Transfer Matrix Method in Sandpile Models

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Abstract

We present a transfer matrix method which is particularly useful for solving some classes of sandpile models. The method is then used to solve the deterministic nonabelian sandpile models for N=2 and N=3. The possibility of generalization to arbitrary N is discussed briefly.

numbers: 11.30.Er, 14.80.Er
I. INTRODUCTION

Sandpile model is the simplest model of self-organized criticality (SOC). This kind of models were first discussed by Bak, Tang, and Wiensenfeld [1]. They used numerical method to study some of the sandpile models and observed that the models automatically evolve into a self-organized critical state when they possess $1/f$ spectra in both spatial and temporal distributions of certain physical quantities. They suspected that SOC maybe an universal characteristic underlying the the nonlinear dispersive systems such as earth quakes, forest fire, turbulence, etc, which are prevalent in nature.

The simplest sandpile model is a cellular automata on an one dimensional lattice with a height number $h_i$ assigned to each site. There are two basic operations of the model - dropping and toppling. Dropping means that one sand is added at some site of the lattice, i.e. $D_i : h_i \rightarrow h_i + 1$. Toppling occurs when a slope (defined as the difference in height between adjacent sites) exceeds some critical value. If toppling occurs at one site, some sands at the site will be moved to the other sites which may trigger further topplings.

Though the rules of evolution of sandpile model are typically very simple, it is hard to solve them analytically when the degrees of freedom become very large. Most researcher handle the models by numerical simulation [2] [3] [4]. If the rules are such that the evolution of the system is independent of the order of the droppings, then the model is called abelian. For a large class of abelian models some exact results have been obtained by Dhar et al. [5] The non-abelian ones are harder to solve and there exists little exact result. In a previous work [6], one of the authors investigated a class of non-abelian sandpile models and was able to solve the model in the deterministic case when the sand is dropped at a fixed site.

In this note, we wish to present a new method of solving this class of sandpile models. We consider the one dimensional case and label the sites from left to right as 1 to $L$. The sand is dropped only at the site 1. If the slope at a site exceeds a given number $N$, then the sand will topple to the right. Let the slope $\sigma_i = h_i - h_{i+1}$, then the toppling rule is “if $\sigma_i > N$, then $\sigma_{i-1} \rightarrow \sigma_{i-1} + N$, $\sigma_i \rightarrow \sigma_i - (N + 1)$, and $\sigma_{i+N} \rightarrow \sigma_{i+N} + 1$”. The rule should be
modified when toppling occurs near the boundary. The condition at the left boundary is trivial. When sands reach beyond the right boundary they drop out from the system, i.e. we keep $h_i = 0$ for $i > L$. A state in which all $\sigma_i \leq N$ is called a stable state. Toppling stops when a stable state is reached. Each dropping and subsequent toppling processes will result in the transition from one stable state to another. Since there are only a finite number of different states in the system, after dropping enough sand at site 1, the system will step into a cycle called the limit cycle. For the system that we consider here there is only one limit cycle in the problem. The number of different states in the limit cycle is $N^L$. In the following, all the states we will refer to are the stable states.

The method to be introduced here is particularly useful for those models in which the structure of the limit cycles has been worked out. Once the cycle structure is known, its information can be succinctly summarized in a matrix which we call the transfer matrix by analogy with the similar matrix in statistical mechanics. For the deterministic model defined above, the structure of the limit cycle has been worked out in ref. [6]. Therefore we should use it as our main example, the method may be applicable to a much wider classes of models.

II. THE DEFINITION OF TRANSFER MATRIX

A state is in the limit circle if and only if

1. There exists at least one site $i$ for any consecutive $N$ sites such that $\sigma_i = N$.

2. There exists a site $i$ satisfying $L - \sigma_L \leq i < L$ such that $\sigma_i = N$.

Since the first condition is for $N$ consecutive sites, there is no constraint in consecutive $N - 1$ sites. Choose the complete set of states for consecutive $N - 1$ sites as a basis. For example in the $N = 2$ model, the states in the basis is $|0\rangle, |1\rangle, |2\rangle$. For $N = 3$ model, the basis is of the form $|33\rangle, |a3\rangle, |3a\rangle, |ab\rangle$, where $a, b$ belongs to $\{0, 1, 2\}$. For arbitrary $N$,
there are \((N+1)^{(N-1)}\) states in the basis. Define the transfer matrix as a mapping between 

\[ |\eta_i\rangle \equiv |\sigma_i\sigma_{i+1} \cdots \sigma_{i+N-2}\rangle \quad \text{and} \quad |\eta_{i+1}\rangle \equiv |\sigma_{i+1}\sigma_{i+2} \cdots \sigma_{i+N-1}\rangle \] 

such that

\[
\langle \eta_i | T | \eta_{i+1} \rangle = \begin{cases} 
1 & \text{if the sequence } \sigma_i\sigma_i + 1 \cdots \sigma_{i+N-1} \text{ satisfies constrain } 1 \\
0 & \text{otherwise}
\end{cases}.
\] (1)

For example, in the \(N=2\) model, if \(\sigma_{i+1} = 2\), the \(\sigma_i\) could be 0, 1, 2, so \(\langle 0 | T | 2 \rangle = \langle 1 | T | 2 \rangle = \langle 2 | T | 2 \rangle = 1\). If \(\sigma_i = 2\), then \(\sigma_{i+1}\) could be 0, 1, , 2, so that \(\langle 2 | T | 0 \rangle = \langle 2 | T | 1 \rangle = 1\). For other cases, \(\langle f | T | i \rangle = 0\).

Let \(|0\rangle = \text{col}(1,0,0)\), \(|1\rangle = \text{col}(0,1,0)\), \(|2\rangle = \text{col}(0,0,1)\), then we can write the transfer matrix in the following form:

\[
T = \begin{bmatrix} 
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 
\end{bmatrix}.
\] (2)

Then the number of states starting with \(|\sigma_i \cdots \sigma_{i+N-2}\rangle\), and ending with \(|\sigma_j \cdots \sigma_{j+N-2}\rangle\) in one limit circle is \(\langle \sigma_i \cdots \sigma_{i+N-2} | T^{(j-i)} | \sigma_j \cdots \sigma_{j+N-2} \rangle\).

The transfer matrix contains all the information about the time average of space dependent functions, such as the one point function \(\langle \sigma_i \rangle\), and the two point function \(\langle \sigma_i \sigma_j \rangle\). Here \(\langle \cdot \cdot \cdot \rangle\) means the average in time over a limit circle. Define \(I = \{ \text{ the complete set of states } |\sigma_{L-N+2} \cdots \sigma_L\rangle \text{ allowed by the boundary condition (2) } \}, \ A = \{ \text{ complete set of states } |\sigma_1 \cdots \sigma_{N-1}\rangle \text{ in general } \}\). The one point function can be evaluated in terms of the summation of all states in a limit circle, i.e.

\[
\langle \sigma_i \rangle = \frac{1}{N_L} \sum_{\eta_i} \sum_{\alpha \in A} \sum_{\beta \in I} \langle \alpha | T^{i-1} | \eta_i \rangle \sigma_i \langle \eta_i | T^{L-i} | \beta \rangle. = \frac{1}{N_L} \sum_{\alpha \in A} \sum_{\beta \in I} \langle \alpha | T^{i-1} E T^{L-i} | \beta \rangle.
\] (3)

where we have introduced the ”evaluation” matrix \(E = \sum_{\eta_i} |\eta_i\rangle \sigma_i \langle \eta_i|\) to signify the fact that the sum over \(\eta_i\) can be expressed as a matrix independent of the position \(i\). Similarly, the two point function is

\[
\langle \sigma_i \sigma_j \rangle = \frac{1}{N_L} \sum_{\eta_i,\eta_j} \sum_{\alpha \in A} \sum_{\beta \in I} \langle \alpha | T^{i-1} | \eta_i \rangle \sigma_i \langle \eta_i | T^{j-i} | \eta_j \rangle \sigma_j \langle \eta_j | T^{L-j} | \beta \rangle
\]

\[
= \frac{1}{N_L} \sum_{\alpha \in A} \sum_{\beta \in B} \langle T^{i-1} E T^{j-i} E T^{L-j} | \beta \rangle.
\] (4)
If the power of $T$ can be evaluated, then it is easy to get these functions. The power of $T$ can be evaluated by the diagonalization of $T$. Alternatively, the direct multiplication of $T$ will show iterative relations which maybe easier to solve and will be used in the following section.

III. $N = 2$ SANDPILE MODEL

For the $N = 2$ model, $T^n$ has the general form

$$T^n = \begin{bmatrix} a_n & a_n & b_n \\ a_n & a_n & b_n \\ b_n & b_n & 2a_n + b_n \end{bmatrix}$$  \hspace{1cm} (5)

The iterative relation can be obtained by $T^{n+1} = TT^n$, it is $a_{n+1} = b_n$ and $b_{n+1} = 2a_n + b_n$. Therefore both of them satisfy the iterative relation

$$u_{n+1} = u_n + 2u_{n-1}.$$ \hspace{1cm} (6)

The initial condition of these equations can be calculated from $T, T^2, T^3$ by direct multiplication. They are $a_1 = 0$, $a_2 = 1$, $a_3 = 1$ and $b_1 = 1$, $b_2 = 1$, $b_3 = 3$. The general form of $a_n$ and $b_n$ are $a_n = b_{n-1} = \frac{1}{3} \left( 2^{n-1} - (-1)^{n-1} \right)$. Substitute $a_n$ and $b_n$ into (3),(4), the results are (for $i < j$)

$$\langle \sigma_i \rangle = 3/2 + (-1)^i 2^{-i-1}$$ \hspace{1cm} (7)

$$\langle \sigma_i \sigma_j \rangle = 9/4 + (-1)^{-i+j} 2^{i-j-1} + 3 (-1)^{-i} 2^{-i-2} + (-1)^{j} 2^{-j-1}$$ \hspace{1cm} (8)

Then two point correlation function is (for $i < j$)

$$\langle \sigma_i \sigma_j \rangle_c = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle = (-1)^{-i+j} 2^{i-j-1} + (-1)^{j+1} 2^{-j-2} + (-1)^{i+j+1} 2^{-i-j-2}$$ \hspace{1cm} (9)

These results agree with [6], however the derivation here is much simpler and more general. The three point function can be evaluated in a similar way. We got the three point function (for $i < j < k$)
\[ \langle \sigma_i \sigma_j \sigma_k \rangle = 3 (-1)^{-j+k} 2^{j-k-2} + (-1)^{-k} 2^{-k-1} + 3 (-1)^{-i+j} 2^{i-j-2} + 9 (-1)^{-i} 2^{-i-3} + 3 (-1)^{-j} 2^{-j-2} + (-1)^{-i+k} 2^{i-k-1} + (-1)^{i-j+k} 2^{-i+j-k-2} + 27/8. \] (10)

The three point correlation function is (for \( i < j < k \))

\[ \langle \sigma_i \sigma_j \sigma_k \rangle_c = \langle \sigma_i \sigma_j \sigma_k \rangle - \langle \sigma_i \sigma_j \rangle_c \langle \sigma_k \rangle - \langle \sigma_i \sigma_k \rangle_c \langle \sigma_j \rangle - \langle \sigma_j \sigma_k \rangle_c \langle \sigma_i \rangle - \langle \sigma_i \rangle_c \langle \sigma_j \rangle \langle \sigma_k \rangle \\
= (-1)^k 2^{-k-3} + (-1)^{k+j} 2^{-k-j-2} + (-1)^{i+k+1} 2^{-i-k-2} \]
\[ + (-1)^{i+k} 2^{-i-k-3} + (-1)^{i+k+j} 2^{-i-k-j-2} + (-1)^{-i+k+j+1} 2^{i-k-j-1} \] (11)

Noting that \( h_i = \sum_{k=i+1}^{L} \sigma_i \), we also get the L dependence of the correlation function of height exactly as follows

\[ \frac{1}{L} \sum_{i=1}^{L} \left( \langle h_i^2 \rangle - \langle h_i \rangle^2 \right) = \frac{5L}{12} - \frac{1}{36} + \frac{2}{5L} - \frac{1}{9 \cdot 2^{2L}} - \frac{2}{9L \cdot 2^{2L}} - \frac{1}{9 \cdot 2^{4L}} - \frac{8}{45L \cdot 2^{4L}} - \frac{L}{6 \cdot 2^{6L}} \] (12)

For large \( L \), \( (1/L) \sum_{i=1}^{L} \left( \langle h_i^2 \rangle - \langle h_i \rangle^2 \right) \propto L \), which agrees with what is expected qualitatively from the random walk argument in [7].

**IV. N = 3 SANDPILE**

For the case of \( N = 3 \), the situation is a little more complicated. The size of the matrix \( T \) become larger. \( T \) is a \( 4^2 \) by \( 4^2 \) matrix and the calculation is somewhat more elaborate. The process of solving the case of \( N = 3 \) will show more structure of the transfer matrix which will be useful when dealing with the cases of \( N \geq 3 \). The elements of \( T \) for \( N = 3 \) are \( \langle ab | T | cd \rangle = \delta_{bc} (1 - (1 - \delta_{3a})(1 - \delta_{3b})(1 - \delta_{3d})) \) by the constraint ([I]).

There is an obvious block structure of \( T \) for \( N = 3 \) model. In order to represent this block structure succinctly, we shall adopt the following basis which can be generalized to arbitrary \( N \) later. Let \( b_i \) be the column vector with component \( (b_i)_j = \delta_{ij} \). Assign \( b_1 \cdots b_9 \) to \( |ab\rangle \), \( b_{10} \cdots b_{12} \) to \( |a3\rangle \), \( b_{13} \cdots b_{15} \) to \( |3b\rangle \), and \( b_{16} \) to \( |33\rangle \), where \( a, b = 0, 1, 2 \). The order of blocks is given by the successive sequence of a binary expression if one replaces 3 by 1 and \( a,b \).
by 0. The relative order between different $|ab\rangle$s or $|a3\rangle$s, $|3a\rangle$s is not important, but we take the order as the number sequence in base 3 for convenience. Let $A = \{|ab\rangle|a, b = 0, 1, 2\}$, $B = \{|a3\rangle|a = 0, 1, 2\}$, $C = \{|3a\rangle|a = 0, 1, 2\}$, and $D = \{|33\rangle\}$ to be the subspace of original space spanned by whole vector in the basis.

The block structure for $N = 3$ model becomes explicit in $T^n$, $n \geq 3$, i.e., the matrix of $T^n$ composes of the “saturated” block. A matrix is called “saturated” if all the elements are equal. Let $\Sigma^{m \times n}$ be the “saturated” matrix of $m$ rows and $n$ columns with all elements equal to 1. Then $T^n$ can be expressed as

$$
T^n = \begin{bmatrix}
A_9(|ab\rangle) & B_3(|a3\rangle) & C_3(|3a\rangle) & D_1(|33\rangle) \\
A_9(|ab\rangle) & b_{n-2}\Sigma^9 & a_{n-1}\Sigma^9 & b_{n-1}\Sigma^9 & b_{n-1}\Sigma^9 \\
B_3(|a3\rangle) & b_{n-1}\Sigma^3 & a_n\Sigma^3 & b_n\Sigma^3 & b_n\Sigma^3 \\
C_3(|3a\rangle) & a_{n-1}\Sigma^3 & c_n\Sigma^3 & a_n\Sigma^3 & a_n\Sigma^3 \\
D_1(|33\rangle) & b_{n-1}\Sigma^3 & a_n\Sigma^3 & b_n\Sigma^3 & b_n\Sigma^3 
\end{bmatrix},
$$

(13)

where $a_n, b_n, c_n$ are called block coefficients satisfying the same iterative relation

$$
u_{n+1} = u_n + 3u_{n-1} + 9u_{n-2}
$$

(14)

but with different initial conditions. The initial conditions are

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 4$$

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 4$$

$$c_2 = 1, \quad c_3 = 1, \quad c_4 = 7.$$

The block structure helps in calculation. If the trivial $\Sigma^{m \times n}$ matrix is ignored, we can use a 4 by 4 matrix instead of 16 by 16 to represent it. This is the basic idea of ‘reduced transfer matrix’ which will be discussed later.

The one point function and two point function can be worked out in the same way as the $N = 2$ case. They are

$$
\langle \sigma_i \rangle = 2 + \frac{3^{-i}(\omega^i + \overline{\omega}^i)}{2}
$$

(15)

$$
\langle \sigma_i \sigma_j \rangle = \frac{1}{4} \left( [3^{-i}\omega^{2i-j} + 2\omega^{i-j} + 3\omega^{-j} + 4\omega^{-i}] + \text{h.c.} \right) + 4,
$$

(16)
where \( \omega = -1 + \sqrt{-2} \). The two point correlation function is (for \( i < j \))

\[
\langle \sigma_i \sigma_j \rangle_c = \langle \sigma_i \rangle \langle \sigma_j \rangle - \frac{1}{4} \left[ 3^{-i} \omega^{2i-j} + \left( 2 - 3^{-i} \right) \omega^{i-j} - \omega^{-j} - \omega^{-i-j} \right] + \text{h.c}. \tag{17}
\]

Since the structure of the block submatrix in the transfer matrix is simple, the original matrix can be reduced to a simpler ‘reduced transfer matrix’ to represent the iteration relation. We use capital letter to label the block, for example, \( T_{AB} \) means the submatrix which maps the subset \( A \) of basis into \( B \). The transfer matrix has the following special property which produces the block structure in \( T^n \) for \( n \geq 3 \). The sum of the elements in a row of a submatrix, \( T_{AB} \), is independent of which row one sums over. Define the sum of the elements in any of the row of \( T_{AB} \) as \( \tilde{T}_{AB} \). Note that we have managed to reduce each submatrix \( T_{AB} \) to a number \( \tilde{T}_{AB} \). Using these reduced matrices, one can simplify the multiplication between \( T \) and \( T^n (n \geq 3) \) by inventing a reduced matrix \( R^n \) corresponding to each \( T^n \) by ignoring the trivial \( \Sigma \) matrix. Denote the block coefficient of \( T^n_{JK} \) as \( R^n_{JK} \). Note that \( R^n \) is a 4 by 4 matrix for \( N = 3 \) case. One can easily show that \( R^{n+1} = \tilde{T} R^n \).

The \( \tilde{T} \) for \( N = 3 \) model is

\[
\tilde{T} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 3 & 1 \\
3 & 1 & 0 & 0 \\
0 & 0 & 3 & 1
\end{bmatrix} \tag{18}
\]

Multiply Eqn. (18) and the reduced matrix corresponding to Eqn. (13) we can get the iterative relation between \( a_n, b_n \) and \( c_n \) easily. The reduced transfer matrix simplifies the original 16 by 16 matrix into to 4 by 4 matrix without losing the information of iterative relation. By reduced transfer matrix, the iteration between \( a_n, b_n, c_n \) can be written as

\[
a_{n+1} = a_n + 3c_n \\
b_{n+1} = 3a_n + b_n \\
c_{n+1} = 3a_{n-1} + a_n
\]
It can be shown that \( a_n, b_n, c_n \) satisfying Eqn.(14). To solve this equation, we have to solve the polynomial equation \( x^3 = x^2 + 3x + 9 \).

For arbitrary \( N \), the reduced transfer matrix is a \( 2^{(N-1)} \) by \( 2^{(N-1)} \) matrix, and they take the form

\[
\begin{bmatrix}
\Delta_1^N & \Delta_2^N & \Delta_3^N & \cdots & \Delta_{2^{N-2}}^N \\
\Delta_1^N & \Delta_2^N & \Delta_3^N & \cdots & \Delta_{2^{N-2}}^N
\end{bmatrix}
\]

(19)

where the \( \Delta_i^N \) and \( \Delta_i'^N \) are submatrices of the reduce transfer matrix with 2 columns and \( 2^{N-2} \) rows. \( (\Delta_i^N)_{pq} = \delta_{ip}(N\delta_{q1} + \delta_{q2}) \). \( (\Delta_i'^N)_{pq} = \delta_{ip}(\delta_{q2}) \). The eigenvalues of reduce transfer matrix plays an important role in solving the model. It can be proved that the eigenvalues of reduced matrix are roots of the equation \( x^N = \sum_{i=0}^{N-1} N^i x^{N-1-i} \). The correlation function can be expressed as the combination of these eigenvalues. This work is in progress.

V. DISCUSSIONS

We have introduced the concept of transfer matrix into the study of steady properties of sandpile models. This is closely related to the Hamiltonian formulation of the usual statistical mechanics. Indeed, one may regard the formulas for the correlation function such as equation (3) and (4) as a “path integral” expressions in a discrete formulation. The transfer matrix plays the role of the evolution operator.

The usefulness of the transfer matrix formulation was illustrated by deriving the one-, two- and three-point correlation functions for a deterministic sandpile model with the critical slope \( N = 2 \) as well as the one- and two-point functions for the same model with \( N = 3 \). In the latter case, we found that the two-point function decreases exponentially as the separation of the two point increases with a correlation length of \( (\ln \sqrt{3})^{-1} \) in the unit of lattice spacing. For the \( N = 2 \) case, the correlation length is \( (\ln 2)^{-1} \). It will be interesting to see if the correlation length become infinity in the large \( N \) limit so that the self-organize criticality in the spatial correlation is restored in the limit.
This work is supported in part by grants from the National Science Council of Taiwan-Republic of China under the contract number NSC-83-0208-M001-069 and the contract number NSC-83-0208-M007-117T.
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