The growth of the density fluctuations in the scale-invariant theory: one more challenge for dark matter

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\textbf{ABSTRACT}

The growth of the density fluctuations is considered to be an important cosmological test. In the standard model, for a matter dominated universe, the growth of the density perturbations evolves with redshift $z$ like $(1 + z)^s$ with $s = 1$. This is not fast enough to form galaxies and to account for the observed present-day inhomogeneities. This problem is usually resolved by assuming that at the recombination epoch the baryons settle down in the potential well of the dark matter previously assembled during the radiation era of the universe. This view is challenged in the present paper by using the recently proposed model of a scale-invariant framework for cosmology that enlarges the invariance group subtending the theory of the gravitation.

From the continuity equation, the Euler and Poisson equations written in the scale-invariant framework, the equation governing the growth of density fluctuations $\delta$ is obtained. Starting from $\delta = 10^{-5}$ at a redshift around 1000, numerical solutions for various density background are obtained. The growth of density fluctuations is much faster than in the standard EdS model. The $s$ values are in the range from 2.7 to 3.9 for $\Omega_m$ between 0.30 and 0.02. This enables the density fluctuations to enter the nonlinear regime with $\delta > 1$ long before the present time, typically at redshifts of about 10, without requiring the presence of dark matter.

\textbf{Keywords:} Cosmology: theory - dark matter - Galaxies: formation

1. INTRODUCTION

When the cosmological constant $\Lambda$ is considered to be equal to zero, the equations of General Relativity are scale invariant, while for a non-zero $\Lambda$ the equations do not have the property of scale invariance, which is present in Maxwell equations. The framework developed by Dirac (1973) and Canuto et al. (1977), the so-called co-tensorial calculus based on Weyl’s geometry (see also Bouvier & Maeder (1978)) allows for accommodation of the scale invariance of the empty space together with a non-zero cosmological constant. The constant $\Lambda$ represents the energy density of the empty space and the specific hypothesis is made that the properties of the empty space, at macroscopic scales, are scale invariant (Maeder 2017a). At the quantum level, this probably does not apply, however in the same way as one may use Einstein’s theory at large-scales, even if it does not apply at the quantum level, one may consider that the large-scale
empty space is scale-invariant. Note that the equation of the vacuum $P_{\text{vac}} = -\varphi_{\text{vac}}$ is precisely that permitting $\varphi_{\text{vac}}$ to remain constant for an adiabatic expansion or contraction (Carroll et al. 1992). The corresponding cosmological equations have been formulated with the Robertson-Walker metric and the weak field approximation has been studied (Maeder 2017a,c). In this approximation, the equation of motion contains a (small) additional term to Newtonian gravitation.

In this context, it is worth to recall the interesting remark by Bondi (1990), who pointed out that “Einstein’s disenchantment with the cosmological constant was partially motivated by a desire to preserve scale invariance of the empty space Einstein equations”. The developments briefly mentioned above, appear as particularly worth to be explored in the present cosmological context, where according to current results 95% of the matter-energy in the universe is in an unknown form. A number of positive observational tests have already been successfully passed, both in cosmology and in the dynamics of galaxies, see Maeder (2017a,b,c, 2018). The cosmological tests considered here concern the growth of the density fluctuations in the scale-invariant theory. The main points to be examined are the following ones. What are the predictions of the scale-invariant theory concerning the growth of perturbations of density during the accelerated expansion? Is dark matter needed to account for the observed present inhomogeneities? At what redshifts do most galaxies form in the above context?

The theory of the formation of the large-scale structure has been studied in details by Peebles (1980), where previous references may be found. Among recent syntheses, one may mention Coles & Lucchin (2002), Durrer (2008) and Theuns (2016) and those mentioned below. In the standard theory, the sub-horizon baryonic density perturbations do not grow during the radiation dominated era and only the perturbations larger than a minimum mass (the Silk mass) are not rapidly erased by photon diffusion (Silk 1968). At the time of recombination, at a redshift $z \approx 10^3$, the observed temperature fluctuations $\delta T/T$ of the CMB are of the order of $10^{-5}$. At this time, the fluctuations of the baryonic density $\delta \rho/\rho$ are of the same order, since up to recombination the matter and radiation were closely coupled. From this time onward, the growth of the density fluctuations $\delta \equiv \delta \rho/\rho$ in the linear regime is determined by the well-known equation (Peebles 1980),

$$\ddot{\delta} + 2H \dot{\delta} = 4\pi G \rho \delta,$$

where $H = \dot{a}/a$ is the Hubble “constant” at the time considered, $\rho$ the pressure and $a(t)$ the expansion factor. In the standard model for a dust Universe ($p = 0$), $\delta$ goes like $t^{2/3} \propto a \propto (1+z)^{-1}$. Thus, starting from perturbations with an amplitude of about $10^{-5}$, a growth by a factor of $10^3$ would lead to fluctuations of order $10^{-2}$ at the present epoch, which is by several orders of magnitude smaller than the non-linear structures presently observed in the Universe (Ostriker 1993; Theuns 2016).

While baryonic perturbations do not amplify during the radiation era, the perturbations of the assumed dark matter (subject to gravitational interaction) are growing during the radiation era. Then, it is usually considered in the standard theory that after recombination the baryons are falling into the potential wells previously created by the dark matter. Their growth towards the present highly contrasted structures is strongly boosted. The cold dark matter (CDM) scenario and its related astronomical tests have been studied by Ostriker (1993), while numerical simulations, techniques, and results have been reviewed by Bertschinger (1998) and more recently by Sommerville & Dave (2015).

The research presented in the current paper questions the need for dark matter in agreement with previous works (Maeder 2017c, 2018) that were showing that, in the scale invariant context, dark matter is not needed to account for the high velocities observed in clusters of galaxies, the high rotation velocities of stars at the edge of spiral galaxies, or the growth of the “vertical” velocity dispersion of stars in the Galaxy as correlated to their age. In Section 2, the expressions of the basic equations of continuity, of Poisson and Euler within the scale-invariant theory are derived. In Section 3, these three equations are applied to density perturbations. In Section 4, numerical solutions of the equation for the growth of density fluctuations are obtained and discussed. Section 5 gives the conclusions.

2. THE EQUATIONS OF EULER, CONTINUITY, AND POISSON IN THE SCALE-INVARIANT CONTEXT

Classically the equations of Euler and Poisson, as well as the continuity equation, form the basis for the study of the evolution of the density fluctuations in the Newtonian approximation. The first two equations are obtained in the weak field approximation of the general field equation in Riemann geometry. The Weyl integrable geometry is appropriate to express a scale-invariant form of the general field equation of gravitation (Canuto et al. 1977). To pass from the framework of general relativity (where symbols are denoted with a prime) to the scale-invariant framework (where symbols are denoted without a prime), there is the following relation between the line elements

$$ds' = \lambda ds,$$

with $\lambda = \lambda(t)$.  \hspace{1cm} (2)
Here, \( \lambda \) is the scale factor which is a function of the time \( t \) only, to preserve the homogeneity and isotropy of space. The assumption of the scale invariance of the empty space applied to the general scale covariant field equation leads to some differential relations for \( \lambda(t) \) (Maeder 2017a),

\[
3 \frac{\dot{\lambda}^2}{c^2 \lambda^2} = \lambda^2 \Lambda \quad \text{and} \quad 2 \frac{\ddot{\lambda}}{c^2 \lambda} - \frac{\dot{\lambda}^2}{c^2 \lambda^2} = \lambda^2 \Lambda.
\]  

(3)

This implies \( \lambda = t_0/t \), where \( t_0 \) is the present time. These expressions are independent of the matter content of the system considered, in the same way as the cosmological constant is the same whatever the material content is.

2.1. The matter equivalent of motion

The equivalent of Newton equation in the scale-invariant framework was derived from the weak field approximation of the geodesic equation (Maeder & Bouvier 1979; Maeder 2017c). This geodesic equation was itself derived from an action principle by Dirac (1973), see also Bouvier & Maeder (1978) for its interpretation in Weyl’s geometry. The metric is essentially the Minkowski metric of the empty space. The equation of motion in spherical coordinates is,

\[
\frac{d^2 r}{dt^2} = - \frac{G M}{r^2} \frac{r}{\lambda} + \kappa(t) \frac{d r}{d t}.
\]  

(4)

The equation contains an additional term proportional to the velocity vector. Thus, in the case of an expansion of a system, there is an additional increase of the expansion; in the case of an infall, the collapse is accelerated. The term \( \kappa(t) \) is called the coefficient of metrical connection. Within the general framework it is usually a four-component field \( \kappa_{\nu} \), however, due to considerations of isotropy and homogeneity of the 3D space only the time-dependent \( \kappa_0 \) component is non-zero (for details see Maeder (2017a)). Therefore, \( \kappa_0 \) is denoted by \( \kappa(t) = -\dot{\lambda}/\lambda \) and thus \( \kappa(t) = 1/t \).

This means that the additional effects were larger at earlier epochs and that they are decreasing as time is passing. In the context of weak fields, as stated by Maeder & Bouvier (1979), the \( g_{\mu \nu} \) and \( \kappa_{\nu} \) terms, as well as the time \( t \) are those of Special Relativity appropriate to the empty space. The time \( t \) is the time in a comoving system, that is to say, the cosmic time. Any observer attached to a comoving galaxy will measure the same age of the Universe. This time starts at the Big Bang with \( t_{\text{in}} = 0 \) and now it is 13.8 Gyr (different units may be considered). This timescale evidently satisfies the above equations (3) with the currently measured cosmological constant \( \Lambda \).

Interestingly enough, the above modified Newton equation, as well as the Euler equation, can also be derived in a rather simple way by the application of simple scale transformations. From Equation (2), one also has

\[
r' = \lambda r,
\]  

(5)

since \( \lambda \) is a function of time only, (as discussed above, symbols with a prime refer to the standard GR-framework, symbols without a prime are related to the scale-invariant framework). With the above relations, the velocities vectors in the two frameworks are related as follows:

\[
\frac{d\bar{v}}{dt} = \frac{d\bar{v}'}{dt} - \frac{1}{\lambda} \frac{d}{dt} \left( \frac{\dot{\lambda}}{\lambda} \bar{r} \right) = \frac{\dot{\lambda}}{\lambda} \bar{r} - \bar{v} = \bar{v} - \kappa \bar{r}.
\]  

(6)

The operator \( \nabla \) transforms like \( \nabla' = \nabla / \lambda \). From the scale invariance of the energy-momentum tensor, there are the following relations for the pressure and the density \( p = \rho' \lambda^2 \) and \( \rho = \rho' \lambda^2 \) (Canuto et al. 1977; Maeder 2017a).

2.2. The equivalent of the Euler equation

To derive the scale-invariant form of Euler equation consider the time derivatives (see the Appendix II in Maeder & Bouvier (1979)),

\[
\frac{d^2 \bar{r}}{dt^2} = \frac{d}{dt} \left( \frac{d\bar{r}}{dt} \right) = \frac{1}{\lambda} \frac{d^{2}\lambda}{dt^{2}} \left( \lambda \bar{r} \right) = \frac{1}{\lambda} \frac{d^{2}\lambda}{dt^{2}} \left( \bar{r} - \kappa \bar{r} \right) = \frac{1}{\lambda} \frac{d^{2}\lambda}{dt^{2}} \left( \bar{r} - \kappa \bar{r} + \kappa^2 r \right),
\]

where \( \kappa = -\dot{\lambda}/\lambda \) and \( \ddot{\lambda}/\lambda = 2\kappa^2 \) have been employed, (see Equation (25) in (Maeder 2017a)) or equivalently that \( \dot{\kappa} = -\kappa^2 \). The standard form of the Euler equation is:

\[
\frac{d\bar{v}'}{dt'} = -\nabla' \Phi - \frac{1}{\rho'} \nabla' p' + \frac{1}{3} \lambda r^2.
\]  

(7)
where for consistency with the equation of motion the $\Lambda$ term (Mavrides 1973) has been included as well. Here, the effect of the cosmological constant $\Lambda$ within the general relativity framework has been included. The new form of the Euler equation can be obtained by substituting $\ddot{\varphi} = \dot{\varphi} - \kappa \dot{\varphi}$ (6) in the usual Euler equation (7) along with the above mentioned scaling properties for $r$, $\varrho$, and $p$, using $\nabla \lambda = 0$, and the scale invariance of the gravitational potential $\Phi' = \Phi$ (see Eq. (22) in Maeder (2017c)),

$$\lambda \frac{d\ddot{\varphi}'}{dt'} = \left( \ddot{r} - \kappa \dot{r} + \kappa^2 \varphi \right) = -\nabla \Phi - \frac{1}{\rho} \nabla p + \frac{1}{3} \lambda^2 \Lambda \varphi,$$

$$\ddot{r} = -\nabla \Phi - \frac{1}{\rho} \nabla p + \frac{1}{3} \Lambda \lambda^2 \varphi + \left( \kappa \dot{r} - \kappa^2 \varphi \right).$$

Using the expressions (22) and (23) in (Maeder 2017a), $\kappa = -\kappa^2 = -\lambda^2 \Lambda / 3$, one derives:

$$\ddot{r} = -\nabla \Phi - \frac{1}{\rho} \nabla p + \kappa^2 \varphi + \left( \kappa \dot{r} - \kappa^2 \varphi \right),$$

and by canceling the $\kappa^2$ terms the equation becomes:

$$\frac{d\ddot{\varphi}}{dt} = -\nabla \Phi - \frac{1}{\rho} \nabla p + \kappa \dot{\varphi}.$$  

Finally, the new Euler equation takes the form,

$$\frac{d\ddot{\varphi}}{dt} = \frac{\partial \ddot{\varphi}}{\partial t} + \left( \ddot{\varphi} \cdot \nabla \right) \varphi = -\nabla \Phi - \frac{1}{\rho} \nabla p + \kappa \dot{\varphi}. \quad (8)$$

The equation of motion contains an additional term proportional to the velocity, as seen above in Equation (4). In the case of an outflow expansion, this term opposes the gravitational deceleration, while for an infall of matter, like for the growth of density fluctuations, the additional term contributes to an infall enhancement. This term plays a great role in the enhancement of density fluctuations in the scale-invariant theory.

2.3. The equivalent of the continuity and Poisson equations

Starting from the standard equations, where the symbols with primes are associated with the standard framework based on general relativity, the equation of continuity and Poisson equations are:

$$\frac{\partial \varrho'}{\partial t'} + \nabla \cdot (\varrho' \varphi') = 0, \quad (9)$$

$$\nabla^2 \varphi' = \Delta' \varphi' = 4\pi G \varrho'.$$  

(10)

With the above transformations and scaling properties for $r' = \lambda r$, $\varrho = \lambda^2 \varrho'$ and $p = \lambda^2 p'$ and $\nabla \lambda = 0$ the equation of continuity becomes:

$$\frac{1}{\lambda} \frac{\partial}{\partial t} \left( \frac{\varrho}{\lambda^2} \right) + \frac{1}{\lambda} \nabla \cdot \left( \frac{\varrho}{\lambda^2} \left( \ddot{r} + \frac{\kappa}{\lambda} \right) \right) = 0,$$  

(11)

$$\frac{1}{\lambda^2} \left[ \frac{\partial p}{\partial t} - 2 \frac{\lambda}{\rho} \ddot{r} + \nabla \cdot (\rho \varphi + \frac{\kappa}{\lambda} \varphi \cdot (\rho \ddot{r}) \right] = 0.$$  

(12)

In the last term, the expression $\nabla \cdot (\rho \dot{r}) = \dot{r} \cdot \nabla \rho + \rho \nabla \cdot \ddot{r} = \dot{r} \cdot \nabla \rho + 3 \rho$ can be used. Then the scale-invariant continuity equation becomes:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \dot{r}) = \kappa \left[ \rho + \dot{r} \cdot \nabla \rho \right].$$  

(13)

Evidently, if $\lambda$ is a constant ($\kappa = 0$), the equation is the usual one. There is an additional term proportional to the density which in conventional treatments of such equations can be viewed as related to sinks and sources while the second term is related to the density gradient and thus can be viewed as diffusion term from point of view of transport theory. However, it is important to remember that the additional terms come from the scaling, which was recalled above, and are related to the corresponding conservation laws (Maeder 2017a).
Let us now turn to the Poisson equation. The potential $\Phi$ considered refers to the Newtonian contribution only and not to the additional term that the scale-invariant equation of motion is containing (see Sec. 2.2 by Maeder (2017c)). There, one has also seen the interesting property that the potential is an invariant, i.e. $\Phi' = \Phi$. Thus, (10) is now:

$$\frac{1}{\lambda^2} \Delta \Phi' = 4\pi G \frac{\rho}{\lambda^2},$$

and noticeably the Poisson equation stays the same,

$$\nabla^2 \Phi = \Delta \Phi = 4\pi G \rho.$$

For homogeneous density distribution $\rho$, the potential $\Phi$ and its gradient are ($\nabla \rho = 0$): $\Phi = \frac{2\pi}{3} G \rho r^2$, and $\nabla \Phi = \frac{4\pi}{3} G \rho \vec{r}$, like in the usual Newtonian case.

3. THE APPLICATION OF THE THREE EQUATIONS TO DENSITY PERTURBATIONS IN THE SCALE-INvariant THEORY

In the standard case, the theory has been developed by several authors, here the methods described by Peebles (1980) and Theuns (2016) will be followed. In the expanding Universe, the position of a point is $\vec{r} = a \vec{x}$, where $a = a(t)$ is the expansion factor and $\vec{x}$ is the co-moving coordinate. Then the velocity of the considered point is

$$\vec{v} = \dot{\vec{r}} = \dot{a} \vec{x} + a \dot{\vec{x}},$$

where the first term is the Hubble velocity,

$$\dot{a} \vec{x} = Ha \vec{x} = H \vec{r}, \quad \text{with} \quad H = \dot{a}/a,$$

and $\vec{v}_p = a \dot{\vec{x}}$ is the peculiar velocity. Since functions can be expressed either via $\vec{x}$ coordinates or via $\vec{r}$ then the corresponding differential operators will be:

$$\nabla = \frac{\partial}{\partial x}, \quad \nabla a = 0, \quad \nabla_r = \frac{1}{a} \nabla, \quad \nabla_r^2 = \frac{1}{a^2} \nabla^2,$$

$$\left( \frac{\partial}{\partial t} \right)_r = \left( \frac{\partial}{\partial t} \right)_x - \left( H \vec{x} \cdot \vec{\nabla} \right)$$

This choice guarantees that $(\frac{\partial \vec{r}}{\partial t})_r = 0$ when $\vec{r} = a \vec{x}$ is used. Notice that all the previously discussed operators $\nabla$ were actually $\nabla_r$ type operators.

3.1. The equation of continuity for the perturbation

Now, consider a density function $\rho$ related to the unperturbed “background” density $\rho_b$ and expressed as

$$\rho(\vec{x}, t) = \rho_b(t)(1 + \delta(\vec{x}, t)), \quad (20)$$

where $\delta(\vec{x}, t)$ is the density perturbation. Notice that the continuity equation (13) is expressed in $\vec{\nabla}$ symbols with respect to $\vec{r}$. By switching from $\vec{r}$ to $\vec{x}$ and inserting the above expression of $\rho(\vec{x}, t)$ in the continuity equation one obtains:

$$\left( \left( \frac{\partial}{\partial t} \right)_x - H \vec{x} \cdot \vec{\nabla} \right) \rho_b(t)(1 + \delta) + \frac{1}{a} \vec{\nabla} \cdot (\rho_b(t)(1 + \delta)(\dot{a} \vec{x} + \vec{v}_p))$$

$$= \kappa \left[ \rho_b(t)(1 + \delta) + \vec{x} \cdot \vec{\nabla} (\rho_b(t)(1 + \delta)) \right].$$

By expanding the terms in this equation the expression becomes:

$$(1 + \delta) \left( \frac{\partial \rho_b}{\partial t} \right)_x + \rho_b \left( \frac{\partial \delta}{\partial t} \right)_x - H \vec{x} \cdot \vec{\nabla} (1 + \delta) \rho_b$$

$$+ \frac{\dot{a}}{a} (1 + \delta) \rho_b \vec{\nabla} \cdot \vec{x} + \frac{\dot{a}}{a} \vec{x} \cdot \vec{\nabla} (1 + \delta) \rho_b$$

$$+ \frac{\vec{v}_p}{a} (1 + \delta) \cdot \vec{\nabla} \rho_b + \frac{\rho_b}{a} \vec{\nabla} \cdot (1 + \delta) \vec{v}_p$$

$$= \kappa \left[ (1 + \delta) \rho_b + \vec{x} \cdot \vec{\nabla} \rho_b(1 + \delta) \right].$$
The third and fifth terms cancel while the sixth term is zero since the background density is homogeneous, thus one has

\[
(1 + \delta) \left( \frac{\partial \varrho_b}{\partial t} \right)_x + \varrho_b \left( \frac{\partial \delta}{\partial t} \right)_x + \frac{\dot{\varrho}_b}{a} (1 + \delta) \varrho_b \vec{x} + \frac{\varrho_b}{a} \vec{\nabla} \cdot (1 + \delta) \vec{v}_p
= \kappa \left( (1 + \delta) \varrho_b + \vec{x} \cdot \vec{\nabla} \varrho_b \right).
\]

(23)

If one considers homogeneous background density and only the terms independent of the perturbation, that is to say, independent of \(\delta\) and \(\vec{v}_p\), then one gets:

\[
\dot{\varrho}_b + H \varrho_b \vec{\nabla} \cdot \vec{x} = \kappa \varrho_b \quad \text{or} \quad \dot{\varrho}_b + 3H \varrho_b = \kappa \varrho_b.
\]

(24)

This equation expresses the evolution of the background density. In the standard case, the right-hand side is zero. The same relationship, but multiplied by \(\delta\), is contained in Equation (23). This allows further simplifications in this equation, which now only remains as:

\[
\varrho_b \left( \frac{\partial \delta}{\partial t} \right)_x + \frac{\varrho_b}{a} \vec{\nabla} \cdot (1 + \delta) \vec{v}_p = \kappa \vec{x} \cdot \vec{\nabla} \varrho_b \delta,
\]

(25)

where the derivative of \(\delta\) is at constant \(x\). With further manipulations using that \(\varrho_b\) is homogeneous and the expressing \(\vec{v}_p = a \vec{x}\), this equation finally becomes:

\[
\dot{\delta} + (1 + \delta) \vec{x} \cdot \vec{\nabla} \delta = \kappa \vec{x} \cdot \vec{\nabla} \delta,
\]

(26)

where the partial time derivative \(\partial \delta / \partial t\) has been combined with the term \(\dot{\delta} \vec{x} \cdot \vec{\nabla} \delta\) to the full time derivative \(\dot{\delta}\).

Let us examine this equation, which depends on the density gradient \(\vec{\nabla} \delta\) of the perturbation. The product \(\vec{x} \cdot \vec{\nabla} \delta\) expresses the projection of the density gradient along the vector \(\vec{x}\), corresponding to the displacement of a piece of the fluid element initially at rest in the comoving medium. Indeed, one is considering a particular element of the background medium of density \(\varrho_b\): from its initial position, this fluid element starts moving with a velocity \(\vec{v}_p\) towards the nascent density fluctuation considered as spherically symmetric. Thus, the product \(\vec{x} \cdot \vec{\nabla} \delta\) representing the above projection is positive, since the directions of both vectors are the same. Therefore, one has:

\[
\vec{x} \cdot \vec{\nabla} \delta \approx x \frac{\partial \delta}{\partial x} = \frac{\partial \ln \delta}{\partial \ln x} \delta \equiv n \delta.
\]

(27)

Alternatively, one can consider separation of the spatial and time dependence of \(\delta\) such that \(\delta(t, \vec{x}) = \delta(t) \delta(\vec{x})\). The assumption of separability applies only to pressureless perturbations, which is the case considered in this paper. Then \(n\) can be defined as the order of homogeneity according to the Euler’s theorem for homogeneous functions \((\vec{x} \cdot \vec{\nabla} \delta(\vec{x}) = n \delta(\vec{x}))\). Thus, \(n\) encodes information only about the geometric shape of \(\delta(\vec{x})\) while the magnitude of \(\delta(\vec{x})\) is arbitrary. From what follows, the anticipated typical value is \(n = 2\), but in Sec. 4 the effects of nearby values such as \(n = 1, 3\) and \(5\) will be explored as well. The linearized equation for the density fluctuation finally becomes:

\[
\dot{\delta} + (1 + \delta) \vec{x} \cdot \vec{\nabla} \delta = n \kappa(t) \delta.
\]

(28)

This equation shows that positive values of \(n\), as indicated above, lead to a growth of the density fluctuations. In the standard case, the right-hand side is zero. Thus, scale invariance leads to an additional growth of the density fluctuations, which depends on the slope \(n\) of the density gradient. The amplitude of this term is also dependent on the epoch because of \(\kappa = 1/t\).

The outer layers of clusters of galaxies may be represented by an isothermal polytrope, \(i.e.,\) with a polytropic index equal to 5. The distribution of the density in the outer layers of such a spherical polytrope follows a law of the form \(\varrho \sim 1/r^2\) (Chandrasekhar 1960). This density profile has been confirmed by extended numerical simulations, leading to the so-called NFW profile, which in the outer layers of a cluster is very close to the \(\varrho \sim 1/r^2\) profile (Navarro et al. 1995). Within the core of a cluster at equilibrium, the density law is slightly less steep with a density law between laws with \(r^{-2}\) and \(r^{-1}\) (Navarro et al. 1995; Ettori 2000). In the expressions of these laws, \(r = 0\) at the center of the perturbation and \(\varrho\) is decreasing outwards. Thus, usually within this system of coordinates where \(\varrho \sim 1/r^2\), homogeneous functions with a negative \(n\) would have to be considered. However, the vector \(\vec{x}\) defined in the above
developments goes oppositely, towards the central perturbation. Thus, in the direction of the vector $\mathbf{x}$, the density is growing like $x^2$ when approaching the central perturbation, where $x$ is the modulus of $\mathbf{x}$. Therefore, the density profiles to be considered here are homogeneous functions with a positive value of $n$, of about $n = +2$. This central profile corresponds to the equilibrium structure, which is not necessarily fully reached during the short formation process under study, and it is interesting to consider several values of $n$ in the numerical models of a spherical perturbation with a radial infall motion directed towards the center.

3.2. The Poisson equation for the perturbation

It has been discussed above that in the scale-invariant context the Poisson equation remains the same, see Equation (15). The density perturbations influence the Newtonian potential $\Phi$. With relation (18), the Poisson equation for the perturbed density becomes:

$$\nabla^2 \Phi = 4\pi G\rho_b(1 + \delta)$$

Let’s define a potential $\Psi$ due to the perturbation of density:

$$\nabla^2 \Psi = 4\pi G a^2 \rho_b \delta.$$  \hspace{1cm} (30)

Thus, the total potential $\Phi$ may be expressed as:

$$\nabla^2 \Phi = \nabla^2 \Psi + 4\pi G a^2 \rho_b.$$ \hspace{1cm} (31)

Up to a sign, the Newtonian force due to the total potential becomes:

$$\mathbf{\nabla} \Phi = \mathbf{\nabla} \Psi + \frac{4\pi}{3} G \rho_b a^2 \mathbf{x}.$$ \hspace{1cm} (32)

It will be useful to have a specific potential $\Psi$ attached to the perturbation of the background density in view of the Euler equation describing the motions associated with perturbations.

3.3. The Euler equation for the perturbation in the expanding Universe

In an expanding Universe with perturbed density fluctuation, after taking care that the derivative $\frac{\partial}{\partial t}$ is taken at constant $\mathbf{x}$, the scale-invariant equation corresponding to the Euler equation (8) becomes:

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \left( \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla \Phi - \frac{1}{\rho} \nabla \rho + \kappa \mathbf{v}.$$  \hspace{1cm} (33)

Which when re-expressed in the comoving $\mathbf{x}$ coordinates and in terms of the potential $\Psi$ is:

$$\left( \frac{\partial}{\partial t} - H \mathbf{x} \cdot \nabla \right) (\dot{\mathbf{a}} \mathbf{x} + \dot{\mathbf{v}}_p) + \frac{1}{a} [(\dot{\mathbf{a}} \mathbf{x} + \dot{\mathbf{v}}_p) \cdot \nabla] (\dot{\mathbf{a}} \mathbf{x} + \dot{\mathbf{v}}_p) =$$

$$-\frac{1}{a} \left( \nabla \Psi + \frac{4\pi}{3} G \rho_b a^2 \mathbf{x} \right) + \kappa(t)(\dot{\mathbf{a}} \mathbf{x} + \dot{\mathbf{v}}_p) - \frac{1}{a\rho} \nabla \rho.$$ \hspace{1cm} (34)

By expanding the various terms in the above equation and considering a polytropic gas $p = p(\rho)$, where $\nabla \rho = v_s^2 \nabla \rho$ and $v_s$ is the sound velocity, one gets:

$$\dot{a} \mathbf{x} + \dot{v}_p = H \dot{a} \mathbf{x} - H(\mathbf{x} \cdot \nabla) \dot{v}_p + \frac{1}{a} \left[ \dot{a} \mathbf{x} + \dot{a}(\mathbf{x} \cdot \nabla) \dot{v}_p + \dot{a} \dot{v}_p + (\mathbf{v}_p \cdot \nabla) \dot{v}_p \right] =$$

$$\frac{1}{a} (\nabla \Psi + \frac{4\pi}{3} G \rho_b a^2 \mathbf{x}) - \frac{v_s^2}{a} \frac{\nabla \rho}{\rho} + \kappa(t)(\dot{a} \mathbf{x} + \dot{v}_p).$$ \hspace{1cm} (35)

On the left side, the third term cancels with the fifth one, the fourth term cancels with the sixth one. Note that the scale-invariant cosmological models obey the following equation (see Equation (29) in Maeder (2017a)):

$$-\frac{4\pi G \rho_b}{3} = \frac{\dot{a}}{a} + \frac{\dot{\lambda}}{a \lambda} \text{, or } \frac{\dot{a}}{a} = -\frac{4\pi G \rho_b a}{3} + \dot{\kappa}(t).$$
Such equation can also be obtained by setting \( v_p = 0 \) and demanding that the resulting equation is true for any \( \vec{x} \). This brings further simplifications in Equation (35) where the terms \( \kappa(t) \vec{a} \vec{x} \) and \( \dot{\vec{a}} \vec{x} \) on both sides will cancel as well,

\[
\dot{\vec{v}}_p + H \vec{v}_p + \frac{1}{a} (\vec{v}_p \cdot \nabla) \vec{v}_p = -\frac{\vec{\nabla} \Psi}{a} - \frac{v_p^2}{\varrho} \vec{\nabla} \varrho \quad + \kappa(t) \vec{v}_p. \tag{36}
\]

By using \( \vec{v}_p = a \dot{\vec{x}} \) and \( \dot{\vec{v}}_p = \dot{a} \dot{\vec{x}} + a \ddot{\vec{x}} \) this equation becomes:

\[
\ddot{\vec{x}} + 2H \dot{\vec{x}} + (\dot{\vec{x}} \cdot \vec{\nabla}) \vec{x} = -\frac{\vec{\nabla} \Psi}{a^2} - \frac{v_p^2}{a^2} \frac{\vec{\nabla} \varrho}{\varrho} + \kappa(t) \dot{\vec{x}}. \tag{37}
\]

This is the expression of the Euler equation applied to the perturbation of the density in the comoving \( \vec{x} \) coordinates system. It differs from the standard case by the last term on the right. In the above expression, the pressure term is indicated. However, we have already discussed that the approximation made in the equation of continuity is valid for pressureless perturbations. Therefore in what follows we will consistently consider the perturbations in a dust universe where the negligible pressure term is absent.

### 3.4. The evolution of the density contrast in the scale-invariant theory

The three equations (28), (31), and (37) form the necessary basis for the study of the evolution of perturbation in a dust Universe. In more general cases, for example for the radiative era, an energy equation must be added (Peebles 1980; Theuns 2016). The scale-invariant effects appear through the terms containing \( \kappa(t) \), which are present in Equations (28) and (37). Let’s concentrate on small perturbations and examine the linear approximation of these three equations. Under this hypothesis, Equation (28) may be simplified:

\[
\vec{\nabla} \cdot \vec{x} = \frac{1}{(1 + \delta)} \left( n \kappa(t) \delta - \dot{\delta} \right) \approx n \kappa(t) \delta - \dot{\delta} + O(\delta^2) \tag{38}
\]

Neglecting terms of order higher than on \( \delta \), the continuity equation for the perturbation further simplifies. Finally, in a dust universe, the three equations become:

\[
\ddot{\delta} + \vec{\nabla} \cdot \dot{\vec{x}} = n \kappa(t) \delta, \tag{39}
\]

\[
\vec{\nabla}^2 \Psi = 4\pi G a^2 \varrho_0 \delta, \tag{40}
\]

\[
\ddot{\vec{x}} + 2H \dot{\vec{x}} + (\dot{\vec{x}} \cdot \vec{\nabla}) \vec{x} = -\frac{\vec{\nabla} \Psi}{a^2} + \kappa(t) \dot{\vec{x}}. \tag{41}
\]

Where the approximation \( \frac{1}{\varrho} \vec{\nabla} \varrho = \frac{1}{(1 + \delta)} \vec{\nabla} (1 + \delta) \approx \vec{\nabla} \delta \) has been utilized. Now, consider the time derivative of (39) and take the divergence of (41) to obtain:

\[
\ddot{\delta} + \vec{\nabla} \cdot \ddot{\vec{x}} = n \kappa \delta + n \kappa \dot{\delta}, \tag{42}
\]

\[
\vec{\nabla} \cdot \ddot{\vec{x}} + 2H \vec{\nabla} \cdot \dot{\vec{x}} + \vec{\nabla} \cdot (\dot{\vec{x}} \cdot \vec{\nabla}) \vec{x} = -\frac{\vec{\nabla}^2 \Psi}{a^2} + \kappa(t) \vec{\nabla} \cdot \dot{\vec{x}}. \tag{43}
\]

Combining these two equations and using (38) to justify neglecting \( \vec{\nabla} \cdot (\dot{\vec{x}} \cdot \vec{\nabla}) \vec{x} \approx O(\delta^2) \), then one arrives at

\[
\ddot{\delta} + (2H - (1 + n) \kappa) \dot{\delta} = 4\pi G \varrho_0 \delta + 2n \kappa (H - \kappa) \delta. \tag{44}
\]

This is the basic equation for the growth of density perturbations in the scale-invariant theory. The term \( \kappa \) multiplying \( \dot{\delta} \) originates both from the continuity and Euler equations. On the right, the term \( \kappa \) multiplying \( H \) comes from the continuity equation, while the products \( \kappa^2 \) comes from both equations. The terms with \( \kappa = -\lambda/\lambda \) are absent in the standard models (there \( \lambda = 1 \)) where one has the well-known equation (Peebles 1980):

\[
\ddot{\delta} + 2H \dot{\delta} = 4\pi G \varrho_0 \delta. \tag{45}
\]
For the standard EdS model of a matter dominated Universe, where \( a(t) \propto t^{2/3} \), this last equation predicts a growth of the perturbation \( \delta \propto t^{2/3} \propto a(t) \propto (1 + z)^{-1} \). Thus, starting from the perturbation of the CMB \( \delta \approx 10^{-5} \) at a redshift \( z \approx 10^3 \), the density fluctuations should be of the order of \( 10^{-2} \) at the present time. This is clearly much smaller and thus in strong contradiction with the observed density structures in today’s Universe, a fact which is often considered as one of the most convincing argument in favor of the existence of dark matter (Theuns 2016).

3.5. The coefficients in the equation for the density perturbations

The differential equation (44), which determines the evolution of density perturbations in the scale-invariant context, is a homogeneous differential equation of the second order with non-constant coefficients. It may be written as

\[
\ddot{\delta} + A \dot{\delta} = B \delta, 
\]

with \( A = 2H - (1 + n)\kappa \) and \( B = 4\pi G \rho_m + 2n\kappa(H - \kappa) \).

The coefficient \( A \) expresses the damping factor. In the standard case, where \( A = 2H \), this term contributes negatively to the second derivative \( \ddot{\delta} \), thus acting as a damping factor. The reason is that the Hubble expansion gives an opposing effect to the accumulation of matter on a particular point of space; thus, it does not favor the growth of density fluctuations. As a consequence, the growth of the density perturbation in the standard case of an expanding universe tends to follow a power law instead of an exponential growth as in the static case. In the scale-invariant case, the additional acceleration term in Equations (4) and (8) also favors a power law, however with a much larger power than in the standard EdS case. If coefficient \( A \) is negative, which is generally the case as seen in Fig. 1, it acts as a growth factor. There the additional \( \kappa \) term contributes to increase the infall and this effect overcomes the opposing effect of the Hubble expansion during most of the evolution (Fig. 2).

The coefficient \( B \) contributes, if positive, to the growth of the density fluctuations. The term \( 4\pi G \rho_m(t) \) expresses the effect of the Newtonian acceleration which always favors the accumulation of matter for a positive density fluctuation. The second term results from the effects of the continuity and Euler equations. Its contribution depends on the sign of \( H - \kappa \), which in a given model changes according to the epochs as shown in Fig. 2. From this figure, one can see that during most of the evolution of the density perturbations, the contribution of the \( H - \kappa \) term is negative.

Since the coefficients are not constant, there is no general expression for the solutions of Equation (46). As stated in Sec. 2.1, the time \( t \) in the modified Euler equation and in the continuity equation is the cosmic time, i.e the time in co-moving galaxies. Also, note that the above choice is also consistent with a timescale of the empty scale-invariant cosmological model that starts at time \( t = 0 \). One should also notice that these equations are “gauge invariant” (e.g.
for \( \lambda \rightarrow \hat{\lambda} \) as long as one takes care to use the correct functional forms of the corresponding quantities, like \( H, \kappa \) and so on for the chosen time gauge.

For the purpose of the easier comparison with observations, however, the constants \( A \) and \( B \) and their ingredients \( H, \kappa \) and \( 4\pi G\varrho_0 \) have to be consistently expressed in the cosmic time \( \tau \), which is different from the arbitrary timescale, say \( t \), used in the early applications of the scale-invariant cosmological models. The expression giving the cosmic time \( t \) as a function of \( \tau \) is:

\[
t = \frac{\tau - \tau_m}{\tau_0 - \tau_m} \Delta_0, \quad \text{with} \quad \tau_m = \tau_0 \Omega_m^3,
\]

where \( \Omega_m = \varrho / \varrho_c \) is the usual density parameter with \( \varrho_c = 3H_0^2/(8\pi G) \). The present time \( \tau_0 \) is fixed to \( \tau_0 = 1 \) in dimensionless time units \( \Delta_0 = 1 \) (\( \tau \)-scale). To get the cosmic time in years, \( \Delta_0 \) may be taken equal to \( 13.8 \times 10^9 \) yrs. (Frieman et al. 2008). The analytical expressions of the scale factor \( a(\tau) \) and \( H(\tau) \) for flat scale-invariant models have been given by Jesus (2017), they exactly correspond to the numerical solutions developed in cosmological models by Maeder (2017a). In the \( \tau \)-scale, the age of the universe is \( \tau_{\text{tot}} = 1 - \tau_m = 1 - \Omega_m^3 \). To express the Hubble “constant” in the cosmic timescale \( t \) consider the following relations:

\[
H(t) = \frac{1}{a} \frac{da}{dt} = \frac{1}{a} \frac{d}{d\tau} \frac{da}{d\tau} = H(\tau) \frac{d\tau}{dt} = H(\tau)(1 - \Omega_m^3) \times \Delta_0^{-1}, \quad \text{with} \quad H(\tau) = \frac{2\tau^2}{\tau^3 - \Omega_m^3},
\]

according to the expression of \( H(\tau) \) given by Jesus (2017). To evaluate the Hubble constant in \([s^{-1}]\) units at a given epoch, the value of \( H(t) \) in Equation (49) should be divided by the current age of the universe \( \Delta_0 = 4.355 \times 10^{17} \) s. There is also a hidden timescale in the term \( 4\pi G\varrho_0 \), since \( \sqrt{G/\varrho} \) is the inverse of the oscillation timescale of the system of density \( \varrho \). For a density parameter \( \Omega_m \) the present-day density of the Universe writes as:

\[
\varrho_m(t_0) = \Omega_m \varrho_c(t_0) = \Omega_m \frac{3H_0^2(t_0)}{8\pi G} = \Omega_m \frac{3H_0^2(\tau_0)}{8\pi G}(1 - \Omega_m^3)^2 \times \Delta_0^{-2}.
\]

These expressions in (50) and (49) show that the Hubble “constant” \( H \) behaves as co-scalar of rank 1, while the density \( \varrho \) behaves as co-scalar of rank 2; furthermore, based on (44) one see that \( \delta \) is a in-scalar according to the terminology discussed in Maeder (2017a) and Maeder & Bouvier (1979).

Since in the scale-invariant cosmological models the matter density obeys the conservation law \( \varrho_m(\tau)a^3(\tau)\lambda(\tau) = \text{const.} \) as shown by Equation (61) in (Maeder 2017a), then \( \varrho_m(\tau_0)a^3(\tau_0)\lambda(\tau_0) = \varrho_m(\tau_0)a^3(\tau_0)\lambda(\tau_0) = \varrho_m(\tau_0)\lambda(\tau_0) \), because \( \lambda(\tau_0) / \lambda(\tau) = \tau \) and \( a(\tau_0) = 1 \) in the \( \tau \)-scale one gets:

\[
4\pi G\varrho_m(\tau) = \frac{4\pi G\varrho_m(\tau_0)\tau}{a(\tau)^3}.
\]
This result is consistent with the integration of (24) using the exact solution by Jesus (2017) when the evolution of the background density in the cosmological models has been calculated within the τ-scale. Then by using \( \varrho_m(t) = \varrho_m(\tau)(1 - \frac{1}{2} \Omega_m^2) \Delta_0^{-2} \) the term \( 4\pi \varrho_m(t) \) at a certain cosmic time \( t \) can be written as:

\[
4\pi G \varrho_m(t) = 4\pi G \Omega_m \varrho_c(\tau_0)(1 - \frac{1}{2} \Omega_m^2) \frac{\tau(t)}{a^3(\tau(t))} = \frac{3}{2} \Omega_m H_0^2(\tau_0) \frac{(1 - \frac{1}{2} \Omega_m^2) \tau(t)}{a^3(\tau(t))}.
\]

Furthermore, by using \( H_0^2(\tau_0) = 4/(1 - \Omega_m) \) according to the expression for \( H(\tau) \) and with the analytical solution \( a(\tau) = (\tau^2 - \Omega_m^2)^{2/3} \) obtained by Jesus (2017), one finally has

\[
4\pi G \varrho_m(t) = \frac{6 \Omega_m \tau(t)}{(\tau_0 + \Omega_m)^2} (1 - \frac{1}{2} \Omega_m^2) \Delta_0^{-2},
\]

as a consistent expression of the density term in the Newtonian framework with the cosmic time as an independent variable. One can see that the references to the cosmic time entering in Equation (48) occur through the expression of the Hubble parameter given by Equation (49).

4. NUMERICAL RESULTS FOR THE EVOLUTION OF THE DENSITY FLUCTUATIONS

Let’s examine the growth of the density fluctuations from the initial conditions in the CMB to the present time for several choices of parameters and initial conditions.

4.1. Numerical values of the coefficients

The exact value of \( \Omega_m \) in the scale-invariant framework is unknown for now. Therefore, one shall explore a large range of density parameters, say from \( \Omega_m = 0.02 \) to 0.50. Let us first examine the numerical values of parameters \( H, \kappa, A \) and \( B \) in the scale-invariant context for a chosen, rather conservative value of \( \Omega_m = 0.10 \), a value smaller than the one currently adopted since there is likely no need of dark matter within the scale-invariant context (Maeder 2017c). Fig. 2 shows the values of the Hubble constant \( H(t) \) and metrical function \( \kappa(t) \) as functions of \( \log(1 + z) \) in this example. One can see that the value of \( \kappa(t) \) is always rather close to \( H(t) \), being higher by about 0.176 dex for redshifts \( z \) larger than about 100. The separation between the two curves diminishes down to zero at \( z + 1 = 1.978 \), where the two curves cross each other and then \( H(t) \) becomes slightly larger than \( \kappa(t) \). Since \( H \) and \( \kappa \) are co-scalar/co-vector of the same rank, one can use the analytical expressions in Jesus (2017) to obtain the value of \( \tau \) when the crossing occurs, \( \tau_\kappa = (1 + \sqrt{5})/2 \Omega_m^{1/3} \) in the \( \tau \)-scale, and then the reciprocal of the scale factor \( a(\tau) \) at that time gives \( z + 1 = 1/a(\tau) \). This crossing point and its relevant values can be used to assess the accuracy of any numerical solutions.

The trends of \( H \) and \( \kappa(t) \), illustrated in Fig. 2 for the same model as above, determine the trends for the parameters \( A \) and \( B \) (Fig. 1). The solutions of the equation (46) depend on the discriminant \( \Delta \) of the equation. From (46), one has \( \Delta = \sqrt{A^2 + 4B} \) and as illustrated in the example of Fig. 1, the sum \( A^2 + 4B \) is always positive, which means that the solutions of the differential equation are real. Let’s concentrate on the growing solution, as is usually the case in the study of fluctuations. Parameter \( A = 2H - 4\kappa \) is negative in the range of redshifts considered since \( 2\kappa \) is always bigger than \( H \). While in the standard EdS model, the parameter \( A \) is positive and acts as a damping effect, now the opposite is happening: \( A \) is negative and favors the growth of the density fluctuations. In the example of Fig. 1, parameter \( B = 4\pi G \varrho_m + 6\kappa(H - \kappa) \) is negative for values above \( z + 1 = 3.16 \), the second term being larger than the first one at the low density considered. However, the constructive effect of parameter \( A \) dominates with respect to \( B \) for the growth of the density fluctuations.

4.2. The initial conditions

The amplitudes of the CMB fluctuations are about \( \delta = 10^{-5} \) at the epoch of recombination that occurs at a redshift of the order of \( z = 10^3 \). A value \( \delta = 1.1 \cdot 10^{-5} \) is estimated at low angular separations, while larger values (up to about \( 8 \cdot 10^{-5} \)) are found at larger separations (Hu & Dodelson 2002; Planck Collaboration 2014; Clements 2017). The estimate of the recombination epoch depends on the model and on the treatment of the physics of the recombination processes, for example, the recombination is estimated to occur at a redshift \( z = 1376 \) by Durrer (2008). In the scale-invariant model, due to the difference of the scale-factors between the ΛCDM and the scale-invariant model (Maeder 2017b), it could be earlier, i.e around \( z = 1680 \). Evidently, the earlier is the start of the growing process, the larger
is the final growth of the density perturbations. Here, in order to be rather conservative let’s keep the start of the growth at an initial redshift $z_{in} = 1376$ with an initial value of $\delta = 10^{-5}$. However, the overall results are the same if one considers $z_{in} = 1000$ instead. Even, tests with $z_{in} = 3000$ and $z_{in} = 500$ are shown in Fig. 3 to demonstrate the stability of the results.

To integrate a second-degree differential equation (46), one also needs the value of the derivative $\dot{\delta}$ at the starting point. For this purpose, consider different approaches to estimate the derivative $\dot{\delta}$ at the initial point. During the radiative era, the density fluctuations with a mass larger than the Silk mass survive without growing or declining. Thus, let us first assume that one has $\ddot{\delta} = 0$ initially. Thus, Equation (46) becomes

$$A \dot{\delta} \approx B \delta, \quad \frac{\delta_2}{\delta_1} \approx e^{\frac{4}{3} \Delta t},$$

(54)

where $\Delta t$ the initial time step $\Delta t = t_2 - t_1$ for the corresponding values of the perturbations $\delta_2$ and $\delta_1$. Let us consider a numerical example with the model for $\Omega_m = 0.1$ and parameter $n = 2$. At time $t_1$ corresponding to $z + 1 = 1377$, we have $\delta_1 = 10^{-5}$, $A = -3.2773 \cdot 10^4$ and $B = -2.5779 \cdot 10^8$. For a time step $\Delta t = 5.598 \cdot 10^{-7}$ in the cosmic timescale, we have $\frac{B}{A} \Delta t = 4.404 \cdot 10^{-3}$ and thus $\delta_2 = 1.0044 \delta_1$ at redshift $z_2 + 1 = 1366.986$. Thus, for a ratio of the redshifts $\frac{z_2 + 1}{z_1 + 1} = 1.00733$, we have a corresponding ratio of the perturbations $\frac{\delta_2}{\delta_1} = 1.0044$, which implies an initial slope $s$ of the relation $\log \delta$ vs. $\log(z + 1)$ equal to $s = 0.60$, a bit lower than the slope $s = 1$ of the classical EdS model for the growth of perturbations.

Let us consider Equation (39) which says that the growth of $\delta$ results from the motion gradient and from the cosmological effect of scale invariance. Let us assume that the motion is initially negligible at the very beginning of the matter dominated era and that the growth is only due to the cosmological effect of scale invariance:

$$\dot{\delta} \approx n \kappa(t) \delta, \quad \frac{\delta_2}{\delta_1} \approx e^{n \kappa \Delta t}.$$  

(55)

Let us make an estimate for the same case as above. The cosmic time $t$ corresponding to $z + 1 = 1377$ is $t = 5.0853 \cdot 10^{-5}$ (with $t_0 = 1$). For the same $\Delta t$, we have $n \kappa(t) \Delta t = 2.202 \cdot 10^{-2}$ and $e^{n \kappa \Delta t} = 1.0223$, thus we have at time $t_2$ a
perturbation $\delta_2 = 1.0223 \cdot 10^{-5}$. This ratio has to be compared to $\frac{z_2+1}{2}$ = 1.00733 and this gives a slope $s = 3.03$, about three times larger than the classical EdS slope.

Thus, in what follows we shall adopt for the initial value of $\dot{\delta}$ the value corresponding to $s = 1$ of the EdS model. We will see that the true value is likely closer to the second estimate rather than to the first one. However, this rather conservative choice $s = 1$ allows us to not overestimate the initial growth of the density fluctuations. Also, as we see in Fig. 4, for a relatively large range of initial slopes, the models converge towards the same slope, typically when $\delta$ becomes larger than about $10^{-4}$ depending on the models. Finally, we note that the oscillations occurring in the radiative era also provide significant values of $\dot{\delta}$ (Coles & Lucchin 2002), which may also contribute to “the launch” of the growth of the density fluctuations.

4.3. The almost linear growth of the density fluctuations

As discussed above, in the first approach we are choosing the low slope present in the standard EdS model as the initial value of $\dot{\delta}(z_m)$. The EdS model behaves like $\delta \propto t^{2/3} \propto 1/(z + 1)$ and the corresponding slope ($s = 1$ in the log $\delta$ vs. log($z + 1$)) is illustrated in Fig. 4 in the case of $n = 2$. Scale-invariant solutions for the growth of the density fluctuations have been calculated for $\Omega_m = 0.02, 0.10, 0.30$ and 0.50, in order to encompass a large range of possibilities. Interestingly enough, after having followed for a short time the initial slope of the EdS model, the resulting scale-invariant solutions show a much faster growth of the perturbation $\delta$. Some small deviations from the linearity appear below $\delta \approx 10^{-4}$, where the slope is slightly lower since it is still influenced by the initial value adopted. Above a value of about $\delta \approx 10^{-4}$, these steep growths follow almost linear relations in the log $\delta$ vs. log($z + 1$) plot with slopes $s$ that we will call the convergence slopes. The relations are not strictly linear, but very close to it, the deviations being larger for the case $n = 1$ at the lowest redshifts as illustrated in Fig. 3. This indicates that in general the growth of the density fluctuations is very close to a power law dependence like $\delta \sim \left(\frac{1}{(z+1)}\right)^s$. The convergence slopes of the linear relations in the log scales are $s = 1.95, 2.22, 2.62$ and 2.90 for $\Omega_m = 0.50, 0.30, 0.10$ and 0.02 respectively, in the case $n = 2$. For the same values of $\Omega_m$ and $n = 3$, these slopes are 2.25, 2.70, 3.35 and 3.91.
We notice that lower matter density experiences a faster growth of the density perturbations. This results from the changes of $H$ and $\kappa$ with $\Omega_m$ intervening in the leading parameters $A$ and $B$. In the early evolution of the perturbation, the values of $H$ and $\kappa$ are larger (factor about 5) in model with $\Omega_m = 0.50$ than with 0.02, while the ratio between $H$ and $\kappa$ has small variations. The square of this factor (of $\sim 5$) intervenes in parameter $B$, while the effect is linear for $A$. Thus, the effect of a larger $\Omega_m$ is more prominent in the term $B$ that is contributing negatively to the perturbation than in the term $A$ which is contributing positively. This accounts for the slower growth of the perturbations in higher density models, despite the higher Newtonian attraction (which varies linearly with $\Omega_m$).

Another way to see and understand the contra-intuitive behavior of the growth of the perturbations with respect to $\Omega_m$ is to notice that $\kappa > H$ for most of the early epoch as shown in Fig. 2. Thus for $n \geq 1$, the term $A < 0$ and therefore has an anti-damping effect until the $\kappa \sim H$ epoch. The $B$ term, on the other hand, is negative at the very early times $t \gtrsim 0$ due to the $\kappa^2$ term. This $\kappa^2 = 1/t^2$ term is the same for any $\Omega_m$. However, $H$ is bigger for smaller $\Omega_m$ as seen in Table 1 (Maeder 2017a). Therefore, the smaller the $\Omega_m$, the smaller the negative term $B$, since $H$ appears as a positive term in this negative $B$ expression. Thus, the growth is controlled by the $A$ term while the suppressing effects of the negative $B$ term are smaller for smaller $\Omega_m$ values.

We also find that even if there are significant differences in the initial slope the evolution is resulting in very little differences with respect to the corresponding convergence slopes. For example, a model for $\Omega_m = 0.10$ with $n = 2$ with an initial slope 5 times smaller than the corresponding convergence slope shows deviations from the convergence slope smaller than 1% in the range of log $\delta$ from -3 to -1. With the initial slope of the EdS model, a density fluctuations $\delta = 1$ is reached for redshifts between $(z + 1) = 2.7$ and 18 for $n = 2$, depending on the $\Omega_m$ values (Fig. 4). During the subsequent evolution towards lower redshifts, the growth of the density fluctuations become non-linear and may lead to the present day inhomogeneous Universe.

4.4. Discussion of the tests and results

We now discuss the dependence of the evolution of the perturbations as a function of the various parameters characterizing the models: the matter density in the Universe $\Omega_m$, the slope $n$ of the distribution of density in the forming cluster, the initial values of $\delta$ at $z$ equal to about 1000 and the value of this initial $z$ redshift.
The effects of different slopes \( n \) of the density distribution in the infalling body are illustrated in Fig. 3, which show results for \( n = 1, 2, 3 \) and 5 with still the initial EdS slopes. We see that steeper density gradients produce a faster growth of the density fluctuations and thus an earlier entering into the nonlinear domain, which marks the time of galaxy formation. For \( \Omega_m = 0.10 \), from a value \( n = 1 \) to \( n = 5 \), the redshift when \( \delta = 1 \) is reached changes from \((z+1) = 2.2\) to about 88 (Fig. 3). Whether due to local effects the representative value of \( n \) may vary in forming clusters of galaxies is an open question, beyond the scope of the present work. But, it opens the possibility that, in some cases, galaxies form very early in the Universe!

The initial value of \( \delta \) does not influence the relative growth over a given interval of redshift. For example, a fluctuation \( \delta = 10^{-4} \) at \( z = 1000 \) will evolve all the way being always a factor of 10 larger than an initial fluctuation \( \delta = 10^{-5} \), at least as long as in the linear regime. Thus, the relative growths of the linear fluctuations predicted by Equation (46) are also scale invariant since \( \delta \) is a in-scalar. The value of the initial slope has been discussed in Sec. 4.2.

Finally, we also point out that the value of the initial \( z \) has little influence on the convergence slope. This is nicely illustrated by the two thin black broken lines in Fig. 3. For the purpose of the demonstration, one model is starting with a fluctuation \( \delta = 10^{-5} \) at \((z+1) = 3000\) and another one at \((z+1) = 500\). This is to be compared to the standard case with \((z+1) = 1377\). We notice an almost parallel evolution of the growing perturbations in the three models.

Consistently with previous remarks, the value for the convergence slopes are almost independent of the initial redshift, whatever is its particular value. Fig. 5 illustrates the results for models with \( n = 2 \) and different \( \Omega_m \) values starting at \( z = 1376 \). For this case, discussed in Sec. 4.3, the convergence slope determined for the models of various density parameters (Fig. 4) has been used to fix the initial derivative or slope of the corresponding model for the growth of density fluctuations. Fig. 5 shows that the relations are linear from the initial redshift to the present epoch. Globally, the results confirm those of Fig. 4, with the differences that due to the initial faster growth, the amplitudes \( \delta = 1 \) are reached a bit earlier. Values \( \delta = 1 \) are reached for values between \((z+1) = 4.0\) and 29.3 for the different \( \Omega_m \) (Fig. 5). Evidently, the simple extrapolation of the scale-invariant models up to the present time would lead to enormous values of the density fluctuations \( \delta \). However, this does not apply, because in the non-linear regime many other effects intervene. On one side, the growth may become initially faster, but also at some stage pressure effects may intervene limiting the growth of density fluctuations.

The general result is that the scale-invariant dynamics permits the growth of the density fluctuations from \( \delta \approx 10^{-5} \) on the last scattering surface to the significant inhomogeneities present in today’s Universe. In realistic cases, the growth of the density fluctuations is very fast, the regime where \( \delta > 1 \) is typically entered at redshifts of a few tens depending on the average background density.

5. CONCLUSIONS

In this paper, the continuity equation, the equations of Euler and Poisson have been expressed in the scale-invariant context. These equations were then used to obtain Equation (44), which governs the evolution of the density perturbations in a matter-dominated universe. Numerical solutions are obtained for an initial amplitude \( \delta = 10^{-5} \) at redshift \( z \approx 10^3 \) and various density parameters. In the standard model of a matter-dominated Universe, the fluctuations of density \( \delta \) grow like \((\frac{1}{1+z})^{s} \) with \( s = 1 \). In the scale-invariant case, the growth of density fluctuations is much faster. The values of \( s \) range from about 2.2 to 2.9 for \( \Omega_m \) between 0.30 and 0.02. This enables the density fluctuations to enter the regime with \( \delta > 1 \) long before the present time, typically at redshifts of an order 10 or more.

Thus, to form the galaxies and clusters of galaxies, the baryons do not need to settle down in a potential well previously installed by an unknown form of dark matter component, since the growth of the density fluctuations in the scale-invariant context is significant and fast enough. This result is in agreement with previous works (Maeder 2017c, 2018) showing that dark matter is not needed to account for the high velocities observed in clusters of galaxies, the high rotation velocities of stars at the edge of spiral galaxies, or the growth of the “vertical” velocity dispersion of stars as correlated to their age within our Galaxy.

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