Stochastic Dynamics of a Vortex Loop.
Thermal Equilibrium

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Abstract. We study stochastic behavior of a single vortex loop appeared in imperfect Bose gas. Dynamics of Bose-condensate is supposed to obey Gross-Pitaevskii equation with additional noise satisfying fluctuation-dissipation relation. The corresponding Fokker-Planck equation for probability functional has a solution

\[ P(\{\psi(r)\}) = N \exp(-H(\{\psi(r)\})/T), \]

where \( H(\{\psi(r)\}) \) is a Ginzburg-Landau free energy. Considering a vortex filaments as a topological defects of the field \( \psi(r) \) we derive a Langevin-type equation of motion of the line with correspondingly transformed stirring force. The respective Fokker-Planck equation for probability functional \( P(\{s(\xi)\}) \) in vortex loop configuration space is shown to have a solution of the form

\[ P(\{s(\xi)\}) = N \exp(-H(\{s\})/T), \]

where \( N \) is a normalizing factor and \( H(\{s\}) \) is energy of vortex line configurations. In other words a thermal equilibrium of Bose-condensate results in a thermal equilibrium of vortex loops appeared in Bose-condensate. Some consequences of that fact and possible violations are discussed.

1 Introduction and scientific background

Quantized vortices appeared in quantum fluids have been an object of intensive study for many years (see for review and bibliography the famous book by Donnelly[1]). The greatest success in investigations of dynamics of quantized vortices has been achieved in relatively simple cases such as vortices in rotating helium (where they form a vortex array orientated along an axis of rotation) or vortex rings. However these simple cases are rather exception than a rule. Due to extremely involved dynamics initially straight lines or rings evolve to form highly entangled chaotic structure. Thus a necessity of use of statistic methods to describe chaotic vortex loop configurations arises. A most tempting way is to treat "gas" vortices as a kind of excitation and to use thermodynamic methods. One of first examples of that way was an use of the Landau criterium for critical velocity where vortex energy and momentum were applied to relation having pure thermodynamic sense. More extended examples would be the famous Kosterlitz-Thouless description of 2D vortices or its 3D variant intensively elaborated currently (for review and bibliography see e.g.[2]).

In the examples above and in many other it is assumed that chaotic vortex loop configurations are in a thermal equilibrium and their statistics obeys the Gibbs distribution. That supposition is based on fundamental physical principles and can be justified in a standard way considering vortex lops as a subsystem submerged into a thermostat and exchanging energy with the latter. The role of
thermostat in the case of vortices is played by the other excitations (phonons and rotons) of an underlying physical field. In the case of Bose-Einstein condensate (BEC), which we consider in this paper, that field is an order parameter $\psi(r,t)$. Vortex lines are just the crossings of surfaces where both real and imaginary part of $\psi(r,t)$ vanish. Excitations of order parameter (phonons and rotons) interact with vortices driving the latter to statistic distribution which in accordance with general principles should be the Gibbs distribution.

It is well known however that the Gibbs distribution can be alternatively obtained in the frame of some reduced model like kinetic equations or Fokker-Planck equation. That way of course is not of such great generality as a principle of maximum entropy, but instead it allows us to clarify the mechanisms of how the Gibbs distribution is established and to discuss possible deviations and violations of equilibrium state. We choose that way to examine how a thermal equilibrium of chaotic vortex loops follows from a thermal equilibrium of BEC.

2 Langevin equation

We perform our consideration on the basis of the Gross-Pitaevskii model\[3],[4]. To describe stochastic behavior of BEC let us add to the right-hand side of the Gross-Pitaevskii equation the random stirring force $\zeta(x,t)$.

$$\frac{\hbar}{m} \frac{\partial \psi}{\partial t} = -(\Lambda + i) \frac{\delta H(\psi)}{\delta \psi^*} + \zeta(x,t). \tag{1}$$

Here $H\{\psi\}$ is the Ginzburg-Landau free energy functional

$$H\{\psi\} = \int d^3x \left[ \frac{\hbar^2}{2m^2} |\nabla \psi|^2 - \frac{\mu}{m} |\psi|^2 + \frac{V_0}{2m} |\psi|^4 \right]. \tag{2}$$

The thermal noise obeys the following fluctuation-dissipation theorem

$$\langle \zeta(x_1,t_1) \zeta^*(x_2,t_2) \rangle = \frac{2k_B T \Lambda}{(m/\hbar)} \delta(x_1 - x_2) \delta(t_1 - t_2). \tag{3}$$

The stochastic problem introduced by (1)-(3) has a solution describing thermal equilibrium, where probability of some configuration of $\psi$-filed is proportional to $\exp(-H\{\psi(r)\}/T)$. This readily follows from the correspondent Fokker-Planck equation\[5].

Let’s go on to the problem of obtaining of the vortex dynamics appearing in BEC, whose own dynamics obeys relations (1)-(3). That problem, as well as more general problems of that kind has been considered many times by many methods\[6] starting from pioneering papers by Pitaevskii\[7] and Fetter\[8]. We develop one more method, neither too rigorous nor principally new but convenient for purpose of this paper. Details of that method will be published elsewhere, here we briefly describe it to an extent to comprehend the put goal.

As any motion of topological defects the vortex line dynamics is determined by the one of the underlying field theory. On the other hand if one ignores a presence of other excitations except of the vortices one can say that all motion of
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BEC is determined by the one of quantum vortices. Thus there is mutual correspondence and the order parameter $\psi(x, t)$ can be considered as some functional of a whole vortex loop configuration $\psi(x | \{s(\xi, t)\})$. Temporal dependance of field $\psi$ is connected with motion of lines and its rate of change (at some point $x$) is expressed by the following chain rule:

$$\frac{\partial \psi(x, t)}{\partial t} = \int \frac{\delta \psi(x | \{s(\xi, t)\})}{\delta s(\xi', t)} \frac{\partial s(\xi', t)}{\partial t} d\xi'. \tag{4}$$

Vortices in quantum fluids are very slender tubes (except of vicinity of phase-transition) and their dynamics is analogous to the one of the strings. That implies that we have to aim our efforts at integrating out radial degrees of freedom. It can be reached by the following procedure. Let us further multiply equation(1) by $\frac{1}{A^2 + 1} \frac{\delta \psi^*}{\delta s(\xi_0, t)}$, where $\xi_0$ some chosen point on the curve. Combining the result obtained with the complex conjugate and integrating over whole space we have

$$\frac{\hbar}{m} \int d^3x \left( \frac{\Delta - i \frac{\partial \psi^*}{\partial s(\xi_0, t)}}{\Delta + 1 \frac{\partial s(\xi_0, t)}{\partial t}} + \frac{\Delta + i \frac{\partial \psi}{\partial s(\xi_0, t)}}{\Delta + 1 \frac{\partial s(\xi_0, t)}{\partial t}} \right) = \int d^3x \left( \frac{\delta H(\psi, \psi^*)}{\delta s(\xi_0, t)} \frac{\delta \psi^*}{\delta s(\xi_0, t)} + \frac{\delta H(\psi, \psi^*)}{\delta s(\xi_0, t)} \frac{\delta \psi}{\delta s(\xi_0, t)} \right) + \int d^3x \left( \frac{\Delta - i}{\Delta + 1} \frac{\delta \psi^*}{\delta s(\xi_0, t)} + \frac{A + i}{\Delta + 1} \frac{\delta \psi}{\delta s(\xi_0, t)} \right). \tag{5}$$

The first integral in the right-hand side of (5) expresses a chain rule for functional derivative $\frac{\delta H(s)}{\delta s(\xi_0, t)}$ where $H(s)$ is the energy of moving BEC expressed via vortex line position. Consequently considering them to be very slender tubes (which is justified when the radius of the curvature $R$ is much larger of the core size $r_0$) and neglecting an energy associated with the core, the quantity $H(s)$ is just kinetic energy of the superfluid flow created by vortices (see e.g. [9],[10])

$$H(s) = \frac{\rho_s \kappa}{8\pi} \int \int \frac{s'(\xi) s'(\xi')}{|s(\xi) - s(\xi')|} d\xi d\xi'. \tag{6}$$

Here $s'(\xi)$ is tangent vector, double integration is performed along a whole line, $\kappa$ is a quantum of circulation equal $2\pi\hbar/m$. Calculation of functional derivative $\frac{\delta H(s)}{\delta s(\xi_0, t)}$ is straightforward and leads to result

$$\frac{\delta H(s)}{\delta s(\xi_0, t)} = \rho_s \kappa s'(\xi_0) \times B(\xi_0). \tag{7}$$

Quantity $B(\xi_0)$ is the velocity of the line element $s'(\xi_0)$ expressed by well known Biot-Savart law. The terms in the left-hand side of equation (5) (first line) can be evaluated in general form by observing that the major contribution into integrals appears from vicinity of the vortex filament (see e.g.[11]). Thus to
evaluate integral we replace $\psi(x \mid s(\xi, t))$ by $\psi_v(x_{\perp}) = \psi_v(s(\xi_{cl}, t) - x)$ where $\psi_v$ is well studied 2D vortex and integration over $d^3x$ by $d^2x_{\perp} d\xi_{cl}$. Functional derivative $\delta \psi^*/\delta (\xi', t)$ should be evaluated by a following rule: $\delta \psi^*/\delta s(\xi', t) = \nabla_{\perp} \psi_v(x_{\perp}) \delta (\xi' - \xi_{cl})$. Here $\xi_{cl}$ is the label of point of the line closest to point $x$. Using the said above and calculating integrals of squared gradients of $\psi_v(x_{\perp})$ we conclude that left-hand side of equation (5) transforms into

$$\frac{\hbar}{m} \frac{2 \pi \rho_s}{A^2 + 1} \dot{s} (\xi_0) \times s'(\xi_0) + \frac{\hbar}{m} \frac{2 \pi \rho_s \sigma A}{A^2 + 1} \dot{s} (\xi_0).$$  \tag{8}$$

Let us now discuss the rest terms of equation (5) including random force $\zeta(x, t)$ (the third line). Consequently considering that the all motion of BEC is connected to motion of line, we have to consider Langevin force $\zeta(x, t)$ as some secondary quantity stemming from random displacements of filaments. Connection between displacements (random) of filaments $\delta s$ and deviations (random) of $\delta \psi(x, t)$ may be written in form similar to (4) with formal substitution $\partial \psi(x, t)/\partial t \rightarrow \delta \psi(x, t)$ and $s \rightarrow \delta s$. Taking into account that $\delta \psi(x, t) = \zeta(x, t) \delta t$, and $\delta s = \zeta(\xi, t) \delta t$ we conclude that quantities $\zeta(x, t)$ and $\zeta(\xi, t)$ are connected to each other by a chain rule

$$\zeta(x, t) = \int r \frac{\delta \psi(x \mid \{s(\xi, t)\})}{\delta s(\xi', t)} \zeta(\xi', t) d\xi'.$$  \tag{9}$$

That implies that to take into consideration random displacements of line we have to change the last term in equation (5) by the one similar to (8) with substitution $s (\xi_0) \rightarrow \zeta(\xi_0, t)$. Gathering all terms we obtain a vector equation, which can be resolved up to tangential velocity $s_{\perp} (\xi_0)$ along the curve. The latter does not have any physical meaning and can be removed by suitable parameterization of the label variable $\xi$. Solving that vector equation we arrive at

$$\dot{s} (\xi_0) = \frac{1 + A^2}{1 + A^2 \sigma^2} B(\xi_0) + \frac{(1 + A^2) \Lambda \sigma}{1 + A^2 \sigma^2} s'(\xi_0) \times B(\xi_0) + (m/\hbar) \zeta(\xi_0, t).$$  \tag{10}$$

Equation (10) describes motion of vortex line in terms of the line itself. It is remarkable fact (not obvious in advance that noise $\zeta(\xi_0, t)$ acting on line is also additive (does not depend on line variables).

The last effort we have to do is to ascertain both the statistic properties of noise $\zeta(\xi_0, t)$ and its intensity. Shortly, it can be done by comparison of equation (8) with substitution $s (\xi_0) \rightarrow \zeta(\xi_0, t)$ with the last term of equation (5). Clearly the former appeared as result of transformation of the latter. Equating them and taking the scalar productions of both parts of the resulting relation we arrive at

$$\langle \eta_1 (\xi_1, t_1) \eta_2 (\xi_2, t_2) \rangle = \frac{k_B T}{\rho_s \pi (\hbar/\rho) (A^2 + 1) \Lambda \sigma} \frac{1}{1 + A^2 \sigma^2} \delta(\xi_1 - \xi_2) \delta(t_1 - t_2) \delta_{\eta_1, \eta_2}. \tag{11}$$
Here $\zeta_{\eta_1}$ and $\zeta_{\eta_2}$ are components of random velocities in $\eta_1, \eta_2$ directions lying in the plain normal to the line.

Thus starting from dynamics of BEC (equation (1)) with the fluctuation-dissipation theorem (2) we derive equation (10) describing motion of vortex line in terms of line itself with the additive noise obeying the fluctuation-dissipation theorem (11). These relations complete a stochastic problem of quantized vortex dynamics under thermal noise stemming from the one stirring the underlying field. In the next section we demonstrate that this problem has an equilibrium solution given by Gibbs distribution $\exp(-H\{s\}/k_B T)$, where $H\{s\}$ - functional of energy due to vortex loop (equation (6)) and $T$ - temperature of Bose-condensate.

3 Fokker-Planck equation

To show it we, first, derive the Fokker-Planck equation corresponding to Langevin type dynamics obeyed (10) and (11). Let us introduce probability distribution functional (PDF)

$$P(\{s(\xi), t\}) = \langle \delta(s(\xi) - s(\xi, t)) \rangle.$$

Here $\delta$ is delta functional in space of vortex loop configurations. Averaging is fulfilled over ensemble of random force. The Fokker-Planck equation can be derived in standard way (see e.g. [12])

$$\frac{\partial P}{\partial t} + \int d\xi \frac{\delta}{\delta s(\xi)} \left[ \frac{1 + A^2}{1 + A^2 \sigma^2} B(\xi) + \frac{(1 + A^2)A \sigma}{1 + A^2 \sigma^2} s'(\xi) \times B(\xi) \right] P + \int \int d\xi d\xi' \frac{k_B T}{2\pi \rho s(h/m)^2} \frac{(A^2 + 1) A \sigma}{1 + A^2 \sigma^2} \delta(\xi - \xi') \delta(t_1 - t_2) \delta_{\eta_1 \eta_2} \frac{\delta}{\delta s(\xi)} \frac{\delta}{\delta s(\xi')} P = 0$$

Equation (13) possesses the equilibrium solution in form of the Gibbs distribution $P(\{s(\xi)\}) = N \exp(-H\{s\}/k_B T)$, where $N$ is a normalizing factor. Let us show that the first integral term vanishes identically for that solution. To do it we exploit relation (7) and parametrization of label variable $\xi$ in which velocity $\cdot s(\xi_0)$ is normal to the line. Using a tensor notation we rewrite the first term in integrand in form (we omit the coefficient and factor $\exp(-H\{s\}/k_B T)$)

$$\epsilon_{\alpha \beta \gamma} \left\{ \frac{\delta s'_{\beta}(\xi)}{\delta s_{\alpha}(\xi)} \frac{\delta H(s)}{\delta s_{\alpha}(\xi, t)} + s'_{\beta}(\xi) \frac{\delta^2 H(s)}{\delta s_{\alpha}(\xi, t) \delta s_{\alpha}(\xi)} + s'_{\beta}(\xi) \frac{\delta H(s)}{\delta s_{\alpha}(\xi, t)} \frac{\delta H(s)}{\delta s_{\alpha}(\xi, t)} \right\}$$

The functional derivative $\delta s'_{\beta}(\xi)/\delta s_{\alpha}(\xi) \propto \delta_{\beta \alpha}$ therefore all terms vanish due to symmetry. Thus the reversible term gives no contribution to flux of probability (in the configuration space) equation (13), one says it is divergence free. Furthermore exploiting again relation (7) one convinces himself that second (dissipative) term in (13) and third (due to stirring force) term exactly compensate each other (locally) as it should be in the thermal equilibrium.
4 Possible violation of thermal equilibrium.

Thus we have proved that the thermal equilibrium of BEC results in the thermal equilibrium of vortex loop. We are now in position to discuss how it can be destroyed. Analyzing the proof one can see that the following steps were crucial.

1. Additive white noise $\zeta(x, t)$ acting on field $\psi(x, t)$ is transformed into additive white noise $f(\xi, t)$ acting on vortex line position $s(\xi, t)$. 2. Intensity of noise $f(\xi, t)$ expressed by (11) is that it locally compensates dissipative flux of probability distribution functional in the Fokker-Planck equation (13).

That observation points out how the thermal equilibrium in space of vortex loops can be destroyed for real vortex tangles appeared e.g. in counterflowing HeII or formed in quenched superfluids. Being a macroscopical objects vortex loops inevitably undergo large scale perturbations generated e.g. by nonuniform flow or by action of other vortex loops, randomly placed with respect to the studied loop. One more essential source of large scale perturbations might be long wave instabilities of vortex filament motion. That type of random action drastically differs from $\delta$-correlated in $\xi$ space thermal noise considered above. It obviously cannot compensate dissipative flux of probability, which is proportional curvature and acts accordingly in small scales. Instead the following scenario seems to be realized [13]. Due to nonlinear character of the equation of motion the large-scale perturbations on an initially smooth filament interact creating higher harmonics. They in turn generate harmonics with larger $\kappa$, where $\kappa$ is one-dimensional wave vector arisen in 1D Fourier transform (with respect to label variable $\xi$) of quantity $s(\xi, t)$. One can say that an additional curvature created by large scale stirring force propagates in region of small scales. In real space that corresponds to entangling of vortex loop and creation of vortex line segments with large local curvature. Then dissipative processes come into play, their role now is that they remove from the system harmonics with very large wave vectors $\kappa$. So the Kolmogorov cascade-like solution with a flux of a curvature in space of $\kappa$ is established and, as a result, the stochastic distribution is far from equilibrium. That scenario is quite similar to the one which is realized in classical turbulence or in the so called weak (wave) turbulence. A difference is that if in case of the wave and classical turbulence) an exchange an energy between harmonics is realized, whereas in our case there is exchange a curvature $\kappa^4 s(\kappa) s(-\kappa)$.

The scenario described above is especially relevant in the so called low-temperature superfluid turbulence case, when the normal component is very small and the usual (considered here) dissipation monotonically increasing with the curvature is absent. Some recent experiments and numerical simulations [14] show that vortex tangle in HeII decays at extremely low temperature (about 1 mK), where dissipation due to normal component is negligibly small. Obviously some other strong mechanisms of dissipation must take place. It can be e.g. emission of phonons and rotons from speedy moving parts of line or just collapse of ”hairpin” segments of the filament. That mechanisms are concentrated on very small scales (of order of the vortex core size) or equivalently in region of very large wave numbers $\kappa$. Therefore regions of the pumping and the sink
of an additional curvature are greatly remote in \(\kappa\) space and a system must be essentially nonequilibrium.

One more reason of a violation of thermal equilibrium might be a reconnection of lines. In real vortex tangle consisting of many loops the vortex filaments undergo frequent collisions and reconnections. Just after reconnection there appear kinks on the curves disappearing later on. From mathematical point of view a kink on the curve can be described as a discontinuity of tangent vector \(s'(\xi, t)\) which has Fourier transform of type \(s'(\kappa) \propto \kappa^{-1}\). Thus the reconnection processes supply the selected curve with discontinuities having a spectrum \(s(\kappa) \propto \kappa^{-2}\). Taking into account a random nature of vortex line collision the reconnection processes can be modelled, in some measure as a random stirring of filament with spectrum of type \(\langle \zeta(\kappa)\zeta(-\kappa) \rangle \propto \kappa^{-4}\). Remember now that the establishing of a thermal equilibrium requires that the random force correlation function is proportional \(\propto \delta(\xi_1 - \xi_2)\) and, consequently the spectrum does not depend on wave number \(\kappa\), \(\langle \zeta(\kappa)\zeta(-\kappa) \rangle = \text{const} \). Therefore the colored noise \(\langle \zeta(\kappa)\zeta(-\kappa) \rangle \propto \kappa^{-4}\) coming from reconnection processes can also lead to the Kolmogorov type nonequilibrium state. To clarify which of mechanisms forming nonequilibrium state prevails in real vortex tangle one has to investigate much more involved problem.

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