BLACK HOLE ENTROPY WITH AND WITHOUT LOG CORRECTION IN LOOP QUANTUM GRAVITY

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Abstract

Earlier calculations of black hole entropy in loop quantum gravity have given a term proportional to the area with a correction involving the logarithm of the area when the area eigenvalue is close to the classical area. However the calculations yield an entropy proportional to the area eigenvalue with no such correction when the area eigenvalue is large compared to the classical area.

Keywords: Black holes, loop quantum gravity, entropy

Introduction

Black holes were found to satisfy an area law, with the area tending to increase quite generally. This was reminiscent of entropy, and since black hole horizons imply limitations on information, it was argued that the area could indeed be a measure of some black hole entropy.

Scalar field theory in a black hole background was then found to lead to an emission of particles at a temperature

\[ T = \frac{\hbar \kappa}{2\pi} \]

related to the surface gravity \( \kappa \). This established a precise connection of black holes with thermodynamics.

Consider the grand partition function for euclidean charged black holes:

\[ Z_{\text{grand}} \equiv e^{-\frac{\mathcal{A}+\mathcal{S}+\phi Q}{\hbar}} = e^{-I/\hbar}. \]  

The functional integral over all configurations consistent with appropriate boundary conditions is semiclassically approximated by the integrand, which involves the action \( I \). For a euclidean Reissner-Nordström black hole,

\[ I = \frac{1}{2}\mathcal{\beta}(M - Q\phi) = \frac{A}{4}. \]

Hence

\[ M = T(S + \frac{I}{h}) + \phi Q = T(S + \frac{A}{4\hbar}) + \phi Q. \]  

There is a formula called the Smarr formula:

\[ M = \frac{\kappa A}{4\pi} + \phi Q = T \frac{A}{2\hbar} + \phi Q. \]  

It implies that \( S = \frac{A}{4\pi} \).

For the extremal black hole, \( r_+ = r_-, Q = M, \phi = 1 \) This is of special interest: the topology changes discontinuously in the passage from a (euclidean) non-extremal to an extremal black hole. The action is

\[ I = \frac{1}{2}\beta(M - Q\phi) = 0, \]  

\[ M = T(S + \frac{I}{h}) + \phi Q = TS + M \Rightarrow S = 0 \]  

Here, the quantum theory is based exclusively on the extremal topology.

An alternative quantization involves a sum over topologies. Here a temperature \( \beta^{-1} \) and a chemical potential \( \phi \) are specified as inputs at the boundary of manifold, and the mass \( M \) and the charge \( Q \) of the black hole are calculated as functional integral averages. The definition of extremality \( Q = M \) is imposed on these averages. This is extremalization after quantization, as
opposed to quantization after extremalization. The partition function is of the form
\[ \sum_{\text{topologies}} \int d\mu e^{-I}, \]  
with \( I \) appropriate for non-extremal/extremal topology. The semiclassical approximation involves replacing the integral by the maximum value of the integrand. That occurs for the non-extremal case, so that once again \( S = \frac{\hbar}{4}\). 

**Counting states in loop quantum gravity**

A framework for the calculation of black hole entropy is provided by loop quantum gravity or the quantum geometry approach.

Quantum states are built up by associating spin variables with “punctures” on a horizon. The entropy is obtained by counting all possible configurations of punctures consistent with a given horizon area \( A \), i.e., a particular eigenvalue of the area operator.

A generic configuration has \( s_j \) punctures with spin \( j, j = 1/2, 1, 3/2 \ldots \)
\[ 2 \sum_j s_j \sqrt{j(j+1)} = A, \]  
\( A \) being the eigenvalue of the horizon area operator in units with
\[ 4\pi\gamma\ell_p^2 = 1, \]  
\( \gamma \) being the Barbero-Immirzi parameter, \( \ell_p \) the Planck length. There is a spin projection constraint
\[ \sum m = 0 \text{ mod } \frac{k}{2}, \text{ all punctures}, \]  
\[ m \in \{-j, -j+1, \ldots, j\} \text{ for puncture with spin } j \]  
(12)
Here \( k \) is an integer representing the level of the Chern-Simons theory, equal to the classical horizon area in the units defined above.

For simplicity, we first consider spin \( 1/2 \) on each puncture. The punctures have to be considered as distinguishable. If the number of punctures is \( p \), all with spin \( 1/2 \),
\[ 2p \sqrt{\frac{3}{4}} = A, \]  
so if we neglect the spin projection constraint, the entropy is
\[ \ln 2^p = p \ln 2 = \frac{A \ln 2}{\sqrt{3}} = \frac{A \ln 2}{4 \sqrt{3} \pi \gamma \ell_p^2}. \]  
This involves \( \gamma \), which can be chosen to yield the desired Bekenstein-Hawking entropy \( \frac{A}{4\gamma} \)
\[ \gamma = \frac{\ln 2}{\sqrt{3}}. \]  
(15)
This assumes only \( j = 1/2 \) at each puncture. For a general configuration, there may be \( s_j \) punctures with spin \( j \) with different \( j \).
\[ N = \prod_j (2j+1)^{s_j} \]  
(16)
if the \( \sum m \) constraint is neglected. Here the first factor is the number of ways of choosing the locations of spins, the second factor counts the number of spin states at different punctures. One must sum \( N \) over all nonnegative \( s_j \) consistent with a given \( A \). To estimate the sum by maximizing \( \ln N \), one has to vary \( s_j \) at fixed \( A \). The simplified Stirling approximation \( \ln p! \approx p \ln p – p \) yields
\[ \ln N = \sum_j s_j \ln \frac{2j+1}{s_j} + \left( \sum_j s_j \right) \ln \left( \sum_j s_j \right), \]  
(17)
\[ \delta \ln N = \sum_j \delta s_j \left[ \ln(2j+1) - \ln s_j + \ln \sum_k s_k \right]. \]  
(18)
With some Lagrange multiplier \( \lambda \) to enforce the area constraint,
\[ \ln(2j+1) - \ln s_j + \ln \sum_k s_k = \lambda \sqrt{j(j+1)}, \]  
(19)
\[ s_j = (2j+1) \exp \left[ -\lambda \sqrt{j(j+1)} \right] \sum_k s_k. \]  
(20)
Summing over \( j \) yields
\[ \sum_j (2j+1) \exp \left[ -\lambda \sqrt{j(j+1)} \right] = 1, \]  
(21)
which determines \( \lambda \approx 1.72 \) and
\[ \ln N = \lambda A/2. \]  
(22)
To make this \( \frac{A}{4\gamma} \) the Barbero-Immirzi parameter
\[ \gamma = \lambda/(2\pi) \approx 0.274. \]  
(23)
Summing over \( s_j \) may raise this value, the projection constraint will lower it.

To enforce the neglected constraint \( \sum m = 0 \), note that if \( p \) is odd, there is no such state, but if \( p \) is even, the number of such states is \( pC_p/2 \). For large \( p \), one may
use the full Stirling approximation: \( \ln p! \approx p \ln p - p + \frac{1}{2} \ln(2\pi p) \),
\[
\ln \frac{p!}{(p/2)!((p/2)!)} \approx p \ln 2 - \frac{1}{2} \ln p, \tag{24}
\]
\[
S \approx \frac{A}{4\ell_p^2} - \frac{1}{2} \ln A. \tag{25}
\]

The spin projection constraint contains a mod \( \frac{1}{2} k \), which has been ignored so far. But it becomes relevant for large eigenvalues of the area operator at a fixed classical area. In the pure spin 1/2 case, with an even number \( p \) of such spins, the total number of spin states is
\[
2^p = \sum_{r=0}^{p} p!C_r. \tag{26}
\]

Let \( p > k = 2n \) and for simplicity, let \( k \) be even. Define \( \ell \) by
\[
\frac{p}{2} = \ell \mod n, \quad 0 \leq \ell \leq n - 1. \tag{27}
\]

If \( \ell \) spins are up and \( p - \ell \) spins are down,
\[
\sum m = \ell - \frac{p}{2}
\]
which is zero mod \( n \), as required by spin projection condition. There are also other possibilities: \( \ell + n, \ell + 2n, \ldots, p - \ell \).

The total number of ways for spin projection zero modulo \( k/2 = n \) is
\[
N = pC_{\ell+n} + pC_{\ell-n} + \ldots + pC_{p-\ell}. \tag{28}
\]

Now, for \( 0 < s < n - 1 \), one has the identity
\[
(1 + e^{2is\pi/n})^p = \sum_{r=0}^{p} e^{2is\pi/n} p!C_r. \tag{29}
\]

Hence,
\[
e^{-2dN\pi/n}(1 + e^{2is\pi/n})^p = \sum_{r=0}^{p} e^{2is\pi/n} p!C_r. \tag{30}
\]

We have to add these equations for all values of \( s \).

Only those coefficients of \( p!C_r \) survive which have \( r = \ell \) modulo \( n \):
\[
\sum_{s=0}^{n-1} e^{-2dN\pi/n}(1 + e^{2is\pi/n})^p = nN. \tag{31}
\]

For fixed \( n \) and large \( p \), the sum is dominated by the term of highest magnitude. But
\[
1 + e^{2is\pi/n} = 2 \cos(s\pi/n)e^{is\pi/n}. \tag{32}
\]
The highest magnitude occurs for \( s = 0 \) and
\[
N \approx 2^p/n, \tag{33}
\]
other terms being exponentially suppressed for large \( p \) at fixed \( n \). Now the area eigenvalue is
\[
4\pi\gamma\ell_p^2 \sqrt{3p} \equiv A, \tag{34}
\]
and
\[
S = \log N = A \frac{\log 2}{4\pi\gamma\ell_p^2} - \log n. \tag{35}
\]

For fixed \( n \sim \) classical horizon area, this goes like \( A \).

In earlier calculations, \( n \sim p \) and this argument does not hold.

Now we come to the case of arbitrary spins. Let \( s_{jm} \) be the number of punctures with spin quantum numbers \( j, m \) in a certain configuration. The no. of all spin states is
\[
\sum_{j,m} (\sum_{s_{jm}} s_{jm})! \prod_{j,m}(s_{jm}^{-1}). \tag{36}
\]
Not all are allowed by the spin projection condition
\[
\sum_{j,m} ms_{jm} = 0, \tag{37}
\]
where strict equality will be imposed at first, other possibilities modulo \( n \) being taken into account later. States with definite area eigenvalue \( A \) have
\[
\sum_{j,m} 8\pi\gamma\ell_p^2 \sqrt{j(j+1)s_{jm}} = A. \tag{38}
\]

To maximize the probability of a configuration \( \{s\} \), one must maximize the combinatorial factor for \( \{s\} \) or its logarithm:
\[
(\sum \delta s) \ln \sum s - \sum (\delta s \ln s) = 0, \tag{39}
\]
where the simplified version of Stirling’s approximation i.e., without the square root factor is used. This relation is subject to
\[
\sum m\delta s = 0, \quad \sum \sqrt{j(j+1)}\delta s = 0. \tag{40}
\]
With two Lagrange multipliers, one finds
\[
\frac{s_{jm}}{\sum s} = e^{-\lambda} \sqrt{j(j+1)}^{-\alpha m}. \tag{41}
\]
It follows that
\[
1 = \sum s_{jm} e^{-\lambda} \sqrt{j(j+1)}^{-\alpha m}. \tag{42}
\]
Later we shall need a non-vanishing value of combinatorial factor for

deal with the integration because it is not consistent with the area

tion sum implies that

The first order variation vanishes and second order

variations are kept:

\[
\frac{(\sum s)!}{\prod s!} \exp\left[ -\frac{\sum (\delta s)^2}{2\bar{s}} + \frac{\sum (\delta s)^2}{2\bar{s}} \right].
\]

(46)

If the second term in the exponent were absent, each

\(\delta s \approx s - \bar{s}\) would produce on integration a factor

\[\sqrt{2\pi\bar{s}},\]

to be compared to a similar factor in the denominator. Note that the second term in the exponent produces a zero mode given by \(\delta s \propto s\), but this is eliminated from the integration because it is not consistent with the area constraint.

Now there are two constraints on the \(\delta s\), so two factors are missing in the numerator. One has instead a factor \(\sqrt{2\pi\sum \bar{s}}\) in the numerator. It is easy to see that each \(\delta s \propto A\), so each factor \(\propto \sqrt{A}\), yielding a resultant factor \(\bar{A}\).

The number of states with spin projection zero is thus

\[
N_0 = \frac{1}{\sqrt{A}} \exp\left( \frac{\lambda A}{8\pi\gamma \ell_p^2} \right),
\]

(47)

where constant factors have been ignored and \(\lambda\) is determined by the condition

\[
\sum_{jm} e^{-\lambda\sqrt{\lambda + 1}} = 1,
\]

(48)
given above.

To take into account the possibility of \(\sum m s_{jm}\) being equal to zero modulo \(n\), we let the spin projection be \(M\), say.

It is necessary to restore \(\alpha \neq 0\). The condition

\[
1 = \sum_{jm} e^{-\lambda \sqrt{\lambda + 1} - \alpha m}
\]

(49)
cannot determine both parameters, but can be solved in principle for \(\lambda(\alpha)\). Note that \(\bar{s}\) now depends on \(\alpha\) and the exponential factor in the number of configurations changes to

\[
\exp\left( \frac{\lambda(\alpha) A}{8\pi\gamma \ell_p^2} + \alpha M \right)
\]

(50)
The projection constraint now takes the form

\[
\sum_m e^{-\lambda \sqrt{\lambda + 1} - \alpha m} = \frac{M}{\sum \bar{s}}.
\]

(51)
So although \(\alpha \neq 0\), it is small for \(M \ll A\):

\[
\lambda'(0) = 0,
\]

(52)

\[
\frac{M}{\lambda'(8\pi\gamma \ell_p^2)} = -\alpha \lambda''(0).
\]

(53)
Furthermore,

\[
\frac{\lambda(\alpha) A}{8\pi\gamma \ell_p^2} + \alpha M = (\lambda(0) - \alpha^2 \lambda''(0)) \frac{A}{8\pi\gamma \ell_p^2}
\]

\[
= (\lambda(0) - \frac{\alpha^2}{2} \lambda''(0)) \lambda(0) \frac{A}{8\pi\gamma \ell_p^2} - \frac{M^2}{2\lambda''(0)} \frac{A}{8\pi\gamma \ell_p^2}
\]

(54)

Note that \(\lambda(0)\) is what was called \(A\) earlier. Since

\[
\lambda''(0) = \frac{\sum m^2 e^{-\lambda(0) \sqrt{\lambda + 1}}}{\sum \sqrt{\lambda + 1} e^{-\lambda(0) \sqrt{\lambda + 1}}},
\]

(55)
which is positive, independent of \(A, M\) and \(\sim o(1)\), we can write

\[
N_M = N_0 e^{-4\pi\gamma \ell_p^2 M^2 / \lambda''(0) A}.
\]

(56)
Since \(M = 0 \mod n\), we have to sum \(N_M\) over the values \(rn\), where \(r = 0, \pm 1, \pm 2, \ldots\), and there arises a factor

\[
\sum_r e^{-4\pi\gamma \ell_p^2 r^2 / \lambda''(0) A}
\]

(57)
which, on approximation by an integral over \(r\), is seen to involve a factor \(\sqrt{A}/n\), cancelling the square root in
\[ \text{for largest values of } | \cos \frac{a \pi}{k+2} |. \text{ The number of punctures } p \text{ occurs only in the exponent:} \]

\[ N = \frac{4}{k+2} \sin^2 \frac{\pi}{k+2} [2 \cos \frac{\pi}{k+2}]^p. \]  

(66)

The area \( \propto p \): \[
\log N \propto \text{area}. \]  

(67)

The argument can be extended, as in the U(1) case, to general spins.

\[ N = \frac{2}{k+2} \frac{(\sum_j n_j)!}{\prod_j n_j!} \sum_a \sin^2 \frac{a \pi}{k+2} F(\cos \frac{a \pi}{k+2}), \]  

(68)

with

\[ f_j(\cos \frac{a \pi}{k+2}) \equiv \frac{\sin \frac{a \pi(2j+1)}{k+2}}{\sin \frac{a \pi j}{k+2}} \]  

(69)

and

\[ F(\cos \frac{a \pi}{k+2}) \equiv \prod_j |f_j|^{n_j}. \]  

(70)

Let us first consider \( k \) becoming large, so that the sum over \( a \) can be treated as an integral. As \( a \) is varied, the integrand

\[ \sin^2 \frac{a \pi}{k+2} [F(\cos \frac{a \pi}{k+2})] \]  

(71)

attains its maximum when

\[ 2 \cot \frac{a \pi}{k+2} = \sin \frac{a \pi}{k+2} F'. \]  

(72)

At this maximum, \( a \) satisfies

\[ \left( \frac{a \pi}{k+2} \right)^2 \approx 2 \frac{F(1)}{F'(1)}. \]  

(73)

which is small because \( F' \) contains \( n_j \). As \( \frac{a \pi}{k+2} \) is small, the integrand is approximated as

\[
\left( \frac{a \pi}{k+2} \right)^2 [F(1) - \frac{1}{2} \frac{a \pi}{k+2} F'']. \]  

(74)

The width of the peak is estimated from the second derivative

\[ \frac{2 \pi^2}{(k+2)^2} F' - \frac{5 \pi^4 a^2}{(k+2)^4} F'' + \frac{\pi^6 a^4}{(k+2)^6} F''', \]  

(75)

which, for large \( n_j \), simplifies at the maximum to

\[ -4 \frac{\pi^2}{(k+2)^2} F. \]  

(76)
Consequently the width $\sigma$ of the peak is given by

$$\sigma^2 = \frac{(k + 2)^2}{\pi^2} \frac{F}{F'}.$$  

(77)

and the integral can be approximated by

$$\frac{2(k + 2)}{\sqrt{\pi}} \frac{F}{F'}^{3/2} F.'$$

(78)

The number of states is

$$N = \frac{4}{\sqrt{\pi}} \left(\sum n_j f_j^{-1} \right)^{-3/2} \prod_j [f_j]^n_j.$$  

(79)

To maximize this number, one sets

$$\delta \log N = 0$$

(80)

when the numbers $n_j$ of punctures with spin $j$ are varied, subject to the constraint of fixed area

$$\sum_j \sqrt{j(j + 1)} \delta n_j = 0.$$  

(81)

One obtains for large $n_j$

$$\log f_j + \log \sum_k n_k - \log n_j - \lambda \sqrt{j(j + 1)} = 0,$$

(82)

where $\lambda$ is a Lagrange multiplier.

$$n_j = \left(\sum_k n_k f_k e^{-1} \sqrt{k(k + 1)}\right)^{1/2},$$

(83)

whence, for consistency,

$$\sum_j n_j f_j e^{-1} \sqrt{j(j + 1)} = 1.$$  

(84)

$j$ goes from $\frac{1}{2}$ to $\frac{1}{2}$ for level $k$, infinite for large $k$ and $f_j$ reduces to $2j + 1$ as $\frac{1}{2} \sqrt{n} \rightarrow 0$ for large area. This relation yields the same $\lambda$ as before.

Taking the above distribution, one easily sees that

$$\log N = -\frac{\lambda A}{8\pi y^2} - \frac{3}{2} \log A$$

(85)

as each $n_j$ goes like the area for large area.

If however $k$ is fixed and the area made large, the sum over the finite number of values of $a$ can be considered term by term. For each $a$, maximization of $N$ with respect to $n_j$ is as above, but $(\frac{f_j}{\sqrt{\pi}})^{3/2}$ does not appear:

$$\log N = -\frac{\lambda A}{8\pi y^2}$$

(86)

for finite $k$. Now $\lambda$ depends on $k$ and also on $a$. In summation of $N$ over $a$, highest $\lambda$ dominates and has to be maximized over $a$, which determines the relevant value(s) of $a$. No log corrections appear because the $(\frac{f_j}{\sqrt{\pi}})^{3/2}$ factor, which appeared with $a$ taken to be continuous, does not appear in this case of finite $k$ and discrete $a$.

These calculations are for most probable distribution, but the sum over all distributions can be estimated. Correction factors proportional to area from factorials and from integrations approximating sums over $n_j$ cancel out because there is only the area constraint.

**Conclusion**

- The laws of black hole mechanics suggested that $S \propto A$.
- Euclidean gravity indicated that $S = \frac{A}{4\pi} = \frac{A}{4\pi}$.
- Loop quantum gravity has indicated that $S = \frac{A}{4\pi} - \frac{3}{2} \ln A$ for area eigenvalues $A$ close to the classical horizon area, with a suitable choice of the Barbero-Immirzi parameter.
- However, $S = \frac{A}{4\pi}$ for large $A$ and fixed classical area again with a suitable choice of the Barbero-Immirzi parameter.

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