A GENERALIZED GAUSS CURVATURE FLOW RELATED TO THE ORLICZ-MINKOWSKI PROBLEM

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ABSTRACT. In this paper a generalized Gauss curvature flow about a convex hypersurface in the Euclidean $n$-space is studied. This flow is closely related to the Orlicz-Minkowski problem, which involves Gauss curvature and a function of support function. Under some appropriate assumptions, we prove the long-time existence and convergence of this flow. As a byproduct, two existence results of solutions to the even Orlicz-Minkowski problem are obtained, one of which improves the known result.

1. INTRODUCTION

Let $M_0$ be a smooth, closed, strictly convex hypersurface in the Euclidean space $\mathbb{R}^n$, which encloses the origin and is given by a smooth embedding $X_0 : S^{n-1} \to \mathbb{R}^n$. Consider a family of closed hypersurfaces $\{M_t\}$ with $M_t = X(S^{n-1}, t)$, where $X : S^{n-1} \times [0, T) \to \mathbb{R}^n$ is a smooth map satisfying the following initial value problem:

$$\frac{\partial X}{\partial t}(x, t) = -f(\nu)K(x,t)\frac{\langle X, \nu \rangle}{\varphi(\langle X, \nu \rangle)} \eta(t)\nu + X,$$

$$X(x, 0) = X_0(x).$$

(1)

Here $f$ is a given positive smooth function on the unit sphere $S^{n-1}$, $\nu$ is the unit outer normal vector of the hypersurface $M_t$ at the point $X(x, t)$, $K(x, t)$ is the Gauss curvature of $M_t$ at $X(x, t)$, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$, $\varphi$ is a positive smooth function defined in $(0, +\infty)$, $\eta$ is a scalar function to be determined in order to keep $M_t$ normalized in a certain sense, and $T$ is the maximal time for which the solution exists.

When $f \equiv 1$ and $\varphi(s) = s$, flow (1) is a normalized Gauss curvature flow. The study of Gauss curvature flow was initiated by Firey [14] for modeling shape change of tumbling stones. Since then, various isotropic and anisotropic geometric flows involving Gauss curvature have been extensively studied, see e.g. [1] [3] [15] [18] [21] [11] [13] [45] and references therein. For isotropic curvature flows, whether the limiting
hypersurfaces are spheres or not is an important issue, see results obtained in e.g. [2, 7, 13]. For anisotropic curvature flows, the limiting hypersurfaces are usually smooth solutions to kinds of Minkowski type problems in convex geometry, providing alternative methods of solving elliptic Monge-Ampère type equations, see e.g. [8, 9, 10, 11, 29, 35, 34].

The generalized anisotropic Gauss curvature flow (1) is closely related to the Orlicz-Minkowski problem arising in modern convex geometry. In fact, by our main Theorem 1 below, the support function \( h \) of the limiting hypersurface of this flow provides a smooth solution to the Monge-Ampère type equation

\[
  c \varphi(h) \det(\nabla^2 h + hI) = f \quad \text{on} \quad S^{n-1}
\]

for some positive constant \( c \). Here \( h \) is a function defined on \( S^{n-1} \), \( \nabla^2 h = (\nabla_{ij} h) \) is the Hessian matrix of covariant derivatives of \( h \) with respect to an orthonormal frame on \( S^{n-1} \), and \( I \) is the unit matrix of order \( n-1 \). Equation (2) is just the smooth case of Orlicz-Minkowski problem.

The Orlicz-Minkowski problem is a basic problem in the Orlicz-Brunn-Minkowski theory in convex geometry. This theory is the recent development of the classical Brunn-Minkowski theory, and has attracted great attention from many scholars, see for example [16, 17, 20, 23, 24, 26, 27, 28, 38, 44, 46, 49] and references therein. The Orlicz-Minkowski problem is a generalization of the classical Minkowski problem, and it asks what are the necessary and sufficient conditions for a Borel measure on the unit sphere \( S^{n-1} \) to be a multiple of the Orlicz surface area measure of a convex body in \( \mathbb{R}^n \). This problem is equivalent to solving equation (2) for some support function \( h \) and constant \( c \) in smooth case. When \( \varphi(s) = s^{1-p} \), Eq. (2) reduces to the \( L_p \)-Minkowski problem, which has been extensively studied, see e.g. [5, 6, 12, 22, 25, 28, 31, 32, 36, 37, 39, 47, 48] and Schneider’s book [40], and corresponding references therein. For a general \( \varphi \), several existence results have been known, see [4, 19, 24, 30].

In this paper we are concerned with the long-time existence and convergence of flow (1) for origin-symmetric convex hypersurfaces. The special case when \( \varphi(s) = s^{1-p} \) with \( p > -n \) was first studied by Bryan, Iwaki and Scheuer [8], and then by Sheng and Yi [42] using a different flow.

In order to study the general case, we need to impose some constraints on \( \varphi \). Two common assumptions are as follows:

(A) \( \varphi \) is a continuous and positive function defined in \((0, +\infty)\) such that \( \phi(s) = \int_0^s 1/\varphi(\tau) \, d\tau \) exists for every \( s > 0 \) and is unbounded as \( s \to +\infty \); Or

(B) \( \varphi \) is a continuous and positive function defined in \((0, +\infty)\) such that for every \( s > 0 \), \( \phi(s) = \int_s^{+\infty} 1/\varphi(\tau) \, d\tau \) exists, and for \( s \) near \( 0 \), \( \phi(s) \leq Ns^p \) for some positive constant \( N \) and some number \( p \in (-n, 0) \).
One can easily see that the special case $\varphi(s) = s^{1-p}$ satisfies (A) when $p > 0$, and (B) when $-n < p < 0$. In fact, these two assumptions were used in [4, 19, 24, 30] to prove existence results of equation (2) by variational methods.

As mentioned above, $\eta(t)$ in (1) is used to keep $M_t$ normalized in a certain sense. In this paper, we find that flow (1) will evolve for a long time if the volume of the convex body bounded by $M_t$ remains unchanged. This requires $\eta$ to be given by

$$\eta(t) = \frac{\int_{S^{n-1}} \rho(u,t)^n du}{\int_{S^{n-1}} f(x) h(x,t)/\varphi(h) dx},$$

where $\rho(\cdot,t)$ and $h(\cdot,t)$ are the radial function and support function of the convex hypersurface $M_t$ respectively. See section 2 for these definitions and computations. Similar $\eta(t)$ was used by Chen, Huang and Zhao [9] to study a geometric flow related to the $L_p$ dual Minkowski problem.

When $f$ is even, namely $f(-x) = f(x)$ for any $x \in S^{n-1}$, we obtain the following long-time existence and convergence of flow (1).

**Theorem 1.** Assume $M_0$ is a smooth, closed, origin-symmetric, uniformly convex hypersurface in $\mathbb{R}^n$. If $f$ is a smooth and even function on $S^{n-1}$, and $\varphi \in C^\infty(0, +\infty)$ satisfies (A) or (B), then flow (1) has a unique smooth solution $M_t$ for all time $t > 0$. Moreover, when $t \to \infty$, a subsequence of $M_t$ converges in $C^\infty$ to a smooth, closed, origin-symmetric, uniformly convex hypersurface, whose support function is a smooth even solution to equation (2) for some positive constant $c$.

The study of flow (1) is inspired by [8, 9, 10, 34] where various Minkowski type problems were studied via different geometric flows. Our paper provides the first example of Gauss curvature flows related to the Orlicz-Minkowski problem. As an application, we have

**Corollary 1.** Assume $f$ is a smooth and even function on $S^{n-1}$. If $\varphi$ is a smooth function satisfying (A) or (B), then there exists a smooth even solution to equation (2) for some positive constant $c$.

The result of Corollary 1 with assumption (A) was obtained by Haberl, Lutwak, Yang and Zhang [19, Theorem 2] for even measures. The result of Corollary 1 with assumption (B) was obtained by Bianchi, Böröczky and Colesanti [4] for $L_{n/p}$ functions (not necessarily even), where two more assumptions on $\varphi$ were needed: $\lim_{s \to 0^+} \varphi(s) = 0$ and $\varphi$ is monotone increasing.

This paper is organized as follows. In section 2, we give some basic knowledge about the flow (1). In section 3, the long-time existence of flow (1) will be proved. First, under assumptions (A) or (B), we derive uniform positive upper and lower bounds for support functions of $\{M_t\}$. Then, the bounds of principal curvatures are derived via proper auxiliary functions and delicate computations. So the long-time existence follows by standard arguments. In section 4, by considering a related geometric functional, we prove that a subsequence of $\{M_t\}$ converges to a smooth solution to equation (2), completing the proof of Theorem 1.
2. Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, and $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$. Assume $M$ is a smooth closed uniformly convex hypersurface in $\mathbb{R}^n$. Without loss of generality, we may assume that $M$ encloses the origin. The support function $h$ of $M$ is defined as

$$h(x) := \max_{y \in M} \langle y, x \rangle, \quad \forall x \in \mathbb{S}^{n-1},$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. And the radial function $\rho$ of $M$ is given by

$$\rho(u) := \max \{ \lambda > 0 : \lambda u \in M \}, \quad \forall u \in \mathbb{S}^{n-1}.$$ 

Note that $\rho(u)u \in M$.

Denote the Gauss map of $M$ by $\nu_M$. Then $M$ can be parametrized by the inverse Gauss map $X: \mathbb{S}^{n-1} \to M$ with $X(x) = \nu_M^{-1}(x)$. The support function $h$ of $M$ can be computed by

$$(4) \quad h(x) = \langle x, X(x) \rangle, \quad x \in \mathbb{S}^{n-1}.$$ 

Note that $x$ is just the unit outer normal of $M$ at $X(x)$. Let $e_{ij}$ be the standard metric of the sphere $\mathbb{S}^{n-1}$, and $\nabla$ be the corresponding connection on $\mathbb{S}^{n-1}$. Differentiating (4), we have

$$\nabla_i h = \langle \nabla_i x, X(x) \rangle + \langle x, \nabla_i X(x) \rangle.$$ 

Since $\nabla_i X(x)$ is tangent to $M$ at $X(x)$, we have

$$\nabla_i h = \langle \nabla_i x, X(x) \rangle.$$ 

It follows that

$$(5) \quad X(x) = \nabla h + hx.$$ 

By differentiating (5) twice, the second fundamental form $A_{ij}$ of $M$ can be computed in terms of the support function, see for example [13],

$$(6) \quad A_{ij} = \nabla_{ij} h + h e_{ij},$$

where $\nabla_{ij} = \nabla_i \nabla_j$ denotes the second order covariant derivative with respect to $e_{ij}$. The induced metric matix $g_{ij}$ of $M$ can be derived by Weingarten’s formula,

$$(7) \quad e_{ij} = \langle \nabla_i x, \nabla_j x \rangle = A_{ik} A_{lj} g^{kl}.$$ 

The principal radii of curvature are the eigenvalues of the matrix $b_{ij} = A^{ik} g_{jk}$. When considering a smooth local orthonormal frame on $\mathbb{S}^{n-1}$, by virtue of (6) and (7), we have

$$(8) \quad b_{ij} = A_{ij} = \nabla_{ij} h + h \delta_{ij}.$$ 

We will use $b^{ij}$ to denote the inverse matrix of $b_{ij}$. The Gauss curvature of $X(x) \in M$ is given by

$$K(x) = \left[ \det(\nabla_{ij} h + h \delta_{ij}) \right]^{-1}. $$
From the evolution equation of $X(x,t)$ in flow (1), we derive the evolution equation of the corresponding support function $h(x,t)$:

$$\frac{\partial h}{\partial t}(x,t) = -\eta(t)f(x)K\rho/\varphi(h) + h(x,t), \quad x \in S^{n-1}. \tag{9}$$

Denote the radial function of $M_t$ by $\rho(u,t)$. From (5), $u$ and $x$ are related by

$$\rho(u) = \nabla h(x) + h(x)x. \tag{10}$$

Let $x = x(u,t)$, by (10), we have

$$\log \rho(u,t) = \log h(x,t) - \log \langle x,u \rangle.
$$

Differentiating the above identity, we have

$$\frac{1}{\rho(u,t)} \frac{\partial \rho(u,t)}{\partial t} = \frac{1}{h(x,t)} \left( \nabla h \cdot \frac{\partial x(u,t)}{\partial t} + \frac{\partial h(x,t)}{\partial t} \right) - \frac{u}{\langle x,u \rangle} \cdot \frac{\partial x(u,t)}{\partial t}
$$

$$= \frac{1}{h(x,t)} \frac{\partial h(x,t)}{\partial t} + \frac{1}{h(x,t)} [\nabla h - \rho(u,t)u] \cdot \frac{\partial x(u,t)}{\partial t}
$$

$$= \frac{1}{h(x,t)} \frac{\partial h(x,t)}{\partial t}.
$$

The evolution equation of radial function then follows from (9),

$$\frac{\partial \rho}{\partial t}(u,t) = -\eta(t)f(x)K\rho/\varphi(h) + \rho(u,t), \tag{11}$$

where $K$ denotes the Gauss curvature at $\rho(u,t)u \in M_t$ and $f$ takes value at the unit normal vector $x(u,t)$.

We use $\text{vol}(t)$ to denote the volume of the convex body bounded by the hypersurface $M_t$. From

$$\text{vol}(t) = \frac{1}{n} \int_{S^{n-1}} \rho(u,t)^n \, du,$$

we have the following computations:

$$\partial_t \text{vol}(t) = \int_{S^{n-1}} \rho(u,t)^{n-1} \partial_t \rho \, du,
$$

$$= \int_{S^{n-1}} \rho^n \, du - \eta(t) \int_{S^{n-1}} \rho^n f(x)K/\varphi(h) \, du,
$$

$$= \int_{S^{n-1}} \rho^n \, du - \eta(t) \int_{S^{n-1}} f(x)h/\varphi(h) \, dx.
$$

If we take $\eta(t)$ as in (3), there is

$$\partial_t \text{vol}(t) \equiv 0, \tag{12}$$

namely the volume of the convex body bounded by $M_t$ remains unchanged.
3. LONG-TIME EXISTENCE OF THE FLOW

In this section, we will give a priori estimates about support functions and obtain the long-time existence of flow (1) under assumptions of Theorem 1.

In the following of this paper, we always assume that $M_0$ is a smooth, closed, origin-symmetric, uniformly convex hypersurface in $\mathbb{R}^n$, $f$ is a smooth, positive and even function on $S^{n-1}$, and $\varphi \in C^\infty(0, +\infty)$ satisfies (A) or (B). $h : S^{n-1} \times [0, T) \to \mathbb{R}$ is a smooth solution to the evolution equation (9) with the initial $h(\cdot, 0)$ the support function of $M_0$. Here $T$ is the maximal time for which the solution exists. Let $M_t$ be the convex hypersurface determined by $h(\cdot, t)$, and $\rho(\cdot, t)$ be the corresponding radial function.

We first prove the uniform positive upper and lower bounds of $h(\cdot, t)$ for $t \in [0, T)$.

**Lemma 1.** When $\varphi$ satisfies (A). There exists a positive constant $C$ independent of $t$, such that for every $t \in [0, T)$

\begin{equation}
\frac{1}{C} \leq \rho(\cdot, t) \leq C \text{ on } S^{n-1}.
\end{equation}

It means that

\begin{equation}
\frac{1}{C} \leq h(\cdot, t) \leq C \text{ on } S^{n-1}.
\end{equation}

**Proof.** Let

\[ J(t) = \int_{S^{n-1}} \phi(h(x, t))f(x) \, dx, \quad t \geq 0. \]

We claim that $J(t)$ is non-increasing. In fact, recalling (9), we have

\[ J'(t) = \int_{S^{n-1}} \phi'(h(x, t)) \partial_t h f(x) \, dx \]

\[ = \int_{S^{n-1}} \left[ -\eta(t) f(x) Kh/\varphi(h) + h \right] f(x) \varphi(h) \, dx \]

\[ = \int_{S^{n-1}} f(x) h/\varphi(h) \, dx - \eta(t) \int_{S^{n-1}} f(x)^2 Kh/\varphi(h)^2 \, dx. \]

By the definition of $\eta(t)$ in (3), there is

\[ \eta(t) = \frac{\int_{S^{n-1}} h/K \, dx}{\int_{S^{n-1}} f(x)h/\varphi(h) \, dx}. \]

Hence

\[ J'(t) \int_{S^{n-1}} f(x)h/\varphi(h) \, dx \]

\begin{equation}
\begin{aligned}
&= \left( \int_{S^{n-1}} f(x)h/\varphi(h) \, dx \right)^2 - \int_{S^{n-1}} h/K \, dx \cdot \int_{S^{n-1}} f(x)^2 Kh/\varphi(h)^2 \, dx \\
&= \left( \int_{S^{n-1}} \sqrt{h/K} \cdot f \sqrt{Kh}/\varphi(h) \, dx \right)^2 - \int_{S^{n-1}} h/K \, dx \cdot \int_{S^{n-1}} f^2 \sqrt{Kh}/\varphi(h)^2 \, dx \\
&\leq 0,
\end{aligned}
\end{equation}
where the last inequality is due to the Hölder’s inequality. Therefore, $J(t)$ is non-increasing.

For each $t$, write

$$R_t = \max_{u \in \mathbb{S}^{n-1}} \rho(u, t) = \rho(u_t, t)$$

for some $u_t \in \mathbb{S}^{n-1}$. Since $M_t$ is origin-symmetric, we have by the definition of support function that

$$h(x, t) \geq R_t |\langle x, u_t \rangle|, \quad \forall x \in \mathbb{S}^{n-1}.$$  

Now we have the following estimates:

$$J(0) \geq J(t) \geq f_{\min} \int_{\mathbb{S}^{n-1}} \phi(h(x, t)) \, dx$$

$$\geq f_{\min} \int_{\mathbb{S}^{n-1}} \phi(R_t |\langle x, u_t \rangle|) \, dx$$

$$= f_{\min} \int_{\mathbb{S}^{n-1}} \phi(R_t |x_1|) \, dx.$$  

Denote $S_1 = \{x \in \mathbb{S}^{n-1} : |x_1| \geq 1/2\}$, then

$$J(0) \geq f_{\min} \int_{\mathbb{S}^{n-1}} \phi(R_t/2) \, dx$$

$$= f_{\min} \phi(R_t/2) |S_1|,$$

which implies that $\phi(R_t/2)$ is uniformly bounded from above. By assumption (A), $\phi(s)$ is strictly increasing and tends to $+\infty$ as $s \to +\infty$. Thus $R_t$ is uniformly bounded from above.

Recalling $\text{vol}(t) \equiv \text{vol}(0)$ by (12), one can easily obtain the uniform positive lower bound of $\rho(\cdot, t)$. In fact, by the concept of minimum ellipsoid of a convex body, there exists a positive constant $C_n$ depending only on $n$, such that

$$\text{vol}(t) \leq C_n R_t^{n-1} \cdot \min_{u \in \mathbb{S}^{n-1}} \rho(u, t).$$

Thus the uniform positive lower bound of $\rho(\cdot, t)$ follows from their uniform upper bound. \qed

**Lemma 2.** When $\varphi$ satisfies (B). There exists a positive constant $C$ independent of $t$, such that for every $t \in [0, T)$

$$(16) \quad 1/C \leq h(\cdot, t) \leq C \text{ on } \mathbb{S}^{n-1}.$$  

It means that

$$(17) \quad 1/C \leq \rho(\cdot, t) \leq C \text{ on } \mathbb{S}^{n-1}.$$  

Proof. Let

$$J(t) = \int_{\mathbb{S}^{n-1}} \phi(h(x, t)) f(x) \, dx, \quad t \geq 0.$$  

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Note that $\phi'(s) = -1/\varphi(s)$. We have

\[
J'(t) = \int_{\mathbb{S}^{n-1}} \phi'(h(x,t)) \partial_t hf(x) \, dx \\
= -\int_{\mathbb{S}^{n-1}} [-\eta(t)f(x)K\varphi(h) + h]f(x)/\varphi(h) \, dx \\
= \eta(t) \int_{\mathbb{S}^{n-1}} f(x)^2 K\varphi(h)^2 \, dx - \int_{\mathbb{S}^{n-1}} f(x)h/\varphi(h) \, dx \\
\geq 0,
\]

where the last inequality is due to (15). Hence $J(t)$ is non-decreasing, and (18)

\[J(t) \geq J(0) > 0.\]

Let $a$ be a positive number to be determined. Write

\[S_t = \{ x \in \mathbb{S}^{n-1} : h(x,t) \geq a \} \]

Then

\[
\int_{S_t} \phi(h(x,t)) f(x) \, dx \leq \int_{S_t} \phi(a) f(x) \, dx \leq \phi(a) \| f \|_{L^1(\mathbb{S}^{n-1})}.
\]

By $\lim_{s \to +\infty} \phi(s) = 0$, one can take a sufficiently large $a$, such that

\[
\int_{S_t} \phi(h(x,t)) f(x) \, dx < J(0)/2, \quad \forall t \in [0,T).
\]

Note that $a$ depends only on $f$ and $\varphi$.

Recall assumption (B), when $s$ near 0, $\phi(s) \leq Ns^p$ for some positive constant $N$ and some number $p \in (-n, 0)$. Since $\phi$ is smooth in $(0, +\infty)$, one can easily see that there exists a positive number $\tilde{N}$ such that

\[
\phi(s) \leq \tilde{N}s^p, \quad \forall s \in (0,a).
\]

Now we can estimate $J(t)$ as follows:

\[
J(t) = \left( \int_{\mathbb{S}^{n-1} \setminus S_t} + \int_{S_t} \right) \phi(h(x,t)) f(x) \, dx \\
\leq \tilde{N} \int_{\mathbb{S}^{n-1} \setminus S_t} h(x,t)^p f(x) \, dx + J(0)/2 \\
\leq \tilde{N} f_{\text{max}} \int_{\mathbb{S}^{n-1}} h(x,t)^p \, dx + J(0)/2,
\]

which together with (18) implies that

\[
\int_{\mathbb{S}^{n-1}} h(x,t)^p \, dx \geq \frac{J(0)}{2\tilde{N} f_{\text{max}}}.
\]

Noting $h(\cdot,t)$ is even, $p \in (-n, 0)$ and $\text{vol}(t) \equiv \text{vol}(0)$, by a simple computation, one can see that $h(\cdot,t)$ has uniform positive upper and lower bounds. \qed
Since $h(\cdot, t)$ is the support function, it is easy to obtain gradient estimates from the bounds of $h(\cdot, t)$. In fact, by the equality $\rho^2 = h^2 + |\nabla h|^2$, we have from the previous lemmas that

**Corollary 2.** Under the assumptions of Theorem 1, we have

$$|\nabla h(x, t)| \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),$$

where $C$ is a positive constant depending only on constants in Lemmas 1 and 2.

To obtain the long-time existence of the flow (1) or (9), we further need to establish uniform upper and lower bounds for the principal curvature.

In the rest of this section, we take a local orthonormal frame \(\{e_1, \cdots, e_n-1\}\) on $S^{n-1}$ such that the standard metric on $S^{n-1}$ is \(\delta_{ij}\). Double indices always mean to sum from 1 to $n-1$. For convenience, we also write $\psi = 1/\varphi, \quad F = \eta(t)f(x)K(x)h\psi(h)$.

By Lemmas 1 and 2, for any $t \in [0, T)$, $h(\cdot, t)$ always ranges within a bounded interval $I' = \{1/C, C\}$, where $C$ is the constant in these two lemmas.

We first derive the upper bound for the Gaussian curvature.

**Lemma 3.** Under the assumptions of Theorem 1, we have

$$K(x, t) \leq C, \quad \forall (x, t) \in S^{n-1} \times [0, T),$$

where $C$ is a positive constant independent of $t$.

**Proof.** Consider the following auxiliary function:

$$Q(x, t) = \frac{1}{h - \varepsilon_0} (F - h) = \frac{-h_t}{h - \varepsilon_0},$$

where $\varepsilon_0$ is a positive constant satisfying $\varepsilon_0 < \min_{S^{n-1} \times [0, T]} h(x, t)$.

Recalling that $F = \eta(t)f(x)K(x)h\psi(h)$ and that $h$ has uniform positive upper and lower bounds, the upper bound of $K(x, t)$ follows from that of $Q(x, t)$. Hence we only need to derive the upper bound of $Q(x, t)$.

First we compute the evolution equation of $Q(x, t)$. Note that

$$\nabla_i Q = \frac{F_i - h_i}{h - \varepsilon_0} - \frac{F - h}{(h - \varepsilon_0)^2} h_i,$$

and

$$\nabla_{ij} Q = \frac{F_{ij} - h_{ij}}{h - \varepsilon_0} - \frac{(F - h)h_{ij}}{(h - \varepsilon_0)^2} - \frac{(F_j - h_j)h_i + (F - h)h_{ij}}{(h - \varepsilon_0)^2} + \frac{2(F - h)h_i h_j}{(h - \varepsilon_0)^3}$$

$$= \frac{F_{ij} - h_{ij}}{h - \varepsilon_0} - \frac{(F - h)h_{ij}}{(h - \varepsilon_0)^2} - \frac{\nabla_i Q h_j}{h - \varepsilon_0} - \frac{\nabla_j Q h_i}{h - \varepsilon_0}.$$
There is also that

\[
\frac{\partial Q}{\partial t} = \frac{F_t - h_t}{h - \varepsilon_0} + \frac{h_t^2}{(h - \varepsilon_0)^2} = \frac{F_t}{h - \varepsilon_0} + Q + Q^2.
\]

We obtain the evolution equation of \( Q(x, t) \):

\[
\frac{\partial Q}{\partial t} - F b^{ij} \nabla_{ij} Q = \frac{1}{h - \varepsilon_0} (F_t - F b^{ij} F_{ij}) + Q + Q^2
\]

\[
+ (Q + 1) \frac{F b^{ij} h_{ij}}{h - \varepsilon_0} + \frac{\nabla_i Q F b^{ij} h_j}{h - \varepsilon_0} + \frac{\nabla_j Q F b^{ij} h_i}{h - \varepsilon_0}.
\]

Now we need to compute the evolution equation of \( F \). From the fact

(19) \[ \frac{\partial K}{\partial b_{ij}} = -K b^{ij}, \]

we have

\[
f(x) h \psi(h) \eta(t) \frac{\partial K}{\partial t} = -F b^{ij} (h_{ij} + \delta_{ij} h_t)
\]

\[
= -F b^{ij} (h_t)_{ij} - F b^{ij} \delta_{ij} h_t
\]

\[
= -F b^{ij} (-F + h)_{ij} - F b^{ij} \delta_{ij} h_t
\]

\[
= F b^{ij} F_{ij} - F b^{ij} b_{ij} + F^2 b^{ij} \delta_{ij}.
\]

Then there is

\[
F_t = f(x) h \psi(h) \eta(t) \frac{\partial K}{\partial t} + K(x, t) f(x) \frac{\partial}{\partial t} (h \psi(h) \eta(t))
\]

\[
= F b^{ij} F_{ij} - F b^{ij} b_{ij} + F^2 b^{ij} \delta_{ij} + K(x, t) f(x) \frac{\partial}{\partial t} (h \psi(h) \eta(t)).
\]

Thus we obtain

\[
\frac{\partial F}{\partial t} - F b^{ij} \nabla_{ij} F = -F(n - 1) + F^2 b^{ij} \delta_{ij} + K(x, t) f(x) \frac{\partial}{\partial t} (h \psi(h) \eta(t)).
\]
At a spatial maximal point of $Q(x, t)$, if we take an orthonormal frame such that $b_{ij}$ is diagonal, we have

$$\frac{\partial Q}{\partial t} - b^{ii} F \nabla_i Q$$

$$\leq \frac{1}{h - \varepsilon_0} (F_t - b^{ii} F \nabla_i F) + Q + Q^2 + \frac{F b^{ii} h_{ii}}{h - \varepsilon_0} + \frac{Q F b^{ii} h_{ii}}{h - \varepsilon_0}$$

$$= \frac{1}{h - \varepsilon_0} [-F b^{ii} b_{ii} + F^2 b^{ii} \delta_{ii} + K(x, t) f(x) \frac{\partial}{\partial t} (h\psi(h)\eta(t))]$$

$$+ Q + Q^2 + \frac{F b^{ii} (b_{ii} - h \delta_{ii})}{h - \varepsilon_0} + \frac{Q F b^{ii} (b_{ii} - h \delta_{ii})}{h - \varepsilon_0}$$

$$= \frac{F^2}{h - \varepsilon_0} \sum_i b^{ii} + Q + Q^2 + \frac{1}{h - \varepsilon_0} K(x, t) f(x) \frac{\partial}{\partial t} (h\psi(h)\eta(t))$$

$$- \frac{h F}{h - \varepsilon_0} \sum_i b^{ii} + \frac{Q F (n - 1)}{h - \varepsilon_0} - \frac{Q F h}{h - \varepsilon_0} \sum_i b^{ii}$$

$$\leq F Q \left(1 - \frac{h}{h - \varepsilon_0}\right) \sum_i b^{ii} + C_1 Q + C_2 Q^2$$

$$+ \frac{1}{h - \varepsilon_0} K(x, t) f(x) \frac{\partial}{\partial t} (h\psi(h)\eta(t)).$$

Since

$$\frac{\partial \eta(t)}{\partial t} = -\frac{\int_{S^{n-1}} \rho^n \, d\mu}{\left[\int_{S^{n-1}} h\psi(h) f(x) \, dx\right]^2} \int_{S^{n-1}} f(x) [\psi'(h) h + \psi(h)] h_t \, dx$$

$$\leq C_3 Q,$$

where $C_3$ is a positive constant depending on $\|f\|_{C(S^{n-1})}$, $\|\varphi\|_{C^1(\mathbb{R})}$ and $\|h\|_{C^1(S^{n-1} \times [0, T])}$. Therefore we have

$$\frac{\partial}{\partial t} (h\psi(h)\eta(t)) = h\psi(h) \frac{\partial \eta(t)}{\partial t} + [\psi'(h) h + \psi(h)] \eta(t) h_t$$

$$\leq C_4 Q.$$

For $Q$ large enough, there is

$$1/C_0 K \leq Q \leq C_0 K,$$

and

$$\sum_i b^{ii} \geq (n - 1) K \frac{1}{\pi^{1/4}}.$$

Hence, for large $Q$, we obtain

$$\frac{\partial Q}{\partial t} \leq C_1 Q^2 (C_2 - \varepsilon_0 Q \frac{1}{\pi^{1/4}}) < 0.$$

Then the upper bound of $K(x, t)$ follows. \(\square\)
Now we can estimate lower bounds of principal curvatures $\kappa_i(x,t)$ of $M_t$ for $i = 1, \cdots, n - 1$.

**Lemma 4.** Under the assumptions of Theorem 7 we have

$$\kappa_i \geq C, \quad \forall (x,t) \in S^{n-1} \times [0,T),$$

where $C$ is a positive constant independent of $t$.

**Proof.** Consider the auxiliary function

$$w(x,t) = \log \lambda_{\max}(b_{ij}) - A \log h + B|\nabla h|^2,$$

where $A, B$ are constants to be determined. $\lambda_{\max}$ is the maximal eigenvalue of $b_{ij}$.

For any fixed $t$, we assume that $\max_{S^{n-1}} w(x,t)$ is attained at $q \in S^{n-1}$. At $q$, we take an orthogonal frame such that $b_{ij}(q,t)$ is diagonal and $\lambda_{\max}(q,t) = b_{11}(q,t)$. Now we can write $w(x,t)$ as

$$w(x,t) = \log b_{11} - A \log h + B|\nabla h|^2.$$

We first compute the evolution equation of $w$. Note that

$$\frac{\partial \log b_{11}}{\partial t} - b^{ii} F \nabla_i \log b_{11} = b^{11} \left( \frac{\partial b_{11}}{\partial t} - F b^{ii} \nabla_i b_{11} \right) + F b^{ii}(b^{11})^2 (\nabla_i b_{11}),$$

$$\frac{\partial \log h}{\partial t} - b^{ii} F \nabla_i \log h = \frac{1}{h} \left( \frac{\partial h}{\partial t} - F b^{ii} \nabla_i h \right) + \frac{F b^{ii} h_i^2}{h^2},$$

$$\frac{\partial |\nabla h|^2}{\partial t} - b^{ii} F \nabla_i |\nabla h|^2 = 2 h_k \left( \frac{\partial h_k}{\partial t} - F b^{ii} \nabla_i h_k \right) - 2 F b^{ii} h_i^2,$$

we have

$$\frac{\partial w}{\partial t} - F b^{ii} \nabla_i w = b^{11} \left( \frac{\partial b_{11}}{\partial t} - F b^{ii} \nabla_i b_{11} \right) + F b^{ii}(b^{11})^2 (\nabla_i b_{11})$$

$$- \frac{A}{h} \left( \frac{\partial h}{\partial t} - F b^{ii} \nabla_i h \right) - \frac{F b^{ii} h_i^2}{h^2}$$

$$+ 2 B h_k \left( \frac{\partial h_k}{\partial t} - F b^{ii} \nabla_i h_k \right) - 2 B F b^{ii} h_i^2.$$

(20)

Let

$$M = \log [hf(x)\psi(h)\eta(t)],$$

then

$$\log F = \log \mathcal{K} + M.$$

Differentiating the above equation, we have by (19) that

$$\nabla_k \frac{F}{F} = \frac{1}{\mathcal{K} b_{ij}} \nabla_k b_{ij} + \nabla_k M = -b^{ij} \nabla_k b_{ij} + M_k,$$

and

$$\nabla_k \frac{F}{F} \frac{F}{F^2} - \frac{\nabla_k F \nabla_i F}{F^2} = -b^{ij} \nabla_k b_{ij} + b^{ij} b^{ij} \nabla_k b_{ij} \nabla_l b_{ij} + \nabla_k M.$$
Recalling the evolution equation of $h$, we have

$$\frac{\partial h}{\partial t} - F b^{ij} \nabla_{ij} h = - F + h - F b^{ij} (b_{ij} - \delta_{ij} h)$$

$$= - F n + h + F h \sum_i b^{ii}, \quad (21)$$

and

$$\frac{\partial h_k}{\partial t} - F b^{ii} \nabla_{ii} h_k = - F_k + h_k - F b^{ij} \nabla_k b_{ij} + F h_k \sum_i b^{ii}$$

$$= - M_k F + h_k + F h_k \sum_i b^{ii}. \quad (22)$$

We also have

$$\frac{\partial h_{kl}}{\partial t} = - \nabla_{kl} F + h_{kl}$$

$$= - \frac{\nabla_k F \nabla_l F}{F} + F b^{ij} \nabla_{kl} b_{ij} - F b^{ij} \delta_{ij} \nabla_{kl} F_{ij} - F \nabla_{kl} M + h_{kl}.$$ 

By the Gauss equation, see e.g. [43] for details,

$$\nabla_{kl} h_{ij} = \nabla_{ij} h_{kl} + 2 \delta_{kl} h_{ij} - 2 \delta_{ij} h_{kl} + \delta_{kj} h_{il} - \delta_{li} h_{kj},$$

or

$$\nabla_{kl} b_{ij} = \nabla_{ij} b_{kl} + \delta_{kl} b_{ij} - \delta_{ij} h_{kl} + \delta_{kj} h_{il} - \delta_{li} h_{kj}.$$ 

Then

$$\frac{\partial h_{kl}}{\partial t} = F b^{ij} \nabla_{ij} h_{kl} + 2 \delta_{kl} F b^{ij} h_{ij} - F b^{ij} \delta_{ij} h_{kl} + F b^{ik} h_{il} - F b^{il} h_{kj}$$

$$- \frac{\nabla_k F \nabla_l F}{F} - F b^{ij} b^{ij} \nabla_{kl} b_{ij} - F \nabla_{kl} M + h_{kl}.$$ 

Hence

$$\frac{\partial b_{kl}}{\partial t} = F b^{ij} \nabla_{ij} b_{kl} + \delta_{kl} F b^{ij} (b_{ij} - h \delta_{ij}) - F b^{ij} \delta_{ij} (b_{kl} - h \delta_{kl})$$

$$+ F b^{ik} h_{il} - F b^{il} h_{kj} + b_{kl} - h \delta_{kl} + (- F + h) \delta_{kl}$$

$$- \frac{\nabla_k F \nabla_l F}{F} - F b^{ij} b^{ij} \nabla_{kl} b_{ij} - F \nabla_{kl} M$$

$$= F b^{ij} \nabla_{ij} b_{kl} + \delta_{kl} F (n-2) - F b^{ij} \delta_{ij} b_{kl} + F b^{ik} h_{il} - F b^{il} h_{kj} + b_{kl}$$

$$- \frac{\nabla_k F \nabla_l F}{F} - F b^{ij} b^{ij} \nabla_{kl} b_{ij} - F \nabla_{kl} M.$$
When \( k = l = 1 \), we have

\[
\frac{\partial b_{11}}{\partial t} = Fb_{i11}\nabla_{ii}(b_{11}) + F(n-2) - F \sum_i b_{1i} - b_{11} + b_{11} - F_1^2 - Fb_{i1j} \nabla_1(b_{ij})^2 - FM_{11}.
\]

(23)

Inserting (21), (22) and (23) into (20), we obtain

\[
\frac{\partial w}{\partial t} - Fb_{i11}\nabla_{ii}w \\
\leq b_{11}F(n-2) + 1 + A - b_{11}FM_{11} + \frac{AFn}{h} - AF \sum_i b_{ii} - 2BFh_kM_k \\
+ 2B|\nabla h|^2 + 2BF|\nabla h|^2 \sum_i b_{ii} - 2BF \sum_i b_{ii} + 4BFh(n-1) \\
= b_{11}F(n-2) - b_{11}FM_{11} - 2BFh_kM_k - (A - 2B|\nabla h|^2)F \sum_i b_{ii} \\
- (A - 2B|\nabla h|^2) - 2BF \sum_i b_{ii} + 4BFh(n-1) + \frac{AFn}{h}.
\]

If we let \( A \) satisfy

\[
A \geq 2B \max_{S^{n-1} \times [0,T]} |\nabla h|^2 + 1,
\]

then

\[
\frac{\partial w}{\partial t} - Fb_{i11}\nabla_{ii}w \\
\leq b_{11}F(n-2) - b_{11}FM_{11} - 2BFh_kM_k \\
- 2BF \sum_i b_{ii} + 4BFh(n-1) + \frac{AFn}{h}.
\]

(24)

Now, we estimate \(-b_{11}FM_{11} - 2BFh_kM_k\). Since

\[
M = \log[hf(x)\psi(h)\eta(t)] \\
= \log f(x) + \log h(x) + \log \psi(h) + \log \eta(t),
\]

then

\[
\nabla_k M = \frac{f_k}{f} + \frac{h_k}{h} + \frac{\psi'}{\psi} h_k,
\]

and

\[
\nabla_{11} M = \frac{f_{11}}{f} - \frac{f^2}{f^2} + \frac{h_{11}}{h} - \frac{h^2}{h^2} + \frac{\psi h_{11}^2 + \psi' h_{11}}{\psi} - \frac{(\psi')^2 h^2}{\psi^2}.
\]
Therefore, we obtain
\[-2Bh_k M_k = -2Bh_k \left( \frac{f_k}{f} + \frac{h_k}{h} + \frac{\psi'}{\psi} h_k \right) \]
\[\leq 2B \left( \frac{\|\nabla f\|\|\nabla h\|}{f} + \frac{\|\nabla h\|^2}{h} + \frac{\|\nabla h\|^2|\psi'|}{\psi} \right) \]
\[\leq c_1 B, \]
where \(c_1\) is a positive constant depending on upper and lower bounds of \(f, \varphi(h)\) and \(h\), and upper bounds of their first order derivatives. We also have
\[-b_{11} M_{11} = -b_{11} \left( \frac{f_{11}}{f} - \frac{f_1^2}{f^2} - \frac{h_1^2}{h^2} + \frac{\psi'' h_1^2}{h} - \frac{(\psi')^2 h_1}{h^2} \right) + \frac{b_{11} \psi'(b_{11} - h)}{h} - \frac{b_{11} b_{11} - h}{h} \]
\[\leq c_2 b_{11} + c_3, \]
where \(c_2, c_3\) are positive constants depending on \(\|\varphi\|_{C^2(I)}, \|f\|_{C^2(S^{n-1})}, \|h\|_{C^1(S^{n-1} \times [0,T])}\), and lower bounds of \(\varphi(h), f\) and \(h\). Thus, we have proved that
\[-b_{11} FM_{11} - 2BFh_k M_k \leq F(c_1 B + c_2 b_{11} + c_3). \]

Now, from (24) we have
\[\frac{\partial w}{\partial t} - Fb_{ii} \nabla_{ii} w \leq F(c_1 B + c_2 b_{11} + c_3) - 2BF \sum_i b_{ii} + 4BFh(n-1) + \frac{AFn}{h}. \]

If we take \(B = 1\), then for \(b_{ii}\) large enough, there is
\[\frac{\partial w}{\partial t} - Fb_{ii} \nabla_{ii} w \leq F(c_1 + c_2 b_{11} + c_3) - 2F \sum_i b_{ii} + 4Fh(n-1) + \frac{AFn}{h} < 0, \]
which implies that
\[\frac{\partial w}{\partial t} < 0. \]
Therefore \(w\) has a uniform upper bound, and so does \(\lambda_{\text{max}}(b_{ij})\). The conclusion of this lemma then follows. \(\square\)

Combining Lemma 3 and Lemma 4, we see that the principal curvatures of \(M_t\) has uniform positive upper and lower bounds. This together with Lemmas 1 and 2 implies that the evolution equation (9) is uniformly parabolic on any finite time interval. Thus, the result of [33] and the standard parabolic theory show that the smooth solution of (9) exists for all time. And by these estimates again, a subsequence of \(M_t\) converges in \(C^\infty\) to a positive, smooth, uniformly convex hypersurface \(M_\infty\) in \(\mathbb{R}^n\). Now to complete the proof of Theorem 1 it remains to check the support function of \(M_\infty\) satisfies Eq. (2).
4. CONVERGENCE OF THE FLOW

In this section, we will complete the proof of Theorem 1. Let \( \tilde{h} \) be the support function of \( M_\infty \). We need to prove that \( \tilde{h} \) is a solution to the following equation

\[
(25) \quad c \varphi(h) \det(\nabla^2 h + hI) = f \text{ on } \mathbb{S}^{n-1}
\]

for some positive constant \( c \).

As before, we define the functional

\[
J(t) = \int_{\mathbb{S}^{n-1}} \phi(h(x),t) f(x) \, dx, \quad t \geq 0.
\]

By the assumptions on \( \phi \), and Lemmas 1 and 2, there exists a positive constant \( C \) which is independent of \( t \), such that

\[
(26) \quad J(t) \leq C, \quad \forall t \geq 0.
\]

We also note that, by proofs of these two lemmas, \( J(t) \) is non-increasing when \( \varphi \) satisfies (A), and non-decreasing when \( \varphi \) satisfies (B).

We now begin the proof with the assumption (A). Recalling \( J'(t) \leq 0 \) for any \( t > 0 \). From

\[
\int_0^t [-J'(t)] \, dt = J(0) - J(t) \leq J(0),
\]

we have

\[
\int_0^\infty [-J'(t)] \, dt \leq J(0),
\]

This implies that there exists a subsequence of times \( t_j \to \infty \) such that

\[
-J'(t_j) \to 0 \quad \text{as} \quad t_j \to \infty.
\]

Recalling (15):

\[
J'(t_j) \int_{\mathbb{S}^{n-1}} f(x) h/\varphi(h) \, dx
\]

\[
= \left( \int_{\mathbb{S}^{n-1}} \sqrt{\tilde{h}/\tilde{K}} \cdot f \sqrt{\tilde{K}h/\varphi(h)} \, dx \right)^2 - \int_{\mathbb{S}^{n-1}} \tilde{h}/\tilde{K} \, dx \cdot \int_{\mathbb{S}^{n-1}} f^2 \tilde{K}h/\varphi(h)^2 \, dx.
\]

Since \( h \) and \( \mathcal{K} \) have uniform positive upper and lower bounds, by passing to the limit, we obtain

\[
\left( \int_{\mathbb{S}^{n-1}} \sqrt{\tilde{h}/\tilde{K}} \cdot f \sqrt{\tilde{K}h/\varphi(h)} \, dx \right)^2 = \int_{\mathbb{S}^{n-1}} \tilde{h}/\tilde{K} \, dx \cdot \int_{\mathbb{S}^{n-1}} f^2 \tilde{K}h/\varphi(h)^2 \, dx,
\]

where \( \tilde{K} \) is the Gauss curvature of \( M_\infty \). By the equality condition for the Hölder’s inequality, there exists a constant \( c \geq 0 \) such that

\[
c^2 \tilde{h}/\tilde{K} = f^2 \tilde{K}h/\varphi(h)^2 \text{ on } \mathbb{S}^{n-1},
\]

namely

\[
c \varphi(\tilde{h})/\tilde{K} = f \text{ on } \mathbb{S}^{n-1},
\]
which is just equation (25). Note $\tilde{h}$ and $\tilde{K}$ have positive upper and lower bounds, $c$ should be positive.

For the proof with the assumption (B). Recalling $J'(t) \geq 0$ for any $t > 0$. By estimate (26),
\[ \int_0^t J'(t) \, dt = J(t) - J(0) \leq J(t) \leq C, \]
which leads to
\[ \int_0^\infty J'(t) \, dt \leq C. \]
This implies that there exists a subsequence of times $t_j \to \infty$ such that
\[ J'(t_j) \to 0 \text{ as } t_j \to \infty. \]
Now using almost the same arguments as above, one can prove $\tilde{h}$ solves Eq. (25) for some positive constant $c$. The proof of Theorem 1 is completed.

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