Investigation of stress concentration at corner points for orthotropic plate bending problem

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Abstract. This article deals with the bending problem for an orthotropic semi-infinite plate strip when three edges of the plate are hinged and the fourth edge goes to infinity. The plate is loaded with distributed load of intensity \( q(y) \). A. Nadai’s approach is applied, which says that to obtain the solution at a far distance from the edge, it is necessary to solve the problem of cylindrical bending. The generalized shearing forces on the fixed edge are investigated.

1. Introduction
In the Cartesian coordinate system, we explore a semi-infinite plate strip of constant thickness \( 2h \) occupying the area \( 0 \leq x < \infty, 0 \leq y \leq b, -h \leq z \leq h \). A distributed load of intensity \( q(y) \) is acting on the plate [3].

We assume that the material of the plate, due to its elastic properties, has three planes of symmetry, i.e., the plate is orthotropic. If these planes are taken as coordinate axes, then the relationships between the components of stress and strain can be expressed by the equations

\[
\sigma_x = E'_x \varepsilon_x + E''_x \varepsilon_y, \quad \sigma_y = E'_y \varepsilon_y + E''_y \varepsilon_x, \quad \sigma_{xy} = G \varepsilon_{xy}. \tag{1}
\]

We assume that the perpendicular to the middle surface of the plate linear elements remains straight and normal to the curved surface of the plate after bending.

We use the following expressions of the strain components:

\[
\varepsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_y = \frac{\partial v}{\partial x} - z \frac{\partial^2 w}{\partial y^2}, \quad \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y}. \tag{2}
\]

When investigating the bending of plates, we consider only the perpendicular displacement [6]. Consequently, for the bending and twisting moments, we have [2]

\[
M_x = \int_{-h}^{h} z \sigma_x \, dz = -D_2 \frac{\partial^2 w}{\partial x^2} - D_1 \frac{\partial^2 w}{\partial y^2}, \quad M_y = \int_{-h}^{h} z \sigma_y \, dz = -D_2 \frac{\partial^2 w}{\partial y^2} - D_1 \frac{\partial^2 w}{\partial x^2},
\]

\[
H = -\int_{-h}^{h} z \sigma_{xy} \, dz = 2D_{xy} \frac{\partial^2 w}{\partial x \partial y}. \tag{3}
\]
where the coefficients, which characterizing the mechanical properties are:

\[ E'_x = \frac{E_x}{1 - \nu_{xy}^2}, \quad E'_y = \frac{E_y}{1 - \nu_{xy}^2}, \quad E'' = \frac{E_y}{1 - \nu_{xy}^2}, \]

\[ D_x = \frac{2E_h^3}{3}, \quad D_y = \frac{2E_h^3}{3}, \quad D_1 = \frac{2E''h^3}{3}, \quad D_{xy} = \frac{2Gh^3}{3}. \]

Substituting (3) into the equations of equilibrium [8], we obtain the equation of bending of orthotropic plates

\[ D_x \frac{\partial^4 w}{\partial x^4} + 2H_1 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q(y), \]

where

\[ H_1 = D_1 + 2D_{xy}. \]

In the specific case of isotropic plate, we have

\[ E'_x = E'_y = \frac{E}{1 - \vartheta^2}, \quad E'' = \frac{\partial E}{1 - \vartheta^2}, \quad G = \frac{E}{2(1 + \vartheta)}, \]

\[ D_x = D_y = H_1 = D = \frac{2Eh^3}{3(1 - \vartheta^2)} \]

and obtain the equation of bending of plates in the well-known form \( \Delta \Delta w = q/D \).

2. Satisfaction of the boundary conditions

It is assumed that the edges of the plate \( y = 0, b \) are hinged which allows us to obtain the solution of equation (6) in the form

\[ w = \sum_{n=1}^{\infty} f_n(x) \sin(\lambda_n y), \quad q = \sum_{n=1}^{\infty} q_n \sin(\lambda_n y), \quad \lambda_n = \frac{\pi n}{b}. \]

Substituting (7) into (5), we obtain the fourth-order inhomogeneous differential equation [7]

\[ D_x f_n^{IV} - 2H_1 \lambda_n^2 f_n^{II} + D_y \lambda_n^4 f_n = q_n. \]

We seek a homogeneous solution of the differential equation in the form

\[ f_n(x) = Ae^{r_1 \lambda_n x}. \]

The characteristic equation of (8) becomes

\[ r^4 - 2H_1 r^2 + D_y = 0, \]

\[ r^2 = \frac{H_1 \pm \sqrt{H_1^2 - D_x D_y}}{D_x}. \]

We calculate its four roots and obtain the general solution of the problem of bending of an orthotropic semi-infinite plate strip with two hinged edges in the form

\[ f_n(x) = A_n e^{-r_1 \lambda_n x} + B_n e^{-r_2 \lambda_n x} + C_n e^{r_1 \lambda_n x} + D_n e^{r_2 \lambda_n x} + \frac{q_n}{D \lambda_n^4}. \]
Applying the Nadai approach, which says that, at a far distance from the edge \( x = 0 \), the solution must tend to the solution of the problem of cylindrical bending,

\[
\lim_{x \to \infty} f_n(x) = \frac{q_n}{D\lambda_n^4},
\]

we obtain

\[
f_n(x) = A_n e^{-r_1 \lambda_n x} + B_n e^{-r_2 \lambda_n x} + \frac{q_n}{D\lambda_n^4}. \quad (14)
\]

The unknown coefficients \( A_n \) and \( B_n \) are determined from the boundary conditions on \( x = 0 \). First, we examine the expression \( H_1^2 - D_x D_y \), which is obtained for the root values of the solution of the differential equation. We consider the following three cases.

a. \( H_1^2 - D_x D_y > 0 \). Therefore

\[
f_n(x) = A_n e^{-r_1 \lambda_n x} + B_n e^{-r_2 \lambda_n x}, \quad r_{1,2} = \sqrt{\frac{H_1 \pm \sqrt{H_1^2 - D_x D_y}}{D_x}}. \quad (15)
\]

We assume that the edge \( x = 0 \) is hinged and have the boundary conditions: \( w = 0, \frac{\partial^2 w}{\partial x^2} = 0 \). Satisfying the boundary conditions, we find the unknown coefficients and determine the plate deflection

\[
w(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{D_y \lambda_n} \left[ \frac{H_1 - \sqrt{H_1^2 - D_x D_y}}{2\sqrt{H_1^2 - D_x D_y}} \exp \left( -\sqrt{\frac{H_1 + \sqrt{H_1^2 - D_x D_y}}{D_x}} \lambda_n x \right) \right.
- \frac{H_1 + \sqrt{H_1^2 - D_x D_y}}{2\sqrt{H_1^2 - D_x D_y}} \exp \left( -\sqrt{\frac{H_1 - \sqrt{H_1^2 - D_x D_y}}{D_x}} \lambda_n x \right) + 1 \right] \sin(\lambda_n y). \quad (16)
\]

The generalized shearing forces for orthotropic materials have the form

\[
V_x(x, y) = -\frac{\partial}{\partial x} \left( D_x \frac{\partial^2 w}{\partial x^2} + H_2 \frac{\partial^2 w}{\partial y^2} \right), \quad V_y(x, y) = -\frac{\partial}{\partial y} \left( D_y \frac{\partial^2 w}{\partial y^2} + H_2 \frac{\partial^2 w}{\partial x^2} \right), \quad (17)
\]

where

\[
H_2 = H_1 + 2D_{xy} = D_1 + 4D_{xy}.
\]

Therefore, for generalized shearing forces, we obtain

\[
V_x(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{2B \lambda_n} \frac{1}{D_y \sqrt{D_x}} \left( \sqrt{H_1 + B[D_x D_y - H_2(H_1 - B)]} \exp \left( -\sqrt{\frac{H_1 + B}{D_x}} \lambda_n x \right) \right.
+ \sqrt{H_1 - B[D_x D_y - H_2(H_1 + B)]} \exp \left( -\sqrt{\frac{H_1 - B}{D_x}} \lambda_n x \right) \sin(\lambda_n y),
\]

\[
V_y(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{2B \lambda_n} \left[ 2B - (B + 2D_{xy}) \exp \left( -\sqrt{\frac{H_1 + B}{D_x}} \lambda_n x \right) \right.
- (2H_1 + B + 2D_{xy}) \exp \left( -\sqrt{\frac{H_1 - B}{D_x}} \lambda_n x \right) \cos(\lambda_n y),
\]

where we used the following notation: \( B = \sqrt{H_1^2 - D_x D_y} \).
b. \( H_1^2 - D_x D_y = 0 \). This coincides with the case of isotropic material \([4]\). We seek \( f_n(x) \) in the form \( f_n(x) = A_n e^{-\lambda_n x} \). If the plate is hinged on the edge \( x = 0 \), then we obtain

\[
w(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{D\lambda_n} \left[ 1 - e^{-\lambda_n x} - \frac{\lambda_n x}{2} e^{-\lambda_n x} \right] \sin(\lambda_n y),
\]

and for generalized shearing forces, we obtain

\[
V_x(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{2\lambda_n} [(3 - \nu) e^{-\lambda_n x} + (1 - \nu) \lambda_n x e^{-\lambda_n x}] \sin(\lambda_n y),
\]

\[
V_y(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{2\lambda_n} [2(1 - e^{-\lambda_n x}) + (1 - \nu) \lambda_n x e^{-\lambda_n x}] \cos(\lambda_n y).
\]

c. \( H_1^2 - D_x D_y < 0 \). The solution (15) has the form

\[
f_n(x) = e^{-\alpha \lambda_n x} [A_n \sin(\beta \lambda_n x) + B_n \cos(\beta \lambda_n x)] + \frac{q_n}{D_y \lambda_n^2},
\]

where

\[
\alpha = \frac{\sqrt{D_x D_y} - H_1}{2D_x}, \quad \beta = \frac{\sqrt{D_x D_y} - H_1}{2D_x}.
\]

Satisfying the boundary conditions of hinge joining, we obtain

\[
w(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{D_y \lambda_n^2} \left( 1 - \frac{\alpha^2 - \beta^2}{2\beta \alpha} \sin(\beta \lambda_n x) + \cos(\beta \lambda_n x) \right) e^{-\alpha \lambda_n x} \sin(\lambda_n y).
\]

For generalized shearing forces, we obtain the expression

\[
V_x(x, y) = \sum_{n=1}^{\infty} q_n \frac{H_2}{\lambda_n D_y} \frac{\alpha^2 + \beta^2}{2\alpha \beta} \left[ 1 + \frac{D_x}{H_2} (\alpha^2 + \beta^2) \right] \beta \cos(\beta \lambda_n x)
\]

\[
+ \left[ 1 - \frac{D_x}{H_2} (\alpha^2 + \beta^2) \right] \alpha \sin(\beta \lambda_n x) e^{-\alpha \lambda_n x} \sin(\lambda_n y),
\]

\[
V_y(x, y) = \sum_{n=1}^{\infty} \frac{q_n}{\lambda_n D_y} \frac{1}{2\alpha \beta} \left( \alpha \beta D_y - 2\alpha \beta D_y e^{-\alpha \lambda_n x} \cos(\beta \lambda_n x) \right)
\]

\[
+ [H_2 (\alpha^2 + \beta^2) - D_y (\alpha^2 - \beta^2)] e^{-\alpha \lambda_n x} \sin(\beta \lambda_n x) \cos(\lambda_n y).
\]

3. Analysis of generalized shearing forces

The calculations for different materials showed that \( H_1^2 - D_x D_y < 0 \) for all materials, i.e., the third case. For generalized shearing forces in the middle of the fixed edge, we have

\[
V_x(0, b/2) = \sum_{n=1}^{\infty} q_n \frac{H_2}{\lambda_n D_y} \frac{\alpha^2 + \beta^2}{2\alpha \beta} \left[ \frac{D_x}{H_2} (\alpha^2 + \beta^2) + 1 \right] \sin \frac{\pi n}{2}.
\]

If we introduce a new coefficient \( \gamma \) equal to

\[
\gamma = \frac{H_2}{D_y} \frac{\alpha^2 + \beta^2}{2\alpha \beta} \left[ \frac{D_x}{H_2} (\alpha^2 + \beta^2) + 1 \right],
\]
Table 1. Generalizing shearing force in the middle of the fixed edge.

| (x, y) | (0, 0) | (0, b/2) | (0, b) |
|--------|--------|----------|--------|
| $V_x(x, y)$ | 0 | $\gamma \sum_{n=1}^{\infty} q_n \frac{\pi n}{2} \sin \frac{\pi n}{2}$ | 0 |
| $V_y(x, y)$ | $-\sum_{n=1}^{\infty} q_n \frac{2n}{\lambda_n}$ | 0 | $-\sum_{n=1}^{\infty} q_n \frac{2n}{\lambda_n} \cos(\pi n)$ |

Table 2. Values of $\gamma$ for the investigated materials.

| Material | Fiber | Fiberglass AG-4 [1] | Fiberglass anisotropic materials [1] | Tornel 40 [10] | Plywood, 3 ply and 5 [10] |
|----------|-------|---------------------|-------------------------------------|---------------|----------------------------|
| $\gamma$ | 1.87  | 1.22                | 1.26                                | 0.59          | 2.33                       |

Table 3. Values of $V_x/q_0b$ for investigated materials.

| Material | Mechanical characteristic coefficients | $V_x/q_0b$ |
|----------|---------------------------------------|------------|
| Kevlar 49 [9] | $E_x = 81.8 \times 10^9$ Pa, $E_y = 5.1 \times 10^9$ Pa, $\nu_{xy} = 0.31$, $\nu_{yx} = 0.019$, $G = 1.82 \times 10^9$ Pa | 0.46 |
| Fiberglass AG-4 [1] | $E_x = 2.1 \times 10^9$ Pa, $E_y = 1.6 \times 10^9$ Pa, $\nu_{xy} = 0.07$, $\nu_{yx} = 0.092$, $G = 0.42 \times 10^9$ Pa | 0.305 |
| Fiberglass anisotropic materials [1] | $E_x = 3.05 \times 10^9$ Pa, $E_y = 1.88 \times 10^9$ Pa, $\nu_{xy} = 0.195$, $\nu_{yx} = 0.12$, $G = 0.49 \times 10^9$ Pa | 0.315 |
| Tornel 40: Carbon [10] | $E_r = 1.61 \times 10^9$ Pa, $E_z = 23 \times 10^9$ Pa, $\nu_{rz} = 0.021$, $\nu_{rz} = 0.3$, $G_{rz} = 2 \times 10^9$ Pa | 0.15 |
| Plywood, 3 and 5 plies [10] | $E'_{ex} = 1.4 \times 10^9$ Pa, $E'_{yx} = 0.117 \times 10^9$ Pa, $E''_{ex} = 0.054 \times 10^9$ Pa, $G = 0.12 \times 10^9$ Pa | 0.58 |

then

$$V_x(0, \frac{b}{2}) = \gamma \sum_{n=1}^{\infty} q_n \frac{\pi n}{2} \sin \frac{\pi n}{2}.$$  \hspace{1cm} (27)

The solution of shearing forces depends on the coefficient $\gamma$ (which depends only on the characteristic coefficients of the mechanical properties of the material) in the middle of the fixed edge (see table 1).

For the materials under study, the values of $\gamma$ are given in table 2.

Table 3 shows the values of generalized shearing forces $V_x$ for different materials with $q = q_0 = \text{const}$.

The case $H_2^2 - D_xD_y > 0$ will be obtained if we can find a material for which $E_2 > E_1$ and $G > E_2/6$.

The following two cases of the load were considered to calculate the generalized shearing...
forces:

1) \( q = q_0 = \text{const}, \quad \text{hence} \quad V_y(0, 0) = -\frac{q_0 b}{2\pi \sum_{n=1}^{\infty} \frac{1}{n}}, \quad V_y(0, b) = -\frac{q_0 b}{2\pi \ln 2}, \)

2) \( q = q_0 \sin(\lambda_1 y), \quad \text{hence} \quad V_y(0, 0) = -\frac{q_1 b}{2\pi}, \quad V_y(0, b) = \frac{q_1 b}{2\pi}. \)

Conclusions

In this article, the values of generalized shearing forces for orthotropic semi-infinite plate strip are obtained. At the corners of the fixed edge, we have \( V_x = 0, \) and at the middle points, we have values different from zero. Thus, for the bending problem of an orthotropic semi-infinite plate strip with three hinged edges, we obtain that the generalized shearing forces are concentrated on the fixed edge. For an isotropic semi-infinite plate strip \((b = 10a)\), the equation of static equilibrium is obtained. Therefore, as compared to V. V. Vasil'ev's work [5] who studied a simply supported plate, the transformation of Kirchhoff and Thomson–Tait is true in this case.

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