The Geometry of $G_2$, Spin$(7)$, and Spin$(8)$-models

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Abstract

We study the geometry of elliptic fibrations given by Weierstrass models resulting from Step 6 of Tate’s algorithm. Such elliptic fibrations have a discriminant locus containing an irreducible component $S$, over which the generic fiber is of Kodaira type $I^*_0$. In string geometry, these geometries are used to geometrically engineer $G_2$, Spin$(7)$, and Spin$(8)$ gauge theories. We give sufficient conditions for the existence of crepant resolutions. When they exist, we give a complete description of all crepant resolutions and show explicitly how the network of flops matches the Coulomb branch of the associated gauge theories. We also compute the triple intersection numbers in each chamber. Physically, they correspond to the Chern-Simons levels of the gauge theory and depend on the choice of a Coulomb branch. We determine the representations associated with these elliptic fibrations by computing intersection numbers with fibral divisors and then interpreting them as weights of a representation. For a five-dimensional gauge theory, we compute the number of hypermultiplets in each representation by matching the triple intersection numbers with the superpotential of the theory.
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1 Introduction

The existence and geometric properties of crepant resolutions for Weierstrass models over higher-dimensional bases are an important but still incomplete aspects of the theory of elliptic fibrations. This study has many applications, such as in the theory of Calabi-Yau varieties, string geometry, and the classification of superconformal field theories. For examples of crepant resolutions of Weierstrass models, see [9, 11, 21, 23, 25–28, 41, 43, 46].

The goal of this paper is to study the geometry of crepant resolutions of Weierstrass models corresponding to Step 6 of Tate’s algorithm [57]. Such a Weierstrass model has a discriminant locus $\Delta$ containing a nonsingular and irreducible divisor $S$ of the base $B$ such that the geometric fiber over the generic point of $S$ is of Kodaira type $I_0^*$ (whose dual graph is the affine Dynkin diagram of type $\tilde{D}_4$). The divisor $S$ appears with multiplicity six in the discriminant $\Delta$ and the remainder of the discriminant $\Delta' = \Delta S^{-6}$ is typically singular. The Kodaira fibers over generic points of $\Delta'$ are of type $I_1$ and the divisors $S$ and $\Delta'$ do not intersect transversally. At the intersection of $S$ and $\Delta'$, we have a “collision of singularities”1 of type $I_1 + I_0^*$ yielding non-Kodaira fibers whose structures are explained in detail in later sections. Such a collision is not allowed in Miranda’s models since the fiber $I_1$ has an infinite $j$-invariant while $I_0^*$ can only take finite values for its $j$-invariant.

1.1 $I_0^*$ as the most versatile Kodaira type

The fiber $I_0^*$ has at least three unique properties that distinguish it from all other Kodaira fibers. Firstly, like smooth elliptic curves, the $I_0^*$ fiber can have a $j$-invariant of any value in the ground field of the elliptic fibration. On the other hand, all other singular Kodaira fibers have a $j$-invariant taking the values 0 or 1728, with the exceptions of the Kodaira fiber $I_{n>0}^*$ and $I_{n>0}$, whose $j$-invariant has a pole and takes an infinite value.

Secondly, while all the other Kodaira fibers have at most two splitting types—split and non-split—the fiber $I_0^*$ distinguishes itself by having three splitting types—split, semi-split, and non-split—corresponding to three distinct Lie algebras—namely $G_2$, $B_3$, and $D_4$. As we assume that the Mordell-Weil group is trivial, these three Lie algebras correspond to the simply-connected groups $G_2$, Spin(7), and Spin(8). It is also more natural to distinguish two different versions of the non-split case: there are two possible Galois groups, which give rise to distinct fiber degenerations, as we shall explain below.

Lastly, the fiber of type $I_0^*$ plays a central role in Miranda’s models [45] and in the classification of “non-Higgsable clusters” [47]. The fiber $I_0^*$ appears in collisions of the types $j = 0$ (namely $II + I_0^*$ and $IV + I_0^*$) and $j = 1728$ (namely $III + I_0^*$) [45]. These collisions are building blocks for the non-Higgsable clusters of elliptically fibered Calabi-Yau threefolds [47]. For example, an isolated curve of self-intersection $-4$ can support a Kodaira fiber of type $I_0^*$ and an associated Lie algebra $\mathfrak{so}_8$ with a trivial representation [47]. There are also three building blocks consisting of two or three rational curves of negative self-intersection intersecting transversally. We write $(-n_1, -n_2, -n_3, \ldots, -n_r)$ for a chain of $r$ rational curves $C_i$, with $1 \leq i \leq r$, where $C_i^2 = -n_i$ and two consecutive curves in the chain intersect transversally. A $(-2, -3)$-chain corresponds to the collision of two Kodaira fibers of type $III$ and $I_0^*$, yielding a semi-simple Lie algebra $\mathfrak{sp}_1 \oplus \mathfrak{g}_2$ [47]. A $(-2, -2, -3)$-chain corresponds to the chain

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1When two irreducible components $\Delta_1$ and $\Delta_2$ of the discriminant locus intersect, if we denote by $T_1$ and $T_2$ the Kodaira type of their respective generic fibers, their intersection is called a collision of singularity of type $T_1 + T_2$. The collision is said to be transverse when $\Delta_1$ and $\Delta_2$ intersect transversally. The type of the generic fiber over their intersection is usually not one of the Kodaira type. The possible types are classified for Miranda’s models [45,56], which are regularizations of Weierstrass models that gives flat elliptic fibrations such that the $j$-invariant is a morphism. Miranda’s models only allow transverse collisions.
of Kodaira fibers II+IV+I∗0, supporting the Lie algebra \( su_2 \oplus g_2 \) [47]; finally, a \((-2,-3,-2)\)-chain corresponds to the chain of Kodaira fibers III+I∗0+III, yielding a Lie algebra \( su_2 \oplus so_7 \oplus su_2 \) [47].

| Fiber type | Dual graph | Splitting type | Field extension | \( \kappa' : \kappa \) |
|------------|------------|----------------|-----------------|------------------|
| I∗0ns \( \Gamma_2 \) | ![](image) | ![](image) | 3 or 6 |
| I∗ss \( \Gamma_3 \) | ![](image) | ![](image) | 2 |
| I∗s \( \Gamma_4 \) | ![](image) | ![](image) | 1 |

Table 1: Fibers of type I∗0.

### 1.2 Geometric fibers, ground fields, and field extensions

The geometry and topology of an I∗0-model is subtle. To fully appreciate the geometry of a Weierstrass model describing the fiber \( I_{0}^{\ast} \), one has to take into account the scheme structure of the generic fiber. This scheme structure contains more detailed information than can be seen by the type of the geometric fiber and impacts the values of the topological invariants. Fiber types, generic fibers, and geometric irreducibility play a central role in this paper. See Appendix C of [21] for a review.

The type of a singular fiber depends on the ground field used to define its scheme structure. For the fiber over the generic point of an irreducible component of the discriminant locus, the natural ground field is the residue field of the generic point [42, §3.1.2, Remark 1.17]. This is because in scheme theory, the fiber over a point \( p \) is by definition \( Y \times_B \text{Spec} \kappa(p) \) [42, §3.1.2, Definition 1.13] and the second projection \( Y_p \to \text{Spec} \kappa(p) \) gives the fiber \( Y_p \) the structure of a scheme defined with the residue field \( \kappa(p) \) as its ground field. As the residue field \( \kappa(p) \) is not necessarily an algebraically closed field, the fiber type of \( Y_p \) as a scheme over \( \kappa(p) \) does not always match its Kodaira type. We recall that a Kodaira fiber is by definition a geometric fiber.\(^2\) The Kodaira type of \( Y_p \) is seen only after a field extension causing all components of the fiber to become geometrically irreducible. In the case of Weierstrass models coming from Tate’s algorithm, the required field extension is carefully described in Tate’s algorithm to be the splitting field of an appropriate cubic or quadratic polynomial defined from the Weierstrass coefficients. For elliptic fibrations, the type of a generic fiber \( Y_p \) that

\(^2\)A geometric fiber is such that its components are all geometrically irreducible, i.e. they do not factor into more components even after a field extension [42, §3.2, Definition 2.8].
is not geometric is either an affine Dynkin diagram of type $\tilde{A}_1$, or a twisted affine Dynkin diagram of type $\tilde{G}_2$, $\tilde{D}_{3+n}$, $\tilde{F}_4$, or $\tilde{C}_{2+n}$.

1.3 $G_2$, Spin(7), and Spin(8)-models

A fiber $I_0^\kappa$ consists of a rational curve of multiplicity two intersecting transversally with four other rational curves. Its dual graph is the affine Dynkin diagram $\tilde{D}_4$. One of the four is the component touching the section of the elliptic fibration. The points of intersection of the other three nodes with the node of multiplicity two can be described by a cubic polynomial that is essentially the auxiliary polynomial $P(T)$. The elliptic fibration is called the $G_2$, Spin(7), or Spin(8)-model, respectively, if $P(T)$ has no $\kappa$-rational roots, a unique $\kappa$-rational root, or three distinct $\kappa$-rational roots. The generic fibers over $S$ are then denoted by

$$I_0^{\text{ns}}, I_0^{\text{ss}}, \text{ and } I_0^s,$$

where “ns”, “ss”, and “s” stand for “non-split”, “semi-split”, and “split” [9]. The fibers $I_0^{\text{ns}}, I_0^{\text{ss}}, \text{ and } I_0^s$ defined with respect to the residue field $\kappa$ are called arithmetic fibers; this distinguishes them from their geometric fiber $I_0^\kappa$ defined in the splitting field of $P(T)$. The Galois group $\text{Gal}(\kappa'/\kappa)$ is trivial for Spin(8)-models, $\mathbb{Z}/2\mathbb{Z}$ for Spin(7)-models, and can be either the symmetric group $S_3$ or the cyclic group $\mathbb{Z}/3\mathbb{Z}$ for $G_2$-models. Thus, the Galois group provides a finer invariant than the number of rational solutions of $P(T)$. We introduce the notion of $G_2^{S_3}$-models and $G_2^{\mathbb{Z}/3\mathbb{Z}}$-models to distinguish between the two cases of a $G_2$-model as they have different fiber structures. One can think of the $G_2^{\mathbb{Z}/3\mathbb{Z}}$-model as a specialization of the $G_2^{S_3}$-model in which the discriminant $\Delta(P)$ of the auxiliary polynomial $P(T)$ is a perfect square in the residue field $\kappa$.

1.4 Crepant resolutions, flops, hyperplane arrangements, and Coulomb branches

Each crepant resolution of a singular Weierstrass model is a relative minimal model (in the sense of the Minimal Model Program) over the Weierstrass model [44]. When the base of the fibration is a curve, the Weierstrass model has a unique crepant resolution. When the base is of dimension two or higher, a crepant resolution does not always exist; furthermore, when it does, it is not necessarily unique. Different crepant resolutions of the same Weierstrass model are connected by a finite sequence of flops. Following F-theory, we attach to a given elliptic fibration a Lie algebra $\mathfrak{g}$, a representation $\mathbf{R}$ of $\mathfrak{g}$, and a hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$. The Lie algebra $\mathfrak{g}$ and the representation $\mathbf{R}$ are determined by the fibers over codimension-one and codimension-two points, respectively, of the base in the discriminant locus. The hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$ is defined inside the dual fundamental Weyl chamber of $\mathfrak{g}$ (i.e. the dual cone of the fundamental Weyl chamber of $\mathfrak{g}$), and its hyperplanes are the set of kernels of the weights of $\mathbf{R}$.

The network of flops is studied using the hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$ inspired from the theory of Coulomb branches of five-dimensional supersymmetric gauge theories with eight supercharges [36]. The network of crepant resolutions is isomorphic to the network of chambers of the hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$ defined by splitting the dual fundamental Weyl chamber of the Lie algebra $\mathfrak{g}$ by the hyperplanes dual to the weights of $\mathbf{R}$. The hyperplane arrangement $I(\mathfrak{g}, \mathbf{R})$, its relation to the Coulomb branches of supersymmetric gauge theories and the network of crepant resolutions are studied in [17, 19, 20, 26, 27, 34, 36].

The algorithm that we use to determine the representation $\mathbf{R}$ works for any base of dimension two or higher and does not require us to impose the Calabi-Yau condition. The algorithm consists of three steps. We start by identifying those vertical curves that are relative extremal curves appearing
over divisors of \( S \), over which the irreducible components of the generic fiber of \( S \) degenerate. We then associate to each of these curves a weight computed geometrically as minus of the intersections of the curve with the fibral divisors. Lastly, the representation \( R \) is then determined from these weights using the notion of saturated set of weights introduced by Bourbaki [10].

We show that each \( G_2 \)-model has a unique crepant resolution, each \( \text{Spin}(7) \)-model has two crepant resolutions connected by a simple flop, and each \( \text{Spin}(8) \)-model has six distinct crepant resolutions forming a cycle—under conditions that ensure the existence of crepant resolutions. Figures 2, 3, and 4 on page 15 depict the structure of the set of crepant resolutions in relationship to the geometry of the Coulomb branch.

When the Weierstrass coefficients vanish to high order along the component \( S \) of the discriminant locus, \( \mathbb{Q} \)-factorial terminal singularities are obstructions to the existence of a crepant resolution depending on the dimension of the base \( B \). See section 4.3 for the case of the \( G_2 \)-model and section 5.1 for the case of terminal singularities with the \( \text{Spin}(7) \)-model. In contrast, crepant resolutions were recently shown to exist in [23] for \( F_4 \)-models for generic coefficients of arbitrary valuations, as long as the restrictions of Step 7 of Tate’s algorithm are satisfied. We also show by direct inspection that each of the crepant resolutions we obtained determines a flat fibration. \( \mathbb{Q} \)-factorial terminal singularities have been discussed recently in F-theory in [4].

In this paper, we consider elliptic fibrations over bases of arbitrary dimensions. For F-theory and M-theory applications, we focus mostly on compactifications yielding five and six-dimensional gauge theories. In particular, we do not discuss Sen’s weak coupling limit of these theories. However, we point out that the weak coupling limit of \( G_2 \), \( \text{Spin}(7) \), and \( \text{Spin}(8) \)-models gives a local \( \mathfrak{so}(8) \)-gauge theory realized by eight D7 branes on top of an O7 orientifold [24, Table 4]. Such an \( \mathfrak{so}(8) \)-gauge theory can also be constructed using K-theory as in [15, §4.2.4].

### 1.5 Non-Kodaira fibers and Tate’s algorithm in higher codimension

The study of the fiber structure of \( G_2 \), \( \text{Spin}(7) \), and \( \text{Spin}(8) \)-models is surprisingly rich in non-Kodaira fibers. We get eight types of non-Kodaira fibers in the fiber degenerations of these elliptic fibrations. They are listed in Figure 1. To put this in perspective, we recall that in the theory of Miranda’s models, there are only seven non-Kodaira fibers across all possible collisions, while here, the \( \text{Spin}(7) \)-model on its own already produces six non-Kodaira fibers. Miranda has observed that non-Kodaira fibers of Miranda’s models of elliptic threefolds [45] are contractions of Kodaira fibers (see also [14]). Here we see that this is also true in higher codimension for the crepant resolutions of the Weierstrass model resulting from Step 6 of Tate’s algorithm.

With a careful study of the crepant resolutions of the Weierstrass models resulting from Step 6 of Tate’s algorithm, it is possible to predict (for Step 6) the possible higher codimension fibers from the valuations of the Weierstrass equation coefficient. In many cases, these valuations do not completely determine the fiber type: different crepant resolutions of the same Weierstrass model can have distinct fiber types over points of codimension two or higher.

### 1.6 Road map to the rest of the paper

The rest of the paper is organized as follows. In section 2, we introduce some basic definitions and fix our conventions in section 2.1 and review Step 6 of Tate’s algorithm in section 2.2. In section 3 we summarize the main mathematical results of the paper. We first derive canonical forms for \( G_2 \), \( \text{Spin}(7) \), and \( \text{Spin}(8) \)-models in section 3.1 by scrutinizing Step 6 of Tate’s algorithm. We distinguish between two types of \( G_2 \)-models using the Galois group of the splitting field of the associated...
polynomial $P(T)$ used in Step 6 of Tate’s algorithm. They have distinct fiber structures and $j$-invariants over $S$. We also distinguish between two Weierstrass models for Spin(7)-models by their fiber degenerations and their $j$-invariants over $S$. We then study the existence of crepant resolutions for these canonical forms and determine the fiber structure for each resolution. We compute the Chern-Simons coefficients as the triple intersection numbers of the fibral divisors in section 3.6. We study $G_2$-models in section 4, Spin(7)-models in section 5, and Spin(8)-models in section 6. In section 7, we apply the results collected in the previous sections to describe the physics of the $I^0_0$ models.

In section 7.1, we compute the one-loop prepotential and compute the number of hypermultiplets transforming in each irreducible representations of the gauge group. In section 7.2, we also count the number of representations of each model by comparing the triple intersection numbers and the cubic prepotential, and show in section 7.2 that the number of representations found are compatible with an anomaly-free six-dimensional theory. The counting matches the number found by Grassi and Morrison [31] using six-dimensional anomaly cancellation conditions and Witten’s quantization method (as generalized by Aspinwall, Katz and Morrison) [6].

2 Preliminaries

In this section, we introduce our conventions and some basic definitions in section 2.1 and review Step 6 of Tate’s algorithm in section 2.2.

2.1 Definitions and conventions

By a variety, we mean an irreducible algebraic variety over the complex numbers. Given a morphism $Y \to B$, we denote by $Y_p$ the fiber of $Y$ over a point $p$. Let $S = V(s) \subset B$ be an effective Cartier divisor of $B$ given by the zero scheme of a section $s$ of a line bundle $\mathcal{S}$. We assume that $S$ is a smooth irreducible variety and denote its generic point by $\eta$. The function field at $\eta$ is a local field $\mathcal{O}_{B,\eta}$ with maximal ideal $m_\eta$ and residue field $\kappa_\eta = \mathcal{O}_{B,\eta} / \mathcal{O}_{B,\eta}$. When there is no possibility for confusion, we simply write $\kappa_\eta$ as $\kappa$. The triplet $(\mathcal{O}_{B,\eta}, m_\eta, \kappa_\eta)$ defines a discrete valuation ring induced by $\mathcal{S}$, and we denote its valuation by $v_S$ and take $s$ as a uniformizing parameter. A function $f \in \mathcal{O}_{B,\eta}$ has valuation $n$ (i.e. $v_S(f) = n$) if and only if $n$ is the order of the zero of $f$ at $s = 0$; if $f$ has a pole at $s = 0$, the valuation is negative. Furthermore, by definition, $v_S(0) = \infty$ and $v_S(1) = 0$.

A genus-one fibration is a surjective proper morphism $\varphi : Y \to B$ between algebraic varieties such that the generic fiber is a smooth curve of genus one. An elliptic fibration is a genus-one fibration endowed with a rational section, which is a rational map $\sigma : B \to Y$ such that $\varphi \circ \sigma$ is the identity when restricted to the domain of $\sigma$. We will also assume that the base $B$ is a smooth (quasi)-projective variety, and that the Mordell-Weil group of the fibration (i.e. the group of rational sections) is trivial. The locus $\Delta$ of points $p$ such that the fiber $Y_p$ is singular is called the discriminant locus of the fibration. Note that a smooth fiber over a closed point is characterized (up to isomorphism) by its $j$-invariant.

Any elliptic fibration over a smooth variety defined with an algebraically closed field is birational to a (possibly singular) Weierstrass model [16, 49, 50]. For Weierstrass models, we use the notation of Tate as presented in Deligne’s formulaire [16]. Tate’s algorithm determines the type of the geometric fiber over the special point of a Weierstrass model over a discrete valuation ring by manipulating the coefficients of the Weierstrass model [57]. There are well-known formulas to compute the discriminant locus $\Delta$ and the $j$-invariant of a Weierstrass model [16]. The type of singular fibers over a generic point of a divisor $S$ of the base are classified by Kodaira and Néron. We denote the type of the geometric generic fiber over an irreducible component of the discriminant locus by one of the Kodaira...
symbols $I_n$, IV, III, II, $I_n^*$, IV*, III*, II* [38].

An algebraic $d$-cycle of a Noetherian scheme $X$ is an element of the free group $Z_d(X)$ generated by irreducible algebraic subvarieties of $X$ of dimension $d$. Following Kodaira [38], we introduce the following definition.

**Definition 2.1** (Fiber type). The *type* of an algebraic 1-cycle $\Theta \in Z_1(X)$ with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$ consists of the isomorphism class of each irreducible curve $\Theta_i$, together with the topological structure of the reduced polyhedron $\sum_i \Theta_i$.

The topological structure of the polyhedron $\sum_i \Theta_i$ is characterized by the underlying set of the scheme-theoretic intersection $\Theta_i \cap \Theta_j$ ($i \neq j$) of the irreducible components $\Theta_i$.

**Definition 2.2** (Dual graph). Given an algebraic one-cycle $\Theta$ with irreducible decomposition $\Theta = \sum_i m_i \Theta_i$, we associate a weighted graph (called the *dual graph* of $\Theta$) such that

- the vertices are the irreducible components $\Theta_i$ of the fiber,
- the weight of the vertex corresponding to an irreducible component $\Theta_i$ is its multiplicity $m_i$,
- the vertices corresponding to the irreducible components $\Theta_i$ and $\Theta_j$ ($i \neq j$) are connected by $\Theta_{i,j} = \deg(\Theta_i \cap \Theta_j)$ edges.\(^3\)

The type of the geometric fiber over a codimension one point of a minimal elliptic fibration is called the *Kodaira type* of the fiber. As shown by Kodaira [38] and Néron [51], there are 10 Kodaira types, and we denote them with the notation of Kodaira. The dual graph of a Kodaira fiber is always an affine Dynkin diagram of type ADE (see Table 2), while the dual graph of a generic fiber itself can be the affine Dynkin diagram of a non-simply-laced simple Lie algebra.

Let $G$ be a simply-connected simple Lie group with Dynkin diagram $\mathfrak{g}$. We denote by $\overline{\mathfrak{g}}$ the affine Dynkin diagram that reduces, upon removal of its extra node, to the Dynkin diagram $\mathfrak{g}$. Following Carter [13], we write $\overline{\mathfrak{g}}'$ for its Langlands dual, namely, the twisted Dynkin diagram whose Cartan matrix is the transpose of the Cartan matrix of $\overline{\mathfrak{g}}$. In particular, $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{g}}'$ are distinct only when $\mathfrak{g}$ is not simply laced (that is, when $\mathfrak{g}=\mathfrak{B}_k, \mathfrak{C}_k, \mathfrak{G}_2$, or $\mathfrak{F}_4$).

| Tate’s Step | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------------|---|---|---|---|---|---|---|---|---|----|
| Kodaira’s symbol | $I_0$ | $I_{n>0}$ | II | III | IV | $I_0^*$ | $I_{n>0}^*$ | IV* | III* | II* |
| Néron’s type | $A$ | $B_n$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5,n$ | $C_6$ | $C_7$ | $C_8$ |
| Dual graph | $-\tilde{A}_{n-1}$ | $\tilde{A}_1$ | $\tilde{A}_1$ | $\tilde{A}_2$ | $\tilde{D}_4$ | $\tilde{D}_{4+n}$ | $\tilde{E}_6$ | $\tilde{E}_7$ | $\tilde{E}_8$ |
| $v(\Delta)$ | 0 | $-n$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $v(j)$ | 0 | $-n$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2: Kodaira-Néron classification. When working over an algebraically closed field, the fiber type is determined uniquely by the valuation of the discriminant $\Delta$ and the $j$-invariant.

**Definition 2.3** ($\mathcal{K}$-model). Let $\mathcal{K}$ be a fiber type. Let $S \subset B$ be a smooth divisor of a projective variety $B$. An elliptic fibration $\varphi : Y \to B$ over $B$ is said to be a $\mathcal{K}$-*model* if the discriminant locus $\Delta(\varphi)$ contains as an irreducible component a divisor $S \subset B$ such that the generic fiber over $S$ is of type $\mathcal{K}$ and any other fiber away from $S$ is irreducible.

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\(^3\)The degree $\deg(\alpha_0)$ of a zero-cycle $\alpha_0$ is defined by passing to the Chow ring and using the degree defined in [29, Definition 1.4].
The locus of points in the base that lie below singular fibers of a non-trivial elliptic fibration is a Cartier divisor $\Delta$ called the discriminant locus of the elliptic fibration. We denote the irreducible components of the reduced discriminant by $\Delta_i$. If the elliptic fibration is minimal, the type of the fiber over the generic point of $\Delta_i$ of $\Delta$ has a dual graph that is an affine Dynkin diagram $\overline{g}_t^i$. If the generic fiber over $\Delta_i$ is irreducible, $g_\text{i}$ is the trivial Lie algebra since $\overline{g}_t^i = A_0$. The Lie algebra $g$ associated with the elliptic fibration $\varphi: Y \to B$ is then the direct sum $g = \oplus_i g_i$, where the Lie algebra $g_i$ is such that the affine Dynkin diagram $\overline{g}_t^i$ is the dual graph of the fiber over the generic point of $\Delta_i$.

When an elliptic $\varphi: Y \to B$ has trivial Mordell-Weil group, the compact Lie group $G$ associated with the elliptic fibration $\varphi$ is semisimple and simply connected and is given by the formula $G = \exp \left( \bigoplus_i g_i \right)$, where the index $i$ runs over all the irreducible components of the reduced discriminant locus, the Lie algebra $g_i$ such that the affine Dynkin diagram $\overline{g}_t^i$ is the dual graph of the fiber over the generic point of the irreducible component $\Delta_i$ of the reduced discriminant of the elliptic fibration, and $\exp \left( \bigoplus_i g_i \right)$ is the compact simply connected Lie group whose Lie algebra is $\bigoplus_i g_i$.

**Definition 2.4** (G-model). Let $G$ be a compact, simply connected Lie group. An elliptic fibration $\varphi: Y \to B$ with an associated Lie group $G$ and trivial Mordell-Weil group is called a $G$-model.

Given a $G$-model, the generic fiber over $S$ is an affine Dynkin diagram. The generic fiber degenerates into a fiber of different type over points of codimension two in the base.

Given an elliptic fibration $\varphi: Y \to B$, if $S$ is an irreducible component of the discriminant locus, the generic fiber over $S$ can degenerate further over subvarieties of $S$. We distinguish between two types of degenerations. A degeneration is said to be arithmetic if it modifies the type of the fiber without changing the type of the geometric fiber. A degeneration is said to be geometric if it modifies the geometric type of the fiber.

The irreducible curves of the degenerations over codimension-two loci give weights of a representation $R$. However, they only give a subset of weights. Hence, we need an algorithm that retrieves the full representation $R$ given only a few of its weights. This problem can be addressed systematically using the notion of a saturated set of weights introduced by Bourbaki [10, Chap.VIII.§7. Sect. 2].

**Definition 2.5** (Saturated set of weights). A set $\Pi$ of integral weights is saturated if for any weight $\varpi \in \Pi$ and any simple root $\alpha$, the weight $\varpi - i\alpha$ is also in $\Pi$ for any $i$ such that $0 \leq i \leq \langle \varpi, \alpha \rangle$. A saturated set has highest weight $\lambda$ if $\lambda \in \Lambda^+$ and $\mu < \lambda$ for any $\mu \in \Pi$.

**Definition 2.6** (Saturation of a subset). Any subsets $\Pi$ of weights is contained in a unique smallest saturated subset. We call it the saturation of $\Pi$.

**Proposition 2.7.**

(a) A saturated set of weights is invariant under the action of the Weyl group.

(b) The saturation of a set of weights $\Pi$ is finite if and only if the set $\Pi$ is finite.

(c) A saturated set with highest weight $\lambda$ consists of all dominant weights lower than or equal to $\lambda$ and their conjugates under the Weyl group.

**Proof.** See [35, Chap. III §13.4].

**Theorem 2.8.** Let $\Pi$ be a finite saturated set of weights. Then there exists a finite dimensional $g$-module whose set of weights is $\Pi$. 
Proof. [10, Chap.VIII.§7. Sect. 2, Corollary to Prop. 5].

Definition 2.9 (Weight vector of a vertical curve). Let $C$ be a vertical curve, i.e. a curve contained in a fiber of the elliptic fibration. Let $S$ be an irreducible component of the reduced discriminant of the elliptic fibration $\varphi : Y \to B$. The pullback of $\varphi^*S$ has irreducible components $D_0, D_1, \ldots, D_n$, where $D_0$ is the component touching the section of the elliptic fibration. The weight vector of $C$ over $S$ is by definition the vector $w_S(C) = (-D_1 \cdot C, \ldots, -D_n \cdot C)$ of intersection numbers $D_i \cdot C$ for $i = 1, \ldots, n$.

Definition 2.10 (Representation of a $G$-model). To a $G$-model, we associate a representation $R$ of the Lie algebra $g$ as follows. The weight vectors of the irreducible vertical rational curves of the fibers over codimension-two points form a set $\Pi$ whose saturation defines uniquely a representation $R$ by Theorem 2.8. We call this representation $R$ the representation of the $G$-model.

Definition 2.11 (Weierstrass model). Consider a variety $B$ endowed with a line bundle $\mathcal{L} \to B$. A Weierstrass model $\mathcal{E} \to B$ over $B$ is a hypersurface cut out by the zero locus of a section of the line bundle of $\mathcal{O}(3) \otimes \pi^* \mathcal{L}^6$ in the projective bundle $\mathbb{P}(\mathcal{O}_B \oplus \mathcal{L} \otimes \mathcal{L}^3) \to B$. We denote by $\mathcal{E}(1)$ the dual of the tautological line bundle of the projective bundle, and denote by $\mathcal{E}(n)$ ($n > 0$) its $n$th-tensor product. The relative projective coordinates of the $\mathbb{P}^2$ bundle are denoted by $[x : y : z]$. In particular, $x$ is a section of $\mathcal{E}(1) \otimes \pi^* \mathcal{L}^2$, $y$ is a section of $\mathcal{E}(1) \otimes \pi^* \mathcal{L}^3$, and $z$ is a section of $\mathcal{E}(1)$. Following Tate and Deligne’s notation, the defining equation of a Weierstrass model is

$$
\mathcal{E} : \quad zy(y + a_1x + a_3z) - (x^3 + a_2x^2z + a_4xz^2 + a_6z^3) = 0,
$$

where the coefficient $a_i$ ($i = 1, 2, 3, 4, 6$) is a section of $\mathcal{L}^i$ on $B$. Such a hypersurface is an elliptic fibration since over the generic point of the base, the fiber is a nonsingular cubic planar curve with a rational point ($x = z = 0$). We use the convention of Deligne’s formulaire [16]:

$$
b_2 = a_1^2 + 4a_2, \quad b_4 = a_1a_3 + 3a_4, \quad b_6 = a_3^2 + 4a_6, \quad b_8 = b_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2,
$$

$$
c_4 = b_2^2 - 24b_4, \quad c_6 = -b_2^3 + 36b_2b_4 - 216b_6,
$$

$$
\Delta = -b_2^6b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \quad j = \frac{c_4^3}{\Delta},
$$

These quantities are related by the relations

$$
4b_8 = b_2b_6 - b_4^2 \quad \text{and} \quad 1728\Delta = c_4^3 - c_6.
$$

The discriminant locus is the subvariety of $B$ cut out by the equation $\Delta = 0$, and is the locus of points $p$ of the base $B$ such that the fiber over $p$ (i.e. $Y_p$) is singular. Over a generic point of $\Delta$, the fiber is a nodal cubic that degenerates to a cuspidal cubic over the codimension-two locus $c_4 = c_6 = 0$. Up to isomorphisms, the $j$-invariant $j = \frac{c_4^3}{\Delta}$ uniquely characterizes nonsingular elliptic curves.

Definition 2.12 (Resolution of singularities). A resolution of singularities of a variety $Y$ is a proper birational morphism $\varphi : \tilde{Y} \to Y$ such that $\tilde{Y}$ is nonsingular and $\varphi$ is an isomorphism away from the singular locus of $Y$. In other words, $\tilde{Y}$ is nonsingular and if $U$ is the singular locus of $Y$, $\varphi$ maps $\varphi^{-1}(Y \setminus U)$isomorphically onto $Y \setminus U$.

Definition 2.13 (Crepant birational map). A birational map $\varphi : \tilde{Y} \to Y$ between two algebraic varieties with $\mathbb{Q}$-Cartier canonical classes is said to be crepant if it preserves the canonical class, i.e. $K_{\tilde{Y}} = \varphi^*K_Y$. 

9
2.2 Step 6 of Tate’s algorithm

Tate’s algorithm consists of eleven steps (see [57], [55, §IV.9], [52], and [56, §4.8]). Step 6 of Tate’s algorithm characterizes the fiber of Kodaira type $I_0^s$. We start with a general Weierstrass equation

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3.$$  

Néron proved in [51] that a Weierstrass model over a discrete valuation ring has a special fiber of type $I_0^s$ (denoted by $C_4$ in Néron’s notation) if and only if the discriminant

$$\Delta (x^3 + a_2x^2 + a_4x + a_6) = -4a_2^3a_6 + a_2a_4^2 + 18a_2a_4a_6 - 4a_4^3 - 27a_6^2$$

has valuation 6 with respect to $S$, and the valuation of the Weierstrass coefficients satisfies the following inequalities:

$$v_S(a_1) \geq 1, \quad v_S(a_2) \geq 1, \quad v_S(a_3) \geq 2, \quad v_S(a_4) \geq 2, \quad v_S(a_6) \geq 3. \quad (2.1)$$

This case is further studied by Tate in Step 6 of his algorithm [57]. Tate defines the following auxiliary cubic polynomial in the polynomial ring $\kappa[T]$ to be

$$P(T) = T^3 + a_{2,1}T^2 + a_{4,2}T + a_{6,3}, \quad (2.2)$$

where $a_{i,j} = a_i/s^j$. The splitting field of the cubic $P(T)$ in $\kappa$ is denoted by $\kappa'$. The discriminant $\Delta(P)$ of $P(T)$ is

$$\Delta(P) := 4a_{2,1}^3a_{6,3} - a_{2,1}a_{4,2}^2 - 18a_{2,1}a_{4,2}a_{6,3} + 4a_{4,2}^3 + 27a_{6,3}^2.$$  

The polynomial $P(T)$ is separable in $\kappa$ if and only if $P(T)$ has three distinct roots in $\kappa'$. This is the case if and only if the discriminant $\Delta(P)$ of $P(T)$ has valuation zero. This condition is equivalent to Néron’s requirement discussed above. In view of the inequalities in equation (2.1), the discriminant of the full Weierstrass equation has valuation 6 if and only if the valuation of $\Delta(P)$ is zero. The type of the geometric fiber is the same as the type of the fiber as seen in the splitting field $\kappa'$ of the polynomial $P(T)$ in $\kappa$. The type of the special fiber as a scheme over $\kappa$ depends on the degree $[\kappa' : \kappa]$ of the field extension $\kappa'/\kappa$:

- $[\kappa' : \kappa] = 6 \implies I_{0,ns}^s$ with Galois group $S_3$ and dual graph $\overline{G}_2^t$,
- $[\kappa' : \kappa] = 3 \implies I_{0,ss}^s$ with Galois group $\mathbb{Z}/3\mathbb{Z}$ and dual graph $\overline{G}_2^t$,
- $[\kappa' : \kappa] = 2 \implies I_{0,ss}^s$ with Galois group $\mathbb{Z}/2\mathbb{Z}$ and dual graph $\overline{B}_3^t$,
- $[\kappa' : \kappa] = 1 \implies I_{0,ss}^s$ with trivial Galois group and dual graph $\overline{D}_4$,

where “ns”, “ss”, and “s” stand for non-split, semi-split, and split, respectively [9].

If the discriminant of $P(T)$ does not have a $\kappa$-rational root, then the fiber is of type $I_{0,ns}^s$ with dual graph $\overline{G}_2^t$. If $P(T)$ has a unique $\kappa$-root, then the fiber is of type $I_{0,ss}^s$ with dual graph $\overline{B}_3^t$. If $P(T)$ has three $\kappa$-roots, then the fiber is of type $I_{0,ss}^s$ with dual graph $\overline{D}_4$.

There are two cases of fibers $I_{0,ns}^s$, depending on whether the Galois group of the field extension $\kappa'/\kappa$ is either $\mathbb{Z}/3\mathbb{Z}$ or $S_3$. The two cases differ by the behavior of the discriminant of $P(T)$ in view of the following well-known theorem.

---

4In F-theory, Tate’s algorithm is discussed in [9, 37, 48], but usually focuses on the minimal valuations of the coefficients of the Weierstrass equation. Hence, we use instead the original paper of Tate [57], which contains some typos that are listed and corrected in in [52].

5In the notation of Liu [42, §10.2], the fibers $I_{0,ns}^s$, $I_{0,ss}^s$, and $I_{0,s}^s$ are denoted by $I_{0,3}^s$, $I_{0,2}^s$, and $I_{0,1}^s$, respectively.
Lemma 2.14 (Galois group of a cubic polynomial). The Galois group of the splitting field of a separable cubic polynomial $P(T)$, defined over a field $\kappa$ of characteristic different from 2 and 3, is

1. $S_3$ if and only if the $P(T)$ is $\kappa$-irreducible and its discriminant is not a perfect square.

2. $\mathbb{Z}/3\mathbb{Z}$ if and only if $P(T)$ is $\kappa$-irreducible and its discriminant is a perfect square.

3. $\mathbb{Z}/2\mathbb{Z}$ if and only if $P(T)$ factorizes into a linear factor and a $\kappa$-irreducible quadric.

4. the trivial group if and only if $P(T)$ factorizes into three linear factors over $\kappa$.

Proof. See [40, Chap. VI §2].

Lemma 2.14 provides a direct route to the classification of fibers of type $I^*_0$, $I^{ss}_0$, and $I^{ns}_0$ using the Galois group of the splitting field of $P(T)$. A more refined classification will also take into account the values of the $j$-invariant. In contrast to other singular Kodaira fibers, the fiber of type $I^*_0$ can take any finite value of the $j$-invariant. However, for the arithmetic fibers, there are some restrictions. For example, a fiber of type $I^{ss}_0$ cannot have $j = 0$.

3 Summary of results

In this section, we summarize the main mathematical results of the paper. In section 7, we discuss the physical implications of our results for five- and six-dimensional gauge theories obtained from of F-theory and M-theory compactifications.

3.1 Canonical forms and crepant resolutions for $G_2$, Spin(7), and Spin(8)-models

We summarize the canonical forms for $G_2$, Spin(7), and Spin(8)-models from Step 6 of Tate’s algorithm. These forms are derived in Theorems 4.1, 5.1, and 6.1. We assume that the Mordell-Weil group of the elliptic fibrations is trivial. Using Step 6 of Tate’s algorithm, we obtain the Weierstrass model describing a Kodaira fiber of type $I^*_0$ over the generic point of $S = V(s)$. This model takes one of the following forms below, which are organized by the fiber type when seen as a scheme over the residue field of $S$, as well as the value of the $j$-invariant:

- $G^S_2$: $y^2z - x^3 - s^2f_{xz}^2 - s^3gz^3 = 0$, $fg \neq 0$, $j \neq 0, 1728$, \hspace{1cm} (3.1)

where $4f^3 + 27g^2$ is not a perfect square modulo $s$.

- $G^{\mathbb{Z}/3\mathbb{Z}}_2$: $y^2z - x^3 - s^2f_{xz}^2 - s^3gz^3 = 0$, $fg \neq 0$, $j \neq 0, 1728$, \hspace{1cm} (3.2)

where $4f^3 + 27g^2$ is a perfect square modulo $s$. In particular, the following $G^{\mathbb{Z}/3\mathbb{Z}}_2$-model is uniquely specified by the valuations of the coefficients:

- $G^{\mathbb{Z}/3\mathbb{Z}}_2$: $y^2z - x^3 - s^{3+\alpha}f_{xz}^2 - s^3gz^3 = 0$, $\alpha \in \mathbb{Z}_{\geq 0}$, $j = 0$, \hspace{1cm} (3.3)

where $g$ is not a cube modulo $s$;

- $\text{Spin}(7)$: $y^2z - (x^3 + a_{2,1}s^2z + a_{4,2}s^2xz^2 + a_{6,4}s^{4+\beta}z^3) = 0$, $\beta \in \mathbb{Z}_{\geq 0}$, $j \neq 0, 1728$, \hspace{1cm} (3.4)

where $a_{2,1}$ is not proportional to $s$;

- $\text{Spin}(7)$: $y^2z - x^3 - s^2f_{xz}^2 - s^{4+\beta}gz^3 = 0$, $\beta \in \mathbb{Z}_{\geq 0}$, $j = 1728$, \hspace{1cm} (3.5)
where $f$ is not a square modulo $s$;

- Spin(8): $zy^2 = (x - sx_1z)(x - sx_2z)(x + sx_3z) - s^{2+\alpha}Qz$, $Q = rx^2 + qsxz - s^2tz^2$, $\alpha \in \mathbb{Z}_{\geq 0}$, (3.6)

where $v_S((x_1 - x_2)(x_1 - x_3)(x_2 - x_3)) = 0$ and $(r,q,t) \neq (0,0,0)$. The $j$-invariant of a Spin(8)-model is $j = 1728\frac{A^2}{B^2 - x^2} = 1728 - \frac{A^2}{A_3 - B^2}$, where $A = -16(-x_1^2 - x_2^2 - x_3^2 + x_1x_2 + x_1x_3 + x_2x_3)$ and $B = 32(-2x_1 + x_2 + x_3)(x_1 - 2x_2 + x_3)(x_1 + x_2 - 2x_3)$. In particular, $j = 0$ when $A = 0$ and $j = 1728$ when $B = 0$.

In contrast to the usual forms seen in the F-theory literature, we do not restrict to the minimal values for the valuations of the coefficients. Allowing more general valuations enables a richer set of behaviors for the degeneration and the $j$-invariant. In the case of Spin(7), depending on the dimension of the base, the valuations can become an obstruction to the existence of a crepant resolution when they are too big.

### 3.2 Crepant resolutions

The following sequences of blowups provide crepant resolutions for the Weierstrass models defined in section 3.1. These are, however, valid only under some conditions which will be discussed in Theorems 4.2, 5.2, and 6.2. In some cases, the non-minimal valuations obstruct the existence of a crepant resolution. We assume that the coefficients of the Weierstrass models are general except for their valuations with respect to $S$. To prove smoothness, we also have to impose some light conditions on the coefficients that are usually left unspecified in the F-theory literature.

\[
\begin{align*}
G_2 & \quad X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(y,e_1|e_2)} X_2 \\
\text{Spin}(7) & \quad X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(y,e_1|e_2)} X_2 \xleftarrow{(x-x_1s,e_3|e_3)} X_3 \xleftarrow{(x-x_3s,e_2|e_4)} X_4 \\
\text{Spin}(8) & \quad X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(y,e_1|e_2)} X_2 \xleftarrow{(x-x_1s,e_3|e_3)} X_3 \xleftarrow{(x-x_3s,e_2|e_4)} X_4
\end{align*}
\] (3.7)

For Spin(8), $i,j$ are two distinct elements of $\{1,2,3\}$ and hence define six distinct crepant resolutions. We have two distinct crepant resolutions for Spin(7), and a unique one for $G_2$.

### 3.3 Hyperplane arrangement $I(\mathfrak{g}, R)$

Let $\mathfrak{g}$ be a semi-simple Lie algebra and $R$ a representation of $\mathfrak{g}$. The kernel of each weight $\varpi$ of $R$ defines a hyperplane $\varpi^\perp$ through the origin of the Cartan sub-algebra of $\mathfrak{g}$.

**Definition 3.1.** The hyperplane arrangement $I(\mathfrak{g}, R)$ is defined inside the dual fundamental Weyl chamber of $\mathfrak{g}$, i.e., the dual cone of the fundamental Weyl chamber of $\mathfrak{g}$, and its hyperplanes are the set of kernels of the weights of $R$.

For each $G$-model, we associate the hyperplane arrangement $I(\mathfrak{g}, R)$ using the representation $R$ induced by the weights of vertical rational curves produced by degenerations of the generic fiber over codimension-two points of the base. We then study the incidence structure of the hyperplane arrangement $I(\mathfrak{g}, R)$.
Proposition 3.2. The weights of the vertical curves over codimension-two points are

| Locus         | G₂          | Spin(7)               | Spin(8)               |
|---------------|-------------|-----------------------|-----------------------|
| Weight(s)     | V(s,4f²+2g²) | V(s,a₄2₁) - 4a₄2     | V(s,d)                |
| R             | 2           | 0 1 - 2               | ±1 0 - 1              |
| L(g,R)        | I(G₂,7)     | (B₃, 7 ⊗ 8)           | I(D₄, 8_v ⊗ 8_c ⊗ 8_s) |
| # of Chambers | 1           | 2                     | 6                     |

The chambers are illustrated in Figures 2, 3, and 4 on page 15 and described in the following theorem.

Theorem 3.3. The hyperplane arrangement I(G₂, 7) has a unique chamber. The hyperplane arrangement I(B₃, 7 ⊗ 8) has two chambers whose incidence graph is the Dynkin diagram A₂. The hyperplane arrangement I(D₄, 8_v ⊗ 8_c ⊗ 8_s) has six chambers whose incidence graph is a hexagon (the affine Dynkin diagram A₆).

Proof. A hyperplane Ï(w) (the kernel of a weight Ï) intersects the interior of the dual fundamental Weyl chamber of g if and only if when written in the basis of positive simple roots, at least two of its coefficients have different signs. The hyperplane arrangement I(G₂, 7) has a unique chamber as none of its weights have coefficients of different signs in the basis of positive simple roots. The hyperplane arrangement I(B₃, 7 ⊗ 8) has two chambers separated by the kernel of the unique (up to a sign) weight of the representation 7 ⊗ 8; this representation is not in the cone generated by the positive simple roots, namely [1 0 -1]. For Spin(8), each of the representations 8_v, 8_c, and 8_s has a unique weight (up to a sign) not in the cone generated by the positive simple roots. These are the following weights, written respectively for 8_v, 8_c, and 8_s in the basis of fundamental weights:

\[ Ï_v = [0 0 -1 1], \quad Ï_c = [-1 0 0 1], \quad Ï_s = [-1 0 1 0]. \]

Let \( φ = (φ_1, φ_2, φ_3, φ_4) \) ∈ h be a vector of the dual fundamental Weyl chamber written in the basis of simple coroots. The linear forms corresponding to the weights Ï_v, Ï_c, and Ï_s are

\[ L_1 = Ï_v, \quad L_2 = Ï_c, \quad L_3 = Ï_s. \]

The set of chambers of I(D₄, 8_v ⊗ 8_c ⊗ 8_s) is in bijection with the set of all possible signs of \( (L_1, L_2, L_3) \). Two chambers are adjacent to each other when the signs of \( (L_1, L_2, L_3) \) differ by one entry only. Since \( L_1 + L_3 = L_2 \), we can neither get the sign vector \((-+−)\) nor \((+++)\). It is easy to see that all the six remaining arrangements of signs are possible. In total, there are six chambers determined by the signs of \( L_1, L_2, \) and \( L_3 \):

1. \((-−−)\) \( φ_1 > φ_3 > φ_4 \), 2. \((-+++)\) \( φ_3 > φ_1 > φ_4 \), 3. \((++−)\) \( φ_3 > φ_4 > φ_1 \),
4. \((+++)\) \( φ_4 > φ_3 > φ_1 \), 5. \((+++)\) \( φ_4 > φ_1 > φ_3 \), 6. \((−−+)\) \( φ_1 > φ_4 > φ_3 \).

Two chambers are said to be adjacent when they differ by the sign of only one \( L_i \). It follows that there are six chambers organized as in Figure 4 on page 15.
### 3.4 Matching the crepant resolutions and the chambers of the hyperplane arrangement

For each crepant resolution, we study the fiber structure and compute geometrically the weights of vertical curves. These weights uniquely determine a chamber. $G_2$-models only have one crepant resolution. For $\text{Spin}(7)$-models, the crepant resolution $Y^\pm$ corresponds to $\pm [1,0,-1]$. For $\text{Spin}(8)$-models, the vertical curves give the weights $\pm L_1$, $\pm L_2$, and $\pm L_3$. The explicit matching of the chambers and the crepant resolutions is given in Table 3.

| Resolutions | Chambers          |
|-------------|-------------------|
| 1 $Y^{(2,3)}$ | $(- - -)$ ($\phi_1 > \phi_3 > \phi_4$) |
| 2 $Y^{(3,2)}$ | $(- - +)$ ($\phi_3 > \phi_1 > \phi_4$) |
| 3 $Y^{(3,4)}$ | $(- + +)$ ($\phi_3 > \phi_4 > \phi_1$) |
| 4 $Y^{(4,3)}$ | $(+ + +)$ ($\phi_4 > \phi_3 > \phi_1$) |
| 5 $Y^{(4,2)}$ | $(+ + -)$ ($\phi_4 > \phi_1 > \phi_3$) |
| 6 $Y^{(2,4)}$ | $(+ - -)$ ($\phi_1 > \phi_4 > \phi_3$) |

Table 3: Matching the crepant resolutions of the $\text{Spin}(8)$-model with the chambers of the hyperplane arrangement $I(D_3, 8_v \oplus 8_s \oplus 8_c)$. 

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Figure 1: Non-Kodaira fibers appearing in the fiber structures of $G_2$, $\text{Spin}(7)$, and $\text{Spin}(8)$-models
Figure 2: Chambers of the hyperplane arrangement $I(G_2, 7)$ or equivalently, the Coulomb phases of a $G_2$ gauge theory with matter in the representation 7. There is a unique chamber since the non-zero weights of 7 are the short roots of $G_2$.

Figure 3: Chambers of the hyperplane arrangement $I(B_3, 8)$ or equivalently, the Coulomb phases of a Spin(7) gauge theory with matter in the representation 8. The only weight defining an interior wall is the weight $[1, 0, -1]$ of the representation 8.

Figure 4: Chambers of the hyperplane arrangement $I(D_4, 8_v \oplus 8_s \oplus 8_c)$ or equivalently, the Coulomb phases of a Spin(8) gauge theory with matter in the representation $8_v \oplus 8_s \oplus 8_c$. The signs in the figure are those of linear forms induced by the weights $L_1 = [0, 0, -1, 1]$, $L_2 = [-1, 0, 0, 1]$, and $L_3 = [-1, 0, 1, 0]$. 

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3.5 Fiber degenerations

The $G_2^{S_3}$-model is the generic case of a fiber of type $I^*_6$ and has the simple fiber structure presented in Figure 5. In codimension two, the generic fiber with dual graph $\tilde{G}_2^t$ degenerates to an incomplete affine Dynkin diagram of type $\tilde{D}_5$, whose dual graph is a Dynkin diagram of type $D_4$. The fiber degenerates further in codimension three to a fiber of type $I^*_2-2-3$, which can be understood as an incomplete fiber of type $\tilde{E}_6$. The $G_2^{Z/3Z}$-model can be realized in various ways. The model is defined in equation (4.2), and has a fiber structure with an enhancement in codimension two from $\tilde{G}_2^t$ to an incomplete $\tilde{E}_6$. This can be understood as a specialization of $G_2^{S_3}$ in which the codimension two fiber $\tilde{D}_5$ does not appear and the fiber degenerates directly to an incomplete $\tilde{E}_6$. The fiber structure of the $G_2^{Z/3Z}$-model defined in equation (4.3) is presented in Figure 7 and has two distinct fibers in codimension-two, in contrast to the other $G_2$-model, which has only one type of specialization in codimension two.

The generic fiber structure of a Spin(7)-model is presented in Figure 3.5. The fiber structure depends on the valuations ($v_S(a_2) \geq 1$ and $v_S(a_6) \geq 4$) and the choice of a crepant resolution. The valuations can be organized into four different cases, and we have two possible choice of crepant resolutions. Thus, there are eight distinct types of fiber structures. Two resolutions related by a flop have distinct fibers in codimension two, three, or four, depending on the valuations. We organize this information by grouping the flops together. In one of the two possible resolutions, both the $C_2$ and $C_3$ components of the generic $\tilde{B}_3^t$ fiber degenerate, while in the flop, only $C_3$ degenerates.

The first case is in Figure 3.5, and corresponds to the lowest possible values for the valuations of $a_2$ and $a_6$, namely $v_S(a_2) = 1$ and $v_S(a_6) = 4$. All the other cases are specialization of this case, obtained by skipping some of the intermediate steps in the degenerations as forced by the valuations. The fiber structure in Figure 9 is for the case ($v_S(a_2) = 1, v_S(a_6) \geq 5$), Figure 10 for the case ($v_S(a_2) = 2, v_S(a_6) = 4$), and finally Figure 11 for the case ($v_S(a_2) \geq 2, v_S(a_6) \geq 5$). Each of these Weierstrass models has two possible resolutions related by a flop, and each possible resolution has a different fiber in codimension two or three.

For the Spin(8)-models, the fiber structure is the split case of a fiber of type $I^*_6$ and is presented in Figures 12 and 13. The most generic case for the Spin(8)-models is when $\alpha = 0$. In such a model, the fiber structure degenerates into $\tilde{D}_5$ in codimension two, $\tilde{E}_6$ or an incomplete $\tilde{D}_6$ in codimension three, and an incomplete $\tilde{E}_7$ in codimension four. The fiber structure for $\alpha > 0$ is presented in Figure 13, in which the fiber structure skips $\tilde{D}_5$ and degenerates directly into either an incomplete $\tilde{D}_6$ in codimension two or an incomplete $\tilde{E}_7$ in codimension three. In contrast to Spin(7)-models, the fiber type does not depend on the choice of the crepant resolution. Even though different crepant resolutions are differentiated by the way the curves split, they give the same fiber types.
Figure 5: Geometric fiber degeneration of $\tilde{G}_2^{S_3}$-model with valuation (2, 3, 6).

Figure 6: Geometric fiber degeneration of $\tilde{G}_2^{\mathbb{Z}/3\mathbb{Z}}$ with valuation ($\geq 3, 3, 6$).

Figure 7: Geometric fiber degeneration for a $\tilde{G}_2^{\mathbb{Z}/3\mathbb{Z}}$-model with valuation (2, 3, 6) and special configuration for $f$ and $g$ such that $\Delta' = 4f^3 + 27g^2$ is a perfect square in the residue field $\kappa(\eta)$. We assume that the base $B$ is a surface to avoid $\mathbb{Q}$-factorial terminal singularities.
Figure 8: Geometric fiber degeneration of a Spin(7)-model with $v_S(a_2) = 1$ and $v_S(a_6) = 4$. When $v_S(a_2) > 1$, the degeneration graph contracts to the middle row. When there are multiple fibers, the one on the left corresponds to the monomial resolution and the one on the right to its flop.
Figure 9: Geometric fiber degeneration of a Spin(7)-model with $v_S(a_2) = 1$ and $v_S(a_6) \geq 5$.

Figure 10: Geometric fiber degeneration of a Spin(7)-model with $v_S(a_2) \geq 2$ and $v_S(a_6) = 4$.

Figure 11: Geometric fiber degeneration of a Spin(7)-model with $v_S(a_2) \geq 2$ and $v_S(a_6) \geq 5$. 
Figure 12: Fiber degeneration of a Spin(8)-model with $\alpha = 0$.

Figure 13: Fiber degeneration of a Spin(8)-model with $\alpha > 0$. 

$\widetilde{D}_4$  Degenerate $\widetilde{D}_5$  Degenerated $\widetilde{E}_7$
3.6 Triple intersection numbers

Two crepant resolutions of the same Weierstrass model have the same Euler characteristic [8]. In this subsection, we compute a topological invariant that is dependent on the choice of the crepant resolution. Let \( D_0, D_1, \ldots, D_r \) be the fibral divisors of an elliptic fibration over a base \( B \) of dimension \( d \). By definition, they are the irreducible components of

\[
\varphi^* S = m_0 D_0 + m_1 D_1 + \cdots + m_r D_r,
\]

where \( m_i \) is the multiplicity of \( D_i \) and \( D_0 \) is the divisor touching the section of the elliptic fibration. It is useful to introduce a polynomial ring \( A_*(Y)[\phi_0, \ldots, \phi_r] \) over the Chow ring \( A_*(Y) \) of \( Y \).

We define the polynomial \( \mathcal{F} \) in \( A_*(B)[\varphi_0, \ldots, \varphi_r] \) via a pushforward as

\[
\mathcal{F} := \varphi_*(D_0\phi_0 + D_1\phi_1 + \cdots D_r\phi_r)^3.
\]

Hence, if \( M \) is an element of \( A_{d-3}(B) \), we have

\[
\int_Y (D_0\phi_0 + D_1\phi_1 + \cdots D_r\phi_r)^3 \cdot \varphi^* M = \int_B \varphi_*(D_0\phi_0 + D_1\phi_1 + \cdots D_r\phi_r)^3 \cdot M = \int_B \mathcal{F} \cdot M.
\]

The polynomial \( \mathcal{F} \) is called the triple intersection polynomial of the elliptic fibration. When the base is a surface, its coefficients are numbers.

A Weierstrass model of a \( G_2 \)-model has a unique crepant resolution and therefore a unique possible triple intersection polynomial. There are two distinct crepant resolutions \( Y^* \) for the Weierstrass model corresponding to a \( \text{Spin}(7) \)-model. These two crepant resolutions also have different triple intersection polynomials \( \mathcal{F}^\pm_{\text{Spin}(7)} \). The Weierstrass model of a \( \text{Spin}(8) \)-model has six distinct crepant resolutions \( Y^{(i,j)} \) with \( \{i, j \neq i \} \subset \{2, 3, 4\} \), and we will compute all six different polynomials \( \mathcal{F}^{(i,j)}_{\text{Spin}(8)} \).

**Theorem 3.4.** The triple intersection numbers of a \( G_2 \), \( \text{Spin}(7) \), or \( \text{Spin}(8) \)-model defined by the sequence of blowups listed in section 3.2 are

\[
\mathcal{F}_{G_2} = -4S(S - L)\phi_0^3 + 3S(S - 2L)\phi_0^2\phi_1 + 3LS\phi_0\phi_1^2 \\
- 4S(S - L)\phi_1^3 + 12S(S - 3L)\phi_1^2\phi_2 + 9S(2S - 3L)\phi_1^2\phi_2 - 27S(S - 2L)\phi_1\phi_2^2
\] (3.9)

\[
\mathcal{F}^\pm_{\text{Spin}(7)} = -4S(S - L)\phi_0^3 + 3S(S - 2L)\phi_0^2\phi_2 + 3LS\phi_0\phi_2^2 - 4S(S - L)\phi_2^3 \\
- 3S(S - 2L)(4\phi_3^2 + 4\phi_1\phi_3 + \phi_3^2)\phi_2 + 3S(2S - 3L)(\phi_1 + 2\phi_3)\phi_2^2 \\
- 4LS\phi_3^3 - 8LS\phi_3^2 + 12S(S - 2L)\phi_1\phi_3^2
\] (3.10)

\[
\mathcal{F}^\mp_{\text{Spin}(7)} = -4S(S - L)\phi_0^3 + 3S(S - 2L)\phi_0^2\phi_2 + 3LS\phi_0\phi_2^2 - 4S(S - L)\phi_2^3 \\
- 3S(S - 2L)(4\phi_3^2 + 4\phi_1\phi_3 + \phi_3^2)\phi_2 + 3S(2S - 3L)(\phi_1 + 2\phi_3)\phi_2^2 \\
- 4S(S - L)\phi_3^3 + 4S(S - 4L)\phi_3^3 + 12S(S - 2L)\phi_1\phi_3^2
\] (3.11)

\[
\mathcal{F}^{(i,j)}_{\text{Spin}(8)} = -4S(S - L)\phi_0^3 + 3S(S - 2L)\phi_0^2\phi_1 + 3LS\phi_0\phi_1^2 - 4S(S - L)\phi_1^3 \\
+ 3S(2S - 3L)(\phi_2 + \phi_3 + \phi_4)\phi_1 - 3S(S - 2L)(\phi_2 + \phi_3 + \phi_4)^2 - 4\phi_3^2 \phi_1 \\
- 4LS\phi_1^3 - 2S^2\phi_3^2 - 4S(S - L)\phi_3^3 + 6S(S - 2L)(\phi_j\phi_k^2 + \phi_i\phi_k^2 + 6\phi_i\phi_j^2).
\] (3.12)
We see that \( \mathcal{F}_{\text{Spin}(7)}^+ \) are related via
\[
\mathcal{F}_{\text{Spin}(7)}^+ = \pi_{13} \mathcal{F}_{\text{Spin}(7)}^+ + 4S(2L - S)(\phi_1 - \phi_3)^3,
\] (3.13)
where \( \pi_{13} \) is the permutation \( \phi_1 \leftrightarrow \phi_3 \).

**Lemma 3.5.** If a \( G \)-model is a Calabi-Yau threefold, \( c_1 = L = -K \). Furthermore, denoting by \( g \) the genus of \( S \), the triple intersection numbers are
\[
\begin{align*}
\mathcal{F}_{G_2} &= -8(g - 1)\phi_0^3 + 3\phi_1\phi_0^2 (4g - 4 - S^2) - 3\phi_1^2\phi_0 (2g - 2 - S^2) \\
&- 8(g - 1)\phi_1^3 + 24 (3g - 3 - S^2) \phi_2^3 - 27(4g - 4 - S^2) \phi_1^2\phi_2 + 9(6g - 6 - S^2) \phi_1\phi_2^2 \\
\mathcal{F}_{\text{Spin}(7)}^+ &= -8(g - 1)\phi_0^3 + 3 (4g - 4 - S^2)\phi_0^2\phi_2 - 3(2g - 2 - S^2)\phi_0\phi_2^2 - 8(g - 1)\phi_2^3 \\
&- 3(4g - 4 - S^2)(4\phi_2 + 4\phi_1\phi_3 + 3\phi_2^2)\phi_2 + 3(6g - 6 - S^2)(\phi_1 + 2\phi_3)\phi_2^2 \\
&+ 4\phi_2^3(2g - S^2 - 2) + 8\phi_3^2(2g - S^2 - 2) + 12\phi_1\phi_2^2(4g - 4 - S^2) \\
\mathcal{F}_{\text{Spin}(7)}^- &= -8(g - 1)\phi_0^3 + 3(4g - 4 - S^2)\phi_0^2\phi_2 - 3(2g - 2 - S^2)\phi_0\phi_2^2 - 8(g - 1)\phi_2^3 \\
&- 3(4g - 4 - S^2)(4\phi_2 + 4\phi_1\phi_3 + 3\phi_2^2)\phi_2 + 3(6g - 6 - S^2)(\phi_3 + 2\phi_1)\phi_2^2 \\
&- 8(g - 1)\phi_3^3 + 4S(S - 4L)\phi_3^1 + 12(4g - 4 - S^2)\phi_1\phi_3^2 \\
\mathcal{F}_{\text{Spin}(8)}^{(i,j)} &= -8(g - 1)\phi_0^3 + 3(4g - 4 - S^2)\phi_0^2\phi_1 - 3(2g - 2 - S^2)\phi_0\phi_1^2 - 8(g - 1)\phi_1^3 \\
&+ 3(6g - g - S^2)(\phi_2 + \phi_3 + 4\phi_4)\phi_1^2 - 3(4g - 4 - S^2)\phi_1^2(\phi_2 + \phi_3 + 4\phi_4)^2 - 4\phi_k^2 \phi_1 \\
&4(2g - 2 - S^2)\phi_3^1 - 2S^2\phi_3^2 - 8(g - 1)\phi_k^3 - 6(4g - 4 - S^2)(\phi_j\phi_k^2 + \phi_i\phi_j^2 + 6\phi_i\phi_k^2),
\end{align*}
\] (3.14, 3.15, 3.16, 3.17)
where \( (i, j, k) \) is a permutation of \((1, 2, 3)\).

Lemma 3.5 is a direct specialization of Theorem 3.4. Theorem 3.4 is proven using the following pushforward theorems (see [21] for examples of such computations).

**Theorem 3.6** (Esole–Jefferson–Kang, see [21]). Let the nonsingular variety \( Z \subset X \) be a complete intersection of \( d \) nonsingular hypersurfaces \( Z_1, \ldots, Z_d \) meeting transversally in \( X \). Let \( E \) be the class of the exceptional divisor of the blowup \( f : \tilde{X} \rightarrow X \) centered at \( Z \). Let \( \tilde{Q}(t) = \sum_a f^* Q_a t^a \) be a formal power series with \( Q_a \in A_*(X) \). We define the associated formal power series \( Q(t) = \sum_a Q_a t^a \) whose coefficients pullback to the coefficients of \( \tilde{Q}(t) \). Then the pushforward \( f_* \tilde{Q}(E) \) is:
\[
f_* \tilde{Q}(E) = \sum_{\ell=1}^d Q(Z_\ell) M_\ell, \quad \text{where} \quad M_\ell = \prod_{m=1}^d \frac{Z_m}{Z_m - Z_\ell}.
\]

The second pushforward theorem deals with the projection from the ambient projective bundle to the base \( B \) over which the Weierstrass model is defined. Let \( \mathcal{V} \) be a vector bundle of rank \( r \) over a nonsingular variety \( B \). The Chow ring of a projective bundle \( \pi : \mathbb{P}(\mathcal{V}) \rightarrow B \) is isomorphic to the module \( A_*(B)[\zeta] \) moded out by the relation [29, Remark 3.2.4, p. 55]
\[
\zeta^r + c_1(\pi^* \mathcal{V}) \zeta^{r-1} + \cdots + c_i(\pi^* \mathcal{V}) \zeta^{r-i} + \cdots + c_r(\pi^* \mathcal{V}) = 0, \quad \zeta = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)).
\]
Theorem 3.7 (See [21] and [1, 2, 22, 30]). Let \( \mathcal{L} \) be a line bundle over a variety \( B \) and \( \pi : X_0 = \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^\otimes 2 \oplus \mathcal{L}^\otimes 3] \rightarrow B \) a projective bundle over \( B \). Let \( \overline{Q}(t) = \sum_a \pi^* Q_a t^a \) be a formal power series in \( t \) such that \( Q_a \in A_4(B) \). Define the auxiliary power series \( Q(t) = \sum_a Q_a t^a \). Then

\[
\pi_* \overline{Q}(H) = -2 \left. \frac{Q(H)}{H^2} \right|_{H=-2L} + 3 \left. \frac{Q(H)}{H^2} \right|_{H=-3L} + \frac{Q(0)}{6L^2},
\]

where \( L = c_1(\mathcal{L}) \) and \( H = c_1(\mathcal{O}_{X_0}(1)) \) is the first Chern class of the dual of the tautological line bundle of \( \pi : X_0 = \mathbb{P}[\mathcal{O}_B \oplus \mathcal{L}^\otimes 2 \oplus \mathcal{L}^\otimes 3] \rightarrow B \).

Lemma 3.5 is a specialization to the case of Calabi-Yau threefolds and will play an important role in section 7.1. For each model and each irreducible representation \( R_i \) induced by its fiber degeneration, we compute the number of hypermultiplets transforming in the representation \( R_i \). The results are given by Proposition 7.1 by comparing the triple intersection numbers given in Lemma 3.5 with the one-loop prepotential computed in section 7.1:

- **G2**: \( n_7 = -10(g-1) + 3S^2 \), \( n_{14} = g \),
- **Spin(7)**: \( n_7 = S^2 - 3(g-1) \), \( n_8 = 2S^2 - 8(g-1) \), \( n_{21} = g \),
- **Spin(8)**: \( n_{8c} = n_{8a} = n_{8c} = S^2 - 4(g-1) \), \( n_{28} = g \).

### 3.7 Euler characteristics and Hodge numbers

The results of this subsection are proven in [21] and used in section 7 to check the cancellation of anomalies in the six-dimensional supersymmetric theory.

**Theorem 3.8** (Euler characteristics). *Smooth elliptic fibrations \( Y \rightarrow B \) defined as crepant resolutions of the Weierstrass models given in section 3.2 have the following Euler characteristics:

- **G2**: \( \chi(Y) = 12 \int_B \frac{L + 2SL - S^2}{(1 + S)(1 + 6L - 3S)} c(TB) \)
- **Spin(7)**: \( \chi(Y) = 4 \int_B \frac{3L + (12L^2 + LS - 5S^2) + 5(3L - 2S)(2L - S)S}{(1 + S)(-1 - 4L + 4S)(-1 + 6L + 2S)} c(TB) \)
- **Spin(8)**: \( \chi(Y) = 12 \int_B \frac{L + 3SL - 2S^2}{(1 + S)(1 + 6L - 4S)} c(TB), \)

where \( L = c_1(\mathcal{L}) \), \( S \) is the class of the divisor \( S \), \( c(TB) \) is the total Chern class of the tangent bundle of the base \( B \) of the Weierstrass model, and \( \int_B A = \int A \cap B \) is the degree in the Chow ring of the base \( B \).

We determine the Euler characteristic for a \( d \)-dimensional base \( B \) via the coefficient of \( t^d \), after substituting \( L \rightarrow tL \), \( S \rightarrow tS \), and replacing \( c(TB) \) by the Chern polynomial \( c_1(TB) = 1 + c_1 t + c_2 t^2 + \cdots + c_d t^d \) with \( c_d = c_d(TB) \). We give the results for threefolds and fourfolds below.

**Lemma 3.9.** If the base is a surface, \( Y \) is a threefold, and the Euler characteristic for each model is

| Model   | Formula                        |
|---------|--------------------------------|
| G2      | \( 12(c_1 L - 6L^2 + 4LS - S^2) \) |
| Spin(7) | \( 4(3c_1 L - 18L^2 + 16LS - 2S^2) \) |
| Spin(8) | \( 12(c_1 L - 6L^2 + 6LS - 2S^2) \) |
We specialize to the Calabi-Yau case by requiring $L = -K_B$. In the case of a Calabi-Yau threefold, we have the following result.

**Lemma 3.10.** For Calabi-Yau threefolds, let $S$ be the divisor over which we have the fiber $I_0$. If $g$ denotes the genus of $S$ and $K$ denotes the canonical class of the base, then the Euler characteristic for each model is

| Model     | $\chi$     |
|-----------|------------|
| $G_2$     | $-60K^2 + 96(1-g) + 36S^2$ |
| Spin(7)   | $-60K^2 + 128(1-g) + 44S^2$ |
| Spin(8)   | $-60K^2 + 144(1-g) + 48S^2$ |

**Theorem 3.11.** For Calabi-Yau threefolds, let $S$ be the divisor over which we have the fiber $I_0$. If $g$ denotes the genus of $S$ and $K$ denotes the canonical class of the base, then the Hodge numbers $h^{1,1}$ and $h^{1,2}$ for each model are

| Model     | $h^{1,1}$       | $h^{1,2}$       |
|-----------|-----------------|-----------------|
| $G_2$     | $13 - K^2$      | $13 + 29K^2 - 18S^2 + 48(g-1)$ |
| Spin(7)   | $14 - K^2$      | $14 + 29K^2 - 22S^2 + 64(g-1)$ |
| Spin(8)   | $15 - K^2$      | $15 + 29K^2 - 24S^2 + 72(g-1)$ |

**Lemma 3.12.** If $Y$ is a fourfold, then the Euler characteristic for each model is

| Model     | $\chi$     |
|-----------|------------|
| $G_2$     | $12(-6c_1L^2 + 4c_1LS - c_1S^2 + c_2L + 36L^3 - 42LS^2 + 17LS^2 - 2S^3)$ |
| Spin(7)   | $4(-18c_1L^2 + 16c_1LS - 5c_1S^2 + 3c_2L + 108L^3 - 166LS^2 + 89LS^2 - 15S^3)$ |
| Spin(8)   | $12(-6c_1L^2 + 6c_1LS - 2c_1S^2 + c_2L + 36L^3 - 60LS^2 + 34LS^2 - 6S^3)$ |

**Lemma 3.13.** If the base is a threefold and $Y$ is a Calabi-Yau fourfold, the Euler characteristic for each model is

| Model     | $\chi$     |
|-----------|------------|
| $G_2$     | $-12(c_2K + 30K^3 + 38K^2S + 16KS^2 + 2S^3)$ |
| Spin(7)   | $-12(c_2K + 30K^3 + 50K^2S + 28KS^2 + 5S^3)$ |
| Spin(8)   | $-12(c_2K + 30K^3 + 54K^2S + 32KS^2 + 6S^3)$ |

## 4 G2-models

A $G_2$-model is a Weierstrass model with a geometric fiber of type $I_0$ over the generic point of a divisor $S$ of the base such that the auxiliary polynomial $P(T)$ is $\kappa$-irreducible. We distinguish two types of $G_2$-models, depending on the Galois group $\text{Gal}(\kappa'/\kappa)$. If the discriminant $\Delta(P)$ of the associated cubic polynomial $P(T)$ is a perfect square modulo $s$, then the Galois group is $\mathbb{Z}/3\mathbb{Z}$, and we call such a model a $G_2^{S^3}$-model. Otherwise the Galois group is the symmetric group $S_3$, and the model is called a $G_2^{S^1}$-model. Geometrically, these two types have different fiber structures from codimension-two.

**Theorem 4.1** (Canonical form for $G_2$-models). A $G_2^{S^3}$-model can always be written in the following canonical form

$$G_2^{S^3} : \quad y^2z = x^3 + s^2fxz^2 + s^3gz^3, \quad v(f) \geq 0, \quad v(g) = 0.$$  

The polynomial $P(T) = T^3 + fT + g$ is an irreducible cubic in $\kappa[T]$ and $\Delta(P) = 4f^3 + 27g^2$ is its discriminant. If $\Delta(P)$ is not a perfect square in $\kappa$, the Galois group $\text{Gal}(\kappa'/\kappa)$ is $S_3$. Otherwise, it is $\mathbb{Z}/3\mathbb{Z}$. The $j$-invariant is $j = 1728\frac{4f^3}{T^3 + 27g^2}$, which varies over $S$. 

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\textbf{Proof.} Directly follows from Step 6 of Tate’s algorithm and Lemma 2.14.

The generic case of a fiber of type \( I_0^* \) is when the Galois group \( \text{Gal}(\kappa'/\kappa) \) is the symmetric group \( S_3 \) and requires \( v(f) = 0 \), i.e.

\[
G_2^{S_3} \quad y^2 z = x^3 + s^2 f x z^2 + s^3 g z^3, \quad v(f) = v(g) = 0.
\]

If we increase the valuation of \( f \), the Galois group is automatically \( \mathbb{Z}/3\mathbb{Z} \) since \( \Delta(P) \) will be a perfect square modulo \( s \). Note that \( g \) cannot be a perfect cube modulo \( s \) because otherwise, \( P(T) = T^3 + g \) will have three \( \kappa \)-roots and the model would be a \( \text{Spin}(8) \)-model instead of a \( G_2 \)-model. Hence, we have in this case

\[
G_2^{\mathbb{Z}/3\mathbb{Z}} \quad y^2 z = x^3 + s^{3+\alpha} f x z^2 + s^3 g z^3, \quad \alpha \geq 0, \quad v(f) = v(g) = 0, \quad g \text{ is not a cube modulo } s. \quad (4.2)
\]

In section 4.2, we prove that this \( G_2^{\mathbb{Z}/3\mathbb{Z}} \)-model is compatible with a crepant resolution after some mild assumptions. Moreover, we allow \( \Delta(P) \) to be a perfect square modulo \( s \) using more complicated coefficients. For example, a Weierstrass model inspired by well-known examples in number theory is

\[
G_2^{\mathbb{Z}/3\mathbb{Z}} \quad y^2 = x^3 + s^2(-3ar + sq)x + s^3(a^2r + ar^2 + st). \quad (4.3)
\]

This model suffers from \( \mathbb{Q} \)-factorial terminal singularities obstructing the existence of a crepant resolution if the base is of dimension three or higher. When the base is a surface, this model does have a crepant resolution, and its fiber structure (see Figure 7) is much richer than that of equation (4.2) (see Figure 6). The fiber structure of the crepant resolution of a \( G_2^{S_3} \)-model described by equation (4.1) is presented in Figure 5.

Finally, we point out that all the different \( G_2 \)-models discussed have (at best) a unique crepant resolution. In other words, a smooth \( G_2 \)-model does not have flops. This can be explained numerically by studying the vertical curves produced by the codimension-two degenerations. They have intersection numbers with the fibral divisors corresponding to weights of the representation \( 7 \) of \( g_2 \); thus, the corresponding hyperplane arrangement \( I_{(g_2, 7)} \) has only one chamber.

**Theorem 4.2** (Crepant resolutions for \( G_2 \)). Assuming that \( V(s) \) and \( V(g) \) intersect transversally, the sequence of blowups that define a crepant resolution of the Weierstrass model \( E_0 : y^2 - (x^3 + s^2 f x + s^3 g) = 0 \) is given to be

\[
X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(y,e_1|e_2)} X_2.
\]

The proper transform of the Weierstrass model is the vanishing locus of

\[
F = e_2 y^2 - e_1 (x^3 + s^{2+\alpha} e_1^2 e_2^\alpha f x + s^3 g), \quad \alpha \geq 0.
\]

The relative projective coordinates are \([ e_1 e_2 x : e_1^2 e_2^2 y : z ][ x : e_2 y : s ][ y : e_1 ]\).

**Proof.** We check smoothness in charts. By our assumptions, \((x, y, s, g)\) forms a regular sequence and can be extended to a local set of coordinates. This will allow us to take derivatives with respect to \( s \) and \( g \). The first blowup is done in three charts.

1. \((x, y, s) \rightarrow (xy, y, ys)\) The defining equation is

\[
F_{(1)} = 1 - y(x^3 + s^{2+\alpha} f x + s^3 g).
\]

There are no singularities left in this chart since \( F \) and \( \partial_y F \) cannot both vanish at the same time. The center of the second blowup is not visible in this chart since \( y \neq 0 \).
2. \((x, y, s) \rightarrow (x, xy, xs)\)

\[ F(2) = y^2 - x(1 + s^{2+\alpha} f + s^3 g). \]

There is a singularity at \((y, x, 1 + s^{2+\alpha} f + s^3 g)\). The exceptional divisor is \(V(x)\). Hence, in this chart, the second blowup has center \((x, y)\) and requires two charts.

(a) \((x, y) \rightarrow (x, yx)\)

\[ F_{(2,1)} = xy^2 - (1 + s^{2+\alpha} f + s^3 g). \]

\(V(F_{(2,1)})\) is smooth in this chart since \(\partial_x F_{(2,1)}\), \(\partial_y F_{(2,1)}\) (or \(\partial_y F_{(2,1)}\)), and \(F_{(2,1)}\) cannot all vanish at the same time.

(b) \((x, y) \rightarrow (xy, x)\)

\[ F_{(2,2)} = y - x(1 + s^{2+\alpha} f + s^3 g). \]

\(V(F_{(2,2)})\) is smooth since \(\partial_y F_{(2,2)}\) is a unit.

3. \((x, y, s) \rightarrow (sx, sy, s)\)

\[ F(3) = y^2 - s(x^3 + s^\alpha fx + g). \]

We have double point singularities at \((y, s, x^3 + fx + g)\). The exceptional divisor is \(V(s)\); hence, the second blowup has center \((y, s)\), which requires two charts.

(a) \((y, s) \rightarrow (y, ys)\)

\[ F_{(3,1)} = y - s(x^3 + s^\alpha fx + g). \]

This is smooth, as can be demonstrated by taking the derivative with respect to \(y\).

(b) \((y, s) \rightarrow (ys, s)\)

\[ F_{(3,2)} = sy^2 - (x^3 + s^\alpha fx + g). \]

Since \(\partial_y F_{(3,2)}\) is a unit, there are no singularities.

\[ \square \]

4.1 \(G^{S_3}_2\)-model

We recall from Theorem 4.2 that, after the blowup, the elliptic fibration is cut out by

\[ F = e_2 y^2 - e_1(x^3 + s^{2+\alpha} e_1 e_2 e_\alpha + s^3 g), \quad \alpha \geq 0. \]

The relative projective coordinates are \([e_1 e_2 x : e_1 e_2^2 y : z][x : e_2 y : s][y : e_1]\). The irreducible components of the generic fiber over \(S\) are \(C_0, C_1,\) and \(C_2\). The curve \(C_\alpha\) is also the generic fiber of the fibral divisor \(D_a\) \((a = 0, 1, 2)\), which are given by

\[
\begin{align*}
D_0 : \quad & s = ze_2 y^2 - e_1 x^3 = 0 \\
D_1 : \quad & e_1 = e_2 = 0 \\
D_2 : \quad & e_2 = x^3 + s^2 fx z^2 + s^3 g z^3 = 0.
\end{align*}
\]

(4.5)

In the Chow ring \(A(Y)\), the divisors \(D_a\) are of classes\(^6\)

\[
[D_0] = [S] - [E_1], \quad [D_1] = [E_1] - [E_2], \quad [D_2] = 2[E_2] - [E_1].
\]

\(^6\)The fibral divisor \(D_1\) is the Cartier divisor \(V(e_1)\) while the Cartier divisor \(V(e_2)\) is \(D_1 + D_2\). Hence to the class of \(D_2\) is \([V(e_2) - V(e_1)] = E_2 - (E_1 - E_2)\).
The curve $C_0$ is the only one that touches the section of the Weierstrass model. The curves $C_0$ and $C_1$ are smooth geometrically irreducible rational curves. Hence, the divisors $D_0$ and $D_1$ are $\mathbb{P}^1$-bundles over $S$:

$$D_0 = \mathbb{P}_S[\mathcal{L} \oplus \mathcal{O}_S], \quad D_1 = \mathbb{P}_S[\mathcal{L}^\otimes 2 \oplus \mathcal{I}].$$

The curve $C_2$ splits into three geometrically irreducible rational curves in the splitting field of the polynomial $P(T)$. The divisor $D_2$ is a triple cover of the $\mathbb{P}^1$-bundle $\mathbb{P}_S[\mathcal{L}^\otimes 3 \oplus \mathcal{I}^\otimes 2]$ ramified over $V(s, 4f^3 + 2g^2)$. Over the generic point of $V(s, 4f^3 + 2g^2)$, the curve $C_2$ factorizes into a line and a double line. The full generic fiber over $V(s, 4f^3 + 2g^2)$ is an incomplete $\tilde{D}_5$, while the full generic fiber over $V(s, 4f^3 + 2g^2)$ is an incomplete $\tilde{B}_4$ if the dimension of the base is three or higher. Over the generic point of $V(s, f, g)$, $C_2$ degenerates further into a triple line $3C'_2$, where

$$C'_2 : e_2 = x = f = g = 0.$$  

It follows that the full fiber structure is an incomplete $\tilde{E}_6$, formed by three rational curves of multiplicity 1, 2, and 3, namely $C_0 + 2C_12 + 3C'_2$. The proper morphism $f : D_2 \rightarrow S$ has a Stein factorization

$$f : D_2 \overset{f'}{\rightarrow} S' \overset{\pi}{\rightarrow} S,$$

where $\pi : S' \rightarrow S$ is a finite map, as well as a triple cover of $S$ with ramification divisor $4f^3 + 27g^2$. The proper morphism $f' : D_2 \rightarrow S'$ has connected fibers, and endows $D_2$ with the structure of a $\mathbb{P}^1$-bundle over $S'$ such that

$$D_2 \cong \mathbb{P}_{S'}[\pi^* (\mathcal{L}^\otimes 3 \oplus \mathcal{I}^\otimes 2)] \rightarrow S'.$$

The node $C'_2$ has the quasi-minuscule weight $[1, -2]$, whose Weyl orbit corresponds to the non-zero weights of the fundamental representation $7$ of $G_2$. It follows that the representation associated with the $G_2$-model is the direct sum of the adjoint representation (which is always present) and the fundamental representation $7$.

### 4.2 $G_2^{Z/3Z}$-model

A $G_2$-model over $S = V(s)$ with Galois group $Z/3Z$ is given by the Weierstrass model

$$\mathcal{E}_0 : \quad y^2 = x^3 + s^{3+\alpha}fx + s^3g, \quad \alpha \geq 0,$$

where $g$ is not a perfect square modulo $s$. This $G_2^{Z/3Z}$ is a specialization of $G_2^{Z3}$, obtained by increasing the valuation of $c_4$. The $G_2^{Z/3Z}$-model distinguishes itself by the behavior of its $j$-invariant over $S$ and its fiber degeneration.

**Lemma 4.3.** The value of the $j$-invariant for the generic fiber over $S$ is 0.

**Proof.** An elliptic fibration $y^2 = x^3 + Fx + G$ has a $j$-invariant $j = 1728\frac{4F^2}{(4F^3 + 27G^2)}$. In this case, since $v_S(F^3) > v_S(G^2)$, $j = 0$ over the generic point of $S$. \hfill \square

The crepant resolution of this elliptic fibration follows the same blowup as in the general case. But the proper transform is now

$$Y : \quad e_2y^2 = e_1(x^3 + e_2^{1+\alpha}e_1^{1+\alpha}s^{3+\alpha}fx + s^3g).$$

In $X_2$, the projective coordinates are

$$[e_1e_2x : e_1e_2^2y : z = 1][x : e_2y : s][y : e_1].$$
The divisor $D_2$ is now simply

$$D_2 : \quad e_2 = x^3 + s^3 g = 0.$$  

The divisor $D_2$ is still a triple cover of the $P^1$-bundle $P_S[\mathcal{L} \otimes \mathcal{S}^2 \otimes \mathcal{S}^2]$, except now branched at $V(s, g)$. The node $C_2$ splits into three irreducible components in the splitting field of the polynomial $T^3 - g$. Hence, the Galois group of the splitting field is $\mathbb{Z}/3\mathbb{Z}$.

The arithmetic degeneration only exits to a $D_4$, while the geometric degeneration to an incomplete $\mathbb{E}_6$ over $V(s, g)$. The node $C'_2$ has the quasi-minuscule weight $[2 - 1]$, whose Weyl orbit corresponds to the non-zero weights of the fundamental representation $7$ of $G_2$. Thus, the representation $7$ is quasi-minuscule.

If the base is of dimension three or higher, there is an arithmetic degeneration

$$\overline{G}_2 \rightarrow \overline{D}_4,$$

as the $\overline{G}_2$ becomes a fiber $\overline{D}_4$ over the loci where the polynomial $x^3 + s^3 g$ splits completely. An example of such a loci is over the intersection of $S$ with the locus

$$V(\xi^3 + g + sr),$$

where $\xi$ is a section of the line bundle $\mathcal{L}^2 \otimes \mathcal{S}^{-1}$.

**Example 4.4.** If $B = \mathbb{P}^3$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^3}(4)$, and $\mathcal{S} = \mathcal{O}_{\mathbb{P}^3}(2)$, then $g$ is a section of $\mathcal{O}(18)$ and $\xi$ is a section of $\mathcal{O}(6)$. Such a curve depends on twenty parameters, and one can easily write a family of them passing through any arbitrary point of $S$.

The same weight $[2 - 1]$ of the representation $7$ appears for the arithmetic degeneration, i.e.

$$\overline{G}_2 \rightarrow \overline{D}_4 \implies \text{weight } [2 - 1] \text{ of the representation } 7.$$  

We remark that this is one third of the weight corresponding to a generic fiber of $D_1$.

### 4.3 $G_2^{\mathbb{Z}/3\mathbb{Z}}$ and terminal singularities

The $G_2^{\mathbb{Z}/3\mathbb{Z}}$-model we have considered in the previous section has a crepant resolution for a base of arbitrary dimension. It was uniquely defined by the valuation of the Weierstrass coefficients $(v_S(c_4) \geq 3, v_S(c_6) = 3)$. In this section, we explore an example of a $G_2^{\mathbb{Z}/3\mathbb{Z}}$-model that has the same valuations as the $G_2^{S_3}$ ($v_S(c_4) = 2, v_S(c_6) = 3$) model. The Galois group $\mathbb{Z}/3\mathbb{Z}$ is enforced by requiring that the discriminant $\Delta(P)$ is a perfect square in the residue field $\kappa(\eta)$. However, we will encounter non-trivial $\mathbb{Q}$-factorial terminal singularities when the base is of dimension three or higher.

Consider the Weierstrass model given by

$$E_0: \quad y^2 = x^3 + s^2(-3ar + sq)x + s^3(a^2r + ar^2 + st). \quad (4.6)$$

The auxiliary polynomial $P(T)$ and its discriminant $\Delta(P)$ are given by

$$P(T) = T^3 - 3arT + ar(a + r), \quad \text{with} \quad \Delta(P) = 27a^2r^2(a - r)^2.$$  

---

This model is inspired by the family of cubics $F = x^3 - bx + bc$, which has Galois group $\mathbb{Z}/3\mathbb{Z}$ when $4b - 27c^2$ is a perfect square as it ensures that its discriminant $\Delta(F) = b^2(4b - 27c^2)$ is a perfect square. A famous example of this type is the family of cubics $F_m = x^3 + mx^2 + (m - 3)x + 1$ introduced by Shank in the definition of the simplest cubic fields [54]: after completing the cube in $x$, $F_m$ takes the form $F = x^3 - bx + bc$ with $b = (m^2 + 3m + 9)/3$ and $c = (2m + 3)/9.$
Note that $P(T)$ is irreducible and $\Delta(P)$ is a perfect square. Hence, the splitting field of $P(T)$ has Galois group $\mathbb{Z}/3\mathbb{Z}$, which means we have a $G_2^{\mathbb{Z}/3\mathbb{Z}}$-model.

Using the same sequence of blowups as in section 3.2, we have

$$\mathcal{E}_2 : \quad e_2 y^2 = e_1 \left( x^3 + s^2 (-3ar + se_1e_2q)x + s^3 (a^2r + ar^2 + se_1e_2t) \right).$$

The irreducible components of the generic fiber over $S$ are

$$D_0 : \quad s = ze_2y^2 - e_1x^3 = 0$$
$$D_1 : \quad e_1 = e_2 = 0$$
$$D_2 : \quad \frac{e_2}{e_1} = x^3 - 3ars^2x + s^3ar(a + r) = 0.$$

The curve $C_2$ (i.e. the generic fiber of $D_2$) is irreducible over $\kappa$. The degeneration of the fibers are characterized by the irreducible components of $\Delta(P) = 27a^2r^2(a - r)^2$. Along $ar = 0$, $P(T)$ specializes to a perfect cube $P(T) = T^3$. This means all three geometric components of $C_2$ coincide, yielding the fiber of type $1-2-3$, which is an incomplete $\widetilde{E}_6$. Along $a-r = 0$, $P(T) = (T-a)^2(T+2a)$. This means two of the three geometric components of $C_2$ coincide, giving an incomplete fiber of type $\widetilde{D}_5$. Finally, along $a+r = 0$, $P(T) = T(T^2+3a^2)$. This means $P(T)$ factorizes into a linear term and an irreducible quadratic term, giving a fiber of type $I_0^{ss}(\widetilde{B}_3)$. We have three codimension two loci over which the fiber $C_2$ splits into a double curve and a curve. These are geometric degenerations to an incomplete $\widetilde{E}_6$ along $V(s,a)$, $V(s,r)$, and $V(s,a+r)$. If the base is of dimension three or higher, the loci $V(s,a)$, $V(s,r)$, $V(s,a-r)$, and $V(s,a+r)$ intersect on a codimension three loci at $V(s,a,r)$. Over $V(s,a+r)$, the generic fiber is of type $1-2-3$. If the base $B$ is a surface, $\mathcal{E}_2 \rightarrow \mathcal{E}_0$ is a crepant resolution. If the base has dimension three or higher, we can rewrite the equation as

$$\mathcal{E}_2 : \quad e_2(y^2 - s^4e_1^2t) = e_1 \left( x^3 + s^2 (-3ar + se_1e_2q)x + s^3ar(a + r) \right),$$

where the singularity is in the patch $e_1s \neq 0$. Analytically, this is a binomial hypersurface of type $V(u_1u_2u_3 - u_1u_2w_3)$.

There are terminal singularities along the codimension four loci $(x, y^2 - e_1^4t, e_2, a, r)$. By the Grothendick-Samuel’s theorem, $\mathcal{E}_2$ does not have a crepant resolution if the base is of dimension three or higher. This is because $\mathcal{E}_2$ is locally a complete intersection nonsingular up to codimension three with terminal singularities in codimension four.

## 5 Spin(7)-models

We recall that a fiber is of type $I_0^{ss}$ if its dual graph is the Dynkin diagram of type $\widetilde{B}_3^t$ and its geometric fiber is the Kodaira fiber of type $I_0^*$ (with dual graph $\widetilde{D}_4$). Tate’s algorithm is performed with respect to the valuation ring associated to a smooth divisor $S = V(s)$ with generic point $\eta$ and residue field $\kappa$. The fiber over $\eta$ is of Kodaira type $I_0^*$ and the auxiliary polynomial of Step 6 is $P(T) = T^3 + a_{21}T^2 + a_{42}T + a_{63}$. The arithmetic fiber is of type $I_0^{ss}$ when the splitting field of $P(T)$ defines a quadratic field extension $\kappa'$ of the residue field $\kappa$. This means that $P(T)$ has a unique $\kappa$-rational solution and that the Galois group $\text{Gal}(\kappa'/\kappa)$ is the cyclic group $\mathbb{Z}/2\mathbb{Z}$.

We give convenient canonical forms for Spin(7)-models in Theorem 5.1. All the canonical forms we deal with have the $\kappa$-rational point of $P(T)$ at the origin $T = 0$. We distinguish between two cases by the value of the $j$-invariant at the generic point $\eta$ in $S$. 

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\textbf{Theorem 5.1} (Canonical forms for Spin(7)-models).

- A Spin(7)-model such that \( j(\eta) \neq 0, 1728 \) always has a canonical form that can be written as
  \[ \text{Spin}(7) \quad \mathcal{E}_0 = V(zy^2 - x^3 - a_{2,1} sx^2 z - s^2 a_{4,2} xz^2 - s^4 + \beta a_{6,4} \beta z^3), \quad j \neq 0, 1728, \quad \beta \in \mathbb{Z}_{\geq 0}, \]
  with
  \[ P(T) = T(T^2 + a_{2,1} T + a_{4,2}), \quad \Delta(P) = a_{1,2}^2(a_{2,1}^2 - 4a_{4,2}), \]
  where \( a_{2,1} \) is not zero modulo \( s \) and \( a_{2,1}^2 - 4a_{4,2} \) is not a perfect square modulo \( s \).

- A Spin(7)-model with \( j(\eta) = 1728 \) can always be written in such a way that \( v_S(a_1) \geq 1, v_S(a_2) \geq 2, v_S(a_3) \geq 2, v_S(a_4) = 2, \) and \( v_S(a_6) \geq 4 \). Thus, it can be put in the canonical form
  \[ \text{Spin}(7) \quad \mathcal{E}_0 = V(zy^2 - x^3 - s^2 a_{4,2} xz^2 - s^4 + \beta a_{6,4} \beta z^3), \quad j = 1728, \quad \beta \in \mathbb{Z}_{\geq 0}, \]
  with
  \[ P(T) = T(T^2 + a_{4,2}), \quad \Delta(P) = -4a_{4,2}^3, \]
  where \( a_{4,2} \) is not a perfect square modulo \( s \).

\textit{Proof.} Without loss of generality, we can solve the arithmetic condition requiring \( P(T) \) to have a \( \kappa \)-rational point, by requiring \( v_S(a_6) \geq 4 \). This is essentially the same as performing a translation that puts the unique \( \kappa(\eta) \)-rational root of \( P(T) \) at \( T = 0 \). We then have to restrict the valuation of \( a_4 \) to be exactly two; otherwise, \( v(a_4) \geq 3 \), the discriminant of \( P(T) \) will be zero in \( \kappa(\eta) \), and \( P(T) \) will have a double root.

The discriminant \( \Delta(P) \) depends only on \( a_{4,2} \) and \( a_{2,1} \). Since the valuation of \( a_4 \) is fixed, it is interesting to explore how the geometry depends on the valuation of \( a_2 \). The valuation of \( a_2 \) characterizes two distinct types of Spin(7)-models as the \( j \)-invariant and the residual discriminant \( \Delta(P) \) have different behavior when \( v_S(a_2) = 1 \) or \( v_S(a_2) \geq 2 \). The \( j \)-invariant takes the generic value \( j = 1728 \) over the generic point of \( S \) when \( v_S(a_2) \geq 2 \) and varies over \( S \). Moreover, the residual discriminant \( \Delta(P) \) is composed of two distinct components when \( v_S(a_2) = 1 \), and the two components coincide when \( v_S(a_2) \geq 2 \). To avoid a trivial Galois group, we assume that the discriminant of the quadratic part of \( P(T) \) is not a perfect square. When \( v_S(a_2) \geq 2 \), we can also complete the cube in \( x \), and take the canonical form to be in a short Weierstrass form without changing the conditions \( v_S(a_4) = 2 \) and \( v_S(a_6) \geq 4 \).

\( \square \)

\subsection{5.1 First crepant resolution of the general Spin(7)}

We first consider a resolution of the Spin(7)-model obtained by blowing up reduced monomial ideals. In the following section, we consider a flop of that resolution. We assume the Theorem 5.1 hold.

\textbf{Theorem 5.2.} Let \( \mathcal{E}_0 \to B \) be a Spin(7)-model given in Theorem 5.1. Let \( Y^+ \) be the proper transform of \( \mathcal{E}_0 \) after the following sequence of blowups starting with the ambient space \( X_0 = \mathbb{P}_B [\mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3} \oplus \mathcal{O}_B] \):

\[ X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(y,e_1|e_2)} X_2 \xleftarrow{(x,e_2|e_3)} X_3^+. \]

If \( V(a_{4,2}) \) and \( V(a_{6,4}) \) are smooth hypersurfaces in \( B \) that meet \( S = V(s) \) transversally, then \( Y^+ \to \mathcal{E}_0 \) is either a crepant resolution or a crepant partial resolution with terminal \( \mathbb{Q} \)-factorial singularities, as indicated in the table below.
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
\(\dim B\) & \(v_S(a_6) = 4\) & \(v_S(a_6) = 5\) & \(v_S(a_6) \geq 6\) \\
\hline
2 & Crepant resolution & Crepant resolution & Term. Q-factor. Sing. \\
\hline
3 & Crepant resolution & Term. Q-factor. Sing. & Term. Q-factor. Sing. \\
\hline
\end{tabular}
\end{table}

Proof of the theorem. Assume that \(V(s)\) and \(V(a_{4,2})\) are smooth varieties intersecting transversally. For notational simplicity, we take \(b = a_{4,2}\) and \(c = a_{6,4} + \beta\). When \(\alpha = 0\), we take \(a = a_{2,1}\); however, if \(\alpha > 0\), we complete the cube in \(x\) such that \(a_2 = 0\). To examine singularities, it is enough to work in the patch \(z = 1\) since the section \((z = x = 0)\) is always smooth. To check smoothness, we work in local patches. The defining equation is

\[ F = y^2 - (x^3 + axs^2 + bs^2x + cs^{4+\beta}), \quad \beta \geq 0, \]

where \(a = 0\) if \(v(a_2) \geq 2\). The variable \(b\) cannot be identically zero for Spin(7), as otherwise, the polynomial \(P(T)\) will have a double root modulo \(s\). The first blowup is centered at \((x, y, s)\) and requires the following three charts:

1. \((x, y, s) \rightarrow (xy, y, sy)\)

   \[ F_{(1)} = 1 - y(x^3 + axs^2 + bs^2x + cy^{1+\beta}s^{4+\beta}). \]

   \(V(F_{(1)})\) is smooth as there is no solution for \(F_{(1)} = \partial_x F_{(1)} = \partial_y F_{(1)} = \partial_s F_{(1)} = 0\).

2. \((x, y, s) \rightarrow (x, yx, sx)\)

   \[ F_{(2)} = y^2 - x(1 + as + bs^2 + cs^{4+\beta}x^{1+\beta}). \]

   \(V(F_{(2)})\) has a double point singularity at \((y, x, 1 + as + bs^2)\). The second blowup is centered at \((x, y)\) and is implemented in two charts:

   a) \((x, y) \rightarrow (x, yx)\)

   \[ F_{(2,1)} = xy^2 - (1 + as + bs^2 + cs^{4+\beta}x^{1+\beta}). \]

   \(V(F_{(2,1)})\) is smooth since \(F_{(2,1)} = \partial_y F_{(2,1)} = \partial_x F_{(2,1)} = 0\) does not have a solution.

   b) \((x, y) \rightarrow (xy, y)\)

   \[ F_{(2,2)} = y - x(1 + as + bs^2 + cs^{4+\beta}x^{1+\beta}y^{1+\beta}). \]

   This is smooth as \(\partial_y F_{(2,2)}\) and \(\partial_x F_{(2,2)}\) cannot vanish simultaneously.

3. \((x, y, s) \rightarrow (sx, sy, s)\)

   \[ F_{(3)} = y^2 - s(x^3 + ax^2 + bx + cs^{1+\beta}), \quad \beta \geq 0. \]

   We have a singularity at \((y, s, x(x^2 + ax + b))\). The second blowup is centered at \((y, s)\). We have to perform the next blowup in two patches.

   a) \((y, s) \rightarrow (y, ys)\)

   \[ F_{(3,1)} = y - s(x^3 + ax^2 + bx + cy^{1+\beta}s^{1+\beta}), \quad \beta \geq 0. \]

   When \(\beta > 0\), \(V(F_{(3,1)})\) is smooth since \(F_{(3,1)} = \partial_y F_{(3,1)} = \partial_x F_{(3,1)} = 0\) cannot vanish simultaneously. When \(\beta = 0\), there is a singularity at \(x = y = 1 - cs^2 = b = 0\). This singularity has a crepant resolution by blowing up \((x, y)\).

   i. \((x, y) \rightarrow (x, xy)\)

   \[ F_{(3,1,1)} = y - s(x^2 + ax + b + cx^2y^{1+\beta}s^{1+\beta}), \quad \beta \geq 0. \]

   \(V(F_{(3,1,1)})\) is smooth since \(\partial_y F_{(3,1,1)}\) and \(\partial_x F_{(3,1,1)}\) cannot vanish at the same time.
We thus have the following conclusions:

1. If \( \beta = 0 \), \( Y^+ \) is smooth if \( V(c) \) is smooth.
2. If \( \beta = 1 \) and \( \dim B = 2 \), \( Y^+ \) is smooth.
3. If \( \beta = 1 \) and \( \dim B \geq 3 \), \( Y^+ \) is factorial with terminal singularities at \((x,y,b,c,s)\). Hence, \( Y^+ \) does not have a crepant resolution.
4. If \( \beta > 1 \) and \( \dim B \geq 2 \), \( Y^+ \) is factorial with terminal singularities at \((y,x,b,s)\). Hence, \( Y^+ \) does not have a crepant resolution.

We thus have the following conclusions:8

\[ F_{(3,1,2)} = 1 - s(y^2x^3 + ayx^2 + bx + cy^\beta s^{1+\beta}), \quad \beta \geq 0. \]

\( V(F_{(3,1,2)}) \) is smooth since \( F_{(3,1,2)}, \partial_b F_{(3,1,2)}, \) and \( \partial_s F_{(3,1,2)} \) cannot vanish at the same time.

(b) \( (y,s) \to (sy,s) \)

\[ F_{(3,2)} = sy^2 - (x^3 + ax^2 + bx + cs^{1+\beta}). \]

We still have a singularity at \((y,s,x,b)\). The next blowup is centered at \((s,x)\). We again have two patches to consider.

i. \( (x,s) \to (x,sx) \)

\[ F_{(3,2,1)} = sy^2 - (x^2 + ax + b + cx^\beta s^{1+\beta}). \]

This is smooth as it is linear in \( b \), which can be taken as a local parameter in the base.

ii. \( (x,s) \to (x,s) \)

The proper transform is

\[ Y^+: F_{(3,2,2)} = y^2 - (s^2x^3 + ax^2 + bx + cs^\beta). \]

When \( \beta = 0 \), \( Y^+ \) is smooth as it is linear in \( c \), which can be used as a local parameter of the base, assuming \( V(c) \) is smooth. When \( \beta > 0 \), we have \( F_{|s=0} = y^2 - bx \), which is irreducible. If we localize at \( V(s) \), \( F = 0 \) is just a redefinition of \( c \) and we trivially have a UFD. It follows by Nagata’s criterion of factoriality that \( Y^+ \) is factorial.

When \( \beta = 1 \), \( Y^+ \) has terminal singularities at \((x,y,b,c,s)\), which is in codimension 4. When \( \beta > 1 \), \( Y^+ \) has terminal singularities at \((x,y,b,s)\), which is in codimension 3.

We thus have the following conclusions:8

The relative projective coordinates of \( X_0 \) over \( B \), \( X_{i+1} \) over \( X_i \) \((i = 0, 1)\), and \( X_3^+ \) over \( X_2 \) are respectively

\[ [e_1e_2^2e_3^2x : e_1e_2^2e_3^2y : z = 1], \quad [e_3x : e_2e_3y : s], \quad [y : e_1], \quad [x : e_2]. \]

The proper transform of \( e_0^2 \) is

\[ Y^+: e_2y^2 = e_1(e_3^2x^3 + a_{2,1+\alpha}s^{1+\alpha}e_1e_2^2e_3^2x^2 + a_{4,2}s^2x + a_{6,4+\beta}e_3^2e_2^2e_1^{1+\beta}s^{4+\beta}). \]

---

8We recall that factoriality is an obstruction for crepant resolutions in presence of terminal singularities.
We now explore the fiber structure of the smooth elliptic fibration $\varphi: Y^+ \to B$ obtained by the crepant resolution. Denoting $C_a$ as the irreducible components of the fiber over the generic point $\eta$ of $S$, we have

$$\varphi^*(\eta) = C_0 + C_1 + 2C_2 + C_3.$$ 

This curve is a scheme with respect to the residue field $\kappa(\eta)$. The curve $C_0$ is the one touching the section of the elliptic fibration. These curves are generic fibers for the fibral divisors $D_a$, which are defined as the irreducible components of

$$\varphi^*(S) = D_0 + D_1 + 2D_2 + D_3.$$ 

The fibral divisors are

$$D_0: \quad s = e_2y^2 - e_1e_3^3x^3 = 0$$
$$D_1: \quad e_3 = e_2(y^2 - a_6e_1^2s^4) - a_42s^2e_1x = 0$$
$$D_2: \quad e_1 = e_2 = 0$$
$$D_3: \quad \frac{e_2}{e_1} = e_3^2x^2 + a_{21}s_3x + a_{42}s^2 = 0.$$

We observe that $D_0$ and $D_2$ are respectively isomorphic to the projective bundles $\mathbb{P}_S[\theta_S \otimes \mathcal{L}]$ and $\mathbb{P}_S[\mathcal{L}^2 \otimes \mathcal{I}]$, while the fibers of the divisors $D_1$ and $D_3$ can degenerate over higher codimension loci. The generic fiber (over $S$) of the divisor $D_3$ is not geometrically irreducible; after a field extension, it splits into two rational curves. The divisor $D_3$ is a double cover of $\mathbb{P}_S[\mathcal{L}^3 \otimes \mathcal{I}^2]$ branched along $V(s, a_{21}^2 - 4a_{42})$. We note that $a_{21}^2 - 4a_{42}$ is one of the components of the discriminant $\Delta(P)$ of the associated polynomial $P(T)$.

The only fiber components that can degenerate are $C_1$ and $C_3$. The degeneration of the generic fiber are on the components of $\Delta(P)$, and are given by

$$V(a_4)\begin{cases} C_3 \to C_1 + C'_3 \\ C_1 \to C_1 + C'_1 \end{cases}, \quad V(a_2^2 - 4a_4)\begin{cases} C_3 \to 2C''_3 \\ C_1 \to C_1 \end{cases},$$

$$V(a_2, a_4)\begin{cases} C_3 \to 2C_{13} \\ C_1 \to C_1 + C'_1 \end{cases}, \quad V(a_4, a_6)\begin{cases} C_3 \to C_1 + C'_3 \\ C_1 \to C_1 + 2C''_1 \end{cases}, \quad V(a_2, a_4, a_6)\begin{cases} C_3 \to 2C_{13} \\ C_1 \to C_1 + 2C''_1 \end{cases},$$

where

$$\begin{cases} C_{13}: e_2 = e_3 = 0, \\ C'_3: e_2 = e_3x + a_2s = 0, \\ C''_3: e_2 = e_3x + \frac{1}{2}a_2s = 0, \\ C'_1: e_3 = y^2 - a_6e_1^2s^4 = 0 \\ C''_1: e_3 = y = 0. \end{cases}$$

Since the divisor $D_0$, $D_i$ ($i = 1, 2, 3$) satisfy the linear relation $D_0 + D_1 + 2D_2 + D_3 \equiv 0$, it is enough to consider the weights with respect to $D_i$ ($i = 1, 2, 3$). Using the fiber structure, it is easy to evaluate the weights for each curve by solving the linear relations below at the level of intersection numbers.

$$C_{13} \equiv C''_3 \equiv C'_3 \equiv \frac{1}{2}C_3, \quad C'_1 \equiv C_1 - \frac{1}{2}C_3, \quad C''_1 \equiv \frac{1}{2}C_1 - \frac{1}{4}C_3.$$
The proper transform is $E$. In this section, we construct a flop of the crepant resolution obtained in the previous subsection. The relative projective coordinates of the ideal. The singular scheme of $Q$ defines $\mathbb{C}$. Recall that for the first resolution, the center of the blowup is the ideal $(e_2, x)$. In the spirit of Atiyah’s flop, this time we blowup the non-Cartier Weil divisor $(e_2, Q)$, i.e. $D_3$:

\[
X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow X_3
\]

The proper transform is

\[
Y^{-}\left\{ e_2(y^2 - a_6e_1s^4) - e_1xQ = 0 \\
Qe_3 - (x^2 + a_2sx + a_4s^2) = 0
\right\};
\]

the relative projective coordinates of $X_i \rightarrow X_{i-1}$ ($i = 1, 2, 3$) are

\[
[e_1e_2e_3x : e_1e_2^2e_3^2y : z = 1] \quad [x : e_2e_3y : s] \quad [y : e_1] \quad [Q : e_2];
\]

and the irreducible components of the generic fibers are

\[
C_0: \quad s = e_2y^2 - e_1xQ = Qe_3 - x^2 = 0, \quad C_1: \quad e_1 = e_2 = 0,
\]

\[
C_2: \quad \frac{e_2}{e_1} = x = Qe_3 - a_4s^2 = 0, \quad C_3: \quad e_3 = x^2 + a_2sx + a_4s^2 = e_2(y^2 - a_6e_1s^4) - e_1xQ = 0.
\]

They degenerate as follows

\[
V(a_4)\left\{ \begin{array}{c}
C_2 \rightarrow C_{23} \\
C_3 \rightarrow C_{23} + C_3^{(1)} + C_3^{(2)}
\end{array} \right\}, \quad V(a_2^2 - 4a_4)\left\{ \begin{array}{c}
C_2 \rightarrow C_2 \\
C_3 \rightarrow 2C_3'
\end{array} \right\}, \quad V(a_2, a_4)\left\{ \begin{array}{c}
C_2 \rightarrow C_{23} \\
C_3 \rightarrow 2C_{23} + 2C_3^{(1)}
\end{array} \right\},
\]

\[
V(a_4, a_6)\left\{ \begin{array}{c}
C_2 \rightarrow C_{23} \\
C_3 \rightarrow C_{23} + 2C_3^{(1')} + C_3^{(2)}
\end{array} \right\}, \quad V(a_2, a_4, a_6)\left\{ \begin{array}{c}
C_2 \rightarrow C_{23} \\
C_3 \rightarrow 2C_{23} + 4C_3^{(1')}
\end{array} \right\},
\]

The new weight is $[1, 0, -1]$.  

5.2 Second crepant resolution of $\text{Spin}(7)$

In this section, we construct a flop of the crepant resolution obtained in the previous subsection. The flop appears after the second blowup. The proper transform of $\mathcal{E}_0$ after the second blowup is $\mathcal{E}_2$, which can be suggestively rewritten as

\[
Y^{-}\left\{ e_2(y^2 - a_6e_1s^4) - e_1xQ = 0 \\
Q - (x^2 + a_2sx + a_4s^2) = 0
\right\};
\]

The first equation emphasizes that $\mathcal{E}_2$ has double point singularities, while the second equation defines $Q$, which is used in our next blowup so that it formally resembles a blowup of a monomial ideal. The singular scheme of $\mathcal{E}_2$ is supported along $(e_2, e_1x, Q, y^2 - a_6e_1s^4)$, which is in the patch $se_1 \neq 0$. Recall that for the first resolution, the center of the blowup is the ideal $(e_2, x)$. In the spirit of Atiyah’s flop, this time we blowup the non-Cartier Weil divisor $(e_2, Q)$, i.e. $D_3$:
where

\[ C_{23} : \quad e_2 = e_3 = x = 0, \quad C_{3}^{(1)} : \quad e_3 = x = y^2 - a_6 e_1 s^4 = 0, \quad C_{3}^{(1')} : \quad e_3 = x = y = 0, \]

\[ C_{3}^{(2)} : \quad e_3 = x + a_2 s = e_2 (y^2 - a_6 e_1 s^4) + a_2 s e_1 Q = 0, \]

\[ C_{3}' : \quad e_3 = x + \frac{1}{2} a_2 s = e_2 (y^2 - a_6 e_1 s^4) - e_1 x Q = 0. \]

The weights of each of these curves with respect to the divisors \( D_i \) \((i = 0, 1, 2, 3)\) are

\[
\begin{array}{cccc}
D_0 & C_0 & C_2 & C_1 & C_3 \\
D_1 & 0 & 2 & -1 & 0 \\
D_2 & -1 & 2 & -1 & 2 \\
D_3 & 0 & 0 & -2 & 4
\end{array}
\begin{array}{cccc}
C_{23} & C_3' & C_{3}^{(2)} & C_{3}^{(1)} & C_{3}^{(1')} \\
0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & -2 & 1 \\
-1 & -1 & -1 & 0 & 0 \\
0 & 2 & 2 & -1 & 0
\end{array}
\]

### 5.3 Weights and representations

Letting \((a, b, c)\) denote a weight expressed in the basis of simple roots and \([a b c]\) denote a weight expressed in the basis of fundamental weights, the weights of the representations \(7\) and \(8\) are given below.

| Representation 7 of \(B_3\) | Representation 8 of \(B_3\) |
|-----------------------------|-----------------------------|
| \(1\ 0\ 0\) | \(0\ 0\ 1\) | \((1/2, 1, 3/2)\) |
| \(-1\ 1\ 0\) | \(0\ 1\ -1\) | \((1/2, 1, 1/2)\) |
| \(0\ -1\ 2\) | \(-1\ 1\ 1\) | \((1/2, 0, 1/2)\) |
| \(0\ 0\ 0\) | \(-1\ 0\ 1\ 1\ 0\ -1\) | \((-1/2, 0, 1/2)(1/2, 0, -1/2)\) |
| \(0\ 1\ -2\) | \(-1\ 1\ -1\) | \((-1/2, 0, -1/2)\) |
| \(-1\ -1\ 0\) | \(0\ -1\ -1\) | \((-1/2, -1, -1/2)\) |
| \(-1\ 0\ 0\) | \(0\ 0\ -1\) | \((-1/2, -1, -3/2)\) |

For the first crepant resolution, we obtained the following two weights from the fiber degenerations:

\[ \begin{bmatrix} 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \]

The weight \( \begin{bmatrix} 0 & -1 & 2 \end{bmatrix} \) is a weight of the representation \(7\) (the vector representation of \(B_3\)). The weight \( \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \) is weight of the representation \(8\) (the spin representation of \(B_3\)). The vector representation of \(B_3\) is quasi-miniscule while the spin representation is minuscule; as their name indicates, they are of dimensions seven and eight, respectively.

| Locus | Spin(7) with \(v_S(a_2) = 1\) | Spin(7) with \(v_S(a_2) \geq 2\) |
|-------|-----------------------------|-----------------------------|
| \(V(a_{4,2})\) | \(V(a_{2,1}^2 - 4a_{4,2})\) | \(V(a_{4,2})\) |
| Weights | \(0\ 1\ -2\) | \(\pm 1\ 0\ -1\) | \(0\ 1\ -2\) and \(-1\ 0\ 1\) |
| Representations | \(7\) | \(8\) | \(7\) and \(8\) |

### 6 Spin(8)-model

The Weierstrass model of a Spin(8)-model is described by Step 6 of Tate’s algorithm with the additional arithmetic condition that \(P(T)\) has three distinct \(\kappa\)-rational solutions, where \(\kappa\) is the residue field of the generic point \(\eta\) of \(S = V(s)\) and \(s\) is a section of \(\mathcal{O}_B(S) = \mathcal{S}\).
Theorem 6.1 (Canonical form for Spin(8)-models). The Weierstrass model for a Spin(8)-model can be written as

\[ \mathcal{E}_0 : \quad z^2 y = (x - sx_1 z)(x - sx_2 z)(x + sx_3 z) - s^{2+\alpha} Q z, \quad Q = r x^2 + q s x z - s^2 t z^2, \quad \alpha \in \mathbb{Z}_{\geq 0}, \]

where \((r, q, t) \neq (0, 0, 0)\) on the divisor \(S\) and the coefficients \(s, x, r, q,\) and \(t\) are sections of the line bundles given below.

|   | \(s\) | \(x\) |
|---|---|---|
| \(\mathcal{J}\) | \(\mathcal{L}^\oplus 2 \otimes \mathcal{J}^{-1}\) |
| \(\mathcal{L}^\oplus 2 \otimes \mathcal{J}^{-\alpha}\) | \(\mathcal{L}^\oplus 2 \otimes \mathcal{J}^{\oplus(-2-\alpha)}\) |
| \(\mathcal{L}^\oplus 1 \otimes \mathcal{J}^{\oplus(-3-\alpha)}\) | \(\mathcal{L}^\oplus 1 \otimes \mathcal{J}^{\oplus(-2-\alpha)}\) |
| \(\mathcal{L}^\oplus 6 \otimes \mathcal{J}^{\oplus(-2-\alpha)}\) |

\(Q\) cannot be identically zero since otherwise, the Mordell-Weil group will contain at least a torsion subgroup \(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\). Moreover, \(d = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)\) cannot be identically zero on \(S\).

Proof. By Step 6 of Tate’s algorithm, the Weierstrass coefficients have following valuations along \(S\):

\[ v_S(a_1) \geq 1, \quad v_S(a_2) \geq 1, \quad v_S(a_3) \geq 2, \quad v_S(a_4) \geq 2, \quad v_S(a_6) \geq 3. \]

By definition, a Spin(8)-model is such that the cubic polynomial \(P(T) = T^3 + a_2 T^2 + a_4 T + a_6\) factorizes in \(\kappa\). That is,

\[ P(T) = (T - x_1)(T - x_2)(T - x_3), \quad x \in \kappa. \]

The discriminant of \(P(T)\) is a perfect square \(\Delta(P) = d^2\) with

\[ d = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) . \]

The polynomial \(P(T)\) has three distinct roots in \(\kappa\) if and only if \(d\) is nonzero modulo \(s\). Working backwards, from \(P(T)\), we can then compute the Weierstrass coefficients \(a_2, a_4,\) and \(a_6\) to be

\[ a_2 = -(x_1 + x_2 + x_3)s + s^2 r', \quad a_4 = s^2(x_1 x_2 + x_1 x_3 + x_2 x_3) + s^3 q', \quad a_6 = s^3 x_1 x_2 x_3 + s^4 t'. \]

We complete the square in \(T\) and obtain \(a_1 = a_3 = 0;\) this modifies \(r', q',\) and \(t'\) accordingly. We then define \(\alpha\) to be the highest power of \(s\) that we can factor out of \(r', q',\) and \(t',\) i.e. \(r' = s^{\alpha} r, q' = s^{\alpha} q,\) and \(t' = t^{s^{\alpha}}\). This explains the canonical form of the equation. We require \(Q\) to be nonzero, as otherwise, the Mordell-Weil group is non-trivial.

\(6.1\) Crepant resolution of singularities

Theorem 6.2 (Crepant resolutions for Spin(8)-models). Assuming that \(V(x_1)\) are smooth varieties intersecting two by two transversally, the following sequence of blowups defines a crepant resolution of the normal form of a Spin(8)-model given by Theorem 6.1:

\[ X_0 \xleftarrow{(x,y,s|e_1)} X_1 \xleftarrow{(y,e_1|e_2)} X_2 \xleftarrow{(x-x_1 s, e_2|e_3)} X_3 \xleftarrow{(x-x_1 s, e_2|e_4)} X_4, \]

where \(X_0 = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{L}^\oplus 2 \oplus \mathcal{L}^\oplus 3)\). The proper transform of \(\mathcal{E}_0\) is

\[ Y^{(i+1,j+1)} : \quad \left\{ \begin{array}{l}
  e_2 (y^2 - e_1 e_3 e_2 e_1^{2+\alpha s} Q) = e_1 u_i u_j (x - x_k s) \\
  e_3 u_i = (x - x_i s) \\
  e_4 u_j = (x - x_j s),
\end{array} \right. \]

where the relative projective coordinates are

\[ [e_1 e_2 e_3^2 e_4 x : e_1 e_2^2 e_3^2 e_4 y : z = 1] [e_3 x : e_3 e_2 e_4 y : s] [y : e_1] [u_i : e_2 e_4] [u_j : e_2]. \]

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Proof. The Kodaira fiber $I^s_0$ over the generic point of $S$ is seen after the first two blowups:

$$X_0 = \mathbb{P}^2[\mathcal{O}_B \oplus \mathcal{L}^\oplus 2 \oplus \mathcal{L}^\oplus 3] \xrightarrow{(x,y,s|e_1)} X_1 \xleftarrow{(y,e_1|e_2)} X_2.$$ 

The proper transform of $\mathcal{E}_0$ is

$$\mathcal{E}_2 : e_2(y^2 - e_2^2 e_1^{2+\alpha} s^{2+\alpha} Q) = e_1(x - x_1 s)(x - x_2 s)(x - x_3 s).$$

In $X_2$, the projective coordinates of the fiber of $X_0$ and the successive blowup maps are

$$[e_1 e_2 x : e_1 e_2^2 y : z = 1][x : e_2 y : s][y : e_1].$$

After the second blowup, the variety is smooth up to codimension two. The generic fibers of the fibral divisors of this partial resolution are

$$C_0 : s = e_2 y^2 - e_1 x^3 = 0$$
$$C_1 : e_1 = e_2 = 0$$
$$C_2^{(i)} : e_2 = (x - x_i s) = 0, \quad i = 1, 2, 3.$$

Their dual graph is the affine Dynkin diagram $\tilde{D}_4$. The component $C_1$ has multiplicity two, where one comes from the exceptional divisor $e_1 = 0$, and the other from the exceptional divisor $e_2 = 0$. The $C_0$ component is the proper transform of the original elliptic fiber and is the only one touching the section $x = z = 0$ of the elliptic fibration. The $C_1$ component is the central node of the affine Dynkin diagram $\tilde{D}_4$, and $C_2^{(i)}$ are the remaining nodes. Over $V(x_1 - x_j)$, the components of $C_2^{(i)}$ and $C_2^{(j)}$ coincide. These three subvarieties intersect along the subvariety $V(x_1 - x_2, x_2 - x_3)$, over which the three external nodes $C_2^{(i)}$ coincide. There are leftover families of double point singularities in codimension three. $\mathcal{E}_2$ has terminal singularities in codimension three along the three loci

$$S_3^{(3)} = V \left( y^2 - e_2^x e_1^{2+\alpha} s^{2+\alpha} Q, e_2, x - x_j s, x_j - x_k \right) = V \left( y^2 - e_2^x e_1^{2+\alpha} s^{2+\alpha} Q \right) \cap C_2^{(j)} \cap C_2^{(k)},$$

where $(i, j, k)$ is a permutation of $(1, 2, 3)$. These singularities are located in the patch $ze_1 s \neq 0$.

These singularities have crepant resolutions obtained by blowing up two of the three Weil divisors $C_2^{(i)}$. Thus, there are six possible choices. Since the singularities are located in the patch $ze_1 s \neq 0$, we note that $\mathcal{E}_2$ has the structure of the binomial variety

$$V(v_1 v_2 - w_1 w_2 w_3),$$

which was studied in [28]. This binomial variety has six small resolutions whose flop diagram is a hexagon (a Dynkin diagram of type $\tilde{A}_5$) [28].

The following is a proof that we have a resolution by inspecting the singularities chart by chart.

Proof in charts.

$$F = y^2 - (x - x_1 s)(x - x_2 s)(x - x_3 s) + s^{2+\alpha}(px^2 + qsx + s^2 t).$$

We assume that $V(x_1), V(x_2),$ and $V(x_3)$ are smooth varieties intersecting two by two transversally. The idea of the proof is the following. Working in charts. The first blowup has center $(x, y, s)$ and requires three charts. If we call the exceptional divisor $E_1 = V(e_1)$, the second blowup is centered at $V(y, e_1)$ and requires two charts. We will show that after two blowups, the proper transform of $F$ describes a smooth variety or a binomial variety of the type $V(u_1 u_2 - w_1 w_2 w_3)$, which will require two more blowups that can be done in six different ways.
1. \((x, y, s) \rightarrow (x y, y, s y)\)

\[ F_{(1)} = 1 - y(x - x_1)(x - x_2)(x - x_3) + s^{2+\alpha}y^{2+\alpha}(px^2 + qx + s^2 t). \]

which is smooth since the system of equations \(\partial_x F = \partial_y F = \partial_s F = F = 0\) has no solutions.

2. \((x, y, s) \rightarrow (x, yx, sx)\)

\[ F_{(2)} = y^2 - x(1 - x_1)(1 - x_2)(1 - x_3) + s^{2+\alpha}x^{2+\alpha}(p + qs + s^2 t). \]

The exceptional divisor is \(V(x)\). Hence the second blowup is centered at \((x, y)\) and requires two charts.

(a) \((x, y) \rightarrow (x, yx)\)

\[ F_{(2,1)} = x(y^2 + s^{2+\alpha}x^\alpha(p + qs + s^2 t)) - (1 - x_1)(1 - x_2)(1 - x_3). \]

This has the singularities of the binomial variety \(V(u_1 u_2 - w_1 w_2 w_3)\).

(b) \((x, y) \rightarrow (xy, y)\)

\[ F_{(2,2)} = y\left(1 + s^{2+\alpha}x^{2+\alpha}y^{\alpha}(p + qs + s^2 t)\right) - x(1 - x_1)(1 - x_2)(1 - x_3). \]

When \(\alpha > 0\), there are no singularities left. However, when \(\alpha = 0\), we still have double point singularities in the patch \(x \neq 0\), and \(F_{(2,2)}\) can be replaced by \(V(u_1 u_2 - w_1 w_2 w_3)\).

3. \((x, y, s) \rightarrow (xs, ys, s)\)

\[ F_{(3)} = y^2 - s(x - x_1)(x - x_2)(x - x_3) + s^{2+\alpha}(px^2 + qx + t). \]

The exceptional divisor is \(V(s)\). Hence, the second blowup is centered at \((y, s)\) in this chart. We then blowup \((y, s)\), which requires two charts.

(a) \((y, s) \rightarrow (y, sy)\).

\[ F_{(3,1)} = y(1 + s^{2+\alpha}y^\alpha(px^2 + qx + t)) - (x - x_1)(x - x_2)(x - x_3). \]

\(F_{(3,1)}\) has the singularities of the binomial variety \(V(u_1 u_2 - w_1 w_2 w_3)\).

(b) \((y, s) \rightarrow (ys, s)\).

\[ F_{(3,2)} = s(y^2 + s^\alpha(px^2 + qx + t)) - (x - x_1)(x - x_2)(x - x_3), \]

which is again of the binomial variety \(V(u_1 u_2 - w_1 w_2 w_3)\).

After two blowups, if there are singularities left, they are those of the binomial variety

\[ V(u_1 u_2 - w_1 w_2 w_3), \]

whose toric description is a triangular prism. A crepant resolution of this binomial variety is given by a sequence of two blowups corresponding to the subdivision of the triangular prism into two tetrahedrons [28]. Blowup of \((u_1, w_i)\) with \((u_1, w_i) \rightarrow (u_1 w_i, w_i)\) gives

\[ u_1 u_2 - w_j w_k = 0. \]

The other patch \((u_1, w_i) \rightarrow (u_1, w_i u_1)\) is trivially smooth. Likely, blowing up \((u_1, w_j)\) with \((u_1, w_j) \rightarrow (u_1 w_j, w_j)\) gives

\[ u_1 u_2 - w_k = 0, \]

which is smooth. The other patch is also trivially smooth. This resolution is \(Y^{i+1,j+1}\). The graph of their flops is an affine Dynkin diagram \(\tilde{A}_5\) (a hexagon).
6.2 Fiber structure and degenerations

The fibral divisors of the elliptic fibration $Y^{(i+1,j+1)}$ for $\alpha = 0$ are

\begin{align*}
D_0 & \quad C_0 & : & s = z e_2 y^2 - e_1 x^3 = 0 \\
D_1 & \quad C_1 & : & e_1 = e_2 = e_3 u_1 - (x - x_1 s) = e_4 u_2 - (x - x_2 s) = 0 \\
D_{i+1} & \quad C_{2}^{(i)} & : & e_3 = x - x_i s = e_4 u_j - (x - x_j s) = z e_2 (y^2 - e_1^2 s^2 Q) - e_1 u_i u_j (x - x_k s) = 0 \\
D_{j+1} & \quad C_{2}^{(j)} & : & e_4 = x - x_j s = e_3 u_i - (x - x_i s) = z e_2 (y^2 - e_1^2 s^2 Q) - e_1 u_i u_j (x - x_k s) = 0 \\
D_{k+1} & \quad C_{2}^{(k)} & : & e_2 = x - x_k s = e_3 u_i - (x - x_i s) = e_4 u_2 - (x_k - x_j s) = 0.
\end{align*}

If $\alpha > 0$, $Q$ is replaced by $e_4^\alpha e_3^\alpha e_2^\alpha e_1^2 s^{2+\alpha} Q$, which is zero for all the fibral divisors $D_0$. Note that this will carry through equations (6.3), (6.4), (6.5) and (6.6). Even though the generic fiber over $S$ has geometric components, this is not necessarily carried over to its degenerations.

When $\alpha = 0$, the generic point of $V(x_i - x_j) \cap S$ (see equation (6.4)) contains the irreducible component $C_{2}^{(x)}$, which is not geometrically irreducible but splits into two geometrically irreducible curves after a quadratic field extension. The fiber is of type $I_1^{ns}$ with its dual graph of type $\overline{B}_4^t$.

Over codimension-two points (the three irreducible components of $V(d)$), we have

\begin{align}
& V(x_i - x_j) \cap S & \quad V(x_i - x_k) \cap S & \quad V(x_j - x_k) \cap S \\
& \begin{cases} C_{2}^{(i)} \to C_{2}^{(i,j)} + C_{2}^{(x)} \\
C_{2}^{(j)} \to C_{2}^{(i,j)} \\
C_{2}^{(k)} \to C_{2}^{(i,k)} \end{cases} & \begin{cases} C_{2}^{(i)} \to C_{2}^{(i,k)} + C_{2}^{(x)} \\
C_{2}^{(j)} \to C_{2}^{(i,k)} \\
C_{2}^{(k)} \to C_{2}^{(i,k)} \end{cases} & \begin{cases} C_{2}^{(j)} \to C_{2}^{(j,k)} + C_{2}^{(x)} \\
C_{2}^{(k)} \to C_{2}^{(j,k)} \end{cases}.
\end{align}

Over codimension-three points (the common intersection of the three components of $V(d)$), we have

\begin{align}
& V(x_1 - x_2, x_2 - x_3) \cap S \\
& \begin{cases} C_{2}^{(i)} \to C_{2}^{(1,2,3)} + C_{2}^{(i,j)} + C_{2}^{(x)} \\
C_{2}^{(j)} \to C_{2}^{(1,2,3)} + C_{2}^{(i,j)} \\
C_{2}^{(k)} \to C_{2}^{(1,2,3)} \end{cases},
\end{align}

with the components of the fiber defined as

\begin{align*}
C_{2}^{(x)} : \quad & e_3 = x_1 - x_j = x - x_i s = u_j = y^2 - e_1^2 s^2 Q = 0 \\
C_{2}^{(i,j)} : \quad & e_3 = e_4 = x_i - x_j = x - x_i s = e_2 (y^2 - e_1^2 s^2 Q) - e_1 u_i u_j (x_i - x_k) s = 0 \\
C_{2}^{(i,j)'} : \quad & e_3 = e_4 = x_i - x_j = x - x_i s = y^2 - e_1^2 s^2 Q = 0 \\
C_{2}^{(x)'} : \quad & e_3 = x_1 - x_k = x - x_i s = e_4 u_j - (x_i - x_j) s = y^2 - e_1^2 s^2 Q = 0 \\
C_{2}^{(x)''} : \quad & e_3 = x_i - x_j = x - x_i s = u_2 = y^2 - e_1^2 s^2 Q = 0 \\
C_{2}^{(i,k)} : \quad & e_3 = e_2 = x_i - x_j = x - x_i s = e_4 u_2 - (x_1 - x_2) s = 0 \\
C_{2}^{(j,k)} : \quad & e_4 = e_2 = x_j - x_k = x - x_j s = e_3 u_i - (x_j - x_i) s = 0 \\
C_{2}^{(j)'} : \quad & e_4 = x_j - x_k = x - x_j s = y^2 - e_1^2 s^2 Q = 0 \\
C_{2}^{(1,2,3)} : \quad & e_2 = e_3 = e_4 = x - x_k s = x_i - x_j = x - x_k = 0.
\end{align*}

Over $V(x_i - x_j) \cap S$, we have a fiber of type $I_1^{ns}$ with dual graph of type $\overline{B}_4^t$. The non-geometrically irreducible node is $C_{2}^{(x)}$ whereas the geometric fiber is a full $\overline{D}_5$. Over $V(x_1 - x_2, x_2 - x_3) \cap S$, the fiber is of type $IV^{ns}$ with dual graph of type $\overline{B}_4^t$ and geometric dual graph of type $\overline{E}_6$. When $Q(x_i, s, z) = 0$, the fibers $I_1^{ns}$ and $IV^{ns}$ degenerate further along the codimension three locus $V(s, x_i - x_j, x_i^2 + x_i r + t)$ in the base $B$ when the curve is $C_{2}^{(x)}$. The degenerations are illustrated in Figure 12 and Figure 13, respectively, for $\alpha = 0$ and $\alpha > 0$. 

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6.3 Flops and representations

For $Y^{2,3}$, the curves obtained by analyzing the fiber structure have the following geometric weights.

\[
\begin{array}{cccccc}
\alpha_0 & D_0 & \begin{pmatrix}
C_0 & C_1 & C_2^{(1)} & C_2^{(2)} & C_2^{(3)}
\end{pmatrix} & \begin{pmatrix}
C_2^{(1,2)} & C_2^{(1')} & C_2^{(1,3)} & C_2^{(1'')} & C_2^{(2,3)} & C_2^{(2')}
\end{pmatrix} \\
\alpha_2 & D_1 & \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
\alpha_1 & D_2 & \begin{pmatrix} -1 & 2 & -1 & -1 & -1 \end{pmatrix} & \begin{pmatrix} -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix} \\
\alpha_3 & D_3 & \begin{pmatrix} 0 & -1 & 2 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \end{pmatrix} \\
\alpha_4 & D_4 & \begin{pmatrix} 0 & -1 & 0 & 0 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 0 & 2 & -2 & 2 & -2 \end{pmatrix} \\
\end{array}
\quad (6.7)
\]

To express the intersection numbers with the fibral divisors, we introduce the convention
\[
\epsilon_0 = (1, 0, 0, 0, 0), \quad \epsilon_1 = (0, 1, 0, 0, 0), \quad \epsilon_2 = (0, 0, 1, 0, 0), \quad \epsilon_3 = (0, 0, 0, 1, 0), \quad \epsilon_4 = (0, 0, 0, 0, 1).
\]

Since the central node of the $D_4$ diagram corresponds to the node $C_1$ of the resolution, in order to match the convention we use for weights, the intersection numbers $(w_0, w_1, w_2, w_3)$ correspond to the weight $[w_2, w_1, w_3]$ of $D_4$:
\[
(w_0, w_1, w_2, w_3, w_4) = \sum_{a=0}^{4} w_a \epsilon_a \rightarrow [w_2, w_1, w_3, w_4]. \quad (6.8)
\]

Since weights with their appropriate multiplicities sum to zero, i.e. $w_0 + 2w_1 + w_2 + w_3 + w_4 = 0$, we have a bijection with the inverse map
\[
[w_1, w_2, w_3, w_4] \rightarrow (−2w_2 − w_1 − w_3 − w_4, w_2, w_1, w_3, w_4).
\]

For the resolution $Y^{(i+1,j+1)}$, by a direct generalization from equation (6.7), we have the geometric weights
\[
C_2^{(i,j)} \rightarrow -\epsilon_1 + 2\epsilon_{j+1}, \quad C_2^{(i')} \rightarrow 2\epsilon_{i+1} - 2\epsilon_{j+1}, \quad C_2^{(i,k)} \rightarrow -\epsilon_1 + 2\epsilon_{k+1},
\]
\[
C_2^{(i'')} \rightarrow 2\epsilon_{i+1} - 2\epsilon_{k+1}, \quad C_2^{(j,k)} \rightarrow -\epsilon_1 + 2\epsilon_{k+1}, \quad C_2^{(j')} \rightarrow 2\epsilon_{j+1} - 2\epsilon_{k+1}.
\]

When $\alpha = 0$, these curves are not geometrically irreducible, as each curve splits into two irreducible curves, each having the same intersection numbers as the fibral divisors corresponding to half of those of $C_2^{(i')}, C_2^{(i'')}$, and $C_2^{(j')}$. When $\alpha > 0$, the curves $C_2^{(i')}, C_2^{(i'')}$, and $C_2^{(j')}$ are double curves and the intersection numbers of the corresponding reduced curves are also half of those of $C_2^{(i')}$, $C_2^{(i'')}$, and $C_2^{(j')}$. Hence, for any $\alpha$, we end up with the following intersection numbers:
\[
\epsilon_{i+1} - \epsilon_{j+1}, \quad \epsilon_{i+1} - \epsilon_{k+1}, \quad \epsilon_{j+1} - \epsilon_{k+1}.
\]

These are, up to a sign, permutations of
\[
(0, 0, 0, 1, -1), \quad (0, 0, 1, 0, -1), \quad (0, 0, 1, -1, 0).
\]

Following the dictionary given by equation (6.8), we get
\[
\begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 & 0 \end{pmatrix},
\]

which are the weights of the minuscule representations $\mathbf{8}_v$, $\mathbf{8}_c$, and $\mathbf{8}_s$, respectively. Hence, each resolution gives the representation $\mathbf{8}_v \oplus \mathbf{8}_c \oplus \mathbf{8}_s$. 

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The hexagon of crepant resolutions is isomorphic to the chamber structure of the hyperplane arrangement

\[ I(D_4, 8_v \oplus 8_c \oplus 8_s), \]

where \( 8_v \) is the vector representation, and \( 8_c \) and \( 8_s \) are the two irreducible spinor representations. Each of these three irreducible representations is minuscule of dimension eight, and their highest weights are respectively \( \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \), and \( \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \). Note that these weights are related by simple involutions: \( 8_v \leftrightarrow 8_c \) by the involution \( \varpi_1 \leftrightarrow \varpi_3 \), and \( 8_v \leftrightarrow 8_c \) by the involution \( \varpi_3 \leftrightarrow \varpi_4 \).

The weights of the representations \( 8_v, 8_c \) and \( 8_s \) are given below with the following conventions: \((a, b, c, d)\) is a weight of \( D_4 \) expressed in the basis of simple roots while \( [a \ b \ c \ d] \) is a weight of \( D_4 \) written in the basis of fundamental weights.

| Weight system of \( 8_v \) of \( D_4 \) | Weight system of \( 8_c \) of \( D_4 \) | Weight system of \( 8_s \) of \( D_4 \) |
|-------------------------------|-------------------------------|-------------------------------|
| \( \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \) | \( \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \) | \( \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix} \) |
| \( \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \) | \( \begin{pmatrix} 0 & 0 & -1 & 0 \end{pmatrix} \) |
| \( \begin{pmatrix} 1, 1, \frac{1}{2}, \frac{1}{2} \\ 0, 1, \frac{1}{2}, \frac{1}{2} \\ 0, 0, \frac{1}{2}, \frac{1}{2} \end{pmatrix} \) | \( \begin{pmatrix} \frac{1}{2}, 1, 1, \frac{1}{2} \\ \frac{1}{2}, 1, 0, \frac{1}{2} \\ \frac{1}{2}, 0, 0, \frac{1}{2} \end{pmatrix} \) | \( \begin{pmatrix} \frac{1}{2}, 1, 1, \frac{1}{2} \\ \frac{1}{2}, 1, 0, \frac{1}{2} \\ \frac{1}{2}, 0, 0, \frac{1}{2} \end{pmatrix} \) |
| \( \begin{pmatrix} 0, 0, -1, \frac{1}{2} \\ 0, -1, \frac{1}{2}, \frac{1}{2} \\ -1, -1, -\frac{1}{2}, -\frac{1}{2} \end{pmatrix} \) | \( \begin{pmatrix} -\frac{1}{2}, 0, 0, \frac{1}{2} \end{pmatrix} \) | \( \begin{pmatrix} -\frac{1}{2}, 0, 0, \frac{1}{2} \end{pmatrix} \) |

7 Application to five-dimensional supergravity theories.

In this section, we use the information we gathered from the geometry of \( G_2 \), \( \text{Spin}(7) \), and \( \text{Spin}(8) \)-models to explore the corresponding gauge theories in M-theory and F-theory compactifications.

We first consider M-theory compactified on a Calabi-Yau threefold \( Y \) elliptically fibered over a smooth rational surface \( B \) of canonical class \( K \). The divisor \( S \) over which the generic fiber is of Kodaira type \( I_0^* \) is now a smooth curve of genus \( g \) and self-intersection \( S^2 \).

The compactification of M-theory on a Calabi-Yau threefold \( Y \) yields a five dimensional supergravity theory with eight supercharges coupled to \( h^{1,1}(Y) \) vector multiplets and \( h^{2,1}(Y) + 1 \) neutral hypermultiplets [12]. The gravitation multiplet also contains a gauge field called the graviphoton. The dynamics of the vector multiplets and the graviphoton are derived from a real function called the prepotential. After integrating out massive charged vector and matter fields, the prepotential receives a one-loop quantum correction protected from additional quantum corrections by supersymmetry. The vector multiplets transform in the adjoint representation of the gauge group while the hypermultiplets transform in representation \( R = \bigoplus_i R_i \) of the gauge group, where \( R_i \) are irreducible representations.

The Coulomb branches of the theory correspond to the chambers of the hyperplane arrangement \( I(\mathfrak{g}, R) \). By matching the crepant resolutions with the chambers of \( I(\mathfrak{g}, R) \), we determine
which resolutions correspond to which phases of the Coulomb branch. The triple intersection numbers of the fibral divisors correspond to the coefficient of the Chern-Simons couplings of the five-dimensional gauge theory and can be compared with the Intrilligator–Morrison–Seiberg (IMS) prepotential, which is the one-loop quantum contribution to the prepotential of the five-dimensional gauge theory. Since in field theory the Chern-Simons couplings are linear in the numbers $n_{R_i}$ of hypermultiplets transforming in the irreducible representation $R_i$, such that $R = \oplus_i R_i$, computing the triple intersection numbers provides a way to determine the numbers $n_{R_i}$ from the topology of the elliptic fibration. We observe by direct computation in each chamber that the numbers we find do not depend on the choice of the crepant resolution. This idea of using the triple intersection numbers to determine the number of multiplets transforming in a given representation was used previously in [25] for $SU(n)$-models and most recently for $F_4$-models in [23]. This technique has been advocated by Grimm and Hayashi in [32]. The number of representations we find using this procedure satisfy the anomaly cancellation equations of a six-dimensional gauge theory with eight supercharges and the same matter content. One can also determine them geometrically using either Witten’s quantization formula for the $G_2$ and Spin(7)-models, or the usual intersecting brane methods for the Spin(8)-model.

7.1 Coulomb branches of $G_2$, Spin(7), and Spin(8) of $D = 5$ $N = 1$ gauge theories

The Intrilligator–Morrison–Seiberg (IMS) prepotential is the one-loop quantum contribution to the prepotential of a five-dimensional gauge theory with the matter fields in the representations $R_i$ of the gauge group. Let $\phi$ denote an element of the Cartan subalgebra of the Lie algebra $g$, $\alpha$ the fundamental roots, $\varpi$ the weights of $R_i$, and $\langle \varpi, \phi \rangle$ the evaluation of a weight $\varpi$ on an element $\phi$ of the Cartan subalgebra. The Intrilligator–Morrison–Seiberg (IMS) prepotential is [36]

$$6 \mathcal{F}_{IMS} = \frac{1}{2} \left( \sum_{\alpha} |\langle \alpha, \phi \rangle|^{\beta} - \sum_{i} \sum_{\varpi R_i} n_{R_i} |\langle \varpi, \phi \rangle|^{\beta} \right).$$

The full prepotential also contains a contribution proportional to the third Casimir invariant of the Lie algebra $g$; for simple groups, it is only nonzero for $SU(N)$ groups with $N \geq 3$.

For a given choice of a Lie algebra $g$, choosing a dual fundamental Weyl chamber resolves the absolute values in the sum over the roots. We then consider the arrangement of hyperplanes $\langle \varpi, \phi \rangle = 0$, where $\varpi$ runs through all the weights of all the representations $R_i$. They define the hyperplane arrangement $I(g, R = \oplus_i R_i)$ restricted to the dual fundamental Weyl chamber. If none of these hyperplanes intersect the interior of the dual fundamental Weyl chamber, we can safely remove the absolute values in the sum over the weights. Otherwise, we have hyperplanes partitioning the fundamental Weyl chamber into subchambers. Each of these subchambers is defined by the signs of the linear forms $\langle \varpi, \phi \rangle$. Two such subchambers are adjacent when they differ by the sign of a unique linear form.

Each of the subchambers is called a Coulomb phase of the gauge theory. The transition from one chamber to an adjacent chamber is a phase transition that geometrically corresponds to a flop between different crepant resolutions of the same singular Weierstrass model. The number of chambers of such a hyperplane arrangement is physically the number of phases of the Coulomb branch of the gauge theory.

For $G_2$ with the adjoint representation $14$ and the fundamental representation $7$, the one-loop prepotential is

$$6 \mathcal{F}_{IMS} = -8 \phi_1^3 (n_{14} + n_7 - 1) + 9 \phi_2 \phi_1^2 (-2n_{14} + n_7 + 2) + 3 \phi_2^2 \phi_1 (8n_{14} - n_7 - 8) - 8(n_{14} - 1) \phi_2^3. \quad (7.1)$$
For Spin(7) with the adjoint representation 21, the vector representation 7, and the spin representation 8, the prepotential depends on the choice of $\text{sign}(\phi_1 - \phi_3) = \pm$, and is given by

$$6\mathcal{F}^\pm_{\text{IMS}} = -(n_8 \pm n_8 + 8n_{21} - 8)\phi_1^3 - (8n_7 + n_8 + n_8 + 8n_{21} - 8)\phi_3^3 - 3n_8(1 \pm 1)\phi_1^2\phi_3 - 3n_8(1 \pm 1)\phi_1\phi_3^2$$

$$+ 3(-n_7 + n_8 + n_{21} - 1)\phi_1^2\phi_2 + 3(n_7 - n_8 + n_{21} - 1)\phi_1\phi_2^2 + 6n_8\phi_1\phi_2\phi_3$$

$$- 8(n_{21} - 1)\phi_3^2 + 12(n_7 - n_{21} + 1)\phi_2\phi_3^2 - 6(n_7 - 3n_{21} + 3)\phi_2^2\phi_3.$$  (7.2)

Finally, for Spin(8) with the adjoint representation 28, the vector representation 8_v, and the two spin representations 8_s and 8_c, we have six chambers. Each chamber is uniquely defined by the ordering of $(\phi_1, \phi_3, \phi_4)$. For example, the first chamber is defined by $\phi_1 > \phi_3 > \phi_4$, and the prepotential is

$$6\mathcal{F}^{(1)}_{\text{IMS}} = -2(n_{8_v} + n_{8_s} + 4n_{28} - 4)\phi_1^3 - 2(n_{8_v} + 4n_{28} - 4)\phi_3^3 - 8(n_{28} - 1)\phi_1^2\phi_3 - 8(n_{28} - 1)\phi_2^3$$

$$+ 3\phi_2((-n_{8_v} + n_{8_s} + n_{8_c})\phi_1^2 + (n_{8_v} + n_{8_s} - n_{8_c})\phi_3^2 + (n_{8_v} - n_{8_s} + n_{8_c})\phi_1\phi_3)$$

$$+ 6(n_{8_v}\phi_1\phi_4 + n_{8_s}\phi_1\phi_3 + n_{8_c}\phi_3\phi_4)\phi_2 - 6(n_{8_v}\phi_1^2 + n_{8_s}\phi_1\phi_3^2 + n_{8_c}\phi_3^2)$$

$$+ 3(n_{8_v} - n_{8_s} - n_{8_c} + 2n_{28} - 2)\phi_2^2\phi_1 + 3(-n_{8_v} + n_{8_s} - n_{8_c} + 2n_{28} - 2)\phi_2^2\phi_3$$

$$+ 3(-n_{8_v} - n_{8_s} + n_{8_c} + 2n_{28} - 2)\phi_2^2\phi_4.$$  (7.3)

The other chambers have the same prepotential up to a permutation of $\phi_1$, $\phi_2$, and $\phi_3$, as in equation (3.8).

### 7.2 Counting hypermultiplets with triple intersection numbers

**Proposition 7.1.** In the case of a Calabi-Yau threefold, the number of each representations derived by matching the triple intersection numbers of a $G_2$, Spin(7), or Spin(8)-model and the one-loop prepotential does not depend on the choice of a crepant resolution and are given by

$\begin{align*}
G_2: \quad n_7 &= 3S^2 - 10(g - 1), \\
Spin(7): \quad n_7 &= S^2 - 3(g - 1), \quad n_8 = 2S^2 - 8(g - 1), \\
Spin(8): \quad n_{8_v} &= n_{8_s} = n_{8_c} = S^2 - 4(g - 1),
\end{align*}$

$n_{14} = g, \quad n_{21} = g, \quad n_{28} = g.$

**Proof.** Direct comparison of equations (7.1), (7.2), (7.2) with Lemma 3.5 after imposing $\phi_0 = 0$. 

We checked that the numbers computed in Proposition 7.1 satisfy the genus formula of Aspinwall–Katz–Morrison, here they are derived from the triple intersection numbers. The same numbers were computed by Grassi and Morrison using Witten’s genus formula.

The advantage of our method is that we do not use the degeneration loci of the curves and hence the computation is the same even if the fiber degeneration is not generic. This method also provides the number of charged hypermultiplets from a purely five-dimensional point of view, thereby avoiding a six-dimensional argument based on cancellations of anomalies and the subtleties of the Kaluza-Klein circle compactification [33]. The same method was used in [25] for SU(N)‐models and in [23] for $\text{F}_4$-models.

The representation induced by the weights of vertical curves over codimension-two points is not always physical as it is possible that no hypermultiplet is charged under that representation. In such a case, the representation is said to be “frozen” [23].

In all cases, the adjoint representation is always frozen when the curve $S$ is a smooth rational curve ($g = 0$) [58]. For a $G_2$-model, the fundamental representation is frozen if and only if $g = 3k + 1$.
and $S^2 = 10k$ with $k$ in $\mathbb{Z}_{\geq 0}$. For a Spin(7)-model, the vector representation is frozen when $S^2 = 3(g - 1)$, whereas the spin representation is frozen when $S^2 = 4(g - 1)$, and both representations are simultaneously frozen when $S^2 = g - 1 = 0$. For a Spin(8)-model, all representations are frozen if and only if $g = 0$ and $S^2 = -4$. In this case, one can check that there are no curves carrying the weights of the representations: this corresponds to the well-known non-Higgsable model with a rational curve of self-intersection $-4$ [47].

### 7.3 Anomaly cancellation for $G_2$, Spin(7), and Spin(8) $D = 6$ $\mathcal{N} = 1$ gauge theories

Consider an $\mathcal{N} = 1$ six-dimensional theory coupled to $n_T$ tensor multiplets, $n_V$ vectors, and $n_H$ hypermultiplets. We assume that there is a simple gauge group $G$, with Lie algebra $\mathfrak{g}$. The action of the gauge group on a hypermultiplet is characterized by a weight vector determining the charge of the hypermultiplet. It follows that hypermultiplets are organized into representations of the Lie algebra $\mathfrak{g}$. In our convention, a neutral hypermultiplet has a zero weight. The number of zero weights of a representation $R$ is denoted by $\dim R_0$. The difference

$$ (\dim R - \dim R_0) $$

is called the charge dimension of the representation $R$ [31]. We denote by $n_R$, the multiplicity of the representation $R_i$ and $n_H^0$, the number of neutral hypermultiplets. The total number of charged hypermultiplets is [31]

$$ n_{H}^{ch} = \sum_i n_R (\dim R_i - \dim R_{i,0}), $$

where $\dim R_{i,0}$ is the number of zero weights in the representation $R_i$. The total number of hypermultiplets is then

$$ n_H = n_H^0 + n_H^{ch}. $$

We can compute $n_H^0$ and $n_T$ from the Hodge numbers of the Calabi-Yau threefold and its base $B$, which we assume is a rational surface [31,53]:

$$ n_V = \dim G, \quad n_H^0 = h^{2,1}(Y) - 1, \quad n_T = h^{1,1}(B) - 1 = 9 - K^2. $$

To check the gravitational anomaly, we only need to compute the number of charged hypermultiplets. The pure gravitational anomaly $\mathfrak{tr} R^4$ is canceled by the vanishing of its coefficient

$$ n_H - n_V + 29n_T - 273 = 0. $$

If the gauge group is a simple group $G$, the remainder of the anomaly polynomial is [53]

$$ I_8 = \frac{9 - n_T}{8} (\mathfrak{tr} R^2)^2 + \frac{1}{6} X^{(2)} \mathfrak{tr} R^2 - \frac{2}{3} X^{(4)}, \quad (7.4) $$

where

$$ X^{(n)} = \mathfrak{tr}_{\text{adj}} F^n - \sum_R n_R \mathfrak{tr}_R F^n. $$

Choosing a reference representation $\mathbf{F}$, we have

$$ X^{(2)} = \left( A_{\text{adj}} - \sum_R n_R A_R \right) \mathfrak{tr}_F F^2, \quad X^{(4)} = \left( B_{\text{adj}} - \sum_R n_R B_R \right) \mathfrak{tr}_F F^4 + \left( C_{\text{adj}} - \sum_R n_R C_R \right) (\mathfrak{tr}_F F^2)^2, $$

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where the coefficients \( A_R, B_R, \) and \( C_R \) are defined by the trace identities

\[
tr_R F^2 = A_R tr F^2, \quad tr F^4 = B_R tr F^4 + C_R (tr F^2)^2.
\]

In a theory with at least two quartic Casimirs, to satisfy the anomaly cancellation conditions, the coefficient of \( tr_F F^4 \) must vanish,

\[
B_{adj} - \sum_R n_R B_R = 0,
\]

and the anomaly cancellation polynomial in equation (7.4) needs to factorize. In this paper, the reference representation will always be the vector representation. Trace identities for \( G_2 \) with the adjoint (14) and the vector representation (7) are [7]

\[
tr_{14} F^2 = 4 tr_7 F^2, \quad tr_7 F^4 = \frac{1}{4} (tr_7 F^2)^2, \quad tr_{14} F^4 = \frac{5}{2} (tr_7 F^2)^2.
\]

Trace identities for \( B_3 \) with the adjoint (21), the vector (7), and the spinor representation (8) are [7]

\[
tr_{21} F^2 = 5 tr_7 F^2, \quad tr_{21} F^4 = -tr_7 F^4 + 3(tr_7 F^2)^2,
\]

\[
tr_8 F^2 = tr_7 F^2, \quad tr_8 F^4 = -\frac{1}{2} tr_7 F^4 + \frac{3}{8} (tr_7 F^2)^2.
\]

Finally, trace identities for \( D_4 \) with the adjoint (28), the vector representation (8_v), and the two spinor representations (8_s) and (8_c) are [7]

\[
tr_{28} F^2 = 6 tr_8_s F^2, \quad tr_{8_s} F^2 = tr_8_s F^2, \quad tr_8_s F^2 = tr_{8_s} F^2,
\]

\[
tr_{28} F^4 = 3(tr_{8_s} F^2)^2, \quad tr_{8_s} F^4 = -\frac{1}{2} tr_{8_s} F^4 + \frac{3}{8} (tr_{8_s} F^2)^2, \quad tr_{8_s} F^4 = -\frac{1}{2} tr_{8_s} F^4 + \frac{3}{8} (tr_{8_s} F^2)^2.
\]

We use the vector representation as the reference representation in all cases, and recall that the number of representations derived from the triple intersection numbers are

- \( G_2 \): \( n_7 = -10(g - 1) + 3S^2 \), \( n_{14} = g \),
- \( \text{Spin}(7) \): \( n_7 = S^2 - 3(g - 1) \), \( n_8 = 2S^2 - 8(g - 1) \), \( n_{21} = g \),
- \( \text{Spin}(8) \): \( n_{8_s} = n_{8_s} = n_{8_c} = S^2 - 4(g - 1) \), \( n_{28} = g \).

For each model, we feed these numbers into the anomaly polynomial.

| \( g \) | \( G_2 \) | \( B_3 \) | \( D_4 \) |
|--------|--------|--------|--------|
| \( R \)  | 14    | 21    | 28  |
| \( 8_s \) | 7     | 7     | 8   |
| \( 8_c \) | 8     | 8     | 8   |

The result for each model is

\[
X^{(2)} = 3 K \cdot S \cdot tr_V F^2, \quad X^{(4)} = -\frac{3}{4} S^2 (tr_V F^2)^2,
\]

and the anomaly cancellation polynomial factorizes as [53]

\[
I_8 = \frac{K^2}{8} (tr R^2)^2 + \frac{1}{2} K \cdot S \cdot tr_V F^2 (tr R^2) + \frac{1}{2} S^2 (tr_V F^2)^2
\]

\[
= \frac{1}{2} \left[ \frac{1}{2} c_1(B) \ tr R^2 - S \ tr_7 F^2 \right]^2.
\]

Since \( I_8 \) factorizes, the anomaly can be canceled by the Green-Schwarz mechanism.

\(^9\)In a theory with only one quartic Casimir, we simply have \( B_R = 0 \).
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