Causal completions as Lorentzian pre-length spaces

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Abstract
In this work we revisit the notion of the (future) causal completion of a globally hyperbolic spacetime and endow it with the structure of a Lorentzian pre-length space. We further carry out this construction for a certain class of generalized Robertson-Walker spacetimes.

Keywords Lorrenzian length space · Causal boundary · Warped product

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1 Introduction

There is no doubt that Roger Penrose is one of the precursors of the mathematical foundations of causal theory in general relativity. His famous notes on causality [1] have become a classic textbook and a solid starting point for graduate students and researchers alike. In this regard, one of Penrose’s great achievements in mathematical relativity is the formal study of the asymptotic structure of spacetime, which arose with the introduction of the conformal compactification and Penrose diagrams [2]. It is through the notion of conformal infinity that the notion of a black hole is abstracted, thus allowing the development of geometric and causal theoretic methods in their analysis. In spite of its success and widespread use, the conformal compactification approach to the asymptotic structure of spacetime has some downsides. Most notably, given a spacetime, there is not a straightforward way to decide whether it admits a conformal compactification or not, or even in the affirmative case, a canonical way of constructing such a compactification. This issue was also tackled by Penrose. In their seminal work, Geroch, Kronheimer and Penrose [3] provided an alternative way to deal with the structure at infinity of a spacetime that relies exclusively on the causal structure of a distinguishing spacetime. Their construction follows in spirit the classical constructions in elementary geometry, where an ideal point (or point at infinity) is attached to a family of curves having a common end. Instead of using parallel rays (as is the case say, in hyperbolic geometry) they considered the causal structure of spacetime and declared that two curves have a common point at future (past) infinity if their corresponding chronological past (future) sets agree. Though elegant and simple at first sight, the construction of the so called causal boundary of a spacetime involves many subtleties, specially when trying to endow it with some additional structure, like a topology or causal relations (see [4, 5] and references therein for an up to date account). Recently, significant results have been accomplished in this regard, most notably the introduction of a notion of black hole based on the causal boundary [6].

On the other hand, we have witnessed in the past few years a surge in the use of non-smooth geometric methods in mathematical relativity. Diverse settings such as cone structures [7, 8], $C^0$ metrics [9–13] and Lorentzian length spaces –to mention just a few– have proven useful in exploring scenarios where (metric) smoothness can not be guaranteed, as the ones linked to recent astronomical observations [14, 15]. As a matter of fact, the use of non-smooth methods is not new. In the context of causality, Penrose and Kronheimer were the first to provide an abstract framework that does not require a metric structure at all [16]. Their notion of causal space lies at the foundations of the theory of Lorentzian pre-length spaces first introduced by Kunzinger and Sämann [17]. The purpose of this work is to present the future (or past) causal completion of a globally hyperbolic spacetime as a Lorentzian pre-length space, thus adding an interesting source of examples to this rapidly growing field [18–24].

This work is organized as follows. In Sect. 2 we establish the basic facts about the causal completion and Lorentzian pre-length spaces, as well as the notation that will be used throughout this work. In Sect. 3 we prove that the future (past) causal completion admits a natural Lorentzian pre-length structure. Finally, in Sect. 4 we exhibit this structure in a class of warped product spacetimes.
2 Preliminaries

2.1 Causal completions

Let \((M, g)\) be a strongly causal spacetime and \(\ll, \preceq\) its usual chronological and causal relations, that is, \(p \ll q\) if and only if there exists a smooth future-directed timelike curve that joins \(p\) to \(q\), whereas \(p \preceq q\) if and only \(p = q\) or if there exists a smooth future-directed causal curve between these points. We define the chronological (causal) future and past sets in the standard way:

\[
I^+(p) = \{ q \in M \mid p \ll q \}, \quad J^+(p) = \{ q \in M \mid p \preceq q \},
I^-(p) = \{ q \in M \mid q \ll p \}, \quad J^-(p) = \{ q \in M \mid q \preceq p \}.
\]

A sequence of points \(\{x_n\}\) is called a future-directed chain if \(x_n \ll x_{n+1}\) for all \(n\) and past directed if \(x_{n+1} \ll x_n\) for all \(n \in \mathbb{N}\). Moreover, it will be called inextensible if \(\{x_n\}\) is not convergent. A subset \(P \subseteq M\) is called past set if \(P = I^-(P)\).

We can define the notions of future sets in a time dual way. We will say that \(P\) is an indecomposable past set (or IP for short) if \(P\) cannot be written as \(P = P_1 \cup P_2\) where \(P_1, P_2 \subset P\) are proper past subsets of \(P\). As it turns out, there are only two classes of indecomposable past sets: (1) the chronological past of points \(I^-(p)\), which will be called proper indecomposable past sets (PIP) and (2) the chronological past of inextensible future-directed chains \(I^-(\{x_n\})\) which will be called terminal indecomposable past sets (TIP). Refer to Thrs. 2.1 and 2.3 in [3] for a detailed analysis.

The future causal completion \(\hat{M}\) of \((M, g)\) is the set of all indecomposable past sets (IPs). Observe that since \((M, g)\) is strongly causal, it is past distinguishing. Hence if \(I^-(p) = I^-(q)\), then \(p = q\). Thus, we have that any point \(p \in M\) determines a unique PIP. The future causal boundary \(\hat{\partial}M\) is then identified with the TIPs. Thus we have

\[
IPs \equiv PIPs \cup TIPs, \quad \hat{M} \equiv M \cup \hat{\partial}M.
\]

In a similar way we can define the past causal completion as \(\check{M} \equiv IFs\) and the past causal boundary as \(\check{\partial}M = TIFs\). That is,

\[
IFs \equiv PIFs \cup TIFs, \quad \check{M} \equiv M \cup \check{\partial}M.
\]

Since both \(\hat{M}\) and \(\check{M}\) include a copy of the spacetime \((M, g)\), it is natural trying to define the total causal completion of \((M, g)\) as \(\hat{M} \cup \check{M}\), where the PIF \(I^+(p)\) is identified with the PIF \(I^+(p)\)—since both represent the point \(p \in M\). However, it was observed early on that such a construction may lead to inconsistencies, as in some cases further identifications on the boundaries \(\check{\partial}M\) and \(\hat{\partial}M\) ought to take place [3, 5, 25, 26]. Thus finding an approach to constructing the causal boundary that works in...
full generality proved to be a delicate task, which was completed only recently [4]. In order to avoid such intricacies, we will focus only on the future causal completion \( \hat{M} \).

Notice that, as the notation suggests, the relation \( \hat{\ll} \) on \( \hat{M} \) given by

\[
P \hat{\ll} Q \iff \exists q \in Q \setminus P \text{ such that } P \subset I^-(q)
\]

is indeed transitive. In fact, if \( P \hat{\ll} Q \hat{\ll} R \) there exist \( q \in Q \setminus P \) and \( r \in R \setminus Q \) such that \( P \subset I^-(q) \subset Q \subset I^-(r) \), thus \( P \hat{\ll} R \).

Hence, we can think of \( \hat{\ll} \) as providing a chronological structure on \( \hat{M} \)[27]. The chronological future and past sets so induced will be denoted by \( \hat{I}^+(P) \) and \( \hat{I}^-(P) \).

Further, we can endow \( \hat{M} \) with a sequential topology which is compatible with the chronology \( \hat{\ll} \) just defined.\(^1\) Thus, consider the limit operator \( \hat{L}_{\text{chr}} \) over the sequences of past sets given by

\[
P \in \hat{L}_{\text{chr}}(\{P_n\}) \iff P \subset LI(\{P_n\}) \text{ and it is maximal in } LS(\{P_n\})^2
\]

and define the future chronological topology \( \hat{T}_{\text{chr}} \) by its closed subsets as follows: a subset \( C \subset \hat{M} \) is closed if and only if for any sequence \( \sigma \subset C \) the inclusion \( \hat{L}_{\text{chr}}(\sigma) \subset C \) holds. Thus we have the following result (see the proof of Thrm. 3.27 in [4])

**Theorem 1** Let \((M, g)\) be a strongly causal spacetime and \( \hat{M} \) its future causal completion endowed with the chronological structure induced by (1) and the topology induced from the chronological limit (2). Then:

(i) The inclusion \( M \hookrightarrow \hat{M} \) is continuous. Moreover, the restriction of the chronological topology to \( M \) is the manifold topology.

(ii) The future causal completion is complete: any future directed chain \( \{P_n\} \) in \( \hat{M} \) converges in \( \hat{T}_{\text{chr}} \). In particular, any future inextensible timelike curve \( \gamma \) on \( M \) has an endpoint in \( \hat{M} \).

(iii) The sets \( \hat{I}^+(P) \) and \( \hat{I}^-(P) \) are open for all \( P \in \hat{M} \).

(iv) \((\hat{M}, \hat{T}_{\text{chr}})\) is a \( T_1 \) topological space.

Since there are examples in which \((\hat{M}, \hat{T}_{\text{chr}})\) is not Hausdorff, we can not aim to furnish a smooth manifold structure on it for such cases. However, some extra structure can be added in particular cases. For instance, a linear connection can be defined on \( \hat{M} \) provided that \((M, g)\) is static and spherically symmetric [28].

### 2.2 Lorentzian pre-length spaces

In their remarkable paper [16], Kronheimer and Penrose developed the notion of a causal space by abstracting the fundamental properties of the chronological and

---

\(^1\) Recall that on any spacetime \((M, g)\) the chronological sets \( I^+(p) \), \( I^-(p) \) are open in the manifold topology.

\(^2\) The set theoretical inferior and superior limits of subsets are defined as \( LI(\{P_n\}) = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} P_m \)
and \( LS(\{P_n\}) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} P_m \), respectively.
causal relations. Such an axiomatic approach to causality has proven useful in many circumstances, for instance in the causet approach to quantum gravity [29]. In recent times, the search for applying synthetic geometrical methods to mathematical relativity sparked a renewed interest in developing causality in an abstract setting. Lorentzian pre-length spaces are a refinement of the notion of causal spaces that incorporates a time distance function, thus providing an analog for the length structure that serves as a building block for the well established synthetic theory in metric spaces [30].

As in the seminal work by Kunzinger and Sämann [17] we define a Lorentzian pre-length space \((X, d, \ll, \leq, \tau)\) as a metric space \((X, d)\) along with two relations \(\ll, \leq\)—named chronological and causal—and a time separation function \(\tau : X \times X \to [0, \infty]\) that satisfy the following axioms:

1. \(\leq\) is a pre-order,
2. \(\ll\) is a transitive relation contained in \(\leq\),
3. \(\tau\) is a lower semi-continuous function —with respect to \(d\)— satisfying
   - \(\tau(x, z) \geq \tau(x, y) + \tau(y, z)\) for all \(x \leq y \leq z\),
   - \(\tau(x, y) > 0\) if and only if \(x \ll y\).

As immediate consequences of the definition we have two of the most important features of the causal structure of a spacetime: (i) the chronological sets \(I^+(p), I^-(p)\) are open, and (ii) either \(x \leq y \ll z\) or \(x \ll y \leq z\) implies \(x \ll z\). These properties, along some additional structure, enable us to build a causality—an even a causal hierarchy—that closely resembles the usual causal structure of a smooth spacetime [17, 31]. Examples of Lorentzian pre-length spaces include a wide variety of geometric structures, like cones [7, 8], causally plain \(C^0\) spacetimes [10], contact structures [23] and Lorentzian taxicab-type spaces [22].

### 3 Lorentzian pre-length space structure on \(\hat{M}\)

As we discussed in the previous section, the topological space \((\hat{M}, \hat{T}_{chr})\) might not be metrizable in general. Thus, in order to construct a Lorentzian pre-length space structure on it, a refinement on the topology is in order. In a recent work [32], Costa, Flores and Herrera studied the so called closed limit topology \(\hat{T}_c\) (or CLT for short). The open Hausdorff limit operator

\[
\hat{L}_H(\{P_n\} = \{P \in \hat{M} | P = LI(\{P_n\}) = LS(\{P_n\})\}
\]

generates the closed sets in \(\hat{T}_c\) as follows: a subset \(C \subset \hat{M}\) is closed if and only if \(\hat{L}_H(\{P_n\}) \subset C\) for every sequence \(\{P_n\} \subset C\). We now summarize the relevant

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3 This is commonly known as the push up property.

4 Notice that the standard Hausdorff limit operator is defined over closed sets. However, both operators are shown to define the same topology (see Sect. 5 in [32]).

5 Observe that \#L_H(\sigma) \in \{0, 1\} and thus \(\hat{T}_c\) is a first order topology (see [33]). In other words, the following equivalence holds

\[
\{P_n\} \to \hat{T}_c P \iff \hat{L}_H(\{P_n\}) = \{P\}.
\]
features of the CLT topology in globally hyperbolic spacetimes (refer to Thrm 4.1 and 4.2 in [32])

**Theorem 2** If \((M, g)\) is a globally hyperbolic spacetime, then the following statements hold for the topological space \((\hat{M}, \hat{T}_c)\):

1. The natural inclusion \(i : M \to \hat{M}\) given by \(i(p) = I^-(p)\) is an open continuous map. Moreover, \(i(M)\) is an open dense subset of \(\hat{M}\), the induced topology on \(M\) is the manifold topology, \(\partial M\) is closed and \((\hat{M}, \hat{T}_c)\) is second countable.
2. The chronological sets \(\hat{I}^\pm(P)\) are open subsets for all \(P \in \hat{M}\).
3. Any future directed chain \(\{P_n\} \subset M\) converges in \(\hat{T}_c\).
4. The topological space \((\hat{M}, \hat{T}_c)\) is metrizable.

Notice that in general, the chronological topology is coarser than CLT. Moreover, they coincide when their corresponding limit operators agree. The next result provides necessary and sufficient conditions for this (see [32, Thrm. 5.3])

**Theorem 3** Let \(\hat{M}\) be a future completion endowed with the limit operators \(\hat{L}_H\) and \(\hat{L}_{chr}\). Both limit operators coincide if and only if, the chronological topology \(\hat{T}_{chr}\) is Hausdorff.

Recall that \(\hat{M}\) is endowed with the chronological relation \(\hat{\ll}\) given by (1). Thus, \(\hat{\ll}\) naturally induces an associated causal relation \(\leq_o\) in \(\hat{M}\) (see [34, Definition 2.22]) by setting

\[
P \leq_o Q \iff \hat{I}^-(P) \subset \hat{I}^-(Q) \text{ and } \hat{I}^+(Q) \subset \hat{I}^+(P).
\]

By construction, this relation is reflexive, transitive and contains \(\hat{\ll}\), thus it is a causal relation according to the definition of a pre-length space.

The causal relation \(\leq_o\), though constructed by a standard procedure, it may seem artificial at first sight. The simple inclusion of past sets provides right away a pre-order in \(\hat{M}\). That is, the relation \(\hat{\leq}\) on \(\hat{M}\) given by

\[
P \hat{\leq} Q \iff P \subset Q.
\]

might be considered as well [35]. Notice that since \(\hat{\leq}\) is a partial order in \(\hat{M}\), then the causal space \((\hat{M}, \hat{\ll}, \hat{\leq})\) satisfies the causality axiom.

The following well known result will be used extensively from now on. We include its proof here for completeness (see for example [32, Proposition 2.11])

**Proposition 4** Let \((M, g)\) be is a globally hyperbolic spacetime. If \(P \in \hat{\partial} M\) then for any \(p \in M\) we have \(P \hat{\leq} I^-(p)\).  

---

[6] Equivalently, if \(P, Q \in M\) are IPs such that \(P \in \hat{\partial} M\) and \(P \hat{\leq} Q\), then \(Q \in \hat{\partial} M\).
Proof Suppose by contradiction that $P \subset I^-(p)$ and consider $\gamma : [a, b) \to M$ an inextensible future directed timelike curve such that $P = I^-(\gamma)$. Since $P$ is a TIP then we must have that $P$ is a proper subset of $I^-(p)$, thus, if we take $\gamma(a)$ then we must have that $\gamma(s) \in J^+(\gamma(a)) \cap I^-(p) \subset J^+(\gamma(a)) \cap J^-(p)$ for all $s \in (a, b)$ and this gives an inextensible timelike curve imprisoned in a compact subset thus contradicting strong causality. \hfill \Box

We now show that the causal relations $\leq_0$ and $\leq$ coincide when $(M, g)$ is globally hyperbolic.

**Proposition 5** Let $(M, g)$ be a globally hyperbolic spacetime. For all $P, Q \in \hat{M}$, $P \leq Q$ if and only if $P \leq_0 Q$.

**Proof** Suppose first that $P \leq Q$, i.e. $P \subset Q$. If $Q$ is a TIP then by Proposition 4 $\hat{I}^+(Q) = \emptyset$ and we have $\hat{I}^+(Q) \subset \hat{I}^+(P)$. If $Q = I^-(q)$ is a PIP, consider $R \in \hat{I}^+(Q)$, which means there exists $r \in R \setminus Q$ such that $Q \subset I^-(r)$ and since $P \subset Q \subset I^-(r)$ with $r \in R \setminus P$ we have $R \in \hat{I}^+(P)$. Thus $\hat{I}^+(Q) \subset \hat{I}^+(P)$.

Similarly, consider $R \ll P$, and $r \in P \setminus R$ with $R \subset I^-(p)$. Since $P \subset Q, p$ also belongs to $Q$ and we get $R \ll Q$, that is, $R \in \hat{I}^-(Q)$. This means that $\hat{I}^-(P) \subset \hat{I}^-(Q)$ and we have $P \leq_0 Q$.

On the other hand, let us suppose $P \leq_0 Q$, that is,

$\hat{I}^+(Q) \subset \hat{I}^+(P)$ and $\hat{I}^-(P) \subset \hat{I}^-(Q)$.

Take $p_0 \in P$, then there exists $q_0 \in P \setminus I^-(p_0)$ with $I^-(p_0) \subset I^-(q_0)$ therefore $I^-(p_0) \ll P$, that is, $I^-(p_0) \in \hat{I}^-(P) \subset \hat{I}^-(Q)$. Thus, there exists $q \in Q \setminus I^-(p_0)$ such that $I^-(p_0) \subset I^-(q) \subset Q$. If $\{q_n\}$ is a future directed chain generating $I^-(p_0)$ we have that $p_n \in I^-(q)$ for $n$ large enough and then $p_0 \in I^-(q) = J^-(q)$, since $M$ is causally simple, being globally hyperbolic. In consequence, $p_0 \leq q \ll q_m$, for some $m$ large enough, where $\{q_n\}$ is a future-directed chain that generates $Q = I^-(\{q_m\})$.

Then, $p_0 \in Q$ and $P \subset Q$, which by definition means $P \ll Q$.

As expected, the relations $\ll$ and $\leq$ naturally extend those of $M$.

**Proposition 6** Let $(M, g)$ be globally hyperbolic. For all $p, q \in M$, $I^-(p) \ll I^-(q)$ if and only if $p \ll q$. Moreover, $I^-(p) \leq I^-(q)$ if and only if $p \ll q$.

**Proof** Assume first that $I^-(p) \ll I^-(q)$. This implies, by definition, that there is an $r \in I^-(q) \setminus I^-(p)$ such that $I^-(p) \subset I^-(r)$. Then $p \ll r \ll q$ which implies $p \ll q$.

Conversely, if $p \ll q$ then $p \neq q$ because $(M, g)$ is distinguishing. Consider $r \in M$ with $p \ll r \ll q$. Thus, $r \in I^-(q) \setminus I^-(p)$ with $I^-(p) \subset I^-(r)$, that is, $I^-(p) \ll I^-(q)$. This concludes the first part of the proof.

Now let us suppose that $I^-(p) \leq I^-(q)$, that is, $I^-(p) \subset I^-(q)$. For $p \in M$, we take $\{p_n\}$ a future-directed chain that generates $I^-(p)$, that is, a sequence with $p_n \ll p_{n+1}$ for all $n \in \mathbb{N}$ and $p_n \to p$. Notice that for all $n \in \mathbb{N}$ we have that $p_n \ll p$, that is, $p_n \in I^-(p)$ and so $p_n \in I^-(q)$. Therefore $p_n \to p \in I^-(q) = J^-(q)$. Therefore, $p \ll q$.

On the other hand, assume $p \leq q$. It suffices to prove that $I^-(p) \subset I^-(q)$. If $x \ll p$, by the push-up property we get $x \ll q$. Therefore, $I^-(p) \subset I^-(q)$.

\hfill \Box
So far, we have exhibited a metric topology, as well as chronological and causal relations in the future causal completion $\hat{M}$\!. In the remaining of this section we will be dealing with the construction of an adequate time separation $\hat{\tau}$ for $\hat{M}$\!. As a first step, notice that if $Q \in \hat{M}$ is generated by a future-directed chain $\{q_n\}$ and $p \in Q$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $p \ll q_n \ll q_{n+1}$\!. By the reverse triangle inequality of $\tau$ we have

$$\tau(p, q_{n+1}) \geq \tau(p, q_n) + \tau(q_n, q_{n+1}) > \tau(p, q_n).$$

Hence, the sequence of real numbers $\{\tau(p, q_n) : p \ll q_n\}$ is strictly increasing. Consequently, in view of Proposition 4 we define $\hat{\tau}$ as follows: for $p \in M$, $P, Q \in \hat{M}$

(i) $\hat{\tau}(P, Q) := 0$ if $P \in \hat{\partial}M$,

(ii) $\hat{\tau}(I^-(p), Q) := \sup\{\tau(p, q_n) : Q = I^-(\{q_n\})\}$.

Since the chain $\{q_n\}$ that generates a past set $Q$ might not be unique, we first have to show that $\hat{\tau}$ is well defined. First note that if $\hat{\tau}(P, Q) = 0$ there is nothing to prove.

Alternatively, if $\{q_n\}$ and $\{r_m\}$ are two different future-directed chains generating $Q$, it is enough to show that

$$\sup\{\tau(p, q_n) : Q = I^-(\{q_n\})\} = \sup\{\tau(p, r_m) : Q = I^-(\{r_m\})\}.$$

For any $p \in Q$ there exists $q_n \in Q$ with $p \ll q_n$, and hence there exists $r_k \in Q$ with $q_n \ll r_k$. Thus $p \ll q_n \ll r_k$ and by the reverse triangle inequality

$$\sup\{\tau(p, r_m)\} \geq \tau(p, r_k) \geq \tau(p, q_n) + \tau(q_n, r_m) > \tau(p, q_n).$$

Hence $\sup\{\tau(p, r_m)\}$ is an upper bound for $\{\tau(p, q_n)\}$ and thus

$$\sup\{\tau(p, q_n) : Q = I^-(\{q_n\})\} \leq \sup\{\tau(p, r_m) : Q = I^-(\{r_m\})\}.$$

Similarly,

$$\sup\{\tau(p, r_m) : Q = I^-(\{r_m\})\} \leq \sup\{\tau(p, q_n) : Q = I^-(\{q_n\})\}.$$

We now notice that $\hat{\tau}$ extends $\tau$ to the future causal completion $\hat{M}$\!.

**Proposition 7** If $P = I^-(p)$ and $Q = I^-(q)$, then $\hat{\tau}(P, Q) = \tau(p, q)$.

**Proof** By definition

$$\hat{\tau}(P, Q) = \sup\{\tau(p, q_n) : Q = I^-(\{q_n\})\}.$$

Since $Q = I^-(q)$ is a PIP, the chain $\{q_n\}$ must converge to $q$. Moreover, the sequence $\{\tau(p, q_n)\}$ is non decreasing and since $(M, g)$ is globally hyperbolic, $\tau$ is a continuous function. Therefore
\[ \hat{\tau}(P, Q) = \sup \{ \tau(p, q_n) : Q = I^-(\{q_n\}) \} = \lim_{n \to \infty} \tau(p, q_n) = \tau(p, q). \] 

\[ \square \]

The following series of lemmas is intended to show that \( \hat{\tau} \) defines a time separation function on \( (\hat{M}, \hat{T}_c) \) when \( (M, g) \) is a globally hyperbolic spacetime.

**Lemma 8 (Positivity).** If \( P, Q \in \hat{M} \), then \( \hat{\tau}(P, Q) > 0 \) if and only if \( P \ll Q \).

**Proof** We begin assuming that \( P \ll Q \). Then \( P = I^-(p) \) for some \( p \in M \) by Proposition 4. By definition, there exists \( q \in Q \setminus I^-(p) \) such that \( I^-(p) \subset I^-(q) \), then \( p \leq q \). Moreover, if \( \{q_n\} \) is a future directed chain generating \( Q \) then there exists \( q_m \in \{q_n\} \) such that \( p \leq q \ll q_n \) for all \( n \geq m \) which implies by the reverse triangle inequality

\[ \tau(p, q_n) \geq \tau(p, q) + \tau(q, q_n) > 0. \]

Therefore

\[ \hat{\tau}(I^-(p), Q) = \sup \{ \tau(p, q_n) : Q = I^-(\{q_n\}) \} > 0. \]

Conversely, assume that \( \hat{\tau}(P, Q) > 0 \), which by the definition of \( \hat{\tau} \) means that \( P \) is a PIP. Suppose that \( P = I^-(p) \). If \( \hat{\tau}(I^-(p), Q) < +\infty \), for any \( \epsilon > 0 \) there exists \( q_n \) with

\[ \tau(p, q_n) > \hat{\tau}(I^-(p), Q) - \epsilon. \]

If we take \( \epsilon = \frac{\hat{\tau}(I^-(p), Q)}{2} \), we have that \( \tau(p, q_n) > 0 \) which implies \( p \ll q_n \ll q_{n+1} \).

Hence, \( q_n \in Q \setminus I^-(p) \) with \( I^-(p) \subset I^-(q_n) \) and then \( I^-(p) \ll Q \).

Finally, if \( \hat{\tau}(I^-(p), Q) = +\infty \), for any \( N \in \mathbb{N} \) there exists \( q_m \in \{q_n\} \) such that

\[ \tau(p, q_m) \geq N > 0, \]

which implies \( p \ll q_n \) for all \( n \geq m \). Then \( I^-(p) \ll Q \).

\[ \square \]

The following result is needed in the proof of the reverse triangle inequality.

**Lemma 9** If \( p \notin Q \) or \( p \in \partial Q \), then \( \hat{\tau}(I^-(p), Q) = 0 \).

**Proof** Let \( \{q_n\} \) be a future-directed chain generating \( Q \in \hat{M} \). Proceeding by contradiction, let us suppose that \( \hat{\tau}(I^-(p), Q) > 0 \) then there exists \( q_k \in \{q_n\} \) with \( \tau(p, q_k) > 0 \), but this only occurs if and only if \( p \ll q_k \), which is a contradiction because \( p \notin Q = I^-(Q) \) or \( p \in \partial Q \), and in either case, there can be no such \( q_k \).

\[ \square \]

**Lemma 10 (Reverse triangle inequality).** If \( P, Q, R \in \hat{M} \) are such that \( P \preceq Q \preceq R \), then

\[ \hat{\tau}(P, R) \geq \hat{\tau}(P, Q) + \hat{\tau}(Q, R). \]
**Proof** We proceed case by case. If \( P, Q, R \) are all TIPs then

\[
\hat{\tau}(P, R) = 0 \geq \hat{\tau}(P, Q) + \hat{\tau}(Q, R) = 0.
\]

Moreover, if \( P \) is a TIP by Proposition 4, \( Q \) and \( R \) are TIPs as well. Also, if \( P, Q, R \) are all PIPs, that is, \( P = I^-(p), Q = I^-(q) \) and \( R = I^-(r) \) for some \( p, q, r \in M \) then the reverse triangle inequality of \( \tau \) gives us the result. Thus, we have only two cases left, namely

(i) \( P \lessgtr Q \lessgtr R \) with \( P = I^-(p), Q = I^-(q), R \in \partial M \);

(ii) \( P \lessgtr Q \lessgtr R \) with \( P = I^-(p), Q, R \in \partial M \).

For case (i), notice that if \( p \in \partial R \) then \( q \notin R \). Moreover, if \( p, q \in \partial R \), then \( p \rightarrow q \)\(^7\) follows. Thus we only have the following possibilities:

(a) \( p \rightarrow q \) and \( p, q \in \partial R \),

(b) \( p \ll q \) with \( q \in \partial R \),

(c) \( p \ll q \) and \( q \in R \),

(d) \( p \rightarrow q, p \in R \) and \( q \in \partial R \),

(e) \( p \rightarrow q \) and \( p, q \in R \).

In case (a) observe that \( \tau(p, q) = 0, \hat{\tau}(I^-(p), R) = \hat{\tau}(I^-(q), R) = 0 \) in virtue of Lemma 9. Thus the triangle inequality holds trivially.

Now, for case (b) consider \( R = I^-(\{r_m\}) \). Since \( p \in R \) and \( \hat{\tau}(I^-(q), R) = 0 \), take a future directed timelike sequence \( \{q_n\} \) such that \( q_n \in I^-(q) \) and \( q_n \rightarrow q \). Thus, \( p \lessgtr q_n \lessgtr r_m \) for some \( n \) and \( m \) natural numbers and this leads to

\[
\tau(p, r_m) \geq \tau(p, q_n) + \tau(q_n, r_m) \geq \tau(p, q_n)
\]

by the reverse triangle inequality of \( \tau \). Since \((M, g)\) is globally hyperbolic, the time separation function \( \tau \) is continuous, thus

\[
\hat{\tau}(I^-(p), R) \geq \hat{\tau}(I^-(p), I^-(q)) + 0 = \hat{\tau}(I^-(p), I^-(q)) + \hat{\tau}(I^-(q), R).
\]

For (c) it is enough to prove

\[
\hat{\tau}(I^-(p), R) \geq \tau(p, q) + \hat{\tau}(I^-(q), R).
\]

Let \( \{r_m\} \) be a future-directed chain generating \( R \). By definition

\[
\hat{\tau}(I^-(p), R) := \sup\{\tau(p, r_m) : R = I^-(\{r_m\})\},
\]

\[
\hat{\tau}(I^-(q), R) := \sup\{\tau(q, r_m) : R = I^-(\{r_m\})\}.
\]

We know that \( I^-(p) \lessgtr I^-(q) \) if and only if \( p \leq q \). Therefore \( p \leq q \ll r_m \) for large \( m \) and we have

\[
\tau(p, r_m) \geq \tau(p, q) + \tau(q, r_m),
\]

\(^7\) Recall that the horismos relation \( p \rightarrow q \) is defined as \( p \leq q \) and \( p \ll q \).
which in turn implies

\[ \hat{\tau}(I^{-}(p), R) \geq \tau(p, q) + \hat{\tau}(I^{-}(q), R). \]

In case (d) we have that \( \hat{\tau}(I^{-}(q), R) = 0 = \hat{\tau}(I^{-}(p), I^{-}(q)) \) and since \( p \in R \) we have that \( p \ll r_m \) for some \( m \in \mathbb{N} \) which leads to \( \tau(p, r_m) > 0 \) and this gives \( \hat{\tau}(I^{-}(p), R) > 0 = \hat{\tau}(I^{-}(p), I^{-}(q)) + \hat{\tau}(I^{-}(q), R) \).

In order to prove (e) note that \( \hat{\tau}(I^{-}(p), I^{-}(q)) = 0 \) and following the same argument as in point (c) we have that \( p \leq q \ll r_m \) for large \( m \) and thus \( \tau(p, r_m) \geq \tau(q, r_m) \) by the reverse triangle inequality in \((M, g)\). Thus, we have \( \hat{\tau}(I^{-}(p), R) \geq \hat{\tau}(I^{-}(q), R) \) which is the reverse triangle inequality in this case.

Similarly, for case (ii) notice that \( R \neq \hat{I}^{+}(Q) \), thus we have the following possibilities:

(a) \( p \in \partial Q \cap \partial R \),
(b) \( p \in \partial Q \) and \( p \in R \),
(c) \( p \in Q \cap R \).

In case (a) observe that all the quantities involved are zero and thus the reverse triangle inequality holds trivially.

In case (b) we have that \( \hat{\tau}(I^{-}(p), Q) = 0 \) and \( \hat{\tau}(Q, R) = 0 \) and since \( p \in R \) we have that for any future directed sequence \( \{r_m\} \) that generates \( R \) we have \( p \ll r_m \) for \( m \geq m_0 \). The latter implies that \( \hat{\tau}(I^{-}(p), R) \geq \tau(p, r_m) > 0 \) and thus the reverse triangle inequality holds trivially again.

Finally, for (c) we have to prove that

\[ \hat{\tau}(I^{-}(p), R) \geq \hat{\tau}(I^{-}(p), Q) \]

since \( \hat{\tau}(Q, R) = 0 \), by definition. In order to verify this, take \( \{q_n\} \) and \( \{r_m\} \) future-directed chains generating \( Q \) and \( R \), respectively. Observe that \( Q \subset R \) since they are causally related in \( \hat{M} \), so there are \( n, m_n \in \mathbb{N} \) such that \( p \ll q_n \ll r_{m_n} \). Therefore, we have \( \tau(p, r_{m_n}) \geq \tau(p, q_n) \) for large \( n \), and thus \( \hat{\tau}(I^{-}(p), R) \geq \hat{\tau}(I^{-}(p), Q) \) holds.

All that is left for \( \hat{\tau} \) to be a time separation function is for it to be lower semicontinuous.

**Lemma 11** (Lower semicontinuity). The function \( \hat{\tau} \) is lower semicontinuous in \( \hat{M} \), i.e. for all \( P, Q \in \hat{M} \), if \( (P_n) \) and \( (Q_n) \) are sequences that converge to \( P \) and \( Q \) with respect to CLT, respectively, then \( \liminf \hat{\tau}(P_n, Q_n) \geq \hat{\tau}(P, Q) \).

**Proof** If \( \hat{\tau}(P, Q) = 0 \), there is nothing to prove. Otherwise, let \( \hat{\tau}(P, Q) > 0 \) and thus \( P \ll Q \) and \( P = I^{-}(p) \) for some \( p \in M \) by Proposition 4.

First we prove that \( \hat{\tau} \) is lower semicontinuous on the second entry. Let \( \{q_n\} \) be a future-directed chain generating \( Q \) and \( \{Q_m\} \) a sequence of IPs converging to \( Q \) in the CLT topology. By definition of \( \hat{\tau} \), for each \( \delta > 0 \) there exists \( N_1 > 0 \) such that \( p \ll q_{N_1} \) and

\[ \tau(p, q_{N_1}) > \hat{\tau}(I^{-}(p), Q) - \delta. \]
Given that \( Q = L_H(\{Q_m\}) \) and \( q_{N_1} \in Q \), there exists \( N_2 > 0 \) such that \( q_{N_1} \in Q_m \) for all \( m \geq N_2 \). If \( \{r_{m,n}\} \) is a future chain generating \( Q_m \), for \( n \) large enough we have that \( p \ll q_{N_1} \ll r_{m,n} \) for all \( m \geq N_2 \) and

\[
\hat{\tau}(I^-(p), Q_m) \geq \tau(p, r_{m,n}) > \tau(p, q_{N_1}) > \hat{\tau}(I^-(p), Q) - \delta.
\]

We now have to prove that \( \hat{\tau} \) is lower semicontinuous in the first entry. Assume that \( \hat{\tau}(I^-(p), Q) \in (0, \infty) \) and let \( \delta > 0 \). Thus, let \( \{P_m\} \) be a sequence such that \( P_m \to I^-(p) \) in the CLT topology. We have to prove that \( \hat{\tau}(P_m, Q) > \hat{\tau}(I^-(p), Q) - \delta \) for large \( m \). Observe that \( \delta M \) is a closed subset in \( \hat{T}_c \) and thus \( P_m = I^-(p_m) \) for large \( m \). Moreover, convergence implies that \( p_m \in Q \) as well for large \( m \). Note that there exists \( N_1 \in \mathbb{N} \) such that \( \hat{\tau}(I^-(p), Q) - \delta/2 < \tau(p, q_{N_1}) \) by the definition of \( \hat{\tau}(I^-(p), Q) \). The continuity of \( \tau \) and the convergence of \( \{p_m\} \) to \( p \) ensures \( |\tau(p, q_{N_1}) - \tau(p_m, q_{N_1})| < \delta/2 \) for large \( m \). Therefore, we have the following inequalities

\[
\hat{\tau}(I^-(p), Q) - \delta/2 < \tau(p, q_{N_1}) < \tau(p_m, q_{N_1}) + \delta/2
\]

and thus we have

\[
\hat{\tau}(I^-(p), Q) - \delta < \tau(p_m, q_{N_1}) < \hat{\tau}(I^-(p_m), Q)
\]

for large \( m \). Hence lower semicontinuity in the first entry follows from the definition of \( \hat{\tau}(I^-(p_m), Q) \).

The case \( \hat{\tau}(I^-(p), Q) = \infty \) is analogous.

The following theorem summarizes the results of the previous lemmas.

**Theorem 12** Let \((M, g)\) be a globally hyperbolic space-time. Then \((\hat{M}, d_c, \llaw, \ll, \hat{\tau})\) is a Lorentzian pre-length space.

It is important to note that the relations \( \llaw, \ll \) as well as the CLT topology are not affected by a conformal change on the spacetime metric \( g \). However, this is not the case for the time separation \( \tau \). Thus, it is expected to have different Lorentzian pre-length structures in the future causal completion within the same conformal class.

### 4 Applications

In this section we show that a class of warped product spacetimes satisfies the condition of Theorem 3 and hence its associated chronological and CLT topologies coincide. This result can be used to carry out explicit calculations regarding the pre-length structure of the future causal completion.\(^8\) Finally, as an illustrative example we consider the particular case of de Sitter spacetime.

\(^8\) A thorough description of the future causal boundary of such spacetimes can be found in [36].

\( \llaw \) Springer
Recall that a Lorentzian warped product is a manifold \( M = (a, b) \times S, (-\infty \leq a < b \leq +\infty) \) furnished with a metric of the form

\[
g = -dt^2 + \alpha(t)h,
\]

where, \((S, h)\) is a Riemannian manifold and \(\alpha\) is a smooth positive function over \((a, b)\).

The chronological relation on these spacetimes can be characterized as follows (see [37, Sect. 2.2]):

**Lemma 13** Let \((M, g)\) be a Lorentzian warped spacetime. If \((t_0, x_0), (t_1, x_1)\) points in \(M\), then,

\[
(t_0, x_0) \ll (t_1, x_1) \iff d(x_0, x_1) < \int_{t_0}^{t_1} \frac{ds}{\sqrt{\alpha(s)}},
\]

where \(d\) is the distance induced in \(S\) by the metric \(h\).

The future causal completion can be characterized depending on the value of \(\int_c^b \frac{ds}{\sqrt{\alpha(s)}}\) for some \(c \in (a, b)\). Indeed we have\(^9\).

**Theorem 14** Let \(M = (a, b) \times S\) with \(g = -dt^2 + \alpha(t)h\) be a warped spacetime and \((S, d)\) a locally compact metric space. Then,

1. If \(\int_c^b \frac{ds}{\sqrt{\alpha(s)}} = \infty\), then, the future causal boundary \(\hat{\partial}M\) is an infinite null cone with base \(\partial BS \setminus \partial CS\) with apex in \(i^+\) and timelike lines over each point in \(\partial CS\) and final point in \(i^+\). Moreover, \(\hat{M}\) is homeomorphic to \(M \cup ((a, b) \times \partial CS) \cup ((a, b) \times \partial BS) \cup i^+\).
2. If \(\int_c^b \frac{ds}{\sqrt{\alpha(s)}} < \infty\), then, \(\hat{\partial}M\) is a copy of the Cauchy completion \(SC\) of \((S, h)\) and timelike lines over each point in \(\partial CS\) that finish in the same point at the copy at infinity of \(SC\). Moreover, \(\hat{M}\) is homeomorphic to \(M \cup ((a, b) \times \partial CS) \cup (\{b\} \times SC)\).

Let \((M, g)\) be a globally hyperbolic warped product spacetime. Then the Riemannian manifold \((S, h)\) is complete as can be seen in [39, Theorem 3.68] and hence the Cauchy boundary \(\partial CS = \emptyset\). Therefore, if in addition

\[
\int_c^b \frac{ds}{\sqrt{\alpha(s)}} < \infty,
\]

then by the previous theorem we have that the future causal completion is characterized as \(\hat{M} = M \cup (\{b\} \times S)\) and the causal boundary consists of a copy of \(S\) at \(b\). As a consequence, any TIP \(P \in \hat{\partial}M\) can be identified with a set of the form \(P = I^- (b, x), x \in S\).\(^10\)

---

\(^9\) Here \(\partial BS\) and \(\partial CS\) are the Busemann and (metric) Cauchy boundaries or \(M\). Refer to [36–38] for a detailed account on the structure and topology of \(\hat{M}\).

\(^10\) Here, \(I^- (b, x) := \{(t_0, x_0) \in M | \int_{t_0}^b \frac{ds}{\sqrt{\alpha(s)}} < \int_c^b \frac{ds}{\sqrt{\alpha(s)}} - d(x_0, x)\}\)
Fig. 1 Case (1). $P$ is proper and $Q$ is terminal

**Proposition 15** Let $(M, g)$ be a globally hyperbolic warped product spacetime with
\[ \int_{c}^{b} \frac{ds}{\sqrt{\alpha(s)}} < \infty \]
for some $c \in (a, b)$. Then the chronological topology $\hat{T}_{\text{chr}}$ in $\hat{M}$ is Hausdorff.

**Proof** Recall that in $i(M)$, the chronological topology agrees with the topology of the space-time $M$, which is Hausdorff. Thus there are only two cases left to consider:

1. $P$ is PIP and $Q$ is a TIP (Fig. 1),
2. $P$ and $Q$ are both TIPs (Fig. 2).

**Case 1.** Consider $p = (t_1, x_1)$ a point in $M$ and $P = I^{-}(p)$ its corresponding past set; and take $Q = (b, x_2) \in \hat{\partial}M$. We know, by the definition of the chronological relation in $M$, that for $t_2 > t_1$, we have

$$(t_1, x_1) \ll (t_2, x_1)$$

because

$$0 = d(x_1, x_1) < \int_{t_1}^{t_2} \frac{ds}{\sqrt{\alpha(s)}}.$$

Note that $(t_2, x_2) \in Q$ since $(t_2, x_2) \ll (t, x_2)$ for all $t > t_2$. Hence, there exists $q_n \in Q$ with $(t_2, x_2) \ll q_n$ and as a consequence $I^{-}(t_2, x_2) \ll Q$. Therefore $Q \in I^{+}(I^{-}(t_2, x_2))$. On the other hand, since $(t_1, x_1) \ll (t_2, x_1)$ then $P \ll I^{-}(t_2, x_1)$, and by definition $P \in \hat{I}^{-}(I^{-}(t_2, x_1))$. Now we need to prove

$$\hat{I}^{+}(I^{-}(t_2, x_2)) \cap \hat{I}^{-}(I^{-}(t_2, x_1)) = \emptyset.$$ 

By contradiction, suppose there exists $R = I^{-}(r, y) \in \hat{M}$, with $r \in (a, b)$, such that

$$R \in \hat{I}^{+}(I^{-}(t_2, x_2)) \cap \hat{I}^{-}(I^{-}(t_2, x_1)) = \emptyset.$$ 

\(\square\) Springer
Fig. 2 Case 2. $P$ and $Q$ are both terminal

Note that $r < b$ because otherwise $R$ and $Q$ would be two TIPs with $R \ll Q$, which cannot occur since $(M, g)$ is globally hyperbolic. Thus, $R$ is proper and

$$I^-(t_2, x_2) \ll I^-(r, y) \ll I^-(t_2, x_1),$$

which happens by Proposition 6 if and only if

$$(t_2, x_2) \ll (r, y) \ll (t_2, x_1),$$

and by transitivity

$$(t_2, x_2) \ll (t_2, x_1),$$

that is,

$$0 \leq d(x_2, x_1) < \int_{t_2}^{t_1} \frac{ds}{\sqrt{\alpha(s)}} = 0,$$

a contradiction. This concludes case 1.

**Case 2.** Consider $P, Q \in \partial M$ with $P \neq Q$. We can represent these sets as

$$P = I^-(b, x_1), \quad Q = I^-(b, x_2).$$

Since $\int_c^b \frac{ds}{\sqrt{\alpha(s)}} < +\infty$ for some $c \in (a, b)$ we can take $t_1, t_2 \in (a, b)$ such that

$$\int_{t_1}^b \frac{ds}{\sqrt{\alpha(s)}} < \frac{1}{3}d(x_1, x_2) \quad \text{and} \quad \int_{t_2}^b \frac{ds}{\sqrt{\alpha(s)}} < \frac{1}{3}d(x_1, x_2)$$

which implies $(t_1, x_1) \notin Q$ and $(t_2, x_2) \notin P$, respectively.
Without loss of generality suppose $t_1 < t_2$. We know that
\[ P \in \hat{I}^+(I^-(t_2, x_1)), \quad Q \in \hat{I}^+(I^-(t_2, x_2)), \]
so it is enough to prove that these sets are disjoint, i.e.
\[ \hat{I}^+(I^-(t_2, x_1)) \cap \hat{I}^+(I^-(t_2, x_2)) = \emptyset. \]

We proceed by contradiction. Since $i(M)$ is dense in $\hat{M}$ we can take $R = I^-(r, y) \in \hat{M}$ such that
\[ R \in \hat{I}^+(I^-(t_2, x_1)) \cap \hat{I}^+(I^-(t_2, x_2)), \]
whence
\[ I^-(t_2, x_1) \ll I^-(r, y) \quad \text{and} \quad I^-(t_2, x_2) \ll I^-(r, y) \]
by Proposition 6 if and only if
\[ (t_2, x_1) \ll (r, y) \quad \text{and} \quad (t_2, x_2) \ll (r, y), \]
which happens if and only if
\[ d(x_1, y) < \int_{t_2}^{r} \frac{ds}{\sqrt{\alpha(s)}} \quad \text{and} \quad d(x_2, y) < \int_{t_2}^{r} \frac{ds}{\sqrt{\alpha(s)}}. \]
where adding both inequalities and using the triangle inequality for $d$ we get
\[ d(x_1, x_2) \leq d(x_1, y) + d(x_2, y) < 2 \int_{t_2}^{r} \frac{ds}{\sqrt{\alpha(s)}} \]
and so
\[ \int_{t_2}^{b} \frac{ds}{\sqrt{\alpha(s)}} < \int_{t_2}^{r} \frac{ds}{\sqrt{\alpha(s)}}, \]
hence, irrespective if $r$ is finite or infinite, we get a contradiction. \hfill $\Box$

The next result is an immediate consequence of Proposition 15 and Theorem 3.

**Corollary 16** Let $(M, g)$ be a globally hyperbolic warped product spacetime with
\[ \int_{c}^{b} \frac{ds}{\sqrt{\alpha(s)}} < \infty \quad \text{for some} \quad c \in (a, b). \]
Then $\hat{T}_{chr} = \hat{T}_c$.

In the particular case of $M = \mathbb{R} \times S$ we have that the time separation function $\hat{\tau}$ when evaluated in $P = I^-(p)$ and $Q$ a TIP is either zero or infinity.

**Proposition 17** If $P, Q \in \hat{M}$ with $Q \in \hat{\partial}M$ then either $\hat{\tau}(P, Q) = 0$ or $\hat{\tau}(P, Q) = \infty$. 

$\square$ Springer
Proof If \( P \prec M \) or \( P \in \hat{\partial}M \) then \( \hat{\tau}(P, Q) = 0 \) and we are done. Now let us assume that \( P = I^-(p) \prec M = I^-(\infty, a) \) and that \( \{q_n\} \) is a future-directed chain generating \( Q \). By chronology, there exists \( q_N \in Q \) with \( p \ll q_N \ll q_n \) for all \( n > N \) and since \( M \) is globally hyperbolic, \( \tau(p, q_N) < \infty \). Given that \( \hat{\tau} \) does not depend on the choice of the chain that generates \( Q \), therefore we can choose a chain such that

\[
q_n = (t_n, a),
\]

which is based on the same spatial constant coordinate \( a \in S \) as \( Q \). From here, since \( Q \) is terminal, \( q_n \) does not converge in \( M \) and then \( t_n \to \infty \). This implies\(^{11}\)

\[
\tau(p, q_n) = \tau(p, q_N) + \tau(q_N, q_n) > \tau(q_N, q_n) \geq t_n - t_N.
\]

Thus \( \hat{\tau}(P, Q) = \infty \) for any \( P \) a PIP and \( Q \) a TIP chronologically related. \( \square \)

Remark 1 It is important to mention that even though the future causal completion \( \hat{\partial} \) is a conformal invariant, the time separation function \( \hat{\tau} \) – and thus the Lorentzian pre-length structure– is not. For instance, consider the globally hyperbolic spacetime \( M = (0, 1) \times \mathbb{R} \) with the usual Lorentzian metric \( g = -dt^2 + dx^2 \). It is easy to verify that \( \hat{\tau}(I^-(p), Q) \) is either zero or a finite positive number for any \( p \in M \) and \( Q \in \hat{\partial}M \). However, if we consider the conformal factor \( \Omega(t, x) = \frac{1}{1-t} \) we have that \( (M, \Omega^2 g) \) takes the warped product form \((0, +\infty) \times \mathbb{R} \) with metric \(-ds^2 + e^{2s}dx^2 \), where \( ds = dt/(1-t) \). Thus by Proposition 17 we have that for this spacetime \( \hat{\tau}_0(I^-(p), Q) \) is either zero or infinity for \( p \in M \) and \( Q \in \hat{\partial}M \).

We conclude this section by providing an explicit example. Consider de Sitter spacetime \( S^4_1 \), the Lorentzian space form of constant sectional curvature \( K = 1 \). It can be realized as the hyperboloid

\[-v^2 + w^2 + x^2 + y^2 + z^2 = 1\]

in flat five-dimensional Minkowski space \( \mathbb{R}^5_1 = (\mathbb{R}^5, L) \). Recall that \( S^4_1 \) can also be described as the globally hyperbolic warped product

\((\mathbb{R} \times S^3, -dt^2 + \cosh^2(t)g_{S^3})\),

where \( g_{S^3} \) is the standard round metric on \( S^3 \). Thus,

\[
\int_0^\infty \frac{ds}{\sqrt{\alpha(s)}} = \int_0^\infty \frac{ds}{\cosh(s)} = \frac{\pi}{2} < +\infty.
\]

Moreover, by Theorem 14 the future causal boundary \( \hat{\partial}S^4_1 \) is a copy at infinity of the base manifold \( S^3 \), a picture that agrees with its standard future conformal infinity \( I^+ \) [40].

\(^{11}\) Note that if \( \gamma : [t_1, t_2] \to M \) is a vertical curve, that is, \( \gamma(t) = (t, a) \) with \( a \in S \) fixed, then \( L_g(\gamma) = \int_{t_1}^{t_2} |\dot{\gamma}| dt = t_2 - t_1 \). This indicates that the g-length of \( \gamma \) grows to infinity as \( t_2 \to \infty \).
Let \( p \leq q \) be two causally related points in \( S^4_1 \). Since the time separation function is given by

\[
\cosh(\tau(p, q)) = L(p, q),
\]

then

\[
\tau(p, q) = t_2 - t_1
\]

for \( p = (t_1, x), q = (t_2, x) \) two points on the same geodesic normal to \( S^3 \). Thus, if \( t_2 \to \infty \) then \( \tau(p, q) \to \infty \). If we consider a PIP \( P = I^-(t, x) \in \hat{S}^4_1 \) and a TIP \( Q = I^-(\infty, y) \in \hat{\partial S}^4_1 \), then by definition

\[
\hat{\tau}(P, Q) = \sup \{ \tau(p, q_n) : Q = I^-(\{q_n\}) \},
\]

where \( \{q_n\} \) is a future-directed chain generating \( Q \). Choose a chain such that

\[
q_n = (t_n, y),
\]

which is based on the same spatial coordinate \( y \in S^3 \). From here, since \( Q \) is terminal, \( \{q_n\} \) does not converge in \( S^4_1 \) and then \( t_n \to \infty \).

We know that if \( p = (t, x) \notin Q \) or \( p \in \partial Q \) we have \( \hat{\tau}(I^-(p), Q) = 0 \) by Lemma 9. On the other hand, if \( p \in Q \), then by the proof of Proposition 17 we have

\[
\hat{\tau}(P, Q) = \infty.
\]

In summary, the time separation function \( \hat{\tau} \) for the future causal completion of de Sitter spacetime is given by

\[
\hat{\tau}(P, Q) = \begin{cases} 
\tau(p, q) & \text{if } P = I^-(p), Q = I^-(q) \\
0 & \text{if } P \hat{\prec} Q \\
\infty & \text{if } P \hat{\ll} Q \text{ and } Q \in \hat{\partial S}^4_1.
\end{cases}
\]

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**Declarations**

**Conflict of interest** The author declare they have no conflict of interests.
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