Control the movement of an inverted pendulum by using a first-order type Takagi-Sugeno-Kang fuzzy controller

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Abstract. The article considers the possibility of using a fuzzy first-order Takagi-Sugeno-Kang type controller to control the movement of an inverted pendulum in order to achieve dynamic stabilization of its unstable equilibrium. The structure of linguistic variables and the base of rules for such a fuzzy controller are proposed. Graphs of changes in the controlled quantities (horizontal carriage displacement and angle of deviation from the vertical) and the control action are constructed, they illustrating the onset of dynamic stabilization of the unstable equilibrium of an inverted pendulum.

Introduction
The problem of inverted pendulum unstable equilibrium dynamic stabilization has been developed in the theory of automatic control, since this object can be considered as an idealization of unstable mechanical systems. Over the past decades, theoretical studies of this problem have been carried out on the basis of modern microprocessor tools and software.

A large number of domestic and foreign publications devoted to the development of algorithms for dynamic stabilization of unstable equilibrium have appeared. This is due to the fact that the class of real control objects with a similar mathematical model has now expanded significantly. In cosmonautics an inverted pendulum, you can consider the movement of a rocket during vertical takeoff [1]. In cybernetics, a similar model is used to describe walking robots, which are called "biped walking machines" [2].

1. Mathematical model of an inverted pendulum
An inverted pendulum is a rod that is hinged on a movable carriage [3, 4, 5]. The center of gravity of the rod is located above the support point, so it is in a state of unstable equilibrium. The rod easily deviates from the equilibrium position by a certain angle. The carriage must be subjected to some action in the direction opposite to the deflection of the rod to return the pendulum to its vertical position. The carriage will make some movement along the horizontal axis.

The rod passes the equilibrium position and deviates in the other direction under external influence. In order for the rod to be in a strictly vertical position, the external action must be oscillatory. The task of inverted pendulum unstable equilibrium dynamic stabilization is to keep the angle of the rod deviation from the vertical in a small neighborhood of its equilibrium point. The pendulum oscillates near an unstable equilibrium position against the background of cart oscillations. A theoretical prediction of this phenomenon was made by Stephenson in 1908. [5]. A physical explanation for the dynamic stabilization of an inverted pendulum was proposed by academician Kapitsa in 1951. [3, 4].

To create a mathematical model of dynamic stabilization of the unstable equilibrium of an inverted pendulum, consider the forces acting on the carriage and the rod separately (fig. 1).
Let's assume that the cart can move along the axis Ox. In this case it is sufficient to take into account the projections of the active forces only on the axis Ox. The rod makes an oscillatory movement in the vertical direction, so it is necessary to take into account the projection of the active forces as an axis Ox and Oy. The following system of equations is valid for the carriage-rod system [3, 4]:

\[
\begin{align*}
    m \frac{d^2x(t)}{dt^2} &= x(t) - L \cdot \sin(\varphi(t)) = H(t), \\
    m \frac{d^2\varphi(t)}{dt^2} &= L \cdot \cos(\varphi(t)) = V(t) - mg, \\
    J \frac{d^2\varphi(t)}{dt^2} &= L \cdot H(t) \cdot \cos(\varphi(t)) + L \cdot V(t) \cdot \sin(\varphi(t)), \\
    M \frac{d^2x(t)}{dt^2} &= u(t) - H(t) - F \cdot \frac{dx(t)}{dt},
\end{align*}
\]

(1)

where \( M \) – carriage mass; \( m \) – rod mass; \( F \) – coefficient of viscous friction when moving the carriage; \( L \) – distance between the axis of the pendulum and the center of its mass; \( J \) – moment of inertia of the pendulum relative to the center of mass; \( u(t) \) – force applied to the cart (control); \( x(t) \) – cart offset coordinate; \( \varphi(t) \) – angle of the rod deviation from the vertical; \( H(t) \) – horizontal reaction force on the pendulum axis; \( V(t) \) – vertical reaction force on the pendulum axis; \( g \) – acceleration of gravity.

After converting first and second equations of system (1), we get the expression:

\[
\begin{align*}
    \left\{ \begin{array}{l}
        m \ddot{x}(t) + mL\varphi^2(t) \cdot \sin \varphi(t) - mL\dot{\varphi}(t) \cdot \cos \varphi(t) = H(t), \\
        -mL\dot{\varphi}^2(t) \cdot \cos \varphi(t) - mL\varphi(t) \cdot \sin \varphi(t) = V(t) - mg.
    \end{array} \right.
\]

(2)

Substitute \( H(t) \) from first equation into second equation of system (2) and get the following expression:

\[
(M + m)\ddot{x}(t) + F\ddot{x}(t) + mL\dot{\varphi}^2(t) \sin \varphi(t) - mL\dot{\varphi}(t) \cos \varphi(t) = u(t),
\]

(3)

Substitute \( H(t) \) and \( V(t) \) from system (2) respectively into third equation of system (1) and get the expression:

\[
(J + mL^2)\ddot{\varphi}(t) - mL\ddot{x}(t) \cdot \cos \varphi(t) = mgL \cdot \sin \varphi(t).
\]

(4)

Nonlinear equations (3) and (4) can be linearized. We can decompose the \( \sin \varphi \) and \( \cos \varphi \) functions into Maclaurin series. In this case, it is advisable to use only the first members of the series (\( \sin \varphi \rightarrow \varphi \); \( \cos \varphi \rightarrow 1 \)). The mathematical model of inverted pendulum unstable equilibrium dynamic stabilization will have the following form:

\[
\begin{align*}
    \left\{ \begin{array}{l}
        (J + mL^2)\ddot{\varphi}(t) - mL\varphi(t) = mL\ddot{x}(t), \\
        (M + m)\ddot{x}(t) + F\ddot{x}(t) - mL\dot{\varphi}(t) = u(t).
    \end{array} \right.
\]

(5)
2. Mathematical model of fuzzy controller operation

The fuzzy controller is a fuzzy output system with two input variables \( x \) and \( \varphi \) (\( x \) is the horizontal displacement of the carriage, \( \varphi \) is the angle of deviation of the rod from the vertical) and one output \( u \) (control action on the carriage) [6]. The diagram of the control system for dynamic stabilization of the unstable equilibrium of an inverted pendulum using a fuzzy controller is shown in Fig. 2.

The control system for dynamic stabilization of unstable equilibrium has a negative feedback. The control object is an inverted pendulum. The fuzzy controller in this system generates the law of the control action \( u(t) \) on the carriage.

One of the new types of fuzzy controllers is the Takagi-Sugeno-Kang type. It was first proposed in 1985 by Japanese mathematicians Takagi, Sugeno, and Kang [7]. A feature of such fuzzy controllers is the use of the functional dependence rule on input variables in the conclusion. The most commonly used functional dependency is in the form of the first degree polynomial. The fuzzy controller in this case will be the first-order Takagi-Sugeno-Kang type.

The input linguistic variables of the fuzzy controller \( x \) and \( \varphi \) are represented as term sets. The structure of term sets of linguistic variables includes two far terms and one or more middle terms. The most widespread are term sets with two far and three middle terms. In this case, the linguistic variables \( x \) and \( \varphi \) will take five terms (values): \( NB \) – negative big, \( NS \) – negative small, \( Z \) – neutral (zero), \( PS \) – positive small, \( PB \) – positive big. Each term is defined by the membership function, which determines the degree of relation to this term.

The input linguistic variables \( x \) and \( \varphi \) and the output linguistic variable \( u \) are linked by a rule base that contains fuzzy statements implications. Each rule contains a condition, conclusion, and weight. The first-order Takagi-Sugeno-Kang fuzzy controller rule base contains expressions of the form:

\[
(x = X_1) \land (\varphi = P_1) \Rightarrow (u = \alpha_{10} + \alpha_{11}x + \alpha_{12}\varphi), \omega_1,
\]

... \( (x = X_S) \land (\varphi = P_S) \Rightarrow (u = \alpha_{S0} + \alpha_{S1}x + \alpha_{S2}\varphi), \omega_S, \)

where \( X_1, X_2, X_3, X_4, X_5 \) – terms of the input linguistic variable \( x \) \( (X_1 – NB, X_2 – NS, X_3 – Z, X_4 – PS, X_5 – PB) \); \( P_1, P_2, P_3, P_4, P_5 \) – terms of the input linguistic variable \( \varphi \) \( (P_1 – NB, P_2 – NS, P_3 – Z, P_4 – PS, P_5 – PB) \); \( S \) – size of the rule base; \( \alpha_{10}, \alpha_{20}, \ldots, \alpha_{S0} \) – the free coefficients of the linear dependence; \( \alpha_{11}, \alpha_{21}, \ldots, \alpha_{S1} \) – the coefficients of the linear dependence at variable \( x \); \( \alpha_{12}, \alpha_{22}, \ldots, \alpha_{S2} \) – the coefficients of the linear dependence at variable \( \varphi \); \( \omega_1, \omega_2, \ldots, \omega_S \) – rule weight.

The size of the rule base \( S \) for a first-order Takagi-Sugeno-Kang fuzzy controller is determined by the formula:

\[
S = n^N,
\]

where \( n \) – number of terms describing the term sets of linguistic variables of a fuzzy controller; \( N \) – number of input variables of the fuzzy controller.

For a fuzzy controller of the control system for dynamic stabilization of an unstable equilibrium of an inverted pendulum that has two input variables \( (N = 2) \) with term sets containing five terms \( (n = 5) \), the size of the rule base \( S \) will be 25 \( (S = 5^2) \).
The process of fuzzy inference in the fuzzy controller type Takagi-Sugeno-Kang of the first order contains three stages [6]: 1) input variables fuzzification; 2) determination of the rules truth degree; 3) output variables defuzzification.

The fuzzification process consists in determining the values of the membership functions for each term of the input linguistic variables. For this purpose, the values of \( x \) and \( \varphi \) - elements of the sets \( X \) and \( P \) of the corresponding input linguistic variables are determined. Then all the values of the membership functions are calculated \( \mu_{NB}(x), \mu_{NS}(x), \mu_{Z}(x), \mu_{PS}(x), \mu_{NB}(\varphi), \mu_{NS}(\varphi), \mu_{Z}(\varphi), \mu_{PS}(\varphi), \mu_{PB}(\varphi) \).

The \( i \)-th rule conditions truth degree the \( d(R_i) \) is calculated using the formula as a conjunction of fuzzy statements:

\[
d(R_i) = \min \left( \mu_{X_i}(x), \mu_{P_i}(\varphi) \right),
\]

where \( x \) и \( \varphi \) - fuzzy linguistic variables; \( X_i, P_j \) - terms of the corresponding linguistic variables; \( \mu_{X_i}(x) \) – term \( X_i \) membership function; \( \mu_{P_j}(\varphi) \) – term \( P_j \) membership function.

The defuzzification process consists of defining specific numeric values of the output variable \( u \). Defuzzification of the output variable \( u \) in fuzzy first-order Takagi-Sugeno-Kang controller is based on a modified center of gravity method. The \( u \) value is calculated using the formula:

\[
u = \left( \sum_{i=1}^{S} d(R_i) \cdot (\alpha_{i0} + \alpha_{i1}x + \alpha_{i2}\varphi) \cdot \omega_i \right) \left( \sum_{i=1}^{S} d(R_i) \right)^{-1}.
\]

The operation of a fuzzy Takagi-Sugeno-Kang controller of the first order can be represented in the form of the diagram shown in Fig. 3.

### Figure 3. The scheme of fuzzy controller

#### 3. The inverted pendulum movement control

Consider an inverted pendulum with a carriage mass \( M \) equal to 0.455 kg and a rod mass \( m \) equal to 0.215 kg. The distance from the pendulum axis to the center of mass of the rod \( L \) is 0.291 m, and the length of the rod \( l \) is 0.582 m. The natural oscillation frequency \( \omega_0 \) is 5 rad/s [8].

The term sets of input linguistic variables \( x \) and \( \varphi \) are shown in Fig. 4 and 5.
The names of the terms included in the rules, the coefficients of input variables $x$ and $\varphi$ linear dependence, and the rules weights are shown in Table 1.

**Table 1.** The rule base of a Takagi-Sugeno-Kang of the first order fuzzy controller type

| $i$ | $x$ | $\varphi$ | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\omega$ | $i$ | $x$ | $\varphi$ | $\alpha_0$ | $\alpha_1$ | $\alpha_2$ | $\omega$ |
|-----|-----|-----------|-----------|-----------|-----------|---------|-----|-----|-----------|-----------|-----------|-----------|---------|
| 1   | $NB$ | $NB$      | 0.223     | 0.039     | 0.148     | 0.887   | 14  | $Z$ | $PS$      | -0.132    | -0.020    | -0.101    | 0.636   |
| 2   | $NB$ | $NS$      | 0.424     | -0.007    | -0.279    | 0.897   | 15  | $Z$ | $PB$      | -0.260    | -0.035    | -0.149    | 0.638   |
| 3   | $NB$ | $Z$       | 0.398     | 0.000     | 0.200     | 0.836   | 16  | $PS$| $NB$      | 0.478     | 0.055     | -0.167    | 0.484   |
| 4   | $NB$ | $PS$      | -0.212    | -0.041    | -0.298    | 0.458   | 17  | $PS$| $NS$      | -0.072    | -0.058    | 0.185     | 0.610   |
| 5   | $NB$ | $PB$      | 0.220     | -0.083    | -0.056    | 0.936   | 18  | $PS$| $Z$       | -0.128    | -0.104    | -0.180    | 0.943   |
| 6   | $NS$ | $NB$      | 0.356     | -0.027    | 0.223     | 0.117   | 19  | $PS$| $PS$      | 0.263     | -0.083    | 0.096     | 0.656   |
| 7   | $NS$ | $NS$      | 0.125     | -0.008    | 0.317     | 0.060   | 20  | $PS$| $PB$      | -0.157    | 0.071     | 0.255     | 0.083   |
| 8   | $NS$ | $Z$       | -0.231    | 0.098     | 0.110     | 0.501   | 21  | $NB$| $NB$      | -0.146    | -0.081    | 0.175     | 0.469   |
| 9   | $NS$ | $PS$      | 0.220     | -0.083    | -0.011    | 0.896   | 22  | $PB$| $NS$      | 0.073     | 0.047     | 0.248     | 0.275   |
| 10  | $NS$ | $PB$      | 0.195     | -0.077    | 0.163     | 0.156   | 23  | $PB$| $Z$       | 0.496     | 0.035     | -0.054    | 0.573   |
| 11  | $Z$  | $NB$      | -0.279    | -0.087    | -0.128    | 0.229   | 24  | $PB$| $PS$      | 0.177     | -0.036    | -0.156    | 0.392   |
| 12  | $Z$  | $NS$      | -0.243    | 0.070     | 0.164     | 0.960   | 25  | $PB$| $PB$      | 0.151     | 0.060     | -0.315    | 0.610   |
| 13  | $Z$  | $Z$       | 0.350     | 0.010     | -0.134    | 0.272   |      |      |            |           |           |           |         |

Plot the output variable $u(t)$ for the Takagi-Sugeno-Kang type first-order fuzzy controller (Fig. 6).

**Figure 4.** Term-set of linguistic variable $x$  
**Figure 5.** Term-set of linguistic variable $\varphi$  

**Figure 6.** $u(t)$ graph
The $u(t)$ graph is characterized by fading harmonic oscillations (Fig. 6), which indicates the onset of inverted pendulum unstable equilibrium dynamic stabilization. Control of inverted pendulum unstable equilibrium dynamic stabilization based on Takagi-Sugeno-Kang type fuzzy controllers is displayed using graphs $x(t)$ and $\phi(t)$. The graph of the function $x(t)$ describing the carriage displacement of the inverted pendulum is shown in Fig. 7.

At the initial point in time $x(0) = 0$. As a result of external influence, the carriage begins to make a complex oscillatory movement. At the time interval $[0; 1.5]$, the effect of external influence on the carriage movement is observed, and at the time interval $[1.5; 5]$, the carriage switches to self-oscillation mode.

The graph of the function $\phi(t)$ describing the deviation of the rod of an inverted pendulum from the vertical position for a fuzzy controller for a Takagi-Sugeno-Kang type fuzzy controller is shown in Fig. 8.
Under the influence of external influence, the rod, as well as the carriage, performs a complex oscillatory movement. At the time interval [0; 3], the influence of external influence is observed. The rod, as well as the carriage, switches to the self-oscillation mode at the time interval [3; 5].

**Conclusion**

A fuzzy first-order Takagi-Sugeno-Kang type controller successfully manages to dynamically stabilize the unstable equilibrium of an inverted pendulum. The obtained graphs of changes in the horizontal displacement of the carriage x and the angle of deviation of the rod from the vertical \( \phi \) demonstrate the onset of dynamic stabilization over a period of time [3; 5].

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