Optimal reconstruction of the states in qutrits system

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Based on mutually unbiased measurements, an optimal tomographic scheme for the multiqutrit states is presented explicitly. Because the reconstruction process of states based on mutually unbiased states is free of information waste, we refer to our scheme as the optimal scheme. By optimal we mean that the number of the required conditional operations reaches the minimum in this tomographic scheme for the states of qutrit systems. Special attention will be paid to how those different mutually unbiased measurements are realized; that is, how to decompose each transformation that connects each mutually unbiased basis with the standard computational basis. It is found that all those transformations can be decomposed into several basic implementable single- and two-qutrit unitary operations. For the three-qutrit system, there exist five different mutually unbiased-bases structures with different entanglement properties, so we introduce the concept of physical complexity to minimize the number of nonlocal operations needed over the five different structures. This scheme is helpful for experimental scientists to realize the most economical reconstruction of quantum states in qutrit systems.

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I. INTRODUCTION

The quantum state of a system is a fundamental concept in quantum mechanics, and a quantum state can be described by a density matrix, which contains all the information one can obtain about that system. A main task for implementing quantum computation is to reconstruct the density matrix of an unknown state, which is called quantum state reconstruction or quantum state tomography[1, 2]. The technique was first developed by Stokes to determine the polarization state of a light beam[3]. Recently, Minimal qubit tomography process has been proposed by Řeháček et al, where only four measurement probabilities are needed for fully determining a single qubit state, rather than the six probabilities in the standard procedure[4]. But the implementation of this tomography process requires measurements of N-particle correlations[5]. The statistical reconstruction of biphotons states based on mutually complementary measurements has been proposed by Bogdanov et al[6, 7]. Ivanov et al proposed a method to determine an unknown mixed qutrit state from nine independent fluorescence signals[8]. Moreva et al paid attention to experimental problem of the realization of the optimal protocol for polarization ququarts state tomography[9]. In 2009, Taguchi et al developed the single scan tomography of spatial three-dimensional (qutrits) state based on the effect of realistic measurement operators[10]. Allevi et al studied the implementation of the reconstruction of the Wigner function and the density matrix for coherent and thermal states by by switching on/off single photon avalanche photodetectors[11].

In order to obtain the full information about the system we need to perform a series of measurements on a large number of identically prepared copies of the system. These measurement results are not independent of each other, so there is redundancy in these results in the previously used quantum tomography processes[12], which causes a resources waste. If we remove this redundancy completely, the reconstruction process will become an optimal one. So, to design an optimal set of measurements for removing the redundancy is of fundamental significance in quantum information processing.

Mutually unbiased bases (MUBs) have been used in a variety of topics in quantum mechanics[13–36]. MUBs are defined by the property that the squared overlap between a vector in one basis and all basis vectors in the other bases are equal. That is to say the detection over a particular basis state does not give any information about the state if it is measured in another basis. Ivanović first introduced the concept of MUBs to the problem of quantum state determination[13], and proved the existence of such bases in the prime-dimension system by an explicit construction. Then it has been shown by Wooters and Fields that measurements in this special class of bases, i.e. mutually unbiased measurements (MUMs) provide a minimal as well as optimal way of complete specification of an unknown density matrix[14]. They proved that the maximal numbers of MUBs is \( d + 1 \) in prime-dimension system. This result also applies to the prime-power-dimension system.

MUBs play a special role in determining quantum states, such as it forms a minimal set of measurement bases and provides an optimal way for determining a quantum state[13–16] etc. Recently an optimal tomographic reconstruction scheme was proposed by Klimov et al for the case of determining a state of multiqubit quantum system based on MUMs in trapped ions system[37]. However, the use of three-level systems instead of two-level systems has been proven to be more secure against a symmetric attack on a quantum key distribution protocol with MUMs than the currently existing measurement protocol[38, 39]. Quantum tomography in high dimensional (qudit) systems has been proposed and the number of required measurements is \( d^{2n} - 1 \) with \( d \) being the dimension of the qudit system and \( n \) being the number of the qudits[12]. This tomography process is not an optimal one,

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II. MUTUALLY UNBIASED BASES AND MUTUALLY UNBIASED MEASUREMENTS

As shown by Wootters, Fields [14] and Klappenecker, Rötteler [19], in finite field language, the first MUB in a $d = p^n \ (p \neq 2)$-dimensional quantum system is the standard basis $B^0$ given by the vector $(a_k^{(i)}) = \delta_{kl}$, $k,l \in F_{p^n}$, where the superscript denotes the basis, $k$ the vector in the basis, $l$ the component and $F_{p^n}$ is the field with $p^n$ elements. The other $d$ MUBs are denoted by $B^r$ which consists of vectors $(a_k^{(r)})_l$ defined by [14]:

$$
(a_k^{(r)})_l = \left(1/\sqrt{d}\right)\omega^{Tr(r^{i^2}+k^2)}, \ r, k, l \in F_{p^n}, \ r \neq 0. \ 
$$

Here $\omega = \exp(2\pi i/p)$ and $Tr \theta = \theta + \theta^p + \theta^{p^2} + \cdots + \theta^{p^{n-1}}$. The set of mutually unbiased projectors can be given by $P_k^{(r)} = |a_k^{(r)}\rangle\langle a_k^{(r)}|$. It is worth noticing that $|a_k^{(r)}\rangle$ contains the computational basis $B^0$. Here $Tr(P^{(s)}_kP^{(r)}_l) = (1/d)(1-\delta_{s\theta}+d\delta_{s\theta,\delta_{jk}})$. Then the measurement probabilities given by $\omega_k^{(r)} = Tr(P^{(r)}_k)$ completely determine the unknown density operator of a $d$-dimensional system [13]:

$$
\rho = \sum_{r=0}^{d-1} \omega^{r^2} |a_k^{(r)}\rangle\langle a_k^{(r)}| - I. \ 
$$

For instance, in a qutrit system, there are three MUBs besides the computational basis $B^0 = \{\{0\}, \{1\}, \{2\}\}$, in the following form with $\omega = \exp(2\pi i/3)$:

$$
B^1 : \{|a_0^{(1)}\rangle\} = \{(1/\sqrt{3})(0 + [1] + [2]), \ 
|a_1^{(1)}\rangle\} = \{(1/\sqrt{3})(0 + \omega[1] + \omega^*[2]), \ 
|a_2^{(1)}\rangle\} = \{(1/\sqrt{3})(0 + \omega^*[1] + [2])\}; \ 
$$

$$
B^2 : \{|a_0^{(2)}\rangle\} = \{(1/\sqrt{3})(\omega[0] + [1] + [2]), \ 
|a_1^{(2)}\rangle\} = \{(1/\sqrt{3})(0 + [1] + [2]), \ 
|a_2^{(2)}\rangle\} = \{(1/\sqrt{3})(0 + [1] + \omega[2])\}; \ 
$$

$$
B^3 : \{|a_0^{(3)}\rangle\} = \{(1/\sqrt{3})(\omega^*[0] + [1] + [2]), \ 
|a_1^{(3)}\rangle\} = \{(1/\sqrt{3})(0 + \omega^*[1] + [2]), \ 
|a_2^{(3)}\rangle\} = \{(1/\sqrt{3})(0 + [1] + \omega[2])\}. \ 
$$

III. RECONSTRUCTION PROCESS FOR AN ARBITRARY SINGLE QUTRIT STATE

An unknown single qutrit state can be expressed as [12, 40]:

$$
\rho = (1/3) \sum_j r_j \lambda_j, \text{ where } \lambda_j \text{ is an identity operator and the other } \lambda_j \text{ are the SU(3) generators [41]. The general method to reconstruct the qutrit state is to measure the expectation values of the } \lambda \text{ operators [12], where } r_j = \langle \lambda_j \rangle = Tr[\rho \lambda_j]. \text{ Thus one will find that the number of required measurements is 8. However, if we choose the MUBs to determine the qutrit state, the number of needed MUBs is only 4 rather than 8 of Ref. [12]. The four optimal set of MUBs have been presented by Eqs. (1a) (1b) plus the standard computational basis in the preceding section. Each of the three MUBs in Eqs. (1a-1c) is related with the standard computational
basis by a unitary transformation. These transformations have been listed in Table I. Here, \( F \) denotes the Fourier transformation:

\[
F|j\rangle = (1/\sqrt{3}) \sum_{i=0}^{2} \exp(2\pi i lj/3)|i\rangle, j = 0, 1, 2, \quad (2)
\]

\( R \) denotes a phase operation:

\[
R = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|, \quad (3)
\]

and the Controlled gate is

\[
X|i\rangle|j\rangle = |i\rangle|j \oplus i\rangle. \quad (4)
\]

Where \( \oplus \) denotes the difference \( j - i \) modulo 3. If there are \( n \) qutrits, the number of MUM is \( 3^n + 1 \), which is far less than \( 3^{2n} - 1 \) in Ref. [12]. That is to say the use of MUMs can represent a considerable reduction in the operations and time required for performing the full state determination [37].

IV. RECONSTRUCTION PROCESS FOR AN ARBITRARY TWO-QUTRIT STATE

Now if we further extend one-qutrit case to two-qutrit case, the density matrix can be expressed as: \( \rho_{12} = (1/9) \sum_{j,k=0}^{8} r_{jk} \lambda_j \otimes \lambda_k \), where \( r_{jk} = \langle \lambda_j \otimes \lambda_k \rangle \). If we use the general method in Ref. [12] to fully determine the state, \( d^{2n} - 1 = 3^4 - 1 = 80 \) measurements will be needed. So much measurements will inevitably introduce redundant information of the state, which is obviously a resource waste. So here we will take advantage of the MUMs to reconstruct the two-qutrit state. It is easy to find that the nine elements of \( F_9 \) (finite field) are \( \{0, \alpha, 2\alpha, 1 + \alpha, 1 + 2\alpha, 2, 2 + \alpha, 2 + 2\alpha\} \) by using the irreducible polynomials \( \theta^2 + \theta + 2 = 0 \) [14]. Here we use the representation \( \{|0\rangle, |\alpha\rangle, |2\alpha\rangle \cdots |2 + 2\alpha\rangle\} \) as the standard basis.

One can find that there will be only \( d^2 + 1 = 3^2 + 1 = 10 \) MUMs to be done, which is much less than 80 of Ref. [12]. It means that the operations and time needed for the whole state determination is greatly reduced. The decompositions for all the MUMs of the two-qutrit system have been listed in Table I.

V. THE PHYSICAL COMPLEXITY FOR IMPLEMENTING THE MUMS IN THE THREE-QUTRIT SYSTEM

In general, the fidelity of single logic gates can be greater than 99%, but nonlocal gates have a relatively lower fidelity. The fidelity of a practical CNOT gate can reach a value up to 0.926 for trapped ions system in Lab [43]. Klimov et al have introduced the concept of the physical complexity of each set of MUBs as a function of the number of nonlocal gates needed for implementing the MUMs [37]. Here the fidelity value of the CNOT gates for qubits systems also can be used to evaluate the physical complexity of the MUBs of qutrits systems. Why can we say so is because of the following point. Although the systems involved here are three-state ones, all the operations used in our reconstruction process can be decomposed into effective two-state operations. So the complexity of the current tomography scheme is proportional to the number of the nonlocal gates used (\( C \sim 6 \)) for two qutrits system. As shown in Ref. [44], the only MUB structure for a two-qutrit system is (4, 6), where 4 is the number of the separable bases and 6 is the number of the bipartite entangled bases.

However in three-qutrit case, there are five sets of MUBs with different structures, namely \( \{(0, 12, 16), (1, 9, 18), (2, 6, 20), (3, 3, 22), (4, 0, 24)\} \) [21, 44]. It is easy to see that the (0, 12, 16) set of MUBs has the minimum physical complexity. We say that the optimal set of the MUBs is (0, 12, 16). The decompositions of the MUBs in the three-qutrit case for (0, 12, 16) are listed in Table II. So the set of MUBs (0, 12, 16) has a complexity \( C < 44 \), which is a very important value in experimental realization of it.

For the multi-qutrit system \( n > 3 \), it is not easy to get all the sets of MUBs, and it is even more difficult to get the explicit decompositions of the optimal set of MUBs. Nevertheless, the results for the two-qutrit and three-qutrit cases have provided the experimentalists valuable references.

VI. CONCLUSION

We have explicitly presented an optimal tomographic scheme for the single-qutrit states, two-qutrit states and three-qutrit states based on the MUMs. Because the MUBs based state reconstruction process is free of information waste, the minimal number of required conditional operations are needed. So we call our qutrits tomographic scheme the op-
timal one. Here, we explicitly decompose each measurement into several basic single- and two-qudit operations. Furthermore, all these basic operations have been proven implementable [43]. The physical complexity of a set of MUBs also has been calculated, which is an important threshold in experiment. We hope these decompositions can help the experimental scientists to realize the most economical reconstruction of quantum states in qudits systems in lab.

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### Table III: The decompositions of MUBs (0,12,16) for three-qudit system

| Basis | Decompositions | Basis | Decompositions |
|-------|----------------|-------|----------------|
| 1     | $F_1^{-1}X_3F_2^{-1}F_3^{-1}$ | 15    | $F_1^{-1}R_2X_2$ |
| 2     | $F_1^{-1}R_1^{-1}F_1^{-1}R_2^{-1}X_3F_2^{-1}F_3^{-1}$ | 16    | $F_1^{-1}R_1^{-1}X_3F_2^{-1}F_3^{-1}$ |
| 3     | $F_1^{-1}X_1R_3$ | 17    | $F_1^{-1}R_1^{-1}X_3F_2^{-1}$ |
| 4     | $F_2^{-1}R_2X_3F_2^{-1}X_1R_3F_3^{-1}$ | 18    | $F_1^{-1}R_1^{-1}X_1^{-1}R_3F_3^{-1}$ |
| 5     | $F_1^{-1}X_2F_3^{-1}$ | 19    | $F_1^{-1}R_1^{-1}X_3F_2^{-1}F_3^{-1}$ |
| 6     | $F_1^{-1}R_2X_3F_1^{-1}R_1X_1^{-1}R_3F_3^{-1}$ | 20    | $F_1^{-1}X_1F_1^{-1}R_1^{-1}$ |
| 7     | $F_2^{-1}X_2^{-1}R_1^{-1}F_1^{-1}R_1^{-1}X_1^{-1}R_3F_3^{-1}$ | 21    | $F_1^{-1}X_2^{-1}F_1^{-1}R_1^{-1}F_3^{-1}$ |
| 8     | $F_1^{-1}X_1^{-1}R_1^{-1}F_2^{-1}R_2^{-1}F_3^{-1}$ | 22    | $F_1^{-1}X_2^{-1}F_1^{-1}X_3^{-1}F_3^{-1}$ |
| 9     | $F_1^{-1}X_1F_2^{-1}R_2^{-1}X_3F_2^{-1}F_3^{-1}$ | 23    | $F_1^{-1}R_1^{-1}X_1^{-1}R_3F_3^{-1}$ |
| 10    | $F_1^{-1}R_1X_2F_2^{-1}R_2^{-1}X_3F_3^{-1}$ | 24    | $F_1^{-1}R_1^{-1}X_2^{-1}R_3F_3^{-1}$ |
| 11    | $F_1^{-1}R_2X_3F_2^{-1}R_2^{-1}X_3F_3^{-1}$ | 25    | $F_1^{-1}R_1^{-1}X_2^{-1}F_2^{-1}F_3^{-1}$ |
| 12    | $F_1^{-1}R_1^{-1}X_1X_2^{-1}F_2^{-1}R_2F_3^{-1}$ | 26    | $F_1^{-1}R_1^{-1}X_1X_2^{-1}F_2^{-1}R_2F_3^{-1}$ |
| 13    | $F_1^{-1}X_1F_2^{-1}R_2^{-1}X_3F_2^{-1}F_3^{-1}$ | 27    | $F_1^{-1}R_1^{-1}X_1X_2^{-1}F_2^{-1}F_3^{-1}$ |
| 14    | $F_1^{-1}R_1X_2F_2^{-1}X_3F_3^{-1}F_1^{-1}$ | 28    | $F_1^{-1}R_1^{-1}X_1F_2^{-1}F_3^{-1}$ |

### References

[1] G. M. D’Ariano, M. G. A. Paris, and M. F. Sacchi, Advances in Imaging and Electron Physics 128, 205 (2003).
[2] G. M. D’Ariano, M. G. A. Paris, and M. F. Sacchi, Lecture Notes in Physics, vol. 649, Springer- Verlag, Berlin, 7 (2004).
[3] G. G. Stokes, Trans. Cambridge Philos. Soc. 9, 399(1852).
[4] J. Réháček, Phys. Rev. A 70, 052321 (2004).
[5] A. Ling et al., Phys. Rev. A 74, 022309 (2006).
[6] Y. I. Bogdanov et al., JETP Letters, 78 (6), 352 (2003).
[7] Y. I. Bogdanov et al., Phys. Rev. A 70, 042303 (2004).
[8] P. A. Ivanov and N. V. Vitanov, Opt. Commun. 264, 368 (2006).
[9] E. V. Moreva et al., arXiv:0811.1927v2 (2008).
[10] G. Taguchi et al., Phys. Rev. A 80, 062102 (2009).
[11] A. Allevi et al., Phys. Rev. A 80, 022114 (2009).
[12] R. T. Thew et al., Phys. Rev. A 66, 012303 (2002).
[13] I. D. Ivanović, J. Phys. A: Math. Gen. 14, 3241 (1981).
[14] W. K. Wootters, B. D. Fields, Ann. Phys. (NY) 191, 363 (1989).
[15] S. Bandyopadhyay et al., Algorithmica, 34, 512 (2002).
[16] M. Revzen, arXiv:0912.5433v1 (2009).
[17] J. Lawrence et al., Phys. Rev. A 65, 032320 (2002).
[18] S. Chaturvedi, Phys. Rev. A 65, 044301 (2002).
[19] A. Klappenecker and M. Rötteler, 7th International Conference on Finite Fields and Applications, Fq7, Lecture Notes in Computer Science 2948, 262 (2004).
[20] A. B. Klimov et al., J. Phys. A 38, 2747 (2005).
[21] J. L. Romero et al., Phys. Rev. A 72, 062310 (2005).
[22] M. Planat and H. Rosu, Eur. Phys. J. D 36, 133 (2005).
[23] T. Durt, J. Phys. A 38, 5267 (2005).
[24] A. O. Pittenger and M. H. Rubin, J. Phys. A 38, 6005 (2005).
[25] A. B. Klimov et al., J. Phys. A 39, 14477 (2006).
[26] W. K. Wootters, Found. Phys. 36, 112 (2006).
[27] A. J. Scott, J. Phys. A 39, 13507 (2006).
[28] A. B. Klimov et al., J. Phys. A 40, 3987 (2007).
[29] I. Bengtsson et al., J. Math. Phys. 48, 052106 (2007).
[30] M. A. Jafarizadeh et al., arXiv:0801.3100v1 (2008).
[31] S. Brierley and S. Weigert, Phys. Rev. A 78, 042312 (2008).
[32] M. R. Kibler, Phys. Rev. A 42, 353001 (2009).
[33] P. Jaming et al., arXiv:0902.0882v2 (2009).
[34] S. Brierley, S. Weigert, I. Bengtsson, arXiv:0907.4097v1 (2009).
[35] A. Ambainis, arXiv:0909.3720v1 (2009).
[36] D. M. Appleby, arXiv:0909.5233v1 (2009).
[37] A. B. Klimov et al., Phys. Rev. A 77, 060303(R) (2008).
[38] D. Bruss, C. Macchiavello, Phys. Rev. Lett. 88, 127901 (2002).
[39] J. N. Cerf et al., Phys. Rev. Lett. 88, 127902 (2002).
[40] D. F. V. James et al., Phys. Rev. A 64, 052312 (2001).
[41] C. M. Caves and G. J. Milburn, Opt. Commun. 179, 439 (2000).
[42] A. B. Klimov et al., Phys. Rev. A 67, 062313 (2003).
[43] M. Riebe et al., Phys. Rev. Lett. 97, 220407 (2006).
[44] J. Lawrence, Phys. Rev. A 70, 012302 (2004).