DISCRETE VARIANTS OF BRUNN-MINKOWSKI TYPE INEQUALITIES

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Abstract. We present an alternative, short proof of a recent discrete version of the Brunn-Minkowski inequality due to Lehec and the second named author. Our proof also yields the four functions theorem of Ahlswede and Daykin as well as some new variants.

1. Introduction

Correlation inequalities such as the Fortuin-Kasteleyn-Ginibre (FKG) inequality are of use in the analysis of several models in Probability Theory and Statistical Physics (see, e.g., Grimmett [5, 6]). These inequalities are closely related to the following four functions theorem of Ahlswede and Daykin [1]:

**Theorem 1.1.** Suppose \( f, g, h, k : \mathbb{Z}^n \to [0, \infty) \) satisfy

\[
f(x)g(y) \leq h(x \wedge y)k(x \vee y) \quad \forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{Z}^n
\]

where \( x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n)) \), and \( x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n)) \). Then

\[
(\sum_{x \in \mathbb{Z}^n} f(x))\left(\sum_{x \in \mathbb{Z}^n} g(x)\right) \leq \left(\sum_{x \in \mathbb{Z}^n} h(x)\right)\left(\sum_{x \in \mathbb{Z}^n} k(x)\right).
\]

Theorem 1.1 is usually formulated under the additional assumption that \( f, g, h, k \) are all supported in the discrete cube \( \{0, 1\}^n \). It was suggested by Gozlan, Roberto, Samson, and Tetali [4] that Theorem 1.1 is connected with a discrete variant of the Brunn-Minkowski inequality, recently proven by Lehec and the second named author [8, Theorem 1.4], which is the case \( \lambda = 1/2, n = 1 \) of the following theorem:

**Theorem 1.2.** Let \( \lambda \in [0, 1] \) and suppose \( f, g, h, k : \mathbb{Z}^n \to [0, \infty) \) satisfy

\[
f(x)g(y) \leq h([\lambda x + (1-\lambda)y])k([(1-\lambda)x + \lambda y]) \quad \forall x, y \in \mathbb{Z}^n
\]

where \( [x] = ([x_1], \ldots, [x_n]) \) and \( \lfloor x \rfloor = ([x_1], \ldots, [x_n]) \). Then

\[
(\sum_{x \in \mathbb{Z}^n} f(x))\left(\sum_{x \in \mathbb{Z}^n} g(x)\right) \leq \left(\sum_{x \in \mathbb{Z}^n} h(x)\right)\left(\sum_{x \in \mathbb{Z}^n} k(x)\right).
\]

Here \( \lfloor r \rfloor = \max\{m \in \mathbb{Z} ; m \leq r \} \) is the lower integer part of \( r \in \mathbb{R} \) and \( \lceil r \rceil = -\lfloor -r \rfloor \) the upper integer part. A standard limiting argument (see [4, Section 2.3]) leads from the case
\( \lambda = 1/2, h = k \) of Theorem 1.2 to the case \( \lambda = 1/2 \) of the Brunn-Minkowski inequality in its multiplicative form:

\[
\text{Vol}\left( \frac{A + B}{2} \right) \geq \sqrt{\text{Vol}(A)\text{Vol}(B)},
\]

where \( A + B = \{x + y; x \in A, y \in B\} \), where \( A, B \subseteq \mathbb{R}^n \) are any Borel-measurable sets, and \( \text{Vol}(\cdot) \) stands for the \( n \)-dimensional Lebesgue volume. The proof in [8] for the case \( n = 1, \lambda = 1/2 \) admits a straightforward generalization to the more general case described above, and it relies on the language of Stochastic Analysis. An alternative argument using ideas from the theory of Optimal Transport was given by Gozlan, Roberto, Samson and Tetali [4].

Our goal in this note is to provide a unified proof of Theorem 1.1 and Theorem 1.2, which is perhaps as elementary as the original proof of the four functions theorem by Ahlswede and Daykin [1]. The first issue that we would like to address, is the identification of the relevant common features of operations such as

(1) \( x \wedge, x \vee, \left\lfloor \frac{x + y}{2} \right\rfloor, \left\lceil \frac{x + y}{2} \right\rceil, \lfloor \lambda x + (1 - \lambda)y \rfloor, \lceil \lambda x + (1 - \lambda)y \rceil, \ldots \)

that are defined for \( x, y \in \mathbb{Z}^n \), with \( 0 < \lambda < 1 \). Our observation is that these operations \( T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n \) satisfy two axioms:

(P1) Additivity: \( T(x + z, y + z) = T(x, y) + z \) for all \( z \in \mathbb{Z}^n \).

(P2) Monotonicity: there exist a total additive ordering \( \leq \) on \( \mathbb{Z}^n \), with respect to which \( T \) is non-decreasing in each of its two entries, i.e., if \( x_1 \leq y_1 \) and \( x_2 \leq y_2 \) then \( T(x_1, x_2) \leq T(y_1, y_2) \).

Recall that a total ordering \( \preceq \) on \( \mathbb{Z}^n \) (or on \( \mathbb{R}^n \)) is a binary relation which is reflexive, antisymmetric and transitive, such that for any distinct \( x, y \), either \( x \preceq y \) or else \( y \preceq x \). An ordering \( \preceq \) is additive if for all \( x, y, z \),

\[
x \preceq y \implies x + z \preceq y + z.
\]

The standard lexicographic order relation on \( \mathbb{Z}^n \) (or on \( \mathbb{R}^n \)) is an example for an additive, total order. Given an additive, total ordering \( \preceq \) on \( \mathbb{R}^n \) and an invertible, linear map \( L : \mathbb{R}^n \to \mathbb{R}^n \) we may construct another additive, total ordering \( \preceq_L \) by requiring that \( x \preceq_L y \) if and only if \( Lx \preceq_L Ly \). The standard lexicographic order on \( \mathbb{Z}^n \) attests to the fact that all of the examples in (1) satisfy properties (P1) and (P2). We prove the following:
Theorem 1.3. Let $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ satisfy properties (P1) and (P2). Suppose $f, g, h, k : \mathbb{Z}^n \to [0, \infty)$ satisfy
\[
f(x)g(y) \leq h(T(x, y))k(x + y - T(x, y)) \quad \forall x, y \in \mathbb{Z}^n.
\]
Then
\[
\left( \sum_{x \in \mathbb{Z}^n} f(x) \right) \left( \sum_{x \in \mathbb{Z}^n} g(x) \right) \leq \left( \sum_{x \in \mathbb{Z}^n} h(x) \right) \left( \sum_{x \in \mathbb{Z}^n} k(x) \right).
\]

Clearly Theorem 1.1 and Theorem 1.2 follow from Theorem 1.3.

See also Borell [2, Theorem 2.1] for Brunn-Minkowski type inequalities for operations other than Minkowski sum with monotonicity properties.

One can relax the monotonicity property (P2) of the map $T$ in Theorem 1.3 by replacing it with the following property, which requires no ordering at all. We formulate this property, as well as our next theorem, in greater generality, with $\mathbb{Z}^n$ replaced by a finitely generated abelian group $G$, and $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ replaced by $T : G \times G \to G$. It is well-known that any such $G$ is isomorphic to $\mathbb{Z}^n \times (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})$ for some primes $p_1, \ldots, p_k$. Therefore, the requirement of a total additive ordering on $G$, as in (P2) and Theorem 1.3, would force $G$ to be isomorphic to $\mathbb{Z}^n$.

(P2') For every finite set $A \subseteq G$ with at least two elements and all $z \in G$, there exist distinct $x, y \in A$ such that for $A_1 = A \setminus \{x\}$, $A_2 = A \setminus \{y\}$, and $A_3 = A \setminus \{x, y\}$, the following conditions hold:
(a) either $T(x, z - y) \notin T(A_i, z - A_i)$ or $T(y, z - x) \notin T(A_i, z - A_i)$ for $i \in \{1, 2\}$,
(b) both $T(x, z - y) \notin T(A_3, z - A_3)$ and $T(y, z - x) \notin T(A_3, z - A_3)$.

Here $T(A_i, z - A_i) = \{T(x, z - y); x, y \in A_i\}$.

We prove the following:

Theorem 1.4. Let $(G, +)$ be a finitely generated abelian group, and $T : G \times G \to G$ satisfy properties (P1) and (P2'). Suppose $f, g, h, k : G \to [0, \infty)$ satisfy
\[
f(x)g(y) \leq h(T(x, y))k(x + y - T(x, y)) \quad \forall x, y \in G.
\]
Then
\[
\left( \sum_{x \in G} f(x) \right) \left( \sum_{x \in G} g(x) \right) \leq \left( \sum_{x \in G} h(x) \right) \left( \sum_{x \in G} k(x) \right).
\]
The next two sections are devoted to the proofs of the above theorems. We additionally include a final section with commentary on the applicability of this work to related inequalities, such as the ones proven by Cordero-Erausquin and Maurey [3] and by Iglesias, Yepes Nicolás and Zvavitch [7].

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2. Proof of Theorem 1.4

The core of this paper is the proof of Theorem 1.4 given in this section. We begin with the following elementary fact:

Fact 2.1. Suppose $a, b, c, d \geq 0$. If $ab \leq cd$ and $\max\{a, b\} \leq \max\{c, d\}$ then $a + b \leq c + d$.

Proof. Pick $A \geq a, B \geq b$ such that $\max\{A, B\} \leq \max\{c, d\}$ and $AB = cd = r$. Then $(A - B)^2 \leq (c - d)^2$ and so $(a + b)^2 \leq (A + B)^2 = 4r + (A - B)^2 \leq 4r + (c - d)^2 = (c + d)^2$. □

Recall that under the assumptions of Theorem 1.4 we have $f, g, h, k : G \to [0, \infty)$ satisfying

$$f(x)g(y) \leq h(T(x, y))k(x + y - T(x, y)) \forall x, y \in G.$$  \hfill (2)

For $j, z \in G$ denote $F_z(j) = f(j)g(z - j)$ and $H_z(j) = h(j)k(z - j)$. Note that, by (2),

$$F_z(j) \leq H_z(T(j, z - j)).$$ \hfill (3)

We claim that for all $i, j, z \in G$ we have

$$F_z(i)F_z(j) \leq H_z(T(i, z - j))H_z(T(j, z - i)).$$ \hfill (4)

Indeed, by (2) and (P1) we have

$$F_z(i)F_z(j) = f(i)g(z - i)f(j)g(z - j) = f(i)g(z - j)f(j)g(z - i)$$
$$\leq h(T(i, z - j))k(z + (i - j) - T(i, z - j))h(T(j, z - i))k(z + (j - i) - T(j, z - i))$$
$$= h(T(i, z - j))k(z - T(i, z - j))h(T(j, z - i))k(z - T(j, z - i))$$
$$= H_z(T(i, z - j))H_z(T(j, z - i)).$$

Proof of Theorem 1.4. We need to prove that

$$\sum_{j, z \in G} F_z(j) \leq \sum_{j, z \in G} H_z(j).$$
Fix $z \in G$. It is sufficient to prove that for every finite set $A \subseteq G$,
\[
\sum_{j \in A} F_z(j) \leq \sum_{j \in T(A, z-A)} H_z(j).
\]
(5)

We proceed to prove so by induction on $n = |A|$.

**Induction base:** For $n = 0$ the statement is vacuous, as the empty sum equals zero. For $n = 1$ the statement holds by (3).

**Induction step:** Assume $n \geq 2$ and that the statement holds for all $m \leq n - 1$. Let $A \subseteq G$ with $|A| = n$. By assumption, there exist distinct $x, y \in A$ such that assertions (a) and (b) in (P2') are satisfied. By switching $x$ with $y$ if necessary, we may assume that $F_z(x) \leq F_z(y)$. By (4) we have
\[
F_z(x)F_z(y) \leq H_z(T(x, z-y))H_z(T(y, z-x)).
\]
(6)

**Case 1:** Assume $F_z(y) \geq \max\{H_z(T(x, z-y)), H_z(T(y, z-x))\}$. Then, it follows from (6) that
\[
F_z(x) \leq \min\{H_z(T(x, z-y)), H_z(T(y, z-x))\}.
\]
(7)

The induction hypothesis for $A_1 = A \setminus \{x\}$ tells us that
\[
\sum_{j \in A_1} F_z(j) \leq \sum_{j \in T(A_1, z-A_1)} H_z(j).
\]
(8)

By adding inequalities (7) and (8), and using property (a) in (P2'), we obtain the desired inequality (5).

**Case 2:** Assume $F_z(y) \leq \max\{H_z(T(x, z-y)), H_z(T(y, z-x))\}$. Since $F_z(x) \leq F_z(y)$, we may apply (6) and Fact 2.1 and obtain
\[
F_z(x) + F_z(y) \leq H_z(T(x, z-y)) + H_z(T(y, z-x)).
\]
(9)

Note that $T(x, z-y) \neq T(y, z-x)$ as $T(y, z-x) - T(x, z-y) = y - x \neq 0$. Therefore, by combining (9) with the induction hypothesis for $A_3 = A \setminus \{x, y\}$ and property (b) in (P2'), we deduce (5). This completes the proof. \qed

3. **Proof of Theorem 1.3**

Theorem 1.3 is an immediate consequence of Theorem 1.4 due to the following observation:

**Lemma 3.1.** Suppose $T : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ satisfies properties (P1) and (P2). Then $T$ satisfies property (P2').
Proof. Let $k \geq 2$ and $z \in \mathbb{Z}^n$. Suppose $A = \{x_1, \ldots, x_k\}$, where $x_1 < \cdots < x_k$. Here $a \preceq b$ means that $a \leq b$ and $a \neq b$. Set $x = x_1, y = x_k$, and recall that $A_1 = A \setminus \{x\}$, $A_2 = A \setminus \{y\}$, and $A_3 = A \setminus \{x, y\}$. By properties (P1) and (P2), we have

\[
\max\{T(w_1, z - v_1) - T(v_2, z - w_2); v_1, v_2, w_1, w_2 \in A_1\} = \\
\max\{T(w_1, z - v_1); v_1, w_1 \in A_1\} - \min\{T(v_2, z - w_2); v_2, w_2 \in A_1\} = \\
T(x_k, z - x_2) - T(x_2, z - x_k) = x_k - x_2 \prec x_k - x_1 = T(x_k, z - x_1) - T(x_1, z - x_k).
\]

Therefore, either $T(x_1, z - x_k) \notin T(A_1, z - A_1)$ or $T(x_k, z - x_1) \notin T(A_1, z - A_1)$. A similar argument shows that the same holds for $A_2$. This verifies condition (a) of property (P2').

Next, we verify condition (b) of property (P2'). Note that if $k = 2$ then $A_3 = \emptyset$, and hence the condition holds trivially. Otherwise, letting $v = \min\{x_{i+1} - x_i; i \in \{1, \ldots, k - 1\}\} > 0$, we see that

\[
\max_{v, w \in A_3} T(v, z - w) = T(x_{k-1}, z - x_2) \\
\preceq T(x_k - v, z - x_1 - v) = T(x_k, z - x_1) - v \prec T(x_k, z - x_1).
\]

Therefore, $T(x_k, z - x_1) \notin T(A_3, z - A_3)$. Similarly, $T(x_1, z - x_k) \notin T(A_3, z - A_3)$, which verifies condition (b) of property (P2'), and completes the proof. \(\square\)

4. Related inequalities

4.1. Continuous Brunn-Minkowski type inequalities. The classical Brunn-Minkowski inequality states that for any two non-empty Borel-measurable subsets of $\mathbb{R}^n$, one has

\[
\text{Vol}(A + B)^{1/n} \geq \text{Vol}(A)^{1/n} + \text{Vol}(B)^{1/n}.
\]

In its equivalent dimension-free form, it states that for any $\lambda \in [0, 1]$,

\[
\text{Vol}(\lambda A + (1 - \lambda)B) \geq \text{Vol}(A)^\lambda \text{Vol}(B)^{1-\lambda}.
\]

A functional form of the Brunn-Minkowski inequality, known as the Prékopa-Leindler inequality, states that for any Borel functions $f, g, h : \mathbb{R}^n \to [0, \infty)$ and any $\lambda \in [0, 1]$ such that $f(x)^\lambda g(y)^{1-\lambda} \leq h(\lambda x + (1 - \lambda)y)$ for all $x, y \in \mathbb{R}^n$, we have

\[
\left(\int_{\mathbb{R}^n} f(x) \, dx\right)^\lambda \left(\int_{\mathbb{R}^n} g(x) \, dx\right)^{1-\lambda} \leq \int_{\mathbb{R}^n} h(x) \, dx.
\]

See, e.g., the first pages in Pisier [9] for proofs of these inequalities. When $\lambda = 1/2$ and $h = k$, the analogy between Theorem 1.2 and the Prékopa–Leindler inequality is evident, see
[4, Section 2.3] for a standard limiting argument that leads from Theorem 1.2 to (10). For \( \lambda \neq 1/2 \), a similar limiting argument leads to a weighted variant of the Prékopa–Leindler inequality due to Cordero-Erausquin and Maurey [3]:

**Theorem 4.1.** Let \( \lambda \in [0, 1] \). Suppose \( f, g, h, k : \mathbb{R}^n \to [0, \infty) \) are measurable functions satisfying
\[
 f(x)g(y) \leq h(\lambda x + (1 - \lambda)y)k((1 - \lambda)x + \lambda y) \quad \forall x, y \in \mathbb{R}^n.
\]
Then
\[
 \left( \int_{\mathbb{R}^n} f(x) \, dx \right) \left( \int_{\mathbb{R}^n} g(x) \, dx \right) \leq \left( \int_{\mathbb{R}^n} h(x) \, dx \right) \left( \int_{\mathbb{R}^n} k(x) \, dx \right).
\]

Note that for \( \lambda = 1/2 \) and \( h = k \), Theorem 4.1 coincides with (10). We omit the details of the standard limiting argument leading from Theorem 1.2 to Theorem 4.1, as they are almost identical to the argument in [4, Section 2.3]. Another inequality in the spirit of Theorem 4.1 is the following limit case of Theorem 1.1. Again, the limiting argument is standard and it is omitted.

**Theorem 4.2.** Suppose \( f, g, h, k : \mathbb{R}^n \to [0, \infty) \) are Borel functions satisfying
\[
 f(x)g(y) \leq h(x \wedge y)k(x \vee y) \quad \forall x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n
\]
where \( x \wedge y = (\min(x_1, y_1), \ldots, \min(x_n, y_n)) \), and \( x \vee y = (\max(x_1, y_1), \ldots, \max(x_n, y_n)) \).
Then
\[
 \left( \int_{\mathbb{R}^n} f(x) \, dx \right) \left( \int_{\mathbb{R}^n} g(x) \, dx \right) \leq \left( \int_{\mathbb{R}^n} h(x) \, dx \right) \left( \int_{\mathbb{R}^n} k(x) \, dx \right).
\]

4.2. **A discrete Brunn-Minkowski inequality.** Recently, the following inequality was proven by Iglesias, Yepes Nicolás and Zvavitch [7]:

**Theorem 4.3.** Let \( \lambda \in [0, 1] \). For any two bounded non-empty sets \( K, L \subseteq \mathbb{R}^n \), we have
\[
 G_n(\lambda K + (1 - \lambda)L + (-1, 1)^n)^{1/n} \geq \lambda G_n(K)^{1/n} + (1 - \lambda)G_n(L)^{1/n},
\]
where \( G_n(M) \) denotes the number of lattice points in \( M \subseteq \mathbb{R}^n \).

We recover a multiplicative version of Theorem 4.3 for \( \lambda = 1/2 \): Let \( f(x) = 1_K(x) \), \( g(x) = 1_L(x) \) be the indicator functions of \( K \) and \( L \), and \( h(x) = k(x) = 1_{1/2(K+L+[-1,1]^n)} \).
Note that for every \( x \in K \) and \( y \in L \), we have
\[
 \left\lfloor \frac{x + y}{2} \right\rfloor \in \frac{K + L}{2} - [0, 1/2]^n \quad \text{and} \quad \left\lceil \frac{x + y}{2} \right\rceil \in \frac{K + L}{2} + [0, 1/2]^n,
\]

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which implies that \( f(x)g(y) \leq h\left(\left\lfloor \frac{x+y}{2} \right\rfloor \right)k\left(\left\lfloor \frac{x+y}{2} \right\rfloor \right) \) for all \( x, y \in \mathbb{Z}^n \). By Theorem 1.2, we have

\[
\sqrt{G_n(K)G_n(L)} \leq G_n\left(\frac{K + L + [-1, 1]^n}{2}\right),
\]

as follows also from Theorem 4.3.

4.3. A remark on periodic functions. We say that a function \( \varphi : \mathbb{Z}^n \to [0, \infty) \) is coordinate-wise periodic with respect to \( (p_1, \ldots, p_n) \in \mathbb{Z}_+^n \) if

\[
\varphi(x_1 + k_1p_1, x_2 + k_2p_2, \ldots, x_n + k_np_n) = \varphi(x_1, \ldots, x_n) \quad \forall (x_1, \ldots, x_n), (k_1, \ldots, k_n) \in \mathbb{Z}^n.
\]

Note that \( p_i = 0 \) for some \( i \) simply means that \( \varphi \) is not assumed to be periodic with respect to the \( i^{th} \) coordinate. We prove the following simple extension of Theorem 1.2:

**Proposition 4.4.** Let \( \lambda \in [0, 1] \). Suppose \( f, g, h, k : \mathbb{Z}^n \to [0, \infty) \) are coordinate-wise periodic with respect to \( (p_1, \ldots, p_n) \in \mathbb{Z}_+^n \), and

\[
(11) \quad f(x)g(y) \leq h\left(\left\lfloor \lambda x + (1-\lambda)y \right\rfloor \right)k\left(\left\lfloor (1-\lambda)x + \lambda y \right\rfloor \right) \quad \forall x, y \in \mathbb{Z}^n.
\]

Then

\[
\left(\sum_{x \in G} f(x)\right)\left(\sum_{x \in G} g(x)\right) \leq \left(\sum_{x \in G} h(x)\right)\left(\sum_{x \in G} k(x)\right),
\]

where \( G = (\mathbb{Z}/p_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p_n\mathbb{Z}) \), and the definitions of \( f, g, h, k \) carry over from \( \mathbb{Z}^n \) to \( G \) in the obvious manner.

**Proof.** We assume that \( p_i > 0 \) for all \( i \) (the proof can be easily modified if \( p_i = 0 \) for some \( i \)). Observe that the function \( e^{-\|x\|_1} = e^{-((|x_1|+\cdots+|x_n|)} \) satisfies

\[
(12) \quad e^{-\|x\|_1}e^{-\|y\|_1} \leq e^{-\|\lambda x + (1-\lambda)y\|_1}e^{-\|\lambda x + (1-\lambda)y\|_1} \quad \forall x, y \in \mathbb{Z}^n.
\]

Indeed, for any real number \( a_i \) between \( x_i, y_i \in \mathbb{Z} \) we have

\[
|a_i| + |x_i + y_i - a_i| \leq |x_i| + |y_i|.
\]

By applying this for \( a_i = \lfloor \lambda x_i + (1-\lambda)y_i \rfloor \) and summing over \( i \) we obtain (12). Define \( \tilde{f}(x) = f(x)e^{-\varepsilon\|x\|_1}, \tilde{g}(x) = g(x)e^{-\varepsilon\|x\|_1}, \tilde{h}(x) = h(x)e^{-\varepsilon\|x\|_1}, \) and \( \tilde{k}(x) = k(x)e^{-\varepsilon\|x\|_1} \). From (11) and (12), we see that

\[
\tilde{f}(x)\tilde{g}(y) \leq \tilde{h}\left(\left\lfloor \lambda x + (1-\lambda)y \right\rfloor \right)\tilde{k}\left(\left\lfloor (1-\lambda)x + \lambda y \right\rfloor \right).
\]

Therefore, by Theorem 1.2, we have

\[
(13) \quad \left(\sum_{x \in \mathbb{Z}^n} \tilde{f}(x)\right)\left(\sum_{x \in \mathbb{Z}^n} \tilde{g}(x)\right) \leq \left(\sum_{x \in \mathbb{Z}^n} \tilde{h}(x)\right)\left(\sum_{x \in \mathbb{Z}^n} \tilde{k}(x)\right).
\]
Next, note that for any function $\varphi : \mathbb{Z}^n \to [0, \infty)$ which is coordinate-wise periodic with respect to $(p_1, \ldots, p_n)$, the function $\tilde{\varphi}(x) = \varphi(x)e^{-\varepsilon \|x\|_1}$ satisfies

\begin{equation}
\lim_{\varepsilon \to 0} \left( \left( \frac{\varepsilon}{2} \right)^n \prod_{i=1}^n p_i \right) \sum_{x \in \mathbb{Z}^n} \tilde{\varphi}(x) = \sum_{x \in G} \varphi(x).
\end{equation}

Indeed,

\[
\sum_{x \in \mathbb{Z}^n} \tilde{\varphi}(x) = \sum_{j_1=0}^{p_1-1} \cdots \sum_{j_n=0}^{p_n-1} \varphi(j_1, \ldots, j_n) \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} e^{-\varepsilon \| (j_1, \ldots, j_n) + (k_1 p_1, \ldots, k_n p_n) \|_1}
\]

\[
= \sum_{j_1=0}^{p_1-1} \cdots \sum_{j_n=0}^{p_n-1} \varphi(j_1, \ldots, j_n) \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} e^{-\varepsilon |(j_1 + k_1 p_1)|} \cdots e^{-\varepsilon |(j_n + k_n p_n)|}
\]

\[
= \sum_{j_1=0}^{p_1-1} \cdots \sum_{j_n=0}^{p_n-1} \varphi(j_1, \ldots, j_n) \prod_{i=1}^n \left( \frac{2}{\varepsilon p_i} + O(1) \right),
\]

where $O(1)$ is an abbreviation for a quantity that remains bounded as $\varepsilon \to 0$, uniformly in $j_1, \ldots, j_n$. Going back to (13) and applying (14), we complete the proof. \hfill \Box

\begin{thebibliography}{9}

[1] R. Ahlswede and D. E. Daykin, An inequality for the weights of two families of sets, their unions and intersections, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete \textbf{43} (1978), no. 3, 183–185.

[2] C. Borell, Convex set functions in $d$-space, Period. Math. Hungar. \textbf{6} (1975), no. 2, 111–136.

[3] D. Cordero-Erausquin and B. Maurey, Some extensions of the Prékopa-Leindler inequality using Borell’s stochastic approach, Stud. Math. \textbf{238} (2017), no. 3, 201–233.

[4] N. Gozlan, C. Roberto, P.-M. Samson, and P. Tetali, Transport proofs of some discrete variants of the Prékopa-Leindler inequality, arXiv:1905.04038 (2019).

[5] G. Grimmett, Percolation, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 321, Springer-Verlag, Berlin, 1999.

[6] G. Grimmett, Probability on graphs, Institute of Mathematical Statistics Textbooks, vol. 1, Cambridge University Press, Cambridge, 2010, Random processes on graphs and lattices.

[7] D. Igelsias López, J. Yepes Nicholás, and A. Zvavitch, Brunn-minkowski type inequalities for the lattice point enumerator, arXiv:1911.12874 (2019).

[8] B. Klartag and J. Lehec, Poisson processes and a log-concave Bernstein theorem, Stud. Math. \textbf{247} (2019), no. 1, 85–107.

[9] G. Pisier, The volume of convex bodies and Banach space geometry, Cambridge Tracts in Mathematics, vol. 94, Cambridge University Press, Cambridge, 1989.

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