Vacuum oscillations of quasi degenerate solar neutrinos

Riccardo Barbieri

Scuola Normale Superiore and INFN, sezione di Pisa, I-56126 Pisa, Italia

Graham G. Ross

Department of Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK

and

Alessandro Strumia

Dipartimento di fisica, Università di Pisa and INFN, sezione di Pisa, I-56126 Pisa, Italia

Abstract

The atmospheric neutrino oscillations and the vacuum oscillation solution of the solar neutrino problem can be consistently described by a doubly or triply degenerate neutrino spectrum as long as the high level of degeneracy required is not spoiled by radiative corrections. We show that this is the case for neutrino mass matrices generated by symmetries. This imposes a strong constraint on the mixing angles and requires the mixing should be close to bi-maximal. We briefly discuss the relevance of our results for the measurability of the neutrino spectrum.

1 Introduction

Observations of atmospheric and solar neutrinos provide very significant indications that neutrinos oscillate between different mass eigenstates $m_i$. As a result, progress has been made in experimentally determining these masses as well as the related mixing parameters. In what follows we assume just 3 light Majorana neutrinos and we use a standard notation for the leptonic mixing matrix

$$V = R_{23}(\theta_{23}) \text{diag}(1, e^{i\phi}, 1) R_{13}(\theta_{13}) R_{12}(\theta_{12})$$

where $R_{ij}(\theta_{ij})$ represents a rotation by $\theta_{ij}$ in the $ij$ plane. Within this framework and with this notation, the present situation can be summarised as follows:

- It is very likely, although still awaiting confirmation, that
  $$\Delta m_{23}^2 = m_{\text{atm}}^2 = 10^{-(3\pm2)} \text{eV}^2,$$
  $$\sin^2 2\theta_{23} \gtrsim 0.8$$

- It is not unlikely that
  $$\Delta m_{12}^2 = m_{\text{sun}}^2 \lesssim 10^{-4} \text{eV}^2,$$
  $$\sin^2 \theta_{13} \lesssim 0.1$$

with

$$3 \cdot 10^{-6} \text{eV}^2 \lesssim m_{\text{sun}}^2 \lesssim 10^{-5} \text{eV}^2,$$
$$2 \cdot 10^{-3} < \sin^2 2\theta_{12} < 2 \cdot 10^{-2}$$

or

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Prior to the discussion of the experimental potential in this area there is, however, one relevant theoretical problem. The VO of solar neutrinos can be consistently described by the ‘degenerate’ or the ‘pseudo-Dirac’ spectra only as long as the high level of degeneracy required is not spoiled by radiative corrections. We investigate this question in this paper, concentrating on the form of neutrino mass matrices motivated by symmetries, as previously suggested. These mixing matrices are all characterized by having $\theta_{12} = \pi/4$ and $\theta_{13} = 0$. We will show that these mass matrices have a sufficient degree of stability against radiative corrections to make the VO solution consistent with both doubly and triply degenerate neutrino spectra. In conjunction with the experimental requirement that $\theta_{23}$ is approximately $\pi/4$, this leads to the so called bimaximal mixing matrix. Similar conclusions have also been reached. Finally we will consider the necessary and sometimes sufficient conditions needed to determine the full neutrino spectrum.

2 Radiative corrections to the neutrino mass textures

We concentrate on neutrino mass matrices, in the charged lepton flavour basis, of the form

$$M_{\nu} = m \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} R_{23}^{T}(\theta_{23})$$

(5)

where: i) for the ‘degenerate’ case, $m$ is the common neutrino mass, $m \gtrsim m_{\text{atm}}$, and $z = e^{i\delta}(1 + \delta)$ with $2\delta + \delta^2 = (m_{\text{atm}}/m)^2$; ii) for the ‘pseudo-Dirac’ case, $m = m_{\text{atm}}$ and $|z|$ is negligibly small.

In both cases the small splitting necessary to describe the VO of solar neutrinos is neglected. It could come from an explicit extra term in the MSW or vacuum oscillation (VO) solutions of the solar neutrino problem.

It is extremely important to know, with a minimum of theoretical bias, which spectrum is realized in nature. However, at the moment we only know that the heaviest neutrino weights less than a few eVs from direct $\beta$-decay searches or from astrophysical and cosmological data. Three different areas of experimental developments can have an impact on this issue:

1. The determination of the actual solution of the solar neutrino problem.
2. The neutrinoless double-beta decay searches.
3. The cosmological signals of a neutrino rest mass.

10^{-5} \text{eV}^2 \lesssim m_{\text{atm}}^2 \lesssim 10^{-4} \text{eV}^2, \\
0.6 \lesssim \sin^2 2\theta_{12} < 0.95 \\
\text{or} \\
5 \cdot 10^{-11} \text{eV}^2 \lesssim m_{\text{sun}}^2 \lesssim 10^{-9} \text{eV}^2, \\
\sin^2 2\theta_{12} \gtrsim 0.6

(4b)

(4c)

corresponding respectively to the small angle MSW (SAMSW), large angle MSW (LAMSW) or vacuum oscillation (VO) solutions of the solar neutrino problem.

It is clearly of great importance to confirm or disprove this picture and further constrain the allowed range of the parameters.

Even accepting (3), which we do hereafter, the neutrino spectrum is not determined. As is well known, three different possibilities exist:

1. “degenerate”:

$$m_1 \approx m_2 \approx m_3 \gtrsim m_{\text{atm}}$$

2. “pseudo-Dirac”:

$$m_1 \approx m_2 \approx m_{\text{atm}} \gg m_3$$

3. “hierarchical”:

$$m_3 \approx m_{\text{atm}} \gg m_2 \approx m_{\text{sun}} \gtrsim m_1$$

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1. The determination of the actual solution of the solar neutrino problem.
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*We remind that in this paper we are considering the case there are just three light neutrinos. Of course our analysis would require revision if this proves not to be the case.
The very fact that this last symmetry is compatible with the Yukawa couplings (the charged fermion masses) of the Standard Model or of the Minimal Supersymmetric Standard Model makes it clear that radiative corrections will not destabilize the vacuum oscillation solution is quite stable against radiative corrections. It is most convenient to discuss the symmetries in the basis rotated by $\theta_{23}$ in which the neutrino mass matrix has the form

$$\tilde{M}_\nu^0 = R_{\tau}^T(\theta_{23})M_\nu R_{\tau}(\theta_{23}) = m \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \quad (9)$$

This is invariant under a U(1) rotation under which the states 1, 2 and 3 have charges $+1$, $-1$ and 0 respectively. Now consider the effect of the radiative corrections. If they preserve the U(1) they will leave the zero structure of the mass matrix intact and in turn this leaves one degenerate pair of neutrinos. It is useful to rewrite (9) in terms of the matrix $\tilde{I}_\tau$ defined by

$$\tilde{I}_\tau = R_{\tau}^T I_\tau R_{\tau} = \mathbb{I} + \epsilon X$$

where

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & s_{23}^2 & c_{23}s_{23} \\ 0 & c_{23}s_{23} & c_{23}^2 \end{pmatrix} \quad (10)$$

and $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$. In terms of $\tilde{I}_\tau$ the renormalized mass matrix in the same basis is given by

$$\tilde{M}_\nu = \tilde{I}_\tau \tilde{M}_\nu^0 \tilde{I}_\tau \quad (11)$$

Let us consider the order at which the degeneracy of the light neutrinos is lifted. Since $\tilde{I}_\tau$ comes from wave function renormalization, only its diagonal elements are invariant under the U(1) discussed above. Thus U(1) breaking effects arise through the elements $X_{23}$, $X_{32}$. But these matrix elements, being off-diagonal, would remove the degeneracy of the eigenvalues at $O(\epsilon)$ only if $\epsilon \simeq |z - 1|$, which is not the case, as required by the atmospheric neutrino anomaly.

### 3 Discussion of the results

We turn now to the quantitative determination of these effects. To do this is useful to rewrite (9) in terms of the matrix

$$I_\tau = V^\dagger I_\tau V \quad (12)$$

as

$$M_\nu = V^* \cdot I_\tau^T M_\nu^0 I_\tau \cdot V^\dagger = V^* \cdot M_\nu' \cdot V^\dagger \quad (13)$$
where, by explicit calculation,
\[
\frac{M'_\nu}{m} = \begin{pmatrix}
-1 - \epsilon s^2_{23} & 0 & \epsilon'(-1 + z)/4 \\
0 & 1 + \epsilon s^2_{23} & \epsilon'(-1 - z)/4 \\
\epsilon'(-1 + z)/4 & \epsilon'(-1 - z)/4 & 1 + 2\epsilon c^2_{23}
\end{pmatrix},
\]
\[\epsilon' \equiv \sqrt{2} \epsilon \sin 2\theta_{23}.\]
Eqs. (13) and (14) are the basic expressions for the renormalized neutrino mass matrix in the flavour basis. The renormalization of the ‘pseudo-Dirac’ spectrum is immediately obtained by setting \(z = 0\) in (14). This gives
\[M_\nu(z = 0) = m R_{23}(\theta_{23}).\]  

which keeps the original form, as anticipated by our symmetry arguments, with (small) renormalizations of the angle \(\theta_{23}\) and of the overall scale \(m\).

For the degenerate case, \(z = \epsilon'\phi(1 + \delta)\), it is best to work with the hermitian matrix
\[M'_\nu M'^\dagger_\nu = V^* M'^\dagger_\nu M'_\nu V^T\]
where, from (14)
\[M'_\nu M'^\dagger_\nu = m'^2 \begin{pmatrix}
1 & 0 & i\epsilon' e^{-i\phi/2} \sin \frac{\phi}{2} \\
0 & 1 & -i\epsilon' e^{-i\phi/2} \cos \frac{\phi}{2} \\
\hbox{h.c.} & \hbox{h.c.} & 1 + 2\delta
\end{pmatrix}
\]
with \(m' = m(1 + \epsilon s^2_{23})\) and up to irrelevant corrections. The eigenvalues of this matrix, i.e. the renormalized squared neutrino masses are
\[m^2_1 = m'^2\]
\[m^2_2 = m'^2(1 + \delta - (\delta^2 + \epsilon'^2)^{1/2})\]
\[\approx m'^2(1 - \epsilon'^2/2\delta)\]
\[m^2_3 \approx m'^2(1 + 2\delta).\]  

As anticipated by our symmetry argument the degeneracy between the light states is lifted at \(O(\epsilon^2)\). Note that had we dropped \(\delta\) in \(m^2_2\) the correction would have been of order \(\epsilon\). The inclusion of the atmospheric mass splitting is crucial and explains why our conclusions about the compatibility of a ‘degenerate’ neutrino spectrum with VO solar oscillations differ from (13).

To the extent (13) represents the exact initial condition for \(M'_\nu\), these renormalized eigenvalues would give
\[\frac{m^2_{\text{sun}}}{m^2} = \frac{\epsilon^2}{\delta} \sin^2 2\theta_{23},\]
\[\frac{m^2_{\text{atm}}}{m^2} = 2\delta\]  
i.e.
\[m^2_{\text{sun}} m^2_{\text{atm}} = 2m^4 \epsilon^2 \sin^2 2\theta_{23}\]  

More generally there could be a splitting of the original unrenormalized eigenvalues, which makes (14) a rough estimate of a lower bound on \(m^2_{\text{sun}} m^2_{\text{atm}}\), barring strong accidental cancellations. Numerically, choosing \(\Lambda = 10^{5\div 16} \text{ GeV}\)
\[\frac{m^2_{\text{sun}} m^2_{\text{atm}}}{(1 \div 20) \times 10^{-11} \text{ eV}^4} \gtrsim \left( \frac{m}{\text{eV}} \right)^4 \{1, (\frac{2}{\cos^2 \beta})^2\}\]  
to be compared with
\[m^2_{\text{sun}} m^2_{\text{atm}} \exp \sim 10^{-11\div 14} \text{ eV}^4.\]  

Eq. (21) is our main result and clearly shows that even a threefold degenerate spectrum, with the neutrino mass matrix following from an underlying symmetry, is compatible with the VO solution of the solar neutrino problem. The radiative correction due to the \(\tau\) Yukawa coupling is actually a candidate for generating the VO \(m^2_{\text{sun}}\) splitting. If this is the case, i.e. the bound in (21) is saturated, a reduction of the uncertainty in the right handed side of (22) would allow a rather precise determination of the average neutrino mass \(m\). At present the lower bound for \(m\) is given by \(m_{\text{atm}} = (0.03 \div 0.1)\ \text{ eV}.\) The upper bound follows from (21). Note that values of cosmological interest, \(\sum m_\nu \sim \text{ eV},\) cannot be safely excluded on the basis of (21).

It is of interest to note that \(\theta_{12} = \pi/4\) and \(\theta_{13} = 0\) have been both necessary to avoid corrections of order \(\epsilon\) to \(m^2_{\text{sun}}\) that would drastically change our conclusions (on the contrary the value of \(\theta_{23}\) does not crucially affect the magnitude of the radiative corrections). At first sight it could appear that the complex 12 rotation necessary to re-diagonalize the RGE-corrected mass matrix [17]
\[U_{12}(\phi/2) = \text{diag}(i, 1, 1) R_{12}(\phi/2) \text{ diag}(-i, 1, 1)\]
induces a too large renormalization of \(\theta_{12}\) unless the phase \(\phi\) is very small. This would be a problem for

\[\text{The signs of the mass squared splittings, } m^2_{\text{sun}} \text{ and } m^2_{\text{atm}}, \text{ are irrelevant.}\]
\[\text{For simplicity, we are neglecting the running of the } \tau \text{ Yukawa coupling. In the MSSM with moderate tan } \beta \text{ and for } \Lambda \approx 2 \times 10^{16} \text{ GeV, our approximation is } \sim 2 \text{ times larger than the exact result.}\]
the model in [9], where $\phi \approx \pi/2$. However this complex rotation does not affect $\theta_{12} = \pi/4$, as may be shown by means of the identity

$$U_{12}(\phi/2)R_{12}(\pi/4) = R_{12}(\pi/4)\text{diag}(e^{i\phi/2}, e^{-i\phi/2}, 1).$$

Consequently the small RGE effects only induce small plex rotation does not affect $\theta$ shown by means of the identity the model in [8], where two degenerate neutrinos have also been studied in [12, 5, 10 , where different correlation presented in eq. (35) of [13] is an equivalent to eq. (19) in [10]. The apparently different solutions, must lie very close to the bimaximal mixing solution 1.

Given that experiments indicate a large $\theta_{23} \approx \pi/4$ and disfavour a large $\theta_{13}$, we conclude that the degenerate case generating a VO solution to the solar neutrino problem, stable against radiative corrections, must lie very close to the bimaximal mixing solution.

Up to now we have concentrated on the VO solution. Our analysis immediately applies to the SAMSW and LAMSW cases as well. In such cases, however, the required level of degeneracy is not incompatible even with a splitting of $O(\epsilon)$. Hence no significant restriction on the mixing parameters arises.

In conclusion, all the three favorite oscillation solutions of the solar neutrino problem (SAMSW, LAMSW or VO) are compatible with all the three possible spectra of neutrinos (‘degenerate’, ‘pseudo-Dirac’, or ‘hierarchical’). The eventual identification of VO as the true solution of the solar neutrino problem would not imply a hierarchical spectrum of neutrinos. Even with a triply degenerate spectrum, the

\*Radiative corrections to the mixing angles in presence of two degenerate neutrinos have also been studied in [4, 5, 6, 7, 3]. The fact that, with a ‘pseudo-Dirac’ spectrum and appropriate correlations between the mixing angles, a cancellation of the $O(\epsilon)$ corrections takes place has been also observed in [4]. Although these delicate cancellations happen in very narrow regions of the mixing angles $|\theta_{13}| < m_{\text{sun}}^2/4m^2$, $|\theta_{12} - \pi/4| < m_{\text{sun}}^2/4m^2$ we consider this situation of physical interest because mixing parameters inside these narrow regions are motivated by symmetries 1, 2, 3.

\*We have shown that the radiative corrections to the solar splitting vanish at leading order in $\lambda$, if $V = R_{23}(\theta_{23}) \cdot R_{13}(0) \cdot R_{12}(\pi/4)$. The same thing happens for a more general $V = R_{12}(\Delta \theta_{12}) \cdot R_{23}(\theta_{23}) \cdot R_{12}(\pi/4)$, since the rotation $R_{12}(\Delta \theta_{12})$ is irrelevant as long as the $\mu$ and $e$ Yukawa couplings are neglected. It seems not unconvincing that even a $V$ of this form might result as a consequence of an approximate symmetry, although this is not the case in the models in 4, 5, 6, 7, 3.

When rewritten in the standard parametrization $\lambda V \lambda' \text{corresponds to having } \theta_{13} \neq 0 \text{ with } \sin\theta_{13} = \tan \theta_{23} \tan(\theta_{12} - \pi/4)$, a relation equivalent to eq. (19) in [6]. The apparently different correlation presented in eq. (35) of [3] is an equivalent parametrization of the same $V$.

known radiative corrections are not too large if the mixing angles have certain values motivated by symmetries. In the next section we discuss how the true neutrino spectrum could be identified by conceivable experiments.

4 Will the neutrino spectrum ever be measured?

The conclusions of the previous section make even more acute the problem of the possible experimental determination of the neutrino spectrum.

If $\theta_{13}$ is non-zero, due to matter effects, the sign of the atmospheric mass splitting might be measurable by the study of $\nu_\mu \rightarrow \nu_e$ and $\nu_\mu \rightarrow \bar{\nu}_e$ oscillations in a long baseline experiment [15], using a $\nu$ beam generated by a neutrino factory. In turn this would allow to discriminate between a ‘hierarchical’ spectrum (where $m^2_3 \gg m^2_{1,2}$) and a ‘pseudo-Dirac’ spectrum (where $m^2_{1,2} \gg m^2_3$).

The $0\nu, 2\beta$ decay searches have set a strong constraint on the modulus of the relevant element of the neutrino mass matrix

$$|M_{\nu_{ee}}| = \left|c_{13}(s^2_{12}m_1 + s^2_{12}m_2e^{2i\varphi_2} + s^2_{13}m_3e^{2i\varphi_3})\right|$$

where $\varphi_i$ are arbitrary phases. At the moment, taking into account the uncertainty on the nuclear matrix element, it is $|M_{\nu_{ee}}| < (0.2 \pm 0.4) \text{ eV}$ [16]. The sensitivity of $0\nu, 2\beta$ experiments is thought to be improvable by about one order of magnitude [17]. A signal for a neutrino mass might also be obtained from studies of large scale structures in the universe, together with accurate measurements of anisotropies in the temperature of the Cosmic Background Radiation. With the standard cosmological model as reference paradigm, a sensitivity to a total neutrino mass $\sum_\nu m_\nu \gtrsim 0.3 \text{ eV}$ is thought to be achievable [18]. The impact of these searches on the issue under consideration can be summarized as follows, as explained below:

1. Finding a $0\nu, 2\beta$ and/or a cosmological signal, at the level specified above, would prove the ‘degenerate’ or the ‘pseudo-Dirac’ spectrum.

Different signals can discriminate between 'degenerate' or 'pseudo-Dirac' spectra and/or imply a specific solution of the solar neutrino problem.
Finding $|M_\nu|_{ee} > 0.01$ eV and/or $\sum \nu m_\nu > 0.3$ eV would be against the `hierarchical’ spectrum since, in such a case

$$|M_\nu|_{ee} \leq |s_{13}^2 m_{\text{atm}} + m_{\text{sun}}| < 0.01 \text{ eV}$$

and

$$\sum \nu m_\nu \approx m_{\text{atm}} < 0.1 \text{ eV}$$

upon use of (23). Specifically, finding

$$|M_\nu|_{ee} > m_{\text{atm}} \quad \text{and/or} \quad \sum \nu m_\nu > 2 m_{\text{atm}}$$

would prove the threefold degenerate spectrum. Furthermore, finding

$$0.01 \text{ eV} < |M_\nu|_{ee} < m_{\text{atm}}$$

and/or

$$\sum \nu m_\nu > 3 |M_\nu|_{ee}$$

would exclude the SAMSW solution of the solar neutrino problem. Finally, the only existence of a bound on $|M_\nu|_{ee}$, $|M_\nu|_{ee} < m_{\text{atm}}$, together with SAMSW for solar neutrinos would prove the ‘hierarchical’ spectrum, since, for ‘degenerate’ or ‘pseudo-Dirac’ neutrinos, the smallness of $\theta_{12}$ implies

$$|M_\nu|_{ee} \approx \max \nu m_\nu \geq m_{\text{atm}}.$$ 

In summary, disappointing as it may be, the experimental distinction between the different neutrino spectra may be hard to achieve in absence of a $0\nu, 2\beta$ or a cosmological signal and with the solution of the solar neutrino problem proven to be either LAMS W or VO. However, the VO solution of the solar neutrino problem will be incompatible with a degenerate spectrum if supersymmetry with $\tan \beta \gtrsim 10$ will be discovered. Of course it is also possible that solar oscillations are not due to one of the three standard solutions and/or that the LSND anomaly [19] will be confirmed.

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