Dilaton Contact Terms in the Bosonic and Heterotic Strings

Mark D. Doyle†‡

Joseph Henry Laboratories
Jadwin Hall
Princeton University
Princeton, NJ 08544 USA

Dilaton contact terms in the bosonic and heterotic strings are examined following the recent work of Distler and Nelson on the bosonic and semirigid strings. In the bosonic case dilaton two-point functions on the sphere are calculated as a stepping stone to constructing a ‘good’ coordinate family for dilaton calculations on higher genus surfaces. It is found that dilaton-dilaton contact terms are improperly normalized, suggesting that the interpretation of the dilaton as the first variation of string coupling breaks down when other dilatons are present. It seems likely that this can be attributed to the tachyon divergence found in [1]. For the heterotic case, it is found that there is no tachyon divergence and that the dilaton contact terms are properly normalized. Thus, a dilaton equation analogous to the one in topological gravity is derived and the interpretation of the dilaton as the string coupling constant goes through.

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† mdd@puhep1.princeton.edu
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1. Introduction

Recently there has been much progress towards a non-perturbative definition of string theory with the introduction of the matrix models and topological field theories. While these two approaches seem vastly dissimilar on the surface, it has become abundantly clear over the past year or so that there are deep connections between them. It is also clear that some of the deeper insights into the nature of string theory are given in terms of the geometry of Riemann surfaces. In fact, many of the surprising features of string theory are found to have a simple, concise, and natural explanations when phrased in geometrical terms. Similarly, a geometric approach to topological gravity based on $N = 2$ semirigid geometry $[2][3][4][5]$ has shed light on the nature of the contact interactions that give rise to the correlation functions and the recursion relations in topological gravity. The primary focus of this recent work has been to establish the ‘puncture’ and ‘dilaton equations’ of $[6]$ and $[7]$ in this geometrical framework. This approach is modeled on the dilaton contact terms in the bosonic string$[1]$. Here the ‘dilaton equation’ is the well-known low-energy theorem that the zero-momentum dilaton couples to the string coupling constant. The reason for this is that the (zero-momentum) dilaton in the bosonic string and all of the operators in topological gravity (except the puncture operator) are BRST-exact and naïvely decouple$[8]$. But, as pointed out in $[9]$, we must be careful. We are concerned here with the ‘equivariant’, or relative, BRST-cohomology and while the states we are considering are BRST-trivial, they are not trivial in the equivariant cohomology. In the full (absolute) cohomology, states are trivial if they can be written as the BRST-operator $Q$ acting on another state. However, we are really interested in states that obey an equivariance condition

\[(b_0 - \bar{b}_0)|\psi\rangle = 0.\]

In the equivariant cohomology states satisfying the equivariance condition are in the same class if they differ by $Q$ acting on a state that also obeys the equivariance condition. Thus a state that is trivial in the full cohomology (i.e., it can be written as $Q$ acting on another state) may not be trivial in the equivariant cohomology if the state it is $Q$ of does not satisfy the equivariance condition. This is precisely the case of the dilaton in critical string theory and the operators in topological gravity. Below we will see how the failure of this equivariance condition prevents states like the dilaton from decoupling.

A careful analysis of the geometry pertinent to the contact interactions in the bosonic and semirigid strings has been carried out in $[1]$ and $[3]$. In both cases, the analysis was
carried out in an operator formalism \cite{10,9,11,2}, in which correlation functions are calculated by associating each amplitude to a punctured (Semirigid Super-) Riemann surface with operators inserted at the punctures. Normal ordering requires the introduction of a coordinate at the puncture. This is particularly important for the dilaton and the other operators of topological gravity, because it is the ghost insertions due to the normal-ordering prescription that actually give non-zero answers. The calculations of \cite{1} and \cite{3} depended implicitly on the existence of what might be called a ‘good’ coordinate family.

Imagine inserting an operator onto a Riemann surface. To do so requires the introduction of a coordinate family that is appropriate for the entire moduli space of the surface, as the point moves around. In addition, while any coordinate family would give equivalent integrated answers, we would like to make our coordinate family as convenient as possible. In particular, we are concerned with integrating over the location of one of the punctures (the one where the dilaton is inserted) while holding everything else fixed. That is, we would like to integrate over the position of the dilaton while keeping the moduli of the surface and the other punctures fixed. In \cite{1}, the properties of a convenient coordinate family for establishing the dilaton equation were outlined. Firstly, we should take advantage of simplicity offered by using a holomorphic coordinate family wherever possible. It would be wonderful if we could use an entirely holomorphic family; however, such global holomorphic family of coordinates do not exist in general. Furthermore, it would easier to integrate over the location of the puncture holding the other moduli fixed, if we use coordinates for the puncture that are independent of the moduli for locations of the other punctures and for the surface itself. This can be done in regions of moduli space where the puncture is far from the others. However, this property cannot be maintained when two punctures approach each other and the coordinates for one puncture will depend on the location of the other puncture \cite{1}. In such a region of moduli space, the coordinates will be given by a plumbing fixture construction in which a ‘standard’ universal three-punctured sphere with coordinates is sewn to the rest of the surface. The colliding of two punctures is then replaced by the two punctures being located on a sphere that is pinching off from the rest of the surface. The trick is to interpolate smoothly between these two desirable coordinate systems. This interpolation gives rise to the non-analytic behavior that ultimately gives rise to the contact interaction in this viewpoint. The strength of this formalism is that the interpolation allows the delta-function contact terms to be smoothed out away from the region of moduli space where the two points actually collide. In \cite{1} and \cite{3}, the dilaton contact terms were calculated only in one patch of moduli space describing the approach.
of one puncture to the fixed location of another puncture. All of the other moduli and their ghost insertions were suppressed. Although it is quite reasonable that one should be able to do this, it is not completely clear that a global coordinate family with these properties exists. One aim of this paper is to establish the existence of ‘good’ coordinate families on general Riemann surfaces. This is done by performing a ‘pants’ decomposition of the surface into a set of punctured spheres and giving a prescription for constructing a suitable family on each sphere. Since different pants decompositions correspond to different cells of moduli space, it still remains to show that the families on each of the cells can be glued together continuously to give a global family. Below we sketch how this may be accomplished.

As a stepping stone towards this goal, we will calculate two-point functions of dilatons on the sphere in the bosonic string. This requires the introduction of a coordinate family describing the insertion of an operator on a three-punctured sphere. We will see that such a family is easily given and that it is easily generalized to give building blocks for global families for higher genus surfaces. The dilaton-dilaton calculation on the sphere is interesting in its own right because we will see that dilaton-dilaton contact terms are normalized incorrectly with respect to dilaton-strong physical state contact terms to give a correct dilaton equation. This was first noted by Distler and Nelson [12]. Thus it seems that while the insertion of a zero-momentum dilaton into a correlation function of strong physical states (see below) behaves like the first variation of the string coupling constant, inserting a second dilaton is not the same as a second variation. While it seems likely that tachyonic divergence found in [1] is the source of this puzzling observation, its full significance is unclear and merits further consideration.

To add further weight to this assertion, we go on to examine the heterotic string where one might expect the tachyon problem to go away. Indeed, it is found that the analogous dilaton equation goes through and that dilaton-dilaton contact terms behave nicely. The calculation here is done using a one patch computation following closely the work in [1] and [3] (although it contains several new features) and makes use of the general framework of [11] which contains a concise exposition of the operator formalism in both the bosonic and heterotic strings and of dilatons in the heterotic string. We hope that the reader finds this approach to be more than satisfactory after the arguments made in the bosonic string.

In the first part of the paper we review the general framework for examining contact terms in the operator formalism developed in [3] and [12]. We then go on to re-examine some issues in the bosonic string by considering dilaton two-point functions on the sphere.
where we explicitly give one global coordinate family for the entire moduli space. We then describe a prescription for constructing a ‘good’ coordinate family on higher genus surfaces by generalizing the coordinate family used on sphere in our two-point calculations. In the second part of the paper, we examine the dilaton contact terms in the heterotic string in which there are new features. Before doing this we review and develop the necessary heterotic geometry and the fermionic operator formalism. Finally, we end with a brief discussion of some of the significant issues raised our results.

2. General Framework

Since we will be examining several different calculations, it is perhaps best to begin with a brief sketch of the general philosophy behind the calculations that we have done. The general framework is that of the operator formalism as developed in [10][13][9][11] and the reader should look there for a more detailed exposition.

Recall that the correlation functions in bosonic string theory can be calculated as path integrals over Riemann surfaces on which operators have been inserted. The path integral can be reduced to an integral of a differential form over the compactified moduli space of a punctured Riemann surface (the punctures arising from the insertions of the operators). As is well-known, the operators in the path integral are represented by vertex operators carrying the appropriate quantum numbers. The vertex operators require normal-ordering and it is necessary to introduce coordinates at the punctures to do this. A particularly nice normal-ordering prescription was given by Polchinski [14]. There one uses the ‘flattest possible coordinate’ at the puncture to normal-order the operators and one finds that the ghost insertions that form the measure are modified by the normal-ordering, giving the so-called \( \hat{b} \)-prescription. Most operators in the bosonic string are unaffected by these modified insertions, but certain ‘frame-dependent’ operators like the dilaton require them. The insertion of these operators is dependent on the prescription used to normal-order them. In fact, it is seen that the dilaton couples to the scalar curvature of the surface precisely through these modified insertions: The curvature gives rise to a mixing of the holomorphic and antiholomorphic coordinates in this ‘flattest’ coordinate system. This can all be elegantly restated and understood through the operator formalism in the way described below. In [9] Nelson gave a beautiful geometric interpretation of the above results and it is this treatment that we follow.
The operator formalism gives a general prescription for constructing the appropriate measures on moduli space associated to particular correlation functions. The idea is to cut disks out around each operator insertion and perform the path integral over the rest of the surface. This information is then represented by a state $\langle \Sigma |$, which encodes all of the information from the rest of the surface into a wave function on the boundaries of the excised disks. This is ideal for our calculations of contact terms because choosing a ‘good’ coordinate family (as in the introduction) allows us to focus only on the region of interest in moduli space, namely, when the two operators are coming close together. The rest of the surface and the other operators are all held fixed in our state as we integrate over the location of an operator insertion. To be more explicit, we give the prescription for constructing the measure in the bosonic string\[9\]. We leave the heterotic generalization for Section 5.

We start with a genus $g$ Riemann surface with $s$ punctures and its moduli space $\mathcal{M}_{g,s}$. In addition, we require a coordinate at each puncture to normal-order our insertions, and, therefore, we consider an infinite-dimensional augmented moduli space of punctured Riemann surfaces with coordinates at each puncture, denoted $\mathcal{P}_{g,s}$. We can also use these coordinates to excise the disks around each puncture. $\mathcal{P}_{g,s}$ is a fiber bundle over $\mathcal{M}_{g,s}$, where the projection $\pi : \mathcal{P}_{g,s} \to \mathcal{M}_{g,s}$ is just forgetting about the coordinates at the punctures. We can construct a measure on $\mathcal{M}_{g,s}$ from a naturally defined one on $\mathcal{P}_{g,s}$ by pulling it back via a section $\sigma$ of $\pi : \mathcal{P}_{g,s} \to \mathcal{M}_{g,s}$. Unfortunately, there is no global holomorphic section; however, it is possible to find local sections that differ by $U(1)$ phases across patch boundaries, and the resulting measure on $\mathcal{M}_{g,s}$ will be independent of the choice of section if the states we are inserting obey certain conditions (at least up to total derivatives). Most physical states in string theory are ‘strong physical states (SPS)’ which satisfy

$$L_n |\psi\rangle = \bar{L}_n |\psi\rangle = 0, \ n \geq 0$$
$$b_n |\psi\rangle = \bar{b}_n |\psi\rangle = 0, \ n \geq 0.$$  \hspace{1cm} (2.1)

Strong physical states have the wonderful property that the measure constructed with them does not depend in any way on the choice of coordinate slice. However, this is overly restrictive, and we would like to insert many interesting states (i.e., dilatons) that do not obey these conditions. In \[9\]it is shown that states obeying a weaker set of conditions,

$$\langle L_0 - \bar{L}_0 |\psi\rangle = 0$$
$$\langle b_0 - \bar{b}_0 |\psi\rangle = 0,$$  \hspace{1cm} (2.2)

\[5\]
can still be inserted. These comprise the ‘weak physical state conditions (WPSC)’ and they are just the condition that the measure formed with such a state be independent of the $U(1)$ phase jumps across patch boundaries. States obeying (2.2), but not (2.1), are precisely the ‘frame-dependent’ states of Polchinski. This has interesting consequences for the dilaton which is the primary focus of this paper.

Since the descriptions of the measures on $\mathcal{M}_{g,s}$ and $\mathcal{P}_{g,s}$ are more than adequately described in other places [9] [10] [1] [11], we will only give a brief description for completeness. Then we will specialize it to our purposes. The measure on $\mathcal{P}_{g,s}$, in abbreviated form $\tilde{\Omega}$, is

$$\tilde{\Omega}(\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_{3g-3+s}) = \langle \Sigma, z_1, \ldots, b[v_1]b[v_2] \cdots b[v_{3g-3+s}]|\psi_1\rangle_{P_1} \otimes \cdots \otimes |\psi_s\rangle_{P_s}. \tag{2.3}$$

The $\tilde{V}_i$ are tangent vectors to $\mathcal{P}_{g,s}$ and the $v_i$ are abstract Virasoro generators that correspond to the $\tilde{V}_i$. The $v_i$ act on $\mathcal{P}_{g,s}$ through ‘Schiffer variations’ and can thus be associated with tangent vectors to $\mathcal{P}_{g,s}$. See [1] and [11] for details. The notation $b[v]$ corresponds to

$$\oint b_z(z) z^v(z) dz,$$

in which the contour integral is performed on a contour surrounding the puncture. To get a measure $\Omega$ on $\mathcal{M}_{g,s}$, we use the coordinate family to pullback the measure $\tilde{\Omega}$,

$$\Omega = \sigma^* \tilde{\Omega}.$$

This is given in the standard way by

$$\Omega(V_1, V_2, \ldots, V_{3g-3+s}) = \tilde{\Omega}(\sigma_*(\tilde{V}_1), \sigma_*(\tilde{V}_2), \ldots, \sigma_*(\tilde{V}_{3g-3+s})). \tag{2.4}$$

Here the vectors $\tilde{V}_i$ are any vectors that project down to the $V_i$, the vectors tangent to $\mathcal{M}_{g,s}$. It can be shown that the measure obtained in this way is independent of the choices made if all of the operators are SPS. For WPS it is found that the measure will only change by a global total derivative if the coordinate slice differs by $U(1)$ phases across patch boundaries, and, hence, the integrated answers will be unaffected by the choices made. Furthermore, it can be shown that if one of the states $|\psi\rangle = |Q\lambda\rangle$, the measure is

$^1$ We have suppressed the antiholomorphic parts and also the fact that the vector fields $v_i$ are really $N$-tuples of vector fields.

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just the exterior derivative, $d$, of the corresponding measure formed with $|\lambda\rangle$, and, thus, that $Q$ acts as the exterior derivative on $\mathcal{P}_{g,s}$.

In this paper we are focusing on the dilatons and their contact terms and the above formalism is readily adapted to our purpose. It is well-known that dilatons measure background curvature and that they couple to the Euler characteristic of the surface. We would like to investigate whether a ‘dilaton equation’ similar to that of (4) is valid in the bosonic and heterotic strings. Heuristically, we have,

$$\langle DO_1 O_2 \ldots O_s \rangle = -2\pi i (2g - 2 + s) \langle O_1 O_2 \ldots O_s \rangle,$$

(2.5)

where the $O_i$ maybe SPS’s or other dilatons. The $(2g - 2)$ arises from integrating the dilaton over the surface and the additional $s$ arises from contact interactions of the dilaton and the other operators. (The factor of $-2\pi i$ is conventional and could be absorbed by a rescaling of the dilaton.)

In the bosonic string the zero-momentum dilaton can be written as

$$(Q + \bar{Q})(c_0 - \bar{c}_0)|0\rangle = (c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|0\rangle.$$  

(2.6)

Naïvely this state should just decouple from all correlation functions since it is BRST-exact, and the BRST-operator $Q$ corresponds to the exterior derivative on $\mathcal{P}_{g,s}$. However, although the dilaton itself satisfies the WPSC, $(c_0 - \bar{c}_0)|0\rangle$ is not annihilated by $(b_0 - \bar{b}_0)$ and so while the measure for the dilaton is locally a total derivative, it is sensitive to the $U(1)$ phase jumps across the patch boundaries. Thus it is not a global total derivative and the contributions at the patch boundaries prevents the dilaton from decoupling; in fact, the boundary contributions build up the Euler characteristic of the surface [9]. Closely related to the dilaton is the state

$$(Q + \bar{Q})(c_0 + \bar{c}_0)|0\rangle$$

(2.7)

Here $(c_0 + \bar{c}_0)|0\rangle$ is annihilated by $(b_0 - \bar{b}_0)$, and is insensitive to the $U(1)$ phases. Hence, the state (2.7) does indeed give rise to global total derivatives and it decouples from all correlation functions. Thus, we are free to add this state to our dilaton state and work with the new dilaton state

$$|D\rangle = 2c_1 c_{-1}|0\rangle.$$  

(2.8)

Correlation functions computed with this state are the same as computed with (2.6) because the measures formed from them differ by a genuine global total derivative. This purely
holomorphic dilaton was used in [3] to avoid the tachyonic divergence found in [1] which resulted from the fusion of the holomorphic and antiholomorphic pieces in the state (2,0).

The bosonic case was mostly investigated in [1]. It was found that the contact interaction for a dilaton and an SPS depended on the plumbing fixture coordinates (see below) used to describe the collision of the two operators, and a physical ‘long thin tube’ prescription was advocated for getting the proper contact term. However, for two dilatons, the contact contribution is $3/2$ the contact contribution of dilaton-SPS, and the dilaton equation does not hold for general correlation functions containing multiple dilatons.\footnote{This failure of the dilaton equation in the bosonic string was lurking in [1], but it was not made explicit because of the focus on the tachyon divergence there. However, as pointed out in [3], one can work with the purely holomorphic dilaton and avoid the divergence. In addition, another one patch calculation in [12] also found this factor of 3/2 and we will later see it explicitly in the calculation on the sphere.} This was the main motivation behind the construction of the semi-rigid approach to topological gravity. There almost all of the operators are WPS’s and the situation is much cleaner: There is no residual choice for the insertion coordinates and the dilaton equation is always obeyed. The situation in the heterotic string is in between the other two cases. There is no tachyon divergence and the dilaton equation is always obeyed after making the residual choice for the plumbing fixture coordinates.

Before presenting our calculations, we should indicate how this formalism can be adapted to the calculation of contact terms and the dilaton equation. The basic idea of the dilaton equation is to integrate over the position of the dilaton and reduce the $n+1$-point function to an $n$-point function. In the version of the calculations developed for the semi-rigid and bosonic strings in [1][2][3], we restrict ourselves to the region of moduli space where the dilaton approaches one particular operator and, working in that one patch only, integrate only over the moduli corresponding to the relative location of the dilaton with respect to the other operator. This implicitly depends on the existence of a ‘good’ coordinate family. Below, we will give a prescription for constructing such a family in the bosonic string. To get a dilaton-dilaton contact term, it is necessary to insert one of the dilatons with a coordinate family representing a curved background. Above it was mentioned that on a Riemann surface curvature can be locally represented as a mixing of the holomorphic and antiholomorphic coordinates describing the location of a point in this one patch where the calculation is being done. This was the method used in [1] and we will extend it here to the heterotic case. For the bosonic string, we will only be working on
the sphere where the curvature is hidden in the non-analytic transition function between the two coordinate patches that are needed to cover the sphere. We refer to this as the ‘global’ picture of curvature because the curvature is not described in a local manner. The advantage of this description is that we can unambiguously integrate over the entire sphere, whereas in the local calculation we are restricted to one patch. Furthermore, we will see that this local description can interfere with the identification of certain total derivatives with respect to the moduli associated to the location of the insertions. But all is not lost and we can still extract all of the information that we will need.

When the operators are far apart, it is appropriate to use as coordinates on the moduli space the positions of the the operators themselves. In this region of moduli space the coordinates at one puncture is independent of the moduli of the others. However, when the two operators approach each other we are better served by making a conformal transformation to a surface in which the two operators are pinching off from the rest of the surface. In this region, we are forced to use coordinates for one puncture that depend on the moduli of the other puncture. The appropriate coordinates are the pinching parameter $q$ associated to the plumbing fixture and the location of where the degeneration is occurring. To specialize even more, we will use the fixture depicted in Fig. 1. In this case, integrating over $q$ will correspond to integrating over the position of the dilaton inserted at $P$ and will leave us with the other operator always at the point $Q$. Our approach is then to choose a coordinate slice that smoothly interpolates between the two pictures. Furthermore, by choosing a ‘good’ coordinate family, we can keep all of the other moduli associated to the surface and the locations of the other operators fixed as we integrate over the location of the dilaton. As mentioned above, the operator formalism allows us to summarize all of this in to a state $\langle \Sigma |$ and we will never have to explicitly display the dependence on the moduli not associated to the locations of the operators that we are considering. This is a substantial simplification. As explained in [1], our interpolation will have the effect of smoothing the delta-function contact interaction out away from $q = 0$ to an annulus in the $q$-plane.

Finally, the above formalism can be extended to the heterotic string. There we are interested in integrating over the moduli space of super-punctured Super-Riemann surfaces, $\hat{M}_{g,s}$. Again, a measure on $\hat{M}_{g,s}$ can be constructed by introducing the augmented moduli space $\hat{P}_{g,s}$ of punctured surfaces with superconformal coordinates at the punctures and using a section to pullback the naturally defined measure. The heterotic string contains new features that do not appear in either the semi-rigid or bosonic cases and so we will develop the necessary formalism in more detail later in the paper.
Fig. 1: Bosonic plumbing fixture with coordinates. The top figure shows coordinates that are appropriate for when \( P \) and \( Q \) are far apart. The coordinates at \( P \) are independent of the location of \( Q \). The lower figure depicts \( P \) and \( Q \) on a sphere pinched off from the rest of the surface, appropriate for when they are close together. Now the coordinates at \( P \) explicitly depend on the coordinates at \( Q \). The plumbing fixture places \( Q \) at 0 and is sewn to the original position of \( Q \) on the surface. This ensures that after integrating over \( P \) we are left with insertions at the original location of \( Q \).

3. Bosonic Dilaton Two-Point Functions

In the bosonic string, we are able to come up with a ‘good’ coordinate slice for all of the moduli space \( \mathcal{M}_{0,4} \) and use this to calculate the two-point functions on the sphere of dilatons and strong physical states. The nice feature is that we can give a global family and not just restrict ourselves to a calculation in one patch, as was done in [1]. This will also give us a basic building block from which coordinate families appropriate for higher genus surfaces can be constructed.

3.1. The Geometry for the Sphere

We are interested in computing two-point functions on the sphere. To this end, we choose a convenient coordinate slice for the four-punctured sphere as follows: The sphere needs two coordinate patches to cover it. The northern hemisphere has coordinate \( z \), while the southern hemisphere has coordinate \( w = -1/z \). Now, letting \( \tilde{r} \) (resp. \( r \)) be the modulus for the location of the point \( P \) (resp. \( Q \)), we can linearly interpolate between the
two hemispheres across the equator by letting \( h(|\tilde{r}|) \) (resp. \( g(|r|) \)) be any function that smoothly interpolates between 1 and 0 as \( \tilde{r} \) (resp. \( r \)) goes from 0 to \( \infty \), Fig. 2. Then

\[
\begin{align*}
\zeta_P(\cdot) &= h(z - \tilde{r}) + (1 - h)\tilde{r}^2(-1/z + 1/\tilde{r}) \\
\zeta_Q(\cdot) &= g(z - r) + (1 - g)r^2(-1/z + 1/r)
\end{align*}
\]

(3.1)
give coordinates centered at \( P \) and \( Q \) respectively, and appropriate for both hemispheres. The curvature of the sphere manifests itself through the non-holomorphic behavior of the interpolating functions \( g \) and \( h \). This is fine as long as \(|r - \tilde{r}| > \epsilon \) for some \( \epsilon > 0 \). But in the neighborhood of \( P \) and \( Q \) close together the coordinates should go over to ones that represent the conformally equivalent picture of a sphere containing the two punctures pinching off from the rest of the surface.

![Fig. 2: The interpolating functions \( f \) and \( g \). \( h \) has the same behavior as \( g \), but its argument is \( \tilde{r} \). Any smooth functions that run between 0 and 1 could be used.](image)

The plumbing fixture is the standard construction to represent the pinching off of a sphere from the rest of a surface. Computing contact terms requires a three-punctured sphere with coordinates pinching off from the rest of the sphere. In \([1]\) a physically motivated choice for choosing appropriate coordinates was given. We will use the more general choice given in \([3]\) with the three punctures placed at 0 where the coordinates are

\[
\xi + \tilde{a}_2\xi^2 + \tilde{a}_3\xi^3 + \cdots,
\]

(3.2)
at 1 with coordinates

\[
(\xi - 1) + a(\xi - 1)^2 + a_3(\xi - 1)^3 + \cdots,
\]

(3.3)
and at $\infty$ with coordinate $1/\xi$. We choose the most general coordinates that vanish at the points $P$ and $Q$ since this is where our operators will be inserted. The coefficients $a$ and $\tilde{a}$ in the expansions of the coordinates will be determined by requiring that dilaton-strong physical state contact terms have the correct normalization; the higher coefficients drop out in dilaton calculations. The point at $\infty$ is sewn via the standard plumbing fixture onto the point $Q$ by the identification

$$\xi = \zeta_Q(\cdot)/q.$$ (3.4)

So for small $|r - \tilde{r}|$ the coordinates at $P$ and $Q$ in (3.3) and (3.2) become

$$\phi_P(\cdot) = (\zeta_Q(\cdot)/q - 1) + a(\zeta_Q(\cdot)/q - 1)^2 + \cdots$$
$$\phi_Q(\cdot) = \zeta_Q(\cdot)/q + \tilde{a}(\zeta_Q(\cdot)/q)^2 + \cdots.$$ (3.5)

Essentially the pinching parameter $q$ and $r$ have replaced $\tilde{r}$ and $r$ as the moduli describing the locations of the two points. The relationship between $q$ and $r$ and $\tilde{r}$ can be found by demanding that $\phi_P(P) = 0$. Thus,

$$q = \zeta_Q(P) = \frac{(\tilde{r} - r)((1 - g)r + g\tilde{r})}{\tilde{r}}.$$ (3.6)

Finally, the two coordinate systems are joined by using yet another function $f(|r - \tilde{r}|)$ that smoothly interpolates from 0 to 1 as $|r - \tilde{r}|$ goes from 0 to $\infty$, Fig. 2. Again we choose a linear interpolation. It is important to realize that we cannot interpolate between just any two coordinates because there is a phase that cannot be removed. (Hence the factors of $r^2$ and $\tilde{r}^2$ in (3.1).) The phase is found by expressing $\zeta_Q$ in terms of $\zeta_P$ and rewriting $\phi_P$ in terms of $\zeta_P$. The final result is

$$\varphi = \frac{((g - 1)r^2 - g\tilde{r}^2)}{\tilde{r}(r - \tilde{r})((1 - g)r + g\tilde{r})}$$
$$\sigma_P(\cdot) = f\varphi \zeta_P + (1 - f)(\varphi \zeta_P + (a \varphi^2 - \frac{h(1 - g)r^2}{q\tilde{r}^3} + \frac{g(1 - h)}{q\tilde{r}})\zeta_P^2 + \cdots)$$
$$\sigma_Q(\cdot) = f(\zeta_Q/q) + (1 - f)(\zeta_Q/q + \tilde{a}(\zeta_Q/q)^2 + \cdots)$$ (3.7)

where $\varphi$ is the relative phase for the coordinates at $P$. These are the coordinates for the holomorphic sector. The antiholomorphic coordinates are simply the barred version of this.

As a warm-up (and because we will need it later) we will calculate the one-point function of the dilaton on the sphere using the linear interpolation in (3.1). A different
interpolation was used in [1]. Since we are only inserting one operator, the $\zeta_Q(\cdot)$ coordinate alone is all that is needed. The pushforwards are calculated by differentiating the coordinate with respect to $r$ and $\vec{r}$ and using

$$
\zeta_Q^* \left( \frac{\partial}{\partial r} \right) = \frac{\partial \zeta_Q}{\partial r} \frac{\partial}{\partial \zeta_Q} + \frac{\partial \bar{\zeta}_Q}{\partial r} \frac{\partial}{\partial \bar{\zeta}_Q},
$$

(3.8)

Taking the derivatives and re-expressing the result in terms of $\zeta_Q$ gives

$$
\frac{\partial \zeta_Q}{\partial r} = -1 + \frac{2}{r} (1 - g) \zeta_Q - \frac{2g(1 - g)}{r^2} \zeta_Q^2 + \frac{|r|g'(|r|)}{2r^2} \zeta_Q^2 + \cdots
$$

$$
\frac{\partial \zeta_Q}{\partial \vec{r}} = \frac{g'(|r|)}{2|r|} \zeta_Q^2 + \cdots.
$$

Thus the corresponding $b$-insertions are

$$
b[\zeta_Q^* \left( \frac{\partial}{\partial r} \right)] = b_{-1} + \cdots
$$

$$
b[\zeta_Q^* \left( \frac{\partial}{\partial \vec{r}} \right)] = -\frac{g'(|r|)}{2|r|} b_1 + \cdots
$$

and, hence, the one-point function is given by

$$
\langle D \rangle = \int dr \land d\vec{r} \langle \Sigma | b_{-1} \left( -\frac{g'(|r|)}{2|r|} \right) b_1 (-2)c_{-1}c_1 |0\rangle
$$

$$
= \int dr \land d\vec{r} \frac{g'(|r|)}{|r|} Z
$$

$$
= -2i \int |r| \land d\theta g'(|r|) Z
$$

$$
= -2\pi i (-2)Z,
$$

(3.9)

where we recall that $g$ runs from 1 to 0 as $|r|$ runs from 0 to $\infty$. ($Z$ is the partition function on the sphere and the insertions for the three conformal Killing vectors have been suppressed.) This is just what was expected since the sphere has $2g - 2 = -2$. Also note that, as in [1], the form of $g$ didn’t matter and that the measure was a total derivative. Of particular importance for our later calculations is the fact that the support of the measure is compact. For it is restricted to the region in which $g'$ is non-zero. This region can be chosen to be an annulus in the $r$-plane, and the measure is identically zero away from this annulus. Thus when integrating, we can use Stokes’ Theorem and contributions will come from the boundary of the annulus. Similar things will happen in the two-point function calculations.
3.2. Two-Point Functions on the Sphere

The computation for the two-point functions is more or less identical to the one for the one-point function. However, now there are four pushforwards to compute and we must use the \( \sigma_P \) and \( \sigma_Q \) coordinates in (3.7). The resulting \( b \)-insertions are

\[
b[\sigma_*(\frac{\partial}{\partial r})] = \frac{1}{q} b_{-1}^{(Q)} + \cdots
\]

\[
b[\sigma_*(\frac{\partial}{\partial \tilde{r}})] = \left( \frac{a(r - \tilde{r})}{2|r - \tilde{r}|} f'(|r - \tilde{r}|) + \frac{hm(r - \tilde{r})^2}{2n|r - \tilde{r}|} f'(|r - \tilde{r}|) - \frac{gm\tilde{r}^2(r - \tilde{r})^2}{2n^2|r - \tilde{r}|} f'(|r - \tilde{r}|) +\right) b_1^{(P)} + \frac{(1 - f)r^3\tilde{r}^2(r - \tilde{r})m}{2n^3|r|} g'|(|r|) + \frac{a(1 - f)r^2\tilde{r}(r - \tilde{r})}{2mn|r|} g'(|r|) b_1^{(Q)} + \frac{1}{q} b_{-1}^{(Q)} + \cdots
\]

\[
b[\sigma_*(\frac{\partial}{\partial \tilde{r}})] = \varphi b_{-1}^{(P)} + \cdots
\]

\[
b[\sigma_*(\frac{\partial}{\partial \tilde{r}})] = \left( -\frac{a(r - \tilde{r})}{2|r - \tilde{r}|} f'(|r - \tilde{r}|) - \frac{hm(r - \tilde{r})^2}{2n|r - \tilde{r}|} f'(|r - \tilde{r}|) + \frac{gm\tilde{r}^2(r - \tilde{r})^2}{2n^2|r - \tilde{r}|} f'(|r - \tilde{r}|) - \frac{fmr^2(r - \tilde{r})}{2n|r|} h'(|r|) \right) b_1^{(P)} + \varphi \bar{b}_{-1}^{(P)} - \frac{\tilde{a}(r - \tilde{r})}{2|r - \tilde{r}|} f'(|r - \tilde{r}|) b_1^{(Q)} + \cdots
\]  

(3.10)

where we have defined

\[
m = (1 - g)r + g\tilde{r}
\]

\[
n = (1 - g)r^2 + g\tilde{r}^2
\]  

(3.11)

for convenience (and \( \bar{\varphi} \), resp. \( \bar{q} \), is the complex conjugate of \( \varphi \), resp. \( q \)). The derivatives are all with respect to the argument shown. And, finally, the dots represent \( b \)-insertions that do not contribute to the dilaton-SPS and dilaton-dilaton two-point functions.

We will calculate the dilaton-SPS in two ways. The first is by putting the dilaton at \( P \) and the SPS at \( Q \), and the second by switching them. With the dilaton at \( P \) and the SPS

---

3 The derivatives and algebra are straightforward but lengthy. These pushforwards (and the heterotic ones also) were calculated with the help of Mathematica.
at $Q$, we need to pick off the contribution of the measure proportional to $b_{-1}^{(P)} b_{1}^{(P)} b_{-1}^{(Q)} b_{1}^{(Q)}$:

$$b[\sigma_*(\frac{\partial}{\partial r})] b[\sigma_*(\frac{\partial}{\partial \tilde{r}})] b[\sigma_*(\frac{\partial}{\partial r})] b[\sigma_*(\frac{\partial}{\partial \tilde{r}})] =$$

$$\frac{1}{|q|^2} \left( \frac{anf'(|r - \tilde{r}|)}{2\tilde{r}m|r - \tilde{r}|} - \frac{g\tilde{r}(r - \tilde{r})f'(|r - \tilde{r}|)}{2n|r - \tilde{r}|} - \frac{fh'(|\tilde{r}|)}{2|\tilde{r}|} + \frac{h(r - \tilde{r})f'(|r - \tilde{r}|)}{2\tilde{r}|r - \tilde{r}|} \right) \times b_{-1}^{(P)} b_{1}^{(P)} b_{-1}^{(Q)} b_{1}^{(Q)}$$

(3.12)

and, hence, we need to integrate

$$\int d\tilde{r} \wedge d\tilde{r} \wedge dr \wedge d\tilde{r} \frac{1}{|q|^2} \left( \frac{anf'(|r - \tilde{r}|)}{2\tilde{r}m|r - \tilde{r}|} - \frac{g\tilde{r}(r - \tilde{r})f'(|r - \tilde{r}|)}{2n|r - \tilde{r}|} - \frac{fh'(|\tilde{r}|)}{2|\tilde{r}|} + \frac{h(r - \tilde{r})f'(|r - \tilde{r}|)}{2\tilde{r}|r - \tilde{r}|} \right)$$

$$b_{-1}^{(P)} b_{1}^{(P)} b_{-1}^{(Q)} b_{1}^{(Q)} |D|^P \otimes |\psi|^Q$$

(3.13)

over the moduli space. The factors of $(1/\varphi)^{L_0}$ and $q^{L_0}$ are needed because we have inserted our states using (3.7), and we would like to compare this with the insertions of states in the coordinates $\zeta_P$ and $\zeta_Q$ in (3.1)[1]. The dilaton has $L_0 = \bar{L}_0 = 0$ but the state $b_{-1} \tilde{b}_{-1} |\psi\rangle$ has $L_0 = \bar{L}_0 = 1$, leaving

$$\int d\tilde{r} \wedge d\tilde{r} \wedge dr \wedge d\tilde{r} \left( \frac{anf'(|r - \tilde{r}|)}{2\tilde{r}m|r - \tilde{r}|} - \frac{g\tilde{r}(r - \tilde{r})f'(|r - \tilde{r}|)}{2n|r - \tilde{r}|} - \frac{fh'(|\tilde{r}|)}{2|\tilde{r}|} + \frac{h(r - \tilde{r})f'(|r - \tilde{r}|)}{2\tilde{r}|r - \tilde{r}|} \right) b_{-1}^{(P)} b_{1}^{(P)} b_{-1}^{(Q)} b_{1}^{(Q)} |D|^P \otimes |\psi|^Q$$

(3.14)

The above can be written

$$\int d\tilde{r} \wedge d\tilde{r} \wedge dr \wedge d\tilde{r} \frac{\partial}{\partial \tilde{r}} \left( - \frac{anf}{\tilde{r}m(r - \tilde{r})} - \frac{hf}{\tilde{r}} + \frac{g\tilde{r}f}{n} \right) b_{-1}^{(P)} b_{1}^{(P)} b_{-1}^{(Q)} b_{1}^{(Q)} 2c_1 c_{-1} |0|^P \otimes |\psi|^Q,$$

or, erasing the $|0\rangle$ at P,

$$\int d \left( 2 \left( - \frac{anf}{\tilde{r}m(r - \tilde{r})} + \frac{hf}{\tilde{r}} - \frac{g\tilde{r}f}{n} \right) \right) d\tilde{r} \wedge dr \wedge d\tilde{r} b_{-1}^{(Q)} b_{1}^{(Q)} |\psi|^Q.$$  

(3.15)

With care this form can be integrated using Stokes’ Theorem. The purpose of the interpolation in (3.7) is to smooth out the delta-function contact term at $\tilde{r} = r$. In fact, the function $f(|r - \tilde{r}|)$ acts as a small distance cut-off and we should change to the more natural variable $y = \tilde{r} - r$ in place of $\tilde{r}$. Then we can integrate over $y$ leaving us with a two-form corresponding to the insertion of the SPS. Making the change of variables in (3.15) gives

$$\int d \left[ 2 \left( - \frac{a(r^2(1 - g) + g(y + r)^2)f(|y|)}{y + r} \right) + \frac{h(|y + r|)f(|y|)}{y + r} - \frac{g(y + r)f(|y|)}{(r^2(1 - g) + g(y + r)^2)} \right] dy \wedge dr \wedge d\tilde{r} b_{-1}^{(Q)} b_{1}^{(Q)} |\psi|^Q.$$  

(3.16)
To apply Stokes’ Theorem it is necessary to determine the support of the original four-form in (3.14) and what the appropriate boundaries are for each term in the three-form. As in the one-point calculation, the form we have to calculate is compactly supported on several annuli. Each term in (3.14) is either proportional to $f'(|r - \tilde{r}|)$ or $h'(|\tilde{r}|) = h'(|y + r|)$. The support of $f'(|r - \tilde{r}|)$ is confined to the small annulus around $y = 0$ shown in Fig. 3. Thus terms in (3.16) that come from terms in (3.14) proportional to $f'(|y|)$ should be integrated around the boundaries of this annulus and only a pole at $y = 0$ can contribute. Furthermore, it should be noted that $f = 0$ on the inner boundary and $f = 1$ on the outer. The support of $h'(|\tilde{r}|)$ is centered around $\tilde{r} = 0$ and so in the $y$-plane this is an annulus around the point $y = -r$, also depicted in Fig. 3. Since $h = 1$ for small $\tilde{r}$ and $h = 0$ for large $\tilde{r}$, the inner contour has $h = 1$ and the outer $h = 0$.

Fig. 3: Regions of integration in the $y$-plane. The forms that are being integrated only have support in the annular regions. Stokes’ Theorem is used to rewrite them as contour integrals.

We can now apply Stokes’ Theorem. The first and third terms (3.16) come from something proportional to $f'(|y|)$ in (3.14) and, hence, have support only on the small annulus around the origin in the $y$-plane. The middle term comes from something proportional to both $f'(|y|)$ and $h'(|y + r|)$ and so has support on both the annulus around the origin and the annulus around the point $y = -r$. Hence, poles at 0 and $-r$ contribute for this term. The integrals are now straightforward and the result is

$$-2\pi i (-2a - 2) \int dr \wedge d\tilde{r} y^{(Q)} \delta^{(Q)} |\psi\rangle^Q.$$ (3.17)

This corresponds to

$$\langle D\psi \rangle = -2\pi i (-2a - 2) \langle \psi \rangle,$$ (3.18)
which is the dilaton equation on the sphere if we choose $a = -1/2$. This is the residual choice of coordinates mentioned earlier.

Since our plumbing fixture was asymmetric, we should see what happens if we put the dilaton at $Q$ and the SPS at $P$. This isn’t really an independent check, since we will determine $\tilde{a}$ instead of $a$. The real question is will the resulting values of $a$ and $\tilde{a}$ will give the right contribution for two dilatons on the sphere.

The piece of the measure proportional to $b_{-1}^{(P)}\bar{b}_{-1}^{(Q)}b_{-1}^{(Q)}b_{1}^{(Q)}$ is

$$|\varphi|^2 \left( -\frac{g'(|r|)}{2r} - \frac{\tilde{a}r f'(|r - \tilde{r}|)}{2m|r - \tilde{r}|} + \frac{\tilde{a}(1 - f)r\tilde{r}g'(|r|)}{2m^2 |r|} \right) b_{-1}^{(P)}\bar{b}_{-1}^{(Q)}b_{-1}^{(Q)}b_{1}^{(Q)}$$

which again gives rise to a total derivative (after changing variables to $y$ in favor of $r$ ($\tilde{r}$) in the first (second) term), with the integral now

$$\int \left( 2 \left( \frac{\tilde{a}g(1 - f)\tilde{r}}{(\tilde{r} + y(g - 1)(\tilde{r} - y))} + \frac{g}{(y - \tilde{r})} \right) d\tilde{r} \wedge d\overline{\tilde{r}} \wedge dy \right) +$$

$$\int \left( 2 \left( \frac{\tilde{a}(1 - g)f(y + r)}{(r + gy)r} + \frac{\tilde{a}f(y + r)^2}{(r + gy)yr} \right) dy \wedge dr \wedge d\tilde{r} \right) b_{-1}^{(P)}\bar{b}_{-1}^{(Q)}|\psi\rangle^P.$$

This time the factor $|\varphi|$ was cancelled by the factors of $(1/\varphi)^{L_0}$ and $(1/\bar{\varphi})^{\bar{L}_0}$ since the strong physical state is located at $P$. The integrals are again straightforward once it is noted that $g'(|y - \tilde{r}|)$ has support on an annulus centered at $y = \tilde{r}$, as shown in Fig. 3. It should be noted that in the first term integrating over $y$ leaves a two-form proportional to $d\tilde{r} \wedge d\overline{\tilde{r}}$ corresponding to the insertion of the SPS at the point $P$, while integrating the second term leaves us with a two-form proportional to $dr \wedge d\tilde{r}$. However, this is not a problem since the second term is purely a contact term and appears only when the two points coincide. With this in mind, we find

$$\langle \psi D \rangle = -2\pi i(2\tilde{a} - 2)\langle \psi \rangle \quad (3.19)$$

and we have to choose $\tilde{a} = 1/2$ to get the dilaton equation. These choices of $a$ and $\tilde{a}$ agree with the physically motivated ones in [1], although one must be careful about signs when comparing the two calculations.

The two-dilaton calculation is similar to the above calculation, only more involved. Because of its length, we have moved it to Appendix A and only give the result here. One can readily recognize the structure of each term in the two-point function. We see again that we have a total derivative and that, after integrating over $y$, we are left with the
measure for the insertion of one dilaton which we recognize from the one-point calculation. The resulting form can be completely, unambiguously integrated. The result is

$$\langle DD \rangle = -2\pi i (2(\bar{a} - a + a\bar{a}) - 2)\langle D \rangle$$

$$= -2\pi i (3/2 - 2)\langle D \rangle.$$  \hfill (3.20)

The 3/2 spoils the dilaton equation and agrees with the calculations in [1] and [12], which were done along the lines that we will use for the heterotic string.

As mentioned earlier, this failure of the dilaton equation is somewhat mysterious. Comparing (3.20) and (3.18), we see that the 3/2 results from extra terms proportional to $\bar{a}$, or terms that depend on the coordinates at $Q$ on the three-punctured sphere. In the heterotic case, terms like this are present, but they are killed by the integration over the odd moduli. This is also the case with potentially divergent contributions that are akin to the tachyon divergence. The low-energy theorem for the zero-momentum dilaton usually identifies it with string coupling constant. More explicitly, inserting a dilaton is supposed to correspond to $\lambda \frac{\partial}{\partial \lambda}$. We have seen that

$$\langle DO_{i_1} \cdots O_{i_n} \rangle = (2g - 2 + n)\langle O_{i_1} \cdots O_{i_n} \rangle,$$

where the $O_i$ are SPS’s. Thus, as expected, the insertion of one dilaton corresponds to the first variation with respect to the string coupling constant. However, we have also seen that when two dilatons are inserted the result is

$$\langle DDO_{i_1} \cdots O_{i_n} \rangle = (2g - 2 + n + 3/2)\langle DO_{i_1} \cdots O_{i_n} \rangle.$$  

If the insertion of a second dilaton were to correspond to the second variation of the string coupling constant, we would have expected a 1 in place of the 3/2. Possibly, this surprising result is in some way due to the tachyon. As further support for this, we will find that the heterotic string behaves nicely, with no tachyon divergence and with a 1 instead of a 3/2.

4. ‘Good’ Coordinate Families on Higher Genus Surfaces

In the previous section, we were able to calculate two-point functions on the sphere because we were able to provide a ‘good’ coordinate family for the moduli space $M_{0,4}$. This section is devoted to sketching how this can be used as a building block to provide
a ‘good’ coordinate family on a higher genus surface. We begin by examining the
what made the coordinate family given in (3.7) ‘good’. For large \( q \) (and large \(| r - \tilde{r} | \)) the
points \( P \) and \( Q \) are widely separated and the coordinate \( \sigma_P(\cdot) \) becomes independent of
the moduli for \( Q \) (except for the \( r \) dependence in the overall phase which is irrelevant
here). In this region, the coordinates look like coordinates one would choose if \( P \) were
the only puncture on the sphere. These coordinates interpolate between the coordinates
\( z \) which is good for everywhere but the south pole and \(-1/z\) at the south pole. On the
other hand, \( \sigma_P(\cdot) \) is constructed to go over to the plumbing fixture coordinates when the
\( P \) and \( Q \) approach each other. In this region, the coordinate for \( P \) depends essentially on
the moduli associated to the location of \( Q \). Finally, since the sphere has no moduli, \( \sigma_P(\cdot) \)
is trivially independent of the moduli associated to the unpunctured surface.

We would like to now generalize this coordinate family to higher genus surfaces. The
key is to decompose the surface via a ‘pants’ decomposition into a set of three-punctured
spheres on which we can easily give ‘good’ coordinate families for the insertion of another
puncture. Then, we must show that we can glue these local coordinate families together
into a global one that covers all of moduli space. Thus we will demonstrate the existence
of a ‘good’ coordinate family that was presumed to exist in the one patch calculations in
[1] and [3]. One could then go on to calculate dilaton correlation functions on higher genus
surfaces.

A ‘pants’ decomposition of an unpunctured genus \( g \) surface is accomplished by choosing
a maximal set of \( 3g - 3 \) non-intersecting closed geodesics on the surface, as shown in
Fig. 4. These curves decompose the surface into \( 2g - 2 \) pants-shaped regions which can
be thought of as three-punctured spheres. The original surface can be reconstructed by
sewing together these spheres using the plumbing fixture construction. In this section,
it helpful to use a modified plumbing fixture. Instead of joining together \( z_1 \) and \( z_2 \) by
\( z_1 z_2 = q \) as we did earlier, we will take \( q \) to be a modulus of a two-punctured sphere and,
choosing the coordinate near the north pole of the two-punctured sphere \( w \) and the coordinate
near the south pole \( q/w \), sew it between the two three-punctured spheres by \( z_1 w = 1 \)
and \( z_2 q/w = 1 \). Thus when a puncture \( P \) on one of pants-shaped regions approaches a
boundary, it moves from a three-punctured sphere onto the two-punctured sphere that

\[ \text{I would like to thank Jacques Distler for pointing this out.} \]
Fig. 4: Pants decomposition of a genus three surface.

Fig. 5: Sewn spheres in one region of a pants-decomposition. The three-punctured sphere is one of the pants-shaped regions of the surface, while the two-punctured spheres contain the moduli of the plumbing fixtures that sew the region to the rest of the surface. The ‘x’ represents an insertion. As the location of the insertion moves into one of the dotted circles, it moves through the plumbing fixture onto the adjacent sphere.

Handles are created by sewing together two punctures on the same sphere. Varying the $q_i$ in the plumbing fixtures then corresponds to varying the $3g - 3$ moduli of the surface. Actually, we have to consider all distinct pants decompositions of a surface, each
giving rise to a cell in moduli space. The boundaries between the cells can be thought of as corresponding to replacing two three-punctured spheres by one four-punctured sphere. This is depicted for a genus two surface in Fig. 6.

![Fig. 6: Different cells of a pants decomposition for a genus two surface.](image)

The dotted lines in the upper diagrams represent the sewings that correspond to the pants-decompositions shown in the lower diagrams. (The two-punctured spheres for the sewing have been suppressed.) The middle diagram is the boundary between the two decompositions.

With these preliminaries out of the way, we can now start to think about integrating a single puncture over a surface. To do this, we have seen that we need a ‘good’ coordinate family over the entire surface. The pants-decomposition has provided us with a set of punctured spheres all sewn together. We will thus be able to integrate the puncture over the entire surface if we can provide a ‘good’ coordinate family over each of the punctured spheres that behaves nicely as we move from one sphere to another, and if the resulting family is continuous across the cell boundaries of moduli space (different pants-decompositions correspond to different cells). The first part can be accomplished by an easy generalization of the coordinates used in section 3. And the second part will require a ‘good’ coordinate family on $\mathcal{M}_{0,n}$. But this too is an easy generalization of the coordinate family on $\mathcal{M}_{0,4}$.

For simplicity, we will focus on a sphere on which each of its punctures is sewn to a puncture on another sphere. Our discussion could be expanded to include the case where two punctures on a sphere are sewn together to form a handle. Our sphere is sewn to three
two-punctured spheres each of which contain one of the moduli of the surface (i.e., a $q_i$). When the puncture $P$ is far from one of the sewn punctures, its coordinates are the same as before:

$$h(|r|)(z - r) + (1 - h(|r|))r^2\left(-\frac{1}{z} + \frac{1}{r}\right). \quad (4.1)$$

This is independent of the moduli of the surface. On the other hand, when $P$ approaches any one of the sewn punctures, we can think of it as having moved through the neck of the plumbing fixture onto a two-punctured sphere. In this region the coordinate is

$$g_i(|r_i|)(w_i - r_i) + (1 - g_i(|r_i|))r_i^2\left(-\frac{q_i}{w_i} + \frac{1}{r_i}\right), \quad (4.2)$$

and it depends on one of the $q_i$. It should be noted, however, that even in this region the coordinates are independent of most of the surface’s moduli. In (4.2), $w_i$ is a coordinate for one of the $3g - 3$ two-punctured spheres, and $r_i$ is the location of the puncture on this plumbing fixture sphere. Finally, to get a ‘good’ coordinate family in this region, we just have to interpolate between the all of the different behaviors. So there will be three interpolating functions that will depend on the differences between $r$ and positions of the three sewn functions. Then the coordinate family can be patched together using these functions; each interpolation will connect a family like the one in (4.1) to one as in (4.2). This is somewhat messy to write down, but the principle is clear enough. One should picture the puncture wandering around on a three-punctured sphere and moving over to one of the two-punctured spheres through the neck of a plumbing fixture when it wanders too close to one of the sewn punctures.

Another essential ingredient is a coordinate family appropriate for $\mathcal{M}_{0,n}$. This is needed for both when there are other operators inserted onto the surface and for demonstrating that our family is continuous across the cell boundaries in moduli space. But this presents no new problems. We proceed as before, except that now we have to interpolate whenever any one of the $n$ punctures approaches another. Again this would be messy to write down, but the principle is clear enough. The important point is that a ‘good’ coordinate family can be written done as before, one in which the coordinates for our inserted puncture is independent of the moduli for the other punctures when it is far from them, and in which, as two punctures collide, it goes over to the coordinates given by the plumbing fixture.

As mentioned earlier, the cell boundaries can be represented as a transition between two different pants-decompositions through a multi-punctured sphere. Our coordinate
family behaves smoothly through this transition and it always remains ‘good’, so we have a prescription that works globally on the moduli space. Finally, if there were other punctures on the surface, then we could just use the coordinate family for $\mathcal{M}_{0,n}$ on the many-punctured spheres that now result from the pants-decomposition. This would have extra interpolations as the puncture $P$ neared the other punctures.

To recap, our prescription for constructing a ‘good’ coordinate family on a surface for inserting a puncture is given by first decomposing the surface into a set of punctured spheres that are sewn together by using a plumbing fixture with two-punctured spheres. Coordinates are then chosen by merely interpolating between the coordinates appropriate for the different regions of the surface as the point $P$ wanders around the surface and appropriate for regions where $P$ approaches other punctures. By construction, the coordinate family is independent of the moduli that are in some sense ‘far’ from $P$. The family is continuous between cell boundaries because it is well-behaved on the many-punctured spheres that provide the transition between different pants-decompositions.

One could now go on and calculate dilaton correlation functions. It is clear that they would go through as for the two-point function on the sphere. Once again the measure will be compactly supported in the regions where the interpolations are taking place and Stokes’ Theorem will be easily applied. With this sketch of how one could put the bosonic calculations on firmer footing complete, we now turn to dilaton contact terms in the heterotic string.

5. Dilaton Contact Terms in the Heterotic String

5.1. The Geometry

Now let us turn to the heterotic case. Once again we adopt the viewpoint of the contact interaction as being the degeneration of a surface with the points corresponding to our two operators pinching off from the rest of the surface. The coordinates for this region of moduli space are now provided by the standard heterotic plumbing fixture. A new feature is that the three-punctured super-sphere is no longer rigid: it has one odd modulus associated to it. Again we will match the coordinates appropriate for this region onto the coordinates appropriate away from the compactification divisor. In this case, we will match up to a superconformal normal-ordered family of coordinates. We will then calculate the pushforwards necessary for the construction of a good string measure.
Before proceeding, we will give a brief review of the necessary geometry. For a more complete treatment see [15]. We recall that a super-Riemann surface (SRS’s) can be constructed via a patch definition. That is, locally our surface looks like a region of the $\mathbb{C}^{1|1}$ plane for which we use the coordinates $(z, \theta; \bar{z}, \bar{\theta})$. Since we are interested in the heterotic string we will only consider the holomorphic sector and we can always obtain the antiholomorphic sector by taking the complex conjugate and setting all odd variables to 0. These local patches can now be glued together to build up a SRS using superconformal transition functions. These are transformations under which the super-derivative $D_{\theta} = \partial_{\theta} + \theta \partial_{z}$ transforms homogeneously. This requirement means that the new $z', \theta'$ must satisfy

$$D_{\theta} z' = \theta' D_{\theta} \theta'.$$  \hfill (5.1)

To calculate correlation functions it is necessary to introduce punctures on the SRS. In this paper we need only consider super-punctures and not spin punctures since we will only consider operators in the Neveu-Schwarz sector. In fact, in order to construct a measure, we will need our punctures to come equipped with a superconformal coordinate that vanishes at the point. We can thus specify a point on the SRS by giving an even function $\sigma = \sigma(z, \theta)$ and recover the puncture as the point where $\sigma$ and $D_{\theta} \sigma$ vanish. Given a coordinate $\sigma$ we can also construct its odd partner, which we denote by $\bar{\sigma}$, by requiring that $\sigma$ and $\bar{\sigma}$ are superconformally related to $z$ and $\theta$, i.e.,

$$D_{\theta} \sigma = \bar{\sigma} D_{\theta} \bar{\sigma}$$

holds. Then the puncture is given by the vanishing of $\sigma$ and $\bar{\sigma}$ \footnote{The vanishing of $\bar{\sigma}$ is equivalent to the vanishing of $D_{\theta} \sigma$ for they are related by the multiplication of a non-vanishing function. However, it is important that we use $\bar{\sigma}$ and not $D_{\theta} \sigma$ as the coordinates of the point in order to maintain superconformal invariance.}. For completeness, we note that if

$$\sigma(z, \theta) = f(z) + \theta \alpha(z),$$  \hfill (5.2)

then

$$\bar{\sigma}(z, \theta) = \beta(z) + \theta g(z)$$  \hfill (5.3)

with

$$g(z)^2 = f'(z)$$

$$\beta(z) = \frac{\alpha(z)}{g(z)}.$$  \hfill (5.4)
(This is true as long $\alpha(z)$ and $\alpha'(z)$ are proportional to the same odd parameter, which will always be the case for us.) The prime denotes differentiation with respect to $z$. We will sometimes denote the location of the puncture as $[\sigma]$ and we will always just give $\sigma$ as the coordinates of the point, it being understood that we mean the pair $(\sigma, \bar{\sigma})$. If we choose $\sigma = (z - u + \xi \theta)$, then we call such a family ‘superconformal normal ordering’ (SCNO) 

The moduli space of a genus $g$ SRS with $s$ super-punctures is denoted by $\hat{M}_{g,s}$ and is $(3g - 3 + s|2g - 2 + s)$ dimensional. We also need the augmented moduli space, $\hat{P}_{g,s}$, which is the infinite dimensional moduli space of punctured SRS’s with a choice of superconformal coordinate centered at each puncture. There is a natural projection $\pi: \hat{P}_{g,s} \to \hat{M}_{g,s}$ given by forgetting the coordinate at the puncture. A coordinate family is then given by a slice $\sigma: \hat{M}_{g,s} \to \hat{P}_{g,s}$. The measure on $\hat{M}_{g,s}$ is obtained by using $\sigma$ to pull back a measure that is naturally defined on $\hat{P}_{g,s}$. The measure is defined in a way similar to (2.3) for the bosonic case, except here we have odd tangent vectors as well as even ones and the Virasoro action on $P_{g,s}$ is extended to an action of the Neveu-Schwarz algebra on $\hat{P}_{g,s}$. Following the conventions of [11], we take the generators of this algebra to be

$$
l_n \leftrightarrow -z^{n+1} \frac{\partial}{\partial z} - \frac{1}{2} (n + 1) z^n \theta \frac{\partial}{\partial \theta} \quad \quad \quad (5.5)
$$

$$
g_k \leftrightarrow \frac{1}{2} z^{k+\frac{1}{2}} \left( \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z} \right).
$$

The factor of $1/2$ in the definition of the $g_k$ is conventional and it will show up later. Again, we associate states in a Hilbert space to a triple of a surface with a puncture and a superconformal coordinate at the puncture. The action of the generators in (5.5) corresponds to an action on the Hilbert space of states by

$$
\langle \Sigma, z - \epsilon z^{n+1} + \frac{1}{2} \epsilon \theta z^{k+\frac{1}{2}} | = \langle \Sigma, z| (1 + \epsilon L_n + \epsilon G_k) + \cdots. \quad (5.6)
$$

The measure, on $\hat{P}_{g,s}$, is given by (again, in abbreviated form)

$$
\tilde{\Omega}(V_1, V_2, \ldots, \tilde{V}_{3g-3+s}, \tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_{2g-2+s}) = 
\langle \Sigma, (z_1, \theta_1), \ldots | B[v_1] \ldots B[v_{3g-3+s}] | \delta[B[v_1]] \ldots \delta[B[v_{2g-2+s}]] | \psi_1 \rangle_{P_1} \otimes \cdots \otimes | \psi_s \rangle_{P_s}. \quad (5.7)
$$

Here $B(z, \theta) = b(z) + \theta \beta(z)$ and $v(z, \theta) = v_1(z) + \theta v_2(z)$ ($\nu(z, \theta) = \nu_1(z) + \theta \nu_2(z)$) is an even (odd) vector field. The notation $B[v]$ corresponds to

$$
\int d\zeta d\theta B(z, \theta) v(z, \theta).
$$
A similar expression holds for $B[\nu]$. See [11] and [13] for more details. The pulled back measure on $\hat{\mathcal{M}}_{g,s}$ is exactly analogous to the bosonic one given in (2.4). It is shown in [11] that this measure changes only by a total derivative for different choices of $\sigma$ if the states that are inserted obey the WPSC, (2.2). Thus, integrated answers are independent of the choices made. In addition, it is shown that $Q$ again acts as the exterior derivative on $\hat{\mathcal{P}}_{g,s}$.

As in the bosonic case, this general formalism is easily adapted to the computation of heterotic dilaton contact terms. Once again contact terms are best dealt with by making a superconformal transformation from the picture of the two operators colliding to one where they are pinched off from the rest of the surface. There are a few differences here though. We have to use a plumbing fixture suitable for the heterotic string and, more importantly, there is an odd modulus associated to the three-punctured super-sphere that has to be considered.

To begin with, we look at the three-punctured super-sphere with coordinates $w$ and $\xi$. The standard (bosonic) three-punctured sphere is rigid, i.e., it has no moduli associated with it. This is because we can use $SL(2, \mathbb{C})$ invariance to carry our three marked points into three standard points, say 0, 1, and $\infty$. On the super-sphere we have $Osp(2,1)$ invariance and this can be used to fix the bosonic coordinates of our three marked points to be 0, 1, and $\infty$. However, we can only fix two of the three fermionic coordinates. This means that the third unfixed fermionic coordinate is a modulus. The dimension of the moduli space, $\hat{\mathcal{M}}_{0,3}$, is thus $(0|1)$. Using $Osp(2,1)$ freedom, we can locate our three points at $(0,0)$, $(1,\tau)$ and $(\infty,0)$. $\tau$ is the leftover odd modulus of the three-punctured super-sphere. Our standard plumbing fixture will be to locate the point $P$ at 1 and $Q$ at 0 and sew the point at $\infty$ onto the rest of the surface. Good coordinates at $\infty$ are simply $-1/w$ and $\xi/w$. This is will make the sewing simple. Since the states will be inserted at $P$ and $Q$, we should use the most general superconformal holomorphic coordinates the vanish at 1 and 0. At 0, we use the coordinates

$$w + \tilde{a}_1 \tau w \xi + \tilde{a}_2 w^2 + \tilde{a}_3 \tau w^2 \xi + \cdots \quad (5.8)$$

(we give just the even coordinate) and at $(1,\tau)$ we use

$$(w-1+\tau \xi)+a_1 \tau (w-1+\tau \xi)(\xi-\tau)+a_2 (w-1+\tau \xi)^2+a_3 \tau (w-1+\tau \xi)^2(\xi-\tau)+\cdots \quad (5.9)$$

The coefficients $\tilde{a}_i$ and $a_i$ for $i \geq 3$ turn out not to affect the dilaton calculations.
This fixture is attached to the rest of the surface by using the standard sewing prescription. If \((x, \xi')\) and \((y, \psi)\) are the coordinates of the two regions of \(C^{1|1}\) that are being sewn together, then they are joined by

\[
xy = -t^2, \quad x\psi = -t\xi', \quad y\xi' = t\psi,
\]

where \(t\) is the sewing modulus. Notice it is \(t^2\) that plays the role that \(q\) played in the bosonic case. Also, the counting of the moduli works out: \(P\) and \(Q\) each have an even and an odd modulus and they are replaced by the moduli of the attachment point (which we again take to be \(Q\)) and the one even modulus of the sewing and the odd modulus of the three-punctured super-sphere, \(\tau\).

Ideally, one would like to do the dilaton two-point function on the sphere in a way analogous to the bosonic case. However, it turns out to be too tedious and complicated. Thus, we will proceed by carrying out a local, one patch calculation, and assume that a ‘good’ coordinate system similar to the one constructed in Section 4 could also be constructed for the heterotic case. One shortcoming of this approach is that we will miss total derivatives with respect to the moduli of \(Q\), preventing us from easily calculating everything that we would like. However, all is not lost and we will still be able to show that the tachyon divergence is absent. Moreover, by using arguments involving the decoupling of genuine BRST-exact states (which we know to be true on general grounds), we will be able to demonstrate that the dilaton equation works in the heterotic string, even when more than one dilaton is in the correlation function.

In the bosonic case, the dilaton was given in (2.6). Similarly, for the heterotic string the dilaton is given by

\[
|D\rangle = 2(Q + \bar{Q})(c_0 - \bar{c}_0)\delta(\gamma_{1/2})|0\rangle
\]

It is convenient to work with \(|D_1\rangle\) and \(|D_2\rangle\), defined in [1], instead, where

\[
|D_1\rangle = Qc_0\delta(\gamma_{1/2})|0\rangle = -\frac{1}{2}c_1\gamma_{-1/2}\delta(\gamma_{1/2})|0\rangle
\]

\[
|D_2\rangle = \bar{Q}\bar{c}_0\delta(\gamma_{1/2})|0\rangle = -2c_1\bar{c}_1\bar{c}_{-1/2}\beta_{-1/2}\delta(\gamma_{1/2})|0\rangle.
\]

Then \(|D\rangle\) is simply \(^6\)

\[
|D\rangle = 2(|D_1\rangle + |D_2\rangle).
\]

\(^6\) The factor of 2 is conventional and is chosen so that the one-point function of \(|D\rangle\) on the sphere is normalized to \(4\pi i Z = -2\pi i \chi Z\) as in [1]. This differs from the conventions in [11].
It was shown in [11] that the orthogonal combination $|D_1⟩ - |D_2⟩$ is a global total derivative and decouples from all correlation functions. This is exactly analogous to the situation described earlier in the discussion of the bosonic string dilaton. Once again, this state is $Q$ of something that obeys the WPSC and hence gives rise to a globally defined total derivative. Thus, one can work equally well with $|D⟩$, $4|D_1⟩$, or $4|D_2⟩$ since they all differ by adding in multiples of $|D_1⟩ - |D_2⟩$ which contributes a genuine global total derivative. Notice that $|D_1⟩$ and $|D_2⟩$ are distinct operators whereas in the bosonic string the corresponding states were merely the barred and un-barred versions of the same state, $2c_1c_{-1}|0⟩$ and $2\bar{c}_1\bar{c}_{-1}|0⟩$. In the bosonic case, it was the contact interaction between the holomorphic and antiholomorphic pieces of the dilaton that produced the tachyonic divergence [1]. In [3] it was seen that this can be avoided by working with the purely holomorphic dilaton only. From this one might expect that the tachyon divergence would appear in the $|D_1⟩|D_2⟩$ contact term. This is precisely the term that will be easiest to calculate (as well as $|D_2⟩|D_1⟩$), and, indeed divergent terms appear. However, they will be killed by the integration over the odd moduli and there is no tachyon divergence.

It is somewhat unfortunate that the local representation of the curvature used below turns out to obscure the total derivatives in the moduli of the point $Q$ appearing in the $|D_1⟩|D_1⟩$ or $|D_2⟩|D_2⟩$ contact terms which are known to be present on general grounds. In particular, it is found that the $|D_1⟩|D_1⟩$ and $|D_2⟩|D_2⟩$ terms depend on the higher curvature coefficients in the local expansion of the coordinates introduced below, while the $|D_1⟩|D_2⟩$ and $|D_2⟩|D_1⟩$ terms do not. But this is not a serious hindrance since we can use the decoupling of $|D_1⟩ - |D_2⟩$ to our advantage, as will be seen below.

5.2. Coordinates

With these preliminaries out of the way, we can now adapt the formalism to our specific calculation. We work in one patch where the coordinates interpolate from those appropriate to the two states far apart to those appropriate for them approaching each other. Things will be simplest if we put the curvature around only one of the points ($Q$ here). Then integrating over the position of $P$ will merely give us only the contact term since the dilaton doesn’t couple to flat backgrounds. So, using the coordinates $(z, \theta)$ for the worldsheet and letting $(\vec{r}, \vec{\rho})$ and $(r, \rho)$ be the positions of $P$ and $Q$, respectively, we choose coordinates

$$\zeta_P(\cdot) = z(\cdot) - \vec{r} + \vec{\rho}\theta(\cdot)$$  

(5.13)
at $P$ and

$$
\zeta_Q(\cdot) = (z(\cdot) - r + \rho\theta(\cdot)) + \rho r R_1 (z(\cdot) - r + \rho\theta(\cdot)) (\theta(\cdot) - \rho) + \\
\bar{r} R_2 (z(\cdot) - r + \rho\theta(\cdot))^2 + \rho \bar{r} R_3 (z(\cdot) - r + \rho\theta(\cdot))^2 (\theta(\cdot) - \rho) + \\
\bar{r} R_4 (z(\cdot) - r + \rho\theta(\cdot))^3 + \cdots
$$

(5.14)

at $Q$, where the coefficients $R_i$ make our coordinate slice non-holomorphic. What we have given is a small $r$ expansion of the coordinates in a curved background in which the mixing of holomorphic and antiholomorphic coordinates is akin to the effect of curvature in Polchinski’s scheme [14]. This is our local curvature picture, and the fact that we have done a small $r$ expansion will obscure some total derivatives that we know are there on general grounds. In the next section, we will find that $R_1$ and $R_2$ are related to the scalar curvature by demanding that the one-point function $\langle (D_1 - D_2) \rangle$ vanish on the sphere and that the one-point function of $\langle (D_1 + D_2) \rangle$ be properly normalized. The higher curvature coefficients cannot be determined in this way. And, as stated above, it turns out that only the $|D_1)|D_2\rangle$ and $|D_2)|D_1\rangle$ terms are independent of the $R_i, i \geq 3$.

When the points are close together, coordinates should go over to those given by the plumbing fixture. The sewing in (5.10) gives (see Fig. 7)

$$
w = \frac{\zeta_Q}{t^2}, \quad \xi = -\frac{\bar{\zeta}_Q}{t}.
$$

(5.15)

Substituting this into the coordinates given in (5.8) and (5.9) results in

$$
\phi_Q(\cdot) = \frac{\zeta_Q(\cdot)}{t^2} + \bar{a}_1 \tau \frac{\zeta_Q(\cdot) \bar{\zeta}_Q(\cdot)}{t^3} + \bar{a}_2 \frac{\zeta_Q(\cdot)^2}{t^4} + \cdots
$$

(5.16)

$$
\phi_P(\cdot) = \Sigma(\cdot) + a_1 \tau \Sigma(\cdot) \bar{\Sigma}(\cdot) + a_2 \Sigma(\cdot)^2 + \cdots,
$$

with

$$
\Sigma(\cdot) = \left( \frac{\zeta_Q(\cdot)}{t^2} - 1 - \frac{\tau \bar{\zeta}_Q(\cdot)}{t} \right)
$$

(5.17)

$$
\bar{\Sigma}(\cdot) = \left( \frac{\bar{\zeta}_Q(\cdot)}{t} + \tau \right).
$$

Also, there is a sign that has been absorbed by redefining the arbitrary odd-indexed $a_i$'s and $\bar{a}_i$'s.

Since we are just trying to pick out the contact term in this calculation, we will take $r, \rho, t, \text{and } \tau$ to be the moduli. Thus we have to eliminate $\bar{r}$ and $\bar{\rho}$ in favor of $t$ and $\tau$ in $\zeta_P$. This is done by again demanding that $\phi_P(P) = 0$ or that $\zeta_Q(P) = t^2$ and $\bar{\zeta}_Q(P) = -\tau t$. In
Fig. 7: Heterotic plumbing fixture with coordinates. The construction is similar to that in Fig. 1, but now there is an odd modulus, $\tau$, associated with the three-punctured super-sphere.

In addition, the contact terms will only be proportional to the curvature in the lowest order so we will only keep terms that are linear in $R_1$ and $R_2$. We find that

$$
\tilde{r} = r + t^2 + t\rho\tau + \tilde{r}R_1t^3\rho\tau - \tilde{r}R_2t^3\rho\tau - \tilde{r}R_2t^4 + \cdots
$$

$$
\tilde{\rho} = \rho - \tau t + \tilde{r}R_1\rho t^2 + \tilde{r}R_2\rho t^3 + \cdots.
$$

(5.18)

With the above definitions and relations in mind, the coordinate slice for our calculation will just be the linear interpolation between the coordinate regions in (5.13), (5.14), and (5.16). The interpolation function is $f(|t|)$ and goes smoothly from 0 to 1 as $|t|$ goes from 0 to $\infty$. Thus,

$$
\sigma_P(\cdot) = \frac{f(|t|)}{t^2} \zeta_P(\cdot) + (1 - f(|t|))\phi_P(\cdot)
$$

$$
\sigma_Q(\cdot) = \frac{f(|t|)}{t^2} \zeta_Q(\cdot) + (1 - f(|t|))\phi_Q(\cdot).
$$

(5.19)

The reader is reminded that there are odd coordinate functions that go along with these expressions that can be found using (5.2)-(5.4). One final note: We have been displaying the coordinates for the holomorphic sector. The appropriate expressions for the antiholomorphic sector are obtained by simply setting all odd parameters to zero in the holomorphic expressions and complex-conjugating the resulting expressions.

5.3. The Measure

With coordinates in hand, we can now calculate the $b$-insertions that are required to produce the measure. The method is identical to that in the bosonic case, with appropriate
generalizations. The pushforwards are given by

$$\sigma_*(\frac{\partial}{\partial m}) = \frac{\partial \sigma}{\partial m} \frac{\partial}{\partial \sigma} + \frac{\partial \bar{\sigma}}{\partial m} \frac{\partial}{\partial \bar{\sigma}} + \frac{\partial \bar{\sigma}}{\partial m} \frac{\partial}{\partial \bar{\sigma}}$$  \hspace{1cm} (5.20)$$

where \(m\) is one of \(r, \bar{r}, \rho, t, \bar{t}, \) or \(\tau\). There is no \(\bar{\sigma}\) term since we are doing the heterotic string. The corresponding operator insertions that form the measure are found by folding pushforwards with the \(B\)’s. Since the resulting insertions are quite long, we have displayed them in Appendix B. In addition, the calculation is done at \(r = \bar{r} = 0\) for convenience. Thus the pushforwards are calculated by first differentiating and then setting \(r\) and \(\bar{r}\) to 0. Finally, the insertions appropriate for each case of interest can be identified and then picked out of the measure\(^7\). The insertions for the various states are

- \(b_{-1} \bar{b}_{-1} \delta[\beta_{-\frac{3}{2}}]\) for \(|\psi\rangle\)
- \(b_{-1} \beta_{\frac{1}{2}} \delta[\beta_{-\frac{3}{2}}]\) for \(|D_1\rangle\).
- \(b_{-1} \bar{b}_{-1} \bar{\delta}[\beta_{-\frac{3}{2}}]\) for \(|D_2\rangle\).

These are the only combinations of ghost insertions that contribute. (Although later we will see that there are terms with additional derivatives of delta-functions that also contribute. These additional contributions are formally equivalent to the above insertions.) It should be noted that for \(|\psi\rangle\) (a SPS), the integration over the odd moduli would give zero if it were not for the fact that the state \(\langle \Sigma | \) itself depends on the odd moduli. In fact, using (5.6), \(\langle \Sigma, \sigma_P \rangle = \langle \Sigma, \bar{\sigma}_{P_0} \rangle \left(1 + 2G_{-\frac{1}{2}}^F - 2t\tau G_{-\frac{1}{2}} + \cdots \right) t^{2L_0} \bar{t}^{2L_0},\) \hspace{1cm} (5.22)$$

where \(\bar{\sigma}_{P_0}\) is \(\sigma_P\) with \(\rho\) and \(\tau\) set to zero and with an overall factor of \(t^2\) removed. This is because the coordinates in (5.19) and (5.13) and (5.14) differ by an overall factor of \(t^2\). These factors of \(t^{2L_0}\) matter for a SPS because the state \(b_{-1}^{(Q)} \bar{b}_{-1}^{(Q)} \delta[\beta_{-\frac{3}{2}}] |\psi\rangle\)

is not annihilated by \(L_0\) and in fact has \(L_0 = 1/2\) and \(\bar{L}_0 = 1\). The dilaton, on the other hand, is annihilated by the \(L_0\) and \(\bar{L}_0\) and no such scale factors are necessary. Furthermore, the \(G_{-\frac{1}{2}}\) combines with the \(\delta[\beta_{-\frac{3}{2}}]\) to give a picture changing operator when inserting a

\(^7\) There are many terms that contribute and \textit{Mathematica} was quite useful here as well.
SPS. However, there are no contributions involving the picture-changing operator for the dilatons [11]. Finally, the expansion of the state was done at \( r = \bar{r} = 0 \).

As an example, let’s calculate the insertions for \(|\psi\rangle, |D_1\rangle, \) and \(|D_2\rangle \) in the background given in (5.14). We find (renaming \( \zeta, \sigma \))

\[
\begin{align*}
\sigma_*\left[ \frac{\partial}{\partial r} \right] &= -1 + R_2 \sigma^2 + \cdots \\
\sigma_*\left[ \frac{\partial}{\partial r} \right] &= R_2 \sigma^2 + R_1 \rho \sigma \bar{\sigma} - 1 + \cdots \\
\sigma_*\left[ \frac{\partial}{\partial \rho} \right] &= \rho + \bar{\sigma} + \cdots,
\end{align*}
\]

where the -1 in the \( \bar{r} \) pushforward is from the barred part and the terms from derivatives of \( \bar{\sigma} \) have been omitted (they always just give the second half of the superconformal vector fields and we can read off the insertions without considering them). These give rise to

\[
\begin{align*}
B[\sigma_*\left[ \frac{\partial}{\partial r} \right]] &= -b_{-1} + R_2 \bar{b}_1 + \cdots \\
B[\sigma_*\left[ \frac{\partial}{\partial r} \right]] &= R_2 b_1 - 2R_1 \rho \beta_{\frac{1}{2}} - \bar{b}_{-1} + \cdots \\
B[\sigma_*\left[ \frac{\partial}{\partial \rho} \right]] &= -\rho b_{-1} + 2\beta_{-\frac{1}{2}} + \cdots,
\end{align*}
\]

where we have kept only the contributing terms. Picking off the measure contribution for \(|D_1\rangle \) and \(|D_2\rangle \) (see (5.21)), we have

\[
\begin{align*}
(-b_{-1})(-2\rho \beta_{\frac{1}{2}})\delta[2\beta_{-\frac{1}{2}}]|D_1\rangle \\
&= -\rho R_1 b_{-1} \beta_{\frac{1}{2}} \delta[\beta_{-\frac{1}{2}}]|D_1\rangle \\
&= \frac{1}{2} \rho R_1 b_{-1} \beta_{\frac{1}{2}} \delta[\beta_{-\frac{1}{2}}]|D_1\rangle \\
&= \frac{1}{2} \rho R_1 |0\rangle,
\end{align*}
\]

and

\[
\begin{align*}
(R_2 \bar{b}_1)(-\bar{b}_{-1})\delta[-\rho b_{-1} + 2\beta_{-\frac{1}{2}}]|D_2\rangle \\
&= \frac{1}{4} R_2 \bar{b}_1 \bar{b}_{-1} \rho b_{-1} \delta'[\beta_{-\frac{1}{2}}]|D_2\rangle \\
&= -\frac{1}{2} \rho R_2 \bar{b}_1 \bar{b}_{-1} \delta'[\beta_{-\frac{1}{2}}]|D_2\rangle \\
&= -\frac{1}{2} \rho R_2 |0\rangle.
\end{align*}
\]
Thus, $R_1 = -R_2$ will enforce the decoupling of $(|D_1⟩ - |D_2⟩)$ which we know to be true on general grounds \([11]\). Furthermore, $R_1$ and $R_2$ are proportional to the scalar curvature since $2(|D_1⟩ + |D_2⟩)$ is the dilaton.

In addition, we will need the insertions for a SPS in this background. We have, from (5.21) and (5.23),

$$(-b_{-1})(-\bar{b}_{-1})\delta[2\beta_{-\frac{3}{2}}]|\psi\rangle = \frac{1}{2} b_{-1}\bar{b}_{-1}\delta[\beta_{-\frac{3}{2}}]|\psi\rangle. \quad (5.26)$$

The conventional factor of 2 may seem out of place, but it will be cancelled by a similar factor of 2 in the expansion of the state $⟨\Sigma, ζ_Q|$ when the $\rho$ integral is done. These factors of 2 appear because of our convention in defining the $g_k$ in (5.14).

### 6. The Dilaton Equation

All the ingredients have now been assembled to demonstrate the dilaton equation in the heterotic string. We are to compare

$$\int [dt \wedge d\tau][d\rho|\Sigma, σ_P, σ_Q|…B[σ_*(\frac{∂}{∂t})]B[σ_*(\frac{∂}{∂\tau})]\delta[B[σ_*(\frac{∂}{∂ρ})]]]$$

$$\times B[σ_*(\frac{∂}{∂r})]B[σ_*(\frac{∂}{∂\tau})]\delta[B[σ_*(\frac{∂}{∂\rho})]]|Φ_1⟩^P \otimes |Φ_2⟩^Q$$

with the state gotten by integrating over $t, \tau$, and $r$,

$$\int [dr \wedge d\tau|\Sigma, ζ_Q|…B[ζ_*(\frac{∂}{∂r})]B[ζ_*(\frac{∂}{∂\tau})]\delta[B[ζ_*(\frac{∂}{∂ρ})]]|Φ_2⟩^Q. \quad (6.2)$$

Physically this corresponds to integrating over the position of the operator at $P$. The dots represent the insertions for the moduli not associated to the locations of the points $P$ and $Q$ and the $Φ_i$ will be one of $|ψ⟩$ (a SPS), $|D_1⟩$, or $|D_2⟩$. Actually, of the eight possibilities involving at least one dilaton, only four can be easily obtained using the way that we have represented the curvature on the surface. These are $|D_1⟩^P \otimes |ψ⟩^Q$, $|D_2⟩^P \otimes |ψ⟩^Q$, $|D_1⟩^P \otimes |D_2⟩^Q$, and $|D_2⟩^P \otimes |D_1⟩^Q$. They only depend on the coefficients $R_1$ and $R_2$ in (5.14). The other contact terms all depend on the higher $R_i$, a signal that in expanding around $r = 0$, we have made certain total derivatives in $r$ and $\tau$ difficult to see. It should be noted that the same problem occurs in the bosonic string if one tries to do that calculation in the same way. We consider each of the four terms that we can easily calculate in turn.
6.1. Dilaton-Strong Physical State

As in the bosonic case, the \(|D_1|^P \otimes |\psi|^Q\) and \(|D_2|^P \otimes |\psi|^Q\) calculations determine the values of \(a_1\) and \(a_2\) that are needed to make the dilaton equation work for correlation functions with one dilaton and the rest SPS’s. This is exactly analogous to the calculation done in the background in (5.14), which determined the relationship between \(R_1\) and \(R_2\), but now the insertions appropriate for the coordinates in (5.19) are used. The corresponding ghost insertions are given in Appendix B.

The insertions that reduce \(|D_1|^P \otimes |\psi|^Q\) to \(|0|^P \otimes |\psi|^Q\) are simply those given in (5.21),

\[
b_{-1}^{(P)} \delta_{-\frac{1}{2}}^{(P)} \frac{\delta[\beta^{(P)}_1]}{b_{-1}^{(Q)} b_{-1}^{(Q)} \delta[\beta^{(Q)}_1]}.
\]

Thus, (6.1) becomes

\[
\int [dt \wedge d\overline{t}] [d\tau \wedge d\overline{\tau}] |d\rho]
\times \left\langle \Sigma, \sigma_P, \sigma_Q \right| \left( -\frac{a_1 \tau f'(|t|)}{2 |t|^5} \right) b_{-1}^{(P)} \beta_{-\frac{1}{2}}^{(P)} \delta[\beta^{(P)}_1] b_{-1}^{(Q)} b_{-1}^{(Q)} \delta[\beta^{(Q)}_1] |D_1|^P \otimes |\psi|^Q. \tag{6.3}
\]

This expression is to be integrated over \(t, \overline{t}\), and \(\tau\), leaving the SPS inserted at \(Q\) with its ghost insertions. To show the dilaton equation, this state is to be compared with the one corresponding to the SPS inserted with the coordinate \(\zeta_Q\) in (5.14). Recalling (5.22) and the discussion following it, and using (5.12), (6.3) becomes

\[
\int [dt \wedge d\overline{t}] [d\tau \wedge d\overline{\tau}] |d\rho] \left\langle \Sigma, \overline{\sigma}_{Q_0} \right| (1 + 2 \rho G_{-\frac{1}{2}}^{(Q)}) \cdots
\times \left( \frac{a_1 \tau f'(|t|)}{2 |t|} \right) \left( \frac{1}{2} b_{-1}^{(P)} \beta_{-\frac{1}{2}}^{(P)} \delta[\beta^{(P)}_1] b_{-1}^{(Q)} b_{-1}^{(Q)} \delta[\beta^{(Q)}_1] c_1^{(P)} \gamma_{-1/2}^{(P)} \delta(\gamma_{1/2}) |0|^P \otimes |\psi|^Q \right. \tag{6.4}
\]

where the factor of 1/2 is picked out so that the measure for the SPS insertion agrees with that given in (5.26), and the dots represent suppressed ghost insertions. The \(b\)-insertions at \(P\) reduce the dilaton to the vacuum, which can be erased. An additional subtlety is that we have to move \(\rho \tau\) to the front of the expression so that we can integrate over \(\tau\). Thus we move \(\rho G_{-\frac{1}{2}}^{(Q)}\) through the ghost insertions, move the \(\tau\) through the \(G_{-\frac{1}{2}}^{(Q)}\), picking up a minus sign and then move the \(\rho \tau\) back through the ghost insertions. This results in

\[
- \int [dt \wedge d\overline{t}] [d\tau \wedge d\overline{\tau}] d\rho \rho \tau \left\langle \Sigma, \overline{\sigma}_{Q_0} \right| \left( \frac{a_1 f'(|t|)}{2 |t|} \right) \left( \frac{1}{2} (2 G_{-\frac{1}{2}}^{(Q)} b_{-1}^{(Q)} b_{-1}^{(Q)} \delta[\beta^{(Q)}_1]) |\psi|^Q. \tag{6.5}
\]
The insertions at $Q$ are the same as the ones that would result from using the coordinate $\zeta_Q$. Furthermore, we can integrate over $\tau$ and do the $t$ integral, recalling that $dt \wedge d\bar{t} = -2i|t|d|t| \wedge d\theta$. The result is

$$2\pi ia_1 \int [dr \wedge d\bar{r}]d|\rho| \langle \Sigma, \zeta_Q | \frac{1}{2}b^{(Q)}_{-1} \bar{b}^{(Q)}_{-1} \delta[\beta_{-\frac{1}{2}}]\rangle^Q.$$

Similarly, $|D_2\rangle_P \otimes |\psi\rangle^Q$ gives

$$\int [dt \wedge d\bar{t}]d\tau [dr \wedge d\bar{r}]d|\rho| \langle \Sigma, \bar{\sigma}_Q | \left( \frac{-a_2\tau f'(|t|)}{8|t|^3} \right) b^{(P)}_1 \bar{b}^{(P)}_{-1} \delta[\beta_{-\frac{1}{2}}] b^{(Q)}_{-1} \bar{b}^{(Q)}_{-1} \delta[\beta_{-\frac{1}{2}}] |D_2\rangle_P \otimes |\psi\rangle^Q$$

$$= \int [dt \wedge d\bar{t}]d\tau [dr \wedge d\bar{r}]d|\rho| \rho \tau \langle \Sigma, \bar{\sigma}_Q | \left( \frac{a_2\tau f'(|t|)}{2|t|} \right) \frac{1}{2} (2G^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{-1} \bar{b}^{(Q)}_{-1} \delta[\beta_{-\frac{1}{2}}] |\psi\rangle^Q$$

$$= 2\pi ia_2 \int [dr \wedge d\bar{r}]d|\rho| \langle \Sigma, \zeta_Q | \frac{1}{2}b^{(Q)}_{-1} \bar{b}^{(Q)}_{-1} \delta[\beta_{-\frac{1}{2}}]\rangle^Q. \tag{6.6}$$

Recalling that the dilaton $|D\rangle = 2(|D_1\rangle + |D_2\rangle)$, the state $4|D_1\rangle$, and the state $4|D_2\rangle$ are all equivalent in correlation functions, we see that the choice $a_1 = a_2 = -1/4$ will give the correct normalization for the dilaton contact term in (2.3), at least when all of the other operators are SPS’s. The final piece is to check that the dilaton-dilaton contact terms behave appropriately. This is done in the following section.

### 6.2. Dilaton-Dilaton

We now turn to $|D_1\rangle_P \otimes |D_2\rangle^Q$ and $|D_2\rangle_P \otimes |D_1\rangle^Q$. Both calculations are presented so that we can check that our answer is consistent, i.e., that it does not matter which dilaton we put at $P$ and which we put at $Q$. Also, it allows us to explicitly check the decoupling of $|D_1\rangle - |D_2\rangle$. The calculations are almost identical to those of the last section, except that there are now many more ghost insertions that contribute and each can be gotten by selecting the insertions from the pushforwards in several different ways.

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There are five different combinations of ghost insertions that appear in the measure that reduce $|D_1⟩^P \otimes |D_2⟩^Q$ to $|0⟩^P \otimes |0⟩^Q$:

\[
\begin{align*}
&b^{(P)}_{-\frac{1}{2}} \beta^{(P)}_{-\frac{1}{2}} \delta^{[\beta^{(P)}]} \left( b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]} \\
&b^{(P)}_{-\frac{1}{2}} \beta^{(P)}_{-\frac{1}{2}} \delta^{[\beta^{(P)}]} \left( b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]} \\
&b^{(P)}_{-\frac{1}{2}} \beta^{(P)}_{-\frac{1}{2}} \delta^{[\beta^{(P)}]} \left( b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]} \\
&b^{(P)}_{-\frac{1}{2}} \beta^{(P)}_{-\frac{1}{2}} \delta^{[\beta^{(P)}]} \left( b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]} \\
&b^{(P)}_{-\frac{1}{2}} \beta^{(P)}_{-\frac{1}{2}} \delta^{[\beta^{(P)}]} \left( b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]}.
\end{align*}
\]

All other possible ghost insertions do not appear for the coordinate system that we are using. Each of the above combinations also occurs several times in the expansion of the measure. After carefully accounting for all relative minus signs for ordering and moving $\rho \tau$ to the front, and using the formal rules $x\delta'(x) = -\delta(x)$ and $x^2\delta''(x) = 2\delta(x)$\(^8\), we find that (5.1) becomes

\[
\int \left[ dt \wedge d\overline{\tau} \right] \left[ dr \wedge d\overline{\rho} \right] \langle \Sigma, \sigma_Q \rangle
\]

\[
\times \left( \frac{\rho a_1 R_2 f'(|t|)}{4|t|} + \cdots \right) \left( \frac{\rho a_1 R_2 f''(|t|)}{8|t|} \right) b^{(P)}_{-\frac{1}{2}} \beta^{(P)}_{-\frac{1}{2}} \delta^{[\beta^{(P)}]} \left( b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]} \langle \Sigma, \sigma_Q \rangle
\]

\[
\int \left[ dt \wedge d\overline{\tau} \right] \left[ dr \wedge d\overline{\rho} \right] \langle \Sigma, \sigma_Q \rangle
\]

\[
\left( \frac{\rho a_1 R_2 f'(|t|)}{8|t|} \right) b^{(P)}_{-\frac{1}{2}} \beta^{(P)}_{-\frac{1}{2}} \delta^{[\beta^{(P)}]} \left( b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]} \langle \Sigma, \sigma_Q \rangle
\]

\[
= \int \left[ dt \wedge d\overline{\tau} \right] \left[ dr \wedge d\overline{\rho} \right] \langle \Sigma, \sigma_Q \rangle \left( \frac{\rho a_1 R_2 f''(|t|)}{2|t|} \right) \frac{R_2}{4} b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \langle \Sigma, \sigma_Q \rangle
\]

where, again, the factor of $R_2/4$ is chosen to agree with the insertions for $|D_2⟩$ in (5.23). Performing the integrations over $t$ and $\tau$ is now straightforward, with the result

\[
2\pi i a_1 \int \left[ dr \wedge d\overline{\rho} \right] \langle \Sigma \rangle \left( \frac{\rho R_2}{4} b^{(Q)}_{-\frac{1}{2}} b^{(Q)}_{0} \right) \delta^{[\beta^{(Q)}]} \langle D_2⟩^Q.
\]
and, after integrating, we see that the contact term between two dilatons is indeed normalized properly.

Comparing this result to (6.6), we see that the dilaton equation holds in the heterotic string, even when other dilatons are in the correlation functions.

Comparing this to (6.7), we again see that the dilaton-dilaton contact terms are properly normalized. Thus, the dilaton equation in the bosonic string is worthy of further investigation. In this paper we have further examined the properties of dilaton contact terms in string theories. In the case of the bosonic strings we sketched how to construct ‘good’ coordinate families for doing the kinds of calculations contained in this paper on higher genus surfaces. This addresses some of the tacit assumptions in [1]. In addition, we are left with the fact that the dilaton equation in the bosonic string

7. Conclusions

Dilatons and contact terms have had a long and sometimes confusing history, and while this work has shed some light on old issues, it also raises some interesting new points worthy of further investigation. In this paper we have further examined the properties of dilaton contact terms in string theories. In the case of the bosonic strings we sketched how to construct ‘good’ coordinate families for doing the kinds of calculations contained in this paper on higher genus surfaces. This addresses some of the tacit assumptions in [1]. In addition, we are left with the fact that the dilaton equation in the bosonic string

\[
\int [dt \wedge d\tau] [dr \wedge d\rho] \epsilon(\Sigma, \sigma_Q) \left( -\frac{\rho a_2 R_1 f'(|t|)}{4|t|} + \cdots \right)
\]

\[
\times b^{(P)}_1 b^{-1)}_1 \delta'[\beta^{(P)}_1] b^{(Q)}_1 \beta^{(Q)}_1 \delta[\beta^{(Q)}_1] D_2 \otimes D_1^Q
\]

\[
= \int [dt \wedge d\tau] [dr \wedge d\rho] \epsilon(\Sigma, \sigma_Q) \left( \frac{\rho a_2 R_1 f'(|t|)}{2|t|} \right)
\]

\[
\times b^{(P)}_1 b^{-1)}_1 \delta'[\beta^{(P)}_1] b^{(Q)}_1 \beta^{(Q)}_1 \delta[\beta^{(Q)}_1] c_1^{(P)} c^{-1)}_1 \beta^{(P)}_1 \delta[\beta^{(P)}_1] D_2 \otimes D_1^Q
\]

\[
= \int [dt \wedge d\tau] [dr \wedge d\rho] \epsilon(\Sigma, \sigma_Q) \left( -\frac{\rho a_2 f'(|t|)}{2|t|} \right) (-R_1) b^{(Q)}_1 \delta[\beta^{(Q)}_1] D_1^Q,
\]

and, after integrating,

\[
2\pi a_2 \int [dr \wedge d\tau] [d\rho] \epsilon(\Sigma, \sigma_Q) \left( -R_1 \right) b^{(Q)}_1 \delta[\beta^{(Q)}_1] D_1^Q.
\]

Comparing this to (6.7), we again see that the dilaton-dilaton contact terms are properly normalized. Thus, the dilaton equation holds in the heterotic string, even when other dilatons are in the correlation functions.
fails when there is more than one dilaton in a correlation function. It is not clear what the origin of this failure is, but one could speculate the it is due to the tachyon. This is also somewhat dismaying since it implies that while the insertion of one dilaton in a correlation function corresponds to the first variation with respect to the string coupling constant, the insertion of two dilatons is not just the second variation of the string coupling constant. Although we do not take this point up here, we feel that this merits further examination.

In the heterotic case we have seen the absence of the tachyonic divergence and that the dilaton equation is valid. This would seem to strengthen the idea that the failure of the dilaton equation in the bosonic string is in some way due to the tachyon. Thus it would seem that the dilaton can really be viewed as the operator which corresponds to varying the string coupling constant in the heterotic string. In addition, although we have not done so here, it would be straightforward to extend the method for constructing a ‘good’ coordinate family in the bosonic string to the heterotic case.

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Appendix A. Bosonic Dilaton Two-Point Function

We present here the calculation for two dilatons on the sphere. Reading off the $b_{-1}^{(P)} b_{-1}^{(Q)} b_{1}^{(P)} b_{1}^{(Q)}$ contribution from (B.10)(and with some help from Mathematica) gives

\[
\left( - \frac{h(r - \tilde{r}) f'(|r - \tilde{r}|) g'(|r|)}{4\tilde{r} |r| \tilde{r} - r} + \frac{\tilde{a}(1 - f) h r (r - \tilde{r}) f'(|r - \tilde{r}|) g'(|r|)}{4m^2 r |\tilde{r} - r|} \right) + \frac{\tilde{a}(1 - f) hr^2 f'(|r - \tilde{r}|) g'(|r|)}{4m^2 n^2 |\tilde{r} - r|} - \frac{\tilde{a}(1 - f) gr^2 (r - \tilde{r})^2 f'(|r - \tilde{r}|) g'(|r|)}{2m^2 n^2 |\tilde{r} - r|} + \frac{\tilde{a}(1 - f) g r^2 (r - \tilde{r})^2 (\tilde{r} + r) f'(|r - \tilde{r}|) g'(|r|)}{4m^2 n^2 |\tilde{r} - r|}
\]

\[
\frac{\tilde{a} f r f'(|r - \tilde{r}|) h'(|\tilde{r}|)}{4m |\tilde{r}| |\tilde{r} - r|} + \frac{g r (\tilde{r} + r) f'(|r - \tilde{r}|) g'(|r|)}{4m^2 |\tilde{r} - r|} - \frac{\tilde{a} f r g'(|r|) h'(|\tilde{r}|)}{4m |\tilde{r}|} + \frac{\tilde{a}(f - 1) f r g'(|r|) h'(|\tilde{r}|)}{4m^2 |\tilde{r}|}
\]

(A.1)
This can be rewritten as

\[
\left[ \frac{\partial}{\partial \tilde{r}} \left( -\frac{h f}{\tilde{r}} \right) \frac{\partial}{\partial \tilde{r}} \left( -\frac{g}{\tilde{r}} \right) + \frac{\partial}{\partial \tilde{r}} \left( \frac{f g \tilde{r}}{n} \right) \frac{\partial}{\partial \tilde{r}} \left( -\frac{g}{\tilde{r}} \right) + \frac{\partial}{\partial \tilde{r}} \left( \frac{-a n f}{m \tilde{r} (\tilde{r} - r)} \right) \frac{\partial}{\partial \tilde{r}} \left( -\frac{g}{\tilde{r}} \right) + \frac{\partial}{\partial \tilde{r}} \left( -\frac{a \tilde{a} (f - f^2/2) r (\tilde{r} + r)}{m^2 (\tilde{r} - r)} \right) \frac{\partial}{\partial \tilde{r}} \left( -\frac{g}{\tilde{r}} \right) + \frac{\partial}{\partial \tilde{r}} \left( \frac{\tilde{a} f^2 h r}{2 m^2} \right) \frac{\partial}{\partial \tilde{r}} \left( -\frac{h}{\tilde{r}} \right) + \frac{\partial}{\partial \tilde{r}} \left( \frac{-\tilde{a} f^2 \tilde{r}}{2 m (\tilde{r} - r)} \right) \right)
\]

\[
\left[ \frac{\partial}{\partial \tilde{r}} \left( -\frac{g}{\tilde{r}} \right) \right]
\]

where \( v = (mn - gr \tilde{r} (\tilde{r} - r)) \). The integral arising from these insertions can be re-written as the sum of three pieces:

\[
\int \frac{\partial}{\partial \tilde{r}} \left( -\frac{g}{\tilde{r}} \right) d \left[ \left( \frac{-a n f}{\tilde{r} m (\tilde{r} - r)} + \frac{h f}{\tilde{r} n} - \frac{g \tilde{r} f}{n} \right) d \tilde{r} \wedge dr \wedge d\tilde{r} \right] b^{(P)} b^{(Q)} b_1 b_{-1} |D|^P \otimes |D|^Q,
\]

(A.3)

\[
\int \frac{\partial}{\partial \tilde{r}} \left( -\frac{h}{\tilde{r}} \right) d \left[ \left( \frac{-\tilde{a} f^2 \tilde{r}}{2 m (\tilde{r} - r)} \right) d \tilde{r} \wedge dr \wedge d\tilde{r} \right] b^{(P)} b_1 b_{-1} b_1 |D|^P \otimes |D|^Q,
\]

(A.4)

and

\[
\int \frac{\partial}{\partial \tilde{r}} \left( -\frac{g}{\tilde{r}} \right) d \left[ \left( \frac{-a \tilde{a} (f - f^2/2) r (\tilde{r} + r)}{m^2 (\tilde{r} - r)} + \frac{-\tilde{a} f (1 - f) h r}{m^2} + \frac{-\tilde{a} f^2 h r}{2 m^2} + \frac{-\tilde{a} \tilde{a} f^2 r (f - f^2/2) v}{m^2 n^2 (\tilde{r} - r)} \right) d \tilde{r} \wedge dr \wedge d\tilde{r} \right] b^{(P)} b_1 b_{-1} b_1 |D|^P \otimes |D|^Q.
\]

(A.5)

The first term, (A.3), is the same as the contributions in (B.15) that arose in the dilaton-strong physical state contribution. Changing variables to \( y \) in favor of \( \tilde{r} \) and integrating as before just leaves

\[
-2\pi i (-2a - 2) \int dr \wedge d\tilde{r} \frac{\partial}{\partial \tilde{r}} \left( \frac{-g}{\tilde{r}} \right) b^{(Q)} b_1 |D|^Q
\]

The remaining integral is just that in (B.3) for the one-point function of the dilaton. If this were the only contribution, then we would have the dilaton equation with the proper normalization (cf. (B.17)). However, there are the terms in (A.4) and (A.5) which are purely contact terms and they spoil the dilaton equation. It is not clear why these terms arise, but they are there and they agree with the local calculation of Distler and Nelson [12]. The terms in (A.4) and (A.5) can be integrated as the other terms have been. In (A.4), we eliminate \( r \) in favor of \( y \) and integrate (using the same contours as in Fig. 3) and find

\[
-2\pi i \tilde{a} \int d\tilde{r} \wedge d\tilde{r} \frac{\partial}{\partial \tilde{r}} \left( \frac{-h}{\tilde{r}} \right) b^{(P)} b_1 |D|^P
\]

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which is just the proper measure for the insertion of a dilaton at $P$ using the coordinates $\zeta_P$ in (3.4) instead of at $\zeta_Q$ at $Q$. This shouldn’t be bothersome because this is a contact term that occurs when $P$ and $Q$ coincide. Finally, of the four terms in (A.5), the middle two integrate to 0 and the other two give

$$-2\pi i(2a\bar{a} + \bar{a}) \int dr \wedge d\tau \frac{\partial}{\partial r} \left( \frac{-g}{r} \right) b^{(Q)}_{-1} b^{(Q)}_1 |D\rangle^Q.$$  

This is, of course, an insertion of the dilaton at $Q$. Putting these three pieces together gives the dilaton two-point function

$$\langle DD\rangle = -2\pi i(2(\bar{a} - a + a\bar{a}) - 2)\langle D\rangle = -2\pi i(3/2 - 2)\langle D\rangle.$$  

which was the result stated in (3.20).

### Appendix B. Heterotic Ghost Insertions

We present here the ghost insertions that result from computing the pushforwards of the tangent vectors associated to the moduli corresponding to the locations of the points $P$ and $Q$ by the coordinate family given in (5.19). They were computed with the help of Mathematica. The measure is formed by multiplying these six contributions together (including delta functions for the insertions for $\rho$ and $\tau$). In the text we only used the contributions to the measure appropriate for the insertion of strong physical states and the dilatons. The insertions are (keeping only the terms with $R_1$ and $R_2$, and recalling that they are evaluated at $r = \bar{r} = 0$)

$$B[\sigma_* \left( \frac{\partial}{\partial t} \right)] = - \left( \frac{2}{t} \right) b^{(P)}_{-1} - \left( \frac{2}{t} + \frac{4p}{t} \right) b^{(P)}_0 - \left( \frac{2p}{t} - \frac{4p^2}{2t} + \frac{a_2 |t| f'(|t|)}{2t} \right) b^{(P)}_1 + \left( \frac{2\tau}{t} + \frac{4p\tau}{t} \right) \beta^{(P)}_{-1} + \left( \frac{2p\tau}{t} + \frac{4p\tau}{t} - \frac{12pp_1\tau}{t} + \frac{a_1 |t| f'(|t|)}{t} \right) \beta^{(P)}_0 - \left( \frac{2}{t} \right) b^{(Q)}_0 - \left( \frac{2}{t} + \frac{\bar{a}_2 |t| f'(|t|)}{2t} \right) b^{(Q)}_1 + \left( \frac{2\bar{p}_1\tau}{t} - \frac{\bar{a}_1 |t| f'(|t|)}{2t} \right) \beta^{(Q)}_0 - \left( \frac{a_2 f'(|t|)}{2|t|} \right) b^{(P)}_1 - \left( \frac{\bar{a}_2 f'(|t|)}{2|t|} \right) b^{(Q)}_1 + \ldots$$  

(B.1)
\[
B[\sigma_\nu(\frac{\partial}{\partial r})] = - \left( \frac{a_2 t f'(|t|)}{2|t|} \right) b_1^{(P)} + \left( \frac{a_1 t \tau f'(|t|)}{|t|} \right) \beta^{(P)}_\frac{1}{2} - \left( \frac{\bar{a}_2 t f'(|t|)}{2|t|} \right) b_1^{(Q)} + \\
\left( \frac{\bar{a}_1 t \tau f'(|t|)}{|t|} \right) \beta^{(Q)}_\frac{1}{2} - \left( \frac{2}{t} \right) \tilde{b}_1^{(P)} - \left( \frac{2}{t} + \frac{4p}{t} \right) \tilde{b}_0^{(P)} - \\
\left( \frac{2p}{t} - \frac{2p^2}{t} + \frac{a_2 |t| f'(|t|)}{2t} \right) b_1^{(P)} - \left( \frac{2}{t} \right) \tilde{b}_1^{(Q)} - \\
\left( \frac{2\bar{p}}{t} + \frac{\bar{a}_2 |t| f'(|t|)}{2t} \right) \tilde{b}_1^{(Q)} + \ldots
\]

\[
B[\sigma_\nu(\frac{\partial}{\partial \tau})] = - (\tau) b_{-1}^{(P)} - (2p \tau + 2p_1 \tau) b_0^{(P)} + (2p^2 \tau - 2pp_1 \tau - p_1^2 \tau) b_1^{(P)} - \\
(2) \beta^{(P)}_{-\frac{1}{2}} - (2p - 2p_1) \beta^{(P)}_\frac{1}{2} - (\bar{p}^2 \tau) b_1^{(Q)} + (2\bar{p}_1) \beta^{(Q)}_\frac{1}{2} + \ldots
\]

\[
B[\sigma_\nu(\frac{\partial}{\partial r})] = - \left( \frac{1}{t^2} \right) b_1^{(P)} - \left( \frac{2p}{t^2} \right) b_0^{(P)} + \left( \frac{2p^2}{t^2} \right) b_1^{(P)} + \left( \frac{2p_1 \tau}{t^2} \right) \beta^{(P)}_{-\frac{1}{2}} - \left( \frac{6pp_1 \tau}{t^2} \right) \beta^{(P)}_{\frac{1}{2}} - \\
\left( \frac{1}{t^2} \right) b_{-1}^{(Q)} - \left( \frac{2\bar{p}}{t^2} \right) b_0^{(Q)} + \left( \frac{2\bar{p}^2}{t^2} \right) b_1^{(Q)} + \left( \frac{2\bar{p}_1 \tau}{t^2} \right) \beta^{(Q)}_{-\frac{1}{2}} - \left( \frac{6\bar{p}p_1 \tau}{t^2} \right) \beta^{(Q)}_{\frac{1}{2}} + \\
(R_2 \tau^2) \bar{b}_{-1}^{(P)} + (2R_2 (1 - f) \bar{t}^2 + 2R_2 p \bar{f}^2) \tilde{b}_0^{(P)} + \\
(R_2 (1 - f) \bar{t}^2 + 2R_2 (1 + f) p \bar{t}^2 - 2R_2 p^2 \bar{t}^2) b_1^{(P)} + (R_2 \bar{t}^2) \tilde{b}_1^{(Q)} + \ldots
\]

\[
B[\sigma_\nu(\frac{\partial}{\partial \tau})] = (R_2 \tau^2 - 2R_1 \bar{t} \tau \rho) b_{-1}^{(P)} + \\
(2R_2 (1 - f) \bar{t}^2 + 2R_2 p \bar{t}^2 - 2R_1 (1 - f) \bar{t} \bar{t} - 4R_1 \bar{t} \rho \bar{t} \rho + \\
2R_1 \bar{t} \rho \bar{t} \rho) b_0^{(P)} + \\
(R_2 (1 - f) \bar{t}^2 + 2R_2 (1 + f) p \bar{t}^2 - 2R_2 p^2 \bar{t}^2 - 2R_1 (1 - f) \bar{t} \bar{t} \rho + \\
4R_1 \bar{t}^2 \bar{t} \rho + 2R_1 \bar{t} \rho \bar{t} \rho - R_1 f \bar{t} \rho + 2R_1 \bar{t} \rho \bar{t} \rho) b_1^{(P)} - \\
(2R_1 \bar{t} \rho + 2R_2 \bar{t}^2 \rho + 2R_2 p \bar{t}^2 \rho) \beta^{(P)}_{\frac{1}{2}} + \\
(-2R_1 \bar{t} \rho + 2R_1 f \bar{t} \rho - 2R_1 \bar{t} \rho \rho - 2R_2 (1 - f) \bar{t}^2 \tau - 2R_2 p \bar{t}^2 \tau - 2R_2 p \bar{t}^2 \tau - \\
4R_2 f \bar{t} \rho \bar{t}^2 \tau + 6R_2 p \bar{t} \rho \bar{t}^2 \tau) \beta^{(P)}_{\frac{1}{2}} + (R_2 \bar{t}^2 + 2R_1 \bar{t} \rho \bar{t} \rho) b_1^{(Q)} - (2R_1 \bar{t} \rho \bar{t} \rho) \beta^{(Q)}_{\frac{1}{2}} - \\
\left( \frac{1}{t^2} \right) \tilde{b}_1^{(P)} - \left( \frac{2p}{t^2} \right) \tilde{b}_0^{(P)} + \left( \frac{2p^2}{t^2} \right) \tilde{b}_1^{(P)} - \left( \frac{1}{t^2} \right) \tilde{b}_{-1}^{(Q)} - \left( \frac{2\bar{p}}{t^2} \right) \tilde{b}_0^{(Q)} + \\
\left( \frac{2\bar{p}}{t^2} \right) \tilde{b}_1^{(Q)} + \ldots
\]
where $p = a_2(1 - f)$, $p_1 = a_1(1 - f)$, $\tilde{p} = \tilde{a}_2(1 - f)$, and $\tilde{p}_1 = \tilde{a}_1(1 - f)$. The dots represent higher terms ($b_n, n \geq 2; \beta_n, n \geq 3/2$) that do not contribute to the dilaton calculations considered here.
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