Reliability analysis of karst roof stability based on strength reduction method

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Abstract: The roof stability of karst caves is of great concern for engineers when they are conducting construction work in karst areas. Accurately determining the factor of safety (FOS) of the cave roof against instability is generally difficult because of the uncertainties involved in the rock properties. In this paper, a reliability method is introduced to access the stability of cave roofs considering the effect of uncertainties, where the strength reduction method is used to compute the FOS of the roof. To facilitate its practical application, an interface program is developed to conduct the reliability analysis automatically by utilizing an existing deterministic numerical package. Both the size of the cave and the thickness of the roof affect the plastic zones in the rock. The stability of the roof will increase with the increase in the roof thickness as the shape of the cave is fixed, and it will increase with the increase in the cave height when the roof thickness and width of the cave are fixed. The reliability of the roof stability can be assessed efficiently using the response surface method as described in this paper. Unlike in the FOS-based method, the effect of uncertainties in the rock properties can be quantitatively considered in the reliability-based method, which may be more informative for decision-making.

Keywords: Karst caves; Roof stability; Strength reduction method; Reliability analysis

1. Introduction
Karst landforms are widely distributed in southwest China because of the special geological conditions and abundant rainfall in the region. When an expressway is constructed in karst area, the thickness between the roof of karst cave and the ground surface is important to the stability of the cave. In geotechnical engineering, empirical methods have been suggested to assess the stability of cave roofs [1,2]. However, such methods often heavily rely on the experience of engineers and could be conservative. To overcome the limitations of empirical methods, mechanics-based methods have been suggested to analyze the roof stability of karst caves [3-7], which are often based on limit equilibrium analysis. To consider the uncertainties in rock properties, methods have also been developed to study the probability of collapse of karst caves due to roof instability [8-11].

Unlike the limit equilibrium methods, the strength reduction method can be used for geotechnical stability analysis without the assumption of failure mechanism of geotechnical problem under investigation [12,13] and has been found useful for analyzing the stability of karst roofs [6]. However,
the effect of uncertain rock properties is not considered. The possible reason is that the strength reduction method is often conducted using a standalone numerical package, which does not have the capability of geotechnical reliability analysis. The purpose of this paper is to propose an efficient method to determine the safety thickness of karst roof by using the strength reduction method considering uncertain rock properties. The structure of this paper is as follows. First, the numerical model that is used to calculate the factor of safety (FOS) of the roof stability of a karst cave based on the strength reduction method is described. Then, the method to analyze the reliability of a karst cave against roof instability is introduced. Finally, the suggested method is used to analyze the stability of a karst roof, and factors that affect the reliability of the karst roof are investigated.

2. Strength reduction method for karst roof stability analysis

In the numerical model, the rock mass is modeled as a Mohr–Coulomb material. In the strength reduction method, the FOS is defined as the ratio of the actual shear strength of roof strata to the reduced shear strength at critical failure [14], which can be determined through a trial-and-error approach. To determine the FOS of the karst roof, the trial values of the shear strength parameters are defined as follows:

\[
\begin{align*}
c_{\text{trial}} &= \frac{c}{FOS_{\text{trial}}} \tan \phi \\
\phi_{\text{trial}} &= \arctan \left( \frac{c}{FOS_{\text{trial}}} \right)
\end{align*}
\]

where \(c_{\text{trial}}\) and \(\phi_{\text{trial}}\) denote the soil cohesion and the friction angle under the limit equilibrium state after reduction, respectively, and \(FOS_{\text{trial}}\) is a trial value of the FOS. If the numerical model converges based on these trial parameters, then the shear strength parameters can be further reduced. If the numerical model does not converge, then the shear strength parameters should be increased. The FOS of karst roof is the trial value of the scaling factor at which the numerical model is at the boundary between convergence and non-convergence. In this paper, the strength reduction method embedded in FLAC3D is used to calculate the FOS of the karst cave.

3. Suggested method for karst roof reliability analysis

3.1 First-order reliability method

Let \(x\) denote the uncertain shear strength parameters involved in the karst cave stability analysis. The performance function can then be written as follows:

\[
g(x) = FOS - 1
\]

In this paper, the first-order reliability method (FORM) [15] is used to calculate the reliability index of the karst cave, where the reliability index \(\beta\) and failure probability \(p_f\) can be calculated as follows: [16]

\[
\beta = \min_{g(x) = 0} \sqrt{y^TR^{-1}y}
\]

\[
p_f = 1 - \Phi(\beta)
\]

where \(y\) is the reduced variables of \(x\), and \(R\) is the correlation matrix of random variables. As can be seen from Eq. (4), the reliability index of the karst cave can be found through solving a constraint minimization problem. In Eq. (4), the minimization point of \(y\) is often called the design point, which is denoted as \(y_d\) herein.

3.2 Response surface method

In this paper, the FOS of the karst cave is calculated with the strength reduction method, which makes it difficult to solve the minimization problem in Eq. (4) directly. To decouple the numerical simulation and the reliability analysis, the response surface method is utilized, where a simple polynomial surrogate model is built to approximate the numerical model in the reliability analysis. In this study, the response surface method suggested in Zhang et al. [17] is adopted, where the performance function is approximated as follows:

\[
G(y) \approx b_0 + \sum_{i=1}^{k} b_i y_i + \sum_{i=1}^{k} b_{k+1} y_i^2
\]
where $y_i$ is the $i^{th}$ reduced variable, $k$ is the dimension of $x$, and $b_i (i = 0, 1, 2, ..., 2k)$ are the coefficients to be calibrated.

To calibrate the $2k + 1$ coefficients in Eq. (6), the FOS of the karst cave can be first determined with the strength reduction method at $2k + 1$ points. Let $y_c = \{y_{c1}, y_{c2}, ..., y_{cn}\}$ denote the center of the calibration points with the rest of the calibration points being $\{y_{c1} \pm m, y_{c2}, ..., y_{cn}\}, \{y_{c1}, y_{c2} \pm m, ..., y_{cn}\}, ... \{y_{c1}, y_{c2}, ..., y_{cn} \pm m\}$, respectively, where $m$ is a parameter that determines the distance among different calibration points. After an initial value of $y_d$ is found, one should check if convergence is achieved. If not, then $y_d$ is iterated as the new sampling center $y_c$ and the sampling range is updated for a new $\beta$ until the difference between the current and previous $\beta$ is smaller than 0.01. Thus, the final solution of $\beta$ is obtained.

4. Interface program for automatic reliability analysis

To facilitate the application of the suggested method, an interface program was developed in MATLAB such that the reliability analysis can be conducted automatically. The workflow of the interface program is as follows:

1. Determine the calibration points in the space of $y$ and convert them into points in $x$ in MATLAB. The coordinates of these points in original space can be written into text files, which can be read by FLAC$^{3D}$.

2. Invoke FLAC$^{3D}$ in MATLAB to evaluate the values of FOS of the karst cave at these calibration points through the command “flac3d700_gui.exe” in MATLAB.

3. Build the response surface based on calibration points in the reduced space and the corresponding values of the FOS of the karst cave at these points.

4. Solve $\beta$ and $y_d$ with MATLAB through the response surface.

5. Check if convergence is achieved. If not, update the response surface for new $\beta$ and $y_d$ until convergence is achieved.

5. Application example

5.1. Engineering background

To demonstrate the usefulness of the suggested method, the roof stability of karst caves in Guangna Expressway, which is located in Yunnan Province of China, is analyzed here. The lithology of the stratum that the highway crosses during its construction is mainly carbonate and elastic, where hidden karst is relatively developed with a height of around 2–6 m and a diameter of around 4 m. Figure 1 shows the geometry model to assess the roof stability of the karst caves along this highway by simplifying the karst cave as an ellipsoid, where $w$ denotes the width of the cave, $h$ denotes the height of the cave, and $H$ denotes the thickness of the roof. Table 1 shows the typical values of the rock properties considered in the numerical model, which are determined based on data from the standard for classification of engineering rock mass [18].

![Figure 1. Geometry of the karst model.](image-url)
Table 1. Mechanical parameters of surrounding rock.

| Elastic modulus (MPa) | Density (kg/m³) | Tensile strength (Mpa) | Poisson’s ratio | Cohesion (kPa) | Friction angle (º) |
|-----------------------|----------------|------------------------|-----------------|---------------|-------------------|
| 2000                  | 3              | 14                     | 0.3             | 100           | 30                |

5.2. Mechanism of roof instability

To analyze the mechanism of roof instability, Figures 2(a), 2(b), and 2(c) show the plastic zones in the rock mass as the thickness of the roof varies and when a cave with a size of \( w = 4 \text{ m} \) and \( h = 2 \text{ m} \) is considered. In Figure 2(a), the thickness of the roof is 0.5 m, and the plastic zone is fully developed and connected to the ground surface, thereby implying that the failure of the roof is quite likely to happen. The FOS of the roof in such a case is 0.85. For comparison, the plastic zones in Figures 2(b) and 2(c) where the roof thickness are 1.0 and 1.5 m, respectively, are much smaller and are not connected to the ground surface. As such, the cave should be still stable in these two cases. The values of FOS of the roof in these two cases are 1.20 and 1.50, respectively. For comparison, Figures 3(a)–(c) and Figures 4(a)–(c) also show how the plastic zones in the rock mass vary with the thickness of the roof of the cave for caves of other sizes. The phenomena in Figures 3(a)–(c) and 4(a)–(c) are similar to those observed in Figures 2(a)–(c), i.e., the sizes of the plastic zones decrease as the thickness of the roof increases, indicating that a cave with the same size will be less likely to fail when it is at a greater depth.

![Figure 2](image1.png)

**Figure 2.** Distributions of plastic zone in karst roofs at different thicknesses for \( w = 4 \text{ m} \), \( h = 2 \text{ m} \)

![Figure 3](image2.png)

**Figure 3.** Distributions of plastic zone in karst roofs at different thicknesses for \( w = 4 \text{ m} \), \( h = 4 \text{ m} \)

![Figure 4](image3.png)

**Figure 4.** Distributions of plastic zone in karst roofs at different thicknesses for \( w = 4 \text{ m} \), \( h = 6 \text{ m} \)

In Figures 2(a), 3(a), and 4(a), the caves have the same roof thickness of \( H = 0.5 \text{ m} \) and the same width of \( w = 4 \text{ m} \). The heights of these caves, however, are different, which are \( h = 2 \text{ m} \), \( h = 4 \text{ m} \), and \( h = 6 \text{ m} \), respectively. A comparison of these three figures shows that the sizes of the plastic zones also decrease as the height of the case increases. Therefore, the values of the FOS of the cave also increase with the height of the cave. A similar phenomenon can also be observed by comparing Figures 2(b)–
4(b) and by comparing Figures 2(c)–4(c). Hence, a cave with a greater size may not necessarily be more dangerous.

5.3. Reliability analysis
In the above stability analysis, the uncertainties in the rock properties are not considered. To quantify the effect of such uncertainties, the stability of the karst roof is also analyzed using the response surface method described in Section 3.2. In the reliability analysis, the mean and the coefficient of variation of cohesion $c$ ($x_1$) are 100 kPa and 0.3, respectively, and the mean and the coefficient of variation of internal friction angle $\varphi$ ($x_2$) are 30° and 0.2, respectively. The statistics of the rock properties are determined with reference to the local experience. Both parameters are assumed log-normally distributed and statistically independent. To demonstrate the suggested reliability method, the reliability of the karst roof for the case of height $h = 2$ m and roof thickness $H = 1$ m is first analyzed, where $m = 1$ is used for the calibration of the first response surface and $m = 0.5$ for the rest. Table 3 shows the results during the iteration process. This table shows that the algorithm converges in two steps, thereby indicating that the response surface is very efficient. At the design point, the absolute values of the coordinates of different variables in the reduced space represent its importance to the reliability of the problem. The more it deviates from zero, the more it is important to the reliability of the stability of the roof. As can be seen from Table 3, the absolute values of the coordinates of the cohesion and the friction angle are 0.637 and 0.301, respectively, at the design point, thereby indicating that the uncertainty in the cohesion is more important to the reliability of the karst roof.

| Table 2. Iteration of reliability calculation for karst roof ($b = 2$ m, $H = 1$ m). |
|-----------------|--------------------------|-----------------|-----------------|
| step size | Iteration step | Reduced space | Original space | $\beta$ |
| $m = 1$ | 1 | $y_1 : 0.000$ | $0.000$ | $x_1 : 100.00$ | 30.000 | 0.69 |
| | | $y_4 : -0.654$ | -0.244 | $x_4 : 80.380$ | 28.536 |
| $m = 0.5$ | 2 | $y_1 : -0.654$ | -0.244 | $x_1 : 80.380$ | 28.536 | 0.70 |
| | | $y_4 : -0.637$ | -0.301 | $x_4 : 80.890$ | 28.194 |

5.4. Factors affecting the reliability of karst roofs
To investigate the factors that affect the reliability of karst roofs, Figure 5 shows how the reliability index of the karst cave changes with the thickness of the roof and the height of the cave where the width of the cave is fixed at $w = 4$ m. As can be seen from this figure, when the thickness of the roof is the same, the reliability index of the cave increases with the height of the cave. When the height of the cave is the same, the reliability index of the cave increases with the thickness of the roof. It is also interesting to observe that when the thickness is small, the change of the reliability index with the height of the cave is more obvious, thereby indicating that the reliability index of the cave is more sensitive to the height of the cave when the thickness of the roof is smaller.
6. Conclusions

In this paper, a reliability-based method is used to analyze the stability of karst cave roofs. The conclusions obtained from this paper are as follows:

5.5. Reliability-based assessment of roof stability

The roof stability is first examined based on the FOS-based method. Figure 6 shows how the FOS of the roof changes with the height of the cave and the thickness of the roof, where the mean values of the cohesion and the friction angle are used for FOS calculation. The target FOS is 1.25 based on the local design code [1,2]. On the basis of such a criterion, caves with a thickness less than 1.0 m and a height less than 2.0 m will be considered as not stable.

To show the effect of uncertainties on the assessment of roof stability, the roof stability is also assessed through results from the reliability analysis. The Geotechnical Engineering Office of Hong Kong [19] and the Australian Geomechanics Society [20] recommend that the target failure probability of the road without geogrid mitigation is in the range of $10^{-3}$–$10^{-5}$ per year. Note that the failure probability calculated in this paper does not mean the probability of failure in a year. As an illustration, suppose the target failure probability is $10^{-3}$, which corresponds to a target reliability index of 3.0 according to Eq. (5). After this target reliability index line is plotted in Figure 7, the caves with a thickness less than 2 m and a height less than 4 m will be considered as not stable. The results of the FOS-based method and the reliability-based method are different; the uncertainties involved in the cave stability evaluation are more rigorously considered in the reliability-based method than in the FOS-based method.

Figure 5. Relationships between $H$ and $\beta$ under different cave shapes ($w = 4$ m)

Figure 6. Roof stability assessment using FOS-based method.

Figure 7. Roof stability assessment using reliability method.
Both the size of the cave and the thickness of the roof affect the plastic zones in the rock. The stability of the roof will increase with the roof thickness as the shape of the cave is fixed and will increase with the height of the cave when the roof thickness and width of the cave are fixed.

(2) The reliability of the roof stability can be assessed efficiently using the first-order reliability method based-response surface method as described in this paper.

(3) Unlike in the FOS-based method, the effect of uncertainties in the rock properties can be quantitatively considered in the reliability-based method, which may be more informative for decision-making.

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