Induced quantum numbers of a magnetic vortex at nonzero temperature

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Abstract

The phenomenon of the finite-temperature induced quantum numbers in fermionic systems with topological defects is analyzed. We consider an ideal gas of two-dimensional relativistic massive electrons in the background of a defect in the form of a pointlike magnetic vortex with arbitrary flux. This system is found to acquire, in addition to fermion number, also orbital angular momentum, spin, and induced magnetic flux, and we determine the functional dependence of the appropriate thermal averages and correlations on the temperature, the vortex flux, and the continuous parameter of the boundary condition at the location of the defect. We find that nonnegativeness of thermal quadratic fluctuations imposes a restriction on the admissible range of values of the boundary parameter. The long-standing problem of the adequate definition of total angular momentum for the system considered is resolved.

1 Introduction

Quantum fermionic systems in different nontrivial topological backgrounds (kinks, vortices, monopoles, skyrmions etc.) can possess rather unusual properties (e.g. fractionization of quantum numbers) [1, 2], see reviews in Refs. [3, 4], and an interest to finite-temperature effects in such systems [5] has been recently revived [6, 7, 8, 9]. In particular, planar systems with a topological defect in the form of a pointlike magnetic vortex deserve a thorough
examination, since they may be relevant for the description of some condensed matter phenomena, including superfluidity and superconductivity \[10, 11, 12\], as well as have various applications in particle physics, cosmology, and astrophysics \[13, 14\]. On the other hand, these systems can be of a certain conceptual importance, providing a field-theoretical manifestation of the famous Bohm-Aharonov effect \[15\]: they involve second-quantized fermions interacting with a vector potential which is caused by a magnetic flux from the inaccessible for the fermions region.

A study of quantum numbers which are induced in the Bohm-Aharonov manner (i.e., by a vector potential of a magnetic vortex) started in Refs.\[16, 17\]. It was shown for a particular choice of the boundary condition at the location of the defect that electric charge \[18\], magnetic flux \[19\], and angular momentum \[20\] are induced in the vacuum of quantized massive fermions. The induced vacuum quantum numbers under the most general set of boundary conditions which are compatible with self-adjointness of the pertinent Dirac Hamiltonian were obtained in Refs.\[21, 22, 23\]. The finite-temperature induced charge was examined in Ref.\[24\]. Following this line, we consider other finite-temperature induced quantum numbers in the present paper.

We start with the operator of the second-quantized fermion field in a static background,

\[
\Psi(x, t) = \sum \int_{(E_\lambda > 0)} e^{-iE_\lambda t} \langle x|\lambda \rangle a_\lambda + \sum \int_{(E_\lambda < 0)} e^{-iE_\lambda t} \langle x|\lambda \rangle b_\lambda^+, \tag{1.1}
\]

where \(a_\lambda^+\) and \(a_\lambda\) (\(b_\lambda^+\) and \(b_\lambda\)) are the fermion (antifermion) creation and destruction operators satisfying anticommutation relations,

\[
[a_\lambda, a_\lambda^+] = [b_\lambda, b_\lambda^+] = \langle \lambda|\lambda' \rangle, \tag{1.2}
\]

and \(\langle x|\lambda \rangle\) is the solution to the stationary Dirac equation,

\[
H \langle x|\lambda \rangle = E_\lambda \langle x|\lambda \rangle, \tag{1.3}
\]

\(H\) is the Dirac Hamiltonian, \(\lambda\) is the set of parameters (quantum numbers) specifying a one-particle state, and \(E_\lambda\) is the energy of the state; symbol \(\sum\) means the summation over discrete and the integration (with a certain measure) over continuous values of \(\lambda\). Ground state \(|\text{vac}\rangle\) of the second-quantized theory is defined as

\[
a_\lambda|\text{vac}\rangle = b_\lambda|\text{vac}\rangle = 0. \tag{1.4}
\]

Let \(J\) be an operator commuting with the Hamiltonian in the first-quantized theory,

\[
[J, H]_\pm = 0. \tag{1.5}
\]

In the case of unbounded operators, commutation of their resolvents is implied, or, to be more specific, it is sufficient to require that operators \(H\) and \(J\) have a common set of eigenfunctions, i.e., relation

\[
J < x|\lambda > = j_\lambda < x|\lambda > \tag{1.6}
\]
holds as well as Eq. (1.3). Eigenfunctions \(\langle x|\lambda \rangle\) satisfy the conditions of completeness and orthonormality; in general, normalization to a delta function is implied. Thus, in the second-quantized theory, the operators of the dynamical variables (physical observables) corresponding to \(H\) and \(J\) can be diagonalized:

\[
\hat{P}^0 \equiv \frac{i}{4} \int d^d x \left\{ [\Psi^+(x,t), \partial_t \Psi(x,t)]_+ - [\partial_t \Psi^+(x,t), \Psi(x,t)]_- \right\} = \\
= \sum \int\! E_{\lambda} \left[ a_{\lambda}^+ a_{\lambda} - b_{\lambda}^+ b_{\lambda} - \frac{1}{2} \text{sgn}(E_{\lambda}) \right], \tag{1.7}
\]

and

\[
\hat{M} \equiv \frac{1}{2} \int d^d x \left[ \Psi^+(x,t), J\Psi(x,t) \right]_+ = \sum \int\! j_{\lambda} \left[ a_{\lambda}^+ a_{\lambda} - b_{\lambda}^+ b_{\lambda} - \frac{1}{2} \text{sgn}(E_{\lambda}) \right], \tag{1.8}
\]

d is the space dimension.

Let us define partition function

\[
Z(\beta, \mu_J) = Sp \exp \left[ -\beta \left( \hat{P}^0 - \mu_J \hat{M} \right) \right], \quad \beta = (k_B T)^{-1}, \tag{1.9}
\]

where \(T\) is the equilibrium temperature, \(k_B\) is the Boltzmann constant, \(\mu_J\) is the generalized chemical potential, and \(Sp\) is the trace or the sum over the expectation values in the Fock state basis created by operators in Eq. (1.2). Although this sum becomes divergent in the limit of infinite space volume, this will not bother us, since the partition function plays a merely supplementary role. The quantities of physical interest are obtained by taking derivatives of \(\ln Z(\beta, \mu_J)\), and these latter may appear to be finite in the infinite volume limit.

In particular, one can define an average of operator \(\hat{M}\) over the grand canonical ensemble

\[
\langle \hat{M} \rangle_{\beta, \mu_J} \equiv \frac{1}{\beta} \frac{\partial}{\partial \mu_J} \ln Z(\beta, \mu_J) = Z^{-1}(\beta, \mu_J) Sp \hat{M} \exp \left[ -\beta \left( \hat{P}^0 - \mu_J \hat{M} \right) \right]. \tag{1.10}
\]

Computing averages

\[
\langle a_{\lambda}^+ a_{\lambda} \rangle_{\beta, \mu_J} = \left\{ \exp[\beta(E_{\lambda} - \mu_J j_{\lambda})] + 1 \right\}^{-1}, \quad E_{\lambda} > 0 \\
\langle b_{\lambda}^+ b_{\lambda} \rangle_{\beta, \mu_J} = \left\{ \exp[\beta(-E_{\lambda} + \mu_J j_{\lambda})] + 1 \right\}^{-1}, \quad E_{\lambda} < 0, \tag{1.11}
\]

and using the explicit form of \(\hat{P}^0\) and \(\hat{M}\) in terms of the creation and destruction operators, one gets the spectral integral representation of average (1.10) (see, e.g. Ref. [5]),

\[
\langle \hat{M} \rangle_{\beta, \mu_J} = -\frac{1}{2} \int_{-\infty}^{\infty} dE \tau_J(E) \tanh \left[ \frac{1}{2} \beta(E - \mu_J j) \right], \tag{1.12}
\]

where the appropriate spectral density is

\[
\tau_J(E) = \frac{1}{\pi} Im Tr J(H - E - i0)^{-1}, \tag{1.13}
\]
$Tr$ is the trace of an integro-differential operator in the functional space: $Tr \, U = \int d^d x \, tr \langle x|U|x \rangle$; $tr$ denotes the trace over spinor indices only; note that the functional trace should be regularized and renormalized by subtraction, if necessary.

Taking $J = I$, where $I$ is the unit matrix in the space of Dirac matrices, one gets $\hat{M} = \hat{N}$, where $\hat{N}$ is the fermion number operator in the second-quantized theory, then $\mu_I$ is the usual chemical potential. In the $d = 1$ case fermion number is the only observable which is conserved in addition to energy. In more than one dimensions there are more conserved observables. In particular, in the $d = 2$ case, in addition to energy and fermion number, also total angular momentum is conserved when the system is rotationally invariant.

Now let us consider an observable which is not conserved and denote the appropriate operator in the first-quantized theory by $\Omega$. Then the corresponding operator in the second-quantized theory,

$$\hat{O} = \frac{1}{2} \int d^d x [\Psi^+, \Omega \Psi]_-, \quad (1.14)$$

is not diagonalizable. Nevertheless, its average over the grand canonical ensemble can be defined in a manner similar to Eq. (1.10),

$$\langle \hat{O} \rangle_{\beta, \mu_J} \equiv Z^{-1}(\beta, \mu_J) Sp \langle \hat{O} \rangle \exp \left[ -\beta \left( \hat{P}^0 - \mu_J \hat{M} \right) \right]. \quad (1.15)$$

One can get an appropriate spectral integral representation,

$$\langle \hat{O} \rangle_{\beta, \mu_J} = -\frac{1}{2} \int_{-\infty}^{\infty} dE \, \tau_\Omega(E) \tanh \left[ \frac{1}{2} \beta (E - \mu_J) \right], \quad (1.16)$$

where

$$\tau_\Omega(E) = \frac{1}{\pi} Im \, Tr \, \Omega(H - E - i0)^{-1}. \quad (1.17)$$

In the present paper we shall be dealing with the averages over the canonical ensemble:

$$M(T) \equiv \langle \hat{M} \rangle_{\beta, \mu_J = 0}, \quad O(T) \equiv \langle \hat{O} \rangle_{\beta, \mu_J = 0}. \quad (1.18)$$

In addition to them we shall be considering also such quantities as the correlation of the conserved and nonconserved observables

$$\Delta (T; \hat{O}, \hat{M}) \equiv \langle \hat{O} \hat{M} \rangle_{\beta, \mu_J = 0} - \langle \hat{O} \rangle_{\beta, \mu_J = 0} \langle \hat{M} \rangle_{\beta, \mu_J = 0} \quad (1.19)$$

and the quadratic fluctuation of the conserved observable

$$\Delta (T; \hat{M}, \hat{M}) \equiv \langle \hat{M}^2 \rangle_{\beta, \mu_J = 0} - \left( \langle \hat{M} \rangle_{\beta, \mu_J = 0} \right)^2 \quad (1.20)$$

Using Eqs. (1.12) and (1.16), one can get the spectral integral representation for Eqs. (1.19) and (1.20):

$$\Delta (T; \hat{O}, \hat{M}) = \frac{1}{\beta} \left( \frac{\partial}{\partial \mu_J} \langle \hat{O} \rangle_{\beta, \mu_J} \right) \bigg|_{\mu_J = 0} = \frac{1}{4} \int_{-\infty}^{\infty} dE \, \tau_{\Omega J}(E) \tanh^2 \left( \frac{1}{2} \beta E \right), \quad (1.21)$$
\[ \Delta(T; \hat{M}, \hat{M}) = \frac{1}{\beta} \left( \frac{\partial}{\partial \mu_j} \langle \hat{M} \rangle_{\beta, \mu_j} \right) \bigg|_{\mu_j=0} = \frac{1}{4} \int_{-\infty}^{\infty} dE \tau f(E) \text{sech}^2 \left( \frac{1}{2} \beta E \right), \quad (1.22) \]

where the appropriate spectral densities are obtained from Eqs. (1.17) and (1.13) by inserting an additional power of operator \( J \) into the trace.

### 2 Observables of the planar fermionic system in the background of a magnetic vortex defect

We consider a spinor field which is quantized in the background of a static magnetic field in 2 + 1-dimensional space-time. The Dirac Hamiltonian takes form

\[ H = -i \alpha \left[ \partial - ieV(x) \right] + \beta m, \quad (2.1) \]

where \( V(x) \) is the vector potential of the field strength \( B(x) = \partial \times V(x) \). The Clifford algebra in this case has two inequivalent irreducible representations which can be differed in the following way:

\[ \alpha^1 \alpha^2 \beta = is, \quad s = \pm 1. \quad (2.2) \]

Choosing the \( \beta \) matrix in the diagonal form,

\[ \beta = \sigma_3, \quad (2.3) \]

one gets

\[ \alpha^1 = -e^{\frac{i}{2} \sigma_3 \chi_s} \sigma_2 e^{-\frac{i}{2} \sigma_3 \chi_s}, \quad \alpha^2 = s e^{\frac{i}{2} \sigma_3 \chi_s} \sigma_1 e^{-\frac{i}{2} \sigma_3 \chi_s}, \quad (2.4) \]

where \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are the Pauli matrices, and \( \chi_1 \) and \( \chi_{-1} \) are the parameters varying in interval \( 0 < \chi_s < 2\pi \) to go over to the equivalent representation. Note also that in odd-dimensional space-time the \( m \) parameter in Eq. (2.1) can take both positive and negative values; a change of sign of \( m \) corresponds to going over to the inequivalent representation.

If a magnetic field is invariant under rotations of the two-dimensional space around its origin, then one has

\[ (x \times \partial) [\partial \times V(x)] = 0, \quad (2.5) \]

and a generator of rotations takes form

\[ J = -i x \times [\partial - ieV(x)] + \frac{1}{2} s \beta + e \int_0^r d\varphi (\partial \times V(x)), \quad (2.6) \]

\( r = \sqrt{(x^1)^2 + (x^2)^2} \) and \( \varphi = \arctan(x^2/x^1) \) are the polar coordinates. One can easily verify that operator \( J [2.6] \) commutes with operator \( H [2.1] \).

In the right hand side of Eq. (2.6), the first two terms represent the orbital and the spin parts of the angular momentum of the charged matter field, whereas the last term represents the angular momentum of the background field. In the nonsingular long-range gauge

\[ x \cdot V(x) = 0, \quad (2.7) \]
one gets
\[ x \times V(x) = \int_0^r dr \, r \left[ \partial \times V(x) \right], \tag{2.8} \]
and Eq. (2.6) takes form
\[ J = -i \, x \times \partial + \frac{1}{2} s \beta. \tag{2.9} \]

The above is relevant for the case of an extensive configuration of the background magnetic field (see, e.g., Ref. [25]). Turning now to the case of the background in the form of a magnetic vortex defect, let the central region (e.g., a disc of radius \( \delta \)) be impenetrable for the charged matter and the background field strength be nonvanishing only in this region (i.e., the region of the defect). Then the angular momentum operator outside the central region consists of two parts, orbital and spin,
\[ J = -i \, x \times [\partial - i e V(x)] + \frac{1}{2} s \beta. \tag{2.10} \]

As is well known (see, e.g., Ref. [15]), due to nonvanishing flux of the background field in the inner region,
\[ \Phi = \int_0^\delta dr \, r \left[ \partial \times V(x) \right], \tag{2.11} \]
the vector potential cannot be made vanishing everywhere in the outer region. In particular, in the gauge (2.7) one gets at \( r > \delta \):
\[ V^1(x) = -\Phi r^{-1} \sin \varphi, \quad V^2(x) = \Phi r^{-1} \cos \varphi, \tag{2.12} \]
and Eq. (2.10) takes form
\[ J = -i \, x \times [\partial - i e \Phi] + \frac{1}{2} s \beta. \tag{2.13} \]

Thus, contrary to the case of the extensive background field configuration when the angular momentum is quantized in half-integer values,
\[ j = n + \frac{1}{2}, \tag{2.14} \]
(this is evident in the gauge (2.7), see Eq. (2.9)), in the case of the vortex defect, the angular momentum is quantized in units
\[ j = n + \frac{1}{2} - e \Phi, \tag{2.15} \]
and this results in such fascinating quantum-mechanical concepts as anyons and fractional statistics [26]. However, in the latter case one can take operator
\[ J' = -i \, x \times [\partial - i e V(x)] + \frac{1}{2} s \beta + \Xi, \tag{2.16} \]
as well as an operator of conserved quantity; here \( \Xi \) is an arbitrary constant. In particular, choosing \( \Xi = e \Phi \), one gets in the gauge (2.7) the same expression as Eq. (2.9) and, consequently, half-integer eigenvalues. The arguments in favour of such a definition of the angular momentum operator are given in Ref. [27]. Not going into details of the discussion at the quantum-mechanical level, we would like to emphasize that the problem of the proper definition of the angular momentum operator might be resolved in the framework of the second-quantized theory at nonzero temperature. Indeed, the quadratic fluctuation (1.22) of the physically meaningful observable has to be nonnegative, and this, as we shall see in Section 6, imposes a definite restriction on the choice of the appropriate operator in the first-quantized theory.

Turning to the nonconserved observables, it is natural to consider, in the capacity of \( \Omega \), the orbital angular momentum operator,

\[
\Lambda = -i \mathbf{x} \times [\partial - ie \mathbf{V}(\mathbf{x})],
\]

and the spin operator,

\[
\Sigma = \frac{1}{2} s \beta.
\]

In addition to these, we consider also operator

\[
\Omega = \frac{e^2}{4\pi} \mathbf{x} \times \alpha,
\]

which corresponds to the observable with the physical meaning of the induced magnetic flux multiplied by \( e \). Really, the latter quantity at finite temperature is

\[
O(T) = e \int_{\delta}^{\infty} dr r B^{(I)}(r),
\]

where the induced magnetic field strength \( B^{(I)} \) is rotationally invariant and satisfies Maxwell equation

\[
\partial_r [r B^{(I)}(r)] = \mathbf{x} \times \mathbf{j}(\mathbf{x}),
\]

with induced current

\[
\mathbf{j}(\mathbf{x}) = -\frac{e}{2} tr \left< \mathbf{x} \left| \alpha \tanh \left( \frac{1}{2} \beta H \right) \right| \mathbf{x} \right>.
\]

Solving Eq. (2.21) and substituting the solution into Eq. (2.20), one gets

\[
O(T) = \frac{e}{4\pi} \int_{r>\delta} d^2 x [\mathbf{x} \times \mathbf{j}(\mathbf{x})],
\]

which is the thermal average of operator \( \hat{O}(1.14) \) constructed from \( \Omega(2.19) \).

In the present paper we shall compute thermal averages of the above observables, as well as correlations of conserved and nonconserved observables and quadratic fluctuations of

\footnote{Note that \( e^2 \) has dimension of mass in the case of \( 2 + 1 \)-dimensional space-time. Hence \( \Omega(2.19) \) is dimensionless, as well as \( \Lambda(2.17) \) and \( \Sigma(2.18) \).}
conserved observables; note that the average and fluctuation of charge (i.e., fermion number times $e$) have been computed earlier [24]. Thermal characteristics of the quantized fermionic matter in the background of a magnetic vortex defect depend both on vortex flux (2.11) and the parameter of the boundary condition for the matter field at the edge of the defect.

### 3 Spectral densities and traces of resolvents

In order to compute thermal characteristics one has to determine spectral densities \( \tau_\Lambda(E) \), \( \tau_\Sigma(E) \), \( \tau_\Omega(E) \), \( \tau_J(E) \), which are imaginary parts of the appropriate functional traces, see, e.g., Eqs.(1.13) and (1.17). Actually, integrals over the real energy spectrum can be transformed into integrals over a contour on the complex energy plane, thus yielding a representation of thermal characteristics through the traces directly. In particular, one gets

\[
M(T) = -\frac{1}{2} \int_C \frac{d\omega}{2\pi i} \tanh \left( \frac{1}{2} \beta \omega \right) \text{Tr} J(H - \omega)^{-1},
\]

and

\[
\Delta(T; \hat{M}, \hat{M}) = \frac{1}{4} \int_C \frac{d\omega}{2\pi i} \text{sech}^2 \left( \frac{1}{2} \beta \omega \right) \text{Tr} J^2(H - \omega)^{-1},
\]

and similarly for other characteristics; here \( C \) is the contour \((-\infty + i0, +\infty + i0)\) and \((+\infty - i0, -\infty - i0)\) in the complex \( \omega \)-plane.

The kernel of the resolvent (the Green’s function) of the Dirac Hamiltonian in the coordinate representation is defined as

\[
G^\omega(r, \varphi; r', \varphi') = \langle r, \varphi | (H - \omega)^{-1} | r', \varphi' \rangle .
\]

Using Eqs.(2.3) and (2.4), one can expand Eq.(3.3) in modes in the following form

\[
G^\omega(r, \varphi; r', \varphi') = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{in(\varphi - \varphi')} \begin{pmatrix} a_n(r; r') & d_n(r; r') e^{-i(s\varphi' - \chi_s)} \\ b_n(r; r') e^{i(s\varphi - \chi_s)} & c_n(r; r') e^{is(\varphi - \varphi')} \end{pmatrix}.
\]

In the background of magnetic vortex defect (2.12), Hamiltonian (2.1) takes form

\[
H = -i\alpha^r \partial_\varphi - ir^{-1} \alpha^\varphi (\partial_\varphi - i e \Phi) + \beta m,
\]

where

\[
\alpha^r = \alpha^1 \cos \varphi + \alpha^2 \sin \varphi, \quad \alpha^\varphi = -\alpha^1 \sin \varphi + \alpha^2 \cos \varphi.
\]

If a size of the defect is neglected (\( \delta \to 0 \)), then a parameter of the boundary condition at the location of the defect (at \( r = 0 \)) exhibits itself as a parameter of a self-adjoint extension of the Hamiltonian operator. Partial Hamiltonians are essentially self-adjoint for all \( n \), with the exception of \( n = n_0 \), where

\[
n_0 = [[e \Phi]] + \frac{1}{2} - \frac{1}{2}s,
\]
\([u]\) is the integer part of quantity \(u\) (i.e., the largest integer which is less than or equal to \(u\)). The partial Hamiltonian for \(n = n_0\) requires a self-adjoint extension according to the Weyl-von Neumann theory of self-adjoint operators (see, e.g., Ref. [28]). Appropriately, radial components \(a_n, b_n, c_n,\) and \(d_n\) in Eq. (3.14) with \(n \neq n_0\) are regular at \(r \to 0\) and \(r' \to 0\), whereas those with \(n = n_0\) satisfy conditions (for details see Ref. [24]):

\[
\begin{align*}
\cos\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) & \lim_{r \to 0} |m| r F a_n(r; r') = - \text{sgn}(m) \sin\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r' \to 0} |m| r^{1-F} b_n(r; r') \\
\cos\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) & \lim_{r \to 0} |m| r F b_n(r; r') = - \text{sgn}(m) \sin\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r' \to 0} |m| r^{1-F} c_n(r; r') \\
\cos\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) & \lim_{r' \to 0} |m| r' F a_n(r; r') = - \text{sgn}(m) \sin\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r \to 0} |m| r^{1-F} d_n(r; r') \\
\cos\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) & \lim_{r' \to 0} |m| r' F b_n(r; r') = - \text{sgn}(m) \sin\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r \to 0} |m| r^{1-F} c_n(r; r')
\end{align*}
\] (3.8)

and

\[
\begin{align*}
\cos\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) & \lim_{r' \to 0} |m| r' F a_n(r; r') = - \text{sgn}(m) \sin\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r \to 0} |m| r^{1-F} b_n(r; r') \\
\cos\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) & \lim_{r' \to 0} |m| r' F b_n(r; r') = - \text{sgn}(m) \sin\left(\frac{s \Theta}{2} + \frac{\pi}{4}\right) \lim_{r \to 0} |m| r^{1-F} c_n(r; r') \\
\end{align*}
\] (3.9)

where

\[
\text{sgn}(u) = \begin{cases} 
1, & u > 0 \\
-1, & u < 0
\end{cases}
\]

\(\Theta\) is the self-adjoint extension parameter, and

\[
F = s [e \Phi] + \frac{1}{2} - \frac{1}{2} s,
\] (3.10)

\([u] = u - [u]\) is the fractional part of quantity \(u\), \(0 \leq [u] < 1\); note here that Eqs. (3.8) and (3.9) imply that \(0 < F < 1\), since in the case of \(F = \frac{1}{2} - \frac{1}{2} s\) all radial components obey the condition of regularity at \(r \to 0\) and \(r' \to 0\). Note also that Eqs. (3.8) and (3.9) are periodic in \(\Theta\) with period \(2\pi\).

The radial components of the resolvent kernel have been determined in Ref. [24], and we list them in Appendix A.

Let us consider quantities

\[
\int_{0}^{\infty} d\varphi \, tr \left[ \lambda G^{\omega}(r, \varphi; r', \varphi) \right] = \sum_{n=-\infty}^{\infty} \left[ (n - e \Phi) a_n(r; r') + (n + s - e \Phi) c_n(r; r') \right],
\] (3.11)

\[
\int_{0}^{\infty} d\varphi \, tr \left[ \Sigma G^{\omega}(r, \varphi; r', \varphi) \right] = \frac{1}{2} s \sum_{n=-\infty}^{\infty} [a_n(r; r') - c_n(r; r')],
\] (3.12)

\[
\int_{0}^{\infty} d\varphi \, tr \left[ \Omega G^{\omega}(r, \varphi; r', \varphi) \right] = \frac{e^2}{4 \pi} \sqrt{s} r \sum_{n=-\infty}^{\infty} [b_n(r; r') + d_n(r; r')],
\] (3.13)

\[
\int_{0}^{\infty} d\varphi \, tr \left[ \Sigma J G^{\omega}(r, \varphi; r', \varphi) \right] = \frac{1}{2} s \sum_{n=-\infty}^{\infty} \left[ (n - e \Phi + \frac{1}{2} s) [a_n(r; r') - c_n(r; r')] \right],
\] (3.14)

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\[ \int_0^\infty d\varphi \, \text{tr} \left[ J^2 G^\omega (r, \varphi; r', \varphi) \right] = \sum_{n=-\infty}^{\infty} \left( n - e\Phi + \frac{1}{2}s \right)^2 \left[ a_n(r;r') + c_n(r;r') \right], \quad (3.15) \]

\[ \int_0^\infty d\varphi \, \text{tr} \left[ \Omega J G^\omega (r, \varphi; r', \varphi) \right] = \frac{e^2}{4\pi^2} \left[ n - e\Phi + \frac{1}{2}s \right] \left[ b_n(r;r') + d_n(r;r') \right], \quad (3.16) \]

where operators \( J, \Lambda, \Sigma, \Omega \) are given by Eqs. (2.13), (2.17)-(2.19), correspondingly. Using the explicit form of \( a_n, b_n, c_n, d_n \) given in Appendix A, we perform summation over \( n \) and get in the case of \( r' > r \) and \( Imk > |Re k| \) (see Appendix B):

\[ \int_0^{2\pi} d\varphi \, \text{tr} \left[ \Lambda G^\omega (r, \varphi; r', \varphi) \right] = \frac{s \sin(F\pi)}{\pi} m \int_0^\infty dy \, \exp \left[ -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right] \left[ K_F(y) - K_{1-F}(y) \right] + \]

\[ + \frac{2s \sin(F\pi)}{\pi \tan \nu_\omega + e^{iF\pi}} \left[ -F(\omega + m) \tan \nu_\omega K_F(\kappa r) K_F(\kappa r') + (1-F)(\omega - m)e^{iF\pi} K_{1-F}(\kappa r) K_{1-F}(\kappa r') \right], \quad (3.17) \]

\[ \int_0^{2\pi} d\varphi \, \text{tr} \left[ \Sigma G^\omega (r, \varphi; r', \varphi) \right] = sm K_0(\kappa |r - r'|) - \frac{s \sin(F\pi)}{4F(1-F)\pi} m \int_0^\infty dy \, \exp \left[ -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right] \times \]

\[ \times \left\{ \frac{e^y}{y} \int_0^\infty du \, e^{-u} \left[ (1-F)K_F(u) + FK_{1-F}(u) \right] - (2F - 1)K_F(y) - K_{1-F}(y) \right\} \]

\[ + \frac{s \sin(F\pi)}{\pi \tan \nu_\omega + e^{iF\pi}} \left[ (\omega + m) \tan \nu_\omega K_F(\kappa r) K_F(\kappa r') - (\omega - m)e^{iF\pi} K_{1-F}(\kappa r) K_{1-F}(\kappa r') \right], \quad (3.18) \]

\[ \int_0^{2\pi} d\varphi \, \text{tr} \left[ \Omega G^\omega (r, \varphi; r', \varphi) \right] = \frac{e^2 s \sin(F\pi)}{4\pi^2} \left[ \int_0^\infty dy \, \exp \left[ -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right] \left[ K_F(y) - K_{1-F}(y) \right] + \right. \]

\[ + \frac{e^2 s \sin(F\pi)}{2\pi^2 (\tan \nu_\omega + e^{iF\pi})} \kappa r \left[ (\omega + m) K_{1-F}(\kappa r) K_F(\kappa r') - (\omega - m)e^{iF\pi} K_F(\kappa r) K_{1-F}(\kappa r') \right], \quad (3.19) \]
\[
\int_0^{2\pi} d\varphi \; tr \left[ \Sigma J G^\omega (r, \varphi; r', \varphi) \right] = \frac{\omega}{2} K_0(\kappa |r - r'|) - \frac{\sin(F\pi)}{2\pi} \int_0^\infty dy \exp \left( -\frac{\kappa^2 rr'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) \times \\
\left\{ \frac{\omega}{4F(1 - F)} e^y \int_0^\infty du e^{-u} [(1 - F)K_F(u) + FK_{1-F}(u)] - \left[ \frac{F - \frac{1}{2}}{2F(1 - F)} + m \right] [K_F(y) - K_{1-F}(y)] \right\} - \\
- \frac{2 (F - \frac{1}{2})^2 \sin(F\pi)}{\pi (\tan \nu \omega + e^{iF\pi})} \left[ (\omega + m) \tan \nu \omega K_F(\kappa r)K_F(\kappa r') + (\omega - m) e^{iF\pi} K_{1-F}(\kappa r)K_{1-F}(\kappa r') \right],
\]

(3.20)

\[
\int_0^{2\pi} d\varphi \; tr \left[ J^2 G^\omega (r, \varphi; r', \varphi) \right] = \frac{\omega}{2} \left[ K_0(\kappa |r - r'|) + 4 \frac{\kappa^2 rr'}{|r - r'|} K_1(\kappa |r - r'|) \right] - \\
- \frac{\sin(F\pi)}{\pi} \int_0^\infty dy \exp \left( -\frac{\kappa^2 rr'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) \times \\
\left\{ \frac{\omega}{2F(1 - F)} \left( 1 + \frac{1}{4y} \right) e^y \int_0^\infty du e^{-u} [(1 - F)K_F(u) + FK_{1-F}(u)] - \\
- \left[ \frac{F - \frac{1}{2}}{4} + y \right] \left( \frac{F - \frac{1}{2}}{2F(1 - F)} + m \right) [K_F(y) - K_{1-F}(y)] - \omega [(1 - F)K_F(y) + FK_{1-F}(y)] \right\} + \\
+ \frac{2 (F - \frac{1}{2})^2 \sin(F\pi)}{\pi (\tan \nu \omega + e^{iF\pi})} \left[ (\omega + m) \tan \nu \omega K_F(\kappa r)K_F(\kappa r') + (\omega - m) e^{iF\pi} K_{1-F}(\kappa r)K_{1-F}(\kappa r') \right],
\]

(3.21)

\[
\int_0^{2\pi} d\varphi \; tr \left[ \Omega J G^\omega (r, \varphi; r', \varphi) \right] = \frac{e^2 \kappa r + r'}{4\pi^2 \kappa r |r - r'|} K_1(\kappa |r - r'|) - \\
- \frac{e^2 \sin(F\pi)}{8F(1 - F)\pi^2} \int_0^\infty dy \left( 1 + \frac{\kappa^2 rr'}{4y^2} + \frac{r^2 - r'^2}{4rr'} \right) \exp \left( -\frac{\kappa^2 rr'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) \times \\
\left\{ e^y \int_0^\infty du e^{-u} [(1 - F)K_F(u) + FK_{1-F}(u)] - (2F - 1) y[K_F(y) - K_{1-F}(y)] \right\} + \\
+ \frac{e^2 \sin(F\pi)}{4\pi^2} \int_0^\infty dy \exp \left( -\frac{\kappa^2 rr'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) [(1 - F)K_F(y) + FK_{1-F}(y)] - \\
- \frac{e^2 (F - \frac{1}{2}) \sin(F\pi)}{2\pi^2 (\tan \nu \omega + e^{iF\pi}) \kappa r} \left[ \tan \nu \omega K_{1-F}(\kappa r)K_F(\kappa r') - e^{iF\pi} K_F(\kappa r)K_{1-F}(\kappa r') \right],
\]

(3.22)
where $\kappa = -ik$, $K_\rho(u)$ is the Macdonald function of order $\rho$, and $\tan \nu_\omega$ is given by Eq. (A.13). The first terms in the right hand sides of Eqs. (3.18), (3.20) - (3.22) diverge in the limit $r' \to r$. These terms coincide with expressions corresponding to the case of absence of the vortex defect (see also Appendix B):

\[ \int_0^{2\pi} d\varphi \, tr \left[ \Sigma G^\omega(r, \varphi; r', \varphi) \right] |_{e\Phi=0} = smK_0(\kappa|r - r'|), \quad (3.23) \]

\[ \int_0^{2\pi} d\varphi \, tr \left[ J G^\omega(r, \varphi; r', \varphi) \right] |_{e\Phi=0} = \frac{\omega}{2} K_0(\kappa|r - r'|), \quad (3.24) \]

\[ \int_0^{2\pi} d\varphi \, tr \left[ J^2 G^\omega(r, \varphi; r', \varphi) \right] |_{e\Phi=0} = \frac{e^2}{4\pi} \kappa \frac{r + r'}{|r - r'|} K_1(\kappa|r - r'|), \quad (3.25) \]

\[ \int_0^{2\pi} d\varphi \, tr \left[ \Omega J G^\omega(r, \varphi; r', \varphi) \right] |_{e\Phi=0} = \frac{e^2}{4\pi} \kappa \frac{r + r'}{|r - r'|} K_1(\kappa|r - r'|), \quad (3.26) \]

and, otherwise,

\[ \int_0^{2\pi} d\varphi \, tr \left[ \Lambda G^\omega(r, \varphi; r', \varphi) \right] |_{e\Phi=0} = \int_0^{\infty} dr \, r \int_0^{2\pi} d\varphi \, tr \left[ \Omega G^\omega(r, \varphi; r', \varphi) \right] |_{e\Phi=0} = 0. \quad (3.27) \]

Thus, quantities (3.17) - (3.22) are made finite in the limit $r' \to r$ by subtracting expressions corresponding to the case of absence of the vortex defect. Integrating over radial variables, we get the renormalized traces:

\[ Tr \Lambda(H - \omega)^{-1} \equiv \int_0^{\infty} dr \int_0^{2\pi} d\varphi \, tr \left[ \Lambda G^\omega(r, \varphi; r, \varphi) \right] = \frac{s}{\omega^2 - m^2} \left[ F^2(\omega + m)\tan \nu_\omega - (1 - F)^2(\omega - m)e^{iF\pi} \tan \nu_\omega + e^{iF\pi} \right] - \frac{2}{3} \left( F - \frac{1}{2} \right) F(1 - F)\omega, \quad (3.28) \]

\[ Tr \Sigma(H - \omega)^{-1} \equiv \int_0^{\infty} dr \int_0^{2\pi} d\varphi \{ tr \left[ \Sigma G^\omega(r, \varphi; r, \varphi) \right] - tr \left[ \Sigma G^\omega(r, \varphi; r, \varphi) \right] |_{e\Phi=0} \} = \frac{1}{2} \frac{s}{\omega^2 - m^2} \left[ F(\omega + m)\tan \nu_\omega - (1 - F)(\omega - m)e^{iF\pi} \tan \nu_\omega + e^{iF\pi} \right] - F(1 - F)m, \quad (3.29) \]
\[ Tr \Omega(H - \omega)^{-1} \equiv \int_0^\infty dr \int_0^{2\pi} d\varphi \, \text{tr} \left[ \Omega \, G^\omega(r, \varphi; r, \varphi) \right] = \]
\[ = -\frac{e^2}{6\pi} \frac{sF(1 - F)}{\omega^2 - m^2} \frac{(1 + F) \tan \nu_\omega - (2 - F)e^{iF\pi}}{\tan \nu_\omega + e^{iF\pi}}, \quad (3.30) \]

\[ Tr \Sigma J(H - \omega)^{-1} \equiv \int_0^\infty dr \int_0^{2\pi} d\varphi \left\{ \text{tr} \left[ \Sigma J \, G^\omega(r, \varphi; r, \varphi) \right] - \text{tr} \left[ \Sigma J \, G^\omega(r, \varphi; r, \varphi) \right] \right|_{\Phi=0} \right\} = \]
\[ = \frac{1}{2} \frac{F - \frac{1}{2}}{\omega^2 - m^2} \frac{F(\omega + m) \tan \nu_\omega - (1 - F)(\omega - m)e^{iF\pi}}{\omega^2 - m^2} + \frac{1}{4} \frac{F(1 - F)}{\omega^2 - m^2} \left[ \omega - \frac{4}{3}(F - \frac{1}{2})m \right], \quad (3.31) \]

\[ Tr J^2(H - \omega)^{-1} \equiv \int_0^\infty dr \int_0^{2\pi} d\varphi \left\{ \text{tr} \left[ J^2 \, G^\omega(r, \varphi; r, \varphi) \right] - \text{tr} \left[ J^2 \, G^\omega(r, \varphi; r, \varphi) \right] \right|_{\Phi=0} \right\} = \]
\[ = -\frac{(F - \frac{1}{2})^2}{\omega^2 - m^2} \frac{F(\omega + m) \tan \nu_\omega + (1 - F)(\omega - m)e^{iF\pi}}{\omega^2 - m^2} + \frac{1}{2} \frac{F(1 - F)}{\omega^2 - m^2} \left\{ \left[ \frac{1}{2} - F(1 - F) \right] \omega - \frac{4}{3}(F - \frac{1}{2})m \right\}, \quad (3.32) \]

\[ Tr \Omega J(H - \omega)^{-1} \equiv \int_0^\infty dr \int_0^{2\pi} d\varphi \left\{ \text{tr} \left[ \Omega J \, G^\omega(r, \varphi; r, \varphi) \right] - \text{tr} \left[ \Omega J \, G^\omega(r, \varphi; r, \varphi) \right] \right|_{\Phi=0} \right\} = \]
\[ = \frac{e^2}{8\pi} \frac{F(1 - F)}{\omega^2 - m^2} \frac{F(1 + F) \tan \nu_\omega + (1 - F)(2 - F)e^{iF\pi}}{\omega^2 - m^2} - F(1 - F)\omega, \quad (3.33) \]

where the integration is performed at \( \text{Re} \, \kappa > |\text{Im} \, \kappa| \) and, then, is continued analytically to half-plane \( \text{Re} \, \kappa > 0 \) \( (\text{Im} \, \kappa > 0) \) which corresponds to the whole plane of complex \( \omega \). For completeness, we present here the result of Ref.\[24]:

\[ Tr \, (H - \omega)^{-1} \equiv \int_0^\infty dr \int_0^{2\pi} d\varphi \left[ \text{tr} \, G^\omega(r, \varphi; r, \varphi) - \text{tr} \, G^\omega(r, \varphi; r, \varphi) \right|_{\Phi=0} \right] = \]
\[ = -\frac{1}{\omega^2 - m^2} \frac{F(\omega + m) \tan \nu_\omega + (1 - F)(\omega - m)e^{iF\pi}}{\omega^2 - m^2} - F(1 - F)\omega. \quad (3.34) \]

There are remarkable relations among different traces. In particular, summing Eqs.\[3.28\] and \[3.29\] we get

\[ Tr \, J(H - \omega)^{-1} = -s \left( F - \frac{1}{2} \right) Tr \, (H - \omega)^{-1} + \frac{sF(1 - F)}{\omega^2 - m^2} \left[ \frac{1}{3} \left( F - \frac{1}{2} \right) \omega + \frac{1}{2}m \right]. \quad (3.35) \]
We list also some other relations:

\[
\begin{align*}
\text{Trace (3.30)} & \text{ is expressed through traces (3.29) and (3.34):} \\
\frac{2\pi m}{e^2} \text{Tr} \Omega(H-\omega)^{-1} = \frac{s}{4} \text{Tr} (H-\omega)^{-1} - \left( F - \frac{1}{2} \right) \text{Tr} \Sigma(H-\omega)^{-1} + \\
& \quad + \frac{1}{4} sF(1-F) \left[ \omega + \frac{2}{3} \left( F - \frac{1}{2} \right) m \right]. \quad \text{(3.36)}
\end{align*}
\]

We list also some other relations:

\[
\begin{align*}
\text{Tr} J^2(H-\omega)^{-1} &= \left( F - \frac{1}{2} \right)^2 \text{Tr} (H-\omega)^{-1} + \frac{1}{2} \frac{F(1-F)}{\omega^2 - m^2} \left[ F(1-F)\omega - \frac{4}{3} \left( F - \frac{1}{2} \right) m \right], \quad \text{(3.37)} \\
\text{Tr} \Lambda J(H-\omega)^{-1} &= -s \left( F - \frac{1}{2} \right) \text{Tr} \Lambda(H-\omega)^{-1} - \\
& \quad - \frac{1}{3} \frac{F(1-F)}{\omega^2 - m^2} \left\{ \frac{1}{2} - F(1-F)\omega + \left( F - \frac{1}{2} \right) m \right\}, \quad \text{(3.38)} \\
\text{Tr} \Sigma J(H-\omega)^{-1} &= -s \left( F - \frac{1}{2} \right) \text{Tr} \Sigma(H-\omega)^{-1} + \frac{1}{4} \frac{F(1-F)}{\omega^2 - m^2} \left[ \omega + \frac{2}{3} \left( F - \frac{1}{2} \right) m \right], \quad \text{(3.39)} \\
\text{Tr} \Omega J(H-\omega)^{-1} &= -s \left( F - \frac{1}{2} \right) \text{Tr} \Omega(H-\omega)^{-1} + \frac{e^2}{12\pi} \frac{F(1-F)}{\omega^2 - m^2} \left[ 1 + \frac{1}{2} F(1-F) \right]. \quad \text{(3.40)}
\end{align*}
\]

## 4 Averages

Similar to Eq. (3.1), we get the thermal averages of orbital angular momentum

\[
L(T) = -\frac{1}{2} \int_C \frac{d\omega}{2\pi i} \tanh \left( \frac{1}{2} \beta \omega \right) \text{Tr} \Lambda(H-\omega)^{-1}, \quad \text{(4.1)}
\]

and spin

\[
S(T) = -\frac{1}{2} \int_C \frac{d\omega}{2\pi i} \tanh \left( \frac{1}{2} \beta \omega \right) \text{Tr} \Sigma(H-\omega)^{-1}, \quad \text{(4.2)}
\]

where \( C \) is the contour \((-\infty + i0, +\infty + i0)\) and \((+\infty - i0, -\infty - i0)\) in the complex \( \omega \)-plane. Using Eqs. (3.28) and (3.29) and deforming the contour around the cuts and poles on the real axis, we obtain the following expressions for the averages as real integrals:

\[
\begin{align*}
L(T) &= s \frac{\sin(F\pi)}{\pi} \int_0^\infty \frac{du}{u^2 + u + 1} \tanh \left( \frac{1}{2} \beta m \sqrt{u + 1} \right) \times \\
& \quad F^2 u^F A + (1-F)^2 u^{1-F} A^{-1} + u \left\{ \frac{1}{2} - F(1-F) \right\} \left( u^F A + u^{1-F} A^{-1} \right) - (2F - 1) \cos(F\pi) \right\} \\
& \quad + \frac{u^F A - u^{1-F} A^{-1} + 2 \cos(F\pi)^2 + 4(u + 1) \sin^2(F\pi)}{[u^F A - u^{1-F} A^{-1} + 2 \cos(F\pi)]^2 + 4(u + 1) \sin^2(F\pi)} \\
& \quad + \frac{s}{4} \left[ 1 - \text{sgn}(A) \right] \frac{[1 - 2F(1-F)]E_{BS} + (2F - 1)m}{(2F - 1)E_{BS} + m} \tanh \left( \frac{1}{2} \beta E_{BS} \right), \quad \text{(4.3)}
\end{align*}
\]
\[ S(T) = -s \frac{\sin(F\pi)}{2\pi} \int_0^\infty \frac{du}{u\sqrt{u}+1} \tanh\left( \frac{1}{2} \beta m \sqrt{u} + 1 \right) \times \]
\[ \times \frac{Fu^F A - (1 - F)u^1 - F A^{-1} + u \left[ \frac{1}{2} u^F A + \frac{1}{2} u^1 - F A^{-1} - (2F - 1) \cos(F\pi) \right]}{[u^F A - u^1 - F A^{-1} + 2 \cos(F\pi)]^2 + 4(u + 1) \sin^2(F\pi)} \]
\[ - \frac{s}{8} [1 - \text{sgn}(A)] \frac{E_{BS} + (2F - 1)m}{(2F - 1)E_{BS} + m} \tanh\left( \frac{1}{2} \beta m \right) + \frac{s}{4} F(1 - F) \tanh\left( \frac{1}{2} \beta m \right), \quad (4.4) \]

where
\[ A = 2^{1 - 2F} \frac{\Gamma(1 - F)}{\Gamma(F)} \tan\left( \frac{s \Theta}{2} + \frac{\pi}{4} \right), \quad (4.5) \]

\[ \Gamma(u) \text{ is the Euler gamma function, } E_{BS} \text{ is the energy of the bound state in the one-particle spectrum, which is determined as a real root of algebraic equation (for details see Ref.}\ [22]\]
\[ \frac{(1 - m^{-1}E_{BS})^F}{(1 + m^{-1}E_{BS})^{1 - F}} A = -1; \quad (4.6) \]

note that the bound state exists at \( \cos \Theta < 0 \) (\( A < 0 \)), and its energy is zero at \( A = -1 \), and, otherwise, one has \( 0 < |E_{BS}| < |m| \) and
\[ \text{sgn}(E_{BS}) = \frac{1}{2} \text{sgn}(m) [\text{sgn}(1 + A^{-1}) - \text{sgn}(1 + A)]. \quad (4.7) \]

Summing Eqs. (4.3) and (4.4) we get the thermal average of total angular momentum, which can be written in the form (compare with Eq. (3.35)):
\[ M(T) = -s \left( F - \frac{1}{2} \right) N(T) + \frac{s}{4} F(1 - F) \tanh\left( \frac{1}{2} \beta m \right), \quad (4.8) \]

where the thermal average of fermion number (i.e. electric charge divided by \( e \)) is given by expression (see Ref. [24]):
\[ \frac{(1 - m^{-1}E_{BS})^F}{(1 + m^{-1}E_{BS})^{1 - F}} A = -1; \quad (4.6) \]

and
\[ N(T) = -\frac{\sin(F\pi)}{\pi} \int_0^\infty \frac{du}{u\sqrt{u}+1} \tanh\left( \frac{1}{2} \beta m \sqrt{u} + 1 \right) \times \]
\[ \times \frac{Fu^F A - (1 - F)u^1 - F A^{-1} + u \left[ \frac{1}{2} u^F A + \frac{1}{2} u^1 - F A^{-1} - (2F - 1) \cos(F\pi) \right]}{[u^F A - u^1 - F A^{-1} + 2 \cos(F\pi)]^2 + 4(u + 1) \sin^2(F\pi)} \]
\[ - \frac{1}{4} [1 - \text{sgn}(A)] \tanh\left( \frac{1}{2} \beta E_{BS} \right). \quad (4.9) \]

Note that both \( L(T) \) and \( S(T) \) are infinite \( \left( \text{divergent as integral } \int_0^\infty \frac{du}{u} \right) \) at half-integer values of \( e\Phi \), unless \( A = 0 \) or \( A^{-1} = 0 \). However, this divergence cancels in the sum, and one gets
\[ M(T)\big|_{F=\frac{1}{2}} = \frac{s}{16} \tanh\left( \frac{1}{2} \beta m \right). \quad (4.10) \]
In the cases of $A = 0$ and $A^{-1} = 0$ expressions (4.3) and (4.4) simplify
\[ L(T) = \frac{s}{2} \left( F - \frac{1}{2} \pm \frac{1}{2} \right)^2 \tanh \left( \frac{1}{2} \beta m \right), \quad \Theta = \pm s \frac{\pi}{2} \text{ (mod } 2\pi) , \] (4.11)
and
\[ S(T) = -\frac{s}{4} \left( F - \frac{1}{2} \pm \frac{1}{2} \right)^2 \tanh \left( \frac{1}{2} \beta m \right), \quad \Theta = \pm s \frac{\pi}{2} \text{ (mod } 2\pi) . \] (4.12)

In the limit $T \to 0$ ($\beta \to \infty$) we get the results of Ref. [23]:
\[ L(0) = \frac{s \text{ sgn}(m)}{2\pi} \int_{-\infty}^{\infty} \frac{dv}{\sqrt{v-1}} \frac{F^2 v^{-1+F} A + 1 - 2F(1 - F) + (1 - F)^2 v^{-F} A^{-1}}{v^F A + 2 + v^{1-F} A^{-1}} , \] (4.13)
and
\[ S(0) = \frac{1}{4} s \text{ sgn}(m) \left[ F(1 - F) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dv}{\sqrt{v-1}} \frac{F v^{-1+F} A + 1 + (1 - F)v^{-F} A^{-1}}{v^F A + 2 + v^{1-F} A^{-1}} \right] . \] (4.14)

In the high-temperature limit the averages tend to zero
\[ L(T \to \infty) = \left\{ \begin{array}{l}
\frac{s \text{ sgn}(m)}{2\pi} \frac{\sin(F\pi)}{\Gamma(1+F)} \frac{1 - 2F(1 - F)}{1 - 2F} \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \left( \frac{|m|}{kBT} \right)^{1-2F} , \quad 0 < F < \frac{1}{2} \\
\frac{s \text{ sgn}(m)}{2\pi} \frac{\sin(F\pi)}{\Gamma(2-F)} \frac{1 - 2F(1 - F)}{2F - 1} \cot \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \left( \frac{|m|}{kBT} \right)^{2F-1} , \quad \frac{1}{2} < F < 1
\end{array} \right. , \] (4.15)
and
\[ S(T \to \infty) = \left\{ \begin{array}{l}
\frac{-s \text{ sgn}(m)}{4\pi} \frac{\sin(F\pi)}{\Gamma(1+F)} \frac{1}{1 - 2F} \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \left( \frac{|m|}{kBT} \right)^{1-2F} , \quad 0 < F < \frac{1}{2} \\
\frac{-s \text{ sgn}(m)}{4\pi} \frac{\sin(F\pi)}{\Gamma(2-F)} \frac{1}{2F - 1} \cot \left( \frac{\Theta}{2} + \frac{\pi}{4} \right) \left( \frac{|m|}{kBT} \right)^{2F-1} , \quad \frac{1}{2} < F < 1
\end{array} \right. \] (4.16)

In conclusion of this section, let us consider the thermal average of induced flux (times $e$)
\[ O(T) = -\frac{1}{2} \int_C \frac{d\omega}{2\pi i} \tanh(\beta\omega) \text{Tr} \Omega(H - \omega)^{-1} , \] (4.17)
where $\Omega$ is given by Eq. (2.19). Using trace identity (3.36), we get
\[ O(T) = \frac{e^2}{2\pi m} \left[ \frac{s}{4} N(T) - \left( F - \frac{1}{2} \right) S(T) + \frac{s}{12} \left( F - \frac{1}{2} \right) F(1 - F) \tanh \left( \frac{1}{2} \beta m \right) \right] , \] (4.18)
or
\[
O(T) = -\frac{e^2}{4\pi m} sF(1 - F) \left\{ \frac{\sin(F\pi)}{\pi} \int_0^\infty \frac{du}{u\sqrt{u+1}} \tanh \left( \frac{1}{2} \beta m \sqrt{u+1} \right) \times \\
\times \frac{u^F A - u^{1-F} A^{-1} - 2u \cos(F\pi)}{[u^F A - u^{1-F} A^{-1} + 2\cos(F\pi)]^2 + 4(u+1)\sin^2(F\pi)} + \\
+ \frac{1}{2} [1 - \text{sgn}(A)] \frac{m}{(2F - 1)E_{BS} + m} \tanh \left( \frac{1}{2} \beta E_{BS} \right) + \frac{1}{3} \left( F - \frac{1}{2} \right) \tanh \left( \frac{1}{2} \beta m \right) \right\}. \quad (4.19)
\]

Unlike \(L(T)\) and \(S(T)\), \(O(T)\) is finite at half-integer values of \(e\Phi:\)
\[
O(T)|_{F=\frac{1}{2}} = -\frac{e^2}{32\pi m} \left\{ [1 - \text{sgn}(\cos \Theta)] \tanh \left( \frac{1}{2} \beta m \sin \Theta \right) + \\
+ \frac{\sin 2\Theta}{2\pi} \int_1^\infty \frac{dv}{v(v-1)} \tanh \left( \frac{1}{2} \beta m \sqrt{v} \right) \right\}, \quad (4.20)
\]

In the cases of \(A = 0\) and \(A^{-1} = 0\), Eq. (4.19) takes form
\[
O(T) = -\frac{e^2}{12\pi m} sF(1 - F) \left( F - \frac{1}{2} \pm \frac{3}{2} \right) \tanh \left( \frac{1}{2} \beta m \right), \quad \Theta = \pm \frac{\pi}{2} \text{ (mod } 2\pi). \quad (4.21)
\]

Note that relation (4.18) at zero temperature was first obtained in Ref. [23], and expression (4.19) at zero temperature takes form (see Ref. [22])
\[
O(0) = -\frac{e^2}{4\pi|m|} sF(1 - F) \left\{ \frac{1}{3} (F - \frac{1}{2}) + \frac{1}{2\pi} \int_1^\infty \frac{dv}{v(v-1)} \frac{u^F A - u^{1-F} A^{-1}}{u^F A + 2 + u^{1-F} A^{-1}} \right\}. \quad (4.22)
\]

In the high-temperature limit we get
\[
O(T \to \infty) = -\frac{e^2}{8\pi k_B T} sF(1 - F) \left\{ \frac{1}{2} [1 - \text{sgn}(A)] \frac{E_{BS}}{(2F - 1)E_{BS} + m} + \frac{1}{3} (F - \frac{1}{2}) + \\
+ \frac{\sin(F\pi)}{\pi} \int_0^\infty \frac{du}{u} \frac{u^F A - u^{1-F} A^{-1} - 2u \cos(F\pi)}{[u^F A - u^{1-F} A^{-1} + 2\cos(F\pi)]^2 + 4(u+1)\sin^2(F\pi)} \right\}. \quad (4.23)
\]

5 Correlations

Similar to Eq. (3.2), we get the thermal correlations of fermion number with orbital angular momentum
\[
\Delta(T; \hat{L}, \hat{N}) = \frac{1}{4} \int \frac{d\omega}{2\pi i} \text{sech}^2 \left( \frac{1}{2} \beta \omega \right) Tr \Lambda(H - \omega)^{-1}, \quad (5.1)
\]
and fermion number with spin

\[ \Delta(T; \hat{S}, \hat{N}) = \frac{1}{4} \int_C \frac{d\omega}{2\pi i} \text{sech}^2 \left( \frac{1}{2\beta \omega} \right) Tr \Sigma(H - \omega)^{-1}, \] (5.2)

where contour \( C \) is defined as above. Using Eqs.(3.28) and (3.29) and deforming the contour around the cuts and poles on the real axis, we obtain the following expressions for the correlations as real integrals:

\[ \Delta(T; \hat{L}, \hat{N}) = -\frac{s \sin(F \pi)}{2\pi} \int_0^\infty \frac{du}{u} \text{sech}^2 \left( \frac{1}{2\beta m \sqrt{u + 1}} \right) \times \]

\[ \times \frac{F^2 u^F A - (1 - F)^2 u^{1-F} A^{-1} - u[1 - 2F(1 - F)] \cos(F \pi)}{[u^F A - u^{1-F} A^{-1} + 2 \cos(F \pi)]^2 + 4(u + 1) \sin^2(F \pi)} - \frac{s}{8} \left[ 1 - \text{sgn}(A) \right] \frac{[1 - 2F(1 - F)] E_{BS} + (2F - 1)m}{(2F - 1) E_{BS} + m} \text{sech}^2 \left( \frac{1}{2\beta E_{BS}} \right) + \]

\[ + \frac{s}{6} \left( F - \frac{1}{2} \right) F(1 - F) \text{sech}^2 \left( \frac{1}{2\beta m} \right), \] (5.3)

and

\[ \Delta(T; \hat{S}, \hat{N}) = \frac{s \sin(F \pi)}{4\pi} \int_0^\infty \frac{du}{u} \text{sech}^2 \left( \frac{1}{2\beta m \sqrt{u + 1}} \right) \times \]

\[ \times \frac{F^2 u^F A - (1 - F)^2 u^{1-F} A^{-1} - u \cos(F \pi)}{[u^F A - u^{1-F} A^{-1} + 2 \cos(F \pi)]^2 + 4(u + 1) \sin^2(F \pi)} + \]

\[ + \frac{s}{16} \left[ 1 - \text{sgn}(A) \right] \frac{E_{BS} + (2F - 1)m}{(2F - 1) E_{BS} + m} \text{sech}^2 \left( \frac{1}{2\beta E_{BS}} \right), \] (5.4)

where \( A \) is defined by Eq.(4.5), and \( E_{BS} \) is determined as a real root of Eq.(4.6).

It should be noted that, if one takes operator

\[ \Lambda' = \Lambda + e \Phi = -i \mathbf{x} \times \partial, \] (5.5)

and defines the corresponding operator in the second-quantized theory,

\[ \hat{L}' = \frac{1}{2} \int d^2x \left[ \Psi^+, \Lambda' \Psi \right]_-, \] (5.6)

then correlation \( \Delta(T; \hat{L}', \hat{N}) \) is infinite. To see this, let us consider quantity (compare with
Eq. (3.17)

\[
\int_0^{2\pi} d\varphi \text{tr} \left[ \mathcal{N} G^\omega (r, \varphi; r', \varphi) \right] = \frac{2 \sin(F\pi)}{\pi (\tan \nu_\omega + e^{iF\pi})} \times \\
\times \left[ n_0(\omega + m) \tan \nu_\omega K_F(\kappa r)K_F(\kappa r') + (n_0 + s)(\omega - m)e^{iF\pi}K_{1-F}(\kappa r)K_{1-F}(\kappa r') \right] - \\
- \frac{\sin(F\pi)}{2F(1-F)\pi} \omega \int_0^\infty dy \exp \left( -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2r r'} y \right) \left\{ e^{\Phi} \int_y^\infty du e^{-u[(1-F)K_F(u) + + FK_{1-F}(u)]} - \left[ (2F - 1) \left( n_0 + \frac{1}{2} s \right) + \frac{1}{2} s \right] [K_F(y) - K_{1-F}(y)] \right\} + 2\omega e^{\Phi} K_0(\kappa|r - r'|).
\]

(5.7)

The last term in Eq. (5.7) diverges in the limit \( r' \to r \), and this divergence cannot be compensated by subtraction (as is the case for Eqs. (3.18), (3.20) - (3.22)), because Eq. (3.21) holds. Since this divergence is proportional to \( \omega \), it does not contribute to average \( L'(T) \), but does contribute to correlation \( \Delta(T; \hat{L}', \hat{N}) \) yielding a term which is \(-\frac{1}{2} e^\Phi \text{sech}^2 \left( \frac{1}{2} \beta m \right) \) times infinity. Thus, if one accepts finiteness of correlations as physically plausible condition, then one has to favour gauge-invariant definition of orbital angular momentum, i.e. to choose \( \hat{A}(3.17) \) instead of \( \hat{A}(5.5) \).

Contrary to the case of averages \( L(T) \) and \( S(T) \), correlations \( \Delta(T; \hat{L}, \hat{N}) \) and \( \Delta(T; \hat{S}, \hat{N}) \) are finite at half-integer values of \( e^\Phi \):

\[
\Delta(T; \hat{L}, \hat{N}) \bigg|_{F=\frac{1}{2}} = -\Delta(T; \hat{S}, \hat{N}) \bigg|_{F=\frac{1}{2}} = -\frac{\sin(2\Theta)}{32\pi} \int_1^\infty dv \text{sech}^2 \left( \frac{1}{2} \beta m \sqrt{v} \right) - \\
- \frac{\sin \Theta}{16} \left[ 1 - \text{sgn}(\cos \Theta) \right] \text{sech}^2 \left( \frac{1}{2} \beta m \sin \Theta \right). 
\]

(5.8)

In the case of \( A = 0 \) and \( A^{-1} = 0 \) expressions (5.3) and (5.4) simplify:

\[
\Delta(T; \hat{L}, \hat{N}) = \frac{s}{6} \left( F - \frac{1}{2} \pm \frac{1}{2} \right) \left[ 1 + 2 \left( F - \frac{1}{2} \pm \frac{1}{2} \right)^2 \right] \text{sech}^2 \left( \frac{1}{2} \beta m \right), \quad \Theta = \pm \frac{\pi}{2} \text{(mod 2}\pi),
\]

and

\[
\Delta(T; \hat{S}, \hat{N}) = -\frac{s}{4} \left( F - \frac{1}{2} \pm \frac{1}{2} \right) \text{sech}^2 \left( \frac{1}{2} \beta m \right), \quad \Theta = \pm \frac{\pi}{2} \text{(mod 2}\pi).
\]

(5.9)

(5.10)

In the limit \( T \to 0 \) (\( \beta \to \infty \)) correlations (5.3) and (5.4) tend exponentially to zero for almost all values of \( \Theta \) with the exception of one corresponding to the zero bound state energy, \( E_{BS} = 0 \) (\( A = -1 \)):

\[
\Delta(0; \hat{L}, \hat{N}) = -\frac{1}{2} \Delta(0; \hat{S}, \hat{N}) = \begin{cases} 
0, & A \neq -1 \\
-\frac{s}{2} \left( F - \frac{1}{2} \right), & A = -1.
\end{cases}
\]

(5.11)
In the high-temperature limit the correlations tend to finite values:

\[
\Delta(\infty; \hat{L}, \hat{N}) = -\frac{s}{8} \left[1 - \text{sgn}(A)\right] \frac{1 - 2F(1 - F)E_{BS} + (2F - 1)m}{(2F - 1)E_{BS} + m} + \frac{s}{6} \left(F - \frac{1}{2}\right) F(1 - F) - \frac{s \sin(F\pi)}{2\pi} \int_0^\infty \frac{du}{u} \frac{u^2 F A - (1 - F)^2 u^{1-F} A^{-1} - u \left[1 - 2F(1 - F)\right] \cos(F\pi)}{u^2 A - u^{1-F} A^{-1} + 2 \cos(F\pi)^2 + 4(u + 1) \sin^2(F\pi)},
\]

and

\[
\Delta(\infty; \hat{S}, \hat{N}) = \frac{s}{16} \left[1 - \text{sgn}(A)\right] \frac{E_{BS} + (2F - 1)m}{(2F - 1)E_{BS} + m} + \frac{s \sin(F\pi)}{4\pi} \int_0^\infty \frac{du}{u} \frac{F u^F A - (1 - F) u^{1-F} A^{-1} - u \cos(F\pi)}{u^2 A - u^{1-F} A^{-1} + 2 \cos(F\pi)^2 + 4(u + 1) \sin^2(F\pi)}.
\]

Summing Eqs. (5.3) and (5.4), we get the thermal correlation of two conserved observables, total angular momentum and fermion number, which can be recast in the form:

\[
\Delta(T; \hat{M}, \hat{N}) = -s \left(F - \frac{1}{2}\right) \Delta(T; \hat{N}, \hat{N}) - \frac{s}{12} \left(F - \frac{1}{2}\right) F(1 - F) \text{sech}^2 \left(\frac{1}{2} \beta m\right),
\]

where

\[
\Delta(T; \hat{N}, \hat{N}) = \frac{\sin(F\pi)}{2\pi} \int_0^\infty \frac{du}{u} \text{sech}^2 \left(\frac{1}{2} \beta m \sqrt{u} + 1\right) \times
\]

\[
F u^F A + (1 - F) u^{1-F} A^{-1} - u(2F - 1) \cos(F\pi)
\]

\[
\left[\frac{u^2 A - u^{1-F} A^{-1} + 2 \cos(F\pi)^2 + 4(u + 1) \sin^2(F\pi)}{u^2 A - u^{1-F} A^{-1} + 2 \cos(F\pi)^2 + 4(u + 1) \sin^2(F\pi)}\right] + \frac{1}{8} \left[1 - \text{sgn}(A)\right] \text{sech}^2 \left(\frac{1}{2} \beta E_{BS}\right) - \frac{1}{4} F(1 - F) \text{sech}^2 \left(\frac{1}{2} \beta m\right)
\]

is the quadratic fluctuation of fermion number which was first computed in Ref. [24]. Note that correlation \(\Delta(T; \hat{M}, \hat{N})\) vanishes at half-integer values of e\(\Phi\).

Using trace identities (3.38) and (3.39), we get the thermal correlations of total angular momentum with orbital angular momentum

\[
\Delta(T; \hat{L}, \hat{M}) = -s \left(F - \frac{1}{2}\right) \Delta(T; \hat{L}, \hat{N}) + \frac{1}{24} \left[1 - F(1 - F)\right] F(1 - F) \text{sech}^2 \left(\frac{1}{2} \beta m\right),
\]

and total angular momentum with spin

\[
\Delta(T; \hat{S}, \hat{M}) = -s \left(F - \frac{1}{2}\right) \Delta(T; \hat{S}, \hat{N}) - \frac{1}{16} F(1 - F) \text{sech}^2 \left(\frac{1}{2} \beta m\right).
\]

Using trace identity (3.36), we get the thermal correlation of fermion number with induced flux multiplied by e

\[
\Delta(T; \hat{O}, \hat{N}) = \frac{e^2}{2\pi m} \left[\frac{s}{4} \Delta(T; \hat{N}, \hat{N}) - \left(F - \frac{1}{2}\right) \Delta(T; \hat{S}, \hat{N}) - \frac{s}{16} F(1 - F) \text{sech}^2 \left(\frac{1}{2} \beta m\right)\right].
\]
or in the explicit form

\[
\Delta(T; \hat{O}, \hat{N}) = \frac{e^{2}\sin(F\pi)}{2(2\pi)^2} \frac{sF(1 - F)}{m} \int_{0}^{\infty} \frac{du}{u} \text{sech}^{2}\left(\frac{1}{2}\beta m \sqrt{u} + 1\right) \times \\
\times \frac{u^{FA} + u^{1-F}A^{-1}}{[u^{FA} - u^{1-F}A^{-1} + 2\cos(F\pi)]^{2} + 4(u + 1)\sin^{2}(F\pi)} + \\
+ \frac{e^{2}}{16\pi} [1 - \text{sgn}(A)] \frac{sF(1 - F)}{(2F - 1)E_{BS} + m} \text{sech}^{2}\left(\frac{1}{2}\beta E_{BS}\right) - \frac{e^{2}sF(1 - F)}{16\pi m} \text{sech}^{2}\left(\frac{1}{2}\beta m\right).
\]

(5.19)

At half-integer values of \(e\Phi\) we get

\[
\Delta(T; \hat{O}, \hat{N})\bigg|_{F=\frac{1}{2}} = \frac{s e^{2}\cos \Theta}{4(4\pi)^{2}m} \int_{1}^{\infty} \frac{dv}{\sqrt{v} - 1} \text{sech}^{2}\left(\frac{1}{2}\beta m \sqrt{v}\right) + \\
+ \frac{s e^{2}}{64\pi m} [1 - \text{sgn}(\cos \Theta)] \text{sech}^{2}\left(\frac{1}{2}\beta m \sin \Theta\right) - \frac{s e^{2}}{64\pi m} \text{sech}^{2}\left(\frac{1}{2}\beta m\right).
\]

(5.20)

Correlation \(\Delta(T; \hat{O}, \hat{N})\) vanishes in the cases of \(A = 0\) and \(A^{-1} = 0\). In the zero-temperature limit we get

\[
\Delta(0; \hat{O}, \hat{N}) = \begin{cases} 
0, & A \neq -1 \\
\frac{e^{2}sF(1 - F)}{8\pi m}, & A = -1.
\end{cases}
\]

(5.21)

In the high-temperature limit correlation (5.19) tends to a finite value

\[
\Delta(\infty; \hat{O}, \hat{N}) = \frac{e^{2}}{16\pi} [1 - \text{sgn}(A)] \frac{sF(1 - F)}{(2F - 1)E_{BS} + m} - \frac{e^{2}sF(1 - F)}{16\pi m} + \\
+ \frac{e^{2}\sin(F\pi)}{2(2\pi)^{2}} \frac{sF(1 - F)}{m} \int_{0}^{\infty} \frac{du}{u} \frac{u^{FA} + u^{1-F}A^{-1}}{[u^{FA} - u^{1-F}A^{-1} + 2\cos(F\pi)]^{2} + 4(u + 1)\sin^{2}(F\pi)}.
\]

(5.22)

Using trace identity (3.40), we get the thermal correlation of total angular momentum with induced flux multiplied by \(e\)

\[
\Delta(T; \hat{O}, \hat{M}) = -s \left(F - \frac{1}{2}\right) \Delta(T; \hat{O}, \hat{N}).
\]

(5.23)

Thus, this correlation vanishes at half-integer values of \(e\Phi\).

6 Nonnegativeness of quadratic fluctuations

As we have seen in the previous section, a gauge-invariant definition of angular momentum is required by the finiteness of the correlation of angular momentum with fermion number. In the present section we shall consider further restrictions which are imposed by the nonnegativeness of quadratic fluctuations.
With the use of trace identity \([3.37]\), the thermal quadratic fluctuation of total angular momentum, Eq.\((6.2)\), is expressed through that of fermion number:

\[
\Delta(T; \hat{M}, \hat{M}) = \left( F - \frac{1}{2} \right)^2 \Delta(T; \hat{N}, \hat{N}) - \frac{1}{8} F^2 (1 - F)^2 \text{sech}^2 \left( \frac{1}{2} \beta m \right), \tag{6.1}
\]

where \(\Delta(T; \hat{N}, \hat{N})\) is given by Eq.\((5.15)\). The thermal quadratic fluctuation of fermion number was analyzed in detail in Ref.\([24]\). In particular, in the high-temperature limit we get

\[
\Delta(\infty; \hat{N}, \hat{N}) = \begin{cases}
\frac{1}{4} (1 - F)^2, & \Theta \neq s \pi / 2 \pmod{2\pi} \\
\frac{1}{4} F^2, & \Theta = s \pi / 2 \pmod{2\pi} \\
\frac{1}{4} F^2, & \Theta \neq -s \pi / 2 \pmod{2\pi} \\
\frac{1}{4} (1 - F)^2, & \Theta = -s \pi / 2 \pmod{2\pi}
\end{cases}, \quad 0 < F \leq \frac{1}{2}
\]

\[
\Delta(\infty; \hat{M}, \hat{M}) = \begin{cases}
\frac{1}{8} (1 - F)^2 \left( (1 - F)^2 - \frac{1}{2} \right), & \Theta \neq s \pi / 2 \pmod{2\pi} \\
\frac{1}{8} F^2 \left( F^2 - \frac{1}{2} \right), & \Theta = s \pi / 2 \pmod{2\pi} \\
\frac{1}{8} F^2 \left( F^2 - \frac{1}{2} \right), & \Theta \neq -s \pi / 2 \pmod{2\pi} \\
\frac{1}{8} (1 - F)^2 \left( (1 - F)^2 - \frac{1}{2} \right), & \Theta = -s \pi / 2 \pmod{2\pi}
\end{cases}, \quad \frac{1}{2} \leq F < 1
\]

which is obviously positive. Substituting Eq.\((5.2)\) into Eq.\((6.1)\), we get

\[
\Delta(\infty; \hat{M}, \hat{M}) = \begin{cases}
\frac{1}{8} (1 - F)^2 \left( (1 - F)^2 - \frac{1}{2} \right), & \Theta \neq s \pi / 2 \pmod{2\pi} \\
\frac{1}{8} F^2 \left( F^2 - \frac{1}{2} \right), & \Theta = s \pi / 2 \pmod{2\pi} \\
\frac{1}{8} F^2 \left( F^2 - \frac{1}{2} \right), & \Theta \neq -s \pi / 2 \pmod{2\pi} \\
\frac{1}{8} (1 - F)^2 \left( (1 - F)^2 - \frac{1}{2} \right), & \Theta = -s \pi / 2 \pmod{2\pi}
\end{cases}, \quad 0 < F \leq \frac{1}{2}
\]

which is positive at \(0 < F < 1 - 2^{-1/2}\) and \(2^{-1/2} < F < 1 \left( \Theta \neq \pi / 2 \pmod{\pi} \right)\) only, and is nonpositive otherwise.

The cause of negativeness of \(\Delta(\infty; \hat{M}, \hat{M})\) is quite understandable from the mathematical point of view. Although operator \(J^2\), as a square of self-adjoint operator \(J\), is nonnegative definite, appropriate spectral density \(\tau_{J^2}(E)\) might be not, and the latter results in negativity of \(\Delta(T; \hat{M}, \hat{M})\), see Eq.\((3.21)\). Nonnegativeness of \(\tau_{J^2}(E)\) is rooted in the procedure of taking a functional trace of the resolvent kernel, with regularization and renormalization involved in the case of infinite space: initially, the trace is positive but divergent (see Eq.\((3.21)\) at \(r' \rightarrow r\)), the comparative trace which corresponds to the case of absence of the defect is also positive but divergent (see Eq.\((3.25)\) at \(r' \rightarrow r\)), then the difference of the above two traces appears to be finite but not positive. Namely the same is the mechanism of the appearance of the negative vacuum energy density, which is widely known as the Casimir effect \([29]\): the vacuum energy density in an infinite space bounded by two parallel plates is positive but divergent, the vacuum energy density in an infinite unbounded space is subtracted, and the result is finite but negative. There is a physically plausible interpretation...
of negativeness of the vacuum energy density, linking it to a force of attraction between two plates. Returning now to negativeness of the thermal quadratic fluctuation, we have not found, up to now, any physically plausible interpretation of this mathematically feasible effect. Thus, we have to stick to the paradigm that the quadratic fluctuation of the physically meaningful observable is to be nonnegative.

Let us recall that, in the case of planar rotationally symmetric system considered in the present paper, we have two conserved observables which are fermion number and total angular momentum. Fermion number is defined uniquely: operator $\hat{N}$ in the second-quantized theory corresponds to unity operator $I$ in the first-quantized theory. As to total angular momentum, situation is different: so far we have chosen operator $\hat{M}$ corresponding to $J_{2.10}$, but, equally as well, the choice can be a superposition of $\hat{M}$ and $\hat{N}$, $\hat{M} + \Xi \hat{N}$, see Eq.$(2.16)$, with $\Xi$ being a function of the parameters of the vortex defect. We shall use this ambiguity and fix $\Xi$ by the requirement that the quadratic fluctuation of improved total angular momentum behave qualitatively in a similar manner as that of fermion number.

First, it is straightforward to get general relation

$$
\Delta(T; \hat{M} + \Xi \hat{N}, \hat{M} + \Xi \hat{N}) = \left( F - \frac{1}{2} - s\Xi \right)^2 \Delta(T; \hat{N}, \hat{N}) - \frac{1}{2} \left[ s\Xi \left( F - \frac{1}{2} \right) + \frac{s}{4} F(1 - F) \right] F(1 - F) \text{sech}^2 \left( \frac{1}{2} \beta m \right). \quad (6.4)
$$

Then, one should note that, without a loss of generality, all possible values of $\Xi$ can be restricted to interval $0 \leq |\Xi| \leq 1/2$, since shift $\Xi \to \Xi \pm 1$ yields the same spectrum of the angular momentum operator in the first-quantized theory. Also, in view of the Bohm-Aharonov effect [15], we suppose that $\Xi$ depends on fractional part of $e\Phi$ rather than $e\Phi$ itself, or, in other words, it depends on $F(3.10)$. Since $F \to 1 - F$ as $s \to -s$, parameter $\Xi$ has to depend on $s(F - 1/2)$. Taking all the above into account, we fix $\Xi = \Xi_F$, where

$$
\Xi_F = \begin{cases} 
-\frac{1}{2} s \text{sgn} \left( F - \frac{1}{2} \right), & F \neq \frac{1}{2} \\
\frac{1}{2}, & F = \frac{1}{2}
\end{cases}, \quad (6.5)
$$

and define the improved total angular momentum operator as

$$
\hat{R} = \hat{M} + \Xi_F \hat{N}. \quad (6.6)
$$

The corresponding quadratic fluctuation takes form

$$
\Delta(T; \hat{R}, \hat{R}) = \left( \left| F - \frac{1}{2} \right| + \frac{1}{2} \right)^2 \Delta(T; \hat{N}, \hat{N}) + \frac{1}{4} \left[ \frac{1}{3} \left( F - \frac{1}{2} \right) - \frac{1}{2} F(1 - F) \right] F(1 - F) \text{sech}^2 \left( \frac{1}{2} \beta m \right). \quad (6.7)
$$

The high-temperature limit of Eq.$(6.7)$ is positive, and the behaviour of $\Delta(T; \hat{R}, \hat{R})$ at finite temperatures is qualitatively the same as that of $\Delta(T; \hat{N}, \hat{N})$. 23
In the present paper we continue a study of the properties of an ideal gas of two-dimensional relativistic massive electrons in the background of a static magnetic vortex defect, which was started in Ref.[24]. We find that this system at thermal equilibrium acquires, in addition to fermion number considered in Ref.[24], the following nontrivial characteristics: orbital angular momentum (4.3), spin (4.4), and induced magnetic flux (times e) (4.19). The local features of the field strength in the interior of the vortex are exhibited by self-adjoint extension parameter Θ which labels boundary conditions at the location of the vortex defect, and arbitrary values of vortex flux Φ are permitted; our results are periodic in Θ with period $2\pi$ at fixed Φ and periodic in Φ with period $e^{-1}$ at fixed Θ. Orbital angular momentum and spin are odd and induced flux is even under transition to the inequivalent representation of the Clifford algebra ($s \rightarrow -s$ or $m \rightarrow -m$). In the zero-temperature limit the results of Refs.[23] and [22] are recovered, see Eqs.(4.13), (4.14) and (4.22), whereas in the high-temperature limit all averages vanish, see Eqs.(4.15), (4.16) and (4.23).

The key point in this study is played by Section 3, where the appropriately renormalized traces of the resolvent operator are obtained, see Eqs.(3.28)-(3.33). Thermal averages are then computed as integrals over a contour in the complex energy plane, see Eqs.(4.1), (4.2) and (4.17). Moreover, the knowledge of the resolvent traces allows one to compute also thermal correlations of conserved and nonconserved observables and thermal quadratic fluctuations of conserved observables. In particular, we have computed the correlations of fermion number with orbital angular momentum, spin and induced flux multiplied by $e$, see Eqs.(5.3), (5.4) and (5.19). These correlations vanish at zero temperature unless the bound state energy in the one-particle spectrum vanishes ($A = -1$), see Eqs.(5.11) and (5.21). The high-temperature limits of the correlations are given by Eqs.(5.12), (5.13) and (5.22). Note that correlations (5.3) and (5.4) are even and correlation (5.19) is odd under transition to the inequivalent representation of the Clifford algebra ($s \rightarrow -s$ or $m \rightarrow -m$).

It should be emphasized that, owing to the gauge-invariant definition of orbital angular momentum (2.17), the correlation of orbital angular momentum with fermion number is finite. As it is shown in Section 5, another definition of orbital angular momentum (see Eq.(5.5)) results in the infinity of the appropriate correlation, which can be regarded as unphysical.

To illustrate the behaviour of an average and a correlation as functions of the boundary parameter Θ, we depict these quantities on Figs.1-3 for one observable, induced flux multiplied by $e$ (given by operator $\hat{O}$ (1.14) with $\Omega$ (2.19)), at several values of the vortex flux. Here quantities $|m|e^{-2}\hat{O}(T)$ and $sme^{-2}\Delta(T; \hat{O}, \hat{N})$ are along the ordinate axes, and quantity $s\Theta\pi^{-1}$ is along the abscissa axes. Values $(k_B T/|m|) = 5^{-1}$, 1, 5 correspond to two dashed (with longer and shorter dashes) and one dotted lines, and values $T = 0$ and $T = \infty$ correspond to solid lines; the latter cannot lead to confusion, since, as it has been already noted, the average at $T = \infty$ vanishes everywhere, while the correlation at $T = 0$ vanishes almost everywhere with the exception of one point ($A = -1$). Our plots correspond to three values of $F$ (3.10) from interval $0 < F \leq 1/2$, whereas interval $1/2 < F < 1$ can be considered by taking in view that $|m|e^{-2}\hat{O}(T) \rightarrow -|m|e^{-2}\hat{O}(T)$ and $sme^{-2}\Delta(T; \hat{O}, \hat{N}) \rightarrow sme^{-2}\Delta(T; \hat{O}, \hat{N})$ at $F \rightarrow 1 - F$ and $\Theta \rightarrow -\Theta$.

As is seen from Figs.1-3, the average at zero temperature is characterized by a jump with
a cusp at the point corresponding to the zero bound state energy \((A = -1)\). As temperature increases, this jump is smoothed out, while extremum evolves close to \(\Theta = s_1^F (\text{mod} \ 2\pi)\) \((A^{-1} = 0)\) in the case of \(0 < F \leq 1/2\) and to \(\Theta = -s_1^F (\text{mod} \ 2\pi)\) \((A = 0)\) in the case of \(1/2 \leq F < 1\). As temperature departs from zero, the correlation develops a maximum at \(A = -1\), which persists in the case of \(F = 1/2\) and is shifted to the left in the case of \(0 < F < 1/2\) and to the right in the case of \(1/2 < F < 1\), with further increase of temperature. At non-zero temperature the correlation becomes negative at \(\cos \Theta > 0\) \((A > 0)\), i.e. in the case of absence of the bound state in the one-particle spectrum.

Due to rotational invariance of the system considered, there is an additional to fermion number conserved observable — total angular momentum. It is natural to take in this capacity the sum of orbital angular momentum and spin: \(\hat{M} = \hat{L} + \hat{S}\). The appropriate average is related to the average fermion number, see Eqs.(4.8) and (4.9), and the correlation with fermion number is related to the quadratic fluctuation of fermion number, see Eqs.(5.14) and (5.15). The latter relations are a consequence of trace identity (3.35), and, using trace identities (3.38)-(3.40), we get the correlations of total angular momentum with orbital angular momentum, spin and induced flux times \(e\), see Eqs.(5.16),(5.17) and (5.23). Using trace identity (3.37), we get the quadratic fluctuation of total angular momentum (6.1). However, the negativeness of this fluctuation signifies that the above defined total angular momentum cannot be regarded as a physically meaningful observable.

To remedy the situation, in Section 6 we introduce improved total angular momentum as a sum of naive total angular momentum and fermion number with the dependent on the vortex flux coefficient, see Eqs.(6.5) and (6.6); the quadratic fluctuation of the improved observable is given by Eq.(6.7). On Figs.4-6 we plot averages and quadratic fluctuations of naive \((\hat{M})\) and improved \((\hat{R})\) total angular momenta for several values of \(F\) from interval \(0 < F \leq 1/2\); interval \(1/2 < F < 1\) can be considered by taking in view that the averages multiplied by \(s\) and the fluctuations are invariant under \(F \rightarrow 1 - F\) and \(\Theta \rightarrow -\Theta\). As before, two dashed (with longer and shorter dashes) and one dotted lines correspond to values \((k_B T/|m|) = 5^{-1}, 1, 5\), and solid lines correspond to values \(T = 0, \infty\). In the \(F \neq 1/2\) case, both fluctuations at extremely small temperatures possess a peak at \(A = -1\) which is smoothed out as temperature increases; incidentally, the minimum evolves to the left of point \(A^{-1} = 0\) in the case of \(0 < F < 1/2\) and to the right of point \(A = 0\) in the case of \(1/2 < F < 1\). In contrast to the \(\hat{R}\)-fluctuation, the \(\hat{M}\)-fluctuation is obviously negative in the vicinity of this minimum, see Figs.4 and 5. In the \(F = 1/2\) case, the behaviour of two fluctuation is completely different, with the \(\hat{M}\)-fluctuation being independent of \(\Theta\) and negative, see Fig.6 and explicit expression (6.1). The behaviour of the \(\hat{R}\)-fluctuation is qualitatively the same as that of the fermion number fluctuation studied in detail in Ref.[24]. Thus, we conclude that a physically meaningful observable is the improved total angular momentum \((\hat{R})\) rather than the naive one \((\hat{M})\).

At this point it is appropriate to discuss a very tiny effect which is common for fluctuations of fermion number and improved total angular momentum. Namely, both fluctuations at rather small temperatures become negative with extremely small absolute values in the case of absence of the bound state in the one-particle spectrum, i.e., at \(\cos \Theta > 0\) \((A > 0)\). Fig.7 illustrates this fact for the \(\hat{R}\)-fluctuation. To be more specific, in the \(F \neq 1/2\) case, this fluctuation at \(T = |m|/(5k_B)\) attains value \(-10^{-4}\) at \(F = 0.1\) \((F = 0.9)\) and value \(-3 \times 10^{-4}\) at \(F = 0.3\) \((F = 0.7)\) in region \((\text{mod}2\pi) < \Theta < s_1^F (\text{mod}2\pi)\) \((-s_1^F (\text{mod}2\pi) < \Theta < s_1^F (\text{mod}2\pi)\).
The radial components of the resolvent kernel of Appendix A (project 2.7/00152). This work was partially supported by the State Foundation for Basic Research of Ukraine.

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Appendix A

The radial components of the resolvent kernel of $H$ (3.3) are presented in the following way, type 1 ($l = s(n - n_0) > 0$):

$$a_n(r; r') = \frac{i\pi}{2} (\omega + m) \left[ \theta(r - r') H^{(1)}_{l-F}(kr) J_{l-F}(kr') + \theta(r' - r) J_{l-F}(kr) H^{(1)}_{l-F}(kr') \right], \quad (A.1)$$

$$b_n(r; r') = \frac{i\pi}{2} k \left[ \theta(r - r') H^{(1)}_{l+1-F}(kr) J_{l-F}(kr') + \theta(r' - r) J_{l+1-F}(kr) H^{(1)}_{l+1-F}(kr') \right], \quad (A.2)$$

$$c_n(r; r') = \frac{i\pi}{2} (\omega - m) \left[ \theta(r - r') H^{(1)}_{l+1-F}(kr) J_{l+1-F}(kr') + \theta(r' - r) J_{l+1-F}(kr) H^{(1)}_{l+1-F}(kr') \right], \quad (A.3)$$

$$d_n(r; r') = \frac{i\pi}{2} k \left[ \theta(r - r') H^{(1)}_{l+F}(kr) J_{l+F}(kr') + \theta(r' - r) J_{l+F}(kr) H^{(1)}_{l+F}(kr') \right]; \quad (A.4)$$

type 2 ($l' = -s(n - n_0) > 0$):

$$a_n(r; r') = \frac{i\pi}{2} (\omega + m) \left[ \theta(r - r') H^{(1)}_{l'+F}(kr) J_{l'+F}(kr') + \theta(r' - r) J_{l'+F}(kr) H^{(1)}_{l'+F}(kr') \right], \quad (A.5)$$

$$b_n(r; r') = -\frac{i\pi}{2} k \left[ \theta(r - r') H^{(1)}_{l'-1+F}(kr) J_{l'+F}(kr') + \theta(r' - r) J_{l'-1+F}(kr) H^{(1)}_{l'+F}(kr') \right]. \quad (A.6)$$
\[ c_n(r; r') = \frac{i\pi}{2}(\omega - m) \left[ \theta(r - r') H_{\nu + F}^{(1)}(kr_j) J_{\nu - F}(kr') + \theta(r' - r) J_{\nu + F}(kr) H_{\nu - F}^{(1)}(kr') \right], \quad (A.7) \]

\[ d_n(r; r') = -i\frac{\pi}{2} k \left[ \theta(r - r') H_{\nu + F}^{(1)}(kr_j) J_{\nu - F}(kr') + \theta(r' - r) J_{\nu + F}(kr) H_{\nu - F}^{(1)}(kr') \right]; \quad (A.8) \]

\[ \text{type 3 (}n = n_0): \]

\[ a_{n_0}(r; r') = i\frac{\pi}{2} \frac{\omega + m}{\sin \nu \omega + \cos \nu \omega e^{i\pi}} \left\{ \theta(r - r') H_{\nu - F}^{(1)}(kr) [\sin \nu \omega J_{\nu - F}(kr') + \cos \nu \omega J_{\nu - F}(kr')] + \right. \]
\[ + \left. \theta(r' - r) [\sin \nu \omega J_{\nu - F}(kr) - \cos \nu \omega J_{\nu + F}(kr)] H_{\nu - F}^{(1)}(kr') \right\}, \quad (A.9) \]

\[ b_{n_0}(r; r') = i\frac{\pi}{2} \frac{k}{\sin \nu \omega + \cos \nu \omega e^{i\pi}} \left\{ \theta(r - r') H_{\nu - F}^{(1)}(kr) [\sin \nu \omega J_{\nu - F}(kr') + \cos \nu \omega J_{\nu - F}(kr')] + \right. \]
\[ + \left. \theta(r' - r) [\sin \nu \omega J_{\nu - F}(kr) - \cos \nu \omega J_{\nu + F}(kr)] H_{\nu - F}^{(1)}(kr') \right\}, \quad (A.10) \]

\[ c_{n_0}(r; r') = i\frac{\pi}{2} \frac{\omega - m}{\sin \nu \omega + \cos \nu \omega e^{i\pi}} \left\{ \theta(r - r') H_{\nu - F}^{(1)}(kr) [\sin \nu \omega J_{\nu - F}(kr') - \cos \nu \omega J_{\nu + F}(kr')] + \right. \]
\[ + \left. \theta(r' - r) [\sin \nu \omega J_{\nu - F}(kr) - \cos \nu \omega J_{\nu + F}(kr)] H_{\nu - F}^{(1)}(kr') \right\}, \quad (A.11) \]

\[ d_{n_0}(r; r') = i\frac{\pi}{2} \frac{k}{\sin \nu \omega + \cos \nu \omega e^{i\pi}} \left\{ \theta(r - r') H_{\nu - F}^{(1)}(kr) [\sin \nu \omega J_{\nu - F}(kr') - \cos \nu \omega J_{\nu + F}(kr')] + \right. \]
\[ + \left. \theta(r' - r) [\sin \nu \omega J_{\nu - F}(kr) + \cos \nu \omega J_{\nu + F}(kr)] H_{\nu - F}^{(1)}(kr') \right\}. \quad (A.12) \]

Here \( k = \sqrt{\omega^2 - m^2} \) and a physical sheet is chosen as \( 0 < \text{Arg} k < \pi \) \( (Imk > 0) \), \( \theta(u) = \frac{1}{2}[1 + \text{sgn}(u)] \), \( J_\rho(u) \) is the Bessel function of order \( \rho \), \( H_\rho^{(1)}(u) \) is the first-kind Hankel function of order \( \rho \), and

\[ \tan \nu \omega = \frac{k^{2F}}{\omega - m} \text{sgn}(m)(2|m|)^{1-2F} \frac{\Gamma(1 - F)}{\Gamma(F)} \tan \left( \frac{\Theta}{2} + \frac{\pi}{4} \right). \quad (A.13) \]

In the absence of the vortex defect the radial components take form:

\[ a_n(r; r')|_{\Phi = 0} = i\frac{\pi}{2}(\omega + m) \left[ \theta(r - r') H_n^{(1)}(kr_j) J_n(kr') + \theta(r' - r) J_n(kr) H_n^{(1)}(kr') \right], \quad (A.14) \]

\[ b_n(r; r')|_{\Phi = 0} = i\frac{\pi}{2} k \left[ \theta(r - r') H_{n+s}^{(1)}(kr_j) J_n(kr') + \theta(r' - r) J_{n+s}(kr) H_n^{(1)}(kr') \right], \quad (A.15) \]
of the resolvent kernel is given by expression

\[ c_n(r; r')|_{e\Phi=0} = \frac{i\pi}{2} (\omega - m) \left[ \theta(r - r')H^{(1)}_{n+s}(kr)J_{n+s}(kr') + \theta(r' - r)J_{n+s}(kr)H^{(1)}_{n+s}(kr') \right], \]

(A.16)

and

\[ d_n(r; r')|_{e\Phi=0} = \frac{i\pi}{2} k \left[ \theta(r - r')H^{(1)}_{n}(kr)J_{n}(kr') + \theta(r' - r)J_{n}(kr)H^{(1)}_{n}(kr') \right]. \]

(A.17)

Note that all radial components behave asymptotically at large distances as outgoing waves.

Appendix B

Let us consider quantity (3.11). The contribution of the regular (types 1 and 2) components of the resolvent kernel is given by expression

\[
\sum_{n \neq n_0} [(n - e\Phi)a_n(r; r') + (n + s - e\Phi)c_n(r; r')] = \\
= s \sum_{l \geq 1} [(l - F)(\omega + m)I_{l-F}(kr)K_{l-F}(kr') + (l + 1 - F)(\omega - m)I_{l+1-F}(kr)K_{l+1-F}(kr')] - \\
- s \sum_{l' \geq 1} [(l' + F)(\omega + m)I_{l'+F}(kr)K_{l'+F}(kr') + (l' - 1 + F)(\omega - m)I_{l'-1+F}(kr)K_{l'-1+F}(kr')],
\]

(B.1)

where \( \kappa = -ik \) and we used relations

\[ J_\rho(ikr) = e^{i\rho \frac{\pi}{2}} I_\rho(\kappa r) \quad \text{and} \quad H^{(1)}(ikr') = \frac{2}{i\pi} e^{-i\rho \frac{\pi}{2}} K_\rho(\kappa r), \]

which are valid at \( \text{Re} \kappa > 0 \); here \( I_\rho(u) \) is the modified Bessel function of order \( \rho \), and

\[ K_\rho(u) = \frac{\pi}{2 \sin(\rho \pi)} \left[ I_{-\rho}(u) - I_\rho(u) \right]. \]

Using relation (see, e.g., Ref. [30])

\[ I_\rho(\kappa r)K_\rho(\kappa r') = \frac{1}{2} \int_0^\infty \frac{dy}{y} \exp \left( -\frac{\kappa^2 y}{2} - \frac{r^2 + r'^2}{2y} \right) I_\rho(y), \quad \text{Re} \kappa^2 > 0, \]

(B.2)

\[ \sum_{l \geq 1} (l + \rho)I_{l+\rho}(y) = \frac{1}{2} y \left[ I_{\rho}(y) + I_{\rho+1}(y) \right], \]

(B.3)

we perform summation in Eq. (B.1) and get in the case of \( \text{Re} \kappa > |\text{Im} \kappa| \):

\[
\sum_{n \neq n_0} [(n - e\Phi)a_n(r; r') + (n + s - e\Phi)c_n(r; r')] = \\
= \frac{s \sin(F\pi)}{\pi} \omega \int_0^\infty dy \exp \left( -\frac{\kappa^2 y}{2} - \frac{r^2 + r'^2}{2y} \right) [K_F(y) - K_{1-F}(y)] + \\
+ sF(\omega + m)I_F(\kappa r)K_F(\kappa r') - s(1 - F)(\omega - m)I_{1-F}(\kappa r)K_{1-F}(\kappa r'). \]

(B.4)
The contribution of the irregular (type 3) components is given by expression

\[(n_0 - e\Phi) a_{n_0}(r; r') + (n_0 + s - e\Phi) c_{n_0}(r; r') = \frac{2s \sin(F\pi)}{\pi (\tan\nu_\omega + e^{iF\pi})} \times \]

\[-F(\omega + m) \tan \nu_\omega K_F(kr)K_F(kr') + (1 - F)(\omega - m)e^{iF\pi}K_{1-F}(kr)K_{1-F}(kr') - sF(\omega + m)I_F(kr)K_F(kr') + s(1 - F)(\omega - m)I_{1-F}(kr)K_{1-F}(kr'). \] (B.5)

Summing Eqs. [3.4] and [3.5], we get Eq. [3.17].

Computation of other quantities, Eqs. [3.12]-[3.16], is similar to the above. As an illustration, let us scrutinize the procedure of computation of the last quantity, Eq. [3.16], which is the most tedious one.

The contribution of the regular components is

\[\frac{e^2}{4\pi} sr \sum_{n \neq n_0} \left( n - e\Phi + \frac{1}{2}s \right) [b_n(r; r') + d_n(r; r')] = \]

\[= -\frac{e^2}{4\pi} \kappa r \left\{ \sum_{l \geq 1} \left( l - F + \frac{1}{2} \right) [I_{l+1-F}(kr)K_{l-F}(kr') - I_{l-F}(kr)K_{l+1-F}(kr')] - \right. \]

\[- \sum_{l' \geq 1} \left( l' + F - \frac{1}{2} \right) [I_{l'+1+F}(kr)K_{l'+F}(kr') - I_{l'+F}(kr)K_{l'-1+F}(kr')] \} . \] (B.6)

Using recurrence relation

\[u \partial_u I_\rho(u) = \pm \rho I_\rho(u) + u I_{\rho \pm 1}(u),\]

we get

\[\frac{e^2}{4\pi} sr \sum_{n \neq n_0} \left( n - e\Phi + \frac{1}{2}s \right) [b_n(r; r') + d_n(r; r')] = \]

\[= \frac{e^2}{4\pi} \sum_{l \geq 1} \left\{ \left( l - F + \frac{1}{2} \right) [(l - F)I_{l-F}(kr)K_{l-F}(kr') + (l - 1 - F)I_{l+1-F}(kr)K_{l+1-F}(kr')] + \right. \]

\[+ \left( l + F - \frac{1}{2} \right) [(l + F)I_{l+1+F}(kr)K_{l+1+F}(kr') + (l - 1 + F)I_{l'-1+F}(kr)K_{l'-1+F}(kr')] \} - \]

\[- \frac{e^2}{4\pi} r \partial_r \sum_{l \geq 1} \left\{ \left( l - F + \frac{1}{2} \right) [I_{l-F}(kr)K_{l-F}(kr') - I_{l+1-F}(kr)K_{l+1-F}(kr')] - \right. \]

\[- \left( l + F - \frac{1}{2} \right) [I_{l+F}(kr)K_{l+F}(kr') - I_{l-1+F}(kr)K_{l-1+F}(kr')] \} = \]
\[
\sum_{l \geq 1} e^2 \int_0^{\infty} dy \exp \left( -\frac{\kappa^2 rr'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) \left\{ \frac{3}{2} I_{l+1-F}(y) + \frac{1}{2} I_{l+1-F}(y) + \frac{1}{2} I_{l-F}(y) + \frac{3}{2} I_{l-1-F}(y) + \frac{3}{2} I_{l+1-F}(y) + \frac{3}{2} I_{l-2+F}(y) + \frac{1}{2} I_{l+F}(y) + (l - 1 - F) I_{l-1-F}(y) - (l + 1 + F) I_{l+1-F}(y) + (l - F) I_{l-F}(y) - (l + 2 - F) I_{l-2+F}(y) + (l - 1 + F) I_{l-1+F}(y) - (l + 1 + F) I_{l+1+F}(y) + (l + 2 + F) I_{l-2+F}(y) - (l - 1) I_{l-1+F}(y) + \right. \\
\left. + (l - 1) I_{l-1-F}(y) - (l + 1) I_{l+1-F}(y) + (l + 2) I_{l-2+F}(y) - (l + 1) I_{l+1+F}(y) + (l + 2) I_{l-2+F}(y) - (l + 1) I_{l+1+F}(y) + \right. \\
\left. + \left(\frac{\kappa^2 rr'}{y^2} + \frac{r^2 - r'^2}{rr'} \right) \left( (1 - F) I_{l-E}(y) + F I_{l}(y) + \frac{1}{2} I_{l-F}(y) + \frac{1}{2} I_{l+1-F}(y) + \frac{1}{2} I_{l+1+F}(y) + \frac{1}{2} I_{l-F}(y) + \frac{1}{2} I_{l+1-F}(y) \right) \right\}, \quad \text{(B.7)}
\]

where the second equality is obtained with the use of representation [B.2] and recurrency relation

\[2\rho I_{\rho}(u) = u [I_{\rho-1}(u) - I_{\rho+1}(u)].\]

Now the summation over \( l \) can be performed with the use of Eq.(B.3) and relation (see Ref.[30])

\[
\sum_{l=1}^{\infty} I_{l+\rho}(y) = -\frac{2}{\rho} \left[ e^y \int_0^{\infty} du \ e^{-u} I_{\rho}(u) - y I_{\rho}(y) - y I_{\rho+1}(y) \right], \quad \text{Re } \rho > -1. \quad \text{(B.8)}
\]

Using again the recurrency relation and the relation between the Macdonald and the modified Bessel functions, we get

\[
\frac{e^2}{4\pi sr} \sum_{n \neq n_0} \left( n - e\Phi + \frac{1}{2}s \right) [b_n(r;r') + d_n(r;r')] =
\]

\[
= \frac{e^2 \sin(\kappa \pi)}{8F(1-F)\pi^2} \int_0^{\infty} dy \left( 1 + \frac{\kappa^2 rr'}{4y^2} \right) \exp \left( -\frac{\kappa^2 rr'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) \times
\]

\[
\times \left\{ e^y \int_0^{\infty} du \ e^{-u} [(1 - F) K_F(u) + F K_{1-F}(u)] + (2F - 1) y [K_{F}(y) - K_{1-F}(y)] \right\} +
\]

\[
+ \frac{e^2}{8\pi} \int_0^{\infty} dy \exp \left( -\frac{\kappa^2 rr'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) \left\{ \frac{2\sin(\kappa \pi)}{\pi} [(1 - F) K_{F}(y) + F K_{1-F}(y)] - \frac{1}{y} \left( F - \frac{1}{2} \right) [F I_{F}(y) - (1 - F) K_{1-F}(y)] + \frac{1}{2} \left( \frac{\kappa^2 rr'}{y^2} + \frac{r^2 - r'^2}{rr'} \right) \left( F - \frac{1}{2} \right) [I_{F}(y) - I_{1-F}(y)] \right\}. \quad \text{(B.9)}
\]
The contribution of the irregular components is

\[
\frac{e^2}{4\pi sr} \left( n_0 - e \Phi + \frac{1}{2} s \right) \left[ b_n(r; r') + d_n(r; r') \right] = \frac{-e^2 \left(F - \frac{1}{2}\right) \sin(F \pi)}{2\pi^2 (\tan \nu + e^{iF \pi}) \kappa r} \times 
\times \left[ \tan \nu \omega K_{1-F}(\kappa r) K_F(\kappa r') - e^{iF \pi} K_F(\kappa r) K_{1-F}(\kappa r') \right] + 
\frac{e^2}{4\pi} \left(F - \frac{1}{2}\right) \kappa r \left[I_{1+F}(\kappa r) K_F(\kappa r') - I_{-F}(\kappa r) K_{1-F}(\kappa r') \right]. \quad (B.10)
\]

The terms containing the \( I_n \)-functions are cancelled in the sum of Eqs. (B.9) and (B.10). Decomposing the integral over \( u \) as \( \int_{y_0}^{\infty} = \int_{0}^{y_0} - \int_{y_0}^{\infty} \), we get the following expression for this sum:

\[
\frac{e^2}{4\pi sr} \sum_{n=-\infty}^{\infty} \left( n - e \Phi + \frac{1}{2} s \right) \left[ b_n(r; r') + d_n(r; r') \right] = 
= \frac{e^2}{4\pi} \int_{0}^{\infty} dy \left( 1 + \frac{\kappa^2 r r'}{4y^2} + \frac{r^2 - r'^2}{4rr'} \right) \exp \left( -\frac{\kappa^2 r r'}{2y} - \frac{(r - r')^2}{2rr'} \right) y - 
- \frac{e^2 \sin(F \pi)}{8F(1 - F)^{\pi^2}} \int_{0}^{\infty} dy \left( 1 + \frac{\kappa^2 r r'}{4y^2} + \frac{r^2 - r'^2}{4rr'} \right) \exp \left( -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2rr'} \right) x 
\times \left\{ e^y \int_{y}^{\infty} du e^{-u} [ (1 - F) K_F(u) + F K_{1-F}(u) ] - (2F - 1)y[K_F(y) - K_{1-F}(y)] \right\} + 
+ \frac{e^2 \sin(F \pi)}{4\pi^2} \int_{0}^{\infty} dy \exp \left( -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2rr'} \right) [(1 - F) K_F(y) + F K_{1-F}(y)] - 
- \frac{e^2 \left(F - \frac{1}{2}\right) \sin(F \pi)}{2\pi^2 (\tan \nu + e^{iF \pi}) \kappa r} \left[ \tan \nu \omega K_{1-F}(\kappa r) K_F(\kappa r') - e^{iF \pi} K_F(\kappa r) K_{1-F}(\kappa r') \right]. \quad (B.11)
\]

Using relation (see, e.g., Ref.[31])

\[
\int_{0}^{\infty} dy y^{s-1} \exp(-py - qy^{-1}) = 2 \left( \frac{q}{p} \right)^{\frac{s}{2}} K_s(2\sqrt{pq}),
\]

we express the first integral over \( y \) in Eq. (B.11) through the Macdonald function and get Eq. (3.22).

In the case of the absence of the vortex defect one uses summation formulae

\[
\sum_{n=-\infty}^{\infty} I_n(y) = e^y, \quad \sum_{n=-\infty}^{\infty} n I_n(y) = 0.
\]
Thus, in particular, we get

\[
\frac{e^2}{4\pi} \sum_{n=-\infty}^{\infty} \left( n + \frac{1}{2} \right) \left[ b_n(r; r') + d_n(r; r') \right] \bigg|_{\Phi=0} =
\]

\[
= -\frac{e^2}{4\pi} kr \sum_{n=-\infty}^{\infty} \left( sn + \frac{1}{2} \right) \left[ I_{sn+1}(kr) K_{sn}(kr') - I_{sn}(kr) K_{sn+1}(kr') \right] =
\]

\[
= \frac{e^2}{4\pi} \sum_{n=-\infty}^{\infty} \left( sn + \frac{1}{2} \right) \left\{ sn I_{sn}(kr) K_{sn}(kr') + (sn + 1) I_{sn+1}(kr) K_{sn+1}(kr') - r \frac{\partial}{\partial r} [I_{sn}(kr) K_{sn}(kr') - I_{sn+1}(kr) K_{sn+1}(kr')] \right\} =
\]

\[
= \frac{e^2}{16\pi} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dy \exp \left( -\frac{\kappa^2 r r'}{2y} - \frac{r^2 + r'^2}{2rr'} y \right) \left( sn + \frac{1}{2} \right) \times
\]

\[
\times \left\{ I_{sn-1}(y) - I_{sn+1}(y) + I_{sn}(y) - I_{sn+2}(y) + \left( \frac{\kappa^2 r r'}{y^2} + \frac{r^2 - r'^2}{rr'} \right) [I_{sn}(y) - I_{sn+1}(y)] \right\} =
\]

\[
= \frac{e^2}{4\pi} \int_0^{\infty} dy \left( 1 + \frac{\kappa^2 r r'}{4y^2} + \frac{r^2 - r'^2}{4rr'} \right) \exp \left( -\frac{\kappa^2 r r'}{2y} - \frac{(r - r')^2}{2rr'} y \right), \quad (B.12)
\]

which is Eq. (3.26).

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Figure 1: $F = 0.1$
Figure 2: $F = 0.3$
Figure 3: $F = 0.5$
Figure 4: $F = 0.1$
Figure 5: \( F = 0.3 \)
Figure 6: $F = 0.5$
Figure 7: a) $F = 0.1$, b) $F = 0.3$, c) $F = 0.5$