The Existence of Global Weak Solutions for Singular Parabolic System of $p$-Laplacian Type

P R Akbar$^a$, Corina Karim$^b$ and R Bagus E W$^c$
Department of Mathematics, Brawijaya University, Indonesia
E-mail: $^a$rayamath07@student.ub.ac.id, $^b$co_mathub@ub.ac.id, $^c$rbagus@ub.ac.id
* Corresponding author

Abstract. We study the existence of global weak solutions for Cauchy-Dirichlet problem for $p$-Laplacian systems in the singular case. Here, we treat the existence of global weak solutions using Galerkin method where the initial data belonging to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$. The main result is to show the validity of theorem of global weak solutions for Cauchy-Dirichlet problem for singular case only.

1. Introduction
We will study the existence of weak solutions for Cauchy-Dirichlet problem for evolutional $p$-Laplacian systems. Let $\Omega$ be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, with smooth boundary $\partial \Omega$, and let $\frac{2m}{m+2} < p < 2$. For a map $u : (0,T) \times \Omega \to \mathbb{R}^n$, $z = (t,x) = (t,x_1,x_2,...,x_n)$, $u = u(z) = (u^1(z),...,u^m(z))$, we consider $p$-Laplacian type as below

\[
\begin{aligned}
\partial_t u - \text{div}(|Du|^{p-2}Du) &= 0 \quad \text{in } (0,T) \times \Omega, \\
u(0,x) &= u_0(x) \quad \text{on } \partial \Omega \times (0,T) \times \Omega,
\end{aligned}
\]

where $u_0(x)$ is any given initial data belonging to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^m)$.

Some researchers have discussed about weak solutions of $p$-Laplace equation. In the scalar case, when the unknown in (1) function are real valued, was studied by DiBenedetto et al. ([1], [2], [3]) used Hölder regularity. In the vectorial case, a few result have been known on the Hölder regularity of solutions ([4], [5], [6]), whose proof is using Campanato’s perturbation. While in the global existence of weak solutions to the heat flow of $p$-harmonic maps, for $2 \leq p < \infty$ has been researched by Chen et al. [7] whose proof is using Galerkin method and monotonicity trick.

In 2013, Misawa [8] using a geometrical progression based on an intrinsic scaling to the evolutionary $p$-Laplace operator in case $p \geq 2$ and [9] was using his method to treat the local boundedness of weak solutions for singular parabolic systems of $p$-Laplacian type only in the singular case. However, the Hölder estimate of solutions for singular case was settled by [10] and the Gradient of its solutions also studied by [11], whose proof based on the intrinsic scaling for both case. Indeed, [9] also studied about the local boundedness of weak solutions for singular parabolic systems of $p$-Laplacian type and in late 2019, [12] studied the local Hölder regularity of weak solutions for degenerate type.
In 2017, Corina and Misawa [13] has been studied the existence of solutions for a Cauchy-Dirichlet problem with constant coefficient using variational method in case $\frac{2m}{m+2} < p < \infty$. In this paper, we will study the existence of global weak solutions for singular parabolic system of $p$-Laplacian type in case $\frac{2m}{m+2} < p < 2$ using Galerkin method.

In particular, according to Evans [14], in Galerkin method we will reformulate (1) as a variational problem in the space variables $x$, where the initial data $u_0(x) \in W^{1,p}(\Omega, \mathbb{R}^m)$. We define Galerkin approximation:

$$u_k(0,x) = \sum_{i=1}^{k} g_{ik}(0)w_i(x),$$

then substitution to (1) where $u_k$ is a function in a finite-dimension space of functions such that

$$\partial_t u_k - \text{div}(|Du_k|^{p-2}Du_k) = 0$$

is orthogonal to this space for each $t$.

When one studies the existence of weak solutions of singular parabolic $p$-Laplacian systems, one needs to invoke a definition of weak solutions itself. The weak solution is defined as usual.

**Definition 1. (Weak Solution)**

A vector-valued function $u$ is a weak solution of (1), if and only if $u \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^m)) \cap L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^m))$ and satisfies

$$\int_{(0,T) \times \Omega} \partial_t u \cdot \varphi + |Du|^{p-2}Du \cdot D\varphi \, dz = 0,$$

for all $\varphi \in L^p(0,T;W^{1,p}_0(\Omega,\mathbb{R}^m))$ with $\partial_t \varphi \in L^2(\Omega,\mathbb{R}^m)$ and $T > 0$.

Our main theorem in this paper is the following:

**Theorem 1. (Existence of global weak solutions)**

Let $\Omega$ be a bounded domain in $\mathbb{R}^m$ with smooth boundary $\partial \Omega$. Then for any initial data, there exists a weak solution of (1) from $\Omega$ into $\mathbb{R}^m$.

2. Result

We will treat the existence of a weak solution of (1) by using Galerkin approximation. First, we approximate a solution to (1) by the solutions to the following equation

$$\begin{cases}
\partial_t u_k - \text{div}(|Du_k|^{p-2}Du_k) = 0 & \text{in } (0,T) \times \Omega, \\
u_k(0,x) = u_0(x) & \text{on } \partial \Omega \times (0,T) \times \Omega,
\end{cases}$$

for fixed $k \geq 1$ and $u_0 \in W^{1,p}(\Omega,\mathbb{R}^m)$. We will use Galerkin method to prove the existence of weak solution of the equation (3).

Secondly, we will prove the energy inequality for weak solution $u_k$, which has an important role in the limiting process.

**Theorem 2. (Energy Inequality Theorem)**

Let $\frac{2m}{m+2} < p < 2$. Let $u_0 \in W^{1,p}_0(\Omega,\mathbb{R}^m)$. Then there exists a unique $u_k \in L^\infty(0,T;W^{1,p}(\Omega,\mathbb{R}^m))$ such that

$$\int_0^T \int_\Omega |\partial_t u_k|^{2} \, dx \, dt + \sup_{0 < t < T} \int_\Omega \frac{1}{p} |Du_k(t)|^p \, dx \leq \int_\Omega \frac{1}{p} |Du_0|^p \, dx.$$
such that we determine $(0,T]$ such that $g > 0$.

Let $T$ (Existence of ODE Theorem)

Theorem 3.

guarantees that there exists at least one time local solution of (6). Hence we have the following theorem.

Integrating (9) over $\Omega$  such that

$$\int_{\Omega} \sum_{j=1}^{k} g_{ik}(t) w_{j}(x) \, dx = \sum_{i=1}^{k} g_{ik}(t) w_{i}(x)$$

such that we determine $\{g_{ik}(t)\}$ as an ordinary differential equation (ODE),

$$\frac{d g_{ik}(t)}{dt} = -\int_{\Omega} \left| \sum_{j=1}^{k} g_{ik}(t) D w_{j}(x) \right|^2 \sum_{j=1}^{k} g_{ik}(t) D w_{j}(x) \cdot D w_{j} dx$$

$$\iff \int_{\Omega} \partial_{t} u_{k} \cdot w_{j} + |Du_{k}|^{p-2} Du_{k} \cdot D w_{j} dx = 0$$

for $1 \leq j \leq k$, with initial value data $g_{ik}(0) = c_{i}$, $i = 1,...,k$, and the sequence $\{c_{i}\} \subset \mathbb{R}$ is defined as

$$u_{k}(0,x) = \sum_{i=1}^{k} c_{i} w_{i}(x) \in W_{0}^{1,p}(\Omega,\mathbb{R}^{m}).$$

The equation (7) is equivalent to

$$u_{k}(0,x) \to u_{0}(x) \text{ strongly in } W_{0}^{1,p}(\Omega,\mathbb{R}^{m}) \text{ as } k \to \infty,$$

because the initial data $u_{0}(x) \in W_{0}^{1,p}(\Omega,\mathbb{R}^{m})$ and $\{w_{i}\}$ is a base in $W_{0}^{1,p}(\Omega,\mathbb{R}^{m})$.

The right hand side of (6) is continuous with respect to the variable $D w_{j}(x)$. Thus Peano’s theorem guarantees that there exists at least one time local solution of (6). Hence we have the following theorem.

Theorem 3. (Existence of ODE Theorem)

Let $T > 0$. Then there exists a unique $g(s) = g_{k}(t) = (g_{1k}(t),...,g_{kk}(t)) \in C^{1}((0,T];\mathbb{R}^{k})$ of ODE (6)-(7) on $(0,T]$ such that

$$\int_{0}^{T} \int_{\Omega} \left| \partial_{t} u_{k} \right|^{2} dx dt + \sup_{0 < t < T} \int_{\Omega} \frac{1}{p} |Du_{k}(t)|^{p} dx \leq \int_{\Omega} \frac{1}{p} |Du_{0}|^{p} dx.$$ (8)

Proof. Multiply (6) by $\frac{d g_{ik}(t)}{dt}$ and take summation over $j = 1,2,...,k$ to have

$$\sum_{j=1}^{k} \left| \frac{d g_{ik}(t)}{dt} \right|^{2} = -\int_{\Omega} \left| \sum_{j=1}^{k} g_{ik}(t) D w_{j}(x) \right|^2 \sum_{j=1}^{k} g_{ik}(t) D w_{j}(x) \cdot \sum_{j=1}^{k} \frac{d g_{ik}(t)}{dt} D w_{j} dx$$

$$\int_{\Omega} \left| \frac{d u_{k}}{dt} \right|^{2} dx = -\int \frac{d}{dt} \left| \sum_{j=1}^{k} g_{ik}(t) D w_{j}(x) \right|^{p} dx.$$ (9)

Integrating (9) over $t \in (0,T]$ to obtain

$$\int_{0}^{T} \int_{\Omega} \left| \partial_{t} u_{k} \right|^{2} dx dt = -\int_{\Omega} \frac{1}{p} |Du_{k}(t)|^{p} dx + \int_{\Omega} \frac{1}{p} |Du_{0}|^{p} dx.$$ (10)
By (8), \( \sum_{j=1}^{k} \left| \frac{d g_j(T)}{dt} \right|^2 \) is bounded. Then we can solve (6) with initial value \( u_k(T,x) \) to have solution of (6) in \([0,T+\delta]\) for some \( \delta > 0 \). Repeat the process by the continuity method to get the global in time existence of ODE in (6) such that \( u_k \in L^\infty(0,\infty; W^{1,p}_0(\Omega, \mathbb{R}^m)) \), \( \partial_t u_k \in L^2((0, \infty) \times \Omega, \mathbb{R}^m) \) and

\[
\int_0^\infty \int_\Omega |\partial_t u_k|^2 \, dx \, dt + \sup_{0 < t < \infty} \int_\Omega \frac{1}{p} |Du_k(t)|^p \, dx \leq \int_\Omega \frac{1}{p} |Du_0|^p \, dx.
\tag{11}
\]

Meanwhile, by Poincaré’s inequality, from (8) we obtain that

\[ \partial_t u_k \text{ is bounded in } L^2((0,T) \times \Omega, \mathbb{R}^m) \]

and

\[ u_k \text{ is bounded in } L^\infty(0,T; W^{1,p} (\Omega, \mathbb{R}^m)) \]

and thus, there exists a subsequence \([u_k]\), denoted by the same notation as above, to get that as \( k \to \infty \), and a limit map \( u \in L^\infty(0,T; W^{1,p} (\Omega, \mathbb{R}^m)) \),

\[ \partial_t u_k \to \partial_t u \text{ weakly in } L^2((0,T) \times \Omega, \mathbb{R}^m) \]
\[ u_k \to u \text{ weakly* in } L^\infty((0,T) \times \Omega; W^{1,p} (\Omega, \mathbb{R}^m)) \]
\[ Du_k \to Du \text{ weakly* in } L^\infty((0,T) \times \Omega; L^p (\Omega, \mathbb{R}^m)) \]
\[ \tag{15} \]

and

\[ u_k \to u \text{ strongly in } L^p((0,T) \times \Omega, \mathbb{R}^m). \]

From (12) and (13) that for every \( k \geq 1 \), the problem (3) has an energy inequality (4).

Here, the boundary condition is understood to hold \( u(t) - u_0(t) \) in the trace sense of \( W^{1,p} (\Omega, \mathbb{R}^m) \) for almost every \( t \in (0,T) \), \( \lim_{t \to 0^+} |u(t) - u(0)| = 0 \). In fact, by construction, for every \( k \geq 1 \) we also have \( u_k(t) - u_0(t) \in W^{1,p}_0 (\Omega, \mathbb{R}^m) \) a.e \( t \in (0,T) \). By using (15), the strong convergence of \( u_k \) to \( u \) in \( L^1((0,T) \times \Omega, \mathbb{R}^m) \) as \( k \to \infty \) and applying the Lebesgue convergence theorem to have

\[ \int_{\Omega} |u_k(t) - u(t)|^p \, dx \to 0, \text{ as } k \to \infty, \]

i.e \( u_k(t) \to u(t) \) weakly in \( W^{1,p} (\Omega, \mathbb{R}^m) \) a.e \( t \in (0,T) \). Then, by Mazur’s theorem we have that \( u_k(t) - u(t) \in W^{1,p}_0 (\Omega, \mathbb{R}^m) \) a.e \( t \in (0,T) \).

Next, We also state a strong convergence theorem below, which also plays crucial role in the limiting process.

**Theorem 4. (Strong Convergence Theorem)**

Let \( T > 0 \). For \( k = 1, 2, \ldots, \) let \( u_k \in L^\infty(0,T; W^{1,p}_0 (\Omega, \mathbb{R}^m)) \) such that \( \partial_t u_k \in L^2((0,T) \times \Omega, \mathbb{R}^m) \) and \( u_k \) be a weak solution of

\[ \partial_t u_k - \text{div}(|Du_k|^{p-2} Du_k) = 0. \]

Assume the boundary condition \( u_k(t) - u(t) \in W^{1,p}_0 (\Omega, \mathbb{R}^m) \) almost every \( t \in (0,T) \). Let \( \{u_k\}_{k=1}^{\infty} \) is bounded in \( L^\infty(0,T; W^{1,p} (\Omega, \mathbb{R}^m)) \), \( \{\partial_t u_k\}_{k=1}^{\infty} \) is bounded in \( L^2((0,T) \times \Omega, \mathbb{R}^m) \). Then \( \{u_k\}_{k=1}^{\infty} \) is precompact in \( L^q_{loc}(0,T; W^{1,q} (\Omega, \mathbb{R}^m)) \), for each \( 1 \leq q < p \).
Proof. Basically, we follow the argument in ([7], Theorem 2.1, pp. 31-33). However, we need some care for our operator, because of a different algebraic inequality for \( p \geq 2 \) and for \( \frac{2m}{m+2} < p < 2 \). First we let \( B \subset \mathbb{R}^m \) be a domain compactly contained in \( \Omega \). For simplicity we assume that \( Q = (0,T) \times B \) for \( T > 0 \). Letting for a positive \( \delta < 1 \) and \( k \geq 1 \), for \( \forall T > 0 \) we define

\[
S_{\delta,k} := \{ z \in Q; |u_k(z) - u(z)| > \delta \}.
\]

From the strong convergence of \( \{u_k\} \) that \( n \)-dimensional Lebesgue measure of subset \( S_{\delta,k} \) in \( \mathbb{R}^m \) converges to zero, such that for any positive \( \delta < 1 \), and thus, taking \( q, 1 \leq q < p \), obtain from Hölder’s inequality that

\[
\int_{S_{\delta,k}} |Du_k - Du|^q \, dz \leq \left( \int_{S_{\delta,k}} |Du_k - Du|^p \, dz \right)^{\frac{q}{p}} \left( \int_{S_{\delta,k}} 1^{\frac{p}{p-q}} \, dz \right)^{\frac{p-q}{p}} \]

\[
\leq \left( \int_{S_{\delta,k}} |Du_k - Du|^p \, dz \right)^{\frac{q}{p}} |S_{\delta,k}|^{1-\frac{q}{p}}.
\]

(16)

The term in bracket in the right-hand side of (16) is bounded to

\[
2^{p-1} \left( \int_{S_{\delta,k}} |Du_k|^p \, dz + \int_{S_{\delta,k}} |Du|^p \, dz \right) \leq 2^{p-1} \left( \int_{S_{\delta,k}} |Du_k|^p \, dz + \int_{S_{\delta,k}} |Du|^p \, dz \right) \leq 2^{p-1} (M^p + M^p) = 2^{p} M^p.
\]

(17)

Claim that \( \forall \varepsilon > 0 \), there exists \( \delta > 0 \), \( k_0 \geq 0 \) such that for any \( k \geq k_0 \),

\[
\int_{Q\setminus S_{\delta,k}} |Du_k - Du|^p \, dz \leq \varepsilon,
\]

(18)

where the set \( Q\setminus S_{\delta,k} := \{ z \in Q; |u_k(z) - u(z)| \leq \delta \} \). Recall that following algebraic inequalities for \( \frac{2m}{m+2} < p < 2 \),

\[
(|P|^{p-2}P - |Q|^{p-2}Q) \cdot (P - Q) \geq C |P - Q|^2 ((|P| + |Q|)^{p-2} - 1),
\]

(19)

\[
|(P|^{p-2}P - |Q|^{p-2}Q)| \leq C |P - Q|^{p-1}.
\]

(20)

Assume that \( u_k - u \) satisfies the zero boundary condition. Then use a test function \( \varphi = \eta(u_k - u) \) to have

\[
\int_{Q} |Du_k|^{p-2}Du_k \cdot D\varphi \, dz = \int_{Q\setminus S_{\delta,k}} |Du_k|^{p-2}Du_k \cdot D(u_k - u) \, dz
\]

\[
+ \int_{S_{\delta,k}} |Du_k|^{p-2}Du_k \cdot D \left( \frac{\delta(u_k - u)}{|u_k - u|} \right) \, dz.
\]

(21)

For some constant \( C \),

\[
C \int_{Q\setminus S_{\delta,k}} |Du_k - Du|^p \, dz
\]

\[
\leq \left( \int_{Q\setminus S_{\delta,k}} (|Du_k| + |Du|)^p \, dz \right)^{\frac{1}{2}} \left( \int_{Q\setminus S_{\delta,k}} |Du_k - Du|^2 (|Du_k| + |Du|)^{p-2} \, dz \right)^{\frac{1}{2}}.
\]

(22)
By (17), the first term in the bracket in (22) is bounded by $2^p M^p$.

For the second term, we claim that $\forall \epsilon > 0$, there exist $\delta > 0$, $k_0 \geq 0$ such that for any $k \geq k_0$,

$$
\int_{Q \setminus S_{\delta,k}} |D u_k - D u|^2 (|D u_k| + |D u|)^{p-2} \, dz \leq \epsilon.
$$

(23)

Use algebraic inequality as in (17) such that

$$
\int_{Q \setminus S_{\delta,k}} |D u_k - D u|^2 (|D u_k| + |D u|)^{p-2} \, dz 
\leq C \int_{Q \setminus S_{\delta,k}} (|D u_k|^{p-2} D u_k - |D u|^{p-2} D u) \cdot (D u_k - D u) \, dz 
\leq C \int_{Q} |D u_k|^{p-2} D u_k \cdot D (\eta(u_k - u)) \, dz - C \int_{S_{\delta,k}} |D u_k|^{p-2} D u_k \cdot D (\frac{\delta(u_k - u)}{|u_k - u|}) \, dz 
= I + II + III.
$$

(24)

Next, solve each term in (24) separately and choose $\delta > 0$ to be small enough to obtain (14). From (13) and (15) it follows that

$$
D u_k \rightarrow D u \text{ strongly in } L^q(0,T : W^{1,q}(B,\mathbb{R}^m)).
$$

Since $B \subset \Omega$ arbitrary, then

$$
D u_k \rightarrow D u \text{ strongly in } L^q_{loc}(0,T; W^{1,q}(\Omega,\mathbb{R}^m)),
$$

where $1 \leq q < p$.

Now, claim that

$$
|D u_k|^{p-2} D u_k \rightharpoonup |D u|^{p-2} D u
$$

weakly in $L^q(0,T : W^{1,q}(B,\mathbb{R}^m))$, $1 < q < p$.

Proof. Assume $\varphi \in L^p(0,T; W^{1,p}(B,\mathbb{R}^m))$.

$$
\left| \int_0^T \int_B |D u_k|^{p-2} D u_k \cdot D \varphi - |D u|^{p-2} D u \cdot D \varphi \, dz \right|
\leq \int_0^T \int_B \left( |D u_k|^{p-2} D u_k - |D u|^{p-2} D u \right) \cdot D \varphi \, dz
\leq \int_0^T \int_B \left( \sum_{\alpha=1}^m \sum_{i=1}^n \int_0^T \left( |D u_k|^{p-2} D^\alpha u_{ik} - |D u|^{p-2} D^\alpha u_{ik} \right) \cdot D^\alpha \varphi \, dz \right)
\leq \int_0^T \int_B \left( \sum_{\alpha=1}^m \sum_{i=1}^n \left( |D u_k|^{p-2} D^\alpha u_{ik} - |D u|^{p-2} D^\alpha u_{ik} \right) \cdot D^\alpha \varphi \, dz \right)
\leq \int_0^T \int_B \left( |D u_k|^{p-2} D u_k - |D u|^{p-2} D u \right) dz \cdot \int_0^T \int_B |D \varphi| \, dz.
$$
By inequality (23), to have
\[
\int_0^T \int_B \left| |Du_k|^{p-2} Du_k - |Du|^{p-2} Du \right| dz \cdot \int_0^T \int_B |D\varphi| dz \\
\leq C \|D\varphi\|_{L^\infty} \int_0^T \int_B |Du_k - Du|^{p-1} dz,
\]
then by Hölder’s inequality, choose \( \alpha = \frac{q}{p-1} \) and \( \beta = \frac{q}{q-(p-1)} \) where \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \), to obtain
\[
C \|D\varphi\|_{L^\infty} \int_0^T \int_B |Du_k - Du|^{p-1} dz \\
\leq C \|D\varphi\|_{L^\infty} \left( \int_0^T \int_B |Du_k - Du|^{(p-1)/\alpha} \right)^{\frac{1}{\alpha}} \left( \int_0^T \int_B 1^{\frac{1}{\beta}} \right)^{\frac{1}{\beta}} \\
\leq C \|D\varphi\|_{L^\infty} \left( \int_0^T \int_B |Du_k - Du|^q \right)^{\frac{p-1}{q}} \left( \int_0^T \int_B 1^{q-(p-1)} \right)^{\frac{q-(p-1)}{q}} \\
\leq C \|D\varphi\|_{L^\infty} \left( \int_0^T \int_B |Du_k - Du|^q \right)^{\frac{p-1}{q}} |Q|^{\frac{q-(p-1)}{q}}. 
\] (26)
since \( Du_k \to Du \) strongly in \( L^q(0,T; W^{1,q}(B,\mathbb{R}^n)) \) for any \( 1 \leq q < p \), then we have (25).

Lastly, we will prove the main theorem in this paper.

**Proof of main theorem.** Fix \( T > 0 \). Using Galerkin method as in (4) such that
\[
\frac{dg_{ik}(t)}{dt} = -\int_\Omega \left| \sum_{i=1}^k g_{ik}(t) Dw_i(x) \right|^{p-2} \sum_{i=1}^k g_{ik}(t) Dw_i(x) \cdot Dw_j dx 
\] (27)
and consider the expansion
\[
\varphi(t,x) = \sum_{i=1}^\infty \tilde{\varphi}_i(t) w_i(x); \quad \{\tilde{\varphi}_i(t)\} \subset L^p(0,T),
\]
where \( \{w_i\} \) is a fundamental system, dense in \( L^p(0,T; W^{1,p}_0(\Omega,\mathbb{R}^m)) \) and orthonormal in \( L^2(\Omega,\mathbb{R}^m) \).

Denoting
\[
\psi_l(t,x) = \sum_{i=1}^l \tilde{\varphi}_i(t) w_i(x).
\]
Let \( l \leq k \) be a positive integer, multiply (27) by \( \tilde{\varphi}_i, i = 1,2,\ldots \), then sum resulting equalities over \( i \) from 1 to \( l \) and integrate in \( t \) over \( (0,T) \) to have, for \( l \geq k \),
\[
\int_0^T \int_\Omega \partial_t u_k \sum_{i=1}^l \tilde{\varphi}_i w_i + |Du_k|^{p-2} Du_k \sum_{i=1}^l \tilde{\varphi}_i Dw_i dz = 0. 
\] (28)
As $k \to \infty$ and $l \to \infty$ in (28) to have

$$\int_{0}^{T} \int_{\Omega} \frac{\partial_t u \cdot \varphi}{\partial t} + |Du|^{p-2} Du \cdot D\varphi \, dz = 0,$$

(29)

for any $\varphi \in L^p(0,T; W^{1,p}_0(\Omega, \mathbb{R}^m))$.

References

[1] Chen Y Z and DiBenedetto E 1992 Hölder estimates of solutions of singular parabolic equations with measurable coefficients *Arch. Ration. Mech. Anal.* 118 257-271.

[2] DiBenedetto E 1986 On the local behaviour of solutions of degenerate parabolic equations with measurable coefficients *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* 3 487-535.

[3] DiBenedetto E 1993 *Degenerate Parabolic Equations* (New York Springer-Verlag) p xv+387.

[4] Campanato S 1992 On nonlinear parabolic systems in divergence form Hölder continuity and partial Hölder continuity of the solution *Ann. Math. Pura Appl.* 137 83-122.

[5] Chen Y Z and DiBenedetto E 1989 Boundary estimates for solutions of nonlinear degenerate parabolic systems *J. Reine Angew Math.* 395 102-131.

[6] Choe H J 1992 Hölder continuity of solutions of certain degenerate parabolic systems *Nonlinear Analysis, Theory, Methods and Applications* 18 (3) 235-243.

[7] Chen Y, Hong M C and Hungerbühler N 1994 Heat flow of $p$-harmonic maps with values into spheres *Mathematische Zeitschrift* 125 25-35.

[8] Misawa M 2013 A Hölder estimate for nonlinear parabolic systems of $p$-Laplacian type *J. Differential Equations* 254 847-878.

[9] Karim C 2018 Local Boundedness of weak solutions for singular parabolic systems of $p$-Laplacian type *The Australian Journal of Mathematical Analysis and Applications* 15 (2-8) 1-5.

[10] Karim C and Misawa M 2015 Hölder regularity for singular parabolic systems of $p$-Laplacian type *Advances in Differential Equations* 20 (7-8) 741-772.

[11] Karim C and Misawa M 2016 Gradient Hölder regularity for nonlinear parabolic systems of $p$-Laplacian type *Differential and Integral Equations* 29 (3-4) 201-228.

[12] Karim C 2019 Local Hölder Regularity of Weak Solutions for Degenerate Parabolic Type *AIP Conference Proceedings* 2192 05003-1 â–Å 05003-4.

[13] Karim C and Misawa M 2017 Existence of Global Weak Solutions for Cauchy-Dirichlet Problem for Evolutional $p$-Laplacian Systems *American Institute of Physics* 1913 020001-1 â–Å 020001-4.

[14] Evans L C 1998 *Partial Differential Equations* (USA : American Mathematical Society) p 353-354.