SOME WEAK INDIVISIBILITY RESULTS IN ULTRAHOMOGENEOUS METRIC SPACES.

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Abstract. We study the validity of a partition property known as weak indivisibility for the integer and the rational Urysohn metric spaces. We also compare weak indivisibility to another partition property, called age-indivisibility, and provide an example of a countable ultrahomogeneous metric space which is age-indivisible but not weakly indivisible.

1. Introduction.

The purpose of this article is the study of certain partition properties of particular metric spaces, called ultrahomogeneous. A metric space is ultrahomogeneous when every isometry between finite metric subspaces of $X$ can be extended to an isometry of $X$ onto itself. For example, when seen as a metric space, any Euclidean space $\mathbb{R}^n$ has this property. So does the separable infinite dimensional Hilbert space $\ell_2$ and its unit sphere $S^\infty$. Another less known example of ultrahomogeneous metric space, though recently a well studied object (see [U08]), is the Urysohn space, denoted $U$: up to isometry, it is the unique complete separable ultrahomogeneous metric space into which every separable metric space embeds (here and in the sequel, all the embeddings are isometric, that is, distance preserving). This space also admits numerous countable analogs. For example, for various countable sets $S$ of positive reals (see [DLPS07] for the precise condition on $S$), there is, up to isometry, a unique countable ultrahomogeneous metric space into which every countable metric space with distances in $S$ embeds. When $S = \mathbb{Q}$ or $\mathbb{N}$ this gives raise to the spaces denoted respectively $U_\mathbb{Q}$ (the rational Urysohn space) and $U_\mathbb{N}$ (the integer Urysohn space). Recently, separable ultrahomogeneous metric spaces have been at the center of active research because of a remarkable connection between their combinatorial behavior when submitted to finite partitions and the dynamical properties of their isometry group. For example, consider the following result. Call a metric space $Z = (Z,d_Z)$ age-indivisible if for every finite metric subspace $Y$ of $Z$ and every partition $Z = B \cup R$ (thought as a coloring of the points of $Z$ with two colors, blue and red), the space $Y$ embeds in $B$ or $R$.

Theorem (Folklore). The spaces $U_\mathbb{Q}$ and $U_\mathbb{N}$ are age-indivisible.

There are at least two directions for possible generalizations. First, one may ask what happens if instead of coloring the points of, say, the space $U_\mathbb{Q}$, we color the isometric copies of a fixed finite metric subspace $X$ of $U_\mathbb{Q}$. We will not touch this subject here but Kechris, Pestov and Todorcevic showed in [KPT05] that the
answer to this question (obtained by Nešetřil in [N07]) has spectacular consequences on the groups iso(U_Q) and iso(U) of surjective self-isometries of U_Q and U. For example, every continuous action of iso(U) (equipped with the pointwise convergence topology) on a compact topological space admits a fixed point.

Another direction of generalization is to ask whether any of those spaces is indivisible, that is, whether B or R necessarily contains not only a copy of a fixed finite Y but of the whole space itself. However, it is known that any indivisible metric space must have a bounded distance set. Therefore, the spaces U_Q and U_N are not indivisible. Still, in this article, we investigate whether despite this obstacle, a partition result weaker than indivisibility but stronger than age-indivisibility holds. Call a metric space X weakly indivisible when for every finite metric subspace Y of X and every finite partition X = B ∪ R, either Y embeds in B or X embeds in R. Building on techniques developed in [LN08] and [NS-], we prove:

**Theorem 1.** The space U_N is weakly indivisible.

As for U_Q, we are not able to prove or disprove weak indivisibility but we obtain the following weakening. If X is a metric space, Y ⊂ X and ε > 0, (Y)_ε denotes the set (Y)_ε = \{x ∈ X : ∃y ∈ Y \text{ } d^X(x, y) ≤ ε\}.

**Theorem 2.** Let U_Q = B ∪ R and ε > 0. Assume that there is a finite metric subspace Y of U_Q that does not embed in B. Then U_Q embeds in (R)_ε.

This in turn leads to the following partition result for U:

**Theorem 3.** Let U = B ∪ R and ε > 0. Assume that there is a compact metric subspace K of U that does not embed in (B)_ε. Then U embeds in (R)_ε.

Note that those results do not answer the following: for a countable ultrahomogeneous metric space is weak indivisibility a strictly stronger property than age-indivisibility? In the last section of this paper, we answer that question by producing an example of countable ultrahomogeneous metric space which is age-indivisible but not weakly indivisible. To our knowledge, this is even one of the first two known examples of a countable ultrahomogeneous relational structure witnessing that weak indivisibility and age-indivisibility are distinct properties (the other example will appear in [LNS-]). Let E_Q be the class of all finite metric spaces X with distances in Q which embed isometrically into the unit sphere S_∞ of ℓ_2 with the property that \{0_{ɛ_2}\} ∪ X is affinely independent. It is known that there is a unique countable ultrahomogeneous metric space S_∞Q whose class of finite metric spaces is exactly E_Q, and that the metric completion of S_∞Q is S_∞ (for a proof, see [NVT06] or [NVT-]).

**Theorem 4.** The space S_∞Q is age-indivisible.

**Theorem 5.** The space S_∞Q is not weakly indivisible.

The proof of each of those results requires the use of a deep theorem: the proof of Theorem 7 is based on a central result of Matoušek and Rödl in Euclidean Ramsey theory, while the proof of Theorem 8 relies on a strong form of the Odell-Schlumprecht distortion theorem in Banach space theory.

The paper is organized as follows. In Section 2 we prove Theorem 1. In section 3 we prove Theorem 2. Theorem 3 is proved in Section 4 and Theorems 4 and 5 are proved in Section 5.
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2. Proof of Theorem 6

The purpose of this section is to prove Theorem 6. In fact, we prove a slightly stronger result. We mentioned in introduction that there are various countable sets $S$ of positive reals for which there is, up to isometry, a unique countable ultrahomogeneous metric space into which every countable metric space with distances in $S$ embeds. It can be proved that when $p \in \mathbb{N}$, the integer interval $\{1, \ldots, p\}$ is such a set. The corresponding countable ultrahomogeneous metric space is denoted $U_p$.

**Theorem 6.** Let $U_p = B \cup R$. Assume that there is $p \in \omega$ such that $U_p$ does not embed in $B$. Then $U_p$ embeds in $R$.

The rest of this section is devoted to a proof of Theorem 6. We fix $p \in \mathbb{N}$ as well as a partition $U_p = B \cup R$ such that $U_p$ does not embed in $B$. Our goal is to prove that $U_p$ embeds into $R$. Let $m := \lceil p/2 \rceil$ (the least integer larger or equal to $p/2$). Recall that if $Y \subset U_p$, the set $(Y)_{\varepsilon}$ is defined by

$$(Y)_{\varepsilon} = \{x \in X : \exists y \in Y : d^X(x, y) \leq \varepsilon\}.$$

In particular, if $x \in U_p$, the set $(\{x\})_{m-1}$ denotes the set of all elements of $U_p$ at distance $\leq m - 1$ from $x$. We are going to construct $\bar{U} \subset R$ isometric to $U_p$ recursively such that for every $x \in \bar{U}$,

$$(\{x\})_{m-1} \cap \bar{U} \subset R.$$

More precisely, fix an enumeration $\{x_n : n \in \mathbb{N}\}$ of $U_p$. We are going to construct $\{\tilde{x}_n : n \in \mathbb{N}\} \subset U_p$ recursively together with a decreasing sequence $(D_n)_{n \in \mathbb{N}}$ of metric subspaces of $U_p$ such that $x_n \mapsto \tilde{x}_n$ is an isometry and, for every $n \in \mathbb{N}$, each $D_n$ is isometric to $U_p \setminus \{x_k : k \leq n\} \subset D_n$, and $(\{\tilde{x}_n\})_{m-1} \cap D_n \subset R$. To do so, we will need the notion of Katětov map as well as several technical lemmas.

**Definition 1.** Given a metric space $X = (X, d^X)$, a map $f : X \to (0, +\infty)$ is Katětov over $X$ when

$$\forall x, y \in X, \quad |f(x) - f(y)| \leq d^X(x, y) \leq f(x) + f(y).$$

Equivalently, one can extend the metric $d^X$ to $X \cup \{f\}$ by defining, for every $x, y$ in $X$, $d^X(x, f) = f(x)$ and $d^X(y, f) = d^X(x, y)$. The corresponding metric space is then written $X \cup \{f\}$. The set of all Katětov maps over $X$ is written $E(X)$. For a metric subspace $X$ of $Y$ and a Katětov map $f \in E(X)$, the orbit of $f$ in $Y$ is the set $O(f, Y)$ defined by

$$O(f, Y) = \{y \in Y : \forall x \in X : d^Y(y, x) = f(x)\}.$$  

Here, the concepts of Katětov map and orbit are relevant because of the following standard reformulation of the notion of ultrahomogeneity, which will be used in the sequel:
Lemma 1. Let $X$ be a countable metric space. Then $X$ is ultrahomogeneous iff for every finite subspace $F \subset X$ and every Katetov map $f$ over $F$, if $F \cup \{f\}$ embeds into $X$, then $O(f, X) \neq \emptyset$.

Proof. For a proof of that fact in the general context of relational structures, see for example [F00]. For a proof in the particular context of metric spaces, see [NVT06] or [NVT12].

Lemma 2. Let $G$ be a finite subset of $U_N$, and $g$ a Katetov map with domain $G$ and with values in $\mathbb{N}$. Then there exists an isometric copy $C$ of $U_N$ inside $U_N$ such that:

(i) $G \subset C$,
(ii) $O(g, C) \subset B$ or $O(g, C) \subset R$.

In words, Lemma 2 asserts that going to a subcopy of $U_N$ if necessary, we may assume that the orbit of $g$ is completely included in one of the parts of the partition. Observe that as a metric space, the orbit $O(g, C)$ is isometric to $U_n$ where $n = 2 \min g$ (Indeed, it is countable ultrahomogeneous with distances in $\{1, \ldots, n\}$ and embeds every countable metric space with distances in $\{1, \ldots, n\}$).

Proof. The proof of Lemma 2 can be found in [NS4]. More precisely, Lemma 2 can be obtained by combining Lemma 2 [NS4] and Lemma 3 [NS4] after having replaced $U_p$ by $U_N$ in those statements. The proof of Lemma 3 [NS4] is elementary, while the proof of Lemma 2 [NS4] represents the core of [NS4]. Those two proofs can be carried out without modification once $U_p$ has been replaced by $U_N$. \qed

Lemma 3. Let $G_0 \subset G$ be finite subsets of $U_N$, and let $\mathcal{G}$ a finite family of Katetov maps with domain $G$ and such that for all $g, g' \in \mathcal{G}$:

$$\max(|g - g'| \mid G_0) = \max|g - g'|,$$
$$\min((g + g') \mid G_0) = \min(g + g'),$$
$$\min(g \mid G_0) = \min(g).$$

Then there exists an isometric copy $C$ of $U_N$ inside $U_N$ such that:

(i) $G \cap C = G_0$,
(ii) $\forall g \in \mathcal{G}$ $O(g \mid G_0, C) \subset O(g, U_N)$.

Proof. Lemma 3 is also a modified version of a result proved in [NS4], namely Lemma 5. Like in the case of Lemma 2 the proof of Lemma 5 [NS4] can be carried out without modification once $U_p$ has been replaced by $U_N$. \qed

2.1. Construction of $\tilde{x}_0$ and $D_0$. First, pick an arbitrary $u \in U_N$ and consider the map $g : \{u\} \rightarrow \mathbb{N}$ defined by $g(u) = m$. By Lemma 2 find an isometric copy $C$ of $U_N$ inside $U_N$ such that:

(i) $u \in C$,
(ii) $O(g, C) \subset B$ or $O(g, C) \subset R$.

Note that since $g$ has minimum $m$, the orbit $O(g, C)$ is isometric to $U_{2m}$ and therefore contains a copy of $U_p$. Hence, because $U_p$ does not embed in $B$, the inclusion $O(g, C) \subset B$ is excluded and we really have $O(g, C) \subset R$. Let $\tilde{x}_0 \in O(g, C)$ and for every $k \leq m$ let $g_k : \{u, \tilde{x}_0\} \rightarrow \mathbb{N}$ be such that $g_k(u) = m$ and $g_k(\tilde{x}_0) = k$. The sets $G_0 = \{\tilde{x}_0\}$ and $G = \{u, \tilde{x}_0\}$, and the family $\mathcal{G} = \{g_k : k \leq m\}$ satisfy the hypotheses of Lemma 3 which allows to obtain an isometric copy $D_0$ of $U_N$ inside $C$ such that:
Indeed, let consequently have Lemma 2 [LN08] and is not included here. □

This appears at the very beginning of Proposition 5. Its proof is an easy modification of Proof. Lemma 4.

proofs are the result of Theorem 1 as well as the following technical lemma:

Consider the map $\hat{x_n}$ such that:

(i) $\{\hat{x_0}, \ldots, \hat{x_n}\} \subset C_t$.
(ii) $\forall k \leq m \quad O(g_k \cap \{\hat{x_0}\}, D_0) \subset O(g_k, C)$.

Note that for every $k \leq m$, we have $O(g_k, C) \subset O(g, C) \subset R$. Therefore, in $D_0$, all the spheres around $\hat{x_0}$ with radius $k \leq m$ are included in $R$. So

$$\{(\hat{x_0})_{m-1} \cap D_0 \subset R. □$$

2.2. Induction step. Assume that we constructed $\{\hat{x_k} : k \leq n\} \subset U_N$ together with a decreasing sequence $(D_k)_{k \leq n}$ of metric subspaces of $U_N$ such that $x_k \mapsto \hat{x_k}$ is an isometry (recall that $\{x_n : n \in N\}$ is the enumeration of $U_N$ we fixed at the beginning of the proof), each $D_k$ is isometric to $U_N$, $\{\hat{x_k} : k \leq n\} \subset D_n$ and $\{(\hat{x_k})_{m-1} \cap D_n \subset R$ for every $k \leq n$. We are going to construct $\hat{x_{n+1}}$ and $D_{n+1}$.

Consider the map $f : \{\hat{x_0}, \ldots, \hat{x_n}\} \longrightarrow N$ where

$$\forall k \leq n \quad f(\hat{x_k}) = dU_N(x_k, x_{n+1}).$$

Consider the set $G$ defined by

$$\{g \in E(\{\hat{x_0}, \ldots, \hat{x_n}\}) : \forall k \leq n \quad (|f(\hat{x_k} - g(\hat{x_n}))| \leq m - 1 \quad \text{and} \quad g(\hat{x_k}) \geq m)\}.$$

This set is finite and a repeated application of Lemma 2 allows to construct an isometric copy $C$ of $U_N$ inside $U_N$ such that:

(i) $\{\hat{x_0}, \ldots, \hat{x_n}\} \subset C$,
(ii) $\forall g \in G, \quad O(g, C) \subset B$ or $R$.

Note that since every $g \in G$ has minimum $m$, the orbit $O(g, C)$ is isometric to $U_{2m}$ and therefore contains a copy of $U_p$. Because $U_p$ does not embed in $B$, we consequently have

$$\forall g \in G, \quad O(g, C) \subset R.$$

Let $\hat{x_{n+1}} \in O(f, C)$. We claim that $\hat{x_{n+1}}$ is as required. Note that, because $\hat{x_{n+1}} \in O(f, C)$, we have

$$\forall k \leq n \quad dU_N(\hat{x_{n+1}}, \hat{x_k}) = f(\hat{x_k}) = dU_N(x_k, x_{n+1}).$$

Therefore, $x_k \mapsto \hat{x_k}$ is an isometry. Next we prove that $\{(\hat{x_n})_{m-1} \subset R$. Indeed, let $y \in \{(\hat{x_n})_{m-1}$. If $dU_N(\hat{x_k}, y) \geq m$ for every $k \leq n$, then the map $dU_N(\cdot, y)$ is in $G$ and so $y \in O(dU_N(\cdot, y), C) \subset R$. Otherwise, we have $dU_N(\hat{x_k}, y) \leq m$ for some $k \leq n$ and $y \in \{(\hat{x_k})_{m-1} \subset R$. □

3. Proof of Theorem 2

The purpose of this section is to prove Theorem 2. The main ingredients of the proofs are the result of Theorem H as well as the following technical lemma:

Lemma 4. Let $q \in N$ be positive. Then there is an isometric copy $U_{n/q}^* \subset U_N$ in $U_N$ such that for every subspace $V$ of $U_{n/q}^*$ isometric to $U_{n/q}^*$, the set $(V)_{1/2q}$ includes an isometric copy of $U_Q$.

Proof. Lemma H is a modified version of a result proved in [LN08], whose statement appears at the very beginning of Proposition 5. Its proof is an easy modification of Lemma 2 [LN08] and is not included here. □
Claim 2. The map \( z \), above a copy of \( \mathbb{Z} \) integer. The function

Consider the set \( X \) It obviously has rational distances. W e are going to show that it is as required.

Proof. Let \( q \in \mathbb{N} \) be positive. Then there is an isometric copy \( \tilde{U}^*_{N/q} \) of \( U^*_{N/q} \) in \( U \) such that for every subspace \( \tilde{V} \) of \( \tilde{U}^*_{N/q} \) isometric to \( U^*_{N/q} \), the set \( (\tilde{V})_{1/2q} \) includes an isometric copy \( U \) of \( U_\mathbb{Q} \). Notice that \( \tilde{U} \subset (\tilde{V})_{1/2q} \subset (\tilde{V})_\varepsilon \subset (R)_\varepsilon \).

4. PROOF OF THEOREM 3

The purpose of this section is to prove Theorem 3. As for Theorem 2, we will use the result of Theorem 1 as well as several technical lemmas. The first one can be seen as a version of Lemma 4 in the context of the space \( U \):

Lemma 5. Let \( q \in \mathbb{N} \) be positive. Then there is an isometric copy \( \tilde{U}^*_{N/q} \) of \( U^*_{N/q} \) in \( U \) such that for every subspace \( \tilde{V} \) of \( \tilde{U}^*_{N/q} \) isometric to \( U^*_{N/q} \), the set \( (\tilde{V})_{1/2q} \) includes an isometric copy \( U \).

Proof. Lemma 5 is a direct consequence of Lemma 4 and of the fact that \( U \) is the metric completion of \( U_\mathbb{Q} \).

The second lemma we will need states that in \( U \), the copies of the compact space \( K \) can be captured by a single finite metric subspace of \( U \):

Lemma 6. There is a finite metric space \( Y \) of \( U \) with rational distances such that \( K \) embeds in \((\tilde{Y})_\varepsilon \) for every subspace \( \tilde{Y} \) of \( U \) isometric to \( Y \).

Proof. Using compactness of \( K \), find a finite subspace \( Z \) of \( K \) such that \( K \subset (Z)_\varepsilon/2 \).

Claim 1. The space \( K \) embeds in \((\tilde{Z})_\varepsilon \) for every subspace \( \tilde{Z} \) of \( U \) isometric to \( Z \).

Proof. This follows from ultrahomogeneity of \( U \): if \( \tilde{Z} \) is a subspace of \( U \) isometric to \( Z \), let \( \phi : Z \to \tilde{Z} \) be an isometry. By ultrahomogeneity of \( U \), find \( \Phi : U \to U \) extending \( \phi \). Then \( \Phi(K) \) is isometric to \( K \) and is included in \( \Phi((Z)_{\varepsilon/2}) = (\Phi(Z))_{\varepsilon/2} = (\tilde{Z})_{\varepsilon/2} \).

Therefore, the space \( Z \) is almost as required except that it may not have rational distances. To arrange that, consider \( q \in \mathbb{N} \) large enough so that \( 1/q < \varepsilon/2 \). For a number \( \alpha \), let \( \lfloor \alpha \rfloor_q \) denote the smallest number \( \geq \alpha \) of the form \( l/q \) with \( l \) integer. The function \( \lfloor \cdot \rfloor_q \) is subadditive and increasing. Hence, the composition \( d^Z_q = \lfloor \cdot \rfloor_q \circ d^Z \) is a metric on \( Z \). Let \( Y \) be defined as the metric space \((Z, d^Z_q)\). It obviously has rational distances. We are going to show that it is as required.

Consider the set \( X = Z \times \{0, 1\} \) and define

\[
\delta((z, i), (z', i')) = \begin{cases} 
\lfloor d^Z_q(z, z') \rfloor_q & \text{if } i = i' = 0, \\
n d^Z_q(z, z') & \text{if } i = i' = 1, \\
n d^Z_q(z, z') + \varepsilon/2 & \text{if } i \neq i'.
\end{cases}
\]

In spirit, the structure \((X, \delta)\) is obtained by putting a copy of \( Y \) \((= (Z, d^Z_q))\) above a copy of \( Z \) such that the distance between any point \((z, 0) \in Z \times \{0\}\) and its counterpart \((z, 1) \in Z \times \{1\}\) is \( \varepsilon/2 \).

Claim 2. The map \( \delta \) is a metric on \( X \).
Proof. The maps \( d^Z \) and \([d^Z]_q\) being metrics on \( Z \times \{0\} \) and \( Z \times \{1\} \), it suffices to verify that the triangle inequality is satisfied on triangles of the form \( \{(x,0),(y,0),(z,1)\} \) and \( \{(x,1),(y,1),(z,0)\} \), with \( x,y,z \in Z \).

Assume that \( x,y,z \in Z \), and consider the triangle \( \{(x,0),(y,0),(z,1)\} \). Then
\[
\delta((x,1),(z,0)) = d^Z(x,z) + \frac{\varepsilon}{2} \\
\leq d^Z(x,y) + d^Z(y,z) + \frac{\varepsilon}{2} \\
\leq [d^Z(x,y)]_q + d^Z(y,z) + \frac{\varepsilon}{2} \\
\leq \delta((x,1),(y,1)) + \delta((y,1),(z,0)).
\]
Similarly,
\[
\delta((y,1),(z,0)) \leq \delta((y,1),(x,1)) + \delta((x,1),(z,0)).
\]
And finally,
\[
\delta((x,1),(y,1)) = [d^Z(x,y)]_q \\
\leq d^Z(x,y) + \frac{1}{q} \\
\leq d^Z(x,y) + \frac{\varepsilon}{2} \\
\leq d^Z(x,z) + d^Z(z,y) + \frac{\varepsilon}{2} \\
\leq d^Z(x,z) + \frac{\varepsilon}{2} + d^Z(z,y) + \frac{\varepsilon}{2} \\
\leq \delta((x,1),(z,0)) + \delta((z,0),(y,1)).
\]
Next, consider the triangle \( \{(x,0),(y,0),(z,1)\} \). We have
\[
\delta((0,0),(z,1)) = d^Z(x,z) + \frac{\varepsilon}{2} \\
\leq d^Z(x,y) + d^Z(y,z) + \frac{\varepsilon}{2} \\
\leq \delta((x,0),(y,0)) + \delta((y,0),(z,1)).
\]
Similarly,
\[
\delta((y,0),(z,1)) \leq \delta((y,0),(x,0)) + \delta((x,0),(z,1)).
\]
Finally,
\[
\delta((x,0),(y,0)) = d^Z(x,y) \\
\leq d^Z(x,z) + d^Z(z,y) \\
\leq d^Z(x,z) + \frac{\varepsilon}{2} + d^Z(z,y) + \frac{\varepsilon}{2} \\
\leq \delta((x,0),(z,1)) + \delta((z,1),(y,0)). \tag*{□}
\]
Denote the space \((X,\delta)\) by \( X \). Recall that every finite metric space embeds isometrically in \( U \). Hence, without loss of generality, we may suppose \( Y \subset X \subset U \).

We claim that \( Y \) is as required. By construction, the space \( Y \) is a finite subspace of \( U \) with rational distances. Observe that \( X \subset (Y)_{\varepsilon/2} \). Assume that a subspace \( \tilde{Y} \) of \( U \) is isometric to \( Y \). By an argument similar to the one used in Claim 1, the space \( X \) embeds in \( (\tilde{Y})_{\varepsilon/2} \). Thus, because \( Z \) embeds in \( X \), the set \((\tilde{Y})_{\varepsilon/2}\) contains
a copy of \( Z \), call it \( \tilde{Z} \). By Claim 1, the set \( (\tilde{Z})_\varepsilon \) contains a copy of \( K \), call it \( \tilde{K} \). Then
\[
\tilde{K} \subset (\tilde{Z})_\varepsilon \subset ((\tilde{Y})_{\varepsilon/2})_{\varepsilon/2} \subset (\tilde{Y})_\varepsilon.
\]
This finishes the proof of Lemma 6. \( \square \)

Proof of Theorem 5. Choose \( q \in \mathbb{N} \) large enough so that \( 1/2q \leq \varepsilon \) and all distances appearing in \( Y \) are integer multiples of \( 1/q \). The partition \( U = B \cup R \) induces a partition of \( U_{\varepsilon/2}^* \) provided by Lemma 6. Note that \( Y \) does not embed in \( B \): indeed, if a subspace \( \tilde{Y} \) of \( B \) were isometric to \( Y \), then \( (\tilde{Y})_\varepsilon \subset (B)_\varepsilon \) and by Lemma 6, the space \( K \) would embed in \( (B)_\varepsilon \), which is not the case. Therefore, by weak indivisibility of \( U_{\varepsilon/2}^* \), there is a subspace \( \tilde{V} \) of \( U_{\varepsilon/2}^* \) isometric to \( U_{\varepsilon/2}^* \) such that \( \tilde{V} \subset R \). By construction of \( U_{\varepsilon/2}^* \), the set \( (\tilde{V})_{1/2q} \) includes an isometric copy \( \tilde{U} \) of \( U \). Notice that \( \tilde{U} \subset (\tilde{V})_{1/2q} \subset (\tilde{V})_\varepsilon \subset (R)_\varepsilon \). \( \square \)

5. Age-indivisibility does not imply weak indivisibility

In this section, we prove results that are slightly stronger than Theorems 7 and 8. In what follows, the set \( S \) is a fixed dense subset of \([0,2]\). Let \( \mathcal{E}_S \) be the class of all finite metric spaces \( X \) with distances in \( S \) which embed isometrically into the unit sphere \( S_\infty^{\mathbb{R}} \) of \( \ell^2 \) with the property that \( \{0_{\ell^2}\} \cup X \) is affinely independent.

Claim 3. There is a unique countable ultrahomogeneous metric space \( S_\infty^S \) whose class of finite metric spaces is exactly \( \mathcal{E}_S \). Moreover, the metric completion of \( S_\infty^S \) is \( S_\infty \).

Proof. See [NVT06] or [NVT-]. \( \square \)

We show:

Theorem 7. The space \( S_\infty^S \) is age-indivisible.

Theorem 8. The space \( S_\infty^S \) is not weakly indivisible.

The proof of those results are provided in Subsection 5.1 and Subsection 5.2 respectively.

5.1. The space \( S_\infty^S \) is age-indivisible. Let \( Y \) be a finite metric subspace of \( S_\infty^S \). We need to show:

Claim 4. There is a finite metric subspace \( Z \) of \( S_\infty^S \) such that for every partition \( Z = B \cup R \), the space \( Y \) embeds in \( B \) or \( R \).

The main ingredient of the proof is the following deep result due to Matoušek and Rödl:

Theorem 9 (Matoušek-Rödl [MR95]). Let \( X \) be an affinely independent finite metric subspace of \( S_\infty \) with circumradius \( r \), and let \( \alpha > 0 \). Then there is a finite metric subspace \( Z \) of \( S_\infty \) with circumradius \( r + \alpha \) such that for every partition \( Z = B \cup R \), the space \( X \) embeds in \( B \) or \( R \).

What we need to prove is that in the case where \( X = Y \), we may arrange \( Z \) to be a subspace of \( S_\infty^S \) (that is, with distances in \( S \) and \( \{0_{\ell^2}\} \cup Z \) affinely independent). We will make use of the following facts along the way:
Theorem 10 (Schoenberg \cite{Schoenberg}). Let $X = \{x_k : 1 \leq k \leq |G|\}$ be a finite set and let $\delta : X^2 \to \mathbb{R}$ satisfying:

(i) for every $x \in X$, $\delta(x, x) = 0$,

(ii) for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$.

Then $(X, \delta)$ is isometric to a subset of $l_2$ iff

$$\max \left\{ \sum_{1 \leq i < j \leq n} \delta(x_i, x_j)^2 x_i x_j : \sum_{k=1}^n x_k^2 = 1 \text{ and } \sum_{k=1}^n x_k = 0 \right\} \leq 0.$$ 

Moreover, $(X, \delta)$ is isometric to an affinely independent subset of $l_2$ iff the inequality is strict.

Claim 5. Let $X$ be a finite affinely independent metric subspace of $S^\infty$ with circumradius $r$. Then there is $\varepsilon > 0$ such that for every $\delta : X^2 \to \mathbb{R}$ satisfying

(i) for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$,

(ii) $|\delta - d^X| < \varepsilon$,

the space $(X, \delta)$ is an affinely independent metric subspace of $S^\infty$.

Proof. Direct from Theorem 10 and from the fact that the map $M \mapsto Q_M$ is continuous, where for a matrix $M = (m_{ij})_{1 \leq i, j \leq n}$,

$$Q_M = \max \left\{ \sum_{1 \leq i < j \leq n} m_{ij} x_i x_j : \sum_{k=1}^n x_k^2 = 1 \text{ and } \sum_{k=1}^n x_k = 0 \right\}. \quad \square$$

Claim 6. Let $X$ be a finite metric subspace of $S^\infty$ with circumradius $r$. Let $\varepsilon > 0$. Then $(X, d^X + \varepsilon)$ is Euclidean, affinely independent with circumradius at most $r + \varepsilon$.

Proof. Let $V$ be the affine space spanned by $X$. Choose $(e_x)_{x \in X}$ a family of pairwise orthogonal vectors in $V^\perp$. For $x \in X$, set $\tilde{x} = x + \sqrt{\varepsilon/2} \ e_x$. Then the set $\{\tilde{x} : x \in X\}$ is affinely independent and is isometric to $(X, d^X + \varepsilon)$. Its circumradius is at most $r + \varepsilon$ because it is contained in the ball centered at the circumcenter of $X$ and with radius $r + \varepsilon$. \square

Claim 7. Let $X$ be an affinely independent subspace of $S^\infty$. Then $X \cup \{0_{l_2}\}$ is affinely independent iff the circumradius of $X$ is $< 1$.

Proof. Let $V$ be the affine space spanned by $X$. Then the set $S^\infty \cap V$ is the circumscribed sphere of $X$ in $V$. It has radius $< 1$ if $0_{l_2}$ does not belong to $V$. \square

Proof of Claim 6. First, we show that there is an affinely independent finite metric subspace $Z_0$ of $S^\infty$ with circumradius $< 1$ such that for every partition $Z_0 = B \cup R$, $Y$ embeds in $B$ or $R$:

Let $r$ denote the circumradius of $Y$. Because $Y$ is a subspace of $S^\infty \subset S^\infty$, the space $Y \cup \{0_{l_2}\}$ is affinely independent and by Claim 6 we have $r < 1$. By Claim 6 fix $\varepsilon > 0$ such that $r + 2\varepsilon < 1$ and such that for every map $\delta : X^2 \to \mathbb{R}$ satisfying

(i) for every $x, x' \in X$, $\delta(x, x) = 0$ and $\delta(x', x) = \delta(x, x')$,

(ii) $|\delta - d^X| < \varepsilon$,

the space $(Y, \delta)$ is still Euclidean and affinely independent. Fix $\alpha > 0$ such that $\alpha < \varepsilon$. By choice of $\varepsilon$, the space $(Y, d^Y - \varepsilon)$ is still Euclidean and affinely independent. It should be clear that its circumradius is at most $r$. Apply Theorem 10 to produce a finite metric subspace $T$ of $S^\infty$ with circumradius $r + \alpha$ such that
for every partition $T = B \cup R$, the space $(Y, d^Y - \varepsilon)$ embeds in $B$ or $R$. Set $Z_0 = (T, d^T + \varepsilon)$. We claim that $Z_0$ is as required.

Indeed, by Claim [6] $Z_0$ is Euclidean, affinely independent, and its circumradius is at most $r + \alpha + \varepsilon < r + 2\varepsilon < 1$. Next, if $Z_0 = B \cup R$, this partition induces a partition $T = B \cup R$. By construction of $T$, there is a subspace $\tilde{Y}$ of $T$ isometric to $(Y, d^Y - \varepsilon)$ contained in $B$ or $R$. Note that in $Z_0$, the metric subspace supported by $\tilde{Y}$ is isometric to $(Y, d^Y - \varepsilon + \varepsilon) = Y$.

Consider the space $Z_0$ we just constructed. Using Claim [5] as well as the denseness of $S$, we may modify slightly all the distances in $Z_0$ that are not in $S$ and turn $Z_0$ into an affinely independent subspace $Z$ of $S^\infty$ with distances in $S$ and circumradius $< 1$. By Claim [7] the space $\{0, \ell\} \cup Z$ is affinely independent. Therefore, $Z$ embeds in $S^\infty$. Finally, note that since all the distances of $Z_0$ that were already in $S$ did not get changed, the copies of $Y$ in $Z_0$ remain unaltered when passing to $Z$. It follows that for every partition $Z = B \cup R$, the space $Y$ embeds in $B$ or $R$. □

5.2. The space $S^\infty_S$ is not weakly indivisible. The starting point of our proof of Theorem 8 is the following theorem:

**Theorem 11** (Odell-Schlumprecht [OS94]). There is a partition $S^\infty = B \cup R$ and $\varepsilon > 0$ such that

(i) For every linear subspace $V$ of $\ell_2$ with $\dim V = \infty$, $S^\infty \cap V \not\subset (B)_\varepsilon$.

(ii) For every linear subspace $V$ of $\ell_2$ with $\dim V = \infty$, $S^\infty \cap V \not\subset (R)_\varepsilon$.

In response to an inquiry of the authors, Thomas Schlumprecht [S08] indicated that the method that was used to prove Theorem 11 in [OS94] (where the statement is proved first in another Banach space known as the Schlumprecht space, and then transferred to $\ell_2$), can be adapted to show that $\dim V = \infty$ may be replaced by $\dim V = 2$ in (i):

**Theorem 12** (Odell-Schlumprecht). There is a partition $S^\infty = B \cup R$ and $\varepsilon > 0$ such that

(i) For every linear subspace $V$ of $\ell_2$ with $\dim V = 2$, $S^\infty \cap V \not\subset (B)_\varepsilon$.

(ii) For every linear subspace $V$ of $\ell_2$ with $\dim V = \infty$, $S^\infty \cap V \not\subset (R)_\varepsilon$.

We are going to show how this result almost directly leads to Theorem 8. Consider the partition of $S^\infty$ provided by Theorem 12. It should be clear that it induces a partition of $S^\infty_S$.

**Claim 8.** $S^\infty_S = B \cup R$ witnesses that $S^\infty_S$ is not weakly indivisible.

The proof makes use of the following fact, which we prove for completeness:

**Claim 9.** Let $Y \subset S^\infty$ be isometric to $S^\infty$. Then there is a closed linear subspace $V$ of $\ell_2$ with $\dim V = \infty$ such that $Y = V \cap S^\infty$.

**Proof.** Consider $V$ the closed linear span of $Y$ in $\ell_2$. Consider also the set $W = \{\lambda y : \lambda \in \mathbb{R}, y \in Y\}$. We will be done if we show $V = W$. Clearly, $W \subset V$. For the reverse inclusion, observe that because $Y$ is closed (it is isometric to a complete metric space), the set $W$ is closed. Therefore, it is enough to show that all the finite linear combinations of elements of $V$ that have norm 1 are in $Y$, ie for every $y_1, \ldots, y_n \in Y$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i y_i \neq 0_{\ell_2}$,

$$\frac{\sum_{i=1}^{n} \lambda_i y_i}{\left\|\sum_{i=1}^{n} \lambda_i y_i\right\|} \in Y.$$
We proceed by induction on \( n \). For \( n = 2 \), we first consider the case \( \lambda_1 = \lambda_2 = 1 \). Note that \( y_1 \) and \( y_2 \) cannot be antipodal (otherwise \( y_1 + y_2 = 0_{\ell_2} \)), and that \( \frac{y_1 + y_2}{\|y_1 + y_2\|} \) can be characterized metrically in terms of \( y_1 \) and \( y_2 \). For example, it is the unique geodesic middle point of \( y_1 \) and \( y_2 \) in the intrinsic metric on \( S^\infty \). Since the intrinsic metric can be defined in terms of the usual Hilbertian metric on \( S^\infty \), this point must belong to \( Y \). By a usual middle-point-type argument, it follows that the entire geodesic segment between \( y_1 \) and \( y_2 \) is contained in \( Y \). Using then that \( Y \) is closed under antipodality (because \( Y \) being isometric to \( S^\infty \) any \( y \in Y \) must have a point at distance 2), as well as a middle-point-type argument again, the entire great circle through \( y_1 \) and \( y_2 \) is contained in \( Y \). That finishes the case \( n = 2 \). Assume that the property is proved up to \( n \geq 2 \). Fix \( y_1, \ldots, y_n \in Y \) and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \). Then writing

\[
z = \sum_{i=1}^n \lambda_i y_i \|\sum_{i=1}^n \lambda_i y_i\|
\]

the vector

\[
\frac{\sum_{i=1}^{n+1} \lambda_i y_i}{\|\sum_{i=1}^{n+1} \lambda_i y_i\|}
\]

is a linear combination of \( z \) and \( y_{n+1} \) with norm 1. Therefore, it is of the form

\[
\frac{\alpha z + \beta y_{n+1}}{\|\alpha z + \beta y_{n+1}\|}.
\]

By induction hypothesis, \( z \) is in \( Y \). So again by induction hypothesis (case \( n = 2 \)),

\[
\frac{\alpha z + \beta y_{n+1}}{\|\alpha z + \beta y_{n+1}\|} \in Y.
\]

Therefore,

\[
\frac{\sum_{i=1}^{n+1} \lambda_i y_i}{\|\sum_{i=1}^{n+1} \lambda_i y_i\|} \in Y.
\]

**Proof of Claim \( \mathbb{S} \)** Let \( W \) be a linear subspace of \( \ell_2 \) with \( \dim W = 2 \). By compactness of \( S^\infty \cap W \) and denseness of \( S^\infty_2 \) in \( S^\infty \), there is \( X \subset S^\infty_2 \) finite such that \( S^\infty \cap W \subset (X)_{\ell_2} \). Let \( X \) denote the metric subspace of \( S^\infty_2 \) supported by the set \( X \). Then \( X \) does not embed in \( B \) because otherwise, there would be a linear subspace \( V \) of \( \ell_2 \) with \( \dim V = 2 \) such that \( S^\infty \cap V \subset (B)_{\ell_2} \), violating (i) of Theorem \( [12] \). On the other hand, \( S^\infty_2 \) cannot embed in \( R \): let \( Y \subset S^\infty_2 \) be isometric to \( S^\infty_2 \). Then \( S^\infty \), the closure \( Y \) of \( Y \) is isometric to \( S^\infty \). By Claim \( \mathbb{S} \) there is a closed linear subspace \( V \) of \( \ell_2 \) with \( \dim V = \infty \) such that \( Y = V \cap S^\infty \). By (ii) of Theorem \( [12] \) \( Y \not\subset (R)_{\ell_2} \). Therefore \( Y \not\subset R \).

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