1. Introduction

In 1970s-1980s, Lepowsky, Primc and Wilson studied the integrable highest weight modules called standard modules using vertex operators to obtain a number of combinatorial identities. See e.g. [14, 20]. Lepowsky and Primc showed that the structure of the standard module is completely determined by the structure of its vacuum space which is isomorphic to a coset subspace of the standard module. Let \( \mathfrak{g} \) be the simple Lie algebra \( \mathfrak{sl}_2 \) and \( \mathfrak{h} \) be its Cartan subalgebra. The corresponding affine Lie algebras are given by

\[
\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus Cc \oplus Cd, \quad \tilde{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus Cc \oplus Cd,
\]

where \( c \) is the canonical central element and \( d \) is the degree operator. They constructed combinatorial bases of the coset space \( \tilde{\mathfrak{g}} \supset \mathfrak{h} \) of standard \( \tilde{\mathfrak{g}} \)-modules build upon Fourier coefficients of vertex operators. This work was generalized by Georgiev to the higher rank case \( \mathfrak{g} = \mathfrak{sl}_{n+1} \). In [13], he first constructed bases for the principal subspaces introduced and studied by Feigin and Stoyanovsky [9, 10]. Then he did for certain subspaces of the standard modules called parafermionic spaces by generalizing the \( Z \)-algebra approach of Lepowsky, Primc and Wilson. As a consequence, he obtained a fermionic character formula of the standard module using vertex operators to obtain a number of combinatorial identities. See e.g. [19, 20].

Recently, based on the study of the construction of principal subspaces for untwisted affine Lie algebras (see e.g. [2, 3, 13]), Butorac, Kozić and Primc obtained combinatorial bases of standard modules for all untwisted affine Lie algebras in [4]. Furthermore, by considering parafermionic spaces, they settled the Kuniba-Nakanishi-Suzuki conjecture [17]. On the other hand, Okado and the author constructed bases of standard modules for twisted affine Lie algebras except for type \( A_{2l} \) by using the results for principal subspaces obtained by Butorac and Sadowski [5]. They were thereby able to show partly fermionic character formulas for twisted cases conjectured in [15].

The aim of this paper is to calculate the fermionic character formula of the standard module for the twisted affine Lie algebra of type \( A_{2l} \), that is, to complete the proof of [15] conjecture 5.3 for all twisted affine Lie algebras. In order to do that, we first need to obtain the quasi-particle basis of the principal subspace of the standard module by generalizing the seminal works of Calinescu, Lepowsky, Milas and Penn [6, 7] to higher rank and level cases. In the same way as [24], we are able to construct the
combinatorial basis of the standard module. In addition to these, we construct the parafermionic basis of the parafermionic space. Finally, we calculate fermionic character formulas for the principal subspace and the parafermionic space as well as the standard module.

2. Preliminaries

2.1. Lattice vertex operator algebras and twisted modules. Let \( g \) be a complex simple Lie algebra of type \( A_{2l} \), and \( \alpha_i \) (\( i = 1, 2, \ldots, 2l \)) be its simple roots. We recall the vertex algebraic construction associated to the root lattice of \( g \) given by

\[
L = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_{2l}.
\]

Its Dynkin diagram and the Dynkin automorphism are given in Table 1.

![Dynkin Diagram and Automorphism](image)

**Table 1.** Dynkin diagram of \( g \) and the automorphism

Thus the automorphism \( \nu \) of \( L \) is determined by

\[
\nu(\alpha_i) = \alpha_{2l-i+1}.
\]

Let \( \langle \cdot, \cdot \rangle \) be a nondegenerate invariant symmetric bilinear form on \( g \). Using this form, we identify the Cartan subalgebra \( h \) of \( g \) with its dual \( h^* \), so that under this identification we have \( L \subset h \). We fix the bilinear form \( \langle \cdot, \cdot \rangle \) so that we have \( \langle \alpha, \alpha \rangle = 2 \) if \( \alpha \) is a root. Note that

\[
\langle \nu \alpha, \alpha \rangle \in 2\mathbb{Z} \quad \text{for all } \alpha \in L,
\]

under the assumptions in Section 2 of [18]. Therefore we choose \( r \) to be 4 rather than 2. In fact, the period of \( \nu \) is allowed to be larger than the order of \( \nu \) in our setting (see the remark in Section 2 of [18]). Let \( \zeta \) be the imaginary unit.

Following [6, 7, 18], we define the functions \( C_0, C : L \times L \to \mathbb{C}^\times \) by

\[
C_0(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle}, \quad C(\alpha, \beta) = \prod_{j=0}^{3} (-\zeta^j)^{\langle \nu^j \alpha, \beta \rangle}.
\]

These functions are bilinear into the abelian group \( \mathbb{C}^\times \) and \( \nu \)-invariant. Since \( C_0 \) and \( C \) satisfy

\[
C_0(\alpha, \alpha) = C(\alpha, \alpha) = 1
\]

for all \( \alpha \in L \), they determine uniquely two central extensions of \( L \) by \( \langle \zeta \rangle \) with commutator maps \( C_0 \) or \( C \) denoted by \( \tilde{L} \) or \( \tilde{L}_\nu \),

\[
1 \longrightarrow \langle \zeta \rangle \longrightarrow \tilde{L} \ (\text{or } \tilde{L}_\nu) \longrightarrow L \longrightarrow 1
\]

where \( \longrightarrow \) stands for the projection to \( L \). Namely, we have \( aba^{-1}b^{-1} = C_0(\alpha, \beta, \gamma) \) for \( a, b \in \tilde{L} \) (resp. \( \tilde{L}_\nu \)). Let each commutator map correspond to 2-cocycle \( \epsilon_{C_0} \) and \( \epsilon_{C} \). That is \( \epsilon_{C_0} \) satisfies

\[
\epsilon_{C_0}(\alpha, \beta)\epsilon_{C_0}(\alpha + \beta, \gamma) = \epsilon_{C_0}(\beta, \gamma)\epsilon_{C_0}(\alpha, \beta + \gamma), \quad \frac{\epsilon_{C_0}(\alpha, \beta)}{\epsilon_{C_0}(\beta, \alpha)} = C_0(\alpha, \beta).
\]

We choose our 2-cocycle \( \epsilon_{C_0} \) to be

\[
\epsilon_{C_0}(\alpha_i, \alpha_j) = \begin{cases} 1 & \text{if } i \leq j \\ (-1)^{\langle \alpha_i, \alpha_j \rangle} & \text{if } i > j \end{cases}.
\]

This 2-cocycle satisfies

\[
\epsilon_{C_0}(\alpha, \beta)^2 = 1, \quad \epsilon_{C_0}(\alpha, \beta) = \epsilon_{C_0}(\nu \alpha, \nu \beta).
\]

Also, the 2-cocycle \( \epsilon_{C} \) is given by

\[
\epsilon_{C_0}(\alpha, \beta) = (-\zeta)^{\langle \nu^{-1} \alpha, \beta \rangle} \epsilon_{C}(\alpha, \beta).
\]
Using these 2-cocycles, we obtain two normalized sections \( e : L \to \hat{L} \) (resp. \( \hat{L} \)) by
\[
e : L \to \hat{L} \text{ (resp. } \hat{L})
\]
with \( e_0 = 1, \overline{e_\alpha} = \alpha \) and \( e_\alpha e_\beta = \epsilon e_0(\alpha, \beta)e_{\alpha + \beta} \) (resp. \( e_0(\alpha, \beta)e_{\alpha + \beta} \)).

According to [3], there exists an automorphism \( \hat{\nu} \) of \( \hat{L} \) such that
\[
\hat{\nu}a = \nu a, \quad \hat{\nu}a = a \text{ if } \nu a = a.
\]
In order to write down this automorphism explicitly, we will use the following notation defined in [7]. Set
\[
\alpha_i^{(j)} = \alpha_i + \cdots + \alpha_i+j-1 = \sum_{k=0}^{j-1} \alpha_{i+k}
\]
for the simple roots of \( \mathfrak{g} \). We have the set of roots of \( \mathfrak{g} \) as follows.
\[
\Delta = \{ \pm \alpha_i^{(j)} \mid 1 \leq i \leq 2l, 1 \leq j \leq 2l - i + 1 \}.
\]
For this notation, we have
\[
\nu(\alpha_i^{(j)}) = \alpha_{2l-i-j+2}^{(j)}.
\]
Thus \( \alpha_i^{(2l-2i+2)} \) (\( i = 1, \ldots, l \)) are invariant under the automorphism \( \nu \). From [3, 7], we can choose \( \hat{\nu} \) to be
\[
\hat{\nu}(e_{\pm \alpha_i}) = -e_{\pm \alpha_{2l-i-j+2}} \text{ if } i \notin \{l,l+1\},
\]
\[
\hat{\nu}(e_{\pm \alpha_i}) = \pm \zeta e_{\pm \alpha_{i+1}},
\]
\[
\hat{\nu}(e_{\pm \alpha_i}) = \pm \zeta e_{\pm \alpha_i}.
\]
As in [7], we say that \( \alpha_i^{(j)} \) contains \( \alpha_m \) if \( i \leq m \leq i + j - 1 \). Now we have the following proposition (See [7]).

**Proposition 1.** The automorphism \( \hat{\nu} \) of \( \hat{L} \) is completely determined as follows.

\[
\hat{\nu}(e_{\pm \alpha_i}) = -e_{\pm \alpha_{2l-i-j+2}} \text{ if } \alpha_i^{(j)} \text{ does not contain } \alpha_l \text{ or } \alpha_{l+1}, \quad (4)
\]
\[
\hat{\nu}(e_{\pm \alpha_i}) = \pm \zeta e_{\pm \alpha_{i+1}} \text{ if } \alpha_i^{(j)} \text{ contains exactly one of } \alpha_l \text{ or } \alpha_{l+1}, \quad (5)
\]
\[
\hat{\nu}(e_{\pm \alpha_i}) = e_{\pm \alpha_{2l-i-j+2}} \text{ if } \alpha_i^{(j)} \text{ contains both } \alpha_l \text{ and } \alpha_{l+1}. \quad (6)
\]

From Proposition 4, we have \( \hat{\nu}(c_{\alpha_i^{(2l-2i+2)}}) = e_{\alpha_i^{(2l-2i+2)}} \) for \( i = 1, \ldots, l \). In other words, our automorphism \( \hat{\nu} \) satisfies (2) as desired. The map \( \hat{\nu} \) is also an automorphism of \( \hat{L}_\nu \) satisfying (2). We have \( \hat{\nu}^4 = 1 \).

We have the affine Lie algebra corresponding to \( \mathfrak{h} \) by
\[
\mathfrak{h} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c
\]
with Lie bracket given by
\[
[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} c, \quad [\mathfrak{h}, c] = 0
\]
for \( m, n \in \mathbb{Z}, \alpha, \beta \in \mathfrak{h} \). This affine Lie algebra has the \( \mathbb{Z} \)-gradation called the weight grading given by
\[
\text{wt}(\alpha \otimes t^m) = -m, \quad \text{wt}(c) = 0
\]
for \( m \in \mathbb{Z} \) and \( \alpha \in \mathfrak{h} \). Consider the subalgebras \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes t^{\pm 1} \mathbb{C}[t^{\pm 1}] \). Then the Heisenberg subalgebra of \( \hat{\mathfrak{h}} \) is given by \( \hat{\mathfrak{h}}_\mathbb{Z} = \hat{\mathfrak{h}}_+ \oplus \hat{\mathfrak{h}}_- \oplus \mathbb{C}c \). We introduce the induced \( \hat{\mathfrak{h}} \)-module
\[
M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c)} \mathbb{C} \simeq \text{Sym}(\hat{\mathfrak{h}}^+),
\]
where \( \mathfrak{h} \otimes \mathbb{C}[t] \) acts trivially on \( \mathbb{C} \) and \( c \) acts as 1. \( M(1) \) is an irreducible \( \hat{\mathfrak{h}}_\mathbb{Z} \)-module and \( \mathbb{Z} \)-graded. Then we consider the induced \( \hat{L} \)-module
\[
\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\mathfrak{g}]} \mathbb{C} \simeq \mathbb{C}[L].
\]
For \( a \in \hat{L} \), we write \( i(a) = a \otimes 1 \in \mathbb{C}\{L\} \). The space \( \mathbb{C}\{L\} \) is \( \mathbb{Z} \)-graded by letting weight be
\[
\text{wt } i(a) = \frac{1}{2}(\overline{a}, a), \quad \text{wt}(1) = 0
\]
for $a \in \hat{L}$. The action of $\hat{L}, \mathfrak{h}$ on $\mathbb{C}\{L\}$ is given by

$$a \cdot \iota(b) = \iota(ab), \quad h \cdot \iota(a) = \langle h, \mathfrak{h} \rangle \iota(a)$$

for $a, b \in \hat{L}, h \in \mathfrak{h}$. We also define the operator $z^h$ on $\mathbb{C}\{L\}$ by

$$z^h \iota(a) = z^{\langle h, \mathfrak{h} \rangle} \iota(a).$$

We set

$$V_L = M(1) \otimes \mathbb{C}\{L\} \simeq \text{Sym}(\mathfrak{h}^-) \otimes \mathbb{C}\{L\}$$

and $1 = 1 \otimes \iota(1)$. Note that $V_L$ has the tensor product grading and is a tensor product of $\hat{L}$-module on which $\hat{L}$ acts by its action on the second component.

We consider the vertex operator acting on $V_L$. For $a \in \mathfrak{h}, m \in \mathbb{Z}$, we set $\alpha(m) = \alpha \otimes t^m$ and

$$\alpha(z) = \sum_{m \in \mathbb{Z}} \alpha(m) z^{-m-1}.$$ As in [11], we define

$$Y(\iota(a), z) = \frac{1}{\circ} \sum_{m \geq 0} -\text{mod}(m) \circ a \circ z^m$$

for $a \in \hat{L}$, where $\circ$ means the normally ordered product in which the operators $\mathfrak{h}(m)$ for $m < 0$ are placed to the left of the operators $\mathfrak{h}(m)$ for $m > 0$. More generally, for a vector $v = \beta_1(-n_1) \cdots \beta_m(-n_m) \otimes \iota(a) \in V_L$ with $\beta_1, \ldots, \beta_m \in \mathfrak{h}, n_1, \ldots, n_m \geq 0$ and $a \in \hat{L}$ we set

$$Y(v, z) = -\text{mod}_{n_1} -\text{mod}_{n_2} \cdots -\text{mod}_{n_m} \circ \iota(a) \circ z^{-m}.$$

This gives a well-defined linear map

$$V_L \to (\text{End } V_L) [[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{m \in \mathbb{Z}} v_m z^{-m-1}, \quad v_m \in \text{End } V_L.$$ Set

$$\omega = \frac{1}{2} \sum_{i=1}^{2l} \gamma_i(-1) \gamma_i(-1) 1 \in V_L,$$

where $\{\gamma_i\}_{i=1}^{2l}$ is an orthonormal basis of $\mathfrak{h}$.

By Chapter 8 in [12], $(V_L, Y, 1, \omega)$ becomes a simple vertex operator algebra associated to the positive definite even lattice $L$ equipped with central charge equal to rank $L = 2l$.

Now we extend the automorphism $\nu$ of $\hat{L}$ to an automorphism of $V_L$ and also denote it by $\nu$ (cf. [8]). Note that the automorphism $\nu$ of $L$ acts in a natural way on $\mathfrak{h}$ and $M(1)$, preserving the grading. We have

$$\nu(u \cdot m) = \nu(u) \cdot \nu(m)$$

for $u \in \mathfrak{h}$ and $m \in M(1)$. Also the automorphism $\nu$ of $\hat{L}$ is extended to $\mathbb{C}\{L\}$ satisfying conditions

$$\nu(h \cdot \iota(a)) = \nu(h) \cdot \nu(\iota(a)), \quad \nu(z^h \cdot \iota(a)) = z^{\nu(h)} \cdot \nu(\iota(a))$$

and $\nu(a \cdot \iota(b)) = \nu(a) \cdot \nu(\iota(b))$ for $h \in \mathfrak{h}$ and $a, b \in \hat{L}$. We take $\nu$ on $V_L$ to be $\nu \otimes \nu$. It follows that $\nu$ is an automorphism of the vertex operator algebra $V_L$ which preserves the grading on $V_L$.

We construct the $\nu$-twisted module for $V_L$ by following [6] [8] [18]. For $j \in \mathbb{Z}$, set

$$\mathfrak{h}_{(j)} = \{ h \in \mathfrak{h} | \nu h = \zeta^j h \} \subset \mathfrak{h},$$

so that we have

$$\mathfrak{h} = \bigoplus_{j \in \mathbb{Z}/4\mathbb{Z}} \mathfrak{h}_{(j)}.$$ Here we identify $\mathfrak{h}_{(j \text{ mod } 4)}$ with $\mathfrak{h}_{(j)}$. Associated to this decomposition, we define a Lie algebra

$$\hat{\mathfrak{g}}[\nu] = \bigoplus_{m \in \mathbb{Z}} \mathfrak{h}_{(4m)} \otimes t^m \oplus \mathbb{C}c$$

with Lie bracket given by

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} c, \quad [\hat{\mathfrak{g}}[\nu], c] = 0.$$
for $m, n \in \frac{1}{4}\mathbb{Z}$ and $\alpha \in \mathfrak{h}_{(4m)}$ and $\beta \in \mathfrak{h}_{(4n)}$. As in the untwisted case, we give the $\frac{1}{4}\mathbb{Z}$-gradation to $\hat{\mathfrak{h}}[\nu]$ by
\[
\text{wt}(\alpha \otimes t^m) = -m, \quad \text{wt}(c) = 0.
\]
Consider the subalgebras $\hat{\mathfrak{h}}[\nu]^\pm = \bigoplus_{m \geq 0} \mathfrak{h}_{(4m)} \otimes t^m$. Then the Heisenberg subalgebra of $\hat{\mathfrak{h}}[\nu]$ is obtained by $\hat{\mathfrak{h}}[\nu]_{\frac{1}{4}\mathbb{Z}} = \hat{\mathfrak{h}}[\nu]^+ \oplus \hat{\mathfrak{h}}[\nu]^- \oplus \mathbb{C}c$. The induced $\hat{\mathfrak{h}}[\nu]$-module is obtained in similar to (7), that is we have
\[
S[\nu] = U(\hat{\mathfrak{h}}[\nu]) \otimes_U (\bigoplus_{m \geq 0} \mathfrak{h}_{(4m)} \otimes \mathbb{C}c) \simeq \text{Sym}(\hat{\mathfrak{h}}[\nu]^-)
\]
where $\bigoplus_{m \geq 0} \mathfrak{h}_{(4m)} \otimes t^m$ acts trivially on $\mathbb{C}$ and $c$ acts as 1. This module is also an irreducible $\hat{\mathfrak{h}}[\nu]_{\frac{1}{4}\mathbb{Z}}$-module.

We continue to follow 6 [18]. Let $P_j$ be the projection from $\mathfrak{h}$ onto $\mathfrak{h}_{(j)}$ for $j \in \mathbb{Z}/4\mathbb{Z}$. We set
\[
N = (1 - P_0)\mathfrak{h} \cap L = \{ \alpha \in L \mid \langle \alpha, \mathfrak{h}_{(0)} \rangle = 0 \}.
\]
Explicitly, we have
\[
N = \text{Span}_\mathbb{Z} \{ \alpha_i - \alpha_{2i-1} | 1 \leq i \leq l \}.
\]
See 7. Let $\tilde{N}$ be the subgroup of $\hat{L}_\nu$ obtained by pulling back the subgroup $N$ of $L$. By Proposition 6. 1 in [18], there exists a unique homomorphism $\tau : \tilde{N} \to \mathbb{C}^\times$ such that
\[
\tau(\zeta) = \zeta, \quad \tau(\nu \hat{a}^{-1}) = \zeta^{-\sum_{j=0}^{3} (\nu \hat{a})^j/2}
\]
for $a \in \hat{L}_\nu$. Let $\mathbb{C}_\tau$ be the one-dimensional $\tilde{N}$-module $\mathbb{C}$ with this character $\tau$ and consider the induced $\hat{L}_\nu$-module
\[
U_T = \mathbb{C}[\hat{L}_\nu] \otimes_{\mathbb{C}[\tilde{N}]} \mathbb{C}_\tau \simeq \mathbb{C}[L/N].
\]
\[
\hat{L}_\nu \text{ and } \mathfrak{h}_{(0)} \text{ act on } U_T \text{ as}
\]
\[
\begin{align*}
a \cdot b \otimes t &= ab \otimes t, \\
\alpha \cdot a \otimes t &= \langle \alpha, \pi \rangle a \otimes t
\end{align*}
\]
for $a, b \in \hat{L}_\nu$, $t \in \mathbb{C}_\tau$, $\alpha \in \mathfrak{h}_{(0)}$ and we set $[\alpha, a] = \langle \alpha, \pi \rangle a$. The operator $z^h$ on $U_T$ is also defined by
\[
z^h \cdot a \otimes t = z^{(h, \pi)} a \otimes t
\]
for $h \in \mathfrak{h}_{(0)}$. For $h \in \mathfrak{h}_{(0)}$ such that $\langle h, L \rangle \subset \mathbb{Z}$, we define the operator $\zeta^h$ on $U_T$ by $\zeta^h \cdot a \otimes t = \zeta^{(h, \pi)} a \otimes t$. Then for $a \in \hat{L}_\nu$ we have $z^h a = az^{h+\langle h, \pi \rangle}$ and $\zeta^h a = a\zeta^{h+\langle h, \pi \rangle}$. Moreover, as an operator on $U_T$, we have
\[
\hat{\nu}a = a\zeta^{-\sum_{j=0}^{3} (\nu \hat{a})^j/2}.
\]
Then $U_T$ is decomposed into
\[
U_T = \bigoplus \alpha \in P_0 L U_\alpha,
\]
where $U_\alpha = \{ u \in U_T | h \cdot u = \langle h, \alpha \rangle u \text{ for } h \in \mathfrak{h}_{(0)} \}$ satisfying $a \cdot U_\alpha \subset U_{\alpha+\pi(\alpha)}$ for $a \in \hat{L}_\nu$. Define the $\frac{1}{4}\mathbb{Z}$-gradation on $U_T$ by
\[
\text{wt}(u) = \frac{1}{2} \langle \alpha, \alpha \rangle
\]
calculated by $d$ for $u \in U_\alpha$ and $\alpha \in P_0 L$. Set $\hat{\mathfrak{h}}[\nu] = \hat{\mathfrak{h}}[\nu] \oplus \mathbb{C}d$. Then $U_T$ becomes an $\hat{\mathfrak{h}}[\nu]$-module by letting $\hat{\mathfrak{h}}[\nu]_{\frac{1}{4}\mathbb{Z}} \subset \hat{\mathfrak{h}}[\nu]$ act trivially. We set
\[
V_T^+ = S[\nu] \otimes U_T \simeq \text{Sym}(\hat{\mathfrak{h}}[\nu]^-) \otimes \mathbb{C}[L/N]
\]
to be a tensor product of $\hat{\mathfrak{h}}[\nu]$-module and $1_T = 1 \otimes (c_0 \otimes 1)$. Note that $\hat{L}_\nu$ acts by its action on the second component and $V_T^+$ is graded by weights described above.

Next we consider the $\hat{\nu}$-twisted vertex operator by following §2 of [6]. For $\alpha \in \mathfrak{h}$ and $j \in \mathbb{Z}/4\mathbb{Z}$, let $\alpha_{(j)}$ stand for $P_j \alpha \in \mathfrak{h}$. Set
\[
\alpha^\hat{\nu}(m) = \alpha_{(4m)} \otimes t^m, \quad \alpha^\hat{\nu}(z) = \sum_{m \in \frac{1}{4}\mathbb{Z}} \alpha^\hat{\nu}(m)z^{-m-1}
\]
By using (11), we have becomes well-defined operator on $z$

Set $\Delta z$

Note that $\Delta z$

For $v$ acting on $V_L$, we set

The map $W : V_L \to \text{End} V_L^T[[z^{1/2}, z^{-1/2}]]$ gives well-defined linear operator on $V_L^T$ as in the untwisted case. Recall the operator $\Delta_z$ on $V_L$ defined in [6] which is obtained from an orthonormal basis $\{\gamma_l\}_{l=1}^{2l}$ of $h$.

Set

where constants $c_{mnj}$ are given by

Note that $\Delta_z$ is independent of the choice of the orthonormal basis. Since $c_{00j} = 0$ for all $j$, the map $e^{\Delta_z} v$ becomes well-defined operator on $V_L$ and we have $e^{\Delta_z} v \in V_L[z^{-1}]$ for all $v \in V_L$. Then, for $v \in V_L$, we have the $\nu$-twisted vertex operator as

Recall the operator $\hat{\Delta}$ defined in [6] which is obtained from an orthonormal basis $\{\gamma_l\}_{l=1}^{2l}$ of $h$.

By $\hat{S} [11, 12, 13]$, $(V_L^T, \hat{Y}^\nu)$ has the structure of an irreducible $\nu$-twisted $V_L$-module. We have the twisted Jacobi identity

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have

$$Y^\nu(\nu^j u, z_0) v, z_2)$$

for $u, v \in V_L$. The commutator formula of twisted vertex operator (cf. [11]) is derived from this identity. By using [11], we have
We define the operators $L^\vartheta(m)$ for $m \in \mathbb{Z}$ by

$$Y^\vartheta(\omega, z) = \sum_{m \in \mathbb{Z}} L^\vartheta(m) z^{-m-2}.$$  

These operators have the commutation relation

$$[L^\vartheta(m), L^\vartheta(n)] = (m - n)L^\vartheta(m + n) + \frac{l}{6}(m^3 - m)\delta_{m+n,0}$$

for $m, n \in \mathbb{Z}$. That is $\{L^\vartheta(m) \mid m \in \mathbb{Z}\}$ generates the Virasoro vertex algebra submodule. By Proposition 6.3 of [8], we have

$$\langle \cdot, \cdot \rangle|_{L^\vartheta} = \left(\frac{\langle \alpha, \alpha \rangle}{2} + \frac{l}{8}\right)u$$

for $1 \in S[\nu]$, $u \in U_\alpha \subset V_L^T$ and

$$[\vartheta(0), \alpha^\vartheta(m)] = -ma^\vartheta(m)$$

for $\alpha \in h(4m)$, $m \in \frac{1}{4}\mathbb{Z}$. Therefore by using the weight gradings on $S[\nu]$ and $U_T$, we have

$$L^\vartheta(0)v = \left(wt(v) + \frac{l}{8}\right)v$$

for a homogeneous vector $v \in V_L^T$.

### 2.2. Representation of twisted affine Lie algebras of type $A_{2l}^{(2)}$ on $V_L^T$. The aim of this section is to recall the twisted vertex operator construction of affine Lie algebras $A_{2l}^{(2)}$ and give their representation on $V_L^T$. We have the root space decomposition

$$\mathfrak{g} = h \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}x_\alpha,$$

where $x_\alpha$ is a root vector such that $[h, x_\alpha] = \alpha(h)$ for $h \in h$. Now, we normalize a root vector $x_\alpha$ so that we have

$$[x_\alpha, x_\beta] = \begin{cases} 
\epsilon_{C_\alpha}(\alpha, -\alpha)\alpha & \text{if } \alpha + \beta = 0 \\
\epsilon_{C_\beta}(\alpha, \beta)x_{\alpha + \beta} & \text{if } \alpha + \beta \in \Delta \\
0 & \text{otherwise}.
\end{cases}$$

Then the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ reads as

$$\langle h, x_\alpha \rangle = 0, \quad \langle x_\alpha, x_\beta \rangle = \begin{cases} 
\epsilon_{C_\alpha}(\alpha, -\alpha) & \text{if } \alpha + \beta = 0 \\
0 & \text{if } \alpha + \beta \neq 0
\end{cases}$$

As in [6], [7], [15], we introduce the map $\varphi : (\mathbb{Z}/4\mathbb{Z}) \times L \to \langle \zeta \rangle$ which is defined by the condition

$$\varphi^j(e_\alpha) = \varphi(j, \alpha)j(e_{\nu^j\alpha}).$$

From the calculation of Proposition [1], we have

$$\varphi(j, \pm \alpha_{i}^{(j)}) = (-1)^j \quad \text{if } \alpha_{i}^{(j)} \text{ does not contain } \alpha_l \text{ or } \alpha_{l+1},$$

$$\varphi(j, \pm \alpha_{i}^{(j)}) = (\pm \zeta)^j \quad \text{if } \alpha_{i}^{(j)} \text{ contains exactly one of } \alpha_l \text{ or } \alpha_{l+1},$$

$$\varphi(j, \pm \alpha_{i}^{(j)}) = 1 \quad \text{if } \alpha_{i}^{(j)} \text{ contains both } \alpha_l \text{ and } \alpha_{l+1}$$

for $j \in \mathbb{Z}/4\mathbb{Z}$. The automorphism $\nu$ of $h$ is lifted to an automorphism of $\mathfrak{g}$ by

$$\nu^j x_\alpha = \varphi(j, \alpha)x_{\nu^j\alpha}.$$
Note that $\nu^4 = 1$ and $\nu$ preserves $[\cdot, \cdot]$ (resp. $\langle \cdot, \cdot \rangle$). Now, the automorphism $\nu$ of $g$ is explicitly given by

$$
\nu \left( x_{\pm\alpha^{(j)}} \right) = -x_{\pm\alpha^{(j)}} \quad \text{if } \alpha^{(j)} \text{ does not contain } \alpha_i \text{ or } \alpha_{i+1},
$$

$$
\nu \left( x_{\pm\alpha^{(j)}} \right) = \pm \zeta x_{\pm\alpha^{(j)}} \quad \text{if } \alpha^{(j)} \text{ contains exactly one of } \alpha_i \text{ or } \alpha_{i+1},
$$

$$
\nu \left( x_{\pm\alpha^{(j)}} \right) = x_{\mp\alpha^{(j)}} \quad \text{if } \alpha^{(j)} \text{ contains both } \alpha_i \text{ and } \alpha_{i+1}.
$$

For $j \in \mathbb{Z}$ set

$$
\mathfrak{g}_{(j)} = \{ x \in \mathfrak{g} \mid \nu x = \zeta^j x \}.
$$

Now, based on (21)-(23) we obtain $\mathfrak{g}_{(j)}$ ($j \in \mathbb{Z}/4\mathbb{Z}$) as follows.

$$
\mathfrak{g}_{(0)} = \bigoplus_{i=1}^l \left( \mathbb{C} x_{\alpha^{(2i-2i+2)}} \oplus \mathbb{C} (\alpha_i + \alpha_{2i-1}) \oplus \mathbb{C} x_{-\alpha^{(2i-2i+2)}} \right) \oplus \bigoplus_{i=1}^{l-1} \bigoplus_{j=1}^{l-j} \mathbb{C} \left( x_{\chi^{(j)}} - x_{\chi^{(j)}} \right),
$$

$$
\mathfrak{g}_{(1)} = \bigoplus_{i=1}^l \left( \mathbb{C} (x_{\alpha^{(i-1)}} + x_{\alpha^{(i+1)}}) \oplus \mathbb{C} (x_{-\alpha^{(i-1)}} - x_{-\alpha^{(i+1)}}) \right),
$$

$$
\mathfrak{g}_{(2)} = \bigoplus_{i=1}^l \mathbb{C} (\alpha_i - \alpha_{2i-1}) \oplus \bigoplus_{i=1}^{l-1} \bigoplus_{j=1}^{l-j} \mathbb{C} \left( x_{\chi^{(j)}} + x_{\chi^{(j)}} \right),
$$

$$
\mathfrak{g}_{(3)} = \bigoplus_{i=1}^l \left( \mathbb{C} (x_{\alpha^{(i-1)}} - x_{\alpha^{(i+1)}}) \oplus \mathbb{C} (x_{-\alpha^{(i-1)}} + x_{-\alpha^{(i+1)}}) \right).
$$

The twisted affine Lie algebra $\tilde{\mathfrak{g}}[\nu]$ associated to $\mathfrak{g}$ and $\nu$ is given by

$$
\tilde{\mathfrak{g}}[\nu] = \bigoplus_{m \in \mathbb{Z}/4\mathbb{Z}} \mathfrak{g}_{(4m)} \otimes t^m \oplus \mathbb{C} c
$$

with Lie bracket

$$
[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m \delta_{m+n, 0} c, \quad [\tilde{\mathfrak{g}}[\nu], c] = 0
$$

for $m, n \in \mathbb{Z}/4\mathbb{Z}$, $x \in \mathfrak{g}_{(4m)}$ and $y \in \mathfrak{g}_{(4n)}$. We also define the Lie algebra $\tilde{\mathfrak{g}}[\nu]$ by

$$
\tilde{\mathfrak{g}}[\nu] = \tilde{\mathfrak{g}}[\nu] \oplus \mathbb{C} d,
$$

where $d$ is the degree operator such that

$$
[d, x \otimes t^m] = m x \otimes t^m
$$

for $m \in \mathbb{Z}/4\mathbb{Z}$, $x \in \mathfrak{g}_{(4m)}$ and $[d, c] = 0$. This Lie algebra $\tilde{\mathfrak{g}}[\nu]$ ( or $\tilde{\mathfrak{g}}[\nu]$) is isomorphic to the twisted affine Lie algebra of type $A_{2l}^{(2)}$. See Table 2 for its Dynkin diagram.

```
  0 1 2 3
0 1 2 3
```

**Table 2.** Twisted affine Dynkin diagram of type $A_{2l}^{(2)}$

**Theorem 2.** ([(6) Theorem 3.1], [(11) Theorem 3], [(18) Theorem 9.1]) The representation of $\tilde{\mathfrak{g}}[\nu]$ on $V_L^T$ extends uniquely to a Lie algebra representation of $\tilde{\mathfrak{g}}[\nu]$ on $V_L^T$ such that

$$(x_{\alpha})_{(4m)} \otimes t^m \mapsto Y_{\alpha}^\rho(m)$$

for all $m \in \mathbb{Z}/4\mathbb{Z}$ and $\alpha \in L_2 = \{ \alpha \in L \mid \langle \alpha, \alpha \rangle = 2 \}$. Moreover $V_L^T$ is irreducible as a $\tilde{\mathfrak{g}}[\nu]$-module.

Therefore $V_L^T$ is an integrable highest weight module of highest weight $\Lambda_0$, where $\Lambda_0$ is the fundamental weight such that $\langle \Lambda_0, c \rangle = 1$ and $\langle \Lambda_0, h(0) \rangle = 0 = \langle \Lambda_0, d \rangle$. A highest weight vector is $1_T$. 
2.3. Standard module. From Theorem 2 we have the basic \( \hat{g}[\nu] \)-module \( L(\Lambda_0) \simeq V^T_L \). We also have the following formulas on \( V^T_L \).

\[
e_\alpha d e^{-1}_\alpha = d + \alpha - \frac{1}{8} \left( \sum_{j=0}^{3} \nu^j \alpha, \alpha \right) c, \tag{24}
\]

\[
e_\alpha h e^{-1}_\alpha = h - \alpha(h) c, \tag{25}
\]

\[
e_\alpha h(m) e^{-1}_\alpha = h(m) \text{ for } j \neq 0, \tag{26}
\]

\[
e_\alpha Y^\beta_m(m) e^{-1}_\alpha = C(\alpha, \beta) Y^\beta_m(m - \langle \alpha, \beta(0) \rangle). \tag{27}
\]

Next we consider the standard \( \hat{g}[\nu] \)-module \( L(k\Lambda_0) \) of higher level \( k \), namely the integrable highest weight \( \hat{g}[\nu] \)-module of highest weight \( k\Lambda_0 \). Since we have \( L(\Lambda_0) \simeq V^T_L \), we can realize \( L(k\Lambda_0) \) as a submodule of the tensor product of \( k \) copies of \( V^T_L \) as

\[
L(k\Lambda_0) \simeq U(\hat{g}[\nu]) \cdot v_0 \subset (V^T_L)^\otimes k,
\]

where \( v_0 = 1_T \otimes \cdots \otimes 1_T \) is a highest weight vector of \( L(k\Lambda_0) \). The action of \( \hat{g}[\nu] \) on \( L(k\Lambda_0) \) is given by coproduct

\[
\Delta^{(k-1)}(x) = x \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes x \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes x,
\]

where each term has \( k \) components. It is also true for the vertex operators. For a positive integer \( n \), we set

\[
x^\nu_{n\alpha}(z) = [\Delta^{(k-1)}(Y^\nu(\iota(e_\alpha), z))]^n. \tag{28}
\]

Note that \( x^\nu_{(k+1)\alpha}(z) = 0 \) because of the null identity on \( V_L \) such that \( e^{\Delta(\iota(e_\alpha)^2)} \cdot 1 = 0 \). We also define a component operator \( x^\nu_{n\alpha}(m) \) by

\[
x^\nu_{n\alpha}(z) = \sum_{m \in \frac{1}{k} \mathbb{Z}} x^\nu_{n\alpha}(m) z^{-m-n(\alpha, \alpha)/2}. \tag{29}
\]

We recall the Cartan subalgebra and simple roots for twisted affine Lie algebras introduced in [16, §8.3]. The Cartan subalgebra is obtained by

\[
h_{(0)} \oplus \mathbb{C} c \oplus \mathbb{C} d.
\]

Chevalley generators \( h_i \) (0 \( \leq i \leq l \)) are given as follows.

\[
\begin{cases}
h_0 = -\sum_{i=1}^{2l} \alpha_i \\
h_i = \alpha_i + \alpha_{2l-i+1} & \text{for } i = 1, \ldots, l-1 \\
h_l = 2(\alpha_l + \alpha_{l+1}).
\end{cases}
\]

We know that \( \hat{g}[\nu] \) contains a finite-dimensional simple Lie algebra \( \hat{g}_{(0)} \) (it coincides with the notation \( \hat{g}_0 \) used in [16]). Note that the Dynkin diagram of \( \hat{g}_{(0)} \) is of type \( B_l \). One can take the set of simple roots of \( \hat{g}_{(0)} \) as that of the image under the projection \( P_0 \) to \( h_{(0)} \). Thus we can take it as \( \{ (\alpha_1)_{(0)}, \ldots, (\alpha_l)_{(0)} \} \).

Set \( Q = \bigoplus_{i=1}^{l} \mathbb{Z}\alpha_i \subset L \). Note that \( Q_{(0)} = P_0 Q \) should be understood as the root lattice of \( \hat{g}[\nu] \).

Following [24], we define the adjoint action of \( e_\alpha \) on \( \hat{g}[\nu] \) by

\[
e_\alpha c e^{-1}_\alpha = c, \tag{30}
\]

\[
e_\alpha d e^{-1}_\alpha = d + \alpha - \frac{1}{2} (\alpha_{(0)}, \alpha_{(0)}) c, \tag{31}
\]

\[
e_\alpha h e^{-1}_\alpha = h - \alpha(h) c \text{ for } h \in h_{(0)}, \tag{32}
\]

\[
e_\alpha h(m) e^{-1}_\alpha = h(m) \text{ for } m \neq 0, \tag{33}
\]

\[
e_\alpha x^\nu_m(m) e^{-1}_\alpha = C(\alpha, \beta) x^\nu_m(m - \langle \alpha, \beta(0) \rangle). \tag{34}
\]

We have its action on \( (V^T_L)^\otimes k \) by \( e_\alpha \mapsto e_\alpha \otimes \cdots \otimes e_\alpha \), and hence also on \( L(k\Lambda_0) \). These calculations are based on [23, 27] on \( V^T_L \). This action corresponds to the translation operator of the affine Weyl group of \( \hat{g}[\nu] \). See Section 1.5 of [4] for untwisted cases.

Next lemma reveals the relation between the twisted vertex operators for positive and negative roots. The proof is completely parallel to [19, Theorem 5.6], [25, Theorem 6.4] or [24, Lemma 3].
Lemma 3. We renormalize the twisted vertex operator $x_{\alpha}^\nu(z)$ as $\tilde{x}_{\alpha}^\nu(z) = 4\sigma(\alpha)^{-1}x_{\alpha}^\nu(z)$ for $\alpha \in L_2$. Then, for $p, q \geq 0$ such that $p + q = k$, we have
\[
\frac{1}{p!} E^{-}(\alpha, z)(z\tilde{x}_{\alpha}^\nu(z))^p E^{+}(\alpha, z) = \frac{1}{q!} \epsilon_C(\alpha, -\alpha)^{-q}(z\tilde{x}_{-\alpha}^\nu(z))^q e_{\alpha} z^{\alpha(0) + \frac{k(n(\alpha), n(\alpha))}{z}}
\] (35)
as an operator on $(V^T)^{\otimes k}$ or $L(k\Lambda_0)$. In particular, (35) can be rewritten as
\[
E^{-}(\alpha, z)\exp(z\tilde{x}_{\alpha}^\nu(z)) E^{+}(\alpha, z) = \exp(\epsilon_C(\alpha, \alpha)^{-1} z\tilde{x}_{-\alpha}^\nu(z)) e_{\alpha} z^{\alpha(0) + \frac{k(n(\alpha), n(\alpha))}{z}}.
\] (36)

3. PRINCIPAL SUBSPACE

Let $\Delta_+$ be the set of positive roots. We set
\[
\mathfrak{n} = \bigoplus_{\alpha \in \Delta_+} \mathbb{C}x_{\alpha}.
\]
The algebra $\mathfrak{n}$ is the nilradical of the Borel subalgebra. Consider its twisted affinization
\[
\hat{\mathfrak{n}}[\nu] = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{n}(4m) \otimes t^m \oplus \mathbb{C}
\]
and its subalgebra
\[
\mathfrak{P}[\nu] = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}} \mathfrak{n}(4m) \otimes t^m.
\]
In [6, 7, 20, 21], the principal subspace $W(k\Lambda_0)$ of $L(k\Lambda_0)$ is defined by
\[W(k\Lambda_0) = U(\mathfrak{P}[\nu]) \cdot v_0.
\]

3.1. Twisted quasi-particle. In this section, we introduce the twisted quasi-particle and its monomials. We define the twisted quasi-particle of color $i$, charge $n$ and energy $-m$ for each simple root $\alpha_i$, $n \in \mathbb{N}$ and $m \in \frac{1}{2}\mathbb{Z}$ as the coefficient $x_{n\alpha_i}^\nu(m)$ in (39).

From [5], we review twisted quasi-particle monomials. A twisted quasi-particle monomial is defined by
\[
b = b(\alpha_1) \cdots b(\alpha_k) = x_{n_{i_1}^{(1)}}^\nu \cdots x_{n_{i_1}^{(1)}+n_{i_2}^{(1)}}^\nu \cdots x_{n_{i_1}^{(1)}+n_{i_2}^{(1)}+\ldots+n_{i_l}^{(1)}}^\nu
\] (37)
with $1 \leq n_{i_1}^{(1)}, \ldots, n_{i_l}^{(1)} \leq \ldots \leq n_{i_1}^{(1)}, n_{i_2}^{(1)}, \ldots, n_{i_l}^{(1)}$ for each $i$. For [8, 7], we set the sequences $\mathcal{R}', \mathcal{E}$ as
\[
\mathcal{R}' = (n_{i_1}^{(1)}, \ldots, n_{i_1}^{(1)}, \ldots, n_{i_l}^{(1)}, \ldots, n_{i_l}^{(1)}), \quad \mathcal{E} = (m_{i_1}^{(1)}, \ldots, m_{i_1}^{(1)}, \ldots, m_{i_l}^{(1)}, \ldots, m_{i_l}^{(1)}).
\]
They are called charge-type, energy-type respectively. For these charge-type and energy-type, we define total-charge and total-energy by
\[
\text{chg } b = \sum_{i=1}^l \sum_{p=1}^{n_{i_1}^{(1)}} n_{p,i}, \quad \text{en } b = \sum_{i=1}^l \sum_{p=1}^{n_{i_1}^{(1)}} m_{p,i}.
\]
The dual-charge-type
\[
\mathcal{R} = (r_1^{(1)}, \ldots, r_1^{(s_1)}, \ldots, r_l^{(1)}, \ldots, r_l^{(s_1)})
\]
is defined in the way that $(r_1^{(1)}, \ldots, r_1^{(s_1)})$ is the transposed partition of $(n_{i_1}, \ldots, n_{i_l}^{(1)})$ for each $i$. We also define the color-type $\mathcal{C}$ by
\[
\mathcal{C} = (r_1, \ldots, r_1)
\]
where $r_i, n_{p,i}, r_i^{(s_1)}$ are related as
\[
r_i = \sum_{p=1}^{n_{i_1}^{(1)}} n_{p,i} = \sum_{i=1}^l r_i^{(s_1)}.
\] (38)
We denote the set of all quasi-particle monomials of the form [8, 7] by $\mathcal{M}_{Q^p}$. In this definition, $r_i^{(s_1)}$ stands for the number of quasi-particles of color $i$ and charge greater then or equal to $s$ in the monomial $b$. Remark that $r_i^{(s_1)} = 0$ for $k < s$ since we have $x_{k+1}\alpha_i^\nu(z) = 0$ on $L(k\Lambda_0)$ for $\alpha \in L$. The charge-type and the dual-charge-type of [8, 7] can be replaced by the $l$-tuple of Young diagrams, so that $i$-th diagram correspond to $b(\alpha_i)$. That is, the $i$-th diagram has $r_i^{(s_1)}$ boxes in the $s$-th row and $n_{p,i}$ boxes in the
For two monomials \( R \) and \( R' \) with \( R' = (r'_{1,1}, \ldots, r'_{1,1}) \), we write \( R < R' \) if there exists \( i \) and \( s \) such that \( r'_{j} = r_{j}^{(1)} \), \( n_{t,j} = n_{t,j} \) for \( j < i, 1 \leq t \leq r(j) \) and \( n_{1,i} = \bar{n}_{1,i}, n_{2,i} = \bar{n}_{2,i}, \ldots, n_{s-1,i} = \bar{n}_{s-1,i}, n_{s,i} = \bar{n}_{s,i} \) or \( n_{t,i} = \bar{n}_{t,i} \) for \( 1 \leq t \leq r(i) \), \( r'_{i} < r_{i}^{(1)} \). We apply this order "<" to energy-types and color-types. For two monomials \( b \) and \( b' \) with charge-types \( R \) and \( R' \), energy-type \( E \) and \( E' \), respectively, we write \( b < b' \) if

1. \( R' < R \) or
2. \( R' = R \) and \( E < E' \).

By [4, 5], we have the linear order "<" on the set of monomials in \( M_{QP} \).

Next, we consider relations among twisted quasi-particles which give the basis conditions that will be described later. The following lemma is proved in the same way as [6, Lemma 3.2] and is a special case of [7, Lemma 2.1, Lemma 2.2].

**Lemma 4.**

(i) \( Y_{\alpha_i} \tilde{\psi}(m) = 0 \) for \( i \notin \{l, l+1\} \) and \( m \in \frac{1}{2} + \frac{1}{2} \mathbb{Z} \)

(ii) \( Y_{\alpha_i} \tilde{\psi}(m) = Y_{\alpha_{i+1}} \tilde{\psi}(m) = 0 \) for \( m \in \frac{1}{2} \mathbb{Z} \)

(iii) \( Y_{\alpha_{2l-i+1}} \tilde{\psi}(m) = (-1)^{2m+1} Y_{\alpha_{l}} \tilde{\psi}(m) \) for \( i \notin \{l, l+1\} \) and \( m \in \frac{1}{2} \mathbb{Z} \)

(iv) \( Y_{\alpha_{l+1}} \tilde{\psi}(m) = \zeta^{4m} Y_{\alpha_{l}} \tilde{\psi}(m) \) for \( m \in \frac{1}{4} + \frac{1}{2} \mathbb{Z} \)

**Proof.** We show that (i) holds for \( \alpha_i \) with \( i \notin \{l, l+1\} \). By taking \( j = 2 \) and \( v = e(\alpha_i) \) in (20) and using Proposition 1, we have

\[
Y^{\tilde{\psi}}(e(\alpha_i), z) = Y^{\tilde{\psi}}(e(\alpha_i), z) \big|_{z \rightarrow -z} \frac{1}{z}.
\]

As the component operator, we have

\[
\sum_{m \in \frac{1}{2} \mathbb{Z}} Y_{\alpha_i} \tilde{\psi}(m) z^{-m-1} = \sum_{m \in \frac{1}{2} \mathbb{Z}} (-1)^{2m} Y_{\alpha_{l}} \tilde{\psi}(m) z^{-m-1}.
\]

Thus we obtain the statement by taking the residue of the both sides of (40). (ii) is obtained from the same argument.

Then we take \( j = 1 \) and \( v = e(\alpha_i) \) (\( i \neq l, l+1 \)) in (20). From Proposition 1, we have

\[
Y^{\tilde{\psi}}(-e(\alpha_{2l-i+1}), z) = Y^{\tilde{\psi}}(e(\alpha_i), z) \big|_{z \rightarrow -z} \zeta z.
\]

Thus we obtain

\[
- \sum_{m \in \frac{1}{2} \mathbb{Z}} Y_{\alpha_{2l-i+1}} \tilde{\psi}(m) z^{-m-1} = \sum_{m \in \frac{1}{2} \mathbb{Z}} \zeta^{4m} Y_{\alpha_{l}} \tilde{\psi}(m) z^{-m-1}.
\]
This implies (iii). We are able to prove (iv) in the same argument by taking \( j = 1 \) and \( \nu = i(e_{\alpha_i}) \) in \cite{20}.

For every simple root \( \alpha_i \), consider the one-dimensional subalgebra of \( \mathfrak{g} \)
\[
\mathfrak{n}_{\alpha_i} = \mathbb{C} x_{\alpha_i},
\]
and its affinizations
\[
\mathfrak{n}_{\alpha_i}[\nu] = \bigoplus_{m \in \mathbb{Z}} (\mathfrak{n}_{\alpha_i})_{(4m)} \otimes e^m.
\]
Set
\[
U = U(\mathfrak{n}_{\alpha_i}[\nu]) \cdots U(\mathfrak{n}_{\alpha_i}[\nu])
\]
to be the subspace of \( U(\mathfrak{g}[\nu]) \). From Lemma \ref{lem:5} we can prove the following lemma by arguing as in \cite{13} Lemma 3.1.

**Lemma 5.** \( W(k\Lambda_0) = U(\mathfrak{n}[\nu]) \cdot v_0 = U \cdot v_0 \)

**Proof.** Lemma \ref{lem:4} implies that the subalgebras \( U(\mathfrak{n}_{\alpha_i}[\nu]), \ldots, U(\mathfrak{n}_{\alpha_i}[\nu]) \) span the principal subspace \( W(k\Lambda_0) \). Then the statement that the ordered set \( U \) spans \( W(k\Lambda_0) \) is showed as in the proof of \cite{13} Lemma 3.1. \( \square \)

From the definition of the twisted vertex operators on \( L(k\Lambda_0) \), we have the following lemma.

**Lemma 6.** For fixed \( 1 \leq n_2 \leq n_1 \), \( N = 0, 1, \ldots, 2n_2 - 1 \) and \( i = 1, \ldots, l \), we have
\[
\left( \frac{d}{dz} \right)^N x_{n_2\alpha_i}(z) x_{n_1\alpha_i}(z) = A_N(z) x_{(n_1+1)\alpha_i}(z) + B_N(z) \frac{d}{dz} x_{(n_1+1)\alpha_i}(z)
\]
where \( A_N(z) \) and \( B_N(z) \) are some formal series with coefficients in the set of quasi-particle polynomials.

**Proof.** The relation \eqref{eq:28} implies that we have
\[
(n + 1) \left( \frac{d}{dz} x_{\alpha_i}(z) \right) x_{\alpha_i}(z) = \frac{d}{dz} x_{(n+1)\alpha_i}(z).
\]
By using the Leibniz rule and \eqref{eq:28}, we have
\[
\left( \frac{d}{dz} \right)^N x_{n_2\alpha_i}(z) = \sum_{k_1 + \cdots + k_{n_2} = N} \binom{N}{k_1, \ldots, k_{n_2}} \left( \frac{d}{dz} \right)^{k_1} x_{\alpha_i}(z) \cdots \left( \frac{d}{dz} \right)^{k_{n_2}} x_{\alpha_i}(z).
\]
For \( N = 0, 1, \ldots, 2n_2 - 1 \), we obtain the statement by combining the above formula, \eqref{eq:28} and \eqref{eq:41}. \( \square \)

We set
\[
\rho_i = \frac{1}{2} \langle (\alpha_i)(0), (\alpha_i)(0) \rangle.
\]
By combining Lemma \ref{lem:4} the next lemma can be proved in the same way as Lemma 4.1 in \cite{5}.

**Lemma 7.** Let \( 1 \leq n_2 \leq n_1 \) be fixed. For \( i = 1, \ldots, l \) and fixed \( j \in \rho_i n_2 + \frac{1}{2} \mathbb{Z} \), \( M \in \rho_i(n_1 + n_2) + \frac{1}{2} \mathbb{Z} \), the \( 4\rho_i n_2 \) monomials from the set
\[
A = \{ x_{n_2\alpha_i}(j) x_{n_1\alpha_i}(M - j), x_{n_2\alpha_i}(j - \frac{1}{2}) x_{n_1\alpha_i}(M - j + \frac{1}{2}), \ldots,
\]
\[
\ldots, x_{n_2\alpha_i}(j - \frac{4\rho_i n_2 - 1}{2}) x_{n_1\alpha_i}(M - j + \frac{4\rho_i n_2 - 1}{2}) \}
\]
can be expressed as a linear combination of monomials from the set
\[
\{ x_{n_2\alpha_i}(s) x_{n_1\alpha_i}(t) \mid s + t = M \} \setminus A
\]
and monomials which have as a factor the quasi-particle \( x_{(n_1+1)\alpha_i}(j') \) for \( j' \in \rho_i(n_1 + 1) + \frac{1}{2} \mathbb{Z} \).
Proof. For $N = 0, 1, \ldots, 4\rho_n n_2 - 1$, we consider the expansion of formal series
\begin{align}
\frac{1}{N!} \left( \frac{d}{dz} \right)^N x^\rho_{n_2 \alpha_i}(z) x^\rho_{n_1 \alpha_i}(z) = \sum_{M \in \rho_i(n_1+n_2)+\frac{1}{2}Z} \left( \begin{array}{c} N \vspace{1mm} \end{array} \begin{array}{c} s-n_2 \vspace{1mm} \end{array} \right) x^\rho_{n_2 \alpha_i}(s) x^\rho_{n_1 \alpha_i}(t) z^{-M-n_2-n_1-N}. \quad (43)
\end{align}

Note that the energy $m$ for quasi-particle with color $i$ and charge $n$ goes over $\rho_i n + \frac{1}{2}Z$ because of Lemma 4. Then for fixed $j \in \rho_i n_2 + \frac{1}{2}Z, M \in \rho_i(n_1+n_2) + \frac{1}{2}Z$, we can separate the right hand side of (43) by
\begin{align}
\left( \begin{array}{c} -j-n_2 \vspace{1mm} \end{array} \begin{array}{c} N \vspace{1mm} \end{array} \right) x^\rho_{n_2 \alpha_i}(j) x^\rho_{n_1 \alpha_i}(M-j) + \left( \begin{array}{c} -j-n_2 + \frac{1}{2} \vspace{1mm} \end{array} \begin{array}{c} N \vspace{1mm} \end{array} \right) x^\rho_{n_2 \alpha_i} \left( j - \frac{1}{2} \right) x^\rho_{n_1 \alpha_i} \left( M - j + \frac{1}{2} \right) + \cdots \\
+ \left( \begin{array}{c} -j-n_2 + \frac{4\rho_n n_2-1}{2} \vspace{1mm} \end{array} \begin{array}{c} N \vspace{1mm} \end{array} \right) x^\rho_{n_2 \alpha_i} \left( j - \frac{4\rho_n n_2-1}{2} \right) x^\rho_{n_1 \alpha_i} \left( M - j + \frac{4\rho_n n_2-1}{2} \right) + \text{other terms.} \quad (44)
\end{align}

From Lemma 6 we also have
\begin{align}
\left( \begin{array}{c} j \vspace{1mm} \end{array} \begin{array}{c} n_1+1 \alpha_i \vspace{1mm} \end{array} \right) = A_N(z) x^\rho_{n_1+1 \alpha_i}(z) + B_N(z) \frac{d}{dz} x^\rho_{n_1 \alpha_i}(z). \quad (45)
\end{align}

Thus we obtain the linear equation
\begin{align}
\left( \begin{array}{c c c}
\begin{array}{c c c}
-\frac{j-n_2}{N} & -\frac{j-n_2+1}{2} & \cdots & -\frac{j-n_2+\frac{4\rho_n n_2-1}{2}}{2} \\
\begin{array}{c c c}
-\frac{j-n_2}{N} & -\frac{j-n_2+1}{2} & \cdots & -\frac{j-n_2+\frac{4\rho_n n_2-1}{2}}{2} \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\frac{j-n_2}{4\rho_n n_2-1} & \frac{j-n_2}{4\rho_n n_2-1} & \cdots & \frac{j-n_2}{4\rho_n n_2-1} \\
\end{array}
\end{array}
\end{array}
\right)
\left( \begin{array}{c}
x^\rho_{n_2 \alpha_i} \left( j \right) x^\rho_{n_1 \alpha_i} \left( M-j \right) \\
x^\rho_{n_2 \alpha_i} \left( j - \frac{1}{2} \right) x^\rho_{n_1 \alpha_i} \left( M-j + \frac{1}{2} \right) \\
x^\rho_{n_2 \alpha_i} \left( j - \frac{4\rho_n n_2-1}{2} \right) x^\rho_{n_1 \alpha_i} \left( M-j + \frac{4\rho_n n_2-1}{2} \right)
\end{array}
\right) = X,
\end{align}

where $X$ is a vector whose components are linear combinations of monomials from the set \{ $x^\rho_{n_2 \alpha_i}(s) x^\rho_{n_1 \alpha_i}(t) \mid s + t = M \} \setminus A$, and monomials which have as a factor the quasi-particle $x^\rho_{(n_1+1) \alpha_i}(j')$ for $j' \in \rho_i(n_1+1)$.

Now, the statement follows from the regularity of the coefficient matrix and it can be proved in the same way as Lemma 4.1 in [4]. 

Let $b$ be a quasi-particle monomial in $M_{QP}$. From Lemma 4 the $4\rho_i n_2$ monomials
\begin{align}
x^\rho_{n_2 \alpha_i}(m) x^\rho_{n_1 \alpha_i}(m') v, x^\rho_{n_2 \alpha_i} \left( m - \frac{1}{2} \right) x^\rho_{n_1 \alpha_i} \left( m' + \frac{1}{2} \right) v, \ldots \\
\ldots, x^\rho_{n_2 \alpha_i} \left( m - \frac{4\rho_i n_2 - 1}{2} \right) x^\rho_{n_1 \alpha_i} \left( m' + \frac{4\rho_i n_2 - 1}{2} \right) v
\end{align}

such that $n_2 < n_1$ can be expressed as a linear combination of the monomials
\begin{align}
x^\rho_{n_2 \alpha_i}(s) x^\rho_{n_1 \alpha_i}(t) v \text{ with } s \leq m - 2\rho_i n_2, \quad t \geq m + 2\rho_i n_2
\end{align}

and monomials which have a factor quasi-particle $x^\rho_{(n_1+1) \alpha_i}(j)$, $j \in \rho_i(n_1+1) + \frac{1}{2}Z$. If $n_2 = n_1$, then the monomials
\begin{align}
x^\rho_{n_1 \alpha_i}(m) x^\rho_{n_1 \alpha_i}(m') v \text{ such that } m' - \rho_i n_1 < m \leq m'
\end{align}

can be expressed as a linear combination of the monomials
\begin{align}
x^\rho_{n_1 \alpha_i}(s) x^\rho_{n_1 \alpha_i}(t) v \text{ with } s \leq t - \rho_i n_1
\end{align}

and monomials which have a factor quasi-particle $x^\rho_{(n_1+1) \alpha_i}(j)$, $j \in \rho_i(n_1+1) + \frac{1}{2}Z$.

The next lemma reveals the relations among quasi-particle monomials with different colors.

Lemma 8. Let $P(z_{i_1}^{(1)}, \ldots, z_{1,1})$ be the polynomial defined by
\begin{align}
P(z_{i_1}^{(1)}, \ldots, z_{1,1}) = \prod_{i=1}^l r_i^{(1)} \prod_{q=1}^{r_{i_1}^{(1)}-1} 3 \left( 1 - \zeta_{i_1}^{(1)} \frac{z_{q,i_1-1}}{z_{p,i}} \right)^{-\langle \rho_i^{(1)} \alpha_i, \alpha_i-1 \rangle \min\{n_{p,i},n_{q,i-1}\}} \\
= \prod_{i=1}^l r_i^{(1)} \prod_{q=1}^{r_{i_1}^{(1)}-1} \left( 1 - \zeta_{i_1}^{(1)} \frac{z_{q,i_1-1}}{z_{p,i}} \right)^\min\{n_{p,i},n_{q,i-1}\}.
\end{align}
Then we have
\[
P(z_{\ell_{1}}, \ldots, z_{1,1}) \prod_{1 \leq s < t \leq \ell^{(1)}} \left(1 + \frac{z_{s,t}^{\nu}}{z_{t,s}^{\nu}}\right)^{n_{s,t}} x_{\rho_{t_{1}}, \alpha}^{\nu}(z_{\ell_{1}}, \ldots) \cdots x_{\rho_{1}, \alpha}^{\nu}(z_{1,1}) v_{0} \]
\[
\in \left[\prod_{i=1}^{l} \prod_{p=1}^{r_{i}^{(1)}} \frac{z_{n_{p}, i}^{\rho} - \frac{1}{2} \sum_{q=1}^{r_{i}^{(1)}} \min\{n_{p}, n_{q, i-1}\}}{z_{i, n_{p}}^{\rho}} \right]^{\nu} \prod_{t=1}^{r_{i}^{(1)}} \frac{z_{t, 0}^{\nu} - \frac{1}{2} \sum_{p=1}^{r_{t}^{(1)}} \min\{n_{p}, n_{t, i-1}\}}{z_{t, 1}^{\nu}} W(k\Lambda_{0})[[\frac{z_{r_{i}^{(1)}, 1}}{z_{1, r_{i}^{(1)}}}, \ldots, \frac{z_{1, 1}}{z_{1, 1}}]].
\]

Proof. By using \[16,\] we have
\[
E^{-}(\alpha, z)E^{+}(\alpha, z)e_{\alpha}z^{\alpha(0)} E^{-}(\beta, w)E^{+}(\beta, w)e_{\beta}w^{\beta(0)}
\]
\[
= \sum_{j=0}^{3} E^{-}(\alpha, z)E^{+}(\beta, w)E^{+}(\alpha, z)z^{\alpha(0)} e_{\beta}w^{\beta(0)}
\]
\[
= \prod_{j=0}^{3} (z^{\frac{1}{2}} - \zeta^{\frac{1}{2}} w^{\frac{1}{2}})^{-\nu_{\alpha, \beta}} E^{-}(\alpha, z)E^{+}(\beta, w)E^{+}(\alpha, z)z^{\alpha(0)} w^{\beta(0)}
\]
on $V_{L}^{\nu}$, where we recall that $z^{h}a = az^{h+\langle h, \nu \rangle}$ for $h \in \mathfrak{h}_{(0)}$, $a \in \hat{L}_{\nu}$. Therefore we have
\[
\begin{align*}
(1) & \quad x_{\rho_{i}, \alpha}(z)x_{\rho_{i}^{(1)}, \alpha}^{\nu}(w)v_{0} \in W(k\Lambda_{0})[[\frac{z_{\nu}}{z_{\nu}}, \frac{w}{w}]] \\
(2) & \quad (z^{\frac{1}{2}} + \frac{w}{z^{\frac{1}{2}}})^{\nu_{\rho_{i}, \alpha}} x_{\rho_{i}, \alpha}(z)x_{\rho_{i}^{(1)}, \alpha}^{\nu}(w)v_{0} \in W(k\Lambda_{0})[[\frac{z_{\nu}}{z_{\nu}}, \frac{w}{w}]], \\
(3) & \quad \prod_{j=0}^{3} (z^{\frac{1}{2}} - \zeta^{\frac{1}{2}} w^{\frac{1}{2}})^{-\nu_{\alpha, \beta}} x_{\rho_{i}, \alpha}(z)x_{\rho_{i}^{(1)}, \alpha}^{\nu}(w)v_{0} \in W(k\Lambda_{0})[[\frac{z_{\nu}}{z_{\nu}}, \frac{w}{w}]]
\end{align*}
\]
\[
\Leftrightarrow \left(1 - \frac{w}{z^{\frac{1}{2}}}\right)^{-\nu_{\alpha, \beta}} x_{\rho_{i}, \alpha}(z)x_{\rho_{i}^{(1)}, \alpha}^{\nu}(w)v_{0} \in \left[z^{-\frac{1}{2}} \left[\prod_{j=0}^{3} (z^{\frac{1}{2}} - \zeta^{\frac{1}{2}} w^{\frac{1}{2}})^{-\nu_{\alpha, \beta}}\right] W(k\Lambda_{0})[[\frac{z_{\nu}}{z_{\nu}}, \frac{w}{w}]]
\]
We can prove the lemma by applying these relations to the generating function $x_{\rho_{i}, \alpha}^{\nu}(z_{r_{i}^{(1)}, 1}) \cdots x_{\rho_{i}, \alpha}^{\nu}(z_{1, 1})$. \[\square\]

3.2. Quasi-particle bases of principal subspace. In view of Lemma \[7\] and \[8\] we consider the following conditions (C1)-(C3) for the modes $m$ in $x_{\rho_{i}, \alpha}(m)$ in \[37\].

(C1) $m_{p, i} \in \rho_{i}n_{p, i} + \frac{1}{2} \mathbb{Z}$ for $1 \leq p \leq \ell^{(1)}, 1 \leq i \leq l$,

(C2) $m_{p, i} \leq -(2p - 1)\rho_{i}n_{p, i} + \frac{1}{2} \sum_{q=1}^{r_{i}^{(1)}} \min\{n_{p, i}, n_{q, i-1}\}$ for $1 \leq p \leq \ell^{(1)}, 1 \leq i \leq l$,

(C3) $m_{p+1, i} \leq m_{p, i} - 2\rho_{i}n_{p, i}$ if $n_{p+1, i} = n_{p, i}$ for $1 \leq p \leq \ell^{(1)} - 1, 1 \leq i \leq l$.

We set
\[
B_{W} = \bigcup_{0 \leq r_{1}^{(k)} \leq \cdots \leq r_{l}^{(1)}} \{b\text{ as in }[37]\text{ satisfying (C1) - (C3)}\}
\]
where $r_{0}^{(1)} = 0$. Then by using the same proof as in \[13\] Theorem 4.1, we can prove the following proposition (cf. \[5\]).

Proposition 9. The set
\[
B_{W} = \{bv_{0} \mid b \in B_{W}\}
\]
spans $W(k\Lambda_{0})$.

Proof. Since we have
\[
U \cdot v_{0} = W(k\Lambda_{0})
\]
from Lemma \[5\] we should show that every vector $bv_{0}$ from $U \cdot v_{0}$ is a linear combination of vectors from $B_{W}$. The condition (C1) follows from Lemma \[4\]. Using the same argument as in \[13\] and the fact that for fixed charge-type and total-energy, the set of quasi-particles is upper bounded with respect to our order,
we can derive the remaining conditions. All vectors \( b_{n0} \) are expressed as a combination of vectors which satisfy the weaker condition

\[
(C2)' \quad m_{p,i} \leq -((\alpha_i)(0), (\alpha_{i-1})(0)) \sum_{q=1}^{r_{i-1}} \min\{n_{p,i}, n_{q,i-1}\} + \frac{\delta_{i,l}(p-1)n_{p,l} - \rho_i n_{p,i}}{2}.
\]

where the first term of the right hand side of \( (C2)' \) is equal to 0 when \( i = 1 \). This condition is a consequence of Lemma 5. In fact, for a quasi-particle \( b \) which contradicts the condition \( (C2)' \), by taking residue of the generating function of \( b \) in Lemma 5 we have that \( b \) is a linear combination of quasi-particles \( b' \) larger than \( b \) since the residue of the right hand side is equal to 0. Thus by induction, we can show the claim. From Lemma 7 in the case that \( n_2 < n_1 \), we can increase an energy of quasi-particles. Therefore we are able to strengthen \( (C2)' \) to \( (C2) \) by induction. Finally, Lemma 7 in the case of \( n_2 = n_1 \) implies \( (C3) \).

### 3.3. Proof of linear independence.

To prove the linear independence of the set \( B_W \), we recall the map \( \Delta_T \) introduced in [6, 22]. Let \( \lambda_i \) \( (i = 1, \ldots, 2l) \) be the fundamental weights of \( g \). Set

\[
\Delta_T(\lambda_i, -z) = \zeta(\lambda_i)(0) z^r(\lambda_i)(0) E^+(-\lambda_i, z).
\]

Note that \( \zeta(\lambda_i)(0) \) and \( z(\lambda_i)(0) \) are operators on \( U_T \) and thus on \( V^T_L \), and \( E^+(-\lambda_i, z) \) also on \( V^T_L \), so that we have

\[
\Delta_T(\lambda_i, -z) \in \text{End} \ V^T_L\{[z^\pm \lambda]\}.
\]

From [21] Proposition 3.4, the commutation relation (16) also holds for \( \Delta_T(\lambda_i, -z) \). The constant term of \( \Delta_T(\lambda_i, -z) \) is denoted by \( \Delta^T(\lambda_i, -z) \). Following [6, 21], we set \( \theta_j : L \to \mathbb{C}^\times \) as the character of the root lattice \( L \) to be

\[
\theta_j(\alpha_i) = -\zeta, \quad \theta_j(\alpha_{2l-1-i}) = \zeta, \quad \theta_j(\alpha_i) = 1 \text{ for } i \notin \{j, 2l-j+1\}.
\]

Define the Lie algebra automorphism \( \tau_{\lambda_j, \theta_j} : \mathfrak{h}[v] \to \mathfrak{h}[v] \) by

\[
\tau_{\lambda_j, \theta_j}(x^\alpha(m)) = \theta_j(\alpha) x^\alpha(m + \langle \alpha(0), \lambda_j \rangle).
\]

For \( a \in U(\mathfrak{h}[v]) \), we have

\[
\Delta^T(\lambda_j, -z)(a \cdot 1_T) = \tau_{\lambda_j, \theta_j}(a) 1_T
\]

on \( V^T_L \). Using [17], we have

\[
[\Delta^T(\lambda_i, -z), a^\alpha(m)] = \begin{cases} 0 & (m \geq 0), \\ \langle \alpha(4m), \lambda_i \rangle (-4m) z^m \Delta^T(\lambda_i, -z) & (m < 0). \end{cases}
\]

Therefore we obtain

\[
[\Delta^T(\lambda_j, -z), a^\alpha(m)] = 0
\]

as an operator on \( V^T_L \) for \( a^\alpha(m) \in \mathfrak{h}[v] \). Furthermore, for a bijection map \( e_{\alpha_i} \), we have

\[
[\Delta^T(\lambda_j, -z), e_{\alpha_i}] = 0 \quad (i \neq j)
\]

from \( z(\lambda_i)(0) e_{\alpha_i} = e_{\alpha,i} z^r(\lambda_i)(0) \) and the fact that we define \( \Delta^T(\lambda_j, -z) \) as the constant term.

We consider the Georgiev-type projection (cf. [5, 13]). We realize \( W(k\Lambda_0) \) as a subspace of the \( k \) fold tensor product of \( W(\Lambda_0) \) in the basic module \( V^T_L \). Namely, we have

\[
W(k\Lambda_0) \subset W(\Lambda_0) \otimes \cdots \otimes W(\Lambda_0) \subset (V^T_L)^\otimes k.
\]

The projection \( \pi_R \) is defined by

\[
\pi_R : W(k\Lambda_0) \to W(\Lambda_0)_{(r^1_1, \ldots, r^1_k)} \otimes \cdots \otimes W(\Lambda_0)_{(r^1_1, \ldots, r^1_k)}
\]

for a dual-charge-type \( R \) given by

\[
R = \left(r^1_1, \ldots, r^1_k; \ldots; r^1_1, \ldots, r^1_k\right).
\]

where \( W(\Lambda_0)_{(r^1_1, \ldots, r^1_k)} \) is the subspace of \( W(\Lambda_0) \) spanned by the vectors whose charges are \( r^1_1, \ldots, r^1_k \) for \( 1 \leq s \leq k \).

By using the relation \( x^\phi_{2\alpha}(z) = 0 \) on \( V^T_L \), we have

\[
\pi_R x^\phi_{n, \alpha_{1,1}}(z, r^1) \cdots x^\phi_{n, 1, \alpha_2}(z, 1) = 0
\]
where $C_R$ is the constant term given by $\prod_{i=1}^{l} \prod_{p=1}^{(i)} (n_{p,i})!$ and

$$0 \leq n_{p,i}^{(k)} \leq n_{p,i}^{(k-1)} \leq \cdots \leq n_{p,i}^{(1)} \leq 1$$

for $1 \leq p \leq r_1^{(1)}$, $1 \leq i \leq l$. Note that the projection of $b_{v_0}$ with $b$ as in (37) is obtained by a coefficient of this projection.

Now, we fix $s \leq k$ and consider the map

$$\Delta_T^s(\lambda_j, -z)_s = 1 \otimes \cdots \otimes 1 \otimes \Delta_T^s(\lambda_j, -z) \otimes 1 \otimes \cdots \otimes 1$$

which only acts on the $s$-th tensor component. From (40), we have

$$\Delta_T^s(\lambda_j, -z)_s \pi_R \tilde{x}^p_{\alpha_1}(z_{r_1^{(1)}}, i) \cdots \tilde{x}^p_{\alpha_1}(z_{1,1}) v_0$$

$$= (-\zeta)^{(r_1^{(s)})} C_R \left( Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{r_1^{(s)})}) \cdots Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{1,1}) \cdots Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{1,1}) 1_T \right)$$

$$\otimes \cdots \otimes$$

$$\left( Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{r_1^{(s)})}) \cdots Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{1,1}) \cdots Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{1,1}) 1_T \right) \prod_{i=1}^{l} \prod_{p=1}^{(i)} \alpha_{p,i}$$

$$\otimes \cdots \otimes$$

$$Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{r_1^{(s)})}) \cdots Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{1,1}) \cdots Y^\tilde{\rho}(\lambda(e_{ai_1}), z_{1,1}) 1_T \right).$$

Thus by taking the corresponding coefficient, we have

$$\Delta_T^s(\lambda_j, -z)_s \pi_R b_{v_0} = (-\zeta)^{(r_1^{(s)})} \pi_R b^+ v_0,$$

where $b^+ = (b_{ai_1}) \cdots (b_{ai_j}) \cdots (b_{ai_1})$ such that

$$b^+ (\alpha_j) = x^p_{n_{r_j^{(1)}}, \alpha_j} (m_{r_j^{(1)}}, i) \cdots x^p_{n_{r_j^{(1)}}, \alpha_j} (m_{r_j^{(1)}}, j) x^p_{n_{r_j^{(1)}}, \alpha_j} (m_{r_j^{(1)}}, j) + \frac{1}{2} \cdots x^p_{n_{1,1}, \alpha_j} (m_{1,1} + \frac{1}{2}).$$

For $b$ as in (37), we set $s = n_{1,1}$, $d = -2m_{1,1} - s$. From (40), we get

$$\Delta_T^s(\lambda_j, -z)_s \pi_R b_{v_0} = (-\zeta)^{(d_1^{(s)})} \pi_R x^p_{n_{r_1^{(1)}}, \alpha_1} (m_{n_{r_1^{(1)}}, i}) \cdots x^p_{n_{r_1^{(1)}}, \alpha_1} (m_{n_{r_1^{(1)}}, i}) \cdots x^p_{n_{r_1^{(1)}}, \alpha_1} (m_{n_{r_1^{(1)}}, i}) + \frac{1}{2} \cdots x^p_{n_{2,1}, \alpha_1} (m_{2,1} + \frac{1}{2}) 1_T \otimes \cdots \otimes 1_T \otimes Y^\tilde{\rho}_{\alpha_1} (m_{2,1} + \frac{1}{2}) 1_T$$

$$\otimes \cdots \otimes Y^\tilde{\rho}_{\alpha_1} (m_{2,1} + \frac{1}{2}) 1_T).$$

Since $Y^\tilde{\rho}_{\alpha_1} (m_{2,1} + \frac{1}{2}) 1_T = \frac{1}{2} e_{\alpha_1}$, by using (27) we have

$$\zeta^{(d_1^{(s)})} C_R^{-1} \Delta_T^s(\lambda_j, -z)_s^{d_1^{(s)}} \pi_R b_{v_0} = \left( \frac{1}{2} \right)^s C_{\alpha_1} (\alpha_1, \alpha_2) \sum_{p=1}^{n_{1,1}^{(1)}} n_{p,2} C_R^{-1} (1 \otimes \cdots \otimes 1 \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_1})$$

$$\pi_R x^p_{n_{r_1^{(1)}}, \alpha_1} (m_{n_{r_1^{(1)}}, i}) \cdots x^p_{n_{r_1^{(1)}}, \alpha_1} (m_{n_{r_1^{(1)}}, i}) \cdots x^p_{n_{r_1^{(1)}}, \alpha_1} (m_{n_{r_1^{(1)}}, i}) \cdots x^p_{n_{2,1}, \alpha_1} (m_{2,1} + \frac{1}{2}) 1_T \otimes \cdots \otimes Y^\tilde{\rho}_{\alpha_1} (m_{2,1} + \frac{1}{2}) 1_T,$$

where $\mathcal{R}^-$ is the dual-charge-type given by

$$\mathcal{R}^- = (r_1^{(1)}, \cdots, r_1^{(k)}; \cdots; r_1^{(s)} - 1, 1, 0, \ldots, 0)$$
By applying the above trick to 
recall that the following relation holds
\[ A \]
where
\[ b \]
a \( \in \)
\[ m \]
Note that these
\[ \alpha \]
We recall that
\[ \nu \]
the following lemma. Since relations between twisted vertex operators also holds, the proof is parallel to
\[ \text{Lemma 11}. \]
\[ \text{Spanning sets for standard modules.} \]
We should prove the linear independence of
\[ B \]
\[ \text{obtained by the projection of a vector in} \]
\[ R \]
\[ \text{determines the dual-charge-type} \]
\[ A \]
\[ \text{element of} \]
\[ B \]
\[ \text{for} \] \( 1 \leq p \leq r_{(1)}^{(1)} \). Since the map \( e_{\alpha} \) is injective, it implies that \( \Delta_{\nu}(\lambda_{1} - \nu)_{R}^{d} \pi_{R} bv_{0} \) is obtained from the projection of a vector \( b'v_{0} \in B_{W} \) which less than \( bv_{0} \) with our linear order.

We can continue to apply this trick for \( n_{2,1}, \ldots, n_{r(1),1} \) and \( i = 2, \ldots, l \). Then the image of \( bv_{0} \) is also obtained by the projection of a vector in \( B_{W} \) which less than that one.

**Theorem 10.** The set \( B_{W} \) is a basis of \( W(k\Lambda_{0}) \).

**Proof.** We should prove the linear independence of \( B_{W} \). We consider a linear combination of vectors in \( B_{W} \),
\[ \sum_{a \in A} c_{a}a_{a}v_{0} = 0 \]
for a finite set \( A \). Let \( b \) be a quasi-particle in \( B_{W} \) with the minimal charge-type \( R' \). The charge-type \( R' \) determines the dual-charge-type \( R \) and the projection \( \pi_{R} \). For a quasi-particle \( b_{\nu} \) with the charge-type \( R' \) greater than \( R' \), we have \( \pi_{R} b_{\nu} v_{0} = 0 \) from the definition of the projection. Therefore we have
\[ \sum_{a \in A'} \pi_{R} c_{a} a_{a} v_{0} = 0, \]
where \( A' \) is a subset of \( A \) such that \( b_{a} \) has the charge-type \( R' \) for \( a \in A' \). Namely, all vectors in \( \{b_{a}v_{0} \mid a \in A' \} \) have the same color-charge-type. We assume that \( \overline{b}_{0} \) is the smallest vector in \( \{b_{a}v_{0} \mid a \in A' \} \). By applying the above trick to \( b_{0}v_{0} \), we can reduce the component operators from \( b_{0}v_{0} \) one by one. Note that all monomial vectors \( b_{a}v_{0} \) such that \( \overline{b} < b_{a} \) will be annihilated in this step. Thus we have that the coefficient \( \pi_{\nu} \) of \( \overline{b} \) is equal to zero. Continuing the process, we can show that all coefficients are zero. \( \square \)

### 4. Bases of Standard Modules

#### 4.1. Spanning sets for standard modules.

To obtain a basis of the standard module, we introduce the following lemma. Since relations between twisted vertex operators also holds, the proof is parallel to that of Lemma 5 of [24].

**Lemma 11.**

1. \( L(k\Lambda_{0}) = U(\mathfrak{h}[v^{-}])QW(k\Lambda_{0}) \)

2. \( L(k\Lambda_{0}) = QW(k\Lambda_{0}) \)

From Lemma 11, we know that the set \( QB_{W} \) spans \( L(k\Lambda_{0}) \), but it is not its basis. For example, we recall that the following relation holds
\[ x_{\alpha_{1}} \left( -\frac{1}{2} \right) \cdot v_{0} = \frac{1}{2} c_{\alpha_{1}} \cdot v_{0} \]
on \( L(\Lambda_{0}) \). To get a canonical basis, we consider the following set.
\[ B_{H} = \left\{ h_{\alpha_{1}} \cdots h_{\alpha_{l}} \mid h_{\alpha_{i}} = \alpha_{\nu}^{\alpha_{i}} (-m_{t_{i},i})^{n_{i,1}} \cdots \alpha_{\nu}^{\alpha_{i}} (-m_{1,i})^{n_{i,1}}, i = 1, \ldots, l \right\} \]
We recall that \( \alpha_{\nu}(m) \) is given by \( \alpha_{(4m)} \otimes t^{m} \). Since we have
\[ h_{(1)} = h_{(3)} = \{0\} \]
for our automorphism \( \nu \), we have \( \alpha_{\nu}(m) = 0 \) unless \( m \in \frac{1}{2} \mathbb{Z} \). Then we introduce the linear order on \( B_{H} \).
A element of \( B_{H} \) has a datum \( (n_{t_{i},i}, n_{1,i} \cdots ; n_{1,i}, n_{1,1}) \) or \( (m_{t_{i},i}, m_{1,i} \cdots ; m_{1,i}, m_{1,1}) \).
We can define the order "<" for such datum in the same way as the charge-type $\mathcal{R}'$. For two elements $h = h_{\alpha_1} \cdots h_{\alpha_1}$, $\overline{h} = \overline{h}_{\alpha_1} \cdots \overline{h}_{\alpha_1} \in B_H$ of fixed degree, we write $h < \overline{h}$ if

(1) $\langle m_{t_i}, \ldots, n_{1,1} \rangle < \langle \overline{m}_{t_i}, \ldots, \overline{n}_{1,1} \rangle$ or

(2) $\langle m_{t_i}, \ldots, n_{1,1} \rangle = \langle \overline{m}_{t_i}, \ldots, \overline{n}_{1,1} \rangle$ and $\langle m_{t_i}, \ldots, m_{1,1} \rangle < \langle \overline{m}_{t_i}, \ldots, \overline{m}_{1,1} \rangle$.

By combining the order defined in Section 3.1, we generalize this order to the set

$$\{ e_{\mu, \overline{h}b}v_0 \mid \mu \in Q, h \in B_H, b \in M_{QP}' \}$$

where $M_{QP}'$ is the subset of $M_{QP}$ with no $x_i^{\rho}(m)$ for $i = 1, \ldots, l$. For two such vectors $e_{\mu, \overline{h}b}v_0, e_{\mu, \overline{h}b}v_0$ in $M_{QP}$ of fixed degree and $\mathfrak{h}(0)$-weight, we denote the color-type of $b, \overline{b}$ by $\mathcal{C}, \overline{\mathcal{C}}$ respectively. Then we write $e_{\mu, \overline{h}b}v_0 < e_{\mu, \overline{h}b}v_0$ if one of the following conditions holds.

(1) $\text{chg } b > \text{chg } \overline{b}$,

(2) $\text{chg } b = \text{chg } \overline{b}$ and $\mathcal{C} < \overline{\mathcal{C}}$,

(3) $\mathcal{C} = \overline{\mathcal{C}}$ and $b < \overline{b}$,

(4) $\mathcal{C} = \overline{\mathcal{C}}$, $b = \overline{b}$ and $b < \overline{h}$,

(5) $b = \overline{b}$ and $h < \overline{h}$.

Note that $M_{QP}$ is upper bounded with respect to this order. By induction in our order "<", the following proposition is proved in the same way as Proposition 6 in [24] (cf. [1] Lemma 2.3).

**Proposition 12.** The set $\mathcal{B}_L = \{ e_{\mu, h}b(v_0) \mid \mu \in Q, h \in B_H, b \in B_W \cap M_{QP}' \}$ spans $L(k\Lambda_0)$.

### 4.2. Combinatorial bases of the standard module

Consider the decomposition

$$L(k\Lambda_0) = \bigoplus_{s \in \mathbb{Z}} L(k\Lambda_0)s, \quad \text{where } L(k\Lambda_0)s = \bigoplus_{s_2, \ldots, s_l \in \mathbb{Z}} L(k\Lambda_0)s_{s_1} \cdots s_{s_l}.$$

From this decomposition we have the Georgiev-type projection

$$\pi_{\mathcal{R}_{\alpha_1}} : L(k\Lambda_0) \rightarrow L(\Lambda(0))_{r_1}^{(1)} \otimes \cdots \otimes L(\Lambda(0))_{r_l}^{(k)}$$

for a fixed dual-charge-type $\mathcal{R}_{\alpha_1} = (r_1^{(1)}, r_1^{(2)}, \ldots, r_1^{(k)})$ for the color 1 and $r_1 = \sum_{s=1}^k r_1^{(s)}$. We also have the decomposition and the Georgiev-type projection $\pi_{\mathcal{R}_{\alpha_1}}$ for the color $i = 2, \ldots, l$. We will use these projections to prove Theorem 13. These projections are naturally generalized to

$$L(k\Lambda_0)[[w_1^{\frac{1}{2}}, \ldots, w_{\overline{1},1}^{\frac{1}{2}}, w_{r_1^{(1)},1}^{\frac{1}{2}}, \ldots, w_{r_1^{(k)},1}^{\frac{1}{2}}]]$$

and denoted by $\pi_{\mathcal{R}_{\alpha_1}}$. Set $\alpha^\varphi(w_\omega) = \sum_{m>0} \alpha^\varphi(m)w^{-m-1}$. We consider the vector

$$e_{\mu, \alpha^\varphi(-m_{t_i}^{(1)}, \ldots, m_{r_1^{(1)},1}^{(1)}, \ldots, x^\varphi_{r_1^{(1)},1}^{(1)} \cdots m_{r_1^{(k)},1}^{(k)}, \ldots, x^\varphi_{r_1^{(k)},1}^{(k)} \cdots m_{1,1}^{(1)}, \ldots, m_{1,1}^{(1)} \cdots m_{1,1}^{(k)}} v_0$$

with dual-charge-type $\mathcal{R} = (\mathcal{R}_{\alpha_1}, \ldots, \mathcal{R}_{\alpha_1})$. We know that the image of this vector with respect to $\pi_{\mathcal{R}_{\alpha_1}}$ is obtained by the coefficient of the corresponding projection of the generating function

$$e_{\mu, \alpha^\varphi(w_1^{\frac{1}{2}}, \ldots, w_{\overline{1},1}^{\frac{1}{2}}, \ldots, x^\varphi_{r_1^{(1)},1}^{(1)} \cdots m_{r_1^{(k)},1}^{(k)}, \ldots, x^\varphi_{r_1^{(k)},1}^{(k)} \cdots m_{1,1}^{(1)}, \ldots, m_{1,1}^{(1)} \cdots m_{1,1}^{(k)}} v_0.$$

In [24], we defined the generalized twisted vertex operator for an elements of an extension of the weight lattice $P$ of $\mathfrak{g}$ to prove the linear independence of $\mathcal{B}_L$. In this paper, we continue to use the operator $\Delta_L^\varphi$ to prove the following theorem.

**Theorem 13.** The set $\mathcal{B}_L$ is a basis of $L(k\Lambda_0)$.

**Proof.** We prove the linear independence of $\mathcal{B}_L$. Consider a linear combination of vectors in $\mathcal{B}_L$,

$$\sum_{e_{\mu, h}b} c_{\mu, h}e_{\mu, h}b v_0 = 0$$

of fixed degree and $\mathfrak{h}(0)$-weight. From [24], $e_{\mu}$ is the bijection such that it maps the weight space $V_{\rho}$ to $V_{\rho+k\alpha}$ for $\mathfrak{h}(0)$-weight $\rho$. Hence we may assume that a summand in [55] with the maximal charge of color 1, $\text{chg}_1 b = r_1$, has $\mu$ with $\alpha_1$ coordinate zero. That is, we assume that summands in [55] has the form
(A) $e_{\mu}h b v_0$ with $\text{chg}_i b = r_1$ and $\mu = c_0 \alpha_1 + \cdots + c_2 \alpha_2$, or 
(B) $e_{\mu}h b v_0$ with $\text{chg}_i b < r_1$ and $\mu = \bar{c}_1 \alpha_1 + \cdots + \bar{c}_1 \alpha_1$, where $\bar{c}_1 > 0$.

Among the vectors $v = e_{\mu}h b v_0$ with $\text{chg}_i b = r_1$, we choose a vector with the maximal charge-type $R'_{\alpha_1}$ and corresponding dual-charge-type

$$R_{\alpha_1} = \left( r_1^{(1)}, \ldots, r_1^{(k-1)}, 0 \right)$$

for the color $i = 1$, where $r_1 = r_1^{(1)} + \cdots + r_1^{(k-1)}$. Note that $r_1^{(k)} = 0$ for a vector in $\mathcal{B}_L$. Since the action of $e_{\alpha}$ on $L(k \Lambda_0)$ is given by

$$e_{\alpha} \cdot v_0 = e_{\alpha} 1_T \otimes \cdots \otimes e_{\alpha} 1_T,$$

we have

$$e_{\mu}h b v_0 \in \bigoplus_{s_1, \ldots, s_k \in \mathbb{Z}} L(\Lambda_0)_{s_1} \otimes \cdots \otimes L(\Lambda_0)_{s_k}.$$ 

Therefore we have $\pi_{R_{\alpha_1}}(e_{\mu}h b v_0) = 0$ for the vector of the form (B). After applying the projection $\pi_{R_{\alpha_1}}$ to the sum (55), we have only the summands of the form (A). From (47), (48), we can apply $\Delta_T(\lambda_1, -z)$ to $\pi_{R_{\alpha_1}}(e_{\mu}h b v_0)$ and it affect only $b$. Now we choose the smallest monomial $b$ in the summands of the form (A). Using the same way to the proof of Theorem 10, we can reduce the color 1 quasi-particles from $e_{\mu}h b v_0$ one by one. Then we have the vector $e_{\mu}h b v_0$ with $\text{chg}_i b' = 0$. By applying the same trick for the color $i = 2, \ldots, l$, we have $c_{\mu, h, b} = 0$. Continuing this process, we can show the linear independence of $\mathcal{B}_L$.

\[\square\]

5. PARAFERMIONIC BASES

In this section we define the parafermionic space as in [4, 24]. By the same argument in [24], we obtain the parafermionic bases of the parafermionic space.

5.1. Vacuum space and twisted $Z$-operators. We denote the vacuum space of the standard module $L(k \Lambda_0)$ by $L(k \Lambda_0)^{\hat{b}[\nu]^+}$. That is we have

$$L(k \Lambda_0)^{\hat{b}[\nu]^+} = \{ v \in L(k \Lambda_0) \mid \hat{b}[\nu]^+ \cdot v = 0 \}. \tag{56}$$

From the Lepowsky-Wilson theorem [21] (A5.3) we have canonical isomorphism of $d$-graded linear spaces

$$U(\hat{b}[\nu]^-) \otimes L(k \Lambda_0)^{\hat{b}[\nu]^+} \xrightarrow{\cong} L(k \Lambda_0)$$

$$\hat{h} \otimes u \mapsto \hat{h} \cdot u$$

where $U(\hat{b}[\nu]^-) \simeq \text{Sym}(\hat{b}[\nu]^\ast)$ is the Fock space of level $k$ for the Heisenberg subalgebra $\hat{b}[\nu]_{1, \mathbb{Z}}$ with the action of $c$ being the multiplication by scalar $k$. This isomorphism gives the direct decomposition

$$L(k \Lambda_0) = L(k \Lambda_0)^{\hat{b}[\nu]^+} \oplus \hat{b}[\nu]^+ \cdot U(\hat{b}[\nu]^-) L(k \Lambda_0)^{\hat{b}[\nu]^+}. \tag{58}$$

Then we obtain the projection

$$\pi^{\hat{b}[\nu]^+} : L(k \Lambda_0) \to L(k \Lambda_0)^{\hat{b}[\nu]^+}$$

from (58). We define the $Z$-operator by

$$Z_{n, \alpha}(z) = E^-(\alpha, z)^{n/k} x^\alpha (z) E^+(\alpha, z)^{n/k}$$

for a quasi-particle of charge $n$ and a root $\alpha$. Note that the action of $Z$-operators commute with the action of the Heisenberg subalgebra $\hat{b}[\nu]_{1, \mathbb{Z}}$ on the standard module $L(k \Lambda_0)$. More generally, we should define the $Z$-operators for quasi-particles of charge-type $R' = (n_{l_1^2}, \ldots, n_{l_1^1}, 1)$. For the twisted vertex operator $x^{\hat{b}[\nu]}_{R'}(z_{l_1^2}, \ldots, z_{l_1^1}) = x^{\hat{b}[\nu]}_{n_{l_1^2}, \alpha_1} (z_{l_1^2}) \cdots x^{\hat{b}[\nu]}_{n_{l_1^1}, \alpha_1} (z_{l_1^1})$ of charge-type $R'$, we define

$$Z_{R'}(z_{l_1^2}, \ldots, z_{l_1^1}) = E^- (\alpha_1, z_{l_1^2}, \ldots, z_{l_1^1})^{n_{l_1^2}/k} \cdots E^- (\alpha_1, z_{l_1^1})^{n_{l_1^1}/k} x^{\hat{b}[\nu]}_{R'}(z_{l_1^2}, \ldots, z_{l_1^1}) \times E^+ (\alpha_1, z_{l_1^2}, \ldots, z_{l_1^1})^{n_{l_1^2}/k} \cdots E^+ (\alpha_1, z_{l_1^1})^{n_{l_1^1}/k}. \tag{59}$$

For convenience, we write this formal Laurent series by

$$Z_{R'}(z_{l_1^2}, \ldots, z_{l_1^1}) = \sum_{m_{l_1^2}, \ldots, m_{l_1^1} \in \mathbb{Z}} Z_{R'}(m_{l_1^2}, \ldots, m_{l_1^1}) z_{l_1^2}^{-m_{l_1^2}} \cdots z_{l_1^1}^{-m_{l_1^1}}.$$
Since $Z$-operators are well-defined on vacuum space and we can express quasi-particle monomials in terms of $Z$-operators by reversing \((\ref{eq:relation})\), we have
\[
\pi^\beta[v]^+ x_{R'}^\beta(z_{i(1)}, \ldots, z_{1,1}) v_0 \mapsto Z_{R'}(z_{i(1)}, \ldots, z_{1,1}) v_0.
\]

Now, Theorem\[13\] implies

**Theorem 14.** The set of vectors
\[
e^\mu_n \pi^\beta[v]^+ (b) v_0 = e^\mu \pi R^\beta (n_{i(1)}, \ldots, n_{1,1}) v_0
\]
such that $\mu \in \mathbb{Q}$ and $b \in B_W \cap M_{Q', P}$ with the charge-type $R'$ and the energy-type $(n_{i(1)}, \ldots, n_{1,1})$ is a basis of the vacuum space $L(k\Lambda_0)^\beta[v]^+$.

The proof is parallel to that of Theorem 3.1 of \[14\].

### 5.2. Parafermionic space and parafermionic current

The parafermionic space and parafermionic current.

We have the projective representation of $Q$ on $L(k\Lambda_0)$. This gives a diagonal action of the sublattice $k\Lambda_0 \subset Q$ by
\[
k\alpha \mapsto \rho(k\alpha) = e_{\alpha} \otimes \cdots \otimes e_{\alpha}.
\]

Note that this action satisfies $\rho(k\alpha) : L(k\Lambda_0)^\beta[v]^+ \rightarrow L(k\Lambda_0)^\beta[v]^+$ for a weight $\mu$. We define the parafermionic space of the highest weight $k\Lambda_0$ as the space of $kQ$-coinvariants in the $kQ$-module $L(k\Lambda_0)^\beta[v]^+$:
\[
L(k\Lambda_0)|_{kQ} := L(k\Lambda_0)^\beta[v]^+/\text{Span}_\mathbb{C} \{ (\rho(k\alpha) - 1) \cdot v \mid \alpha \in Q, v \in L(k\Lambda_0)^\beta[v]^+ \}.
\]

We have the canonical projection
\[
\pi_{kQ}^\beta : L(k\Lambda_0)^\beta[v]^+ \rightarrow L(k\Lambda_0)|_{kQ}.
\]

The composition $\pi_{kQ}^\beta \circ \pi^\beta[v]^+$ is denoted by $\pi$. Then we have
\[
L(k\Lambda_0)|_{kQ} \cong \bigoplus_{\mu \in k\Lambda_0 + Q/kQ} L(k\Lambda_0)^\beta[v]^+.
\]

For every root $\beta$, we define the parafermionic current of charge $n$ by
\[
\Psi_{n, \beta}^\beta(z) = Z_{n, \beta}(z)^{-n\beta(0)/k} e^{-n\beta/k},
\]
where $\epsilon_{\beta} : L(k\Lambda_0) \rightarrow \mathbb{C}^\times$ is given by
\[
\epsilon_{\beta} u = C(\alpha, \beta) u \quad \text{for } u \in L(k\Lambda_0)_{\mu}.
\]

The commutativity with the action of Heisenberg subalgebra of $Z$-operator implies that the parafermionic current preserves the vacuum space $L(k\Lambda_0)^\beta[v]^+$. We rewrite the commutation relation \[(\ref{eq:relation})\] by
\[
x_{n, \beta}^\beta(z) e_{\alpha} = C(\alpha, \beta)^{-1} \epsilon_{\alpha} x_{n, \beta}^\beta(z)^{\epsilon_{\alpha}(\beta(0))}.
\]

Using this relation and the one between $z^\mu$ and $e_{\alpha}$, we have
\[
[\rho(k\alpha), \Psi_{n, \beta}^\beta(z)] = 0,
\]
where the map $\epsilon_{\beta}$ contribute to vanishing constant term $C(\alpha, \beta)$. Therefore $\Psi_{n, \beta}^\beta(z)$ is well-defined on the parafermionic space $L(k\Lambda_0)|_{kQ}$. For a quasi-particle of charge-type $R' = (n_{i(1)}, \ldots, n_{1,1})$, the parafermionic current of charge-type $R'$ is also defined by
\[
\Psi_{R'}(z_{i(1)}, \ldots, z_{1,1}) = Z_{R'}(z_{i(1)}, \ldots, z_{1,1}) z_{i(1)}^{\alpha_{\alpha(0)}/k} \cdots z_{1,1}^{\alpha_{\alpha(0)}/k} e_{\alpha_{i(1)}} z_{1,1}^{\alpha_{\alpha(0)}/k} \cdots e_{\alpha_{1}}.
\]

Note that the parafermionic current of charge-type $R'$ also commute with the diagonal action $\rho(k\alpha)$ for $\alpha \in Q$. As in the $Z$-operator, we set
\[
\Psi_{R'}(z_{i(1)}, \ldots, z_{1,1}) = \sum_{m_{i(1)}, \ldots, m_{1,1}} \Psi_{R'}^\beta(m_{i(1)}, \ldots, m_{1,1}) z_{i(1)}^{m_{i(1)}} \cdots z_{1,1}^{m_{1,1}}
\]
where the summation is over all sequences $(m_{i(1)}, \ldots, m_{1,1})$ such that $m_{p,i} \in \frac{1}{2} \mathbb{Z} + \frac{n_{i,\alpha(0)+\mu}}{k}$ on the $\mu$-weight space $L(k\Lambda_0)^\beta[v]^+$.
Then we introduce several lemmas which give the parafermionic bases of the space $L(k\Lambda_0)_{kQ}^{\beta|\nu)^+}$. First, next lemma associates the coefficients of $\mathcal{Z}$-operators with those of parafermionic currents (cf. [24 Lemma 10]).

**Lemma 15.** For a simple root $\beta$, $m \in \frac{1}{2}\mathbb{Z}$ and weight $\mu$ we have

$$\mathcal{Z}_\beta(m)\big|_{L(k\Lambda_0)_{\mu}^{\beta|\nu)^+}} = C(\beta, \mu)\psi_{\beta}(m + \langle \beta(0), \mu \rangle / k)\big|_{L(k\Lambda_0)_{\mu}^{\beta|\nu)^+}}.$$

Next, we consider the relation between different parafermionic currents. The following lemmas are obtained by direct computation. The proof are parallel to that of Lemma 3.2 and 3.3 of [4] respectively.

**Lemma 16.** For a simple root $\beta$ and a positive integer $n$,

$$\hat{\Psi}_{n\beta}(z) = \left( \prod_{1 \leq p < s \leq n} \prod_{i=0}^{3} (z_+^{i,\beta} - \zeta^i z_p^{i,\beta})^{(n,\beta)} / k \right) \Psi_{\beta}(z_n) \cdots \Psi_{\beta}(z_1) \bigg|_{z_n = \cdots = z_1 = z}.$$

Note that we use the fact that $C(\beta, \beta) = 1$ for a simple root in the proof of the lemma [10]. For simplicity, we set

$$\Psi_{n_1 \beta_1, \ldots, n_t \beta_t}(z_t, \ldots, z_1) = \mathcal{Z}(n_1, \ldots, n_t)(z_t, \ldots, z_1) \prod_{i=1}^t (-n_i(\beta_i(0))/k - n_i/k)$$

for a given simple roots $\beta_t, \ldots, \beta_1$ and charges $n_t, \ldots, n_1$.

**Lemma 17.**

$$\hat{\Psi}_{n_1 \beta_1, \ldots, n_t \beta_t}(z_t, \ldots, z_1) = \left( \prod_{1 \leq p < s \leq t} C(\beta_s, \beta_p)^{n_s n_p / k} \prod_{i=0}^{3} (z_+^{i,\beta_s} - \zeta^i z_p^{i,\beta_p})^{(n_s,\beta_s, n_p,\beta_p)} / k \right) \hat{\Psi}_{n_1 \beta_1}(z_t) \cdots \hat{\Psi}_{n_t \beta_t}(z_1).$$

From Theorem [14] we have

**Theorem 18.** For the highest weight $k\Lambda_0$, the set of vectors

$$\pi_{kQ}^{\beta|\nu)^+} \mathcal{Z}_{R_\alpha}(m_{1,1}, \ldots, m_{1,1}) v_0 = \psi_{R_\alpha}(m_{1,1}, \ldots, m_{1,1}) v_0$$

is a basis of the paragfermionic space $L(k\Lambda_0)_{kQ}^{\beta|\nu)^+}$, where $\mathcal{Z}_{R_\alpha}(m_{1,1}, \ldots, m_{1,1}) v_0$ is a vector that appears in the basis of the vacuum space $L(k\Lambda_0)_{kQ}^{\beta|\nu)^+}$.

6. THE FERMIONIC CHARACTER FORMULA

By using the quasi-particle bases and the parafermionic bases, we can calculate the character of the principal subspace and the parafermionic space. Furthermore, combining the main theorem and Lepowsky-Wilson theorem [57], we will obtain the fermionic character formula of the standard module.

6.1. Principal subspace. On the principal subspace $W(k\Lambda_0)$, we use the weight gradation by $L^\beta(0)$ or equivalent to $d$. That is, we have

$$[L^\beta(0), x_\alpha^\beta(m)] = \left( -m - 1 + \frac{\langle \alpha, \alpha \rangle}{2} \right) x_\alpha^\beta(m) = -m x_\alpha^\beta(m).$$

We define the character of the principal subspace $W(k\Lambda_0)$ by

$$\text{ch } W(k\Lambda_0) = \sum_{m, r_1, \ldots, r_l \geq 0} \dim W(k\Lambda_0)(m, r_1, \ldots, r_l) y_1^{r_1} \cdots y_l^{r_l},$$

where $W(k\Lambda_0)(m, r_1, \ldots, r_l)$ is the weight subspace spanned by monomial vectors of the weight $k\Lambda_0 - m\delta + r_1 \alpha_1 + \cdots + r_l \alpha_l$ with respect to $\mathfrak{h}_0 \oplus \mathbb{C}d$.

For an arbitrary quasi-particle monomial of the form [57], we define the sequence $\mathcal{P}_i = (p_i^{(1)}, \ldots, p_i^{(k)})$ by $p_i^{(s)} = r_i^{(s)} - r_i^{(s+1)}$ for $i = 1, \ldots, l$, $s = 1, \ldots, k$ so that $p_i^{(s)}$ stand for the number of quasi-particles of
color $i$ and charge $s$ in the monomial $[22]$. Set $P = (P_t, \ldots, P_1)$. We then rewrite the condition (C2) on the energies in terms of $P$. For a fixed charge-type $\mathcal{R}' = (n_{r_1}, \ldots, n_{1,1})$, we have

$$
\sum_{p=1}^{r_{(1)}} \rho_i(2p-1)n_{p,i} = \rho_i \sum_{s,t=1}^{k} \min\{s,t\} p_i^{(s)} p_i^{(t)}.
$$

Then we also have

$$
\frac{1}{2} \sum_{p=1}^{r_{(1)}} \sum_{q=1}^{r_{(q)}} \min\{n_{p,t}, n_{q,t-1}\} = \frac{1}{2} \sum_{s,t=1}^{k} \min\{s,t\} p_i^{(s)} p_i^{(t)}.
$$

These expression are proved by induction on the level of the standard module (cf. [13]). We write

$$
P = \prod_{i=1}^{l} \prod_{s,t=1}^{k} (q^\frac{i}{2})_{p_i^{(s)}} p_i^{(t)}.
$$

where the sum goes over all finite sequences $P$ of $k$ nonnegative integers.

6.2. Parafermionic space. Using the corresponding result of the principal subspace, we calculate the character of the parafermionic space. For untwisted Lie algebras, we use the parafermionic grading operator defined by (3.35) in [23]. We should modify the grading operator $L^{\beta}(0)$ as well. But we do not find the coset Virasoro algebra construction [23] §3 for \nu-twisted $V_L$-module. Therefore we replace the grading operator $L^{\beta}(0)$ by $D$ defined by

$$
D = -d - D^{\nu} +, \quad D^{\nu} = \frac{\langle \mu(0), \mu(0) \rangle}{2k}
$$

for a weight $\mu$ (cf. [24]). For a simple root $\beta \in L$ and $m \in \frac{1}{2} \mathbb{Z}$, we have

$$
[D, x^\beta_\nu(m)] = \left(-m - \frac{\langle \beta(0), \beta(0) \rangle}{2k}\right) x^\beta_\nu(m).
$$

Then we also have

$$
[D, \psi^\beta_\nu(m)] = \left(-m - \frac{\langle \beta(0), \beta(0) \rangle}{2k}\right) \psi^\beta_\nu(m).
$$

The conformal energy of $\psi^\beta_\nu(m)$ is defined as the coefficient of the right hand side and denoted by

$$
\text{en} \psi^\beta_\nu(m) = -m - \frac{\langle \beta(0), \beta(0) \rangle}{2k}.
$$

Now we can compute the conformal energies of $\psi^\beta_{\nu_1,\beta}(m)$ and $\psi^\beta_{n_1,\beta_1,\ldots,\beta_1}(m_1, \ldots, m_1)$ in the same way as [24] Lemma 14].

Lemma 20. For a simple root $\beta$ and charge $n$, we have

$$
\text{en} \psi^\beta_{n\beta}(m) = -m - \frac{n^2 \langle \beta(0), \beta(0) \rangle}{2k}.
$$

Moreover, for simple roots $\beta_1, \ldots, \beta_1$ and charges $n_1, \ldots, n_1$, we have

$$
\text{en} \psi^\beta_{n_1,\beta_1,\ldots,\beta_1}(m_1, \ldots, m_1) = \sum_{i=1}^{t} \left(\text{en} \psi^\beta_{n_1,\beta_1}(m_1) - \sum_{p=1}^{i-1} \frac{n_1(\beta_1)(0), n_p(\beta_p)(0)}{k}\right).
$$
Using \(31\), we find that \([D, \rho(k\alpha)] = 0\) for \(\alpha \in Q\). Thus the grading operator \(D\) is well-defined on the parafermionic space \(L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[v]^+}\).

We define the character of the parafermionic space \(L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[v]^+}\) by
\[
\text{ch} \ L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[v]^+} = \sum_{m,r_1,\ldots,r_i \geq 0} \dim(L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[v]^+})_{(m,r_1,\ldots,r_i)} q^m y_1^{r_1} \cdots y_i^{r_i},
\]
where \((L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[v]^+})_{(m,r_1,\ldots,r_i)}\) is the weight subspace spanned by monomial vectors of conformal energy \(m\) and color-type \(C = (r_i, \ldots, r_i)\).

Consider the quasi-particle monomial
\[
x_{n_{r_1}}^{(1)} \alpha_1(m_{r_1}, l) \cdots x_{n_{1},r}^{(1)} \alpha_1(m_{r_1}, l) \cdots x_{n_{1},r}^{(1)} \alpha_1(m_{r_1}, l),\]
with charge-type \(\mathcal{C}' = (n_{r_1}, l, \ldots, n_{1}, 1)\), dual-charge-type \(\mathcal{R} = (r_i^{(1)}, \ldots, r_i^{(k-1)}, 0)\). Note that \(p_i^{(k)}\) (or equivalently \(r_i^{(k)}\)) is equal to zero for all \(i = 1, \ldots, l\) in \(64\). To emphasize the dependence of \(k\), we replace \(\mathcal{P}\) by \(\mathcal{P}^{(k-1)}\). We consider the parafermionic basis vector
\[
\psi_{n_{r_1}}^{(1)} \alpha_1, \ldots, n_{1}, \alpha_1(m_{r_1}, l, \ldots, m_{1})
\]
which correspond to the monomial \(64\). From lemma \(20\) the conformal energy of this current is given by
\[
-\sum_{i=1}^{l} \left( \sum_{s=1}^{r_i^{(1)}} m_{s,i} + \sum_{s=1}^{r_i^{(1)}} n_{s,i}^2 \rho \right) + \sum_{s=1}^{r_i^{(1)}} \frac{s-2n_{s,i}n_{s,i}\rho}{k} + \sum_{j=1}^{i-1} \sum_{l=1}^{r_i^{(1)}} \frac{\langle n_{s,i}(\alpha)(0), n_{t,j}(\alpha)(0) \rangle}{k}
\]
\[
= -\sum_{i=1}^{l} \sum_{s=1}^{r_i^{(1)}} m_{s,i} - \frac{1}{2} \sum_{i,j=1}^{l} \sum_{s=1}^{r_i^{(1)}} \sum_{t=1}^{r_j^{(1)}} \frac{\langle n_{s,i}(\alpha)(0), n_{t,j}(\alpha)(0) \rangle}{k}
\]
\[
= -\sum_{i=1}^{l} \sum_{s=1}^{r_i^{(1)}} m_{s,i} - \frac{1}{2} \sum_{i,j=1}^{l} \sum_{s=1}^{k-1} \sum_{t=1}^{k-1} \frac{s-t}{k} \langle \alpha(0), \alpha(0) \rangle p_i^{(s)} p_j^{(t)},
\]
where we use the fact that
\[
\sum_{s=1}^{r_i^{(1)}} n_{s,i} = \sum_{s=1}^{k-1} s p_i^{(s)}.
\]
Combining Theorem \(19\) and the contribution of the conformal shift, we obtain the character of the parafermionic space \(L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[v]^+}\).

**Theorem 21.** For affine Lie algebras \(A_{2l}\), we obtain
\[
\text{ch} \ L(k\Lambda_0)_{kQ}^{\hat{\mathfrak{h}}[v]^+} = \sum_{\mathcal{P}^{(k-1)}} \frac{q^{\sum_{i=1}^{l} \langle \alpha_i(0), (\alpha_i)(0) \rangle \sum_{s,t=1}^{k-1} D_s^{(k)} p_i^{(s)} p_j^{(t)}}}{\prod_{i=1}^{l} \prod_{s=1}^{k-1} (q^{\mathfrak{h}} p_i^{(s)})} \prod_{i=1}^{l} \prod_{s=1}^{k-1} (q^{\mathfrak{h}} p_i^{(s)})
\]
where the sum runs over all sequences \(\mathcal{P}^{(k-1)}\) of \(l(k-1)\) nonnegative integers and
\[
D_{s,t} = \min\{s, t\} - \frac{st}{k}.
\]

**6.3. Standard module.** Finally, we calculate the character of the standard module \(L(k\Lambda_0)\). The character of the standard module is defined in the same way as that of principal subspace. We see that the character formula of the standard module is given as follows.

**Theorem 22.** For affine Lie algebras \(A_{2l}\), we have
\[
\text{ch} \ L(k\Lambda_0) = \frac{1}{\prod_{i=1}^{l} (q^{\mathfrak{h}})^{\infty}} \sum_{\eta \in Q(0)} q^{\eta(\eta)/2k} \prod_{i=1}^{l} \prod_{p^{(k-1)}} \sum_{\mathfrak{h}} q^{\mathfrak{h} \sum_{i,j=1}^{l} \langle \alpha_i(0), (\alpha_i)(0) \rangle \sum_{s,t=1}^{k-1} D^{(k)} p_i^{(s)} p_j^{(t)}}}{\prod_{i=1}^{l} \prod_{s=1}^{k-1} (q^{\mathfrak{h}} p_i^{(s)})}
where \( \eta_i \in \mathbb{Z}(\alpha_i)_0 \) and the sum \( \sum_{p^{(k-1)}} \) runs over all sequences \( p^{(k-1)} \) of \( l(k-1) \) nonnegative integers satisfying

\[
\sum_{i=1}^{l} \sum_{s=1}^{k-1} s p_i^{(s)}(\alpha_i)_0 \in \eta + kQ_0.
\]

In fact, using Lepowsky-Wilson theorem \([57]\), we have the following relation for the character formula

\[
\text{ch } L(k\Lambda_0) = \frac{1}{\prod_{s=1}^{l} (q^\frac{1}{2})_\infty} \text{ch } L(k\Lambda_0) \hat{b}[\nu]_+.
\]

Since the basis of the vacuum space \( L(k\Lambda_0) \hat{b}[\nu]_+ \) is given in Theorem \([13]\) we are able to calculate \( \text{ch } L(k\Lambda_0) \hat{b}[\nu]_+ \) in the same way as \([24]\) Theorem 17.

**Acknowledgments**

The author would like to thank Masato Okado for helpful comments and discussion on this research. This work is supported by JST, the establishment of university fellowships towards the creation of science technology innovation, Grant Number JPMJFS2138.

**References**

[1] G. E. Andrews, The theory of partitions. Addison-Wesley 1976.
[2] M. Butorac, A note on principal subspaces of the affine Lie algebras in types \( D, E \) and \( F \), preprint [arXiv:1902.10794].
[3] M. Butorac and S. Kozić, Principal subspaces for the affine Lie algebras in type \( D, E \) and \( F \), preprint [arXiv:1902.10794].
[4] M. Butorac and S. Kozić, Parafermionic bases of standard modules for affine Lie algebras, Mathematical Zeitschrift (2020), published online, https://doi.org/10.1007/s00209-020-02639-w.
[5] M. Butorac and C. Sadowski, Combinatorial bases of principal subspaces of modules for twisted affine Lie algebras of type \( A^{(2)}_{2n-2} \), \( D^{(2)}_4 \), \( E_6^{(2)} \) and \( D^{(2)}_4 \), New York J. Math. 25 (2019), 79-106.
[6] C. Calinescu, J. Lepowsky, A. Milas, Vertex-algebraic structure of principal subspaces of basic \( A^{(2)}_{2n} \)-modules, I. Internat. J. Math. 25 (2014), 1450063.
[7] C. Calinescu, A. Milas, M. Penn, Vertex algebraic structure of principal subspaces of basic \( A^{(2)}_{2n} \)-modules, J. Pure Appl. Algebra 220 (2016), 1752-1784.
[8] C. Dong and J. Lepowsky, The algebraic structure of relative twisted vertex operators, J. Pure Appl. Algebra 110 (1996) 259-295.
[9] B. Feigin and A. Stoyanovsky, Quasi-particles models for the representations of Lie algebras and geometry of flag manifold, [arXiv:hep-th/9308079].
[10] B. Feigin and A. Stoyanovsky, Functional models for representations of current algebras and semi-infinite Schubert cells (Russian), Funktsional Anal i Prilozhen, 28 (1994), 68-90; translation in: Funct. Anal. Appl., 28 (1994), 55-72.
[11] I. Frenkel, J. Lepowsky and A. Meurman, Vertex operator calculus, in Mathematical Aspects of String Theory, ed. S.-T. Yau (World Scientific, Singapore, 1987), pp. 150-188.
[12] I. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Mathematics, Vol. 134 (Academic Press, 1988).
[13] G. Georgiev, Combinatorial construction of modules for infinite dimensional Lie algebras, I. Principal subspace, J. Pure Appl. Algebra 112 (1996), 247-286.
[14] G. Georgiev, Combinatorial construction of modules for infinite-dimensional Lie algebras, II, Parafermionic space, [arXiv:algq-9504024].
[15] G. Hatayama, A. Kuniba, M. Okado, T. Takagi and T. Tsujii, Path, Crystals and Fermionic Formulae, MathPhys Odyssey 2001, 205-272, Prog. Math. Phys. 23, Birkhäuser Boston, MA, 2002.
[16] V.G. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge University Press, Cambridge, 1990.
[17] A. Kuniba, T. Nakanishi and J. Suzuki, Characters in conformal field theories from thermodynamic Bethe Ansatz, Modern Phys. Lett. A8 (1993), 1649-1659.
[18] J. Lepowsky, Calculus of twisted vertex operators, Proc. Nat. Acad. Sci. USA, 82 (1985), 8295-8299.
[19] J. Lepowsky and M. Primc, Structure of the standard modules for the affine Lie algebra \( \hat{A}^{(1)}_1 \), Contemporary Math., 46, Amer. Math. Soc, Providence, RI, 1985.
[20] J. Lepowsky and R.L. Wilson, Construction of the affine Lie algebra \( \hat{A}^{(1)}_1 \), Comm. Math. Phys. 62 (1978), 43-53.
[21] J. Lepowsky and R.L. Wilson, The structure of standard modules, I: Universal algebras and the Rogers-Ramanujan identities, Invent. Math. 77 (1984), 199-290.
[22] H.-S. Li, The physics superselection principle in vertex operator algebra theory, J. Algebra 196 (1997) 436-457.
[23] H.-S. Li, On abelian coset generalized vertex algebras, Commum, Contemp. Math. 03, No. 02, (2001), 287-340.
[24] M. Okado and R. Takenaka, Parafermionic bases of standard modules for twisted affine Lie algebras of type \( A^{(2)}_{2l-1} \), \( D^{(2)}_{l+1} \), \( E^{(2)}_6 \) and \( D^{(2)}_{4} \), [arXiv:2109.08892].
[25] M. Primc, Vertex operator construction of standard modules for \( \hat{A}^{(1)}_1 \), Pacific J. Math., 162 (1994), 143-187.
