BIFURCATIONS OF NON SMOOTH VECTOR FIELDS ON \( \mathbb{R}^2 \) BY GEOMETRIC SINGULAR PERTURBATIONS

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ABSTRACT. Our object of study is non smooth vector fields on \( \mathbb{R}^2 \). We apply the techniques of geometric singular perturbations in non smooth vector fields after regularization and a blow−up. In this way we are able to bring out some results that bridge the space between non−smooth dynamical systems presenting typical singularities and singularly perturbed smooth systems.

1. Introduction

This work fits within the geometric study of Singular Perturbation Problems expressed by vector fields on \( \mathbb{R}^2 \). We study the phase portraits of certain non−smooth planar vector fields having a curve \( \Sigma \) as the discontinuity set. We present some results in the framework developed by Sotomayor and Teixeira in [10] (and extended in [9]) and establish a bridge between those systems and the fundamental role played by the Geometric Singular Perturbation (abbreviated by GSP) Theory. This transition was introduced in papers like [2] and [7], in dimensions 2 and 3 respectively. Results in this context can be found in [8]. We deal with non−smooth vector fields presenting structurally unstable configurations and we prove that these structurally unstable configurations are carried over the GSP Problem associated. Some good surveys about GSP Theory are [3] and [4], among others.

Let \( \mathcal{U} \subseteq \mathbb{R}^2 \) be an open set and \( \Sigma \subseteq \mathcal{U} \) given by \( \Sigma = f^{-1}(0) \), where \( f : \mathcal{U} \to \mathbb{R} \) is a smooth function having \( 0 \in \mathbb{R} \) as a regular value (i.e. \( \nabla f(p) \neq 0 \), for any \( p \in f^{-1}(0) \)). Clearly \( \Sigma \) is the separating boundary of the regions \( \Sigma_+ = \{ q \in \mathcal{U} | f(q) \geq 0 \} \) and \( \Sigma_- = \{ q \in \mathcal{U} | f(q) \leq 0 \} \). We can assume that \( \Sigma \) is represented, locally around a point \( q = (x,y) \), by the function \( f(x,y) = x \).

Designate by \( \mathcal{X}^r \) the space of \( C^r \)−vector fields on a compact set \( K \subset \mathcal{U} \) endowed with the \( C^r \)−topology with \( r \geq 1 \) large enough or \( r = \infty \). Call

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\(\tilde{\Omega}^r = \tilde{\Omega}^r(K, f)\) the space of vector fields \(Z : K \setminus \Sigma \to \mathbb{R}^2\) such that

\[
Z(x, y) = \begin{cases} 
X(x, y), & \text{for } (x, y) \in \Sigma_+,
Y(x, y), & \text{for } (x, y) \in \Sigma_-,
\end{cases}
\]

where \(X = (f_1, g_1), \ Y = (f_2, g_2)\) are in \(X^r\). The trajectories of \(Z\) are solutions of \(\dot{q} = Z(q)\), which has, in general, discontinuous right–hand side.

In what follows we will use the notation

\[
X.f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad Y.f(p) = \langle \nabla f(p), Y(p) \rangle.
\]

We distinguish the following regions on the discontinuity set \(\Sigma\):

\begin{itemize}
  \item \(\Sigma_1 \subseteq \Sigma\) is the sewing region if \((X.f)(Y.f) > 0\) on \(\Sigma_1\).
  \item \(\Sigma_2 \subseteq \Sigma\) is the escaping region if \((X.f) > 0\) and \((Y.f) < 0\) on \(\Sigma_2\).
  \item \(\Sigma_3 \subseteq \Sigma\) is the sliding region if \((X.f) < 0\) and \((Y.f) > 0\) on \(\Sigma_3\).
\end{itemize}

Consider \(Z \in \tilde{\Omega}^r\). The sliding vector field associated to \(Z\) is the vector field \(Z^s\) tangent to \(\Sigma_3\) and defined at \(q \in \Sigma_3\) by \(Z^s(q) = m - q\) with \(m\) being the point of the segment joining \(q + X(q)\) and \(q + Y(q)\) such that \(m - q\) is tangent to \(\Sigma_3\) (see Figure 1). It is clear that if \(q \in \Sigma_3\) then \(q \in \Sigma_2\) for \(-Z\) and then we can define the escaping vector field on \(\Sigma_2\) associated to \(Z\) by \(Z^e = -(-Z)^s\). The sewing vector field associated to \(Z\) is the vector field \(Z^w\) defined in \(q \in \Sigma_1\) as an arbitrary convex combination of \(X(q)\) and \(Y(q)\), i.e., \(Z^w(q) = \lambda X(q) + (1 - \lambda)Y(q)\) where \(\lambda \in [0, 1]\). In what follows we use the notation \(Z^\Sigma\) for all these cases.

Let \(\Omega^r = \Omega^r(K, f)\) be the space of vector fields \(Z : K \to \mathbb{R}^2\) such that \(Z \in \tilde{\Omega}^r\) and \(Z(q) = Z^\Sigma(q)\) for all \(q \in \Sigma\). We write \(Z = (X, Y)\), which we will accept to be multivalued in points of \(\Sigma\). The basic results of differential equations, in this context, were stated by Filippov in [5]. Related theories can be found in [6, 10, 12].

![Figure 1. Filippov’s convention.](image)

An approximation of the non–smooth vector field \(Z = (X, Y)\) by a 1–parameter family \(Z_\epsilon\) of smooth vector fields is called an \(\epsilon\)–regularization of \(Z\). We give the details about this process in Section 3. A transition function is used to average \(X\) and \(Y\) in order to get a family of smooth vector
fields that approximates $Z$. The main goal of this process is to deduce certain dynamical properties of the non-smooth dynamical system (abbreviated by NSDS) from the regularized system. The regularization process developed by Sotomayor and Teixeira produces a singular problem for which the discontinuous set is a center manifold. Via a blow up we establish a bridge between NSDS and the geometric singular perturbation theory.

Roughly speaking, the main results of this paper are the following:

**Theorem 1.** Consider $Z(x,y) = Z_\lambda(x,y) = (X(x,y),Y_\lambda(x,y)) \in \Omega^\epsilon$, where $\lambda \in (-1,1) \subset \mathbb{R}$, a non-smooth planar vector field and $\Sigma$ identified with the $y$-axis. Let the trajectories of $X$ be transverse to $\Sigma$ and $Y_0$ presenting either a hyperbolic saddle $q \in \Sigma$ or a hyperbolic focus $q \in \Sigma$ or a $\Sigma$-cusp point $q$. Then there exists a singular perturbation problem

\begin{align*}
\rho' &= \alpha(r,\rho,y,\lambda), \\
y' &= r\beta(r,\rho,y,\lambda),
\end{align*}

with $r \geq 0$, $\rho \in (0,\pi)$, $y \in \Sigma$ and $\alpha$ and $\beta$ of class $C^\epsilon$ such that the unfolding of $\Sigma$ produces the same topological behaviors as the unfolding of the corresponding normal forms of $Z_\lambda$ presented in subsection 5.1.

**Theorem 2.** Consider $Z(x,y) = Z_\mu(x,y) = (X_\mu(x,y),Y_\mu(x,y)) \in \Omega^\epsilon$, where either $\mu = \lambda \in \mathbb{R}$ or $\mu = (\lambda,\epsilon) \in \mathbb{R}_2$, a non-smooth planar vector field and $\Sigma$ identified with the $y$-axis. Consider that $q = (x_q,y_q) \in \Sigma$ is a $\Sigma$-fold point of both $X_\mu$ and $Y_\mu$ when $\mu = 0$ or $\mu = (0,0)$. Then there exists a singular perturbation problem

\begin{align*}
\rho' &= \alpha(r,\rho,y,\lambda), \\
y' &= r\beta(r,\rho,y,\lambda),
\end{align*}

with $r \geq 0$, $\rho \in (0,\pi)$, $y \in \Sigma$ and $\alpha$ and $\beta$ of class $C^\epsilon$ such that the following statements holds:

(a): For all small neighborhood $U$ of $q$ in $\Sigma$ the region $(\Sigma_2 \cup \Sigma_3) \cap (U - \{y_q\})$ is homeomorphic to the slow manifold $\alpha(0,\rho,y,\lambda) = 0$ of (2) where $y \in (U - \{y_q\})$.

(b): The vector field $Z_{\Sigma'}$, on $(\Sigma_2 \cup \Sigma_3) \cap (U - \{y_q\})$, and the reduced problem of (2), with $y \in (U - \{y_q\})$, are topologically equivalent.

(c): The slow manifold $\alpha(0,\rho,y,0) = 0$ of (2), where $y = y_q$, has just an horizontal component, i.e., $\alpha(0,\rho,y_q,0) = 0$ can be identified with $\{(\rho,y) | \rho \in (0,\pi), y = y_q\}$. Moreover, this configuration is structurally unstable.

The unfolding of (2) produces the same topological behaviors as the unfolding of the corresponding normal forms of $Z_\lambda$ presented in Table 1 and in (12).

Observe that Theorem 2 generalize the Theorem 1.1 of [8], because here we allow that $X.f(q) = Y.f(q) = 0$.

The paper is organized as follows: in Section 2 we give the basic theory about Non-Smooth Vector Fields on the Plane, in Section 3 we give the theory about the regularization process, in Section 4 we present the GSP Theory, in Section 5 we present the singularities treated in Theorem 1 and
give its normal forms, in Section 6 we present the singularities treated in Theorem 2 and give its normal forms and in Section 7 we prove Theorems 1 and 2.

2. Preliminaries

We say that \( q \in \Sigma \) is a \( \Sigma \)-regular point if

1. \( X.f(q)Y.f(q) > 0 \) or
2. \( X.f(q)Y.f(q) < 0 \) and \( Z_{\Sigma}(q) \neq 0 \) (that is \( q \in \Sigma_2 \cup \Sigma_3 \) and it is not a singular point of \( Z_{\Sigma} \)).

The points of \( \Sigma \) which are not \( \Sigma \)-regular are called \( \Sigma \)-singular. We distinguish two subsets in the set of \( \Sigma \)-singular points: \( \Sigma^t \) and \( \Sigma^p \). Any \( q \in \Sigma^p \) is called a pseudo equilibrium of \( Z \) and it is characterized by \( Z_{\Sigma}(q) = 0 \). Any \( q \in \Sigma^t \) is called a tangential singularity and is characterized by \( Z_{\Sigma}(q) \neq 0 \) and \( X.f(q)Y.f(q) = 0 \) (\( q \) is a contact point of \( Z_{\Sigma} \)).

A tangential singularity \( q \in \Sigma^t \) is a \( \Sigma \)-fold point of \( X \) if \( X.f(q) = 0 \) but \( X^2.f(q) = X.(X.f)(q) \neq 0 \). Moreover, \( q \in \Sigma \) is a visible (resp. invisible) \( \Sigma \)-fold point of \( X \) if \( X.f(q) = 0 \) and \( X^2.f(q) > 0 \) (resp. \( X^2.f(q) < 0 \)).

We say that \( q \in \Sigma^t \) is a \( \Sigma \)-cusp point of \( X \) if \( X.f(q) = 0 \), \( X^2.f(q) = 0 \) but \( X^3.f(q) \neq 0 \). Moreover, \( q \in \Sigma \) is a natural (resp. inverse) \( \Sigma \)-cusp point of \( X \) if \( X.f(q) = 0 \), \( X^2.f(q) = 0 \) and \( X^3.f(q) > 0 \) (resp. \( X^3.f(q) < 0 \)).

A pseudo equilibrium \( q \in \Sigma^p \) is a \( \Sigma \)-saddle provided one of the following condition is satisfied: (i) \( q \in \Sigma_2 \) and \( q \) is an attractor for \( Z^\Sigma \) or (ii) \( q \in \Sigma_3 \) and \( q \) is a repeller for \( Z^\Sigma \). A pseudo equilibrium \( q \in \Sigma^p \) is a \( \Sigma \)-repeller (resp. \( \Sigma \)-attractor) provided \( q \in \Sigma_2 \) (resp. \( q \in \Sigma_3 \)) and \( q \) is a repeller (resp. attractor) equilibrium point for \( Z^\Sigma \).

3. Regularization

In this section we present the concept of \( \epsilon \)-regularization of non-smooth vector fields. It was introduced by Sotomayor and Teixeira in [10]. The regularization gives the mathematical tool to study the stability of these systems, according with the program introduced by Peixoto. The method consists in the analysis of the regularized vector field which is a smooth approximation of the non-smooth vector field. Using this process we get a 1-parameter family of vector fields \( Z_\epsilon \in \mathbb{R}^r \) such that for each \( \epsilon_0 > 0 \) fixed it satisfies that:

1. \( Z_{\epsilon_0} \) is equal to \( X \) in all points of \( \Sigma_+ \) whose distance to \( \Sigma \) is bigger than \( \epsilon_0 \);
2. \( Z_{\epsilon_0} \) is equal to \( Y \) in all points of \( \Sigma_- \) whose distance to \( \Sigma \) is bigger than \( \epsilon_0 \).

**Definition 1.** A \( C^\infty \) function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) is a transition function if \( \varphi(x) = -1 \) for \( x \leq -1 \), \( \varphi(x) = 1 \) for \( x \geq 1 \) and \( \varphi'(x) > 0 \) if \( x \in (-1,1) \). The
\( \epsilon \)-regularization of \( Z = (X, Y) \) is the 1-parameter family \( Z_\epsilon \in \mathcal{X}^r \) given by

\[
Z_\epsilon(q) = \left( \frac{1}{2} + \frac{\varphi_\epsilon(f(q))}{2} \right) X(q) + \left( \frac{1}{2} - \frac{\varphi_\epsilon(f(q))}{2} \right) Y(q),
\]

with \( \varphi_\epsilon(x) = \varphi(x/\epsilon) \), for \( \epsilon > 0 \).

4. Singular Perturbations

**Definition 2.** Let \( U \subseteq \mathbb{R}^2 \) be an open subset and take \( \epsilon \geq 0 \). A singular perturbation problem in \( U \) (SP–Problem) is a differential system which can be written like

\[
\begin{align*}
x' &= dx/d\tau = l(x, y, \epsilon), \\
y' &= dy/d\tau = \epsilon m(x, y, \epsilon)
\end{align*}
\]

or equivalently, after the time re-scaling \( t = \epsilon \tau \)

\[
\begin{align*}
\dot{x} &= edx/dt = l(x, y, \epsilon), \\
\dot{y} &= dy/dt = m(x, y, \epsilon),
\end{align*}
\]

with \((x, y) \in U\) and \( l, m \) smooth in all variables.

The understanding of the phase portrait of the vector field associated to a SP–problem is the main goal of the geometric singular perturbation theory (GSP–theory). The techniques of GSP–theory can be used to obtain information on the dynamics of \( [3] \) for small values of \( \epsilon > 0 \), mainly in searching limit cycles. System \( [3] \) is called the fast system, and \( [4] \) the slow system of SP–problem. Observe that for \( \epsilon > 0 \) the phase portraits of the fast and the slow systems coincide. For \( \epsilon = 0 \), let \( S \) be the set

\[
S = \{(x, y) : f(x, y, 0) = 0\}
\]

of all singular points of \( [3] \). We call \( S \) the slow manifold of the singular perturbation problem and it is important to notice that equation \( [4] \) defines a dynamical system, on \( S \), called the reduced problem:

\[
f(x, y, 0) = 0, \quad \dot{y} = g(x, y, 0).
\]

Combining results on the dynamics of these two limiting problems, with \( \epsilon = 0 \), one obtains information on the dynamics of \( X_\epsilon \) for small values of \( \epsilon \). We refer to \( [4] \) for an introduction to the general theory of singular perturbations. Related problems can be seen in \( [1], [3] \) and \( [11] \).

5. Boundary Bifurcations

Consider \( Z = (X, Y) \). In this section we assume that the trajectories of the smooth vector field \( X \) is transversal to \( \Sigma \) and that \( Y \) has either a hyperbolic saddle or a hyperbolic focus or \( \Sigma \)–cusp point in \( \Sigma \). This configuration is clearly structurally unstable. We present here its normal forms and unfoldings.
5.1. **Codimension One Normal Forms.** Take $\Sigma$ as the $y$–axis, i.e., $f(x, y) = x$ and consider the parameter $\lambda \in (-1, 1)$.

- **Regular–saddle:** Assume that $X$ is transversal to $\Sigma$ and that $Y$ has a hyperbolic saddle in $\Sigma$. The following normal form generically unfolds this configuration.

$$Z(x, y) = Z_\lambda(x, y) = \begin{cases} 
  X(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{for } (x, y) \in \Sigma_+, \\
  Y_\lambda(x, y) = \begin{pmatrix} -y \\ -x - \lambda \end{pmatrix}, & \text{for } (x, y) \in \Sigma_-.
\end{cases}$$

- **Regular–focus:** Assume that $X$ is transversal to $\Sigma$ and that $Y$ has a hyperbolic focus in $\Sigma$. The following normal form generically unfolds this configuration.

$$Z(x, y) = Z_\lambda(x, y) = \begin{cases} 
  X(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{for } (x, y) \in \Sigma_+, \\
  Y_\lambda(x, y) = \begin{pmatrix} x + y + \lambda \\ -x + y - \lambda \end{pmatrix}, & \text{for } (x, y) \in \Sigma_-.
\end{cases}$$

- **Regular–cusp:** Assume that $X$ is transversal to $\Sigma$ and that $Y$ has a $\Sigma$–cusp point. The following normal form generically unfolds this configuration.

$$Z(x, y) = Z_\lambda(x, y) = \begin{cases} 
  X(x, y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{for } (x, y) \in \Sigma_+, \\
  Y_\lambda(x, y) = \begin{pmatrix} -y^2 + \lambda \\ 1 \end{pmatrix}, & \text{for } (x, y) \in \Sigma_-.
\end{cases}$$

5.2. **Regular–saddle Bifurcation.** Consider the regular–saddle normal form given in the previous subsection. The regularized vector field becomes

$$\begin{align*}
\dot{x} &= \frac{1 - y}{2} + \varphi\left(\frac{x}{\epsilon}\right) \frac{1 + y}{2}, \\
\dot{y} &= \frac{1 - x - \lambda}{2} + \varphi\left(\frac{x}{\epsilon}\right) \frac{1 + x + \lambda}{2}.
\end{align*}$$

where $\varphi(x/\epsilon)$ is the transition function. Making the polar blow up

$$x = r \cos \theta \quad \text{and} \quad \epsilon = r \sin \theta,$$

where $\theta = (2\rho + \pi)/4$, we obtain

$$\begin{align*}
r \dot{\theta} &= -\sin \theta \left(\frac{1 - y}{2} + \varphi(\cot \theta) \frac{1 + y}{2}\right), \\
\dot{y} &= \frac{1 - r \cos \theta - \lambda}{2} + \varphi(\cot \theta) \frac{1 + r \cos \theta + \lambda}{2}.
\end{align*}$$
Remark 1. We use the new variable \( \theta = (2\rho + \pi)/4 \), with \( \rho \in (0, \pi) \), because in the coordinates \((r,\theta)\) the transition function \( \varphi \) is constant when \( \theta \in (0, \pi/4) \cup (3\pi/4, \pi) \). So, in the next Figures in the text, we can draw the slow manifold and slow dynamics with \( \rho \in (0, \pi) \).

In the blowing up locus \( r = 0 \) the fast dynamics is determined by the system
\[
\theta' = -\sin \theta \left( \frac{1-y}{2} + \varphi(\cot \theta)\frac{1+y}{2} \right), \quad y' = 0;
\]
and the slow dynamics on the slow manifold is determined by the reduced system
\[
\frac{-1+y}{2} + \varphi(\cot \theta)\frac{-1-y}{2} = 0, \quad \dot{y} = \frac{1-\lambda}{2} + \varphi(\cot \theta)\frac{1+\lambda}{2}.
\]

We remark that the slow manifold is implicitly defined by \((-1+y)/2 + \varphi(\cot \theta)(-1-y)/2 = 0\) and do not depends on the parameter \( \lambda \) (see Figure 2). Moreover, \( y(\theta) \) defined in this way is such that
\[
\lim_{\theta \to \pi/4} y(\theta) = +\infty \quad \text{and} \quad \lim_{\theta \to 3\pi/4} y(\theta) = 0.
\]

![Figure 2](image-url)

**Figure 2.** In this figure is pictured the slow manifold to the left and the case \( \lambda < 0 \) at the right. In both we consider \( \rho \in (0, \pi) \).

By other hand, the dynamics on the slow manifold depends on \( \lambda \). In fact, if either \( \lambda > 0 \) or if \( \lambda = 0 \) then \( \dot{y} \neq 0 \) (see Figure 3) and if \( \lambda > 0 \) so \( \dot{y} \) has an unique repeller critical point \( P \) (see Figure 2) given implicitly by the equation \( \varphi(\cot \theta) = (-1+\lambda)/(1+\lambda) \).

In Figure 2 and in the next ones, double arrow over a curve means that it is a trajectory of the fast dynamical system, and simple arrow means that it is a trajectory of the one dimensional slow dynamical system.

5.3. Regular−focus Bifurcation. Consider the regular−focus normal form given in Subsection 5.1. The regularized vector field becomes
Figure 3. In this figure is pictured the case $\lambda = 0$ to the left and the case $\lambda > 0$ at the right. In both we consider $\rho \in (0, \pi)$.

\[
\dot{x} = \frac{1 + \lambda + x + y}{2} + \varphi \left( \frac{x}{\rho} \right) \frac{1 - \lambda - x - y}{2},
\]
\[
\dot{y} = \frac{1 - \lambda - x + y}{2} + \varphi \left( \frac{x}{\rho} \right) \frac{1 + \lambda + x - y}{2}.
\]

Similarly to the previous case, considering the polar blow-up given in (5), where $\theta = (2\rho + \pi)/4$, we get

\[
\begin{align*}
\dot{r} & = -\sin \theta \left( \frac{1 + \lambda + y + r \cos \theta}{2} + \varphi(\cot \theta) \frac{1 - \lambda - y - r \cos \theta}{2} \right), \\
\dot{\theta} & = \frac{1 - \lambda + y - r \cos \theta}{2} + \varphi(\cot \theta) \frac{1 + \lambda - y + r \cos \theta}{2}.
\end{align*}
\]

Putting $r = 0$, the fast dynamics is determined by the system

\[
\begin{align*}
\theta' & = \sin \theta \left( \frac{-1 - \lambda - y}{2} + \varphi(\cot \theta) \frac{-1 + \lambda + y}{2} \right), \\
y' & = 0;
\end{align*}
\]

and the slow dynamics on the slow manifold is determined by the reduced system

\[
\begin{align*}
-\frac{1 - \lambda - y}{2} + \varphi(\cot \theta) \frac{-1 + \lambda + y}{2} & = 0, \\
\dot{y} & = \frac{1 - \lambda + y}{2} + \varphi(\cot \theta) \frac{1 + \lambda - y}{2}.
\end{align*}
\]

In this case, the slow manifold depends of the parameter $\lambda$. In fact, it is given implicitly by $(-1 - \lambda - y)/2 + \varphi(\cot \theta)(-1 + \lambda + y)/2 = 0$ (see Figure 4). The slow manifold $y(\theta)$, given in the previous equation, satisfies

\[
\lim_{\theta \to \frac{\pi}{4}} y(\theta) = -\infty \quad \text{and} \quad \lim_{\theta \to \frac{3\pi}{4}} y(\theta) = -\lambda.
\]

We give now the dynamics on the slow manifold. If $\lambda < 0$ so $\dot{y} \neq 0$ (see Figure 4), if $\lambda > 0$ so $\dot{y}$ has an unique critical point $P$, given implicitly as the solution of $\varphi(\cot \theta) = (-1 + \lambda + y)/(1 + \lambda - y)$, which is an attractor (see Figure 5) and if $\lambda = 0$ so $\dot{y} \neq 0$ (see Figure 5).
5.4. Regular–Cusp Bifurcation. Consider the regular–cusp normal form given in Subsection 5.1. The regularized vector field becomes

\[
\begin{align*}
\dot{x} &= \frac{1 + \lambda - y^2}{2} + \varphi\left(\frac{x}{\epsilon}\right) \frac{1 - \lambda + y^2}{2}, \\
\dot{y} &= \varphi\left(\frac{x}{\epsilon}\right).
\end{align*}
\]

Making the polar blow–up given in (5), where \( \theta = (2\rho + \pi)/4 \), we get

\[
\begin{align*}
\dot{r} &= \sin \theta \left( \varphi(\cot \theta)(-1 + \lambda - y^2) - 1 - \lambda + y^2 \right), \\
\dot{\theta} &= \varphi(\cot \theta).
\end{align*}
\]

Putting \( r = 0 \) the fast dynamics is determined by the system

\[
\theta' = \frac{\sin \theta}{2} (\varphi(\cot \theta)((-1 + \lambda - y^2) - 1 - \lambda + y^2)), \quad y' = 0;
\]
and the slow dynamics on the slow manifold is determined by the reduced system

$$\varphi(\cot \theta)(-1 + \lambda - y^2) - 1 - \lambda + y^2 = 0, \quad \dot{y} = \varphi(\cot \theta).$$

Observe that the slow manifold depends on the parameter $\lambda$. We can obtain the explicit form. In fact, the slow manifold is composed by two branches (see Figure 6):

$$y^\lambda(\theta) = \pm \sqrt{\frac{\lambda(1 - \varphi(\cot \theta)) + 1 + \varphi(\cot \theta)}{1 - \varphi(\cot \theta)}}.$$  

The slow manifold satisfies the properties:

(i) $\lim_{\theta \to \pi/4} y^\lambda(\theta) = \pm \infty$;

(ii) If $\lambda < 0$ there exists $\theta^* \in (\pi/4, 3\pi/4)$ (respectively, $\rho^* \in (0, \pi)$) such that $y^\lambda(\theta^*) = 0$ and the slow manifold is not defined for $\theta \in (\rho^*, \pi)$. For $\theta \in (\pi/4, \theta^*)$ there exist homeomorphisms $\xi_\pm$ between each branch of the slow manifold and $\mathbb{R}^*$. That is, for each $z \in \mathbb{R}^*$ there exists $\theta(z) \in (\pi/4, \theta^*)$ such that $y^\lambda_\pm(\theta(z)) = z$;

(iii) If $\lambda \geq 0$ there exist homeomorphisms $\xi_\pm$ between each branch of the slow manifold and $\mathbb{R}^*$. That is, for each $z \in \mathbb{R}^*$ there exists $\theta(z) \in (\pi/4, 3\pi/4)$ such that $y^\lambda_\pm(\theta(z)) = z$.

In fact, the item (i) is a straightforward calculus. In order to prove the item (ii) observe Expression (6). Let $\theta^*$ be such that $\varphi(\cot \theta^*) = (1 + \lambda)(\lambda - 1)$. We have,

$$y^\lambda_\pm(\theta^*) = 0$$

and the radical in (6) is negative for $\theta \in (\theta^*, \pi)$.

We define the maps:

$$\xi_\pm : \mathbb{R}^* \to (\pi/4, \theta^*)$$

$$z \mapsto \theta(z) = \cot^{-1}\left(\frac{\varphi^{-1}\left(z^2 \pm \frac{\lambda - 1}{1 - \lambda + z^2}\right)}{1 - \lambda + z^2}\right).$$

Given $z \in \mathbb{R}^*$ if we put $\xi(z) = \theta(z)$ in (7) we get $y^\lambda_\pm(\theta(z)) = z$. Note that $\xi_\pm$ are homeomorphisms.

The proof of the item (iii) is analogous.

The dynamics in the slow manifold is given by $\dot{y} = \varphi(\cot \theta)$. So there exists a unique critical point $p$ given implicitly as the solution of $\varphi(\cot \theta_p) = 0$. Note that this critical point is a repeller because $\varphi(\cot \theta) < 0$ for $\theta < \theta_p$ and $\varphi(\cot \theta) > 0$ for $\theta > \theta_p$. See Figure [6]
In this section we analyze the dynamics of a NSDS around a point $q$ which is $\Sigma$–fold of both $X$ and $Y$. We say that $q$ is a Fold–Fold singularity of $Z \in \Omega^r$. We divide the fold–fold singularities in types according with the sign of $X^2.f(q)$ and $Y^2.f(q)$:

(a) **Elliptic case**: $X^2.f(q) > 0$ and $Y^2.f(q) < 0$. See Figure 7 (a).

(b) **Hyperbolic case**: $X^2.f(q) < 0, Y^2.f(q) > 0$. See Figure 7 (b).

(c.1) **Parabolic visible case**: $X^2.f(q) > 0, Y^2.f(q) > 0$. See Figure 7 (c.1).

(c.2) **Parabolic invisible case**: $X^2.f(q) < 0, Y^2.f(q) < 0$. See Figure 7 (c.2).

Note that, we can define a first return map $\psi_Z$ only in the elliptic case. Take $\Sigma$ as the $y$–axis, i.e., $f(x,y) = x$ and consider the parameter $\lambda \in (-1,1)$. The generic normal forms of the hyperbolic and parabolic fold–fold singularities are given in Table 1. The normal form of the elliptic fold–fold singularity is given in Subsection 6.4.

In the next three subsections we study the dynamics of the hyperbolic and parabolic fold–fold singularities via geometric singular perturbations.
6.1. **Hyperbolic Case.** Consider the normal form of the hyperbolic fold—fold singularity given in Table [1]. The regularized vector field is given by

\[
\begin{align*}
\dot{x} &= -\frac{\lambda}{2} + \varphi \left( \frac{x}{\epsilon} \right) \left( -\frac{\lambda + 2y}{2} \right), \\
\dot{y} &= -1.
\end{align*}
\]

By the polar blow up we get

\[
\begin{align*}
\dot{r} \theta &= \sin \theta \left( \frac{\lambda}{2} + \varphi(\cot \theta) \frac{\lambda - 2y}{2} \right), \\
\dot{y} &= -1
\end{align*}
\]

where \( \theta = (2\rho + \pi)/4 \).

Putting \( r = 0 \) the fast dynamics is determined by the system

\[
\theta' = \sin \left( \frac{\lambda}{2} + \varphi(\cot \theta) \left( \frac{\lambda - 2y}{2} \right) \right), \quad y' = 0;
\]

and the slow dynamics on the slow manifold is determined by the reduced system

\[
\frac{\lambda}{2} + \varphi(\cot \theta) \left( \frac{\lambda - 2y}{2} \right) = 0, \quad \dot{y} = -1.
\]

In this case we obtain the explicit expression for the slow manifold:

\[
y(\theta) = \frac{\lambda(1 + \varphi(\cot \theta))}{2\varphi(\cot \theta)}.
\]

Observe that, the slow manifold \( y(\theta) \) is not defined for \( \theta_0 \) such that \( \varphi(\cot \theta_0) = 0 \). So, for \( \lambda \neq 0 \), \( y(\theta) \) have two branches and satisfies:
(a) \( \lim_{\theta \to \theta_0^-} y(\theta) = -\infty \) for \( \lambda < 0 \) and \( \lim_{\theta \to \theta_0^+} y(\theta) = +\infty \) for \( \lambda > 0 \);

(b) \( \lim_{\theta \to \theta_0^-} y(\theta) = +\infty \) for \( \lambda < 0 \) and \( \lim_{\theta \to \theta_0^+} y(\theta) = -\infty \) for \( \lambda > 0 \);

(c) For \( \lambda = 0 \) the slow manifold is given implicitly by \( y\varphi(\cot \theta) = 0 \), that is, \( \{ (\theta, y) \mid \theta = \theta_0 \} \cup \{ (\theta, y) \mid y = 0 \} \) is the slow manifold.

![Diagram of slow manifolds](image)

**Figure 8.** Slow manifold depending of the parameter \( \lambda \).

The dynamics on the slow manifold is given by \( \dot{y} = -1 \). Therefore, do not exist critical points. See Figure 8.

6.2. **Parabolic visible case.** Consider the normal form of the parabolic visible fold—fold singularity given in Table I. The regularized vector field is

\[
\begin{align*}
\dot{x} &= -\frac{\lambda}{2} + \varphi \left( \frac{x}{\epsilon} \right) \left( -\frac{\lambda + 2y}{2} \right), \\
\dot{y} &= \varphi \left( \frac{x}{\epsilon} \right).
\end{align*}
\]

By the polar blow up we get
\[ r \dot{\theta} = \sin \theta \left( \frac{\lambda}{2} + \varphi(\cot \theta) \left( \frac{\lambda - 2y}{2} \right) \right), \]

\[ \dot{y} = \varphi(\cot \theta). \]

where \( \theta = (2\rho + \pi)/4. \)

Putting \( r = 0 \) the fast dynamics is determined by the system

\[ \theta' = \sin \theta \left( \frac{\lambda}{2} + \varphi(\cot \theta) \left( \frac{\lambda - 2y}{2} \right) \right), \quad y' = 0; \]

and the slow dynamics on the slow manifold is determined by the reduced system

\[ \frac{\lambda}{2} + \varphi(\cot \theta) \left( \frac{\lambda - 2y}{2} \right) = 0, \quad \dot{y} = \varphi(\cot \theta). \]

The analysis is similar to the hyperbolic case. In the present case the dynamics on the slow manifold is given by \( \dot{y} = \varphi(\cot \theta) \). So, for \( \lambda = 0 \), the straight line \( \theta = \theta_0 \) is composed by critical points. See Figure 9.

\[ \text{Figure 9. Bifurcation Diagram of the Parabolic Visible Fold–Fold Singularity.} \]
6.3. Parabolic invisible case. For this case, we get one different topological type of bifurcation. Consider the normal form of the parabolic invisible fold—fold singularity given in Table I. The regularized vector field is

\[
\begin{align*}
\dot{x} &= -\frac{\lambda}{2} \varphi \left( \frac{x}{\epsilon} \right) + \frac{-\lambda + 2y}{2}, \\
\dot{y} &= -1.
\end{align*}
\]

By the polar blow up we get

\[
\begin{align*}
r\dot{\theta} &= \frac{\sin \theta}{2} \left( \lambda \varphi (\cot \theta) + \lambda - 2y \right), \\
\dot{y} &= -1.
\end{align*}
\]

where \( \theta = (2\rho + \pi)/4. \)

Putting \( r = 0 \) the fast dynamics is determined by the system

\[
\theta' = \frac{\sin \theta}{2} \left( \lambda \varphi (\cot \theta) + \lambda - 2y \right), \quad y' = 0;
\]

and the slow dynamics on the slow manifold is determined by the reduced system

\[
\sin \theta \left( \frac{\lambda}{2} \varphi (\cot \theta) + \left( \frac{\lambda - 2y}{2} \right) \right) = 0, \quad \dot{y} = -1.
\]

We have the explicit expression for the slow manifold in this case:

\[
y(\theta) = \frac{\lambda}{2} (1 + \varphi (\cot \theta)).
\]

The analysis is similar to the previous cases and the bifurcation diagram is expressed in Figure 10.

6.4. Elliptic case. In this case, associated with the non—smooth vector fields, there exist the first return map \( \psi_Z(p) \). Therefore, we need to analyze the structural stability of this one dimensional diffeomorphism.

Consider \( Z \) presenting an elliptic fold—fold singularity, \( f(x, y) = x \) and

\[(8) \quad Z_\lambda(x, y) = \begin{cases} 
X_\lambda(x, y) = (y - \lambda, 1), & \text{for } (x, y) \in \Sigma_+, \\
Y_\lambda(x, y) = (y, -1), & \text{for } (x, y) \in \Sigma_-.
\end{cases}\]

The regularized vector field is

\[
\begin{align*}
\dot{x} &= -\varphi \left( \frac{x}{\epsilon} \right) \left( \frac{\lambda}{2} + \frac{2y - \lambda}{2} \right), \\
\dot{y} &= \varphi \left( \frac{x}{\epsilon} \right).
\end{align*}
\]

By the polar blow up we get
\[
\frac{r}{\dot{\theta}} = \sin \theta \left( \frac{\lambda \varphi(\cot \theta)}{2} + \left( \frac{\lambda - 2y}{2} \right) \right),
\]

\[
\dot{y} = \varphi(\cot \theta)
\]

where \( \theta = (2\rho + \pi)/4 \).

Putting \( r = 0 \) the fast dynamics is determined by the system

\[
\theta' = \sin \theta \left( \frac{\lambda \varphi(\cot \theta)}{2} + \left( \frac{\lambda - 2y}{2} \right) \right), \quad y' = 0;
\]

and the slow dynamics on the slow manifold is determined by the reduced system

\[
\frac{\lambda \varphi(\cot \theta)}{2} + \left( \frac{\lambda - 2y}{2} \right) = 0, \quad \dot{y} = \varphi(\cot \theta).
\]

In this case, for \( \lambda = 0 \), we only have sewing region on the non-smooth manifold \( \Sigma \). The explicit expression for the slow manifold is

\[
y(\theta) = \frac{\lambda (1 + \varphi(\cot \theta))}{2}
\]
and there exist one critical point which is attractor if $\lambda < 0$ and repeller if $\lambda > 0$. See Figure 11.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{bifurcation_diagram.png}
\caption{Bifurcation Diagram of the 1-parameter Elliptic Fold–Fold Singularity.}
\end{figure}

Differently of the hyperbolic and parabolic cases, the unfolding (8) does not give the generic unfolding of a non–smooth vector field presenting a elliptic fold–fold singularity.

Let $Z_\lambda = (X_\lambda, Y)$ be as in (8). The expression of its first return map is

\begin{equation}
\psi_{Z_\lambda}(y) = \gamma_Y \circ \gamma_{X_\lambda}(y) = y - 2\lambda,
\end{equation}

where $\gamma_{X_\lambda}(p)$ (respectively $\gamma_Y(p)$) is the first return to $\Sigma$ of the trajectory of $X_\lambda$ (respectively $Y$) that passes through $p$.

Therefore, we conclude that the critical point of $\psi_{Z_\lambda}$ is not hyperbolic. In order to obtain the generic unfolding of this case we need to unfold the first return map. So, the unfolding of the elliptic fold–fold singularity depends on two parameters. The first, $\lambda$, is responsible by the displacement of one fold along the $y$–axis and another one, $\varepsilon$, for the unfolding of $\psi_{Z_\lambda}$.

Consider the $\varepsilon$–perturbation of the smooth vector field $Y$, given by:
\[
Y_\epsilon(x, y) = g_1^\epsilon(t(x, y)) \frac{\partial}{\partial x} + g_2^\epsilon(t(x, y)) \frac{\partial}{\partial y}
\]
\[
= g_1^\epsilon(t) \frac{\partial}{\partial x} + g_2^\epsilon(t) \frac{\partial}{\partial y}.
\]

So, the flows of \(X_\lambda\) and \(Y_\epsilon\) are:

\[
\phi_t^{X_\lambda}(x_0, y_0) = (x_0 + (y_0 - \lambda)t + t^2/2, y_0 + t),
\]

\[
\phi_t^{Y_\epsilon}(x_0, y_0) = \left( x_0 + \int_0^t g_1^\epsilon(s) \, ds, y_0 + \int_0^t g_2^\epsilon(s) \, ds \right).
\]

Let \(t^* \in \mathbb{R}^*\) and \(t_1 = 2(\lambda - y_0)\) such that

\[
\int_0^{t^*} g_1^\epsilon(s) \, ds = 0
\]

and \(\phi_{t_1}^{X_\lambda}(0, y_0) = (0, -y_0 + 2\lambda) \in \Sigma\).

Observe that there exist \(t^*\) as in Equation (11), because 0 is an elliptic fold singularity for \(Y\).

We suppose that \(g_i^\epsilon(\cdot), i = 1, 2\) satisfies:

(a) \(g_i\) are \(C^r\) functions for \(i = 1, 2\);

(b) \(Y_\epsilon f(0, 0) = g_1^\epsilon(0) = 0\);

(c) \(Y_\epsilon^2 f(0, 0) = g_2^\epsilon(0) \frac{d}{dx} g_1^\epsilon(0) + g_2^\epsilon(0) \frac{d}{dy} g_1^\epsilon(0) \neq 0\);

(d) \(\int_0^{t^*} g_2^\epsilon(s) \, ds = (\epsilon - 2)y + O(y^2)\).

The smooth vector fields \(X_\lambda, Y_\epsilon\) exhibited in (8), (10), respectively, supply the unfolding of \(\psi_Z\):

\[
\psi_{Z_\lambda, \epsilon}(y) = \phi_{t^*}^{Y_\epsilon} \circ \phi_{t_1}^{X_\lambda}(0, y) = (1 - \epsilon)y - 2\lambda + 2\lambda \epsilon + O(y^2).
\]

Therefore, the generic unfolding for the non-smooth vector field \(Z\) with the origin is an elliptic fold-singularity is \(Z_{\lambda, \epsilon} = (X_\lambda, Y_\epsilon)\) where

\[
Z_{\lambda, \epsilon}(x, y) = \begin{cases} 
X_\lambda(x, y) = (y - \lambda, 1), & \text{if } (x, y) \in \Sigma_+,
Y_\epsilon(x, y) = (g_1^\epsilon(x, y), g_2^\epsilon(x, y)), & \text{if } (x, y) \in \Sigma_-
\end{cases}
\]

and the smooth function \(g_i^\epsilon, i = 1, 2\) satisfies the conditions (a), (b), (c) and (d) given previously.

7. Conclusion

We note that: if for any \(q \in \Sigma\) we have that \(Xf(q) \neq 0\) or \(Yf(q) \neq 0\) then, by Theorem 1.1 of [8], there exists a singular perturbation problem such that the sliding region is homeomorphic to the slow manifold and the sliding vector field is topologically equivalent to the reduced problem. This
7.1. **Proof of Theorem 1.** In face of Theorem 1.1 of [8], this theorem is the subject of section 5. Moreover, as we give the topological behavior of the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ it is easy to construct the bifurcation diagram of [4].

7.2. **Proof of Theorem 2.** In this theorem we extend Theorem 1.1 of [8] considering that can exists a point $q$ such that $X.f(q) = Y.f(q) = 0$, $X^2.f(q) \neq 0$ and $Y^2.f(q) \neq 0$. In this way, $q$ is a $\Sigma-$fold point of both $X$ and $Y$.

Consider a NSDS $Z_\lambda = (X,Y)$ where $\lambda \in \mathbb{R}$ is a parameter. If with the variation of $\lambda \in (-\varepsilon, \varepsilon)$, the following behaviors are observable then we consider that $Z_\lambda$ presents a bifurcation, where $\varepsilon > 0$ and small. Consider $\lambda^+ \in (0, \varepsilon)$ and $\lambda^- \in (-\varepsilon, 0)$. The behaviors are:

(i): A change of stability on $\Sigma$, i.e., where $Z_{\lambda^+}$ has a sliding region $\Sigma_3$, the non-smooth vector field $Z_{\lambda^-}$ has an escaping region $\Sigma_2$.

(ii): A change of stability on $\dot{y}_\lambda$, i.e., there are components of $\Sigma$ such that the induced flow on the slow manifold is such that $\dot{y}_{\lambda^+} > 0$ and $\dot{y}_{\lambda^-} < 0$.

(iii): A change of stability of the $\Sigma-$singularity, i.e., $Z_{\lambda^+}$ presents a $\Sigma-$attractor and $Z_{\lambda^-}$ presents a $\Sigma-$repeller.

(iv): A change of orientation on $\Sigma_1$ (the sewer region), i.e., $Z_{\lambda^+}$ and $Z_{\lambda^-}$ presents distinct orientations on $\Sigma_1$.

In face of these previous observations, Theorem 2 follows straightforward from section 6. Note that, as we give the topological behavior of the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ it is easy to construct the bifurcation diagram of [2] when $\lambda \in \mathbb{R}$.

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