Focus-style proof systems and interpolation for the
alternation-free $\mu$-calculus

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Abstract

In this paper we introduce a cut-free sequent calculus for the alternation-free fragment of the modal $\mu$-calculus. This system allows for cyclic proofs and uses a simple focus mechanism to control the unravelling of fixpoints along infinite branches. We show that the proof system is sound and complete and apply it to prove that the alternation-free fragment has the Craig interpolation property.

1 Introduction

In this paper we present a circular proof system for the alternation-free fragment of the modal $\mu$-calculus and use this system to proof Craig interpolation for the alternation-free fragment.

1.1 The alternation-free $\mu$-calculus

The modal $\mu$-calculus, introduced by Kozen [21], is a logic for describing properties of processes that are modelled by labelled transition systems. It extends the expressive power of propositional modal logic by means of least and greatest fixpoint operators. This addition permits the expression of all monadic second-order properties of transition systems [18]. The $\mu$-calculus is generally regarded as a universal specification language, since it embeds most other logics that are used for this purpose, such as LTL, CTL, CTL* and PDL.

The alternation-free $\mu$-calculus is a fragment of the $\mu$-calculus in which there is no interaction between least and greatest fixpoint operators. It can be checked that the translations of both CTL and PDL into the $\mu$-calculus yield alternation-free formulas. Over tree structures, or when restricted to bisimulation-invariant properties, the expressive power of the alternation-free $\mu$-calculus corresponds to monadic second-order logic where the quantification is restricted to sets that are finite, or in a suitable sense well-founded [30, 13]. For more restricted classes of structures, such as for instance infinite words, it can be shown that the alternation-free fragment already has the same expressivity as the full $\mu$-calculus [20, 17].

Many theoretical results on the modal $\mu$-calculus depend on the translation from formulas in the $\mu$-calculus to automata [18, 30]. The general idea is to construct for every formula an automaton that accepts precisely the pointed structures where the formula is true. For the alternation-free fragment the codomain of this translation can be taken to consist of weak alternating automata [14, 17]. These are parity automata for which the assignment of priorities to states is restricted such that all states from the same strongly connected component have the same priority.
1.2 A cyclic focus system for the alternation-free $\mu$-calculus

In the theory of the modal $\mu$-calculus automata- and game-theoretic approaches have long been at the centre of attention. Apart from the rather straightforward tableau games by Niwiński & Walukiewicz [31] there have for a long time been few successful applications of proof-theoretic techniques. This situation has changed with a recent breakthrough by Afshari & Leigh [1], who obtain completeness of Kozen’s axiomatization of the modal $\mu$-calculus using purely proof-theoretic arguments. The proof of this result can be taken to consist of a series of proof transformations: First, it starts with a successful infinite tableau in the sense of [31]. Second, one then adds a mechanism for annotating formulas that was developed by Jungteerapanich and Stirling [19, 33] to detect after finitely many steps when a branch of the tableau tree may develop into a successful infinite branch, thus obtaining a finite but cyclic tableau. Third, Afshari & Leigh show how to apply a series of transformations to this finite annotated tableau to obtain a proof in a cyclic sequent system for the model $\mu$-calculus. Fourth, and finally, this proof can be turned into a Hilbert-style proof in Kozen’s axiomatization.

In this paper we present an annotated cyclic proof system for the alternation-free $\mu$-calculus that corresponds roughly to the annotated tableaux of Jungteerapanich and Stirling mentioned in the second step above. But, whereas in the system for the full $\mu$-calculus these annotations are sequences of names for fixpoint variables, for the alternation-free fragment it suffices to annotate formulas with just one bit of information. We think of this bit as indicating whether a formula is in what we call in focus or whether it is unfocused. We use this terminology because our proof system for the alternation-free $\mu$-calculus is a generalization of the focus games for weaker fixpoint logics such as LTL and CTL by Lange & Stirling [24]. These are games based on a tableau such that at every sequent of the tableau there is exactly one formula in focus. In our system we generalise this so that a proof node may feature a set of formulas in focus.

Our system contains only two proof rules that manipulate annotations. The first is the rule that unfolds least fixpoints. Whenever one is unfolding a least fixpoint formula that is in focus at the current sequent then its unfolding in the sequent further away from the root needs to be unfocused. Unfolding greatest fixpoints has no influence on the annotations. The other rules that affects the focus is a special focus rule. It can only be applied if the current sequence does not contain any formula that is in focus. The rule then simply continues the proof search with the same formulas but now they are all in focus.

The design of the annotation mechanisms in the tableau by Jungteerapanich & Stirling and in the focus system from this paper are heavily influenced by ideas from automata theory. It was already observed by Niwiński & Walukiewicz [31] that a tree automaton can be used that expects precisely the trees that encode successful tableaux. This automaton is the product of a tree automaton checking for local consistency of the tableau and a deterministic automaton over infinite words that detects whether every branch in the tableau is successful. That a branch in a tableau is successful means that it carries at least one trail of formulas where the most significant fixpoint that is unravelled infinitely often is a greatest fixpoint. It is relatively straightforward to give a nondeterministic automaton that detects successful branches, but the construction needs a deterministic automaton, which is obtained using the Safra construction [32]. The crucial insight of Jungteerapanich & Stirling [19, 33] is that this deterministic automaton that results from the Safra construction can be encoded inside the tableau by using annotations of formulas.

The relation between detecting successful branches in a proof and the determinization of automata on infinite words can also be seen more directly. In the proof system annotations are used to detect whether a branch of the proof carries at least one trail such that the most significant fixpoint that is unfolded infinitely often on the trail is a greatest fixpoint. This is analogous to a problem that arises when one tries to use the powerset construction to construct an equivalent deterministic automaton from a given non-deterministic parity automaton operating on infinite words. The problem there is to determine whether a sequence of macrostates of the deterministic automaton carries a run of the
original non-deterministic automaton that satisfies the parity condition. It is possible to view the
annotated sequents of Jungteerapanich & Stirling as a representation of the Safra trees which provide
the states of a deterministic Muller automaton that one obtains when determinizing a non-deterministic
parity automaton [19, sec. 4.3.5].

For alternation-free formulas it is significantly simpler to detect successful branches, because one
can show that the fixpoints that are unravelled infinitely often on a trail of alternation-free formulas
are either all least or all greatest fixpoints. One can compare the problem of finding such a trail
to the problem of recognising a successful run of a non-deterministic weak stream automaton in the
macrostates of a determinization of the automaton. In fact the focus mechanism from the proof
that we develop in this paper can also be used to transform a non-deterministic weak automaton
into an equivalent deterministic co-Büchi automaton. This relatively simple construction is a special
case of Theorem 15.2.1 in [9], which shows that every non-deterministic co-Büchi automaton can be
transformed into an equivalent deterministic co-Büchi automaton.

1.3 Interpolation for the alternation-free \(\mu\)-calculus

We apply the proof system introduced in this report to prove that the alternation-free \(\mu\)-calculus
has Craig’s interpolation property. This means that for any two alternation-free formulas \(\varphi\) and \(\psi\)
such that \(\varphi \rightarrow \psi\) is valid there is an interpolant \(\chi\) of \(\varphi\) and \(\psi\) in the alternation-free \(\mu\)-calculus. An
interpolant \(\chi\) of \(\varphi\) and \(\psi\) is a formula which contains only propositional letters that occur in both \(\varphi\)
and \(\psi\) such that both \(\varphi \rightarrow \chi\) and \(\chi \rightarrow \psi\) are valid.

Basic modal logic [16] and the full \(\mu\)-calculus [8] have Craig interpolation. In fact both formalisms
enjoy a even stronger property called uniform interpolation, where the interpolant \(\chi\) only depends on
\(\varphi\) and the set of propositional letters that occur in \(\psi\) (but not on the formula \(\psi\) itself). Despite these
strong positive results, interpolation is certainly not guaranteed to hold for fixpoint logics. For instance,
even Craig interpolation fails for weak temporal logics or epistemic logics with a common knowledge
modality [25, 34]. Moreover, one can show that uniform interpolations fails for both PDL and for
the alternation-free \(\mu\)-calculus [8]. The argument relies on the observation that uniform interpolation
corresponds to the definability of bisimulation quantifiers. But, adding bisimulation quantifiers to
PDL, or the alternation-free fragment, allows the expression of arbitrary fixpoints and thus increases
the expressive power to the level of the full \(\mu\)-calculus. It is still somewhat unclear whether PDL has
Craig interpolation. Various proofs have been proposed, but they have either been retracted or still
wait for a proper verification [3, 4].

The uniform interpolation result for the modal \(\mu\)-calculus has been generalised to the wider setting
of coalgebraic fixpoint logic [26, 12], but the proofs known for these results are all automata-theoretic
in nature. Recently, however, Afshari & Leigh [2] pioneered the use of proof-theoretic methods in
fixpoint logics, to prove, among other things, a Lyndon-style interpolation theorem for the (full) modal
\(\mu\)-calculus. Their proof, however, does not immediately yield interpolation results for fragments of
the logic; in particular, for any pair of alternation-free formulas of which the implication is valid,
their approach will yield an interpolant inside the full \(\mu\)-calculus, but not necessarily one that is itself
alternation free. It is here that the simplicity of our focus-style proof system comes in.

Summarising our interpolation proof for the alternation-free \(\mu\)-calculus, we base ourselves on Maehara’s method, adapted to the setting of cyclic proofs. Roughly, the idea underlying Maehara’s method
is that, given a proof \(\Pi\) for an implication \(\varphi \rightarrow \psi\) one defines the interpolant \(\chi\) by an induction on
the complexity of the proof \(\Pi\). The difficulty in applying this method to cyclic proof systems is that
here, some proof leaves may not be axiomatic and thus fail to have a trivial interpolant. In particular,
a discharged leaf indicates an infinite continuation of the current branch. Such a leaf introduces a
fixpoint variable into the interpolant, which will be bound later in the induction. The crux of our
proof, then, lies in the the way that we handle the additional complications that arise in correctly
managing the annotations in our proof system, in order to make sure that these interpolants belong
1.4 Overview

This paper is organized as follows: The preliminaries about the syntax and semantics of the \( \mu \)-calculus and its alternation-free fragment are covered in Section 2. In Section 3 we present our version of the tableau games by Niwiński & Walukiewicz that we use later as an intermediate step in the soundness and completeness proofs for our proof system. In Section 4 we introduce our focus system for the alternation-free \( \mu \)-calculus and we prove some basic results about the system. The sections 5 and 6 contain the proofs of soundness and completeness of the focus system. In Section 7 we show how to use the focus system to prove interpolation for the alternation-free \( \mu \)-calculus.
2 Preliminaries

We first fix some terminology related to relations and trees and then discuss the syntax and semantics of the $\mu$-calculus and its alternation-free fragment.

2.1 Relations and trees

Given a binary relation $R \subseteq S \times S$, we let $R^{-1}$, $R^+$ and $R^*$ denote, respectively, the converse, the transitive closure and the reflexive-transitive closure of $R$. For a subset $S \subseteq T$, we write $R[S] := \{ t \in T \mid Rst \text{ for some } s \in S \}$; in the case of a singleton, we write $R[s]$ rather than $R[\{s\}]$. Elements of $R(s)$ and $R^{-1}(s)$ are called, respectively, successors and predecessors of $s$. An $R$-path of length $n$ is a sequence $s_0s_1 \cdots s_n$ (with $n \geq 0$ such that $Rs_i s_{i+1}$ for all $0 \leq i < n$); we say that such a path leads from $s_0$ to $s_n$. Similarly, an infinite path starting at $s$ is a sequence $(s_n)_{n \in \omega}$ such that $Rs_i s_{i+1}$ for all $i < \omega$.

A structure $T = (T, R)$, with $R$ a binary relation on $T$, is a tree if there is a node $r$ such that for every $t \in T$ there is a unique path leading from $r$ to $t$. The node $r$, which is then characterized as the only node in $T$ without predecessors, is called the root of $T$. Every non-root node $u$ has a unique predecessor, which is called the parent of $u$; conversely, the successors of a node $u$ are sometimes called its children. If $R^+tu$ we call $u$ a descendant of $t$ and, conversely, $t$ an ancestor of $u$; in case $R^*tu$ we add the adjective ‘proper’. If $s$ is an ancestor of $t$ we define the interval $[s, t]$ as the set of nodes on the (unique) path from $s$ to $t$. A branch of a tree is a path that starts at the root. A leaf of a tree is a node without successors. For nodes of a tree we will generally use the letters $s, t, u, v, \ldots$, for leaves we will use $l, m, \ldots$. The depth of a node $u$ in a finite tree $T = (T, R)$ is the maximal length of a path leading from $u$ to a leaf of $T$. The hereditarily finite part of a tree $T = (T, R)$ is the subset $HF(T) := \{ t \in T \mid R^*[t] \text{ is finite} \}$.

A tree with back edges is a structure of the form $(T, R, c)$ such that $c$ is a partial function on the collection of leaves, mapping any leaf $l \in \text{Dom}(c)$ to one of its proper ancestors; this node $c(l)$ will be called the companion of $l$.

2.2 The modal $\mu$-calculus and its alternation-free fragment

In this part we review syntax and semantics of the modal $\mu$-calculus and discuss its alternation-free fragment.

2.2.1 The modal $\mu$-calculus

Syntax The formulas in the modal $\mu$-calculus are generated by the grammar

$$\varphi ::= p \mid \mathbf{F} \mid \bot \mid T \mid (\varphi \lor \varphi) \mid (\varphi \land \varphi) \mid \square \varphi \mid \lozenge \varphi \mid \mu x \varphi \mid \nu x \varphi,$$

where $p$ and $x$ are taken from a fixed set $\text{Prop}$ of propositional variables and in formulas of the form $\mu x.\varphi$ and $\nu x.\varphi$ there are no occurrences of $x$ in $\varphi$. We write $\mu\text{ML}$ for the set of formulas in the modal $\mu$-calculus.

Formulas of the form $\mu x.\varphi$ ($\nu x.\varphi$) are called $\mu$-formulas ($\nu$-formulas, respectively); formulas of either kind are called fixpoint formulas. The operators $\mu$ and $\nu$ are called fixpoint operators. We use $\eta \in \{\mu, \nu\}$ to denote an arbitrary fixpoint operator and write $\overline{\mu} := \nu$ if $\eta = \mu$ and $\overline{\nu} := \mu$ if $\eta = \nu$. Formulas that are of the form $\square \varphi$ or $\lozenge \varphi$ are called modal. Formulas of the form $\varphi \land \psi$ or $\varphi \lor \psi$ are called boolean. Formulas of the form $p$ or $\mathbf{F}$ for some $p \in \text{Prop}$ are called literals and the set of all literals is denoted by $\text{Lit}$; a formula is atomic if it is either a literal or an atomic constant, that is, $T$ or $\bot$.  

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We use standard terminology for the binding of variables by the fixpoint operators and for substitutions. In particular we write \( FV(\varphi) \) for the set of variables that occur freely in \( \varphi \) and \( BV(\varphi) \) for the set of all variables that are bound by some fixpoint operator in \( \varphi \). We do count occurrences of \( x \) as free occurrences of \( x \). Unless specified otherwise, we assume that all formulas \( \varphi \in \mu ML \) are tidy in the sense \( FV(\varphi) \cap BV(\varphi) = \emptyset \). Given formulas \( \varphi \) and \( \psi \) and a propositional variable \( x \) such that there is no occurrences of \( x \) in \( \varphi \), we let \( \varphi[\psi/x] \) denote the formula that results from substituting all free occurrences of \( x \) in \( \varphi \) by the formula \( \psi \). We only apply this substitution in situations where \( FV(\psi) \cap BV(\varphi) = \emptyset \). This guarantees that no variable capture will occur. If the variable that is substituted is clear from the context we also write \( \varphi(\psi) \) for \( \varphi[\psi/x] \). An important use of substitutions of formulas are the unfolding of fixpoint formulas. Given a fixpoint formula \( \xi = \eta x. \chi \) its unfolding is the formula \( \chi[\xi/x] \).

Given a formula \( \varphi \in \mu ML \) we define its negation \( \overline{\varphi} \) as follows. First, we define the boolean dual \( \varphi^0 \) of \( \varphi \) using the following induction.

\[
\begin{align*}
\bot^0 & := T, & T^0 & := \bot, \\
(\overline{\varphi \lor \psi})^0 & := \varphi^0 \land \psi^0, & (\varphi \land \psi)^0 & := \varphi^0 \lor \psi^0, \\
(\square \varphi)^0 & := \varphi^0, & (\Box \varphi)^0 & := \varphi^0, \\
(\mu x. \varphi)^0 & := \nu x. \varphi^0 & (\mu x. \varphi)^0 & := \mu x. \varphi^0.
\end{align*}
\]

Based on this definition, we define the formula \( \overline{\varphi} \) as the formula \( \varphi^0[p = \overline{p} \mid p \in FV(\varphi)] \) that we obtain from \( \varphi^0 \) by replacing all occurrences of \( p \) with \( \overline{p} \), and vice versa, for all free proposition letters \( p \) in \( \varphi \).

Observe that if \( \varphi \) is tidy then so is \( \overline{\varphi} \).

For every formula \( \varphi \in \mu ML \) define the set \( \text{Clos}_0(\varphi) \) as follows

\[
\begin{align*}
\text{Clos}_0(p) & := \emptyset, & \text{Clos}_0(\overline{\varphi}) & := \emptyset, \\
\text{Clos}_0(\psi_0 \land \psi_1) & := \{ \psi_0, \psi_1 \}, & \text{Clos}_0(\psi_0 \lor \psi_1) & := \{ \psi_0, \psi_1 \}, \\
\text{Clos}_0(\Box \varphi) & := \{ \psi \}, & \text{Clos}_0(\varphi') & := \{ \psi \}, \\
\text{Clos}_0(\mu x. \psi) & := \{ \psi[\mu x. \psi/x] \}, & \text{Clos}_0(\nu x. \psi) & := \{ \psi[\nu x. \psi/x] \}.
\end{align*}
\]

If \( \psi \in \text{Clos}_0(\varphi) \) we sometimes write \( \varphi \rightarrowC \psi \). Moreover, we define the closure \( \text{Clos}(\varphi) \subseteq \mu ML \) of \( \varphi \) as the least set \( \Sigma \) containing \( \varphi \) that is closed in the sense that \( \text{Clos}_0(\psi) \subseteq \Sigma \) for all \( \psi \in \Sigma \). We define \( \text{Clos}(\Phi) = \bigcup_{\varphi \in \Phi} \text{Clos}(\varphi) \) for any \( \Phi \subseteq \mu ML \). It is well known that \( \text{Clos}(\Phi) \) is finite iff \( \Phi \) is finite.

A trace is a sequence \( (\varphi_n)_{n<\kappa} \), with \( \kappa \leq \omega \), of formulas such that \( \varphi_n \rightarrowC \varphi_{n+1} \), for all \( n \) such that \( n+1 < \kappa \). If \( \tau = (\varphi_n)_{n<\kappa} \) is an infinite trace, then there is a unique formula \( \varphi \) that occurs infinitely often on \( \tau \) and is a subformula of \( \varphi_n \) for cofinitely many \( n \). This formula is always a fixpoint formula, and where it is of the form \( \varphi_\tau = \eta x. \psi \) we call \( \tau \) an \( \eta \)-trace. A proof that there exists a unique such fixpoint formula \( \varphi \) can be found in Proposition 6.4 of [24], but the observation is well-known in the literature and goes back at least to [10]. A formula \( \varphi \in \mu ML \) is guarded if in every subformula \( \eta x. \psi \) of \( \varphi \) all free occurrences of \( x \) in \( \psi \) are in the scope of a modality. It is well known that every formula can be transformed into an equivalent guarded formula, and not hard to verify that all formulas in the closure of a guarded formula are also guarded.

**Semantics** The semantics of the modal \( \mu \)-calculus is given in terms of Kripke models \( S = (S, R, V) \), where \( S \) is a set whose elements are called worlds, points or states, \( R \subseteq S \times S \) is a binary relation on \( S \) called the accessibility relation and \( V : \text{Prop} \rightarrow \mathcal{P}S \) is a function called the valuation function. The meaning \( [\varphi]^S \subseteq S \) of a formula \( \varphi \in \mu ML \) relative to a Kripke model \( S = (S, R, V) \) is defined by
induction on the complexity of $\varphi$:

\[
[p]^S := V(p) \quad [\top]^S := S \setminus V(p) \\
[\varnothing]^S := \varnothing \quad [\perp]^S := S \\
[\varphi \lor \psi]^S := [\varphi]^S \cup [\psi]^S \\
[\varphi \land \psi]^S := [\varphi]^S \cap [\psi]^S \\
[\forall x.\varphi]^S := \bigcap \{U \subseteq S \mid [\varphi]^S_{[x=S]} \subseteq U\} \\
[\exists x.\varphi]^S := \bigcup \{U \subseteq S \mid [\varphi]^S_{[x=S]} \subseteq U\} \\
[\forall \psi]^S := S \setminus V(p) \\
[\exists \psi]^S := S
\]

Here, $S[x \mapsto U]$ for some $U \subseteq S$ denotes the model $(S, R, V')$, where $V'(x) = u$ and $V'(p) = V(p)$ for all $p \in \text{Prop}$ with $p \neq x$. We say that $\varphi$ is true at $s$ if $s \in [\varphi]^S$. A formula $\varphi \in \mu\text{ML}$ is valid if $[\varphi]^S = S$ holds in all Kripke models $S = (S, R, V)$ and two formulas $\varphi, \psi \in \mu\text{ML}$ are equivalent if $[\varphi]^S = [\psi]^S$ for all Kripke models $S$.

Alternatively, the semantics of the $\mu$-calculus is often given in terms of a so-called evaluation or model checking game. Let $\xi \in \mu\text{ML}$ be a $\mu$-calculus formula, and let $S = (S, R, V)$ be a Kripke model. The evaluation game $E(\xi, S)$ is the following infinite two-player game. Its positions are pairs of the form $(\varphi, s) \in \text{Clo}(\xi) \times S$, and its ownership function and admissible rules are given in Table 1. For the winning conditions of this game, consider an infinite match of the form $\Sigma = (\varphi_n, s_n)_{n<\omega}$; then we define the winner of the match to be Eloise if the induced trace $(\varphi_n)_{n<\omega}$ is a $\nu$-trace, and Abelard if it is a $\mu$-trace. It is well-known that this game can be presented as a parity game, and as such it has positional determinacy.

| Position | Player | Admissible moves |
|----------|--------|------------------|
| $(p, s)$ with $p \in FV(\xi)$ and $s \in V(p)$ | $\forall$ | $\varnothing$ |
| $(p, s)$ with $p \in FV(\xi)$ and $s \notin V(p)$ | $\exists$ | $\varnothing$ |
| $(\top, s)$ with $p \in FV(\xi)$ and $s \in V(p)$ | $\exists$ | $\varnothing$ |
| $(\perp, s)$ with $p \in FV(\xi)$ and $s \notin V(p)$ | $\forall$ | $\varnothing$ |
| $(\varphi \land \psi, s)$ | $\exists$ | $\{(\varphi, s), (\psi, s)\}$ |
| $(\varphi \lor \psi, s)$ | $\forall$ | $\{(\varphi, s), (\psi, s)\}$ |
| $(\forall \psi, s)$ | $\exists$ | $\{(\varphi, t) \mid sRt\}$ |
| $(\exists \psi, s)$ | $\forall$ | $\{(\varphi, t) \mid sRt\}$ |
| $(\forall x.\varphi, s)$ | $-$ | $\{(\forall x.\varphi/x, s)\}$ |

Table 1: The evaluation game $E(\xi, S)$

### 2.2.2 The alternation-free fragment

As mentioned in the introduction, the alternation-free fragment of the modal $\mu$-calculus consists of relatively simple formulas, in which the interaction between least- and greatest fixpoint operators is restricted. There are various ways to formalise this intuition. Following the approach by Niwiński [29], we call a formula $\xi$ alternation free if it satisfies the following: if $\xi$ has a subformula $\eta x.\varphi$ then no free occurrence of $x$ in $\varphi$ can be in the scope of an $\eta$-operator. An inductive definition of this set can be given as follows.

**Definition 2.1.** By a mutual induction we define the alternation-free $\mu$-calculus $\mathcal{L}$, and, for a subset $Q \subseteq \text{Prop}$ and $\eta \in \{\mu, \nu\}$, its noetherian $\eta$-fragment over $Q$, $N_0^\eta(Q)$.

\[
\mathcal{L} \ni \varphi := \bot \mid T \mid p \mid \top \mid (\varphi_0 \land \varphi_1) \mid (\varphi_0 \lor \varphi_1) \mid \forall \varphi \mid \forall x.\varphi \mid \nu p.\varphi_p \mid \nu p.\varphi_p^\nu \\
N_0^\mu(\mathcal{L}) \ni \varphi := \bot \mid T \mid q \mid (\forall \varphi \land \varphi_1) \mid (\forall \varphi \lor \varphi_1) \mid \forall \varphi \mid \forall x.\varphi \mid \nu p.\varphi_p \mid \nu p.\varphi_p^\nu \\
N_0^\nu(\mathcal{L}) \ni \varphi := \bot \mid T \mid q \mid (\forall \varphi \land \varphi_1) \mid (\forall \varphi \lor \varphi_1) \mid \forall \varphi \mid \forall x.\varphi \mid \nu p.\varphi_p \mid \nu p.\varphi_p^\nu
\]

We assume familiarity with such games, see the appendix for some definitions.
where \( p \in \text{Prop} \), \( q \in Q \), \( \varphi_p \in N^0_Q(L) \) for \( P \subseteq \text{Prop} \), and \( \psi \in L \) is such that \( \text{FV}(\psi) \cap Q = \emptyset \). Here and in the sequel we shall write \( p \) for \( \{p\} \) and \( Q \) for \( Q \cup \{q\} \).

Throughout the text we shall simply refer to elements of \( L \) as \textit{formulas}.

The intuition underlying this definition is that \( N^0_Q(L) \) consists of those alternation-free formulas in which free variables from \( Q \) may not occur in the scope of an \( \mu \)-operator. The name ‘noetherian’ refers to a semantic property that characterize the \( N^0_Q(L) \) formulas \([13]\): if a formula \( \varphi \in N^0_Q(L) \) is satisfied at the root of a tree model \( T \), then it is also true in a variant of \( T \) where we restrict the interpretation of the proposition letters in \( Q \) to noetherian subtrees of \( T \), i.e., subtrees without infinite paths.

**Example 2.2.** For some examples of alternation-free formulas, observe that \( \text{Proposition 2.3.} \) For some examples of alternation-free formulas, observe that

**Proposition 2.3.** Let \( \xi \) be an alternation-free formula. Then
  1) its negation \( \overline{\xi} \) is alternation free;
  2) if \( \xi \) is a fixpoint formula, then its unfolding is alternation free;
  3) every subformula of \( \xi \) is alternation free;
  4) every formula in \( \text{Clos}(\xi) \) is alternation free;
  5) there is an alternation-free guarded formula \( \xi' \) that is equivalent to \( \xi \).

**Proof.** Item 2) is immediate by Proposition 2.5 and Proposition 2.4. For item 5) a careful inspection will reveal that the standard procedure for guarding formulas (see \([35, 22, 5]\)) transforms alternation-free formulas to guarded alternation-free formulas. The other items can be proved by routine arguments.

**Proposition 2.4.**
  1) If \( Q \) and \( Q' \) are sets of proposition letters with \( Q \subseteq Q' \), then \( N^0_Q(L) \subseteq N^0_{Q'}(L) \).
  2) \( L = N^0_Q(L) \).

**Proof.** Item 1) can be proved by a straightforward induction on the complexity of formulas in \( N^0_Q(L) \); we leave the details for the reader. A similar induction shows that \( N^0_Q(L) \subseteq L \), for any set \( Q \) of variables; clearly this takes care of the inclusion \( \subseteq \) in item 2).

This leaves the statement that \( L \subseteq N^0_Q(L) \), which we prove by induction on the complexity of \( L \)-formulas. We confine our attention here to the case where \( \varphi \in L \) is a fixpoint formula, say, \( \varphi = \lambda p.\varphi' \). But then it is obvious that \( \text{FV}(\varphi) \cap \{p\} = \emptyset \), so that \( \varphi \in N^0_Q(L) \) by definition of the latter set. It follows that \( \varphi \in N^0_Q(L) \) by item 1).

The following proposition states some useful closure conditions on sets of the form \( N^0_Q(L) \).

**Proposition 2.5.** Let \( \chi \) and \( \xi \) be formulas in \( L \), let \( x, y \) be variables, and let \( Q \) be a set of variables. Then the following hold:
  1) if \( \xi \in N^0_Q(L) \) and \( y \notin \text{FV}(\xi) \), then \( \xi \in N^0_{Qy}(L) \);
  2) if \( \chi \in N^0_{Qx}(L) \), \( \xi \in N^0_Q(L) \) and \( \xi \) is free for \( x \) in \( \chi \), then \( \chi[\xi/x] \in N^0_Q(L) \);
3) if \( \eta x \chi \in N^0_Q(L) \) then \( \chi[\eta x \chi/x] \in N^0_Q(L) \).

Proof. We prove item 1 of the proposition by a straightforward induction on the complexity of \( \xi \). We only cover the case of the induction step where \( \xi \) is of the form \( \lambda z \xi' \). Here we distinguish cases. If \( \text{FV}(\xi) \cap Q = \emptyset \) then we find \( \text{FV}(\xi) \cap (Q \cup \{ y \}) = \emptyset \) since \( y \notin \text{FV}(\xi) \) by assumption. Here it is immediate by the definition of \( N^0_{Q\eta}(L) \) that \( \xi \) belongs to it.

If, on the other hand, we have \( \text{FV}(\xi) \cap Q \neq \emptyset \), then we can only have \( \xi \in N^0_Q(L) \) if \( \lambda = \eta \). We now make a further case distinction: if \( y = z \) then we have \( \xi' \in N^0_{Q\eta}(L) \) so that also \( \xi \in N^0_{Q\eta}(L) \). If \( y \) and \( z \) are distinct variables, then it must be the case that \( \xi' \in N^0_Q(L) \); since we clearly have \( y \notin \text{FV}(\xi') \) as well, the inductive hypothesis yields that \( \xi' \in N^0_{Q\eta}(L) \). But then we immediately find \( \xi \in N^0_{Q\eta}(L) \) by definition of the latter set.

For the proof of item 2 we proceed by induction on the complexity of \( \chi \). Again, we only cover the inductive case where \( \chi \) is a fixpoint formula, say, \( \chi = \lambda y.\chi' \). We make a case distinction. First assume that \( \psi \notin \text{FV}(\chi) \); then we find \( \chi[\psi/x] = \chi \), so that \( \chi[\psi/x] \in N^0_{\eta Q}(L) \) by assumption. It then follows that \( \chi[\psi/x] \in N^0_Q(L) \) by Proposition 2.4.

Assume, then, that \( \psi \in \text{FV}(\chi) \); since \( \chi \in N^0_{Q\eta}(L) \) this can only be the case if \( \lambda = \eta \), and, again by definition of \( N^0_{Q\eta}(L) \), we find \( \chi' \in N^0_{Q\eta}(L) \). Furthermore, as \( \xi \) is free for \( x \) in \( \chi \), the variable \( y \) cannot be free in \( \xi \), so that it follows by item 1 and the assumption that \( \xi \in N^0_Q(L) \) that \( \xi \in N^0_{Q\eta}(L) \). We may now use the inductive hypothesis on \( \chi' \) and \( \xi \), to find that \( \chi'[\psi/x] \in N^0_{Q\eta}(L) \); and from this we conclude that \( \chi[\psi/x] \in N^0_Q(L) \) by definition of \( N^0_Q(L) \).

Finally, item 3 is immediate by item 2.

The next observation can be used to simplify the formulation of the winning conditions of the evaluation game for alternation-free formulas somewhat. It is a direct consequence of results in [23], so we confine ourselves to a proof sketch.

**Proposition 2.6.** For any infinite trace \( \tau = (\varphi_n)_{n<\omega} \) of \( L \)-formulas the following are equivalent:

1) \( \tau \) is an \( \eta \)-trace;

2) \( \varphi_n \) is an \( \eta \)-formula, for infinitely many \( n \);

3) \( \varphi_n \) is an \( \Pi \)-formula, for at most finitely many \( n \).

Proof (sketch). Let \( \xi = \eta z.\xi' \) be the characteristic fixpoint formula of \( \tau \), i.e., \( \xi \) is the unique formulas that occurs infinitely often on \( \tau \) and that is a subformula of almost all formulas on \( \tau \). Clearly it suffices to prove that almost every fixpoint formula on \( \tau \) is an \( \eta \)-formula as well.

To show why this is the case, it will be convenient to introduce the following notation. We write \( \psi \rightarrow_{\eta} \varphi \) if there is a sequence \( (\chi_0)_{0 \leq i \leq n} \) such that \( \psi = \chi_0, \varphi = \chi_n, \chi_i \rightarrow_{\eta} \chi_{i+1} \) for all \( i < n \), and every \( \chi_i \) is of the form \( \chi_i'[p/x] \) for some formula \( \chi_i' \) and some \( x \in \text{FV}(\chi_i') \). Then it readily follows from the definitions that \( \xi \rightarrow_{\eta} \varphi_n \) for almost every formula \( \varphi_n \) on \( \tau \). The key observation in the proof is now that if \( \xi \) is alternation-free, and \( \varphi \) is a fixpoint formula such that \( \xi \rightarrow_{\eta} \varphi \), then \( \varphi \) is an \( \eta \)-formula.

To be more precise we first show that

for all \( \varphi \) with \( \xi \rightarrow_{\eta} \varphi \) there is some \( \varphi^0 \in N^0_Q(L) \) such that \( z \in \text{FV}(\varphi^0) \) and \( \varphi = \varphi^0[\xi/z] \). \( \)(1)\n
We prove this claim by induction on the length of the path \( \xi \rightarrow_{\eta} \varphi \). In the base case we have \( \varphi = \xi \) and we let \( \varphi^0 := \xi[z/x] \).

In the inductive step there is some \( \chi \) such that \( \xi \rightarrow_{\eta} \chi \rightarrow_{\eta} \varphi \). By the inductive hypothesis there is some \( \chi^0 \in N^0_Q(L) \) such that \( z \in \text{FV}(\chi^0) \) and \( \chi = \chi^0[\xi/z] \). We distinguish cases depending on the main connective of \( \chi \). Omitting the boolean and modal cases we focus on the case where \( \chi \) is a fixpoint formula, and we further distinguish cases depending on whether \( \chi = \xi \) or not.

If \( \chi = \xi \) then \( \varphi = \xi'[\xi/z] \). Because \( \xi \) is alternation free we know that \( \xi' \in N^0_Q(L) \). We can thus let \( \varphi^0 := \xi'[z/x] \).
If $\chi = \lambda y.\chi'$ but $\chi \neq \xi$ then we have $\varphi = \chi'[\chi/y]$. From the inductive hypothesis we get that $\chi = \chi^\circ[\xi/z]$ for some $\chi^\circ \in N^\circ_2(\mathcal{L})$ with $z \in FV\chi^\circ$. Because $\chi \neq \xi$ it follows from $\chi = \lambda y.\chi'$ and $\chi = \chi^\circ[\xi/z]$ that $\chi^\circ = \lambda y.\rho$ for some $\rho$ with $\chi' = \rho[\xi/z]$. Hence, $\varphi = \rho[\xi/z][\chi/y]$. Because $z \not\in FV(\chi)$ and $y \not\in FV(\xi)$ (because $BV(\chi) \cap FV(\xi) = \emptyset$) we may commute these substitutions (cf. Proposition 3.11 in [23]). Hence $\varphi = \rho[\chi/y][\xi/z]$, and we may set $\varphi^\circ := \rho[\chi/y]$. Because $\chi^\circ = \lambda y.\rho$ and $z \in FV(\chi^\circ)$ it follows that $z \neq y$ and that $z \in FV(\rho)$. Thus also $z \in FV(\rho[\chi/y])$. Lastly, it follows from $\chi^\circ = \lambda y.\rho$, $\chi^\circ \in N^\circ_2(\mathcal{L})$ and $z \in FV(\rho)$ that $\rho \in N^\circ_2(\mathcal{L})$. It is not hard to see that $N^\circ_2(\mathcal{L})$ is closed under substitution with the alternation free formula $\chi$, where $z \not\in FV(\chi)$ and thus $\rho[\chi/y] \in N^\circ_2(\mathcal{L})$. This finishes the proof of \(\square\).

The claim about fixpoint formulas $\varphi$ such that $\xi \Rightarrow^{\xi}_\mathcal{L} \varphi$ can be derived from \(\square\) as follows. Assume that $\varphi$ is of the form $\varphi = \lambda y.\rho$, then if $\lambda y.\rho = \varphi^\circ[\xi/z]$ with $z \in FV(\varphi^\circ)$ and $\varphi \neq \xi$ then it must be the case that $\varphi^\circ = \lambda y.\rho^\circ$, and because $\varphi^\circ \in N^\circ_2(\mathcal{L})$ and $z \in FV(\rho^\circ)$ this is only possible if $\lambda = \eta$. That is, $\varphi$ is an $\eta$-formula as required. \(\square\)
3 Tableaux and tableau games

In this section we define a tableau game for the alternation-free \( \mu \)-calculus that is a adaptation of the tableau game by Niwiński and Walukiewicz [31]. We also show that the tableau game is adequate with respect to the semantics in Kripke frames, meaning that Abelard has a winning strategy in the tableau game for some tableau of some formula iff the formula is valid. The soundness and completeness proofs for the focus system of this paper rely on this result. There we will exploit that proofs in the focus system closely correspond to winning strategies for one of the two players in the tableau game.

3.1 Tableaux

We first define the notion of a tableau, which are the graphs over which the tableau game is played. The nodes of a tableau for some formula \( \varphi \) are labelled with sequents, which are sets of formulas from the closure of \( \varphi \).

A sequent \( \Phi \subseteq L \) is a finite set of formulas. When writing sequents we often leave out the braces, meaning that we write for instance \( \varphi_1, \ldots, \varphi_i \) for the sequent \( \{\varphi_1, \ldots, \varphi_i\} \). Given a sequent \( \Phi \) we write \( \varphi_1, \ldots, \varphi_i, \Phi \) for the sequent \( \{\varphi_1, \ldots, \varphi_i\} \cup \Phi \). Our tableau is defined such that it makes sense to read the formulas in a sequent \( \Phi \) disjunctively. We show below that Abelard has a winning strategy in the tableau for some sequent if the disjunction of its formulas are valid. This is different from the satisfiability tableaux in [31], where sequents are read conjunctively.

The tableau system is based on the rules in Figure 1. Each rule has one conclusion and a finite, possibly zero, number of premises. We call a rule applicable to some sequent if the sequent is of a form that matches the conclusion of the rule.

\[
\begin{align*}
\frac{\mu, \varphi, \Psi}{\Phi} & \quad \frac{\varphi, \psi, \Phi}{\varphi \lor \psi, \Phi} & \frac{\varphi, \Phi}{\varphi \land \psi, \Phi} \\
\frac{\varphi_1, \Phi}{\Psi, \Box \varphi_1, \ldots, \Box \varphi_n, \Box \Phi} & \quad \frac{\mu \varphi / x, \Phi}{\mu \varphi, \Phi} & \frac{\nu \varphi / x, \Phi}{\nu \varphi, \Phi}
\end{align*}
\]

Figure 1: Rules of the tableaux system

Here are some first remarks on the tableau rules. Both the rules \( \text{Ax1} \) and \( \text{Ax2} \), which have no premises, are also called axioms. The boolean rules \( R_\lor \) and \( R_\land \) are standard proof rules for the disjunction and conjunction. The number of premises of the modal rule \( M \) is not fixed, but depends on the number of box formulas in the conclusion; as a special case, if the conclusion contains no box formula at all, then the rule has an empty set of premises, similar to an axiom. Furthermore, the rule \( M \) has as its side condition \( (\dagger) \) that \( \Psi \) is a set of atomic formulas that is locally falsifiable, i.e., \( \Psi \) does not contain \( \top \) and there is no proposition letter \( p \) such that both \( p \) and \( \varphi \) belong to \( \Psi \). Finally, the fixpoint rules are simple unfolding rules for the least and greatest fixpoint operators.

**Definition 3.1.** A tableau is a quintuple \( T = (V, E, \Phi, Q, v_I) \), where \( V \) is a set of nodes, \( E \) is a binary relation on \( V \), \( v_I \) is the initial node or root of the tableau, and both \( \Phi \) and \( Q \) are labelling functions. Here \( \Phi \) maps every node \( v \) to a non-empty sequent \( \Phi_v \), and

\[
Q : V \to \{\text{Ax1}, \text{Ax2}, R_\lor, R_\land, M, R_\mu, R_\nu\}
\]

associates a proof rule \( Q_v \) with each node \( v \) in \( V \). Tableaux are required to satisfy the following coherence conditions:
1. If a node is labelled with the name of a proof rule then it has as many successors as the proof rule has premises, and the annotated sequents at the node and its successors match the specification of the proof rules in Figure 1.

2. A node can only be labelled with the modal rule $M$ if its side condition $(\dagger)$ is met.

A tableau is a tableau for a sequent $\Phi$ if $\Phi$ is the sequent of the root of the tableau. Observe that it follows from condition 2 in Definition 3.1 that if a node $u$ is labelled with $M$, then no other rule is applicable.

**Proposition 3.2.** There is a tree-based tableau for every sequent $\Phi$.

**Proof.** This can be proved in a straightforward step-wise procedure in which we construct the tree underlying $T$ by repeatedly extending it at non-axiomatic leaves using any of the proof rules that are applicable at that leaf. This generates a possibly infinite tree that is a tableau because in every sequent there is at least one rule applicable. Note that $M$ can be applied in sequents without modal formulas, in which case it has no premises and thus creates a leaf of the tableau. $\square$

A crucial aspect of tableaux for the $\mu$-calculus is that one has to keep track of the development of individual formulas along infinite paths in the tableau. For this purpose we define the notion of a trail in a path of the tableau.

**Definition 3.3.** Let $T = (V, E, \Phi, Q, v_I)$ be a tableau. For all nodes $u, v \in V$ such that $Euv$ we define the active trail relation $A_{u,v} \subseteq \Phi_u \times \Phi_v$ and the passive trail relation $P_{u,v} \subseteq \Phi_u \times \Phi_v$, both of which relate formulas in the sequents at $u$ and $v$. The idea is that $A$ connects the active formulas in the premise and conclusion, whereas $P$ connects the side formulas. Both relations are defined via a case distinction depending on the rule that is applied at $u$:

- **Case $Q_u = R_{\lor}$:** Then $\Phi_u = \{\varphi \lor \psi\} \cup \Psi$ and $\Phi_v = \{\varphi, \psi\} \cup \Psi$ for some sequent $\Psi$. We define $A_{u,v} = \{(\varphi \lor \psi, \varphi), (\varphi \lor \psi, \psi)\}$ and $P_{u,v} = \Delta_{\varphi}$, where $\Delta_{\varphi} = \{\varphi | \varphi \in \Psi\}$.

- **Case $Q_u = R_{\land}$:** In this case $\Phi_u = \{\varphi \land \psi\} \cup \Psi$ and $\Phi_v = \{\chi\} \cup \Psi$ for some sequent $\Psi$ and $\chi$ such that $\chi = \psi$ if $v$ corresponds to the left premise of $R_{\land}$ and $\chi = \varphi$ if $v$ corresponds to the right premise. In both cases we set $A_{u,v} = \{\varphi \land \psi, \chi\}$ and $P_{u,v} = \Delta_{\varphi}$.

- **Case $Q_u = M$:** Then $\Phi_u = \Psi \cup \{\Box \varphi_1, \ldots, \Box \varphi_n\} \cup \bigvee \Phi$ and $\Phi_v = \{\varphi\} \cup \Phi$ for some sequent $\Phi$ and locally falsifiable set of literals $\Psi \subseteq \text{Lit}$. We can thus define $A_{u,v} = \{\Box \varphi, \varphi | \varphi \in \Phi\}$ and $P_{u,v} = \emptyset$.

- **Case $Q_u = R_{\neg}$:** Then $\Phi_u = \{\mu x.\varphi\} \cup \Psi$ and $\Phi_v = \{\varphi[\mu x.\varphi/x]\} \cup \Psi$ for some sequent $\Psi$. We define $A_{u,v} = \{\mu x.\varphi, \varphi[\mu x.\varphi/x]\}$ and $P_{u,v} = \Delta_{\varphi}$.

- **Case $Q_u = R_{\nu}$:** Then $\Phi_u = \{\nu x.\varphi\} \cup \Psi$ and $\Phi_v = \{\varphi[\nu x.\varphi/x]\} \cup \Psi$ for some sequent $\Psi$. We define $A_{u,v} = \{\nu x.\varphi, \varphi[\nu x.\varphi/x]\}$ and $P_{u,v} = \Delta_{\varphi}$.

Note that it is not possible that $Q_u = Ax1$ or $Q_u = Ax2$ because $u$ is assumed to have a successor.

Finally, for all nodes $u$ and $v$ with $Euv$, the general trail relation $T_{u,v}$ is defined as $T_{u,v} := A_{u,v} \cup P_{u,v}$. $\triangle$

**Definition 3.4.** Let $T = (V, E, \Phi, Q, v_I)$ be a tableau. A path in $T$ is simply a path in the underlying graph $(V, E)$ of $T$, that is, a sequence $\pi = (v_{i,n})_{n<\kappa}$, for some ordinal $\kappa$ with $0 < \kappa \leq \omega$, such that $E_{v_{i,n}}$ for every $i$ such that $i + 1 < \kappa$. A trail on such a path $\pi$ is a sequence $(\varphi_{i,n})_{n<\kappa}$ of formulas such that $(\varphi_{i,n})_{n<\kappa}$ for every $i$ such that $i + 1 < \kappa$. An infinite trail $(\varphi_{i,n})_{n<\omega}$ is a $\nu$-trail if there is some $i \in \omega$ such that $\varphi_j$ is not a $\mu$-formula for any $j \geq i$. $\triangle$

**Remark 3.5.** Although our tableaux are very much inspired by the ones introduced by Niwiński and Walukiewicz, there are some notable differences in the actual definitions. In particular, the fixpoint rules in our tableaux simply unfold fixpoint formulas; that is, we omit the mechanism of definition.
lists. Furthermore, since we are working with alternation-free formulas, our definition of successful paths can be simplified somewhat, cf. Proposition 2.6. Some minor differences are that we always decompose formulas until we reach literals, and that our tableaux are not necessarily tree-based. 

It is easy to see that because of guardedness, we have the following.

**Proposition 3.6.** Let $\pi$ be an infinite path in a tableau $T$, and let $(\varphi_n)_{n<\omega}$ be a trail on $\pi$. Then

1) $\pi$ witnesses infinitely many applications of the rule $\mathbf{M}$;
2) there are infinitely many $i$ such that $(\varphi_i, \varphi_{i+1}) \in A_{v_i,v_{i+1}}$.

Before we move on to the definition of tableau games, we need to have a closer look at trails. Recall that for any two nodes $u, v \in V$, the trail relation $T_{u,v}$ is the union of an active and a passive trail relation, and that the passive relation is always a subset of the diagonal relation on formulas. As a consequence, we may reduce any trail $(\varphi_n)_{n<\kappa}$ on a path $\pi = (v_n)_{n<\kappa}$ simply by omitting all $\varphi_{i+1}$ from the sequence for which $(\varphi_i, \varphi_{i+1})$ belongs to the passive trail relation $P_{v_i,v_{i+1}}$.

**Definition 3.7.** Let $\tau = (\varphi_n)_{n<\kappa}$ be a trail on the path $\pi = (v_n)_{n<\kappa}$ in some tableau $T$. Then the condensed trail $\tilde{\tau}$ is obtained from $\tau$ by omitting all $\varphi_{i+1}$ from $\tau$ for which $(\varphi_i, \varphi_{i+1})$ belongs to the passive trail relation $P_{v_i,v_{i+1}}$.

It is not difficult to see that condensed trails are *traces*, and that it follows from Proposition 3.6 that the active reduct of an infinite trail is infinite.

### 3.2 Tableau games

We are now ready to introduce the *tableau game* $G(T)$ that we associate with a tableau $T$. We will first give the formal definition of this game, and then provide an intuitive explanation; Appendix A contains more information on infinite games. We shall refer to the two players of tableau games as *Eloise* and *Abelard*.

**Definition 3.8.** Given a tableau $T = (V, E, \Phi, Q, v_I)$, the tableau game $G(T)$ is the (initialised) board game $G(T) = (V, E, O, M_v, v_I)$ defined as follows. $O$ is a partial map that assigns a player to some positions in $V$; the player $O(v)$ will then be called the *owner* of the position $v$. More specifically, Eloise owns all positions that are labelled with one of the axioms, $\text{Ax}_1$ or $\text{Ax}_2$, or with the rule $R_\text{Ax}$; Abelard owns all position labelled with $\mathbf{M}$; $O$ is undefined on all other positions. In this context $v_I$ will be called the *initial* or *starting* position of the game.

The set $M_v$ is the *winning condition* of the game (for Abelard); it is defined as the set of infinite paths through the graph that carry a $v$-trail.

A *match* of the game consists of the two players moving a token from one position to another, starting at the initial position, and following the edge relation $E$. The owner of a position is responsible for moving the token from that position to an adjacent one (that is, an $E$-successor); in case this is impossible because the node has no $E$-successors, the player gets stuck and immediately loses the match. For instance, Eloise loses as soon as the token reaches an axiomatic leaf labelled $\text{Ax}_1$ or $\text{Ax}_2$; similarly, Abelard loses at any modal node without successors. If the token reaches a position that is not owned by a player, that is, a node of $T$ that is labelled with the proof rule $R_\lor$, $R_\land$, or $R_\neg$, the token automatically moves to the unique successor of the position. If neither player gets stuck, the resulting match is infinite; we declare Abelard to be its winner if the match, as an $E$-path, belongs to the set $M_v$, that is, if it carries a $v$-trail.

Finally, we say that a position $v$ is a *winning* position for a player if they have a way of playing the game that guarantees they win the resulting match, no matter how their opponent plays. For a formalisation of these concepts we refer to the Appendix.
Remark 3.9. If $T$ is tree-based the notion of a strategy can be simplified. The point is that in this case finite matches can always be identified with their last position, since any node in a tree corresponds to a unique path from the root to that node. It follows that any strategy in such a game is positional (that is, the move suggested to the player only depends on the current position). Moreover, we may identify a strategy for either player with a subtree $S$ of $T$ that contains the root of $T$ and, for any node $s$ in $S$, (1) it contains all successors of $s$ in case the owner of $s$ owns the position $s$ himself, while (2) it contains exactly one successor of $s$ in case the player’s opponent owns the position $s$. \hfill\blacktriangleleft

The observations below are basically due to Niwiński & Walukiewicz [31].

Theorem 3.10 (Determinacy). Let $T$ be a tableau for a sequent $\Phi$. Then at any position of the tableau game for $T$ precisely one of the players has a winning strategy.

Proof. The key observation underlying this theorem is that tableau games are regular. That is, using the labelling maps $Q$ and $\Sigma$ of a tableau $T$, we can find a finite set $C$, a colouring map $\gamma : V \to C$, and an $\omega$-regular subset $L \subseteq C^*$ such that $M_L = \{ (v_n)_{n \in \omega} \in \text{InfPath}(T) \mid (\gamma(v_n))_{n \in \omega} \in L \}$. The determinacy of $G(T)$ then follows by the classic result by Büchi & Landweber [6] on the determinacy of regular games. We skip further details of the proof, since it is rather similar to the analogous proof in [31]. \hfill\blacktriangle

For the Adequacy Theorem below we do provide a proof, since our proof is somewhat different from the one by Niwiński and Walukiewicz, and we will need an adaptation of our proof further on.

Theorem 3.11 (Adequacy). Let $T$ be a tableau for a sequent $\Phi$. Then Eloise (Abelard, respectively) has a winning strategy in $G(T)$ iff the formula $\bigvee \Phi$ is refutable (valid, respectively).

Proof. Fix a sequent $\Phi$ and a tableau $T$ for $\Phi$. We will prove the following statement:

\[ \text{Eloise has a winning strategy in } G(T) \text{ iff } \Phi \text{ is refutable.} \tag{2} \]

The theorem follows from this by the determinacy of $G(T)$.

For the left to right implication of (2), fix a tableau $T = (V, E, \Phi, Q, v_f)$; it will be convenient to assume that $T$ is tree-based. This is without loss of generality: if the graph underlying $T$ does not have the shape of a tree, we may simply continue with its unravelling.

Let $f$ be a winning strategy for Eloise in the game $G(T)$; recall that we may think of $f$ as a subtree $T_f$ of $T$. We will first define the pointed model in which the sequent $\Phi$ can be refuted. We define a state to be a maximal path in $T_f$ which does not contain any modal node, with the possible exception of its final node $\text{last}(\pi)$. Note that by maximality, the first node of a state is either the root of $T$ or else a successor of a modal node. Given a state $\pi = v_0 \cdots v_k$ and a formula $\varphi$, we say that $\varphi$ occurs at $\pi$, if $\varphi \in \bigcup \Phi_{v_i}$. We let $S_f$ denote the collection of all states, and define an accessibility relation $R_f$ on this set by putting $R_f(\pi, \rho)$ if the first node of $\rho$ is an $E$-successor of the last node of $\pi$. Note that this can only happen if $\text{last}(\pi)$ is modal. Finally, we define the valuation $V_f$ by putting $V_f(\rho) := \{ \pi \mid \rho \notin \Phi_{\text{last}(\pi)} \}$, and we set $S_f := (S_f, R_f, V_f)$.

In the sequel we will need the following observation; we leave its proof as an exercise.

Claim 1. Let $\varphi \in \Phi_{v_j}$ be a non-atomic formula, where $v_j$ is some node on a finite path $\pi = (v_i)_{i < k}$. If $\pi$ is a state, then the formula is active at some node $v_m$ on $\pi$, with $j \leq m < k$.

Now let $\pi_0$ be any state of which $\text{first}(\pi_0)$ is the root of $T$. We will prove that the pointed model $S_f, \pi_0$ refutes $\Phi$ by showing that

for every $\varphi \in \Phi$, the position $(\varphi, \pi_0)$ is winning for Abelard in $E(\bigvee \Phi, S_f)$. \tag{3}

To prove this, we will provide Abelard with a winning strategy in the evaluation game $E(\bigvee \Phi, S_f) \oplus (\varphi, \pi_0)$, for each $\varphi \in \Phi$. Fix such a $\varphi$, and abbreviate $E := E(\bigvee \Phi, S_f) \oplus (\varphi, \pi_0)$. The key idea is that, while
Case playing \( \mathcal{E} \), Abelard maintains a private match of the tableau game \( \mathcal{G}(\mathbb{T}) \), which is guided by Eloise’s winning strategy \( f \) and such that the current match of \( \mathcal{E} \) corresponds to a trail on this \( \mathcal{G}(\mathbb{T}) \)-match.

For some more detail on this link between the two games, let \( \Sigma = (\varphi_0, \pi_0)(\varphi_1, \pi_1) \cdots (\varphi_n, \pi_n) \) be a match of \( \mathcal{E} \). We will say that a \( \mathcal{G}(\mathbb{T}) \)-match \( \pi \) is linked to \( \Sigma \) if the following holds. First, let \( i_1, \ldots, i_k \) be such that \( 0 < i_1 < \cdots < i_k \leq n \) and \( \varphi_{i_1-1}, \ldots, \varphi_{i_k-1} \) is the sequence of all modal formulas among \( \varphi_0, \ldots, \varphi_{n-1} \). Then we require that \( \pi \) is the concatenation \( \pi = \pi_0 \circ \cdots \circ \pi_{i_k-1} \circ \rho \), where \( \rho \subseteq \pi_n \), and that the sequence \( \varphi_0 \cdots \varphi_n \) is the active condensation of some trail on \( \pi \).

Clearly then the matches that just consist of the initial positions of \( \mathcal{E} \) and \( \mathcal{G}(\mathbb{T}) \), respectively, are linked. Our proof of \( \text{Claim} \) is based on the fact that Abelard has a strategy that keeps such a link throughout the play of \( \mathcal{E} \). As the crucial observation underlying this strategy, the following claim states that Abelard can always maintain the link for one more round of the evaluation game.

**Claim 2.** Let \( \Sigma = (\varphi_0, \pi_0)(\varphi_1, \pi_1) \cdots (\varphi_n, \pi_n) \) be some \( \mathcal{E} \)-match and let \( \pi \) be an \( f \)-guided \( \mathcal{G}(\mathbb{T}) \)-match that is linked to \( \Sigma \). Then the following hold.

1) If \( (\varphi_n, \pi_n) \) is a position for Abelard in \( \mathcal{E} \), then he has a move \( (\varphi_{n+1}, \pi_{n+1}) \) such that some \( f \)-guided extension \( \pi' \) of \( \pi \) is linked to \( \Sigma \cdot (\varphi_{n+1}, \pi_{n+1}) \).

2) If \( (\varphi_n, \pi_n) \) is not a position for Abelard in \( \mathcal{E} \), then for any move \( (\varphi_{n+1}, \pi_{n+1}) \) there is some \( f \)-guided extension \( \pi' \) of \( \pi \) that is linked to \( \Sigma \cdot (\varphi_{n+1}, \pi_{n+1}) \).

**Proof of Claim** Let \( \Sigma \) and \( \pi \) be as in the formulation of the claim. Then \( \pi = \pi_0 \circ \cdots \circ \pi_{i_k-1} \circ \rho \), where \( \rho \subseteq \pi_n \) and \( i_1, \ldots, i_k \) are such that \( 0 < i_1 < \cdots < i_k \leq n \) and \( \varphi_{i_1-1}, \ldots, \varphi_{i_k-1} \) is the sequence of all modal formulas among \( \varphi_0, \ldots, \varphi_{n-1} \). Furthermore \( (\varphi_i)_{i=0}^{n} = \tau \) for some trail \( \tau \) on \( \pi \). Write \( \rho = \psi_1 \cdots \psi_l \), then \( \rho = \pi_n \) if \( v_l \) is modal.

We prove the claim by a case distinction on the nature of \( \varphi_n \). Note that \( \varphi_n \in \Phi_{v_l} \), and that by Claim 1 there is a node \( v_i \) on the path \( \pi_n \) such that \( i_k \leq i \) and \( \varphi_n \) is active at \( v_i \).

**Case** \( \varphi_n = \psi_0 \lor \psi_1 \) for some formulas \( \psi_0, \psi_1 \). The position \( (\varphi_n, \pi_n) \) in \( \mathcal{E} \) then belongs to Abelard. As \( \psi_0 \lor \psi_1 \) is the active formula at the node \( v_i \) in \( \mathbb{T} \), this means that \( Q_{v_i} = R_{\lambda}, \) so that \( v_i \), as a position of \( \mathcal{G}(\mathbb{T}) \), belongs to Eloise. This means that in \( \mathcal{E} \), Abelard may pick the formula \( \psi_j \), which is associated with the successor \( v_{i+1} \) of \( v_i \) on \( \pi_n \). Note that, since \( \pi_n \) is part of the \( f \)-guided match \( \pi \), this successor is the one that is picked by Eloise in \( \mathcal{G}(\mathbb{T}) \) at the position \( v_i \) in the match \( \pi \).

We define \( \Sigma' := \Sigma \cdot (\psi_j, \pi_n) \), \( \rho' := \psi_0 \circ \cdots \circ \psi_{i_k-1} \circ \rho \), where \( \rho' := \rho \cdot \psi_1 \cdots \psi_l \cdot v_i \cdot v_{i+1} \). Observe that since \( v_{i+1} \) lies on the path \( \pi_n \), we still have \( \rho' \subseteq \pi_n \). Furthermore, it is obvious that \( \rho' \) extends \( \tau \) via a number of passive trail steps, i.e., where \( \varphi_n \) is not active, until \( \varphi_n \) is the active formula at \( v_l \); from this it easily follows that \( \tau = \tau \cdot \psi_j = \varphi_0 \cdots \varphi_n \cdot \psi_j \). Furthermore, since the position \( v_{i+1} \) of \( v_i \) lies on the path \( \pi_n \), it was picked by Eloise’s winning strategy in \( \mathcal{G}(\mathbb{T}) \) at the position \( v_i \) in the match \( \pi \); this means that the match \( \rho' \) is still \( f \)-guided.

**Case** \( \varphi_n = \psi_0 \lor \psi_1 \) for some formulas \( \psi_0, \psi_1 \). The position \( (\varphi_n, \pi_n) \) in \( \mathcal{E} \) then belongs to Eloise, so suppose that she continues the match \( \Sigma \) by picking the formula \( \psi_j \). In this case we have \( Q_{v_n} = R_{\lambda} \), so that \( v_i \) has a unique successor \( v_{i+1} \) which features both \( \psi_0 \) and \( \psi_1 \) in its label set. This means that if we define \( \Sigma' := \Sigma \cdot (\psi_j, \pi_n) \), \( \rho' := \psi_0 \circ \cdots \circ \psi_{i_k-1} \circ \rho \), where \( \rho' := \rho \cdot \psi_1 \cdots \psi_l \cdot v_i \cdot v_{i+1} \) and \( \rho' := \pi \cdot \varphi_n \cdots \varphi_n \cdot \psi_j \), it is not hard to see that \( \Sigma' \) and \( \rho' \) are linked, with \( \rho' \) the witnessing trail on \( \rho' \).

**Case** \( \varphi_n = \eta.\psi \) for some binder \( \eta \), variable \( x \) and formula \( \psi \). The match \( \Sigma \) is then continued with the automatic move \( (\eta.\psi' / \psi / x) \cdot \pi_{n+1} \cdot \psi_n \). This case is in fact very similar to the one where \( \varphi \) is a disjunction, so we omit the details.
Case $\varphi_n = \Box \psi$ for some formula $\psi$. Then the position $(\varphi_n, \pi_n)$ belongs to Abelard: he has to come up with an $R_f$-successor of the state $\pi_n$. Since $\Box \psi$ is active in it, the node $v_i$ must be modal, in the sense that $Q_{v_i} = \mathbf{M}$. By the definition of a state this can only be the case if $v_i$ is the last node on the path/state $\pi_n$; recall that in this case we have $\rho = \pi_n$. Let $u \in E[v_i]$ be the successor of $v$ associated with $\psi$, and let $\pi_{n+1}$ be any state with $\text{first}(\pi_{n+1}) = u$. It follows by definition of $R_f$ that $\pi_{n+1}$ is a successor of $\pi_n$ in the model $\mathcal{S}_f$. This $\pi_{n+1}$ will then be Abelard’s (legitimate) pick in $\mathcal{E}$ at the position $(\Box \psi, \pi_{n+1})$.

Define $\Sigma' := \Sigma \cdot (\psi, \pi_{n+1})$, $\pi' := \pi \cdot v_{i+1} \cdots v_i u$ and $\rho' := \tau \cdot \varphi_n \cdots \varphi_0 \cdot \psi$. Then we find that $\pi' = \pi_0 \circ \cdots \circ \pi_{i-1} \circ \rho \circ \rho'$, where $\rho'$ is the one-position path $u$. Clearly then $\rho' \sqsubseteq \pi_{n+1}$. Furthermore, it is easy to verify that $\hat{\pi}' = \hat{\tau} \cdot \psi = \varphi_0 \cdots \varphi_n \psi$. This means that $\Sigma'$ and $\pi'$ are linked, as required.

Case $\varphi_n = \Diamond \psi$ for some formula $\psi$. As in the previous case this means that $v_i$ is a modal node, and $v_i = \text{last}(\pi_n)$. However, the position $(\varphi_n, \pi_n)$ now belongs to Eloise; suppose that she picks an $R_f$-successor $\pi_{n+1}$ of $\pi_n$. Let $u := \text{first}(\pi_{n+1})$, then it follows from the definition of $R_f$ that $u$ is an $E$-successor of $v_i$. As such, $u$ is a legitimate move for Abelard in the tableau game.

It then follows, exactly as in the previous case, that $\pi' := \pi \cdot v_{i+1} \cdots v_i u$ is linked to $\Sigma' := \Sigma \cdot (\psi, \pi_{n+1})$.

This finishes the proof of the claim.

On the basis of Claim 2 we may assume that Abelard indeed uses a strategy $f'$ that keeps a link between the $\mathcal{E}$-match and his privately played $f$-guided $\mathcal{G}($T$)$-match. We claim that $f'$ is actually a winning strategy for him. To prove this, consider a full $f'$-guided match $\Sigma$; we claim that Abelard must be the winner of $\Sigma$. This is easy to see if $\Sigma$ is finite, since it follows by the first item of the Claim that playing $f'$, Abelard will never get stuck.

This leaves the case where $\Sigma$ is infinite. Let $\Sigma = (\varphi_n, s_n)_{n<\omega}$; it easily follows from Claim 2 that there must be an infinite $f$-guided $\mathcal{G}($T$)$-match $\pi$, such that the sequence $(\varphi_n)_{n<\omega}$ is the active condensation of some trail on $\pi$. Since $\pi$ is guided by Eloise’s winning strategy $f$ this means that all of its trails are $\nu$-trails; but then obviously $(\varphi_n)_{n<\omega}$ is a $\nu$-trace, meaning that Abelard is the winner of $\Sigma$ indeed.

The implication from left to right in (2) is proved along similar lines, so we permit ourselves to be a bit more sketchy. Assume that $\Phi$ is refuted in some pointed model $(\mathcal{S}, s)$. Then by the adequacy of the game semantics for the modal $\rho$-calculus, Abelard has a winning strategy $f$ in the evaluation game $\mathcal{E}(\mathcal{V} \varphi, \mathcal{S})$ initialised at position $(\mathcal{V} \Phi, s)$. Without loss of generality we may assume $f$ to be positional, i.e., it only depends on the current position of the match.

The idea of the proof is now simple: while playing $\mathcal{G}($T$)$, Eloise will make sure that, where $\pi = v_0 \cdots v_k$ is the current match, every formula in $\Phi_{v_k}$ is the endpoint of some trail, and every trail $\tau$ on $\pi$ is such that its condensed trace $\hat{\tau}$ is the projection of an $f$-guided match of $\mathcal{E}(\mathcal{V} \varphi, \mathcal{S})$ initialised at position $(\varphi, s)$ for some $\varphi \in \Phi$. To show that Eloise can maintain this condition for the full duration of the match, it suffices to prove that she can keep it during one single round. For this proof we make a case distinction, as to the rule applied at the last node $v_k$ of the partial $\mathcal{G}($T$)$-match $\pi = v_0 \cdots v_k$.

The proof details are fairly routine, so we confine ourselves to one case, leaving the other cases as an exercise.

Assume, then, that $v_k$ is a conjunctive node, that is, $Q_{v_k} = R_\wedge$. This node belongs to Eloise, so as her move she has to pick an $E$-successor of $v_k$. The active formula at $v_k$ is some conjunction, say, $\psi_0 \wedge \psi_1 \in \Phi_{v_k}$. By the inductive assumption there is some trail $\tau = \varphi_0 \cdots \varphi_k$ on $\pi$ such that $\varphi_k = \psi_0 \wedge \psi_1$, and there is some $f$-guided $\mathcal{E}$-match of which $\tau$ is the projection, i.e., it is of the form $\Sigma = (\varphi_0, s_0) \cdots (\varphi_k, s_k)$. Now observe that in $\mathcal{E}$, the last position of this match, viz., $(\varphi_k, s_k) =$
$(\psi_0 \land \psi_1, s_k)$, belongs to Abelard. Assume that his winning strategy $f$ tells him to pick the formula $\psi_j$ at this position, then in the tableau game, at the position $v_k$, Eloise will pick the $E$-successor $u_j$ of $v_k$ that is associated with the conjunct $\psi_j$. That is, she extends the match $\pi$ to $\pi':=\pi \cdot u_j$.

To see that Eloise has maintained the invariant, consider an arbitrary trail on $\pi'$; clearly such a trail is of the form $\sigma' = \sigma \cdot \psi$, for some trail $\sigma$ on $\pi$, and some formula $\psi \in \Phi_{u_j}$. It is not hard to see that either $\text{last}(\sigma) = \psi_0 \land \psi_1$ and $\psi = \psi_j$, or else $\text{last}(\sigma) = \psi$. In the first case $\hat{\sigma}'$ is the match $(\varphi_0, s_0) \cdots (\varphi_k, s_k) \cdot (\psi_j, s_k)$; in the second case we find that $\hat{\sigma}' = \hat{\sigma}$ so that for the associated $f$-guided $E$-match we can take any such match that we inductively know to exist for $\sigma$.

**Corollary 3.12.** Let $T$ and $T'$ be two tableaux for the same sequent. Then Abelard has a winning strategy in $G(T)$ iff he has a winning strategy in $G(T')$. 

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4 The focus system

In this section we introduce our annotated proof systems for the alternation-free $\mu$-calculus. We consider two versions of the system, which we call Focus and Focus$_\infty$, respectively. Proofs in Focus$_\infty$ are infinite trees that are closely related to winning strategies of Abelard in the tableau game. The focus mechanism that is implemented by the annotations of formulas helps ensuring that all the infinite branches in a Focus$_\infty$ proof are of the right shape. The other proof system Focus can be seen as a finite variant of Focus$_\infty$. Proofs in this systems are finite trees. However, the system is circular in that it contains a discharge rule that allows to discharge a leaf of the tree if the same sequent as the sequent at the leaf is reached again closer to the root of the tree. A crucial observation, which is established in Proposition 4.8 below, is that every infinite Focus$_\infty$-proof can be turned into a finite Focus-proof, by using the discharge rule to detach infinite branches.

4.1 Basic notions

In this first part of this section we provide the definition of the proof systems Focus and Focus$_\infty$.

An annotated formula is a pair $(\varphi, a) \in \mathcal{L} \times \{f, u\}$; we usually write $\varphi^a$ instead of $(\varphi, a)$ and call $a$ the annotation of $\varphi$. A formula that is annotated with $f$ is called in focus. We use $a, b, c, \ldots$ as symbols to range over the set $\{f, u\}$. A finite set of annotated formulas is called an annotated sequent; we shall use the letters $\Sigma, \Gamma, \Delta, \ldots$ for annotated sequents. Given a finite set of formulas $\Phi \subseteq \mathcal{L}$, we define $\Phi^a$ to be the annotated sequent $\Phi^a := \{\varphi^a \mid \varphi \in \Phi\}$. We use analogous notation for annotated sequents as we introduced for sequents in Section 3. For instance, we omit braces and write $\bowtie \Sigma$ for the sequent $\bowtie \varphi^a := \{\varphi^a \mid \varphi^a \in \Sigma\}$. In practice we will often be sloppy and refer to annotated sequents as sequents.

The proof rules of our focus proof systems Focus and Focus$_\infty$ are given in Figure 2. The rules $\text{Ax1}, \text{Ax2}, R_\land, R_\lor,$ and $R_\nu$ are direct counterparts of the tableau rules with the same names, the only difference concerning the annotation of formulas. In general we use the same terminology when talking about rules in the focus system as we introduced for tableau rules in Section 3.

\[
\begin{align*}
\frac{\varphi^a, \psi^a, \Sigma}{\varphi \lor \psi} & \quad \text{Ax1} \\
\frac{\varphi^a, \psi^a, \Sigma}{(\varphi \land \psi)^a, \Sigma} & \quad \text{Ax2} \\
\frac{\mu x.\varphi^a/x^a, \Sigma}{\varphi^a, \Sigma} & \quad R_\mu \\
\frac{\nu x.\varphi^a/x^a, \Sigma}{\varphi^a, \Sigma} & \quad R_\nu \\
\frac{(\Sigma)}{\varphi^a, \Sigma} & \quad W \\
\frac{\Sigma}{\Phi^a} & \quad F \\
\end{align*}
\]

Figure 2: Proof rules of the focus system

Here are some more specific comments about the individual proof rules. The boolean rules ($R_\land$ and $R_\lor$) are fairly standard; observe that the annotation of the active formula is simply inherited by its subformulas. The fixpoint rules ($R_\mu$ and $R_\nu$) simply unfold the fixpoint formulas; note, however, the difference between $R_\mu$ and $R_\nu$ when it comes to the annotations: in $R_\nu$, the annotation of the active $\mu$-formula remains the same under unfolding, while in $R_\mu$, the active $\mu$-formula loses focus when it gets unfolded.
The box rule $R_\Box$ is the standard modal rule in one-sided sequent systems; the annotation of any formula in the consequents and its derived formula in the antecedent are the same. $R_\Box$ can be seen as a simplified version of $M$, differing from it in two ways. First, it can only be applied to a sequent that contains only modal formulas; and second, it has one premise only, related to the single box formula in the consequent. Since we have an explicit weakening rule $W$ the effect of $R_\Box$ is that of a restricted $M$ where we only continue with one of the premises of $M$ and discard all others. This reflects the fact that proofs in the focus system are related to winning strategies of Abelard in the tableau game, which select only one successor of $M$.

The rule $W$ is a weakening rule, which has as its side condition (†) that the formula $\varphi^x$ does not belong to the context; that is, we require $W$ to properly weaken the sequent. Next to $R_\mu$, the focus rule $F$ is the only rule that changes the annotations of formulas. Finally, the discharge rule $D$ is a special proof rule that allows us to discharge an assumption if it is repeating a sequent that occurs further down in the proof. Every application $D^x$ of this rule is marked by a so-called discharge token $x$ that is taken from some fixed infinite set $\mathcal{D} = \{x, y, z, \ldots\}$. In Figure 2 this is suggested by the notation $[\Sigma]^x$. The precise conditions under which $D^x$ can be employed are explained in Definition 4.1 below.

**Definition 4.1.** A pre-proof $\Pi = (T, P, \Sigma, R)$ is a quadruple such that $(T, P)$ is a, possibly infinite, tree with nodes $T$ and parent relation $P$; $\Sigma$ is a function that maps every node $u \in T$ to a non-empty annotated sequent $\Sigma_u$; and

$$R : T \to \{\text{Ax1, Ax2, } R_r, R_\mu, R_\Box, R_\Box, W, F\} \cup \{D^x \mid x \in \mathcal{D}\} \cup \mathcal{D} \cup \{\star\},$$

is a map that assigns to every node $u$ of $T$ its label $R(u)$, which is either (i) the name of a proof rule, (ii) a discharge token or (iii) the symbol $\star$.

To qualify as a pre-proof, such a quadruple is required to satisfy the following conditions:

1. If a node is labelled with the name of a proof rule then it has as many children as the proof rule has premises, and the annotated sequents at the node and its children match the specification of the proof rules in Figure 2.

2. If a node is labelled with a discharge token or with $\star$ then it is a leaf. We call such nodes non-axiomatic leaves as opposed to the axiomatic leaves that are labelled with a proof rule that has no premises.

3. For every leaf $l$ that is labelled with a discharge token $x \in \mathcal{D}$ there is exactly one node $u$ in $\Pi$ that is labelled with $D^x$. This node $u$, as well as its (unique) child, is a proper ancestor of $l$ and satisfies $\Sigma_u = \Sigma_l$. In this situation we call $l$ a discharged leaf, and $u$ its companion; we write $c$ for the function that maps a discharged leaf $l$ to its companion $c(l)$.

4. If $l$ is a discharged leaf with companion $c(l)$ then every annotated sequent at a node on the path from $c(l)$ to $l$ contains at least one formula that is in focus.

Non-axiomatic leaves that are not discharged, are called open; the sequent at an open leaf is an open assumption of the pre-proof. We call a pre-proof a proof in Focus if it is finite and does not have any open assumptions.

A infinite branch $\beta = \beta_0, \beta_1, \cdots$ is successful if there is some $i$ such that for all $j \geq i$ the annotated sequent at $\beta_j$ contains at least one formula in focus. A pre-proof is a proof in Focus when it does not have any non-axiomatic leaves and all its infinite branches are successful.

An unannotated sequent $\Phi$ is derivable in Focus (in Focus) if there is a Focus proof (a Focus proof, respectively) such that $\Phi^f$ is the annotated sequent at the root of the proof.

Note that condition (4) from Definition 4.1 implies that the focus rule is never applied on the path to a discharged leaf from its companion, because every sequent on this path contains a formula that is in focus. We leave it for the reader to verify that by guardedness there is at least one application of $R_\Box$ on such a path.
4.2 Basic observations

We now provide some preliminary results about our systems Focus\(_{\infty}\) and Focus. Our first result states that, basically, Focus\(_{\infty}\) and Focus can be seen as the infinitary and circular version of the same proof system.

Theorem 4.2. A annotated sequent \(\Gamma\) is provable in Focus iff it is provable in Focus\(_{\infty}\).

The two directions of this theorem are proved in Propositions 4.3 and 4.8.

Proposition 4.3. If an annotated sequent \(\Gamma\) is provable in Focus then it is provable in Focus\(_{\infty}\).

Proof. Let \(\Pi = (T, P, \Sigma, R)\) be a proof of \(\Gamma\) in Focus. We define a proof \(\Pi' = (T', P', \Sigma', R')\) of \(\Gamma\) in Focus\(_{\infty}\). The idea is to unravel the proof \(\Pi\) at discharged leaves. To this aim first define the relation \(L\) on \(T\) such that \(Luv\) holds iff either \(Puv\) or \(u\) is a discharged leaf and \(v = c(u)\). Let \(A\) be the set of all finite paths \(\pi\) in \(L\) that start at the root \(r\) of \((T, P)\). Formally, \(\pi = \pi_0, \ldots, \pi_n\) is in \(A\) iff \(\pi_0 = r\) and \(L\pi_i \pi_{i+1}\) for all \(i \in \{0, \ldots, n-1\}\). For any path \(\pi = \pi_0, \ldots, \pi_n \in A\) define \(last(\pi) = \pi_n\).

Consider the set \(S = D \cup \{D^x \mid x \in D\}\). We can think of \(S\) as the set of all labels of rules that take care of discharged leaves in \(\Pi\). We want to get rid of these in \(\Pi'\). We then define \(T' = \{\pi \in A \mid R(last(\pi)) \notin S\}\) and set \(P'\pi\rho\) for \(\pi, \rho \in T'\) iff \(\rho = \pi \cdot \alpha_1 \cdots \alpha_n\) with \(n \geq 1\) and \(\alpha_i \in S\) for all \(i \in \{1, \ldots, n-1\}\). Moreover, we set \(\Sigma_n' = \Sigma_{last(\pi)}\) and \(R'(\pi) = R(last(\pi))\).

Note that for every node \(v \in T\) we can define a unique \(L\)-path \(\pi^v = \pi^v_0 \cdots \pi^v_n\) with \(\pi^v_0 = v, R(\pi^v) \notin S\) and \(R(\pi^v) \in S\) for all \(i \in \{0, \ldots, n-1\}\). This path is unique because every node \(w\) with \(R(w) \in S\) has a unique \(L\)-successor. To see that this path is always finite assume for contradiction that it is infinite, that is, it never reaches a node \(\pi^v_n\) such that \(R(\pi^v) \notin S\). Because \(T\) is finite it would follow that from some point on the path visits only nodes that it visit infinitely often. Hence, there must then be some discharged leaf such that the infinite path visits all the nodes between mentioned leaf and its companion. But as we saw, this would mean that the path passes a node \(w\) with \(R(w) = R_\infty \notin S\); thus the path is finite after all. Finally, observe that by the definition of the rules in \(S\) we have \(\Sigma_v = \Sigma_{\pi^v_n}\) for every such path \(\pi^v\).

Then observe that for every \(\rho \in T'\) the assignment \(v \mapsto \rho \cdot \pi^v\) (where this composition of paths identifies \(last(\rho)\) with \(\pi^v_0\)) defines a bijective correspondence between the \(L\)-successors of \(last(\rho)\) and the \(P'\)-successors of \(\rho\). Moreover, we have that \(\Sigma_v = \Sigma_{\pi^v_n}\). It follows from these observations that \(\Pi'\) is a pre-proof.

It remains to verify that all infinite branches \(\beta = \beta_0, \beta_1, \cdots\) in \(\Pi'\) are successful. Clearly any such \(\beta\) gives rise to an infinite \(L\)-path \(\alpha = \alpha_0\alpha_1 \cdots\) in \(\Pi\) such that there are indices \(h_0 < h_1 < \cdots\) with \(last(\beta_k) = \alpha_{h_k}\) for all \(k \in \omega\). Because \(T\) is finite, from some point on \(\alpha\) only passes nodes that are between some discharged leaf and its companion node. By condition 4 from Definition 4.1 it follows that from then on the sequent at the node in the path always contains a focused formula. It is clear that then the same property also holds for \(\beta\).

The converse direction of the previous proposition, which is proven in Proposition 4.8 below, requires some additional preliminary observations.

Proposition 4.4. Let \(\Phi\) be the set of formulas that occur in the annotated sequent \(\Sigma_v\) at the root of some pre-proof \(\Pi = (T, P, \Sigma, R)\). Then all formulas that occur annotated in \(\Sigma_v\) for any \(t \in T\) are in \(Clos(\Phi)\).

Proof. This is an easy induction on the depth of \(t\) in the tree \((T, P)\). It amounts to checking that if the formulas in the conclusion of any of the rules from Figure 2 are in \(Clos(\Phi)\) then so are the formulas at any of the premises.

Proposition 4.5. For every successful infinite branch \(\beta = \beta_0, \beta_1, \cdots\) in a Focus\(_{\infty}\)-proof \(\Pi = (T, P, \Sigma, R)\) there are infinitely many \(i\) such that \(R(\beta_i) = R_\infty\).
Proof. We first argue that there are infinitely many \( i \) such that \( \beta_i \notin \{W, F\} \). Assume for a contradiction that there is some \( i \) such that \( R(\beta_j) \in \{W, F\} \) for all \( j \geq i \). Because \( \beta \) is successful there is a \( k \) such that every sequent at \( \beta_j \) for \( j \geq k \) contains at least one formula in focus. This means that after \( k \) the rule \( F \) is never applicable, since it requires that all formulas are unfocused. Hence, it would follow that \( R(\beta_j) = W \) for all \( j \geq \max(i, k) \). But this is impossible because \( W \) strictly decreases the number of annotated formulas in the sequent and therefore it is only applicable finitely many times in a row.

Because there are infinitely many \( i \) such that \( \beta_i \notin \{W, F\} \) it follows that there are infinitely many applications of \( R \) to a subset of \( \{R_s, R_v, R_u, R_w, R_C\} \). This is only possible if there are infinitely many applications of \( R_u \) or \( R_v \) as well. But then because we are working under the assumption that all formulas are guarded it follows that there are infinitely many applications of \( R_C \).

Definition 4.6. A node \( m \) in a pre-proof \( \Pi = (T, P, \Sigma, R) \) is called a successful repeat if it has a proper ancestor \( t \) such that \( \Sigma_t = \Sigma_m \), \( R(t) \neq D \), and for every node \( v \in [t, r] \), the annotated sequent \( \Sigma_v \) contains a formula in focus. The node \( t \) with this property is called a witness to the successful-repeat status of \( m \).

Note that the definition of a successful repeat implies that \( F \) cannot be applied on the path between the successful repeat and its witness.

Proposition 4.7. Every successful branch \( \beta = \beta_0 \beta_1 \cdots \) in a \( \text{Focus}_\infty \)-proof \( \Pi = (T, P, \Sigma, R) \) contains a successful repeat.

Proof. First note that by definition of \( \text{Focus}_\infty \)-proofs, there are no applications of the discharge rule in \( \Pi \). In addition, since \( \beta \) is successful there is a \( k \) such that \( \beta_j \) contains a formula in focus for all \( j \geq k \).

Furthermore, by Proposition 4.4 all sequents along \( \beta \) are subsets of the finite set \( \text{Clos}(\Phi) \times \{u, f\} \), where \( \Phi \) is the set of formulas in the annotated sequent at the root of \( T \). It then follows by the pigeonhole principle that \( \beta \) must have a successful repeat.

Proposition 4.8. If an annotated sequent \( \Gamma \) is provable in \( \text{Focus}_\infty \) then it is provable in Focus.

Proof. Assume that \( \Pi = (T, P, \Sigma, R) \) is a proof for the annotated sequent \( \Gamma \) in \( \text{Focus}_\infty \). If \( \Pi \) is finite we are done, so assume otherwise; then by König’s Lemma the set \( B^\infty \) of infinite branches of \( \Pi \) is nonempty.

Because of Proposition 4.4 we may define for every infinite branch \( \tau \in B^\infty \) the number \( l(\tau) \in \omega \) as the least number \( n \in \omega \) such that \( \tau(n) \) is a successful repeat. This means that \( \tau(l(\tau)) \) is the first successful repeat on \( \tau \). Our first claim is the following:

there is no pair \( \sigma, \tau \) of infinite branches such that \( \sigma(l(\sigma)) \) is a proper ancestor of \( \tau(l(\tau)) \).  
(4)

To see this, suppose for contradiction that \( \sigma(l(\sigma)) \) is a proper ancestor of \( \tau(l(\tau)) \), then \( \sigma(l(\sigma)) \) actually lies on the branch \( \tau \). But this would mean that \( \sigma(l(\sigma)) \) is a successful repeat on \( \tau \), contradicting the fact that \( \tau(l(\tau)) \) is the first successful repeat on \( \tau \).

Our second claim is that

the set \( Y := \{ t \in T \mid t \) has a descendant \( \tau(l(\tau)) \), for some \( \tau \in B^\infty \} \) is finite.  
(5)

For a proof of (5), assume for contradiction that \( Y \) is infinite. Observe that \( Y \) is in fact (the carrier of) a subtree of \( (T, P) \), and as such a finitely branching tree. It thus follows by König’s Lemma that \( Y \) has an infinite branch \( \sigma \), which is then clearly also an infinite branch of \( \Pi \). Consider the node \( s := \sigma(l(\sigma)) \). Since \( \sigma \) is infinite, it passes through some proper descendant \( t \) of \( s \). This node \( t \) then belongs to the set \( Y \), so that by definition it has a descendant of the form \( \tau(l(\tau)) \) for some \( \tau \in B^\infty \). But then \( \sigma(l(\sigma)) \) is a proper ancestor of \( \tau(l(\tau)) \), which contradicts our earlier claim (4). It follows that the set \( Y \) is finite indeed.
Let $\gamma$ be a proof in $\Gamma$. For all nodes $u,v \in T$ such that $u \leq v$, we define the active trail relation $\mathcal{A}_{u,v} \subseteq \Phi_u \times \Phi_v$ and the passive trail relation $\mathcal{P}_{u,v} \subseteq \Phi_u \times \Phi_v$ by a case distinction depending on the rule that is applied at $u$:

**Definition 4.9.** Let $\Pi = (T, \Phi, \Sigma, \mathcal{R})$ be a proof in $\text{Focus}_\infty$. Similarly as for tableaux we define active and passive trail relations in a $\text{Focus}_\infty$-proof $\Pi = (T, \Phi, \Sigma, \mathcal{R})$. For all nodes $u,v \in T$ such that $P_{uv}$ we define the active trail relation $\mathcal{A}_{u,v} \subseteq \Phi_u \times \Phi_v$ and the passive trail relation $\mathcal{P}_{u,v} \subseteq \Phi_u \times \Phi_v$ by a case distinction depending on the rule that is applied at $u$:

**4.3 Trails in the focus system**

In the last part of this section we consider trails that run in branches of $\text{Focus}_\infty$-proofs. The notion is analogous to the trails in tableaux from Definition 3.3.

Let $\Pi = (T, \Phi, \Sigma, \mathcal{R})$ be a proof in $\text{Focus}_\infty$. Similarly as for tableaux we define active and passive trail relations in a $\text{Focus}_\infty$-proof $\Pi = (T, \Phi, \Sigma, \mathcal{R})$. For all nodes $u,v \in T$ such that $P_{uv}$ we define the active trail relation $\mathcal{A}_{u,v} \subseteq \Phi_u \times \Phi_v$ and the passive trail relation $\mathcal{P}_{u,v} \subseteq \Phi_u \times \Phi_v$ by a case distinction depending on the rule that is applied at $u$: 

Note that it obviously follows from $\hat{\gamma}$ that the set 
$$\hat{Y} := \{ \tau(l(\tau)) \mid \tau \in B^\infty \}$$

is finite as well. Recall that every element $l \in \hat{Y}$ is a successful repeat; we may thus define a map $c : \hat{Y} \to T$ by setting $c(l)$ to be the first ancestor $t$ of $l$ witnessing that $l$ is a successful repeat. Finally, let $\text{Ran}(c)$ denote the range of $c$.

We are almost ready for the definition of the finite tree $(T', P')$ that will support the proof $\Pi'$ of $\Gamma$; the only thing left to care of is the well-founded part of $\Pi'$. For this we first define $Z$ to consist of those successors of nodes in $Y$ that generate a finite subtree:

$$Z := R[Y] \cap HF(T, P),$$

then it is easy to show that the collection $R^*[Z]$ of descendants of nodes in $Z$ is finite.

With the above definitions we have all the material in hands to define a $\text{Focus}$-proof $\Pi' = (T', P', \Sigma', \mathcal{R}')$ of $\Gamma$. The basic idea is that $\Pi'$ will be based on the set $Y \cup R^*[Z]$, with the nodes in $\hat{Y}$ providing the discharged assumptions of $\Pi'$. Note however, that for a correct presentation of the discharge rule, every companion node $u$ of such a leaf in $\hat{Y}$ needs to be provided with a successor $u^+$ that is labelled with the same annotated sequent as the companion node and the leaf.

First of all we set

$$T' := Y \cup R^*[Z] \cup \{ u^+ \mid u \in \text{Ran}(c) \}$$

and

$$P' := \{(u,v) \in P \mid u \in T' \setminus \text{Ran}(c) \text{ and } v \in T' \}
\cup \{(u,u^+) \mid u \in \text{Ran}(c) \}
\cup \{(u^+,v) \mid u \in \text{Ran}(c), (u,v) \in P \}$$

The point of adding the nodes $u^+$ is to make space for applications of the rule $D^x$ at companion nodes. Furthermore, we put

$$\Sigma'(u) := \begin{cases} \Sigma(u) & \text{if } u \in T' \\ \Sigma(t) & \text{if } u = t^+ \text{ for some } t \in \text{Ran}(c). \end{cases}$$

Finally, for the definition of the rule labelling $R'$, we introduce a set $A := \{ x_u \mid u \in \text{Ran}(c) \}$ of discharge tokens, and we define

$$R'(u) := \begin{cases} R(u) & \text{if } u \in T' \setminus (\hat{Y} \cup \text{Ran}(c)) \\ x_{c(l)} & \text{if } u = l \in \hat{Y}, \\ D^x & \text{if } u \in \text{Ran}(c) \\ R(t) & \text{if } u = t^+ \text{ for some } t \in \text{Ran}(c). \end{cases}$$

It is straightforward to verify that with this definition, $\Pi'$ is indeed a $\text{Focus}$-proof of the sequent $\Gamma$. 

**4.3 Trails in the focus system**

In the last part of this section we consider trails that run in branches of $\text{Focus}_\infty$-proofs. The notion is analogous to the trails in tableaux from Definition 3.3.
A tree there is at most one unique path between any two nodes.

**Proposition 4.11.** Every infinite branch in a graph \(G\) is the identity relation, and if \(Puv\) and there is a path from \(w\) to \(v\) in \(G\) then we set \(T_{u,v} = T_{w,v} \cup T_{w,v}\), where we write \(\top\) for the composition of relations. This is well-defined because in a tree there is at most one unique path between any two nodes.

The central observation about the focus mechanism is that it enforces that every infinite branch in a Focus, proofs contains a successful trail:

**Proposition 4.11.** Every infinite branch in a Focus, proof \(\Pi\) carries a successful trail.

**Proof.** Consider an infinite branch \(\alpha\) in some Focus, proof \(\Pi\). Then \(\alpha\) is successful, so that we may fix an \(i\) such that \(\alpha_j\) contains a formula in focus for every \(j \geq i\). We claim that

\[
\text{for every } j \geq i \text{ and } \psi^j \in \Sigma_{\alpha_j+1}, \text{ there is some } \chi^j \in \Sigma_{\alpha_j} \text{ such that } (\chi^j, \psi^j) \in T_{\alpha_j, \alpha_j+1}. \tag{6}
\]

To see this, let \(j \geq i\) and \(\psi^j \in \Sigma_{\alpha_j+1}\). It is obvious that there is some annotated formula \(\chi^a \in \Sigma_j\) with \((\chi^a, \psi^j) \in T_{\alpha_j, \alpha_j+1}\). The key observation is that in fact \(a = f\), and this holds because otherwise the focus rule would have been applied when moving from \(\alpha_j\) to \(\alpha_{j+1}\), which is not possible because there is some formula in focus at \(\alpha_j\).

Now consider the graph \((V, E)\) where

\[
V \subseteq \{ (j, \varphi) \mid i \leq j < \omega \text{ and } \varphi^j \in \Sigma_{\alpha_j} \},
\]

and

\[
E \subseteq \{ (j, \varphi, (j+1, \psi)) \mid (\varphi, \psi) \in T_{\alpha_j, \alpha_{j+1}} \}.
\]

This graph is directed, acyclic, infinite and finitely branching, so by a (variation of) König’s Lemma there is an infinite path \(p^I_0 p^I_1 \cdots\) in this graph. This path is a path on \(\alpha\) because the formulas are related by the trail relation. The trail is successful because it is not possible to unravel a least fixpoint and end up with a formula of the from \(p^I_{j+1}\), simply because the rule \(R_{\mu}\) attaches the label \(\mu\) to the unravelling of the fixpoint.\(\square\)
5 Soundness

In this section we show that our proof systems are sound, meaning that any provable formula is valid. Because of the adequacy of the tableaux game that was established in Theorem 4.11, it suffices to show that for every provable formula Abelard has a winning strategy in some tableau for this formula. Moreover, we only need to consider proofs in Focus because by Theorem 4.2, every formula that is provable in Focus is also provable in Focus∞.

Theorem 5.1. If ϕ is provable in Focus∞, then there is some tableau T for ϕ such that Abelard has a winning strategy in T.

Proof. Assume there is a proof Π = (T, P, Σ, R) of {ϕ/} in Focus∞. We are going to construct a tableau T = (V,E,Φ,Q, vI) and a winning strategy for Abelard in G(T). Our construction will be such that (V,E) is a potentially infinite tree, of which the winning strategy S ⊆ V for Abelard is a subtree, as in Remark 5.9.

The construction of T and S proceeds via an induction that starts from the root and in every step adds children to one of the nodes in the subtree S that is not yet an axiom. Nodes of T that are not in S are always immediately completely extended using Proposition 5.2. Thus, they do not have to be treated in the inductive construction. The construction of S is guided by the structure of Π.

In addition to the tableau T, we will construct a function g : S → T mapping those nodes of T that belong to the strategy S to nodes of Π. This functions will satisfy the following three conditions, which will allow us to lift the successful trails from Π to S:

1. If Euν then P*(g(v))g(u).
2. If ϕa ∈ Σg(u) then ϕ ∈ Φg.
3. If Euν and (ψb,ϕc) ∈ TΠ g(v),g(u) then (ψ,ϕ) ∈ TΠ u,v.

We now describe the iterative construction of the approximating objects T1, S1 and g1 for all i ∈ ω, which in the limit will yield T, S and g. Each T1 will be a pre-tableau, that is, an object as defined in Definition 5.1 except that we do not require the rule labelling to be defined for every leaf of the tree. Leaves without labels will be called undetermined, and the basic idea underlying the construction is that each step will take care of one undetermined leaf.

We will make sure that in each step i of the construction, the entities T1, S1 and g1 satisfy the conditions 1, 2 and 3 and moreover ensure that all undetermined leaves of T1 belong to S1. It is easy to see that then also S and g satisfy these conditions.

In the base case we let T0 be the node vI labelled with just ϕ at the root of the tableau. We let g0(v0) be the root of the proof Π. The strategy S0 just contains the node v0.

In the inductive step we assume that we have already constructed a pre-tableau T1, a subtree S1 corresponding to Abelard’s strategy and a function g1 : S1 → T satisfying the above conditions 1, 2, 3.

To extend these objects further we fix an undetermined leaf l of S1. We may choose l such that its distance to the root of T1 is minimal among all the undetermined leaves of T1. This is necessary to ensure that every undetermined leaf gets treated eventually and thus the trees S and T in the limit do not contain any undetermined leaves. We distinguish cases depending on the rule that is applied in Π at g1(l).

Case R(g1(l)) = Axi1 or R(g1(l)) = Axi2: In this case we may simply label the node l with the corresponding axiom, while apart from this, we do not change T1, S1 or g1. Note that l will remain an (axiomatic) leaf of the tableau T.

Case R(g1(l)) = R∨: If the rule applied at g1(l) is R∨ with principal formula, say, (ϕ ∨ ψ)a, then we apply the disjunction rule at l. This is possible because by condition 2, the formula ϕ ∨ ψ occurs at l,
as it occurs in \( g_i(l) \). We extend \( T_i \), \( S_i \) and \( g_i \) accordingly, meaning that \( T_{i+1} \) is \( T_i \) extended with one node \( v \) that is labelled with the premise of the application of the disjunction rule, \( S_{i+1} \) is \( S_i \) extended to contain \( v \) and \( g_{i+1} \) is just like \( g_i \) but additionally maps \( v \) to the successor node of \( g_i(l) \) in \( T \). It is easy to check that with these definitions, the conditions 1–3 are satisfied.

**Case** \( R(g_i(l)) = R_\land \): In the case where \( R_\land \) is applied at \( g_i(l) \) with to a principal formula \((\varphi \land \psi)^a\) it follows that \( g_i(l) \) has a child \( t_\varphi \) for \( \varphi^a \) and a child \( t_\psi \) for \( \psi^a \). By condition 2 it follows that \( \varphi \land \psi \in \Phi_t \). We can then apply the conjunction rule at \( l \) to the formula \( \varphi \land \psi \) and obtain two new premises \( v_\varphi \) and \( v_\psi \) for each of the conjuncts. \( T_{i+1} \) is defined to extend \( T_i \) with these additional two children. We let \( S_{i+1} \) include both nodes \( v_\varphi \) and \( v_\psi \) as the conjunction rule belongs to Eloise in the tableaux game. Moreover, \( g_{i+1} \) is the same as \( g_i \) on the domain of \( g_i \), while it maps \( v_\varphi \) to \( t_\varphi \) and \( v_\psi \) to \( t_\psi \). It is easy to check that the conditions 1–3 are satisfied.

**Case** \( R(g_i(l)) = R_\lor \): We want to match this application of \( R_\lor \) in \( T \) with an application of the rule \( M \) in the tableau system. To make this work, however, two difficulties need to be addressed.

The first issue is that to apply the rule \( M \) in the tableau system, every formula in the consequent must be either atomic or modal, whereas the sequent \( \Phi_t \) may contain boolean or fixpoint formulas.

The second difficulty is that the rule \( R_\lor \) in the focus proof system has only one premise, whereas the tableau rule \( M \) has one premise for each box formula in the conclusion.

To address the first difficulty we step by step apply the Boolean rules \( (R_\lor, R_\land) \) to break down all the Boolean formulas in \( \Phi_t \) and the fixpoint rules \( (R_\mu, R_\nu) \) to unfold all fixpoint formulas. Because the rule \( R_\land \) is branching this process generates a subtree \( T_l \) at \( l \) such that all leaves of \( T_l \) contain literals and modal formulas only. Moreover, any modal formula from in \( \Phi_t \) is still present in \( \Phi_m \) for any such leaf \( m \) because modal formulas are not affected by the application of Boolean or fixpoint rules.

We add all nodes of \( T_l \) to the strategy \( S_\land \), and we define \( g_{i+1}(v) := g_i(v) \) for any \( v \) in this subtree. To see that this does not violate condition 2 or 3 note that all formulas in \( \Sigma_{g_i(l)} \) are modal and so, as we saw, remain present throughout the subtree.

We then want to expand any leaf in \( T_l \) by applying the modal rule \( M \). To see how this is done, fix such a leaf \( m \). Applying the modal rule of the tableau system at \( m \) generates a new child \( n_\varphi \) for every box formula \( \square \varphi \in \Phi_t \). At this point we have to solve our second difficulty mentioned above, which is to select one child \( n_\varphi \) to add into \( S_{i+1} \) and finish the construction of the tableau for all other children.

To select the appropriate child of \( m \), consider the unique box formula \( \square \varphi \) such that \( \square \varphi^a \in \Sigma_{g_i(l)} \) for some \( a \in \{ f, u \} \) — such a formula exists because \( R_\lor \) is applied at \( g_i(l) \). By condition 2 we then have \( \varphi \in \Phi_t \) and from this it follows, as we saw already, that \( \varphi \in \Phi_m \). We select the child \( n_\varphi \) of \( m \) to be added to \( S_{i+1} \) and set \( g_{i+1}(n_\varphi) = t \), where \( t \) is the unique child of \( g_i(l) \) in \( T \). It is not hard to see that this definition satisfies the conditions 2 and 3, because all diamond formulas in \( \Sigma_{g_i(l)} \) are also in \( \Phi_t \) and thus are still present in \( \Phi_m \).

We still need to deal with the other children of \( m \), since these are still undetermined but not in \( S_{i+1} \), something we do not allow in our iterative construction. To solve this issue we simply use Proposition 4.2 to obtain a new tree-shaped tableau \( T_k \) for any such child \( k \) of \( m \) with \( k \neq n_\varphi \). For the definition of \( T_{i+1} \) we append \( T_k \) above the child \( k \). Hence, the only undetermined leaf that is left above \( m \) in \( T_{i+1} \) is the node \( n_\varphi \), which belongs to \( S_{i+1} \).

**Case** \( R(g_i(l)) = R_\mu \) or \( R(g_i(l)) = R_\nu \): The case for the fixpoint rules is similar to the case for \( R_\lor \), we just apply the corresponding fixpoint rule on the tableau side.

**Case** \( R(g_i(l)) = W \): Note that in this case the sequent \( \Sigma_t \), associated with the successor node \( t \) of \( g(l) \), being the premise of an application of the weakening rule, is a (proper) subset of the consequent sequent \( \Sigma_{g_i(l)} \). In this case we simply define \( T_{i+1} := T_i \) and \( S_{i+1} := S_i \), but we modify \( g_i \) so that \( g_{i+1} : S_{i+1} \rightarrow T \) maps \( g_{i+1}(l) = t \) and \( g_{i+1}(k) = g_i(k) \) for all \( k \neq l \). This clearly satisfies condition 1. To see that it satisfies the other two conditions we use the facts that \( \Sigma_t \subseteq \Sigma_{g_i(l)} \), and that the trail relation for the weakening rule is trivial.

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However, after applying this step we still have that \( l \) is an undetermined leaf of \( T_{i+1} \). Thus the construction does not really make progress in this step and one might worry that not all undetermined leaves get eventually. We address this matter further below.

**Case** \( R(g_i(l)) = F \): The case for the focus change rule \( F \) is analogous to the previous case for the weakening rule \( W \). The fact that the annotations of formulas change has no bearing on the conditions.

We now address the problem that in the cases for \( W \) and \( F \), we do not extend \( T_i \) at its undetermined leaf \( l \). Thus, without further arguments it would seem possible that the construction loops through these cases without ever making progress at the undetermined leaf \( l \). To see that this can not happen note first that in each of these cases we are moving on in the proof \( \Pi \) in the sense that \( g_{i+1}(l) \neq g_i(l) \) and \((g_i(l), g_{i+1}(l)) \in P \). Thus, if we would never make progress at \( l \) this means that we would need to follow an infinite path in \( \Pi \) such that at every node on this path is either labelled with \( W \) or with \( F \). However, this would contradict Proposition 4.5 because every infinite branch in \( \Pi \) is successful.

It remains to be seen that \( S \) is a winning strategy for Abelard. It is clear that Abelard wins all finite matches that are played according to \( S \) because by construction all leaves in \( S \) are axioms. To show that all infinite matches are winning, consider an infinite path \( \beta = \beta_0\beta_1\ldots \) in \( S \). We need to show that \( \beta \) contains a \( \nu \)-trail. Using condition 1 it follows that there is an infinite path \( \alpha = \alpha_0\alpha_1\ldots \) in \( \Pi \) such that for every \( i \in \omega \) we have that \( g(\beta_i) = \alpha_{k_i} \) for some \( k_i \in \omega \). Moreover, if \( i \leq j \) then \( k_i \leq k_j \).

By Proposition 4.11 the infinite path \( \alpha \) contains a successful trail \( \varphi_{a_0}^a\varphi_{a_1}^a \ldots \). With condition 6 it follows that then \( \varphi_{k_0}\varphi_{k_1}\varphi_{k_2} \ldots \) is a trail in \( \beta \). Because the trail in \( \alpha \) was successful in the sense of Definition 4.9 it follows that the trail in \( \varphi_0\varphi_1 \ldots \) in \( \beta \) is a \( \nu \)-trail. 

\[ \square \]
6 Completeness

In this section we show that the focus systems are complete, that is, every valid formula is provable. As for the soundness argument in the previous section, we rely on Theorem 3.11 which states that Abelard has a winning strategy in any tableau for a given valid formula, and on Theorem 4.2 which claims that every formula that is provable in Focus\(_\infty\) is also provable in Focus. Thus, it suffices to show that winning strategies for Abelard in the tableau game can be transformed into Focus\(_\infty\)-proofs.

Theorem 6.1. If Abelard has a winning strategy in some tableau game for a formula \(\varphi\) then \(\varphi\) is provable in Focus\(_\infty\).

Proof. Let \(T = (V, E, \Phi, Q, v_T)\) be a tableau for \(\varphi\) and let \(S\) be a winning strategy for Abelard in \(T\). Because of Proposition 3.2, Corollary 3.12 and Remark 3.9 of we may assume that \(T\) is tree based, with root \(v_T\), and that \(S \subseteq V\) is a subtree of \(T\). We are going to construct a Focus\(_\infty\)-proof \(\Pi = (T, P, \Sigma, R)\) for \(\varphi_f\).

We construct the pre-proof \(\Pi\) of \(\varphi_f\) together with a function \(g: S \rightarrow T\) in such a way that the following conditions are satisfied:

1. If \(Evu\) then \(P^+ g(v)g(u)\).
2. For every \(v \in S\) and every infinite branch \(\beta = \beta_0\beta_1 \cdots\) in \(\Pi\) with \(\beta_0 = g(v)\) there is some \(i \in \omega\) and some \(u \in S\) such that \(Evu\) and \(g(u) = \beta_i\).
3. For every \(\varphi \in \Phi_v\) there is a unique \(a_\varphi \in \{f, u\}\) such that \(\varphi^{a_\varphi} \in \Sigma_{g(v)}\).
4. If \(Evu\) and \((\varphi, \psi) \in T_{v,u}\) then \((\varphi^{a_\varphi}, \psi^{a_\varphi}) \in T_{g(v),g(u)}\).
5. If \(Evu,\) and \(s\) and \(t\) are nodes on the path from \(g(v)\) to \(g(u)\) such that \(P^+ st, (\chi^a, \varphi^f) \in T_{g(v),s}\) for some \(a \in \{f, u\}\) and \((\varphi^f, \psi^a) \in T_{s,t}\), then \(\chi = \varphi\) and \(\chi\) is a \(\mu\)-formula.
6. If \(\alpha\) is an infinite branch of \(\Pi\) and \(F\) is applicable at some node on \(\alpha\), then \(F\) is applied at some later node on \(\alpha\).

The purpose of these conditions is that they allow us to prove later that every branch in \(\Pi\) is successful.

We construct \(\Pi\) and \(g\) as the limit of finite stages, where at stage \(i\) we have constructed a finite pre-proof \(\Pi_i\) and a partial function \(g_i: S \rightarrow \Pi_i\). At every stage we make sure that \(g_i\) and \(\Pi_i\) satisfy the following conditions:

7. All open leaves of \(\Pi_i\) are in the range of \(g_i\).
8. For all nodes \(v \in S\) of \(S\) such that \(g_i\) is defined on \(v\) we have that

\[\Phi_v = \{ \varphi \mid \varphi^a \text{ is in the sequent at } g_i(v) \text{ in } \Pi_i \text{ for some } a \in \{f, u\} \}.\]

In the base case we define \(\Pi_0\) to consist of just one node \(r\) that is labelled with the sequent \(\varphi_f\). The partial function \(g_0\) maps \(r\) to \(v_T\). Clearly, this satisfies the conditions 7 and 8.

In the inductive step we consider any open leaf \(m\) of \(\Pi_i\), which has a minimal distance from the root of \(\Pi_i\). This ensures that in the limit every open leaf is eventually treated, so that \(\Pi\) will not have any open leaves. By condition 7 there is a \(u \in S\) such that \(g(u) = m\).

Our plan is then to extend the proof \(\Pi_i\) at the open leaf \(m\) to mirror the rule that is applied at \(u\) in \(T\). In general this is possible because by condition 8 the formulas in the annotated sequent at \(m = g_i(u)\) are the same as the formulas at \(u\). All children of \(u\) that are in \(S\) should then be mapped by \(g_{i+1}\) to new open leaves in \(\Pi_{i+1}\). This guarantees that condition 7 is satisfied at step \(i + 1\) and because we are going to simulate the rule in the tableau by rules in the focus system we ensure that
condition 5 holds at these children as well. Clearly, the precise definition of \( \Pi_{i+1} \) depends on the rule applied at \( u \). Before going into the details we address two technical issues that feature in all the cases.

First, to ensure that condition 5 is satisfied by our construction we will apply \( F \) at \( m \), whenever it is applicable. Thus, we need to check whether all formulas in the sequent of \( m \) are annotated with \( u \). If this is the case then we apply the focus rule and proceed with the premise \( n \) of this application of the focus rule. Otherwise we just proceed with \( n = m \). Note that in either case the sequent at \( n \) contains the same formulas as the sequent at \( m \) and if \( n \neq m \) then the trace relation relates the formulas at \( n \) in an obvious way to those at \( m \).

The second technical issue is that to ensure condition 4 we may need to apply \( W \) to the new leaves of \( \Pi_{i+1} \). To see how this is done assume we have already extended \( \Pi_i \) and obtained a new leaf \( v \) which we would like to add into the range of \( g_{i+1} \). The annotated sequent at \( v \), however, might contain both instances \( \varphi^f \) and \( \varphi^u \) of some formula \( \varphi \), which would violate condition 3. To take care of this we apply \( W \) to get rid of the unfocused occurrence \( \varphi^u \). in fact, we might need to apply \( W \) multiple times to get rid of all unfocused duplicates of formulas. In the following we will refer to the node of the proof, that is obtained by repeatedly applying \( W \) in this way at an open leaf \( l \), as the normalization of \( l \).

We are now ready to discuss the main part of the construction, which is based on a case distinction depending on the rule \( Q(u) \) that is applied at \( u \).

**Case** \( Q(u) = Ax1 \) or \( Q(u) = Ax2 \): In this case we can just apply the corresponding rule at \( m = g(u) \). We might need to apply \( W \) to get rid of side formulas that where present in the tableau. There is no need to extend \( g \).

**Case** \( Q(u) = R_v \): In this case we can just apply \( R_v \) at \( m \). This generates a new open leaf \( l \) which corresponds to the successor node \( v \) of \( u \) in the tableau. We define \( g_{i+1} \) such that it maps \( v \) to the normalization of \( l \).

**Case** \( Q(u) = R_\Box \): In this case we also apply \( R_\Box \) in the focus system at \( m \). This generates two successors which we can associate with the two children of \( u \), both of which must be in \( S \). Thus, \( g_{i+1} \) will map the children of \( u \) to the normalizations of the successors we have added to \( m \).

**Case** \( Q(u) = M \): In this case we want to apply the rule \( R_\Box \) in the focus system. However, the sequent \( \Sigma_m \) might contain multiple box formulas, whereas \( R_\Box \) can only be applied to one of those. To select the proper formula \( \Box \varphi^u \in \Sigma_m \) we use the fact that the successors of \( u \) are indexed by the box formulas in \( \Phi_u \), and that the strategy \( S \) contains precisely one these successors. That is, let \( \Box \varphi^u \in \Sigma_m \) be such that its associated successor \( v_{\Box} \) of \( u \) belongs to \( S \). Then apply \( W \) at \( m \) until we have removed all formulas from the sequent that are not diamond formulas and that are distinct from \( \Box \varphi \). Once this is done the sequent only contains annotated versions of the diamond formulas from \( \Phi_u \) plus an annotated version of the formula \( \Box \varphi \). We can then apply \( R_\Box \) and obtain a new node \( l \) and we set \( g_{i+1}(v_{\Box}) = l \).

**Case** \( Q(u) = R_\Diamond \) or \( Q(u) = R_\Diamond \): This is analogous to the case for \( R_\Box \). Note, however, that the application of the fixpoint rules in the focus system has an effect on the annotation.

We define the function \( g : S \rightarrow T \) as the limit of the maps \( g_i \). To see that \( g \) is actually a total function, first observe that for every \( v \in S \) and \( i \in \omega \) either \( v \) is already in the domain of \( g_i \), in which case it is in the domain of \( g \), or there is some node \( u \) on the branch leading to \( v \) that is mapped by \( g_i \) to an open leaf of \( \Pi_i \). Eventually, the proof is extended at this leaf because in every step we treat an open leaf that is maximally close to the root. It is easy to check that in every step, when we extend the proof \( \Pi_j \) at some open leaf, we also move forward on the branches of \( T \) that run through \( v \). Iterating this reasoning shows that eventually \( v \) must be added to the domain of some \( g_j \).

We now show that \( g \), together with \( \Pi \), satisfies the conditions 1–4. To start with, it is clear from the step-wise construction of \( g \) and \( \Pi \) that condition 1 is satisfied.

Condition 2 holds because all trees \( \Pi_i \) are finite. Thus, on every infinite branch of \( \Pi \) there are infinitely many nodes that are a leaf in some \( \Pi_i \) and by condition 7 each of these nodes is in the range
of $g_t$ and thus of $g$.

Condition 3 is obviously satisfied at the root of $\Pi$. It is satisfied at all other nodes because of condition 5 and because we make sure that we only add nodes to the domain of $g$ that are normalized, using the procedure described above.

To see that condition 4 is satisfied by $\Pi$ and $g$ one has to carefully inspect each case of the inductive definition of $\Pi$. This is tedious but does not give rise to any technical difficulties.

To check condition 5 note that if $(\varphi^f, \psi^m) \in T_s$ then the trace from $\varphi^f$ to $\psi^m$ must loose its focus at some point on the path from $s$ to $t$. The only case of the inductive construction of $\Pi$ where this is possible is the case where $\mathsf{Q}(u) = R_u$ and in this case the formula that losse its focus is the active formula, which is then a $\mu$-formula and already present at the open leaf that we are extending.

For condition 6 first observe that if $F$ is applicable at some node that is an open leaf of some $\Pi$, then it will be applied immediately when this open leaf is taken care of. Moreover, it is not hard to see that if $F$ becomes applicable at some node $v$ during some stage $i$ of the construction of $\Pi$, then it will remain applicable at every node that is added above $v$ at this stage. This applies in particular to the new open leaves that get added above $v$, and so the focus rule will be applied to each of these at a later stage of the construction.

It remains to be shown that every infinite branch in $\Pi$ is successful. Let $\beta = \beta_0 \beta_1 \cdots$ be such a branch. We claim that

$$a_n = f \text{ for all } n > l. \quad (7)$$

and to prove (7) we will link $\beta$ to a match in $S$. Observe that because of condition 2 we can ‘lift’ $\beta$ to a branch $\alpha = \alpha_0 \alpha_1 \cdots$ in $S$ such that there are $0 = k_0 \leq k_1 \leq k_2 \leq \cdots$ with $g(\alpha_i) = \beta_k$ for all $i < \omega$.

Because $\alpha$, as a match of the tableau game, is won by Abelard, it contains a $\nu$-trace $\varphi_0 \varphi_1 \cdots$. This trace being a $\nu$-trace means that there is some $m \in \omega$ such that $\varphi_h$ is a $\mu$-formula for no $h \geq m$. We then use condition 4 to obtain a trace $\psi_0 \psi_1 \cdots$ in $\beta$ such that $\varphi_i = \psi_h$. Now distinguish cases.

First assume that there is an application of the focus rule at some $\beta_i$, with $l \geq k_i$. Then at $\beta_{i+1}$ all formulas are in focus and thus in particular the annotation $a_{t+1}$ of the formula $\psi_{t+1}$ must be equal to $f$. We show that

$$a_n = f \text{ for all } n > l. \quad (8)$$

Assume for contradiction that this is not the case and let $n$ be the smallest number larger than $l$ such that $a_t = u$; since $a_{t+1} = f$ we find that $n > l + 1$, and by assumption on $n$ we have $a_{t-1} = f$. Now let $h$ be such that $\beta_{n-1}$ and $\beta_n$ are on the path between $g(\alpha_h) = \beta_k$ and $g(\alpha_{h+1}) = \beta_{k+1}$; since $k_m \leq l \leq n - 1$ it follows that $h \geq m$. But then by condition 5 $\varphi_h$ must be a $\mu$-formula, which contradicts our observation above that $\varphi_h$ is not a $\mu$-formula for any $h \geq m$. This proves (8), which means that for every $n > l$, the formula $\psi_n$ is in focus at $\beta_n$. From this $\Pi$ is immediate.

If, on the other hand, there is no application of the focus rule on $\beta_{k_m} \beta_{k_{m+1}} \cdots$ then it follows by condition 6 that the focus rule is not applicable at any sequent $\beta_l$ with $l \geq k_m$. In other words, all these sequents contain a formula in focus, which proves (7) indeed.
7 Interpolation

In this section we will show that the alternation-free fragment of the modal \( \mu \)-calculus enjoys the Craig interpolation property. To introduce the actual statement that we will prove, consider an implication of the form \( \varphi \to \psi \), with \( \varphi, \psi \in \mathcal{L} \). First of all, we may without loss of generality assume that \( \varphi \) and \( \psi \) are guarded, so that we may indeed take a proof-theoretic approach using the Focus system. Given our interpretation of sequents, we represent the implication \( \varphi \to \psi \) as the sequent \( \varphi, \psi \), and similarly, the implications involving the interpolant \( \theta \) can be represented as, respectively, the sequents \( \varphi, \theta \) and \( \theta, \psi \). What we will prove below is that for an arbitrary derivable sequent \( \Sigma \), and an arbitrary partition \( \Sigma = \Sigma^L, \Sigma^R \) of \( \Sigma \), there is an interpolant \( \theta \) such that the sequents \( \Sigma^L, \theta \) and \( \Sigma^R, \theta \) are both provable.

Before we can formulate and prove our result, we need some preparation. First of all, we will assume that in our Focus proofs every application of the discharge rule discharges at least one assumption, i.e., every node in the proof that is labelled with the discharge rule is the companion of at least one leaf. It is easy to see that we can make this assumption without loss of generality — we leave the details to the reader.

Second, our proof will comprise a proof transformation of a Focus-derivation of the sequent \( \Sigma \) into derivations of the mentioned sequents \( \Sigma^L, \theta \) and \( \Sigma^R, \theta \). Note however, that the latter derivations may use two additional proof rules, viz., the generalized focus rule \( F^+ \) and the unfocus rule \( U \) given in Figure 3.

\[
\frac{\Phi f, \Sigma}{\Phi^+, \Sigma} F^+ \quad \frac{\Phi u, \Sigma}{\Phi f, \Sigma} U
\]

Figure 3: Two additional proof rules

In words, the rule \( F^+ \) generalises the ordinary focus rule \( F \) in that it does not require all formulas in the consequent to be unfocused, and that it does not require all unfocused formulas in the consequent to be put in focus. Similar to this, but in the opposite direction, the unfocus rule \( U \) is used to drop the focus from any number of formulas in the conclusion.

**Definition 7.1.** The systems \( \text{Focus}^+ \) and \( \text{Focus}_\infty^+ \) are the extensions with the rules \( F^+ \) and \( U \) of, respectively, the systems \( \text{Focus} \) and \( \text{Focus}_\infty \).

For both systems the notion of a (pre-)proof (with or without assumptions) are defined as in Definition 4.1 with the following adaptations. Condition 4 is modified so that it does not allow for the application of \( F^+ \) or \( U \) on the path to a leaf from its companion. Analogously, a successful infinite branch in \( \text{Focus}_\infty^+ \)-proofs only allows for finitely many applications of \( F \), \( F^+ \) and \( U \) (and as before for \( \text{Focus}_\infty \)-proofs, the nodes on such a branch eventually always feature a formula in focus).

Clearly the proof systems \( \text{Focus}^+ \) and \( \text{Focus}_\infty^+ \) are only useful as tools in an interpolation proof if they are sound.

**Proposition 7.2.** Let \( \varphi \in \mathcal{L} \) be provable in either \( \text{Focus}^+ \) or \( \text{Focus}_\infty^+ \). Then \( \varphi \) is valid.

**Proof.** (Sketch) To see this one can check that the argument from Proposition 4.3 and Section 5 also works for these systems. For Proposition 4.3 this works because this proposition just depends on the match between the constraint on the path between a discharged leaf and its companion in \( \text{Focus}_\infty^+ \)-proofs and the constraint on successful infinite paths in \( \text{Focus}_\infty^+ \)-proofs. It thus follows that the soundness of \( \text{Focus}_\infty^+ \) entails the soundness of \( \text{Focus}^+ \). The soundness of \( \text{Focus}_\infty^+ \) can be proved similarly to the soundness of \( \text{Focus}_\infty \) in Theorem 5.1. This proof of this theorem can be adapted directly to \( \text{Focus}_\infty^+ \), however, this requires analogues of Propositions 4.5 and 4.11 for \( \text{Focus}_\infty^+ \).

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Proposition 4.5: This proposition needs that on every infinite branch there are only finitely many applications of trivial focus-management rules that do not change any of the formulas in the sequent. This is guaranteed by the condition on infinite branches in Focus_\infty proofs.

Proposition 4.11: This proposition relies on the property that on every infinite branch, from some moment on, every formula in focus at some node on the branch, has a predecessor in the trail relation that is in focus at the parent of mentioned node. This also holds for infinite branches in Focus_\infty-proofs because these can only have finitely many applications of rules that lack this property, viz., F and F^+.

Furthermore, it will be convenient for us to fine-tune the notion of a partition in the following way.

Definition 7.3. A partition of a set \( A \) is a non-empty finite tuple \((A_1, \ldots, A_n)\) of pairwise disjoint subsets of \( A \) such that \( \bigcup_{i=1}^n A_i = A \). A binary partition of \( A \) may be denoted as \( A^L \mid A^R \); in this setting we may refer to the members of \( A^L \) and \( A^R \) as being left and right elements of \( A \), respectively.

Finally, to formulate the condition on an interpolant, note that we may identify the vocabulary of a sequent \( \Sigma \) simply with the set \( \text{FV}(\Sigma) \) of free variables occurring in \( \Sigma \). Our interpolation result can then be stated as follows:

Theorem 7.4 (Interpolation). Let \( \Pi \) be a proof of some sequent \( \Phi \), and let \( \Phi^L \mid \Phi^R \) be a partition of \( \Phi \). Then there are a formula \( \theta \) with \( \text{FV}(\theta) \subseteq \text{FV}(\Phi^L) \cap \text{FV}(\Phi^R) \), and Focus_\infty-proofs \( \Pi^L, \Pi^R \), all effectively obtainable from \( \Pi, \Phi^L \) and \( \Phi^R \), such that \( \Pi^L \) derives the sequent \( \Phi^L, \theta \) and \( \Pi^R \) derives the sequent \( \Phi^R, \theta \).

The remainder of this section contains the proof of this theorem. We first consider the definition of interpolants for the conclusion of a single proof rule, under the assumption that we already have interpolants for the premises. We then show in Proposition 7.8 that this definition is well-behaved. We need some additional auxiliary definitions.

In this section it will be convenient to define the negation of \( \theta \) in a slightly simpler manner than in section 2. This is possible since the bound variables of \( \theta \) will be taken from the set \( \mathcal{D} \) of discharge tokens, which is disjoint from the collection of variables used in the formulas featuring in \( \Pi \).

Definition 7.5. Given a formula \( \phi \) such that \( \text{BV}(\phi) \subseteq \mathcal{D} \), we define the formula \( \overline{\phi} \) as follows. For atomic \( \phi \) we define

\[
\overline{x} := \begin{cases} x & \text{if } x \in \mathcal{D} \\ \overline{\phi} & \text{otherwise}, \end{cases}
\]

and then we inductively continue with

\[
\begin{align*}
\overline{\phi \land \psi} & := \overline{\phi} \lor \overline{\psi} \\
\overline{\phi \lor \psi} & := \overline{\phi} \land \overline{\psi} \\
\overline{\mu x. \phi} & := \nu x. \overline{\phi} \\
\overline{\nu x. \phi} & := \mu x. \overline{\phi}
\end{align*}
\]

It is not hard to see that \( \overline{\theta} = \theta \) precisely if \( \text{FV}(\theta) \) does not contain any discharge token from \( \mathcal{D} \) as a free variable. For atomic formulas \( \phi \) that are not of the form \( x \in \mathcal{D} \) we will continue to write \( \overline{\phi} \) rather than \( \overline{x} \).

Definition 7.6. A formula is basic if it is either of the form \( p \) or \( \overline{p} \) for some proposition letter \( p \), or of the form \( x, x_0 \land x_1, x_0 \lor x_1, \Diamond x \) or \( \Box x \), where \( x, x_0 \) and \( x_1 \) are discharge tokens.

Definition 7.7. Let \( R \) be some derivation rule, let
be an instance of \( R \), and let \( \Sigma^L | \Sigma^R \) be a partition of \( \Sigma \). By a case distinction as to the nature of the rule \( R \) we define a basic formula \( \chi(x_0, \ldots, x_{n-1}) \), together with a partition \( \Sigma^L_i | \Sigma^R_i \) for each \( \Sigma_i \). Here the variables \( x_0, \ldots, x_{n-1} \) correspond to the premises of the rule.

**Case** \( R = \text{Ax1} \). Let \( \Sigma \) be of the form \( \Sigma = \{ p, \overline{p} \} \), and observe that since there are no premises, we only need to define the formula \( \chi \). For this purpose we make a further case distinction as to the exact nature of the partition.

- If \( \Sigma^L | \Sigma^R = p^a | \overline{p}^b \), define \( \chi := \overline{p} \).
- If \( \Sigma^L | \Sigma^R = \overline{p}^a | p^b \), define \( \chi := p \).
- If \( \Sigma^L | \Sigma^R = p^a, \overline{p}^b | \emptyset \), define \( \chi := \bot \).
- If \( \Sigma^L | \Sigma^R = \emptyset | p^a, \overline{p}^b \), define \( \chi := \top \).

**Case** \( R = \text{Ax2} \). Here \( \Sigma \) must be of the form \( \Sigma = \{ \top \} \), and, as in the case of the other axiom, we only need to define the formula \( \chi \) since there are no premises. We make a further case distinction.

- If \( \Sigma^L | \Sigma^R = \top^a | \emptyset \), define \( \chi := \bot \).
- If \( \Sigma^L | \Sigma^R = \emptyset | \top^a \), define \( \chi := \top \).

**Case** \( R = R_\land \). We distinguish cases, as to which side the active formula \((\varphi_0 \land \varphi_1)^a\) belongs to.

**Subcase** \((\varphi_0 \land \varphi_1)^a \in \Sigma^L \). We may then represent the partition of \( \Sigma \) as \((\varphi_0 \land \varphi_1)^a, \Sigma_0 | \Sigma_1 \).

Here we define \( \chi(x_0, x_1) := x_0 \lor x_1 \), and we partition the premises of \( R_\land \) as, respectively, \( \varphi_0^a, \Sigma_0 | \Sigma_1 \setminus \{ \varphi_1^a \} \) and \( \varphi_1^a, \Sigma_0 | \Sigma_1 \setminus \{ \varphi_0^a \} \).

**Subcase** \((\varphi_0 \land \varphi_1)^a \in \Sigma^R \). We may now represent the partition of \( \Sigma \) as \( \Sigma_0 | \Sigma_1, (\varphi_0 \land \varphi_1)^a \).

Now we define \( \chi(x_0, x_1) := x_0 \land x_1 \), and we partition the premises of \( R_\land \) as, respectively, \( \Sigma_0 \setminus \{ \varphi_0^a \} | \Sigma_1, \varphi_0^a \) and \( \Sigma_0 \setminus \{ \varphi_1^a \} | \Sigma_1, \varphi_1^a \).

**Case** \( R = R_0 \). We only consider the case where the active formula \((\varphi_0 \lor \varphi_1)^a\) belongs to \( \Sigma^L \) (the other case is symmetric). We may then represent the partition of \( \Sigma \) as \((\varphi_0 \lor \varphi_1)^a, \Sigma_0 | \Sigma_1 \).

Here we define \( \chi(x_0) := x_0 \), and we partition the premise of \( R_0 \) as \( \varphi_0^a, \varphi_1^a, \Sigma_0 | \Sigma_1 \setminus \{ \varphi_0^a, \varphi_1^a \} \).

**Case** \( R = R_3 \). We distinguish cases, as to whether the active formula \( \Box \varphi^a \) belongs to \( \Sigma^L \) or to \( \Sigma^R \).

**Subcase** \( \Box \varphi^a \in \Sigma^L \). We may then represent the partition of \( \Sigma \) as \( \Box \varphi^a, \emptyset \Sigma_0 | \Sigma_1 \). We define \( \chi := \Box x_0 \) and we partition the premise of \( R_3 \) as \( \varphi, \Sigma_0 | \Sigma_1 \setminus \{ \varphi \} \).

**Subcase** \( \Box \varphi^a \in \Sigma^R \). We may then represent the partition of \( \Sigma \) as \( \Sigma_0 | \Sigma_1, \Box \varphi^a \).

Now we define \( \chi := \Box x_0 \) and we partition the premise of \( R_3 \) as \( \Sigma_0 \setminus \{ \varphi \} | \Sigma_1, \varphi \).

**Case** \( R = R_\mu \). We only consider the case where the active formula \( \mu x. \varphi^a \) belongs to \( \Sigma^L \) (the other case is symmetric). We may then represent the partition of \( \Sigma \) as \( \mu x. \varphi^a, \Sigma_0 | \Sigma_1 \).

Here we define \( \chi(x_0) := x_0 \), and we partition the premise of \( R_\mu \) as \( \varphi(\mu x. \varphi)^a, \Sigma_0 | \Sigma_1 \setminus \{ \varphi(\mu x. \varphi)^a \} \).

**Case** \( R = R_\nu \). The definitions are analogous to the case of \( R_\mu \).

**Case** \( R = W \). We only consider the case where the active formula \( \varphi^a \) belongs to \( \Sigma^L \) (the other case is symmetric). We may then represent the partition of \( \Sigma \) as \( \varphi^a, \Sigma_0 | \Sigma_1 \). Here we define \( \chi(x_0) := x_0 \), and we partition the premise of \( W \) as \( \Sigma_0 | \Sigma_1 \).
Case $R = F$. Let $\Sigma = \Phi^u$, we may then represent the partition of $\Sigma$ as $\Phi^u_0 | \Phi^u_1$. In this case we define $\chi(x_0) := x_0$, and we partition the premise of $F$ as $\Phi^f_0 | \Phi^f_1$.

Case $R = D$. In this case the premise and the conclusions are the same, and so we also partition the premise in the same way as the conclusion. Furthermore, we define $\chi := x_0$.

Proposition 7.8 (Interpolation Transfer). Let

\[
\begin{array}{c}
\Sigma_0 \\
\vdots \\
\Sigma_{n-1}
\end{array}
\]

be an instance of some derivation rule $R \neq D$, let $\Sigma^L_i | \Sigma^R_i$ be a partition of $\Sigma$, and let $\chi$ and $\Sigma^L_i | \Sigma^R_i$, for $i = 0, \ldots, n - 1$ be as in Definition 7.7. Then the following hold.

1) $FV(\Sigma^L_i) \subseteq FV(\Sigma^R_i)$ where $K \in \{L, R\}$.
2) For any sequence $\theta_0, \ldots, \theta_{n-1}$ of formulas and any $b \in \{u, f\}$ there are derivations $\Xi^L$ and $\Xi^R$:

\[
\begin{array}{c}
\Sigma^L_0, \theta_0^b \\
\vdots \\
\Sigma^L_{n-1}, \theta_{n-1}^b \\
\Sigma^L, \chi(\theta_0, \ldots, \theta_{n-1})^b
\end{array}
\quad
\begin{array}{c}
\Sigma^R_0, \theta_0^b \\
\vdots \\
\Sigma^R_{n-1}, \theta_{n-1}^b \\
\Sigma^R, \chi(\theta_0, \ldots, \theta_{n-1})^b
\end{array}
\]

Provided that $R \neq F$, these derivations satisfy the following conditions:

a) $\Xi^L$ and $\Xi^R$ do not involve the rules $F$, $F^+$ or $U$.

b) If, for some $i$, the assumption $\Sigma^L_i, \theta_i^b$ contains a formula in focus, then so does every sequent in $\Xi^L$ on the path to this assumption.

c) If, for some $i$, the assumption $\Sigma^R_i, \theta_i^b$ contains a formula in focus, then so does every sequent in $\Xi^L$ on the path to this assumption.

Proof. The proof of both parts proceeds via a case distinction depending on the proof rule $R$, following the case distinction in Definition 7.7. Part 1 easily follows from a direct inspection. For part 2 we restrict attention to some representative cases.

Below we use $(W)$ as a ‘proof rule’ in the sense that, in a proof, we draw the configuration $\Gamma_t : (W)$ to indicate that either $\Gamma_t$ is a proper subset of $\Gamma_s$, in which case we are witnessing an application of the weakening rule at node $s$, or else there is only one single node $s = t$ labelled with $\Gamma_s = \Gamma_t$.

Case $R = Ax1$. As an example consider the case where the partition is such that $\Sigma^L | \Sigma^R = p^a | \overline{p}^a$. Then we have by definition that $\chi = \overline{p}$ and hence we need to supply proofs for the annotated sequents $\Sigma^L, \chi^b = p^a, \overline{p}^a$ and $\Sigma^R, \chi^b = \overline{p}^a, p^a$. Both of these can easily be proved with the axiom Ax1.

As a second example consider the case where the partition is such that $\Sigma^L | \Sigma^R = p^a, \overline{p}^a | \emptyset$. Then we have that $\chi = \bot$ and hence need to provide proofs for the sequents $\Sigma^L, \chi^b = p^a, \overline{p}^a, \bot^b$ and $\Sigma^R, \chi^b = \bot^b$. The latter is proved with Ax2 and for the former we use the proof:

\[
\begin{array}{c}
p^a, \overline{p}^a, \bot^b \quad Ax1 \\
p^a, \overline{p}^a, \bot^b \quad W
\end{array}
\]
\textbf{Case }R = R_\wedge. \text{ First assume that the active formula } (\phi_0 \land \phi_1)^a \text{ belongs to } \Sigma^L. \text{ We may then represent the partition of } \Sigma \text{ as } (\phi_0 \land \phi_1)^a, \Sigma_0 \mid \Sigma_1. \text{ For the claim of the proposition, the following derivations suffice:}

\[
\begin{array}{c}
\Sigma_0, \phi_0^a, \phi_1^b, \theta_0^b \quad \text{W} \\
\Sigma_0, \phi_0^a, \phi_1^b, \theta_0^b \quad \text{R}_\vee \\
\Sigma_0, \phi_0^a, (\theta_0 \lor \theta_1)^b \quad \text{R}_\wedge
\end{array}
\]

\[
\begin{array}{c}
\Sigma_1 \setminus \{\phi_0^a, \phi_1^b\}, \theta_0^b \quad \text{W} \\
\Sigma_1 \setminus \{\phi_0^a, \phi_1^b\}, \theta_0^b \quad \text{R}_\vee \\
\Sigma_1, (\theta_0 \land \theta_1)^b \quad \text{R}_\wedge
\end{array}
\]

\[
\Sigma_0, (\phi_0 \land \phi_1)^a, (\theta_0 \lor \theta_1)^b
\]

We then consider the other possibility, where the active formula \((\phi_0 \land \phi_1)^a\) belongs to \(\Sigma^R\). We may represent the partition of \(\Sigma\) as \((\phi_0 \land \phi_1)^a, \Sigma_1\). Now the following derivations suffice:

\[
\begin{array}{c}
\Sigma_0 \setminus \{\phi_0^a\}, \theta_0^b \quad \text{(W)} \\
\Sigma_0, \theta_0^b \quad \text{R}_\wedge
\end{array}
\]

\[
\begin{array}{c}
\Sigma_1, \phi_0^a, \phi_1^b, \theta_0^b \quad \text{W} \\
\Sigma_1, \phi_0^a, (\theta_0 \lor \theta_1)^b \quad \text{R}_\vee \\
\Sigma_1, (\phi_0 \land \phi_1)^a, (\theta_0 \lor \theta_1)^b
\end{array}
\]

\[
\begin{array}{c}
\Sigma_1 \setminus \{\phi_0^a, \phi_1^b\}, \theta_0^b \quad \text{(W)} \\
\Sigma_1, \theta_0^b \quad \text{R}_\wedge
\end{array}
\]

\[
\begin{array}{c}
\Sigma_1 \setminus \{\phi_0^a, \phi_1^b\}, \theta_0^b \quad \text{(W)} \\
\Sigma_1, \theta_0^b \quad \text{R}_\wedge
\end{array}
\]

\[
\Sigma_1, (\phi_0 \lor \phi_1)^a
\]

\textbf{Case }R = R_\vee. \text{ We only consider the case where the active formula } (\phi_0 \lor \phi_1)^a \text{ belongs to } \Sigma^L \text{ (the other case is similar). We may then represent the partition of } \Sigma \text{ as } (\phi_0 \lor \phi_1)^a, \Sigma_0 \mid \Sigma_1. \text{ The two derivations below then suffice to prove the proposition:}

\[
\begin{array}{c}
\phi_0^a, \phi_1^a, \Sigma_0, \theta_0^b \quad \text{R}_\vee \\
(\phi_0 \lor \phi_1)^a, \Sigma_0, \theta_0^b
\end{array}
\]

\[
\begin{array}{c}
\Sigma_1 \setminus \{\phi_0^a, \phi_1^b\}, \theta_0^b \quad \text{(W)} \\
\Sigma_1, \theta_0^b \quad \text{(W)} \\
\Sigma_1, (\phi_0 \lor \phi_1)^a
\end{array}
\]

\textbf{Case }R = R_\exists. \text{ We only consider the case where the active formula } \Box \phi^a \text{ belongs to } \Sigma^L \text{ (the other case is similar). We may then represent the partition of } \Sigma \text{ as } \Box \phi^a, \Diamond \Sigma_0 \mid \Diamond \Sigma_1. \text{ The two derivations below then suffice to prove the proposition:}

\[
\begin{array}{c}
\phi^a, \Sigma_0, \theta_0^b \quad \text{R}_\Box \\
\Box \phi^a, \Diamond \Sigma_0, \Diamond \theta_0^b
\end{array}
\]

\[
\begin{array}{c}
\Sigma_1 \setminus \{\phi^a\}, \theta_0^b \quad \text{(W)} \\
\Diamond \Sigma_1, \Box \theta_0^b \quad \text{R}_\Box
\end{array}
\]

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Case $R = R_\mu$. We only consider the case where the active formula $\mu x. \varphi^a$ belongs to $\Sigma^L$ (the other case is similar). We may then represent the partition of $\Sigma$ as $\mu x. \varphi^a, \Sigma_0 \mid \Sigma_1$. The two derivations below then suffice to prove the proposition:

$$
\frac{\varphi(\mu x. \varphi)^u, \Sigma_0, \theta_0^b}{\mu x. \varphi^a, \Sigma_0, \theta_0^b} R_\mu
$$

$$
\frac{\Sigma_1 \setminus \{ \varphi(\mu x. \varphi)^u \}, \theta_1^b}{\Sigma_1, \theta_0^b} (W)
$$

Case $R = F$. Let $\Sigma = \Phi^u$, we may then represent the partition of $\Sigma$ as $\Phi_0^u \mid \Phi_1^u$. The proposition is then witnessed by the following proofs:

$$
\frac{\Phi_0^f, \theta_0^b}{\Phi_0^u, \theta_0^b} F^+
$$

and

$$
\frac{\Phi_1^f, \theta_0^b}{\Phi_1^u, \theta_0^b} F^+
$$

To finish the proof of Proposition 7.8, we need to check that each of the proofs given above satisfies the conditions (a) - (c). Condition (a) can be verified by a direct inspection. One may also verify the conditions (b) and (c) directly, using the observation that for any node $t$ in the pre-proofs $\Pi^L$ and $\Pi^R$, if some formula occurring at a child of $t$ is annotated with $f$, then also some formula at $t$ is annotated with $f$.

To prove Theorem 7.4 we assemble the interpolant $\theta$ by an induction on the tree that underlies the proof $\Pi$, where most cases of the inductive step are covered by Definition 7.7 and Proposition 7.8. The main difficulty is treating the cases for discharged leafs and the discharge rule. The idea is to introduce a fresh variable as the interpolant of a discharged leaf and to then bind the variable with a fixpoint operator at the step that corresponds to the application of the discharge rule at the companion of the leaf. We need to ensure that this can be done in such that the interpolant stays alternation-free. The key notion that allows us to organize the introduction of fixpoint operators to the interpolant are the fixpoint colourings from Definition 7.13 below. The fixpoint colouring specifies for every node in $\Pi$ whether the application of the discharge rule at the node should be either a least fixpoint $\mu$ or a greatest fixpoint $\nu$. Before we can discuss this notion we need to show that the partition of $\Phi^L \mid \Phi^R$ of the root of $\Pi$ can be extended in a well-behaved way to all nodes of the proof.

**Definition 7.9.** Let $\Pi = (T, P, R, \Sigma)$ be a proof. A **nodewise partition** of $\Pi$ is a pair $(\Sigma^L, \Sigma^R)$ of labellings such that, for every $t \in T$, the pair $\Sigma^L_t \mid \Sigma^R_t$ is a partition of $\Sigma_t$. Such a partition is **coherent** if it agrees with the derivation rules applied in the proof, as expressed by Definition 7.7.

**Proposition 7.10.** Let $\Pi$ be a proof of some sequent $\Gamma$ and let $(\Gamma^L, \Gamma^R)$ be a partition of $\Gamma$. Then there is a unique coherent nodewise partition $(\Sigma^L, \Sigma^R)$ of $\Pi$ such that $\Sigma^L_r = \Gamma^L$ and $\Sigma^R_r = \Gamma^R$, where $r$ is the root of $\Pi$.

**Proof.** Immediate by the definitions.
We shall refer to the nodewise partition given in Proposition 7.10 as being induced by the partition of the root sequent.

**Definition 7.11.** Let \( \Pi = (T, P, R, \Sigma) \) be a proof and let \( (\Sigma^L, \Sigma^R) \) be a coherent nodewise partition of \( \Pi \). This partition is called balanced if \( \Sigma^L_1 = \Sigma^L_{c(l)} \) and \( \Sigma^R_1 = \Sigma^R_{c(l)} \), for every discharged leaves \( l \) of \( \Pi \).

In words, a coherent nodewise partition is balanced if it splits the sequents of any discharged leaf in exactly the same manner as it splits the leaf’s companion node. As a corollary of the following proposition, for every partition \((\Gamma^L, \Gamma^R)\) of a provable sequent \( \Gamma \) we can find a proof on which the induced partition is balanced.

**Proposition 7.12.** Let \( \Pi \) be a proof of some sequent \( \Gamma \), and let \((\Gamma^L, \Gamma^R)\) be a partition of \( \Gamma \). Then there is some finite proof \( \Pi' \) of \( \Gamma \) such that the nodewise partition on \( \Pi' \), induced by \((\Gamma^L, \Gamma^R)\), is balanced.

**Proof.** Let \( \bar{\Pi} \) be the full unravelling of \( \Pi \) into a Focus\( \infty \)-proof according to Proposition 4.3 and extend the nodewise partition of \( \Pi \) to \( \bar{\Pi} \) in the obvious way. Using the same strategy as in the proof of Proposition 4.3 we may ‘cut off’ \( \bar{\Pi} \) to a balanced proof \( \Pi' \).

**Definition 7.13.** Let \( \Pi = (T, P, R, \Sigma) \) be a proof of some sequent \( \Gamma \), and let \((\Sigma^L, \Sigma^R)\) be a nodewise partition of \( \Gamma \). A fixpoint colouring for \((\Sigma^L, \Sigma^R)\) is a map \( \eta : T \rightarrow \{\mu, \nu, \check{\mu}, \check{\nu}\} \), satisfying the conditions below (where we write \( T_{\mu} := \eta^{-1}(\mu) \), etc.):
1) \( T_{\nu} \) consists of those nodes that belong to no set of the form \([c(l), l]\);
2) for every discharged leaf \( l \) of \( \Pi \) we have either \([c(l), l] \subseteq T_{\mu} \) or \([c(l), l] \subseteq T_{\nu} \);
3) if \( t \in T_{\nu} \) then \( \Sigma^L_t \) contains a focused formula, and if \( t \in T_{\mu} \) then \( \Sigma^R_t \) contains a focused formula.
We usually write \( \eta_t \) rather than \( \eta(t) \) and refer to \( \eta_t \) as the fixpoint type of \( t \). Nodes in \( T_{\nu}, T_{\mu} \) and \( T_{\nu} \) will sometimes be called transparent, magenta and navy, respectively.

**Proposition 7.14.** Let \((\Sigma^L, \Sigma^R)\) be a balanced nodewise partition of some proof \( \Pi \). Then there is a fixpoint colouring \( \eta \) for \((\Sigma^L, \Sigma^R)\).

For a proof of Proposition 7.14 we need the following definition and auxiliary proposition.

**Definition 7.15.** Let \( u_0 \) and \( u_1 \) be two nodes of some proof \( \Pi \). We call \( u_0 \) and \( u_1 \) closely connected if there is a non-axiomatic leaf \( l \) such that \( u_0, u_1 \in [c(l), l] \). The relation of being connected is the reflexive/transitive closure of that of being closely connected.

The relation of being connected is easily seen to be an equivalence relation, which refines the partition induced by the fixpoint colouring; note that transparent nodes are only connected to themselves. Furthermore, as we will see, the partition induced by the connectedness relation refines the fixpoint colouring mentioned in Proposition 7.14. Here is the key observation that makes this possible.

**Proposition 7.16.** Let \((\Sigma^L, \Sigma^R)\) be a balanced nodewise partition of some proof \( \Pi = (T, P, R, \Sigma) \), and let \( u \) and \( v \) be connected nodes of \( \Pi \). Then, for \( K \in \{L, R\} \), we have
\[
\Sigma^K_u \text{ contains a formula in focus iff } \Sigma^K_v \text{ contains a formula in focus.} \tag{9}
\]

**Proof.** Fix \( K \in \{L, R\} \). We first consider one direction of the equivalence in (9), for a special case.

**Claim 1.** For any two nodes \( u, v \) in \( \Pi \) such that \( Puv \) and \( R_u \neq F \), the following holds:
\[
\text{if } \Sigma^K_u \text{ contains a formula in focus, then so does } \Sigma^K_v. \tag{10}
\]
This claim can be proved in a case distinction as to the proof rule $R_u$, through a straightforward inspection.

We then show that $\mathfrak{I}$ holds in the following restricted case. Note that Claim 1 may be applied on any companion-leaf path since the focus rule is never applied on such a path.

**Claim 2.** Let $u$ and $v$ be nodes in $\Pi$ such that $v$ is a discharged leaf and $u \in [c(v), v]$. Then $u$ and $v$ satisfy $\mathfrak{I}$.

**Proof of Claim** Assume first that $\Sigma^K_u$ contains a formula in focus. Iteratively applying Claim 1 backwards along the path $[c(v), u]$ we find that $\Sigma^K_u$ contains a formula in focus. But then the same applies to $\Sigma^K_v$: since $(\Sigma^L, \Sigma^R)$ is balanced we have $\Sigma^K_u = \Sigma^K_v$. For the other direction assume that $\Sigma^K_u$ contains a formula in focus. Again with claim 1 applied iteratively, now backwards along the path $[u, v]$, we show that $\Sigma^K_u$ must contain a formula in focus as well.

Finally, it is immediate by Claim 2 and the definitions that $\mathfrak{I}$ holds in case $u$ and $v$ are closely connected, and from this an easy induction shows that $\mathfrak{I}$ holds as well if $u$ and $v$ are merely connected.

**Proof of Proposition 7.14.** Let $(\Sigma^L, \Sigma^R)$ be a balanced nodewise partition of some proof $\Pi$. First define $\eta_u = \checkmark$ for every node $u$ that does not lie on any path to a discharged leaf from its companion node.

Then, consider any equivalence class $C$ of the connectedness relation defined in Definition 7.15 such that $C \cap T_\checkmark = \emptyset$, and make a case distinction. If every node $u$ in $C$ is such that $\Sigma^K_u$ contains a formula in focus, then we map all $C$-nodes to $\mu$.

If, on the other hand, some node $u$ in $C$ is such that $\Sigma^K_u$ contains no formula in focus, we reason as follows. Since $\eta_u \neq \checkmark$, $u$ must lie on some path to a non-axiomatic leaf $l$ from its companion node $c(l)$. By the conditions on a successful proof, $\Sigma_u$ must contain some formula in focus, and so this formula must belong to $\Sigma^K_u$. It then follows from Proposition 7.16 that every node in $C$ has a right formula in focus. In this case we map all $C$-nodes to $\nu$.

With this definition it is straightforward to verify that $\eta$ is a fixpoint colouring for $\Sigma^L_{\|} \Sigma^R$. QED

We will now see how we can read off interpolants from a balanced nodewise partition and an associated fixpoint colouring. Basically, the idea is that with every node of the proof we will associate a formula that can be seen as some kind of ‘preliminary’ interpolant for the partition of the sequent of that node.

**Definition 7.17.** Let $(\Sigma^L, \Sigma^R)$ be a balanced nodewise partition of some proof $\Pi$, and let $\eta$ be some fixpoint colouring for $(\Sigma^L, \Sigma^R)$. By induction on the depth of nodes we will associate a formula $\theta(s)$ with every node $s$ of $\Pi$. The bound variables of these formulas, if any, will be supplied by the discharge tokens used in $\Pi$.

For the definition of $\theta(s)$, inductively assume that $\theta(t)$ has already been defined for all proper descendants of $s$. We distinguish cases depending on whether $s \in \text{Ran}(c)$ and on whether $s$ is a discharged leaf:

**Case $s \in \text{Dom}(c).** In this case we consider the discharge token $x_{c(s)}$ associated with the companion of $s$ as a variable and define

$\theta(s) := x_{c(s)}$.

**Case $s \notin \text{Dom}(c)$ and $s \notin \text{Ran}(c).** Note that this case includes the situation where $s$ is an axiomatic leaf, which is one of the base cases of the induction.
Let $R = R_s$ be the derivation rule applied at the node $s$, and assume that $s$ has successors $v_0, \ldots, v_{n-1}$. Let $\chi_s(x_0, \ldots, x_{n-1})$ be the basic formula provided by Definition 7.17. Inductively we assume formulas $\theta(v_i)$ for all $i < n$, and so we may define

$$\theta(s) := \chi_s(\theta(v_0), \ldots, \theta(v_{n-1})).$$

**Case** $s \in \text{Ran}(c)$. In this case the rule applied at $s$ is the discharge rule, with discharge token $x_s$, $s$ has a unique child $s'$, and, obviously, we have $\eta_s \in \{\mu, \nu\}$. We define

$$\theta(s) := \eta_s x_s \theta(s').$$

In this case we bind the variable $x_s$, which was introduced at the leaves discharged by $s$.

Finally we define

$$\theta_\Pi := \theta(r),$$

where $r$ is the root of $\Pi$.

We will prove a number of statements about these interpolants $\theta(s)$, for which we need some auxiliary definitions. We call a node $u$ a proper connected ancestor of $s$, notation: $P^+_{\Pi} us$, if $u$ is both connected to and a proper ancestor of $s$. For a node $s$ in $\Pi$ we then define

$$X(s) := \{x_u \mid u \in \text{Ran}(c) \text{ and } P^+_{\Pi} us\}.$$

Intuitively, $X(s)$ can be seen as the set of discharge tokens that may occur as free variables in the interpolant $\theta(s)$. Furthermore, we call a node special if it is not connected to its parent, or if has no parent at all (that is, it is the root of $\Pi$). Observe that in particular all nodes in $T_r$ are special.

**Proposition 7.18.** The following hold for every node $s$ in $\Pi$:

1) if $R_s \neq D$ then $X(s) = X(v)$ for every $v \in P(s)$ that is connected to $s$;

2) if $R_s = D$ then $X(s) = X(s') \setminus \{x_s\}$, where $s'$ is the unique child of $s$;

3) if $s$ is special then $X(s) = \emptyset$.

**Proof.** For item 1) the key observation is that if $R_s \neq D$, and $v$ is connected to $s$, then $s$ and $v$ have exactly the same connected strict ancestors. From this it is immediate that $X(s) = X(v)$.

In case $R_s = D$, then $s$ is connected to its unique child $s'$ — here we use the fact that every application of the discharge rule discharges at least one leaf, so that $s'$ actually lies on some path from $s$ to a leaf of which $s$ is the companion. But if $s$ and $s'$ are connected, then they have the same connected strict ancestors, with the obvious exception of $s$ itself. From this item 2) follows directly.

Item 3) follows from the definition of $X(s)$ and the observation that if $s$ is special then it has no proper connected ancestors.

Our next claim is that the interpolant $\theta_\Pi$ is of the right syntactic shape, in that it is alternation free and only contains free variables that occur in both $\Sigma_L^L$ and $\Sigma_R^R$, where $r$ is the root of $\Pi$.

**Proposition 7.19.** The following hold for every node $s$ in $\Pi$:

1) $\text{FV}(\theta(s)) \subseteq \left(\text{FV}(\Sigma_L^L) \cap \text{FV}(\Sigma_R^R)\right) \cup X(s)$;

2) $\theta(s) \in N^L_{\varphi}(\mathcal{L})$ if $\eta_s \in \{\mu, \nu\}$;

3) $\theta(s) \in N^L_{\varphi}(\mathcal{L}) = N^R_{\varphi}(\mathcal{L})$ if $s$ is special.

**Proof.** We prove the first two items by induction on the depth of $s$ in $\Pi$, making the same case distinction as in Definition 7.17.
Case $s \in \text{Dom}(c)$. In this case $s$ is a discharged leaf, and we have $\theta(s) = x_{c(s)}$, so that $FV(\theta(s)) = \{x_{c(s)}\} \subseteq X(s)$ because the companion $c(s)$ of $s$ must be a proper ancestor of $s$ and by definition $c(s)$ is connected to $s$. Moreover, we clearly find $\theta(s) \in N^N_{X(s)}(L)$.

Case $s \not\in \text{Dom}(c)$ and $s \not\in \text{Ran}(c)$. Assume that $t$ has children $v_0, \ldots, v_{n-1}$, then we have $\theta(s) = \chi_s(\theta(v_0), \ldots, \theta(v_{n-1}))$, where $\chi_s(x_0, \ldots, x_{n-1})$ is the basic formula provided by Definition 7.7.

For item 1) we now reason as follows:

$$FV(\theta(s)) = \bigcup_i FV(\theta(v_i))$$

$$\subseteq \bigcup_i \left( (FV(\Sigma^L_{v_i}) \cap FV(\Sigma^R_{v_i})) \cup X(v_i) \right)$$ (definition $\theta(s)$)

$$\subseteq \bigcup_i \left( (FV(\Sigma^L_{v_i}) \cap FV(\Sigma^R_{v_i})) \cup X(s) \right)$$ (induction hypothesis)

$$\subseteq \left( FV(\Sigma^L_s) \cap FV(\Sigma^R_s) \right) \cup X(s)$$ (Proposition 7.18(1))

which suffices to prove item 1).

For item 2) we first show that if $\eta_s \in \{\mu, \nu\}$ then $\theta(s) \in N^N_{X(s)}(L)$. Assume that $\eta_s \in \{\mu, \nu\}$. We claim that

$$\theta(v_i) \in N^N_{X(v_i)}(L) \quad \text{for all } i < n.$$ (11)

To see that this is the case fix $i$ and distinguish cases depending on whether $v_i$ is special or not. If $v_i$ is special then we reason as follows:

$$FV(\theta(v_i)) \subseteq \left( FV(\Sigma^L_{v_i}) \cap FV(\Sigma^R_{v_i}) \right) \cup X(s)$$ (induction hypothesis)

$$= FV(\Sigma^L_{v_i}) \cap FV(\Sigma^R_{v_i})$$ (Proposition 7.18(1))

$$\subseteq FV(\Sigma^L_s) \cap FV(\Sigma^R_s)$$ (Proposition 7.8(1))

so that $FV(\theta(v_i)) \cap X(s) = \emptyset$. From this \[11\] is immediate by the definitions.

On the other hand, if $v_i$ is not special then by definition it is connected to $s$. It follows that $\eta_{v_i} = \eta_s \in \{\mu, \nu\}$ and thus we obtain by the inductive hypothesis that $\theta(v_i) \in N^N_{X(v_i)}(L)$. But since $s \not\in \text{Ran}(c)$ we have $R_s \neq D$ and so by Proposition 7.18(2) we find $X(v_i) = X(s)$. This finishes the proof of \[11\].

To show that $\theta(s) \in N^N_{X(s)}(L)$ recall that $\theta(s) = \chi_s(\theta(v_0), \ldots, \theta(v_{n-1}))$. Because of \[11\] it suffices to check that $N^N_{X(s)}(L)$ is closed under the schema $\chi_s$. But since $\chi_s$ is a basic formula, this is immediate by the definitions.

Case $s \in \text{Ran}(c)$. In this case the rule applied at $s$ is the discharge rule, with discharge token $x_s$, $s$ has a unique child $s'$, $\eta_s \in \{\mu, \nu\}$ and by definition $\theta(s) = \eta_s x_s. \theta(s')$. To prove item 1) we can then reason as follows:

$$FV(\theta(s)) = FV(\theta(s')) \setminus \{x_s\}$$ (definition $\theta(s)$)

$$\subseteq \left( \left( FV(\Sigma^L_s) \cap FV(\Sigma^R_s) \right) \cup X(s') \right) \setminus \{x_s\}$$ (induction hypothesis)

$$\subseteq \left( \left( FV(\Sigma^L_s) \cap FV(\Sigma^R_s) \right) \cup X(s') \right) \setminus \{x_s\}$$ (Proposition 7.8(1))

$$\subseteq \left( FV(\Sigma^L_{s'}) \cap FV(\Sigma^R_{s'}) \right) \cup X(s)$$ (Proposition 7.13(2))

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To check item 2, note that \( \eta_s \in \{\mu, \nu\} \), because \( s \) itself is on the path from \( t \) to any of the leaves that it discharges, and that \( \eta_s = \eta_s \) because \( s' \) is connected to \( s \). By the inductive hypothesis we find that \( \theta(s') \in N_{\Sigma(s')}^N(\ell) \), so that it is clear from the definitions that \( \theta(s) \in N_{\Sigma(s') \setminus \{x_s\}}^N(\ell) \).

It follows that \( \theta(s) \in N_{\Sigma(s)}^N(\ell) \), since \( X(s) = X(s') \setminus \{x_s\} \) by Proposition 7.18(3).

This finishes the proof of the first two items of the proposition.

For item 3, let \( s \) be special. It is then immediate from item 2 and Proposition 7.18(3) that \( \theta(s) \in N_{\Sigma(s)}^N(\ell) \). The statement then follows by the observation of Proposition 7.22 that \( N_{\Sigma}^N(\ell) = L = N_{\Sigma}^N(\ell) \).

Proposition 7.23 is the key technical result of our proof. In its formulation we need the following.

**Definition 7.20.** Let \( \Pi = (T, P, \Sigma, R) \) be some proof. A **global annotation** for \( \Pi \) is a map \( a : T \to \{u, f\} \); the dual of the global annotation \( a \) is the map \( \overline{\pi} \) given by

\[
\overline{\pi}(t) := \begin{cases} f & \text{if } a(t) = u \\ u & \text{if } a(t) = f. \end{cases}
\]

A global annotation \( a \) is **consistent** with a fixpoint colouring \( \eta \) if it satisfies \( a(t) = u \) if \( \eta_t = \mu \) and \( a(t) = f \) if \( \eta_t = \nu \).

Note that the conditions on an annotation \( a \) to be consistent with a fixpoint colouring \( \eta \) only mentions the nodes in \( T_\mu \) and \( T_\nu \); the annotation \( a(t) \) can be arbitrary for \( t \in T_\chi \).

For the final part of the interpolation argument we need a general observation about the result of applying a substitution to (all formulas in a) proof. First we need some definitions.

**Definition 7.21.** Let \( \Sigma \) be an annotated sequent. We define \( BV(\Sigma) = \bigcup \{ BV(\psi) \mid \psi^\alpha \in \Sigma \} \), and, for any formula \( \varphi \) such that \( FV(\varphi) \cap BV(\Sigma) = \emptyset \), we set

\[
\Sigma[\varphi/x] := \{ (\psi[\varphi/x])^\alpha \mid \psi^\alpha \in \Sigma \}.
\]

Furthermore, where \( \Pi = (T, P, R, \Sigma) \) is some proof, we let \( \Pi[\varphi/x] \) denote the labelled tree \( \Pi[\varphi/x] := (T, P, R, \Sigma') \) which is obtained from \( \Pi \) by replacing every annotated sequent \( \Sigma_i \) with \( \Sigma_i[\varphi/x] \).

**Proposition 7.22.** Let \( \Pi \) be a **Focus** \( \Sigma \)-proof of a sequent \( \Sigma \) with open assumptions \( \{\Gamma_i \mid i \in I\} \), and let \( \varphi \) be a formula such that \( FV(\varphi) \cap BV(\Sigma) = \emptyset \). Then \( \Pi[\varphi/x] \) is a well-formed **Focus** \( \Sigma \)-proof of the sequent \( \Sigma[\varphi/x] \), with open assumptions \( \{\Gamma_i[\varphi/x] \mid i \in I\} \).

**Proof.** (Sketch) One may show that \( BV(\chi) \subseteq BV(\psi) \) for every \( \chi \in Clos(\psi) \), by an induction on the length of the trace from \( \psi \) to \( \chi \) witnessing that \( \chi \in Clos(\psi) \). Because every formula \( \chi \) that occurs in one of the sequents of \( \Pi \) belongs to the closure of \( \Sigma \) it follows that \( BV(\chi) \subseteq BV(\Sigma) \) and hence all the substitutions are well-defined. Moreover, one can check that all the proof rules remain valid if one performs the same substitution uniformly on all the formulas in the conclusion and the premises. It should also be also clear that the global conditions on proofs are not affected by the substitution.

**Proposition 7.23.** Let \( \Sigma^L, \Sigma^R \) be a balanced nodewise partition of some proof \( \Pi \), let \( \eta \) be some fixpoint colouring for \( \Sigma^L, \Sigma^R \), and let \( a : T \to \{u, f\} \) be a global annotation that is consistent with \( \eta \). Then we can effectively construct **Focus** \( \Pi^L \) and \( \Pi^R \) of the sequents \( \Sigma^L \), \( (\theta_{\Pi})^{a(s)} \) and \( \Sigma^R, (\theta_{\Pi})^{\overline{a(s)}} \), respectively, where \( r \) is the root of \( \Pi \).
Proof. For every node $s$ of $\Pi$ we will construct two proofs with open assumptions, $\Pi^L_s$ and $\Pi^R_s$, for the sequents $\Sigma^L_s, \theta(s)^{(c)}$ and $\Sigma^R_s, \theta(s)^{(c)}$, respectively. We will make sure that the only open assumptions of these proofs will be associated with leaves $l$ of which the companion node $c(l)$ is a proper connected ancestor of $s$. We define $\Pi^L_s$ and $\Pi^R_s$ as labelled trees that satisfy conditions 1 and 2 from Definition 4.1. We check the other conditions in subsequent claims. The definition of $\Pi^L_s$ and $\Pi^R_s$ proceeds by induction on the depth of $s$ in the tree $\Pi$, where we make the same case distinction as in Definition 7.17.

Case $s \in \text{Dom}(c)$. In this case we let $\Pi^L_s$ and $\Pi^R_s$ be the leaves that are labelled with the discharge variable $x_{c(s)}$ and the sequents $\Sigma^L_s, \theta(s) = \Sigma^L_{c(s)}, x_{c(s)}^{a(l)}$ and $\Sigma^R_s, \theta(s) = \Sigma^R_{c(s)}, x_{c(s)}^{a(l)}$, respectively. Note that here we are creating an open assumption that is labelled with a discharge token and not with $\star$. This open assumption will be discharged later when the induction is at the node $c(s)$.

Case $s \notin \text{Dom}(c)$ and $s \notin \text{Ran}(c)$. The basic strategy in this case is to use Proposition 7.8 to extend the proofs $\Pi^L_s$ and $\Pi^R_s$. The details depend on the global annotation $a$. We only consider the subcases where $a(s)$ is distinct from $a(v)$ for at least one child $v$ of $s$. The case where $a(s) = a(v)$ for all $v \in P(s)$ is similar, but easier.

Subcase $a(s) = a, \text{ but } a(v) = f$, for some $v \in P(s)$. As a representative example of this, consider the situation where $R_s$ is binary, and $a(s) = a(v_0) = u$, while $a(v_1) = f$, where $v_0$ and $v_1$ are the two successors of $s$.

We first consider the proof $\Pi^L_{v_0}$. Inductively we assume labelled trees $\Pi^L_{v_0}$ and $\Pi^R_{v_1}$ for, respectively, the sequents $\Sigma^L_{v_0}, \theta(v_0)^{u}$ and $\Sigma^R_{v_1}, \theta(v_1)^{f}$. Combining these with the proof with assumptions $\Xi^L$ from Proposition 7.8 we then define $\Pi^L_s$ to be the following labelled tree:

$$
\begin{align*}
\Pi^L_{v_0} & \quad \Sigma^L_{v_0}, \theta(v_0)^{u} \\
\Sigma^L_{v_1}, \theta(v_1)^{f} & \quad \Sigma^L_{v_1}, \theta(v_1)^{u} \\
& \quad \Xi^L \\
& \quad \Sigma^L_s, \chi_1(\theta(v_0), \theta(v_1))^u
\end{align*}
$$

A similar construction works for $\Pi^R_s$. Inductively we are given proofs $\Pi^R_{v_0}$ and $\Pi^R_{v_1}$ for, respectively, the sequents $\Sigma^R_{v_0}, \theta(v_0)^{f}$ and $\Sigma^R_{v_1}, \theta(v_1)^{u}$. Together with the proof $\Xi^R$ that we obtain from Proposition 7.8 we can define $\Pi^R_s$ as follows:

$$
\begin{align*}
\Pi^R_{v_0} & \quad \Sigma^R_{v_0}, \theta(v_0)^{f} \\
\Sigma^R_{v_1}, \theta(v_1)^{u} & \quad \Sigma^R_{v_1}, \theta(v_1)^{f} \\
& \quad \Xi^R \\
& \quad \Sigma^R_s, \chi_1(\theta(v_0), \theta(v_1))^f
\end{align*}
$$

Subcase $a(s) = f$, but $a(v) = u$, for some $v \in P(s)$. Similarly as in the previous subcase, we consider a representative example where $s$ has two successors, $v_0$ and $v_1$, but now $a(s) = a(v_0) = f$, while $a(v_1) = u$. Inductively we are provided with labelled trees $\Pi^L_{v_0}$ and $\Pi^L_{v_1}$ for, respectively, the sequents $\Sigma^L_{v_0}, \theta(v_0)^f$ and $\Sigma^L_{v_1}, \theta(v_1)^u$. Combining these with the proof with assumptions $\Xi^L$, which we obtain by Proposition 7.8 we then define $\Pi^L_s$ to be the following labelled tree:
\[
\begin{array}{c}
\Pi_{v_0}^L \\
\Sigma_{v_0}^L, \theta(v_0)^f \\
\hline
\Sigma_{v_1}^L, \theta(v_1)^u \\
\Sigma_{v_1}^L, \theta(v_1)^f \\
\Sigma_{v}^L, \chi_s(\theta(v_0), \theta(v_1))^f \\
\end{array}
\]

Again, a similar construction works for \(\Pi^R\).

**Case** \(s \in \text{Ran}(c)\). In this case the rule applied at \(s\) is the discharge rule; let \(x_s, s'\) and \(\eta_s\) be as in the corresponding case in Definition 7.14.

Note that by the assumption on \(a\) we have that \(a(s) = a(s')\) and \(a(s) = a(l)\) for any discharged leaf \(l\) such that \(c(l) = s\). Furthermore, there are only two possibilities: either \(a(s) = u\) and \(\eta_s = \mu\), or \(a(s) = f\) and \(\eta_s = \nu\). We cover both cases at once but first consider the definition of \(\Pi^L\). Inductively we have a proof \(\Pi'_L\) of \(\Sigma_s^L, \theta(s')^a(s')\). Note that \(\Sigma_s^L = \Sigma_s^L\), because the discharge rule is applied at \(s\).

Let \((\Pi')^L := \Pi_L^L[\eta, x_s, \theta(s')/x_s]\); that is, \((\Pi')^L\) is the labelled tree \(\Pi_s^L\), with all occurrences of \(x_s\) replaced by the formula \(\eta, x_s, \theta(s')\). That this is a well-defined operation on proofs follows from Proposition 7.22. However, we need to make sure that \(\text{FV}(\eta, x_s, \theta(s')) \cap BV(\Sigma_s^L, \theta(s')^a(s')) = \emptyset\). This follows with item [1] of Proposition 7.19 and the observations that the variables in \(X(s')\) do not occur as bound variables in any of the formulas in \(\Sigma_s^L\) nor in \(\theta(s')\). Note that \((\Pi')^L\) has the open assumption \(\Sigma_s^L, (\eta, x_s, \theta(s'))^a(s)\) instead of \(\Sigma_s^L, x_s^a\).

To obtain \(\Pi_s^L\) from \((\Pi')^L\), add one application of the fixpoint rule for \(\eta, x_s, \theta(s')\), followed by an application of the discharge rule for the discharge token \(x_s\):

\[
\begin{array}{c}
[\Sigma_s^L, (\eta, x_s, \theta(s'))^a(s)]_{x_s} \\
(\Pi')^L \\
\hline
\Sigma_s^L, (\theta(s)[\eta, x_s, \theta(s')/x_s])^a(s) \\
\Sigma_s^L, (\eta, x_s, \theta(s'))^a(s) \\
\Sigma_s^L, (\eta, x_s, \theta(s'))^a(s) \\
\end{array}
\]

The application of the rule \(R_{\eta_s}\) is correct because if \(\eta_s = \mu\) then \(a(s) = u\). Thus, the unfolded fixpoint formula in the premise of the application of \(R_{\eta_s}\) is still annotated with \(a(s)\). If \(\eta_s = \nu\) then the unfolded fixpoint stays annotated with \(a(s)\) because \(R_v\) does not change the annotation of its principal formula. Also note that the proof \(\Pi_s^L\) no longer contains open assumptions that are labelled with the token \(x_s\).

A similar construction can be used to define \(\Pi^R\). By induction there is a proof \(\Pi^R\) of \(\Sigma^R, \theta(s')^{\Xi(s')}\). As before we use Proposition 7.22 to substitute all occurrences of \(x_s\) with \(\eta, x_s, \theta(s')\) in the proof \(\Pi^R\) to obtain a proof \((\Pi')^R := \Pi^R[\eta, x_s, \theta(s')/x_s]\). Note that \((\Pi')^R\) has the open assumption \(\Sigma^R, (\eta, x_s, \theta(s'))^{\Xi(s')}\) instead of \(\Sigma^R, x_s^{\Xi(s')}\). We then construct the proof \(\Pi_s^R\) as follows:
Claim the root of the proof tree \( \Pi \) may assume the existence of an injection for every node \( s \), \( \Pi \) contains, for every node \( s \) in \( \Pi \), some substitution instance of \( \Pi \) as a subproof. In particular, we may assume the existence of an injection \( f \) mapping \( \Pi \)-nodes to \( \Pi \)-nodes, in such a way that \( f(s) \) is the root of the proof tree \( \Pi_s \) for every node \( s \) of \( \Pi \). A similar observation holds for the proof \( \Pi^R \).

CLAIM 1. For all nodes \( s \) in \( \Pi \) the following hold.

1) \( \Pi^L \) is a \textit{Focus} \( \tau \)-proof for the sequent \( \Sigma^L, \theta(s)^{a(s)} \), with assumptions \( \{\Sigma^L, x_{c(l)} | P^+c(l)s \} \) such that additionally for every node \( t' \) that is on a path from the root \( f(s) \) of \( \Pi^L \) to one of its open assumptions the following hold:

(a) the annotated sequent at \( t' \) contains at least one formula that is in focus;
(b) the rule applied at \( t' \) is not \( F, F^+ \) or \( U \).

2) \( \Pi^R \) is a \textit{Focus} \( \tau \)-proof for the sequent \( \Sigma^R, \theta(s)^{\pi(s)} \), with assumptions \( \{\Sigma^R, x_{c(l)} | P^+c(l)s \} \) such that additionally for every node \( t' \) that is on a path from the root \( f(s) \) of \( \Pi^R \) to one of its open assumptions it holds that:

(a) the annotated sequent at \( t' \) contains at least one formula that is in focus;
(b) the rule applied at \( t' \) is not \( F, F^+ \) or \( U \).

Proof of Claim As mentioned, our proof proceeds by induction on the complexity of the proof \( \Pi \) or, to be somewhat more precise, by induction on the depth of \( s \) in \( \Pi \). Here we will use the same case distinction as the construction of \( \Pi^L \) and \( \Pi^R \). We focus on the proof \( \Pi^L \), the case of \( \Pi^R \) being similar.

First we make an auxiliary observation that will be helpful for understanding our proof:

\[
[\Sigma^R, \theta(s)^{\pi(s)} \mapsto_{s} \eta \mapsto_{s} (\Pi^L)^{R}]
\]

\[
\Sigma^R, (\theta(s')) \mapsto_{s} (\Pi^L)^{R}
\]

\[
\Sigma^R, (\theta(s') \mapsto_{s} \eta \mapsto_{s} (\Pi^L)^{R})
\]

\[
\Sigma^R, (\theta(s') \mapsto_{s} \eta \mapsto_{s} (\Pi^L)^{R})
\]

Note that if \( \eta = \mu \), then \( \eta = \nu \), \( a(s) = f \) and \( \eta(s) = u \). Therefore, the application of the rule \( R_\mu \) above has the right annotation at the unfolded fixpoint.

We now check that \( \Pi^L \) and \( \Pi^R \) are indeed \textit{Focus} \( \tau \)-proofs of, respectively, the sequents \( \Sigma^L, \theta(r)^{a(s)} \) and \( \Sigma^R, \theta(r)^{\pi(s)} \), where \( r \) is the root of \( \Pi \). Note that whereas we are proving statements about \( \Pi^L \) and \( \Pi^R \), our proof is by induction on the complexity of the original proof \( \Pi \). In the formulation of the inductive hypothesis it is convenient to allow for proofs in which some open assumptions are already labelled with a discharge token instead of with \( \ast \). (In the end of the induction this makes no difference because \( \Pi^L \) and \( \Pi^R \) do not have any open assumption.) With this adaptation we will establish the claim below.

Before going into the details we observe that, given the inductive definition of the proof \( \Pi^L \), it contains, for every node \( s \) in \( \Pi \), some substitution instance of \( \Pi^L \) as a subproof. In particular, we may assume the existence of an injection \( f \) mapping \( \Pi \)-nodes to \( \Pi \)-nodes, in such a way that \( f(s) \) is the root of the proof tree \( \Pi^L \) for every node \( s \) of \( \Pi \). A similar observation holds for the proof \( \Pi^R \).

We now turn to the inductive proof of the Claim proper. It is obvious from the construction that the root \( f(s) \) of \( \Pi^L \) is labelled with the annotated sequent \( \Sigma^L, \theta(s)^{a(s)} \), and it is not hard to see that
the open assumptions of this proof are indeed of the form claimed above. To show that \( \Pi^L_s \) is indeed a \( \text{Focus}^+ \)-proof we need to check the conditions from Definition 4.1.

Condition 1 which requires the annotated sequents to match the applied proof rule at every node, can be easily verified by inspecting the nodes that are added in each step of the construction of \( \Pi^L_s \). Similarly, it is clear that only leaves get labelled with discharge tokens and thus condition 2 is satisfied.

It is also not too hard to see that all non-axiomatic leaves that are not open assumptions are discharged. This is just our (already established) claim that all open assumptions of \( \Pi^L_s \) are in the set \( \{ \Sigma^L_s, \nu^s(l) \mid \text{P}^+c(l)s \text{ and } \text{P}^+s\ell \} \). This means that condition 3 is satisfied. (Note that it is here where we conveniently allow for open leaves that are labelled with a discharge token rather than with \( * \).

It is left to consider condition 4, modified to account for \( \text{Focus}^+ \)-proofs. Let \( l \) be some leaf of \( \Pi^L_s \), then we need to consider arbitrary nodes on the path from \( c(l) \) to \( l \). The only interesting case of this is where \( c(l) = s \), and we discharge the open leaves that are labelled with \( x_s \). Here condition 4 is a special case of the statements (1)a) and (1)b) that we are about to prove.

To prove the parts (1)a) and (1)b) of the inductive statement, let \( t' \) be a node on a path from the root \( f(s) \) of \( \Pi^L_s \) to one of its open assumptions. We now make our case distinction.

**Case** \( s \in \text{Dom}(c) \). In this case \( \Pi^L_s \) contains \( f(s) \) as its single node, and so (1)a) follows by (12), while (1)b) is obvious by construction.

**Case** \( s \notin \text{Dom}(c) \) and \( s \notin \text{Ran}(c) \). Let \( v_0, \ldots, v_{n-1} \) be the children of \( s \) (in \( \Pi \)). Then by construction \( \Pi^L_s \) consists of the pre-proofs \( \Pi^L_{v_0}, \ldots, \Pi^L_{v_{n-1}} \), linked to the root \( f(s) \) via an instance \( \Xi^L \) of Proposition 7.8 in such a way that (i) all open leaves of \( \Pi^L_s \) belong to one of the \( \Pi^L_{v_i} \) where \( s \) and \( v_i \) are connected, and (ii) \( \Pi^L_{v_i} \) is directly pasted to the corresponding leaf of \( \Xi^L \) in case \( s \) and \( v_i \) are connected (that is, no focus or unfocus rule are needed). Concerning the position of the node \( t' \) in \( \Pi^L_s \) it follows from (i) and (ii) that there is a child \( v = v_i \) of \( s \), which is connected to \( s \) and such that \( t' \) either lies (in the \( \Pi^L_{v_i} \)-part of \( \Pi^L_s \)) on the path from \( f(v) \) to an open leaf, or on the path in \( \Pi^L_s \) from \( f(s) \) to \( f(v) \). Since the first case is easily taken care of by the inductive hypothesis, we focus on the latter. It follows from (ii) that the full path from \( f(s) \) to \( f(v) \) is taken from the pre-proof \( \Xi^L \) as provided by Proposition 7.8. But then both (1)a) and (1)b) are immediate by item (2)a) and (b) from mentioned proposition, given the fact that by (12) the node \( f(v) \) features a formula in focus. (Note that the rule applied at \( s \) in \( \Pi^L_s \) is not the focus rule since \( s \in T^f \cup T^\nu \) and thus \( \Sigma_s \) contains a formula in focus.)

**Case** \( s \in \text{Ran}(c) \). Let \( s^+ \) be the unique successor of \( s \) in \( \Pi \). Then by construction \( \Pi^L_s \) consists of a substitution instance of \( \Pi^L_{s^+} \), connected to \( f(s) \) via the application of the rules \( \text{R}_{s^+} \) (at the unique successor of \( f(s) \)) and \( \text{D}^s \) (at \( f(s) \) itself). Clearly then there are two possible locations for the node \( t' \). If \( t' \) is situated in the sub-tree rooted at \( f(s^+) \), then (1)a) and (1)b) follow from the inductive hypothesis (note that when we apply a substitutions to the derivation \( \Pi^L_{s^+} \) we do not change the proof rules or alter the annotations). On the other hand, the only two nodes of \( \Pi^L_s \) that do not belong to mentioned subtree are \( f(s) \) itself and its unique child. These nodes carry the same sequent label, and so in this case (1)a) follows from (12). Finally, (1)b) is obvious since we already saw that the rules applied in \( \Pi^L_s \) at \( f(s) \) and its successor are \( \text{D}^s \) and \( \text{R}_{s^+} \), respectively.

This finishes the proof of the Claim.

Finally, the proof of the Proposition is immediate by these claims if we consider the case \( s = r \), where \( r \) denotes the root of the tree.
8 Conclusion & Questions

In this paper we saw that the idea of placing formulas in focus can be extended from the setting of logics like LTL and CTL to that of the alternation-free modal \( \mu \)-calculus: we designed a very simple and natural, cut-free sequent system which is sound and complete for all validities in the language consisting of all (guarded) formulas in the alternation-free fragment \( L \) of the modal \( \mu \)-calculus. We then used this proof system Focus to show that the alternation-free fragment enjoys the Craig Interpolation Theorem. Clearly, both results add credibility to the claim that \( L \) is an interesting logic with good meta-logical properties.

Below we list some questions for future research.

1. Probably the most obvious question is whether the restriction to guarded formulas can be lifted. In fact, we believe that the focus proof system, possibly with some minor modifications in the definition of a proof, is also sound and complete for the full alternation-free fragment. To prove this observation, one may bring ideas from Friedmann & Lange into our definition of tableaux and tableau games. Note that, in any case, our interpolation theorem applies to the full language of \( L \).

2. Another question is whether we may tidy up the focus proof system, in the same way that Afshari & Leigh did with the Jungteerapanich-Stirling system. As a corollary of this it should be possible to obtain an annotation-free sequent system for the alternation-free fragment of the \( \mu \)-calculus, and to prove completeness of Kozen’s (Hilbert-style) axiomatisation for \( L \).

3. Moving in a somewhat different direction, we are interested to see to which degree the focus system can serve as a basis for sound and complete derivation systems for the alternation-free validities in classes of frames satisfying various kinds of frame conditions.

4. We think it is of interest to see which other fragments of the modal \( \mu \)-calculus enjoy Craig interpolation. A very recent result by L. Zenger shows that the fragments \( \Sigma^\mu_1 \) and \( \Pi^\mu_1 \) consisting of, respectively, the \( \mu \)-calculus formulas that only contain least- or greatest fixpoint operators, each have Craig interpolation. Clearly, a particular interesting question would be whether our focus system can be used to shed some light on the interpolation problem for propositional dynamic logic (see the introduction for some more information) and other fragments of the alternation-free \( \mu \)-calculus. Looking at fragments of the modal \( \mu \)-calculus that are more expressive than \( L \), an obvious question is whether every bounded level of the alternation hierarchy admits Craig interpolation.

5. Finally, the original (uniform) interpolation proof for the full \( \mu \)-calculus is based on a direct automata-theoretic construction. Is something like this possible here as well? That is, given two modal automata \( A_\varphi \) and \( A_\psi \) corresponding to \( L \)-formulas \( \varphi \) and \( \psi \), can we directly construct a modal automaton \( B \) which serves as an interpolant for \( A_\varphi \) and \( A_\psi \) (so that we may obtain an \( L \)-interpolant for \( \varphi \) and \( \psi \) by translating the automaton \( B \) back into \( L \))? Recall that the automata corresponding to the alternation-free \( \mu \)-calculus are so-called weak modal parity automata.
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A Infinite games

In this brief appendix we give the basic definitions of infinite two-player games. We fix two players that we shall refer to as \( \exists \) (female) and \( \forall \) (male).

A two-player game is a quadruple \( \mathcal{G} = (V, E, O, W) \) where \( (V, E) \) is a graph, \( O \) is a map \( O : V \to \{ \exists, \forall \} \), and \( W \) is a set of infinite paths in \( (V, E) \). We denote \( \mathcal{G}_\Pi := O^{-1}(\Pi) \). An initialised game is a pair consisting of a game \( \mathcal{G} \) and an element \( v \) of \( V \); such a pair is usually denoted as \( \mathcal{G}@v \).

We will refer to \( (V, E) \) as the board or arena of the game. Elements of \( V \) will be called positions, and \( O(v) \) is the owner of \( v \). Given a position \( v \) for player II \( \in \{ \exists, \forall \} \), the set \( E[v] \) denotes the set of moves that are legitimate or admissible to II at \( v \). The set \( W \) is called the winning condition of the game.

A match of an initialised game consists of the two players moving a token from one position to another, starting at the initial position, and following the edge relation \( E \). Formally, a match or play of the game \( \mathcal{G} = (V, E, O, W) \) starting at position \( v_I \) is simply a path \( \pi \) through the graph \( (V, E) \) such that \( \text{first}(\pi) = v_I \). Such a match \( \pi \) is full if it is maximal as a path, that is, either finite with \( E[\text{last}(\pi)] = \emptyset \), or infinite. The owner of a position is responsible for moving the token from that position to an adjacent one (that is, an \( E \)-successor); in case this is impossible because the node has no \( E \)-successors, the player gets stuck and immediately loses the match. If neither player gets stuck, the resulting match is infinite; we declare \( \exists \) to be its winner if the match, as an E-path, belongs to the set \( W \). Full matches that are not won by \( \exists \) are won by \( \forall \).

Given these definitions, it should be clear that it does not matter which player owns a state that has a unique successor; for this reason we often take \( O \) to be a partial map, provided \( O(v) \) is defined whenever \( |E[v]| \neq 1 \).

A position \( v \) is a winning position for a player if they have a way of playing the game that guarantees they win the resulting match, no matter how their opponent plays. To formalise this, we let \( PM_\Pi \) denote the collection of partial matches \( \pi \) ending in a position \( \text{last}(\pi) \in V_\Pi \), and define \( PM_\Pi[v] \) as the set of partial matches in \( PM_\Pi \) starting at position \( v \). A strategy for a player \( P \) is a function \( f : PM_P \to V \); if \( f(\pi) \not\in E[\text{last}(\pi)] \), for some \( \pi \in PM_P \), we say that \( f \) prescribes an illegitimate move in \( \pi \). A match \( \pi = (v_0)_{i<\kappa} \) is guided by a \( P \)-strategy \( f \) if \( f(v_0v_1 \cdots v_{n-1}) = v_n \) for all \( n < \kappa \) such that \( v_0 \cdots v_{n-1} \in PM_P \). A position \( v \) is reachable by a strategy \( f \) is there is an \( f \)-guided match \( \pi \) with \( v = \text{last}(\pi) \). A \( P \)-strategy \( f \) is legitimate from a position \( v \) if the moves it prescribes to \( f \)-guided partial matches in \( PM_P[v] \) are always legitimate, and winning for \( P \) from \( v \) if in addition \( P \) wins all \( f \)-guided full matches starting at \( v \). When defining a strategy \( f \) for one of the players in a board game, we can and in practice will confine ourselves to defining \( f \) for partial matches that are themselves guided by \( f \). A position \( v \) is a winning position for player \( P \in \{ \exists, \forall \} \) if \( P \) has a winning strategy in the game \( \mathcal{G}@v \); the set of these positions is denoted as \( \text{Win}_P(\mathcal{G}) \). The game \( \mathcal{G} \) is determined if every position is winning for either \( \exists \) or \( \forall \).

A strategy is positional if it only depends on the last position of a partial match, i.e., if \( f(\pi) = f(\pi') \) whenever \( \text{last}(\pi) = \text{last}(\pi') \); such a strategy can and will be presented as a map \( f : V_P \to V \).

A priority map on the board \( V \) is a map \( \Omega : V \to \omega \) with finite range. A parity game is a board game \( \mathcal{G} = (V, E, O, W_\Omega) \) in which the winning condition \( W_\Omega \) is given as follows. Given an infinite match \( \pi \), let \( \text{inf}(\pi) \) be the set of positions that occur infinitely often in \( \pi \); then \( W_\Omega \) consists of those infinite paths \( \pi \) such that \( \max(\Omega[\text{inf}(\pi)]) \) is even. Such a parity game is usually denoted as \( \mathcal{G} = (V, E, O, \Omega) \). The following fact, independently due to Emerson & Jutla \[11\] and Mostowski \[27\], will be quite useful to us.

**Fact A.1** (Positional Determinacy). Let \( \mathcal{G} = (G, E, O, \Omega) \) be a parity game. Then \( \mathcal{G} \) is determined, and both players have positional winning strategies.