A sufficient condition for a graph with boxicity at most its chromatic number

∗Akira Kamibeppu

Abstract

A box in Euclidean $k$-space is the Cartesian product of $k$ closed intervals on the real line. The boxicity of a graph $G$, denoted by box($G$), is the minimum nonnegative integer $k$ such that $G$ can be isomorphic to the intersection graph of a family of boxes in Euclidean $k$-space. In this paper, we present a sufficient condition for a graph $G$ under which box($G$) ≤ $\chi(G)$ holds, where $\chi(G)$ denotes the chromatic number of $G$. Bhowmick and Chandran [2] proved that box($G$) ≤ $\chi(G)$ holds for a graph $G$ with no asteroidal triples. We prove that box($G$) ≤ $\chi(G)$ holds for a graph $G$ in a special family of circulant graphs with an asteroidal triple.

Keywords: boxicity; chromatic number; maximum degree; interval graph; split graph.

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1 Introduction and Preliminaries

A box in Euclidean $k$-space is the Cartesian product of $k$ closed intervals on the real line. The boxicity of a graph $G$, denoted by box($G$), is the minimum nonnegative integer $k$ such that $G$ can be isomorphic to the intersection graph of a family of boxes in Euclidean $k$-space. The concept of boxicity of graphs was introduced by Roberts [13]. It has applications in some research fields, for example, a problem of niche overlap in ecology (see [14] for detail). It is known that boxicity of graphs have relationships with some graph invariants. Chandran, Francis and Sivadasan [3] proved that box($G$) ≤ $2\Delta_G^2$ holds for a graph $G$, where $\Delta_G$ is the maximum degree of $G$. In fact, they also showed in the proof that box($G$) ≤ $2\chi(G^2)$ holds for a graph $G$, where $G^2$ is the graph obtained from $G$ by adding edges whose endvertices have common neighbors in $G$ and the symbol $\chi(G^2)$ means the chromatic number of $G^2$. Moreover they conjectured that box($G$) is $O(\Delta_G)$. Esperet [9] improved the previous upper bound for boxicity, that is, proved that box($G$) ≤ $\Delta_G^2 + 2$ holds for a graph $G$. Adiga, Bhowmick and Chandran [1] disproved the above conjecture. In fact, they proved that there exist graphs with boxicity $\Omega(\Delta_G \log \Delta_G)$. Before these results appear, Chandran and Sivadasan [8] presented chordal graphs, circular arc graphs, circular-arc graphs, and chordal bipartite graphs.

∗Academic Assembly Institute of Science and Engineering, Shimane University, Matsue, Shimane 690-8504, Japan.
E-mail address: kamibeppu@riko.shimane-u.ac.jp
astroidal triple free graphs, co-comparability graphs and permutation graphs as examples whose boxicity are bounded above by a linear function of their maximum degree. These results are based on their main result which states box(G) ≤ tw(G) + 2 for a graph G, where tw(G) is the treewidth of G.

So far some researchers found relationships between boxicity and chromatic number. Chandran, Das and Shah [3] proved that if the boxicity of a graph G with n vertices is equal to n/2 − s for s ≥ 0, then χ(G) ≥ n/(2s + 2) holds. This result implies that if the boxicity of a graph is close to n/2, then its chromatic number is large. Esperet [10] proved that box(G) ≤ χ_a(G)(χ_a(G) − 1) holds for a graph G with χ_a(G) ≥ 2, where χ_a(G) is the acyclic chromatic number of G. Bhowmick and Chandran [2] proved that box(G) ≤ χ(G) holds for a graph G with no asteroidal triples. We remark that box(G) ≤ χ(G) does not hold in general. We consider the graph H obtained from a balanced complete bipartite graph with at least 10 vertices by removing a perfect matching. Then box(H) > 2 = χ(H) holds (see [3] for detail). Recently, Chandran Chandran, Mathew and Rajendraprasad [6] remarked that almost all graphs have boxicity more than their chromatic number, which is based on the probabilistic method, but the family of graphs with boxicity at most their chromatic number is not narrow. In this paper, we present a sufficient condition for a graph G under which box(G) ≤ χ(G) holds. Moreover we show that box(G) ≤ χ(G) holds for a graph G in a special family of circulant graphs with an asteroidal triple. Other results appear in [5, 7].

All graphs are finite, simple and undirected in this paper. We use V(G) for the vertex set of a graph G and E(G) for the edge set of the graph G. An edge of a graph with endvertices u and v is denoted by uv. Then u is called a neighbor of v (or v is called a neighbor of u). The set of all neighbors of a vertex v in G is denoted by N_G(v), or briefly by N(v). A map c : V(G) → {1, . . . , k} is called a k-coloring of a graph G if c(u) ̸= c(v) holds whenever uv is an edge of G. The chromatic number of G, denoted by χ(G), is the minimum positive integer k such that the graph G has a k-coloring. A k-coloring of G gives a partition of V(G) into k independent sets, called color classes, so we use the notation \{V_i\}_{i=1}^k for a k-coloring of G with color classes V_1, . . . , V_k.

The following is a useful result to calculate boxicity of graphs.

**Theorem 1.1** ([13]). Let G be a graph. Then box(G) ≤ k holds if and only if there exist k interval graphs H_1, . . . , H_k with vertex set V(G) such that E(G) = E(H_1) ∩ · · · ∩ E(H_k) holds.

## 2 Split Interval Graphs

For a k-coloring \{V_i\}_{i=1}^k of a graph G, we will consider a supergraph H_i of G which accompanies a color class V_i so that E(G) = E(H_1) ∩ · · · ∩ E(H_k) holds. Nonadjacent vertices of H_i might have at least two common neighbors in V_i, which causes induced cycles of H_i, so making the set V(H_i) \ V_i complete is a natural way to avoid their cycles. The resulting graph is called a split graph (see Theorem 2.1 below).

A graph is chordal if the graph contains no induced cycles other than triangles. A graph is called a split graph if the graph and its complement are chordal. The following is a characterization of split graphs.
Theorem 2.1 ([11]). A graph $G$ is a split graph if and only if $V(G)$ can be partitioned into an independent set and a clique.

In this paper, $P_{uv}$ denotes a path between vertices $u$ and $v$ of a graph $G$, if exists. A triple of vertices $u$, $v$ and $w$ of $G$ is said to be asteroidal if there exist paths $P_{uw}$, $P_{uv}$ and $P_{vu}$ in $G$ such that $N_G(w) \cap V(P_{uw}) = \emptyset$, $N_G(u) \cap V(P_{vw}) = \emptyset$ and $N_G(v) \cap V(P_{wu}) = \emptyset$ hold. We introduce a characterization of interval graphs.

Theorem 2.2 ([12]). A graph $G$ is an interval graph if and only if $G$ is chordal and has no asteroidal triples.

Lemma 2.3. Let $G$ be a split graph with a partition of $V(G)$ into an independent set $S$ and a clique $K$. If for any triple of vertices $u$, $v$ and $w$ in $S$ two vertices $x$ and $y$ in \{ $u$, $v$, $w$ \} satisfy $N_G(x) \supseteq N_G(y)$, the graph $G$ is an interval graph.

Proof. It is sufficient by Theorem 2.2 to show that any triple of vertices of the split graph $G$ is not asteroidal since the family of split graphs is a special class of chordal graphs. For any triple of vertices $u$, $v$ and $w$ of $G$, if one of them, say $u$, is in the clique $K$, then $N_G(u)$ contains vertices of any path $P_{vw}$. So in this case the triple of vertices is not asteroidal. Hence we may assume that the triple of vertices $u$, $v$ and $w$ of $G$ are in the independent set $S$. By our assumption, if $N_G(u) \supseteq N_G(v)$ holds, $N_G(u)$ contains a vertex in any path $P_{vw}$. Hence our assumption tells us that the triple is also not asteroidal. □

3 Our Results

The following is our main result in this paper.

Theorem 3.1. If a graph $G$ has a $\chi(G)$-coloring $\{V_i\}_{i=1}^{\chi(G)}$, where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n(i)}\}$, with the property that, for each vertex $v_{i,j}$ in $V(G)$, there exists a subset $X_{i,j}$ of $V(G) \setminus V_i$ containing $N_G(v_{i,j})$ such that

(i) $X_{i,1} \supseteq \cdots \supseteq X_{i,k(i)}$ and $X_{i,k(i)+1} \subseteq \cdots \subseteq X_{i,n(i)}$ hold for each $i$, where $k(i) \in \{1, 2, \ldots, n(i)\}$, and

(ii) either $v_{i,s} \notin X_{j,t}$ or $v_{j,t} \notin X_{i,s}$ holds for any pair of nonadjacent vertices $v_{i,s}$ and $v_{j,t}$ in $G$,

then the inequality $\text{box}(G) \leq \chi(G)$.

Note. In (i), ‘$k(i) = n(i)$’ means that only the sequence $X_{i,1} \supseteq \cdots \supseteq X_{i,n(i)}$ is required. It also has a similar meaning in Corollary 3.3 below.

Proof. Let $\{V_i\}_{i=1}^{\chi(G)}$ be a $\chi(G)$-coloring of the graph $G$ with the property. We define the supergraph $H_i$ of $G$ for each color $i \in \{1, 2, \ldots, \chi(G)\}$ as follows:

$$V(H_i) = V(G), \ E(H_i) = E(G) \cup \bigcup_{k=1}^{n(i)} \{v_{i,k}x \mid x \in X_{i,k}\} \cup \{xy \mid x, y \in V(G) \setminus V_i\}. $$
By construction the graph $H_i$ is a split graph with the partition of $V(H_i)$ into the independent set $V_i$ and the clique $V(G) \setminus V_i$ for each $i \in \{1, 2, \ldots, \chi(G)\}$. For any triple of vertices $v_{i,p}$, $v_{i,q}$ and $v_{i,r}$ in $V_i$, we can find two vertices $v_{i,s}$ and $v_{i,t}$ in $\{v_{i,p}, v_{i,q}, v_{i,r}\}$ such that $X_i,s \supseteq X_i,t$ by condition (i). Hence we see that $H_i$ is an interval graph by Lemma 2.3 since $X_i,j$ is the same as $N_{H_i}(v_{i,j})$.

We note that each edge of $G$ is an edge of $H_i$ for each $i \in \{1, 2, \ldots, \chi(G)\}$ by definition. Let $u$ and $v$ be nonadjacent vertices in $G$. We write $u = v_{i,k}$ and $v = v_{j,l}$ under the coloring $\{V_i\}_{i=1}^{\chi(G)}$. If $i = j$, then $u$ and $v$ are in $V_i$, and hence they are not adjacent in $H_i$. In what follows, we assume $i \neq j$. By condition (ii), $u$ and $v$ are nonadjacent in $H_i$ if $v_{j,l} \notin X_i,k$ holds, otherwise $u$ and $v$ are nonadjacent in $H_j$ since $v_{i,k} \notin X_{j,l}$ holds. Hence we have $E(G) = E(H_i) \cap \cdots \cap E(H_{\chi(G)})$. This completes the proof of our theorem by Theorem 1.1.

Every optimal vertex coloring $c$ of a complete multipartite graph satisfies the condition of Theorem 3.1. For each vertex $v$ of the graph, take a set $X_{c(v),v}$ as the set of all neighbors of $v$. Paths and cycles also have an optimal coloring with the condition of Theorem 3.1. Needless to say, we see that the boxicity of these graphs are at most their chromatic number respectively without applying Theorem 3.1.

**Remark 3.2.** The family of graphs with vertex colorings that satisfy the condition of Theorem 3.1 is not contained by the family of graphs with no asteroidal triples. Consider the graph in Figure 1. It is easy to check that the triple of vertices with degree 1 in the graph forms an asteroidal triple. Under the vertex coloring $c$ of the graph in Figure 1, we define $X_{c(v),v}$ as the set of all neighbors of $v$.

![Figure 1: A graph with an asteroidal triple.](image)

The condition which the colorings of complete multipartite graphs or the graph in Figure 1 satisfy is stronger than the condition which Theorem 3.1 claims.

**Corollary 3.3.** If a graph $G$ has a $\chi(G)$-coloring $\{V_i\}_{i=1}^{\chi(G)}$ with the property that, for each color $i \in \{1, 2, \ldots, \chi(G)\}$,

$$N_G(v_{i,1}) \supseteq \cdots \supseteq N_G(v_{i,k(i)+1}) \subseteq \cdots \subseteq N_G(v_{i,n(i)})$$

hold, where $V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n(i)}\}$ and $k(i) \in \{1, 2, \ldots, n(i)\}$, the inequality box($G$) $\leq \chi(G)$ holds.

**Proof.** Define $X_{i,k}$ as $N_G(v_{i,k})$ for each vertex $v_{i,k} \in V(G)$. For any pair of nonadjacent vertices $v_{i,k}$ and $v_{j,l}$, we note that $v_{i,k}$ is not in $N_G(v_{j,l})$ (and $v_{j,l}$ is also not in $N_G(v_{j,l})$). The corollary follows from Theorem 3.1. □

4
4 Boxicity of Circulant Graphs

In this section we present a family of graphs with an asteroidal triple and show that box(G) ≤ χ(G) for a graph G in the family. We define the graph G_{a,b}, where a ≥ 2b, as follows: V(G_{a,b}) = \{0, 1, \ldots, a - 1\} and uv ∈ E(G_{a,b}) if and only if u ∈ \{v + b, v + b + 1, \ldots, v + a - b\} with addition modulo a. For example, G_{a,1} and G_{a,2} are isomorphic to a complete graph and the complement of a cycle respectively. The graph G_{a,b} is a member of the family of circulant graphs. It is known that α(G_{a,b}) = b and χ(G_{a,b}) ≥ a/b hold (see [15], page 44 for detail), where α(G_{a,b}) denotes the independence number of G_{a,b}.

**Theorem 4.1.** For a circulant graph G_{nb,b} with n ≥ 2 and b ≥ 1, the inequality box(G_{nb,b}) ≤ χ(G_{nb,b}) holds.

**Proof.** We arrange all vertices 0, 1, \ldots, nb - 1 of G_{nb,b} clockwise in this order. We see that every set of consecutive b vertices forms a maximum independent set of G_{nb,b}. We also note that n = nb/b ≤ χ(G_{nb,b}) ≤ n, that is, χ(G_{nb,b}) = n holds since V(G_{nb,b}) can be partitioned into n maximum independent sets with b vertices.

We write v_{i,j} for the vertex \((i - 1)b + j - 1\) in V(G_{nb,b}), where \(i \in \{1, 2, \ldots, n\}\) and \(j \in \{1, \ldots, b\}\). We note that v_{0,j} and v_{n+1,j} are identified with v_{n,j} and v_{1,j} in G_{nb,b} respectively. Let V_{i} be the set \(\{v_{i,1}, \ldots, v_{i,b}\}\). We see that \(\{V_{i}\}_{i=1}^{n}\) becomes an n-coloring of G_{nb,b}. We also see that

\[ N(v_{i,j}) = \{v_{i-1,1}, \ldots, v_{i-1,j}\} \cup \{v_{i+1,j}, \ldots, v_{i+1,b}\} \cup U_{i} \]

holds where \(U_{i} = V(G_{nb,b}) \setminus (V_{i-1} \cup V_{i} \cup V_{i+1})\). We consider cases whether b is even or not. In either case our statement follows from Theorem 3.1.

**Case (I):** We assume that b is even.

We define the subset \(X_{i,j}\) of \(V(G_{nb,b}) \setminus V_{i}\) for the vertex \(v_{i,j}\) of G_{nb,b} as follows:

\[ X_{i,j} = \begin{cases} N(v_{i,j}) \cup \{v_{i-1,j}, \ldots, v_{i-1,b/2}\} & \text{if } 1 \leq j \leq b/2, \\ N(v_{i,j}) \cup \{v_{i+1,b/2+1}, \ldots, v_{i+1,j}\} & \text{if } b/2 < j \leq b. \end{cases} \]

We can check that

\[ X_{i,1} \supseteq \cdots \supseteq X_{i,b/2} \supseteq X_{i,b/2+1} \supseteq \cdots \supseteq X_{i,b} \]

hold for each color i.

We remark that the vertex \(v_{i,j}\) of G_{nb,b} is not adjacent to every vertex in

\[ \{v_{i-1,j+1}, \ldots, v_{i-1,b}\} \cup V_{i} \cup \{v_{i+1,j+1}, \ldots, v_{i+1,j-1}\} \]

for \(2 \leq j \leq b - 1\). The vertex \(v_{i,1}\) is not adjacent to every vertex in \((V_{i-1} \setminus \{v_{i-1,1}\}) \cup V_{i}\) and \(v_{i,b}\) is not adjacent to every vertex in \(V_{i} \cup (V_{i+1} \setminus \{v_{i+1,b}\})\).

**Subcase (I-i):** We assume that \(1 \leq j \leq b/2\) holds.

Note that \(v_{i,j}\) and \(v_{i-1,k}\) are nonadjacent in G_{nb,b}, where \(j + 1 \leq k \leq b\). If \(b/2 + 1 \leq k \leq b\), then \(v_{i-1,k} \notin X_{i,j}\) holds. If \(j + 1 \leq k \leq b/2\), then we can check that \(v_{i,j} \notin X_{i-1,k}\) holds.
We also note that \( v_{i,j} \) and \( v_{i+1,j} \) are nonadjacent, where \( 1 \leq l \leq j - 1 \) and \( j \neq 1 \). Then \( v_{i+1,l} \notin X_{i,j} \) holds.

**Subcase (I-ii):** We assume that \( b/2 < j \leq b \) holds.

Note that \( v_{i,j} \) and \( v_{i-1,k} \) are nonadjacent in \( G_{nb,b} \), where \( j + 1 \leq k \leq b \) and \( j \neq b \). Then \( v_{i-1,k} \notin X_{i,j} \) holds.

Note that \( v_{i,j} \) and \( v_{i+1,l} \) are also nonadjacent, where \( 1 \leq l \leq j - 1 \). If \( 1 \leq l \leq b/2 \), then \( v_{i+1,l} \notin X_{i,j} \) holds. If \( b/2 < l \leq j - 1 \), we see that \( v_{i,j} \notin X_{i+1,l} \) holds.

**Case (II):** We assume that \( b \) is odd.

There is no essential difference between Case (I) and (II). The difference is only the definition of the subset \( X_{i,j} \) of \( V(G_{nb,b}) \setminus V_i \) by subscript. The reader will be able to check that

\[
X_{i,1} \supseteq \cdots \supseteq X_{i,[b/2]-1} \supseteq X_{i,[b/2]} \subseteq X_{i,[b/2]+1} \subseteq \cdots \subseteq X_{i,b}
\]

holds under the following definition of the subset \( X_{i,j} \) of \( V(G_{nb,b}) \setminus V_i \) for the vertex \( v_{i,j} \) of \( G_{nb,b} \):

\[
X_{i,j} = \begin{cases} 
N(v_{i,j}) \cup \{v_{i-1,j}, \ldots, v_{i-1,[b/2]}\} & \text{if } 1 \leq j \leq \lceil b/2 \rceil, \\
N(v_{i,j}) \cup \{v_{i+1,[b/2]}, \ldots, v_{i+1,j}\} & \text{if } \lceil b/2 \rceil < j \leq b.
\end{cases}
\]

The reader also see in the same way that either \( v_{i,j} \notin X_{s,t} \) or \( v_{s,t} \notin X_{i,j} \) holds for any pair of nonadjacent vertices \( v_{i,j} \) and \( v_{s,t} \) in \( G_{nb,b} \).

We remark that a circulant graph \( G_{a,b} \) has an asteroidal triple for \( a \geq 3b \) and \( b \geq 3 \). In fact, the triple of vertices \( u = 1, v = \lceil b/2 \rceil \) and \( w = b \) becomes an asteroidal triple. We can find paths \( P_{uv} \) with green edges, \( P_{vu} \) with blue edges and \( P_{uw} \) with red edges that we desire as in Figure 2 below.

![Figure 2: The graph \( G_{a,b} \) with an asteroidal triple.](image)

Next we consider a circulant graph \( G_{a,b} \) for the other cases \( a = nb + r \) and \( 1 \leq r < b \). We partition \( V(G_{nb+r,b}) \) into \( n \) sets with consecutive \( b \) vertices and a set with consecutive \( r \) vertices so that we have an \((n+1)\)-coloring of \( G_{nb+r,b} \). When we take the same strategy in Theorem 4.1 for this \((n+1)\)-coloring, the resulting family \( \{X_{i,j}\} \) of sets does not satisfy condition (ii) of Theorem 3.1 except for the case \( r = b-1 \) because it has a color class with less than \( b-1 \) vertices. See Figure 3 and 4 below, where two vertices connected with the dashed line segment mean a pair of nonadjacent vertices in \( G_{nb+r,b} \). Let \( V_i \) be the unique color class with \( r \) vertices. For example in Figure 3 with the assumption \( r < \lceil b/2 \rceil \), we notice that
\( \bullet v_{i-1, \lfloor b/2 \rfloor + r} \) and \( v_{i+1, \lfloor b/2 \rfloor - r} \) are nonadjacent in \( G_{nb+r,b} \).

\( \bullet v_{i-1, \lfloor b/2 \rfloor + r} \in X_{i+1, \lfloor b/2 \rfloor - r} \) and \( v_{i+1, \lfloor b/2 \rfloor - r} \in X_{i-1, \lfloor b/2 \rfloor + r} \) hold.

Figure 3: The graph \( G_{nb+r,b} \) with \( r < \lfloor b/2 \rfloor \).

Figure 4: The graph \( G_{nb+r,b} \) with \( \lceil b/2 \rceil \leq r < b - 1 \).

Also see Figure 4 with the assumption \( \lceil b/2 \rceil \leq r < b - 1 \). If \( n \geq b - r - 1 \), we can partition \( V(G_{nb+r,b}) \) into \( (b - r) \) sets with consecutive \( (b - 1) \) vertices and \( (n - b + r + 1) \) sets with consecutive \( b \) vertices each of which will be a color class of an \( (n + 1) \)-coloring of \( G_{nb+r,b} \).

**Theorem 4.2.** For a circulant graph \( G_{nb+r,b} \) with \( n \geq 2 \), \( b \geq 2 \) and \( 1 \leq r < b \), if \( n \geq b - r - 1 \), the inequality box\( (G_{nb+r,b}) \leq \chi(G_{nb+r,b}) \) holds.

**Proof.** We arrange all vertices \( 0, 1, \ldots, nb + r - 1 \) of \( G_{nb+r,b} \) clockwise in this order. Let \( k = n - b + r + 1 \). The symbol \( v_{i,j} \) means the vertex of \( G_{nb+r,b} \) defined as follows:

\[
v_{i,j} = \begin{cases} 
(i - 1)b + j - 1 & \text{if } (i, j) \in \{1, \ldots, k\} \times \{1, \ldots, b\}, \\
(i - 1)(b - 1) + j - 1 + k & \text{if } (i, j) \in \{k + 1, \ldots, n + 1\} \times \{1, \ldots, b - 1\}.
\end{cases}
\]

Moreover we define

\[
V_i = \begin{cases} 
\{v_{i,1}, \ldots, v_{i,b}\} & \text{if } i \in \{1, \ldots, k\}, \\
\{v_{i,1}, \ldots, v_{i,b-1}\} & \text{if } i \in \{k + 1, \ldots, n + 1\}.
\end{cases}
\]

We notice that \( \{V_i\}_{i=1}^{n+1} \) becomes an \( (n+1) \)-coloring of \( G_{nb+r,b} \). Since \( n + r/b = (nb + r)/b \leq \chi(G_{nb+r,b}) \) holds, we conclude that \( \chi(G_{nb+r,b}) = n + 1 \) holds.
Let $|V_i|$ be the cardinality of $V_i$, that is, $|V_i| \in \{b - 1, b\}$. In what follows, we always identify $V_0$ and $V_{n+2}$ with $V_{n+1}$ and $V_1$ respectively. For the vertex $v_{i,j}$ of $G_{nb+r,b}$, we define the subset $X_{i,j}$ of $V(G_{nb+r,b}) \setminus V_i$ to be the union $N(v_{i,j}) \cup Y_{i,j}$, where

(i) when $|V_i| = b$ is even,

$$Y_{i,j} = \begin{cases}
\{v_{i-1,j}, \ldots, v_{i-1,|V_i|/2}\} & \text{if } 1 \leq j \leq |V_i|/2 \text{ and } |V_{i-1}| = b, \\
\{v_{i-1,\max\{1,j-1\}}, \ldots, v_{i-1,|V_i|/2-1}\} & \text{if } 1 \leq j \leq |V_i|/2 \text{ and } |V_{i-1}| = b - 1, \\
\{v_{i+1,|V_i|/2+1}, \ldots, v_{i+1,\min\{j,|V_{i+1}|}\}\} & \text{if } |V_i|/2 < j \leq |V_i|,
\end{cases}$$

(ii) when $|V_i| = b - 1$ is even,

$$Y_{i,j} = \begin{cases}
\{v_{i-1,j}, \ldots, v_{i-1,|V_i|/2}\} & \text{if } 1 \leq j \leq |V_i|/2 \text{ and } |V_{i-1}| = b, \\
\{v_{i-1,\max\{1,j-1\}}, \ldots, v_{i-1,|V_i|/2-1}\} & \text{if } 1 \leq j \leq |V_i|/2 \text{ and } |V_{i-1}| = b - 1, \\
\{v_{i+1,|V_i|/2+2}, \ldots, v_{i+1,\min\{j+1,|V_{i+1}|\}\}} & \text{if } |V_i|/2 < j \leq |V_i|,
\end{cases}$$

(iii) when $|V_i| = b$ is odd,

$$Y_{i,j} = \begin{cases}
\{v_{i-1,j}, \ldots, v_{i-1,\lfloor|V_i|/2\rfloor}\} & \text{if } 1 \leq j \leq \lfloor|V_i|/2\rfloor \text{ and } |V_{i-1}| = b, \\
\{v_{i-1,\max\{1,j-1\}}, \ldots, v_{i-1,\lfloor|V_i|/2\rfloor-1}\} & \text{if } 1 \leq j \leq \lfloor|V_i|/2\rfloor \text{ and } |V_{i-1}| = b - 1, \\
\{v_{i+1,\lfloor|V_i|/2\rfloor+1}, \ldots, v_{i+1,\min\{j+1,|V_{i+1}|\}\}} & \text{if } \lfloor|V_i|/2\rfloor < j \leq |V_i|,
\end{cases}$$

(iv) when $|V_i| = b - 1$ is odd,

$$Y_{i,j} = \begin{cases}
\{v_{i-1,j}, \ldots, v_{i-1,\lfloor|V_i|/2\rfloor}\} & \text{if } 1 \leq j \leq \lfloor|V_i|/2\rfloor \text{ and } |V_{i-1}| = b, \\
\{v_{i-1,\max\{1,j-1\}}, \ldots, v_{i-1,\lfloor|V_i|/2\rfloor-1}\} & \text{if } 1 \leq j \leq \lfloor|V_i|/2\rfloor \text{ and } |V_{i-1}| = b - 1, \\
\{v_{i+1,\lfloor|V_i|/2\rfloor+1}, \ldots, v_{i+1,\min\{j+1,|V_{i+1}|\}\}} & \text{if } \lfloor|V_i|/2\rfloor < j \leq |V_i|.
\end{cases}$$

It is easy to check that

$$X_{i,1} \supseteq \cdots \supseteq X_{i,|V_i|/2} \text{ and } X_{i,|V_i|/2+1} \subseteq \cdots \subseteq X_{i,|V_i|}$$

hold for a color class $V_i$ with an even number of vertices, and

$$X_{i,1} \supseteq \cdots \supseteq X_{i,\lfloor|V_i|/2\rfloor-1} \supseteq X_{i,\lfloor|V_i|/2\rfloor} \subseteq X_{i,\lfloor|V_i|/2\rfloor+1} \subseteq \cdots \subseteq X_{i,|V_i|}$$

holds for a color class $V_i$ with an odd number of vertices.

Note that all vertices which are nonadjacent to $v_{i,j}$ are within $V_{i-1} \cup V_i \cup V_{i+1}$. In what follows, we always assume that $v_{i,j}$ is neither adjacent to $v_{i-1,s}$ nor $v_{i+1,t}$.

**Case (I):** We assume that $j \in \{1, 2, \ldots, \lfloor|V_i|/2\rfloor\}$.

We may assume that $|V_{i-1}| = b$. When $|V_{i-1}| = b - 1$, we should replace the ordered triple $(j + 1, \lfloor|V_i|/2\rfloor, \lfloor|V_i|/2\rfloor + 1)$ in the following argument with $(j, \lfloor|V_i|/2\rfloor - 1, \lfloor|V_i|/2\rfloor)$.

We notice that $s \in \{j + 1, j + 2, \ldots, |V_{i-1}|\}$. If $j + 1 \leq s \leq \lfloor|V_i|/2\rfloor$, we see that $v_{i,j} \not\in X_{i-1,s}$ holds since $\lfloor|V_i|/2\rfloor \leq \lfloor|V_{i-1}|/2\rfloor$. If $\lfloor|V_i|/2\rfloor + 1 \leq s \leq |V_{i-1}|$, then $v_{i-1,s} \not\in X_{i,j}$ holds. Also note that $v_{i+1,t} \not\in X_{i,j}$ holds.

**Case (II):** We assume that $j \in \{\lfloor|V_i|/2\rfloor + 1, \ldots, |V_i|\}$ and $|V_i|$ is odd.
Again first we assume that $|V_i| = b$. When $|V_i| = b - 1$, we should replace the ordered triple $(j - 1, [\lfloor |V_i|/2 \rfloor - 1, \lfloor |V_i|/2 \rfloor])$ in the following argument with $(j, \lfloor |V_i|/2 \rfloor, \lfloor |V_i|/2 \rfloor + 1)$.

We see that $t \in \{1, 2, \ldots, j - 1\}$. If $1 \leq t \leq \lfloor |V_i|/2 \rfloor - 1$, then $v_{i+1,t} \notin X_{i,j}$ holds. If $\lfloor |V_i|/2 \rfloor \leq t \leq j - 1$, we can check that $v_{i,j} \notin X_{i+1,t}$ holds since $\lfloor |V_{i+1}|/2 \rfloor \leq \lfloor |V_i|/2 \rfloor$ holds. We also see that $v_{i-1,s} \notin X_{i,j}$ holds.

**Case (III):** We assume that $j \in \{\lfloor |V_i|/2 \rfloor + 1, \ldots, |V_i|\}$ and $|V_i|$ is even.

If $|V_i| = b$, we should just replace the ordered pair $(\lfloor |V_i|/2 \rfloor - 1, \lfloor |V_i|/2 \rfloor)$ in Case (II) under $|V_i| = b$ with $(\lfloor |V_i|/2 \rfloor, \lfloor |V_i|/2 \rfloor + 1)$. If $|V_i| = b - 1$, we should replace the ordered triple $(j - 1, \lfloor |V_i|/2 \rfloor - 1, \lfloor |V_i|/2 \rfloor)$ in Case (II) under $|V_i| = b$ with $(j, \lfloor |V_i|/2 \rfloor + 1, \lfloor |V_i|/2 \rfloor + 2)$.

Thus our claim follows from Theorem 3.1.

**Question.** Does the inequality $\text{box}(G_{a,b}) \leq \chi(G_{a,b})$ hold for a circulant graph $G_{a,b}$ with $a \geq 2b$ in general?

For $b \geq 5$, the circulant graph $G_{2b+1,b}$ does not satisfy the condition $n \geq b - r - 1$ in Theorem 4.2, but $\text{box}(G_{2b+1,b}) \leq \chi(G_{2b+1,b})$ holds. In fact, $G_{2b+1,b}$ is isomorphic to a cycle with $2b + 1$ vertices.

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