RIGOROUS UNDERSTANDING OF THE LINEAR RESPONSE OF PLASMAS IN THE FRAMEWORK OF TEMPERED DISTRIBUTION THEORY

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Abstract. For the linearized Vlasov equation, we establish a mathematically rigorous characterization of the specific (unique) solution which physically describes the response of a hot plasma to an external small-amplitude electromagnetic disturbance. This allows us to define the plasma conductivity operator \( \sigma \) which gives the current density \( \mathbf{J} = \sigma \mathbf{E} \) induced in the plasma by the electric field \( \mathbf{E} \) of the disturbance. With this aim we study first a simple model problem with damping and explain the relationship between the general solutions of the Cauchy problem with given data and the tempered solution which is unique. We apply them to the specific case of a hot uniform non-magnetized non-relativistic plasma (linear Landau damping).

1. Introduction

A plasma is a collection of a sufficiently large number of electrically charged particles of various species (electrons, protons, and ions of different elements), subject to electromagnetic fields. In kinetic theory, the configuration of a plasma is specified by a family of functions \( F_s : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}_+ \), labeled by the index of particle species \( s \in S \) and defined so that \( F_s(t, x, p) \) gives the density of particles of the species \( s \) at the time \( t \), position \( x \) and relativistic momentum \( p \).

We consider a given stationary configuration \( \{ F_{0,s}(x, p) \} \in S \) of the plasma in presence of a prescribed magnetic field \( B_0(x) \). Physically, the application of a small-amplitude electromagnetic perturbation induces a change the distribution functions \( F_s \), which represents the response of the plasma to the imposed electromagnetic disturbance. One expects that the response of the plasma is uniquely determined by and depends continuously on the imposed perturbation. A precise mathematical analysis of the response of a plasma is important since that is the basis for the construction of constitutive relations for plasma waves.

If \( E \) and \( B \) are the electric and magnetic field of the imposed perturbation, and \( F_s = F_{0,s} + f_s \) is the perturbed distribution function for the particle species \( s \in S \), the dynamics of a small perturbation \( f_s \) is, in general, governed by the linearized relativistic Vlasov equation [1, 3, 4, 8],

\[
\frac{\partial f_s}{\partial t} + v(p) \cdot \nabla_x f_s + q_s \left( \frac{v(p)}{c} \times B_0 \right) \cdot \nabla_p f_s = -q_s \left( E + \frac{v(p)}{c} \times B \right) \cdot \nabla_p F_{0,s},
\]

where \( v(p) = p/m_s \gamma(p) \) is the relativistic velocity, \( \gamma(p) = \left( 1 + p^2/m_s^2 c^2 \right)^{1/2} \), \( c \) is the speed of light, \( m_s \) is the mass of particles of the species \( s \) and \( q_s \) their electric charge (c.g.s. units are used throughout the paper).
The corresponding perturbation in the electric current density induced by the electromagnetic disturbance reads

\[
j(t, x) = \sum_{s \in S} q_s \int_{\mathbb{R}^3} v f_s(t, x, v) dv.
\]

Since \( f_s \) depends linearly on the electromagnetic field \((E, B)\) and the magnetic field depends linearly on the electric field via the Faraday law,

\[
\partial_t B + c \nabla \times E = 0,
\]

we expect that the induced current density \( j \) can be expressed as a linear operator acting on the electric-field perturbation, namely,

\[
j = \sigma(E).
\]

This is referred to as the linear constitutive relation of the plasma and the operator \( \sigma \) is the conductivity operator, which fully describes the linear plasma response (the induced charge density is related to the induced current density \( j \) by the charge continuity equation.)

One should note however that, for the operator \( \sigma \) to be well-defined, the perturbation \( f_s(t, x, p) \) of the distribution function and the perturbation \( B(t, x) \) of the magnetic field should be uniquely determined, given an electric field disturbance \( E(t, x) \). A quick inspection of equation (1) reveals that this is not the case since the first-order partial differential operator on the left-hand side has a non-trivial null space (as it is the case for any Cauchy problem). Analogously the magnetic field is not uniquely determined by the Faraday law (3): If \( B(t, x) \) is a solution, then so is \( B(t, x) + B_1(x) \), with \( B_1 \) depending only on the position \( x \).

The linearized Vlasov equation (1) alone does not determine a unique solution unless a Cauchy datum is given at a point \( t_0 \). In fact, the null space of the operator

\[
\mathcal{L} = \partial_t + v(p) \cdot \nabla_x + q_s \left( \frac{v(p)}{c} \times B_0 \right) \cdot \nabla_p,
\]

comprises all functions that are constant along the characteristics \((t, x(t), p(t))\),

\[
\frac{dx}{dt} = v(p), \quad \frac{dp}{dt} = q_s \frac{p \times B_0(x)}{m_s \gamma c}.
\]

Physically, the latter system of ordinary differential equations describes the motion of plasma particles of the species \( s \) subject only to the background magnetic field (to be referred to as unperturbed orbits).

The Cauchy problem addressed above is well posed and it can be solved by integration along the characteristics. This solution, however, mixes the response of the plasma with the evolution of the initial condition. Being interested in the former only, the initial time \( t_0 \) in the physics literature is pushed to \( t_0 \to -\infty \) where both the initial condition for \( f_s \) and the electromagnetic perturbation are assumed to vanish. As a result, one obtains the formal solution

\[
f_s(t, x, p) = \int_{-\infty}^{t} h_s(s, X_s(s, t, x, p), P_s(s, t, x, p)) ds,
\]

where \( h_s(t, x, p) \) is the function on the right-hand side of equation (1) and the curve \( s \mapsto (X_s(s, t, x, p), P_s(s, t, x, p)) \) is the solution of the equations for the characteristics subject to the final conditions \( X_s(t, t, x, p) = x, \ P_s(t, t, x, p) = p. \)
Generally non-uniqueness of solutions of the linearized Vlasov equation, due to the non-trivial kernel of the linearized Vlasov operator, as well as the convergence of the integral in (5) are not discussed in the physics literature.

Nevertheless, it is very classical in physics to speak, as the title of this paper mentions, of the 'plasma response', implying uniqueness, hence selecting the only relevant solution (5) and its associated Cauchy datum.

This paper is a first step toward a rigorous mathematical characterization of the unique relevant solution of the linearized Vlasov equation which describes the physical response of the plasma. It is called in the sequel the causal solution (in agreement with the physics terminology) and it can be understood through the limit absorption principle. Specifically, the idea is to prove that the linearized Vlasov equation with the addition of a constant damping term has a unique solution in the space $S'$ of tempered distributions and this unique solution approaches the causal solution in $S'$ when dissipation parameter goes to zero.

We complete this program in detail and compute the conductivity operator in the case of a homogeneous non-magnetized plasma in one spatial dimension (which is the standard paradigm for linear Landau damping), and prove that the operator $\sigma$ amounts to the sum of a Fourier multiplier homogeneous of degree zero away from $k = 0$, $(\omega, k)$ being the Fourier variables (frequency and wave vector), and of a remainder operator localized in a neighborhood of $k = 0$. The Fourier multiplier agrees with the known physics result.

Although limited to a simple plasma equilibrium, these result supports the physics works that rely on the pseudo-differential form of the conductivity operator [7, 5, 6, 2]. More precisely, even if $\sigma$ is rigorously not a pseudo-differential, it is close enough if the spectrum of the electromagnetic disturbance vanishes for $k = 0$.

Partial results (not included in this paper) indicate that the foregoing conclusion applies to the case of a relativistic plasma, both with and without equilibrium magnetic field, in dimension $d = 3$.

For the case of an homogeneous non-magnetized plasma, we consider the standard Landau damping problem in the non-relativistic one-dimensional case, for which we obtain the causal solution in the following theorem, cf. section 2 for notations and precise definitions. Let $d = 1$, $B_0 = 0$, $\gamma(v) = 1$, and let the background distribution be constant in space, i.e., $F_{0,s} = F_{0,s}(v)$, with $F_{0,s} \in S(R)$. Given $E = E(t, x)$ in $S(R^2)$ and $\nu > 0$, we consider the equation

\begin{equation}
L_\nu f_{s,\nu} := \partial_t f_{s,\nu} + \nu f_{s,\nu} + v\partial_x f_{s,\nu} = -\frac{q_s}{m_s} EF'_{0,s}(v),
\end{equation}

together with the functions

\begin{equation}
f_{s,\nu}(t, x, v) := -(q_s/m_s)F'_{s,0}(v) \int_{-\infty}^{t} e^{\nu(s-t)}E(s, x - v \cdot (t - s))ds,
\end{equation}

and

\begin{equation}
f(s, t, x, v) := -(q_s/m_s)F'_{s,0}(v) \int_{-\infty}^{t} E(s, x - v \cdot (t - s))ds.
\end{equation}

Equation (5) with $f_{s,\nu}$ gives the current density

\begin{equation}
\begin{split}
j_\nu(t, x) := \sigma_\nu(E)(t, x) = -\sum_s \frac{q_s^2}{m_s} \int_{\mathbb{R}} vF'_{s,0}(v) \int_{-\infty}^{t} e^{\nu(s-t)}E(s, x - v \cdot (t - s))dsv,
\end{split}
\end{equation}
which defines the conductivity operator with damping \( \sigma_v \). Let \( \chi \in C_0^\infty(\mathbb{R}) \) be an arbitrary cut-off function with \( \chi(0) = 1 \) and we also introduce the following Fourier multiplier

\[
(9) \quad \mathcal{F}(\sigma_{1-\chi}(E))(\omega, k) = (1 - \chi(k)) \hat{\sigma}_{\text{ph}}(\omega, k) \hat{E}(\omega, k),
\]

where \( \mathcal{F} \) denotes the Fourier transform, \( \hat{\sigma}_{\text{ph}} \) is the physical conductivity tensor (explicit form given in proposition 4.5 below).

**Theorem 1.1.** The functions and operators defined above have the following properties:

(i) For all \( E \in \mathcal{S}(\mathbb{R}^2) \), \( f_{\nu} \) belongs to \( \mathcal{S}(\mathbb{R}^3) \) and is the unique solution in \( \mathcal{S}' \) of equation (6). Moreover, for \( \nu \to 0^+ \), \( f_{\nu} \to f_s \) in the topology of \( \mathcal{S}' \), with \( f_s \in C_b^\infty(\mathbb{R}^3) \). Notice that \( f_s(t, x, \cdot) \in \mathcal{S}(\mathbb{R}) \) for every \((t, x) \in \mathbb{R}^2 \) and is a classical solution of equation (6) with \( \nu = 0 \).

(ii) The operator \( \sigma_v \) is a continuous linear operator from \( \mathcal{S}(\mathbb{R}^2) \) to \( \mathcal{S}(\mathbb{R}^2) \) for all \( \nu > 0 \). When \( \nu \to 0^+ \), \( j_{\nu} \to j = \sigma(E) \) in \( \mathcal{S}' \) and \( j \) belongs to \( C_b^\infty(\mathbb{R}^2) \). The limits \( j \) and \( f_s \) satisfy equation (3).

(iii) For any \( E \in \mathcal{S}(\mathbb{R}^2) \) such that \( E(\omega, k) = 0 \) in \( \text{supp} \chi \), \( \sigma(E) = \sigma_{1-\chi}(E) \).

**Remark 1.** The last item of theorem 1.1 shows that the operator \( \sigma_{\chi} := \sigma - \sigma_{1-\chi} \) is well defined and is zero when the Fourier transform in \( x \) of \( E \) is 0 for small values of \( k \). An expression of \( \sigma_{\chi} \) is also available (see proposition 4.3).

In section 2 we set the notations and preliminary definitions. In section 3, a general characterization of the plasma response is given in a simple case study. The rest of the paper is dedicated to the proof of theorem 1.1.

### 2. Notation and Basic Definitions

We shall work with tempered distributions on the Euclidean space-time \( \mathbb{R}^{1+d} \). Let us denote by \( \mathcal{S}(\mathbb{R}^d) \) the Schwartz space of rapidly decreasing functions on \( \mathbb{R}^d \). Those are functions \( \varphi \in C^\infty(\mathbb{R}^d) \) for which

\[
\sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta \varphi(x)| < \infty.
\]

Semi-norms in \( \mathcal{S} \) are defined by

\[
||\varphi||_k = \max_{|\alpha| + |\beta| \leq k} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial_x^\beta \varphi(x)|, \quad k \in \mathbb{N}.
\]

With those semi-norms, the Schwartz space is a Fréchet space and its topological dual \( \mathcal{S}'(\mathbb{R}^d) \) is the space of tempered distributions, namely, the space of continuous linear functionals : \( \mathcal{S}(\mathbb{R}^d) \to \mathbb{C} \). In this case continuity of a linear functional \( u \) means that there exists an integer \( k > 0 \) and a constant \( C_k > 0 \) such that \( u(\varphi) \leq C_k ||\varphi||_k \).

For the Fourier transform of a function \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) we write

\[
\hat{\varphi}(\eta) = \int_{\mathbb{R}^d} \varphi(y)e^{-iy \cdot \eta} \, dy.
\]

The Fourier transform \( \mathcal{F} : \varphi \mapsto \hat{\varphi} \) is continuous from \( \mathcal{S} \) into itself and extends to \( \mathcal{S}' \) by duality. Specifically, this means

\[
\hat{u}(\varphi) = u(\hat{\varphi}).
\]
That $\hat{u}$ is a continuous linear functional follows from the continuity of $u$ and of the Fourier transform $\varphi \to \hat{\varphi}$.

We shall also need to draw the connection to physics results: for the Fourier transform in time and space of a function $\varphi = \varphi(t, x)$ in $S(R^{1+d})$, we also adopt

$$\hat{\varphi}(\omega, k) = \int_{R^4} \varphi(t, x)e^{i\omega t - ik \cdot x}dt dx,$$

where, $(t, x)$ are physical coordinates in the Euclidean space time, $\omega$ is the angular frequency and $k$ is the wave vector. The standard form is recovered upon introducing normalized coordinates $(y, \eta) \in R^{1+d} \times R^{1+d}$ with $y = (t/T, x/L)$ and $\eta = (-\omega T, kL)$, where $T$ and $L$ are normalization scales. With some abuse of notation, we write $\varphi(y) = \varphi(t, x)$ and $\hat{\varphi}(\eta) = \hat{\varphi}(\omega, k)$, thus implying the normalization and the sign change in $\omega$.

Our main results require a few basic definitions on Fourier multipliers.

**Definition 1** (Fourier multipliers). A tempered distribution $a \in S'(R^d \times R^d)$ defines a continuous operator $A : S(R^d) \to S'(R^d)$ by,

$$\langle Au, \varphi \rangle = (2\pi)^{-d} \int e^{iy \cdot \eta} a(y, \eta) \hat{\varphi}(\eta) \varphi(y) dy,$$

with integral in the sense of distributions (i.e., $\langle Au, \varphi \rangle = (2\pi)^{-d} \langle a, \hat{\varphi} \varphi e^{iy \cdot \eta} \rangle$). The operator $A$ is referred to as Fourier multiplier.

We shall see that Fourier multipliers are relevant to the case of a uniform plasma equilibrium.

### 3. Characterization of the response operator: a simple case study

In this section, we give a mathematically precise characterization of what has been informally called “response” in sections and for a simple case study.

Given $v \in S(R^{1+d})$, we consider the equation

$$\partial_t u(t, x) = v(t, x), \quad u(0, \cdot) = u_0 \in C^\infty_b(R^d),$$

where $C^\infty_b(R^d)$ is the space of smooth bounded functions with bounded derivatives. All solutions of this problem are

$$u(t, x) = u_0(x) + \int_0^t v(s, x) ds,$$

and we have $u \in C^\infty(R^{1+d}).$

**Proposition 3.1.** In addition, $u \in C^\infty_b(R^{1+d}).$

**Proof.** Considering the derivatives $\partial_t^q \partial_x^p u(t, x)$, for $q > 1$ one has $\partial_t^q \partial_x^p u(t, x) = \partial_t^{q-1} \partial_x^p v(t, x)$ with $v \in S(R^{1+d})$, while for $q = 0$ and for every $m > 1$ we have

$$|\partial_x^p u(t, x)| \leq |\partial_x^p u_0(x)| + \sup_{s \in R} \left(1 + s^2\right)^m \partial_x^p v(s, x) \int_{-\infty}^{+\infty} \frac{ds}{(1 + s^2)^m},$$

with $\partial_x^p u_0$ bounded by hypothesis. \qed
If we think of \( u \) as the response of a localized perturbation \( v \), causality requires that \( u \to 0 \) for \( t \to -\infty \) since the perturbation decreases exponentially in time. The limit for \( t \to -\infty \) of the solution exists since \( v \in S(\mathbb{R}^{1+d}) \) and thus

\[
\int_{-\infty}^{0} v(s, \cdot) ds
\]

is finite. This requirement selects uniquely the initial condition \( u_0 \), namely,

\[
u_0(x) = \int_{-\infty}^{0} v(s, x) ds,
\]

thereby establishing uniqueness of the response.

**Proposition 3.2.** For \( v \in S(\mathbb{R}^{1+d}) \), there exists a unique solution in \( C^\infty_c(\mathbb{R}^{1+d}) \) of the equation \( \partial_t u = v \) such that \( \lim_{t \to -\infty} u(t, x) = 0 \) pointwise in \( x \), and that is given by the function

\[
u(t, x) = \int_{-\infty}^{t} v(s, x) ds,
\]

in \( C^\infty_c(\mathbb{R}, S(\mathbb{R}^d)) \). This is referred to as the causal solution.

**Proof.** The initial condition

\[
u_0(x) = \int_{-\infty}^{0} v(s, x) ds,
\]

belongs to \( S(\mathbb{R}^d) \) and thus to \( C^\infty_c(\mathbb{R}^{1+d}) \). Proposition 3.1 gives us a unique solution in \( C^\infty_c(\mathbb{R}^{1+d}) \), which is

\[
u(t, x) = \left( \int_{0}^{0} + \int_{0}^{t} \right) v(s, x) ds = \int_{-\infty}^{t} v(s, x) ds.
\]

For every integers \( q \geq 0 \) and \( t \in \mathbb{R} \), \( \partial^q_t \nu(t, \cdot) \in S(\mathbb{R}^d) \) and \( u \) satisfies the condition \( \lim_{t \to -\infty} u(t, x) = 0 \). As for the uniqueness, if \( u_* \in C^\infty_c(\mathbb{R}^d) \) is another initial condition such that the limit \( t \to -\infty \) of the corresponding solution vanishes, then

\[
0 = u_*(x) + \int_{0}^{-\infty} v(s, x) ds = u_*(x) - u_0(x),
\]

which shows that \( u_* = u_0 \).

The causal solution is bounded and thus defines a distribution in \( S' \) given by

\[
\langle u, \varphi \rangle = \int_{\mathbb{R}^{1+d}} u(t, x) \varphi(t, x) dt dx, \quad \forall \varphi \in S(\mathbb{R}^{1+d}).
\]

We consider now the map \( v \mapsto u \) acting : \( S(\mathbb{R}^{1+d}) \to S'(\mathbb{R}^{1+d}) \).

**Proposition 3.3.** The maps \( S(\mathbb{R}^{1+d}) \ni v \mapsto u \in S'(\mathbb{R}^{1+d}) \) amounts to a continuous linear operator, namely, there are integers \( m, \mu \) and a constant \( K_{m,\mu} \) such that

\[
|\langle u, \varphi \rangle| \leq K_{m,\mu} \|v\|_{2\mu} \|\varphi\|_{2m}.
\]

In addition the operator \( v \mapsto u \) is continuous : \( S(\mathbb{R}^{1+d}) \to L^\infty_c(\mathbb{R}^{1+d}) \).

**Proof.** That \( u \) defined above is a tempered distribution is a classical result: the integral is well-defined and one only needs to check the continuity of the functional; with that aim, for \( m > (1 + d)/2 \) one has

\[
|\langle u, \varphi \rangle| \leq \|u\|_{L^\infty(\mathbb{R}^{1+d})} \sup_{y \in \mathbb{R}^{1+d}} \left| (1 + y^2)^m \varphi(y) \right| \int_{\mathbb{R}^{1+d}} \frac{dy}{(1 + y^2)^m}.
\]
and

$$\sup_{y \in \mathbb{R}^{1+d}} |(1 + y^2)^m \varphi(y)| \leq C_m \|\varphi\|_{2m},$$

which shows continuity with respect to a set of semi-norms of $\varphi$ in $S$. We now need to look at the modulus of continuity which depends on the $L^\infty$ norm of the causal solution $u \in C_b^\infty(\mathbb{R}^{1+d})$. As in proposition 3.3

$$|u(t, x)| \leq \sup_{t \in \mathbb{R}} |(1 + t^2)^\mu v(t, x)| \int_{-\infty}^{+\infty} \frac{ds}{(1 + s^2)^\mu}, \quad \mu > 1/2,$$

and thus

$$\|u\|_{L^\infty(\mathbb{R}^{1+d})} \leq \left( \int_{-\infty}^{+\infty} \frac{ds}{(1 + s^2)^\mu} \right) \sup_{(t, x) \in \mathbb{R}^{1+d}} |(1 + t^2)^\mu v(t, x)|.$$

Since

$$\sup_{(t, x) \in \mathbb{R}^{1+d}} |(1 + t^2)^\mu v(t, x)| \leq \sup_{(t, x) \in \mathbb{R}^{1+d}} |(1 + t^2 + x^2)^\mu v(t, x)| \leq C_\mu \|v\|_{2\mu},$$

we have

$$|\langle u, \varphi \rangle| \leq K_{m, \mu} \|v\|_{2\mu} \|\varphi\|_{2m},$$

for $m > (1 + d)/2$ and $\mu > 1/2$. \(\square\)

This linear continuous operator describes the causal response of the simple equation under consideration to a perturbation $v$. The following simple result provides us with a characterization of causal solutions that can be used for more general problems.

**Proposition 3.4.** The regularized problem $\partial_t u^{\nu} + \nu v^{\nu} = v$ in $S'(\mathbb{R}^{1+d})$ with $\nu > 0$ and $v \in S(\mathbb{R}^{1+d})$ has a unique solution $u^{\nu} \in S'(\mathbb{R}^{1+d})$, which in fact belongs to $S(\mathbb{R}^{1+d})$. Explicitly, the solution is given by

$$u^{\nu}(t, x) = \int_{-\infty}^{t} e^{-\nu(t-t')} v(t', x) dt'.$$

Furthermore, there exists $u_* \in S'(\mathbb{R}^{1+d})$ such that $u^{\nu} \to u_*$ in $S'(\mathbb{R}^{1+d})$ as $\nu \to 0^+$. The limit $u_*$ is equal to the unique causal solution $u$ obtained in proposition 3.2.

**Remark 2.** The integral in the definition of $u^{\nu}$ is absolutely convergent as $t - t' \geq 0$ on the domain of integration and $v(\cdot, x) \in L^1(\mathbb{R})$. It is, however, not obvious that $u^{\nu}$ belongs to $S(\mathbb{R}^{1+d})$. This is proven by showing that the equation $\partial_t u^{\nu} + \nu v^{\nu} = v$ has a unique solution in $S(\mathbb{R}^{1+d})$ and that the Fourier transform of such a unique solution is equal to the Fourier transform of $\int_{-\infty}^{t} e^{-\nu(t-t')} v(t', x) dt'$.

**Proof.** If $u$ is a solutions in $S'$ of the regularized equation, its Fourier transform satisfies

$$-i(\omega + i\nu)\hat{u}^{\nu} = \hat{v}.$$

For $\nu > 0$, this has one and only one solution

$$\hat{u}^{\nu}(\omega, k) = i \frac{\hat{v}(\omega, k)}{\omega + i\nu}.$$
and we have $\hat{\omega} \in \mathcal{S}(\mathbb{R}^{1+d})$ since $(\omega + iv)^{-n}$ is smooth and polynomially bounded for $\omega \in \mathbb{R}$ and for all integers $n > 0$. Hence, its inverse Fourier transform belongs to $\mathcal{S}(\mathbb{R}^{1+d})$. We recall that
\[
u(t, x) = \frac{1}{(2\pi)^{1+d}} \hat{\nu}(t, x),
\]
so that, if $\hat{\varphi}(x) = \varphi(-x),$
\[
\langle \nu', \varphi \rangle = \frac{1}{(2\pi)^{1+d}} \hat{\nu}'(t, x) = \langle \hat{\nu}', \varphi \rangle = \langle \hat{\nu}'(t, x), \varphi \rangle.
\]
Let us introduce, for every $\varphi \in \mathcal{S}(\mathbb{R}^{1+d})$, the function $\hat{\psi} \in \mathcal{S}(\mathbb{R})$ given by
\[
\hat{\psi}(\omega) = \int_{\mathbb{R}^d} \hat{\nu}(\omega, k) \hat{\varphi}(\omega, k) dk,
\]
\[
\nu'(\omega, k) = (2\pi)^{-1} \hat{\nu}(\omega, k) \hat{\varphi}(\omega, k) \in \mathcal{S}(\mathbb{R}^{1+d}) \text{ since } \hat{\nu}(\omega, k) \in \mathcal{S}(\mathbb{R}^d)
\]
and note that the function $\hat{\psi} \in \mathcal{S}(\mathbb{R})$ is smooth and polynomially bounded.

We deduce
\[
\langle \nu', \varphi \rangle = \int_{\mathbb{R}} \hat{\psi}(\omega) d\omega.
\]
However, the sequence $\{\nu'\}_\nu \in \mathcal{S}(\mathbb{R}^{1+d})$ is not bounded in $\mathcal{S}$. In order to take the limit, we use the identity
\[
\frac{i}{\omega + iv} = \int_0^{+\infty} e^{(\omega + iv)t} dt,
\]
and note that the function $(t, \omega) \mapsto e^{i(\omega + iv)t} \hat{\psi}(\omega)$ belongs to $L^1(\mathbb{R} \times \mathbb{R})$ so that, by Fubini’s theorem,
\[
\langle \nu', \varphi \rangle = \int_0^{+\infty} e^{-vt} \int_{-\infty}^{+\infty} e^{i\omega t} \hat{\psi}(\omega) d\omega dt = 2\pi \int_0^{+\infty} e^{-vt} \varphi(-t) dt.
\]
Also the Fourier inversion theorem gives
\[
\psi(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \hat{\psi}(\omega) d\omega
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t} \int_{\mathbb{R}^d} \hat{\nu}(\omega, k) (2\pi)^{-1} \hat{\varphi}(\omega, k) dk d\omega
\]
\[
= \frac{1}{(2\pi)^{d+2}} \int_{-\infty}^{+\infty} e^{-i\omega t} \int_{\mathbb{R}^{1+d}} e^{-i(k \cdot x_1 - \omega t_1)} v(t_1, x_1) dt_1 dx_1
\]
\[
\quad \times \int_{\mathbb{R}^{1+d}} e^{i(k \cdot x_2 - \omega t_2)} \varphi(t_2, x_2) dt_2 dx_2 d\omega
\]
\[
= \frac{1}{(2\pi)^d} \int_{-\infty}^{+\infty} e^{-i\omega t} \int_{\mathbb{R}^{1+d}} e^{i\omega t_1} v(t_1, x) dt_1 \int_{\mathbb{R}} e^{-i\omega t_2} \varphi(t_2, x) dt_2 dxd\omega
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}^{1+d}} v(t', x) \varphi(t' - t, x) dt' dx,
\]
and this yields
\[
\langle \nu', \varphi \rangle = \int_0^{+\infty} \int_{\mathbb{R}^{1+d}} e^{-vt'} v(t', x) \varphi(t' + t'', x) dt' dx dt''.
\]
By the change of variables $t'' = t - s$, $t' = s$, one has

$$\langle u', \varphi \rangle = \int_{\mathbb{R}^{1+d}} \int_{-\infty}^{t} e^{-\nu(t-s)} v(s, x) \varphi(t, x) ds \, dt \, dx,$$

which shows that the distribution $u'$ is regular and equal to

$$u'(t, x) = \int_{-\infty}^{t} e^{-\nu(t-s)} v(s, x) ds,$$

as claimed. In addition, with $u$ defined in proposition 3.3,

$$\lim_{\nu \to 0^+} \langle u', \varphi \rangle = \int_{\mathbb{R}^{1+d}} \int_{-\infty}^{t} v(s, x) \varphi(t, x) ds \, dt \, dx = \int u(t, x) \varphi(t, x) dt \, dx,$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^{1+d})$, that is,

$$u' \to u, \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^{1+d}),$$

and the limit is the causal solution of proposition 3.2. \hfill \Box

The characterization given by proposition 3.4 is essentially a limiting absorption principle where $\nu$ plays the role of the absorption coefficient. The advantage is that it can indeed be generalized in order to include physically relevant situations, as described in the next section, where this characterization of the response via the regularized equation is applied to a physically relevant case.

4. Uniform isotropic plasmas in one spatial dimension: the standard linear Landau damping

Here we consider in detail the case of a non-magnetized non-relativistic plasma in one spatial dimension and for a single particle species. This is the textbook example for linear Landau damping. Equation (1) reduces to

$$\partial_t f(t, x, v) + v \partial_x f(t, x, v) = -(q/m) E(t, x) F'(v),$$

where $F \in \mathcal{S}(\mathbb{R})$ is the equilibrium distribution function, and $E \in \mathcal{S}(\mathbb{R}^2)$ is the electric field perturbation. The species index $s$ is dropped for simplicity. In this case the linearized Vlasov operator is the free advection operator

$$\mathcal{L}_0 = \partial_t + v \partial_x,$$

and its null space comprises all the functions such that

$$f(t, x, v) = h(x - vt, v),$$

or, if $h(x, v)$ has a Fourier transform in $x$,

$$\hat{f}(\omega, k, v) = 2\pi \hat{h}(k, v) \delta(\omega - kv).$$

We consider a regularized version of the advection operator and then pass to the limit to recover a solution of the original problem (limiting absorption principle).
4.1. Regularized problem. In this section we prove the existence and uniqueness of the solution of the regularized problem \([10]\), where by regularization we mean the addition of the damping term to the operator \(L_0\) thus obtaining

\[
L_\nu = \partial_t + \nu + v\partial_x.
\]

We prove the following existence and uniqueness result.

**Proposition 4.1.** Let \(E \in \mathcal{S}(\mathbb{R}^2)\) and \(F \in \mathcal{S}(\mathbb{R})\). Then,

\[
L_\nu f_\nu = -(q/m)EF',
\]

has a unique solution in \(\mathcal{S}'(\mathbb{R}^3)\). This is the unique (classical) solution of the Cauchy problem

\[
L_\nu f_\nu(t, x, v) = -(q/m)E(t, x)F'(v),
\]

with initial condition

\[
f_\nu(0, x, v) = -(q/m)F'(v) \int_{-\infty}^{0} e^{\nu(s-t)} E(s, x - v \cdot (t-s)) ds,
\]

at \(t = 0\). Furthermore, one has \(f_\nu \in \mathcal{S}(\mathbb{R}^3)\).

**Remark 3.** In proposition \([11]\) we distinguish between the equation in \(\mathcal{S}'\) and in classical sense.

**Proof.** For \(F \in \mathcal{S}(\mathbb{R}), E \in \mathcal{S}(\mathbb{R}^2)\) and \(\nu > 0\), by Fourier transform we obtain the unique solution in \(\mathcal{S}'\) of the problem

\[
L_\nu f_\nu(\cdot, v) = -(q/m)E(\cdot)F'(v),
\]

which is given by

\[
f_\nu(0, x, v) = -(q/m)E(k, v) \int_{-\infty}^{0} e^{\nu(s-t)} E(s, x - v \cdot (t-s)) ds,
\]

at \(t = 0\). Furthermore, one has \(f_\nu \in \mathcal{S}(\mathbb{R}^3)\).

By inspection of expression \([11]\) we also notice that \(f_\nu \in \mathcal{S}(\mathbb{R}^3)\), but the sequence \(f_\nu\) is not uniformly bounded in \(\mathcal{S}\).

On the other side, one can consider the initial value problem in \(C^1(\mathbb{R}, \mathcal{S}'(\mathbb{R}))\) for each \(v \in \mathbb{R}\)

\[
\begin{cases}
\partial_t f_\nu(t, x, v) + v\partial_x f_\nu(t, x, v) + \nu f_\nu(t, x, v) = -(q/m)E(t, x)F'(v),
\end{cases}
\]

\[
f_\nu(0, x, v) = f^{(0)}_\nu(x, v).
\]

We seek the Cauchy condition such that its unique solution is in \(\mathcal{S}'(\mathbb{R}^2)\). By partial Fourier transform in space, this can be rewritten as

\[
\begin{cases}
\partial_t \hat{f}_\nu(t, k, v) + kv \hat{f}_\nu(t, k, v) + \nu \hat{f}_\nu(t, k, v) = -(q/m)\hat{E}(t, k)F'(v),
\end{cases}
\]

\[
\hat{f}_\nu(0, k, v) = \hat{f}^{(0)}_\nu(k, v),
\]

and the solution is

\[
\hat{f}_\nu(t, k, v) = e^{-\nu(t+kv)} \left[ \hat{f}^{(0)}_\nu(k, v) - (q/m)\hat{E}(k, v) \int_{0}^{t} e^{(\nu+ikv)s} \hat{E}(s, k) ds \right].
\]

Since \(\hat{E} \in \mathcal{S}(\mathbb{R}^2)\), the integral in \(s\) gives a continuous and bounded function, and yet, in general, the solution blows up for \(t \rightarrow -\infty\) as \(e^{-\nu t}\). It follows that we have \(\hat{f}_\nu \in \mathcal{S}'\) if and only if the initial condition satisfy

\[
\hat{f}^{(0)}_\nu(k, v) = -(q/m)F'(v) \int_{-\infty}^{0} e^{(\nu+ikv)s} \hat{E}(s, k) ds.
\]
The corresponding solution amounts to
\[ \tilde{f}_\nu(t, k, v) = -(q/m) F'(v) \int_{-\infty}^{t} e^{(\nu + ikv)(s-t)} \tilde{E}(s, k) ds, \]
and this is the unique solution in \( S'(\mathbb{R}^3) \). Upon inserting the full Fourier transform of \( E(t, x) \), one can check that this is just an alternative form of (11).

The inversion of partial Fourier transform gives the solution in the physical space
\[ f_\nu(t, x, v) = -(q/m) F'(v) \int_{-\infty}^{t} e^{\nu(s-t)} E(s, x - v \cdot (t-s)) ds, \]
and we note that
\[ X(s, t, x, v) = x - v \cdot (t-s), \quad V(s, t, x, v) = v, \]
is just the solution of the equations for the characteristics of \( \mathcal{L}_\nu \) integrated backward in time from \((t, x, v)\).

The key point of this construction is that the requirement \( f_\nu \in S'(\mathbb{R}^3) \) selects a specific initial condition for the Cauchy problem, thus uniquely determining the response of the regularized operator.

**Remark 4.** For any other initial condition \( f^{(0)}_\nu \in S'(\mathbb{R}) \), the solution belongs to \( C^1(\mathbb{R}, S'(\mathbb{R})) \), but it is not tempered in time, as it grows exponentially for \( t \to -\infty \).

We can compute the electric current density via equation (2), namely,
\[ j_{\nu}(t, x) = q \int_{\mathbb{R}} v f_{\nu}(t, x, v) dv, \]
and \( j_{\nu} \in S(\mathbb{R}^2) \). The map \( E \mapsto j_{\nu} = \sigma_{\nu}(E) \) defines a linear continuous operator \( \sigma_{\nu} : S(\mathbb{R}^2) \to S(\mathbb{R}^2) \) which is given by the Fourier multiplier
\[ \hat{j}_{\nu}(\omega, k) = \hat{\sigma}_{\nu}(\omega, k) \hat{E}(\omega, k), \quad \hat{\sigma}_{\nu}(\omega, k) = -\frac{q^2}{m} \int_{\mathbb{R}} \frac{v F'(v)}{\omega - kv + i\nu} dv. \]
The continuity of \( \sigma_{\nu} \) in particular follows from the estimate
\[ \left| \partial_\alpha^\alpha \partial_\beta^\beta \hat{\sigma}_{\nu}(\omega, k) \right| \leq C_{\nu^\alpha m^\beta+1} \int_{\mathbb{R}} |v^{\beta+1} F'(v)| dv, \]
for any non-negative integers \( \alpha, \beta \), where the constant \( C \) depends only on \( q^2/m, \alpha, \) and \( \beta \). We observe that this estimate is not uniform in \( \nu \) as expected, since \( f_{\nu} \) is not uniformly bounded in \( S \).

**4.2. Limiting absorption principle.** We apply now the limiting absorption principle, that is, we consider the limit of the distribution \( f_{\nu} \) and current \( j_{\nu} \) for \( \nu \to 0^+ \).

**Proposition 4.2.** The solution \( f_{\nu} \) defined in (12) with \( E \in S(\mathbb{R}^2) \) and \( F \in S(\mathbb{R}) \) has a pointwise limit
\[ f(t, x, v) = -(q/m) F'(v) \int_{-\infty}^{t} E(s, x - v \cdot (t-s)) ds, \]
which is in \( C^\infty_b(\mathbb{R}^3) \) and for every \((t, x) \in \mathbb{R}^2 \) we have \( f(t, x, \cdot) \in S(\mathbb{R}), \mathcal{L}_0 f = -(q/m) EF' \).
Proposition 4.3. Let $j_\nu$ be defined by equation (13) where $f_\nu$ is given in proposition 4.2 with $E \in \mathcal{S}(\mathbb{R}^2)$ and $F \in \mathcal{S}(\mathbb{R})$. Then, the function defined by 

$$j(t, x) = -\frac{q^2}{m} \int_{D_t} vF'(v)E(s, x - v \cdot (t - s))dsdv,$$

with $D_t = [-\infty, t] \times \mathbb{R}$, belongs to $C_b^\infty(\mathbb{R}^2)$ and $j_\nu(t, x) \to j(t, x)$ pointwise in $\mathbb{R}^2$. The map $\sigma : E \mapsto j = \sigma(E)$ is a linear continuous operator from $\mathcal{S}(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$.

Proof. Let us first show that $j \in C_b^\infty(\mathbb{R}^2)$. With this aim we consider the function 

$$E(t, x, s, v) = vF'(v)E(s, x - v \cdot (t - s)),$$

for $(t, x) \in \mathbb{R}^2$ and $(s, v) \in D_t$. Then $E$ is of class $C^\infty$ and for all $\alpha, \beta \in \mathbb{N}_0$ we have 

$$|\partial_\alpha^\alpha \partial_\beta^\beta E(t, x, s, v)| \leq |v^{\alpha+1}F'(v)| \cdot |\partial_\alpha^\alpha \partial_\beta^\beta E(s, x - v(t - s))|$$

$$\leq \frac{|v^{\alpha+1}F'(v)|}{(1 + s^2)^m} \sup_{t, x}|(1 + t^2)^m \partial_\alpha^\alpha \partial_\beta^\beta E(t, x)|$$

$$\leq \frac{|v^{\alpha+1}F'(v)|}{(1 + s^2)^m} \|E\|_{\alpha+\beta+m},$$

where $\|E\|_k$ denotes the standard norms in $\mathcal{S}$, cf. section 2. For $m > 1/2$ this upper bound is integrable on $\mathbb{R}^2$ and thus we can differentiate under the integral sign and obtain 

$$|\partial_\alpha^\alpha \partial_\beta^\beta j(t, x)| \leq \left( \int_{\mathbb{R}^2} \frac{|v^{\alpha+1}F'(v)|}{(1 + s^2)^m}dsdv \right) \|E\|_{\alpha+\beta+m},$$

uniformly in $\mathbb{R}^2$. In the upper bound we have extended the integration domain from $D_t$ to the whole space $\mathbb{R}^2$. This proves the claim $j \in C_b^\infty(\mathbb{R}^2)$.

As a tempered distribution, $j$ acts on a test function $\psi \in \mathcal{S}(\mathbb{R}^2)$ by integration 

$$\langle j, \psi \rangle = \int_{\mathbb{R}^2} j(t, x)\psi(t, x)dtdx,$$
and since \( j \) is uniformly bounded,
\[
|⟨j, ψ⟩| \leq \|j\|_{L^∞(R^2)} \|ψ\|_{L^1(R^2)} \leq C \|E\|_m \cdot \|ψ\|_n,
\]
where \( m,n \in \mathbb{N} \), with \( m \) defined above and \( n > 1 \). We deduce that the map \( E \mapsto j \) from \( S \rightarrow S' \) is continuous.

At last we address the pointwise convergence of \( j_ν \) to \( j \). Since
\[
e^{ν(s-t)} \leq 1, \quad \text{for} \ (s,v) \in D_t,
\]
we have
\[
e^{ν(s,t)}|E(t,x,s,v)| \leq \frac{|νF'(v)|}{(1 + s^2)^m} \|E\|_m
\]
and for \( m > 1/2 \) the bound is in \( L^1 \) and we can pass the limit under the integral sign obtaining \( j(t,x) = \lim j_ν(t,x) \).

For an explicit calculation of the conductivity operator we consider the limit \( ν → 0^+ \) in Fourier space. We consider \( j_ν \) as a tempered distribution acting on \( ψ ∈ S(R^2) \) by
\[
(⟨j_ν, ψ⟩) = -i(q^2/m) \int_{R^2×R} νF'(v) \hat{E}(ω,k)ψ(ω,k) \frac{dωdkdv}{ω - kv + iν},
\]
where the integral is absolutely convergent and defines a linear continuous operator from \( S(R^2) \) to \( C \), hence \( j_ν ∈ S'(R^2) \). We now want to pass to the limit for \( ν → 0^+ \).

We will make also use of the Hilbert transform which is defined by
\[
\mathcal{H}(φ)(x) = \frac{1}{π} \text{p.v.} \int \frac{φ(y)}{x - y} dy = \frac{1}{π} (\text{p.v.} \frac{1}{x} * φ)(x),
\]
for a function \( φ ∈ C_0^∞ \). We have the following properties of the Hilbert transform which we give without proof.

**Proposition 4.4.** The Hilbert transform \( \mathcal{H} \) defined above can be extended to compactly supported distributions. It can be extended to functions in \( S(R) \) through the equality
\[
\mathcal{H}(φ)(x) = \frac{1}{π} \int_{R} \frac{1}{2u} [φ(x-u) - φ(x+u)] du.
\]
It extends also as an isometry from \( L^2(R) \) into itself and
\[
\mathcal{H}(u') = (\mathcal{H}(u))', \quad \forall u ∈ L^2(R).
\]
In addition, \( \mathcal{H}(u') = (\mathcal{H}(u))^{' \prime} \) where \( (·)' \) denotes the distributional derivative. This implies that the Hilbert transform acts on Sobolev spaces \( H^k(R) \) as an isometry, i.e., \( \mathcal{H} : H^k(R) → H^k(R) \) for every non-negative integer \( k \). Particularly if \( u ∈ S(R) \), then \( \mathcal{H}(u) ∈ H^{k+∞}(R) \).

Let us introduce a scale length \( λ > 0 \) and a cutoff function \( χ ∈ C_0^∞(R) \), \( χ(z) = 1 \) for \( |z| ≤ 1/2 \) and \( \text{supp} \, χ ⊆ (-1, 1) \). Let \( G(v) = νF'(v)/n \) where \( n > 0 \) is the uniform background plasma density and let \( ω_p^2 = 4πq^2n/m \) be the squared plasma frequency of the considered species.

**Proposition 4.5.** Let \( E ∈ S(R^2) \), \( F ∈ S(R) \) and let \( j_ν ∈ S(R^2) \) be the current density uniquely defined in proposition [4.3] with Fourier transform in \( S' \) given in equation [(13)].
We define the operator \( \lambda \) so that we can take the limit one rewrites \( \lambda \sigma \) in the definition of \( \sigma \).

Remark and for \( vF \) version (15) for its action as a tempered distribution. Using Fourier multiplier associated with the conductivity operator and on the expression (16).

Proof of proposition 4.5. This provides a “global” expression for the conductivity operator which is not a Fourier multiplier. We have 4\( \pi i \omega \) this is a Fourier multiplier and is continuous from \( S(\mathbb{R}^2) \) to \( S'(\mathbb{R}^2) \). Then

(ii) One has

\[ j_\nu \to j \text{ in } S'(\mathbb{R}^2) \text{ for } \nu \to 0^+, \text{ and } j = \sigma E, \]

where \( j \) is the pointwise limit constructed in proposition 4.3. This defines a linear continuous operator \( \sigma : S(\mathbb{R}^2) \to S'(\mathbb{R}^2) \).

(iii) For every \( \lambda > 0 \), define \( \sigma_{\lambda,\chi} := \sigma - \sigma_{\lambda,1-\chi} \). One has the identity,

\[ \langle \sigma_{\lambda,\chi}(E), \hat{\psi} \rangle = -\frac{i\omega^2}{4\pi} \int_{\mathbb{R}^2} \chi(k\lambda)G(v) \left[ \pi \mathcal{H} \left( \hat{E}(\cdot, k) \right)(kv) - i\pi \hat{E}(kv, k) \right] dk dv. \]

Remark 5. The conductivity \( \hat{\sigma}_{ph} \) is exactly the same as that obtained formally in the physics literature. We have 4\( \pi i \omega \hat{\sigma}_{ph} = \omega^2 \mathcal{E}(\omega/k) \) with \( s \in C^\infty(\mathbb{R}) \).

Remark 6. One should notice that the result is valid for any non-negative constant \( \lambda \). For \( \lambda \to +\infty \) the Fourier multiplier \( \sigma_{\lambda,1-\chi} \) converges to zero, while the integrand in the definition of \( \sigma_{\lambda,\chi} \) is bounded by an \( L^1 \)-function uniformly for \( \lambda \in [\lambda_0, +\infty) \), so that we can take the limit \( \lambda \to +\infty \) with the result that

\[ \langle \sigma(E), \hat{\psi} \rangle = -\frac{i\omega^2}{4\pi} \int_{\mathbb{R}^2} G(v) \left[ \pi \mathcal{H} \left( \hat{E}(\cdot, k) \right)(kv) - i\pi \hat{E}(kv, k) \right] dk dv. \]

This provides a “global” expression for the conductivity operator which is not a Fourier multiplier.

Proof of proposition 4.5. For this proof, we rely on the expression [14] for the Fourier multiplier associated with the conductivity operator and on the expression [15] for its action as a tempered distribution. Using \( vF'(v) = nG(v) \) and \( \omega^2 \), one rewrites

\[ \hat{\sigma}_{\nu}(\omega, k) = -\frac{i\omega^2}{4\pi} \int_{\mathbb{R}^2} G(v) \frac{\phi(\omega, k)}{\omega - kv + iv} dv, \]

and for \( \phi \in S(\mathbb{R}^2) \) we define

\[ I_{\nu}(\phi) = -\frac{i\omega^2}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}} \chi(k\lambda)G(v) \frac{\phi(\omega, k)}{\omega - kv + iv} d\omega dk dv. \]

The limit of these expressions is deduced from classical arguments of distribution theory, as follows.

For \( k \neq 0 \), define the two functions \( G_0 \) and \( G_1 \) elements of \( C^\infty \) by

\[ G_0(\frac{\omega}{k}, u) = \frac{1}{2} \left[ G(\frac{\omega}{k} - u) + G(\frac{\omega}{k} + u) \right], \]

\[ uG_1(\frac{\omega}{k}, u) = \frac{1}{2} \left[ G(\frac{\omega}{k} - u) - G(\frac{\omega}{k} + u) \right]. \]
Similarly, for all \((\omega, k, v) \in \mathbb{R}^3\) we define

\[
\phi_0(kv, \omega - kv, k) = \frac{1}{2}(\phi(\omega, k) + \phi(2kv - \omega, k)),
\]

\[
\phi_1(kv, \omega - kv, k) = \frac{\phi(\omega, k) - \phi_0(kv, \omega - kv, k)}{\omega - kv} = \int_0^1 \phi_0(\omega + 2(kv - \omega)s, k) ds.
\]

One has the identities:

1. for \(k \neq 0\),

\[
\hat{\sigma}_\nu(\omega, k) = -i \frac{\omega^2}{4\pi} \int \frac{G_0(\frac{\omega}{k}, u) + uG_1(\frac{\omega}{k}, u)}{ku + i\nu} du,
\]

2. for all \(\phi \in \mathcal{S}(\mathbb{R}^2)\),

\[
I_\nu(\phi) = -i \frac{\omega^2}{4\pi} \int_{\mathbb{R}^2} x(\lambda)G(v) \times \frac{\phi_0(kv, \omega - kv, k) + (\omega - kv)\phi_1(kv, \omega - kv, k)}{\omega - kv + i\nu} dw dk dv
\]

\[
= -i \frac{\omega^2}{4\pi} \int_{\mathbb{R}^2} x(\lambda)G(v) \left[ \int_{\mathbb{R}} \frac{\phi_0(kv, \omega, k) + i\nu\phi_1(kv, \omega, k)}{\omega + i\nu} d\omega \right] dk dv.
\]

We shall then compute the limit, when \(\nu \to 0^+\), of the integrals

\[
A_\nu(k; \omega) = \int_{\mathbb{R}} \frac{G_0(\frac{\omega}{k}, u) + uG_1(\frac{\omega}{k}, u)}{ku + i\nu} du,
\]

\[
B_\nu(k; v) = \int_{\mathbb{R}} \frac{\phi_0(kv, \omega, k) + i\nu\phi_1(kv, \omega, k)}{\omega + i\nu} d\omega.
\]

As \(G_0, G_1\) are even functions in \(u\), and \(\phi_0, \phi_1\) are even functions in \(\omega\), one deduces

\[
kA_\nu(k; \omega) = \int_{\mathbb{R}} \frac{k^2u^2G_1(\frac{\omega}{k}, u) - ikuG_0(\frac{\omega}{k}, u)}{ku^2 + \nu^2} du
\]

\[
= \int_{\mathbb{R}} \frac{k^2u^2G_1(\frac{\omega}{k}, u)}{ku^2 + \nu^2} du - ik \int_{\mathbb{R}} \frac{G_0(\frac{\omega}{k}, vt)}{kt^2 + 1} dt,
\]

\[
B_\nu(k; v) = \int_{\mathbb{R}} \frac{\omega^2\phi_1(kv, \omega, k) - i\nu\phi_0(kv, \omega, k)}{\omega^2 + \nu^2} d\omega
\]

\[
= \int_{\mathbb{R}} \frac{\omega^2\phi_1(kv, \omega, k)}{\omega^2 + \nu^2} d\omega - i \int_{\mathbb{R}} \frac{\phi_0(kv, \omega, k)}{t^2 + 1} dt.
\]

We observe that \(\frac{k^2u^2}{ku^2 + \nu^2}\) and \(\frac{\omega^2}{\omega^2 + \nu^2}\) are uniformly bounded by 1. Moreover, \(t \mapsto \frac{1}{t^2 + 1}\) is in \(L^1(\mathbb{R})\), and, for \(k \neq 0\), \(t \mapsto \frac{1}{kt^2 + 1}\) belongs to \(L^1(\mathbb{R})\). Using the dominated convergence theorem, and the values \(\int_{\mathbb{R}} \frac{dt}{t^2 + \gamma} = \pi\), \(\int_{\mathbb{R}} \frac{dt}{kt^2 + \gamma} = \frac{\pi}{k}\), one has

\[
kA_\nu(k; \omega) \to \int_{\mathbb{R}} G_1(\frac{\omega}{k}, u) du - i\pi G_0(\frac{\omega}{k}, 0),
\]

\[
B_\nu(k; v) \to \int_{\mathbb{R}} \phi_1(kv, \omega, k) d\omega - i\pi \phi_0(kv, 0, k),
\]

hence, upon choosing \(\phi = \hat{E}\psi\), the expressions of item (i) and (iii) of proposition

\[\text{[Proposition 4.5]}\]
As for convergence in item (ii), let us split the regularized current into two contributions

\[
\langle \hat{j}_\nu, \psi \rangle = -\frac{q^2}{m} \int_{\mathbb{R}^2 \times \mathbb{R}} (1 - \chi(k\lambda)) \nu F'(v) \frac{\hat{E}(\omega, k)\psi(\omega, k)}{\omega - kv + iv} d\omega dk dv
\]

\[
= -\frac{q^2}{m} \int_{\mathbb{R}^2 \times \mathbb{R}} \chi(k\lambda) \nu F'(v) \frac{\hat{E}(\omega, k)\psi(\omega, k)}{\omega - kv + iv} d\omega dk dv
\]

where \( \phi = \hat{E}\psi \). The first integral is supported away from the singular point \( k = 0 \), and we can use the same argument to pass the limit under the integral, while the limit of \( I_\nu(\phi) \) has been computed above. Therefore, we have

\[
\langle \hat{j}_\nu, \psi \rangle \xrightarrow{\nu \to 0^+} \int_{\mathbb{R}^2} ((1 - \chi(k\lambda)) \hat{\sigma}_{\text{ph}}(\omega, k)\hat{E}(\omega, k) d\omega dk + \langle \sigma_{\lambda, \chi}(E), \psi \rangle.
\]

Since \( (1 - \chi(k\lambda))/k \) is bounded by \( 2\lambda \), the Fourier multiplier \( \sigma_{\lambda, 1 - \chi} \) amounts to a linear continuous operator from \( S \to S' \). Using the properties of the Fourier transform we have

\[
\lim_{\nu \to 0^+} \langle \hat{j}_\nu, \psi \rangle = \lim_{\nu \to 0^+} \langle j_\nu, \psi \rangle = \langle j, \psi \rangle = \langle \hat{j}, \psi \rangle,
\]

where \( j = \sigma(E) \) is constructed in proposition 4.3. We deduce that \( \sigma_{\lambda, \chi} = \sigma - \sigma_{\lambda, 1 - \chi} \) is the difference between linear continuous operator from \( S \to S' \) and thus is a linear and continuous operator. \( \square \)

The operator \( \sigma_{\lambda, \chi} \) does not play any role when the electric field perturbation is supported away from \( k = 0 \), i.e., for non-static fields. More precisely we have the following result.

**Corollary 4.6.** If \( E \in \mathcal{S}(\mathbb{R}^2) \) is such that \( \hat{E}(\omega, k) = 0 \) for \( |k| \leq 1/\lambda \), then \( \hat{j} \in C^\infty(\mathbb{R}^2) \) and

\[
\hat{j}(\omega, k) = \hat{\sigma}_{\text{ph}}(\omega, k)\hat{E}(\omega, k),
\]

which expresses the usual Ohm’s law for a uniform plasma.

**Proof.** We observe that, for every \( \psi \in \mathcal{S}(\mathbb{R}^2) \),

\[
\langle \sigma_{\lambda, \chi}(E), \psi \rangle = \lim_{\nu \to 0^+} \frac{-iq^2}{m} \int_{\mathbb{R}^2 \times \mathbb{R}} \chi(k\lambda) \nu F'(v) \frac{\hat{E}(\omega, k)\psi(\omega, k)}{\omega - kv + iv} d\omega dk dv,
\]

and by hypothesis \( \chi(k\lambda)\hat{E}(\omega, k) = 0 \) for all \((\omega, k) \in \mathbb{R}^2\). Hence, \( \sigma_{\lambda, \chi}(E) = 0 \) and

\[
\hat{\sigma}(E) = \sigma_{\lambda, 1 - \chi}(E) = (1 - \chi(k\lambda))\hat{\sigma}_{\text{ph}}(\omega, k)\hat{E}(\omega, k),
\]

and by hypothesis \( (1 - \chi(k\lambda))\hat{E}(\omega, k) = \hat{E}(\omega, k) \). The fact that \( \hat{j} \) is in \( C^\infty \) follows from the properties of the Hilbert transform summarized in proposition 4.3 that imply in particular, \( \mathcal{H}(G) \in H^\infty(\mathbb{R}) \). \( \square \)
4.3. **Proof of theorem 1.1** We collect at last the partial results of this section and give the proof of theorem 1.1 stated in the introduction.

We start with item (i). The fact that $f_{s,\nu}$ belongs to $\mathcal{S}(\mathbb{R}^3)$ and is the unique solution of equation (6) in $\mathcal{S}'(\mathbb{R}^3)$ is proven in proposition 4.1. Pointwise convergence $f_{s,\nu} \to f_s$ is established in proposition 4.2 where $f_s$ is given in equation (8) and solves equation (6) with $\nu = 0$. Proposition 4.2 also gives $f_{s,\nu}(t,x,\cdot) \in \mathcal{S}(\mathbb{R})$ for every $(t,x) \in \mathbb{R}^2$. The fact that $f_{s,\nu} \to f_s$ in the topology of $\mathcal{S}'$, follows from pointwise convergence and the fact that $f_{s,\nu}, f_s \in C^\infty_b$, since

$$|\langle f_{s,\nu} - f, \phi \rangle| = \int_{\mathbb{R}^3} |f_{s,\nu} - f_s| \phi \, dt \, dx \, dv \leq \|f_{s,\nu} - f_s\|_{L^\infty(\mathbb{R})} \|\phi\|_{L^1(\mathbb{R}^3)},$$

for all $\phi \in \mathcal{S}(\mathbb{R}^3)$.

As for item (ii), the continuity of $\sigma_\nu$ from $\mathcal{S}$ into itself follows from equation (14) and comments thereon. Pointwise convergence of $j_\nu$ is proven in proposition 4.3 together with the continuity of $\sigma$ from $\mathcal{S} \to \mathcal{S}'$, while the convergence in $\mathcal{S}'$ is item (ii) of proposition 4.5. That the limit current $j$ and the limit function $f_s$ are related by equation (2) follows from proposition 4.3.

At last, item (iii) of the theorem is corollary 4.6 which follows from proposition 4.5 where the expression of the conductivity operator are given explicitly.

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