Compact Packings of the Plane with Three Sizes of Discs

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Abstract
A compact packing is a set of non-overlapping discs where all the holes between discs are curvilinear triangles. There is only one compact packing by discs of size 1. There are exactly nine values of $r$ which allow a compact packing by discs of sizes 1 and $r$. We prove here that there are exactly 164 pairs $(r, s)$ allowing a compact packing by discs of sizes 1, $r$ and $s$.

Keywords Circle packing · Compact packing · Triangulated packing

1 Introduction
A set of interior-disjoint discs is called a packing. Packings are of special interest to model the structure of materials, e.g., crystals or granular materials, and the goal in this context is to understand which typical or extremal properties have the packings (see, e.g., [13, 16, 18]). In 1964, Fejes Tóth coined the notion of compact packing [7]: this is a packing whose contact graph (the graph which connects the center of mutually tangent discs) is triangulated. Equivalently, all its holes are curvilinear triangles.

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Fig. 1 An example of compact packing for each of the nine possible values of $r < 1$ which allow a compact packing by discs of sizes 1 and $r$. They are all periodic, with the parallelogram showing a fundamental domain. The top-right word over $\{1, r\}$ codes the corona of a small disc (see Sect. 2). A label in c1–c9 is assigned to each case (top-left), followed by a letter in brackets which refers to its type (see Appendix B).

There is only one compact packing with all the discs of the same size, called the hexagonal compact packing: the disc centers are located on the triangular grid. In [15], it is proven that there are exactly nine values of $r$ which allow a compact packing with discs of size (radius) 1 and $r$. Figure 1 depicts an example of compact packing for each case. All these packings already appeared in [7], except $c_5$ which later appeared in [16] and $c_2$ which was new at that time.

Recently, it was proven in [17] that there are at most 11462 pairs $(r, s)$ which allow a compact packing by discs of sizes 1, $r$ and $s$. The author provided several examples and suggested that a complete characterization could be beyond the actual capacity of computers. We here overcome this limitation and we prove:

**Theorem 1.1** There are exactly 164 pairs $(r, s)$ which allow a compact packing by discs of sizes 1, $r$ and $s$, with $0 < s < r < 1$. 
Fig. 2 Some compact packings with three sizes of discs. They are all periodic, with the parallelogram showing a fundamental domain. The top-left number refers to numbers in Appendix A and the two top-right words over \{1, r, s\} codes the coronas of both a small and a medium disc (see Sect. 2). The letter in brackets refers to the type of the compact packing (see Appendix B).

All the values $r$ and $s$ are algebraic and their minimal polynomials (which can be quite complicated) are given in the supplementary materials (as well as numerical approximations). Figure 2 depicts an example of compact packing for nine of these 164 cases. The full list is in Appendix A. In each case we found a periodic compact packing, so that it suffices to give its fundamental domain (as done in Figs. 1 and 2). It is of course difficult to convince oneself with the naked eye that the depicted packings are really compact. However, we shall see that their combinatorics is sufficient to ensure that they are indeed compact. Note also that, in many cases, more compact packings than the only one depicted are possible (sometimes much more—as for the surprising number 83 in Fig. 2). This is discussed in Appendix B.

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Many compact packings with three sizes of discs can be found by hand. Those on the first line in Fig. 2, for example, are easily derived from compact packings with two
sizes of discs. Some others were already known. Number 107 appears in [7, p. 187]. Numbers 99, 104 and 143–146 appear in [17]. Number 51 can be found on a street pavement in Weggis, Switzerland (Rigiblickstrasse 1). The challenge is to find them all. As we shall see, this is mathematically rather simple, but the profusion of cases and the complexity of the calculations make it a computational challenge.

Before we continue, let us give a motivation to study compact packings. A central problem in packing theory is to find the maximal density over a set $\mathcal{P}$ of packings, defined by

$$\delta(\mathcal{P}) := \sup_{P \in \mathcal{P}} \limsup_{k \to \infty} \frac{\text{area of the } k \times k \text{ square covered by } P}{k^2}.$$  

The maximal density over packings with only one size of disc has been proven to be $\pi/\sqrt{12} \simeq 0.9069$, attained for the unique possible compact packing (the hexagonal one) [6]. With two sizes of discs, the maximal density turns out to be known only for sizes which allow a compact packing. Namely, all the compact packings in Fig. 1 have been proven to maximize the density among the packings with the same sizes of discs [11,12,14], except $c_5$ and $c_9$ which are still in the running. Compact packings thus seem to be good candidates to provably maximize the density. So, what about the 164 new cases with three discs? Some can be easily ruled out. Number 33, for example (depicted top-right in Fig. 2), has small discs which can be inserted in the holes between two large and a medium discs: this increases the density but yields a non-compact packing. We shall therefore consider only so-called saturated packings, i.e., such that no further discs can be added. Does it suffice?

**Open Question 1.2** Assume that a finite set of different discs allow a saturated compact packing. Consider all the packings by these discs: is the maximal density achieved over compact packings only?

The densest compact packings in cases 1–18 turn out to have only two sizes of discs, so that we do not get anything new. Cases 24, 29–34 and 37–44 are ruled out because the densest compact packings are not saturated. There are still 130 candidates left…

The compact packings depicted in Appendix A are actually the densest among the possible compact packings, except in cases 1–18 where the densest compact packings have only two discs (this can be checked with the help of Appendix B. As already mentioned, they are all periodic. On the one hand, it is a chance because it makes it easy to describe them. On the other hand, it is a bit disappointing because one of our main goals to extend [15] from two to three sizes of discs was to find a densest aperiodic packing. This could indeed have been put forward as a rather simple explanation of aperiodic structure of materials known as quasicrystals. Maybe a few extra disk sizes would be enough?

**Open Question 1.3** Is there a finite set of different discs which allow compact packings and such that the densest one is aperiodic?

To finish with density of compact packings, let us say a word about higher dimensional packings, namely sphere packings. The notion of density is easily generalized
in $\mathbb{R}^n$. Then, a sphere packing in $\mathbb{R}^n$ is said to be compact if its contact graph is a homogeneous simplicial complex of dimension $n$. This coincides with the previous notion for $n = 2$. For $n = 3$, it means that it can be seen as a tiling by tetrahedra which can intersect only on a full face, a full edge or a point. The case of packings with only one size of sphere has been extensively studied (see, e.g., [2]). The maximal density is known in dimension 3 [10], 8 [23] and 24 [1]. In dimension 8 and 24, the maximal density turns out to be achieved by a compact packing! This is not the case in dimension 3, but there is no compact packing in this case (regular tetrahedra do not tile the space). Compact packings are thus still good candidates to provably maximize the density in higher dimensions. In particular, what about the compact packings with two sizes of spheres studied in [8]?

The paper is organized as follows. Section 2 sets out some notations and introduces the important notion of corona. In Sect. 3, the overall strategy is presented. As we shall see, the proof heavily relies on computer, and we shall explain it in details how in order to make it reproducible. All the computations were made with the open-source software SageMath [22] on our modest laptop, an Intel Core i5-7300U with 4 cores at 2.60 GHz and 15, 6 Go RAM. Sects. 4–9 are dedicated to the proof itself (more details are given in Sect. 3). An example of compact packing for each possible size is given in Appendix A, while the variety of compact packings which is possible for each of these sizes is discussed in Appendix B. Last, Appendix C provides indications on the code used on computer to find possible sizes, which can be found in supplementary materials.

2 Notations

The large disc is assumed to have size 1. The size of the medium and small discs are denoted by $r$ and $s$, $0 < s < r < 1$. We also call $x$-disc a disc of size $x$. In a compact packing, the corona of a disc is the set of discs it is tangent to. We shall consider coronas up to isometries. A corona is said to be small, medium or large depending whether the surrounded disc is small, medium or large. We shall also call them s-, r- and l-coronas. The coding of a corona is a word over the alphabet $\{1, r, s\}$: each letter corresponds to a disc and gives its size with two letters being neighbors if and only if the corresponding discs are tangent. We may use an exponent when there are many consecutive letters in the coding, for example $11r^{12}s$ denotes the corona $11r^{12}s$ (12 consecutive s-discs). Any circular permutation or reversal of this word corresponds to the same corona: we shall usually use the lexicographically smallest coding. Given pairwise tangent discs of size $x$, $y$ and $z$, $\hat{x}yz$ denotes the non-oriented angle between the segments which connect the center of the disc of size $y$ to the two other centers. Figure 3 illustrates this.

3 Overall Strategy

We here sketch the overall strategy to prove Theorem 1.1.
3.1 Coronas (Sects. 4 and 6)

First, independently of the values of \( r \) and \( s \), we find a finite set of candidates coronas containing at least all the \( s \)- and \( r \)-coronas which can appear in a compact packing by three sizes of discs. This is easy for \( s \)-coronas because an \( s \)-disc is surrounded by at most six discs (with equality only if the six discs are small). But since an \( r \)-disc can be surrounded by arbitrarily many \( s \)-discs for \( s \) small enough, there are infinitely many \( r \)-coronas. However, we are interested not in any \( r \)-corona but in those which do appear in a compact packing by three sizes of discs. Such a packing also has an \( s \)-corona which shall yield a lower bound on the ratio \( s/r \) (Lemma 6.1), hence on the maximal number of \( s \)-discs in an \( r \)-corona and therefore on the possible \( r \)-coronas.

3.2 Equations (Sects. 5 and 6)

Then, we associate with each candidate corona an algebraic equation satisfied by any values of \( r \) and \( s \) which allow a compact packing containing this corona (if any). The basic idea is that the sum of the angles between the center of consecutive discs in the corona and the center of the surrounded disc is \( 2\pi \). For example, the \( s \)-corona 1rsrs (Fig. 3) yields the equation \( \hat{1} + \hat{r} + \hat{s} + \hat{s} + \hat{s} = 2\pi \). This equation can then be transformed into an algebraic equation by taking the cosine of both sides and using some trigonometry and algebra (at the cost of some parasitic solutions that must be eliminated afterward).

3.3 Computations (Sects. 7–9)

We can now solve the systems of algebraic equations associated with each pair of candidate \( s \)- and \( r \)-coronas. We need to perform exact solving because we want to be sure that discs are really tangent when they should be. Exact solving of systems of algebraic equations is not a trivial task, but it is a classic of computer algebra and various ingenious algorithms do exist. Once \( r \) and \( s \) are known, we can compute all the coronas they indeed allow, i.e., the “local structure” of packings.

Fig. 3 The \( s \)-corona 1rsrs and the angle \( r \hat{s} 1 \)
3.4 Packings (Sects. 7–9 and Appendix A)

Last, we have to determine which sets of coronas indeed allow a compact packing of the whole plane. Determining whether three discs allow a compact packing of the plane amounts to determine whether the 10 different triangles which connect the centers of three mutually tangent discs can tile the plane, i.e., whether isometric copies of these triangles can cover the whole plane so that the intersection of any two triangles is either empty, or a vertex, or a complete edge. This could be hard since the general issue of whether a given finite set of polygons can tile the plane is undecidable [20]. This has to do with the existence of finite sets of tiles which can tile the plane but only aperiodically (as few as two tiles suffice [19]). However, all the cases of Theorem 1.1 actually allow a periodic compact packing with the three sizes of discs that we managed to find by hand (it is a purely combinatorial issue once the set of possible coronas is known).

3.5 Computational Issues

The polynomial systems associated with pairs of candidates s- and r-coronas can be quite hard to solve. For example, the pair 11rrs/11rrs 12 yields two bivariate polynomials of degree 28 and 416, with the latter being 1.4 Mo when written to a plain text file. Of course, since it depends on the computer and algorithm used to solve them, it is a subjective issue. The approach in [17] yields similar polynomial systems, but the attempt to solve them by computing Gröbner basis with the open-source software Singular [4] succeeded only on a few systems. In a private communication, Bruno Salvy showed us that Gröbner basis could be computed more efficiently with the algorithm [5]. Still, some polynomial systems seem out of reach (moreover, the source code of this latter algorithm is not available). With SageMath [22], all the computations we eventually perform take less than one hour on our laptop. For each pair $P$ and $Q$ of integer polynomials in $r$ and $s$ associated with two coronas, we proceed as follows.

1. We use the hidden variable method (see, e.g., Sect. 3.5 of [3] or Sect. 9.2.4 of [24]) to compute a set which contains the roots of $P$ and $Q$. Recall that the resultant of two univariate polynomials is a scalar which is equal to zero iff the two polynomials have a common root. If we see $P$ and $Q$ as polynomials in $s$ with coefficients in $\mathbb{Z}[r]$ ($r$ is the “hidden variable”), then their resultant is a polynomial in $r$ which has a root $r_0$ iff $P(r_0, s)$ and $Q(r_0, s)$ have a common root $s$. Computing the roots of this resultant thus yields the first entries of all the solution $(r, s)$ of $P = Q = 0$. We similarly get the second entries by exchanging the roles of $r$ and $s$. The cartesian product of these two sets, with the condition $0 < s < r < 1$, yields the wanted set. Wrong pairs must then be filtered.

2. We first filter using interval arithmetic, that is, numerical computations with exact error bounds. This allows to be sure to never reject a true solution. Namely, we check the equation on the s- and r-coronas (the original non-algebraic equation on angles) and the existence of at least one 1-corona (similar equation on angles). We also search for all the coronas compatible with the values of $r$ and $s$. 

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3. We then filter exactly. This is the more time-consuming part. Actually, with the default 53-bits precision of SageMath, all the pairs which successfully passed the interval arithmetic filtering also passed this one. We check all the coronas found by the previous filtering. This time, we have to check the equations associated with coronas because no exact computation can be performed on the non-algebraic angles. The way we proceed shall be detailed after we formalize how an algebraic equation is associated with a corona (Sect. 5).

3.6 Combinatorial Issues

The computational tactic discussed above does not suffice to solve all the polynomial systems. For example, computing the resultant for the polynomials associated with the pair 11rrs/11rrs122 exhausted the memory of our laptop. Moreover, the number of candidates pairs of coronas (hence of polynomial systems) to check is huge. Indeed, in Sects. 4 and 6, trying to bound this number as sharply as possible (with no more algebra than in [15]), we get 16805 pairs. Actually, a key factor to prove Theorem 1.1 is to split the compact packings in three combinatorial classes:

1. The packings where no small and medium disks are adjacent (Sect. 7). These packings are said to be large separated (the large discs form the phase interface). The problem primarily reduces to the already solved problem of finding compact packings with two sizes of discs. This class yields 18 cases of Theorem 1.1 (numbers 1–18 in Appendix A).

2. The packings with two different s-coronas other than ssssss (Sect. 8). This allows to use two s-coronas instead of s- and r-coronas with minor changes in the strategy. The point is that there are much less s-coronas than r-coronas, and the associated equations are much simpler (mainly because there are fewer discs in an s-corona). This class yields only one case of Theorem 1.1 (number 19 in Appendix A).

3. The packings with only one s-corona other than ssssss (Sect. 9). Here we have to consider the 16805 pairs of s- and r-coronas and their complicated associated equations. However, the hypothesis that there is only one s-corona (besides ssssss, which is always possible) greatly helps. Indeed, it yields a strong (but simple) constraint on the two neighboring discs of each s-disc in an r-corona (namely, if the sequence xsy appears in the r-corona, then xry must appear in the s-corona). Moreover, as a rule of thumb, r-coronas associated with complicated equations contain many s-discs, thus many such constraints to satisfy: this often (more than nine time out of ten) rules out a pair without any computation! This class yields 145 cases of Theorem 1.1 (numbers 20–164 in Appendix A).

3.7 Other Optimizations

Many other optimizations could be designed. We actually introduced many ones that we later removed, because there is a trade-off between the time required to check the computations and the details required to explain the optimizations. The above strategy, with its computational and combinatorial optimizations, allows to check all the cases
in less than one hour of computation on our laptop. The code is in Python, using powerful SageMath functions: it has only slightly more than 700 lines, comments included (see Appendix C).

4 Small Coronas

An s-corona contains at most six discs, with equality only if there are only s-discs. There are thus finitely many different s-coronas. We want here to be as precise as possible in order to reduce the number of cases to be considered afterward. Let us define for \( \vec{k} = (k_1, \ldots, k_6) \in \mathbb{N}^6 \) the function

\[
S_{\vec{k}}(r, s) := k_1 \widehat{s} + k_2 \widehat{sr} + k_3 \widehat{ss} + k_4 \widehat{sr} + k_5 \widehat{ss} + k_6 \widehat{sss}.
\]

This function counts the angles to pass from disc to disc in an s-corona. To each s-corona corresponds a vector \( \vec{k} \), called its angle vector, which satisfies

\[
S_{\vec{k}}(r, s) = 2\pi. \tag{1}
\]

We have to find the possible values \( \vec{k} \) for which the equation \( S_{\vec{k}}(r, s) = 2\pi \) admits a solution \( 0 < s < r < 1 \). The angles which occurs in \( S_{\vec{k}}(r, s) \) decrease with \( s \), except \( \widehat{sss} \), and increase with \( r \). This yields the following inequalities, strict except for the s-corona ssssss:

\[
S_{\vec{k}}(r, s) \leq \lim_{s \to 0} S_{\vec{k}}(r, s) = k_1 \pi + k_2 \pi + k_3 \frac{\pi}{2} + k_4 \pi + k_5 \frac{\pi}{2} + k_6 \frac{\pi}{3},
\]

\[
S_{\vec{k}}(r, s) \geq \inf_{r \to 1} \lim_{s \to 1} S_{\vec{k}}(r, s) = \lim_{r \to 1} S_{\vec{k}}(r, s)
\]

\[
= k_1 \frac{\pi}{3} + k_2 \frac{\pi}{3} + k_3 \frac{\pi}{3} + k_4 \frac{\pi}{3} + k_5 \frac{\pi}{3} + k_6 \frac{\pi}{3}.
\]

The existence of \( (r, s) \) such that \( S_{\vec{k}}(r, s) = 2\pi \) thus yields the inequalities

\[
k_1 + k_2 + k_3 + k_4 + k_5 + k_6 < 6 < 3k_1 + 3k_2 + \frac{3}{2} k_3 + 3k_4 + \frac{3}{2} k_5 + k_6,
\]

except for the s-corona ssssss \((k_1 = \cdots = k_5 = 0 \text{ and } k_6 = 6)\). An exhaustive search on computer yields 383 possible values for \( \vec{k} \). For each of them, one shall check that there indeed exists a coding over \( \{1, r, s\} \) with this angle vector.\(^1\) This is the case if the graph depicted in Fig. 4 contains a cycle such that \( \vec{k} \) counts the number of times each edge appears in this cycle. If this cycle contains a loop, then either it contains only this loop or this loop must be accessible from another vertex. This yields the conditions:

\(^1\) For example, \( \vec{k} = (0, 3, 0, 0, 0, 0) \) corresponds to three angles \( \widehat{sr} \) around an s-disc: this is combinatorially impossible.
Fig. 4  The cycles of this graph encode coronas associated with $k$

**Table 1** The 55 possible s-coronas, besides sssss

| rtrr | rtrrs | rtrss | rttrs | rtrr | rtrss | rttrs | rtrs | rr | rtrss |
|------|--------|--------|-------|------|--------|-------|-----|----|-------|
| 11111 | 1111s  | 111ss  | 11s1s | 1111 | 11sss | 1ss1s | 11s1 | 11 | 11ss  |
| 1111r | 111rs  | 11rss  | 11rs1s | 111r | 1r1ss | 1rs1s | 1r1r | 1r | 1r1ss |
| 111rr | 111rs  | 1r1ss | 1r1s1s | 11rr | 1r1r | 1rss | 1r1r | 1r | 1rr   |
| 11lrr | 1r1rs  | r1rss  | rrs1s | 1r1r | 1r1r | 1r1rs | 1r1r | 1r | r1rs  |
| lrrrr | r1r1s  | lrrrs  | lrrs  |  | | | | | |

Those on the first line have no 1-disc and those on the second line no r-disc. In each column, the codings are all equal if we replace each 1 by an r (this is used in Lemma 6.1)

\[
\begin{align*}
(k_1 = 0) &\lor (k_2 \neq 0 \land k_3 \neq 0) \lor (k_2 = k_3 = k_4 = k_5 = k_6 = 0), \quad (2) \\
(k_4 = 0) &\lor (k_2 \neq 0 \land k_5 \neq 0) \lor (k_1 = k_2 = k_3 = k_5 = k_6 = 0), \quad (3) \\
(k_6 = 0) &\lor (k_3 \neq 0 \land k_5 \neq 0) \lor (k_1 = k_2 = k_3 = k_4 = k_5 = 0). \quad (4)
\end{align*}
\]

The cycle must do $k_0 := \min(k_2, k_3, k_5)$ round trips around the three vertices, and possibly some round trips between two vertices which use an even number of times the edge which connects these vertices. This yields the conditions:

\[
k_2 - k_0 \in 2\mathbb{N}, \quad k_3 - k_0 \in 2\mathbb{N}, \quad k_5 - k_0 \in 2\mathbb{N}.
\]  

All the previous conditions eventually lead to 56 possible angle vectors. Table 1 gives, for each, a coding of the corresponding corona.

---

2 An angle vector could correspond to several coronas. For example, $(0, 2, 2, 0, 2, 0)$ corresponds to 1rs1s, 1r1rs, 1rs1r and 1rs1rs. But this does not happen here.
5 Polynomial Associated with a Small Corona

We here associate with an s-corona a polynomial equation in \( r \) and \( s \) so that any pair \((r, s)\) for which the corona can be geometrically represented corresponds to a solution of this equation. First take the cosines of both sides of (1) and fully expand the left-hand side. Subtract \( \cos(2\pi) = 1 \) to both sides. This yields a polynomial equation in cosines and sines of the angles occurring in \( S_k \). The cosine law then allows to replace each cosine by a rational fraction in \( r \) and \( s \). Namely, the cosines of \( \hat{1}s, \hat{1}s, \hat{1}s, \hat{r}s, \hat{r}s \), and \( \hat{s}s \) are respectively replaced by

\[
1 - \frac{2}{(1+s)^2}, \quad \frac{1}{1+s} - \frac{2r}{(r+s)(1+s)}, \quad \frac{s}{1+s}, \quad 1 - \frac{2r^2}{(r+s)^2}, \quad \frac{s}{r+s}, \quad \frac{1}{2}.
\]

The squares of the sines follow (same order):

\[
\frac{4s(s+2)}{(s+1)^4}, \quad \frac{4rs(r+s+1)}{(s+1)^2(r+s)^2}, \quad \frac{2s+1}{(s+1)^2}, \quad \frac{4s(2r+s)^2}{(r+s)^4}, \quad \frac{r(r+2s)}{(r+s)^2}, \quad \frac{3}{4}.
\]

We express sines (same order) using auxiliary variables \( X_1 \) through \( X_6 \):

\[
\frac{2X_1}{(s+1)^2}, \quad \frac{2X_2}{(s+1)(r+s)}, \quad \frac{X_3}{s+1}, \quad \frac{2rX_4}{(r+s)^2}, \quad \frac{X_5}{r+s}, \quad \frac{X_6}{2},
\]

where the squares of the auxiliary variables \( X_1, \ldots, X_6 \) are (same order):

\[
s(s+2), \quad rs(r+s+1), \quad 2s+1, \quad s(2r+s), \quad r(r+2s), \quad 3.
\]

This yields a polynomial system in \( r, s \) and the \( X_i \)'s.

One can use the above expression of \( X_i^2 \) as a polynomial in \( r \) and \( s \) to remove any power \( k \geq 2 \) of \( X_i \). We then successively eliminate each \( X_i \) as follows. We write the equation \( AX_i + B = 0 \), where \( A \) and \( B \) do not contain any occurrence of \( X_i \). We then multiply both sides by \( AX_i - B \), so that no solution is lost. This yields the new equation \( A^2X_i^2 - B^2 = 0 \), where \( X_i^2 \) can be replaced by its expression as a polynomial in \( r \) and \( s \). In short:

\[
AX_i + B = 0 \rightarrow A^2X_i^2 - B^2 = 0.
\]

Doing this successively for each \( X_i \) eventually yields the wanted polynomial equation in \( r \) and \( s \). We simplify it by removing multiplicities of factors as well as factors which have clearly no roots \( 0 < s < r < 1 \).

For the 10 s-coronas without r-disc, we get a polynomial in \( s \) (Table 2). Each of the 10 s-coronas without large disc yields the same polynomial as the corona where \( r \) has been replaced by 1, with the variable \( s/r \) instead of \( s \). The 35 remaining s-coronas yield an explicit polynomial which can be rather complex (Table 3 gives the degrees). For example, the s-corona 11rs yields:
Table 2  s-coronas without r-discs, associated polynomials and root s ∈ (0, 1)

|   |      |                          |  |
|---|------|--------------------------|---|
| 11111 | 5s^4 + 20s^3 + 10s^2 - 20s + 1 | 0.701 |
| 1111s | s^4 - 10s^2 - 8s + 9         | 0.637 |
| 111ss | s^8 - 8s^7 - 44s^6 - 232s^5 - 482s^4 - 24s^3 + 388s^2 - 120s + 9 | 0.545 |
| 1s1ls | 8s^3 + 3s^2 - 2s - 1         | 0.533 |
| 1111  | s^2 + 2s - 1                 | 0.414 |
| 11s   | 9s^4 - 12s^3 - 26s^2 - 12s + 9 | 0.386 |
| 1s1ss | s^4 - 28s^3 - 10s^2 + 4s + 1 | 0.349 |
| 11s   | 2s^2 + 3s - 1                | 0.280 |
| 111   | 3s^2 + 6s - 1                | 0.154 |
| 11ss  | s^2 - 10s + 1                | 0.101 |

Table 3  Degree of the polynomial in r and s associated with each s-corona

|   |   |       |   |   |   |       |   |   |
|---|---|-------|---|---|---|-------|---|---|
| l1r | 2 | l1rr | 4 | l1rsl | 6 | lrs | 8 | l11r1r | 12 |
| l1ls | 2 | lrrss | 4 | l11l1r | 7 | l1rs | 8 | l11rrr | 12 |
| lrsr | 2 | l1rls | 6 | l1rlr | 7 | lrs | 8 | l11lrs | 18 |
| l1lr | 3 | l1rs | 6 | lrs1s | 7 | l1lrr | 10 | l1rrs | 18 |
| l1r | 3 | l1rsr | 6 | l1lrs | 8 | l1rr | 10 | l1rrs | 24 |
| lrr | 3 | l1rls | 6 | l1lrs | 8 | l1rs | 10 | l1r | 11 |
| lrr | 3 | l1rs | 6 | l1rss | 8 | lrls | 10 | l1rr | 28 |

\[
r^2 s^4 - 2r^2 s^3 - 2rs^4 - 23r^2 s^2 - 28rs^3 + s^4 - 24r^2 s^2 - 58rs^2 - 2s^3 + 16r^2 - 8rs + s^2.
\]

If discs of sizes s, r and 1 are compatible with a corona, then \((r, s)\) is a solution of the equation associated with this corona. But some solutions of this equation may not correspond to a corona: they are parasitic solutions. The first reason is that taking the cosines of the equations on angles can lead to solutions which are true only modulo \(2\pi\). Such cases are however easily detected just by a numerical computation. The second reason is that, to eliminate the \(X_i\)’s, we multiply by factors \(AX_i - B\) which introduce new solutions. To detect whether a given pair \((r, s)\) comes from these multiplying factors, we check by interval arithmetic whether \(AX_i - B\) could be equal to zero, and only if it does (it is very rare), then we substitute the exact values of \(r\) and \(s\) in the initial equation in \(r, s\) and the \(X_i\)’s (this is more time-consuming).

6 Medium Coronas and Associated Polynomials

Since \(s\) can be arbitrarily smaller than \(r\), there can be infinitely many \(s\)-discs in an \(r\)-corona. Let us however see that this cannot happen in a compact packing.
Lemma 6.1 The ratio \( s/r \) is uniformly bounded from below in compact packings with three sizes of discs.

Proof Consider a compact packing with three sizes of discs. It contains an \( s \)-disc and not only \( s \)-discs, thus an \( s \)-corona other than \( ss \)ssss. By replacing each \( 1 \) by an \( r \) in the coding of this \( s \)-corona, Table 1 shows that we get the coding of a new \( s \)-corona. In this new corona, the ratio \( s/r \) is smaller. Indeed, the \( 1 \)-discs have been “deflated” into \( r \)-discs, so that the perimeter of the corona decreased, whence the size of the surrounded small disc too. But there is at most 10 possible ratios \( s/r \) for an \( s \)-corona without large discs: they correspond to the values computed in Table 2 for compact packing with two sizes of discs (the smallest is \( 5 - 2\sqrt{6} \approx 0.101 \)).

This lemma ensures that the number of \( s \)-discs in an \( s \)-corona is uniformly bounded in compact packings. There are thus only finitely many different \( r \)-coronas in compact packings. To find them all, we proceed similarly as for \( s \)-coronas. We define for \( 0 < s < r < 1 \) and \( \vec{l} = (l_1, \ldots, l_6) \in \mathbb{N}^6 \) the function
\[
M_{\vec{l}}(r, s) := l_1 \vec{1}r + l_2 \vec{r}r + l_3 \vec{1}rs + l_4 \vec{r}rr + l_5 \vec{r}rs + l_6 \vec{s}rs.
\]

We have to find the possible values \( \vec{l} \) for which the equation \( M_{\vec{l}}(r, s) = 2\pi \) admits a solution \( 0 < s < r < 1 \). Actually, since the solution should correspond to an \( r \)-corona which occurs in a packing, we can assume that it satisfies \( s/r \geq \alpha \), where \( \alpha \) is the lower bound on \( s/r \) given by the \( s \)-coronas which occur in this packing. The angles which occur in \( M_{\vec{l}}(r, s) \) decrease with \( r \) (except \( \vec{r}r \)) and increase with \( s \). This yields the following inequalities, strict except for the \( r \)-corona \( rrrrrr \):
\[
M_{\vec{l}}(r, s) \leq \sup_{r \to s} \lim_{s \to r} M_{\vec{l}}(r, s) = l_1 \pi + l_2 \frac{\pi}{2} + l_3 \frac{\pi}{2} + l_4 \frac{\pi}{3} + l_5 \frac{\pi}{3} + l_6 \frac{\pi}{3},
\]
\[
M_{\vec{l}}(r, s) \geq \inf_{s/r \to \alpha} \lim_{r \to s} M_{\vec{l}}(r, s) = l_1 \frac{\pi}{3} + l_2 \frac{\pi}{3} + l_3 u_\alpha + l_4 \frac{\pi}{3} + l_5 u_\alpha + l_6 v_\alpha,
\]
where \( \vec{1}rs \) has been bounded from below by \( \vec{r}rs \) for any \( r \), and the limits \( u_\alpha \) and \( v_\alpha \) of \( \vec{r}rs \) and \( \vec{s}rs \) when \( s/r \to \alpha \) are obtained via the cosine law:
\[
u_\alpha := \arccos \left( \frac{1}{1 + \alpha} \right) \quad \text{and} \quad v_\alpha := \arccos \left( 1 - \frac{2\alpha^2}{(1 + \alpha)^2} \right).
\]
The existence of \((r, s)\) such that \( M_{\vec{l}}(r, s) = 2\pi \) thus yields the inequalities
\[
l_1 + l_2 + l_4 + \frac{3}{\pi} (l_3 u_\alpha + l_5 u_\alpha + l_6 v_\alpha) < 6 < 3l_1 + \frac{3}{2} l_2 + \frac{3}{2} l_3 + l_4 + l_5 + l_6,
\]
except for the \( r \)-corona \( rrrrrr \) \((l_1 = \cdots = l_5 = 0 \text{ and } l_6 = 6)\). We also impose
\[
l_1 + l_2 + l_4 + \frac{1}{2} (l_3 + l_5) < 6,
\]
which says that an \( r \)-corona (other than \( rrrrrr \)) contains at most five \( r \)- or 1-discs. An exhaustive search on computer, which also checks whether there indeed exists a coding.
Table 4 Each $s$-corona whose 1-discs have been deflated in $r$-discs (first line) yields a lower bound $\alpha$ on $s/r$ in any compact packing which contains it (second line), and thus an upper bound on the number of possible $r$-coronas in this packing (third line)

| $rrrr$ | $rrrs$ | $rrss$ | $rrrs$ | $rr$ | $rrss$ | $rrss$ | $rrr$ | $rrs$ | $rrss$ |
|--------|--------|--------|--------|-----|--------|--------|------|------|--------|
| 0.701  | 0.637  | 0.545  | 0.533  | 0.414 | 0.386  | 0.349  | 0.280 | 0.154 | 0.101  |

for each possible $l$, eventually yields the possible $r$-coronas for each value of $\alpha$. Table 4 gives the numbers of such coronas.

Since any $r$-corona for some lower bound $\alpha$ also appears for a smaller $\alpha$, there are at most 1654 different $r$-coronas. Now, each $s$-corona in the $k$-th column of Table 1 and each $r$-corona in the same $k$-th column of Table 4 form a pair which could appear in the same compact packing. The total number of pairs, namely 16805, is thus the element-wise product of the vectors $(8, 10, 6, 6, 6, 3, 3, 6, 4, 3)$ and $(84, 94, 130, 143, 197, 241, 272, 386, 889, 1654)$.

A polynomial in $r$ and $s$ is associated with each medium corona as done for the small coronas in Sect. 5. The cosine law yields the cosines of $\hat{r}1, \hat{rr}, \hat{rs}, \hat{rrr}, \hat{rrs}$ and $\hat{srs}$ (in this order):

$$1 - \frac{2}{(1 + r)^2}, \frac{r}{1 + r}, 1 - \frac{2s}{(r + s)(1 + r)}, \frac{1}{2}, \frac{r}{r + s}, 1 - \frac{2s^2}{(r + s)^2}.$$  

The squares of the sines follow (same order):

$$\frac{4r(r + 2)}{(r + 1)^4}, \frac{2r + 1}{(r + 1)^2}, \frac{4rs(r + s + 1)}{(r + 1)^2(r + s)^2}, \frac{3}{4}, \frac{s(2r + s)}{(r + s)^2}, \frac{4r(r + 2)s^2}{(r + s)^4}.$$  

We express sines (same order) using auxiliary variables:

$$\frac{2X_7}{(r + 1)^2}, \frac{X_8}{r + 1}, \frac{2X_2}{(r + 1)(r + s)}, \frac{X_6}{2}, \frac{X_4}{r + s}, \frac{2sX_5}{(r + s)^2},$$

where $X_1, \ldots, X_6$ are defined in Sect. 5 and $X_7$ and $X_8$ have respective squares $r(r + 2)$ and $2r + 1$.

The $X_i$’s elimination eventually yields a bivariate polynomial in $r$ and $s$. These polynomials are usually larger than those associated with $s$-coronas (because there are generally more discs—up to 33—in a medium corona than in a small one). The degree can be as high as 416 (for the $s$-corona $11r_{rs12}$) and the mean degree is 57.88 (standard deviation 50.16). Computing them all for the 1654 $r$-coronas takes 2 h 21 min on our laptop and yields a 35 Mo file (this is actually not necessary, as we shall see).
7 Large Separated

A compact packing by discs of sizes 1, \( r \) and \( s \) is said to be large separated if it does not contain any adjacent s- and r-discs, i.e., there is always a large disc between a small disc and a medium one. Any s-corona has only s- and 1-discs: this yields one of the 10 equations in \( s \) given by Table 2. Any r-corona has only r- and 1-discs: this also yields one of these 10 equations, up to the replacement of each s by an \( r \). With \( s < r \), this yields \( \binom{10}{2} = 45 \) possible pairs \((r, s)\).

For each of these 45 pairs, we compute all the possible coronas, first by interval arithmetic and then exactly, as explained Sect. 3. This takes around 4 min on our laptop. We find a periodic packing with all the sizes of discs for 18 of these 45 cases (numbered 1 to 18 in Appendix A). They turn out to be exactly those with a 1-corona which contains both an s- and an r-disc and can act as a pivotal point to connect s- and r-discs of the packing.

What about the 27 other cases? One could imagine packings in which s- and r-discs are separated by more 1-discs. The point of the following lemma is to show that this would need a very special 1-corona, namely one with an r-disc and three consecutive 1-discs:

Lemma 7.1 Assume that disc sizes are such that none of the following is possible:

1. An \( s \)-corona with an \( r \)-disc or conversely.
2. A 1-corona with both an \( s \)-disc and an \( r \)-disc.
3. A 1-corona with an \( r \)-disc and three consecutive 1-discs.

Then, no compact packing with all the sizes of discs is possible.

Proof Let us call critical a 1-disc which has a neighboring r-disc. We claim that any neighboring 1-disc \( D' \) of a critical disc \( D \) is critical itself. Indeed, consider the two discs which are neighbors of both \( D \) and \( D' \). They cannot be both 1-discs, otherwise together with \( D' \) they would form three consecutive 1-disks in the corona of \( D \), which the third hypothesis rules out. None of them can be an s-disc because \( D \) would have an s-disc and an r-disc in its corona, which the second hypothesis rules out. Thus, at least one is an r-disc: \( D' \) is critical as claimed.

Now, assume that such a packing exists and get a contradiction. Consider a pair of closest r- and s-discs (according the graph distance in the contact graph of the packing). The first hypothesis ensures that they are not neighbors. Consider a shortest path between them. It has only 1-disk. Walking from the r-disc to the s-disc along this path, the first disc is critical by definition, and all the following ones also according to the above claim. But the last one has a neighboring s-disc: this contradicts the second hypothesis.

Lemma 7.1 rules out 20 of the 27 remaining cases (see, e.g., Fig. 5). Actually, Lemma 7.1 is purely combinatorial and still holds under any permutation of the disc types. In particular, exchanging types s and r rules out 13 of these 27 remaining cases: nine are also ruled out by the original lemma, while four are newly ruled out. On the whole it rules out \( 20 + 4 \) of the 27 cases, so that there are still three remaining cases, depicted in Fig. 6. They satisfy the two first hypotheses of Lemma 7.1 but not the third one. The following lemma rules them out:
Fig. 5 A case (given by all the possible coronas) ruled out by Lemma 7.1

Fig. 6 Three cases (one per line, given by all the possible coronas) which do not allow any compact packing of the plane with all the three sizes of discs

Fig. 7 In the three cases whose coronas are depicted in Fig. 6, any 1-disc of a (hypothetical) packing is either critical or surrounded only by critical discs

Lemma 7.2 None of the cases depicted in Fig. 6 allow a compact packing of the plane with all the three sizes of discs.

Proof We keep the notion of critical disc introduced in the proof of Lemma 7.1. Here, only the first and second hypotheses of Lemma 7.1 are fulfilled because each case has the corona 111rr1r (fourth position on each line in Fig. 6). A 1-disc to a critical disc is no more necessarily critical itself. However, we claim that if a 1-disc has a critical neighbor, then all its neighbors are critical discs (see below). The proof of Lemma 7.1 then goes the same way. On a path of 1-discs going from an r-disc to an s-disc, the first 1-disc is critical (the initial r-disc is a neighbor) and any non-critical one (if any) appears between two critical ones. In particular, the last 1-disc is critical (the final s-disc is a neighbor), which contradicts the second hypothesis of Lemma 7.1 and proves Lemma 7.2.

Let us prove the above claim (disc labels refer to Fig. 7). Consider a 1-disc a which is not critical but has a critical neighbor b. The neighbors c and d of both a and b are neither s-disc because of the second hypothesis of Lemma 7.1, nor r-disc because a is not critical. They are thus 1-discs. Only one of the coronas depicted in Fig. 6 is possible for b: 111rr1r. This corona ensures that the neighbor of b and c is an r-disc,
as well as the neighbor of $b$ and $d$. In particular, both $c$ and $d$ are critical, and the
second hypothesis of Lemma 7.1 ensures that they do not have a neighboring $s$-disc.
The neighbor $e$ of $c$ and $a$ is thus a 1-disc, as well as the neighbor $f$ of $d$ and $a$. The
corona of $a$ now contains five 1-discs: according to Fig. 6, a sixth 1-disc $g$ completes
this corona. The three consecutive neighbors $e$, $a$ and $b$ of $c$ ensure that its corona is
$111rr1r$, so that the neighbor of $c$ and $e$ is an $r$-disc. The same holds for each disc of
the corona of $a$: the neighbor (other than $a$) of two consecutive discs of the corona is
an $r$-disc. This proves that all the discs of the corona of $a$ are critical.

\section{Two small coronas}

We consider compact packings for pairs $(r, s)$ which allow at least two different $s$-
coronas other than ssssss. We want to find $(r, s)$ by solving the polynomial system
associated with these two $s$-coronas. Since there are only 55 $s$-coronas, there are at
most $\binom{55}{2} = 1485$ such systems. Moreover, the equations associated with $s$-coronas
are much simpler.

If two different $s$-coronas contain only $s$- and 1-discs, then they characterize two
different values of $s$. Similarly, if they contain only $s$- and $r$-discs, then they characterize
two different values of $s/r$. We thus rule out these pairs. We get 1395 systems of two
polynomial equations in $r$ and $s$. We then follow the computational tactic explained
in Sect. 3.

The hidden variable method fails on the three pairs $rrs/1rss$, $rsss/1111r$ and
$1rss/1111r$. The computation of the algebraic roots of the resultants indeed raises
(various) exceptions. This could be overcome, but we just compute roots with interval
arithmetic instead. Indeed, this yields 179 pairs $(r, s)$ of intervals which are all ruled
out by the interval arithmetic filtering.

The hidden variable method works on the 1392 other pairs and yields 13239 pairs
$(r, s)$, with $0 < s < r < 1$. Arithmetic filtering rules out all but 37 of them. The exact
filtering validate all these pairs and yields all the possible coronas in each case. We
find a periodic packing for only one of these 37 cases, namely the pair of $s$-coronas
$1srrs/1s1ss$ (number 19 in Appendix A). Its coronas are depicted in Fig. 8.

Actually, the case $1srrs/1s1ss$ is the only one with an $s$-corona which contains both
an $r$-disc and a 1-disc and can act as a pivotal point to connect $r$- and 1-discs of the
packing. In the 36 other cases, $r$- and 1-discs are never in contact: only $s$-separated
packings are possible. The situation is thus very similar to Sect. 7 and Lemma 7.1
again helps. Precisely, the permutation $(s, r, 1)$ on the disc types in Lemma 7.1 rules out
32 of these 36 cases and the permutation $(s, 1, r)$ rules out the four last ones (as well as
28 already ruled out).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig8}
\caption{The only case which allows two small coronas and a valid packing}
\end{figure}
Actually, even if the only remaining case allows the two small coronas 1srrs and 1s1ss, only the first one can appear in a valid packing:

**Proposition 8.1** There is no compact packing with three sizes of discs which contains both the small coronas 1srrs and 1s1ss.

**Proof** There is no other small corona (besides ssssss) compatible with the values of r and s characterized by these two coronas, and the only compatible large corona is s_{12}. Assume that a small corona 1s1ss appears (Fig. 9, left). Each of the two large discs of this corona must be surrounded by small discs (Fig. 9, center). A small disc with a factor s1ss in its corona (one of them is pointed in Fig. 9, left) must have a corona 1s1ss. This yields a third large disc, also surrounded by small discs (Fig. 9, right). This argument can be repeated on each small disc with factor s1ss in its corona (pointed in Fig. 9, right), forbidding any medium disc to ever appear.

\[\square\]

9 One Small Corona

The remaining class is the main one. The packings contain adjacent s- and r-discs (otherwise they are large separated) and allow only one s-corona (besides ssssss). In particular, any such packing has an r-corona which contains an s-disc. We shall here always assume that in the pair of s- and r-corona used to get the equations in r and s, the r-corona contains an s-disc. This indeed yields a strong combinatorial constraint. Consider, for example, the pair 11rrs/11rrs_{12} (computing the resultants for the associated equations exhausted the memory of our laptop, as mentioned in Sect. 3). Whenever, in the r-corona, there is an s-disc between an x-disc and an y-disc, the corona of this s-disc must contain an r-disc between an x-disc and an y-disc. Here, this shows that the coding of the s-corona must contain the factors rrs, srs and sr1. However, neither srs nor sr1 appear in 11rrs. This pair can thus be ruled out without any further computation.

Recall that an angle vector can generally have different codings (not for s-coronas, however). In the above example, 11rrs_{12} is the unique coding of (1, 1, 1, 1, 1, 11). But the angle vector (0, 0, 4, 0, 6, 10), for example, admits 1022 different codings. To rule
a case out, each of these codings must be checked. Formally, a small angle vector $\vec{k}$ is said to cover a medium angle vector $\vec{l}$ if there exists a coding of $\vec{l}$ such that, for any factor $xsy$ of this coding, the (unique) coding of $\vec{k}$ contains $xry$. One also says that $\vec{k}$ pre-covers $\vec{l}$ if there exists a coding of $\vec{l}$ such that, for any factor $xs$ of this coding, the coding of $\vec{k}$ contains $x$. This latter condition is weaker but it can be directly checked on the angle vector (which precisely counts the factors of length 2), hence faster.

Consider the 16805 candidates pairs small/medium angle vectors. Keeping only those where the s-corona contains an r-disc and the r-corona an s-disc reduces to 12265 pairs. Checking the pre-covering condition reduces to 2889 pairs. Checking then the covering condition reduces to 803 pairs. These 803 pairs correspond to 192 different medium coronas out of 1654 initially, and the associated equations are generally much simpler.\(^3\)

The hidden variable method fails on 23 of these 803 pairs. For eight of them, the computation of the algebraic roots of the resultants raises (various) exceptions. Again, we compute the roots with interval arithmetic and filter them with interval arithmetic. Only one pair remains: 1rr1s/11rrs. Lemma 9.2 rules out this pair with a combinatorial argument. The 15 other pairs fail because the two resultants are zero, that is, there is a continuum of possibles pairs $(r, s)$. We rule them out as follows. In 13 out of these 15 cases, neither the s- nor the r-corona does contain a 1-disc. In order to allow a packing with three sizes of discs, there must be an s-corona or an r-corona with a 1-disc. If there is such a corona with an s-disc and an r-disc adjacent, then this case appears in the initial list of 16805 candidates pairs and is handled elsewhere. Otherwise, there is either an s-corona with only 1- and s-discs, or an r-corona with only 1- and r-discs. In the former case, the corona appears in Table 2 and characterizes s. The same holds in the latter case, with r instead of s. In both cases, and for each of these 10 new possible coronas, we consider the polynomial system formed by the equation associated to the new corona and the two ones associated with the initial pair of corona. We find (using Gröbner basis) that none of these systems does have a solution. These 13 cases are thus ruled out. The two remaining cases are 1rr1r1srs and 11r/111s1s. The first case is not possible: the s-corona 1rr tells us that the s-discs must be in the interstices between one 1-disc and two r-discs, but then the r-corona 1r1srs should be an r-corona 1r1r with s-discs in the interstices, but the r-corona 1r1r is forbidden ($r$ should be arbitrarily small). In the second case, the s-corona 11r tells us that the s-discs must be in the interstices between two 1-discs and one r-disc, but then the r-corona 111s1s is not completed because one could add s-discs between the central r-discs and two consecutives 1-discs, yielding the r-corona 1s1s1s1s. This is thus a subcase of 11r1s1s1s1s, which is handled elsewhere. All the cases where the hidden variable method fails are thus ruled out.

The hidden variable method works on the 780 other pairs and yields 56968 algebraic pairs $(r, s)$, with $0 < s < r < 1$. Arithmetic filtering rules out all but 202 of them. The exact filtering rules out 27 of these pairs. One is the pair 1srrs/rrsrsss which turns out to also allow the s-corona 1s1ss: this is actually the only one case which allows a packing Sect. 8 (number 19 in Appendix A). The 26 other ones are detected as “duplicates”.

\(^3\) Computing the 192 polynomials takes 2 min on our laptop and yields a 256 Ko file. The mean degree is 14, the maximum one 80 for 11rrsrrss. This has to be compared with the statistics provided for all the 1654 polynomials at the end of Sect. 6.
once all the possible coronas are computed, that is, there is another pair which yields the same values of \((r, s)\) and the same set of possible coronas. The 175 other cases are validated and all the possible coronas computed. Computing exactly all the large coronas is the most time-consuming part of all the paper (30 min on our laptop, i.e., half of the total computation time).

We find a periodic packing for 145 of these 175 cases (numbers 20–164 in Appendix A). We shall rule out the other cases by two combinatorial lemmas. The first one rules out 24 cases:

**Lemma 9.1** If a compact packing contains an \(s\)-corona \(1rss, 11rss, 1rrss\) or \(1srss\), then it must contain another \(s\)-corona (other than \(ssssss\)).

**Proof** The proof does not rely on the value of \(r\) and \(s\). The four cases are similar and depicted in Fig. 10. Polygons around letters link the text and the figure. Consider an \(s\)-disc and an \(s\)-disc in its corona. The \(s\)-corona is determined and yields in the corona of the \(s\)-disc a factor (\(1ss1\) in the first case, \(rssr\) in the other ones) which appears neither in these four \(s\)-coronas nor in \(ssssss\).

The second lemma rules out the six last pairs, as well as the pair \(1rr1s/11rrs\), for which no exact filtering was performed (because the hidden variable method yields only interval for \(r\) and \(s\)):

**Lemma 9.2** The small/medium coronas \(1rsrs/1rr1ss\), \(11rr/11rrs\), \(1rr1s/11rrs\), \(rrrr/1rsrsr\), \(rrrrs/11rssr\), \(rrrs/11rssr\) and \(rrrs/11rssr\) do not allow a packing with all the three sizes of discs (without any other small corona, except \(ssssss\)).

**Proof** We check the seven cases one by one, using the values of \(r\) and \(s\) to determine (with a computer) all the possible coronas and then relying on a short combinatorial argument (illustrated next to it).

\(1rsrs/1rr1ss\). The values of \(r\) and \(s\) allow no other \(r\)-corona. In the \([s]\)-corona, the \([r]\)-disc has three neighbor \(s\)-discs. This is incompatible with the \(r\)-corona \(1rr1ss\) (Fig. 11).
**11rr/11rrs.** The values of $r$ and $s$ allow no other $r$-corona. In the corona of the $r$-disc, the corona of the $\bar{r}$-disc enforces the factor $1rr1$ in the corona of the $\bar{r}$-disc. This is incompatible with the $r$-corona $11rrs$ (Fig. 12).

**1rrls/11rrs.** The values of $r$ and $s$ allow no other $r$-corona. In the corona of the $r$-disc, the corona of the $\bar{r}$-disc enforces the factor $srrs$ in the corona of the $\bar{r}$-disc. This is incompatible with the $r$-corona $11rrs$ (Fig. 13).

**rrr/s/1rsrsr.** The values of $r$ and $s$ also allow no other $r$-corona. In the $s$-corona, there are two symmetric ways to draw the corona of the $\bar{s}$-disc. Once it is done, the corona of the neighboring $\bar{r}$-disc is determined. The coronas of the two $\bar{r}$-discs are also determined. This enforces two $1$-discs in the $\bar{r}$-corona. This is incompatible with the $r$-corona $1rsrsr$ (Fig. 14).

**rrrs/11rssr.** The values of $r$ and $s$ also allow the $r$-coronas $rsrsrss$ and $1111r$, as well as the $1$-corona $1r1r1r$. Exchanging types $r$ and $1$ in Lemma 7.1 ensures that any packing with three sizes of discs must contain the $r$-corona $11rssr$. Consider an $\bar{r}$-corona $11rssr$. The corona of the $\bar{r}$-disc must be $11rssr$ and its position is determined.
This enforces a factor $srrs$ in the $s$-corona. This is incompatible with the $s$-corona $rrrr$ (Fig. 15).

**rrrs/11rssr.** The values of $r$ and $s$ also allow the $r$-coronas $rrrsrsss$, $rrsrss$, and $111rr$, as well as the $1$-coronas $11r11rr$ and $111r1rr$. The argument works exactly as for the previous case and forces an impossible factor $srrs$ in the $s$-corona.

**rrrs/11rssr.** The values of $r$ and $s$ also allow the $r$-coronas $rrsrsrssss$, $rrsrsrsss$, $rrsrsrssss$, $rrssrssss$, $rrsrssr$, $rrrsrss$, $rrrssr$, $rrssr$, $rrrsrr$, and $111r$. The $r$-coronas $1rrsrrrrr$ and $rrsrsrssss$ are actually impossible because the $s$-corona $rrs$ forbid three consecutive $s$-discs in any $r$-corona. Exchanging types $r$ and $1$ in Lemma 7.1 ensures that any packing with three sizes of discs must contain an $r$-corona $1rrsrrss$ or $11rsrr$. In both $r$-coronas, the $s$-corona $rrr$ yields a factor $1sr$ in the $r$-corona. This is incompatible with the allowed $r$-coronas ($1rrsrrss$ has been discarded) (Fig. 16). □

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