Perturbative PDF of the total magnetization of the 4D Ising model

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ABSTRACT: We compute, at one loop in perturbation theory, the probability density function of the total magnetization $M$ of the Ising model on the 4-torus and the 4-sphere. We develop a single perturbative expansion that is valid in the symmetric phase as well as the broken symmetry phase, provided that the correlation length is large compared to the system size $L$. We find that, at the critical point, for large system size in lattice units, the PDF approaches $p(M) \sim \exp(-f(L)M^4)$. Consequently, the critical value of the Binder cumulant of the total magnetization is $U = 1 - 4\Gamma(5/4)^2 \over 3\Gamma(3/4)^2$. We validate our results by comparison with Monte Carlo simulation.
1 Main result

When the system size $L$ is large, the probability distribution of the total magnetization $M$ of the Ising model changes qualitatively between the two phases of the model. In the symmetric phase, if the correlation length $\xi$ is large in lattice units, but small compared to the system size, the distribution is well approximated by a zero-mean normal. In the broken symmetry phase, still for $1 \ll \xi \ll L$, the distribution is well approximated by a mixture of two normals centered at non-zero values $\pm M_0$.

In the opposite regime, $1 \ll L \ll \xi$, in two and three dimensions, the probability distribution of $M$ is a non-trivial function of $M$. It is not immediately clear if the same is also true in four dimensions. Since the field theory that describes the Ising critical point in four dimensions becomes weakly coupled at low energy, one may expect the probability distribution of $M$ to remain a zero-centered normal even in this regime. This is in fact the case for the total magnetization of a subsystem of intermediate size: large in lattice units, but small compared to $L$ [1]. Here we show that the same is not true for the total magnetization of the entire system. We compute the logarithm of the PDF of $M$ at one-loop in 4 dimensions, and show that, for $1 \ll L \ll \xi$, the distribution is not normal, but rather of the form $-\log p(M) \sim f(L)M^4$. We find that corrections to this form vanish very slowly with increasing system size, like $1/\log L$.

When the correlation length is large in lattice unit, the Ising model in 4 dimensions is described by the $\phi^4$ field theory:

$$S[\phi] \equiv \int_{\mathcal{M}} \left( \frac{1}{2} \phi (\Delta + r_0) \phi + \frac{1}{24} u_0 \phi^4 \right). \quad (1.1)$$

Here we take the manifold $\mathcal{M}$ to be a 4-torus of radius $R$, i.e. $x_i \sim x_i + 2\pi R$. In this field theory context, we define the PDF of the average magnetization $m = M/V$ as:

$$p(m) \equiv \frac{1}{Z} \int [D\phi] \delta \left( m - \frac{1}{V} \int_{\mathcal{M}} \phi \right) e^{-S[\phi]}, \quad (1.2)$$
Figure 1. Binder cumulant of the total magnetization of the 4D Ising model. The left plot gives a broad picture, the right plot shows the critical region close-up. The shaded regions display the one-sigma confidence intervals obtained from Monte Carlo simulation. The center lines are obtained from a 3-parameter fit of the perturbative result (1.3). The horizontal dashed line indicates the critical value $J_c = 0.1496938$. The fit has $J_c = 0.1496938$, $J - J_c = -0.027 \cdot r$, $g = 0.43$, with $r$, $g$ defined at the renormalization scale $L = 2\pi R = 8$.

and we evaluate it at one loop in perturbation theory, obtaining:

$$-\log p(m) = \text{const.} + V\left(\frac{1}{2}m^2 - \frac{g}{6} (rR^2) - 1 + O(g^2)\right) +$$

$$\frac{2\pi^2}{9} g m^4 \left(1 - \frac{g}{2} f_2 (rR^2) + O(g^3)\right) +$$

$$\frac{8\pi^4}{81} g^3 m^6 R^2 \left(f_3 (rR^2) + O(g)\right) + O(g^4 m^8) \right).$$

(1.3)

Here the coefficients $r$ and $g$ are the renormalized counterparts to $r_0$ and $u_0$; the functions $f_1$, $f_2$ and $f_3$ are plotted in fig. 2, and an explicit expression (2.15) is given below.

The dimension-2 coupling $r$ controls the cross-over between the symmetric and broken-symmetry phases. The expression (1.3) is valid for $rR^2 \gtrsim -1/2$. For lower values of $r$, the perturbative vacuum becomes unstable, invalidating the perturbative expansion. For this reason, the functions $f_i$ diverge as their argument approaches -1. On the other hand, $f_1$, $f_2$, $f_3$ all go to zero as their argument approaches positive infinity, and hence for large $r$ the probability distribution of $m$ is a zero-centered normal:

$$p(m) \sim N e^{-\frac{1}{2}V r m^2} \quad \text{for} \quad rR^2 \gg 1.$$  

(1.4)

The dimensionless coupling $g$ is the parameter of the perturbative expansion. The
The functions $f_i$ that appear in (1.3). The functions go to zero as $x \to \infty$, and diverge for $x \to -1$ ($x \to -4$ for the sphere), signaling an instability of the perturbative vacuum.

expansion is valid for $g \lesssim 1$, provided $rR^2$ is sufficiently far from the bound discussed above.

In order to describe how $p(m)$ depends on $R$ at fixed bare couplings $r_0$, $u_0$, it is necessary to account for renormalization effects. The specific renormalization scheme we used is described in section 2, and (2.13) gives expression for $p(m)$ evaluated at a generic renormalization scale $\mu$. However, for simplicity, we chose to evaluate (1.3) at the scale $\mu^2 = r + R - 2$. This choice of $\mu$ is optimal for the reliability of perturbation theory, because it avoids the emergence of large logarithms over the widest possible range of parameters.

At one loop, the Callan-Symanzik equations for $r$ and $g$ are:

$$
\mu \frac{dg}{d\mu} = g^2, \quad \mu \frac{dr}{d\mu} = \frac{1}{3} gr.
$$

(1.5)

These can be integrated and combined with the condition $\mu^2 = r + R^{-2}$ to obtain a system of equations connecting the renormalized couplings at two different values of $R$:

$$
g(R_1) = g(R_2) = 1 - \frac{1}{2} g(R_1) \log \frac{r(R_2) + R_2^2}{r(R_1) + R_1^2}.
$$

(1.6)

A few solutions to this system of equations are shown in fig. 3. Within the perturbative regime $g \lesssim 1$, $rR^2 \gtrsim -1/2$, the coupling $r$ varies little with $R$, and, as is clear from the differential form (1.5), the sign of $r$ is always preserved. Thus we conclude that the critical point is at $r = 0$, and the symmetric phase is realized for $r > 0$.

\footnote{There are of course many alternative, arguably simpler, solutions that differ by sub-leading orders in an expansion in $g(R_1)$. The one displayed here is the exact solution to (1.5).}
Figure 3. A few solutions to (1.6) with $g(R_1) = 0.45$. In the left plot, the shaded region shows where the perturbative vacuum becomes unstable, invalidating the perturbative expansion. Similarly, in the right plot, the lines become dotted outside of the perturbative region. Note how the sign of $r$ is preserved and $r = 0$ is a solution.

At the critical point, the renormalized coupling $g$ follows the simpler Callan-Symanzik equation:

$$\frac{1}{g(R_2)} - \frac{1}{g(R_1)} = \log \frac{R_2}{R_1},$$

(1.7)

and hence, as system size grows, the renormalized coupling $g$ goes to zero as $1/\log(R)$. In this regime, the quartic term in (1.3) dominates all the others. This is perhaps most evident if the PDF is expressed in terms of the rescaled quantity $\bar{m} = g^4 m$, whose variance remains finite as $g \to 0$. Thus we conclude that, at the critical point, for sufficiently large system size:

$$- \log p(m) \sim \text{const} + \frac{2\pi^2}{9} V g m^4.$$

(1.8)

In Monte Carlo simulations, the qualitative behavior of the distribution of the magnetization is often characterized by measuring the so-called Binder cumulant [1]:

$$U = 1 - \frac{\langle m^4 \rangle}{3 \langle m^2 \rangle^2}.$$

(1.9)

This quantity is constructed to be independent of the overall scale of $m$, and to be zero if $m$ is normally distributed. From (1.8) we conclude that, on a 4-Torus, at the critical point:

$$U = 1 - \frac{4 \Gamma \left( \frac{5}{4} \right)^2}{3 \Gamma \left( \frac{3}{4} \right)^2} = 0.27052 \ldots$$

(1.10)
In fig. 1, we show a comparison of the Binder cumulant computed from (1.3) near the critical point, and the results of Monte Carlo simulation of the 4D Ising model. The agreement is excellent except for the smallest system size $L = 8$. Notice how slowly the finite size Binder cumulant approaches the asymptotic value (1.10).

2 Derivation on the 4-torus

We now describe briefly how the result (1.3) is obtained. The perturbative approach is similar to the computation of the effective action, as in e.g. [2], except that we are interested in the whole probability distribution of $m$, instead of just the expected value. The main difficulty lies in evaluating the loop integrals at finite size.

The average magnetization $m$ is proportional to the zero-momentum mode of the field:

$$m = \frac{1}{V} \int d^4 x \; \phi(x) = \frac{1}{V} \phi_{n=0}.$$  (2.1)

Because of the delta function in (1.2), the zero-mode becomes an external field, whereas all other modes are still part of the functional integral. Separating the zero-mode from the other modes in the action yields:

$$S(m, \phi) =$$

$$= V \left( \frac{1}{2} r_0 m^2 + \frac{u_0}{24} m^4 \right) + \frac{1}{2V} \sum_n \left( \frac{n^2}{R^2} + r_0 + \frac{1}{2} u_0 m^2 \right) \phi_n \phi_n +$$

$$+ \frac{u_0}{6V^2} \sum_{n_1, n_2} \phi_{n_1} \phi_{n_2} \phi_{-n_1-n_2} + \frac{u_0}{24V^3} \sum_{n_1, n_2, n_3} \phi_{n_1} \phi_{n_2} \phi_{n_3} \phi_{-n_1-n_2-n_3},$$  (2.2)

where all summations now are over $\mathbb{Z}^4 \setminus \{0\}$.

We introduce renormalized couplings

$$u_0 = u(1 + u\delta u); \quad r_0 = r(1 + u\delta r),$$  (2.3)

and we obtain the following edges and vertices in the diagrammatic expansion:

\begin{align*}
n_1 \quad &-\quad n_2 \quad \frac{V}{n_1^2 R^{-2} + r} \delta_{n_1+n_2} \quad \times \quad n_1 \quad - \frac{um^2}{4V} \delta_{n_1+n_2} \\
\quad &\quad \times \quad \frac{u}{6V^2} \delta_{n_1+n_2+n_3} \quad n_1 \quad \times \quad \frac{u}{24V^3} \delta_{n_1+n_2+n_3+n_4}
\end{align*}

plus additional vertices associated with the counterterms $\delta r, \delta u$ which we do not list for brevity.
The logarithm of the probability distribution of $m$ is the sum of all connected diagrams:

$$\log p(m) = \text{const} - V \left( \frac{1}{2} m^2 + \frac{u}{24} m^4 \right) + \cdots,$$

where again we left out all diagrams involving the counterterms for brevity.

Retaining only one-loop diagrams, and employing a heat kernel regulator we have:

$$-\log p(m) = \text{const} + V \left( \frac{1}{2} m^2 \left( r + u \left( \delta r + \frac{1}{2} I_1 \right) + O(u^2) \right) + \frac{u}{24} m^4 \left( 1 + u \left( \delta u - \frac{3}{2} I_2 \right) + O(u^2) \right) + \frac{u^3}{48} m^6 \left( I_3 + O(u) \right) + O(u^4 m^8) \right),$$

where

$$I_k = \frac{1}{V} \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-s(n^2 R^{-2} + r)} (n^2 R^{-2} + r)^k = \frac{1}{(k-1)!} \left( -\frac{\partial}{\partial r} - s \right)^{k-1} I_1$$

The summation $I_1$ can be simplified substantially using the following trick:

$$I_1 = \frac{1}{V} \int_s^\infty dt \sum_{n \in \mathbb{Z} \setminus \{0\}} e^{-t(n^2 R^{-2} + r)}$$

$$= \frac{1}{V} \int_s^\infty dt \ e^{-tr} \left( \sum_{m=-\infty}^{\infty} e^{-tm^2 R^{-2}} \right)^4 - 1 \right)$$

where:

$$\theta(z) = \theta_3(0; e^{-z}) = \sum_{m=-\infty}^{\infty} e^{-m^2 z}.$$
asymptotic behavior of \( \theta(z) \) for \( z \to 0 \), which can be obtained from its definition using the Euler-Maclaurin formula:

\[
\theta(z) \sim \sqrt{\frac{\pi}{z}} + o(z^k) \quad \text{for} \quad z \to 0.
\] (2.8)

Thus we have:

\[
I_1 = \frac{1}{16\pi^2} \left( \int_s^\infty \frac{dt}{t} e^{-t\mu^2} t^{-2} \left( 1 + \frac{t}{\mu^2} - r \right) + \frac{1}{R^2} f(rR^2, \mu R) + O(s) \right)
\]

\[
= \frac{1}{16\pi^2} \left( \frac{1}{s} + r \left( \log s\mu^2 + \gamma_E \right) - \mu^2 + \frac{1}{R^2} f(rR^2, \mu R) + O(s) \right),
\] (2.9)

where \( \mu \) is a renormalization scale that can be chosen at will, and:

\[
f(\tilde{r}, \tilde{\mu}) = \int_0^\infty dz e^{-z\tilde{r}} \left( \frac{\theta(z)^4 - 1}{\pi^2} - \frac{1 + \frac{\tilde{\mu}^2 - \tilde{r}}{z^2}}{e^{-z(\mu^2 - r)}} \right).
\] (2.10)

We set the counterterms to:

\[
\delta r = -\frac{1}{32\pi^2} \left( \frac{1}{s} + r \left( \log s\mu^2 + \gamma_E - 1 \right) \right),
\] (2.11)

\[
\delta u = -\frac{3}{32\pi^2} \left( \log s\mu^2 + \gamma_E + 1 \right).
\] (2.12)

The Callan-Symanzik equations (1.5) follow from this subtraction choice. Finally, we obtain:

\[
- \log p(m) = \text{const} + V \left( \frac{1}{2} m^2 \left( r + \frac{u}{32\pi^2 R^2} f(rR^2, \mu R) + (r - \mu^2) R^2 \right) + O(u^2) \right) + \frac{u}{24} m^4 \left( 1 + \frac{3u}{32\pi^2} f^{(1,0)}(rR^2, \mu R) + O(u^2) \right) + \frac{u^3}{48} m^6 \left( \frac{R^2}{32\pi^2} f^{(2,0)}(rR^2, \mu R) + O(u) \right) + O(u^4 m^8),
\] (2.13)

from which (1.3) is obtained by setting \( g = \frac{3u}{16\pi^2} \) and \( \mu^2 = r + R^{-2} \). This last choice is motivated as follows. The subtraction in (2.10) is similar to:

\[
\int_0^\infty dz \frac{e^{-az} - e^{-bz}}{z} = \log \frac{b}{a}.
\] (2.14)

When the large-\( z \) asymptotic behavior of the two terms is not well matched, the integral becomes large in magnitude, making the perturbative expansion less reliable. With the choice \( \mu^2 = r + R^{-2} \), both terms in (2.10) have the same asymptotic behavior \( \sim e^{-z(r+1)} \), thus avoiding the large log problem over the widest possible range of parameters.
For completeness, let us display explicitly the functions \( f_i \) that parametrize (1.3):

\[
\begin{align*}
    f_1(\bar{r}) &= f(\bar{r}, \sqrt{\bar{r} + 1}) = \int_0^\infty dz \, e^{-z \bar{r}} \left( \frac{\theta(z)^4}{\pi^2} - \frac{(1 + z)e^{-z}}{z^2} \right), \\
    f_2(\bar{r}) &= f^{(1,0)}(\bar{r}, \sqrt{\bar{r} + 1}) = \int_0^\infty dz \, e^{-z \bar{r}} \left( \frac{\theta(z)^4}{\pi^2} - \frac{e^{-z}}{z^2} \right), \\
    f_3(\bar{r}) &= f^{(2,0)}(\bar{r}, \sqrt{\bar{r} + 1}) = \int_0^\infty dz \, e^{-z \bar{r}} z^2 \frac{\theta(z)^4}{\pi^2} - 1,
\end{align*}
\]

and \( \theta \) is given by (2.7).

### 3 Results for the 4-sphere

It is possible to obtain \( p(m) \) on the 4 sphere as well, with similar methods. Here we highlight the main differences from the 4 torus.

Curved manifolds allow for an additional renormalizable coupling:

\[
\Delta S = \int_M \frac{1}{2} \xi_0 \phi^2 \mathcal{R},
\]

where \( \mathcal{R} \) is the scalar curvature and \( \xi_0 \) is a dimensionless bare coupling. The free theory is Weyl invariant if \( \xi_0 = \frac{1}{6} \). In the presence of interactions, the coupling needs to be renormalized, and its renormalized counterpart \( \xi \) becomes a running coupling. At one loop, the Callan-Symanzik equation for \( \xi \) is:

\[
\frac{d\xi}{d\mu} = \frac{1}{3} g \left( \xi - \frac{1}{6} \right).
\]

From this expression it seems that \( \xi \) can be set to the critical value \( \frac{1}{6} \) at all energy scales. However, this turns out to be an illusion: at higher orders in perturbation theory the Callan-Symanzik equation becomes inhomogeneous [3]. Therefore, on a curved manifold, \( \xi \) is simply another free parameter of the scalar field theory.

The probability distribution of the total magnetization on the 4-sphere is still given by (1.3), with the substitution:

\[
r R^2 \to \frac{\xi}{12} + r R^2,
\]

where \( R \) is now the radius of the sphere, and with the functions \( f_i \) defined as:

\[
\begin{align*}
    f_1(\bar{r}) &= \left(2 - \bar{r} \right) \int_0^\infty dz \, e^{-z(\bar{r} + 4)} H(z) + \frac{7}{3}, \\
    f_2(\bar{r}) &= \int_0^\infty dz \, e^{-z(\bar{r} + 4)} \left(1 + z(2 - \bar{r}) \right) H(z) + \frac{6}{\bar{r} + 4}, \\
    f_3(\bar{r}) &= \int_0^\infty dz \, e^{-z(\bar{r} + 4)} \left(2 + z(2 - \bar{r}) \right) z H(z) + \frac{\bar{r} + 10}{(\bar{r} + 4)^2},
\end{align*}
\]
where:

\[ H(z) = \sum_{\ell=1}^{\infty} (2\ell + 3)e^{-z(\ell(\ell+3)-4)} - \frac{1}{z}. \] (3.8)

The functions \( f_i \) for the sphere are also displayed in fig. 2. They diverge for \( \bar{r} \to -4 \), signaling the instability of the perturbative vacuum, and they go to zero for \( \bar{r} \to \infty \).

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