Logarithmic $\hat{\mathfrak{sl}}(2)$ CFT models from Nichols algebras: I

A M Semikhatov and I Yu Tipunin

Lebedev Physics Institute, 53 Lelinsky prosp., Moscow 119991, Russia

E-mail: asemikha@gmail.com and tipunin@pli.ru

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Abstract

We construct chiral algebras that centralize rank-2 Nichols algebras with at least one fermionic generator. This gives ‘logarithmic’ $W$-algebra extensions of a fractional-level $\hat{\mathfrak{sl}}(2)$ algebra. We discuss crucial aspects of the emerging general relation between Nichols algebras and logarithmic conformal field theory (CFT) models: (i) the extra input, beyond the Nichols algebra proper, needed to uniquely specify a conformal model; (ii) a relation between the CFT counterparts of Nichols algebras connected by Weyl groupoid maps; and (iii) the common double bosonization $U(\mathfrak{X})$ of such Nichols algebras. For an extended chiral algebra, candidates for its simple modules that are counterparts of the $U(\mathfrak{X})$ simple modules are proposed, as a first step toward a functorial relation between $U(\mathfrak{X})$ and $W$-algebra representation categories.

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1. Introduction

Logarithmic models of two-dimensional conformal field theory (CFT) with $\hat{s}(2)$ symmetry have been addressed in [1–7]. Here, we approach logarithmic $\hat{s}(2)$ models from the theory of Nichols algebras [8–21]. The major advantages are that (i) we then have a recipe for seeking the extended chiral algebra: as (the maximal local algebra in) the kernel of Nichols algebra generators represented by screening operators [22], and (ii) very strong hints about the chiral algebra representation theory come from the Nichols-algebra side.

Our starting point is therefore brazenly algebraic—a braiding matrix $(q_{i,k})_{1 \leq i \leq \operatorname{rank}}, 1 \leq k \leq \operatorname{rank}$, which is just a collection of factors defining the diagonal braiding of generators of a Nichols algebra. Finite-dimensional Nichols algebras with diagonal braiding have been impressively classified [17]; in the spirit of [23, 24], this calls for translating the available structural results into the
language of logarithmic CFT models. We here go beyond the case of rank-1 Nichols algebras [25], and in rank 2 select two braiding matrices

\[
Q_a = \begin{pmatrix} -1 & \frac{1}{q} & \frac{1}{q^2} \\ (1)q^{-1} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q_b = \begin{pmatrix} -1 & \frac{1}{q} & \frac{1}{q^2} \\ (1)q^{-1} & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad q = e^{\pi i/p} \quad (1.1)
\]

with an integer \( p \geq 2 \).

A characteristic result deduced from (1.1) is a triplet–triplet extended algebra \( \mathcal{W}(2, (2p)_{2p}) \) that centralizes the Nichols algebra (the one corresponding to \( Q_a \) with \( j = 0 \), for definiteness): it extends \( \widehat{s}\ell\ell(2) \) at the level \( k = \frac{1}{p} - 2 \) and is generated by dimension-2\( p \) fields

\[
\mathcal{W}^+(z) = \mathcal{F}_1(z), \quad \mathcal{W}^-(z) = s^- (2, p) \mathcal{F}_{-1}(z),
\]

where \( \mathcal{F}_a(z) \) are \( \widehat{s}\ell\ell(2) \) primaries of charge \( \pm 1 \), constructed in a free-field realization associated with the chosen braiding matrix, and \( s^- (2, p) \) is an \( \widehat{s}\ell\ell(2) \) singular vector operator.

The algebra is triplet–triplet because \( \mathcal{W}^\pm(z) \), together with \( \mathcal{W}^0(z) \), make up a triplet, and in addition each \( \mathcal{W}^{\pm, 0, -}(z) \) is part of a triplet under the zero-mode subalgebra of \( \widehat{s}\ell\ell(2) \).

Other results include a similar construction for the second braiding matrix, the ‘triplet–multiplet’ \( W \)-algebras that extend some other fractional-level \( \widehat{s}\ell\ell(2) \), and links between the extended algebras and certain Hopf-algebraic counterparts. The reader is also invited to ‘Logarithmic \( \widehat{s}\ell\ell(2) \) CFT models from Nichols algebras. \( N \)’ with \( N > 1 \) for modular transformations of the extended (pseudo)characters, fusion of representations, etc. Part of our effort, in this paper especially, is given to grasping the general principles governing how a given Nichols algebra can be ‘mapped’ into a logarithmic CFT.

‘Non-Nichols-algebra’ data for logarithmic CFTs. We refrain from claiming, except in private, that a collection of roots of unity \( q_{ij} \) contains exactly as much information as is needed for uniquely reconstructing a CFT model. What precisely the extra input consists of was a major part of the intrigue in writing this paper. One such input is \( j \) in each case in (1.1). When realizing the \( q_{ij} \) as the braiding of screening operators, the range of \( j \) is promoted from \( \mathbb{Z}_2 \) to \( \mathbb{Z} \) (hence an arbitrariness), with drastic consequences for logarithmic theories with different \( j \) (we actually consider \( j = 0, 1, 2, \ldots \) for \( Q_a \) and \( j = -1, 0, 1, 2, \ldots \) for \( Q_b \)). The extended algebra is triplet–triplet for a single value of \( j \) in each case, and triplet–multiplet for other \( j \).

More on the relevant integers: \( p, j, \) and \( p' \). The theory of Nichols algebras only requires that for each case in (1.1), \( q^2 \) be a primitive \( p \)th root of unity. If \( q = e^{\pi i/p} \) (with an odd \( p \) coprime with \( p \)) is chosen instead of \( q \) in (1.1), then the screening operators whose braiding reproduces the braiding matrix depend on two full-fledged coprime integers, \( p \) and \( p' := jp + p \). This is how \( (p, p') \)-type logarithmic models appear in our context. To somewhat reduce technical details, we restrict ourself to the case \( p' = jp + 1 \); the relevant \( \widehat{s}\ell\ell(2) \) machinery is then structurally the same as for the general \( p' \) coprime with \( p \), but slightly simpler to follow.

Thus, setting

\[
q = e^{\pi i/p} \quad (1.2)
\]

in (1.1) and introducing the \( j \) parameter, we effectively deal with \( 2p \)th roots of unity of the form \( e^{\pi i \frac{mp + j}{p}} \), which is the source of \( (p, jp + 1) \) logarithmic models that occur in what follows.
Nichols algebras and fermionic screenings. The Nichols algebras associated with braiding matrices in (1.1) are selected from the list of about two dozen rank-2 Nichols algebras with diagonal braiding [26] based on two properties:

1) these are not isolated but ‘serial’ algebras, existing for each integer \( p \geq 2 \) (to become the \( p \) in the labeling of logarithmic CFT models), and
2) at least one generator of the Nichols algebra has the self-braiding factor \( q_{i,i} = -1 \), which is standardly rephrased by saying that this generator is a fermion.

Because Nichols algebra generators \( F_j \) are identified with screening operators, the \( F_i \) with \( q_{i,i} = -1 \) are generally referred to as fermionic screenings. Fermionic screenings occur in CFT models rather frequently, but are nevertheless slightly disquieting from a ‘conceptual’ standpoint because of the lack of an obvious relation to root systems: while some other ‘good’ systems of screenings are reproducible by rescaling classic root systems, typically by factors like \( 1/\sqrt{p} \), those with fermionic screenings are not. With the power of Nichols algebras, however, this complication passes unnoticed; moreover, there is a procedure to extract a (generalized) Cartan matrix \( A = (a_{i,j}) \) from a braiding matrix (dependent on several ‘\( p \)-type’ parameters in general). Such a Cartan matrix is an important part of the connection between Nichols algebras and conformal models, as we see shortly.

The classes to which braiding matrices (1.1) belong are defined by the conditions [26]

\[
Q_a : \quad q_{11} = -1, \quad q_{12}q_{21}q_{22} = 1, \quad q_{12}q_{21} \in \mathbb{R}_p, \\
Q_s : \quad q_{11} = -1, \quad q_{12}q_{21} \in \mathbb{R}_p, \quad q_{22} = -1, \\
\overline{Q}_a : \quad q_{12}q_{21}q_{11} = 1, \quad q_{12}q_{21} \in \mathbb{R}_p, \quad q_{22} = -1,
\]

where \( \mathbb{R}_p \) is the set of primitive \( p \)th roots of unity, \( p \geq 2 \), and where we temporarily (until (1.8)) add the third case that is merely a 1 ↔ 2 relabeling of the first. In each case, the associated Cartan matrix is the \( A_2 \) one, \( (a_{i,j}) = \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right) \). The Nichols algebra has dimension \( 4p \) in each case.

From braiding to screenings. The point of contact with CFT, already mentioned in the foregoing, is the interpretation of Nichols algebra generators as screenings,

\[
F_i = \oint e^{\alpha_i \cdot \varphi}, \quad 1 \leq i \leq \text{rank} \equiv \theta,
\]

where \( \varphi(z) \) is a \( \theta \)-plet of scalar fields with standard normalization, the dot denotes Euclidean scalar product, and \( \alpha_i \in \mathbb{C}^\theta \) are called the momenta of the screenings. The relation between the screening momenta and the braiding matrix is postulated [24] in the form of \( 1/\theta(\theta + 1) \) equations

\[
e^{\pi \alpha_i \cdot \alpha_j} = q_{j,i}, \quad e^{2\pi \alpha_i \cdot \alpha_j} = q_{j,k}q_{k,j},
\]

and the \( \theta^2 - \theta \) logical ‘or’ conditions

\[
a_i \cdot a_i \cdot a_i = 2a_i \cdot a_j \sqrt{(1 - a_{i,j})a_i \cdot a_i} = 2
\]

imposed for each pair \( i \neq j \) and involving the Cartan matrix \( a_{i,j} \) associated with the given braiding matrix.

For the braiding matrices defined by each line in (1.3), we solve equations (1.5)-(1.6) (with \( \theta = 2 \)) for the scalar products of the momenta, with \( q \) as in (1.1). The respective
solutions are
\[ R_a(j) : \quad \alpha_1 \cdot \alpha_1 = 1, \quad \alpha_1 \cdot \alpha_2 = -\frac{jp+1}{p}, \quad \alpha_2 \cdot \alpha_2 = 2 \frac{jp+1}{p}, \]
\[ R_s(j) : \quad \alpha_1 \cdot \alpha_1 = 1, \quad \alpha_1 \cdot \alpha_2 = \frac{jp+1}{p}, \quad \alpha_2 \cdot \alpha_2 = 1, \]
\[ R_s(j) : \quad \alpha_1 \cdot \alpha_1 = 2 \frac{jp+1}{p}, \quad \alpha_1 \cdot \alpha_2 = -\frac{jp+1}{p}, \quad \alpha_2 \cdot \alpha_2 = 1, \] (1.7)

with an arbitrary \( j \in \mathbb{Z} \) in each case. These three systems of \( \mathbb{C}^2 \) vectors are related by Weyl-groupoid pseudoreflections, which map as (‘1’ with respect to \( \alpha_1 \) and ‘2’ with respect to \( \alpha_2 \))

![Diagram](image)

\( (1.8) \)

(in our case of a classic, \( A_2 \), Cartan matrix, it is the same for the entire orbit, and therefore the Weyl groupoid becomes the corresponding Weyl group).

The third, ‘opposite asymmetric’ case does not have to be considered separately. In what follows, we therefore speak about two—‘asymmetric’ and ‘symmetric’—systems of the screening momenta, braiding matrices, braided spaces \( X_a \) and \( X_s \), Nichols algebras \( B(X_a) \) and \( B(X_s) \), etc; in the narrow, literal sense, the symmetry is with respect to the anti-diagonal of \( Q_s \) in (1.1), but it also shows up on the CFT side, as we describe below.

2 \( \neq \) 3: From parafermions to \( \widehat{\mathfrak{sl}}(2) \). Taking rank \( = 2 \) means that the kernel of two screenings is defined more or less uniquely in a two-boson space; there, the kernel contains parafermionic fields \( j^+ (z) \) and \( j^- (z) \)—generators of the \( \widehat{\mathfrak{sl}}(2)/u(1) \) coset theory, which are mutually nonlocal. Apparently, one can proceed within the theory of parafermions and their ‘representations,’ but this is a challenge we are not up to, in this paper at least. Instead, we introduce a third, ‘auxiliary’ scalar field representing the \( u(1) \) in the above coset, with the OPE

\[ \chi(z) \chi(w) = 2k \log(z - w), \] (1.9)

where \( k = \frac{1}{p} + j - 2 \) in the asymmetric case and \( k = \frac{1}{p} + j - 1 \) in the symmetric case, and use it to convert the parafermions into level-\( k \) \( \widehat{\mathfrak{sl}}(2) \) currents

\[ J^\pm (z) = \xi^\pm j^\pm (z) e^{\pm \frac{i}{2} \chi(z)}. \]

We can then use the full power of the \( \widehat{\mathfrak{sl}}(2) \) representation theory, but the price is the occurrence of twisted (spectral-flow transformed) \( \widehat{\mathfrak{sl}}(2) \) representations, because the locality requirement alone leaves a size-\( \mathbb{Z} \) arbitrariness in how the third scalar enters the relevant operators.

Eliminating \( \chi(z) \) in order to return to a ‘clean’ Nichols-algebra-motivated two-boson conformal model is not difficult at the level of formulas (for generators of the extended algebra, etc), and can be quite interesting at the level of characters (cf [27]), but the meaning

---

1 We use noncanonical normalizations. The canonical ones are of course easy to restore, but at the expense of factors like \( \sqrt{\pm 1} \), entailing either a choice for the sign of \( k \) or unnecessary imaginary units.
of ‘algebras’ generated by mutually nonlocal fields would then have to be worked out first, before the ‘three-boson’ results in this paper can be restated in that language.

From \( \hat{sl}(2)_k \) to \( W \) algebras. The \( \hat{sl}(2)_k \) algebra is used as a seed to grow extended algebras living in the kernel of the screenings inside \( \hat{sl}(2)_k \) representations. The relevant representations are Wakimoto-type modules [28, 29]. These modules are related by intertwining maps, which can also be associated with screenings [30], and the intersection of the screening kernels we are interested in is the socle of such modules (cf [4]).

In the kernel, we select mutually local fields, and among them, \( \hat{sl}(2)_k \) primaries with the minimal Sugawara dimension as generators of the extended algebra—a \( W \)-algebra of mutually local fields in the kernel. The generators come in a triplet \( W^+(z), W^0(z), \) and \( W^-(z) \) and, moreover, each of these is also a multiplet with respect to the zero-mode \( \hat{sl}(2)_k \) subalgebra of \( \hat{sl}(2)_k \). We construct the \( W \)-algebra generators as explicitly as the celebrated construction [31] of \( sl(2) \) singular vectors allows. The \( W \)-algebras constructed in the asymmetric and symmetric realizations are isomorphic and are in fact related as two ‘rebosonizations’ of the Wakimoto representation, as we discuss in section 5.5.1.

For the asymmetric realization (the top row in (1.7)) for definiteness, with \( j \geq 0 \), the three generators have the (Sugawara) dimension \( 2jp(2jp + 1) \) and each is part of a \((4jp + 3)\)-plet with respect to the zero-mode \( \hat{sl}(2)_k \) algebra. The cases \( j = 0 \) (dimension \( 2p \) and zero-mode triplets) and \( j \geq 1 \) are essentially different. For \( j = 0 \), the \( W^+(z) \) field is a pure exponential, and there exists a ‘long’ screening mapping as \( W^+(z) \xrightarrow{\hat{e}} W^0(z) \xrightarrow{\hat{e}} W^-(z); \text{ for } j \geq 1 \), by contrast, all three fields (denoted as \( W^+(z), W^0(z), \) and \( W^-(z) \)) have the form of a differential polynomial times an exponential and the long screening does not map that way. Another characteristic illustration of the difference is the result of Hamiltonian reduction to single-boson models: to the \((p, 1)\) triplet algebra [33, 34, 22, 35] for \( j = 0 \) and to the triplet algebras of \((p, p')\) logarithmic models introduced in [36] (also see [35, 37]) for \( j \geq 1 \) (with \( p' = p + 1 \) due to our choice of \( q \) in (1.2), as discussed above).

Our ‘\( W \)’-style notation for the triplet–triplet algebra (corresponding to \( j = 0 \) in the asymmetric realization) is \( W(2, (2p)^{3}\times 3) \), where 2 is the dimension of the energy–momentum tensor and 2\( p \) is the dimension of all the \( W^{+,0,-}(z) \) fields. The first multiplicity of 3 is for the superscript of the \( W^{+,0,-}(z) \) fields, and the second, for the fact that each of these fields is a triplet under the zero-mode \( \hat{sl}(2)_k \).

\( \begin{align*}
\chi(z)\chi(w) & = (2 + kn)k \log(z - w), \quad n \in \mathbb{Z}.
\end{align*} \)

The parafermions would then be ‘dressed’ not into \( J^\pm(z) \) but into local fields \( J^\pm(z) \) with the OPEs \( \mathcal{O}^\pm(z), \mathcal{O}^\pm(w) \propto (z-w)^n \) and \( J^+(z), J^-(w) = \text{const} \cdot (z-w)^{-n-2} + \cdots \). For \( n = 1 \), in particular, \( J^+(z) \) and \( J^-(z) \) generate the \( N = 2 \) super-Virasoro algebra. The ‘parafermion core’ of all these theories with different \( n \) is the same. The value \( n = 0 \) chosen in (1.9) and corresponding to \( \hat{sl}(2)_k \) is the lowest in the sense that taking \( n \leq -1 \) leads to unconventional theories where \( J^+(z) \) (and \( J^-(z) \)) has a nonvanishing OPE with itself. We do not return to this point here and stay with (1.9).

\( \begin{align*}
\text{The screenings that are identified with Nichols algebra generators are sometimes conventionally called ‘short’ screenings, as opposed to ‘long’ screenings, which centralize the nonextended algebra (Virasoro in [22], \( W_k \) in [32], and \( \hat{sl}(2)_k \) in this paper).}
\end{align*} \)

\( \begin{align*}
\text{This already shows that when it comes to comparing appropriate categories, the extra data used explicitly or implicitly in ‘reconstructing’ a logarithmic model from a Nichols algebra can play a decisive role: the categories compare reasonably well for \((p, 1)\) models [38–40] and somewhat worse for \((p, p')\) models [36, 41–44]. Modular transformations are remarkably robust, however: they are the same on the CFT and Hopf-algebraic sides of \((p, p')\) models [36, 41].}
\end{align*} \)
Choosing an \( \mathcal{H}/YD \) category. The relation between \( \mathcal{B}(X) \) and the extended algebra starts working at its full strength when lifted to the level of a correspondence between the categories of their representations. On the Nichols-algebra side, the relevant representations are Yetter–Drinfeld \( \mathcal{B}(X) \) modules. The class of models that we wish to have in CFT should be ‘rational in a reasonably broad sense’ (certainly with finitely many simple modules and possibly with the \( C_2 \) cofiniteness property [45], etc; ultimately, the rigorous framework of [46] must be kept in mind). To avoid ‘strongly nonrational’ effects, the scalar fields must take values in a torus; in other words, a particular lattice vertex-operator algebra should be selected.

On the Nichols-algebra side, the corresponding finiteness/discreteness is controlled by choosing a nonbraided Hopf algebra \( H \) used for ‘YDinization,’ i.e., representing all relevant braided spaces (starting with \( X \)) as objects in \( \mathcal{H}/YD \) (the category of Yetter–Drinfeld \( H \)-modules). Such an \( H \) is by no means unique [47], and its choice is another input beyond the Nichols algebra itself, because \( \mathcal{B}(X) \) is independent of an \( H \) such that \( X \in \mathcal{H}/YD \), but the resulting representation categories acquire such a dependence. For diagonal braiding, we have \( H = k\Gamma \) for an Abelian group \( \Gamma \). The choice of \( \Gamma \) is in turn related to the representation of screenings in terms of free bosons in (1.4): once scalar fields are introduced, there is a zero-mode operator \( \phi^{(i)}(z) \) for each scalar \( \phi^{(i)}(z) = \bar{\phi}^{(i)} + \phi^{(i)}_0 \log z + \sum_{n \in \mathbb{Z}} \frac{z^{-n}}{n!} \phi^{(i)}_n \), and operators \( e^{i\alpha_x \phi^{(i)}} \), with \( \alpha \) determined by the chosen lattice vertex-operator algebra, can be considered generators of \( \Gamma \).

We do not quite study \( \mathcal{B}(X) \mathcal{YD} \) for \( X \in \mathcal{H}/YD \) in this paper, however, the reason being that, the results in [25] notwithstanding, we made a much faster progress within a more ‘old-fashioned’ setting of \( U(X) \)-modules for a double bosonization \( U(X) = \mathcal{B}(X^\ast) \otimes \mathcal{B}(X) \otimes H \) of our Nichols algebras.

Double bosonization: \( U(X) \). Double bosonization of braided Hopf algebras was discussed previously in more general contexts in [48, 49]; it is to produce a nonbraided Hopf algebra from a braided Hopf algebra \( R \) in \( \mathcal{H}/YD \), its dual \( R^\ast \), and the Hopf algebra \( H \) itself, under a number of conditions satisfied by all the ingredients. The construction ‘doubles’ the standard Radford bosonization (byproduct formula) [50] whereby the smash product \( R \# H \) is endowed with the structure of a Hopf algebra. For a Nichols algebra \( \mathcal{B}(X) \) with diagonal braiding and for a commutative cocommutative \( H = k\Gamma \), the list of double-bosonization conditions is somewhat reduced, the most essential remaining one being the symmetricity of the braiding in the standard sense \( q_{i,j} = q_{j,i} \). Double bosonization then yields a (nonbraided) Hopf algebra

\[
U(X) = \mathcal{B}(X^\ast) \otimes \mathcal{B}(X) \otimes k\Gamma,
\]

which contains the ‘single’ bosonizations \( \mathcal{B}(X) \# k\Gamma \) and \( \mathcal{B}(X^\ast) \# k\Gamma \) as Hopf subalgebras, where the prime indicates that the \( k\Gamma \) action and coaction are changed by composing each with the antipode (they remain a left action and a left coaction for \( H = k\Gamma \)).

This double bosonization previously appeared in [51, 19, 52]. We arrive at (1.10) in a way that may be interesting in and of itself, the key observation being that even without the assumption that \( X \) is in \( \mathcal{H}/YD \) (but with diagonal braiding), there is an associative algebra structure on \( \mathcal{U}(X) = \mathcal{B}(X^\ast) \otimes \mathcal{B}(X) \otimes k\Gamma \) for an Abelian group \( \Gamma \) that is read off from the monodromy \( \Psi^2 : X \otimes X \to X \otimes X \). This associative algebra is only ‘half the way’ to (1.10) because, not surprisingly, \( \Gamma \) is ‘too coarse’ to endow \( (X, \Psi) \) with the structure of a Yetter–Drinfeld module, and for essentially the same reason, \( \mathcal{U}(X) \) is not a Hopf algebra. In fact, \( \Gamma \) is generated by the squares of the generators of a ‘minimal’ \( \Gamma \) featuring in (1.10), and \( \mathcal{U}(X) \) can be considered a subalgebra in \( \mathcal{U}(X) \).

5 Our base field is \( \mathbb{C} \), but we write \( k\Gamma \) for (perhaps misinterpreted) esthetic reasons.
Applying (1.10) to the two (nonisomorphic) rank-2 Nichols algebras studied here, the ‘asymmetric’ $B(X_a)$ and ‘symmetric’ $B(X_s)$ yields two $64p^d$-dimensional Hopf algebras $U(X_a)$ and $U(X_s)$. These turn out to be isomorphic as associative algebras:

$$\tilde{\sigma} : U(X_a) \sim U(X_s).$$

Moreover, the two coproducts, $\Delta_a$ and $\Delta_s$, are related by a similarity transformation: there exists an invertible element $\Phi \in U(X_a) \otimes U(X_s)$ satisfying an appropriate cocycle condition such that

$$\Phi^{-1} \Delta_a(\tilde{\sigma}(x)) \Phi = (\tilde{\sigma} \otimes \tilde{\sigma}) \circ \Delta_a(x)$$

for all $x \in U(X_s)$. Hence, in particular, the representation categories of $U(X_a)$ and $U(X_s)$ are equivalent as monoidal categories.

The representation theory of $U(X)$ is expected, at the very least, to produce hints as to the representation theory on the CFT side; but the hope—and the subject of future research—is that the category of $U(X)$-modules is equivalent to the representation category of a chiral algebra. Here is a subtlety, however, which has no clear-cut analogue in the $W_{p,1}$/Virasoro case.

The $W(2, (2p)^{3 \times 2})$ algebra. Only a subalgebra in $W(2, (2p)^{3 \times 3})$ commutes with the generators of the Abelian group $\Gamma$ that fixes the underlying Yetter–Drinfeld category or, equivalently, allows construction of the double bosonization. This subalgebra does not contain $J^+(z)$ and $J^-(z)$ currents, but contains their squares (normal-ordered products) $(J^+(z))^2$ and $(J^-(z))^2$. For consistency, the fields sitting in the middle of each triplet with respect to the zero-mode $s\ell(2)$ also have to be dropped. The triplets then become ‘doublets’ with respect to $(J^z(z))^0$, and the notation for this algebra is $W(2, (2p)^{3 \times 2})$.

The representation theory of $U(X)$ (developed to some depth in [53]) and the representation theory of $W(2, (2p)^{3 \times 2})$ (rudimentary) show as much agreement as the ‘rudimentary’ one allows. In particular, there are natural candidates for irreducible $W(2, (2p)^{3 \times 2})$ modules. The irreducible modules are divided into spectral flow orbits, which are in a $1 : 1$ correspondence (not only in number, but also as $\Gamma$-modules) with the $U(X)$ simple modules.

It is for the representation category of $W(2, (2p)^{3 \times 2})$ that we expect a number of ‘good’ properties resembling those of $W_{p,1}$-models [54]. Although the existing data is scarce, it is very tempting to conjecture that much information about the representation category of $W(2, (2p)^{3 \times 2})$ (including fusion and, possibly, modular transformations) can be obtained from investigating the $U(X)$ category.

Notation and conventions. The braiding matrix elements are defined such that

$$\Psi : F_i \otimes F_k \mapsto q_{i,k} F_k \otimes F_i, \quad 1 \leq i, k \leq \text{rank}. \tag{1.13}$$

The notations for conformal fields, such as $\varphi_1(z)$ and $\varphi_2(z)$, are ‘local’ to each case, asymmetric (section 4) and symmetric (section 5); they are for different scalar fields, whose OPEs are related to the respective system of scalar products in (1.7).

Our $\hat{\mathfrak{sl}}(2)$ conventions are given in section B.1 in the appendix.

We use $q$-integers, factorial, and binomials defined as

$$\langle n \rangle \equiv \frac{q^{2n} - 1}{q^2 - 1}, \quad \langle n \rangle! = \langle 1 \rangle \cdots \langle n \rangle, \quad \binom{m}{n} = \frac{\langle m \rangle!}{\langle n \rangle! \langle m - n \rangle!} \quad (m \geq n \geq 0),$$

all of which are assumed specialized to $q = q$ in (1.2).
2. Nichols algebra: the ‘asymmetric’ case

We explicitly describe the Nichols algebra $\mathfrak{B}(X)$ corresponding to the first line in (1.3): we take $X = X_4$ to be a braided vector space with basis $B, F$ and with the diagonal braiding in this basis specified by the braiding matrix

$$(q_{ij}) = \begin{pmatrix} -1 & \xi^{-1}q^{-1} \\ \xi q^{-1} & q \end{pmatrix}$$

(2.1)

(the conventions are such that, e.g., $\Psi(B \otimes F) = \xi^{-1}q^{-1}F \otimes B$). Here, $\xi$ is any $2p$th root of unity, $\xi^{2p} = 1$; this is an ‘inessential’ variable, which can be eliminated (‘set equal to 1’) by a twist map associated with a certain 2-cocycle [15]. In the applications in what follows, we restrict to the case where $\xi^2 = 1$ anyway, but we nevertheless keep $\xi$ in this section, to add some flavor to the explicit formulas.

2.1. $\mathfrak{B}(X)$: a presentation

For the braiding matrix in (2.1), the Nichols algebra $\mathfrak{B}(X)$ is the quotient [20] (also see [55])

$$\mathfrak{B}(X) = T(X)/([F, [F, B]], B^2, F^p)$$

(2.2)

if $p \geq 3$ and

$$\mathfrak{B}(X) = T(X)/(B^2, [B, F]^2, F^2)$$

(2.3)

if $p = 2$, with dim $\mathfrak{B}(X) = 4p$ in all cases; here, square brackets denote $q$-commutators, $[F, B] = FB - \xi q^{-1}BF$ and $[F, [F, B]] = F[F, B] - \xi q[F, B]F$. The double commutator in (2.2) therefore represents the relation

$$\xi^2BF^2 - \xi(q + q^{-1})FBF + F^2B = 0.$$  

(2.4)

This implies that

$$BFBF - \xi^{-2}FBFB = 0.$$  

(2.5)

For $p = 2$, this last relation is a rewriting of $[B, F]^2 = 0$ under the condition that $B^2 = F^2 = 0$.

The multiplication on $T(X)$ and $\mathfrak{B}(X)$ understood in (2.2) and the subsequent formulas is by ‘concatenation’ $X^{\otimes m} \otimes X^{\otimes n} \rightarrow X^{\otimes (m+n)}$, which simply maps $(x_1, \ldots, x_m) \otimes (y_1, \ldots, y_n) \mapsto (x_1, \ldots, x_m, y_1, \ldots, y_n)$; comultiplication is then by ‘deshuffling.’

2.2. $\mathfrak{B}(X)$ as a subalgebra

Another description of any Nichols algebra $\mathfrak{B}(X)$, which is in fact the description in terms of screening operators, is not as a quotient of but as a subspace in $T(X)$—as the algebra generated by basis elements of $X$ under the shuffle product [11, 23]. The two realizations of $\mathfrak{B}(X)$ are related by the total braided symmetrizer map $\mathfrak{S}_n : X^{\otimes n} \rightarrow X^{\otimes n}$ in each grade. It is an isomorphism precisely because in the presentation $\mathfrak{B}(X)^{(n)} = T(X)^{(n)}/J^{(n)}$, the ideal $J^{(n)}$ is the kernel of $\mathfrak{S}_n$.

6 The symbol $\mathfrak{B}$ chosen for a fermionic generator may not suggest the best mnemonics; the association, possibly somewhat far-fetched, is with the physicists’ notation $b, c$ for a pair of fermionic (‘ghost’) fields. Indeed, a $C$ appears as a pair to $B$ in what follows.
2.2.1. In this language, the \( \mathfrak{B}(X) \) in section 2.1 is the linear span of the \( 4p \) PBW elements

\[
V_n = \frac{1}{(n)!} F * F * \cdots * F, \quad 0 \leq n \leq p - 1
\]

(with \( r_0 = 1 \)),

\[
FB_n = \frac{1}{(n-1)!} F * F * \cdots * F * B, \quad 1 \leq n \leq p,
\]

\[
XB_n = \frac{1}{(n-2)!} \left( F * F * \cdots * F * B - \xi^{-1} q F * F * \cdots * F * B \right), \quad 2 \leq n \leq p + 1,
\]

\[
BFB_n = \frac{1}{(n-3)!} F * F * \cdots * F * B * F * B, \quad 3 \leq n \leq p + 2,
\]

where \( * \) denotes the shuffle product associated with the given braiding,

\[
(x_1, \ldots, x_m) \otimes (y_1, \ldots, y_n) \mapsto \bigoplus_{i=1}^{m} (x_1, \ldots, x_m, y_1, \ldots, y_n)
\]

(our conventions are just as in [23], except that \( * \) was not used for the shuffle product there).

In terms of the concatenation product, we have

\[
\gamma_0 \equiv F^n, \quad FB_n = \sum_{i=1}^{n} \xi^{n-i} q^{n-i+1} F^{i-1} B F^{n-i},
\]

\[
XB_n = \sum_{i=2}^{n} \xi^{n-i-1} q^{n-i-1} (1 - q^2)(i-1) F^{i-1} B F^{n-i},
\]

and some lower-degree elements are given by

\[
FB_1 = B, \quad FB_2 = \xi q^{-1} BF + FB, \quad FB_3 = \xi^2 q^{-2} BFF + \xi q^{-1} BFB + FFB,
\]

\[
XB_2 = q^{-1} \xi^{-1} (1 - q^2) FBF, \quad XB_3 = (1 - q^2) FBF + q^{-1} \xi^{-1} (1 - q^4) FFB,
\]

\[
BFB_3 = (1 - q^{-2}) BFB.
\]

The shuffle multiplication table in the above basis is evaluated as

\[
\gamma_0 \ast \gamma_0 = \binom{n+m}{n} \gamma_{n+m}, \quad \gamma_0 \ast FB_n = \binom{n+m-1}{n} FB_{n+m},
\]

\[
\gamma_0 \ast XB_n = \binom{n+m-2}{n} XB_{n+m}, \quad \gamma_0 \ast BFB_n = \binom{n+m-3}{n} BFB_{n+m},
\]

\[
FB_n \ast \gamma_0 = q^{1-m} \xi q^{-1} \binom{n+m-2}{n-1} XB_{n+m}, \quad FB_n \ast FB_m = \binom{n+m-1}{n-1} FB_{n+m},
\]

\[
FB_n \ast XB_m = q^{m-1} \xi^{-1} \binom{n+m-3}{n-1} FB_{n+m}, \quad (in \ particular, FB_n \ast FB_1 = 0),
\]

\[
FB_n \ast XB_m = -q^{m-1} \xi^{-1} \binom{n+m-3}{n-1} FB_{n+m}, \quad FB_n \ast BFB_m = 0,
\]

\[
XB_n \ast \gamma_0 = q^{m-2} \xi^{-1} \binom{n+m-2}{n-2} XB_{n+m}, \quad XB_n \ast FB_m = 0,
\]

\[
XB_n \ast XB_m = q^{m-1} \xi^{-1} \binom{n+m-3}{n-2} FB_{n+m}, \quad XB_n \ast XB_m = 0,
\]

\[
XB_n \ast BFB_m = 0, \quad BFB_n \ast \gamma_0 = \xi^{-2m} \binom{n+m-3}{n-3} BFB_{n+m},
\]

and all other products with \( BFB_n \) vanish.
Although this is obvious from the general theory, we note explicitly that the relations by which the quotient is taken in (2.2) are now ‘resolved’—hold identically—due to the properties of the shuffle product; in particular, the \( s \)-form of ‘Serre relation’ (2.4) holds identically:
\[
\xi^2 B * F * F - \xi (q + q^{-1}) F * B * F + F * F * B = 0.
\]

2.2.2. With multiplication given by the shuffle product, comultiplication is by deconcatenation (see [23]). For the above basis elements, their deconcatenation can be calculated from the definition, for example,
\[
\Delta_{FBF_3} = (1 - q^{-2})(BFB \otimes I + BF \otimes B + B \otimes FB + B \otimes I \otimes BFB)
\]
\[
= BFB_3 \otimes I + I \otimes BFB_3 + XB_2 \otimes FB_1 + \xi q^{-1} I \otimes XB_2 + \xi^{-1} (q - q^{-1}) FB_2 \otimes FB_1.
\]
The result is
\[
\Delta_{FB} = \sum_{i=0}^{n} \xi_i \otimes FB_{n-i},
\]
\[
\Delta_{FB} = \sum_{i=0}^{n-1} \xi_i \otimes FB_{n-i} + \sum_{i=1}^{n} \xi^{n-i} q^{n-i+1} FB_i \otimes FB_{n-i},
\]
\[
\Delta_{XB} = \sum_{i=0}^{n-2} \xi_i \otimes XB_{n-i} + \sum_{i=1}^{n} \xi^{n-i} q^{n-i+1} XB_i \otimes FB_{n-i} + \sum_{i=1}^{n-1} \xi^{-1} q^{2n-3} (q^{-2} - 1) \xi_i \otimes FB_{n-i},
\]
\[
\Delta_{F BF_3} = \sum_{i=0}^{n-2} \xi i \otimes F B F_{n-i} + \sum_{i=1}^{n} \xi^{2n-2} FB_i \otimes FB_{n-i} + \sum_{i=2}^{n-1} \xi^{-1} q^{n-i-1} XB_i \otimes FB_{n-i} - \sum_{i=1}^{n-2} \xi^{n-i-1} q^{n-i+1} FB_i \otimes XB_{n-i} + \sum_{i=2}^{n-1} \xi^{n-i-2} q^{n-i+2} (q^2 - 1) (i - 1) FB_i \otimes FB_{n-i}.
\]

2.2.3. The antipode is given by ‘half-twist,’ the Matsumoto lift of the longest element in the symmetric group [23], which is evaluated for the above basis elements as
\[
S(I_i) = (-1)^i q^{(n-1)} I_i,
\]
\[
S(B) = -B,
\]
\[
S(FB_i) = (-1)^i q^{(n-4)(n-1)} FB_i + (-1)^i q^{(n-4)(n-1)+1} FB_i, \quad 2 \leq n \leq p,
\]
\[
S(XB_i) = (-1)^i q^{(n-2)(n-1)} XB_i + (-1)^i q^{-1} q^{(n-2)(n-1)-1} (1 - q^2) (n - 1) FB_i,
\]
\[
S(FB_{n-i}) = (-1)^i q^{(n-5)(n-2)} FB_{n-i}.
\]

2.3. Vertex operators and Yetter–Drinfeld \( \mathcal{B}(X) \)-modules

For any pair of integers \( \tilde{r} \) and \( \tilde{s} \), we introduce a one-dimensional braided vector space \( Y^{[\tilde{r}, \tilde{s}]} \), with a fixed basis vector \( V^{[\tilde{r}, \tilde{s}]} \) (called a vertex operator) such that
\[
\Psi : B \otimes V^{[\tilde{r}, \tilde{s}]} \mapsto q^{\tilde{r}} V^{[\tilde{r}, \tilde{s}]} \otimes B, \quad V^{[\tilde{r}, \tilde{s}]} \otimes B \mapsto q^{\tilde{s}} B \otimes V^{[\tilde{r}, \tilde{s}]}.
\]
\[
\Psi : F \otimes V^{[\tilde{r}, \tilde{s}]} \mapsto q^{\tilde{r}} V^{[\tilde{r}, \tilde{s}]} \otimes F, \quad V^{[\tilde{r}, \tilde{s}]} \otimes F \mapsto q^{\tilde{s}} F \otimes V^{[\tilde{r}, \tilde{s}]}.
\]
Each space \( \mathcal{B}(X) \otimes Y^{[\tilde{r}, \tilde{s}]} \otimes \mathcal{B}(X) \otimes Y^{[\tilde{r}, \tilde{s}]} \otimes \ldots \otimes \mathcal{B}(X) \otimes Y^{[\tilde{r}, \tilde{s}]} \) is a Yetter–Drinfeld \( \mathcal{B}(X) \)-module (an ‘\( N \)-vertex’ Yetter–Drinfeld module in [23]) under the left adjoint action and coaction given by deconcatenation up to the first \( Y \) space. We here consider only one-vertex modules; the \( \mathcal{B}(X) \) action and coaction are then those in (A.1); coaction therefore reduces to
just the comultiplication in section 2.2.2, and the adjoint action can be calculated using the formulas in sections 2.2.1–2.2.3. The result is

\[ B \triangleright e_0 V^{[\ell, \mathfrak{r}], [\ell, \mathfrak{s}]} = \xi^{1-n}q^{1-n} \mathfrak{r}^{\mathfrak{r}} \mathfrak{s}^{\mathfrak{s}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} + (1 - q^{-2\mathfrak{r} + 2\mathfrak{s}}) \xi^{-n}q^{-n} \mathfrak{r}^{\mathfrak{r}} \mathfrak{s}^{\mathfrak{s}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} + (1 - q^{-2\mathfrak{s} + 2\mathfrak{r}}) \xi^{-n}q^{-n} \mathfrak{s}^{\mathfrak{s}} \mathfrak{r}^{\mathfrak{r}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} , \]

and

\[ F \triangleright e_0 V^{[\ell, \mathfrak{r}], [\ell, \mathfrak{s}]} = (n + 1) (1 - q^{2(n+\mathfrak{r} + \mathfrak{s})}) \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} , \]

\[ F \triangleright F^{[\ell, \mathfrak{r}], [\ell, \mathfrak{s}]} = (n) (1 - q^{2(n+\mathfrak{r} + \mathfrak{s}-1)}) \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} , \]

\[ F \triangleright X^{[\ell, \mathfrak{r}], [\ell, \mathfrak{s}]} = (n) (1 - q^{2(n+\mathfrak{r} + \mathfrak{s}-2)}) \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} , \]

\[ F \triangleright F^{[\ell, \mathfrak{r}], [\ell, \mathfrak{s}]} = (n - 1) (1 - q^{2(n+\mathfrak{r} + \mathfrak{s}-3)}) \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} \mathfrak{B}^{\mathfrak{B}} , \]

where the range of \( n \), unless specified explicitly, is in each case as indicated in section 2.2.1.

We see that in the action on one-vertex modules, \( \mathfrak{r} \) and \( \mathfrak{s} \) enter only modulo \( p \). We encounter this again in section 4.6.2.

3. Nichols algebra: the ‘symmetric’ case

We explicitly describe \( \mathfrak{B}(X) \) in the case corresponding to the second line in (1.3); we take \( X = X_1 \) to be a braided vector space with basis \( F_1, F_2 \) and with diagonal braiding in this basis specified by the braiding matrix

\[ (q_{ij}) = \begin{pmatrix} -1 & -\xi^{-1}q \\ -\xi q & -1 \end{pmatrix} , \]

(3.1)

where we allow \( \xi \) to be any 2\( p \)th root of unity, \( \xi^{2p} = 1 \) (an ‘inessential’ variable, eliminated by a twist map [15]).

3.1. \( \mathfrak{B}(X) \): a presentation

The Nichols algebra \( \mathfrak{B}(X) \) associated with (3.1) is the quotient [20] (also see [55])

\[ \mathfrak{B}(X) = T(X) / \langle F_1^2, (F_1 F_2)^p - \xi^{-1} (F_2 F_1)^p, F_2^2 \rangle , \quad \dim \mathfrak{B}(X) = 4p . \]

3.2. \( \mathfrak{B}(X) \) as a subalgebra

Another description of \( \mathfrak{B}(X) \) in section 3.1, which is in fact the description in terms of screening operators, is not as a quotient of but as a subspace in \( T(X) \). The product and coproduct in \( \mathfrak{B}(X) \) are then the shuffle product and deconcatenation (see [11, 23]). We now describe these in some detail.

3.2.1. The total symmetrizer map. Let \( a \) and \( b \) denote two elements with respectively the same braiding as \( F_1 \) and \( F_2 \), but without any algebraic constraints. Then the map \( x \mapsto \mathfrak{S}_x \) by the total braided symmetrizer in each graded component is as follows (see [23] for our conventions on \( \mathfrak{S} \)). Any noncommutative monomial containing \( a^2 \) or \( b^2 \) is mapped to zero, and the ‘alternating’ monomials

\[ \mathfrak{a} a = (ab)^n , \quad \mathfrak{b} a = (ba)^n , \quad \mathfrak{a} a = (ab)^n a , \quad \mathfrak{b} a = (ba)^n b \]
map as
\[
AB_n \mapsto (1 - q^2)^{n-1} (n - 1)! (AB_n - \xi^{-r} q^n BA_n),
\]
\[
BA_n \mapsto (1 - q^2)^{n-1} (n - 1)! (BA_n - \xi^r q^n AB_n),
\]
\[
AB_{n+1} \mapsto (1 - q^2)^n (n)! AB_n,
\]
\[
BA_{n+1} \mapsto (1 - q^2)^n (n)! BA_n.
\]
Hence, \(\mathcal{S}_{2r+1} AB_r\) and \(\mathcal{S}_{2r+1} BA_r\) are nonzero only for \(0 \leq r \leq p - 1\), and \(\mathcal{S}_{2r} AB_r\) and \(\mathcal{S}_{2r} BA_r\) are nonzero only for \(1 \leq r \leq p - 1\), but only the linear combination \(AB_p + \xi^{-p} BA_p\) for \(r = p\). The algebra is therefore a linear span of the \(4p\) elements
\[
1,
AB_r, \quad 1 \leq r \leq p - 1, \quad BA_r, \quad 1 \leq r \leq p - 1,
AB_p + \xi^{-p} BA_p,
AB_r, \quad 0 \leq r \leq p - 1, \quad BA_r, \quad 0 \leq r \leq p - 1.
\]

3.2.2. Shuffle product. The shuffle multiplication table of the above basis elements is as follows:
\[
AB_r \ast AB_s = \binom{r + s}{s} AB_{r+s},
\]
\[
AB_r \ast BA_s = \xi^r q^s \binom{r + s - 1}{s} AB_{r+s} + \xi^{-r} q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
AB_r \ast AB_A = \xi^r q^s \binom{r + s}{s} AB_{r+s}, \quad AB_r \ast AB_A = \xi^{-r} q^s \binom{r + s}{s-1} BA_{r+s},
\]
\[
BA_r \ast AB_A = \xi^r q^s \binom{r + s - 1}{s} BA_{r+s} + \xi^{-r} q^s \binom{r + s - 1}{s-1} AB_{r+s},
\]
\[
BA_r \ast BA_A = \xi^r q^s \binom{r + s}{s} BA_{r+s}, \quad BA_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s-1} AB_{r+s},
\]
\[
AB_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad AB_r \ast BA_A = \xi^r q^s \binom{r + s}{s-1} BA_{r+s},
\]
\[
AB_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad AB_r \ast AB_A = \xi^r q^s \binom{r + s}{s-1} BA_{r+s},
\]
\[
BA_r \ast AB_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s} + \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
BA_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad BA_r \ast BA_A = \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
AB_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad AB_r \ast BA_A = \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
AB_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s} + \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
BA_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad BA_r \ast BA_A = \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
AB_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad AB_r \ast BA_A = \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
BA_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s} + \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
AB_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad AB_r \ast BA_A = \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
BA_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s} + \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
\[
AB_r \ast BA_A = \xi^{-r} q^s \binom{r + s}{s} AB_{r+s}, \quad AB_r \ast BA_A = \xi^r q^s \binom{r + s - 1}{s-1} BA_{r+s},
\]
Due to the binomial coefficient vanishing, no elements outside the ranges specified in (3.2) occur in the right-hand sides. For \(AB_r + \xi^{-p} BA_r\), strictly speaking, all products must be listed separately, which is easy because all of them are zero; these zero products are also reproduced by taking the appropriate linear combinations of the above formulas, with due care; for example, it follows that \((AB_r + \xi^{-r} BA_r) \ast AB_A = (1 + q^s) \binom{r + s}{s} AB_{r+s}\), which vanishes at \(r = p\) for all \(s \geq 0\) already because \(1 + q^p = 0\).
3.2.3. The deconcatenation coproduct is given by quite evident formulas,

\[ \Delta A_{B_1} = 1 \otimes A_{B_1} + a \otimes A_{B_{1-1}} + A_{B_1} \otimes A_{B_{1-1}} + \ldots + A_{B_{A-1}} \otimes b + A_{B_1} \otimes 1, \]
\[ \Delta A_{B_1} = 1 \otimes A_{B_1} + a \otimes A_{B_{1-1}} + A_{B_1} \otimes A_{B_{1-1}} + \ldots + A_{B_{A-1}} \otimes a + A_{B_1} \otimes 1, \]

and similarly for \( a_{B_1 \otimes B_2} \). The formula for \( \Delta (a_{B_1} + \xi^{-p}a_{B_1}) \), once again, follows by extending \( \Delta a_{B_1} \) and \( \Delta A_{B_1} \) to \( r = p \) and combining them appropriately.

3.2.4. The antipode, given by the ‘half-twist’ [23] (the Matsumoto lift of the longest element in the symmetric group), acts on the basis monomials as

\[ S(a_{B_1}) = (-1)^r \xi^{-r} q^2 a_{B_1}, \quad S(A_{B_1}) = (-1)^r q^{(r+1)} A_{B_1}, \quad S(a_{B_{1-1}}) = (-1)^{r+1} q^{(r+1)} a_{B_{1-1}}, \]

\( S(b_{B_1}) = (-1)^r q^{(r+1)} b_{B_1} \) and \( S(a_{B_1} + \xi^{-p}a_{B_1}) = a_{B_1} + \xi^{-p}a_{B_1} \).

3.3. Vertex operators and Yetter–Drinfeld \( \mathcal{B}(X) \)-modules

For any pair of integers \( i_1 \) and \( i_2 \), we introduce a one-dimensional braided vector space \( Y^{[i_1, i_2]} \), with a fixed basis vector \( V^{[i_1, i_2]} \) such that

\[ \Psi : a \otimes V^{[i_1, i_2]} \mapsto q^2 V^{[i_1, i_2]} \otimes a, \quad V^{[i_1, i_2]} \otimes a \mapsto q^2 a \otimes V^{[i_1, i_2]}, \]

\[ \Psi : b \otimes V^{[i_1, i_2]} \mapsto q^2 V^{[i_1, i_2]} \otimes b, \quad V^{[i_1, i_2]} \otimes b \mapsto q^2 b \otimes V^{[i_1, i_2]}. \]

Each space \( \mathcal{B}(X) \otimes Y^{[i_1, i_2]} \otimes \mathcal{B}(X) \otimes Y^{[i_1, i_2]} \otimes \cdots \otimes \mathcal{B}(X) \otimes Y^{[i_1, i_2]} \) is a Yetter–Drinfeld \( \mathcal{B}(X) \)-module (an ‘N-vertex’ Yetter–Drinfeld module) [23]. We here consider only one-vertex modules; the \( \mathcal{B}(X) \) action and coaction are then those in (A.1). The coaction is therefore literally the same as in section 3.2.3, and the (adjoint) action evaluates as

\[ a \triangleright A_{B_1} V^{[i_1, i_2]} = \xi^{-n} q^n (1 - q^{2i_1}) A_{B_{1-1}} V^{[i_1, i_2]}, \]

\[ a \triangleright A_{B_{1-1}} V^{[i_1, i_2]} = (1 - q^{2i_1+n}) A_{B_{1-1}} V^{[i_1, i_2]}, \]

\[ a \triangleright A_{B_1} V^{[i_1, i_2]} = 0, \]

\[ a \triangleright A_{B_1} V^{[i_1, i_2]} = (1 - q^{2i_1+n+1}) A_{B_1} V^{[i_1, i_2]} + \xi^{-1} q^{n+1} (q^{2i_1} - 1) A_{B_{1-1}} V^{[i_1, i_2]}, \]

\[ b \triangleright a_{B_1} V^{[i_1, i_2]} = (1 - q^{2i_1+n}) A_{B_{1-1}} V^{[i_1, i_2]}, \]

\[ b \triangleright a_{B_1} V^{[i_1, i_2]} = \xi^{-n} q^n (1 - q^{2i_1+n}) A_{B_{1-1}} V^{[i_1, i_2]}, \]

\[ b \triangleright A_{B_1} V^{[i_1, i_2]} = (1 - q^{2i_1+n+1}) A_{B_{1-1}} V^{[i_1, i_2]} + (1 - q^{2i_1+n+1}) A_{B_{1-1}} V^{[i_1, i_2]}, \]

\[ b \triangleright a_{B_1} V^{[i_1, i_2]} = 0. \]

4. From \( \mathcal{B}(X) \) to extended chiral algebras: asymmetric case

We take two screening operators \( B \) and \( F \) with the respective momenta \( \alpha_1 \) and \( \alpha_2 \) defined by the first line in (1.7). Then the corresponding braiding matrix is \( Q_n = Q_n(j, \alpha) \) in (1.1). Then we seek the kernel \( \ker B \cap \ker F \) in Sections 4.1 and 4.2, we fix our conventions for scalar fields and, for the convenience of the reader, recall the relevant points from [24]: a Virasoro algebra and parafermionic fields in the kernel of the screenings; the nonlocal parafermionic fields are converted into decent \( \mathfrak{s} \ell (2) \) currents by introducing an ‘auxiliary’ third scalar. In sections 4.3 and 4.4, we find more fields in the kernel by looking at certain representations of this \( \mathfrak{s} \ell (2) \) and finally use the locality requirement and propose the triplet–triplet and triplet–multiplet algebras, \( W(2, (2p+3) \times 3) \) and \( W(2, (4p^2 + 2p) \times 2 \times 3) \), which logarithmically extend the \( \mathfrak{s} \ell (2) \) algebra. In section 4.5, we briefly consider the Hamiltonian reduction of the obtained
W algebras to the \((p,1)\) and \((p,p')\) triplet algebras, in fact reproducing the constructions of the latter given in [22] and [36]. In section 4.6, we outline the construction of some W-modules.

4.1. Scalar fields and a Virasoro algebra

We introduce two scalar fields \(\varphi_1(z)\) and \(\varphi_2(z)\) with the OPEs determined by scalar products in the first line in (1.7):

\[
\varphi_1(z)\varphi_1(w) = \log(z-w), \quad \varphi_1(z)\varphi_2(w) = \left(-\frac{1}{p} - j\right)\log(z-w),
\]

\[
\varphi_2(z)\varphi_2(w) = \left(\frac{2}{p} + 2j\right)\log(z-w).
\]

It is readily verified that with these OPEs, the kernel of our two screenings

\[
B = \oint e^{\varphi_1} \quad \text{and} \quad F = \oint e^{\varphi_2}
\]

contains the energy–momentum tensor\(^7\)

\[
T(z) = -\frac{1}{k}\partial\varphi_1\partial\varphi_1(z) - \frac{1}{k}\partial\varphi_2\partial\varphi_2(z) - \frac{1}{2k(k+2)}\partial\varphi_2\partial\varphi_2(z)
\]

\[
- \partial^2\varphi_1(z) - \frac{1}{2(k+2)}\partial^2\varphi_2(z),
\]

where we set

\[
k := \frac{1}{p} + j - 2. \quad (4.1)
\]

This energy–momentum tensor represents the Virasoro algebra with the central charge

\[
c = \frac{3k}{k+2} - 1. \quad (4.2)
\]

It is useful to introduce \(\omega_1, \omega_2 \in \mathbb{C}^2\) as ‘fundamental weights’ \((\omega_i \cdot \alpha_j = \delta_{ij})\) with respect to \(\alpha_1\) and \(\alpha_2\) defined by scalar products in the first line in (1.7):

\[
\omega_1 = \frac{1}{k}(-2\alpha_1 - \alpha_2), \quad \omega_2 = \frac{1}{k}\left(-\alpha_1 - \frac{1}{k+2}\alpha_2\right).
\]

It follows that

\[
\omega_1 \cdot \omega_1 = -\frac{2}{k}, \quad \omega_1 \cdot \omega_2 = -\frac{1}{k}, \quad \omega_2 \cdot \omega_2 = -\frac{1}{k(k+2)}.
\]

Anticipating the appearance of a third scalar, we also consider a three-dimensional space spanned by \(\alpha_1, \alpha_2,\) and \(\alpha_3\), where \(\alpha_3\) has zero scalar products with \(\alpha_1\) and \(\alpha_2\) and is normalized by the condition read off from (1.9), \(\alpha_3\alpha_3 = 2k\). Then the corresponding ‘fundamental weight’ is \(\omega_3 = \frac{1}{k}\alpha_3\).

We slightly abuse the notation and write \(\omega_i(z)\) for the appropriate linear combinations of the fields,

\[
\omega_1(z) = \frac{1}{k}(-2\varphi_1(z) - \varphi_2(z)), \quad \omega_2(z) = \frac{1}{k}\left(-\varphi_1(z) - \frac{1}{k+2}\varphi_2(z)\right), \quad \omega_3(z) = \frac{1}{2k}\chi(z).
\]

\(^7\) We tend to write \(AB(z)\) for the normal-ordered product of two fields \(A(z)\) and \(B(z)\), and \(ABC(z)\), etc, for nested normal-ordered products \(A(BC(z))\), etc. The convention is not always obeyed, however; a notable case where it is violated is in expressions involving exponentials: for example, we write \((\chi(z) + j\varphi_1(z))e^{\chi(z)}\), whereas nested normal products are in fact understood in all cases, such as \((j\varphi_1j\varphi_2'))(z))\). Insisting on the rigorous writing would make many formulas incomprehensible.
4.2. Parafermionic fields in ker $B \cap \ker F$ and the $\widehat{sl}(2)_k$ algebra

The form of the central charge in (4.2) immediately suggests that the kernel must contain generators of the $\widehat{sl}(2)/\mu(1)$ coset; indeed, these are given by

\[
\begin{align*}
J^+(z) &= e^{\omega_1(z)}, \\
J^-(z) &= -(\partial \varphi_1 \partial \varphi_1(z) + \partial \varphi_1 \partial \varphi_2(z) + (k + 1) \partial^2 \varphi_1(z)) e^{-\omega_1(z)}. 
\end{align*}
\]  

(4.3)

4.2.1. A long screening. The fields $T(z)$ and $J^\pm(z)$ are in the kernel not only of the two ‘short’ screenings $B$ and $F$ but also of the ‘long’ screening

\[
\mathcal{E} = \oint e^{-\frac{i}{\mu}z^2} = \oint e^{-\frac{\mu}{i}z^2}.
\]

The long screening is readily verified to commute with both short screenings:

\[
[\mathcal{E}, B] = 0, \quad [\mathcal{E}, F] = 0.
\]

4.2.2. The $\widehat{sl}(2)$ currents. The fields $J^\pm(z)$ are nonlocal with respect to one another (their OPEs contain noninteger powers $(z - w)^{\pm j_p/(j_1 - j_2 + 1)}$, and in fact represent a coset theory $\widehat{sl}(2)/\mu(1)$. To deal with local fields, we introduce a third, auxiliary scalar $\chi(z)$ with OPE (1.9) and construct the $\widehat{sl}(2)_k$ currents (with our conventions given in section B.1 in the appendix)

\[
\begin{align*}
J^+(z) &= J^+(z) e^{\frac{1}{2} \partial \chi(z)}, \\
J^0(z) &= \frac{1}{2} \partial \chi(z), \\
J^-(z) &= J^-(z) e^{-\frac{1}{2} \partial \chi(z)}. 
\end{align*}
\]  

(4.4)

The associated Sugawara energy–momentum tensor (B.1) is then evaluated as

\[
T_{\text{long}}(z) = T(z) + \frac{1}{4k} \partial \chi \partial \chi(z).
\]

4.2.3. Orthogonalizing the scalar fields. It is useful to pass to a triplet of pairwise OPE-orthogonal scalar fields, $(\varphi_1(z), \varphi_2(z), \chi(z)) \rightarrow (\alpha(z), \varphi(z), \chi(z))$, where

\[
\begin{align*}
\alpha(z) &= -2\varphi_1(z) - \varphi_2(z), \\
\varphi(z) &= \varphi_2(z)
\end{align*}
\]

are mutually orthogonal (have a regular OPE) and $\alpha(z)$ is normalized as

\[
\alpha(z) a(z) = -2k \log(z - w).
\]

We keep the notation $\omega(z)$ introduced in section 4.1; we then have the OPEs

\[
\alpha(z) \omega_1(w) = -2 \log(z - w), \quad \alpha(z) \omega_2(w) = -\log(z - w).
\]

Whenever the context requires this, we understand $\omega_1(z)$ and $\omega_2(z)$ to be reexpressed in terms of the new fields, as

\[
\omega_1(z) = \frac{1}{k} \alpha(z), \quad \omega_2(z) = \frac{1}{2k} \alpha(z) + \frac{1}{2(k + 2)} \varphi(z).
\]

Then the $\widehat{sl}(2)_k$ currents in (4.4) take the form [56]

\[
\begin{align*}
J^+(z) &= e^{\omega_1(z) + 2\omega_2(z)}, \\
J^0(z) &= \frac{1}{2} \partial \chi(z), \\
J^-(z) &= \left((k + 2) T_m(z) - \frac{1}{4} \partial \alpha \partial \alpha(z) + \frac{k + 1}{2} \partial^2 \alpha(z)\right) e^{-\omega_1(z) - 2\omega_2(z)}, 
\end{align*}
\]  

(4.5)
where
\[ T_m(z) := \frac{1}{4(k+2)} \partial \psi \partial \psi(z) + \frac{k+1}{2(k+2)} \partial^2 \psi(z). \] (4.6)
This \( T_m(z) \) is an energy–momentum tensor with central charge
\[ c_m = 13 - 6(k + 2) - \frac{6}{k + 2}, \] (4.7)
i.e., it satisfies the OPE
\[ T_m(z)T_m(w) = \frac{c_m/2}{(z-w)^4} + \frac{2T_m(w)}{(z-w)^2} + \frac{\partial T_m(w)}{z-w} \]
encoding a Virasoro algebra. We note for the future use that primary fields of this Virasoro algebra and their conformal dimensions are
\[ V^m_{\ell s}(z) = e^{(i/2)(1-s)} + \frac{e^{i\ell}}{z(w)} \] and
\[ \Delta^m_{\ell s} = \frac{\ell^2 - 1}{4} (k + 2) + \frac{s^2 - 1}{4(k+2)} + \frac{1}{2}(1-sr), \] (4.8)
We also note that the Sugawara energy–momentum tensor is now reexpressed as a sum of three energy–momentum tensors,
\[ T_{\text{Sug}}(z) = T_m(z) + \left( -\frac{1}{4k} \partial \alpha \partial \alpha(z) + \frac{1}{2} \partial^2 \alpha(z) \right) + \frac{1}{4k} \partial \chi \partial \chi(z), \]
with the respective central charges \( c_m, 6k + 1, \) and 1.

4.2.4. The fact that the \( \psi(z) \) field enters the \( \hat{s}(2) \) currents in (4.5) only through \( T_m(z) [56] \) can be viewed, depending on one’s taste, as either (i) a technicality or (ii) an important structural piece; we switch between the two standpoints at will.

(i) We can of course assume that \( T_m(z) \) is expressed through a free scalar, as in (4.6), and continue working with the three scalars \( \alpha(z), \psi(z), \) and \( \chi(z); \) an enveloping algebra of \( \hat{s}(2) \) is then selected from differential polynomials in \( \partial \alpha(z), \partial \psi(z), \partial \chi(z), \) and \( e^{\pm \frac{i}{2}(\chi(z) + \frac{1}{2} \psi(z))} \) by taking the kernel of both screenings
\[ B = \oint e^{-\frac{i}{2}\psi - \frac{1}{2}\alpha}, \quad F = \oint e^\psi. \]

(ii) Equivalently, and more interestingly, we can recall that the kernel of the single screening \( F \) selects differential polynomials in \( T_m(z) \) (the enveloping of the Virasoro algebra) from the algebra of differential polynomials in \( \partial \psi(z). \) We can therefore ‘use up’ the \( F \) screening to forget about the \( \psi(z) \) scalar and deal with the \( \hat{s}(2) \) currents in (4.5) expressed in terms of two scalars \( \alpha(z) \) and \( \chi(z) \) and an ‘abstract’ energy–momentum tensor—not a free scalar—with central charge (4.7). The remaining screening,
\[ B = \oint V^m_{2,1} e^{-\frac{i}{2}w}, \] (4.9)
then serves to select the enveloping algebra of \( \hat{s}(2)_k \) from the algebra of differential polynomials in \( \partial \alpha(z), \partial \chi(z), T_m(z), \) and \( e^{\pm \frac{i}{2}(\chi(z) + \frac{1}{2} \psi(z))} \). In (4.9), \( V^m_{2,1}(z) \) is the ‘21’ primary field of the Virasoro algebra with central charge \( c_m; \) it is defined by the OPE
\[ T_m(z)V^m_{2,1}(w) = \Delta^m_{2,1} V^m_{2,1}(w) + \frac{\partial V^m_{2,1}(w)}{z-w}, \quad \Delta^m_{2,1} = \frac{1}{4}(3k + 4), \]
and the differential equation\(^8\)
\[ \partial^2 V^m_{2,1}(z) - (k + 2) T_m V^m_{2,1}(z) = 0. \]

\(^8\) This equation is needed, in particular, in verifying that (4.9) is a screening for the \( \hat{s}(2) \) algebra in (4.5).
In what follows, we speak of the \( \varphi \) sector as the matter theory; the name is motivated by the relation to Hamiltonian reduction, as we see below.

4.3. More of \( \ker B \cap \ker F \)

Another piece of the kernel is easy to find. It contains the fields

\[
\mathcal{F}_h(z) = e^{2h(\omega(t) + \omega(z))},
\]

for any \( h = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \) Each \( \mathcal{F}_h(z) \) is an \( \hat{s} \ell(2) \) primary,

\[
J_+ \mathcal{F}_h(z) = 0, \quad J_0 \mathcal{F}_h(z) = h\mathcal{F}_h(z), \quad J_- \mathcal{F}_h(z) = 0,
\]

and generates a ‘horizontal’ \((2h + 1)\)-plet under the action of the zero-mode \( \ell(2) \) algebra:

\[
(J_0^-)^{2h+1} \mathcal{F}_h(z) = 0, \quad (J_0^+)^{2h} \mathcal{F}_h(z) \neq 0.
\]

Of course, the entire \( \hat{s} \ell(2) \) module generated from \( \mathcal{F}_h(z) \) is in the kernel.

If the matter theory is singled out as explained above, we reexpress \( \mathcal{F}_h(z) \) as

\[
\mathcal{F}_h(z) = V_{1,1,2h+1}^m(z) e^{2\Delta_0(z) + \frac{k}{2} z},
\]

where \( V_{1,1,2h+1}^m(z) \) are \( T_m(z) \)-primary fields of dimension \( \Delta_{1,1}^m \) (see \((4.8)\)). The Sugawara dimension of \( \mathcal{F}_h(z) \) is of course

\[
\Delta_j(h) = \frac{h(h+1)}{k+2} = \frac{h(h+1)p}{jp+1}.
\]

4.4. Even more of \( \ker B \cap \ker F \) and the extended algebras

We temporarily fix a positive integer or half-integer \( h \). It is easy to verify that \( \mathcal{F}_{-h}(z) \) is in \( \ker R \), but not in \( \ker F \). The intersection of the two kernels is to be found deeper in the Wakimoto-type free-field module \([28, 29]\) associated with \( \mathcal{F}_{-h}(z) \). The actual picture depends on the value of \( j \). We consider the special case \( j = 0 \) separately and then discuss the cases \( j > 0 \). When we speak of nonvanishing and vanishing singular vectors in what follows, we refer to the typical picture in figure 1. The Verma-module embedding pattern on the left changes in Wakimoto modules to the one on the right. In particular, the ‘reversal’ of an arrow leading from the top means that the corresponding singular vector vanishes.

4.4.1. The case \( j = 0 \) and a triplet–triplet algebra

In the \( \hat{s} \ell(2)_k \) Verma module whose highest-weight vector has the same charge (eigenvalue of \( J_0^0 \)) and dimension as those of \( \mathcal{F}_{-h}(z) \), there are two basic singular vectors, which happen to lie on the same level (see figure 2; our notation for \( \hat{s} \ell(2) \) singular vectors is explained in appendix B):

- \( s^+(1, 2hp + 1) \) with (charge, level) relative to those of the highest-weight vector equal to \((-1, 2hp)\), and
- \( s^-(2h, p) \) with the relative charge and level \((2h, 2hp)\).

In the free-field module that we actually have, \( s^+(1, 2hp + 1) \mathcal{F}_{-h}(z) \) vanishes; this is shown in figure 2 with a dashed line. On the other hand, the \( s^- (2h, p) \) singular vector evaluated on \( \mathcal{F}_{-h}(z) \) does not vanish and is in the kernel of both screenings. The zero-mode \( s\ell(2) \) algebra produces a \((2h + 1)\)-plet from it, terminating in the grade next to the one with the vanishing singular vector.
Figure 1. Left: embedding diagram of an $\hat{s}(2)$ Verma module. The highest-weight vector is at the top. Arrows are drawn toward singular vectors (submodules). Right: the corresponding Wakimoto module. The same subquotients are ‘glued’ to one another in a different manner. Black dots show the socle of the module.

Figure 2. Left: the $\hat{s}(2)$ primary $\mathcal{F}_h(z)$, equation (4.10), is an element of a $(2h + 1)$-plet under the action of the zero-mode $s(2)$ algebra. Right: $\mathcal{F}_{-h}(z)$ for $h > 0$ and some structures in the associated module. The open dot shows a ‘cosingular’ vector in the grade where the $s^-(1, 2hp + 1)$ singular vector vanishes. The horizontal arrow is a map by $E^{2h}$. The right black dot is a nonvanishing singular vector, from which the zero-mode $s(2)$ algebra generates a $(2h + 1)$-plet, of the same Sugawara dimension as in the left diagram. The dimensions of $\mathcal{F}_h$ and $\mathcal{F}_{-h}$ compare as

$$\frac{h(h+1)}{2} - \frac{h(-h+1)}{2} = 2hp.$$ 

The value $j = 0$ is singled out by the fact that the long screening is then a generator of a Lie algebra $s(2)$ rather than of a quantum $s(2)$ group, as for other integer $j$. Mapping $\mathcal{F}_h(z)$ by the long screening produces just the $s^-(2h, p)$ singular vector:

$$(E)^{2h}\mathcal{F}_h(z) = s^-(2h, p)\mathcal{F}_{-h}(z).$$ (4.12)

Because the long screening is in the ‘matter’ sector, equation (4.12) can also be written as

$$(E)^{2h}\mathcal{F}_h(z) = SV_{2p-1,2h+1}^{\text{m}}(z) e^{h(\omega_1(z)+2\omega_2(z))},$$
where \( SV_{2p-1, 2h+1}^m(z) \) is the corresponding Virasoro singular vector evaluated on (the field corresponding to) the Virasoro primary state \( V_{2p-1, 2h+1}^m(z) \), which occurs here because

\[
\mathcal{F}_{-h}(z) = V_{2p-1, 2h+1}^m(z) e^{-h\omega_1(z) + 2\omega_0(z)}.
\]

We also note the matter dimension of the fields in the two \((2h+1)\)-plets:

\[
\Delta_{1, 2h+1} = h(p(h + 1) - 1). \quad (4.13)
\]

Continuing with the embedding diagrams of Wakimoto-type modules allows describing all of the socle (the black dots in figure 1, right, plus similar dots in other Wakimoto modules, which increase in number as we go down the diagram), but we stop here because our main task now is to propose generators (minimal-dimension fields) of the maximum local algebra in the kernel.

The mutual (non)locality of \( \mathcal{F}_n(z) \) and \( \mathcal{F}_n'(w) \) is measured by the powers \((z - w)^{2h/p}\) in their operator product. We choose \( h \) such that these exponents be integer for all, integer and half-integer, \( h' \). This means taking integer \( h \), and the local algebra generators are those with the smallest positive integer \( h = 1 \), \( \mathcal{F}_1(z) \) and \( s^-(2, p) \mathcal{F}_{-1}(z) \). These fields are then the leftmost and the rightmost operators in a triplet under the action of \( \mathcal{E} \). In addition, each of these fields is the rightmost element in a triplet with respect to the action of the zero-mode \( s(t) \).

For each integer \( p \geq 2 \), we propose the algebra \( \mathbf{W}(2, (2p) \times 3 \times 3) \) generated by the fields

\[
\mathcal{W}^+(z) = \mathcal{F}_1(z), \quad \mathcal{W}^0(z) = \mathcal{E} \mathcal{W}^+(z), \quad \mathcal{W}^-(z) = \mathcal{E} \mathcal{W}^0(z) = s^-(2, p) \mathcal{F}_{-1}(z),
\]

together with the corresponding \( s(t) \) triplets \((J_0^-)^i \mathcal{W}^{k, 0}(z), i = 0, 1, 2\), as a ‘logarithmic’ extension of the \( s(t) \) algebra at the level \( k = \frac{1}{p} - 2 \). To be more explicit, we recall that the \( \mathcal{F}_{\pm 1}(z) \) are here given by

\[
\mathcal{F}_1(z) = e^{2\omega_1(z) + 2\omega_0(z)} V_{1, 3}^m(z) e^{\omega_1(z) + 2\omega_0(z)},
\]

\[
\mathcal{F}_{-1}(z) = e^{-2\omega_1(z) - 2\omega_0(z)} V_{2p-1, 3}^m(z) e^{-\omega_1(z) - 2\omega_0(z)}, \quad k = \frac{1}{p} - 2.
\]

Conjecturally, \( \mathbf{W}(2, (2p) \times 3 \times 3) \) contains all mutually local fields in \( \ker B \cap \ker F \) (in particular, the \( \mathcal{F}_n(z) \) with integer \( n \geq 2 \) and their images \( \mathcal{E}^m \mathcal{F}_n(z) \), \( 1 \leq m \leq 2n \), under the long screening).

Each field \( \mathcal{W}^0(z) = \mathcal{W}^+(0, 0)(z) \) also belongs to a triplet \((J_0^-)^3 \mathcal{W}^0(z), J_0^- \mathcal{W}^0(z), \mathcal{W}^0(z)\) under the zero-mode \( s(t) \) algebra, as in figure 2 (where we now set \( h = 1 \)). In particular,

\[
(J_0^-)^3 \mathcal{W}^+(z)
\]

\[
= \Big( \frac{1}{2} \partial \psi \partial \psi(z) - \partial^2 \psi(z) + \partial \psi \partial a(z) + \frac{1}{2} \partial a(z) \partial a(z) - \partial^2 a(z) \Big) e^{-2\omega_1(z) + 2\omega_0(z) - 2\omega_0(z)}.
\]

The fields \( \mathcal{W}^+(z), \mathcal{W}^0(z), \mathcal{W}^-(z) \) (and the entire ‘zero-mode’ triplets) have the Sugawara dimension (see (4.11))

\[
\Delta_0(1) = 2p.
\]

An example of the triplet–triplet algebra generators is given in section C.1 in the appendix.

**Remark: a doublet.** We note that setting \( h = \frac{1}{2} \) instead of \( h = 1 \) in (4.12) yields a doublet of dimension \( -\frac{2p}{3} \) fields \( \mathcal{F}_1(z) \) and \( \mathcal{E} \mathcal{F}_1(z) = s^-(1, p) \mathcal{F}_{-1}(z) \) (each of which also belongs to a zero-mode \( s(t) \) doublet); they can be regarded as generators of an ‘almost local’ doublet.
algebra—an analogue of a pair of fermions (derivatives of the symplectic fermions $\psi^{\pm}(z)$) well known from the logarithmic $(p = 2, 1)$ ‘matter’ models$^9$.

4.4.2. Cases $j \geq 1$ and triplet–multiplet algebras. For $j \geq 1$, we repeat the construction in section 4.4.1 mutatis mutandis, noting from the start that with $k = j - 2 + \frac{1}{p}$, mutual (non)locality of vertex operators is measured by the powers $(z - w)^{2p/(pj+1)}$ in their OPEs. In seeking the local algebra in the kernel of the screenings, we therefore start with the $\hat{s}_\ell(2)$ modules generated from $F_{\pm h}(z)$ with $h = jp + 1$. (4.16)

In figure 3, we represent $F_{jp+1}(z)$ with the top right corner. It has a vanishing singular vector $s^+(zp, 3p+1)(z)$ (the top left open circle) and the nonvanishing singular vector $W^+(z) = s^-(zp, 3p)F_{jp+1}(z)$, (4.17)

from which (due to another singular vector vanishing) the zero-mode $s\ell(2)$ algebra generates a $(4jp+3)$-plet. It is easy to see that $W^+(z)$ has the structure

$$W^+(z) = \mathcal{D}^+(z) e^{i^{(p\omega_1(z)+2)(jp+1)}\omega_1(z)+2(2jp+1)\omega_3(z)},$$

where $\mathcal{D}^+(z)$ is a degree-$jp(3p - 1)$ differential polynomial in the fields (and the exponential has the Sugawara dimension $jp^2 + jp + 2p$). All black dots in figure 3 are in $\ker B \cap \ker F$, but we select $W^+(z)$ as an extended algebra generator.

Figure 3 is a ‘refinement’ of the left part of figure 2 for $h$ in (4.16), $j \geq 1$. Instead of the right part of figure 2, we then have figure 4, where the top right corner represents $F_{-jp-1}(z)$ and the bottom right corner is the nonvanishing singular vector

$$W^-(z) = s^+(3jp + 2, p)F_{-jp-1}(z)$$

which lies in $\ker B \cap \ker F$ and has the structure

$$W^-(z) = \mathcal{D}^-(z) e^{i^{(3jp+2)\omega_1(z)-2(2j(p+1))\omega_2(z)+2(2jp+1)\omega_3(z)}},$$

9 Hamiltonian reduction (see section 4.5.1 below) of $\mathcal{F}_{1/2}(z)$ and $\mathcal{E}\mathcal{F}_{1/2}(z)$ gives a doublet of ‘matter’ fields of dimension $2\omega^2$, which are the $\partial\psi^{\pm}(z)$ for $p = 2$. 

Figure 3. Relevant structures in the Wakimoto-type $\hat{s}\ell(2)$ module associated with the field $F_{jp+1}(z)$, which sits at the top right corner. Vanishing singular vectors are shown with open dots. The charges (eigenvalues of $J_0^0$) of the states are indicated. Two arrows show singular vectors, $s^\pm$, in the corresponding Verma module; the right singular vector is nonvanishing in our free-field realization. The dotted arrow shows the relative level (difference of Sugawara dimensions) of the two floors.
where $\mathcal{P}^-(z)$ is a differential polynomial in the fields of the degree $(p-1)(3jp + 2)$ (and the exponential has the Sugawara dimension $jp^2 + 3jp + 2$).

The zero-mode $\widehat{\mathfrak{sl}}(2)$ algebra generates a $(4jp + 3)$-plet from $\mathcal{W}^-(z)$ as well as from $\mathcal{W}^+(z)$. Because the Sugawara dimensions of the top right corners in figures 3 and 4 are

$$\dim \mathcal{F}_{jp+1}(z) = jp^2 + 2p \quad \text{and} \quad \dim \mathcal{F}_{-jp-1}(z) = jp^2,$$

the $(4jp + 3)$-plets in figures 3 and 4 have the same Sugawara dimension $4jp^2 + 2p$.

Without drawing another picture, we briefly describe the relevant structure of the $\widehat{\mathfrak{sl}}(2)$-module associated with $\mathcal{F}_0(z) = 1$ (the unit operator). There, the singular vector

$$\mathcal{W}^0(z) = s^- (2jp + 1, 2p) \mathcal{W}^0(z), \quad a = 1, 0, -1.$$  \hspace{1cm} (4.19)

is nonvanishing in the free-field realization and is also the rightmost element of a zero-mode $(4jp + 3)$-plet located at the same Sugawara dimension as the two $(4jp + 3)$-plets containing $\mathcal{W}^+(z)$ and $\mathcal{W}^-(z)$.

We summarize our findings as the following conjecture on the extended algebra. For fixed integers $p \geq 2$ and $j \geq 1$, let

$$r_a = (2-a)jp + 1 - a, \quad \text{and} \quad s_a = (2+a)p \quad \text{for} \quad a = 1, 0, -1.$$

The three dimension-$(4jp^2 + 2p)$ fields (4.17), (4.19), (4.18), which can also be written as

$$\mathcal{W}^a(z) = s^- (r_a, s_a) \mathcal{F}_{a(jp+1)}(z), \quad a = 1, 0, -1,$$  \hspace{1cm} (4.20)

together with the entire $(4jp + 3)$-plets $(\mathcal{J}_0^-)^i/\mathcal{W}^i(z), 0 \leq i \leq 4jp + 2$, generate a $\mathcal{W}$-algebra of mutually local fields in the kernel of the two screenings.

We call this $\mathcal{W}$-algebra the triplet–multiplet algebra, $\mathcal{W}(2, (4jp^2 + 2p)^{3\times(4jp+3)})$, although its triplet structure is somewhat more elusive than that of the $\mathcal{W}(2, (2p)^{3\times3})$ algebra in section 4.4.1: the long screening does not map between elements of the triplet.
4.5. Hamiltonian reduction

The choice of scalar fields made in section 4.2.3 implies that the Hamiltonian reduction of \( J^{±,0}(z) \) and \( W_{±}^{±,0}(z) \) is obtained by simply setting \( a(z) = \chi(z) = 0 \), leaving us with only the matter field \( \phi(z) \).

4.5.1. \( j = 0 \): the \((p, 1)\) triplet algebra. Setting \( a(z) = \chi(z) = 0 \) reduces the three fields in (4.14) to three fields generating the triplet \( W_{p,1} \) algebra, exactly as it was obtained in [22]. (In particular, formula (4.13) with \( h = 1 \) gives the dimension \( 2p + 1 \) of the triplet algebra generators.)

Moreover, much as the \( s\ell(2) \) currents were expressed in terms of the ‘matter’ energy–momentum tensor \( T_{\alpha\beta}(z) \) and the two additional scalars \( \chi(z) \) and \( a(z) \) in (4.5), simple analysis of the construction in (4.14) readily shows how to ‘pack’ the generators \( W_{p,1} \) algebra, properly dressed with the additional scalars, into the \( W_{2p}^{±,0}(z) \) generators of \( W(2, (2p)^{3\times3}) \) (we do not need this construction in this paper, however).

4.5.2. \( j > 0 \): the \((p, p')\) algebras. Setting \( a(z) = \chi(z) = 0 \) in the expressions for \( W_{p}^{±,0}(z) \) in (4.20) reduces them to the corresponding three fields generating the triplet \( W_{p_+, p_-} \) algebra obtained in [36, section 4.2.1.]:

\[
\begin{align*}
W^+(z) &= \mathcal{P}^{+}_{jp(3p-1)}(z)e^{\varphi(z)}, \\
W^0(z) &= \mathcal{P}^{0}_{(3p-1)(2jp+1)}(z), \\
W^-(z) &= \mathcal{P}^{-}_{(p-1)(3jp+2)}(z)e^{-\varphi(z)},
\end{align*}
\]

where \( \mathcal{P}^{m,0}_{(3p-1)(2jp+1)}(z) \) are differential polynomials of the indicated degree \( m \) in \( \varphi(z) \), \( n \geq 1 \). In terms of \( p_+ := p \) and \( p_- := jp + 1 \), the degrees of these polynomials are \( jp(3p-1) = (p_- - 1)(3p_+ - 1) \), \( (2p - 1)(2jp + 1) = (2p_+ - 1)(2p_- - 1) \), and \( (p-1)(3jp+2) = (p_+ - 1)(3p_- - 1) \), which coincides with what we had in [36]. As regards the exponentials in the above formulas, they are also the same as in [36], where the scalar field was normalized canonically, giving rise to the factors \( \pm\sqrt{p_+ p_-} = \pm\sqrt{p \cdot p \cdot (\frac{2}{3} + 2j)} \) in the exponents.

We also recall from [36] that the OPE of \( W^+(z) \) and \( W^-(z) \) is

\[
W^+(z)W^-(w) = \frac{S_{p_+, p_-}(T)}{(z - w)^{2p_+ - 3p_- + 1}} + \text{less singular terms}, \tag{4.21}
\]

where \( S_{p_+, p_-}(T) \) is the vacuum singular vector—the polynomial of degree \( \frac{1}{2}(p_+ - 1)(p_- - 1) \) in \( T \) and \( \varphi T \), \( n \geq 1 \), such that \( S_{p_+, p_-}(T) = 0 \) is the polynomial relation for the energy–momentum tensor in the \( (p_+, p_-) \) Virasoro minimal model. This degree is \( \frac{1}{2}jp(p - 1) \) in our current case.

4.6. \( W(2, (2p)^{3\times3}) \) highest-weight states

We return to the triplet–triplet algebra (in particular, we now have \( k = \frac{1}{p} - 2 \)) and construct a class of its highest-weight states.

4.6.1. For integer \( r, s, \) and \( \vartheta \), we set

\[
V_{r,s,\vartheta}(z) = e^{\vartheta a_0(z)}e^{-\frac{1}{p}a_2(z)}e^{-\frac{1-2rs}{p}a_1(z)},
\]
This is a twisted relaxed $\widehat{\mathfrak{sl}}(2)$ highest-weight state of twist $\vartheta - 1$: the top modes $J^+_n$ that do not annihilate it act as

\[
J^+_n V_{s,r,\vartheta} (z) = V_{s,r+p,\vartheta} (z), \\
J^-_n V_{s,r,\vartheta} (z) = -\frac{r(p + r - s)}{p^2} V_{s,r-p,\vartheta} (z).
\]

The Sugawara dimension of $V_{s,r,\vartheta} (z)$ is

\[
\Delta_{s,r;\vartheta} = \frac{(s + \vartheta - 1)^2 - 4r(\vartheta - 1)}{4p} + \frac{1}{2}(1 - s - \vartheta^2).
\]

An extra vanishing condition, $J^-_{1,\vartheta} V_{s,r,\vartheta} (z) = 0$, occurs whenever

\[
r = 0 \quad \text{or} \quad r = s - p.
\]

In these two cases, the corresponding operators $V_{s,r,\vartheta} (z)$ are twisted highest-weight states:

\[
V_{0,0,\vartheta} (z) = e^{\frac{1}{4z} \omega_1 (z) + \frac{1}{2z^2} \omega_0 (z)} \| \lambda^+ (1, 1; \vartheta) \|,
\]

\[
V_{s,1-s,\vartheta} (z) = e^{\frac{1}{z} \omega_1 (z) + \frac{1}{2z^2} \omega_0 (z) + \frac{2}{z} \omega_1 (z) + \frac{2}{z} \omega_0 (z)} \| \lambda^- (1, 1; \vartheta) \|
\]

(with $\lambda^\pm (r, s)$ defined in section B.3 in the appendix). The occurrence of a twisted highest-weight state is illustrated in the left part of figure 5. We also note that the matter dimension of each of these two states is $\Delta_{s,1}^m$ (see (4.8)).

4.6.2. We reparameterize $V_{s,r,\vartheta} (z)$ by defining

\[
V_{s,r,\vartheta}^{n,m} [n,m] (z) = V_{s-2p, r+2p, \vartheta} (z)
\]

\[
= e^{\frac{1}{z} \omega_1 (z) + \frac{1}{z} \omega_0 (z) + \frac{1}{z} \omega_1 (z) + \frac{1}{z} \omega_0 (z)} \| \lambda_{(1, 1; \vartheta)} (z) \|
\]

where now

\[
1 \leq s \leq p, \quad 0 \leq r \leq p - 1, \quad \nu, \mu = 0, 1, \quad n, m, \vartheta \in \mathbb{Z}.
\]
This allows representing the $W(2, (2p)^3)\times 3$ action very conveniently. It follows that $V^\nu,\mu_{s,r,\theta}[n, m](z)$ is always annihilated by $W^+_{i, \theta}$ with $i \geq \theta + s - pv = 2p(n + 1)$ and by $V^+_{i, \theta}$ with $i \geq \theta - s + pv + 2pm$. The top modes of $W^+$ and $W^-$ that do not annihilate $V^\nu,\mu_{s,r,\theta}[n, m](z)$ identically act as

$$W^-_{i, \theta}(2p(n + 1) - p - s + pv - 2p(n + 1))V^\nu,\mu_{s,r,\theta}[n, m](z) = V^\nu,\mu_{s,r,\theta}[n + 1, m](z), \quad (4.23)$$

$$W^-_{i, \theta}(2p(n + 1) - s + pv - 2pm - i)V^\nu,\mu_{s,r,\theta}[n, m](z) = \left(\frac{2p - 1}{p - 1}\right)^{\frac{p - 1}{i}} \prod_{i=1}^{p-1} (s - pv - 2pm - i) (s - pv + 2pm + i)$$

$$\times V^\nu,\mu_{s,r,\theta}[n - 1, m](z). \quad (4.24)$$

The $n$ label changes as $n \mapsto n \pm 1$ in these formulas, and it follows that a $W(2, (2p)^3)$ module generically contains the direct sum of subspaces spanned by the $V^\nu,\mu_{s,r,\theta}[n, m](z)$ over all $n$. We now detect cases where such a module is reducible and then identify a submodule in it.

4.6.3. For $n = 0$ and $\nu = 0$, annihilation conditions hold in the form

$$W^-_{i, \theta}V^\nu,\mu_{s,r,\theta}[0, m](z) = 0, \quad i \geq \theta + s - 2p,$$

and for $n = 0$ and $\nu = 1$,

$$W^-_{i, \theta}V^\nu,\mu_{s,r,\theta}[0, m](z) = 0, \quad i \geq \theta.$$

Hence, for each $s, 1 \leq s \leq p - 1$, a submodule is generated from $V^\nu,\mu_{s,r,\theta}[0, m](z)$. For $\nu = 0$, this submodule is illustrated in the right part of figure 5.

We return to the consideration of these vertex operators in section 7, where we also discuss the relation to the $U(X)$ algebra.

5. From $\mathfrak{b}(X)$ to extended chiral algebras: symmetric case

We take two screening operators $F_1$ and $F_2$ with the momenta defined by the second line in (1.7). The corresponding braiding matrix $Q_{\nu} = Q_{\nu}(j, p)$ is then the one in (1.1). We seek the kernel $\ker F_1 \cap \ker F_2$. We fix our notation in section 5.1, identify the parafermions in the kernel and convert them into $\hat{sl}(2)$ currents in section 5.2, and then use the $\hat{sl}(2)$ representation theory to find other pieces of the kernel: the ‘easy’ one in section 5.3 and the ‘difficult’ in section 5.4, where we identify the extended algebra generators.

5.1. Scalar fields and a Virasoro algebra

We introduce two scalar fields $\varphi_1(z)$ and $\varphi_2(z)$ with the OPEs determined by scalar products in the second line in (1.7):

$$\varphi_1(z)\varphi_1(w) = \log(z - w), \quad \varphi_1(z)\varphi_2(w) = \left(\frac{1}{p} + j\right) \log(z - w),$$

$$\varphi_2(z)\varphi_2(w) = \log(z - w).$$

It follows from these OPEs that the kernel of the two screenings

$$F_1 = \oint e^{\varphi_1} \quad \text{and} \quad F_2 = \oint e^{\varphi_2}$$

contains the energy–momentum tensor

$$T(z) = -\frac{1}{2k(k + 2)} \partial \varphi_1 \partial \varphi_1(z) + \frac{k + 1}{k(k + 2)} \partial \varphi_1 \partial \varphi_2(z) - \frac{1}{2k(k + 2)} \partial \varphi_2 \partial \varphi_2(z)$$

$$- \frac{1}{2(k + 2)} \partial^2 \varphi_1(z) - \frac{1}{2(k + 2)} \partial^2 \varphi_2(z). \quad (5.1)$$

$$25$$
where we set

\[ k := \frac{1}{p} + j - 1. \tag{5.2} \]

In terms of this \( k \), the central charge of \( T(z) \) is expressed as in (4.2).

It is convenient in what follows to introduce ‘fundamental weights’ \( \omega_1, \omega_2 \in \mathbb{C}^2 \) for the vectors \( \alpha_1 \) and \( \alpha_2 \) in the second line in (1.7):

\[ \omega_1 = \frac{1}{k(k+2)}(-\alpha_1 + (k+1)\alpha_2), \quad \omega_2 = \frac{1}{k(k+2)}((k+1)\alpha_1 - \alpha_2). \]

Then

\[ \omega_1 \cdot \omega_1 = -\frac{1}{k(k+2)}, \quad \omega_1 \cdot \omega_2 = \frac{k+1}{k(2+k)}, \quad \omega_2 \cdot \omega_2 = -\frac{1}{k(k+2)}. \]

Anticipating the appearance of a third scalar field, we pass from \( \mathbb{C}^2 \) to \( \mathbb{C}^3 \) by adding the vector \( \alpha_3 \), such that \( \alpha_3 \cdot \alpha_1 = \alpha_3 \cdot \alpha_2 = 0 \) and \( \alpha_3 \cdot \alpha_3 = 2k \), and the corresponding ‘fundamental weight’ \( \omega_3 = \frac{1}{2k} \alpha_3 \). With a slight abuse of notation, we write \( \omega_3(z) \) for the corresponding linear combinations of our scalar fields:

\[ \omega_1(z) = \frac{1}{k(k+2)}(-\varphi_1(z) + (k+1)\varphi_2(z)), \quad \omega_2(z) = \frac{1}{k(k+2)}((k+1)\varphi_1(z) - \varphi_2(z)), \]

and \( \omega_3(z) = \frac{1}{2k} \chi(z) \).

5.1.1. A long screening. The above fields \( T(z) \) and \( j^+(z) \) are also in the kernel of the ‘long’ screening

\[ \mathcal{E} = \oint \partial \varphi_1 e^{-\frac{1}{\kappa}(\varphi_1 + \varphi_2)}. \]

(Up to a coefficient, \( \mathcal{E} \) is equal to \( \oint (a_1 \partial \varphi_1 + a_2 \partial \varphi_2) e^{-\frac{1}{\kappa}(\varphi_1 + \varphi_2)} \) for any \( a_1 \neq a_2 \), because \( \oint (\partial \varphi_1 + \partial \varphi_2) e^{-\frac{1}{\kappa}(\varphi_1 + \varphi_2)} = 0 \).) It follows that

\[ [\mathcal{E}, F_1] = 0, \quad [\mathcal{E}, F_2] = 0. \]

5.2. Parafermionic fields in \( \ker F_1 \cap \ker F_2 \) and \( \widehat{\mathfrak{s}l}(2)_k \)

The kernel of the two screenings contains parafermionic fields

\[ j^+(z) = \partial \varphi_1(z) e^{-\omega_1(z) + \omega_2(z)}, \]

\[ j^-(z) = \partial \varphi_2(z) e^{\omega_1(z) - \omega_2(z)} \tag{5.3} \]

(we note that \( j^+(z) \) is \( F_1 \)-exact and \( j^-(z) \) is \( F_2 \)-exact). These nonlocal fields can be dressed into \( \widehat{\mathfrak{s}l}(2) \) currents by introducing an auxiliary scalar with the OPE in (1.9) (of course, with \( k \) given by (5.2)):

\[ J^+(z) = \partial \varphi_1(z) e^{-\omega_1(z) + \omega_2(z) + 2\omega_2(z)}, \]

\[ J_0(z) = \frac{1}{2} \chi(z), \]

\[ J^-(z) = \partial \varphi_2(z) e^{\omega_1(z) - \omega_2(z) - 2\omega_2(z)}. \]
5.3. More of ker $F_1 \cap \ker F_2$

Another part of ker $F_1 \cap \ker F_2$ is easy to find. For any $h = 1, 2, \ldots$, the field

$$G_h(z) = e^{\hbar h_0(z) + \hbar h_2(z)} = e^{\frac{\hbar}{2} \Phi(z) + \frac{\hbar}{2} \Phi(z)}$$

is in the kernel of both screenings, is a relaxed highest-weight state,

$$J^- G_h(z) = J^0 G_h(z) = J^+ G_h(z) = 0,$$

and is the central element in a $(2h + 1)$-plet with respect to the zero-mode $\mathfrak{s}\mathfrak{l}(2)$ algebra:

$$(J^0)^h G_h(z) \neq 0, \quad (J^+)^{h+1} G_h(z) = 0, \quad h \geq 1.$$  

In particular, $(J^0)^h G_h(z)$ is a highest-weight state and $(J^0)^h G_h(z)$ is a $\vartheta = 1$ twisted highest-weight state, which we represent as

$$\begin{array}{ccc}
J^0 & G_h & J^0 \\
\downarrow & \downarrow & \downarrow \\
F & F & F
\end{array} \quad (5.4)$$

The Sugawara dimension of the fields in this $(2h + 1)$-plet is $\frac{h(h+1)}{k+2} = \frac{h(h+1)p}{(j+1)p+1}$.

5.4. Even more of ker $F_1 \cap \ker F_2$ and the extended algebras

The field $G_h(z)$ with a negative integer $h$ is not in the kernel of either $F_1$ or $F_2$, but is simply related to fields that are in one of these kernels, as we now describe.

In what follows, we have to consider $G_h(z)$ and $G_{-h}(z)$ simultaneously; we therefore assume $h$ to be a positive integer parameter from now on, and define

$$L_{-h}(z) = \frac{1}{\prod_{i=0}^{h-1} (-h-i)} (J^0)^h G_{-h}(z) = e^{-2\hbar h_0(z) - 2\hbar h_2(z)} \in \ker F_1,$$

$$R_{-h}(z) = \frac{1}{\prod_{i=0}^{h-1} (-h-i)} (J^+)^h G_{-h}(z) = e^{-2\hbar h_0(z) + 2\hbar h_2(z)} \in \ker F_2$$

(with $L_{-h}(z) \notin \ker F_2$ and $R_{-h}(z) \notin \ker F_1$). These two states are identified with a highest-weight state and a $\vartheta = 1$ twisted highest-weight state as

$$L_{-h}(z) \equiv \{-h\}, \quad R_{-h}(z) \equiv \left\{ h + \frac{k}{2}; 1 \right\}, \quad (5.5)$$

which we show with corners in the diagram

$$\begin{array}{ccc}
L_{-h} & \uparrow & \K \, \quad \downarrow & \K \quad \downarrow & \downarrow \\
-\h & \downarrow & \uparrow & \uparrow & \downarrow & \h
\end{array} \quad (5.6)$$

where $\K = (J_0^0)^h G_{-h}(z)$ and $\K = (J_0^0)^h G_{-h}(z)$, and $-\h$ and $\h$ indicate the charges (eigenvalues of $J^0$) of the appropriate states.

The intersection of the kernels is to be sought deeper in the module whose top is shown above. The actual picture depends on $j$, and we here restrict ourselves to $j \geq -1$; the case $j = -1$ is special.
5.4.1. The case \( j = -1 \) and a triplet–triplet algebra. We recall that \( h = 1, 2, \ldots \). For \( j = -1 \), with \( k + 2 = \frac{5}{2} \), the Verma-module highest-weight state \(|-h\rangle\) has singular vectors \( s^+ (1, 2hp + 1) \) and \( s^- (2h, p) \), on the relative level \( 2hp \) both and at the respective charge grades \(-h - 1\) and \( h \). In our free-field realization, the first of these vanishes, which we show with the left dashed arrow in figure 6, but the second one is nonvanishing (the north-west–south-east counterpart of the right picture in figure 2, but we must not forget that the screenings in the two cases are different.

Continuing the embedding diagrams of Wakimoto modules, it is not difficult to describe \( \ker F_1 \cap \ker F_2 \) quite explicitly, but we here stop at the level in the module where the fields presumably generating the extended algebra are located. For this, guided by mutual locality, we set \( h = 1 \) in the above formulas; the \( (2h + 1) \)-plets are then triplets, which we now write in more detail.

The ‘plus’ triplet (equation (5.4)) is

\[
\begin{align*}
\mathcal{W}_+^+(z) &= \frac{1}{2} J_0^+ \mathcal{G}_1(z) = \frac{1}{2} e^{2\omega_2(z) + 2\omega_1(z)}, \\
J_0^- \mathcal{W}_+^+(z) &= \mathcal{G}_1(z) = e^{\omega_1(z) + \omega_2(z)}, \\
(J_0^-)^2 \mathcal{W}_+^+(z) &= J_0^- \mathcal{G}_1(z) = e^{2\omega_2(z) - 2\omega_1(z)}.
\end{align*}
\]

The action of the long screening generates the ‘middle’ triplet

\[
\begin{align*}
\mathcal{W}_0^0(z) &= \mathcal{E} \mathcal{W}_+^+(z), \\
J_0^- \mathcal{W}_0^0(z) &= \mathcal{E} J_0^- \mathcal{W}_+^+(z), \\
(J_0^-)^2 \mathcal{W}_0^0(z) &= \mathcal{E} (J_0^-)^2 \mathcal{W}_+^+(z).
\end{align*}
\]

Figure 6 is a symmetric counterpart of the right picture in figure 2, but we must not forget that the screenings in the two cases are different.
The grade of $\mathcal{W}^0(z)$ coincides with the grade of the level-$2p$ singular vector $s^-(1,2p)$ in the $\mathfrak{sl}(2)$ Verma module $\mathcal{M}_0$ with zero weight, and $(J^0_0)^2 \mathcal{W}^0(z)$ is one grade to the right of the level-$2p$ singular vector $s^+(2,p+1)$ in $\mathcal{M}_0$; both these singular vectors vanish in our free-field realization (and hence the ‘middle’ triplet is not in the Wakimoto-type module associated with $\mathcal{M}_0$).

Acting on (5.8) with the long screening generates the ‘minus’ triplet (figure 6):

$$\mathcal{W}^-(z) = E \mathcal{W}^0(z) = s^-(2, p) e^{-2\omega_0(z) - 2\omega_1(z)},$$

$$J^0_0 \mathcal{W}^-(z) = E J^0_0 \mathcal{W}^0(z),$$

$$(J^0_0)^2 \mathcal{W}^-(z) = E (J^0_0)^2 \mathcal{W}^0(z) = s^+(2, p; 1) e^{-2\omega_0(z) + 2\omega_1(z)}$$

(with the equalities to singular vectors holding up to nonzero factors). We propose these dimension-$2p$ fields, together with the $\mathfrak{sl}(2)$ currents, as the triplet–triplet algebra generators in the ‘symmetric’ realization. Conjecturally, the algebra contains all mutually local fields in $F_1 \cap \ker F_2$.

An example of the triplet–triplet algebra in the ‘symmetric’ realization is given in section C.3 in the appendix.

5.4.2. Cases $j \geq 0$ and triplet–multiplet algebras. We briefly discuss the local algebra in the kernel for $j \geq 0$ in the symmetric realization. The relevant operator products contain characteristic factors $(z - w)^{\frac{5j+2}{j^2+1}}$; for locality, we therefore choose the smallest $h$ ensuring integers in the exponent,

$$h = (j + 1)p + 1.$$  

In (5.4) with this $h$, the state $(J^0_0)^h G_h(z)$ (the top right corner) has a nonvanishing singular vector

$$\mathcal{W}^+(z) = s^-(h - 1, 3p) (J^0_0)^h G_h(z) = h! s^-(h - 1, 3p) e^{2\omega_0(z) + 2\omega_1(z)},$$

which is in the grade with charge $2h - 1$ and at the level $3p^2(j + 1)$ relative to the top. Also, the left corner in diagram (5.4) is a visualization of the fact that $(J^0_0)^h G_h$ also has a vanishing singular vector $s^+(2h + 1, 1) = (J^0_0)^{h(j+1)}$; in the corresponding Verma module, this state has an $s^+(h - 1, 3p + 1)$ singular vector, located at the charge $-2h$ and on the same level $3p^2(j + 1)$ relative to the top as $\mathcal{W}^+(z)$. This same singular vector is ‘seen’ from $\mathcal{W}^+(z)$, and its vanishing makes $\mathcal{W}^+(z)$ the rightmost element in a $(4h - 1)$-plet under the zero-mode $\mathfrak{sl}(2)$.

We have thus found a $(4h - 1)$-plet at the absolute level (Sugawara dimension)

$$\frac{h(h+1)}{2} + 3p^2(j + 1) = 2p(2p(j + 1) + 1).$$

We next find a $(4h - 1)$-plet, at the same absolute level, in the module whose top is shown in (5.6). With the chosen $h$, $L_{-h}$ has the vanishing singular vector $s^+(1, 2p + 1)$ and the nonvanishing one $s^-(3h - 1, p)$. These are shown in figure 7, where another crucial piece is the vanishing singular vector $s^+(h - 1, 3p + 1)$ and, importantly, the ‘mirror images’ of all these singular vectors—twisted singular vectors in the twisted module associated with $\mathcal{R}_{-h}(z)$. The result is that

$$\mathcal{W}^-(z) = s^-(3h - 1, p) L_{-(j+1)p-1}(z) \in \ker F_1 \cap \ker F_2$$

is the rightmost element of a $(4h - 1)$-plet, at the same Sugawara dimension $2p(2p(j + 1) + 1)$ as $\mathcal{W}^+(z)$.

In this case, there is also a nonvanishing singular vector in the module associated with the unit operator:

$$\mathcal{W}^0(z) = s^-(2h - 1, 2p) 1(z) \in \ker F_1 \cap \ker F_2$$

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Figure 7. $h = (j + 1)p + 1$ for $j \geq 0$.

(the arrangement of the relevant singular vectors is similar to the one in figure 7, and we omit the details).

We repeat that each $W^{\pm,0} (z)$ is an $\hat{sl}(2)$ primary state of dimension $2p(2p(j + 1) + 1)$, is part of a $(4(j + 1)p + 3)$-plet, and the three of them, together with the $\hat{sl}(2)$ currents, conjecturally generate a $W$-algebra of local fields in $\ker F_1 \cap \ker F_2$.

5.5. Relation between the symmetric and asymmetric realizations

The symmetric and asymmetric realizations of extended algebras can be mapped onto one another by a nonlocal field transformation. Introducing it invokes the Wakimoto bosonization, and this deserves a terminological comment.

5.5.0. In CFT, representing anything as an exponential of free scalar(s) is standardly called bosonization. A typical example is a free-fermion first-order system of fields $\eta (z)$ and $\xi (z)$ expressed as

$$\eta (z) = e^{-F(z)}, \quad \xi (z) = e^{F(z)},$$

where $F(z)$ is a scalar field with canonical normalization. The concept and the terminology have been extended to representing bosonic first-order systems, with the OPE

$$\beta(z) \gamma(w) = \frac{-1}{z - w},$$

in terms of two scalars [57], even though the procedure is a map from bosons to bosons. By the same token, the Wakimoto free-field construction for $\hat{sl}(2)$ currents is also commonly called bosonization (although it involves by far not only exponentials).
Moreover, the extended algebra generators are also in the image of some level (in this case, $k$) deduced from the context. It is worth noting that the $\beta\gamma$ current is then mapped as

$$f(z) f(w) = \log(z - w).$$

### 5.5.2. Wakimoto $\rightarrow$ symmetric realization.

The $\beta\gamma, \partial f$ system of fields can be embedded into the algebra of fields in section 5.2 by a map $\rho_s[\cdot]$ such that

$$\rho_s[\beta](z) = -\partial \phi_1(z) e^{-\omega_2(z) + 2\omega_3(z)},$$

$$\rho_s[\gamma](z) = e^{\omega_2(z) - \omega_3(z) - 2\omega_1(z)},$$

$$\rho_s[\partial f](z) = \frac{\sqrt{2}}{\sqrt{k+2}} \left( \frac{k+1}{k} \partial \phi_1(z) - \frac{1}{k} \partial \phi_2(z) + \frac{k+2}{2k} \partial \chi(z) \right)$$

$$= \sqrt{2(k+2)} (\partial \omega_2(z) + \partial \omega_3(z)).$$

It is worth noting that the $\beta\gamma$ current evaluates on this state in the Wakimoto bosonization ingredients for $h > 0$, and hence a 'half' of the diagrams in sections 6 and 7 is not expressible in the Wakimoto bosonization, but the $\mathcal{W}$ field (and, for $f > 0$, $\mathcal{W}$) is.

Under $\rho_s[\cdot]$, the currents (5.10) are mapped onto the $\hat{\mathfrak{sl}}(2)$ currents in section 5.2. Moreover, the extended algebra generators are also in the image of $\rho_s[\cdot]$. Indeed, the states at the left and right corners in (5.4) are respectively given by

$$\rho_s[\gamma(z)^{2h} e^{\frac{z}{\sqrt{2}} f(z)}] \quad \text{and} \quad \rho_s[e^{\frac{z}{\sqrt{2}} f(z)}], \quad h > 0.$$  

For $h = 1$, these are respectively $(\mathcal{L}_-^\beta)^2 \mathcal{W}^+(z)$ and $\mathcal{W}^+(z)$, equation (5.7). Next, the highest-weight state in section 5.4 is readily seen to be expressed as

$$\mathcal{L}_- \mathcal{R}^+ = \rho_s[e^{\frac{z}{\sqrt{2}} f(z)}],$$

and hence the $\mathfrak{s}^\ast(2, p)$ singular vector evaluates on this state in the $\beta, \gamma, \partial f$ theory. The same applies to $\mathcal{W}^-(z)$, $\mathcal{W}^0(z)$, and $\mathcal{W}^+(z)$ in section 5.4.2. The twisted highest-weight state $\mathcal{R}_-^+ \mathcal{R}^+$ is not expressible in terms of the Wakimoto bosonization ingredients for $h > 0$, and hence a 'half' of the diagrams in figures 6 and 7 is not expressible in the Wakimoto bosonization, but the $\mathcal{W}^+$ field (and, for $f > 0$, $\mathcal{W}^-$) is.

To construct the inverse map, we have to express the three currents $\partial \phi_1(z), \partial \phi_2(z)$, and $\partial \chi(z)$—which we temporarily denote as $\partial \phi_1^0(z), \partial \phi_2^0(z)$, and $\partial \chi^0(z)$—in terms of three currents in the $\beta, \gamma, \partial f$ theory. Two of these are $\beta\gamma(z)\partial f(z)$, and the third is the $\eta\xi(z)$ current 'hidden' inside the $\beta, \gamma$ system [57]:

$$\partial \phi_1^0(z) = \rho_s[\eta\xi(z)],$$

$$\partial \phi_2^0(z) = \rho_s[(k + 1)\beta\gamma(z) + 2(k + 2)\partial f(z) + (k + 1)\eta\xi(z)],$$

$$\partial \chi^0(z) = \rho_s[2\beta\gamma(z) + \sqrt{2}(k + 2)\partial f(z)].$$

11 A standard notational atrocity committed in formulas (5.10) is the indiscriminate use of 'throughout' symbols for $\hat{\mathfrak{sl}}(2)$ currents; physicists tend to read such notation as the statement that the right-hand sides satisfy an $\hat{\mathfrak{sl}}(2)$ algebra, of some level (in this case, $k$) deduced from the context.
5.5.3. *Wakimoto → asymmetric realization.* The $\beta, \gamma, \partial f$ fields are mapped into the algebra of fields in section 4.2 as

\[
\rho_4[\beta](z) = -e^{i\omega_1(z)+2i\omega_2(z)},
\]

\[
\rho_4[\gamma](z) = \left(\frac{1}{2} \partial \varphi(z) + \frac{1}{2} \partial a(z)\right) e^{-i\omega_1(z)-2i\omega_2(z)},
\]

\[
\rho_4[\partial f](z) = \frac{\sqrt{k+2}}{\sqrt{2}} \left(\frac{1}{k+2} \partial \varphi(z) + \frac{1}{k} \partial \chi(z) + \frac{1}{k} \partial a(z)\right) = \sqrt{2(k+2)}(\partial \omega_2(z) + \partial \omega_3(z)).
\]

Also,

\[
\rho_4[\beta \gamma](z) = -\frac{1}{k} \partial \chi(z) - \frac{k+2}{2k} \partial a(z) - \frac{1}{2} \partial \varphi(z) = -(k+2)\partial \omega_2(z) - 2\partial \omega_3(z).
\]

We omit the maps between relevant vertices, and only give the relations that allow inverting $\rho_4[-]$, where we temporarily write $\varphi_1^2(z), \varphi_2^2(z), \text{and } \chi^a(z)$ for the fields introduced in section 4.1. Then

\[
\partial \varphi_1^2(z) = -\rho_4[\eta \xi](z),
\]

\[
\partial \varphi_2^2(z) = \rho_4[(k+2)\beta \gamma(z) + \sqrt{2(k+2)}\partial f(z) + (k+2)\eta \xi(z)],
\]

\[
\partial \chi^a(z) = \rho_4[2\beta \gamma(z) + \sqrt{2(k+2)}\partial f(z)].
\]  

5.5.4. The map between the symmetric and asymmetric realizations is

\[
\rho_4[-] \circ \rho_4^{-1}[-],
\]

which is a ‘highly nonlocal change of variables’ in the two-boson space (the $\chi(z)$ are the same in (5.11) and (5.12)). We note that $k$ in (5.11) and (5.12) and the related formulas is a common parameter, the $\hat{sl}(2)$ level. It is noninteger in our setting, and therefore the screenings $F = \oint e^{\varphi^1}$ and $F_2 = \oint e^{\varphi^2}$ are undefined in intrinsic $\beta, \gamma$ terms. If one starts with the Wakimoto realization of $\hat{sl}(2)$ and maps it by $\rho_4[-]$ or $\rho_4[-]$, then the appearance of these screenings is ‘accidental.’

6. Double bosonization of $\mathcal{B}(X)$

There is a functorial, vector-space-preserving correspondence between multivertex Yetter–Drinfeld $\mathcal{B}(X)$ modules and modules over a nonbraided Hopf algebra $U(X)$, the double bosonization of $\mathcal{B}(X)$. In this section, we show how $U(X)$ can be constructed in general (sections 6.1–6.4) and evaluate it for our two Nichols algebras (sections 6.5 and 6.6). The two resulting $U(X)$ compare nicely, as we show in section 6.7.

6.1. The dual Nichols algebra

For a Nichols algebra $\mathcal{B}(X)$, we let $\mathcal{B}(X)^\ast$ denote its graded dual. The reader can consider $X$ an object in a rigid braided category; we do not go into the details of (standard) axioms and simply assume all the necessary structures to exist. The first of these is the evaluation $\langle , \rangle : \mathcal{B}(X)^\ast \otimes \mathcal{B}(X) \rightarrow k$, which is diagrammatically denoted by $\langle , \rangle$. By abuse of notation, similar diagrams are used to represent the restriction of $\langle , \rangle$ to $X^\ast \otimes X$; it is extended to tensor
products as $\bigcup$, and so on. The product, coproduct, and braiding in $\mathcal{B}^* = \mathcal{B}(X)^*$ are defined by the respective rules

$$\begin{align*}
\mathcal{B}^* \times \mathcal{B}^* & \rightarrow \mathcal{B}^* \\
\mathcal{B}^* & \rightarrow \mathcal{B}^* \\
\mathcal{B}^* & \rightarrow \mathcal{B}^*
\end{align*}$$

(6.1)

Every (left–left) Yetter–Drinfeld $\mathcal{B}(X)$-module carries a left action of $\mathcal{B}(X)^*$ defined standardly using the $\mathcal{B}(X)$-coaction, as $\alpha y = \langle \alpha, y \rangle\psi_2$, which for one-vertex modules (and, of course, products as $J. Phys. A: Math. Theor.$) reduces to the coproduct on $\mathcal{B}(X)$:

$$\begin{align*}
\begin{array}{c}
\hline
\hline
\hline
\hline
\hline
\hline
\end{array}
\end{align*}$$

(6.2)

On graded components, this gives the maps

$$e_s^{(r)} : X^{*\otimes r} \otimes X^{\otimes s} \otimes Y \rightarrow X^{*\otimes (r-s)} \otimes Y, \quad s \geq r,$$

such that

$$\begin{align*}
\begin{array}{cccc}
X^* & X^* & X & X & Y \\
\otimes & & \otimes & & \otimes \\
\hline
\hline
\hline
\hline
\hline
\hline
\end{array}
\end{align*}$$

(6.3)

for $r = 2$ and totally similarly for all $r \leq s$.

We now discuss the nomenclature of tensor products $X^{*\otimes r} \otimes X^{\otimes s} \otimes Y$. We here label the tensor factors as $(-r, \ldots, -1, 1, \ldots, s; s + 1)$ (the semicolon separates $Y$). The pairing such as in the right-hand side of (6.3) is then conveniently written as $\rho_{(r,s)}$; we slightly abuse the notation by writing $\rho_{(r,s)} : X^{*\otimes r} \otimes X^{\otimes s} \otimes Y \rightarrow X^{*\otimes (r-s)} \otimes X^{\otimes (s-r)} \otimes Y$ (with $r \geq s$ and $s \geq r$) also for the map that should be more rigorously written as $id^{\otimes (r-s)} \otimes \rho_{(r,s)} \otimes id^{\otimes (s-r)}$.

The ‘leg notation’ for elements of the braid group extends to negative labels according to the pattern

$$\begin{align*}
\Psi_{-3} &= id^{\otimes (r-3)} \otimes \Psi \otimes id^{\otimes s} : X^{*\otimes r} \otimes X^{\otimes s} \rightarrow X^{*\otimes r} \otimes X^{\otimes s}, \\
\Psi_{-2} &= id^{\otimes (r-2)} \otimes \Psi \otimes id^{\otimes s} : X^{*\otimes r} \otimes X^{\otimes s} \rightarrow X^{*\otimes r} \otimes X^{\otimes s}, \\
\Psi_{-1} &= id^{\otimes (r-1)} \otimes \Psi \otimes id^{\otimes s} : X^{*\otimes r} \otimes X^{\otimes s} \rightarrow X^{*\otimes (r-1)} \otimes X \otimes X^{*} \otimes X^{\otimes (s-1)}
\end{align*}$$

and, ‘in the middle,’

$$\begin{align*}
\Psi_0 &= id^{\otimes (r-1)} \otimes \Psi \otimes id^{\otimes (r-1)} : X^{*\otimes r} \otimes X^{\otimes s} \rightarrow X^{*\otimes (r-1)} \otimes X \otimes X^{*} \otimes X^{\otimes (s-1)}
\end{align*}$$

(and, of course, $\Psi_1 = id^{\otimes r} \otimes \Psi \otimes id^{\otimes (r-2)}$, and so on).

We let $f_x : X \otimes X^{\otimes r} \otimes Y \rightarrow X^{\otimes (r+1)} \otimes Y$ denote the map such that for any $x \in X$ and $y \in X^{\otimes s} \otimes Y$, $f_x(x) = x \otimes_{\Psi} y$, the adjoint action by $x$. We keep the same notation for the adjoint action map also in the case where some $X^*$ factors precede the $X$ in the tensor product and the rigorous writing should be $id^{\otimes r} \otimes f_x : X^{*\otimes r} \otimes X \otimes X^{\otimes s} \otimes X \otimes X^* \otimes X^{\otimes (s-1)} \rightarrow X^{*\otimes r} \otimes X \otimes X^{\otimes (r+1)} \otimes Y$.

6.2. ‘Commutator’ identities

We now see how the left adjoint action of $\mathcal{B}(X)$ and the above left action of $\mathcal{B}(X)^*$ commute with each other.
6.2.1. We first recall a ‘commutator’ identity for the maps $X^* \otimes X^\otimes s+1 \otimes Y \to X^\otimes s \otimes Y$ effected by $e^{(1)}$ and $f$ (with appropriate subscripts, to be restored momentarily) [23]. We first show the identity in graphic form, with $s = 2$ for definiteness:

$$\text{(6.4)}$$

Each diagram is a map $X^* \otimes X \otimes X^\otimes 2 \otimes Y \to X^\otimes 2 \otimes Y$. In the first diagram in the left-hand side, the adjoint action by $x \in X$ is applied first, and is followed by the action of an element of $X^*$; in the second diagram, the order is reversed, at the expense of a braiding. The second diagram in the left-hand side can of course be rewritten as

$$\text{(6.5)}$$

where $\Psi_0^{-1} : X^* \otimes X \to X \otimes X^*$ appears.

The general (and ‘analytic’) form of (6.4) for maps $X^* \otimes X^\otimes (s+1) \otimes Y \to X^\otimes s \otimes Y$ is [23]

$$\text{(6.5)}$$

where $\mathcal{K}_2(s+1)$ is the monodromy operation

$$\text{(6.6)}$$

6.2.2. Straightforward calculation shows that identity (6.5) generalizes to a ‘commutator’ of $f$ with $e^{(r)}$ as follows (both sides are maps $X^{*\otimes r} \otimes X^\otimes (s+1) \otimes Y \to X^\otimes (s+1-r) \otimes Y$):

$$\text{(6.7)}$$

34
Here, $\Psi^{-1}_{-r+1} \cdots \Psi^{-1}_0 : X^{* \otimes r} \to X \otimes X^{* \otimes r}$; for $r = 3$, for example, this is the map $\delta_{1,1}$. The left-hand side can also be written as $e^{(r)}_1 \circ f_1 - f_{-r} \circ e^{(r)}_2 \circ \Psi_r \cdots \Psi_1$.

To continue with the $r = 3$ example, we write the right-hand side of (6.7) explicitly for $r = 3$ and $s = 3$:

$$
\begin{align*}
\Psi^{-1}_{-3+1} \cdots \Psi^{-1}_0 &= X^{* \otimes 3} \to X \otimes X^{* \otimes 3} \\
\end{align*}
$$

with the two lines corresponding to the two terms in the right-hand side of (6.7).

6.2.3. Diagonal braiding and a $T(X^*) \otimes \mathfrak{B}(X) \otimes k\Gamma$ algebra. We next specify (6.7) to the case of diagonal braiding, defined by (1.13) on $\mathfrak{B}(X)$ generators. First, for the basis $(\hat{F}_i)$ in $X^*$ such that $\langle \hat{F}_i, \hat{F}_j \rangle = \delta_{i,j}$, we apply the right diagram in (6.1) to deduce that the $q_{ij}$ in $\Psi : \hat{F}_i \otimes \hat{F}_j \mapsto q_{i,j} \hat{F}_i \otimes \hat{F}_j$ and $\Psi : \hat{F}_i \otimes \hat{F}_j \mapsto q_{i,j}^{-1} \hat{F}_j \otimes \hat{F}_i$ are given by

$$
q_{ij} = q_{ij}^{-1} \quad \text{and} \quad q_{ij}^{-1} = q_{ij}. \quad (6.9)
$$

Then the monodromy operation evaluates on generators as

$$
\begin{align*}
\mathcal{K}_2(s+1) : \hat{F}_i \otimes \hat{F}_j \otimes \hat{F}_k \otimes \cdots \otimes \hat{F}_{j_i} \otimes y \\
&\mapsto \langle \hat{F}_i, \hat{F}_j \rangle(q_{j_i,i,q_{j_i,j_i}}) \cdots (q_{j_i,j_i,q_{j_i,j_i}})(q_{j_i,j_i,q_{j_i,j_i}}) \hat{F}_i \otimes \cdots \otimes \hat{F}_{j_i} \otimes y \quad (6.10)
\end{align*}
$$

for a homogeneous $y \in Y$.

Let $\Gamma$ be an Abelian group with generators $K_j = 1, \ldots, \theta$, such that their action on $\mathfrak{B}(X) \otimes Y$, interpreted as adjoint action, produces the monodromies as in the last formula:

$$
\begin{align*}
\mathcal{K}_j \hat{F}_j \mathcal{K}_j^{-1} &= q_{i,j} q_{j,i} \hat{F}_j \\
\end{align*}
$$

(and $\mathcal{K}_j \mathcal{K}_j^{-1} = q_{i,j} q_{j,i}$. Commutator identity (6.5) then becomes

$$
\hat{F}_i \hat{F}_j - q_{j,i} \hat{F}_j \hat{F}_i = \delta_{i,j}(1 - K_j). \quad (6.12)
$$

Also setting

$$
\begin{align*}
\mathcal{K}_j \hat{F}_j \mathcal{K}_j^{-1} &= q_{i,j}^{-1} q_{j,i} \hat{F}_j,
\end{align*}
$$

we obtain an associative algebra on $T(X^*) \otimes \mathfrak{B}(X) \otimes k\Gamma$ with relations given by those in $\mathfrak{B}(X)$ and (6.11)–(6.13).

Formula (6.7) becomes the statement that for each $F_j$, the map $T(X^*) \to T(X^*) \otimes k\Gamma$ defined as

$$
\begin{align*}
\langle \hat{F}_i \cdots \hat{F}_n \rangle_{\mathcal{D}_j} &= \hat{F}_i \cdots \hat{F}_n F_j - q_{j,i} \cdots q_{j,j_i} \hat{F}_i \cdots \hat{F}_{j_i} \\
\end{align*}
$$

(in particular, $\hat{F}_i \mathcal{D}_j = \delta_{i,j}(1 - K_j)$) is a braided right derivation:

$$
\begin{align*}
\langle \hat{F}_i \cdots \hat{F}_n \rangle_{\mathcal{D}_j} &= \hat{F}_i \cdots \hat{F}_n (\hat{F}_i \mathcal{D}_j) + q_{j,i} (\langle \hat{F}_i \cdots \hat{F}_n \rangle_{\mathcal{D}_j}) \hat{F}_i. \quad (6.15)
\end{align*}
$$

35
6.2.4. The ‘half-way algebra’ $\mathcal{B}(X)^* \otimes \mathcal{B}(X) \otimes k\Gamma$. We verify in the particular cases studied in this paper that the associative algebra structure on $T(X^*) \otimes \mathcal{B}(X) \otimes k\Gamma$ pushes forward to $\mathcal{B}(X^*) \otimes \mathcal{B}(X) \otimes k\Gamma$, i.e., to the quotient $\mathcal{B}(X^*) = T(X^*)/J^*$ in the first factor (the general proof must be possible by borrowing some relevant steps from [52]). This algebra is only half the way from $\mathcal{B}(X)$ to its double bosonization because the Abelian group $\Gamma$ read off from monodromy, not braiding, is too coarse to yield a Hopf algebra.

6.3. The case of an $H^\Psi YD$ braiding

We next assume that the braiding $\Psi$ is the one in the Yetter–Drinfeld category $H^\Psi YD$ for some Hopf algebra $H$ (see section A.2.1 in the appendix). Monodromy—double braiding—then evaluates as

$$\Psi^2 : x \otimes y \mapsto x_{r(1)} \, y_{r(1)} \, s(x_{r(1)}) \triangleright x_{r(1)} \otimes y_{r(1)} \, \triangleright y_{r(1)}.$$ 

From now on, we assume $H$ to be commutative and cocommutative. Then the last formula is simplified to

$$\Psi^2 : x \otimes y \mapsto y_{r(1)} \, x_{r(1)} \, \triangleright x_{r(1)} \otimes y_{r(1)} \, \triangleright y_{r(1)}.$$ 

This is to be used in the definition of $K_2$, equation (6.6). We ask when the right-hand side can be interpreted ‘nonsymmetrically’ with respect to $x$ and $y$, as an operation acting on $y$, not involving any action on $x$. This is the case if Sommerhäuser’s condition [49]

$$u_{r(1)} \triangleright v \otimes u_{r(1)} = v_{r(1)} \otimes v_{r(1)} \, \triangleright u$$

is imposed for all relevant $u$ and $v$ (we actually need the cases where $u \otimes v$ is in $\mathcal{B}(X) \otimes \mathcal{B}(X)$, $\mathcal{B}(X) \otimes Y$, and $Y \otimes \mathcal{B}(X)$). With (6.16), indeed, the double braiding takes the form

$$\Psi^2 : x \otimes y \mapsto x_{r(1)} \otimes y_{r(1)} \triangleright x_{r(1)} \otimes y_{r(1)} \triangleright y.$$ 

We assume (6.16) to hold from now on. Before proceeding, we make two brief remarks:

1. It suffices to impose (6.16) on the generators, because if this condition holds for pairs $(y, x)$ and $(z, x)$, then it also holds for $(y \otimes z, x)$:

   $$(y \otimes z)_{r(1)} \triangleright x \otimes (y \otimes z)_{r(1)} = y_{r(1)} \otimes z_{r(1)} \triangleright x \otimes y_{r(1)} \otimes z_{r(1)},$$

   where the last equality is of course valid because $H$ is cocommutative.

2. In terms of the braiding matrix entries, condition (6.16) implies that $q_{i,j} = q_{j,i}$. In particular, for braiding matrices (2.1) and (3.1), this means that $\xi^2 = 1$, which brings us back to the braiding matrices in (1.1).

For a commutative and cocommutative $H$, $H = k\Gamma$ (see section A.2.3 in the appendix), double braiding (6.17), with $x = F_i$, becomes

$$\Psi^2 : F_i \otimes y \mapsto F_i \otimes (g_i)^2 \triangleright y.$$ 

Also interpreting the $\Gamma$ action as an adjoint action (i.e., writing $g_i \cdot F_j = g_i F_j g_i^{-1}$), we see that $K_i$ in (6.11)–(6.13) are given by

$$K_i = (g_i)^2.$$
Hence, there is an associative algebra structure on \( U(X) = \mathcal{B}(X^*) \otimes \mathcal{B}(X) \otimes k\Gamma \) with \( \mathcal{B}(X^*) \otimes \mathcal{B}(X) \otimes k\Gamma \) a subalgebra in it.

We summarize our findings as follows.

6.4. Double bosonization: the Hopf algebra \( U(X) \)

For a braided vector space \( X \in \mathcal{BV}^D \) with a chosen basis \( F_i \) (and the dual basis \( \widehat{F}_i \) in \( X^* \)) and a symmetric braiding matrix \((q_{ij})\) in this basis, the algebra \( U(X) \) on generators \( F_i, \widehat{F}_i, g_i \) \((i = 1, \ldots, \theta)\) contains \( \mathcal{B}(X^*) \) and \( \mathcal{B}(X) \) as subalgebras and, in addition, has the relations
\[
g_i F_j g_i^{-1} = q_{ij} F_j, \\
\widehat{F}_i F_j - q_{ij} F_j \widehat{F}_i = \delta_{ij}(1 - (g_j)^2), \tag{6.18}
\]
Moreover, \( U(X) \) is a Hopf algebra, with the coproducts such that all \( g_i \in \Gamma \), \( i = 1, \ldots, \theta \), are group-like and
\[
\Delta : F_i \mapsto g_j \otimes F_j + F_j \otimes 1 \quad \Delta : \widehat{F}_i \mapsto g_j \otimes \widehat{F}_j + \widehat{F}_j \otimes 1.
\]
and with the antipode
\[
S(F_i) = -g_i^{-1} F_i, \quad S(\widehat{F}_i) = -g_i^{-1} \widehat{F}_i.
\]
The formula for \( \Delta(F_i) = \Delta(F_i \# 1) \) is nothing but the Radford formula for \( \mathcal{B}(X) \# k\Gamma \). Hence, in particular, the relations in \( \mathcal{B}(X) \) are compatible with the coproduct (the corresponding ideal is a Hopf ideal). The formula for \( \Delta(\widehat{F}_i) \), similarly, is the Radford formula for \( \mathcal{B}(X^*) \# k\Gamma \) with the \( k\Gamma \) action and coaction changed by composing each with the antipode; for a commutative cocommutative Hopf algebra, this still gives a left action and a left coaction. It therefore only remains to verify that cross-commutator (6.18) is compatible with the above coproduct. This is straightforward.

We also note that the \( q \)-commutator in (6.18) can be conveniently ‘straightened out’ by defining \( \phi_i = g_i^{-1} \widehat{F}_i \). Then
\[
F_i \phi_j - \phi_j F_i = \delta_{ij}(g_i - g_i^{-1}),
\]
and we also have
\[
\Delta(\phi_i) = 1 \otimes \phi_i + \phi_i \otimes g_i^{-1}, \quad S(\phi_i) = -\phi_i g_i.
\]

6.5. The \( U(X_q) \) algebra

6.5.1. We consider the graded dual \( \mathcal{B}(X^*) \) of the Nichols algebra in section 2 and define elements \( \widehat{B}, \widehat{F} \in X^* \) by requiring that the only nonzero evaluations that they have with the PBW basis in \( \mathcal{B}(X) \) be \( (\widehat{B}, B) = 1 \) and \( (\widehat{F}, F) = 1 \) (recall that using the notation for the PBW basis in section 2.2.1, \( B = rB_1 \) and \( F = r_1 \)).

The quotient by the kernel of a bilinear form is known to be another characterization of a Nichols algebra. Also, from (6.1), the braiding matrix in the basis \( (\widehat{B}, \widehat{F}) \) coincides with (2.1). Therefore, \( \mathcal{B}(X^*) \) is isomorphic to \( \mathcal{B}(X) \) and is the quotient of \( T(X^*) \) by the ideal generated by
\[
\widehat{B}^2, \widehat{F}^2, \text{ and } \xi^2 \widehat{B} \widehat{F} \widehat{F} - \xi (q + q^{-1}) \widehat{F} \widehat{B} \widehat{F} + \widehat{F} \widehat{F} \widehat{B}. \tag{6.19}
\]
12 That the elements in (6.19) are in the kernel of the form is in fact obvious for \( \widehat{B}^2 \) and \( \widehat{F}^2 \); for the last element in (6.19), it is easy to verify its vanishing on all degree-three elements of the PBW basis in \( \mathcal{B}(X) \) (see section 2.2.1): \( \text{FB}_3 = \xi^2 q^{-2} BFF + \xi q^{-1} FBF + FFB, \text{ xB}_3 = (1 - q^2) FBF + q^{-1} \xi^{-1}(1 - q^4) FFB, \text{ and xFB}_3 = (1 - q^{-2}) BFB. \)
The dual algebra \( \mathfrak{B}(X)^* \) acts on each multivertex \( \mathfrak{B}(X) \) -module in accordance with (6.2). Claiming this requires verifying that the action by elements (6.19) commute(s) with the left adjoint action of \( \mathfrak{B}(X) \). We show this.

For the last element in (6.19), we take the general ‘commutator’ formula (6.7) with \( r = 3 \) (in which case it takes the graphic form that differs from (6.8) only in the number of strands ‘inside the loop’). The two brackets in the right-hand side of (6.7) then become \((id + \Psi_{-1} + \Psi_{-1} \Psi_{-2}) (id + \Psi_{-2} + \Psi_{-2} \Psi_{-1})\) (using the conventions set in section 6.1), and it is straightforward to verify that both vanish when applied to \( \xi^2 \hat{B} \otimes \hat{F} \otimes \hat{F} - \xi (q + q^{-1}) \hat{F} \otimes \hat{B} \otimes \hat{F} + \hat{F} \otimes \hat{F} \otimes \hat{B} \).

For \( \hat{F} \), similarly, the two elements of the braid group algebra that occur in applying (6.7) are \((id + \Psi_{-1} + \Psi_{-2} + \cdots + \Psi_{-1} \Psi_{-2} \cdots \Psi_{p+1})\) and \((id + \Psi_{-p+1} + \Psi_{-p+1} \Psi_{p+2} + \cdots + \Psi_{-p+1} \cdots \Psi_{-1})\). Both are immediately seen to vanish when applied to \( \hat{F} \hat{B} \).

For \( \hat{B} \), totally similarly but even simpler, everything reduces to the ‘basic property of a fermion’ \((id + \Psi) \hat{B} \otimes \hat{B} = 0\).

The ‘half-way algebra’ \( \mathfrak{B}(X)^* \otimes \mathfrak{B}(X) \otimes \mathbb{R}^\Gamma \) in (6.11)–(6.13) is therefore an associative algebra on generators \( B, F, K_B, K_F, \hat{B}, \hat{F} \) with the relations
\[
\begin{align*}
K_B B K_B^{-1} &= B, & K_B F K_B^{-1} &= \xi^{-2} q^{-2} F, \\
K_F B K_F^{-1} &= \xi^2 q^{-2} B, & K_F F K_F^{-1} &= q^4 F, \\
BB + BB &= 1 - K_B, & \hat{B}F - \xi q^{-1} \hat{F}B &= 0, \\
\hat{F}B - \xi^{-1} q^{-1} \hat{F}B &= 0, & \hat{F} - q \hat{F}F &= 1 - K_F, \\
K_B \hat{B}K_B^{-1} &= \hat{B}, & K_B \hat{F}K_B^{-1} &= \xi^2 q^2 \hat{F}, \\
K_F \hat{B}K_F^{-1} &= \xi^{-2} q^2 \hat{B}, & K_F \hat{F}K_F^{-1} &= q^{-2} \hat{F}, \\
B^2 &= 0, & F^p &= 0, & \xi^2 BFF - \xi (q + q^{-1}) FBF + FFB &= 0, \\
\hat{B}^2 &= 0, & \hat{F}^p &= 0, & \xi^2 \hat{B}\hat{F}\hat{F} - \xi (q + q^{-1}) \hat{F} \hat{B}\hat{F} + \hat{F} \hat{F} \hat{B} &= 0
\end{align*}
\]

(and we can also consistently impose the relations \( K_B^q = 1 \) and \( K_F^q = 1 \)).

To obtain a Hopf algebra, as explained above, we set \( \xi^2 = 1 \), as is required by condition (6.16), and take an Abelian group \( \Gamma \) such that the braided vector space \( X \) become an object in \( \mathcal{U} \). We choose \( \Gamma \) to be the Abelian group with two generators \( K \) and \( k \), with
\[
k^{2p} = 1, \quad K^{2p} = 1,
\]
acting and coacting as
\[
\begin{align*}
k \circ B &= -B, & k \circ F &= \xi q F, \\
k \circ F &= \xi q B, & K \circ F &= q^{-2} F, \\
B \mapsto k^{-1} \otimes B, & F \mapsto K^{-1} \otimes F.
\end{align*}
\]

Then the ‘half-way algebra’ is a subalgebra in the Hopf algebra \( \mathcal{U}(X) = \mathfrak{B}(X)^* \otimes \mathfrak{B}(X) \otimes k\Gamma \), embedded via \( K_B = k^{-2} \) and \( K_F = K^{-2} \). The generators \( \phi_i \) introduced in section 6.4 are now \( k \hat{B} \) and \( K \hat{F} \). We change the normalization in order to obtain more conventional commutation relations in what follows: we set
\[
C = \frac{-1}{q - q^{-1}} k \hat{B} \quad \text{and} \quad E = \frac{1}{q - q^{-1}} K \hat{F}.
\]

The Hopf algebra \( \mathcal{U}(X) \) with the generators chosen this way is fully described below for the convenience of further reference.
6.6. The Hopf algebra \( \mathbf{U}(X_\lambda) \). It follows that the double-bosonization algebra \( \mathbf{U}(X_\lambda) \) is the algebra on generators \( B, F, k, K, C, E \) with the following relations. First, \( \mathbf{U}(X_\lambda) \) contains a \( \mathcal{U}_q\mathfrak{sl}(2) \) subalgebra (which is also a Hopf subalgebra) generated by \( E, K, \) and \( F \), with the relations

\[
K F = q^{-2} F K, \quad E F - F E = K - K^{-1} = q - q^{-1}, \quad K E = q^2 E K,
\]

\[
F^p = 0, \quad E^p = 0, \quad K^{2p} = 1.
\]  

Next, \( \mathcal{U}_q\mathfrak{sl}(2) \) and \( k \) generate a subalgebra, denoted by \( \mathcal{U}_q\mathfrak{sl}(2) \) in what follows, with further relations

\[
k E = \xi q^{-1} E k, \quad k F = \xi q F k, \quad k^2 = 1, \quad k K = K k.
\]  

The other relations in \( \mathbf{U}(X_\lambda) \) are

\[
k B = - B k, \quad K B = \xi q B K, \quad k C = - C k, \quad K C = \xi q^{-1} C K,
\]

\[
B^2 = 0, \quad B C - C B = \frac{k - k^{-1}}{q - q^{-1}}, \quad C^2 = 0,
\]

\[
F C - C F = 0, \quad B E - E B = 0,
\]

\[
F F B - \xi (q + q^{-1}) F B F + B F F = 0, \quad E E C - \xi (q + q^{-1}) E C E + C E E = 0.
\]

The coproduct, antipode, and counit are given by

\[
\Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(E) = E \otimes K + 1 \otimes E,
\]

\[
\Delta(B) = B \otimes 1 + k^{-1} \otimes B, \quad \Delta(C) = C \otimes k + 1 \otimes C,
\]

\[
S(B) = - k B, \quad S(F) = - K F, \quad S(C) = - C K^{-1}, \quad S(E) = - E K^{-1},
\]

\[
\varepsilon(B) = 0, \quad \varepsilon(F) = 0, \quad \varepsilon(C) = 0, \quad \varepsilon(E) = 0,
\]

with \( k \) and \( K \) group-like.

6.6. The \( \mathbf{U}(X_\lambda) \) algebra

In the graded dual \( \mathcal{B}(X_\lambda)^* \), we define \( \hat{a}, \hat{b} \in X_\lambda^* \) by requiring that the only nonzero evaluations that they have with the PBW basis in \( \mathcal{B}(X_\lambda) \) be \( \langle \hat{a}, a \rangle = 1 \) and \( \langle \hat{b}, b \rangle = 1 \). In accordance with (6.9), the braiding matrix of \( \langle \hat{a}, \hat{b} \rangle \) coincides with (3.1). It is also easy to see that the coproduct in section 3.2.3 immediately implies the relations

\[
\hat{a}^2 = 0, \quad \hat{b}^2 = 0, \quad (\hat{a} \hat{b})^p - \xi^{-p} (\hat{b} \hat{a})^p = 0.
\]

The 'half-way algebra,' in addition to the relations in \( \mathcal{B}(X_\lambda) \) and \( \mathcal{B}(X_\lambda^*) \), has the relations

\[
\hat{a} a + a \hat{a} = 1 - K_a, \quad \hat{a} b + \xi q b \hat{a} = 0, \quad \hat{b} a + \xi^{-1} q a \hat{b} = 0, \quad \hat{b} b + b \hat{b} = 1 - K_b,
\]

where

\[
K_a a K_a^{-1} = a, \quad K_b b K_b^{-1} = q^2 b,
\]

\[
K_a a K_b^{-1} = q^2 a, \quad K_b b K_a^{-1} = b
\]

(and we can set \( K_a^0 = 1 \) and \( K_b^0 = 1 \)).
6.6.1. The Hopf structure of $U(X_i)$. To obtain a Hopf algebra, as in 6.4, we assume that $\xi^2 = 1$ in accordance with condition (6.16) and take $\Gamma$ to be the Abelian group with generators $K_1$ and $K_2$, $K_{1p} = 1$, acting and coacting on the basis $F_1 = a$ and $F_2 = b$ in $X_i$ as

$$
\begin{align*}
K_1 \cdot F_1 &= -F_1, & K_1 \cdot F_2 &= -\xi q^{-1}F_2, \\
K_2 \cdot F_1 &= -\xi q^{-1}F_1, & K_2 \cdot F_2 &= -F_2,
\end{align*}
$$

and $F_i \mapsto K_i^{-1} \otimes F_i$. In accordance with 6.4, the double bosonization $U(X_i) = \mathfrak{B}(X_i^*) \otimes \mathfrak{B}(X_i) \otimes k\Gamma$ is then the algebra on generators $F_1$, $F_2$, $\phi_1$, $\phi_2$, $K_1$, $K_2$ with the relations

$$
K_1^{2p} = 1, \quad K_2^{2p} = 1, \quad (F_i K_1)^p = (F_i K_2)^p = 0,
$$

$$
K_1 F_1 K_1^{-1} = -F_1, \quad K_1 F_2 K_1^{-1} = -\xi q^{-1}F_1,
$$

$$
F_1 \phi_1 - \phi_1 F_2 = K_1 - K_1^{-1}, \quad F_2 \phi_2 - \phi_2 F_2 = K_2 - K_2^{-1},
$$

and with the Hopf-algebra structure defined by

$$
\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \Delta(\phi_i) = \phi_i \otimes K_i + 1 \otimes \phi_i, \\
S(F_i) = -K_i F_i, \quad S(\phi_i) = -\phi_i K_i^{-1},
$$

$$
\varepsilon(F_i) = 0, \quad \varepsilon(\phi_i) = 0,
$$

where $\Delta$ and $\varepsilon$ are the comultiplication and counit, respectively. These relations, together with the relations (6.16), ensure that $U(X_i)$ is a Hopf algebra.

6.7. Isomorphism

From sections 6.5.4 and 6.6.1, we have double bosonizations $U(X_i)$ and $U(X_i)$ of the ‘asymmetric’ and ‘symmetric’ Nichols algebras. They turn out to be ‘essentially the same’—related somewhat simpler than their CFT counterparts in section 5.5.

(1) The algebras $U(X_i)$ and $U(X_i)$ are isomorphic as associative algebras. Explicitly, the isomorphism $\sigma : U(X_i) \rightarrow U(X_i)$ is given by

$$
\sigma(F_1) = (q - q^{-1}) (ξBF - q^{-1}FB), \quad \sigma(F_2) = -(q - q^{-1}) Ck^{-1},
$$

$$
\sigma(\phi_1) = EC - ξqCE, \quad \sigma(\phi_2) = Bk,
$$

$$
\sigma(K_1) = Kk, \quad \sigma(K_2) = k^{-1}
$$

and the inverse map is

$$
\sigma^{-1}(B) = -\frac{1}{(q - q^{-1})^2} (F_1 F_2 + ξq F_2 F_1), \quad \sigma^{-1}(F) = \phi_2 K_2,
$$

$$
\sigma^{-1}(C) = -\frac{1}{q - q^{-1}} F_2 K_2^{-1}, \quad \sigma^{-1}(E) = -q^{-1}(\phi_1 \phi_2 + ξ q \phi_2 \phi_1),
$$

$$
\sigma^{-1}(K) = K_1 K_2, \quad \sigma^{-1}(k) = K_2^{-1}.
$$

(2) The two coalgebra structures, $\Delta_a$ defined in (6.23) and $\Delta_s$ defined in (6.26), are related as stated in (1.12) with

$$
\Phi = 1 \otimes 1 + (q - q^{-1}) Bk \otimes Ck^{-1}.
$$
(3) The antipodes are related as $US_a(\hat{\sigma}(x))U^{-1} = \hat{\sigma}S_a(x)$ with $U = 1 - (q - q^{-1})BCk$.

In proving that $\hat{\sigma}$ is an algebra morphism, we must of course verify that relations are mapped into relations. It is immediate to see that $\hat{\sigma}(F_1F_2)\hat{\sigma}(F_2F_1)$ vanishes due to the ‘$FFB$’ relation in (6.22) (and its consequence (2.5)). To see how $(F_1F_2)^p - \xi^{-p}(F_2F_1)^p$ is mapped by $\hat{\sigma}$, we first inductively establish the identities

$$\hat{\sigma}: \frac{1}{(q - q^{-1})^{2n}}((F_1F_2)^n - \xi^n(F_2F_1)^n) \mapsto (-1)^nF^n + (-1)^n\xi^nq^{-2n+1}(q^n + 1)CBF^n k^{-1}$$

$$\quad + (-1)^{n+1}\xi^{n+1}q^{-2n}(q^n + 1)CBF^{n-1} k^{-1},$$

whence it indeed follows that the right-hand side vanishes at $n = p$ due to the relations in $U(X_a)$ and the fact that $q^p + 1 = 0$. For the $\phi_i$, everything is totally similar.

That the above $\Phi$ does the job in (1.12) is verified on the $U(X_s)$ generators straightforwardly. The map relating the antipodes is then standard, $U = \Phi^{(1)}S(\Phi^{(2)})$ [58].

6.7.1. The isomorphism of associative algebras $U(X_s) \simeq U(X_a)$ is not unexpected if we recall that the Nichols algebras $B(X_s)$ and $B(X_a)$ are related by a Weyl pseudoreflection, which (at the level of bosonizations $B(X) \# k\Gamma$, to be precise) roughly amounts to the following procedure [20]: pick up a $B(X)$ generator $F_i$; drop $F_i$ (the corresponding one-dimensional subspace) and ‘add’ the dual $\tilde{F}_i$ instead; and replace each $F_i$, $i \neq \ell$, with $(ad_{F_j})^{m_0}F_j$, where ‘max’ means taking the maximum power (of the braided adjoint) that does not yet lead to identical vanishing. For $B(X_s)$, whose fermionic generators $F_1$ and $F_2$ are shown on the left in figure 8, this means dropping $F_1$, introducing $B = \tilde{F}_1$ instead, and replacing $F_2$ with $F = [F_1, F_2]_q$ (a braided commutator); these $B$ and $F$ are generators of $B(X_s)$. In the double bosonization of each Nichols algebra, the generators are the original $B(X)$ generators and their opposite ones, and the two Nichols algebras related by a Weyl reflection yield two systems of generators in the same $U(X)$. 

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6.7.2. We note that figure 8 is reproduced in the structure of equations (5.11) and (5.12).

6.8. Simple $U(X)$-modules

In view of the statements in 6.7, the representation theories of $U(X_2)$ and $U(X_3)$ are equivalent. We choose $U(X_2)$ with $\xi = 1$ and let it be denoted simply by $U(X)$ (the algebra with $\xi = -1$ has an equivalent representation category [59]).

We quote some of the results established in [53]. The algebra $U(X)$ has $4p^3$ simple modules, which are labeled as

$$Z_{a,b}^{\alpha,\beta}, \quad \alpha, \beta = \pm, \quad s = 1, \ldots, p, \quad r = 0, \ldots, p - 1,$$

and have the dimensions

$$\text{dim } Z_{a,b}^{\alpha,\beta} = \begin{cases} 2s - 1, & r = 0, \\ 2s + 1, & r = s, \\ 4s, & r \neq 0, s, \\ 4p, & 1 \leq r \leq p - 1, \\ 1 \leq s \leq p - 1, \\ s = p. \end{cases} \quad (6.29)$$

On the highest-weight vector of $Z_{a,b}^{\alpha,\beta}$, $k$ and $K$ have the respective eigenvalues

$$\beta q^{-r} \quad \text{and} \quad \alpha q^{r-1}. \quad (6.30)$$

7. $U(X)$ and the $W(2, (2p)^{3\times3})$ algebra

We consider the triplet–triplet $W$-algebra $W(2, (2p)^{3\times3})$ (which corresponds to $j = 0$ in the ‘asymmetric’ case, for definiteness). Introducing an Abelian group $\Gamma$ such that $X \in \mathcal{D}$ implies an effect that has no clear analogue in the known $W_{p,1}/$Virasoro case: not all of the $W$-algebra commutes with $\Gamma$.

7.1. $\Gamma$ in terms of free fields

In the asymmetric free-field realization, the generators of $\Gamma$ can be represented in terms of zero modes of the fields as

$$k = e^{-i\pi \Phi}, \quad K = e^{-i\pi \Psi}, \quad K = e^{-i\pi \Phi}$$

(see the field redefinition in section 4.2.3). It follows that $k$ anticommutes with $J^+(z)$ and $J^-(z)$.  

7.2. $W(2, (2p)^{3\times3}) \subset W(2, (2p)^{3\times3})$

To maintain the idea that the algebras ‘on the Hopf side’ and ‘on the CFT side’ centralize each other, we have to choose a subalgebra in $W(2, (2p)^{3\times3})$ that commutes with $\Gamma$. We recall from section 4.4.1 that the (not necessarily minimal) set of fields generating the $W(2, (2p)^{3\times3})$ algebra was

$$J_0^-, J_0^+ W^a(z), \quad J_0^+ W^a(z), \quad a = +, 0, -,$$

which gave rise to $J^-(z), J^0(z), J^+(z)$ in their OPEs. The subalgebra that centralizes $\Gamma$ is generated by the fields

$$J_0^-, J_0^+ W^a(z), \quad W^a(z), \quad a = +, 0, -,$$

and

$$(J^-(z))^2, \quad J^0(z), \quad (J^+(z))^2, \quad T_{\text{Sug}}(z).$$
((J^+(z))^2 do occur in the OPEs of \( J^+J^- \), \( \mathcal{W}^+(z) \), and \( \mathcal{W}^-(z) \), but we do not discuss the minimal set of generators now, emphasizing the ‘J-squaredness’ instead). We let this algebra be denoted by \( \mathcal{W}(2, (2p)^{\times 3 \times 2}) \), because with the middle terms dropped, the triplets under the horizontal \( s(2) \) become ‘doublets’ with respect to zero modes of \((J^+(z))^2\) and \((J^-(z))^2\).

The representation theories of \( \mathcal{W}(2, (2p)^{\times 3 \times 2}) \) and \( \mathcal{W}(2, (2p)^{\times 3 \times 3}) \) are not very different. It remains to be seen whether/how reintroducing \( J^+(z) \) and \( J^-(z) \) as such, not their squares, spoils some presumably nice properties of the \( \mathcal{W}(2, (2p)^{\times 3 \times 2}) \) theory.

7.3. The \( \mathcal{W}(2, (2p)^{\times 3 \times 2}) \) action on vertices

We recall the vertex operators \( V^{r,\mu}_{s,\nu}[n, m](z) \) introduced in section 4.6.2. We saw there that acting with \( \mathcal{W}^+ \) and \( \mathcal{W}^- \) maps over the values of \( n \). We next observe that the modes of \((J^+(z))^2\) and \((J^-(z))^2\) map over the values of \( m \).

The annihilating conditions with respect to the modes of \((J^+(z))^2\) and \((J^-(z))^2\) are
\[
(J^+(z))^2_{2\ell, -1 + \ell} V^{r,\mu}_{s,\nu}[n, m](z) = 0, \quad \ell \geq 0,
\]
\[
(J^-(z))^2_{3 - 2\ell, 2\ell} V^{r,\mu}_{s,\nu}[n, m](z) = 0, \quad \ell \geq 0,
\]
and the maximum modes that are generically nonvanishing act as
\[
((J^+)^2)_{2\ell, 2\ell - 1} V^{r,\mu}_{s,\nu}[n, m + 1](z),
\]
\[
((J^-)^2)_{2\ell, -2\ell} V^{r,\mu}_{s,\nu}[n, m](z) = \left( \frac{r - s}{p} + \mu + \nu + 2n + 2m \right) \left( \frac{r - s}{p} + \mu + \nu + 1 + 2n + 2m \right)
\]
\[
\times \left( \frac{r}{p} + \mu + 2m \right) \left( \frac{r}{p} + \mu + 1 + 2m \right) V^{r,\mu}_{s,\nu}[n, m - 1](z).
\]
The vanishing conditions for the brackets in the last formula, as well as in (4.24), allow making some simple observations about the occurrence of \( \mathcal{W}(2, (2p)^{\times 3 \times 2}) \) submodules.

7.4. \( \mathcal{W}(2, (2p)^{\times 3 \times 2}) \) submodules and the correspondence with \( \mathcal{U}(X) \) representations

For the algebra \( \mathcal{V} = \mathcal{W}(2, (2p)^{\times 3 \times 2}) \), we describe a set of its (conjecturally simple) modules \( \mathcal{Z}^{r,\mu}_{s,\nu} \) and set them in correspondence. We set \( \alpha = (-1)^v \) and \( \beta = (-1)^\mu \) (with \( \nu \) and \( \mu \) taking values 0 or 1 hereafter in this subsection.

Let \( \mathcal{F}^{r,\mu}_{s,\nu} \) be the space spanned by \( P(\partial \varphi, \partial a, \partial \chi) V^{r,\mu}_{s,\nu}[n, m](z) \), where \( n, m \in \mathbb{Z} \) and \( P \) are differential polynomials. It bears a natural \( \mathcal{V} \)-action and, as a \( \mathcal{V} \)-module, plays the role of a Verma module, with the \( V^{r,\mu}_{s,\nu}[n, m](z) \), \( n, m \in \mathbb{Z} \), being the extremal vectors. The \( \mathcal{V} \)-module structure of \( \mathcal{F}^{r,\mu}_{s,\nu} \) depends on the parameters \( 1 \leq s \leq p \) and \( 0 \leq r \leq p - 1 \), and we now list some characteristic irreducibility/reducibility cases.

1. For \( s = p \) and \( r \neq 0 \), we conclude from sections 4.6.2 and 7.3 that each extremal vector is reachable by the \( \mathcal{V} \) action from any other extremal vector; we conjecture that \( \mathcal{Z}^{r,\mu}_{p,0} = \mathcal{F}^{r,\mu}_{p,0} \) is irreducible in this case.
2. For \( 1 \leq s \leq p - 1 \) and \( r \neq 0 \), \( r \neq s \), it follows from section 4.6.2 that extremal vectors with \( n < 0 \) are unreachable from extremal vectors with \( n \geq 0 \). There is therefore a proper submodule \( \mathcal{Z}^{r,\mu}_{s,\nu} \subset \mathcal{F}^{r,\mu}_{s,\nu} \) generated from \( V^{r,\mu}_{s,\nu}(0, 0)(z) \).
3. For \( 1 \leq s \leq p - 1 \) and \( r = s \), it follows from section 4.6.2 that extremal vectors with \( n < 0 \) are unreachable from the vectors with \( n \geq 0 \); from section 7.3, moreover, we see that in the case \((v = 0, \mu = 0)\), extremal vectors with \( n < -m \) are unreachable from the vectors with \( n \geq -m \), and in the three remaining cases \((v = 0, \mu = 1), (v = 1, \mu = 0), \) and \((v = 1, \mu = 1)\), extremal vectors with \( n < -m - 1 \) are unreachable from those with
Winding the way toward the Morita equivalence of categories. Moreover, it can be expected that
in any of the three cases ($v = 0$, $\mu = 1$), ($v = 1$, $\mu = 0$), and ($v = 1$, $\mu = 1$). The submodule is
the intersection of two submodules, respectively represented by dots to the right of (and including)
the tilted line $n + m = -1$ and by dots above (and including) the horizontal line $n = 0$. (In the case
($v = 0$, $\mu = 0$), the tilted line is $n + m = 0$.)

Figure 9. The filled dots are the extremal vectors $V_{\nu,\mu}^\alpha[n, m](z)$ that are in the submodule $Z_{s,0,\vartheta}^{\alpha,\beta}$
in any of the three cases ($v = 0$, $\mu = 1$), ($v = 1$, $\mu = 0$), and ($v = 1$, $\mu = 1$). The submodule is
the intersection of two submodules, respectively represented by dots to the right of (and including)
the tilted line $n + m = -1$ and by dots above (and including) the horizontal line $n = 0$. (In the case
($v = 0$, $\mu = 0$), the tilted line is $n + m = 0$.)

$n \geq -m - 1$. The smallest submodule $Z_{s,0,\vartheta}^{\alpha,\beta} \subset F_{s,0,\vartheta}^{\alpha,\beta}$ has the extremal vectors as shown
in figure 9, and can be generated from $V_{s,0,\vartheta}^0[0, 0](z)$.

(4) For $1 \leq s \leq p - 1$ and $r = 0$, it follows from section 4.6.2 that extremal vectors with
$n < 0$ are unreachable from the vectors with $n \geq 0$, and from section 7.3, that extremal vectors with $m < 0$ are unreachable from the vectors with $m \geq 0$. (The picture is similar
to that in figure 9, but with the tilted line becoming the vertical line $m = 0$.) In this case,
a proper submodule $Z_{s,0,\vartheta}^{\alpha,\beta} \subset F_{s,0,\vartheta}^{\alpha,\beta}$ also can be generated from $V_{s,0,\vartheta}^0[0, 0](z)$.

In cases (1), (2), (3), and (4), the respective submodules $Z_{s,0,\vartheta}^{\alpha,\beta}$, $Z_{s,0,\vartheta}^{\alpha,\beta}$, $Z_{s,0,\vartheta}^{\alpha,\beta}$, and $Z_{s,0,\vartheta}^{\alpha,\beta}$
are conjecturally simple $\mathcal{W}$-modules. We also conjecture that the $Z_{s,0,\vartheta}^{\alpha,\beta}$, with $1 \leq s \leq p$,
$0 \leq r \leq p - 1$, $\vartheta \in \mathbb{Z}$, and $\alpha, \beta = \pm$ are all simple modules of $\mathcal{W}$ (we do not define $Z_{p,0,\vartheta}^{\alpha,\beta}$
here; the case $s = p, r = 0$ will be considered elsewhere).

The correspondence with the simple $U(X)$-modules is suggested by the $\Gamma$-action on the
$\mathcal{W}$-modules. The $\Gamma$-generators represented as in (7.1) act on the extremal states as

$$kV_{s,0,\vartheta}[n, m](z) = (-1)^s q^{-s}V_{s,0,\vartheta}^{\alpha,\beta}[n, m](z),$$
$$KV_{s,0,\vartheta}[n, m](z) = (-1)^s q^{-s}V_{s,0,\vartheta}^{\alpha,\beta}[n, m](z),$$

which is to be compared with (6.30). Because $\mathcal{W}$ and $U(X)$ commute, every vector in
$Z_{s,0,\vartheta}^{\alpha,\beta}$ is a highest-weight vector of the $U(X)$-module $Z_{p,0,\vartheta}^{\alpha,\beta}$. This suggests the correspondence
$Z_{s,0,\vartheta}^{\alpha,\beta} \rightarrow Z_{p,0,\vartheta}^{\alpha,\beta}$ (for $\vartheta \in \mathbb{Z}$) between $\mathcal{W}$- and $U(X)$-modules (quite similar to [38, 39]), thus
paving the way toward the Morita equivalence of categories. Moreover, it can be expected that
$\mathcal{W}$-fusion of the $Z_{s,0,\vartheta}^{\alpha,\beta}$ is closely related to tensor products of the $Z_{p,0,\vartheta}^{\alpha,\beta}$ modules.
8. Conclusions and an outlook

The project pursued in this paper is by no means completed at this point. We described two Nichols algebras $\mathcal{B}(X_0)$ and $\mathcal{B}(X_3)$ quite explicitly, and calculated their action on one-vertex Yetter–Drinfeld modules, but we have not yet considered their Yetter–Drinfeld categories in greater detail (including the multivertex modules). We identified generators of extended chiral algebras and found candidates for irreducible representations in the ‘best’ case ($j = 0$ in the asymmetric realization), but we have not yet computed the characters of these representations. We outlined the Hamiltonian reduction of the extended algebras to the previously known one-boson ‘logarithmic’ chiral algebras, but we have not explicitly described how the fields of these last algebras are combined and ‘dressed’ by the additional scalars into the extended algebras discussed in this paper. We derived an ordinary Hopf algebra $U(X)$ from a $\mathcal{B}(X)$ by double bosonization (and noticed that $U(X)$ is ‘the same’ for $\mathcal{B}(X_0)$ and $\mathcal{B}(X_3)$), but we have not yet articulated the functorial properties of the correspondence between $\mathcal{B}(X)$ and $U(X)$, for $X \in \mathcal{YD}$. We noticed an encouraging correspondence between simple $U(X)$ and $W(2, (2p)^{x \times 2})$ modules, but we have not yet stated the expected categorial equivalence, with the spectral flow taken into account.

Further prospects are therefore numerous and can be rather exciting.

1. The characters of representations of the extended algebras constructed here can be calculated relatively straightforwardly; in addition to their ‘nominal’ significance, they may encode some nontrivial combinatorial properties, being related to generation functions of planar partitions subjected to some set of constraints [60].

2. The associative algebra isomorphism $\sigma$ and the twist $\Phi$ in section 6.7 can be regarded as algebraic counterparts of the ‘nonlocal change of variables’ $\rho_0[\cdot] \circ \rho_j^{-1}[\cdot]$ in section 5.5. The similarity may not necessarily be superficial.

3. With the $(p, p')$ logarithmic minimal models [36] deducible by Hamiltonian reduction from a theory with manifest $\hat{sl}(2)$ symmetry, it is interesting how much this last can help in elucidating a number of subtle properties of the $(p, p')$ models [42, 43, 61]. In particular, setting $p = 2$ and $j = 1$ in the ‘asymmetric’ case gives an $\hat{sl}(2)_{-1}$ model (with central charge $-1$), with the underlying $(2, 3)$ minimal model (also see [62]).

4. The case of three fermionic screenings, for which some Nichols algebra details were already worked out in [21] (where the Nichols algebra generators were not called screenings, however), is certainly interesting from the CFT standpoint [63]. In higher rank, moreover, Nichols algebras may depend on several ‘$p$-type’ parameters, which is another interesting possibility of going beyond logarithmic CFTs based on rescalings of classic root lattices.

5. Going beyond finite-dimensional Nichols algebras—extending them by divided powers of nonfermionic generators—is certainly of interest in logarithmic CFTs (cf [64]), and possibly also in the theory of Nichols algebras. A double bosonization of our $\mathcal{B}(X_0)$ and $\mathcal{B}(X_3)$ ‘with divided powers’ can be regarded as a Lusztig-type extension of $U(X)$.

6. Nichols algebras (and their Yetter–Drinfeld modules) have already appeared in [65], in the context of integrable deformations of CFTs. Links with the results in this paper and possible generalizations are quite intriguing.

7. Lattice models related to logarithmic CFTs (see [66–68] and the references therein) are another direction where the ‘braided’ standpoint may be welcome. A spin chain based on our $U(X)$ can also be useful in describing the spin quantum Hall effect [69].

8. An intriguing development is to investigate modular properties of the extended-algebra characters and compare them with the modular group representation realized on the center of the Hopf algebra $U(X)$. Based on our previous experience [54, 36], the
exact coincidence of modular group representations can be expected here (which would
nevertheless be a nontrivial result). An even more ambitious program is to compare the
modular group representation generated from $W$-algebra characters with the one realized
in braided terms, much in the spirit of the recent development in [70, 71]. A higher-genus
mapping class group is closely related to $n$th cohomology spaces of $U(X)$ with coefficients
in $U(X)^{\otimes} \otimes Y_1 \otimes \cdots \otimes Y_M$, where the $Y_i$ (vertex-operator algebra representations and
simultaneously Nichols algebra representations) label boundary conditions imposed at
the holes of the Riemann surface. It would be very interesting to reformulate this in the
framework of the corresponding Nichols algebra (cf the complex constructed in [23]) by
taking its multiple tensor products with itself and with the dual.

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Appendix A. Some constructions and conventions for Nichols algebras

A.1. $\mathfrak{B}(X)$ action and coaction on multivertex Yetter–Drinfeld modules

We use the standard graphic notation for basic Hopf-algebra structures and braiding. The
diagrams are read from top down; the convention for braiding is \( / \) (and \( \backslash \) is the inverse
braiding). All our conventions are fully explained in [23]. A warning (also articulated in [23])
is that diagrams of two types are actually in use: those where a line denotes a Hopf algebra
(such as $\mathfrak{B}(X)$) or its module, etc, and those where a line is a copy of a braided space (such
as $X$).

We let $X = (X, \Psi)$ denote a braided vector space and $\mathfrak{B}(X)$ its Nichols algebra. The
reader can always regard $X$ as an object in a braided category such that the braiding induced
on $X$ coincides with $\Psi$.

For any braided vector spaces $Y_i$ in the same category, there is a Yetter–Drinfeld $\mathfrak{B}(X)$-
module structure on $\mathfrak{B}(X) \otimes Y_1 \otimes \mathfrak{B}(X) \otimes Y_2 \otimes \cdots \otimes \mathfrak{B}(X) \otimes Y_N$, given by the left adjoint action
and by the coaction via deconcatenation up to the first $Y$ space (the $N$-vertex Yetter–Drinfeld
module, considered in more detail in [23]). For one-vertex modules, in particular, the action
and coaction are

$$
\begin{align*}
\begin{array}{c}
\text{and} \\
\end{array}
\end{align*}
\tag{A.1}
$$

where the black horizontal strip indicates that the $\mathfrak{B}(X)$ (co)action applies to the tensor product
as a whole—the tensor product of a copy of $\mathfrak{B}(X)$ (single vertical line) and $V^{(\alpha,\beta)}$ (double
vertical line).
A.2 A (very) brief reminder on $H \#_H \mathcal{YD}$ and Radford’s biproduct formula

A.2.1. For an (ordinary) Hopf algebra $H$ and its module comodule $U$, the left–left Yetter–Drinfeld axiom is

$$(h \triangleright u)_{(-1)} \otimes (h \triangleright u)_{(2)} = h' \triangleright u_{(-1)}, s(h'') \otimes h'' \triangleright u_{(2)},$$

where $h \mapsto h' \otimes h''$ is the $H$ coproduct and $s$ is the antipode of $H$, and $h \triangleright u$ and $\delta : u \mapsto u_{(-1)} \otimes u_{(2)}$ define the left $H$-module and left $H$-comodule structures. The category $H \#_H \mathcal{YD}$ of left–left Yetter–Drinfeld $H$-modules is pre-braided, and braided if $s$ is bijective, with the braiding and its inverse given by

$$\Psi : \mathcal{U} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{U} : u \otimes v \mapsto u_{(-1)} \triangleright v \otimes u_{(2)},$$

$$\Psi^{-1} : \mathcal{V} \otimes \mathcal{U} \to \mathcal{U} \otimes \mathcal{V} : v \otimes u \mapsto v_{(2)} \otimes s^{-1}(v_{(-1)}) \triangleright u. \quad (A.2)$$

A.2.2. For a Hopf-algebra object $\mathcal{R}$ in $H \#_H \mathcal{YD}$, the smash product $\mathcal{R} \# H$ is made into an ordinary Hopf algebra by Radford’s formula [50], dubbed bosonization when rediscovered in [72] (and actually placed into the context of braided categories there). The multiplication in $\mathcal{R} \# H$ is the standard

$$(r \# h)(t \# g) = r(h' \triangleright t) \# h'' g,$$

where $h \triangleright t$ is the left $H$-action on its (Yetter–Drinfeld) modules and $h \rightarrow h' \otimes h''$ is the coproduct of $H$, and Radford’s coproduct is

$$r \# h \rightarrow (r' \# r_{(-1)} h') \otimes (r_{(1)} \# h''),$$

where $r \mapsto r_{(-1)} \otimes r_{(2)}$ is the $H$-coaction and $r \mapsto r \otimes r \in \mathcal{R} \otimes \mathcal{R}$ is the coproduct of $\mathcal{R}$. The bialgebra is furthermore made into a Hopf algebra by defining the antipode

$$S(r \# h) = (1 \# s(r_{(-1)} h))(S(r_{(2)}) \# 1),$$

where $s$ is the antipode of $H$ and $S$ is the antipode of $\mathcal{R}$.

A.2.3. A special case of $H \#_H \mathcal{YD}$ is where $H$ is commutative and cocommutative, $H = k \Gamma$ with a finite Abelian group $\Gamma$. Then Yetter–Drinfeld $H$-modules are just $\Gamma$-graded vector spaces $\mathcal{X} = \bigoplus_{g \in \Gamma} \mathcal{X}_g$ with the left comodule structure $\delta : x \mapsto g \otimes x$ for all $x \in \mathcal{X}_g$, and with $\Gamma$ acting on each $\mathcal{X}_g$. The action is diagonalizable, and hence $\mathcal{X} = \bigoplus_{\chi \in \hat{\Gamma}} \mathcal{X}_\chi$, where $\hat{\Gamma}$ is the group of characters of $\Gamma$ and $\mathcal{X}_\chi = \{ x \in \mathcal{X} \mid g \triangleright x = \chi(g)x \text{ for all } g \in \hat{\Gamma} \}$. Then

$$\mathcal{X} = \bigoplus_{g \in \Gamma, \chi \in \hat{\Gamma}} \mathcal{X}_g^{\chi},$$

where $\mathcal{X}_g^{\chi} = \mathcal{X}_\chi \cap \mathcal{X}_g$. Therefore, each Yetter–Drinfeld $H$-module $\mathcal{X}$ has a basis $(x_a)$ such that, for some $g_a \in \Gamma$ and $\chi_a \in \hat{\Gamma}$, $\delta x_a = g_a \otimes x_a$ and $g \triangleright x_a = \chi_a(g)x_a$ for all $g$. Braiding (A.2) then becomes $\Psi : x_a \otimes x_b \mapsto \chi_b(g_a)x_b \otimes x_a$. For the Nichols algebra generators $F_i \in \mathcal{X}$, in particular, with

$$\delta F_i = g_i \otimes F_i \quad \text{and} \quad g \triangleright F_i = \chi_i(g)F_i,$$

we recover braiding (1.13) with $q_{i,k} = \chi_i(g_k)$.  

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Appendix B. Twisted Verma modules and singular vectors of \( \hat{\mathfrak{s}\ell}(2) \)

B.1. The \( \hat{\mathfrak{s}\ell}(2) \) algebra

Our conventions for the \( \hat{\mathfrak{s}\ell}(2) \) algebra are

\[
\begin{align*}
[J^0_m, J^\pm_n] &= \pm j^\pm_{m+n}, \\
[J^0_m, J^0_n] &= \frac{k}{2} m \delta_{m+n,0}, \\
[J^+_m, J^-_n] &= km \delta_{m+n,0} + 2 j^0_{m+n},
\end{align*}
\]

where \( j^\pm_0(z) = \sum_{m \in \mathbb{Z}} j^\pm_m z^{-n-1} \). The zero modes \( j^\pm_0 \) generate an \( \mathfrak{s}\ell(2) \) Lie algebra, which we call the zero-mode \( \hat{\mathfrak{s}\ell}(2) \) to distinguish it from other classical and quantum \( \mathfrak{s}\ell(2) \) algebras.

The Sugawara energy–momentum tensor constructed from the \( \hat{\mathfrak{s}\ell}(2) \) currents is

\[
T_{\text{Sug}}(z) := \frac{1}{2(k+2)} (J^+ J^- (z) + J^- J^+ (z) + j^0 j^0 (z)). \tag{B.1}
\]

Several different energy–momentum tensors occur in the text, and we refer to dimensions of (primary) fields determined by the OPE with \( T_{\text{Sug}}(z) \) as the Sugawara dimensions.

B.2. Twisted Verma modules of \( \hat{\mathfrak{s}\ell}(2) \)

We fix our conventions regarding twisted Verma modules [73]. For \( \lambda \in \mathbb{C} \) and \( \vartheta \in \mathbb{Z} \), the twisted Verma module \( \mathcal{M}_{\lambda, \vartheta} \) is freely generated by \( J^+_{\leq \vartheta-1}, J^-_{\leq \vartheta}, \) and \( j^0_0 \) from a twisted highest-weight vector \( |\lambda; \vartheta\rangle \) defined by the conditions

\[
\begin{align*}
J^+_\vartheta |\lambda; \vartheta\rangle &= j^+_{\vartheta+1} |\lambda; \vartheta\rangle = J^-_{\vartheta-1} |\lambda; \vartheta\rangle = 0, \\
\left( j^0_0 + \frac{k}{2} \vartheta \right) |\lambda; \vartheta\rangle &= \lambda |\lambda; \vartheta\rangle.
\end{align*}
\]

Setting \( \vartheta = 0 \) gives the usual ("untwisted") Verma modules. We write \( |\lambda\rangle = |\lambda; 0\rangle \) and, similarly, \( \mathcal{M}_\lambda = \mathcal{M}_{\lambda, 0} \). The highest-weight vector \( |\lambda; \vartheta\rangle \) of a twisted Verma module has the Sugawara dimension

\[
\Delta_{\lambda, \vartheta} = \frac{\lambda^2 + \lambda k + 2}{k+2} - \vartheta \lambda + \frac{k}{4} \vartheta^2. \tag{B.3}
\]

We write \( |\alpha\rangle \equiv |\lambda; \vartheta\rangle \) whenever conditions (B.2) are satisfied for a state \( |\alpha\rangle \).

We introduce the (charge, dimension) bigrade for vectors in a Verma module \( \mathcal{M}_\lambda \) in an obvious way, by assigning the grade \( (\lambda, \Delta_\lambda) \) to the highest-weight vector \( |\lambda\rangle \) and setting \( \text{gr} \lambda^+_{\vartheta-n} = (1, n) \), \( \text{gr} \lambda^-_{\vartheta-n} = (-1, n) \), and \( \text{gr} j^0_{n} = (0, n) \). Then, e.g., \( \text{gr} f^+_{\vartheta} |\lambda\rangle = (\lambda + 1, \Delta_\lambda + 1) \). For the twisted highest-weight state \( |\lambda; \vartheta\rangle \), we have

\[
\text{gr} |\lambda; \vartheta\rangle = \left( \lambda, -\vartheta \frac{k}{4}, \Delta_{\lambda, \vartheta} \right).
\]

Twists, although producing nonequivalent modules, do not alter the submodule structure, and we can therefore reformulate a classic result as follows.

B.3.

**Theorem** ([74, 31]). A singular vector exists in the twisted Verma module \( \mathcal{M}_{\lambda, \vartheta} \) of \( \hat{\mathfrak{s}\ell}(2)_k \) if and only if \( \lambda \) can be written as \( \lambda = \lambda^+ (r, s) \) or \( \lambda = \lambda^- (r, s) \) with \( r, s \in \mathbb{N} \), where

\[
\begin{align*}
\lambda^+ (r, s) &= \frac{r-1}{2} - (k+2) \frac{s-1}{2}, \quad &\lambda^- (r, s) &= -\frac{r+1}{2} + (k+2) \frac{s}{2}.
\end{align*}
\]
Whenever \( \lambda = \lambda^+(r, s) \) or \( \lambda = \lambda^-(r, s) \), the corresponding singular vector is given by

\[
\mathbf{s}^+ (r, s; \vartheta) = (J_{\vartheta}^-)_{r+1} (J_{\vartheta}^-)_{r+2} \cdots (J_{\vartheta}^-)_{r+k} \mathbf{s}^+ (J_{\vartheta}^-)^{r+3(k+2)} \mathbf{s}^+ (J_{\vartheta}^-)^{r+4(k+2)} \mathbf{s}^+ \cdots
\]

or

\[
\mathbf{s}^- (r, s; \vartheta) = (J_{\vartheta}^-)_{r-1} (J_{\vartheta}^-)_{r+1} \cdots (J_{\vartheta}^-)_{r+k} \mathbf{s}^- (J_{\vartheta}^-)^{r+3(k+2)} \mathbf{s}^- (J_{\vartheta}^-)^{r+4(k+2)} \mathbf{s}^- \cdots
\]

(B.4)

The dependence of \( \lambda^\pm (r, s) \) on \( k \) is not indicated for the brevity of notation.

B.3.1. We recall that the above formulas yield polynomial expressions in the currents via repeated application of the formulas

\[
(J_0^+) \lambda = -\alpha (\alpha - 1) J_0^- \lambda (J_0^-)^{\alpha - 2} - 2 \alpha J_0^0 \lambda (J_0^-)^{\alpha - 1} + J_0^+ \lambda (J_0^-)^{\alpha},
\]

\[
(J_0^-) \lambda = -\alpha (\alpha - 1) J_0^+ \lambda (J_0^+)^{\alpha - 2} - 2 \alpha J_0^- \lambda (J_0^+)^{\alpha - 1} + J_0^- \lambda (J_0^+)^{\alpha},
\]

(B.6)

and, when necessary, of their images under the Lie algebra homomorphism \( J_0^+ \mapsto J_{\vartheta}^+, \quad J_0^- \mapsto J_{\vartheta}^- \), \( J_0^0 \mapsto J_{\vartheta}^0 + \frac{1}{k} \vartheta \delta_{k, 0}, \quad J_{\vartheta}^0 \mapsto J_{\vartheta}^- \), Formulas (B.6) are derived for positive integer \( \alpha \) and are then continued to arbitrary complex \( \alpha \).

B.3.2. Singular vectors \( \mathbf{s}^\pm (r, s; \vartheta) = \mathbf{s}^\pm (r, s; \vartheta | \lambda) \) constructed on a twisted highest-weight state \( | \lambda; \vartheta \rangle \) lie in the grades

\[
\text{gr} \mathbf{s}^+ (r, s; \vartheta | \lambda) = \left( \lambda - r - \frac{k}{2} \vartheta, \Delta_{\lambda; \vartheta} + r(s - 1 + \vartheta) \right),
\]

\[
\text{gr} \mathbf{s}^- (r, s; \vartheta | \lambda) = \left( \lambda + r - \frac{k}{2} \vartheta, \Delta_{\lambda; \vartheta} + r(s - \vartheta) \right).
\]

B.3.3. For \( s = 1 \), the above singular vectors do not require any algebraic rearrangements and take the simple form

\[
\mathbf{s}^+ (r, 1; \vartheta | \lambda) = (J_{\vartheta}^-)^r | \lambda; \vartheta \rangle, \quad \mathbf{s}^- (r, 1; \vartheta | \lambda) = (J_{\vartheta}^0)^r | \lambda; \vartheta \rangle.
\]

Appendix C. Triplet–triplet algebra: examples

C.1. Asymmetric realization

To give examples of \( \mathbf{W} (2, (2p)^{3\times 3}) \) algebras, we first recall that the \( \mathcal{W}^+ (\zeta) \) generator is always given by

\[
\mathcal{W}^+ (\zeta) = \mathbf{e}^{\partial \theta / \partial \zeta} + \mathbf{e}^{-\partial \theta / \partial \zeta}
\]

and the entire multiplet in the left part of figure 2 is

\[
J_0^- \mathcal{W}^+ (\zeta) = (\partial \psi (\zeta) + \partial a (\zeta)) \mathbf{e}^{\partial \theta / \partial \zeta},
\]

\[
(J_0^-)^2 \mathcal{W}^+ (\zeta) = (\frac{1}{2} \partial \psi \partial \psi (\zeta) - \partial^2 \psi (\zeta) + \partial \psi \partial a (\zeta) + \frac{1}{2} \partial a \partial a (\zeta) - \partial^2 a (\zeta)) \mathbf{e}^{\partial \theta / \partial \zeta} + \frac{1}{2} \mathbf{e}^{-\partial \theta / \partial \zeta} (\zeta + a (\zeta)).
\]
C.1.1. For \( p = 2 \), the other two fields in (4.14) are given by (omitting the conventional (\( z \)) argument of gradients of the scalars)

\[
\begin{align*}
\mathcal{W}^0(z) &= \left( -\frac{1}{3} \partial^3 \varphi + 2 \partial^2 \varphi \partial \varphi - \frac{1}{3} (\partial \varphi)^3 \right) e^{-\frac{1}{8} \chi(z)} e^{-\frac{i}{2} a(z)}, \\
\mathcal{W}^-(z) &= (8 \partial^2 \varphi + 8(\partial \varphi)^2) e^{-2\varphi(z)} - \frac{1}{8} \chi(z) e^{-\frac{i}{2} a(z)}.
\end{align*}
\]

Also for \( p = 2 \), the middle terms in the two zero-mode triplets are given by

\[
\begin{align*}
J_0 \mathcal{W}^0(z) &= (\partial^2 \varphi)^2 + 3 \partial^3 \varphi \partial \varphi - 2 \partial^2 \varphi (\partial \varphi)^2 + \left[ -\frac{1}{3} \partial^3 \varphi + 2 \partial^2 \varphi \partial \varphi - \frac{4}{3} (\partial \varphi)^3 \right] \partial a - \frac{1}{6} \partial^4 \varphi
\end{align*}
\]

and

\[
\begin{align*}
J_0 \mathcal{W}^-(z) &= (4 \partial^3 \varphi + 8 \partial^2 \varphi \partial a - 8(\partial \varphi)^3 + 8(\partial \varphi)^2 \partial a) e^{-2\varphi(z)},
\end{align*}
\]

and the leftmost terms, by somewhat lengthier expressions.

C.1.2. For \( p = 3 \), the \( \mathcal{W}^0(z) \) and \( \mathcal{W}^-(z) \) fields in (4.14) are

\[
\begin{align*}
\mathcal{W}^0(z) &= \left( -\frac{1}{10} \partial^5 \varphi + \frac{3}{4} \partial^3 \varphi \partial^2 \varphi + \frac{3}{8} \partial^4 \varphi \partial \varphi - \frac{25}{8} (\partial^2 \varphi)^2 \partial \varphi - \frac{9}{4} \partial^3 \varphi (\partial \varphi)^2 \\
&\quad + \frac{25}{4} \partial^2 \varphi (\partial \varphi)^3 - \frac{81}{20} (\partial \varphi)^5 \right) e^{-\frac{1}{8} \chi(z)} e^{-\frac{i}{2} a(z)},
\end{align*}
\]

\[
\begin{align*}
\mathcal{W}^-(z) &= \left( \partial^3 \varphi - \frac{117}{8} (\partial^2 \varphi)^2 + 27 \partial^3 \varphi \partial \varphi - 54 \partial^2 \varphi (\partial \varphi)^2 - \frac{81}{2} (\partial \varphi)^4 \right) e^{-3\varphi(z)} - \frac{1}{8} \chi(z) e^{-\frac{i}{2} a(z)}.
\end{align*}
\]

C.2. Symmetric realization

For the ‘symmetric’ realization in section 5, the expanded expressions for the algebra generators are much more bulky than for the ‘asymmetric’ realization, and we therefore restrict ourself to only \( p = 2 \) for illustration.

The field \( \mathcal{W}^-(z) \) (see (5.9)) is

\[
\begin{align*}
\mathcal{W}^-(z) &= \left( -\partial^3 \varphi_1 + 42 (\partial^2 \varphi_1)^2 + 24 \partial^2 \varphi_1 \partial^2 \varphi_2 - 24 \partial^3 \varphi_1 \partial \varphi_1 - 12 \partial^3 \varphi_1 \partial \varphi_2 \\
&\quad + 24 (\partial \varphi_1)^2 \partial^2 \varphi_2 + 24 \partial^2 \varphi_1 (\partial \varphi_1)^2 + 24 \partial^2 \varphi_1 \partial \varphi_1 \partial \varphi_2 + 24 \partial^2 \varphi_1 (\partial \varphi_2)^2 \\
&\quad + 24 (\partial \varphi_1)^2 + 48 (\partial \varphi_1)^3 \partial \varphi_2 + 24 (\partial \varphi_1)^2 (\partial \varphi_2)^2 \right) e^{-\frac{1}{8} \chi(z)} e^{-\frac{i}{2} a(z)}.
\end{align*}
\]

It looks simpler in the Wakimoto realization in section 5.5

\[
\begin{align*}
\mathcal{W}^-(z) &= (18 a \beta \beta - 12 \partial^2 \beta \beta - 24 \partial f \partial \beta \beta + 24 \partial^2 f \beta \beta + 24 \partial f \partial f \beta \beta) e^{-2f(z)}.
\end{align*}
\]

Also, the middle element of the ‘middle’ triplet is explicitly given by

\[
\begin{align*}
J_0 \mathcal{W}^0(z) &= 8(\partial \varphi_1)^4 + 16 (\partial \varphi_1)^3 \partial \varphi_2 - 16 \partial \varphi_1 (\partial \varphi_2)^3 + 24 \partial \varphi_1 \partial^2 \varphi_2 \partial \varphi_2 - 4 \partial \varphi_1 \partial \varphi_2 \\
&\quad - 8(\partial \varphi_2)^4 - 24 \partial^2 \varphi_1 (\partial \varphi_1)^2 - 24 \partial^2 \varphi_1 \partial \varphi_1 \partial \varphi_2 + 6(\partial^2 \varphi_1)^2 + 24 \partial^2 \varphi_2 (\partial \varphi_2)^2 \\
&\quad - 6(\partial^2 \varphi_2)^2 + 8 \partial^3 \varphi_1 \partial \varphi_1 + 4 \partial^3 \varphi_1 \partial \varphi_2 - 8 \partial^3 \varphi_2 \partial \varphi_2 - \partial^4 \varphi_1 + \partial^4 \varphi_2.
\end{align*}
\]

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