Asymptotics of recurrence coefficients for the Laguerre weight with a singularity at the edge

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Abstract In this paper, we study the asymptotics of the leading coefficients and the recurrence coefficients for the orthogonal polynomials with respect to the Laguerre weight with singularity of root type and jump type at the soft edge via the Deift-Zhou steepest descent method. The asymptotic formulas of the leading coefficients and the recurrence coefficients for large n are described in terms of a class of analytic solutions to the the $\sigma$-form of the Painlevé II equation and the Painlevé XXXIV equation.

Keywords: Asymptotic; Leading coefficients; Recurrence coefficients; Riemann-Hilbert approach

Mathematics subject classification 2010: 41A60; 34M55; 33C45

1 Introduction and statement of results

We consider the orthogonal polynomials $p_n(x)$ satisfied the following relationship

$$\int_{0}^{+\infty} p_n(x)p_m(x)w(x)dx = \delta_{n,m}, \quad (1.1)$$

with the weight perturbed by a Fisher-Hartwig singularity

$$w(x) := w(x; \alpha, \beta, \omega, \mu) = x^\alpha e^{-x|x-\mu|^{2\beta}} \begin{cases} 1, & 0 < x \leq \mu \\ \omega, & x > \mu \end{cases}, \quad (1.2)$$

where $\alpha > -1$, $\beta > -\frac{1}{2}$, $\omega \in \mathbb{C} \setminus (-\infty, 0)$ and $\mu = 4n + 4^{2/3}n^{1/3}s$. It is seen that the weight $w(x)$ given as (1.2) has Fisher-Hartwig singularity of root type and jump type at $x = \mu$, where $2^{2/3}n^{2/3} \left( \frac{\mu}{4n} - 1 \right) \to s$ with finite $s$ as $n \to \infty$.

Let $\gamma_n$ be the leading coefficient of the orthonormal polynomials with respect to (1.2) and

$$p_n(x) = \gamma_n \pi_n(x), \quad (1.3)$$

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then we have the three-term recurrence relations

\[ z\pi_n(x) = \pi_{n+1}(x) + a_n\pi_n(x) + b_n^2\pi_{n-1}(x) \]  

(1.4)

with \( \pi_{-1}(x) = 0 \) and \( \pi_0(x) = 1 \).

When \( \beta = 0 \) and \( \omega = 1 \) in weight (1.2), the (1.2) is the Laguerre weight; see [22] [14]. Based on Deift-Zhou steepest descent method, the Laguerre-type weight \( w(x) = x^\alpha e^{-Q(x)} \) were considered in [23], where \( Q(x) \) is a polynomial with positive leading coefficient, and global asymptotic expansions of the Laguerre polynomials were studied in [21]. The recurrence coefficients of orthogonal polynomials may be related to the solutions of the Painlevé equations; see [6] for the Laguerre-type weight perturbed by a factor \( e^{-s/x} \) with \( s > 0 \), and [10] for semi-classical Laguerre polynomials.

The discontinuous weights were considered by the ladder operator approach in [4], see also [1] [18] for the discontinuous Laguerre weight with a jump, and [3] [20] for the Hermite weight with an isolated zero and the Gaussian weight with two jumps respectively. In [18], Lyu and Chen studied the weight given in (1.2) in the case \( \beta = 0, \omega = 0 \) and \( \alpha = O(n) \) as \( n \to \infty \) or \( \alpha \) is finite. The logarithmic derivative of the limit of the probability of the largest eigenvalue for large \( n \) was described respecting certain solutions of the Jimbo-Miwa-Okamoto \( \sigma \)-form of the Painlevé II equation

\[
(\sigma'')^2 + 4(\sigma')^3 - 4s(\sigma')^2 + 4\sigma'\sigma - (2\beta)^2 = 0,
\]

(1.5)

with \( \beta = 0 \), and the asymptotics of the recurrence coefficients were also obtained.

In [15], Its, Kuijlaars and Östensson investigated perturbed Gaussian unitary ensemble (pGUE) with singularity of root type at \( \sqrt{2n} + \frac{s}{\sqrt{2n^{1/6}}} \) for bounded \( s \) based on the Deift-Zhou steepest descent method. They obtained the asymptotic formulas of the limiting eigenvalue correlation kernel as \( n \to \infty \) characterized in terms of a solution to the Painlevé XXXIV (P_{34} for short) equation

\[
u_{ss} = \frac{u_s^2}{2u} + 4u^2 + 2su - \frac{(2\beta)^2}{2u}.
\]

(1.6)

Later, the pGUE with a jump at \( \sqrt{2n} + \frac{s}{\sqrt{2n^{1/6}}} \) was studied in [25] and [3], and with singularity of root type and jump type at \( \sqrt{2n} + \frac{s}{\sqrt{2n^{1/6}}} \) were investigated in [24]. In [24], they found that the asymptotic formulas of the recurrence coefficients and Hankel determinant associated with the weight can be characterized in terms of a class of solutions to the equations (1.5) and (1.6). The pGUE with singularity of root type and jump type at \( \sqrt{2n} + \frac{s}{\sqrt{2n^{1/6}}} \) for the case \( 2\beta \in \mathbb{N} \) and \( \omega = 0, 1 \) was considered in [13] earlier by applying the Okamoto \( \tau \)-function theory.

In this paper, we investigate the asymptotic formulas of the leading coefficients and recurrence coefficients for the perturbed Laguerre weight with singularity of root type and jump type at \( 4n + 4^{2/3}n^{1/3}s \) defined as (1.2) by Deift-Zhou steepest decent method.
1.1 Statement of main results

The asymptotic formulas of leading coefficients and recurrence coefficients are characterized as follows.

**Theorem 1.** Let $\alpha > -1$, $\beta > -\frac{1}{2}$, $\omega \in \mathbb{C} \setminus (-\infty, 0)$, $\mu = 4n + 4n^{1/3}s$, $\gamma_n$ be the leading coefficients in (1.3), and $a_n$ and $b_n$ be the recurrence coefficients in (1.4). Then the following holds

$$
\gamma_{n-1} = \frac{n^{-n^{-\frac{2}{3}} - \beta + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} e^n}}{\sqrt{2\pi}} \left[ 1 + \frac{\sigma(s)}{2^{1/3}n^{1/3}} + \frac{\sigma^2(s) + 2(\alpha + 2\beta - 1)u(s)}{2^{5/3}n^{2/3}} + O(n^{-1}) \right],
$$

$$
\gamma_n = \frac{n^{-n^{-\frac{2}{3}} - \beta - \frac{1}{2} + \frac{1}{2} e^n}}{\sqrt{2\pi}} \left[ 1 + \frac{\sigma(s)}{2^{1/3}n^{1/3}} + \frac{\sigma^2(s) + 2(\alpha + 2\beta + 1)u(s)}{2^{5/3}n^{2/3}} + O(n^{-1}) \right],
$$

$$
a_n = 2 - \frac{2^{1/3}u(s)}{n^{2/3}} + O(n^{-1}),
$$

$$
b_n = 1 - \frac{2^{1/3}u(s)}{n^{2/3}} + O(n^{-1}),
$$

where $\sigma(s) = \sigma(s; \beta, \omega, \mu)$ is the analytic solutions to the Jimbo-Miwa-Okamoto $\sigma$-form of the Painlevé II equation (1.5) and $u(s) = u(s; \beta, \omega, \mu)$ is the analytic solutions to the $P_{34}$ equation (1.6).

When $\beta = 0$, the asymptotic formulas of the recurrence coefficients have been studied by Lyu and Chen; see [18]. It is seen that the parameter $\omega$ in (1.2) is not appeared in equations (1.5) and (1.6). A similar phenomenon can be found in [11 25 19].

**Theorem 2.** Let $\pi_n(z)$ be the monic orthogonal polynomials defined as (1.3), then

$$
\pi_n(4nz) = \frac{n^n e^{2n(z - \sqrt{z(z - 1)})(\sqrt{z + \sqrt{z - 1}})2^{2n+\alpha+2\beta}}}{2^{n+2\beta}e^n} \left\{ \frac{\sigma(s)}{2^{n+\frac{4}{3}z + \frac{2}{3}(z - 1)^{\frac{1}{3}+\beta}}} \frac{1}{n^\frac{4}{3}} + O \left( \frac{1}{n} \right) \right\}
$$

$$
\left[ \frac{2\sigma^2(s) - 2u(s) - s}{2^{n+\frac{4}{3}z + \frac{2}{3}(z - 1)^{\frac{1}{3}+\beta}}} + \frac{(\alpha + 2\beta)(2u(s) + s)}{2^{n+\frac{4}{3}z + \frac{2}{3}(z - 1)^{\frac{1}{3}+\beta}}} \right] \frac{1}{n^\frac{4}{3}} + O \left( \frac{1}{n} \right)
$$

is valid uniformly on compact subsets of $\mathbb{C} \setminus [0, 1]$, where $arg \; z \in (-\pi, \pi)$ and $arg(z - 1) \in (-\pi, \pi)$.

The paper is organized as follows. In Section 2, we state a nonlinear steepest descent analysis of the RH problem for the orthogonal polynomial associated with the weight (1.2). In Section 3, we give the proofs of the theorems 1 and 2.
2 Nonlinear steepest descent analysis of orthogonal polynomials

In this section, we take a nonlinear steepest descent analysis of the Riemann-Hilbert problem for $Y(z; \alpha, \beta, \omega, \mu)$ associated with the weight (1.2) via the Deift-Zhou method; see [7], [8], and [9].

2.1 Riemann-Hilbert problem for orthogonal polynomials

(Y1) $Y(z; \alpha, \beta, \omega, \mu)(Y(z) \text{ for short})$ is analytic in $\mathbb{C} \setminus [0, +\infty)$.

(Y2) $Y(z)$ satisfies the following jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (0, +\infty),$$

where the weight $w(x)$ is given in (1.2).

(Y3) The asymptotic behavior of $Y(z)$ at infinity is

$$Y(z) = \left( I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + O(z^{-3}) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad z \to \infty.$$  \hspace{1cm} (2.2)

(Y4) The behavior of $Y(z)$ at the point 0 is

$$Y(z) = \begin{cases} O\left( \begin{pmatrix} 1 & |z|^\alpha \\ 1 & |z|^\alpha \end{pmatrix} \right), & z \to 0, \quad \alpha < 0, \\ O\left( \begin{pmatrix} 1 & \log |z| \\ 1 & \log |z| \end{pmatrix} \right), & z \to 0, \quad \alpha = 0, \\ O\left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right), & z \to 0, \quad \alpha > 0. \end{cases}$$

(Y5) The behavior of $Y(z)$ at the point $\mu$ is

$$Y(z) = \begin{cases} O\left( \begin{pmatrix} 1 & |z-\mu|^{2\beta} \\ 1 & |z-\mu|^{2\beta} \end{pmatrix} \right), & z \to \mu, \quad -\frac{1}{2} < \beta < 0, \\ O\left( \begin{pmatrix} 1 & \log |z-\mu| \\ 1 & \log |z-\mu| \end{pmatrix} \right), & z \to \mu, \quad \beta = 0, \\ O\left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right), & z \to \mu, \quad \beta > 0. \end{cases}$$

According to [11], the unique solution of the Riemann-Hilbert problem for $Y(z)$ is given by

$$Y(z) = \begin{pmatrix} \pi_n(z) \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi i} \int_0^{+\infty} \frac{\pi_n(x)w(x)}{x-z} \, dx \\ -\gamma_{n-1}^2 \int_0^{+\infty} \frac{\pi_{n-1}(x)w(x)}{x-z} \, dx \end{pmatrix},$$

where the monic orthogonal polynomial $\pi_n(z)$ and leading coefficients $\gamma_n$ are defined as (1.3) and (1.4) respectively.
2.2 The first transformation: $Y \rightarrow T$

Before making the transformation, there is need to introduce some auxiliary functions. The density function respond to the external field $4x$ with $x > 0$ is $\varphi(x) = \frac{2}{\pi} \sqrt{\frac{1-x}{x}}$, $x \in (0, 1)$, see [21, (3.14)] and [23, 26]. Then we define

$$g(z) = \int_0^1 \ln(z - x) \varphi(x) dx,$$

and

$$\phi(z) = 2 \int_1^z \sqrt{\frac{1-x}{x}} dx = 2 \left[ \sqrt{z(z-1)} - \ln \left( \sqrt{z-1} + \sqrt{z} \right) \right]$$

with $\arg z \in (-\pi, \pi)$, $\arg(z-1) \in (-\pi, \pi)$ and $z \in \mathbb{C}\setminus(-\infty, 1]$. The $g(z)$ and $\phi(z)$ are related by the condition $2g(z) + 2\phi(z) - 4z - l = 0$, $z \in \mathbb{C}\setminus(-\infty, 1]$, where $l = -2(1 + 2 \ln 2)$; see [21] for more details.

Now, we introduce the first transformation

$$T(z) = (4n)^{-(n+\frac{a}{2}+\beta)} e^{-\frac{1}{2} n \sigma_3} Y(4nz) e^{n \left( \frac{1}{2} - g(z) \right) \sigma_3 (4n)^{\left( \frac{a}{2} + \beta \right) \sigma_3}}, \quad z \in \mathbb{C}\setminus[0, +\infty), \quad (2.8)$$

where the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $T(z)$ is analytic in $\mathbb{C}\setminus[0, +\infty)$, and the behavior of $T(z)$ at infinity is $T(z) = I + O(1/z)$, $z \to \infty$.

2.3 The second transformation: $T \rightarrow S$

We introduce the transformation as

$$S(z) = \begin{cases} 
T(z), & \text{for } z \text{ outside the lens region}, \\
T(z) \begin{pmatrix} 1 & 0 \\ -z-\alpha(t-z) & 2\beta e^{2n\phi(z)} \end{pmatrix}, & \text{for } z \text{ in the upper lens region}, \\
T(z) \begin{pmatrix} 1 & 0 \\ z-\alpha(t-z) & 2\beta e^{2n\phi(z)} \end{pmatrix}, & \text{for } z \text{ in the lower lens region}.
\end{cases}$$

Then

$$S(z) \text{ is analytic in } \mathbb{C}\setminus\Sigma_S, \text{ where the contours } \Sigma_S \text{ are depicted in Figure 1}.$$
(S2) $S(z)$ satisfies the following jump conditions

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ z^{-\alpha}(t-z)^{-2\beta}e^{2n\phi(z)} & 0 \end{pmatrix}, \quad z \text{ on lens}, \quad (2.10)$$

and for $t < 1$

$$S_+(x) = S_-(x) \begin{cases} \begin{pmatrix} 0 & x^\alpha|x-t|^{2\beta} \\ -x^{-\alpha}|x-t|^{-2\beta} & 0 \end{pmatrix}, & x \in (0, t), \\ \begin{pmatrix} e^{2n\phi(x)} & \omega x^\alpha|x-t|^{2\beta} \\ 0 & e^{2n\phi_-(x)} \end{pmatrix}, & x \in (t, 1), \\ \begin{pmatrix} 1 & \omega x^\alpha|x-t|^{2\beta}e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix}, & x \in (1, +\infty), \end{cases} \quad (2.11)$$

and for $t > 1$

$$S_+(x) = S_-(x) \begin{cases} \begin{pmatrix} 0 & x^\alpha|x-t|^{2\beta} \\ -x^{-\alpha}|x-t|^{-2\beta} & 0 \end{pmatrix}, & x \in (0, 1), \\ \begin{pmatrix} 0 & x^\alpha|x-t|^{2\beta}e^{-2n\phi(x)} \\ -x^{-\alpha}|x-t|^{-2\beta}e^{2n\phi(x)} & 0 \end{pmatrix}, & x \in (1, t), \\ \begin{pmatrix} 1 & \omega x^\alpha|x-t|^{2\beta}e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix}, & x \in (t, +\infty). \end{cases} \quad (2.12)$$

(S3) The behavior of $S(z)$ at infinity is $S(z) = I + O(1/z)$, $z \to \infty$.

(S4) The behaviors of $S(z)$ at $z = 0$ and $z = t$ are the same as those of $T(z)$.

2.4 Global Parametrix

It is seen that the jump matrix for $S$ tends to the identity on $\Sigma_S \setminus [0, t]$ as $n \to \infty$, the limiting Riemann-Hilbert problem $N(z)$ can be as follows:

(N1) $N(z)$ is analytic in $\mathbb{C} \setminus [0, t]$.

(N2) $N(z)$ satisfies the jump condition

$$N_+(x) = N_-(x) \begin{pmatrix} 0 & x^\alpha|x-t|^{2\beta} \\ -x^{-\alpha}|x-t|^{-2\beta} & 0 \end{pmatrix}, \quad x \in (0, t). \quad (2.13)$$

(N3) As $z \to \infty$, $N(z) = I + O(z^{-1})$.

It is seen that a solution of the RH problem for $N(z)$ can be constructed as

$$N(z) = d(\infty)^{\sigma_3}N_0(z)d(z)^{-\sigma_3}, \quad (2.14)$$
where \( N_0(z) = \left( \frac{\rho(z) + \rho^{-1}(z)}{2} - \frac{\rho(z) - \rho^{-1}(z)}{2} \right) \), \( \rho(z) = \left( \frac{z-t}{z} \right)^{1/4} \), and \( d(z) \) is the Szegő function, which is analytic in \( C \setminus [0, t] \) and satisfies \( d_+(x)d_-(x) = x^\alpha |x-t|^{2\beta}, x \in (0, t) \); for Szegő function see [22]. Then \( d(z) \) can be written explicitly as

\[
d(z) = z^{\frac{\beta}{2}} (z-t)^\beta \left( \frac{2z - t + 2\sqrt{z(z-t)}}{t} \right)^{-\left(\frac{\alpha}{2} + \beta\right)}
\]

with \( d(\infty) = (t/4)^{\alpha/2 + \beta} \), where \( \arg z \in (-\pi, \pi) \), \( \arg(z-t) \in (-\pi, \pi) \) and the logarithm function \( \log z \) takes the principle branch such that \( \arg z \in (-\pi, \pi) \); see [17] for a relevant construction.

### 2.5 Local parametrix at \( z = t \)

The construction of a local parametrix \( P^{(0)}(z) \) at a neighborhood \( U(0, r) \) can be refered to [23], where the center of \( U(0, r) \) is 0 with the radius \( r \), here \( r \) is small and positive. Hence, We concentrate on \( z = t \).

Note that \( t = \frac{\mu}{4n} = 1 + \frac{s}{2n^{1/3}} \), we construct a local parametrix \( P^{(1)}(z) \) at a neighborhood \( U(1, r) \), centered at 1 with the radius \( r \). \( P^{(1)}(z) \) is analytic in \( U(1, r) \setminus \Sigma_S \), satisfies the same jump condition as that of \( S(z) \), \( \Sigma_S \cap U(1, r) \), \( P^{(1)}(z) \) satisfies the matching condition \( P^{(1)}(z)N^{-1}(z) = I + O(n^{-1/3}) \) on \( \partial U(1, r) \), and has the same behavior at \( z = t \) as that of \( S(z) \).

In order to construct the \( P^{(1)}(z) \), we need to introduce the RH problem for the \( P_{34} \) equation expressed as follows; see [15], [16], and [24]. Let \( \Phi(\zeta) := \Phi(\zeta; s) \), Then

\[
\Phi(\zeta) \text{ is analytic in } C \setminus \bigcup_{i=1}^4 \Sigma_i; \text{ for the contours } \Sigma_i \text{ cf. Figure 2.}
\]
(Φ2) \( \Phi(\zeta) \) satisfies the jump conditions
\[
\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Sigma_1,
\]
\[
\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{2\beta \pi i} & 1 \end{pmatrix}, \quad \zeta \in \Sigma_2,
\]
\[
\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \zeta \in \Sigma_3,
\]
\[
\Phi_+(\zeta) = \Phi_-(\zeta) \begin{pmatrix} 1 & 0 \\ e^{-2\beta \pi i} & 1 \end{pmatrix}, \quad \zeta \in \Sigma_4.
\]

(Φ3) The behavior of \( \Phi(z) \) at infinity is
\[
\Phi(\zeta) = \left( \begin{array}{ccc}
1 & 0 \\
\im s_1(s) & 1
\end{array} \right) \left[ I + \frac{1}{\zeta} \begin{pmatrix}
m_1(s) & -\im s_1(s) \\
-\im s_1(s) & -m_1(s)
\end{pmatrix} + O(\zeta^{-2}) \right] \times \zeta^{-\frac{1}{2} \sigma_3} I + i \sigma_1 \zeta^{-\theta(\zeta; s) \sigma_3} \sqrt{2},
\]
where \( \theta(\zeta; s) = \frac{2}{3} s^{3/2} + s \zeta^{1/2} \), here arg \( \zeta \in (\pi, \pi) \).

(Φ4) As \( \zeta \to 0 \), the behavior of \( \Phi(z) \) is for \(-\frac{1}{2} < \beta < 0 \)
\[
\Phi(\zeta) = O\left( \begin{array}{cc}
|\zeta|^\beta & |\zeta|^{-\beta} \\
|\zeta|^{-\beta} & |\zeta|^\beta
\end{array} \right)
\]
and for \( \beta \geq 0 \)
\[
\Phi(\zeta) = \begin{cases}
O\left( \begin{array}{cc}
|\zeta|^\beta & |\zeta|^{-\beta} \\
|\zeta|^{-\beta} & |\zeta|^\beta
\end{array} \right), & \zeta \in \Omega_1 \cup \Omega_4, \\
O\left( \begin{array}{cc}
|\zeta|^{-\beta} & |\zeta|^\beta \\
|\zeta|^\beta & |\zeta|^{-\beta}
\end{array} \right), & \zeta \in \Omega_2 \cup \Omega_3.
\end{cases}
\]

It is seen from [15, 24] that for \( \beta > -\frac{1}{2} \), \( \omega \in \mathbb{C} \setminus (-\infty, 0) \), there exits unique solution to the Riemann-Hilbert problem for \( \Phi(\zeta; s) \), and \( \sigma(s) = m_2(s) + \frac{\pi^2}{4} \) and \( u(s) = -\sigma'(s) = -m_2'(s) - \frac{\pi}{2} \) satisfy the equations \([1.5]\) and \([1.6]\) respectively.

Define the conformal mapping as
\[
f(z) = \left( \frac{3}{2} \phi(z) \right)^{2/3}
\]
with \( f(z) = 2^{2/3}(z - 1) + O(z - 1)^2 \), \( z \to 1 \), and
\[
E(z) = N(z) z^{\frac{2}{3} \sigma_3} (z - t)^{\beta \sigma_3} \frac{1}{\sqrt{2}} (I - i \sigma_1) \left[ n^{2/3} (f(z) - f(t)) \right]^{\sigma_3/4}.
\]

Then, we introduce the following local parametrix
\[
P^{(1)}(z) = E(z) \Phi \left( n^{2/3} (f(z) - f(t)) ; n^{2/3} f(t) \right) e^{n \phi(z) \sigma_3 (z - t)^{\sigma_3}}.
\]

It is easily seen that the matching condition \( P^{(1)}(z) N^{-1}(z) = I + O \left( n^{-1/3} \right) \) is fulfilled.
2.6 The final transformation: $S \to R$

![Figure 3: The jump contours $\Sigma_R$ for the RH problem $R$.](image)

The final transformation is defined as follows

$$R(z) = \begin{cases} 
    S(z)N^{-1}(z), & z \in \mathbb{C} \setminus \{U(0, r) \cup U(1, r) \cup \Sigma_S\}, \\
    S(z)(P^{(0)})^{-1}(z), & z \in U(0, r) \setminus \Sigma_S, \\
    S(z)(P^{(1)})^{-1}(z), & z \in U(1, r) \setminus \Sigma_S.
\end{cases} \quad (2.26)$$

Then, $R(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$, satisfies the jump conditions $R_+(z) = R_-(z)J_R(z)$, $z \in \Sigma_R$, and tends to $I$ as $z \to \infty$; for jump contours $\Sigma_R$ see Figure 3. Note that as $n \to \infty$, $J_R(z) = 1 + O(n^{-1/3})$ for $z \in \partial U(1, r)$. Therefore, as $n \to \infty$, $R(z) = I + O(n^{-1/3})$, where it is uniform for $z$ in whole complex plane.

3 Proof of the main theorems

In this section, we prove the theorem 1 and theorem 1. A similar derivation can be found in [25], [24] and [17].

3.1 Proof of Theorem 1

It is seen, from (2.2) and (2.5), that the leading coefficients of orthogonal polynomials are given by

$$\gamma_{n-1}^2 = -\frac{1}{2\pi i} (Y_1)_{21}, \quad \gamma_n^2 = -\frac{1}{2\pi i} (Y_1)_{12}^{-1}, \quad (3.1)$$

see [9] (3.12) and [2] (5.7)]. Tracing back the transformations (2.8) (2.9) and (2.26) yields to

$$Y(4nz) = (4n)^{\frac{\alpha}{2} + \beta} e_n^{\frac{4n}{l} \sigma_3} R(z) N(z) e^{n g(z) \sigma_3 - \frac{4n}{l} \sigma_3} (4n)^{-\frac{\alpha}{2} + \beta} \sigma_3. \quad (3.2)$$

Denote $N(z)$ and $R(z)$ as

$$N(z) = I + \frac{N_1}{z} + \frac{N_2}{z^2} + O(1/z^3), \quad z \to \infty, \quad (3.3)$$

$$R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + O(1/z^3), \quad z \to \infty. \quad (3.4)$$
Note the fact that 

\[ e^{n g(z) \sigma_1 z^{-n \sigma_3}} = I + G_2/z^2 + O(z^{-3}) \quad \text{as} \quad z \to \infty, \]

with \( G_2 = -\frac{n}{2} \int_{-1}^{1} x^2 \varphi(x) dx \sigma_3 \), we from (3.2) have

\[
Y_1 = 4n(4n)^{(n+\frac{2}{n}+\beta)\sigma} e^{\frac{1}{4} nl\sigma_3} (R_1 + N_1) e^{-\frac{1}{4} nl\sigma_3} (4n)^{-\frac{2}{n}+\beta)\sigma_3}, \quad (3.5)
\]

\[
Y_2 = 16n^2(4n)^{(n+\frac{2}{n}+\beta)\sigma} e^{\frac{1}{4} nl\sigma_3} (R_2 + G_2 + N_2 + R_1 N_1) e^{-\frac{1}{4} nl\sigma_3} (4n)^{-\frac{2}{n}+\beta)\sigma_3}. \quad (3.6)
\]

Next, we expand \( N(z) \) and \( R(z) \) as \( z \to \infty \). From (2.14), we have

\[
N(z) = I + \frac{N_1}{z} + \frac{N_2}{z^2} + O(z^{-3}), \quad (3.7)
\]

with

\[
N_1 = \left( \frac{t}{4} \right)^{(\frac{2}{n}+\beta)\sigma_3} \left[ -t \sigma_2 + \frac{(2\beta - \alpha)t \sigma_3}{4} \right] \left( \frac{t}{4} \right)^{-\frac{1}{4}(\frac{2}{n}+\beta)\sigma_3}, \quad (3.8)
\]

\[
N_2 = \left( \frac{t}{4} \right)^{(\frac{2}{n}+\beta)\sigma_3} \left[ \frac{i(\alpha - 2\beta)^2 \sigma_1}{16} - \frac{t^2 \sigma_2}{8} + \frac{(10\beta - 3\alpha)t^2 \sigma_3}{32} \right.
\]

\[
\left. + \frac{(\alpha^2 - 4\alpha\beta + 4\beta^2 + 1)t^2 I}{32} \right] \left( \frac{t}{4} \right)^{-\frac{1}{4}(\frac{2}{n}+\beta)\sigma_3}, \quad (3.9)
\]

where the Pauli matrices \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are given in (2.8), and we have made use of the facts that as \( z \to \infty \)

\[
d(z) = \left( \frac{t}{4} \right)^{\frac{2}{n}+\beta} \left[ 1 + \frac{(\alpha - 2\beta) t}{4z} + \frac{(\alpha^2 - 4\alpha\beta + 4\beta^2 + 3\alpha - 10\beta) t^2}{32z^2} \right] + O(z^{-3}), \]

\[
N_0(z) = I + \frac{\alpha}{4z} \sigma_2 + \frac{\beta}{32} I + O(z^{-3}).
\]

Now we are in position to estimate the behavior of \( R(z) \) as \( z \to \infty \). In view of (2.20), (2.24) and (2.25), we get the following expression of the jump for \( R(z) \) on \( \partial U(1, r) \) as \( n \to \infty \)

\[
J_R(z) = N(z) z^{\frac{2}{n} \sigma_3} (z - t)^{\beta \sigma_3}
\]

\[
\times \left[ I + \frac{(F + \varphi_1) \sigma_3 - i \varphi_1 \sigma_1}{n^{1/3}} + \frac{-\tau_3(z; t) \sigma_2 + \tau_4(z; t) I}{n^{2/3}} + O(n^{-1}) \right]
\]

\[
\times (z - t)^{-\beta \sigma_3} z^{-\frac{2}{n} \sigma_3} N^{-1}(z), \quad (3.10)
\]

where

\[
F(z; t, n) = \frac{s^2}{4(f(z) - f(t))^{1/2}}, \quad \varphi_1(z; t) = \frac{m_2(s)}{2(f(z) - f(t))^{1/2}}, \quad (3.11)
\]

\[
\varphi_2(z; t) = -\frac{m^2(s) + m^2(s)}{2(f(z) - f(t))^{1/2}}, \quad \tau_3(z; t) = F \varphi_1 - \varphi_2, \quad \tau_4(z; t) = F \varphi_1 + \frac{F^2}{2}, \quad (3.12)
\]

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Note the fact that $N_0(z)$ can be written in the following form $N_0(z) = \frac{I-i\sigma_1}{\sqrt{2}}\rho(z)^{-\sigma_3}\frac{I+i\sigma_1}{\sqrt{2}}$, see [25 (106)]. We get on $\partial U(1, r)$

$$J_R(z) = d(\infty)^{\sigma_3} \frac{I-i\sigma_1}{\sqrt{2}} \left[ I + \frac{\Theta_1(z)}{n^{1/3}} + \frac{\Theta_2(z)}{n^{2/3}} + O(n^{-1}) \right] \frac{I+i\sigma_1}{\sqrt{2}} d(\infty)^{-\sigma_3}, \ (3.13)$$

where

$$\Theta_1(z) = \tau_1(z;t)\sigma_+ + \tau_2(z;t)\sigma_- + i\varphi_1 \left[ \frac{i\xi^2(z;t) - i\xi^{-2}(z;t)}{2}\sigma_3 - \frac{\xi^2(z;t) + \xi^{-2}(z;t) - 2}{2}\rho(z)^{-\sigma_3}\sigma_1\rho(z)^{\sigma_3} \right],$$

$$\Theta_2(z) = \tau_3(z;t)\sigma_3 + \tau_4(z;t)I + \frac{\xi^2(z;t) + \xi^{-2}(z;t) - 2}{2}\tau_3(z;t)\sigma_3 + \frac{i\xi^2(z;t) - i\xi^{-2}(z;t)}{2}\tau_3(z;t)\rho(z)^{-\sigma_3}\sigma_1\rho(z)^{\sigma_3},$$

with $\sigma_+ = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$, $\sigma_- = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ and

$$\xi(z;t) = z^2(z-t)\frac{d(z)}{I}, \quad \tau_1(z;t) = -\rho^{-2}(z)(F + 2\varphi_1)i, \quad \tau_2(z;t) = \rho^2(z)Fi. \ (3.14)$$

We write

$$R(z) = d(\infty)^{\sigma_3} \frac{I-i\sigma_1}{\sqrt{2}} \left[ I + \frac{R^{(1)}(z)}{n^{1/3}} + \frac{R^{(2)}(z)}{n^{2/3}} + O(n^{-1}) \right] \frac{I+i\sigma_1}{\sqrt{2}} d(\infty)^{-\sigma_3}. \ (3.15)$$

Then, $R^{(1)}(z)$ and $R^{(2)}(z)$ satisfy the following scalar Riemann-Hilbert problems

$$R^{(1)}_+(z) - R^{(1)}_-(z) = \Theta_1(z), \quad R^{(2)}_+(z) - R^{(2)}_-(z) = \Theta_2(z) + R^{(1)}_-(z)\Theta_1(z). \ (3.16)$$

Note that (3.14) and (2.15), one can obtain

$$\xi(z;t) = 1 + \frac{(\alpha + 2\beta)}{\sqrt{t}}\sqrt{z-t} + \frac{(\alpha + 2\beta)^2}{2t}(z-t) + O((z-t)^{3/2}), \quad z \to t,$n

$$\xi^2(z;t) + \xi^{-2}(z;t) - 2 = O(z-t), \quad z \to t$$

$$\xi^2(z;t) - \xi^{-2}(z;t) = \frac{4(\alpha + 2\beta)}{\sqrt{t}}\sqrt{z-t} + O((z-t)^{3/2}), \quad z \to t.$$n

It is seen, from (3.12) and (3.14), that $\tau_2(z;t)$ is analytic at $z = t$, and $\tau_1(z;t), \tau_3(z;t)$ and $\tau_4(z;t)$ have a simple pole at $z = t$. Let $k_2(s) = \lim_{z \to t} \tau_2(z;t)$, $k_3(s) = \text{Res}_{z=t} \tau_3(z;t)$ with $j = 1, 3, 4$. Hence, there have

$$R^{(1)}(z) = \begin{cases} \frac{k_1(s)}{z-t}\sigma_+ - \Theta_1(z), & z \in U(1, r), \\ \frac{k_3(s)}{z-t}\sigma_+, & z \not\in U(1, r), \end{cases} \quad \text{(3.17)}$$

$$R^{(2)}(z) = \frac{k_3(s)\sigma_3 + k_4(s)I + k_5(s)\sigma_+ - k_1(s)k_2(s)\sigma_-\sigma_+}{z-t}, \quad z \not\in U(1, r). \quad \text{(3.18)}$$
where $k_5(s)$ is related to the residue of the $\xi(z; t)$ in $R^{(1)}(z)\Theta_1(z)$ and $\Theta_2(z)$. Employing (3.12) and (3.14), we have

\[
\begin{align*}
  k_1(s) &= -\frac{is}{2^{1/3}} + O(n^{-\frac{2}{3}}), \quad k_2(s) = \frac{is^2}{2^{2/3}} + O(n^{-\frac{2}{3}}), \\
  k_3(s) &= \frac{4s^2\sigma(s) - s^2\sigma(s) - 4u(s) - 2s}{2^{1/3}} + O(n^{-\frac{2}{3}}), \\
  k_4(s) &= \frac{s^2\sigma(s)}{2^{1/3}} + O\left(n^{-\frac{2}{3}}\right), \\
  k_5(s) &= -2^{-\frac{2}{3}}i(\alpha + 2\beta)\left(u(s) + \frac{s}{2}\right) + O\left(n^{-\frac{2}{3}}\right).
\end{align*}
\] (3.19)

Expanding $\frac{1}{z-t}$ for $z$ at infinity, together with (3.17), (3.18), $R_j$ in (3.4) and (3.15), we get for $j = 1, 2$,

\[
R_j = d(\infty)^{x_\alpha} \left[ \frac{k_1(s)(i\sigma_3 + \sigma_1)}{2n^{1/3}} + \frac{k_5(s)(i\sigma_3 + \sigma_1) - (k_1(s)k_2(s) + 2k_3(s))\sigma_2}{2n^{2/3}} + O(n^{-1}) \right] d(\infty)^{-x_\alpha}. \tag{3.20}
\]

From $N_1$ in (3.7), $R_1$ in (3.20), and $t = 1 + \frac{s}{2^{1/3}n^{2/3}}$, we have

\[
\begin{align*}
  \gamma_{n-1}^2 &= -\frac{1}{2\pi i(4n)^{2n+\alpha+2\beta-1}e^{it}} \left( \frac{4}{t} \right)^{\alpha+2\beta} \\
  &\times \left[ -\frac{it}{4} + \frac{k_1(s)}{2n^{1/3}} + \frac{-i(2k_3(s) + k_1(s)k_2(s)) + k_5(s)}{2n^{2/3}} + O(n^{-1}) \right], \tag{3.21}
\end{align*}
\]

\[
\begin{align*}
  \gamma_{n}^2 &= -\frac{1}{2\pi i(4n)^{2n+\alpha+2\beta-1}e^{it}} \left( \frac{4}{t} \right)^{\alpha+2\beta} \\
  &\times \left[ \frac{it}{4} + \frac{k_1(s)}{2n^{1/3}} + \frac{i(2k_3(s) + k_1(s)k_2(s)) + k_5(s)}{2n^{2/3}} + O(n^{-1}) \right]. \tag{3.22}
\end{align*}
\]

Hence, from (3.19), we can complete the proof of (1.7) and (1.8) given in Theorem 1.

The recurrence coefficients $a_n$ as seen in (1.4) can be expressed as

\[
a_n = (Y_1)_{11} + \frac{(Y_2)_{12}}{(Y_1)_{12}}, \quad b_n = (Y_1)_{12}(Y_1)_{21}, \tag{3.23}
\]

see [9] (3.12)], where $Y_1$ and $Y_2$ are defined as (2.2). In view of (3.5) and (3.6), there have

\[
a_n = \frac{1}{4n} \left[ (N_1)_{11} + (R_1)_{11} + \frac{(N_2)_{12} + (R_2)_{12} + (R_1)_{11}(N_1)_{12} + (R_1)_{12}(N_1)_{22}}{(N_1)_{12} + (R_1)_{12}}. \right. \tag{3.24}
\]

In view of (3.7) and (3.20), (3.19) and $t = 1 + \frac{s}{2^{1/3}n^{2/3}}$, the equation (3.24) is simplified to

\[
\frac{a_n}{4n} = \frac{t}{2} + \frac{k_1(s)^2 + k_1(s)k_2(s) + 2k_3(s)}{n^{2/3}} + O(n^{-1}). \tag{3.25}
\]
Thus, (1.9) in theorem 1 follows from (3.25) and (3.19). It follows from (3.5) that

\[ b_n^2 = 16n^2 \left( (N_{12})_1 + (N_{12})_2 + (N_{12})_1(R_{12})_2 + (R_{12})_1(R_{12})_1 \right). \]  

(3.26)

From (3.7), (3.20), (3.19), and \( t = 1 + \frac{s}{2n^{2/3}} \), we have

\[ b_n = 4n \left[ \frac{t^2}{16} + \frac{(2k_3(s) + k_1(s)k_2(s))t + k_2^2(s)}{4n^{2/3}} + O(n^{-1}) \right]^{1/2}. \]  

(3.27)

Note that (3.19), the proof of (??) in theorem 1 is completed.

### 3.2 Proof of Theorem 2

From (3.2) and (2.5), we have

\[ \pi_n(4nz) = (4n)^n e^{\eta(s)} (R_{12}N_{11} + R_{12}N_{21}). \]  

(3.28)

When \( z \not\in \overline{U(1, r)} \), according to (3.15), (3.17) and (3.18), there have

\[ R(z) = d(\infty)^{\sigma_3} \frac{I - i\sigma_1}{\sqrt{2}} \left[ I + \frac{k_1(s)\sigma_+}{(z - t)n^{1/3}} \right. \]
\[ + \left. \frac{(k_3(s)\sigma_3 + k_4(s))I + k_5(s)\sigma_+ - k_1(s)k_2(s)\sigma_-\sigma_+}{(z - t)n^{2/3}} \right] + O(n^{-1}) \]
\[ \frac{I + i\sigma_1}{\sqrt{2}} d(\infty)^{-\sigma_3}. \]

Note that (2.14) and (2.6), we can complete the proof of the theorem 2.

### Acknowledgements

This work was supported by the Science Foundation of Education Department of Jiangxi Province under grant number GJJ170937 and the National Natural Science Foundation of China under grant number 11801376.

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