Part of the D-dimensional anharmonic oscillator spectra

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Abstract

The pseudoperturbative shifted -$l$ expansion technique PSLET [12,16] is generalized for states with arbitrary number of nodal zeros. Interdimensional degeneracies, emerging from the isomorphism between angular momentum and dimensionality of the central force Schrödinger equation, are used to construct part of the $D$-dimensional anharmonic oscillator bound-state spectra. PSLET results are found to compare excellently with those from a series [7], exact and open perturbation [9] solutions.
1 Introduction

Besides its importance in quantum field theory and chemical physics, quantal isotropic anharmonic oscillator

\[ V(r) = \alpha_0 q^2 + \alpha q^4, \]  

is also a popular theoretical laboratory for examining the validity of approximation techniques [1-12]. Therefore, it has been a subject of an enormous number of papers (long list of these could be found in [3]). However, most of these studies are devoted to one spatial dimension (1D). Nevertheless, there has been a considerable interest, since the end of the 1960s as a result of studies in the far infrared and microwave regions, in the analysis of two- and three-dimensional (2D and 3D, respectively) anharmonic oscillators [6, and references therein]. Among the most recent ones, to the best of our knowledge, exist the work of Lakshmanan et al.[6] and Taseli [7]. They have treated the 2D- and 3D- anharmonic oscillators via a phase-integral [6] (PIM) and a series solution, weighted by an appropriate function, [7] (SSM) methods. Hence, further studies of such oscillators should be of great interest.

In many problems, on the other hand, the Hamiltonian does not contain any physical parameter suitable for a perturbation expansion treatment. More often, the Hamiltonian contains physical parameters, but, typically, zeroth-order solutions for especial values of these are not tractable or good starting approximations. Moreover, it is well known that the implementation of Rayleigh-Schrödinger perturbation theory, or even naive perturbation series, expresses the eigenvalue of the AHO as a formal power series in \(\alpha\) which is quite often divergent, or at best asymptotic, for every \(\alpha \neq 0\). One has therefore to resort to summation tools to sum up such series [12-15]. Hence,
apparently artificial perturbation theories have been devised and proven to be ways to make progress \[1,9,10,12,16-21, \text{ etc.}\]. Yet, in the simplest case, analytical calculations can aid numerical studies in areas where numerical techniques might not be controlled. For example, when bound - state wave functions with arbitrary nodal zeros are in demand for certain singular potentials \((\text{a next level of complexity})\), analytical solutions can supply a basis for numerical calculations.

In a preceding paper \[12\], we have treated the nodeless states of the AHO (1) via a pseudoperturbation shifted - \(l\) expansion technique (PSLET), where \(l\) denotes the angular momentum quantum number. Successfully, the same recipe has been applied to quasi - relativistic harmonic oscillator \[16\], and spiked harmonic oscillator, etc \[17\].

Encouraged by its satisfactory performance in handling nodeless states, we feel tempted to generalize PSLET recipe \((\text{in section 2})\) for states with arbitrary number of nodal zeros, \(k \geq 0\). Moreover, in the underlying time - independent radial Schrödinger equation, in \(\hbar = m = 1\) units,

\[
\left[ -\frac{1}{2} \frac{d^2}{dq^2} + \frac{l(l+1)}{2q^2} + V(q) \right] \Psi_{k,l}(q) = E_{k,l} \Psi_{k,l}(q), \tag{2}
\]

the isomorphism between orbital angular momentum \(l\) and dimensionality \(D\) invites interdimensional degeneracies \[7,22-24\]. Which, in effect, allows us to generate the ladder of excited states for any given \(k\) and nonzero \(l\) from the \(l=0\) result, with that \(k\), by the transcription \(D \rightarrow D + 2l\). That is, if \(E_{k,l}(D)\) is the eigenvalue in \(D\) - dimensions, then

\[
E_{k,l}(2) \equiv E_{k,l-1}(4) \equiv \cdots \equiv E_{k,1}(2l) \equiv E_{k,0}(2l + 2) \tag{3}
\]

for even \(D\), and
\[ E_{k,l}(3) \equiv E_{k,l-1}(5) \equiv \cdots \equiv E_{k,1}(2l+1) \equiv E_{k,0}(2l+3) \]  \hspace{1cm} (4)

for odd \(D\). For more details the reader may refer to refs [7,22,25]. We therefore calculate, in section 3, the energies for 2D - and 3D - anharmonic oscillators, for a given number of nodes \(k\) and different values of \(l\), and construct part of its \(D\)-dimensional bound-state spectra. We compare our results with those from PIM [6], SSM [7], exact and an open perturbation solutions [9]. Section 4 is devoted for concluding remarks.

2 The generalized PSLET

Although some of the following expressions have appeared in previous articles [12,16,17], we would like to repeat them to make this article self contained.

We simply start with shifting the angular momentum quantum number \(l\) in (2) through \(\bar{l} = l - \beta\) and use \(1/\bar{l}\) as a pseudoperturbation expansion parameter. Where \(\beta\) is a suitable shift introduced to remove the poles that would emerge, at lowest orbital states with \(l = 0\), in our expansions below. Hence, equation (2) reads

\[
\left\{ -\frac{1}{2} \frac{d^2}{dq^2} + \frac{\bar{l}^2 + (2\beta + 1)\bar{l} + \beta(\beta + 1)}{2q^2} + \frac{\bar{l}^2}{Q} V(q) \right\} \Psi_{k,l}(q) = E_{k,l} \Psi_{k,l}(q), \quad (5)
\]

where \(Q\) is a constant that scales the potential \(V(q)\) at large - \(l_D\) limit (the pseudoclassical limit [22]) and is set, for any specific choice of \(l_D\) and \(k\), equal to \(\bar{l}^2\) at the end of the calculations. Here \(l_D = l + (D - 3)/2\), to incorporate the interdimensional degeneracies associated with the isomorphism between angular momentum and dimensionality \(D\). Hence, \(\bar{l} \rightarrow \bar{l} = l_D - \beta\) throughout this paper. Next, we shift the origin of the coordinate system through
$x = \bar{l}^{1/2}(q - q_o)/q_o$, where $q_o$ is currently an arbitrary point to be determined below. Expansions about this point, $x = 0$ (i.e. $q = q_o$), localize the problem at $q_o$ and the derivatives, in effect, contain information not only at $q_o$ but also at any point on $q$-axis, in accordance with Taylor’s theorem. Equation (5) thus becomes

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{q_o^2}{l} \bar{V}(x(q)) \right] \Psi_{k,l}(x) = \frac{q_o^2}{l} E_{k,l} \Psi_{k,l}(x),$$

with

$$\frac{q_o^2}{l} \bar{V}(x(q)) = q_o^2 \left[ \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \right] + \bar{l}^{1/2} B_1 x + \sum_{n=0}^{\infty} v^{(n)}(x) \bar{l}^{-n/2},$$

where

$$v^{(0)}(x) = B_2 x^2 + \frac{2\beta + 1}{2},$$

$$v^{(1)}(x) = -(2\beta + 1)x + B_3 x^3,$$

$$v^{(n)}(x) = B_{n+2} x^{n+2} + (-1)^n (2\beta + 1) \frac{n+1}{2} x^n + (-1)^n \frac{\beta(\beta + 1)}{2} (n-1) x^{(n-2)} ; \quad n \geq 2,$$

$$B_n = (-1)^n \frac{(n+1)}{2} + \left( \frac{d^n V(q_o)}{dq_o^n} \right) \frac{q_o^{n+2}}{n!Q}.$$
It is then convenient to expand $E_{k,l}$ as

$$E_{k,l} = \sum_{n=-2}^{\infty} E_{k,l}^{(n)} \bar{l}^{-n}. \quad (12)$$

Equation (6), along with (7)-(11), is evidently the one-dimensional Schrödinger equation for a perturbed harmonic oscillator

$$\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} w^2 x^2 + \varepsilon_o + P(x) \right] X_k(x) = \lambda_k X_k(x), \quad (13)$$

where $w^2 = 2B_2$, and $P(x)$ represents the remaining terms in $\text{Eq.}(6)$ as infinite power series perturbations to the harmonic oscillator. One would then imply that

$$\begin{align*}
\lambda_k &= \bar{l} \left[ \frac{1}{2} + \frac{q_o^2 V(q_o)}{Q} \right] + \left[ \frac{2\beta + 1}{2} + \frac{\beta(\beta + 1)}{2\bar{l}} \right] + \lambda_k^{(0)} + \sum_{n=2}^{\infty} \lambda_k^{(n)} \bar{l}^{-n}, \quad (15) \\
\varepsilon_o &= \bar{l} \left[ \frac{1}{2} + \frac{q_o^2 V(q_o)}{Q} \right] + \left[ \frac{2\beta + 1}{2} + (k + \frac{1}{2})w \right] \\
&+ \frac{1}{\bar{l}} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_k^{(0)} \right] + \sum_{n=2}^{\infty} \lambda_k^{(n-1)} \bar{l}^{-n},
\end{align*}$$

and

$$\lambda_k = q_o^2 \sum_{n=-2}^{\infty} E_{k,l}^{(n)} \bar{l}^{-(n+1)}. \quad (16)$$

Hence, equations (15) and (16) yield
E^{(-2)}_{k,l} = \frac{1}{2q_o^2} + \frac{V(q_o)}{Q} \quad (17)

E^{(-1)}_{k,l} = \frac{1}{q_o^2} \left[ \frac{2\beta + 1}{2} + (k + \frac{1}{2})w \right] \quad (18)

E^{(0)}_{k,l} = \frac{1}{q_o^2} \left[ \frac{\beta(\beta + 1)}{2} + \lambda_k^{(0)} \right] \quad (19)

E^{(n)}_{k,l} = \lambda_k^{(n)}/q_o^2; \quad n \geq 1. \quad (20)

Where \( q_o \) is chosen to minimize \( E^{(-2)}_{k,l} \), i.e.

\[ \frac{dE^{(-2)}_{k,l}}{dq_o} = 0 \quad \text{and} \quad \frac{d^2E^{(-2)}_{k,l}}{dq_o^2} > 0. \quad (21) \]

Hereby, \( V(q) \) is assumed to be well behaved so that \( E^{(-2)}_{k,l} \) has a minimum \( q_o \) and there are well-defined bound states. Equation (21) in turn gives, with \( \bar{l} = \sqrt{Q} \),

\[ l_D - \beta = \sqrt{q_o^3V'(q_o)}. \quad (22) \]

Consequently, the second term in Eq.(7) vanishes and the first term adds a constant to the energy eigenvalues. It should be noted that the energy term \( \bar{l}^2E^{(-2)}_{k,l} \) corresponds roughly to the energy of a classical particle with angular momentum \( L_z=\bar{l} \) executing circular motion of radius \( q_o \) in the potential \( V(q_o) \).

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It thus identifies the zeroth-order approximation, to all eigenvalues, as a classical approximation and the higher-order corrections as quantum fluctuations around the minimum \( q_0 \), organized in inverse powers of \( \bar{l} \). The next correction to the energy series, \( \bar{l}E_{k,l}^{(-1)} \), consists of a constant term and the exact eigenvalues of the harmonic oscillator \( w^2x^2/2 \). The shifting parameter \( \beta \) is determined by choosing \( \bar{l}E_{k,l}^{(-1)}=0 \). This choice is physically motivated. In addition to its vital role in removing the singularity at \( l = 0 \), it also requires the agreements between PSLET eigenvalues and eigenfunctions with the exact well known ones for the harmonic oscillator and Coulomb potentials. Hence

\[
\beta = -\left[ \frac{1}{2} + (k + \frac{1}{2})w \right], \quad w = \sqrt{3 + \frac{q_o V''(q_o)}{V'(q_o)}} \quad (23)
\]

where primes of \( V(q_o) \) denote derivatives with respect to \( q_o \). Then equation (6) reduces to

\[
\left[ -\frac{1}{2} \frac{d^2}{dx^2} + \sum_{n=0}^{\infty} v^{(n)}l^{-n/2} \right] \Psi_{k,l}(x) = \left[ \sum_{n=1}^{\infty} q_o^2 E_{k,l}^{(n-1)}l^{-n} \right] \Psi_{k,l}(x). \quad (24)
\]

Setting the wave functions with any number of nodes as

\[
\Psi_{k,l}(x(q)) = F_{k,l}(x) \exp(U_{k,l}(x)), \quad (25)
\]

equation (24) readily transforms into the following Riccati equation:

\[
F_{k,l}(x) \left[ -\frac{1}{2} \left( U_{k,l}''(x) + U_{k,l}'(x)U_{k,l}'(x) \right) + \sum_{n=0}^{\infty} v^{(n)}(x)l^{-n/2} \right.
\]

\[
- \sum_{n=1}^{\infty} q_o^2 E_{k,l}^{(n-1)}l^{-n} \right] - F_{k,l}'(x)U_{k,l}'(x) - \frac{1}{2} F_{k,l}''(x) = 0, \quad (26)
\]
where the primes denote derivatives with respect to $x$. It is evident that this equation admits solution of the form

$$U'_{k,l}(x) = \sum_{n=0}^{\infty} U^{(n)}_{k}(x) \bar{l}^{-n/2} + \sum_{n=0}^{\infty} G^{(n)}_{k}(x) \bar{l}^{-(n+1)/2}, \quad (27)$$

$$F_{k,l}(x) = x^k + \sum_{n=0}^{k-1} \sum_{p=0}^{n} d_{p,k}^{(n)} x^p \bar{l}^{-n/2}, \quad (28)$$

where

$$U^{(n)}_{k}(x) = \sum_{m=0}^{n+1} D_{m,n,k} x^{2m-1} ; \quad D_{0,n,k} = 0, \quad (29)$$

$$G^{(n)}_{k}(x) = \sum_{m=0}^{n+1} C_{m,n,k} x^{2m}. \quad (30)$$

Substituting equations (27) - (30) into equation (26) implies

$$F_{k,l}(x) \left[ -\frac{1}{2} \sum_{n=0}^{\infty} \left( U^{(n)}_{k} \bar{l}^{-n/2} + G^{(n)}_{k} \bar{l}^{-(n+1)/2} \right) + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} U^{(m)}_{k} U^{(n-m)}_{k} \bar{l}^{-n/2} + G^{(m)}_{k} G^{(n-m)}_{k} \bar{l}^{-(n+1)/2} \right]$$

$$+ 2U^{(m)}_{k} G^{(n-m)}_{k} \bar{l}^{-(n+1)/2} + \sum_{n=0}^{\infty} v^{(n)} \bar{l}^{-n/2} - \sum_{n=1}^{\infty} q_0 E^{(n-1)}_{k,l} \bar{l}^{-n}$$

$$- F'_{k,l}(x) \left[ \sum_{n=0}^{\infty} \left( U^{(n)}_{k} \bar{l}^{-n/2} + G^{(n)}_{k} \bar{l}^{-(n+1)/2} \right) \right] - \frac{1}{2} F''_{k,l}(x) = 0 \quad (31)$$

The above procedure obviously reduces to the one described by Mustafa and
Odeh [12,16,17], when $k = 0$. Moreover, the solution of equation (31) follows from the uniqueness of power series representation. Therefore, for a given $k$ we equate the coefficients of the same powers of $\bar{l}$ and $x$, respectively. For example, when $k = 1$ one obtains

$$D_{1,0,1} = -w, \quad U_1^{(0)}(x) = -w x, \quad (32)$$

$$C_{1,0,1} = -\frac{B_3}{w}, \quad a_{0,1}^{(1)} = -\frac{C_{0,0,1}}{w}, \quad (33)$$

$$C_{0,0,1} = \frac{1}{w} (2C_{1,0,1} + 2\beta + 1), \quad (34)$$

$$D_{2,2,1} = \frac{1}{w} \left( \frac{C_{1,0,1}^2}{2} - B_4 \right), \quad (35)$$

$$D_{1,2,1} = \frac{1}{w} \left( \frac{5}{2} D_{2,2,1} + C_{0,0,1} \ C_{1,0,1} - \frac{3}{2} (2\beta + 1) \right), \quad (36)$$

$$E_{1,l}^{(0)} = \frac{1}{q_o^2} \left( \frac{\beta (\beta + 1)}{2} + a_{0,1}^{(1)} \ C_{1,0,1} - \frac{3}{2} \frac{D_{1,2,1}}{2} - \frac{C_{0,0,1}^2}{2} \right), \quad (37)$$

eetc. Here, we reported the nonzero coefficients only. One can then calculate the energy eigenvalues and eigenfunctions from the knowledge of $C_{m,n,k}$, $D_{m,n,k}$, and $a_{p,k}^{(n)}$ in a hierarchical manner. Nevertheless, the procedure just described is suitable for a software package such as MAPLE to determine
the energy eigenvalue and eigenfunction corrections up to any order of the pseudoperturbation series (12).

Although the energy series, equation (12), could appear divergent, or, at best, asymptotic for small $\bar{l}$, one can still calculate the eigenenergies to a very good accuracy by forming the sophisticated $[N,M]$ Padé approximation [1]

$$P_N^M(1/\bar{l}) = (P_0 + P_1/\bar{l} + \cdots + P_M/\bar{l}^M)/(1 + q_1/\bar{l} + \cdots + q_N/\bar{l}^N)$$

to the energy series (12). The energy series (12) is calculated up to $E^{(8)}_{k,l}/\bar{l}^8$ by

$$E_{k,l} = \bar{l}^2 E^{(-2)}_{k,l} + E^{(0)}_{k,l} + \cdots + E^{(8)}_{k,l}/\bar{l}^8 + O(1/\bar{l}^9),$$

(38)

and with the $P_4^4(1/\bar{l})$ Padé approximant it becomes

$$E_{k,l}[4,4] = \bar{l}^2 E^{(-2)}_{k,l} + P_4^4(1/\bar{l}).$$

(39)

Our recipe is therefore well prescribed.

3 D - anharmonic oscillator spectra

In this section we consider the phenomenologically useful and methodically challenging AHO interactions (1), and illustrate the above mentioned procedure.

The substitution of (1) in (23), for $k \geq 0$, implies

$$w = \sqrt{\frac{8\alpha_o q_o + 24\alpha q_o^3}{2\alpha_o q_o + 4\alpha q_o^3}},$$

(40)
and Eq.(22) yields

\[ l_D + \frac{1}{2} \left( 1 + (2k + 1) \sqrt{\frac{8\alpha_o q_o + 24\alpha q_o^3}{2\alpha_o q_o + 4\alpha q_o^3}} \right) = q_o^2 \sqrt{2\alpha_o + 4\alpha q_o^2}. \] (41)

Once \( q_o \) is determined (often numerically) the coefficients \( C_{m,n,k}, D_{m,n,k}, \) and \( q_{p,k}^{(n)} \) are obtained in a sequential manner. Then, the eigenvalues, equation (38), and eigenfunctions (25), along with (27)-(30), are calculated in one batch for each value of \( k, D, l, \alpha_o, \) and \( \alpha. \)

Tables 1 and 2 show the 2D - and 3D - AHO energies, respectively, for different values of \( k, l, \) and \( g = \alpha/2. \) PSLET results, \( E_P, \) compare excellently with those reported by Taseli [7] via a series solution weighted by an appropriate function, \( E_{SS}. \) The 3D - AHO energies for \( k = 1 \) and \( l = 0, \) or equivalently 1D - AHO third excited state, are displayed along with those of Bessis and Bessis [9], \( E_{BB}, \) via an open perturbation technique and the exact ones, \( E_{ex}, \) using Bargman representation [26] (direct numerical integrations) for different anharmonicities. Obviously, our results are in quantitative and/or qualitative agreements with the other ones. They are also in good agreement with those reported by Kleinert [3] via a variational perturbation approach.

Clearly, the accuracy of PSLET increases for larger \( k \) and/or \( l. \) The \( P_4(1/l) \) Padé approximant enhances the accuracy, although its effect is not dramatic for weak anharmonicities. Hence, one proceeds with confidence and obtain, via (39), the 3D - energies (table 4) for states with \( k = 1 \) and \( l = 1, 5, 10. \) In doing so, one should keep the stability of the Padé sequence in point (for more details on this issue the reader may refer to [16]). The same recipe is used to calculate the 2D - AHO energies (table 5) for states with \( k = 0, 1 \) and \( l = 0, 1, 5, 10. \) Then the D - dimensional spectra follow from the
implicated wisdom of (3) and (4) (i.e.: the 2D - and 3D - AHO energies are the basic ingredients for the construction of the D - dimensional ones [22]).

4 Concluding remarks

In this work we have generalized PSLET [12,16,17] for states with arbitrary number of nodal zeros, $k \geq 0$. Starting with the ”radical” central force problem, represented by the radial Schrödinger equation, and generalizing the angular momentum to the D - Dimensional one (i.e.: $l \rightarrow l_D = l + (D - 3)/2$), we have treated the AHO in D - dimensions. The comparison between PSLET results with the other ones, including direct numerical integration, is readily satisfactory.

Although we have used Padé approximants to improve the numerical performance of PSLET, it is by no means clear whether the Padé approximants are necessarily the most effective ones. It has been suggested by Weniger [14] that much better results could be obtained via a class of sequence transformations [27] ( The details of which could be found in [27-30, and references therein]). However, such interesting investigations already lie beyond the scope of the attendant proposal.

Finally, the applicability of PSLET extends far beyond the present D - dimensional anharmonic oscillator model. It could be applied to angular momentum states of multi - electron atoms [23,24], quark - antiquark models [31], 2D - Hydrogenic donor states in an arbitrary magnetic fields [32], 2D - magnetoe excitons [33], two - electron quantum dots [34], etc.
References

[1] C M Bender and S A Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, (McGraw - Hill, New York, 1978).

[2] G A Arteca, F M Fernandez, and E A Castro, *Large Order Perturbation Theory and Summation Methods in Quantum Mechanics*, (Springer - Verlag, Berlin, 1990).

[3] H Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer - Physics*, (World Scientific, Singapore, 1995) 2nd ed.

[4] T Kunihiro, Phys. Rev. **D 57**, R2035 (1998).

[5] W Janks and H Kleinert, Phys. Rev. Lett. **75**, 2787 (1995).

[6] M Lakshmanan et al, Phys. Rev. **A 49**, 3296 (1994).

[7] H Taseli, J. Math. Chem. **20**, 235 (1996).

[8] C M Bender and T T Wu, Phys. Rev. **184**, 1231 (1969).

[9] N Bessis and G Bessis, J. Math. Phys. **38**, 5483 (1997).

[10] M Znojil, J. Math. Phys. **38**, 5087 (1997).

[11] M Znojil, J. Phys. **A 32**, 7419 (1999).

[12] A N Drozdov, J. Phys. **A 28**, 445 (1995).

[13] O Mustafa and M Odeh, Eur. Phys. J. **B 15**, 143 (2000).

[14] B Simon, Bull. Am. Math. Soc. **24**, 303 (1991).

[15] E J Weniger, Phys. Rev. Lett. **77**, 2859 (1996).

[16] I A Ivanov, Phys. Rev. **A 54**, 81 (1996).
[16] O Mustafa and M Odeh, J. Phys. A 32, 6653 (1999).
[17] O Mustafa and M Odeh, J. Phys. B 32, 3055 (1999).
[18] S A Maluendes et al, Phys. Rev. D 34, 1835 (1986).
[19] T Barakat, M Odeh, and O Mustafa, J. Phys. A 31, 3469 (1998).
[20] O Mustafa and S C Chhajlany, Phys. Rev. A 50, 2926 (1994).
  O Mustafa and M Odeh, J. Phys. A 33, (in press) (2000).
  O Mustafa and M Odeh, Commun. Theor. Phys. 33, 469 (2000).
[21] S A Maluendes, F M Fernandez, and E A Castro, Phys. Lett. A 124, 215 (1987).
[22] D R Herschbach et al, *Dimensional Scaling in Chemical Physics*, (Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993).
[23] M Dunn and D K Watson, Few - Body Systems 21, 187 (1996).
[24] M Dunn and D K Watson, Phys. Rev. A 59, 1109 (1999).
[25] D R Herschbach, J. Chem. Phys. 84, 838 (1986).
[26] F T Hioe and E W Montroll, J. Math. Phys. 16, 1945 (1975).
[27] E J Weniger, Comput. Phys. Rep. 10, 189 (1989).
[28] E J Weniger and J Cizek, Comput. Phys. Commun. 59, 471 (1990).
[29] E J Weniger, Phys. Rev. A 56, 5165 (1997).
[30] U D Jentschura et al., Comput. Phys. Commun. 116, 28 (1999).
[31] O Mustafa and R Sever, Phys. Rev. A 44, 4142 (1991).
[32] O Mustafa, J. Phys.: Condens. Matter 8, 8073 (1996).

[33] L Quiroga et al., J. Phys.: Condens. Matter 7, 7517 (1995).

[34] J L Zhu et al., Phys. Rev. B 58, 13755 (1998).
Table 1: 2D anharmonic oscillator (1) energies, in $\hbar = m = 1$ units. Where $\alpha_0 = 1/2$ and $g = \alpha/2$ to recover Taseli’s results, $E_{SSM}$, via series solution [7]. $E_P$ represents PSLET results, Eq.(38), and $E[4, 4]$ shows the $P^4_4(1/\bar{l})$ Padé approximant, Eq.(39).

| $g$  | $k$  | $l$  | $2E_P$          | $2E[4, 4]$         | $E_{SSM}$         |
|------|------|------|-----------------|-------------------|------------------|
| $10^{-4}$ | 0    | 0    | 2.000199955022 | 2.000199955022  | 2.000199955022  |
|      | 5    | 12.00419695953 | 12.00419695953 | 12.00419695953  |
| 1    | 1    | 8.002398591662 | 8.002398591662 | 8.002398591662  |
| 3    | 16.00958904459 | 16.00958904459 | 16.00958904459  |
| 1    | 0    | 2.947835      | 2.952052        | 2.952050        |
|      | 2    | 10.390626203  | 10.390627276    | 10.390627295    |
|      | 4    | 19.217523488  | 19.2175234955   | 19.2175234959   |
| 1    | 1    | 15.48277174   | 15.48277148     | 15.48277158     |
| $10^4$ | 0    | 50.75164      | 50.54788        | 50.54804        |
|      | 4    | 368.0300833   | 368.030082436   | 368.030082448   |
| 1    | 205.3783     | 205.3774       | 205.3777        |
| 2    | 394.577414   | 394.577403     | 394.577407     |
Table 2: Same as Table 1 for 3D anharmonic oscillator (1) energies.

| \( g \) | \( k \) | \( l \) | \( 2E_P \)       | \( 2E\{4,4\} \)   | \( E_{SSM} \)   |
|-------|-----|-----|----------------|-------------------|---------------|
| \( 10^{-4} \) | 0   | 0   | 3.0003748969   | 3.0003748969     | 3.0003748969 |
|       | 10  |     | 23.014356719   | 23.014356719     | 23.014356719 |
|       | 1   | 0   | 4.648511       | 4.648815         | 4.648813      |
|       |     | 1   | 8.380337       | 8.38034245       | 8.38034253    |
|       | 5   |     | 26.528917558   | 26.528917558     | 26.528917558 |
|       | 1   | 3   | 27.898417763   | 27.898417756     | 27.898417760 |
| 20    | 0   | 1   | 19.783266      | 19.7832518       | 19.7832519    |
|       | 2   |     | 30.057200      | 30.057199029     | 30.057199045 |
|       | 5   |     | 65.961500037   | 65.96150003049   | 65.96150003068 |
|       | 1   | 1   | 44.209282      | 44.209279007     | 44.209279973  |
| \( 10^3 \) | 0   | 0   | 38.092         | 38.086822         | 38.086833     |
|       | 0   | 3   | 149.439046     | 149.439045568     | 149.439045581 |
Table 3: Three-dimensional energies for the $k = 1$ and $l = 0$ state or equivalently one-dimensional third excited state energies for $V(q) = \frac{q^2}{2} + \alpha q^4$. $E_{BB}$ denotes Bessis and Bessis results [9] and the exact ones $E_{\text{exact}}$, reported therein, from direct numerical integrations for different anharmonicities.

| $\alpha$ | $E_P$    | $E[4, 4]$ | $E_{\text{ex}}$ | $E_{BB}$  |
|----------|----------|-----------|------------------|-----------|
| 0.002    | 3.53674413 | 3.536744133 | 3.53674413 | 3.53674  |
| 0.01     | 3.67109494  | 3.67109494  | 3.67109494 | 3.67109  |
| 0.1      | 4.6288828   | 4.6288828   | 4.62888281 | 4.62884  |
| 0.3      | 5.79657376  | 5.79657363  | 5.79657363 | 5.79679  |
| 0.5      | 6.578402    | 6.578402    | 6.57840195 | 6.57953  |
| 0.7      | 7.193266    | 7.193265    | 7.19326528 | 7.19549  |
| 1        | 7.942405    | 7.942404    | 7.94240399 | 7.94630  |
| 2        | 9.727325    | 9.727322    | 9.72732319 | 9.73596  |
| 50       | 27.192660   | 27.192638   | 27.1926458 | 27.2473  |
| 1000     | 73.419158   | 73.419089   | 73.419114  | 73.5805  |
| 8000     | 146.745600  | 146.745461  | 146.745512 | 147.0714 |
Table 4: Three-dimensional energies for states with $k = 1$ and $l = 1, 5, 10$ for $V(q) = \frac{q^2}{2} + \alpha q^4$. Only $E[4, 4]'s$ are listed for different anharmonicities.

| $\alpha$ | $l = 1$ | $l = 5$ | $l = 10$ |
|----------|---------|---------|---------|
| 0.01     | 4.76645813712 | 9.289594583372 | 15.233049583486 |
| 0.1      | 6.176138 | 12.89579856 | 22.309686916 |
| 0.5      | 8.93090 | 19.3542918 | 34.30436531 |
| 1        | 10.83313 | 23.7006578 | 42.25455311 |
| 50       | 37.4108 | 83.258353 | 149.9642236 |
| 1000     | 101.07403 | 225.231013 | 405.9901767 |
Table 5: Two-dimensional energies for states with $k = 0, 1$ and $l = 0, 1, 5, 10$ for $V(q) = \frac{q^2}{2} + \alpha q^4$. Only $E_{[4, 4]}$'s are listed for different anharmonicities.

| $k$ | $\alpha$ | $l = 0$       | $l = 1$       | $l = 5$       | $l = 10$       |
|-----|----------|---------------|---------------|---------------|---------------|
| 0   | 0.01     | 1.0191783021  | 2.056555600   | 6.372257220   | 12.0962676139707 |
| 0.1 | 1.150188 | 2.4144303     | 8.29606606    | 16.976887733  |
| 0.3 | 1.33966 | 2.895905      | 10.53678440   | 22.227347003  |
| 0.5 | 1.4760  | 3.231453      | 12.01658310   | 25.611647809  |
| 0.7 | 1.5866  | 3.499749      | 13.17451055   | 28.235574336  |
| 1   | 1.7242  | 3.830324      | 14.58077151   | 31.403160969  |
| 50  | 5.512   | 12.6399       | 50.376652     | 110.564887242 |
| 1   | 0.01     | 3.129048426   | 4.21691935465 | 8.71236579303 | 14.62528496652399 |
| 0.1 | 3.876642 | 5.3954269     | 12.01280444   | 21.323943303  |
| 0.3 | 4.8105  | 6.80426       | 15.6275228    | 28.28370297   |
| 0.5 | 5.4412  | 7.74139       | 17.9691075    | 32.72664974   |
| 0.7 | 5.9389  | 8.47655       | 19.7879425    | 36.15909940   |
| 1   | 6.5466  | 9.3708        | 21.9862477    | 40.29318318   |
| 50  | 22.267  | 32.237        | 77.122811     | 142.8926171   |