The first coefficient of Homflypt and Kauffman polynomials: Vertigan proof of polynomial complexity using dynamic programming
by Józef H.Przytycki

Abstract. We describe the polynomial time complexity algorithm for computing first coefficients of the skein (Homflypt) and Kauffman polynomial invariants of links, discovered by D.Vertigan in 1992 but never published.

1. Introduction

We showed in [P-P-2] that an essential part of the Jones-type polynomial link invariants can be computed in subexponential time. This is in a sharp contrast to the result of Jaeger, Vertigan and Welsh [JVW] that computing the whole polynomial and most of its evaluations is \#P-hard and is conjectured to be of exponential complexity.

Motivated by [P-P-2], Dirk Vertigan described the polynomial time complexity algorithm for computing first coefficients of the skein (Homflypt) and Kauffman polynomials of links\(^1\). The polynomial time complexity of other coefficients follows easily from the first coefficient. We express the time complexity of our algorithms as a function of the number of crossings, \(n\), and we assume that the number of link components, \(\text{com}(L)\), of a link \(L\) is less than or equal to the number of crossings.

The skein (Homflypt) polynomial, \(P_L(a, z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]\), of oriented links in \(R^3\) is defined recursively as follows [HOMFLY, PT]:

\[
\begin{align*}
(\text{i}): & \quad P_{\text{trivial knot}}(a, z) = 1, \\
(\text{ii}): & \quad aP_{L_+}(a, z) + a^{-1}P_{L_-}(a, z) = zP_{L_0}(a, z).
\end{align*}
\]

Let \(\text{com}(L)\) denote the number of components of \(L\) then \(z^{\text{com}(L)-1}P_L(a, z) \in \mathbb{Z}[a^{\pm 1}, z]\) and it can be written as \(\sum_{i=0}^{\text{com}(L)} P_{2i}(a)z^{2i} \in \mathbb{Z}[a^{\pm 1}, z]\).

\(^1\)On 13 Jan 1992 we got an e-mail from Paul Seymour, editor of Proceedings to which [P-P-2] was submitted informing as that: “The referee for your paper on polynomials for the Seattle meeting has done some further work of his own, extending the results in your paper, and now he is worried that he has abused his position as referee for his own gain. I asked him to summarize his results and send them to me, and told him I would pass them on to you. So please, what are your reactions? Do you have any objections to the referee publishing the stuff below as his own work?” We were very enthusiastic about the referee’s result but he somehow never published the paper, and we included his description in the appendix of our preprint [P-P-1].
Theorem 1.1 (Vertigan).

\( P_{2i}(a) \) can be computed in polynomial time. More precisely: let \( D \) be a diagram of \( L \) with \( n \) crossings then the time complexity of computing \( P_{2i}(a)(L) \) is \( O(n^{2+3i}) \).

In fact Vertigan announced \( O(n^{2+2i}) \) time algorithm but the proof is more involved than that of Theorem 1.1, in which case one easily reduces the theorem for \( P_{2i}(a) \) for links to the result for \( P_0(a) \) for knots. We describe the case of \( P_0(a) \) first.

2. Computation of \( P_0(a) \)

Theorem 2.1. Let \( n \) denote the number of crossings of a knot diagram then \( P_0(a) \) can be computed in quadratic time (i.e. in \( O(n^2) \) time).

Proof. Let \( D \) be an oriented knot diagram with \( n \) crossings. We can think of \( D \) as a 4-regular graph (any crossing is a vertex of valency four). Choose a point inside any edge of \( D \) and order them according to the orientation of the knot. So we get points \( b_0, b_1, ..., b_{m-1}, b_m = b_0 \) where \( m \) is the number of edges of \( D \) (in fact \( m = 2n \)). We think of \( b_0 \) as a base point of \( D \). Let \( D_{i,j}, (0 \leq i < j \leq m) \), denote the part of \( D \) between points \( b_i \) and \( b_j \) (with the convention that \( D_{0,m} \) denotes \( D \)). Further let \( \hat{D}_{i,j} \) denote the closure of \( D_{i,j} \), that is we join, in \( \hat{D}_{i,j} \), \( b_j \) with \( b_i \) by an overpass (an arc going above the rest of the diagram), compare Fig.2.1.

\[
\begin{align*}
\text{Fig. 2.1; } D & = D_{0,10} = \hat{D}_{0,10}, D_{2,9} \text{ and } \hat{D}_{2,9} \\
\end{align*}
\]

Lemma 2.2. If \( j - i \leq 3 \) then \( \hat{D}_{i,j} \) represents the unknot.

Proof. \( D_{i,j} \), for \( j - i \leq 3 \), can have at most one crossing and \( \hat{D}_{i,j} \) can be drawn with no more than one crossing. Therefore \( \hat{D}_{i,j} \) represents the unknot.

Notice that \( \hat{D}_{i,i+4} \) cannot represent a nontrivial knot neither. \( \hat{D}_{i,i+5} \) can represent only a trefoil knot or the unknot (compare Section 5).
To continue the proof of Theorem 2.1 first observe that if diagrams $D_+, D_-$ and $D_0$ form a skein triplet then the skein relation for the skein polynomial $P(a, z)$ reduces, for $P_0(a)$, to the formula:

$$a P_0(a)(D_+) + a^{-1} P_0(a)(D_-) = \begin{cases} 
P_0(a)(D_0) & \text{in the case of a selfcrossing} \\
0 & \text{in the case of a crossing between different components} 
\end{cases}$$

For the trivial link of $n$ components, $T_n$, one has $P_0(a)(T_n) = (a + a^{-1})^{n-1}$. Now consider $D_{i,j}$ which has a crossing (otherwise $\hat{D}_{i,j}$ represents the unknot). Let $q$ be the first crossing in $D_{i,j}$ after $b_i$. Without lost of generality we can assume that the arc $b_i, b_{i+1}$ is involved in the crossing (otherwise $D_{i,j} = D_{i+1,j}$). We have two possibilities:

(i) the arc $b_i, b_{i+1}$ is an overpass and then $\hat{D}_{i,j} = \hat{D}_{i+1,j}$, or
(ii) the arc $b_i, b_{i+1}$ is an underpass and in that case we consider the skein triplet $\hat{D}_{i,j}, \hat{D}_{i,j}', \hat{D}_{i,j}^0$ where the second element of the triplet is obtained from the first by changing at $q$ the undercrossing to the overcrossing and the third by smoothing it at $q$. The important observation here is that $\hat{D}_{i,j}' = \hat{D}_{i+1,j}$ and that $\hat{D}_{i,j}^0$ is a two component link composed of $\hat{D}_{i+1,k}$ and $\hat{D}_{k+1,j}$ where $i < k \leq j$ and $q$ is the crossing between arcs $b_i, b_{i+1}$ and $b_k, b_{k+1}$ (compare Fig.2.2). The first coefficient of a two component link can be easily computed from that of the components (see [L-M] or formula 3.1). Therefore we get:

$$a^\epsilon(q) P_0(a)(\hat{D}_{i,j}) + a^{-\epsilon(q)} P_0(a)(\hat{D}_{i+1,j}) =$$

$$(-a^{-2})^{\ell k(\hat{D}_{i,j})} (a + a^{-1}) P_0(a)(\hat{D}_{i+1,k}) P_0(a)(\hat{D}_{k+1,j})$$

where $\epsilon(q)$ is the sign of the crossing $q$ and $\ell k(L)$ the global linking number of the link $L$.

(i) and (ii) allow as to reduce the computation of $P_0(a)(\hat{D}_{i,j})$ to that of $P_0(a)(\hat{D}_{s,t})$ with $i < s$ and $t \leq j$. Furthermore we know the value of $P_0(a)(\hat{D}_{i,j})$ to be equal to 1 for $j - i \leq 3$ by Lemma 2.2. Therefore we can find the value of $P_0(a)(\hat{D}_{i,j})$ for any $0 \leq i < j \leq m$, including that for $D = D_{0,m} = \hat{D}_{0,m}$, in at most $m^2/2 = 2n^2$ steps. This completes the proof of Theorem 2.1. \qed
Note that we do not address technical details of complexity of presenting the computed polynomial in the ordered form. One can improve constant by considering $D$ or its mirror image $\bar{D}$ and observing that $D$ or $\bar{D}$ can be changed to a descending diagram by switching no more than $\frac{n^2}{2}$ crossings.

3. Computation of $P_{2i}(a)$.

To finish the proof of Theorem 1.1 first observe that Theorem 2.1 can be extended to the case of a link by the Lickorish-Millett formula [L-M]:

3.1.

For a link $L$ of $\text{com}(L)$ components $K_1, K_2, ... K_{\text{com}(L)}$

$$P_0(a)(L) = (-a^{-2})^{k(L)}(a + a^{-1})^{\text{com}(L) - 1}\prod_{i=1}^{\text{com}(L)} P_0(a)(K_i)$$

We assume that the number of components of a link is not too big with respect to the number of crossings. It remains to see that one can find $P_{2i+2}(a)$ in $O(n^{2+3(i+1)})$ time assuming that $P_{2i}(a)$ can be found in $O(n^{2+3i})$ time. We use the generalization of Formula 3.1 to any coefficient $P_{2i}(a)$:

3.2.

$$P_{2i+2}(a)(L) = (-a^{-2})^{k(L)}(a + a^{-1})^{\text{com}(L) - 1}\prod_{j=1}^{\text{com}(L)} P_{2i+2}(a)(K_j) + \sum_{j=1}^{n'} P_{2i}(a)(L_j)$$

where $n'$ denotes the number of crossings between different components of the considered digram of $L$ (therefore $n' \leq n$) and $L_j$’s are certain $n - 1$ crossing $\text{com}(L) - 1$ component link diagrams obtained from $L$. 
Formula 3.2 follow from the recursive relation:

$$aP_{2n+2}(a)(D_+) + a^{-1}P_{2n+2}(a)(D_-) = \begin{cases} P_{2n+2}(a)(D_0) & \text{in the case of a selfcrossing} \\ P_{2n}(a)(D_0) & \text{in the case of a crossing} \\ \end{cases}$$

between different components

Then we proceed exactly as in the proof of Theorem 2.1 except that for the value of $P_{2i+2}(a)(\hat{D}_{i,j}^0)$ one has to use formula 3.2 instead of 3.1.

4. COEFFICIENTS OF THE KAUFFMAN POLYNOMIAL, $F_L(a, z)$.

The Vertigan algorithm can be used also to compute first coefficients of the Kauffman polynomial, $F_L(a, z)$, in polynomial time. One can write $z^{\text{com}(L)-1}F_L(a, z)$ as $\sum_{i=0}^{N} F_i(a)z^i$.

**Theorem 4.1** (Vertigan). $F_i(a)$ can be computed in polynomial time. More precisely: let $D$ be a diagram of $L$ with $n$ crossings, then the time complexity of computing $F_i(a)(L)$ is $O(n^{2+2i})$.

**Proof.** (sketch) The main point of the proof is the observation that $F_0(a)(L) = P_0(a)(L)$ (compare [Pr] or [Li]). The additional information needed in the proof is the skein relation connecting coefficients of the Kauffman polynomial of diagrams $D_+, D_-, D_0$ and $D_\infty$:

4.2.

$$a^{w(D_+)}aF_{i+2}(a)(D_+) + a^{w(D_-)}a^{-1}F_{i+2}(a)(D_-) = \begin{cases} a^{w(D_0)}F_{i+2}(a)(D_0) + a^{w(D\infty)}F_{i+1}(a)(D\infty) & \text{in the case of a selfcrossing} \\ a^{w(D_0)}F_i(a)(D_0) + a^{w(D\infty)}F_i(a)(D\infty) & \text{in the case of a mixed crossing} \end{cases}$$

where $D_+, D_-$ and $D_0$ are consistently oriented diagrams. For $D_\infty$ we can choose any orientation which agrees with that of $D_+\text{ outside components involved in the crossing.} w(D)$ is the planar writhe (or Tait number) of $D$ equal to the algebraic sum of signs of crossings.

5. POLYNOMIALS OF VIRTUAL DIAGRAMS.

As a comment to the note after Lemma 2.2 one should stress that $D_{i,i+4}$ from Figure 5.1 cannot be obtained from any diagram $D$, so formally if $j-i=4$ then $\hat{D}_{i,i+4}$ represents the unknot. Only $\hat{D}_{i,i+5}$ can represent a trefoil (as illustrated in Figure 5.2).
Fig. 5.1; $D_{i,+4}$, $\hat{D}_{i,+4}$

Fig. 5.2; $D = D_{0,6}$, $D_{0,5}$ and $\hat{D}_{0,5}$

However, more possibilities arrive if we allow virtual diagrams (as introduced by Kauffman [Kau]). It may be interesting to use Vertigan algorithm for skein (Homflypt) and Kauffman polynomials of virtual knots.

6. DYNAMIC PROGRAMMING

The method of dynamic programming, used in Vertigan algorithm is not familiar in knot theory circles, thus we give a short, historically based, introduction to the topic.

From [CLR].

R.Bellman began the systematic study of dynamic programming in 1955. The word “programming,” both here and in linear programming, refers to the use of a tabular solution method. Although optimization techniques incorporating elements of dynamic programming were known earlier, Bellman provided the area with a solid mathematical basis (Richard Bellman [Be]).

Dynamic programming is effective when a given subproblem may arise from more than one partial set of choices; the key technique is to
store, or “memorize,” the solution to each such subproblem in case it should reappear. ...this simple idea can easily transform exponential-time algorithms into polynomial-time algorithms.

Example: Longest common subsequence. 

$O(mn)$-time algorithm for the longest-common-subsequence problem seems to be a folk algorithm.

In a longest-common-subsequence problem, we are given two sequences $X = (x_1, x_2, ..., x_m)$ and $Y = (y_1, y_2, ..., y_n)$ and wish to find a maximum-length common subsequence of $X$ and $Y$.

Another example of dynamic programming is used in H. Morton’s algorithm computing the Homflypt polynomial of closed $k$ braids (fixed $k$) in polynomial time with respect to the number of crossings $\text{M-S}$.

7. Knotoids of Vladimir Turaev

One should mention here that the theory of Knotoids introduced by V. Turaev in 2010 $\text{Tur}$ is, at least in its pictographic form, very much related to Vertigan approach to compute first coefficients of the Homflypt and Kauffman polynomials.

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References

[Be] R. Bellman, Dynamic Programming, Princeton University Press, 1957.

[CLR] T. H. Cormen, C. E. Leiserson, R. L. Rivest, Introduction to Algorithms, The MIT Press, 1989, Chapter IV(16): Dynamic Programming.

[HOMFLY] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, A. Ocneanu, A new polynomial invariant of knots and links. Bull. Amer. Math. Soc., 12, 1985, 239-249.

[JVW] F. Jaeger, D. L. Vertigan, D. J. A. Welsh, On the Computational Complexity of the Jones and Tutte Polynomials, Math. Proc. Camb. Phil. Soc. 108, 1990, 35-53.

[Kau] L. Kauffman, A Survey of Virtual Knot Theory, Proceedings of in Proceedings of Knots in Hellas 98, World Sci. Pub. 2000, 143-202.

[Li] W. B. R. Lickorish, Polynomials for links, Bull. London Math. Soc. 20, 1988, 227-294.

[L-M] W. B. R. Lickorish, K. Millett, A polynomial invariant of oriented links, Topology 26(1), 1987, 107-141.
H. R. Morton, H. B. Short, Calculating the 2-variable polynomial for knots presented as closed braids, *J. Algorithms* 11(1), 1990, 117–131.

T. M. Przytycka, J. H. Przytycki, Subexponentially Computable Truncations of Jones-type Polynomials, with Appendix on Vertigan’s Algorithm, Technical Report 22, Odense University, October 1992.

T. M. Przytycka, J. H. Przytycki, Subexponentially computable truncations of Jones-type polynomials, in “Graph Structure Theory”, Contemporary Mathematics 147, 1993, 63-108.

J. H. Przytycki, Survey on recent invariants in classical knot theory, Warsaw University Preprints 6,8,9; 1986; a part of: *Knots: combinatorial approach to the knot theory*, Script, Warsaw, August 1995, (in Polish, English translation (extended) in preparation; to be published by Cambridge University Press, 2018); the Survey was put on arXiv in 2008: e-print: [http://front.math.ucdavis.edu/0810.4191](http://front.math.ucdavis.edu/0810.4191)

J. H. Przytycki, P. Traczyk, Invariants of links of Conway type. *Kobe J. Math.* 4, 1987, 115-139.

V. Turaev, Knotoids, *Osaka Journal of Mathematics*, 49, 2012, 195-223; e-print: [arXiv:1002.4133 [math.GT]](http://arxiv.org/abs/1002.4133)

D. Vertigan, letter of March 26, 1992.

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