A Note on Smarr Relation and Coupling Constants

Shi-Qian Hu\textsuperscript{1}, \textsuperscript{\&} Xiao-Mei Kuang\textsuperscript{1}, \textsuperscript{\dag} and Yen Chin Ong\textsuperscript{1,2}, \textsuperscript{\&}

\textsuperscript{1}Center for Gravitation and Cosmology, College of Physical Science and Technology, Yangzhou University, Yangzhou 225009, China
\textsuperscript{2}Nordita, KTH Royal Institute of Technology & Stockholm University, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden

The Smarr relation plays an important role in black hole thermodynamics. It is often claimed that the Smarr relation can be written down simply by observing the scaling behavior of the various thermodynamical quantities. We point out that this is not necessarily so in the presence of dimensionful coupling constants, and discuss the issues involving the identification of thermodynamical variables.

I. SMARR RELATION AND THE FIRST LAW

The fact that black holes behave like a thermodynamical system has dramatically changed our understanding of black holes ever since its conception in 1973 \cite{1}. For an asymptotically flat Kerr-Newman black hole, the first law of black hole mechanics takes the form

\[ dM = TdS + ΦdQ + ΩdJ, \]  \hspace{1cm} (1)

where \( M \) denotes the ADM mass of the black hole, \( S \) its Bekenstein-Hawking entropy, \( T \) its Hawking temperature, \( Q \) its electrical charge and \( J \) its angular momentum. The first law thus relates the various differential quantities. In some applications, one would like to work directly with the black hole parameters instead of their differentials. Fortunately, there is the Smarr relation \cite{2}:

\[ M = 2TS - ΦQ + 2ΩJ, \]  \hspace{1cm} (2)

where \( Φ \) denotes the electrical potential, while \( Ω \) denotes the angular velocity of the black hole.

Smarr relations such as this have been widely studied in the literature, beyond the Kerr-Newman family. A good rule of thumb for writing down the Smarr relation for a given black hole is to look at the scaling (i.e. the dimensions) of the various thermodynamical quantities. See, e.g., Sec.2 of \cite{3}. For example, in 4-dimensions, and in the units \( ħ = k_B = c = 1 \), we have \( M, Q \propto L, \) and \( J, S \propto L^2 \), where \( L \) is a length scale. Due to Euler’s theorem of quasi-homogeneous function (see below), we can simply write down \( M = M(S, Q, J) \) as

\[ 1 \cdot M = 2 \frac{∂M}{∂S} S + 1 \cdot \frac{∂M}{∂Q} Q + 2 \cdot \frac{∂M}{∂J} J. \]  \hspace{1cm} (3)

From the first law (and the chain rule), one could identify the various partial derivatives and arrive at the Smarr relation, Eq.(2). Similarly, in the extended black hole thermodynamics in which the negative cosmological constant is treated as a thermodynamical variable (a pressure) \cite{4, 5}, the Smarr relation picks up an additional term \(-2(∂M/∂Λ)Λ = -2VP \), where \( V \) is the “thermodynamical volume”. This identification gives rise to an entire enterprise of “black hole chemistry” \cite{6}. Note that the coefficient \(-2 \) comes simply from the dimension of the cosmological constant: \( Λ \propto L^{-2} \) (which is true in all dimensions). This suggests that if one has a dimensionless thermodynamical variable, say \( N \propto L^0 \), it would not appear in the Smarr relation because the aforementioned rule of thumb would simply yield 0 \( \cdot (∂M/∂N) \).

In this work, we look into the subtleties of such seemingly straightforward statements, and found that contrary to folklore, it is possible for the coefficients that appear in the Smarr relation to differ from the scaling power of the corresponding thermodynamical variables. This is perhaps known to workers in the field, but it is worth emphasizing explicitly, since the counter-example discussed below leads to interesting questions regarding the identification of the thermodynamical variables. Before we get into the issue, it is useful to first review Euler’s theorem for quasi-homogeneous functions.

II. EULER’S THEOREM FOR QUASI-HOMOGENEOUS FUNCTIONS

Let \( \{χ^1, \cdots, χ^n\} \) be a set of real variables. Let \( \{κ_i\}_{i=1,\cdots,n} \) be a set of weights (\( κ_i \in \mathbb{R} \)). A function \( \mathcal{F}(χ^1, \cdots, χ^n) : \mathbb{R}^n \rightarrow \mathbb{R} \) is called a quasi-homogeneous equation of degree \( r \) if under re-scaling by a scale factor \( α > 0 \), one has

\[ \mathcal{F}(ακ_1χ^1, \cdots, ακ_nχ^n) = α^r \mathcal{F}(χ^1, \cdots, χ^n). \]  \hspace{1cm} (4)

A theorem by Euler states that a differentiable quasi-homogeneous function satisfies

\[ \sum_{i=1}^n κ_i χ^i \frac{∂\mathcal{F}}{∂χ^i} = r\mathcal{F}. \]  \hspace{1cm} (5)

In fact, this is both a sufficient and necessary condition for \( \mathcal{F} \) to be quasi-homogeneous \cite{7, 8}. If all the weights...
are equal to 1, then the function is said to be homoge-
nous.

To see how this leads to the Smarr relation, let us con-
sider a simple example: a 4-dimensional asymptotically
flat Reissner-Nordström black hole, whose metric coeffi-
cient

\[ -g_{tt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \]

yields

\[ M = \frac{r_+}{2} \left( 1 + \frac{Q^2}{r_+^2} \right), \]

where \( r_+ \) denotes the outer (event) horizon. Equivalently,
its ADM mass can be taken as a function of the thermo-
dynamical variables \( S = \pi r_+^2 \) and \( Q \):

\[ M(S, Q) = \frac{1}{2} \sqrt{\frac{S}{\pi}} \left( 1 + \frac{\pi Q^2}{S} \right). \]

We note that under re-scaling,

\[ M(\alpha^2 S, \alpha Q) = \frac{\alpha}{2} \sqrt{\frac{S}{\pi}} \left( 1 + \frac{\pi \alpha^2 Q^2}{\alpha^2 S} \right) = \alpha M(S, Q). \]

Therefore \( M \) is a quasi-homogeneous function with de-
gree \( r = 1 \) (i.e. \( M \) is homogeneous), and Euler’s theorem
gives the Smarr relation

\[ M = 2 \frac{\partial M}{\partial S} S + \frac{\partial M}{\partial Q} Q = 2TS + \Phi Q. \]

Note that for Smarr relation to hold, \( r \) should be a fixed
nonzero number in Eq.(4): there exist known black hole
solutions that do not satisfy the standard Smarr relation
precisely because this condition is not satisfied. For ex-
ample, a nonlinear electromagnetic field gave rise to a
charged black hole whose metric coefficient is given by

\[ -g_{tt} = 1 - \frac{2M}{r} \left[ \frac{Mr^3}{(r^2 + q^2)^2} - \frac{q^2 r^3}{2(r^2 + q^2)^2} \right], \]

where \( M \) and \( q \) are, respectively, the mass and charge of
the black hole. This is a charged version of the regular
Bardeen solution [9, 10], in fact the first regular exact
black hole solution known [11]. It is straightforward to
check that \( M = \alpha' M \) under the transformation \((A, q) \rightarrow (\alpha^2 A, \alpha^q) \) if and only if \( x = 2y \) and \( r = y \), for all \( y \),
including \( y = 0 \). Indeed, since \( A \) has dimension \( L^2 \) and
\( q \) has dimension \( L \), if the standard Smarr relation holds,
then one should have

\[ 2A \frac{\partial M}{\partial A} + q \frac{\partial M}{\partial q} = M. \]

However, explicit calculation shows that the LHS of
Eq.(12) is identically 0, which would be absurd since \( M \)
is the black hole mass. The Smarr relation can neverthe-
less be generalized to accommodate black holes coupled
with nonlinear electromagnetic fields [12–14].

Let us now proceed to discuss a peculiar case in which
the folklore rule does not work, but more interestingly,
it leads us to question which quantities are the correct
thermodynamical variables.

III. SMARR RELATION AND AXIONIC COUPLING

In [15], an asymptotically anti-de Sitter charged flat
black hole coupled with \( k \)-essence field was studied in
the context of holography. The action of the theory is the

\[ I = \int d^4x \sqrt{-g} \left[ \kappa (R - 2\Lambda) - \mathcal{K}(X_1, X_2) \right], \]

where \( \Lambda = -3/l^2 \) is the cosmological constant, and the
\( k \)-essence term

\[ \mathcal{K}(X_1, X_2) = \sum_{i=1}^{2} \left[ \frac{1}{2} \nabla^\mu \phi_i \nabla_\mu \phi_i + \gamma \left( \frac{1}{2} \nabla^\mu \phi_i \nabla_\mu \phi_i \right)^k \right] \]

(14)

is the Lagrangian of two axion fields \( \phi_i \), with \( X_i := (1/2)\nabla^\mu \phi_i \nabla_\mu \phi_i \), distributed homogeneously along the co-
ordinates of the planar horizon \( x_i \), specifically \( \phi_i = \lambda x_i \).
The action thus consists of a standard kinetic term and a
nonlinear one, with a coupling constant \( \gamma \), which is
dimensionful, usually taken to be positive to avoid phan-
том instability.

The metric component of the black hole reads (here we
ignore the magnetic charge for simplicity)

\[ -g_{tt} = \frac{r^2}{r^2} - \frac{2M}{r} - \frac{\lambda^2}{2k} + \gamma \frac{2k(2k - 3)}{4k^3} \frac{Q^2}{r^2}. \]

The physical mass and physical charge that satisfy the
first laws are: \( \mathcal{M} = 4\pi \sigma M, \mathcal{Q} = \sigma Q \). If the horizon is
compact (toral), then \( \sigma \) is the dimensionless area of the
horizon (analogous to 4r for a 2-sphere). In the planar
limit, both \( \sigma \) and the physical mass \( \mathcal{M} \) tend to infinity,
keeping the parameter \( M \) finite. Similarly for the elec-
trical charge. That is, \( M, \mathcal{Q} \) are the mass and charge
density parameters. The nonlinear power \( k \) should be
bounded below, \( k > 3/2 \), to ensure a proper AdS asymp-
totic behavior with a well-defined mass [15].

The authors of [15] emphasized the importance of treat-
ing \( \lambda \) as the axionic charge when investigating phase tran-
sitions, as well as in the holographic contexts. In fact the
linear term and the nonlinear terms in the \( k \)-essence La-
grangian \( \mathcal{K}(X_1, X_2) \) gave rise to two physical axionic
charges: \( \mathcal{Q}_i = -\sigma \lambda \) and \( \mathcal{Q}_{i,k} = -\sigma \gamma \lambda^k \).

We can write the physical mass in terms of other black
hole parameters:

\[ \mathcal{M}(\mathcal{A}, \mathcal{Q}, \mathcal{Q}_i, \mathcal{Q}_{i,k}) = 2\sqrt{\frac{\mathcal{A}}{3\sigma}} - \frac{\mathcal{Q}^2}{2\sigma^2} + \frac{\mathcal{Q}_{i,k}^2}{\sigma^2 \gamma (2k - 3)} \left( \frac{\mathcal{A}}{\sigma} \right)^{1-k} + \frac{\mathcal{Q}^2}{4\sigma \mathcal{A}}, \]

(16)

where \( \mathcal{A} = \sigma r^2 \) is the area of the event horizon.

Under the transformation

\((\mathcal{A}, \mathcal{Q}, \mathcal{Q}_i, \mathcal{Q}_{i,k}) \rightarrow (\alpha^2 \mathcal{A}, \alpha^2 \mathcal{Q}, \alpha^2 \mathcal{Q}_i, \alpha^2 \mathcal{Q}_{i,k})\),

one can check that \( \mathcal{M} \) is homogeneous, i.e.,

\[ \mathcal{M}(\alpha^2 \mathcal{A}, \alpha^2 \mathcal{Q}, \alpha^2 \mathcal{Q}_i, \alpha^2 \mathcal{Q}_{i,k}) = \alpha \mathcal{M}(\mathcal{A}, \mathcal{Q}, \mathcal{Q}_i, \mathcal{Q}_{i,k}), \]

where
if and only if $x = 0$ and $y = k - 1$. The fact that $x = 0$ is expected since $\mathcal{D}_i$ is dimensionless. However, $\mathcal{D}_{i,k}$ has the same dimension as $\gamma$, which from Eq.(15), can be obtained to be $L^{2k-2}$. That is, dim($\mathcal{D}_{i,k}$) = $L^{2k-2} \neq L^0$. Consequently, the Smarr relation has a term

$$(k - 1)\mathcal{D}_{i,k} \frac{\partial \mathcal{M}}{\partial \mathcal{D}_{i,k}} \subset \mathcal{M}. \tag{17}$$

If one naively writes down the Smarr relation following the folklore, one would have written down a wrong term

$$(2k - 2)\mathcal{D}_{i,k} \frac{\partial \mathcal{M}}{\partial \mathcal{D}_{i,k}} \tag{18}$$

instead, which is twice as large.

The reason for this is that a dimensionful coupling constant $\gamma$ appears in the same term as $\mathcal{D}_{i,k}$ in Eq.(16), but $\gamma$ itself is not a thermodynamical variable so it does not contribute to power of $a$ under the scaling transformation. It is not easy to know this a priori from the metric tensor Eq.(15) since Smarr relation requires one to first identify the correct physical quantities that appear in the first law of black hole mechanics (mistaking a mere mass parameter as the true physical mass do sometimes happen, which would lead to incorrect conclusion when dealing with physical processes [16]). Such a phenomenon does not occur in, say, AdS-Kerr black hole, despite its physical mass $E := M/(1 - a^2/l^2)$ involves two other dimensionful parameters: the rotation parameter $a$ and cosmological constant length scale $l$. Whether or not $a$ and $l$ are thermodynamical variables are irrelevant here, since their dimensions cancel in taking the ratio.

Since $\gamma$ only appears together with $\mathcal{D}_{i,k}$, an alternative interpretation would be to take $\gamma$ as the thermodynamical variable, instead of $\mathcal{D}_{i,k}$ [17]. We would then obtain a Smarr relation in which the “folklore method” does lead to the correct form

$$\mathcal{M} = 2TS + \Phi \mathcal{D} - 2PV + (2k - 2)\varphi \gamma, \tag{19}$$

with $\varphi := \partial \mathcal{M} / \partial \gamma$.

IV. DISCUSSION

In this short note we point out that the usual “folklore method”, which allows one to write down the Smarr relation of a black hole by merely inspecting the dimension of the thermodynamical variables, may fail if the terms contain dimensionful coupling constants (which are not thermodynamical variable). This means that one has to exercise extra caution in the presence of such constants. We provide a concrete example to illustrate this: a charged AdS flat black hole coupled to $k$-essence fields.

In this case it is nevertheless possible to “save” the folklore method by re-interpreting the coupling constant $\gamma$ as a thermodynamical variable, while treating one of the axionic charge, $\mathcal{D}_{i,k}$, as a non-thermodynamic parameter. In this interpretation, we can consider varying the coupling “constant”, much like varying the cosmological “constant” to produce the pressure term.

Since there are two axionic charges $\mathcal{D}_{i,k}$ and $\mathcal{D}_i$. If we treat $\mathcal{D}_{i,k}$ as a mere parameter, it makes sense that $\mathcal{D}_i$ should be treated on an equal footing. Since it is not accompanied by a coupling constant (more precisely, one may say that its coupling constant is unity, which unlike $\gamma$, cannot be promoted to be a parameter), this explains why it does not appear in the Smarr relation Eq.(19). However, note that both axionic charges appear in the first law of the axionic charged black hole (see [15], though the pressure term is not considered therein)

$$d\mathcal{M} = TdS + \Phi d\mathcal{D} + VdP + \sum_{i=1}^{2} \left[ \hat{\Psi}_i d\mathcal{D}_i + \hat{\Psi}_{i,k} d\mathcal{D}_{i,k} \right], \tag{20}$$

where $\hat{\Psi}_i := \partial \mathcal{M} / \partial \mathcal{D}_i$, and $\hat{\Psi}_{i,k} := \partial \mathcal{M} / \partial \mathcal{D}_{i,k}$. If we treat $\gamma$ as thermodynamical variable instead, the $\mathcal{D}_i$ term would be completely absent in the first law [17]:

$$d\mathcal{M} = TdS + \Phi d\mathcal{D} + VdP + \varphi d\gamma. \tag{21}$$

Of course, mathematically both interpretations are consistent. Physically however, if we treat $\gamma$ as the thermodynamical variable, one must ask why the axionic charges are not on the same footing as the electric charge $\mathcal{D}$ and the mass $\mathcal{M}$ (the gravitational charge). One peculiarity is notable: $Q_i$ is essentially $\lambda$, and $Q_{i,k}$ is essentially $\lambda^k$ (upto some multiplicative factors), but $\lambda$ is the same dilation parameter that occurs in the definition of the axion fields $\phi_i := \lambda x_i$. It is interesting that the same parameter gives rise to two distinct axionic charges, which are independently treated as thermodynamical variables. Perhaps this somehow justifies why we should not treat it on the same footing as $\mathcal{M}$ and $\mathcal{D}$; but maybe this is just a red herring.

Does the validity of the “folklore method” actually give us more insight into the physics (of which parameters are better chosen as the “right” thermodynamical variables)? We leave this puzzle open for future investigations.

Acknowledgments

XMK is supported by the Natural Science Foundation of China (grant No.11705161) and Natural Science Foundation of Jiangsu Province (grant No.BK20170481). YCO is supported by the National Natural Science Foundation of China (grant No.11705162) and the Natural Science Foundation of Jiangsu Province (No.BK20170479). YCO thanks Nordita, where part of this work was carried out, for hospitality during his summer visit and participation in the Lambda program. The authors also thank all members of Center for Gravitation and Cosmology (CGC) of Yangzhou University (http://www.cgc-yzu.cn) for discussions and various supports.
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