On Fiber Bundles and Quaternionic Slice Regular Functions

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Received: 16 October 2021 / Accepted: 1 June 2022 / Published online: 21 June 2022
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Abstract
The papers (González-Cervantes in Adv Appl Clifford Algebras 31:55, 2021; González-Cervantes Complex Variables and Elliptic Equations 2021) are the first works to apply the theory of fiber bundles in the study of the quaternionic slice regular functions. The main goal of the present work is to extend the results given in González-Cervantes (2021), where the quaternionic right linear space of quaternionic slice regular functions was presented as the base space of a fiber bundle. When the quaternionic right linear space of quaternionic slice regular functions is associated to certain domains then this paper shows that the elements of total space, given in González-Cervantes (2021), are defined from a pair of harmonic functions and a pair of orthogonal vectors. Simplifying the computations presented in González-Cervantes (2021), where each element of the total space is formed by two pair of conjugate harmonic functions and a pair of orthogonal unit vectors. This work also gives some interpretations of the behavior of the zero sets of some quaternionic slice regular polynomials in terms of the theory of fiber bundles.

Keywords
Quaternionic slice regular functions · Fiber bundles · Harmonic functions · Quaternionic slice regular polynomials · Zero sets

Mathematics Subject Classification Primary 30G35; Secondary 46M20

Communicated by Irene Sabadini.

This work was partial supported by CONACYT and by Instituto Politécnico Nacional (grant numbers SIP20221274).

This article is part of the topical collection “Higher Dimensional Geometric Function Theory and Hypercomplex Analysis” edited by Irene Sabadini, Michael Shapiro and Daniele Struppa.

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1 Introduction

The theory of fiber bundle in algebraic topology arises in 1950 and N. E. Steenrod published the first textbook on fiber bundles in 1951, see [26]. One the first applications of this theory was to give a mathematical interpretation of many physical phenomena, see [3, 4, 27].

In addition, paper [20] shows that the quaternionic right linear space of slice regular functions is the total space of a fiber bundle intrinsically defined from the Representations Theorem and Splitting Lemma. What is more, the quaternionic slice regular functions is defined on the total space of some sphere bundles in paper [21].

Let us recall that given a two-dimensional harmonic function \( u \) defined on a domain one can find a holomorphic function such that \( \text{Re} f = u \). Particularly, if \( D \) is a disk then the Schwarz’s formula helps us to obtain \( f \). As a consequence of the above results this work shows a simplified version of the computations presented in [20] and a supplement to [2], where the relationship between the harmonicity with the slice regularity was studied.

The zero sets of the quaternionic slice regular polynomials was studied in [12, 15, 17]. Now this paper interprets a relationship between some quaternionic slice regular polynomials with their zero sets in terms of the theory of fiber bundles.

The structure of the paper is as follows: Sect. 2 shows some basic facts about the conjugate harmonic functions, the zero sets of complex polynomials and the theory presented in [20]. Section 3 has three subsections, the first one presents a simplified version of the fiber bundle induced by the theory of quaternionic slice regular functions. The relationship of some slice regular polynomials with its zero sets is studied using the fiber bundles theory in Sect. 3.2. Finally Sect. 3.3 shows some conclusions and future works.

2 Preliminaries

Below we give basic definitions and facts on the harmonic functions, the zero set of complex polynomials and on the fiber bundles. These notions will be used throughout the whole paper.

2.1 Basic Definitions and Facts On the Harmonic Functions and Complex Polynomials

Let us recall that given a harmonic function \( u \) defined on a simply connected domain \( G \subset \mathbb{C} \) then a conjugate harmonic of \( u \) is

\[
v = \int_{(x_0, y_0)}^{(x, y)} (-u_y dx + u_x dy),
\]

and it is uniquely determined up to an additive constant, see [25] and if we assume that \( G \) is the disk \(|z| \leq \rho\) then any holomorphic function with the real part \( u \) is given
by

\[ \frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{it}) \frac{\rho e^{it} + z}{\rho e^{it} - z} dt + i\lambda, \]  

(2.2)

for \(|z| < \rho\), where \(\lambda \in \mathbb{R}\), see the Schwarz’s Formula in [1].

On the other hand, given a \(n\)-tuple \(A\) of complex numbers there exists a unique monic complex polynomial \(f\) of degree \(n\) such that the components of \(A\) are the zeros of \(f\). What is more, the mapping from the \(n\)-tuples of complex numbers to the set of complex monic polynomials of degree \(n\): \(A \mapsto f\) is a bijective mapping.

In addition, the Gauss–Lucas Theorem shows that \(Z_f \subset Kull(Z_f)\) for any complex polynomial \(f\), where \(Kull(Z_f)\) is the convex hull of \(Z_f\) that is the intersection of all convex set that contain \(Z_f\), see [1, 23].

### 2.2 Rudiments of Quaternionic Analysis and Fiber Bundles

The skew-field of quaternions, denoted by \(\mathbb{H}\), consists of \(q = x_0 + x_1e_1 + x_2e_2 + x_3e_3\) where \(x_0, x_1, x_2, x_3 \in \mathbb{R}\) and \(e_1^2 = e_2^2 = e_3^2 = -1\), \(e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = e_1, e_3e_1 = -e_1e_3 = e_2\). The sets \([e_1, e_2, e_3]\) and \([1, e_1, e_2, e_3]\) are the standard basis of \(\mathbb{R}^3\) and \(\mathbb{H}\), respectively. The vector part of \(q \in \mathbb{H}\) is \(q = x_1e_1 + x_2e_2 + x_3e_3\) and the real part is \(q_0 = x_0\). The conjugate quaternionic of \(q\) is \(\bar{q} = q_0 - q\) and its norm is \(\|q\| := \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}\).

The quaternionic unit open ball is \(\mathbb{B}^4(0, 1) := \{q \in \mathbb{H} \mid \|q\| < 1\}\). Usually, given \(q \in \mathbb{H}\) and \(\rho > 0\) denote \(\mathbb{B}^4(q, \rho) = \{r \in \mathbb{H} \mid \|q - r\| < \rho\}\). The unit spheres in \(\mathbb{R}^3\) and in \(\mathbb{H}\) are \(S^2 := \{q \in \mathbb{R}^3 \mid \|q\| = 1\}\) and \(S^3 := \{q \in \mathbb{H} \mid \|q\| = 1\}\), respectively.

The set \(T\) consists of \((i, j) \in S^2 \times S^2\) such that \((i, j)\) is co-oriented with the standard basis of \(\mathbb{R}^3\).

Due to \(i^2 = -1\) for all \(i \in S^2\) we see that \(\mathbb{C}(i) := \{x + iy \mid x, y \in \mathbb{R}\} \cong \mathbb{C}\) as fields. What is more, if \(q \neq 0\) then \(q\) can be rewritten by \(x + \mathbf{I}_q y\) where \(x, y \in \mathbb{R}\) and \(\mathbf{I}_q := \|q\|^{-1}q \in S^2\). If \(q = 0\) then choose \(y = 0\).

Given \(u \in S^3\), the mapping \(q \mapsto u\bar{q}\) for all \(q \in \mathbb{R}^3\) is a quaternionic rotation that preserves \(\mathbb{R}^3\), see [22]. Define \(R_u : T \rightarrow T\) by \(R_u(i, j) := (u\bar{u}, u\bar{j})\) for all \((i, j) \in T\) and consider the norm \(\|\cdot\|_{S^2}\) in \(T\).

A real differentiable function \(f : \Omega \rightarrow \mathbb{H}\), where \(\Omega \subset \mathbb{H}\) is an open set, is called left slice regular function, or slice regular function on \(\Omega\), if

\[ \bar{\partial}_i f |_{\Omega \cap C(i)} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f |_{\Omega \cap C(i)} = 0 \text{ on } \Omega \cap C(i), \]

for all \(i \in S^2\). The derivative of \(f\), or Cullen’s derivative, is \(f' = \partial_k f |_{\Omega \cap C(i)} = \frac{\partial}{\partial x} f |_{\Omega \cap C(i)} = \partial_x f |_{\Omega \cap C(i)}\). The quaternionic right linear space of the slice regular functions on \(\Omega\) is denoted by \(S\mathcal{R}(\Omega)\), see [7, 9, 10, 12, 16]. A set \(U \subset \mathbb{H}\) is called axially symmetric slice domain, or axially symmetric s-domain, if \(U \cap \mathbb{R} \neq \emptyset, U_1 = \).
\( U \cap \mathbb{C}(i) \) is a domain in \( \mathbb{C}(i) \) for all \( i \in \mathbb{S}^2 \), and \( x + iy \in U \) with \( x, y \in \mathbb{R} \) implies \( \{x + jy \mid j \in \mathbb{S}^2\} \subset U \).

Given \( i \in \mathbb{S}^2 \) an axially symmetric s-domain \( \Omega \subset \mathbb{H} \) shall be called \( i \)-simply connected axially symmetric s-domain if \( \Omega_i \subset \mathbb{C}(i) \) is a simply connected domain. From the symmetric property of \( \Omega \) one sees that \( \Omega_j \subset \mathbb{C}(j) \) is a simply connected domain for all \( j \in \mathbb{S}^2 \). The axially symmetric s-domains \( \Omega, \Xi \subset \mathbb{H} \) are \( i \)-conformally equivalent iff there exists \( \alpha \in S^R(\Omega) \) such that \( Q_{i,j}[\alpha] : \Omega_i \to \Xi_i \) is a biholomorphism.

Let us mention two properties of the quaternionic slice regular functions.

Splitting Lemma. Given an axially symmetric s-domain \( \Omega \subset \mathbb{H} \) and \( f \in S^R(\Omega) \).

For every \( i, j \in \mathbb{S} \), orthogonal to each other, there exist \( F, G \in \text{Hol}(\Omega_i) \), holomorphic functions, such that \( f|_{\Omega_i} = F + Gj \) on \( \Omega_i \), see [10].

Representation Formula. Given an axially symmetric s-domain \( \Omega \subset \mathbb{H} \) and \( f \in S^R(\Omega) \). For every \( q = x + I_q y \in \Omega \) with \( x, y \in \mathbb{R} \) and \( I_q \in \mathbb{S}^2 \) one has that

\[
    f(x + I_q y) = \frac{1}{2}[f(x + iy) + f(x - iy)] + \frac{1}{2} I_q i[f(x - iy) - f(x + iy)],
\]

for all \( i \in \mathbb{S}^2 \), see [7].

From the previous results one has the following operators:

- \( Q_{i,j} : S^R(\Omega) \to \text{Hol}(\Omega_i) + \text{Hol}(\Omega_j) \) given by \( Q_{i,j}[f] = f \mid_{\Omega_i} = f_1 + f_2 j \) for all \( f \in S^R(\Omega) \) where \( f_1, f_2 \in \text{Hol}(\Omega_i) \), space of holomorphic function on \( \Omega_i \).

- \( P_{i,j} : \text{Hol}(\Omega_i) + \text{Hol}(\Omega_j) \to S^R(\Omega) \) defined by

\[
    P_{i,j}[g](q) = \frac{1}{2} [(1 + I_q i)g(x - yi) + (1 - I_q i)g(x + yi)],
\]

for all \( g \in \text{Hol}(\Omega_i) + \text{Hol}(\Omega_j) \) where \( q = x + I_q y \in \Omega \).

What is more,

\[
    P_{i,j} \circ Q_{i,j} = \mathcal{I}_{S^R(\Omega)} \quad \text{and} \quad Q_{i,j} \circ P_{i,j} = \mathcal{I}_{\text{Hol}(\Omega_i) + \text{Hol}(\Omega_j)},
\]

(2.4)

where \( \mathcal{I}_{S^R(\Omega)} \) and \( \mathcal{I}_{\text{Hol}(\Omega_i) + \text{Hol}(\Omega_j)} \) are the identity operators in \( S^R(\Omega) \) and in \( \text{Hol}(\Omega_i) + \text{Hol}(\Omega_j) \), respectively, see [19].

Given \( g \in S^R(\Omega) \) and \( (i, j) \in T \), the real components of \( Q_{i,j}[g] \) are given by

\[
    D_1[g, i, j] := \frac{Q_{i,j}[g] + \bar{Q}_{i,j}[g]}{2} , \quad D_2[g, i, j] := -\frac{Q_{i,j}[g]i + \bar{Q}_{i,j}[g]j}{2} , \quad D_3[g, i, j] := -\frac{Q_{i,j}[g]j + \bar{Q}_{i,j}[g]i}{2} , \quad D_4[g, i, j] := \frac{Q_{i,j}[g]ij + \bar{Q}_{i,j}[g]ji}{2} ,
\]

(2.5)
i.e.,

\[
    Q_{i,j}[g] = D_1[g, i, j] + D_2[g, i, j]i + D_3[g, i, j]j + D_4[g, i, j]ij .
\]

(2.6)
Let us recall several properties of the quaternionic slice regular functions: For any \( f, g \in \mathcal{SR}(\Omega) \) define

\[
f \cdot_i g := P_{i,j}[f_1 g_1 + f_2 g_2 j],
\]

where \( Q_{i,j}[f] = f_1 + f_2 j \) and \( Q_{i,j}[g] = g_1 + g_2 j \) with \( f_1, f_2, g_1, g_2 \in \text{Hol}(\Omega_4) \). Also note that

\[
f + ifi = 2f_1 \quad \text{and} \quad f - ifi = 2f_2 j,
\]

on \( \Omega_4 \). What is more, if \( f, g \in \mathcal{SR}(\mathbb{B}^4(0,1)) \) there exist \( (a_n), (b_n) \subset \mathbb{H} \) such that

\[
f(q) = \sum_{n=0}^{\infty} q^n a_n, \quad g(q) = \sum_{n=0}^{\infty} q^n b_n \quad \text{and} \quad f \ast g(q) := \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n} a_k b_{n-k},
\]

for all \( q \in \mathbb{B}^4(0,1) \), see [7]. By \( \mathcal{PSR}(\mathbb{H}) \) we mean the set of the monic quaternionic slice regular polynomials, i.e., \( f \in \mathcal{PSR}(\mathbb{H}) \) iff there exists \( a_0, a_1, \ldots, a_{n-1} \in \mathbb{H} \) such that

\[
f(q) = a_0 + qa_1 + \cdots + q^{n-1}a_{n-1} + q^n.
\]

Denote \( \mathcal{SR}_c(\Omega) := \mathcal{SR}(\Omega) \cap C(\overline{\Omega}, \mathbb{H}) \).

On the other hand, in [5] it is found that a fiber bundle is denoted as \( (X, P, B, F) \), where \( X, B \) and \( F \) are Hausdorff spaces and are called total space, base space and the fiber space, respectively. There exists a topological group \( K \), called structure group, acting on \( F \) as a group of homeomorphisms. The bundle projection \( P : X \to B \) satisfies that for each element of \( B \) has a neighborhood \( U \subset B \) and a homeomorphism \( \varphi : U \times F \to P^{-1}(U) \), called a trivialization over \( U \), such that \( P \circ \varphi(b, y) = b \) for all \( b \in U \) and \( y \in F \). The family of all trivializations, \( \Phi \), satisfy that

1. If \( \varphi : U \times F \to P^{-1}(U) \) belongs to \( \Phi \) and if \( V \subset U \) then \( \varphi|_{V \times F} \) belongs to \( \Phi \).
2. If \( \varphi, \psi \in \Phi \) are trivializations over \( U \subset B \) then there exists a map \( \psi : U \to K \)
   such that \( \psi(u, y) = \varphi(u, \varphi(u)(y)) \) for all \( u \in U \) and \( y \in F \).
3. \( \Phi \) is a maximal family that satisfies the previous facts.

A continuous map \( S : B \to X \) such that \( P \circ S(x) = x \) for all \( x \in B \) is a section of \( (X, P, B, F) \), see [26] and an arbitrary map \( \mathcal{R} : A \to B \), where \( A \) is a nonempty set, induce the pullback fiber bundle \( (\mathcal{R}^*(X), P', A, F) \), where \( \mathcal{R}^*(X) := \{(a, x) \in A \times X \mid P(x) = \mathcal{R}(a)\} \) and \( P'(a, x) = a \) for all \( (a, x) \in \mathcal{R}^*(X) \), see [5, 18].

A morphism between two fiber bundles

\[
\Gamma : (X_1, P_1, B_1, F_1) \to (X_2, P_2, B_2, F_2)
\]
consists of a pair of continuous maps $\Gamma_1 : X_1 \to X_2$ and $\Gamma_2 : B_1 \to B_2$ such that the diagram

$$
\begin{array}{ccc}
P_2 & \xrightarrow{\quad} & B_2 \\
X_2 & \xleftarrow{\Gamma_1} & X_1 \\
\uparrow & & \uparrow \\
P_1 & \xleftarrow{\quad} & B_1
\end{array}
$$

commutes and there exists a morphism $\Gamma^{-1} : (X_2, P_2, B_2, F_2) \to (X_1, P_1, B_1, F_1)$ such that $\Gamma \circ \Gamma^{-1}$ and $\Gamma^{-1} \circ \Gamma$ are the identity morphisms, then $\Gamma$ is an isomorphism, see [5, 18].

An interesting application of the previous theory in the quaternionic slice regular functions was presented in [20] showing that $(\mathcal{H}(S_\Omega), P_\Omega, \mathcal{SR}_c(\Omega), T)$ is a kind of bi-sphere bundle, where $\Omega \subset \mathbb{H}$ is a bounded axially symmetric s-domain, the set of pairs of conjugated harmonic functions on

$$
S_\Omega := \{(x, y) \in \mathbb{R}^2 \mid \text{there exists } i \in S^2 \text{ such that } x + yi \in \Omega\}
$$

that are continuous on $S_\Omega$ is denoted by $\text{Harm}_c^2(S_\Omega)$,

$$
\mathcal{H}(S_\Omega) := \left\{ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (i, j) \right) \mid (a, b), (c, d) \in \text{Harm}_c^2(S_\Omega), (i, j) \in T \right\}.
$$

The projection bundle $P_\Omega : \mathcal{H}(S_\Omega) \to \mathcal{SR}_c(\Omega)$ is

$$
P_\Omega \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (i, j) \right) := P_{i, j}[a + bi + cj + dij],
$$

and the structure group is $K = \{R_u \mid u \in S^3\}$. The trivializations are given by

$$
\varphi_u[f, (i, j)] := \left( \begin{array}{c}
D_1[f, ui\bar{u}, uj\bar{u}] \\
D_2[f, ui\bar{u}, uj\bar{u}] \\
D_3[f, ui\bar{u}, uj\bar{u}] \\
D_4[f, ui\bar{u}, uj\bar{u}]
\end{array} \right), (ui\bar{u}, uj\bar{u}),
$$

for all $(f, (i, j)) \in U \times T$, where $U \subset \mathcal{SR}_c(\Omega)$ is a neighborhood and $u \in S^3$.

### 3 Main Results

#### 3.1 A Fiber Bundle Over the Quaternionic Slice Regular Functions

Paper [2] presents several relationships between the harmonic function theory with the slice regular functions. This subsection presents a consequence of the harmonicity in the interpretation of the quaternionic slice regular functions in terms of fiber bundles expanding the results obtained in [20].
Definition 3.1 Given \( i \in S^2 \) and let \( \Omega \subset \mathbb{H} \) be a \( i \)-simply connected bounded axially symmetric s-domain.

1. The set \( \text{CHarm}(\Omega) \) is formed by harmonic functions \( a \) on \( \Omega \) such that \( a, a_x \) and \( a_y \) are continuous on \( \mathring{\Omega} \). Given \( a, b \in \text{CHarm}(\Omega) \) define \( a \sim b \) iff there exits a real constant \( \lambda \) such that \( a = b + \lambda \). Define

\[
N_{\text{CHarm}}(a) := \|a\|_\infty + \|a_x\|_\infty + \|a_y\|_\infty.
\]

for all \( a \in \text{CHarm}(\Omega) \) and in \( \text{CHarm}(\Omega) / \sim \) consider the norm

\[
N_1([a]) := \inf \{ N_{\text{CHarm}}(b) \mid b \in [a] \},
\]

for all \([a] \in \text{CHarm}(\Omega) / \sim \) and define

\[
X_\Omega := (\text{CHarm}(\Omega) / \sim) \times (\text{CHarm}(\Omega) / \sim) \times T,
\]

equipped with the norm

\[
N_{X_\Omega}([a], [c], (i, j)) = N_1([a]) + N_1([c]) + \|\langle i, j \rangle\|_{\mathbb{R}^6}.
\]

2. The function set \( \text{CSR}(\Omega) \) consists of \( f \in \mathcal{SR}(\Omega) \) such that \( f \) and \( f' \) are continuous on \( \mathring{\Omega} \). Given \( f, g \in \text{CSR}(\Omega) \) define \( f \sim g \) iff there exists \( c \in \mathbb{H} \) such that \( f = g + c \).

Consider \( B_\Omega = \text{CSR}(\Omega) / \sim \) equipped with the norm

\[
N_{B_\Omega}([f]) := \inf \{ \|g\|_\infty \mid g \in [f] \}, \ \forall [f] \in B_\Omega.
\]

3. Given \([a], [c], (i, j) \) \( \in X_\Omega \) define

\[
P_\Omega ([a], [c], (i, j)) := [ P_{i,j} (a + cj + i \int_{(x_0, y_0)}^{(x, y)} -(a + cj)x \, dx + (a + cj)y \, dy ) ].
\]

where the integration is on any path contained in \( S_\Omega \) from \( (x_0, y_0) \) to \( (x, y) \).

4. Given \((i, j) \in T \) set

\[
S_{\Omega,i,j}([f]) := (\ [D_1(f, i, j)], [D_3(f, i, j)], (i, j) ),
\]

for all \([f] \in B_\Omega.\)

5. Let \( U \subset B_\Omega \) be a neighborhood and \( u \in S^3 \). Denote

\[
\varphi_u([f], (i, j)) := ([D_1(f, ui\bar{u}, uj\bar{u})], [D_3(f, ui\bar{u}, uj\bar{u})], (ui\bar{u}, uj\bar{u})) ,
\]

for all \(([f], (i, j) \in U \times T.\)

Remark 3.2 Note the equivalence relations established to obtain \( X_\Omega \) and \( B_\Omega \) allow us to see that \( P_\Omega \) is well-defined and to use \( T \) as our fiber space.
Proposition 3.3 \((X_\Omega, P_\Omega, B_\Omega, T)\) is a fiber bundle where the structure group is \(K = \{R_u \mid u \in \mathbb{S}^3\}\) and the family of trivializations is \(\Phi = \{\varphi_u \mid u \in \mathbb{S}^3\}\).

Proof The normed space \(X_\Omega, B_\Omega\) and \(T\) are Hausdorff spaces. Given \(a, c, p, r \in \text{CHarm}(\Omega)\) and \((i, j), (k, l) \in T\) from Proposition 3.4 presented in [20] one obtains

\[
\|P_{i,j}(a + bi + cj + dij) - P_{k,1}(p + qk + rl + skl)\|_\infty \\
\leq 2(\|a - p\|_\infty + \|b - q\|_\infty + \|c - r\|_\infty + \|d - s\|_\infty + \|(i, j) - (k, l)\|_{\mathbb{R}^6})
\]

(\(\|p\|_\infty + \|q\|_\infty + \|r\|_\infty + \|s\|_\infty + 1\)),

where the harmonic functions \(b, d, q\) and \(r\) are given in terms of \(a, c, p\) and \(r\) according to (2.1), respectively. As \(\Omega \subset \mathbb{H}\) is a bounded set there exist two constants \(k_1, k_2 > 0\) such that

\[
\|b - q\|_\infty \leq k_1(\|a_x - p_x\|_\infty + \|a_y - p_y\|_\infty)
\]

\[
\|d - s\|_\infty \leq k_1(\|c_x - r_x\|_\infty + \|c_y - r_y\|_\infty)
\]

and

\[
\|P_{i,j}(a + bi + cj + dij) - P_{k,1}(p + qk + rl + skl)\|_\infty \\
\leq k_2(\|a - p\|_\infty + \|a_x - p_x\|_\infty + \|a_y - p_y\|_\infty + \|c - r\|_\infty + \|c_x - r_x\|_\infty
\]

\[
+ \|c_y - r_y\|_\infty + \|(i, j) - (k, l)\|_{\mathbb{R}^6})(\|p\|_\infty + \|p_x\|_\infty + \|p_y\|_\infty + \|r\|_\infty
\]

\[
+ \|r_x\|_\infty + \|r_y\|_\infty + 1) = (N_{\text{CHarm}(\Omega)}(a - p) + N_{\text{CHarm}(\Omega)}(c - r)
\]

\[
+ \|(i, j) - (k, l)\|_{\mathbb{R}^6})(N_{\text{CHarm}(\Omega)}(p) + N_{\text{CHarm}(\Omega)}(r) + 1).
\]

By the properties of the infimum of a set one obtains that

\[
N_{B_\Omega}(P_\Omega([a], [c], (i, j)) - P_\Omega([p], [r], (k, l)))
\leq k_2[N_{X_\Omega}([a] - [p], [c] - [r], (i - k, j - l))](N_1([p]) + N_1([r]) + 1).
\]

Therefore, \(P_\Omega : X_\Omega \rightarrow B_\Omega\) is a continuous mapping.

Analogously to the computations presented in Proposition 3.6 given in [20] one obtains that \(K = \{R_u \mid u \in \mathbb{S}^3\}\) equipped with the composition is a topological group which acts on \(T\) as a group of homeomorphisms.

Let \(U \subset B_\Omega\) be a neighborhood. The operator \(\varphi_u : U \times T \rightarrow P^{-1}(U)\) is a homeomorphism.

1.- \(\varphi_u\) is a bijective operator. If \(\varphi_u([f], (i, j)) = \varphi_u([g], (k, l))\) then \((i, j) = (k, l)\) and there exists \(\lambda_1, \lambda_2 \in \mathbb{R}\) such that

\[
D_1(f, uiu, uju) = D_1(g, uiu, uju) + \lambda_1,
\]

\[
D_3(f, uiu, uju) = D_3(g, uiu, uju) + \lambda_2
\]

and using (2.1) one concludes that \(Q_{uiu, uju}[f] = Q_{uiu, uju}[g] + c\), where \(c \in \mathbb{H}\) and applying \(P_{uiu, uju}\) one obtains that \([f] = [g]\).
Given \(([a], [c], (i, j)) \in P^{-1}(U)\). Denote \((k, l) = (\bar{a}iu, \bar{a}ju) \in T\) and
\[
f = P_{uk\bar{a}, ul\bar{a}}(a + cu\bar{u}) + uk\bar{a} \int \limits_{(x_0, y_0)}^{(x, y)} -(a + cu\bar{u})_y dx + (a + cu\bar{u})_x dy,
\]
that satisfy \(\varphi_u([f], (k, l)) = ([a], [c], (i, j))\).

2.- The inverse operator of \(\varphi_u\) is given by
\[
\varphi_u^{-1}([a], [c], (k, l)) = (P_\Omega([a], [c], (\bar{a}ku, \bar{a}lu)), (\bar{a}ku, \bar{a}lu)).
\]

3.- Continuity of \(\varphi_u\). Given \([f], [g] \in U\) and \((i, j), (k, l) \in \mathbb{S}^2\), from Proposition 3.5 presented in [20] one has that
\[
\|D_1(f, ui\bar{u}, uj\bar{u})(x, y) - D_1(g, uk\bar{u}, ul\bar{u})(x, y)\|
+ \|D_3(f, ui\bar{u}, uj\bar{u})(x, y) - D_3(g, uk\bar{u}, ul\bar{u})(x, y)\|
+ \|(ui\bar{u}, uj\bar{u}) - (uk\bar{u}, ul\bar{u})\|_{\mathbb{R}^6}
\leq 4\|f - g\|_{\infty} + 4\|\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y} - g'(x + ky)\|
+ (2\|g'\|_{\infty} + 1)\|(i, j) - (k, l)\|_{\mathbb{R}^6}.
\]

Repeating the previous computations for \(f', g'\) and recalling that this derivative on each slice \(\Omega_i\) is given by \(\frac{\partial}{\partial x}\), or equivalently by \(-i\frac{\partial}{\partial y}\), one concludes that
\[
\|D_1(f, ui\bar{u}, uj\bar{u}, x, y) - D_1(g, uk\bar{u}, ul\bar{u}, x, y)\|
+ \|D_3(f, ui\bar{u}, uj\bar{u}, x, y) - D_3(g, uk\bar{u}, ul\bar{u}, x, y)\|
+ \|(ui\bar{u}, uj\bar{u}) - (uk\bar{u}, ul\bar{u})\|_{\mathbb{R}^6}
\leq 4\|f' - g'\|_{\infty} + 4\|\frac{\partial g'}{\partial x} + i\frac{\partial g'}{\partial y} - g'(x + ky)\|
+ (2\|g'\|_{\infty} + 1)\|(i, j) - (k, l)\|_{\mathbb{R}^6}.
\]

and
\[
\|D_1(f, ui\bar{u}, uj\bar{u}, y, x) - D_1(g, uk\bar{u}, ul\bar{u}, y, x)\|
+ \|D_3(f, ui\bar{u}, uj\bar{u}, y, x) - D_3(g, uk\bar{u}, ul\bar{u}, y, x)\|
+ \|(ui\bar{u}, uj\bar{u}) - (uk\bar{u}, ul\bar{u})\|_{\mathbb{R}^6}
\leq 4\|f' - g'\|_{\infty} + 4\|\frac{\partial g'}{\partial x} + i\frac{\partial g'}{\partial y} - g'(x + ky)\|
+ (2\|g'\|_{\infty} + 1)\|(i, j) - (k, l)\|_{\mathbb{R}^6}.
\]

Therefore
\[
\|D_1(f, ui\bar{u}, uj\bar{u}, x, y) - D_1(g, uk\bar{u}, ul\bar{u}, x, y)\|
+ \|D_3(f, ui\bar{u}, uj\bar{u}, x, y) - D_3(g, uk\bar{u}, ul\bar{u}, x, y)\|
+ \|(ui\bar{u}, uj\bar{u}) - (uk\bar{u}, ul\bar{u})\|_{\mathbb{R}^6}
\]
\[
\begin{align*}
+ \|D_1(f, u\bar{u}, u\bar{j}u)_x(x, y) - D_1(g, u k\bar{u}, u l\bar{u})_x(x, y)\| \\
+ \|D_3(f, u\bar{u}, u\bar{j}u)_x(x, y) - D_3(g, u k\bar{u}, u l\bar{u})_x(x, y)\| \\
+ \|u(\bar{u}, u\bar{j}u) - (u k\bar{u}, u l\bar{u})\|_{\mathbb{R}^6} \\
+ \|D_1(f, u\bar{u}, u\bar{j}u)_y(x, y) - D_1(g, u k\bar{u}, u l\bar{u})_y(x, y)\| \\
+ \|D_3(f, u\bar{u}, u\bar{j}u)_y(x, y) - D_3(g, u k\bar{u}, u l\bar{u})_y(x, y)\| \\
+ \|u(\bar{u}, u\bar{j}u) - (u k\bar{u}, u l\bar{u})\|_{\mathbb{R}^6} \\
\leq 4\|f - g\|_{\infty} + 4\|g(x + iy) - g(x + k)\| \\
+ (2\|g\|_{\infty} + 1)\|u i, j\| - (u k, l)\|_{\mathbb{R}^6} \\
+ 4\|f' - g'\|_{\infty} + 4\|g'(x + iy) - g'(x + k)\| \\
+ (2\|g'\|_{\infty} + 1)\|u i, j\| - (u k, l)\|_{\mathbb{R}^6} \\
+ 4\|f'' - g''\|_{\infty} + 4\|g''(x + iy) - g''(x + k)\| \\
+ (2\|g''\|_{\infty} + 1)\|u i, j\| - (u k, l)\|_{\mathbb{R}^6}.
\end{align*}
\]

If there is a sequence \( (f_n) \in C\mathcal{SR}(\Omega) \) and \( g \in C\mathcal{SR}(\Omega) \) such that \( \|f_n - g\|_{\infty} \to 0 \), then from the Splitting Lemma on each slice we have two sequences of holomorphic functions that converge uniformly to holomorphic functions in the compact set \( \bar{\Omega} \) and from the well-known theorem of Weierstrass we have that there exists the uniform convergence between its derivatives. From Representation Theorem one has that \( \|f_n' - g''\|_{\infty} \to 0 \). Thus from the previous reasoning and the properties of the infimum of a set one obtains the continuity of \( \phi_u \) by sequences.

On the other hand, using (2.1), (2.4) and (2.6) one has that

\[
P_{\Omega} \circ \phi_u([f], (i, j)) = P_{\Omega} ([D_1(f, u\bar{u}, u\bar{j}u)], [D_3(f, u\bar{u}, u\bar{j}u)], (u\bar{u}, u\bar{j}u)) = [f].
\]

for all \( ([f], (i, j)) \in U \times T \).

Note that, given \( u, v \in \mathbb{S}^3 \), \( (i, j) \in T \) and \n\[
[f] = [P_{u\bar{u}, v\bar{j}u}((a + cu\bar{j}u) + u\bar{u} \int_{(x_0, y_0)}^{(x, y)} (a + cu\bar{j}u) dx + (a + cu\bar{j}u) dy)] \in B_{\Omega},
\]

with \( a, c \in CHarm^2(S_\Omega) \), one has that

\[
\phi_u([f], (i, j)) = (\phi_v([f], R_p(i, j)), (v(pv)u(i\bar{v}v)\bar{u}, v(pv)j(\bar{v}v)\bar{u})) = \phi_v([f], R_p(i, j)),
\]

where \( p = \bar{v}u \in \mathbb{S}^3 \).

\[\square\]

**Remark 3.4** In \((X_{\Omega}, P_{\Omega}, B_{\Omega}, T)\) the fibers of \( B_{\Omega} \) are

\[
(P_{\Omega})^{-1}([f]) := \{([D_1(f, i, j)], [D_3(f, i, j)], (i, j)) \mid (i, j) \in T\},
\]

where \([f] \in B_{\Omega}\).
Proposition 3.5 The operator \( S_{\Omega,1,j} \) is a section of \((X_\Omega, P_\Omega, B_\Omega, T)\) for all \((i, j) \in T\).

Proof From formulas (2.1), (2.4) and (2.6) one has that
\[
P_\Omega \circ S_{i,j}([f]) := ([D_1(f, i, j)], [D_3(f, i, j)], (i, j)) = [f].
\]
for all \([f] \in B_\Omega\) and the continuity of \(S_{i,j}\) is proved in the same way as the continuity of \(\varphi_u\) in the previous proposition. \(\square\)

Corollary 3.6 Given \(f \in CSR(\Omega)\).
1. For each \((i, j) \in T\) there are unique \([a], [c] \in CHarm(S_\Omega)\) such that \([(a), [c], (i, j)] \in P^{-1}([f])\).
2. \(Q_{i,j}(f) \in Hol(\Omega_4)\) if and only if there exists \(a \in CHarm(S_\Omega)\) such that \((a), [0], (i, j)) \in P^{-1}([f])\).
3. If \(f\) is an intrinsic slice regular functions on \(\Omega\), see [9, 19], if and only if there exists \(a \in CHarm(S_\Omega)\) such that \([(a), 0, (i, j)) \in P^{-1}([f])\) for all \((i, j) \in T\).

Proof These facts follow from the properties of \(P_{i,j}\) and \(Q_{i,j}\). \(\square\)

Remark 3.7 It is important to comment that Propositions 3.3 and 3.5 and Corollary 3.6 are deeply analogous to Propositions 3.6 and 3.8 and Corollary 3.9 given in [20], respectively. But the sentences presented in this paper simplify the representation of each element of the total and the base spaces using the formula (2.1), and to achieve this goal more requirements were established in our functions such as two equivalence relations and some conditions in the partial derivatives.

The following operations are necessary to show some algebraic properties of \(P_\Omega\).

Definition 3.8 Given \([(a), [c], (i, j)), ([p], [r], (i, j)) \in X\). Define
\[
([a], [c], (i, j)) + ([p], [r], (i, j)) := ([a + p], [c + r], (i, j)),
\]
\[
\mathcal{D}([a], [c], (i, j)) := ([a], [c], (i, j)),
\]
\[
\mathcal{R}_u\left(([a], [c], (i, j))\right) := ([a], [c], (u\bar{i}u, u\bar{j}u))
\]
where \(u \in S^3\). For \(f, g \in CSR(\Omega)\) denote \([f] + [g] = [f + g]\) and \([f]' = [f']\).

Proposition 3.9 Given \((i, j) \in T\) and \(A, B \in X_\Omega\). One obtains that
\[
P_\Omega(A + B) = P_\Omega(A) + P_\Omega(B),
\]
\[
(P_\Omega(A))' = P_\Omega(D(A)),
\]
\[
\mathcal{R}_u(A) = P_{u\bar{i}u, u\bar{j}u}[uQ_{i,j}[P_\Omega(A)]].
\]

Proof These facts are consequences of the previous definition. \(\square\)

Remark 3.10 The operations in Definition 3.8 can be represented by pullbacks in a similar way to Remark 3.12 of [20]. Analogously to Proposition 3.14 given in [20], one can see that the isomorphism of two quaternionic right linear space of slice regular
functions associated to i-simply connected bounded axially symmetric s-domain that are i-conformally equivalents becomes an isomorphisms of the fiber bundles given in Proposition 3.3.

On the other hand, we shall see a version of the Schwarz’s Formula for quaternionic slice regular functions to get an idea of how to define a fiber bundle associated to the slice regular functions using (2.2).

**Proposition 3.11** Given \( f \in \mathcal{SR}_c(\mathbb{B}^4(0, \rho)) \) and \( i, j \in \mathbb{S}^2 \) there exist \( a, c \in CHarm(\mathbb{B}^2(0, \rho)) \) and \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that

\[
 f(q) = \frac{1}{2\pi} \int_0^{2\pi} (\rho e^{it} + q) \ast (\rho e^{it} - q)^{-\ast} (a(\rho \cos t, \rho \sin t) + c(\rho \cos t, \rho \sin t)j)dt \\
 + \lambda_1 i + \lambda_2 ij,
\]

for all \( q \in \mathbb{B}^4(0, \rho) \) or equivalently

\[
 f(q) = u_{0,a,c} + \sum_{n=1}^{\infty} q^n u_{n,a,c}, \quad \forall q \in \mathbb{B}^4(0, \rho),
\]

where the quaternionic coefficients are given by

\[
 u_{0,a,c} = \frac{1}{2\pi} \int_0^{2\pi} (a(\rho \cos t, \rho \sin t) + c(\rho \cos t, \rho \sin t)j)dt,
\]

\[
 u_{n,a,c} = \frac{1}{\pi} \int_0^{2\pi} (\rho e^{it})^{-n} (a(\rho \cos t, \rho \sin t) + c(\rho \cos t, \rho \sin t)j)dt,
\]

for all \( n \in \mathbb{N} \).

**Proof** Note that \( Q_{i,j}[f] = f_1 + f_2j \), where \( f_1, f_2 \in Hol(\mathbb{B}^4(0, \rho)_i) \) and \( f_1, f_2 \in C(\mathbb{B}^4(0, \rho), \mathbb{C}) \). Denote \( a = \text{Re}(f_1) \) and \( c = \text{Re}(f_2) \). Thus from (2.2) there exist \( \lambda_1, \lambda_2 \in \mathbb{R} \) such that

\[
 Q_{i,j}[f](z) = f_1(z) + f_2(z)j = \frac{1}{2\pi} \int_0^{2\pi} \left( a(\rho e^{it})\rho e^{it} + z \right) \left( c(\rho e^{it})\rho e^{it} - z \right) dt + i\lambda \lambda_1 i + \lambda_2 ij
\]

and applying operator \( P_{i,j} \) one get that

\[
 f(q) = \frac{1}{2\pi} \int_0^{2\pi} P_{i,j} \left( \frac{\rho e^{it} + z}{\rho e^{it} - z} \right) (q)(a(\rho \cos t, \rho \sin t) + c(\rho \cos t, \rho \sin t)j)dt \\
 + \lambda_1 i + \lambda_2 ij, \quad \forall q \in \mathbb{B}^4(0, \rho).
\]
Identity (3.1) follows from

\[ P_{1,j} \left( \frac{pe^{it} + z}{pe^{it} - z} \right) (q) = (pe^{it} + q) \ast (pe^{it} - q)^{-*}, \quad \forall q \in \mathbb{B}^4(0, \rho). \]

On the other hand, for \(|z| < \rho\) we see that

\[ \frac{pe^{it} + z}{pe^{it} - z} = \left( 1 + \frac{z}{pe^{it}} \right) \left( \frac{1}{1 - \frac{z}{pe^{it}}} \right) = 1 + \sum_{n=0}^{\infty} z^n \frac{2}{\rho^n e^{it}}. \]

Therefore

\[ P_{1,j} \left( \frac{pe^{it} + z}{pe^{it} - z} \right) (q) = 1 + \sum_{n=0}^{\infty} q^n 2(\rho e^{it})^{-n}, \quad \forall q \in \mathbb{B}^4(0, \rho). \]

The uniform convergence of previous series implies that

\[ f(q) = \frac{1}{2\pi} \int_0^{2\pi} \left( a(\rho \cos t, \rho \sin t) + c(\rho \cos t, \rho \sin t) j \right) dt \]

\[ + \sum_{n=0}^{\infty} q^n \frac{1}{\pi} \int_0^{\pi} (\rho e^{it})^{-n} \left( a(\rho \cos t, \rho \sin t) + c(\rho \cos t, \rho \sin t) j \right) dt, \]

for all \( q \in \mathbb{B}^4(0, \rho) \). \( \square \)

**Definition 3.12** Given \( \mathbb{B}^2(0, \rho) = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\|_2 < \rho\} \) we shall consider \( C\text{Harm}(\mathbb{B}^2(0, \rho)), C\mathbb{S}R(\mathbb{B}^4(0, \rho)) \) and given \( a, c \in \text{Harm}_c(\mathbb{B}^2(0, \rho)) \) and \((i, j) \in T\) define

\[ P_\rho (a, c, (i, j)) = \frac{1}{2\pi} \int_0^{2\pi} P_{1,j} \left( \frac{pe^{it} + z}{pe^{it} - z} \right) \left( a(\rho \cos t, \rho \sin t) + c(\rho \cos t, \rho \sin t) j \right) dt, \]

and for all \((i, j) \in T\). Also denote

\[ \psi_{\rho,u}(f, (i, j)) = (D_1(f, uiu, uj\bar{u}), D_3(f, ui\bar{u}, uj\bar{u}), (uiu, uj\bar{u})), \]

for all \( f \in C\mathbb{S}R(\mathbb{B}^4(0, \rho)), (i, j) \in T \) and all \( u \in \mathbb{S}^2 \).

**Remark 3.13** One can define suitable norms and equivalence relations in \( C\text{Harm}(\mathbb{B}^2(0, \rho)) \) and in \( C\mathbb{S}R(\mathbb{B}^4(0, \rho)) \), in a similar way to Definition 3.1, to find a fiber bundle \((X_\rho, P_\rho, B_\rho, T)\). Note that, one has a total space \( X_\rho \) from \( C\text{Harm}(\mathbb{B}^2(0, \rho)) \) and a base space \( B_\rho \) from \( C\mathbb{S}R(\mathbb{B}^4(0, \rho)) \). Moreover, the operator \( P_\rho \) allows us to find a bundle projection \( P_\rho \) while from \( \psi_{\rho,u} \) one has an idea to define the trivializations and
the sections. Clearly, the structure group is $K = \{ R_u \mid u \in S^3 \}$ and the proofs of these facts shall be similar to proof of Proposition 3.3. In addition, some properties of $(X_\rho, P_\rho, B_\rho, T)$ such the representations of the sections, the algebraic properties of $P_\rho$, the interpretations of the operations in the base space in terms of pullback bundles and so on are obtained from a similar way to those presented in Proposition 3.11, Remark 3.12 and Proposition 3.14 given in [20]. That’s way we do not write more details of these properties.

**Remark 3.14** The details of the fiber bundle $(X_\rho, P_\rho, B_\rho, T)$ are omitted since given $i \in S^2$ then an $i$-simply connected bounded axially symmetric s-domain $\Omega \subset \mathbb{H}$ and $\mathbb{B}^4(0, \rho)$ are $i$-conformally equivalents as a consequence from the Riemann’s conformal mapping Theorem applied in the slices $\Omega_i$ and $\mathbb{B}^4(0, \rho)_i$ and from Proposition 3.14 given [20] one obtains an isomorphism between the fiber bundles $(X_\Omega, P_\Omega, B_\Omega, T)$ and $(X_{\mathbb{B}^4(0, \rho)}, P_{\mathbb{B}^4(0, \rho)}, B_{\mathbb{B}^4(0, \rho)}, T)$.

### 3.2 Fiber Bundles and the Zero Sets of Some Slice Regular Functions

This subsection shall be generalize the bijective mapping from the $n$-tuples of complex numbers to the set of complex monic polynomials of degree $n$, to some quaternionic slice regular polynomials using the theory of fiber bundles.

**Definition 3.15** The set $SRB(\mathbb{B}^4)$ consists of quaternionic slice regular functions on $\mathbb{B}^4$ such that

$$f(q) = \sum_{n=0}^{\infty} q^n r_n, \quad \forall q \in \mathbb{B}^4,$$

where the set $\{r_n \mid n \in \mathbb{N} \cup \{0\}\} \subset \mathbb{H}$ satisfy that $\{r_n \mid n \in \mathbb{N} \cup \{0\}\}$ contains a basis of $\mathbb{R}^3$. The set $\mathcal{P}SRB(\mathbb{H})$ is made up of $f \in \mathcal{P}SR(\mathbb{H})$ such that the set of vector parts of coefficients of $f$ contains a basis of $\mathbb{R}^3$.

Given a family of sets $\Lambda = \{A_i \subset \mathbb{C}(i) \mid i \in S^2\}$. The slice kull generated by $\Lambda$ is defined by

$$SKull(\Lambda) := \bigcup_{i \in S^2} Kull(A_i),$$

where $Kull(A_i)$ is the kull generated by $A_i$ in $\mathbb{C}(i)$. This concept can be justified by the slice topology studied in [13] that extends the concept of slice regularity from a convenient topology in $\mathbb{H}$. In our case, we only are going to use $SKull(\Lambda)$ to explain a consequence of the Gauss-Lucas theorem in the elements of the base and total spaces of two fiber bundles induced by some slice regular polynomials.

**Proposition 3.16** Some properties of $SRB(\mathbb{B}^4)$.

1. Given $f, g \in SRB(\mathbb{B}^4)$. If there exist $i, i' \in S^2$ different unit vectors, $c \in \mathbb{C}(i)$ and $c' \in \mathbb{C}(i')$ such that

$$f = g \bullet_{i,j} (1 + cj) \quad \text{and} \quad f = g \bullet_{i',j'} (1 + c'j'),$$

then $f = g \bullet_{i,j} (1 + cj)$. In particular, if $i = i'$ then $f = g$. This implies that $f \in SRB(\mathbb{B}^4)$.
where \( j \in S^2 \) is orthogonal to \( i \) and \( j' \in S^2 \) is orthogonal to \( i' \), then \( f = g \) on \( \mathbb{B}^4 \).

2. Consider \( f, g \in \text{PSRB}(\mathbb{H}) \) and \( i, i' \in S^2 \) such that the zeros of \( f \pm if \) in \( \mathbb{C}(i) \) are the same zeros of \( g \pm ig \) in \( \mathbb{C}(i) \) including their multiplicities and the same relationship between \( f \pm if' \) and \( f \pm ig' \) in \( \mathbb{C}(i') \). Then \( f = g \).

**Proof** 1. If \( f = g \cdot i_{i,j} \) and \( f = g \cdot i_{i',j'} \). Therefore, from definition (2.7) and identities (2.8) one has that

\[
\begin{align*}
f + if &= g + ig, \quad f - if = cg - icg, \quad \text{on } \mathbb{C}(i). \\
f + i'f' &= g + i'g', \quad f - i'f' = c'g - i'c'g', \quad \text{on } \mathbb{C}(i').
\end{align*}
\]

Then \( 2f = (1 + c)g + i(1 - c)gi \) on \( \mathbb{C}(i) \) and \( 2f = (1 + c')g + i'(1 - c')gi' \) on \( \mathbb{C}(i') \). Thus denoting \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) and \( g(q) = \sum_{n=0}^{\infty} q^n b_n \) for all \( q \in \mathbb{B}^4 \) one get that

\[
\sum_{n=0}^{\infty} z^n 2a_n = \sum_{n=0}^{\infty} z^n (1 + c)b_n + \sum_{n=0}^{\infty} z^n i(1 - c)b_n i
\]

for all \( z \in \mathbb{C}(i) \) and

\[
\sum_{n=0}^{\infty} w^n 2a_n = \sum_{n=0}^{\infty} w^n (1 + c)b_n + \sum_{n=0}^{\infty} w^n i'(1 - c')b_n i'
\]

for all \( w \in \mathbb{C}(i') \). Representation theorem allows to see that

\[
\sum_{n=0}^{\infty} q^n 2a_n = \sum_{n=0}^{\infty} q^n((1 + c)b_n + i(1 - c)b_n i)
\]

and

\[
\sum_{n=0}^{\infty} q^n 2a_n = \sum_{n=0}^{\infty} q^n((1 + c')b_n + i'(1 - c')b_n i')
\]

for all \( q \in \mathbb{B}^4 \). Therefore

\[
(1 + c)b_n + i(1 - c)b_n i = (1 + c')b_n + i'(1 - c')b_n i'
\]

for all \( n \in \mathbb{N} \) and as a basis of \( \mathbb{R}^3 \) is contained in \( \{b_n \mid n \in \mathbb{N} \cup \{0\} \} \) then

\[
(1 + c)x + i(1 - c)xi = (1 + c')x + i'(1 - c')xi'
\]

for all \( x \in \mathbb{R}^3 \) and choosing some vectors \( x \) one has \( c = c' = 1 \). Thus \( f = g \cdot i_{i,j} \) (1 + j), i.e., \( f = g \).
2. Given \( f = f_1 + f_2 \mathbf{j}, \quad g = g_1 + g_2 \mathbf{j} \in \mathcal{PSRB}(\mathbb{H}) \), where \( f_1, f_2, g_1, g_2 \in \text{Hol}(\mathbb{C}(i)) \) are complex polynomials. Note that \( f_1 \) and \( g_1 \) are monic polynomials. Due to \( Z_f \cap \mathbb{C}(i) = Z_g \cap \mathbb{C}(i) \) and \( Z_f \cap \mathbb{C}(i') = Z_g \cap \mathbb{C}(i') \) there exist \( c \in \mathbb{C}(i) \) and \( c' \in \mathbb{C}(i') \) such that
\[
 f = g \bullet_{i,j} (1 + c \mathbf{j}) \quad \text{and} \quad f = g \bullet_{i',j'} (1 + c' \mathbf{j}').
\]
From the previous fact \( f = g \).

\[\square\]

**Remark 3.17** To show the importance of the function set \( \mathcal{SRB}(\mathbb{H}^4) \) in the previous proposition, let’s look at the following:

1. If \( f \) is an intrinsic slice regular function on \( \mathbb{H}^4(0, 1) \), see [9], there exists a sequence of real numbers \( (\lambda_n)_{n \in \mathbb{N} \cup \{0\}} \) such that \( f(q) = \sum_{n=0}^{\infty} q^n \lambda_n \). Thus \( f = f \bullet_{i,j} (1 + c \mathbf{j}) \) for all \( i, j \in \mathbb{S}^2 \), orthogonal to each other, and all \( c \in \mathbb{C}(i) \).

2. If \( i, j, k, l \in \mathbb{S}^2 \) with \( i \) and \( j \), orthogonal to each other, and the same for the pair of vectors \( k \) and \( l \) such that \( j = kl \). Then the functions \( f(q) = q(1 + j) \) and \( g(q) = q(1 - j) \) are different and meet that \( f = g \bullet_{i,j} (1 - j) \) and \( f = g \bullet_{k,l} (1 - l) \).

On the other hand, given \( i, j \in \mathbb{S}^2 \), orthogonal to each other, the slice regular polynomials \( f(q) = (q^2 - 1) + (q - 1) \mathbf{j} \) and \( g(q) = (q^2 - 1) + 7(q - 1) \mathbf{j} \) satisfy that \( f = g \bullet_{i,j} (1 + 7 \mathbf{j}) \). Then \( Z_{f \pm i fi} \cap \mathbb{C}(i) = Z_{g \pm i gi} \cap \mathbb{C}(i) \). But \( f \neq g \). That’s why the identities \( Z_{f \pm i fi} \cap \mathbb{C}(i) = Z_{g \pm i gi} \cap \mathbb{C}(i) \) are necessary in the previous proposition to show that \( f = g \).

**Proposition 3.18** Denote
\[
 X = \{ (Z_{f-i fi} \cap \mathbb{C}(i)) \times (Z_{f+i fi} \cap \mathbb{C}(i)) \times (Z_{f-j fj} \cap \mathbb{C}(j)) \times (Z_{f+j fj} \cap \mathbb{C}(j)) \mid f \in \mathcal{PSRB}(\mathbb{H}), \ (i, j) \in T \}
\]
and \( \mathcal{B} = \mathcal{PSRB}(\mathbb{H}) \) both sets equipped with the discrete topology. By \( \mathbf{P} : X \rightarrow \mathcal{B} \) we mean the operator
\[
 \mathbf{P}((Z_{f-i fi} \cap \mathbb{C}(i)) \times (Z_{f+i fi} \cap \mathbb{C}(i)) \times (Z_{f-j fj} \cap \mathbb{C}(j)) \times (Z_{f+j fj} \cap \mathbb{C}(j)) ) = f.
\]
Also given \( u \in \mathbb{S}^3 \) and let \( U \) be a neighborhood in \( \mathcal{PSRB}(\mathbb{H}) \) denote \( \phi_u : U \times T \) given as follows:
\[
 \phi_u(f, (i, j)) = (Z_{f-ult fi u} \cap \mathbb{C}(u i u)) \times (Z_{f+ult fi u} \cap \mathbb{C}(u i u)) \times (Z_{f-j u j fi u} \cap \mathbb{C}(u j u)) \times (Z_{f+j u j fi u} \cap \mathbb{C}(u j u)).
\]
Given \( (i, j) \in T \) define
We shall establish two fiber bundles:

\[ S_{i,j}(f) = (Z_{f-i} \cap \mathbb{C}(i)) \times (Z_{f+i} \cap \mathbb{C}(i)) \times (Z_{f-j} \cap \mathbb{C}(j)) \times (Z_{f+j} \cap \mathbb{C}(j)). \]

Then \((X, P, B, T)\) is a fiber bundle where the structure group is \(K = \{R_u \mid u \in S^3\}\). The family of trivializations is \(\Phi = \{\varphi_u \mid u \in S^3\}\) and a family of sections is \(\{S_{i,j} \mid (i, j) \in T\}\).

**Proof** The fact 2 of the previous proposition establishes the well-definition of the bundle projection \(P\) and also helps us to see that \(P \circ \phi_u(f, (i, j)) = f\) for all \(f \in PSRB(\mathbb{H})\) and all \((i, j) \in T\). Fixing \((i, j) \in T\) and doing \(u = 1\) one sees the section \(S_{i,j}(f) = \phi_1(f, (i, j))\) for all \(f \in PSRB(\mathbb{H})\). \(\square\)

**Remark 3.19** Note that \(X\) and \(B\) are equipped with the discrete topology since the important thing to describe the relationship between some slice regular polynomials with their zero sets.

The mapping from the finite subsets of \(\mathbb{C}\) to the set of the monic complex polynomial \(Z_f \mapsto f\) is a bijective mapping. In this sense Proposition 3.18 presents an extension of this phenomena to \(SRB(\mathbb{H})\) explained in terms of the fiber bundle theory and complements the results of the zero sets of slice regular functions presented in [2, 15]

**Definition 3.20** We shall establish two fiber bundles:

1. Denote \(PSRB(\mathbb{H}) = \{f \in PSR(\mathbb{H}) \mid f' \in PSRB(\mathbb{H})\}\),

\[
X_1 = \left( (Z_{f'_{-i}} \cap \mathbb{C}(i)) \times (Z_{f'_{+i}} \cap \mathbb{C}(i)) \times (Z_{f'_{-j}} \cap \mathbb{C}(j)) \times (Z_{f'_{+j}} \cap \mathbb{C}(j)) \right) \times \left( (Z_{f_{-i}} \cap \mathbb{C}(i)) \times (Z_{f_{+i}} \cap \mathbb{C}(i)) \times (Z_{f_{-j}} \cap \mathbb{C}(j)) \times (Z_{f_{+j}} \cap \mathbb{C}(j)) \right), \]  

\[
(\forall f \in PSRB(\mathbb{H}), (i, j) \in T),
\]

\(B_1 = \{f' \mid f \in PSRB(\mathbb{H})\}\), both equipped with the discrete topology, and define the mapping \(P_1 : X_1 \to B_1\) as follows

\[
P_1 \left( (Z_{f'_{-i}} \cap \mathbb{C}(i)) \times (Z_{f'_{+i}} \cap \mathbb{C}(i)) \times (Z_{f'_{-j}} \cap \mathbb{C}(j)) \times (Z_{f'_{+j}} \cap \mathbb{C}(j)) \right) = f'.
\]

In addition, define

\[
\varphi^1_u(f', (i, j)) = (Z_{f'-ui} \cap \mathbb{C}(ui)) \times (Z_{f'+ui} \cap \mathbb{C}(ui)) \times (Z_{f'-uj} \cap \mathbb{C}(uj)) \times (Z_{f'+uj} \cap \mathbb{C}(uj)),
\]

where \(u \in S^3\) and \(U \subset PSRB(\mathbb{H})\) is a neighborhood. Set \(K = \{R_u \mid u \in S^3\}\) and \(\Phi_1 = \{\varphi^1_u \mid u \in S^3\}\).

2. Consider

\[
X_2 := \{\text{Kull}(Z_f \cap \mathbb{C}(i)) \times \text{Kull}(Z_f \cap \mathbb{C}(j)) \mid f \in PSRB(\mathbb{H}), (i, j) \in T\},
\]
$B_2 := \{ \text{SKull}(\Lambda_f) \mid f \in \text{PSRB}(\mathbb{H}) \}$, both equipped with the discrete topology, and define $P_2 : X_2 \rightarrow B_2$ as follows:

$$P_2(\text{Kull}(Z_f \cap C(i)) \times \text{Kull}(Z_f \cap C(j))) = \bigcup_{v \in S^2} \text{Kull}(Z_f \cap C(v)) = \text{SKull}(\Lambda_f).$$

Given a neighborhood $U \subset B_2$ define $\varphi_u^2 : U \times T$ as follows

$$\varphi_u^2(\text{SKull}(\Lambda_f), (i, j)) = \text{SKull}(\Lambda_f) \cap C(u\bar{i}) \times \text{SKull}(\Lambda_f) \cap C(u\bar{j})$$

$$= \text{Kull}(Z_f \cap C(u\bar{i})) \times \text{Kull}(Z_f \cap C(u\bar{j})).$$

Set $K = \{ R_u \mid u \in \mathbb{S}^3 \}$ and $\Phi_2 = \{ \varphi_u^2 \mid u \in \mathbb{S}^3 \}$.

**Proposition 3.21** ($X_1, P_1, B_1, T$) and ($X_2, P_2, B_2, T$) are fiber bundles and there exists a morphism $\Gamma : (X_1, P_1, B_1, F_1) \rightarrow (X_2, P_2, B_2, F_2)$.

**Proof** ($X_1, P_1, B_1, T$) is a fiber bundle as a consequence of the previous proposition and ($X_2, P_2, B_2, T$) is a fiber bundle directly from its definition. The morphisms are the following:

$$\Gamma_1[(Z_{f'_{-i}f'_{1}} \cap C(i)) \times (Z_{f'+i}f'_{1} \cap C(i)) \times (Z_{f'-j}f'_{1} \cap C(j)) \times (Z_{f'+j}f'_{1} \cap C(j)))]$$

$$= \text{Kull}(Z_f \cap C(i)) \times \text{Kull}(Z_f \cap C(j))$$

and $\Gamma_2[f'] = \text{SKull}(\Lambda_f)$, for all $f \in \text{PSRB}(\mathbb{H})$ and $(i, j) \in T$.

**Remark 3.22** Given $f \in \text{PSRB}(\mathbb{H})$ and $(i, j) \in T$. Then Gauss Lucas Theorem applied in the complex components of $f'_{|\mathbb{C}(i)}$ gives us the following contention

$$Z_{f'_{-i}f'_{1}} \cap C(i) \cap Z_{f'+i}f'_{1} = Z_{f'} \cap C(i) \subset \text{Kull}(Z_f \cap C(i))$$

and

$$Z_{f'} = \bigcup_{i \in S^2} Z_{f'} \cap C(i) \subset \bigcup_{i \in S^2} \text{Kull}(Z_f \cap C(i)).$$

i.e., $Z_{f'} \subset \text{SKull}(\Lambda_f)$, where $\Lambda_f := \{ Z_f \cap C(i) \mid i \in S^2 \}$.

Therefore, Gauss-Lucas Theorem shows us another relationship between $A \in X_1$ and $\Gamma_1(A) \in X_2$ and gives us another point of view of the mapping $f' \mapsto \text{SKull}(\Lambda_f)$ for all $f \in \text{PSRB}(\mathbb{H})$.

It is important to comment that the facts presented in this subsection complement the results presented in [17] which shows a quaternionic version of the Gauss-Lucas Theorem for quaternionic slice regular polynomials.
3.3 Conclusions and Future Works

1. The computations presented in this paper can be repeated for slice regular function on Clifford algebras, for example in Sect. 3 we have to increase the number of harmonic functions to form each element of the total space also increase the dimension of our hyper-sphere.

2. The theory of slice regular functions associated to some sphere bundles was defined in [21]. Thus the present work makes us wonder what kind of sphere bundles allow to obtain a slice regular function from a pair of harmonic functions.

3. The characterization of some quaternionic slice regular function spaces and their properties such as Bergman space, Hardy space and so on, see [6, 8, 11, 14], in terms of the theory of fiber bundles only using pairs of harmonic functions will be interesting.

4. Section 3.2 studies a property of zero sets of some polynomials applying the theory of fiber bundles. But this theory can be used to describe the behavior of the zero sets, the singularities and to give an interpretation of a principle of identity of a set of quaternionic slice regular functions.

5. The generalization of this fiber bundle approach to the theory of slice regular functions over the Octonions 0.

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