Higher order gravity theories and scalar tensor theories

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We generalize the known equivalence between higher order gravity theories and scalar tensor theories to a new class of theories. Specifically, in the context of a first order or Palatini variational principle where the metric and connection are treated as independent variables, we consider theories for which the Lagrangian density is a function $f$ of (i) the Ricci scalar computed from the metric, and (ii) a second Ricci scalar computed from the connection. We show that such theories can be written as tensor-multi-scalar theories with two scalar fields with the following features: (i) the two dimensional σ-model metric that defines the kinetic energy terms for the scalar fields has constant, negative curvature; (ii) the coupling function determining the coupling to matter of the scalar fields is universal, independent of the choice of function $f$; and (iii) if both mass eigenstates are long ranged, then the Eddington post-Newtonian parameter $\gamma$ has value 1/2. Therefore in order to be compatible with solar system experiments at least one of the mass eigenstates must be short ranged.

I. INTRODUCTION AND SUMMARY

A well-known class of theories of gravity that modify general relativity can be obtained by taking the Lagrangian density to be some nonlinear function $f$ of the matter fields. We use units in which $\hbar = c = 1$, and we use the sign conventions of Ref. [1].

The equivalence was first shown, in the context of a specific choice of $f$, by Higgs [2], and later independently by several different researchers including Bicknell [3], Teyssandier and Tourrenc [4], and Whitt [5]. The generalization to arbitrary functions $f$ was first given by Schmidt [6]. Subsequently Wands [7] once again independently discovered the equivalence, and independently derived the result for arbitrary $f$.

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General functions $f(\hat{R})$, the theories [11] and [12] are not equivalent [10].

Modifications of gravity of both types [11] and [12] have been suggested [20] to explain the observed recent acceleration of the Universe’s expansion [16, 17]. However, there are as yet no successful models along these lines. Theories of the type [11] contain a scalar field which couples to matter with the same strength as does gravity, as pointed out by Chiba [18]. These theories are thus ruled out by solar system experiments unless the scalar field is short ranged, and in that case it is difficult to obtain an accelerating Universe [19]. Theories of the type [12] are actually not modifications of gravity at all, despite appearances. Instead, these theories, when written in a canonical form, contain a scalar field with no kinetic energy energy term. When one integrates out the scalar field one obtains general relativity coupled to a modified matter action, not the original matter action $S_m[\bar{g}_{\mu\nu}, \psi_m]$ that appears in Eq. [12].

This paper is based on the observation, due to Nima Arkani-Hamed [21], that theories of the form [12] are unstable to matter loop corrections. Specifically, matter loops will give rise to a correction to the action [12] proportional to the Ricci scalar $\hat{R}$ of the metric $\bar{g}_{\mu\nu}$. In general relativity these corrections merely renormalize Newton’s constant of gravitation. Here, however, such corrections give rise to a Lagrangian density for the gravitational action which is no longer a function of $\hat{R}$ alone, but instead is a function of both $\hat{R}$ and $\hat{R}$, the Ricci scalars.

3 It may be possible to evade this constraint for certain choices of the function $f(\hat{R})$. For example, for $f(\hat{R}) \propto \hat{R}^2(\ln \hat{R})^n$, the potential $V(\Phi)$ for the scalar field is of the form $V(\Phi) \propto 1/\Phi^n$ for large $\Phi$. For this potential, and for the matter coupling for $\Phi$ that follows from the action [12], there exists a regime in which the coupling of the scalar field to large objects is suppressed by nonlinear effects [13]. However, it is not yet clear if this class of “chameleon field” theories give acceptable cosmologies.

4 This statement is valid classically. However quantum mechanically things may be more complicated.
of the connection and of the metric. Thus, the form of the gravitational action \([1.2]\) is not preserved under loop corrections, and there is a fine-tuning problem inherent in assuming an action of the form \([1.2]\).

That fine tuning problem is also apparent in the alternative description of the theory \([1.2]\), derived in Ref. \([20]\), as a type of scalar-tensor type theory without any kinetic energy term for the scalar field \([21]\). The action of that alternative description is

\[
S[\tilde{g}_{\mu\nu}, \Phi, \psi_m] = \int d^4x\sqrt{-\tilde{g}} \left[ \frac{R}{2\kappa^2} - V(\Phi) \right] + S_m[\varepsilon^{2\alpha(\Phi)} g_{\mu\nu}, \psi_m], \quad (1.3)
\]

where \(\Phi\) is the auxiliary scalar field with coupling function \(\alpha(\Phi)\) and potential \(V(\Phi)\), and \(g_{\mu\nu} = \exp[-2\alpha(\Phi)]\tilde{g}_{\mu\nu}\) is the Einstein frame metric. In terms of these variables, the correction term due to matter loops is proportional to \(\tilde{R} = e^{-2\alpha}[R - 6(\nabla\alpha)^2 - 6\Box\alpha]\). Therefore matter loops induce a kinetic term for the field \(\Phi\), which becomes a dynamical field, and the loop-corrected theory is a genuine modification of general relativity, unlike the original theory \([1.2]\) or \([1.3]\).

These considerations suggest that, in the context of the first-order variation formalism, it is natural to consider a more general class of theories of gravity of the form

\[
S[\tilde{g}_{\mu\nu}, \tilde{\nabla}_\mu, \psi_m] = \frac{1}{2\kappa^2} \int d^4x\sqrt{-\tilde{g}} f(\tilde{R}, \tilde{\nabla}) + S_m[\tilde{g}_{\mu\nu}, \psi_m]. \quad (1.4)
\]

Here the Lagrangian density of the gravitational action is some function \(f\) of both the Ricci scalar \(\tilde{R}\) of the metric \(\tilde{g}_{\mu\nu}\) and the Ricci scalar \(\tilde{\nabla}_\mu\) of the connection \(\tilde{\nabla}_\mu\). These theories are stable under loop corrections, which just give rise to corrections to the function \(f(\tilde{R}, \tilde{\nabla})\). The purpose of this paper is to analyze the class of theories \([1.4]\), and to derive for these theories an equivalent description, at the classical level, as tensor-biscalar theories.

### A. Summary of results

We now turn to a description of our results. We define

\[
f_1(\tilde{R}, \tilde{\nabla}) = \frac{\partial f}{\partial \tilde{R}}(\tilde{R}, \tilde{\nabla}), \quad f_2(\tilde{R}, \tilde{\nabla}) = \frac{\partial f}{\partial \tilde{\nabla}}(\tilde{R}, \tilde{\nabla}), \quad (1.5)
\]

and we define the signs

\[
\varepsilon_1 = \text{sign} f_1, \quad \varepsilon_2 = \text{sign} f_2, \quad (1.6)
\]

and

\[
\varepsilon_3 = \text{sign}(f_1 + f_2). \quad (1.7)
\]

Our construction in Sec. \([11]\) below shows that if \(\varepsilon_1, \varepsilon_2\) and \(\varepsilon_3\) are constant in an initial data set, then they will be constant throughout the corresponding future Cauchy evolution. We restrict attention to this portion of phase space, in which the signs \(\varepsilon_1, \varepsilon_2\) and \(\varepsilon_3\) are constants, independent of space and time. The reason for this restriction is that in regions of phase space where one or more of these variables flips sign somewhere in spacetime, we suspect that the theory does not possess a well posed initial value formulation \(^5\). Assuming that \(\varepsilon_1, \varepsilon_2,\) and \(\varepsilon_3\) are constants, we show below that in order for the theory not to contain ghosts or negative energy excitations, we must have

\[
\varepsilon_1 = 1, \quad (1.8)
\]

\[
\varepsilon_2 = -1, \quad (1.9)
\]

and

\[
\varepsilon_3 = 1. \quad (1.10)
\]

Next, we further restrict the region of phase space under consideration, and the set of allowed functions \(f\), as follows. We define \(\Delta(\tilde{R}, \tilde{\nabla})\) to be the determinant of the matrix of second order partial derivatives of \(f\) with respect to \(\tilde{R}\) and \(\tilde{\nabla}\). We assume that there exists an open domain \(D\) of points \((\tilde{R}, \tilde{\nabla})\) for which the following three conditions are satisfied: (i) We have

\[
\Delta(\tilde{R}, \tilde{\nabla}) \neq 0 \quad (1.11)
\]

for all points \((\tilde{R}, \tilde{\nabla})\) in \(D\); (ii) The conditions \([1.8] - [1.10]\) are satisfied for all points \((\tilde{R}, \tilde{\nabla})\) in \(D\); and (iii) Equations \([1.10]\) and \([1.14]\) below define a bijection between \(D\) and some open domain \(D'\) of field values \((\Phi, \Psi)\). Then, the theory \([1.4]\) with initial data in the domain \(D\) is equivalent to a tensor-biscalar theory with field values in the domain \(D'\).

We next describe this tensor-biscalar theory. The action for a general tensor-multi-scalar theory is the following functional of the matter fields \(\psi_m\), the Einstein-frame metric \(g_{\mu\nu}\), and a N-tuple of scalar fields \(\Phi^A = (\Phi^1, \ldots, \Phi^N)\)

\[
S[\tilde{g}_{\mu\nu}, \Phi^A, \psi_m] = \int d^4x\sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2\kappa^2} - V(\Phi^A) \right]
\]

\[
-\frac{1}{2}\gamma_{AB}(\Phi^C)g^{\mu\nu}\tilde{\nabla}_\mu\Phi^A\tilde{\nabla}_\nu\Phi^B + S_m[\varepsilon^{2\alpha(\Phi^A)} g_{\mu\nu}, \psi_m]. \quad (1.12)
\]

\(^5\) The reason for this suspicion is that points in spacetime at which \(\varepsilon_1, \varepsilon_2\) or \(\varepsilon_3\) flip sign are points at which one of the three kinetic terms in the action \([20]\) below vanishes. The initial value formulation is known to break down in other, simpler contexts in which this type of phenomenon occurs. For example, the theory \(S = \int d^4x\sqrt{-\tilde{g}}/4\kappa^2 - (\nabla\Phi)^2 / 2 - \xi R\Phi^2\) has a good initial value formulation in the region of phase space \(\Phi \leq \Phi_c\), where \(\Phi_c = 1/\sqrt{4\kappa^2\xi}\), but not once \(\Phi\) crosses the value \(\Phi_c\), since \(\Phi \equiv \Phi_c; g_{\mu\nu} = \text{anything} \) is a solution of the equations of motion.
This action is characterized by the coupling function \( \alpha(\Phi^4) \) and potential \( V(\Phi^4) \) of the scalar fields, and by the \( \sigma \)-model metric \( \gamma_{AB}(\Phi^C) \). For the theory \( \ref{1.4} \) we have \( N = 2 \), \( \Phi^4 = (\Phi^1, \Phi^2) = (\Phi, \Psi) \), the \( \sigma \)-model metric is given by

\[
\gamma_{AB}(\Phi^C) d\Phi^A d\Phi^B = d\Phi^2 + \cosh^2 \left( \frac{\kappa}{\sqrt{6}} \Phi \right) d\Psi^2,
\]

(1.13)

and the coupling function is

\[
\alpha(\Phi, \Psi) = \frac{\kappa}{\sqrt{6}} \Psi + \ln \cosh \left( \frac{\kappa}{\sqrt{6}} \Phi \right).
\]

(1.14)

The potential is given by the formula

\[
V(\Phi, \Psi) = \frac{\mu^2 \alpha(\Phi, \Psi)}{2\kappa^2} \left[ \varphi f_1(\varphi, \psi) + \psi f_2(\varphi, \psi) - f(\varphi, \psi) \right].
\]

(1.15)

Here we have introduced two additional scalar fields \( \varphi \) and \( \psi \), which are given in terms of \( \Phi \) and \( \Psi \) by the equations

\[
|f_1(\varphi, \psi)| = \exp \left[ -2 \frac{\kappa}{\sqrt{6}} \Psi \right],
\]

(1.16)

and

\[
\frac{|f_2(\varphi, \psi)|}{|f_1(\varphi, \psi)|} = \tanh^2 \left( \frac{\kappa}{\sqrt{6}} \Phi \right).
\]

(1.17)

For specific choices of the function \( f(\bar{R}, \bar{R}) \) it is straightforward to compute the potential \( V \) from Eqs. \ref{1.15} - \ref{1.17}. For example, if \( f = \bar{R} - \bar{R} + \bar{R} \bar{R}/m^2 \) we have \( V = m^2 u(1 - u)/(2\kappa^2) \), where

\[
\alpha = \cosh^2 \left( \frac{\kappa}{\sqrt{6}} \Phi \right) \left[ 1 - \exp \left( 2 \frac{\kappa}{\sqrt{6}} \Psi \right) \right].
\]

(1.18)

\section{II. DERIVATION OF EQUIVALENCE}

Consider the gravitation theory given by the action \ref{1.4}. An action which is equivalent to this at the classical level can be obtained by using the technique of Refs. \ref{5, 8} of introducing auxiliary scalar field(s). Specifically, we introduce two scalar fields \( \varphi \) and \( \psi \), and we define the action

\[
\tilde{S}[\tilde{g}_{\mu\nu}, \tilde{\nabla}_\mu, \varphi, \psi, \psi_m] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left[ f(\varphi, \psi) + (\tilde{R} - \varphi)f_1(\varphi, \psi) + (\tilde{R} - \psi)f_2(\varphi, \psi) \right] + S_m[\tilde{g}_{\mu\nu}, \psi_m].
\]

(2.1)

Then the equations of motion for \( \varphi \) and \( \psi \) enforce \( \varphi = \tilde{R} \), \( \psi = \tilde{R} \), as long as

\[
\det \left[ \frac{\partial^2 f}{\partial \varphi^2 \partial \psi} \frac{\partial^2 f}{\partial \psi^2} \right] \neq 0.
\]

(2.2)

The condition \ref{2.2} will always be satisfied because of our assumption \ref{1.11} above. Therefore the actions \ref{1.4} and \ref{2.1} are classically equivalent.

We write the action \ref{2.1} as \( \tilde{S} = \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 + S_m \), where

\[
\tilde{S}_1 = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \tilde{R} f_1(\varphi, \psi),
\]

(2.3)

\[
\tilde{S}_2 = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \tilde{R} f_2(\varphi, \psi),
\]

(2.4)

and

\[
\tilde{S}_3 = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left[ f - \varphi f_1 - \psi f_2 \right].
\]

(2.5)

The action \( \tilde{S}_1 \) can be rewritten as an Einstein-frame form using the following standard procedure. We define the scalar field \( \rho \) by

\[
f_1(\varphi, \psi) = \varepsilon_1 e^{-2\rho},
\]

(2.6)

where \( \varepsilon_1 = \pm 1 \) [cf. Eq. \ref{1.6} above]. Defining the conformally transformed metric

\[
\tilde{g}_{\mu\nu} = \exp[-2\rho]\tilde{g}_{\mu\nu},
\]

(2.7)

and using \( \tilde{R} = e^{-2\rho} \tilde{R} - 6(\tilde{\nabla}\rho)^2 - 6\tilde{\Box}\rho \), we obtain

\[
\tilde{S}_1[\tilde{g}_{\mu\nu}, \varphi, \psi] = \frac{\varepsilon_1}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left[ \tilde{R} - 6(\tilde{\nabla}\rho)^2 \right].
\]

(2.8)

A similar transformation can be carried out for the action \( \tilde{S}_2 \). We define the scalar field \( \sigma \) by

\[
f_2(\varphi, \psi) = \varepsilon_2 e^{-2\sigma},
\]

(2.9)

where \( \varepsilon_2 = \pm 1 \) [cf. Eq. \ref{1.6} above]. Now the Ricci scalar \( \tilde{R} \) that appears in the action \( \tilde{S}_2 \) depends both on the metric \( \tilde{g}_{\mu\nu} \) and on the connection \( \tilde{\nabla}_\mu \) via

\[
\tilde{R} = \tilde{g}_{\mu\nu} \tilde{R}_{\mu\nu},
\]

(2.10)

where \( \tilde{R}_{\mu\nu} \) is the Ricci tensor of the connection \( \tilde{\nabla}_\mu \). Under conformal transformations of the metric that preserve the connection, this quantity will undergo a simple, multiplicative transformation coming from the first factor in Eq. \ref{2.10}. Thus, if we define

\[
\tilde{g}_{\mu\nu} = \exp[-2\sigma]g_{\mu\nu},
\]

(2.11)

we obtain

\[
\tilde{S}_2[\tilde{g}_{\mu\nu}, \tilde{\nabla}_\mu, \varphi, \psi] = \frac{\varepsilon_2}{2\kappa^2} \int d^4x \sqrt{-\tilde{g}} (\tilde{g}_{\mu\nu} \tilde{R}_{\mu\nu}).
\]

(2.12)
Next, we note that the total action (2.1) depends on the connection $\nabla_\mu$ only through the term (2.12). Also the action (2.12) is just the Palatini-variation version of the Einstein-Hilbert action of general relativity. Therefore, the equation of motion for $\nabla_\mu$ stipulates that $\nabla_\mu$ is the connection $\nabla_\mu$ determined by the metric $\bar{g}_{\mu\nu}$. In the classical theory, we are free to substitute the relation $\nabla_\mu = \nabla_\mu$ back into the action (2.12) to obtain

$$\tilde{S}_1[\bar{g}_{\mu\nu}, \varphi, \psi] = \frac{\varepsilon_2}{2\kappa^2} \int d^4 x \sqrt{-\bar{g}} R,$$  \hspace{1cm} (2.13)

where $R$ is the Ricci scalar of the metric $\bar{g}_{\mu\nu}$. In other words, we obtain an equivalent theory at the classical level by replacing the Palatini-variation action $\tilde{S}_1$ with its standard-variation version $\tilde{S}_2$. Next, we write the action (2.13) in terms of the metric $g_{\mu\nu}$ using the definitions (2.7) and (2.11). This gives

$$\tilde{S}_2[\bar{g}_{\mu\nu}, \varphi, \psi] = \frac{\varepsilon_2}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ e^{2\tau} \bar{R} + 6e^{2\tau} (\nabla \tau)^2 \right],$$  \hspace{1cm} (2.14)

where

$$\tau = \rho - \sigma.$$  \hspace{1cm} (2.15)

Adding this to the action (2.8) and rewriting in terms of the conformally transformed metric

$$g_{\mu\nu} = e^{-2\chi} \bar{g}_{\mu\nu}$$  \hspace{1cm} (2.16)

gives

$$\tilde{S}_1 + \tilde{S}_2' = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left\{ 6\varepsilon_2 e^{2\chi} e^{2\tau} (\nabla \tau)^2 + e^{2\chi}(\varepsilon_1 + \varepsilon_2 e^{2\tau}) \left[ R - 6(\nabla \chi)^2 - 6\Box \chi \right] - 6\varepsilon_1 e^{2\chi}(\nabla \rho)^2 \right\}.$$  \hspace{1cm} (2.17)

Next, it follows from the definitions (2.14), (2.15) and (2.16) of $\rho$, $\sigma$ and $\tau$ and the definition (1.14) of $\varepsilon_3$ that

$$\varepsilon_3 = \text{sign}(\varepsilon_1 + \varepsilon_2 e^{2\tau}).$$  \hspace{1cm} (2.18)

We now choose the conformal factor $e^{2\chi}$ entering into the definition (2.16) of the metric $g_{\mu\nu}$ to satisfy

$$e^{2\chi}|\varepsilon_1 + \varepsilon_2 e^{2\tau}| = 1.$$  \hspace{1cm} (2.19)

Using Eqs. (2.18) and (2.19) the action (2.17) simplifies to

$$\tilde{S}_1 + \tilde{S}_2' = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ \varepsilon_3 R - \frac{6\varepsilon_1 \varepsilon_3}{\varepsilon_1 + \varepsilon_2 e^{2\tau}} (\nabla \rho)^2 + \frac{6\varepsilon_1 \varepsilon_2 e^{2\tau}}{\varepsilon_3 + \varepsilon_2 e^{2\tau}} (\nabla \tau)^2 \right].$$  \hspace{1cm} (2.20)

Now the sign of the coefficient of the Ricci scalar $R$ in the action (2.20) must be positive, otherwise the matter action $S_m$ will contain fields whose kinetic terms have the wrong sign relative to the gravitational action, giving rise to an unstable theory. This yields the condition (1.10). Similarly, requiring that the last two terms in the action (2.20) have their conventional signs and using the relation (2.18) yields the conditions (1.8) and (1.9).

Next, we define the fields $\Phi$ and $\Psi$ by

$$\Phi = \frac{\sqrt{6}}{\kappa} \text{sign}(\tau) \tanh^{-1}(e^{-\tau})$$  \hspace{1cm} (2.21)

and

$$\Psi = \frac{\sqrt{6}}{\kappa} \rho.$$  \hspace{1cm} (2.22)

It follows that

$$\frac{1}{\varepsilon_1 + \varepsilon_2 e^{2\tau}} = \cosh^2(\kappa \Phi/\sqrt{6}),$$  \hspace{1cm} (2.23)

and substituting into the action (2.20) reproduces the first and third terms of the action (2.12). The last term in the action (2.12) is just the fourth term in the action (2.1) where we have used the relation

$$\bar{g}_{\mu\nu} = e^{2\alpha} g_{\mu\nu}$$  \hspace{1cm} (2.24)

between the Jordan-frame metric $\bar{g}_{\mu\nu}$ and the Einstein frame metric $g_{\mu\nu}$. In order to compute the coupling function $\alpha$ that is defined by the relation (2.24), we combine the definitions (2.4) and (2.10) of $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ together with the formula (2.19) for the conformal factor $e^{2\chi}$. This yields

$$e^{2\alpha} = \frac{e^{2\rho}}{\varepsilon_1 + \varepsilon_2 e^{2\tau}}.$$  \hspace{1cm} (2.25)

and combining this with Eqs. (2.22) and (2.23) yields the formula (1.14). Finally, the expression (1.15) for the potential is obtained by combining Eqs. (2.20) and (2.24), and the formulae (1.16) and (1.17) for $\varphi$ and $\psi$ in terms of $\Phi$ and $\Psi$ can be obtained from the definitions (2.6), (2.9), (2.11), (2.21) and (2.22).

Finally, we note that the condition (1.14) together with Eqs. (1.8), (1.9) and (2.18) imply that $\tau \leq 0$, which implies from Eq. (2.21) that

$$\Phi \leq 0.$$  \hspace{1cm} (2.26)

7 If the sign of a scalar-field action relative to the gravitational action is flipped from its usual value, then the Hamiltonian of the theory is unbounded below and one expects the theory to be quantum mechanically unstable. Note however that in such theories the decay timescales may be long enough for the theory to make sense as an effective field theory over a certain range of scales, as suggested in Refs. 21, 22. In this paper we restrict attention to sectors where the signs of the scalar-field actions have the conventional value.
In other words, the two theories are equivalent in the following sense: solutions of the original theory \([1.1]\) can be mapped onto solutions of the tensor-bisclar theory \([2.20]\) that satisfy \([2.20]\). This restriction is somewhat puzzling as the condition \([2.20]\) is clearly not preserved under general dynamical evolution. The resolution of this puzzle is that if one starts with an initial data set satisfying \([2.20]\) and evolves forward using the tensor-bisclar theory until the condition \([2.20]\) is violated, the locus of points at which \(\Phi = 0\) corresponds to a singularity of the original variables \(g_{\mu\nu}, \nabla_{\mu}\). This can be seen from Eq. \([1.14]\), since we have argued that \(\varepsilon_2 = \text{sign}(f_2)\) cannot flip sign, so \(f_1\) must pass through \(\pm \infty\) as \(\Phi\) passes through 0.

We also note that the methods of this paper can be applied to special choices of the function \(f(\hat{R}, \hat{R})\) for which the quantity \(\Delta(\bar{R})\) vanishes identically, cf. Eq. \([1.11]\) above. In such cases one obtains a theory with one scalar field. The choice \(f(\hat{R}, \hat{R}) = \hat{R} + g(\hat{R})\) is equivalent to a theory of the type \([1.14]\) with \(f(\hat{R}) = \hat{R} + g(\hat{R})\). The choice \(f(\hat{R}, \hat{R}) = \hat{R} + g(\hat{R})\) can be obtained from the above analysis with \(\varepsilon_1 = 1, \rho = \varphi = 0\). In this case the formula \([1.15]\) for the potential becomes

\[
V(\Phi) = e^{\frac{2\alpha(\Phi)}{\kappa^2}} [\psi g'(\psi) - g(\psi)]
\]

where \(\tanh^2(\kappa\Phi/\sqrt{6}) = |g'(\psi)|\).

III. DISCUSSION AND IMPLICATIONS

Consider first the compatibility of the theory \([1.14]\) with solar system experiments. Let’s assume that the background, present-day cosmological values of the fields \(\Phi\) and \(\Psi\) are \(\Phi_0\) and \(\Psi_0\). Let’s also assume, initially, that the potential \(V\) is such that the fields \(\delta\Phi = \Phi - \Phi_0\) and \(\delta\Psi = \Psi - \Psi_0\) have small masses and are long ranged on solar system scales. The Eddington parameterized post-Newtonian (PPN) parameter \(\gamma\) for theory is then given by \([2]\)

\[
1 - \gamma = \frac{2\alpha_0^2}{1 + \alpha_0^2},
\]

where

\[
\alpha_0^2 = \frac{2}{\kappa^2} \gamma^{AB} \frac{\partial \alpha}{\partial \Phi^A} \frac{\partial \alpha}{\partial \Phi^B}
\]

evaluated at the present-day background values of the fields. Evaluating the expression \([3.1]\) using Eqs. \([1.14]\) – \([1.18]\) gives \(\gamma = 1/2\). This is in conflict with solar system VLBI observations which show that \([2]\)

\[
|\gamma - 1| \leq 3 \times 10^{-4}.
\]

It follows that the potential \(V(\Phi, \Psi)\) must be such that at least one of the two mass eigenstates for perturbations about \((\Phi_0, \Psi_0)\) is short ranged. Those mass eigenstates are determined from the equations \([2]\)

\[
\gamma_{AB}(\Phi_0^C) = \sum_j \lambda_A^j \lambda_B^j
\]

\[
\nabla_A \nabla_B V(\Phi_0^C) = \sum_j \lambda_A^j \lambda_B^j
\]

where \(j\) labels the eigenstate, \(m_j\) are the masses, and \(\nabla_A\) is the covariant derivative determined by the metric \(\gamma_{AB}\) on field space. The formula \([3.1]\) is then replaced by

\[
\alpha_0^2 = \frac{2}{\kappa^2} \sum_j \left(\gamma^{AB} \lambda_A^j \frac{\partial \alpha}{\partial \Phi^B}\right)^2,
\]

where \(\sum_j\) denotes the sum over the light, long-ranged eigenstates.

If there is one light and one heavy eigenstate, and if \(\chi\) is the angle in field space between the light eigenstate and the \(\partial/\partial \Phi\) direction, then a short calculation using Eqs. \([1.13]\), \([1.14]\) and \([3.6]\) gives

\[
\alpha_0^2 = \frac{\sin \chi + \cos \chi \sinh(\kappa \Phi_0/\sqrt{6})}{3 \cosh^2(\kappa \Phi_0/\sqrt{6})}.
\]

Thus if \(\chi \ll 1\) and \(\kappa \Phi_0 \ll 1\), then \(\alpha_0^2 \ll 1\) and the solar system constraint \([3.8]\) can be evaded. It is possible to find functions \(f(\hat{R}, \hat{R})\) and values of \((\Phi_0, \Psi_0)\) for which \(\chi \ll 1\). This is because \(\chi\) is determined by the values of \(f\) and its derivatives up to third order at the point \((\hat{R}_0, \Phi_0)\) that corresponds to \((\Phi_0, \Psi_0)\), so that there many more free parameters than there are constraints. Thus, there exists a class of gravitation theories of this type that are compatible with solar system observations.

It might be possible to further specialize the choice of \(f(\hat{R}, \hat{R})\) to give a viable model of the recent acceleration of the Universe, along the lines suggested by Carroll et. al. \([12]\). However, in such a case, it would not be completely accurate to call the theory \([1.14]\) a modified theory of gravity, since the theory would contain a scalar field which acts mostly as a source of gravity and does not mediate any long range forces. Rather, the explanation for the acceleration of the Universe would in such a case be partly a modification of gravity and partly a quintessence-type source of gravity.

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