Solving the Braid Word Problem Via the Fundamental Group

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Abstract

The word problem of a group is a very important question. The word problem in the braid group is of particular interest for topologists, algebraists and geometers. In [7] we have looked at the braid group from a topological point of view, and thus using a new computerized representation of some elements of the fundamental group we gave a solution for its word problem. In this paper we will give an algorithm that will make it possible to transform the new presentation from [7] into a syntactic presentation. This will make it possible to computerize the group operation to sets of elements of the fundamental group, called a g-base, which are isomorphic to the braid group. Moreover we will show that it is sufficient enough to look at the syntactic presentation in order to solve the braid word problem, resulting with a better and faster braid word solution.

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Introduction

Let $D$ be a closed disk, and $K = \{k_1, ..., k_n\}$ be $n$ points in $D$. Let $B$ be the group of all diffeomorphisms $\beta$ of $D$ such that $\beta(K) = K$, and $\beta|_{\partial D} = \text{Id}|_{\partial D}$. The braid group is derived from $B$ by identifying two elements if their actions on $\pi_1(D \setminus K, u)$ are equal. To simplify the algorithm, we choose a geometric base of $\pi_1(D \setminus K, u)$, and we look at the action of $\beta \in B$ on the elements of this geometrical base. This yields a group which is isomorphic to the braid group.

In [7] we gave a geometric approach for solving the braid word problem based on this definition of the braid group. Now, we will show a method that will enable us to add to the new computerized presentation of the braid group that was represented in [7] a group structure, which means describing a way for computing the result
of multiplication of two braids only when given their two representations of the g-base.

This yields us with a new syntactic approach for encoding the elements of the g-base of the fundamental group, that will make it possible to improve the running time of the algorithm that solves the braid word problem, resulting with an even more practical algorithm for short braid words.

In section 1, we will give the topological definition of the braid group, based on some facts concerning the fundamental group of a punctured disk, and in the end of this section we will define the braid word problem. In section 2, we will recall the new presentation of the elements of the g-base of the fundamental group that was presented in [7]. In section 3 and 4, we will introduce the syntactic presentation of elements of the g-base and the methods to transform the geometrical presentation into the syntactic and vice versa. Section 5 will be dedicated to adding a group structure on the new presentations. Section 6 will show the improvement of the solution of the braid word problem that was given in [7], and section 7 will give conclusions, future applications of the new presentation, and further plans.

1 The topological definition of the braid group

In the first part of this section we will give the definition of the fundamental group and some aspects of it, while on the second part we will give the topological definition of the braid group which is different then the widely used algebraic definition introduced by Artin [1].

1.1 The fundamental group

Let \( D \) be a topological space. We fix a base point \( u \in D \). A path in \( D \) is a continuous map \( \gamma : [0,1] \to D \). A loop based at \( u \) is a path on \( \overline{D} \) such that \( \gamma(0) = \gamma(1) = u \). Two loops \( \gamma_1 \) and \( \gamma_2 \) are said to be homotopic if there is a continuous map \( G : [0,1] \times [0,1] \to D \) such that \( G(0,t) = \gamma_1(t) \) and \( G(1,t) = \gamma_2(t) \) for all \( 0 \leq t \leq 1 \), and \( G(s,0) = G(s,1) = u \) for all \( 0 \leq s \leq 1 \). Homotopy is an equivalence relation on the set of all loops based at \( u \).

**Definition 1.1** The fundamental group of \( D \) is the set of homotopy classes of loops based at \( u \), and is denoted by \( \pi_1(D,u) \). The operation of concatenation of loops forms a group structure on \( \pi_1(D,u) \).

We will reduce our look at the fundamental group to the case where \( D \) is a closed disk, and \( K = \{ k_1, \ldots, k_n \} \) is a finite set such that \( K \subset int(D) \). When we will refer to \( D \) and \( K \) from practical algorithmic point of view we will reduce our
look even more to the case were $D$ is the closed unit disk and $K$ is a finite set of points ordered from left to right on the x-axis.

**Remark 1.2** It is known that the fundamental group of a punctured disk with $n$ holes is a free group on $n$ generators.

We fix an orientation on $D$. Let $q$ be a simple path connecting $u$ with one of the $k_i$, say $k_{i_0}$, such that $q$ does not meet any other point $k_j$ where $j \neq i_0$. To $q$ we will assign a loop $l(q)$ (which is an element of $\pi_1(D \setminus K, u)$) as follows:

**Definition 1.3** $l(q)$

Let $c$ be a simple loop equal to the (oriented) boundary of a small neighborhood $V$ of $k_{i_0}$ chosen such that $q' = q \setminus (V \cap q)$ is a simple path. Then $l(q) = q' \cup c \cup q'^{-1}$. We will use the same notation for the element of $\pi_1(D \setminus K, u)$ corresponding to $l(q)$.

**Definition 1.4** Let $(T_1, \ldots, T_n)$ be an ordered set of simple paths in $D$ which connect the $k_i$'s with $u$ such that:

1. $T_i \cap k_j = \emptyset$ if $i \neq j$ for all $i, j = 1, \ldots, n$.
2. $\bigcap_{i=1}^{n} T_i = \{u\}$.
3. for a small circle $c(u)$ around $u$, each $u'_i = T_i \cap c(u)$ is a single point and the order in $(u'_1, \ldots, u'_n)$ is consistent with the positive orientation of $c(u)$.

We say that two such sets $(T_1, \ldots, T_n)$ and $(T'_1, \ldots, T'_n)$ are equivalent if $l(T_i) = l(T'_i)$ in $\pi_1(D \setminus K, u)$ for all $i = 1, \ldots, n$. An equivalence class of such sets is called a bush in $D \setminus K$.

**Definition 1.5** A g-base (geometrical base) of $\pi_1(D \setminus K, u)$ is an ordered free base of $\pi_1(D \setminus K, u)$ which has the form $(l(T_1), \ldots, l(T_n))$, where $(T_1, \ldots, T_n)$ is a bush in $D \setminus K$.

We would like to point out a particular g-base which will be used in the paper. Choose $T_i$ to be the straight line connecting $u$ with $k_i$, then we call $(l(T_1), \ldots, l(T_n))$ the standard g-base of $\pi_1(D \setminus K, u)$ and it is shown in the following figure:
We want to point out that the $n$ elements of the g-base generate the fundamental group of the punctured disk $D \setminus K$, and we will call the elements of the standard g-base the standard generators of the fundamental group.

1.2 The braid group

Let $D, K, u$ be as above.

**Definition 1.6** Let $B$ be the group of all diffeomorphisms $\beta$ of $D$ such that $\beta(K) = K$, $\beta|_{\partial D} = \text{Id}|_{\partial D}$. For $\beta_1, \beta_2 \in B$ we say that $\beta_1$ is equivalent to $\beta_2$ if $\beta_1$ and $\beta_2$ induce the same automorphism of $\pi_1(D \setminus K, u)$. The quotient of $B$ by this equivalence relation is called the braid group $B_n[D, K]$ ($n = \#K$). The elements of $B_n[D, K]$ are called **braids**.

**Remark 1.7** For the canonical homomorphism $\psi : B \to \text{Aut}(\pi_1(D \setminus K, u))$ we actually have $B_n[D, K] \cong \text{Im}(\psi)$.

We recall two facts from [section III].

1. If $K' \subset D'$, where $D$ is another disk, and $\#K = \#K'$ then $B_n[D, K] \cong B_n[D', K']$.

2. Any braid $\beta \in B_n[D, K]$ transforms a g-base to a g-base. Moreover, for every two g-bases, there exists a unique braid which transforms one g-base to another.

We distinguish some elements in $B_n[D, K]$ called **half-twists**.

Let $D, K, u$ be as above. Let $a, b \in K$ be two points. We denote $K_{a,b} = K \setminus \{a, b\}$. Let $\sigma$ be a simple path in $D \setminus (\partial D \cup K_{a,b})$ connecting $a$ with $b$. Choose a small regular neighborhood $U$ of $\sigma$ and an orientation preserving diffeomorphism $f : \mathbb{R}^2 \to \mathbb{C}$ such that $f(\sigma) = [-1, 1]$, $f(U) = \{z \in \mathbb{C} \mid |z| < 2\}$. 

\[5\]
Let $\alpha(x), 0 \leq x$ be a real smooth monotone function such that:

$$
\alpha(x) = \begin{cases} 
1 & 0 \leq x \leq \frac{3}{2} \\
0 & 2 \leq x
\end{cases}
$$

Define a diffeomorphism $h : \mathbb{C} \to \mathbb{C}$ as follows: for $z = re^{i\varphi} \in \mathbb{C}$ let $h(z) = re^{i(\varphi + \alpha(r)\pi)}$.

For the set $\{ z \in \mathbb{C} \mid 2 \leq |z| \}$, $h(z) = \text{Id}$, and for the set $\{ z \in \mathbb{C} \mid |z| \leq \frac{3}{2} \}$, $h(z)$ a rotation by $180^\circ$ in the positive direction.

The diffeomorphism $h$ defined above induces an automorphism on $\pi_1(D \setminus K, u)$, that switches the position of two generators of $\pi_1(D \setminus K, u)$, as can be seen in the figure:

Considering $(f \circ h \circ f^{-1})|_D$ (we will compose from left to right) we get a diffeomorphism of $D$ which switches $a$ and $b$ and is the identity on $D \setminus U$. Thus it defines an element of $B_n[D,K]$.

**Definition 1.8** Let $H(\sigma)$ be the braid defined by $(f \circ h \circ f^{-1})|_D$. We call $H(\sigma)$ the positive half-twist defined by $\sigma$.

The half-twists generate $B_n$. In fact, one can choose $n-1$ half-twists that generates $B_n$ (see below):

**Definition 1.9** Let $K = \{k_1, ..., k_n\}$, and $\sigma_1, ..., \sigma_{n-1}$ be a system of simple paths in $D \setminus \partial D$ such that each $\sigma_i$ connects $k_i$ with $k_{i+1}$ and

for all $i, j \in \{1, ..., n-1\}$, $i < j$,

$$
\sigma_i \cap \sigma_j = \emptyset \quad 2 \leq |i - j| \quad \sigma_i \cap \sigma_{i+1} = \{k_{i+1}\} \quad i = 1, ..., n-2.
$$

Let $H_i = H(\sigma_i)$. The ordered system of (positive) half twists $(H_1, ..., H_{n-1})$ are called a frame of $B_n[D,K]$. 

6
Theorem 1.10 If \((H_1,\ldots,H_{n-1})\) is a frame of \(B_n[D,K]\), then \(B_n[D,K]\) is generated by \(\{H_i\}_{i=1}^{n-1}\). Moreover, if \((H_1,\ldots,H_{n-1})\) is a frame of \(B_n[D,K]\), then the set \(\{H_i\}_{i=1}^{n-1}\) with the two relations \(H_iH_j = H_jH_i\) if \(2 \leq |i-j|\) and \(H_iH_{i+1}H_i = H_{i+1}H_iH_{i+1}\) for any \(i = 1,\ldots,n-2\) are sufficient enough to present \(B_n[D,K]\) and therefore this definition and Artin’s definition for the braid group are equivalent.

Proof: See [10].

Definition 1.11 Let \(\sigma_i\) be the straight line connecting \(k_i\) to \(k_{i+1}\), then we call the frame \((H_1,\ldots,H_{n-1})\) the standard frame of \(B_n[D,K]\).

Let \(G = <\gamma_1,\ldots,\gamma_n>\) be the ordered set of elements of the fundamental group that consists of the elements of the g-base. Now, we look at the action of the diffeomorphisms in \(B\) on \(G\). We can decide whether two diffeomorphisms \(\beta_1\) and \(\beta_2\) represent the same braid or not by looking at their action on \(G\), since the automorphism on \(\pi_1(D \setminus K,u)\) is determined by the action on its generators.

Because the set of half-twists in every frame generate \(B_n[D,K]\), it is sufficient to check the action of each half-twist on \(G\).

Proposition 1.12 Let \((H_1,\ldots,H_{n-1})\) be the standard frame of \(B_n[D,K]\) and \(G = <\gamma_1,\ldots,\gamma_n>\) be the ordered set of elements of the g-base for \(\pi_1(D \setminus K,u)\). The action of \(H_i\) on \(G\) is given by \(G' = <\gamma_1,\ldots,\gamma_{i-1},\gamma_{i+1},\gamma_i\gamma_{i+1}\gamma_i^{-1},\gamma_i+2,\ldots,\gamma_n>\).

Proof: Follows immediately from the definition of the half-twist. \(\square\)

The last proposition means that when we work with the g-base elements we actually working with a subset of the fundamental group, which is the conjugacy classes of the elements of the standard g-base. We will exploit this characteristic in the future when we will present the solution to the word problem.

1.3 The braid word problem

Definition 1.13 Let \(b \in B_n\) be a braid. Then it is clear that \(b = \sigma_{i_1}^{e_1} \cdot \ldots \cdot \sigma_{i_l}^{e_l}\) for some sequence of generators, where \(i_1,\ldots,i_l \in \{1,\ldots,n-1\}\) and \(e_1,\ldots,e_l \in \{1,-1\}\). We will call such a presentation of \(b\) a braid word, and \(\sigma_{i_k}^{e_k}\) will be called the \(k^{th}\) letter of the word \(b\). \(l\) is the length of the braid word.

We will distinguish between two relations on the braid words.

Definition 1.14 Let \(w_1\) and \(w_2\) be two braid words. We will say that \(w_1 = w_2\) if they represent the same element of the braid group.
Definition 1.15 Let \( w_1 \) and \( w_2 \) be two braid words. We will say that \( w_1 \equiv w_2 \) if \( w_1 \) and \( w_2 \) are identical letter by letter.

Now, we can introduce the word problem: Given two braid words \( w_1 \) and \( w_2 \), decide whether \( w_1 = w_2 \) or not.

2 The computerized representation of the g-base

In this section, we will describe the way we encode the g-base in \([7]\). It involves some conventions.

Let \( D \) be the closed unit disk, the point \( u \) is the point \((0, -1)\) and the points in \( K \) are on the \( x \)-axis.

In order to encode the path in \( D \), which is an element of the g-base, we will distinguish some positions in \( D \).

Notation 2.1 We will denote by \((i, 1)\) a point close to \( k_i \) but above it, \((i, -1)\) a point close to \( k_i \) but below it, and \((i, 0)\) the point \( k_i \) itself. We will also denote the point \( u \) by \((-1, 0)\) (which is not its position in \( D \), rather only a notation).

To represent a path in \( D \), we will use a linked list which its links are based on the notations above, which represents the position of the path in relation to the points \( u \) and \( k_i \), \( i = 1, ..., n \).

Each link of the list holds the two numbers as described above. We will call them \((\text{point}, \text{position})\).

Example 2.2 The list \((1, 0) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (4, -1) \rightarrow (5, 0)\) represents the following path:

As a rule, we will never connect the point \( u \) to any point \((i, -1)\). This will be done in order to obtain a unique way of representation, and to make the automatic computation of the twists easier.

We will be able to tell whether a path \((-1, 0) \rightarrow (i, 1)\) is passing to the left or to the right of the point \( i \) simply by checking its continuation. If the path is turning to the left \((-1, 0) \rightarrow (i, 1) \rightarrow (i - 1, e)\), then it is passing to the right of the point \( i \), and if the path is turning to the right \((-1, 0) \rightarrow (i, 1) \rightarrow (i + 1, e)\), then it is passing to the left of the point \( i \) (where \( e \in \{-1, 1, 0\} \)).
Example 2.3 The list \((-1, 0) \rightarrow (3, 1) \rightarrow (2, 0)\) represents the following path:

```
  1 2 3 4
  u
```

The list \((-1, 0) \rightarrow (2, 1) \rightarrow (3, 1) \rightarrow (3, -1) \rightarrow (2, 0)\) represents the following path:

```
  1 2 3 4
  u
```

In order to unify our treatment on all the paths of the g-base, we concatenate all of them into one list, which means that after we arrive at the end of one path (i.e. a link \((i, 0)\)), the following link will be \((-1, 0)\) marking the beginning of the next path. For convenience, and not for mathematical reasons, we add the link \((-1, 0)\) at the end of the list.

Example 2.4 The list \((-1, 0) \rightarrow (1, 1) \rightarrow (2, 0) \rightarrow (-1, 0) \rightarrow (1, 0) \rightarrow (-1, 0) \rightarrow (4, 0) \rightarrow (-1, 0) \rightarrow (4, 1) \rightarrow (3, 0) \rightarrow (-1, 0)\) represents the g-base in the following figure (the small circles around the points are omitted):
The solution for the braid word problem given in [7] is based on an algorithm that performs the action of the half-twists on the elements of the g-base, maintaining a reduced form which is unique. If at the end of the process we find that the two g-bases after the action of the two braid words are the same, then the two elements of the braid group represent the same braid, and vice versa. This is without any questions, a geometrical solution, and in the following sections we will present the syntactical solution, using the same methods.

We will lay out an algorithmic way which will make it possible for us to apply the group action of the braid group on the g-bases, thus obtaining a new presentation of the braid group as an ordered set of elements of a g-base with the new operation.

In order to achieve this, we need a method that will enable us to write the elements of the g-base after the action using the elements of the standard g-base, and another method that will make it possible to return from the syntactic writing using the elements of the standard g-base into the representation of its paths. This is going to be the main subject of the next two sections.

3 Transforming the geometric presentation to a syntactic one

Let the standard g-base be \(< γ_1, ..., γ_n >\). In this section we are going to introduce the algorithms that gets as an input a path \(γ ∈ π_1(D \setminus K, u)\) which is an element of the standard g-base after the action of some braid, and returns a word in \(\{γ_1, ..., γ_n, γ_1^{-1}, ..., γ_n^{-1}\}\) that represents the same element \(γ\).

We need more to say that every link \(L\) consists of two numbers (Point,Position) and that by writing \(L\cdot Point\) we mean the Point element in the link \(L\), and by writing \(L\cdot Position\) we mean the Position element of the link \(L\).

3.1 The PathToSyntactic(\(γ\)) algorithm

Algorithm 3.1 PathToSyntactic(\(γ\))

input: \(γ\) - a list that represents an element in \(π_1(D \setminus K, u)\) which is an element of the standard g-base after the action of a braid.

output: a word \(w\) that consists of the generators of the fundamental group from the standard g-base, which is the same element as \(γ\).

PathToSyntactic(\(γ\))

\(w ← Null\)

for all the links \(L\) in \(γ\) do

\(LastLink ← the\ link\ before\ L\)
NextLink ← the link after L
SecondLink ← the link after NextLink

if L.Position = 0 then
    concatenate $\gamma_{L.Point}$ to w
    continue
if L.Position = −1 then
    continue
if L.Position = 1 then
    if NextLink.Point = L.Point − 1 then
        concatenate $\gamma_{L.Point}$ to w
        continue
    if NextLink.Point = L.Point + 1 then
        concatenate $\gamma_{L.Point}^{-1}$ to w
        continue
    if NextLink.Point = L.Point then
        if SecondLink.Point = L.Point − 1 then
            concatenate $\gamma_{L.Point}^{-1}$ to w
            continue
        if SecondLink.Point = L.Point + 1 then
            concatenate $\gamma_{L.Point}$ to w
            continue

l ← the length of w
for all $i$ from $l − 1$ to 1 step −1 do
    t ← the $i$th letter in w
    concatenate to w the letter $t^{-1}$
return w

3.2 Proof of correctness

Proposition 3.2 Let $\gamma$ be the list representing the path. Then, any link in $\gamma$ that has position −1 does not contribute any letter to the word $w$ representing the same elements using the standard $g$-base elements as generators.

Proof: Let us look at a part of the path that goes beneath the points $(i, 0)$. This part consists only on links of the type $(i, −1)$. The link that is before this sublist must be of the type $(j, 1)$, because it can not be the link denoted $(-1, 0)$, which is the first link in the path, and must be followed by convention with a link above a point, nor any of the links $(j, 0)$ since those links represents the end of the path.
We look at the path that this part represents. This is a line going from the one point (say \(i\)) to another (say \(j\)), which goes below the points of \(K\), and therefore below the x-axis. This means that the part of the path represented by those links is homotopically equivalent to two straight lines forming a 'V' shape starting from \((i, 1) \to (-1, 0) \to (j, e)\) where \(e \in \{0, 1\}\). These two lines are represented by two elements of the standard g-base that are added by looking at links just before this section and immediately after it, hence, this part of the path does not contribute any letters to the word \(w\). \(\square\)

**Proposition 3.3** Let \(\gamma\) be the list representing the path. The last link \(L = (i, 0)\) of \(\gamma\) contributes the letter \(\gamma_i\) to \(w\).

*Proof:* The last link in the list representing a path of the g-base is always of the type \((i, 0)\) It represents the loop of the path around the point \(i\). This contributes the element of the standard g-base that corresponds to that loop, which is \(\gamma_i\). \(\square\)

**Proposition 3.4** Let \(L = (i, 1)\) be a link in the list \(\gamma\). The element of the standard g-base that corresponds to \(L\) and therefore we need to add to the word \(w\) is determined by the following rules:

1. If the path goes to the left after the link. Then, we have to add the letter \(\gamma_i\).

2. If the path goes to the right after the link. Then, we have to add the letter \(\gamma_i^{-1}\).

3. If the path goes to the same point then, the letter we have to add is determined by the next link. If the following link goes to the left we have to add the letter \(\gamma_i^{-1}\), and if the following link goes to the right we have to add the letter \(\gamma_i\).

*Proof:* The easiest way to prove this is to look at the homotopy type geometrically. In case 1, the path simply goes from right to left above the point \(i\) which means that homotopically all we need to add is \(\gamma_i\), (see the following figure (a)).

Case 2, is the same as case 1 but on the other direction, which means that homotopically we need to add \(\gamma_i^{-1}\), (see the following figure (b)).

Case 3, is somewhat more complicated. The only reason why we might have two consecutive links over the same point is when we are switching direction around
this point. Therefore if the path goes to the left, it came from the left, and rounded the point \(i\) clockwise (because the initial point was \((i, 1)\)). This means that we must add the letter \(\gamma_i^{-1}\). (see the following figure (c)).

On the other hand, if the path continues to the right, it came from the right, and rounded the point \(i\) counterclockwise (again because the initial point was above \(i\)). This means that we have to add the letter \(\gamma_i\). (see the following figure (d)). □

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example-diagram.png}
\caption{Example diagrams}
\end{figure}

**Proposition 3.5** Let \(w\) be the word resulted by going over the list and \(l\) be the length of \(w\). Let \(w'\) be the word created from \(w\) by taking the first \(l-1\) letters of \(w\). Then, in order to complete the word \(w\) one has to make the word to a conjugacy by concatenating \(w'^{-1}\) to \(w\).

**Proof:** This is true because the path is encoded in such a way that the way back after the circle around the end point is omitted. But, the way back is exactly the opposite of the way to the end point, and therefore can be represented exactly by the negation of the same word. □

**Theorem 3.6** The algorithm \PathToSyntactic(\(\gamma\)) returns a word in the elements of the standard \(g\)-base that represents the same element of the fundamental group represented by \(\gamma\).

□

This concludes the proof of the correctness of the algorithm. In the next section we will compute its complexity.
3.3 Complexity

In this subsection we will compute and prove the complexity and the amount of memory needed for the \texttt{PathToSyntactic}($\gamma$) algorithm.

\textbf{Proposition 3.7} Let \( n \) be the length of the list representing the path \( \gamma \). Then, the complexity of the algorithm \texttt{PathToSyntactic}($\gamma$) is bounded by \( O(n) \) operations.

\textit{Proof:} The algorithm goes over every link in the list exactly once. On each of the links it may at most add one element of the standard g-base to the word \( w \). In the worst case this yields \( n \) operations exactly. Now, at the end of this process the algorithm doubles the result word \( w \) to make it conjugated. The total of operations therefore, is bounded by \( 2n \) which is \( O(n) \). \qed

\textbf{Proposition 3.8} Let \( n \) be the length of the list representing the path \( \gamma \). Then, the amount of memory needed for the algorithm \texttt{PathToSyntactic}($\gamma$) is bounded by \( O(n) \).

\textit{Proof:} Apart from a fixed number of counters and help variables, the algorithm \texttt{PathToSyntactic}($\gamma$) uses only the memory needed to store the word \( w \), which is linear in \( n \). Therefore the amount of memory it needs is bounded by \( O(n) \). \qed

4 Transforming the syntactic presentation to the geometric one

Let the standard g-base be \( < \gamma_1, \ldots, \gamma_n > \). In this section we are going to introduce the algorithm that gets as an input a word in the generators of the fundamental group \( \{ \gamma_1, \ldots, \gamma_n \} \), and will return a list that represents the path in \( \pi_1(D \setminus K, u) \). All paths will be encoded in the way that has been described in section 2.

4.1 The \texttt{SyntacticToPath}(\( w \)) algorithm

\textbf{Algorithm 4.1} \texttt{SyntacticToPath}(\( w \))
input: a word $w$ that consists of the generators of the fundamental group from the standard $g$-base, which is a conjugacy of an element of $\gamma$, written such that $w \equiv Q^{-1}\gamma Q$.

output: a list that represents an element in $\pi_1(D \setminus K, u)$ which is an element of the standard $g$-base after the action of a braid, which is the same as $w$.

`SyntacticToPath(w)`

if $w = $ Null then

    return the standard $g$-base

$L \leftarrow$ the empty list

set first link in $L$ to $(-1, 0)$

for every letter $l$ in the first $\frac{|w| - 1}{2}$ letters of $w$ do

    CurrentPoint $\leftarrow$ the generator index in $l$

    NextPoint $\leftarrow$ the generator index in the letter after $l$

    LastLetter $\leftarrow$ the letter before $l$

    if $l$ is the first letter in $w$ then

        add to $L$ the link $(CurrentPoint, 1)$

        LastPoint $\leftarrow$ CurrentPoint

        continue

    if LastLetter $= \gamma_i^{-1}$ for some $i$ then

        if LastPoint $< CurrentPoint$ then

            for all $c$ from LastPoint + 1 to CurrentPoint − 1 do

                add to $L$ the link $(c, -1)$

            if $l = \gamma_i$ for some $i$ then

                add to $L$ the link $(CurrentPoint, -1)$

                add to $L$ the link $(CurrentPoint, 1)$

                continue

        else

            for all $c$ from LastPoint to CurrentPoint + 1 step −1 do

                add to $L$ the link $(c, -1)$

            if $l = \gamma_i^{-1}$ for some $i$ then

                add to $L$ the link $(CurrentPoint, -1)$

                add to $L$ the link $(CurrentPoint, 1)$

                continue

        else

            if LastPoint $> CurrentPoint$ then

                for all $c$ from LastPoint − 1 to CurrentPoint + 1 step −1 do

                    add to $L$ the link $(c, -1)$

                if $l = \gamma_i^{-1}$ for some $i$ then

                    add to $L$ the link $(CurrentPoint, -1)$

                    continue

            else

                return the standard $g$-base
add to $L$ the link $(\text{CurrentPoint}, 1)$
continue
else
for all $c$ from $\text{LastPoint}$ to $\text{CurrentPoint} - 1$ do
  add to $L$ the link $(c, -1)$
if $l = \gamma_i$ for some $i$ then
  add to $L$ the link $(\text{CurrentPoint}, -1)$
add to $L$ the link $(\text{CurrentPoint}, 1)$
continue
set the last link in $L$ Position to 0
add to $L$ the link $(-1, 0)$
Reduce($L$)
return $L$

### 4.2 Proof of correctness

In this subsection we will prove that the algorithm $\text{SyntacticToPath}(w)$ returns a list representing an element of a g-base which is homotopically equivalent to the element represented by the word $w$ that consists of the generators of the standard g-base.

**Proposition 4.2** Let $w$ be the input word consists of the generators of the fundamental group. Then, in order to compute the path represented by the word $w$ it is sufficient enough to look at the first half of $w$.

**Proof:** The word $w$ represents a conjugated element in the fundamental group of one of the elements of the standard g-base, and written as $w \equiv Q^{-1}\gamma_i Q$ for some element of the fundamental group $Q$. Therefore, the second half of the word $w$ represents exactly the back path of the loop represented by $w$. Since by convention we represent the path only by a sequence of links that start at the base point $u$ and ends at one of the points of $K$, the back path is omitted and therefore, not necessary for the computation. □

**Proposition 4.3** Let $w$ be the input word consists of the generators of the fundamental group. If the first letter in $w$ is $\gamma_i$ then, the second link in the list must be $(i, 1)$ or $(i, 0)$. 
Proof: The element $\gamma_i$ represents a loop that rounds the point $i$. $w = \gamma_i$ then, the list the algorithm has to return is $(-1,0) \to (i,0) \to (-1,0)$. So, we need to check only the case when there are more generators in $w$. In that case, since all the generators in $w$ connect always at $u$, This means that the loop $\gamma_i$ represents the fact that the path goes over the point $i$ and into one of the directions left or right. □

Proposition 4.4 Let $\gamma_i$ and $\gamma_j$ be two consecutive letters in $w$, and suppose $i < j$. Then, one of the following cases must happen:

1. If $\gamma_i$ is in a negative power, and $\gamma_j$ is in a positive power. Then, we have to add the links $(i+1,-1) \to \ldots \to (j,-1) \to (j,1)$.

2. If both $\gamma_i$ and $\gamma_j$ are in a negative power. Then, we have to add the links $(i+1,-1) \to \ldots \to (j-1,-1) \to (j,1)$.

3. If $\gamma_i$ is in a positive power, and $\gamma_j$ is in a negative power. Then, we have to add the links $(i,-1) \to \ldots \to (j-1,-1) \to (j,1)$.

4. If both $\gamma_i$ and $\gamma_j$ are in a positive power. Then, we have to add the links $(i,-1) \to \ldots \to (j,-1) \to (j,1)$.

Proof: The proof follows from the homotopy relation between each of the cases and the list of the links we add.

1. In this case we have a negative loop over the point $i$, which lies to the left of the point $j$, which is rounded by the second loop. This means that we orbit the point $j$ in a counterclockwise direction, after coming from the left. This is homotopic to the list of links $(i+1,-1) \to \ldots \to (j,-1) \to (j,1)$, as can be seen by the following figure:

2. In this case we have a negative loop over the point $i$, which lies to the left of the point $j$, which is rounded by the second negative loop. This means that
we simply have a path going from left to right below the points in between $i$ and $j$. This is homotopic to the list of links $(i+1, -1) \rightarrow \ldots \rightarrow (j-1, -1) \rightarrow (j, 1)$, as can be seen by the following figure:

3. In this case we have a positive loop over the point $i$, which lies to the left of the point $j$, which is rounded by the second negative loop. This means that we orbit the point $i$ counterclockwise and then, we go below the points between $i$ and $j$. This is homotopic to the list of links $(i, -1) \rightarrow \ldots \rightarrow (j-1, -1) \rightarrow (j, 1)$, as can be seen in the following figure:

4. In this case we have a positive loop over the point $i$, which lies to the left of the point $j$, which is rounded by the second positive loop. This means that we orbit both the point $i$ and the point $j$ in a counterclockwise direction. This is homotopic to the list of links $(i, -1) \rightarrow \ldots \rightarrow (j, -1) \rightarrow (j, 1)$, as can be seen by the following figure:

Now, we will consider the case when $j < i$, which means that the path is going from left to right, so we have the following proposition:
Proposition 4.5 Let $\gamma_i$ and $\gamma_j$ be two consecutive letters in $w$, and suppose $j < i$. Then, one of the following cases must happen:

1. If $\gamma_i$ is in a negative power, and $\gamma_j$ is in a positive power. Then, we have to add the links $(i, -1) \rightarrow \ldots \rightarrow (j + 1, -1) \rightarrow (j, 1)$.

2. If both $\gamma_i$ and $\gamma_j$ are in a negative power. Then, we have to add the links $(i, -1) \rightarrow \ldots \rightarrow (j, -1) \rightarrow (j, 1)$.

3. If $\gamma_i$ is in a positive power, and $\gamma_j$ is in a negative power. Then, we have to add the links $(i - 1, -1) \rightarrow \ldots \rightarrow (j, -1) \rightarrow (j, 1)$.

4. If both $\gamma_i$ and $\gamma_j$ are in a positive power. Then, we have to add the links $(i - 1, -1) \rightarrow \ldots \rightarrow (j + 1, -1) \rightarrow (j, 1)$.

Proof: The proof is similar to the proof of the proposition 4.4 above. □

Now, what is left is to notice that the middle letter in $w$ marks the end point of the path. This is due to the convention of the representation of the path that marks the end of the path in a link connected to the point $i$, that is $(i, 0)$. So, what the algorithm does is when finishing passing over the letters, it simply connects the last link to the point it is associated with. In order to comply with the representation of the path’s conventions, the algorithm adds in the end one more link $(-1, 0)$, and reduces its presentation by using the Reduce($L$) algorithm found in [7].

This completes the proof of the correctness of the algorithm, so we get:

Theorem 4.6 The algorithm SyntacticToPath($w$) returns a list representing the the same element of the fundamental group as the word $w$ in the elements of the standard $g$-base represents, where $w \equiv Q^{-1} \gamma Q$.

□

4.3 Complexity

In this subsection we will compute and prove the complexity and the amount of memory needed for the SyntacticToPath($w$) algorithm.
Proposition 4.7  Let \( l \) be the length of the word \( w \) in the standard set of generators of the fundamental group, and \( n \) be the number of points in \( K \). Then, the complexity of the algorithm SyntacticToPath(\( w \)) is bounded by \( O(nl) \) operations.

Proof: The algorithm goes over every letter in the first half of the word \( w \) exactly once. By each letter the algorithm might add at most \( n + 1 \) links to the list, since it might add all the links between \( i \) and \( j \) where \( \gamma_i \) and \( \gamma_j \) are consecutive letters in \( w \), and since \( |i - j| < n + 1 \). At the end of the algorithm it uses the Reduce(\( g \)) algorithm that was presented in [7]. Its complexity is bounded by the length of the list as proven in [7]. So, at most the algorithm complexity is bounded by \( nl \) operations which is \( O(nl) \).

\[ \square \]

Proposition 4.8  Let \( l \) be the length of the word \( w \) in the standard set of generators of the fundamental group, and \( n \) be the number of points in \( K \). Then, The amount of memory needed for the algorithm SyntacticToPath(\( w \)) is bounded by \( O(nl) \).

Proof: Besides a finite predetermined number of variables, the algorithm creates a new linked list, that represents the path. The size of this list was calculated in the proof of the complexity proposition. Therefore, the list might in the worst case have \( (n + 1)l + 2 \) links, which is bounded by \( O(nl) \).

\[ \square \]

5  Group structure over the presentation of the g-base

When we consider the lists that represent elements of the g-base, as described in [7], we get a unique way for presenting elements of the braid group. So, we thought of a way to multiply lists representing different g-bases. By multiplying, or by adding a group structure to the presentation of the g-bases, we mean that when we have two lists \( L_1, L_2 \) representing braids \( \beta_1, \beta_2 \), (that is, two lists that each one of them represents a g-base, which is the result of the action of the braid on the standard g-base), we want to be able to perform a process that will result with a list \( L_3 = L_1L_2 \) that represents a g-base, which is the result of the action of the braid \( \beta_1\beta_2 \) on the standard g-base.

Since multiplying braids is actually the composition of diffeomorphisms, the result on the g-base is a composition of the elements of the g-base. When we have
two lists $L_1$ and $L_2$ that represents the two braids $\beta_1$ and $\beta_2$, we have to look at the action of the first braid on the standard $g$-base $< \gamma_1, ..., \gamma_n >$, resulting with a different $g$-base say $< \gamma'_1, ..., \gamma'_n >$ and than to act with $\beta_2$ on the elements of the new $g$-base, resulting with a third $g$-base say $< \gamma''_1, ..., \gamma''_n >$. The latter is the result of the action of the braid $\beta_1 \beta_2$ on the standard $g$-base.

This is rather difficult to perform from a geometrical point of view, but fortunately easy to perform on the syntactical presentation of the elements of the $g$-base. The action is as follows:

**Algorithm 5.1** Multiply($L_1, L_2$)

**input**: $L_1$ and $L_2$ - two lists representing two $g$-bases associated with $\beta_1$ and $\beta_2$.

**output**: A list representing the $g$-base resulted after the action of the braid $\beta_1 \beta_2$ on the standard $g$-base.

**Multiply($L_1, L_2$)**

Let $n$ be the number of points in $K$.

Break $L_1$ into $n$ sublists $L_1[i]$ each one representing one element of the $g$-base.

Break $L_2$ into $n$ sublists $L_2[i]$ each one representing one element of the $g$-base.

for all $i$ from 1 to $n$ do

\[ w_1[i] \leftarrow \text{PathToSyntactic}(L_1[i]) \]

\[ w_2[i] \leftarrow \text{PathToSyntactic}(L_2[i]) \]

for all $i$ from 1 to $n$ do

replace in $w_1[i]$ every letter $\gamma_j$ with $w_2[j]$, and every letter $\gamma_j^{-1}$ with $w_2[j]^{-1}$

store the result in $w_3[i]$

Eliminate every two consecutive letters in $w_3[i]$ of the type $\gamma_i^e \gamma_i^{-e}$ where $e \in \{1, -1\}$.

\[ L_3[i] \leftarrow \text{SyntacticToPath}(w_3[i]) \]

$L_3$ ← the concatenation of all the paths in $L_3[i]$ into one list representing the resulted $g$-base

return $L_3$

**Proposition 5.2** Let $l_1$ and $l_2$ be the lengths of the lists $L_1$ and $L_2$, suppose $n$ is the number of strings in the braid group. Then, the complexity of the algorithm is bounded by $O(nl_1l_2)$

**Proof**: Breaking the lists into $n$ small lists takes $O(l_1 + l_2)$ actions, since the sum of all the lengths of all the paths $L_1[i]$ is equal to $l_1 + n$ (We add the link $(-1, 0)$ to
each path), and the sum of all the lengths of all the paths $L_2[i]$ is equal to $l_2 + n$, and the action of $\text{PathToSyntactic}(L_1[i])$ and $\text{PathToSyntactic}(L_2[i])$ is linear in the length of all the lists.

The replacement of every letter in $w_1[i]$ with the word $w_2[j]$ is bounded by the multiplication of the length of $w_1[i]$ by $l_2$. Again, since the sum of all the lengths of $w_1[i]$ is $l_1 + n$, and since we do not replace the elements $(-1, 0)$, we result with a boundary of $O(l_1 l_2)$.

The elimination process is linear in the length of $w_3[i]$ since each letter that was inserted to $w_3[i]$ can be extracted only once. If one uses a doubly-connected list or a vector to store the letters, then, moving in both direction on the list is possible resulting a linear time elimination. Note that the size of all the elements $w_3[i]$ is bounded by the previous argument by $O(l_1 l_2)$.

Concatenation of all the paths takes $2n$ operations, since this is only the connection of linked lists, or $O(l_1 l_2)$ if they are stored in a vector.

The only thing we have not checked yet is how long it takes for the algorithm to transform every syntactic word $w_3[i]$ into a list. But, as proved above this is bounded by $O(n l_1 l_2)$.

So, at the end we have reached the boundary of $O(n l_1 l_2)$.

\[ \square \]

**Proposition 5.3** Let $l_1$ and $l_2$ be the lengths of the lists $L_1$ and $L_2$. Then, the amount of memory needed for the algorithm is bounded by $O(l_1 l_2)$.

\[ \text{Proof:} \] The largest structure needed for the algorithm is the resulted linked list $L_3$ which has $O(l_1 l_2)$ links. All the other data structures are lists that the sum of their sizes do not exceed $O(l_1 + l_2)$, and variables that their number is preknown and constant. Therefore, the proposition is proved. \[ \square \]

6 An improvement of the braid word solution

In this section, we will show how one can improve the solution for the braid word problem that was presented in [7]. The main idea of the new solution is to base it on the syntactic presentation of the elements of the g-base. Since every element in the g-base, resulted by an action of a braid, is conjugated to an element of the
standard g-base, it is possible to encode the elements during the computation in a more efficient way.

Unfortunately, the process of performing the action of the diffeomorphisms on the elements of a g-base still takes a long time. But, since the algorithm in [7] is very practical for short words, and since using the new method will improve the running time of the algorithm, we result in an even more practical solution to the word problem.

**Definition 6.1** Let $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ be a g-base. We say that the g-base $\Delta = \langle \gamma'_1, \ldots, \gamma'_n \rangle$ is obtained from $\Gamma$ by the braid move $H_i$ (or $\Gamma$ is obtained from $\Delta$ by the braid move $H_i^{-1}$) if

1. $\gamma'_j = \gamma_j$ for all $j \neq i, i + 1$
2. $\gamma'_i = \gamma_{i+1}$
3. $\gamma'_{i+1} = \gamma_{i+1} \gamma_i \gamma_i^{-1}$

If we consider the elements of the standard frame $H_1, \ldots, H_{n-1}$ as the generator set for the braid group, then the action that $H_i$ performs over a g-base is exactly the $i^{th}$ braid move on it.

Due to the fact that the group operation between two braids $\beta_1$ and $\beta_2$ is a composition of diffeomorphisms, in order to compute the action on the standard g-base (or any g-base for that matter), we need to go over the braid word from the end to the beginning and perform on the g-base at each step the braid move induced by the generator at this position. This is the same as going over the braid word from the beginning to the end, replacing for each letter $H_i$ each appearance of $\gamma_i$ with $\gamma_{i+1}$ and $\gamma_{i+1}$ with $\gamma_{i+1} \gamma_i \gamma_i^{-1}$, or its negative compliant for $H_i^{-1}$.

This yields the following algorithm for encoding the diffeomorphisms action on the standard g-base, using the braid move defined above.

### 6.1 The ProcessWord(w) algorithm

In this section, we will introduce the changed `ProcessWord(w)` algorithm. This algorithm as explained above will process on the syntactic presentation of the g-base.
Algorithm 6.2  ProcessWord\(w\)

**input:** \(w\) - a braid word.

**output:** \(w'\) - a word which represents the g-base elements written in the standard generators of the fundamental group.

We denote \(e \in \{1, -1\}\)

\[
\text{ProcessWord}(w)
\]

\[\Gamma \leftarrow \text{the standard g-base } < \gamma_1, ..., \gamma_n>\]

**for** each letter \(l = H_i^e\) in the braid word \(w\) **from the end to the beginning** **do**

**if** \(e = 1\) **then**

activate the braid move \(H_i\) on \(\Gamma\)

**else**

activate the braid move \(H_i^{-1}\) on \(\Gamma\)

\(\text{Reduce}(\Gamma)\)

return \(\Gamma\)

The action of the braid generator on the g-base, is given as above by maintaining \(n - 2\) elements without a change, replacing one elements position with the other, and conjugating the latter. Therefore, we will show how to encode the conjugacy in a way that will make the computation fast on one hand, and will save a lot of memory on the other.

**Notation 6.3** We will denote \(B_i^j\) the part of the word \(\gamma\) that represents an element of the g-base, that begins at the \(i^{th}\) letter and its length is \(j\) letters.

This will make it possible to reduce the size of the memory needed to represents elements of the changed g-base.

**Example 6.4** Instead of writing the element of the g-base \(\gamma = \gamma_3\gamma_2\gamma_1\gamma_2^{-1}\gamma_3^{-1}\) we can write \(\gamma = \gamma_3\gamma_2\gamma_1 B_i^2\)

This method helps to keep the number of moves and copies we have to make small, although it makes it a bit more difficult to perform the \(\text{Reduce}(\Gamma)\) procedure.

### 6.2 The Reduce\((\Gamma)\) algorithm

The new version of the \(\text{Reduce}(\Gamma)\) algorithm gets as an input an ordered set of syntactic representations of a g-base for the fundamental group, that we get after
the operation of a braid generator on it. The algorithm performs a reduction of these words into the smallest form possible.

Since the fundamental group of a punctured disk is a free group, the only reduction rule available is when we have to consecutive letters $\gamma_i\gamma_i^{-1}$ that we can immediately delete both.

So, what the $Reduce(\Gamma)$ algorithm does is to go over each element in $\Gamma$ and eliminate all the pairs $\gamma_i\gamma_i^{-1}$ and $\gamma_i^{-1}\gamma_i$.

**Definition 6.5** Let $\pi_1(D \setminus K, u)$ be the fundamental group of a punctured disk, and let $w$ be a word in its standard generators. We call $w$ reduced if $w$ does not contain any two consecutive letters of the form $\gamma_i^e\gamma_i^{-e}$ for any $1 \leq i \leq n$ and $e \in \{-1, 1\}$.

### 6.3 Why is that better than before

Since the fundamental group of a punctured disk is a free group on $n$ generators, two reduced words $w$ and $w'$ are the same if and only if $w \equiv w'$ ($w$ and $w'$ are equal letter by letter).

This is easy to check and therefore, we hold a solution for the braid word problem. If one wants to compare two braid words $\beta_1$ and $\beta_2$, he has to compute $\Gamma_1 = ProcessWord(\beta_1)$ and $\Gamma_2 = ProcessWord(\beta_2)$ and compare the elements of $\Gamma_1$ with the elements of $\Gamma_2$.

Denote $\Gamma_1 =< \gamma_{1,1}, \ldots, \gamma_{1,n} >$ and $\Gamma_2 =< \gamma_{2,1}, \ldots, \gamma_{2,n} >$, where $\gamma_{j,i}$ is reduced, than $\beta_1 = \beta_2$ if and only if for every $1 \leq i \leq n$ we have $\gamma_{1,i} \equiv \gamma_{2,i}$.

Although not in complexity, this is faster than the algorithm $ProcessWord(w)$ that was presented in [7], since no maintaining of the linked list in necessary. Since the algorithm there was very practical for short braid words, this one is even more practical.

### 7 Conclusions

Although for very long braid words this algorithm running time is long, due to the fact that the complexity of the g-base grows with the length of the braid word,
for short braid words we obtained a quick algorithm in comparison with other methods.

The worst drawback of this algorithm is the representation of the conjugated elements in $\pi_1(D \setminus K, u)$ to the elements of the standard g-base. Perhaps a new method for encoding these elements will make it possible to reduce the running time of the algorithm even more.

Another thing, that can help to reduce the running time of the algorithm is to add some generators to the standard generators of the fundamental group $\pi_1(D \setminus K, u)$. These elements will make it possible to shorten the length of the words that represents the different elements of the g-base. Some ideas we thought about are to add the elements $\gamma_{i,j} = \gamma_i \cdot \ldots \cdot \gamma_j$. This set of generators will make it possible to write an element of the g-base using only its path turning points, which will shorten the word immensely.

We believe that there is a connection between presentations of two conjugated braid words, therefore we believe that this connection might yield us a practical algorithm for solving the conjugacy problem in the braid group.

The operation of the unprocess of the algorithm here, which means to compute the braid word from a given g-base is even easier than in the case of the other presentation.

Using this new method allows us to leave the geometrical solution in [6], this makes things even easier in order to compute the braid monodromy automatically for some of the cases.

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