ELECTRON THERMAL SELF-ENERGY IN A MAGNETIC FIELD†

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ABSTRACT

Using the general form of the static energy solutions to the Dirac equation with a magnetic field, we calculate a general self-energy matrix in the Furry-picture. In the limit of high temperatures, but even higher magnetic fields, a self-consistent dispersion relation is solved. In contrast to the high temperature limit, this merely results in a small mass shift. The electron anomalous magnetic moment is calculated. The contribution from thermal fermions is found to be different from the corresponding contribution using perturbation theory and plane-wave external states. In the low temperature limit the self-energy is shown to exhibit de Haas–van Alphen oscillations. In the limit of low temperatures and high densities, the self-energy becomes very large.

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1 Introduction

Several compact stellar objects (e.g. neutron stars, white dwarfs and red giants) and cosmological models are characterized by high temperatures and/or densities, and also the presence of a strong background magnetic field. It is thus of great interest to compute the properties of electrons in such extreme environments. Earlier the effective potential has been considered. Here we shall deal with the electron self-energy, important for kinematical issues. Apart from the general self-energy matrix in the Furry-picture, and the treatment of the de Haas–van Alphen oscillations in the self-energy, this work is merely a shorter version of results published elsewhere, to which we refer for further details and references.

2 Furry–Picture Propagator

The Dirac equation for an electron (charge -e) in an external field, with the vector potential $A^\nu$, reads

$$\left( i\partial + eA - m \right)\Psi^{(\pm)}(x) = 0 \ ,$$

where $\kappa$ denotes a complete set of quantum numbers necessary to specify the solutions, and the superscripts $+$ and $-$ are referring to positive and negative energy solutions, respectively. Using a complete set of static energy solutions we may represent the second quantized electron field in the Furry-picture

$$\Psi(x, t) = \sum_\kappa \left[ b_\kappa \Psi^{(+)\kappa}(x, t) + d^{\dagger}_\kappa \Psi^{(-)\kappa}(x, t) \right] \ ,$$

where $(b, d)$ are the standard annihilation operators for particles and anti-particles. The fermion propagator, including the effects of some distribution of particles, can then be constructed explicitly as the expectation value

$$iS(x', x) = \langle T \left[ \Psi(x')\overline{\Psi}(x) \right] \rangle \ .$$

In the case of a static uniform magnetic field in the negative $z$-direction we may choose $A_\nu = (0, 0, Bx, 0)$. The solutions to the Dirac equation must then be of the form

$$\Psi^{(\pm)}_{\zeta, n, p_y, p_z}(x, t) = \frac{1}{N_\kappa} \exp[\pm i(-E_n t + p_y y + p_z z)]V_{n, p_y}(x)u_{\zeta, n, p_y, p_z} \ ,$$

where $\zeta = \pm 1$ is a polarization index, $n = 0, 1, 2, \ldots$, and $p_y, p_z \in \mathbb{R}$. The energy spectrum is given by

$$E_n(B, p_z) = \sqrt{m^2 + p_z^2 + 2eBn} \ .$$
In the chiral representation of the $\gamma$-matrices we have

$$V_{n,p_y}(x) = \text{diag}[I_{n,p_y}(x), I_{n-1,p_y}(x), I_{n-1,p_y}(x), I_{n,p_y}(x), I_{n-1,p_y}(x)]$$

where $I_{n,p_y}(x) = (eB/\pi)^{1/4} \exp[-eB/2(x-p_y/eB)]H_n[\sqrt{2eB(x-p_y/eB)}]$. Here $H_n$ is a Hermite polynomial, and we define $I_{-1,p_y} = 0$. The spinor $u_{\zeta,n,p_y,p_z}$ is independent of $x''$. There is a twofold degeneracy corresponding to $\zeta = \pm 1$ in all but the lowest Landau level $n = 0$. The electron propagator is with this choice of gauge

$$S(x'^\nu, x^\nu) = \sum_{n=0}^\infty \int d^4 k_0 d^3 k y d k_z S_{n,k_0,k_y,k_z}(x', x)$$

$$\times \left[ \frac{1}{k_0^2 - m^2 - k_y^2 - 2eBn + i\varepsilon} + 2\pi i \delta(k_y^2 - m^2 - k_y^2 - 2eBn)f_F(k_0) \right]$$

$$\times \exp[-ik_0(t' - t) + ik_y(y' - y) + ik_z(z' - z)]$$

where $d^n k \equiv d^4 k/(2\pi)^n$, and $f_F(k_0) \equiv \Theta(k_0)f'_F(k_0) + \Theta(-k_0)f_F(k_0)$. In the case of thermal equilibrium the one-particle fermionic distribution is $f_F^\pm(k_0) = 1/(e^{\beta(k_0 \mp \mu)} + 1)$, where $\beta$ is the inverse temperature and $\mu$ is the chemical potential determined by the charge density of electrons and positrons of the system. Not specifying the representation of the $\gamma$-matrices we may write

$$S_{n,k_0,k_y,k_z}(x', x) = (k_0\gamma_0 - k_z\gamma_2 + m)[\sigma_+ I_{n,k_y}(x')I_{n,k_y}(x) + \sigma_- I_{n-1,k_y}(x')I_{n-1,k_y}(x)]$$

$$-i\sqrt{2eB}[\gamma_+ I_{n,k_y}(x')I_{n-1,k_y}(x) - \gamma_- I_{n-1,k_y}(x')I_{n,k_y}(x)]$$

where $\gamma_\pm \equiv (\gamma_x \pm i\gamma_y)/2$, and $\sigma_\pm \equiv (1 \pm \sigma_z)/2$ ($\otimes 1$ when necessary).

### 3 The General Self-energy Matrix

There are two possible contributions to the one-loop electron self-energy. The tadpole contribution is proportional to the total electric charge and current in the medium and thus vanishing in a neutral environment. Therefore we are left only with the 1PI self-energy

$$-i\Sigma(x', x) = (-ie)^2\gamma_\mu iD^{\mu\nu}(x' - x)iS(x', x)\gamma_\nu$$

The photon propagator may be written

$$iD^{\mu\nu}(x) = \int d^4q e^{-iq.x} \left( g^{\mu\nu} - \xi q^\mu q^\nu \frac{\partial}{\partial q^2} \right) \left[ \frac{-i}{q^2 + i\varepsilon} - 2\pi \delta(q^2)f_B(q_0) \right]$$

where the photon distribution function in the case of thermal equilibrium is $f_B(q_0) = 1/(\exp[\beta|q_0|] - 1)$. The effective Dirac equation may then be written

$$\int d^4x \Psi^{(\pm)}_{\zeta,n,p_y,p_z}(x)(i\partial + eA - m)\Psi^{(\pm)}_{\zeta',n,p_y,p_z}(x) =$$
\[
\int d^4x \int d^4x' \tilde{\Psi}^{(+)}_{\zeta,n,p_y,p_z}(x) \Sigma(x,x') \Psi^{(+)}_{\zeta',n',p_y,p_z}(x')
\]
\[
\equiv \frac{1}{N_\kappa} \int dt \, dy \, dz \, \tilde{\Sigma}_N(B,p_z; p_y) u_{\zeta',n',p_y,p_z} \,
\]
(11)

where we have used the general form of the electron wave-functions in Eq.(11) to define a general self-energy matrix \( \tilde{\Sigma}_N \). The space-time integral (i.e. energy—momentum conservation) implies that both sides of Eq.(11) are diagonal in \( n, p_y \) and \( p_z \). On-shell, i.e. with tree-level wave-functions, the effective Dirac equation (11) results in an energy shift \( E^{(1)}_{\zeta,n}(B,p_z) = E_n(B,p_z) + \Delta E_{\zeta,n}(B,p_z) \), where

\[
\Delta E_{\zeta,n}(B,p_z) = \tau_{\zeta,n,p_y,p_z} \tilde{\Sigma}_n(B,p_z; p_y) u_{\zeta',n',p_y,p_z} \,
\]
(12)

that actually is independent of \( p_y \). The on-shell self-energy has also been shown\(^2\) to be independent of the gauge-fixing parameter \( \xi \). If not otherwise stated we shall use the Feynman gauge \( \xi = 0 \).

4 Strong Field Limit

In favorable conditions the gauge-fixing dependence of the self-energy may be neglected also off-shell. Then we may move forward and solve a self-consistent dispersion relation. It is a well-known fact that in the high-temperature limit this is the case. Self consistent dispersion-relations have been solved in the high temperature limit without\(^3\) and also with a magnetic field using the Schwinger proper time method\(^2,6\). The result, rich in physics, contains for example particle as well as hole solutions. With the general self-energy matrix presented here it is possible to perform the same analysis also in the Furry-picture.

On the other hand the Furry-picture is particularly suitable in the strong field limit. In this limit the dominant contribution will come from intermediate electron states in the lowest Landau level \( (n = 0) \) only. In the limit \( \{ eB \gg T^2 \gg m^2, p_z^2, \mu^2 \} \) we find in the lowest Landau level, using only \( n = 0 \) in the electron propagator Eq.(7)

\[
\hat{\Sigma}_0^{\text{vac}} \approx \sigma_+ m \alpha \frac{\ln^2 \left( \frac{2eB}{m^2} + \Delta - 1 \right)}{4\pi},
\]
(13)

\[
\hat{\Sigma}_0^{e^+ e^-} \approx -\sigma_+ m \frac{\alpha}{\pi} \left[ \ln^2 \left( \frac{T}{m} \right) - 2 \ln \left( \frac{T}{m} \right) \ln \left( \frac{eB}{m^2\sqrt{\Delta + 1}} \right) \right],
\]
(14)

\[
\hat{\Sigma}_0^\gamma \approx -\sigma_+ m \frac{2\alpha}{\pi} \ln \left( \frac{T}{m} \right),
\]
(15)

where we have defined \( m^2 \Delta \equiv E^2 - m^2 - p_z^2 \), as the deviation from the on-shell energy eigenvalue. The self-energy \( \hat{\Sigma}_0 = \hat{\Sigma}_0^{\text{vac}} + \hat{\Sigma}_0^{e^+ e^-} + \hat{\Sigma}_0^\gamma \), has been split into its
contributions from the vacuum, thermal fermions, and thermal photons, respectively. Similarly the gauge-fixing dependent parts of the off-shell self-energy are in this limit

\[ \hat{\Sigma}_0^{\text{vac}}(\xi) \propto \xi m \alpha \Delta \ln \left( \frac{2eB}{m^2} \right), \quad (16) \]

\[ \hat{\Sigma}_0^{e^+e^-}(\xi) \propto \xi m \alpha \Delta \left( \frac{mT}{2eB} \right)^2 \ln \left( \frac{2eB}{m^2} \right), \quad (17) \]

\[ \hat{\Sigma}_0^{e^+e^-}(\xi) \propto \xi m \alpha \Delta \ln \left( \frac{T}{m} \right). \quad (18) \]

Keeping only terms \( O[\ln^2(2eB/m^2)] \) and \( O[\ln(2eB/m^2) \ln(T/m)] \), we may thus neglect the gauge fixing dependence. Also notice that all of the three different contributions to the \( \xi \) dependent part of the self-energy are proportional to \( \Delta \). We have thus explicitly shown that the on-shell (i.e. \( \Delta = 0 \)) self-energy is gauge-fixing independent in this limit. The self-consistent dispersion relation obtained from Eq.(11) reads in this limit

\[ (E_{\gamma} - p_z \gamma_z - m) \sigma_\pm u_0 = \delta M_B \sigma_\pm u_0, \quad (19) \]

where we have defined the “thermo-magnetic mass”

\[ \delta M_B = m \alpha / (4\pi) \left[ \ln^2 \left( \frac{2eB}{m^2} \right) + 8 \ln \left( \frac{T}{m} \right) \ln \left( \frac{eB}{m^2} \right) - 4 \ln^2 \left( \frac{T}{m} \right) \right]. \quad (20) \]

The presence of \( \sigma_+ \) just signifies that we have already used that the wave-function in the lowest Landau level is proportional to the matrix \( V_{0,p_y}(x) = I_{0,p_y}(x) \text{ diag}[1, 0, 1, 0] \), according to Eq.(4). The above Eq.(19) is thus equivalent to the tree-level Dirac equation with the mass \( M \equiv m + \delta M_B \). Unlike in the high temperature limit, this self-consistent dispersion relation thus only results in an energy shift

\[ E_{0}^{(1)} = \sqrt{(m + \delta M_B)^2 + p_z^2}. \quad (21) \]

To the lowest order in \( \delta M_B \) this is the same as would have been obtained from Eq.(12). How large can then \( \delta M_B \) be? In order for \( \delta M_B \approx m \) we must have \( eB/m^2 \approx 10^{17} \), i.e. \( B \approx 10^{27} T \), an immensely large field. However, this makes our negligence of the \( \xi \) dependent parts more accurate then it appeared at first when keeping only the leading logarithms. The gauge fixing dependent part of the self-energy is proportional to \( \Delta \approx \delta M_B m/E_0 \), that is small for most magnetic field strengths.

5 Weaker Magnetic Fields

In the case of an arbitrary magnetic field the sum over all Landau levels in the electron propagator has to be considered. This problem has recently been solved. Let
us consider an on-shell electron in the lowest Landau level with vanishing momentum. We may now perform an expansion to linear order in $eB/\lambda^2$, where $\lambda$ must be some energy scale, initially not known in this naive expansion. The result for the contributions to the self-energy from vacuum and thermal photons are, respectively,

$$\Delta E_{0}^{\text{vac}} \simeq -\frac{eB}{2m} \frac{\alpha}{2 \pi} ,$$  \hspace{1cm} (22)

$$\Delta E_{0}^{\gamma} \simeq \frac{m \alpha}{3} \frac{T^2}{m^2} + \frac{eB}{2m} \frac{\alpha T^2}{9}.$$  \hspace{1cm} (23)

The thermal electron contribution is more involved. In the absence of a magnetic field we find the well-known result

$$\Delta E_{0}^{e^+e^-} (B = 0, p_z = 0) \equiv \Delta m^{e^+e^-} = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} dk_0 \Theta(k_0^2 - m^2) f_F(k_0) \frac{k_0 - 2m}{m(k_0 - m)}.$$  \hspace{1cm} (24)

To linear order in $eB$ we find

$$\Delta E_{0}^{e^+e^-} = \Delta m^{e^+e^-} + \frac{eB}{2m} \frac{\alpha}{3\pi} \int_{m}^{\infty} \frac{d\omega}{\sqrt{\omega^2 - m^2}} \left[ \left( \frac{2\omega^2 + 2m\omega - m^2}{m^2} - \frac{m}{(\omega + m)} \right) f_F^+ (\omega) + 2m \frac{df_F^+ (\omega)}{d\omega} \right] +$$

$$\frac{eB}{2m} \frac{\alpha}{3\pi} \int_{m}^{\infty} \frac{d\omega}{\sqrt{\omega^2 - m^2}} \frac{2\omega^3 - 3m^2 \omega - 2m^3}{m^2(\omega + m)} f_F^- (\omega) ,$$  \hspace{1cm} (25)

where some integrations by parts has been performed in order to make the result infrared finite. The spin energy-shift for a particle of charge $-e$, with spin $(s)$ magnetic moment $\mu = -e/(2m)g(s)$ in a magnetic field $B = -B\hat{z}$ is $\Delta E = -\mu \cdot B$. We may thus write

$$\Delta E = \Delta m^{\beta,\mu} - \frac{eB}{2m} \frac{\delta g}{2} ,$$  \hspace{1cm} (26)

where $\Delta m^{\beta,\mu}$ is the thermal mass, and $\delta g/2$ is the anomaly of the magnetic moment. Usually $\delta g/2$ is obtained from the vertex correction (i.e. the triangle diagram). However, in the case of the contribution from thermal electrons the results differ. The discrepancy is caused by the external states being Landau levels, and not plane waves, as assumed when considering the vertex correction. Notice that the anomaly is obtained from the transverse part of the vertex correction, not included in the Ward-identity relating the vertex to the self-energy.

Let us now consider the limit of vanishing temperature, but finite density of electrons. We may then perform the energy integral in Eq.(24) and Eq.(25) to find

$$\Delta E_{0}^{e^-} \simeq \frac{\alpha}{4\pi} \left[ \frac{\mu - 2m}{m} \sqrt{\mu^2 - m^2} - 3m \ln \left( \frac{\mu + \sqrt{\mu^2 - m^2}}{m} \right) \right]$$
This may become very large at high densities, like for example in the case of a neutron star, where $\mu/m \approx 10^2$, and $T/m \approx 1$. Comparing with the high temperature case, we see that the leading terms $m(e\mu/m)^2$ has a similar origin to the corresponding $m(eT/m)^2$ terms. Thus this contribution has been obtained from a “hard dense loop”, and higher loop corrections will only give sub-dominant contributions in powers of $(e^2\mu/m)$.

### 6 De Haas—van Alphen Oscillations

Notice that the self-energy given in Eq. (27) will become divergent as $\mu \to m^+$, $T = 0$. This is not physically acceptable, since the dense and thermal contribution should vanish in this limit. It seems as if our naive expansion in this case has been in powers of $eB/(\mu^2 - m^2)$, and thus is not valid for small densities. On the other hand, if $2eB \approx \mu^2 - m^2$ only a few lower Landau levels will be occupied by electrons. We thus have a situation very similar to the strong field limit, well suited for the explicit spectral decomposition. Split the thermal electron part of the self-energy into its contributions from the different Landau levels:

\[
\Delta E_0^e = \sum_{n=0}^{N} \Delta E_0^e(n) \quad , \quad N = \text{Int}[{(\mu^2 - m^2)}/(2eB)] \quad .
\]

Defining $a_n \equiv m(\omega - m)/2eB - n$, and $\Theta_n = 1 \quad , \quad n \geq 0; \quad \Theta_n = 0 \quad , \quad n \leq -1$, we find

\[
\begin{align*}
\Delta E_0^e(n) &= \frac{\alpha}{\pi} \int_0^{\mu} \frac{d\omega}{\sqrt{\omega^2 - m^2 - 2eBn}} \left\{ \frac{m}{n!} a_n + \Theta_{n-1} \omega - m \right\} \\
&\times \left\{ (-a_n)^{n-1} e^{a_n E_1(a_n - i\epsilon)} + \sum_{l=0}^{n-2} (-a_n)^l (n - 2 - l)! \right\} - \frac{m}{n!} \Theta_{n-1} \Bigg) \quad .
\end{align*}
\]

In the limit $\mu \to m$ only the lowest Landau level will contribute. Using the series expansion of the exponential integral, we find for $\mu^2 - m^2 \ll 2eB$

\[
\Delta E_0^e \simeq -m \frac{\alpha}{\pi} \left\{ \ln \left( \frac{eB}{m^2} \right) \ln \left( \frac{\mu + \sqrt{\mu^2 - m^2}}{m} \right) + \sqrt{2\mu - m} \left[ 2 - \ln \left( \frac{\mu - m}{m} \right) \right] \right\} \quad .
\]

We can thus see that the electron contribution to the self-energy is vanishing as $\mu \to m^+$, $T = 0$, and furthermore that it is non-analytical in $B$. As $(\mu^2 - m^2)/2eB$ is becoming larger, consecutive Landau levels will cross the Fermi-surface, and start contributing. The contribution from the $n$-th Landau level is vanishing as $\mu \to$
\(\sqrt{m^2 + 2eBn}\), so the self-energy is continuous. However, the derivative with respect to the chemical potential is diverging as \(\mu \to \sqrt{m^2 + 2eBn}\). These sharp cusps will be smoothed out at finite temperature. There are thus oscillations in the self-energy at low temperatures, and Fermi-momentum squared of the order of the magnetic field. As a function of \(1/eB\), they have a quasi-periodicity of \(2/(\mu^2 - m^2) = 2\pi/A\), where \(A\) is the area of an extremal cross section of the Fermi-surface. Such oscillations are well-known as de Haas–van Alphen oscillations in the magnetization of a Fermi-gas, but not earlier found in the fermion self-energy.

For \(n \geq 1\), \(a_n\) may become negative. This will cause an imaginary part in the self-energy according to \(E_{1}(-x - i\varepsilon) = -Ei(x) + i\pi\), for \(x > 0\). Let \(\Gamma_{e\rightarrow e\gamma}\) denote the total decay rate for an electron in the Fermi sea decaying into an electron in the lowest Landau level, that not is supposed to be occupied. Then we find

\[
\Gamma_{e\rightarrow e\gamma} = 2\text{Im} \Delta E_{0}^{e-}.
\]  

An ostensible contradiction appears here. The lowest Landau level, always below the Fermi energy, is supposed not to be occupied. However, perturbation theory rests on the assumption on adiabatic turn on/off of the interaction as \(t \to \pm\infty\). The external state is thus supposed to be separated from the heat and charge bath before and after the interaction is taking place, and not affected by the occupation of states in the medium. In Fig. 1 we show the real and imaginary parts of the dense electron contribution to the self-energy as a function of the inverse magnetic field. Unfortunately the cusps in the real part, as Landau levels with \(n \geq 2\) are crossing the Fermi-surface cannot be distinguished on this scale. The cusp in the imaginary part at \(\mu = m + 2eB/2m\) follows from Eq. (29), but has no counterpart in ordinary de Haas–van Alphen oscillations.

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8 References

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Figure 1: The real part and the imaginary part (dotted line) of the electron self-energy exhibits de Haas–van Alphen oscillations at low temperatures. Here $T = 0$ and $\mu = \sqrt{3}m$. Notice the scaling of the real part, that actually is negative.

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