Universal charge-mass relation: From black holes to atomic nuclei

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel
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The cosmic censorship hypothesis, introduced by Penrose forty years ago, is one of the cornerstones of general relativity. This conjecture asserts that spacetime singularities that arise in gravitational collapse are always hidden inside of black holes. The elimination of a black-hole horizon is ruled out by this principle because it would expose naked singularities to distant observers. We test the consistency of this prediction in a gedanken experiment in which a charged object is swallowed by a charged black hole. We find that the validity of the cosmic censorship conjecture requires the existence of a charge-mass bound of the form \( q \leq \mu^{2/3} E_c^{-1/3} \), where \( q \) and \( \mu \) are the charge and mass of the physical system respectively, and \( E_c \) is the critical electric field for pair-production. Applying this bound to charged atomic nuclei, one finds an upper limit on the number \( Z \) of protons in a nucleus of given mass number \( A \): \( Z \leq Z^* = \alpha^{-1/3} A^{1/3} \), where \( \alpha = e^2/\hbar \) is the fine structure constant. We test the validity of this novel bound against the \((Z, A)\)-relation of atomic nuclei as deduced from the Weizsäcker semi-empirical mass formula.

What are the physical limitations on the magnitude of the electric charge of a system characterized by general parameters such as size and mass? In a purely classical context, one can construct a quantity with dimensions of length from the mass and charge parameters of the system: \( R_c \equiv q^2/2\mu c^2 \) [1, 2]. It was shown [3, 4] that classical electrodynamics is a self-consistent theory only in describing the motion of charges with a characteristic radius greater than the classical radius \( R_c \). In fact, all known physical systems are characterized by the relation \( R > R_c \), where \( R \) is the circumscribing radius of the system. This observation can be written as

\[
q \leq (2\mu R)^{1/2}.
\] (1)

(We shall use natural units in which \( G = c = 1 \).)

Black holes, with their extreme gravitational binding, are the only known objects in nature whose size can come close to the limit: an extremal Reissner-Nordström black hole satisfies the relation \( R/R_c = 2 \) (other black holes satisfy \( R/R_c > 2 \)). On the other hand, weakly self-gravitating systems conform to the bound [11] with orders of magnitude to spare. For example, atomic nuclei satisfy the relation \( R/R_c \sim 10^2 - 10^3 \) and are therefore far larger than their classical radius. Thus, even atomic nuclei, the densest composite charged objects in nature (with negligible self-gravity), conform to the bound [11] with more than an order of magnitude to spare. This may suggest that for weakly self-gravitating systems the bound [11] is a bit loose. This observation immediately motivates one to look for a stronger upper limit on the electric charge of a weakly self-gravitating spherical object of given mass and radius.

In the following it will be shown that, the self-consistency of the physics of black holes reveals the existence of a universal charge-mass upper limit of the form:

\[
q \leq \mu^{2/3} E_c^{-1/3}, \tag{2}
\]

where \( E_c \) is the critical electric field for pair-production [3, 6]. This new bound would be stronger than bound [11] for spherical objects with \( \mu \leq 8R^3E_c^2 \).

The influential theorems of Hawking and Penrose [7] demonstrate that spacetime singularities are ubiquitous features of general relativity, Einstein’s theory of gravity. This implies that general relativity itself predicts its own failure to describe the physics of these extreme situations. Nevertheless, the utility of general relativity in describing gravitational phenomena is maintained by the cosmic censorship conjecture [8–10]. The weak cosmic censorship conjecture (WCCC) asserts that spacetime singularities that arise in gravitational collapse are always hidden inside of black holes. This statement is based on the common wisdom that singularities are not pervasive [10].

The validity of the WCCC is a necessary condition to ensure the predictability of the laws of physics [8–10]. The conjecture, which is widely believed to be true, has become one of the cornerstones of general relativity. Moreover, it is being envisaged as a basic principle of nature. However, despite the flurry of research over the years, the validity of this conjecture is still an open question (see e.g. [11, 28] and references therein).

The destruction of a black-hole event horizon is ruled out by this principle because it would expose the inner singularities to distant observers. Moreover, the horizon area of a black hole, \( A_{\text{hor}} \), is associated with an entropy \( S_{\text{BH}} = A_{\text{hor}}/4\hbar \) [29]. Therefore, without any obvious physical mechanism to compensate for the loss of the black-hole enormous entropy, the destruction of the black-hole event horizon would violate the generalized second law of thermodynamics [29]. For these two reasons, any process which seems, at first sight, to remove the black-hole horizon is expected to be unphysical.

According to the uniqueness theorems [30, 34], all sta-
tionary solutions of the Einstein-Maxwell equations are uniquely described by the Kerr-Newman metric which is characterized by three conserved parameters: the gravitational mass $M$, the angular momentum $J$, and the electric charge $Q$. A black-hole solution must satisfy the relation

$$M^2 - Q^2 - a^2 \geq 0$$

where $a \equiv J/M$ is the specific angular momentum of the black hole. Extreme black holes are the ones which saturate the relation \([3]\). As is well known, the Kerr-Newman metric with $M^2 - Q^2 - a^2 < 0$ does not contain an event horizon, and it is therefore associated with a naked singularity rather than a black hole. In this work we inquire into the physical mechanism which protects the black-hole horizon from being eliminated by the absorption of charged objects which may “supersaturate” the extremality condition \([3]\).

One of the earliest attempts to eliminate the horizon of a black hole is due to Wald \([16]\). Wald tried to over-charge an extremal Reissner-Nordström black hole (characterized by $Q = M$) by dropping into it a charged test particle whose charge-to-mass ratio is larger than unity. Wald considered the specific case of a particle which starts falling from spatial infinity (thus, the particles energy-at-infinity is larger than its rest mass). He has shown that this attempt to over-charge the black hole would fail because of the Coulomb potential barrier surrounding the black hole.

A more dangerous version of Wald’s original gedanken experiment is one in which the charged particle is slowly lowered towards the black hole. In this case, the energy delivered to the black hole (the part contributed by the body’s rest mass, see below) can be red-shifted by letting the assimilation point approach the black-hole horizon. On the other hand, the particle’s charge is not redshifted by the gravitational field of the black hole. At a first sight the particle [characterized by a small (redshifted) mass-energy] is not hindered from entering the black hole and removing its horizon, thereby violating cosmic censorship.

Consider a charged body of rest mass $\mu$, electric charge $q$, and proper radius $R$ approaching a charged Reissner-Nordström black hole. (We assume $q > 0$ without loss of generality). The test-particle approximation imposes the constraint $\mu \ll R \ll M$. This guarantees that the object has a negligible self-gravity and that it is much smaller than the scale set by the black hole.

The external gravitational field of the Reissner-Nordström black hole is given by

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2}) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

The black hole’s (event and inner) horizons are located at $r_{\pm} = M \pm (M^2 - Q^2)^{1/2}$.

The total energy $E$ (energy-at-infinity) of the body in a black-hole spacetime is made up of three contributions \([35]\):

- $E_0 = \mu (g_{00})^{1/2}$, the energy associated with the body’s mass (red-shifted by the gravitational field).
- $E_{\text{elec}} = qQ/r$, the electrostatic interaction of the charged body with the external electric field.
- $E_{\text{self}}$, the gravitationally induced self-energy of the charged body. The third contribution, $E_{\text{self}}$, reflects the effect of the spacetime curvature on the particles electrostatic self-interaction. The physical origin of this force is the distortion of the charges long-range Coulomb field by the spacetime curvature. This can also be interpreted as being due to the image charge induced inside the (polarized) black hole \([34, 37]\). The self-interaction of a charged particle in the black-hole spacetime results with a repulsive (i.e., directed away from the black hole) self-force. A variety of techniques have been used to demonstrate this effect in the black-hole spacetime. In particular, one finds \([38, 39]\) $E_{\text{self}} = Mq^2/2r^2$ in the Reissner-Nordström spacetime.

Thus, the total energy of a charged body located at the radial coordinate $r$ in the black-hole spacetime is given by

$$E(r) = \mu \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{1/2} + \frac{qQ}{r} + \frac{Mq^2}{2r^2}.$$  

The radial coordinate $r$ is related to the proper distance $\ell$ above the horizon through the relation

$$\ell(r) = \int_{r_+}^r (g_{rr})^{1/2} dr.$$  

Assuming $r - r_+ \ll r_+ - r_-$ (of course, this assumption can only be valid for non-extremal black holes), one finds:

$$\ell(r) = \frac{2(r_+ - r_+)^{1/2}}{(r_+ - r_-)^{1/2}} \left[1 - \frac{r - r_+}{6(r_+ - r_-)} + \mathcal{O}\left(\frac{r - r_+}{r_+}\right)\right].$$

From \([7]\) one obtains the inverse relation

$$r(\ell) = r_+ + (r_+ - r_-) \frac{\ell^2}{4r_+^2} \left[1 + \mathcal{O}(\ell^2/r_+^2)\right].$$

Note that for extremal black holes Eq. \([6]\) implies $\ell = \infty$ for any point outside the horizon. Thus, Eq. \([7]\) is valid only for non-extremal black holes under the assumption $r - r_+ \ll r_+ - r_-$. For non-extremal black holes this amounts to the assumption $\ell \ll r_+$.

Taking cognizance of Eqs. \([5]\) and \([6]\), one finds that the total energy of a charged particle at a proper distance
\( \ell (\ell \ll r_+) \) above the horizon of a non-extremal black hole is given by:
\[
\mathcal{E}(\ell) = \frac{\mu\ell(r_+ - r_-)}{2r_+^2} + \frac{qQ}{r_+} - \frac{qQ\ell^2(r_+ - r_-)}{4r_+^3} + \frac{Mq^2}{2r_+^2} .
\]  

This expression is actually the effective potential governing the motion of a charged body in the black-hole spacetime. Provided \( qQ > 0 \), it has a maximal height located at \( \ell = \ell^*(\mu, q; M, Q) = \frac{\mu r_+^2}{qQ} \).

The most challenging situation for the cosmic censorship conjecture occurs when the energy-to-charge ratio of the captured particle is as small as possible. This can be achieved if one slowly lowers the body towards the black hole, providing it with the minimal energy \( \mathcal{E}_{\text{min}} = \mathcal{E}(\ell^*) \) required in order to overcome the potential barrier (recall that the effective potential barrier has a maximum located at \( \ell = \ell^* \)). This is also true for any charged object which is released to fall freely from \( \ell > \ell^* \) with the minimally required energy \( \mathcal{E}(\ell^*) \).

The absorption of the charged object by the black hole results with a change \( \Delta M = \mathcal{E}(\ell^*) \) in the black-hole mass (assuming that the energy delivered to the black hole is as small as possible) and a change \( \Delta Q = q \) in its charge. The condition for the black hole to preserve its integrity after the assimilation of the body is:
\[
Q + q \leq M + \mathcal{E}(\ell^*) .
\]  

Substituting \( \mathcal{E}(\ell^*) \) from Eq. (9) one finds a necessary and sufficient condition for removal of the black-hole horizon:
\[
(q - e)^2 + \frac{2e}{M} \left( \mu\ell^* - q^2 - \frac{q\ell^2}{2M} \right) + \frac{q^2}{M} < 0 ,
\]  

where \( r_+ \equiv M \pm \varepsilon \). The expression on the l.h.s. of (11) is minimized for \( q = e + O(\epsilon^2/M) \), yielding
\[
2\mu\ell^* - q^2 - q\ell^2/M < 0 ,
\]  

as a sufficient condition for elimination of the black-hole horizon. Finally, substituting \( \ell^* = \frac{\mu r_+^2}{qQ} \) into (12), one finds
\[
q^3 > \mu^2/E_+ ,
\]  

as a sufficient condition for removal of the black-hole horizon, where \( E_+ = Q/r_+^2 = M^{-1} + O(\epsilon^2/M) \) is the black-hole electric field in the vicinity of its horizon. An assimilation of a charged object satisfying condition (13) by a charged black hole would violate the cosmic censorship conjecture.

At this point it should be emphasized that Schwinger discharge of the black hole (vacuum polarization effects) sets an upper bound on the black-hole electric field [13, 14]:
\[
E_+ \leq E_c = \frac{m_i}{\epsilon h} ,
\]  

where \( e \) is the elementary electric charge and \( m_i \) is the rest mass of the lightest stable charged particle. Thus, the validity of the WCC conjecture (namely, the integrity of the black-hole horizon) requires the existence of a universal charge-mass bound of the form
\[
q \leq \mu^{2/3}E_c^{-1/3} .
\]  

It is worth recalling that the test-particle approximation we have used is valid for objects in the regime \( \mu \ll R \ll E_c^{-1} \). These inequalities are easily satisfied by charged objects with \( q \gg \mu \). The intriguing feature of our derivation is that it uses a principle whose very meaning stems from gravitation (the cosmic censorship principle) to derive a universal bound which has nothing to do with gravitation [written out fully, the bound \( 15 \) would involve \( \hbar \) and \( c \), but not \( G \)]. This provides a striking illustration of the unity of physics.

The lightest charged particle in nature is the electron, and one should therefore take \( m_i \to m_e \) in the bound (16). With this value of \( m_i \) it is straightforward to verify that atomic nuclei conform to the upper bound (15). This in turn guarantees that the absorption of charged nuclei by a black hole would respect the cosmic censorship principle. Yet, we conjecture that charged objects like atomic nuclei and quark nuggets whose size is smaller than the Compton wavelength of the electron would conform to an even tighter bound with \( m_i \to m_p \), where \( m_p \) is the proton’s rest mass. Below we shall discuss and test this conjecture.

Consider a nucleus composed of \( Z \) protons and \( N \) neutrons. We first point out that the Compton wavelength \( \hbar/m_e \) of an electron is much larger than the size of a typical nucleus \( \sim A^{1/3}\hbar/m_p \), where \( A = Z + N \) is the baryon number (this is true for all nuclei with mass numbers \( A \lesssim 10^8 \) [40]). This fact implies that the wavefunction of an electron inside a nucleus is almost identically zero. This in turn implies that the structure of atomic nuclei is mainly determined by the physical properties of the nucleons (protons and neutrons) that are trapped inside the nucleus, whereas the electrons which are almost entirely left outside the nucleus have almost no influence on its internal structure.

Let us assume for a moment that we live in a world in which there are no electrons. Since these light particles are not trapped inside the nucleus, it is reasonable to expect that this assumption will have no significant influence on the internal structure of dense nuclear matter. To be precise, it is well known that electrons do participate in nuclear radioactive processes (e.g., in the beta-decay process), but since they are not trapped inside the nucleus itself they have no significant influence on its internal structure. (The internal structure itself is determined by the protons and neutrons that compose the nuclei.) With this assumption, the critical electric field is given by Eq. (17) with \( m_i \to m_p \).
The charge and mass of a nucleus are given by
\[ q = Ze \] ; \[ \mu = Zm_p + Nm_n - E_B \approx Am_p \] , \hspace{1cm} (16)
where \( E_B \) is the binding energy of the nucleus, which is typically much smaller than its mass. Substituting Eq. \( \text{[16]} \) into the upper bound \( \text{[15]} \), one finds the \((Z,A)\)-inequality:
\[ Z \leq Z^* = \alpha^{-1/3} A^{2/3} \] , \hspace{1cm} (17)
for charged matter of nuclear density, where \( \alpha = e^2/h \approx 1/137 \) is the fine structure constant.

The largest known completely stable nucleus is lead-208 which contains 82 protons and 126 neutrons. This nucleus satisfies the relation \( Z/A^{2/3} \approx 2.33 \) and it therefore conforms to the upper bound \( \text{[17]} \) by a factor of \( \sim 2.2 \). The largest known artificially made nucleus contains 118 protons and a total number of 294 nucleons -- it satisfies the relation \( Z/A^{2/3} \approx 2.67 \) and it therefore conforms to the upper bound \( \text{[17]} \) by a factor of \( \sim 1.9 \).

It is expected that even heavier meta-stable nuclei would be produced in the forthcoming years using accelerator production techniques. Some calculations suggest that nuclei of \( A \approx 300 \) to 476 with low excitation energies may exist for very long times \[ \text{[42]} \]. Could these nuclei be able to threaten the validity of the cosmic censorship conjecture by violating the \((Z,A)\)-bound \( \text{[17]} \)? To answer this question, we shall investigate the \((Z,A)\)-relation of atomic nuclei as deduced from the well-known semi-empirical mass formula \[ \text{[43–46]} \].

The binding energy \( E_B \) of a nucleus (that is, the difference between its mass and the sum of the masses of its individual constituents) is well approximated by the semi-empirical mass formula, also known as Weizsäcker’s formula \[ \text{[43–46]} \] :

\[ E_B(A,Z) = a_V A - a_s A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} \]
\[ - \frac{(A-2Z)^2}{A} + \frac{a_p}{A^{1/2}} \] . \hspace{1cm} (18)

This well-known formula is partially based on theory and partly on empirical measurements. The theory is based on the liquid drop model which treats the nucleus as a drop of incompressible nuclear fluid composed of protons and neutrons (but not electrons!). The five terms on the r.h.s of Eq. \( \text{[18]} \) correspond to the cohesive binding of all the nucleons by the strong nuclear force, the electrostatic mutual repulsion of the protons, a surface energy term, an asymmetry term (derivable from the protons and neutrons occupying independent quantum momentum states) and a pairing term (partly derivable from the protons and neutrons occupying independent quantum spin states) \[ \text{[43–46]} \]. The coefficients in the semi-empirical mass formula are calculated by fitting to experimentally measured masses of nuclei.

By maximizing \( E_B(A,Z) \) with respect to \( Z \), one finds the number of protons of the most stable nucleus of atomic mass \( A \):
\[ Z(A) = \frac{A}{2} \frac{1}{1 + \beta A^{2/3}} \] , \hspace{1cm} (19)
where \( \beta = \frac{4c}{\alpha A} \). (For light nuclei this expression reduces to the canonical relation \( Z = A/2 \).)

The requirement \( Z(A) \leq Z^*(A) \) yields the quadratic equation
\[ 2\beta A^{2/3} - (\alpha A)^{1/3} + 2 \geq 0 \] . \hspace{1cm} (20)

The \textit{experimentally} measured value of \( \beta \) is \( \sim 7.7 \times 10^{-3} \) \[ \text{[43–46]} \]. It is easy to verify that with this value of \( \beta \) the inequality \( \text{[20]} \) holds true for all \( A \) values. It is worth emphasizing, however, that as opposed to the loose bound \( \text{[11]} \) which is respected by all nuclei with more than an order of magnitude to spare, the new bound \( \text{[17]} \) is much stronger — \( Z(A) \) is of the \textit{same} order of magnitude as the upper limit \( Z^*(A) \). In fact, the ratio \( Z(A)/Z^*(A) \) reaches a maximal value of \( \sim 0.56 \).

Strange quark matter consisting of up, down, and strange quarks may have an energy per baryon that is less than that of nuclear matter \[ \text{[47–48]} \] and would then be the true ground state of baryonic matter. The possible existence of metastable or even stable quark nuggets (also known as strangelets) has been widely discussed \[ \text{[40, 49–51]} \]. It is believed that the net electric charge of color-flavor locked strangelets \[ \text{[40, 49–51]} \] is concentrated near their surface. Thus, one expects a charge-mass relation of the form \( Z \propto A^{2/3} \). The upper bound \( \text{[17]} \) limits the allowed value of the proportionality coefficient to be less than \( \alpha^{-1/3} \). The accepted estimate for this constant is \( \sim 0.3 \) \[ \text{[40]} \], which indeed conform to the upper limit \( \text{[17]} \).

In summary, an application of the cosmic censorship principle to a gedanken experiment in which a charged object falls into a black hole, enables us to reveal a universal relation between the maximal electric charge and mass of any spherically symmetric object with negligible self gravity: \( q \leq m^{2/3} \mu^{-1/3} \). For objects with nuclear matter density the upper bound corresponds to \( Z \leq Z^* = \alpha^{-1/3} A^{2/3} \). This relation limits the charges of objects such as atomic nuclei and quark nuggets. For these objects, the new bound is more restrictive than other limits existing in the literature.

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