Distributed Derivative-free Learning Method for Stochastic Optimization over a Network with Sparse Activity

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Abstract

This paper addresses a distributed optimization problem in a communication network where nodes are active sporadically. Each active node applies some learning method to control its action to maximize the global utility function, which is defined as the sum of the local utility functions of active nodes. We deal with stochastic optimization problem with the setting that utility functions are disturbed by some non-additive stochastic process. We consider a more challenging situation where the learning method has to be performed only based on a scalar approximation of the utility function, rather than its closed-form expression, so that the typical gradient descent method cannot be applied. This setting is quite realistic when the network is affected by some stochastic and time-varying process, and that each node cannot have the full knowledge of the network states. We propose a distributed optimization algorithm and prove its almost surely convergence to the optimum. Convergence rate is also derived with an additional assumption that the objective function is strongly concave. Numerical results are also presented to justify our claim.

Index Terms

Stochastic optimization, derivative-free learning, sparse network, convergence analysis, distributed algorithm

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I. INTRODUCTION

We consider the distributed optimization of a network with sparse communication, i.e., nodes are active occasionally in a discrete-time system, so that only a small number of nodes are active at the same time-slot. For example, in the modern communication system, independent mobile phone users communicate with the base station at different time. This model is also important in Internet of Things such as underwater wireless sensor networks, where sensor nodes keep asleep frequently to save battery.

Suppose that the performance of the network is characterized by a global utility function, which is defined as the sum of the local utility functions of all active nodes at one time-slot. Each active node aims to properly control its own action to maximize the global utility. The local utility of any active node is a function of the action of all the active nodes, as well as some stochastic environment state that can be seen as a non-additive stochastic process, e.g., stochastic and time-varying channel gain in wireless communication system. Such stochastic optimization problem is important for the improvement of network performance and has attracted much attention in various field, e.g., radio resource management [2], power control [3], [4], and beamforming allocation [5].

The convex optimization problem is well investigated by applying the typical gradient descent/ascent method [6], under the condition that each node is able to calculate the partial derivative related to its action. Sub-gradient based methods have been proposed to solve distributed optimization of the sum of several convex function, over time-varying [7], [8], [9] or asynchronous [10], [11] networks. In these previous work, each node/agent requires the gradient information of its local function to perform the optimization algorithm.

Stochastic learning schemes based on stochastic gradient descent have been widely studied. In this work we consider a more challenging framework that nodes are unaware of any gradient information. Since the network is distributed by some non-additive stochastic process, our setting is quite practical in the following situations:

- the system is so complex that the closed-form expression of any utility function is unavailable;
- the computation of gradient requires much informational exchange and introduces a huge burden to the network.

A detailed motivating example is presented in Section III to highlight the interest of our setting.
We assume that each active node only has a numerical observation of its local utility, our optimization algorithm should be performed only based on this zeroth-order information. Moreover, we consider a distributed setting such that nodes can only exchange the local utilities with their neighbors in order to estimate the global utility, which make the problem more challenging.

A. Related work

Our derivative-free optimization problem is known as zero-order stochastic optimization and bandit optimization. There are numbers of work based on two-point gradient estimator, e.g., [12], [13], [14], [15], under the assumption that two values of the objective function \( f(a_k^{(1)}; s_k) \) and \( f(a_k^{(2)}; s_k) \) are available under the same stochastic parameter \( s_k \). However such assumption is unrealistic in our setting, e.g., in i.i.d. channel, the value of \( s_k \) change fast, it is impossible to observe two network utilities using different action \( a_k \) while under the same environment state. Therefore, we should propose some gradient estimator only based on a single realization of objection function to estimate the gradient. A classical method was proposed in [16] of which the algorithm is near-optimal: for general convex and Lipschitz objective functions, the resulted optimization error is \( O(K^{-0.25}) \) after a total number of \( K \) iterations. From then on, several advanced methods were proposed (e.g., [17], [18], [19]) to accelerate the convergence speed of the algorithm for the general convex functions or the convex functions with additional assumptions, e.g., smooth or strongly convex. However, the optimal algorithm to address bandit optimization is still unknown. It is worth mentioning that, the optimization error cannot be better than \( O(K^{-0.5}) \) after \( K \) iterations, according to the lower bounds of the convergence rate derived in [20], [17], [13].

Although bandit optimization has attracted much attention in recent years, the existed algorithms are usually centralized and hard be decentralized. In fact, in all the above mentioned references, their algorithms contain the operations of vectors and matrices that require a control center to handle. In our setting, each node only controls its local variable (a coordinate) and may not have the full knowledge of the objective function due to the distributed setting. For example, in the algorithm proposed in [16], the core is to generate a random unit perturbation vector at each iteration, which is the key to ensure that the expectation of the resulted gradient estimator is equal to the gradient of a smoothed version of the objective function by applying Stokes Theorem. This requires a control center as the resulted perturbation vector cannot have a unit norm without such a control center. In our distributed network, each node can only generate
its own random perturbation independently. Different tools are needed to obtain our analytically results: we derive upper bounds for the bias of gradient estimator. Moreover, the existing work in learning community usually focused on the performance after a given number of iterations. However, finite-time horizon is not adapted to wireless networks, as it is usually hard to predict the duration of connection and the total number of iterations. For the above reason, in this work, we aim to propose some optimal solution with asymptotic performance guarantee.

In our recent work [21], we have proposed a learning algorithm named DOSP (distributed optimization algorithm using stochastic perturbation) to solve the above derivative-free optimization problem, however, in a synchronized network with small number of nodes, i.e., nodes are always active and update their action at each time-slot. The basic idea of the DOSP algorithm is to estimate the gradient of the objective function only based the numerical measurement of the objective function. It has been shown that the estimation bias of gradient is vanishing as the number of nodes is finite. The convergence of the DOSP can be proved with the tools of stochastic approximation [22]. This technique is closely related to simultaneous perturbation gradient approximation in [23], [24] and extremum seeking with stochastic perturbation proposed in [25]. Please refer to [21] for the detailed discussion. It is worth mentioning that, sine perturbation based extremum seeking method [26], [27], [28] can be another option to solve the derivative-free optimization problem. However, it is impractical to ensure that the sine function used by each node is orthogonal in a distributed setting, especially when the number of nodes is large.

B. Our contribution

This paper extends our previous results in [21] by considering a more realistic network model, i.e., nodes are sporadically active and the entire network may be of large scale. The achievable value of action of each node is considered as constrained, i.e., belonging to some closed-interval. We present a modified DOSP algorithm with two major differences compared with the original DOSP algorithm: nodes can update their action only when they are active; each node updates its step-size asynchronously, independently, and randomly, according to its times of being active.

This paper focuses on the convergence analysis of the proposed learning algorithm. Convergence rate has also be investigated with an additional assumption that the utility functions are strongly concave. Compared with that in [21], the analysis is much more challenging because of the additional random terms. The network is dynamic as nodes have random activity, its global utility function is harder to be characterized than a fixed network that nodes are always active.
As we try to estimate the gradient using the numerical value of utility function, an essential term to be analyzed is the estimation bias of gradient. In [21], an upper bound of such bias term is proved to be proportional to the vanishing step-size, which is deterministic and identical for all nodes at each iteration. Due to the random activity of each node, the algorithm is performed in an asynchronous manner, i.e., the times of update of each node is random. As a consequence, the step-size of each node (function of times of update) is random and independent, which makes the problem further challenging. We have to resort to some new tools such as concentration inequalities to show that the bias term is vanishing as well. It is notable that our proposed solution can achieve the optimal convergence rate when the objective function is smooth and strongly concave: our achievable optimization error is proved to be $O(K^{-0.5})$, which is the same compared with the lower bounds in [17], [13] in terms of the decreasing order.

The rest of the paper is organized as follows. Section II describes the problem as well as some basic assumptions. Section III provides examples to motivate the interest of the problem. Section IV presents our distributed optimization algorithm, of which the almost sure convergence discussed in Section V. The convergence rate of the proposed learning algorithm is derived in Section VI. Section VII presents some numerical illustrations and Section VIII concludes this paper. Main notations in this paper are listed in Table 1.

II. SYSTEM MODEL AND ASSUMPTIONS

A. Network model

Consider a network $\mathcal{N}$ with $N = |\mathcal{N}|$ nodes and a time-varying directed graph $\mathcal{G}^{(k)} = (\mathcal{N}, \mathcal{E}^{(k)})$ at each discrete time-slot $k$. Note that the edge set $\mathcal{E}^{(k)}$ is a set of pairs of nodes that are able to have direct communication. We can use a communication matrix $\mathbf{E}(k) = [E_{i,j}(k)]_{i,j \in \mathcal{N}}$ to describe the network connectivity, with $E_{i,j}(k) \neq 0$ if and only if $(i, j) \in \mathcal{E}^{(k)}$. In this work, the network topology is assumed to be stochastic, such that any two different nodes can become neighbors with a non-zero constant probability, i.e., $\mathbb{P}(E_{i,j}(k)) > 0$, $\forall i, j \in \mathcal{N}$. It is worth mentioning that such assumption can be naturally satisfied when nodes are moving freely in some closed area.

Suppose that at each discrete time-slot $k$, only a random subset $\mathcal{N}^{(k)} \subseteq \mathcal{N}$ of nodes are active, i.e., perform some action. Introduce a binary variable $\delta_{i,k}$ to indicate whether node $i$ is active or
not at time-slot $k$, i.e.,
\[
\delta_{i,k} = \begin{cases} 
1, & \text{if } i \in \mathcal{N}^{(k)}, \\
0, & \text{else}.
\end{cases}
\]

Define $a_{i,k}$ as the value of the action of node $i$ at time-slot $k$ under the condition that $\delta_{i,k} = 1$. Suppose that the value of $a_{i,k}$ is bounded, i.e., $a_{i,k} \in \mathcal{A}_i = [a_{i,\text{min}}, a_{i,\text{max}}]$. Denote $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$ as the feasible set of the action vector $\mathbf{a}_k = [a_{1,k}, \ldots, a_{N,k}]^T$. Denote
\[
\sigma_a^2 = \max_{i \in \mathcal{N}} \{ a_{i,\text{min}}^2, a_{i,\text{max}}^2 \}.
\]

Introduce $n_k = |\mathcal{N}^{(k)}| = \sum_{i=1}^{N} \delta_{i,k}$ the number of active nodes at time-slot $k$. Mathematically, we assume that:
Assumption 1. The binary variables $\delta_{i,k}$ are i.i.d. with $P(\delta_{i,k} = 1) = q_a$. Then $n_k$ follows a binomial distribution with $E(n_k) = Nq_a = \lambda$. In the situation where $N \to \infty$, the value of $q_a$ is small such that $\lambda \ll \infty$, $n_k$ follows a Poisson distribution with parameter $\lambda$.

Note that we can have a large network with $N \to \infty$, while our results hold for any value of $N$ as long as $\lambda = Nq_a < \infty$.

B. Utility functions

We assume that each active node $i$ with $\delta_{i,k} = 1$ is able to evaluate a pre-defined local utility function $u_i(a_k, \delta_k, S_k)$, which depends on the action vector $a_k$, the activity vector $\delta_k = [\delta_{1,k}, \ldots, \delta_{N,k}]^T$, and is also disturbed by a non-additive stochastic process $S_k$ of the whole network, e.g., stochastic channels in wireless networks. Consider $S_k \in S$ as a stochastic matrix to describe the environment state of the network at any time-slot $k$, which is assumed to be independent and identically distributed (i.i.d.) in this paper. The local utilities of the non-active nodes are not meaningful, thus we define $u_i = 0$ if $\delta_{i,k} = 0$.

The global utility $f(a_k, \delta_k, S_k)$ of the entire network is defined as the sum of local utilities of the active nodes at each time-slot $k$, i.e.,

$$f(a_k, \delta_k, S_k) = \sum_{i \in \mathcal{N}(k)} u_i(a_k, \delta_k, S_k). \quad (2)$$

We are interested in the configuration of the value of $a_{i,k}$ for each node $i$ such that $i \in \mathcal{N}(k)$ at each time-slot $k$, in order to the maximize of the average global utility function

$$F(a_k) = \mathbb{E}_{\delta, S}(f(a_k, \delta_k, S_k)). \quad (3)$$

It is also necessary to define the average global utility function with a given realization of the activity vector $\delta_k$, i.e.,

$$G(a_k, \delta_k) = \mathbb{E}_S(f(a_k, \delta_k, S)). \quad (4)$$

According to Assumption 1 it is easy to deduce that

$$F(a_k) = \sum_{\delta_k \in \mathcal{D}} q_a^{n_k} (1 - q_a)^{N-n_k} G(a_k, \delta_k) \quad (5)$$

with $\mathcal{D} = \{\delta = [\delta_{1,k}, \ldots, \delta_{N,k}]^T : \delta_i \in \{0, 1\}, \forall i\}$.

Assume that at time-slot $k$ each active node $i \in \mathcal{N}(k)$ is able to have a numerical observation $\tilde{u}_{i,k}$ of $u_i(a_k, \delta_k, S_k)$:

$$\tilde{u}_{i,k} = u_i(a_k, \delta_k, S_k) + \eta_{i,k}, \quad (6)$$
where $\eta_{i,k}$ is the additive random noise caused by observation of $u_i$. Such noise is assumed to be statistically independent and have zero mean and bound variance.

**Assumption 2.** For any integer $k$ and $i \in \mathcal{N}(k)$, we have $\mathbb{E}(\eta_{i,k}) = 0$ and $\mathbb{E}(\eta^2_{i,k}) = \sigma^2_\eta < \infty$. Besides, for any $i \neq j$ and $k \neq k'$, we have $\mathbb{E}(\eta_{i,k}\eta_{j,k'}) = \mathbb{E}(\eta_{i,k}\eta_{i,k'}) = 0$.

In order to approximate the global utility of the network, active nodes have to broadcast their observation of local utilities to their active neighbors (other active nodes within transmission range). Without any communication, an active node only knows its local utility. We consider a realistic situation where an active node $i$ can receive $\tilde{u}_{j,k}$ from another active node $j$ only if both of the following events occur: $E1$. node $j$ is a neighbor of node $i$ at time-slot $k$; $E2$. there is no collision or packet loss during the transmission. In other words, node $i$ receives $\tilde{u}_{j,k}$ from a subset $\mathcal{I}^{(i,k)}$ of its active neighbors, with $\mathcal{I}^{(i,k)} \subseteq \mathcal{N}(k) \setminus \{i\}$. Mathematically:

**Assumption 3.** At any time-slot $k$, any active node $i \in \mathcal{N}(k)$ knows the utility $\tilde{u}_{j,k}$ of another active node $j \in \mathcal{N}(k)$ with a constant probability $q_\tau \in (0, 1]$, i.e.,

$$\mathbb{P}(j \in \mathcal{I}^{(i,k)}) = q_\tau, \quad \mathbb{P}(j \notin \mathcal{I}^{(i,k)}) = 1 - q_\tau, \quad \forall j \neq i.$$  

(7)

Note that $q_\tau$ is in fact a joint probability of events $E1$ and $E2$.

In Section IV-A, we will present an efficient way to estimate the global utility $\tilde{f}$ using incomplete information of $\tilde{u}_{i,k}$.

**Remark 1.** Note that it is straightforward to extend the results in this work to a more general case where $\mathbb{P}(j \in \mathcal{I}^{(i,k)})$ is not identical. We assume that $\mathbb{P}(j \in \mathcal{I}^{(i,k)}) = q_\tau$ mainly to lighten the expressions of this paper.

It is worth mentioning that our aforementioned network model can hold in wireless settings. In fact, the wireless link between any two nodes in such a network is affected by fast fading, modeled usually by Rayleigh or Nakagami distribution. This implies that the link changes from one slot to another in an i.i.d. way. If the link is good, then the nodes can communicate and if the link is bad they cannot communicate. As a result, the link qualities in such a time-varying network are reshuffled at each slot.
C. Problem formulation

With the above definition, our problem can be written as

\[
\begin{align*}
\max_a & \quad F(a) = \mathbb{E}_\delta(G(a, \delta)) = \mathbb{E}_{\delta, S}(f(a, \delta, S)) \\
\text{s.t.} & \quad a \in A
\end{align*}
\]  

(8)

We consider a situation where nodes do not have the knowledge of \(S\) to get the closed-form expression of the utility functions. This setting is quite realistic as \(S\) may have large dimension and be constantly time-varying. In this paper, the proposed learning algorithm is performed only with the numerical value of utility function. An motivating example is introduced in Section III.

Denote \(a^* = [a^*_1, \ldots, a^*_N]\) as the optimum solution of the problem. To ensure the existence of \(a^*\), we assume that:

**Assumption 4.** Both \(G(a, \delta)\) and \(F(a)\) are first order and second order differentialable functions of \(a \in A\). The optimal point \(a^*\) exists such that \(\partial F(a^*) / \partial a_i = 0\) and \(\partial^2 F(a^*) / \partial a_i^2 < 0\), \(\forall i \in N\). Besides, \(a^*\) is not on the boundary of \(A\), i.e., \(a^*_i \in (a_{1,\min}, a_{1,\max}), \forall i \in N\). The objective function \(F\) is strictly concave such that

\[(a - a')^T \cdot (\nabla F(a) - \nabla F(a')) < 0, \forall a, a' \in A : a \neq a'.\]

(9)

**Remark 2.** It is worth mentioning that we assumed \((a_{i,k} - a^*_i)^T \frac{\partial}{\partial a_{i,k}} F(a_k) \leq 0, \forall i \in N\) in [1], which has been relaxed by Assumption 4 in this paper.

We have a further assumption to ensure the performance of the proposed derivative-free learning algorithm.

**Assumption 5.** There exists \(\alpha_G \in (0, +\infty)\) such that

\[ \left| \frac{\partial^2}{\partial a_i \partial a_j} G(a, \delta) \right| \leq \alpha_G, \quad \forall i, j \in N^{(k)} \]

(10)

The function \(a \mapsto u_i(a, \delta, S)\) is Lipschitz for any \(\delta\) and \(S\),

\[ \| u_i(a, \delta, S) - u_i(a', \delta, S) \| \leq L_S \| (a - a') \circ \delta \|, \]

(11)

with constant \(L_S < \infty\). Besides, \(L = \sqrt{\mathbb{E}_S(L^2_S)} < \infty\). 

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III. Motivating Example

Recently, derivative-free optimization is of interest in various applications, e.g., management of fog computing in IoT [29], sensor selection for parameter estimation [30], and adversarial machine learning [31]. In this section we provide another motivating example, which of particular interest for the problem considered in this paper.

Consider a power allocation problem in a network with \( N \) transmitter-receiver links. As shown in Figure 1, each link corresponds to a node in our system model. Transmitter \( i \) sends some packet to receiver \( i \) when \( \delta_{i,k} = 1 \). Let \( S_k = [s_{ij,k}]_{i,j \in \mathcal{N}} \) denote the time-varying stochastic channel matrix, each element \( s_{ij,k} \in \mathbb{R}^+ \) represents the channel gain between transmitter \( i \) and receiver \( j \) at time \( k \). Each active transmitter \( i \) sets its transmission power \( p_{i,k} \), the Shannon achievable rate of the link is then given by [32]

\[
r_{i,k} = \log \left( 1 + \frac{s_{ii,k}p_{i,k}}{\sigma^2 + \sum_{j \neq i} \delta_{j,k}s_{ji,k}p_{i,k}} \right). \tag{12}
\]

At each time-slot \( k \), define the global utility, which is widely used in wireless systems, as

\[
y_k(p_k, \delta_k, S_k) = \sum_{i \in \mathcal{N}(k)} (\omega_1 r_{i,k} - \omega_2 p_{i,k}), \tag{14}
\]

where \( \omega_1, \omega_2 \in \mathbb{R}^+ \) are constants and \( \omega_2 p_{i,k} \) denotes the energy costs of the packet transmission.

However, \( y_k \) is not concave of \( p_{i,k}, \forall i \in \mathcal{N}(k) \). For this reason, we have to consider the approximation of \( r_{i,k} \) and some variable change to make the objective function concave, which is a well known problem in the sum rate maximization problem in wireless network [32]. It is common to use change of variable \( p_{i,k} = e^{a_{i,k}} \) and consider the approximation \( y_k \approx f_k = \sum_{i \in \mathcal{N}(k)} u_i(a_k, \delta_k, S_k) \) with [32]

\[
u_i = \omega_1 \log \left( \frac{s_{ii,k}e^{a_{i,k}}}{\sigma^2 + \sum_{j \neq i} \delta_{j,k}s_{ji,k}e^{a_{j,k}}} \right) - \omega_2 e^{a_{i,k}}. \tag{13}
\]

It is straightforward to show that \( \partial^2 f_k/\partial a_{i,k}^2 < 0, \forall i \in \mathcal{N}(k) \), thus the condition (9) in Assumption 4 is satisfied.

In order to perform classical gradient methods, each transmitter should be able to evaluate

\[
\frac{\partial f}{\partial a_{i,k}} = \omega_1 - \sum_{n \in \mathcal{N}(k)} \frac{\omega_1 s_{in,k}e^{a_{i,k}}}{\sigma^2 + \sum_{j \neq n} \delta_{j,k}s_{jn,k}e^{a_{j,k}}} - \omega_2 e^{a_{i,k}}, \tag{14}
\]

of which the calculation requires much information, such as the cross-channel gain \( s_{in,k} \forall n \in \mathcal{N}(k) \setminus \{i\} \), as well as all the interference estimated by each active receiver. All the channel information has to be estimated and exchanged by each active node, which is a huge burden for
the network. Therefore, we desire to propose a distributed optimization algorithm only with the numerical observation of utilities. The framework is distributed such that each node can only know the local utilities of its neighbors and of itself, see Figure 1 for more details.

![Diagram](image)

Figure 1. (I) At time \( k \), link 2 is inactive, each active transmitter \( i \) communicates with its receiver with transmission power \( e^{a_{i,k}} \) and introduces interference to the other links; (II) Each active receiver \( i \) broadcast \( \tilde{u}_{i,k} \) to its neighbors: the green links mean that there is a successful transmission of \( \tilde{u}_{i,k} \); the red links represent the transmission failure caused by collision or packet loss; the gray links mean that link 1 and link \( N \) are not neighbors so exchange of \( \tilde{u}_{i,k} \) between these nodes.

IV. DOSP LEARNING ALGORITHM WITH SPORADIC UPDATES

In this section, we describe our DOSP-S learning algorithm, namely, distributed optimization algorithm using stochastic perturbation with sporadic updates. We start with the approximation of the value of global utility based on the collected local utilities by each active node in Section IV-A, before the presentation of DOSP-S in Section IV-B.

A. Estimation of global utility \( \tilde{f} \)

Recall that the global utility is 0 if no nodes are active, hence we focus on the opposite situation. For any time-slot \( k \) such that \( n_k \geq 1 \), we consider an arbitrary active node \( i \) as reference and denote \( \tilde{f}_{i,k} \) as the numerical value of global utility approximated by node \( i \). If node \( i \) knows the constant probability \( q_r \) of successfully receiving \( \tilde{u}_{j,k} \) from another node \( j \), we can have an unbiased estimation of \( f \) according to the following proposition.
Proposition 1. Suppose that Assumption 3 holds and \( q_r \) is known by all nodes, then each active node \( i \) can estimate

\[
\tilde{f}_{i,k}(a_k, \delta_k, S_k, I^{(i,k)}) = \tilde{u}_{i,k} + \frac{1}{q_r} \sum_{j \in I^{(i,k)}} \tilde{u}_{j,k}
\]

of which the expected value over all possible sets \( I^{(i,k)} \) and the additive noise \( \eta_k \) equals to the global utility function, i.e.,

\[
\mathbb{E}_{I, \eta_k}(\tilde{f}_{i,k}(a_k, \delta_k, S_k, I^{(i,k)})) = f(a_k, \delta_k, S_k).
\]

Proof: Introduce \( \kappa_{i,j,k} \in \{0, 1\} \) with \( \kappa_{i,j,k} = 1 \) if \( j \in I^{(i,k)} \), otherwise \( \kappa_{i,j,k} = 0 \). Then (15) can be re-written as

\[
\tilde{f}_{i,k} = \tilde{u}_{i,k} + \frac{1}{q_r} \sum_{j \in N^k \setminus \{i\}} \kappa_{i,j,k} \tilde{u}_{j,k}
\]

By Assumption 3, we have \( \mathbb{E}(\kappa_{i,j,k}) = \mathbb{P}(\kappa_{i,j,k} = 1) = q_r \). Based on (17), we evaluate

\[
\mathbb{E}_{I}(\tilde{f}_{i,k}) = \tilde{u}_{i,k} + \frac{1}{q_r} \mathbb{E}_{I}(\sum_{j \in N^k \setminus \{i\}} \kappa_{i,j,k} \tilde{u}_{j,k})
\]

\[
= \tilde{u}_{i,k} + \frac{1}{q_r} \sum_{j \in N^k \setminus \{i\}} \mathbb{E}(\kappa_{i,j,k}) \tilde{u}_{j,k} = \sum_{j \in N^k} \tilde{u}_{j,k}.
\]

Since \( \mathbb{E}_{\eta}(\tilde{u}_{j,k}) = u_j(a_k, \delta_k, S_k) + \mathbb{E}_{\eta}(\eta_{j,k}) = u_j(a_k, \delta_k, S_k) \) by Assumption 2, we can easily get \( \mathbb{E}_{I, \eta_k}(\tilde{f}_{i,k}) = f(a_k, \delta_k, S_k) \), which concludes the proof.

Remark 3. In a more general case where the probability of receiving \( \tilde{u}_{j,k} \) from different nodes is not the same, we can have an similar estimator of \( f \) with trivial extension.

Remark 4. In our work, we use only the current information of local utilities to estimate \( f \) without considering any previous utility values. Due to the stochastic environment considered in this work, there could be a significant difference between \( f(a, S_{k-1}) \) and \( f(a, S_k) \) as \( S_{k-1} \) and \( S_k \) are independent. Hence we cannot use the previous values of utilities and apply the compensating scheme as in [33].

B. Learning Algorithm

This section presents our learning algorithm DOSP-S, which is a modified version of the DOSP algorithm in [21]. We first introduce some important parameters to be used in our algorithm, as presented in the following assumption.
Assumption 6. (I). \( \{ \beta_\ell \}_{\ell \geq 0} \) and \( \{ \gamma_\ell \}_{\ell \geq 0} \) are positive vanishing sequences, i.e., \( \beta_\ell = \beta_0 \ell^{-c_1} \) and \( \gamma_\ell = \gamma_0 \ell^{-c_2} \), with \( \beta_0 > 0 \), \( \gamma_0 > 0 \), \( c_1 \in (0.5, 1) \), and \( c_2 \in (0, 1 - c_1) \), such that
\[
\sum_{\ell=1}^{\infty} \beta_\ell \gamma_\ell = \infty \quad \text{and} \quad \sum_{\ell=1}^{\infty} \beta_\ell^2 < \infty;
\] (19)

(II). \( \{ \Phi_{i,k} \}_{i \in \mathcal{N}, k \geq 1} \) are i.i.d. zero-mean random variables, there exist \( \sigma_\Phi > 0 \) and \( \alpha_\Phi > 0 \) such that \( \mathbb{E}(\Phi_{i,k}^2) = \sigma_\Phi^2 \) and \( |\Phi_{i,k}| \leq \alpha_\Phi \). (III). There exists \( K_0 < \infty \) such that
\[
\alpha_\Phi \gamma_\ell \leq \max_{i \in \mathcal{N}} \{|a_{i,\max} - a_i^*|, |a_{i,\min} - a_i^*|\}, \quad \forall \ell \geq K_0.
\] (20)

Since we have \( a_i^* \in (a_{i,\min}, a_{i,\max}) \), \( \forall i \in \mathcal{N} \) in Assumption 4, such \( K_0 < \infty \) always exists to ensure (20).

Denote \( \tilde{a}_{i,k} \) as an intermediate variable. For any active node \( i \) at time-slot \( k \), the learning algorithm is given by
\[
\tilde{a}_{i,k+1} = a_{i,k} + \tilde{\beta}_{i,k} \Phi_{i,k} \tilde{f}_{i,k} (a_k + \tilde{\gamma}_k \circ \Phi_k, \delta_k, S_k),
\] (21)
\[
a_{i,k+1} = \text{Proj}_{i,k+1} (\tilde{a}_{i,k+1}),
\] (22)
in which we use the equivalent step-sizes
\[
\tilde{\beta}_{i,k} = \delta_{i,k} \beta_{i,k}, \quad \tilde{\gamma}_{i,k} = \delta_{i,k} \gamma_{i,k},
\] (23)
where \( \ell_{i,k} \) denotes the index of the step-sizes \( \gamma \) and \( \beta \) to be applied by node \( i \) at iteration \( k \) during the algorithm. In this paper, \( \ell_{i,k} \) is supposed to be generated independently and randomly by each node with
\[
\ell_{i,k} = \tilde{\ell}_{i,k} + \delta_{i,k} : \quad \tilde{\ell}_{i,k} \sim \mathcal{B}(k - 1, q_a).
\] (24)

Notice that \( \mathcal{B} \) represents Binomial distribution. We denote \( \tilde{\beta}_k = [\tilde{\beta}_{1,k}, \ldots, \tilde{\beta}_{N,k}]^T \), \( \tilde{\gamma}_k = [\tilde{\gamma}_{1,k}, \ldots, \tilde{\gamma}_{N,k}]^T \) and \( \circ \) represents the element-wise production of two vectors.

We have to apply the projection of \( \tilde{a}_{i,k} \) as in (22), to ensure that the actually performed action \( a_{i,k} + \tilde{\gamma}_{i,k} \Phi_{i,k} \) belongs to the feasible set \( A_i \). The operator \( \text{Proj}_{i,k} \) is defined as
\[
\text{Proj}_{i,k} (\tilde{a}_{i,k}) = \min \{ \max \{ \tilde{a}_{i,k}, a_{i,\min} + \alpha_\Phi \tilde{\gamma}_{i,k} \}, a_{i,\max} - \alpha_\Phi \tilde{\gamma}_{i,k} \}.
\] (25)

Recall that \( |\Phi_{i,k}| \leq \alpha_\Phi \) in Assumption 6 we have then
\[
a_{i,k} + \tilde{\gamma}_{i,k} \Phi_{i,k} \in [a_{i,k} - \alpha_\Phi \tilde{\gamma}_{i,k}, a_{i,k} + \alpha_\Phi \tilde{\gamma}_{i,k}] \subseteq A_i,
\] (26)
Algorithm 1 DOSP-S for each node $i$

1) Initialize $k = 1$, set step-sizes $\tilde{\beta}_{i,k} = \delta_i \beta_{i,k}$ and $\tilde{\gamma}_{i,k} = \delta_i \gamma_{i,k}$, set the value $a_{i,1}$ randomly from the interval $[a_{i,\text{min}} + \tilde{\gamma}_{i,k} \alpha, a_{i,\text{max}} + \tilde{\gamma}_{i,k} \alpha]$. 

2) If $\delta_{i,k} = 1$
   
   a) Generate a random variable $\Phi_{i,k}$, perform action with value $\tilde{a}_{i,k} = a_{i,k} + \tilde{\gamma}_{i,k} \Phi_{i,k}$; 
   
   b) Estimate $\tilde{u}_{i,k}$, broadcast this value to its active neighbors, and receive $\tilde{u}_{j,k}$ from active neighbors $j \in \mathcal{I}^{(i,k)}$. Calculate $\tilde{f}_{i,k}$ according to (15), i.e., $\tilde{f}_{i,k} = \tilde{u}_{i,k} + q_f^{-1} \sum_{j \in \mathcal{I}^{(i,k)}} \tilde{u}_{j,k}$; 
   
   c) Update $\tilde{a}_{i,k+1}$ using (21), i.e., $\tilde{a}_{i,k+1} = a_{i,k} + \tilde{\beta}_{i,k} \Phi_{i,k} \tilde{f}_{i,k}$. 

3) If $\delta_{i,k} = 0$, then $\tilde{a}_{i,k+1} = a_{i,k}$. 

4) Generate $\tilde{\ell}_{i,k+1} \sim \mathcal{B}(k, q_a)$, set $\tilde{\beta}_{i,k+1} = \delta_{i,k+1} \beta_{i,k+1} + \tilde{\ell}_{i,k} + 1$ and $\tilde{\gamma}_{i,k+1} = \delta_{i,k+1} \gamma_{i,k+1} + \tilde{\ell}_{i,k+1}$. 

5) Update $a_{i,k+1}$ using (22), i.e., $a_{i,k+1} = \text{Proj}_{i,k+1}(\tilde{a}_{i,k+1})$. 

6) $k = k + 1$, go to 2.

which means that the actually performed action always belongs to the feasible set.

The proposed learning algorithm is concluded in Algorithm 1. The main difference between the DOSP-S algorithm and the DOSP algorithm in [21] comes from the network model. Since not all nodes are active at the same time, the step-sizes $\tilde{\beta}_{i,k}$ and $\tilde{\gamma}_{i,k}$ are not updated simultaneously, the analysis becomes more challenging as we will discuss in Section V.

V. ALMOST SURE CONVERGENCE

We investigate the convergence of Algorithm 1 in this section. We mainly need to investigate the divergence

$$d_k = \frac{1}{N} \| a_k - a^* \|^2,$$

which represents the distance between the actual $a_k$ and the optimal point $a^*$. Our aim is to prove that $d_k \to 0$ a.s. Compared with the original DOSP algorithm, the main challenge of the analysis comes from the additional randomness of the network topology, which makes the objective function completely different and more complicated to be characterized. Moreover, the fact that each nodes uses independent and random step-sizes also makes the analysis challenging.

A fundamental step is to learn the relation between $d_{k+1}$ and $d_k$. Similar to the analysis of stochastic approximation, we can write (21) into the generalized Robbins-Monro form (22).
by introducing two noise terms. Denote \( \tilde{g}_{i,k} = \tilde{\beta}_{i,k} \Phi_{i,k} \tilde{g}_{i,k} \) and \( \bar{g}_{i,k} = \mathbb{E}_{\mathbf{S}, \mathbf{\eta}, \mathbf{I}, \mathbf{\Phi}, \mathbf{\delta}, \mathbf{\ell}}(\tilde{g}_{i,k}) \), i.e., the expected value of \( \tilde{g}_{i,k} \) with respect to (w.r.t.) \((S_k, \eta_k, I^{(i,k)}, \Phi_k, \delta_k, \ell_k)\), conditioned by any \( \mathbf{a}_k \in \mathcal{A} \). Rewrite (21) as

\[
\tilde{a}_{i,k+1} = a_{i,k} + \tilde{g}_{i,k} = a_{i,k} + \bar{g}_{i,k} + (\tilde{g}_{i,k} - \bar{g}_{i,k}) = a_{i,k} + \frac{\sigma_a^2}{\bar{\gamma}} \beta \gamma_{i,k} \left( \frac{\partial}{\partial a_{i,k}} F(a_k) + b_{i,k} \right) + e_{i,k},
\]

where we introduce

\[
e_{i,k} = \tilde{g}_{i,k} - \bar{g}_{i,k}, \quad \quad (29)
\]

\[
b_{i,k} = \frac{q_a}{\sigma_a^2 \beta \gamma_{i,k}} \bar{g}_{i,k} - \frac{\partial}{\partial a_{i,k}} F(a_k); \quad \quad (30)
\]

\[
\bar{\gamma}_{k} = \mathbb{E}(\tilde{\beta}_{i,k} \tilde{g}_{i,k}) = \mathbb{E}_{\delta, \ell}(\delta_{i,k} \beta_{\ell_{i,k}} \gamma_{\ell_{i,k}}) \quad \quad (31)
\]

Note that \( e_{i,k} \) is in fact the stochastic noise indicating the difference between the value of a single realization of \( \tilde{g}_{i,k} \) and its average \( \bar{g}_{i,k} \); \( b_{i,k} \) represents the difference between \( \bar{g}_{i,k} \) and \( \partial F/\partial a_{i,k} \).

The average step-size \( \bar{\gamma}_k \) can be evaluated by

\[
\bar{\gamma}_k = \mathbb{E}\left(\tilde{\beta}_{i,k} \tilde{g}_{i,k}\right) = \mathbb{E}_{\delta, \ell}(\delta_{i,k} \beta_{\ell_{i,k}} \gamma_{\ell_{i,k}}) = \mathbb{E}_{\delta, \ell}(\delta_{i,k} \beta_{\ell_{i,k}} \gamma_{\ell_{i,k}}) = \mathbb{E}_{\delta, \ell}(\delta_{i,k} \beta_{\ell_{i,k}} \gamma_{\ell_{i,k}})
\]

which is identical for any node \( i \) at time-slot \( k \), since the statistical property of \( \delta_{i,k} \) and \( \tilde{\ell}_{i,k} \) is assumed to be same for all nodes. Similar to \( \bar{\gamma}_k \), define the following average step-sizes that will be used in our analysis:

\[
\bar{\beta}_k = \mathbb{E}_{\delta, \ell}(\tilde{\beta}_{i,k}) \quad \quad \bar{\gamma}_k = \mathbb{E}_{\delta, \ell}(\tilde{\gamma}_{i,k}) \quad \quad \bar{\beta}_{k}^2 = \mathbb{E}_{\delta, \ell}(\tilde{\beta}_{i,k}^2) \quad \quad \bar{\gamma}_{k}^2 = \mathbb{E}_{\delta, \ell}(\tilde{\gamma}_{i,k}^2)
\]

(33)

Denote \( \tilde{g}_k = [\tilde{g}_{1,k}, \ldots, \tilde{g}_{N,k}]^T \), \( \bar{g}_k = [\bar{g}_{1,k}, \ldots, \bar{g}_{N,k}]^T \), \( b_k = [b_{1,k}, \ldots, b_{N,k}]^T \), \( e_k = [e_{1,k}, \ldots, e_{N,k}]^T \) and \( \nabla F(a_k) = [\frac{\partial}{\partial a_1} F(a_k), \ldots, \frac{\partial}{\partial a_N} F(a_k)]^T \). Then we rewrite (21) into \( \tilde{a}_{k+1} = a_k + \tilde{g}_k \) with

\[
\tilde{g}_k = \sigma_a^2 q_a^{-1} \bar{\gamma}_k (\nabla F(a_k) + b_k) + e_k.
\]

(34)

Based on the above notations, we can find an upper bound of \( d_{k+1} \) as a function of \( d_k \):
Proposition 2. Introduce $\Delta_k = \alpha_0^2 \gamma_0^2 N^{-1} \sum_{i \in N} \delta_{i,k} \iota_{i,k}$ with

$$
\tau_{i,k} = \begin{cases} 
1, & \text{if } \tilde{\tau}_{i,k} < K_0 - 1, \\
0, & \text{otherwise}.
\end{cases}
$$

(35)

Then for any $k \geq K_0$, we have

$$
d_{k+1} \leq d_k + \Delta_{k+1} + \frac{1}{N} \| \hat{g}_k \|^2 + \frac{2}{N} (a_k - a^*)^T \cdot e_k + \frac{2 \sigma^2 \Phi}{q k} \beta \gamma_k (a_k - a^*)^T \cdot (\nabla F(a_k) + b_k). 
$$

(36)

Proof: By definition of $d_k$, we have

$$
d_{k+1} = \frac{\| \mathbf{a}_{k+1} - a^* \|^2}{N} + \frac{\| a_{k+1} - a^* \|^2 - \| \mathbf{a}_{k+1} - a^* \|^2}{N}
\leq \frac{1}{N} \| \mathbf{a}_k + \hat{g}_k - a^* \|^2 + \Delta_{k+1}
= d_k + \frac{1}{N} \| \hat{g}_k \|^2 + \frac{2}{N} (a_k - a^*)^T \cdot \hat{g}_k + \Delta_{k+1}
$$

(37)

where (a) is by the following

$$
\frac{\| a_k - a^* \|^2 - \| \mathbf{a}_k - a^* \|^2}{N} \leq \Delta_k, \quad \forall k \geq K_0,
$$

(38)

with the proof detail in Appendix A. We get (36) by substituting (34) into (37), which concludes the proof.

Our next step is to show the desirable properties of $\beta \gamma_k$, $b_k$, $e_k$, and $\Delta_k$ respectively, before our main convergence result.

Proposition 3. We have

$$
\sum_{k=1}^{\infty} \beta \gamma_k = \sum_{k=1}^{\infty} \beta_k \gamma_k \rightarrow \infty,
$$

(39)

$$
\sum_{k=1}^{\infty} \beta_k^2 \leq \sum_{k=1}^{\infty} \beta^2_k = \sum_{k=1}^{\infty} \beta^2_k < \infty.
$$

(40)

Proof: See Appendix B.

Proposition 3 states that the average step-sizes $\beta \gamma_k$ and $\beta_k$ inherit the property of $\beta_k \gamma_k$ and $\beta_k$, which is essential for the convergence of our DOSP-S learning algorithm.

Theorem 1. If all the assumptions are satisfied, then for any node $i$ and any time-slot $k$, we have

$$
|b_{i,k}| \leq (2 \sigma^2 \Phi)^{-1} \alpha G \alpha^3 w_{i,k}.
$$

(41)
with
\[ w_{i,k} = q_i \beta_i \gamma_{k-1} \sum_{j_1, j_2 \in N} \mathbb{E}_{\delta, \ell} \left( \beta_{i,k} \gamma_{j_1,k} \gamma_{j_2,k} \right). \] (42)

Furthermore, \( w_{i,k} \to 0 \) as \( k \to \infty \). Thus \( |b_{i,k}| \to 0 \).

Proof. See Appendix C. The proof of (41) is mainly by the application of Taylor’s theorem and the mean value theorem. We can see that the estimation bias of gradient \( b_{i,k} \) comes from the second order term of the objective function. The value of \( |b_{i,k}| \) can be bounded as \( |\frac{\partial^2 G}{\partial a_i \partial a_j}| \) is bounded by Assumption 5. The proof of \( w_{i,k} \to 0 \) is challenging, we have used Chernoff’s bound to show that \( \mathbb{E}_{\delta, \ell} (\beta_{i,k} \gamma_{j1,k} \gamma_{j2,k}) \) decreases much faster than \( \beta \gamma \).

Remark 5. In the case where nodes are always active, we get \( w_{i,k} = N^2 \gamma_k \) in our previous work [21]. We can directly have \( w_{i,k} \to 0 \) as \( \gamma_k \to 0 \), given that \( N < \infty \). While in this paper, the analysis of \( w_{i,k} \) is much more complicated due to the asynchronous feature of the algorithm: each node maintains a random and individual step-size \( \tilde{\gamma}_{i,k} \). In (42), \( w_{i,k} \) has complicated form of which the closed expression are hard to derive.

The property of \( e_k \) is stated as follows:

Proposition 4. If all the assumptions are satisfied, then we have \( N^{-1} \left| \sum_{k=1}^{\infty} (a_k - a^*)^T \cdot e_k \right| < \infty \) a.s.

Proof. See Appendix D. The proof is by applying Doob’s martingale inequality, which is a suitable tool in the framework of stochastic approximation.

The property of \( \Delta_k \) is similar to \( e_k \):

Proposition 5. There exist bounded constants \( K_1 \geq K_0 \) and \( \tilde{C} > 0 \), such that \( \mathbb{E}(\Delta_k) \leq \tilde{C} \beta^2_{k-1} \) for any \( K \geq K_1 \). Meanwhile, we have \( |\sum_{k=K}^{\infty} \Delta_k| < \infty \) a.s.

Proof. See Appendix F.

Based on all the above results, we can finally prove the a.s. convergence of our DOSP-S algorithm.

Theorem 2. If all the assumptions are satisfied, then \( a_k \to a^* \) as \( k \to \infty \) almost surely by applying Algorithm 1.
Proof. See Appendix G. Based on (36) and our results that \( N^{-1} \left| \sum_{k=1}^{\infty} (a_k - a^*)^T e_k \right| < \infty \) a.s. and \( \left| \sum_{k=K_1}^{\infty} \Delta_k \right| < \infty \) a.s., we can get \( \lim_{k \to \infty} (\nabla F(a_k) + b_k) = 0 \) a.s. with basic steps as in the framework of stochastic approximation. Meanwhile, we have shown that \( \lim_{k \to \infty} \|b_k\| \to 0 \) in Theorem 1. Thus, \( \nabla F(a_k) \to 0 \) a.s. when \( k \to \infty \).

VI. CONVERGENCE RATE

In this section, we investigate the speed of convergence to optimum of the proposed learning algorithm. Specifically, we derive an upper bound of the average divergence

\[
D_k = N^{-1} \mathbb{E} \left( \|a_k - a^*\|^2 \right).
\]

(43)

Note that the expectation is taken over all the random terms including \( a_k \). An additional assumption is made as follows, which is a common setting in the analysis of the convergence rate [34].

Assumption 7. \( F(a) \) is strongly concave, there exists \( \alpha_F > 0 \) such that

\[
(a - a^*)^T \nabla F(a_k) \leq -\alpha_F \|a - a^*\|_2^2, \quad \forall a \in A.
\]

(44)

As a starting point, we need to find the recurrence relation between \( D_{k+1} \) and \( D_k \).

Lemma 1. Under Assumptions 1-7, \( D_{k+1} \) is upper bounded by a function of \( D_k \) as \( k \geq K_1 \), i.e.,

\[
D_{k+1} \leq (1 - A \theta_k) D_k + B \psi_k \sqrt{D_k} + C \upsilon_k,
\]

(45)

with bounded constants \( A = 2\sigma_F^2 \alpha_F \), \( B = \alpha_G \alpha_F^3 \), \( C = \bar{C} + (1 + q_i^{-1}) \lambda \sigma_F^2 \sigma_N^2 + (1 + (2q_i^{-1} + 5)\lambda + (q_i^{-1} + 5)\lambda^2 + \lambda^3 L^2) \sigma_F^2 \sigma_N^2 \) and vanishing sequences

\[
\theta_k = q_i^{-1} \bar{\gamma}_k, \quad \upsilon_k = \bar{\gamma}_k^2, \quad \psi_k = 2N \bar{\gamma}_k \bar{\gamma}_k^2 + (N - 1) \bar{\gamma}_k^2 + \bar{\gamma}_k^2.
\]

(46) (47)

Proof: See Appendix H.

Based on (45), we can derive the upper bounds of \( D_k \), as stated as follows.

Theorem 3. Introduce \( K_2 \) the minimum value of \( k \geq K_1 \) such that \( \theta_k < A^{-1} \). Define the following parameters:

\[
\chi_k = \frac{1}{\theta_k} - \frac{\psi_k^2 \theta_k}{\psi_k^2 \theta_k + 1}, \quad \epsilon_1 = \max_{k \geq K_0} \chi_k, \quad \epsilon_2 = \max_{k \geq K_0} \frac{\theta_k \upsilon_k}{\psi_k^2}.
\]

(48)
\[ \omega_k = \frac{1}{\theta_k} - \frac{v_{k+1}}{v_k \theta_{k+1}}, \epsilon_3 = \max_{k \geq K_0} \omega_k, \epsilon_4 = \max_{k \geq K_0} \frac{\psi_k^2}{\theta_k v_k}. \] (49)

If \( \epsilon_1 < A \) and \( \epsilon_2 < \infty \), then
\[ D_k \leq \vartheta^2 \psi_k^2 \theta_k^{-2}, \quad \forall k \geq K_0, \] (50)
with
\[ \vartheta \geq \max \left\{ \frac{\theta_K \sqrt{D_{K_0}}}{\psi_K}, \frac{B + \sqrt{B^2 + 4C \varrho_2 (A - \epsilon_1)}}{2 (A - \epsilon_1)} \right\}. \] (51)

If \( \epsilon_3 < A \) and \( \epsilon_4 < \infty \), then
\[ D_k \leq \vartheta^2 v_k \theta_k^{-1}, \quad \forall k \geq K_0, \] (52)
with
\[ \vartheta \geq \max \left\{ \sqrt{\frac{D_{K_0} \theta_K}{v_K}}, \frac{B \sqrt{\varrho_4 + \sqrt{B^2 \varrho_4 + 4C (A - \epsilon_3)}}}{2 (A - \epsilon_3)} \right\}. \] (53)

**Proof:** See Appendix [I].

The general form of the upper bounds of \( D_k \) looks complicated mainly due to the averaged parameters \( \theta_k, v_k, \) and \( \psi_k \). The conditions that \( \epsilon_1 < A \) and \( \epsilon_3 < A \) can be checked numerically for any fix value of \( N, q_a, \) and any sequences \( \{ \beta_k \}_{k \geq 0} \) and \( \{ \gamma_k \}_{k \geq 0} \). Here we focus on the theoretical analysis of: i) decreasing order of \( D_k \); ii) convergence of \( \epsilon_2 \) and \( \epsilon_4 \); iii) convergence of \( \epsilon_1 \) and \( \epsilon_3 \).

We propose first the upper bounds of \( v_k \theta_k^{-1}, \psi_k^2 \theta_k^{-2}, \theta_k v_k \psi_k^{-2}, \) and \( \psi_k^2 \theta_k^{-1} v_k^{-1} \) in the following lemma.

**Lemma 2.** Consider \( \beta_\ell = \beta_0 k^{-c_1} \) and \( \gamma_\ell = \gamma_0 k^{-c_2} \). For any \( \xi \in (0, 1) \) and \( \xi' > 0 \), there exists \( K' \) such that \( \forall k \geq K' \),
\[ \frac{v_k}{\theta_k} < (1 + \xi') (1 - \xi)^{-2c_1} q_a \beta_0^{-1} (q_a k)^{-c_1 + c_2}, \] (54)
\[ \frac{\psi_k^2}{\theta_k^2} < \frac{(1 + \xi')^2}{(1 - \xi)^{2c_1 + 4c_2}} (\lambda + 1)^4 \gamma_0^2 (q_a k)^{-2c_2}, \] (55)
\[ \frac{\psi_k^2}{\theta_k v_k} < \frac{(1 + \xi')^2 (\lambda + 1)^4 \gamma_0}{(1 - \xi)^{2c_1 + 4c_2} q_a \beta_0} (q_a (k - 1) + 1)^{c_1 - 3c_2}, \] (56)
\[ \frac{\theta_k v_k}{\psi_k^2} < \frac{(1 + \xi')^2 q_a \beta_0}{(1 - \xi)^{3c_1 + 4c_2} \lambda^4 \gamma_0^3} (q_a (k - 1) + 1)^{-c_1 + 3c_2}. \] (57)

Both \( \xi \) and \( \xi' \) can be arbitrarily close to 0 as \( K' \to \infty \).

**Proof:** See Appendix [I].
From Lemma 2, we can clearly see that the decreasing order of \( v_k \theta_k^{-1} \) and of \( \psi_k^2 \theta_k^{-2} \) is the same as that of \( \beta_k \gamma k \propto k^{-c_1+c_2} \) and of \( \gamma_k^2 \propto k^{-2c_2} \), respectively. According to (56) and (57), we find that \( \lim_{k \to \infty} \theta_k v_k \psi_k^{-2} < \infty \) and \( \epsilon_2 < \infty \) if and only if \( c_1 > 3c_2 \), whereas \( \lim_{k \to \infty} \psi_k^2 \theta_k^{-1} v_k^{-1} < \infty \) and \( \epsilon_4 < \infty \) if and only if \( c_1 \leq 3c_2 \).

The convergence of \( \chi_k \) and \( \varpi_k \) are discussed in the following lemma, which is more challenging to be justified.

**Lemma 3.** Consider \( \beta_k = \beta_0 k^{-c_1} \) and \( \gamma_k = \gamma_0 k^{-c_2} \), then both \( \chi_k \) and \( \varpi_k \) are bounded. There always exist \( \beta_0 < \infty \) and \( \gamma_0 < \infty \) such that \( \epsilon_1 = \max_{k \geq K_0} \chi_k < A \) and \( \epsilon_3 = \max_{k \geq K_0} \varpi_k < A \).

**Proof:** See Appendix K.

The following theorem concludes our discussion.

**Theorem 4.** Consider \( \beta_k = \beta_0 k^{-c_1} \) and \( \gamma_k = \gamma_0 k^{-c_2} \), if the value of \( \beta_0 \gamma_0 < \infty \) is large enough, then there exists \( \Xi < \infty \), such that

\[
D_k \leq \Xi q_r^{-1} (q_a k)^{-\min \{ 2c_2, c_1 - c_2 \}} , \quad \forall k \geq K_2.
\]

As \( c_1 = 0.75 \) and \( c_2 = 0.25 \), the upper bound of \( D_k \) has the optimum decreasing order, i.e., \( D_k = O(q_r^{-1} (q_a k)^{-0.5}) \).

**Proof:** From Lemma 1 we find that \( q_r \) only affects the constant term \( C \), such that \( C = O(q_r^{-1}) \). We also have the upper bound of \( D_k \) is dominated by a linear function of \( C \) when \( C \) is large by Theorem 3. Thus \( D_k = O(q_r^{-1}) \). Then we consider three situations separately.

Case 1: \( 3c_2 < c_1 \). We have \( \epsilon_2 < \infty \) and \( \epsilon_4 = \infty \). Then only (50) is valid with \( \vartheta < \infty \). We have \( D_k \to O(q_r^{-1} (q_a k)^{-2c_2}) \).

Case 2: \( 3c_2 > c_1 \). We have \( \epsilon_4 < \infty \) and \( \epsilon_2 = \infty \). Then only (52) is valid with \( \varrho < \infty \). We have \( D_k \to O(q_r^{-1} (q_a k)^{-c_1+c_2}) \).

Case 3: \( 3c_2 = c_1 \). Both (50) and (52) are valid, we have \( D_k \to O((q_a k)^{-2c_2}) \) or \( D_k \to O(q_r^{-1} (q_a k)^{-c_1+c_2}) \).

As \( c_1 + c_2 \leq 1 \) and \( c_2 > 0.5 \), it is easy to deduce that \( \min \{ 2c_2, c_1 - c_2 \} \leq 0.5 \), where the equality holds only if \( c_1 = 0.75 \) and \( c_2 = 0.25 \).

**Remark 6.** From \( \left| \frac{\partial^2 G(a, \delta)}{\partial a_i \partial a_j} \right| \leq \alpha_G \) in Assumption 5, one have \( \left| \frac{\partial^2 F(a)}{\partial a_i \partial a_j} \right| \leq \alpha_G \) by definition 4, which means that \( \| \nabla F(a) - \nabla F(a') \| \leq N \alpha_G \| a - a' \| \) and \( | F(a) - F(a') | \leq N \alpha_G \| a - a' \| \).
\[ N\alpha G \|a - a'\|^2/2 \] for any \( a, a' \in A \). Applying Jensen’s inequality, we can then derive the upper bound of optimization error

\[
F(a^*) - \mathbb{E}(F(\frac{1}{K} \sum_{k=1}^{K} a_k)) \leq \frac{1}{K} \sum_{k=1}^{K} (F(a^*) - \mathbb{E}(F(a_k)))
\]

\[
\leq \frac{N\alpha G}{2K} \sum_{k=1}^{K} \|a_k - a^*\|^2 \leq \frac{N\alpha G}{2K} \sum_{k=1}^{K} \Xi' K^{-0.5}
\]

\[
\leq N\alpha G \Xi' K^{-0.5} = O(K^{-0.5}).
\]

Clearly, the optimization error achieved by our proposed solution is \( O(K^{-0.5}) \) when the objective function is smooth and strongly concave.

**VII. Numerical illustration**

This section presents some numerical examples to further illustrate our results.

We consider the power control problem described in Section III. Recall that the network is composed of \( N \) transmitter-receiver link, each link has a probability \( q_a \) to be active at any time-slot, with the the local utility function defined in (13). The power gain is \( s_{ij} = |h_{ij}|^2 \), where \( h_{ij} \), the channel between transmitter \( i \) and receiver \( j \), follows Gaussian distribution with variance \( \sigma_{ii}^2 = 1 \) (direct channel) and \( \sigma_{ij}^2 = 0.1 \) (cross channel). The rest of the system parameters are set as \( \sigma^2 = 0.2 \), \( \omega_1 = 20 \) and \( \omega_2 = 1 \). In the proposed learning algorithm, the random perturbation \( \Phi_{i,k} \in \{-1, 1\} \) is generated as a symmetric Bernoulli random variable.

First, we set \( \beta_\ell = 0.025\ell^{-0.75}, \gamma_\ell = 10\ell^{-0.25} \) and consider \( N = 50, q_a = 0.05 \) and \( q_r = 1 \). We perform a single simulation to show the convergence of the action \( a_{i,k} \) performed by all nodes. The result is shown in Figure 2 which contains \( N = 50 \) curves. We can see that all the curves turn to be close to each other and converge after sufficient number of iterations. Note that the optimum value \( a_i^* \) should be identical for all nodes in this example, as the global utility function has a symmetric shape and the random coefficients are generated using the same mechanism. Because of the sparse activity of nodes, the final index of iteration look large. In fact, the average times of update performed by each node is 2500 when \( k = 5 \times 10^4 \) and \( q_a = 0.05 \).

Second, we investigate the influence of fact that nodes have incomplete knowledge of local utilities. We set \( q_r = \{1, 0.5, 0.1\} \) and the other parameters remain unchanged. In order to show that our algorithm converges to optimum, we consider also the ideal gradient descent method as a reference, with the exact partial derivative obtained by (14). As we have discussed in Section III,
this ideal method requires much informational exchange and may be infeasible in practice. Figure 3 shows the evolution of the average global utility by 100 independent simulations. From the oscillation of the curves, we can see that the objective function is quite sensitive to the stochastic channel and not easy to optimize. We find that the global utility tends to the maximum value in average in all cases. The value of $q_t$ does not seriously affect the convergence speed, when an active node has only 10% opportunity to know the local utility of another active node. The two curves corresponding to $q_t = 1$ and $q_t = 0.5$ are quite close.

Finally, we are interested in the evolution of the average divergence $D_k = N^{-1} \mathbb{E}(\|a_k - a^*\|^2)$. We still use $\beta_\ell = \frac{\ell^{-0.75}}{10Nq_a}$ and $\gamma_\ell = 10\ell^{-0.25}$, while consider various values of $N$, $q_a$, and $q_t$. The result is presented in Figure 4. Note that the optimal point $a^*$ is approximately obtained by applying the ideal gradient method. We plot an additional curve $\Xi k^{-0.5}$ in Figure 4 which represents the theoretical convergence rate when $\beta_\ell \propto \ell^{-0.75}$ and $\gamma_\ell \propto \ell^{-0.25}$, under the condition that the objective function is strongly concave. Note that $\Xi = 50$ is set to facilitate the visual comparison of different curves, as we only focus on the asymptotic decreasing speed.

We can see that all the tails of the curves in Figure 4 are approximately parallel, which means that $D_k \to O(k^{-0.5})$ with different values of $N$, $q_a$, and $q_t$. We can also see the influence of $q_t$ on $D_k$ with fixed $N = 50$ and $q_a = 0.5$: compared with the case where $q_t = 1$, $D_k$ converges slightly slower as $q_t = 0.5$, which confirms our discussion of Figure 3.
VIII. CONCLUSION

In this paper we consider a distributed derivative-free optimization problem in a large network with sparse activity. We propose a learning algorithm to make each active node control its action to maximize the global utility function of the network, which is also affected by some stochastic process. The algorithm is performed only based on the numerical observation of the global utility rather than its gradient. We prove the almost surely convergence of the algorithm with the tools of stochastic approximation and concentration inequalities. The analysis is challenging because of the asynchronous feature of the network. We have also derived the convergence rate of the proposed algorithm. We provide simulation results to corroborate our claim. Both theoretical and numerical results show that our derivative-free learning algorithm can converge at a rate $O(k^{-0.5})$.

APPENDIX

A. Proof of inequality (38)

In this proof, we investigate the property of the projection (25). Define $C_i^* = [a_{i,\min} + \alpha \Phi \gamma_K, a_{i,\max} - \alpha \Phi \gamma_K]$, then (20) implies that $a_i^* \in C_i^*, \forall i \in \mathcal{N}$. Similarly, let $C_{i,k} = [a_{i,\min} + \alpha \Phi \bar{\gamma}_{i,k}, a_{i,\max} - \alpha \Phi \bar{\gamma}_{i,k}]$ for
Figure 4. Evolution of $D_k$ by 50 simulations. We use $\beta_k = \frac{k^{-0.75}}{\ln N}$ and $\gamma_k = 10k^{-0.25}$ and consider various setting of $(N, q_a, q_r)$. We also use an addition curve $O(k^{-0.5})$ to present the theoretical upper bound of $D_k$.

any $i \in \mathcal{N}$ and $k \geq K_0$. Due to the fact that $\gamma_{i,k}^{\star} = \delta_{i,k}\gamma_{\delta_{i,k}} + \tilde{\ell}_{i,k}$, there is not always $a_i^\star \in C_{i,k}$. Different cases must be considered depending on the values of $\delta_{i,k}$ and $\tilde{\ell}_{i,k}$.

Case 1, $\delta_{i,k} = 0$. We have $C_{i,k} = [a_{i,\min} + 0, a_{i,\max} + 0]$, thus $a_i^\star \in C_{i,k}$. By definition, we also have $a_{i,k} = \text{Proj}_{i,k}(\tilde{a}_{i,k}) \in C_{i,k}$. Since the projection decreases the Euclidean distance between $\tilde{a}_{i,k}$ and $a_i^\star$ if $\tilde{a}_{i,k} \notin C_{i,k}$, it is easy to show that $|\text{Proj}_{i,k}(\tilde{a}_{i,k}) - a_i^\star| \leq |\tilde{a}_{i,k} - a_i^\star|$. Hence $(a_i^\star - a_i^\star)^2 - (\tilde{a}_{i,k} - a_i^\star)^2 \leq 0$.

Case 2, $\delta_{i,k} = 1$ and $\tilde{\ell}_{i,k} \leq K_0 - 1$. Then $\gamma_{i,k}^{\star} \leq \gamma_{K_0}$ and $C_i^{\star} \subseteq C_{i,k}$. Thus $a_i^\star \in C_{i,k}$ as in Case 1 and $(a_i^\star - a_i^\star)^2 - (\tilde{a}_{i,k} - a_i^\star)^2 \leq 0$.

Case 3, $\delta_{i,k} = 1$ and $\tilde{\ell}_{i,k} < K_0 - 1$. This case is complicated as $a_i^\star \notin C_{i,k}$. There exist two possible situations: $a_i^\star \in [a_{i,\min}, a_{i,\min} + \alpha \Phi_i^{\gamma_{i,k}}]$ or $a_i^\star \in [a_{i,\max}, a_{i,\max} - \alpha \Phi_i^{\gamma_{i,k}}]$. Here we mainly consider $a_i^\star \in [a_{i,\min}, a_{i,\min} + \alpha \Phi_i^{\gamma_{i,k}}]$ as the analysis of the other situation is similar. We still need to discuss the possible value of $a_i$ in separate situations: i). if $\tilde{a}_{i,k} \in C_{i,k}$, we have $\text{Proj}_{i,k}(\tilde{a}_{i,k}) = \tilde{a}_{i,k}$, then $(a_i^\star - a_i^\star)^2 - (\tilde{a}_{i,k} - a_i^\star)^2 = 0$; ii). if $\tilde{a}_{i,k} > a_{i,\max} - \alpha \Phi_i^{\gamma_{i,k}}$, then $\text{Proj}_{i,k}(\tilde{a}_{i,k}) = a_{i,\max} - \alpha \Phi_i^{\gamma_{i,k}} < \tilde{a}_{i,k}$. Meanwhile $a_i^\star \leq a_{i,\min} + \alpha \Phi_i^{\gamma_{i,k}} < a_{i,\max} - \alpha \Phi_i^{\gamma_{i,k}}$. We get $(a_i^\star - a_i^\star)^2 - (\tilde{a}_{i,k} - a_i^\star)^2 < 0$; iii). if $\tilde{a}_{i,k} < a_{i,\min} + \alpha \Phi_i^{\gamma_{i,k}}$, then $\text{Proj}_{i,k}(\tilde{a}_{i,k}) = a_{i,\min} + \alpha \Phi_i^{\gamma_{i,k}}$. We have $(a_i^\star - a_i^\star)^2 - (\tilde{a}_{i,k} - a_i^\star)^2 \leq (a_i^\star - a_i^\star)^2 = (a_{i,\min} + \alpha \Phi_i^{\gamma_{i,k}} - a_i^\star)^2 \leq \alpha^2 \gamma_0^2$.
In summary, \((a_{i,k} - a^*_i)^2 - (a_{i,k} - a^*_i)^2 \leq \alpha^2_\Phi \gamma^2_0\) in Case 3.

Based on the above discussions, we have the following bound to conclude the three cases\((a_{i,k} - a^*_i)^2 - (a_{i,k} - a^*_i)^2 \leq \alpha^2_\Phi \gamma^2_0 \delta_{i,k}^2\) with \(i, k\) defined in (35). Finally, we get \(\|a_k - a^*\|_N^2 - \|a_k - a^*\|_N^2 \leq \frac{\alpha^2_\Phi \gamma^2_0}{N} \sum_{i \in N} \delta_{i,k}^2\) which concludes the proof.

**B. Proof of Lemma 3**

We first present an important lemma with its proof in Appendix F of our previous work [35].

**Lemma 4.** Consider any sequence \(\{x_k\}\) and let \(\pi_k = \sum_{\ell=1}^k x_\ell (1-p)^{\ell} (k-1)\) with \(p \in [0, 1]\), we always have \(\sum_{k=1}^\infty \pi_k = \sum_{k=1}^\infty x_k\).

Replace \(x_k\) by \(\beta_k^2\) and \(p\) by \(1-q_a\), we get that \(\sum_{k=1}^\infty \beta_k^2 = \sum_{k=1}^\infty \beta_k\) then (39) can be proved as \(\sum_{k=1}^\infty \beta_k \to \infty\); Replace \(x_k\) by \(\beta_k^2\) and \(p\) by \(1-q_a\), we have \(\sum_{k=1}^\infty \beta_k^2 = \sum_{k=1}^\infty \beta_k^2\).

We can finally justify (40) with the assumption that \(\sum_{k=1}^\infty \beta_k^2 < \infty\).

**C. Proof of Theorem 1**

This proof contains two parts. We first show that (41) is an upper bound of \(|b_{i,k}|\) in Appendix C1 then we prove that this upper bound is vanishing in Appendix C2.

1) **Proof of (41)**: As \(b_{i,k}\) describes the difference between \(\tilde{g}_{i,k}\) and \(\partial F(a_k)/\partial a_i\), we start with the derivation of \(\tilde{g}_{i,k}\) by successively taking the expectation of \(\tilde{g}_{i,k}\) w.r.t. multiple stochastic terms \((S, I, \eta, \Phi, \delta, \ell)\), which makes the analysis complicated. By definition, we have

\[
\tilde{g}_{i,k} = \mathbb{E}_{S, I, \eta, \Phi, \delta, \ell} \left( \tilde{f}_{i,k} (a_k + \tilde{\gamma}_k \circ \Phi_k, \delta_k, S_k) \right)
\]

\[
\begin{aligned}
&= \mathbb{E}_{\Phi, \delta, \ell} \left( \tilde{f}_{i,k} (a_k + \tilde{\gamma}_k \circ \Phi_k, \delta_k, S_k) \right) \\
&= \mathbb{E}_{\Phi, \delta, \ell} \left( \tilde{f}_{i,k} (a_k + \tilde{\gamma}_k \circ \Phi_k, \delta_k, S_k) \right) \\
&= \mathbb{E}_{\Phi, \delta, \ell} \left( \tilde{f}_{i,k} (a_k + \tilde{\gamma}_k \circ \Phi_k, \delta_k, S_k) \right) \\
&= \mathbb{E}_{\Phi, \delta, \ell} \left( \tilde{f}_{i,k} (a_k + \tilde{\gamma}_k \circ \Phi_k, \delta_k, S_k) \right) \\
&= \mathbb{E}_{\Phi, \delta, \ell} \left( \tilde{f}_{i,k} (a_k + \tilde{\gamma}_k \circ \Phi_k, \delta_k, S_k) \right)
\end{aligned}
\]

where \((a)\) holds as the stochastic term \(\tilde{f}_{i,k} \Phi_i, k\) generated during the DOSP-S algorithm is independent of \((S, I, \ell, \eta, k)\) caused by the system environment, the unsuccessful packet transmission and the measurement noise; \((b)\) is by taking expectation of \(\tilde{f}_{i,k}\) w.r.t. \((\ell, \eta, k)\),

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which has already been solved in Proposition 1; (c) is by taking expectation of \( f \) w.r.t. \( S \), recall that \( G(\alpha, \delta) = \mathbb{E}_S( f(\alpha, \delta, S)) \) by definition (4); (d) comes from the extension of \( G(\alpha_k + \gamma_k \circ \Phi_k, \delta_k) \) by applying Taylor’s theorem and mean-valued theorem, i.e., there exists \( \bar{\alpha}_k = [\bar{a}_{1,k}, \ldots, \bar{a}_{N,k}]^T \) with \( \bar{a}_{i,k} \in (a_{i,k}, a_{i,k} + \gamma_{i,k} \Phi_{i,k}), \forall i \in \mathcal{N}, \) such that (d) can be satisfied.

We should continue the derivation in (59) by considering the expectation w.r.t. \( (\Phi_k, \delta_k, \ell_k) \). We have

\[
\mathbb{E}_{\Phi, \delta, \ell} \left( \tilde{\beta}_{i,k} \Phi_{i,k} G(\alpha_k, \delta_k) \right) = 0, \tag{60}
\]

as \( \Phi_{i,k} \) is independent of \( (\alpha_k, \delta_k, \tilde{\beta}_{i,k}) \) and \( \mathbb{E}_{\Phi}(\Phi_{i,k}) = 0 \) by Assumption 6. Meanwhile,

\[
\mathbb{E}_{\Phi, \delta, \ell} \left( \tilde{\beta}_{i,k} \Phi_{i,k} \sum_{j \in \mathcal{N}} \bar{\gamma}_{j,k} \Phi_{j,k} \frac{\partial G}{\partial a_j}(\alpha_k, \delta_k) \right)
= \sigma_\Phi^2 \mathbb{E}_{\delta, \ell} \left( \delta_{i,k} \beta_{\ell_{i,k}} \gamma_{\ell_{i,k}} \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right) + 0
= \sigma_\Phi^2 \mathbb{P}(\delta_{i,k} = 1) \mathbb{E}_{\delta, \ell} \left( \beta_{\ell_{i,k}} \gamma_{\ell_{i,k}} \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right) \mathbb{P}(\delta_{i,k} = 1) + 0
= \sigma_\Phi^2 \mathbb{E}_{\ell} \left( \beta_{1+\ell_{i,k}} \gamma_{1+\ell_{i,k}} \right) \mathbb{E}_{\delta} \left( \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right) \mathbb{E}_{\delta} \left( \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right)
= \sigma_\Phi^2 \mathbb{E}_{\ell} \left( \beta_{1+\ell_{i,k}} \gamma_{1+\ell_{i,k}} \right) \mathbb{E}_{\delta} \left( \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right), \tag{61}
\]

in which (a) is again by Assumption 6, i.e., \( \mathbb{E}_{\Phi}(\Phi_{i,k}^2) = \sigma_\Phi^2 \) and \( \mathbb{E}_{\Phi}(\Phi_{i,k} \Phi_{j,k}^2) = 0 \) \( \forall j \neq i \); (b) comes from \( \mathbb{E}_{\delta, \ell} \left( \delta_{i,k} \beta_{\ell_{i,k}} \gamma_{\ell_{i,k}} \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right) \mathbb{P}(\delta_{i,k} = 1) = 0 \); (c) is by the independence of \( \delta_k \) and \( \ell \); (d) holds as \( \mathbb{E}_{\delta} \left( \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right) = \mathbb{E}_{\delta} \left( \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right) \mathbb{P}(\delta_{i,k} = 1) \mathbb{P}(\delta_{i,k} = 1) \), note that \( \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) = 0 \) in the case where \( \delta_{i,k} = 0 \) meaning that \( G \) is not a function of \( a_{i,k} \); (e) comes from

\[
\mathbb{P}(\delta_{i,k} = 1) \mathbb{E}_{\ell}(\beta_{1+\ell_{i,k}} \gamma_{1+\ell_{i,k}}) \text{ and from the relation between } F \text{ and } G \text{ discussed in (5), we have}
\frac{\partial F}{\partial a_i}(\alpha_k) = \frac{\partial}{\partial a_i} \left( \sum_{\delta_k \in \mathcal{D}} q^{n_k}_{\alpha_k} (1 - q_{\alpha_k}) G(\alpha_k, \delta_k) \right) = \sum_{\delta_k \in \mathcal{D}} q^{n_k}_{\alpha_k} (1 - q_{\alpha_k}) \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) = \mathbb{E}_{\delta} \left( \frac{\partial G}{\partial a_i}(\alpha_k, \delta_k) \right).\]

Substituting (60) and (61) into (59), we get

\[
\tilde{g}_{i,k} = \sigma_\Phi^2 q_{\alpha_k}^{-1} \beta \gamma_k \left( \frac{\partial F}{\partial a_i}(\alpha_k) + b_{i,k} \right)
\]

with the bias term \( b_{i,k} = \sum_{j_1, j_2 \in \mathcal{N}} \mathbb{E}_{\Phi, \delta, \ell} \left( q_{\alpha_k} \tilde{g}_{j_1,k} \tilde{g}_{j_2,k} \Phi_{j_1,k} \Phi_{j_2,k} \frac{\partial^2 G}{\partial a_{j_1} \partial a_{j_2}}(\bar{a}_{j_1,k}, \delta_k) \right) \).
As $\left| \frac{\partial^2 G(\tilde{a}_k, \delta_k)}{\partial a_{j_1} \partial a_{j_2}} \right| \leq \alpha_G$ (by Assumption 3) and $|\Phi_{i,k}| \leq \alpha_\Phi$, $\forall \ i \in \mathcal{N}$ (by Assumption 5), it is straightforward to get

$$|b_{i,k}| \leq \frac{\alpha^3 G \sum_{j_1, j_2 \in \mathcal{N}} E_{\delta, \ell} (\tilde{\beta}_{i,k} \tilde{\gamma}_{j_1,k} \tilde{\gamma}_{j_2,k})}{q_a^{-1} \beta \gamma_k} \alpha G w_{i,k}.$$ 

Therefore, $b_{i,k}$ in (62) can be bounded by (41) with $w_{i,k}$ defined in (42), which concludes the first part of the proof.

2) **Proof of $|b_{i,k}| \to 0$:** Our next target is to show $w_{i,k} \to 0$, from which we can directly get $|b_{i,k}| \to 0$. The proof is quite challenging, as $w_{i,k}$ contains a summation of $\mathcal{N}^2$ terms of expectation whose closed form expression are hard to obtain. Moreover, the denominator of $w_{i,k}$ is vanishing, i.e., $\beta \gamma_k \to 0$.

Denote $\mathcal{N}_{-i} = \mathcal{N} \setminus \{i\}$, we evaluate the numerator of $w_{i,k}$:

$$\sum_{j_1, j_2 \in \mathcal{N}_{-i}} E_{\delta, \ell} (\tilde{\beta}_{i,k} \tilde{\gamma}_{j_1,k} \tilde{\gamma}_{j_2,k}) = \sum_{j_1, j_2 \in \mathcal{N}_{-i}} E_{\delta, \ell} (\tilde{\beta}_{i,k} \tilde{\gamma}_{j_1,k} \tilde{\gamma}_{j_2,k})$$

$$+ \sum_{j \in \mathcal{N}_{-i}} E_{\delta, \ell} (2 \tilde{\beta}_{i,k} \tilde{\gamma}_{i,k} \tilde{\gamma}_{j,k} + \tilde{\gamma}_{i,k}^2) + E_{\delta, \ell} (\tilde{\beta}_{i,k} \tilde{\gamma}_{i,k}^2)$$

$$= (N - 1) \left( (N - 2) \beta \gamma_k \gamma_k^2 + 2 \beta \gamma_k \gamma_k + \beta \gamma_k \gamma_k^2 \right), \quad (63)$$

where $\beta \gamma_k^2$, $\gamma_k$, $\gamma_k^2$, and $\beta_k$ are defined in (33). From (63) and the fact that $\gamma_k^2 \leq \overline{\gamma}_k^2$, $w_{i,k}$ can be bounded by

$$w_{i,k} \leq \frac{\beta \gamma_k^2 + 2 (N - 1) \beta \gamma_k \gamma_k + (N - 1)^2 \beta \gamma_k \gamma_k^2}{q_a^{-1} \beta \gamma_k}$$

$$< 2 \lambda \gamma_k + \frac{\beta \gamma_k^2 + (N - 1)^2 \beta \gamma_k \gamma_k}{q_a^{-1} \beta \gamma_k}, \quad (64)$$

note that $(N - 1) q_a < N q_a = \lambda$. The following lemma is useful to find upper bounds of $\overline{\gamma}_k$, $\beta \gamma_k^2$ and $\beta_k \gamma_k^2$.

**Lemma 5.** Consider an arbitrary positive decreasing sequence $\{z_k\}$ and an arbitrary $0 < \xi < 1$.

Denote

$$p_{k, \xi} = \exp \left( -2^{-1} \xi^2 q_a (k - 1) \right), \quad (65)$$

$$\overline{k}_\xi = \lfloor (1 - \xi) q_a (k - 1) \rfloor + 2, \quad (66)$$

Then we have

$$E_{\delta, \ell} (\delta_{i,k} z_{\overline{k}_\xi}) \leq q_a \left( p_{k, \xi} z_{\overline{k}_\xi} \right). \quad (67)$$
Proof. We have,
\[
\mathbb{E}_{\delta, \ell} (\delta_{i,k} z_{\ell_{i,k}}) = \mathbb{P} (\delta_{i,k} = 1) \mathbb{E}_{\delta, \ell} (z_{\ell_{i,k+1}} | \delta_{i,k} = 1)
\]
\[
= q_a \sum_{\ell = 0}^{k-1} \mathbb{P} (\tilde{\ell}_{i,k} = \ell) z_{\ell+1} \overset{(a)}{\leq} q_a z_1 \mathbb{P} (\tilde{\ell}_{i,k} \leq k_\xi - 2)
\]
\[
+ q_a \gamma_{k_\xi} \mathbb{P} (\tilde{\ell}_{i,k} \geq k_\xi - 1) \overset{(b)}{<} q_a \left( p_{k,\xi} z_1 + z_{k_\xi} \right),
\]
(68)
in which \((a)\) is by the fact that \(\gamma_{\ell}\) is a decreasing sequence; \((b)\) is obtained by using Chernoff Bound, \(i.e.,\)
\[
\mathbb{P} (\tilde{\ell}_{i,k} \leq k_\xi - 2) = \mathbb{P} (\tilde{\ell}_{i,k} \leq \lfloor (1 - \xi) q_a (k - 1) \rfloor)
\]
\[
\leq e^{-\frac{1}{2} \xi^2 \mathbb{E} (\tilde{\ell}_{i,k})} = p_{k,\xi},
\]
(69)
and by \(\mathbb{P} (\tilde{\ell}_{i,k} \geq k_\xi - 1) < 1\), which concludes the proof.

Applying Lemma [5] we can obtain the following bounds
\[
\pi_k < q_a (p_{k,\xi} \gamma_1 + \gamma_{k_\xi}); \quad \bar{\gamma}_k < q_a (p_{k,\xi} \gamma_1^2 + \gamma_{k_\xi}^2);
\]
\[
\bar{\beta}_k < q_a (p_{k,\xi} \beta_1 + \beta_{k_\xi}); \quad \bar{\gamma} \bar{\beta}_k < q_a (p_{k,\xi} \beta_1 \gamma_1^2 + \beta_{k_\xi} \gamma_{k_\xi}^2).
\]
(70)
As \(p_{k,\xi}, \beta_k, \) and \(\gamma_k\) are vanishing, (70) implies that \(\pi_k \rightarrow 0, \bar{\gamma}_k \rightarrow 0, \bar{\beta}_k \rightarrow 0, \) and \(\bar{\gamma} \bar{\beta}_k \rightarrow 0.\)

Applying the upper bounds in (70), we have
\[
\overline{\beta} \overline{\gamma}^2 + (N - 1)^2 \overline{\beta} \overline{\gamma}^2 < q_a \left( p_{k,\xi} \beta_1 \gamma_1^2 + \beta_{k_\xi} \gamma_{k_\xi}^2 \right)
\]
\[
+ (N - 1)^2 q_a^2 \left( p_{k,\xi} \beta_1 + \beta_{k_\xi} \right) \left( p_{k,\xi} \gamma_1^2 + \gamma_{k_\xi}^2 \right)
\]
\[
< \left( \lambda^2 (p_{k,\xi} + 2) + q_a \right) \beta_1 \gamma_1^2 p_{k,\xi} + \left( \lambda^2 + q_a \right) \beta_{k_\xi} \gamma_{k_\xi}^2
\]
\[
< \left( 3 \lambda^2 + q_a \right) \beta_1 \gamma_1^2 p_{k,\xi} + \left( \lambda^2 + q_a \right) \beta_{k_\xi} \gamma_{k_\xi}^2,
\]
(71)
where the upper bound is by \(\gamma_{k_\xi} < \gamma_1\) and \(\beta_{k_\xi} < \beta_1,\) as \(k_\xi = \lfloor (1 - \xi) q (k - 1) \rfloor + 2 > 1.\)

Meanwhile, thanks to the fact that \(\beta_k \gamma_k\) is a convex function of \(k,\) we can apply Jensen’s inequality to get the lower bound
\[
\overline{\beta} \overline{\gamma} = q_a \mathbb{E}_{\ell} \left( \beta_{1+\tilde{\ell}_{i,k}} \gamma_{1+\tilde{\ell}_{i,k}} \right) \geq q_a \beta_{k'} \gamma_{k'},
\]
(72)
in which we denote \(k' = 1 + \mathbb{E} (\tilde{\ell}_{i,k}) = 1 + q_a (k - 1).\) Note that \(\beta_{k'}\) and \(\gamma_{k'}\) represent functions of \(k' \in \mathbb{R}^+, \) \(e.g., \beta_{k'} = \beta_0 (k')^{-c_1}.\) Here we slightly abuse the notation as \(\{\beta_{\ell}\}\) and \(\{\gamma_{\ell}\}\) are initially defined as sequences with integer index.

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Due to the independence of \( \Omega_k \), we have \( \omega_{i,k} < \Omega_k \) with

\[
\Omega_k = (3\lambda^2 + q_a) \beta_1 \gamma_1^2 \frac{p_{k,\xi}}{\beta_k' \gamma_k'} + (\lambda^2 + q_a) \frac{\beta_k \gamma_k^2}{\beta_k' \gamma_k'} \\
+ 2\lambda q_a \left( \gamma_1 p_{k,\xi} + \gamma_k \right).
\]

(73)

The last step is to show that \( \Omega_k \to 0 \) considering \( \beta_k = \beta_0 k^{-c_1} \) and \( \gamma_k = \gamma_0 k^{-c_2} \). Since \( \lambda < \infty \), \( q_a \leq 1 \), \( p_{k,\xi} \to 0 \) and \( \gamma_k \to 0 \), we mainly need to check whether \( \frac{p_{k,\xi}}{\beta_k' \gamma_k'} \) and \( \frac{\beta_k \gamma_k^2}{\beta_k' \gamma_k'} \) are vanishing. In fact, we have

\[
\lim_{k \to \infty} \frac{p_{k,\xi}}{\beta_k' \gamma_k'} = \lim_{k \to \infty} \frac{\exp (-2^{-1} q_a k)}{\beta_0 \gamma_0 (1 + [q_a (k - 1)])^{-c_1 - c_2}} = 0,
\]

since the exponential term decreases much faster than \( k^{-c_1 - c_2} \). Meanwhile, we have

\[
\frac{\beta_k \gamma_k^2}{\beta_k' \gamma_k'} = \frac{\beta_0 \gamma_0 (\{(1 - \xi) q_a (k - 1)\} + 2)^{c_1 - c_2}}{\beta_0 \gamma_0 (q_a (k - 1) + 1)^{-c_1 - c_2} \gamma_k'} \\
\leq (\xi) q_a (k - 1) + 1)^{c_1 - c_2} \gamma_k < (1 - \xi)^{1+c_2},
\]

(74)

where \((a)\) is by \(|x| > x - 1, \forall x > 0\); \((b)\) holds for any \( \xi \in (0, 1) \) and \( k \geq 1 \), as

\[
\frac{(1 - \xi) q_a (k - 1) + 1}{q_a (k - 1) + 1} = 1 - \xi + \frac{\xi}{q_a (k - 1) + 1} > 1 - \xi.
\]

From (74), we finally have

\[
\lim_{k \to \infty} \frac{\beta_k \gamma_k^2}{\beta_k' \gamma_k'} \leq \lim_{k \to \infty} \frac{\gamma_k}{(1 - \xi)^{1+c_2}} = 0.
\]

(75)

We have shown that each term of \( \Omega_k \) in (73) is vanishing, hence \( \Omega_k \to 0 \) implying that \( \omega_{i,k} \to 0 \) and \( |b_{i,k}| \to 0 \).

**D. Proof Proposition 4**

We first show that \( \{\sum_{k=K}^{K'} (a_k - a^*)^T \cdot e_k\}_{K' \geq K} \) is martingale, then apply Doob’s martingale inequality [36] to prove Proposition 4. In order to lighten the notations, we introduce \( \mathcal{F}_k = \{S_k, \Phi_k, T_k, \eta_k, \delta_k, \ell_k\} \) to denote the collection of all stochastic terms.

The noise term \( e_k \) has zero mean, since \( \mathbb{E}_{\mathcal{F}} (e_k) = \mathbb{E}_{\mathcal{F}} (\bar{g}_k - \bar{g}_k) = \bar{g}_k - \bar{g}_k = 0, \forall a_k \in \mathcal{A} \). Due to the independence of \( \mathcal{F}_k \) and \( \mathcal{F}_{k'} \) for any \( k \neq k' \), \( e_k \) and \( e_{k'} \) are independent. Hence, the
sequence $\{\sum_{k=K}^{K'} (a_k - a^*)^T \cdot e_k\}_{K' \geq K}$ is martingale. We apply Doob’s martingale inequality to get, $\forall \rho > 0$,
\[
\mathbb{P}\left( \sup_{K' \geq K} \left\| \frac{1}{N} \sum_{k=K}^{K'} (a_k - a^*)^T \cdot e_k \right\| \geq \rho \right) \\
\leq \frac{1}{\rho^2 N^2} \mathbb{E}_F \left( \left\| \sum_{k=K}^{K'} (a_k - a^*)^T \cdot e_k \right\|^2 \right).
\]  
(76)

We need to evaluate
\[
\mathbb{E}_F \left( \left\| \sum_{k=K}^{K'} (a_k - a^*)^T \cdot e_k \right\|^2 \right) = \sum_{k=K}^{K'} \mathbb{E}_F \left( \left\| (a_k - a^*)^T \cdot e_k \right\|^2 \right) \\
\leq \sum_{k=K}^{K'} \mathbb{E}_F \left( \|a_k - a^*\|^2 \|e_k\|^2 \right) \leq \frac{D^2_{\max}}{N} \sum_{k=K}^{K'} \mathbb{E}_F \left( \|\tilde{g}_k - \bar{g}_k\|^2 \right) \\
\leq N D^2_{\max} C' \sum_{k=K}^{K'} \bar{\beta}_k^2 \tag{77}
\]

where (a) comes from $\mathbb{E}(e_{i,k_1} e_{i,k_2}) = 0$ for any $k_1 \neq k_2$; (b) is by Cauchy–Schwarz inequality; in (c) we denote $D^2_{\max} = \max_{j \in \mathcal{N}} \{(a_{i,max} - a_{i,min})^2\}$, then we have $\|a_k - a^*\|^2 \leq N D^2_{\max}$, recall that $a_{i,min} \in [a_{i,min}, a_{i,max}], \forall i \in \mathcal{N}$; (d) is by Lemma 6 stated in what follows, of which the proof is given in Appendix [E]

**Lemma 6.** If all the assumptions are satisfied, then $\mathbb{E}_{\mathbf{s},\Phi,i,\eta,\delta,\tilde{\ell}}(\|\tilde{g}_k\|^2) < N C' \bar{\beta}_k^2$, with $C' =\|(1 + q^ {-1} \lambda) \sigma^2_{\tilde{\Phi}} \sigma^2_{\eta} + (1 + (2 q^ {-1} + 5) \lambda + (q^ {-1} + 5) \lambda^2 + \lambda^3) L^2 \sigma^2_{\tilde{\Phi}} \sigma^2_{\alpha} < \infty$.

Substituting (77) into (76), we get
\[
\mathbb{P}\left( \sup_{K' \geq K} \left\| \frac{1}{N} \sum_{k=K}^{K'} (a_k - a^*)^T \cdot e_k \right\| \geq \rho \right) \leq \frac{D^2_{\max}}{\rho^2} \sum_{k=K}^{K'} \bar{\beta}_k^2.
\]  
(78)

Since $\lim_{K \to \infty} \sum_{k=K}^{K'} \bar{\beta}_k^2 = 0$ by Lemma 3, we can say that $N^{-1} \left\| \sum_{k=K}^{\infty} (a_k - a^*)^T \cdot e_k \right\|$ is bounded a.s. according to (78), Proposition 4 is then proved.

**E. Proof of Lemma 6**

We evaluate the expectation of $\tilde{g}_{i,k}^2$ on all the random terms,
\[
\mathbb{E}_F \left( \tilde{g}_{i,k}^2 \right) = \mathbb{P}(\delta_{i,k} = 1) \mathbb{E}_F \left( \tilde{g}_{i,k}^2 \mid \delta_{i,k} = 1 \right) \\
= q_{i} \mathbb{E}_{\Phi,\tilde{\ell}} \left( \beta_{1+i,k}^2 \Phi_{i,k}^2 \mathbb{E}_{\mathbf{s},\ell,\eta,\delta} \left( \tilde{f}_{i,k}^2 \mid \delta_{i,k} = 1 \right) \right),
\]  
(79)
it is worth mentioning that \( \tilde{\ell}_k, \Phi_k, \eta_k, S_k, \) and \( \delta_k \) are mutually independent. According to definition of \( \bar{f}_{i,k} \) (\ref{eq:17}), as \( \delta_{i,k} = 1, \)

\[
\bar{f}_{i,k}^2 = (\bar{u}_{i,k} + \sum_{j \in \mathcal{N}(k) \setminus \{i\}} \frac{\kappa_{i,j,k}}{q_t} \bar{u}_{j,k})^2 = \bar{u}_{i,k}^2 + \sum_{j \in \mathcal{N}(k) \setminus \{i\}} \frac{\kappa_{i,j,k}^2}{q_t^2} \bar{u}_{j,k}^2 \]

\[
+ \sum_{j \in \mathcal{N}(k) \setminus \{i\}} \frac{2\kappa_{i,j,k}}{q_t} \bar{u}_{i,k} \bar{u}_{j,k} + \sum_{j_1, j_2 \in \mathcal{N}(k), j_1 \neq j_2 \neq i} \frac{\kappa_{i,j_1,k} \kappa_{i,j_2,k}}{q_t} \bar{u}_{j_1,k} \bar{u}_{j_2,k}. \]

Recall that \( \mathbb{E}(\kappa_{i,j,k}) = \mathbb{E}(\kappa_{i,j,k}) = q_t, \forall j \neq i \) and \( \mathbb{E}(\kappa_{i,j_1,k} \kappa_{i,j_2,k}) = q_t^2, \forall j_1 \neq j_2 \neq i, \) \( \kappa_{i,j_1,k} \) and \( \kappa_{i,j_2,k} \) are independent. Similarly, we have \( \mathbb{E}_{\eta}(\bar{u}_{i,k}^2) = \mathbb{E}_{\eta}(\bar{u}_{i,k} + \eta_{i,k})^2 = u_{i,k}^2 + \sigma_\eta^2, \forall i, \) and \( \mathbb{E}_{\eta}(\bar{u}_{i,k} \bar{u}_{j,k}) = u_{i,k}u_{j,k}, \forall i \neq j. \) We can take expectation of \( \bar{f}_{i,k}^2 \) on \( T^{(i,k)} \) and \( \eta_k \) to get

\[
\mathbb{E}_{T^{(i,k)}, \eta_k}(\bar{f}_{i,k}^2 | \delta_{i,k} = 1) = \left(1 + \frac{m}{q_t}\right) \sigma_\eta^2 + u_{i,k}^2 \]

\[
+ \sum_{j \in \mathcal{N}(k) \setminus \{i\}} \left(\frac{1}{q_t} u_{j,k}^2 + 2u_{i,k}u_{j,k}\right) + \sum_{j_1, j_2 \in \mathcal{N}(k), j_1 \neq j_2 \neq i} u_{j_1,k}u_{j_2,k} \quad (80) \]

where we denote \( m = \sum_{j \in \mathcal{N}(k) \setminus \{i\}} 1 = n_k - 1. \)

We then need to find an upper bound of \( u_{i,k}^2 \) and \( u_{j_1,k}u_{j_2,k}. \) For any \( \delta_k, S_k \) and \( j \in \mathcal{N}(k), \) we have,

\[
u_{j,k}^2 = u_{j}^2(\tilde{\alpha}_k, \delta_k, S_k) \leq (|u_{j}(0, \delta_k, S_k)| + L_{S_k}|\tilde{\alpha}_k \circ \delta_k|)^2 \]

\[
\leq L_{S_k}^2 ||\tilde{\alpha}_k \circ \delta_k||^2 \leq L_{S_k}^2 (m + 1) \sigma_a^2 < \infty, \quad (81) \]

where \( (a) \) is by \( (11), \) \( i.e., \) the assumption that \( u_i \) is Lipschitz; \( (b) \) comes from \( u_j(0, \delta_k, S_k) = u_j(0, 0, S_k) = 0, \) as \( \tilde{\alpha}_k = 0 \) also means no nodes perform action; \( (c) \) is by \( ||\tilde{\alpha}_k \circ \delta_k||^2 \leq \sum_{j \in \mathcal{N}} \delta_{j,k}\sigma_a^2 = (m + 1) \sigma_a^2, \) where \( \sigma_a^2 \) is the upper bound of \( \alpha_{i,k} \) defined in \( (1). \) Based on \( (81), \) we can also deduce

\[
u_{j_1,k}u_{j_2,k} \leq |u_{j_1,k}| |u_{j_2,k}| \leq L_{S_k}^2 (m + 1) \sigma_a^2 \quad (82) \]

for any \( j_1, j_2 \in \mathcal{N}(k) \) such that \( j_1 \neq j_2. \)

By substituting \( (81) \) and \( (82) \) into \( (80), \) we get

\[
\mathbb{E}_{T^{(i,k)}, \eta_k}(\bar{f}_{i,k}^2 | \delta_{i,k} = 1) \leq \left(q_t^{-1}m + 1\right) \sigma_\eta^2 \]

\[
+ \left(1 + \left(q_t^{-1} + 2\right)(m + m^2) + m^3\right) L_{S_k}^2 \sigma_a^2. \quad (83) \]

Meanwhile, we have \( L^2 = \mathbb{E}_k(L_{S_k}^2) < \infty \) by Assumption \( 5. \) In both cases where the random variable \( m = \sum_{j \in \mathcal{N}(k) \setminus \{i\}} \delta_{j,k} \) follows a binomial distribution or Poisson distribution, it is easy to
show that \( \mathbb{E}(m) = (N - 1)q_a \leq \lambda \), \( \mathbb{E}(m^2) \leq \lambda^2 + \lambda \), and \( \mathbb{E}(m^3) \leq \lambda^3 + 3\lambda^2 + \lambda \). Thus we can further take the expectation of both sides of (83) on \( S_k \) and \( \delta_k \) to get

\[
\mathbb{E}_{S_k, \eta, \delta}(\tilde{f}_{i,k}^2 | \delta_i, k = 1) \leq (1 + q^{-1}_a) \sigma_{\eta}^2 + (1 + \lambda^3) L^2 \sigma_{a}^2 \\
+ ((5 + 2q^{-1}_a) \lambda + (5 + q^{-1}_a) \lambda^2) L^2 \sigma_{a}^2 = \sigma_{\Phi}^{-2} C',
\]

with \( C' \) defined in Lemma [6].

Finally, by substituting (84) into (79), we get

\[
\mathbb{E}_F(\tilde{g}_{i,k}^2) \leq q_a \mathbb{E}_{\Phi, \tilde{\ell}}(\beta_{1+\tilde{\ell}, k}^2 \Phi_{i,k}^2 \sigma_{\Phi}^{-2} C') \\
= C' q_a \mathbb{E}_{\Phi}(\beta_{1+\tilde{\ell}, k}^2 \sigma_{\Phi}^{-2} \Phi_{i,k}^2) = C' \beta_{k},
\]

note that \( \mathbb{E}_{\Phi}(\Phi_{i,k}^2) = \sigma_{\Phi}^2 \) and \( \beta_{k} = \mathbb{E}(\delta_{i,k} \beta_{i,k}^2) = \mathbb{E}(\delta_{i,k} = 1) \mathbb{E}_{\Phi}(\beta_{i,k}^2) \). In the end, Lemma [6] can be proved since \( \mathbb{E}_F(\|g_k\|^2) = \sum_{i=1}^{N} \mathbb{E}_F(\tilde{g}_{i,k}^2) \leq NC' \beta_{k}^2.

F. Proof of Proposition [5]

By definition, \( \delta_{i,k}, \ell_{i,k} \) takes binary value, we can evaluate

\[
\mathbb{E}(\delta_{i,k}, \ell_{i,k}) = \mathbb{P}(\delta_{i,k} = 1, \ell_{i,k} < K_0 - 1) = \mathbb{P}(\delta_{i,k} = 1) \\
\times \mathbb{P}(\ell_{i,k} < K_0 - 1) \overset{(a)}{\leq} q_a \exp \left( -\frac{(q_a (k - 1) - (K_0 - 1))^2}{2q_a (k - 1)} \right) \\
\leq q_a \exp \left( -\frac{q_a}{2} (k - 1) + K_0 - 1 \right),
\]

where \( (a) \) is by Chernoff’s bound, note that \( \ell_{i,k} \sim \mathcal{B}(k - 1, q_a) \). From (86) and the definition of \( \Delta_k \) in Proposition [2] we get

\[
\mathbb{E}(\Delta_k) \leq \alpha_{\Phi}^2 \gamma_0^2 q_a \exp \left( -\frac{q_a}{2} (k - 1) + K_0 - 1 \right).
\]

Meanwhile, we obtain \( \beta_{k-1} \geq q_a (q_a (k - 2) + 1)^{-2c_2} \) using similar steps as (72). We have

\[
\lim_{k \to \infty} \frac{\exp \left( -\frac{q_a}{2} (k - 1) + K_0 - 1 \right)}{(q_a (k - 2) + 1)^{-2c_2}} = 0,
\]

meaning that the upper bound of \( \mathbb{E}(\Delta_k) \) decreases much faster than the lower bound of \( \beta_{k-1} \). Therefore, there must exist some bounded constants \( K_1 \geq K_0 \) and \( \tilde{C} > 0 \), such that \( \mathbb{E}(\Delta_k) \leq C \beta_{k-1} \), \( \forall K \geq K_1 \).

Denote \( e'_k = \Delta_k - \mathbb{E}(\Delta_k) \), then \( \{\sum_{k=K_1}^{K'} e'_k\} \) for \( k \geq K_1 \) is martingale because of \( \mathbb{E}(e'_k) = 0 \) and the independence of \( \Delta_k \) and \( \Delta_k' \) for any \( k \neq k' \). Obviously, \( 0 \leq \Delta_k \leq \alpha_{\Phi}^2 \gamma_0^2 \), thus \( |e'_k| \leq |\Delta_k| < \infty. \)
We can use Doob’s martingale inequality to prove \(|\sum_{k=K_1}^{\infty} e'_k| < \infty\) a.s., with similar steps as the proof of Proposition 4. In the end, we have
\[
\sum_{k=K_1}^{\infty} \Delta_k = \sum_{k=K_1}^{\infty} \mathbb{E}(\Delta_k) + \sum_{k=K_1}^{\infty} e'_k \leq \sum_{k=K_1}^{\infty} \beta^2_k -1 + \sum_{k=K_1}^{\infty} e'_k < \infty \quad \text{a.s.} 
\]
which \(\sum_{k=1}^{\infty} \beta^2_k < \infty\) by Proposition 3. As \(\Delta_k \geq 0\) by definition, we also have
\[
\left| \sum_{k=K_1}^{\infty} \Delta_k \right| < \infty \quad \text{a.s., which concludes the proof.}
\]

\[G. \text{ Proof sketch of Theorem 2}\]

We perform the summation of (36) from \(k = K_0\) to \(k = K\):
\[
d_{K+1} = d_{K_0} + \frac{2\sigma^2}{\lambda} \sum_{k=K_0}^{K} \beta \gamma_k (a_k - a^*)^T \cdot (\nabla F(a_k) + b_k) \\
+ \frac{1}{N} \sum_{k=K_0}^{K} \|\hat{g}_k\|^2 + \frac{2}{N} \sum_{k=K_0}^{K} (a_k - a^*)^T \cdot e_k + \sum_{k=K_0}^{K} \Delta_{k+1}. 
\]

(89)

According to Lemma 6,
\[
\frac{1}{N} \sum_{k=1}^{\infty} \mathbb{E}(\|\hat{g}_k\|^2) \leq C \sum_{k=1}^{\infty} \beta^2_k < \infty, 
\]

as \(\sum_{k=1}^{\infty} \beta^2_k < \infty\) by Proposition 3. We can deduce that
\[
\frac{1}{N} \sum_{k=K_0}^{\infty} \|\hat{g}_k\|^2 < \infty, \quad \text{a.s.} 
\]

(91)

otherwise (90) cannot hold. Besides we also have \(\frac{2}{N} \left| \sum_{k=K_0}^{K} (a_k - a^*)^T \cdot e_k \right| < \infty \) a.s. and \(\left| \sum_{k=K_0}^{K} \Delta_{k+1} \right| < \infty \) a.s. by Propositions 4 and 5.

From Theorem 1 we know that \(|b_{i,k}| \rightarrow 0, \forall i \in \mathcal{N}\). In other words, for an arbitrary small positive value \(\varepsilon\), there exists \(K'\) such that \(\|\nabla F(a_k) + b_k\| \geq (1 - \varepsilon) \|\nabla F(a_k)\|\). By the concavity of \(F\), we have \((a_k - a^*)^T \cdot \nabla F(a_k) \leq 0\), thus
\[
\frac{2\sigma^2}{\lambda} \sum_{k=1}^{K} \beta \gamma_k (a_k - a^*)^T \cdot (\nabla F(a_k) + b_k) \\
\leq \frac{2\sigma^2}{\lambda} (1 - \varepsilon) \sum_{k=0}^{K} \beta \gamma_k (a_k - a^*)^T \cdot \nabla F(a_k). 
\]

(92)
The following steps of the proof is the same to the classical proof in [37]. The basic idea is that, if $a_k$ does not converge to $a^*$, then due to $\sum_{k=0}^{\infty} \beta \gamma_k \to \infty$, we have

$$\sum_{k=0}^{\infty} \beta \gamma_k (a_k - a^*)^T \cdot \nabla F(a_k) < -\infty,$$

which leads to $\lim_{K \to \infty} d_{K+1} < -\infty$ by the above equations (90), (92), and Proposition 4. However $d_{K+1}$ should be positive by definition. Therefore, there should be $\lim_{k \to \infty} \nabla F(a_k) = 0$ and $\lim_{k \to \infty} a_k = a^*$ a.s., which concludes the proof.

H. Proof of Lemma [7]

The relation between $d_{k+1}$ and $d_k$ has been presented in (36). In this proof, we aim to deduce an upper bound of $D_{k+1} = \mathbb{E}(d_{k+1})$, which should be a function of $D_k = \mathbb{E}(d_k)$. By performing the expectation on all the random terms of (36), we have

$$D_{k+1} \leq D_k + \mathbb{E} \left( \frac{1}{N} \| \hat{g}_k \|^2 + \frac{2}{N} (a_k - a^*)^T \cdot e_k + \Delta_{k+1} \right) + \frac{2 \sigma^2}{\lambda} \beta \gamma_k \mathbb{E} \left( (a_k - a^*)^T \cdot (\nabla F(a_k) + b_k) \right).$$

(94)

Since $\mathbb{E}(e_k) = 0$, $\mathbb{E}(\Delta_{k+1}) \leq C \beta^2_k$ and the upper bound of $\mathbb{E}(\| \hat{g}_k \|^2)$ has been given by Lemma [6] we get

$$\mathbb{E} \left( \frac{1}{N} \| \hat{g}_k \|^2 + \frac{2}{N} (a_k - a^*)^T \cdot e_k + \Delta_{k+1} \right) \leq C \beta^2_k,$$

(95)

with $C = C' + \tilde{C}$. We then need to bound the last term $\mathbb{E}((a_k - a^*)^T \cdot (\nabla F(a_k) + b_k))$. With the bound of $|b_{i,k}|$, we have

$$(a_k - a^*)^T \cdot b_k$$

$$\leq \sum_{i=1}^{N} |a_{i,k} - a_i^*| |b_{i,k}| \leq \frac{3 \alpha_G}{2 \sigma^2} \sum_{i=1}^{N} |a_{i,k} - a_i^*| w_{i,k}$$

$$\leq \frac{3 \alpha_G q_k \psi_k}{2 \sigma^2 \beta \gamma_k} \sqrt{N \sum_{i=1}^{N} (a_{i,k} - a_i^*)^2} = \frac{3 \alpha_G \lambda \psi_k \sqrt{d_k}}{2 \sigma^2 \beta \gamma_k}.$$  

(96)

Note that we use (64) to bound $w_{i,k}$, i.e., $w_{i,k} \leq q_k \beta^{-1} \gamma_k \psi_k$ with $\psi_k$ defined in (47). Since $\mathbb{E}(\sqrt{d_k}) \leq \sqrt{\mathbb{E}(d_k)} = \sqrt{D_k}$, by taking expectation on both sides of (96), we get

$$\mathbb{E}((a_k - a^*)^T \cdot b_k) < \frac{3 \alpha_G \lambda \psi_k}{2 \sigma^2 \beta \gamma_k} \sqrt{D_k}.$$  

(97)
Meanwhile, according to Assumption 7, we have

$$E\left( (a_k - a^*)^T \cdot \nabla F(a_k) \right) \leq -\alpha_F \cdot ND_k$$  \hspace{1cm} (98)

Substituting (95), (97) and (98) into (94), we get

$$D_{k+1} \leq (1 - 2\sigma^2 \alpha_F q_k^{-1} \beta \gamma_k) D_k + \alpha_G a_k^3 \psi_k \sqrt{D_k} + C\beta^2,$$

which concludes the proof.

I. Proof of Theorem 3

We present the proof of (50) and (52) in Appendix 11 and in Appendix 12 respectively.

1) Proof of (50): The proof realized by induction. First of all, we can easily get $D_{K_0} \leq \vartheta^2 \psi^2 \theta_{K_0}^{-2}$ by definition of $\vartheta$. The main problem is to verify whether $D_{k+1} \leq \vartheta^2 \psi^2 \theta_{k+1}^{-2}$ can be obtained from $D_k \leq \vartheta^2 \psi^2 \theta_k^{-2}$, $\forall k \geq K_0$.

Suppose that $D_k \leq \vartheta^2 \psi^2 \theta_k^{-2}$ is true, then by (45) we have

$$D_{k+1} \leq (1 - A\theta_k) \frac{\psi_k^2}{\theta_k^2} \vartheta^2 + B\frac{\psi_k^2}{\theta_k} \vartheta + C\psi_k = 0,$$

as $1 - A\theta_k \geq 0$, $\forall k \geq K_0$. The problem turns to prove the existence of a constant $\vartheta \in \mathbb{R}^+$ that ensures

$$D_{k+1} \leq (1 - A\theta_k) \frac{\psi_k^2}{\theta_k^2} \vartheta^2 + B\frac{\psi_k^2}{\theta_k} \vartheta + C\psi_k \leq \vartheta^2 \frac{\psi_{k+1}^2}{\theta_{k+1}^2},$$

which can be rewrite as

$$(A - \chi_k) \vartheta^2 - B\vartheta - C\psi_k \theta_k \psi_k^{-2} \geq 0,$$

in which $\chi_k = (1 - \frac{\psi_{k+1}^2 \theta_k}{\psi_k^2 \theta_k^{-2}}) \theta_k^{-1}$ as defined in (48). By solving (101), we obtain

$$\vartheta \geq \overline{\vartheta}_k = \frac{B}{2(A - \chi_k)} + \sqrt{\left(\frac{B}{2(A - \chi_k)}\right)^2 + C\psi_k \theta_k \psi_k^{-2}} / (A - \chi_k),$$

as $A - \chi_k > 0$ by assumption and $\vartheta > 0$. The last step is to find an upper bound of $\overline{\vartheta}_k$ that is independent of $k$. By definition (48), we have $\epsilon_1 \geq \chi_k$ and $\epsilon_2 \geq \psi_k \theta_k \psi_k^{-2}$, $\forall k \geq K_0$. According to the monotonicity of $\overline{\vartheta}_k$ w.r.t. $\chi_k$ and $\psi_k \theta_k \psi_k^{-2}$, we have the upper bound of $\overline{\vartheta}_k$, $\forall k \geq K_0$,

$$\overline{\vartheta}_k \leq \frac{B}{2(A - \epsilon_1)} + \sqrt{\frac{B^2}{4(A - \epsilon_1)^2} + \frac{C\epsilon_2}{A - \epsilon_1}}.$$

(102)

We can conclude that, $D_k \leq \vartheta^2 \psi^2 \theta_k$ leads to $D_{k+1} \leq \vartheta^2 \psi^2 \theta_{k+1}^{-2}$ if $\vartheta$ satisfies (51), i.e.,

$$\vartheta \geq \sup_{k \geq K_0} \overline{\vartheta}_k = \frac{B + \sqrt{B^2 + 4C\epsilon_2(A - \epsilon_1)}}{2(A - \epsilon_1)}.$$

(103)
2) Proof of (52): Similar steps can be used to prove (52). First, we have $D_{K_0} \leq \varrho^2 v_{K_0} \theta_{K_0}^{-1}$ by definition of $\varrho$. Then for any $k \geq K_0$, we should show that $D_k \leq \varrho^2 v_k \theta_k^{-1}$ leads to $D_{k+1} \leq \varrho^2 v_{k+1} \theta_{k+1}^{-1}$.

Suppose that $D_k \leq \varrho^2 v_k \theta_k^{-1}$ is true, from (45), we have

$$D_{k+1} \leq (1 - A\theta_k) \frac{v_k}{\theta_k} \varrho^2 + B\psi_k \sqrt{\frac{v_k}{\theta_k}} \varrho + C v_k.$$  

(104)

To show $D_{k+1} \leq \varrho^2 v_{k+1} \theta_{k+1}^{-1}$, the following has to be true:

$$(1 - A\theta_k) \frac{v_k}{\theta_k} \varrho^2 + B\psi_k \sqrt{\frac{v_k}{\theta_k}} \varrho + C v_k \leq \varrho^2 \frac{v_{k+1}}{\theta_{k+1}}.$$  

(105)

We rewrite (105) as

$$(A - \varpi_k) \varrho^2 - B \frac{\psi_k}{\sqrt{\theta_k v_k}} \varrho - C \geq 0,$$  

(106)

where by definition (49), $\varpi_k = \left(1 - \frac{v_{k+1} \theta_{k+1}^{-1}}{v_k \theta_k^{-1}}\right) \theta_k^{-1}$. As $A - \varpi_k \geq 0$ and $\varrho > 0$, (106) can be solved, i.e.,

$$\varrho \geq \overline{\varrho}_k = \frac{B \psi_k}{\sqrt{\theta_k v_k}} \sqrt{\left( \frac{B \psi_k}{\sqrt{\theta_k v_k}} \right)^2 + 4C (A - \varpi_k)} \quad \frac{2 (A - \varpi_k)}{2 (A - \varpi_k)}.$$

Consider $\epsilon_3$ and $\epsilon_4$ given in (49), i.e., $\epsilon_3 \geq \varpi_k$ and $\epsilon_4 \geq \frac{\psi_k^2}{\theta_k v_k}$, $\forall k \geq K_0$. We can derive the upper bound of $\overline{\varrho}_k$,

$$\overline{\varrho}_k \leq \frac{B \sqrt{\epsilon_4} + \sqrt{B^2 \epsilon_4 + 4C (A - \epsilon_3)}}{2 (A - \epsilon_3)} \quad \forall k \geq K_0.$$  

(107)

Therefore, if $\varrho$ satisfies (53), i.e.,

$$\varrho \geq \sup_{k \geq K_0} \overline{\varrho}_k = \frac{B \sqrt{\epsilon_4} + \sqrt{B^2 \epsilon_4 + 4C (A - \epsilon_3)}}{2 (A - \epsilon_3)},$$  

(108)

then (105) is true and $D_{k+1} \leq \varrho^2 v_{k+1} \theta_{k+1}^{-1}$ holds, which concludes the proof.

J. Proof of Lemma 2

Recall that $p_{k,\xi} = e^{-\frac{\xi}{2} q_a (k-1)}$, $\overline{\gamma}_l = q (k - 1) + 1$, and $\overline{k}_\xi = \left[(1 - \xi) q_a (k - 1)\right] + 2$. We use Lemma 5 to get

$$\theta_k < p_{k,\xi} \beta_1 \gamma_1 + \beta_\xi \gamma_\xi, v_k < q_a \left( p_{k,\xi} \beta_1^2 + \beta_\xi^2 \right),$$

$$\psi_k < (3 \lambda^2 + 6 \lambda q_a + q_a) \beta_1 \gamma_1^2 p_{k,\xi} + (\lambda^2 + 2 \lambda q_a + q_a) \beta_\xi \gamma_\xi^2.$$  

(109)
The exponential term \( p_{k, \xi} \) decreases much faster than \( \beta_{k_\xi} \gamma_{k_\xi}, \beta_{k_\xi}^2 \gamma_{k_\xi}^2 \) and \( \beta_{k_\xi}^3 \). Thus, for any \( \xi' > 0 \), there exists \( K' \) such that \( \forall k \geq K' \), one has \( \theta_k < (1 + \xi') \beta_{k_\xi} \gamma_{k_\xi}, v_k < (1 + \xi') q_a \beta_{k_\xi}^2 \), and \( \psi_k < (1 + \xi') (\lambda + 1)^2 \beta_{k_\xi}^2 \gamma_{k_\xi}^2 \) from (109). Meanwhile, similar to (72), by Jensen’s inequality, we have

\[
\theta_k \geq \beta_{k_\xi}^2 \gamma_{k_\xi}; \quad v_k \geq q_a \beta_{k_\xi}^2; \quad \psi_k > \chi^2 \beta_{k_\xi}^2.
\]

Hence, \( v_k \theta_k^{-1} \) can be bounded as follows:

\[
\frac{v_k}{\theta_k} \begin{cases} (a) &< (1 + \xi') q_a \beta_0 \gamma_0 \left( \frac{(1 - \xi) q_a (k - 1) + 1}{q_a (k - 1) + 1} \right)^{-c_1-c_2} \\
\text{and of } & k, \xi, \end{cases}
\]

\[
\times \left( \frac{(1 - \xi) q_a (k - 1) + 1}{k - 1 + 1} \right)^{-c_1+c_2} k^{-c_1+c_2},
\]

\[
\begin{cases} (b) &< (1 + \xi') \beta_0 \gamma_0^{-1} (1 - \xi)^{-2c_1} q_a^{1-c_1+c_2} k^{-c_1+c_2}.
\end{cases}
\]

where \( (a) \) is by \( x - 1 < |x| \leq x, \forall x \geq 0 \) and \( (b) \) is by using \( (\frac{ux+1}{x+1})^{-z} < y^{-z}, \forall 0 < y < 1 \) and \( \forall x, z \in \mathbb{R}^+ \), so that

\[
\left( \frac{(1 - \xi) q_a (k - 1) + 1}{q_a (k - 1) + 1} \right)^{-c_1-c_2} \left( \frac{(1 - \xi) q_a (k - 1) + 1}{k - 1 + 1} \right)^{-c_1+c_2} < (1 - \xi)^{-c_1-c_2} ((1 - \xi) q_a)^{-c_1+c_2} = (1 - \xi)^{-2c_1} q_a^{-c_1+c_2}.
\]

Using similar steps, (55)-(56) can be proved as well.

**K. Proof of Lemma 3**

In order to prove Lemma 3, we mainly need to show that \( \lim_{k \to \infty} \beta_0 \gamma_0 \varpi_k < +\infty \) and \( \lim_{k \to \infty} \beta_0 \gamma_0 \chi_k < +\infty \), as both numerators and denominators of \( \varpi_k \) and \( \chi_k \) are bounded and vanishing. We present a basic lemma in Appendix K1, then the convergence of the upper bounds of \( \varpi_k \) and of \( \chi_k \) are investigated in Appendix K2 and in Appendix K2, respectively.

1) A useful lemma: We mainly prove the following lemma:

**Lemma 7. Consider a sequence \( z_\ell = \ell^{-c} \) with \( c > 0 \). Define \( \bar{z}_k = \sum_{\ell=0}^{k-1} z_{\ell+1} q_{\ell+1}^c (1 - q_a)^{k-\ell-1} (\frac{k-1}{\ell}) \) and

\[
\bar{z}_k = \sum_{\ell=0}^{k-1} \frac{z_{\ell+1}}{\ell+1} q_{\ell+1}^c (1 - q_a)^{k-\ell-1} \left( \binom{k-1}{\ell} \right),
\]

then

\[
\bar{z}_k - \bar{z}_{k+1} < c q_a z'_k.
\]
Proof: We rewrite $\mathcal{z}_{k+1}$ as follows

$$\mathcal{z}_{k+1} = (a) z_1 q_a (1 - q_a)^k + \sum_{\ell=1}^{k-1} z_{\ell+1} q_\ell q_a^{\ell+1} (1 - q_a)^{k-\ell} \left( \frac{k-1}{\ell} \right)$$

$$+ \sum_{\ell=1}^{k-1} z_{\ell+1} q_\ell q_a^{\ell+1} (1 - q_a)^{k-\ell} \left( \frac{k-1}{\ell-1} \right) + z_{k+1} q_a^{k+1}$$

$$= \sum_{\ell=0}^{k-1} ((1 - q_a) z_{\ell+1} + q_\ell z_{\ell+2} q_a^{\ell+1} (1 - q_a)^{k-1-\ell} \left( \frac{k-1}{\ell} \right), \quad (114)$$

where $(a)$ is by $\binom{k}{\ell} = \binom{k-1}{\ell} + \binom{k-1}{\ell-1}$. From $(114)$, we have

$$\mathcal{z}_k - \mathcal{z}_{k+1} = (b) \sum_{\ell=0}^{k-1} \left( z_{\ell+1} - z_{\ell+2} \right) q_a^{\ell+1} (1 - q_a)^{k-1-\ell} \left( \frac{k-1}{\ell} \right)$$

$$< c q_a \sum_{\ell=0}^{k-1} \left( \frac{z_{\ell+1} - z_{\ell+2}}{z_{\ell+1}^{\ell+1}} \right) = c q_a \mathcal{z}_k, \quad (115)$$

where $(b)$ is by $\frac{z_{\ell+1} - z_{\ell+2}}{z_{\ell+1}^{\ell+1}} = 1 - \left( 1 + \frac{1}{\ell+1} \right) < \frac{c}{\ell+1}$, such bound is tight when $\ell$ is large, as $\lim_{x \to 0} \frac{1-(1+\epsilon) \ln x}{\ln x} = 1$. It is worth mentioning that $\mathcal{z}_k - \mathcal{z}_{k+1} > 0$ can be directly proved from $(115)$, since $z_{\ell+1} > z_{\ell+2}, \forall \ell$. Such result is valid for any decreasing sequence.

2) Convergence of $\omega_k$: Applying Lemma (7) and by replacing $z_\ell = \ell^{-c}$ with $\beta_\ell^2 = \beta_0^2 \ell^{-2c_1}$, we have

$$v_k \theta_{k+1} - v_{k+1} \theta_k < (v_k - v_{k+1}) \theta_k$$

$$< 2 c_1 q_a \beta_0^2 \sum_{\ell=0}^{k-1} \frac{\beta_\ell^2}{\ell+1} q_a^{\ell+2} (1 - q_a)^{k-1-\ell} \left( \frac{k-1}{\ell} \right). \quad (116)$$

We use Lemma 5 to get, for any $0 < \xi < 1$,

$$v_k \theta_{k+1} - v_{k+1} \theta_k < 2 c_1 q_a^2 \beta_0^2 \theta_k (p_{k,\xi} + ((1 - \xi) q_a k)^{-2c_1-1}),$$

from which and $(110)$ we can deduce

$$\omega_k < \frac{2 c_1 q_a^2 \beta_0^2 \theta_k (p_{k,\xi} + ((1 - \xi) q_a k)^{-2c_1-1})}{q_a \beta_0^2 (q_a (k - 1) + 1)^{-2c_1} \beta_0 \gamma_0 (q_a k + 1)^{-c_1-c_2}}$$

$$< \frac{2 c_1 q_a (p_{k,\xi} + ((1 - \xi) q_a k)^{-2c_1-1}) \beta_0 \gamma_0 (q_a k + 1)^{-3c_1-c_2}}{\beta_0 \gamma_0 (q_a k + 1)^{-3c_1-c_2}} = \omega_k^+. \quad (117)$$

We have $\lim_{k \to \infty} p_{k,\xi} (q_a k + 1)^{3c_1+c_2} = 0$ and

$$\lim_{k \to \infty} ((1 - \xi) q_a k)^{-2c_1-1} = \begin{cases} 0 & \text{if } c_1 + c_2 < 1 \\ (1 - \xi)^{-2c_1-1} & \text{if } c_1 + c_2 = 1 \end{cases}$$
Hence $\lim_{k \to \infty} \beta_0 \gamma_0 \beta_k \leq 2c_1 q_a (1 - \xi)^{-2c_1 - 1}$ as $c_1 + c_2 \leq 1$. We can deduce that $\beta_0 \gamma_0 \beta_k$ is bounded and $\beta_k < A$ can be true $\forall k$, as long as the value of $\beta_0 \gamma_0$ is large enough.

3) Convergence of $\chi_k$: We can use similar steps to show that $\beta_0 \gamma_0 \chi_k$ is bounded. We need to evaluate

$$\psi_k - \psi_{k+1} < \beta' \gamma^2_k - \beta' \gamma^2_{k+1} + 2N \left( \beta' \gamma_k - \beta' \gamma_{k+1} \right) + (N - 1)^2 \left( \beta' \gamma_k - \beta' \gamma_{k+1} \right)$$

where $(a)$ is obtained by applying Lemma 7, the terms $\beta' \gamma^2_k, \beta' \gamma_{k+1}$ and $\tau_k'$ are defined in the same way as $\alpha_k'$ in (112). We can use Lemma 5 to show (b). Note that the explicit expression of the upper bound is quite long, we introduce a bounded constant $C''$ instead. The bound (118) is reasonably tight as $p_{k, \xi} = \exp \left( -\frac{1}{2} \xi q_a k \right)$ is negligible before $k^{-c_1 - 2c_2 - 1}$ when $k$ goes large.

Based on (118) and (110), we can get

$$\frac{\chi_k}{1 + \psi_{k+1} / \psi_k \theta_{k+1}} = \frac{\psi_k \theta_{k+1} - \psi_{k+1} \theta_k}{\psi_k \theta_{k+1} \theta_k} < \frac{(\psi_k - \psi_{k+1}) \theta_k}{\psi_k \theta_{k+1} \theta_k}$$

where $\frac{\psi_k \theta_{k+1}}{\psi_k \theta_{k+1}} = 2$ and

$$\lim_{k \to \infty} \chi_k^+ = \begin{cases} 0 & \text{if } c_1 + c_2 < 1, \\ \frac{(c_1 + 2c_2) q_a (1 - \xi)^{-c_1 - 2c_2 - 1}}{1 + 2q_a \lambda^{-2}} & \text{if } c_1 + c_2 = 1, \end{cases}$$

we can conclude that $\lim_{k \to \infty} \beta_0 \gamma_0 \chi_k$ is bounded. Therefore $\chi_k < A$ can be true when $\beta_0 \gamma_0$ is large enough.

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