We prove the following results. If $X_3$ is a generic complete intersection Calabi-Yau 3-fold,

(1) then for each natural number $d$ there exists a rational map $c \in Hom_{bir}(\mathbb{P}^1, X_3)$ of $\deg(c(\mathbb{P}^1)) = d$,

(2) further more all such $c$ are immersions satisfying

$$N_{c(\mathbb{P}^1)/X_3} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$  (0.1)
1 Statement

In mirror symmetry there is a general consensus that a “generic” Calabi-Yau 3-fold over \( \mathbb{C} \) should contain and only contain finitely many irreducible rational curves of each degree with respect to the polarization. This is the basis for the calculation of instantons in Mirror symmetry. While we are still in the process of finalizing the precise statement, let us prove it for the complete intersection Calabi-Yau 3-folds in a single projective space.

**Theorem 1.1.**

Let \( X_3 \) be a generic, complete intersection Calabi-Yau 3-folds over \( \mathbb{C} \). Then

1. \( X_3 \) admits an irreducible rational curve \( C \) of each degree,
2. all such \( C \subset X_3 \) are immersions and the normal bundle \( N_{C/X_3} \) is isomorphic to

\[
\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).
\]  

(1.1)

2 Sketch of the proof

2.1 Setting

Throughout the paper rational curves are curves rationally parametrized by \( \mathbb{P}^1 \). The image is called an irreducible rational curve.

Rational curves on projective varieties has been a topic for many decades. The general theory which has its own technique is not the focus of this paper.
Instead we are interested in a specific type of problems with a quite different technique.

**Focus 2.1.** Which generic complete intersection $X$ admits an irreducible rational curve $C$ of each degree?

**Focus 2.2.** Once the first question is affirmative, what is the normal sheaf $N_{C/X}$? (2.1)

Theorem 1.1 tries to answer these two questions in the case of Calabi-Yau 3-folds.

The idea of the work starts from and stays in a down-to-earth setting, which employees linear algebra only. The method first converts the invariant expression of Theorem 1.1 to a variant expression as the content of Theorem 1.1 stays the same. Then it explores the unique linear algebra in the variant setting to reach an algebraic result. At last it converts the algebraic result back to the invariants. In technique the first conversion

\[ \text{Invariant} \Rightarrow \text{Variant} \]  

uses classical geometry. The second conversion

\[ \text{Variant} \Rightarrow \text{invariant} \]  

uses Clemens’ deformation idea [1].

Let’s start with this alternative setting. Let

\[ M_d = (H^0(O_{P^1}(d)))^{\otimes(n+1)} \cong \mathbb{C}^{(n+1)(d+1)}. \]  

The open set $M_{\text{bir},d}$ of $M_d$ represents (but is not equal to)

\[ \{c \in \text{Hom}_{\text{bir}}(P^1, P^n) : \deg(c(P^1)) = d\}. \]

Let

\[ g_i, i = 1, \cdots, r = n - 3 \]
be sections in $H^0(\mathcal{O}_{\mathbb{P}^n}(h_i))$ and
\begin{equation}
\sum_{i=1}^{r} h_i = n + 1. \tag{2.5}
\end{equation}

Let
\begin{equation*}
X_i = \cap_{k=1}^{n-i} \text{div}(g_k).
\end{equation*}

In this paper, the Cartesian product
\begin{equation*}
(g_1, \cdots, g_{n-i})
\end{equation*}
is also called a complete intersection of type $(h_1, \cdots, h_{n-i})$. So
\begin{equation}
X_3 = \cap_{i=1}^{r} \text{div}(g_i) \tag{2.7}
\end{equation}
is a complete intersection Calabi-Yau 3-fold in the usual sense for generic
\begin{equation*}
(g_1, \cdots, g_r) \in \prod_{i+1}^{r} H^0(\mathcal{O}_{\mathbb{P}^n}(h_i)).
\end{equation*}

Choose distinct $h_i d + 1$ points $t_j^i \in \mathbb{P}^1$. Let
\begin{equation}
t^i = (t_1^i, \cdots, t_{h_i d+1}^i) \in \text{Sym}^{h_i d+1}(\mathbb{P}^1). \tag{2.8}
\end{equation}

In the rest of the paper, we’ll use following conventions in affine coordinates.
(a) $t_j^i$ or $t$ denotes a complex number which is a point in an affine open set $\mathbb{C} \subset \mathbb{P}^1$,
(b) $c(t)$ denotes the image
\begin{equation*}
\begin{array}{c}
\mathbb{C} \to \mathbb{C}^{(n+1)(d+1)} \\
t \to c(t),
\end{array}
\end{equation*}
(c) $g_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(h_i))$ is a homogeneous polynomial of degree $h_i$ in $n + 1$ variables.

We should note that in these affine coordinates, the incidence relation $c^*(g_i) = 0$ can be expressed as the composition $g_i(c(t)) = 0$ for all
\begin{equation*}
t \in \mathbb{C}.
\end{equation*}
Let $C_M$ be a system of affine coordinates for $M_d$, which determines an isomorphism

$$M_d \cong \mathbb{C}^{(n+1)(d+1)}.$$ 

We define a system of polynomials in the variable $c \in M_d$

$$g_i(c(t^j)), i = 1, \cdots, r, j = 1, \cdots, h_i d + 1. \quad (2.9)$$

Then the subsets of polynomials

$$g_1, \cdots, g_l, l \leq r$$
give a rise to a holomorphic map $\mu_l$

$$\mu_l : M_d \cong \mathbb{C}^{(n+1)(d+1)} \rightarrow \mathbb{C}^{md+l}, \quad (2.10)$$

where $m_l = \sum_{i=1}^{l} h_i$. We’ll denote

$$I_l = \mu_l^{-1}(0) \quad (2.11)$$

which will be called the incidence scheme of rational curves on the complete intersection of $g_1, \cdots, g_l$. Then the differential map $(\mu_l)$ is represented by the Jacobian matrix of size

$$\left( m_l d + l \right) \times \left( n + 1 \right) \left( d + 1 \right)$$

denoted by $J_l$, which depends on $g_i, t^j$. However once the points $t^j$ are fixed, the matrix $J_l$ is well-defined and varied algebraically on the entire affine space $M_d \times \mathbb{A}$, where $\mathbb{A} \subset \prod_i \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(h_i)))$ is affine. We call it the Jacobian matrix of the incidence scheme $I_l$.

The scheme $I_l$, which could be reducible with multiple dimensions, is the alternative to various moduli spaces of rational curves and maps. In this paper we show a methodology in the calculation of the Jacobian matrix $J_l$ at a point of $I_l$ corresponding to an irreducible rational curve on generic complete intersections. The components containing such points always have the smallest dimension as expected. They correspond to the components that have actual fundamental classes instead of virtual fundamental classes in Mirror symmetry. The methodology is rooted in a specific pattern of the Jacobian matrix $J_l$ modeled on the Vandermonde matrices.

\footnote{For instance, it contains the components of multiple covering maps of $\mathbb{P}^1$.}
In general we are interested in the following questions,
(1) what is the dimension of $I_l$?,
(2) is it reduced?
(3) is it irreducible, and if not, what is the structure of each component?
Answers to these questions would solve some technical problems in Mirror symmetry, that have been left out of the physics’ formulation. Thus it is an alternative to patch some holes in the general mathematical theory. The complete answers to these questions are out of scope of this paper. Here we are going to use linear algebra to explore the dimension of $I_l$ for some complete intersections.

2.2 Existence

In the first part, we prove the existence of irreducible rational curves $C$ on a generic complete intersection Calabi-Yau 3-fold $X_3$. We avoid a direct construction.

(I) First we’ll use the proven existence of irreducible rational curves of arbitrary degrees on a single generic hypersurface of lower degrees (which are Fano). After its extension to a special complete intersection in $\mathbb{P}^n$ by joining more Fano hypersurfaces, we use linear algebra to glue the block matrix for each Fano hypersurface to obtain the non-degeneracy of Jacobian matrix $J_r$. This will show the existence of an irreducible rational curve $C'$ of each degree on a special complete intersection Calabi-Yau 3-fold $X'_3$ with the non-degenerate Jacobian $J'_r$ at the point $(C', X'_3)$.  

(II) Applying Clemens’ deformation idea, this smooth $C'$ is deformed to generic $X_3$ as a different irreducible rational curve $C \subset X_3$. The requirement for such a deformation is the non-degeneracy of Jacobian matrix $J_l$.

2.3 Rigidity

The second step is to show the normal bundle of all such irreducible curves $C \subset X_3$ are split as in (0.1). This rigidity is determined by the dimension of

\[2 \text{ For curves of a fixed and small degree, computer software has been used to find its Jacobian. See p. 295, [2].} \]
the incidence scheme $I_r$. First we have the general study of a uniruled projective variety to deduce that a free rational curve on it has the unobstructed deformation on its generic hypersurfaces. This non-trivial result follows from [7] or [8]. Then we notice that the filtration of complete intersections

$$X_3 \subset \cdots \subset X_0 \cong \mathbb{P}^n$$

(2.12)
corresponds to another filtration of subvarieties

$$I_r|_c \subset I_{r-1}|_c \subset \cdots \subset I_1|_c \subset I_0|_c,$$

(2.13)
where the subscript $|_c$ means the analytic neighborhood around the point $c$ with $c(\mathbb{P}^1) = C$. Then we notice all $X_i$ for $i \neq 3$ are Fano, therefore uniruled. Applying above deformation result for rational curve on uniruled varieties, we obtain a recursive formula

$$\dim(I_r|_c) = \dim(I_{r+1}|_c) - (h_l d + 1).$$

(2.14)
(or equivalently $\dim(H^0(c^*(T_{X_i}))) = \dim(H^0(c^*(T_{X_{i+1}}))) - (h_l d + 1)$.) Applying the Calabi-Yau condition (2.5) we obtain that

$$\dim(I_r|_c) = 4.$$ (2.15)
(or $\dim(H^0(c^*(T_{X_3}))) = 3$).

Then the dimension $\dim(H^0(c^*(T_{X_3}))) = 3$ forces $c$ to be an immersion and furthermore to be rigid.

We organize the rest of the paper as follows. In section 3, we prove the existence, and in section 4 we prove the rigidity. Appendix covers a particular technique in linear algebra for the existence.

### 3 Existence of rational curves

Starting from this section we give technical proofs. This section focuses on the existence.

We resume all notations introduced in section 1. Recall $g_i, i = 1, \cdots, r$ are sections in $H^0(O_{\mathbb{P}^n}(h_i))$ such that $n - 3 = r$ and $n + 1 = \sum_i h_i$. Let

$$X_l = \cap_{i=1}^{n-l} \text{div}(g_i), l \leq r$$

(3.1)
In particular, if
\[(g_1, \cdots, g_r) \in \prod_{i=1}^{r} H^0(\mathcal{O}_{P^7}(h_i))\]
is generic, $X_3$ is a smooth complete intersection Calabi-Yau 3-fold.

In this subsection, we prove part (1) of Theorem 1.1. It asserts the existence of an irreducible rational curve of each degree $d$ on the generic $X_3$.

**Theorem 3.1.** There exist irreducible rational curves of each degree $d$ on the following generic complete intersection Calabi-Yau 3-folds.

1. $(2, 2, 2, 2)$ type in $P^7$ \hspace{1cm} (3.2)
2. $(3, 2, 2)$ type in $P^6$, \hspace{1cm} (3.3)
3. $(3, 3)$ and $(4, 2)$ types in $P^5$ \hspace{1cm} (3.4)
4. $(5)$ type, i.e. quintic 3-fold in $P^4$. \hspace{1cm} (3.5)

The case (3.5) has been proved by H. Clemens [1] and S. Katz ([3]). Let's consider other 3 cases. Suppose that in all 3 cases, there exist a special complete intersection, denoted by sections $g'_1, \cdots, g'_r$ and a smooth rational curve $c_g$ of degree $d$ with $c'_g(g'_i) = 0$ for all $i$ such that the corresponding Jacobian matrix $J_r$ at $c_g$ has full rank. Then we divide the coordinates of $M_d$ to two parts, $c_{\text{ind}}$ and $c_{\text{free}}$. The partial derivatives with respect to $c_{\text{ind}}$ form the maximal minor block $J_{\text{ind}}$ in the Jacobian matrix $J_r$. Thus $J_{\text{ind}}$ is a maximal non-degenerate block. The size of $J_{\text{ind}}$ is therefore

\[\left( (n+1)d + r \right) \times \left( (n+1)d + r \right)\]

Let the remaining coordinates of $M_d$ be $c_{\text{free}}$, and corresponding block matrix in $J_r$ is $J_{\text{free}}$. Hence $(c_{\text{ind}}c_{\text{free}})$ are the affine coordinates of $M_d$ and $(J_{\text{ind}}J_{\text{free}}) = J_r$. Since $J_{\text{ind}}$ evaluated at the special complete intersection $(g'_i)$ and smooth $c_g$ is non-degenerate, by the implicit function theorem in complex analysis, there exists analytic functions near $c_g$, $c_{\text{ind}} = \alpha_{\text{ind}}(c_{\text{free}}, g_1, \cdots, g_r)$ \hspace{1cm} (3.6)

\[g_i(c(t_j^i)) = 0, \text{ for all } i, j\]
where $c = \alpha_{\text{ind}}(c_{\text{free}}, g_1, \ldots, g_r)c_{\text{free}}$. In geometric term, this means that there exists a smooth rational curve $c$ of degree $d$ on each same type of complete intersection (free choice of $g_i$) near the special one ($g'_i$). Since these deformed complete intersections ($g_i$) are all generic in Zariski-topology, we complete the proof of the existence in these cases. So in the following subsections we are going to find such a special complete intersection in each case.

### 3.1 Complete intersection of $(2, 2, 2, 2)$ type in $\mathbb{P}^7$

**Proposition 3.2.** There is a smooth rational curve $C_g$ of each degree $d$ on a special complete intersection $X_3$ of type $(2, 2, 2, 2)$ in $\mathbb{P}^7$ such that $J_4|_{c_g}$ at $(c_g, X_3)$ is non-degenerate, where $c_g$ is the normalization of $C_g$.

**Proof.** Let $[z_0, \ldots, z_7]$ be homogeneous coordinates of $\mathbb{P}^7$. Let $\mathbb{P}^3$ the subspace defined by $z_4 = z_5 = z_6 = z_7 = 0$. First we consider a generic quadric $g \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ in $\mathbb{P}^3$. By [4], Hilbert scheme $\text{Hilb}_{c}(\mathbb{P}^1)(\text{div}(g))$ at a smooth rational curve

$$ c \in \text{Hom}_{\text{bir}}(\mathbb{P}^1, \mathbb{P}^n) $$

of degree $d$ is non-empty and smooth with the expected dimension. Then $I_1$ is smooth at $c$ with expected dimension. Let $J'_1$ be the corresponding Jacobian matrix of $I_1$ in $\mathbb{P}^3$ (as defined in the introduction). By Lemma A.2, it has full rank. In the following we use the “gluing” technique for block matrices to extend the Jacobian matrix $J'_1$ to $\mathbb{P}^7$. Assume $c = [c_0, c_1, c_2, c_3]$. It is extended to a smooth rational curve $c_g$ in $\mathbb{P}^7$ as follows

$$ c_g = [c_0, c_1, c_2, c_3, c_0, c_1, c_2, c_3], \quad \text{(3.7)} $$

which is isomorphic to $c$. We define the special complete intersection of $(2, 2, 2, 2)$ type as follows. Let

$$ g_1 = g(\text{by the extension}), $$

$$ g_2 = g(z_4, z_5, z_2, z_3), $$

$$ g_3 = g(z_0, z_1, z_6, z_7), $$

$$ g_4 = z_6^2 + z_7^2 - z_2^2 - z_3^2. \quad \text{(3.8)} $$
be four quadrics in $\mathbb{P}^7$. Then we have $c'_g(g_i) = 0$ for all $i$.

Let $\theta_1, \theta_3, \theta_5, \theta_7$ be the affine coordinates for 2nd, 4th, 6th and 8th copies $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ in $M_d$. Let $V_0$ be the hyperplane of $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$, whose elements vanish at 0. Let $\theta_0, \theta_2, \theta_4, \theta_6$ be the affine coordinates for $V_0$ in the 1st, 3rd, 5th, 7th components of $M_d$. The we’ll show these variables, that are not all variables of $M_d$, are $c_{\text{ind}}$. This amounts to show that the Jacobian matrix has full rank. Thus we first write down the blocks of Jacobian matrix. We let

\[
B_{11} = \frac{\partial(g_1(c_0(t^1_1)), \ldots, g_1(c_0(t^1_{2d+1}))))}{\partial(\theta_0, \theta_1)}

B_{12} = \frac{\partial(g_1(c_0(t^1_1)), \ldots, g_1(c_0(t^1_{2d+1}))))}{\partial(\theta_2, \theta_1)}

B_{22} = \frac{\partial(g_2(c_0(t^2_1)), \ldots, g_2(c_0(t^2_{2d+1}))))}{\partial(\theta_2, \theta_1)}

B_{23} = \frac{\partial(g_2(c_0(t^2_1)), \ldots, g_2(c_0(t^2_{2d+1}))))}{\partial(\theta_4, \theta_1)}

B_{33} = \frac{\partial(g_3(c_0(t^3_1)), \ldots, g_3(c_0(t^3_{2d+1}))))}{\partial(\theta_4, \theta_6)}

B_{34} = \frac{\partial(g_3(c_0(t^3_1)), \ldots, g_3(c_0(t^3_{2d+1}))))}{\partial(\theta_6, \theta_7)}

B_{42} = \frac{\partial(g_4(c_0(t^4_1)), \ldots, g_4(c_0(t^4_{2d+1}))))}{\partial(\theta_2, \theta_3)}

B_{44} = \frac{\partial(g_4(c_0(t^4_1)), \ldots, g_4(c_0(t^4_{2d+1}))))}{\partial(\theta_6, \theta_7)}
\]

Then one of maximal minor blocks $J_{\text{ind}}$ of the Jacobian matrix $J_4$ ( in $\mathbb{P}^7$) at $c_g$ is formed by the block matrices

\[
J_{\text{ind}} = \begin{pmatrix}
B_{11} & B_{12} & 0 & 0 \\
0 & B_{22} & B_{23} & 0 \\
0 & 0 & B_{33} & B_{34} \\
0 & B_{42} & 0 & B_{44}
\end{pmatrix}
\]

(3.10)

We can verify that $B_{ij}$ in (3.10) satisfy all conditions in Lemma A.2. Furthermore all conditions in Lemma A.6 for $J|_{c_g}$ are also satisfied. By the Lemmas $J|_{c_g}$ is non-degenerate. We complete the proof. 

\[\]

3.2 Complete intersection of $(3, 2, 2)$ type in $\mathbb{P}^6$

Proposition 3.3. There is a smooth rational curve $C_g$ of each degree $d$ on a special complete intersection $X_3$ of type $(3, 2, 2)$ in $\mathbb{P}^6$ such that $J_3|_{c_g}$ at $(c_g, X_3)$ is non-degenerate, where $c_g$ is the normalization of $C_g$. 

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Proof. Let $z_0, \ldots, z_6$ be homogeneous coordinates of $\mathbb{P}^6$. Let $\mathbb{P}^3 \subset \mathbb{P}^6$ be the subspace defined by $z_4 = z_5 = z_6 = 0$. By [4], there exists a generic

$$g \in H^0(\mathcal{O}_{\mathbb{P}^3}(2))$$

such that it contains a smooth rational curve

$$c = [c_0, c_1, c_2, c_3]. \quad (3.11)$$

We define a new smooth rational curve of degree $d$ to be

$$c_g = [c_0, c_1, c_2, c_3, c_0, c_1, c_2]. \quad (3.12)$$

We define hypersurfaces as

$$g_1 = z_0 g(z_0, z_1, z_2, z_3) + (z_0 - z_4) \alpha(z_0, z_1, z_2, z_3)$$
$$g_2 = g(z_4, z_5, z_2, z_3)$$
$$g_3 = g(z_4, z_5, z_6, z_3) \quad (3.13)$$

where $\alpha$ is a generic quadric with respect to $g$. Let $\theta_1, \ldots, \theta_7$ be the affine coordinates for 7 copies $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ in $M_d$. Let

$$t^1 = (t_1^1, \ldots, t_{3d+1}^1) \in \text{sym}^{3d+1}(\mathbb{P}^1)$$
$$t^2 = (t_1^2, \ldots, t_{3d+1}^2) \in \text{sym}^{2d+1}(\mathbb{P}^1)$$
$$t^3 = (t_1^3, \ldots, t_{3d+1}^3) \in \text{sym}^{2d+1}(\mathbb{P}^1). \quad (3.14)$$

We write down the block matrices for the Jacobian matrix. Let

$$B_{1j} = \frac{\partial (g_1(c_g(t_1^1)), \ldots, g_1(c_g(t_{3d+1}^1)))}{\partial \theta_j}$$
$$B_{2j} = \frac{\partial (g_2(c_g(t_1^2)), \ldots, g_2(c_g(t_{2d+1}^2)))}{\partial \theta_j}$$
$$B_{3j} = \frac{\partial (g_3(c_g(t_1^3)), \ldots, g_3(c_g(t_{2d+1}^3)))}{\partial \theta_j}. \quad (3.15)$$

Next we write down the Jacobian matrix directly as

$$J_3 = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\
0 & 0 & B_{23} & B_{24} & B_{25} & B_{26} & 0 \\
0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37}
\end{pmatrix} \quad (3.16)$$

Then we verify $B_{ij}$ satisfy all conditions in Lemmas A.2, A.3. Then we can apply Lemma A.5. We obtain that Jacobian matrix $J_3$ of the incidence scheme has full rank. We complete the proof of this case. \qed
3.3 Complete intersections of \((3, 3)\) and \((4, 2)\) types in \(P^5\)

**Proposition 3.4.** There is a smooth rational curve \(C_g\) of each degree \(d\) on a special complete intersection \(X_3\) of type \((3, 3)\) in \(P^5\) such that \(J_2|_{c_g}\) at \((c_g, X_3)\) is non-degenerate, where \(c_g\) is the normalization of \(C_g\).

**Proof.** Let \(z_0, \ldots, z_5\) be homogeneous coordinates of \(P^5\). Let \(P^3 \subset P^5\) be the subspace defined by \(z_4 = z_5 = 0\). By \([4]\), there exists a generic \(g \in H^0(\mathcal{O}_{P^3}(2))\) such that it contains an irreducible rational curve

\[
c = [c_0, c_1, c_2, c_3]
\]

(3.17)
of degree \(d\). Next we define a smooth rational curve of degree \(d\) in \(P^5\) as

\[
c_g = [c_0, c_1, c_2, c_3, c_1, c_0]
\]

(3.18)

We also define two cubic hypersurfaces

\[
g_1 = z_0 g(z_0, z_1, z_2, z_3) + (z_0 - z_5)\alpha(z_0, z_1, z_2, z_3)
g_2 = z_0 g(z_5, z_4, z_2, z_3) + (z_1 - z_4)\alpha(z_0, z_1, z_2, z_3),
\]

(3.19)

where \(\alpha\) is a generic quadric with respect to \(g\). Let \(\theta_1, \ldots, \theta_6\) be the affine coordinates for 6 copies \(H^0(\mathcal{O}_{P^1}(d))\) in \(M_d\) (all variables of \(M_d\)). Let

\[
t^1 = (t^1_1, \ldots, t^1_{3d+1}) \in \text{sym}^{3d+1}(P^1)
t^2 = (t^2_1, \ldots, t^2_{3d+1}) \in \text{sym}^{3d+1}(P^1).
\]

(3.20)

Let

\[
B_{ij} = \frac{\partial (g_i(c_g(t^i_1)), \ldots, g_i(c_g(t^i_{3d+1})))}{\partial \theta_j}
\]

(3.21)

for \(i = 1, 2, j = 1, \ldots, 6\). Then we directly write down the Jacobian \(J_2\) of the incidence scheme at \(c_g\).

\[
J_2 = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & B_{14} & 0 & B_{16} \\
0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26}
\end{pmatrix}
\]

(3.22)

Similarly conditions of Lemma A.3 are met. By Lemma A.4, it has full rank. We complete the proof in this case. \(\Box\)
Proposition 3.5. There is a smooth rational curve $C_g$ of each degree $d$ on a special complete intersection $X_3$ of type $(4,2)$ in $\mathbb{P}^5$ such that $J_2|_{c_g}$ at $(c_g, X_3)$ is non-degenerate, where $c_g$ is the normalization of $C_g$.

Proof. As in the case 1, $z_0, \cdots, z_5$ are homogeneous coordinates of $\mathbb{P}^5$. Let $\mathbb{P}^4$ be the subspace covered by the coordinates

$$[z_0, z_1, z_2, z_3, z_4, 0].$$

Let $\mathbb{P}^3$ be the subspace covered by coordinates

$$[z_0, z_1, z_2, z_3, 0, 0].$$

By the Mori’s result [6], there is a smooth rational curve $c_s$ of degree $d$ on a generic quartic $g \in H^0(\mathcal{O}_{\mathbb{P}^3}(4))$. Let

$$t^1 = (t^1_1, \cdots, t^1_{4d}) \in \text{sym}^{4d}(\mathbb{P}^1)$$

be generic.

Then the Jacobian matrix $J'$

$$J' = \frac{\partial (g(c_s(t^1_1)), \cdots, g(c_s(t^1_{4d})))}{\partial (\theta_0, \cdots, \theta_3)}$$

has full rank.\footnote{The number of points $t^1_j$ is unusual. This exceptional case holds only for quartic surface in $\mathbb{P}^3$.} Let’s extend it to $\mathbb{P}^4$. Define a new quartic by setting

$g_1 = g + z^3_0 z_4$ in $\mathbb{P}^4$ and new rational curve by setting

$c = [c_0, c_1, c_2, c_3, 0]$ where $[c_0, c_1, c_2, c_3]$ is the Mori’s curve. Next we add one generic point

$t^1_{4d+1} \in \mathbb{P}^1,$

and extend the coordinates of

$H^0(\mathcal{O}_{\mathbb{P}^4}(d))^\oplus 4$
to
\[ H^0(\mathcal{O}_{\mathbb{P}^1}(d))^{\oplus 5}. \]
These add the a new row—the differential of \( dg_1(c(t_{4d+1}^1)) \) to the Jacobian \( J' \), Hence we obtain a new Jacobian matrix \( J_1 \) in \( \mathbb{P}^4 \) at \( c \),
\[
J_1|_{c_g} = \begin{pmatrix}
J' & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}
\tag{3.25}
\]
where
\[
\begin{pmatrix}
B_{21} & B_{22}
\end{pmatrix}
\tag{3.26}
\]
is the differential 1-form \( dg_1(c(t_{4d+1}^1)) \). Hence \( J_1|_{c} \) in \( \mathbb{P}^4 \) has full rank.
To summarize it, we found a special quartic \( g_1 \) in \( \mathbb{P}^4 \) containing a smooth rational curve \( c \) of the given degree and its Jacobian matrix is of full rank.

Next we extend it one more time to \( \mathbb{P}^5 \). Let \( g_1 \) be the extension of original \( g_1 \) to \( \mathbb{P}^5 \). Let \( g_2 = z_0 z_4 + z_1 z_5 \) and the \( c_g = [c_0, c_1, c_2, c_3, 0, 0] \)
be the smooth curve in \( \mathbb{P}^5 \). Thus \( c_g \) lies on the complete intersection of \( g_1, g_2 \).

We add new set of generic \( 2d + 1 \) points in \( \mathbb{P}^1 \),
\[
t^2 = (t_{1}^2, \cdots, t_{2d+1}^2) \in \text{sym}^{2d+1}(\mathbb{P}^1). \tag{3.27}
\]

Then we can write down the Jacobian matrix \( J_2 \) in this case. It is equal to
\[
J_2 = \begin{pmatrix}
J_1 & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\tag{3.28}
\]
where
\[
\begin{pmatrix}
A_{21} & A_{22}
\end{pmatrix}
\tag{3.29}
\]
is the Jacobian matrix at \( c_g \) of \( 2d + 1 \) many functions (in \( c \)),
\[
g_2(c(t_{1}^2)), \cdots, g_2(c(t_{2d+1}^2)) \tag{3.30}
\]
and \( A_{22} \) is the block with respect to \( 2d+2 \) coordinates in last two components of
\[
H^0(\mathcal{O}_{\mathbb{P}^1}(d))^{\oplus 6}. \tag{3.31}
\]
By Lemma A.2, \( A_{22} \) has full rank. Hence the Jacobian matrix \( J_2 \) also has full rank. We complete the proof. \( \square \)

\(^4\) We added one more feature to Mori’s existence. That is the non-degeneracy of the Jacobian matrix.
3.4 Quintic in $\mathbb{P}^4$

This has been proved in [3]. We’ll repeat the same construction in [1], but continue with our method. By the Mori’s result [6], there is a smooth rational curve $c_s$ of degree $d$ on a generic quartic $\text{div}(g)$ in $\mathbb{P}^3$. Now we extend it to $\mathbb{P}^4$ by setting

$$c_g = [c_0, c_1, c_2, c_3, 0], \text{ for } c_s = [c_0, c_1, c_2, c_3],$$

(3.32)

and the new quintic

$$g_1 = lg + qz_4,$$

(3.33)

where $l$ is a generic linear polynomial and $q$ is a generic quartic. By [9], the corresponding Jacobian matrix $J_1$ has full rank.

**Corollary 3.6.** Let $X_3$ be a generic complete intersection Calabi-Yau 3-fold. Then it contains an irreducible rational curve of each degree $d$.

*Proof.* $X_3$ is projectively embedded as one of the complete intersections in Theorem 3.1. Therefore the corollary follows.

\[\square\]

**Remark** In above arguments for the existence, all rational curves are smooth. But non-smooth and irreducible rational curves on a generic complete intersection Calabi-Yau 3-fold do exist.

4 Rigidity of rational curves

4.1 Rational curves with unobstructed deformation

In this subsection we let $Y$ be an arbitrary smooth projective variety over $\mathbb{C}$. We’ll prove a general assertion in Theorem 4.3 about uniruled variety. For a parametrized rational curves on $Y$, the notion of having unobstructed deformation is weaker than being a free morphism. Precisely let $c \in \text{Hom}_{\text{bir}}(\mathbb{P}^1, Y)$. 

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**Definition 4.1.** If the normal sheaf \( c^*(N_{c/P_1}/Y) \) denoted simply by \( N_{c/Y} \) has the vanishing first cohomology, i.e.

\[
H^1(N_{c/Y}) = 0,
\]

then we say \( c \) has unobstructed deformation on \( Y \).

Since \( H^1(N_{c/Y}) = 0 \) is equivalent to \( H^1(c^*(T_Y)) = 0 \), then Definition 4.1 is equivalent to the following splitting

\[
c^*(T_Y) \cong \bigoplus_j \mathcal{O}_{P_1}(a_j), a_j \geq -1.
\]

On the other hand, we define

**Definition 4.2.** If \( c^*(T_Y) \) is generated by global sections, we say \( c \) is a free morphism, and \( Y \) is uniruled.

This is a special case of the more general definition in [5].

It is clear that \( c \) being free is equivalent to the splitting

\[
c^*(T_Y) \cong \bigoplus_j \mathcal{O}_{P_1}(a_j), a_j \geq 0.
\]

So being free is stronger than having unobstructed deformation.

**Theorem 4.3.** Let \( L \cong \mathcal{O}_{P^n}(1)|_Y \) be a very ample line bundle on \( Y \), and \( \dim(Y) \geq 4 \). Let

\[
X = \text{div}(f) \subset Y
\]

where \( f \in H^0(L^h) \) is generic with an \( h \). Let

\[
c : P^1 \to C \subset X \subset Y
\]

be a birational morphism onto an irreducible rational curve \( C \). If \( c \) is free on \( Y \), then \( c \) has unobstructed deformation on \( X \), i.e.

\[
H^1(N_{c/X}) = 0.
\]
Proof. of Theorem 4.3: By the assumption of the theorem, we have a polarization of $Y$ such that

$$Y \subset \mathbb{P}^n$$  \hspace{1cm} (4.5)

is a smooth subvariety of dimension $\geq 4$, and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)|_Y$. Let

$$s \in H^0(\mathcal{O}_{\mathbb{P}^n}(h))$$  \hspace{1cm} (4.6)

be generic. Let $f = s|_Y \in H^0(\mathcal{L}^h)$.

We denote

$$\text{div}(f) = X, \text{div}(s) = Z.$$  \hspace{1cm} (4.7)

By the genericity of $Z$, scheme-theoretically

$$X = Y \cap Z.$$  \hspace{1cm} (4.8)

Let $c : \mathbb{P}^1 \to X$ be generic in $\text{Hom}_{\text{bir}}(\mathbb{P}^1, X)$.

We have a non-commutative diagram of exact sequences

\[
\begin{array}{cccccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & H^0(c^* T_X) & \to & H^0(c^* T_Y) & \xrightarrow{\mu_3} & H^0(c^* N_{X/Y}) & \xrightarrow{\mu_4} & H^1(c^* T_X) & \to & H^1(c^* T_Y) \\
\downarrow & \downarrow & & & & \begin{array}{c} \mu_3 \\ \mu_4 \end{array} & & \downarrow & & \\
0 & H^0(c^* T_Z) & \to & H^0(c^* (T_{\mathbb{P}^n})) & \xrightarrow{\mu_3} & H^0(c^* (N_{Z/\mathbb{P}^n})) & \to & H^1(c^* (T_Z)) \\
& & \downarrow & & & \| & & \downarrow & & \\
& & H^0(c^* (N_{Y/\mathbb{P}^n})) & \xrightarrow{\mu_4} & H^0(c^* (N_{Z/\mathbb{P}^n})) & \to & H^1(c^* (T_Y)) & \to & 0 \\
& & & & & \downarrow & & & & \\
& & & & & H^1(c^* (T_Y)) & & & & \\
\end{array}
\]  \hspace{1cm} (4.9)

Let’s define and analyze the diagram. Including all zero spaces, there are 5 rows and 6 columns. Second and third rows are parts of the long exact sequences from the short exact sequences of sheaves over $\mathbb{P}^1$,

\[
0 \to c^* (T_X) \to c^* (T_Y) \to c^* (N_{X/Y}) \to 0, \hspace{1cm} (4.10)
\]

\[
0 \to c^* (T_Z) \to c^* (T_{\mathbb{P}^n}) \to c^* (N_{Z/\mathbb{P}^n}) \to 0. \hspace{1cm} (4.11)
\]

The third column is the part of the long sequence of the short exact sequence of sheaves over $\mathbb{P}^1$,

\[
0 \to c^* (T_Y) \to c^* (T_{\mathbb{P}^n}) \to c^* (N_{Y/\mathbb{P}^n}) \to 0. \hspace{1cm} (4.12)
\]
The isomorphism in the fourth column is from the adjunction formula.

The homomorphism \( \mu_4 \) is well-defined only if \( c \) has unobstructed deformation on \( Y \). Let’s see this in the following. Because \( c : P^1 \to Y \) is free, therefore \( c \) has unobstructed deformation on \( Y \). So \( H^1(c^*(T_Y)) = 0 \). Therefore the third column exact sequence in (4.9) splits, i.e.

\[
H^0(c^*(T_{P^n})) \cong H^0(c^*(N_{Y/P^n})) \oplus H^0(c^*(T_Y)).
\]  

We define \( \mu_4 \) to be the restriction of \( \mu_3 \) to the subspace, \( H^0(c^*(N_{Y/P^n})) \) i.e.

\[
\mu_4 : H^0(c^*(N_{Y/P^n})) \to H^0(c^*(N_{Z/P^n}))
\]

(But \( \mu_4 \) may not be zero.). In the sense of this splitting, the map \( \mu_3 \) also splits as

\[
\mu_3 = \mu_1 \oplus \mu_4.
\]

Next we go further to use a construction to prove that \( \mu_4 \) is the zero map due to the global generation. This is just a specification of

\[
H^0(c^*(N_{Y/P^n}))
\]

inside of

\[
H^0(c^*(T_Y)).
\]

By (4.13), there are a Zariski open set \( U \subset P^1 \) and a trivial subbundle

\[
E \subset c^*(N_{Y/P^n})
\]

over \( U \) such that

\[
E \oplus c^*(T_Y)|_U = c^*(T_{P^n})|_U
\]

and

\[
E \subset T_Z|_U.
\]

Let

\[
B = \{ \sigma \in H^0(c^*(T_{P^n})) : \sigma|_U \in E \}.
\]

So \( B \) is a subspace of \( H^0(c^*(T_{P^n})) \). It generates a sheaf \( E_{P^1} \) over \( P^1 \) (in general \( B \) could be zero, so did \( E_{P^1} \)). Because

\[
c^*(T_Y), c^*(T_{P^n})
\]
are generated by global section, the sheaf $E_{P^1}$ over $P^1$ satisfies

$$B \simeq H^0(c^*(N_{Y/P^n})).$$ (4.19)

and (4.17) extends to

$$E_{P^1} \subseteq T_Z.$$ (4.20)

We then define $\mu_4$ to be the restriction of $\mu_3$ to $B$ (this is the same as the definition before. But this time, the subspace $H^0(c^*(N_{Y/P^n}))$ is uniquely identified.). By the condition (4.20), $\mu_4$ is the zero map.

Thus

$$\text{Image}(\mu_1) = \text{Image}(\mu_3).$$ (4.21)

Now we consider the third row, an exact sequence in (4.9). Since $Z$ is a generic hypersurface of $P^n$, we can apply Theorem 1.1, [8] (or [7]) to obtain that $H^1(c^*(T_Z)) = 0$. The exactness of the sequence implies that $\mu_3$ is surjective. So is $\mu_1$. Next we shift the focus to the second row in (4.9). We apply $H^1(c^*(T_Y)) = 0$ again to obtain that $H^1(c^*(T_X)) = 0$. Using the exact sequence

$$0 \to T_{P^1} \to c^*(T_X) \to N_{c/X} \to 0.$$ (4.22)

we obtain that

$$H^1(c^*(T_X)) = H^1(N_{c/X}) = 0.$$ (4.23)

\[\square\]

4.2 Free morphism

In this subsection, we continue the existence result — the existence of irreducible rational curves of each degree $d$ on a generic complete intersection Calabi-Yau 3-folds. From now on we change $Y$ to be a complete intersection as follows,

$$Y = \cap_{i=1}^{r-1} \text{div}(g_i)$$ (4.24)

for the fixed set of generic $g_i$. So $Y$, which is equal to $X_4$, is a smooth 4-dimensional complete intersection and Fano. As before for the fixed set of $g_i$,

$$X_3 = \cap_{i=1}^{r} \text{div}(g_i)$$ (4.25)
is a Calabi-Yau 3-fold contained in the Fano 4-fold $Y = X_4$.

We would like to show a generic rational curve $c$ on $X_3$ is free on $Y$, in another word, such rational curves cover $Y$. Thus we assume

**Assumption 4.4.** $c_0 \in \text{Hom}_{\text{bir}}(\mathbb{P}^1, X_3)$ of $\text{deg}(c_0(\mathbb{P}^1)) = d$ exists.

We’ll work in a neighborhood of each component of the incidence schemes around $c_0$.

Let’s resume all the notations in section 2. We let $g_1, \cdots, g_{r-1}$ be fixed (where $r = n - 3$). Then the incidence scheme $I_{r-1}$ is also fixed. Let $I_{r-1}$ be a component of $I_{r-1}$ containing $c_0$. Let $\Gamma$ be an irreducible component of the scheme

$$\{(c, g_r) \in I_{r-1} \times H^0(\mathcal{O}_{\mathbb{P}^n}(h_r)) : c^*(g_r) = 0\} \quad (4.26)$$

containing $c_0$. Let

$$\pi : I_{r-1} \times H^0(\mathcal{O}_{\mathbb{P}^n}(h_r)) \to M_d \quad (4.27)$$

be the projection. Let $\mathcal{I} = \pi(\Gamma)$. So

$$\mathcal{I} \subset I_{r-1}.$$  

Let $0 \in \mathbb{P}^1$ be a generic point. Let $\mathcal{R}$ be the closure of the open scheme,

$$\mathcal{R} = \{(c, y) \in \mathbb{P}(\mathcal{I}) \times Y : y = c(0)\}. \quad (4.28)$$

The equation $y = c(0)$ means that $c$ is regular at $0$.

We would like to show that

**Proposition 4.5.** The open scheme $\hat{\mathcal{R}}$ is a rational map from $\mathbb{P}(\mathcal{I})$ to $Y$ and dominates $Y$.

*Proof.* Let $l$ be an integer satisfying

$$0 \leq l \leq r. \quad (4.29)$$
Let $I_l$ be a component of $I_t$ containing $c_0$. Define
\[ \mathcal{R}_l \subset \mathbb{P}(I_l) \times X_{n-l} \tag{4.30} \]
to be the closure of
\[ \hat{R}_l = \{(c, x) : c(0) = x\}. \tag{4.31} \]
for each $l$.

First we consider the case $l < r - 1$. Suppose $\mathcal{R}_l$ is onto $X_{n-l}$, where $\dim(X_{n-l}) = n - l$. Let $x \in X_{n-l}$ have coordinates $[0, \cdots, 0, 1]$. Then the dimension of the fibre $(\mathcal{R}_l)_x$ of $\mathcal{R}_l$ over the generic point $x \in X_{n-l}$ satisfies
\[ \dim((\mathcal{R}_l)_x) \geq (h_{l+1} + \cdots + h_r)d. \tag{4.32} \]
Hence after imposing $h_{l+1}d + 1$ equations,
\[ g_{l+1}(c(t_{l+1}^1)) = \cdots = g_{l+1}(c(t_{h_{l+1}d+1}^l)) = 0, \]
we obtain that
\[ \dim((\mathcal{R}_{l+1})_x) \geq (h_{l+2} + \cdots + h_r)d - 1 \geq 0. \tag{4.33} \]
for all $x \in X_{n-l-1}$. Let
\[ \text{Proj} : \mathbb{P}(I_l) \to \mathbb{P}^n \]
be the projection. Then the correspondence $\mathcal{R}_{l+1}$ sends $\text{Proj}(\mathcal{R}_{l+1})$ onto $X_{n-l-1}$ (not a map). Next we show $\mathcal{R}_{l+1}$ is a rational map.

Applying Assumption 4.4, there is a
\[ c_0 \in \text{Proj}(\mathcal{R}_{l+1}) \cap M_{\text{bir},d}. \]
Then generic $c \in \text{Proj}(\mathcal{R}_{l+1})$ must be in $M_{\text{bir},d}$ because $M_{\text{bir},d}$ is an open set. Hence the correspondence $\mathcal{R}_{l+1}$ is a rational map. By the induction on the index $l$ for
\[ l = 0 \Rightarrow l = r - 2, \]
we showed that the irreducible component
\[ \hat{R}_{r-1} \]
dominates $Y$, through the rational evaluation map
\[ c \to c(0). \]
To prove Proposition 4.5 it suffices to extend above proof to $l = r$, but the situation is slightly different.

Let
\[ t^r = (t^r_1, \cdots, t^r_{h_r d+1}) \in \text{sym}^{h_r, d+1} (\mathbb{P}^1) \]  
be generic. By the result of above argument, $\hat{R}_{r-1}$ dominates $Y$. So for a generic $y \in Y$,
\[ \dim((R_{r-1})_y) \geq h_r d. \]  

For the fixed $y$, we define $S_y$ to be closure of
\[ S_y \subset I_{r-1} \times H^0(\mathcal{O}_{\mathbb{P}^n_d}(h_r)) \]
\[ S_y = \{(c, g_r) : c \in I_{r-1}, g_r(c(t^r_1)) = \cdots = g_r(c(t^r_{h_r d+1})) = 0\} \]  
where $H^0(\mathcal{O}_{\mathbb{P}^n_d}(h_r))$ is an open set of space of hypersurface $g_r$ satisfying that the Jacobian matrix of the functions in $c$
\[ g_r(c(t^r_1)), \cdots, g_r(c(t^r_{h_r d+1})) \]
at $c \in I_{r-1}$ has full rank. By the inequality (4.35), we count the dimension to obtain the $S_y$ is non-empty. Hence
\[ \pi(S_y) \]  
is non-empty for generic $y$. This shows that the correspondence $R$ is onto $Y$. Now let’s show it is a rational map. Notice a generic $c \in \pi(S_y)$ is a generic element in
\[ \bigcup_{y \in Y} \pi(S) = \mathcal{I}. \]  

Hence it suffices to show there is one regular point for $R$. By Assumption 4.4, there is a $c_0 \in S_y$ such that $c_0$ is in $M_{\text{bir}, d}$, and $c_0(0) = y$ for some $y \in Y$ (not all $y$). Therefore $R$ is regular at ONE point. With the same reason as above, $R$ is a rational map. Thus $R$ dominates $Y$. This completes the proof.

\[ \square \]

**Corollary 4.6.** Let $(c, y) \in R$ be generic. Then $c$ is a free morphism on $Y$. 

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Proof. By Proposition 4.5, there is a Zariski open set \( U \) of \( \mathcal{R} \), such that the projection
\[
\text{Proj} : U \to \text{Proj}(U) \subset Y
\]
is smooth. Hence its differential is onto. Let \( (c, y) \in U \). Then the pull-back of the tangent bundle
\[
c^*(T_Y) \simeq \bigoplus_k \mathcal{O}_{\mathbb{P}^1}(k)
\]
does not have negative summand, i.e. \( k \geq 0 \). Hence a \( c \) gives a free morphism \( \mathbb{P}^1 \to Y \).

4.3 Unobstructed deformation

Notice that there is a degree \( d \) rational curve \( c \in \text{Hom}_{\text{bir}}(\mathbb{P}^1, X_3) \) for a generic \( X_3 \). By corollary 4.6, it is a free morphism in \( Y \), we apply Theorem 4.3. Then the second row of (4.9) implies a recursive formula
\[
dim(H^0(c^*(X_{l+1}))) = \dim(H^0(c^*(X_l))) - h_l d - 1.
\]
(4.40)

Since
\[
\dim(H^0(c^*(X_0))) = (n + 1)d + n,
\]
\[
\dim(H^0(c^*(X_3))) = (n + 1)d + n - \sum_{k=1}^r h_k d - (n - 3) = 3.
\]
(4.41)

Now we consider it from a different point of view. Because \( c \) is a birational map to its image, there are finitely many points \( t_i \in \mathbb{P}^1 \) where the differential map
\[
c_* : T_{t_i} \mathbb{P}^1 \to T_{c(t_i)} Y
\]
is a zero map. Assume its vanishing order at \( t_i \) is \( m_i \). Let
\[
m = \sum_i m_i.
\]
(4.42)

Let \( s(t) \in H^0(\mathcal{O}_{\mathbb{P}^1}(m)) \) such that
\[
div(s(t)) = \Sigma_i m_i t_i.
\]
The sheaf morphism $c_*$ is injective and induces a composed morphism $\xi_s$ of sheaves
\[ T_{P^1} \xrightarrow{c_*} c^*(T_{X_3}) \xrightarrow{\xi_s} c^*(T_{X_3}) \otimes \mathcal{O}_{P^1}(-m). \] (4.43)

It is easy to see that the induced bundle morphism $\xi_s$ is injective. Let
\[ N_m = \frac{c^*(T_{X_3}) \otimes \mathcal{O}_{P^1}(-m)}{\xi_s(T_{P^1})}. \] (4.44)

Then
\[ \dim(H^0(N_m)) = \dim(H^0(c^*(T_{X_3}) \otimes \mathcal{O}_{P^1}(-m))) - 3. \] (4.45)

On the other hand, three dimensional automorphism group of $P^1$ gives a rise to a 3-dimensional subspace $K$ of
\[ H^0(c^*(T_{X_3})). \]

By (4.41), $K = H^0(c^*(T_{X_3}))$. Over each point $t \in P^1$, $K$ spans a one dimensional subspace. Hence
\[ c^*(T_{X_3}) \simeq \mathcal{O}_{P^1}(2) \oplus \mathcal{O}_{P^1}(-k_1) \oplus \mathcal{O}_{P^1}(-k_2), \] (4.46)

where $k_1, k_2$ are some positive integers. This implies that
\[ \dim\left( H^0(c^*T_{X_3} \otimes \mathcal{O}_{P^1}(-m)) \right) = \dim(H^0(\mathcal{O}_{P^1}(2 - m))). \] (4.47)

Then
\[ \dim\left( H^0(c^*T_{X_3} \otimes \mathcal{O}_{P^1}(-m)) \right) = 3 - m. \] (4.48)

Since $\dim(H^0(N_m)) \geq 0$, by the formula (4.45), $-m \geq 0$. By the definition of $m$, $m = 0$. Hence $c$ is an immersion.

Now $N_{c/X_3}$ is a vector bundle. By the Calabi-Yau condition,
\[ N_{c/X_3} \simeq \mathcal{O}_{P^1}(k) \oplus \mathcal{O}_{P^1}(-2 - k) \] (4.49)

where $k$ is non-positive. By Theorem 4.3, $H^1(N_{c/X_3}) = 0$. Hence $k = -1$. Since $X_3$ is generic, the same proof is valid for all $c \in \text{Hom}_{\text{bir}}(P^1, X_3)$. Together with the existence in section 3, we complete the proof of Theorem 1.1.
A Vandermonde type of matrices

In the appendix, we give proofs of lemmas dealing with Vandermonde type of matrices that are products of diagonal matrices and Vandermonde matrices. The main purpose is to use rather elementary technique to glue matrices of smaller sizes, the Vandermonde type of matrices, to obtain a matrix of larger size (which is a Jacobian matrix of a complete intersection). We assume that the linear algebra has the ground field $\mathbb{C}$.

First we introduce and study Vandermonde type of matrices, that are smaller matrices as entries in the large block matrices. Let

$$h \in H^0(\mathcal{O}_{\mathbb{P}^1}(v)) \quad \text{(A.1)}$$

be a polynomial, where $t \in \mathbb{P}^1$ is the variable, and Let

$$t = (t_1, \cdots, t_u) \in Sym^n(\mathbb{P}^1).$$

In the following in order to define matrices, we use the affine expression in subsection 2.1. Let $\mathcal{D}_u$ be a diagonal matrix

$$\mathcal{D}_u = \begin{pmatrix}
    h(t_1) & 0 & \cdots & 0 \\
    0 & h(t_2) & \cdots & 0 \\
    \vdots & \cdots & \ddots & \vdots \\
    0 & \cdots & 0 & h(t_u)
\end{pmatrix} \quad \text{(A.2)}$$

Definition A.1.

$$\mathcal{V}_0(h, t, m) = \mathcal{D}_u \begin{pmatrix}
    t_1^m & \cdots & t_1 \\
    \vdots & \cdots & \vdots \\
    t_u^m & \cdots & t_u
\end{pmatrix} \quad \text{(A.3)}$$

and

$$\mathcal{V}_1(h, t, m) = \mathcal{D}_u \begin{pmatrix}
    t_1^m & \cdots & 1 \\
    \vdots & \cdots & \vdots \\
    t_u^m & \cdots & 1
\end{pmatrix} \quad \text{(A.4)}$$

We call them Vandermonde type of matrices of orders 0 and 1 respectively.
Remark The Jacobian matrices $J_l$ of incidence schemes (we are interested in) are all made of Vandermonde type of matrices.

Lemma A.2. If $h_1, h_2$ are two relatively prime polynomials with distinct zeros, and degrees $\geq d$, then for generic

$$(t^1, t^2) \in \text{Sym}^d(\mathbb{P}^1) \times \text{Sym}^{d+1}(\mathbb{P}^1)$$

the square matrix

$$B = \begin{pmatrix} V_0(h_1, t^1, d) & V_1(h_2, t^1, d) \\ V_1(h_1, t^2, d) & V_0(h_2, t^2, d) \end{pmatrix}$$

is non-degenerate.

Proof. It suffices to prove it for a special $t^i$. Since $h_1$ has degree $\geq d$ and it is relatively prime to $h_2$, we can choose $t_1^1, \ldots, t^1_d$ to be the zeros of $h_1(t)$ and $h_2(t)$ is non-zero at all points of $t^1, t^2$. Then

$$B = \begin{pmatrix} 0 & V_1(h_2, t^1, d) \\ V_1(h_1, t^2, d) & V_0(h_2, t^2, d) \end{pmatrix}$$

Then its determinant is

$$-|V_1(h_2, t^1)||V_1(h_1, t^2)|.$$  \hfill (A.7)

Since $V_1(h_2, t^1), V_1(h_1, t^2)$ are types of Vandermonde matrices, the distinct $t^i_j$ and nonvanishing $h_1, h_2$ imply the non-degeneracy. Therefore $B$ is non-degenerate.

Lemma A.3. Let $h_3 \in H^0(\mathcal{O}_{\mathbb{P}^1}(2d))$ and $h_1, h_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be two vectors not on the same line through the origin. They are also pair wisely relatively prime, and all zeros are distinct. Let

$$(t^1, t^2, t^3) \in \text{sym}^d(\mathbb{P}^1) \times \text{sym}^d(\mathbb{P}^1) \times \text{sym}^{d+1}(\mathbb{P}^1)$$

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be generic. Then the square matrix \((3d + 1) \times (3d + 1)\),
\[
B = \begin{pmatrix}
V_0(h_1, t^1, d) & V_0(h_2, t^1, d) & V_1(h_3, t^1, d) \\
V_0(h_1, t^2, d) & V_0(h_2, t^2, d) & V_1(h_3, t^2, d) \\
V_0(h_1, t^3, d) & V_0(h_2, t^3, d) & V_1(h_3, t^3, d)
\end{pmatrix}
\] (A.8)
is non-degenerate.

Proof. By the linear algebra in Proposition 2.10, \([7]\), we obtain that the matrix
\[
V = \begin{pmatrix}
V_0(h_1, t^1, d) & V_0(h_2, t^1, d) \\
V_0(h_1, t^2, d) & V_0(h_2, t^2, d)
\end{pmatrix}
\] (A.9)
has full rank for any \((t^1, t^2)\) with \(2d\) distinct points in \(\mathbf{P}^1\).

Then we rewrite \(B\) as a block matrix,
\[
B = \begin{pmatrix}
V & \mathcal{A} \\
\mathcal{D} & V_1(h_3, t^3, d)
\end{pmatrix}
\] (A.10)
where
\[
\mathcal{D} = \begin{pmatrix}
V_0(h_1, t^3, d) & V_0(h_2, t^3, d)
\end{pmatrix}
\] (A.11)
and
\[
\mathcal{A} = \begin{pmatrix}
V_1(h_3, t^1, d) \\
V_1(h_3, t^2, d)
\end{pmatrix}
\] (A.12)

Then for generic \(t^1, t^2, t^3\), \(B\) is column-equivalent to
\[
\begin{pmatrix}
V & 0 \\
\mathcal{D} & -\mathcal{D}V^{-1}\mathcal{A} + V_1(h_3, t^3, d)
\end{pmatrix}
\] (A.13)
By choosing the points of \(t^1, t^2\) to be the zeros of \(h_3\) and \(V\) remains invertible, we obtain that
\[
\mathcal{A} = 0.
\]
Thus we have a specialization
\[-\mathcal{D}V^{-1}\mathcal{A} + V_1(h_3, t^3, d) = V_1(h_3, t^3, d)\]
which is non-degenerate. Thus for generic \(t^1, t^2, t^3\),
\[-\mathcal{D}V^{-1}\mathcal{A} + V_1(h_3, t^3, d)\]
is non-degenerate, so is \(B\).
We’ll use two different types matrices $B$ in Lemmas A. 2, A.3 as building blocks to obtain matrices of larger sizes. We’ll show that non-degeneracy of large matrices is the result of that of the smaller, block matrices.

**Lemma A.4.** Let $B_{ij}, i = 1, 2, j = 1, \cdots, 6$ be $(3d + 1) \times (d + 1)$ matrices. Assume

1. \( (B_{13} \ B_{14}) = (B_{23} \ B_{24}) \) \hspace{1cm} (A.14)

2. \( (B_{13} \ B_{14} \ B_{16}) \)

and

\( (B_{11} \ B_{12} - B_{22} \ B_{25}) \)

have full rank,

3. the columns of \( (B_{13} \ B_{14}) \)

span the columns of \( (B_{11} \ B_{12}) \)

Then

\[ J = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0 & B_{16} \\ 0 & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} \end{pmatrix} \] \hspace{1cm} (A.19)

is non-degenerate.

**Proof.** In the following, we apply the column and row operations to the matrix. They will not change its rank. Applying the row operations on the matrix $J$ we obtain it is row-equivalent to

\[ J = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & 0 & B_{16} \\ -B_{11} & B_{22} - B_{12} & 0 & 0 & B_{25} & B_{26} - B_{16} \end{pmatrix} \] \hspace{1cm} (A.20)

By the condition (3), it is column-equivalent to

\[ J = \begin{pmatrix} 0 & 0 & B_{13} & B_{14} & 0 & B_{16} \\ -B_{11} & B_{22} - B_{12} & 0 & 0 & B_{25} & B_{26} - B_{16} \end{pmatrix} \] \hspace{1cm} (A.21)
By the condition (2), it is further column equivalent to
\[
J = \begin{pmatrix}
0 & 0 & B_{13} & B_{14} & 0 & B_{16} \\
-B_{11} & B_{22} - B_{12} & 0 & 0 & B_{25} & 0
\end{pmatrix}
\]  
(A.22)

By the condition (2), we obtain it has full rank.

Lemma A.5. Consider the block matrix
\[
J = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\
0 & 0 & B_{23} & B_{24} & B_{25} & B_{26} & 0 \\
0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37}
\end{pmatrix}
\]  
(A.23)

where the entries in first row are \((3d + 1) \times (d + 1)\) matrices and all the rest are \((2d + 1) \times (d + 1)\) matrices. We assume

(1) \( (B_{24} B_{25} B_{26}) = (B_{34} B_{35} B_{36}) \).  
(A.24)

(2) \( (B_{11} B_{12} B_{15}) \)  
(A.25)

(3) \( (B_{23} -B_{37}) \)  
(A.26)

and

(3) the columns of
\[
( B_{11} B_{12} )
\]  
(A.28)

span the columns of
\[
( B_{13} B_{14} )
\]  
(A.29)

then the matrix \( J \) has full rank.
Proof. The proof uses row and column operations which are the same as those in Lemma A.4. So in the following we just list the equivalent matrices in the reduction.

\[ J = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & B_{24} & B_{25} & B_{26} & 0 \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \]  
(A.30)

\[ \downarrow \text{row operations} \]

\[ \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & -B_{37} \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \]  
(A.31)

\[ \downarrow \text{column operations} \]

\[ J = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & -B_{37} \\ 0 & 0 & 0 & B_{34} & B_{35} & B_{36} & B_{37} \end{pmatrix} \]  
(A.32)

\[ \downarrow \text{column operations} \]

\[ J = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 & B_{15} & 0 & 0 \\ 0 & 0 & B_{23} & 0 & 0 & 0 & -B_{37} \\ 0 & 0 & 0 & B_{34} & 0 & B_{36} & 0 \end{pmatrix} \]  
(A.33)

The last matrix has full rank. \qed

Lemma A.6. Consider the block matrix

\[ J = \begin{pmatrix} B_{11} & B_{12} & 0 & 0 \\ 0 & B_{22} & B_{23} & 0 \\ 0 & 0 & B_{33} & B_{34} \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix}, \]  
(A.34)

where \( B_{ij} \) are non degenerate square matrices of the same size. If

\[ \begin{pmatrix} -B_{22}B_{23}^{-1}B_{33} & B_{34} \\ B_{42} & B_{44} \end{pmatrix} \]  
(A.35)

is non-degenerate, so is \( J \).
Proof. As before we perform row and column operations on \( J \) to obtain

\[
\begin{pmatrix}
B_{11} & 0 & 0 & 0 \\
0 & B_{22} & B_{23} & 0 \\
0 & 0 & B_{33} & B_{34} \\
0 & B_{42} & 0 & B_{44}
\end{pmatrix}
\]  

(A.36)

\[
\downarrow
\]

\[
\begin{pmatrix}
B_{11} & 0 & 0 & 0 \\
0 & 0 & B_{23} & 0 \\
0 & -B_{22}B_{23}^{-1}B_{33} & B_{33} & B_{34} \\
0 & B_{42} & 0 & B_{44}
\end{pmatrix}
\]  

(A.37)

\[
\downarrow
\]

\[
\begin{pmatrix}
B_{11} & 0 & 0 & 0 \\
0 & 0 & B_{23} & 0 \\
0 & -B_{22}B_{23}^{-1}B_{33} & 0 & B_{34} \\
0 & B_{42} & 0 & B_{44}
\end{pmatrix}
\]  

(A.38)

By the assumption,

\[
\begin{pmatrix}
-B_{22}B_{23}^{-1}B_{33} & B_{34} \\
B_{42} & B_{44}
\end{pmatrix}
\]  

(A.39)

is non-degenerate, so is \( J \).

\[\square\]

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