Bethe states of the trigonometric $SU(3)$ spin chain with generic open boundaries

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Abstract

By combining the algebraic Bethe ansatz and the off-diagonal Bethe ansatz, we investigate the trigonometric $SU(3)$ model with generic open boundaries. The eigenvalues of the transfer matrix are given in terms of an inhomogeneous $T - Q$ relation, and the corresponding eigenstates are expressed in terms of nested Bethe-type eigenstates which have well-defined homogeneous limit. This exact solution provides a basis for further analyzing the thermodynamic properties and correlation functions of the anisotropic models associated with higher rank algebras.

Keywords: The quantum spin chain; Bethe ansatz; The Bethe-type eigenstates

1. Introduction

Since the work of Yang and Baxter’s pioneering works [1, 2], the exactly solvable quantum integrable systems [3] have attracted a great deal of interest. Especially the quantum integrable models are very useful in nano-scale systems where alternative approaches involving mean field approximations or perturbations have failed [4, 5]. Moreover, it also play very important role in statistical physics [6], low-dimensional condensed matter physics [7], and even some mathematical areas such as quantum groups [8].

By using the coordinate Bethe ansatz and the algebraic Bethe ansatz [9, 10, 11, 12], both eigenvalues and eigenstates of the integrable models with $U(1)$ symmetry can be given successfully. For the integrable models without $U(1)$ symmetry (with generic boundary conditions), it has been proven that the off-diagonal Bethe ansatz is a powerful method [13, 14, 15, 16, 17, 18, 19, 20] (For further information, we refer the reader to [21]). Based on the inhomogeneous $T - Q$ relation constructed via the off-diagonal Bethe Ansatz, a systematic method [22, 23] for retrieving the Bethe-type eigenstates (Bethe states) of integrable models without obvious reference state

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is also developed by employing certain orthogonal basis of the Hilbert space. It should also be
remark that the separation of variables (SoV) method \[24, 25, 26\] and the modified algebraic
Bethe ansatz method \[27, 28, 29\] were also used to approach the eigenstates of several integrable
models.

With the help of the off-diagonal Bethe ansatz \[21\], the exact energy spectrum of the rational
$SU(3)$ model with unparallel boundary fields has been obtained \[30\]. However, the eigenstates
(or Bethe states) which have played important roles in applications of the model are still missing.
In this paper, the $R$-matrix is the trigonometric one associated with the $SU_q(3)$ algebra and the
boundary reflection matrices are the most generic reflection matrices with non-zero off-diagonal
elements. By combining the nested algebraic Bethe ansatz and off-diagonal Bethe ansatz, we
obtain the eigenvalues which have well-defined homogeneous limit and the corresponding Bethe
states of the transfer matrix of the model. Numerical results for the small size systems show that
the spectrum obtained by the nested Bethe ansatz equations (BAEs) is complete and the Bethe
states are exactly correct.

The paper is organized as follows. In section 2 the $SU_q(3)$-invariant $R$-matrix and correspond-
ing generic integral non-diagonal boundary reflection matrices are introduced. In section 3 we
introduce the vacuum state of the system. In section 4, we construct the Bethe-type eigenstate of
the transfer matrix. In section 5, we introduce the gauge transformation in order to construct the
nested Bethe states. In section 6, the nested Bethe states of the system are obtained. Section 7
gives some discussions.

2. The model

Throughout this paper we adopt the standard notations: for any matrix $A \in \text{End}(V)$, $A_j$ is an
embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as
identity on the other factor spaces. For $R \in \text{End}(V \otimes V)$, $R_{ij}$ is an embedding operator of $R$ in the
tensor space, which acts as identity on the factor spaces except for the $i$-th and $j$-th ones.

The $R$-matrix $R(u) \in \text{End}(V \otimes V)$ used in this paper was first proposed by Perk and Shultz \[31\]
and further studied in \[32, 33, 34, 35, 36\]. We now discuss the trigonometric one associated with the $SU_q(3)$ algebra,

$$
R_{12}(u) = \begin{pmatrix}
  a(u) & b(u) & c(u) & c(u) \\
  b(u) & d(u) & c(u) & c(u) \\
  c(u) & c(u) & a(u) & b(u) \\
  c(u) & c(u) & b(u) & d(u)
\end{pmatrix},
$$

(2.1)

where the matrix elements are

$$
a(u) = \sinh(u + \eta), \quad b(u) = \sinh(u),
$$

$$
c(u) = e^u \sinh \eta, \quad d(u) = e^{-u} \sinh \eta.
$$

(2.2)
Here $R_{12}(u) = P_1 R_{12}(u) P_2$ with $P_1$ being the usual permutation operator and $t_i$ denotes transposition in the $i$-th space. The functions $\rho_1(u), \rho_2(u)$ and the crossing matrix $M$ are given by

$$
\rho_1(u) = -\sinh(u - \eta) \sinh(u + \eta),
\rho_2(u) = -\sinh(u) \sinh(u + 3\eta),
$$

$$
M = \begin{pmatrix}
e^{\eta} & e^{2\eta} \\
0 & 1
\end{pmatrix}.
$$

The $R$-matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$
R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2).
$$

Let us now introduce the reflection matrix $K^-(u)$ and the dual one $K^+(u)$. The former satisfies the reflection equation (RE)

$$
R_{12}(u_1 - u_2)K_1^-(u_1)R_{21}(u_1 + u_2)K_2^-(u_2)
= K_2^-(u_2)R_{21}(u_1 + u_2)K_1^-(u_1)R_{12}(u_1 - u_2),
$$

and the latter satisfies the dual RE

$$
R_{12}(u_2 - u_1)K_1^+(u_1)M_{12}^{-1}R_{21}(-u_1 - u_2 - 3\eta)M_{12}K_2^+(u_2)
= K_2^+(u_2)M_{12}^{-1}R_{21}(-u_1 - u_2 - 3\eta)M_{12}K_1^+(u_1)R_{12}(u_1 - u_2).
$$

We now consider the generic non-diagonal $K$-matrices $K^-(u)$

$$
K^-(u) = \begin{pmatrix}
e^{\alpha \sinh(\xi - u) + c e^{2\alpha} \sinh(2u)} & 0 & 0 \\
0 & e^{\alpha \sinh(\xi - u)} & c_1 \sinh(2u) \\
0 & 0 & e^{-\alpha \sinh(\xi + u)}
\end{pmatrix},
$$

$$
= \begin{pmatrix} k_{11}^- & 0 & 0 \\
0 & k_{22}^- & k_{23}^- \\
0 & k_{12}^- & k_{33}^-
\end{pmatrix},
$$

where the four boundary parameters $c_1, c_2$ and $\xi$ satisfy a constraint

$$
c^2 = c_1 c_2 + c e^\xi.
$$

The dual non-diagonal reflection matrix $K^+(u)$ is given by

$$
K^+(u) = M K^-(u - 3\eta/2)|_{(c_1,c_2),(\xi,\xi')} = \begin{pmatrix} k_{11}^+ & 0 & 0 \\
0 & k_{22}^+ & k_{23}^+ \\
0 & k_{12}^+ & k_{33}^+
\end{pmatrix}.
$$
with the constraint

\[ c'^2 = c_1'c_2' + c'e^e. \]

In order to show the intergrability of the system, we first introduce the “row-to-row” monodromy matrices \( T_0(u) \) and \( \hat{T}_0(u) \)

\[
T_0(u) = R_{0N}(u - \theta_N)R_{0,N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1), \quad (2.10)
\]

\[
\hat{T}_0(u) = R_{10}(u + \theta_1)R_{20}(u + \theta_2) \cdots R_{N0}(u + \theta_N), \quad (2.11)
\]

where \( \{\theta_j, j = 1, \cdots, N\} \) are the inhomogeneous parameters and \( N \) is the number of sites. The one-row monodromy matrices are the \( 3 \times 3 \) matrices in the auxiliary space 0 and their elements act on the quantum space \( \mathbf{V}^{\otimes N} \). For the system with open boundaries, we need to define the double-row monodromy matrix \( \mathcal{T}_0(u) \)

\[
\mathcal{T}_0(u) = T_0(u)K_0(u)\hat{T}_0(u) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) \end{pmatrix}, \quad (2.12)
\]

which satisfies the similar relation as \( (2.6) \)

\[
R_{12}(u_1 - u_2)T_1(u_1)R_{21}(u_1 + u_2)\mathcal{T}_2(u_2) = \mathcal{T}_2(u_2)R_{12}(u_1 + u_2)T_1(u_1)R_{21}(u_1 - u_2). \quad (2.13)
\]

Then the transfer matrix of the system can be constructed as

\[
t(u) = tr_0[K_0^+(u)\mathcal{T}_0(u)] = \left[ k_{11}^+(u)A(u) + \sum_{i,j=1} k_{i+1,j+1}^+(u)D_{ij}(u) \right], \quad (2.14)
\]

where \( k_{ij}^+ \) is the \( K^+ \) matrix element in the \( i \)th row and \( j \)th column. Using Eq. QYBE \( (2.5) \), RE \( (2.6) \) and dual RE \( (2.7) \), we obtain the commutativity of \( t(u) \), which means \( [t(u), t(u')]=0 \). The Hamiltonian of the model is constructed by taking the derivative of the logarithm of the transfer matrix as \( \hat{H} \)

\[
H = \sinh \eta \frac{\partial \ln t(u)}{\partial u} |_{u=0,[\eta]=0}. \quad (2.15)
\]

3. Vacuum state

The block-diagonal structure of the \( K \)-matrix \( (2.3) \) permits us to use the nested algebraic Bethe ansatz to construct the associated Bethe states and obtain the eigenvalues as follows.

We first introduce the reference state \( |\Psi_0\rangle \) as

\[
|\Psi_0\rangle = \bigotimes_{j=1}^{M} |0\rangle_j, \quad |0\rangle_j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (3.1)
\]

It was shown in \( [30] \) that the transfer matrix has the property: \( t(0) \propto \text{id} \), implying that the Hamiltonian given by \( (2.15) \) is actually proportion to \( t'(0) \).
From the relations (2.12), (3.1), the elements of matrix $T$ acting on the reference state $|Ψ₀⟩$ give rise to

$$A(u)|Ψ₀⟩ = \frac{1}{2} (k_{11}^- u a₀(u) b₀(u))|Ψ₀⟩,$$
$$D_{11}(u)|Ψ₀⟩ = \left(\frac{d(2u)k_{11}^- u}{a(2u)} a₀(u) + \left[ \frac{d(2u)}{a(2u)} k_{11}^- u - \frac{d(2u)}{a(2u)} k_{11}^- u \right] b₀(u) \right)|Ψ₀⟩,$$
$$D_{22}(u)|Ψ₀⟩ = \left(\frac{d(2u)k_{11}^- u}{a(2u)} a₀(u) + \left[ \frac{d(2u)}{a(2u)} k_{11}^- u - \frac{d(2u)}{a(2u)} k_{11}^- u \right] b₀(u) \right)|Ψ₀⟩,$$
$$D_{12}(u)|Ψ₀⟩ = k_{12}^- u b₀(u)|Ψ₀⟩,$$
$$B_i(u)|Ψ₀⟩ \neq 0, \quad C_i(u)|Ψ₀⟩ = 0, \quad i = 1, 2, \quad (3.2)$$

where

$$b₀(u) = \prod_{j=1}^{K} \sinh(u + θ_j) \sinh(u - θ_j), \quad a₀(u) = b₀(u + η). \quad (3.3)$$

It is easy to prove that the reference state $|Ψ₀⟩$ is an eigenstate of the transfer matrix

$$t(u)|Ψ₀⟩ = Λ₀ |Ψ₀⟩,$$

$$Λ₀ = k_{11}^- u k_{11}^- u a₀(u) + \frac{d(2u)}{a(2u)} \sum_{i=1}^{2} k_{i+1,i+1}^- u a₀(u) k_{11}^- u (a₀(u) - b₀(u))$$
$$+ \sum_{i,j=1}^{2} k_{i+1,i+1}^- u k_{j+1,j+1}^- u b₀(u). \quad (3.4)$$

4. Bethe state

From (3.2), we see that the operators $B_i(u)$ and $B_2(u)$ acting on the reference state give nonzero values, thus can be regarded as the creation operators of the eigenstates of the system. Following the procedure of the nested algebraic Bethe ansatz, the eigenstates of the transfer matrix can be constructed as

$$|Ψ⟩ = B_{a_0}(u_M)B_{a_0-1}(u_{M-1}) \ldots B_{a_0}(u_1)t^{a_0-1} \ldots t^{a_0} |Ψ₀⟩, \quad (4.1)$$

where we have used the convention that the repeated indices indicates the sum over the values 1,2, and $\{t^{a_0-1} \ldots t^{a_0}\}$ are some undetermined functions of the Bethe roots $|u_i⟩$. In fact, the $\{t^{a_0-1} \ldots t^{a_0}\}$ are the vector components of the nested Bethe states (see below (4.11)). As the transfer matrix (2.14) acting on the assumed states (4.1), we should exchange the positions of the operators $A(u)$, $D_{ij}(u)$ and the operators $B_{a_i}(u)$. Combining the reflection equation (2.13) and the Yang-Baxter
equation 2.5, we can derive following commutation relations

\[
A(u)B_j(\lambda) = \frac{a(\lambda - u) b(\lambda + u)}{b(\lambda - u) a(\lambda + u)} B_j(\lambda) A(u) - \frac{b(2\lambda) c(u - \lambda)}{a(2\lambda) b(\lambda - u)} B_j(u) A(\lambda) - \frac{c(u + \lambda)}{a(u + \lambda)} B_j(u) \hat{D}_{j}(\lambda),
\]

\[
\hat{D}_{ij}(u)B_k(\lambda) = \frac{r^{ij}_k(u + \lambda + \eta)r^{j}_k(u - \lambda)}{b(u + \lambda + \eta)b(u - \lambda)} B_i(\lambda) \hat{D}_{k}(u) - \frac{r^{ij}_k(2u + \eta)c(u - \lambda)}{a(2u)b(u - \lambda)} B_i(\lambda) \hat{D}_{k}(u) - \frac{b(2\lambda)d(u + \lambda)}{a(2\lambda)} \hat{D}_{ij}(u) A(\lambda),
\]

\[
B_i(u)B_j(\lambda) = B_k(\lambda) B_i(u) \frac{r^{ij}_k(u - \lambda)}{a(u - \lambda)},
\]

where

\[
r(u) = \begin{pmatrix}
\sinh(\eta + u) & 0 & 0 & 0 \\
0 & \sinh(u) & e^u \sinh \eta & 0 \\
0 & e^{-u} \sinh \eta & \sinh(u) & 0 \\
0 & 0 & 0 & \sinh(\eta + u)
\end{pmatrix},
\]

and

\[
\hat{D}_{ij}(u) = D_{ij}(u) - \frac{\delta_{ij}d(2u)}{a(2u)} A(u).
\]

Acting the transfer matrix \( t(u) \) on the assumed eigen-state \( \Psi \) and repeatedly using the commutation relations, we obtain

\[
t(u) |\Psi\rangle = \frac{d(2u)}{a(2u)} \sum_{i=2}^{3} \left( k_i^+(u) + k_i^-(u) \right) k_i^+(u) a_0(u) \prod_{i=1}^{M} \frac{a(u_i - u) b(u_i + u)}{b(u_i - u) a(u_i + u)} |\Psi\rangle + b_0(u) \prod_{i=1}^{M} \frac{1}{b(u - u_i) b(u + u_i + \eta)} \hat{\Lambda}(u, \{u_j\}) |\Psi\rangle + \text{unwanted terms}.
\]

The vector components \( \{ F^{\omega_1, \ldots, \omega_M} \} \) in the Bethe state (4.1) allow us to reconstruct the associated Bethe state \( |F\rangle \), and \( \hat{\Lambda}(u, \{u_j\}) \) in the (4.7) is the associated eigenvalue of the nested transfer matrix \( \hat{t}(u, \{u_j\}) \)

\[
\hat{t}(u, \{u_j\}) = e^U \frac{\sinh(2u)}{\sinh(2u + \eta)} \text{tr}_0 \left[ K_0(u)r_0(u + u_M + \eta) \cdots r_0(u + u_1 + \eta) \times K_0(u)r_0(u - u_1) \cdots r_0(u - u_M) \right].
\]
which appears due to the commutation relations \[4.3\] and

\[
\hat{K}^+(u) = e^{-\eta} \left( \begin{array}{cc}
 k_{22}(u) & k_{23}(u) \\
 k_{32}(u)^* & k_{33}(u)^*
\end{array} \right) \nonumber
\]

\[
= \left( e^{-\eta \frac{u}{2}} \sinh(\zeta' + \frac{2}{3} \eta + u) \right) e^{i \eta} c_1' \sinh(-2u - 3 \eta) \nonumber
e^{i \eta} c_2' \sinh(\zeta' - \frac{2}{3} \eta - u) \right). \tag{4.9}
\]

\[
\hat{K}^-(u) = \frac{\sinh(2u + \eta)}{\sinh(2u)} \left( \begin{array}{cc}
 k_{22}(u) - \frac{\eta}{a_{23}(u)} & k_{23}(u) \\
 k_{32}(u)^* - \frac{\eta}{a_{31}(u)} & k_{33}(u)^*
\end{array} \right) \nonumber
\]

\[
= \left( \left( e^{u+u} \sinh(\zeta - u) - c \sinh(\eta) \right) c_2 \sinh(2u + \eta) \right) \nonumber
e^{u-\eta} \sinh(\zeta + u + \eta) - c \sinh(\eta) \right). \tag{4.10}
\]

The eigen-equation

\[
\hat{t}(u, \{u_j\}) \langle F \rangle = \hat{\Lambda}(u, \{u_j\}) \langle F \rangle,
\]

will be determined in section 6. The eigenstate \[\langle F \rangle\] can be decomposed in terms of the basis \[\{|u_1, a_2 \cdots a_M\} | u_i = 1, 2; i = 1, \cdots, M\] of the M-fold tensor space of \[C^2 \otimes C^2 \otimes \cdots \otimes C^2\] as

\[
\langle F \rangle = \sum_{\{u_i\}=1}^2 \bar{F}^{u_i} \cdot \cdot \cdot a_{2} |u_1, a_2 \cdots a_M\rangle, \tag{4.11}
\]

where the vector components \[\bar{F}^{u_i} \cdot \cdot \cdot a_{2} \] will be determined by \[6,1\] lately. The third term in equation \[4.7\] represent the unwanted terms. The unwanted terms should be zero gives rise to the M Bethe roots must satisfy the following Bethe ansatz equations (BAEs)

\[
1 = -K^{(1)}(u) \frac{\sinh(2u_k + 3 \eta) a_0(u_k) Q^{(1)}(u_k - \eta)}{\sinh(2u_k + \eta) b_0(u_k) \hat{\Lambda}(u_k, \{u_j\})}, \quad k = 1 \ldots M, \tag{4.12}
\]

where

\[
Q^{(1)}(u) = \prod_{i=1}^{M} \sinh(u - u_i) \sinh(u + u_i + \eta), \tag{4.13}
\]

\[
K^{(1)}(u) = (e^u \sinh(\zeta - u) + ce^{2u} \sinh 2u) \nonumber
\]

\[
\times (e^{-u+\frac{u}{2}} \sinh(\zeta' + u + \frac{\eta}{2}) - c'e^{-2u+\eta} \sinh(2u + \eta)). \tag{4.14}
\]

5. Gauge transformation

In the previous section, we have reduced constructing eigenstates of the original transfer matrix \[t(u)\] defined by \[\ref{2.13}\] into the Bethe states problem of the nested transfer matrix \[\hat{t}(u, \{u_j\})\] given by \[\ref{4.8}\]. Now we construct the eigenstate \[\langle F \rangle\] of the nested transfer matrix \[\hat{t}(u, \{u_j\})\]. Considering the reflection matrices \[\ref{4.7}\] and \[\ref{3.10}\] have the off-diagonal elements, the \[\hat{t}(u, \{u_j\})\] does not have the obvious reference state, which means the analogy of construction of \[\Psi_0\] is invalid. Thanks to the works \[\ref{2.11}\], we can solve the nested Bethe states problem \[\ref{4.11}\] as follows.

For simplicity, we take the notation \[\lambda = u + \frac{\eta}{2}\] and \[\lambda_j = u_j + \frac{\eta}{2}\]. We recognize the \[\hat{t}(u, \{u_j\})\] as the transfer matrix of the open spin-1/2 chain of length \[M\] with non-diagonal boundary terms.
We first introduce some parameterization:

\[ e^{\theta_-} = 2pc_1, \quad e^{\theta_+} = 2qe'_1, \]

\[
\begin{align*}
2 \sinh^2(\alpha_-) &= -4 + \tilde{\xi}_1^2 + \frac{(4 + \tilde{\xi}_1^2 - \tilde{\xi}_1^2)^2}{16} + \tilde{\xi}_1^2, \\
2 \sinh^2(\beta_-) &= -4 + \tilde{\xi}_2^2 + \frac{(4 + \tilde{\xi}_2^2 - \tilde{\xi}_2^2)^2}{16} + \tilde{\xi}_2^2, \\
\sinh^2(-\beta_+) &= -\frac{1 - q^2}{2} + \sqrt{\frac{(1 - q^2)^2}{4} + q^2 \cosh^2 \zeta}, \\
\sinh^2(-\alpha_+) &= \frac{q^2 \sinh^2 \zeta}{1 + \sinh^2(-\beta_+)} ,
\end{align*}
\]

where

\[
1 = 4p^2c_1c_2, \quad 1 = 4q^2c'_1c'_2, \\
\tilde{\xi}_1 = p(e^{\varepsilon_--2c \sinh(\eta)} - e^{-\varepsilon-}), \quad \tilde{\xi}_2 = p(-e^{\varepsilon_+2c \sinh(\eta)} - e^{-\varepsilon}).
\]

Then the non-diagonal reflection matrices (4.9) and (4.10) read

\[
\tilde{K}^-(\lambda) = \frac{1}{2p} \begin{pmatrix}
\tilde{K}_{11}(\lambda) & \tilde{K}_{12}(\lambda) \\
\tilde{K}_{21}(\lambda) & \tilde{K}_{22}(\lambda)
\end{pmatrix}, \quad \tilde{K}^+(\lambda) = \frac{1}{2q} \begin{pmatrix}
\tilde{K}_{11}^+(\lambda) & \tilde{K}_{12}^+(\lambda) \\
\tilde{K}_{21}^+(\lambda) & \tilde{K}_{22}^+(\lambda)
\end{pmatrix},
\]

where

\[
\begin{align*}
\tilde{K}_{11}(\lambda) &= 2[\sinh(\alpha_-) \cosh(\beta_-) \cosh(\lambda) + \cosh(\alpha_-) \sinh(\beta_-) \sinh(\lambda)], \\
\tilde{K}_{22}(\lambda) &= 2[\sinh(\alpha_-) \cosh(\beta_-) \cosh(\lambda) - \cosh(\alpha_-) \sinh(\beta_-) \sinh(\lambda)], \\
\tilde{K}_{12}(\lambda) &= e^{\theta_-} \sinh(2\lambda), \quad \tilde{K}_{21}(\lambda) = e^{-\theta_-} \sinh(2\lambda), \\
\tilde{K}^+(\lambda) &= \tilde{K}^-(\lambda, \eta, \alpha_-, \beta_-, p, (-\alpha_-, \beta_-, \eta, q), \theta_-).
\end{align*}
\]

By using the above relations, the nested transfer matrix can be expressed as

\[
\tilde{R}(\lambda, \{\lambda_j\}) = \frac{e^{\theta_-} \sinh(2\lambda - \eta)}{\sinh(2\lambda)} \sigma_1(-\lambda_1) \cdots \sigma_M(-\lambda_M) \tilde{T}_0(\lambda) \tilde{T}_0(\lambda) \tilde{T}_0(\lambda) \tilde{T}_0(\lambda)
\]

\[
\times \sigma_1(\lambda_1) \cdots \sigma_M(\lambda_M),
\]

where

\[
\begin{align*}
\sigma(\lambda) &= \begin{pmatrix} e^{\alpha_-} & e^{-\alpha_-} \\
e^{\alpha_+} & e^{-\alpha_+} \end{pmatrix}, \\
\tilde{T}_0(\lambda, \{\lambda_j\}) &= \tilde{R}_{0M}(\lambda + \lambda_M) \cdots \tilde{R}_{01}(\lambda + \lambda_1), \\
\tilde{T}(\lambda, \{\lambda_j\}) &= \tilde{R}_{10}(\lambda - \lambda_1) \cdots \tilde{R}_{10}(\lambda - \lambda_M), \\
\tilde{R}_{0j}(\lambda) &= \begin{pmatrix} \sinh(\lambda + \eta) & 0 & 0 & 0 \\
0 & \sinh(\lambda) & \sinh(\eta) & 0 \\
0 & \sinh(\eta) & \sinh(\lambda) & 0 \\
0 & 0 & 0 & \sinh(\lambda + \eta) \end{pmatrix}.
\end{align*}
\]
Now, we introduce the gauge transformations

\[
X_m(\lambda) = \begin{pmatrix} e^{-[\lambda+(a+m)\eta]} & 1 \\ 1 & e^{-[\lambda+(a-m)\eta]} \end{pmatrix}, \quad Y_m(\lambda) = \begin{pmatrix} e^{-[\lambda+(a-m)\eta]} & 1 \\ 1 & e^{-[\lambda+(a+m)\eta]} \end{pmatrix},
\]

\[
\tilde{X}_m(\lambda) = \frac{e^{i\eta}}{2 \sinh(m\eta)} \begin{pmatrix} 1 & -e^{-[\lambda+(a+m)\eta]} \\ -e^{[\lambda+(a+m)\eta]} & 1 \end{pmatrix},
\]

\[
\tilde{Y}_m(\lambda) = \frac{e^{i\eta}}{2 \sinh(m\eta)} \begin{pmatrix} -1 & e^{-[\lambda+(a-m)\eta]} \\ e^{[\lambda+(a-m)\eta]} & -1 \end{pmatrix},
\]

\[
\alpha \eta = -\theta_+ + \eta + \frac{i \pi}{2}, \quad \theta = \alpha_+ + \beta_+ - i \frac{\pi}{2}.
\]

Now, we construct the eigenstate \( |\mathcal{F} \rangle \) in (4.11) as \[22\]

\[
|\mathcal{F} \rangle = \sigma_1(-\lambda_1) \cdots \sigma_M(-\lambda_M)(w_1, \ldots, w_M; m; \tilde{m})
\]

\[
= \sigma_1(-\lambda_1) \cdots \sigma_M(-\lambda_M)C_m(w_1)C_{m+2}(w_2) \cdots C_{m+2(M-1)}(w_M)|\tilde{m} \rangle,
\]

\[
= \sum_{a_1=1}^{2} \mathcal{F}^{a_1 a_2 \cdots a_M} |a_1, a_2, \ldots, a_M \rangle.
\]

(6.1)

where the \( C_m(\lambda) \) matrix and the reference state are defined as

\[
C_m(\lambda) = \tilde{X}_m(\lambda)\tilde{T}(\lambda)\tilde{K}^-(\lambda)\tilde{T}(\lambda)\tilde{X}_m(-\lambda),
\]

\[
|\tilde{m} \rangle = \bigotimes_{n=1}^{M} e^{-[\lambda_+ + (n+M-n+1)\eta]} |\uparrow \rangle_n + |\downarrow \rangle_n.
\]

(6.2)

6. Nested Bethe states
The first term in (6.5) gives the eigenvalue \( \hat{\lambda} \).

From the reflection equation, we can obtain following commutation relations

\[
\hat{D}_{m-2}(u_2)\hat{C}_{m-2}(u_1) = \sinh(u_1 - u_2 + \eta) \sinh(u_1 + u_2) \frac{\hat{C}_{m-2}(u_1)\hat{D}_{m}(u_2)}{\sinh(u_1 + u_2 + \eta) \sinh(u_1 - u_2 + \eta) \sinh(u_1) \sinh(u_1 + u_2 + \eta) \sinh(\eta)} - \sinh(m\eta - u_1 + u_2) \sinh(u_1) \sinh(u_1 + u_2 + \eta) \sinh(\eta) \frac{\hat{C}_{m-2}(u_2)\hat{D}_{m}(u_1)}{\sinh(u_1 + u_2 + \eta) \sinh(u_1 - u_2 + \eta) \sinh(u_1) \sinh(u_1 + u_2 + \eta) \sinh(\eta)} \quad (6.3)
\]

\[
\hat{A}_m(u_2)\hat{C}_m(u_1) = \sinh(u_1 - u_2 + \eta) \sinh(u_1 + u_2) \frac{\hat{C}_m(u_1)\hat{A}_{m+2}(u_2)}{\sinh(u_1 + u_2 + \eta) \sinh(u_1 - u_2 + \eta) \sinh(u_1) \sinh(u_1 + u_2 + \eta) \sinh(\eta)} + \sinh(\eta) \sinh(u_1 - u_2) \sinh(m\eta - u_2 - u_1) \frac{\hat{A}_{m+2}(u_2)\hat{D}_m(u_1)}{\sinh(u_1 + u_2 + \eta) \sinh(u_1 - u_2 + \eta) \sinh(u_1) \sinh(u_1 + u_2 + \eta) \sinh(\eta)} + \sinh(m\eta + u_1 - u_2) \sinh(\eta) \frac{\hat{A}_{m}(u_1)\hat{C}_{m}(u_2)}{\sinh(u_1 - u_2 + \eta) \sinh(\eta)} \quad (6.4)
\]

Acting the transfer matrix \( \hat{r}(\lambda) \) on the state \( |\mathcal{F}\rangle \) and repeatedly using the commutation relations, we obtain

\[
\hat{r}(\lambda, \{\lambda_j\}) |\mathcal{F}\rangle = \hat{A}(\lambda, \{\lambda_j\}) |\mathcal{F}\rangle + \text{unwanted terms.} \quad (6.5)
\]

The first term in (6.5) gives the eigenvalue \( \hat{\lambda}(\lambda, \{\lambda_j\}) \). [22]

\[
\hat{\lambda}(\lambda) = \frac{\sinh(2\lambda - \eta) \sinh(2\lambda + 2\eta) K^{(2)}(\lambda, \lambda) \hat{a}(\lambda) Q^{(2)}(\lambda - \eta)}{\sinh(2\lambda) \sinh(2\lambda + \eta)} + \sinh(2\lambda - \eta) K^{(3)}(\lambda, \lambda) \hat{a}(\lambda) Q^{(2)}(\lambda + \eta) \sinh(2\lambda + \eta) \cdot \sinh(2\lambda - \eta) \sinh(2\lambda + 2\eta) \hat{a}(\lambda) \hat{a}(\lambda) - \eta \frac{h}{Q^{(2)}(\lambda)}, \quad (6.6)
\]

where

\[
\hat{a}(\lambda) = \sum_{j=1}^{M} \sinh(\lambda + \lambda_j + \eta) \sinh(\lambda - \lambda_j + \eta),
\]

\[
\hat{\lambda}(\lambda) = \sum_{j=1}^{M} \sinh(\lambda - \lambda_j) \sinh(\lambda + \lambda_j),
\]

\[
K^{(2)}(\lambda) = -\frac{1}{pq} e^{\eta} \sinh(\lambda - \lambda_\alpha) \sinh(\lambda - \lambda_\beta) \sinh(\lambda - \alpha_\alpha) \cosh(\lambda - \beta_\beta),
\]

\[
K^{(3)}(\lambda) = -\frac{1}{pq} e^{\eta} \sinh(-\lambda - \eta - \alpha_\alpha) \cosh(-\lambda - \eta - \beta_\beta) \sinh(-\alpha_\alpha) \cosh(-\beta_\beta),
\]

\[
h = \frac{1}{2pq} e^{\eta} \cosh\left[(M + 1)\eta + \alpha_\alpha + \beta_\beta + \alpha_+ + \beta_\alpha\right] - \frac{1}{2pq} e^{\eta} \cosh(\theta_\theta - \theta_\theta),
\]

\[
Q^{(2)}(\lambda) = \sum_{j=1}^{M} \sinh(\lambda - w_j) \sinh(\lambda + w_j + \eta)
\]

\[
= \sum_{j=1}^{M} \sinh(\lambda - g_j - \frac{1}{2} \eta) \sinh(\lambda + g_j + \frac{3}{2} \eta). \quad (6.7)
\]
The second term in (6.5) should be zero, which ensure the $|F\rangle$ is the eigenstate of the nested transfer matrix \[22, 27, 28, 29\]. This requires that the $M$ Bethe roots $g_l$ must satisfy the BAEs

\[
1 + \frac{\sinh(2g_l + 3\eta)}{\sinh(2g_l + 2\eta)\sinh(2g_l + 3\eta)} \frac{Q^{(2)}(g_l + \eta)Q^{(2)}(g_l - \frac{1}{2}\eta)}{Q^{(1)}(g_l + \frac{1}{2}\eta)Q^{(1)}(g_l - \eta)} = -\hbar \frac{\sinh(2g_l + 2\eta)\sinh(2g_l + 3\eta)}{2\sinh(2g_l + 3\eta)} \frac{Q^{(1)}(g_l + \eta)}{Q^{(1)}(g_l + \frac{1}{2}\eta)Q^{(2)}(g_l + \frac{1}{2}\eta)}.
\]

\[l = 1, \ldots, M.\]  

(6.8)
Table 2: Solutions of BAEs (6.10) and (6.11) where $N = 3$ with the parameters $\eta = 0.2, \zeta = 0.1, c = 1, c_1 = -0.3, \zeta' = -0.4, \kappa' = -0.3, e_1 = 0.8940, e_2 = 0.2704, \beta_1 = 1.5708, \beta_2 = 0.8573, \theta_1 = -0.7779 - 1.5708i, \theta_2 = 0.2604 + 3.1416i, p = 1.5418i$ and $\eta = 0.9267$ for the case of $E_a$ is the corresponding eigenenergy. The energy $E_a$ calculated from (6.12) is the same as that from the exact diagonalization of the Hamiltonian (2.15).

| $a_2$          | $a_3$          | $a_4$          | $a_5$          | $a_6$          | $E_a$          | $\eta$ |
|---------------|---------------|---------------|---------------|---------------|---------------|--------|
| 0.3661 + 0.0946i | 0.3661 + 0.0946i | 0.3703 + 0.0878i | 0.3703 + 0.0878i | 0.1701 + 0.0000i | 7.466014      | 1      |
| 0.1708 - 0.0000i | 0.1708 - 0.0000i | 0.3322 - 0.0000i | 0.3322 - 0.0000i | -              | 7.267876      | 2      |
| 0.1692 + 0.0000i | 0.1692 + 0.0000i | 0.1715 + 0.0000i | 0.1715 + 0.0000i | -              | 5.968136      | 4      |
| 0.3703 - 0.0878i | 0.3703 - 0.0878i | -0.2000 + 0.2956i | -0.2000 + 0.2956i | -              | 5.765600      | 5      |
| -0.2000 - 0.3990i | -0.2000 - 0.3990i | -              | -              | -              | 5.855803      | 6      |
| -0.2309 + 0.2518i | 0.0309 + 0.2518i | -              | -0.3877 + 0.2796i | -              | 4.328439      | 7      |
| 0.0085 + 0.1597i | 0.0085 + 0.1597i | 0.0249 - 0.0303i | 0.0249 - 0.0303i | -              | 4.360177      | 8      |
| -0.2000 + 0.0000i | -0.2000 + 0.0000i | 0.0430 - 0.0000i | 0.0430 - 0.0000i | -              | 4.231025      | 9      |
| 0.4212 - 0.3413i | 0.4212 - 0.3413i | -              | -0.1571 + 0.0000i | -              | 3.852817      | 10     |
| 0.0027 + 0.1795i | 0.0027 + 0.1795i | -              | -0.0618 - 0.0000i | -              | 3.373027      | 11     |
| 0.0100 - 0.1375i | 0.0100 - 0.1375i | 0.0813 - 0.0768i | 0.0813 - 0.0768i | -              | 3.961098      | 12     |
| 0.0178 - 0.0000i | 0.0178 - 0.0000i | 0.2020 + 0.0000i | 0.2020 + 0.0000i | -              | 2.673798      | 13     |
| -0.2000 - 0.0879i | -0.2000 - 0.0879i | 0.1721 + 0.0000i | 0.1721 + 0.0000i | -              | 2.370817      | 14     |
| 0.2439 - 0.0000i | 0.2574 - 0.0000i | -              | -0.1286 + 0.0000i | -              | 1.764730      | 15     |
| 0.0348 - 0.0000i | 0.0458 + 0.0000i | -              | -0.0918 + 0.0000i | -              | 1.377880      | 16     |
| 0.1034 - 0.0000i | 0.1099 - 0.0429i | -              | -0.0602 + 0.0000i | -              | 0.769410      | 17     |
| 0.0100 - 0.0336i | 0.0185 - 0.0000i | 0.2863 + 0.0000i | 0.2863 + 0.0000i | -              | 0.033844      | 18     |
| 0.0173 - 0.0000i | 0.1601 + 0.0000i | -              | -0.1375 - 0.0000i | -              | -4.400469     | 19     |
| -0.1000 + 0.0391i | -0.1672 - 0.0000i | -              | -4.127163      | -              | -        |
| -0.1000 + 0.2373i | -0.2000 + 0.0617i | -              | -1.516419      | -              | -        |
| -0.1000 + 0.1444i | -0.2000 + 0.0617i | -              | -1.516419      | -              | -        |
| -0.1000 + 0.0435i | -0.2000 + 0.0617i | -              | -1.516419      | -              | -        |
| -0.1000 + 0.0601i | -0.2000 + 0.0617i | -              | -1.516419      | -              | -        |
| -0.1000 + 0.0418i | -0.2000 + 0.0617i | -              | -1.516419      | -              | -        |
Now we are ready to write out the eigenvalues $\Lambda(u)$ of the transfer matrices $\tau(u)$ as

$$
\Lambda(u) = \frac{\sinh(2u + 3\eta)}{\sinh(2u + \eta)} K^{(1)}(u) a_0(u) \frac{Q^{(1)}(u - \eta)}{Q^{(1)}(u)} + 
\frac{\sinh 2u \sinh(2u + 3\eta)}{\sinh(2u + \eta) \sinh(2u + 2\eta)} K^{(2)}(u) b_0(u) \frac{Q^{(1)}(u + \eta)Q^{(2)}(u + \frac{1}{2}\eta)}{Q^{(1)}(u)Q^{(2)}(u + \frac{1}{2}\eta)} + 
\frac{\sinh 2u}{\sinh(2u + 2\eta)} K^{(3)}(u) b_0(u) \frac{Q^{(2)}(u + \frac{1}{2}\eta)}{Q^{(2)}(u + \frac{1}{2}\eta)} + 
\sinh 2u \sinh(2u + 3\eta) b_0(u) \frac{hQ^{(1)}(u + \eta)}{Q^{(2)}(u + \frac{1}{2}\eta)}, \quad (6.9)
$$

where the $2M$ Bethe roots must satisfy the BAEs (4.12) and (6.8), namely

$$
l = 1, \ldots, M, \quad 1 + \frac{\sinh(2g_l + 3\eta)}{\sinh(2g_l + \eta)} K^{(2)}(g_l + \frac{1}{2}\eta)Q^{(1)}(g_l + \eta)Q^{(2)}(g_l - \frac{1}{2}\eta)}{Q^{(1)}(g_l)Q^{(2)}(g_l + \frac{1}{2}\eta)},
$$

$$
-1 = \frac{\sinh(2g_k + 2\eta)}{\sinh(2g_k + 2\eta)} K^{(1)}(g_k) a_0(g_k) \frac{Q^{(1)}(g_k + \eta)Q^{(2)}(g_k - \frac{1}{2}\eta)}{Q^{(1)}(g_k - \eta)Q^{(2)}(g_k + \frac{1}{2}\eta)}, \quad (6.10)
$$

$$
k = 1, \ldots, M. \quad (6.11)
$$

The eigenvalue the Hamiltonian (2.15) can be obtained by

$$
E = \sinh \eta \frac{\partial \ln \Lambda(u)}{\partial u} |_{u=0,\theta_j=0}. \quad (6.12)
$$

Here we present the numerical results for the $N = 2$ and $N = 3$ in Table 1 and Table 2 respectively. From these Tables, we can see that the eigenvalues obtained from (6.12) are the same as that from the exact diagonalization of the Hamiltonian (2.15). Moreover, numerical results also show that the energy spectrum is complete.

Now, the eigenstate of the system can be written out explicitly as

$$
\Psi = \sum_{\{a_i=1\}}^2 (a_1, a_2, \ldots, a_M) \langle F \rangle B_{a_1}(u_M)B_{a_2}(u_{M-1}) \ldots B_{a_1}(u_1) |\Psi_0\rangle
$$

$$
= \sum_{\{a_i=1\}}^2 \langle F^{a_1a_2\cdots a_M} \rangle B_{a_1}(u_M)B_{a_2}(u_{M-1}) \ldots B_{a_1}(u_1) |\Psi_0\rangle. \quad (6.13)
$$

As we mentioned above, the $F^{a_1\cdots a_M}$ are the vector components of the nested Bethe states (For example, for $N = 2, M = 2$, the state can be expressed by $F^{11} B_1(u_2)B_1(u_1) + F^{12} B_1(u_2)B_1(u_1) + F^{21} B_1(u_2)B_2(u_1) + F^{22} B_2(u_2)B_2(u_1)$). The numerical solutions of the states constructed by (4.1) with vector components $\langle F^{a_1a_2\cdots a_M} \rangle$ given by (6.1) are shown in Table 3 which give rise to the eigenstates of the system. Lastly, we should note that both the Bethe state given by (6.13) and
| $P_{i+1}^m$ | $B_{i+1}(M_i) | P_{i+1}^m | (M_i)^{N+1} | P_{i+1}^m$ | $E_i$ | $H(P_{i+1}) - E_i | P_{i+1}^m$ | $n$ |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| (-0.5508 - 0.0000i) | 1.0677 + 0.0000i | 1.3430 + 0.0000i | -2.5973 + 0.0000i | 0 | 0 | 0 | -1.0388 + 0.0000i | 2.0672 + 0.0000i | 0.0000 + 0.0000i | 2.5979 + 0.0000i | -5.1455 + 0.0000i | 5.3807982858 | $10^{-13}$ | 1 |
| (0.0000 + 6.9054i) | 0.0000 + 45.1930i | 0 | 0 | 0 | 4.8827952486 | $10^{-13}$ | 2 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3.5453692295 | 0 | 3 |
| (0.3696 + 0.4676i) | 0.4754 + 0.2248i | 0.4753 + 0.7129i | 0.4996 + 0.2026i | 0 | 0 | 0 | -1.4922 + 21.7322i | 1.6156 + 23.5296i | 0 | 1.9057 + 27.7531i | -0.4553 + 6.6311i | 2.7734720648 | $10^{-10}$ | 4 |
| (3.1100 + 6.7355i) | -3.8978 + 0.9624i | -2.8400 + 8.8559i | 0.3460 + 0.8335i | -1.0251 - 2.4746i | 0 | 0 | -0.9302 - 2.2454i | 0 | 0 | 2.3345140250 | $10^{-12}$ | 5 |
| (-1.9233 + 0.0000i) | 1.0429 + 0.4627i | 0.2625 + 0.3788i | -0.3661 + 0.0000i | 0 | 0 | 0 | -1.4005 - 0.0000i | 0.8738 + 0.0000i | 0 | -0.0652 + 0.0000i | -0.2482 - 0.0000i | 0.6052190547 | $10^{-10}$ | 6 |
the eigenvalue $\Lambda(n)$ given by (6.9) have well-defined homogeneous limit (i.e., $\theta_j \rightarrow 0$). This implies that in the homogeneous limit, the resulting Bethe states and the eigenvalues give rise to the eigenstates and the corresponding eigenvalues of the model described by the Hamiltonian (2.15).

7. Concluding remarks

In this paper, we have studied the trigonometric $SU(3)$ model with the most generic integrable boundary condition, which is described by the Hamiltonian (2.15) and the corresponding integrable boundary terms are associated with the most generic non-diagonal $K$-matrices given by (6.8)-(6.9). By combining the algebraic Bethe ansatz and the off-diagonal Bethe ansatz, we construct the eigenstates of the transfer matrix in terms of the nested Bethe states given by (4.1) and (6.1), which have well-defined homogeneous limit. The corresponding eigenvalues are given in terms of the inhomogeneous $T - Q$ relation (6.9) and the associated BAEs (6.10)-(6.11).

There are three forms of the off-diagonal reflection matrices $K^\pm$, i.e., the upper block form, the $X$ form and the lower block form (3.10). In this paper, we only consider the case that both the $K^+$ and the $K^-$ have a lower block form. Our method can also be applied to the cases that both the $K^+$ and the $K^-$ have an upper block or $X$ form simultaneously: For the upper $K^+$-matrix block form, the eigenstate of the transfer matrix can be constructed as $C_{ai}(a_M)C_{ai,-}(a_M-1)\ldots C_{ai}(a_1)\mathcal{F}_{aM}^{aM-1\ldots 1}(|\Psi_0\rangle)$, where $|\Psi_0\rangle = (0, 0, 1)^T$, $C_1 = \mathbb{T}_{31}$, $C_2 = \mathbb{T}_{32}$, and $\mathbb{T}_{ij}$ is the element of matrix $\mathbb{T}$ with the $i$th row and the $j$th column. While for the $X$ form, the corresponding eigenstate can be constructed as $X_{ai}(a_M)X_{ai,-}(a_M-1)\ldots X_{ai}(a_1)\mathcal{F}_{aM}^{aM-1\ldots 1}(|\Psi_0\rangle)$.
Ψ_0⟩ = (0, 1, 0)^t, X_1 = T_{21} and X_2 = T_{23}. We can prove that the eigenstate |F⟩ can be constructed by a similar nested Bethe states of an open spin-1/2 chain of length M with non-diagonal boundary terms. However, for the case that the K− and K+ have a different form, it seems that there does not exist an obvious reference state to which the first level algebraic Bethe ansatz can be applied, which leads to the eigenstate problem of the corresponding model being still non-trivial.

The exact solution in this paper provides the basis for further analyzing the thermodynamic properties and the correlation functions of the model. In particular, the explicit expressions (6.13) and (6.1) enable one to calculate the scalar products of Bethe states, by which the correlation behavior of some local operators can be computed further. Moreover, the T − Q relation and the associated BAEs allow one to calculate the boundary energy of the model in the thermodynamic limit.

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