Constraining white dwarf viscosity through tidal heating in detached binary systems

Simone Dall’Osso$^{1,2}$† & Elena M. Rossi$^3$

1 Racah Institute of Physics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel
2 Theoretical Astrophysics, University of Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany
3 Leiden Observatory, Leiden University, P.O. Box 9513, 2300 RA, Leiden, The Netherlands

Submitted: Revised: Accepted:

ABSTRACT

Although the internal structure of white dwarfs is considered to be generally well understood, the source and entity of viscosity is still very uncertain. We propose here to study white dwarf viscous properties using short period (< 1 hr), detached white dwarf binaries, such as the newly discovered ~ 12.8 min system (J0651). These binaries are wide enough that mass transfer has not yet started but close enough that the least massive component is subject to a measurable tidal deformation. The associated tidal torque transfers orbital energy, which is partially converted into heat by the action of viscosity within the deformed star. As a consequence, its outer non-degenerate layers expand, and the star puffs up. We self-consistently calculate the fractional change in radius, and the degree of asynchronism (ratio of stellar to orbital spin) as a function of the viscous time. Specializing our calculations to J0651, we find that the discrepancy between the measured radius of the secondary star and He white dwarf model predictions can be interpreted as tidal inflation if the viscous timescale is either ~ 2 × 10^3 yr or ~ 10^4 yr. Such values point to a non-microscopic viscosity, possibly given by tidally induced turbulence, or by magnetic field stresses with a magnetic field strength of 10^-10 Gauss. Fortunately, these two timescales produce very different degree of asynchronism, with the shortest one, bringing the system much closer to synchronisation. A measurement of the stellar spin can thus univocally determined the mean viscosity. Currently, we may exclude a middle range viscous time of a few 10^3 yr, which would give a radial inflation of ~ 10%, which is not observed. Extrapolating the secondary’s expansion we predict that the star will fill is Roche lobe at a separation which is ~ 1.2 – 1.3 smaller than the current one. Applying this method to a future sample of systems can allow us to learn whether viscosity changes with mass and/or nuclear composition.

Key words: binaries : close, gravitational waves, stars: white dwarfs, stars: interiors, methods: analytical

1 INTRODUCTION

Short-period white dwarf binaries are very interesting systems. They are emitters of gravitational waves (GW), and candidate progenitors of supernovae Ia (Webbink 1984; Iben & Tutukov 1984) and Ia (Bildsten et al. 2007). When their orbital period is of tens of minutes and mass transfer has not yet started, they become clean laboratories to study the reaction of their internal structure to tidal forces. In turn, from this reaction a lot can be learnt on the internal properties of these stellar remnants. Recently, the “Extreme Low Mass (ELM)” Survey (Brown et al. 2010; Kilic et al. 2010) has increased to 24 the number of known systems that will merge within a Hubble time (Kilic et al. 2012). In particular, there are currently three detached systems with orbital period P less than 40 minutes: J0651 ($P \approx 12.75$ min, Brown et al. 2011), J1630 ($P \approx 39$ min, Kilic et al. 2011a) and J0106 ($P \approx 39$ min, Kilic et al. 2011b).

Among these, the best case study is J0651. First, tidal elongation of the secondary was directly measured from the source lightcurve. Second, it is an eclipse system, which allows for a precise and model independent determination of the stellar radii (see Table 1). Finally, the system has the shortest GW-driven merger timescale (a few Myr), which makes evolutionary processes particularly relevant.

We also find very intriguing that the measured radius of the secondary $R_2 = 0.0353 \pm 0.0004 R_\odot$ is 5% larger than that predicted by the best matching He WD model ($\approx 0.0337 R_\odot$), Panei et al. 2007). This discrepancy is significant at the 4σ level. We will show that it can be explained as due to tidal heating with plausible values of the viscous timescale.

† Here and thereafter, we refer to the least massive member of the binary as “secondary”. The most massive is therefore the “primary” member.
There exists a maximum \( \eta \) beyond which the star’s self gravity is not able anymore to counteract the tidal pull, and the star falls apart. This happens at the so called “tidal radius” \( a_T \), defined as \( a_T \approx 2.15 R_2 \eta^{-1/3} \), where \( \eta \approx M_2/M_1 \). The corresponding value of \( \eta_t \approx 0.42 \). For J0651 this separation is only \( \approx 2.8 \times R_2 \), and mass transfer starts well before the system can reach tidal disruption (see Table 1).

We sketch here the derivation of \( a_T(a) \) as a function of the orbital separation. The secondary’s tidal bulge is a result of the work done by the first term in the expansion of the primary’s newtonian potential \( \psi_T \). If we consider a two dimensional problem, in the orbital plane of the binary,

\[
\psi_T = \frac{GM_1}{a} \left( \frac{r}{a} \right)^2 3 \cos^2(\theta) - 1 \frac{1}{2}.
\]

(Alexander 1973 and references therein), where the polar coordinates \((r, \theta)\) are centered in the centre of mass of the secondary, and \( \theta \) is measured from the semi-major axis of the bulge. In eq. 1

\[
0 \leq r \leq R_2.
\]

This tidal pull causes a decrease in the binding energy of the originally unperturbed secondary. In turn, the change in the secondary’s external potential, \( \psi_\ast \),

\[
\psi_\ast = \kappa_2 \left( \frac{R_2}{r} \right)^3 \psi_T,
\]

is a result of the tidal bulge: since the two stars are now closer to each other, the orbital binding energy increases, further drawing from the stellar binding energy.

In the system’s energy budget, the additional energy terms \( \psi_T \) and \( \psi_\ast \) are thus balanced by the decrease in the stellar binding energies. This statement of energy conservation reads,

\[
M_2 \psi_T + M_1 \psi_\ast = \frac{\gamma G M_2^2}{R_2} - \frac{\gamma G M_1^2}{R},
\]

where \( \psi_T \) and \( \psi_\ast \) are evaluated along the line of centres (\( \theta = 0 \)), with \( r = a \) and stellar radius \( R \) (instead of \( R_2 \)) in eq. 2 and with \( r = R \) in eq. 1. The structural constant in the self gravity term (right-hand side) is \( \gamma = 3/(5 - n) = 6/7 \) (cfr. Lai & Shapiro 1995).

A manipulation of eq. 3 leads to the following algebraic equation for \( \eta \),

\[
\kappa_2 (1 + \eta)^6 + q \left( \frac{a}{R_2} \right)^3 (1 + \eta)^3 - q^2 \gamma \left( \frac{a}{R_2} \right)^6 \eta = 0,
\]

For a given polytropic index \( n \) and the mass ratio \( q \), we can solve numerically the above equation to obtain \( \eta(a) \). In Fig. 1b we show our results for the the detached WD-WD system J0651, whose observed parameters are listed in Tab. 1. The agreement of the observed \( \eta \approx 3\% \) with our prediction (black line) is quite satisfactory. Instead, the classical approximation of \( \eta \),

\[
\frac{h}{R_2} = \frac{\psi_T}{(3GM_2/R_2)},
\]

where \( \phi_v \) is not included, fails to describe the final stages of the inspiral (red line).

3 THE SYSTEM DYNAMICS

In this section, we describe the dynamics of a detached binary system. The equations we derive here will allow us to calculate the tidal dissipation and the consequent thermal expansion of the mean radius of the secondary.

2 TIDAL INTERACTION

Let us consider a binary white dwarf system, at a separation where mass transfer has not yet started. For a system like J0651, we will show that this implies periods larger than \( \approx 10 \) min. The primary and the secondary have masses and radii \( M_1 \) and \( M_2 \) and \( R_1 \) and \( R_2 \) respectively. We model the stars as polytropic structures with index \( n = 3/2 \), appropriate for low mass (i.e. non-relativistic) white dwarfs. The global deformability of the secondary is measured by its Love number, \( \kappa_2 \approx 0.15 \), which depends on the star density profile (Love 1909, Verbunt & Hut 1983, Hinderer 2008).

In this section, we report the calculation presented in Paper I of the evolution of the tidal deformation (referred to as “bulge”) of the secondary star as a function of the orbital separation. We work in the limit of small deformations, \( \eta = h/R_2 \ll 1 \), where \( h \) is the “height” of the tidal bulge, i.e. the elongation with respect to the unperturbed radius. The fractional height \( \eta \) is a function of the orbital separation, \( a \), and can be linked to the semi-major axis of the white dwarf by

\[
R(a) = R_2[1 + \eta(a)].
\]
GW-induced tidal torque tracks the acceleration of the orbit, hence the misalignment of the bulge axis with the line of the centres induces a continuous GW emission. The rapid orbital decay causes a continuous GW emission. The GW-induced tidal torque, due only to the WD viscosity, is negligible, viscosity is expected to largely determine the tidal interactions in these systems. Interestingly, J0651 may be in an intermediate regime: its insiprimal timescale, \( t_{GW} \approx 3 \) Myr, is sufficiently short that tidal coupling might result in significant gravitational interaction also from the GW-induced tidal torque. This makes J0651 an excellent case to study tidal interactions in detail, and to constrain WD viscosity.

We generalise here the dynamics described in our previous paper, including the action of the viscous torque \( N_v \), which drains angular momentum from the bulge and transfers it to the star’s rigid rotation (i.e. it increases its spin).

The net torque acting on the bulge, and changing its angular momentum \( J_b \), is

\[
\frac{dJ_b}{dt} = N_T - N_v, \tag{7}
\]

where the tidal torque, \( N_T \), speeds up the bulge and the viscous torque \( N_v \) resists the motion and tends to slow it down. The bulge moment of inertia \( I_b \), is, for small \( \beta \) and small tidal deformation \( \eta \),

\[
I_b = \frac{6 \pi^2 \kappa_2}{(1 + q)^2} \left( \frac{\omega_b}{\omega_*} \right)^4 M_2 R_2^2, \tag{12}
\]

(Paper I eq.4). The change of angular momentum \( J_b = I_b \omega_b \) can therefore be expressed as \( \frac{dJ_b}{dt} = 5 I_b \Dot{\omega}_b \).

Let’s now specify the two terms on the right hand side of eq. \( \boxed{7} \). The tidal torque arises when the bulge angular speed \( \omega_b \neq \omega_* \), which causes a misalignment by an angle \( \beta \) between the bulge axis and the line of the centres. It can be derived from \( N_T = \nabla_\theta \psi_\alpha(a) \times a \), which gives the following expression

\[
N_T \approx N \beta \left( \frac{\omega_b}{\omega_*} \right)^4, \tag{8}
\]

where the numerical constant \( N = 3 \pi \kappa_2 \omega_*^2 M_2 R_2^2/(1 + q)^2 \), and we use the property that for our WD systems \( \beta \ll 1 \) (see Paper I, eq. 3). The effective angle \( \beta \) in the above expression should be regarded as a global property of the star like, e.g., its total deformability \( \kappa_2 \). Finally, the angular momentum which is subtracted from the bulge by \( N_v \) is transferred to the spin,

\[
N_v = \frac{dI_b}{dt} = \frac{J_b}{I_b}, \tag{9}
\]

where with the last step we define the viscous timescale. The WD spin is \( J_s = I_\omega \omega_s \), where the unperturbed moment of inertia is \( I_* \approx 0.2 M_2 R_2^2 \) and \( \omega_s \) is the spin angular frequency.

The ultimate source of angular momentum and energy in tidal interactions is the binary orbit. Its angular momentum \( J_o \) decreases because of the existence of a tidal torque and because it can be transported away by GW emission,

\[
\frac{dJ_o}{dt} = - (N_T + N_{GW}) \!, \tag{10}
\]

where \( J_o = I_\omega \omega_* \),

\[
N_{GW} = \frac{1}{3} I_s \frac{\omega_*}{\omega_s}, \tag{11}
\]

and the orbital moment of inertia is

\[
I_o = M_2 M_1^{2/3} G^{2/3} (1 + q)^{-1} \omega_*^{-4/3}. \tag{12}
\]

At equilibrium the bulge speeds up at the same rate as the orbit shrinks, \( \omega_b = \omega_* \). Using this condition and eqs. \( \boxed{7} \) and \( \boxed{10} \) it is easy to derive that in absence of GW emission \( (N_{GW} = 0) \) the tidal torque, due only to the WD viscosity, is

\[
N_T = \frac{N_v}{1 - x} = N_{T,v}, \tag{12}
\]

By transferring to it angular momentum extracted from the orbit.

---

**Table 1.** The detached WD system J0651. List of the measured parameters, which we use in this paper (Brown et al. 2011). The orbital separations at which (depending on \( t_v \)) the secondary will fill its Roche lobe, \( a_{\text{Roche}} \), are theoretically derived in Sec. 4.3. The numbers in parenthesis indicate 1σ errors on the last digit. The current orbital separation, \( a \) and \( a_{\text{Roche}} \) are expressed in units of \( R_2 \).

| Parameter | Value |
|-----------|-------|
| \( M_2 \) | 0.25 \( M_\odot \) |
| \( M_1 \) | 0.55 \( M_\odot \) |
| \( q = M_2/M_1 \) | 0.45 |
| \( R_2 \) | 0.0353(4) \( R_\odot \) |
| \( \eta \) | 0.033 |
| \( a_{\text{Roche}} \) | 3.76 \( R_2 \) if \( t_v = 170 \) kyr |
| \( a_{\text{Roche}} \) | 3.96 \( R_2 \) if \( t_v = 11 \) kyr |

**Figure 1.** Tidal deformation as a function of the orbital frequency, for the secondary WD. The black line is our calculation (eq.\( \boxed{4} \), while the red line is the classical approximation (eq.\( \boxed{5} \). The system parameters are those of J0651 (see Table 1) and its observed tidal deformation is marked in blue, where we also indicated the error bars at one σ. The chosen adiabatic index is \( n = 3/2 \).
where
\[ x = \frac{15I_b}{I_0} \approx \frac{90\pi^2\kappa_2 R_2^8 \omega_0^{16/3}}{(1 + q)^{5/3}G^{8/3}M_2^2 M_1^{2/3}}. \]

When, in addition, a dynamical component of the tidal torque arises due to the emission of GWs \((N_{T,GW})\), the total tidal torque can be expressed as
\[ N_T = N_{T,v} + N_{T,GW} = N_{T,v} \left( 1 + \frac{5t_v}{t_{GW}} \right) = N_{T,v} (1 + \tau) \],
(13)

where the last step defines the parameter \(\tau = 5t_v/t_{GW}\), measuring the relative contribution of \(N_{T,GW}\) to the total tidal torque. In J0651 for example, we obtain \(\tau \approx 0.2\) for \(t_v = 10^5\) yrs, scaling linearly with the poorly constrained value of the viscous time. Thus, the tidal torque would be dominated by viscous effects as long as \(t_v \ll 10^5\) yrs, say, while for larger values of \(t_v\), the GW-induced torque becomes progressively important, and even dominant at \(t_v \gtrsim 5 \times 10^5\) yrs. For wider systems, on the other hand, \(N_{T,GW}\) is generally expected to be unimportant, due to the steep increase of \(t_{GW}\) with the orbital period.

Finally, we can evaluate the relative importance of the tidal torque versus GW emission for causing the orbital decay,
\[ \frac{N_{GW}}{N_T} = \frac{\tau}{x(1 + \tau)(1 - \alpha)} \],
(14)

where \(\alpha = \omega_\alpha/\omega_0\) measures the asynchronism of the secondary. For low mass systems with orbital periods longer than \(\approx 10\) minutes, \(x \ll 1\) and \(\tau \lesssim 1\). For J0651 we get \(x \approx 4 \times 10^{-3}\), thus, as long as \(t_v > 2 \times 10^5\) yrs, the effect of the tidal torque on the orbital evolution is at best marginal, even if \(\alpha = 0\). However, as we will see below, the asynchronism is expected to be very close to unity if \(t_v \lesssim 10^4\) yrs. Hence, GW emission is always dominant in determining the measured orbital shrinkage of J0651.

4 \ THE EFFECTS OF VISCOSITY

The WD internal viscosity plays an all-important role in determining its tidal evolution. Viscous dissipation of the bulge energy causes a progressive synchronisation of the WD spin with the orbital period. The released heat in turn affects the internal energy balance and can even induce measurable changes of the WD global properties, such as its outer radius or its luminosity.

4.1 Degree of asynchronism

As we discussed in detail in sec. 3 the internal viscous torque changes the spin of the star by transferring its angular momentum, drained from the tidal bulge. The temporal evolution of the asynchronism \(\alpha\) is
\[ \dot{\alpha} = \frac{\dot{\omega}_\alpha}{\omega_\alpha} = \frac{N_{T,v}}{t_e} \left( 1 - x \right), \]
(15)

which requires to simultaneously track the spin and orbital evolution. The former is governed by eq. (9) and depends only on the viscous torque,
\[ \dot{\omega}_\alpha = \frac{N_{T,v}}{I_0} \left( 1 - x \right), \]
(16)

where we used eq. (12) and the definition \(dJ_\alpha/dt = I_\alpha \dot{\omega}_\alpha\). The orbital angular frequency, instead, can change both under the action of GW emission and of the tidal torque (eq. (10)). Since, \(dJ_\alpha/dt = -1/3I_\alpha \dot{\omega}_\alpha\), we can derive that,

\[ \frac{d\omega_\alpha}{\omega_\alpha} = \frac{\dot{\alpha}}{\alpha} = \frac{Q}{t_v} \left( \frac{1 - x}{1 + (1 + \tau)(1 - \alpha)x/\tau} \right) \approx \frac{Q}{t_v} \left( 1 - \frac{x}{\tau} \right), \]

where \(Q = 30\pi^2 / [ (1 + q)^2 \omega_0^2] \).

Finally, from eqs. (16) and (17) and after some manipulation we obtain the temporal evolution of \(\alpha\),
\[ \dot{\alpha} = \frac{(1 - \alpha)}{t_v} Q (1 - x) \omega_0^4 - \frac{\alpha}{t_{GW}} - \frac{(1 - \alpha)}{t_v} \frac{x(1 + \tau)}{5}, \]
(18)

where \(Q = 30\pi^2 / [ (1 + q)^2 \omega_0^2] \).

In general, as long as \(t_{GW} \gg t_v\), the asynchronism decreases (\(\dot{\alpha} > 0\)) and \(\alpha\) reaches the asymptotic solution of eq. (18) at a given orbit, before the system changes its separation. On the other hand, for \(t_{GW} \ll t_v\) the viscosity is too slow to be able to efficiently transfer angular momentum to the star spin before the orbital separation changes. As a consequence, the asynchronism progressively increases (\(\dot{\alpha} < 0\)). This latter case is relevant, for instance, for double neutron stars at coalescence (Paper I), which never reach tidal locking (e.g. Bildsten & Cutler 1992). For systems considered in this paper, instead, \(\alpha\) changes on a viscous timescale and it is possible to describe its evolution as a function of \(\omega_\alpha\), as a series of equilibrium solutions given by eq. (18),
\[ \frac{d\omega_\alpha}{\omega_\alpha} = \frac{\dot{\omega}_\alpha}{\omega_\alpha} = \frac{Q t_{GW} (1 - x) (1 - \alpha)}{t_v [1 + (1 + \tau)(1 - \alpha)x/\tau]} \omega_\alpha^{2 - \alpha} - \frac{\alpha}{\omega_\alpha}, \]
(19)

where the time evolution of \(\omega_\alpha\) is given by eq. (17). The solution is shown in Fig. 2 for four values of the viscous timescale, \(t_v = 11\) kys, 33 kys, 170 kys and 2 Myrs. We start with \(\omega_\alpha \approx 0\) at large separation, motivated by the long, ~hrs or more, spin periods of isolated WD dwarfs (Spruit 1998 and references therein). This figure shows that, only for very short viscous timescale \(t_v < 10^4\) yr, tidal synchronisation is (nearly) reached before the two stars come into contact (for J0651 the latter happens at \(P \approx 9\) min, if \(t_v \approx 11\) kys, or at \(P \approx 9.7\) min, if \(t_v \approx 170\) kys), see § 4.3. The leftmost vertical line marks the current orbital frequency of J0651: its degree of synchronisation is very sensitive to the value of the viscous timescale, going from \(< 1\%\) if \(t_v > 10^6\) yrs, to \(\sim 80\%\) if \(t_v \lesssim 10^4\) yr range.
it would also cause a smaller value of\( \alpha \).\footnote{\( t_v \) is used before (see figure caption for details). Viscosity has a double role in determining the energy dissipation rate, at a given specific value of \( t_v \). The resulting evolution of the asynchronism is determined by eq. 19, the associated energy dissipation rate can be obtained straightforwardly. The resulting evolution of \( W \) with the orbital period is plotted in Fig. 3 for the same four values of the viscous timescale used before (see figure caption for details). Viscosity has a double role in determining the energy dissipation rate, at a given \( \omega_0 \). The viscous timescale appears explicitly in eq. 20 through the term \( N_0 \) (see eq. 9), and implicitly, through the term \( (1 - \alpha) \). While decreasing the viscous time would increase the coefficient in eq. 20, it would also cause a smaller value of \( (1 - \alpha) \), because spin synchronisation would be more efficient. The value of \( W \) will be determined by a trade-off between these two competing effects. To illustrate this, we show in Fig. 4 the dependence of \( W \) on the viscous timescale, with labels on different branches of the curve, indicating the terms that determine the dependence on viscosity. As expected from the previous discussion, \( W \) displays a maximum around a specific value of \( t_v \) for \( t_v \approx 35 \) kyr, decays at both ends of the considered range. In particular, we obtain \( W_{\text{max}} \approx 2.5 \times 10^{32} \) erg s\(^{-1}\) for \( t_v \approx 35 \) kyr, corresponding to \( (dR/R)_{\text{max}} \approx 0.095 \).

\subsection{4.2 Internal heating}

The viscous torque also drains energy from the tidal bulge along with angular momentum. While part of this energy is transferred to the WD spin another part is released locally, heating the inner layers of the star. The rate of heat release will thus be given by the difference between the rate of energy extraction from the bulge and the rate of energy transfer to the spin.

\[ W = \frac{N_v \omega_0 - N_c \omega_c^*}{N_v \omega_0} \approx \frac{N_{T_v}(1 - x)}{t_v} \omega_c^* (1 - \alpha) = \frac{c_2}{t_v} \omega_c^* (1 - \alpha)^2 \]

where \( c_2 = 6\pi^2 R_v^2 (1 - x) M_2 R_2^2 \left[ \frac{\omega_c^* (1 + \alpha)}{\omega_c^* (1 + \alpha)} - \frac{\omega_c^* (1 + \alpha)}{\omega_c^* (1 + \alpha)} \right]^2 \) depends only on stellar parameters and we used the expression for the lag angle (cf. Paper I),

\[ \beta = 2\pi^2 \frac{\omega_c^*}{(1 + \alpha)} (1 - \alpha) \]

Once the evolution of the asynchronism is determined by eq. 19, the associated energy dissipation rate can be obtained straightforwardly. The resulting evolution of \( W \) with the orbital period is plotted in Fig. 3 for the same four values of the viscous timescale used before (see figure caption for details). Viscosity has a double role in determining the energy dissipation rate, at a given \( \omega_0 \). The viscous timescale appears explicitly in eq. 20 through the term \( N_0 \) (see eq. 9), and implicitly, through the term \( (1 - \alpha) \). While decreasing the viscous time would increase the coefficient in eq. 20, it would also cause a smaller value of \( (1 - \alpha) \), because spin synchronisation would be more efficient. The value of \( W \) will be determined by a trade-off between these two competing effects. To illustrate this, we show in Fig. 4 the dependence of \( W \) on the viscous timescale, with labels on different branches of the curve, indicating the terms that determine the dependence on viscosity. As expected from the previous discussion, \( W \) displays a maximum around a specific value of \( t_v \) for \( t_v \approx 35 \) kyr, decays at both ends of the considered range. In particular, we obtain \( W_{\text{max}} \approx 2.5 \times 10^{32} \) erg s\(^{-1}\) for \( t_v \approx 35 \) kyr, corresponding to \( (dR/R)_{\text{max}} \approx 0.095 \).

\subsection{4.3 Inflation of the secondary white dwarf}

The viscously dissipated energy ultimately inflates the outer non-degenerate layers of the white dwarf. These are in radiative and hydrostatic equilibrium, which combined with the equation of state for ideal gases gives the following relation between the density \( \rho \) and the temperature \( T \),

\[ \rho \approx 1.52 \times 10^{-21} \left( \frac{L}{L_\odot} \right)^{1/2} T^{3.25} \text{ g cm}^{-3} \]

(21)

where \( L \) is the WD luminosity, and here and in the following we adopt a WD composition with 90% He and 10% metals. The deep interior of the star is instead electron degenerate and the transition between the two regimes occurs at the layer where density and temperature are related by

\[ \rho \approx 10^{-8} \rho_\odot (1 - \alpha)^{1.5} \text{ g cm}^{-3} \]

(22)

(see Shapiro & Teukolsky 1983). The non-degenerate layer is quite thin, and it can be shown that the transition radius, \( R_t \), and the stellar radius \( R_2 \) are related by

\[ R_2 = \frac{AR_t}{R_t T_\odot - A} \]

where \( A \approx 2.64 \times 10^{-16} M_2/R_2 \). Combining eq. 22 and eq. 23 we can express the transition temperature in terms of the WD luminosity, \( L \sim T_\odot^{1.5} \). Therefore, a perturbation \( W \) to the WD luminosity causes a fractional upward change in the transition temperature given by \( W/L = 3.5 \times dT_\odot/T_\odot \).

The tidally-induced increase of the WD internal luminosity will affect the radiative and hydrostatic equilibrium of the non-degenerate envelope, inducing a change in its radial extension, hence in the WD radius \( R_2 \). The variation of the external radius as a consequence of a tidal dissipation rate \( W \) can be calculated from eq. 23

\[ dR_2 = \left[ \frac{\partial R_2}{\partial T_\odot} \right] R_t dT_\odot + \left( \frac{\partial R_2}{\partial R_e} \right) _{T_\odot} dR_e , \]

(24)

from where it is straightforward to derive

\[ \left( \frac{\partial R_2}{\partial T_\odot} \right) _{R_t} = R_2 L_\odot , \left( \frac{\partial R_2}{\partial R_e} \right) _{T_\odot} = \left( \frac{R_2}{R_\odot} \right)^2 . \]
The relation between $T_*$ and $R_1$ needs instead knowledge of the density radial profile in the envelope, which can be derived from the radiative transfer equation together with eq. 22

$$\frac{dp}{dr} = -3.25 \frac{\rho}{T} r^2.$$  

(25)

Therefore,

$$\frac{dR_1}{dT_*} = \left(\frac{dp}{dr}\right)^{-1} \frac{dR}{dT_*} = -\frac{3}{6.5} \frac{R_2^2}{A}$$  

(26)

where we used eq. 22 to evaluate $d\rho_r/dT_*$. We can finally insert all pieces in eq. 24 and derive the fractional increase of the stellar radius,

$$\frac{dR_2}{R_2} = \frac{7}{13} \frac{R_2 T_*}{T} \approx \frac{1.9}{A} R_2 T_* W$$  

(27)

The set of equations which comprises eq. 18, eq. 19, eq. 20 and eq. 27 allow us to self consistently track the viscous inflation of the mean stellar radius as a function of the orbital period. In Fig. 5 we show the radial inflation of the star as a function of the system separation for increasingly large viscous times.

The secondary’s radius in J0651 has been measured precisely: from the eclipses of its lightcurve, Brown et al. (2011) derive $R_2 = 0.0353 \pm 0.0004 R_\odot$, noting that this differs by 5% from the $0.0337 R_\odot$ radius predicted by the best-matching ELM He WD model (Panci et al. 2007). Given the high accuracy of the radius determination, this difference is significant at the 4σ level. If taken at face value this discrepancy can be interpreted as due to tidal heating of the white dwarf, with two possible values of the viscous time ($\approx 170$ kyr and $\approx 11$ kyr, cf. Fig. 5, upper panel) reflecting the double role of viscosity in determining $W$ (see discussion in section 4.2, and Fig. 3). For a WD mass of $\approx 0.25 M_\odot$ and radius $\approx 2.5 \times 10^6$ cm (Brown et al. 2011), these timescales correspond to an average $\mu \approx 9 \times 10^7$ g cm$^{-1}$ s$^{-1}$ and $\approx 1.5 \times 10^{11}$ g cm$^{-1}$ s$^{-1}$, respectively. This is orders of magnitude larger than the microscopic viscosity of the degenerate plasma (Kopal 1968, Durisen 1973), but somewhat smaller that the estimated viscosity of high-mass stars in X-ray binaries with circularized orbits (Suntanyo 1974). While fluid turbulence might play a role in WD interiors (Iben, Tutukov & Fedorova 1998), it is also plausible that the viscosity associated to a properly oriented internal magnetic field of $\approx 10^{-10} - 10^{-11}$ Gauss can suffice to reach this level of viscous dissipation (cf. Suntanyo 1974, Smarr & Blandford 1976).

Although the viscous timescale is not uniquely determined by the amount of dissipation, this degeneracy is completely removed by a joint determination of the degree of synchronisation of the white dwarf (Fig. 5 lower panel). In the above interpretation the secondary’s spin in J0651 would be expected to be $\approx 10$ times slower than the orbital period for the longer value of $t_\nu$, or to differ from it by only $\approx 20\%$, in the alternative case. Viscous dissipation would also enhance the WD luminosity by $\lesssim 0.05 L_\odot$, which corresponds to a $\sim (30-50)\%$ perturbation of its intrinsic luminosity. Hence, along with a $\sim 5\%$ increase of the outer radius, we would expect a $\sim (5-10)\%$ increase of the effective temperature.

The above argument can be reversed to conclude that if the coefficient of dynamic viscosity were $\mu > 10^{12}$ g cm$^{-1}$ s$^{-1}$, as previously proposed in the literature, then the secondary’s spin would be expected to be almost exactly synchronous, the amount of tidal dissipation would be much smaller than stated above and the inflation of the white dwarf envelope essentially negligible. Therefore measuring the spin of the tidally perturbed secondary in J0651 will allow a straightforward check of these predictions, reaching unambiguous conclusions about the magnitude and nature of the internal viscosity of white dwarfs.

In Figure 5 we plot also the fractional size of the Roche lobe with respect to $R_2$: $(r_L - R_2)/R_2$, where $r_L \approx 0.31 a$ (Eggelton 1983). We find that, due to the inflation of the outer envelope caused by tidal heating, the secondary fills its Roche lobe $(dR_2 \approx r_L)$ when the binary has a period of $P \approx 9.7$ min and $P \approx 9$ min, for $t_\nu \approx 1.7 \times 10^7$ yr and $\approx 1.1 \times 10^7$ yr respectively. This is well before the tidal radius ($P \approx 56$ s) as defined in sec. 2.

5 DISCUSSION ON J0651 AND CONCLUSIONS

We have thoroughly addressed the consequences of tidal interactions in detached white dwarf binaries, developing a simple and general formalism that includes the effects of viscosity and of GW-driven orbital evolution. The main physical implications of our scenario were illustrated quantitatively with a focus on short period systems, where tidal effects are most pronounced.

When applied to J0651, an ELM system with an orbital period of only $\sim 12.8$ min, our analysis shows the full potential of tidal interactions to reveal the internal properties of white dwarfs. We have demonstrated that the large tidal deformation of the secondary star in J0651 can only be explained by including, in the tidal interaction energy, a term that depends on the star’s “deformability”, measured by the Love number ($h_2$). This provides a direct observational test of the formulae derived here and in Paper I, hence also of the expressions for the tidal radius (see § 2). This approach will make it possible to probe the white dwarf internal structure when a sample of tidally distorted objects will be available, ranging in mass and orbital separations.

Further developing on the system’s dynamics we have illustrated the role of the internal viscous torque, and of the GW-driven
torque, in driving tidal coupling, hence the exchange of angular momentum between the orbit and the secondary star, and the double role of viscous dissipation in determining the synchronisation causes the expansion of the non-degenerate stellar envelope, which momentum between the orbit and the secondary star, and the double role of viscous dissipation in determining the synchronisation causes the expansion of the non-degenerate stellar envelope, which

**Figure 6. Upper panel:** fractional change of the outer radius of the secondary in J0651, as a function of the viscous timescale in unit of Myr. The vertical dashed lines mark the two values τ_v = 1.7 × 10^5 yr and τ_v = 1.1 × 10^4 yr, that give a radial inflation of 5% (horizontal dashed line). **Lower panel:** Asynchronism as a function of viscous time in Myr, for J0651. The horizontal dashed lines mark the asynchronism expected for the two viscous timescales (vertical dashed lines) for which the radial inflation of the secondary in J0651 is 5% (see upper panel).

**REFERENCES**

Bildsten, L., & Cutler, C. 1992, Ap. J., 400, 175
Bildsten, L., Shen, K. J., Weinberg, N. N., & Nelemans, G. 2007, Ap. J. Lett., 662, L95
Brown, W. R., Kilic, M., Allende Prieto, C., & Kenyon, S. J. 2010, Ap. J. Lett., 737, L23
Brown, W. R., Kilic, M., Hermes, J. J., et al. 2011, Ap. J. Lett., 737, L23
Dall’Osso, S., & Rossi, E. M. 2013, Mon. Not. Roy. Astro. Soc., 428, 518
Durisen, R. H. 1973, Ap. J., 183, 215
Eggleton, P. 1983, Ap. J., 268, 368
Hinderer, T. 2008, Ap. J., 677, 1216
Iben, I., Jr., & Tutukov, A. V. 1984, Ap. J. Suppl., 54, 335
Iben, I., Jr., Tutukov, A. V., & Fedorova, A. V. 1998, Ap. J., 503, 344
Kilic, M., Brown, W. R., Allende Prieto, C., Kenyon, S. J., & Paneli, J. A. 2010, ApJ, 716, 122
Kilic, M., Brown, W. R., Hermes, J. J., et al. 2011a, Mon. Not. Roy. Astro. Soc., 418, L157
Kilic, M., Brown, W. R., Kenyon, S. J., et al. 2011b, Mon. Not. Roy. Astro. Soc., 413, L101
Kilic, M., Brown, W., Allende Pietro, C., Kenyon, S. J.; Heinke, C. O.; Ageros, M. A. & Kleinman, S. J., 2012, Ap. J., 751, 141
Kopal, Z. 1968, Ap&SS, 1, 179
Love, A. E. H. 1909, Mon. Not. Roy. Astro. Soc., 69, 476
Panei, J. A., Althaus, L. G., Chen, X., & Han, Z. 2007, Mon. Not. Roy. Astro. Soc., 382, 779
Press, W. H., Smarr, L. L., & Wiita, P. J. 1975, Ap. J. Lett., 202, L135
Shapiro, S. L., & Teukolsky, S. A. 1983, Research supported by the National Science Foundation. New York, Wiley-Interscience, 1983, 663 p.,
Smarr, L. L., & Blandford, R. 1976, Ap. J., 207, 574
Sutantyo, W. 1974, Astron. Astrophys., 35, 251
Verbunt, F., & Hut, P. 1983, Astron. Astrophys., 127, 161
Webbink, R. F. 1984, Ap. J., 277, 355
Zahn, J. P. 1966, Annales d’Astrophysique, 29, 313
Zahn, J.-P. 1977, Astron. Astrophys., 57, 383

**ACNOWLEDGEMENTS**

SD was supported by an ERC Advanced Research Grant and by the Israeli Center for Excellence for High Energy Astrophysics.

**ACKNOWLEDGEMENTS**

SD was supported by an ERC Advanced Research Grant and by the Israeli Center for Excellence for High Energy Astrophysics.