Crosscap Number and the Partial Order on Two-Bridge Knots

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Abstract

We consider the relationship between the crosscap number $\gamma$ of knots and the partial order on the set of all prime knots, which is defined as follows. For two knots $K$ and $J$, we say $K \geq J$ if there exists an epimorphism $f : \pi_1(S^3 - K) \to \pi_1(S^3 - J)$. We prove that if $K$ and $J$ are 2-bridge knots and $K \succ J$, then $\gamma(K) \geq 3\gamma(J) - 4$. We show that the inequality is sharp and that it does not hold in general for all prime knots. A similar result relating the genera of two knots has been proven by Suzuki and Tran. Namely, if $K$ and $J$ are 2-bridge knots and $K \succ J$, then $g(K) \geq 3g(J) - 1$, where $g(K)$ denotes the genus of the knot $K$.

Dedicated to our friend and colleague, Mark Kidwell, 1948–2019.

1 Introduction

The crosscap number, or non-orientable genus, of a knot $K$ in the 3-sphere, denoted here as $\gamma(K)$, was introduced by Clark in [4]. It is defined as the smallest first Betti number of any embedded, compact, connected, nonorientable surface in $S^3$ that spans $K$. For convenience, the crosscap number of the unknot is defined to be 0. Clark observed that a knot has crosscap number equal to 1 if and only if it bounds a Möbius band and hence is a $(2,2k+1)$-cable of some knot (the centerline of the Möbius band). Thus the $(2,2k+1)$-torus knot has crosscap number equal to 1 for all $k$. 

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The crosscap number is not additive with respect to connected sum, however, Clark showed that for knots $K_1$ and $K_2$

$$\gamma(K_1) + \gamma(K_2) - 1 \leq \gamma(K_1 \sharp K_2) \leq \gamma(K_1) + \gamma(K_2).$$

(1)

Clark also showed, that for any knot $K$

$$\gamma(K) \leq 2g(K) + 1,$$

(2)

where $g(K)$ is the genus of $K$. Clark asked if equality ever occurs in (2) and in [16], Murakami and Yasuhara showed that this is the case for the knot $7_4$. This required them to to compute $\gamma(7_4) = 3$, a computation that has now been made much easier by the work of Bessho [3] and Hirasawa and Teragaito [9], who, taken together, show how to compute the crosscap number of any 2-bridge knot. The crosscap number has also been computed for any torus knot [21], any alternating knot [14], or any pretzel knot, [11]. For alternating links, $K$, lower and upper bounds for $\gamma(K)$ that depend on the Jones polynomial of $K$ are given in [12]. For alternating knots with 12 or fewer crossings, these bounds are often sufficient to determine $\gamma(K)$.

Building on the results of [9], we prove the following theorem.

**Theorem 1.** If $K$ and $J$ are 2-bridge knots and $K > J$, then

$$\gamma(K) \geq 3\gamma(J) - 4.$$

Here, by $K \geq J$, we mean that there exists an epimorphism $f : \pi_1(S^3 - K) \to \pi_1(S^3 - J)$. This defines a partial order on the set of all prime knots, [19]. A similar result relating the genera of the two knots has been proven by Suzuki and Tran, [20]. Namely, if $K$ and $J$ are 2-bridge knots and $K > J$, then $g(K) \geq 3g(J) - 1$, where $g(K)$ denotes the genus of the knot $K$.

Note that if the hypothesis of Theorem 1 is relaxed to $K \geq J$, then the result is not true. Because $K \geq K$ for every knot $K$, if the Theorem were true, it would imply that $\gamma(K) \geq 3\gamma(K) - 4$, or equivalently, that $2 \geq \gamma(K)$ for every knot $K$, which is false. Moreover, in the example immediately following the proof of Theorem 1 we exhibit infinitely many pairs of 2-bridge knots $K$ and $J$ with $K > J$ and $\gamma(K) = 3\gamma(J) - 4$. Hence, the inequality given in Theorem 1 is sharp. We also point out that if the assumption that $K$ and $J$ are 2-bridge knots is dropped, the result is no longer true. In [10], the authors show that 11a239 (which is not a 2-bridge knot) is greater than the knot 6_3, which is 2-bridge. Moreover, the crosscap number of both knots is 3, contradicting the conclusion of Theorem 1.

The crosscap number of 11a239 can be found in KnotInfo [15].

It is not hard to see that the $(2, 2k + 1)$-torus knot is greater than or equal to the $(2, 2j + 1)$-torus knot whenever $2j + 1$ divides $2k + 1$. To do this, consider the “standard” diagram of the $(2, 2k + 1)$-torus knot as the closure of a 2-string braid with axis $A$. This diagram is rotationally symmetric around $A$, and the quotient space obtained from the exterior of the knot by identifying points $x$ and $y$ if $y$ is obtained by rotating $x$ through $2\pi/r$ radians around $A$, where $r = (2k + 1)/(2j + 1)$, is the exterior of the $(2, 2j + 1)$-torus knot. This quotient map induces an epimorphism on the fundamental groups associated to the two knots. Note that Theorem 1 now asserts that $1 \geq 3 \cdot 1 - 4$, which is true.
In Section 2, we summarize the results of [9] that will be needed in this paper. Their analysis depends heavily on representing 2-bridge knots by continued fractions. In Section 3, we recall the definition of the depth of a rational number defined by [9] and show how to use it together with data associated to the even continued fraction expansion of \( p/q \) to prove Theorem 1.

2 Continued Fractions, 2-Bridge Knots, and Crosscap Number

There is a well-established connection between 2-bridge knots, 4-plat diagrams, and continued fractions. We recall the bare rudiments of this theory and refer the reader to [5] or [18] for more information. Every 2-bridge knot or link has a 4-plat diagram as in Figure 1. Here a box labeled \( a_i \) denotes \( |a_i| \) half-twists between the two strands entering and exiting the box, with the further convention that the twists are right-handed if \( i \) is odd and left-handed if \( i \) is even. Note that a negative number of right-handed twists is equal to the opposite number of left-handed twists and vice-versa. In particular, if \( a_i > 0 \) for all \( i \), or \( a_i < 0 \) for all \( i \), then the diagram in Figure 1 is alternating and minimal. That is, it represents a knot with crossing number equal to \( |a_1| + |a_2| + \cdots + |a_k| \). See [5] for a more extensive description of this topic. The reader is also referred to the paper [13] by Kauffman and Lambropoulou. While not one of the (historically) first papers on the subject, their paper is noteworthy for its completeness, approach, historical description of the topic, and extensive bibliography.

The sequence \( a_1, a_2, \ldots, a_k \) gives rise to the reduced fraction \( \frac{p}{q} \), via the continued fraction,

\[
\frac{p}{q} = [a_1, a_2, \ldots, a_k] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_k}}}}.
\]

This associates to each 4-plat diagram of a 2-bridge knot \( K \), a rational number \( p/q \), with \( 0 < p < q \), \( q \) odd, and \( \gcd(p, q) = 1 \). (In the case of a 2-bridge link, \( p \) will be odd and \( q \) even.) Clearly, we can work backwards from any such fraction to obtain a 2-bridge knot, which we denote \( K_{p/q} \). Moreover, the fractions \( p/q \) and \( p'/q' \) represent ambient isotopic 2-bridge knots if and only if \( q' = q \) and either \( p' \equiv p \pmod{q} \) or \( q'p \equiv \pm 1 \pmod{q} \). Because we will also consider a 2-bridge knot \( K_{p/q} \) and its mirror image, \( K_{-p/q} \), as equivalent, the equivalence on fractions becomes \( q' = q \) and either \( p' \equiv \pm p \pmod{q} \) or \( q'p \equiv \pm 1 \pmod{q} \). Note that \( \gamma(K_{p/q}) = \gamma(K_{-p/q}) \).

In Figure 2, we see that the knot depicted in Figure 1 can be viewed as the boundary of a surface obtained by plumbing together \( k \) twisted bands, where the \( i \)-th band has \( a_i \) half-twists. Hence, each band is either an annulus, if \( a_i \) is even, or a Möbius band, if \( a_i \) is odd, and the surface is orientable if and only if each \( a_i \) is even. The surface strongly deformation retracts to the cores of the bands and hence has first Betti number equal to \( k \). If \( a_i \) is even for all \( i \), then the plumbed surface is orientable. However, in this case, Clark [4] credits Mark Kidwell for the observation that introducing a single
crossing to the knot diagram by means of a Reidemeister Type I move, corresponds to adding a Möbius band to the Seifert surface, thus producing a nonorientable spanning surface for the same knot. This is illustrated in the last frame of Figure 2.

These surfaces are the natural starting point in trying to find a surface with minimal genus (in the orientable case) or minimal first Betti number (in the nonorientable case). As it happens, in both cases, we need look no further. Gabai proved that the minimal genus orientable surface for $K_{p/q}$ is the surface that arises from the continued fraction expansion for $p/q$ which contains all even and nonzero entries [6]. Similarly, Bessho [3] proved that the surface given by a shortest–length continued fraction expansion containing an odd integer, realizes the crosscap number. Moreover, if a shortest–length continued fraction produces an orientable surface, then adding a Möbius band à la Kidwell, produces a surface realizing the crosscap number.

The connection between these surfaces associated to $K_{p/q}$ and the length of continued fraction expansions of $p/q$ motivates deeper understanding of continued fraction expansions. Given a fraction $p/q$, there are infinitely many ways to express it as a continued fraction. In an attempt to use only even integers (as mentioned in the result of Gabai), one can obtain the following result (see Lemma 2.1 in [7]).

Lemma 1. Suppose $0 < p < q$, $\gcd(p,q) = 1$, $p$ is even and $q$ is odd. Then there exists a unique continued fraction $p/q = [a_1, a_2, \ldots, a_{2n}]$ where $a_i \neq 0$ and $a_i$ is even for all $i$. We call this the even continued fraction for $p/q$. 
Finding this even continued fraction expansion for a given $p/q$ is a straightforward exercise that involves the Euclidean Algorithm (again, see [7] for details). By comparison, finding a shortest-length continued fraction expansion associated to $p/q$ is more difficult and such an expansion is not necessarily unique. Hirasawa and Teragaito [9] prove that a continued fraction expansion can be shortened if and only if the expansion has one or more of the following three characteristics:

- Some $a_i$ in the continued fraction expansion is 0.
- Some $a_i$ in the continued fraction expansion is $\pm 1$.
- There exists a substring of $a_i$’s of the form $(2, -2)$ or $(2, 3, -3, 2)$, and so on, where the entries in the substring alternate in sign, the substring begins and ends with elements of magnitude 2, and all other elements of the substring, of which there can be any number, including none, have magnitude 3. (We warn the reader that this notation does not agree with that used by Hirasawa and Teragaito, who use a subtractive form of the continued fraction expression, rather than the additive form that we are using.)

Hirasawa and Teragaito define three shortening moves that can be applied to a continued fraction expansion in each of the three cases above. Of the three shortening moves, we will only need the exact description of the first move. The following Lemma is proven in [9].

**Lemma 2.** A zero can be removed from a continued fraction expansion as follows,

$$p/q = r + [a_1, a_2, \ldots, a_{i-1}, 0, a_{i+1}, a_{i+2}, \ldots, a_n] = r + [a_1, a_2, \ldots, a_{i-1} + a_{i+1}, a_{i+2}, \ldots, a_n],$$

for all $1 < i < n$. (Here $r$ is any integer. We typically omit $r$ if $r = 0$.)

The main result of [9] is the following algorithm for computing $\gamma(K_{p/q})$, which rests on the above discussion.

**Algorithm 1** (Hirasawa–Teragaito). To compute the crosscap number $\gamma(K)$ of the 2-bridge knot $K$:

1. Choose some fraction $p/q$ that represents $K$.
2. Choose some continued fraction expansion $p/q = r + [a_1, a_2, \ldots, a_n]$ of length $n$ for $p/q$. This expansion can be shortened if and only if the expansion has one or more of the three characteristics mentioned above.
3. Through repeated application of the three shortening moves, transform the given continued fraction expansion into a shortest length continued fraction expansion. (We continue here to denote a shortest continued fraction expansion as $r + [a_1, a_2, \ldots, a_n]$.)
4. Once a shortest length continued fraction expansion for $p/q$ has been found, if some $a_i$ is odd or equal to $\pm 2$, then $\gamma(K_{p/q}) = n$, otherwise $\gamma(K_{p/q}) = n + 1$.

As an example, consider the knot $7_4$ which is the 2-bridge knot $K_{4/15}$. Note that $4/15 = [4, -4]$. Because none of the entries are 0, $\pm 1$, or $\pm 2$, this continued fraction cannot be made shorter using the three shortening moves and hence is shortest. It follows that $\gamma(K_{4/15}) = 3$. A nonorientable spanning surface with minimal first Betti number is shown in Figure 2. We invite the reader to compare this with
the much more difficult computation given in [16] which required proving that the crosscap number is not 2 by first showing that the Goeritz matrix of any knot with crosscap number equal to 2 has a certain form, and then showing that this form cannot be achieved by the knot 7_4.

Computing the crosscap number as described in Algorithm 1 has the slight drawback that no direct algorithm is given for finding a shortest–length continued fraction, although it is not hard to imagine an algorithmic scheme for the application of the three shortening moves. In this paper, we discuss an alternative approach to computing the crosscap number using the unique even continued fraction expansion of p/q. This approach avoids the need to find a shortest-length representative. As a result, we are able to give, later in the paper, Algorithm 2 for computing crosscap number that we believe is more direct, simpler, and faster than Algorithm 1. Moreover, Algorithm 2 also facilitates the comparison of the crosscap numbers of 2-bridge knots that are related by the partial order on all prime knots. While somewhat off the main course of this paper, and not needed for the proof of Theorem 1, we describe in Section 3 a very direct way to find a shortest continued fraction expansion of 7_4. However, computer experimentation suggests that this approach is no faster than Algorithm 2.

As already mentioned in the Introduction, we may define a partial order on the set of prime knots by declaring \( K \geq J \) if there is an epimorphism \( \phi : \pi_1(S^3 - K) \to \pi_1(S^3 - J) \). Note that every knot is greater than or equal to both itself and the unknot. In [17], Ohtsuki, Riley and Sakuma systematically construct epimorphisms between 2-bridge knots. In particular, they show that if \( J \) corresponds to the fraction \( p/q = [a] \), and \( K \) corresponds to the fraction \( p'/q' = [b] \), where the vector \( b \) has the form

\[
b = (\epsilon_1 a, 2c_1, \epsilon_2 a^{-1}, 2c_2, \epsilon_3 a, 2c_3, \ldots, \epsilon_n a^{(-1)^{n-1}}),
\]

with \( \epsilon_i \in \{-1, 1\} \), \( c_i \in \mathbb{Z} \), and furthermore, if \( c_i = 0 \) then \( \epsilon_i = \epsilon_{i+1} \), then there is a branch fold map from the complement of \( K \) onto the complement of \( J \). This map induces an epimorphism from the group of \( K \) to the group of \( J \). Here we use \((a, b)\) to represent the concatenation of the vectors \( a \) and \( b \), denote the reverse of the vector \( a \) as \( a^{-1} \), and denote by \( 2c_i \) the vector \((2c_i)\) of length one. Ohtsuki, Riley and Sakuma’s result motivates the following definition.

**Definition 1.** Let \( a, b \) be vectors. We say that \( b \) admits a parsing with respect to \( a \) if

\[
b = (\epsilon_1 a, 2c_1, \epsilon_2 a^{-1}, 2c_2, \epsilon_3 a, 2c_3, \ldots, \epsilon_n a^{(-1)^{n-1}}),
\]

where \( \epsilon_i \in \{-1, 1\} \) for all \( i \), and each \( c_i \) is an integer. Furthermore, if \( c_i = 0 \) then \( \epsilon_i = \epsilon_{i+1} \).

It is easy to prove that if \( b \) parses with respect to \( a \), then this parsing is unique. If \( b \) parses with respect to \( a \), then we call the vectors, \( 2c_i \), a-**connectors**, or more simply, **connectors**. The a-connectors separate the a-tiles: \( \epsilon_1 a, \epsilon_2 a^{-1}, \ldots, \epsilon_n a^{(-1)^{n-1}} \). The construction given by Ohtsuki, Riley, and Sakuma has been improved to the following theorem, which forms the basis of our overall approach to the proof of Theorem 1.

**Theorem 2** (Ohtsuki-Riley-Sakuma [17], Agol [1], Aimi-Lee-Sakai-Sakuma [2]). Let \( K \) and \( J \) be 2-bridge knots, corresponding to the fractions \( p'/q' \) and \( p/q \), respectively, where \( p/q = [a] \). Then \( K \geq J \) if and only if \( p'/q' \) can be expressed as \( p'/q' = [b] \), where \( b \) parses with respect to \( a \) and contains an odd number of a-tiles.

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3 Depth

There is a beautiful correspondence between continued fraction expansions of \( p/q \) and edge paths from \( 1/0 \) to \( p/q \) in the Farey Graph shown in Figure 3, where the graph is embedded in the Poincaré Disk (together with the circle at infinity). Note that two fractions, \( a/b \) and \( c/d \), in \( \mathbb{Q} \cup \{-1, 1\} \) are connected by an edge in the Farey Graph if and only if \( ad - bc = \pm 1 \). Recall that the mediant, or child, of the two fractions \( a/b \) and \( c/d \) is defined to be \( \frac{a+c}{b+d} \) and, therefore, it is connected by an edge to each of its parents, \( a/b \) and \( c/d \). The entire graph can be generated recursively by taking mediants, starting from \( 1/0 \) and \( 0/1 \). (To generate the negative rational numbers, we start with \((-1)/0\) and \(0/1\).)

Given a continued fraction expansion \( p/q = r + [a_1, a_2, \ldots, a_n] \) let \( p_0/q_0 = r \) and, for \( i > 0 \), define \( p_i/q_i = r + [a_1, a_2, \ldots, a_i] \). This defines the edge path

\[
\frac{1}{0} \rightarrow \frac{p_0}{q_0} \rightarrow \frac{p_1}{q_1} \rightarrow \cdots \rightarrow \frac{p_n}{q_n},
\]

which always proceeds, repeatedly, from parent to child. At the vertex \( p_{i-1}/q_{i-1} \), the path turns through \( |a_i| \) triangles on the left side of the path if \((-1)^i a_i < 0 \) and on the right side if \((-1)^i a_i > 0 \). (If \( a_i = 0 \), the path goes backwards along the most recent edge.) Because of this, the entries \( a_i \) are sometimes called the turning numbers. For example, the path from \( 1/0 \) to \( 3/4 \) corresponding to the expansion \( 3/4 = [2, -2, 2] \) is highlighted in Figure 3. Consecutive edges are always separated by
2 triangles on the left because the turning numbers alternate in sign. In general, the turning switches from one side of the path to the other when \(a_i\) and \(a_{i+1}\) have the same sign, and stays on the same side of the path when \(a_i\) and \(a_{i+1}\) have opposite signs. Perhaps the best reference for this material is Hatcher’s excellent introduction to number theory from a geometric point of view [8].

Closely related to the length of the shortest path from \(1/0\) to the fraction \(p/q\) is the depth of \(p/q\), which is defined in [9] as follows.

**Definition 2.** The depth of \(p/q \in \mathbb{Q} \cup \{1/0\}\), denoted as \(d(p/q)\), is zero if \(p/q \in \mathbb{Z} \cup \{1/0\}\). Otherwise, the depth of \(p/q\) is one more than the minimum of the depths of its two parents.

It follows directly from the definition that the depth of a child minus the depth of its parent is always zero or one. The following fact regarding depth is proven in [9].

**Lemma 3.** Suppose \(p/q = r + [a_1, a_2, \ldots, a_n]\) defines the edge path

\[
\frac{1}{0} \rightarrow \frac{p_0}{q_0} \rightarrow \frac{p_1}{q_1} \rightarrow \cdots \rightarrow \frac{p_n}{q_n},
\]

from \(1/0\) to \(p/q\). The path is a shortest path from \(1/0\) to \(p/q\) if and only if \(d(p_i/q_i) = i\) for all \(i \geq 0\).

Given a vector \(a = (a_1, a_2, \ldots, a_m)\), we define its depth as \(d(a) = d(p/q)\) where \(p/q = a = [a_1, a_2, \ldots, a_m]\).

Once the depth of \(p/q\) has been determined, we can compute \(\gamma(K_{p/q})\) from the even continued fraction expansion of \(p/q\) using the following Proposition (which is a restatement of Hirasawa and Teragaito’s work).

**Proposition 1.** Let \(p/q = a = [a_1, a_2, \ldots, a_{2n}]\) be the unique even continued fraction expansion of \(p/q\). Then

\[
\gamma(K_{p/q}) = \begin{cases} 
  d(p/q), & \text{if } a_i = \pm 2 \text{ for some } i, \\
  d(p/q) + 1, & \text{otherwise}.
\end{cases}
\]

**Proof.** If \(|a_i| \geq 4\) for all \(i\), then none of Hirasawa and Teragaito’s three shortening moves apply and hence this continued fraction expansion is shortest and defines a shortest path from \(1/0\) to \(p/q\). Thus by Lemma 3 it follows that \(d(p/q) = 2n\). It now follows from Algorithm 1 that \(\gamma(K_{p/q}) = 2n + 1 = d(p/q) + 1\). Otherwise, some \(a_i = \pm 2\). If the even continued fraction is shortest, then again, we obtain \(\gamma(K_{p/q}) = d(p/q)\). If the even continued fraction expansion is not shortest, then the shortest one must contain an odd entry because the even continued fraction expansion is unique. If the length of the shortest continued fraction is \(m\), then by Algorithm 1 we have that \(\gamma(K_{p/q}) = m\) and, by Lemma 3, we have that \(m = d(p/q)\). Hence \(\gamma(K_{p/q}) = d(p/q)\).

The depth of a fraction \(p/q\) can be determined from a 2-complex associated to an edge path. Suppose

\[
\frac{1}{0} \rightarrow \frac{0}{1} \rightarrow \frac{p_1}{q_1} \rightarrow \cdots \rightarrow \frac{p_n}{q_n} = \frac{p}{q}
\]
is the edge path determined by the fraction \( p/q = [a_1, a_2, \ldots, a_n] \). If we think of the vertices along the path as recursively generated by taking mediants, then we can readily think of the path as lying in a 2-complex, made up of triangles that are added one at a time each time a mediant is introduced. We start with the initial vertices \( \{1/0, 0/1\} \). Taking the mediant of these two vertices gives \( 1/1 \) and the first triangle is defined to be the one with vertices \( \{1/0, 0/1, 1/1\} \). Next we take the mediant of \( 0/1 \) and \( 1/1 \) to obtain \( 1/2 \) and the second triangle has vertices \( \{0/1, 1/1, 1/2\} \). As we continue taking mediants, we eventually reach \( p_1/q_1 = 1/a_1 \). (If \( a_1 < 0 \), then we must think of the initial pair of vertices as \( \{(-1)/0, 0/1\} \).) Each time we introduce a new mediant, we attach a triangle along one of its edges to the already existing 2-complex.

For example, consider \( 10/23 = [2, 4, -2, 2] \). This defines the edge path

\[
\frac{1}{0} \rightarrow \frac{0}{1} \rightarrow \frac{1}{2} \rightarrow \frac{4}{9} \rightarrow \frac{7}{16} \rightarrow \frac{10}{23}.
\]

The path and its associated 2-complex are shown in Figure 4. Above or below the fraction at each vertex, we display the depth of the fraction. The depths are computed recursively using Definition 2. In particular, the depth of \( 10/23 \) is 3. Notice that the edge path defined by \( [2, 4, -2, 2] \) is not shortest; the depth does not increase by 1 along the last edge of the path. A shorter path is given by \( [2, 3, 3] \), which starts the same, but then goes from \( 1/2 \) to \( 3/7 \) to \( 10/23 \).

![Figure 4: The path associated to \([2, 4, -2, 2]\) with auxiliary data and depths displayed.](image)

Given \( p/q = [a_1, a_2, \ldots, a_n] \), we will now show that one can compute the depth of \( p/q \) directly from the turning numbers rather than having to draw the 2-complex as in Figure 4 and then compute the depth of every vertex in the 2-complex. To do this, we introduce the **auxiliary data**, \( \{d_i, e_i\} \) at each turning number \( a_i \), as follows. Consider the two edges that terminate at vertex \( p_i/q_i \) and which originate at the two parents of this fraction. One of these edges is in the edge path, but the other is not. Define \( d_i \) to be the change in depth along the edge from \( p_{i-1}/q_{i-1} \) to \( p_i/q_i \) and \( e_i \) to be the change in depth
along the edge from the other parent. It follows that both $d_i$ and $e_i$ lie in $\{0, 1\}$ because each records
the change in depth along an edge that points from a parent to a child in the Farey Graph. Thus, the possible values for $\{d_i, e_i\}$ are, a priori, limited to $\{\{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}\}$. However, because our edge paths begin with $1/0 \rightarrow 0/1$ we will not encounter a triangle with all integral vertices and, therefore, auxiliary data $\{0,0\}$ will not occur in this context. In Figure 4 we have written the pairs $\{d_i, e_i\}$ in the relevant triangle at each vertex along the path. (We have written 11 for $\{1,1\}$, etc.).

Clearly, the depth of $p/q$ is the sum $d_1 + d_2 + \cdots + d_n$ because the depth along the path starts at zero
and each $d_i$ records the change in depth along one edge in the path. If $p/q = [a_1, a_2, \ldots, a_n]$, then we
will assume that each $a_i$ is even and nonzero. It is now easy to show that $\{d_1, e_1\} = \{1, 1\}$ if $|a_1| = 2
\text{ and } \{d_1, e_1\} = \{1, 0\}$ if $|a_1| \geq 4$. Thus, we can determine the initial auxiliary data from $a_1$. The next
result tells us how $\{d_i, e_i\}$, the sign of $a_ia_{i+1}$, and $|a_{i+1}|$ determine $\{d_{i+1}, e_{i+1}\}$. This will allow for
a recursive calculation of all the auxiliary data, and hence the depth of $p/q$. For example, consider
$92/125 = [2, -2, 2, 4, -4, 2]$. According to Lemma 4 $\{d_1, e_1\} = 11$ because $a_1 = 2$. Since $a_1a_2 = -4$,
we look in the last row of the table and the column headed by $|a_{i+1}| = 2$ and $a_ia_{i+1} < 0$, to find
that $\{d_2, e_2\} = 01$. Continuing in this way, we see that the associated sequence of auxiliary data is
$\{11, 01, 01, 10, 10, 11\}$ from which we conclude that the depth of 92/125 is $1 + 0 + 0 + 1 + 1 + 1 = 4$.

**Lemma 4.** Suppose that $p/q = [a_1, a_2, \ldots, a_{2n}]$ where $a_i$ is nonzero and even for all $i$. Define $\{d_1, e_1\}$ to be 11 if $|a_1| = 2$ or 10 if $|a_1| \geq 4$. Then $\{d_{i+1}, e_{i+1}\}$ for $i \geq 1$ is determined recursively by the following table

| $d_i e_i$ | $|a_{i+1}| = 2 \& a_ia_{i+1} > 0$ | $|a_{i+1}| = 2 \& a_ia_{i+1} < 0$ | $|a_{i+1}| \geq 4$ |
|---------|-------------------------------|-------------------------------|-----------------|
| 01      | 10                            | 01                            | 10              |
| 10      | 11                            | 11                            | 10              |
| 11      | 11                            | 01                            | 10              |

**Proof.** We derive the various outcomes in the table for the the case where $d_1e_1 = 11$ by using Figure 5.
Here we have imagined the last triangle in the 2-complex before advancing one more edge in the path.
The last triangle is assumed to have auxiliary data 11 and so the depths of its vertices are labeled
accordingly, in this case, $d, d-1$, and $d-1$. The turning due to the sign of $a_ia_{i+1}$ and the magnitude
of $a_{i+1}$, then leads us to the next value of auxiliary data. Note that both this figure and its mirror
image are needed in general. Similar arguments (and figures) handle the other two cases, where $d_i e_i$ is
either 01 or 10.

Considering auxiliary data provides a straightforward proof of the following result which will be used
in the proof of Theorem 1.

**Lemma 5.** Suppose $a$ is a vector of nonzero even entries. Then $d(a) = d(-a)$ and $d(a) = d(a^{-1})$.

**Proof.** First note that the auxiliary data associated to $-a$ is exactly the same as that associated to $a$. Thus both have the same depth. Next, it is well-known that both $a$ and $a^{-1}$ correspond to the
same 2-bridge knot $K$. This is because the 4-plat diagrams determined by $a$ and $a^{-1}$, respectively,
are related by a 180 degree rotation. Thus both \( a \) and \( a^{-1} \) determine knots with the same crosscap number. Because one has an entry of \( \pm 2 \) if and only if the other one does, Proposition \( 1 \) implies they must have the same depth.

Returning to our definition of the auxiliary data, we introduce two definitions that are slight variations. Given a vector \( a \) all of whose entries are nonzero and even, we can get a different sequence of auxiliary data by still using the table in Lemma \( 4 \) but by starting with different initial data. Define \( d(a, 01) \) to be the depth derived by starting with initial data 01, computing all subsequent auxiliary data according to the table in Lemma \( 4 \), and then summing the first entry of each data pair. Similarly, let \( d(a, 10) \) be the analogous depth obtained by using 10 as the initial data.

**Lemma 6.** Let \( a \) be a vector all of whose entries are even and nonzero. Then

- \( d(a, 11) - d(a, 01) \in \{0, 1\} \),
- \( d(a, 11) - d(a, 10) \in \{-1, 0\} \), and
- \( d(a, 10) - d(a, 01) \in \{0, 1, 2\} \).

**Proof.** First note that the third conclusion follows from the first two because

\[
d(a, 10) - d(a, 01) = d(a, 10) - d(a, 01) + d(a, 11) - d(a, 11).
\]
To prove the first two assertions, we induct on the length of \( a \). If \( a = (a_1) \), the result is trivial. Now assume that \( a \) has length bigger than 1 and that the result is true for any shorter vector. Let \( a = (a_1, a_2, \ldots, a_{2n}) \) and \( a' = (a_2, a_3, \ldots, a_{2n}) \). We consider three cases, each corresponding to one of the three (output) columns of Lemma 4.

For the first case, if \( |a_2| = 2 \) and \( a_1 a_2 > 0 \), then by considering the three possible initial values for the auxiliary data \( d_1 e_1 \) we obtain that \( d(a, 11) = 1 + d(a', 11) \), \( d(a, 10) = 1 + d(a', 11) \), and \( d(a, 01) = 0 + d(a', 10) \). Forming the appropriate differences and using the inductive hypothesis immediately gives the result. The proofs of the remaining two cases are similar.

Returning to Algorithm 1, we now offer, in passing, a more direct method of finding a shortest continued fraction expansion for any fraction \( \frac{p}{q} \). Given \( \frac{p}{q} \), first note that it is straightforward to find its two parents in the Farey graph. To do this, first use the Euclidean Algorithm to express the greatest common divisor of \( p \) and \( q \) as \( \pm 1 = aq - bp \). Next, let \( c = p - a \) and \( d = q - b \). Thus the mediant of \( \frac{a}{b} \) and \( \frac{c}{d} \) is equal to \( \frac{p}{q} \). Moreover, \( ad - bc = a(q - b) - b(p - a) = aq - bp = \pm 1 \). Thus the fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) are connected by an edge in the Farey Graph. A feature of the Farey Graph is that if \( \frac{a}{b} \) and \( \frac{c}{d} \) are parents of \( \frac{p}{q} \), then, in fact, one of the parents is a parent of the other parent. Thus, of the two parents of \( \frac{p}{q} \), one is a grandparent of \( \frac{p}{q} \). (Exceptions occur with \( \pm 1/1 \) which have no grandparents.) Starting from \( \frac{p}{q} \) we can form a path back to \( 1/0 \) by repeatedly moving to the grandparent of \( \frac{p}{q} \), as long as it exists. Simply find the two parents and then take the one with the smallest denominator. Do this until grandparents do not exist, at which point we are only one edge away from \( 1/0 \). Because this approach is not central to this paper, we leave it to the reader to show that this generates a shortest path from \( 1/0 \) to \( \frac{p}{q} \) and hence an algorithmic improvement to Step 3 of Algorithm 1. For example, consider \( 34/149 \). The grandparent edge path is

\[
\begin{array}{cccccccc}
1 & 0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 5 & \rightarrow & 13 & \rightarrow & 34 & \rightarrow & 149.
\end{array}
\]

giving a depth of 5.

**Lemma 7.** Suppose \( a \) and \( b \) are both vectors of all even, nonzero entries and that \( c \) is an integer with \( |c| \geq 4 \). If \( A = (a, c, b) \), then

\[
d(A) = 1 + d(a) + d(b).
\]

*Proof.* First note that the vector \( a \) contributes \( d(a) \) to the depth of \( A \). Let \( d_i e_i \) be the auxiliary data of \( A \) at \( c \). Because \( |c| \geq 4 \), it follows from Lemma 4 that \( d_i e_i = 10 \). Thus, the entry \( c \) contributes +1 to the depth of \( A \). We also have that

\[
d_{i+1} e_{i+1} = \begin{cases} 
11 & \text{if } |b_1| = 2, \\
10 & \text{if } |b_1| \geq 4.
\end{cases}
\]

Thus, the first element \( b_1 \) of \( b \) has the same auxiliary data as it would if only its depth were being computed. So \( b \) contributes \( d(b) \) to the depth of \( A \). Therefore, \( d(A) = 1 + d(a) + d(b) \).

The last result needed for Theorem is the following.
Lemma 8. Suppose \( a \) and \( b \) are both vectors of all even, nonzero entries and that \( c \) is any integer. Let \( A = (a, 2c, b) \), where we further assume that if \( c = 0 \), then the last entry of \( a \) is equal to the first entry of \( b \). Then

\[
d(A) \geq d(a) + d(b) - 1.
\]

Proof. We break the proof into multiple cases.

Case I: \( c = 0 \)
Let \( A' \) be obtained from \( A \) by removing \( 2c \) as in Lemma 2. If \( a = (a_1, a_2, \ldots, a_m) \) and \( b = (a_m, b_2, \ldots, b_n) \), then \( A' = (a_1, a_2, \ldots, a_{m-1}, 2a_m, b_2, \ldots, b_n) \). We also have \( d(A) = d(A') \) since these vectors determine the same fraction. Now \( |2a_m| \geq 4 \) because \( |a_m| \geq 2 \). Therefore, by Lemmas 5 and 7 we have

\[
d(A) = d(A') = 1 + d((a_1, a_2, \ldots, a_{m-1})) + d((b_n, b_{n-1}, \ldots, b_2)).
\]

Removing the last entry from \( a \) or \( b^{-1} \) reduces its depth by at most 1, therefore

\[
d(A) \geq 1 + d(a) - 1 + d(b^{-1}) - 1 = d(a) + d(b) - 1.
\]

Case II: \( |2c| \geq 4 \)
Lemma 7 implies that \( d(A) = 1 + d(a) + d(b) \geq d(a) + d(b) - 1 \).

Case III: \( 2c = \pm 2 \).
The sequence of auxiliary data for \( A \) begins with the sequence of auxiliary data for \( a \) then has the auxiliary data for the entry \( 2c \), which we denote \( d'e' \), and then has auxiliary data some \( ij \in \{01, 10, 11\} \) for the first entry \( b_1 \) of \( b \). If we were to compute \( d(b) \) then the initial auxiliary data for \( b_1 \) should be 11 if \( |b_1| = 2 \) or 10 if \( |b_1| \geq 4 \). First consider the case where \( |b_1| = 2 \), Now,

\[
d(A) = d(A) + d' + d(b, ij)
= d(a) + d' + d(b) + d(b, ij) - d(b, 11).
\]

If \( ij = 10 \), then by Lemma 6, the difference of the last two terms is either \(-1\) or 0, and because \( d' \) is equal to either \( 0 \) or \( 1 \), we obtain the desired result. Similar arguments apply if \( ij \neq 01 \) or \( |b_1| \geq 4 \). \( \square \)

Lemmas 4, 5, 8 allow us to compare the depth of a vector \( a \) to the depth of a vector \( b \) that parses with respect to \( a \). For example, consider \( a = [2, 2, -2, 2] \) which has auxiliary data \( \{11, 11, 01, 01\} \). Note that \( d(a) = d(a^{-1}) = 2 \) but that the auxiliary data for \( a^{-1} \) is \( \{11, 01, 01, 10\} \). Different types of connectors in a vector that parses with respect to \( a \) change the auxiliary data and depth in ways reflected by the lemmas. For example, \( b = (a, -2, a^{-1}) \) has auxiliary data \( \{11, 11, 01, 01, 01, 01, 01, 01, 10\} \). This data starts with the auxiliary data for \( a \) but ends with data for \( a^{-1} \) that assumes initial data of 01 (thus contributing \( d(a^{-1}, 01) \) to the computation of depth). Since \( d(b) = 3 = d(a) + d(a^{-1}) - 1 \), this example illustrates that the inequality in Lemma 8 is sharp.

We are now prepared to prove Theorem 1.
Theorem 1. Suppose $K$ and $J$ are 2-bridge knots with $K > J$. Then

$$\gamma(K) \geq 3\gamma(J) - 4.$$ 

Proof. We may assume that $J$ corresponds to the fraction $p/q$ with $0 < p < q$, $\gcd(p,q) = 1$, $p$ is even, and $q$ is odd. Let $p/q = a = [a_1, a_2, \ldots, a_{2n}]$ be the unique even continued fraction expansion of $p/q$. Because $K > J$, it follows from Theorem 2 that $K$ corresponds to a fraction $p'/q'$ with continued fraction expansion given by

$$p'/q' = b = [\epsilon_1 a, 2c_1, \epsilon_2 a^{-1}, 2c_2, \ldots, 2c_{2k}, \epsilon_{2k+1} a],$$

where $\epsilon_i \in \{-1, 1\}$ for all $i$, each $c_i$ is an integer, and if $c_i = 0$, then $\epsilon_i = \epsilon_{i+1}$. Moreover, $2k + 1 \geq 3$ because $K \neq J$. If some connectors, $2c_i$, are equal to zero, then we will collapse them by Lemma 2, to arrive at a shorter vector $b'$, which will then have all even and nonzero entries and hence will be the unique continued fraction expansion for $p'/q' = b'$. As both vectors determine the same fraction, $d(b) = d(b')$.

We first note that if $a$ contains an entry equal to $\pm 2$, then so does $b'$. Hence if $\gamma(J) = d(a)$, then $\gamma(K) = d(b')$. To see this, assume that $a$ contains an entry that is $\pm 2$. If this entry is not the first or last entry of $a$, then it appears somewhere in $b$ and continues to appear in $b'$ even after any zero-connectors are eliminated. If instead, the first or last entry of $a$ is $\pm 2$, then so is either the first or last entry of $b$, respectively, and hence also, $b'$. Thus $b'$ contains an entry that is $\pm 2$. The second part of the conclusion follows from Proposition 1.

We now consider two major cases.

Case I: The vector $a$ contains some entry equal to $\pm 2$.

By our previous observation, we have that $\gamma(J) = d(a)$ and $\gamma(K) = d(b) = d(b')$. We can build $b$ in $2k$ steps by starting with $a$ and then repeatedly appending $(2c_i, \epsilon_{i+1} a^{\pm 1})$. Applying Lemma 8 a total of $2k$ times, and using Lemma 5 we obtain $d(b) \geq (2k + 1)d(a) - 2k$. It then follows that

$$\gamma(K) \geq (2k + 1)\gamma(J) - 2k \geq (2k + 1)(\gamma(J) - 1) - 1 \geq 3\gamma(J) - 4$$

since $2k + 1 \geq 3$.

Case II: No entry of the vector $a$ is equal to $\pm 2$.

Because every entry of $a$ has magnitude 4 or more, every entry has auxiliary data of 10. Therefore, $d(a) = |a|$, where $|a|$ is the length of the vector $a$, and moreover $\gamma(J) = d(a) + 1 = |a| + 1$. We now break this case into two subcases.

Case IIa: No entry of the vector $a$ is equal to $\pm 2$ but some connector in $b$ is equal to $\pm 2$.

Because some connector in $b$ is $\pm 2$, after collapsing any zero-connectors, it is still the case that $b'$ contains an entry equal to $\pm 2$. Thus $\gamma(K) = d(b')$. We also have that every entry of $b'$, except possibly
some connectors, has magnitude at least 4. Thus, all non-connector entries of $b'$ have auxiliary data of 10. Because any connector of magnitude 2 follows an element of magnitude of 4 or more, it has auxiliary data of 11. Therefore, $d(b') = |b'|$. If $z$ is the number of zero-connectors in $b$, then we are assuming that $z \leq 2k - 1$ and hence that $-2z \geq -4k + 2$. Since $2k + 1 \geq 3$, we now have

$$\gamma(K) = |b'|$$

$$= |b| - 2z$$

$$= (2k + 1)|a| + 2k - 2z$$

$$\geq (2k + 1)|a| - 2k + 2$$

$$\geq (2k + 1)(\gamma(J) - 2) + 3$$

$$\geq 3\gamma(J) - 4$$

**Case IIb:** No entry of the vector $a$ is equal to $\pm 2$ and no connector in $b$ is equal to $\pm 2$.

In this case, the vector $b$ does not contain an entry equal to $\pm 2$ and this remains true after collapsing any connectors which are equal to zero. Thus $\gamma(J) = d(a) + 1$ and $\gamma(K) = d(b') + 1$. Every entry of both $a$ and $b'$ have magnitude 4 or more. Therefore, we have $d(a) = |a|$ and $d(b') = |b'|$. Assuming that $b$ has $z$ zero-connectors with $z \leq 2k$, we have

$$\gamma(K) = |b'| + 1$$

$$= |b| - 2z + 1$$

$$= (2k + 1)|a| + 2k - 2z + 1$$

$$\geq (2k + 1)|a| - 2k + 1$$

$$\geq (2k + 1)(\gamma(J) - 2) + 2$$

$$\geq 3\gamma(J) - 4$$

We now give examples to show that the inequality in Theorem 1 is sharp. Let $J_n$ be the 2-bridge knot corresponding to the fraction $\frac{2n}{4n^2 + 1} = [2n, 2n]$ for $n > 1$ and $K_n$ the 2-bridge knot corresponding to $\frac{32n^3 + 60n}{64n^2 + 20n^2 + 1} = [2n, 2n, 0, 2n, 2n, 0, 2n, 2n] = [2n, 4n, 4n, 2n]$. This gives infinitely many different pairs of knots, $(K_n, J_n)$, with $K_n > J_n$ by Theorem 2. Propositions 1 and 4 tell us that $\gamma(K_n) = 5$ and $\gamma(J_n) = 3$. Because $5 = 3 \cdot 3 - 4$, we see that the inequality of Theorem 1 is sharp.

Finally, we summarize our work in the following algorithm to compute the crosscap number of any 2-bridge knot $K$.

**Algorithm 2.** To compute the crosscap number $\gamma(K_{p/q})$ of the 2-bridge knot $K_{p/q}$, where $0 < p < q$, $\gcd(p, q) = 1$, $p$ is even and $q$ is odd.

1. Compute the unique even continued fraction expansion $p/q = [a_1, a_2, \ldots, a_{2n}]$ where each $a_i$ is even and nonzero.
2. Compute the auxiliary data and depth of \( p/q, d(p/q) \), using Lemma 4.

3. If some \( a_i = \pm 2 \) for some \( i \), then \( \gamma(K) = d(p/q) \), otherwise \( \gamma(K) = d(p/q) + 1 \).

A Jupyter notebook containing a Python program that implements Algorithm 2 is included with the arXiv upload of this paper.

4 Computations

Using our implementation of Algorithm 2, we computed the crosscap number of every 2-bridge knot with 16 or fewer crossings. Our computations agree with the data in [15], for knots up to 12 crossings (the limit of KnotInfo’s table). A table of crosscap numbers is given in [9] for 2-bridge knots with 12 or fewer crossings. Our computations agree with their data as well. In Table 1 we give the distribution of crosscap numbers for all 2-bridge knots up to 16 crossings. It was proven in [16] that for any knot \( K \)

\[
\gamma(K) \leq \left\lfloor \frac{cr(K)}{2} \right\rfloor,
\]

where \( cr(K) \) is the crossing number of \( K \) and \( \lfloor x \rfloor \) is the floor of \( x \), that is, the largest integer less than or equal to \( x \). This inequality is confirmed by our computations and is borne out in Table 1.

Table 1: Crosscap numbers by crossing number for 2-bridge knots with 16 or fewer crossings.

| \( cr(K) \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total |
|------------|---|---|---|---|---|---|---|---|-------|
| 3          | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1     |
| 4          | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1     |
| 5          | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2     |
| 6          | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 3     |
| 7          | 1 | 2 | 4 | 0 | 0 | 0 | 0 | 0 | 7     |
| 8          | 0 | 3 | 7 | 2 | 0 | 0 | 0 | 0 | 12    |
| 9          | 1 | 3 | 12| 8 | 0 | 0 | 0 | 0 | 24    |
| 10         | 0 | 4 | 17| 21| 3 | 0 | 0 | 0 | 45    |
| 11         | 1 | 4 | 26| 43| 17| 0 | 0 | 0 | 91    |
| 12         | 0 | 5 | 33| 78| 53| 7 | 0 | 0 | 176   |
| 13         | 1 | 5 | 44| 127|39 |0 | 0 | 0 | 352   |
| 14         | 0 | 6 | 53| 194|278|150|12|0 | 693   |
| 15         | 1 | 6 | 68| 280|526|419|87|0 | 1387  |
| 16         | 0 | 7 | 79| 389|889|989|375|24|2752  |
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