On a quaternionic Maxwell equation for the
time-dependent electromagnetic field in a
chiral medium

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November 2, 2018

Abstract
Maxwell’s equations for the time-dependent electromagnetic field
in a homogeneous chiral medium are reduced to a single quaternionic
equation. Its fundamental solution satisfying the causality principle is
obtained which allows us to solve the time-dependent chiral Maxwell
system with sources.

1 Introduction
We consider Maxwell’s equations for the time-dependent electromagnetic
field in a homogeneous chiral medium and show their equivalence to a sin-
gle quaternionic equation. This result generalizes the well known (see \[13], \[6], \[10]) quaternionic reformulation of the Maxwell equations for non-chiral media. Nevertheless the new quaternionic differential operator is essentially different from the quaternionic operator corresponding to the non-chiral case. We obtain a fundamental solution of the new operator in explicit form satisfying the causality principle. Its convolution with a quaternionic function representing sources of the electromagnetic field gives us a solution of the inhomogeneous Maxwell system in a whole space.

2 Maxwell’s equations for chiral media

Consider time-dependent Maxwell’s equations

\[
\text{rot } \overrightarrow{E}(t, x) = -\partial_t \overrightarrow{B}(t, x), \tag{1}
\]

\[
\text{rot } \overrightarrow{H}(t, x) = \partial_t \overrightarrow{D}(t, x) + \overrightarrow{j}(t, x), \tag{2}
\]

\[
\text{div } \overrightarrow{E}(t, x) = \frac{\rho(t, x)}{\varepsilon}, \quad \text{div } \overrightarrow{H}(t, x) = 0 \tag{3}
\]

with the Drude-Born-Fedorov constitutive relations corresponding to the chiral media \[2], \[11], \[12] \n
\[
\overrightarrow{B}(t, x) = \mu(\overrightarrow{H}(t, x) + \beta \text{rot } \overrightarrow{H}(t, x)), \tag{4}
\]

\[
\overrightarrow{D}(t, x) = \varepsilon(\overrightarrow{E}(t, x) + \beta \text{rot } \overrightarrow{E}(t, x)), \tag{5}
\]

where \(\beta\) is the chirality measure of the medium. \(\beta, \varepsilon, \mu\) are real scalars assumed to be constants. Note that the charge density \(\rho\) and the current density \(\overrightarrow{j}\) are related by the continuity equation \(\partial_t \rho + \text{div } \overrightarrow{j} = 0\).

Incorporating the constitutive relations \[4], \[5] into the system \[1]-\[3] we arrive at the main object of our study, the time-dependent Maxwell system for a homogeneous chiral medium

\[
\text{rot } \overrightarrow{H}(t, x) = \varepsilon(\partial_t \overrightarrow{E}(t, x) + \beta \partial_t \text{rot } \overrightarrow{E}(t, x)) + \overrightarrow{j}(t, x), \tag{6}
\]

\[
\text{rot } \overrightarrow{E}(t, x) = -\mu(\partial_t \overrightarrow{H}(t, x) + \beta \partial_t \text{rot } \overrightarrow{H}(t, x)), \tag{7}
\]
$\text{div} \, \vec{E}(t, x) = \frac{\rho(t, x)}{\varepsilon}, \quad \text{div} \, \vec{H}(t, x) = 0. \quad (8)$

Application of rot to (6) and (7) allows us to separate the equations for $\vec{E}$ and $\vec{H}$ and to obtain in this way the wave equations for a chiral medium

$\text{rot rot} \, \vec{E} + \varepsilon \mu \partial_t^2 \vec{E} + 2\beta \varepsilon \mu \partial_t^2 \text{rot rot} \, \vec{E} = -\mu \partial_t \vec{j} - \beta \mu \partial_t \text{rot} \, \vec{j}, \quad (9)$

$\text{rot rot} \, \vec{H} + \varepsilon \mu \partial_t^2 \vec{H} + 2\beta \varepsilon \mu \partial_t^2 \text{rot rot} \, \vec{H} = \text{rot} \, \vec{j}. \quad (10)$

It should be noted that when $\beta = 0$, (9) and (10) reduce to the wave equations for non-chiral media but in general to the difference of the usual non-chiral wave equations their chiral generalizations represent equations of fourth order.

3 Some notations from quaternionic analysis

We will consider biquaternion-valued functions defined in some domain $\Omega \subset \mathbb{R}^3$. On the set of continuously differentiable such functions the well known Moisil-Teodoresco operator is defined by the expression $D = i_1 \frac{\partial}{\partial x_1} + i_2 \frac{\partial}{\partial x_2} + i_3 \frac{\partial}{\partial x_3}$ (see, e.g., [5]), where $i_k$, $k = 1, 2, 3$ are basic quaternionic imaginary units. Denote $D_\alpha = D + \alpha$, where $\alpha \in \mathbb{C}$ and $\text{Im} \, \alpha \geq 0$. The fundamental solution for this operator is known [9] (see also [10]):

$K_\alpha(x) = -\text{grad} \, \Theta_\alpha(x) + \alpha \Theta_\alpha(x) = (\alpha + \frac{x}{|x|^2} - i\alpha \frac{x}{|x|})\Theta_\alpha(x), \quad (11)$

where $i$ is the usual complex imaginary unit commuting with $i_k$, $x = \sum_{k=1}^{3} x_k i_k$ and $\Theta_\alpha(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}$. Note that $K_\alpha$ fulfills the following radiation condition at infinity uniformly in all directions

$$(1 + \frac{i x}{|x|}) \cdot K_\alpha(x) = o\left(\frac{1}{|x|}\right), \quad \text{when } |x| \to \infty \quad (12)$$

which is in agreement with the Silver-Müller radiation conditions [8].
4 Field equations in quaternionic form

In this section we rewrite the field equations from Section 2 in quaternionic form.

Let us introduce the following quaternionic operator

\[ M = \beta \sqrt{\varepsilon \mu} \partial_t D + \sqrt{\varepsilon \mu} \partial_t - i D \]

and consider the purely vectorial biquaternionic function

\[ \vec{V}(t, x) = \vec{E}(t, x) - i \sqrt{\mu \varepsilon} \vec{H}(t, x). \]

Proposition 1 The quaternionic equation

\[ M \vec{V}(t, x) = -\sqrt{\mu \varepsilon} \vec{j}(t, x) - \beta \sqrt{\mu \varepsilon} \partial_t \rho(t, x) + \frac{i \rho(t, x)}{\varepsilon} \]

is equivalent to the Maxwell system (6)-(8), the vectors \( \vec{E} \) and \( \vec{H} \) are solutions of (6)-(8) if and only if the purely vectorial biquaternionic function \( \vec{V} \) defined by (14) is a solution of (15).

Proof. The scalar and the vector parts of (15) have the form

\[ -\beta \sqrt{\varepsilon \mu} \partial_t \text{div} \vec{E} + \sqrt{\mu \varepsilon} \text{div} \vec{H} + i(\text{div} \vec{E} + \beta \mu \partial_t \text{div} \vec{H}) = -\beta \sqrt{\mu \varepsilon} \partial_t \rho + \frac{i \rho}{\varepsilon}, \]

\[ \beta \sqrt{\varepsilon \mu} \partial_t \text{rot} \vec{E} + \sqrt{\varepsilon \mu} \partial_t \vec{E} - \sqrt{\mu \varepsilon} \text{rot} \vec{H} - i(\text{rot} \vec{E} + \beta \mu \partial_t \text{rot} \vec{H} + \mu \partial_t \vec{H}) = -\sqrt{\mu \varepsilon} \vec{j}. \]

The real part of (17) coincides with (9) and the imaginary part coincides with (7). Applying divergence to the equation (17) and using the continuity equation gives us

\[ \partial_t \text{div} \vec{H} = 0 \quad \text{and} \quad \partial_t \text{div} \vec{E} = \frac{1}{\varepsilon} \partial_t \rho. \]

Taking into account these two equalities we obtain from (16) that the vectors \( \vec{E} \) and \( \vec{H} \) satisfy equations (8). \( \blacksquare \)
It should be noted that for $\beta = 0$ from (13) we obtain the operator which was studied in [7] with the aid of the factorization of the wave operator for non-chiral media

$$\varepsilon \mu \partial_t^2 - \Delta_x = (\sqrt{\varepsilon \mu} \partial_t + iD)(\sqrt{\varepsilon \mu} \partial_t - iD).$$

In the case under consideration we obtain a similar result. Let us denote by $M^*$ the complex conjugate operator of $M$:

$$M^* = \beta \sqrt{\varepsilon \mu} \partial_t D + \sqrt{\varepsilon \mu} \partial_t + iD.$$  

For simplicity we consider now a sourceless situation. In this case the equations (9) and (10) are homogeneous and can be represented as follows

$$MM^* \overrightarrow{U}(t, x) = 0,$$

where $\overrightarrow{U}$ stands for $\overrightarrow{E}$ or for $\overrightarrow{H}$.

5 Fundamental solution of the operator $M$

We will construct a fundamental solution of the operator $M$ using the results of the previous section and well known facts from quaternionic analysis. Consider the equation

$$(\beta \sqrt{\varepsilon \mu} \partial_t D + \sqrt{\varepsilon \mu} \partial_t - iD)f(t, x) = \delta(t, x).$$

Applying the Fourier transform $\mathcal{F}$ with respect to the time-variable $t$ we obtain

$$(\beta \sqrt{\varepsilon \mu} i \omega D + \sqrt{\varepsilon \mu} i \omega - iD)F(\omega, x) = \delta(x),$$

where $F(\omega, x) = \mathcal{F}\{f(t, x)\} = \int_{-\infty}^{\infty} f(t, x)e^{-i \omega t}dt$. The last equation can be rewritten as follows

$$(D + \alpha)(\beta \sqrt{\varepsilon \mu} \omega - 1)iF(\omega, x) = \delta(x),$$

where $\alpha = \frac{\sqrt{\varepsilon \mu} \omega}{\beta \sqrt{\varepsilon \mu} \omega - 1}$. The fundamental solution of $D_\alpha$ is given by (11), so we have

$$ (\beta \sqrt{\varepsilon \mu} \omega - 1)iF(\omega, x) = (\alpha + \frac{x}{|x|^2} - i \alpha \frac{x}{|x|})\Theta_\alpha(x),$$

from where

$$5$$
\[ F(\omega, x) = \left[ \frac{i \sqrt{\varepsilon \mu \omega}}{(\beta \sqrt{\varepsilon \mu \omega} - 1)^2} \left( 1 - \frac{ix}{|x|} \right) + \frac{ix}{|x|^2} \frac{1}{\beta \sqrt{\varepsilon \mu \omega} - 1} \right] e^{i|x| \sqrt{\varepsilon \mu \omega}} \frac{1}{4\pi |x|}. \]

We write it in a more convenient form

\[ F(\omega, x) = \left( \frac{1}{(\omega - a)^2} A(x) + \frac{1}{\omega - a} B(x) \right) E(x) e^{i\omega t}, \]

where \( a = \frac{1}{\beta \sqrt{\varepsilon \mu}} \), \( c(x) = \frac{|x|}{\beta^2 \sqrt{\varepsilon \mu}} \), \( E(x) = \frac{e^{i|x|}}{4\pi |x|} \),

\[ A(x) = \frac{i}{\beta^2 \varepsilon \mu} \left( 1 - \frac{ix}{|x|} \right), \quad B(x) = \frac{i}{\beta \sqrt{\varepsilon \mu}} \left( \frac{1}{\beta} \left( 1 - \frac{ix}{|x|} \right) + \frac{x}{|x|^2} \right). \]

In order to obtain the fundamental solution \( f(t, x) \) we should apply the inverse Fourier transform to \( F(\omega, x) \). Among different regularizations of the resulting integral we should choose the one leading to a fundamental solution satisfying the causality principle, that is vanishing for \( t < 0 \). Such an election is done by introducing of a small parameter \( y > 0 \) in the following way

\[ f(t, x) = \lim_{y \to 0} \mathcal{F}^{-1} \{ F(z, x) \} \]

where \( z = \omega - iy \). This regularization is in agreement with the condition \( \text{Im} \alpha \geq 0 \). We have

\[ \mathcal{F}^{-1} \{ F(z, x) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{(\omega - a_y)^2} A(x) + \frac{1}{\omega - a_y} B(x) \right) E(x) e^{i\omega t} e^{i\omega t} d\omega \]

where \( a_y = a + iy \). Expression (19) includes two integrals of the form

\[ I_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} e^{i\omega t} d\omega, \quad k = 1, 2 \]

where \( c = c(x) \). We have

\[ I_k = \frac{1}{2\pi} \sum_{j=0}^{\infty} \left( \frac{(ic)^j}{j!} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \right)^{j+k}. \]
Denote

\[ I_{k,j}(t) = \int_{-\infty}^{\infty} \frac{e^{i\omega t} d\omega}{(\omega - a_y)^{j+k}}. \]

For \( k = 1 \) and \( j = 0 \) we obtain (see, e.g., [3, Sect. 8.7])

\[ I_{1,0}(t) = 2\pi i H(t) e^{ia_y t} \]

where \( H \) is the Heaviside function. For all other cases, that is for \( k = 1 \) and \( j = 0 \), \( 1, \infty \) and for \( k = 2 \) and \( j = 0, \infty \) we have that \( j + k \geq 2 \) and the integrand in (20) has a pole at the point \( a_y \) of order \( j + k \). Using a result from the residue theory [4, Sect. 4.3] we obtain

\[ I_{k,j}(t) = 2\pi i \text{Res}_{a_y} \frac{e^{i\omega t}}{(\omega - a_y)^{j+k}} \quad \text{for} \quad t \geq 0 \quad \text{and} \quad j + k \geq 2. \]

Consider

\[ \text{Res}_{a_y} \frac{e^{i\omega t}}{(\omega - a_y)^{j+k}} = \frac{1}{(j + k - 1)!} \lim_{\omega \to a_y} \frac{\partial^{j+k-1}}{\partial \omega^{j+k-1}} e^{i\omega t} = \frac{(it)^{j+k-1} e^{ia_y t}}{(j + k - 1)!} \quad \text{for} \quad t \geq 0 \]

and \( j + k \geq 2 \).

For \( t < 0 \) we have that \( I_{k,j}(t) \) is equal to the sum of residues with respect to singularities in the lower half-plane \( y < 0 \) which is zero because the integrand is analytic there. Thus we obtain

\[ I_{k,j}(t) = 2\pi i H(t) \frac{(it)^{j+k-1}}{(j + k - 1)!} e^{ia_y t}. \]

Substitution of this result into (20) gives us

\[ I_1 = iH(t)e^{ia_y t} \sum_{j=0}^{\infty} \frac{(-ct)^j}{j!} \quad \text{and} \quad I_2 = -H(t)e^{ia_y t} \sum_{j=0}^{\infty} \frac{(-ct)^j}{j!(j+1)!}. \]

Now using the series representations of the Bessel functions \( J_0 \) and \( J_1 \) (see e.g. [14, Chapter 5]) we obtain

\[ I_1 = iH(t)e^{ia_y t} J_0 \left( 2\sqrt{ct} \right) \quad \text{and} \quad I_2 = -H(t) \sqrt{\frac{t}{c}} e^{ia_y t} J_1 \left( 2\sqrt{ct} \right). \]

Substituting these expressions in (19) and then in (18) we arrive at the following expression for \( f \):

\[ 7 \]
Finally we rewrite the obtained fundamental solution of the operator $M$ in explicit form:

$$f(t, x) = H(t) e^{iat} E(x) \left( -A(x) \sqrt{\frac{t}{c}} J_1 \left( 2\sqrt{ct} \right) + iB(x) J_0 \left( 2\sqrt{ct} \right) \right).$$

Let us notice that $f$ fulfills the causality principle requirement which guarantees that its convolution with the function from the right-hand side of (15) gives us the unique physically meaningful solution of the inhomogeneous Maxwell system (6)-(8) in a whole space.

Acknowledgement 2 The authors wish to express their gratitude to CONACYT for the support of this work via the grant Catedra Patrimonial No. 010286 and a Research Project.

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