Superinflation, quintessence, and nonsingular cosmologies

E. Gunzig1,2, A. Saa1,3,4, L. Brenig1,5, V. Faraoni1,6, T.M. Rocha Filho7 and A. Figueiredo7.
1 RgpR, Université Libre de Bruxelles, CP 231, 1050 Bruxelles, Belgium
2 Instituts Internationaux de Chimie et de Physique Solvay, CP231, 1050 Bruxelles, Belgium
3 Departamento de Física Fonamental, Universitat de Barcelona, Av. Diagonal 647, 08028 Barcelona, Spain
4 Departamento de Matemática Aplicada, IMECC–UNICAMP, CP6065, 13081-970 Campinas, SP, Brazil
5 Service de Physique Statistique, Université Libre de Bruxelles, CP231, 1050 Bruxelles, Belgium
6 INFN-Laboratori Nazionali di Frascati, Box 13, 00044 Frascati, Roma, Italy
7 Instituto de Física, Universidade de Brasília, 70.910-900 Brasília, DF, Brazil

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The dynamics of a universe dominated by a self-interacting nonminimally coupled scalar field are considered. The structure of the phase space and complete phase portraits are given. New dynamical behaviors include superinflation ($H > 0$), avoidance of big bang singularities through classical birth of the universe, and spontaneous entry into and exit from inflation. This model is promising for describing quintessence as a nonminimally coupled scalar field.

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A dynamical system approach to a self-consistent nonsingular cosmological history is presented in the framework of the classical Einstein equations with a nonminimally coupled scalar field. The description of the matter content of the cosmos with a single scalar is appropriate during important epochs of the history of the universe [1]. A crucial ingredient of the physics of scalar fields in curved spaces is their nonminimal coupling to the Ricci scalar $R$ of spacetime, which is required by first loop corrections [2], by specific particle theories [3], and by scale-invariance arguments at the classical level [4]. It is well known that nonminimal coupling dictates the success or failure of inflationary models [5]; more generally, it turns out to strongly affect the cosmic dynamics, which is qualitatively richer than in the minimally coupled case. We show, indeed, that nonminimal coupling leads to new dynamical behaviors, such as a regime that we propose to call superinflation ($H > 0$), which cannot be achieved with minimal coupling [6], spontaneous entry into and exit from inflation, with or without a cosmological constant, and a possible model for quintessence. Spontaneous superinflation provides a classical alternative, which is impossible with minimal coupling, to semiclassical [7,8] and quantum [9] birth of the universe.

We consider the nonminimally coupled theory described by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left( -\frac{\kappa}{\kappa} + g_{\mu\nu} \partial_\mu \psi \partial_\nu \psi - 2V + \kappa R \psi^2 \right),$$

(1)

where $\kappa \equiv 8\pi G$ ($G$ being Newton’s constant), $\xi$ is the nonminimal coupling constant, and a cosmological constant $\Lambda$, if present, is incorporated in the scalar field potential $V(\psi)$. We use the full conserved scalar field stress-energy tensor

$$T_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \xi (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) (\psi^2) + \xi G_{\mu\nu} \psi^2$$

and $\kappa / \kappa$ the Einstein tensor), thereby avoiding the widespread effective coupling $\kappa_{\text{eff}} = \kappa (1 - \kappa \xi \psi^2)^{-1}$ in the Einstein equations $G_{\mu\nu} = \kappa T_{\mu\nu}$. We study the dynamics of a spatially flat Friedmann-Robertson-Walker universe with line element $ds^2 = dt^2 - a^2(\tau) (dx^2 + dy^2 + dz^2)$. This yields the trace equation

$$R = -\kappa (\sigma - 3p),$$

and the constraint $3H^2 = \kappa \sigma$ (which guarantees that the energy density $\sigma \geq 0$), and the Klein-Gordon equation, respectively

$$6 \left[ 1 - \xi (1 - 6\xi) \kappa \psi^2 \right] (H + 2H^2) - \kappa (6\xi - 1) \dot{\psi}^2 - 4\kappa V + 6\kappa \xi \psi \frac{dV}{d\psi} = 0,$$

(3)

$$\frac{9\kappa}{2} \dot{\psi}^2 + 6\xi \kappa H \dot{\psi} - 3H^2 (1 - \kappa \xi \psi^2) + \kappa V = 0,$$

(4)

$$\ddot{\psi} + 3H \dot{\psi} + \xi R \psi + \frac{dV}{d\psi} = 0.$$  

(5)

The (time-dependent) equation of state of the $\psi$ field, rather than being imposed a priori, follows self-consistently from the dynamics. The system (3)-(5) reduces to a two-dimensional set of first order equations for $\psi$ and $H \equiv \dot{a}/a$ (note that this dimensional reduction is not possible for spatially curved universes [8,10]). One has

$$\ddot{\psi} = -6\xi H \dot{\psi} \pm \frac{1}{2\kappa} \sqrt{G(H, \psi),}$$

(6)

$$\dot{H} = \frac{1}{1 + \kappa (6\xi - 1) \psi^2} \left[ 3(2\xi - 1) H^2 + 3\xi (6\xi - 1) (4\xi - 1) \kappa H \dot{\psi}^2 + \xi (6\xi - 1) H \psi \sqrt{G} 

+ (1 - 2\xi) \kappa V (\psi) - \kappa \xi \psi \frac{dV}{d\psi} \right],$$

(7)
\( \mathcal{G}(H, \psi) = 8\kappa^2 \left[ \frac{3H^2}{\kappa} - V(\psi) + 3\xi(6\xi - 1)H^2\psi^2 \right] \). (8)

Due to the energy constraint (4), the trajectories are restricted to a two-dimensional manifold \( \Sigma \) in the three-dimensional \((H, \psi, \dot{\psi})\) phase space, possibly with "holes" (dynamically forbidden regions) corresponding to \( \mathcal{G}(H, \psi) < 0 \) (cf. Eq. (6)). \( \Sigma \) is composed of two sheets corresponding to the positive or negative sign in Eq. (6). The two sheets smoothly join on the boundary \( \mathcal{G} = 0 \) of the dynamically forbidden region.

In the following, for simplicity, we project the dynamics of the phase space onto the \((H, \psi)\) plane, but the true nature of \( \Sigma \) should always be kept in mind. We now restrict to the widely used potential

\[
V(\psi) = \frac{3\alpha}{\kappa} \psi^2 - \frac{\Omega}{4} \psi^4 - \frac{9\omega}{\kappa^2},
\]

consisting of a mass term, a quartic self-coupling and, possibly, a cosmological constant term. For consistency with previous works [8] we use the dimensionless symbols \( \alpha \equiv \kappa m^2/6 \) (\( m \) being the scalar field mass) and \( \omega \equiv -\kappa^2\Lambda/9 \).

The fixed points of the system (3)-(5) are the de Sitter solutions with constant scalar field

\[
H^2 = \frac{3(\alpha^2 - \Omega\omega)}{\kappa(\Omega - 6\xi\alpha)}, \quad \psi^2 = \frac{6(\alpha - 6\xi\omega)}{\kappa(\Omega - 6\xi\alpha)}
\]

(\( \Omega \neq 6\alpha\xi \)), and \((H, \psi) = \left( \pm \sqrt{-3\omega/\kappa}, 0 \right) \). The fixed points (10) exist also for \( \omega = 0 \), due to the presence of the matter field \( \psi \) (in this case, the two points \((\pm \sqrt{-3\omega/\kappa}, 0)\) collapse into the Minkowski space fixed point \((0, 0)\)). We only present here the case of conformal coupling, \( \xi = 1/6 \).

The function

\[
L(\psi, \dot{\psi}) = \frac{1}{2} \dot{\psi}^2 + \frac{\alpha}{4} \psi^4 - \frac{3\omega}{\kappa} \psi^2 + V(\psi)
\]

is such that \( dL/d\tau = -3H\dot{\psi}^2 \) along the trajectories. For \( H > 0 \), \( L \) is a Lyapunov function in a region containing the origin; the solutions are then confined by the closed lines of constant \( L \), implying asymptotic convergence to the fixed points on the \( H \) axis. This behavior is confirmed by exhaustive numerical simulations reported in the following. We first exclude a cosmological constant by setting \( \omega = 0 \). The phase portrait qualitatively differs according to the ratio \( \Omega/\alpha \).

FIG. 1. Qualitative phase portrait for the system (3)-(5). Shadowed regions correspond to the dynamically forbidden regions \( \mathcal{G}(H, \psi) < 0 \), cf. Eq. (6)), and the dots to the fixed points. Figures a-e, were obtained by using \( \Omega/\alpha \), respectively, 2, 5, 3/2, 3/2, and 1/2, and \( \omega = 0 \). Figure f corresponds to the case \( \omega = -1/10 \) and \( \Omega/\alpha = 3/2 \).

The case \( \Omega = 2\alpha \): The Minkowski space \((H, \psi, \dot{\psi}) = (0, 0, 0)\) is a fixed point, attractive for \( H > 0 \) and repulsive for \( H < 0 \); the projections of the de Sitter spaces \((\pm H_0, \pm \psi_0, 0)\) are saddle points, i.e. they possess attractive and repulsive eigendirections in the phase space as shown by the arrows in (Fig. 1a). The new kind of solutions, only present in this case \( \Omega = 2\alpha \) and unavoidably missed in the approach using \( \kappa_{\text{eff}}(\tau) \)

\[
H(\tau) = \sqrt{\frac{C}{2}} \tanh \left( \sqrt{2C} \tau \right), \quad \psi = \pm \psi_0 \equiv \pm \sqrt{\frac{6}{\kappa}}
\]

(12)

(where \( C = H + 2H^2 = -R/6 \) is constant), corresponds to heteroclinic straight lines connecting de Sitter fixed points, starting along the repulsive eigendirection of one of them and ending along the attractive eigendirection of the other (Fig. 1a). They are tangent to the boundary of the forbidden regions at \((H, \psi) = (0, \pm \psi_0)\). For \( |H| > \sqrt{C}/2 \), another straight line solution is obtained from the general form

\[
H(\tau) = \sqrt{\frac{C}{2}} \frac{w_1 e^{Ct/2} - w_2 e^{-Ct/2}}{w_1 e^{Ct/2} + w_2 e^{-Ct/2}}, \quad \psi = \psi_0,
\]

(13)

where \( w_1 \) and \( w_2 \) are integration constants. The nonsingular solutions (12) connect a contracting \((\tau \to -\infty)\) de Sitter regime to an expanding one \((\tau \to +\infty)\) and passes
through a minimum nonvanishing value of the scale factor \(\tau = 0\).

In addition to these straight lines, there exist other heteroclinic solutions: one starting at \((H, \psi) = (0, 0)\) and ending at \((-H_0, \psi_0)\), and another one from \((H_0, \psi_0)\) to \((0, 0)\). A third solution starts at the expanding de Sitter point and goes to infinity, while another one comes from infinity and arrives to the contracting de Sitter point. The phase portrait is symmetric about the origin.

Near the Minkowski fixed point \((0, 0, 0)\), numerical analysis confirms the peculiar behavior suggested by the Lyapunov function: orbits approaching this point with positive \(H\) are attracted to it, bouncing back and forth infinitely many times off the \(G = 0\) boundary in the \((H, \psi)\) projection (Fig. 1a). In the space \((H, \psi, \dot{\psi})\) these orbits are seen to spiral down on a cone towards its apex at the origin. The cone results from the union of the two sheets in the vicinity of the origin. Along the spiral, the orbit passes almost periodically from one sheet to the other with period \(\tau_{\text{bounce}} = 2\pi m\) (\(\tau_{\text{bounce}}\) is obtained by an asymptotic analysis of the system \((3)-(5)\). Typically, after a few bounces, the period coincides with \(\tau_{\text{bounce}}\) with good accuracy). A similar behavior for \(\Omega < 0\) was reported in the earlier numerical analysis of \([10]\), but using the effective coupling \(\kappa_{\text{eff}}(\tau)\) and the variables \(\psi\) and \(\dot{\psi}\).

In the \(H < 0\) half-plane, the situation is reversed: orbits starting within \(H < 0\) are repelled by the origin and depart from it bouncing off the \(G = 0\) boundary.

The case \(\Omega > 2\alpha\): the situation (Fig. 1b) is analogous to the previous one, but now the straight heteroclinic solutions are missing, and are replaced by the solution starting at the contracting de Sitter fixed point and escaping to infinity, and by the solution coming from infinity and arriving to the expanding de Sitter point. The two-sheeted structure of \(\Sigma\) implies that no actual intersections occur between different orbits in Fig. 1b, which live in different sheets but are projected in the same plane.

The case \(\alpha < \Omega < 2\alpha\): as shown in Fig. 1c, there are no straight heteroclinic lines but new interesting features emerge. A new heteroclinic solution appears starting from the origin and ending in the expanding de Sitter fixed point. As in the previous case, the quadrant \(\psi > 0\), \(H < 0\) is obtained from the \(\psi > 0\), \(H > 0\) one by reflection about the \(\psi\)-axis and time-reversal.

The crucial feature of this case is the appearance of a dense set of homoclinic solutions (Fig. 1d) departing from the origin with negative \(H\) and returning to it with positive \(H\), going around the forbidden region. Superinflation plays a central role along these orbits: only a regime with \(\dot{H} > 0\) permits the smooth transition from an initial contracting \(H < 0\) phase to an expanding one \(H > 0\). This transition occurs at the nonvanishing minimum of the scale factor. The behavior of this family of homoclinics, as well as of the other solutions, is universal: they rapidly converge in the spiraling region near the origin, irrespective of initial conditions. As all of these homoclinics originate around Minkowski fixed point due to its instability with respect to perturbations with \(H < 0\), this dynamics constitute a classical alternative to the previously proposed semiclassical birth of the universe from empty space \([7,8]\).

The case \(0 < \Omega < \alpha\): the fixed points \((10)\) disappear and the only bounded solutions are the homoclinics associated with the origin (see Fig. 1e). This situation is therefore the most favorable for the classical spontaneous exit from empty Minkowski fixed point.

Analogous results hold also with a small and positive cosmological constant \(\Lambda\) (see, for instance, Fig. 1f), with the phase portrait being classified according to \(\Omega / (\alpha - \omega)\), but the fixed point \((0, 0, 0)\) of the \(\omega = 0\) case splits into two de Sitter fixed points with memory of the previous stability properties. Now, the approximate period between two consecutive bounces is

\[
\tau_{\text{bounce}} = \frac{2\pi}{\sqrt{m^2 - \frac{1}{\Omega^2\alpha^2}}.}
\]

![FIG. 2. The plane \((H, \psi)\) for the \(\omega = 0\) case and the equation of state. The Darkest region corresponds to the superinflation regime \((H > 0)\). In the region \(H < 0\) and \(H > 0\), the equation of state associated with the bouncing solution \(\psi\) passes periodically through radiation domination (crossing the \(H\)-axis), matter domination \((p = 0)\), and re-acceleration (between \(G = 0\) and \(\dot{a} = 0\) lines). Asymptically, as \(\tau \to +\infty\), the universe becomes matter dominated \(\alpha(\tau) \propto \tau^{2/3}\) and tends to infinite dilution.](image)
the line $\dot{H} = 0$ (or parts of it) belong to the dynamically accessible region $G \geq 0$ of the $(H, \psi)$ plane. This implies that, for an arbitrary potential $V(\psi)$, superinflation corresponds to $\psi dV/d\psi \leq 0$. For our particular potential (9), this requires $\Omega > 0$. The superinflationary behavior occurs only once along each homoclinic and brings the solution from the primordial Minkowski fixed point neighborhood to the succession of eras corresponding to different equations of state (during each bounce), towards infinite dilution and equation of state $p = 0$. Indeed, asymptotic analysis for $\tau \to +\infty$ and for any value of $\alpha$ and $\Omega$ (with $\omega = 0$) shows that the scale factor $a(\tau)$ exhibits oscillations of concavity corresponding to accelerated and decelerated epochs. These oscillations are damped as $\tau \to +\infty$; in this regime, $a(\tau) \propto R^{2/3}$ and the universe becomes matter dominated.

While it is not claimed here that the evolution of our universe is modeled by an entire orbit of the system (3)-(5) on the accessible manifold $\Sigma$, the application to specific eras of the cosmological history is intriguing. Indeed, the bounces reported above (during which $\dot{H} < 0$), one encounters, respectively, radiation domination crossing the $H$-axis, matter domination ($p = 0$), acceleration (a possible quintessence model?) until the next bounce in the $(H, \psi)$ projection, where this sequence is reversed. If we identify one period as our cosmological history, then the reported accelerated expansion of the universe today [11] suggests to locate our epoch in the sector between the line $H = \sqrt{\alpha} \psi$ and the $G = 0$ boundary. The identification of the age of the universe ($\sim 10^{17}$ s) with $\tau_{\text{bounce}}$ would then yield the scalar field mass $m \simeq 10^{-13}$ eV, which is suggestive of an axion [1] or of an ultralight pseudo-Goldstone boson (that the quintessence field should be very light was already suggested [12]).

Although our simplified model is not a quintessential scenario, its features are promising for the description of quintessence as a nonminimally coupled scalar field. Furthermore, the main properties of the model presented here remain unaltered for a large class of potentials $V(\psi)$. Finally, the fact that the dynamics of the system (3)-(5) are confined to the two-dimensional smooth manifold $\Sigma$ indicates the absence of chaos, a conclusion supported by further analysis. This is not the case, a priori, of a spatially curved universe and/or of multiple matter fields.

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\* Electronic addresses: egunzig@ulb.ac.be, asaa@fnn.ub.es, lbrenig@ulb.ac.be, vfaraoni@ulb.ac.be, marciano@fis.unb.br, annibal@fis.unb.br

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