Abstract: A vast amount of information about distance based graph invariants is contained in the Hosoya polynomial. Such an information is helpful to determine well-known distance based molecular descriptors. The Hosoya index or $Z$-index of a graph $G$ is the total number of its matching. The Hosoya index is a prominent example of topological indices, which are of great interest in combinatorial chemistry, and later on it applies to address several chemical properties in molecular structures. In this article, we investigate Hosoya properties (Hosoya polynomial, reciprocal Hosoya polynomial and Hosoya index) of the commuting graph associated with an algebraic structure developed by the symmetries of regular molecular gones (constructed by atoms with regular atomic-bonding).

Keywords: geodesic, Hosoya polynomial, reciprocal Hosoya polynomial, Hosoya index, molecular structure, network

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1 Introduction

A numeral quantity which captures the symmetry of a molecular structure is known as a topological index. In fact, a topological index is a numerical characterization of a chemical graph and it provides a mathematical function of the structure in quantitative structure-activity relationship (QSAR)/quantitative structure-property relationship (QSPR) studies. It correlates certain physico chemical properties such as boiling point, stability and strain energy of chemical compounds of a molecular structure (graph). Several properties of chemical compounds in a molecular structure can be determined with the assistance of mathematical languages rendered by various types of topological indices. Harold Wiener, introduced the concept of first (distance based) topological index while working on the boiling point of Paraffin and named this index as path number (Wiener, 1947). Next off, it was renamed as the Wiener index, and that was the time theory of topological indices began.

We consider simple and connected graph (chemical structure) $G$ with vertex set $V(G)$ and edge set $E(G)$. We denote the two adjacent vertices $u$ and $v$ in $G$ as $u \sim v$ and non-adjacent vertices as $u \nabla v$. Starting from a vertex $u$ and ending at a vertex $v$ in $G$, a shortest alternating sequences of vertices and edges without repetition of any vertex is known as a $u-v$ geodesic. The number of edges in a $u-v$ geodesic is denoted by $d(u,v)$, and is called the distance between $u$ and $v$ in $G$. The sum of two graphs $G_1$ and $G_2$, denoted by $G_1 + G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and an edgset $E(G_1) \cup E(G_2) \cup \{u \sim v: u \in V(G_1) \land v \in V(G_2)\}$. The maximum distance of a vertex $v$ among all of its distances with all the vertices of $G$ is called the eccentricity of $v$, denoted by $ecc(v)$. The number $diam(G) = \max_{v \in V(G)} ecc(v)$ is the diameter of $G$.

1.1 Hosoya properties

Many chemist used the concept of counting polynomial, introduced by Polya (1936), in order to obtain the molecular
orbits of unsaturated hydrocarbons. In this regard, the spectra of the characteristic polynomial of graphs were studied extensively. In 1988, Hosoya used this concept to find the polynomials of many chemical structures (Hosoya, 1986b), known as Hosoya polynomials and seeks a lot of attention afterwards. In 1996, the Hosoya polynomial was renamed as Wiener polynomial by Sagan et al. (1996), but now a days, majority of researchers used the term Hosoya polynomial. The Hosoya polynomial provides a pile of information about distance based graph invariants. The relationship between the Hosoya polynomial and the hyper Wiener index was observed by Cash (2002). Several other applications of extended Wiener indices were found the polynomials of many chemical structures (Hosoya, 1988). In 2008, Deng investigated the problem with respect to the Hosoya index for various graphs.

1.2 Group of symmetries and commuting graph

Group of symmetries finds its notable use in the theory of molecular vibrations and electron structures. Due to their noteworthy employment in chemical structures, in the context of topological indices, we consider the group of symmetries of a regular molecular polygon (also called a regular molecular $n$-gon constructed on $n \geq 2$ atoms with their regular atomic-bonding). A regular molecular $n$-gon is a molecular structure whose corners are atoms and sides are atom-bonds of same length, and each internal angle between atom-bonds is of the measurement $\pi - \frac{2k\pi}{n}$ radian. The group of symmetries of a regular molecular $n$-gon consists of $2n$ elements, which are $n$ rotations about its center through an angle of $\frac{2k\pi}{n}$ radian, where $k = 0, 1, \ldots, n-1$, either all clockwise or all anti-clockwise) and $n$ reflections (for even $n$, the reflections through a line joining the mid-points of the opposite atom-bonds or through a line joining two opposite atoms; and for odd $n$, the reflections through those lines which join an atom with the mid-point of the opposite atom-bond).

Symbolically, the group of symmetries is denoted by $D_n$ and is called the dihedral group (a group theoretic name) of order $2n$ (Majeed, 2013). If we denote a rotation by ‘$a$’ and a reflection by ‘$b$’, then $2n$ elements of $D_n$ are $a, a^2, \ldots, a^{n-1}, a^n = e$ and $b, ab, a^2b, \ldots, a^{n-1}b$ (here $e$ is the identical shape molecular $n$–gon). In generating form, $D_n$ can be represented as follows:

$$D_n = \{a, b \mid a^n = b^2 = e, ab = ba^{-1}\}.$$ 

The center of $D_n$ is:

$$\zeta(D_n) = \begin{cases} \{e\}, & \text{when } n \text{ is odd}, \\ \{e, a^2\}, & \text{when } n \text{ is even}. \end{cases}$$

Let $\Omega_1 = \{e, a, a^2, \ldots, a^{n-1}\}$, $\Omega_2 = \{b, ab, a^2b, \ldots, a^{n-1}b\}$ and $\Omega_3 = \Omega_1 - \zeta(D_n)$. Then $|\Omega_1| = n = |\Omega_3|$ and

$$|\Omega_2| = \begin{cases} n-1, & \text{when } n \text{ is odd}, \\ n-2, & \text{when } n \text{ is even}. \end{cases}$$

In the case of even value of $n \geq 4$, we partitioned $\Omega_2$ into $\frac{n}{2}$ two element subsets $\Omega_{2i} = \{a^i b, a^{n-i} b\}$, $0 \leq i \leq \frac{n}{2} - 1$, so that $\Omega_2 = \bigcup_{i=0}^{\frac{n}{2} - 1} \Omega_{2i}$.

The commuting graph of a non-abelian group $G$ is denoted by $\Gamma_G = C(\Gamma, \Omega)$ with vertex set $\Omega \subseteq \Gamma$. For two
unlike elements $x, y \in \Omega, x \sim y$ in $\Gamma$. The notion of commuting graphs on non-central elements of a group has been studied by many researchers – see for instance Ali et al. (2016) and Bunday (2006), and the references therein. The commuting graph on $D_n$ is defined by Ali et al. (2016) in the following result.

Proposition 1
For all $n \geq 3$, let $\Gamma = \mathcal{C}(D_n, D_n)$ be a commuting graph on $D_n$, then:

$$\Gamma_n = \begin{cases} K_1 \cup \left( K_{\frac{n}{2}} \cup N_{\frac{n}{2}} \right), & \text{when } n \text{ is odd,} \\ K_2 \cup \left( K_{\frac{n}{2}} \cup \frac{n}{2} K_2 \right), & \text{when } n \text{ is even.} \end{cases}$$

Here $K_1$ is the trivial graph, $K_p$ is a complete graph on $p$ vertices, $N_t$ is a null graph on $t$ vertices, and $\frac{n}{2} K_2$ is the union of $\frac{n}{2}$ copies of $K_2$.

2. Hosoya polynomials

In this section, we find Hosoya polynomial and reciprocal status Hosoya polynomial of $\Gamma$.

2.1 Hosoya polynomial

Our first two results of this section provide the coefficients for the Hosoya polynomial of the commuting graph on $D_n$.

Proposition 2
Consider the commuting graph $\Gamma = \mathcal{C}(D_n, D_n)$ associated with the group $\Gamma = D_n$, for odd values of $n \geq 3$. Then:

$$d(\Gamma_n, k) = \begin{cases} \frac{2n}{n(n+1)}, & \text{for } k = 0, \\ \frac{n(n+1)}{2}, & \text{for } k = 1, \\ \frac{3n(n-1)}{2}, & \text{for } k = 2. \end{cases}$$

Proof. Note that $\text{diam}(\Gamma_n) = 2$. So, we have to find $d(\Gamma_n, 0), d(\Gamma_n, 1)$ and $d(\Gamma_n, 2)$. Consider the set $V_p$ of all the pairs of vertices (same and distinct) of $\Gamma_n$, then:

$$|V_p| = \left( \frac{\Gamma_n}{2} \right)^2 + \Gamma_n = n(2n+1).$$

Let $S(\Gamma_n, k) = \{ (l, m); l, m \in V(\Gamma_n) \mid d(l, m) = k \}$ and $d(\Gamma_n, k) = |S(\Gamma_n, k)|$.

Thus:

$$V_p = S(\Gamma_n, 0) \cup S(\Gamma_n, 1) \cup S(\Gamma_n, 2).$$

Since, $d(l, l) = 0$ for all $l \in V(\Gamma_n)$, so $S(\Gamma_n, 0) = \{ (l, l); l \in V(\Gamma_n) \} = V(\Gamma_n)$. Thus, $d(\Gamma_n, 0) = 2n$. By Proposition 1, $\Gamma_n = K_1 \cup \left( K_{\frac{n}{2}} \cup N_{\frac{n}{2}} \right)$ with $V(K_1) = \zeta(\Gamma)$, $V(K_{\frac{n}{2}}) = \Omega_3$, and $V(N_{\frac{n}{2}}) = \Omega_2$. Therefore:

$$S(\Gamma_n, 1) = \{ (l, m); l \in \zeta(\Gamma), m \in \Omega_3 \} \cup \{ (l, m); l \in \zeta(\Gamma), m \in \Omega_2 \} \cup \{ (l, m); l, m \in \Omega_3 \text{ and } l \neq m \}.$$

Accordingly,

$$d(\Gamma_n, 1) = n-1+n+\left( \frac{n-1}{2} \right) = \frac{n(n+1)}{2}.$$ 

By Eq. 1, we have $|V_p| = d(\Gamma_n, 0) + d(\Gamma_n, 1) + d(\Gamma_n, 2)$. Therefore:

$$d(\Gamma_n, 2) = |V_p| - d(\Gamma_n, 0) - d(\Gamma_n, 1) = n(2n+1) - 2n - \frac{n(n+1)}{2} = \frac{3n(n-1)}{2}.$$ 

Proposition 3
Consider the commuting graph $\Gamma_n$ associated with the group $\Gamma = D_n$ for even values of $n \geq 4$. Then:

$$d(\Gamma_n, k) = \begin{cases} \frac{2n}{n(n+4)}, & \text{for } k = 0, \\ \frac{n(n+4)}{2}, & \text{for } k = 1, \\ \frac{3n(n-2)}{2}, & \text{for } k = 2. \end{cases}$$

Proof. As $\text{diam}(\Gamma_n) = 2$, so we have to find the coefficients $d(\Gamma_n, 0), d(\Gamma_n, 1)$ and $d(\Gamma_n, 2)$. If $V_p$ denotes the set of all the pairs of vertices (same and distinct) of $\Gamma_n$, then $|V_p| = n(2n+1)$, by Proposition 2. Let $S(\Gamma_n, k) = \{ (l, m); l, m \in V(\Gamma_n) \mid d(l, m) = k \}$, then $d(\Gamma_n, k) = |S(\Gamma_n, k)|$ and

$$V_p = S(\Gamma_n, 0) \cup S(\Gamma_n, 1) \cup S(\Gamma_n, 2).$$
Since, \(d(l,l)=0\) \(\forall l \in V(\Gamma_g)\), so \(S(\Gamma_g,0) = \{(l,l)\}\).
\(l \in V(\Gamma_g) = V(\Gamma_g)\). Thus, \(d(\Gamma_g,0) = 2n\). By Proposition 1, \(\Gamma_g = K_2 + (K_{m_1} \cup \overset{n}{\cup} K_{m_2})\) with \(V(K_{m_1}) = \zeta(\Gamma)\), \(V(K_{m_2}) = \Omega_g\)
and \(V(K_{\frac{n}{2}}) = \bigcup_{i=0}^{n} \Omega_g = \Omega_2\). Thus:
\[
S(\Gamma_g,1) = \{(l,m); l \in \zeta(\Gamma), m \in \Omega_g\}
\cup \{(l,m); l \in \zeta(\Gamma), m \in \Omega_g\}
\cup \{(l,m); l \in \zeta(\Gamma), m \in \Omega_g, \text{and} l \neq m\}
\cup \bigcup_{i=0}^{n} \{(l,m); m \in \Omega_g, \text{and} l \neq m\}.
\]
Accordingly,
\[
d(\Gamma_g,1) = 2(n-2) + 2(n) + 1 + \frac{n(n-2)}{2} + \frac{n(1)}{2}
= 2n - 4 + \frac{5n}{2} + 1 + \frac{n(n-2)(n-3)}{2} = \frac{n(n+4)}{2}.
\]
From Eq. 2, we have \(|V_{p_c}| = d(\Gamma_g,0) + d(\Gamma_g,1) + d(\Gamma_g,2)\).
Therefore
\[
d(\Gamma_g,2) = |V_{p_c}| - d(\Gamma_g,0) - d(\Gamma_g,1)
= n(2n+1) - 2n - \frac{n(n+4)}{2}
= \frac{3n(n-2)}{2}.
\]
The Hosoya polynomials of the commuting graph on 
\(D_n\), for odd and even values of \(n \geq 3\), is obtained in the following result.

**Theorem 4**
For \(n \geq 3\), let \(\Gamma_g\) be the commuting graph on \(\Gamma = D_n\), then:
\[
H(\Gamma_g,x) = \begin{cases} \frac{n}{2} \left\{3(n-1)x^2 + (n+1)x + 4\right\}, & \text{for odd } n, \\ \frac{n}{2} \left\{3(n-2)x^2 + (n+4)x + 4\right\}, & \text{for even } n. \end{cases}
\]

*Proof.* Using the coefficients \(d(\Gamma_g,k)\), computed in Propositions 2 and 3, in the formula of the Hosoya polynomial, we have:

For odd \(n\):
\[
H(\Gamma_g,x) = d(\Gamma_g,0)x^0 + d(\Gamma_g,1)x^1 + d(\Gamma_g,2)x^2
= (2n)x^0 + \left(\frac{n(n+1)}{2}\right)x^1 + \left(\frac{3n(n-1)}{2}\right)x^2
= \frac{n}{2}\left\{3(n-1)x^2 + (n+1)x + 4\right\}.
\]

For even \(n\):
\[
H(\Gamma_g,x) = d(\Gamma_g,0)x^0 + d(\Gamma_g,1)x^1 + d(\Gamma_g,2)x^2
= (2n)x^0 + \left(\frac{n(n+1)}{2}\right)x^1 + \left(\frac{3n(n-2)}{2}\right)x^2
= \frac{n}{2}\left\{3(n-2)x^2 + (n+4)x + 4\right\}.
\]

### 2.2 Reciprocal status Hosoya polynomial

Firstly, we find the reciprocal status of each vertex of the commuting graph on \(D_n\), for odd and even values of \(n \geq 3\), in the following two results, respectively.

**Proposition 5**
If \(l\) is a vertex in the commuting graph on \(D_n\) for odd values of \(n \geq 3\), then:
\[
r(l) = \begin{cases} 2n-1, & \text{whenever } l \in \zeta(\Gamma_g), \\ \frac{3n-1}{2}, & \text{whenever } l \in \Omega_g, \\ n, & \text{whenever } l \in \Omega_2. \end{cases}
\]

*Proof.* By Proposition 1, the commuting graph on \(D_n\) is \(K_2 + (K_{m_1} \cup K_{m_2})\) with the vertex set \(\zeta(D_n) \cup \Omega_g \cup \Omega_2\). Accordingly, we have:
Whenever \(l \in \zeta(D_n)\); \(ecc(l) = 1\), and by the definition of reciprocal status, we get
\[
r(s(l)) = \frac{1}{1}(n+1) = 2n-1.
\]
Whenever \(l \in \Omega_g\); \(ecc(l) = 2\), and by the definition of reciprocal status, we get
\[
r(s(l)) = \frac{1}{2}(n-1) + \frac{1}{2}n = \frac{3n}{2} - 1.
\]
Whenever \( l \in \Omega_2 \): \( \text{ecc}(l) = 2 \), and by the definition of reciprocal status, we get
\[
rs(l) = \left( \frac{1}{1} \right) + \left( \frac{1}{2} \right)(2n - 2) = n.
\]

**Proposition 6**

For even values of \( n \geq 4 \), if \( l \) is any vertex of the commuting graph on \( D_n \), then:
\[
rs(l) = \begin{cases} 
2n - 1, & \text{whenever } l \in \zeta(D_n), \\
3n - 1, & \text{whenever } l \in \Omega_1, \\
n + 1, & \text{whenever } l \in \Omega_2.
\end{cases}
\]

**Proof.** By Proposition 1, the commuting graph on \( D_n \) is \( K_1 + (K_{2n} \cup K_2) \) with the vertex set \( \zeta(D_n) \cup \Omega_1 \cup \Omega_2 \).

Accordingly, we have:
Whenever \( l \in \zeta(D_n) \): \( \text{ecc}(l) = 1 \), and by the definition of reciprocal status, we get
\[
rs(l) = \left( \frac{1}{1} \right) + n - 2 + \left( \frac{n}{2} \right) = 2n - 1.
\]

Whenever \( l \in \Omega_1 \): \( \text{ecc}(l) = 2 \), and by the definition of reciprocal status, we get
\[
rs(l) = \left( \frac{1}{1} \right) \left( n - 3 + 2 \right) + \left( \frac{1}{2} \right) \left( \frac{n}{2} \right) = \frac{3n}{2} - 1.
\]

Whenever \( l \in \Omega_2 \): \( \text{ecc}(l) = 2 \), and by the definition of reciprocal status, we get
\[
rs(l) = \left( \frac{1}{1} \right) \left( 3 + \left( \frac{1}{2} \right) \right) \left( n - 2 \right) + \left( \frac{2n}{2} - 2 \right) = n + 1.
\]

The next two results provide the reciprocal status Hosoya polynomial of the commuting graph on \( D_n \).

**Theorem 7**

For odd \( n \geq 3 \), if \( \Gamma_o \) is the commuting graph on \( \Gamma = D_n \). Then:
\[
H_{rs}(\Gamma_o) = \left( n - 1 \right) \left( x^{\frac{2n}{2}} + nx^{\frac{1}{2n}} + \left( \frac{n - 1}{2} \right) \right)^{3n - 2}.
\]

**Proof.** Proposition 5 implies that there are three types \((a - b, a - c, b - b)\) of edges in \( \Gamma_o \) according to the reciprocal statuses of their end vertices, where \( a = 2n - 1 \), \( b = \frac{3n}{2} - 1 \), \( c = n \). Table 1 shows the edge partition accordingly.

By using the edge partition, given in the Table 1, in the formula of the reciprocal status Hosoya polynomial, we have:
\[
H_{rs}(G) = \sum_{E_{a,b}} x^{a+b} + \sum_{E_{a,c}} x^{a+c} + \sum_{E_{b,b}} x^{b+b}
\]
\[
= \left( n - 1 \right) \left( x^{\frac{2n}{2} - 1} + nx^{\frac{1}{2n} - 1} + \left( \frac{n - 1}{2} \right) \right)^{3n - 2}
\]
\[
= \left( n - 1 \right) \left( x^{\frac{2n}{2} - 1} + nx^{\frac{1}{2n} - 1} + \left( \frac{n - 1}{2} \right) \right)^{3n - 2}.
\]

**Theorem 8**

For even \( n \geq 4 \), if \( \Gamma_o \) is the commuting graph on \( \Gamma = D_n \). Then:
\[
H_{rs}(\Gamma_o) = \left( n - 2 \right) \left( x^{\frac{2n}{2} - 1} + \left( 2n - 4 \right) x^{\frac{2n}{2} - 1} + x^{\frac{1}{2n} - 1} + 2nx^{\frac{1}{2n} - 1} + \frac{n}{2} x^{2n - 2} \right).
\]

**Proof.** Proposition 6 implies that there are five types \((a - a, a - b, b - b, b - c, c - c)\) of edges in \( \Gamma_o \) accordingly, the edge partition is given in the Table 2, according to the reciprocal statuses of their end vertices, where \( a = \frac{2n}{2} - 1 \), \( b = 2n - 1 \), \( c = 1 \).

**Table 1:** Edge partition of \( \Gamma_o \) according to reciprocal statuses

| Edges type | Partition of edge set | Number of edges |
|------------|-----------------------|----------------|
| \( a - b \) | \( uv \in E(\Gamma_o) | rs(u) = a, rs(v) = b \) | \( |E_{a,b}| = n - 1 \) |
| \( a - c \) | \( uv \in E(\Gamma_o) | rs(u) = a, rs(v) = c \) | \( |E_{a,c}| = n \) |
| \( b - b \) | \( uv \in E(\Gamma_o) | rs(u) = b, rs(v) = b \) | \( |E_{b,b}| = \frac{n - 1}{2} \) |
| Type of edges | Partition of edge set | Number of edges |
|--------------|-----------------------|----------------|
| $a - a$      | $E_{a,a} = \{uv \in E(\Gamma_{a}) | rs(u) = a, rs(v) = a\}$ | $|E_{a,a}| = \binom{n-2}{2}$ |
| $a - b$      | $E_{a,b} = \{uv \in E(\Gamma_{a}) | rs(u) = a, rs(v) = b\}$ | $|E_{a,b}| = 2(n-2)$ |
| $b - b$      | $E_{b,b} = \{uv \in E(\Gamma_{a}) | rs(u) = b, rs(v) = b\}$ | $|E_{b,b}| = 1$ |
| $b - c$      | $E_{b,c} = \{uv \in E(\Gamma_{a}) | rs(u) = b, rs(v) = c\}$ | $|E_{b,c}| = 2n$ |
| $c - c$      | $E_{c,c} = \{uv \in E(\Gamma_{a}) | rs(u) = c, rs(v) = c\}$ | $|E_{c,c}| = \frac{n}{2}$ |

By using the edge partition of $\Gamma_{a}$, given in the Table 2, in the formula of the reciprocal Hosoya polynomial, we have:

$$H_{\mathbb{C}}(\mathbb{G}) = \sum_{E_{a,a}} x^{2} + \sum_{E_{a,b}} x^{b+1} + \sum_{E_{b,b}} x^{b+1} + \sum_{E_{b,c}} x^{b+c} + \sum_{E_{c,c}} x^{c+c}$$

$$= \left(\frac{n-2}{2}\right)x^{2} + 2(n-2)x^{2n-1} + x^{2n-1} + 2nx^{n-1} + \left(\frac{n}{2}\right)x^{n+1}$$

$$= \left(\frac{n-2}{2}\right)x^{3n-2} + (2n-4)x^{2n-2} + x^{4n-2} + 2nx^{3n} + \left(\frac{n}{2}\right)x^{2n+2}.$$

### 3 Hosoya index

In this section, we investigate the Hosoya index of the commuting graph of the dihedral group. The largest possible value of the Hosoya index, on a graph with $n$ vertices, is given by the complete graph $K_n$. Generally, the Hosoya index of a complete graph $K_n$, $n \geq 1$ is:

$$1 + \sum_{k=1}^{\left[\frac{2n}{2}\right]} \prod_{i=0}^{k-1} \binom{n-2i}{2}$$

which can be viewed in accordance with the number of non-empty matchings given in Table 3, where $m_k$ denotes the number of matchings of cardinality $k$, $1 \leq k \leq \left[\frac{n}{2}\right]$.

For all odd values of $n \geq 3$, the Hosoya index is computed in the following result.

**Theorem 9**

Let $\Gamma_{a}$ be a commuting graph on $\Gamma = D_n$, for odd $n \geq 3$. Then the Hosoya index of $\Gamma_{a}$ is:

$$1 + \left(\frac{n}{2}\right) + n + \sum_{k=2}^{\left[\frac{n}{2}\right]} \prod_{i=0}^{k-1} \frac{1}{k} \prod_{i=0}^{k-1} \left(\frac{n-2i}{2}\right) + n \prod_{i=0}^{k-1} \left(\frac{n-2i-1}{2}\right).$$

**Proof.** As $\Gamma_{a} = K_{1} + (K_{n} \cup N_{n})$ with $V(\Gamma_{a}) = \zeta(D_{n}) \cup \Omega_{1} \cup \Omega_{2}$, by Proposition 1, so there are three types of edges in $\Gamma_{a}$:

Type-1: $u \sim v$ for $u, v \in \Omega_{1}$;

Type-2: $u \sim v$ for $u \in \Omega_{1}$ and $v \in \zeta(D_{n})$;

Type-3: $u \sim v$ for $u \in \Omega_{2}$ and $v \in \zeta(D_{n})$.

As $\zeta(D_{n}) \cup \Omega_{2} = \Omega_{1}$ induces a complete subgraph $K_{n}$ on $n$ vertices in $\Gamma_{a}$, so there are two types of matchings in $\Gamma_{a}$:

(M₁) Matchings of edges of Type-1 and Type-2;

(M₂) Matchings of edges of Type-1 and Type-3.

The number of matchings of each type can be found as follows:

(M₁) Since the edges of Type-1 and Type-2 are the edges of complete graph $K_{n}$ induced by the vertices in $\zeta(D_{n}) \cup \Omega_{1} = \Omega_{1}$, so the number of matchings in this type can be obtained by counting the matchings in $K_{n}$ for all $n \geq 3$, which are given in Table 4, where $m_k$ denotes the number of matchings of order $k$ for $1 \leq k \leq \left[\frac{n}{2}\right]$.

(M₂) Each matching of this type can be obtained by adding one edge of Type-3 into each matching of the edges of Type-1. As each edge of Type-1 is an edge of a complete graph $K_{n}$ induced by the vertices in $\Omega_{1}$, so each matching of the edges of Type-1 is actually the matching in a complete graph $K_{n}$. The numbers of such matchings are listed in Table 5, where $m_k$ denotes the number of matchings of order $k$ for $1 \leq k \leq \left[\frac{n}{2}\right]$. 


Table 3: The number of non-empty matchings in a complete graph $K_n$

| $K_n$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $\ldots$ | $m_k$ |
|-------|-------|-------|-------|-------|----------|-------|
| $K_2$ | [2]   |       |       |       |          |       |
| $K_3$ | [3]   | [2]   |       |       |          |       |
| $K_4$ | [4]   | [2]   | [2]   | [2]   |          |       |
| $K_5$ | [5]   | [2]   |       |       |          |       |
| $K_6$ | [6]   | [2]   | [2]   | [2]   |          |       |
| $K_7$ | [7]   | [2]   |       |       |          |       |
| $K_8$ | [8]   | [2]   | [2]   | [2]   |          |       |
| $K_9$ | [9]   | [2]   |       |       |          |       |
| $\vdots$ | :     | :     | :     | :     | $\ddots$ | :     |
| $K_n$ | $n$   | [2]   | [2]   | [2]   | $\frac{n-2}{2}$ | $\frac{n-4}{2}$ |

Table 4: The number of non-empty matchings in a complete graph $K_n$ for odd $n \geq 3$

| $K_n$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $\ldots$ | $m_k$ |
|-------|-------|-------|-------|-------|----------|-------|
| $K_3$ | [3]   |       |       |       |          |       |
| $K_5$ | [5]   | [2]   |       |       |          |       |
| $K_7$ | [7]   | [2]   |       |       |          |       |
| $K_9$ | [9]   | [2]   |       |       |          |       |
| $\vdots$ | :     | :     | :     | :     | $\ddots$ | :     |
| $K_n$ | $n$   | [2]   |       |       | $\frac{n-2}{2}$ | $\frac{n-4}{2}$ |

$\prod_{i=0}^{k} \frac{1}{n-2i}$
Now, since there are \( n \) edges of Type-3, so the required matchings can be obtained as follows:

**Matchings of order 1:** These are \( n \) such matchings corresponding to \( n \) edges of Type-3.

**Matchings of order 2:** Each of these matchings can be obtained by adding one edge of Type-3 into each matching of order 1 in \( K_{n-1} \). There are \( n \) edges of Type-3 and \( \left( \begin{array}{c} n-1 \\ 2 \end{array} \right) \) matchings of order 1 in \( K_{n-1} \), by Table 5. Therefore, by the rule of product, the number of matchings of order 2 is:

\[
\frac{1}{2} n \left( \begin{array}{c} n-1 \\ 2 \end{array} \right) \left( \begin{array}{c} n-3 \\ 2 \end{array} \right)
\]

**Matchings of order 3:** Each of these matchings can be obtained by adding one edge of Type-3 into each matching of order 2 in \( K_{n-1} \). There are \( n \) edges of Type-3 and \( \frac{1}{3} \left( \begin{array}{c} n-1 \\ 2 \end{array} \right) \left( \begin{array}{c} n-3 \\ 2 \end{array} \right) \left( \begin{array}{c} n-5 \\ 2 \end{array} \right) \) matchings of order 2 in \( K_{n-1} \), by Table 5. Therefore, by the rule of product, the number of matchings of order 3 in \( K_{n-1} \), by Table 5. Therefore, by the rule of product, the number of matchings of order 4 is:

\[
\frac{1}{3} \frac{n-1}{2} \left( \begin{array}{c} n-3 \\ 2 \end{array} \right) \left( \begin{array}{c} n-5 \\ 2 \end{array} \right)
\]

**Matchings of order \( k \):** Generally, each matching of order \( k \) can be obtained by adding one edge of Type-3 into each matching of order \( k-1 \) in \( K_{n-1} \). There are \( n \) edges of Type-3 and \( \frac{1}{k-1} \prod_{i=0}^{k-2} \left( \begin{array}{c} n-2i-1 \\ 2 \end{array} \right) \) matchings of order \( k-1 \) in \( K_{n-1} \), by Table 5. Therefore, by the rule of product, the number of matchings of order \( k \) is:

\[
\frac{1}{k-1} \prod_{i=0}^{k-2} \left( \begin{array}{c} n-2i-1 \\ 2 \end{array} \right)
\]

Now, by the rule of sum, the total number of matchings (matchings \( M_1 \) + matchings \( M_2 \)) in \( \Gamma_g \) of each order can be counted as follows:

The number of matching of order 1 is:

\[
\left( \begin{array}{c} n \\ 2 \end{array} \right) + n
\]

The number of matching of order 2 is:

\[
\frac{1}{2} \left( \begin{array}{c} n \\ 2 \end{array} \right) + n \left( \begin{array}{c} n-1 \\ 2 \end{array} \right)
\]

**Table 5:** The number of non-empty matchings in a complete graph \( K_{n-1} \) for odd \( n \geq 3 \)

| \( K_n \) | \( m_1 \) | \( m_2 \) | \( m_3 \) | \( m_4 \) | \( \ldots \) | \( m_k \) |
|---|---|---|---|---|---|---|
| \( K_2 \) | \( \frac{2}{2} \) | \( \frac{2}{2} \) | | | | |
| \( K_4 \) | \( \frac{4}{2} \) | \( \frac{1}{2} \left( \begin{array}{c} 4 \\ 2 \end{array} \right) \) | \( \frac{1}{2} \left( \begin{array}{c} 4 \\ 2 \end{array} \right) \) | | | |
| \( K_6 \) | \( \frac{6}{2} \) | \( \frac{1}{2} \left( \begin{array}{c} 6 \\ 2 \end{array} \right) \) | \( \frac{1}{3} \left( \begin{array}{c} 6 \\ 2 \end{array} \right) \) | \( \frac{1}{2} \left( \begin{array}{c} 6 \\ 2 \end{array} \right) \) | | |
| \( K_8 \) | \( \frac{8}{2} \) | \( \frac{1}{2} \left( \begin{array}{c} 8 \\ 2 \end{array} \right) \) | \( \frac{1}{3} \left( \begin{array}{c} 8 \\ 2 \end{array} \right) \) | \( \frac{1}{4} \left( \begin{array}{c} 8 \\ 2 \end{array} \right) \) | \( \frac{1}{2} \left( \begin{array}{c} 8 \\ 2 \end{array} \right) \) | |
| \( K_{n-1} \) | \( \frac{n-1}{2} \) | \( \frac{1}{2} \left( \begin{array}{c} n-1 \\ 2 \end{array} \right) \) | \( \frac{1}{3} \left( \begin{array}{c} n-1 \\ 2 \end{array} \right) \) | \( \frac{1}{4} \left( \begin{array}{c} n-1 \\ 2 \end{array} \right) \) | \( \frac{1}{k-1} \prod_{i=0}^{k-2} \left( \begin{array}{c} n-2i-1 \\ 2 \end{array} \right) \) | \( \frac{1}{k-1} \prod_{i=0}^{k-2} \left( \begin{array}{c} n-2i-1 \\ 2 \end{array} \right) \) | \( \frac{1}{k-1} \prod_{i=0}^{k-2} \left( \begin{array}{c} n-2i-1 \\ 2 \end{array} \right) \) | \( \frac{1}{k-1} \prod_{i=0}^{k-2} \left( \begin{array}{c} n-2i-1 \\ 2 \end{array} \right) \) | \( \frac{1}{k-1} \prod_{i=0}^{k-2} \left( \begin{array}{c} n-2i-1 \\ 2 \end{array} \right) \) | | | | | |
The number of matching of order 3 is:
\[
\frac{1}{3} \binom{n}{2} \binom{n-2}{2} \binom{n-4}{2} + \frac{1}{2} \binom{n-1}{2} \binom{n-3}{2}.
\]

The number of matching of order 4 is:
\[
\frac{1}{4} \binom{n}{2} \binom{n-2}{2} \binom{n-4}{2} \binom{n-6}{2} + \frac{1}{3} \binom{n-1}{2} \binom{n-3}{2} \binom{n-5}{2}.
\]

Generally, the number of matching of order \(k\) is:
\[
\frac{1}{k} \prod_{i=0}^{k-1} \binom{n-2i}{2} + \frac{n}{k-1} \prod_{i=0}^{k-2} \binom{n-2i-1}{2}
\]
where \(2 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor\).

Hence, the Hosoya index of \(\Gamma_g\) is:
\[
1 + \binom{n}{2} + n + \sum_{k=3}^{n} \left( \frac{1}{k} \prod_{i=0}^{k-1} \binom{n-2i}{2} + \frac{n}{k-1} \prod_{i=0}^{k-2} \binom{n-2i-1}{2} \right).
\]

When \(n = 2\), then \(\Gamma = D_2\) is an abelian group and so the commuting graph \(\Gamma_g\) is the complete graph \(K_2\). Hence, by Table 3, the Hosoya index of \(\Gamma_g\) is \(1 + m_1 + m_2 = 10\).

The following result provides the Hosoya index of the commuting graph of \(D_n\) for even values of \(n > 3\).

**Theorem 10**

Let \(\Gamma_g\) be a commuting graph on \(\Gamma = D_n\) for even \(n > 3\). Then the Hosoya index of \(\Gamma_g\) is:
\[
1 + \sum_{k=1}^{n} m_1^k + \sum_{k=1}^{n} m_2^k + \sum_{k=1}^{n} m_3^k + \sum_{k=1}^{n} m_4^k + \sum_{k=1}^{n} m_5^k + \sum_{k=1}^{n} m_6^k,
\]
where:
\[
m_1^k = \frac{1}{k} \prod_{i=0}^{k-1} \binom{n-2i}{2},
\]
\[
m_2^k = 2^n m_2^k = n(n-1),
\]
\[
m_3^k = \binom{n}{2},
\]
\[
m_4^k = 2^n m_4^k = 2n,\]
\[
m_5^k = n(n-2),
\]
\[
m_6^k = \frac{n(n-1)}{2} + n(n-3) + \cdots + \frac{n(n-1)}{2}.
\]

\[
m_k^s = n\left[ \frac{2}{k-1} \prod_{i=0}^{k-2} \binom{n-2i-2}{2} + \frac{n-1}{k-2} \prod_{i=0}^{k-3} \binom{n-2i-2}{2} \right]
\]
for \(3 \leq k \leq \frac{n}{2}\).

\[
m_k^s = \frac{n(n-1)}{k-2} \prod_{i=0}^{k-3} \binom{n-2i-2}{2},
\]
\[
m_2^s = 2m(n-1),
\]
\[
m_3^s = 2m(n-1),
\]
\[
m_4^s = n\left( \frac{n-3}{2} + \frac{n-3}{k-2} \right),\]
\[
m_5^s = \sum_{i=1}^{n} \prod_{j=0}^{i-1} \binom{n-2i}{2} \binom{n}{2k-j} \right)
\]
for \(2 \leq k \leq n\).

**Proof.** By Proposition 1, \(\Gamma_g = K_2 + (K_{2 \omega_1} \cup \frac{n}{2} K_2)\) with \(V(\Gamma_g) = \zeta(D_2) \cup \Omega_2 \cup \Omega_2\), where \(\Omega_2 = \bigcup_{i=0}^{n-3} \Omega_i^2\). Thus, we have following types of edges in \(\Gamma_g\).

Type-1: \(x \sim y\) for \(x, y \in \Omega_1\);

Type-2: \(x \sim y\) for \(x = y \in \zeta(D_2)\);

Type-3: \(x \sim y\) for \(x \in \Omega_1\) and \(y \in \zeta(D_2)\);

Type-4: \(x \sim y\) for \(x \in \Omega_2\) and \(y \in \zeta(D_2)\);

Type-5: \(x \sim y\) for \(x, y \in \Omega_2 \subset \Omega_2\), where \(0 \leq i \leq \frac{n}{2} - 1\).

According to these types, following seven types of matchings between the edges of \(\Gamma_g\) exist:

- \((m^1)\) Matchings between the edges of Type-1, Type-2 and Type-3;
- \((m^2)\) Matchings between the edges of Type-4;
- \((m^3)\) Matchings between the edges of Type-5;
- \((m^4)\) Matchings between the edges of Type-1 and Type-4;
- \((m^5)\) Matchings between the edges of Type-3 and Type-4;
- \((m^6)\) Matchings between the edges of Type-1, Type-2 and the edges of Type-5.
The number of all these types of matchings is computed as follows:

(m') Since \( \zeta(D)_i \cup \Omega_i = \Omega_i \) induces a complete graph \( k_i \), so the edges of Type-1, Type-2 and Type-3 are the edges of \( k_i \), and all the matchings between these edges are counted in Table 6, where \( m_k^i \) denotes the number of matchings of order \( k \), where \( 1 \leq k \leq \frac{n}{2} \).

(m') Let \( m_k^i \) denotes the number of matchings of order \( k \), for \( k = 1, 2 \).

For \( m_1^i \): The number of matchings of order 1 is equal to the number of edges of Type-4, which is 2n. Hence \( m_1^i = 2n \).

For \( m_2^i \): Let \( e = x \sim y \) be an edge of Type-4 with \( x \in \Omega_i \) for fixed \( 0 \leq i \leq \frac{n}{2} - 1 \) and \( y \in \zeta(D)_i \). Then, together with the edge \( e \), each edge of Type-4 having one end in \( \Omega_i \) and one end in \( \zeta(D)_i \) form a matching of order 2. Therefore \( m_2^i = \frac{1}{2} (4(n-1) \times 2) = n(n-1) \). There is no matching of order greater than two in this case.

(m') There are \( \frac{n}{2} \) edges of Type-5, and no two of them shares a common vertex. Therefore, there exists a matching of every order \( k \) such that \( 1 \leq k \leq \frac{n}{2} \). Let \( m_k^i \) denotes the number of matchings of order \( k \). Then:

\[
m_k^i = \binom{n/2}{k},
\]

(m') Let \( m_k^i \) denotes the number of matchings of order \( k \), for \( 1 \leq k \leq \frac{n}{2} + 1 \). Then \( m_k^i = 0 \) in this case. In \( \Gamma_i \), no edge of Type-2 shares a common vertex with any edge of Type-4. Therefore, in this case we can get a matching by combining any matching of the edges of Type-1 with each matching of the edges of Type-4. As there are \( m_j^i \) matchings of order \( j \), \( 1 \leq j \leq \frac{n-2}{2} \), between the edges of Type-1, which are the edges of a complete graph \( K_{n-2} \), where each \( m_j^i \) can be obtained from Table 6 and there are \( m_k^i = 2n \) and \( m_k^i = n(n-1) \) matching of order 1 and 2, respectively, between the edges of Type-4. Therefore, by the rule of product, we get:

\[
m_k^i = m_k^i \times m_1^i = 2nm_k^i,
\]

for \( 3 \leq k \leq \frac{n}{2} \):

\[
m_k^i = m_k^i \times m_{k-1}^i + m_k^i \times m_{k-2}^i
\]

\[=2nm_{k-1}^i+n(n-1)m_{k-2}^i.
\]

and for \( k = \frac{n}{2} + 1 \):

\[
m_k^i = m_k^i \times m_{k-2}^i = n(n-1)m_{k-2}^i.
\]

(m') Let \( m_k^i \) denotes the number of matchings of order \( k \) for \( k = 1, 2 \). Then \( m_1^i = 0 \) in this case. Here, we can use matchings of order 1 only between the edges of Type-4. For otherwise, we cannot use any edge of Type-3 because both types of edges commonly shared the vertices in \( \zeta(D)_i \). Hence, we can get matchings of order 2 only in this case. Let \( M = \{e = x \sim y\} \) be a matching of order 1 between the edges of Type-4 with \( x \in \Omega_i \) for \( 0 \leq i \leq \frac{n}{2} - 1 \) and \( y \in \zeta(D)_i \). Then every edge of Type-3, not adjacent with \( y \) can contribute to form a matching of order 2. Since there are \( n-2 \) such edges of Type-3 each of which can be used in any of \( 2n \) matchings of order 1 between the edges of Type-4, so by the rule of product, we have:

\[
m_2^i = 2n(n-2).
\]

(m') Let \( m_k^i \) denotes the number of matchings of order \( k \) for \( 1 \leq k \leq \frac{n}{2} \). Then \( m_k^i = 0 \) in this case, because to find matchings, both the matchings of order 1 and 2 between the edges of Type-4 will be used, and any matching of order \( j \), \( 1 \leq j \leq \frac{n}{2} - 1 \) between the edges of Type-3 will be used. Accordingly, by counting these matchings with the rule of product, we get:

\[
m_k^i = 4 \times 1 \times \left( \frac{n}{2} - 1 \right) \times n = 2n \left( \frac{n}{2} - 1 \right).
\]

and for \( 3 \leq k \leq \frac{n}{2} \),

\[
m_k^i = n \left[ \frac{2}{k-1} \times \left( \frac{n-1}{k-1} \right) + \frac{2}{k-2} \times \left( \frac{n-2}{k-2} \right) \right].
\]

(m') Since the edges of Type-1, Type-2 and Type-3 are the edges of a complete graph \( k_n \) induced by \( \zeta(D)_i \cup \Omega_i = \Omega_i \), so in this case we find the matchings between the edges of Type-5 and the edges of \( k_n \). Let \( m_k^i \) be the number of matchings of order \( k \). Then \( m_k^i = 0 \). Since no edge of Type-5 share a common vertex with any edge of \( k_n \), so corresponds to each matching of the edges of Type-5, every matching of the edges of \( k_n \) can be used to find a matching in this case. As there are \( m_k^i \) matchings
of order $1 \leq t \leq \frac{n}{2}$ between the edges of $k_2$, counted
in Table 6, and there are $m_1 = \binom{n}{2}$ matchings
of order $1 \leq j \leq \frac{n}{2}$ between the edges of Type-5. Thus,
the maximum order of a matching in this case is
\( \frac{n}{2} + \frac{n}{2} = n \). Therefore, we can find $m_k^j$ for $2 \leq k \leq n$ as
follows:
\[
m_k^j = m_1^j m_2^j,
\]
\[
m_k^j = m_1^j m_2^j + m_3^j m_4^j,
\]
\[
m_k^j = m_1^j m_2^j + m_3^j m_4^j + m_5^j m_6^j,
\]
\[
m_k^j = m_1^j m_2^j + m_3^j m_4^j + m_5^j m_6^j + m_7^j m_8^j,
\]
and so on, generally, we have:
\[
m_k^j = \sum_{i=1}^{k-1} m_i^j m_{i+1}^j.
\]

Hence, by the rule of sum, the Hosoya index of $G'$ is:
\[
1 + \sum_{i=1}^{n} \text{the number of matchings } (m^i) = 1 + \sum_{i=1}^{n} m^i = m_1 + m_2 + \sum_{i=3}^{n} m^i = m_2 + \sum_{i=3}^{n} m^i,
\]
with
\[
m_1 = \frac{1}{k} \prod_{i=0}^{k-1} \left( n - 2i \right),
\]
\[
m_2 = 2n, m_2^j = n(n-1),
\]
\[
m_k^j = \begin{cases} n(n-1) & \text{if } 3 \leq k \leq \frac{n}{2}, \\
\end{cases}
\]
\[
m_3^j = 2n \left( \frac{n-2}{2} \right),
\]
\[
m_4^j = n \left[ \frac{2}{k-1} \prod_{i=0}^{k-1} \left( n - 2i - 2 \right) + \frac{1}{k-2} \prod_{i=0}^{k-3} \left( n - 2i - 2 \right) \right],
\]
for $3 \leq k \leq \frac{n}{2}$.

\[
m_3^j = \frac{n(n-1)}{k-2} \prod_{i=0}^{k-1} \left( n - 2i - 2 \right),
\]
\[
m_2^j = 2n(n-1),
\]

**Table 6**: The number of non-empty matchings in a complete graph $K_n'$ for even $n \geq 2$

| $K_n$ | $m_1^n$ | $m_2^n$ | $m_3^n$ | $m_4^n$ | ... | $m_k^n$ |
|------|--------|--------|--------|--------|------|--------|
| $K_2$ | 2 | 2 |
| $K_4$ | 4 | \(\frac{4}{2} \) | \(\frac{2}{2} \) | \(\frac{2}{2} \) |
| $K_6$ | 6 | \(\frac{6}{2} \) | \(\frac{3}{2} \) | \(\frac{2}{2} \) |
| $K_8$ | 8 | \(\frac{8}{2} \) | \(\frac{4}{2} \) | \(\frac{2}{2} \) |
| ... | ... | ... | ... | ... | ... | ... |
| $K_{n,2}$ | \(\frac{n}{2} \) | \(\frac{n}{2} \) | \(\frac{n-2}{2} \) | \(\frac{n-4}{2} \) | ... | \(\frac{1}{k} \prod_{i=0}^{k-1} \left( n - 2i \right) \) |
\[ m_2^k = 2n \binom{n - 1}{k - 1} \]

\[ m_3^k = n \left( \frac{n - 1}{2} \right) + \frac{n - 1}{k - 1} + 2 \binom{n - 2}{k - 2} \]

for \( 3 \leq k \leq \frac{n}{2} \).

\[ m_i^k = \sum_{j=1}^{k-i} \prod_{s=1}^{j-1} \binom{n - 2i}{2} \binom{n - 2}{k - j} \]

for \( 2 \leq k \leq n \).

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