Abstract

We have calculated the decay amplitude for the process $K_S \to \gamma \gamma$ at one loop order in chiral perturbation theory. As a new improvement we have included the weak mass term which is only relevant for processes with external fields in the final state. This term was ignored in earlier publications for this decay. We find that the inclusion of $G'_8$ brings the theoretical decay rate into a good agreement with experiment.

Key words: rare decay, non-leptonic kaon decay, chiral symmetry

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1. Introduction

The non-leptonic kaon decay $K_S \to \gamma \gamma$ provides a good testing bed for the effective lagrangian method at one loop order. The reason hinges in the fact that to first order in perturbation theory, short-distance effects are suppressed and the decay amplitude to one-loop order in chiral effective theory is free from unknown low energy effective constants (LEC’s). For a good recent review on the weak chiral lagrangian see [1]. The branching ratio with respect to the $\pi^+ \pi^-$ channel is theoretically obtained in [2]. The decay rate is also evaluated in an independent work [3]. The theoretical result at $p^4$ gives $BR(K_S \to \gamma \gamma) = 2.1 \times 10^{-6}$ [2, 3]. This finding is in good agreement with experimental measurement of NA31 that obtained $BR(K_S \to \gamma \gamma) = (2.4 \pm .9) \times 10^{-6}$ [4] and with that of KLOE that measured $BR(K_S \to \gamma \gamma) = (2.26 \pm .12) \times 10^{-6}$ [5]. On the other hand, the most recent measurement from

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NA48, obtained $BR(K_S \to \gamma\gamma) = 2.71 \times 10^{-6}$ [6] with a total uncertainty of about 3%. The latter experiment opens up the possibility of a sizable correction of order 30% from two-loop effects. In view of this observation it is deemed interesting to study the effect of the higher order corrections for this decay [7]. The leading two-loop divergences for the octet part of the non-leptonic weak sector are already available in [8].

The application of weak lagrangian is also extended to other rare process, namely, $K \to \pi\gamma\gamma$ in [9, 10, 11]. The decays $K \to 2\pi$ and $K \to 3\pi$ are studied at one loop order in many places, e.g. see [12, 13, 14].

Given all these, we now turn to the main point which motivates the present work. In [15] it is demonstrated that the weak mass term appearing in the lowest order weak lagrangian has no contribution to the physical amplitude when there is no external fields in the decay or when the interaction does not carry four-momentum. Moreover, it is explicitly shown in [14] that these effects can be reconstructed from redefining the weak effective constants of order $p^4$ in the decay $K \to 3\pi$. In fact the presence of both strong and weak effective constants of $p^4$ order made this procedure possible. This type of relations cannot be generalized to the amplitude at order $p^4$ in the presence of external fields as in the case of non-leptonic kaon decay to two photons, because there are no tree diagram of next-to-leading order for this process. For a detailed discussion on the contribution of the weak mass term on the $K \to \pi\pi$ amplitude we suggest [16] and references therein. We therefore have recalculated the one loop order amplitude for the decay $K_S \to \gamma\gamma$ and have also taken into account the weak mass term. The only weak effective constants involved at order $p^4$ are $G_8$, $G_{27}$ and $G_8'$. It is easy then to see that the $G_8'$ effect has an essential contribution at order $p^4$ to the decay and cannot be disregarded.

The organization of this letter is as follows. We provide a brief introduction to the weak and strong chiral lagrangian at leading order in Section 2. In Section 3 the kinematics is described. Sections 4 and 5 are devoted to our analytical and numerical results respectively. Finally we conclude in the last section.

2. The ChPT Lagrangians

We apply effective lagrangians in order to study the low energy dynamics of the strong and weak interactions. The lagrangian we use is the lowest order chiral lagrangian. The expansion parameter is in terms of external
momentum "\(p\)" and quark masses, "\(m_q\)". Quark masses are counted of order \(p^2\) due to the lowest order mass relation \(m^2_\pi = B_0(m_u + m_d)\). Here we only present the leading order strong and weak chiral lagrangian. The leading order lagrangian which is of order \(p^2\), assumes the form

\[
\mathcal{L}_2 = \mathcal{L}_{S2} + \mathcal{L}_{W2}. \tag{1}
\]

\(\mathcal{L}_{S2}\) refers to the strong sector with \(\Delta S = 0\) and \(\mathcal{L}_{W2}\) stands for the weak part with \(\Delta S = \pm 1\). For the strong part we use

\[
\mathcal{L}_{S2} = \frac{F_0^2}{4} \langle u^\dagger u + \chi_\pm \rangle, \tag{2}
\]

where \(F_0\) is the pion decay constant at chiral limit and we define the matrices \(u^\mu\) and \(\chi_\pm\) as following

\[
u^\mu = i u^\dagger D^\mu U u = u^\dagger, \quad u^2 = U, \quad \chi_\pm = u^\dagger \chi u^\dagger \pm u \chi^\dagger u. \tag{3}\]

The matrix \(U \in SU(3)\) contains the octet of light pseudo-scalar mesons with its exponential representation given in terms of meson fields matrix as

\[
U(\phi) = \exp(i \sqrt{2} \phi / F_0), \tag{4}
\]

where

\[
\phi(x) = \begin{pmatrix}
\pi^+ \\
\pi^- \\
K^+
\end{pmatrix} = \begin{pmatrix}
\frac{\pi_3}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} \\
-\frac{\pi_3}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} \\
K^0
\end{pmatrix}. \tag{5}
\]

We use the method of external fields discussed in [17]. The external fields are then defined through the covariant derivatives as

\[
D^\mu U = \partial^\mu U - ir^\mu U + iU l^\mu. \tag{6}
\]

The right-handed and left-handed external fields are expressed by \(r^\mu\) and \(l^\mu\) respectively. In the present work we set

\[
r^\mu = l^\mu = e A^\mu \begin{pmatrix}
2/3 \\
-1/3 \\
-1/3
\end{pmatrix}. \tag{7}
\]
The electron charge is denoted by $e$ and $A_\mu$ is the classical photon field. The Hermitian $3 \times 3$ matrix $\chi$ involves the scalar (s) and pseudo-scalar external densities and is given by $\chi = 2B_0(s + ip)$. The constant $B_0$ is related to the pion decay constant and quark condensate. For our purpose it suffices to write

$$\chi = 2B_0 \begin{pmatrix} m_u \\ m_d \\ m_s \end{pmatrix}. \quad (8)$$

The $\Delta S = \pm 1$ part of the weak effective lagrangian contains both the $\Delta I = 1/2$ piece and the $\Delta I = 3/2$ transition and has the form $[18]$

$$L_{W2} = CF_0^4 \left[ G_8 \langle \Delta_{32} u_\mu u^\mu \rangle + G'_8 \langle \Delta_{32} \chi_+ \rangle \\
+ G_{27} t^{ij,kl} \langle \Delta_{ij} u_\mu \rangle \langle \Delta_{kl} u^\mu \rangle \right] + h.c, \quad (9)$$

where the coefficient $C$ is defined as

$$C = -\frac{3}{5} \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \approx -1.09 \times 10^{-6} \ GeV^{-2} \quad (10)$$

and the matrix $\Delta_{ij}$ is given by

$$\Delta_{ij} = u \lambda_{ij} u^\dagger, \quad (\lambda_{ij})_{ab} = \delta_{ia} \delta_{jb}. \quad (11)$$

The nonzero components of the tensor $t^{ij,kl}$ are

$$t^{21,13} = t^{13,21} = \frac{1}{3}, \quad t^{22,23} = t^{23,22} = -\frac{1}{6},$$

$$t^{23,33} = t^{33,23} = \frac{1}{6}, \quad t^{23,11} = t^{11,23} = \frac{1}{3}. \quad (12)$$

The constant $F_0$ is the pion decay constant at chiral limit.

### 3. Kinematics

The decay amplitude of $K_S \rightarrow \gamma \gamma$ with the following momentum assignment

$$K_S(p) \rightarrow \gamma(k_1)\gamma(k_2), \quad (13)$$

...
has the form

\[ A(K_S \to \gamma \gamma) = M_{\mu\nu}(k_1, k_2) \epsilon_1^\mu(k_1) \epsilon_2^\nu(k_2), \]

(14)

where \( \epsilon_1^\mu \) and \( \epsilon_2^\nu \) are the polarization four-vectors of the outgoing photons carrying momentum \( k_1 \) and \( k_2 \) respectively. Due to the gauge invariance, Lorentz symmetry and Bose symmetry, \( M_{\mu\nu}(k_1, k_2) \) takes on the specific form

\[ M_{\mu\nu}(k_1, k_2) = F(p^2) (k_{1\nu} k_{2\mu} - k_{1\mu} k_{2\nu} g_{\mu\nu}). \]

(15)

Where \( p = k_1 + k_2 \) and \( k_1^2 = k_2^2 = 0 \) for photons with on-shell masses. The decay width for a decay with two particles in the final state reads

\[ \Gamma(K_S \to \gamma \gamma) = \frac{1}{16\pi m_{K}} |F(p^2 = m_{K}^2)|^2. \]

(16)

4. Analytical Results

The decay amplitude gets no tree-level contribution of order \( p^2 \) and \( p^4 \). This is because all the particles involved here are neutral particles. Thus, the leading non-zero part of the amplitude originates from loop diagrams constructed out of strong and weak lagrangians of order \( p^2 \). The relevant Feynman diagrams for this decay is depicted in Fig. 1. Since tree diagrams are absent here we therefore expect that the sum of all the Feynman diagrams ends up finite, i.e. all infinities from loop integrals cancel. This is indeed proven by our explicit calculation. We present our result in a form that full agreement with the earlier results given in [2, 3] can be simply understood besides an extra term in our expression followed by the coupling constant \( G' \) which is new. We use the lowest order relations \( m_{\pi}^2 = B_0(m_u + m_d) \) and \( m_{K}^2 = B_0(m_s + m_u) \) to replace quark masses with the mesons masses. The following analytical result is achieved

\[
F(p^2) = \frac{C(G_8 + G_{27} - 4/3G'_{8})}{2\pi} \alpha_{em} F_{\pi} \left( \frac{p^2 - m_{\pi}^2}{p^2} \right) \left( 1 + \frac{m_{\pi}^2}{p^2} \log^2 \left( \frac{\sqrt{1 - 4m_{\pi}^2/p^2} - 1}{\sqrt{1 - 4m_{\pi}^2/p^2} + 1} \right) \right) - (\pi \to K) \]

(17)

using the program FORM [19]. It should be noticed that for \( p^2 = m_{K}^2 \), the contribution of \( G'_{8} \) reduces the magnitude of \( F(m_{K}^2) \).
5. Numerical results

In this section we estimate the decay rate for two sets of inputs. We use the physical values instead of lowest order values for masses and pion decay constants in doing numerics. This is because the difference stands on the higher chiral order. We use then $F_\pi = 0.092$ GeV for the pion decay constant, $m_\pi = 0.1395$ GeV and $m_K = 0.4936$ GeV for the charged pion and kaon mass respectively. The quantities $G_8$ and $G_{27}$ are determined at one-loop ChPT order by performing direct fit to the experimental data in decays $K \rightarrow \pi\pi\pi$ and $K \rightarrow \pi\pi$. For detailed discussions we refer the reader to [14]. The values of $G_8$ and $G_{27}$ which are presented in [14] read

$$G_8 = 5.49 \pm 0.02 \quad G_{27} = 0.392 \pm 0.002$$

We quote these values as ”Set 1” hereafter. There is also another estimate for these quantities based on a hadronic model including a $Q_2$ penguin-like contribution that obtained [20]

$$G_8 = 6.0 \pm 1.7 \quad G_{27} = 0.35 \pm 0.15.$$  

We quote these values as ”Set 2” hereafter. This set of values inherits large uncertainties, however, we shall use them for the sake of comparison. There is one study concerning the determination of the quantity $G'_8$ employing a hadronic model at next to leading order in large $N_c$ [21]. They obtained

$$G'_8 = 0.9 \pm 0.1$$

(20)
We set $G'_8 = 0$ in Eq. 17 and use the values of $G_8$ and $G_{27}$ given in Eq. 18 and Eq. 19 to obtain the theoretical decay rate

$$\Gamma(K_S \rightarrow \gamma\gamma)_{th} = (2.879 \pm 0.02) \times 10^{-20} \text{ GeV, \quad Set 1}$$

$$\Gamma(K_S \rightarrow \gamma\gamma)_{th} = (3.401 \pm 1.828) \times 10^{-20} \text{ GeV. \quad Set 2} \quad (21)$$

The estimated errors are due to the uncertainties in $G_8$ and $G_{27}$. As we expect from the formula in Eq. 17, a non-zero value for $G'_8$ lowers the value of the decay rate such that we obtain

$$\Gamma(K_S \rightarrow \gamma\gamma)_{th} = (1.817 \pm 0.165) \times 10^{-20} \text{ GeV, \quad Set 1}$$

$$\Gamma(K_S \rightarrow \gamma\gamma)_{th} = (2.237 \pm 1.494) \times 10^{-20} \text{ GeV. \quad Set 2} \quad (22)$$

We compare our theoretical result with the averaged experimental measurements provided in [22]

$$\Gamma(K_S \rightarrow \gamma\gamma)_{exp} = (2.115 \pm 0.136) \times 10^{-20} \text{ GeV.} \quad (23)$$

Using the values for the theoretical decay rate given in Eq. 22 and the experimental total decay rate $\Gamma(K_S)_{exp} = (7.385 \pm 0.026) \times 10^{-15}$ [22] GeV, it is possible to obtain the branching ratio as

$$BR(K_S \rightarrow \gamma\gamma)_{th} = (2.46 \pm 0.22) \times 10^{-6}, \quad Set 1$$

$$BR(K_S \rightarrow \gamma\gamma)_{th} = (3.02 \pm 2.02) \times 10^{-6}. \quad Set 2 \quad (24)$$

The experimental branching ratio provided in [22] reads

$$BR(K_S \rightarrow \gamma\gamma)_{exp} = (2.63 \pm 0.17) \times 10^{-6}. \quad (25)$$

6. Conclusion

In this letter we have recalculated the $K_S \rightarrow \gamma\gamma$ at one-loop order and in addition have added a contribution due to the weak mass term in the leading order of weak action which had been ignored in the previous works. We obtained the full result for the decay rate at one-loop order. There are three effective constants involved, namely, $G_8$, $G_{27}$ and $G'_8$. We have evaluated the decay rate using two sets of inputs for $G_8$ and $G_{27}$ and one single value for $G'_8$. It is realized that the effect of the weak mass term is important and in order for a one-loop result to reach the corresponding experimental data it is necessary to take it into account.
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