MODIFIED JACOBI FORMS OF INDEX ZERO

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Abstract. By modifying a slash operator of index zero we define modified Jacobi forms of index zero. Such forms play a role of generating nearly holomorphic modular forms of integral weight. Furthermore, by observing a relation between the coefficients of Fourier development of a modified Jacobi form we construct a family of finite-dimensional subspaces.

1. Introduction

Let \( \mathcal{H} \) denote the complex upper half-plane \( \{ \tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0 \} \). The letters \( \tau \) and \( z \) will always stand for variables in \( \mathcal{H} \) and \( \mathbb{C} \), respectively. For fixed integers \( k \) and \( m \) \((\geq 0)\) we define two slash operators on a function \( \phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \) as

\[
(\phi|_{k,m} \begin{bmatrix} a & b \\ c & d \end{bmatrix})(\tau, z) := (c\tau + d)^{-k} e^{m\left(-cz^2/c\tau + d\right)} \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) \quad \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{R})\right)
\]

\[
(\phi|_m [\lambda : \mu])(\tau, z) := e^{m(\lambda^2\tau + 2\lambda z + \lambda \mu)} \phi(\tau, z + \lambda \tau + \mu) \quad (\lambda, \mu \in \mathbb{R}^2)
\]

where \( e^m(x) := \exp(2\pi mx) \) (and \( e(x) := e^1(x) \)). Then we have the relations

\[
(\phi|_{k,m} M)|_{k,m} M' = \phi|_{k,m}(MM') \quad (M, M' \in \text{SL}_2(\mathbb{R}))
\]

\[
(\phi|_m X)|_{m} X' = e^{m(\det(XX'))} \phi|_m(X + X') \quad (X, X' \in \mathbb{R}^2)
\]

\[
(\phi|_{k,m} M)|_m (XM) = (\phi|_m X)|_{k,m} M \quad (M \in \text{SL}_2(\mathbb{R}), \ X \in \mathbb{R}^2).
\]

(\( \mathbb{I} \) \( \S 1 \)). From now on, we let \( \Gamma_1 := \text{SL}_2(\mathbb{Z}) \) throughout this paper. A Jacobi form of weight \( k \) and index \( m \) on a subgroup \( \Gamma \subset \Gamma_1 \) of finite index is a holomorphic function \( \phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C} \) satisfying

(i) \( \phi|_{k,m} M = \phi \) for all \( M \in \Gamma \),

(ii) \( \phi|_m X = \phi \) for all \( X \in \mathbb{Z}^2 \),

(iii) for each \( M \in \Gamma_1 \), \( \phi|_{k,m} M \) has a Fourier development of the form

\[
\sum_{n, r \in \mathbb{Z}} c(n, r) q^n \zeta^r \quad (q := e(\tau), \ \zeta := e(z)).
\]

The \( \mathbb{C} \)-vector space of all such functions \( \phi \) is denoted by \( J_{k,m}(\Gamma) \). Then, as is well-known, \( \sum_{k, m} J_{k,m}(\Gamma) \) forms a bigraded ring (\( \mathbb{II} \) Theorem 1.5). And, Eichler-Zagier further developed the following theorems.

**Theorem 1.1 (\( \mathbb{II} \) Theorem 1.3).** Let \( \phi \in J_{k,m}(\Gamma) \) and \( (\lambda, \mu) \in \mathbb{Q}^2 \). Then the function

\[
(\phi|_m [\lambda : \mu])(\tau, 0) = e^m(\lambda^2\tau) \phi(\tau, \lambda \tau + \mu)
\]

is a modular form of weight \( k \) on some subgroup of \( \Gamma \) of finite index depending only on \( \Gamma \) and \( (\lambda, \mu) \) (in the sense of \( \mathbb{II} \)).

**Theorem 1.2 (\( \mathbb{II} \) Theorem 1.1).** \( J_{k,m}(\Gamma) \) is a finite-dimensional space.
On the other hand, a Jacobi form of weight \( k \) and index 0 is independent of \( z \) by the third condition for a Jacobi form, and hence it is simply an ordinary modular form of weight \( k \) in \( \tau \). In this paper we shall first modify the slash operator described in \( \text{(1.1)} \) when \( m = 0 \), and define so called modified Jacobi forms of weight \( k \) and index 0. And we shall obtain an analogue of Theorem 1.1 (Theorem 3.2). This construction of modified Jacobi forms is in fact motivated by the Weierstrass \( \sigma \)-function, which will give us Klein forms as nearly holomorphic modular forms of integral weight by virtue of Theorem 3.2 (Remark 3.5).

We shall further construct certain subspaces of modified Jacobi forms of index 0 and show that they are indeed finite-dimensional (Theorem 4.6), which could be an analogue of Theorem 1.1 (Theorem 3.2).

2. Modification of a slash operator

For a lattice \( L = [\omega_1, \omega_2] = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \) in \( \mathbb{C} \) the Weierstrass \( \wp \)-function is defined by

\[
\wp(z, L) := \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),
\]

and the Weierstrass \( \sigma \)-function is defined as

\[
\sigma(z, L) := z \prod_{\omega \in L - \{0\}} \left( 1 - \frac{z}{\omega} \right)e^{(z/\omega) + (z/\omega)^2/2}.
\]

Taking the logarithmic derivative we come up with the Weierstrass \( \zeta \)-function

\[
\zeta(z, L) := \frac{\sigma'(z, L)}{\sigma(z, L)} = \frac{1}{z} + \sum_{\omega \in L - \{0\}} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).
\]

Differentiating the function \( \zeta(z + \omega, L) - \zeta(z, L) \) for \( \omega \in L \) results in 0 because \( \frac{d}{dz}\zeta(z, L) = -\wp(z, L) \) and the \( \wp \)-function is periodic with respect to \( L \). Hence there is a constant \( \eta(\omega, L) \) satisfying \( \zeta(z + \omega, L) = \zeta(z, L) + \eta(\omega, L) \). So we define the Weierstrass \( \eta \)-function by \( \mathbb{R} \)-linearity, namely, if \( z = r_1 \omega_1 + r_2 \omega_2 \) with \( r_1, r_2 \in \mathbb{R} \), then

\[
\eta(z, L) := r_1 \eta(\omega_1, L) + r_2 \eta(\omega_2, L).
\]

We further define a function \( \psi(z, L) \) on \( \mathbb{C} \) by

\[
\psi(z, L) := \begin{cases} 
-1 & \text{if } z \in L - 2L \\
1 & \text{otherwise}.
\end{cases}
\]

If \( X = (\lambda \mu) \in \mathbb{R}^2 \) is fixed, then the value \( \psi(\lambda \tau + \mu, [\tau, 1]) \) does not depend on \( \tau \in \mathcal{H} \). Therefore we will simply write \( \psi(X) \) for \( \psi(\lambda \tau + \mu, [\tau, 1]) \).

**Lemma 2.1.** Let \( L \) be a lattice in \( \mathbb{C} \).

(i) \( \sigma(z, L), \eta(z, L) \) and \( \psi(z, L) \) are homogeneous of degree 1, \(-1\) and 0, respectively. Namely, for any \( \lambda \in \mathbb{C} - \{0\} \) we have

\[
\sigma(\lambda z, \lambda L) = \lambda \sigma(z, L), \quad \eta(\lambda z, \lambda L) = \frac{1}{\lambda} \eta(z, L) \quad \text{and} \quad \psi(\lambda z, \lambda L) = \psi(z, L).
\]

(ii) If \( L = [\omega_1, \omega_2] \) and \((\lambda \mu) \in \mathbb{Z}^2 \), then

\[
\psi(\lambda \omega_1 + \mu \omega_2, L) = (-1)^{\lambda \mu + \lambda + \mu}.
\]

(iii) Let \( X \in \mathbb{R}^2 \) and \( M \in \text{SL}_2(\mathbb{Z}) \). Then \( \psi(X M) = \psi(X) \).

**Proof.** One can obtain (i) and (ii) directly from the definitions of \( \sigma(z, L), \eta(z, L) \) and \( \psi(z, L) \).
Now, let $X = (\lambda \mu) \in \mathbb{R}^2$ and $M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$. We derive that
\[
\psi(XM) = \psi((\lambda a + \mu c)\tau + (\lambda b + \mu d), [\tau, 1]) \text{ for any } \tau \in \mathcal{H}
\]
\[
= \psi((\lambda a + \mu c)\tau + (\lambda b + \mu d), [\tau + b, c\tau + d]) \text{ by the fact } [a\tau + b, c\tau + d] = [\tau, 1]
\]
\[
= \psi(\lambda(x + b) + \mu(c\tau + d), [\tau + b, c\tau + d])
\]
\[
= \psi\left(\lambda \tau + b \over c\tau + d + \mu, \left[\frac{\lambda \tau + b}{c\tau + d}, 1\right]\right) \text{ by (i)}
\]
\[
= \psi(\lambda\tau + \mu, [\tau, 1]) = \psi(X).
\]
This proves (iii). \qed

**Lemma 2.2** (Legendre Relation). Let $L = [\omega_1, \omega_2]$ be a lattice in $\mathbb{C}$ with $\omega_1/\omega_2 \in \mathcal{H}$. Then we have
\[
\eta(\omega_2, L)\omega_1 - \eta(\omega_1, L)\omega_2 = 2\pi i.
\]

**Proof.** See \[\textit{p}. 241 or \[\textit{p}. 41. \]

Let $k$ be an integer and $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ be a function. We define two operators $|\phi\rangle_k$ and $|\phi\rangle_k'$ on $\phi$ as follows.
\[
|\phi\rangle_k M(\tau, z) := (\phi|k,0\rangle M)(\tau, z) \quad (M \in \text{SL}_2(\mathbb{R}))
\]
\[
|\phi\rangle_k'[\lambda \mu](\tau, z) := \left(\psi(\lambda \mu) \exp(\eta(\lambda\tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda\tau + \mu)))\right)^k
\]
\[
\times (\phi|0\rangle [\lambda \mu](\tau, z) \quad ((\lambda \mu) \in \mathbb{R}^2).
\]

**Proposition 2.3.**

(i) For any $X, X' \in \mathbb{R}^2$ we get
\[
(\phi|\eta\rangle_k X)|\eta\rangle_k X' = \left(\psi(X)\psi(X')\psi(X + X')e\left(\frac{1}{2} \det(X')X\right)\right)^k (\phi|\eta\rangle_k(X + X')).
\]

In particular, if $X, X' \in \mathbb{Z}^2$, then
\[
(\phi|\eta\rangle_k X)|\eta\rangle_k X' = \phi|\eta\rangle_k(X + X').
\]

(ii) For any $M \in \text{SL}_2(\mathbb{Z})$ and $X \in \mathbb{R}^2$ we have
\[
(\phi|\eta\rangle_k X)|\eta\rangle_k(XM) = (\phi|\eta\rangle_k X)|\eta\rangle_k M.
\]

**Proof.** (i) Let $X = (\lambda \mu)$ and $X' = (\lambda' \mu')$. If we set $\omega = \lambda\tau + \mu$ and $\omega' = \lambda'\tau + \mu'$, then
\[
(\phi|\eta\rangle_k X)|\eta\rangle_k X'
\]
\[
= \left(\psi(X)\psi(X')\psi(X + X')e\left(\frac{1}{2} \det(X')X\right)\right)^k \phi(\tau, z + \omega)
\]
\[
= \left(\psi(X)\psi(X')\psi(X + X')e\left(\frac{1}{2} \det(X')X\right)\right)^k \phi(\tau, z + \omega + \omega')
\]
\[
= \left(\psi(X)\psi(X')\psi(X + X')e\left(\frac{1}{2} \det(X')X\right)\right)^k \phi(\tau, z + \omega + \omega')
\]
\[
= \left(\psi(X)\psi(X')\psi(X + X')e\left(\frac{1}{2} \det(X')X\right)\right)^k \phi(\tau, z + \omega + \omega')
\]
\[
= \left(\psi(X)\psi(X')\psi(X + X')e(\pi i(\lambda'\mu - \lambda\mu'))\right)^k \phi(\tau, z + \omega + \omega')
\]
\[
= \left(\psi(X)\psi(X')\psi(X + X')e(\pi i(\lambda'\mu - \lambda\mu'))\right)^k \phi(\tau, z + \omega + \omega')
\]
This yields the first part of (i). Moreover, if $X, X' \in \mathbb{Z}^2$, then
\[
\psi(X)\psi(X')\psi(X + X')e^\pi i(\lambda'\mu - \lambda\mu')
\]
\[
= (-1)^{\lambda\mu + \lambda + \mu}(-1)^{\lambda'\mu + \lambda' + \mu'}(-1)^{(\lambda + \lambda')(\mu + \mu')} + (\lambda + \lambda')(\mu + \mu')(\lambda + \lambda')(\mu + \mu')(-1)^{\lambda\mu - \lambda\mu'} = 1 \text{ by Lemma 2.1(ii)},
\]
which proves the second part of (i).

(ii) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = (\lambda \mu)$ and $(\lambda' \mu') = XM$. Then

\[(\phi|_k^\prime M)|_k'(XM)\]
\[= \left( \psi(XM) \exp(\eta(\lambda' \tau + \mu', [\tau, 1])(z + \frac{1}{2}(\lambda' \tau + \mu'))) \right)^k ((\phi|_{k,0}M)|_0(XM))\]
\[= \left( \psi(X) \exp(\eta(\lambda \tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda \tau + \mu'))) \right)^k ((\phi|_{0,0}X)|_{k,0}M) \text{ by Lemma 2.1(iii) and (1.2).} \]

On the other hand, we achieve that

\[(\phi|_k^\prime X)|_k' M \]
\[= \left( \psi(X) \exp(\eta(\lambda \tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda \tau + \mu'))) \right)^k ((\phi|_{0}X)|_{k,0}M) \]
\[= \left( \psi(X) \exp(\eta(\lambda(\alpha \tau + b) + \mu(\tau + c) + d), [\alpha \tau + b, \tau + c] + d)(z + \frac{1}{2}(\lambda(\alpha \tau + b) + \mu(\tau + c) + d))) \right)^k \]
\[\times (\phi|_{0}X)|_{k,0}M \text{ by Lemma 2.1(i)} \]
\[= \left( \psi(X) \exp(\eta(\lambda' \tau + \mu', [\tau, 1])(z + \frac{1}{2}(\lambda' \tau + \mu'))) \right)^k ((\phi|_{0}X)|_{k,0}M) \text{ by the fact } [\alpha \tau + b, \tau + c + d] = [\tau, 1]. \]

Hence we obtain the assertion (ii). □

Define a function $\rho(\tau, z)$ by

\[\rho(\tau, z) := \exp(\frac{1}{2}\eta(1, [\tau, 1])z^2 - \pi iz).\]

Since $\eta(1, [\tau, 1])$ has the expansion

\[\eta(1, [\tau, 1]) = \frac{(2\pi i)^2}{12} \left( -1 + 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \right)\]

\text{[\textit{ib} p. 249]}, $\rho(\tau, z)$ is a holomorphic function on $\mathcal{H} \times \mathbb{C}$.

**Definition 2.4.** A modified Jacobi form of weight $k$ and index 0 on a subgroup $\Gamma \subset \Gamma_1$ of finite index is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ satisfying

(i) $\phi|_k^\prime M = \phi$ for all $M \in \Gamma$,
(ii) $\phi|_0^\prime X = \phi$ for all $X \in \mathbb{Z}^2$,
(iii) for each $M \in \Gamma_1$, the function $\rho^k(\phi|_k^\prime M)$ has a Fourier development of the form

\[\sum_{n \geq 0} \sum_{|r| \leq r_0(n)} c(n, r)q^n z^r\]

for some increasing sequence $\{r_0(n)\}_{n \geq 0}$ of nonnegative integers such that

\[\frac{r_0(n)}{n} \to 0 \text{ as } n \to \infty.\]

Here we allow that $c(n, r)$ could be zero even if $|r| \leq r_0(n)$.

In what follows, we denote the $\mathbb{C}$-vector space of all such functions $\phi$ by $\langle J_k(\Gamma) \rangle$. And we let $J_* (\Gamma) := \sum_{k \in \mathbb{Z}} J_k(\Gamma)$.

**Proposition 2.5.** If $\Gamma$ is a subgroup of $\Gamma_1$ of finite index, then $J_* (\Gamma)$ is a graded ring.
Proof. Let $\phi \in J_k(\Gamma)$ and $\phi' \in J_{k'}(\Gamma)$ for some $k, k' \in \mathbb{Z}$. If $M \in \Gamma$ and $X \in \mathbb{Z}^2$, then one can readily show by the definitions of slash operators that $(\phi\phi')_{k+k'}M = \phi\phi'$ and $(\phi\phi')_{k+k'}X = \phi\phi'$.

Now let $M \in \Gamma_1$ and suppose that $\rho^{k}(\phi|_kM)$ and $\rho^{k'}(\phi'|_{k'}M)$ have Fourier developments

$$
\sum_{n \geq 0} \sum_{|r| \leq r_0(n)} c(n, r)q^r \zeta^n \quad \text{and} \quad \sum_{m \geq 0} \sum_{|s| \leq s_0(m)} d(m, s)q^s \zeta^m
$$

respectively. Here $\{r_0(n)\}_{n \geq 0}$ and $\{s_0(m)\}_{m \geq 0}$ are increasing sequences of nonnegative integers such that both $r_0(n)/n$ and $s_0(m)/m$ tend to 0 as $n, m \to \infty$. We then have

$$
\rho^{k+k'}(\phi\phi')'_{k+k'}M) = \rho^{k}(\phi|_kM)\rho^{k'}(\phi'|_{k'}M) \quad \text{by the definition of } |'|
$$

$$
= \sum_{\ell=n+m, n \geq 0, m \geq 0} \sum_{|r| \leq r_0(n), |s| \leq s_0(m)} c(n, r)d(m, s)q^{r+s}\zeta^n \zeta^m.
$$

Note that for a given $\ell \geq 0$ there are only finitely many nonnegative integers $n$ and $m$ such that $\ell = n + m$. So there are only finitely many integers $\ell$ which contribute terms of the form $q^\ell \zeta^n$. Furthermore, we have

$$
\frac{|\ell|}{\ell} \leq \frac{r_0(n) + s_0(m)}{n + m} \leq 2 \max \left( \frac{r_0(n + m)}{n + m}, \frac{s_0(n + m)}{n + m} \right) \to 0 \text{ as } n + m \to \infty.
$$

Hence $\phi\phi'$ satisfies the third condition for a modified Jacobi form of weight $k+k'$. This proves the proposition.

\[\square\]

Proposition 2.6. Let $\phi$ be a modified Jacobi form of weight $k$. Assume that the function $z \mapsto \phi(\tau_0, z)$ is not identically zero for a fixed point $\tau_0 \in \mathfrak{H}$. Let $F$ be a fundamental domain for the torus $\mathbb{C}/[\tau_0, 1]$, whose boundary does not have any zeros of $\phi(\tau_0, z)$, such $F$ always exists). Then $\phi(\tau_0, z)$ has exactly $-k$ zeros (counting multiplicity) in $F$.

Proof. It follows from the second condition for a modified Jacobi form $\phi$ that

$$
\left(\psi(\lambda \mu) \exp(\eta(\lambda \tau + \mu, [\tau, 1]) (z + \frac{1}{2}(\lambda \tau + \mu)) \right)^k \phi(\tau, z + \lambda \tau + \mu) = \phi(\tau, z) \quad ((\lambda \mu) \in \mathbb{Z}^2). \tag{2.1}
$$

Differentiating the above equation with respect to $z$ we have

$$
\left(\psi(\lambda \mu) \exp(\eta(\lambda \tau + \mu, [\tau, 1]) (z + \frac{1}{2}(\lambda \tau + \mu)) \right)^k \left((k\eta(\lambda \tau + \mu, [\tau, 1]) + \phi_z(\tau, z + \lambda \tau + \mu)\right) = \phi_z(\tau, z) \tag{2.2}
$$

where $\phi_z = \frac{d}{dz}\phi$. Dividing the equation (2.2) by (2.1) we obtain

$$
k\eta(\lambda \tau + \mu, [\tau, 1]) + \frac{\phi_z}{\phi}(\tau, z + \lambda \tau + \mu) = \frac{\phi_z}{\phi}(\tau, z).
$$

If we put $(\lambda \mu) = (1 \ 0)$ and $(0 \ 1)$ in the previous equation, then we get that

$$
k\eta(\tau, [\tau, 1]) + \frac{\phi_z}{\phi}(\tau, z + \tau) = \frac{\phi_z}{\phi}(\tau, z) \text{ and } k\eta(1, [\tau, 1]) + \frac{\phi_z}{\phi}(\tau, z + 1) = \frac{\phi_z}{\phi}(\tau, z), \tag{2.3}
$$

respectively. Now, we set $\partial F = \partial F_1 + \partial F_2 + \partial F_3 + \partial F_4$ as follows:

[Diagram of boundary conditions]

\[\square\]
By the Residue Theorem we derive that the number of zeros of $\phi(\tau_0, z)$ in $F$ is equal to
\[
\frac{1}{2\pi i} \oint_{\partial F} \frac{\phi_z}{\phi}(\tau_0, z)dz = \left( \frac{1}{2\pi i} \oint_{\partial F_1} \frac{\phi_z}{\phi}(\tau_0, z)dz + \frac{1}{2\pi i} \oint_{\partial F_2} \frac{\phi_z}{\phi}(\tau_0, z)dz \right) \\
+ \left( \frac{1}{2\pi i} \oint_{\partial F_1} \frac{\phi_z}{\phi}(\tau_0, z)dz - \frac{1}{2\pi i} \oint_{\partial F_1} \frac{\phi_z}{\phi}(\tau_0, z + \nu_0)dz \right) \\
+ \left( -\frac{1}{2\pi i} \oint_{\partial F_4} \frac{\phi_z}{\phi}(\tau_0, z + 1)dz + \frac{1}{2\pi i} \oint_{\partial F_4} \frac{\phi_z}{\phi}(\tau_0, z)dz \right) \\
= \frac{1}{2\pi i} \oint_{\partial F_1} k\eta(\tau_0, [\tau_0, 1])dz + \frac{1}{2\pi i} \oint_{\partial F_4} k\eta(1, [\tau_0, 1])dz \text{ by } (2.3) \\
= \frac{k}{2\pi i} (\eta(\tau_0, [\tau_0, 1]) - \tau_0\eta(1, [\tau_0, 1])) = -k \text{ by Lemma 2.2}
\]
This completes the proof. □

**Corollary 2.7.** Let $\Gamma$ be a subgroup of $\Gamma_1$ of finite index. For all positive integers $k$ we have $J_k(\Gamma) = \{0\}$.

**Proof.** Assume that there exists a nonzero element $\phi$ of $J_k(\Gamma)$ for some $k > 0$. Take a point $\tau_0 \in \mathcal{H}$ such that $\phi(\tau_0, z)$ is not identically zero. And, consider a fundamental domain $F$ for $\mathbb{C}/[\tau_0, 1]$ whose boundary has no zeros of $\phi(\tau_0, z)$. Then $\phi(\tau_0, z)$ has $-k < 0$ zeros in $F$ by Proposition 2.6 which is impossible. Hence $J_k(\Gamma) = \{0\}$ for all $k > 0$, as desired. □

3. **Construction of nearly holomorphic modular forms**

In this section we shall show that through a modified Jacobi form one can generate nearly holomorphic modular forms of integral weight.

**Lemma 3.1.** If $(\lambda \quad \mu) \in \mathbb{R}^2$, then
\[
\exp(\eta(\lambda\tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda\tau + \mu)))\rho(\tau, z)\rho(\tau, z + \lambda\tau + \mu)^{-1} = e^{(\frac{1}{2}\mu(1 - \lambda))q^{(1-\lambda)/2}\zeta^{-\lambda}}.
\]

**Proof.** We achieve that
\[
\exp(\eta(\lambda\tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda\tau + \mu)))\rho(\tau, z)\rho(\tau, z + \lambda\tau + \mu)^{-1} \\
= \exp \left( (\lambda\eta(\tau, [\tau, 1]) + \mu\eta(1, [\tau, 1]))(z + \frac{1}{2}(\lambda\tau + \mu)) + \frac{1}{4}\eta(1, [\tau, 1])z^2 - \pi iz \\
- \frac{1}{2}\eta(1, [\tau, 1])(z + \lambda\tau + \mu)^2 + \pi i(z + \lambda\tau + \mu) \right) \\
= \exp \left( (\tau\eta(1, [\tau, 1]) - \eta(\tau, [\tau, 1]))(-\lambda - \frac{1}{2}\lambda^2\tau - \frac{1}{2}\lambda\mu) + \pi i\lambda\tau + \pi i\mu \right) \\
= \exp \left( 2\pi i(-\lambda - \frac{1}{2}\lambda^2\tau - \frac{1}{2}\lambda\mu) + \pi i\lambda\tau + \pi i\mu \right) \text{ by Lemma 2.6} \\
= e^{(\frac{1}{2}\mu(1 - \lambda))q^{(1-\lambda)/2}\zeta^{-\lambda}}
\]
as desired. □

**Theorem 3.2.** Let $\Gamma$ be a subgroup of $\Gamma_1$ of finite index. Let $\phi \in J_k(\Gamma)$ for some integer $k$. Then for $X \in \mathbb{Q}^2$ the function
\[
\phi_X(\tau) := (\phi|_k X)(\tau, 0)
\]
is a nearly holomorphic modular form (that is, holomorphic on $\mathcal{H}$ and meromorphic at every cusp) of weight $k$ on some subgroup of $\Gamma$ of finite index depending only on $\Gamma$ and $X$. 
Thus the above observations indicate that \( \phi \) \( X \in \mathbb{Z}^2 \) we have

\[
\phi_{X+X'}(\tau) = (\phi|\tau|^r(X+X'))(\tau,0) = (\phi|\tau|^r(X+X))(\tau,0)
\]

\[
(\psi(X')\psi(X)|\psi(X') + X)e\left(\frac{1}{2} \det \left( X_{X'} \right) \right)^{-k} ((\phi|\tau|^rX')|\tau|^rX)(\tau,0) \text{ by Proposition 2.3(i)}
\]

\[
(\psi(X)\psi(X)e\left(\frac{1}{2} \det \left( X_{X'} \right) \right)^{-k} ((\phi|\tau|^rX')|\tau|^rX)(\tau,0) \text{ because } X' \in \mathbb{Z}^2
\]

\[
e^k \left( - \frac{k}{2} \det \left( X_{X'} \right) \right) \phi_X(\tau) \text{ from the second condition for } \phi(\tau,z).
\]

On the other hand, for any \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \) we deduce that

\[
(c\tau + d)^{-k} \phi_X \left( \frac{a\tau + b}{c\tau + d} \right) = ((\phi|\tau|^r X)|\tau|^r M)(\tau,0) \text{ by the definitions of } \phi_X \text{ and slash operators}
\]

\[
= ((\phi|\tau|^r X)|\tau|^r (XM))(\tau,0) \text{ by Proposition 2.3(ii)}
\]

\[
= (\phi|\tau|^r(XM))(\tau,0) \text{ by the first condition for } \phi(\tau,z)
\]

\[
= \phi_{XM}(\tau).
\]

Thus the above observations indicate that \( \phi_X(\tau) \) behaves like a modular form with respect to the congruence subgroup

\[
\left\{ M \in \Gamma' : XM \equiv X \pmod{2\mathbb{Z}^2}, \frac{k}{2} \det \left( X_{XM} - X \right) \in \mathbb{Z} \right\}
\]

which contains \( \Gamma \cap \Gamma(\frac{2N^2}{\gcd(N,k)}) \) if \( X \in N^{-1}\mathbb{Z}^2 \) for some integer \( N \geq 1 \) and \( \Gamma(\frac{2N^2}{\gcd(N,k)}) \) is the principal congruence subgroup of level \( \frac{2N^2}{\gcd(N,k)} \).

Next, let \( M = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_1 \) and suppose that \( \rho^k(\phi|\tau|^r M) \) has a Fourier development

\[
\sum_{n \geq 0} \sum_{|\tau| \leq r_0(n)} c(n,r)q^n \zeta^r
\]

such that \( r_0(n)/n \to 0 \) as \( n \to \infty \). Then we get that

\[
(c\tau + d)^{-k} \phi_X \left( \frac{a\tau + b}{c\tau + d} \right)
\]

\[
= ((\phi|\tau|^r X)|\tau|^r M)(\tau,0) \text{ by the definitions of } \phi_X \text{ and slash operators}
\]

\[
= ((\phi|\tau|^r X)|\tau|^r (XM))(\tau,0) \text{ by Proposition 2.3(ii)}
\]

\[
= \left( \psi(XM) \exp\left( \frac{1}{2} \eta(\lambda'\tau + \mu', [\tau,1])(\lambda'\tau + \mu') \right) \right)^k (\phi|\tau|^r M)(\tau,\lambda'\tau + \mu') \text{ where } (\lambda' \mu') = XM
\]

\[
= \left( \psi(XM) \exp\left( \frac{1}{2} \eta(\lambda'\tau + \mu', [\tau,1])(\lambda'\tau + \mu') \right) \right)^k \left( \rho^{-k} \sum_{n \geq 0} \sum_{|\tau| \leq r_0(n)} c(n,r)q^n \zeta^r \right)(\tau,\lambda'\tau + \mu')
\]

\[
= \left( \psi(XM) e(\frac{1}{2} \mu'(1 - \lambda'))q^{\lambda'(1 - \lambda')/2} \right) \sum_{n \geq 0} \sum_{|\tau| \leq r_0(n)} c(n,r)e(\mu' r)q^n + \lambda' r \text{ by Lemma 5.1 (3.1)}
\]

Now, we observe that

\[
n + \lambda' r \geq n - |\lambda'|r_0(n) = n \left( 1 - \frac{|\lambda'|}{r_0(n)} \right) \to \infty \text{ as } n \to \infty,
\]
from which it follows that for a given rational number $\ell$ there are only finitely many $n$ and $r$ which contributes the term $q^\ell$ in (3.1). Hence $\phi_X(\tau)$ is meromorphic at each cusp. This completes the proof.

We are ready to introduce well-known Klein forms in view of Weierstrass $\sigma$-function which will be a concrete example of modified Jacobi form.

**Lemma 3.3.**

(i) Let $L$ be a lattice in $\mathbb{C}$. If $\omega \in L$, then

$$
\frac{\sigma(z + \omega, L)}{\sigma(z, L)} = \psi(\omega, L) \exp(\eta(z + \frac{1}{2}\omega)).
$$

(ii) The function $\sigma(\tau, z) := \sigma(z, [\tau, 1])$ has the infinite product expansion

$$
\sigma(\tau, z) = -\frac{1}{2\pi i} \rho(\tau, z)(1 - \zeta) \prod_{n=1}^{\infty} \frac{(1 - q^n\zeta)(1 - q^n\zeta^{-1})}{(1 - q^n)^2}.
$$

**Proof.** See [5] p. 241, p. 247 or [7] p. 44, p. 53.

**Theorem 3.4.** The function $\sigma(\tau, z)$ belongs to $J_{-1}(\Gamma_1)$.

**Proof.** First, we note that $\sigma(\tau, z)$ is a holomorphic function by Lemma 3.3(ii).

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$. Then we derive

$$(\sigma|_{-1} M)(\tau, z) = (c\tau + d)\sigma\left(\frac{z}{c\tau + d}, \left[\begin{array}{c} a\tau + b \\ c\tau + d \end{array}\right], 1\right)$$

$$= \sigma(z, [a\tau + b, c\tau + d]) \text{ by Lemma 2.1(i)}$$

$$= \sigma(\tau, z) \text{ by the fact } [a\tau + b, c\tau + d] = [\tau, 1].$$

And, for $X = (\lambda \mu) \in \mathbb{Z}^2$ we achieve that

$$(\sigma|_{-1} X)(\tau, z) = \left(\psi(\lambda \mu) \exp(\eta(\lambda\tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda\tau + \mu)))\right)^{-1} \sigma(\tau, z + \lambda\tau + \mu)$$

$$= \sigma(\tau, z) \text{ by Lemma 3.3(i)}. $$

Finally, let $M \in \Gamma_1$. Then we have

$$\rho(\tau, z)^{-1}(\sigma|_{-1} M)(\tau, z) = \rho(\tau, z)^{-1}\sigma(\tau, z) \text{ by the first part of the proof}$$

$$= -\frac{1}{2\pi i} (1 - \zeta) \prod_{n=1}^{\infty} \frac{(1 - q^n\zeta)(1 - q^n\zeta^{-1})}{(1 - q^n)^2} \text{ by Lemma 3.3(ii).}$$

Hence it is easy to check that $\sigma(\tau, z)$ satisfies the third condition for a modified Jacobi form by adopting the argument in the proof of Proposition 2.5. Therefore $\sigma(\tau, z)$ belongs to $J_{-1}(\Gamma_1)$.

**Remark 3.5.** It follows that if $(\lambda \mu) \in N^{-1}\mathbb{Z}^2$ for some integer $N \geq 1$, then the function

$$\xi_{(\lambda \mu)}(\tau) := (\sigma|_{-1} [\lambda \mu])(\tau, 0) = \psi(\lambda \mu) \exp(-\frac{1}{2} \eta(\lambda\tau + \mu, [\tau, 1])(\lambda\tau + \mu)) \sigma(\tau, \lambda\tau + \mu)$$

is a nearly holomorphic modular form of weight $-1$ on $\Gamma(2N^2)$ by Theorem 3.2. This function is called a Klein form (indexed by $(\lambda \mu)$) whose infinite product expansion is

$$\xi_{(\lambda \mu)}(\tau) = e(\mu(\lambda - 1)/2)q^{\lambda(\lambda-1)/2}(1 - e(\mu)q^\lambda)^{\infty} \prod_{n=1}^{\infty} \frac{(1 - e(\mu)q^{n+\lambda})(1 - e(-\mu)q^{n-\lambda})}{(1 - q^n)^2}$$

if $(\lambda \mu) \notin \mathbb{Z}^2$ (and identically zero otherwise). The modularity of a product of finitely many Klein forms on $\Gamma(N)$ are intensively studied by Kubert and Lang ([3]). On the other hand, in a recent paper [2] the authors present a sufficient condition for a product of Klein forms to be a nearly holomorphic modular form on $\Gamma_1(N)$.
4. Finite-dimensional subspaces

In this section we shall consider a family of finite-dimensional subspaces of $J_k(\Gamma_1)$.

**Definition 4.1.** Let $k$ be an integer. For each integer $m > 0$ we let $J_k^m$ be the subspace of $J_k := J_k(\Gamma_1)$ consisting of $\phi$ for which $\rho^k \phi = \sum_{n \geq 0} \sum_{|r| \leq r_0(n)} c(n, r) q^n z^r$ and

\[
\min\{n - r_0(n) : n \geq 0\} + \frac{k}{8} \geq -m. \tag{4.1}
\]

**Remark 4.2.** (i) Note that since

\[
n - r_0(n) = n \left(1 - \frac{r_0(n)}{n}\right) \to \infty \text{ as } n \to \infty,
\]

the minimum of $n - r_0(n)$ exists. Thus we have a filtration $J_k^1 \subseteq J_k^2 \subseteq \cdots$ and $J_k = \bigcup_{m=1}^{\infty} J_k^m$. (ii) Since $\rho^{-1} \sigma$ has the following Fourier development

\[
-\frac{1}{2\pi i} (1 - \zeta) \prod_{n=1}^{\infty} \frac{(1 - q^n \zeta)(1 - q^n \zeta^{-1})}{(1 - q^n)^2} = -\frac{1}{2\pi i} \left\{ (1 - \zeta) + (-\zeta^{-1} + 3 - 3\zeta + \zeta^2)q \right. \\
+(-3\zeta^{-1} + 9 - 9\zeta + 3\zeta^2)q^2 \\
+(-\zeta^{-2} + 22\zeta - 9\zeta^{-1} + 22\zeta + 9\zeta^2 - \zeta^3)q^3 \\
+(51 - 51\zeta - 22\zeta^{-1} + 22\zeta^2 + 3\zeta^2 - 3\zeta^3)q^4 \\
+(9\zeta^{-2} - 51\zeta^{-1} + 108 - 108\zeta + 51\zeta^2 - 9\zeta^3)q^5 + \cdots \bigg\},
\]

one can readily check that the Weierstrass $\sigma$-function in fact lies in $J_k^1$.

**Proposition 4.3.** Let $\phi \in J_k$ for some integer $k$ with $\rho^k \phi = \sum_{n,r} c(n, r) q^n z^r$. Then we have

\[
c(n, r) = (-1)^k c(n - r\lambda - \frac{1}{2}k(\lambda^2 + \lambda), r + \lambda k) \tag{4.2}
\]

for all integers $n$, $r$ and $\lambda$.

**Proof.** If $(\lambda, \mu) \in \mathbb{Z}^2$, then we derive that

\[
\sum_{n,r} c(n, r) q^n z^r = \rho^k \phi(\tau, z) = \rho^k (\phi''\left[\lambda, \mu\right])
\]

\[
= \rho(\tau, z)^k \left( \psi(\lambda, \mu) \exp(\eta(\lambda \tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda \tau + \mu))) \right)^k \phi(\tau, z + \lambda \tau + \mu)
\]

\[
= \rho(\tau, z)^k \left( \psi(\lambda, \mu) \exp(\eta(\lambda \tau + \mu, [\tau, 1])(z + \frac{1}{2}(\lambda \tau + \mu))) \right)^k \rho(\tau, z + \lambda \tau + \mu)^{-k} (\rho^k \phi)(\tau, z + \lambda \tau + \mu)
\]

\[
= \left( \psi(\lambda, \mu) e\left(\frac{1}{2}\mu(1 - \lambda)\right) q^{\lambda(1 - \lambda)/2} z^{-\lambda} \right)^k \sum_{n, r} c(n, r) q^{n + r\lambda} z^r \text{ by Lemma 3.1}
\]

\[
= (-1)^k \sum_{n, r} c(n, r) q^{n + r\lambda + \lambda(k(1 - \lambda)/2) z^{-\lambda}} \text{ by Lemma 2.1(ii)}
\]

\[
= (-1)^k \sum_{n', r'} c(n' - r'\lambda - \frac{1}{2}k(\lambda^2 + \lambda), r' + \lambda k) q^{n'} z^{r'}
\]

by letting $r' := r - \lambda k$ and $n' := n + r'\lambda + \frac{1}{2}k(\lambda^2 + \lambda)$.

We get the assertion by comparing the coefficients of Fourier developments. \qed

**Remark 4.4.** If we put $\lambda = 1$ in (4.2), then we have

\[
c(n, r) = (-1)^k c(n - r - k, r + k). \tag{4.3}
\]
Let \( \phi \in J_k \) for some integer \( k \) with \( \rho^k \phi = \sum_{n,r} c(n,r)q^n \zeta^r \). Then, for \( X = (u \ v) \in \mathbb{Q}^2 \) we defined in Theorem 4.2

\[
\phi_X(\tau) := (\phi''_k X)(\tau, 0) 
\]

\[
= \left\{ \left( \psi(u \ v) \exp(\eta(u\tau + v, [\tau, 1])(z + \frac{1}{2}(u\tau + v))) \right) \rho(\tau, z + u\tau + v)^{-k}(\rho^k \phi)(\tau, z + u\tau + v) \right\}(\tau, 0) 
\]

\[
= \left\{ \left( \psi(u \ v)e(\frac{1}{2}v(1 - u))q^{(n-1)/2}q^{-u} \right)^k \sum_{n,r} c(n,r)e(vr)q^{n+ur}\right\}(\tau, 0) \text{ by Lemma 3.1} 
\]

\[
= \left( \psi(u \ v)e(\frac{1}{2}v(1 - u)) \right)^k \sum_{n,r} c(n,r)e(vr)q^{n+ur+k} \tag{4.4} 
\]

**Proposition 4.5.** Let \( X, Y \in \mathbb{Q}^2 \) with \( X \equiv Y \pmod{\mathbb{Z}^2} \). Then \( \phi_X(\tau) = \xi \phi_Y(\tau) \) for some root of unity \( \xi \).

**Proof.** If \( X = (u \ v) \) and \( Y = (u \ v + 1) \), then one can readily get from the expression \[(4.4)\] that \( \phi_X(\tau) = \xi \phi_Y(\tau) \) for some root of unity \( \xi \).

Now, let \( X = (u \ v) \) and \( Y = (u + 1 \ v) \). We obtain from the expression \[(4.4)\] that

\[
\phi_Y(\tau) = \left( \psi(u + 1 \ v)e(-\frac{1}{2}vu) \right)^k \sum_{n,r} c(n,r)e(vr)q^{n+ur+k} \tag{4.4} 
\]

by letting \( r' := r - k \) and \( n' := n + r' + k \)

\[
= \left( \psi(u + 1 \ v)e(-\frac{1}{2}vu) \right)^k e(vk)(-1)^k \sum_{n',r'} c(n',r')e(vr')q^{n'+ur'+k} \tag{4.4} 
\]

by Remark 4.4

This proves the proposition. \( \square \)

Now we are ready to prove our main theorem about dimension.

**Theorem 4.6.** Let \( k < 0 \) and \( m > 0 \) be integers. Then \( J_k^m \) is finite-dimensional.

**Proof.** Pick any \(-k + 1\) distinct pairs of \( X_j = (u_j \ v_j) \in \mathbb{Q}^2 \) with \( 0 < u_j, v_j < 1 \). Let \( \phi \in J_k^m \) with \( \rho^k \phi = \sum_{n,r} c(n,r)q^n \zeta^r \). Then the functions

\[
\phi_{X_j}(\tau) = (\phi''_k X_j)(\tau, 0) \quad (j = 1, \ldots, -k + 1) 
\]

are nearly holomorphic modular forms on some congruence subgroups \( \Gamma_j \) depending on \( X_j \) by Theorem 3.2

If \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \), then we have

\[
(c\tau + d)^{-k}\phi_{X_j} \left( \frac{a\tau + b}{c\tau + d} \right) = (\phi''_k X_j)(\tau, 0) \text{ by the definitions of slash operators} 
\]

\[
= (\phi''_k X_j)(\tau, 0) \text{ by Proposition 2.3(ii)} 
\]

\[
= (\phi''_k X_j)(\tau, 0) \text{ by the first condition for a modified Jacobi form} 
\]

\[
= \phi_{X_j M}(\tau). \tag{4.5} 
\]

Set \( (u'_j \ v'_j) := X_j M \). We may assume that \( 0 \leq u'_j, v'_j < 1 \) to estimate \( \text{ord}_q \phi_{X_j M}(\tau) \) by Proposition 4.5

Note that from the expression \[(4.4)\] one can deduce

\[
\phi_{X_j M}(\tau) = \xi \sum_{n \geq 0} \sum_{|r| \leq \rho(n)} c(n,r)e(v'_j r)q^{n+u'_j r + ku'_j (1-u'_j)/2} 
\]
for some root of unity \( \xi \). It follows that

\[
\text{ord}_q \phi_{X,M}(\tau) \geq \min\{n + u_j' r + k u_j'(1 - u_j')/2 : n \geq 0, |r| \leq r_0(n)\}
\geq \min\{n - r_0(n) + k/8 : n \geq 0\} \text{ because } 0 \leq u_j', v_j' < 1
\geq -m \text{ by the condition 4.4 for } \phi \in J^m_k.
\]  \hspace{1cm} (4.6)

On the other hand, we consider a map

\[
g : J^m_k \to \bigoplus_{j=1}^{k+1} M_{k+12m}(\Gamma_j)
\phi \mapsto (\phi_{X_j}(\tau)\Delta(\tau)^m)_{j=1}^{k+1}
\]

where \( M_{k+12m}(\Gamma_j) \) is the space of modular forms of weight \( k + 12m \) on \( \Gamma_j \), and \( \Delta(\tau) := (2\pi i)^{12}q \prod_{n=1}^{\infty}(1 - q^n)^{24} \) is the modular discriminant function. As is well-known, each \( M_{k+12m}(\Gamma_j) \) is of finite dimension \([6]\) Theorem 2.23) and \( \Delta(\tau) \) is a modular form of weight 12 on \( \Gamma_1 \) which does not vanish on \( \mathcal{H} \) and has \( \text{ord}_q \Delta(\tau) = 1 \) \([7]\) Chapter I 3\). Hence the identity \([15.5]\) and the inequality \([4.6]\) imply that \( g \) is well-defined.

If \( \phi \) and \( \phi' \) are two distinct elements of \( J^m_k \), then there exists a point \( \tau_0 \in \mathcal{H} \) such that the function \( (\phi - \phi')(\tau_0, z) \) is not identically zero and has no zeros on the boundary of the fundamental domain generated by \( \tau_0 \) and 1 \((\text{for the torus } \mathbb{C}/[\mathbb{Z}, 1])\). Suppose that \( \phi \) and \( \phi' \) have the same image via \( g \). Then for every \( j \)

\[
0 = (\phi_{X_j}\Delta^m - \phi'_{X_j}\Delta^m)(\tau_0) = (\phi_{X_j} - \phi'_{X_j})(\tau_0)\Delta(\tau_0)^m.
\]

Since \( \Delta(\tau_0) \neq 0 \), we get

\[
0 = (\phi_{X_j} - \phi'_{X_j})(\tau_0) = (\phi'_{X_j})f_jX_j - \phi'_{X_j}f_jX_j)(\tau_0, 0)
= \left(\psi(u_j v_j) \exp(\eta(u_j \tau_0 + v_j, [\tau_0, 1])\frac{1}{2}(u_j \tau_0 + v_j))\right)^k (\phi - \phi')(\tau_0, u_j \tau_0 + v_j).
\]

This implies \( (\phi - \phi')(\tau_0, u_j \tau_0 + v_j) = 0 \). Thus the function \( (\phi - \phi')(\tau_0, z) \) has at least \(-k + 1\) distinct zeros in the fundamental domain \( \mathcal{F} \) generated by \( \tau_0 \) and 1. (When \( (\phi - \phi')(\tau_0, z) \) has zeros on the boundary \( \partial \mathcal{F} \), we slightly move \( \mathcal{F} \) into a new domain \( \mathcal{F}' \) so that the points \( u_j \tau_0 + v_j \) still lie inside \( \mathcal{F}' \) and the boundary \( \partial \mathcal{F}' \) has no zeros of \( (\phi - \phi')(\tau_0, z) \).) But, this contradicts Proposition \([20]\) Therefore, \( g \) is injective; hence we obtain

\[
\dim_{\mathbb{C}}J^m_k \leq \sum_{j=1}^{-k+1} \dim_{\mathbb{C}}M_{k+12m}(\Gamma_j) < \infty
\]
as desired. \( \square \)

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