Groups with Boundedly Finite Conjugacy Classes of Commutators

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Abstract. In 1954 B. H. Neumann discovered that if $G$ is a group in which all conjugacy classes are finite with bounded size, then the derived group $G'$ is finite. Later (in 1957) Wiegold found an explicit bound for the order of $G'$. We study groups in which the conjugacy classes containing commutators are finite with bounded size. We obtain the following results.

Let $G$ be a group and $n$ a positive integer.

If $|x^G| \leq n$ for any commutator $x \in G$, then the second derived group $G''$ is finite with $n$-bounded order.

If $|x^G| \leq n$ for any commutator $x \in G$, then the order of $\gamma_3(G')$ is finite and $n$-bounded.

1. Introduction

Given a group $G$ and an element $x \in G$, we write $x^G$ for the conjugacy class containing $x$. Of course, if the number of elements in $x^G$ is finite, we have $|x^G| = [G : C_G(x)]$. A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size. One of B. H. Neumann’s discoveries was that in a BFC-group the derived group $G'$ is finite [3]. It follows that if $|x^G| \leq n$ for each $x \in G$, then the order of $G'$ is bounded by a number depending only on $n$. A first explicit bound for the order of $G'$ was found by J. Wiegold [7], and the best known was obtained in [1] (see also [4] and [6]).

In the present article we deal with groups $G$ such that $|x^G| \leq n$ whenever $x$ is a commutator, that is, $x = [x_1, x_2]$ for suitable $x_1, x_2 \in G$. Here and throughout the article we write $[x_1, x_2]$ for $x_1^{-1}x_2^{-1}x_1x_2$. As usual, we denote by $G'$ the derived group of $G$ and by $G''$ the derived group of $G'$. (the second derived group of $G$).
**Theorem 1.1.** Let \( n \) be a positive integer and \( G \) a group in which \(|x^G| \leq n\) for any commutator \( x \). Then \(|G''|\) is finite and \( n \)-bounded.

Further, we consider groups \( G \) in which \(|x^G'| \leq n\) whenever \( x \) is a commutator.

**Theorem 1.2.** Let \( n \) be a positive integer and \( G \) a group in which \(|x^G'| \leq n\) for any commutator \( x \). Then \(|\gamma_3(G')|\) is finite and \( n \)-bounded.

Here \( \gamma_3(G') \) denotes the third term of the lower central series of \( G' \). We do not know whether under hypothesis of Theorem 1.2 the second derived group \( G'' \) must necessarily be finite. Note that under hypothesis of Theorem 1.1 \( \gamma_3(G) \) can be infinite. This can be shown using any example of an infinite torsion-free metabelian group whose commutator quotient is finite (see for instance [2]).

We make no attempts to obtain good bounds for \(|G''|\) in Theorem 1.1 and \(|\gamma_3(G')|\) in Theorem 1.2. The proofs given here yield bounds \( n^{54n^{14}} \) and \( n^{12n^{10}} \), respectively. The bounds however do not look realistic at all.

## 2. Proofs

Let \( G \) be a group generated by a set \( X \) such that \( X = X^{-1} \). Given an element \( g \in G \), we write \( l_X(g) \) for the minimal number \( l \) with the property that \( g \) can be written as a product of \( l \) elements of \( X \). Clearly, \( l_X(g) = 0 \) if and only if \( g = 1 \). We call \( l_X(g) \) the length of \( g \) with respect to \( X \).

**Lemma 2.1.** Let \( H \) be a group generated by a set \( X = X^{-1} \) and let \( K \) be a subgroup of finite index \( m \) in \( H \). Then each coset \( Kb \) contains an element \( g \) such that \( l_X(g) \leq m - 1 \).

**Proof.** If \( b \in K \), the result is obvious. Therefore we assume that \( b \not\in K \). Choose \( g \in Kb \) in such a way that \( s = l_X(g) \) is as small as possible and suppose that \( s \geq m \). Write \( g = x_1 \cdots x_s \) with \( x_i \in X \) and set \( y_j = x_1 \cdots x_j \) for \( j = 1, \ldots, s \). Since \( s \) is the minimum of lengths of elements in \( Kb \), it follows that none of the elements \( y_1, \ldots, y_s \) lies in \( K \). Thus, these \( s \) elements belong to the union of at most \( m - 1 \) right cosets of \( K \) and we conclude that \( Ky_i = Ky_j \) for some \( 1 \leq i < j \leq s \). It is now easy to see that the element \( h = y_1x_{j+1} \cdots x_s \) belongs to \( Kb \) while \( l_X(h) < l_X(g) \). This is a contradiction with the choice of \( g \). \( \square \)

In the sequel the above lemma will be used in the situation where \( H \) is the derived group of a group \( G \) and \( X \) is the set of commutators in \( G \). Therefore we will write \( l(g) \) to denote the smallest number such that the element \( g \in G' \) can be written as a product of as many commutators.
Recall that if $H$ is a group and $a \in H$, the subgroup $[H, a]$ is generated by all commutators of the form $[h, a]$, where $h \in H$. It is well-known that $[H, a]$ is always normal in $H$. Recall that in any group $G$ the following “standard commutator identities” hold.

1. $[x, y]^{-1} = [y, x]$;
2. $[xy, z] = [x, z]^y[y, z]$;
3. $[x, yz] = [x, z][x, y]^z$.

In what follows the above identities will be used without explicit references.

We will now fix some notation and hypothesis.

**Hypothesis 2.2.** Let $G$ be a group and $K$ a subgroup containing $H = G'$. Let $X$ denote the set of commutators in $G$ and suppose that $C_K(x)$ has finite index at most $n$ in $K$ for each $x \in X$. Let $m$ be the maximum of indices of $C_H(x)$ in $H$, where $x \in X$. Suppose further that $a \in X$ and $C_H(a)$ has index precisely $m$ in $H$. Choose $b_1, \ldots, b_m \in H$ such that $l(b_i) \leq m - 1$ and $a^H = \{a^{b_i}; i = 1, \ldots, m\}$. (The existence of such elements is guaranteed by Lemma 2.1.) Set $U = C_K(\langle b_1, \ldots, b_m \rangle)$.

**Lemma 2.3.** Assume Hypothesis 2.2. Then for any $x \in X$ the subgroup $[H, x]$ has finite $m$-bounded order.

**Proof.** Choose $x \in X$. Since $C_H(x)$ has index at most $m$ in $H$, by Lemma 2.1 we can choose elements $y_1, \ldots, y_m$ such that $l(y_i) \leq m - 1$ and $[H, x]$ is generated by the commutators $[y_i, x]$. For each $i = 1, \ldots, m$ write $y_i = y_{i1} \cdots y_{i(m-1)}$, where $y_{ij} \in X$. The standard commutator identities show that $[y_i, x]$ can be written as a product of conjugates in $H$ of the commutators $[y_{ij}, x]$. Let $h_1, \ldots, h_s$ be the conjugates in $H$ of elements from the set $\{x, y_{ij}; 1 \leq i, j \leq m\}$. Since $C_H(h)$ has finite index at most $m$ in $H$ for each $h \in X$, it follows that $s$ is $m$-bounded. Let $T = \langle h_1, \ldots, h_s \rangle$. It is clear that $[H, x] \leq T'$ and so it is sufficient to show that $T'$ has finite $m$-bounded order. Observe that $C_H(h_i)$ has finite index at most $m$ in $H$ for each $i = 1, \ldots, s$. It follows that the center $Z(T)$ has index at most $m^s$ in $T$. Thus, Schur’s theorem [5 10.1.4] tells us that $T'$ has finite $m$-bounded order, as required. \[\square\]

Note that the subgroup $U$ has finite $n$-bounded index in $K$. This follows from the facts that $l(b_i) \leq m - 1$ and $C_K(x)$ has index at most $n$ in $K$ for each $n \in K$.

The next lemma is somewhat analogous with Lemma 4.5 of Wiegold [7].
Lemma 2.4. Assume Hypothesis 2.2. Suppose that \( u \in U \) and \( ua \in X \). Then \( [H, u] \leq [H, a] \).

Proof. Since \( u \in U \), it follows that \( (ua)^{b_i} = u a^{b_i} \) for each \( i = 1, \ldots, m \). Therefore the elements \( u a^{b_i} \) form the conjugacy class \( (ua)^{H} \). For an arbitrary element \( g \in H \) there exists \( h \in \{b_1, \ldots, b_m\} \) such that \( (ua)^g = u a^h \) and so \( a^g a^g = u a^h \). Therefore \( [u, g] = a^g a^{-g} \in [H, a] \). The lemma follows.

Proposition 2.5. Assume Hypothesis 2.2 and write \( a = [d, e] \) for suitable \( d, e \in G \). There exists a subgroup \( U_1 \leq U \) with the following properties.

1. The index of \( U_1 \) in \( K \) is \( n \)-bounded;
2. \( [H, U_1'] \leq [H, a]^{d^{-1}} \);
3. \( [H, [U_1, d]] \leq [H, a] \).

Proof. Set \( U_1 = U \cap U^{d^{-1}} \cap U^{d^{-1}e^{-1}} \).
Since the index of \( U \) in \( K \) is \( n \)-bounded, we conclude that the index of \( U_1 \) in \( K \) is \( n \)-bounded as well. Choose arbitrarily elements \( h_1, h_2 \in U_1 \). Write
\[
[h_1 d, e h_2] = [h_1, h_2]^{d}[d, h_2][h_1, e]^{d h_2}[d, e]^{h_2}
\]
and so
\[
[h_1 d, e h_2]^{h_2^{-1}} = [h_1, h_2]^{d h_2^{-1}}[d, h_2]^{h_2^{-1}}[h_1, e]^{d}[d, e].
\]
Denote the product \( [h_1, h_2]^{d h_2^{-1}}[d, h_2]^{h_2^{-1}}[h_1, e]^{d} \) by \( u \). Thus, the right hand side of the above equality is \( u a \) while, obviously, on the left hand side we have a commutator. Let us check that \( u \in U \). We see that \( [h_1, h_2]^{d h_2^{-1}} \in U_1^{d h_2^{-1}} \leq U \) because \( U_1^{d} \leq U \). By the same reason, \( [d, h_2]^{h_2^{-1}} \in U \). Finally, \( [h_1, e]^{d} \in U_1^{d}U_1^{e} \leq U \) so indeed \( u \in U \). By Lemma 2.4 \( [H, u] \leq [H, a] \). This holds for any choice of \( h_1, h_2 \in U_1 \).
In particular, taking \( h_1 = 1 \) we see that \( [H, [d, h_2]^{h_2^{-1}}] \leq [H, a] \) while taking \( h_2 = 1 \) we conclude that \( [H, [h_1, e]^{d}] \leq [H, a] \). It now follows that \( [H, [h_1, h_2]^{d h_2^{-1}}] \leq [H, a] \). Since \( [H, a] \) is normal in \( H \), we have \( [H, [h_1, h_2]] \leq [H, a]^{d^{-1}} \) and so \( [H, U_1'] \leq [H, a]^{d^{-1}} \), which proves that \( U_1 \) has property 2. Examine again the inclusion \( [H, [d, h_2]^{h_2^{-1}}] \leq [H, a] \). Since \( [H, a] \) is normal in \( H \), it follows that \( [H, [U_1, d]] \leq [H, a] \). Therefore \( U_1 \) has property 3 as well. The proof is now complete.

We are ready to prove our main results.

Proof of Theorem 1.1. Recall that \( G \) is a group in which \( |x^G| \leq n \) for any commutator \( x \). We need to show that \( |G''| \) is finite and \( n \)-bounded.
We denote by $X$ the set of commutators in $G$ and set $H = G'$. Let $m$ be the maximum of indices of $C_H(x)$ in $H$, where $x \in X$. Of course, $m \leq n$. Choose $a \in X$ such that $C_H(a)$ has index precisely $m$ in $H$. Choose $b_1, \ldots, b_m \in H$ such that $l(b_i) \leq m - 1$ and $a^H = \{a^{b_i}; i = 1, \ldots, m\}$. Set $U = C_G((b_1, \ldots, b_m))$. Note that the index of $U$ in $G$ is $n$-bounded. Applying Proposition 2.5 with $K = G$ we find a subgroup $U_1$, of $n$-bounded index, such that $[H, U_1] \leq \langle [H, a]^G \rangle$. Since the index of $U_1$ in $G$ is $n$-bounded, we can find $n$-boundedly many commutators $c_1, \ldots, c_s \in X$ such that $H = \langle c_1, \ldots, c_s, H \cap U_1 \rangle$. Let $T$ be the normal closure in $G$ of the product of the subgroups $[H, a]$ and $[H, c_i]$ for $i = 1, \ldots, s$. By Lemma 2.3 each of these subgroups has $n$-bounded order. Our hypothesis is that each of them has at most $n$ conjugates. Thus, $T$ is a product of $n$-boundedly many finite subgroups, normalizing each other and having $n$-bounded order. We conclude that $T$ has finite $n$-bounded order. Therefore it is sufficient to show that the second derived group of the quotient $G/T$ has finite $n$-bounded order. So we pass to the quotient $G/T$. To avoid complicated notation the images of $G$, $H$ and $X$ will be denoted by the same symbols. We observe that the derived group of $HU_1$ is contained in $Z(H)$. This follows from the facts that $HU_1$ is generated by $c_1, \ldots, c_s$ and $U_1$ and modulo $T$ we have $c_1, \ldots, c_s \in Z(H)$ and $U_1 \leq Z(H)$.

Let $\mathcal{X}$ denote the family of subgroups $S \leq G$ with the following properties.

1. $H \leq S$;
2. $S' \leq Z(H)$;
3. $S$ has finite index in $G$.

We already know that $\mathcal{X}$ is non-empty since it contains $HU_1$. Choose $J \in \mathcal{X}$ of minimal possible index $j$ in $G$. Since the index of $U_1$ in $G$ is $n$-bounded, the index $j$ is $n$-bounded, too. We will now use induction on $j$. If $j = 1$, then $J = G$ and $H \leq Z(H)$. So $G'' = 1$ and we have nothing to prove. Thus, we assume that $j \geq 2$.

Again, we take a commutator $a_0 \in X$ such that $C_H(a_0)$ has maximal possible index in $H$ and write $a_0 = [d, e]$ for suitable $d, e \in G$. If both $d$ and $e$ belong to $J$, we conclude (since $J' \leq Z(H)$) that $H$ is abelian and $G'' = 1$. Thus, assume that at least one of them, say $d$, is not in $J$. We will use Proposition 2.5 with $K = G$. It follows that there is a subgroup $V$ of $n$-bounded index in $G$ such that $[H, V, d] \leq [H, a_0]$. Replacing if necessary $V$ by $V \cap J$, without loss of generality we can assume that $V \leq J$. Let $L = J(d)$. Note that $L' = J'[J, d]$. Let $1 = g_1, \ldots, g_l$ be a full system of representatives of the right cosets of $V$ in $J$. Then $[J, d]$ is generated by $[V, d]^{g_1}, \ldots, [V, d]^{g_l}$ and $[g_1, d], \ldots, [g_l, d]$. This is
straightforward from the fact that \([vg, d] = [v, d]g[g, d]\) for any \(g, v \in G\). Next, for each \(i = 1, \ldots, t\) set \(x_i = [g_i, d]\). Let \(R\) be the normal closure in \(G\) of the product of the subgroups \([H, a_0]^g\) and \([H, x_i]\) for \(i = 1, \ldots, t\). By Lemma 2.3 each of these subgroups has \(n\)-bounded order. Our hypothesis is that each of them has at most \(n\) conjugates. Thus, \(R\) is a product of \(n\)-boundedly many finite subgroups, normalizing each other and having \(n\)-bounded order. We conclude that \(R\) has finite \(n\)-bounded order. We see that \([H, L]\) \(\leq\) \(R\). Since \(d \notin J\), the index of \(L\) in \(G\) is strictly smaller than \(j\). Therefore, by induction on \(j\), the second derived group of \(G/R\) is finite with bounded order. Taking into account that also \(R\) is finite with bounded order, we deduce that \(G''\) is finite with bounded order. The proof is now complete. \(\square\)

**Proof of Theorem 1.2.** Recall that \(G\) is a group in which \(|xG'| \leq n\) for any commutator \(x\). We need to prove that \(\gamma_3(G')\) is finite with \(n\)-bounded order. As before, we write \(X\) for the set of commutators in \(G\) and \(H\) for the derived group. Choose a commutator \(a \in X\) such that \(C_H(a)\) has maximal possible index in \(H\). We will use Proposition 2.5 with \(K = H\). It follows that \(H\) contains a subgroup \(U_1\) of finite \(n\)-bounded index such that \([H, U_1'] \leq [H, a]^d\) for some \(d \in G\). Write \(b_0 = a^{-1}\) and so \([H, U_1] \leq [H, b_0]\). Since the index of \(U_1\) in \(H\) is \(n\)-bounded, we can find \(n\)-boundedly many commutators \(b_1, \ldots, b_s \in X\) such that \(H = \langle b_1, \ldots, b_s, U_1 \rangle\). Let \(T\) be the product of the subgroups \([H, b_i]\) for \(i = 0, 1, \ldots, s\). By Lemma 2.3 each of these subgroups has \(n\)-bounded order. All of them are normal in \(H\) and so \(T\) is normal in \(H\) and has finite \(n\)-bounded order. The center of \(H/T\) contains the images of \(U_1'\) and \(b_1, \ldots, b_s\). It follows that the quotient of \(H/T\) over its center is abelian. Therefore \(\gamma_3(H) \leq T\), which completes the proof. \(\square\)

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