A HIERARCHIC ARRAY OF INTEGRABLE MODELS

Peter G. O. Freund
Enrico Fermi Institute and Department of Physics
University of Chicago, Chicago, IL 60637

and

Anton V. Zabrodin
Institute of Chemical Physics
Kosygina Str. 4, SU-117334, Moscow, Russia

ABSTRACT

Motivated by Harish-Chandra theory, we construct, starting from a simple CDD-pole $S$-matrix, a hierarchy of new $S$-matrices involving ever “higher” (in the sense of Barnes) gamma functions. These new $S$-matrices correspond to scattering of excitations in ever more complex integrable models. From each of these models, new ones are obtained either by “$q$-deformation”, or by considering the Selberg-type Euler products of which they represent the “infinite place”. A hierarchic array of integrable models is thus obtained. A remarkable diagonal link in this array is established. Though many entries in this array correspond to familiar integrable models, the array also leads to new models. In setting up this array we were led to new results on the $q$-gamma function and on the $q$-deformed Bloch-Wigner function.

\footnote{Work supported in part by the NSF: PHY-91-23780}
1. INTRODUCTION

The factorized $S$-matrices, which describe the scattering of excitations, are known for many $(1 + 1)$-dimensional integrable models (e.g. XXZ, XYZ models and their generalizations [1], continuum sine-Gordon theory [2], etc...). The Jost functions for the corresponding input 4-point amplitudes have been identified [3],[4] with the explicitly calculable Harish-Chandra $c$-functions of certain, in general quantum symmetric spaces.

Here we wish to draw attention to unexpected connections between various integrable models, which arise when certain coupling parameters assume special rational values. These integrable models then naturally arrange themselves into an “array of models”. By moving from one site of this array to one of its nearest neighbors, the just-mentioned connections between known models come into full play. Sooner or later though, such moves to nearest neighbors in this “array of models” leads us outside the realm of the known and suggest the existence of new integrable models. The hierarchic array is presented in figures 1 and 2, and it leaves, as we shall see, a number of open questions. The very existence of this array is our main result. It implies hierarchical relations between different integrable models, which look both promising and instructive.

2. SINE-GORDON THEORY AT CERTAIN RATIONAL VALUES OF THE COUPLING CONSTANT

To begin with, consider soliton-soliton scattering in the sine-Gordon model. Up to a finite number of, for us irrelevant factors, the Jost function for this process is given by the infinite product [2]

$$A(u) = \prod_{k=1}^{\infty} \frac{\Gamma(2k \frac{8\pi}{\gamma} + i \frac{8u}{\gamma})\Gamma(1 + 2k \frac{8\pi}{\gamma} + i \frac{8u}{\gamma})}{\Gamma((2k + 1) \frac{8\pi}{\gamma} + i \frac{8u}{\gamma})\Gamma(1 + (2k - 1) \frac{8\pi}{\gamma} + i \frac{8u}{\gamma})}$$  \hspace{1cm} (1a)

where $u$ is the relative rapidity,

$$\frac{\gamma}{8\pi} = \frac{\beta^2}{8\pi} \frac{\beta^2}{1 - \beta^2}$$ \hspace{1cm} (1b)

and $\beta$ is the sine-Gordon coupling constant. Now fix

$$\frac{\beta^2}{8\pi} = \frac{2}{3}, \hspace{1cm} \frac{\gamma}{8\pi} = 2.$$ \hspace{1cm} (2a)
Then

\[ A(u) = K \frac{\tilde{c}_2(\frac{u}{\pi})}{\tilde{c}_2(\frac{u}{\pi} + \frac{1}{2})}, \quad (3a) \]

with \( K \) a constant and

\[ \tilde{c}_2(x) = \frac{\Gamma_2(\frac{ix}{2})}{\Gamma_2(\frac{x}{2}) \Gamma_2(\frac{ix}{2} + \frac{1}{2})} \quad (3b) \]

where \( \Gamma_2 \) is the “higher” gamma-function [5] related to Barnes’ \( G \)-function [6] by

\[ \Gamma_2(x) = (2\pi)^{-x/2} e^{-x^2/2} G(x - 1). \quad (4) \]

The equations (3) are remarkable on two counts:

i) the infinite product of eq. (1a) has disappeared (it is “hidden” in \( \Gamma_2 \)), and

ii) the function \( \tilde{c}_2 \) which has appeared, is strongly reminiscent (see eq. (3b)) of the Harish-Chandra \( c \)-function [7] of the real hyperbolic plane \( (\text{SL}(2, \mathbb{R})/\text{SO}(2)) \)

\[ c(x) = \frac{\Gamma(\frac{x}{2})}{\Gamma(\frac{x}{2}) \Gamma(\frac{x}{2} + \frac{1}{2})} \quad (5) \]

The difference between \( c(x) \) and \( \tilde{c}_2(x) \) is visible from eqs. (5) and (3b): the gamma function \( \Gamma(x) \) is replaced by the “higher” gamma function \( \Gamma_2(x) \). A similar \( \Gamma \to \Gamma_2 \) transition occurs not only for \( \beta^2/8\pi = 2/3 \), Eq. (2a), but more generally, also for

\[ \frac{\beta^2}{8\pi} = 1 - \frac{1}{m}, \quad \frac{\gamma}{8\pi} = m - 1 \quad m = 4, 5, 6 \ldots \quad (2b) \]

These are the well-known rational values of \( \beta^2/8\pi \) at which the spectrum truncates, as noted by Leclair [2].

The sine-Gordon soliton-soliton \( S \)-matrix is essentially [4] the \( q \to 1 \) limit of the \( S \)-matrix \( S_1^{(2)} \) for the scattering of two (dressed) excitations in the XYZ model. In terms of the parameter \( l \) introduced in [4], the regime (2) corresponds to the rational values

\[ l = \frac{1}{m - 1} \quad m = 3, 4, 5, \ldots \quad (6) \]
These values of $l$ are special for the transformation \[ l \rightarrow l' = \frac{1}{n} \quad n \rightarrow n' = \frac{1}{l} = m - 1 \] (7)

which takes one from the magnetic model $\mathcal{M}_n$ (corresponding to the $Z_n$-Baxter model) with parameters $q, l$ to the $\mathcal{M}_{n'}$ model with parameters $q, l'$, while leaving the $S$-matrix unchanged. It is for and only for the values (6) of $l$ that, starting from physical values of $l$ and $n$ (i.e. integer $n \geq 2$), the transformation (7) yields a new physical pair of values $l'$ and $n'$ ($n'$ also integer). As will be argued in Section 5 the symmetric space underlying this argument is the double cosetings of a Kac-Moody symmetry and the transformation (7) corresponds to the familiar rank-level duality [8]. The XYZ-like $\mathcal{M}_n$ models yield, for $l$ given by (6), an $S$-matrix similar to (3), but involving a $q$-deformed $\Gamma_2$ function. The situation is similar to “$S$-wave” scattering on ordinary rank one symmetric spaces [9]. There the Jost function is given [10] by the ratio of two ordinary gamma functions (eq. (5) above), which when $q$-deformed is still meaningful: it corresponds [4] to the scattering of two first-level excitations in the $n \rightarrow \infty$ limit of the $\mathcal{M}_n$ model (the $R$-matrix in this limit has been obtained recently by Shibukawa and Ueno [11]), or in an alternative interpretation \cite{3},\cite{4} to the scattering of two dressed excitations in the spin $1/2$ XXZ chain. There is much more to this analogy. To understand this, we first present some old, as well as some new, results on higher gamma functions.

3. ORDINARY AND HIGHER GAMMA FUNCTIONS AND THEIR q-DEFORMATIONS. THE q-BLOCH-WIGNER FUNCTION

The Weierstrass product for the ordinary gamma function is \cite{12}:

\[ \Gamma(x) = \gamma(x) \prod_{a=1}^{\infty} (x + a)^{-1} \] (8a)

with prefactor

\[ \gamma(x) = \frac{1}{x} e^{-Cx}, \] (8b)

$C =$ Euler-Mascheroni constant, and Weierstrass factors

\[ w_a(x) = ae^{x/a} \] (8c)
The gamma function has half the poles of the function $\frac{\pi}{\sin \pi x}$, to which it is related by

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} = \frac{1}{x} \prod_{a \in \mathbb{Z}^*} (x + a)^{-1}a$$  \hspace{1cm} (9)$$

($\mathbb{Z}^*$ = set of nonvanishing integers). $\Gamma(x)$ obeys the functional equation

$$\Gamma(x + 1) = x\Gamma(x)$$  \hspace{1cm} (10)$$
as well as its higher “multiplication” generalization due to Gauss. The relations (8)-(11) allow for two types of generalization relevant for us.

**A) The higher gamma functions**

With the notations

$$\Gamma(x) = G_1(x), \hspace{0.5cm} x = G_0(x)$$  \hspace{1cm} (11)$$
equation (10) can be written as

$$G_1(x + 1) = G_0(x)G_1(x)$$  \hspace{1cm} (12)$$
which invites the recursive generalization

$$G_{n+1}(x + 1) = G_n(x)G_1(x)$$  \hspace{1cm} (13)$$

These equations, along with some other requirements (viz. $G_0(x) = x$, $G_n(1) = 1$, existence of all derivatives of $G_n$ and the condition that the $(n + 1)^{st}$ derivative of $\log G_n(x)$ should be non-negative for real $x \geq 1$ [13]) defines a unique sequence of meromorphic functions $G_n(x)$, $n = 0, 1, 2, \ldots$. $G_2(x)$ is Barnes’ $G$-function [6]. Let us call $n$ the *height* of the higher $G$-function $G_n$. Nowadays it is customary to work not directly with the $G_n$ functions themselves, but with the _higher gamma_ functions $\Gamma_n$ which differ from the $G_n$ by some prefactors [5]. For instance $\Gamma_2(x)$ is related to $G_2 \equiv G$ via eq. (4). We shall also say that the higher gamma function $\Gamma_n$ has **height** $n$.

Like the ordinary gamma functions, the functions $\Gamma_n$ also admit Weierstrass product representations of the form

$$\Gamma_n(x) = \gamma_n(x) \prod_{a=1}^{\infty} (x + a)^{a^{n-1}}e^{w^{(n)}_a(x)}$$  \hspace{1cm} (14)$$
with $\gamma_n, w_a^{(n)}$ given in ref. [5]. Notice that $(\Gamma_n(x))^{-1}$ has poles at the negative integers. The pole at $x = -a$ being of order $a^{n-1}$. We can rewrite eq. (15) as an $n$-fold infinite product

$$
\Gamma_n(x) = \hat{\gamma}_n(x) \prod_{a_1=0}^{\infty} \prod_{a_2=0}^{\infty} \prod_{a_{n-1}=0}^{\infty} \prod_{a_n=1}^{\infty} (x + a_1 + a_2 + \ldots + a_n) \hat{w}_{a_1...a_n}^{(n)}(x) \tag{15}
$$

The factors $\hat{\gamma}_n$, $\hat{w}_{a_1...a_n}^{(n)}$ can be found from the references [5], [6]. Yet another form of the infinite products involves intermediate lower gamma functions instead of $x + a_1 + \ldots + a_n = \Gamma_0(x + a_1 + \ldots + a_n)$. For instance, we can write

$$
\Gamma_2(x) = \hat{\gamma}_2(x) \prod_{a=1}^{\infty} \left[ \Gamma_1(x + a) \right]^{-1} \hat{w}_a^{(2)}(x) \tag{16}
$$

More generally $\Gamma_n$ is an $(n-k)$-fold infinite product of $\Gamma_k$ functions or their inverses, of course with appropriate prefactors and Weierstrass factors.

Just as the ordinary gamma function is “half” of a trigonometric function, eq. (9), so the higher gamma function $\Gamma_2(x), \Gamma_3(x), \ldots$ are also related to “higher” trigonometric functions $\Lambda_2(x), \Lambda_3(x), \ldots$ Eq. (9) generalizes to

$$
\Lambda_n(x) = \Gamma_n(x) \left[ \Gamma_n(-x) \right]^{-(-1)^{n+1}} = \exp \left[ -\pi \int_0^x t^{n-1} \cot \pi t \, dt \right]
$$

$$
n = 2, 3, \ldots \tag{17}
$$

Notice the alternation of product and ratio of $\Gamma_n$’s in the definition of $\Lambda_n$. For $n = 1, \Lambda_1(x)$ is defined [5] as $\pi \exp [-\pi \int_0^x \cot \pi t \, dt] = \pi (\sin \pi x)^{-1}$ which according to Eq. (9) is equal to $\Gamma(x) \Gamma(1-x)$. The higher trigonometric function $\Lambda_n(x)$ is meromorphic of order $n$ and obeys [5]

$$
\log \Lambda_n(x) = \frac{\Gamma(n)}{(2\pi i)^{n-1}} [-\zeta(n) + D_n(x)] \tag{18}
$$

where $\zeta(n)$ is the Riemann zeta function and

$$
D_n(x) = \sum_{k=1}^{n} \frac{(2\pi i x)^{n-k}}{(n-k)!} \text{Li}_k(e^{-i2\pi x}) - \frac{(2\pi i x)^n}{2n!} \tag{19}
$$

is closely related, as noted by Rovinskii [5], to the Bloch-Wigner-Ramakrishnan function

$$
D_n(y) = \text{Re} \left[ i^{n+1} \left( \sum_{k=1}^{n} \frac{(-\log |y|)^{n-k}}{(n-k)!} \text{Li}_k(y) + \frac{\log |y|^n}{2n!} \right) \right] \tag{20}
$$
The function $D_2(y)$, the original Bloch-Wigner function, appears in the calculation of volumes in 3-d hyperbolic (Bolyai-Lobachewsky) geometry [15].

The Euler polylogarithm $\text{Li}_k(x)$, which appears in eqs. (19) and (20) is defined as

$$\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}, \quad (21)$$

It will be important for us that the higher gamma functions play the same role in the functional equation of Selberg zeta functions [16], that the ordinary gamma functions plays in the functional equation of the Riemann zeta function [17]. Selberg zeta functions are associated to double coset spaces $M = \Gamma \backslash G/K$, rank $G/K = 1$, where $K$ is the maximal compact subgroup of $G$ and $\Gamma$ is a suitable discrete subgroup of $G$. To these data one associates, in the simplest case of the trivial representation of $\Gamma$, the Selberg zeta function $Z_M(s)$, defined as usual, by an Euler product. For instance, for $G = SO(2,1)$, $K = SO(2)$, this Euler product is [16], [18]

$$Z_M(z) = \prod_{p \in P} \prod_{k=0}^{\infty} \left(1 - e^{-\tau(p)(s+k)}\right). \quad (22)$$

Here $P$ is the set of all primitive closed geodesics on $M$ (a closed geodesic is primitive if it is not the multiple traversal of another closed geodesic) and $\tau(p)$ is the length of the primitive closed geodesic $p$. Unlike the prime ideals over which the (simple) Euler product is taken in the cases of the Riemann and Dedekind zeta functions, for the Selberg zeta function the product runs over “primitive elements” (geodesics) and is a double product. Like the Riemann zeta function, the Selberg zeta function also obeys a functional equation. For instance, for $G = SO(2n,1)$, $K = SO(2n)$ [19], if we define [13], [18], [20]

$$\check{Z}_M(s) = Z_\infty(s)Z_M(s)$$

$$Z_\infty(s) = [\Gamma_{2n}(s)\Gamma_{2n}(s+1)]^{(-1)^{n+1}volM} \quad (23)$$

($Z_M$ as in eq. (22)), then the functional equation takes the simple form

$$\check{Z}_M(2n - 1 - s) = \check{Z}_M(s). \quad (24)$$

This parallels the Riemann case, where

$$\check{\zeta}(s) = \zeta_\infty(s)\zeta(s) \quad \zeta_\infty(s) = \pi^{-s/2}\Gamma\left(s\frac{1}{2}\right) \quad (25)$$
and the functional equation reads
\[ \hat{\zeta}(1 - s) = \hat{\zeta}(s). \] (26)

The second relevant generalization of this “gamma hierarchy” concerns q-deformations.

B) q-deformed gamma, higher gamma and Bloch-Wigner functions

The q-deformations of \( G_0(x) = \Gamma_0(x) = x \) and of \( G_1(x) = \Gamma_1(x) = \Gamma(x) \) are classical [21]

\[ \Gamma_{0,q}(x) = [x]_q = \frac{1 - qx}{1 - q} \] (27a)

and

\[ \Gamma_{1,q}(x) = (1 - q)^{1-x} \prod_{a=0}^{\infty} [1 + a]_q [x + a]_q^{-1}. \] (27b)

Notice the simplicity of (27b), as compared to eq. (8). The q-deformation of eq. (9) we find to be

\[ \Gamma_{1,q}(x) \Gamma_{1,q}(1 - x) = i(1 - q)(q; q)_{\infty} q^{\frac{1}{8} - \frac{x}{2}} \frac{1}{\vartheta_1(\pi(x|\tau))} \] (28a)

where

\[ q = e^{i2\pi\tau} \]

\[ (q; q)_{\infty} = \prod_{a=1}^{\infty} (1 - q^a) \] (28b)

\[ \vartheta_1(u|\tau) = 2q^{1/8} \sin u \prod_{n=1}^{\infty} (1 - 2q^n \cos 2u + q^{2n})(1 - q^n). \]

In the Jacobi theta function \( \vartheta_1 \) we have used \( q \) related to \( \tau \) as in (28b) rather than the often encountered \( \hat{q} = e^{i\pi\tau} \). Askey [22] has already pointed out that \( \Gamma_{1,q}(x) \Gamma_{1,q}(1 - x) \) should involve Jacobi theta functions. Our equation (28) makes this fact explicit. The Jacobi \( \vartheta_1 \)-function is thus the q-deformation of the trigonometric sine function.
Higher gamma and trigonometric functions can be similarly $q$-deformed. For instance, we define the $q$-deformed higher trigonometric function as (see eq. (17))

$$
\Lambda_{2,q}(x) = \prod_{k=1}^{\infty} \frac{\Gamma_{1,q}(-x+k)}{\Gamma_{1,q}(x+k)}(1-q)^{-2x}.
$$

(29)

The logarithm of $\Lambda_{2}$ will then correspond to a $q$-deformed $D_{2}$ function (Eq. (19)), itself closely related to a $q$-deformed Bloch-Wigner function $D_{2,q}$. We shall refer to $\log \Lambda_{2,q}$ as a $q$-deformed Bloch-Wigner function, though, strictly speaking, it is $D_{2}$ and not $D_{2}$ whose $q$-deformation this provides. A similar $q$-deformed Bloch-Wigner function was considered in a different approach by Bloch and Zagier [23]. For the $q$-analogue of the $n=2$ case of eqs. (18), (19) we then readily find

$$
\log \Lambda_{2,q}(x) = \frac{1}{1-q} \int_0^q \frac{dz}{z} \left\{ \log \left[ i q^{-1/8} z^{-1/2} (q; q)_{\infty} \vartheta_1 \left( \frac{\log z}{2i} | \tau \right) \right] \right\}
$$

(30)

with $\tau, (q; q)_{\infty}$ and $\vartheta_1$ as in (28b) and $\int d_q z$ standing for Jackson $q$-integration [21]. The $q$-integral of $z^{-1} \log \vartheta_1$ is thus the $q$-deformation of the dilogarithmic expression of eq. (19). Similar considerations apply to the $q$-deformation of the higher trigonometric functions and of the higher polylogarithms.

4. THE ARRAY OF GAMMA AND ZETA FUNCTIONS AND THEIR SYMMETRIC SPACES

With the principals all in place, we can now arrange them in an array from which their relations become clear. In this section we present the array as a mathematical construct. In the next section we shall show that to each “site” in this array corresponds a physical scattering problem with “geometric” Jost function. The array is presented in Fig. 1.

The pivotal column of this array is the third column. Each of the entries in this column is a gamma function, the height of which increases in unit steps as we move down the column. The entry at height one (second row) is $\Gamma_1$, the building block à la Harish-Chandra-Bhamamurthy-Gindikin-Karpelevič of the $c$-function of an ordinary symmetric space (see eq. (5) above and ref. [7]). The function $\Gamma_1$ has this property because $\pi^{-s/2} \Gamma_1(s/2)$ is the gamma factor (i.e. the local factor $\zeta_{\infty}(s)$ at the infinite place [10], [24]) in the function $\hat{\zeta}(s)$, (eq. (25)), in terms of which the functional equation for the Riemann zeta function takes the simple form (26).
Descending one step down the third column lands us at the function $\Gamma_2$, itself, as we have seen, the gamma factor for the Selberg zeta function $Z_M(s)$ of a 2-dimensional Riemann surface $M_2$. Further down at height $2n$ we encounter the gamma factors of Selberg zeta functions of the $2n$ dimensional domains $M_{2n} = \Gamma \backslash SO(2n,1)/SO(2n)$. Each of these zeta functions is an Euler product, and the corresponding Euler factor at primitive element $p$ (at height 1 this primitive element is a prime number, at height 2 a primitive geodesic, ...) is displayed in the second column. In the first column we display the Euler product, gamma factor included, i.e. what we called $\tilde{\zeta}(s)$, $\tilde{Z}_M(s)$ in section 2. Finally, in the fourth column we present the $q$-deformation of the gamma function which appears in the third column at the same height (i.e. in the same row).

The new feature in all this is, that the Euler factor (in the second column) at height $h$, coincides, up to a constant factor, with the inverse of the $q$-deformed gamma function (in the fourth column) at height $h-1$, provided the deformation parameter $q$ at height $h-1$ is set equal to the inverse of the “norm” of the primitive element at height $h$. (By norm of a primitive element we mean the prime number $p$ itself at height $h = 1$, $\exp(\tau(p))$ for a primitive closed geodesic $p$ of length $\tau(p)$ at height $h = 2$, etc...) This is shown in fig. 1 by the diagonal arrows. These diagonal arrows express a generalization of the $p$-adics-quantum-group connection [25], [26], [3], [4].

In the array we only showed explicitly heights 0, 1, 2. At larger heights the same pattern is followed, with the remark that for odd dimensional spaces, the vanishing Euler characteristic leads to a simplified functional equation [16], [19], [20].

We can now translate all this into statements about integrable models.

5. THE ARRAY OF INTEGRABLE MODELS

Starting again at height 1 in the pivotal third column, we encounter the building block of the Jost function for “$S$-wave” scattering in the hyperbolic plane $SL(2,\mathbb{R})/SO(2)$, or equivalently for the scattering of dressed excitations in the Heisenberg XXX chain. Next to it (in the second column) is its adelic partner $\zeta_p(x)$, the building block of the Jost function

$$c_p(x) = \frac{\zeta_p(ix)}{\zeta_p(ix + 1)}$$  \hspace{1cm} (31)
for “$S$-wave” scattering on the $p$-adic hyperbolic plane $SL(2, \mathbb{Q}_p)/SL(2, \mathbb{Z}_p)$ \[10\], i.e. the Bethe lattice with incidence number $p + 1$. The Euler product $\hat{\zeta}(x)$ builds the “$S$-wave” scattering on the adelic hyperbolic plane $SL(2, \mathbb{A})/SL(2, \mathbb{Z}_p)$ \[10\], or the closely related scattering on the fundamental domain of $SL(2, \mathbb{Z})$ on the real hyperbolic plane \[27\]. We see thus that height 1 (second row) in the array corresponds to $SL(2)$, at least as far as the first three columns are concerned. The $q$-deformation in the fourth column is the Macdonald-Harish-Chandra \[25\] $c$-function for root system $A_1$, again $SL(2)$. This, as we saw in our earlier work \[4\], is modeled in the scattering of two (first level) excitations in the $n \to \infty$ limit of the $\mathcal{M}_n$ model, as well as \[3\] in the $S$-wave scattering on quantum hyperboloids, or equivalently, in the dressed excitation scattering in the spin $1/2$ XXZ model. So the full height 1 row in the array can be interpreted in terms of ordinary, or as in the fourth column, quantum symmetric spaces and modeled on specific physical systems.

Now let us move down to height 2. In the third column the entry is now a $\Gamma_2$ function. As we saw in eqs. (3) this $\Gamma_2$ builds – much in the way that $\Gamma_1$ did at height 1 – the Jost function for soliton-soliton scattering in the sine-Gordon model at the values (2) of the coupling parameter. What is the symmetric space here? The $\Gamma_2$ function is an infinite product of $\Gamma_1$ factors. This suggests a Gindikin-Karpelevič type product \[7\] over infinitely many roots, therefore a Kac-Moody symmetry $\hat{\mathcal{A}}_k^1$. The level $k$ is related to the parameter $l$. Specifically $k = l^{-1}$. When eqs. (2a) or (6) are imposed this yields $k = m-1$ (remember, $m-1$ is a positive integer). The transformation (7) then interchanges $l^{-1} = k =$ level with $n =$ rank $+1$. In other words, it becomes rank-level duality \[8\] as already mentioned in section 2. The level $k$ is also related to an $SU(2)\tilde{q}$ with deformation parameter $\tilde{q}$, a root of unity $\tilde{q}^{k+2} = 1$ \[28\]. We are dealing with the $c$-function obtained from the zonal spherical functions of this symmetric space. Though the symmetric space itself is infinite-dimensional, it is truncated to a single “radial” dimension by the double coseting involved in going to zonal spherical functions.

From this connection with 1-loop Kac-Moody algebras it also follows that the height $h$ (of a gamma function) introduced above can be interpreted as a “loop number”. More specifically, the usual symmetric space of height 1 corresponds to zero loop number, the height 2 Kac-Moody symmetric space just mentioned to loop number 1 (1-loop algebras). Height 3 is then to be associated with “2-loop-algebras”, etc... Height zero corresponds to loop
number -1.

The $\Gamma_2$ function, as we saw, can be $q$-deformed and the corresponding $\Gamma_{2,q}$ function (fourth column) “builds” the Jost function for the scattering of dressed excitations in the XYZ model, the true $n = 2$ case, for the special values (6) of the parameter $l$ (which as in ref. [4] is given by $l = \frac{\gamma}{\pi \tau}$ with $\gamma$ the anisotropy parameter, and $\tau$ the modular parameter).

Moving now to the second column at height two, we find the Euler partners of $\Gamma_2$ in the Selberg zeta function $\hat{Z}_M$ for which $\Gamma_2$ provides the gamma factor $Z_\infty$. Again, the diagonal connection in the array shows that this yields the dressed excitation scattering in the XXZ model (or more generally in the $\cal M_\infty$ model) as in the fourth column at height 1. Finally, at the “adelic-like” position (first column) at height 2 (third row) we encounter an $S$-matrix built out of Selberg zeta-functions $\hat{Z}_M$ the same way that the adelic $S$-matrix at height 1 was built out of $\hat{\zeta}$ functions. The $S$-matrix build out of $\hat{Z}_M$ factors is new and its physical realization is not known to us.

We can obviously continue this way, down the array and encounter higher ($n > 2$) gamma functions and their $q$-deformations, Euler partners and $\hat{Z}_M$ functions. They correspond to rank one ordinary or quantum symmetric spaces of higher dimensionality. There is the possibility that the corresponding physical models are also higher dimensional. This would be very interesting.

At this point, however, we prefer to take two steps back and move up to height zero (first row). Here $\Gamma_0(x) = x$ is the building block of a Blaschke-CDD factor

$$c_{\text{Blaschke-CDD}}(k) = \Gamma_0(ik - a) = ik - a$$  \hspace{1cm} (32)

whence the $S$-matrix

$$S_{\text{Blaschke-CDD}} = \frac{\Gamma_0(ik - a)}{\Gamma_0(-ik - a)}.$$  \hspace{1cm} (33)

By $q$-deforming it we get the type of factor we encountered at height 1 in the second column ("$S$-wave" scattering on Bethe lattice). This is again the diagonal connection in the array at work. Just as in Section 3, the Euler partners and adelic counterparts of the Blaschke factor, if at all meaningful, are not known to us. We are now in measure to produce the array of integrable models. It is presented in Fig. 2.

This array is the main result of this section. As was emphasized, the array pivots around its third column. Moving down this column means enlarging
the symmetric spaces underlying these models. Moving to the right (to the fourth column) produces spin chains by $q$-deformation. Moving left (to the second column) yields “Euler partners” of the pivotal model of the same height, and finally (first column) “adelic-like” models. Already at height 2 these are new models. Similarly the height 3, 4, ... models are yet to be physically identified. But they are there!

6. CONCLUSIONS

We have found here a hierarchical array of gamma functions and, in effect, of symmetric spaces. A symmetric space can be $q$-deformed (move from the third to the fourth column) or extended while maintaining rank 1 (move down the ordinary third column, or down the quantum fourth column). One can consider its Euler partners (move from the third to the second column) and its “adelic-like” counterpart (move from the second to the first column). To each site in the array we found a corresponding physical model at special rational values of some parameter. Pushing things to the extreme, we were led to $S$-matrices for which the underlying physical model is as yet unknown: those involving Selberg zeta functions and those at height $h > 2$.

The mathematical novelties we encountered were: the $q$-deformed Bloch-Wigner function(eq. (29), (30)), the new $q$-deformed trigonometric function eq. (28), and most importantly the array of gamma functions and symmetric spaces of section 2.

Acknowledgements: We wish to thank Spencer Bloch for very valuable conversations. A. V. Zabrodin wishes to thank the Mathematical Disciplines Center at the University of Chicago for its hospitality and support.

REFERENCES

1. Zamolodchikov, A.B.: Comm. Math. Phys. 69, 165 (1979); Kulish, P.P., and Reshetikhin, N. Yu.: Soviet Phys. JETP 53, 108 (1981).

2. Zamolodchikov, A. B., and Zamolodchikov, Al. B.: Ann. Phys. (N.Y.) 120, 253 (1979); Leclair, A., Phys. Lett. B230, 103 (1989).

3. Zabrodin, A. V., Mod. Phys. Lett. A7, 441 (1992).
4. Freund, P. G. O., and Zabrodin, A. V., Phys. Lett. **284B**, 283 (1992); Comm. Math. Phys, in press.

5. Kurokawa, N., Proc. Japan Acad. **67A**, 61 (1991); Rovinskii, M. Z., Funct. Analysis and Appl. **25**, 74 (1991).

6. Barnes, E. W., Quart. Journal Pure and Appl. Math. **31**, 264 (1900).

7. Helgason, S.: *Topics in Harmonic Analysis on Homogenous Spaces*, Birkhäuser, Basel, 1981.

8. Frenkel, I., Lect. Notes Math. no. 933 Springer, Berlin 1982, p. 71; Jimbo M., and Miwa, T., Adv. Stud. Pure Math. **4**, 97 (1984).

9. Olshanetsky, M. A., and Perelomov, A. M.: Phys. Reports **94**, 313 (1983); Wehrhahn, R.F.: Phys. Rev. Lett. **65**, 1294 (1990).

10. Freund, P. G. O., Phys. Lett. **257B**, 119 (1991).

11. Shibukawa, Y., and Ueno, K., Waseda preprint, 1992.

12. Whittaker, E. T., and Watson, G. N., *A Course of Modern Analysis*, Cambridge University Press, Cambridge, 14th ed. 1927.

13. Vigneras, M. F., Astérisque **61**, 235 (1979).

14. Bloch, S. Lecture Notes, Univ. of Cal., Irvine (1977), unpublished; D. Ramakrishnan, Contemp. Math. **55**, 371 (1986); D. Zagier, Math. Ann. **286**, 613 (1990).

15. Milnor, J., L’Enseignement Math. **29**, 281 (1983).

16. Selberg, A., J. Indian Math. Soc. **20** 47 (1956).

17. Ireland, K., and Rosen, M., *A Classical Introduction to Modern Number Theory*, Springer, N.Y., 1980.

18. Cartier, P., and Voros, A.: in *The Grothendieck Festschrift*, vol. 2, Birkhäuser, Basel, 1990, p. 1.

19. Gangolli, R., Illinois J. Math. **21**, 1 (1977).
20. Kurokawa, N., Contemp. Math. **83**, 133 (1989).

21. Gasper, G., and Rahman, M., *Basic Hypergeometric Series*, Cambridge Univ. Press, Cambridge, 1990.

22. Askey, R., Applicable Analysis **8**, 125 (1978).

23. Bloch, S. private communication; Zagier, D., Math. Ann. **286**, 613 (1990).

24. Langlands, R.P, *Euler Products*, Yale Univ. Press, New Haven, 1971.

25. Macdonald, I.G.: in *Orthogonal Polynomials: Theory and Practice*, P. Nevai ed., Kluwer Academic Publ., Dordrecht, 1990; Queen Mary College preprint 1989.

26. Freund, P.G.O.: in *Superstrings and Particle Theory*, L. Clavelli and B. Harms eds., World Scientific, Singapore, 1990.

27. Faddeev, L.D., and Pavlov, B. S., Sem. Steklov Math. Inst. Leningrad **27**, 161 (1972).

28. Moore, G., and Seiberg, N., Comm. Math. Phys. **123**, 177 (1989); Alvarez-Gaumé, L., Gomez, C., and Sierra, G., Phys. Lett. **220B**, 142 (1989).
Figure 1

NOTE: The diagonal arrows, referred to in the text, run from entries in the fourth column to entries in the second column with the same number of underlines (i.e., one row below). These arrows will be displayed explicitly in the preprints mailed out by ordinary, rather than electronic, mail.
| ? | ← | ? | ← | Blaschke-CCD Factor | → | “$S$”-wave scattering on Bethe lattice |
| ↓ | ↓ | ↓ | ↓ | ↓ | ↓ |
| “$S$”-wave scattering on adelic hyperbolic plane; scattering on fundamental domain $\text{SL}(2, \mathbb{Z}) \setminus \text{SL}(2, \mathbb{R})/\text{SO}(2, \mathbb{R})$ | ← | “$S$”-wave scattering on real hyperbolic plane; dressed excitation scattering in the XXX model | ← | “$S$”-wave scattering on quantum hyperboloid; dressed excitation scattering in the XXX model and $\mathcal{M}_\infty$ models |
| ↓ | ↓ | ↓ | ↓ | ↓ |
| ? | ← | “$S$”-wave scattering on quantum hyperboloid; dressed excitation scattering in the XXZ and $\mathcal{M}_\infty$ models | ← | Soliton scattering of sine-Gordon model at special rational values of the coupling parameter | → | dressed excitation scattering in the XYZ model for $l = 1/n$ |
| ↓ | ↓ | ↓ | ↓ | ↓ |

NOTE: The diagonal arrows, referred to in the text, run from entries in the fourth column to entries in the second column with the same number of underlines (i.e., one row below). These arrows will be displayed explicitly in the preprints mailed out by ordinary, rather than electronic, mail.