SPECTRAL MEASURES GENERATED BY ARBITRARY AND RANDOM CONVOLUTIONS

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Abstract. We study spectral measures generated by infinite convolution products of discrete measures generated by Hadamard triples, and we present sufficient conditions for the measures to be spectral, generalizing a criterion by Strichartz. We then study the spectral measures generated by random convolutions of finite atomic measures and rescaling, where the digits are chosen from a finite collection of digit sets. We show that in dimension one, or in higher dimensions under certain conditions, “almost all” such measures generate spectral measures, or, in the case of complete digit sets, translational tiles. Our proofs are based on the study of self-affine spectral measures and tiles generated by Hadamard triples in quasi-product form.

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1. Introduction

Let \( \mu \) be a compactly supported Borel probability measure on \( \mathbb{R}^d \) and let \( \langle \cdot , \cdot \rangle \) and \( \langle \cdot , \cdot \rangle_{L^2(\mu)} \) denote respectively the standard inner product on \( \mathbb{R}^d \) and \( L^2(\mu) \). The measure \( \mu \) is called a spectral measure if there exists a countable set \( \Lambda \subset \mathbb{R}^d \), called spectrum of the measure \( \mu \), such that the collection of exponential functions \( E(\Lambda) := \{ e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda \} \) forms an orthonormal basis for \( L^2(\mu) \). If we define the Fourier transform of \( \mu \) to be

\[
\hat{\mu}(\xi) = \int e^{-2\pi i \langle \xi, x \rangle} d\mu(x),
\]

then \( E(\Lambda) \) is an orthonormal basis for \( \mu \) if and only if

(i) (Mutual orthogonality) \( \hat{\mu}(\lambda - \lambda') = 0 \) for all \( \lambda \neq \lambda' \in \Lambda \).

(ii) (Completeness) If \( \langle f , e^{2\pi i \langle \lambda, x \rangle} \rangle_{L^2(\mu)} = 0 \) for all \( \lambda \in \Lambda \), then \( f = 0, \mu\text{-a.e.} \)

If only condition (i) is satisfied, then we say that \( \Lambda \) is a mutually orthogonal set.

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Classical spectral measures were first introduced by Fuglede \[Fug74\] when he studied his famous conjecture stating that $\chi_{\Omega} dx$ is a spectral measure if and only if $\Omega$ is a translational tile. Although the conjecture was proven to be false eventually in general \[Tao04, KM06\], this conjecture has generated a lot of interest (see \[LW96, LW97b, IKT01, IK13, JP99, Kol00\] and the reference therein) and it is related to the construction of Gabor and wavelet bases \[LW03, Wan02\]. The studies entered into the realm of fractals when Jorgensen and Pedersen discovered that some singular fractal measures can also be spectral \[JP98\]. Since then, singular spectral measures has been an active research topic which involves constructing new examples \[LW02, Str00, DJ07\], classifying classes of measures which are spectral \[HL07, Dai12\] and classifying their possible spectra \[DHS09, DHL13\]. It was surprising to find that the convergence of the associated Fourier series is uniform in the space of continuous functions \[Str06\]. All the constructions of singular spectral measures, in the literature, to the best of our knowledge, are based on the Hadamard triple assumption.

**Definition 1.1.** Let $R \in M_d(\mathbb{Z})$ be an $d \times d$ expansive matrix (expansive means that all eigenvalues have modulus strictly greater than 1) with integer entries. Let $B, L \subset \mathbb{Z}^d$ and $0 \in B \cap L$ be finite sets of integer vectors with $N := \#B = \#L$ ($\#$ denotes the cardinality). We say that the system $(R, B, L)$ forms a Hadamard triple if the matrix

$$H = \frac{1}{\sqrt{N}} \left[ e^{2\pi i (R^{-1}b, \ell)} \right]_{\ell \in L, b \in B}$$

is unitary, i.e., $H^* H = I$.

Given a discrete set $A \subset \mathbb{R}^d$, we define the discrete measure on $A$ by

$$\delta_A = \frac{1}{\#A} \sum_{a \in A} \delta_a$$

where $\delta_a$ is the Dirac mass at $a$. From a direct observation, we can easily see that $(R, B, L)$ forms a Hadamard triple if and only if the discrete measure $\delta_{R^{-1}B}$ is a spectral measure with spectrum $L$. Singular spectral measures have been constructed by infinite convolutions of these discrete measures. To put it in the most general sense, suppose that we are given a sequence of Hadamard triples $(R_i, B_i, L_i), i = 1, 2, \ldots$ Then we define

$$R_n = R_n \ldots R_1$$

and the probability measure induced by these triples as

$$\mu = \mu(R, B_i) = \delta_{R_1^{-1}B_1} * \delta_{R_2^{-1}B_2} * \ldots * \delta_{R_n^{-1}B_n} * \ldots,$$

assuming the infinite convolution product is weakly convergent to a Borel probability measure.

It is easy to show that the measure has an infinite mutually orthogonal set

$$\Lambda = L_1 + R_1^T L_2 + \ldots + (R_1^T R_2^T \ldots R_{n-1}^T) L_n + \ldots$$

The spectral property of these measures was first studied by Strichartz \[Str00\], in which the sequence \{(R_i, B_i)\} was called a compatible tower, and it has received a lot of attention recently since all measures arising from factorization of Lebesgue measure on $[0,1]^d$ are of this type \[GL14\] and it gives rise to spectral measures with support of arbitrary dimensions \[DS15\]. We also note that if all $R_i = R$ and $B_i = B$ for some expanding matrix $R \in M_d(\mathbb{Z})$ and $B \subset \mathbb{R}^d$, then the measure

$$\mu = \mu_{R, B} = \delta_{R^{-1}B} * \delta_{R^{-2}B} * \ldots * \delta_{R^{-n}B} * \ldots$$
is reduced to the self-affine measure generated by the maps $τ_i(x) = R^{-1}(x + b)$, see [Hut81]. It was recently proved by the authors, by suitably modifying $L$, that all self-affine measures generated by Hadamard triples are spectral measures [DL15 DHL15]. In this paper, we study the spectral property of these arbitrary convolution and then a special case of random convolution with finitely many choices of Hadamard triples, chosen in a random order.

**Arbitrary convolutions.** We first generalize the Strichartz criterion for $Λ$ in (1.3) to be a spectrum for $μ$. For $μ$ and $Λ$ in (1.2) and (1.3), we define

$$μ_n = δ_{R_1^{-1}B_1} * δ_{R_2^{-1}B_2} * ... * δ_{R_n^{-1}B_n}, \text{ } μ > n = δ_{R_{n+1}^{-1}B_{n+1}} * δ_{R_{n+2}^{-1}B_{n+2}} * ...$$

and

$$K_n = \left\{ \sum_{k=n+1}^{∞} R_k^{-1}b_k : b_k ∈ B_k \right\}, B_n = \left\{ \sum_{k=1}^{n} R_k^{-1}b_k : b_k ∈ B_k \right\}.$$ 

Hence, $K_0 = \bigcup_{b ∈ B_n} (b_n + K_n)$ and $K_0, B_n, K_n$ are respectively the support of $μ, μ_n$ and $μ > n$. We will also use the notation $T(\{R_i, B_i\})$ for the support $K_0$ of the measure $μ$.

We say that $μ$ satisfies the no overlap condition if

$$μ((b_n + K_n) ∩ (b_n' + K_n)) = 0, \text{ for all } b_n \neq b_n' ∈ B_n, \text{ for all } n ∈ N.$$ 

For $Λ$ in (1.3), we also define its $n^{th}$-level approximation.

$$Λ_n = L_1 + R_1^T L_2 + ... + R_1^T R_2^T ... R_{n-1}^T L_n.$$ 

It is easy to see that $#B_n = #Λ_n = \prod_{i=1}^{n} N_i := M_n$. From this, we consider the following matrices

$$F_n = \frac{1}{√M_n} [\hat{μ} > n(Λ)|e^{-2πi(b, λ)}]|_{λ ∈ Λ_n, b ∈ B_n}.$$ 

Recall that the singular values of $F_n$ are the eigenvalues of $F_n^∗ F_n$ and we denote by $σ(F_n)$ the set of all singular values of $F_n$.

Our first main result is as follows:

**Theorem 1.2.** Suppose that the measure $μ$ in (1.2) satisfies the no-overlap condition and that $μ$ is compactly supported. If $inf_n min σ(F_n) > 0$, then $Λ$ is a spectrum for $μ$.

In particular, if $inf_n inf_λ |μ > n(λ)| > 0$, then $Λ$ is a spectrum for $μ$.

We remark that the assumption that the measure $μ$ is compactly supported ensures that the family of step functions is dense in $L^2(μ)$ and the no-overlap condition is also necessary to ensure that $μ(K_n) = 1/M_n$. In fact, if the no-overlap condition is not satisfied, $μ$ can be non-spectral (see Example 1.8).

**Random Convolutions.** In the second part of the paper, we consider $R_i = R$ for all $i$ with $R$ is a fixed integral expanding matrix. Let also $B(1), ..., B(N)$ be a finite collection of sets in $\mathbb{R}^d$, with $0 ∈ B(i)$ and $#B(i) = M ≤ |det R|$, for all $i$, so that $(R, B(i), L)$ form Hadamard triples for all $i$. Note that the set $L$ is the same for all $i$.

Let $ω = ω_1 ω_2 ...$ be an infinite word in $\{1, ..., N\}^N$. The measure $μ$ in (1.2) is now read as a random convolution of discrete measures scaled by $R$.

$$μ_ω = μ(ω, R) := δ_{R^{-1}B(ω_1)} * δ_{R^{-2}B(ω_2)} * ...$$ 

(1.5)
Some special cases of these measures were studied by He et al. \[AHL15a, AHL15b\]. We will see that spectral measures exist in abundance in the setting of random convolutions. To be precise, we treat \(\omega_n\) as independent random variables with values \(1, \ldots, N\), with equal probability \(1/N\), and \(P\) is the product probability on \({1, \ldots, N}\)^N. i.e.

\[
P(\omega_1 = i_1, \ldots, \omega_k = i_k) = \frac{1}{N^k}, \quad \forall k \in \mathbb{N}, \ i_1, \ldots, i_k \in \{1, \ldots, N\}.
\]

The main important observation is that measures in \((1.5)\) can be put together in the fibres of the self-affine measures generated by a Hadamard triple in quasi-product form.

**Definition 1.3.** Given the Hadamard triples \((R, B(i), L)\), \(i = 1, \ldots, N\) and \(#B(i) = M\). We associate the matrix \(R\) and the sets \(B\) and \(L\) with the following form:

\[
R = \begin{bmatrix} R_1 & 0 \\ C & R \end{bmatrix},
\]

where \(R_1 \in M_r(\mathbb{Z})\), \(R \in M_d(\mathbb{Z})\) and \(C \in M_{d,s}(\mathbb{Z})\). Let

\[
B = \left\{ \begin{bmatrix} a_i \\ d_{i,j} \end{bmatrix} : i \in \{1, \ldots, N\}, d_{i,j} \in B(i) \right\}
\]

where \(a_i \in \mathbb{Z}^r\), \(d_{i,j} \in \mathbb{Z}^d\) and \(a_1 = d_{1,1} = 0\), for all \(i, j\).

Suppose \(L = L_1 \times L\) with \(L_1 \subset \mathbb{Z}^r\), \(L \subset \mathbb{Z}^d\) and \((R_1, B_1) := \{a_i : 1 \leq i \leq N\}, L_1\) is a Hadamard triple. Then we say that \((R, B, L)\) is in quasi-product form on \(\mathbb{R}^{d+r}\) associated with \((R, B(i), L)\), \(i = 1, \ldots, N\). The self-affine measure associated with \((R, B, L)\) is the measure defined by

\[
\mu_{R, B} = \delta_{R^{-1}B} \ast \delta_{R^{-2}B} \ast \ldots \ast \delta_{R^{-n}B} \ast \ldots
\]

We denote also by \(\mu_1\) the self-affine measure associated with \((R_1, B_1)\) defined in \((1.4)\).

**Theorem 1.4.** Let \((R, B(i), L), i = 1, \ldots, N\), be the Hadamard triples and \((R, B, L)\) be the triple in quasi-product form associated with \((R, B(i), L)\) in Definition \((1.3)\). Assume \(\Lambda_1\) is a spectrum for \(\mu_1\) and let \(\Lambda_2\) be a subset of \(\mathbb{R}^d\). Then \(\Lambda_1 \times \Lambda_2\) is a spectrum for \(\mu\) if and only if \(\Lambda_2\) is a spectrum for \(\mu_\omega\), \(P\)-almost surely.

With the theorem above, we will construct a spectrum of the form \(\Lambda_1 \times \Lambda_2\) for \(\mu_{R, B}\) in some associated quasi-product form. Under two different assumptions, we have the following conclusion:

**Theorem 1.5.** Let \((R, B(i), L), i = 1, \ldots, N,\) be the Hadamard triples. Assume that one of the following condition holds:

(i) the Hadamard triples \((R, B(i), L)\) are on \(\mathbb{R}^1\), i.e. \(R\) is an integer.

(ii) Each \(B(i)\) is a complete set representative of \(R\)

Then there exists a set \(\Lambda\) such that \(\Lambda\) is a spectrum for \(\mu_\omega\), for \(P\)-almost every \(\omega\).

Moreover, in the case (ii), there exists a lattice \(\Gamma\) such that the support \(T(\{R, B(i)\}_k)\) of the measure \(\mu_\omega\) tiles \(\mathbb{R}^d\) by \(\Gamma\) and \(\mu_\omega\) is the normalized Lebesgue measure on \(T(\{R, B(i)\}_k)\), for \(P\)-almost every \(\omega = (i_1i_2 \ldots)\).

**Definition 1.6.** We say that a Lebesgue measurable set \(T\) tiles \(\mathbb{R}^d\) by a set \(T\), if \((T + t)_{t \in T}\) is a partition of \(\mathbb{R}^d\), up to Lebesgue measure zero.
The proof for the first case involves one of the canonical spectra in spectral measures theory. These are studied in [DJ06, DJ07, DJ09]. We call it here the dynamically simple spectra (Definition 3.8). We will summarize this in a separate study in the appendix of this paper. Theorem 1.5 perhaps hints towards a conjecture about random convolutions.

**Conjecture 1.7.** Let \((R, B(i), L), i = 1, ..., N\) be the Hadamard triples on \(\mathbb{R}^d\). Then some associated quasi-product form admits a spectrum of the form \(\Lambda_1 \times \Lambda_2\) and hence \(\Lambda_2\) is a spectrum for \(\mu_\omega\), \(\mathbb{P}\)-almost surely.

Theorem 1.5 showed that the conjecture is true on \(\mathbb{R}^1\) and in the case when we can construct a quasi-product form self-affine tile. In the end of the introduction, we illustrate Theorem 1.5 by an example. It is very interesting to notice, that some simple infinite convolution products, are not spectral. This sheds some light on our results that show that “almost every” infinite convolution is a spectral measure. However, not all of them as we see in the next example.

**Example 1.8.** Let \(R = 2\) and \(B(0) = \{0, 1\}\) and \(B(1) = \{0, 3\}\). As each \(B(i)\) is a complete residue modulo 2. Theorem 1.5 shows that, almost surely,

\[
\mu_\omega = \delta_{B(\omega_1)/2} * \delta_{B(\omega_2)/2} * ... 
\]

is a spectral measure with a common spectrum \(\mathbb{Z}\). However, if we consider a special case with \(\omega = 01111...\), we see that the measure

\[
\mu_\omega = \delta_{\{0,1\}/2} \ast L_{[0,3/2]},
\]

where \(L_{[0,3/2]}\) is the normalized Lebesgue measure supported on the interval \([0,3/2]\). Thus, in the first level, the no-overlap condition is not satisfied. Moreover, the measure \(\mu_\omega\) is absolutely continuous with respect to the Lebesgue measure, but it is not spectral as the density is not uniformly distributed [DL14]. Despite this specific example, the measures \(\mu_\omega\) are spectral, for almost all \(\omega \in \{0,1\}^\mathbb{N}\), by Theorem 1.5.

One may also refer to [AHL15a, AHL15b] for some deterministic examples in which the random convolution is spectral everywhere. However, strong assumption on \(L\) is required and it does not cover Example 1.8.

We organize our paper as follows: we study arbitrary convolutions in Section 2 and random convolutions in Section 3. In the appendix, we study the dynamically simple spectrum used in Section 3.

## 2. Arbitrary convolutions

Given a sequence of Hadamard triples \(\{(R_i, B_i, L_i)\}\) with measures \(\mu\) defined in (1.2), its Fourier transform is easily computed as

\[
\hat{\mu}(\xi) = \prod_{n=1}^{\infty} \hat{\delta}_{B_i}((R_n^T)^{-1}\xi).
\]

We first note that

**Lemma 2.1.** The set \(\Lambda\) in (1.3) is a mutually orthogonal set for \(\mu\).
Proof. This was proved in Strichartz [Str00, Theorem 2.7]. In short, it follows from the fact that the Hadamard matrices \( H_n = \frac{1}{\sqrt{N}} \left[ e^{2\pi i(R^{-1}b,\ell)} \right]_{\ell \in L_i, b \in B_i} \) have mutually orthogonal rows, and so does the matrix
\[
\frac{1}{\sqrt{M_n}} \left( e^{-2\pi i(R^{-n}b,\lambda)} \right)_{\lambda \in \Lambda_n, b \in B_n}.
\]

Recall that we can write the support of \( \mu_0, K_0 \), as
\[
K_0 = \bigcup_{b \in B_n} (b + K_n).
\]
Denote by \( K_b = b + K_n \) and by \( 1_{K_b} \) the characteristic function of \( K_b \). Let
\[
S_n = \left\{ \sum_{b \in B_n} w_b 1_{K_b} : w_b \in \mathbb{C} \right\}.
\]
\( S_n \) denotes the collection of all \( n \)th level step functions on \( K_0 \). As
\[
K_n = \bigcup_{b \in B_{n+1}} (R^{-1}b + K_{n+1})
\]
and \( 0 \in B_n \) for all \( n \), we have \( S_1 \subset S_2 \subset \ldots \). Let also
\[
S = \bigcup_{n=1}^{\infty} S_n.
\]

**Lemma 2.2.** If \( \mu \) is compactly supported, then \( S \) forms a dense set of functions in \( L^2(\mu) \).

**Proof.** Take first a continuous function \( f \) on \( K_0 \) and \( \epsilon > 0 \). Since \( K_0 \) is compact, the function \( f \) is uniformly continuous. We can find \( m \) large enough such that the diameter of all sets \( K_b, b \in B_m \), is small enough so that \( |f(x) - f(y)| < \epsilon \) for all \( x, y \in K_b \). Consider \( g = \sum_{b \in B_n} f(b) 1_{K_b} \). It is easy to see that \( \sup_{x \in K_0} |f(x) - g(x)| < \epsilon \). Hence, \( S \) is uniformly dense in \( C(K_0) \). As \( \mu \) is a regular Borel measure, \( S \) is dense in \( L^2(\mu) \). \( \square \)

**Lemma 2.3.** Let \( f = \sum_{b \in B_n} w_b 1_{K_b} \in S_n \) and let \( w = (w_b)_{b \in B_n} \). Denote by \( \| \cdot \| \) the Euclidean norm on \( \mathbb{C}^{M_n} \). Then
\[
\int |f|^2 d\mu = \frac{1}{M_n} \sum_{b \in B_n} |w_b|^2 = \frac{1}{M_n} \|w\|^2.
\]
(2.2)
\[
\int f(x)e^{-2\pi i(\lambda, x)} d\mu(x) = \frac{1}{M_n} \mu^{>}_n(\lambda) \sum_{b \in B_n} w_b e^{-2\pi i(b, \lambda)}.
\]
(Recall that \( M_n = N_1 \ldots N_n \)). Moreover,
\[
\sum_{\lambda \in \Lambda_n} \left| \int f(x)e^{-2\pi i(\lambda, x)} d\mu(x) \right|^2 = \frac{1}{M_n} \|F_n w\|^2.
\]
(2.4)
This implies that $\mu(K_b) \geq 1/M_n$ for all $b \in B_n$. On the other hand, because of the no-overlap condition and (2.1),

$$1 = \mu(K_0) = \mu \left( \bigcup_{b \in B_n} K_b \right) = \sum_{b \in B_n} \mu(K_b).$$

If $\mu(K_b) > 1/M_n$ for some $b \in B_n$, then $\sum_{b \in B_n} \mu(K_b) > 1$, which is a contradiction. Hence, all $K_b, b \in B_n$ have the same $\mu$-measure $1/M_n$ and (2.2) follows from a direct computation. For (2.3), we note that (2.5) now becomes

$$\frac{1}{M_n} = \mu(K_b) = \int 1_{K_b}(x)d(\mu_n * \mu_{>n}(x)) = \frac{1}{M_n} + \frac{1}{M_n} \sum_{b', b' \neq b} \int 1_{b+K_n}(b' + y)d\mu_{>n}(y)$$

since supp $\mu_{>n} = K_n$. Thus, $\int 1_{b+K_n}(b' + y)d\mu_{>n}(y) = 0$ and $1_{b+K_n}(b' + y) = 0$ $\mu_{>n}$-a.e. Hence,

$$\int f(x)e^{-2\pi i(\lambda, x)}d\mu(x) = \sum_{b \in B_n} w_b \int 1_{K_b}(x)e^{-2\pi i(\lambda, x)}d(\mu_n * \mu_{>n}(x))$$

$$= \sum_{b \in B_n} w_b \int \int 1_{b+K_n}(x + y)e^{-2\pi i(\lambda, x+y)}d\mu_n(x)d\mu_{>n}(y).$$

$$= \sum_{b \in B_n} w_b\frac{1}{M_n} \int \sum_{b' \in B_n} 1_{b+K_n}(b' + y)e^{-2\pi i(\lambda, b' + y)}d\mu_{>n}(y)$$

$$= \sum_{b \in B_n} w_b\frac{1}{M_n}e^{-2\pi i(\lambda, b)} \int e^{-2\pi i(\lambda, y)}d\mu_{>n}(y)$$

$$= \frac{1}{M_n} \mu_{>n}(\lambda) \sum_{b \in B_n} w_b e^{-2\pi i(b, \lambda)}.$$

Thus (2.3) follows. Finally, we have

$$\sum_{\lambda \in \Lambda_n} \left| \int f(x)e^{-2\pi i(\lambda, x)}d\mu(x) \right|^2 = \frac{1}{M_n} \sum_{\lambda \in \Lambda_n} \left| \mu_{>n}(\lambda) \right|^2 \left| \sum_{b \in B_n} w_b e^{-2\pi i(b, \lambda)} \right|^2$$

$$= \frac{1}{M_n} \sum_{\lambda \in \Lambda_n} \sum_{b \in B_n} w_b |\mu_{>n}(\lambda)|e^{-2\pi i(b, \lambda)} \left| \sum_{b \in B_n} w_b e^{-2\pi i(b, \lambda)} \right|^2 = \frac{1}{M_n} \|F_n w\|^2,$$

and (2.4) follows. \qed

We are now ready to prove our first theorem. We recall a standard fact of matrix analysis: If $A$ is a self-adjoint matrix and $\lambda_{\text{min}}$ is its minimum eigenvalue, then

$$\lambda_{\text{min}} = \min_{\|w\|=1} \langle Aw, w \rangle.$$
Proof of Theorem 1.2. Suppose first that \( \sigma := \inf_n \min \sigma(F_n) > 0 \). For all \( f \in S_n \), by equations (2.2) and (2.4) in Lemma 2.3,
\[
\sum_{\lambda \in \Lambda_n} \left| \int f(x)e^{-2\pi i \langle \lambda, x \rangle}d\mu(x) \right|^2 = \frac{1}{M_n} \| F_n w \|^2 = \frac{1}{M_n} (F_n^* F_n w, w) \geq \frac{1}{M_n} \sigma \| w \|^2 = \sigma \int |f|^2 d\mu.
\]
As \( f \in S_n \subset S_m \) for all \( m > n \), we apply \( f \) to the inequality for \( m \) and obtain
\[
\sum_{\lambda \in \Lambda_m} \left| \int f(x)e^{-2\pi i \langle \lambda, x \rangle}d\mu(x) \right|^2 \geq \sigma \int |f|^2 d\mu.
\]
Taking \( m \) to infinity and using the fact that \( \Lambda = \bigcup_{m=1}^{\infty} \Lambda_m \), we have
\[
\sum_{\lambda \in \Lambda} \left| \int f(x)e^{-2\pi i \langle \lambda, x \rangle}d\mu(x) \right|^2 \geq \sigma \int |f|^2 d\mu.
\]
As \( S \) forms a dense set, the above inequality is actually true for any \( f \in L^2(\mu) \). This establishes the completeness of \( \Lambda \) in \( L^2(\mu) \).

Let \( \delta = \inf_n \inf_{\lambda \in \Lambda_n} |\widehat{\mu}_{>n}(\lambda)| \). We now prove the special case. This follows from a direct observation that
\[
\| F_n w \|^2 \geq \delta^2 \| H_n w \|^2 = \delta^2 \| w \|^2
\]
where \( H_n = \frac{1}{\sqrt{M_n}} [e^{-2\pi i \langle b, \lambda \rangle}]_{\lambda \in \Lambda_n, b \in B_n} \) is a Hadamard matrix. Hence, \( \sigma \geq \delta^2 > 0 \). \( \square \)

Remark 2.4. We note that the theorem generalizes the result of Strichartz [Str00 Theorem 2.8], which asserted that if the Hadamard triples \((R_i, B_i, L_i)\) are chosen only from finitely many choices, and the zero sets \( Z_i \) of the functions
\[
m_{B_i}(x) = \frac{1}{N_i} \sum_{b \in B_i} e^{2\pi i \langle b, x \rangle}
\]
are separated from the set
\[
\Gamma_n = (R_n^T)^{-1} (L_1 + R_1^T L_2 + \ldots + (R_1^T R_2^T \ldots R_{n-1}^T) L_n)
\]
by a distance \( \delta > 0 \), uniformly in \( n \). Then the measure \( \mu(R_i, B_i) \) is a spectral measure. Indeed, this assumption implies \( \inf_n \inf_{\lambda \in \Lambda_n} |\widehat{\mu}_{>n}(\lambda)|^2 > 0 \). To see this, we note that
\[
|\widehat{\mu}_{>n}(\lambda)|^2 = \prod_{k=n+1}^{\infty} |m_{B_k}( (R_k^T)^{-1} \lambda) |^2.
\]
As there are only finitely many \( B_i, \Gamma_n \) lies inside a compact set independent of \( n \). From the fact that \( m_{B_i}(0) = 1 \) and that \( (R_k^T)^{-1} \lambda \) decays to zero exponentially, we can find a \( k_1 \), independent of \( \lambda \in \Lambda_n \) such that \( \prod_{k=n+k_1+1}^{\infty} |m_{B_k}( (R_k^T)^{-1} \lambda) |^2 \) is uniformly bounded below by some constant \( c > 0 \). For the first \( k_1 \) terms, the assumption on \( Z_i \) guarantees they are bounded away from \( \delta^{k_1} \), for some \( \delta' > 0 \). Thus,
\[
\inf_n \inf_{\lambda \in \Lambda_n} |\widehat{\mu}_{>n}(\lambda)|^2 \geq \delta^{k_1} c > 0.
\]
In this section, we study random convolutions of discrete measures generated by Hadamard triples. We first show that quasi-product forms generate Hadamard triples.

**Proposition 3.1.** If \((R, B, L)\) is in quasi-product form as in Definition 1.3 then \((R, B, L)\) is a Hadamard triple on \(\mathbb{R}^{d+r}\).

**Proof.** We have that \(R^{-1}\) is of the form

\[
R^{-1} = \begin{bmatrix} R_1^{-1} & 0 \\ D & R_1^{-1} \end{bmatrix},
\]

for some matrix \(D\). Consider \(b = \begin{bmatrix} a_i \\ d_{i,j} \end{bmatrix} \neq b' = \begin{bmatrix} a'_i \\ d'_{i,j} \end{bmatrix}\) and

\[
A((i,j),(i',j')) := \sum_{\ell \in L} e^{-2\pi i \langle R^{-1}(b-b'), \ell \rangle} = \sum_{\ell_1 \in L_1} \sum_{\ell_2 \in L_2} e^{-2\pi i (\langle R_1^{-1}(a_i-a_i'), \ell_1 \rangle + \langle D(a_i-a_i'), \ell_2 \rangle + \langle R_1^{-1}(d_{i,j}-d'_{i,j}), \ell_2 \rangle)} = \left( \sum_{\ell_1 \in L_1} e^{-2\pi i \langle R_1^{-1}(a_i-a_i'), \ell_1 \rangle} \right) \cdot \left( \sum_{\ell_2 \in L_2} e^{-2\pi i (\langle D(a_i-a_i'), \ell_2 \rangle + \langle R_2^{-1}(d_{i,j}-d'_{i,j}), \ell_2 \rangle)} \right).
\]

If \(i \neq i'\) then \(A((i,j),(i',j')) = 0\) because \((R_1, B_1, L_1)\) is a Hadamard triple. If \(i = i'\), then

\[
A((i,j),(i',j')) = N_1 \sum_{\ell_2 \in L_2} e^{2\pi i R_2^{-1}(d_{i,j}-d'_{i,j}) \ell_2} = 0,
\]

because \((R_2, B_2(i), L_2)\) are Hadamard triples for all \(i\). This shows that the matrix \(\left[ e^{-2\pi i \langle R^{-1}b, \ell \rangle} \right]_{\ell \in L, b \in B}\) has mutually orthogonal rows and hence \((R, B, L)\) is a Hadamard triple on \(\mathbb{R}^{d+r}\). \(\square\)

We now derive and collect the necessary information about the self-affine measure generated by the quasi-product form in Definition 1.3. These properties were all considered in [DJ07]. First, we note that

\[
R^{-1} = \begin{bmatrix} R_1^{-1} & 0 \\ -R_1^{-1}CR_1^{-1} & R_1^{-1} \end{bmatrix}
\]

and, by induction,

\[
R^{-k} = \begin{bmatrix} R_1^{-k} & 0 \\ D_k & R_1^{-k} \end{bmatrix}, \text{ where } D_k := -\sum_{j=0}^{k-1} R^{-1(k-j)}CR_1^{-1(k-j)}.
\]

The support of the self-affine measure \(\mu\) defined by \(R\) and \(B\) is given by

\[
T(R, B) = \left\{ \sum_{k=1}^{\infty} R^{-k}b_k : b_k \in B \right\}.
\]
Therefore any element \((x, y)^T \in T(\mathbb{R}, B)\) can be written in the following form
\[
x = \sum_{k=1}^{\infty} R_1^{-k} a_{i_k}, \quad y = \sum_{k=1}^{\infty} D_k a_{i_k} + \sum_{k=1}^{\infty} R^{-k} d_{i_k,j_k}.
\]

Let \(X_1\) be the attractor (in \(\mathbb{R}^r\)) associated to the IFS defined by the pair \((R_1, B_1)\), i.e.,
\[
X_1 = T(R_1, B_1) = \left\{ \sum_{k=1}^{\infty} R_1^{-k} b_k : b_k \in B_1 \right\}.
\]

Let \(\mu_1\) be the (equal-weighted) invariant measure associated to this pair.

For each sequence \(\omega = (i_1i_2 \ldots) \in \{1, \ldots, N\}^\mathbb{N} = \{1, \ldots, N\} \times \{1, \ldots, N\} \times \ldots\), define the map \(\pi : \Omega_1 \to X_1\) by
\[
\pi(\omega) = \sum_{k=1}^{\infty} R_1^{-k} a_{i_k}.
\]

As \((R_1, B_1)\) forms a Hadamard triple with \(L_1\), the measure \(\mu_1\) has the no-overlap property \cite[Theorem 1.7]{DL15}. It implies that for \(\mu_1\)-a.e. \(x \in X_1\), there is a unique \(\omega\) such that \(\pi(\omega) = x\). We define this as \(\pi^{-1}(x)\). This establishes a bijective correspondence, up to measure zero, between the set \(\Omega_1 := \{1, \ldots, N\}^\mathbb{N}\) and \(X_1\). For details about the correspondence, one can refer to \cite[Section 1.4]{Kig01}. The measure \(\mu_1\) from \(X_1\) is pulled back to the product measure \(\mathbb{P}\) defined in \cite{DL15}, i.e.
\[
\mu_1 = \mathbb{P} \circ \pi^{-1}.
\]

For \(\omega = (i_1i_2 \ldots)\) in \(\Omega_1\), define
\[
\Omega_2(\omega) := \{(d_{i_1,j_1}, d_{i_2,j_2} \ldots d_{i_n,j_n} \ldots) : j_k \in \{1, \ldots, N\}\}.
\]

For \(\omega \in \Omega_1\), define \(g(\omega) := \sum_{k=1}^{\infty} D_k a_{i_k}\). Also define
\[
X_2(\omega) := \left\{ \sum_{k=1}^{\infty} R_2^{-k} d_{i_k,j_k} : d_{i_k,j_k} \in B(i_k) \right\}.
\]

Note that \(T(\mathbb{R}, B)\) takes the following form
\[
T(\mathbb{R}, B) = \{(\pi(\omega), g(\omega) + y)^T : \omega \in \Omega_1, y \in X_2(\omega)\}.
\]

For \(x \in X_1\), up to a \(\mu_1\)-measure zero set, \(F\), we can write \(\omega = \pi^{-1}(x) = (i_1i_2\ldots)\) and we can define \(\mu_x^2\) to be the infinite convolution product defined by \(\mu_\omega\) in \cite{DL15}, i.e.
\[
\mu_x^2 = \mu_\omega = \delta_{R_2^{-1}B_2(i_1)} * \delta_{R_2^{-2}B_2(i_2)} * \ldots.
\]

with the support of \(\mu_x^2\) equal to \(X_2(x) := X_2(\pi^{-1}(x))\).

The following lemmas, established in \cite{DL07}, are the key identities for our analysis.

**Lemma 3.2.** \cite[Lemma 4.4]{DL07} For any bounded Borel functions on \(\mathbb{R}^d\), the self-affine measure \(\mu_{\mathbb{R}, B}\) satisfies
\[
\int_{T(\mathbb{R}, B)} f d\mu_{\mathbb{R}, B} = \int_{X_1} \int_{X_2(x)} f(x, y + g(x)) \, d\mu_x^2(y) \, d\mu_1(x).
\]
Lemma 3.3. \[DJ07\] Lemma 4.5] If \( \Lambda_1 \) is a spectrum for the measure \( \mu_1 \), then

\[
F(y) := \sum_{\lambda_1 \in \Lambda_1} |\widehat{\mu_{RB}}(x + \lambda_1, y)|^2 = \int_{X_1} |\widehat{\mu_s^2}(y)|^2 d\mu_1(s), \quad (x \in \mathbb{R}^r, y \in \mathbb{R}^{d-r}).
\]

We recall also the Jorgensen-Pedersen Lemma about checking when \( \Lambda \) is a spectrum for \( \mu \).

Lemma 3.4. \[JP98\] \( \Lambda \) is a spectrum for a probability measure \( \mu \) on \( \mathbb{R}^d \) if and only if

\[
Q(\xi) := \sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi + \lambda)|^2 \equiv 1.
\]

Moreover, if \( \Lambda \) is an orthogonal set, then \( Q \) is an entire function on \( \mathbb{C}^d \) with \( 0 \leq Q(x) \leq 1 \) for \( x \in \mathbb{R}^d \).

Proof of Theorem 1.4. Assume that \( \Lambda_2 \) is a spectrum for \( \mu_\omega \) for \( \mathbb{P} \)-a.e. \( \omega \). Let

\[
E = \{ \omega : \Lambda_2 \text{ is a spectrum for } \mu_\omega \}.
\]

Then \( \mathbb{P}(E \cap (\Omega_1 \setminus F)) = 1 \). The set of points \( x \in \Omega_1 \) such that \( \Lambda_2 \) is a spectrum for \( \mu_2^x \) is exactly equal to \( \pi^{-1}(x) \in E \cap (\Omega_1 \setminus F) \). This shows \( \Lambda_2 \) is a spectrum for \( \mu_2^x \) for \( \mu_1 \)-a.e. \( x \). We now check that \( \Lambda_1 \times \Lambda_2 \) is a spectrum for \( \mu_{RB} \). For \( (x, y) \in \mathbb{R}^{d+r} \) we have, with Lemma 3.3 and Fubini’s theorem,

\[
\sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} |\widehat{\mu_{RB}}(x + \lambda_1, y + \lambda_2)|^2 = \sum_{\lambda_2 \in \Lambda_2} \int_{X_1} |\widehat{\mu_2^x}(y + \lambda_2)|^2 d\mu_1(x)
\]

\[
= \int_{X_1} \sum_{\lambda_2 \in \Lambda_2} |\widehat{\mu_2^x}(y + \lambda_2)|^2 d\mu_1(x) = \int_{X_1} 1 = 1.
\]

Thus \( \Lambda_1 \times \Lambda_2 \) is a spectrum for \( \mu_{RB} \).

For the converse, assume \( \Lambda_1 \times \Lambda_2 \) is a spectrum for \( \mu_{RB} \). Take \( \lambda_1 \neq \lambda'_1 \) in \( \Lambda_1 \) and \( \lambda_2, \lambda'_2 \) in \( \Lambda_2 \). Then

\[
0 = \widehat{\mu_{RB}}(\lambda_1 - \lambda'_1, \lambda_2 - \lambda'_2) = \int_{X_1} e^{-2\pi i (\lambda_1 - \lambda'_1) \cdot x} \frac{|\widehat{\mu_2^x}(\lambda_2 - \lambda'_2)|^2}{d\mu_1(x)}.
\]

But \( \Lambda_1 - \lambda'_1 \) is a spectrum for \( \mu_1 \) so \( \mu_2^x(\lambda_2 - \lambda'_2) = 0 \) for \( \mu_1 \)-a.e. \( x \). Since \( \Lambda_2 \) is countable, this implies that \( \Lambda_2 \) is an orthogonal set for \( \mu_2^x \), for \( \mu_1 \)-a.e. \( x \). and in particular

\[
(3.4) \quad \sum_{\lambda_2 \in \Lambda_2} |\widehat{\mu_2^x}(y + \lambda_2)|^2 \leq 1 \text{ for all } y \in \mathbb{R}^{d-r} \text{ and } \mu_1 \text{-a.e. } x.
\]

Then

\[
1 \leq \sum_{\lambda_1 \in \Lambda_1} \sum_{\lambda_2 \in \Lambda_2} |\widehat{\mu_{RB}}(x + \lambda_1, y + \lambda_2)|^2 = \sum_{\lambda_2 \in \Lambda_2} \sum_{\lambda_1 \in \Lambda_1} \left| \int_{X_1} e^{-2\pi i ((\lambda_1 + x) \cdot t)} \widehat{\mu_2^x}(y + \lambda_2) d\mu_1(t) \right|^2
\]

\[
= \sum_{\lambda_2 \in \Lambda_2} \int_{X_1} |\widehat{\mu_2^x}(y + \lambda_2)|^2 d\mu_1(t) \text{ (by the Parseval equality for the spectrum } \Lambda_1) 
\]

\[
= \int_{X_1} \sum_{\lambda_2 \in \Lambda_2} |\widehat{\mu_2^x}(y + \lambda_2)|^2 d\mu_1(t) \leq \int_{X_1} 1 = 1.
\]
But combining with (3.4), we get that, for a fixed $y$,

$$\sum_{\lambda_2 \in \Lambda_2} |\hat{\mu}_2^2(y + \lambda_2)|^2 = 1$$

for $\mu_1$-a.e. $t$. As $Q(y) := \sum_{\lambda_2 \in \Lambda_2} |\hat{\mu}_2^2(y + \lambda_2)|^2$ is a continuous function, by Lemma 3.3 taking a countable dense set of $y$, we get that $\Lambda_2$ is a spectrum for $\mu_2^2$, for $\mu_1$-a.e. $t$ and this means $\Lambda_2$ is a spectrum, $\mathcal{P}$-almost surely. \hfill $\square$

Remark 3.6. Here are some remarks about the properties of extreme cycles.

In rest of the section, we will prove the spectral property result in Theorem 1.5. To this end, we need to analyze a dynamical system generated by the Hadamard triple, for a detailed account of this dynamical system see [DJ06, DJ07].

**Definition 3.5.** Let $(R, B, L)$ be a Hadamard triple. We define the function

$$m_B(x) = \frac{1}{\# B} \sum_{b \in B} e^{2\pi i (b, x)}, \quad (x \in \mathbb{R}^d).$$

The Hadamard triple condition implies that $\delta_{R^{-1}B}$ is a spectral measure with a spectrum $L$ and $m_{R^{-1}B}$ is the Fourier transform of the Dirac measure. Lemma 3.3 implies that

$$(3.5) \quad \sum_{\ell \in L} |m_B((R^T)^{-1}(x + \ell))|^2 = 1, \text{ or } \sum_{\ell \in L} |m_B(\tau_\ell(x))|^2 = 1,$$

where we define the maps

$$\tau_\ell(x) = (R^T)^{-1}(x + \ell), \quad (x \in \mathbb{R}^d, \ell \in L), \text{ and } \tau_{\ell_1...\ell_m} = \tau_{\ell_1} \circ ... \circ \tau_{\ell_m}.$$

A closed set $K$ in $\mathbb{R}^d$ is called invariant (with respect to the system $(R, B, L)$) if, for all $x \in K$ and all $\ell \in L$

$$m_B(\tau_\ell(x)) > 0 \implies \tau_\ell(x) \in K.$$

We say that the transition, using $\ell$, from $x$ to $\tau_\ell(x)$ is possible, if $\ell \in L$ and $m_B(\tau_\ell(x)) > 0$. A compact invariant set is called minimal if it does not contain any proper compact invariant subset.

For $\ell_1, ..., \ell_m \in L$, the cycle $C(\ell_1, ..., \ell_m)$ is the set

$$C(\ell_1, ..., \ell_m) = \{x_0, \tau_{\ell_m}(x_0), \tau_{\ell_{m-1}\ell_m}(x_0), ..., \tau_{\ell_2...\ell_m}(x_0)\},$$

where $x_0 := \phi(\ell_1, ..., \ell_m)$ is the fixed point of the map $\tau_{\ell_1...\ell_m}$, i.e. $\tau_{\ell_1...\ell_m}(x_0) = x_0$. The cycle $C(\ell_1, ..., \ell_m)$ is called an extreme cycle for $(R, B, L)$ if $|m_B(x)| = 1$ for all $x \in C(\ell_1, ..., \ell_m)$.

**Remark 3.6.** Here are some remarks about the properties of extreme cycles.

(i) For any extreme cycles, the only possible transition is from $x_0$ to $\tau_{\ell_m}(x_0)$ since $|m_B(\tau_{\ell_m}(x_0))| = 1$ and (3.5) implies all other must be zero.

(ii) Given any $c$ in an extreme cycle $C$, we can always find another point in this cycle $c'$ such that $c = \tau_\ell(c')$ for a unique digit $\ell$ defining the cycle. Iterating the process, for any $n \geq 1$, $c = \tau_{\ell_0...\ell_{n-1}}(c')$ for some $c' \in C$. Rewriting the relation, we have

$$(3.6) \quad c' = (R^T)^n(-c) + \ell_0 + R^T\ell_1 + ... + (R^T)^{n-1}\ell_{n-1}.$$
(iii) If 0 ∈ B, we have

\[ \langle R^n b, c \rangle \in \mathbb{Z}, \ \forall n \geq 0 \text{ and } b \in B. \]

First, \(|m_B(c)| = 1\) implies that \(\langle b, c \rangle \in \mathbb{Z}\) for all \(b \in B\) (we have equality in a triangle inequality so all the terms of the sum that defines \(m_B\) must be equal to 1, since \(0 \in B\)). In general, from (3.6) and the fact that \(c'\) is an extreme cycle point, \(m_B(c') = 1\) and \(\langle b, c' \rangle \in \mathbb{Z}\). As we know \(\langle b, \ell \rangle \in \mathbb{Z}\) so we must have \(\langle R^n b, c \rangle = \langle b, (R^T)^n c \rangle \in \mathbb{Z} \).

The following theorem shows the structure of minimal compact invariant sets and we will use it throughout the rest of the paper.

**Theorem 3.7. [CCR96 Theorem 2.8]** Let \(\mathcal{M}\) be a minimal compact invariant set contained in the zero set of an entire function \(h\) on \(\mathbb{R}^d\).

(i) There exists a proper rational subspace \(V\) (can be \(\{0\}\)) invariant for \(R^T\) such that \(\mathcal{M}\) is contained in the union \(\mathcal{R}\) of finitely many translates of \(V\).

(ii) This union contains the translates of \(V\) by the elements of a cycle \(C(\ell_1, \ldots, \ell_m)\) in \(\mathcal{M}\), and \(h\) is zero on \(x + V\) for all \(x \in \mathcal{C}(\ell_1, \ldots, \ell_m)\).

(iii) If the hypothesis “(H) modulo \(V\)” is satisfied, i.e., \((R^T)^{-1}(e_1 - e'_1) + (R^T)^{-2}(e_2 - e'_2) + \cdots + (R^T)^{-p}(e_p - e'_p) \in V\) implies \(e_1 - e'_1, e_2 - e'_2, \ldots, e_p - e'_p \in V\) for all \(e_1, \ldots, e_p, e'_1, \ldots, e'_p \in L\), then

\[ \mathcal{R} = \{x_0 + V, \tau_{\ell_m}(x_0) + V, \ldots, \tau_{\ell_2...\ell_m}(x_0) + V\} \]

where \(x_0 = \varphi(\ell_1, \ldots, \ell_m)\) and every possible transition from a point in \(\mathcal{M} \cap (\tau_{\ell_{q-1}...\ell_m}(x_0) + V)\) leads to a point in \(\mathcal{M} \cap (\tau_{\ell_q...\ell_m}(x_0) + V)\) for all \(1 \leq q \leq m\), with \(\ell_0 = \ell_m\).

(iv) The union \(\mathcal{R}\) is invariant.

In particular, from (3.5), extreme cycles are clearly compact invariant sets which correspond to the case \(V = \{0\}\) (if needed, we can always take the entire function \(h\) to be 0, in Theorem 3.7). However, the extreme cycles are not the only minimal compact invariant sets (see [DJ07] for some examples). We isolate this special case in the following definition.

**Definition 3.8.** We say that the Hadamard triple \((R, B, L)\) is dynamically simple if the only minimal compact invariant set are extreme cycles. For a Hadamard triple \((R, B, L)\), the orthonormal set \(\Lambda\) generated by extreme cycles is the smallest set such that

(i) it contains \(\mathcal{C}\) for all extreme cycles \(\mathcal{C}\) for \((R, B, L)\)

(ii) it satisfies \(R^T \Lambda + L \subset \Lambda\).

When this set \(\Lambda\) is a spectrum (see Theorem 3.9 below), we call it the dynamically simple spectrum.

More generally, the **set generated by an invariant subset** \(A\) of \(\mathbb{R}^d\), is the smallest set which contains \(-A\) and satisfies (ii).

**Theorem 3.9.** Let \((R, B, L)\) be a dynamically simple Hadamard triple. Then the orthonormal set \(\Lambda\) generated by extreme cycles is a spectrum for the self-affine measure \(\mu_{R,B}\) and \(\Lambda\) is explicitly given by

\[ \Lambda = \{\ell_0 + R^T \ell_1 + \cdots + (R^T)^{n-1} \ell_{n-1} + (R^T)^n (-c) : \ell_0, \ldots, \ell_{n-1} \in L, n \geq 0, c \text{ are extreme cycle points}\}. \]

Moreover, if \((R, B, L)\) is a Hadamard triple on \(\mathbb{R}^1\), it must be dynamically simple.
Proof. This theorem combines results in [DJ06, DJ07, DJ09]. An independent proof will be given in the appendix of this paper. □

Our main theorem leading to main conclusion in the introduction is the following:

**Theorem 3.10.** Assume that the Hadamard triple \((R, B, L)\) is in a quasi-product form defined in Definition 3.8 with \(C = 0\) and that \((R, B, L)\) is dynamically simple. Let \(\Lambda_1\) be the orthonormal set generated by extreme cycles for \((R_1, B_1, L_1)\) and suppose that \(\Lambda_2\) is the set generated by those cycles which are extreme for all triples \((R, B(i), L_i)\), \(i = 1, \ldots, N_1\). Then \(\Lambda_1 \times \Lambda_2\) is a spectrum for \((R, B, L)\) and \(\Lambda_2\) is a spectrum for \(\mu_\omega\) for \(\mathbb{P}\)-almost every \(\omega\).

**Proof.** As \((R, B, L)\) is dynamically simple, we can define \(\Lambda\) to be the dynamically simple spectrum for the quasi-product form \((R, B, L)\). We need to show that \(\Lambda = \Lambda_1 \times \Lambda_2\). In the proof, it is worth to note that \(|m_B(x)| = 1\) if and only if \((b, x) \in \mathbb{Z}\) for all \(b \in B\), since \(0 \in B\).

We show first that \(\Lambda \subseteq \Lambda_1 \times \Lambda_2\). Property (ii) in Definition 3.8 shows that the sets \(\Lambda_1\) and \(\Lambda_2\) satisfy \(R_1^T \Lambda_1 + L_1 \subseteq \Lambda_1\) and \(R^T \Lambda_2 + L \subseteq \Lambda_2\). With \(R = \begin{bmatrix} R_1 & 0 \\ 0 & R \end{bmatrix}\), it is clear that

\[
R^T(\Lambda_1 \times \Lambda_2) + (L_1 \times L_2) \subseteq \Lambda_1 \times \Lambda_2.
\]

Thus we only have to show that \(\Lambda_1 \times \Lambda_2\) contains \(-C\) for all extreme cycles \(C\) for \((R, B, L)\) and then it follows from definition of \(\Lambda\) that \(\Lambda \subseteq \Lambda_1 \times \Lambda_2\).

Let \(C = \{x_0, x_1, \ldots, x_{p-1}\}\) be such an extreme cycle of \((R, B, L)\). Then there exists \(\ell = (\ell_1, \ell_2) \in L_1 \times L_2\) such that \(x_{k+1} = (R_1^T)^{-1}(x_k + (\ell_1, \ell_2)^T)\) for all \(k = 0, \ldots, p-1\) and \(x_p = x_0\). Writing \(x_k = (x_k^{(1)}, x_k^{(2)})\), we must have \(x_{k+1}^{(1)} = (R_1^T)^{-1}(x_k^{(1)} + l_1)\) and \(x_{k+1}^{(2)} = (R^T)^{-1}(x_k^{(2)} + l_2)\) for all \(k = 0, \ldots, p-1\). Thus the first components form a cycle for \((R_1, B_1, L_1)\) and the second components form a cycle for \((R_2^T, B(i), L)\) for all \(i\). From the property of extreme cycle, we have that \(\langle b, x_k \rangle \in \mathbb{Z}\) for all \(b \in B\) and \(k = 0, 1, \ldots, p-1\). Therefore,

\[
\langle a_i, x_k^{(1)} \rangle + \langle d_{i,j}, x_k^{(2)} \rangle \in \mathbb{Z}
\]

for all \(1 \leq i \leq N, 1 \leq j \leq M\). Since \(d_{i,1} = 0\), we must have \(\langle a_i, x_0^{(1)} \rangle \in \mathbb{Z}\) for \(1 \leq i \leq N\) and therefore \(\langle d_{i,j}, x_0^{(2)} \rangle \in \mathbb{Z}\) for \(1 \leq j \leq M\). This shows that \(C_1 = \{x_0^{(1)}, x_1^{(1)}, \ldots, x_{p-1}^{(1)}\}\) is an extreme cycle for \((R_1, B_1, L_1)\), and \(C_2 = \{x_0^{(2)}, x_1^{(2)}, \ldots, x_{p-1}^{(2)}\}\) is an extreme cycle for all \((R_2^T, B(i), L)\). Hence, \(-C_1 \subseteq \Lambda_1\), \(-C_2 \subseteq \Lambda_2\), and \(-C \subseteq (-C_1) \times (-C_2) \subseteq \Lambda_1 \times \Lambda_2\). Since \(\Lambda\) is the smallest set which is invariant under \(R^T \Lambda + L\) and which contains \(-C\) for all extreme cycles \(C\), we must have \(\Lambda \subseteq \Lambda_1 \times \Lambda_2\).

Next, we show that \(\Lambda_1 \times \Lambda_2 = \Lambda\). It suffices to show that \(\Lambda_1 \times \Lambda_2\) forms an orthogonal set for \(\mu_{R,B}\). Indeed, \(\Lambda\) is a spectrum for \(\mu_{R,B}\) by Theorem 3.9. This means that \(\Lambda\) is a maximal orthogonal set (i.e., if \(\lambda' \notin \Lambda\), the exponential \(e^{2\pi i \langle \lambda', x \rangle}\) cannot be orthogonal to all exponentials with frequencies in \(\Lambda\)). But \(\Lambda \subseteq \Lambda_1 \times \Lambda_2\) and \(\Lambda_1 \times \Lambda_2\) is a mutually orthogonal set, we must have \(\Lambda = \Lambda_1 \times \Lambda_2\).
To show that $\Lambda_1 \times \Lambda_2$ forms an orthogonal set for $\mu_{R,B}$. We first note that as $\Lambda_1$ is a spectrum for the measure $\mu_1 = \mu(R_1, B_1)$ by Theorem 3.9, $\Lambda_1$ is a mutually orthogonal set for $\mu_1$. Hence,
\begin{equation}
\hat{\mu}_1(\lambda_1 - \lambda_1') = 0, \forall \lambda_1 \neq \lambda_1' \in \Lambda_1.
\end{equation}
We now show that $\Lambda_2$ is a mutually orthogonal set for all $\mu_x$. Indeed, for all $x = x(i_1, i_2, \ldots)$,
\begin{equation}
\hat{\mu}_x^{(2)}(\lambda_2 - \lambda_2') = \prod_{k=1}^{\infty} m_{B_2(i_k)}((R^T)^{-k}(\lambda_2 - \lambda_2'))
\end{equation}
where $\lambda_2 \neq \lambda_2' \in \Lambda_2$. They can be written as
\begin{equation}
\lambda_2 = \ell_0 + R^T \ell_1 + \ldots + (R^T)^{m-1} \ell_{m-1} + (R^T)^m(-x_0),
\end{equation}
\begin{equation}
\lambda_2' = \ell_0' + R^T \ell_1' + \ldots + (R^T)^{m'-1} \ell_{m'-1}' + (R^T)^{m'}(-x_0'),
\end{equation}
with $\ell_i, \ell_i' \in L$, $x_0, x_0'$ extreme cycle points for $(R, B(i), L)$. From (3.16), for any $p \geq 1$, we can write
\begin{equation}
-x_0 = (R^T)^k(-x_k) + \alpha_p + R^T \alpha_{p-1} + \ldots + (R^T)^{k-1} \alpha_{p-k}.
\end{equation}
Using (3.13) in (3.11), we can write $\lambda_2$ with as many digits as we want. Similarly, we can do this for case of $\lambda_2'$ in (3.12) and therefore we can take $m = m'$, and, as $\lambda_2, \lambda_2'$ are distinct elements, we can assume that there exists $n < m$ such that $\ell_0 = \ell_0', \ldots, \ell_{n-1} = \ell_{n-1}', \ell_n \neq \ell_n'$. Then
\begin{equation}
m_{B_2(i_{n+1})}((R^T)^{-n-1}(\lambda_2 - \lambda_2')) = m_{B_2(i_n)}((R^T)^{-1}(\ell_n - \ell_n') + M_0 + (R^T)^{m-n-1}(x - x')),
\end{equation}
where $M_0 \in \mathbb{Z}^d$ and $x, x'$ are extreme cycle points. From integral periodicity of $m_{B_2(i_n)}$ and (3.7), the above quantity is equal to $m_{B_2(i_n)}((R^T)^{-1}(\ell_n - \ell_n')) = 0$ by the Hadamard triple assumption. This implies from (3.10) that $\hat{\mu}_x^{(2)}(\lambda_2 - \lambda_2') = 0$.

If now $(\lambda_1, \lambda_2) \neq (\lambda_1', \lambda_2') \in \Lambda_1 \times \Lambda_2$, we have, by Lemma 3.2
\begin{equation}
\langle e^{2\pi i((\lambda_1, \lambda_2), (x, y))}, e^{2\pi i((\lambda_1', \lambda_2'), (x, y))} \rangle_{L^2(\mu_{R,B})} = \int e^{2\pi i((\lambda_1 - \lambda_1', \lambda_2 - \lambda_2'), (x, y))} d\mu_B(x, y)
\end{equation}
\begin{equation}
= \int \int e^{2\pi i((\lambda_1 - \lambda_1')x + (\lambda_2 - \lambda_2')y)} d\mu_x^{(2)}(y) d\mu_1(x)
\end{equation}
\begin{equation}
= \int e^{2\pi i((\lambda_1 - \lambda_1', x) \hat{\mu}_x^{(2)}(\lambda_2 - \lambda_2') d\mu_1(x)}.
\end{equation}
As $\Lambda_2$ is a mutually orthogonal set for $\mu_x^{(2)}$, the term above is equal to 0 if $\lambda_2 \neq \lambda_2'$. And if $\lambda_2 = \lambda_2'$, we must have $\lambda_1 \neq \lambda_1'$ and hence $\hat{\mu}_1(\lambda_1 - \lambda_1') = 0$. Thus $\Lambda_1 \times \Lambda_2$ forms an orthogonal set and hence completes the proof that $\Lambda_1 \times \Lambda_2$ is a dynamically simple spectrum for $(R, B, L)$.

Finally, by Theorem 3.9, $\Lambda_1 \times \Lambda_2$ is a spectrum for the self-affine measure $\mu_{R,B}$. Therefore, it follows from Theorem 1.4 that $\Lambda_2$ is $P$-almost surely a spectral measure for $\mu_\omega$. \hfill \Box

We now present the proof of Theorem 1.5.

**Proof of Theorem 1.5** (when (i) holds, i.e., the Hadamard triples $(R, B(i), L)$ are on $\mathbb{R}^1$). We pick a number $p \in \mathbb{N}$ such that $pN \neq R$. Define the matrix
\[
R = \begin{bmatrix}
pN & 0 \\
0 & R
\end{bmatrix}.
\]
Let $\tilde{B}(i) = B(i(\text{mod } N))$ for all $i \in \{0, 1, ..., pN - 1\}$. Here, $i(\text{mod } N)$ is the remainder when $i$ is divided by $N$. Let 

$$\tilde{B} := \left\{ (i, d)^T : i \in \{0, 1, \ldots, pN - 1\}, d \in \tilde{B}(i) \right\}.$$ 

Let $\tilde{L} := \{0, \ldots, pN - 1\} \times L$. In each of the coordinates, they form Hadamard triples on $\mathbb{R}^1$ and hence must be dynamically simple by Theorem 3.9. We now show that $(\mathcal{R}, \tilde{B}, \tilde{L})$ is also dynamically simple, so that Theorem 3.10 is applicable.

Let $\mathcal{M}$ be a minimal compact invariant set. Assume that $\mathcal{M}$ is infinite and $\mathcal{M}$ is not an extreme cycle. Then, by Theorem 3.7, there is a subspace $V \neq \mathbb{R}^2$, invariant for $\mathcal{R}$, such that $\mathcal{M}$ is contained in a union of finitely many translates of $V$. Since $V$ is invariant for $\mathcal{R}$, $pN \neq R$ and $V \neq \{0\}$ ($V = \{0\}$ corresponds to the extreme cycles), the only options are $V = \mathbb{R} \times \{0\}$ or $V = \{0\} \times \mathbb{R}$. We show that the first case is impossible while the second case implies all $\tilde{B}_j$ are the same, which means that Theorem 1.5 holds trivially.

**Case (i)** $V = \mathbb{R} \times \{0\}$. A direct check shows that the hypothesis “(H) modulo $V$” in Theorem 3.7(iii) is satisfied with $\tilde{L}$. (See for example [DJ07, Proposition 3.7] for an analogous proof). Applying now Theorem 3.7(iii), we deduce the existence of an $\tilde{L}$-cycle $(x_0, y_0)$, with digits $(i_1, \ell_1), \ldots, (i_m, \ell_m)$ ($\ell_j \in L$, $i_j \in \{0, 1, \ldots, pN - 1\}$) such that

$$\mathcal{M} \subset \bigcup_{k=1}^{m} (\tau(i_k, \ell_k) \cdots \tau(i_m, \ell_m)(x_0, y_0) + V) =: \mathcal{R}$$

and $\mathcal{R}$ is invariant. Moreover, every possible transition from $\tau(i_k, \ell_k) \cdots (i_m, \ell_m)(x_0, y_0) + V$ leads to a point in $\tau(i_k-1, \ell_k-1) \cdots (i_m, \ell_m)(x_0, y_0) + V$ for $1 \leq k \leq m$, where $(i_0, \ell_0) := (i_m, \ell_m)$.

Let $(x, y_0) \in (x_0, y_0) + V$. Let $\ell \neq \ell_m$. Then $\tau(i', \ell)(x, y_0) \notin \tau(i_m, \ell_m)(x_0, y_0) + V$, thus the transition is not possible so $m_{\tilde{B}}(\tau(i', \ell)(x, y_0)) = 0$ which means

$$\sum_{k=0}^{pN-1} \sum_{d \in \tilde{B}(k)} e^{2\pi i (k \frac{x + i' y_0}{pN} + d \frac{\nu_0 + \ell}{R})} = 0.$$ 

Let $x' := \frac{x + i' y_0}{pN}$. Then

$$0 = \sum_{k=0}^{pN-1} e^{2\pi i k x'} \sum_{d \in \tilde{B}(k)} e^{2\pi i d \frac{\nu_0 + \ell}{R}}.$$ 

Since $x'$ can be any real number, the coefficients of this polynomial must be zero:

$$\sum_{d \in \tilde{B}(k)} e^{2\pi i d \frac{\nu_0 + \ell}{R}} = 0,$$

by the linear independence of the trigonometric polynomials $e^{2\pi i k x'}$, $k = 0, 1, \ldots, pN - 1$. This means that $m_{\tilde{B}(k)}(\tau_\ell(y_0)) = 0$ for all $\ell \neq \ell_m$ and hence $|m_{\tilde{B}(k)}(\tau_{\ell_m}(y_0))| = 1$. Since $0 \in \tilde{B}(k)$ we have equality in the triangle inequality, so $m_{\tilde{B}(k)}(\tau_{\ell_m}(y_0)) = 1$. We can do this for all the points in the cycle $C_2 := \{y_k := \tau_{\ell_k} \cdots \tau_{\ell_m}(y_0) : 1 \leq k \leq m\}$, and we conclude that $C_2$ is an extreme $L$-cycle for all $\tilde{B}(k)$. 
We now consider \( v := (x, y_k) \in \mathcal{M} \) and the possible transition from \( v \), which must be of the form \((i', \ell_{k-1})\). From the extreme cycle property, we have \( m_{\tilde{B}(k)}(y_{k-1}) = 1 \) and hence

\[
0 \neq m_{\tilde{B}}(\tau(i', \ell_{k-1})(x, y_k)) = \frac{1}{pN} \sum_{j=0}^{pN-1} e^{2\pi i j \frac{x+1+i'}{pN}} m_{\tilde{B}(k)}(y_{k-1}) = \sum_{j=0}^{pN-1} e^{2\pi i j \frac{x+1+i'}{pN}}.
\]

This holds if and only if \( x + i' \notin \mathbb{Z} \) or \( x + i' \) is a multiple of \( pN \). As \( \mathcal{M} \) is infinite and \( \mathcal{M} \subset T(\mathbb{R}, \tilde{L}) = [0, 1] \times T(\mathbb{R}, L) \), we can assume that \( x \notin \mathbb{Z} \). In this case, when \( i' = 0 \),

\[
m_{\tilde{B}}(\tau(0, \ell_{k-1})(x, y_k)) = \sum_{j=0}^{pN-1} e^{2\pi i j \frac{x}{pN}} \neq 0.
\]

Hence the transition is possible and we conclude that \( (x/pN, y_{k-1}) \) is in \( \mathcal{M} \). Iterate this step by replacing \((x, y_k)\) with \((x/pN, y_{k-1})\). Taking the limit and using the compactness of \( \mathcal{M} \), we obtain that \( \mathcal{M} \) contains \((0, y_k)\) for all \( k \). But that means that \( \mathcal{M} \) contains an \( \tilde{L} \)-cycle which is extreme for \( \tilde{B} \), and by minimality, it has to be equal to the extreme cycle. That is a contradiction. Thus, \( V \) cannot be \( \mathbb{R} \times \{0\} \).

**Case (ii) \( V = \{0\} \times \mathbb{R} \).** As before, “(H) modulo \( V \)” in Theorem 3.7(iii) is satisfied and Theorem 3.7 implies that there exists \((x_0, y_0)\), and an \( \tilde{L} \)-cycle, with digits \((i_1, \ell_1), \ldots, (i_m, \ell_m)\) such that

\[
M \subset \bigcup_{k=1}^{m} \left( \tau(i_k, \ell_k) \cdots \tau(i_m, \ell_m)(x_0, y_0) + V \right) =: \mathcal{R},
\]

\( \mathcal{R} \) is invariant and every possible transition from \( \tau(i_k, \ell_k) \cdots \tau(i_m, \ell_m)(x_0, y_0) + V \) leads to a point in \( \tau(i_{k-1}, \ell_{k-1}) \cdots \tau(i_m, \ell_m)(x_0, y_0) + V \) for \( 1 \leq k \leq m \), where \((i_0, \ell_0) := (i_m, \ell_m)\).

Take \( i' \neq i_m \) in \( \{0, \ldots, pN - 1\} \) and \( y \in \mathbb{R} \). The transition from \((x_0, y)\) to \( \tau(i', \ell)(x_0, y) \) is not possible and thus \( m_B(\tau(i', \ell)(x_0, y)) = 0 \). Then with \( y' = (y + \ell)/R \),

\[
0 = \sum_{j=0}^{pN-1} \sum_{d \in \tilde{B}(j)} e^{2\pi i j \frac{x+1+i'}{pN}} e^{2\pi i d y' R} = \sum_{d \in \cup_{j} \tilde{B}(j)} \sum_{\{j: d \in \tilde{B}(j)\}} e^{2\pi i j \frac{x+1+i'}{pN}} e^{2\pi i d y' \frac{1}{pN}}.
\]

Then, all the coefficients are zero so, for all \( d \in \cup_{j=1}^{N} \tilde{B}(j) \), and all \( i' \neq i_m \),

\[
\sum_{j: d \in \tilde{B}(j)} e^{2\pi i j \frac{x+1+i'}{pN}} = 0.
\]

But \( 0 \in \tilde{B}(j) \) for all \( j \) so

\[
\sum_{j=0}^{pN-1} e^{2\pi i j \frac{x+1+i'}{pN}} = 0
\]

for all \( i' \neq i_m \). As the same time, this implies that \( \sum_{j=0}^{pN-1} e^{2\pi i j \frac{x+1+i_m}{pN}} = 1 \) and hence \( x_0 \equiv (-i_m) \mod {pN} \). Since \( x_0 \in [0, 1] \) (by \( \mathcal{M} \subset T(\mathbb{R}, \tilde{L}) = [0, 1] \times T(\mathbb{R}, L) \)) we obtain that \( x_0 = 0 \) or \( x_0 = 1 \).
If $x_0 = 0$, then the digits corresponding to this cycle are $m = 1$ and $i_1 = 0$. Then we have
\[ \sum_{j:d \in B_2(j)} e^{2\pi i d \cdot x} = 0 \]
for all $i' \neq i_1 = 0$ and all $d \in \bigcup_{j=1}^{N} \tilde{B}(j)$. Let $A_d(x) := \sum_{j:d \in B_2(j)} x^j$. Then $A_d(e^{2\pi i x}) = 0$ for all $i' \in \{1, \ldots, pN - 1\}$. Therefore $A_d$ is divisible by $1 + x + \cdots + x^{pN-1}$ and this implies that $A_d(x) = 1 + x + \cdots + x^{pN-1}$. So every $d \in \bigcup_{j=1}^{N} \tilde{B}(j)$ appears in all $\tilde{B}(j)$. But this means that all the sets $\tilde{B}(j)$ are equal and so all $\mu_\omega = \mu_{R, \tilde{B}(0)}$ which is the self-affine spectral measure. The conclusion holds trivially. Similarly, the case $x_0 = 1$ follows from the same argument, the cycle has digits $m = 1$ and $i_1 = pN - 1$.

Now, we can see that the only minimal compact invariant sets are extreme cycles. By Theorem [3.10] with $\Lambda_2$ as defined in its hypothesis, we have that $\mu_x^2$ has spectrum $\Lambda_2$ for $\mu$-a.e. $x$. Note that $\mu_1$ is the Lebesgue measure. Then $\mu_\omega$ has spectrum $\Lambda_2$ for $\tilde{\mathbb{P}}$-a.e. $\omega \in \{0, \ldots, pN - 1\}^N$, where $\tilde{\mathbb{P}}$ is the product probability measure on $\{0, 1, \ldots, pN - 1\}^N$ that assigns equal probabilities $\frac{1}{pN}$ to each digit $0, 1, \ldots, pN - 1$. Consider now the map
\[ \Phi: \{0, 1, \ldots, pN - 1\}^N \to \{0, 1, \ldots, N - 1\}^N, \quad \Phi(i_1i_2\ldots) = (i_1(\text{mod} \ N), i_2(\text{mod} \ N), \ldots). \]
By checking on cylinder sets, note that for any Borel subset of $\{0, 1, \ldots, N - 1\}^N$,
\[ \mathbb{P}(E) = \tilde{\mathbb{P}}(\Phi^{-1}(E)). \]
Also, note that for $\omega = i_1i_2\cdots \in \{0, 1, \ldots, pN - 1\}^N$, we have $\mu_\omega = \mu_{\Phi(\omega)}$, because $\tilde{B}_2(i) = B(i(\text{mod} \ N))$. Then
\[ \mathbb{P}(\omega: \mu_\omega \text{ has spectrum } \Lambda_2) = \tilde{\mathbb{P}}(\omega: \mu_{\Phi(\omega)} \text{ has spectrum } \Lambda_2) = \tilde{\mathbb{P}}(\omega: \mu_\omega \text{ has spectrum } \Lambda_2) = 1. \]
This completes the proof. \qed

Proof of Theorem [1.3] (when (ii) holds, i.e., each $B(i)$ is a complete set representative of $R$). Consider
\[ R = \begin{bmatrix} N & 0 \\ 0 & R \end{bmatrix}, \]
and
\[ \tilde{B} := \{(i, d)^T : i \in \{0, 1, \ldots, N - 1\}, d \in B(i)\}. \]
In this case, the attractor $T(R, \tilde{B})$ defined in [3.2] is a self-affine tile and it admits a lattice tiling of the form $\mathbb{Z} \times \hat{\Gamma}$ for some lattice $\hat{\Gamma}$ (See e.g. [2, 15, Proposition 4.4, Claim]). Hence, it admits a spectrum of the form $\mathbb{Z} \times \Gamma$ with $\Gamma$ a dual lattice of $\hat{\Gamma}$. Hence, Theorem [1.3] shows that $\Gamma$ is almost surely a spectrum for $\mu_\omega$.

The final statement in Theorem [1.3] will be proved via the following general lemma. \qed

Lemma 3.11. Let $\mu$ be a Borel, compactly supported probability measure on $\mathbb{R}^d$. Suppose $\mu$ is spectral and the spectrum is a full-rank lattice $\Gamma$. Then $\mu$ is the Lebesgue measure with support $T$ which tiles $\mathbb{R}^d$ by the dual lattice $\hat{\Gamma}$.
Proof. Let $\Gamma = AZ^d$ for some integer $d \times d$ non-singular matrix $A$. The dual lattice is $\tilde{\Gamma} = (A^T)^{-1}Z^d$. We change the variable to reduce the problem to the case when $\Gamma = Z^d$. Define the Borel probability measure $\nu$ by

$$\int f(x) d\nu(x) = \int f(A^T x) d\mu(x),$$

for all continuous functions $f$ on $\mathbb{R}^d$. Then $\nu$ has spectrum $Z^d$. By Lemma 3.4

$$\sum_{n \in Z^d} |\hat{\nu}(\xi + n)|^2 = 1.$$  

Thus,

$$\int_{\mathbb{R}^d} |\hat{\nu}(\xi)|^2 d\xi = \int_{[0,1)^d} \sum_{n \in Z^d} |\hat{\nu}(\xi + n)|^2 d\xi = 1.$$

This shows that $\nu$ is absolutely continuous with respect to the Lebesgue measure. As $\nu$ is a spectral measure, we must have $\nu = \frac{1}{\text{Leb}(S)} \chi_S dx$ for some measurable set $S$ (Theorem 1.5 in [DL14]). $S$ is therefore a spectral set with spectrum $Z^d$. By the well-known theorem of Fuglede [Fug74], $S$ is a translational tile by tiling set $Z^d$. This implies that $\mu$ is the normalized Lebesgue measure on the set $T := (A^T)^{-1}S$, which tiles $\mathbb{R}^d$ by $(A^T)^{-1}Z^d = \tilde{\Gamma}$.

See also [DJ13, Theorem 2.4] for a variation of the proof. $\square$

4. Appendix: dynamically simple spectrum

We will prove Theorem 3.9 in this section. We let $\Lambda$ be the orthonormal set generated by the extreme cycles for $(R, B, L)$ and

$$\Lambda' = \{\ell_0 + R^T \ell_1 + \ldots + (R^T)^{n-1} \ell_{n-1} + (R^T)^n (-c) : \ell_0, \ldots, \ell_{n-1} \in L, n \geq 0, c \text{ are extreme cycle points} \}$$

the set given in Theorem 3.9. We first prove from definition that they are the same.

Lemma 4.1. $\Lambda = \Lambda'$. In fact,

$$\Lambda = R^T \Lambda + L.$$

Proof. It is clear that $R^T \Lambda' + L \subset \Lambda'$. Also, for any extreme cycle points $c$, there exists unique $\ell$ such that $c' = \tau_\ell(c)$ is an extreme cycle point. Hence, $-c = \ell + R^T (-c')$, which implies $\Lambda'$ contains all extreme cycles. By definition, $\Lambda \subset \Lambda'$. On the other hand, since $-c \in \Lambda$, the invariance implies that $\ell_m + R^T (-c) \in \Lambda$ for all $\ell_n \in L$. Inductively, $(R^T)^2 (-c) + R^T \ell_n + \ell_{n-1} \in \Lambda$, and in the end,

$$\ell_0 + R^T \ell_1 + \ldots + (R^T)^n (-c) \in \Lambda,$$

for all $n$. Thus $\Lambda' \subset \Lambda$. This shows $\Lambda = \Lambda'$. From the definition of $\Lambda'$, it is clear that $\Lambda = R^T \Lambda + L$. $\square$

From now on, we will work on the expression $\Lambda'$, and for simplicity, we still write it as $\Lambda$. We first show the mutually orthogonality of $\Lambda$ in $\mu_{R,B}$.

Proposition 4.2. $\Lambda$ is a mutually orthogonal set in $\mu_{R,B}$.

Proof. We need to see whether

$$(4.1) \quad \hat{\mu}(\lambda - \lambda') = \prod_{k=1}^{\infty} m_B((R^T)^{-k}(\lambda - \lambda'))$$
is zero whenever \( \lambda \neq \lambda' \in \Lambda \). Now, they can be written as
\[
\lambda = \ell_0 + R^T \ell_1 + \ldots + (R^T)^{m-1} \ell_{m-1} + (R^T)^m (-c),
\]
(4.2)
\[
\lambda' = \ell'_0 + R^T \ell'_1 + \ldots + (R^T)^{m'-1} \ell'_{m'-1} + (R^T)^{m'} (-c'),
\]
(4.3)
with \( \ell_i, \ell'_i \in L \), \( c, c' \) extreme cycle points for \((R, B, L)\) (they may be from different cycles). From (3.6), for any \( p \geq 1 \), we can write
\[
-c = (R^T)^k (-c_k) + \alpha_p + R^T \alpha_{p-1} + \ldots + (R^T)^{k-1} \alpha_{p-k}
\]
for some digits \( \alpha_i \) in \( L \) and another extreme cycle point \( c_k \). Using (4.4) in (4.2), we can write \( \lambda \) with as many digits as we want. Similarly, we can do it for case of \( \lambda' \) in (4.3). As \( \lambda, \lambda' \) are distinct elements, we can assume for some \( m = m' \) that there exists \( n < m \) such that \( \ell_0 = \ell'_0, \ldots, \ell_{n-1} = \ell'_{n-1}, \ell_n \neq \ell'_n \).

\[
m_B((R^T)^{-n-1}(\lambda - \lambda')) = m_B((R^T)^{-1}(\ell_n - \ell'_n) + M_0 + (R^T)^{m-n-1}(x - x')),
\]

where \( M_0 \) is some integer vector in \( \mathbb{Z}^d \) and \( x, x' \) are extreme cycle points. From the integral periodicity of \( m_B \) and (5.7), \((b, M) \in \mathbb{Z} \) and \((b, (R^T)^{m-n-1}(x - x')) \in \mathbb{Z} \), The term above is equal to \( m_B((R^T)^{-1}(\ell_n - \ell'_n)) = 0 \) by the Hadamard triple assumption. This implies from (4.1) that \( \hat{\mu}(\lambda - \lambda') = 0 \).

We need an easy geometric lemma.

**Lemma 4.3.** Suppose that \((R, B, L)\) is a Hadamard triple and we define \( \tau_\ell(x) = (R^T)^{-1}(x + \ell) \). Let \( B_r \) be the closed Euclidean ball centered at origin. Then for \( r \) sufficiently large,
\[
\bigcup_{\ell \in L} \tau_\ell(B_r) \subset B_r.
\]

**Proof.** Denote by \( | \cdot | \) the Euclidean distance on \( \mathbb{R}^d \). Since \( R \) is expansive, there exists \( 0 < c > 1 \) such that \( |(R^T)^{-1}v| \leq c|v| \) for all \( v \in \mathbb{R}^d \). Let
\[
M = \max\{|(R^T)^{-1}\ell| : \ell \in L\}
\]
and let \( r > cM/(1-c) \). Then for all \( \ell \in L \) and \( |x| \leq r \), we have
\[
|\tau_\ell(x)| \leq c(r + M) < r.
\]
Hence, \( \tau_\ell(B_r) \subset B_r \). This proves the lemma. \( \square \)

**Proof of Theorem 3.9.** As mutually orthogonality has been established in Proposition 4.2, we just need to show that the set \( \Lambda \) generated by the extreme cycle is complete. By Jorgensen-Pedersen Lemma (Lemma 3.4), we need to show
\[
Q_\Lambda(\xi) := \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + \lambda)|^2 = 1, \forall \xi \in \mathbb{R}^d.
\]
In fact, \( Q_\Lambda \leq 1 \) is well-known by mutually orthogonality. We just need to see whether \( Q_\Lambda \geq 1 \). To do this, we define the *Ruelle transfer operator*
\[
\mathcal{R} f(\xi) := \sum_{\ell \in L} |m_B(\tau_\ell(\xi))|^2 f(\tau_\ell(\xi)).
\]
Using Lemma 4.3, we choose $r$ large enough such that the closed ball $B_r$ satisfies

$$
\bigcup_{\ell \in L} \tau_{\ell}(B_r) \subset B_r
$$

and let $c_r = \min_{\xi \in B_r} Q_\Lambda(\xi)$. Then $\mathcal{R} c_r = c_r$. On the other hand, as $\Lambda$ satisfies $\mathcal{R}^T \Lambda + L = \Lambda$ by Lemma 4.1, we have

$$
Q_\Lambda(\xi) = \sum_{\ell \in L} \sum_{\lambda \in \Lambda} |\hat{\mu}(\xi + R^T \lambda + \ell)|^2
$$

$$
= \sum_{\ell \in L} \sum_{\lambda \in \Lambda} |m_B((R^T)^{-1}(\xi + \ell))|^2 |\hat{\mu}((R^T)^{-1}(\xi + \ell) + \lambda)|^2 = (\mathcal{R} Q_\Lambda)(\xi).
$$

Thus, if we define

$$
f_n = Q_\Lambda - c_r,
$$

then $\mathcal{R} f_n = f_n$ and $f_n$ is an entire function. Consider the set in $B_r$ for which $Q_n$ attains minimum, i.e.,

$$
\mathcal{M}_0 = \{\xi \in B_r : f_n(\xi) = 0\}.
$$

We note that $\mathcal{M}_0$ is a compact invariant set in $B_r$. To show the invariance, we suppose $\xi \in \mathcal{M}$ and $|m_B(\tau_{\ell}(\xi))| > 0$. As

$$
0 = f_n(\xi) = \sum_{\ell \in L} |m_B(\tau_{\ell}(\xi))|^2 f_n(\tau_{\ell}(\xi))
$$

and $f \geq 0$, $|m_B(\tau_{\ell}(\xi))|^2 f_n(\tau_{\ell}(\xi)) = 0$ and hence $f_n(\tau_{\ell}(\xi)) = 0$. Because of (4.5), $\tau_{\ell}(\xi) \in B_r$.

Take a minimal compact invariant set $\mathcal{M} \subset \mathcal{M}_0 \subset B_R$. The crux of the proof is to note that the dynamically simple Hadamard triple assumption forces $\mathcal{M}$ to be an extreme cycle. But extreme cycles are contained in $\Lambda$, this in turn shows that there are some points (indeed the whole extreme cycle) $x_0 \in \Lambda \cap \mathcal{M}_0$. By mutual orthogonality, $Q_\Lambda(x_0) = 1$,

$$
f_n(x_0) = 0, \text{ and } c_r = Q_\Lambda(x_0) = 1.
$$

Hence, $\min_{\xi \in B_r} Q_\Lambda(\xi) = 1$. But $r$ can be arbitrarily large this shows $Q_\Lambda(\xi) \geq 1$ for $\xi \in \mathbb{R}^d$.

Finally, on $\mathbb{R}^1$, the zero set of an entire function must be a discrete set, showing that any minimal invariant set contained in $\mathcal{M}_0$ must be discrete, which must be an extreme cycle as the subspace can only be $V = \{0\}$ by Theorem 3.7.

The idea of cycle points is also related to the integer points inside a self-affine fractal. A study in this direction can be found in [GY06].

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