Two-dimensional Poisson Trees converge to the Brownian web

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Abstract The Brownian web can be roughly described as a family of coalescing one-dimensional Brownian motions starting at all times in $\mathbb{R}$ and at all points of $\mathbb{R}$. It was introduced by Arratia; a variant was then studied by Tóth and Werner; another variant was analyzed recently by Fontes, Isopi, Newman and Ravishankar. The two-dimensional Poisson tree is a family of continuous time one-dimensional random walks with uniform jumps in a bounded interval. The walks start at the space-time points of a homogeneous Poisson process in $\mathbb{R}^2$ and are in fact constructed as a function of the point process. This tree was introduced by Ferrari, Landim and Thorisson. By verifying criteria derived by Fontes, Isopi, Newman and Ravishankar, we show that, when properly rescaled, and under the topology introduced by those authors, Poisson trees converge weakly to the Brownian web.

Keywords and phrases Brownian web, Poisson tree, Donsker’s invariance principle, coalescing one-dimensional Brownian motions

1991 Mathematics Subject Classification 60K35, 60F17

1 Introduction and results

The Two-dimensional Poisson Tree Let $S$ be a two-dimensional homogeneous Poisson process of parameter $\lambda$. $S$ is a random subset of $\mathbb{R}^2$, $s \in S$ has coordinates $s_1, s_2$. 

♦ Partially supported by CNPq grants 520811/96-8, 300576/92-7 and 662177/96-7 (PRONEX) and FAPESP grant 99/11962-9

† Research supported in part by Foundation of Beijing Education Bureau and FAPESP through grant 01/02577-6.
For $x = (x_1, x_2) \in \mathbb{R}^2$, $t \geq x_2$ and $r > 0$, let $M(x, t, r)$ be the following rectangle
\[
 M(x, t, r) := \{(x'_1, x'_2) : |x'_1 - x_1| \leq r, \ x_2 \leq x'_2 \leq t\}. \tag{1.1}
\]
As $t$ grows, the rectangle gets longer. The first time $t$ that $M(x, t, r)$ hits some (or another, when $x \in S$) point of $S$ is called $\tau(x, S, r)$; this is defined by
\[
 \tau(x, S, r) := \inf\left\{ t > x_2 : M(x, t, r) \cap (S \setminus \{x\}) \neq \emptyset \right\}. \tag{1.2}
\]
The hitting point is the point $\alpha(x) \in S$ defined by
\[
 \alpha(x) := M(x, \tau(x, S, r), r) \cap (S \setminus \{x\}), \tag{1.3}
\]
which consists of a unique point almost surely. If $x = s$ some $s \in S$, we say that $\alpha(x) = \alpha(s)$ is the *mother* of $s$ and that $s$ is a *daughter* of $\alpha(s)$. Let $\alpha^0(x) = x$ and iteratively, for $n \geq 1$, $\alpha^n(x) = \alpha(\alpha^{n-1}(x))$. For the case of $x = s$, $\alpha^n(x) = \alpha^n(s)$ is the $n$th grand mother of $s$.

Now let $G = (V, E)$ be the random directed graph with vertices $V = S$ and edges $E = \{(s, \alpha(s)) : s \in S\}$. This model was proposed by Ferrari, Landim and Thorisson \[5\] who also proved that $G$ has a unique connected component. It is straightforward to see that there are no circuits in the graph $G$ which is then called the two-dimensional *Poisson tree*.

This model is related to a model of drainage networks of Gangopadhyay, Roy and Sarkar \[10\], which can be viewed as a discrete space, long range version of the two-dimensional Poisson tree.

**Brownian web** Arratia \[2, 3\], and later Tóth and Werner \[16\] constructed random processes that formally correspond to coalescing one-dimensional Brownian motions starting from every space-time point. Fontes, Isopi, Newman and Ravishankar \[7\] (see also \[8\]) extended their work by constructing and characterizing the so-called *Brownian web* as a random variable taking values in a metric space whose points are compact sets of paths. The space is given as follows.

Let $(\mathbb{R}^2, \rho)$ be the completion (or compactification) of $\mathbb{R}^2$ under the metric $\rho$ defined as
\[
 \rho(x, y) = \frac{\left| \tanh(x_1) - \tanh(y_1) \right|}{1 + |x_2|} \vee \frac{\left| \tanh(x_2) - \tanh(y_2) \right|}{1 + |y_2|}, \tag{1.4}
\]
for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2 := [-\infty, \infty]^2$, namely, the image of $\mathbb{R}^2$ under the mapping
\[
 x = (x_1, x_2) \mapsto (\Phi(x_1, x_2), \Psi(x_2)) := \left( \frac{\tanh(x_1)}{1 + |x_2|}, \tanh(x_2) \right). \tag{1.5}
\]

For $t_0 \in [-\infty, \infty]$, let $C[t_0]$ denote the set of functions $f$ from $[t_0, \infty]$ to $[-\infty, \infty]$ such that $\Phi(f(t), t)$ is continuous. Define
\[
 \Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}, \tag{1.6}
\]
where \((f, t_0) \in \Pi\) then represents a path in \(\mathbb{R}^2\) starting at \((f(t_0), t_0)\).

For \((f(t_0), t_0) \in \Pi\), we denote by \(\bar{f}\) the function that extend \(f\) to all \([-\infty, \infty]\) by setting it equal to \(f(t_0)\) for all \(t < t_0\). We take

\[
d((f_1(t_1), (f_2(t_2))) = (\sup_t |\Phi(\bar{f}_1(t), t) - \Phi(\bar{f}_2(t), t)|) \vee |\Psi(t_1) - \Psi(t_2)|, \tag{1.7}
\]

then we get a metric space \((\Pi, d)\) of paths with specified starting points in space-time. It is straightforward to check that \((\Pi, d)\) is a complete separable metric space.

Let now \(\mathcal{H}\) denote the set of compact subset of \((\Pi, d)\), with \(d_{\mathcal{H}}\) the induced Hausdorff metric, i.e.,

\[
d_{\mathcal{H}}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1, g_2). \tag{1.8}
\]

\((\mathcal{H}, d_{\mathcal{H}})\) is also a complete separable metric space. Denote by \(\mathcal{F}_\mathcal{H}\) the corresponding Borel \(\sigma\)-algebra generated by \(d_{\mathcal{H}}\).

The Brownian web is characterized in [7] as a \((\mathcal{H}, \mathcal{F}_\mathcal{H})\)-valued random variable \(\bar{W}\) (or its distribution \(\mu_{\bar{W}}\)). Define the finite-dimensional distributions of \(\bar{W}\) as the induced probability measures \(\mu_{\bar{W}}(x_1, t_1; \ldots; x_n, t_n)\) on the subsets of paths starting from any finite deterministic set of points \((x_1, t_1), \ldots, (x_n, t_n)\) in \(\mathbb{R}^2\).

Given \(t_0 \in \mathbb{R}, t > 0, a < b,\) and a \((\mathcal{H}, \mathcal{F}_\mathcal{H})\)-valued random variable \(V\), let \(\eta_V(t_0, t; a, b)\) be the \(\{0, 1, 2, \ldots, \infty\}\)-valued random variable giving the number of distinct points in \(\mathbb{R} \times \{t_0 + t\}\) that are touched by paths in \(V\) which also touch some point in \([a, b] \times \{t_0\}\).

The following is the characterization Theorem of the Brownian web of Fontes, Isopi, Newman and Ravishankar in [7].

**Theorem A** There is an \((\mathcal{H}, \mathcal{F}_\mathcal{H})\)-valued random variable \(\bar{W}\) whose distribution \(\mu_{\bar{W}}\) is uniquely determined by the following three properties:

1. (o) from any deterministic point \((x, t)\) in \(\mathbb{R}^2\), there is almost surely a unique path \(W_{x, t}\) in \(\bar{W}\) starting from \((x, t)\).
2. (i) for any deterministic \(n\) and \((x_1, t_1), \ldots, (x_n, t_n)\), the joint distribution of \(W_{x_1, t_1}, \ldots, W_{x_n, t_n}\) is that of coalescing Brownian motions (with unit diffusion constant), and
3. (ii) for any deterministic, dense countable subset \(\mathcal{D}\) of \(\mathbb{R}^2\), almost surely, \(\bar{W}\) is the closure in \((\mathcal{H}, d_{\mathcal{H}})\) of \(\{W_{x, t} : (x, t) \in \mathcal{D}\}\).

The \((\mathcal{H}, \mathcal{F}_\mathcal{H})\)-valued random variable \(\bar{W}\) given in Theorem A is called standard Brownian web. Alternative characterizations can be found in [7] as well.

**Main result: invariance principle** The Poisson tree induces sets of continuous paths as follows. For any \(s = (s_1, s_2) \in S\), the Poisson process with parameter \(\lambda\), we define the path \(X^s\) in \(\mathbb{R}^2\) as the linearly interpolated line composed by all edges \(\{(\alpha^{n-1}(s), \alpha^n(s)) : n \in \mathbb{N}\}\) of the Poisson tree \(G\). Clearly, \(X^s \in C[s_2] \times \{s_2\} \subset \Pi\). Let

\[
X := \{X^s : s \in S\}, \tag{1.9}
\]

which we also call the Poisson web.
By the definition of the Poisson tree, \( X \) depends on \( \lambda > 0 \) and \( r > 0 \). In case of necessity, we denote it by \( X(\lambda, r) \). Take \( \lambda = \lambda_0 = \sqrt{3}/6 \), \( r = r_0 = \sqrt{3} \), and let

\[
X_1 := X(\lambda_0, r_0). \tag{1.10}
\]

\[
X_\delta := \{(\delta x_1, \delta^2 x_2) \in \mathbb{R}^2 : (x_1, x_2) \in X_1\}, \tag{1.11}
\]

for \( \delta \in (0, 1] \). Namely, \( X_\delta \) is the diffusive rescaling of \( X_1 \).

Another family of Poisson trees \( Y_\delta, \delta \in (0, 1] \), is defined as

\[
Y_\delta := X(\lambda(\delta), r(\delta)), \tag{1.12}
\]

with \( \lambda(\delta) = \delta^{-1} \), \( r(\delta) = (3\delta/2)^{1/3} \) for \( \delta \in (0, 1] \). It is straightforward to verify the following lemma.

**Lemma 1.1** For any \( \delta \in (0, 1] \), almost surely, the closures of the Poisson trees \( X_\delta \) and \( Y_\delta \) defined in (1.10), (1.11) and (1.12) are compact subsets of \( (\Pi, d) \).

By Lemma 1.1, the closure of \( X_\delta \) (resp. \( Y_\delta \)), also denoted by \( X_\delta \) (resp. \( Y_\delta \)), which is obtained by adding all the paths of the form \((f, t_0)\) with \( t_0 \in [\infty, \infty] \) and \( f \equiv \infty \) or \( f \equiv -\infty \), is an \((H, \mathcal{F}_H)\)-valued random variable.

Our main result is a proof that \( X_\delta \) and \( Y_\delta \) converge in distribution to the Brownian web characterized in Theorem A. Comparing with the classical Donsker’s invariance principle \[4\] for a single path, we call it the invariance principle in the web case.

**Theorem 1.1** Each of the rescaled Poisson trees \( X_\delta \) and \( Y_\delta \) converges in distribution to the standard Brownian web as \( \delta \to 0 \).

**Background** Arratia’s construction of a system of coalescing one dimensional Brownian motions starting from every space-time point in \( \mathbb{R}^2 \) \[3\] begins with coalescing Brownian motions starting from a countable dense subset of \( \mathbb{R}^2 \). A kind of semicontinuity condition is then imposed in order to get paths starting from every point of \( \mathbb{R}^2 \), and also to insure a certain flow condition, which in particular yields a unique path starting at each space-time point. A variant of that construction was done by Tóth and Werner \[16\], with a different kind of semicontinuity condition, also yielding uniqueness, in work where the final object was auxiliary in the definition and study of the self-repelling motion. The construction of Fontes, Isopi, Newman and Ravishankar \[7, 8\] was done with the aim of providing a set up for weak convergence. Their choice of \( (H, \mathcal{F}_H) \) as sample space, with its good topological properties, was inspired by the set up of Aizenman and Burchard \[1\] for the study of scaling limits of critical statistical mechanics models. This choice sets the stage for the derivation of criteria for characterization of and weak convergence to the constructed object, the Brownian web \[7, 8\]. These criteria were then verified for rescaled coalescing one dimensional random walks starting from every space-time point in \( \mathbb{Z} \times \mathbb{R} \) \[7, 8\]. The main part of the proof of Theorem 1.1 consists of the verification of those
criteria for $X_\delta$ and $Y_\delta$. The construction of Fontes, Isopi, Newman and Ravishankar begins as those of Arratia and of Tóth and Werner, with coalescing Brownian motions starting from a countable dense subset of $\mathbb{R}^2$. In order to have paths starting at every point, instead of imposing a semicontinuity condition, in particular disregarding flow/uniqueness issues, they take the closure in path space of the initial countable collection of paths. The resulting set of paths, the Brownian web, turns out to be almost surely compact, a property which allows it to live in $(\mathcal{H}, \mathcal{F}_H)$. Within the topological framework of that space, suitable criteria for characterization of and weak convergence to the Brownian web are then derived. The compactness of the Brownian web is its main distinguishing feature vis-a-vis the previous constructions. One other such feature is the occurrence in the Brownian web of multiple space-time points, where more than one path start out from. Multiple points, with alternative definitions but the same nature, are however a feature of all the constructions. The semicontinuity conditions of Arratia and of Tóth and Werner eliminate, each in its own ad hoc way, all but one of the paths starting out from those points. This elimination is at the root of the noncompactness of their resulting sets of paths.

We next state one of the above mentioned weak convergence criteria. Let $D$ be a countable dense set of points in $\mathbb{R}^2$.

**Theorem B** [7] Suppose $X_1, X_2, \ldots$ are $(\mathcal{H}, \mathcal{F}_H)$-valued random variables with noncrossing paths. If, in addition, the following three conditions are valid, the distribution of $X_n$ converges to the distribution $\mu_W$ of the standard Brownian web.

1. There exist $\theta^n_1, \ldots, \theta^n_m \in X_n$ such that for any deterministic $y_1, \ldots, y_m \in D$, $\theta^{y_1}_n, \ldots, \theta^{y_m}_n$ converge in distribution as $n \to \infty$ to coalescing Brownian motions (with unit diffusion constant) starting at $y_1, \ldots, y_m$.

2. $\limsup_{n \to \infty} \sup_{(a,t_0) \in \mathbb{R}^2} \mathbb{P}(\eta_{X_n}(t_0, t; a, a + \epsilon) \geq 2) \to 0$ as $\epsilon \to 0$;

3. $\epsilon^{-1} \limsup_{n \to \infty} \sup_{(a,t_0) \in \mathbb{R}^2} \mathbb{P}(\eta_{X_n}(t_0, t; a, a + \epsilon) \geq 3) \to 0$ as $\epsilon \to 0$.

To prove the main result, we show in Section 2 that the Poisson webs $X_\delta$ and $Y_\delta$ satisfy the hypothesis of Theorem B. The verification of $I_1$, on Subsection 2.1, relies on a comparison with independent paths and on the almost sure coalescence of the Poisson web paths with each other. See Lemma 2.5.

In Subsection 2.2, an FKG inequality enjoyed by the distribution of a single Poisson web path (Lemma 2.8) and the $O(t^{-1/2})$ decay of the coalescence time of two such paths (Lemma 2.10), combined with $I_1$, yield both $B_1$ and $B_2$. An analogous argument, which relies as well on an FKG property of the constituent paths, also holds for ordinary coalescing one dimensional random walks starting from all space time points, and can be used, together with the analogue of Lemma 2.5 (which follows immediately from Donsker’s invariance principle in this case), to establish their convergence (when suitably rescaled) to the Brownian web. The arguments for the Poisson web, which are similar in spirit to the ones for coalescing random walks, are nonetheless much more involved for the former case than the ones for the latter case — these are essentially immediate.
Working out a second example of a process in the basin of attraction of the Brownian web (the first example, just mentioned above, being ordinary one dimensional coalescing random walks) that is natural on one side, and that requires substantial technical attention on another side, is the primary point of this paper. Its main result may have an applied interest, e.g. in the context of drainage networks. The convergence results here may lead to rigorous/alternative verification of some of the scaling theory for those networks. See [14].

Ordinary one dimensional coalescing random walks starting from all space time points have also been proposed as model of a drainage network [15], so the latter remark applies to them as well. Another application would be in obtaining aging results from the scaling limit results for systems that could be modelled by Poisson webs, like drainage networks. For the relation between aging and scaling limits, see e.g. [9], [6], [7], [8] and references therein.

2 Proofs

Coalescing random walks Let $S$ be the Poisson process with parameter $\lambda > 0$, fix some $r > 0$. For any $x = (x_1, x_2) \in \mathbb{R}^2$, let $\tau^n(x) = [\alpha^n(x)]_2$, $n \geq 0$, be the second coordinate of $\alpha^n(x)$ and consider $\{\xi^x(t) : t \geq x_2\}$ as the continuous time Markov process defined by

$$\xi^x(t) = [\alpha^n(x)]_1, \text{the first coordinate of } \alpha^n(x); \ t \in [\tau^n(x), \tau^{n+1}(x)), \ n \geq 0. \quad (2.1)$$

We remark that for any fixed $(x^i)_{i=1}^m$, with $x^i = (x^i_1, x^i_2) \in \mathbb{R}^2$ for $i = 1, \ldots, m$, $\{\{\xi^x(t) : t \geq x^i_2\} : \text{for any fixed } x = (x_1, x_2) \in \mathbb{R}^2, \text{the marginal distribution of } \xi^x(\cdot) \text{ is that of a continuous time random walk starting at time } x_2 \text{ in position } x_1, \text{ which, at exponentially distributed random waiting times of mean } (2r\lambda)^{-1}, \text{ chooses a point uniformly in interval } [x_1 - r, x_1 + r] \text{ and then jumps to that point. The interaction appears when two walks are located at points } x_1 \in \mathbb{R} \text{ and } y_1 \in \mathbb{R} \text{ at some time } t_0: \text{ if } |x_1 - y_1| \leq 2r, \text{ and the Poisson process } S \text{ makes that } \alpha(x_1, t_0) = \alpha(y_1, t_0) = s = (s_1, s_2) \in S, \text{ then both walks jump to the same position } s_1 \in \mathbb{R} \text{ at time } \tau(x_1, t_0) = \tau(y_1, t_0) = s_2 \text{ and coalesce. We note that the finite system of coalescing random walks } \{\{\xi^x(t) : t \geq x^i_2\} : i = 1, \ldots, m\} \text{ is also strong Markov.}

For $x = (x_1, x_2) \in \mathbb{R}^2$, let $x_\delta = (\delta^{-1}x_1, \delta^{-2}x_2), \delta \in (0, 1]$. For the single random walk starting at $x = (x_1, x_2)$, $\xi^x(\cdot)$, defined in the last paragraph, the diffusive rescaling is

$$\xi^x_\delta(t) := \delta \xi^{x_\delta}(\delta^{-2}t), \text{ for } t \geq x_2; \ \delta \in (0, 1]. \quad (2.2)$$

Since Theorems A and B apply to continuous paths only, we need to replace the original processes by their linearly interpolated versions:

$$\tilde{\xi}^x_\delta(t) = \delta \left\{ \xi^{x_\delta}(\tau^n(x_\delta)) + \frac{\delta^{-2}t - \tau^n(x_\delta)}{\tau^{n+1}(x_\delta) - \tau^n(x_\delta)} \left( \xi^{x_\delta}(\tau^{n+1}(x_\delta)) - \xi^{x_\delta}(\tau^n(x_\delta)) \right) \right\}. \quad (2.3)$$
for \( t \geq x_2 \) such that \( \delta^{-2} t \in [\tau^n(x_\delta), \tau^{n+1}(x_\delta)) \), \( n \geq 0 \); \( \delta \in (0, 1] \), \( x \in \mathbb{R}^2 \). Denote by \( \bar{\xi}_\delta^x \) the corresponding continuous path in \( \mathbb{R}^2 \) and note that \( \bar{\xi}_\delta^x \) is just \( X^s \) in (1.9) with \( s \in S \). It is straightforward to see that \( \bar{\xi}_\delta^x \in X_\delta \), the Poisson web defined by (1.11), if and only if \( x_\delta \in S \).

Let

\[
\theta_\delta^x := \begin{cases} 
\bar{\xi}_\delta^x & \text{if } x_\delta \in S \\
\xi_\delta^{(d[\alpha(x_\delta)], \delta_2[\alpha(x_\delta)])} & \text{otherwise.}
\end{cases} 
\] (2.4)

In this way, for all \( x \in \mathbb{R}^2 \) and \( \delta \in (0, 1] \), \( \theta_\delta^x \in X_\delta \). Note that the paths defined by (2.3) and (2.4) depend on the choice of \( \lambda > 0 \) and \( r > 0 \). In case of necessity, we denote them by \( \xi_\delta^x(\lambda, r) \) and \( \theta_\delta^x(\lambda, r) \).

The following is an application of the classical Donsker’s theorem [4] in our case.

**Lemma 2.1** If \( \lambda = \lambda_0 = \sqrt{3}/6 \), \( r = r_0 = \sqrt{3} \), then \( \bar{\xi}_\delta^x \) converges in distribution as \( \delta \to 0 \) to \( B^x \), the Brownian path with unit diffusion coefficient starting from space-time point \( x = (x_1, x_2) \in \mathbb{R}^2 \).

For any \( x^1, \ldots, x^m \in \mathbb{R}^2 \), \( m \in \mathbb{N} \), regard \( (\bar{\xi}_\delta^{x^1}_\delta, \ldots, \bar{\xi}_\delta^{x^m}_\delta) \) and \( (\theta_\delta^{x^1}_\delta, \ldots, \theta_\delta^{x^m}_\delta) \) as random variables in the product metric space \((\Pi^m, d^m)\), where \( d^m \) is a distance on \( \Pi^m \) such that the topology generated by it coincides with the corresponding product topology. Here we choose and define

\[
d^m[(\xi^1_\delta, \ldots, \xi^m_\delta), (\zeta^1_\delta, \ldots, \zeta^m_\delta)] = \max_{1 \leq i \leq m} d(\xi^i_\delta, \zeta^i_\delta),
\] (2.5)

for all \( (\xi^1_\delta, \ldots, \xi^m_\delta), (\zeta^1_\delta, \ldots, \zeta^m_\delta) \in \Pi^m \), where \( d \) was defined in (1.1). The next result follows immediately from the definition.

**Lemma 2.2**

\[
\mathbb{P}_\lambda\{d^m[(\bar{\xi}_\delta^{x^1}_\delta, \ldots, \bar{\xi}_\delta^{x^m}_\delta), (\theta_\delta^{x^1}_\delta, \ldots, \theta_\delta^{x^m}_\delta)] \geq \epsilon\} \to 0, \text{ as } \delta \to 0
\] (2.6)

for all \( \epsilon > 0 \), \( \lambda > 0 \), \( r > 0 \), and \( x^1, \ldots, x^m \in \mathbb{R}^2, m \in \mathbb{N} \), where \( \mathbb{P}_\lambda \) is the probability distribution of \( S \), the Poisson process with parameter \( \lambda \).

For the collection of Poisson trees \( Y_\delta \)'s, we define the path \( \bar{\xi}_\delta^x \) as

\[
\bar{\xi}_\delta^x(t) := \xi_1^x(t)(\lambda(\delta), r(\delta)), \quad \forall \ x \in \mathbb{R}^2, \ \delta \in (0, 1],
\] (2.7)

the rescaled continuous path defined in (2.3) with \( (\lambda, r) = (\lambda(\delta), r(\delta)) = (\delta^{-1}, (3\delta/2)^{1/3}) \).

We also define the path

\[
\bar{\theta}_\delta^x := \theta_1^x(\lambda(\delta), r(\delta)), \quad \forall \ x \in \mathbb{R}^2, \ \delta \in (0, 1],
\] (2.8)

so that \( \bar{\theta}_\delta^x \in Y_\delta \).

For the same reasons as in Lemma 2.1 and Lemma 2.2 we have:
Lemma 2.3 \( \bar{\zeta}_x \) converges in distribution as \( \delta \to 0 \) to \( B^x \) for all space-time point \( x = (x_1, x_2) \in \mathbb{R}^2 \).

Lemma 2.4

\[
P\{d^m[(\bar{\zeta}_{\delta}^{x_1}, \ldots, \bar{\zeta}_{\delta}^{x_m}), (\bar{\theta}_{\delta}^{y_1}, \ldots, \bar{\phi}_{\delta}^{y_m})] \geq \epsilon \} \to 0, \quad \text{as} \quad \delta \to 0 \tag{2.9}
\]
for all \( \epsilon > 0 \) and \( x^1, \ldots, x^m \in \mathbb{R}^2, m \in \mathbb{N} \), where \( d^m \) is defined in \((2.3)\).

2.1 Convergence in finite-dimensional cases: verification of condition \( I_1 \).

In this subsection, we begin to prove Theorem 1.1. In our proofs, we will mainly verify the corresponding conditions of Theorem B for the Poisson trees \( X_\delta, \delta \in (0, 1] \), because of the essential similarity with the Poisson trees \( Y_\delta, \delta \in (0, 1] \); some remarks will be given for the case of the latter processes.

Let \( D \) be a countable dense set of points in \( \mathbb{R}^2 \). As pointed in the last subsection, for any \( y \in \mathbb{R}^2 \) and \( \delta \in (0, 1] \), as single-paths, \( \theta_y^\delta \in X_\delta, \vartheta_y^\delta \in Y_\delta \). In this subsection, to prove condition \( I_1 \), we will show that, for any \( y_1, y_2, \ldots, y_m \in D \), \((\theta_{\delta}^{y_1}, \ldots, \theta_{\delta}^{y_m}) \) and \((\vartheta_{\delta}^{y_1}, \ldots, \vartheta_{\delta}^{y_m}) \) converge in distribution as \( \delta \to 0 \) to coalescing Brownian motions (with unit diffusion constant) starting at \( y_1, \ldots, y_m \). Actually, by Lemma 2.2 and Lemma 2.4, we only need to prove the following.

Lemma 2.5 \((\bar{\xi}^{y_1}_\delta, \ldots, \bar{\xi}^{y_m}_\delta) \) and \((\bar{\zeta}^{y_1}_\delta, \ldots, \bar{\zeta}^{y_m}_\delta) \) converge in distribution as \( \delta \to 0 \) to coalescing Brownian motions (with unit diffusion constant) starting at \( y_1, \ldots, y_m \in D \).

For the finite system of coalescing random walks defined in the last subsection, Ferrari, Landim and Thorisson [5] proved that, for any \( x^1, x^2 \in \mathbb{R}^2 \), the random walks \( \xi^{x_1}(t) \) and \( \xi^{x_2}(t) \), \( t \geq x_1^2 \lor x_2^2 \) will meet and then coalesce almost surely. This also follows from Lemma 2.10 below. The following is a corollary of this result.

Lemma 2.6 For any \( \lambda > 0, \ r > 0 \), we have

\[
\lim_{\sigma \to \infty} \mathbb{P}_\lambda\{\bar{d}(\bar{\tau}^{x_1}_1, \bar{\tau}^{x_2}_1) \geq \sigma \} = 0, \tag{2.10}
\]
for all \( x^1, x^2 \in \mathbb{R}^2 \), where \( \bar{d} \) is a function defined on \( \Pi^2 \) such that for any \((f_1, t_1), (f_2, t_2) \in \Pi \)

\[
\bar{d}((f_1, t_1), (f_2, t_2)) := \sup_{t} |\hat{f}_1(t) - \hat{f}_2(t)| \lor |t_1 - t_2|. \tag{2.11}
\]

Now, for any \( y^1, \ldots, y^m \in D \), let \( B^y, \ldots, B^y \) be \( m \) independent Brownian paths starting at space-time points \( y^1, \ldots, y^m \), respectively. As Arratia did in [2], we construct the one-dimensional coalescing Brownian motions starting at \( y^1, \ldots, y^m \) by defining a continuous function \( f \) from \( \Pi^m \) to \( \Pi^m \).
Let $\gamma_0 = \min(y_2^1, \ldots, y_2^m)$, define the stopping time

$$\gamma_1 = \min\{t > \gamma_0 : \exists 1 \leq i, j \leq m, \text{such that } t \geq y_2^i \vee y_2^j, \text{ and } B^{y^i}(t) = B^{y^j}(t)\}. \tag{2.12}$$

At time $\gamma_1$, let $(i, j)$ be the pair such that $B^{y^i}(\gamma_1) = B^{y^j}(\gamma_1)$ and $B^{y^i}(t) < B^{y^j}(t)$ for all $t \in [y_2^i \vee y_2^j, \gamma_1)$, note that such pair $(i, j)$ is unique almost surely and we denote $I_1 = \{i\}$, $J_1 = \{j\}$. Denote by $p_1 := B^{y^i}(\gamma_1)$ the position of the first coalescence. Now, we renew the system by resetting $B^{y^i}(t) = B^{y^j}(t)$ for all $t \geq \gamma_1$.

For all $1 \leq k \leq m - 2$, after we have defined $\gamma_1, \ldots, \gamma_k$ and renewed the system $k$ times, we define

$$\gamma_{k+1} := \min\left\{ t > \gamma_k : \exists 1 \leq i, j \leq m, \text{such that } t \geq y_2^i \vee y_2^j, \text{ and } B^{y^i}(t) = B^{y^j}(t), \forall t' \in [y_2^i \vee y_2^j, t]. \right\} \tag{2.13}$$

Let $(i, j)$ be the (almost surely) unique pair such that $B^{y^i}(\gamma_{k+1}) = B^{y^j}(\gamma_{k+1})$, and $B^{y^i}(t) < B^{y^j}(t)$ for all $t \in [y_2^i \vee y_2^j, \gamma_{k+1})$. Let

$$I_{k+1} = \{i' : \exists \epsilon > 0, \text{such that } B^{y^i}(t) = B^{y^i}(t), \forall t \in [\gamma_{k+1} - \epsilon, \gamma_{k+1})\};$$

$$J_{k+1} = \{j' : \exists \epsilon > 0, \text{such that } B^{y^j}(t) = B^{y^j}(t), \forall t \in [\gamma_{k+1} - \epsilon, \gamma_{k+1})\}.$$

We renew the system for the $k + 1$-st time by resetting $B^{y^i}(t) = B^{y^{i+1}}(t)$ for all $j \in I_{k+1} \cup J_{k+1} \setminus \{i_{k+1}\}$ and $t \geq \gamma_{k+1}$, where $i_{k+1} \in I_{k+1}$ satisfying that, for any other $i' \in I_{k+1}$, $B^{y^i}(t) \geq B^{y^{i+1}}(t)$ for all $t \in [y_2^{i+1} \vee y_2^{j+1}, \gamma_{k+1})$. Denote by $p_{k+1} := B^{y^{i+1}}(\gamma_{k+1})$ the position of the $k + 1$-st coalescence.

By the basic properties of one-dimensional Brownian motion, we have that

$$-\infty < \gamma_0 < \gamma_1 < \ldots < \gamma_{m-1} < \infty \quad \text{and} \quad \rho := \min\{|p_k - p_{k'}| : 1 \leq k, k' \leq m - 1\} > 0 \tag{2.14}$$

almost surely. That is, let $C \subset \Pi^m$ be the set of all $m$-dimensional continuous paths starting at space-time points $y^1, \ldots, y^m \in \mathbb{R}^2$ satisfying condition (2.14). Then, we have

$$\mathbb{P}\{(B^{y^1}, \ldots, B^{y^m}) \in C\} = 1. \tag{2.15}$$

The resulting system after $m - 1$ steps of renewing is the so-called one-dimensional coalescing Brownian motions starting at space-time points $y^1, \ldots, y^m$, which is denoted by $f(B^{y^1}, \ldots, B^{y^m})$, a function of the $m$ independent Brownian motions $B^{y^1}, \ldots, B^{y^m}$.

For any $m$ distinct points $y^1, \ldots, y^m \in \mathcal{D}$ and $\delta \in (0, 1]$. Let $\zeta_{\delta}^1, \ldots, \zeta_{\delta}^m$ be the $m$ rescaled continuous random paths defined in (2.23) from the same Poisson process with $\lambda = \lambda_0 = \sqrt{3}/6$ and $r = r_0 = \sqrt{3}$. Having $(\zeta_{\delta}^1, \ldots, \zeta_{\delta}^m)$ as a random element in $\Pi^m$, we want to define a function $f_\delta$ of it to $\Pi^m$. This is our main idea for the verification of condition $I_1$: we define what we call “$\delta$-coalescence” of the random paths $\zeta_{\delta}^1, \ldots, \zeta_{\delta}^m$. 

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renew the system by resetting \( \bar{\xi}_{\delta} \) in such a way that, in the system (a): There exists \( n \) such that \( \bar{\xi}_{\delta} \) point of the original (before rescaling) random walk of \( \gamma \) the whole system step by step as follows. Let \( \delta > \sqrt{\gamma} \),

\[
\delta := \inf \{ \delta \in \mathbb{R} : \exists i, j \text{ such that } \delta - 2 \gamma, \delta - 2 \gamma - 1 \}
\]

Suppose that \( \bar{\xi}_{\delta} \) in involved are independent.

In case (a), suppose that \( \delta - 2 \gamma \) and \( \bar{\xi}_{\delta} \) meet and coalesce at space-time point \( (p_{\delta, 1}, \gamma_{\delta, 1}) \).

In such a way that, in the system \( f_{\delta}(\bar{\xi}_{\delta}^{y_1}, \ldots, \bar{\xi}_{\delta}^{y_m}) \), before any \( \delta \)-coalescence, the paths involved are independent.

Similarly as we have done with \( f \) in the preceding paragraphs, we define \( f_{\delta} \) by renewing the whole system step by step as follows. Let \( \gamma_{\delta, 0} = \min(y_2, \ldots, y_m) \). From now on, we assume that \( \delta > 0 \) is close enough to 0, so that, in particular, the following stopping time is well-defined.

\[
\gamma_{\delta, 1} = \inf \{ t > \gamma_{\delta, 0} : \exists i, j \leq m, \text{ such that } t \geq y_2, y_m, |\bar{\xi}_{\delta}^{y_i}(t) - \bar{\xi}_{\delta}^{y_j}(t)| < 2\sqrt{\delta} \}, \quad (2.16)
\]

where \( \bar{\xi}_{\delta}^{y_i}(t), 1 \leq i \leq m \) is the rescaled random walk defined in (2.12).

Suppose that \( (i, j) \) is the (almost surely) unique pair such that \( |\bar{\xi}_{\delta}^{y_i}(\gamma_{\delta, 1}) - \bar{\xi}_{\delta}^{y_j}(\gamma_{\delta, 1})| \leq 2\sqrt{\delta} \) and \( \bar{\xi}_{\delta}^{y_i}(t) - \bar{\xi}_{\delta}^{y_j}(t) > 2\sqrt{\delta} \) for all \( t \in [y_2, \gamma_{\delta, 1}] \). We denote it by \( (i_1, j_1) \), and let \( \mathcal{I}_{\delta, 1} := \{ i_1 \}, \mathcal{J}_{\delta, 1} := \{ j_1 \} \). We renew the system according to the following two cases. Case (a): There exists \( n' \geq 1 \) such that \( \delta - 2 \gamma_{\delta, 1} = \tau^{n'}(y_{\delta}^{j_1}) \). Case (b): There exists \( n'' \geq 1 \) such that \( \delta - 2 \gamma_{\delta, 1} = \tau^{n''}(y_{\delta}^{j_1}) \).

Recall that \( y_{\delta}^{j_1} = (\delta - 2 y_{\delta}^{j_1}, \delta - 2 y_{\delta}^{j_1}) \in \mathbb{R}^2 \), i.e., the starting space-time point of the original (before rescaling) random walk of \( \bar{\xi}_{\delta}^{y_1} \).

In case (a), suppose that \( \delta - 2 \gamma_{\delta, 1} = \tau^{n'}(y_{\delta}^{j_1}) \in [\tau^k(y_{\delta}^{j_1}), \tau^{k'+1}(y_{\delta}^{j_1})] \) for some \( k' \geq 0 \). We renew the system by resetting \( \bar{\xi}_{\delta}^{y_1}(t) \) for \( t \in \langle \delta^2, \tau^k(y_{\delta}^{j_1}), \delta^2 \tau^{k'+1}(y_{\delta}^{j_1}) \rangle \) and resetting \( \bar{\xi}_{\delta}^{y_i}(t) \) for \( t \in \langle \delta, \tau, \delta^2 \rangle \).
for all \( t \geq \delta^2 T_{n-1}(y_{\delta}^i) \) as follows (see Figure 1(a)):

\[
\bar{\xi}_{\delta}^{y_{\delta}^{i_1}}(t) = \begin{cases} \\
\delta \cdot \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^k(y_{\delta}^{i_1})) \\
\delta \cdot \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^k(y_{\delta}^{i_1})) + \delta \cdot \frac{\delta^{-2}t - \tau_{n-1}(y_{\delta}^{i_1})}{\tau_{n+1}(y_{\delta}^{i_1}) - \tau_{n-1}(y_{\delta}^{i_1})} \\
\cdot \left[ \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^k(y_{\delta}^{i_1})) - \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^k(y_{\delta}^{i_1})) \right]
\end{cases}, \quad t \in [\delta^2 T_k(y_{\delta}^{i_1}), \gamma_{\delta,1}) \tag{2.17}
\]

\[
\bar{\xi}_{\delta}^{y_{\delta}^{i_1}}(t) = \begin{cases} \\
\bar{\xi}_{\delta}^{y_{\delta}^{i_1}}(t) \\
\delta \cdot \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^{n-1}(y_{\delta}^{i_1})) + \delta \cdot \frac{\delta^{-2}t - \tau_{n-1}(y_{\delta}^{i_1})}{\tau_{n}(y_{\delta}^{i_1}) - \tau_{n-1}(y_{\delta}^{i_1})} \\
\cdot \left[ \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^n(y_{\delta}^{i_1})) - \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^n(y_{\delta}^{i_1})) \right]
\end{cases}, \quad t \in [\delta^2 T_{n-1}(y_{\delta}^{i_1}), \gamma_{\delta,1}) \tag{2.18}
\]

In the renewed system, the paths \( \bar{\xi}_{\delta}^{y_{\delta}^{i_1}} \) and \( \bar{\xi}_{\delta}^{y_{\delta}^{i_1}} \) meet and coalesce at time \( \gamma_{\delta,1} \).

In case (b), suppose that \( \delta^{-2}\gamma_{\delta,1} = \tau^{n''}(y_{\delta}^{i_1}) \in [\tau^{n''}(y_{\delta}^{i_1}), \tau^{k''+1}(y_{\delta}^{i_1})] \) for some \( k'' \geq 0 \). We renew the system by resetting \( \bar{\xi}_{\delta}^{y_{\delta}^{i_1}}(t) \) for all \( t \geq \delta^2 T_{k''}(y_{\delta}^{i_1}) \) as follow (see Figure 1(b)).

\[
\bar{\xi}_{\delta}^{y_{\delta}^{i_1}}(t) = \begin{cases} \\
\bar{\xi}_{\delta}^{y_{\delta}^{i_1}}(t) \\
\delta \cdot \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^{k''}(y_{\delta}^{i_1})) + \delta \cdot \frac{\delta^{-2}t - \tau^{k''}(y_{\delta}^{i_1})}{\tau_{n''}(y_{\delta}^{i_1}) - \tau^{k''}(y_{\delta}^{i_1})} \\
\cdot \left[ \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^{k''}(y_{\delta}^{i_1})) - \xi_{\delta}^{y_{\delta}^{i_1}}(\tau^{k''}(y_{\delta}^{i_1})) \right]
\end{cases}, \quad t \in [\delta^2 T_{k''}(y_{\delta}^{i_1}), \gamma_{\delta,1}) \tag{2.19}
\]

Denote by \( p_{\delta,1} := \xi_{\delta}^{y_{\delta}^{i_1}}(\gamma_{\delta,1}) \) the position of the first \( \delta \)-coalescence, and finally let \( K_1 = \{1, 2, \ldots, m\} \setminus \{j_1\} \).

For all \( 1 \leq k \leq m - 2 \), after we have defined \( \gamma_{\delta,1}, \ldots, \gamma_{\delta,k}; K_1, \ldots, K_k \) and renewed the system \( k \) times, we define

\[
\gamma_{\delta,k+1} := \inf \left\{ t > \gamma_{\delta,k} : \exists i, j \in K_k, \text{ such that } t \geq y_{i_2} \lor y_{j_2}, |\xi_{\delta}^{y_{i_1}}(t) - \xi_{\delta}^{y_{j_1}}(t)| < 2\sqrt{3}\delta \text{ and } \xi_{\delta}^{y_{i_1}}(t') - \xi_{\delta}^{y_{j_1}}(t') > 2\sqrt{3}\delta, \forall t' \in [y_{i_2} \lor y_{j_2}, t) \right\} \tag{2.20}
\]

Let \( (i, j) \) be the (almost surely) unique pair such that \( |\xi_{\delta}^{y_{i_1}}(\gamma_{\delta,k+1}) - \xi_{\delta}^{y_{j_1}}(\gamma_{\delta,k+1})| < 2\sqrt{3}\delta \) and \( \xi_{\delta}^{y_{i_1}}(t) - \xi_{\delta}^{y_{j_1}}(t) > 2\sqrt{3}\delta \) for all \( t \in [\gamma_{\delta,k}, \gamma_{\delta,k+1}) \). Let

\[
\mathcal{I}_{\delta,k+1} = \{i' : \exists \epsilon > 0, \text{ such that } \bar{\xi}_{\delta}^{y_{i_1}}(t) = \bar{\xi}_{\delta}^{y_{j_1}}(t), \forall t \in [\gamma_{\delta,k+1} - \epsilon, \gamma_{\delta,k+1})\};
\]
\[ J_{\delta,k+1} = \{ j' : \exists \epsilon > 0, \text{ such that } \xi_{\delta}^{j'}(t) = \xi_{\delta}^{j'}(t), \forall t \in [\gamma_{\delta,k+1} - \epsilon, \gamma_{\delta,k+1}) \}. \]

Let \( i_{k+1} \in I_{\delta,k+1} \) (resp. \( j_{k+1} \in J_{\delta,k+1} \)) satisfy that, for any other \( i' \in I_{\delta,k+1} \) (resp. \( j' \in J_{\delta,k+1} \)), \( \tilde{\xi}^{j'}(t) \geq \xi^{j_{k+1}}(t) \) (resp. \( \tilde{\xi}^{j'}(t) \geq \xi^{j_{k+1}}(t) \)) for all \( t \in [y_{2}^{j_{k+1}} \vee y_{2}^{i'}, \gamma_{\delta,k+1}) \) (resp. \( t \in [y_{2}^{j_{k+1}} \vee y_{2}^{i'}, \gamma_{\delta,k+1}) \)).

Now, suppose that at time \( \gamma_{\delta,k_0} \), \( 1 \leq k_0 \leq k \) (resp. \( \gamma_{\delta,k_1} \), \( 1 \leq k_1 \leq k \)), all paths of \( \{ \tilde{\xi}^{i} : i \in I_{\delta,k+1} \} \) (resp. \( \{ \xi^{j'} : j \in J_{\delta,k+1} \} \)) have coalesced into one path. Then we renew the system in two steps. Firstly, we renew \( \tilde{\xi}_{\delta}^{j_{k+1}} \) and \( \tilde{\xi}_{\delta}^{j_{k+1}} \). Because it is only necessary to renew \( \tilde{\xi}_{\delta}^{j_{k+1}} \) and \( \tilde{\xi}_{\delta}^{j_{k+1}} \) after time \( \gamma_{\delta,k_0} \) and \( \gamma_{\delta,k_1} \), respectively, we deal with that as we did in equations (2.17), (2.18) and (2.19) for the two-path system \( (\xi_{\delta}^{(p_{\delta,k_0},\gamma_{\delta,k_0})}, \xi_{\delta}^{(p_{\delta,k_1},\gamma_{\delta,k_1})}) \) starting from space-time points \( (p_{\delta,k_0}, \gamma_{\delta,k_0}), (p_{\delta,k_1}, \gamma_{\delta,k_1}) \in \mathbb{R}^2 \). Secondly, we renew the system by resetting \( \tilde{\xi}_{\delta}^{i}(t) = \tilde{\xi}_{\delta}^{j_{k+1}}(t) \) for all \( i \in I_{\delta,k+1} \) and \( t \geq \gamma_{\delta,k_0} \), and resetting \( \tilde{\xi}_{\delta}^{j'}(t) = \tilde{\xi}_{\delta}^{j_{k+1}}(t) \) for all \( j \in J_{\delta,k+1} \) and \( t \geq \gamma_{\delta,k_1} \). Let \( K_{k+1} = K_k \setminus \{ j_{k+1} \} \), and \( p_{\delta,k+1} := \tilde{\xi}_{\delta}^{j_{k+1}}(\gamma_{\delta,k+1}) \), the position of the \((k+1)\)th \( \delta \)-coalescence.

We denote the resulting object after renewing \( m - 1 \) times by \( f_{\delta}(\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}) \) and, with that, finish the definition of the function \( f_{\delta} \). Clearly, for \( \delta \in (0,1] \) small enough, we have

\[ -\infty < \gamma_{\delta,0} < \gamma_{\delta,1} < \ldots < \gamma_{\delta,m-1} < \infty \]

(2.21) and the function \( f_{\delta} \) is well defined almost surely.

Now, suppose that \( \tilde{\xi}_{\delta}^{i} \) has the same distribution as \( \tilde{\xi}^{i} \), \( 1 \leq i \leq m \), and, as a random element in \( \Pi^m \), \( (\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}) \) has independent components. It is easy to see that the function \( f_{\delta} \) is also well defined for the random paths \( (\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}) \). Let \( C_{\delta} \subset \Pi^m \) be such that

\[ \mathbb{P}\{ (\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}) \in C_{\delta} \} = 1 \]

(2.22) and, on \( C_{\delta} \), \( f_{\delta} \) is well defined.

**Lemma 2.7** Let \( (\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}) \) is the \( m \) rescaled continuous random paths defined in (2.3) from the same Poisson process with \( \lambda = \lambda_0 = \sqrt{3}/6 \) and \( r = r_0 = \sqrt{3} \), and \( (\xi_{\delta}^{1}, \ldots, \xi_{\delta}^{m}) \) have independent components and \( \tilde{\xi}_{\delta}^{i} \) have the same distribution as \( \xi_{\delta}^{i} \) for all \( 1 \leq i \leq m \). Then,

a) \( f_{\delta}(\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}) \) has the same distribution as \( f_{\delta}(\xi_{\delta}^{1}, \ldots, \xi_{\delta}^{m}) \).

b) \( f_{\delta}(\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}) \) converges in distribution to \( f(B_{\delta}^{1}, \ldots, B_{\delta}^{m}) \) as \( \delta \to 0 \).

c) for any \( \epsilon > 0 \),

\[ \mathbb{P}\{d_{\text{res}}(f_{\delta}(\tilde{\xi}_{\delta}^{1}, \ldots, \tilde{\xi}_{\delta}^{m}), (\xi_{\delta}^{1}, \ldots, \xi_{\delta}^{m})) \geq \epsilon \} \to 0, \quad \text{as } \delta \to 0, \]

(2.23) where \( d_{\text{res}} \) was defined in (2.4).
Proof. a) Immediate from the definition of $f_\delta$. We only need to prove b) and c).

By Lemma 2.1 and independence, $(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta)$ converges in distribution to $(B^y_1, \ldots, B^y_m)$ as $\delta \to 0$. By an extended continuous mapping theorem of Mann and Wald [12], Prohorov [13] (see also Theorem 3.27 of [11]), we only need to prove that, for any $c = (c^1, \ldots, c^m) \in C$, if $c_\delta = (c^1_\delta, \ldots, c^m_\delta) \in C_\delta$ such that $d^m(c_\delta, c) \to 0$ as $\delta \to 0$, then $d^m(f_\delta(c_\delta), f(c)) \to 0$ as $\delta \to 0$. It is straightforward to check this by the definitions of $f_\delta$, $f$, and $d^m$; then we get b).

Since the function $\tanh(t)$ is Lipschitz continuous, we have

$$d^m[f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta), (\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta)] \leq C_L d^m[f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta), (\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta)],$$

(2.24)

where $0 < C_L < \infty$ is the corresponding Lipschitz constant, and $d^m$ is defined from $d$ in the same way as we did in (2.4).

Now, if $\tilde{d}^m[f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta), (\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta)] \geq C_L \cdot \epsilon$, then by the definition of function $f_\delta$, there should exist some $1 \leq k \leq m - 1$ such that

$$\tilde{d}(\tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta, \tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta) \geq C_L \cdot \epsilon / n(j_k),$$

(2.25)

where $n(j_k)$ is the number of times that path $\tilde{\xi}^{y_{jk}}_\delta$ was finally renewed in the process of the definition of $f_\delta$. Obviously, $1 \leq n(j_k) \leq m - 1$.

By the basic property of coalescence, we have

$$\tilde{d}(\tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta, \tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta) \leq \tilde{d}(\tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta, \tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta).$$

(2.26)

Then, by (2.24), (2.25), (2.26), the strong Markov property of the coalescing random walks, and also the stationarity of the Poisson process, we have

$$\mathbb{P}\{d^m[f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta), (\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta)] \geq \epsilon\} \leq \mathbb{P}\{\tilde{d}^m[f_\delta(\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta), (\tilde{\xi}^y_\delta, \ldots, \tilde{\xi}^y_\delta)] \geq C_L \cdot \epsilon\} \leq \sum_{k=1}^{m-1} \mathbb{P}\{\tilde{d}(\tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta, \tilde{\xi}^{(p_{\delta,k},\gamma_{\delta,k})}_\delta) \geq C_L \cdot \epsilon / n(j_k)\} \leq (m-1)\mathbb{P}\{\tilde{d}(\tilde{\xi}^{(0,0)}_\delta, \tilde{\xi}^{(2\sqrt{3}\delta,0)}_\delta) \geq C_L \cdot \epsilon / (m-1)\} = (m-1)\mathbb{P}_{\lambda_1}\{\tilde{d}(\tilde{\xi}^{(0,0)}_\delta, \tilde{\xi}^{(2\sqrt{3}\delta,0)}_\delta) \geq \delta^{-1} \cdot C_L \cdot \epsilon / (m-1)\}.$$  

Now, by Lemma 2.6 we get c), and the proof is finished. □

Lemma 2.5 is an immediate consequence of Lemma 2.1. Thus, condition $I_1$ for the Poisson web $X_\delta$, $\delta \in (0, 1)$ follows from Lemma 2.2.

Remark: To verify condition $I_1$ for the Poisson trees $X_\delta$, $\delta \in (0, 1)$, in the definition of the function $f_\delta$, one should use “$\delta$-coalescence” when the distance of two rescaled random walks is less than $2\tau(\delta) = 2(3\delta/2)^{1/3}$. Recall that for the Poisson trees $X_\delta$, $\delta \in (0, 1)$, we use it when that distance is less than $2\sqrt{3}\delta$.  

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2.2 Verification of conditions $B_1$ and $B_2$

Consider the Poisson process $S$ with parameter $\lambda > 0$ and the corresponding Poisson tree $X := X(\lambda, r)$ defined in (1.9) with respect to some fixed $r > 0$. Given $t_0 \in \mathbb{R}$, $t > 0$, $a, b \in \mathbb{R}$ with $a < b$, let $\eta_X(t_0, t; a, b)$ be the $\{0, 1, 2, \ldots, \infty\}$-valued random variable defined before the statement of Theorem $A$. Let $\bar{\eta}_X(t_0, t; a, b)$ be another $\{0, 1, 2, \ldots, \infty\}$-valued random variable defined as the number of distinct points $y, \eta$ before the statement of Theorem $A$. Let $\bar{\Pi}$ denote the space of paths where $\bar{\eta}_X(t_0, t; a, b)$ is the Markov process defined in (2.1). It is straightforward to see that, for any fixed $n \in \mathbb{N}$,

$$\bar{\eta}_X(t_0, t; a, b) \geq n \Rightarrow \eta_X(t_0, t; a - 2r, b + 2r) \geq n \Rightarrow \bar{\eta}_X(t_0, t; a - 4r, b + 4r) \geq n. \quad (2.28)$$

This implies that, to verify conditions $B_1$, $B_2$ for Poisson trees $X_\delta$ and $Y_\delta$, we only need to verify the following $B'_1$ and $B'_2$ respectively.

$$(B'_1) \quad \limsup_{n \to \infty} \mathbb{P}(\bar{\eta}_{X_\delta}(0, t; 0, \epsilon) \geq 2) \to 0 \text{ as } \epsilon \to 0;$$

$$(B'_2) \quad \epsilon^{-1} \limsup_{n \to \infty} \mathbb{P}(\bar{\eta}_{X_\delta}(0, t; 0, \epsilon) \geq 3) \to 0 \text{ as } \epsilon \to 0+$$

for any sequence of positive numbers $(\delta_n)$ such that $\lim_{n \to \infty} \delta_n = 0$, where $\bar{\eta}_X = \bar{\eta}_{X_\delta}$ or $\bar{\eta}_{Y_\delta}$, and we have used the space homogeneity of the Poisson point process to eliminate the $\sup_{(a, t_0) \in \mathbb{R}^2}$ and put $a = t_0 = 0$.

Here, we firstly introduce an FKG inequality for probability measures on the path space, which will play an important role in our proofs. Let $\xi = \xi^{(0,0)}$ be the random path starting at the origin defined in (2.27); denote by $\bar{\Pi}$ the space of paths where $\xi$ takes value. We define a partial order "\leq" on $\bar{\Pi}$ as follows. Given $\pi_1, \pi_2 \in \bar{\Pi},$

$$\pi_1 \preceq \pi_2 \text{ if and only if } \pi_1(t) - \pi_1(s) \leq \pi_2(t) - \pi_2(s) \text{ for all } t \geq s \geq 0. \quad (2.29)$$

Define increasing events in $\bar{\Pi}$ as usual. Denote by $\mu_\xi$ the distribution of $\xi$ on $\bar{\Pi}$.

**Lemma 2.8 (FKG Inequality)** $\mu_\xi$ satisfies the FKG inequality, namely, for any increasing events $A, B \subseteq \bar{\Pi}$, $\mu_\xi(A \cap B) \geq \mu_\xi(A)\mu_\xi(B)$.

**Proof.** Let $Z_i, i \in \mathbb{N}$ be an i.i.d. family of random variables with uniform distribution on the interval $[-r, r]$. Let $\{N(t) : t \geq 0\}$ be a one-dimensional Poisson process with parameter $2r\lambda$. Assume that $\{N(t) : t \geq 0\}$ is independent of $Z_i, i \in \mathbb{N}$. Define the random process $\{Y(t) : t \geq 0\}$ as

$$Y(t) = \sum_{i=0}^{N(t)} Z_i, \quad \forall \ t \geq 0,$$

where $Z_0 \equiv 0$. Then $Y$ has the same distribution as $\xi$.

Now, for any given $n \in \mathbb{N}$, we define a discrete time random walk $Y_n$ such that $Y_n$ converges in distribution to $Y$ as $n \to \infty$. Let $J_{n,i}, i \in \mathbb{N}$ be an i.i.d. family of random
variables such that \( \mathbb{P}(J_{n,i} = 1) = 1 - \mathbb{P}(J_{n,i} = 0) = \frac{2r\lambda}{n} \); let \( Z_{n,i} = Z_i \cdot J_{n,i} \). Let \( l_n := 1/n \) be the unit length of time. Define \( Y_n \) as

\[
Y_n(t) = \sum_{i=0}^{\lfloor nt \rfloor} Z_{n,i}, \quad \forall \ t \geq 0,
\]

where \( Z_{n,0} \equiv 0 \) and \( \lfloor nt \rfloor \) is the integer part of \( nt \).

For any given \( t \geq 0 \), define the random variable

\[
N_n(t) := |\{m : Z_{n,m} \neq 0 \text{ and } 1 \leq m \leq nt\}|.
\]

Noticing that \( N_n(t) \) converges in distribution to \( N(t) \) as \( n \rightarrow \infty \), it is straightforward to check that \( Y_n \) converges in distribution to \( Y \) as \( n \rightarrow \infty \). Denote by \( \mu_{Y_n} \) the distribution of \( Y_n \).

Considering the configuration space \( \Omega_n := [-r, r]^N \), let \( \mu_n = \Pi_{z \in \mathbb{R}} \mu_{z_{n,i}} \) be a product probability measure on \( \Omega_n \), where \( \mu_{z_{n,i}} \) is the distributions of \( Z_{n,i} \). Define the map \( \psi_n : \Omega_n \rightarrow \bar{\Pi} \) by \( \psi_n(z_1, \ldots, z_i, \ldots) = \pi \) such that

\[
\pi(t) = \sum_{i=0}^{\lfloor nt \rfloor} z_i, \quad \forall \ t \geq 0,
\]

where \( z_0 \equiv 0 \).

Obviously, \( \psi_n \) is an increasing map from \( \Omega_n \) to \( \bar{\Pi} \) (under the standard partial order on \( \Omega_n \) and the order \( \leq \) on \( \bar{\Pi} \)) and \( \psi_n(Z_{n,1}, \ldots, Z_{n,i}, \ldots) \) has the same distribution as \( Y_n \).

For any increasing events \( A, B \) contained in \( \bar{\Pi} \), it is straightforward to check that \( \psi_n^{-1}(A), \psi_n^{-1}(B) \) are increasing and \( \psi_n^{-1}(A \cap B) = \psi_n^{-1}(A) \cap \psi_n^{-1}(B) \). So,

\[
\mu_{Y_n}(A \cap B) = \mu_n(\psi_n^{-1}(A \cap B)) = \mu_n(\psi_n^{-1}(A) \cap \psi_n^{-1}(B)) \\
\geq \mu_n(\psi_n^{-1}(A)) \mu_n(\psi_n^{-1}(B)) = \mu_{Y_n}(A) \mu_{Y_n}(B).
\]

(2.30)

Here we used the FKG inequality for the product measure \( \mu_n \) on \( \Omega_n \) in the third step. Thus we get the FKG inequality for \( \mu_{Y_n} \). The Lemma follows immediately by taking \( n \rightarrow \infty \). \( \square \)

Let \( \xi^{(0,0)}, \xi^{(r,0)}, r \geq r \), be two random walks defined in \( \{2,3\} \) from \( S \). Note that \( \{\xi^{(0,0)}, \xi^{(r,0)}\} \) makes of a simple system of coalescing random walks starting at the space-time points \((0,0),(r,0)\). Denote by \( \Delta_\gamma \) the difference between the two walks, then, \( \Delta_\gamma \) makes of a jump process in \([0, \infty)\) with absorbing state 0: In \( x \in [r, \infty) \), it has rates

\[
(2r + x \wedge (2r)) \lambda
\]

and jump laws

\[
\nu_x := \frac{2r - x \wedge (2r)}{2r + x \wedge (2r)} \delta_{\{x\}} + \frac{2(x \wedge (2r))}{2r + x \wedge (2r)} U[r - x \wedge (2r), r],
\]

(2.31)

where \( \delta_{\{x\}} \) is the usual dirac measure and \( U[r - x, r] \) is the uniform distribution on \([r - x, r]\). Note that the process \( \Delta_\gamma, r \geq r \) will never visit \((0, r)\).

Denote by \( \Xi_\gamma, r \geq r \) the one-dimensional random walk starting at \( r \) with uniform rate \( 3r \lambda \) and uniform jumps in \([−r, r]\). Let \( T = \inf \{t > 0 : \Xi_\gamma(t) \leq 0\} \), firstly, we have
Lemma 2.9 There exists a constant $c_1 > 0$ such that $\mathbb{P}(T > t) \leq c_1/\sqrt{t}$ for any $t > 0$, where $c_1$ depends on $r, \lambda$ and $\gamma$.

Proof. This is certainly a well known result. On not having found a reference, we briefly sketch an argument, which is rather standard. We give the result for the embedded chain of $\Xi_\gamma$, which we denote $\tilde{\Xi}_\gamma$. The result for $\Xi_\gamma$ follows in a standard way. We use Skorohod’s representation of $\tilde{\Xi}_\gamma$ as

$$\tilde{\Xi}_\gamma(n) = B(S_n),$$

(2.32)

where $B(\cdot)$ is a Brownian motion of unit diffusion coefficient started at $\gamma$, and $S_1, S_2, \ldots$ are the partial sums of i.i.d. random variables with a positive exponential moment (one of which has the distribution of the exit time of a standard Brownian motion $B'$ from the interval $(-R, R)$, with $R$ a random variable uniformly distributed in $(0, r)$ and independent of $B'$. Let $\tilde{T} = \inf\{n > 0 : \tilde{\Xi}_\gamma(n) \leq 0\}$. Then $\tilde{T} > n$ implies that $M(S_n) > -r$, where $M(\cdot)$ is the running maximum of $B(\cdot)$. Thus

$$\mathbb{P}(\tilde{T} > n) \leq \mathbb{P}(M(\mu n) > -r) + \mathbb{P}(S_n < \mu n),$$

(2.33)

for all $\mu > 0$, and the well known distribution of $M$, along with the exponential moment condition of the increments of $S_n$, via the standard large deviation estimate for sums of i.i.d. random variables with that condition, imply the claim of the lemma for $\mathbb{P}(\tilde{T} > n)$. □

For the jump process $\Delta_{2r}$, let $T = \inf\{t > 0 : \Delta_{2r}(t) = \xi^{(0,0)}(t) - \xi^{(2r,0)}(t) = 0\}$, we also have

Lemma 2.10 There exists a constant $c_2 > 0$ such that $\mathbb{P}(T > t) \leq c_2/\sqrt{t}$ for any $t > 0$, where $c_2$ depends on $r$ and $\lambda$.

Proof. Let $\Delta'_\gamma$, $\gamma \geq r$, be the jump process in $[0, \infty)$ starting at $\gamma$: In $x \in [r, \infty)$, it has uniform rate $3r\lambda$ (be independent of $x$) and has jump laws given in (2.31) (dependent on $x$). Note that all $\Delta'_\gamma$, $\gamma \geq r$ can be coupled together such that, for any $\gamma \leq \gamma'$,

$$\mathbb{P}(\Delta'_\gamma(t) \leq \Delta'_\gamma(t), \forall t \geq 0) = 1.$$  

(2.34)

On the other hand, let $T'_\gamma$ be the first hitting time to 0 of $\Delta'_\gamma$, we have clearly

$$\mathbb{P}(T > t) \leq \mathbb{P}(T'_\gamma > t), \forall t > 0, \gamma \geq 2r.$$  

(2.35)

Now, we can prove the Lemma by two steps, in the first step, we couple $\Delta'_\gamma$ with a process $\Delta_n$, which also has absorbing state 0, such that $T' \leq T_n$, the first hitting time to 0 of $\Delta_n$, with probability one. In the second step, we prove that $\mathbb{P}(\tilde{T} > t) \leq c_2/\sqrt{t}$ for any $t > 0$.

Step 1. For any $x \in [r, 2r)$, let $p_x$ be the probability of the process $\Delta'_\gamma$ jump to 0 in two steps, it is straightforward to calculate by (2.31) that

$$p_x = \frac{2x}{2r + x} \int_{r-x}^{2r-x} \frac{12r - y}{x(2r + y)} dy = \frac{4r}{2r + x} \ln \frac{4r - x}{3r - x} - \frac{r}{2r + x}.$$
Let $p = \inf_{r \leq x < 2r} p_x$, obviously, $p > 0$, and let $q = 1 - p$.

Let us consider the behavior of the jump process $\Delta'_x$ for some $x \in [r, 2r)$ in the coming two clock ticks. Obviously, it should behave as the following three cases. a) jumps to 0 at the first clock tick with probability $\frac{2r-x}{2r+x}$ and then stays there; b) stays in $[r, 2r)$ at the first clock tick and jumps to 0 at the second clock tick, note that the probability of this case is $p_x$ given above; c) does not jump to 0 at both the two clock ticks. Note that in case c) the process should stay in $[r, 2r)$.

Let $\bar{\Delta}$ be a jump process in $[0, \infty)$ with absorbing state 0 as follows. It starts at $4r$ and behaves as $\Delta'_x$ in $[2r, \infty)$. Once it hits $[r, 2r)$, assume that it be in some $x \in [r, 2r)$, it stays at $x$ and waits two clock ticks (with rate $3r\lambda$, the same rate as the process $\Delta'_x$) and then, either jumps to 0 with probability $p$ or jumps back to $4r$ with probability $q$.

By the comments and setting in the last two paragraphs, one may couple $\Delta'_x$ and $\bar{\Delta}$ together such that
\[ P(T_{4r} \leq \bar{T}) = 1. \] (2.36)

Note that the above coupling only guarantees (2.36) but the stochastic domination between $\Delta'_x$ and $\bar{\Delta}$.

**Step 2.** Let $T_{[r,2r)}$ be the first hitting time to $[r, 2r)$ of $\Delta'_x$, let $\tau_1, \tau_2$, independent of $T_{[r,2r)}$, be two i.i.d. waiting times with rate $3r\lambda$. Let $\bar{T} = T_{[r,2r)} + \tau_1 + \tau_2$ and $\bar{T}_i, i \in \mathbb{N}$ be a series of independent copies of $\bar{T}$. By the definition of the process $\bar{\Delta}$, $\bar{T}$ has the same distribution as
\[ \bar{T} := \sum_{i=1}^{N(p)} \bar{T}_i, \] (2.37)
where $N(p)$ is geometric(p), the geometric distribution with parameter $p$, and independent of $\bar{T}_i, i \in \mathbb{N}$. At this point, the claim of the lemma should be pretty clear, and could be argued in the following way. Consider $G(s) := \sum_{n \geq 1} P(\bar{T} > n) s^n$, $|s| < 1$. From the relationship of $G$ and the moment generating function of $\bar{T}$, together with (2.37), we readily get the following.
\[ G(s) \leq \text{const}/\sqrt{1 - s} \] (2.38)
for $|s| < 1$, from which there follows
\[ n P(\bar{T} > n) e^{-1} \leq G(1 - 1/n). \] (2.39)
(2.39) and (2.38) now yield the lemma. □

Now, we begin to verify conditions $B_1$ and $B_2$. By Lemma 2.5, it is straightforward to get that, for both Poisson trees $X_\delta$ and $Y_\delta$,
\[ \limsup_{n \to \infty} P(\eta_{\delta_n}(0, t; 0, \epsilon) \geq 2) = 2\phi(\epsilon/\sqrt{2t}) - 1, \] (2.40)
where $\delta_n$ is any sequence of positive numbers converging to 0 as $n \to \infty$, $\eta_{\delta} = \eta_{X_{\delta}}$ or $\eta_{Y_{\delta}}$, and $\phi(x)$ is the standard normal distribution function. By (2.28) we also have
\[ \limsup_{n \to \infty} P(\bar{\eta}_{\delta_n}(0, t; 0, \epsilon) \geq 2) = 2\phi(\epsilon/\sqrt{2t}) - 1. \] (2.41)
This gives $B_1$ and $B'_1$.

Verifying $B'_2$ for the Poisson web $X_\delta$ is equivalent to checking that for any $t > 0$

$$
\epsilon^{-1} \limsup_{N \to \infty} \mathbb{P}(\bar{\eta}_{X_1}(0, tN; 0, \epsilon\sqrt{N}) \geq 3) \to 0 \text{ as } \epsilon \to 0+, \quad (2.42)
$$

where $X_1$ is defined in (1.10).

For that, fix $t > 0$. On the Poisson field $S$ with parameter $\lambda = \lambda_0 = \sqrt{3}/6$, choose $r = r_0 = \sqrt{3}$, and then define $X_1$ as in (1.10). We first condition the probability in (2.42) on the set of points of intersection, in decreasing order, of the paths $\xi^s$, $s \in S$, with $[0, \epsilon\sqrt{N}]$, denoted $\{K_1, \ldots, K_J\}$, where $J, K_1, \ldots, K_J$ are random variables, with $J$ an integer which can equal 0 (in which case set of intersection points is empty by convention). We note that by the definition of $\xi^s$, $s \in S$, no two distinct $K_i$'s can be at distance smaller than $r_0$. For $\{x_1, \ldots, x_n\} \subset [0, \epsilon\sqrt{N}]$, let $\xi_j := \xi(x_j, 0), 1 \leq j \leq n$ as in (2.1). Let $\eta' = \eta'(x_1, \ldots, x_n) = |\{\xi_j(tN) : 1 \leq j \leq n\}|$ (conventioned to be 0 if $\{x_1, \ldots, x_n\} = \emptyset$. Clearly, $J, K_1, \ldots, K_J$ depend only on the points of $S$ below time 0. Thus, since $\eta'$ depends only on the points of $S$ above and at time 0 for all $\{x_1, \ldots, x_n\} \subset [0, \epsilon\sqrt{N}]$, given $J = n, K_1 = x_1, \ldots, K_J = x_n$ the probability in (2.42) equals

$$
\mathbb{P}(\eta' \geq 3). \quad (2.43)
$$

We derive below an upper bound for (2.43) which is independent of $\{x_1, \ldots, x_n\}$ (see (2.44) and (2.45)). First, we enlarge, if necessary, the set $\{x_1, \ldots, x_n\}$ to make sure that $x_1 = 0, x_n = \epsilon\sqrt{N}$, and $r_0 \leq x_j - x_{j-1} \leq 2r_0$. This also ensures that $n \leq \epsilon\sqrt{N}/r_0 + 1$, and the enlargement can only increase (2.43).

Now for the bound. If $\eta' \geq 3$, then there should be some $1 \leq j \leq n - 1$ such that $\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN)$. Hence,

$$
\begin{align*}
\mathbb{P}(\eta' \geq 3) \leq & \sum_{j=2}^{n-1} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN)) \\
= & \sum_{j=2}^{n-1} \int_{\Pi_j} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN)|\xi_j = \pi)\mu_{\xi_j}(d\pi) \\
= & \sum_{j=2}^{n-1} \int_{\Pi_j} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN)|\xi_j = \pi)\mathbb{P}(\xi_j(tN) < \xi_n(tN)|\xi_j = \pi)\mu_{\xi_j}(d\pi),
\end{align*}
$$

(2.44)

where $\Pi_j$ is the state space of $\xi_j$, and $\mu_{\xi_j}$ its distribution. In the latter equality, we used the independence of $\xi_{j-1}(tN) < \xi_j(tN)$ and $\xi_j(tN) < \xi_n(tN)$ conditioned on $\xi_j = \pi$.

We argue below that $\mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN)|\xi_j = \pi)$ decreases in $\pi$ and $\mathbb{P}(\xi_j(tN) < \xi_n(tN)|\xi_j = \pi)$ increases in $\pi$. This and the FKG Inequality for $\mu_{\xi_j}$ (Lemma 2.8) imply
that the right hand side of (2.44) is bounded above by

\[
\sum_{j=2}^{n-1} \int_{\Pi_j} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) | \xi_j = \pi) \mu_{\xi_j}(d\pi) \\
\cdot \int_{\Pi_j} \mathbb{P}(\xi_j(tN) < \xi_n(tN) | \xi_j = \pi) \mu_{\xi_j}(d\pi) \\
= \sum_{j=2}^{n-1} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN)) \mathbb{P}(\xi_j(tN) < \xi_n(tN)) \\
\leq \sum_{j=2}^{n-1} \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN)) \mathbb{P}(\xi_j(tN) < \xi_n(tN)) \\
\leq (n-1) \mathbb{P}(\xi^{(0,0)}(tN) < \xi^{(2r_0,0)}(tN)) \mathbb{P}(\xi^{(0,0)}(tN) < \xi^{(\epsilon\sqrt{N},0)}(tN)) \\
\leq \frac{\epsilon}{r_0} \mathbb{P}(T > tN) \mathbb{P}(T_{\epsilon,N} > tN),
\]

where \( \mathcal{T} \) is the time when \( \xi^{(0,0)} \) and \( \xi^{(2r_0,0)} \) meet and coalesce, and \( T_{\epsilon,N} \) is the analogue time for \( \xi^{(0,0)} \) and \( \xi^{(\epsilon\sqrt{N},0)} \).

To see that \( \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) | \xi_j = \pi) \) decreases in \( \pi \), it is enough to consider \( \pi_1 \leq \pi_2 \) such that \( \pi_2(s) = \pi_1(s) \) for \( 0 \leq s < t_0 \) and \( \pi_2(s) = \pi_1(s) + a \) for \( s \geq t_0 \), for some \( 0 < t_0 < tN \) and \( 0 < a < r_0 \). As observed by Rongfeng Sun (see Acknowledgements below), we can couple \( \mathbb{P}(\cdot | \xi_j = \pi_1) \) and \( \mathbb{P}(\cdot | \xi_j = \pi_2) \) by shifting all the Poisson points of \( S \) at and above time \( t_0 \) by \( a \) units to right, keeping the remaining points still. Since this operation preserves the event \( \{\xi_{j-1}(tN) < \xi_j(tN)\} \), we get the claimed monotonicity of \( \mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) | \xi_j = \pi) \). The argument for the monotonicity of \( \mathbb{P}(\xi_j(tN) < \xi_n(tN) | \xi_j = \pi) \) is similar.

Now, let us consider the items at the last line of equation (2.45). By Lemma 2.5,

\[
\limsup_{N \to \infty} \mathbb{P}(T_{\epsilon,N} > tN) = \mathbb{P}(T_{\epsilon,B} > t),
\]

where \( T_{\epsilon,B} \) is the time when two i.i.d. Brownian motions starting at the same time at distance \( \epsilon \) apart meet and coalesce. Thus the latter probability is an \( O(\epsilon) \) for every \( t > 0 \) fixed. By Lemma 2.10 \( \mathbb{P}(T > tN) \leq c_2/\sqrt{tN} \). These estimates imply that

\[
\limsup_{N \to \infty} \mathbb{P}(\bar{\eta}_{X_{\delta}}(0,tN;0,\epsilon\sqrt{N}) \geq 3) \leq \limsup_{N \to \infty} \frac{\epsilon\sqrt{N}}{r_0} \mathbb{P}(T > tN) \mathbb{P}(T_{\epsilon,N} > tN) = O(\epsilon^2),
\]

and we get \( B'_\delta \) for \( X_\delta \).

Finally, we verify \( B'_\delta \) for Poisson web \( Y_\delta \). Fix \( t > 0 \). Given \( \epsilon > 0 \) and \( \{\delta_i > 0, i \in \mathbb{N}\} \) such that \( \delta_i \to 0 \) as \( i \to \infty \).

Given \( \delta_i \), on the Poisson field \( S \) with parameter \( \lambda(\delta_i) = \delta_i^{-1} \), choose \( r = r(\delta_i) = (3\delta_i/2)^{1/3} \), and then define \( Y_\delta = X(\lambda(\delta_i), r(\delta_i)) \) as in (1.12). As we did for \( X_\delta \), conditioned on the set of points of intersection of the paths \( \xi^s, s \in S \), with \( [0, \epsilon] \), define \( \eta'' \) be
the number of the corresponding remaining paths in time $t$. An analogous procedure gives that

$$
P(\eta'' \geq 3) \leq \frac{\epsilon}{r(\delta_i)} P(T_{\delta_i}) > t) P(T_{\epsilon,i} > t),$$

where $T_{\delta_i}$ is the time when $\xi^{(0,0)}$ and $\xi^{(2r(\delta_i),0)}$ meet and coalesce, and $T_{\epsilon,i}$ is the analogue time for $\xi^{(0,0)}$ and $\xi^{(\epsilon,0)}$. By Lemma 2.5, for any fixed $t$, $\limsup_{i \to \infty} P(T_{\epsilon,i} > t) = O(\epsilon)$, so, to verify $B'_2$ for the Poisson web $Y_\delta$, it is sufficient to prove that

$$
P(T_{\delta_i} > t) = O(r(\delta_i)). \quad (2.46)$$

As random walks in $[0, \infty)$, $\xi^{(0,0)}$ and $\xi^{(2r(\delta_i),0)}$ have rate $2r(\delta_i)\lambda(\delta_i) = 2(3/2)^{1/3} \cdot \delta_i^{-2/3}$ (which tends to $\infty$ as $i \to \infty$). Let $\{\xi^{(0,0)'}, \xi^{(r(\delta_i),0)'}\}$ be another (coalescing) random walk system, and the only difference from $\{\xi^{(0,0)}, \xi^{(r(\delta_i),0)}\}$ is that, as single walks, $\xi^{(0,0)'}$ and $\xi^{(2r(\delta_i),0)'}$ have unit rate. Let $T'_{\delta_i}$ be the corresponding coalescence time for $\{\xi^{(0,0)'}, \xi^{(r(\delta_i),0)'}\}$. It is clear that

$$
P(T_{\delta_i} > t) = P(T'_{\delta_i} > t[2r(\delta_i)\lambda(\delta_i)]) = P(T'_r > 2(3/2)^{1/3}t \delta_i^{-2/3}). \quad (2.47)$$

Using Lemma 2.10 for $T'_{\delta_i}$, we get (2.46), and then get $B'_2$ for $Y_\delta$.

**Acknowledgements**

This work was begun when one of us (X.-Y. W.) was visiting the Statistics Department of the Institute of Mathematics and Statistics of the University of São Paulo. He is thankful to the probability group of IME-USP for hospitality. We would like to thank A. Sarkar for introducing us to drainage networks, and C. M. Newman for comments on the background paragraph of earlier versions. We much thank Rongfeng Sun for pointing out important corrections on Subsection 2.2 of an earlier version, as well as for subsequent discussions on our fixing of them.

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