Evaluation parameters and Bethe roots for the six vertex model at roots of unity

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Abstract

We propose an expression for the current form of the lowering operator of the $sl_2$ loop algebra symmetry of the six vertex model (XXZ spin chain) at roots of unity. This operator has poles which correspond to the evaluation parameters of representation theory which are given as the roots of the Drinfeld polynomial. We explicitly compute these polynomials in terms of the Bethe roots which characterize the highest weight states for all values of $S^z$. From these polynomials we find that the Bethe roots satisfy sum rules for each value of $S^z$.

Keywords: Bethe’s ansatz, loop algebra, quantum spin chains

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I. INTRODUCTION

Recently it has been demonstrated\cite{ref1} that the six vertex model with periodic boundary conditions specified by a $2^L \times 2^L$ transfer matrix $T(v; \gamma)$

$$T_{\{\mu\},\{\mu'\}}(v; \gamma) = \sum_{\lambda_i=\pm1} W_{\mu_1,\mu'_1}(\lambda_1, \lambda_2) W_{\mu_2,\mu'_2}(\lambda_2, \lambda_3) \cdots W_{\mu_L,\mu'_L}(\lambda_L, \lambda_1) \quad (1.1)$$

with

$$W_{\mu_i,\mu'_i}(\lambda_i, \lambda_{i+1}) = \frac{1}{2} (a + b) \delta_{\mu_i,\mu'_i} \delta_{\lambda_i,\lambda_{i+1}} + \frac{1}{2} (a - b) \sigma^z_{\mu_i,\mu'_i} \sigma^z_{\lambda_i,\lambda_{i+1}}$$

$$+ c (\sigma^+_{\mu_i,\mu'_i} \sigma^-_{\lambda_i,\lambda_{i+1}} + \sigma^-_{\mu_i,\mu'_i} \sigma^+_{\lambda_i,\lambda_{i+1}}) \quad (1.2)$$

where $\sigma^i$ are the Pauli matrices and

$$a = i \sinh \frac{1}{2} (v - i\gamma), \ b = -i \sinh \frac{1}{2} (v + i\gamma), \ c = -i \sinh i\gamma \quad (1.3)$$

has an $sl_2$ loop algebra symmetry when $\gamma$ satisfies the “root of unity condition”

$$\gamma = \gamma_0 = \frac{m\pi}{N} \quad (1.4)$$

with $m$ and $N$ relatively prime. The operator

$$S^z = \frac{1}{2} \sum_{j=1}^L \sigma^z_j \quad (1.5)$$

commutes with the transfer matrix $T(v; \gamma)$ and in the sector $S^z \equiv 0 \pmod{N}$ the $sl_2$ loop algebra symmetry was proven in ref\cite{ref1} by demonstrating that the operators with $q = e^{i(v-\gamma_0)}$

$$S^{\pm(N)} = \sum_{1 \leq j_1 < \cdots < j_N \leq L} q^{(N/2)\sigma^z} \otimes \cdots \otimes q^{(N/2)\sigma^z} \otimes \sigma_{j_1}^\pm \otimes q^{((N-2)/2)\sigma^z} \otimes \cdots$$

$$\otimes q^{((N-2)/2)\sigma^z} \otimes \sigma_{j_2}^\pm \otimes q^{((N-4)/2)\sigma^z} \otimes \cdots \otimes \sigma_{j_N}^\pm \otimes q^{-(N/2)\sigma^z} \otimes \cdots \otimes q^{-(N/2)\sigma^z} \quad (1.6)$$

and

$$T^{\pm(N)} = \sum_{1 \leq j_1 < \cdots < j_N \leq L} q^{-((N/2)\sigma^z)} \otimes \cdots \otimes q^{-((N/2)\sigma^z)} \otimes \sigma_{j_1}^\pm \otimes q^{-((N-2)/2)\sigma^z} \otimes \cdots$$

$$\otimes q^{-(N-2)/2)\sigma^z} \otimes \sigma_{j_2}^\pm \otimes q^{-(N-4)/2)\sigma^z} \otimes \cdots \otimes \sigma_{j_N}^\pm \otimes q^{(N/2)\sigma^z} \otimes \cdots \otimes q^{(N/2)\sigma^z} \quad (1.7)$$

commute (anticommute) with the transfer matrix if $N - m$ is even (odd) and satisfy the defining relations for the Chevalley generators of the $sl_2$ loop algebra. In the sectors where $S^z \equiv r \neq 0 \pmod{N}$ with $1 \leq r \leq N - 1$ operators such as $(T^\pm)^r(S^-)^r S^{-(N)}$ commute (anticommute) with the transfer matrix and the $sl_2$ loop algebra symmetry was inferred from computer computations.

The eigenvalues of the transfer matrix are polynomials in the variable $e^v$. In the limit $v \to \pm i\gamma$ the transfer matrix reduces to
$$T(v; \gamma) \rightarrow (\sinh i\gamma)^L \Pi_{\pm} \left[ I - \frac{v \mp i\gamma}{\sinh i\gamma} \left( H_{\text{XXZ}} + \frac{L \cos \gamma}{2} \right) + O((v \mp i\gamma)^2) \right]$$  \hspace{1cm} (1.8)$$

where $\Pi_{\pm}$ is the left (right) shift operator whose eigenvalues are $e^{\mp iP}$ where $P$ is the momentum of the state and

$$H_{\text{XXZ}} = -\frac{1}{2} \sum_{j=1}^{L} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right)$$  \hspace{1cm} (1.9)$$

with

$$\Delta = -\cos \gamma$$  \hspace{1cm} (1.10)$$

is the Hamiltonian of the XXZ spin chain. Therefore every degeneracy of the transfer matrix (polynomial in $e^v$) eigenvalues implies a corresponding degeneracy of the (numerical) Hamiltonian eigenvalues. However, the converse need not be true and the Hamiltonian (1.9) is known to have degeneracies such as that caused by invariance under parity which the transfer matrix does not have.

The existence of this symmetry means that the space of eigenstates states of the transfer matrix can be decomposed into finite dimensional representations of the $\mathfrak{sl}_2$ loop algebra. Each finite dimensional representation has a vector $|\Omega>\,\,$ with the property that

$$S^{+(N)}|\Omega> = T^{+(N)}|\Omega> = 0$$ \hspace{1cm} (1.11)$$

and we define $S^z_{\text{max}}\,$ by

$$S^z|\Omega> = S^z_{\text{max}}|\Omega>.$$ \hspace{1cm} (1.12)$$

In analogy with the finite Lie algebra $\mathfrak{sl}_2\,$ we call this (following ref²) the highest weight vector of the representation.

A fundamental property of all affine Lie algebras is that they may be defined either by a Chevalley basis or by a mode basis. In the mode basis the elements of the $\mathfrak{sl}_2$ loop algebra are $e(n), \, f(n)$ and $h(n),\,$ where $n$ is an integer, which satisfy the commutation relations

$$[e(m), f(n)] = h(m + n)$$
$$[e(m), h(n)] = -2e(m + n)$$
$$[f(m), h(n)] = 2f(m + n)$$ \hspace{1cm} (1.13)$$

and there is the relation to the Chevalley basis of

$$e(0) = T^{-(N)} , \quad e(-1) = S^{-(N)}$$
$$f(0) = T^{+(N)} , \quad f(1) = S^{+(N)}.$$ \hspace{1cm} (1.14)$$

The theory of finite dimensional representations of affine Lie algebras has been extensively studied in²⁴⁵ in terms of this mode basis where it is shown that all irreducible finite dimensional representations are tensor products evaluation representations. These evaluation representations are specified by vectors $|a_j, m_j>$ where for all integer $n$ (positive, negative or zero)
where \( a_j \) are called evaluation parameters and \( e_{m_j}, f_{m_j}, h_{m_j} \) are a spin \( m_j \) representation of \( sl_2 \). The evaluation parameters are shown to be the roots of the Drinfeld polynomial \( P_\Omega(z) \) defined as

\[
P_\Omega(z) = \prod_j (z - a_j)^{m_j}
\]  
(1.16)

where the \( a_j \) are distinct and

\[
P_\Omega(z) = \sum_{r \geq 0} \mu_r(-z)^r
\]  
(1.17)

and for each highest weight state \( |\Omega> \) we compute \( \mu_r \) from the eigenvalue equation

\[
\frac{T^{+(N)r}}{r!} S^{-(N)r} |\Omega> = \mu_r |\Omega>
\]

(1.18)

If the roots of the Drinfeld polynomial (1.17) are all distinct only spin 1/2 representations occur. In this case if we call \( d_\Omega \) the degree of the Drinfeld polynomial then the number of evaluation parameters is \( d_\Omega \) and the number of states with spin \( S^z = S^z_{\text{max}} - lN \) is the binomial coefficient

\[
\binom{d_\Omega}{l} = \text{number of degenerate eigenvalues}
\]

(1.19)

In the mode basis we define the generating function (currents) for sufficiently large \( z \) by

\[
E^-(z) = \sum_{n=0}^{\infty} e(n) z^{-n}.
\]

(1.20)

From the definition of evaluation representation this current will have poles when \( z = a_j \). In order to show that only spin 1/2 representations occur it is sufficient to demonstrate that there are only single poles in the current \( E(z) \).

To compute the Drinfeld polynomial from the definition (1.17) we need to be in possession of the highest weight vectors. For a finite Lie algebra these highest weight vectors cannot be computed from symmetry considerations alone. However for affine Lie algebras if the sets of evaluation parameters are all distinct then the algebra will be powerful enough to determine the highest weight vectors for all multiplets except singlets.

For the six vertex model, however, we may make use of the fact noted in ref.s\textsuperscript{1-3} that the highest weight vectors of the \( sl_2 \) loop algebra are identical with the eigenvectors which are obtained from the solution of Bethe’s equation for \( j = 1, \cdots, L - |S^z| \)

\[
\left( \frac{\sinh \frac{1}{2}(v_j + i\gamma)}{\sinh \frac{1}{2}(v_j - i\gamma)} \right)^L = \prod_{l=1 \atop i \neq j}^{L - |S^z|} \frac{\sinh \frac{1}{2}(v_j - v_l + 2i\gamma)}{\sinh \frac{1}{2}(v_j - v_l - 2i\gamma)}
\]

(1.21)
In fact there are two distinct ways of specifying the eigenvectors in terms of the solutions (Bethe roots) of the equation (1.21); the coordinate Bethe ansatz which uses explicit forms of the wavefunctions in coordinate space and the algebraic Bethe ansatz which produces the wave functions by exhibiting an operator which creates the wave functions by acting on the state of all spins up.

In this paper we use the algebraic Bethe ansatz. In this method of solution of the six vertex model we introduce the \((2 \times 2)\) monodromy matrix (of operators)

\[
M_{\lambda,\lambda'}(v; \gamma) = \sum_{\lambda_i = \pm 1}^{n} W_{\mu_1,\mu'_1}(\lambda) W_{\mu_2,\mu'_2}(\lambda_2, \lambda_3) \cdots W_{\mu_L,\mu'_L}(\lambda_L, \lambda')
\]

for which we will use the conventional notation

\[
M(v; \gamma) = \begin{pmatrix} A(v; \gamma) & B(v; \gamma) \\ C(v; \gamma) & D(v; \gamma) \end{pmatrix}.
\]

The transfer matrix (1.1) is

\[
T(v; \gamma) = A(v; \gamma) + D(v; \gamma).
\]

and here and in the rest of the paper we make (for convenience) the restriction that \(L\) be even and thus the eigenvalues of \(S_z\) are integer.

Fundamental to the algebraic Bethe’s ansatz are the “commutation” relations of the operators \((2^L \times 2^L\) matrices) \(A(v; \gamma), B(v; \gamma), C(v; \gamma), D(v; \gamma)\) of which for our purposes here we will need the following three

\[
[B(v), B(v')] = 0
\]

\[
A(v)B(v') = f(v - v')B(v')A(v) + g(v' - v)B(v)A(v')
\]

\[
D(v)B(v') = f(v' - v)B(v')D(v) + g(v - v')B(v)D(v')
\]

where

\[
f(v) = -\frac{\sinh \frac{1}{2}(v + 2i\gamma)}{\sinh(v/2)}
\]

\[
g(v) = -\frac{\sinh i\gamma}{\sinh(v/2)}
\]

and note the periodicity properties

\[
B(v - 2\pi i) = -B(v), \quad C(v - 2\pi i) = -C(v), \quad A(v - 2\pi i) = A(v), \quad D(v - 2\pi i) = D(v).
\]

The eigenvectors of the six vertex model for all states where the solutions \(v_k\) to Bethe’s equation (1.21) do not lead to undefined factors of 0/0 are given in this notation by

\[
|\{v_k\} > = \prod_{k=1}^{n} B(v_k; \gamma) |0 >
\]
where \( |0 > \) is the unique state with all spins up with
\[
S^z|0 > = \frac{L}{2}|0 >
\]
and
\[
S^z|\{v_k\} > = (\frac{L}{2} - n)|\{v_k\} > .
\]

The corresponding eigenvalue of the transfer matrix is
\[
t(v) = \sinh^L \frac{1}{2} (v - i\gamma) \prod_{j=1}^{n} f(v - v_j) + \sinh^L \frac{1}{2} (v + i\gamma) \prod_{j=1}^{n} f(v_j - v)
\]

If \( \gamma \) is a “generic” value then the eigenstates (1.31) are complete and there are no factors of \( \frac{0}{0} \) in Bethe’s equation (1.21). However, when the root of unity condition (1.4) holds it was seen on refs.1-3 that factors of \( \frac{0}{0} \) do occur in (1.21) because of solutions which approach the exact complete N strings first found by Baxter where
\[
v_k = v_0 - 2ik\gamma_0 \quad \text{for} \quad k = 0, 1, \cdots, N - 1.
\]

In ref.3 we studied these string solutions of Bethe’s equation (1.21) in the limit as \( \gamma \rightarrow \gamma_0 \) and found a set of equations which determines the parameter \( v_0 \) of the exact complete N string (1.35) in this limiting sense. However, this is not the only way to proceed because in the direct coordinate space solution for the eigenfunctions it was shown by Baxter in24 that there are certain equations which vanish automatically independent of \( v_0 \) and hence solutions of the string form (1.35) exist where \( v_0 \) is arbitrary. It is thus to be expected that this arbitrary parameter should be closely related to the variable \( z \) in the current operator (1.20) of the affine Lie algebra.

This phenomenon of the automatic vanishing of certain constraint equations in the coordinate space Bethe’s ansatz is mirrored in the algebraic Bethe’s ansatz by the vanishing of the operator24
\[
\prod_{k=0}^{N-1} B(v - 2ik\gamma_0) = 0
\]
which by (1.31) should create the exact complete N string (1.35).

We denote the operator which replaces the vanishing string creation operator (1.36) when the root of unity condition (1.4) holds by \( B^{(N)}(v) \). It has the property
\[
T(v) \prod_{j=1}^{l} B^{(N)}(v'_j) \prod_{k=1}^{n} B(v_k)|0 > = (-1)^{l(N-m)} \prod_{j=1}^{l} B^{(N)}(v'_j) T(v) \prod_{k=1}^{n} B(v_k)|0 >
\]
for all \( v, v'_j \) and integer \( l \geq 1 \). We find (1.37) is satisfied for \( B^{(N)}(v) \) of the form
\[
B^{(N)}(v) = \sum_{k=0}^{N-1} \left( \prod_{l=0}^{k-1} B(v - 2il\gamma_0) \right) \left( B_{\gamma}(v - 2ik\gamma_0) + \frac{X(v - 2ik\gamma_0)}{Y(v)} B_{\nu}(v - 2ik\gamma_0) \right) \times \left( \prod_{l=k+1}^{N-1} B(v - 2il\gamma_0) \right)
\]
where \( B_\gamma(v) \) and \( B_v(v) \) specify derivatives of \( B(v) \) with respect to \( \gamma \) and \( v \) respectively. This operator, with \( X(v) \) and \( Y(v) \), arbitrary satisfies the commutation relations

\[
[B^{(N)}(v), B(v')] = 0 \quad (1.39)
\]

\[
[B^{(N)}(v), B^{(N)}(v')] = 0 \quad (1.40)
\]

for all values of \( v \) and \( v' \).

The functions \( X(v) \) and \( Y(v) \) depend on the highest weight vector on which the operator acts. Denoting by \( v_k \) the solution of Bethe’s equation \( (1.21) \) which specifies the highest weight vector we find

\[
X(v) = 2i \sum_{l=0}^{N-1} \frac{l \sinh \frac{L}{2}(v - (2l + 1)i\gamma_0)}{\prod_{k=1}^{L} \sinh \frac{1}{2}(v - v_k - 2il\gamma_0) \sinh \frac{1}{2}(v - v_k - 2i(l + 1)\gamma_0)} (1.41)
\]

and

\[
Y(v) = \sum_{l=0}^{N-1} \frac{\sinh \frac{L}{2}(v - (2l + 1)i\gamma_0)}{\prod_{k=1}^{L} \sinh \frac{1}{2}(v - v_k - 2il\gamma_0) \sinh \frac{1}{2}(v - v_k - 2i(l + 1)\gamma_0)} (1.42)
\]

Because the Chevalley generators \( S^{\pm(N)} \) and \( T^{\pm(N)} \) (anti) commute with \( T(v) \) the mode operators \( e(n) \) and \( f(n) \) and their generating functions will (anti) commute with \( T(v) \) and will therefore obey \( (1.37) \). We propose that the solution \( B^{(N)}(v) \) \( (1.38) \) of \( (1.37) \), \( (1.39) \) and \( (1.40) \) is proportional to the current \( E^\gamma(z) \). Therefore the zeroes of the function \( Y(v) \) are the roots of the Drinfeld polynomial computed in a fashion totally independent of the definition \( (1.17) \).

We will prove in section two that \( B^{(N)}(v) \) satisfies \( (1.37) \). In section 3 where we will study the polynomial \( Y(v) \) and compute its degree for the various classes of states specified by

\[
S^z \equiv r \pmod{N} \quad (1.43)
\]

studied in refs. \( 3 \). We will also see that this polynomial leads to sum rules for each value of \( S^z \) which are not only valid for roots of unity \( (1.44) \) but which hold for all real values of \( \gamma \). Examples of these sum rules are

\[
S^z = 0 : \quad \frac{L}{2} \sum_{k=1}^{L} v_k = 0, \text{ or } \pi i \quad (1.44)
\]

\[
S^z = 1 : \quad \left\{ L - (e^{i\gamma} + e^{-i\gamma}) \sum_{k=1}^{L} e^{v_k} \right\} \prod_{k=1}^{L} e^{v_k} = \left\{ L - (e^{i\gamma} + e^{-i\gamma}) \sum_{k=1}^{L} e^{-v_k} \right\} \prod_{k=1}^{L} e^{-v_k} \quad (1.45)
\]

\[
S^z = 2 : \quad \left( L(L - 1) - 2L(e^{i\gamma} + e^{-i\gamma}) \sum_{k=1}^{L} e^{v_k} \right.
\]

\[
+ (e^{2i\gamma} + e^{-2i\gamma}) \sum_{k=1}^{L} e^{2v_k} + (e^{i\gamma} + e^{-i\gamma})^2 \left( \sum_{k=1}^{L} e^{v_k} \right)^2 \right) \prod_{k=1}^{L} e^{v_k}
\]
\[
\begin{align*}
&= \left( L(L-1) - 2L(e^{i\gamma} + e^{-i\gamma}) \sum_{k=1}^{L-2} e^{-v_k} \right. \\
&\quad \left. + (e^{2i\gamma} + e^{-2i\gamma}) \sum_{k=1}^{L-2} e^{-2v_k} + (e^{i\gamma} + e^{-i\gamma})^2 \left( \sum_{k=1}^{L-2} e^{-v_k} \right)^2 \right) \prod_{k=1}^{L-2} e^{-v_k}.
\end{align*}
\]

The sum rule (1.44) is the XXZ version of a sum rule in the XYZ model\cite{27} and has recently been studied numerically\cite{28,29}. The sum rules for \( S^z \neq 0 \) have not previously been seen.

In section 4 we discuss the relation between these algebraic Bethe’s ansatz computations and the theory of finite dimensional representations of affine Lie algebras and quantum groups at roots of unity. Concluding remarks are in sec. 5.

II. DERIVATION OF THE OPERATOR \( B(N)(v) \)

We begin the study of the operator \( B(N)(v) \) given by (1.38) by proving the commutation relations (1.39) and (1.40) which do not depend on the form of \( X(v) \) and \( Y(v) \). We will then use the defining property (1.37) to determine the functions \( X(v) \) and \( Y(v) \) given by (1.41) and (1.42).

A. Proof of (1.39)

Write the operator \( B(N)(v) \) of (1.38) as

\[
B(N)(v) = B_1^{(N)}(v) + B_2^{(N)}(v) / Y(v)
\]

with

\[
B_1^{(N)}(v) = \sum_k^N \left( \prod_{l=0}^{k-1} B(v - 2il\gamma_0) \right) \left( B_\gamma(v - 2ik\gamma_0) \right) \left( \prod_{l=k+1}^{N-1} B(v - 2il\gamma_0) \right)
\]

\[
B_2^{(N)}(v) = \sum_k^N \left( \prod_{l=0}^{k-1} B(v - 2il\gamma_0) \right) \left( X(v - 2ik\gamma_0) \right) \left( \prod_{l=k+1}^{N-1} B(v - 2il\gamma_0) \right).
\]

Then by differentiating (1.25) with respect to \( v \) we find

\[
[B_v(v), B(v')] = 0
\]

and hence

\[
[B_2^{(N)}(v), B(v')] = 0.
\]

It also follows from (1.25) that

\[
\left[ \prod_{l=0}^{N-1} B(u_l), B(v') \right] = 0.
\]
Differentiating this with respect to $\gamma$ and then setting $u_l = v - 2il\gamma_0$ we find

$$[B_1^{(N)}(v), B(v')] + \left[ \prod_{l=0}^{N-1} B(v - 2il\gamma_0), B_\gamma(v') \right] = 0. \quad (2.7)$$

Both terms in the second commutator are zero because of the vanishing condition (1.36) and thus

$$[B_1^{(N)}(v), B(v')] = 0. \quad (2.8)$$

Combining (2.5) and (2.8) we obtain the desired result (1.39).

**B. Proof of (1.40)**

Using the decomposition (2.1) we see that the desired commutation relation (1.40) will follow from the three separate commutation relations

$$[B_j^{(N)}(v), B_{j'}^{(N)}(v')] = 0 \quad \text{for} \quad j, j' = 1, 2 \quad (2.9)$$

To prove (2.9) for $j = j' = 1$ we note that from (1.25) it follows that

$$\left[ \prod_{l=0}^{N-1} B(u_l), \prod_{l=0}^{N-1} B(w_l) \right] = 0 \quad (2.10)$$

and thus

$$\frac{d^2}{d\gamma^2} \left[ \prod_{l=0}^{N-1} B(u_l), \prod_{l=0}^{N-1} B(w_l) \right] = 0 \quad (2.11)$$

Then if we first carry out the differentiations, then set $u_l = v - 2il\gamma_0$, $w_l = w - 2il\gamma_0$ and then use the vanishing condition (1.36) we see that the only non-vanishing terms are those where one derivative acts on a $B(u_l)$ and one acts on a $B(w_l)$ and thus it follows that

$$[B_1^{(N)}(v), B_1^{(N)}(w)] = 0. \quad (2.12)$$

To prove (2.9) for $j = 1$, $j' = 2$ we differentiate (2.10) with respect to $\gamma$ and $w_l$ to find

$$\frac{d^2}{d\gamma dw_l} \left[ \prod_{l=0}^{N-1} B(u_l), \prod_{l=0}^{N-1} B(w_l) \right]$$

$$= \frac{d}{d\gamma} \left[ \prod_{l=0}^{N-1} B(u_l), \left( \prod_{l=0}^{k-1} B(w_l) \right) B_w(w_k) \left( \prod_{l=k+1}^{N-1} B(w_l) \right) \right] = 0 \quad (2.13)$$

Then as before carry out the differentiation with respect to $\gamma$. set $v_l = v - 2il\gamma_0$, $w_l = w - 2il\gamma_0$, and use the vanishing condition (1.36) to obtain

$$[B_1^{(N)}(v), \left( \prod_{l=0}^{k-1} B(w_l) \right) B_w(w_k) \left( \prod_{l=k+1}^{N-1} B(w_l) \right)] = 0 \quad (2.14)$$

from which (2.9) follows for $j = 1$, $j' = 2$.

Finally we note that (2.9) for $j = j' = 2$ follows immediately from (1.25). Thus we have proven the desired commutation relation (1.40).
C. Derivation of \(X(v)\) and \(Y(v)\)

The functions \(X(v)\) and \(Y(v)\) are uniquely determined by the defining property of the operator \(B^{(N)}(v)\) \([1.37]\) with \(l = 1\). We begin the derivation by proving the following four relations:

**Proposition**

\[
\begin{align*}
(1) & \quad A(v)B_v(v') \prod_{l=1}^{N-1} B(v' - 2il\gamma_0) = (-1)^{N+m}B_v(v') \left( \prod_{l=0}^{N-1} B(v' - 2il\gamma_0) \right) A(v) \\
& \quad + \frac{(-1)^{N+m}}{2 \sinh \frac{1}{2}(v' - v)} B(v) \left( \prod_{l=0 \atop l \neq 0}^{N-1} B(v' - 2il\gamma_0) \right) A(v') \\
& \quad - \frac{(-1)^{N+m}}{2 \sinh \frac{1}{2}(v' - v - 2i\gamma_0)} B(v) \left( \prod_{l=0 \atop l \neq N-1}^{N-1} B(v' - 2il\gamma_0) \right) A(v' - 2i\gamma_0) \quad (2.15)
\end{align*}
\]

\[
\begin{align*}
(2) & \quad D(v)B_v(v') \prod_{l=1}^{N-1} B(v' - 2il\gamma_0) = (-1)^{N+m}B_v(v') \left( \prod_{l=0}^{N-1} B(v' - 2il\gamma_0) \right) D(v) \\
& \quad + \frac{(-1)^{N+m}}{2 \sinh \frac{1}{2}(v' - v)} B(v) \left( \prod_{l=0 \atop l \neq 0}^{N-1} B(v' - 2il\gamma_0) \right) D(v') \\
& \quad - \frac{(-1)^{N+m}}{2 \sinh \frac{1}{2}(v' - v - 2(N-1)i\gamma_0)} B(v) \left( \prod_{l=0 \atop l \neq N-1}^{N-1} B(v' - 2il\gamma_0) \right) D(v' - 2(N-1)i\gamma_0) \quad (2.16)
\end{align*}
\]

\[
\begin{align*}
(3) & \quad A(v)B_1^{(N)}(v') = (-1)^{N+m}B_1^{(N)}(v') A(v) \\
& \quad + i \sum_{k=0}^{N-1} \frac{(-1)^{N+m}}{2 \sinh \frac{1}{2}(v' - v - 2ik\gamma_0)} B(v) \left( \prod_{l=0 \atop l \neq k}^{N-1} B(v' - 2il\gamma_0) \right) A(v' - 2ik\gamma_0) \quad (2.17)
\end{align*}
\]

and

\[
\begin{align*}
(4) & \quad D(v)B_1^{(N)}(v') = (-1)^{N+m}B_1^{(N)}(v') D(v) \\
& \quad - i \sum_{k=0}^{N-1} \frac{(-1)^{N+m}}{2 \sinh \frac{1}{2}(v' - v - 2ik\gamma_0)} B(v) \left( \prod_{l=0 \atop l \neq k}^{N-1} B(v' - 2il\gamma_0) \right) D(v' - 2ik\gamma_0) \quad (2.18)
\end{align*}
\]

**Proof**

We begin by iterating the basic relations of the algebraic Bethe’s ansatz \([1.25]-[1.27]\) to obtain
\[ A(v) \prod_{l=1}^{n} B(u_l) = \Lambda \left( \prod_{l=1}^{n} B(u_l) \right) A(v) + B(v) \sum_{k=1}^{n} \Lambda_k \left( \prod_{l=1, l \neq k}^{n} B(u_l) \right) A(u_k) \]  
\tag{2.19}

and

\[ D(v) \prod_{l=1}^{n} B(v_l) = \tilde{\Lambda} \left( \prod_{l=1}^{n} B(u_l) \right) D(v) + B(v) \sum_{k=1}^{n} \tilde{\Lambda}_k \left( \prod_{l=1, l \neq k}^{n} B(u_l) \right) D(u_k) \]  
\tag{2.20}

where

\[ \Lambda = \prod_{l=1}^{n} f(v - u_l), \quad \Lambda_k = g(u_k - v) \prod_{l=1, l \neq k}^{n} f(u_k - u_l) \]  
\tag{2.21}

\[ \tilde{\Lambda} = \prod_{l=1}^{n} f(u_l - v), \quad \tilde{\Lambda}_k = g(v - u_k) \prod_{l=1, l \neq k}^{n} f(u_l - u_k) \]  
\tag{2.22}

To prove (2.13) we start with (2.19) and differentiate with respect to \( u_1 \) to obtain

\[
\begin{align*}
A(v)B_v(u_1) \prod_{l=2}^{n} B(u_l) \\
&= \Lambda B_v(u_1) \left( \prod_{l=2}^{n} B(u_l) \right) A(v) + B(v) \Lambda_1 \left( \prod_{l=2}^{n} B(u_l) \right) A_v(u_1) \\
&+ B(v) \sum_{k=2}^{n} \Lambda_k B_v(u_1) \left( \prod_{l=2, l \neq k}^{n} B(u_l) \right) A(u_k) \\
&+ \frac{\partial \Lambda}{\partial v_1} \left( \prod_{l=1}^{n} B(u_l) \right) A(v) + B(v) \sum_{k=1}^{n} \frac{\partial \Lambda_k}{\partial v_1} \left( \prod_{l=1, l \neq k}^{n} B(u_l) \right) A(u_k)
\end{align*}
\]  
\tag{2.23}

where

\[
\begin{align*}
\frac{\partial \Lambda}{\partial u_1} &= -f_v(v - u_1) \prod_{l=2}^{n} f(v - u_l) \\
\frac{\partial \Lambda_1}{\partial u_1} &= g_v(v_1 - v) \prod_{l=2}^{n} f(u_1 - u_l) + g(u_1 - v) \sum_{k=2}^{n} f_v(u_1 - u_k) \prod_{l=2, l \neq k}^{n} f(u_1 - u_l) \\
\frac{\partial \Lambda_k}{\partial u_1} &= -g(u_k - v)f_v(u_k - u_1) \prod_{l=2, l \neq k}^{n} f(v_k - v_l) \text{ for } k \neq 1.
\end{align*}
\]  
\tag{2.24}

\tag{2.25}
\tag{2.26}

Now after differentiating set \( n = N \) and \( u_l = v' - 2i(l-1)\gamma_0 \). Then

\[
\begin{align*}
f(-2i\gamma_0) &= f(2(N-1)i\gamma_0) = 0 \tag{2.27}
\end{align*}
\]

\[
\begin{align*}
f_v(-2i\gamma_0) &= f_v(2(N-1)i\gamma_0) = \frac{1}{2\sinh i\gamma_0} \tag{2.28}
\end{align*}
\]
and therefore

\[
\Lambda = \prod_{l=1}^{N} f(v - u_l) = \prod_{l=1}^{N} \frac{\sinh \frac{1}{2}(v - v' + 2i\gamma_0)}{\sinh \frac{1}{2}(v - v' + 2i(l - 1)\gamma_0)} = (-1)^N \frac{\sinh \frac{1}{2}(v - v' + 2iN\gamma_0)}{\sinh \frac{1}{2}(v - v')}
\]

\[
= (-1)^{N+m}
\]

(2.29)

\[
\Lambda_k = 0 \quad \text{for} \quad k = 1, \cdots, N
\]

(2.30)

\[
\frac{\partial \Lambda}{\partial u_1} = \frac{(-1)^{N+m}}{2 \sinh \frac{1}{2}(v' - v)}
\]

(2.31)

\[
\frac{\partial \Lambda_2}{\partial u_1} = \frac{(-1)^{N+m+1}}{2 \sinh \frac{1}{2}(v' - v - 2i\gamma_0)}
\]

(2.32)

\[
\frac{\partial \Lambda_k}{\partial u_1} = 0 \quad \text{for} \quad k = 3, \cdots, N
\]

(2.33)

and using the vanishing condition (1.36) we find that the only terms in (2.23) which do not vanish are the first term and the last term with \( k = 1, 2 \). Thus we find that (2.23) reduces to (2.15) as desired.

To prove (2.17) we similarly differentiate (2.19) with respect to \( \gamma \) and then set \( n = N \) and \( u_l = v' - 2(l - 1)i\gamma_0 \). The derivative of the left hand side of (2.19) is

\[
A(v) \sum_{k=1}^{N} \left( \prod_{l=0}^{k-1} B(v' - 2i\gamma_0) \right) B_\gamma(v' - 2ik\gamma_0) \left( \prod_{l=k+1}^{N-1} B(v' - 2i\gamma_0) \right) = A(v)B_1^{(N)}(v')
\]

(2.34)

where the term with \( A_\gamma(v_0) \) vanishes because of the condition (1.36). To differentiate the right hand side of (2.19) we use both the vanishing condition (1.36) and conditions (2.29) and (2.30) on \( \Lambda \) and \( \Lambda_k \). Thus we find that the derivative of (2.19) with respect to \( \gamma \) reduces to

\[
A(v)B_1^{(N)}(v') = (-1)^{N+m}B_1^{(N)}(v')A(v) + B(v) \sum_{k=1}^{N} \frac{\partial \Lambda_k}{\partial \gamma} \left( \prod_{l=1}^{N} B(v' - 2i\gamma_0) \right) A(v' - 2ik\gamma_0).
\]

(2.35)

By differentiating (2.21) with respect to \( \gamma \) and noting that

\[
f_\gamma(v) = -i \frac{\cosh \frac{1}{2}(v + 2i\gamma)}{\sinh(v/2)}
\]

(2.36)

we find

\[
\frac{\partial \Lambda_k}{\partial \gamma} = i \frac{(-1)^{N+m+1}}{\sinh \frac{1}{2}(v' - v - 2i(k - 1)\gamma_0)}
\]

(2.37)

we find that (2.33) reduces to (2.17) as desired.

The proof of (2.16) and (2.18) follows in an similar manner by differentiating (2.20)

QED
We now use (2.13)-(2.18), the definition (2.4) of $B^{(N)}(v)$ in terms of $B^{(N)}_1(v)$ and $B^{(N)}_2(v)$ to find

$$A(v)B^{(N)}(v') = (-1)^{N+m}B^{(N)}(v')A(v)$$

$$-B(v) \sum_{k=0}^{N-1} \left\{ \frac{X(v' - 2i(k - 1)\gamma_0) - X(v' - 2ik\gamma_0)}{Y(v')} \right\}$$

$$\times \left( \prod_{l=1, l \neq k}^{N} B(v' - 2il\gamma_0) \right) \left( -1 \right)^{N+m} A(v' - 2ik\gamma_0)$$

$$D(v)B^{(N)}(v') = (-1)^{N+m}B^{(N)}(v')D(v)$$

$$-B(v) \sum_{k=0}^{N-1} \left\{ \frac{X(v' - 2i(k + 1)\gamma_0) - X(v' - 2ik\gamma_0)}{Y(v')} \right\} + 2i$$

$$\times \left( \prod_{l=1, l \neq k}^{N} B(v' - 2il\gamma_0) \right) \left( -1 \right)^{N+m} D(v' - 2ik\gamma_0)$$

These two operators may now be applied to the Bethe vector (1.31). Then we use (2.19) and (2.20) to move $A(v' - 2ik\gamma_0)$ and $D(v' - 2ik\gamma_0)$ to the right to act on $|0>$, use the vanishing condition (1.36) on all terms generated which contain $B(v' - 2ik\gamma_0)$ and then add the two expressions together using the definition of the transfer matrix in terms of $A(v)$ and $D(v)$ (1.24) to find

$$T(v)B^{(N)}(v') \prod_{k=1}^{n} B(v_k)|0> = (-1)^{N+m}B^{(N)}(v')T(v) \prod_{k=1}^{n} B(v_k)|0>$$

$$-B(v) \sum_{k=1}^{n-1} \left\{ \frac{X(v' - 2i(k - 1)\gamma_0) - X(v' - 2ik\gamma_0)}{Y(v')} \right\}$$

$$\times \left( \prod_{l=1, l \neq k}^{n} B(v' - 2il\gamma_0) \right) \left( -1 \right)^{N+m}\tilde{a}(v' - 2ik\gamma_0)$$

$$\times \left( \prod_{l=1, l \neq k}^{n} B(v' - 2il\gamma_0) \right) \left( -1 \right)^{N+m}\tilde{d}(v' - 2ik\gamma_0)$$

where

$$\tilde{a}(v) = \sinh^{l} \frac{1}{2}(v - i\gamma_0) \prod_{k=1}^{n} f(v - v_k)$$

$$\tilde{d}(v) = \sinh^{l} \frac{1}{2}(v + i\gamma_0) \prod_{k=1}^{n} f(v_k - v).$$

In order for the defining relation for the operator $B^{(N)}(v)$ to hold for $l = 1$ the term proportional to $B(v)$ in (2.40) must vanish. Thus we find that $X(v)$ and $Y(v)$ must satisfy the following functional equation for $k = 0, \cdots, N - 1$.
\[
\left( \frac{X(v - 2i(k - 1)\gamma_0) - X(v - 2ik\gamma_0)}{Y(v)} - 2i \right) \tilde{a}(v - 2ik\gamma_0) + \left( \frac{X(v - 2i(k + 1)\gamma_0) - X(v - 2ik\gamma_0)}{Y(v)} + 2i \right) \tilde{d}(v - 2ik\gamma_0) = 0. \tag{2.43}
\]

It is easily verified that this equation (and the periodicity condition \(X(v + 2iN\gamma_0) = X(v)\)) is satisfied if

\[
\frac{X(v)}{Y(v)} - \frac{X(v - 2i\gamma_0)}{Y(v)} = -2i \left( \frac{N \tilde{a}(v)}{\sum_{l=0}^{N-1} \tilde{a}(v - 2il\gamma_0) - 1} \right) \tag{2.44}
\]

with

\[
\tilde{a}(v) = \sinh^L \frac{1}{2}(v - i\gamma_0) \prod_{k=1}^{n} \frac{1}{\sinh \frac{1}{2}(v_k - v) \sinh \frac{1}{2}(v_k - v + 2i\gamma_0)}. \tag{2.45}
\]

The recursion relation (2.44) only defines \(X(v)\) up to an additive function with the periodicity \(X_0(v - 2i\gamma_0) = X_0(v)\). But any such function \(X_0(v)\) gives a vanishing contribution when used in the expression for \(B^{(N)}(v)\) (1.38) by use of the relation which follows from vanishing condition (1.36) that

\[
\sum_{k=0}^{N-1} B_0(v - 2ik\gamma_0) \prod_{l=0}^{N-1} B(v - 2il\gamma_0) = 0. \tag{2.46}
\]

Therefore we easily verify by substitution that the only solution (2.44) which gives a non-vanishing contribution to \(B^{(N)}(v)\) is

\[
X(v) = 2i \sum_{k=0}^{N-1} k \tilde{a}(v - 2ik\gamma_0). \tag{2.47}
\]

Therefore it also follows that

\[
Y(v) = \sum_{k=0}^{N-1} \tilde{a}(v - 2ik\gamma_0) \tag{2.48}
\]

and thus (1.41) and (1.42) are proven.

Finally we note that \(Y(v)\) has a close connection with the eigenvalues of the transfer matrix. To see this we note that from (1.34) we find

\[
(-1)^n \frac{t(v)}{\prod_{k=1}^{n} \sinh \frac{1}{2}(v - v_k + 2i\gamma_0) \sinh \frac{1}{2}(v - v_k - 2i\gamma_0)} = \frac{\sinh^L \frac{1}{2}(v - i\gamma_0)}{\prod_{k=1}^{n} \sinh \frac{1}{2}(v - v_k - 2i\gamma_0) \sinh \frac{1}{2}(v - v_k)} + \frac{\sinh^L \frac{1}{2}(v + i\gamma_0)}{\prod_{k=1}^{n} \sinh \frac{1}{2}(v - v_k + 2i\gamma_0) \sinh \frac{1}{2}(v - v_k)}. \tag{2.49}
\]

Thus if we replace \(v\) by \(v - 2il\gamma_0\) and sum on \(l\) from 0 to \(N - 1\) we find that the two terms on the right hand side are equal by periodicity and thus we obtain

\[
(-1)^n \frac{1}{2} \sum_{l=0}^{N-1} \frac{t(v - 2il\gamma_0)}{\prod_{k=1}^{n} \sinh \frac{1}{2}(v - v_k - 2(l - 1)i\gamma_0) \sinh \frac{1}{2}(v - v_k - 2(l + 1)i\gamma_0)} = Y(v) \tag{2.50}
\]

where we have used the expression for \(Y(v)\) (1.42).
D. Multiple excitations

It remains to demonstrate that the operator $B^{(N)}(v)$ which has been shown to satisfy the defining equation (1.37) for $l = 1$ in fact satisfies (1.37) for all integer $l > 1$. Consider first $l = 2$. Then using (2.40) to commute $T(v)$ to the right past $B^{(N)}(v'_1)$ we obtain

$$T(v)B^{(N)}(v'_1)B^{(N)}(v'_2)\prod_{k=1}^n B(v_k)|0> =$$

$$+(-1)^{N+m}B^{(N)}(v'_1)T(v)B^{(N)}(v'_2)\prod_{k=1}^n B(v_k)|0>$$

$$-B(v)\sum_{k=0}^{N-1}\left\{\left(\frac{X(v'_1 - 2i(k-1)\gamma_0) - X(v'_1 - 2ik\gamma_0)}{Y(v'_1)} - 2i\right)\right.$$

$$\times\left(\prod_{l=1,\neq k} B(v'_1 - 2il\gamma_0)\right) (-1)^{N+m}A(v'_1 - 2ik\gamma_0)\frac{2\sinh\frac{1}{2}(v'_1 - v - 2ik\gamma_0)}{Y(v'_1)} B^{(N)}(v'_2)\prod_{k=1}^n B(v_k)|0>$$

$$+ \left(\frac{X(v'_1 - 2i(k+1)\gamma_0) - X(v'_1 - 2ik\gamma_0)}{Y(v'_1)} + 2i\right)$$

$$\times\left(\prod_{l=1,\neq k} B(v'_1 - 2il\gamma_0)\right) (-1)^{N+m}D(v'_1 - 2ik\gamma_0)\frac{2\sinh\frac{1}{2}(v'_1 - v - 2ik\gamma_0)}{Y(v'_1)} B^{(N)}(v'_2)\prod_{k=1}^n B(v_k)|0> \right\}.$$  

In the first term on the right hand side of (2.51) we use (2.40) once again to (anti) commute $T(v)$ past $B^{(N)}(v'_2)$ without picking up any additional terms. In the remaining terms we commute $A(v'_1)$ and $D(v'_1)$ to the right past $B^{(N)}(v'_2)$ using (2.38) and (2.39). The vanishing condition (1.36) shows that only the first terms in (2.38) and (2.39) contribute and thus we find

$$T(v)B^{(N)}(v'_1)B^{(N)}(v'_2)\prod_{k=1}^n B(v_k)|0> =$$

$$+(-1)^{2(N+m)}B^{(N)}(v'_1)B^{(N)}(v'_2)T(v)\prod_{k=1}^n B(v_k)|0>$$

$$+(-1)^{N+m}B(v)B^{(N)}(v'_2)\sum_{k=0}^{N-1}\left\{\left(\frac{X(v'_1 - 2i(k-1)\gamma_0) - X(v'_1 - 2ik\gamma_0)}{Y(v'_1)} - 2i\right)\right.$$

$$\times\left(\prod_{l=1,\neq k} B(v'_1 - 2il\gamma_0)\right) (-1)^{N+m}A(v'_1 - 2ik\gamma_0)\frac{2\sinh\frac{1}{2}(v'_1 - v - 2ik\gamma_0)}{Y(v'_1)} \prod_{k=1}^n B(v_k)|0>$$

$$+ \left(\frac{X(v'_1 - 2i(k+1)\gamma_0) - X(v'_1 - 2ik\gamma_0)}{Y(v'_1)} + 2i\right)$$

$$\times\left(\prod_{l=1,\neq k} B(v'_1 - 2il\gamma_0)\right) (-1)^{N+m}D(v'_1 - 2ik\gamma_0)\frac{2\sinh\frac{1}{2}(v'_1 - v - 2ik\gamma_0)}{Y(v'_1)} \prod_{k=1}^n B(v_k)|0> \right\}.$$  

(2.52)

The terms to the right of $B^{(N)}(v'_2)$ are now exactly the terms shown to vanish in the previous
section and thus (1.37) holds to \( l = 2 \). Iteration of this argument proves (1.37) for arbitrary \( l \).

III. PROPERTIES OF \( X(V), Y(V) \) AND THE SUM RULES

The operator \( B^{(N)}(v) \) has the possibility of having poles at the poles of \( X(v) \) or the zeroes of \( Y(v) \).

A. The poles of \( X(v) \)

The function \( X(v) \) (1.41) has poles at

\[
v = v_j + 2ip\gamma_0 \quad \text{where} \quad p = 0, \cdots, N - 1
\]

As \( v \to v_j + 2ip\gamma_0 \) for \( p \geq 1 \) \( X(v) \) behaves as

\[
X(v) \to \frac{2i(p - 1) \sinh L \frac{1}{2}(v_j + i\gamma_0)}{\sinh \frac{1}{2}(v_j + 2ip\gamma_0 - v) \prod_{k \neq j}^n \sinh \frac{1}{2}(v_j - v_k) \prod_{k}^n \sinh \frac{1}{2}(v_k - v_j) - 2i\gamma_0)} \]

which upon using the Bethe’s equation (1.21) reduces to

\[
X(v) \to \frac{-2i \sinh^L \frac{1}{2}(v_j + i\gamma_0)}{\sinh \frac{1}{2}(v_j + 2i\gamma_0 - v) \prod_{k \neq j}^n \sinh \frac{1}{2}(v_j - v_k) \prod_{k}^n \sinh \frac{1}{2}(v_k - v_j) - 2i\gamma_0) \} \]

As \( v \to v_j \) we have

\[
X(v) \to \frac{2i(N - 1) \sinh L \frac{1}{2}(v_j + i\gamma_0)}{\sinh \frac{1}{2}(v_j - v) \prod_{k \neq j}^n \sinh \frac{1}{2}(v_j - v_k) \prod_{k}^n \sinh \frac{1}{2}(v_k - v_j) - 2i\gamma_0) \}
\]

Using these limiting forms in the expression for the operator \( B^{(N)}(v) \) and using (2.46) it follows that as \( v \to v_j + 2il\gamma_0 \) for any \( l \) that

\[
B^{(N)}(v) \to \frac{2iN \sinh L \frac{1}{2}(v_j + i\gamma_0)}{\sinh \frac{1}{2}(v_j - v) \prod_{k \neq j}^n \sinh \frac{1}{2}(v_j - v_k) \prod_{k}^n \sinh \frac{1}{2}(v_k - v_j) - 2i\gamma_0) \times B_v(v_j) \prod_{l=1}^{N-1} B(v_j - 2il\gamma_0) \]

This operator vanishes when applied to the Bethe state because of the vanishing condition (1.36) and therefore \( B^{(N)}(v) \) does not have a pole at \( v = v_j + 2ip\gamma_0 \) even though \( X(v) \) does.
B. Properties of $Y(v)$

The function $Y(v)$ defined by (1.42) also appears to have poles at $v \to v_j + 2ip\gamma_0$. In this limit

$$
Y(v) \to \frac{\sinh^L \frac{1}{2}(v_j - i\gamma_0)}{\sinh \frac{1}{2}(v_j + 2ip\gamma_0 - v) \prod_{k \neq j} \sinh \frac{1}{2}(v_k - v_j) \prod_{k} \sinh \frac{1}{2}(v_k - v_j + 2i\gamma_0)} + \frac{\sinh^L \frac{1}{2}(v_j + i\gamma_0)}{\sinh \frac{1}{2}(v_j + 2ip\gamma_0 - v) \prod_{k \neq j} \sinh \frac{1}{2}(v_k - v_j) \prod_{k} \sinh \frac{1}{2}(v_k - v_j - 2i\gamma_0).}
$$

(3.6)

However by use of the Bethe equation (1.21) we see that the residue at this pole vanishes. Therefore $Y(v)$ in fact has no poles and therefore must be a Laurent polynomial in $e^v$. Moreover it follows from the periodicity condition $Y(v + 2i\gamma_0) = Y(v)$ that $Y(v)$ is in fact a Laurent polynomial in

$$z = e^{Nv}. \quad (3.7)$$

To further study the properties of the Laurent polynomial we introduce the positive and negative degrees of the polynomial defined by the behavior of $Y(v)$ as $v \to \pm \infty$ as

$$Y(v) \sim C_{\pm} e^{\pm Nd_{\pm} v} \text{ as } v \to \pm \infty. \quad (3.8)$$

We may thus define the Drinfeld polynomial as

$$P_{\Omega}(z) = e^{d_- Nv} Y(v). \quad (3.9)$$

This is a polynomial in $z$ with degree

$$d = d_+ + d_- \quad (3.10)$$

This limiting behavior of $Y(v)$ is obtained from (1.42). There still are several cases to consider depending on the number of Bethe roots $v_k$ which take on the value of $\pm \infty$.

Consider first the case of no infinite $v_k$. Then as $v \to \pm \infty$ we find that

$$Y(v) \sim 2^{-(L-2n)} \sum_{l=0}^{N-1} C_{\pm}(l) e^{(\frac{L}{2} - n - l) |v|} \sum_{j=0}^{N-1} e^{-(\frac{L}{2} - n - l) i\gamma_0 (2j + 1)}$$

$$= 2^{-(L-2n)} C_{\pm}(r) Ne^{(\frac{L}{2} - n - r)} \quad (3.11)$$

where

$$\frac{L}{2} - n - r \equiv 0 \pmod{N} \quad \text{and} \quad r = 0, 1, \cdots, N - 1 \quad (3.12)$$

and the first few $C_{\pm}(j)$ are
\[ C_{\pm}(0) = \prod_{k=1}^{n} e^{\pm v_k} \] (3.13)

\[ C_{\pm}(1) = \left( -L + (e^{i\gamma_0} + e^{-i\gamma_0}) \sum_{k=1}^{n} e^{\pm v_k} \right) \prod_{k=1}^{n} e^{\pm v_k} \] (3.14)

\[ C_{\pm}(2) = \left( \frac{1}{2}L(L-1) - L(e^{i\gamma_0} + e^{-i\gamma_0}) \sum_{k=1}^{n} e^{\pm v_k} \right. \\
+ \frac{1}{2}(e^{2i\gamma_0} + e^{-2i\gamma_0}) \sum_{k=1}^{n} e^{2v_k} + \frac{1}{2}(e^{i\gamma_0} + e^{-i\gamma_0})^2(\sum_{k=1}^{n} e^{\pm v_k})^2 \right) \prod_{k=1}^{n} e^{\pm v_k} \] (3.15)

Therefore for the case of no infinite roots \( v_k \) we have

\[ d_+ = d_- = \left( \frac{L}{2} - n - r \right)/N = \left[ \frac{S^z}{N} \right] \] (3.16)

where in the last line we have used (1.33) and \([x]\) denotes the greatest integer in \( x \). Thus the degree of the Drinfeld polynomial for highest weight states with no infinite roots is

\[ d = 2\left[ \frac{S^z}{N} \right]. \] (3.17)

When \( S^z \equiv 0 \pmod{N} \) all evidence in refs.\[1\] is that infinite roots never occur. However for all other cases there are indeed infinite roots and the evidence of refs.\[1\] is that there are two cases to distinguish

(A) The pair \( v = \infty \) and \( -\infty \) occurs \( p \) times.

(B) The single root \( v_k = \infty \) or \( -\infty \) occurs \( s \) times.

For case (A) we let \( p \) pairs of roots go to \( \pm \infty \) in (1.41) and (1.42) and define

\[ \lim \frac{X(v)}{Y(v)} = \frac{X_p(v)}{Y_p(v)} \] (3.18)

where \( X_p(v) \) and \( Y_p(v) \) are obtained from (1.41) and (1.42) by merely omitting the infinite roots (and thus replacing \( n \) by \( n - 2p \)).

The previous argument generalizes by replacing \( n \) by \( n - 2p \) and thus we find

\[ Y_p(v) \sim 2^{-(t-2n+4p)}C_{\pm}(r)Ne^{\frac{1}{2}(L/2-n-r+2p)} \] (3.19)

where

\[ \frac{L}{2} - n - r + 2p \equiv 0 \pmod{N}. \] (3.20)

From the data of table 10 of ref.\[2\] for \( N = 3 \) we find that for \( S^z \equiv 1 \pmod{3} \) that \( p = 2 \) and for \( S^z \equiv 2 \pmod{3} \) that \( p = 1 \). This gives \( r = p \) for \( N = 3 \). If this holds generally for all \( N \) then we find from (3.20) that the degrees of \( Y_p(v) \) are

\[ d_+ = d_- = \left[ \frac{S^z}{N} \right] + 1 \] (3.21)

and thus the degree of the Drinfeld polynomial is
\[ d = 2\lfloor S^z/N \rfloor + 2 \]  

(3.22)

For case (B) we let \( s \) roots \( v_k \to \infty \) or \(-\infty\) in (1.41) and (1.42) to find

\[
X(v)_{\pm s} = 2\pi \sum_{l=0}^{N-1} \frac{e^{\mp s(2l+1)i\gamma_0} \sinh \frac{L}{2}(v - (2l + 1)i\gamma_0)}{\prod_{k=1}^{n-s} \sinh \frac{1}{2}(v - v_k - 2il\gamma_0) \sinh \frac{1}{2}(v - v_k - 2i(l + 1)\gamma_0)} \]

(3.23)

and

\[
Y(v)_{\pm s} = \sum_{l=0}^{N-1} \frac{e^{\pm s(2l+i\gamma_0)} \sinh \frac{L}{2}(v - i\gamma_0(2l + 1))}{\prod_{k=1}^{n-s} \sinh \frac{1}{2}(v_k - v - 2il\gamma_0) \sinh \frac{1}{2}(v_k - v - 2i(l + 1)\gamma_0)} \]

(3.24)

Now the limiting behaviors as \( v \to \pm \infty \) are no longer the same and we find from (3.24) that as \( v \to \infty \)

\[
Y_{\pm s}(v) \sim 2^{-(L-2n-2s)}C_\pm(r_+)N e^{\frac{i}{\lambda}(\frac{L}{2} - n - r_++s)}
\]

(3.25)

where

\[
\frac{L}{2} - n - r_+ + s \equiv 0 \pmod{N}.
\]

(3.26)

and similarly as \( v \to -\infty \)

\[
Y_{\pm s}(v) \sim 2^{-(L-2n-2s)}C_\pm(r_-)N e^{\frac{i}{\lambda}(\frac{L}{2} - n - r_-+s)}
\]

(3.27)

with

\[
\frac{L}{2} - n - r_- + s \equiv 0 \pmod{N}.
\]

(3.28)

In ref. we found that for \( S^z = 1 \pmod{3} \) that \( s = 2 \) and if \( S^z = 2 \pmod{3} \) that \( s = 1 \). From this we find for \( Y_+ \) if \( S^z \equiv 1 \pmod{3} \) that \( r_+ = 2, \ r_- = 1 \) and if \( S^z \equiv 2 \pmod{3} \) that \( r_+ = 1, \ r_- = 2 \). Thus we find for this (and for all other \( N \) with the same relation of \( r_\pm \) to \( S^z \) that

\[
d_+ = \lfloor S^z/N \rfloor + 1, \ d_- = \lfloor S^z/N \rfloor
\]

(3.29)

and thus the degree of the Drinfeld polynomial is

\[
d = 2\lfloor S^z/N \rfloor + 1.
\]

(3.30)

C. Sum rules for Bethe roots

In general the Laurent polynomial \( Y(v) \) has \( d_+ + d_- \geq 1 \). However if the highest weight state has no infinite roots and if

\[
0 \leq S^z \leq N - 1
\]

(3.31)
then we see from (3.13)-(3.15) that \( Y(v) \) is a constant which must equal the limiting values obtained from \( v \to \pm \infty \). Thus it follows that when (3.31) holds we have \( N \) distinct sum rules

\[ C_+(S^z) = C_-(S^z). \]  

(3.32)

For example we find from (3.13)-(3.15) that

\[ S^z = 0 : \sum_{k=0}^{L/2} v_k = 0, \text{ or } \pi i \]  

(3.33)

\[ S^z = 1 : \{ L - (e^{i\gamma_0} + e^{-i\gamma_0}) \sum_{k=1}^{L/2-1} e^{vk} \} \prod_{k=1}^{L/2-1} e^{vk} = \{ L - (e^{i\gamma_0} + e^{-i\gamma_0}) \sum_{k=1}^{L/2-1} e^{-vk} \} \prod_{k=1}^{L/2-1} e^{-vk} \]  

(3.34)

\[ S^z = 2 : \left( \frac{1}{2} L(L-1) - L(e^{i\gamma_0} + e^{-i\gamma_0}) \sum_{k=1}^{n} e^{vk} \right. \]
\[ + \left. \frac{1}{2} (e^{2i\gamma_0} + e^{-2i\gamma_0}) \sum_{k=1}^{n} e^{2vk} + \frac{1}{2}(e^{i\gamma_0} + e^{-i\gamma_0})^2 (\sum_{k=1}^{n} e^{vk})^2 \right) \prod_{k=1}^{n} e^{vk} \]
\[ = \left( \frac{1}{2} L(L-1) - L(e^{i\gamma_0} + e^{-i\gamma_0}) \sum_{k=1}^{n} e^{-vk} \right. \]
\[ + \left. \frac{1}{2} (e^{2i\gamma_0} + e^{-2i\gamma_0}) \sum_{k=1}^{n} e^{-2vk} + \frac{1}{2}(e^{i\gamma_0} + e^{-i\gamma_0})^2 (\sum_{k=1}^{n} e^{-vk})^2 \right) \prod_{k=1}^{n} e^{-vk} \]  

(3.35)

These sum rules have been derived under the assumption that \( \gamma_0 \) satisfies the root of unity (rationality) condition (1.4). However, for fixed \( S^z \), these sum rules hold for all \( N \) such that \( S^z \leq N - 1 \) and will therefore hold for all \( \gamma \) such that \( \gamma/\pi \) is irrational. Moreover the sum rule will also hold at all rational values of \( \gamma/\pi \) for the Bethe’s roots which are obtained by continuity from the irrational values of \( \gamma/\pi \) provided that all the \( v_k \) are finite. Thus we have proven the general sum rules (1.44)-(1.46) given in the introduction.

It is worth noting that these sum rules follow merely from the properties of the function \( Y(v) \) as defined in terms of the Bethe roots by (1.12). The relation of \( Y(v) \) to the operator \( B^{(N)}(v) \) has not been used in this derivation and in principle these sum rules, which to the authors knowledge are new, could have been derived without the knowledge of the \( sl_2 \) loop algebra symmetry which forms the starting point for this present paper.

**IV. REPRESENTATION THEORY**

The mathematical theory of finite dimensional representations of affine Lie algebras\[4, 11\] and the demonstration of the \( sl_2 \) loop algebra symmetry in the six vertex model at roots of unity in the sector \( S^z \equiv 0 \) (mod \( N \)) makes no use of the algebraic Bethe’s ansatz. On the other hand the computation of the \( B^{(N)}(v) \) in this paper makes no use of representation theory. Nevertheless the conjecture that

\[ B^{(N)}(v) = N(z)E^-(z), \]  

(4.1)

where \( N(z) \) is a scalar with no poles implies that the two methods are part of the same subject and that by combining them we are able to obtain results which have so far been inaccessible to either method in isolation.
One such result is the explicit expression for the Drinfeld polynomial

\[ P_\Omega(z) = e^{d_{-Nv}Y(v)}. \]  

(4.2)

Furthermore if the roots of \( Y(v) \) are distinct the current \( E^{-}(z) \) will be given solely in terms of the residue at the poles \( z = a_j \) where \( P_\Omega(a_j) = 0 \). Therefore it must follow that

\[ \frac{B^{(N)}(v)}{N(z)}|\Omega> = \sum_{j=1}^{d_\Omega} \frac{zE^{(N)}_\Omega(j)}{P'_\Omega(a_j)(z - a_j)}|\Omega> \]

(4.3)

When the roots \( P_\Omega \) are distinct representation theory says that only spin 1/2 representations occur in the degenerate multiplets and therefore

\[ E^{(N)2}_\Omega(j)|\Omega> = 0. \]

(4.4)

From this we see that if we denote the spin in the multiplet by

\[ S^z = S^z_{\text{max}} - Nl \quad 0 \leq l \leq d_\Omega \]

(4.5)

that the multiplicity is given by the binomial coefficient given in the introduction

\[ \text{multiplicity} = \binom{d_\Omega}{l}. \]

(4.6)

This is in complete agreement with ref. for all sectors not only for \( S^z \equiv 0 \) (mod \( N \)) where the symmetry algebra was proven but in all other sectors where a projection was needed in order to obtain the algebra. Thus the mechanism of the infinite roots which automatically appears in the algebraic Bethe ansatz makes explicit the projection mechanism only effected on the computer when using the Chevalley basis for the loop algebra generators. We also note from (4.3) that the operator \( B^{(N)}(v) \) acting on \( |\Omega> \) will create a vector space of dimension \( d_\Omega \). For the case \( N = 3 \) this dimension has been computed from the coordinate space Bethe’s ansatz by Braak and Andre.

\[ \begin{align*}
\text{V. CONCLUSIONS}
\end{align*} \]

The results of the previous section are obtained by using the representation theory of affine Lie algebras to provide existence theorems for algebraic Bethe’s ansatz computations. It is clearly desirable to avoid representation theory altogether and to explicitly compute the normalizing constant \( N(z) \) and the residue operators \( E^{(N)}_\Omega(j) \) explicitly from \( B^{(N)}(v) \). Conversely it is desirable to compute the expression (1.42) for the Drinfeld polynomial directly from representation theory without the use of the algebraic Bethe’s ansatz. These can perhaps be viewed as mathematical problems.

But from the physical point of view perhaps the most interesting question is to inquire into the physical meaning and significance of the Drinfeld polynomial (1.42). This polynomial has been seen previously in our study of the limiting form of Bethe’s ansatz as \( \gamma \rightarrow \gamma_0 \) (see the function \( K(\alpha_j) \) of eqn. 2.18 of ref.). In addition very closely related expressions have been seen in the computation of the free energy of the superintegrable chiral Potts
model (see eqn. 4.5 of ref. 31) and in the study of the RSOS models (see eqn. 3.29 of ref. 32). The evaluation parameters, which are the roots of the Drinfeld polynomial, must have a further physical significance which is yet to be discovered.

Finally it needs to be noted that everything in this paper can be extended from the spin 1/2 XXZ model to any model based on a quantum group just as the loop symmetry was extended in ref. 33.
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