I = 2 Pion Scattering Length from a Coarse Anisotropic Lattice Calculation

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Abstract

Using the tadpole improved clover Wilson quark action on coarse anisotropic lattices, the $\pi\pi$ scattering length in the $I = 2$ channel is calculated within quenched approximation. We show that such a calculation is feasible using small lattices on small computers provided that the finite volume and finite lattice spacing errors are under control. Our results are extrapolated towards the chiral, infinite volume and continuum limit. Comparisons of our results with previous lattice results from JLQCD collaboration, the new results from E865 experiment, and the results from Chiral Perturbation Theory are made. Good agreements are found.

Key words: $\pi\pi$ scattering length, lattice QCD, improved actions.
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1 Introduction

It has become clear that anisotropic, coarse lattices and improved lattice actions are ideal candidates for lattice QCD calculations on small computers\footnote{This work is supported by the National Natural Science Foundation of China (NSFC) and Pandeng fund.}
They are particularly advantageous for heavy objects like the glueballs, one meson states with nonzero spatial momenta and multi-meson states with or without spatial momenta. The gauge action employed is the tadpole improved gluonic action on anisotropic lattices [2]. Using this action, glueball and hadron spectrum has been studied within quenched approximation [3–9].

In this letter, we report our results on the pion-pion scattering lengths within quenched approximation using relatively small lattices. Lattice calculations of pion scattering lengths have been performed by various authors using symmetric lattices without the improvement [10–12]. With the symmetric lattices and Wilson action, large lattices (typically 24^3 64 or larger) have to be used which require substantial amount of computer resources. It gets even more challenging if the chiral, infinite volume and continuum limit extrapolation is to be made [12]. In this letter, we show that such a calculation is feasible using relatively small lattices (8^3 40 or so) on small computers with the tadpole improved anisotropic lattice actions. Our final extrapolated result on the pion-pion scattering length in the I = 2 channel is compared with previous lattice result [12], the new experimental result from E865 collaboration and Chiral Perturbation Theory.

The fermion action used in this calculation is the tadpole improved clover Wilson action on anisotropic lattices [13,14]. Among the parameters which appear in the fermion matrix, the so-called bare velocity of light ν is tuned non-perturbatively using the single pion dispersion relations [14]. The tadpole improved tree-level values are used for other parameters. In the fermion matrix, the bare quark mass dependence is singled out so that one could utilize the shifted structure of the matrix to solve for quark propagators at various values of valance quark mass at the cost of solving only the lightest one, using the so-called Multi-mass Minimal Residual (M^3 R) algorithm [15–17].

2 Formulation to extract the scattering lengths

In order to calculate hadron scattering lengths on the lattice, or the scattering phase shifts in general, one uses Lüscher’s formula which relates the exact energy level of two hadron states in a finite box to the scattering phase shift in the continuum. In the case of pion-pion scattering, this formula relates the exact two pion energy E_{\pi\pi}^{(I)} in a finite box of size L and isospin I channel to the corresponding scattering length a_{0}^{(I)} in the continuum [18]:

\begin{equation}
E_{\pi\pi}^{(I)} - 2m_\pi = -\frac{4\pi a_0^{(I)}}{m_\pi L^3} \left[ 1 + c_1 \frac{a_0^{(I)}}{L} + c_2 \left( \frac{a_0^{(I)}}{L} \right)^2 \right] + O(L^{-6}) ,
\end{equation}
where \( c_1 = -2.837297 \) and \( c_2 = 6.375183 \) are numerical constants. The formula suffers dramatic changes in the quenched approximation\footnote{The authors would like to thank Professor C. Bernard for bringing our attention to this issue.} as discussed in [19]. In the \( I = 0 \) channel, one-loop quenched chiral perturbation theory gives anomalous contributions to the two-pion energy that are of order \( L^0 = 1 \) and of order \( L^{-2} \) due to \( \eta' \) loops. In the \( I = 2 \) channel, these enhanced contributions are absent.

To measure the pion mass \( m_\pi \) and to extract the exact energy level \( E^{(I)}_{\pi\pi} \) of two pions with zero relative momentum, appropriate correlation functions are constructed from the corresponding operators in the \( I = 2 \) channel. We have used the operators proposed in Ref. [11]. Numerically, it is more advantageous to construct the ratio of the correlation functions defined above. It is argued [20,19] that one should use the linear fitting function:

\[
R^{I=2}(t) \equiv \frac{C^{I=2}_{\pi\pi}(t)/(C_\pi(t)C_\pi'(t))^{T>\sim >t\sim 1}}{1 - \delta E^{(2)}_{\pi\pi} t},
\]

where \( C^{I=2}_{\pi\pi}(t) \) and \( C_\pi(t) \) are the two and one pion correlation functions and \( \delta E^{(2)}_{\pi\pi} = E^{(2)}_{\pi\pi} - 2m_\pi \) is the energy shift. Two pion correlation function, or equivalently, the ratio \( R(t) \) constructed above can be transformed into products of quark propagators using Wick’s theorem [11]. The \( I = 2 \) two pion correlation function is given by two contributions which are termed Direct and Cross contributions [10,11]. The two pion correlation function in the \( I = 0 \) channel is, however, more complicated which involves vacuum diagrams that require to compute the quark propagators for wall sources placed at \textit{every} time-slice, a procedure which is more time-consuming than the \( I = 2 \) channel.

### 3 Simulation details

Simulations are performed on several PCs and workstations. Configurations are generated using the pure gauge action with fixed anisotropy \( \xi = 5 \) for \( 4^340, 6^340 \) and \( 8^340 \) lattices at the gauge coupling \( \beta = 1.7, 2.2, 2.4 \) and \( 2.6 \). The spatial lattice spacing \( a_s \) is roughly between 0.19fm and 0.39fm while the physical size of the lattice ranges from 0.8fm to 3.2fm. For each set of parameters, several hundred decorrelated gauge field configurations are used to measure the fermionic quantities. Statistical errors are all analyzed using the usual jack-knife method.

Quark propagators are measured using the Multi-mass Minimal Residue algorithm for 5 different values of bare quark mass using wall sources to enhance
the signal [10–12]. Periodic boundary condition is applied to all three spatial directions while in the temporal direction, Dirichlet boundary condition is utilized.

The single pseudo-scalar and vector meson correlation functions at zero spatial momentum and three lowest lattice momenta, namely (100), (110) and (111) are constructed from the corresponding quark propagators. Using the anisotropic lattices, we are able to obtain decent effective mass plateaus for these energy levels. The parameter $\nu$, also known as the bare velocity of light, that enters the fermion matrix is determined non-perturbatively using the single pion dispersion relations as described in Ref. [14].

Two pion correlation functions and the ratio $\mathcal{R}(t)$ are constructed for all $\kappa$, $\beta$ and $L$ values as a function of the temporal separation $t$. Then, the linear fit (2) is performed. Again, due to the usage of the anisotropic lattices, we obtained reasonable signal for the energy shift with a typical error around ten percent. The energy shifts $\delta E_{\pi\pi}^{(2)}$ are substituted into Lüscher’s formula to solve for the scattering length $a_0^{(2)}$ for a given set of parameters. From these results, we could perform an extrapolation towards the chiral, infinite volume and zero lattice spacing limit.

In the chiral limit, the $\pi\pi$ scattering length in the $I = 2$ channel is given by the current algebra result due to Weinberg [21] in full QCD:

$$a_0^{(2)} = -\frac{1}{16\pi} \frac{m_\pi}{f_\pi^2},$$

where $f_\pi \sim 93\text{MeV}$ is the pion decay constant. Chiral Perturbation Theory results to one-loop and two-loop order have been calculated [22,23]. However, the one-loop and two-loop numerical results on the pion-pion scattering length in the $I = 2$ channel do not differ from the current algebra value substantially. Complication arises in the quenched approximation. In principle, the quenched scattering lengths becomes divergent in the chiral limit [19]. However, these divergent terms only become numerically important when the pion mass is close to zero. For the parameters used in our simulation, these terms seem to be numerically small in the $I = 2$ channel and we could not observe any sign of divergence from our data.

In early lattice calculations [10–12], both the mass and the decay constant of the pion were calculated on the lattice. Then, the lattice results of $m_\pi$ and $f_\pi$ were substituted into the current algebra result (3) to obtain a prediction of the scattering length. This is to be compared with the scattering length obtained from the energy shifts on the lattice and Lüscher’s formula. In these studies, some discrepancies between the lattice results and the chiral results were observed. The disadvantage of the this procedure is the following: first,
it is difficult to make a direct chiral extrapolation; second, the lattice results of the decay constant are usually much less accurate, both statistically and systematically, than the mass values. In fact, most of the discrepancies are due to inaccuracy of the decay constants, as realized in [12]. In Ref. [12], the authors proposed to use the quantity $a_0^{(2)}/m_\pi$, which is much better behaved in the chiral limit. They found that the results obtained using this quantity is in much better agreement with both current algebra and the experiment. In this letter, we use a similar but dimensionless quantity $F = a_0^{(2)}/m_\rho/m_\pi$, which in the chiral limit reads:

$$ F \equiv \frac{a_0^{(2)} m_\rho^2}{m_\pi} = \frac{1}{16\pi} \frac{m_\rho^2}{f_\pi^2} \sim -1.3638 \ , \tag{4} $$

where the final numerical value is obtained by substituting in the experimental values for $m_\rho$ and $f_\pi$. On the lattice, the scattering length $a_0^{(2)}$ is extracted from Lüscher’s formula. More importantly, the mass of the pion and the rho can be obtained with good accuracy on the lattice. So, the factor $F$ can be calculated on the lattice with good precision without the lattice calculation of $f_\pi$. The error of the factor $F$ obtained on the lattice will mainly come from the error of the scattering length $a_0^{(2)}$, or equivalently, the energy shift $\delta E^{(2)}_{\pi\pi}$. Since we have calculated the factor $F$ for 5 different values of valance quark mass, we could perform a chiral extrapolation and extract the result of $F$ in the chiral limit. Comparisons with Weinberg’s result (4) and the experiment will offer us a cross check among different methods.

As an illustration in the upper half of Fig. 1, we show the chiral extrapolation of the quantity $F$ as a function of the pseudo-scalar mass squared ($m_\pi^2$) for the simulation on $4^3 40$ lattices at $\beta = 2.2$. The fitting quality for the chiral extrapolation of our data at other simulation parameters are quite similar. The pseudo-scalar and vector meson mass squared $m_\pi^2$ and $m_\rho^2$ are also shown in the lower half of the figure as a function of $1/(2\kappa)$, which linearly depends on the valance quark mass. It is seen that meson mass squared depends on the valance quark mass linearly. The data points for the factor $F$ are also shown in the lower half of the plot. The fitting quality for the pion, rho and the factor $F$ is reasonable. Admittedly, it is a bit astonishing to observe such a linear behavior since our data were obtained at relatively heavy pion mass values. Typical $m_\pi/m_\rho$ values are above 0.7. We do not have a theoretical explanation for this behavior at the moment. Using our data, we have tried to make the chiral extrapolation by adding a term proportional to $m_\pi^4$ to our fitting function. However, this does not improve the quality of the fit and the fitted coefficient of the $m_\pi^4$ term turns out to be consistent with zero with a large error. Therefore, in the following we will only quote our results using a simple linear extrapolation.

After the chiral extrapolation, we now turn to study the finite volume effects
Fig. 1. Chiral extrapolation for the quantity \( F \) for our simulation results at \( \beta = 2.2 \) on \( 4^340 \) lattices. In the lower half of the plot, the pseudo-scalar and vector meson mass squared are plotted as open squares and open hexagons, respectively, as functions of \( 1/(2\kappa) \). The straight lines represent the corresponding linear fit for them. Also shown in the lower half as open circles are the results for the factor \( F \). In the upper half of the plot, the same quantity \( F \) is shown as a function of \( m^2_\pi \). The straight line shows the linear extrapolation towards the chiral limit \( m^2_\pi = 0 \), where the extrapolated result is also shown. As a comparison, the corresponding experimental result [24] for this quantity is shown as a shaded band.

of the simulation. According to formula (1), the quantity \( F \) obtained on finite lattices differ from its infinite volume value by corrections of the form \( 1/L^3 \). However, it was argued in Ref.[20,11,19] that in a quenched calculation, the form of Lüscher’s formula is invalidated by finite volume corrections of the form \( 1/L^5 \) instead of \( 1/L^6 \). This would mean that the factor \( F \) receives finite volume correction of the form \( 1/L^2 \). In our simulation, however, we were unable to judge from our data which extrapolation is more convincing. We therefore performed our infinite volume extrapolation in both ways, calling them scheme I (extrapolating according to \( 1/L^3 \)) and scheme II (extrapolating according to \( 1/L^2 \)), respectively. Extrapolation in these two different schemes yields compatible results within statistical errors. The fitting quality of Scheme II is somewhat, but not overwhelmingly, better than that of Scheme I. In Fig. 2, we show the infinite volume extrapolation according to Scheme I for the simulation points at \( \beta = 2.6, 2.4, 2.2 \) and 1.7. The extrapolated results are shown with blue open squares at \( L = \infty \), together with the corresponding errors. The straight lines represent the linear extrapolation in \( 1/L^3 \). It is seen that, on physically small lattices, e.g. those with \( \beta = 2.6 \) and \( \beta = 2.4 \), the
Fig. 2. Infinite volume extrapolation in Scheme I for the quantity \( F = a_0^2 \frac{m_\rho^2}{m_\pi} \) for our simulation results at \( \beta = 2.6, 2.4, 2.2 \) and 1.7. The straight lines represent the corresponding linear extrapolation in \( 1/L^3 \). The extrapolated result is also shown, together with its error.

finite volume correction is much more significant than larger lattices. This is also reflected by the slopes of the linear fits. The infinite volume extrapolation in Scheme II is similar.

Finally, we can make an extrapolation towards the continuum limit by eliminating the finite lattice spacing errors. Since we have used the tadpole improved clover Wilson action, all physical quantities differ from their continuum counterparts by terms that are proportional to \( a_s \). The physical value of \( a_s \) for each value of \( \beta \) can be found from Ref. [4,9]. This extrapolation is shown in Fig. 3 where the results from the chiral and infinite volume extrapolation discussed above are indicated as data points in the plot for all 4 values of \( \beta \) that have been simulated. In the lower/upper half of the plot, results in Scheme I/II are shown. The straight lines show the extrapolation towards the \( a_s = 0 \) limit and the extrapolated results are also shown together with the experimental result from Ref.[24] which is shown as the shaded band. For comparison with chiral perturbation theory, Weinberg’s result (4) and the results from Chiral perturbation theory are also shown as the red and magenta points respectively. It is seen that our lattice calculation gives a compatible result for the quantity \( F \) when compared with the experiment. The statistical error of our final result is about the same as that of Ref. [12]. We think the result is promising since we have shown that such a calculation can be obtained on relatively small lattices and limited computer resources. Another encouraging sign is that all
Fig. 3. Continuum extrapolation for the quantity $a_0^{(2)} m_{\rho}^2/m_\pi$ obtained from our simulation results at $\beta = 2.6, 2.4, 2.2$ and $1.7$. The results for both Scheme I and II are shown. The straight lines represent the linear extrapolation in $a_s/r_0$. The extrapolated results are also shown, together with the experimental result from Ref. [24] indicated by the shaded band. For comparison, Weinberg’s result (4) and the results from Chiral perturbation theory are also shown as the red and magenta points respectively.

Data points, even the one at a lattice spacing of 0.4 fm, show little dependence on $a_s$ indicating that the $O(a_s)$ lattice effects are small, presumably due to the tadpole improvement of the action. The slope of our linear extrapolation is small. This is very helpful in cutting down the error of the final extrapolated result. In Ref. [12], continuum extrapolation was also performed for Wilson fermions without the improvement. There, a larger slope of the linear extrapolation is seen, see Fig. 4 of Ref [12]. This magnifies the error of the final extrapolated result when compared with the errors of the results at finite lattice spacings.

To summarize, we obtain from the linear continuum extrapolation the following result for the quantity $F$:

$$
\frac{a_0^{(2)} m_{\rho}^2}{m_\pi} = -1.05(23) \text{ for Scheme I ,}
$$

$$
\frac{a_0^{(2)} m_{\rho}^2}{m_\pi} = -1.41(28) \text{ for Scheme II .}
$$

(5)
If we substitute in the mass of the mesons from the experiment, we obtain the quantity $a_0^{(2)} m_{\pi}$:

\[
a_0^{(2)} m_{\pi} = -0.0342(75) \text{ for Scheme I},
\]
\[
a_0^{(2)} m_{\pi} = -0.0459(91) \text{ for Scheme II}.
\]

This result is compatible with previous lattice result using Wilson fermions on large lattices [12]. Current algebra prediction (3) yields a value of $a_0^{(2)} m_{\pi} = -0.046$. This quantity has been calculated in Chiral Perturbation Theory to one-loop order with the result: $a_0^{(2)} m_{\pi} = -0.042$ [22] and recently to two-loop order [23,25]. The final result from Chiral Perturbation Theory gives: $a_0^{(2)} m_{\pi} = -0.0444(10)$, where the error comes from theoretical uncertainties. On the experimental side, a new result from E865 collaboration [24] claims $a_0^{(2)} m_{\pi} = -0.036(9)$. It is encouraging to find out that our lattice results in both schemes are compatible with the experiment. Our result in Scheme II also agrees with the Chiral Perturbation Theory results very well while our result in Scheme I is barely within one standard deviation of the chiral results.

4 Conclusions

In this letter, we have calculated pion-pion scattering lengths in isospin $I = 2$ channel using quenched lattice QCD. It is shown that such a calculation is feasible using coarse, anisotropic, small lattices with limited computer resources like several personal computers and workstations. The calculation is done using the tadpole improved clover Wilson action on anisotropic lattices. The anisotropy helps to enhance the temporal resolution of correlation functions while the improvement helps to cut down the finite lattice spacing errors on coarse lattices. Simulations are performed on lattices with various sizes, ranging from 0.8fm to about 3fm and with different value of lattice spacing. The infinite volume extrapolation is explored in two different schemes which yields compatible final results. The lattice result for the scattering length is extrapolated towards the chiral and continuum limit where a result consistent with the experiment is found. Comparisons with previous lattice results and the Chiral Perturbation Theory are also made with encouraging results. Our data and results also suggest that, using improved gluonic and fermionic actions, the lattice spacing errors for the scattering length is under control even on coarse lattices of $a_s \sim 0.4$fm.

Finally, our method for calculating the pion-pion scattering length discussed here can be easily generalized to calculate the scattering lengths of other hadrons, or in other channels, e.g. $I = 0$ channel where extra care has to be taken due to enhanced terms coming from quenched chiral loops. The method
can also be applied to calculate the scattering phase shift at non-zero spatial lattice momenta, where presumably larger lattices are needed.

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