Majorana path integral for nonequilibrium dynamics of two-level systems

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We present a new field-theoretic approach to analyze non-equilibrium dynamics of two-level systems (TLS), which is based on a correspondence between a driven TLS and a Majorana fermion field theory coupled to bosonic fields. This approach allows us to calculate analytically properties of non-linear TLS dynamics with an arbitrary accuracy. We apply our method to analyze specific TLS dynamics under a monochromatic periodic drive that is relevant to the problem of decoherence in Josephson junction qubits. It is demonstrated that the method gives the precise positions of the resonance peaks in the non-linear dielectric response function that are in agreement with numerical simulations.

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I. INTRODUCTION

A driven two-level-system (TLS) represents a canonical dynamical system that features a rich variety of interesting non-linear phenomena including various resonance effects, quantum interference phenomena due to level crossings, coherent destruction of tunnelling, etc (see e.g., Ref. [1]). Despite the formal simplicity of its formulation, this quantum dynamical system does not admit an exact solution in a closed analytical form for an arbitrary external drive and one often resorts to numerical simulations or various approximation schemes for its analysis. There are deep mathematical reasons for the lack of our ability to solve the corresponding differential equations, which go back to the old works on Riccatti differential equations and Lie theory. The mathematical problem itself has a wide spectrum of applications in technology and physics ranging from technologically important nuclear magnetic resonance spectroscopy\textsuperscript{a} to Maxwell-Bloch theory of two-level lasers\textsuperscript{b}, non-equilibrium superconductivity\textsuperscript{c}, and fundamental field-theoretical models such as the Wess-Zumino-Witten theory\textsuperscript{d}. Furthermore, with the increased technological ability to fabricate and control quantum systems, new realizations of the model arise on a continuous basis, such as, for example, artificial “atoms” interacting with strongly oscillating fields\textsuperscript{a}. This variety of new applications has motivated focused theoretical researches of the model recently, see e.g., Refs. [2]-[11] that studied a weak driving limit with small number of photons and Ref. [12] that investigated the regime of strong driving with large photon numbers, just to name a few relevant papers.

Another important field, where the problem of driven TLS dynamics is of great importance, is quantum computing. There are various realizations of qubits\textsuperscript{13}-\textsuperscript{15}, which in the course of quantum evolution may exhibit interesting non-linear phenomena, such as interference between multiple Landau-Zener transitions at a level crossing, where adiabatic evolution between them results in an oscillatory qubit magnetization in the regime of strong qubit driving\textsuperscript{16-12}. Physics of driven TLS shows up in qubits also from a different perspective: Low-energy charge defects are widely believed to be the dominant source of dephasing in superconducting Josephson junction qubits\textsuperscript{16}. There, interactions between the charged TLS defect and an applied electric field gives rise to the same problem of a non-equilibrium TLS under a periodic monochromatic drive. In general driven TLS was a subject of intensive studies during the last decade, different aspects of which are presented in e.g., Refs.[13]-[23].

In most cases mentioned above, the basic problem that we are actually interested in is summarized by the simple Hamiltonian, $H = -\Delta(t) \cdot \hat{\sigma}$, which leads to the following evolution operator $\hat{U}(\tau)$ and the “partition function” $Z$

$$\hat{U}(\tau) = \hat{T} \exp \left\{ i \int_0^\tau dt \Delta(t) \cdot \hat{\sigma} \right\}, \quad Z = \text{tr} \hat{U}(\tau). \quad (1)$$

Here $\hat{T}$ is the time ordering operator, $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ is the vector of Pauli matrices, and $\Delta(t) = [\Delta_1(t), \Delta_2(t), \Delta_3(t)]$ is the three-component external driving field.

In this paper, we propose an analytical Majorana field-theory approach, which is capable of describing arbitrary external driving fields. Though the approach can be applied to the most general case however we consider following form of the periodic driving field: $\Delta(t) = [\Delta_1(t), \Delta_2(t), \Delta_3(t)]$, where $\Delta_3$ is time independent constant (this assumption does not break the generality as it corresponds to transforming to a frame with static $\Delta_3$) and we also assume that the two time-varying components have a finite number of harmonics, $\Delta_i(t) = \epsilon_i + \sum_n A_n \cos(\omega_n t)$, $i = 1, 2$. Let us first summarize three steps that one has to follow to analyze this quantum dynamical system within our approach: (1) Fourier transform the fields:

$$\Delta_i(\omega) = \epsilon_i \delta(\omega) + \sum_{n, \sigma = \pm} A_n \delta(\omega + \sigma \omega_n); \quad (2)$$

(2) Write exact Dyson equations for the fermionic Green’s function, $\mathcal{K}(\omega, \omega')$ in a Majorana field theory;
(3) Solve the resulting equations recursively. The following presentation deciphers this prescription and provides a specific example of its use for the most experimentally relevant (but analytically unsolvable) case of a simple monochromatic drive:

\[ \Delta_1(t) = \frac{\varepsilon}{2} + A \cos(\omega t), \quad \Delta_2(t) = 0, \quad \Delta_3(t) = \frac{\Delta}{2} = \text{const}, \]

were \( \varepsilon \) is an energy splitting, \( w \) and \( A \) are the frequency and amplitude of an external field. Our main result is a practically-useful continued fraction representation of the correlation function, \( K(t, t') = \text{tr}(\hat{\sigma}_3(t)\hat{\sigma}_3(t')) \), that contains its exact spectrum.

**II. ACTION IN TERMS OF MAJORANA FERMIONS**

Our work is based on the observation that the expression for the “partition function,” \( Z \), resembles the Green’s function of a spinning particle passing in a one-dimensional space from point \( x(t = 0) = 0 \) to \( x(t = \tau) \) in Feynman path integral quantization approach \( \text{[24]} \), where \( \Delta(t) \) has a meaning of the velocity, \( \dot{x}(t) \), of the spinning particle. Subsequently, the conversion to Majorana fields is achieved by using the approach of Refs. \( \text{[24,25]} \), which provides a simple prescription: the Pauli matrices \( \hat{\sigma}_i \), \( \hat{\sigma}_3 = i\hat{\sigma}_1\hat{\sigma}_2 \) should be replaced by the Majorana fields \( \xi_\mu(t), \xi_\nu(t) \) respectively. In other words, Pauli matrices can be regarded as a quantized version of path-integral Majorana fields. We note that Majorana fermion is its own antiparticle, i.e., its creation and annihilation operators are identical. As fields, they can be described by real valued Grassmann variables \( \xi_\mu(t), \xi_\nu(t) \) denoted below as \( \xi(t) \). The spin dynamics has been investigated in a different context using Majorana fermion representation in Ref. \( \text{[27]} \). Similar approaches have been employed previously in Refs. \( \text{[28,29]} \).

A cornerstone of this work is that the “partition function,” \( Z \), i.e., the trace of the evolution operator can be exactly reproduced within a field theory of three Majorana fermions \( \xi_1(t), \xi_2(t), \text{ and } \xi_3(t) \), defined by the functional integral

\[ Z = \frac{1}{Z_0} \int \mathcal{D}\xi_1(t)\mathcal{D}\xi_2(t)\mathcal{D}\xi_3(t)e^{iS(\{\xi\}, \Delta)}. \]

where \( Z_0 = \text{Det} \left[ \frac{d}{dt} \right] \), and the action [here we set \( \Delta_3(t) \equiv \Delta/2 = \text{const} \)] has the form:

\[ iS(\{\xi\}, \Delta) = \int_0^\tau dt \left[ \frac{1}{4} \xi_\mu(t)\dot{\xi}_\mu(t) + \frac{1}{4} \xi_3(t)\dot{\xi}_3(t) \right. \]

\[ + \left. \Delta_\mu(t)\xi_\mu(t)\xi_\nu(t) - \frac{1}{4} \epsilon_{\mu\nu} \xi_\mu(t)\xi_\nu(t)\Delta \right]. \]

In Eq. \( \text{[5]} \) \( \mu, \nu = 1, 2, \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0 \) and the summation over repeated indices is implied.

**Equivalence of the driven TLS [3] and the Majorana field theory [5] can be shown in three steps:**

(i) First we expand the exponent of the interaction term in the expression of the partition function, \( Z \) [see Eq. \( \text{[4]} \)]

\[ Z = \frac{1}{Z_0} \sum_{n=0}^\infty \int \mathcal{D}\xi_1(t)\mathcal{D}\xi_3(t) \int_0^\tau \frac{dt_1}{\int_0^t dt_2 \cdots \int_0^{t_{n-1}} dt_n} \]

\[ (\Delta_\mu(t_1)\xi_\mu(t_1)\xi_\nu(t_1) - \frac{1}{4} \epsilon_{\mu\nu} \xi_\mu(t_1)\xi_\nu(t_1)\Delta_3) \]

\[ \times \exp \left\{ \int_0^\tau dt \left[ \frac{1}{4} \xi_\mu(t)\dot{\xi}_\mu(t) + \frac{1}{4} \xi_3(t)\dot{\xi}_3(t) \right] \right\}. \]

(ii) At the second stage we make use of the free Green’s function of the Majorana fields (see details in Appendix A)

\[ \langle \xi_\mu(t_1)\xi_\nu(t_2) \rangle = \frac{1}{Z_0} \int \mathcal{D}\xi_2(t)\xi_\nu(t_1)\xi_\nu(t_2) \]

\[ \times \exp \left\{ \int_0^\tau dt \left[ \frac{1}{4} \xi_\mu(t)\dot{\xi}_\mu(t) + \frac{1}{4} \xi_3(t)\dot{\xi}_3(t) \right] \right\}. \]

This form of the Green’s function follows straightforwardly from the identity \( \frac{d}{dt} \text{sign}[t - t'] = \delta[t - t'] \). Then, by use of the Wick’s theorem, one can replace correlation functions of the Majorana fields in the expression \( \text{[6]} \) by the trace of the product of Pauli matrices. This procedure yields

\[ \frac{1}{Z_0} \int \mathcal{D}\xi_1(t_1)\mathcal{D}\xi_3(t_2) \cdots \mathcal{D}\xi_n \]

\[ \times \exp \left\{ \int_0^\tau dt \left[ \frac{1}{4} \xi_\mu(t)\dot{\xi}_\mu(t) + \frac{1}{4} \xi_3(t)\dot{\xi}_3(t) \right] \right\} \]

\[ = \sum_{\text{all pairings}} (-1)^P \delta_{\mu_1\mu_1} \delta_{\mu_2\mu_2} \cdots \delta_{\mu_n/2\mu_n/2} \]

\[ = \text{Tr}[\sigma_\mu \sigma_\mu \cdots \sigma_\mu], \]

where \( P \) is the number of permutations needed for obtaining the set of indices \( \mu_1\mu_1, \mu_2\mu_2 \cdots \mu_n/2\mu_n/2 \) from \( \mu_1\mu_2 \cdots \mu_n \) (note that \( n \) here is even). It is also important to note that \( \text{sign}[t - t'] \) factors in the Majorana fermion Green’s function \( \text{[7]} \) disappear in the expression above due to the time ordering of fields in Eq. \( \text{[5]} \). Similar situation is with the functional integral over the fields \( \xi_3 \), which satisfy the condition \( \xi_3(t)\xi_3(t') = \text{sign}[t - t'] \). This suggests that the fields \( \xi_3(t) \) simply can be effectively dropped from Eq. \( \text{[5]} \), as the whole integral gives one. Finally, the expression \( \text{[8]} \) shows that one can replace Majorana fields, \( \xi_\mu(t) \), by the corresponding Pauli matrices, \( \sigma_\mu \).

(iii) Upon replacing the fields \( \xi_\mu(t) \) in Eq. \( \text{[5]} \) by the Pauli matrices, \( \sigma_\mu \), dropping \( \xi_3(t) \) and subsequently collecting the obtained series back to the exponent one will obtain \( Z = \text{Tr}\hat{T} \exp \left\{ i \int_0^\tau dt (\Delta(t) \cdot \hat{\sigma}) \right\} \). In derivation of this formula we also use the relation \( \hat{\sigma}_3 = i\hat{\sigma}_1\hat{\sigma}_2 \).
As we mentioned above, action (5) can be interpreted as the path-integral quantization of relativistic spinning particle in a fixed gauge in an external magnetic field $\Delta(t) = \Delta/2$. 

Finally, Gaussian integration over the fields $\xi_\mu(t)$, $\mu = 1, 2$ gives the partition function in the form $Z = \int D[\xi_3(t)] \exp \{ i S_3[\xi_3(t), \Delta(t)] \}$, where the effective action for the $\xi_3(t)$ field reads

$$i S_3[\xi_3, \Delta(t)] = \frac{1}{4} \int_0^T dt \xi_3 + 2 \int_0^T dt dt' \xi_3(t) \times$$

$$\left[ \Delta_+(t') G_-(t'-t') \Delta_-(t) + \Delta_-(t') G_+(t'-t') \Delta_+ (t) \right] \xi_3(t).$$

Here $G_{\pm}(t'-t') = \frac{1}{2} \delta^{\pm}i \Delta(t'-t) \text{sign}[t-t']$ is the Green’s function of the differential operator $(\partial_t \pm i \Delta)$, while $\Delta_\pm(t) = \Delta_1(t) \mp i \Delta_2(t)$. Eq. (9) is one of the new results presented here.

For our purposes it is convenient to express $S_3$ through the Fourier images of the fields and functions in Eq. (9). Then, in the limit $\tau \to \infty$, we obtain for $S_3$

$$S_3[\xi_3, \Delta(t)] = -\frac{\pi}{2} \int d\omega d\omega' \xi_3(\omega) K^{-1}(\omega, \omega') \xi_3(\omega'),$$

(10)

where $K(\omega, \omega') = [\omega \delta(\omega + \omega') + G(\omega, \omega')]^{-1}$, and $G(\omega, \omega')$ is an antisymmetric kernel given by

$$G(\omega, \omega') = \int d\omega_1 \frac{\Delta_{+}(\omega_1) \Delta_{-}(\omega + \omega_1 + \omega')(\omega - \omega')}{(\omega_1 + \omega + \Delta)(\omega_1 + \omega' + \Delta)},$$

(11)

Due to the novelty of our approach, we first briefly derive several established results and then turn to the main problem of dephasing in the presence of a monochromatic drive.

III. APPLICATION TO SUPERCONDUCTIVITY AND VERIFICATION OF THE BCS RESULT

The aim of the present section is to apply the developed technique to study the nonperturbative properties of the pairing Hamiltonian. The textbook expression of the partition function of the pairing model is given in terms of a functional integral with respect to Grassmann fields, $\bar{c}_\sigma(t)$, $c_\sigma(t)$, $\sigma = \uparrow, \downarrow$, with the action including four-fermionic pairing interaction. The standard approach to treat this action is to decouple the interaction term by introducing a set of bosonic fields, $\Delta(t)$, $\overline{\Delta}(t)$, over which one will have an additional functional integral. Partition function corresponding to the zero-particle and paired sectors of the pairing Hamiltonian reduces to

$$Z_{BCS} = \int D[\Delta, \overline{\Delta}, c, c^+ \exp \{ -S_{BCS} \},$$

$$S_{BCS} = \int dt \{ \sum_k \bar{\psi}_k (\partial_\tau + h_k) \psi_k + \frac{1}{g} \Delta \}$$

(12)

where $g$ is the interaction constant,

$$\psi_k = \left( \begin{array}{c} c_{k, \uparrow} \\ \bar{c}_{-k, \downarrow} \end{array} \right)$$

(13)

defines the Nambu spinor and

$$h_k = \epsilon_k \sigma_3 + \Delta_1 \sigma_1 + \Delta_2 \sigma_2$$

(14)

is the matrix Hamiltonian, where $\Delta = \Delta_1 - i \Delta_2$, $\overline{\Delta} = \Delta_1 + i \Delta_2$. Remarkably, the form of the operator $h_k$ is very much reminiscent to our driven Hamiltonian (6), but with the third component, $\epsilon_k$, being a time independent constant. Therefore it is straightforward to employ the above developed technique of Majorana field theory to represent the exact BCS partition function, $Z_{BCS}$ as a functional with respect to one specie Majorana fermion, $\xi_3(t)$, and Hubbard-Stratonovich boson fields, $\Delta$ and $\overline{\Delta}$, see Eq. (10). If one is interested for example in calculation of $Z_{BCS}$, then, by integrating out $\xi_3(t)$ one produces an effective bosonic action:

$$Z_{BCS} = \int D[\Delta] e^{S_{eff}(\Delta)}$$

(15)

$$S_{eff} = -\int d\omega \frac{\Delta^2(\omega)}{g} + \frac{1}{2} \text{tr} \log [\omega \delta(\omega + \omega') + G(\omega, \omega')]$$

where $G(\omega, \omega')$ is a functional of $\Delta(t)$ and $\overline{\Delta}(t)$, and is defined by Eq. (11). Note that the functional integral Eq. (15) is formally nothing but a nonlinear functional determinant written in energy (rather than imaginary time) space. Variation of the effective action with respect to the bosonic fields gives the equation of motion, $\frac{\delta S_{eff}}{\delta \Delta_{\pm}(\omega)} = 0$, which in turn yields the general gap equation

$$-\frac{2\Delta_{\pm}(\omega)}{g} + \frac{1}{2} \int d\omega' K(\omega, \omega')$$

(16)

$$\times \frac{\Delta_{\pm}(\omega + \omega' + \epsilon_k)(\omega + \omega' + \epsilon_k)}{(\omega + \omega + \epsilon_k)(\omega + \omega') + \epsilon_k} = 0,$$

written in the energy representation.

The pairing hamiltonian itself is designed to describe the superconducting phase, where the order parameter, $\Delta$, is different from zero. The gap equation (16), i.e. the equation for the Fourier-transformed bosonic Hubbard-Stratonovich field, $\Delta(\omega)$, describes the dynamics of the order parameter in the global gauge symmetry broken phase.
The BCS mean-field solution corresponds to the choice \( \Delta_{1}^0(\omega) = \Delta_{0}^0 \delta(\omega) \), where \( \Delta_{0}^0 \) is \( \omega \) independent. In other words it assumes that the time-dependence of the order parameter is unimportant. Then, from Eq. (11) we find that

\[
G_{MF}(\omega, \omega') = \frac{\omega \Delta_{0}^2}{\epsilon_{k}^2 + 2 \Delta_{0}^2 - \omega^2} \delta(\omega + \omega').
\]

(17)

and

\[
\mathcal{K}_{MF}(\omega, \omega') = \frac{\omega (\epsilon_{k}^2 + 2 \Delta_{0}^2 - \omega^2) \delta(\omega + \omega')}{\epsilon_{k}^2 - \omega^2}.
\]

(18)

Making use of Eqs. (17) and (18), and substituting them into (16), we obtain

\[
-\frac{2}{g} + N(0) \int d\omega \frac{1}{\epsilon_{k}^2 + 2 \Delta_{0}^2 - \omega^2} = 0,
\]

(19)

which reproduces the standard BCS gap equation with \( N(0) \) being the approximated to a constant density of states.

**IV. NON-DISSIPATIVE TWO-LEVEL SYSTEMS**

Neglecting environmental effects on the driven TLS we consider the Hamiltonian \( H = -\Delta(t) \hat{\sigma} \), with the driving fields given by Eq. (3). Our goal is the calculation of spin-spin correlation function

\[
K(t, t') = \text{tr} \left[ U(-\infty, t) \hat{\sigma}_{3} U(t, t') \hat{\sigma}_{3} U(t', \infty) \right]
\]

(20)

where \( U(t, t') \) is an evolution operator defined in (1). According to the representation developed above, this function is identical to the correlation function, \( K(t, t') = \langle \xi_{3}(t) \xi_{3}(t') \rangle \), of the Majorana field, \( \xi_{3}(t) \), calculated with the use of action \( \mathcal{S}_{3}[\xi_{3}, \Delta] \) from Eq. (11). Importantly, in this formulation, the field \( \xi_{3}(t) \) is the representative of the Pauli matrix \( \hat{\sigma}_{3} \).

An observable of practical interest is the additional Majorana field, \( \eta \), and then decouple the quadratic in \( \Delta_{1}(t) \) term. In this new action,

\[
i \tilde{S}_{3}[\xi_{3}(t), \eta(t)] = \int_{0}^{\tau} dt \left[ \frac{1}{4} \xi_{3} \dot{\xi}_{3} + \frac{1}{4} \eta \dot{\eta} + i \Delta(t) \xi_{3} \eta \right],
\]

(22)

the field \( \Delta_{1}(t) \) plays the role of a “gauge field.” Then we introduce variables \( \xi_{\pm} = (\xi_{3} \pm \eta)/2 \) and eliminate the “gauge field,” \( i \Delta_{1}(t) = W^{-1} \partial_{w} W \) with \( W(t) = \exp[i \int^{t} \partial_{w} \Delta_{1}(t') dt] \) from the action by rescaling the fields, \( \hat{\xi}_{\pm} = W \hat{\xi} \). In this new fields action Eq. (4) acquires the form of a free field theory, \( i \tilde{S}_{3} \left[ \hat{\xi}_{\pm}(t) \right] = \frac{1}{2} \int_{0}^{\tau} dt \left[ \xi_{+}^{\dagger} \dot{\xi}_{-} + \xi_{-}^{\dagger} \dot{\xi}_{+} \right] \), but the correlation function now transforms into

\[
K(t, t') = 2 \Re \left[ \langle \hat{\xi}_{-}^{\dagger}(t) \hat{\xi}_{+}(t') \rangle e^{2i \int_{0}^{\tau} dt' \Delta_{1}(t') \right]
\]

\[
= \text{sign}[t - t'] \cos \left[ 2 \int_{0}^{\tau} dt' \Delta_{1}(t') \right].
\]

(23)

This known expression can also be obtained directly from the original \( T \)-ordered exponent.

If \( \Delta \neq 0 \), it is convenient to perform calculations in frequency space. From Eq. (11), it follows that the Fourier image of \( K(t, t') \) is nothing but the inverse of the operator, \( \mathcal{K}(\omega, \omega') \). Therefore, the problem reduces to the calculation of this inverse matrix.

Let us calculate \( \mathcal{K}(\omega, \omega') \) for the particular driving field in Eq. (3). Our strategy will be to firstly calculate \( G(\omega, \omega') \) from its definition (11), by inserting there the Fourier transformed driving field \( \Delta_{1}(\omega) = \Delta_{0} \delta(\omega) = [\epsilon/2] \delta(\omega) + A \delta(\omega - w) + \delta(\omega + w) \). Then, integration over \( \omega_{1} \) yields

\[
G(\omega, -\omega') = \beta_{0}(\omega, w) \delta(\omega - \omega') + \sum_{k=1,2,\sigma=\pm} \beta_{k}(\omega, \sigma w) \delta(\omega + \sigma k w - \omega')
\]

(24)

where

\[
\beta_{0}(\omega, w) = \varepsilon^{2} f[\omega] + A^{2} (f[\omega - w] + f[\omega + w]),
\]

\[
\beta_{1}(\omega, w) = \varepsilon A f[\omega] + f[\omega + w],
\]

\[
\beta_{2}(\omega, w) = A^{2} (f[\omega - w] + f[\omega + w])
\]

(25)

with \( f[x] = 2x/(\pi^{2} - \Delta^{2}) \). Secondly, in order to find the inverse of \( \mathcal{K}^{-1} \) (i.e. \( \mathcal{K} \)), we use the identity \( \mathcal{K} = \mathcal{K}^{\dagger} \).  

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**FIG. 1**: Diagrammatic representation for the matrix elements of \( G \). Short \( [\beta_{1}(\omega, \pm w)] \) and long \( [\beta_{2}(\omega, \pm w)] \) arrows represent \( \omega \to \omega \pm w \) and \( \omega \to \omega \pm 2w \) matrix elements respectively.
satisfy certain functional relations. We identify these
Fourier transform of this expression yields the correlation
will be determined below \[\text{see Eq. (30)}\]. Importantly,
structure of powers of \(\beta\) and \(\omega\) elements of the form
representation for the elements of the matrix \(K\) as the analytic continuation of the internal part. For
our purpose it is convenient to develop a diagrammatic representation for the elements of the matrix \(K^{-1}\) and
its powers. We denote the powers \(\beta_0(\omega)\) of the diagonal
elements of \(K^{-1}\) as empty circle, \(\bigcirc\), elements \(\beta_1(\omega, -w)\) and \(\beta_1(\omega, w)\) as left and right arrows from \(\omega\) to \(\omega - w\) and
\(\omega + w\) respectively. Similar arrows, but between \(\omega\) and \(\omega \pm 2w\), will represent elements \(\beta_2(\omega, \pm w)\) (see Fig. 1).

Matrix elements of \(K\) can be found by analyzing the
structure of powers of \(1 - K^{-1}\). They have the following form
\[
K(\omega, \omega') = B(\omega, w)\delta(\omega - \omega') + \sum_{n=1, \sigma = \pm}^{\infty} \Gamma_n(\omega, \sigma w)\delta(\omega + \sigma nw - \omega'),
\]
where diagonal, \(B\), and off-diagonal, \(\Gamma_n\), elements of \(K\)
will be determined below [see Eq. (31)]. Importantly,
Fourier transform of this expression yields the correlation
function, \(K(t, t')\), which in time representation depends on the difference, \(t - t'\), due to the presence of \(\delta(\omega - \omega')\) in (26). One can write Dyson

type equation for the diagonal elements represented dia-
grammatically in Fig. 2, where the full circles and the
thick lines represent full series of diagrams presented in
Figs. 3 and 4 respectively, and the empty circle, marked
as \(B_0(\omega, w)\), represents the full sum of bare diagonal el-
ments of the form \([1 + 2\omega - \beta_0(\omega, w)]^k\), leading to
\[
B_0(\omega, w) = \sum_{k=0}^{\infty} [1 + 2\omega - \beta_0(\omega, w)]^k = \frac{1}{\beta_0(\omega, w) - 2\omega}.
\]

Functional relations pertinent to the diagrammatic series of
Figs. 2 and 4 read
\[
B(\omega, w) = B_0(\omega, w) - B_0(\omega, w)B(\omega, w) - \sum_{k=1, 2, \sigma = \pm}^{\infty} \beta_k(\omega, \sigma w)C(\omega + \sigma w, -\sigma w)B(\omega + \sigma kw, w).
\]

Similarly, the off-diagonal terms in Eq. (26), that contain
\(\delta(\omega = \omega' \pm w)\) satisfy the relations following from Fig. 4
\[
C(\omega, w) = -\beta_1(\omega, w) - \beta_2(\omega, w)C(\omega + 2w, -w) - \sum_{k=1, 2, \sigma = \pm}^{\infty} \beta_k(\omega, \sigma w)C(\omega + \sigma w, -\sigma w)B(\omega + \sigma kw, w).
\]

In Eqs. (29) and (28), \(C(\omega, \pm w)\) represents the fully
dressed hopping matrix element of \((1 - K^{-1})^{-1}\), that cor-
responds to the transition from \(\omega\) to \(\omega \pm w\).

Finally, all remaining terms, \(\Gamma_n(\omega, w)\), in Eq. (30)
can be expressed via \(B(\omega + nw, w)\) as follows
\[
\Gamma_n(\omega, w) = B(\omega, w) \prod_{k=1}^{n} C[\omega \pm (k - 1)w]B(\omega \pm kw, w).
\]

Solving Eq. (28) with respect to \(B(\omega, w)\), one obtains
it in terms of \(B(\omega \pm w, w)\). This relation provides a possibility
to generate continued fraction form of the solution
for \(B(\omega, w)\). Iterations in Eqs. (28) and (29) lead to the
relations
\[
C_0(\omega, w) = -\beta_1(\omega, w), \quad C_{m+1}(\omega, w) = -\beta_1(\omega, w) - C_m(\omega + w, -w)\beta_2(\omega, w) \times B_m(\omega + 2w, w) - C_m(\omega, w)\beta_2(\omega - w, w)B_m(\omega - w, w),
\]
\[
B_m(\omega + w, w) = -2w + \beta_0(\omega, w) - \sum_{k=1, 2, \sigma = \pm}^{\infty} \beta_k(\omega, \sigma w)C_m(\omega + \sigma w, -\sigma w)B_m(\omega + \sigma w, w).
\]
While the solution of Eqs. (28) and (29) is simply defined by the infinite number of iterations \( B(\omega, w) = \lim_{n \to \infty} B_n(\omega, w), \quad C(\omega, w) = \lim_{n \to \infty} C_n(\omega, w), \) which is the continued fraction representation of the diagonal and off-diagonal matrix elements of \( K(\omega, \omega'). \) Analytical expressions for diagonal elements of \( K \) obtained within two- and three-iteration approximation are presented in Appendix B.

Evaluating the matrix elements of \( K(\omega) = \int d\omega' K(\omega, \omega'), \) which is the Fourier image of the correlation function \( K(t, 0) \), we find the spectrum of frequencies which contribute here. It is clear that all the matrix elements, \( \Gamma_n(\omega, \pm w), \) of \( K(\omega, \omega') \), defined by Eq. (30), will contribute to \( K(\omega) \) substantially. It is also clear that due to the periodicity of \( B(\omega, w) + \sum_{n=1}^{\infty} \Gamma_n(\omega, \pm w) \), as a function of \( \omega \), which defines the Fourier image of the correlation function, \( K(t, t') \), only a part of terms in the sum give essential contribution in the particular region of \( \omega \). Contribution of the tail becomes progressively smaller. We analytically calculate the first four elements of this sum, which gives the main contribution into the spin-spin correlation function in the region \( 0 < \omega/\Delta < 10 \) for the particular choices of the parameters \( \Delta, w, a, \) and \( A \). We have restricted ourselves within the fourth iteration level of the solution of Eqs. (28) for the same values of parameters. This means that we cut the exact continued fraction representation of \( K(\omega, \omega') \) after four fractions, as we checked that five and more iterations do not affect the result for these parameters in the plotted range of \( \omega/\Delta \). Comparison of our analytical expression for \( K(\omega) \) with numerically evaluated solution of the corresponding Schrödinger equation are presented in Fig. 5 for various values of model parameters. Note that in these plots we took into account a finite relaxation rate, \( \tilde{\gamma} \), in the Majorana fermion Green’s function, which is needed to ensure the causality. More specifically we chose \( \tilde{\gamma}/\Delta \sim 10^{-2} \) to ensure exponential decay of the Green’s function at times \( t > \tilde{\gamma}^{-1} \).

V. SUMMARY

In summary, we have developed a new technique for studies of general non-equilibrium two-level systems. The technique is based on the mapping to a Majorana fermion field theory coupled to a scalar field. We have applied the technique to study the dynamics of two-level systems with driving fields given by Eq. (3). Our analytical result for the nonlinear dielectric response function in energy space is shown to be in good agreement with the numerically evaluated solution of the time-dependent Schrödinger equation. We see that positions of the resonance peaks in \( K(\omega) \) are in agreement with the results of exact numerical simulations.

Our technique allows generalization to dissipative two
level systems. We can consider decoherence being caused by the dissipative environment and also generated by dissipative elements in superconducting electronic circuits elements. One can extend our approach and include relaxation and dephasing times ($T_1$ and $T_2$) into consideration. However, these problems as well as the comparison of our approach to the well known comprehensive method based on the Bloch-Redfield equations is a subject of further studies.

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VI. APPENDIX A

Here we will present some of the basic properties of Majorana fields, and will calculate the Green’s function of a free non-interacting Majorana fermions. As fields, Majorana fermions are described by real Grassmann variables, $\xi$, with following properties:

$$\{\xi_\mu, \xi_\nu\} = 0, \quad \xi_\mu^+ = \xi_\mu,$$  \hspace{1cm} (32)

where curly brackets stand for an anticommutator.

Integration rules over Grassmann variables are very simple. Namely

$$\int d\xi_\mu = 0, \quad \int d\xi_\mu \xi_\nu = \delta_{\mu \nu}. \hspace{1cm} (33)$$

Using these rules of integration, one can prove that a Gaussian integral over Majorana fermions is expressed in terms of the determinant of the quadratic form. But contrary to ordinary fermionic integrals, it is equal to the square root of the determinant:

$$\int \mathcal{D}\xi \exp \left\{ \sum_{\mu, \nu=1}^N \xi_\mu A_{\mu \nu} \xi_\nu \right\} = \sqrt{\text{Det}[A_{\mu \nu}]} \hspace{1cm} (34)$$

This expression is correct both, for matrices and for differential operators.

In order to calculate the Green’s function, $\langle \xi_\mu(t)\xi_\nu(t') \rangle$, of non-interacting Majorana fermions, we introduce a generating functional

$$\mathcal{Z}(\eta_\mu) = \int \mathcal{D}\xi_\mu \exp \left\{ \int dt \left[ \frac{1}{4} \xi_\mu \dot{\xi}_\mu + \xi_\mu \eta_\mu \right] \right\}. \hspace{1cm} (35)$$

From this expression it is clear that

$$\langle \xi_\mu(t)\xi_\nu(t') \rangle = \frac{1}{\mathcal{Z}_0} \left. \frac{\partial^2 \mathcal{Z}(\eta_\mu)}{\partial \eta_\mu(t) \partial \eta_\nu(t')} \right|_{\eta_\mu = 0}. \hspace{1cm} (36)$$

So we need to calculate $\mathcal{Z}(\eta_\mu)$. Calculation is straightforward. One can find from Eq. (35)

$$\mathcal{Z}(\eta_\mu) = \int \mathcal{D}\xi_\mu \times \exp \left\{ \int dt \left[ \frac{1}{8} (\xi_\mu + 2\eta_\mu d^{-1}_t) d_t (\xi_\mu - 2d^{-1}_t \eta_\mu) + \frac{1}{8} (\xi_\mu - 2\eta_\mu d^{-1}_t) d_t (\xi_\mu + 2d^{-1}_t \eta_\mu) - \eta_\mu d^{-1}_t \eta_\mu \right] \right\},$$

where $d^{-1}_t = \frac{1}{2} \text{sign}[t - t']$ is the Green’s function of the differential operator, $d_t = \frac{\partial}{\partial t}$. According to (34), the Gaussian integral over $\xi_\mu \pm 2\eta_\mu d^{-1}_t$ yields $\mathcal{Z}_0 = \sqrt{\text{Det}[d_t]^{-1}}$, which is a C-number, and we obtain $\mathcal{Z}(\eta_\mu) = \mathcal{Z}_0 \exp \left\{ -\eta_\mu d^{-1}_t \eta_\mu \right\}$. Now, differentiating $\mathcal{Z}(\eta_\mu)$ twice with respect to $\eta_\mu$, and taking the limit $\eta_\mu = 0$, we reproduce the formula from the main text:

$$\langle \xi_\mu(t_1)\xi_\nu(t_2) \rangle = \frac{1}{\mathcal{Z}_0} \int \mathcal{D}\xi(t) \xi_\mu(t_1)\xi_\nu(t_2) \times \exp \left[ \frac{1}{4} \int dt \xi_\mu \dot{\xi}_\mu \right] = \delta_{\mu \nu} \text{sign}[t_1 - t_2]. \hspace{1cm} (38)$$

VII. APPENDIX B

In this Appendix we present approximate expressions for solution of Eqs. (28) and (29), which where obtained within two and three iterations (i.e. by cutting the continued fraction representation after two and three fractions).

Within two-iteration approximation we have for the diagonal element, $B_1(\omega, w)$, of matrix $K$:

$$B_1(\omega, w) = \frac{1}{-2\omega + \beta_0(\omega, w) + \frac{\beta_1(\omega, w)}{2(\omega + w) + \beta_0(\omega + w)} + \frac{\beta_2(\omega, w)}{2(\omega + 2w) + \beta_0(\omega + 2w)} + \frac{\beta_3(\omega, w)}{2(\omega + 3w) + \beta_0(\omega + 3w)}}, \hspace{1cm} (39)$$

Similarly, for hopping element $C_1(\omega, w)$ we get from Eq. (31)

$$C_1(\omega, w) = -\beta_1(\omega, w) + \frac{\beta_2(\omega, w) \beta_1(\omega + 2w, -w)}{-2(\omega + 2w) + \beta_0(\omega + 2w)} + \frac{\beta_1(\omega, -w) \beta_2(\omega - w, w)}{-2(\omega - w) + \beta_0(\omega - w)} \hspace{1cm} (40)$$
This yields the following expression for diagonal elements of $K$ in a three-iteration approximation

$$B_2(\omega, w) = \frac{1}{-2\omega + \beta_0(\omega, w) - C_1(\omega, w)B_1(\omega, \omega + w) - C_1(\omega, -w)B_1(\omega, \omega - w) + \beta_2(\omega, w)B_1(\omega, \omega + 2w) + \beta_2(\omega, -w)B_1(\omega, \omega - 2w)}$$

Finally, within the same level of approximation one has for hopping elements $C_2(\omega, w)$:

$$C_2(\omega, w) = -\beta_1(\omega, w) - \beta_2(\omega, w)C_1(\omega + 2w, -w)B_0(\omega + 2w, w) - C_1(\omega, -w)\beta_2(\omega - w, w)B_0(\omega - w, w). \quad (41)$$

As we see from Eq. (41), one can continue this procedure to higher levels of iteration to analytically calculate the survival probability of the spin with an arbitrary accuracy.

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