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HILBERT DOMAINS QUASI-ISOMETRIC TO NORMED VECTOR SPACES

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ABSTRACT. We prove that a Hilbert domain which is quasi-isometric to a normed vector space is actually a convex polytope.

1. INTRODUCTION

A Hilbert domain in $\mathbb{R}^m$ is a metric space $(C, d_C)$, where $C$ is an open bounded convex set in $\mathbb{R}^m$ and $d_C$ is the distance function on $C$ — called the Hilbert metric — defined as follows.

Given two distinct points $p$ and $q$ in $C$, let $a$ and $b$ be the intersection points of the straight line defined by $p$ and $q$ with $\partial C$ so that $p = (1 - s)a + sb$ and $q = (1 - t)a + tb$ with $0 < s < t < 1$. Then

$$d_C(p, q) := \frac{1}{2} \ln [a, p, q, b],$$

where

$$[a, p, q, b] := \frac{1 - s}{s} \times \frac{t}{1 - t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points $(a, p, q, b)$.

We complete the definition by setting $d_C(p, p) := 0$.

The metric space $(C, d_C)$ thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of $\mathbb{R}^m$ and in which the affine open segments joining two points of the boundary $\partial C$ are geodesics that are isometric to $(\mathbb{R}, | \cdot |)$.

For further information about Hilbert geometry, we refer to [4, 5, 9, 11] and the excellent introduction [15] by Socié-Méthou.

The two fundamental examples of Hilbert domains $(C, d_C)$ in $\mathbb{R}^m$ correspond to the case when $C$ is an ellipsoid, which gives the Klein model of $m$-dimensional hyperbolic geometry (see for

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example ([13], first chapter]), and the case when the closure $\overline{C}$ is a $m$-simplex for which there exists a norm $\|\cdot\|_C$ on $\mathbb{R}^m$ such that $(\overline{C}, d_C)$ is isometric to the normed vector space $(\mathbb{R}^m, \|\cdot\|_C)$ (see [8], pages 110–113 or [14], pages 22–23).

Much has been done to study the similarities between Hilbert and hyperbolic geometries (see for example [7], [16] or [1]), but little literature deals with the question of knowing to what extent a Hilbert geometry is close to that of a normed vector space. So let us mention three results in this latter direction which are relevant for our present work.

**Theorem 1.1** ([10], Theorem 2). A Hilbert domain $(\mathcal{C}, d_C)$ in $\mathbb{R}^m$ is isometric to a normed vector space if and only if $\mathcal{C}$ is the interior of a $m$-simplex.

**Theorem 1.2** ([6], Theorem 3.1). If $\mathcal{C}$ is an open convex polygonal set in $\mathbb{R}^2$, then $(\mathcal{C}, d_C)$ is Lipschitz equivalent to Euclidean plane.

**Theorem 1.3** ([2], Theorem 1.1. See also [17]). If $\mathcal{C}$ is an open set in $\mathbb{R}^m$ whose closure $\overline{\mathcal{C}}$ is a convex polytope, then $(\mathcal{C}, d_C)$ is Lipschitz equivalent to Euclidean $m$-space.

In light of these three results, it is natural to ask whether the converse of Theorem 1.3 — which generalizes Theorem 1.2 in higher dimensions — holds. In other words, if a Hilbert domain $(\mathcal{C}, d_C)$ in $\mathbb{R}^m$ is quasi-isometric to a normed vector space, what can be said about $\mathcal{C}$? Here, by quasi-isometric we mean the following (see [3]):

**Definition 1.1.** Given real numbers $A \geq 1$ and $B \geq 0$, a metric space $(S, d)$ is said to be $(A, B)$-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ if and only if there exists a map $f : S \rightarrow V$ such that

$$\frac{1}{A} d(p, q) - B \leq \|f(p) - f(q)\| \leq Ad(p, q) + B$$

for all $p, q \in S$.

We can now state the result of this paper which asserts that the converse of Theorem 1.3 is actually true:

**Theorem 1.4.** If a Hilbert domain $(\mathcal{C}, d_C)$ in $\mathbb{R}^m$ is $(A, B)$-quasi-isometric to a normed vector space $(V, \|\cdot\|)$ for some real constants $A \geq 1$ and $B \geq 0$, then $\mathcal{C}$ is the interior of a convex polytope.

2. Proof of Theorem 1.4

The proof of Theorem 1.4 is based on an idea developed by Förtsch and Karlsson in their paper [10]. It needs the following fact due to Karlsson and Noskov:

**Theorem 2.1** ([12], Theorem 5.2). Let $(\mathcal{C}, d_C)$ be a Hilbert domain in $\mathbb{R}^n$ and $x, y \in \partial \mathcal{C}$ such that $[x, y] \not\subseteq \partial \mathcal{C}$. Then, given a point $p_0 \in \mathcal{C}$, there exists a constant $K(p_0, x, y) > 0$ such that for any sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $\mathcal{C}$ that converge respectively to $x$ and $y$ in $\mathbb{R}^m$ one can find an integer $n_0 \in \mathbb{N}$ for which we have

$$d_C(x_n, y_n) \geq d_C(x_n, p_0) + d_C(y_n, p_0) - K(p_0, x, y)$$

for all $n \geq n_0$. 
Now, here is the key result which gives the proof of Theorem 1.4:

**Proposition 2.1.** Let \((\mathcal{C}, d_{\mathcal{C}})\) be a Hilbert domain in \(\mathbb{R}^n\) which is \((A, B)\)-quasi-isometric to a normed vector space \((V, \|\cdot\|)\) for some real constants \(A \geq 1\) and \(B \geq 0\). Then, if \(N = N(A, \|\cdot\|)\) denotes the maximum number of points in the ball \(\{v \in V \mid \|v\| \leq 2A\}\) whose pairwise distances with respect to \(\|\cdot\|\) are greater than or equal to \(1/(2A)\), and if \(X \subseteq \partial \mathcal{C}\) is such that \([x, y] \not\subseteq \partial \mathcal{C}\) for all \(x, y \in X\) with \(x \neq y\), we have

\[
\text{card}(X) \leq N.
\]

**Proof.**

Let \(f : \mathcal{C} \rightarrow V\) such that

\[
\frac{1}{A} d_{\mathcal{C}}(p, q) - B \leq \|f(p) - f(q)\| \leq Ad_{\mathcal{C}}(p, q) + B
\]

for all \(p, q \in \mathcal{C}\).

First of all, up to translations, we may assume that \(0 \in \mathcal{C}\) and \(f(0) = 0\).

Then suppose that there exists a subset \(X\) of the boundary \(\partial \mathcal{C}\) such that \([x, y] \not\subseteq \partial \mathcal{C}\) for all \(x, y \in X\) with \(x \neq y\) and \(\text{card}(X) \geq N + 1\). So, pick \(N + 1\) distinct points \(x_1, \ldots, x_{N+1}\) in \(X\), and for each \(k \in \{1, \ldots, N + 1\}\), let \(\gamma_k : [0, +\infty) \rightarrow \mathcal{C}\) be a geodesic of \((\mathcal{C}, d_{\mathcal{C}})\) that satisfies \(\gamma_k(0) = 0\), \(\lim_{t \to +\infty} \gamma_k(t) = x_k\) in \(\mathbb{R}^n\) and \(d_{\mathcal{C}}(0, \gamma_k(t)) = t\) for all \(t \geq 0\).

This implies that for all integers \(n \geq 1\) and every \(k \in \{1, \ldots, N + 1\}\), we have

\[
\left\| \frac{f(\gamma_k(n))}{n} \right\| \leq A + \frac{B}{n}
\]

from the second inequality in Equation 2.1 with \(p = \gamma_k(n)\) and \(q = 0\).

On the other hand, Theorem 2.1 yields the existence of some integer \(n_0 \geq 1\) such that

\[
d_{\mathcal{C}}(\gamma_i(n), \gamma_j(n)) \geq 2n - K(0, x_i, x_j)
\]

for all integers \(n \geq n_0\) and every \(i, j \in \{1, \ldots, N + 1\}\) with \(i \neq j\), and hence

\[
\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{2}{A} - \frac{1}{n} \left( \frac{K(0, x_i, x_j)}{A} + B \right)
\]

from the first inequality in Equation 2.1 with \(p = \gamma_i(n)\) and \(q = \gamma_j(n)\).

Now, fixing an integer \(n \geq n_0 + AB + \max\{K(0, x_i, x_j) \mid i, j \in \{1, \ldots, N + 1\}\}\), we get

\[
\left\| \frac{f(\gamma_k(n))}{n} \right\| \leq 2A
\]

for all \(k \in \{1, \ldots, N + 1\}\) by Equation 2.2 together with

\[
\left\| \frac{f(\gamma_i(n))}{n} - \frac{f(\gamma_j(n))}{n} \right\| \geq \frac{1}{2A}
\]

for all \(i, j \in \{1, \ldots, N + 1\}\) with \(i \neq j\) by Equation 2.3.

But this contradicts the definition of \(N = N(A, \|\cdot\|)\).

Therefore, Proposition 2.1 is proved. \(\Box\)

**Remark.** Given \(v \in V\) such that \(\|v\| = 2A\), we have \(\|v\| = 2A\) and \(\|v - (-v)\| = 2\|v\| = 4A \geq 1/(2A)\), which shows that \(N \geq 2\).

The second ingredient we will need for the proof of Theorem 1.4 is the following:
Proposition 2.2. Let \( C \) be an open bounded convex set in \( \mathbb{R}^2 \).
If there exists a non-empty finite subset \( Y \) of the boundary \( \partial C \) such that for every \( x \in \partial C \) one can find \( y \in Y \) with \([x, y] \subseteq \partial C\), then the closure \( \overline{C} \) is a convex polygon.

Proof. Assume \( 0 \in C \) and let us consider the continuous map \( \pi : R \rightarrow \partial C \) which assigns to each \( \theta \in R \) the unique intersection point \( \pi(\theta) \) of \( \partial C \) with the half-line \( R^+ \cdot (\cos \theta, \sin \theta) \).

For each pair \((x, y) \in \partial C \times \partial C\), denote by \( A(x, y) \subseteq \partial C \) the arc segment defined by \( A(x, y) = \pi([\theta_1, \theta_2]) \), where \( \theta_1 \) and \( \theta_2 \) are the unique real numbers such that \( \pi(\theta_1) = x \) and \( \pi(\theta_2) = y \) with \( \theta_1 \in [0, 2\pi) \) and \( \theta_1 \leq \theta_2 < \theta_1 + 2\pi \).

Before proving Proposition 2.2, notice that adding a point of \( \partial C \) to \( Y \) does not change \( Y \)'s property at all, and therefore we may assume that \( \text{card}(Y) \geq 2 \).

So, write \( Y = \{x_1, \ldots, x_n\} \) with \( x_1 = \pi(\theta_1), \ldots, x_n = \pi(\theta_n) \), where \( \theta_1 \in [0, 2\pi) \) and \( \theta_1 < \cdots < \theta_n < \theta_{n+1} := \theta_1 + 2\pi \), and let \( x_{n+1} := \pi(\theta_{n+1}) = x_1 \).

Fix \( k \in \{1, \ldots, n\} \) and pick an arbitrary \( x \in A(x_k, x_{k+1}) \setminus \{x_k, x_{k+1}\} \).

By hypothesis, one can find \( y \in Y \) with \([x, y] \subseteq \partial C\).

Then the convex set \( C \) is contained in one of the two open half-planes in \( \mathbb{R}^2 \) bounded by the line passing through the points \( x \) and \( y \), and hence either \( A(x, y) = [x, y] \), or \( A(y, x) = [x, y] \).

Since \( x_k \in A(y, x) \) and \( x_{k+1} \in A(x, y) \), we then have \( x_k \in [x, y] \) or \( x_{k+1} \in [x, y] \), which yields \( A(x_k, x) = [x_k, x] \) or \( A(x, x_{k+1}) = [x, x_{k+1}] \).

Conclusion: \( A(x_k, x_{k+1}) = S_k \cup S_{k+1} \), where \( S_k = \{x \in A(x_k, x_{k+1}) | A(x_k, x) = [x_k, x]\} \) and \( S_{k+1} = \{x \in A(x, x_{k+1}) | A(x, x_k+1) = [x, x_{k+1}]\} \).

Now, the set \( S_k \) (resp. \( S_{k+1} \)) satisfies \([x_k, x] \subseteq S_k \) (resp. \([x, x_{k+1}] \subseteq S_{k+1} \)) whenever \( x \in S_k \) (resp. \( x \in S_{k+1} \)).

So, if we consider \( \alpha_0 := \max\{\theta \in [\theta_k, \theta_{k+1}] | A(x_k, \pi(\theta)) = [x_k, \pi(\theta)]\} \), we have \( S_k = [x_k, \pi(\alpha_0)] \) and \( S_{k+1} = [\pi(\alpha_0), x_{k+1}] \).

Hence, \( A(x_k, x_{k+1}) \) is the union of the two affine segments \([x_k, \pi(\alpha_0)] \) and \([\pi(\alpha_0), x_{k+1}] \).

Finally, since \( \partial C = \bigcup_{k=1}^n A(x_k, x_{k+1}) \), this implies that \( \partial C \) is the union of \( 2n \) affine segments in \( \mathbb{R}^2 \), and thus \( \overline{C} \) is a convex polygon. \( \square \)

Before proving Theorem 4.4, let us recall the following useful result, where a convex polyhedron in \( \mathbb{R}^m \) is the intersection of a finite number of closed half-spaces:

Theorem 2.2 ([4], Theorem 4.7). Let \( P \) be a convex set in \( \mathbb{R}^m \) and \( p \in \hat{P} \).
Then \( P \) is a convex polyhedron if and only if all its plane sections containing \( p \) are convex polyhedra.

Proof of Theorem 4.4.
Let \( (C, \mathcal{C}) \) be a non-empty Hilbert domain in \( \mathbb{R}^m \) that is \((A, B)\)-quasi-isometric to a normed vector space \((V, \|\cdot\|)\) for some real constants \( A \geq 1 \) and \( B \geq 0 \).

According to Theorem 2.2, it suffices to prove Theorem 4.3 for \( m = 2 \) since any plane section of \( C \) gives rise to a 2-dimensional Hilbert domain which is also \((A, B)\)-quasi-isometric to \((V, \|\cdot\|)\).

So, let \( m = 2 \), and consider the set \( \mathcal{E} = \{X \subseteq \partial C | [x, y] \nsubseteq \partial C \text{ for all } x, y \in X \text{ with } x \neq y\} \).

It is not empty since \([x, y] \in \mathcal{E} \) for some \( x, y \in \partial C \) with \( x \neq y \) (indeed, \( C \) is a non-empty open set in \( \mathbb{R}^2 \)), which implies together with Proposition 2.1 that \( n = \max\{\text{card}(X) | X \in \mathcal{E} \} \) does exist and satisfies \( 2 \leq n \leq N \) (recall that \( N \geq 2 \)).
Then pick $Y \in \mathcal{E}$ such that $\text{card}(Y) = n$, write $Y = \{x_1, \ldots, x_n\}$, and prove that for every $x \in \partial C$ one can find $k \in \{1, \ldots, n\}$ such that $[x, x_k] \subseteq \partial C$.

Owing to Proposition \ref{prop:convex_polygon}, this will show that $\overline{C}$ is a convex polygon.

So, suppose that there exists $x_0 \in \partial C$ satisfying $[x_0, x_k] \nsubseteq \partial C$ for all $k \in \{1, \ldots, n\}$, and let us find a contradiction by considering $Z := Y \cup \{x_0\}$.

First, since $x_0 \neq x_k$ for all $k \in \{1, \ldots, n\}$ (if not, we would get an index $k \in \{1, \ldots, n\}$ such that $[x_0, x_k] = \{x_0\} \subseteq \partial C$, which is false), we have $x_0 \notin Y$. Hence $\text{card}(Z) = n + 1$.

Next, since $Y \in \mathcal{E}$ and $[x_0, x_k] \nsubseteq \partial C$ for all $k \in \{1, \ldots, n\}$, we have $Z \in \mathcal{E}$.

Therefore, the assumption of the existence of $x_0$ yields a set $Z \in \mathcal{E}$ whose cardinality is greater than that of $Y$, which contradicts the very definition of $Y$.

Conclusion: $\overline{C}$ is a convex polygon, and this proves Theorem \ref{thm:convex_polygon}.

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