EQUIVALENT AND ATTAINED VERSION
OF HARDY’S INEQUALITY IN \( \mathbb{R}^n \)

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Abstract. We investigate connections between Hardy’s inequality in the whole space \( \mathbb{R}^n \) and embedding inequalities for Sobolev-Lorentz spaces. In particular, we complete previous results due to [1] Alvino and [29] Talenti by establishing optimal embedding inequalities for the Sobolev-Lorentz quasinorm \( \| \nabla \cdot \|_{p,q} \) also in the range \( p < q < \infty \), which remained essentially open since [1]. Attainability of the best embedding constants is also studied, as well as the limiting case when \( q = \infty \). Here, we surprisingly discover that the Hardy inequality is equivalent to the corresponding Sobolev-Marcinkiewicz embedding inequality. Moreover, the latter turns out to be attained by the so-called “ghost” extremal functions of [6, Brezis-Vazquez], in striking contrast with the Hardy inequality, which is never attained. In this sense, our functional approach seems to be more natural than the classical Sobolev setting, answering a question raised in [6].

1. Introduction

The classical Hardy inequality for smooth compactly supported functions in \( \Omega \subseteq \mathbb{R}^n \) and for \( 1 < p < n \), reads as follows

\[
(H_p) \quad \left( \frac{n-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} dx \leq \int_{\Omega} |\nabla u|^p dx
\]

where the constant in the left hand side of \((H_p)\) is sharp for any sufficiently smooth domain containing the origin. Actually, Hardy proved in 1925 the one dimensional version of \((H_p)\), see [19, 11] for a historical insight into the subject. The original result has been extended and generalized by many authors in several directions which break through different aspects of Analysis, Geometry and PDE, among which we mention [23, 26, 21, 14, 18, 8, 15, 17, 22].

While much progress has been achieved in understanding \((H_p)\) and its generalizations, a basic question raised by Brezis and Vazquez in [6] on the attainability of the best constant in \((H_p)\) has not been given a full answer yet. Indeed, in [6, 23] it was found that additional lower order terms are admissible on the left hand side of \((H_p)\) as long as \( \Omega \) stays bounded, and an extensive literature has been devoted to searching for such remainder terms in Hardy and Hardy-type inequalities, see [15, 8] and references therein. This phenomenon yields an obstruction to...
the attainability of the best constant in \( \mathcal{H}_p \), provided the domain \( \Omega \) contains the origin.

When \( \Omega = \mathbb{R}^n \) the existence of a suitable class of remainders has been recently established in \([8, 28]\), see also \([16]\). As mentioned, the presence of remainders prevents that the Hardy inequality is attained, and we refer also to the recent papers \([12, 13]\) for a deeper understanding of this phenomenon. In particular, the Euler-Lagrange equation corresponding to the equality case in Hardy’s inequality has no solution in the Sobolev space \( \mathcal{D}^{1,p}(\mathbb{R}^n) \), defined as the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm \( \|\nabla \cdot\|_p \), however it is explicitly solved by a class of functions which do not belong to this space. The lack of a proper function space setting was pointed out in \([6]\) and has inspired our work since the very beginning.

Another interesting aspect of \( \mathcal{H}_p \) is its equivalence to the optimal Sobolev embedding for the space \( \mathcal{D}^{1,p}(\mathbb{R}^n) \) in the context of Lorentz spaces, namely

\[
(A_{p,p}) \quad \|u\|_{p^{*},p} \leq S_{n,p} \|\nabla u\|_p
\]

which was obtained by Alvino in \([1]\), see also Peetre \([27]\). The constant

\[
S_{n,p} = \frac{n}{p} \frac{\Gamma(1+n/2)^2}{\sqrt{\pi}} = \frac{n}{p} \frac{\omega_n}{n}
\]

is best possible and the embedding given by \( (A_{p,p}) \) is optimal in regards to the target space \( L^{p^{*},p} \) which is smallest among all rearrangements invariant spaces \([3, 14]\) (\( \Gamma \) denotes the standard Euler Gamma function and \( \omega_n \) stands for the measure of the unit sphere in \( \mathbb{R}^n \)). In this sense, \( (A_{p,p}) \) yields the optimal version of the Sobolev embedding theorem.

The equivalence between \( \mathcal{H}_p \) and \( (A_{p,p}) \) is a consequence of the Pólya-Szego principle and the Hardy-Littlewood inequality by which the left hand side of \( \mathcal{H}_p \) does not increase under radially decreasing symmetrization and it is equal to the left hand side of \( (A_{p,p}) \) when \( u \) is radially decreasing.

Alvino in \([1]\) proved actually the following inequalities

\[
(A_{p,q}) \quad \|u\|_{p^{*},q} \leq \frac{n}{p} \frac{\omega_n^{q/2}}{n-p} \|\nabla u\|_{p,q}, \quad 1 \leq p < n
\]

related to the Sobolev-Lorentz embedding

\[
D_H^{1,q} L^{p,q}(\mathbb{R}^n) \hookrightarrow L^{p^{*},q}(\mathbb{R}^n)
\]

with the restriction

\[
1 \leq q \leq p,
\]

see also \([7]\). The homogeneous Sobolev-Lorentz space \( D_H^{1,q} L^{p,q}(\mathbb{R}^n) \) is obtained as the closure of smooth compactly supported functions with respect to the Lorentz quasi-norm \( \|\nabla \cdot\|_{p,q} \). Note that the validity of the embedding \([11]\) for \( 1 \leq q \leq +\infty \) is well known in the literature of interpolation theory: more direct and short proofs can be found in \([29, 24]\).
Let us point out that the embedding constant in $(A_{p,q})$ is sharp, and it does not depend on the second Lorentz index $q$. Moreover, up to a normalizing factor, it turns out to be the Hardy constant.

**Main results.**

Our first goal is to extend the validity of the embedding inequality $(A_{p,p})$ to the values $p < q \leq \infty$, still preserving the optimal constant, thus completing the results of Alvino [1] and Talenti [29] to the whole range $1 \leq q \leq \infty$.

In the case $q = \infty$ the functional setting is somewhat delicate, as no Meyer-Serrin type result holds for the corresponding homogeneous spaces. Thus, let us define for $1 \leq p < \infty$ and $1 \leq q \leq \infty$ the space

$$D^1_W L^{p,q}(\mathbb{R}^n) := \{ u \in L^{p^*,q}(\mathbb{R}^n) : |\nabla u| \in L^{p,q}(\mathbb{R}^n) \}.$$ 

Then it turns out that for $q < \infty$

$$D^1_H L^{p,q}(\mathbb{R}^n) = D^1_W L^{p,q}(\mathbb{R}^n) =: D^1 L^{p,q}(\mathbb{R}^n)$$

whereas for $q = \infty$ one has

$$D^1_H L^{p,\infty}(\mathbb{R}^n) \subsetneq D^1_W L^{p,\infty}(\mathbb{R}^n),$$

see [10] and Section 2 for more details.

**Theorem 1.** Let $1 \leq p < n$, $p < q \leq \infty$. Then the following inequality holds for any $u \in D^1_W L^{p,q}(\mathbb{R}^n)$

$$(A_{p,q}) \quad \| u \|_{p^*,q} \leq \frac{p}{n-p} \omega_n^{-\frac{1}{n}} \| \nabla u \|_{p,q}$$

where the constant $\frac{p}{n-p} \omega_n^{-\frac{1}{n}}$ is sharp.

Then, surprisingly, we establish the equivalence between $(A_{p,\infty})$ and $(H_p)$.

**Theorem 2.** Let $1 \leq p < n$. Then, Hardy’s inequality

$$(H_p) \quad \left( \frac{n-p}{p} \right)^p \int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx \leq \int_{\mathbb{R}^n} |\nabla u|^p \, dx$$

holds for any $u \in D^1_{W} L^{p}(\mathbb{R}^n)$ if and only if the Sobolev-Marcinkiewicz embedding inequality

$$(A_{p,\infty}) \quad \| v \|_{p^*,\infty} \leq \frac{p}{n-p} \omega_n^{-\frac{1}{n}} \| \nabla v \|_{p,\infty}$$

holds for any $v \in D^1_W L^{p,\infty}(\mathbb{R}^n)$. 
Finally we study the attainability of \((A_{p,q})\). In particular, in the limiting case \(q = \infty\), \((A_{p,\infty})\) turns out to be attained, in striking contrast to Hardy’s inequality, regardless of their equivalence as established in Theorem 2.

**Theorem 3.** Let \(1 \leq p < n\) and \(1 \leq q \leq \infty\). Then, the sharp constant in \((A_{p,q})\) is attained if and only if \(q = +\infty\). Moreover, an extremal function for \((A_{p,\infty})\) in \(\mathcal{D}_W L^{p,\infty}\) is given by

\[
\psi(x) = |x|^{-\frac{n-p}{p}}
\]

**Remark:** The extremal function in Theorem 3 is exactly the “ghost” extremal function of [6, Brezis-Vazquez].

**Overview.**

In Section 2 we recall for convenience some well known facts and prove a few preliminary results. Then, in Section 3 we prove Theorem 1 by showing that \((A_{p,q})\) for \(p < q \leq \infty\) can be obtained as a consequence of \((A_{p,p})\), which is actually equivalent to Hardy’s inequality. The proof relies on suitable scaling properties whereas the sharpness of the embedding constants is proved by inspection. As a byproduct of Theorem 3, the sharp Marcinkiewicz type inequality \((A_{p,\infty})\) in \(\mathcal{D}_W L^{p,\infty}\) is a consequence of \((A_{p,p})\), that is of the Hardy inequality \((H_p)\). In Section 4, we surprisingly prove also the converse, namely that the validity of \((A_{p,\infty})\) in \(\mathcal{D}_W L^{p,\infty}\) implies Hardy’s inequality \((H_p)\) in \(\mathcal{D}^{1,p}\). In Section 5 we prove that the best constant in \((A_{p,q})\) is never attained as long as \(q < \infty\), and then attained at the endpoint of the Lorentz scale for \(q = \infty\). This is in striking contrast with Hardy’s inequality which is never attained though being equivalent to \((A_{p,\infty})\).

In this sense our functional framework, namely the Sobolev-Marcinkiewicz space \(\mathcal{D}_W L^{p,\infty}\), seems to be qualified as more natural than the classical \(\mathcal{D}^{1,p}\) setting in the tradition of Hardy type inequalities. This phenomenon throws light on the importance of considering the couple (inequality, functional setting), as the whole information retained can be differently shared between the two components through equivalent versions. Finally, in the Appendix we recall and adapt to our situation a by now standard technique to reduce the embedding problems to the radial case, as initially developed by [2, Alvino-Lions-Trombetti].

**2. Preliminaries**

For convenience of the reader, let us briefly recall some basic facts on Lorentz spaces [20] which will be widely used throughout the paper.

For a measurable function \(u : \Omega \to \mathbb{R}^+\), let \(u^*\) denote its decreasing rearrangement which is defined as the distribution function of the distribution function \(\mu_u\) of \(u\), namely

\[
u^*(s) = |\{t \in [0, +\infty) : \mu_u(t) > s\}| = \sup\{t > 0 : \mu_u(t) > s\}, \quad s \in [0, |\Omega|]
\]
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whereas the spherically symmetric rearrangement $u^\#(x)$ of $u$ can be defined as

$$u^\#(x) = u^*(\omega_n |x|^n), \quad x \in \Omega^\#$$

where $\Omega^\# \subset \mathbb{R}^n$ is the open ball with center in the origin which satisfies $|\Omega^\#| = |\Omega|$ and $\omega_n$ is the area of the unit sphere of $\mathbb{R}^n$. Clearly, $u^*$ is a nonnegative, non-increasing and right-continuous function on $[0, \infty)$. Moreover, the (nonlinear) rearrangement operator enjoys the following properties:

i) Positively homogeneous: $(\lambda u)^* = |\lambda|u^*$, $\lambda \in \mathbb{R}$;

ii) Sub-additive: $(u + v)^*(t + s) \leq u^*(t) + v^*(s)$, $t, s \geq 0$;

iii) Monotone: $0 \leq u(x) \leq v(x)$ a.e. in $\Omega \Rightarrow u^*(t) \leq v^*(t)$, $t \in (0, |\Omega|)$;

iv) $u$ and $u^*$ are equidistributed and in particular (Cavalieri’s principle)

$$\int_\Omega A(|u(x)|) \, dx = \int_0^{[\Omega]} A(u^*(s)) \, ds$$

for any continuous function $A : [0, \infty] \to [0, \infty]$, nondecreasing and such that $A(0) = 0$;

v) The following inequality holds (Hardy-Littlewood):

$$\int_\Omega u(x)v(x) \, dx \leq \int_0^{[\Omega]} u^*(s)v^*(s) \, ds$$

provided the integrals involved are well defined.

vi) The map $u \mapsto u^*$ preserves Lipschitz regularity, namely

$$^* : \text{Lip}(\Omega) \to \text{Lip}(0, |\Omega|)$$

Then, the Lorentz space $L^{p,q}(\Omega)$ is a rearrangement invariant Banach space which can be defined as follows

$$L^{p,q}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} \mid \|u\|_{p,q} := \left( \int_0^{[\Omega]} (u^*(t)t^{1/p})^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}$$

where the quantity $\|u\|_{p,q}$ is a quasi-norm which admits an equivalent norm. One clearly has $L^{p,p} = L^p$ and furthermore, with respect to the second index, Lorentz spaces satisfy the following inclusions (Lorentz scale)

$$L^{p,q_1} \subseteq L^{p,q_2}, \quad \text{if } 1 \leq q_1 < q_2 \leq \infty$$

For $q = \infty$ we obtain the so-called Marcinkiewicz or weak-$L^p$ space, which is defined as follows

$$\|u\|_{p,\infty} := \sup_{t>0} t^{\frac{1}{p}} u^*(t)$$
Notice that in particular one has $L^{p^*,q} \subseteq L^{p^*,q} = L^{p^*}$.

Sobolev-Lorentz spaces generalize classical Sobolev spaces. First order Sobolev-Lorentz spaces can be defined either as the closure of smooth compactly supported functions $u$ with respect to the norm $\| \nabla u \|_{p,q} + \| u \|_{p,q}$, or as the set of functions in $L^{p,q}(\Omega)$ whose distributional gradient also belongs to $L^{p,q}$. We refer for the general theory on Sobolev-Lorentz spaces to [10] and references therein, and to [9] for more general Sobolev spaces realized on rearrangement invariant Banach spaces.

Here we focus on homogeneous Sobolev-Lorentz spaces defined for $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$ by

$$D^1_{H,L^{p,q}} = \text{cl}\{ u \in C_c^\infty(\mathbb{R}^n) : \| \nabla u \|_{p,q} < \infty \}$$

Since $D^1_{H,L^{p,q}} \hookrightarrow L^{p^*,q}$, as a consequence of [1, 29], one may also consider the alternative definition given by

$$D^1_{W,L^{p,q}} = \{ u \in L^{p^*,q}(\mathbb{R}^n) : \| \nabla u \|_{p,q} < \infty \}.$$ 

It turns out that the two spaces coincide as long as $q < \infty$ [10, Section 4]:

$$D^1_{W,L^{p,q}} = D^1_{H,L^{p,q}} = D^1_{L^{p,q}}$$ 

whereas in the limiting case $q = \infty$, we have

$$D^1_{H,L^{p,\infty}} \subsetneq D^1_{W,L^{p,\infty}}$$

A function belonging to $D^1_{W,L^{p,\infty}} \setminus D^1_{H,L^{p,\infty}}$ is given by $u(x) = |x|^{-\frac{n-p}{p}}$ ([11] Prop. 4.7]).

Sobolev-Lorentz spaces enjoy invariance properties by scaling. As a consequence, inequalities $(A_{p,q})$ and in particular the Hardy inequality $(H_p)$ are invariant under the action of the group of dilations, as established in the following

**Proposition 1.** Let $\lambda > 0$, $1 \leq p < n$, $1 \leq q \leq \infty$ and $u_\lambda(x) := u(\lambda x)$. Then, the following quotients

$$\frac{\| \nabla u_\lambda \|_{p,q}}{\| u_\lambda \|_{p^*,q}}, \quad u \in D^1_{L^{p,q}}(\mathbb{R}^n)$$

and

$$\frac{\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} \, dx}{\| \nabla u_\lambda \|^p_p}, \quad u \in D^1_{L^p}(\mathbb{R}^n)$$

are constant with respect to $\lambda$.

**Proof.** Let us first consider the case $q < \infty$. We have $u_\lambda^*(|x|) = u^*(\lambda|x|)$, $u_\lambda^*(t) = u^*(\lambda^n t)$ and

$$\| u_\lambda \|^q_{p^*,q} = \int_0^{+\infty} u_\lambda^*(t) \frac{1}{t^p} \, dt \frac{1}{t} = \int_0^{+\infty} u^*(\tau) \frac{1}{\tau^{n-p}} \, d\tau = \frac{\lambda^{-\frac{nq}{p}}}{\lambda^{\frac{n}{p}}} \| u \|^q_{p^*,q}$$
From $\nabla u_\lambda(x) = \lambda \nabla u(x)$, we have $|\nabla u_\lambda|^p(|x|) = \lambda |\nabla u|^p(\lambda |x|)$ and $|\nabla u_\lambda|^*(t) = \lambda |\nabla u|^*(\lambda^n t)$. Hence

$$
\|\nabla u_\lambda\|_{p,q}^q = \int_0^{+\infty} \left[ |\nabla u_\lambda|^*(t) t^{\frac{1}{p}} \right]^q \frac{dt}{t} = \int_0^{+\infty} \left[ |\nabla u|^*(\tau)(\lambda^{-n}\tau)^{\frac{1}{p}} \right]^q \frac{d\tau}{\tau}
$$

and the first claim follows as $(-\frac{n}{p} + 1) = -\frac{n}{p^*}$.

When $q = \infty$ we have,

$$
\|u_\lambda\|_{p^*,\infty} = \sup_t t^{1/p^*} u_\lambda^*(t) = \lambda^{-n/p^*} \sup\tau^{1/p^*} u^*(\tau) = \lambda^{-n/p^*} \|u\|_{p^*,\infty}
$$

as well as

$$
\|\nabla u_\lambda\|_{p,\infty} = \sup_t t^{1/p} \nabla u_\lambda^*(t) = \lambda^{-n/p+1} \sup\tau^{1/p} \nabla u^*(\tau) = \lambda^{-n/p+1} \|u\|_{p,\infty}
$$

and the first claim follows as above.

The second claim follows by observing that

$$
\int_{\mathbb{R}^n} \frac{|v_\lambda|^p}{|x|^p} \, dx = \lambda^{-n+p} \int_{\mathbb{R}^n} \frac{|v|^p}{|x|^p} \, dx
$$

\[\square\]

### 3. Proof of Theorem \[\square\]

Next we will prove that the sharp embedding inequality $(A_{p,p})$ implies all the embedding inequalities $(A_{p,q})$, $p < q \leq +\infty$, still preserving the sharp embedding constants. The proof strongly relies on the reduction to the radial case, which is a rather delicate issue in the case $q > p$: indeed, the argument used in \[\square\] for $q < p$, is based on a generalization of the Pólya-Szegő result, which cannot be applied here. However, following the approach of \[\square\], one can prove that for any $u \in C^\infty_0(\mathbb{R}^n)$ there exists $v \in D^{1,2}L^{p,q}(\mathbb{R}^n)$, namely $v \in D^1L^{p,q}(\mathbb{R}^n)$ and it is radial and monotone decreasing, such that

$$
\|v\|_{p^*,q} \geq \|u\|_{p^*,q} \quad \text{and} \quad \|\nabla v\|_{p,q} \leq \|\nabla u\|_{p,q}
$$

This fact allows to restrict to radial decreasing functions also in the case $p < q < +\infty$. Though the argument is standard by now, we outline the details in the Appendix.

The proof of Theorem \[\square\] is based on scaling arguments. Let us divide the proof into two steps.

**Step 1.** The case $p < q < \infty$. 

Let \( u \in \mathcal{D}^1L^{p,q}(\mathbb{R}^n) \) such that \( u = u^\# \) and define the radially decreasing function

\[
v(x) := \left[ u(|x|^\frac{p}{q}) \right]^\frac{q}{p} = \left[ u^\ast \left( |x|^{\frac{p}{q}} \right) \right]^\frac{q}{p}
\]

so that

\[
v^\#(|x|) = \left[ u(|x|^\frac{p}{q}) \right]^\frac{q}{p}, \quad v^\ast(t) = v^\#(\left( \frac{t}{\omega_n} \right)^{1/n}) = \left[ u \left( \left( \frac{t}{\omega_n} \right)^{\frac{p}{q}} \right) \right]^\frac{q}{p} = \left[ u^\ast \left( \left( \frac{t}{\omega_n} \right)^{\frac{p}{q}} \right) \right]^\frac{q}{p}
\]

One has \( v \in L^{p^*,p}(\mathbb{R}^n) \), indeed

\[
\|v\|_{p^*,p} = \left\{ \int_0^\infty \left[ u^\ast(t) t^{1/p^*} \right] \frac{dt}{t} \right\}^{1/p}
\]

\[
= \left\{ \int_0^\infty \left[ u^\ast \left( \frac{t}{\omega_n} \right)^{\frac{p}{q}} t^{1/p^*} \right] \frac{dt}{t} \right\}^{1/p}
\]

\[
= \left\{ \int_0^\infty \left[ u^\ast \left( \frac{t}{\omega_n} \right)^{\frac{p}{q}} t^{\frac{n-p}{pq}} \right] \frac{dt}{t} \right\}^{1/p}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \left\{ \int_0^\infty \left[ u^\ast \left( \frac{t}{\omega_n} \right)^{\frac{1}{p^*}} \right] t^{\frac{n-p}{pq}} \frac{dt}{t} \right\}^{1/p}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \omega_n^{\frac{n-p}{pq}} \left\{ \int_0^\infty \left[ u^\ast \left( \frac{\tau}{\omega_n} \right) \right] t^{\frac{n-p}{pq}} \frac{d\tau}{\tau} \right\}^{1/p}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \omega_n^{\frac{n-p}{pq}} \|u\|_{p^*,q}^{\frac{q}{p}}
\]

Moreover, \( v \in \mathcal{D}^{1,p}(\mathbb{R}^n) \) as one has

\[
|\nabla v(x)| = |u(|x|^\frac{p}{q})|^{\frac{n-2}{p}} |\nabla u(|x|^\frac{p}{q})||x|^{\frac{n}{q}}
\]

so that

\[
\|\nabla v\|_p = \left\{ \int_{\mathbb{R}^n} |\nabla v|^p dx \right\}^{1/p}
\]

\[
= \left\{ \int_{\mathbb{R}^n} \left[ u(|x|^\frac{p}{q}) \right]^{\frac{n-2}{p}} |\nabla u(|x|^\frac{p}{q})||x|^{\frac{n}{q}} dx \right\}^{1/p}
\]

here the fact \( q > p \) is crucial. Next apply the (generalized) Hardy-Littlewood inequality to have

\[
\int_{\mathbb{R}^n} |f(x)g(x)h(x)| dx \leq \int_0^{\infty} f^\ast(t) g^\ast(t) h^\ast(t) dt = \int_{\mathbb{R}^n} f^\sharp(x) g^\sharp(x) h^\sharp(x) dx
\]
Since \( q > p \) the following hold

\[
\left( |u(x|^\frac{p}{n})|^{\frac{q-p}{p-n}} \right)^\frac{1}{q} (|y|) = |u|^{\frac{q-p}{n}} (|y|^\frac{p}{n})
\]
\[
\left( |x|^\frac{q-p}{n} \right)^\frac{1}{q} (|y|) = |y|^{\frac{q-p}{n}}
\]
\[
\|\nabla u(x|^\frac{p}{n})\|^2 (|y|) = |\nabla u|^2(|y|^\frac{p}{n})
\]

Thus

\[
\|\nabla v\|_p \leq \left\{ \int_{\mathbb{R}^n} \left[ |u|^{\frac{q-p}{p-n}} (|x|^\frac{p}{n}) |\nabla u|^2(|x|^\frac{p}{n}) |x|^\frac{q-p}{n} \right]^p dx \right\}^{1/p}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \left\{ \int_{\mathbb{R}^n} \left[ |u|^{q-p} \cdot |y|^{\frac{(n-p)(q-p)}{qp}} \left( |\nabla u|^2\right)^p(|y|) |y|^{\frac{n-q-p}{q}} \right] dy \right\}^{1/p}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \left\{ \int_{\mathbb{R}^n} \left[ |u|^{q-p} \cdot |y|^{\frac{(n-p)(q-p)}{qp}} \left( |\nabla u|^2\right)^p(|y|) |y|^{\frac{n-q-p}{q}} \right] dy \right\}^{1/p}
\]

\[
\leq \left( \frac{q}{p} \right)^{\frac{1}{p}} \left\{ \int_{\mathbb{R}^n} \left[ |u|^{q-p} \cdot |y|^{\frac{(n-p)(q-p)}{qp}} \left( |\nabla u|^2\right)^p(|y|) |y|^{\frac{n-q-p}{q}} \right] dy \right\}^{1/p}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \left\{ \int_{\mathbb{R}^n} \left[ |u|^{q-p} \cdot |y|^{\frac{(n-p)(q-p)}{qp}} \left( |\nabla u|^2\right)^p(|y|) |y|^{\frac{n-q-p}{q}} \right] dy \right\}^{1/p}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \left\{ \left( \frac{q-p}{q} \right)^{\frac{1}{q}} \int_{\mathbb{R}^n} \left( \nabla u \right)^q |y|^{\frac{n-q-p}{q}} dt \right\}^{\frac{1}{q}}
\]

\[
= \left( \frac{q}{p} \right)^{\frac{1}{p}} \left\{ \left( \frac{q-p}{q} \right)^{\frac{1}{q}} \int_{\mathbb{R}^n} \left( \nabla u \right)^q |y|^{\frac{n-q-p}{q}} dt \right\}^{\frac{1}{q}}
\]

\[
(5) = \left( \frac{q}{p} \right)^{\frac{1}{p}} \omega_n^{\frac{(n-p)(q-p)}{np^2}} \|u\|_{p \cdot q} \|\nabla u\|_{p \cdot q}
\]

Now combine Alvino’s inequality

\[
\|v\|_{\omega^1/p} \leq \frac{p}{n-p} \omega_n^{1/n} \|\nabla v\|_p
\]
with (4) and (5) to obtain

\[
\|u\|_{p^*, q} = \left( \frac{p}{q} \right)^{\frac{1}{q}} \omega_n^{\frac{(q-n)(n-p)}{nq}} \|v\|_{p^*, p}^{\frac{q}{p}} \\
\leq \left( \frac{p}{q} \right)^{\frac{1}{q}} \omega_n^{\frac{(q-n)(n-p)}{nq}} \left( \frac{p}{n-p} \right)^{\frac{q}{np}} \omega_n^{\frac{q-n}{nq}} \|\nabla v\|_{p^*}^{\frac{q}{n}} \\
\leq \left( \frac{p}{n-p} \right)^{\frac{q}{nq}} \omega_n^{\frac{q-n}{nq}} \|u\|_{p^*, q}^{\frac{q-n}{nq}} \cdot \|\nabla u\|_{p, q}^{\frac{q}{n}}
\]

and thus our claim

\[
\|u\|_{p^*, q} \leq \frac{p}{n-p} \omega_n^{\frac{1}{p}} \|\nabla u\|_{p, q}
\]

**Step 2.** The case \( q = \infty \).

Let \( u \in D_0^1 L^{p, \infty}(\mathbb{R}^n) \). Let us define the auxiliary function

\[
v(r) = r^{n/p^*} u^\sharp(r)
\]

Then

(6) \[
\|u\|_{p^*, \infty} = \sup_{t>0} t^{1/p^*} u^\star(t) = \omega_n^{1/p^*} \sup_{r>0} r^{n/p^*} u^\sharp(r) = \omega_n^{1/p^*} \|v\|_{\infty}
\]

Since \( v \) has finite \( L^\infty \) norm, it coincides with the limit of the \( L^\gamma \) norm of \( v \), as \( \gamma \to +\infty \). For \( 1 < \tilde{p} < n \) by applying inequality (A\( p, q \)) we have

\[
\|v\|_{\gamma} = \int_0^{+\infty} (u^\sharp(r))^\gamma r^{\frac{n-\gamma}{p}} \frac{dr}{r} = \int_0^{+\infty} \left[ u^\sharp(r) r^{\frac{n-\gamma}{p}} + \frac{1}{r} \right]^\gamma \frac{dr}{r} \\
= \int_0^{+\infty} \left[ u^\sharp(r) r^{n/p^*} \right]^\gamma \frac{dr}{r} = \left[ n \omega_n^{\gamma/p^*} \right]^{-1} \|u\|_{p^*, \gamma}^{\gamma} \quad \text{(where} \quad \frac{n}{\tilde{p}} = \frac{n}{p} + \frac{1}{\gamma} \text{)}
\]

\[
\leq \left[ n \omega_n^{\gamma/p^*} \right]^{-1} \left[ \frac{\tilde{p}}{n-p} \omega_n^{1/n} \right]^\gamma \|\nabla u\|_{p, \gamma}^{\gamma} \\
= \left[ n \omega_n^{\gamma/p^*} \right]^{-1} \left[ \frac{\tilde{p}}{n-p} \omega_n^{1/n} \right]^\gamma \left[ n \omega_n^{\gamma/p^*} \right] \int_0^{1} \left[ \nabla u^\sharp r^{n/p} \right]^\gamma \frac{dr}{r} \\
= \omega_n^{-\gamma/p^* + 1/\tilde{p}} \left[ \frac{\tilde{p}}{n-p} \omega_n^{1/n} \right]^\gamma \int_0^{1} \left[ \nabla u^\sharp r^{n/p} \right]^\gamma \frac{dr}{r} \\
= \omega_n^{-\gamma/p^* + 1/\tilde{p}} \left[ \frac{\tilde{p}}{n-p} \omega_n^{1/n} \right]^\gamma \|\nabla u^\sharp r^{n/p}\|_{\gamma}^{\gamma}
\]
Combining the last inequality with (34) we obtain
\[ \|u\|_{p^*,\infty} = \omega_n^{1/p^*} \|v\|_{\infty} = \omega_n^{1/p^*} \lim_{\gamma \to +\infty} \|v\|_{\gamma} \]
\[ \leq \omega_n^{1/p^*} \lim_{\gamma \to +\infty} \omega_n^{-1/p^* + 1/n} \frac{\tilde{p}}{n-p} \|\nabla u|^{r/n/p}\|_{\gamma} \]
\[ = \omega_n^{1/p^* - 1/n} \frac{p}{n-p} \|\nabla u|^{r/n/p}\|_{\infty} \]
since \( \tilde{p} \to p \) and \( \tilde{p}^* \to p^* \), as \( \gamma \to \infty \). Recalling that
\[ \|\nabla u|^{r/n/p}\|_{\infty} = \omega_n^{-1/p} \|\nabla u\|_{p,\infty} \]
the claim follows.

3.1. **Best constants.** The proof of Theorem 1 will be complete if we prove that
the constant
\[ \frac{\rho - 1}{p - 1} \frac{1}{n} \]
appearing in (A_{p,q}) is sharp for any \( 1 \leq p < q \leq +\infty \).
Notice that for \( q = +\infty \), the sharpness will be a consequence of the attainability
of the constant which will be proved in Section 5, hence we consider only the
case \( q < +\infty \). For this purpose we have to check that the maximizing sequence
introduced in [1] for the case \( 1 \leq q \leq p \) actually works also in the case \( p < q \leq +\infty \).
Consider the radial decreasing function
\[ v_\varepsilon(x) := \begin{cases} 
|x|^{\frac{2-p}{p} + \varepsilon}, & \text{if } |x| < 1 \\
1 - \left(\frac{n-p}{p} - \varepsilon\right) (|x| - 1), & \text{if } 1 \leq |x| < 1 + \frac{1}{n-p} 
\end{cases} \]
whose gradient is given by
\[ |\nabla v_\varepsilon(x)| := \begin{cases} 
\left(\frac{n-p}{p} - \varepsilon\right) |x|^{-\frac{2}{p} + \varepsilon}, & \text{if } |x| < 1 \\
\left(\frac{n-p}{p} - \varepsilon\right), & \text{if } 1 \leq |x| < 1 + \frac{1}{n-p} 
\end{cases} \]
which is a decreasing radial function. One has
\[ \|\nabla v_\varepsilon\|_{p,q}^q = n \omega_n^{\frac{n}{q}} \left(\frac{n-p}{p} - \varepsilon\right)^q \frac{1}{q} \left[1 + \frac{p}{q} \left(\frac{1}{n-p} - 1\right)\right]^{\frac{n-p}{p}} - \frac{p}{q} \]
and
\[ \|v_\varepsilon\|_{p^*,q}^q = n \omega_n^{\frac{n}{q}} \frac{1}{q} + n \omega_n^{\frac{n}{q}} \int_1^{1+\frac{1}{n-p} - \varepsilon} \left[1 - \left(\frac{n-p}{p} - \varepsilon\right) (r-1)^{\frac{n-p}{p}} - 1\right] \frac{n-p}{p} dr 
\leq n \omega_n^{\frac{n}{q}} \frac{1}{q} + n \omega_n^{\frac{n}{q}} \int_1^{1+\frac{1}{n-p} - \varepsilon} r^{\frac{n-p}{p} - 1} 
\leq n \omega_n^{\frac{n}{q}} \frac{1}{q} + n \omega_n^{\frac{n}{q}} \frac{pq}{n-p} \left[ \left(\frac{1}{n-p} - 1\right)\right]^{\frac{n-p}{p} - 1} - 1 \]
so that
\[
\frac{\|\nabla v_\varepsilon\|_{p,q}^q}{\|v_\varepsilon\|_{p,q}^p} \geq \omega_n^{\frac{q}{p}} \left( \frac{n-p}{p} - \varepsilon \right)^q \frac{1}{q} \left( \frac{\|\nabla v\|_{p,q}}{\|\nabla v\|_{p,q}} - \frac{n-p}{p} \right) 
\]
\[
\frac{1}{q} + \frac{m}{n-p} \left[ \frac{1}{n-p} \varepsilon + 1 \right] \frac{n-p}{p} - 1
\]
\[
\longrightarrow \omega_n^{\frac{q}{p}} \left( \frac{n-p}{p} \right)^q
\]
as \varepsilon \to 0, which is our thesis.

4. Proof of Theorem 2
As byproduct of Theorem 1 we have proved the implication \((H_p) \Rightarrow (A_{p,\infty})\). We next prove the converse.

Suppose that \((A_{p,\infty})\) holds, namely
\[
\|v\|_{p,\infty} \leq \frac{p}{n-p} \omega_n^{\frac{1}{n}} \|\nabla v\|_{p,\infty}, \quad v \in \mathcal{D}_1^1 L^p, \infty, (\mathbb{R}^n)
\]
Then we want to prove Hardy’s inequality \((H_p)\) for any function \(u \in \mathcal{D}^{1,p}\). Actually, thanks to the Pólya-Szego and Hardy-Littlewood inequalities, we may restrict ourselves to prove the validity of \((H_p)\) for any \(u \in \mathcal{D}^{1,p}\), that is by density, the class of radially decreasing Lipschitz function with compact support, such that \(\|\nabla u\|_p < +\infty\). By Proposition 1, we may also assume that \(u\) has support in \(B_1\), the unit ball centered at the origin.

Let us define an auxiliary radial function \(v\) as follows
\[
v(r) = \int_r^1 \rho^{\frac{n}{p}} \int_0^1 |u'(t)|^p t^{n-1} dt d\rho
\]
Then \(v \in C^1(B_1 \setminus \{0\})\) and
\[
v'(r) = -\rho^{\frac{n}{p}} \int_r^1 |u'(t)|^p t^{n-1} dt, \quad v(1) = 0, \quad \lim_{r \to 0^+} v(r) = +\infty
\]
so that \(v\) is radially decreasing and also \(|v'| = |\nabla v|\) is radially decreasing.

Hence
\[
\|\nabla v\|_{p,\infty} = \omega_n^{\frac{1}{p}} \sup_{B_1} |\nabla v|^p |x|^p = \omega_n^{\frac{1}{p}} \sup_{0<r<1} |v'| r^{\frac{n}{p}}
\]
\[
= \omega_n^{\frac{1}{p}} \int_0^1 |u'|^p d\rho = \omega_n^{\frac{1}{p}-1} \|\nabla u\|_p
\]
Moreover, \( v \in L^{p^*, \infty} \) since

\[
\|v\|_{p^*, \infty} = \frac{1}{\omega_n} \sup_{0 < r < 1} |v| r^{\frac{2-n}{p}} = \frac{1}{\omega_n} \sup_{0 < r < 1} r^{\frac{2-n}{p}} \int_1^r \rho^{\frac{n}{p}} \int_0^1 |u'(t)| t^{n-1} dt d\rho
\]

\[
= \frac{1}{\omega_n} \sup_{0 < r < 1} r^{\frac{2-n}{p}} \int_1^r |u'(t)| t^{n-1} dt \int_0^1 \rho^{\frac{n}{p}} d\rho
\]

\[
= \frac{p}{n-p} \frac{1}{\omega_n} \sup_{0 < r < 1} r^{\frac{2-n}{p}} \int_1^r |u'(t)| t^{n-1} \left( \rho^{\frac{n}{p}} - t^{\frac{n}{p}} \right) dt
\]

\[
\leq \frac{p}{n-p} \frac{1}{\omega_n} \sup_{0 < r < 1} \int_1^r |u'(t)| t^{n-1} dt = \frac{p}{n-p} \frac{1}{\omega_n} \||\nabla u||_p^p < \infty
\]

thus we have proved \( v \in D^{1, W^{1, \infty}}(\mathbb{R}^n) \).

Now the idea is to estimate from below the norm \( \|v\|_{p^*, \infty} \) with the left hand side of Hardy’s inequality involving the function \( u \). Since

\[
\|v\|_{p^*, \infty} = \frac{1}{\omega_n} \sup_{0 < r < 1} |v| r^{\frac{2-n}{p}}
\]

we have to estimate the quantity

\[
|v| r^{\frac{2-n}{p}} = r^{\frac{2-n}{p}} \int_1^r \rho^{\frac{n}{p}} \int_0^1 |u'(t)| t^{n-1} dt d\rho
\]

from below with Hardy’s integral involving \( u \). Since \(-u'(t) = |u'(t)| \), we have

\[
(9) \quad u^p(\rho) = \left[ \int_0^1 |u'| dt \right]^p = \left[ \int_0^1 |u'| t^{\frac{p-1}{p}} t^{-\frac{n}{p}} dt \right]^p
\]

\[
\leq \int_0^1 |u'| t^{\frac{n-1}{p}} dt \left[ \int_0^1 t^{-\frac{n}{p}} dt \right]^{p-1}
\]

\[
\leq \left( \frac{p}{n-p} \right)^{p-1} \rho^{-\frac{n-p}{p}} (p-1) \int_0^1 |u'| t^{\frac{n-1}{p}} dt.
\]

A key step in the proof is now the evaluation of the following limit, obtained applying de l’Hôpital’s theorem (note that we have an indefinite form \( \infty/\infty \), since
\( n > p \): 

\[
\lim_{r \to 0^+} r^{\frac{n-p}{p}} \int_r^1 z^{-\frac{n}{p}} \int_z^1 u^p(\rho) \rho^{n-p-1} d\rho dz \\
= \lim_{r \to 0^+} \int_r^1 z^{-\frac{n}{p}} \int_z^1 u^p(\rho) \rho^{n-p-1} d\rho dz \\
= \frac{p}{n-p} \lim_{r \to 0^+} r^{-\frac{n}{p}} \int_r^1 u^p(\rho) \rho^{n-p-1} d\rho \\
= \frac{p}{n-p} \int_0^1 u^p(\rho) \rho^{n-p-1} d\rho
\]

and thanks to (9), we have

\[
\int_0^1 u^p(\rho) \rho^{n-p-1} d\rho = \frac{n-p}{p} \lim_{r \to 0^+} r^{\frac{n-p}{p}} \int_r^1 z^{-\frac{n}{p}} \int_z^1 u^p(\rho) \rho^{n-p-1} d\rho dz \\
\leq \frac{n-p}{p} \sup_{0 < r < 1} r^{\frac{n-p}{p}} \int_r^1 z^{-\frac{n}{p}} \int_z^1 u^p(\rho) \rho^{n-p-1} d\rho dz \\
\leq \left( \frac{p}{n-p} \right)^{p-2} \sup_{0 < r < 1} r^{\frac{n-p}{p}} \int_r^1 z^{-\frac{n}{p}} \int_z^1 \rho^{\frac{n}{p} - 1} \rho^{n-p-1} \int_\rho^1 |u'|^p t^{\frac{p-1}{p}} \rho dt d\rho dz \\
= \left( \frac{p}{n-p} \right)^{p-2} \sup_{0 < r < 1} r^{\frac{n-p}{p}} \int_r^1 z^{-\frac{n}{p}} \int_z^1 \rho^{\frac{n}{p} - 2} \int_\rho^1 |u'|^p t^{\frac{p-1}{p}} t r t^{\frac{p-1}{p}} \rho dt \\
= \left( \frac{p}{n-p} \right)^{p-1} \sup_{0 < r < 1} r^{\frac{n-p}{p}} v(r)
\]

By Fubini’s theorem, we reverse the order of integration in the last integral to get

\[
\left( \frac{p}{n-p} \right)^{p-2} \sup_{0 < r < 1} r^{\frac{n-p}{p}} \int_r^1 z^{-\frac{n}{p}} \int_z^1 |u'|^p t^{\frac{p-1}{p}} \rho^{n-p-2} d\rho dt dz \\
\leq \left( \frac{p}{n-p} \right)^{p-1} \sup_{0 < r < 1} r^{\frac{n-p}{p}} \int_r^1 z^{-\frac{n}{p}} \int_z^1 |u'|^p t^{\frac{p-1}{p}} t \rho^{n-p-1} \rho dt \\
= \left( \frac{p}{n-p} \right)^{p-1} \sup_{0 < r < 1} r^{\frac{n-p}{p}} v(r)
\]

where we have used the fact \( \frac{p-1}{p} n + \frac{n}{p} - 1 = n - 1 \).
We conclude the proof by applying the embedding inequality (7),
\[
\int_{\mathbb{R}^n} \frac{u^p}{|x|^p} \, dx = \omega_n \int_0^1 \int_0^1 u^p(\rho) \rho^{n-p-1} \, d\rho \\
\leq \omega_n \left( \frac{p}{n-p} \right)^{p-1} \sup_{0<r<1} r^{\frac{n-p}{p}} v(r) \\
= \omega_n \left( \frac{p}{n-p} \right)^{p-1} \omega_n^{-\frac{n}{p}} \|v\|_{p^*,\infty} \\
\leq \omega_n^{1-\frac{1}{p}} \left( \frac{p}{n-p} \right)^{p} \omega_n^{-\frac{1}{p}} \|v\|_{p,\infty} \\
= \omega_n^{1-\frac{1}{p}} \left( \frac{p}{n-p} \right)^{p} \|\nabla v\|_{p,\infty} \\
= \left( \frac{p}{n-p} \right)^{p} \|\nabla u\|_{p}^{p},
\]
thus Hardy’s inequality.

5. Proof of Theorem 3

Here we discuss the attainability of the sharp embedding constant in \((A_{p,q})\). Observe that for \(q = +\infty\), the best embedding constant in \((A_{p,\infty})\) is attained by the function \(\psi = |x|^{-\frac{n-p}{p}}\), which is radially decreasing together with the gradient \(|\nabla \psi| = \frac{n-p}{p} |x|^{n/p}\). Hence
\[
\|\psi\|_{p^*,\infty} = \omega_n^{1/p^*} \sup_{r>0} r^{\frac{n-p}{p^*}} = \omega_n^{1/p^*} \\
\|\nabla \psi\|_{p,\infty} = \omega_n^{1/p} \frac{n-p}{p} \sup_{r>0} r^{\frac{n}{p}} = \omega_n^{1/p} \frac{n-p}{p}.
\]
Note that actually we have a whole family of extremal functions, due to the invariance by dilation proved in Proposition 1. Moreover, there are plenty of maximizers for \((A_{p,\infty})\) in \(D^{1}_{W}L^{p,q}(\mathbb{R}^n)\), since it is enough to have a local asymptotic behavior as \(\psi\).

We next consider the case \(p < q < +\infty\). We will argue by contradiction, proving that the sharp embedding constant is never attained. Let us suppose the inequality \((A_{p,q})\) is attained at some \(q < \infty\). Following the lines of Section 3 we have at least one radially decreasing maximizer \(u \in D^{1}L^{p,q}(\mathbb{R}^n)\), namely a function \(u\) such that
\[
\|u\|_{p^*,q} = \frac{p}{n-p} \omega_n^{1/q} \|\nabla u\|_{p,q}.
\]
Next define
\[
v(x) := \left[ u(|x|^{\frac{n}{p}}) \right]^{\frac{p}{q}} = \left[ u^{*}(\omega_n^{\frac{p}{q}} |x|) \right]^{\frac{p}{q}}
\]
so that
\[ v^\sharp(|x|) = \left[ u(|x|^{\frac{1}{n}}) \right]^{\frac{q}{p}}, \quad v^*(t) = u^\sharp \left( \left( \frac{t}{\omega_n} \right)^{1/n} \right) = \left[ u \left( \left( \frac{t}{\omega_n} \right)^{\frac{1}{n}} \right) \right]^{\frac{q}{p}} = \left[ v^* \left( \frac{t^\sharp}{\omega_n} \right)^{\frac{q}{p}} \right]^{\frac{q}{p}}. \]

By (4), one has \( v \in L^{p^\ast,p}(\mathbb{R}^n) \) with
\[ \| v \|_{p^\ast,p} = \left( \frac{q}{p} \right)^{\frac{1}{p}} \omega_n^{\frac{p-q}{np^2}} \| u \|_{p^\ast,q}^{\frac{q}{p}}. \]
and \( \nabla v \in L^p \), with
\[ \| \nabla v \|_p \leq \left( \frac{q}{p} \right)^{\frac{1}{p}} \omega_n^{\frac{p-q}{np^2}} \| u \|_{p^\ast,q} \cdot \| \nabla u \|_{p,q}. \]

By (10) we obtain
\[ \| \nabla v \|_p \leq \left( \frac{q}{p} \right)^{\frac{1}{p}} \left( \frac{p}{n-p} \right)^{\frac{q}{p}} \omega_n^{\frac{p-q}{np^2}} \| \nabla u \|_{p^\ast,q} \]
and in turn
\[ \| v \|_{p^\ast,p} = \left( \frac{q}{p} \right)^{\frac{1}{p}} \left( \frac{p}{n-p} \right)^{\frac{q}{p}} \omega_n^{\frac{p-q}{np^2}} \| \nabla u \|_{p^\ast,q} \]
\[ \geq \left( \frac{q}{p} \right)^{\frac{1}{p}} \left( \frac{p}{n-p} \right)^{\frac{q}{p}} \omega_n^{\frac{p-q}{np^2}} \left( \frac{q}{p} \right)^{\frac{1}{p}} \left( \frac{p}{n-p} \right)^{\frac{q}{p}} \| \nabla v \|_p \]
\[ = \frac{p}{n-p} \omega_n^{\frac{q}{np^2}} \| \nabla v \|_p. \]
This directly implies
\[ \| v \|_{p^\ast,p} = \frac{p}{n-p} \omega_n^{-\frac{q}{np^2}} \| \nabla v \|_p \]
and, since \( v = v^\sharp \) one has
\[ \int_{\mathbb{R}^n} \frac{v^p}{|x|^p} \, dx = \frac{p}{n-p} \int_{\mathbb{R}^n} |\nabla v|^p \, dx \]
which contradicts the non-attainability of Hardy’s inequality.

**Appendix A. Reduction to the radial case**

Here we follow [2]. Let \( u \in \mathcal{D}^1 L^{p,q} \), \( u \neq 0 \), smooth and compactly supported. Notice that because of the invariance by dilation, we can also prescribe the measure of the support. We aim at proving that there exists \( v \in \mathcal{D}^1 L^{p,q} \) such that \( \| v \|_{p^\ast,q} \geq \| u \|_{p^\ast,q} \) and \( \| \nabla v \|_{p,q} \leq \| \nabla u \|_{p,q} \). This yields the following maximization problem
\[ \max \{ \| v \|_{p^\ast,q} : v \in W^1_0 L^{p,q}(\Omega), |\nabla v| \leq f \in L^{p,q} \text{ a.e. in } \Omega, f^* = |\nabla u|^* \} \geq \| u \|_{p^\ast,q}. \]
(the last inequality is trivial: set $v \equiv u$, $f \equiv |\nabla u|$). It is known that for any $f \geq 0$, $f \in L^{p,q}(\Omega)$ there exists a maximal nonnegative sub-solution $v \in W^{1}L^{p,q}_{0}(\Omega)$ of the problem
\begin{equation}
|\nabla v| \leq f
\end{equation}
(see [21] Prop. 7.2, p. 164 where the statement is proved for $f \in W^{1,p}_{0}$ but it can be generalized thanks to the monotonicity of the decreasing rearrangement).

Consider the maximization problem
\[
I(u) = \left\{ \sup_{v : v \text{ enjoys (11)}} \|v\|_{p^*,q} : f \geq 0, f \in L^{p,q} \text{ and } f^\sharp = |\nabla u|^t \right\}
\]
It was proved in [18] that if $v$ satisfies (11), with $f \geq 0$, $f \in L^{p,q}$ and $f^\sharp = |\nabla u|^t$, then
\[
v^*(t) \leq \frac{1}{n\omega_{n}^{1/n}} \int_{t}^{[\Omega]} s^{1/n} F(s) \frac{ds}{s}
\]
for some positive $F \in L^{p,q}(0,|\Omega|)$ such that $F(\omega_{n}|x|^n) \prec |\nabla u|^t(|x|)$ where, we recall
\[
f \prec g \iff \begin{cases} \int_{0}^{t} f^*(s)ds \leq \int_{0}^{t} g^*(s)ds, & \text{for } t \in [0,|\Omega|] \\ \int_{0}^{[\Omega]} f^*(s)ds = \int_{0}^{[\Omega]} g^*(s)ds & \int_{0}^{[\Omega]} g^*(s)ds \end{cases}
\]
The relation $f \prec g$ it is known as the Hardy-Littlewood-Polya relation between $f$ and $g$. In particular one has $f^{**}(t) \leq g^{**}(t)$ for any $t$ (see [18, 3, 2] for the definition of $\prec$ and its properties). It turns out that
\[
I(u) \leq J(u)
\]
where $J(u)$ is the following relaxed maximization problem
\[
J(u) = \left\{ \sup_{w : w(|x|) = \frac{1}{n\omega_{n}^{1/n}} \int_{[\Omega]} F(s) s^{1/n} \frac{ds}{s}, F \in L^{p,q}(0,|\Omega|) : F(\omega_{n}|x|^n) \prec |\nabla u|^t(|x|), F \geq 0} \right\}
\]
By direct calculations we have
\begin{equation}
\|w\|_{p^*,q} \leq C_{n,q}\|F\|_{p,q} \leq C\|\nabla u\|_{p,q}
\end{equation}
Consider the following class
\[
K(|\nabla u|^t) := \{ F \in L^{p,q}(0,|\Omega|) : F \geq 0, F(\omega_{n}|x|^n) \prec |\nabla u|^t(|x|) \}
\]
for which the following properties are proved in [2]:
\begin{itemize}
\item $K(|\nabla u|^t)$ is a convex, weakly compact and closed set in $L^{p,q}(0,|\Omega|)$;
\item $K(|\nabla u|^t)$ is the weak closure, in $L^{p,q}(0,|\Omega|)$, of the set of positive functions $f$ such that $f^\sharp = |\nabla u|^t$;
\end{itemize}
any extreme point of $K(|\nabla u|^2)$ (namely, any $F$ such that do not exist $F_1, F_2 \in K, F_1 \neq F_2$, for which $F = \frac{F_1 + F_2}{2}$) is equi-measurable with $|\nabla u|$ with $F^* = |\nabla u|^*$.

(actually the result in [2] is proved in $L^p(\Omega)$, but it can be straightforward generalized).

Thanks to the previous properties, if $w_j$ is any maximizing sequence for $J(u)$, then $|\nabla w_j(|x|)| = F_j(\omega_n|x|)$ is uniformly bounded in $L^{p,q}(\Omega)$ (or, equivalently, $F_j(s)$ is uniformly bounded in $L^{p,q}(0, |\Omega|)$ so that $F_j(s)$ converges weakly in $L^{p,q}(0, |\Omega|)$ to some $F_0(s) \in K(|\nabla u|^2)$, and $F_j(\omega_n|x|)$ converges weakly in $L^{p,q}(\Omega)$ to $F_0(\omega_n|x|)$.

As a consequence, up to a subsequence, $\{w_j\}$ converges pointwise if $x \neq 0$, and also weakly into $L^{p,q}(\Omega)$, to the associated function $w_0 \in L^{p,q}(\Omega)$:

$$w_j(|x|) \overset{L^{p,q}(\Omega)}{\to} w_0(|x|) = \frac{1}{n \omega_n} \int_{\omega_n|x|^n} F_0(s) s^{1/n} \frac{ds}{s}$$

Once we prove that $w_j \to w_0$ in $L^{p,q}(\Omega)$, then $w_0$ will be a maximum point for $J(u)$, and hence an extreme point of $K(|\nabla u|^2)$. As a consequence, $|\nabla w_0|^2(|x|) = F_0^*(\omega_n|x|) = |\nabla u|^2(|x|)$ and thus $|\nabla w_0|^*(s) = F_0^*(s) = |\nabla u|^*(s)$, and our claim follows:

$$w_0 \in D^{1,2}L^{p,q} : \|w_0\|_{p,q} = J(u) \geq I(u) \geq \|u\|_{p,q}, \|\nabla w_0\|_{p,q} = \|\nabla u\|_{p,q}$$

In order to prove the strong convergence of $w_j$ in $L^{p,q}(\Omega)$, let us focus on its gradient $F_j(\omega_n|x|)$ whose Lorentz quasi-norm is given by

$$\|\nabla w_j\|_{L^{p,q}(\Omega)}^q = \|F_j\|_{L^{p,q}(0, |\Omega|)}^q = \int_{0}^{\Omega} [F_j^* t^q]^{q/t} dt$$

Let us recall the maximal function defined for a measurable function $f$ as follows

$$f^{**}(t) := \frac{1}{t} \int_{0}^{t} f^*(s) \, ds$$

which defines an equivalent norm in $L^{p,q}(\Omega)$, as long as $p > 1$, by

$$\|f\|_{p,q} = \|t^{1 - \frac{1}{p}} f^{**}(t)\|_{L^q(0, |\Omega|)}$$

Since $F_j(\omega_n|x|) \prec |\nabla u|^2(|x|)$, we have

$$F_j^*(t) \leq F_j^{**}(t) \leq |\nabla u|^{**}(t), \quad (F_j^q)^*(t) = (F_j^q)^*(t) \leq (F_j^q)^*(t) \leq (|\nabla u|^q)^*(t),$$

and hence

$$(F_j^q)^*(t) t^{q-1} \leq (|\nabla u|^q)^*(t) t^{q-1} \in L^1(0, |\Omega|), \quad \text{since } |\nabla u| \in L^{p,q}$$

On the other hand, since $F_j(s)$ converges weakly to $F_0(s)$ in $L^{p,q}(0, |\Omega|)$, we get

$$\int_{0}^{\Omega} F_j^*(t) dt = \int_{0}^{\Omega} F_j^*(t) \cdot 1 dt \to \int_{0}^{\Omega} F_0^*(t) dt$$
so that, up to a subsequence, $F_j^*(t)$ converges in measure and a.e. to $F_0^*(t)$. We can then apply the Lebesgue dominated convergence theorem to the sequence $(F_j^*)^*(t)t^{\frac{2}{p}-1}$, obtaining
\[
\|F_j\|_{L^{p,q}(0,|\Omega|)} \to \|F_0\|_{L^{p,q}(0,|\Omega|)}
\]
and eventually $|\nabla w_j| = F_j^*(\omega_n|x|^n) \to |\nabla w_0| = F_0^*(\omega_n|x|^n)$ strongly in $L^{p,q}(\Omega)$. From the embedding $W_0^{1,L^{p,q}} \hookrightarrow L^{p',q}$, we have $w_j \to w_0$ strongly in $L^{p',q}(\Omega)$.

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