid in the form of an evolving cosmological constant or a quintessential [Peebles et al. (1988)] type of scalar field. Those unknown mysterious fluid which has the property that the positive energy density and sufficient negative pressure, known as dark energy (DE) [Padmanabhan (2003); Sahni et al. (2000)] in which the potential dominates over the kinetic term. In present time, DE related problems are most interesting research topic of theoretical physics [Weinberg (1989)]. There are several interesting form of solution of this type problem such as phantom [Caldwell (2002); Fu et al. (2008)] tachyon scalar field [Sen (2002); Balart et al. (2007); Farajollahi et al. (2011); del Campo et al. (2009)], hessence [Wei et al. (2005)], dilaton scalar field [Morris (2012); Marcus (1990)], K-essence scalar field [Armendariz - Picon et al. (2001); Bouhmadi-Lpez et al. (2010); Malquarti et al. (2003)], DBI essence scalar field [Spalinski (2007); Martin et al. (2008)] and many others.

Another unknown missing matter component of the universe is known as the dark matter (DM) which holds together the galaxy clusters. DM is also needed to explain the current large scale structure of the universe. It can be predicted that in cosmic concordance ΛCDM model, the Universe is formed of 26% matter (baryonic + dark matter) and ~ 74% of a smooth vacuum energy component, whereas the thermal CMB component contributes only about 0.01%, however, its angular power spectrum of temperature encode important information about the structure formation process and other cosmic observables.

Loop Quantum Gravity (LQG) theory has been basically trying to quantize the gravity with a non-perturbative and background independent way. As a result, the quantum effect of our universe quite comfortably describe by LQG [Ashtekar et al. (2004); Rovelli (1998)]. The theory and principles of LQG when combined with cosmological framework then it creates a new theoretical framework, named as Loop Quantum Cosmology(LQC) [Bojowald (2001, 2002, 2003, 2008); Ashtekar (2007); Ashtekar et al. (2003, 2006, 2008)]. LQC is basically based
on discrete quantum geometry instead of classical space-time continuum. Friedmann equation is modified by adding a term quadratic in density to describe the effect of LQG. In LQC, the standard Friedmann equation is modified with the help of the non-perturbative effects which leads to the correction term $\frac{\rho^2}{\rho_c}$ and which leads to the result of mechanically bouncy universe when the matter energy density reaches to the level of Plank density $\rho_c$. In 2005, Bojowald [Bojowald (2005)] reviewed to give an overview and summary of the current status of the research work on LQC in detail and that review was also modified by Bojowald (2008). A valuable report about the existing state of art on LQC is discussed in Ashtekar et al. (2011). Recently, Sadjadi [Sadjadi (2013)] has been discussed about the related study on LQC like a super acceleration and its possible phase transitions, i.e., the crossing of the phantom divide line $\omega = -1$.

In observational study, the theoretical models and bounds of the parameters are tested by the combinations of different observations astrophysical data repeatedly. The observational facts are not explained properly by standard big bang cosmology with perfect fluid. Even though in Einstein’s gravity, the cosmological constant $\Lambda$ (which has the equation of state $w_\Lambda = -1$) allows the cosmic acceleration at late times, but till now there is no proof of the origin of $\Lambda$ and the observational bounds on $\Lambda$ are incompatible with theoretical predictions in vacuum state.

For flat universe, if we assume the universe is filled with dust-like matter and dark energy, then we need to know $\Omega_m$ of the dust-like matter and $H(z)$ to a very high accuracy in order to get a handle on $\Omega_X$ or $w_X$ of the dark energy [Choudhury et al. (2007); Padmanabhan et al. (2003)]. From observations, this can be a fairly degeneracy for determining $w_X(z)$. For $z > 0.01$, TONRY data set with the 230 data points [Tonry et al. (2003)] with the 23 points from Barris et al. [Barris et al. (2004)] are still valid. For $1 < z < 1.6$, the “gold” sample of Riess et al. [Riess et al. (2004)] with 156 data points are valid. In the flat FRW universe, one finds $\Omega_\Lambda + \Omega_m = 1$, which are currently favoured by CMBR data (for recent WMAP results, see Spergel et al. (2003)). For the most recent Riess data set gives a best-fit value of $\Omega_m$ to be $0.31 \pm 0.04$, which matches with the value $\Omega_m = 0.29^{+0.05}_{-0.03}$ obtained by Riess.
et al [Riess et al (1998)]. In comparison, the best-fit \( \Omega_m \) for flat models was found to be \( 0.31 \pm 0.08 \) [Choudhury et al (2007)]. The best-fit constant equation of state parameter \( w \) for Union 2 data sample gives \( w = -0.997^{+0.050}_{-0.054} \) (stat) \( ^{+0.077}_{-0.083} \) (stat+sys together) for a flat universe, or \( w = -1.038^{+0.056}_{-0.059} \) (stat) \( ^{+0.093}_{-0.097} \) (stat+sys together) with curvature [Amanullah et al (2010)]. Now, Chaplygin gas is the more effective candidate of dark energy with equation of state \( p = -B/\rho \) [Kamenshchik et al (2001)] with \( B > 0 \). It has been generalized to the form \( p = -B/\rho^\alpha \) [Gorini et al (2003)] and thereafter modified to the form \( p = A \rho - B/\rho^\alpha \) [Debnath et al (2004)]. The MCG best fits with the 3 year WMAP and the SDSS data with the choice of parameters \( A = 0.085 \) and \( \alpha = 1.724 \) [Lu et al (2008)] which are improved constraints than the previous ones \(-0.35 < A < 0.025 \) [Dao-Jun et al (2005)].

In this work, we assume the FRW universe in Loop Quantum Cosmology (LQC) model filled with the dark matter and the MCG type dark energy. We present the Hubble parameter in terms of the observable parameters \( \Omega_{m0} \) and \( H_0 \) with the redshift \( z \) and the other parameters like \( A, B, w_m, \omega \) and \( \alpha \) in Section 3. From Stern data set (12 points), we obtain the bounds of the arbitrary parameters by minimizing the \( \chi^2 \) test in Subsection 3.1. The best-fit values of the parameters are obtained by 66%, 90% and 99% confidence levels. Next due to joint analysis with BAO and CMB observations, we also obtain the bounds and the best fit values of the parameters \( (A, B) \) by fixing some other parameters \( H_0, \Omega_{m0}, w_m, \omega \) and \( \alpha \) at their most suitable values in Subsection 3.2 and Subsection 3.3 respectively. From the best fit of distance modulus \( \mu(z) \) for our theoretical MCG model in LQC with SNe Type Ia union2 sample 552 data from [Amanullah et al (2010)] in Subsection 3.4, we conclude that our model is in agreement with the union2 sample data. After that in section 4 we consider the SNe Type Ia Riess 292 data from [Riess et al (2004, 2007); Astier et al (2006)] and examine the bounds of the arbitrary parameters \( A & B \) by minimizing the \( \chi^2 \) test for 66%, 90% and 99% confidence levels by fixing \( H_0, \Omega_{m0}, w_m, \omega \) and \( \alpha \) at their most suitable values and then we draw the distance modulus \( \mu(z) \) for our theoretical MCG model in LQC with SNe Type Ia Riess 292 data from [Riess et al (2004, 2007); Astier et al (2006)] in Subsection 4.1 and also concluded that our model is in agreement with the Riess 292 sample data.
of singularities of this scenario have been studied in Section 5 and finally, the concluding remarks of the paper are summarized in Section 6.

2. BASIC EQUATIONS AND SOLUTIONS FOR GCCG IN LQC

In recent years, loop quantum gravity (LQG) is an outstanding effort to describe the quantum effect of our universe. Nowadays several dark energy models are studied in the framework of LQC. Till now, Quintessence and phantom dark energy models [Wu et al. (2008); Chen et al. (2008)] have been studied in the cosmological evolution in LQC. Then Modified Chapingly Gas coupled to dark matter in the universe and it was described in the framework LQC by Jamil et al. [Jamil et al. (2011)] who resolved the famous cosmic coincidence problem in modern cosmology. Some authors have studied the model with an interacting phantom scalar field with an exponential potential and deduced that the dark energy dominated future singularities have been appearing in the standard FRW cosmology but some of these singularities may be avoided by loop quantum effects.

We consider the flat homogeneous and isotropic universe described by FRW metric, so the modified Einstein’s field equations in LQC are given by [Jamil et al. (2011)]

\[ H^2 = \frac{\rho}{3} \left( 1 - \frac{\rho}{\rho_c} \right) \]  

(1)

and

\[ \dot{H} = -\frac{1}{2}(\rho + p) \left( 1 - \frac{2\rho}{\rho_c} \right) \]  

(2)

where \( H \) is the Hubble parameter defined as \( H = \frac{\dot{a}}{a} \) with \( a \) is the scale factor. Where \( \rho_c = \sqrt{3\pi^2\gamma^3G^2\hbar} \) is called the critical loop quantum density, \( \gamma \) is the dimensionless Barbero-Immirzi parameter. Here the universe begins to bounce and then oscillate forever when the energy density of the universe becomes of the same order of the critical density \( \rho_c \). Thus the big bang, big rip and other singularities problems, which could not explained by the Einstein’s cosmology, might solve in LQC. It is to be noted that the parameter \( \gamma \) is fixed in LQC by the requirement of the validity of Bekenstein-Hawking entropy for the Schwarzschild black hole and it has been suggested that \( \gamma \sim 0.2375 \) by the black hole thermodynamics in LQC. The
physical solutions are allowed only when $\rho \leq \rho_c$. For $\rho = \rho_c$, it is called bounce. The maximum value of the Hubble factor $H$ is settled for $\rho_{\text{max}} = \frac{\rho_c}{2}$ and the maximum value of Hubble factor is $\frac{\kappa \rho_c}{12}$.

Here $\rho = \rho_x + \rho_m$ and $p = p_x + p_m$, where $\rho_m$, $p_m$ are the matter-density and pressure contribution of matter respectively and $\rho_x$, $p_x$ are respectively the energy density and pressure contribution of some dark energy. Now we consider the Universe is filled with Generalized Cosmic Chaplygin Gas (GCCG) model whose equation of state (EOS) is given by [Gonzalez-Diaz (2003); Chakraborty et al. (2007)]

$$p_x = -\rho_x^{-\alpha}[C + (\rho_x^{1+\alpha} - C)^{-\omega}]$$

where $C = \frac{A}{1+\omega} - 1$ with $A$ is a constant which can take on both positive and negative values and $-l < \omega < 0$, $l$ being a positive definite constant which can take on values larger than unity. We also consider the dark matter and the dark energy are separately conserved and the conservation equations of dark matter and dark energy (GCCG) are given by

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0$$

and

$$\dot{\rho}_x + 3H(\rho_x + p_x) = 0$$

From first conservation equation (4) we have the solution of $\rho_m$ as

$$\rho_m = \rho_{m0}(1 + z)^{3(1+w_m)}$$

where $p_m = \rho_m w_m$. From the conservation equation (5) we have the solution of the energy density as

$$\rho_x = \left[\left(\frac{A}{1+\omega} - 1\right) + (1 + B(1 + z)^{3(1+\alpha)(1+\omega)})\frac{1}{1+\omega}\right]^\frac{1}{1+\alpha}$$

where $B$ is the integrating constant, $z = \frac{1}{a} - 1$ is the cosmological redshift (choosing $a_0 = 1$) and the first constant term can be interpreted as the contribution of dark energy.
3. Observational Data Analysis

From the solution (7) of GCCG and defining the dimensionless density parameter \( \Omega_m = \frac{\rho_m}{3H_0^2} \), we have the expression for Hubble parameter \( H \) in terms of redshift parameter \( z \) as follows (8 \( \pi G = c = 1 \))

\[
H(z) = \frac{1}{\sqrt{3}} \left[ 3\Omega_m H_0^2 (1 + z)^{3(1+w_m)} + \left( \frac{A}{1+\omega} - 1 \right) (1 + B(1 + z)^{3(1+\alpha)(1+\omega)}) \right]^{\frac{1}{2}}
\]

\[
\times \left[ 1 - \frac{3\Omega_m H_0^2 (1 + z)^{3(1+w_m)}}{\rho_c} + \left( \frac{A}{1+\omega} - 1 \right) (1 + B(1 + z)^{3(1+\alpha)(1+\omega)}) \right]^{\frac{1}{2}}
\]

From equation (8), we see that the value of \( H \) depends on \( H_0, A, B, \Omega_m, w_m, \omega, \alpha, z \). The \( E(z) \) can be written as

\[
E(z) = \frac{H(z)}{H_0}
\]

Now \( E(z) \) contains unknown parameters like \( A, B, \omega \) and \( \alpha \). Now we will fixing two parameters and by observational data set the relation between the other two parameters will obtain and find the bounds of the parameters.

| \( z \)  | \( H(z) \) | \( \sigma(z) \) |
|---------|-----------|-------------|
| 0       | 73        | ± 8         |
| 0.1     | 69        | ± 12        |
| 0.17    | 83        | ± 8         |
| 0.27    | 77        | ± 14        |
| 0.4     | 95        | ± 17.4      |
| 0.48    | 90        | ± 60        |
| 0.88    | 97        | ± 40.4      |
| 0.9     | 117       | ± 23        |
| 1.3     | 168       | ± 17.4      |
| 1.43    | 177       | ± 18.2      |
| 1.53    | 140       | ± 14        |
| 1.75    | 202       | ± 40.4      |
Table 1: The Hubble parameter $H(z)$ and the standard error $\sigma(z)$ for different values of redshift $z$.

In the following subsections, we shall investigate the data analysis mechanism for Stern, Stern+BAO and Stern+BAO+CMB observational data to find some bound of the parameters of GCCG with LQC. We shall use the $\chi^2$ minimization technique (statistical data analysis) to test the theoretical Hubble parameter with the observed data set to get the best fit values of the unknown parameters with different confidence levels.

3.1. Analysis with Stern ($H(z)$-z) Data Set

In 2010, Stern et al. [Stern et al. (2010)] proposed an observed data set which is known as Stern ($H(z)$-z) data set. Stern data set consisted with the observed value of Hubble parameter $H(z)$ and the standard error $\sigma(z)$ for different values of redshift $z$ (twelve data points), which are given in Table 1. Here we use Stern data set (twelve data points) to analyze the model. Before going to apply $\chi^2$ minimization technique, we first form the $\chi^2$ statistics as a sum of standard normal distribution as follows:

$$\chi^2_{\text{Stern}} = \sum \frac{(H(z) - H_{\text{obs}}(z))^2}{\sigma^2(z)}$$

where $H_{\text{obs}}(z)$ and $H(z)$ are observational and theoretical values of Hubble parameter at different redshifts $z$ respectively and $\sigma(z)$ is the corresponding error for the particular observation given in Table 1. Also, the nuisance parameter $H_{\text{obs}}$ can be safely marginalized. Here the present value of Hubble parameter $H_0$ is been settled at $72 \pm 8$ Km s$^{-1}$ Mpc$^{-1}$ with a fixed prior distribution. Now we shall determine the bounds of parameters $A$ and $B$ for different $\alpha$ from minimizing the above distribution $\chi^2_{\text{Stern}}$. Fixing the other parameters $\Omega_{m0}$, $w_m$, $\omega$, $\alpha$, the relation between $A$ and $B$ can be determined by the observational data. The probability distribution function in terms of the parameters $A, B, \Omega_{m0}, w_m, \omega$ and $\alpha$ can be written as

$$L = \int e^{-\frac{1}{2}\chi^2_{\text{Stern}}} P(H_0) dH_0$$

where $P(H_0)$ is the prior distribution function for $H_0$. 
Now, using $\chi^2$ minimization technique, we plot the graph of the unknown parameters $A$ and $B$ for different $\alpha$ and fixing the other parameters for different confidence levels (like 66%, 90% and 99%). The best fit values of the parameters $A$ and $B$ are written in Table 2. It is to be noted that our best fit analysis with Stern observational data support the theoretical range of the parameters.

The 66% (solid, blue), 90% (dashed, red) and 99% (dashed, black) contours for $(A, B)$ are plotted in figures 1, 2 and 3 for different values of $\alpha$. Also the best fit values of $A$ and $B$ are tabulated in Table 2.

| $\alpha$ | $A$       | $B$       | $\chi^2_{min}$ |
|----------|-----------|-----------|----------------|
| 0.0020   | 0.628976  | 5.62894   | 7.09652        |
| 0.0010   | 0.628989  | 5.62894   | 7.09652        |
| 0.0005   | -0.069897 | 5.58487   | 7.09670        |

Table 2: $H(z)$-$z$ (Stern): The best fit values of $A$, $B$ and the minimum values of $\chi^2$ for different values of $\alpha$ and fixed value of other parameters.

### 3.2. Joint Analysis with Stern + BAO Data Sets

Now we use the statistical approach of joint analysis put forward by Eisenstein et al. (2005). The Baryon Acoustic Oscillation (BAO) peak parameter value has been proposed in their method of joint analysis. Sloan Digital Sky Survey (SDSS) survey is one of the primordial redshift survey by which the BAO signal has been directly detected at a scale $\sim 100$ Mpc. In this case, the said analysis is actually the combination of angular diameter distance and Hubble parameter at that redshift. This analysis is independent of the measurement of $H_0$ and not containing any particular dark energy. Here we shall check the parameters $A$ and $B$ with the measurements of the BAO peak at low redshift (with range $0 < z < 0.35$) using standard $\chi^2$ technique. The error, corresponding to the standard deviation, is follow the Gaussian distribution. Low-redshift distance have the ability to measure the Hubble constant $H_0$ directly. It lightly depends on different cosmological parameters and the
equation of state of dark energy. The BAO peak parameter might be defined as

$$A = \frac{\sqrt{\Omega_m}}{E(z_1)} \left( \frac{1}{z_1} \int_0^{z_1} \frac{dz}{E(z)} \right)^{2/3}$$  

(12)

Here $E(z) = H(z)/H_0$ is the normalized Hubble parameter. The redshift $z_1$ is the typical redshift of the SDSS sample whose value is settled as 0.35 and the integration term is the dimensionless comoving distance to the redshift $z_1$. The value of the parameter $A$ for the flat model of the universe is proposed as Eisenstein et al. (2005) $A = 0.469 \pm 0.017$ using SDSS data from luminous red galaxies survey. Now the $\chi^2$ function for the BAO measurement can be written as

$$\chi^2_{BAO} = \frac{(A - 0.469)^2}{(0.017)^2}$$  

(13)

Now the total joint data analysis (Stern+BAO) for the $\chi^2$ function is defined by Wu et al. (2007); Thakur et al. (2009); Paul et al. (2010, 2011); Ghose et al. (2012); Chakraborty et al. (2012)

$$\chi^2_{total} = \chi^2_{Stern} + \chi^2_{BAO}$$  

(14)

According to our analysis the joint scheme gives the best fit values of $A$ and $B$ for different $\alpha$ in Table 3. Finally we draw the contours $A$ vs $B$ for the 66% (solid, blue), 90% (dashed, red) and 99% (dashed, black) confidence limits depicted in figures 4 – 6 for different values of $\alpha$.

| $\alpha$ | $A$    | $B$    | $\chi^2_{min}$  |
|----------|--------|--------|-----------------|
| 0.0020   | 1.4401851 | 5.71536 | 768.073         |
| 0.0010   | 0.0296015 | 5.62625 | 768.073         |
| 0.0005   | -0.666052 | 5.58219 | 768.074         |

Table 3: $H(z)$-z (Stern) + BAO : The best fit values of $A$, $B$ and the minimum values of $\chi^2$ for different values of $\alpha$ and fixed value of other parameters.

3.3. Joint Analysis with Stern + BAO + CMB Data Sets

In this subsection, we shall follow the pathway, proposed by some author Bond et al. (1997); Efstathiou et al. (1999); Nessaeris et al. (2007), using Cosmic Microwave Background
Fig.1-3 show that the variation of $A$ with $B$ for $\Omega_{m0} = 0.0643, w_m = 0.051, \omega = -0.92$ with $\alpha = 0.0020, 0.0010 & 0.0005$ respectively for different confidence levels. The 66% (solid, blue), 90% (dashed, red) and 99% (dashed, black) contours are plotted in these figures for the $H(z)$-$z$ (Stern) analysis.

Fig.4-6 show that the variation of $A$ with $B$ for $\Omega_{m0} = 0.01, w_m = 0.051, \omega = -0.92$ with $\alpha = 0.0020, 0.0010 & 0.0005$ respectively for different confidence levels. The 66% (solid, blue), 90% (dashed, red) and 99% (dashed, black) contours are plotted in these figures for the $H(z)$-$z$(Stern)+BAO joint analysis.
(CMB) shift parameter. The interesting geometrical probe of dark energy can be determined by the angular scale of the first acoustic peak through angular scale of the sound horizon at the surface of last scattering which is encoded in the CMB power spectrum. It is not sensitive with respect to perturbations but are suitable to constrain model parameter. The CMB power spectrum first peak is the shift parameter which is given by

$$\mathcal{R} = \sqrt{\Omega_m} \int_{0}^{z_2} \frac{dz}{E(z)}$$

(15)

where \(z_2\) is the value of redshift at the last scattering surface.

From WMAP7 data of the work of Komatsu et al. \(\text{Komatsu et al. (2011)}\), the value of the parameter has proposed as \(\mathcal{R} = 1.726 \pm 0.018\) at the redshift \(z = 1091.3\). Therefore the \(\chi^2\) function for the CMB measurement can be written as

$$\chi^2_{CMB} = \frac{(\mathcal{R} - 1.726)^2}{(0.018)^2}$$

(16)

Now when we consider three cosmological tests together, the total joint data analysis (Stern+BAO+CMB) for the \(\chi^2\) function may be defined by

$$\chi^2_{TOTAL} = \chi^2_{Stern} + \chi^2_{BAO} + \chi^2_{CMB}$$

(17)

Now the best fit values of \(A\) and \(B\) for joint analysis of BAO and CMB with Stern observational data support the theoretical range of the parameters given in Table 4. The 66\% (solid, blue), 90\% (dashed, red) and 99\% (dashed, black) contours are plotted in figures 7-9 for different values of \(\alpha\).

| \(\alpha\) | \(A\)     | \(B\)     | \(\chi^2_{\text{min}}\) |
|------------|-----------|-----------|------------------------|
| 0.0020     | 1.4816294 | 5.71517   | 9962.75                |
| 0.0010     | 0.0713549 | 5.62606   | 9962.75                |
| 0.0005     | -0.624274 | 5.58201   | 9962.75                |

Table 4: \(H(z)-z\) (Stern) + BAO + CMB : The best fit values of \(A\), \(B\) and the minimum values of \(\chi^2\) for different values of \(\alpha\) and fixed value of other parameters.
3.4. Redshift-Magnitude Observations of Supernovae Type Ia Union2 Sample [From Amanullah et al. (2010)]

The main evidence for the existence of dark energy is provided by the Supernova Type Ia experiments. Two teams of High-z Supernova Search and the Supernova Cosmology Project have discovered several type Ia supernovas at the high redshifts [Perlmutter et al. (1998, 1999); Riess et al. (1998, 2004)] since 1995. The observations directly measure the distance modulus of a Supernovae and its redshift $z$ [Riess et al. (2007); Kowalaski et al. (2008)]. Here we take recent observational data, including SNe Ia which consists of 557 data points and belongs to the Union2 sample [Amanullah et al. (2010)].

Motivated by the work of some authors [Thakur et al. (2009); Paul et al. (2010, 2011); Ghose et al. (2012); Chakraborty et al. (2012)] here we determine distance modulus $d_L(z)$ for our theoretical GCCG in LQC model and tested with the SNe Type Ia data. From the observations, the luminosity distance $d_L(z)$ determines the dark energy density and is defined by

$$d_L(z) = (1 + z)H_0\int_0^z \frac{dz'}{H(z')}$$

and the distance modulus (distance between absolute and apparent luminosity of a distance object) for Supernovas is given by

$$\mu(z) = 5\log_{10} \left[ \frac{d_L(z)/H_0}{1 \text{ MPc}} \right] + 25$$

The best fit of distance modulus as a function $\mu(z)$ of redshift $z$ for our theoretical model and the Supernova Type Ia Union2 sample are drawn in figure 10 for our best fit values of $A$, $B$ with the other previously chosen parameters. In Figure 11, we have shown that the variation of the curves with slightly changes in the value of $A$ and $B$ ($A = 0.001&B = 0.13$ for Black line; $A = 0.03&B = 0.05$ for Red line; $A = 0.002&B = 0.025$ for Green line). From the curves, we see that the theoretical GCCG with LQC is in agreement with the union2 sample data.
1.5
0.0 0.5 1.0 1.5
34 36 38 40 42 44 46

0.0 0.5 1.0 1.5
34 36 38 40 42 44 46

Fig.7

Fig.8

Fig.9

Fig.10

Fig.11

Fig.7-9 show that the variation of $A$ with $B$ for $\Omega_m = 0.01, w_m = 0.051, \omega = -0.92$ with $\alpha = 0.0020, 0.0010 & 0.0005$ respectively for different confidence levels. The 66% (solid, blue), 90% (dashed, red) and 99% (dashed, black) contours are plotted in these figures for the $H(z) - z(Stern)+BAO+CMB$ analysis.

Fig.10 show $\mu(z)$ vs $z$ for our GCCG with LQC (solid red line) and the Union2 sample (dotted points). Fig 11 shows the same for different value of $A & B$ ($A = 0.001 & B = 0.13$ for Black line; $A = 0.03 & B = 0.05$ for Red line; $A = 0.002 & B = 0.025$ for Green line).
4. Analysis with Supernovae Type Ia Data [From Riess et al. (2004, 2007); Astier et al. (2006)]

In this section we analyzed our GCCG with LQC model with same spirit (as stated before in the previous sections) and obtaining the bounds of the arbitrary parameters ($A$ & $B$) by fixing the cosmological parameters around their favorable value with the help of observational Supernovae Type Ia data which belongs to [Riess et al. (2004, 2007); Astier et al. (2006)] and also shown in Table 5 at Appendix. As like Sec. 3.1, here also we are applying $\chi^2$ minimization technique, where the $\chi^2$ statistics is as follows:

$$
\chi^2_{(SNeTypeIa)} = \sum \frac{(H(z) - H_{obs}(z))^2}{\sigma^2(z)}
$$

(20)

where the $H_{obs}(z)$ and $\sigma(z)$ are given in Table 5 and also the probability distribution function can be expressed as

$$
L = \int e^{-\frac{1}{2}\chi^2_{(SNeTypeIa)}} P(H_0) dH_0
$$

(21)

where $P(H_0)$ is the prior distribution function for $H_0$. By using $\chi^2$ minimization technique, here we plot the graph of the unknown parameters $A$ and $B$ for same values of $\alpha$ (as stated above) and fixing the other parameters for their most suitable values and draw for different confidence levels (as 66%, 90% and 99%). The best fit values of the parameters $A$ and $B$ are written in Table 6. It is to be noted that our best fit analysis with SNe Type Ia observational 292 data also support the theoretical range of the parameters. It is also to be observed that for different $\alpha (=0.0020, 0.0010 \& 0.0005)$ the best fit value of $A$ and $B$ are almost same but the value of $\chi^2_{min}$ are different for each cases. The 66% (solid, blue), 90% (dashed, red) and 99% (dashed, black) contours for $(A, B)$ are plotted in figures 12, 13 and 14 for different values of $\alpha$. Also the best fit values of $A$ and $B$ are tabulated in Table 6.

| $\alpha$ | $A$    | $B$    | $\chi^2_{min}$ |
|---------|--------|--------|----------------|
| 0.0020  | 0.304936 | 0.266141 | 87260.15       |
| 0.0010  | 0.304936 | 0.266141 | 86928.93       |
| 0.0005  | 0.304936 | 0.266141 | 86863.26       |
Table 6: $H(z)$-z SNe Type Ia: The best fit values of $A$, $B$ and the minimum values of $\chi^2$ for different values of $\alpha$ and fixed value of other parameters.

### 4.1. Redshift-Magnitude Observational Analysis with Supernovae Type Ia 292 Data [From Riess et al. (2004, 2007); Astier et al. (2006)]

In this subsection we measure the distance modulus (as like Sec. 3.4) of SNe Type Ia 292 data which belongs to [Riess et al. (2004, 2007); Astier et al. (2006)]. Here also we use the same luminosity distance $d_L(z)$ which is defined as

$$d_L(z) = (1 + z)H_0 \int_0^z \frac{dz'}{H(z')}$$

and the distance modulus for SNe Type Ia observational 292 data is given below:

$$\mu(z) = 5 \log_{10} \left[ \frac{d_L(z)/H_0}{1 \text{ MPc}} \right] + 25$$

As stated above, the best fit of distance modulus $\mu(z)$ which is a function of redshift $z$ for our theoretical model and the SNe Type Ia 292 data from [Riess et al. (2004, 2007); Astier et al. (2006)] are drawn in figure 15 with most favorable different values of $A$, $B$ with the previously chosen other parameters. From the curves, we can conclude that the theoretical GCCG with LQC is in agreement with the SNe Type Ia 292 data from [Riess et al. (2004, 2007); Astier et al. (2006)].

### 5. Study of Future Singularities

In recent time, the well established universal fact for any energy dominated model of the universe is intended to the result in future singularity. Without studying of these singularities, the ultimate goal of this study of our model become incomplete. A well known cosmological hypothesis is that the universe dominated by phantom energy ends with a future singularity, which violates the dominant energy condition (DEC), known as Big Rip [Caldwell et al. (2003)]. In 2005, Nojiri, et al. (2005) studied the various types of singularities for an phantom energy
Fig.12-14 show that the variation of $A$ with $B$ for $\Omega_{m0} = 6.43 \times 10^{-8}, w_m = 0.051, \omega = -0.92$ with $\alpha = 0.0020, 0.0010 & 0.0005$ respectively for different confidence levels. The 66% (solid, blue), 90% (dashed, red) and 99% (dashed, black) contours are plotted in these figures for the $H(z)-z$ of SNe Type Ia data.

Fig.15 shows $\mu(z)$ vs $z$ for our GCCG with LQC for SNe Type Ia data(dotted points) Three lines are drawn for different value of $A$ & $B$ ($A = 0.001 & B = 0.13$ for Black line; $A = 0.03 & B = 0.05$ for Red line; $A = 0.002 & B = 0.025$ for Green line).
dominated universe. There are many effective approaches have been adopted by some authors [Sami et al. (2006); Naskar et al. (2007); Samart et al. (2007); Cailleteau et al. (2008); Singh (2009); Corichi et al. (2009); Lamon et al. (2010); Singh et al. (2011)] to study the future singularities. Singularities are basically characterized by the growth of energy and curvature at the time of occurrence of them. It is observed that the quantum effects are not only very dominant near the singularities, they may prevent these singularities. All different types of future singularities for different scenario was discussed by Nojiri et al. (2011). Recently Bamba et al. (2013) studied future singularities in the context of LQC and shown that some of these singularities may be avoided. In this regards some works have been done by Rudra et al. (2012a); Chowdhury et al. (2013); Rudra et al. (2012b). Future singularities are basically classified in four types and in each cases our model have been tested for those scenarios as follows:

- **TYPE-I Singularity (Big Rip):** When $\rho \to \infty$ and $|p| \to \infty$ for $a \to \infty$ and $t \to t_s$.

In this present scenario our predicted model of LQC with GCCG and DM in non interacting scenario have been tested and we have

\[
a \to \infty : \quad \rho_x \to \infty \quad \text{for} \quad 1 + \omega < 0 \Rightarrow |p_x| \to 0
\]

\[
\rho_x \to (C + 1)^{\frac{1}{\alpha+1}} \quad \text{for} \quad 1 + \omega > 0 \Rightarrow |p_x| \to (C + 1)^{\frac{1}{\alpha+1}}
\]

and from the above results we can conclude that there is no possibility of Type-I i.e., “Big Rip” singularity and the result is absolutely accordance with the work of some authors [Gonzalez-Diaz (2003); Bamba et al. (2013); Chowdhury et al. (2013)] who have shown that “Big Rip” can be easily avoided in LQC with non interacting GCCG and DM and produced a singularity free late universe.

- **TYPE-II Singularity (Sudden):** When $\rho \to \rho_s$ and $|p| \to \infty$ for $a \to a_s$ and $t \to t_s$.

In this case we have been again considering our predicted model of LQC with non interacting GCCG and DM and we find that

\[
a \to a_s \sim 0 : \quad \rho_x \to \infty \quad \text{for} \quad 1 + \omega > 0 \Rightarrow |p_x| \to 0
\]

\[
\rho_x \to (C + 1)^{\frac{1}{\alpha+1}} \quad \text{for} \quad 1 + \omega < 0 \Rightarrow |p_x| \to (C + 1)^{\frac{1}{\alpha+1}}
\]
and it can be concluded that there is no possibility of the Type-II or “Sudden” singularity for our predicted model.

- **TYPE-III Singularity (Big Freeze):** When $\rho \to \infty$ and $|p| \to \infty$ if $a \to a_s$ and $t \to t_s$.

In this present condition, it can be quite evidently concluded from our model of LQC with non interacting GCCG and DM that it does not support this Type-III or “Big Freeze” singularity. In this regards there are some works by some authors [Chowdhury et al. (2013); Rudra et al. (2012b)] in supports of this result.

- **TYPE-IV Singularity (Generalized Sudden):** For $t \to t_s$, $a \to a_s$, $\rho \to 0$ and $|p| \to 0$

In this regards we have expressed scale factor $a(t)$ in terms of energy density of GCCG as follows:

$$a = \left[ \frac{B}{(\rho^\alpha - C)^\omega - 1} \right]^{\frac{1}{3(1+\alpha)(1+\omega)}}$$

and therefore it can be easily concluded that this type of singularity is not supported by our predicted LQC model with non interacting GCCG and DM.

### 6. Discussions

We have assumed the FRW universe in loop quantum cosmology (LQC) model filled with the dark matter and the Generalized Cosmic Chaplygin gas (GCCG) type dark energy. We present the Hubble parameter $H(z)$ in terms of the observable parameters $\Omega_{m0}$ and $H_0$ with the redshift $z$ and the other parameters like $A$, $B$, $w_m$, $\omega$ and $\alpha$. From Stern data set (12 points), we have obtained the best fit values of two arbitrary parameters $(A, B)$ in table 2 by fixing other parameters $\Omega_{m0} = 0.0643$, $w_m = 0.051$, $\omega = -0.92$ and $\alpha = 0.002, 0.001, 0.0005$ by minimizing the $\chi^2$ test. The bounds of the parameters $(A, B)$ are obtained by 66%, 90% and 99% confidence levels in figures 1-3. Next due to joint analysis with BAO and CMB observations, we have also obtained the best fit values of the parameters $(A, B)$ by fixing
the other parameters (same values) in tables 3 and 4 respectively. Also the bounds of the parameters \((A, B)\) due to joint analysis with BAO and CMB observations are obtained by 66%, 90% and 99% confidence levels in figures 4-6 and figures 7-9 respectively. From tables 1-3, we see that when \(\alpha = 0.002, 0.001\), the best-fit values of \(A\) and \(B\) are positive for all our observational data. But if \(\alpha = 0.0005\), the best-fit value of \(A\) is negative and best-fit value of \(B\) is still positive for all our observational data. From the best fit value of distance modulus \(\mu(z)\) for our theoretical GCCG model in LQC is drawn in figure 10. Fig 11 shows the same for different value of \(A \& B\) (\(A = 0.001 \& B = 0.13\) for Black line; \(A = 0.03 \& B = 0.05\) for Red line; \(A = 0.002 \& B = 0.025\) for Green line). From the figure, we have concluded that our predicted theoretical GCCG model in LQC permitted the union2 sample data sets of SNe Type Ia. After that we have considered SNe Type Ia Riess 292 data from [Riess et al. (2004, 2007); Astier et al. (2006)] and tested our theoretical GCCG with LQC model by minimizing the \(\chi^2\) test (same as stated above) and obtained the bounds of the parameters \((A, B)\) given in Table 6. When the other parameters are fixed at their suitable value as \(\Omega_m = 6.43 \times 10^{-8}, w_m = 0.051, \omega = -0.92\) with \(\alpha = 0.0020, 0.0010 \& 0.0005\), we have drawn the figure 12-14 respectively and the best fitted values of \(A = 0.304936 \& B = 0.266141\) are almost same in every cases of \(\alpha\). Figure 15 shows the distance modulus \(\mu(z)\) of our theoretical GCCG model in LQC together with Riess 292 data for different favorable values of \(A \& B\) (\(A = 0.001 \& B = 0.13\) for Black line; \(A = 0.03 \& B = 0.05\) for Red line; \(A = 0.002 \& B = 0.025\) for Green line) vs redshift \(z\) and which depicted that our theoretical GCCG model in LQC suitably permitted with the Riess 292 data of SNe Type Ia. From the above results, we can finally conclude that our theoretical GCCG model in LQC is in agreement with the Supernovae Type Ia sample data. In addition, we have also investigated in details about the Future Singularities like Type-I, Type-II, Type-III and Type-IV that may be formed and or avoided in this model and it is found that our model is completely free from any types of future singularities.
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A. Appendix material
Table 5. SNe la 292 Data Set from Riess et al. (2004, 2007); Astier et al. (2006)

| Name     | z   | H(z)  | σ(z) | Type |
|----------|-----|-------|------|------|
| SN90O    | 0.030 | 35.90 | 0.21 | Gold |
| SN90T    | 0.040 | 36.38 | 0.20 | Gold |
| SN90af   | 0.050 | 36.84 | 0.22 | Gold |
| SN91ag   | 0.014 | 34.13 | 0.29 | Gold |
| SN91U    | 0.033 | 35.53 | 0.21 | Gold |
| SN91S    | 0.056 | 37.31 | 0.19 | Gold |
| SN92al   | 0.014 | 34.12 | 0.29 | Gold |
| SN92bo   | 0.017 | 34.70 | 0.26 | Gold |
| SN92bc   | 0.018 | 34.96 | 0.25 | Gold |
| SN92ag   | 0.026 | 35.06 | 0.25 | Silver |
| SN92P    | 0.026 | 35.63 | 0.22 | Gold |
| SN92bg   | 0.036 | 36.17 | 0.20 | Gold |
| SN92bl   | 0.043 | 36.52 | 0.19 | Gold |
| SN92bh   | 0.045 | 36.99 | 0.18 | Gold |
| SN92J    | 0.046 | 36.35 | 0.21 | Gold |
| SN92bk   | 0.058 | 37.13 | 0.19 | Gold |
| SN92au   | 0.061 | 37.31 | 0.22 | Gold |
| SN92bs   | 0.063 | 37.67 | 0.19 | Gold |
| SN92ae   | 0.075 | 37.77 | 0.19 | Gold |
| SN92bp   | 0.079 | 37.94 | 0.18 | Gold |
| SN92br   | 0.088 | 38.07 | 0.28 | Gold |
| SN92aq   | 0.101 | 38.70 | 0.20 | Gold |
| SN93ae   | 0.018 | 34.29 | 0.25 | Gold |
| SN93H    | 0.025 | 35.09 | 0.22 | Gold |
| SN93ah   | 0.028 | 35.53 | 0.22 | Gold |
| SN93ac   | 0.049 | 36.90 | 0.21 | Silver |
| SN93ag   | 0.050 | 37.07 | 0.19 | Gold |
| SN93O    | 0.052 | 37.16 | 0.18 | Gold |
| SN93B    | 0.071 | 37.78 | 0.19 | Gold |
| SN94S    | 0.016 | 34.50 | 0.27 | Gold |
| SN94M    | 0.024 | 35.09 | 0.22 | Gold |
| SN94Q    | 0.029 | 35.70 | 0.21 | Gold |
| SN94T    | 0.036 | 36.01 | 0.21 | Gold |
| SN94C    | 0.051 | 36.67 | 0.17 | Silver |
| SN94B    | 0.089 | 38.50 | 0.17 | Silver |
Table 5—Continued

| Name   | z   | $H(z)$ | $\sigma(z)$ | Type |
|--------|-----|--------|-------------|------|
| SN95K  | 0.478 | 42.48  | 0.23        | Gold |
| SN95ak | 0.021 | 34.70  | 0.24        | Silver |
| SN95E  | 0.011 | 32.95  | 0.35        | Silver |
| SN95bd | 0.015 | 34.07  | 0.28        | Silver |
| SN95ac | 0.049 | 36.55  | 0.20        | Gold |
| SN95ar | 0.465 | 42.81  | 0.22        | Silver |
| SN95as | 0.498 | 43.21  | 0.24        | Silver |
| SN95aw | 0.400 | 42.04  | 0.19        | Gold |
| SN95ax | 0.615 | 42.85  | 0.23        | Gold |
| SN95ay | 0.480 | 42.37  | 0.20        | Gold |
| SN95az | 0.450 | 42.13  | 0.21        | Gold |
| SN95ba | 0.388 | 42.07  | 0.19        | Gold |
| SN95M  | 0.053 | 37.17  | 0.16        | Silver |
| SN95ae | 0.067 | 37.54  | 0.34        | Silver |
| SN95ao | 0.300 | 40.76  | 0.60        | Silver |
| SN95ap | 0.230 | 40.44  | 0.46        | Silver |
| SN96E  | 0.425 | 41.69  | 0.40        | Gold |
| SN96H  | 0.620 | 43.11  | 0.28        | Gold |
| SN96I  | 0.570 | 42.80  | 0.25        | Gold |
| SN96J  | 0.300 | 41.01  | 0.25        | Gold |
| SN96K  | 0.380 | 42.02  | 0.22        | Gold |
| SN96U  | 0.430 | 42.33  | 0.34        | Gold |
| SN96Z  | 0.008 | 32.45  | 0.45        | Silver |
| SN96bo | 0.016 | 33.82  | 0.30        | Silver |
| SN96bv | 0.016 | 34.21  | 0.27        | Silver |
| SN96bk | 0.007 | 32.09  | 0.53        | Silver |
| SN96C  | 0.027 | 35.90  | 0.21        | Gold |
| SN96bl | 0.034 | 36.19  | 0.20        | Gold |
| SN96ab | 0.124 | 39.19  | 0.22        | Gold |
| SN96cf | 0.570 | 42.77  | 0.19        | Silver |
| SN96cg | 0.490 | 42.58  | 0.19        | Silver |
| SN96ci | 0.495 | 42.25  | 0.19        | Gold |
| SN96cl | 0.828 | 43.96  | 0.46        | Gold |
| SN96cm | 0.450 | 42.58  | 0.19        | Silver |
| SN96cn | 0.430 | 42.56  | 0.18        | Silver |
Table 5—Continued

| Name   | $z$  | $H(z)$ | $\sigma(z)$ | Type |
|--------|------|--------|-------------|------|
| SN96V  | 0.024| 35.33  | 0.26        | Silver |
| SN96T  | 0.240| 40.68  | 0.43        | Silver |
| SN96R  | 0.160| 39.08  | 0.40        | Silver |
| SN97eq | 0.538| 42.66  | 0.18        | Gold  |
| SN97ek | 0.860| 44.03  | 0.30        | Gold  |
| SN97ez | 0.778| 43.81  | 0.35        | Gold  |
| SN97as | 0.508| 42.19  | 0.35        | Gold  |
| SN97aw | 0.440| 42.56  | 0.40        | Silver |
| SN97bb | 0.518| 42.83  | 0.31        | Gold  |
| SN97bh | 0.420| 41.76  | 0.23        | Silver |
| SN97bj | 0.334| 40.92  | 0.30        | Gold  |
| SN97ce | 0.440| 42.07  | 0.19        | Gold  |
| SN97cj | 0.500| 42.73  | 0.20        | Gold  |
| SN97do | 0.010| 33.72  | 0.39        | Gold  |
| SN97E  | 0.013| 34.02  | 0.31        | Gold  |
| SN97Y  | 0.016| 34.53  | 0.27        | Gold  |
| SN97cn | 0.017| 34.71  | 0.28        | Gold  |
| SN97dg | 0.029| 36.13  | 0.21        | Gold  |
| SN97F  | 0.580| 43.04  | 0.21        | Gold  |
| SN97H  | 0.526| 42.56  | 0.18        | Gold  |
| SN97I  | 0.172| 39.79  | 0.18        | Gold  |
| SN97N  | 0.180| 39.98  | 0.18        | Gold  |
| SN97O  | 0.374| 43.07  | 0.20        | Silver |
| SN97P  | 0.472| 42.46  | 0.19        | Gold  |
| SN97Q  | 0.430| 41.99  | 0.18        | Gold  |
| SN97R  | 0.657| 43.27  | 0.20        | Gold  |
| SN97ac | 0.320| 41.45  | 0.18        | Gold  |
| SN97af | 0.579| 42.86  | 0.19        | Gold  |
| SN97ai | 0.450| 42.10  | 0.23        | Gold  |
| SN97aj | 0.581| 42.63  | 0.19        | Gold  |
| SN97am | 0.416| 42.10  | 0.19        | Gold  |
| SN97ap | 0.830| 43.85  | 0.19        | Gold  |
| SN97ck | 0.970| 44.13  | 0.38        | Silver |
| SN98ax | 0.497| 42.77  | 0.31        | Silver |
| SN98aw | 0.440| 42.02  | 0.19        | Silver |
| Name   | $z$   | $H(z)$ | $\sigma(z)$ | Type  |
|--------|-------|--------|-------------|-------|
| SN98ay | 0.638 | 43.29  | 0.36        | Silver|
| SN98ba | 0.430 | 42.36  | 0.25        | Gold  |
| SN98be | 0.644 | 42.77  | 0.26        | Silver|
| SN98as | 0.355 | 41.77  | 0.28        | Silver|
| SN98bi | 0.740 | 43.35  | 0.30        | Gold  |
| SN98ac | 0.460 | 41.81  | 0.40        | Gold  |
| SN98M  | 0.630 | 43.26  | 0.37        | Gold  |
| SN98J  | 0.828 | 43.59  | 0.61        | Gold  |
| SN98I  | 0.886 | 42.91  | 0.81        | Silver|
| SN98bp | 0.010 | 33.20  | 0.38        | Gold  |
| SN98ef | 0.017 | 34.18  | 0.26        | Gold  |
| SN98V  | 0.017 | 34.47  | 0.26        | Gold  |
| SN98co | 0.017 | 34.62  | 0.27        | Gold  |
| SN98eg | 0.023 | 35.35  | 0.22        | Gold  |
| SN98cs | 0.032 | 36.08  | 0.20        | Gold  |
| SN98dx | 0.053 | 36.95  | 0.19        | Gold  |
| SN99Q2 | 0.459 | 42.67  | 0.22        | Gold  |
| SN99U2 | 0.511 | 42.83  | 0.21        | Gold  |
| SN99S  | 0.474 | 42.81  | 0.22        | Gold  |
| SN99N  | 0.537 | 42.85  | 0.41        | Gold  |
| SN99M  | 0.493 | 40.42  | 0.60        | Silver|
| SN99fn | 0.477 | 42.38  | 0.21        | Gold  |
| SN99ff | 0.455 | 42.29  | 0.28        | Gold  |
| SN99fj | 0.815 | 43.75  | 0.33        | Gold  |
| SN99fm | 0.949 | 44.00  | 0.24        | Gold  |
| SN99fk | 1.056 | 44.35  | 0.23        | Gold  |
| SN99fw | 0.278 | 41.01  | 0.41        | Gold  |
| SN99cp | 0.010 | 33.56  | 0.37        | Gold  |
| SN99dq | 0.013 | 33.73  | 0.30        | Gold  |
| SN99dk | 0.014 | 34.43  | 0.29        | Gold  |
| SN99aa | 0.015 | 34.58  | 0.28        | Gold  |
| SN99X  | 0.025 | 35.40  | 0.22        | Gold  |
| SN99gp | 0.026 | 35.57  | 0.21        | Gold  |
| SN99cc | 0.031 | 35.84  | 0.21        | Gold  |
| SN99ef | 0.038 | 36.67  | 0.19        | Gold  |
| Name   | $z$  | $H(z)$ | $\sigma(z)$ | Type |
|--------|------|--------|-------------|------|
| SN99fv | 1.199| 44.19  | 0.34        | Gold |
| SN99fh | 0.369| 41.62  | 0.31        | Silver |
| SN99da | 0.012| 34.05  | 0.36        | Silver |
| SN00ec | 0.470| 42.76  | 0.21        | Gold |
| SN00dz | 0.500| 42.74  | 0.24        | Gold |
| SN00ea | 0.420| 40.79  | 0.32        | Silver |
| SN00eg | 0.540| 41.96  | 0.41        | Gold |
| SN00ee | 0.470| 42.73  | 0.23        | Gold |
| SN00eh | 0.490| 42.40  | 0.25        | Gold |
| SN00fr | 0.543| 42.67  | 0.19        | Gold |
| SN00dk | 0.016| 34.41  | 0.27        | Gold |
| SN00B  | 0.019| 34.59  | 0.25        | Gold |
| SN00fa | 0.021| 35.05  | 0.23        | Gold |
| SN00cn | 0.023| 35.14  | 0.22        | Gold |
| SN00bk | 0.026| 35.35  | 0.23        | Gold |
| SN00cf | 0.036| 36.39  | 0.19        | Gold |
| SN00ce | 0.016| 34.47  | 0.26        | Silver |
| SN01iv | 0.397| 40.89  | 0.30        | Silver |
| SN01iw | 0.340| 40.72  | 0.26        | Silver |
| SN01jh | 0.884| 44.22  | 0.19        | Gold |
| SN01hu | 0.882| 43.89  | 0.30        | Gold |
| SN01ix | 0.710| 43.03  | 0.32        | Silver |
| SN01iy | 0.570| 42.87  | 0.31        | Gold |
| SN01jp | 0.528| 42.76  | 0.25        | Gold |
| SN01V  | 0.016| 34.13  | 0.27        | Gold |
| SN01fo | 0.771| 43.12  | 0.17        | Gold |
| SN01fs | 0.873| 43.75  | 0.38        | Silver |
| SN01hs | 0.832| 43.55  | 0.29        | Gold |
| SN01hx | 0.798| 43.88  | 0.31        | Gold |
| SN01hy | 0.811| 43.97  | 0.35        | Gold |
| SN01jb | 0.698| 43.33  | 0.32        | Silver |
| SN01jf | 0.815| 44.09  | 0.28        | Gold |
| SN01jm | 0.977| 43.91  | 0.26        | Gold |
| SN01kd | 0.935| 43.99  | 0.38        | Silver |
| SN02P  | 0.719| 43.22  | 0.26        | Silver |
| Name   | $z$  | $H(z)$ | $\sigma(z)$ | Type   |
|--------|------|--------|-------------|--------|
| SN02ab | 0.422| 42.02  | 0.17        | Silver |
| SN02ad | 0.514| 42.39  | 0.27        | Silver |
| 1997ff | 1.755| 45.35  | 0.35        | Gold   |
| 2002dc | 0.475| 42.24  | 0.20        | Gold   |
| 2002dd | 0.950| 43.98  | 0.34        | Gold   |
| 2003aj | 1.307| 44.99  | 0.31        | Silver |
| 2002fx | 1.400| 45.28  | 0.81        | Silver |
| 2003eq | 0.840| 43.67  | 0.21        | Gold   |
| 2003es | 0.954| 44.30  | 0.27        | Gold   |
| 2003az | 1.265| 44.64  | 0.25        | Silver |
| 2002kc | 0.216| 40.33  | 0.19        | Silver |
| 2003eb | 0.900| 43.64  | 0.25        | Gold   |
| 2003XX | 0.935| 43.97  | 0.29        | Gold   |
| 2002hr | 0.526| 43.08  | 0.27        | Silver |
| 2003bd | 0.670| 43.19  | 0.24        | Gold   |
| 2002kd | 0.735| 43.14  | 0.19        | Gold   |
| 2003be | 0.640| 43.01  | 0.25        | Gold   |
| 2003dy | 1.340| 44.92  | 0.31        | Gold   |
| 2002ki | 1.140| 44.71  | 0.29        | Gold   |
| 2003ak | 1.551| 45.07  | 0.32        | Silver |
| 2002hp | 1.305| 44.51  | 0.30        | Gold   |
| 2002fw | 1.300| 45.06  | 0.20        | Gold   |
| HST04Pat|0.970|44.67   |0.36        |Gold   |
| HST04Mcg|1.370|45.23   |0.25        |Gold   |
| HST05Fer|1.020|43.99   |0.27        |Gold   |
| HST05Koe|1.230|45.17   |0.23        |Gold   |
| HST05Dic|0.638|42.89   |0.18        |Silver |
| HST04Gre|1.140|44.44   |0.31        |Gold   |
| HST04Omb|0.975|44.21   |0.26        |Gold   |
| HST05Red|1.190|43.64   |0.39        |Silver |
| HST05Lan|1.230|44.97   |0.20        |Gold   |
| HST04Tha|0.954|43.85   |0.27        |Gold   |
| HST04Rak|0.740|43.38   |0.22        |Gold   |
| HST05Zwi|0.521|42.05   |0.37        |Silver |
| HST04Hawk|0.490|42.54   |0.24        |Silver |
Table 5—Continued

| Name       | $z$  | $H(z)$ | $\sigma(z)$ | Type  |
|------------|------|--------|-------------|-------|
| HST04Kur   | 0.359| 41.23  | 0.39        | Silver|
| HST04Yow   | 0.460| 42.23  | 0.32        | Gold  |
| HST04Man   | 0.854| 43.96  | 0.29        | Gold  |
| HST05Spo   | 0.839| 43.45  | 0.20        | Gold  |
| HST04Eag   | 1.020| 44.52  | 0.19        | Gold  |
| HST05Gab   | 1.120| 44.67  | 0.18        | Gold  |
| HST05Str   | 1.010| 44.77  | 0.19        | Gold  |
| HST04Sas   | 1.390| 44.90  | 0.19        | Gold  |
| SN88U      | 0.309| 41.43  | 0.36        | Silver|
| SN-03D1au  | 0.504| 42.61  | 0.17        | Gold  |
| SN-03D1aw  | 0.582| 43.07  | 0.17        | Gold  |
| SN-03D1ax  | 0.496| 42.36  | 0.17        | Gold  |
| SN-03D1bp  | 0.346| 41.55  | 0.17        | Silver|
| SN-03D1cm  | 0.870| 44.28  | 0.34        | Gold  |
| SN-03D1co  | 0.679| 43.58  | 0.19        | Gold  |
| SN-03D1ew  | 0.868| 44.06  | 0.38        | Silver|
| SN-03D1fc  | 0.331| 41.13  | 0.17        | Gold  |
| SN-03D1fl  | 0.688| 43.23  | 0.17        | Gold  |
| SN-03D1fq  | 0.800| 43.67  | 0.19        | Gold  |
| SN-03D1gt  | 0.548| 43.01  | 0.18        | Silver|
| SN-03D3af  | 0.532| 42.78  | 0.18        | Gold  |
| SN-03D3aw  | 0.449| 42.05  | 0.17        | Gold  |
| SN-03D3ay  | 0.371| 41.67  | 0.17        | Gold  |
| SN-03D3ba  | 0.291| 41.18  | 0.17        | Silver|
| SN-03D3bh  | 0.249| 40.76  | 0.17        | Gold  |
| SN-03D3cc  | 0.463| 42.27  | 0.17        | Gold  |
| SN-03D3cd  | 0.461| 42.22  | 0.17        | Gold  |
| SN-03D4ag  | 0.285| 40.92  | 0.17        | Gold  |
| SN-03D4at  | 0.633| 43.32  | 0.18        | Gold  |
| SN-03D4aud | 0.468| 42.89  | 0.18        | Silver|
| SN-03D4bcd | 0.572| 43.71  | 0.21        | Silver|
| SN-03D4cn  | 0.818| 43.72  | 0.34        | Silver|
| SN-03D4cx  | 0.949| 43.69  | 0.32        | Gold  |
| SN-03D4cy  | 0.927| 44.74  | 0.41        | Silver|
| SN-03D4cz  | 0.695| 43.21  | 0.19        | Gold  |
Table 5—Continued

| Name    | $z$  | $H(z)$ | $\sigma(z)$ | Type   |
|---------|------|--------|-------------|--------|
| SN-03D4dh | 0.627 | 42.93  | 0.17        | Gold   |
| SN-03D4di | 0.905 | 43.89  | 0.30        | Gold   |
| SN-03D4dy | 0.604 | 42.70  | 0.17        | Gold   |
| SN-03D4fd | 0.791 | 43.54  | 0.18        | Gold   |
| SN-03D4gf | 0.581 | 42.95  | 0.17        | Gold   |
| SN-03D4gg | 0.592 | 42.75  | 0.19        | Gold   |
| SN-03D4gl | 0.571 | 42.65  | 0.18        | Gold   |
| SN-04D1ag | 0.557 | 42.70  | 0.17        | Gold   |
| SN-04D1aj | 0.721 | 43.39  | 0.20        | Silver |
| SN-04D1ak | 0.526 | 42.83  | 0.17        | Silver |
| SN-04D2cf | 0.369 | 41.67  | 0.17        | Gold   |
| SN-04D2fp | 0.415 | 41.96  | 0.17        | Gold   |
| SN-04D2fs | 0.357 | 41.63  | 0.17        | Gold   |
| SN-04D2gb | 0.430 | 41.96  | 0.17        | Gold   |
| SN-04D2gc | 0.521 | 42.62  | 0.17        | Silver |
| SN-04D2gp | 0.707 | 43.42  | 0.21        | Gold   |
| SN-04D2iu | 0.691 | 43.33  | 0.21        | Silver |
| SN-04D2ja | 0.741 | 43.61  | 0.20        | Silver |
| SN-04D3co | 0.620 | 43.21  | 0.18        | Gold   |
| SN-04D3cp | 0.830 | 44.60  | 0.38        | Silver |
| SN-04D3cy | 0.643 | 43.21  | 0.18        | Gold   |
| SN-04D3dd | 1.010 | 44.86  | 0.55        | Silver |
| SN-04D3df | 0.470 | 42.45  | 0.17        | Gold   |
| SN-04D3do | 0.610 | 42.98  | 0.17        | Gold   |
| SN-04D3ez | 0.263 | 40.87  | 0.17        | Gold   |
| SN-04D3fk | 0.358 | 41.66  | 0.17        | Gold   |
| SN-04D3fq | 0.730 | 43.47  | 0.18        | Gold   |
| SN-04D3gt | 0.451 | 42.22  | 0.17        | Silver |
| SN-04D3gx | 0.910 | 44.44  | 0.38        | Silver |
| SN-04D3hn | 0.552 | 42.65  | 0.17        | Gold   |
| SN-04D3is | 0.710 | 43.36  | 0.18        | Silver |
| SN-04D3ki | 0.930 | 44.61  | 0.46        | Silver |
| SN-04D3kr | 0.337 | 41.44  | 0.17        | Gold   |
| SN-04D3ks | 0.752 | 43.35  | 0.19        | Silver |
| SN-04D3lp | 0.983 | 44.13  | 0.52        | Silver |
Table 5—Continued

| Name      | $z$ | $H(z)$ | $\sigma(z)$ | Type   |
|-----------|-----|--------|-------------|--------|
| SN-04D3iu | 0.822 | 43.73  | 0.27        | Gold   |
| SN-04D3ml | 0.950 | 44.14  | 0.31        | Gold   |
| SN-04D3nc | 0.817 | 43.84  | 0.30        | Silver |
| SN-04D3nh | 0.340 | 41.51  | 0.17        | Gold   |
| SN-04D3nr | 0.960 | 43.81  | 0.28        | Silver |
| SN-04D3ny | 0.810 | 43.88  | 0.34        | Silver |
| SN-04D3oe | 0.756 | 43.64  | 0.17        | Gold   |
| SN-04D4an | 0.613 | 43.15  | 0.18        | Gold   |
| SN-04D4bk | 0.840 | 43.66  | 0.25        | Silver |
| SN-04D4bq | 0.550 | 42.67  | 0.17        | Gold   |
| SN-04D4dm | 0.811 | 44.13  | 0.31        | Gold   |
| SN-04D4dw | 0.961 | 44.18  | 0.33        | Gold   |