Quantum Mechanics as a Classical Theory
VII: The Classical Spin Eigenfunctions

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December 13, 2021

Abstract

In this continuation paper the Schrödinger equation for the half-integral spin eigenfunctions is obtained and solved. We show that all the properties already derived using the Heisemberg matrix calculation and Pauli’s matrices are also obtained in the realm of these analytical functions. We also show that Einstein-Bose condensation for fermions is expected. We then conclude this series of two papers on the concept of classical spin.

1 Introduction

In paper VI (hereafter VI) of this series we developed a method by means of which it was possible to derive the classical coordinate-momentum representation for the behavior of particles with half-integral spin.

It was shown there that the matrix representation of this calculation is readily obtained when we pass from the active to the passive views and change the Poisson bracket to its similar matrix commutator. Then all matrix mechanics related to the spin property was derived by straightforward calculations.

We will now make use of the fact that our classical calculations give us the coordinate-momentum representation of the spin property which is suitable to be quantized—in the Schrödinger or Dirac’s sense—to derive the Schrödinger equation for the spin eigenfunction. This will be done in the second section of this paper. This will show that Heisemberg’s matrix calculus and Schrödinger’s analytical equation give identical results.

In the third section the Schrödinger equation for the spin will be solved and we will show that all the properties derived by means of the algebraic calculations are also obtainable with analytical functions.
We will devote the fourth section to show quantitatively that fermionic Einstein-Bose condensation is expected by the present theory.

In the last section we will make our conclusions.

2 The Spin Schrödinger Equation

We are now in position to derive the Schrödinger equation that will enable us to obtain the half-integral spin eigenfunctions.

The two functions

\[ S^2 \text{ and } S_3 \]  

have to be written as operators in a quantization procedure. The spin eigenfunction will be the function that makes these operators diagonal.

The quantization procedure has to be undertaken with much care since the function \( S^2 \) has terms with products of position and momentum operators which do not commute. To see this one needs only to look at this function written in the coordinate-momentum representation

\[ S^2 = \frac{1}{16} \left[ \frac{\alpha}{\beta} (x^2 + y^2)^2 + \frac{\beta}{\alpha} (p_x^2 + p_y^2)^2 + 2 (x^2 + y^2) (p_x^2 + p_y^2) \right]. \]  

This task is greatly simplified if we note that we might write

\[ S^2 = \frac{1}{4} S_0^2, \]  

where

\[ S_0 = \frac{1}{2} \left[ \sqrt{\frac{\alpha}{\beta}} (x^2 + y^2) + \sqrt{\frac{\beta}{\alpha}} (p_x^2 + p_y^2) \right]. \]  

This states the very difference from this problem to the one usually found in textbooks on the solution of orbital angular momentum Schrödinger equations. While in the later case one has only the operator \( L^2 \) with the operator \( L \), its square root, unknown, in the present approach both are known by principle and their relation is defined by equation (3). This means that we will have only to find the function that makes the operators \( S_0 \) and \( S_3 \) diagonal to automatically make \( S^2 \) also diagonal.

The next step is to transform the coordinate-momentum representation of our problem into an operator representation.

We begin with equation (2) and quantize the function \( S_0 \) in the usual way, giving

\[ \hat{S}_0 = \frac{1}{2} \left[ \sqrt{\frac{\alpha}{\beta}} (\hat{x}^2 + \hat{y}^2) + \sqrt{\frac{\beta}{\alpha}} (\hat{p}_x^2 + \hat{p}_y^2) \right], \]  

and also

\[ \hat{S}^2 = \frac{1}{16 \gamma^2} \left[ (\hat{p}_x^2 + \hat{p}_y^2) + \gamma^2 (\hat{x}^2 + \hat{y}^2) \right]^2. \]
where
\[ \gamma = \sqrt{\frac{\alpha}{\beta}}. \]  

(7)

We might develop the product represented in expression (6) as a repeated application of a differential operator to get
\[
\hat{S}^2 = \frac{1}{16\gamma^2} \left\{ \left( \hat{p}_x^2 + \hat{p}_y^2 \right)^2 + \gamma^4 \left( \hat{x}^2 + \hat{y}^2 \right)^2 + 2\gamma^2 \left( \hat{x}^2 + \hat{y}^2 \right) \left( \hat{p}_x^2 + \hat{p}_y^2 \right) \right\} - 4i\hbar\gamma^2 \left( \hat{x}\hat{p}_x + \hat{y}\hat{p}_y \right) - 4\gamma^2\hbar^2
\]

(8)

where the crossed terms in momentum-coordinates were treated using Dirac’s symmetrization procedure according to which
\[
\hat{x}\hat{p}_x \rightarrow \frac{1}{2} \left( \hat{x}^2\hat{p}_x^2 + \hat{p}_x^2\hat{x}^2 \right) = \hat{x}^2\hat{p}_x^2 - 2i\hbar\hat{x}\hat{p}_x - \hbar^2,
\]

or simply developing the squared operator applied upon some function. The first four terms in expression (8) represent \( \hat{S}^2_0 \) which means that we might write this expression as
\[
\hat{S}^2 = \frac{1}{4} \hat{S}^2_0 - \frac{\hbar^2}{4}
\]

(10)
as our final operator \( \hat{S}^2 \).

Looking at expressions (10) we see that, if equation
\[
\hat{S}_0\psi = \hbar\lambda\psi
\]

is satisfied, then the equation
\[
\hat{S}^2\psi = \left( \frac{1}{4} \hat{S}^2_0 - \frac{\hbar^2}{4} \right) \psi = \hbar^2 \left( \lambda^2 - \frac{1}{4} \right) \psi
\]

(12)
is automatically satisfied and we have a relation between the eigenvalues of (11) and (12).

Equation (11) might be written, in a differential form and in rectangular coordinates, as
\[
\frac{1}{2} \left[ -\hbar^2 \frac{\beta}{\alpha} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi + \sqrt{\frac{\alpha}{\beta}} \left( x^2 + y^2 \right) \psi \right] = \hbar\lambda\psi.
\]

(13)

We might now introduce polar coordinates
\[
x = r\cos\theta; \quad y = r\sin\theta
\]

(14)

and
\[
\rho = \left( \frac{\alpha}{\beta\hbar^2} \right)^{1/4} \frac{1}{r}
\]

(15)
to find equation
\[
- \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial \psi}{\partial \rho} \right) + \left( \rho^2 - 2\lambda - \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \right) \psi = 0
\] (16)

together with the equation for $S_3$ given by
\[
- \frac{i\hbar}{2} \frac{\partial \psi}{\partial \theta} = m\psi.
\] (17)

Equation (17) might be solved by putting
\[
\psi (\rho, \theta) = e^{i2m\theta/\hbar} R(\rho)
\] (18)
to get for equation (16)
\[
- \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial R}{\partial \rho} \right) + \left( \rho^2 - 2\lambda + \frac{4m^2}{\rho^2} \right) R = 0.
\] (19)

This equation will be solved in the next section. However, it is noteworthy that, for the wave function to be a continuous function of $\theta$, when a rotation by an angle of $2k\pi$ radians is performed, we must have only half-integral values for $m$.

### 3 Solution of the Equation

The solution of equation (19) follows a standard method which we will only briefly sketch here. We begin by looking at the asymptotic behavior of function $R(\rho)$ for very large values of the variable $\rho$. For this case we put
\[
R(\rho) \xrightarrow{\rho \to \infty} e^{-k\rho^2}
\] (20)
to find
\[
k = +1/2
\] (21)
as the only physically acceptable asymptotic behavior.

For small values of the variable $\rho$ we might try
\[
R(\rho) \xrightarrow{\rho \to 0} \rho^s
\] (22)
to find
\[
s = \pm 2|m|, \pm 2|m| - 1,
\] (23)
giving the four possible asymptotic behaviors
\[
R(\rho) \xrightarrow{\rho \to 0} \rho^{\pm 2|m|}, \rho^{\pm 2|m| - 1},
\] (24)
where the first two choices select the even series by letting \(a_0\) free (and making \(a_1 \equiv 0\)) and the second two choices select the odd series by letting \(a_1\) free (and making \(a_0 \equiv 0\)). We shall choose the plus sign on both cases because the wave function has to be finite at the origin \(\rho = 0\). We then have

\[
R(\rho) \xrightarrow{\rho \to 0} \rho^{2|m|}, \rho^{2|m|-1}.
\]  

(25)

We next search for an expression for all values of \(\rho\) using the series expansion

\[
R(\rho) = \rho^s e^{-\frac{1}{2} \rho^2} \sum_n a_n \rho^n.
\]  

(26)

Substitution of expression (26) into equation (19) gives the following equation

\[
\sum_n \rho^{-2} \left\{ \left[ (s+n)^2 - 4|m|^2 \right] a_n \rho^n - 2[\lambda - (1+s+n)] a_n \rho^{n+2} \right\} = 0,
\]  

(27)

which gives, for the coefficients, the recursion relation

\[
a_{n+2} = \frac{2 [(1+s+n) - \lambda] (n+s+2)^2 - 4|m|^2}{(n+s+2)^2 - 4|m|^2} a_n.
\]  

(28)

At this point it is easy to see that both choices of \(s\) in equation (25) will give exactly the same recursion relation and so, also the same series—direct substitution of \(s = 2|m|\) with \(n\) even and substitution of \(s = 2|m|-1\) with \(n\) odd shows this. We then will work only with the choice

\[
s = 2|m|.
\]  

(29)

If the series in expression (26) does not terminate, its asymptotic behavior when \(n \to \infty\) is

\[
\frac{a_n}{a_{n-2}} \to \frac{2}{n},
\]  

(30)

which is the same as the term \(\rho^n e^\rho^2\) and is not acceptable as a physical asymptotic behavior. Then the series shall terminate; this is accomplished by making the choice

\[
\lambda_N = (1 + 2|m| + N),
\]  

(31)

for some value of \(n = N\). Since \(N\) must be a positive number, we automatically find the relation

\[
|m| \leq \frac{\lambda_N - 1}{2}.
\]  

(32)

The correct eigenfunction of the problem is given by

\[
R(\rho) = \rho^{2|m|} e^{-\frac{1}{2} \rho^2} \sum_{n=0}^{N} a_n \rho^n
\]  

(33)
and is a solution of the system

\[
S_3 \psi = \hbar m \psi ; \quad S^2 \psi = \hbar^2 \left( \frac{\lambda N - 1}{2} \right) \left( \frac{\lambda N + 1}{2} \right) \psi. \tag{34}
\]

Comparing this last expression with

\[
S^2 \psi = \hbar^2 \ell (\ell + 1) \psi, \tag{35}
\]

we find that

\[
\ell = \frac{\lambda N - 1}{2}. \tag{36}
\]

Equation (32) then means that

\[
|m| \leq \ell \tag{37}
\]
as expected.

The multiplicity of our functions might be calculated with the use of the quantum number \(\ell\). One might easily check that this multiplicity is given by

\[
2\ell + 1 = \lambda N. \tag{38}
\]

The final eigenfunction of our problem might be written in the \((r, \theta)\) representation as

\[
\psi(r, \theta) = \left( \frac{\alpha}{\beta \hbar^2} \right)^{|m|/2} r^{2|m|} e^{2i m \theta / \hbar} e^{-\frac{1}{2} \left( \alpha / \beta \hbar^2 \right) r^2} \sum_{n=0}^{N} a_n \left( \frac{\alpha}{\beta \hbar^2} \right)^{n/4} r^n, \tag{39}
\]

where the coefficients \(a_n\) are given by expression (28) and the ratio \(\alpha / \beta\) is a structure constant used to identify the actual particle we are interested in—as seen in expression (6). This structure constant is necessary since our calculations were general and reflect the behavior of any half-integer spin particle. It is then possible to calculate quantities such as the radius of the particle in terms of this constant by performing the integral

\[
\overline{r}_{\ell,|m|}(\alpha / \beta) = \int \psi_{\ell,|m|}(r, \theta) r \psi_{\ell,|m|}(r, \theta) r dr d\theta. \tag{40}
\]

As an example we might find the radius of a half spin particle \((m = 1/2, \ell = 1/2)\) using its density function

\[
d(r) = |\psi(r, \theta)|^2 = N^2 r^2 e^{-\left( \frac{\alpha}{\beta} \right)^{1/2} r^2}, \tag{41}
\]

where \(N\) is a normalization constant. This density is related with the internal structure distribution of the half-integral spin particles and shall not be interpreted as probability distributions—we might get the mass distribution of the electron by multiplying the above expression by its mass, for example.
The mean radius of this particle is

\[
<r_{1/2,1/2}> = \frac{\int_0^\infty r^4 e^{-\frac{\alpha}{\beta} \bar{h} r^2} \, dr}{\int_0^\infty r^3 e^{-\frac{\alpha}{\beta} \bar{h} r^2} \, dr} = \frac{3\sqrt{\pi} \bar{h}}{4} \left( \frac{\alpha}{\beta} \right)^{-1/4},
\]

where we notice that the bigger the structure constant, the smallest the particle.

The first possible values of \( \lambda \) are given in Table I. In this table we show the \( \ell \) value related to the chosen \( \lambda \) value and all possible values of \( |m| \). The values of the cutoff number \( N \) and the multiplicity associated with \( \ell \) is also shown.

It is remarkable that if we try to put odd values for variable \( \lambda \), or else, integer values for variable \( \ell \), we cannot terminate the series in (39), since in this case the cutoff number \( N \) is odd, which is not allowed by our choice of the asymptotic behavior at the origin in expression (29). Since the series does not terminate its asymptotic behavior for large values of variable \( \rho \) is no more given by expression (20) and we must reject these solutions.

We also plot the density distributions for the cases \((\ell = 1/2, |m| = 1/2), (\ell = 7/2, |m| = 7/2)\) and \((\ell = 9/2, |m| = 3/2)\) in figures 1, 2 and 3 respectively.

Figure 1 shows that the \( \ell = 1/2 \) spin particles structure is somewhat like a ring (zero density at the origin) with maximum density at a radius \( r_{\ell=1/2,|m|=1/2} \), depending on the structure constant.

In figure 2 the same behavior of figure 1 is attained but we might see that the distance of the maximum density from the origin is now \( r_{\ell=7/2,|m|=7/2}(\alpha/\beta) \) which is greater than the one for the half spin case. We can visualize this as the increasing of a ‘centrifugal’ force giving the screening of the particle matter distribution—we are considering the same value for the structure constant.

Figure 3 shows that the same ring structure will be present in all functions (they depend on \( r^2|m| \) which always makes the densities tend to zero in the vicinities of the origin). For the cases where the difference in the quantum numbers \( \ell \) and \( |m| \) are different from zero we also find the appearance of nodes reflecting the multiple ring structure of these particles. The number of these concentric rings will be given by the expression \((\ell - |m|) + 1\).

4 Bose-Einstein Condensation

In the previous paper (VI) we have shown that, if the parameter \( \lambda \) has a lower bound, then the phenomenon of Bose-Einstein condensation will be expected for some temperature \( T_{\text{cond}} \). This is precisely what we have found. In Table I we note that the parameter \( \lambda \) has the value \( \lambda_{\text{min}} = 2 \) as a lower bound.

This implies that the fermion shall be supplied with at least the energy

\[
E_{\text{min}} = 2\hbar \omega
\]

(43)

to continue to act as a fermion.

Because of this lower bound for the energy of the fermions we expect that, at some value of the temperature, condensation takes place.
5 Conclusion

In these two papers we aimed at showing that: (1) the concept of spin is not a particularity of the quantum mechanical formalism and might also be represented in the realm of classical mechanics from where we can extract a model for it (or a picture). (2) even in this case ‘space quantization’ might be obtained apart from a constant (phenomenologically obtainable) which we relate to Planck’s constant. (3) the concept of spin is related to the symmetries generated by the Lie algebra associated with its Lie Group (SU(2)); since this group is the same generated by classical phase-space functions obeying the same Lie algebra with the product defined as the Poisson bracket, there is no impossibility in deriving the concept of classical spin. (4) this classical representation of the spin might be ‘quantized’ using traditional methods to derive a Schrödinger equation which is an analytical representation for this quantity. The solution of this equation, when squared, will give us information on the particle internal structure, as for example its mass or charge distribution or its mean radius, in terms of some characteristic constant related to each particle and identifying it. (5) all expected quantum properties already obtained by Heisenberg matrix calculations, using Pauli’s matrices, are also obtained with this method and this result reaffirms the formal identity between Heisenberg’s matrix calculus and Schrödinger’s one. (6) spin is not a characteristic of relativistic calculations although it is better represented in the realm of this theory where Lorentz invariance might be imposed. (7) fermionic Bose-Einstein condensation has a very intuitive explanation by means of the minimum energy the fermions have to possess in order to behave like fermions.

We hope that these calculations will help in clarifying some misconceptions rather diffused in the literature about the classical versus quantum status of half-spin particles and also about the possibility of an analytical representation of the concept of half-spin.

Aside the epistemological aspects, we also hope that the possession of the spin eigenfunctions will help the investigations in areas such as superconductivity, Bose-Einstein condensation and many others.

6 Acknowledgements

The authors wish to thanks the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for sponsoring this research.
Table 1: Values of $\lambda$ or $\ell$ in terms of $m$ and $N$. The multiplicity of each choice is also shown.
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