Complete scalar-flat Kähler metrics on affine algebraic manifolds

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Abstract

Let \((X, L_X)\) be an \(n\)-dimensional polarized manifold. Let \(D\) be a smooth hypersurface defined by a holomorphic section of \(L_X\). We prove that if \(D\) has a constant positive scalar curvature Kähler metric, \(X \setminus D\) admits a complete scalar-flat Kähler metric, under the following three conditions: (i) \(n \geq 6\) and there is no nonzero holomorphic vector field on \(X\) vanishing on \(D\), (ii) an average of a scalar curvature on \(D\) denoted by \(\hat{S}_D\) satisfies the inequality \(0 < 3\hat{S}_D < n(n-1)\), (iii) there are positive integers \(l(>n), m\) such that the line bundle \(K_X^{-l} \otimes L_X^m\) is very ample and the ratio \(m/l\) is sufficiently small.

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1 Introduction

The existence of constant scalar curvature Kähler (cscK) metrics on complex manifolds is a fundamental problem in Kähler geometry. If a complex manifold is noncompact, there are many positive results in this problem. In 1979, Calabi [6] showed that if a Fano manifold has a Kähler Einstein metric, then there is a complete Ricci-flat Kähler metric on the total space of the canonical line bundle. In addition, there exist following generalizations. In 1990, Bando-Kobayashi [5] showed that if a Fano manifold admits an anti-canonical smooth divisor which has a Ricci-positive Kähler Einstein metric, then

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there exists a complete Ricci-flat Kähler metric on the complement (see also [13]). Tian-Yau [12] showed that if a Fano manifold admits an anti-canonical smooth divisor which has a Ricci-flat Kähler metric, then there is a complete Ricci-flat Kähler metric on the complement. In 2002, on the other hand, as a scalar curvature version of Calabi’s result [6], Hwang-Singer [8] showed that if a polarized manifold has a nonnegative cscK metric, then the total space of the dual line bundle admits a complete scalar-flat Kähler metric. However, a similar generalization of Hwang-Singer [8] like Bando-Kobayashi [5] and Tian-Yau [12] is unknown since it is hard to solve a forth order nonlinear partial differential equation.

In this paper, assuming the existence of a smooth hypersurface which admits a constant positive scalar curvature Kähler metric, we will prove the existence of a complete scalar-flat Kähler metric on the complement of this hypersurface by using the results in [1] and [2]. Our proof goes roughly as follows.

Step 1. We show that if the smooth hypersurface has a cscK metric, there is a complete Kähler metric whose scalar curvature decays at a higher order.

Step 2. We show that the existence of a complete Kähler metric whose scalar curvature is sufficiently small implies the existence of a complete scalar-flat Kähler metric.

Step 3. We construct a complete Kähler metric on the complement of the smooth hypersurface, whose scalar curvature is arbitrarily small.

Step 4. Finally, we show the existence of a complete scalar-flat Kähler metric by solving the forth order nonlinear partial differential equation.

Now we describe our strategy which contains the results in the previous papers [1] and [2] more precisely. Let \((X, L_X)\) be a polarized manifold of dimension \(n\), i.e., \(X\) is an \(n\)-dimensional compact complex manifold and \(L_X\) is an ample line bundle over \(X\). Assume that there is a smooth hypersurface \(D \subset X\) with

\[
D \in |L_X|.
\]

Set an ample line bundle \(L_D := \mathcal{O}(D)|_D = L_X|_D\) over \(D\). Since \(L_X\) is ample, there exists a Hermitian metric \(h_X\) on \(L_X\) which defines a Kähler metric \(\theta_X\) on \(X\), i.e., the curvature form of \(h_X\) multiplied by \(\sqrt{-1}\) is positive definite. Then, the restriction of \(h_X\) to \(L_D\) defines also a Kähler metric \(\theta_D\) on \(D\). Let \(\hat{S}_D\) be the average of the scalar curvature \(S(\theta_D)\) of \(\theta_D\) defined by

\[
\hat{S}_D := \frac{\int_D S(\theta_D)\theta_D^{n-1}}{\int_D \theta_D^{n-1}} = \frac{(n-1)c_1(K_D^{-1}) \cup c_1(L_D)^{n-2}}{c_1(L_D)^{n-1}},
\]

where \(K_D^{-1}\) is the anti-canonical line bundle of \(D\). Note that \(\hat{S}_D\) is a topological invariant in the sense that it is representable in terms of Chern classes of the line bundles \(K_D^{-1}\) and \(L_D\). In this paper, we treat the following case:

\[
\hat{S}_D > 0. \quad (1.1)
\]

Let \(\sigma_D \in H^0(X, L_X)\) be a defining section of \(D\) and set \(t := \log \|\sigma_D\|^2_{h_X}\). Following [5],
we can define a complete Kähler metric $\omega_0$ by
$$\omega_0 := \frac{n(n-1)}{\hat{S}_D} \sqrt{-1} \partial \bar{\partial} \exp \left( \frac{\hat{S}_D}{n(n-1)} t \right)$$
on the noncompact complex manifold $X \setminus D$. In addition, since $(X \setminus D, \omega_0)$ is of asymptotically conical geometry (see [5] or Section 4 of [1]), we can define weighted Banach spaces $C^k,\alpha_\delta = C^k,\alpha_\delta(X \setminus D)$ for $k \in \mathbb{Z} \geq 0$, $\alpha \in (0, 1)$ and with a weight $\delta \in \mathbb{R}$ with respect to the distance function $r$ defined by $\omega_0$ from some fixed point in $X \setminus D$. It follows from the construction of $\omega_0$ that $S(\omega_0) = O(r^{-2})$ near $D$.

**Step 1.** The cscK condition implies the following stronger decay property (see [1]).

**Theorem 1.1.** If $\theta_D$ is a constant positive scalar curvature Kähler metric on $D$, i.e., $S(\theta_D) = \hat{S}_D > 0$, we have
$$S(\omega_0) = O(r^{-2-2n(n-1)/\hat{S}_D})$$
as $r \to \infty$.

Thus, the cscK condition implies that $S(\omega_0) \in C^{k,\alpha}_\delta$ for some $\delta > 2$ and any $k, \alpha$.

**Step 2.** To construct a complete scalar-flat Kähler metric on $X \setminus D$, the linearization of the scalar curvature operator plays an important role:
$$L_{\omega_0} = -\mathcal{D}_{\omega_0}^* \mathcal{D}_{\omega_0} + (\nabla^{1,0}_*, \nabla^{0,1} S(\omega_0))_{\omega_0}.$$
Here, $\mathcal{D}_{\omega_0} = \overline{\partial} \circ \nabla^{1,0}$. We will show that if $4 < \delta < 2n$ and there is no nonzero holomorphic vector field on $X$ which vanishes on $D$, then $\mathcal{D}_{\omega_0}^* \mathcal{D}_{\omega_0} : C^{4,\alpha}_\delta \to C^{0,\alpha}_\delta$ is isomorphic. For such operators, we consider the following:

**Condition 1.2.** Assume that $n \geq 3$ and there is no nonzero holomorphic vector field on $X$ which vanishes on $D$. For $4 < \delta < 2n$, the operator
$$L_{\omega_0} : C^{4,\alpha}_{\delta-4} \to C^{0,\alpha}_\delta$$
is isomorphic, i.e., we can find a constant $\hat{K} > 0$ such that
$$||L_{\omega_0} \phi||_{C^{0,\alpha}_\delta} \geq \hat{K} ||\phi||_{C^{4,\alpha}_{\delta-4}}$$
for any $\phi \in C^{4,\alpha}_{\delta-4}$.

In addition, we consider

**Condition 1.3.**
$$||S(\omega_0)||_{C^{0,\alpha}_\delta} < c_0 \hat{K}/2.$$

Here, the constant $c_0$ is defined Lemma 6.2 in [1]. Under these conditions, we have the following result (see [1]):
Theorem 1.4. Assume that \( n \geq 3 \) and there is no nonzero holomorphic vector field on \( X \) which vanishes on \( D \). Assume that \( \theta_D \) is a constant scalar curvature Kähler metric satisfying
\[
0 < \hat{S}_D < n(n-1).
\]
Assume moreover that Condition 1.2 and Condition 1.3 hold, then \( X \setminus D \) admits a complete scalar-flat Kähler metric.

In fact, we can show the existence of a complete scalar-flat Kähler metric on \( X \setminus D \) under the following assumptions: (i) \( n \geq 3 \) and there is no nonzero holomorphic vector field on \( X \) which vanishes on \( D \), (ii) there exists a complete Kähler metric on \( X \setminus D \) which is of asymptotically conical geometry, such that its scalar curvature is sufficiently small and decays at a higher order. So, if there exists a complete Kähler metric on \( X \setminus D \) which is sufficiently close to \( \omega_0 \) at infinity, satisfying Condition 1.2 and Condition 1.3, we can show the existence of a complete scalar-flat Kähler metric on \( X \setminus D \).

Theorem 1.4 is proved by the fixed point theorem on the weighted Banach space \( C^{4,\alpha}(X \setminus D) \) by following Arezzo-Pacard [3], [4] (see also [11]). In general, constants \( c_0, \hat{K} \) which arise in Condition 1.2 and Condition 1.3 depend on the background Kähler metric \( \omega_0 \). In addition, to construct such a Kähler metric, we have to find a complete Kähler metric \( X \setminus D \) whose scalar curvature is arbitrarily small.

Step 3. We consider a degenerate (meromorphic) complex Monge-Ampère equation. Take positive integers \( l > n \) and \( m \) such that the line bundle \( K_X^{-l} \otimes L_X^m \) is very ample. Let \( F \in |K_X^{-l} \otimes L_X^m| \) be a smooth hypersurface defined by a holomorphic section \( \sigma_F \in H^0(X, K_X^{-l} \otimes L_X^m) \) such that the divisor \( D + F \) is simple normal crossing. For a defining section \( \sigma_D \in H^0(X, L_X) \) of \( D \), set
\[
\xi := \sigma_F \otimes \sigma_D^{-m}.
\]
From the result due to Yau [14, Theorem 7], we can solve the following degenerate complex Monge-Ampère equation:
\[
(\theta_X + \sqrt{-1} \partial \bar{\partial} \varphi)^n = \xi^{-1/l} \wedge \bar{\xi}^{-1/l}.
\]
Moreover, it follows from a priori estimate due to Kołodziej [9] that the solution \( \varphi \) is bounded on \( X \). Thus, we can glue plurisubharmonic functions by using the regularized maximum function. To compute the scalar curvature of the glued Kähler metric, we need to study behaviors of higher order derivatives of the solution \( \varphi \). So, we give explicit estimates of them near the intersection \( D \cap F \) (see [2]):

Theorem 1.5. Let \( (z^i)_{i=1}^n = (z^1, z^2, \ldots, z^{n-2}, w_F, w_D) \) be local holomorphic coordinates such that \( \{ w_F = 0 \} = F \) and \( \{ w_D = 0 \} = D \). Then, there exists a positive integer \( a(n) \) depending only on the dimension \( n \) such that
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\[
\begin{align*}
\frac{\partial^2}{\partial z^i \partial z^j} \varphi &= O \left( |w_D|^{-2a(n)m/l} |w_F|^{-2a(n)/l} \right), \\
\frac{\partial^4}{\partial w^2_i \partial w^2_F} \varphi &= O \left( |w_D|^{-2a(n)m/l} |w_F|^{-2a(n)/l} \right), \\
\frac{\partial^4}{\partial w^2_F \partial w^2_D} \varphi &= O \left( |w_D|^{-2a(n)m/l} |w_F|^{-2a(n)/l} \right),
\end{align*}
\]

as \( |w_F| |w_D| \to 0 \), for any \( 1 \leq i, j \leq n - 2 \) and multi-index \( \alpha = (\alpha_1, ..., \alpha_n) \) with \( 0 \leq \sum_i \alpha_i \leq 2 \).

By applying Theorem 1.5 and gluing plurisubharmonic functions, we have the following result (see [2]):

**Theorem 1.6.** Assume that there exist positive integers \( l > n \) and \( m \) such that

\[
a(n)m \frac{l}{2l} < \frac{\hat{S}_D}{n(n - 1)},
\]

and the line bundle \( K^{-l}_X \otimes L^m_X \) is very ample. Here, \( a(n) \) is the positive integer in Theorem 1.5. Take a smooth hypersurface \( F \in |K^{-l}_X \otimes L^m_X| \) such that \( D + F \) is simple normal crossing. Then, for any relatively compact domain \( Y \subset X \setminus (D \cup F) \), there exists a complete Kähler metric \( \omega_F \) on \( X \setminus D \) whose scalar curvature \( S(\omega_F) = 0 \) on \( Y \) and is arbitrarily small on the complement of \( Y \). In addition, \( \omega_F = \omega_0 \) on some neighborhood of \( D \setminus (D \cap F) \).

For example, if the anti-canonical line bundle \( K^{-1}_X \otimes L^m_X \) of the compact complex manifold \( X \) is nef (in particular, \( X \) is Fano), the assumption (1.2) in Theorem 1.6 holds, i.e., we can always find such integers \( l, m \). In this paper, we treat the case that \( K^{-1}_X \) has positivity in the sense of (1.1) and (1.2). From Theorem 1.4, if there exists a complete Kähler metric \( \omega_F \) on \( X \setminus D \) which satisfies Condition 1.2 and Condition 1.3, \( X \setminus D \) admits a complete scalar-flat Kähler metric. In fact, Theorem 1.6 gives a Kähler metric whose scalar curvature is under control. However, the Kähler metric \( \omega_F \) in Theorem 1.6 is not of asymptotically conical geometry (near the intersection of \( D \) and \( F \)). So, when we replace the complete Kähler metric \( \omega_0 \) with \( \omega_F \) obtained in Theorem 1.6 we can not apply Theorem 1.4 to a construction of a complete scalar-flat Kähler metric.

To solve this problem, we consider an average on some closed subset in \( |K^{-l}_X \otimes L^{m+\beta}_X| \). Then, the asymptotically conicalness is recovered and we obtain the first result in this paper:

**Theorem 1.7.** Assume that there are positive integers \( l > n \) and \( m \) such that the line bundle \( K^{-l}_X \otimes L^m_X \) is very ample and

\[
a(n)m \frac{l}{2l} < \frac{\hat{S}_D}{n(n - 1)},
\]

Then, there exists a complete Kähler metric \( \overline{\omega} \) on \( X \setminus D \) satisfies following properties:
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• \( \omega \) is equivalent to \( \omega_0 \), i.e., there is a constant \( C > 0 \) such that

\[
C^{-1} \omega_0 < \omega < C \omega_0.
\]

Moreover, the Kähler metric \( \omega \) is of asymptotically conical geometry.

• Assume that \( n \geq 4 \). If \( \theta_D \) is cscK and \( 0 < \hat{S}_D < n(n - 1) \), the \( C^{k,\alpha} \)-norm of the scalar curvature \( S(\omega) \) of weight \( \delta \in (4, \min\{2n, 2 + 2n(n - 1)/\hat{S}_D\}) \) can be made arbitrarily small.

Thus, we obtain the Kähler metric \( \omega \) which is of asymptotically conical geometry. In addition, the scalar curvature of \( \omega \) is arbitrarily small in the sense of the weight norm.

Step 4. Finally, by applying the similar argument in the proof of Theorem 1.4 to the Kähler metric \( \omega \) obtained in Theorem 1.7, we obtain our main result in this paper:

**Theorem 1.8.** Assume following conditions:

• \( n \geq 6 \) and there is no nonzero holomorphic vector field on \( X \) which vanishes on \( D \).

• The following inequality holds:

\[
0 < 3 \hat{S}_D < n(n - 1).
\]

• There are positive integers \( l > n \) and \( m \) such that the line bundle \( K_X^{-l} \otimes L_X^m \) is very ample and

\[
\frac{a(n)m}{2l} < \frac{\hat{S}_D}{n(n - 1)}.
\]

Then, if \( D \) admits a cscK metric \( \theta_D \), \( X \setminus D \) admits a complete scalar-flat Kähler metric.

In other word, we can solve the following forth order nonlinear partial differential equation:

\[
S(\omega + \sqrt{-1} \partial \bar{\partial} \phi) = 0, \quad \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0, \quad \phi \in C^{4,\alpha}_{\delta-4}
\]

for a weight \( 8 < \delta < \min\{2n, 2 + 2n(n - 1)/\hat{S}_D\} \). The reason why we assume that \( n \geq 6 \) and \( 0 < 3 \hat{S}_D < n(n - 1) \) in Theorem 1.8 is that we need the isomorphic Laplacian \( \Delta_{\omega} \) between higher order weighted Banach spaces.

This paper is organized as follows. In Section 2, we will prove Theorem 1.7. Namely, we recover the asymptotically conicalness by constructing an average metric. In Section 3, we prove Theorem 1.8 i.e., we show the existence of a complete scalar-flat Kähler metric.

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2 Proof of Theorem 1.7

In this section, we prove Theorem 1.7. We construct the complete Kähler metric $\omega_F = \omega_{c,v,\eta}$ whose scalar curvature is arbitrarily small on $X \setminus D$ in [2]. So, see the definitions of the parameters $c,v,\eta,\kappa$ and the functions $\Theta(t), G_\alpha^\beta(b), \tilde{G}_\alpha^\beta(b)$ in Section 2 of [2]. For $\beta \in \mathbb{Z}_{>0}$, take a holomorphic section $\sigma_0 \in H^0(K_X^{-1} \otimes L_X^{m+\beta})$. We may assume that $D + F_0$ is simple normal crossing, where $F_0$ is a smooth hypersurface defined by $\sigma_0$. Let $(\sigma_i)_i \subset H^0(K_X^{-1} \otimes L_X^{m+\beta})$ be an orthonormal basis with respect to the $L^2$ inner product. Take a sufficiently small number $\tau \in \mathbb{R}$. Write $h(l,m,\beta) := \dim H^0(K_X^{-l} \otimes L_X^{m+\beta})$. For $s = (s_i)_i \in \mathbb{D}^{h(l,m,\beta)} := \{ z = (z_i) \in \mathbb{C}^{h(l,m,\beta)} | |z_i| \leq 1 \}$, define a meromorphic section of the line bundle $K_X^{-l} \otimes L_X^m$ by

$$\sigma_s := (\sigma_0 + \tau \sum_{i=1}^{h(l,m,\beta)} s_i \sigma_i) \otimes \sigma_D^{-\beta}.$$ 

Note that by taking a sufficiently small $\tau$, we may assume that $\sigma_s \neq 0$ for any $s \in \mathbb{D}^{h(l,m,\beta)}$. In addition, $\sigma_s \to \sigma_0 \otimes \sigma_D^{-\beta}$ for any $s \in \mathbb{D}^{h(l,m,\beta)}$ as $\tau \to 0$. Let $F_s$ be a smooth hypersurface defined by $\text{div} \sigma_s = F_s - \beta D$. Since $\sigma_0$ contained in $\sigma_s$ is not multiplied by $\tau$, the variation of $\tau$ affects the choice of $F_s$ if $\tau \neq 0$.

By applying Theorem 1.6, we obtain a complete Kähler metric $\omega_{F_s}$ with small scalar curvature for a meromorphic section $\sigma_s \otimes \sigma_D^{-\beta}$ of $K_X^{-1}$ (see [2]). In fact, for a smooth function on $X \setminus (D \cup F_s)$ defined by $b_s := \log ||\sigma_s||^{-2}$, we can obtain a Kähler metric $\sqrt{-1} \partial \bar{\partial} G_\alpha^\beta(b_s)$ on $X$. Directly, we have

$$\sqrt{-1} \partial \bar{\partial} G_\alpha^\beta(b_s) = \left( \frac{1}{e^{-b_s} + v} \right)^{1/\beta} \left( \beta \sqrt{-1} \partial \bar{\partial} b_s + \frac{e^{-b_s}}{e^{-b_s} + v} \sqrt{-1} \partial b_s \wedge \bar{\partial} b_s \right).$$

This metric does not grow near $D$. When we glued plurisubharmonic functions in [2], we considered the Kähler potential $\kappa \Theta(t) + G_\alpha^\beta(b_s)$. In addition, $\lim_{b_s \to -\infty} \sqrt{-1} \partial \bar{\partial} G_\alpha^\beta(b_s) > -\infty$. So, we can construct a complete Kähler metric $\omega_{F_s}$ with small scalar curvature by using the regularized maximum $M_\eta$ (see [7] or [2]) to glue three plurisubharmonic functions $\Theta(t), G_\alpha^\beta(b_t), t + \varphi + c$.

$(X \setminus D, \omega_{F_s})$ is not of asymptotically conical geometry for any $s \in \mathbb{D}^{h(l,m,\beta)}$ (see Remark 4.2 in [2]). To solve this problem, consider an average metric $\overline{\omega}$ defined by

$$\overline{\omega} = \overline{\omega}(c,v,\eta,\tau) := \int_{\mathbb{D}^{h(l,m,\beta)}} \omega_{F_s} d\mu(s).$$

Here, $\mu$ is the Lebesgue probability measure on $\mathbb{D}^{h(l,m,\beta)}$ and $c,v,\eta$ are parameters in the definition of $\omega_F$ in Theorem 1.6 (see [2]). Recall that $\eta = (\eta_1, \eta_2, \eta_3)$ and $\eta_1, \eta_2 = O(c), \eta_3 = O(1)$. To prove that $(X \setminus D, \overline{\omega})$ is of asymptotically conical geometry, it is enough to prove the following lemma:

**Lemma 2.1.** For the Kähler metric $\overline{\omega}$ defined above, we have

$$\overline{\omega} - \omega_0 = O(||\sigma_D||^2)$$

as $\sigma_D \to 0$. 
Proof. The region where \((X \setminus D, \omega_F)\) is not of asymptotically conical geometry is defined by
\[
|\Theta(t) - \tilde{G}^\beta_v(b_s)| < \eta_1 + \eta_2 \tag{2.1}
\]
(see [2]). For sufficiently large \(b_s > 0\), we have \(v^{-1/\beta} b_s \approx G^\beta_v(b_s)\). Here, \(b_s := \log ||\sigma_s||^{-2}\).

From the following inequality
\[
v^{-1/\beta} b_s \approx G^\beta_v(b_s) > (1 - \kappa)\Theta(t) - \eta_1 - \eta_2
\]
obtained by (2.1), we have
\[
||\sigma_s||^2 < \exp \left( -\left( v^{1/\beta} / \beta \right) \right) \left( (1 - \kappa)||\sigma_D||^{-2\tilde{S}/n(n-1)} - \eta_1 - \eta_2 \right).
\]

Take a point \(p \in X \setminus D\) near \(D\). Assume that \(\sigma_s(p) = 0\) for \(s \in \mathbb{D}^{h(l,m,\beta)}\). Then, an element \(\tilde{s} \in \mathbb{D}^{h(l,m,\beta)}\) satisfying the inequality above has to satisfy
\[
||\sigma_s||^2 < \exp \left( -\left( v^{1/\beta} / \beta \right) \right) \left( (1 - \kappa)||\sigma_D||^{-2\tilde{S}/n(n-1)} - \eta_1 - \eta_2 \right).
\]

By considering a suitable unitary transformation \(u = (u_{i,j}) \in U(h(l,m,\beta))\), we can write as \(\sum_{i=1}^{h(l,m,\beta)} (s_i - \tilde{s}_i)\sigma_i(p) = (\sum_{i,j=1}^{h(l,m,\beta)} u_{i,j}(s_i - \tilde{s}_i))\sigma(p)\) for some holomorphic section \(\tilde{\sigma} \in H^0(K_X^{-l} \otimes L_X^{m+\beta})\) such that \(\tilde{\sigma}(p) \neq 0\) and have
\[
\left| \sum_{i,j=1}^{h(l,m,\beta)} u_{i,j}(s_i - \tilde{s}_i) \right|^2 < \frac{\exp \left( -\left( v^{1/\beta} / \beta \right) \right) \left( (1 - \kappa)||\sigma_D||^{-2\tilde{S}/n(n-1)} - \eta_1 - \eta_2 \right)}{\tau^2 ||\tilde{\sigma} \otimes \sigma_D^{-\beta}(p)||^2}. \tag{2.3}
\]

Then, we have the following estimate
\[
\int \partial \Theta(t) \wedge \overline{\partial} \Theta(t) d\mu(s) = \exp \left( O \left( -||\sigma_D||^{-2\tilde{S}/n(n-1)} \right) \right).
\]

Next, we consider the term
\[
\int \partial G^\beta_v(b_s) \wedge \overline{\partial} G^\beta_v(b_s) d\mu(s)
\]
which appears in \(\omega\). From the inequality (2.1), we have
\[
v^{-1/\beta} b_s \approx G^\beta_v(b_s) < (1 - \kappa)\Theta(t) + \eta_1 + \eta_2.
\]

Thus, the following inequality
\[
||\sigma_s||^{-2} < \exp \left( v^{1/\beta} / \beta \right) \left( (1 - \kappa)||\sigma_D||^{-2\tilde{S}/n(n-1)} + \eta_1 + \eta_2 \right)
\]
holds. Thus, we can estimate as follows

\[ \int \partial G^\beta_v(\beta b) \wedge \overline{\partial} G^\beta_v(\beta b) d\mu(s) \leq \exp \left( 2v^{1/\beta}(\eta_1 + \eta_2)/\beta \right) /\tau^2 ||\tilde{\sigma} \otimes \sigma_D^{-\beta}||^2. \tag{2.4} \]

By the definition of \( \overline{\omega} \), we obtain

\[ \overline{\omega} \approx \left( 1 - \exp \left( -||\sigma_D||^{-2S_D/n(n-1)} \right) \right) \omega_0 + O(||\sigma_D||^{2\beta}) \]

near \( D \).

**Proof of Theorem 1.7.** Lemma 2.1 implies that the complete Kähler manifold \( (X \setminus D, \omega) \) is of asymptotically conical geometry. Thus, we will prove that the scalar curvature can be made small arbitrarily. To show this, we take parameters \( c, v, \tau \) and an integer \( \beta \) so that

\[ v^{1/\beta} c = k \log c, \quad \tau^2 = v, \quad \beta > \delta \tag{2.5} \]

for a sufficiently large \( k \in \mathbb{N} \) specified later.

Firstly, from the construction of \( \overline{\omega} \), weight norms of \( S(\omega) \) away from \( D \cup F_0 \) can be made small arbitrarily by taking sufficiently small \( \tau \). To show this, we study a function \( f : \tau \rightarrow \overline{\omega}/\omega^0_{F_0} \). Note that this function is smooth and \( f(0) = 1 \) and \( \omega_{F_0} \) is Ricci-flat away from \( D \cup F_0 \). So, we have \( S(\overline{\omega}) = O(\tau) \) away from \( D \cup F_0 \).

Secondly, we study \( S(\omega) \) near \( F_0 \) and away from \( D \). We can write as

\[ \overline{\omega} = \int (\sqrt{-1} \partial \bar{\partial} M_{c,v,\eta}) \, d\mu, \]

where

\[ M_{c,v,\eta} = \frac{\partial M_{c,v,\eta}}{\partial t_2}(\gamma^\beta + \kappa \omega_0) + \frac{\partial M_{c,v,\eta}}{\partial t_3} \sqrt{-1} \partial \bar{\partial}(t + \varphi) \]

\[ + \left[ \partial \tilde{G}_v^\beta(b_s) + \partial (t + \varphi) \right] \left[ \frac{\partial^2 M_{c,v,\eta}}{\partial t_2 \partial t_3} \right] \left[ \partial \tilde{G}_v^\beta(b_s) + \partial (t + \varphi) \right] t. \]

On this region, we consider a sufficiently small neighborhood of \( F_0 \) by taking a sufficiently large parameter \( c \). So, it is enough to consider the region defined by the following inequality

\[ G_v^\beta(b_s) + \kappa \Theta(t) - \eta_2 > \max\{ \Theta(t) + \eta_1, t + \varphi + c + \eta_3 \}. \]

In addition, since we are considering the region away from \( D \), by taking a sufficiently large parameter \( c \), the inequality above can be rewritten as follows:

\[ G_v^\beta(b_s) + \kappa \Theta(t) - (t + \varphi + c) > \eta_2 + \eta_3. \tag{2.6} \]

So, we have

\[ \sum_{l,m} s_l^2 < \exp \left( -(v^{1/\beta}/\beta)(t + \varphi + c - \kappa \Theta(t) + \eta_2 + \eta_3) \right) /\tau^2 = O(c^{\beta-1}). \]
Recall the definition of $\gamma^\beta_v$ in [2] and the relation of the parameters $c, v$:

$$cv^{1/\beta} = k \log c$$

So, the inequality above (2.6) implies that

$$||\sigma_F||^2 \leq v^{k/\beta}.$$ 

Thus, we don’t have to consider the case that $S(\omega_F) = O(1)$ and we have

$$\varpi = \int \gamma^\beta_v d\mu(s) + \kappa \omega_0 \approx v^{-1/\beta} \sqrt{-1} \partial \bar{\partial} b + \kappa \omega_0.$$  (2.7)

by taking a sufficiently large $c$. Since the Ricci form of $\sqrt{-1} \partial \bar{\partial} (G^\beta_v(\beta b_0) + \kappa \Theta(t))$ is bounded near $F_0$ and away from $D$, we can conclude that $S(\varpi) = O(v^{1/\beta})$.

Thirdly, we study $S(\varpi)$ near $D$. Write

$$\varpi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi.$$ 

By taking the trace with respect to the background metric $\omega_0$, we have

$$\Delta_{\omega_0} \psi = \text{tr}_{\omega_0} \varpi - n.$$ 

To estimate

$$S(\varpi) = S(\omega_0 + \sqrt{-1} \partial \bar{\partial} \psi),$$ 

we study the right hand side in the equation above. Recall the construction of the complete Kähler metric $\omega_{F_s}$. The bounded region where plurisubharmonic functions $\Theta(t), t + \varphi + c$ are glued is defined by following inequalities:

$$\tilde{G}^\beta_v(b_s) + \eta_2 < \max \{\Theta(t) - \eta_1, (t + \varphi + c) - \eta_3\},$$

$$|\Theta(t) - (t + \varphi + c)| < \eta_1 + \eta_3.$$ 

In addition, $\omega_{F_s}$ is written as

$$\omega_{F_0} = \frac{\partial M_{c,v,\eta}}{\partial t_1} \omega_0 + \frac{\partial M_{c,v,\eta}}{\partial t_3} \sqrt{-1} \partial \bar{\partial} (t + \varphi)$$

$$+ \left[ \partial \Theta(t) \partial (t + \varphi) \right] \left[ \frac{\partial^2 M_{c,v,\eta}}{\partial t_1 \partial t_3} \right] \left[ \overline{\partial} \Theta(t) \overline{\partial} (t + \varphi) \right].$$

Recall that $\eta_1 + \eta_3 = O(c)$. So, the inequality

$$|\Theta(t) - (t + \varphi + c)| < \eta_1 + \eta_3$$

implies the following equivalence between complete Kähler metrics:

$$\frac{\partial M_{c,v,\eta}}{\partial t_1} \omega_0 < \omega_{F_s} < 2\omega_0$$

on the region above. Next, we consider the region contained in the other region defined by

$$v^{-1/\beta} \beta b_s \approx G^\beta_v(\beta b_s) > (1 - \kappa) \Theta(t) - \eta_1 - \eta_2.$$
In order to estimate $\text{tr}_{\omega_0} \omega - n$, it is enough to estimate the following terms

$$c^{-1} \int \partial \Theta(t) \wedge \overline{\partial} \Theta(t) d\mu(s), \quad c^{-1} \int \partial G_{\omega}^\beta(\beta b_s) \wedge \overline{\partial} G_{\omega}^\beta(\beta b_s) d\mu(s).$$

Since $c \leq \Theta(t)$ on this region, the first term can be estimated as follows

$$c^{-1} ||\sigma_D||^{-4 \hat{S}_D/n(n-1)} \exp \left( -v^{1/\beta} (1 - \kappa) ||\sigma_D||^{-2 \hat{S}_D/n(n-1)}/\beta \right) / \tau^2 ||\sigma \otimes \sigma_D^{-\beta}||^2 = O(c^{1-(1-\kappa)k/\beta - n(n-1)\beta/\hat{S}_D + \log c})$$

for parameters $\tau^2 = v, cv^{1/\beta} = k \log c$. From the estimate (2.4), the second term can be estimated as follows

$$c^{-1} \int \partial G_{\omega}^\beta(\beta b_s) \wedge \overline{\partial} G_{\omega}^\beta(\beta b_s) d\mu(s) \leq \exp \left( 2v^{1/\beta} (\eta_1 + \eta_2)/\beta \right) / \tau^2 ||\sigma \otimes \sigma_D^{-\beta}||^2 \leq O(c^{-1+2(a_1 + a_2)k/\beta - n(n-1)\beta/\hat{S}_D})$$

Recall the relation between parameters (see Claim 2 in Section 4 of [2]) :

$$1 - \kappa + \kappa a_1 - a_2 = 0.$$

Since the choice of $a_i \in (0,1)$ is independent of $\beta, k$, we can choose sufficiently small $a_i$ and $\kappa$ which is sufficiently close to 1. Thus, we can make the following terms :

$$1 - (1 - \kappa)k/\beta - n(n-1)\beta/\hat{S}_D + \beta
\quad -1 + 2(a_1 + a_2)k/\beta + \beta - n(n-1)\beta/\hat{S}_D$$

negative by taking sufficiently large $\beta$ and $k$. Thus, we can estimate $\Delta_{\omega_0} \psi = \text{tr}_{\omega_0} \omega - n$ near $D$. From the equivalence (2.7), we obtain the following estimate near $F_0$ and away from $D$ :

$$\text{tr}_{\omega_0} \omega - n = O(v^{-1/\beta}).$$

For any weight $\epsilon \in (4, 2n)$, we have the following inequality

$$\Delta_{\omega_0} \tilde{C} \rho^{-\epsilon+2} < -C \rho^{-\epsilon} < \Delta_{\omega_0} v^{1/\beta} \psi < C \rho^{-\epsilon} < -\Delta_{\omega_0} \tilde{C} \rho^{-\epsilon+2}$$

on $X \setminus D$ for some constants $C, \tilde{C} > 0$ depending only on $\epsilon$ and $n$. Here $\rho = ||\sigma_D||^{-\hat{S}_D/n(n-1)}$ is the barrier function defined in Section 5 of [1] (see [5]). Thus, the maximum principle tells us that there is the following $C^{0}_{\epsilon-4}$-estimate of $\psi$ :

$$||\psi||_{C^{0}_{\epsilon-4}} \leq C v^{-1/\beta}. \quad (2.8)$$

Recall the linearization of the scalar curvature operator

$$S(\omega) = S(\omega_0) + L_{\omega_0}(\psi) + Q_{\omega_0}(\psi). \quad (2.9)$$
In addition, the term $Q_{\omega_0}(\psi)$ can be written as

$$Q_{\omega_0}(\psi) = (L_{\omega_0 + s\tau} \mathcal{D}_s - L_{\omega_0})(\psi)$$

for some $s \in [0, 1]$ (see [11] or [1]). Choose

$$\epsilon > \delta + 2.$$

In this case, we can consider that $c \leq \Theta(t) \approx r^2$.

Recall the interior Schauder estimate:

$$||\psi||_{C^{2,\alpha}_{t-\delta}} \leq C(\omega_0)(||\mathcal{D}_s \nabla \mathcal{D}_s - n||_{C^{2,\alpha}_{t-\delta}} + ||\psi||_{C^{2,\alpha}_{t-\delta}}).$$

Here, $C(\omega_0)$ is a positive constant depending only on $\omega_0$. The previous estimate (2.8) implies that $||\psi||_{C^{2,\alpha}_{t-\delta}} = O(v^{-1/\beta})$. Then, the equality (2.9) implies that the norm of scalar curvature of weight $\delta$ is estimated from above by $c^{v \log c - 1}(\log c)^{-1}$. In these settings of parameters, we show finally that the scalar curvature on the region defined by

$$t + \varphi + c - \eta_3 > \max\{\Theta(t) + \eta_1, \tilde{G}_v^\beta(b_s) + \eta_2\}$$

can be estimated in the sense of weighted norms. It follows from the first discussion that $S(\mathcal{D}) = O(\tau)$ on the region above. Then, we have

$$S(\mathcal{D}) = O(\tau) = O\left(c^{-\beta/2}(k \log c)^{\beta/2}\right).$$

Recall that $\Theta(t) \approx r^2 \approx c$ on this region. So, we can estimate the $C^{k,\alpha}$-norm of the function $S(\mathcal{D})(v^2 + 1)^{\delta/2} \approx S(\mathcal{D})e^{\delta/2}$ in the definition of the weighted norm (see [1]) on this region. More precisely, the choice of $\beta$:

$$\beta > \delta$$

implies that we can estimate the $\delta$-weighted norm of the scalar curvature $S(\mathcal{D})$ on the region above. Therefore, from the discussion above, we can conclude that the weight norm of $S(\omega_{t,v,\eta})$ can be made small arbitrarily by taking a sufficiently large parameter $c$ (equivalently, sufficiently small parameters $v, \tau$). In addition, from the linearization of scalar curvatures, the scalar curvature $S(\mathcal{D})$ decays just like $S(\omega_0)$. Thus, we finish the proof of Theorem 1.7.

**Remark 2.2.** If $\theta_D$ is cscK, Theorem 1.1 implies that we have

$$S(\mathcal{D}) = O(||\sigma_D||^{2+2\tilde{S}_D/n(n-1)}) = O(r^{2-2n(n-1)/\tilde{S}_D})$$

near $D$ (see [1]).

**Remark 2.3.** Recall that we choose a parameter $v > 0$ so that the inequality $(||\sigma_F||^{2\beta} + v)^{2/\beta} < ||\sigma_F||^{4\alpha m/4}$ holds on the region defined by

$$\Theta(t) + \eta_1 < \max\{\tilde{G}_v^\beta(b) - \eta_2, (t + \varphi + c) - \eta_3\},$$

$$|\tilde{G}_v^\beta(b) - (t + \varphi + c)| < \eta_2 + \eta_3$$

in [2]. Note that $G_v^\beta(b) \approx \beta^v e^{-1/\beta} b$ for sufficiently large $b > 0$. The choice of parameters $ce^{1/\beta} = k \log c$ in the previous theorem implies that we have $||\sigma_F||^{-2\beta} \approx v^k$. Therefore, we can choose a suitable parameter $v > 0$ so that $(||\sigma_F||^{2\beta} + v)^{2/\beta} < ||\sigma_F||^{4\alpha m/4}$ without contradiction (see Remark 2.5 in [2]).
3 Proof of Theorem 1.8

After this, all weighted Banach spaces $C^{k,a}_\delta = C^{k,a}_\delta(X \setminus D)$ are defined by the fixed Kähler metric $\omega_0$. In Theorem 1.8 we assume that

$$0 < 3\hat{S}_D < n(n-1)$$

and we choose a weight $\delta$ so that

$$8 < \delta < \min\{2n, 2 + 2n(n-1)/\hat{S}_D\} \quad (3.1)$$

and a function

$$\phi D_N D_N \phi$$

is integrable for $\phi \in C^{4,a}_{\delta-4}$ with respect to the volume form $\omega^n$. In addition, we may assume that the integer $a(n)$ in Theorem 1.6 satisfies

$$12/a(n) < \delta - 8 < \min\{2n - 8, 2n(n-1)/\hat{S}_D - 6\}.$$
has an upper bound depending only on $\phi$. We prove this by contradiction. Assume that there exists a sequence $(\tau, v, c) \to (0, 0, \infty)$ such that $\|D_\varphi^s D_\varphi \phi\|_{C^{4,\alpha}_{\delta-4}} \to 0$ for some $\phi \in C^{4,\alpha}_{\delta-4}$ with $\|\phi\|_{C^{4,\alpha}_{\delta-4}} = 1$. By integration by parts, we have

$$\int_{X \setminus D} \phi D_\varphi^* D_\varphi \phi \omega^i = \int_{X \setminus D} |D_\varphi \phi|^2 \omega^i.$$

Recall that $D_\varphi \to D_{\sqrt{-1} \partial \partial \psi(t+\varphi)}$ as $(\tau, v, c) \to (0, 0, \infty)$. We show that

$$\int_{X \setminus D} \phi D_\varphi^* D_\varphi \phi \omega^i \to 0$$

as $(\tau, v, c) \to (0, 0, \infty)$. To see this, we study the volume of the subset $\cup_{s \in \mathbb{D}^+(l,m+1)} F_s$. For $p \in X \setminus D$ close to $F_0$, we can find $s \in \mathbb{D}^+(l,m+1)$ such that $s(p) = 0$. So, we have

$$\|\sigma_0(p)\| \leq ||\sigma_1(p)|| + \tau \sum_{i} s_i \sigma_1(p) || \leq C\tau.$$

On the other hand, $\varphi < v^{-1/\beta} \sqrt{-1} \partial \partial h_0$ near $F_0$. Thus, we have

$$\int_{\cup_{s \in \mathbb{D}^+(l,m+1)} F_s} \omega^i = O(\tau^2 v^{-n/\beta}) = O(v^{1-n/\beta}).$$

It follows from the choice of $v > 0$ in this theorem that the desired convergence above holds as $(\tau, v, c) \to (0, 0, \infty)$ by taking sufficiently large $\beta$. Then, we obtain a holomorphic vector field

$$\nabla^{1,0} \phi = g^{i,j} \overline{\partial \phi} \frac{\partial}{\partial \psi^i} \frac{\partial}{\partial z^j}$$

on $X \setminus (D \cup F_0)$. Here, we write $\sqrt{-1} \partial \partial (t + \varphi) = \sqrt{-1} g_{i,j} dz^i \wedge d\overline{\psi}$. It follows from the definitions of $\phi$ and $g^{i,j}$ that $\nabla^{1,0} \phi$ can be extended to $X$. The decay condition of $\phi$ and the assumption of holomorphic vector fields on $X$ imply that $\phi = 0$. This is contradiction and the resonance theorem (Theorem 3.1) implies that the inverse operator $D_\varphi^* D_\varphi^{-1}$ has an uniform bound. \hfill $\square$

Recall the following relation

$$L_\varphi = -D_\varphi^* D_\varphi + (\nabla^{1,0} \phi, \nabla^{0,1} \phi)_{\varphi}.$$

Thus, Theorem 1.7 and Theorem 3.2 imply that Condition 1.2 holds with respect to $\varphi$.

**Theorem 3.3.** Take parameters so that $v^{1/\beta} c = k \log c$ and $\tau^2 = v$. Assume that $\theta_D$ is $\csc K$ and $D_\varphi^* D_\varphi : C^{4,\alpha}_{\delta-4} \to C^{0,\alpha}_{\delta}$ is isomorphic. Then, we can make the norm of the linear operator $(\nabla^{1,0} \phi, \nabla^{0,1} \phi)_{\varphi} = L_\varphi + D_\varphi^* D_\varphi$ small arbitrarily so that $L_\varphi : C^{4,\alpha}_{\delta-4} \to C^{0,\alpha}_{\delta}$ is isomorphic. Moreover, we can find a constant $\tilde{K} > 0$ such that

$$||L_\varphi \phi||_{C^{0,\alpha}_{\delta}} \geq \tilde{K} ||\phi||_{C^{4,\alpha}_{\delta-4}}$$

for any $c, v, \tau, \phi \in C^{4,\alpha}_{\delta-4}$. 


We need the following lemma:

**Lemma 3.4.** Assume that \( n \geq 5 \) and 

\[
3\hat{S}_D < n(n - 1).
\]

Then, for \( \delta > 0 \), there exists \( c_0 > 0 \) independent of \( \varpi \) such that if \( ||\phi||_{C^{4,\alpha}_{\delta-4}(X \setminus D)} \leq c_0 \), we have

\[
||L_{\varpi_{\phi}} - L_{\varpi}||_{C^{4,\alpha}_{\delta-4} \rightarrow C^{0,\alpha}_s} \leq \frac{\hat{K}}{2}
\]

and \( \varpi_{\phi} = \varpi + \sqrt{-1}d\bar{\partial}\phi \) is positive.

**Proof.** For \( \psi \in C^{4,\alpha}_{\delta-4} \), the following inequality holds:

\[
||(r^2 + 1)^{\delta/2}(g^{i\bar{j}}_{\phi} g^{k\bar{l}}_{\phi} - g^{i\bar{j}}_{\Phi} g^{k\bar{l}}_{\Phi})\psi_{i\bar{j},k\bar{l}}||_{C^{0,\alpha}} \\
\leq ||(r^2 + 1)^{\delta/2}(g^{i\bar{j}}_{\phi} g^{k\bar{l}}_{\Phi} - g^{i\bar{j}}_{\Phi} g^{k\bar{l}}_{\phi})||_{C^{0,\alpha}} ||\psi||_{C^{4,\alpha}_{\delta-4}(X \setminus D)} \\
= ||(r^2 + 1)^{\delta/2}(g^{i\bar{j}}_{\phi} (g^{k\bar{l}}_{\phi} - g^{k\bar{l}}_{\Phi}) + (g^{i\bar{j}}_{\Phi} - g^{i\bar{j}}_{\phi}) g^{k\bar{l}}_{\phi})||_{C^{0,\alpha}} ||\psi||_{C^{4,\alpha}_{\delta-4}(X \setminus D)}.
\]

In addition, we have the following equation:

\[
g^{i\bar{j}}_{\phi} - \frac{1}{g} = g^{-1}(g - g_{\phi})g^{-1}
\]

for \( \phi \in C^{4,\alpha}_{\delta-4} \) such that \( \varpi_{\phi} = \varpi + \sqrt{-1}d\bar{\partial}\phi \) is positive.

It is enough to study the region where \( M_{c,v,\eta} = t + \varphi + c \). The \( C^2 \)-estimate of the degenerate complex Monge–Ampère equation tells us that

\[
g^{i\bar{j}}_{\phi} = O(||\sigma_D||^{-2m/l}) = O(r^{2m/l \times n(n-1)}\hat{S}_D).
\]

Since we have already known the explicit \( C^2,\alpha \)-estimate of the solution of the degenerate complex Monge–Ampère equation from [2], we can estimate the \( C^{0,\alpha} \)-norm of coefficients \( g^{i\bar{j}}_{\phi}, g^{i\bar{j}}_{\phi} \). The hypothesis

\[
a(n) \frac{m}{2l} < \frac{\hat{S}_D}{n(n - 1)}
\]

implies that \( 4 + 3 \times 2m/l \times n(n - 1)/\hat{S}_D - (\delta - 4) < 8 + 12/a(n) - \delta < 0 \). So, the equation (3.2) and the estimate (3.3) implies that the term

\[
||(r^2 + 1)^{\delta/2}(g^{i\bar{j}}_{\phi} (g^{k\bar{l}}_{\phi} - g^{k\bar{l}}_{\Phi}) + (g^{i\bar{j}}_{\Phi} - g^{i\bar{j}}_{\phi}) g^{k\bar{l}}_{\phi})||_{C^{0,\alpha}}
\]

is estimated form above by \( 2c_0 \). By taking a sufficiently small \( c_0 \), we can make the operator norm of \( L_{\varpi_{\phi}} - L_{\varpi} \) small arbitrarily. Thus, we have the desired result. \( \square \)

**Remark 3.5.** The reason why we replace the hypothesis for weights of Banach spaces in the above lemma comes from the \( C^2 \)-estimate of the solution of the degenerate complex Monge–Ampère equation due to Păun [10]. From this, the positivity of \( \varpi_{\phi} \) holds. On the other hand, we need to assume that \( \delta - 4 > 4 \) to control the factor \( (r^2 + 1)^{\delta/2} \). So, the choice of a weight \( \delta \) implies that we need to assume that the dimension \( n \) is greater than 4 and \( \hat{S}_D/n(n-1) \) is smaller than 1/3. In addition, since we need to choose \( \epsilon > \delta + 2 \) in the proof of Theorem [1,7] we need to assume that \( n > 5 \).
Constants $\hat{K}$ and $c_0$ which appear in Theorem 3.3 and Lemma 3.4 respectively, are uniform for parameters $c, v, \tau$. Therefore, Theorem 1.7 implies that Condition 1.3 holds with respect to $\omega$.

**Theorem 3.6.** For the complete Kähler metric $\omega$ above, the inequality

$$||S(\omega)||_{C^{0,\alpha}_\delta} \leq c_0 \hat{K}/2$$

holds by taking suitable parameters $v, c, \tau$.

### 3.2 The fixed point theorem

Finally, we show that the existence of a complete scalar-flat Kähler metric on $X \setminus D$. Following Arezzo-Pacard [3], [4], for the expansion of the scalar curvature

$$S(\omega + \sqrt{-1} \partial \bar{\partial} \phi) = S(\omega) + L_\omega(\phi) + Q_\omega(\phi),$$

we consider the following operator

$$\mathcal{N}(\phi) := -L^{-1}_\omega (S(\omega) + Q_\omega(\phi)) \in C_{\alpha}^{4,\alpha}$$

for $\phi \in C_{\alpha}^{4,\alpha}$ by following Arezzo-Pacard [3], [4] (see also [11]). Lemma 3.4 implies that $\mathcal{N}$ is the contraction map on the neighborhood of the origin of $C_{\delta-4}^{4,\alpha}$ for a suitable weight $\delta$. In [1], we assume that Condition 1.2 and Condition 1.3 hold. Namely, we assume that there exists a complete Kähler metric $\omega_0$ whose scalar curvature is sufficiently small so that the operator $L_{\omega_0}$ has the uniformly bounded inverse. As we have seen, by constructing the Kähler metric $\omega$, Theorem 3.3 and Theorem 3.6 imply that we don’t have to assume that Condition 1.2 and Condition 1.3 hold. The following Proposition implies the existence of a complete scalar-flat Kähler metric.

**Proposition 3.7.** Set

$$U := \left\{ \phi \in C_{\delta-4}^{4,\alpha} : ||\phi||_{C_{\delta-4}^{4,\alpha}} \leq c_0 \right\}.$$

If the assumption in Theorem 1.8 holds, the operator $\mathcal{N}$ is a contraction on $U$ and $\mathcal{N}(U) \subset U$ by taking suitable parameters $c, v, \tau$.

**Proof.** Immediately, we have

$$||\mathcal{N}(\phi)||_{C_{\delta-4}^{4,\alpha}} \leq ||\mathcal{N}(\phi) - \mathcal{N}(0)||_{C_{\delta-4}^{4,\alpha}} + ||\mathcal{N}(0)||_{C_{\delta-4}^{4,\alpha}}.$$

From Lemma 3.4 and the condition $||\phi||_{C_{\delta-4}^{4,\alpha}} \leq c_0$, the same argument in the proof in [1] implies that we obtain the following estimate:

$$||\mathcal{N}(\phi) - \mathcal{N}(0)||_{C_{\delta-4}^{4,\alpha}} \leq || - L^{-1}_\omega (Q_\omega(\phi)) ||_{C_{\delta-4}^{4,\alpha}} \leq \hat{K}^{-1} ||L_{\omega+s\sqrt{-1} \partial \bar{\partial} \phi} - L_\omega||_{C_{\delta-4}^{4,\alpha}} ||\phi||_{C_{\delta-4}^{4,\alpha}}$$

for some $s \in [0, 1]$. Lemma 3.4 implies that we have

$$||\mathcal{N}(\phi) - \mathcal{N}(0)||_{C_{\delta-4}^{4,\alpha}} \leq \frac{1}{2} c_0.$$
Theorem 3.3 and Theorem 3.6 implies that we have
\[ ||N(0)||_{C^{4,\alpha}_{d-4}} = ||L^{-1}_{\omega}(S(\omega))||_{C^{4,\alpha}_{d-4}} \leq \hat{K}^{-1}||S(\omega)||_{C^{0,\alpha}_{d}} \leq \frac{1}{2}c_0. \]
Thus, \( N(\phi) \in U. \)

**Proof of Theorem 1.8.** From the discussion above, there exists a unique \( \phi_\infty := \lim_{i \to \infty} N^i(\phi) \) for any \( \phi \in U \subset C^{4,\alpha}_{d-4} \) satisfying \( \phi_\infty = N(\phi_\infty) \) under the hypothesis in Theorem 1.8. Therefore, \( \omega + \sqrt{-1}\partial\bar{\partial}\phi_\infty \) is a complete scalar-flat Kähler metric on \( X \setminus D. \)

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