Modal Fracture of Higher Groups

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Abstract

In this paper, we examine the modal aspects of higher groups in Shulman’s Cohesive Homotopy Type Theory. We show that every higher group sits within a modal fracture hexagon which renders it into its discrete, infinitesimal, and contractible components. This gives an unstable and synthetic construction of Schreiber’s differential cohomology hexagon. As an example of this modal fracture hexagon, we recover the character diagram characterizing ordinary differential cohomology by its relation to its underlying integral cohomology and differential form data, although there is a subtle obstruction to generalizing the usual hexagon to higher types. Assuming the existence of a long exact sequence of differential form classifiers, we construct the classifiers for circle $k$-gerbes with connection and describe their modal fracture hexagon.

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1 Introduction

There are many situations where cohomology is useful but we need more than just the information of cohomology classes and their relations in cohomology — we need the information of specific cocycles which give rise to those classes and cochains which witness these relations. A striking example of this situation is ordinary differential cohomology. To give a home for calculations done in [4], Cheeger and Simons [3] gave a series of lectures in 1973 defining and studying differential characters, which equip classes in ordinary integral cohomology with explicit differential form representatives. Slightly earlier, Deligne [6] had put forward a cohomology theory in the complex analytic setting which would go on to be called Deligne cohomology. It was later realized that when put in the differential geometric setting, Deligne cohomology gave a presentation
of the theory of differential characters. This combined theory has become known as ordinary differential cohomology.

The ordinary differential cohomology $D_k(X)$ of a manifold $X$ is characterized by its relationship to the ordinary cohomology of $X$ and the differential forms on $X$ by a diagram known as the differential cohomology hexagon or the character diagram [23]:

$$\Lambda^k(X)/\text{im}(d) \xrightarrow{d} \Lambda^{k+1}_\text{cl}(X) \xrightarrow{} H^k(X; \mathbb{R}) \xrightarrow{D_k(X)} H^k(X; \mathbb{R}) \xrightarrow{} H^{k+1}(X; \mathbb{Z})$$

In this diagram, the top and bottom sequences are long exact, and the diagonal sequences are exact in the middle. The bottom sequence is the Bockstein sequence associated to the universal cover short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 0$$

while the top sequence is given by de Rham’s theorem representing real cohomology classes by differential forms.

This sort of diagram is characteristic of differential cohomology theories in general. Bunke, Nikolaus, and Vokel [2] interpret differential cohomology theories as sheaves on the site of smooth manifolds and construct differential cohomology hexagons very generally in this setting:

$$A(\hat{E})^k(X) \xrightarrow{} Z(\hat{E})^k(X)$$

$$H^{k-1}(X; Z(\hat{E})) \xrightarrow{} \hat{E}^k(X) \xrightarrow{} H^{k}(X; Z(\hat{E}))$$

Here, $\hat{E}$ is the differential cohomology theory, $U(\hat{E})$ is its underlying topological cohomology, $Z(\hat{E})$ are the differential cycles, $S$ is the secondary cohomology theory given by flat classes, and $A$ classifies differential deformations (this summary discussion is lifted from [2]). Here, as with ordinary differential cohomology, the top and bottom sequences are exact, and the diagonal sequences are exact in the middle.

The arguments of Bunke, Nikolaus, and Vokel are abstract and modal in character. This is emphasized by Schreiber in his book [20], where he constructs similar diagrams in the setting of an adjoint triple

$$\Pi_\infty \xrightarrow{} \Gamma \xrightarrow{} \mathcal{E}$$

in which the middle functor is fully faithful and the leftmost adjoint $\Pi_\infty$ preserves products. In the case that $\Gamma$ is the global sections functor of an $\infty$-topos $\mathcal{E}$ landing in the $\infty$-topos of homotopy types $\mathcal{H}$, this structure makes $\mathcal{E}$ into a strongly $\infty$-connected $\infty$-topos. In the case that $\mathcal{E}$ is the $\infty$-topos of sheaves of homotopy types on manifolds, the leftmost adjoint $\Pi_\infty$ is given by localizing at the sheaf of real-valued functions; for a representable, this recovers the homotopy type or fundamental $\infty$-groupoid of the manifold. Schreiber shows in Proposition 4.1.17 of [20] that any such adjoint triple gives rise to differential cohomology hexagons, specializing to those of Bunke, Nikolaus, and Vokel in the case that $\mathcal{E}$ is the $\infty$-topos of sheaves of homotopy types on smooth manifolds.
This abstract re-reading of the differential cohomology hexagons shows that there is nothing specifically “differential” about them, and that they arise in situations where there is no differential calculus to be found. Schreiber emphasizes this point in an nLab article [21] where he refigures these hexagons as modal fracture squares. To recover more traditional fracture theorems, Schreiber considers the case where $E = A$-Mod is the ∞-category of modules over an $E_2$-ring $A$, $\Gamma = \Gamma_I$ is the reflection into $I$-nilpotent modules (with $I \subseteq \pi_0 A$ a finitely generated ideal) constructed by Lurie in [14, Notation 4.1.13], and $\Pi_\infty M = M^{\dagger}$ is the $I$-completion constructed in [14, Notation 4.2.3]. The subcategories of $I$-nilpotent and $I$-complete modules are distinct but equivalent, allowing us to see them as a single ∞-category $H$.

In this paper, we will construct the modal fracture hexagon associated to a higher group — a homotopy type which may be delooped — synthetically, working in an appropriately modal homotopy type theory. We will work in Shulman’s flat homotopy type theory [22], a variant of homotopy type theory that adds a comodality $♭$ which may be thought of as the comonad $\Delta \Gamma$ from Diagram 1.0.2. We will equip this type theory equipped with a modality $S$ (which may be thought of as the monad $\Delta \Pi_\infty$) which satisfies a unity of opposites axiom (Axiom 1) implying that $♭$ is left adjoint to $♭$ in the sense that $♭(f X \to Y) = ♭(X \to ♭Y)$.

We can think of this axiom as the internalization of the adjunction $\Delta \Pi_\infty \dashv \Delta \Gamma$ induced by Diagram 1.0.2.

This type theory should have models in all strongly $\infty$-connected geometric morphisms between ∞-toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{S}$ is strongly $\infty$-connected when its inverse image $f^* : \mathcal{S} \to \mathcal{E}$ is fully faithful and has a left adjoint $f_! : \mathcal{E} \to \mathcal{S}$ which preserves finite products. We then have $f = f^* f_!$ and $♭ = f^* f_*$. An ∞-topos $\mathcal{E}$ is strongly $\infty$-connected when its terminal geometric morphism $\Gamma : \mathcal{E} \to \infty\text{Grpd}$ is strongly $\infty$-connected.

Our main theorem (an unstable and synthetic version of Proposition 4.1.17 in [20]) is as follows:

**Theorem 2.4.2.** For a crisp $\infty$-group $G$, there is a modal fracture hexagon:

$$
\begin{array}{ccc}
\pi & \xrightarrow{\theta} & G \\
\downarrow & \searrow & \downarrow \\
♭G & \xrightarrow{♭}\theta & fG \\
\end{array}
$$

where

- $\theta : G \to g$ is the infinitesimal remainder of $G$, the quotient $G/♭G$, and
- $\pi : G \to G$ is the universal (contractible) $\infty$-cover of $G$.

Moreover,

1. The middle diagonal sequences are fiber sequences.
2. The top and bottom sequences are fiber sequences.
3. Both squares are pullbacks.

Furthermore, the homotopy type of $g$ is a delooping of $♭G$:

$$f g = ♭BG.$$

Therefore, if $G$ is $k$-commutative for $k \geq 1$ (that is, admits further deloopings $♭^{k+1}G$), then we may continue the modal fracture hexagon on to $♭^kG$. 
We will define the notions of universal \(\infty\)-cover and of infinitesimal remainder in Sections 2.2 and 2.3 respectively. Our proof of this theorem will make extensive use of the theory of modalities in homotopy type theory developed by Rijke, Shulman, and Spitters [19], as well as the theory of modal étale maps developed in [5] and the theory of modal fibrations developed in [17].

Having proven this theorem, we will turn our attention to providing interesting examples of it. To that end, in Section 3 we will construct ordinary differential cohomology (in the guise of the classifying bundles \(B_k U(1)\) of connections on \(k\)-gerbes with band \(U(1)\), see Definition 3.2.1) in smooth real cohesive homotopy type theory. For this, we assume the existence of a long exact sequence
\[
0 \rightarrow \flat \mathbb{R} \rightarrow \mathbb{R} \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow \cdots
\]
where the \(\Lambda^k\) classify differential \(k\)-forms. It should be possible to construct the \(\Lambda^k\) from the axioms of synthetic differential geometry with tiny infinitesimals, but we do not do so here for reasons of space and self-containment. See Remark 3.1.2 for a full discussion.

Our construction of ordinary differential cohomology is clean, conceptual, and modal. We do not, however, recover exactly the character diagram 1.0.1, because de Rham’s theorem does not hold for all types (see Proposition 3.3.6). In Section 3.3, we do recover a very similar diagram and find that the obstruction to these two diagrams being the same lies in a shifted version of ordinary differential cohomology.

Because the arguments given in Section 3 are abstract and modal in character, they are applicable in other settings. In Section 3.4, we express our construction of ordinary differential cohomology in the abstract setting of a contractible and infinitesimal resolution of a crisp abelian group. In Section 3.5, we briefly describe how to construct combinatorial analogues of ordinary differential cohomology in symmetric simplicial homotopy types, making use of an observation of Lawvere that cocycle classifiers may be constructed using the tinyness of the simplices.

## 2 The Modal Fracture Hexagon

In this section, we will construct the modal fracture hexagon of a higher group.

A higher group \(G\) is a type equipped with a 0-connected delooping \(BG\). An ordinary group \(G\) may be considered as a higher group by taking \(BG\) to be the type of \(G\)-torsors and equating \(G\) with the group of automorphisms of \(G\) considered as a \(G\)-torsor.

The theory of higher groups is expressed in terms of their deloopings: for example a homomorphism \(G \rightarrow H\) is equivalently a pointed map \(BG \rightarrow BH\). See [1] for a development of the elementary theory of higher groups in homotopy type theory.

The modal fracture hexagon associated to a (crisp) higher group \(G\) will factor \(G\) into its universal \(\infty\)-cover \(\tilde{G}\) and its infinitesimal remainder \(\Gamma G\). We will therefore introduce \(\tilde{G}\) and \(\Gamma G\) and prove some lemmas about them which will set the stage for the modal fracture hexagon.

**Notation 1.** We will use the Agda-inspired notation for dependent pair types (also known as dependent sum types) and dependent function types (also known as dependent product types):
\[
(a : A) \times B(a) \equiv \sum_{a:A} B(a)
\]
\[
(a : A) \rightarrow B(a) \equiv \prod_{a:A} B(a).
\]

If \(X\) is a pointed type, we refer to its base point as \(\text{pt}_X : X\). If \(X\) and \(Y\) are pointed types, then we define \(X \rightarrow Y\) to be the type of pointed functions between them:
\[
(X \rightarrow Y) \equiv (f : X \rightarrow Y) \times (f(\text{pt}_X) = \text{pt}_Y).
\]

### 2.1 Preliminaries

In this section, we will review Shulman’s flat type theory [22] and the necessary lemmas.
In constructive mathematics, the proposition that all functions \( \mathbb{R} \to \mathbb{R} \) are continuous is undecided — there are models of constructive set theory (and homotopy type theory) in which every function \( \mathbb{R} \to \mathbb{R} \) is continuous (and, of course, familiar models where there are discontinuous functions \( \mathbb{R} \to \mathbb{R} \)). Since, in type theory, such a function \( f : \mathbb{R} \to \mathbb{R} \) is defined by giving the image \( f(x) \) of a free variable \( x : \mathbb{R} \), we see that in a pure constructive setting, the dependence of terms on their free variables confers a liminal sort of continuity. This is a very powerful observation which, in its various guises, lets us avoid the menial checking of continuity, smoothness, regularity, and so on for various sorts of functions in various models of homotopy type theory. It extends far beyond real valued functions; for example, the assignment of a vector space \( V_p \) to a point \( p \) in a manifold \( M \), constructively, gives a vector bundle \((p : M) \times V_p \to M \) over \( M \) with all its requirements of continuity or smoothness (depending on the model). Since all sorts of continuity (continuity, smoothness, regularity, analyticity) can be captured in various models, Lawvere named the general notion “cohesion” in his paper [10], whose generalization to \( \infty \)-categories in [20] inspired the type theory of [22].

However, not every dependency is cohesive (continuous, smooth, etc.). To enable discontinuous dependencies, then, we must mark our free variables as varying cohesively or not. For this reason, Shulman introduces “crisp” variables, which are free variables in which terms depend discontinuously:

\[ a :: A. \]

Any variable appearing in the type of a crisp variable must also be crisp, and a crisp variable may only be substituted by expressions that only involve crisp variables. When all the variables in an expression are crisp, we say that that expression is crisp; so, we may only substitute crisp expressions in for crisp variables. Constants — like \( 0 : \mathbb{N} \) or \( \mathbb{N} : \text{Type} \) — appearing in an empty context are therefore always crisp. This means that one cannot give a closed form example of a term which is not crisp; all terms with no free variables are crisp. For emphasis, we will say that a term which is not crisp is cohesive. The rules for crisp type theory can be found in Section 2 of [22].

Given this notion of discontinuous dependence of terms on their free variables, we can now define an operation on types which removes the cohesion amongst their points. Given a crisp type \( X \), we have a type \( \flat X \) whose points are, in a sense, the crisp points of \( X \). Since it is free variables that may be crisp, we express this idea by allowing ourselves to assume that a (cohesive) variable \( x : \flat X \) is of the form \( u^x \) for a crisp \( u :: X \). More precisely, whenever we have type family \( C : \flat X \to \text{Type} \), an \( x :: \flat X \), and an element \( f(u) : C(u^x) \) depending on a crisp \( u :: X \), we get an element

\[ (\text{let } u^x \coloneqq x \text{ in } f(u)) : C(x) \]

and if \( x \equiv u^x \), then \( (\text{let } u^x \coloneqq x \text{ in } f(u)) \equiv f(v) \). We refer to this method of proof as “\( \flat \)-induction”. The full rules for \( \flat \) can be found in Section 4 of [22].

We have an inclusion \((\cdot)_p : \flat X \to X \) given by \( x_p \coloneqq \text{let } u^x \coloneqq x \text{ in } u \). Since we are thinking of a dependence on a crisp variable as a discontinuous dependence, if this map \((\cdot)_p : \flat X \to X \) is an equivalence then every discontinuous dependence on \( x :: X \) underlies a continuous dependence on \( x \). This leads us to the following definition:

**Definition 2.1.1.** A crisp type \( X :: \text{Type} \) is **crisply discrete** if the counit \((\cdot)_p : \flat X \to X \) is an equivalence.\(^2\)

Note that this definition is only sensible for crisp types, since we may only form \( \flat X \) for crisp \( X :: \text{Type} \). We would also like a notion of discreteness which applies to any type, and a reflection \( X \to f \text{ of a type into a discrete type. For that reason, we will also presume that there is a modality } f \text{ called the } \text{shape} \text{ (and which we think of as sending a type } X \text{ to its homotopy type or shape } fX) \text{. We refer to the } f \text{-modal types as discrete. To make sure that these two notions of discreteness coincide, we assume the following axiom:}

**Axiom 1** (Unity of Opposites). For any crisp type \( X \), the counit \((\cdot)^f : \flat X \to X \) is an equivalence if and only if the unit \((\cdot)^f : X \to fX \) is an equivalence.

This axiom implies that \( f \) is left adjoint to \( \flat \), at least for crisp maps. In [22], Shulman assumes an axiom \( C0 \) which lets him define the \( f \)-modality as a localization and prove our Unity of Opposites axiom. The two

---

1Note that as these are terms and not free variables, we don’t need to use the special syntax \( a :: A \). The double colon introduces a crisp free variable.

2See Remark 6.13 of [22] for a discussion on some of the subtleties in the notion of crisp discreteness.
Axioms have roughly the same strength, though C0 is slightly stronger since it assumes that \( f \) is an accessible modality.

**Theorem 2.1.2.** Let \( X \) and \( Y \) be crisp types. Then
\[
♭(X \to ♭Y) = ♭(♯X \to Y).
\]

**Proof.** This is Theorem 9.15 of [22]. Note that Axiom C0 is only used via our Unity of Opposites axiom. □

**Remark 2.1.3.** Theorem 2.1.2 justifies the use of the symbol “\( ♭ \)” in flat type theory. If we think of \( ♭X \) as the homotopy type of \( X \), then the adjointness of \( ♭ \) with \( ♬ \) tells us that \( ♭BG \) modulates principal \( G \)-bundles with a homotopy invariant parallel transport — that is, bundles with flat connection. This terminology is due to Schreiber in [20].

We may also define the truncated shape modalities \( ♭_n \) to have as modal types the types which are both \( n \)-truncated and ♭-modal. It is not known whether \( ♭_n X = \| ♭X \|_n \) for general \( X \), but it is true for crisp \( X \) (see Proposition 4.5 of [17]).

We will now prepare ourselves by proving a few preservation properties of the \( ♭ \) comodality and the \( ♬ \) modality. A first time reader may content themselves with the statements of the lemmas, as the proofs are mere technicalities.

**Lemma 2.1.4.** The comodality \( ♭ \) preserves fiber sequences. Let \( f :: X \to Y \) be a crisp map and \( y :: Y \) a crisp point. Then we have an equivalence \( ♭\text{fib}_f(y) = \text{fib}_{♭f}(y^♭) \) such that
\[
\begin{array}{ccc}
♭\text{fib}_f(y) & \cong & \text{fib}_{♭f}(y) \\
\downarrow & & \downarrow \\
\text{fib}_{♭f}(y^♭) & \cong & \text{fib}_f(y^♭)
\end{array}
\]

commutes. In particular, \( ♭\text{fib}_f(y) \to ♭X \to ♭Y \) is a fiber sequence and that the naturality squares give a map of fiber sequences:
\[
\begin{array}{ccc}
♭\text{fib}_f(y) & \to & \text{fib}_f(y) \\
\downarrow & & \downarrow \\
♭X & \to & X \\
\downarrow & & \downarrow \\
♭Y & \to & Y
\end{array}
\]

**Proof.** We begin by constructing the equivalence:
\[
♭\text{fib}_f(y) = \text{fib}(\{(x : X) \times (f(x) = y)\})
\]
\[
= (u : ♭X) \times (\text{let } x^♭ := u \text{ in } ♭f(x) = y^♭)) \quad [22, \text{Lemma. 6.8}]
\]
\[
= (u : ♭X) \times (\text{let } x^♭ := u \text{ in } f(x)^♭ = y^♭)) \quad [22, \text{Theorem. 6.1}]
\]
\[
= (u : ♭X) \times (♭f(u) = y^♭)) \quad [22, \text{Lemma. 4.4}]
\]
\[
= \text{fib}_{♭f}(y^♭).
\]

We will need to understand what this equivalence does on elements \((x, p)^♭\) for \((x, p) :: \text{fib}_f(y)\). The first equivalence in the composite sends \((x, p)^♭\) to \((x^♭, p^♭)\), and no other equivalence affects the first component, so the first component of the result will be \(x^♭\). The second equivalence will send \(p^♭\) to \(\text{ap}_♭(−)^♭ p\), where \(\text{ap}_♭\) is the crisp application function. The next equivalence is given by reflexivity, since \(♭f(x^♭) \equiv f(x)^♭\). In total, then, this equivalence acts as
\[
(x, p)^♭ \mapsto (x^♭, \text{ap}_♭(−)^♭ p).
\]
Now, to show the triangle commutes, it will suffice to show that it commutes for \((x, p) \mapsto \underline{\text{fib}} f(y)\). This is to say, we need to show that sending \((x, p) \mapsto \underline{\text{fib}} f(y)\) through the above equivalence and then into \(\underline{\text{fib}} f(y)\) yields \((x, p) \mapsto \underline{\text{fib}} f(y)\). This is to say, we need to show that sending \((x, p) \mapsto \underline{\text{fib}} f(y)\) through the above equivalence and then into \(\underline{\text{fib}} f(y)\) yields \((x, p) \mapsto \underline{\text{fib}} f(y)\). The map \(\delta\) from \(\underline{\text{fib}}_y f(\underline{\text{fib}} f(y))\) to \(\underline{\text{fib}} f(y)\) sends \((u, q) \mapsto (u, \square_b u \cdot \text{ap}(\cdot))q\), where \(\square_b u : f(u) = b f(u)\) is the naturality square. So, the round trip \(\underline{\text{fib}} f(y) \rightarrow \underline{\text{fib}} f(\underline{\text{fib}} f(y)) \rightarrow \underline{\text{fib}} f(y)\) acts as \((x, p) \mapsto (x, p) \mapsto (x, p) \mapsto (x, \text{ap}(\cdot)p)\).

Now, \(x^b \equiv x\), so it remains to show that \(\square_b (x^b) \cdot \text{ap}(\cdot)p\) is an equivalence. However, the naturality square is defined by \(\square_b (x^b) \equiv \text{refl}(f(x^b)) : f(x^b) = b f(x^b),\) so it only remains to show that the two applications cancel. This can easily be shown by a crisp path induction.

**Lemma 2.1.5.** Let \(f : X \rightarrow Y\) be a crisp map between crisp types. The following are equivalent:

1. For every crisp \(y : Y\), \(\underline{\text{fib}} f(y)\) is discrete.
2. The naturality square

\[
\begin{array}{ccc}
\underline{\text{fib}} f(y) & \xrightarrow{(-)_b} & Y \\
\downarrow & & \downarrow \\
\underline{\text{fib}} f(y) & \xrightarrow{(-)_b} & Y
\end{array}
\]

is a pullback.

**Proof.** We note that the naturality square being a pullback is equivalent to the induced map

\[\underline{\text{fib}} f(u) \rightarrow \underline{\text{fib}} f(u)\]

begin an equivalence for all \(u : b Y\). By the universal property of \(b Y\), we may assume that \(u\) is of the form \(y^b\) for a crisp \(y : Y\). By Lemma 2.1.4, we have that

\[\underline{\text{fib}} f(y) \rightarrow \underline{\text{fib}} f(y)\]

commutes. Therefore, the naturality square is a pullback if and only if for all crisp \(y : Y\), we have that \((-)_b : \underline{\text{fib}} f(y) \rightarrow \underline{\text{fib}} f(y)\) is an equivalence; but this is precisely what it means for \(\underline{\text{fib}} f(y)\) to be discrete. \(\square\)

**Lemma 2.1.6.** Let \(X\) be a crisp type, and let \(a, b : X\) be crisp elements. Then there is an equivalence \(e : (a^b = b^b) \simeq b(a = b)\) together with a commutation of the following triangle:

\[
\begin{array}{ccc}
(a^b = b^b) & \xrightarrow{\text{ap}(\cdot)_b} & (a = b) \\
\downarrow & & \downarrow \\
\underline{b}(a = b) & \xrightarrow{(-)_b} & \underline{b}(a = b)
\end{array}
\]

**Proof.** For the construction of the equivalence \(e\) we refer to [22, Theorem. 6.1]. For the commutativity, we use function extensionality to work from \(u : \underline{b}(a = b)\) seeking \(e^{-1}(u) = u\) and proceed by \(\underline{b}\)-induction and then identity induction in which case both sides reduce to \(\text{refl}\). \(\square\)
**Lemma 2.1.7.** Let $G$ be a crisp higher group; that is, suppose that $BG$ is a crisp, 0-connected type and its base point $pt :: BG$ is also crisp. Then $\flat G$ is also a higher group and we may take

$$\flat \flat G \equiv \flat BG$$

pointed at $pt^\flat$. Furthermore, the counit $(-) : \flat G \rightarrow G$ is a homomorphism delooped by the counit $(-) : \flat BG \rightarrow BG$.

**Proof.** We need to show that $\flat \flatBG$ deloops $\flat G$ via an equivalence $e : \Omega \flat \flatBG = \flat G$, that it is 0-connected, and that looping the counit $(-) : \flat BG \rightarrow BG$ corresponds to the counit $(-) : \flat G \rightarrow G$ along the equivalence $e$.

For the equivalence $e : \Omega \flat \flatBG = \flat G$, we may take the equivalence $(pt^\flat = pt^\flat)$ of Lemma 2.1.6.

The commutation of the triangle

$$(pt^\flat = pt^\flat) \quad \xrightarrow{ap(-)_s} \quad (pt = pt) \quad \xrightarrow{(-)_s} \quad \flat(pt = pt)$$

shows that $(-)_s : \flat BG \rightarrow BG$ deloops $(-)_s : \flat G \rightarrow G$.

To show that $\flat \flatBG$ is connected, we rely on [22, Corollary. 6.7] which says that $\parallel \flatBG \parallel_0 = \parallel BG \parallel_0$, which is $\ast$ by the hypothesis that $BG$ is 0-connected. □

We end with a useful lemma: $\flat$ preserves long exact sequences of groups.

**Lemma 2.1.8.** The comodality $\flat$ preserves crisp short and long exact sequences of groups.

**Proof.** A sequence

$$0 \rightarrow K \rightarrow G \rightarrow H \rightarrow 0$$

of groups is short exact if and only if its delooping

$$BK \cdot \rightarrow BG \cdot \rightarrow BH$$

is a fiber sequence. But $\flat$ preserves crisp fiber sequences by Lemma 2.1.4 and by ?? we have that the fiber sequence

$$\flatBK \cdot \rightarrow \flatBG \cdot \rightarrow \flatBH$$

deloops the sequence $\flatK \rightarrow \flatG \rightarrow \flatH$, so this sequence is also short exact.

Now, a complex of groups

$$\cdots \rightarrow A_{n-1} d \rightarrow A_n d \rightarrow A_{n+1} \rightarrow \cdots$$

satisfying $d \circ d = 0$ is long exact if and only if the sequences

$$0 \rightarrow K_n \rightarrow A_n \rightarrow K_{n+1} \rightarrow 0$$

are short exact, where $K_n := \ker(A_n \rightarrow A_{n+1})$. Now, we have a complex

$$\cdots \rightarrow \flatA_{n-1} b \rightarrow \flatA_n b \rightarrow \flatA_{n+1} b \rightarrow \cdots$$

by the functoriality of $\flat$. Since $\flat$ preserves short exact sequences, the sequences

$$0 \rightarrow \flatK_n \rightarrow \flatA_n \rightarrow \flatA_{n+1} \rightarrow 0$$

are short exact. Now, since $\flat$ preserves fibers we have that

$$\flatK_n = \ker(\flatA_n \rightarrow \flatA_{n+1})$$

so that the $\flat$-ed complex is long exact. □
2.2 The Universal ∞-Cover of a Higher Group

An ∞-cover of a type $X$ is a generalization of the notion of cover from a theory concerning 1-types (the fundamental groupoid of $X$, with the universal cover being simply connected) to arbitrary types (the homotopy type of $X$, with the universal ∞-cover being contractible).

Recall that, classically, a covering map $\pi : \tilde{X} \to X$ satisfies a unique path lifting property; that is, every square of the following form admits a unique filler:

$$
\begin{array}{ccc}
\ast & \xrightarrow{\tilde{x}} & \tilde{X} \\
\downarrow & \swarrow & \searrow \\
\mathbb{R} & \xrightarrow{\gamma} & X
\end{array}
$$

This property can be extended to a unique lifting property against any map which induces an equivalence fundamental groupoids. That is, whenever $f : A \to B$ induces an equivalence $\int_1 f : \int_1 A \to \int_1 B$, every square of the following form admits a unique filler:

$$
\begin{array}{ccc}
A & \xrightarrow{\tilde{\gamma}} & \tilde{X} \\
\downarrow & \swarrow & \searrow \\
B & \xrightarrow{\gamma} & X
\end{array}
$$

For any modality $!$, there is an orthogonal factorization system where $!$-equivalences (those maps $f$ such that $!f$ is an equivalence) lift uniquely against $!$-étale maps ([18, 5]).

**Definition 2.2.1.** A map $f : A \to B$ is $!$-étale for a modality $!$ if the naturality square

$$
\begin{array}{ccc}
A & \xrightarrow{(-)!} & !A \\
f & \downarrow & \downarrow {!f} \\
B & \xrightarrow{(-)!} & !B
\end{array}
$$

is a pullback.

We may single out the covering maps as the $\int_1$-étale maps whose fibers are sets. For more on this point of view, see the last section of [17]. Here, however, we will be more concerned with $\int$-étale maps, which we will call ∞-covers. This notion was called a “modal covering” in [24], and was referred to as an ∞-cover in the setting of ∞-categories by Schreiber in [20].

**Definition 2.2.2.** A map $\pi : E \to B$ is an ∞-cover if the naturality square

$$
\begin{array}{ccc}
E & \xrightarrow{(-)!} & \int E \\
\pi & \downarrow & \downarrow \int\pi \\
B & \xrightarrow{(-)!} & \int B
\end{array}
$$

is a pullback. That is, an ∞-cover is precisely a $\int$-étale map.

A map $\pi : E \to B$ is an $n$-cover if it is $\int_{n+1}$-étale and its fibers are $n$-types. We call a 1-cover just a cover, or a covering map.

Theorem 6.1 of [17] gives a useful way for proving that a map is an ∞-cover.

**Proposition 2.2.3** (Theorem 6.1 of [17]). Let $\pi : E \to B$ and suppose that there is a crisp, discrete type $F$ so that for all $b : B$, $\|\text{fib}_\pi(b) = F\|$. Then $\pi$ is an ∞-cover.

**Example 2.2.4.** As an example of an ∞-cover, consider the exponential map $\mathbb{R} \to S^1$ from the real line to the circle. The fibers of this map are all merely $\mathbb{Z}$, so by Theorem 2.2.3, this map is an ∞-cover. Since $\mathbb{R}$ is contractible, it is in fact the universal ∞-cover of the circle.
Just as the universal cover of a space $X$ is any simply connected cover $\tilde{X}$ of a type $X$ is any contractible cover — contractible in the sense of being $f$-connected, meaning $f^\infty X = \ast$. Since units of a modality are modally connected, we may always construct a universal $\infty$-cover by taking the fiber of the $f$-unit $(-)^f : X \to fX$.

**Definition 2.2.5.** The universal $\infty$-cover of a pointed type $X$ is defined to be the fiber of the $f$-unit:

$$\tilde{X} := \text{fib}((-)^f : X \to fX).$$

Since the units of modalities are modally connected, $\tilde{X}$ is homotopically contractible:

$$f^\infty \tilde{X} = \ast.$$

Let’s take a bit to get an image of the universal $\infty$-cover of a type. The universal $\infty$-cover of a type only differs from its universal cover in the identifications between its points; in other words, it is a “stacky” version of the universal cover.

**Proposition 2.2.6.** Let $X$ be a crisp type. Then the map $\tilde{X} \to \tilde{X}$ from the universal $\infty$-cover of $X$ to its universal cover induced by the square

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi} & X \\
\downarrow & \downarrow & \downarrow \\
\tilde{X} & \xrightarrow{\pi} & fX
\end{array}$$

is $\|\|_0$-connected and $f$-modal. In particular, if $X$ is a set then $\|\tilde{X}\|_0 = \tilde{X}$. Furthermore, its fibers may be identified with the loop space $\Omega((fX)(1))$ of the first stage of the Whitehead tower of the shape of $X$.

**Proof.** We begin by noting that the unique factorization $f : fX \to \|1\|_1 X$ of the $\|1\|_1$ unit $(-)^{1} : X \to \|1\|_1 X$ through the $f$ unit is a $\|\|_1$ unit. We note that $(-)^1 : X \to \|1\|_1 X$ and $|\cdot|_{1} : X \to \|fX\|_1$ have the same universal property: any map from $X$ to a discrete 1-type factors uniquely through them. However, unless $X$ is crisp, we do not know that $\|fX\|_1$ is itself discrete; in general, we can only conclude that there is a map $\|fX\|_1 \to fX$. This is why we must assume that $X$ is crisp. By Proposition 4.5 of [17], $\|fX\|_1$ is discrete and therefore the map $\|fX\|_1 \to f \|1\|_1 X$ is an equivalence. Since $f$ factors uniquely through this map (since $f$ is a 1-type), we see that $f$ is equal to the $\|\|_1$ unit of $fX$ and is therefore a $\|\|_1$ unit. In particular, $f : fX \to \|1\|_1 X$ is 1-connected.

Now we will show that the fibers of the induced map $\tilde{X} \to \tilde{X}$ are 0-connected and discrete. Consider the following diagram:

$$\begin{array}{ccc}
\text{fib}(\rho) & \rightarrow & \ast \\
\downarrow & & \downarrow \\
\tilde{X} & \rightarrow & \text{fib}((\pi p)^{1}) \\
\downarrow & & \downarrow \\
\tilde{X} & \rightarrow & fX \\
\downarrow & & \downarrow \\
\tilde{X} & \rightarrow & f \|1\|_1 X
\end{array}$$

All vertical sequences are fiber sequences, and the bottom two sequences are fiber sequences; therefore, the top sequence is a fiber sequence, which tells us that the fiber over any point $p : \tilde{X}$ is equivalent to $\Omega \text{fib}_f((\pi p)^{1})$. As we have shown that the fibers of $f$ are 1-connected, their loop spaces are 0-connected. And as $f$ is a map between discrete types, its fibers are discrete and so their loop spaces are discrete. Finally, we note that the fiber of the 1-truncation $|\cdot|_{1} : fX \to \|fX\|_1$ is the first stage $(fX)(1)$ of the Whitehead tower of $fX$. 

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We are now ready to prove a simple sort of fracture theorem for any crisp, pointed type. This square will make up the left square of our modal fracture hexagon.

**Proposition 2.2.7.** Let \( X \) be a crisp, pointed type. Then the \( \flat \) naturality square of the universal \( \infty \)-cover \( \pi : X \to X \) is a pullback:

\[
\begin{array}{ccc}
\flat X & \xrightarrow{(-)_\flat} & \infty X \\
\downarrow & & \downarrow \pi \\
\flat X & \xrightarrow{(-)_\flat} & X
\end{array}
\]

**Proof.** By Lemma 2.1.5, it will suffice to show that over a crisp point \( x : X \), \( \text{fib} \pi (x) \) is discrete. But since \( \pi \) is, by definition, the fiber of \((-)_\flat \), we have that

\[ \text{fib}_\pi (x) = \Omega (f X, x^\flat). \]

Since \( f X \) is discrete by assumption, so is \( \Omega (f X, x^\flat) \).

Although \( f \) is not a left exact modality — it does not preserve all pullbacks — it does preserve pullbacks and fibers of \( f \)-fibrations. The theory of modal fibrations was developed in [17]. Included amongst the \( f \)-fibrations are the \( f \)-étale maps, and so \( f \) preserves fiber sequences of \( f \)-étale maps.

**Lemma 2.2.8.** Let \( f : X \to Y \) be an \( \infty \)-cover. Then for any \( y : Y \), the sequence

\[ \text{fib}_f (y) \to f X \xrightarrow{f f} f Y \]

is a fiber sequence.

**Proof.** Since a \( f \)-étale map \( f \) is modal, its étale and modal factors agree (they are equivalently \( f \)), so by Theorem 1.2 of [17], \( f \) is a \( f \)-fibration. The result then follows since \( f \) preserves all fibers of \( f \)-fibrations (see also Theorem 1.2 of [17]).

Importantly, it is also true that the shape of a crisp \( n \)-connected type is also \( n \)-connected by Theorem 8.6 of [17]. It follows that \( f BG \) is a delooping of \( f G \) for crisp higher groups \( G \), and that this can continue for higher deloopings.

**Proposition 2.2.9.** Let \( G \) be a crisp higher group. Then its universal \( \infty \)-cover \( \infty G \) is a higher group and \( \pi : G \to G \) is a homomorphism. Furthermore, if \( G \) is \( k \)-commutative, then so is \( \infty \).

**Proof.** We may define

\[ B^i G \coloneqq \text{fib}((-)_\flat ^i) : B^i G \to \text{fib} B^i G. \]

This lets us extend the fiber sequence:

\[
\begin{array}{ccc}
\infty G & \xrightarrow{\pi} & G \\
\downarrow & & \downarrow (-)_\flat \\
\infty B G & \xrightarrow{B \pi} & BG \\
\downarrow & & \downarrow (-)_\flat \\
\infty B^2 G & \xrightarrow{B^2 \pi} & B^2 G \\
\downarrow & & \downarrow (-)_\flat \\
\vdots
\end{array}
\]

\[ \Box \]
2.3 The Infinitesimal Remainder of a Higher Group

In this section, we will investigate the infinitesimal remainder $\theta : G \to g$ of higher group $G$. The infinitesimal remainder is what is left of a higher group when all of its crisp points have been made equal. Having trivialized all substantial difference between points, we are left with the infinitesimal differences that remain.

**Definition 2.3.1.** Let $G$ be a higher group. Define its infinitesimal remainder to be

$$g \equiv \text{fib}(\mathbb{1}_G : bBG \to BG).$$

Then, continuing the fiber sequence, we have

$$\begin{align*}
\mathbb{1}_G &\xrightarrow{\mathbb{1}_\theta} G \xrightarrow{\theta} g \\
\mathbb{1}B G &\longrightarrow BG
\end{align*}$$

which defines the quotient map $\theta : G \to g$.

**Remark 2.3.2.** By its construction, we can see that $g$ modulates flat connections on trivial principal $G$-bundles, with respect to the interpretation of $bBG$ given in Remark 2.1.3. In the setting of differential geometry, such flat connections on trivial principal $G$-bundles are given by closed $g$-valued 1-forms, where here $g$ is the Lie algebra of the Lie group $G$. In this setting, $\theta$ is the Mauer-Cartan form on $G$. This is why we adopt the name $\theta : G \to g$ for the infinitesimal remainder in general. This can in fact be proven in the setting of synthetic differential geometry with tiny infinitesimals satisfying a principle of constancy using a purely modal argument. See Remark 3.1.2 for a further discussion.

**Remark 2.3.3.** The infinitesimal remainder $g$ is defined as the de Rham coefficient object of $BG$ in Definition 5.2.59 of [20]. Schreiber defined $\mathbb{b}dR X$ for any (crisp) pointed type $X$ as the fiber of $(\mathbb{1}_\theta : bX \to X$, so that $g \equiv \mathbb{b}dR BG$. We focus on the case that $X$ is 0-connected — of the form $BG$ — and so only consider the infinitesimal remainder of a higher group $G$.

While the infinitesimal remainder exists for any (crisp) higher group, it is not necessarily itself a higher group. However, if $G$ is braided, then $g$ will be a higher group.

**Proposition 2.3.4.** If $G$ is a crisp $k$-commutative higher group, then $g$ is a $(k-1)$-commutative higher group. In particular, if $G$ is a braided higher group, then $g$ is a higher group and the remainder map $\theta : G \to g$ is a homomorphism.

**Proof.** We may define

$$B^i g \equiv \text{fib}(\mathbb{1}_{\mathbb{1}^i G} : bB^{i+1}G \to B^{i+1}G),$$

which lets us continue the fiber sequence:

$$\begin{align*}
\mathbb{1}_G &\xrightarrow{\mathbb{1}_\theta} G \xrightarrow{\theta} g \\
\mathbb{1}B G &\longrightarrow BG \xrightarrow{B\theta} Bg \\
\mathbb{1}B^2 G &\longrightarrow B^2 G \xrightarrow{B^2 \theta} B^2 g \\
\cdots
\end{align*}$$

**Remark 2.3.5.** We can see the delooping $B\theta : BG \to Bg$ of the infinitesimal remainder $\theta : G \to g$ as taking the curvature of a principal $G$-bundle, in that $B\theta$ is an obstruction to the flatness of that bundle since

$$bBG \to BG \to Bg$$

is a fiber sequence.
As with any good construction, the infinitesimal remainder is functorial in its higher group. This is defined easily since the infinitesimal remainder is constructed as a fiber.

**Definition 2.3.6.** Let \( f :: G \to H \) be a crisp homomorphism of higher groups with delooping \( Bf : BG \to BH \). Then we have a pushforward \( f_* : g \to h \) given by \((t,p) \mapsto (\flat Bf(t), (ap Bf \circ p) \circ pt_Bf)\). This is the unique map fitting into the following diagram:

\[
\begin{array}{ccc}
\mathbb{g} & \xrightarrow{f_*} & \mathbb{h} \\
\downarrow & & \downarrow \\
\flat BG & \xrightarrow{\flat Bf} & \flat BH \\
\downarrow & & \downarrow \\
BG & \xrightarrow{Bf} & BH
\end{array}
\]

If \( G \) and \( H \) are \( k \)-commutative and \( f \) is a \( k \)-commutative homomorphism, then \( f_* \) admits a unique structure of a \((k-1)\)-commutative homomorphism by defining \( B^{k-1}f_* \) to be the map induced by \( \flat B^k f \) on the fiber.

We record a useful lemma: the fibers of the quotient map \( \theta : G \to g \) are all are identifiable with \( \flat G \).

**Lemma 2.3.7.** Let \( G \) be a crisp higher group. For \( t : g \), we have \( \|\text{fib}_b(t) = \flat G\| \).

**Proof.** By definition, \( t : g \) is of the form \((T, p)\) for \( T : \flat BG \) and \( p : T_b = \text{pt}_{BG} \). Since \( \flat BG \) is \( 0 \)-connected and we are trying to prove a proposition, we may suppose that \( q : T = \text{pt}_{BG} \). We then have that \( t = (\text{pt}_{BG}, (-)_{b^*}(q) \bullet p) \), and therefore:

\[
\text{fib}_b(t) \equiv (g : G) \times ((\text{pt}_{BG}, g) = (\text{pt}_{BG}, (-)_{b^*}(q) \bullet p)) = (g : G) \times (a : \flat G) \times (a_{b^*} \bullet g = (-)_{b^*}(q) \bullet p)
\]

\[
= (g : G) \times (a : \flat G) \times (g = a_{b^*}^{-1}(-)_{b^*}(q) \bullet p) = \flat G.
\]

The infinitesimal remainder is *infinitesimal* in the sense that it has a single crisp point.

**Proposition 2.3.8.** Let \( G \) be a higher group. Then its infinitesimal remainder \( g \) is infinitesimal in the sense that

\[ \flat g = \ast. \]

**Proof.** By Lemma 2.1.4, \( \flat \) preserves the fiber sequence

\[ \mathbb{g} \to \flat BG \to BG. \]

But \( \flat(-)_b : \flat\flat BG \to \flat BG \) is an equivalence by Theorem 6.18 of [22], so \( \flat g \) is contractible.

Despite being infinitesimal, we will see that \( g \) has (in general) a highly non-trivial homotopy type.

**Remark 2.3.9.** The infinitesimal remainder \( g \) is of special interest when \( G \) is a Lie group, since in this case the vanishing of the cohomology groups \( H^*(g; \mathbb{Z}/p) \) for all primes \( p \) is equivalent to the Friedlander-Milnor conjecture. The fact that this conjecture remains unproven is a testament to the intricacy of the homotopy type of the infinitesimal space \( g \).

We note that \( g \) itself represents an obstruction to the discreteness of \( G \).

**Proposition 2.3.10.** A crisp higher group \( G \) is discrete if and only if its infinitesimal remainder \( g \) is contractible.

**Proof.** If \( G \) is discrete, then \((-)_b : g \to G \) is an equivalence and so \((-)_b : \flat BG \to BG \) is an equivalence; this implies that \( g = \ast \). On the other hand, if \( g = \ast \) then \((-)_b : \flat BG \to BG \) is an equivalence and so its action on loops is an equivalence. \( \square \)
Using Proposition 2.2.3, we can quickly show that \( \theta : G \to \mathbb{g} \) is an infinite-cover. This gives us the right hand pullback square in our modal fracture hexagon.

**Proposition 2.3.11.** Let \( G \) be a crisp infinite-group. Then the infinitesimal remainder \( \theta : G \to \mathbb{g} \) is an infinite-cover. In particular, the \( j \)-naturality square:

\[
\begin{array}{ccc}
G & \xrightarrow{\theta} & \mathbb{g} \\
\downarrow \rho & & \downarrow \rho \\
jG & \xrightarrow{j\theta} & j\mathbb{g}
\end{array}
\]

is a pullback. If, furthermore, \( G \) is (crisply) an \( n \)-type, then \( \theta \) is an \( (n+1) \)-cover.

**Proof.** By Proposition 2.2.3, to show that \( \theta : G \to \mathbb{g} \) is an infinite-cover (resp. an \( (n+1) \)-cover) it suffices to show that the fibers are merely equivalent to a crisply discrete type (resp. a crisply discrete \( n \)-type). But by Lemma 2.3.7, the fibers of \( \theta : G \to \mathbb{g} \) are all merely equivalent to \( \flat G \) which is crisply discrete (and, by Theorem 6.6 of [22], if \( G \) is (crisply) an \( n \)-type then so is \( \flat G \)). \( \square \)

There is a sense in which the infinitesimal remainder of a higher group behaves like its Lie algebra. Just as the Lie algebra of a Lie group is the same as the Lie algebra of its universal cover, we can show that the infinitesimal remainder of a higher group is the same as that of its universal infinite-cover.

**Proposition 2.3.12.** Let \( G \xrightarrow{\phi} H \xrightarrow{\psi} K \) be a crisp exact sequence of higher groups. Then

1. \( K \) is discrete if and only if \( \phi^* : \mathbb{g} \to \mathbb{h} \) is an equivalence.
2. \( G \) is discrete if and only if \( \psi^* : \mathbb{h} \to \mathbb{k} \) is an equivalence.

**Proof.** We consider the following diagram in which each horizontal and vertical sequence is a fiber sequence:

\[
\begin{array}{cccc}
\mathbb{g} & \xrightarrow{\mathbb{h}} & \xrightarrow{\mathbb{k}} \\
\downarrow & & \downarrow \\
\flat BG & \xrightarrow{\flat BH} & \xrightarrow{\flat BK} \\
\downarrow & & \downarrow \\
BG & \xrightarrow{BH} & \xrightarrow{BK}
\end{array}
\]

If \( K \) is discrete, then \( k = * \) and so \( \mathbb{g} = \mathbb{h} \). On the other hand, if \( \mathbb{g} = \mathbb{h} \), then the bottom left square of the above diagram is a pullback. Therefore, the it induces an equivalence on the fibers of the horizontal maps:

\[
\begin{array}{cccc}
\flat K & \xrightarrow{\flat BG} & \xrightarrow{\flat BH} \\
\downarrow & & \downarrow \\
K & \xrightarrow{BG} & \xrightarrow{BH}
\end{array}
\]

This shows that \( K \) is discrete.

If \( \psi^* : \mathbb{h} \to \mathbb{k} \) is an equivalence, then its fiber \( \mathbb{g} \) is contractible. Therefore, \( G \) is discrete. On the other hand, if \( G \) is discrete, then the bottom right square is a pullback, and therefore the induced map on vertical fibers is an equivalence. This map is \( \psi^* : \mathbb{h} \to \mathbb{k} \). \( \square \)

**Corollary 2.3.13.** The universal infinite-cover \( \pi : \overset{\infty}{G} \to G \) induces an equivalence \( \overset{\infty}{\mathbb{g}} = \mathbb{g} \) fitting into the following commutative diagram:

\[
\begin{array}{cccc}
\overset{\infty}{\mathbb{g}} & \xrightarrow{\mathbb{g}} & \xrightarrow{*} \\
\downarrow & & \downarrow \\
\flat \overset{\infty}{BG} & \xrightarrow{\flat BG} & \xrightarrow{\flat jBG} \\
\downarrow & & \downarrow \\
\overset{\infty}{BG} & \xrightarrow{BG} & \xrightarrow{jBG}
\end{array}
\]
In particular, this gives us a long fiber sequence

\[
\begin{array}{c}
\vdots \\
\mathbb{B}G \\ B\pi \\
\end{array} \\
\mathbb{B}G \xrightarrow{(-)} \mathbb{G} \xrightarrow{\theta} G \\
\vdots \\
\mathbb{B}G \xrightarrow{\mathbb{B}\pi} \mathbb{B}G \\
\end{array}
\]

which forms the top fiber sequence of the modal fracture hexagon.

### 2.4 The Modal Fracture Hexagon

We have seen the two main fiber sequences

\[
\begin{array}{c}
\mathbb{G} \xrightarrow{\pi} G \xrightarrow{(-)^f} \int G \\
\mathbb{B}G \xrightarrow{\mathbb{B}\pi} \mathbb{B}G \xrightarrow{(-)^f} \int \mathbb{B}G \\
\end{array}
\]

and

\[
\begin{array}{c}
\mathbb{G} \xrightarrow{(-)} G \xrightarrow{\theta} \int g \\
\mathbb{B}G \xrightarrow{\mathbb{B}G} \mathbb{B}G \\
\end{array}
\]

associated to a higher group \( G \). Now, when we apply \( b \) to the left sequence and \( f \) to the right sequence, we find the sequences

\[
\begin{array}{c}
b\mathbb{G} \xrightarrow{b\pi} bG \xrightarrow{(-)^f\circ(-)} \int G \\
b\mathbb{B}G \xrightarrow{b\mathbb{B}\pi} b\mathbb{B}G \xrightarrow{(-)^f\circ(-)} \int \mathbb{B}G \\
\end{array}
\]

and

\[
\begin{array}{c}
b\mathbb{G} \xrightarrow{(-)^f\circ(-)} \int G \xrightarrow{f\theta} \int \mathbb{g} \\
b\mathbb{B}G \xrightarrow{(-)^f\circ(-)} \int \mathbb{B}G \\
\end{array}
\]

which are the same sequence, just shifted over. This gives us the bottom exact sequence of our modal fracture hexagon, reading the sequence on the left. But it also proves that \( \int \mathbb{g} = b\mathbb{B}G \), which gives us the top exact sequence of our modal fracture hexagon by Corollary 2.3.13.

Of course, we need to be able to apply \( f \) freely to fiber sequences to fulfil this argument. But \( f \) is not left exact, and so does not preserve fiber sequences in general. Luckily, Theorem 6.1 of [17] gives us a trick for showing that a map is a \( f \)-fibration which allows us to prove this general lemma.

**Lemma 2.4.1.** Let \( G \) be a crisp higher group (that is, \( B\mathbb{G} \) is a crisply pointed 0-connected type). Then any crisp map \( f :: X \rightarrow B\mathbb{G} \) is a \( f \)-fibration.

**Proof.** Since \( B\mathbb{G} \) is 0-connected, all the fibers of \( f \) are merely equivalent to the fiber \( \text{fib}_f(\text{pt}) \) over the basepoint. Therefore, their homotopy types are merely equivalent to \( \int \text{fib}_f(\text{pt}) \), which is a crisp, discrete type. It follows by Theorem 6.1 of [17] that \( f \) is a \( f \)-fibration. \( \square \)

This means that we can freely apply \( f \) to crisp fiber sequences of 0-connected types. This concludes our proof of the main theorem.

**Theorem 2.4.2.** For a crisp \( \infty \)-group \( G \), there is a modal fracture hexagon:

\[
\begin{array}{c}
\mathbb{G} \xrightarrow{(-)^f} \mathbb{G} \xrightarrow{\theta} G \\
\mathbb{B}G \xrightarrow{\mathbb{B}\pi} \mathbb{B}G \\
\end{array}
\]

where
• $\theta : G \to g$ is the infinitesimal remainder of $G$, the quotient $G \sslash \♭ G$, and
• $\pi : G \to G$ is the universal (contractible) $\infty$-cover of $G$.

Moreover,

1. The middle diagonal sequences are fiber sequences.
2. The top and bottom sequences are fiber sequences.
3. Both squares are pullbacks.

Furthermore, the homotopy type of $g$ is a delooping of $\♭ G$:

$$fg = \♭ B^\infty G.$$

Therefore, if $G$ is $k$-commutative for $k \geq 1$ (that is, admits further deloopings $B^{k+1} G$), then we may continue the modal fracture hexagon on to $B^k G$.

Proof. We assemble the various components of the proof here.

1. The middle diagonal sequences are fiber sequences by definition (see Definition 2.2.5 and Definition 2.3.6).
2. The top sequence was shown to be a fiber sequence in Corollary 2.3.13. We showed that the bottom sequence is a fiber sequence at the beginning of this subsection.
3. The left square was shown to be a pullback in Proposition 2.2.7, and the right sequence in Proposition 2.3.11.

Finally, we calculated the homotopy type of $g$ at the beginning of this subsection.

3 Ordinary Differential Cohomology

In this section, we will use modal fracture to construct ordinary differential cohomology in cohesive homotopy type theory. We will recover a differential hexagon for ordinary differential cohomology which very closely resembles the classical hexagon; however, as de Rham’s theorem does not hold for all types, we will not recover the classical hexagon exactly. For more discussion of these subtleties, see Section 3.3.

In [2], Bunke, Nikolaus, and Vokel show that differential cohomology theories can be understood as spectra in the $\infty$-topos of sheaves on a site of manifolds. Schreiber notes in Proposition 4.4.9 of [20] that the simpler site consisting of Euclidean spaces and smooth maps between them yields the same topos of sheaves, and proves in Proposition 4.4.8 that this $\infty$-topos is cohesive. This topos, and the similar $\infty$-Dubuc topos (called the $\infty$-Cahiers topos in Remark 4.5.6 and $\text{SynthDiff} \times \text{Grpd}$ in Definition 4.5.7 of ibid.), will be our intended model for cohesive homotopy type theory in this section.

The theme of this paper is that the main feature of differential cohomology — the differential cohomology hexagon — is not of a particularly differential character, but arises from the more basic opposition between an adjoint modality $f$ and comodality $\♭$. As we saw in the previous section, in the presence of these (co)modalities, any higher group may be fractured in a manner resembling the differential cohomology hexagon.

We will take a similarly general view in constructing ordinary differential cohomology. The key idea in ordinary differential cohomology is the equipping of differential form data to integral cohomology. We will therefore focus on cohomology theories (in particular, $\infty$-commutative higher groups or connective spectra) which arise by equipping an existing cohomology theory with extra data representing the cocycles. Our exposition will focus on ordinary differential cohomology, but this extra generality will enable us to define combinatorial analogues of ordinary differential cohomology as well (see Section 3.5).
3.1 Assumptions and Preliminaries

For this section, we make the following assumption.

**Assumption 1.** In cohesive homotopy type theory with the axioms of synthetic differential geometry, tiny infinitesimal varieties, and a principle of constancy, we have a contractible and infinitesimal resolution of $U(1)$

$$0 \rightarrow bU(1) \rightarrow U(1) \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \cdots$$

(3.1.1)
given by the differential $k$-form classifiers $\Lambda^k$. That is:

- The $\Lambda^k$ are crisp $\mathbb{R}$-vector spaces.
- The maps $d : \Lambda^k \rightarrow \Lambda^{k+1}$ are crisply $♭\mathbb{R}$-linear (not $\mathbb{R}$-linear!), and the sequence Eq. (3.1.1) is crisply long exact.
- The $\Lambda^k$ are infinitesimal: $♭\Lambda^k = \ast$. Therefore also the closed $k$-form classifiers $\Lambda^k_{\text{cl}} := \ker(d : \Lambda^k \rightarrow \Lambda^{k+1})$ are infinitesimal.

Here, $U(1) \equiv \{ z : \mathbb{C} \mid z \bar{z} = 1 \}$ is the abelian group of units in the smooth complex numbers, which are defined as $\mathbb{C} \equiv \mathbb{R}[i]/(i^2 + 1)$ where $\mathbb{R}$ are the smooth real numbers presumed by synthetic differential geometry.

**Remark 3.1.2.** For reasons of space, we will not justify this assumption in this paper. In forthcoming work, we will show how one can construct the form classifiers and their long exact sequence

$$0 \rightarrow b\mathbb{R} \rightarrow \mathbb{R} \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \rightarrow \cdots$$

(3.1.3)
from the axioms of synthetic differential geometry with tiny infinitesimals and a principle of constancy. Synthetic differential geometry is an axiomatic system for working with nilpotent infinitesimals put forward first by Lawvere [11] and developed by Bunge, Dubuc, Kock, Wraith, and others. It admits a model in sheaves on infinitesimally extended Euclidean spaces, known as the Dubuc topos (or Cahiers topos) [7]; for a review of models see [16]. The Dubuc topos is cohesive, and is our intended model for this section.

It was noted by Lawvere [13] that the exceptional projectivity enjoyed by the infinitesimal interval $\mathbb{D} = \{ \epsilon : \mathbb{R} \mid \epsilon^2 = 0 \}$ was equivalent to the existence of an (external) right adjoint to the exponential functor $X \mapsto X^{\mathbb{D}}$. We will follow Yetter [25] in calling objects $T$ for which the functor $X \mapsto X^T$ admits a right adjoint tiny objects. Lawvere and Kock showed how one could use this “amazing” right adjoint to construct the form classifiers $\Lambda^k$ (see Section I.20 of [9] for a construction of $\Lambda^1$).

However, working with the form classifiers was difficult in synthetic differential geometry since the adjoint which defines them only exists externally. This may be remedied by using Shulman’s Cohesive HoTT, where the $♭$ modality allows for an internalization of the external. This allows us to give a fully internal theory of the form classifiers. We will, however, post-pone a discussion of this internal theory of tiny objects to future work.

The principle of constancy says that if the differential of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ vanishes uniformly, then $f$ is constant. This extra principle has been long considered in synthetic differential geometry (see, for example, the second chapter of [15]), but when combined with real cohesion it implies the exactness of the sequence

$$0 \rightarrow b\mathbb{R} \rightarrow \mathbb{R} \xrightarrow{d} \Lambda^1$$

and so begins the theory of differential cohomology we will see shortly. The interaction with cohesion is non-trivial in many ways for synthetic differential geometry. For example, the principle of constancy in the presence of real cohesion implies the existence of primitives, and the exponential functions $\exp(-) : \mathbb{R} \rightarrow \mathbb{R}^+$ and $\exp(2\pi i \cdot) : \mathbb{R} \rightarrow U(1)$ (where $U(1) := \{ z : \mathbb{C} \mid z \bar{z} = 1 \}$).

**Remark 3.1.4.** Externally, the smooth reals $\mathbb{R}$ correspond to the sheaf of smooth real valued functions, $U(1)$ corresponds to the sheaf of smooth $U(1)$-valued functions, and $\Lambda^k$ is the sheaf sending a manifold to its set of differential $k$-forms.
We have assumed the existence of a crisp long exact sequence of abelian groups in which each of the $\Lambda^i$ are real vector spaces (but $d : \Lambda^i \to \Lambda^{i+1}$ are not $\mathbb{R}$-linear). These are to be the differential form classifiers, which externally are the sheaves of $\mathbb{R}$-valued $n$-forms on manifolds (or infinitesimally extended manifolds).

If we define

$$
\Lambda^n_{cl} := \ker(d : \Lambda^n \to \Lambda^{n+1})
$$

to be the closed $n$-form classifier, then we can reorganize the long exact sequence (3.1.3) into a series of short exact sequences of abelian groups

$$
0 \to \Lambda^n_{cl} \to \Lambda^n \xrightarrow{d} \Lambda^{n+1}_{cl} \to 0.
$$

The first of these short exact sequences is

$$
0 \to \mathbb{B} \mathbb{R} \to \mathbb{R} \xrightarrow{d} \Lambda^1_{cl} \to 0
$$

which we may extend into a long fiber sequence

$$
\begin{array}{c}
\mathbb{B} \mathbb{R} \\
\downarrow
\end{array} \xrightarrow{(-)} \begin{array}{c}
\mathbb{R} \\
\downarrow
\end{array} \xrightarrow{d} \begin{array}{c}
\Lambda^1_{cl} \\
\downarrow
\end{array} \xrightarrow{} \cdots
$$

This shows that $\Lambda^1_{cl}$ is the infinitesimal remainder of the additive Lie group $\mathbb{R}$. Since $\mathbb{R}$ has contractible shape by definition, we see that $\mathbb{R} \xrightarrow{d} \Lambda^1_{cl}$ is the universal $\infty$-cover of $\Lambda^1_{cl}$. This gives us the following theorem, a form of de Rham’s theorem in smooth cohesion.

**Lemma 3.1.5.** The $n$-form classifiers $\Lambda^n$ are contractible.

**Proof.** This follows immediately from the assumption that they are real vector spaces. By Lemma 6.9 of [17], to show that $\Lambda^n$ is contractible it suffices to give for every $\omega : \Lambda^n$ a path $\gamma : \mathbb{R} \to \Lambda^n$ from $\omega$ to $0$. We can of course define

$$
\gamma(t) := t \omega
$$

which gives our desired contraction. □

**Theorem 3.1.6.** Let $\Lambda^n_{cl} := \ker(d : \Lambda^n \to \Lambda^{n+1})$ be the closed $n$-form classifier. Then

$$
\int \Lambda^n_{cl} = \mathbb{B}^n \mathbb{R}.
$$

**Proof.** Since $\Lambda^1_{cl}$ is the infinitesimal remainder of $\mathbb{R}$, this follows from Theorem 2.4.2:

$$
\int \Lambda^1_{cl} = \mathbb{B} \mathbb{R}.
$$

We then proceed by induction. We have a short exact sequence of abelian groups

$$
0 \to \Lambda^n_{cl} \to \Lambda^n \xrightarrow{d} \Lambda^{n+1}_{cl} \to 0.
$$

We note that since $d : \Lambda^n \to \Lambda^{n+1}_{cl}$ is an abelian group homomorphism, all of its fibers are identifiable with the crisp type $\Lambda^n_{cl}$ and therefore, by the “good fibrations” trick (Theorem 6.1 of [17]), it is a $\int$-fibration. Therefore, we get a fiber sequence

$$
\int \Lambda^n_{cl} \to \int \Lambda^n \to \int \Lambda^{n+1}_{cl}.
$$

Now, since $\Lambda^n$ is contractible by Lemma 3.1.5, we see that

$$
\Omega \int \Lambda^{n+1}_{cl} = \int \Lambda^n_{cl}
$$

By inductive hypothesis, $\int \Lambda^n_{cl} = \mathbb{B}^n \mathbb{R}$, so all that remains is to show that $\int \Lambda^{n+1}_{cl}$ is $n$-connected. We will do this by showing that for any $u : \int \Lambda^{n+1}_{cl}$, the loop space $\Omega(\int \Lambda^{n+1}_{cl}, u)$ is $(n - 1)$-connected. By Corollary 9.12 of [22], the $\int$-unit $(-)^\partial : \Lambda^{n+1}_{cl} \to \int \Lambda^{n+1}_{cl}$ is surjective, so there exists an $\omega : \Lambda^{n+1}_{cl}$ with $u = \omega^\partial$. We then have a fiber sequence
\[ \text{fib}_d(\omega) \to \Lambda^n \xrightarrow{d} \Lambda_n^{n+1} \]

which, since \(\Lambda^n \xrightarrow{d} \Lambda_n^{n+1}\) is a \(f\)-fibration descends to a fiber sequence

\[ \int \text{fib}_d(\omega) \to \int \Lambda_n \xrightarrow{f} \int \Lambda_n^{n+1}. \]  

(3.1.7)

Since \(d\) is surjective, there is a \(\alpha : \Lambda^n\) with \(d\alpha = \omega\), and we may therefore contract \(\int \Lambda^n\) onto \(\alpha\). This lets us equate the sequence (3.1.7) with the sequence

\[ \Omega(\int \Lambda_n^{n+1}, \omega^f) \to * \xrightarrow{\omega^f} \int \Lambda_n^{n+1}. \]

But as \(\alpha : \Lambda^n \to \Lambda_n^{n+1}\) is an abelian group homomorphism, its fibers are all identifiable with its kernel \(\Lambda_n^0\); this means that \(\Omega(\int \Lambda_n^{n+1}, u)\) is identifiable with \(\int \Lambda_n^0\), which by inductive hypothesis is \((n-1)\)-connected. \(\square\)

We may understand this theorem as a form of de Rham theorem in smooth cohesive homotopy type theory. We may think of the unit \((-)^f : \Lambda_n^0 \to bB^n R\) as giving the de Rham class of a closed \(n\)-form. That this map is the \(f\)-unit says that this is the universal discrete cohomological invariant of closed \(n\)-forms. Explicitly, if \(E_k\) is a loop spectrum, then \(H^k(\Lambda_n^0, E_k) := ||\Lambda_n^0 \to E_k||_0\). Therefore, if the \(E_k\) are discrete, then any cohomology class \(c : \Lambda_n^0 \to E_k\) factors through the de Rham class \((-)^f : \Lambda_n^0 \to bB^n R\). In this sense, every discrete cohomological invariant of closed \(n\)-forms is in fact an invariant of their de Rham class in discrete real cohomology.

### 3.2 Circle \(k\)-Gerbes with Connection

We can now go about defining ordinary differential cohomology. We understand ordinary differential cohomology as equipping integral cohomology with differential form data. Hopkins and Singer define (Definition 2.4 of [8]) a differential cocycle of degree \(k+1\) on \(X\) to be a triple \((c, h, \omega)\) consisting of an underlying cocycle \(c \in Z^{k+1}(X, Z)\) in integral cohomology, a curvature form \(\omega \in \Lambda_{k+1}^1(X)\), and a monodromy term \(h \in C^k(X, R)\) satisfying the equation \(dh = \omega - c\).

We will follow their lead, at least in spirit. In true homotopy type theoretic fashion, we will define the classifying types first and then derive the cohomology theory by truncation.

**Definition 3.2.1.** We define the classifier \(B_U^k(1)\) of degree \((k+1)\) classes in ordinary differential cohomology to be the pullback:

\[
\begin{array}{ccc}
\text{B}_U^k(1) & \xrightarrow{F} & \Lambda_{n+1}^k \\
\downarrow & & \downarrow \alpha \\
B^{k+1} Z & \xrightarrow{(-)^f} & bB^{k+1} R
\end{array}
\]

Therefore, a cocycle \(\tilde{c} : X \to B_U^k(1)\) in differential cohomology will consist of an underlying cocycle \(c : X \to B^{k+1} Z\), a curvature form \(\omega : X \to \Lambda_{k+1}^n\), together with an identification \(h : c = \omega \) in \(X \to bB^{k+1} R\). Since \(h\) lands in types identifiable with \(\Omega bB^{k+1} R\), which equals \(bB^{k} R\), we may consider it as the monodromy term in discrete real cohomology. We will now set about justifying this terminology.

We may note immediately from this definition that the map \(\text{B}_U^k(1) \to B^{k+1} Z\) (which we may think of as taking the underlying class in ordinary cohomology) is the \(f\)-unit. This means that the underlying cocycle is the universal discrete cohomological invariant of a differential cocycle.

**Lemma 3.2.2.** The pullback square

\[
\begin{array}{ccc}
\text{B}_U^k(1) & \xrightarrow{F} & \Lambda_{n+1}^k \\
\downarrow & & \downarrow \alpha \\
B^{k+1} Z & \xrightarrow{(-)^f} & bB^{k+1} R
\end{array}
\]

is a \(f\)-naturality square. That is, \(B_U^k(1) \to B^{k+1} Z\) is a \(f\)-unit.
Proof. Since $B^k_U(1) \to B^{k+1}Z$ is a map into a $j$-modal type, to show that it is a $j$ unit it suffices to show that it is $j$-connected. Since we have a pullback square, the fibers of $B^k_U(1) \to B^{k+1}Z$ are the same as those of $(-)^j : \Lambda^{k+1}_N \to \flat B^{k+1}R$. But as this map is a $j$-unit, its fibers are $j$-connected. □

The reason for our change of index — defining $B^k_U(1)$ to represent degree $(k + 1)$ classes — is because we would like to think of $B^k_U(1)$ as more directly classifying connections on $k$-gerbes with band $U(1)$. To reify this idea, let’s give the map $B^k_U(1) \to B^kU(1)$ which we think of as taking the underlying $k$-gerbe.

Construction 3.2.3. We construct a map $B^k_U(1) \to B^kU(1)$ which makes the following triangle commute:

$$
\begin{array}{ccc}
B^k_U(1) & \longrightarrow & B^kU(1) \\
\downarrow (-)^j & & \downarrow (-)^j \\
B^{k+1}Z & \longrightarrow & (-)^j
\end{array}
$$

Construction. Since $R \to U(1)$ is the universal $\infty$-cover of $U(1)$, by Corollary 2.3.13, $U(1)$ has the same infinitesimal remainder as $R$, which is $\Lambda^1_\text{cl}$. Therefore, by modal fracture Theorem 2.4.2, we have a pullback square

$$
\begin{array}{ccc}
B^kU(1) & \longrightarrow & B^k\Lambda^1_\text{cl} \\
\downarrow (-)^j & \downarrow & \downarrow \\
\Lambda^N_\text{cl} & \longrightarrow & \flat B^{k+1}R
\end{array}
$$

Now, since we have a series of short exact sequences

$$0 \to \Lambda^n_{\text{cl}} \to \Lambda^n \to \Lambda^{n+1}_{\text{cl}} \to 0$$

we have long fiber sequences

$$
\begin{array}{c}
\Lambda^n_{\text{cl}} \longrightarrow \Lambda^n \longrightarrow \Lambda^{n+1}_{\text{cl}} \\
\biggm\downarrow \quad \quad \quad \quad \biggm\downarrow \\
\text{BA}^n_{\text{cl}} \longrightarrow \text{BA}^n \longrightarrow \ldots
\end{array}
$$

for each $n$. In particular, we have maps $B^n\Lambda^{m+1}_{\text{cl}} \to B^n\Lambda^{m+1}_{\text{cl}}$ for all $n$ and $m$. Taking repeated pullbacks along these maps gives us a diagram

$$
\begin{array}{ccc}
B^k_U(1) & \longrightarrow & \Lambda^{k+1}_{\text{cl}} \\
\downarrow & & \downarrow \quad \quad \downarrow \\
\bullet & \longrightarrow & \text{BA}^k_{\text{cl}} \\
\downarrow & & \downarrow \\
\vdots & \quad \quad \quad \downarrow \\
\bullet & \longrightarrow & B^{k-1}\Lambda^2_{\text{cl}} \\
\downarrow & & \downarrow \\
B^kU(1) & \longrightarrow & B^k\Lambda^1_{\text{cl}} \\
\downarrow & & \downarrow \\
B^{k+1}Z & \longrightarrow & \flat B^{k+1}R
\end{array}
$$

(3.2.4)

The dashed composite in this diagram is what we were seeking to construct. □
Remark 3.2.5. Diagram 3.2.4 shows us that the following square is a pullback:

\[
\begin{array}{ccc}
B^k_U(1) & \longrightarrow & \Lambda^{k+1}_{\text{cl}} \\
\downarrow & & \downarrow \\
B^k U(1) & \longrightarrow & B^k \Lambda^1_{\text{cl}}
\end{array}
\]

If we note that \(B^k \Lambda^1_{\text{cl}}\) is \(B^k u(1)\), we get an alternate definition of \(B^k_U(1)\) by this pullback. This shows that our definition agrees with Schreiber’s Definition 4.4.93 in [20].

We can now see that the map \(B^k_U(1) \to \Lambda^{k+1}_{\text{cl}}\) takes the curvature \((k+1)\)-form. We can justify this by showing that the fiber of this map is \(bB^k U(1)\); in other words, a circle \(k\)-gerbe with connection is flat if and only if its curvature vanishes.

Lemma 3.2.6. The map \(F_{(-)} : B^k_U(1) \to \Lambda^{k+1}_{\text{cl}}\) has fiber \(bB^k U(1)\). Since this map gives an obstruction to flatness, we refer to it as the curvature \((k+1)\)-form.

Proof. By considering the top part from the diagram (3.2.4), we find a pullback square

\[
\begin{array}{ccc}
B^k_U(1) & \longrightarrow & \Lambda^{k+1}_{\text{cl}} \\
\downarrow & & \downarrow \\
B^k U(1) & \longrightarrow & B^k \Lambda^1_{\text{cl}}
\end{array}
\]

For this reason, we get an equivalence on fibers:

\[
\bullet \longrightarrow B^k_U(1) \longrightarrow \Lambda^{k+1}_{\text{cl}}
\]

\[
\downarrow \quad \downarrow \quad \downarrow \\
\downarrow \quad \downarrow \quad \downarrow \\
\downarrow \quad \downarrow \quad \downarrow
\]

\[
bB^k U(1) \longrightarrow B^k U(1) \longrightarrow B^k \Lambda^1_{\text{cl}}
\]

As a corollary, we may characterize the curvature \(F_{(-)} : B^k_U(1) \to \Lambda^{k+1}_{\text{cl}}\) modally.

Corollary 3.2.7. The curvature \(F_{(-)} : B^k_U(1) \to \Lambda^{k+1}_{\text{cl}}\) is a unit for the \((k-1)\)-truncation modality. In particular,

\[
\left\| B^k_U(1) \right\|_j = \Lambda^{k+1}_{\text{cl}}
\]

for any \(0 \leq j < k\).

Proof. As \(\Lambda^{k+1}_{\text{cl}}\) is 0-truncated and so \((k-1)\) truncated, it will suffice to show that \(F_{(-)}\) is \((k-1)\)-connected. But by Lemma 3.2.6, the fiber of \(F_{(-)}\) over any point \(\omega : \Lambda^{k+1}_{\text{cl}}\) is identifiable with \(bB^k U(1)\), which is \((k-1)\)-connected.

Though our notation may have suggested that the \(B^k_U(1)\) form a loop spectrum, they do not. Indeed, \(\Omega B^k U(1) = bB^{k-1} U(1)\), as can be seen by taking loops of the pullback square defining \(B^k_U(1)\) and noting that \(\Lambda^{n+1}_{\text{cl}}\) is a set (0-type). In total,

\[
\pi_* B^k_U(1) = \begin{cases} 
\Lambda^{k+1}_{\text{cl}} & \text{if } * = 0 \\
\ast & \text{if } * = k \\
0 & \text{otherwise.}
\end{cases}
\]

Nevertheless, each \(B^k_U(1)\) is an infinite loop space in its own right.
**Definition 3.2.8.** For \( n, k \geq 0 \), define \( B^n B^k_U(1) \) to be the following pullback:

\[
\begin{array}{ccc}
B^n B^k_U(1) & \longrightarrow & B^n \Lambda_{cl}^{k+1} \\
\downarrow & & \downarrow \\
B^{n+k+1} Z & \longrightarrow & bB^{n+k+1} \mathbb{R}
\end{array}
\]

It is immediate from this definition and the commutation of taking loops with taking pullbacks that \( \Omega B^{n+1} B^k_U(1) = B^n B^k_U(1) \). We have already seen these deloopings before in Diagram (3.2.4):

\[
\begin{array}{ccc}
B^k_U(1) & \longrightarrow & \Lambda_{cl}^{k+1} \\
\downarrow & & \downarrow \\
B B^{k-1} U(1) & \longrightarrow & B \Lambda_{cl}^k \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
B^{k-1} B_U(1) & \longrightarrow & B^{k-1} \Lambda_{cl}^2 \\
\downarrow & & \downarrow \\
B^k U(1) & \longrightarrow & B^k \Lambda_{cl}^1 \\
\downarrow & & \downarrow \\
B^{k+1} Z & \longrightarrow & bB^{k+1} \mathbb{R}
\end{array}
\]

These maps along the left hand side give us maps of loop spectra

\[ B^* B^k U(1) \to \Sigma B^* B^{k-1} U(1) \]

We will see in Section 3.3 that for \( \bullet = 0 \), these maps give obstructions to de Rham’s theorem for general types.

Each \( B^k_U(1) \) is a higher group itself. We may therefore ask: what is it’s infinitesimal remainder?

**Lemma 3.2.9.** The infinitesimal remainder of \( B^k_U(1) \) is the curvature \( F_{(-)} : B^k_U(1) \to \Lambda_{cl}^{k+1} \).

**Proof.** Consider the following diagram:

\[
\begin{array}{ccc}
\{ \text{bdR} BB^k_U(1) \} & \sim & \Lambda_{cl}^{k+1} \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & \ast \\
\downarrow & & \downarrow \\
\{ bBB^k_U(1) \} & \sim & \{ bB \Lambda_{cl}^{k+1} \} \\
\downarrow & & \downarrow \\
B^{k+2} Z & \longrightarrow & bB^{k+2} \mathbb{R} \\
\downarrow & & \downarrow \\
\ast & \longrightarrow & \ast \\
\downarrow & & \downarrow \\
\{ B B^k_U(1) \} & \sim & \{ B \Lambda_{cl}^{k+1} \} \\
\downarrow & & \downarrow \\
B^{k+2} Z & \longrightarrow & bB^{k+2} \mathbb{R}
\end{array}
\]

The diagonal sequences in this diagram are fiber sequences of the \( b \)-counits which define the infinitesimal remainders. Now, since \( bB \Lambda_{cl}^{k+1} = \ast \) since the form classifiers are infinitesimal, we find that the fiber of the \( b \)-counit \( (-)_b : bB \Lambda_{cl}^{k+1} \to B \Lambda_{cl}^{k+1} \) is \( \Omega B \Lambda_{cl}^{k+1} \), which is \( \Lambda_{cl}^{k+1} \).
Now, the frontmost square is a crisp pullback and $\flat$ is left exact, so the middle square is also a pullback. Then, since the diagonal sequences are fiber sequences, the back square is also a pullback. But this shows that the infinitesimal remainder of $B^k_U(1)$ is $\Lambda^{k+1}_{cl}$.

If we take one more fiber, we can continue the diagram to give us the following diagram:

This shows that the infinitesimal remainder $\theta$ is equal, modulo our constructed equivalence, to the curvature $F(\cdot)$.

Now that we know the infinitesimal remainder of $B^k_U(1)$, we are almost ready to understand its modal fracture hexagon. But first, we must understand its universal $\infty$-cover. We will show that the universal $\infty$-cover of $B^k_U(1)$ is an analogous type $B^k_\nabla \mathbb{R}$.

**Definition 3.2.10.** For $n, k \geq 0$, define $B^n B^k_\nabla \mathbb{R}$ to be the universal $\infty$-cover of $B^n \Lambda^{k+1}_{cl}$:

$$
B^n B^k_\nabla \mathbb{R} \xrightarrow{F(-)} B^n \Lambda^{k+1}_{cl} \\
\downarrow \downarrow \\
* \xrightarrow{\flat} b^B^{n+k+1} \mathbb{R}
$$

We refer to the cohomology theories $B^k_\nabla \mathbb{R}$ as *pure differential cohomology*.

Just as we may think of $B^k_U(1)$ as classifying circle $k$-gerbes with connection, we may think of $B^k_\nabla \mathbb{R}$ as classifying affine $k$-gerbes with connection. We can now show that $B^k_\nabla \mathbb{R}$ is the universal $\infty$-cover of $B^k_U(1)$.

**Proposition 3.2.11.** The map $(\omega, p) \mapsto (pt_{B^{k+1}Z}, \omega, \lambda, p) : B^k_\nabla \mathbb{R} \to B^k_\nabla U(1)$ is the universal $\infty$-cover of $B^k_U(1)$.

**Proof.** Consider the following cube:

In this cube, the front and back spaces are pullbacks by definition, and the right face is a pullback because its top and bottom sides are identities. Therefore, the left face is a pullback. Since $B^k_\nabla U(1) \to B^{k+1}Z$ is a $f$-unit by Lemma 3.2.2, this shows that the dashed map is the fiber of a $f$-unit, and therefore the universal $\infty$-cover. 

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Remark 3.2.12. The fiber sequence

\[ B^k_{\nabla} \mathbb{R} \to B^k_{\nabla} U(1) \to B^{k+1} \mathbb{Z} \]

expresses the informal identity

ordinary differential cohomology = pure differential cohomology + ordinary cohomology.

We are now ready to assemble what we have learned into the modal fracture hexagon of \( B^k_{\nabla} U(1) \):

\[ \begin{array}{ccc}
B^k_{\nabla} \mathbb{R} & \xrightarrow{(-)_\nabla} & B^k_{\nabla} U(1) \\
\beta B^k \mathbb{R} & \xrightarrow{(-)_\nabla} & \beta B^k U(1) \\
\beta B^k U(1) & \xrightarrow{(-)_\nabla} & B^{k+1} \mathbb{Z} \\
\beta & \xrightarrow{(-)_\nabla} & \beta B^k \mathbb{R} \\
\beta \beta B^k U(1) & \xrightarrow{(-)_\nabla} \end{array} \]

(3.2.13)

3.3 Descending to Cohomology and the Character Diagram

In this section, we will discuss how the modal fracture hexagon (3.2.13) descends to cohomology. In general, if \( E_* \) is a loop spectrum, then the we may define the cohomology groups of a type valued in \( E_* \) to be the 0-truncated types of maps:

\[ H^k(X; E_\nabla) := \| X \to E_k \|_0. \]

However, these abelian groups are not discrete — externally, they are (possible non-constant) sheaves of abelian groups. We will want the discrete (externally, constant) invariants. With this in mind, we make the following definitions.

Definition 3.3.1. Let \( X \) be a crisp type. We then make the following definitions:

\[ \begin{align*}
H^n(X; Z) &:= \| \flat(X \to B^n Z) \|_0 \\
H^n(X; \flat \mathbb{R}) &:= \| \flat(X \to \flat B^n \mathbb{R}) \|_0 \\
H^n(X; \flat U(1)) &:= \| \flat(X \to \flat B^n U(1)) \|_0 \\
H^{n,k}_\nabla(X; U(1)) &:= \| \flat(X \to \flat B^n B^k_{\nabla} U(1)) \|_0 \\
H^{n,k}_\nabla(X; \mathbb{R}) &:= \| \flat(X \to \flat B^n B^k \mathbb{R}) \|_0 \\
\Lambda^k(X) &:= \| \flat(X \to \Lambda^k) \|_0 \\
\Lambda^k_{cl}(X) &:= \| \flat(X \to \Lambda^k_{cl}) \|_0.
\end{align*} \]

Remark 3.3.2. In full cohesion, it would be better to work with codiscrete cohomology groups, rather than discrete cohomology groups. This way the definition could be given for all types and not just crisp ones. But we will continue to use discrete groups so that we do not need to work with the codiscrete modality \# in this paper.

We note that with these definitions we may reduce the calculation of ordinary differential cohomology for discrete and homotopically contractible types.

Proposition 3.3.3. Let \( X \) be a crisp type and let \( k \geq 1 \).

1. If \( X \) is discrete (that is, \( X = \int X \)), then \( H^{n,k}_\nabla(X; U(1)) = H^{n+k}(X; \flat U(1)) \).

2. If \( X \) is homotopically contractible (that is, \( \int X = \ast \)), then \( H^{n,k}_\nabla(X; U(1)) = H^n(X; \Lambda^k_{cl}) \).

We may make similar calculations for pure differential cohomology:
1. If \( X \) is discrete, then \( H^{n,k}(X; \mathbb{R}) = H^{n+k}_{\text{cl}}(X; \mathbb{R}) \).

2. If \( X \) is homotopically contractible, then \( H^{n,k}_{\text{cl}}(X; \mathbb{R}) = H^n(X; \Lambda_{\text{cl}}^{k+1}) \).

**Proof.** We will only prove the identities for ordinary differential cohomology; the proofs for pure differential cohomology are identical. We take advantage of the adjointness between \( f \) and \( \flat \).

1. Suppose that \( X \) is discrete. Then

\[
H^{n,k}_{\text{cl}}(X; U(1)) = \| \flat(X \to B^n B_{\text{cl}}^k U(1)) \|_0
= \| \flat (fX \to B^n B_{\text{cl}}^k U(1)) \|_0
= \| \flat(X \to \flat B^n B_{\text{cl}}^k U(1)) \|_0
= \| \flat(X \to \flat B^{n+k} U(1)) \|_0
= H^{n+k}(X; \flat U(1))
\]

2. Suppose that \( X \) is homotopically contractible, and let \( i : B^{n+k+1} \mathbb{Z} \to \flat B^{n+k+1} \mathbb{R} \) denote the (delooping of) the inclusion. Then

\[
H^{n,k}_{\text{cl}}(X; U(1)) = \| \flat(X \to B^n B_{\text{cl}}^k U(1)) \|
= \| (\omega : X \to B^n \Lambda_{\text{cl}}^{k+1}) \times (c : X \to B^{n+k+1} \mathbb{Z}) \|
\times (h : i\mathbb{c} = \omega^f)\|_0
= \| (\omega : X \to B^n \Lambda_{\text{cl}}^{k+1}) \times (c : B^{n+k+1} \mathbb{Z}) \|
\times (h : (x : X) \to i\mathbb{c} = \omega^f)\|_0
\]

Since \( X \) is homotopically contractible, we have an equivalence \( e : (X \to \flat B^{n+k+1} \mathbb{R}) \simeq \flat B^{n+k+1} \). We therefore have \( e((-)^f \circ \omega) : \flat B^{n+k+1} \) and for all \( x : X \) a witness \( \omega(x)^f = e((-)^f \circ \omega) \). We may therefore continue:

\[
= \| (\omega : X \to B^n \Lambda_{\text{cl}}^{k+1}) \times (c : B^{n+k+1} \mathbb{Z}) \|
\times (h : (x : X) \to i\mathbb{c} = e((-)^f \circ \omega))\|_0
= \| (\omega : X \to B^n \Lambda_{\text{cl}}^{k+1}) \times (c : B^{n+k+1} \mathbb{Z}) \|
\times (i\mathbb{c} = e((-)^f \circ \omega))\|_0
\]

Now, both \( B^{n+k+1} \mathbb{Z} \) and \( (i\mathbb{c} = e((-)^f \circ \omega)) \) are 0-connected. The latter because it is identifiable with \( \Omega \flat B^{n+k+1} \mathbb{R} \), which is \( \flat B^{n+k} \mathbb{R} \) and so 0-connected for \( k \geq 1 \) and any \( n \). We may therefore continue:

\[
= \| X \to B^n \Lambda_{\text{cl}}^{k+1} \|
= H^n(X; \Lambda_{\text{cl}}^{k+1}).
\]

**Remark 3.3.4.** We note here that since every type \( X \) lives in the center of a fiber sequence

\[
\overset{\sim}{X} \to X \to \flat X
\]

between a homotopically contractible type and a discrete type, we get a Serre spectral sequence converging to the \( k^{\text{th}} \) ordinary differential cohomology of \( X \) with \( E_2 \) page depending on it’s \( \flat U(1) \) cohomology and the \( \Lambda_{\text{cl}}^{k+1} \) valued cohomology of it’s universal \( \infty \)-cover.
Now, since the top, bottom, and diagonal sequences in the modal fracture hexagon (3.2.13) of $Bk\nabla U(1)$ are fiber sequences, when we take $♭$ and 0-truncations we will get long exact sequences. With the above definitions, we get the following diagram:

\[
\begin{array}{cccc}
H_0^{\flat,k}(X;\mathbb{R}) & \to & \Lambda_{cl}^{k+1}(X) \\
H^k(X;\flat\mathbb{R}) & \to & H_{cl}^{k+1}(X;U(1)) & \to & H^{k+1}(X;\flat\mathbb{R}) \\
H^k(X;\flat U(1)) & \beta & H^{k+1}(X;\mathbb{Z})
\end{array}
\]

in which the top and bottom sequences are long exact, and the diagonal sequences are exact in the middle.

This looks very much like the character diagram for ordinary differential cohomology [23] except for two differences:

1. Where we have the pure cohomology $H_0^{\flat,k}(X;\mathbb{R})$, one would normally find $\Lambda_k(X)/\text{im}(d)$, the abelian group which fits into an exact sequence

   \[
   \Lambda^{k-1}(X) \xrightarrow{d} \Lambda^k(X) \to \Lambda^k(X)/\text{im}(d) \to 0.
   \]

2. Where we have $H^{k+1}(X;\flat\mathbb{R})$, which is ordinary (discrete) cohomology with real coefficients, one would normally find the de Rham cohomology $H^{k+1}_{\text{dR}}(X)$. The de Rham cohomology is defined as closed forms mod exact forms, and so $H^{k+1}_{\text{dR}}(X)$ is the abelian group fitting the following exact sequence:

   \[
   \Lambda^k(X) \xrightarrow{d} \Lambda^{k+1}(X) \to H^{k+1}_{\text{dR}}(X) \to 0.
   \]

Both of these discrepancies are instances of de Rham’s theorem that the de Rham cohomology of forms is the (discrete) ordinary cohomology with real coefficients. Classically and externally, this holds for smooth manifolds. We note that de Rham’s theorem cannot hold for all types for rather trivial reasons: the form classifiers are sets, and so $\Lambda^k(X)$ depends only on the set truncation of $X$ whereas $H^k(X;\flat\mathbb{R})$ can depend on the $k$-truncation of $X$.

**Proposition 3.3.6.** The de Rham theorem does not hold for the delooping $♭B\mathbb{R}$ of the discrete additive group of real numbers. Explicitly,

\[
\begin{align*}
H^1_{\text{dR}}(♭B\mathbb{R}) &= 0 \\
H^1(♭B\mathbb{R};♭\mathbb{R}) &= ♭\text{Hom}(♭\mathbb{R},♭\mathbb{R}) \neq 0
\end{align*}
\]

**Proof.** Since $♭B\mathbb{R}$ is 0-connected and the form classifiers are sets, every map $♭B\mathbb{R} \to \Lambda^k$ is constant for all $k$. Therefore,

\[
H^1_{\text{dR}}(♭B\mathbb{R}) = \Lambda^1_{cl}(♭B\mathbb{R})/\Lambda^0(♭B\mathbb{R}) = 0
\]

On the other hand, $H^1(♭B\mathbb{R};♭\mathbb{R}) = ||♭(♭B\mathbb{R} \to ♭B\mathbb{R})||_0$ is the set of group homomorphisms from $♭\mathbb{R}$ to itself (modulo conjugacy, which makes no difference). The identity is not conjugate to 0, and so this group is not trivial.

We can, however, make explicit the obstruction to de Rham’s theorem lying in the first (cohomological) degree pure differential cohomology groups $H^{1,k}_{\text{dR}}(X;\mathbb{R})$. We begin first by trying to construct an exact sequence

\[
\Lambda^k(X) \xrightarrow{d} \Lambda^{k+1}(X) \to H_0^{0,k}(X;\mathbb{R}) \to 0.
\]

Recall Diagram 3.2.4. There is a similar diagram for pure differential cohomology:
for the map $\Lambda^k$ to be the image of $d^k$ to be the image of $d^k$. From this, we see that the surjectivity of the map $\Lambda^k(X)$ to $H^0, k(X; \mathbb{R})$ is determined by the vanishing of the map $H^0, k(X; \mathbb{R}) \to H^1, X; \mathbb{R}$). Furthermore, the version of Diagram 3.3.7 for $k - 1$ shows us that $d : \Lambda^k(X) \to \Lambda^k(X)$ factors through $H^0, k(X; \mathbb{R})$. This means that for the kernel of $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ to be the image of $d : \Lambda^k(X) \to \Lambda^k(X)$, we need for $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ to be surjective; this is controlled by the vanishing of $H^0, k(X; \mathbb{R}) \to H^1, X; k - 2$. In general, we see the obstructions to having exact sequences

$$0 \to H^k(X; \mathbb{R}) \to H^0, k(X; \mathbb{R}) \to \Lambda^k(X) \to H^0, k(X; \mathbb{R}) \to H^1, X; k - 1 \to \cdots$$

From this, we see that the surjectivity of the map $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ is determined by the vanishing of the map $H^0, k(X; \mathbb{R}) \to H^1, X; \mathbb{R}$). Furthermore, the version of Diagram 3.3.7 for $k - 1$ shows us that $d : \Lambda^k(X) \to \Lambda^k(X)$ factors through $H^0, k(X; \mathbb{R})$. This means that for the kernel of $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ to be the image of $d : \Lambda^k(X) \to \Lambda^k(X)$, we need for $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ to be surjective; this is controlled by the vanishing of $H^0, k(X; \mathbb{R}) \to H^1, X; k - 2$. In general, we see the obstructions to having exact sequences

$$\Lambda^k(X) \to \Lambda^k(X) \to H^0, k(X; \mathbb{R})$$

lie in $H^1, k(X; \mathbb{R})$ and $H^1, k - 2(X; \mathbb{R})$.

First cohomological degree pure differential cohomology groups also control obstructions to de Rham’s theorem for general types $X$. By definition we have a fiber sequence $B^k \mathbb{R} \to \Lambda^k \to \mathcal{B} \mathbb{R}$ which may be delooped arbitrarily. We therefore get exact sequences

$$0 \to H^k(X; \mathbb{R}) \to H^0, k(X; \mathbb{R}) \to \Lambda^k(X) \to H^0, k(X; \mathbb{R}) \to H^1, k(X; \mathbb{R}) \to \cdots$$

This exact sequence shows us that the surjectivity of the map $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ is controlled by the vanishing of the map $H^0, k(X; \mathbb{R}) \to H^1, k(X; \mathbb{R})$. Furthermore, in order for the kernel of $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ to be surjective, we need for $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ to be surjective. As we saw above, for the map $\Lambda^k(X) \to H^0, k(X; \mathbb{R})$ to be surjective, we must have that $H^0, k(X; \mathbb{R}) \to H^1, k(X; \mathbb{R})$ vanishes.
Remembering the classical, external differential cohomology hexagon, we are led to the following conjecture:

**Conjecture 3.3.8.** Let \( X \) be a crisp smooth manifold. Then \( H^{1,k}_\nabla(X; \mathbb{R}) \) vanishes for all \( k \).

### 3.4 Abstract Ordinary Differential Cohomology

In the above sections, we constructed ordinary differential cohomology from the assumption of a long exact sequence of form classifiers. Apart from the concrete differential geometric input of the form classifiers, the construction was entirely abstract. In this section, we will describe the abstract ordinary differential cohomology theory from an axiomatic perspective.

The role of the form classifiers will be played by a *contractible and infinitesimal resolution* of a crisp abelian group \( C \).

**Definition 3.4.1.** Let \( C \) be a crisp abelian group. A *contractible and infinitesimal resolution* (CIR) of \( C \) is a crisp long exact sequence

\[
0 \to \♭C \to C \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \cdots
\]

where the \( C_n \) are homotopically contractible — \( \mathbb{S}C_n = \ast \) — and where the kernels \( Z_n \equiv \ker(d : C_n \to C_{n+1}) \) are infinitesimal — \( \♭Z_n = \ast \). We may think of \( C_n \) as the abelian group of \( n \)-cochains, and \( Z_n \) as the abelian group of \( n \)-cocycles.

**Remark 3.4.2.** In an \( \infty \)-topos of sheaves of homotopy types, an abelian group \( C \) in the empty context (which would therefore be crisp) is a sheaf of abelian groups. In this setting, we can understand a contractible and infinitesimal resolution of \( C \) as presenting a cohomology theory on the site. The \( C_n \) are the sheaves of \( n \)-cochains, and the \( Z_n \) the sheaves of \( n \)-cocycles. To suppose that the chain complex \( d : C_n \to C_{n+1} \) is exact is to say that representables have vanishing cohomology. To say that \( Z_n \) is infinitesimal for \( n > 0 \) is to say that there is a unique \( n \)-cocycle on the terminal sheaf, namely 0. To say that the \( C_n \) are contractible may be understood as saying that for any two objects of the site, there is a homotopically unique *concordance* between any \( n \)-cochains on them.

**Remark 3.4.3.** It’s likely that the generality could be pushed even further by taking \( C \) to be a spectrum and giving the following definition of a contractible and infinitesimal resolution of \( C \):

- Two sequences \( C_n \) and \( Z_n \) of spectra, \( n \geq 0 \), with \( C_0 = C \) and \( Z_0 = \♭C \). We may think of \( C_n \) as the spectrum of \( n \)-cochains, and \( Z_n \) as the spectrum of \( n \)-cocycles.

- Fiber sequences \( Z_n \xrightarrow{i_n} C_n \xrightarrow{d} Z_{n+1} \) in which all maps \( d \) are \( \mathbb{S} \)-fibrations, and where \( i_0 : Z_0 \to C_0 \) is \((-1)_0 : \♭C \to C \).

- The \( C_n \) are contractible, and the \( Z_n \) are infinitesimal.

This definition re-expresses the long exact sequence \( C_{n-1} \xrightarrow{d} C_n \xrightarrow{d} C_{n+1} \) in terms of the short exact sequences

\[
0 \to Z_n \to C_n \xrightarrow{d} Z_{n+1} \to 0
\]

where \( Z_n \equiv \ker(d : C_n \to C_{n+1}) \). As we have no concrete examples at this level of generality in mind, we leave the details of this generalization to future work.

For the rest of this section, we fix a crisp abelian group \( C \) and an contractible and infinitesimal resolution of it. We can then prove analogues of the lemmas in the above sections. We begin by an analogue of Theorem 3.1.6.

**Lemma 3.4.4.** Let \( C \) be a crisp abelian group and \( \mathcal{C}_\bullet \) a contractible and infinitesimal resolution of \( C \). Then \( d : C \to Z_1 \) is the infinitesimal remainder of \( C \).
Proof. By hypothesis, we have a short exact sequence

\[ 0 \to \♭\mathbb{C} \to C \xrightarrow{d} Z_1 \to 0. \]

We therefore have a long fiber sequence

\[ C \xrightarrow{d} Z_1 \to \♭\mathbb{C} \to \mathbb{C} \]

which exhibits \( d : C \to Z_1 \) as the infinitesimal remainder of \( C \). \qed

**Theorem 3.4.5.** For \( C \) a crisp abelian group and \( C_* \), a contractible and infinitesimal resolution of \( C \), we have

\[ fZ_n = \♭B^n\mathbb{C}. \]

**Proof.** The same as the proof of Theorem 3.1.6. \qed

We now define analogues of the ordinary differential geometry classifiers \( B^n\mathbb{B}^k_\mathbb{C}U(1) \).

**Definition 3.4.6.** For \( n, k \geq 0 \), define \( B^nD_k \) to be the following pullback:

\[
\begin{array}{ccc}
B^nD_k & \xrightarrow{B^nF^\natural} & B^nZ_{k+1} \\
\downarrow & & \downarrow \text{(-)}^! \\
\mathcal{B}B^{n+k}C & \to & \♭B^{n+k+1}\mathbb{C}
\end{array}
\]

We refer to \( F^\natural : D_k \to Z_{k+1} \) as the curvature.

We begin by noting that \( D_0 \) is simply \( C \). This went without saying before; we refrained from defining \( B^nU(1) \), but if we had it would have been \( U(1) \).

**Lemma 3.4.7.** As abelian groups, \( D_0 = C \).

**Proof.** The defining pullback of \( D_0 \) is

\[
\begin{array}{ccc}
D_0 & \xrightarrow{F^\natural} & Z_1 \\
\downarrow & & \downarrow \text{(-)}^! \\
\mathcal{B}C & \to & \♭B^1\mathbb{C}
\end{array}
\]

But \( Z_1 \) is the infinitesimal remainder of \( C \), so the right square in the modal fracture hexagon of \( C \) shows that \( C \) is the pullback of the same diagram. \qed

We can prove an analogue of Lemma 3.2.2.

**Lemma 3.4.8.** The defining diagram

\[
\begin{array}{ccc}
B^nD_k & \xrightarrow{B^nF^\natural} & B^nZ_{k+1} \\
\downarrow & & \downarrow \text{(-)}^! \\
\mathcal{B}B^{n+k}C & \to & \♭B^{n+k+1}\mathbb{C}
\end{array}
\]

is a \( f \)-naturality square. In particular, \( fD_k = fB^kC \).

**Proof.** Since \( fB^{n+k}C \) is discrete, it suffices to show that the fibers of \( B^nD_k \to fB^{n+k}C \) are \( f \)-connected. But they are equivalent to the fibers of \( \text{(-)}^! : B^nZ_{k+1} \to \♭B^{n+k+1}\mathbb{C} \), which are \( f \)-contractible. \qed

As a corollary, we can deduce an analogue of Proposition 3.2.11.
Corollary 3.4.9. The curvature $F_{(-)} : D_k \to Z_{k+1}$ induces an equivalence on universal $\infty$-covers: $\overline{D}_k = Z_{k+1}$.

Proof. The defining pullback

\[
\begin{array}{ccc}
D_k & \xrightarrow{F_{-}} & Z_{k+1} \\
\downarrow \phi & & \downarrow (-)^f \\
\int B^k C & \xrightarrow{\flat} & \flat B^{k+1} C
\end{array}
\]

induces an equivalence on the fibers of the vertical maps. Since these maps are $f$-units, the fibers are by definition the respective universal $\infty$-covers.

As with $B^k U(1)$, we may see $D_k$ as equipping $k$-gerbes with band $C$ with cocycle data coming from $Z_{k+1}$. We have the analogue of Diagram 3.2.4:

\[
\begin{array}{ccc}
D_k & \xrightarrow{d} & Z_{k+1} \\
\downarrow & & \downarrow \\
B D_k & \xrightarrow{d} & B Z_k \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
B^{k-1} D_1 & \xrightarrow{d} & B^{k-1} Z_2 \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
B^k C & \xrightarrow{d} & B^k Z_1 \\
\downarrow & & \downarrow \\
\int B^k C & \xrightarrow{\flat} & \flat B^{k+1} \mathbb{R}
\end{array}
\]

(3.4.10)

By applying $\flat$ to this diagram and recalling that the $Z_i$ (and therefore their deloopings) are infinitesimal, we see that $\flat D_k = \flat B^k C$. We can use a composite square from this diagram to prove an analogue of Lemma 3.2.6.

Lemma 3.4.11. The fiber of the curvature $F_{(-)} : D_k \to Z_{k+1}$ is $\flat B^k C$. We are therefore justified in seeing the curvature as an obstruction to the flatness of the underlying gerbe.

Proof. The defining pullback

\[
\begin{array}{ccc}
D_k & \xrightarrow{F_{-}} & Z_{k+1} \\
\downarrow \phi & & \downarrow (-)^f \\
\int B^k C & \xrightarrow{\flat} & \flat B^{k+1} C
\end{array}
\]

induces an equivalence on the fibers of the horizontal maps. But the fiber of $\int B^k C \to \flat B^{k+1} C$ is $\flat B^k C$.

Corollary 3.4.12. The curvature $F_{(-)} : D_k \to Z_{k+1}$ is a unit for the $(k-1)$-truncation modality.

Finally, we record an analogue to Lemma 3.2.9.

Lemma 3.4.13. The infinitesimal remainder of $D_k$ is the curvature $F_{(-)} : D_k \to Z_{k+1}$.

Proof. Exactly as Lemma 3.2.9.
We may put these results together to find the modal fracture hexagon of $D_k$.

**Theorem 3.4.14.** The modal fracture hexagon of $D_k$ (Definition 3.4.6) is:

$$
\begin{array}{cccccc}
\text{Z}_k & \xrightarrow{\pi} & \text{Z}_k+1 & \xrightarrow{(-)'} & \text{♭B}^k C & \xrightarrow{jB^k} \text{♭B}^{k+1} C \\
(-), & & \text{♭B}^k C & \xrightarrow{(-)_b} & \text{♭B}^k C & \xrightarrow{(-)'_b} jB^k C
\end{array}
$$

(3.4.15)

### 3.5 A combinatorial analogue of differential cohomology

Our arguments in the preceding sections have been abstract and modal in character. This abstraction means that we can apply these arguments in settings other than differential geometry. In this subsection, we will sketch a combinatorial analogue of differential cohomology taking place in the cohesive $\infty$-topos of symmetric simplicial homotopy types. We will mix internal and external reasoning in sketching the setup.

A symmetric simplicial homotopy type $S$ is an $\infty$-functor $X : \text{Fin}_{>0} \to \mathbf{H}$ from the category of non-empty finite sets into the $\infty$-category of homotopy types. These are the unordered analogue of simplicial homotopy types.

The $\infty$-topos of symmetric simplicial homotopy types is cohesive. The modalities operate on a symmetric simplicial homotopy type $X$ in the following ways:

- $♭X$ is the discrete (0-skeletal) inclusion of $X_0 \equiv X([0])$ the homotopy type of 0-simplices in $X$.
- $ʃX$ is the discrete inclusion of the geometric realization (or colimit) of $X$.
- $♭♭X$ is the codiscrete (0-coskeletal) inclusion of $X_0$.

We have thus far avoided using the codiscrete modality $♭♭$ in this paper, but it plays a crucial role in this section. This is because the $n$-simplex $Δ[n]$ may be defined to be the codiscrete reflection of the $(n + 1)$-element set $[n] \equiv \{0, \ldots, n\}$.

$$Δ[n] := ʃ\bar{[n]}.$$ 

We may therefore axiomatize symmetric simplicial cohesion internally with the following axiom:

**Axiom 2** (Symmetric Simplicial Cohesion). A crisp type $X$ is crisply discrete if and only if it is $ʃ\bar{[n]}$-local for all $n$.

We may therefore define $ʃ = \text{Loc}_{ʃ\bar{[n]}([n];\mathbb{N})}$ to be the localization at the simplices, and the Symmetric Simplicial Cohesion axiom will ensure that $ʃ$ is adjoint to $♭♭$ as required by the Unity of Opposites axiom.

In his paper [12], Lawvere points out that the simplices $Δ[n]$ are tiny, much like the infinitesimal disks in synthetic differential geometry. $Δ[n]$ being tiny means the functor $(-)\bar{Δ}[n]$ admits an external right adjoint. We may refer to this adjoint as $(-)^{\Delta[n]}$, following Lawvere. If $C$ is a crisp codiscrete abelian group, then the external adjointness shows that maps $X \to C^{\Delta[n]}$ correspond to maps $X^{Δ[n]} \to C$, which, since $C$ is codiscrete, correspond to maps $♭X^{Δ[n]} \to C$; but $♭X^{Δ[n]} = X^{Δ[n]}([0]) = X([n])$, so such maps ultimate correspond to maps $X([n]) \to C$ — that is, to $C$-valued $n$-cochains on the symmetric simplicial homotopy type $X$! In total, $C^{\Δ[n]}$ classifies $n$-cochains, much in the way that $Δ^n$ classifies differential $n$-forms. We note that $C^{\Δ[n]}$ inherits the (crisp) algebraic structure of $C$ since $(-)^{\Delta[n]}$ is a right adjoint.

If furthermore $C$ is a ring, then the $C^{\Delta[n]}$ will be modules and since codiscretes are contractible in this topos (by Theorem 10.2 of [22], noting that it satisfies Shulman’s Axiom C2), we see that the $C^{\Delta[n]}$ are contractible. We may use the face inclusions $Δ[n] \to Δ[n + 1]$ to give maps $C^{\Delta[n]} \to C^{\Delta[n+1]}$, and taking their alternating sum gives us a chain complex

$$C \xrightarrow{d} C^{\Delta[1]} \xrightarrow{d} C^{\Delta[2]} \xrightarrow{d} \ldots$$

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Reasoning externally, we can see that this sequence will be exact since the $C$-valued cohomology of the $n$-simplices is trivial. Furthermore, since $♭C_k = bC$ by adjointness ($♭(\star \to C_k) = b(\star \Delta_k \to C)$), we see that the $Z_k$ are infinitesimal:

$$♭Z_k = \ker(♭d : bC_k \to bC_{k+1}) = \star.$$

For this reason, we may make the following assumption in the setting of symmetric simplicial cohesion, mirroring Assumption 1 of the existence of form classifiers in synthetic differential cohesion.

**Assumption 2.** Let $C$ be a codiscrete ring, and define $C_n := C^{\infty}$. Then the sequence

$$0 \to bC \to C \xrightarrow{d} C_1 \xrightarrow{d} C_2 \xrightarrow{d} \cdots$$

forms a contractible and infinitesimal resolution of $C$.

We can now interpret the abstract language of Section 3.4 into the more concrete language of Assumption 2:

- We have begun with a codiscrete abelian group $C$. We note that since $C$ is codiscrete, it is homotopically contractible: $ʃ C = \star$. Therefore, $C = C$.
- The abelian groups $C_k$ are the $k$-cochain classifiers.
- The kernels $Z_k := \ker(d : C_k \to C_{k+1})$ classify $k$-cocycles. Applying Theorem 3.4.5 here shows us that $ʃ Z_k = bB^k C$.

From this, we see that cohomology valued in the discrete group $♭C$ is the universal discrete cohomological invariant of $k$-cocycles value in $C$. This justifies a remark of Lawvere in [12] that the $Z_k$ have the homotopy type of the Eilenberg-MacLane space $K(bC, k)$.

- Since $C$ is contractible, we have that $D_k$ as defined in Definition 3.4.6 is the universal $\infty$-cover $\infty Z_{k+1}$ of $Z_{k+1}$. We see that $D_k$ classifies $(k+1)$-cocycles together with witnesses that their induced cohomology class vanishes in $♭B^{k+1} C$.

The $D_k$ in this setting have more in common with pure differential cohomology $B_{\nabla}^k \mathbb{R}$ than with ordinary differential cohomology $B_{\nabla}^k U(1)$ on account of being contractible. We can remedy this by introducing some new data. Suppose that we have an exact sequence

$$0 \to K \to C \to G \to 0$$

of crisp codiscrete abelian groups. We may then redefine $D_k$ to instead be the following pullback:

$$\begin{array}{ccc}
D_k & \xrightarrow{\partial} & Z_{k+1} \\
\downarrow & & \downarrow \\
♭B^{k+1} K & \xrightarrow{♭} & bB^{k+1} C
\end{array}$$

We will then have $ʃ D_k = bB^{k+1} K$, $\infty D_k = \infty Z_{k+1}$, and $♭D_k = bB^{k} G$, giving us a modal fracture hexagon:

$$\begin{array}{ccc}
\infty Z_{k+1} & \xrightarrow{\pi} & Z_{k+1} \\
♭B^k C & \xrightarrow{♭} & bB^{k+1} C \\
♭B^k G & \xrightarrow{♭} & bB^{k+1} K
\end{array}$$

(3.5.1)
Taking the short exact sequence $0 \to K \to C \to G \to 0$ to be

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{U}(1) \to 0$$

gives us a bona-fide combinatorial analogue of ordinary differential cohomology, fitting within a similar hexagon. However, instead of equipping the integral cohomology of manifolds with differential form data, we are equipping the integral cohomology of symmetric simplicial sets with real cocycle data.

We intend to give this combinatorial analogue of ordinary differential cohomology a fully internal treatment in future work.

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