ON SOME SIGMA MODELS WITH POTENTIALS AND THE KLEIN - GORDON TYPE EQUATIONS

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Abstract

The gauge equivalent counterparts of the some (1+1)-, or (2+0)- dimensional $\sigma$-models are found. Also we have proved the equivalence between the some spin-phonon equations and the Yajima-Oikawa-Ma equations.
1 Introduction

Integrable $\sigma$-models play an important role in modern theoretical and mathematical physics. They arise in gravity theory, extended supergravity, in theory of Anderson localization, the Kaluza-Klein theory, in theory of strings, and superstrings. The simplest nonrelativistic version of the $\sigma$-model is a continuous classical spin Heisenberg model or the $(2+1)$-dimensional isotropic Landau-Lifshitz equation (LLE)

$$S_t = S \wedge \triangle S$$

and its extensions to higher spins and to multidimensions[1]. Then in stationary limit the LLE (1) coincide with the $\sigma$-model equation

$$\triangle S + (\nabla S)^2 S = 0.$$  

Here

$$S^2 = S_3^2 + r^2(S_1^2 + S_2^2) = E = \pm 1,$$

$$\triangle = \frac{\partial^2}{\partial x^2} + \alpha^2 \frac{\partial^2}{\partial y^2}, \quad \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}, \quad i^2 = 1, \quad j^2 = \alpha^2, \quad \alpha^2 = \pm 1, \quad r^2 = \pm 1, \quad E = \pm 1.$$
In this work we consider the some generalizations of the $\sigma$-model (2), namely, the $\sigma$-models with potentials.

Let us now consider the Myrzakulov IX (M-IX) equation [1]

$$iS_t + \frac{1}{2}[S, M_1 S] + A_2 S_x + A_1 S_y = 0$$  \hspace{1cm} (5a)

$$M_{2u} = \frac{\alpha^2}{2i} tr(S[S_x, S_y])$$  \hspace{1cm} (5b)

where $\alpha, b, a = \text{consts}$ and

$$S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2 \quad S^2 = EI, \quad r^2 = \pm 1,$$

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b - a) \frac{\partial^2}{\partial x \partial y} + 4(a^2 - 2ab - b) \frac{\partial^2}{\partial x^2},$$

$$M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(2a + 1) \frac{\partial^2}{\partial x \partial y} + 4a(a + 1) \frac{\partial^2}{\partial x^2},$$

$$A_1 = i\{\alpha(2b + 1)u_y - 2(2ab + a + b)u_x\},$$

$$A_2 = i\{4\alpha^{-1}(2a^2b + a^2 + 2ab + b)u_x - 2(2ab + a + b)u_y\}.$$

This set of equations arises from the compatibility condition of the following linear equations [1]

$$\alpha \Phi_y = \frac{1}{2}[S + (2a + 1)I] \Phi_x$$  \hspace{1cm} (6a)

$$\Phi_t = 2i[S + (2b + 1)I] \Phi_{xx} + W \Phi_x$$  \hspace{1cm} (6b)

with

$$W = 2i\{(2b + 1)(F^+ + F^- S) + (F^+ S + F^-) + (2b - a + \frac{1}{2}) SS_x + \frac{1}{2} S_x + \frac{\alpha}{2} SS_y\},$$

$$F^\pm = A \pm D, \quad A = i[u_y - \frac{2a}{\alpha} u_x], \quad D = i[\frac{2(a + 1)}{\alpha} u_x - u_y].$$

It is well known that equation (5) is gauge and Lakshmanan equivalent to the following Zakharov equation (ZE) [2]

$$iq_t + M_1 q + vq = 0,$$  \hspace{1cm} (7a)

$$ip_t - M_1 p - vp = 0,$$  \hspace{1cm} (7b)

$$M_{2v} = -2M_1(pq),$$  \hspace{1cm} (7c)

The Lax representation of this equation has the form

$$\alpha \Psi_y = B_1 \Psi_x + B_0 \Psi,$$  \hspace{1cm} (8a)

$$\Psi_t = iC_2 \Psi_{xx} + C_1 \Psi_x + C_0 \Psi,$$  \hspace{1cm} (8b)

with

$$B_1 = \begin{pmatrix} a + 1 & 0 \\ 0 & a \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}.$$
\[ C_2 = \begin{pmatrix} b + 1 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & iq \\ ip & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \]

\[ c_{12} = i(2b - a + 1)q_x + i\alpha q_y, \quad c_{21} = i(a - 2b)p_x - i\alpha p_y. \]

Here \( c_{jj} \) is the solution of the following equations

\[ \begin{aligned}
(a+1)c_{11} - \alpha c_{11} &= i[(2b-a+1)(pq)_x + \alpha (pq)_y], \\
ac_{22} - \alpha c_{22} &= i[(a-2b)(pq)_x - \alpha (pq)_y].
\end{aligned} \] (9)

The goal of this work is to explore the some \( \sigma \)-models with potentials derivable from the M-IX equation(5).

2 Gauge equivalence between the some \( \sigma \)-models with potentials and the Klein-Gordon-type equations

It is interesting to note that the equation(5) admits some reductions in 1+1 or 2+0 dimensions. Let us now consider these reductions.

2.1 The Myrzakulov XXXII (M-XXXII) equation

Suppose that now \( \nu = t \) is the some "hidden" parameter, \( S = S(x, y), \ u = u(x, y), \ q = q(x, y), \ p = p(x, y), \ v = v(x, y) \) and at the same time \( \Phi = \Phi(x, y, \nu), \ \Psi = \Psi(x, y, \nu) \). Then equation (5) reduces to the following \( \sigma \)-model with potential

\[ M_1 S + \{ k_1 S_x^2 + k_2 S_x S_y + k_3 S_y^2 \} S + A_2 S_S S_x + A_1 S_S y = 0 \quad (10a) \]

\[ M_2 u = \frac{\alpha^2}{2i} tr(S[S_x, S_y]) \quad (10b) \]

where \( M_1 \) we write in the form

\[ M_1 = k_3 \frac{\partial^2}{\partial y^2} + k_2 \frac{\partial^2}{\partial x \partial y} + k_1 \frac{\partial^2}{\partial x^2}, \]

which is the compatibility condition \( \Phi_{\nu \nu} = \Phi_{\nu y} \) of the set (6) and called the M-XXXII equation[1]. The G-equivalent and L-equivalent counterpart of this equation is given by

\[ M_1 q + vq = 0, \quad M_1 p + vp = 0, \quad M_2 v = -2M_1(pq). \] (11)

This is the some modified complex Klein-Gordon equation (mKGE).

2.2 The Myrzakulov XV (M-XV) equation

Let us consider the case: \( a = b \). Then we obtain the M-XV equation [1]

\[ \alpha^2 S_{yy} - a(a + 1)S_{xx} + \{ \alpha^2 S_y^2 - a(a + 1)S_x^2 \} S + + A_2 S S_x + A_1 S S_y \quad (12a) \]

\[ M_2 u = \frac{\alpha^2}{2i} tr(S[S_x, S_y]) \quad (12b) \]
where $A_j'' = A_j$ as $a = b$. The corresponding mKGE has the form

$$\alpha^2 q_{yy} - a(a+1)q_{xx} + v q = 0, \quad M_2 v = -2[\alpha^2(|q|^2)_{yy} - a(a+1)(|q|^2)_{xx}] \quad (13)$$

The Lax representations of these equations we obtain from (6) and (8) respectively as $a = b$.

### 2.3 The Myrzakulov XIV (M-XIV) equation

Now we consider the reduction: $a = -\frac{1}{2}$. Then the equation (10) reduces to the M-XIV equation\[1\]

$$S_{xx} + 2\alpha(2b+1)S_{xy} + \alpha^2 S_{yy} + \{S_{xx} + 2\alpha(2b+1)S_{xy} + \alpha^2 S_{yy}\}S + A_2'SS_x + A_1'SS_y = 0 \quad (14a)$$

$$\alpha^2 u_{yy} - u_{xx} = \frac{\alpha^2}{2i} tr(S[S_x, S_y]) \quad (14b)$$

where $A_j' = A_j$ as $a = -\frac{1}{2}$. The corresponding gauge equivalent equation is obtained from (11) and looks like

$$q_{xx} + 2\alpha(2b+1)q_{xy} + \alpha^2 q_{yy} + v q = 0 \quad (15a)$$

$$\alpha^2 v_{yy} - v_{xx} = -2\{\alpha^2(pq)_{yy} + 2\alpha(2b+1)(pq)_{xy} + (pq)_{xx}\} \quad (15b)$$

From (6) and (8) we obtain the Lax representations of (14) and (15) respectively as $a = -\frac{1}{2}$.

### 2.4 The Myrzakulov XIII (M-XIII) equation

Now let us consider the case: $a = b = -\frac{1}{2}$. In this case the equations (10) reduce to the $\sigma$-model

$$S_{xx} + \alpha^2 S_{yy} + \{S_{xx} + \alpha^2 S_{yy}\}S + iu_ySS_x + iu_xSS_y = 0 \quad (16a)$$

$$\alpha^2 u_{yy} - u_{xx} = \frac{\alpha^2}{2i} tr(S[S_y, S_x]) \quad (16b)$$

which is the M-XIII equation\[1\]. The equivalent counterpart of the equation (16) is the following equation

$$q_{xx} + \alpha^2 q_{yy} + v q = 0 \quad (17a)$$

$$\alpha^2 v_{yy} - v_{xx} = -2\{\alpha^2(pq)_{yy} + (pq)_{xx}\} \quad (17b)$$

that follows from the mKGE(11).
2.5 The Myrzakulov XII (M-XII) equation

Now let us consider the case: \( a = b = -1 \). In this case the equations (10) reduce to the \( \sigma \)-model - the M-XII equation[1]

\[
S_{YY} + S_Y^2 + iwSS_Y = 0 \quad (18a)
\]

\[
w_Y + w_X + \frac{1}{4i} tr(S[S_X, S_Y]) \quad (18b)
\]

where \( X = 2x \), \( Y = \alpha y \), \( w = -\frac{wY}{\alpha} \). The equivalent counterpart of the equation (18) is the mKGE

\[
q_{YY} + vq = 0 \quad (19a)
\]

\[
v_X + v_Y + 2(pq)_Y = 0. \quad (19b)
\]

that follows from the (11).

2.6 The Myrzakulov XXXI (M-XXXI) equation

This \( \sigma \)-model equation is read as[1]

\[
bS_{\eta\eta} - (b + 1)S_{\xi\xi} \} + \{bS_{\eta\eta} - (b + 1)S_{\xi\xi}\} S + i(b + 1)w_\eta SS_\eta + ibw_\xi SS_\xi = 0 \quad (20a)
\]

\[
w_\xi_\eta = -\frac{1}{4i} tr(S[S_\eta, S_\xi]) \quad (20b)
\]

which is the M-XXXI equation, where \( w = -\alpha^{-1}u \). The equivalent mKGE looks like

\[
(1 + b)q_{\xi\xi} - bq_{\eta\eta} + vq = 0 \quad (21a)
\]

\[
v_{\xi\eta} = -2\{(1 + b)(pq)_{\xi\xi} - b(pq)_{\eta\eta}\} \quad (21b)
\]

So we have found the G-equivalent counterparts of the \( \sigma \)-models with potentials.

3 The other (1+1)-dimensional reductions: gauge equivalence between the M\(_{00}^{\text{VIII}}\)-equation and the Yajima-Oikawa -Ma equation

Let us now we consider the reduction of the M-IX equation (5) as \( a = b = -1 \). We have

\[
iS_t + \frac{1}{2}[S, S_{YY}] + iwS_Y = 0 \quad (22a)
\]

\[
w_X + w_Y + \frac{1}{4i} tr(S[S_x, S_y]) = 0 \quad (22b)
\]

This equation is the Myrzakulov VIII (M-VIII) equation[1]. The G-equivalent and L-equivalent counterpart of equation (22) is given by

\[
iq_t + q_{YY} + vq = 0, \quad (23a)
\]

\[
-ip_t - p_{YY} - vp = 0, \quad (23b)
\]
\[ v_X + v_Y + 2(pq)_Y = 0. \]  

(23c)

Now let us take the case when \( X = t \). Then the M-VIII equation (22) pass to the following \( M^{53}_{00} \) - equation [1]

\[ iS_t + \frac{1}{2}[S, S_Y] + iwS_Y = 0 \]  

(24a)

\[ w_t + w_Y + \frac{1}{4}\{tr(S_Y^2)\}_Y = 0 \]  

(24b)

The \( M^{53}_{00} \) - equation (24) was proposed in [1] to describe a nonlinear dynamics of the compressible magnets (see, Appendix). It is integrable and has the different soliton solutions[1].

In our case equation (23) becomes

\[ iq_t + q_{YY} + vq = 0, \]  

(25a)

\[ ip_t - pv_Y - vp = 0, \]  

(25b)

\[ vt + v_Y + 2(pq)_Y = 0. \]  

(25c)

that is the Yajima-Oikawa equation(YOE)[3]. So we have proved that the \( M^{53}_{00} \) - equation (24) and the YOE (25) is gauge equivalent to each other. The Lax representations of (24) and (25) we can get from (6) and (8) respectively as \( a = b = -1 \) (see, for example, the ref.[1]). Note that our Lax representation for the YOE (25) is different than that which was presented in [3].

Also we would like note that the M-VIII equation (22) we usually write in the following form

\[ iS_t = \frac{1}{2}[S_{\xi\xi}, S] + iwS_{\xi} \]  

(26a)

\[ w_\eta = \frac{1}{4i}\{tr(S[S_{\eta}, S_{\xi}]) \} \]  

(26b)

The gauge equivalent counterpart of this equation is the following ZE[2]

\[ iq_t + q_{\xi\xi} + vq = 0, \]  

(27a)

\[ v_\eta = -2r^2(\bar{q}q)_{\xi}. \]  

(27b)

As \( \eta = t \) equation (26) take the other form of the \( M^{53}_{00} \) - equation

\[ iS_t = \frac{1}{2}[S_{\xi\xi}, S] + iwS_{\xi} \]  

(28a)

\[ w_t = \frac{1}{4i}\{tr(S^2_{\xi})\}_{\xi} \]  

(28b)

Similarly, equation (27) becomes

\[ iq_t + q_{\xi\xi} + vq = 0, \]  

(29a)

\[ v_t = -2r^2(\bar{q}q)_{\xi}, \]  

(29b)

which is called the Ma equation and was considered in [4].
4 Conclusion

We have established the gauge equivalence between the (1+1)-, or (2+0)-dimensional $\sigma$-models and the Klein-Gordon type equations. Also we have proved the equivalence between the some spin-phonon equations and the Yajima-Oikawa-Ma equations.

5 Appendix: On some soliton equations of compressible magnets

Solitons in magnetically ordered crystals have been widely investigated from both theoretical and experimental points of view. In particular, the existence of coupled magnetoelastic solitons in the Heisenberg compressible spin chain has been extensively demonstrated. In [1] we presented a new classes of integrable and nonintegrable soliton equations of spin systems. Below we present some of these nonlinear models of magnets - the some of the Myrzakulov equations(ME), which describe the nonlinear dynamics of compressible magnets.

5.1 The 0-class of spin-phonon systems

The Myrzakulov equations with the potentials have the form:

the $M_{00}^{10}$ - equation:

$$2iS_t = [S, S_{xx}] + (u + h)[S, \sigma_3]$$

the $M_{00}^{20}$ - equation:

$$2iS_t = [S, S_{xx}] + (u\sigma_3 + h)[S, \sigma_3]$$

the $M_{00}^{30}$ - equation:

$$2iS_t = \{[\mu\tilde{S}_x^2 - u + m][S, S_x]\}_x + h[S, \sigma_3]$$

the $M_{00}^{40}$ - equation:

$$2iS_t = n[S, S_{xxxx}] + 2\{[\mu\tilde{S}_x^2 - u + m][S, S_x]\}_x + h[S, \sigma_3]$$

the $M_{00}^{50}$ - equation:

$$2iS_t = [S, S_{xx}] + 2uS_x$$

where $v_0, \mu, \lambda, n, m, a, b, \alpha, \beta, \rho, h$ are constants, $u$ is a scalar function(potential), subscripts denote partial differentiations, $[,]$ $\{,\}$ is commutator (anticommutator),

$$S = \begin{pmatrix} S_3 & \sigma^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2, \quad r^2 = \pm 1 \quad S^2 = I.$$
5.2 The 1-class of spin-phonon systems

The $M_{00}^{11}$ - equation:

$$2i S_t = [S, S_{xx}] + (u + h)[S, \sigma_3]$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S_3)_{xx}$$

the $M_{00}^{12}$ - equation:

$$2i S_t = [S, S_{xx}] + (u + h)[S, \sigma_3]$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S_3)_{xx}$$

the $M_{00}^{13}$ - equation:

$$2i S_t = [S, S_{xx}] + (u + h)[S, \sigma_3]$$
$$u_t + u_x + \lambda(S_3)_x = 0$$

the $M_{00}^{14}$ - equation:

$$2i S_t = [S, S_{xx}] + (u + h)[S, \sigma_3]$$
$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(S_3)_x = 0$$

5.3 The 2-class of spin-phonon systems

The $M_{00}^{21}$ - equation:

$$2i S_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(S^2_3)_{xx}$$

the $M_{00}^{22}$ - equation:

$$2i S_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(S^2_3)_{xx}$$

the $M_{00}^{23}$ - equation:

$$2i S_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]$$
$$u_t + u_x + \lambda(S^2_3)_x = 0$$

the $M_{00}^{24}$ - equation:

$$2i S_t = [S, S_{xx}] + (uS_3 + h)[S, \sigma_3]$$
$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(S^2_3)_x = 0$$
5.4 The 3-class of spin-phonon systems

The $M_{00}^{31}$ - equation:

$$2iS_t = \{(\mu \vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(\vec{S}_x^2)_{xx}$$

the $M_{00}^{32}$ - equation:

$$2iS_t = \{(\mu \vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(\vec{S}_x^2)_{xx}$$

the $M_{00}^{33}$ - equation:

$$2iS_t = \{(\mu \vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$u_t + u_x + \lambda(\vec{S}_x^2)_x = 0$$

the $M_{00}^{34}$ - equation:

$$2iS_t = \{(\mu \vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(\vec{S}_x^2)_x = 0$$

5.5 The 4-class of spin-phonon systems

The $M_{00}^{41}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(\vec{S}_x^2)_{xx}$$

the $M_{00}^{42}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(\vec{S}_x^2)_{xx}$$

the $M_{00}^{43}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$u_t + u_x + \lambda(\vec{S}_x^2)_x = 0$$

the $M_{00}^{44}$ - equation:

$$2iS_t = [S, S_{xxxx}] + 2\{(1 + \mu)\vec{S}_x^2 - u + m)[S, S_x]\}_x$$

$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(\vec{S}_x^2)_x = 0$$
5.6 The 5-class of spin-phonon systems

The $M_{00}^{51}$ - equation:

$$2iS_t = [S, S_{xx}] + 2uS_x$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \lambda(f)_{xx}$$

the $M_{00}^{52}$ - equation:

$$2iS_t = [S, S_{xx}] + 2uS_x$$
$$\rho u_{tt} = \nu_0^2 u_{xx} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \lambda(f)_{xx}$$

the $M_{00}^{53}$ - equation:

$$2iS_t = [S, S_{xx}] + 2uS_x$$
$$u_t + u_x + \lambda(f)_x = 0$$

the $M_{00}^{54}$ - equation:

$$2iS_t = [S, S_{xx}] + 2uS_x$$
$$u_t + u_x + \alpha(u^2)_x + \beta u_{xxxx} + \lambda(f)_x = 0$$

Here $f = \frac{1}{4}tr(S_x^2)$, $\lambda = 1$.

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