ASYMPTOTICS FOR VENTTSSEL’ PROBLEMS FOR OPERATORS IN NON DIVERGENCE FORM IN IRREGULAR DOMAINS

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Abstract. We study a Venttsel’ problem in a three dimensional fractal domain for an operator in non divergence form. We prove existence, uniqueness and regularity results of the strict solution for both the fractal and prefractal problem, via a semigroup approach. In view of numerical approximations, we study the asymptotic behaviour of the solutions of the prefractal problems and we prove that the prefractal solutions converge in the Mosco-Kuwae-Shioya sense to the (limit) solution of the fractal one.

1. Introduction. In this paper we study a parabolic Venttsel’ problem in a three-dimensional fractal domain for an operator in non divergence form. We prove existence, uniqueness and regularity results of the strict solution for both the fractal and prefractal problem, via a semigroup approach. In view of numerical approximations, we study the asymptotic behaviour of the solutions of the prefractal problems and we prove that the prefractal solutions converge in the Mosco-Kuwae-Shioya sense to the (limit) solution of the fractal one.

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The literature on Venttsel’ problems in regular domains is large, we refer to [14] and the references listed in, as to Venttsel problems in fractal domains the first results, to our knowledge, can be found in [32], where a second order operator in divergence form in a two dimensional fractal domain is considered (see also [9] and [10] for the numerical approximation). It is to be pointed out that the extension to the three dimensional case will require different tools to study the asymptotic behaviour of the solutions.

More precisely, we consider the following boundary value problem for a second order operator in non divergence form with Venttsel’s boundary conditions:

\[
(P) \begin{cases}
    u_t(t,P) - Lu(t,P) = f(t,P) & \text{in } [0,T] \times Q \\
    u_t(t,P) - \Delta_S u(t,P) + b(P)u(t,P) = -\frac{\partial u}{\partial n_A} + f(t,P) & \text{on } [0,T] \times S \\
    \frac{\partial u}{\partial n_A} = 0 & \text{on } [0,T] \times \partial Q \setminus S \\
    u(0,P) = 0 & \text{in } Q,
\end{cases}
\]

here \( L \) is an operator in non divergence form, \( Lu = \sum_{i,j=1}^3 a_{ij} \partial_{ij} u + a_0 u, \ a_{ij} \) are symmetric, uniformly Lipschitz functions in \( Q \) satisfying suitable ellipticity conditions (see condition \((H^1)\) in Section 4) and \( a_0 \) is a positive \( L^\infty(Q) \) function, \( Q \) is the three-dimensional domain with lateral boundary \( Q = F \times [0,1], \) where \( F \) is the Koch snowflake; \( \Delta_S \) is the fractal Laplacian on \( S \) (see Theorem 4.5 in Section 4), \( b \) is a continuous strictly positive function on \( S, \ \frac{\partial u}{\partial n_A} \) is the co-normal derivative across \( \partial Q \) to be defined in a suitable sense (see Theorem 8.1), \( f(t,P) \) is a given function in \( C^\theta([0,T];L^2(Q,m)) \), \( \theta \in (0,1) \) and \( m \) is the sum of the three-dimensional Lebesgue measure and of a suitable measure \( g \) supported on \( S \) (see Section 2).

From the point of view of numerical analysis it is also crucial to study the corresponding approximating (prefractal) problems \((P_h)\). To this aim the asymptotic behaviour, as \( h \to \infty, \) of the approximating solutions is studied. More precisely, we consider for each \( h \in \mathbb{N}, \) the prefractal problems

\[
(P_h) \begin{cases}
    (u_h)_t(t,P) - L_h u_h(t,P) = f_h(t,P) & \text{in } [0,T] \times Q_h \\
    \delta_h(u_h)_t(t,P) - \Delta_{S_h} u_h(t,P) + \delta_h b(P) u_h(t,P) = & \text{in } [0,T] \times S_h \\
    -\frac{\partial u_h}{\partial n_h} + \delta_h f_h(t,P) & \text{on } [0,T] \times \partial Q_h \setminus S_h \\
    \frac{\partial u_h}{\partial n_A} = 0 & \text{on } [0,T] \times \partial Q_h \setminus S_h \\
    u_h(0,P) = 0 & \text{in } Q_h
\end{cases}
\]

where \( L_h u = \sum_{i,j=1}^3 a_{ij}^h \partial_{ij} u + a_0^h u, \ a_{ij}^h \) are uniformly Lipschitz functions in \( Q, \) satisfying suitable ellipticity conditions (see condition \((H_h)\) in Section 4), \( a_{ij}^h \) are positive \( L^\infty(Q) \) functions, \( Q_h \) are (invading) domains approximating \( Q, \) that is for all \( h \in \mathbb{N} \ Q_h \subset Q_{h+1}, \) and \( Q_h \to Q, h \to +\infty. \ S_h = F_h \times [0,1] \) are the corresponding approximating polyhedral surfaces, where \( F_h \) is a prefractal curve approximating \( F \) (see Section 2); \( \Delta_{S_h} \) is the piecewise tangential Laplacian defined on \( S_h, \ \frac{\partial u_h}{\partial n_A} \) is the co-normal derivative across \( \partial Q_h \) to be defined in a suitable sense (see Theorem 8.2), \( f_h(t,P) \) is a given function in \( C^\theta([0,T];L^2(Q,m_h)), \ \theta \in (0,1); \ m_h \) is the sum of the three-dimensional Lebesgue measure and of the surface measure \( \delta_h \sigma \) of \( S_h, \) where \( \delta_h \) is a positive constant (see Section 4).

The natural functional setting for these problems is \( L^2(Q,m) \) and \( L^2(Q,m_h), \) respectively because of the presence of the time derivative in the boundary conditions. In this setting, existence and uniqueness results are proved via a semigroup approach, in Section 7 we consider the abstract Cauchy problems \((P)\) and \((P_h)\),
$h \in \mathbb{N}$ and we prove the existence and uniqueness of the strict solutions for $(P)$ and $(P_h)$, for every $h \in \mathbb{N}$, (see Theorem 7.1 and Theorem 7.2).

The study of the asymptotic behaviour of the solutions of problems $(P_h)$, as $h \to \infty$ brings us to the framework of varying Hilbert spaces $L^2(Q,m_h)$ . The convergence of the solutions of problems $(P_h)$ to problem $(P)$ is thus obtained via the Mosco-Kuwae-Shioya convergence of the approximating energy forms $E^{(h)}$ (see [39] and [26]) in varying Hilbert spaces. Actually the convergence of energies implies the convergence of semigroups in a suitable sense, (see Section 7). Finally, in Theorems 8.1 and 8.2 we prove that problems $(P)$ and $(P_h)$ are the strong formulations of the abstract problems $(P_h)$ and $(P)$, respectively.

It is to be pointed out that the proof of the convergence of the energies in the three dimensional case, with respect to the two-dimensional one dealt with in [32], is a delicate issue and it relies on new density results, for functions in the domain of the energy form $E$ related to $(P)$, recently proved in ([29]).

We point out that from our asymptotic analysis it turns out that the solution of the limit fractal problem $(P)$ can be indeed approximated by the prefractal solutions of the corresponding approximating prefractal problems $(P_h)$ (see Theorems 7.4 and 7.5). It is still an open problem, even in the bidimensional case and when the operator is the Laplacian, to obtain a quantitative estimate of the rate of convergence in terms of $h$. Nevertheless our results are a first step towards the study of the numerical approximation of the prefractal problem $(P_h)$ for every fixed $h$.

By proceeding as in [9] and [10] one can perform an a priori error estimate of the numerical approximation by a finite element scheme in space and a finite difference scheme in time. We stress the fact that the domains $Q_h$ are not convex polyhedral domains hence $H^2$ regularity of the solution $u_h$ is deteriorated by the presence of reentrant wedges and vertices. A delicate regularity analysis of the solution $u_h$ is necessary in order to obtain an optimal rate of convergence. This will object of a forthcoming paper.

The outline of the paper is the following: in Section 2 we introduce the geometry of the Koch snowflake and the domains $Q$ and $Q_h$; in Section 3 we introduce the main functional spaces; in Section 4 we define the prefractal and fractal energy forms; in Section 5 we introduce the notion of convergence in varying Hilbert spaces; in Section 6 we recall the density Theorems for functions in the domain of the energy form $E$, we give the definition of Mosco-Kuwae-Shioya convergence of energy forms and we prove the convergence of the approximating energies $E^{(h)}$ to $E$ (see Theorem 6.8); in Section 7 we consider the abstract Cauchy problems in both the fractal and prefractal case and we state the existence and uniqueness Theorem for $(P_h)$ and $(P)$ respectively; we then prove that the solutions of problems $(P_h)$ do converge to the solution of the problem $(P)$ in a suitable sense (see Theorem 7.4); in Section 8 we prove that the solutions of the abstract problems $(P_h)$ and $(P)$ solve in a suitable strong sense problems $(P_h)$ and $(P)$ respectively.

2. Geometry of $Q$, $S$ and $S_h$. We denote by $|P - P_0|$ the Euclidean distance in $\mathbb{R}^n$ and the Euclidean balls by $B(P_0,r) = \{ P \in \mathbb{R}^n : |P - P_0| < r \}$, $P_0 \in \mathbb{R}^n$, $r > 0$.

By the snowflake $F$ we denote the union of three complanar Koch curves $K_i$ (see [12]).

We assume that the junction points $A_1, A_3, A_5$ are the vertices of a regular triangle with unit side length, that is $|A_1 - A_3| = |A_1 - A_5| = |A_3 - A_5| = 1$. One can
define, in a natural way, a finite Borel measure $\mu_F$ supported on $F$ by

$$
\mu_F := \mu_1 + \mu_2 + \mu_3
$$

where $\mu_i$ denotes the normalized $d_f$-dimensional Hausdorff measure, restricted to $K_i$, $i = 1, 2, 3$.

The measure $\mu_F$ is a $d$-measure (see Definition 3.1), that is there exist two positive constants $c_1, c_2$

$$
c_1 r^d \leq \mu_F(B(P,r) \cap F) \leq c_2 r^d, \forall P \in F
$$

where $d = d_f = \frac{\log 4}{\log 3}$ and where $B(P,r)$ denotes the Euclidean ball in $\mathbb{R}^2$. $K_1$ is the uniquely determined self-similar set with respect to four suitable contractions $\psi^{(1)}, \ldots, \psi^{(4)}$, with respect to the same ratio $\frac{1}{3}$ (see Section 3.2 in [15]). Let $V^{(1)}_0 := \{A_1, A_3\}$, $V^{(1)}_{j_1 \ldots j_h} := \psi^{(1)}_{j_1} \circ \ldots \circ \psi^{(1)}_{j_h}(V^{(1)}_0)$ and

$$
V^{(1)}_h := \bigcup_{j_1 \ldots j_h = 1}^4 V^{(1)}_{j_1 \ldots j_h}.
$$

We set $V^{(1)}_* := \bigcup_{h \geq 0} V^{(1)}_h$. It holds that $K_1 = \overline{V^{(1)}_*}$. Let $K^{(0)}_1$ denote the unit segment whose endpoints are $A_1$ and $A_3$ and $K^{(1)}_{j_1 \ldots j_h} := \psi^{(1)}_{j_1} \circ \ldots \circ \psi^{(1)}_{j_h}(K^{(0)}_1)$. For $h > 0$ we denote

$$
F^{(1)}_h = \{\psi^{(1)}_{j_1} \circ \ldots \circ \psi^{(1)}_{j_h}(K^{(0)}_1), j_1, \ldots, j_h = 1, \ldots, 4\}.
$$

We set $K^{(1)}_1 = \bigcup_{j=1}^4 \psi^{(1)}_j(K^{(0)}_1)$, $K^{(h+1)}_1 = \bigcup_{M \in K^{(h)}_1} \bigcup_{j=1}^4 \psi^{(1)}_j(M)$, where $M$ denotes a segment of the $h + 1$-th generation; $K^{(h+1)}_1$ the polygonal curve and $V^{(1)}_{h+1}$ the set of its vertices.

In a similar way, it is possible to approximate $K_2, K_3$ by the sequences $(V^{(2)}_h)_{h \geq 0}$, $(V^{(3)}_h)_{h \geq 0}$, and denote their limits by $V^{(2)}_*, V^{(3)}_*$, and the corresponding polygonal curves $K^{(h+1)}_2, K^{(h+1)}_3$.

In order to approximate $F$, we define the increasing sequence of finite sets of points $V_h := \bigcup_{i=1}^3 V^{(i)}_h, h \geq 1$ and $V_* := \bigcup_{h \geq 1} V_h$. It holds that $V_* = \bigcup_{i=1}^3 V^{(i)}_*$ and $F = \overline{V_*}$. In the following we denote by
The prefractal $F_h$ at the step $h = 2$ and $h = 5$

Figure 3. Surface $S_3$

$F_{h+1} = \bigcup_{i=1}^{3} K_i^{(h+1)}$

the closed polygonal curve approximating $F$ at the $(h + 1)$-th step.

By $S_h$ we denote $F_h \times I$, where $F_h$ is the prefractal approximation of $F$ at the step $h$, $I = [0, 1]$. $S_h$ is a surface of polyhedral type. We give a point $P \in S_h$ the Cartesian coordinates $P = (x, x_3)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of $P$ on the plane containing $F_h$ and $x_3$ is the coordinate of the orthogonal projection of $P$ on the $x_3$-line containing the interval $I$.

By $\Omega_h$ we denote the open bounded two-dimensional domain with boundary $F_h$. By $Q_h$ we denote the domain with $S_h$ as lateral surface and $\tilde{\Omega}_h := (\Omega_h \times \{0\}) \cup (\Omega_h \times \{1\})$ as bases of $Q_h$.

The measure on $S_h$ is

$$d\sigma = dl \times dx_3,$$

where $dl$ is the arc-length measure on $F_h$ and $dx_3$ is the one-dimensional Lebesgue measure on $I$. We introduce the fractal surface $S = F \times I$ given by the Cartesian product between $F$ and $I$. It can be defined on $S$ the finite Borel measure

$$dg = d\mu_F \times dx_3$$

supported on $S$. The measure $g$ is a $d$-measure (see Definition 3.1), that is there exist two positive constants $c_1, c_2$

$$c_1 r^d \leq g(B(P,r) \cap S) \leq c_2 r^d, \forall P \in S$$
Remark 3.3. It is known that the limit exists at quasi every point that does not create ambiguity. Let \( \mathcal{L}_3 \) be the Lebesgue measure space of continuous functions on \( T \) equipped with the functional space of infinitely differentiable functions with compact support \( G \) and \( \cdot \) we denote the Sobolev spaces, possibly fractional (see [40]).

Functional spaces. By \( L^2(\cdot) \) we denote the Lebesgue space with respect to the Lebesgue measure \( \mathcal{L}_3 \) on subsets of \( \mathbb{R}^3 \), which will be left to the context whenever that does not create ambiguity. Let \( T \) be a closed set of \( \mathbb{R}^3 \), by \( C(T) \) we denote the space of continuous functions on \( T \) and \( C^{0,\beta}(T) \) is the space of Hölder continuous functions on \( T, 0 < \beta < 1 \). Let \( G \) be an open set of \( \mathbb{R}^3 \), by \( H^s(G) \), \( s \in \mathbb{R}^+ \) we denote the Sobolev spaces, possibly fractional (see [40]). \( D(G) \) is the space of infinitely differentiable functions with compact support on \( G \).

Definition 3.1. Let \( T \subseteq \mathbb{R}^n \) be a non empty closed set. \( T \) is called \( d \)-set \((0 < d \leq n)\) if there exists a Borel measure \( \mu \) with \( supp \mu = T \) such that for some \( c_1, c_2 > 0 \)
\[
c_1 r^d \leq \mu(B(P, r)) \leq c_2 r^d, P \in T, 0 < r \leq 1
\]
\( \mu \) is called \( d \)-measure on \( T \).

3.1. Sobolev spaces.

Definition 3.2. Let \( \mathcal{F} \) be an open subset in \( \mathbb{R}^3 \). If \( f \in H^s(\mathcal{F}) \), we call trace of \( f \)
\[
\gamma_0 f(P) = \lim_{r \to 0} \frac{1}{|B(P, r)|} \int_{B(P, r) \cap \mathcal{F}} f(Q) d\mathcal{L}_3
\]

Remark 3.3. It is known that the limit exists at quasi every \( P \in \mathcal{F} \) with respect to the \((s,2)\)-capacity (see [1]).

Proposition 3.4. Let \( Q_h \) and \( S_h \) be as above.

Let \( \frac{1}{2} < s < \frac{3}{2} \). Then \( H^{s-\frac{1}{2}}(S_h) \) is the trace space to \( S_h \) of \( H^s(Q_h) \) in the following sense:

1. \( \gamma_0 \) is a continuous and linear operator from \( H^s(Q_h) \) to \( H^{s-\frac{1}{2}}(S_h) \),
2. there is a continuous linear operator \( Ext \) from \( H^{s-\frac{1}{2}}(S_h) \) to \( H^s(Q_h) \), such that \( \gamma_0 \circ Ext \) is the identity operator in \( H^{s-\frac{1}{2}}(S_h) \).

From now on we denote by \( u|_{S_h} \) the trace operator, that is \( u|_{S_h} = \gamma_0 u \).

The following Theorem characterizes the trace on the polyhedral set \( S_h \) of a function belonging to the Sobolev space \( H^3(\mathbb{R}^3) \).

Theorem 3.5. Let \( S_h \) be as above. Let \( u \in H^3(\mathbb{R}^3) \) and \( \delta_h = (3^{1-d})^h \). Then for \( \frac{1}{2} \leq \beta \leq 1 \),
\[
\| u|_{S_h} \|_{L^2(S_h)}^2 \leq \frac{C_\beta}{\delta_h} \| u \|_{H^3(\mathbb{R}^3)}^2,
\]
where \( C_\beta \) is independent from \( h \).

Proof. We adapt the proof from the two dimensional case treated in [8]. Any \( u \in H^3(\mathbb{R}^n) \) can be written in terms of Bessel kernels \( G_\beta \), of order \( \beta \), that is \( u = G_\beta * g \), \( g \in L^2(\mathbb{R}^3) \), (see [45]). Then
\[
\| u|_{S_h} \|_{L^2(S_h)}^2 = \int_{S_h} \int_{\mathbb{R}^3} G_\beta(x-y)g(y)dy|d\sigma \leq \int_{S_h} \int_{\mathbb{R}^3} |G_\beta(x-y)|^2|g(y)|^2dy(\int_{\mathbb{R}^3} |G_\beta(x-y)|^{2(1-a)}dy)d\sigma,
\]
where \(0 < a < 1\) will be chosen later. By using the estimates for the Bessel kernels and Lemma 1 on page 104 in [22], we get
\[
\int_{\mathbb{R}^3} |G_\beta(x-y)|^{2(1-a)}dy \leq C_1
\]
if
\[
3 > 2(3 - \beta)(1 - a),
\]
where \(C_1\) is independent of \(h\).

Moreover, since \(S_h\) is a 2—set, with \(c_2 = C\delta_h^{-1}\), (according to Definition 3.1 we get again from Lemma 1 on page 104 in [22])
\[
\int_{S_h} |G_\beta(x-y)|^{2a}d\sigma \leq C_4\delta_h^{-1},
\]
if
\[
2 > 2a(3 - \beta),
\]
where \(C_4\) is independent of \(h\).

By choosing \(a\) in order to satisfy (3.2) and (3.3), we get
\[
\|u\|_{L^2(S_h)}^2 \leq C_1 \int_{S_h} \left( \int_{\mathbb{R}^3} |G_\beta(x-y)|^{2a} |g(y)|^2 dy \right) d\sigma = C_1 \int_{S_h} \left( \int_{\mathbb{R}^3} |G_\beta(x-y)|^{2a} g(y)^2 dy \right) \leq C_1 C_4 \delta_h^{-1} \int_{\mathbb{R}^3} |g(y)|^2 dy = C_1 C_4 \delta_h^{-1} \|u\|_{L^2(\mathbb{R}^3)},
\]
where \(C_\beta = C_1 C_4\) is independent of \(h\).

The following theorem that characterizes the trace on the set \(S\) of a function belonging to Sobolev spaces \(H_\beta(\mathbb{R}^3)\) is a consequence of Theorem 1 in Chapter 5 of [22] as the fractal \(S\) is a \(d\)—set.

**Theorem 3.6.** Let \(u \in H_\beta(\mathbb{R}^3)\). Then, for \(1 - \frac{d_\beta}{2} < \beta\),
\[
\|u\|_{L^2(S)}^2 \leq C_\beta \|u\|_{H_\beta(\mathbb{R}^3)}^2.
\]

It is possible to prove that the domains \(Q_h\) are \((\epsilon, \delta)\) domains with parameter independent of the increasing number of sides of \(S_h\). We recall an extension Theorem for \((\epsilon, \delta)\) domains (see [20]) adapted to the present case:

**Theorem 3.7.** There exists a bounded linear extension operator \(\text{Ext}_J : H^1(Q_h) \to H^1(\mathbb{R}^3)\), such that
\[
\|\text{Ext}_J v\|_{H^1(\mathbb{R}^3)}^2 \leq C_J \|v\|_{H^1(Q_h)}^2,
\]
with \(C_J\) independent of \(h\).

**Theorem 3.8.** There exists a linear extension operator \(\text{Ext} : H^\beta(\mathbb{R}^3) \to H^\beta(\mathbb{R}^3)\), such that, for any \(\beta > 0\),
\[
\|\text{Ext} v\|_{H^\beta(\mathbb{R}^3)} \leq C_\beta \|v\|_{H^\beta(Q)}
\]
with \(C_\beta\) depending on \(\beta\).
3.2. Besov spaces.

Proposition 3.9. The snowflake $F$ is a $d$-set with $d = d_f$, the Hausdorff measure $\mu_F$ is a $d$-measure and $S$ is a $d$-set with $d = d_f + 1$.

Definition 3.10. We say that if $f \in B^{2,2}_2(S)$ if $f \in L^2(S,g)$ and $\|f\|_{B^{2,2}_2(S)} < +\infty$, where

$$\|f\|_{B^{2,2}_2(S)} = \|f\|_{L^2(S,g)} + \left( \int \int_{|P-P'| < 1} \frac{|f(P) - f(P')|^2}{|P-P'|^{2d_f+1}} \, dg(P)dg(P') \right)^\frac{1}{2}.$$  (3.4)

The following trace Theorem is due to Jonsson-Wallin, for the proof see Theorem 1 of Chapter VII in [22]:

Theorem 3.11. $B^{2,2}_2(S)$ is the trace space of $H^1(Q)$ that is:

1. There exists a linear and continuous operator, (trace operator), $\gamma_0 : H^1(Q) \to B^{2,2}_2(S)$.
2. There exists a linear and continuous operator, (extension operator), $\text{Ext} : B^{2,2}_2(S) \to H^1(Q)$ such that

$$\gamma_0 \circ \text{Ext} = \text{Id}_{B^{2,2}_2(S)}$$

From now on we denote by $u|_S$ the trace operator, that is $u|_S = \gamma_0 u$.

4. Energy forms and semigroups.

4.1. The energy forms $E^{(h)}$. Let $Q, S, Q_h$ and $S_h$ be defined as in Section 2. We consider the energy forms $E_{S_h}$ on $S_h = F_h \times I$, $h \in \mathbb{N}$. By $l$ we denote the arc-length coordinate on each edge $E_l$ and we introduce the coordinate $x_1 = x_1(l)$, $x_2 = x_2(l)$, $x_3 = x_3$ on every affine face $S_h^{(j)}$ of $S_h$. By $dl$ we denote the 1-dimensional measure given by the arc-length $l$, and by $d\sigma$ the surface measure on $S_h^{(j)}$, $d\sigma = dldx_3$ and $\delta_h = (3^{1-d_f})^h$. $E_{S_h}$ is defined by

$$E_{S_h}[u] = \sum_j \left( \int_{S_h^{(j)}} \sigma_h^1 |D_l u|^2 + \sigma_h^2 |\partial_3 u|^2 \right) \, d\sigma,$$

where $\sigma_h^1$, $\sigma_h^2$ are positive constants, $D_l$ denotes the tangential derivative along the prefractal $F_h$, and $u \in H^1(S_h)$. By the Fubini theorem, $E_{S_h}$ can be written in the form

$$E_{S_h}[u] = \sigma_h^1 \int_I \left( \int_{F_h} |D_l u|^2 \, dl \right) \, dx_3 + \sigma_h^2 \int_{F_h} \left( \int_I |\partial_3 u|^2 \, dx_3 \right) \, dl.$$  

We denote by $E_{S_h}(u,v)$ the corresponding bilinear form.

Let us consider now the function space

$$V(Q, S_h) = \{ u \in H^1(Q) : u|_{S_h} \in H^1(S_h) \}.$$

Let us consider the energy forms

$$\tilde{E}^{(h)}[u] = \int_Q \chi_{Q_h} L_h u \, ud\mathcal{L}_3 + E_{S_h}[u|_{S_h}] + \delta_h \int_{S_h} b |u|_{S_h}^2 \, d\sigma$$
defined on $V(Q,S_h)$, where $b \in C(\overline{Q})$, $b > 0$, $L_hu = \sum_{i,j=1}^{3} a_{ij}^h \partial_{ij}u + a_0^h u$, is a non



divergence operator, $a_{ij}^h$ are uniformly Lipschitz continuous functions in $\overline{Q}$ and $a_0^h$ are positive essentially bounded functions in $Q$, $\partial_i$ denotes the partial derivative with respect to $x_i$, $i = 1, 2, 3$ and $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$;


\[
(H_1^h) \quad \begin{cases} a_{ij}^h = a_{ji}^h, & \forall i, j = 1, 2, 3 \\
\exists \lambda > 0 : \\
\sum_{i,j=1}^{3} a_{ij}^h \xi_i \xi_j \geq \lambda \sum_{i=1}^{3} |\xi_i|^2 & \forall (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \end{cases}
\]

We recast $L_hu$ as the sum of an operator in divergence form and a lower order operator, by Leibniz formula:

\[
\sum_{i,j=1}^{3} a_{ij}^h \partial_{ij}u + a_0^h u = \sum_{i,j=1}^{3} \partial_i(a_{ij}^h \partial_j u) - \sum_{i=1}^{3} b_i^h \partial_i u + a_0^h u.
\]

where $b_i^h = \sum_{j=1}^{3} \partial_i(a_{ij}^h)$, we note that $b_i^h \in L^\infty(Q)$. Moreover we assume that

\[
(H_2^h) \quad \inf_Q a_0^h - \frac{1}{\lambda} \sum \|b_i^h\|^2_{L^\infty(Q)} > 0
\]

From now on we consider the following energy forms

\[
\tilde{E}^{(h)}[u] = \int_Q \chi_{Q_h} A^h Dv \cdot DuD\mathcal{L}_3 - \sum_{i=1}^{3} \int_Q \chi_{Q_h} b_i^h \partial_i u DvD\mathcal{L}_3 + E_{S_h}[u|S_h] + \delta_h \int_{S_h} b_i^h \partial_i u|_{S_h}^2 d\sigma + \int_Q \chi_{Q_h} a_0^h u|_{S_h}^2 d\mathcal{L}_3,
\]

(4.5)

where $A^h = [a_{ij}^h], i, j = 1, 2, 3$.

The corresponding bilinear form is

\[
\tilde{E}^{(h)}(u, v) = \int_Q \chi_{Q_h} (A^h Dv \cdot Dv + a_0^h uv) DvD\mathcal{L}_3 - \int_Q \chi_{Q_h} \sum_{i=1}^{3} b_i^h \partial_i u v DvD\mathcal{L}_3 + E_{S_h}(u|S_h, v|S_h) + \delta_h \int_{S_h} bu|S_h, v|S_h d\sigma
\]

(4.6)

defined on $V(Q, S_h) \times V(Q, S_h)$.

In the following we consider also the space $L^2(Q, m_h)$, where $m_h$ is the measure with

\[
dm_h = \chi_{Q_h} D\mathcal{L}_3 + \chi_{S_h} \delta_h d\sigma.
\]

(4.7)

**Theorem 4.1.** The form $\tilde{E}^{(h)}$, defined in (4.6) with domain $V(Q, S_h)$, is a bilinear form in $L^2(Q, m_h)$, and the space $V(Q, S_h)$ is a Hilbert space equipped with the scalar product

\[
(u, v)_{V(Q, S_h)} = \int_Q \chi_{Q_h} DuDvD\mathcal{L}_3 + E_{S_h}(u|S_h, v|S_h) + (u, v)_{L^2(Q, m_h)}.
\]

Moreover from assumption $(H_2^h)$ we deduce that $\tilde{E}^{(h)}[u]$ is a positive definite non-symmetric form. We decompose $\tilde{E}^{(h)}(u, v) = E^{(h)}(u, v) - \int_Q \chi_{Q_h} \sum_{i=1}^{3} b_i^h \partial_i u v D\mathcal{L}_3$ where $E^{(h)}(u, v)$ is a symmetric form in $V(Q, S_h)$.

We note that in assumption $(H_1^h)$ the constant $\lambda$ could depend on $h$, in the asymptotic analysis they cannot degenerate (i.e. they cannot tend to zero ) and it has to be assumed that $\inf_h \lambda_h > 0$. 

4.2. The energy form \( E \). By proceeding as in [15] we construct an energy form on \( F \), by defining a Lagrangian measure \( \mathcal{L}_F \) on \( F \), which has the role of the Euclidean Lagrangian \( d\mathcal{L}(u,v) = Du \, Dv \, dx \). The corresponding energy form on \( F \) is given by

\[
E_F(u,v) = \int_F d\mathcal{L}_F(u,v)
\]

with domain \( \mathcal{D}(F) = \{ u \in L^2(F, \mu_F) : E_F[u] < +\infty \} \) dense in \( L^2(F, \mu_F) \).

**Proposition 4.2.** \( \mathcal{D}(F) \) is a Hilbert space equipped with the following norm

\[
\|u\|_{\mathcal{D}(F)} = (\|u\|_{L^2(F)}^2 + E_F[u])^{\frac{1}{2}}.
\]

We now define the energy form on \( S \) and the fractal Laplacian \( \Delta_S \).

\[
E_S[u] = \int_I E_F[u] \, dx + \int_I \int I |\partial_a u|^2 \, dx \, d\mu_F,
\]

The form \( E_S \) is defined for \( u \in \mathcal{D}(S) \),

\[
\mathcal{D}(S) = \mathcal{C}(S) \cap L^2(0,1; \mathcal{D}(F)) \cap H^1(0,1; L^2(F))^{\| \cdot \|_{\mathcal{D}(S)}},
\]

where \( \| \cdot \|_{\mathcal{D}(S)} \) is the intrinsic norm

\[
\|u\|_{\mathcal{D}(S)} = (E_S[u] + \|u\|_{L^2(S,g)}^2)^{\frac{1}{2}}.
\]

**Proposition 4.3.** \( E_S(u,v) \) with domain \( \mathcal{D}(S) \times \mathcal{D}(S) \) is a closed bilinear form in \( L^2(S,g) \) and \( \mathcal{D}(S) \) is a Hilbert space equipped with the intrinsic norm.

We now give an embedding result for the domain \( \mathcal{D}(S) \). Unlike the two-dimensional case where there is a characterization of the functions in \( \mathcal{D}(F) \) in terms of the so-called Lipschitz spaces (see Theorem 3.1 in [33]), we do not have a characterization for \( \mathcal{D}(S) \), but the following result holds:

**Proposition 4.4.** \( \mathcal{D}(S) \subset B^{2,2}_\beta(S) \), for any \( 0 < \beta < 1 \).

The proof of the following Proposition can be found in [29], Proposition 4.4.

From Proposition 4.3 we have

\[
\text{Theorem 4.5. There exists a unique non positive self-adjoint operator } \Delta_S \text{ on } L^2(S,g) \text{ with domain } \mathcal{D}(\Delta_S) := \{ u \in L^2(S,g) : \Delta_S u \in L^2(S,g) \} \subseteq \mathcal{D}(S) \text{ dense in } L^2(S,g) \text{ such that }
\]

\[
E_S(u,v) = -\int_S \Delta_S u \, v \, dg, \text{ for each } u \in \mathcal{D}(\Delta_S), v \in \mathcal{D}(S).
\]

Now we introduce the energy form on \( Q \). Let us consider the space

\[
V(Q,S) = \{ u \in H^1(Q) : u|_S \in \mathcal{D}(S) \}
\]

Let us consider the energy form

\[
E[u] = \int_Q Lu \, ud\mathcal{L}_3 + E_S[u|_S] + \int_S b|u|_S^2 \, dg
\]

defined on \( V(Q,S) \), where \( b \in \mathcal{C}(\overline{Q}) \), \( b > 0 \), \( Lu = \sum_{i,j=1}^3 a_{ij} \partial_{ij} u + a_0 u \), is a non divergence operator, \( a_{ij} \) are uniformly Lipschitz continuous functions in \( \overline{Q} \) and \( a_0 \) is a positive essentially bounded function in \( Q \), \( \partial_i \) denotes the partial derivative with respect to \( x_i, i = 1, 2, 3 \) and \( \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \):

\[
(H^1) \quad \begin{cases} 
   a_{ij} = a_{ji}, & \forall i,j = 1,2,3 \\
   \exists \lambda > 0 : & \\
   \sum_{i,j=1}^3 a_{ij} \xi_i \xi_j \geq \lambda \sum_{i=1}^3 |\xi_i|^2 & \forall (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3
\end{cases}
\]

We recast \( Lu \) as the sum of an operator in divergence form and a lower order operator, by Leibniz formula:
\[
\sum_{i,j=1}^{3} a_{ij} \partial_{ij} u + a_0 u = \sum_{i,j=1}^{3} \partial_i (a_{ij} \partial_j u) - \sum_{i=1}^{3} b_i \partial_i u + a_0 u.
\]

where \(b_i = \sum_{j=1}^{3} \partial_i (a_{ij})\), we note that \(b_i \in L^{\infty}(Q)\).

Moreover we assume

\[
(H^2) \quad \inf_Q a_0 - \frac{1}{\lambda} \sum_i \|b_i\|_{L^{\infty}(Q)}^2 > 0.
\]

From now on we consider the energy form:

\[
\bar{E}[u] = \int_Q (ADu \cdot Du + a_0|u|^2) d\mathcal{L}_3 - \sum_{i=1}^{3} \int_Q b_i \partial_i u \cdot ud\mathcal{L}_3 + E_S[u|_S] + \int_S |u|_S^2 dg
\]

where \(\mathcal{A} = [a_{ij}]\). We denote by \(L^2(\mathcal{Q}, m)\) the Lebesgue space with respect to the measure \(dm = d\mathcal{L}_3 + dg\).

By \(\bar{E}(u, v)\) we denote the bilinear form

\[
\bar{E}(u, v) = \int_Q (ADu \cdot Dv + a_0uv) d\mathcal{L}_3 - \sum_{i=1}^{3} \int_Q b_i \partial_i u \cdot v d\mathcal{L}_3 + E_S[u|_S, v|_S] + \int_S |u|_S v|_S dg.
\]

defined on \(V(Q, S) \times V(Q, S)\).

**Proposition 4.6.** The form \(\bar{E}\) is a bilinear form on \(L^2(\mathcal{Q}, m)\) and \(V(Q, S)\) is a Hilbert space equipped with the scalar product

\[
(u, v)_{V(Q, S)} = (u, v)_{H^1(Q)} + E_S(u|_S, v|_S) + (u|_S, v|_S)_{L^2(S, g)}
\]

with norm

\[
||u||_{V(Q, S)} = (||u||_{H^1(Q)}^2 + ||u|_S||_{L^2(S)}^2)^{\frac{1}{2}}.
\]

Moreover from assumption \((H^2)\) we deduce that \(\bar{E}[u]\) is a positive definite non-symmetric form. We decompose \(\bar{E}(u, v) = E(u, v) - \int_Q \sum_{i=1}^{3} b_i \partial_i u \cdot v d\mathcal{L}_3\) where \(E(u, v)\) is a symmetric form in \(V(Q, S)\).

4.3. **Semigroups associated with \(\bar{E}\) and \(\bar{E}^{(b)}\).**

**Proposition 4.7.** Since \(\bar{E}(u, v)\) is a closed bilinear form in \(L^2(\mathcal{Q}, m)\) with domain \(V(Q, S)\) dense in \(L^2(\mathcal{Q}, m)\), there exists a unique non positive operator \(\bar{A}\) on \(L^2(\mathcal{Q}, m)\) such that

\[
\bar{E}(u, v) = -\int_\mathcal{Q} \bar{A}u \cdot v d\mathcal{L}_3, u \in \mathcal{D}(\bar{A}), v \in V(Q, S).
\]

\(\bar{A}\) is the infinitesimal generator of a strongly continuous semigroup \(\{T(t)\}_{t \geq 0}\).

**Proposition 4.8.** Let \(\{T(t)\}_{t \geq 0}\) be the semigroup associated with \(\bar{A}\). Then \(\{T(t)\}_{t \geq 0}\) is a contraction analytic semigroup on \(L^2(\mathcal{Q}, m)\).

**Proof.** The proof follows from Chapter 17, Section 6 in [11], since \(\bar{E}[\cdot]\) is weakly coercive that is there exist positive constants \(\lambda_0\) and \(\alpha_0\):
\[ E[u] + \lambda_0 \|u\|_{L^2(Q,m)} \geq \alpha_0 \|u\|_{V(Q,S)}^2. \]

**Proposition 4.9.** Since \( \tilde{E}^h(u,v) \) is a closed bilinear form in \( L^2(Q,m_h) \) with domain \( V(Q,S_h) \) dense in \( L^2(Q,m_h) \), there exists a unique non positive operator \( \tilde{A}^h \) on \( L^2(Q,m_h) \) with \( \mathcal{D}(\tilde{A}^h) \subseteq V(Q,S_h) \) dense in \( L^2(Q,m_h) \), such that
\[
\tilde{E}^h(u,v) = -\int_Q \tilde{A}^h u \cdot v \, dm_h, \quad u \in \mathcal{D}(\tilde{A}^h), \quad v \in V(Q,S_h).
\]

**Proposition 4.10.** Let \( \{T_h(t)\}_{t \geq 0} \) be the semigroup associated with \( \tilde{A}^h \). Then \( \{T^h(t)\}_{t \geq 0} \) is a contraction analytic semigroup on \( L^2(Q,m_h) \).

**Proof.** The proof follows from Chapter 17, Section 6 in [11], since \( \tilde{E}^h[\cdot] \) is weakly coercive that is there exist positive constants \( \lambda_0 \) and \( \alpha_0 \) depending on \( h \) such that
\[
\tilde{E}^h[u] + \lambda_0 \|u\|_{L^2(Q,m_h)} \geq \alpha_0 \|u\|_{V(Q,S_h)}^2. \tag{4.12}
\]

In the following, for every \( \alpha > 0 \), we denote by \( \tilde{G}^h, \tilde{C}^h, \tilde{G}^h_\alpha \) and \( \tilde{G}^h_\alpha \) the resolvents and coresolvents associated with \( \tilde{A} \) and its adjoint \( \tilde{A} \). From Theorem 2.8 in [38] we have that the following equalities hold
\[
\tilde{E}(\tilde{G}_\alpha f, v) + \alpha(\tilde{G}_\alpha f, v)_{L^2(\tilde{Q},m)} = (f,v)_{L^2(\tilde{Q},m)} = \tilde{E}(v, \tilde{G}_\alpha f) + \alpha(v, \tilde{G}_\alpha f)_{L^2(\tilde{Q},m)}
\]
for every \( f \in L^2(Q,m) \) and \( v \in V(Q,S) \), and
\[
\tilde{E}^h(\tilde{G}_\alpha f, v) + \alpha(\tilde{G}^h_\alpha f, v)_{L^2(\tilde{Q},m_h)} = (f,v)_{L^2(\tilde{Q},m_h)} = \tilde{E}^h(v, \tilde{G}^h_\alpha f) + \alpha(v, \tilde{G}^h_\alpha f)_{L^2(\tilde{Q},m_h)}
\]
for every \( f \in L^2(Q,m_h) \) and \( v \in V(Q,S_h) \).

**5. Varying Hilbert spaces.** We introduce the notion of convergence in varying Hilbert spaces; for more details, see [26].

**Definition 5.1.** A sequence of Hilbert spaces \( \{H_h\}_{h \in \mathbb{N}} \) converges to a Hilbert space \( H \) if there exists a dense subspace \( C \subset H \) and a sequence \( \{\Phi_h\}_{h \in \mathbb{N}} \) of linear operators \( \Phi_h : C \to H_h \) such that
\[
\lim_{h \to \infty} \|\Phi_h u\|_{H_h} = \|u\|_H \text{ for any } u \in C.
\]

We set \( \mathcal{H} = (\bigcup_{h=1}^{\infty} H_h) \cup H \).

We now provide the definitions of strong and weak convergence in \( \mathcal{H} \).

**Definition 5.2.** A sequence of vectors \( \{u_h\}_{h \in \mathbb{N}} \) strongly converges to \( u \) in \( \mathcal{H} \) if \( u_h \in H_h \), \( u \in H \) and there exists a sequence \( \{\overline{u}_m\}_{m \in \mathbb{N}} \in C \) tending to \( u \) in \( H \) such that
\[
\lim_{m \to \infty} \|\Phi_h u_m - u_h\|_{H_h} = 0.
\]

**Definition 5.3.** A sequence of vectors \( \{u_h\}_{h \in \mathbb{N}} \) weakly converges to \( u \) in \( \mathcal{H} \), if \( u_h \in H_h \), \( u \in H \) and
\[
(u_h, v_h)_{H_h} \to (u, v)_H
\]
for every sequence \( \{v_h\}_{h \in \mathbb{N}} \) strongly tending to \( v \) in \( \mathcal{H} \).

**Remark 5.4.** Strong convergence implies weak convergence.

**Lemma 5.5.** Let \( \{u_h\}_{h \in \mathbb{N}} \) be a sequence weakly convergent to \( u \) in \( H \), then

- \( \sup_h \|u_h\|_{H_h} < \infty \).
- \( \|u\|_H \leq \lim_{h \to \infty} \|u_h\|_{H_h} \).
\[ u_h \to u \iff \|u\|_H = \lim_{h \to \infty} \|u_h\|_{H_h}. \]

Now we state other characterizations of strong convergence in \( \mathcal{H} \).

**Lemma 5.6.** Let \( u \in H \) and let \( \{u_h\}_{h \in \mathbb{N}} \) be a sequence of vectors \( u_h \in H_h \). Then \( \{u_h\}_{h \in \mathbb{N}} \) strongly converges to \( u \) in \( \mathcal{H} \) if and only if

\[ (u_h v_h)_{H_h} \to (u, v)_H \]

for every sequence \( \{v_h\}_{h \in \mathbb{N}} \) with \( v_h \in H_h \) weakly converging to a vector \( v \) in \( \mathcal{H} \).

**Lemma 5.7.** A sequence of vectors \( \{u_h\}_{h \in \mathbb{N}} \) with \( u_h \in H_h \) strongly converges to a vector \( u \) in \( \mathcal{H} \) if and only if

\begin{itemize}
  \item \( \|u_h\|_{H_h} \to \|u\|_H \)
  \item \( (u_h, \Phi_h \varphi)_{H_h} \to (u, \varphi)_H \) for every \( \varphi \in C \).
\end{itemize}

**Lemma 5.8.** Let \( \{u_h\}_{h \in \mathbb{N}} \) be a sequence with \( u_h \in H_h \). If \( \|u_h\|_{H_h} \) is uniformly bounded, there exists a subsequence of \( \{u_h\}_{h \in \mathbb{N}} \) which weakly converges in \( \mathcal{H} \).

**Lemma 5.9.** For every \( u \in H \) there exists a sequence \( \{u_h\}_{h \in \mathbb{N}}, u_h \in H_h \) strongly converging to \( u \) in \( \mathcal{H} \).

**Definition 5.10.** A sequence of bounded operators \( \{B_h\}_{h \in \mathbb{N}}, B_h \in \mathcal{L}(H_h) \) strongly converges to an operator \( B \in \mathcal{L}(H) \), if for every sequence of vectors \( \{u_h\}_{h \in \mathbb{N}} \) with \( u_h \in H_h \) strongly converging to \( u \) in \( \mathcal{H} \), the sequence \( \{B_h u_h\}_{h \in \mathbb{N}} \) strongly converges to \( Bu \) in \( \mathcal{H} \).

**Definition 5.11.** A sequence of bounded operators \( \{B_h\}_{h \in \mathbb{N}}, B_h \in \mathcal{L}(H_h) \) weakly converges to an operator \( B \in \mathcal{L}(H) \), if for every sequence of vectors \( \{u_h\}_{h \in \mathbb{N}} \) with \( u_h \in H_h \) weakly converging to \( u \) in \( \mathcal{H} \), the sequence \( \{B_h u_h\}_{h \in \mathbb{N}} \) weakly converges to \( Bu \) in \( \mathcal{H} \).

**Proposition 5.12.** A sequence of bounded operators \( \{B_h\}_{h \in \mathbb{N}}, B_h \in \mathcal{L}(H_h) \) strongly converges to an operator \( B \in \mathcal{L}(H) \) if and only if the sequence \( \bar{B}_h \) weakly converges to \( \bar{B} \), where \( B \) and \( \bar{B}_h \) denote the adjoint operators of \( B \) and \( B_h \) respectively.

### 5.1 Convergence of spaces

From now on we put \( H = L^2(Q, m) \), where \( m \) is the measure defined in (4.9), and the sequence \( \{H_h\}_{h \in \mathbb{N}} = \{L^2(Q, m_h) \cap L^2(Q)\}_{h \in \mathbb{N}} \), where \( m_h \) is the measure defined in (4.7), with norms

\[ \|u\|_H^2 = \|u\|_{L^2(Q)}^2 + \|u\|_{L^2(S, g)}^2, \quad \|u\|_{H_h}^2 = \|u\|_{L^2(Q_h)}^2 + \|u\|_{L^2(S_h, \delta_h, d\sigma)}^2 \]

**Proposition 5.13.** Let \( \delta_h = (3^{-d} \bar{h})^{1/3} \). The sequence of Hilbert spaces \( \{H_h\}_{h \in \mathbb{N}} \) converges to the Hilbert space \( H \).

**Proof.** We put \( C = C(\bar{Q}) \) and \( \Phi_h \) the identical operator on \( C(\bar{Q}) \). We have to prove that

\[ \lim_{h \to \infty} \|u\|_{H_h} = \|u\|_H, \text{ for any } u \in C. \]

So we have to prove that

\[ \lim_{h \to \infty} \int_Q \chi_{Q_h} |u|^2 d\mathcal{L}_3 = \int_Q |u|^2 d\mathcal{L}_3 \]

and

\[ \lim_{h \to \infty} \delta_h \int_{F_h} |u|^2 d\mathcal{L}_3 = \int_F |u|^2 d\mathcal{L}_3. \]
The last equality amounts to prove that
\[ \lim_{h \to \infty} \delta_h \int_{F_h} |u|^2 \, dl = \int_F |u|^2 \, d\mu. \]

We note that
\[ \delta_h \int_{F_h} |u|^2 \, dl = \sum_{j=1}^{3-4h} \delta_h \int_{M_j} |u|^2 \, dl, \]
where \( M_j \) denotes a segment of \( h \)-generation.

Since \( u(\cdot, x_3) \) is continuous on \( F_h \) for each \( x_3 \in [0, 1] \), by the mean value Theorem, there exists \( \xi_j \in M_j \) such that
\[ \delta_h \int_{F_h} |u|^2 \, dl = \sum_{j=1}^{3-4h} \delta_h |u(\xi_j, x_3)|^2 3^{-h}. \]

We can write
\[ |\int_F |u(x, x_3)|^2 \, d\mu - \delta_h \int_{F_h} |u(x, x_3)|^2 \, dl| \leq \sum_{j=1}^{3-4h} \frac{|u(P_j, x_3)|^2}{4^h} + |\sum_{j=1}^{3-4h} \delta_h 3^{-h} (|u(P_j, x_3)|^2 - |u(\xi_j, x_3)|^2)|, \]
where \( P_j \) is one of the endpoints of \( M_j \). The first term of right-hand side of the inequality tends to zero as \( h \to \infty \) from the Corollary 3.4 in [34], while the second vanishes since \( |u|^2 \) is uniformly continuous in every \( M_j \). Since
\[ \sup_{x_3 \in [0,1]} \delta_h \int_{F_h} |u|^2 \, dl \leq 3 \|u\|^2_{C(Q)} \]
the thesis follows from dominated convergence theorem.

6. M-convergence of the energy forms and density theorems. In this section we prove the main theorem of this paper, the M-K-S-convergence of the energy forms. The proof relies on some crucial density results for the functions in the domain of the energy form \( E \) specialized to the case of interest, for further details we refer the reader to [29].

6.1. Density theorems for \( \mathcal{D}(S) \). In the notation of [35] we denote by \( W(0,1) \) the following space:
\[ W(0,1) := L^2(0,1; \mathcal{D}(F)) \cap H^1(0,1; L^2(F)). \]

This is a Hilbert space equipped with the norm
\[ \|u\|_{W(0,1)} = \|u\|_{L^2([0,1]; \mathcal{D}(F))} + \|\partial_3 u\|_{L^2([0,1]; L^2(F))}. \]

From theorem 2.1 page 11 of [35] it follows that \( D([0,1]; \mathcal{D}(F)) \) is densely embedded in \( W(0,1) \), that is
\[ \overline{D([0,1]; \mathcal{D}(F))}^{\| \cdot \|_{W(0,1)}} = W(0,1) \]

Proposition 6.1. \( D([0,1]; \mathcal{D}(F)) \subset C(S) \).
(See Proposition 5.2 in [29]).

Theorem 6.2. The space \( D([0,1]; \mathcal{D}(F)) \) is dense in \( \mathcal{D}(S) \) with respect to the intrinsic norm \( \| \cdot \|_{\mathcal{D}(S)} \).
(See Theorem 5.3 in [29]).
6.2. Density theorem for $V(Q,S)$.

**Proposition 6.3.** Let $\beta = \frac{d_f}{2}$. Let $\gamma_0$ and $Ext$ be respectively the trace and the extension operator defined in Section 2. Then

1. If $u \in C(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ then $\gamma_0 u \in C(S) \cap B_{\beta}^2(\mathbb{S})$.
2. If $u \in C(S) \cap B_{\beta}^2(\mathbb{S})$ then $Ext(u) \in C(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$.

(See Proposition 5.5 in [29]).

**Theorem 6.4.** For every $u \in V(Q,S) = \{ u \in H^1(Q) : u|_S \in \mathcal{D}(S) \}$, there exists $\psi_n \in V(Q,S) \cap C(\mathbb{Q})$ such that:

- $\| \psi_n - u \|_{H^1(Q)} \to 0$, for $n \to \infty$
- $\| \psi_n - u \|_H \to 0$, for $n \to \infty$
- $E_S[\psi_n - u] \to 0$, for $n \to \infty$.

(See Theorem 5.4 in [29]).

6.3. M-convergence of the symmetric energy forms. We now give the definition of $M$-convergence of symmetric forms in the case of varying Hilbert space, by using the definition of Kuwae and Shioya in [26]. We put the form

$$E[u] = +\infty, \text{ for every } u \in H \setminus V(Q,S)$$

and

$$E^{(h)}[u] = +\infty, \text{ for every } u \in H_h \setminus V(Q,S_h)$$

**Definition 6.5.** A sequence of symmetric forms $\{ E^{(h)} \}$ $M$-converges to a form $E$ if

1. for every $v_h \in H_h$ weakly converging to $u \in H$ in $\mathcal{H}$
   $$\lim_{h \to \infty} E^{(h)}[v_h] \geq E[u]$$
2. for every $u \in H$ there exists $\{ w_h \}$, with $w_h \in H_h$ strongly converging to $u \in \mathcal{H}$ such that
   $$\lim_{h \to \infty} E^{(h)}[w_h] \leq E[u].$$

**Proposition 6.6.** If $\{ v_h \}_{h \in \mathbb{N}}$ weakly converges to a vector $u$ in $\mathcal{H}$, then $\{ v_h \}_{h \in \mathbb{N}}$ weakly converges to $u$ in $L^2(Q)$ and $\lim_{h \to \infty} \delta_h \int_{S_h} \varphi v_h d\sigma = \int_S \varphi u d\sigma$, for every $\varphi \in C$.

**Proof.** The definition of weak convergence in $\mathcal{H}$ implies that for every $\varphi_h \in H_h$ strongly converging to $\varphi \in H$

$$\lim_{h \to \infty} \left( \int_{Q_h} v_h \varphi_h d\mathcal{L}_3 + \delta_h \int_{S_h} v_h \varphi_h d\sigma \right) = \int_Q u \varphi d\mathcal{L}_3 + \int_S u \varphi d\sigma. \quad (6.13)$$

For every $v \in C$ we set $\varphi_h = w|_{Q_h}$ and $\varphi = w|_{Q}$: $\varphi_h \in H_h$ and $\varphi \in H$. We prove that $\varphi_h$ strongly converges to $\varphi$ in $\mathcal{H}$. This result follows from Lemma 5.7, in fact the first claim holds since

$$\| \varphi_h \|_{H^1} = \int_{Q_h} |w|^2 d\mathcal{L}_3, \quad \| \varphi \|_{H^1} = \int_Q |w|^2 d\mathcal{L}_3$$

and $Q_h$ is a family of sets invading $Q$. By the same argument it follows that

$$(g, \varphi_h)_{H_h} \to (g, \varphi)_H \forall g \in C.$$
From (6.13) and the choice of $\varphi$ and $\varphi$

$$\lim_{h \to \infty} \int_{Q_h} v_h d\mathcal{L}^3 = \int_Q u d\mathcal{L}^3, \forall w \in C$$ (6.14)

The constant sequence $\{w\}$ strongly converges to $w$ in $\mathcal{H}$; choosing $\varphi = w$ in (6.13) and taking into account (6.14), by difference we obtain

$$\lim_{h \to \infty} \delta_h \int_{S_h} w v_h d\sigma = \int_S w u d\sigma.$$

We now prove the weak convergence of $v_h$ to $u$ in $L^2(Q)$. We first prove the convergence for every $\phi \in C(\mathcal{Q})$, then the claim will follow by density.

$$\lim_{h \to \infty} \int_Q \phi d\mathcal{L}^3 = \lim_{h \to \infty} (\int_Q \phi \chi_{Q_h} d\mathcal{L}^3 + \int_Q \phi \chi_{\mathcal{Q} \setminus Q_h} d\mathcal{L}^3) = \int_Q u \phi d\mathcal{L}^3,$$

since $\phi \chi_{\mathcal{Q} \setminus Q_h}$ strongly tends to zero in $\mathcal{H}$ and $\phi \chi_{Q_h}$ strongly converges to $\phi \chi_Q$ in $\mathcal{H}$.

**Proposition 6.7.** If $v_h$ weakly converges to $u$ in $H^1(Q)$ and $b \in C(\mathcal{Q})$, then $\delta_h \int_{S_h} b|v_h|^2 d\sigma \to \int_S b|u|^2 d\sigma$.

**Proof.**

$$|\delta_h \int_{S_h} b|v_h|^2 d\sigma - \int_S b|u|^2 d\sigma| \leq |\delta_h \int_{S_h} b|v_h|^2 d\sigma - \delta_h \int_{S_h} b|u|^2 d\sigma| + |\delta_h \int_{S_h} b|u|^2 d\sigma - \int_S b|u|^2 d\sigma|.$$ (6.15)

The first term in the right hand side of (6.15) can be estimated as follows:

$$|\delta_h \int_{S_h} b|v_h|^2 d\sigma - \delta_h \int_{S_h} b|u|^2 d\sigma| \leq \delta_h \|b\|_{C(\mathcal{Q})}(\|v_h - u\|_{L^2(S_h)}) \|v_h + u\|_{L^2(S_h)} \|v_h\|_{L^2(S_h)} \|u\|_{L^2(S_h)} + \|v_h\|_{L^2(S_h)} \|u\|_{L^2(S_h)}).$$

Since $v_h$ weakly converges in $H^1(Q)$ to $u$, then $v_h$ strongly converges to $u$ in $H^\alpha(Q)$ for every $\alpha \in (0, 1)$. Considering the extension of $(v_h - u)$ to $H^\alpha(\mathbb{R}^3)$, it follows from Theorems 3.5 and 3.8

$$\delta_h \|v_h - u\|_{L^2(S_h)} \leq C_\alpha \|\text{Ext}(v_h - u)\|_{H^\alpha(\mathbb{R}^3)} \leq c \|v_h - u\|_{H^1(Q)}.$$ 

From these inequalities it follows that

$$|\delta_h \int_{S_h} b|v_h|^2 d\sigma - \delta_h \int_{S_h} b|u|^2 d\sigma| \to 0.$$ 

Since $u \in H^1(Q)$ there exists a sequence $\{g_n\} \subset H^1(Q) \cap C(\mathcal{Q})$ such that $\|g_n - u\|_{H^1(Q)} \to 0$ (see Proposition 4.4 in [20]).

Now we estimate the second term on the right hand side of (6.15).

$$|\delta_h \int_{S_h} b|u|^2 d\sigma - \int_S b|u|^2 d\sigma| \leq |\delta_h \int_{S_h} b|u|^2 d\sigma - \delta_h \int_{S_h} b|g_n|^2 d\sigma| + |\delta_h \int_{S_h} b|g_n|^2 d\sigma - \int_S b|g_n|^2 d\sigma| + |\int_S b|g_n|^2 d\sigma - \int_{S_h} b|u|^2 d\sigma|.$$ 

It is possible to estimate from above the first and the third term of the right hand side of this inequality with $\|g_n - u\|_{H^1(Q)}$, and hence we conclude that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that these two terms are less than $c \varepsilon$.

If we choose $n > n_\varepsilon$, the second term in the right-hand side goes to 0 for $h$ tending to $+\infty$, since $H^1_h$ converges to $H$. 

$\Box$
We set \( \sum_\{i,j\} \) to Theorem 6.8. Let \( \delta_h = (3^{1-d})^h \), \( \sigma_1^{h} = \sigma_1c_0(\delta_h)^{-1} \), \( \sigma_2^{h} = \sigma_2c_0\delta_h \). Let us assume that there exists \( M_0 \) such that \( \| a_{ij}^{h} \|_{L^\infty(Q)} \leq M_0 \| a_0 \|_{L^\infty(Q)} \leq M \) for every \( h \in \mathbb{N} \), \( i, j = 1, 2, 3 \) and that \( a_{ij}^{h} \) and \( a_0 \) converge a.e. in \( Q \) to \( a_{ij} \) and \( a_0 \) respectively, then the sequence \( E^{(h)} \) converges in the sense of Mosco, Kuwae, Shiya to the form \( E \).

**Proof. Condition 1.**

Let \( v_h \in V(Q, S_h) \), there exists a \( c \) independent from \( h \) such that

\[
\| v_h \|_{H^1(Q_h)}^2 + ES_h[v_h|S_h] + \delta_h \| v_h \|^2_{L^2(S_h)} \leq C
\]

and then \( \| v_h \|_{H^1(Q)} < C \). For every \( h \in \mathbb{N} \) from Theorem 3.7 there exists a continuous linear operator \( Ext : H^1(Q_h) \to H^1(\mathbb{R}^3) \) such that

\[
\| Extv_h \|_{H^1(\mathbb{R}^3)} \leq c \| v_h \|_{H^1(Q_h)} \leq cC.
\]

Let \( \tilde{v}_h = Extv_h|Q, v_h \in H^1(Q) \) and \( \| \tilde{v}_h \|_{H^1(Q)} \leq cC \), thus there exists a subsequence, still denoted by \( \tilde{v}_h \), weakly converging to \( \tilde{v} \) in \( H^1(Q) \) and hence strongly in \( L^2(Q) \). By Proposition 6.6 it follows that \( v_h \) weakly converges to \( u \) in \( L^2(Q) \).

We want to prove that \( \tilde{v} = u \) a.e. that is \( \int_Q (\tilde{v} - u) \phi dL_3 = 0 \) for each \( \phi \in L^2(Q) \).

\[
\int_Q (\tilde{v} - u) \phi dL_3 = \int_Q \left( \tilde{v} - \tilde{v}_h + \tilde{v}_h - u \right) \phi dL_3 =
\]

\[
\int_Q (\tilde{v} - \tilde{v}_h) \phi dL_3 + \int_{Q_h} (v_h - u) \phi dL_3 + \int_{Q - Q_h} (\tilde{v}_h - u) \phi dL_3.
\]

Since \( \tilde{v}_h \to \tilde{v} \) in \( L^2(Q) \) and \( v_h \) weakly converges to \( u \) in \( L^2(Q) \), it follows that the first two terms of right hand side vanish. Moreover, from Hölder inequality and since \( |Q - Q_h| \to 0 \) for \( h \to \infty \), \( \int_{Q - Q_h} (\tilde{v}_h - u) \phi dL_3 \leq \| \phi \|_{L^2(Q - Q_h)}(\| \tilde{v}_h \|_{L^2(Q)} + \| u \|_{L^2(Q)}) \to 0 \).

Now we prove that

\[
\lim_{h \to \infty} \int_Q \chi_{Q_h}(A^h Dv_h \cdot Du_h + a_0^h |v_h|^2) dL_3 \geq \int_Q (ADu \cdot Du + a_0|u|^2) dL_3.
\]

We set \( \sqrt{A} = [c_{ij}] \) and \( \sqrt{A^h} = [c_{ij}^h] \). From the assumptions it follows that \( |c_{ij}^h| \leq M_1 \) for every \( i, j, c_{ij}^h \to c_{ij} \) a.e.

From Severini-Egorov Theorem it follows that \( \sum_{i,j=1}^3 c_{ij}^h \chi_{Q_h} \) converges quasi-uniformly to \( \sum_{i,j=1}^3 c_{ij} \chi_{Q} \) as well as \( \chi_{Q_h} \sqrt{a_0^h} \to \chi_{Q} \sqrt{a_0} \), from the weak convergence of \( v_h \) to \( u \) in \( H^1(Q) \) we deduce that \( \sum_{i,j=1}^3 c_{ij} \chi_{Q_h} \partial_i v_h \) weakly converges in \( L^2(Q) \) to \( \sum_{i,j=1}^3 c_{ij} \chi_{Q} \partial_j u \) and \( \chi_{Q_h} \sqrt{a_0^h} v_h \) to \( \chi_{Q} \sqrt{a_0} u \). Then

\[
\lim_{h \to \infty} \int_Q \chi_{Q_h}(A^h Dv_h \cdot Dv_h + a_0^h |v_h|^2) dL_3 \geq
\]

\[
\lim_{h \to \infty} \int_Q \chi_{Q_h}(A^h Dv_h \cdot Dv_h dL_3 + \lim_{h \to \infty} \int_Q \chi_{Q_h} a_0^h |v_h|^2)
\]

\[
\lim_{h \to \infty} \int_Q |\chi_{Q_h}(\sqrt{A^h} Dv_h)|^2 dL_3 + \lim_{h \to \infty} \int_Q |\chi_{Q_h} \sqrt{a_0^h} v_h|^2) dL_3 = \]

\[
\]
\[
\lim_{h \to \infty} \sum_{i=1}^{3} \left\| \sum_{j=1}^{3} c_{ij} \chi_{Q_i} \partial_j v_h \right\|_{L^2(Q)}^2 + \lim_{h \to \infty} \left\| \chi_{Q_h} \sqrt{a_0^h v_h^2} \right\|_{L^2(Q)}^2 \geq \\
\sum_{i=1}^{3} \left\| \sum_{j=1}^{3} c_{ij} \chi_{Q_i} \partial_j u \right\|_{L^2(Q)}^2 + \left\| \sqrt{a_0^h u} \right\|_{L^2(Q)}^2
\]

The proof that \( \lim_{h \to \infty} E_{S_h}[v_h] \geq E_S[u] \) follows from Remark 5.1 in [34].

**Condition 2.** We suppose that \( u \in V(Q, S) \).

**Step 1.** We suppose that \( u \in C(\overline{Q}) \), hence \( u \in H \). We extend by continuity \( u \) to \( \overline{h} \) and we put \( \overline{u} \) this extension.

Following the same approach of [31], we introduce a quasi uniform triangulation \( \tau_h \) of \( \overline{f} \) made by equilateral tetrahedron \( T_h^i \) such that the vertices of the prefractal surface \( S_h \) are nodes of the triangulation at the \( h - th \) level. Let \( S_h \) be the space of all the functions being continuous on \( \overline{f} \) and affine on the tetrahedrons of \( \tau_h \). We indicate by \( M_h \) the nodes of \( \tau_h \), that is the set of the vertices of all \( T_h^i \). For a given continuous function \( u \), we denote by \( I_h u \) the function which is affine on every \( T_h^i \in \tau_h \) and which interpolates \( u \) in the nodes \( P_{j,i} \in M_h \cap Q_h \). We put \( w_h = I_h \overline{u} \), and we prove that \( \{w_h\} \) strongly converges in \( \mathcal{H} \) using the Lemma 5.6: we have to prove that \( (w_h, v_h)_{H_h} \to (u, v)_H \) for every \( \{v_h\} \) weakly converging to \( v \) in \( \mathcal{H} \). It holds that

\[
\|w_h - u\|_{H^1(\overline{f})} \to 0
\]

for \( h \) tending to \( \infty \) (see [17]) and hence \( \|w_h - u\|_{H^1(Q)} \to 0 \). From Theorem 3.5, there exists \( c \) independent of \( h \) such that \( \|w_h - u\|_{L^2(S_h)} \leq c(\delta_h)^{-1/2} \|w_h - u\|_{H^1(Q)} \).

\[
0 \leq |(w_h, v_h)_{H_h} - (u, v)_H| = \int_{Q_h} w_h v_h d\mathcal{L}_3 + \delta_h \int_{S_h} w_h v_h d\sigma - \int_{\overline{f}} uv d\mathcal{L}_3 - \int_S uv d\sigma = \\
|[(w_h - u, v_h)_{L^2(Q_h)} + \delta_h \int_{S_h} (w_h - u) v_h d\sigma + (u, v_h)_{H_h} - (u, v)_H]| \leq \\
|[(w_h - u, v_h)_{L^2(Q_h)}| + |(w_h - u) v_h d\mathcal{L}_3| + |(w_h - u, v)_H| \leq \\
\|w_h - u\|_{L^2(Q)} \|v_h\|_{L^2(Q)} + \sqrt{\delta_h} \|w_h - u\|_{L^2(S_h)} \sqrt{\delta_h} \|v_h\|_{L^2(S_h)} + |(u, v)_H| - |(u, v)_H|.
\]

The thesis follows from the weak convergence of \( v_h \) to \( v \) in \( \mathcal{H} \) and from the fact that \( \sqrt{\delta_h} \|w_h - u\|_{L^2(S_h)} \leq c \|w_h - u\|_{H^1(Q)} \).

We now show that the above sequence \( \{v_h\} \) satisfies condition 2 of \( M \)-convergence. In fact \( \lim_{h \to \infty} \delta_h \int_{S_h} b|w_h|^2 d\sigma = \int_S b|u|^2 d\sigma \). From [30] we have \( \lim_{h \to \infty} E_{S_h}[v_h] \leq E_S[u] \).

We prove that

\[
\lim_{h \to \infty} \int_{Q} \chi_{Q_h}(A^h D w_h \cdot D w_h + a^h_0 w_h w_h)d\mathcal{L}_3 \leq \int_{Q} \chi_{Q}(A D u \cdot D u + a_0 uu)d\mathcal{L}_3.
\]

The thesis follows since

\[
\lim_{h \to \infty} \int_{Q} \chi_{Q_h} A^h D w_h \cdot D w_h d\mathcal{L}_3 = \lim_{h \to \infty} \sum_{i=1}^{3} \sum_{j=1}^{3} c_{ij} \chi_{Q_i} \partial_j w_h \right\|_{L^2(Q)}^2 \\
\lim_{h \to \infty} \int_{Q} \chi_{Q_h} a^h_0 w_h w_h d\mathcal{L}_3 = \lim_{h \to \infty} \chi_{Q_h} \sqrt{a^h_0 w_h} \right\|_{L^2(Q)}^2
\]
and, from the assumptions on \( c^h_{ij}, a^0 \) and on \( w_h \), we deduce that \( \sum_{j=1}^3 c^h_{ij} \chi_Q \partial_j w_h \) converges to \( \sum_{j=1}^3 c^h_{ij} \chi_Q \partial_j u \) and \( \chi_Q \sqrt{a^0} w_h \) converges to \( \chi_Q a_0 u \) in \( L^2(Q) \). Then the thesis follows from the limsup properties of the sum.

**Step 2.** If \( u \in V(Q, S) \), but \( u \) is not continuous, from Theorem 6.3 there exists \( \psi_n \in V(Q, S) \) such that \( \psi_n \to u \) in \( H, \| \psi_n - u \|_{V(Q, S)} \to 0 \). Let \( n \in \mathbb{N} \) fixed such that \( \| \psi_n - u \|_{V(Q, S)} \leq \frac{2}{n} \) and \( \| \psi_n - u \|_H \leq \frac{1}{n} \). By \( \psi_n \) we denote a continuous extension in \( \mathcal{T} \).

From Step 1 we have that for every fixed \( n \in \mathbb{N} \) \( \psi_n \) converges to \( \psi_n \) in \( H \) and
\[
\lim_{n \to \infty} E^h[\psi_n] = E[u].
\]

Applying the upper limit for \( n \to \infty \) to both sides of the above inequality we obtain
\[
\limsup_{n \to \infty} E^h[I_h \psi_n] \leq \lim_{n \to \infty} E[\psi_n].
\]

Now we want to apply Corollary 1.5 in [2] for proving that there exists an increasing mapping \( h \to n(h) \), such that, denoting \( w_h = I_h \psi_n(h) \), we have that \( w_h \) converges to \( u \) in \( H \) and \( \limsup_{h \to \infty} E^h[w_h] \leq E[u] \). To this aim we have to prove that
\[
\lim_{n \to \infty} \lim_{h \to \infty} (w_{h,n}, v_h)_{H^1} - (u, v)_H \leq 0,
\]

for every \( \{ v_h \} \) weakly converging to \( v \) in \( H \).

Heuristically, this follows from
\[
\| (w_{h,n}, v_h)_{H^1} - (\tilde{\psi_n}, v)_H \| \leq \| w_{h,n} - \tilde{\psi_n} \|_H \| v \|_H \leq \| w_{h,n} - \tilde{\psi_n} + \tilde{\psi_n} - u \|_H \| v \|_H.
\]

Passing to the upper limit for \( h \to \infty \), we obtain
\[
\limsup_{h \to \infty} \| (w_{h,n}, v_h)_{H^1} - (u, v)_H \| \to 0.
\]

Then Corollary 1.16 in [2] provides the thesis. \( \Box \)

From the previous Theorem we deduce that

**Proposition 6.9.** For every \( u \in V(Q, S) \) there exists a sequence \( w_h \in H_h \) strongly converging to \( u \in H \) such that
\[
\lim_{h \to +\infty} E^h[w_h] = E[u].
\]

6.4. Convergence of the total energy forms \( \bar{E}^h \). We now study the convergence of the energy forms \( \bar{E}^h(u, v) \) to \( E(u, v) \).

Since we are now in the framework of non symmetric forms we make use of the results in [38], [18], [42], [46] and [37] adapted to the present case. Under the hypothesis of Theorem 6.8 the following Theorem holds.

**Theorem 6.10.** Let \( \bar{E}^h \) and \( \bar{E} \) be defined as in (4.5) and (4.8). Let us assume that there exists \( M > 0 \) such that \( \| b_i^h \|_{L^\infty} \leq M, \| a^0_i \|_{L^\infty} \leq M, \| a^0_i \|_{L^\infty} \leq M \) for every \( h \in \mathbb{N}, i, j = 1, 2, 3 \) and that \( b_i^h \) converges a.e. in \( Q \) to \( b_i \), \( a^0_i \) and \( a^0_0 \) converge a.e. in \( Q \) to \( a_{ij} \) and \( a_0 \) respectively, then

1. If \( u_h \in V(Q, S_h) \) weakly converges to \( u \) in \( H \) and \( \lim_{h \to \infty} \| u_h \|_{V(Q, S_h)} < +\infty \) then \( u \in V(Q, S) \).
2. For every \( w \in V(Q, S) \) there exists a sequence \( \{ w_h \} \), \( w_h \in V(Q, S_h) \), strongly converging to \( w \) in \( H \) such that for any \( u_h \in V(Q, S_h) \) weakly converging to \( u \in V(Q, S) \)
\[
\lim_{h \to +\infty} \bar{E}^h(w_h, u_h) = \bar{E}(w, u)
\]
Proof. Let \( u_h \) satisfy the assumptions of condition 1. From the assumptions on \( a_{ij}^h \) and \( a_h^0 \) it follows that there exists a constant \( \Lambda > 0 \) independent of \( h \) such that \( E^h[u_h] \leq \Lambda \| u_h \|^2_{V(Q,S_h)} \). From Theorem 6.10 we have \( E[u] \leq \lim_{h \to \infty} E^h[u_h] \leq A \lim_{h \to \infty} \| u_h \|^2_{V(Q,S_h)} < +\infty \) hence \( u \in V(Q,S) \).

We now prove 2. From Proposition 6.9 and Remark 2.47 in [46] we have that the sequence \( E^h[u] \) satisfies (6.16). It remains to prove 2. for the term \( \sum_i \int_Q \chi_{Q_h} b_i^h \partial_i w_h u_h d\mathcal{L}_3 \) that is

\[
\lim_{h \to \infty} \sum_i \int_Q \chi_{Q_h} b_i^h \partial_i w_h u_h d\mathcal{L}_3 = \int_Q b_i \partial_i u d\mathcal{L}_3.
\]

By proceeding as in step 2 of condition 2. in Theorem 6.10 we consider the sequence \( w_h \), it turns out that \( w_h \in V(Q,S_h) \) and \( w_h \to w \) in \( H^1(Q) \)

\[
\left| \sum_i \int_Q \chi_{Q_h} b_i^h \partial_i w_h u_h - b_i \partial_i u d\mathcal{L}_3 \right| \leq
\sum_i \int_Q |\chi_{Q_h} b_i^h (\partial_i w_h - \partial_i u) + \partial_i (u - u_h) | d\mathcal{L}_3 + \int_Q |b_i| |\partial_i w u| d\mathcal{L}_3 \to 0
\]

Where the limit follows taking into account that \( \| b_i^h \|_{L^\infty} \leq M \) for every \( i \), and that \( b_i^h \to b_i \) \( q \)-uniformly on \( Q \) and that the Lebesgue measure of \( Q - Q_h \) vanishes.

Remark 6.11. We point out that from the hypothesis on \( b_i^h \) in Theorem 6.10 the constants in (4.12) do not depend on \( h \).

Now we state a Theorem that follows from Theorem 6.10, which is a generalization of Theorem 2.4 in [26].

Theorem 6.12. Let \( \tilde{E}(h) \) and \( \tilde{E} \) be the energy forms defined in 4.5 and in 4.8, respectively; under the assumptions of Theorem 6.10 the semigroups \( \{T_h(t)\} \) associated with the form \( \tilde{E}^h \) converge, for every \( t \geq 0 \), to the semigroup \( T(t) \) associated with the form \( \tilde{E} \), in the sense of Definition 5.10.

Proof. In order to prove the convergence of the semigroups we prove that for every \( \alpha > 0 \) \( G^h_\alpha \) strongly converges to \( G_\alpha \) in the sense of Definition 5.10. Taking into account proposition 5.12 we prove that \( \hat{G}^h_\alpha \) weakly converges to \( \hat{G}_\alpha \). We follow the arguments of [42] adapted to the framework of varying Hilbert spaces in [46] (see Theorem 2.41). Let \( u_h \in H_h \) be a sequence weakly converging to \( u \in \mathcal{H} \) in \( \mathcal{H} \). Let \( \alpha > 0 \) we prove that \( (v_h, \hat{G}^h_\alpha u_h)_{\mathcal{H}_h} \to (v, \hat{G}_\alpha u)_{\mathcal{H}} \) for every \( v_h \to v \) in \( \mathcal{H} \). We set

\[
w_h = \hat{G}^h_\alpha u_h, \tag{6.17}\]

it holds that \( \| \hat{G}^h_\alpha \|_{\mathcal{L}(\mathcal{H}_h)} \leq \alpha^{-1} \), from the weak convergence of \( u_h \) there exists \( M > 0 \) such that

\[
\| u_h \|_{\mathcal{H}_h} \leq M
\]

hence for every \( h \),

\[
\| w_h \|_{\mathcal{H}_h} \leq M \alpha^{-1}
\]
and there exists a subsequence still denoted by \( w_h \) weakly converging in \( \mathcal{H} \) to \( \tilde{w} \in H \). We now prove that \( \tilde{w} \in V(Q,S) \) by proving that condition 1. in Theorem 6.10 is satisfied that is

\[
\sup_h \| w_h \|_{V(Q,S_h)} < +\infty
\]

From (4.12) and Remark 6.11 we deduce that

\[
\| w_h \|_{V(Q,S_h)}^2 \leq \frac{1}{\alpha_0} (\tilde{E}^h[w_h] + \lambda_0 \| w_h \|_{H_h}^2) = \frac{1}{\alpha_0} \| w_h \|_{V(Q,S_h)} \tilde{E}^h(w_h/\| w_h \|_{V(Q,S_h)}, w_h) + \lambda_0 (w_h/\| w_h \|_{V(Q,S_h)},w_h)
\]

hence

\[
\| w_h \|_{V(Q,S_h)} \leq \frac{1}{\alpha_0} \sup_{\|z\|_{V(Q,S_h)}=1} (\tilde{E}^h(z,w_h) + \alpha(z,w_h)_{H_h} + (\lambda_0 - \alpha)(z,w_h)_{H_h}) \leq \frac{1}{\alpha_0} \sup_{\|z\|_{V(Q,S_h)}=1} (z,w_h)_{H_h} + |\lambda_0 - \alpha|(z,w_h)_{H_h} \leq \frac{1}{\alpha_0} (M + |\lambda_0 - \alpha|M\alpha^{-1})
\]

hence \( \sup_h \| w_h \|_{V(Q,S_h)} < +\infty \) and \( \tilde{w} \in V(Q,S) \). We now prove that \( \tilde{w} = \hat{G}_\alpha u \); that is \( \tilde{E}(f,\tilde{w}) + \alpha(f,\tilde{w})_H = (f,u)_H \) for every \( f \in V(Q,S) \). From condition 2. in Theorem 6.10 there exists \( f_h \in V(Q,S_h), f_h \to f \) in \( \mathcal{H} \) such that for every \( v \in V(Q,S) \) and for every \( v_h \in V(Q,S_h) \) \( v_h \to v \) in \( \mathcal{H} \) we have \( \lim_{h \to +\infty} \tilde{E}^h(f_h,v_h) = \tilde{E}(f,v) \), choosing \( v_h = w_h \) (see (6.17)) and \( v = \tilde{w} \) we have

\[
\tilde{E}(f,\tilde{w}) + \alpha(f,\tilde{w})_H = \lim_{h \to +\infty} \tilde{E}^h(f_h,w_h) + \alpha(f_h,\tilde{w}_h)_{H_h} = \lim_{h \to +\infty} (f_h,\tilde{u}_h)_{H_h} = (f,u)
\]

From Theorem 2.21 in [46] the convergence of semigroups follows.

7. Existence results for the abstract fractal and prefractal problems and convergence of the solutions. Let us consider

\[
(\mathcal{P}) \quad \begin{cases} \frac{du(t)}{dt} = \tilde{A} u(t) + f(t), & 0 \leq t \leq T \\ u(0) = 0 \end{cases}
\]

and for every \( h \in \mathbb{N} \)

\[
(\mathcal{P}_h) \quad \begin{cases} \frac{du_h(t)}{dt} = \tilde{A}_h u_h(t) + f_h(t), & 0 \leq t \leq T \\ u_h(0) = 0 \end{cases}
\]

where \( \tilde{A} : \mathcal{D}(\tilde{A}) \subset H \to H \) and \( \tilde{A}_h : \mathcal{D}(\tilde{A}_h) \subset H_h \to H_h \) are the infinitesimal generators associated with the energy form \( \tilde{E} \) and \( \tilde{E}^{(h)} \) respectively. From Theorem 4.3.1 page 149 in [36] we deduce the following existence results.

**Theorem 7.1.** Let \( 0 < \theta < 1, f \in C^\theta([0,T];L^2(\bar{Q}, m)) \) and let

\[
u(t) = \int_0^t T(t-s) f(s) ds,
\]

where \( T(t) \) is the analytic semigroup generated by \( \tilde{A} \). Then \( u \) is the unique strict solution of (7.18), that is

\[
u \in C^1([0,T];L^2(\bar{Q}, m)) \cap C([0,T];D(\tilde{A})),
\]

\[
u(t) = \tilde{A} u(t) + f(t), \quad \forall t \in [0,T] \text{ and } u(0) = 0.
\]
and there exists $c$ such that the following inequality holds:
\[
\|u\|_{C^1([0,T];L^2(Q,m))} + \|u\|_{C([0,T];D(A))] \leq c\|f\|_{C^0([0,T];L^2(Q,m))}.
\] (7.21)

**Theorem 7.2.** Let $0 < \theta < 1$, $f_h \in C^0([0,T];H_h)$ and let
\[
u_h(t) = \int_0^t T_h(t-s)f_h(s)ds, \forall h \in \mathbb{N}
\] (7.22)

where $T_h(t)$ is the analytic semigroup generated by $\bar{A}_h$. Then $\nu_h$ is the unique strict solution of (7.19), that is
\[
u_h \in C^1([0,T]; L^2(Q,m_h)) \cap C([0,T]; D(\bar{A}_h)),
\]
and there exists $C$, independent from $h$, such that the following inequality holds:
\[
\|u_h\|_{C^1([0,T];L^2(Q,m_h))} + \|u_h\|_{C([0,T];D(\bar{A}_h))] \leq C\|f_h\|_{C^0([0,T];L^2(Q,m_h))}.
\] (7.23)

This Section is devoted to the study of the behavior of $u_h$ when $h \to \infty$. We denote $K_h = L^2([0,T];H_h)$ and $K = L^2([0,T];H)$. It holds that $K_h$ converges to $K$ in the sense of definition 5.1, where the set $C = C([0,T] \times Q)$ and $\Phi_h$ is the identical operator on $C$. We denote $\mathcal{K} = (\bigcup_{h=1}^{\infty} K_h) \cup K$. Now we give a characterization of the strong convergence in $\mathcal{K}$.

**Proposition 7.3.** A sequence $\{u_h\}$ strongly converges to $u$ in $\mathcal{K}$ if one of the following conditions holds:

1. \[
\int_0^T \|u_h(t)\|_{H_h}^2 dt \to \int_0^T \|u(t)\|_{H}^2 dt, \quad \forall \varphi \in C([0,T] \times \overline{Q}).
\]

2. \[
\int_0^T (u_h(t),v(t))_{H_h} dt \to \int_0^T (u(t),v(t))_{H} dt, \quad \forall \{v_h\} \text{ weakly converging to } v \in \mathcal{K}.
\]

**Theorem 7.4.** Let $u$ and $u_h$ be the solutions of the problems (P) and (P_h) respectively. Let $\delta_h$ be as in Theorem 6.8 and $a^h_{i,j}$, $a^0$ and $b^h$ as in Theorem 6.10. If for every $t \in [0,T]$, $\{f_h(t)\}$ strongly converges to $f(t)$ in $\mathcal{K}$ and there exists a constant $c$ such that
\[
\|f_h\|_{C^0([0,T];H_h)} < c, \forall h \in \mathbb{N}
\] (7.24)
then

1. $\{u_h(t)\}$ converges to $u(t)$ in $\mathcal{K}$, for every fixed $t \in [0,T]$

2. $\{u_h\}$ converges to $u$ in $\mathcal{K}$.

**Proof.** In order to prove 1) we use Lemma 5.6, hence we have to see that for every $t \in [0,T]$
\[
(u_h,v_h)_{H_h} \to (u,v)_H
\]

for every sequence $\{v_h\}$, with $v_h \in H_h$ weakly converging in $\mathcal{K}$ to $v \in H$.

We have
\[
(u_h,v_h)_{H_h} = \int_{Q_h} \int_0^t T_h(t-s)f_h(s,P)dsv_h(P)d\mathcal{L}_3 + \delta_h \int_{S_h} \int_0^t T_h(t-s)f_h(s,P)dsv_h(P)d\sigma =
\]
\[ \int_0^t (T_h(t-s)f_h(s), v_h)_{H_h} \, ds. \]

From Theorem 6.12, since for every \( t \in [0, T] \), \( f_h(t) \to f(t) \) in \( \mathcal{H} \), then
\[
T_h(t)f_h(t) \to T(t)f(t) \text{ in } \mathcal{H};
\]
Moreover, since \( v_h \) weakly converges to \( v \) in \( \mathcal{H} \) for every \( t \in [0, T] \), it follows that
\[
(T_h(t-s)f_h(s), v_h)_{H_h} \to (T(t-s)f(s), v)_{H}.
\]

From Lemma 5.5, the contraction property of \( T_h \) and the assumption (7.24) \( \|f_h\|_{C^\alpha([0,T];H_h)} < c \), we have that there exists a constant \( c \) independent from \( h \) such that
\[
|(T_h(t-s)f_h(s), v_h)_{H_h}| \leq c.
\]
The claim follows from dominated convergence Theorem.

Now we prove 2). We note that
\[
\|u_h(t)\|_{H_h} \leq c_1 \|f_h\|_{C^\alpha([0,T];H_h)} \leq c, \quad \forall t \in [0, T]
\]
where the last inequality follows from (7.23) and (7.24).

Thus the sequence \( \{u_h\} \) is equibounded in \([0, T]\) and from 1)
\[
\|u_h(t)\|_{H_h} \to \|u(t)\|_{H}.
\]

By applying dominated convergence Theorem we obtain that
\[
\|u_h\|_{K_h} \to \|u\|_{K}.
\]
From 1) it follows in particular that for every \( t \in [0, T] \)
\[
(u_h(t), \psi(t))_{H_h} \to (u(t), \psi(t))_{H}, \quad \forall \psi \in C([0, T] \times \overline{Q}).
\]

Since
\[
|(u_h(t), \psi(t))_{H_h}| \leq c\|\psi\|_{C([0,T] \times \overline{Q})},
\]
From the dominated convergence Theorem we have
\[
(u_h, \psi)_{K_h} \to (u, \psi)_{K} \quad \forall \psi \in C([0, T] \times \overline{Q}).
\]

From Proposition 7.3 we proved 2). □

**Theorem 7.5.** With the same assumptions as in Theorem 7.4 we have
1. \( \frac{du_h}{dt} \) weakly converges to \( \frac{du}{dt} \) in \( \mathcal{K} \),
2. \( \{A_h u_h\} \) weakly converges to \( A u \) in \( \mathcal{K} \).

**Proof.** It holds
\[
\sup_{t \in [0,T]} \| \frac{du_h}{dt} \|_{H_h} \leq c
\]
in particular \( \frac{du_h}{dt} \in L^2([0,T];H_h) \) and there exists \( c \) independent from \( h \) such that
\[
\| \frac{du_h}{dt} \|_{L^2([0,T];H_h)} \leq c, \quad \forall h \in \mathbb{N}.
\]
From Lemma 5.8 there exists a subsequence, still denoted by \( \frac{du_h}{dt} \), which weakly converges in \( \mathcal{K} \) to a function \( v \in \mathcal{K} \).

We have to prove that \( v = \frac{du}{dt} \).

From definition of weak convergence we can write
\[
(\frac{du_h}{dt}, w_h)_{K_h} \to (v, w)_{K}
\]
for every sequence \( \{ w_h \} \in K_h, w_h \to w \) in \( \mathcal{K} \).

Choosing \( \{ w_h \} = \{ \varphi(t, P) \} \), where \( \varphi \in C^1([0, T]; C(Q)) \), we have

\[
\lim_{h \to \infty} \int_Q \int_0^T \frac{du_h(t, P)}{dt} \varphi(t, P) dt dm_h = \int_Q \int_0^T v(t, P) \varphi(t, P) dt dm.
\]

We integrate by parts and we obtain

\[
\int_Q \int_0^T \frac{du_h(t, P)}{dt} \varphi(t, P) dt dm_h = \int_Q (u_h(T, P) \varphi(T, P) - u_h(0, P) \varphi(0, P)) dm_h - \int_Q \int_0^T u_h(t, P) \frac{d\varphi(t, P)}{dt} dt dm_h.
\]

Passing to the limit in the first term in the right hand side of this equality for \( h \to \infty \), we obtain, by 1) in Theorem 7.4

\[
\int_Q (u_h(T, P) \varphi(T, P) - u_h(0, P) \varphi(0, P)) dm_h \to \int_Q (u(T, P) \varphi(T, P) - u(0, P) \varphi(0, P)) dm.
\]

It remains to study

\[
\lim_{h \to \infty} \int_Q \int_0^T u_h(t, P) \frac{d\varphi(t, P)}{dt} dt dm_h.
\]

It holds that

\[
\int_0^T \int_Q u_h(t, P) \frac{d\varphi(t, P)}{dt} dt dm_h = (u_h(t), \frac{d\varphi(t)}{dt})_{K_h}
\]

From 2) in Theorem 7.4

\[
(u(t), \frac{d\varphi(t)}{dt})_{K_h} \to (u(t), \frac{d\varphi(t)}{dt})_K,
\]

hence

\[
\int_Q \int_0^T v(t, P) \varphi(t, P) dt dm = \int_Q (u(T, P) \varphi(T, P) - u(0, P) \varphi(0, P)) dm + \int_Q \int_0^T u(t, P) \frac{d\varphi(t, P)}{dt} dt dm,
\]

which implies \( v = \frac{du}{dt} \).

8. **Strong interpretation.**

8.1. **The fractal case.**

**Theorem 8.1.** Let \( u \) be the solution of the problem (7.18). Then for every fixed \( t \in [0, T] \)

\[
\begin{cases}
  u_t(t, P) - Lu(t, P) = f(t, P) & \text{for a.e. } P \in Q \\
  \frac{\partial u}{\partial n} - \in (B^2_\beta(S))' \\
  \frac{\partial u}{\partial n} = 0 & \beta = \frac{d}{2} \beta \\
  u(0, P) = 0 & \text{in } H^{-1/2}(\tilde{\Omega})
\end{cases}
\]

and for every \( z \in \mathcal{D}(S) \)

\[
\langle u_t, z \rangle_{(\mathcal{D}(S))', \mathcal{D}(S)} = -E_S(u|S, z) - \left\langle \frac{\partial u}{\partial n}, z \right\rangle_{(\mathcal{D}(S))', \mathcal{D}(S)} + \langle f, z \rangle_{(\mathcal{D}(S))', \mathcal{D}(S)} - \int_S bu|S z dg,
\]

(8.26)
where \( \frac{\partial u}{\partial n_A} \) is the co-normal derivative, defined as an element of \( (B^2_\beta(S))' \). Moreover \( \frac{\partial u}{\partial n_A} \in C([0,T];B^2_\beta(S))' \).

**Proof.** Let \( \varphi \) be an arbitrary function in \( C_0^\infty(Q) \), by multiplying equation (7.18) in \((P)\) and integrating over \( Q \) we obtain

\[
\int_Q u_t \varphi dm = \int_Q \hat{A}u \varphi dm + \int_Q f \varphi dm.
\]

From (4.11) we have

\[
\int_Q u_t \varphi dm = -\hat{E}(u, \varphi) + \int_Q f \varphi dm.
\]

Since \( \varphi \) is compactly supported on \( Q \), then

\[
\int_Q (ADu \cdot D\varphi + a_0u\varphi) d\mathcal{L}_3 = \int_Q f \varphi d\mathcal{L}_3 - \int_Q u_t \varphi d\mathcal{L}_3 + \sum_{i=1}^3 \int_Q b_i \partial_i u \varphi d\mathcal{L}_3.
\]

Hence it follows that for every fixed \( t \in [0,T] \)

\[
\sum_{i,j=1}^3 \partial_i (a_{ij}(P) \partial_j u(t, P)) = u_t(t, P) - f(t, P) - \sum_{i=1}^3 b_i(P) \partial_i u(t, P) - a_0(P)u(t, P),
\]

in \( L^2(Q) \). Since the right hand side of (8.27) belongs to \( L^2(Q) \), we deduce that

\[
\sum_{i=1}^3 \partial_i (a_{ij}(P) \partial_j u(t, P)) \in C([0,T]; L^2(Q)),
\]

hence \( u \in C([0,T]; V(Q)) \), where

\[
V(Q) = \left\{ u \in H^1(Q) : \sum_{i,j=1}^3 \partial_i (a_{ij} \partial_j u) \in L^2(Q) \right\}.
\]

Here the derivatives are intended in the distributional sense. By proceeding as in [28] one can prove a Green formula for the domain \( Q \). The boundary \( \partial Q \) is an arbitrary closed set in the sense of [21], that is \( \partial Q = S \cup \Omega \), and it supports \( w \) the measure \( \tilde{m} = \chi_S g + \chi_{\Omega} \sigma \). Hence we have:

\[
\begin{aligned}
\left\langle \frac{\partial u}{\partial n_A}, \varphi \right\rangle_{H^{-1/2}(\Omega), H^{1/2}(\tilde{\Omega})} + \left\langle \frac{\partial u}{\partial n_A}, \varphi \right\rangle_{(B^2_\beta(S))', B^2_\beta(S)} = \\
\int_Q ADu(t, P) \cdot D\varphi(P) d\mathcal{L}_3 + \int_Q \sum_{i,j=1}^3 \partial_i (a_{ij}(P) \partial_j u(t, P)) \varphi d\mathcal{L}_3
\end{aligned}
\]

for every \( t \in [0,T] \) and for every \( \varphi \in H^1(Q) \).

By following [32] it holds \( \frac{\partial u}{\partial n_A} \in C([0,T]; (B^2_\beta(S))') \).

Now let \( \psi \) be an arbitrary function in \( V(Q,S) \) for every fixed \( t \in [0,T] \). Multiplying (7.18) and integrating over \( Q \), we obtain

\[
\begin{aligned}
\int_Q u_t \psi d\mathcal{L}_3 + \int_S u_t \psi dg \\
= - \int_Q (ADu \cdot D\psi + a_0u\psi) d\mathcal{L}_3 + \sum_{i=1}^3 \int_Q b_i \partial_i u \psi d\mathcal{L}_3 + \\
- E_S(u|_S, \psi|_S) - \int_S b|_S u|_S \psi|_S dg + \int_Q f \psi d\mathcal{L}_3 + \int_S f|_S \psi|_S dg.
\end{aligned}
\]
Taking into account (8.28), we get
\[ \int_Q u_t \psi d\mathcal{L}_3 + \int_S u_t \psi dg = \left\langle \frac{\partial u}{\partial n_A}, \varphi \right\rangle - \left\langle \frac{\partial u}{\partial n_A}, \psi \right\rangle_{(B^2_0(\Omega))'} - \int_Q (\sum_{i,j=1}^3 \partial_i (a_{ij} \partial_j u) \psi - a_0 u \psi) d\mathcal{L}_3 + \int_S b_i \partial_i u \psi d\mathcal{L}_3 - E_S(u|_S, \psi|_S) - \int_S b u|_S \psi|_S dg + \int_Q f \psi d\mathcal{L}_3 + \int_S f|_S \psi|_S dg. \]

Since \( u_t - \sum_{i,j=1}^3 \partial_i (a_{ij} \partial_j u) + \sum_{i=1}^3 b_i \partial_i u + a_0 u - f = 0 \) a.e. in \( Q \), we have
\[ \int_S u_t \psi dg = - \left\langle \frac{\partial u}{\partial n_A}, \varphi \right\rangle_{H^{-1/2}(\Omega)} - \left\langle \frac{\partial u}{\partial n_A}, \psi \right\rangle_{(B^2_0(\Omega))'} + E_S(u|_S, \psi|_S) - \int_S b u|_S \psi|_S dg + \int_S f|_S \psi|_S dg \] (8.29)

From Proposition 4.4, by proceeding as in Section 6 of [27] with a suitable choice of the function \( \psi \), we have
\[ u_t - \Delta_S u + b u = - \frac{\partial u}{\partial n_A} + f \] (8.30)
in \((\mathcal{D}(S))'\) and \( \frac{\partial u}{\partial n_A} = 0 \) in \( H^{-1/2}(\Omega) \).

**8.2. The prefractal case.**

**Theorem 8.2.** Let \( u_h \) be the solution of problem (7.19). Then we have for every fixed \( t \in [0, T] \)
\[ \begin{align*}
(u_h)_t(t, P) - L_h u_h(t, P) &= f_h(t, P) \quad \text{for a.e.} \ P \in Q \\
\frac{\partial u_h}{\partial n_{A_h}} &\in (H^{-1/2}(\partial Q_h)), \\
u_h(0, P) &= 0 \quad \text{in} H^{1/2}(S_h) \\
\frac{\partial u}{\partial n_{A_h}} &= 0 \quad \text{on} \{0 \in H^{-1/2}(\Omega_h)\}
\end{align*} \]

and
\[ \delta_h(u_h)_t - \Delta_{S_h} u_h + \delta_h b u_h = - \frac{\partial u_h}{\partial n_{A_h}} + \delta_h f_h, \] (8.31)
in \( H^{-1/2}(S_h) \) \( \frac{\partial u_{a_h}}{\partial n_{A_h}} \) is the inward co-normal derivative and \( \Delta_{S_h} \) is the piece-wise tangential Laplacian associated to the Dirichlet form \( E_{S_h} \). Moreover \( \frac{\partial u}{\partial n_{A_h}} \in C([0, T]; (H^{-1/2}(\partial Q_h))). \)

**Proof.** The first equality follows by proceeding as in Theorem 8.1. From this it follows that for every \( t \in [0, T] \)
\[ u_h(t, \cdot) \in V(Q_h) = \left\{ u_h \in H^1(Q) : \sum_{i,j=1}^3 \partial_i (a_{ij} \partial_j u_h) \in L^2(Q_h) \right\}. \]

Proceeding as in section 6.2 of [32] we prove that for every \( t \in [0, T] \), \( \frac{\partial u_h}{\partial n_{A_h}} \in (H^{-1/2}(\partial Q_h)). \) By proceeding as in Theorem 8.1 we can prove that for every \( t \in [0, T] \)
and for every \( z \in V(Q, S_h) \)
\[ \delta_h((u_h(t))_t, z)_{L^2(S_h)} - (\Delta_{S_h} u_h(t), z)_{(H^{1/2}(S_h))'} - (\delta_h b u_h(t), z)_{L^2(S_h)} = - \left\langle \frac{\partial u_h}{\partial n_{A_h}}, z \right\rangle_{(H^{1/2}(\partial Q_h))'} + \delta_h f_h(t, z)_{L^2(S_h)} \]
that is the boundary condition
\[ \delta_h(u_h)_t - \Delta_{S_h} u_h + \delta_h b u_h = - \frac{\partial u_h}{\partial n_{A_h}} + \delta_h f_h \]

holds in the dual of \( H^2(S_h) \) (see [28]) and \( \frac{\partial u_h}{\partial n_{A_h}} = 0 \) in the dual of \( H^2(\tilde{\Omega}_h) \).

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