The complete bipartite graph $K_{4,4}$ is Uniformly Most-Reliable

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In network design, the all-terminal reliability maximization is of paramount importance. In this classical setting, we assume a simple graph with perfect nodes but independent edge failures with identical probability $\rho$. The goal is to communicate $n$ terminals using $e$ edges, in such a way that the connectedness probability of the resulting random graph is maximum. A graph with $n$ nodes and $e$ edges that meets the maximum reliability property for all $\rho \in (0, 1)$ is called uniformly most-reliable $(n, e)$-graph (UMRG). The discovery of these graphs is a challenging problem that involves an interplay between extremal graph theory and computational optimization. Recent works confirm the existence of special cubic UMRGs, such as Wagner, Petersen and Yutsis graphs, and a 4-regular graph $H = C_7$. In a foundational work in the field, Boesch. et. al. state with no proof that the bipartite complete graph $K_{4,4}$ is UMRG. In this paper, we revisit the breakthroughs in the theory of UMRG. A simple methodology to determine UMRGs based on counting trivial cuts is presented. Finally, we test this methodology to mathematically prove that the complete bipartite graph $K_{4,4}$ is UMRG.

Keywords: Graph Theory, Network Reliability, Uniformly Most-Reliable Graphs

1 Motivation

Network reliability analysis deals with probabilistic-based models, where the goal is to determine the probability of correct operation of a system. In its most elementary setting, we are given a simple graph $G$ with perfect nodes but random edge failures with identical and independent probability $\rho$. The all-terminal reliability is the probability that the resulting random graph remains connected. This special measure is known as the all-terminal reliability, and its evaluation belongs to the class of $\mathcal{NP}$-Hard problems, as Provan and Ball (1983) proved. Therefore, the related literature offers pointwise reliability estimations, Cancela et al. (2015), as well as exact (exponential time) reliability evaluation methods (Satyanarayana and Chang, 1983). The reader can find an excellent monograph on the combinatorics of network reliability authored by Colbourn (1987).

Even though the reliability evaluation is useful for optimization and diagnostics, scarce works deal with reliability optimization, which is known as network synthesis. Boesch et al. (2009) offer a survey on this topic.
on challenges in network reliability, and a section is devoted to synthesis. Barrera et al. (2015) develop a reliability maximization subject to budget constraints, including dependent failures and heterogeneous costs in their model. More recently, a general network optimization proposal deals with simultaneous node and edge failures (Pulsipher and Zavala, 2020). Given the hardness of the problem, these authors consider sample average approximation (SAA; see Kleywegt et al. (2002) for details) in order to obtain solutions under small instances. It can be observed that the symmetry in the globally optimum solutions is lost when we deal with non-identical edge failures.

Our goal is to understand the interplay between connectivity theory and network synthesis, and the dialogue between fully deterministic and probabilistic models. We will focus on the corresponding synthesis under the all-terminal reliability setting, where history reveals that optimal graphs are highly symmetric, but it is still enigmatic to determine a full list of them, or even the existence for a fixed number of nodes and edges. In the first non-trivial constructions of optimal graphs, Boesch et al. (1991) state that the bipartite complete graph $K_{4,4}$ is optimal, but the authors do not offer a formal proof. To the best of our knowledge, this is the first work that presents a formal proof of this ancient statement.

Figure 1 presents a graph-constellation with an updated set of the UMRGs found so far as a function of the pairs $(n, e)$. The pairs where a formal inexistence of UMRG is proved are marked with red circles. The reader can appreciate that the smallest counterexample on the existence of UMRG is the pair $(6, 11)$, and it was established by Myrvold (1996).

The family of sparse $(n, n + i)$ graphs are straight lines with unit slope (see Subsection 3.3 for details on sparse UMRGs). The reader can find trees, elementary cycles, balanced $\theta$-graphs and elementary subdivisions of $K_4$. The green squares represent $K_n$ minus an arbitrary matching (see Subsection 3.4 for details on dense UMRGs). Cubic graphs can be found in the straight line with slope $3/2$. These graphs include $K_4$, Wagner, Petersen and Yutsis. It is still an open problem to determine whether Heawood and Cantor-Mobius are UMRG. As far as we know, the only 4-regular graphs include $K_5$, the odd antihole $C_7$, and in this work we include $K_{4,4}$ (see Theorem 1).

The contributions of this work can be summarized by the following items:

- A simple counting methodology is introduced in order to study UMRGs. This methodology considers trivial cuts and Inclusion-Exclusion principle.

- In order to test the effectiveness of our methodology, we formally prove that $K_{4,4}$ is UMRG.

This paper is organized as follows. Section 2 presents a formal definition of UMRG and classical terminology from graph theory. The related work is presented in Section 3. The main contribution is presented in Section 4, where it is formally proved that $K_{4,4}$ is the only uniformly most-reliable $(8, 16)$-graph. Section 5 presents concluding remarks and trends for future work.
The complete bipartite graph $K_{4,4}$ is Uniformly Most-Reliable

Fig. 1: UMRG found so far as a function of $(n, e)$
2 Definitions and terminology

Consider a simple graph \( G = (V, E) \) with perfect nodes and unreliable edges that fail independently with identical failure probability \( \rho \). The all-terminal reliability \( R_G(\rho) \) measures the probability that the resulting random graph remains connected. For convenience, we work with the unreliability \( U_G(\rho) = 1 - R_G(\rho) \). Let us denote \( n = |V| \) and \( e = |E| \) the respective order and size of the graph \( G \). A cutset is an edge-set \( C \subseteq E \) such that the resulting graph \( G - C \) is not connected. Denote by \( m_k(G) \) the number of cutsets with cardinality \( k \). By sum-rule, the unreliability polynomial can be expressed as follows:

\[
U_G(\rho) = \sum_{k=0}^{e} m_k(G)\rho^k(1-\rho)^{e-k}.
\]  

An \((n, e)\)-graph is a graph with \( n \) nodes and \( e \) edges. Since the number of \((n, e)\)-graphs is finite, if we consider a fixed \( \rho \in [0, 1] \), there is at least one graph \( H \) that attains the minimum unreliability, i.e., \( U_H(\rho) \leq U_G(\rho) \) for all \((n, e)\)-graph \( G \). Further, if the previous condition holds for all \( \rho \in [0, 1] \) and all \((n, e)\)-graphs \( G \), then \( H \) is a UMRG.

The edge-connectivity \( \lambda(G) \) is the smallest natural number \( \lambda \) such that \( m_\lambda > 0 \). A trivial cutset is a cutset that includes all the edges adjacent to a fixed node. The degree of a node \( v \in V \) is the number of edges that are incident to \( v \) and is written \( d_v \). A graph is regular if all the nodes have identical degrees. The minimum degree of a graph \( G \) is denoted by \( \delta(G) \). Using trivial cutsets, \( \lambda(G) \leq \delta(G) \). A graph is super-\( \lambda \), or superconnected, if it is \( \lambda \)-regular and further, it has only trivial cutsets: \( m_\lambda = n \). A bridge is a single edge \( uv \) such that \( G - uv \) has more components than \( G \). A cut-point is a node \( v \) such that \( G - \{v\} \) has more components than \( G \). A graph \( G \) with more than two nodes is biconnected if it is connected and it has no cut-points. A tree is an acyclic connected graph and the number of spanning trees of \( G \) is its treewidth, denoted by \( \tau(G) \). A matching is a disjoint (or non-adjacent) set of edges. A perfect matching is a matching that meets all the nodes of a graph. The elementary cycle with \( n \) nodes is denoted by \( C_n \), and \( K_n \) represents the complete graph with \( n \) nodes, where all the nodes are pairwise adjacent. A triangle is a \( C_3 \) while a square is a \( C_4 \). The bipartite complete graph \( K_{n_1,n_2} \) is represented by a bipartition where the node-set \( V = A \cup B \), \( |A| = n_1 \), \( |B| = n_2 \) and every node from \( A \) is linked with every node from \( B \). Additionally, there is no pair of nodes belonging to \( A \) or to \( B \) that are mutually adjacent. A multipartite complete graph \( K_{n_1,\ldots,n_k} \) is defined analogously, where \( V = \bigcup_{i=1}^{k} V_i \) is a partition, \( |V_i| = n_i \) and every node from \( V_i \) is linked with all the remaining nodes from \( V_j \) for all \( j \neq i \). A graph is sparse if \( e = O(n) \), and dense if \( e = O(n^2) \).

3 Related work

The most influential works in the field of UMRGs are here revisited. The etymology and elements on UMRGs, findings of sparse and dense UMRGs, as well as open conjectures, are presented.

3.1 Etymology

The most-reliable graphs for all values of \( \rho \) were called uniformly optimally reliable networks by Boesch \cite{boesch1986} in his seminal work. Later, Wendy Myrvold \cite{myrvold1996} offered a nice survey up to 1996. The author explains that the term uniformly most-reliable graph (UMRG) is adopted to avoid a tongue-twister.
3.2 Elements

Expression (1) suggests a sufficient criterion for a graph $H$ to be UMRG, specifically, $m_k(H) \leq m_k(G)$ for all $k \in \{0, \ldots, e\}$ and for all $(n, e)$-graph $G$. In 1986, Boesch conjectured that the converse holds, and this is yet one of the major open problems in the field (Boesch, 1986). As far as we know, the most recent survey in this topic was written a decade ago by Boesch et al. (2009), and it summarizes different trends in network reliability analysis.

Still today, the search of UMRGs is based on the minimization of all the coefficients $m_k$. This approach is promoted by the following result:

**Proposition 1 (Theorem 1, Bauer et al. (1987))**

(i) If there exists $k$ such that $m_i(H) = m_i(G)$ for all $i < k$ but $m_k(H) < m_k(G)$, then there exists $\rho_0 > 0$ such that $U_H(\rho) < U_G(\rho)$ for all $\rho \in (0, \rho_0)$.

(ii) If there exists $k$ such that $m_i(H) = m_i(G)$ for all $i > k$ but $m_k(H) < m_k(G)$, then there exists $\rho_1 < 1$ such that $U_H(\rho) < U_G(\rho)$ for all $\rho \in (\rho_1, 1)$.

By definition, there are no cutsets with lower cardinality than the edge connectivity $\lambda$. Therefore, $m_i(G) = 0$ for all $i < \lambda$, and by Proposition (i) UMRGs must have the maximum edge-connectivity $\lambda$. Furthermore, the number of cutsets $m_\lambda$ must be minimized. On the other hand, $m_i(G) = \binom{e}{i}$ for all $i > e - n + 1$, since trees are minimally connected with $e = n - 1$ edges. The number of spanning subgraphs with $e - n + 1$ edges is precisely the tree-number $\tau(G)$, so

$$m_{e-n+1}(G) = \binom{e}{e-n+1} - \tau(G).$$

Using Proposition (ii), the maximization of the tree-number is a necessary condition. Prior observations directly connect this network design problem with extremal graph theory:

**Corollary 1 (Necessary Criterion)** If $H$ is UMRG, it must have the maximum tree-number $\tau(H)$, maximum connectivity $\lambda(H)$, and the minimum number of cutsets $m_\lambda(H)$ among all $(n, e)$-graphs with maximum connectivity.

For convenience we say that an $(n, e)$-graph $H$ is $t$-optimal if $\tau(H) \geq \tau(G)$ for every $(n, e)$ graph $G$. Briefly, Corollary claims that UMRG must be $t$-optimal and max-$\lambda$ min-$m_\lambda$.

It is worth remarking that there are $(n, e)$-pairs where a UMRG does not exist. Indeed, Myrvold et al. (1991) proposed an infinite family of counterexamples. The historical conjecture of Leggett and Bedrosian (1965) asserts that $t$-optimal graphs must be almost regular, that is, the degrees differ at most by one. Even though closed formulas are available for the tree-number of specific graphs, the progress on $t$-optimality is effective under special regularity conditions (Cheng, 1981), almost-complete graphs or other special graphs with few edges (Petingi et al., 1998).
3.3 Finding Sparse UMRG

By Corollary 1, Bauer et al. (1985) provide a family of candidates of UMRG. Later works try to find uniformly most-reliable \((n, n+i)\)-graphs for \(i\) small (i.e., sparse graphs), by a simultaneous minimization of all the coefficients \(m_k\). When \(i = 0\) we have \(n = e\), and the elementary cycle \(C_n\) is \(t\)-optimal. All the other graphs with \(n = e\) are not 2-connected, and by direct inspection we can see that \(C_n\) achieves the minimum coefficients \(m_k\). The first non-trivial UMRGs were found by Boesch et al. (1991). A new reading of Bauer et. al. construction lead them to find that \(\theta\)-graphs are \((n, n+1)\) UMRGs, when the path lengths are as even as possible. A more challenging problem is to find \((n, n+2)\) UMRGs. Boesch et. al. minimize the four effective terms \(m_0, m_1, m_2\) and \(m_3\) from Expression (1).

An \((n, n+2)\) max-\(\lambda\)-min-\(\lambda\) graph already minimizes the first three terms. If in addition the tree-number is minimized all the coefficients are simultaneously minimized, and the result must be a UMRG. The merit of the paper Boesch et al. (1991) is to adequately select the feasible graphs from Bauer et al. (1987) that minimizes the tree-number. Observe that \(K_4\) can be partitioned into three perfect matchings, \(PM_1, PM_2\) and \(PM_3\). The result is that we should insert \(n - 4\) points in the six edges of \(K_4\) in such a way that:

(i) the number of inserted nodes in all the edges differ by at most one, and

(ii) if we insert the same number of nodes in two different matchings \(PM_i \neq PM_j\), then the number of nodes in the four edges from \(PM_i \cup PM_j\) are identical.

The resulting \((n, n+2)\)-graph defines, for every \(n \geq 4\), a single graph up to isomorphism. The authors formally prove that the resulting graph is uniformly most-reliable \((n, n+2)\)-graph. Furthermore, inspired by a previous research on \(t\)-optimality in multipartite graphs authored by Cheng (1981), they conjecture that all uniformly most-reliable \((n, n+3)\)-graphs with more than \(6\) nodes are elementary subdivisions of \(K_{3,3}\). This conjecture is correct, and it was proved by Wang (1994).

For every even natural \(n\), Möbius graph \(M_n\) is constructed from the cycle \(C_{2n}\) adding \(n\) new edges joining every pair of opposite nodes. Note that \(M_2 = K_4\) and \(M_3 = K_{3,3}\), are Möbius graphs. More recently, Romero (2017) formally proved using iterative augmentation that \(M_4\), known as Wagner graph, is uniformly most-reliable as well. In this sense, Möbius graphs apparently generalize the particular result for \(K_{3,3}\) and \(K_4\), however \(M_5\) is not UMRG, since Petersen graph is UMRG; see Rela et al. (2018).

3.4 Finding Dense UMRG

So far, UMRGs are fully characterized when few edges are removed to the complete graph \(K_n\). In fact, Kelmans and Chelnokov (1974) formally proved that if an arbitrary matching is removed to \(K_n\), the result is a UMRG. Further research tries to characterize UMRGs when we must remove \(e\) edges meeting the inequality \(n/2 \leq e \leq n\). Kelmans (1996) found a characterization of those graphs having up to \(n\) edges removed from \(K_n\). Petingi et al. (1998) formally proved the \(t\)-optimality of the graphs suggested by Kelmans, for special cases where \(n = 3k\), being \(k\) some positive integer.

3.5 Reliability-Increasing Transformations

An alternative approach to prove the existence of UMRGs is to construct the graphs using reliability-increasing operations. This is the case of dense graphs, using the transformation proposed by Kelmans (1996), called swing surgery. In Canale et al. (2019) a reliability-improving graph transformation is proposed, where non-biconnected graphs can be always transformed into biconnected graphs:
The complete bipartite graph $K_{4,4}$ is Uniformly Most-Reliable

**Lemma 1** For every non-biconnected graph $G$ with $e \geq n$ edges, there exists a biconnected graph $G'$ such that $m_k(G') \leq m_k(G)$ for all $k \in \{0, \ldots, e\}$.

Figures 2 and 3 illustrate the idea of the proof, where the bridges are first included in cycles, and finally cut-points are avoided.

![Fig. 2: Step 1: avoiding the bridge $e = (vw)$.](image1)

![Fig. 3: Step 2: avoiding the cut-node $u$.](image2)

### 3.6 Recent Progress

In the last decade, the progress in this field of UMRG shows to be slow, and few novel results are available. Brown and Cox (2014) proved the inexistence of UMRGs for infinite pairs $(n, e)$. On one hand, they show that this non-existence can be generalized to the case of multigraphs. On the other hand, they formally prove that uniformly least-reliable graphs (ULRGs) always exist among the large family of multigraphs. Recently, Archer et al. (2019) discovered that UMRGs always exist if $n \geq 5$ is odd and $e$ is either $\binom{n}{2} - (n + 1)/2$ or $\binom{n}{2} - (n + 3)/2$, and a construction is also provided.

For practical reasons, the most recent works try to find almost-UMRGs instead. In Bourel et al. (2019), the authors develop a GRASP/VND heuristic in order to find highly reliable cubic graphs. They found new candidates of 3-regular UMRG, such as Yutsis, Heawood and Kantor-Möbius graphs. Canale et al. (2019) mathematically proved that Yutsis is in fact UMRG. Since Heawood and Mobius-Kantor meet the necessary criterion from Proposition 1 and also have largest girth, it is conjectured that these graphs also belong to the set of UMRGs. Rela et al. (2018) proved that Petersen graph is UMRG. Furthermore, the only non-trivial 4-regular UMRG known so far is the complement of a cycle with seven nodes $H = C_7$, also known in the literature as an odd-antihole (Rela et al., 2019). A complementary research is focused on UMRGs under node-failures, and several bipartite and multipartite graphs are UMRGs. The reader is invited to consult S. Yu (2010) and cites therein for further details.
4 The Bipartite Graph $K_{4,4}$ is UMRG

The main result of this work is Theorem 1, where it is formally proved that the bipartite complete graph $K_{4,4}$ is UMRG. Figure 4 shows three representations for $K_{4,4}$. The proof-strategy is presented in Subsection 4.3. Basic terminology is defined in Subsection 4.2. Regular graphs are studied in Subsection ??, while non-regular graphs and Theorem 1 are considered in ??.

![Fig. 4: Three representations of $K_{4,4}$. From left to right, as a bipartite graph, as a circulant one, and as a $(4,4)$ cage.](image)

4.1 Proof-Strategy

An immediate corollary from Expression (1) is that it suffices to prove that $m_k(K_{4,4}) \leq m_k(G)$ for all graph $G$ and all $k \in \{0, \ldots, 16\}$. Unless specified otherwise, our universe, $\Omega$, is the set of all connected $(8,16)$-graphs. In order to state the main ingredients of our proof, first let us recall some intrinsic properties from our candidate graph $K_{4,4}$:

1. $K_{4,4}$ is superconnected, hence it minimizes $m_k$ for all $k \leq 4$.
2. If $G \in \Omega$, clearly $m_k(G) = \binom{16}{k}$ for all $G$ and for all $k \geq 10$. Then, $m_k(K_{4,4}) = m_k(G)$ for all $G \in \Omega$ and all $k \geq 10$.
3. Cheng (1981) states that $K_{4,4}$ is $t$-optimal. Then, $m_9(K_{4,4}) \leq m_9(G)$ for all $G \in \Omega$.

From the previous properties and Lemma 1, it suffices to prove that $m_k(K_{4,4}) \leq m_k(G)$ for all $k \in \{5, 6, 7, 8\}$ and all biconnected graphs $G \in \Omega$. Since $G$ is biconnected, its minimum degree is $\delta(G) \in \{2, 3, 4\}$. Further, in the special case where $\delta(G) = 4$, necessarily $G$ is a 4-regular graph belonging to $G$. Our proof-strategy can be summarized in two steps:

1. **Step 1**: Prove that $m_k(K_{4,4}) \leq m_k(4,4)$ for all biconnected regular graphs $G \in \Omega$. This is formally established in Proposition 3 (Subsection 4.3).
2. **Step 2**: Prove that $m_k(K_{4,4}) \leq m_k(4,4)$ for the remaining biconnected regular graphs $G \in \Omega$ with $\delta(G) \in \{2, 3\}$ (Subsection 4.3). These results are proved separately in Lemmas 2 and 3 for $\delta(G) = 2$ and $\delta(G) = 3$ respectively.

The main characters of our proof is the power of inclusion-exclusion principle, increasing sequences related with binomial coefficients, and a subtle discussion of degree-sequences, whenever necessary. The comparison with regular graphs (Step 1) considers the fact that $K_{n,n}$ is the only triangle-free graph in the class $(n, n^2/4)$; see Mantel (1907) for details. Then, the maximum girth 4 is attained in $\Omega$ by $K_{4,4}$. Step 2 is more involved, and considers non-regular graphs.
4.2 Terminology

The following terminology will be useful for both steps of the proof. If \( A \subset V_G \), let \( [A] \) be the subgraph induced by \( A \) in \( G \) and \( \partial A = \{ uv \in E_G : u \in A, v \notin A \} \). Then,

\[
|\partial A| = \sum_{v \in A} d_v - 2|E[A]|.
\]

in particular if \( G \) is 4-regular, we have

\[
|\partial A| = 4|A| - 2|E[A]|.
\]

Given a set \( A \), let \( A^{(k)} \) denotes all the subsets of \( A \) with cardinality \( k \), i.e. \( A^{(k)} = \{ S \subset A : |S| = k \} \).

- Let \( V_i(G) \) be the set of nodes of degree \( i \), i.e., \( V_i = \{ v \in V : d_v = i \} \).
- Let \( E_v(G) \) be the set of edges incident to \( v \), i.e. \( E_v = \{ uv : uv \in E \} \).
- If \( e \) is an edge, let \( E_e(G) \) be the set of edges adjacent to \( e \), i.e., if \( e = uv \), then \( E_e = E_u \triangle E_v \).
- Let \( M^k(G) \) be the family of cutsets of the graph \( G \), of cardinality \( k \), i.e.

\[
M^k(G) = \{ S \in (EG)^{(k)} : G - S \text{ disconnected} \},
\]

therefore \( m_k(G) = |M^k(G)| \). When the graph \( G \) is clear from the context we will write \( V_i \) and \( M^k \) instead of \( V_i(G) \) and \( M^k(G) \), respectively.

- If \( H \subset G \) is a connected subgraph of \( G \), \( M^k_H(G) \) denotes the cutsets of cardinality \( k \) containing \( \partial H = \partial VH \) but no edge of \( H \), i.e. \( M^k_H(G) = \{ S \in M^k(G) : \partial H \subset S \subset EG \setminus EH \} \). If \( H \) has only one node \( v \) we will write \( M^k_v(G) \), i.e. \( M^k(G) = M^k_E(G) \). Similarly, if \( H \) has two nodes \( v \) and \( w \) we will write \( M^k_{vw}(G) \).

Since any disconnected graph of order \( n \) will have a connected component of at most \( \lfloor n/2 \rfloor \) nodes, then, if \( G \) is a \((8,16)\)-graph, and \( S \in M^k(G) \), then \( G \setminus S \) will have a connected component \( H \) of cardinality 4 at most, therefore, if \( C^i(G) \) is the set of connected subgraphs or order \( i \), then

\[
M^k = \bigcup_{i=1}^{4} \bigcup_{H \in C^i : |\partial H| \leq k} M^k_H.
\]
4.3 Step 1: Regular Graphs

In this section we will find lower bounds for \( m_k(G) \), for any regular \((8, 16)\)-graph \( G \). As a direct consequence of these bounds, we will find the corresponding coefficients for \( K_{4, 4} \) and prove they are optimal.

If \( G \) is 4-regular, \( H \in C^i(G) \) with \( i \leq 4 \) and \( \partial H \leq 8 \), there are precisely four mutually disjoint and exhaustive cases:

- \( H \in C^1 \) and \( |\partial H| = 4 \), for which there are exactly \( |V G| \) possible \( H \)'s.
- \( H \in C^2 \) and \( |\partial H| = 6 \), for which there are exactly \( |E G| \) possible \( H \)'s.
- \( H \in C^3 \) and \( |\partial H| \in \{6, 8\} \), depending on \( |H| \) being a 3-path or a triangle. In the former, there are \( |V G|^{\binom{4}{2}} = 6|V G| \) possibilities. In the latter, the number of possibilities is a function of the number of triangles, that depends on the 4-regular graph \( G \) under consideration. However, this number is strictly positive, unless \( G = K_{4, 4} \).
- \( H \in C^4 \) and \( |\partial H| \in \{4, 6, 8\} \). If \( |\partial H| = 8 \), then \( H \) is either a square or a triangle with a pending node. If \( |\partial H| \in \{4, 6\} \), then \( H \) is either a complete graph minus an edge or the complete graph, respectively.

Therefore, we have the following inclusions, where \( M_i^k \) and \( M_{i,j}^k \) are:

\[
M_5^5 \supseteq M_1^5, \quad M_6^6 \supseteq M_1^6 \cup M_2^6, \quad M_7^7 \supseteq M_1^7 \cup M_2^7, \\
M_8^8 \supseteq M_1^8 \cup M_2^8 \cup M_3^8 \cup M_4^8, \quad M_{i,j}^k = \cup_{H \in C^i \cap \partial H = j} M_H^k.
\]

which are equalities if the graph has girth 4 or greater.

**Proposition 2** If \( G \) is a regular \((8, 16)\)-graph, then

- \( m_5 \geq 8^{\binom{16-4}{1}} = 96 \).
- \( m_6 \geq 8^{\binom{16-4}{2}} + 16 = 544 \).
- \( m_7 \geq 8^{\binom{16-4}{3}} - 16 + 16^{\binom{16-7}{1}} = 1888 \).
- \( m_8 \geq 8^{\binom{16-4}{4}} - \left(8^{\binom{15}{2}} - 16 \right) - 16(16 - 7) + 16^{\binom{16-7}{1}} + 8^{\binom{15}{2}} + 8 \binom{16}{3} + c/2 \), where \( t \) is 1 if \( G \) has triangles and 0 otherwise, while \( c \) is the number of squares.

Moreover, the equalities hold if the graph has girth 4.

**Proof:** The inequality \( m_5 \) follows by (3) and the fact that \( M_1^5 = \{E_v \cup \{e\} : v \in V, e \in E \setminus E_v \} \) and \( u \neq v \Rightarrow M_u^5 \cap M_v^5 = \emptyset \).

The inequality \( m_6 \) follows by (3) and the facts that \( M_1^6 = \{E_v \cup \{e, f\} : v \in V, e, f \in E \setminus E_v \} \), \( M_2^6 = \{E_e : e \in E \} \) and the following intersections are empty: \( M_1^6 \cap M_2^6, M_1^6 \cap M_5^6 \) and \( M_1^6 \cap M_6^6 \), for any two different nodes \( u \) and \( v \) and different edges \( e \) and \( f \).
The complete bipartite graph $K_{4,4}$ is Uniformly Most-Reliable

The inequality $m_7$ follows by \[\text{(3)}\] and the facts that $M_1^7 = \{E_v \cup S : v \in V, S \in (E \setminus E_v)^3\}$, $M_2^7 = \{E_v \cup \{f\} : e \in E, f \notin E_v \cup \{e\}\}$, both $M_1^7 \cap M_2^7$ and $M_1^7 \cap M_2^7$ are empty for any two different edges $e$ and $f$ while $|M_1^7 \cap M_2^7| = 1$ if $uv \in E$ and 0 otherwise.

The inequality $m_8$ follows by \[\text{(3)}\], but it is more involved. Clearly,

\[
\begin{align*}
M_1^8 &= \{E_v \cup S : v \in V \land S \in (E \setminus E_v)^4\}, \\
M_2^8 &= \{E_v \cup \{f, g\} : e \in E \land f, g \in E \setminus E_v\}, \\
M_3^8 &= \{\partial\{u, v, w\} : uw, vw \in E \land uw \notin E\}, \\
M_4^8 &= \{\partial \cup \{e, f\} : |T| \leq 7\} & \text{or} & C_3 \land e, f \notin \partial T \land |\{e, f\} \cap E[T]| \leq 1, \\
M_5^8 &= \{\partial S : \|S\| \leq 3\}.
\end{align*}
\]

These sets are pairwise disjoint, so

$$|M^8| \geq |M_1^8| + |M_2^8| + |M_3^8| + |M_4^8| + |M_5^8|.$$ 

In the following, we bound term-by-term. Beginning with the last one, if $S$ is a square, then, in a regular $(8, 16)$-graph, the complement of the square has four edges as well. Thus, that complement can be a square or a triangle with a pending edge. However, if $G$ has girth 4 or greater, then $S'$ must be a square. So, for each pair of complement squares, we have only one set in $M_1^8$. If the graph $G$ has girth 3, then each set in $M_1^8$ corresponds to one or two squares. In any case, the cardinality is at least $c/2$.

For $M_3^8$, we can bound the cardinality by considering one triangle $T$ and for each edge $g \in T$ we pick the edges $e$ and $f$ among those edges that do not belong to $\partial T \cup \partial T \setminus \{g\}$. Since there are three ways to choose $g$, we have $3\binom{|\partial T| - |\partial T \setminus \{g\}|}{2} = 3\binom{|E| - |\partial T \cup \partial T \setminus \{g\}|}{2}$ ways to choose the edges $e$ and $f$.

The cardinality of $M_3^8$ corresponds to $|V G|$ ways to choose $v$ and $\binom{|H|}{2}$ ways to choose $u, w$ among the neighbours of $v$. Analogously, the cardinality of $M_2^8$ corresponds to $|E G|$ ways to choose $vw$ and $\binom{|E|}{2}$ ways to choose the edges $e, f$ not adjacent with $v$ or $w$.

Finally, for the cardinality of $M_1^8$ observe that $E_v \cup S = E_v \cup S'$ if and only if either $vw' \notin E$ with $E_v = S'$ and $E_v = S$ or $vw' \in E$ and $E_v \cup S = E_v \cup \{e\}$ with $e \in E \setminus E_v$. There are $|E| = \binom{V}{2} - 16$ ways to choose $vw'$ and $|E G|\binom{|E_v| - |E_v \cup \{e\}|}{1}$ ways to choose $vw' \in E$ and $E_v \cup S = E_v \cup \{e\}$.

**Corollary 2** The cutsets $m_k(K_{4,4})$ for $k \in \{5, 6, 7, 8\}$ are 96, 544, 1888 and 4446, respectively.

**Proof:** The coefficients follow directly from Proposition \[\text{(3)}\] and the computation of $t$ and $c$ for $K_{4,4}$. Since $K_{4,4}$ is bipartite, $t = 0$. In order to compute $c$, consider the bipartite representation of $K_{4,4}$ and notice that each square determines two nodes in one stable set and two nodes in the other stable set. Since there are $\binom{V}{2}$ ways to choose 2 nodes out of 4, there are $\binom{V}{2}^2$ ways to choose the cycles.

Recall that by Mantel theorem, $K_{4,4}$ is the only triangle-free graph in $\Omega$. Then:

**Proposition 3** $K_{4,4}$ is the most reliable graph among the regular $(8, 16)$-graphs.

**Proof:** Combining Mantel theorem and Proposition \[\text{(3)}\] $m_k(K_{4,4}) \leq m_k(G)$ for $k \in \{5, 6, 7\}$. If $k = 8$, since any other regular $(8, 16)$-graph has at least one triangle, we have, by Proposition \[\text{(3)}\] that $m_8(G) - m_8(K_{4,4}) \geq 3\binom{V}{2} + c(G) - 18 \geq 66$. 

\[\square\]
4.4 Step 2: Non-Regular Graphs

We will now prove the optimality of $K_{4,4}$ among non-regular $(8,16)$-graphs. First, let us introduce specific notation and bounds. For each vertex $u$ and edge $uv$, we will consider intensive use of bounds for the cardinality of the sets $M_u^k$ and $M_u^{k-1}$, recall that $M_u^k = M_{E_u}^k = \{ S \in M^k : E_{uv} \subset S \}$ and $M_u^{k-1} = M_{E_{uv}}^{k-1} = \{ S \in M^{k-1} : E_{uv} \subset S \}$. Note that both $E_v$ and $E_{uv}$ are cutsets, moreover $E_v \subset M^d_v$ and $E_{uv} \subset M^{d_u+d_v}$.

It is straight to see that for each $v \in V G$ and $uv \in E G$, we have

$$|M_v^k| = \left( |EG| - d_v \right), \quad \text{and} \quad |M_{uv}^k| = \left( |EG| - d_u - d_v + 1 \right).$$

Let us consider the function $g_k(i) = \left( \frac{|EG|}{k-i} \right)$, which is decreasing on $i$, i.e. $g_k(i) \leq g_k(i-1)$, as can easily be verified. Table II shows some values of $g_k(i)$.

| $k \setminus i$ | 1   | 2   | 3   | 4   | 5   | 6   | $m_k(K_{4,4})$ |
|--------------|-----|-----|-----|-----|-----|-----|----------------|
| 5            | 1365| 364 | 78  | 12  | 1   | 0   | 96            |
| 6            | 3003| 1001| 364 | 66  | 11  | 1   | 544           |
| 7            | 5005| 2002| 715 | 220 | 55  | 10  | 1888          |
| 8            | 6435| 3003| 1287| 495 | 165 | 45  | 4446          |

Tab. 1: Values of $g_k(i)$ and $m_k(K_{4,4})$

Let us find some bounds for the intersections of sets $M_u^k$:

$$|M_u^k \cap M_v^k| = \begin{cases} g_k(d_u + d_v) & uv \notin E, \\ g_k(d_u + d_v - 1) & uv \in E. \end{cases} \quad (6)$$

Thus, $g_k(d_u + d_v) \leq |M_u^k \cap M_v^k| \leq g_k(d_u + d_v - 1)$. Analogously,

$$\bigcap_{v \in A} M_v^k = g_k \left( \sum_{v \in A} d_v - |E[G]| \right). \quad (7)$$

From this equation, and the decreasing character of $g_k(i)$ we obtain two bounds for $|\bigcap_{v \in A} M_v^k|:

$$g_k \left( \sum_{v \in A} d_v \right) \leq |\bigcap_{v \in A} M_v^k| \leq g_k \left( \sum_{v \in A} d_v - \min \left( k, \left( \frac{|A|}{2} \right) \right) \right). \quad (8)$$

Finally, given a subset $A$ of nodes of $G$, since $M^k \supset \bigcup_{v \in A} M_v^k$, then, by Inclusion-Exclusion Principle we get:

$$m_k = |M^k| = \bigcup_{v \in V} M_v^k = \sum_{v \in V} |M_v^k| - \sum_{u,v \in A, u \neq v} |M_u^k \cap M_v^k| + \ldots \quad (9)$$

$$\geq \sum_{v \in A} g_k(d_v) - \sum_{uv \in A} g_k(d_u + d_v - 1) + \sum_{uv, w \in A} g_k(d_u + d_v + d_w) - \ldots \quad (10)$$
Remarks 1 Sharper bounds than |10| can be obtained with the following considerations:

1. For any fixed node $u$, the number of terms of the form $g_k(d_u + d_v - 1)$ is at most $d_u$. Thus, in the second term of |10| we can preserve $d_u$ terms of the form $g_k(d_u + d_v - 1)$ and replace the other $(|A| - 1 - d_u)$ by $g_k(d_u + d_v)$. 

2. If we sort the $\binom{|A|}{2}$ numbers $(d_u + d_v)$ in increasing order, we can replace the last term in the second sum $\binom{|A|}{2} - \min(k, \frac{1}{2} \sum_{v \in A} d_u)$ by $g_k(d_u + d_v)$.

In Lemmas |8| and |9| we consider the degree-sequence sorted increasingly, and select the first $h$ nodes. Let us call $V^hG$ to this set. The node-set $V^hG$ is not unique, but the proofs do not depend on the specific selection.

Lemma 2 The coefficients $m_5, \ldots, m_8$ are smaller in $K_{4,4}$ than in any (8,16)-graphs $G$ with $\delta(G) = 2$.

Proof: Let $G$ be an (8,16)-graph with $\delta(G) = 2$. By |10|, with $A = V^1G$, i.e. $A = \{u\}$ with $d_u = 2$, we have $m_k(G) \geq g_k(2) \geq m_k(K_{4,4})$ for $k \in \{5, 6, 7\}$, except for $k = 8$; see Table |1|.

It suffices to prove that $m_8(G) \geq \alpha = m_8(K_{4,4}) = 4446$. First, consider the case where $|V_2(G)| \geq 2$. By |10| with $A = V^2G$, i.e. $A = \{u, v\} \subset V_2(G)$, we have,

$$m_8(G) \geq 2g_8(2) - g_8(2 + 2 - 1) = 2 \times 3003 - 1287 > \alpha.$$ 

If $|V_2(G)| = 1$. Let us denote $g(x)$ for $g_8(x)$. We divide the proof into two cases:

1. If $|V_3| \geq 2$, by |8| and |9| with $A = V^3G$, we discuss according to the set $E[A]$ of edges of the subgraph induced by $A = \{u, v, w\}$ with $d_u = 2$:

$$m_8(G) \geq g(2) + 2g(3)$$

$$+ \begin{cases} 
-2g(2 + 3) - g(3 + 3) + g(2 + 3 + 3) & E[A] = \emptyset, \\
-2g(2 + 3) - g(3 + 3 + 1) + g(2 + 3 + 3 + 1) & E[A] = \{uv\}, \\
-g(2 + 3 - 1) - g(2 + 3) - g(3 + 3) + g(2 + 3 + 3 - 1) & E[A] = \{uw\}, \\
-g(2 + 3 - 1) - g(2 + 3) - g(3 + 3 - 1) - g(2 + 3 + 3 - 2) & E[A] = \{vw\}, \\
-2g(2 + 3 - 1) - g(3 + 3) + g(2 + 3 + 3 - 2) & E[A] = \{uw, vw\}, \\
-2g(2 + 3 - 1) + g(2 + 3 + 3 - 3) & E[A] = \{uw, vw, vw\}, \\
\end{cases}$$

which is greater than $4587 > \alpha$ in the six cases.

2. If $|V_3| \leq 1$, consider all possible degree sequences: $(2, 3, 4, 4, 4, 4, 4, 7), (2, 3, 4, 4, 4, 4, 4, 5, 6), (2, 3, 4, 4, 4, 5, 5, 5), (2, 4, 4, 4, 4, 4, 4, 6), (2, 4, 4, 4, 4, 4, 5, 6)$. For the first three sequences of the form $(2, 3, 4, 4, 4, \ldots)$, consider |10| with $A = V^5G$:

$$m_8(G) \geq g(2) + g(3) + 3g(4) - g(2 + 3 - 1) - 3g(2 + 4 - 1) - 3g(3 + 4 - 1) - 3g(4 + 4 - 1) + 3g(2 + 3 + 4) + 3g(3 + 4 + 4) + g(4 + 4 + 4)$$

$$- 3g(2 + 3 + 4 + 4 - C_2^4) - g(3 + 4 + 4 + 4 - C_2^4) - g(2 + 4 + 4 + 4 - C_2^4) + g(2 + 3 + 4 + 4 + 4) \geq 4595 > \alpha.$$
The terms in the second and last line are null, since the arguments of \( g \) are strictly greater than 8.

The last two sequences have the form \((2, 4, 4, 4, \ldots)\). Then, using \([10]\) with \( A = V^5G \) and Remark \([11]\):

\[
m_8(G) \geq g(2) + 4g(4) - 2g(2+4-1) - 2g(2+4) - 10g(4+4-1) - 4g(2+3 \times 4 - \binom{4}{2}) = 4505 > \alpha,
\]

where the null terms like \( g(2 + 3 + 4) \) or \(-4g(4 \times 4 - \binom{4}{2})\) were discarded.

\[\square\]

**Lemma 3** The coefficients \( m_5 \) to \( m_8 \) are smaller in \( K_{4,4} \) than in any \((8, 16)-\)graphs \( G \) with \( \delta(G) = 3 \).

**Proof:** Let \( G \) be an \((8, 16)-\)graph with \( \delta(G) = 3 \). First, consider \( k \in \{5, 6, 7\} \). We proceed as in the previous lemma, discussing according to the number of degree-3 nodes \( |V_3| \):

1. If \( |V_3| \geq 3 \), then by \([11]\) with \( A = V^3G \): \( m_k(G) \geq 3g_k(3) - 3g_k(3 + 3 - 1) \in \{231, 825, 1980\} \) for \( k \in \{5, 6, 7\} \) respectively, all greater than the corresponding \( m_k(K_{4,4}) \).

2. If \( |V_3| = 2 \), then \( G \) should have at least four nodes of degree 4, i.e., \( |V_4| \geq 4 \). Then, using \([10]\) with \( A = V^5G \), we have \( m_k(G) \geq 2g_k(3) + 4g_k(4) - g_k(3 + 3 - 1) - 6g_k(3 + 4 - 1) - 3g_k(4 + 4 - 1) \), so \( m_k(G) \in \{191, 753, 1972\} \) for \( k \in \{5, 6, 7\} \) respectively, all greater than the corresponding \( m_k(K_{4,4}) \). Again, we discarded null terms like \(-g_k(3 + 3 + 4 + 4 - \binom{4}{2})\).

3. If \( |V_3| = 1 \) the degree-sequence must be \((3, 4, 4, 4, 4, 4, 4, 5)\). We use \([10]\) with \( A = V^7G \), to obtain

\[
m_k(G) \geq g_k(3) + 6g_k(4) - g_k(3 + 4 - 1) - \binom{4}{2}g_k(4 + 4 - 1), \text{ so } m_k(G) \in \{150, 676, 1960\} \text{ for } k \in \{5, 6, 7\} \text{ respectively, all greater than } m_k(K_{4,4}). \text{ The other terms in } [10] \text{ are null, like } g_k(3 + 4 + 4 + 4 - \binom{4}{2}) = g_k(9) \text{ or } g_k(3 + 4 + 4 + 4 + 4 - \binom{4}{2}) = g_k(8). \]

Finally, if \( k = 8 \) we write \( g(x) \) instead of \( g_8(x) \) and \( \alpha = m_8(K_{4,4}) = 4446 \).

Let us discuss according to \( |V_3| \):

1. If \( |V_3| \geq 5 \), then by \([11]\) and \( A = V^5G \), we have:

\[
m_8(G) \geq 5g(3) - 10g(3 + 3 - 1) - 5g(3 + 3 + 3 + 6) = 4560 > \alpha.
\]

2. If \( |V_3| = 4 \) and \( |V_4| \geq 1 \) then, by \([10]\) and \( A = V^5G \), we have:

\[
m_8(G) \geq 4g(3) + g(4) - 6g(3 + 3 - 1) - 4g(3 + 4 - 1) - g(3 + 3 + 3 + 6) - 4g(3 + 3 + 4 + 6) = 4392 < \alpha.
\]

But \( |E[V_3]| \geq 4 \), since \( G \) is biconnected, and:

\[
m_8(G) \geq 4g(3) + g(4) - 4g(3 + 3 - 1) - 2g(3 + 3) - 4g(3 + 4 - 1) - g(3 + 3 + 3 + 4) - 4g(3 + 3 + 4 + 6) = 4676 > \alpha.
\]
3. If $|V_3| = 4$ and $|V_4| = 0$ then, the only possible sequence is $(3, 3, 3, 5, 5, 5, 5)$, so by (10) and $A = V^8G$, and the previous observation about the biconnectivity of $G$ we have:

$$m_8(G) \geq 4g(3) + g(5) - 4g(3 + 3 - 1) - 2g(3 + 3) - 4g(3 + 5 - 1)$$

$$- g(3 + 3 + 3 + 3 - 4) - 4g(3 + 3 + 3 + 5 - 6) = 4514 > \alpha.$$ 

4. If $|V_3| = 3$ and $|V_4| \geq 3$, by (9) with $A = V^8G$ and (1), using that $h = |E[V_3]| \in \{0, 1, 2, 3\}$:

$$m_8(G) \geq 3g(3) + 3g(4) - hg(3 + 3 - 1) - (3 - h)g(3 + 3) - (9 - 2h)g(3 + 4 - 1)$$

$$- 2hg(3 + 4) - 3g(4 + 4 - 1) - 3g(3 + 3 + 3 + 4 - 6) - 9g(3 + 3 + 4 + 4 - 6)$$

$$- g(3 + 3 + 3 + 4 + 4 + 4 - 8) \geq 4599 > \alpha.$$ 

The remaining degree-sequences are $(3, 3, 3, 4, 4, 5, 5, 5)$, $(3, 3, 4, 4, 4, 4, 6)$, $(3, 3, 4, 4, 4, 4, 5, 5)$, $(3, 4, 4, 4, 4, 4, 5, 5)$. Consider the cases separately:

- If the degree-sequence is $(3, 3, 3, 4, 5, 5, 5)$, we apply (9) and (8), with $A = V$ so

$$m_8(G) \geq 3g(3) + 2g(4) + 3g(5) - 3g(3 + 3 - 1) - g(4 + 4 - 1) - 3g(5 + 5 - 1)$$

$$- 6g(3 + 4 - 1) - 9g(3 + 5 - 1) - 6g(4 + 5 - 1) - 2g(3 + 3 + 3 + 4 - 6)$$

$$- 3g(3 + 3 + 3 + 5 - 6) - 3g(3 + 3 + 4 + 4 - 6)$$

$$= 4461 > \alpha.$$ 

For $S \subset A$ with $|S| = 6$, the terms $g(\sum_{v \in S} d_u - 8)$ are null since the greatest is $g(3 + 3 + 3 + 4 + 4 + 5 - 8)$, which is already null.

- If the degree-sequences is $(3, 3, 4, 4, 4, 4, 6)$ we apply (9) and (8), with $A = V$ and consider Remark [8]. We distinguish whether $uv \in EG$ or not, with $V_3 = \{u, v\}$, i.e. whether $h = |E[V_3]| \in \{0, 1\}$ respectively. Indeed, if $uv \in EG$, then the maximum number of edges between $V_3$ and $V_4$ is 4, otherwise this number is at most 6. If $|V_4| = 5$, then $|E[V_4]| \leq 8$ in order to $G$ to be biconnected. Then:

$$m_8(G) \geq 2g(3) + 5g(4) + g(6) - g(3 + 3 - h) - (6 - 2h)g(3 + 4 - 1) - (4 + 2h)g(3 + 4)$$

$$- 8g(4 + 4 - 1) - 2g(3 + 6 - 1) - \left(\frac{4}{2}\right)g(3 + 3 + 4 + 4 - 6);$$

which is 4663 if $h = 0$ and 4615 if $h = 1$, both greater than $\alpha$.

- Finally, if the degree-sequences are $(3, 3, 4, 4, 4, 4, 5, 5)$ or $(3, 4, 4, 4, 4, 4, 4, 5)$, the study is more involved. We will provide lower-bounds for the cardinality of $M' = \bigcup_{v \in VG} M^8_v \cup \bigcup_{v \in EG} M^8_v$, applying Remark [8] to find a bound on $|\bigcup_{v \in VG} M^8_v|$, and detailed analysis for the remaining terms.
By inclusion-exclusion principle:

\[ |M'| = \bigcup_{v \in V_G} M_v^8 + \bigcup_{e \in EG} M_v^8 - \bigcup_{e \in EG} M_v^8 \cap \bigcup_{e \in EG} M_e^8 \]

\[ = \bigcup_{v \in V_G} M_v^8 + \sum_{e \in EG} |M_e^8| - \sum_{e \in EG} \sum_{v \in V_G} |M_v^8 \cap M_e^8|, \]

\[ \geq \bigcup_{v \in V_G} M_v^8 + \sum_{e \in EG} |M_e^8| - 18, \]

where the second equality holds since \( M_e^8 \cap M_v^8 = \emptyset \) for all \( e, f \in E \), while the last inequality is due to the fact that \( M_v^8 \cap M_e^8 \) are empty sets unless \( d_v = 3, e = uv \) with \( d_u = d_w = 4 \) and \( v \) is adjacent to \( u \) or \( w \), which are only 9 cases for \( |V_3| = 1 \) and at most 18 cases for \( |V_3| = 2 \).

By (10), we have

\[ \bigcup_{v \in V_G} M_v^8 \geq 2g(3) + 4g(4) + 2g(5) - f(3 + 3 - 1) - 6g(4 + 4 - 1) - 8g(3 + 4 - 1) \]

\[ - 4g(3 + 5 - 1) - 8g(4 + 5 - 1) - \binom{4}{2}g(3 + 3 + 4 + 4 - 6) = 4255, \]

and

\[ \bigcup_{v \in V_G} M_v^8 \geq g(3) + 6g(4) + g(5) - 6g(3 + 4 - 1) - g(3 + 5 - 1) - 6g(4 + 5 - 1) = 4137, \]

for the second sequence. Now, let us consider the sum \( S = \sum_{e \in EG} |M_e^8| \). If \( e = uv \), then \( |M_e^8| = \binom{16 - d_u - d_v + 1}{d_u - d_v - d_v + 1} \), which has the four possible values shown in Table 2, since there are three possible values of the degrees. Thus, a possible lower bound for that sum \( S \) is the minimum of \( 168a + 36b + 36c + 8d \), subject to \( a + b = 5, 6, d = 9, 10, \) and \( a + b + c + d = 16 \) for the first sequence and to \( a + b = 3, d \leq 5, \) and \( a + b + c + d = 16 \) for the second sequence. These minima are attained by \( a = 0, b = 5, d = 0, c = 11 \) and by \( a = 0, b = 3, d = 5, c = 8 \) respectively, thus \( S \geq 268 \) and \( 436 \) respectively. All the bounds together give us: \( m_8(G) \geq 4255 + 268 - 18 = 4505 \) and, \( m_8(G) \geq 4162 + 436 - 18 = 4580 \), both greater than \( \alpha \).

Theorem 1 \( K_{4,4} \) is uniformly most-reliable.

Proof: \( K_{4,4} \) is 4-regular and superconnected, then, it minimizes \( m_4 \). Clearly, \( m_i = 0 \) for \( i \in \{0, 1, 2, 3\} \), and \( m_i = \binom{16}{i} \) for all \( (8, 16) \)-graphs, when \( i \geq 10 \). By Lemmas 3, Lemma 4, and Corollary 5, coefficients \( m_5 \) to \( m_8 \) are minimized. Cheng [1981] asserts that \( K_{4,4} \) is \( t \)-optimal, so coefficient \( m_9 \) is also minimal. Then, \( K_{4,4} \) minimizes all the coefficients of the Equation (10).
The complete bipartite graph $K_{4,4}$ is Uniformly Most-Reliable

| $d_u$ | $d_v$ | $|M_{uv}^8|$ |
|-------|-------|------------|
| 3     | 4     | 168        |
| 3     | 5     | 36         |
| 4     | 4     | 36         |
| 4     | 5     | 8          |

Tab. 2: Values of $|M_{uv}^8|$ according with $d_u$ and $d_v$

5 Conclusions and Trends for Future Work

Uniformly most-reliable graphs (UMRG) represent a synthesis in network reliability analysis. Finding them is a hard task not well understood. An exhaustive comparison is computationally prohibitive for most cases. Prior works in the field try to globally minimize the cutsets. This methodology provides uniformly most-reliable $(n, n + i)$ graphs for $i \leq 6$. In this paper, we formally proved that the $K_{4,4}$ is UMRG. To the best of our knowledge, this is the second non-trivial 4-regular UMRG in the literature (since $K_5$ is 4-regular) after $H = \overline{C_7}$ proved in Rela et al. (2019).

There are several trends for future work. The problem of $t$-optimality is still not well understood. A more general result of Theorem 1 for $K_{n,n}$ is also challenging. Our methodology could be used for the discovery of other UMRGs. As future work, we want to determine the optimality for $K_{n,n}$ and the maximum girth conjecture posed by Boesch (1986) in his seminal work.

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