Perturbation theory for bright spinor Bose–Einstein condensate solitons

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We develop a perturbation theory for bright solitons of the $F = 1$ integrable spinor Bose–Einstein condensate (BEC) model. The formalism is based on using the Riemann–Hilbert problem and provides the means to analytically calculate evolution of the soliton parameters. Both rank-one and rank-two soliton solutions of the model are obtained. We prove equivalence of the rank-one soliton and the ferromagnetic rank-two soliton. Taking into account a splitting of a perturbed polar rank-two soliton into two ferromagnetic solitons, it is sufficient to elaborate a perturbation theory for the rank-one solitons only. Treating a small deviation from the integrability condition as a perturbation, we describe the spinor BEC soliton dynamics in the adiabatic approximation. It is shown that the soliton is quite robust against such a perturbation and preserves its velocity, amplitude, and population of different spin components, only the soliton frequency acquires a small shift. Results of numerical simulations agree well with the analytical predictions, demonstrating only slight soliton profile deformation.

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I. INTRODUCTION

Bright and dark solitons in quasi-one-dimensional Bose–Einstein condensates (BECs), observed experimentally 1, 2, 3, 4, are expected to be important for various applications in atom optics 5, including atom interferometry, atom lasers, and coherent atom transport. Recent experimental and theoretical advances in BEC soliton dynamics are reviewed in Refs. 6, 7, 8.

Spinor BEC of alkali atoms 9, 10 with a purely optical confinement, along with the two-component condensate 11, 12, 13, represents an example of the condensate with internal degrees of freedom which endow the solitons with vectorial properties. Modulational instability in the spinor BEC model was investigated in Ref. 14, and some exact solutions and their stability were studied in Ref. 15. Vector gap solitons and self-trapped waves were identified in the spinor BEC model loaded into one-dimensional optical lattice potential 16. Recently bright–dark soliton complexes in this model have been found 17 by reducing it to the completely integrable Yajima–Oikawa system 18.

Wadati and co-workers found 19 that the three-component nonlinear equations describing the BEC with the hyperfine spin $F = 1$ admit the reduction to another integrable model – the $2 \times 2$ matrix nonlinear Schrödinger (NLS) equation, after imposing a constraint on the condensate parameters. Both bright 20 and dark 21 solitons possessing properties of true solitons of integrable equations have been found. The formalism of the inverse scattering transform for the matrix NLS equation under non-vanishing boundary conditions was developed in Refs. 22 and extended in Ref. 23 to describe bright spinor BEC soliton dynamics on a finite background. The full-time description of the modulational instability development in the integrable spinor BEC model was given both numerically 15 and analytically 20, 26.

Integrable models provide a very useful proving ground for testing new analytical and numerical approaches to study such a complicated system as the spinor BEC. At the same time, the integrability conditions impose specific restrictions on the parameters of the model which can conflict with actual experimental settings, despite the fact that the effective interaction between atoms in BEC can be tuned, to some extent, by the optically induced Feshbach resonance 24, 25. Besides, in experiment it is impossible to exactly hold the conditions between parameters which assure integrability of the model. Therefore, sufficiently general analytical results concerning the full (nonintegrable) model with realistic parameters would be of importance.

As a step in this direction, in the present paper we develop a perturbation theory for the integrable spinor BEC model. Evidently, small disturbance of the integrability condition can be considered as a perturbation of the integrable model. Our formalism is based on the Riemann–Hilbert (RH) problem associated with the spinor BEC model. The main advantage of the proposed method is its algebraic nature, as distinct from the method using its algebraic nature, as distinct from the method using the Gel’fand–Levitan integral equations 26. The application of the RH problem for treating perturbed soliton dynamics goes back to Refs. 27, 28. The modern version of the perturbation theory in terms of the RH problem has been developed in a series of papers 29, 30, 31, 32, 33, 34, with its most general formulation in Ref. 35. Another version of the soliton perturbation theory (the direct perturbation theory) has been developed on the basis of expanding perturbed solutions into squared eigenfunctions of the linearized soliton equations 36, 37.
As was shown by Wadati and co-workers \[19, 22\], bright solitons in the integrable spinor BEC model can exist in two spin states – ferromagnetic (non-zero total spin) and polar (zero total spin). Energy of the polar soliton is greater than that of the ferromagnetic soliton. Moreover, the polar soliton demonstrates a two-lumped profile in a wide range of its parameters. Our numerical simulations revealed that the polar soliton is unstable under the action of a perturbation of a rather general form and splits into two ferromagnetic solitons. This fact is crucial for the development of a perturbation theory for spinor BEC solitons.

The paper is organized as follows. After formulating the model in Sec. II, we introduce in Sec. III analytic solutions of the associated spectral problem, in order to formulate in Sec. IV the RH problem. Solving this problem, we derive evolution equations for the soliton parameters which exactly account for the perturbation and serve as the generating equations for iterations. Section VII contains a description of the soliton dynamics in the adiabatic approximation of the perturbation theory. We show analytically that a ferromagnetic soliton is quite robust against a small disturbance of the integrability condition, the only manifestation of the perturbation action is a minor shift of the soliton frequency. Numerical simulations of the perturbed spinor BEC equations are in close agreement with the analytical predictions revealing only a small soliton shape distortion and little perturbation-induced radiation. Section VIII concludes the paper.

II. MODEL

We consider an effective one-dimensional BEC trapped in a pencil-shaped region elongated in the $x$ direction and tightly confined in the transversal directions. The assembly of atoms in the hyperfine spin $F = 1$ state is described by a vector order parameter $\mathbf{\Phi}(x,t) = (\Phi_+(x,t), \Phi_0(x,t), \Phi_-(x,t))^T$, where its components correspond to three values of the spin projection $m_F = 1, 0, -1$. The functions $\Phi_\pm$ and $\Phi_0$ obey a system of coupled Gross–Pitaevskii equations \[22, 38\]

$$i\hbar \partial_t \Phi_\pm = -\frac{\hbar^2}{2m} \partial_x^2 \Phi_\pm + (c_0 + c_2)(|\Phi_\pm|^2 + |\Phi_0|^2)\Phi_\pm + (c_0 - c_2)(|\Phi_\pm|^2 - |\Phi_-|^2)\Phi_\mp,$$

$$i\hbar \partial_t \Phi_0 = -\frac{\hbar^2}{2m} \partial_x^2 \Phi_0 + (c_0 + c_2)(|\Phi_\pm|^2 + |\Phi_-|^2)\Phi_0 + c_0|\Phi_0|^2\Phi_0 + 2c_2\Phi_+\Phi_-\Phi_0^*,$$

where the constant parameters $c_0 = (g_0 + 2g_2)/3$ and $c_2 = (g_2 - g_0)/3$ control the spin-independent and spin-dependent interaction, respectively. The coupling constant $g_f (f = 0, 2)$ is given in terms of the s-wave scattering length $a_f$ in the channel with the total hyperfine spin $f$,

$$g_f = \frac{4\hbar^2 a_f}{ma_f^2} \left( 1 - \frac{a_f}{a_\perp} \right)^{-1}.$$

Here $a_\perp$ is the size of the transverse ground state, $m$ is the atom mass, and $C = -\zeta(1/2) \approx 1.46$.

It was noted in \[19\] that Eqs. (2.1) are reduced to an integrable system under the constraint

$$c_0 = c_2 \equiv -c < 0. \tag{2.2}$$

The negative $c_2$ means that we consider the ferromagnetic ground state of the spinor BEC with attractive interactions. The condition \[22\], being written in terms of $g_f$ as $2g_0 = -g_2 > 0$, imposes a constraint on the scattering lengths: $a_\perp = 3Ca_0a_2/(2a_0 + a_2)$. Redefining the function $\Phi$ as $\Phi \to (\phi_+, \sqrt{2}\phi_0, \phi_-)^T$, normalizing the coordinates as $t \to (e/h)t$ and $x \to (2mc/h)x$, and accounting for the constraint \[2.2\], we obtain a reduced system of equations in a dimensionless form:

$$i\partial_t \phi_\pm + \partial_x^2 \phi_\pm + 2(|\phi_\pm|^2 + 2|\phi_0|^2) \phi_\pm + 2\phi_+^2\phi_0^2 = 0, \tag{2.3}$$

$$i\partial_t \phi_0 + \partial_x^2 \phi_0 + 2(|\phi_+|^2 + |\phi_0|^2 + |\phi_-|^2) \phi_0 + 2\phi_+\phi_0\phi_- = 0.$$

After arranging the components $\phi_\pm$ and $\phi_0$ into a $2 \times 2$ matrix $Q$,

$$Q = \begin{pmatrix} \phi_+ & \phi_0 \\ \phi_0 & \phi_- \end{pmatrix}, \tag{2.4}$$

we transform Eqs. (2.3) to the integrable matrix NLS equation

$$i\partial_t Q + \partial_x^2 Q + 2QQ^1Q = 0. \tag{2.5}$$

The matrix NLS equation \[2.5\] appears as a compatibility condition of the system of linear equations \[2.7\]

$$\partial_x \psi = ik[A, \psi] + \hat{Q}\psi, \tag{2.6}$$

$$\partial_t \psi = 2ik[A, \psi] + V\psi. \tag{2.7}$$
where $\Lambda = \text{diag}(-1, -1, 1, 1)$,
\[
\dot{Q} = \begin{pmatrix} 0 & Q \\ -Q^\dagger & 0 \end{pmatrix}, V = 2k\dot{Q} + i \begin{pmatrix} QQ^\dagger & Q_x \\ Q_x^\dagger & -Q^\dagger Q \end{pmatrix},
\]
and $k$ is a spectral parameter. Equation (2.6) (the spectral problem) enables us to determine initial spectral data from the known potential $\dot{Q}_0$, while Eq. (2.7) governs the temporal evolution of the spectral data. A new solution of Eq. (2.6) (and hence of the BEC equations (2.3)) is obtained as a result of the reconstruction of the potential $\dot{Q}$ from the time-dependent spectral data.

### III. JOST AND ANALYTIC SOLUTIONS

To determine the spectral data, we introduce matrix Jost solutions $J_{\pm}(x, k)$ to the spectral problem (2.6) by means of the asymptotes $J_{\pm} \to \mathbb{I}$ as $x \to \pm \infty$. Since $\text{tr} \Lambda = 0$, we have $\det J_{\pm} = 1$ for all $t$. Being solutions of the first-order equation (2.6), the Jost functions are not independent but are interconnected by the scattering matrix $S$:

\[
J_- = J_+ E S_+ E^{-1}, \quad E = \exp(ik\Lambda x), \quad \det S = 1.
\]

Besides, the Jost solutions and the scattering matrix obey the involution property. Indeed, since the potential $\dot{Q}$ is anti-Hermitian, we obtain

\[
J_+^\dagger(k^*) = J_-^{-1}(k). \tag{3.2}
\]

Similarly for the scattering matrix:

\[
S^\dagger(k) = S^{-1}(k). \tag{3.3}
\]

Note that the scattering matrix is defined for real $k$.

For the subsequent analysis, analytic properties of the Jost solutions are of primary importance. Let us represent the matrix Jost solution $J$ as a collection of columns: $J = (J_1^[, 0], J_2^[, 0], J_3^[, 0], J_4^[, 0])$, and consider the first column. Rewriting the spectral equation (2.6) with the corresponding boundary conditions in the form of the Volterra integral equations, we obtain a closed system of equations for entries of the first column:

\[
\begin{align*}
J_{-11} &= 1 + \int_{-\infty}^{x} dx' (\phi_+ J_{-31} + \phi_0 J_{-41})(x'), \\
J_{-21} &= \int_{-\infty}^{x} dx' (\phi_0 J_{-31} + \phi_- J_{-41})(x'), \\
J_{-31} &= -\int_{-\infty}^{x} dx' (\phi_+^* J_{-11} + \phi_0^* J_{-21})(x')e^{2ik(x-x')}, \\
J_{-41} &= -\int_{-\infty}^{x} dx' (\phi_0^* J_{-11} + \phi_-^* J_{-21})(x')e^{2ik(x-x')}.
\end{align*}
\]

The last two integrands point out that the column $J_{-31}^[, 0]$ is analytic in the upper half-plane $\mathbb{C}_+$, where $\text{Im} k > 0$, and continuous on the real axis $\text{Im} k = 0$. This can be proved in the same way as for the scalar NLS equation, under the condition of sufficiently fast decrease of the potential $\dot{Q}$ at infinity. Similarly we obtain that the column $J_{-21}^[, 0]$ is analytic in $\mathbb{C}_+$ as well, while the two other columns $J_{-31}^[, 0]$ and $J_{-41}^[, 0]$ are analytic in the lower half-plane $\mathbb{C}_-$ and continuous on the real axis $\text{Im} k = 0$. As regards the matrix solution $J_+$, its first and second columns $J_{11}^[, 0]$ and $J_{21}^[, 0]$ are analytic in $\mathbb{C}_-$, while the third and forth ones $J_{31}^[, 0]$ and $J_{41}^[, 0]$ are analytic in $\mathbb{C}_+$. Therefore, the matrix function

\[
\psi_+ = (J_{11}^[, 0], J_{21}^[, 0], J_{31}^[, 0], J_{41}^[, 0]) \tag{3.4}
\]

solves the spectral equation (2.6) and is analytic as a whole in $\mathbb{C}_+$.

It is not difficult to see from Eqs. (3.1) and (3.4) that the analytic solution $\psi_+$ can be expressed in terms of the Jost functions and some entries of the scattering matrix:

\[
\psi_+ = J_+ ES_+ E^{-1} = J_- ES_- E^{-1}, \tag{3.5}
\]

where

\[
S_+(k) = \begin{pmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{21} & s_{22} & 0 & 0 \\ s_{31} & s_{32} & 1 & 0 \\ s_{41} & s_{42} & 0 & 1 \end{pmatrix}, \quad S_-(k) = \begin{pmatrix} 1 & 0 & s_{31}^* & s_{41}^* \\ 0 & 1 & s_{32} & s_{42} \\ 0 & 0 & s_{33} & s_{43} \\ 0 & 0 & s_{34} & s_{44} \end{pmatrix}. \tag{3.6}
\]

In writing the expression for $S_-$ we use the involution (3.3). These upper and lower block-triangular matrices $S_\pm$ factorize the scattering matrix (3.6): $S S_\pm = S_\pm$. Besides, it follows from Eq. (3.5) and $\det J_\pm = 1$ that

\[
\det \psi_+ = m_+^{(2)} = m_-^{(2)*}, \tag{3.7}
\]

where $m_+^{(2)}$ ($m_-^{(2)}$) is the second-order principal upper (lower) minor of the scattering matrix.

To obtain the analytic counterpart of $\psi_+$ in $\mathbb{C}_-$, we consider the adjoint spectral equation

\[
\partial_x K_\pm = ik[\Lambda, K_\pm] - K_\pm \ddot{Q} \tag{3.8}
\]

with the asymptotic conditions $K_\pm \to \mathbb{I}$ as $x \to \pm \infty$. The inverse matrix $K_-^{-1}$ can serve as a solution of the adjoint equation (3.8). Now we write a closed system of integral equations for rows of the matrices $K_\pm$. For example, the first row $K_\pm^{-1}$ obeys the equations

\[
\begin{align*}
K_{-11} &= 1 + \int_{-\infty}^{x} dx' (\phi_+^* K_{-13} + \phi_0^* K_{-14})(x'), \\
K_{-12} &= \int_{-\infty}^{x} dx' (\phi_0^* K_{-13} + \phi_-^* K_{-14})(x'), \\
K_{-13} &= -\int_{-\infty}^{x} dx' (\phi_+ K_{-11} + \phi_0 K_{-12})(x')e^{-2ik(x-x')}, \\
K_{-14} &= -\int_{-\infty}^{x} dx' (\phi_0 K_{-11} + \phi_- K_{-12})(x')e^{-2ik(x-x')}.
\end{align*}
\]
It is seen that the row $K_{-[1]}$ is analytic in $\mathbb{C}_-$. Similarly, the second row $K_{-[2]}$ is analytic in $\mathbb{C}_-$, too, and the rows $K_{-[3]}$ and $K_{-[4]}$ are analytic in $\mathbb{C}_+$. For the matrix solution $K_+$ we find that the rows $K_{+[1]}$ and $K_{+[2]}$ are analytic in $\mathbb{C}_+$, while $K_{+[3]}$ and $K_{+[4]}$ are analytic in $\mathbb{C}_-$. Therefore, the matrix function

$$\psi^{-1}_- = (K_{-[1]}, K_{-[2]}, K_{-[3]}, K_{-[4]})^T$$

solves the adjoint equation (3.8) and is analytic as a whole in $\mathbb{C}_-$. Similar to $\psi_+$, the function $\psi^{-1}_-$ is expressed in terms of the Jost solutions and the scattering matrix:

$$\psi^{-1}_- = ET_+E^{-1}J_{+}^{-1} = ET_-E^{-1}J_{-}^{-1},$$

where the matrices $T_{\pm}$,

$$T_+ = \begin{pmatrix} s_{11} & s_{21} & s_{31} & s_{41} \\ s_{12} & s_{22} & s_{32} & s_{42} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_- = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{pmatrix},$$

provide one more factorization of the scattering matrix: $T_- = T_+S$. As in Eq. (3.7), we can write

$$\det \psi^{-1}_- = m^{(2)*}_+ = m^{(2)}_-.$$

Note that the analytic solutions satisfy the involution property as well:

$$\psi^+_+(k) = \psi^+_{-}(k^*).$$

This property can be taken as a definition of the analytic function $\psi^-_-$ from the known analytic function $\psi_+$.

**IV. THE RIEMANN-HILBERT PROBLEM**

Hence, we constructed two matrix functions $\psi_+$ and $\psi^-_-$ which are analytic in complementary domains of the complex plane and conjugate on the real line. Indeed, it follows from Eqs. (3.5) and (3.10) that $\psi_{\pm}$ obey the relation

$$\psi^{-1}_-(k)\psi_+(k) = EG(k)E^{-1}, \quad \text{Im}k = 0,$$

where

$$G = T_+S_+ = T_-S_- = \begin{pmatrix} 1 & 0 & s_{31}^* & s_{41}^* \\ 0 & 1 & s_{32}^* & s_{42}^* \\ s_{31} & s_{32} & 1 & 0 \\ s_{41} & s_{42} & 0 & 1 \end{pmatrix}.$$ (4.2)

Equation (4.1) determines a matrix Riemann-Hilbert problem, i.e. a problem of the analytic factorization of a nondegenerate matrix $G$ in (4.2), given on the real line, into a product of two matrices which are analytic in complementary domains $\mathbb{C}_\pm$. The RH problem (4.1) needs a normalization condition, which is usually taken as

$$\psi_{\pm}(x,k) \rightarrow 1 \quad \text{at} \quad |k| \rightarrow \infty.$$ (4.3)

The analytic matrix functions $\psi_{\pm}$ can be treated as a result of a nonlinear mapping between the potential $Q(x)$ and a set of the spectral data which uniquely characterizes a solution of the RH problem (4.1) and (4.3). Conversely, the potential can be reconstructed from an asymptotic expansion of $\psi_{\pm}(x,k)$ for large $k$. Indeed, writing $\psi_{\pm}$ as

$$\psi_+(x,k) = 1 + k^{-1}\psi^{(1)}_+ + O(k^{-2}),$$

$$\psi^-_1(x,k) = 1 + k^{-1}\psi^{(1)}_++ O(k^{-2})$$

and inserting these expansions into Eqs. (4.6) and (4.8), we obtain

$$Q = -i[A, \psi^{(1)}_+ + i[A, \psi^{(1)}_+].$$ (4.4)

Hence, having solved the RH problem, we can find solutions of the BEC equations.

In general, the matrices $\psi_+$ and $\psi^-_-$ can have zeros $k_j$ and $k_i$ in the corresponding domains of analyticity: $\text{det}\psi_+ (k_j) = 0, k_j \in \mathbb{C}_+$, and $\text{det}\psi^-_-(k_i) = 0, k_i \in \mathbb{C}_-$. In virtue of the involution (3.12), we obtain $k_i = k_j^*$ and equal number $N$ of zeros in both half-planes. The corresponding RH problem is said to be nonregular, or the RH problem with zeros. They are zeros of the RH problem that determine soliton solutions of the BEC equations. It is seen from Eqs. (3.10) and (3.7) that zeros of $\psi_+$ nullify $2 \times 2$ minors of $\psi_+$. Hence, the rank of $\psi_+(k_j)$ can be equal to one or two. It means in turn that there exist one ((11)) or two ((12) and (22)) four-component eigenvectors that correspond to zero eigenvalue of $\psi_+(k_j)$:

$$\psi_+(k_j) |1_j\rangle = 0 \quad \text{for rank}\psi_+(k_j) = 1,$$

$$\psi_+(k_j) |2_j\rangle = 0 \quad \text{for rank}\psi_+(k_j) = 2.$$ (4.5)

The geometric multiplicity of $k_j$ is equal to the dimension of the null space of $\psi_+(k_j)$ (1 or 2 in our case). In this paper, we only consider the case of zeros $k_j$ with its geometric multiplicity equal to the algebraic multiplicity [which is the order of the zero $k_j$ in $\text{det}\psi_+(k)$]. Note that the solution of the RH problem for the general case of zeros with unequal geometric and algebraic multiplicities was elaborated in Ref. [10].

We will solve the matrix non-regular RH problem with zeros $k_1$ and $k_1^*$ by means of its regularization, i.e. by extracting from $\psi_+$ and $\psi^-_-$ rational factors that are responsible for the appearance of zeros. Hence, $\text{det}\psi_+(k_1) = 0$ [and correspondingly $\text{det}\psi^-_-(k_1^*) = 0$]. We need a rational matrix function $\Xi^{-1}(x,k)$ which has a pole in the point $k_1$. Let us take $\Xi^{-1}(x,k)$ in the form

$$\Xi^{-1}(x,k) = 1 + \frac{k_1 - k}{k - k_1} P(r),$$

where

$$P(r) = \sum_{l,m=1}^{r} |l(M^{-1})_lm\langle m|,$$ (4.6)
\(<m| = |m|^\dagger\) due to involution, and \(r = \text{rank } \psi_+(k_1)\). \(P^{(r)}\) is a projector of rank \(r\), \((P^{(r)})^2 = P^{(r)}\), and entries of the \(r \times r\) matrix \(M\) are determined by
\[
(M)_{lm} = \langle l|m\rangle = \sum_{a=1}^{4} (l^a)^\dagger (m)_a.
\]
In the appropriate basis the projector is represented as \(P^{(1)} = \text{diag}(1, 0, 0, 0)\) or \(P^{(2)} = \text{diag}(1, 1, 0, 0)\). This yields
\[
\det \Xi^{-1} = \left(\frac{k - k_1^*}{k - k_1}\right)^r.
\]
Therefore, the product \(\psi_+ (x, k) \Xi^{-1} (x, k)\) is regular in \(k_1\). In the same way, the regularization of \(\psi_-^1\) in the point \(k_1^\dagger\) is performed by the rational function
\[
\Xi (x, k) = 1 - \frac{k_1 - k_1^*}{k - k_1} P^{(r)},
\]
which provides the product \(\Xi \psi_-^1\) to be regular in \(k_1^\dagger\). Therefore, the analytic factors are factorized as
\[
\psi_+ (k) = \tilde{\psi}_+ (k) \Xi (k), \quad \psi_- (k) = \Xi^{-1} (k) \tilde{\psi}_-, \quad (4.8)
\]
with holomorphic functions \(\tilde{\psi}_\pm\) which determine the regular (without zeros) RH problem:
\[
\tilde{\psi}_-^1 (k) \tilde{\psi}_+^1 (k) = \Xi (k) EG(k) E^{-1} \Xi^{-1} (k), \quad k \in \text{Re}. \quad (4.9)
\]

For several pairs of zeros \((k_j, k_j^\dagger), j > 1\), the regularization of the RH problem can be performed in the same step-by-step manner, with the appropriate definition of the eigenvectors within each step. However, for practical calculation of \(N\)-soliton effects it is much more convenient to expand a product of rational factors into simple fractions, thereby transforming the product-type expression into a sum-type one \(52, 40\).

It is easy to find the coordinate dependence of the eigenvectors. Indeed, differentiating \((4.5)\) in \(x\) and in \(t\) with \(j = 1\) and accounting Eqs. \((2.10)\) and \((2.7)\) gives \((|l_1\rangle \equiv |l\rangle)\)
\[
\partial_x |l\rangle = ik_1 \Lambda |l\rangle, \quad \partial_t |l\rangle = 2ik_1^2 |l\rangle, \quad (4.10)
\]
with \(l = 1\) for rank one, and \(l = 1, 2\) for rank two. Hence,
\[
|l\rangle = \exp(ik_1 \Lambda x + 2ik_1^2 \Lambda t) |l^{(0)}\rangle, \quad (4.11)
\]
where \(|l^{(0)}\rangle\) is the coordinate-free four-dimensional vector.

Zeros \(k_j\) and vectors \(|l^{(0)}\rangle\) comprise the discrete data of the RH problem that determine the soliton content of a solution of the BEC equations. The continuous data are characterized by the off-block-diagonal parts of the matrix \(G(k) \quad 42, k \in \text{Re}\), and are responsible for the radiation components. In the following section we concretize the above relations to obtain one-soliton solutions of Eqs. \((2.3)\).

V. SOLITON SOLUTIONS

A. Rank-one soliton

To obtain the rank-one soliton solution of the BEC equations \((2.3)\), we consider the single pair \(k_1\) and \(k_1^\dagger\) of zeros and the eigenvector \(|1\rangle\). In accordance with Eq. \((4.11)\), the eigenvector takes the form
\[
|1\rangle = (e^{-ik_1 x - 2ik_1^2 t}, e^{-ik_1 x - 2ik_1^2 t}, e^{ik_1 x + 2ik_1^2 t}, e^{ik_1 x + 2ik_1^2 t})^T, \quad (5.1)
\]
where \(n_a, a = 1, \ldots, 4\) are complex numbers. The RH data are purely discrete: \(N = 1, G(k) = \mathbb{1}, \psi_\pm = \mathbb{1}\). Hence, the solution of the RH problem is given by the rational function \(\Xi\) in \((4.7)\) with the projector \(P^{(1)}\). The reconstruction formula \((4.4)\) is simplified to
\[
\tilde{Q} = -2\nu [\Lambda, P^{(1)}], \quad (5.2)
\]
where we set \(k_1 = \mu + i\nu\) and the projector \(P^{(1)}\) in \((4.10)\) is explicitly written as
\[
P^{(1)} = \frac{1}{2} \left[ \left( |n_1|^2 + |n_2|^2 \right) \left( |n_3|^2 + |n_4|^2 \right) \right]^{-1/2} \tilde{P}
\times e^{-2i\mu x - 4i(\mu^2 - \nu^2) \text{sech} z}.
\]

Here
\[
\tilde{P}_{ab} = n_a n_b, \quad z = 2\nu (x + 4\mu t) + \rho, \quad e^{2\rho} = \frac{|n_1|^2 + |n_2|^2}{|n_3|^2 + |n_4|^2}.
\]

Hence, it follows from Eq. \((5.2)\) that the soliton solution is given by
\[
Q = 2\nu \Pi^{(1)} e^{-2i\mu x - 4i(\mu^2 - \nu^2) \text{sech} z} \quad (5.3)
\]
with the polarization matrix
\[
\Pi^{(1)} = \frac{1}{2} \left[ \left( |n_1|^2 + |n_2|^2 \right) \left( |n_3|^2 + |n_4|^2 \right) \right]^{-1/2} \left( n_1 n_3^* n_4 n_2^* \right).
\]

Note that \(n_2 n_4^* = n_1 n_3^*\) due to the structure of the matrix \(Q\) in \((2.3)\). Besides, the matrix \(\Pi^{(1)}\) obeys automatically two conditions:
\[
\det \Pi^{(1)} = 0, \quad \Pi^{(1)}_{11}^2 + \Pi^{(1)}_{22}^2 + 2|\Pi^{(1)}_{12}|^2 = 1.
\]

Moreover, it is not difficult to show that the matrix \(\Pi^{(1)}\) depends only on two essential real parameters. Indeed, the rank-one soliton \((5.3)\) can be represented as
\[
Q = 2\nu \left( e^{-i\chi \sin^2 \theta} \cos \theta \sin \theta \right) e^{i\nu \text{sech} z} \left( \cos \theta \sin \phi - e^{i\chi \sin \phi} \cos \theta \right), \quad (5.4)
\]
where
\[
\cos \theta = \frac{|n_1|}{|n_1|^2 + |n_2|^2} = \frac{|n_3|}{|n_3|^2 + |n_4|^2}, \quad \chi = \arg (n_3 - n_4),
\]
and
\[
\cos \phi = \frac{|n_2|}{|n_1|^2 + |n_2|^2} = \frac{|n_4|}{|n_3|^2 + |n_4|^2}.
\]
\[ \varphi = -2\mu x - 4(\mu^2 - \nu^2)t + \phi_\alpha, \]
\[ \varphi_\alpha = \arg(n_1 - n_4) = \arg(n_2 - n_3). \]

The soliton amplitude is determined by the parameter \( \nu \), and its velocity is equal to \( 4\mu \). The parameters \( \rho \) and \( \phi_\alpha \) give the initial position of the soliton center and its initial phase, respectively. The angle \( \theta \) determines the normalized population of atoms in different spin states, while the phase factor \( e^{i\chi} \) is responsible for the relative phases between the components \( \phi_+ \) and \( \phi_- \).

It should be noted for future use that the constant soliton parameters acquire in general a slow \( t \) dependence in the presence of perturbation. This results in a modification of the equations for coordinates:
\[ z = 2\nu(x - \xi(t)), \quad \varphi = -\mu z + \delta(t), \quad (5.5) \]
\[ \xi(t) = -\frac{1}{2\nu} \left( 8 \int_0^t dt' \mu t' \nu(t') + \rho(t) \right), \]
\[ \delta(t) = -2\mu \xi(t) - 4 \int_0^t dt' [\mu^2(t') - \nu^2(t')] + \varphi_\alpha(t). \]

**B. Rank-two soliton**

As before, we begin with the pair \( k_1 \) and \( k_1^* \) of zeros, but now we have two linearly-independent eigenvectors
\[ |1\rangle = \left( e^{-ik_1 x - 2ik_1^2 t} p_1, e^{-ik_1 x - 2ik_1^2 t} p_2, e^{ik_1 x + 2ik_1^2 t} p_3, e^{ik_1 x + 2ik_1^2 t} p_4 \right)^T, \]
\[ |2\rangle = \left( e^{-ik_2 x - 2ik_2^2 t} q_1, e^{-ik_2 x - 2ik_2^2 t} q_2, e^{ik_2 x + 2ik_2^2 t} q_3, e^{ik_2 x + 2ik_2^2 t} q_4 \right)^T, \]
with \( p_a \) and \( q_a \), \( a = 1, \ldots, 4 \), being complex numbers. The rational function \( \Sigma \) is given by Eq. (4.7) with the rank-two projector \( P^{(2)} \). This projector is written in accordance with Eq. (4.0) as
\[ P^{(2)} = \sum_{i,m=1}^2 |m\rangle (M^{-1})_{mi} |l\rangle = (\det M)^{-1} \]
\[ \times \left( M_{22} |1\rangle \langle 1| - M_{12} |1\rangle \langle 2| - M_{21} |2\rangle \langle 1| + M_{11} |2\rangle \langle 2| \right). \]

In this case
\[ M = \left( \begin{array}{cccc} A_1 e^{z'} + B_1 e^{-z'} & A_3 e^{z'} + B_3 e^{-z'} \\ A_3^* e^{z'} + B_3^* e^{-z'} & A_2 e^{z'} + B_2 e^{-z'} \end{array} \right), \]
\[ z' = 2\nu(x + 4\mu t), \]
\[ A_1 = |p_1|^2 + |p_2|^2, \quad B_1 = |p_3|^2 + |p_4|^2, \]
\[ A_2 = |q_1|^2 + |q_2|^2, \quad B_2 = |q_3|^2 + |q_4|^2, \]
\[ A_3 = p_1^* q_1 + p_2^* q_2, \quad B_3 = p_3^* q_3 + p_4^* q_4. \]

Introducing the notations (to reproduce literally the results of Ref. [19])
\[ p_1 q_2 - p_2 q_1 = e^{\rho + i\sigma}, \quad p_3 q_2 - p_2 q_3 = \beta^*, \quad p_1 q_4 - p_4 q_1 = \gamma^*, \]
\[ p_1 q_3 - p_3 q_1 = p_4 q_2 - p_2 q_4 = \alpha^*, \quad (5.7) \]
we write explicitly the projector \( P^{(2)} \) as
\[ P^{(2)} = \left( \begin{array}{cccc} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{12} & P_{22} & P_{14} & P_{24} \\ P_{13} & P_{14} & 1 - P_{11} & -P_{12} \\ P_{14} & P_{24} & -P_{12} & 1 - P_{22} \end{array} \right), \]
where
\[ P_{11} = Z^{-1}(|\alpha|^2 + |\gamma|^2 + e^{2\tau}), \]
\[ P_{22} = Z^{-1}(|\alpha|^2 + |\beta|^2 + e^{2\tau}), \]
\[ P_{12} = -Z^{-1}(\alpha^* \beta + \alpha \gamma^*), \quad (5.8) \]
\[ P_{14} = e^{i\varphi} Z^{-1}(\beta e^z - \alpha^* e^{-z}), \]
\[ P_{13} = e^{i\varphi} Z^{-1}(\beta^* e^{-z} + \alpha^* e^z), \]
\[ P_{24} = e^{i\varphi} Z^{-1}(\gamma e^z + \beta^* e^{-z}), \]
\[ \mathcal{D} = \det \Pi^{(2)}, \quad \varphi = -2\mu x - 4(\mu^2 - \nu^2)t + \sigma, \]
\[ Z = \det M = 1 + e^{2\tau} + |\mathcal{D}|^2 e^{-2\tau}. \]

\( \Pi^{(2)} \) is the polarization matrix \( \Pi^{(2)} = \left( \begin{array}{cc} \beta & \alpha \\ \alpha^* & \gamma \end{array} \right) \) subjected to the normalization condition [19]
\[ 2|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1. \quad (5.10) \]

As a result, we immediately find from Eq. (5.2) with \( P^{(2)} \) the rank-two soliton solution of the BEC equations [2, 3, 19]
\[ Q(x, t) = 4\nu e^{i\varphi} Z^{-1} \left[ \Pi^{(2)} e^z + \sigma_2 \Pi^{(2)*} \sigma_2 \right], \quad (5.11) \]
where \( \sigma_2 \) is the Pauli matrix. Notice that the soliton solution of the matrix NLS equation was previously obtained in Ref. [27] by means of the Gelfand–Levitan integral equations, while our derivation is purely algebraic. The soliton [6, 11] was also derived by Gerdjikov and coworkers via the dressing procedure [11].

We will distinguish between two featured cases of \( \det \Pi^{(2)} \), namely, \( \det \Pi^{(2)} = 0 \) and \( \det \Pi^{(2)} \neq 0 \). These cases display different spin properties. Indeed, the spin density vector \( \vec{f}(x, t) = \text{tr}(Q^\dagger \vec{\sigma} Q) \), where \( \vec{\sigma} \) is the set of the Pauli matrices, is given in general by a spatially odd function
\[ \vec{f}(x, t) = \left( \frac{4\nu}{Z} \right)^2 \left( e^{2\tau} - |\mathcal{D}|^2 e^{-2\tau} \right) \]
\[ \times \left( \begin{array}{c} \alpha \beta + \bar{\alpha} \bar{\beta} + \alpha \gamma^* + \bar{\alpha} \bar{\gamma} \\ i(\alpha \bar{\beta} - \alpha \bar{\beta} + \alpha \gamma^* - \bar{\alpha} \bar{\gamma}) \end{array} \right), \quad (5.12) \]
with absolute value being of the form
\[ |\vec{f}| = \left( \frac{4\nu}{Z} \right)^2 \left| e^{2\tau} - |\mathcal{D}|^2 e^{-2\tau} \right| \left( 1 - 4|\mathcal{D}|^2 \right)^{1/2}. \quad (5.13) \]

Therefore, the total spin vector \( \vec{F} = \int dx \vec{f}(x, t) \) is zero. However, for \( \mathcal{D} = 0 \), as it follows from Eq. (5.12), the
The energy difference between both states with equal atom number density is 

\[ E_{\nu} = 4cN_{\nu}^{f}(\mu^2 - \nu^2/3). \]

In turn, the total number of atoms in the polar state and its energy are given by 

\[ N_{\nu}^{p} = 8\nu, \quad E_{\nu}^{p} = 4cN_{\nu}^{p}(\mu^2 - \nu^2/3). \]

The energy difference between both states with equal amount of atoms is 

\[ E_{\nu}^{f} - E_{\nu}^{p} = -(1/16)c(N_{\nu}^{f})^3 < 0. \]

Hence, the ferromagnetic state is energetically preferable, from the viewpoint of stability, as compared with the polar state.

The atom number density of the polar soliton is described by the function 

\[ n_{p}(z) = \left( \frac{4\nu}{Z} \right)^2 (e^{2z} + 4|\nu|^2 + |\nu|^2 e^{-2z}) \cdot \]  

Figure 1 demonstrates typical profiles of the atom number density function \([5.11]\) for different \(|\nu|\). The two-humped structure becomes more pronounced with decreasing \(|\nu|\). Such a state can be treated as a pair of two ferromagnetic solitons with antiparallel spins \([22]\).

The energy density in the ferromagnetic state is given by 

\[ E_{\nu} = 4cN_{\nu}^{f}(\mu^2 - \nu^2/3). \]

The energy difference between both states with equal amount of atoms is 

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![Figure 1](image1.png)

**FIG. 1:** Profiles of the atom number density function of the polar state: \(|\nu| = 1/8\) (thick line), \(|\nu| = 1/20\) (thin line), \(\nu = 0.5\).

![Figure 2](image2.png)

**FIG. 2:** Splitting of the components \(\phi_{\pm}\) and \(\phi_{0}\) of a perturbed polar soliton and of the atom number density \(n(x,t)\). Here \(|\beta|^2 = 0.7, |\alpha|^2 = |\gamma|^2 = 0.1\). The perturbation is of the form \([6.11]\) and \([7.2]\) with \(\epsilon = 0.1\).

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Second, rank-two solutions were obtained and classified as ferromagnetic and polar solitons. The polar soliton is perturbatively unstable and splits into two rank-two ferromagnetic solitons. Third, we proved equivalence of the rank-one soliton and rank-two ferromagnetic soliton. Therefore, it is sufficient to elaborate a perturbation theory for the more familiar type of solitons – the rank-one soliton [5,3]. This will be done in the following section.

VI. PERTURBATION THEORY FOR THE BRIGHT SPINOR BEC SOLITON

In this section we perform a general analysis of the perturbed spinor BEC equations
\[ i\partial_t \phi_{\pm} + \partial_x^2 \phi_{\pm} + 2 \left( |\phi_{\pm}|^2 + 2|\phi_0|^2 \right) \phi_{\pm} + 2\delta_{\pm}^* \delta_{0} = \epsilon R_\pm, \]  
\[ i\partial_t \phi_0 + \partial_x^2 \phi_0 + 2 \left( |\phi_+|^2 + |\phi_0|^2 + |\phi_-|^2 \right) \phi_0 + 2\delta_+^* \delta_- = \epsilon R_0. \]

Here \( R_\pm \) and \( R_0 \) determine a functional form of a perturbation, and \( \epsilon \) is a small parameter. To distinguish between the ‘integrable’ and ‘perturbative’ contributions, we will assign the symbol \( \delta / \delta t \) to the latter. Hence,
\[ i\delta \hat{Q} / \delta t = \epsilon \hat{R}, \quad \hat{R} = \begin{pmatrix} 0 & R \\ \bar{R}^t & 0 \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} R_+ & R_0 \\ R_0 & R_- \end{pmatrix}. \]

In general, a perturbation causes a slow evolution of the RH data. Indeed, a perturbation leads to a variation \( \delta \hat{Q} \) of the potential entering the spectral equation (2.6), and in turn to a variation of the Jost solutions:
\[ \delta J_{\pm} = i k [\Lambda, \delta J_{\pm}] + \hat{Q} \delta J_{\pm}. \]

Solving this equation gives
\[ \delta J_{\pm} = J_{\pm} E \left( \int_{\pm \infty} dx' E^{-1} J_{\pm}^{-1} \delta \hat{Q} J_{\pm} E \right) E^{-1}. \]

As a result, we find from Eqs. (3.1), (3.5), and (3.10) a variation of the scattering matrix:
\[ \delta S / \delta t = -i\epsilon S_+ \int_{-\infty}^{\infty} dx E^{-1} \psi_+^{-1} \hat{R} \psi_+ E S_-^{-1} = -i\epsilon T_- \int_{-\infty}^{\infty} dx E^{-1} \psi_-^{-1} \hat{R} \psi_- E T_-^{-1}. \]

Here \( S_\pm \) and \( T_\pm \) are the matrices defined in Sect. III. Notice that they are the analytic solutions \( \psi_\pm \) that enter naturally into this equation. Let us denote
\[ \Upsilon\pm(a, b) = \int_a^b dx E^{-1} \psi_\pm E, \quad \Upsilon_\pm(k) = \Upsilon_\pm(-\infty, \infty). \]

Then
\[ \delta S / \delta t = -i\epsilon S_+ \Upsilon_+(k)^{-1} S_+ - i\epsilon T_- \Upsilon_-(k) T_-^{-1}. \]

The matrices \( \Upsilon_\pm \) are interrelated by means of the matrix \( G \) entering the RH problem (4.1):
\[ \Upsilon_+(k) = GT_+(k) G^{-1}. \]  

Eventually, variations of the analytic solutions follow from Eqs. (3.5) and (3.10):
\[ \delta \psi_+ / \delta t = -i\epsilon \psi_+ E H_+ E^{-1}, \quad \delta \psi_- / \delta t = i\epsilon \psi_- E H_- E^{-1} \psi_-^{-1}. \]

Here \( H_\pm \) are the evolution functionals [30, 35] that are defined in terms of \( \Upsilon_\pm \),
\[ H_+ = \Upsilon_+(k) M_1 - \Upsilon_+(x, \infty), \quad H_- = M_1 \Upsilon_-(k) - \Upsilon_-(x, \infty), \quad M_1 = \text{diag}(1, 1, 0, 0), \]

and contain all essential information about a perturbation. In particular, the evolution equations for \( \psi_\pm \) gain additional terms caused by the perturbation and expressed in terms of \( H_\pm \):
\[ \partial_t \psi_+ = 2ik^2[\Lambda, \psi_+] + V \psi_+ - i\epsilon \psi_+ E H_+ E^{-1}, \quad \partial_t \psi_- = 2ik^2[\Lambda, \psi_-] - \psi_- V + i\epsilon \psi_- E H_- E^{-1} \psi_-^{-1}. \]

Besides, the evolution equation for the matrix \( G \) of the RH problem has the form
\[ \partial_t G = 2ik^2[\Lambda, G] - i\epsilon(GH_+ - H_- G). \]

In fact, this equation gives the evolution of the continuous RH data. Note that the involution (6.12) connects \( H_+ \) with \( H_- : H_- = H_+^t, \quad k \in \text{Re}. \)

Now we consider a single rank-one soliton and derive perturbation-induced evolution equations for the discrete RH data, i.e. for the zero \( k_1 \) and the eigenvector \( |\Psi\rangle \). It is more convenient to work with the vector \( |n\rangle = (n_1, n_2, n_3, n_4)^t \) which is constant in the absence of perturbation and acquires slow \( t \) dependence under the action of a perturbation. We start from the equation
\[ \psi_+(k_1)|1\rangle = \psi_+(k_1) \exp \left[ \left( ik_1 x + 2i \int dt k_1^2 \right) \Lambda \right] |p\rangle = 0 \]

which is valid irrespectively of the presence of a perturbation. Here the integral in the exponent accounts for a possible perturbation-induced time dependence of the zero \( k_1 \). Taking the total derivative in \( t \), we obtain
\[ \left( \partial_t \left[ \psi_+(k) e^{ikAx + 2i \int dt k_1^2 \Lambda} \right] \right) + \partial_k \left[ \psi_+(k) e^{ikAx + 2i \int dt k_1^2 \Lambda} \right] \partial_k |p\rangle |k_1 \rangle + \left[ \psi_+(k_1) e^{ik_1 x + 2i \int dt k_1^2 \Lambda} \partial_k |p\rangle \right] |k_1 \rangle = 0. \]
The first term with $\partial_t \psi_+$ is given by Eq. (6.3) which contains the evolution functional $H_+$. Recall that the evolution functional $H_+(k)$ is defined in terms of $\Upsilon_+$ in (6.4), which in turn depends on $\psi_+^{-1}$. Hence, the function $H_+$ is meromorphic in $\mathbb{C}_+$ with the simple pole in $k_1$, where $\psi_+$ has zero:

$$H_+(k) = H_+^{(reg)}(k) + \frac{1}{k-k_1} \text{Res}[H_+(k), k_1].$$

Here $H_+^{(reg)}$ stands for the regular part of $H_+$ in the point $k_1$. Following now the method developed in Refs. [30, 33], we find that the perturbed evolution of the vector $|n\rangle$ is given by

$$\partial_t |n \rangle = i \epsilon e^{-2i \int dt k_1^2 A H_+^{(reg)}(k_1)} e^{2i \int dt k_1^2 |n \rangle}. \quad (6.7)$$

Since the left-hand side of Eq. (6.7) is evidently $x$-independent, we can consider this equation for $x \to +\infty$, where $H_+$ has only two non-zero columns [see Eq. (6.4)]:

$$H_+(x \to +\infty) = \begin{pmatrix} \Upsilon_+^{(1)} & \Upsilon_+^{(2)} & 0 & 0 \end{pmatrix}. \quad (6.8)$$

Hence, Eqs. (6.7), written in components, take the form

$$\begin{align*}
\partial_t n_1 &= i \epsilon (X_{11} n_1 + X_{12} n_2), \\
\partial_t n_2 &= i \epsilon (X_{21} n_1 + X_{22} n_2), \\
\partial_t n_3 &= i \epsilon (X_{31} n_1 + X_{32} n_2), \\
\partial_t n_4 &= i \epsilon (X_{41} n_1 + X_{42} n_2),
\end{align*} \quad (6.9)$$

where for simplicity we use the notation $X_{ab} = \Upsilon_+^{(reg)}(k_1)$, $a, b = 1, 2$, and $X_{ab} = \Upsilon_+^{(reg)}(k_1) e^{-4i \int dt k_1^2}$ for $a = 3, 4$ and $b = 1, 2$. Here $\Upsilon_+^{(reg)}$ is the regular part of $\Upsilon_+$ in the point $k_1$.

$$\Upsilon_+^{(reg)}(k_1) = \int dx E^{-1}(k_1) \{ \tilde{R} (1 - P^{(1)}) + P^{(1)} \tilde{R} P^{(1)} + 2v \alpha [\Lambda, P^{(1)}] \tilde{R} (1 - P^{(1)}) \} E(k_1). \quad (6.10)$$

This relation follows from $\Upsilon_+(k) = \int dx E^{-1}\tilde{\Xi}^{-1} \tilde{R} E$. Note that in virtue of the specific symmetry of the matrix $\tilde{Q}$ (2.8) the entries $\Upsilon_{+32}$ and $\Upsilon_{+41}$ are equal, as well as the entries $\Upsilon_{+14}$ and $\Upsilon_{+23}$.

Now we can derive the evolution equation for the parameters $\theta$ and $\chi$ entering the polarization matrix of the soliton solution (5.4). Indeed, these parameters are defined in terms of $n_a$ which in turn obey Eqs. (6.9). Simple calculation gives

\begin{align*}
\partial_t \cos \theta &= \frac{\epsilon}{2} \left[ e^{\phi \pm i \varphi_0} (X_{31} e^{-i \chi} \cos \theta + X_{41} \sin \theta) - e^{\phi - i \varphi_0} (X_{31}^* e^{i \chi} \cos \theta + X_{41}^* \sin \theta) \right], \quad (6.11) \\
\partial_t \chi &= \frac{\epsilon}{2} \left[ e^{\phi \pm i \varphi_0} (X_{31} e^{-i \chi} - X_{42} e^{i \chi} + (\tan \theta - \cot \theta) X_{41}) - e^{\phi - i \varphi_0} (X_{31}^* e^{i \chi} - X_{42}^* e^{-i \chi} + (\tan \theta - \cot \theta) X_{41}^*) \right] \quad (6.12)
\end{align*}

Just in the same way we obtain evolution equations for the parameters $\varphi_0$ and $\rho$ which are also expressed in terms of $n_a$:

\begin{align*}
\partial_t \varphi_0 &= \frac{\epsilon}{2} \left[ X_{11} + X_{12} e^{-i \chi} + X_{12}^* e^{i \chi} \right] \tan \theta - e^{\phi \pm i \varphi_0} \left( X_{41} \cot \theta + X_{42} e^{i \chi} \right) + e^{\phi - i \varphi_0} \left( X_{41}^* \cot \theta + X_{42}^* e^{-i \chi} \right), \quad (6.13) \\
\partial_t \rho &= \frac{\epsilon}{2} \left[ (X_{11} - X_{12}) \cos^2 \theta + (X_{22} - X_{22}^*) \sin^2 \theta + [(X_{12} - X_{21}) e^{i \chi} - (X_{12}^* - X_{21}) e^{-i \chi}] \sin \theta \cos \theta + e^{\phi \pm i \varphi_0} (X_{31} e^{-i \chi} \cos^2 \theta) + e^{\phi - i \varphi_0} (X_{31}^* e^{i \chi} \cos^2 \theta) + 2 X_{41} \sin \theta \cos \theta + X_{42} e^{i \chi} \sin^2 \theta) \right] \quad (6.14)
\end{align*}

In fact, Eqs. (6.11)–(6.14) are greatly simplified when calculating the functions $X_{ab}$ for a specific perturbation. This will be demonstrated in the next Section.

To derive evolution equation for $k_1$ (and hence for the soliton amplitude $\nu$ and velocity $\mu$), we start from the equation $\partial_t \psi_+(k_1) = 0$. Taking the total derivative in $t$ yields

$$\partial_t (\det \psi_+(k))|_{k_1} + (\partial_t \det \psi_+(k))|_{k_1} \partial_t k_1 = 0. \quad (6.15)$$

In accordance with Eqs. (4.8) and (4.7) we can write

$$\det \psi_+(k) = \frac{k - k_1}{k - k_1^*} \det \tilde{\psi}_+(k),$$

where $\det \tilde{\psi}_+(k_1) \neq 0$ because $\tilde{\psi}_+(k)$ is a solution of the regular RH problem (4.9). Accounting now for the relation

$$\partial_t \det \psi_+(k) = -i \epsilon \text{tr} H_+ \det \psi_+(k),$$

we eventually obtain a simple evolution equation for the zero $k_1$:

$$\partial_t k_1 = \frac{i}{2} \epsilon \text{tr} \text{Res} \{ H_+(k), k_1 \} = \frac{i}{2} \epsilon \text{Res} \{ \tilde{T}_{+11}(k) + \tilde{T}_{+22}(k), k_1 \}. \quad (6.15)$$

Summarizing, Eqs. (6.9), (6.11)–(6.14), and (6.15) determine perturbation-induced evolution of the RH data. It should be stressed that these equations are exact because we did not yet refer to smallness of $\epsilon$ anywhere. At the same time, these equations cannot be directly applied because $\Upsilon_\pm$ entering them depend on unknown solutions $\psi_\pm$ of the spectral problem with the perturbed potential $\tilde{Q}$. To proceed further, we develop, owing to the smallness of $\epsilon$, the adiabatic approximation of the general perturbation theory.
VII. ADIABATIC APPROXIMATION

In the framework of the adiabatic approximation, we assume that the perturbed soliton adjusts its shape to the unperturbed one at the cost of slow evolution of its parameters. Hence, only the discrete RH data are relevant in this approximation, and we can put \( \tilde{\psi}_+ = \mathbb{1} \) for the solution of the regular RH problem (2.2). Therefore, \( \psi_+ = \Xi \). In other words, it is the rational function \( \Xi \) that completely determines soliton dynamics in the adiabatic approximation. In particular, we have

\[
\Upsilon_+ = \int_{-\infty}^{\infty} dx E^{-1} \Xi^{-1} \tilde{R} \Xi E, \quad \Xi^{-1}(k) = \Xi^\dagger(k). \tag{7.1}
\]

As an important example, we consider a perturbation caused by a small disturbance of the integrability condition (2.2). In this case we introduce a small parameter as \( c = c_0 - c_2 \), while the functional form of the perturbations \( R_{\pm,0} \) has the form

\[
R_{\pm,0} = (|\phi_+|^2 + 2|\phi_0|^2 + |\phi_-|^2) \phi_{\pm,0}. \tag{7.2}
\]

Inserting the explicit expressions for the soliton components \( \phi_{\pm,0} \) into this equation gives

\[
R_+ = (2\nu^3 e^{i(\varphi - \chi)} \cos \theta \text{sech}^3 z, \quad R_- = (2\nu^3 e^{i(\varphi + \chi)} \sin \theta \text{sech}^3 z, \quad R_0 = (2\nu^3 e^{i\chi} \cos \theta \sin \theta \text{sech}^3 z. \tag{7.3}
\]

Matrix elements of \( \Upsilon_+ \) which are the main ingredients of the evolution equations for the soliton parameters are found from Eqs. (7.1) and (7.2), and the projector \( P^{(1)} \) is calculated by means of the simple formula

\[
P^{(1)} = |1\rangle\langle 1|, \quad |1\rangle = |1\rangle^\dagger,
\]

which follows from Eq. (6.6). The eigenvector \( |1\rangle \) is given by Eq. (6.7). As a result, matrix elements of the projector are as follows \((P_{ab} = P_{ab}^{(1)}):\)

\[
P_{11}^{(1)} = \frac{1}{2} e^{z \cos^2 \theta \text{sech} z}, \quad P_{12}^{(1)} = \frac{1}{2} e^{-iz \cos \theta \sin \theta \text{sech} z}, \quad P_{13}^{(1)} = \frac{1}{2} e^{-z(\varphi - \chi)} \cos \theta \text{sech}^2 z, \quad P_{14}^{(1)} = \frac{1}{2} e^{-z(\varphi + \chi)} \sin \theta \text{sech}^2 z, \quad P_{21}^{(1)} = \frac{1}{2} e^{iz \cos \theta \sin \theta \text{sech} z}, \quad P_{22}^{(1)} = \frac{1}{2} e^{z \cos^2 \theta \text{sech} z}, \quad P_{23}^{(1)} = \frac{1}{2} e^{z(\varphi - \chi)} \cos \theta \text{sech}^2 z, \quad P_{24}^{(1)} = \frac{1}{2} e^{z(\varphi + \chi)} \sin \theta \text{sech}^2 z, \quad P_{31}^{(1)} = \frac{1}{2} e^{-z(\varphi - \chi)} \cos \theta \text{sech}^2 z, \quad P_{32}^{(1)} = \frac{1}{2} e^{-iz \cos \theta \sin \theta \text{sech} z}, \quad P_{33}^{(1)} = \frac{1}{2} e^{-z \cos^2 \theta \text{sech} z}, \quad P_{34}^{(1)} = \frac{1}{2} e^{-z(\varphi + \chi)} \sin \theta \text{sech}^2 z, \quad P_{41}^{(1)} = \frac{1}{2} e^{iz \cos \theta \sin \theta \text{sech} z}, \quad P_{42}^{(1)} = \frac{1}{2} e^{z(\varphi - \chi)} \cos \theta \text{sech}^2 z, \quad P_{43}^{(1)} = \frac{1}{2} e^{z(\varphi + \chi)} \sin \theta \text{sech}^2 z, \quad P_{44}^{(1)} = \frac{1}{2} e^{z \cos^2 \theta \text{sech} z}.
\]

Now we easily obtain from Eqs. (7.1) and (7.2) that

\[
\text{Res} [\Upsilon_{+11}(k) + \Upsilon_{+22}(k), k_1] = 0.
\]

Therefore, \( \partial_t k_1 = 0 \) in accordance with Eq. (6.13), which means that the soliton amplitude and velocity preserve their initial values. Finding evolution of the other soliton parameters demands knowledge of the regular part of \( \Upsilon_+(k_1) \). Calculation due to Eq. (6.10) gives

\[
X_{11} = X_{12} = X_{21} = X_{22} = 0, \quad X_{31} = (2\nu^2) \exp(-\rho - i\varphi_0 + i\chi) \cos^2 \theta, \quad X_{42} = (2\nu^2) \exp(-\rho - i\varphi_0 - i\chi) \sin^2 \theta, \quad X_{41} = X_{32} = (2\nu^2) \exp(-\rho - i\varphi_0) \cos \theta \sin \theta.
\]

Substituting these functions into Eqs. (6.11)--(6.13), we obtain:

\[
\rho = \text{const}, \quad \theta = \text{const}, \quad \chi = \text{const}, \quad \varphi_0(t) = \varphi_0(0) - 4\nu^2 t.
\]

As a result, within the adiabatic approximation, the only manifestation of the perturbation caused by a small deviation from the integrability condition (2.2) consists in a small shift of the soliton frequency equal to \( 4\nu^2 \). Hence, a ferromagnetic soliton is a pretty robust object against a small disturbance of the integrability condition.

This conclusion has been checked by comparison with direct simulations of the perturbed equations (6.1). The left-hand panel of Fig. 3 demonstrates the evolution of the perturbed \( \phi_+ \) component profile. We see a small profile distortion. Very little energy radiation is emitted to the far field. The same results are valid for the other two components. It is seen from the right-hand panel that there is a good agreement of the predicted linear dependence of the frequency shift on \( \epsilon \) with that obtained numerically.

VIII. CONCLUSION

In the present paper we have developed a perturbation theory for bright solitons of the integrable spinor BEC model. This model is equivalent to the \( 2 \times 2 \) matrix NLS equation and is naturally associated with the matrix RH problem. We have demonstrated the efficiency of the
formalism based on the RH problem, for solving both integrable and nearly integrable versions of the spinor BEC model. We have obtained the rank-one and rank-two soliton solutions of the model. Depending on the spin properties, the rank-two soliton can be of the ferromagnetic type or of the polar type. We have proven that the ferromagnetic soliton is equivalent to the rank-one soliton. As regards the polar soliton, its profile is characterized by a two-humped structure in a wide region of the soliton parameters. We have observed from numerical experiments that the polar soliton is unstable under the action of a perturbation and splits into a pair of ferromagnetic solitons. Owing to this fact, the problem to construct a perturbation theory for the spinor BEC solitons has been reduced to that for the rank-one solitons.

We have derived perturbation-induced evolution equations for the soliton parameters. In the adiabatic approximation of the perturbation theory these equations have been applied to a practically important case of a perturbation caused by a small deviation of the model parameters from those in the integrable case. We have shown a considerable stability of the ferromagnetic soliton in the presence of such a perturbation. Namely, the soliton preserves its amplitude, velocity and spin properties, a small frequency shift being the only manifestation of the perturbed environment. At the same time, the polar soliton solution of the integrable model parameters from those in the integrable case. We have observed from numerical experiments that the polar soliton is unstable under the action of a perturbation and splits into a pair of ferromagnetic solitons. Owing to this fact, the problem to construct a perturbation theory for the spinor BEC solitons has been reduced to that for the rank-one solitons.

Three more points deserve a special comment. First, instability of a perturbed polar soliton and its splitting into ferromagnetic ones have been observed numerically. Analytical study of this phenomenon demands a separate consideration and can be performed, for example, by a stability analysis as developed in Ref. [13]. Second, we have restricted ourselves to the study of the adiabatic approximation of the general perturbation theory. Our equations permit us to go beyond this approximation and take into account the soliton shape distortion effects. However, quantitative characteristics of the first-order effects are too small to be verified experimentally, at least at present. Examples of practical calculations in the first-order approximation can be found in Ref. [13]. Third, the formalism developed here for the single perturbed soliton can be straightforwardly generalized to the case of \( N \) weakly interacting solitons arranged into a train-like configuration. Analysis of the soliton train dynamics by the soliton perturbation theory can be found in Refs. [44, 45] for optical solitons and in Ref. [46] for scalar bright BEC solitons.

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