BILAYERS IN FOUR DIMENSIONS AND SUPERSYMMETRY

Romain ATTAL

Laboratoire de Physique Théorique et Hautes Energies, Universités Pierre et Marie Curie (Paris 6) et Denis Diderot (Paris 7); Unité de recherche associée au CNRS (D0 280).

Abstract: I build $N=1$ superstrings in $\mathbb{R}^2$ out of purely geometric bosonic data. The world-sheet is a bilayer of uniform thickness and the $2D$ supercharge vanishes in a natural way.

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1E-mail: attal@lpthe.jussieu.fr.
2LPTHE tour 16 / 1er étage, Université P. et M. Curie, BP 126, 4 place Jussieu, F 75252 PARIS CEDEX 05 (France).
1 Introduction

The usual approach to superstrings [1] uses anticommuting variables which are not very intuitive objects. In order to understand them better, I have sought for a more pictorial description. The basic idea is to use standard bosonization techniques [2] and to interpret geometrically the compactified bosonic field as kinks in the normal bundle. This is only possible when the space-time is a four-manifold. The resulting model is the following: I consider a bilayer with a uniform thickness living in a four dimensional, flat Euclidean space and choose an action proportional to the total area $A$ of this bilayer. I show that this is a $\sigma$-model, taking values in the projectified normal bundle, which can be fermionized into a worldsheet Dirac fermion coupled to the normal connection [3]. For a particular value of the thickness, related to the string tension, this model is equivalent to a free four-vector Majorana fermion with the orthogonality constraint of a spinning string (the massless Dirac-Ramond equation) [4].

2 Action

Our bilayers are described by:

- a smooth closed orientable $2D$ surface $\Sigma$, with $p$ marked points $S_1 \cdots S_p$ ;
- an immersion $X : \Sigma \rightarrow \mathbb{R}^4$ ;
- a smooth section of the projectified normal bundle induced by $X$ on $\Sigma$ ($Y \in \Gamma(PN_X \Sigma)$ can be singular at the punctures $S_1 \cdots S_p$) ;
- a thickness $2\delta > 0$.

The $S_i$'s are the limits of infinitesimal circles mapped to twisted strings. If $y(P)$ is a unit vector in the line $Y(P)$ ($\forall P \in \Sigma$), the area of the bilayer $(X \pm \delta y)(\Sigma)$ is:

$$A = \int_{\Sigma} d\xi_1 \wedge d\xi_2 \left( (\det[\partial_a(X + \delta y).\partial_b(X + \delta y)])^{1/2} + (\det[\partial_a(X - \delta y).\partial_b(X - \delta y)])^{1/2} \right)$$

which I expand in powers of $\delta$:

$$A = 2 \int_{\Sigma} d\xi_1 \wedge d\xi_2 \left( g^{1/2} \left( 1 + \frac{\delta^2}{2} g^{ab} \partial_a y^\perp \cdot \partial_b y^\perp + \delta^2 \mathcal{R} + \mathcal{O}(\delta^4) \right) \right) . \quad (2)$$

Here, $\xi = (\xi^1; \xi^2)$ is a local coordinate system on $\Sigma$, the dot denotes the standard inner product in $\mathbb{R}^4$, $\partial_a y^\perp$ is the normal part of $\partial_a y$, $g_{ab} = \partial_a X.\partial_b X$, $g = \det[g_{ab}]$, and $\mathcal{R}$ is Ricci's scalar curvature. The $\mathcal{O}(\delta^4)$ terms, containing more derivatives, are irrelevant, and I drop the topological term $\int_{\Sigma} d\xi_1 \wedge d\xi_2 \ g^{1/2} \mathcal{R} = 8\pi(1 - \text{genus}(\Sigma))$. The second term in (2) can be rewritten as follows. Pick a generic $N \in \Gamma(N_X \Sigma)$ with isolated zeros $Z_1 \cdots Z_q$ of indices $\iota_1 \cdots \iota_q$. The normal $n = N/\|N\|$ and binormal $b$ define a right handed orthonormal frame in $N_X \Sigma$ over $\Sigma \setminus \{Z_1 \cdots Z_q\}$, where the normal connection $\nabla^\perp$ is represented by the matrix $\left( \begin{array}{cc} d & -T \\ T & d \end{array} \right)$ with $d = d\xi_1 \partial_1 + d\xi_2 \partial_2$ and $T = b.dn$. If $\theta : \Sigma_Z \rightarrow \mathbb{R}/\pi \mathbb{Z}$ is the angle from $\pm n$ to $Y$, we have:

$$\pm y = \cos \theta \ n + \sin \theta \ b ,$$
Due to the equation of motion (\(d\omega = 0\)) \( (d\omega) = (\partial_1 \theta + T_1) d\xi^2 - (\partial_2 \theta + T_2) d\xi^1 \) if \( g_{ab} = e^{\phi} \delta_{ab} \). I take the action to be \( S = \mu A \), \( \mu \) being the string tension of one layer. In the partition function \( Z(\mathcal{A}) = \int D\theta \, e^{-\mu \delta^2 \int_{\Sigma} \omega \wedge * \omega} \), we sum over the \( \theta \)'s which satisfy \( \int_{\mathcal{Z}} \omega = 0 \), since \( Y \) is regular at these points, and \( \int_{\mathcal{S}_i} \omega = n_i \pi \) (the boundary strings can be twisted). Among these functions, the classical configurations are the solutions of the equation of motion \( d\omega = 0 \) and are parametrized by \( H_1(\Sigma; \mathbb{Z}) \).

### 3 Fermions

Since \( P N_X \Sigma \) is a circle bundle, this system admits kinks and a fermionic representation by holonomies \( \bar{b} \). If \( \gamma : [0; 1] \to \Sigma \) is a path joining 0 to \( P \), we define:

\[
\begin{align*}
    b &= \exp \left( k \int_{\gamma} i d\theta - \omega \right) \quad c = \exp \left( -k \int_{\gamma} i d\theta - \omega \right) \\
    \bar{b} &= \exp \left( k \int_{\gamma} i d\theta + \omega \right) \quad \bar{c} = \exp \left( -k \int_{\gamma} i d\theta + \omega \right).
\end{align*}
\]

Due to the equation of motion \( (d\omega = 0) \), their correlators only depend on \( \gamma \in H_1(\Sigma, P - P_0; \mathbb{Z}) \). In order to recover the same number of degrees of freedom in (7) and (8), we thus obtain \( \frac{1}{Z(\mathcal{A})} \int D\theta \, e^{-\mu \delta^2 \int_{\Sigma} \omega \wedge * \omega} \, b(z)c(0) = \langle b(z)c(0) \rangle \sim z^{-1} \),

on \( \mathbb{C} \) and without the gauge field \( T \), we must fix \( k = (2\pi \mu \delta^2)^{1/2} \), as can be seen after a Gaussian integration. Moreover, for the special value \( k = 1 \), i.e. \( \delta = (2\pi \mu)^{-1/2} = \delta_0 \), there is no quartic term in the fermionic action \( \bar{c} \) and \( \psi = \begin{pmatrix} c \\ b \end{pmatrix} \) satisfies the following equation of motion:

\[
\begin{pmatrix}
    0 & 2\partial + i(T_1 + iT_2) \\
    2\partial + i(T_1 - iT_2) & 0
\end{pmatrix}
\begin{pmatrix}
    c \\
    b
\end{pmatrix}
= (\partial + iT)\psi = 0.
\]

This shows that \( \psi \) is a 2D Dirac spinor and a vector in \( N_X \Sigma \):

\[
\psi \in \Gamma(K^{1/2} \otimes_C N_X \Sigma) \oplus \Gamma(K^{-1/2} \otimes_C N_X \Sigma) .
\]

Here, \( N_X \Sigma \) is viewed as a complex line bundle on \( \Sigma \), \( K \) denotes the canonical line bundle of holomorphic \((1, 0)\)-forms on \( \Sigma \), \( K^{1/2} \) is one of the \( \oplus_{\text{genus}(\Sigma)+1} \) spin structures on \( \Sigma \), \( K^{-1/2} \) is the dual bundle of \( K \) and \( K^{-1/2} = K^{1/2} \otimes_C K^* \). Since the normal connection \( \nabla_{\Sigma} \) is the projection on \( N_X \Sigma \) of the trivial connection \( \nabla \) acting on sections of the total bundle \( X^*(T \mathbb{R}^4) = T \Sigma \oplus \mathbb{R} \otimes N_X \Sigma \), we can replace \( \psi \) by a free four-vector Majorana fermion

\[
\Psi \in \Gamma(K^{1/2} \otimes \mathbb{R} X^*(T \mathbb{R}^4)) \oplus \Gamma(K^{-1/2} \otimes \mathbb{R} X^*(T \mathbb{R}^4)) \text{ and } \partial \Psi = 0,
\]

with the orthogonality constraint \( \Psi.dX = 0 \) to be applied on the Hilbert space in order to recover the same number of degrees of freedom in (7) and (8). We thus obtain
three equivalent descriptions of a fermionic string satisfying the (massless) Dirac-Ramond equation:

1. a $\sigma$-model in $PN_X\Sigma$;
2. $\psi \in \Gamma(K^{1/2} \otimes_C N_X\Sigma) \oplus \Gamma(K^{1/2} \otimes_C N_X\Sigma)$ and $(\not\partial + i\not\mathcal{I})\psi = 0$;
3. $\Psi \in \Gamma(K^{1/2} \otimes_R X^*(T\mathbb{R}^4)) \oplus \Gamma(K^{-1/2} \otimes_R X^*(T\mathbb{R}^4))$, $\Psi$ is real, $\not\partial\Psi = 0$ and $\Psi.dX = 0$.

4 Conclusion

The previous computations suggest a simple picture for superstrings in four dimensions: they are double covers of bosonic strings and the two nearby world-sheets must be separated by $2\delta_0$ in order to have free fields. This suggests that one interpret the tachyonic instability of bosonic strings as a phase transition to a fermionic vacuum.

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