ELEMENTARY INVERSION OF RIESZ POTENTIALS
AND RADON-JOHN TRANSFORMS

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Abstract. New simple proofs are given to some elementary approximate and explicit inversion formulae for Riesz potentials. The results are applied to reconstruction of functions from their integrals over Euclidean planes in integral geometry.

1. Introduction

The Riesz potential of order $\alpha$ of a sufficiently good function $f$ on $\mathbb{R}^n$ is defined by

$$ (I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y) \, dy}{|x - y|^{n-\alpha}}, \quad \gamma_n(\alpha) = \frac{2^n \pi^{n/2} \Gamma(\alpha/2)}{\Gamma((n - \alpha)/2)}, \quad (1.1) $$

$$ \text{Re} \alpha > 0, \quad \alpha \neq n, n + 2, n + 4, \ldots. $$

This operator can be regarded (in a certain sense) as a negative power of “minus-Laplacian”, namely,

$$ I^\alpha = (-\Delta)^{-\alpha/2}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}. \quad (1.2) $$

If $f \in L^p(\mathbb{R}^n)$, then the integral (1.1) converges a.e. provided that $1 \leq p < n/\text{Re} \alpha$. This condition is sharp. Explicit and approximate inversion formulae for $I^\alpha$ are of great importance. We refer to [7, 12, 21, 23, 26, 27, 30] for the basic properties and applications of Riesz potentials. In this article we restrict to real $\alpha \in (0, n)$. This case reflects basic features and is sufficient for integral geometric applications in Section 4.

Our main working tools are a suitably chosen auxiliary function $w(x)$ (we call it a reconstructing function) and its scaled version $w_t(x) = t^{-n}w(x/t)$, $t > 0$. In the modern literature such a function is called a

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wavelet and a convolution \((f \ast w_t)(x)\) is called the continuous wavelet transform of \(f\); cf. [2, 3]. We choose
\[
w(x) = (-\Delta)^m[(1 + |x|^2)^{m-(n+\alpha)/2}], \quad \tilde{w}(x) = (-\Delta w)(x),
\]
m \in \mathbb{N}, and keep this notation throughout the paper.

The main results of the article are presented by Theorem A below and Theorem B in Section 4. The latter gives an example of application of Theorem A to elementary inversion of the \(k\)-plane Radon-John transform in integral geometry and falls into the general scope of the convolution-backprojection method (cf. [15, 22]). Radon transforms and their generalizations are the basic tools in tomography and numerous related areas of pure and applied mathematics; see [1, 5, 6, 7, 8, 9, 10, 11, 14, 15, 20, 18] and references therein.

**Theorem A.** Let \(2m > \alpha, \ f \in L^p(\mathbb{R}^n), 1 \leq p < n/\alpha\). Then
\[
c_{\alpha,m} f = \lim_{t \to 0} (I^\alpha f \ast t^{-\alpha} w_t), \quad c_{\alpha,m} = \gamma_n(2m - \alpha),
\]
\[
d_{\alpha,m} f = \int_0^\infty I^\alpha f \ast \tilde{w}_t \, dt, \quad d_{\alpha,m} = (2m - \alpha) c_{\alpha,m},
\]
where \(\int_0^\infty (\ldots) = \lim_{\varepsilon \to 0} \int_0^\varepsilon (\ldots)\) and \(\gamma_n(\cdot)\) has the same meaning as in (1.1). The limit in both formulae exists in the \(L^p\)-norm and in the a.e. sense. If, moreover, \(f \in C_0(\mathbb{R}^n)\), then it exists in the \(\text{sup}\)-norm.

We recall that \(C_0(\mathbb{R}^n)\) denotes the space of continuous functions on \(\mathbb{R}^n\) vanishing at infinity.

Some comments are in order. Statements of this kind are not new. Regarding (1.4), we observe that approximate inversion of operators of the potential type was initiated by Zavolzhenskii, Nogin and Samko [32, 17, 24, 25, 26]. Conceptually this approach is close to the convolution-backprojection method in tomography [15]. An elegant formula (1.4) is due to Samko [26, p. 325], [24, 25]; see also [19]. Below we give a new simple proof of it.

Regarding (1.5), wavelet-like inversion formulae for Riesz potentials are also well-known; see, e.g., [21, Section 17]. Our aim is to show that elementary reconstructing functions (1.3) work perfectly in the wavelet inversion scheme and the relevant justification is much simpler than in the general theory; cf. [21]. The case \(\alpha = 0\) in (1.5), when \(I^\alpha\) is substituted by the identity operator, represents a variant of Calderón’s reproducing formula [2, 3].

The validity of formulae (1.4) and (1.5) can be formally explained in the language of the Fourier transform. We have
\[
\lim_{t \to 0} (I^\alpha f \ast t^{-\alpha} w_t)(\xi) = \hat{f}(\xi) \lim_{t \to 0} (t|\xi|)^{-\alpha} \hat{w}(t\xi) = c_w \hat{f}(\xi)
\]
provided that the limit $c_w = \lim_{\xi \to 0} |\xi|^{-\alpha} \hat{w}(\xi)$ exists. Moreover, since $\tilde{w}$ in (1.3) is a radial function, we easily get
\[
\left[ \int_0^\infty I^\alpha f * \tilde{w}_t \frac{dt}{t^{1+\alpha}} \right]^\wedge (\xi) = d_w \hat{f}(\xi), \quad d_w = \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \frac{\hat{w}(y) dy}{|y|^{n+\alpha-2}}, \quad (1.7)
\]
where $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere $|x| = 1$.

This simple argument does not work for arbitrary $L^p$-functions. To cover this case, we develop another technique and give the final answer without using the Fourier transform.

2. Preliminaries

We recall some known facts. The Fourier transform of a function $f \in L^1 = L^1(\mathbb{R}^n)$ is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix \cdot \xi} dx, \quad x \cdot \xi = x_1 \xi_1 + \ldots + x_n \xi_n. \quad (2.1)
\]
Let $S = S(\mathbb{R}^n)$ be the Schwartz space of all $C^\infty$ functions which, together with derivatives of all orders, vanish at infinity faster than any inverse power of $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. We endow $S(\mathbb{R}^n)$ with a standard topology generated by the sequence of norms
\[
||\varphi||_m = \max_x (1 + |x|)^m \sum_{|j| \leq m} |(\partial^j \varphi)(x)|, \quad m = 0, 1, 2, \ldots.
\]

The following spaces adapted to Riesz potentials were introduced by Semyanistyi [28]; further generalizations due to Lizorkin and Samko can be found in [13, 26].

Let $\Psi = \Psi(\mathbb{R}^n)$ be the subspace of $S$, consisting of functions $\psi(\xi)$ vanishing at $\xi = 0$ with all derivatives, and let $\Phi = \Phi(\mathbb{R}^n)$ be the Fourier image of $\Psi$. We equip $\Phi$ with the induced topology of $S$. Then $\Phi$ becomes a topological vector space which is isomorphic to $\Psi$ under the action of the Fourier transform. We denote by $\Phi'$ the space of distributions over $\Phi$.

The main reason for introducing the spaces $\Psi$ and $\Phi$ is that $\Psi$ is invariant under multiplication by $|\xi|^\alpha$ for any $\alpha \in \mathbb{C}$ and therefore, the Riesz potential $I^\alpha$ is an automorphism of $\Phi$ and $\Phi'$.

**Proposition 2.1.** [21, p. 21] Let $f \in L^r$, $1 \leq r < \infty$, and $g \in L^p$, $1 \leq p < \infty$. If $f = g$ in the $\Phi'$-sense, then $f = g$ almost everywhere.
3. Proof of Theorem A

The following auxiliary propositions form the heart of the proof.

**Lemma 3.1.** Let \( w \in L^1(\mathbb{R}^n) \) be such that \( h = I^a w \) has an integrable decreasing radial majorant. If \( f \in L^p(\mathbb{R}^n), \) \( 1 \leq p < n/\alpha, \) then

\[
\lim_{t \to 0} (I^\alpha f * t^{-\alpha} w) = c_w(\alpha) f, \quad c_w(\alpha) = \int_{\mathbb{R}^n} h(x) \, dx, \quad (3.1)
\]

where the limit exists in the \( L^p \)-norm and in the a.e. sense. If, moreover, \( f \in C_0(\mathbb{R}^n) \), then the limit exists in the sup-norm.

**Proof.** We have \( t^{-\alpha}(I^\alpha w_t)(x) = t^{-\alpha}h(x/t) = h_t(x) \). Then

\[
I^\alpha f * t^{-\alpha} w_t = f * t^{-\alpha} I^\alpha w_t = f * h_t,
\]

which yields an approximate identity [30, p. 62]. The above application of Fubini’s theorem is permissible because \( (I^\alpha |f|) * |w| < \infty \) a.e. \( \Box \)

There are many ways to choose a reconstructing function \( w \). Usually \( w \) or \( I^\alpha w \) or both are expressed analytically in a pretty complicated way. As we shall see below, an advantage of the choice \( (1.3) \) is that both \( w \) and \( h = I^\alpha w \) are elementary and a constant \( c_w(\alpha) \) can be different from zero.

**Lemma 3.2.** Let \( w(x) = (\Delta)^m [(1 + |x|^2)^{m-(n+\alpha)/2}] \). If \( \alpha > 0 \), then \( w \in L^1(\mathbb{R}^n) \). If, moreover, \( \alpha < \min(n, 2m) \) and \( h = I^\alpha w \), then

\[
h(x) = a_{\alpha,m} (1+|x|^2)^{(\alpha-n)/2-m}, \quad a_{\alpha,m} = 2^{2m-\alpha} \frac{\Gamma((n+2m-\alpha)/2)}{\Gamma((n+\alpha-2m)/2)},
\]

so that \( h \in L^1(\mathbb{R}^n) \) and

\[
c_{\alpha,m} \equiv \int_{\mathbb{R}^n} h(x) \, dx = \frac{2^{2m-\alpha}\pi^{n/2}\Gamma(m-\alpha/2)}{\Gamma((n+\alpha-2m)/2)} = \gamma_n(2m-\alpha);
\]

cf. the normalizing constant in \( (1.4) \). If \( \alpha \) is not an integer, then \( c_{\alpha,m} \neq 0 \) for any \( m > \alpha/2 \). If \( \alpha \in \{1,2,\ldots,n-1\} \), then \( c_{\alpha,m} \neq 0 \) provided, e.g., that \( m = [(\alpha+2)/2] \).

**Proof.** The first statement can be easily checked by differentiation. For instance, we can write \( \Delta \) in polar coordinates to get

\[
w(x) = (L^m \psi)(|x|^2), \quad L = 4r^{1-n/2} \frac{d}{dr}, \quad \psi(r) = (1+r)^{(n+\alpha)/2}.
\]

This gives \( w(x) = O((1 + |x|^2)^{-(\alpha+n)/2}) \in L^1(\mathbb{R}^n) \). To prove the second statement we invoke the Bessel-McDonald kernel

\[
G_\alpha(\xi) = \lambda_\alpha \frac{K_{(n-\alpha)/2}(|\xi|)}{|\xi|^{(n-\alpha)/2}}, \quad \lambda_\alpha = \frac{2^{1-(\alpha+n)/2}}{\pi^{n/2}\Gamma(\alpha/2)}, \quad (3.2)
\]
where $K_\nu(\cdot)$ denotes the modified Bessel function (or the McDonald function) of order $\nu$ with the property

$$K_\nu(z) = K_{-\nu}(z); \quad (3.3)$$

see, e.g., [31]. It is known that

$$G_\alpha(\xi) \leq c_\alpha \begin{cases} 
|\xi|^{(\alpha-n-1)/2}e^{-|\xi|} & \text{if } |\xi| > 1, \\
|\xi|^\alpha & \text{if } |\xi| < 1, \alpha < n, \\
1 & \text{if } |\xi| < 1, \alpha > n, \\
1 + \log(1/|\xi|) & \text{if } |\xi| < 1, \alpha = n, 
\end{cases} \quad (3.4)$$

where $c_\alpha$ is a continuous function of $\alpha$; see, e.g., [21, p. 257], [16, p. 285]. Furthermore, for any $\alpha > 0$ the Fourier transform of $(1 + |x|^2)^{-\alpha/2}$ in the sense of distributions is computed by the formula

$$( (1 + |x|^2)^{-\alpha/2}, \phi(x) ) = (2\pi)^{-n}(G_\alpha(\xi), \hat{\phi}(\xi)), \quad \phi \in S(\mathbb{R}^n), \quad (3.5)$$

[16, Section 8.1]. Let us prove that

$$( I^\alpha w, \phi ) = a_{\alpha,m} \left( (1 + |x|^2)^{(\alpha-2m-\alpha)/2}, \phi(x) \right) \quad (3.6)$$

for any test function $\phi$ in the space $\Phi(\mathbb{R}^n)$. Once this is done, the required pointwise equality will follow owing to Proposition 2.1. We have

$$( I^\alpha w, \phi ) = (w, I^\alpha \phi) = ((1 + |x|^2)^{(2m-n-\alpha)/2}, (-\Delta)^m (I^\alpha \phi)(x))$$

$$= (2\pi)^{-n}(G_{n+\alpha-2m}(\xi), |\xi|^{2m-\alpha}\hat{\phi}(\xi))$$

$$= (2\pi)^{-n} \lambda_{n+\alpha-2m} \left( \frac{K_{m-\alpha/2}(|\xi|)}{|\xi|^{m-\alpha/2}}, |\xi|^{2m-\alpha}\hat{\phi}(\xi) \right).$$

Using (3.3) and (3.5), we continue:

$$( I^\alpha w, \phi ) = \frac{(2\pi)^{-n} \lambda_{n+\alpha-2m}}{\lambda_{2m+n-\alpha}} (G_{2m+n-\alpha}(\xi), \hat{\phi}(\xi))$$

$$= a_{\alpha,m} \left( (1 + |x|^2)^{(\alpha-2m-\alpha)/2}, \phi(x) \right).$$

Evaluation of $c_{\alpha,m}$ is straightforward. To complete the proof, we note that $c_{\alpha,m} \neq 0$ if $(n + \alpha - 2m)/2 \neq 0, -1, -2, \ldots$. The latter is guaranteed under the afore-mentioned choice of $m$. □

**Proof of Theorem A.** The first statement follows immediately from Lemmas 3.1 and 3.2. To prove the second statement, we first note that $\tilde{w} = -\Delta w$ belongs to $L^1$. Hence, the convolution $I^\alpha f * t^{-\alpha} \tilde{w}_t$ is well-defined in the Lebesgue sense and can be written as $f * \hat{h}_t$ with
\( \hat{h} = I^\alpha \tilde{w} = I^\alpha (-\Delta) w. \) Furthermore, for any test function \( \phi \in \Phi(\mathbb{R}^n), \)

\[
(\hat{h}, \phi) = (I^\alpha (-\Delta) w, \phi) = -(I^\alpha w, \Delta \phi) = -a_{\alpha,m} (\Delta [(1 + |x|^{2})^{(\alpha-n)/2-m}], \phi).
\]

Since \( \Delta [(1 + |x|^2)^{(\alpha-n)/2-m}] = O((1 + |x|^2)^{(\alpha-n)/2-m-1}) \in L^1, \) then, by Proposition 2.1, we have a pointwise equality

\[
\hat{h}(x) = -a_{\alpha,m} \Delta [(1 + |x|^2)^{(\alpha-n)/2-m}].
\]

Denote

\[
J_\varepsilon = \int_\varepsilon^\infty \frac{I^\alpha f * \tilde{w}_t}{t^{1+\alpha}} \, dt, \quad \varepsilon > 0.
\]

Then

\[
J_\varepsilon = \int_\varepsilon^\infty \frac{f * \tilde{h}_t}{t} \, dt = f * \psi_\varepsilon, \quad \psi_\varepsilon(x) = \int_\varepsilon^\infty \frac{\tilde{h}_t(x)}{t} \, dt.
\]

Setting \( \tilde{h}(x) = \tilde{h}_0(|x|^2), \) we get

\[
\psi_\varepsilon(x) = \int_\varepsilon^\infty \frac{\tilde{h}_0(|x|^2/t^2)}{t^{n+1}} \, dt = \frac{1}{2 |x|^n} \int_0^{\varepsilon x^2} \tilde{h}_0(r) \, r^{n/2-1} \, dr.
\]

This is a scaled version of the function \( \psi(x) = 2^{-1}|x|^{-n} \int_0^{|x|^2} \tilde{h}_0(r) \, r^{n/2-1} \, dr. \) Observing that

\[
\tilde{h}_0(r) = -a_{\alpha,m} L [(1 + r)^{(\alpha-n)/2-m}], \quad L = 4r^{1-n/2} \frac{d}{dr} r^{n/2} \frac{d}{dr},
\]

we have

\[
\psi(x) = - \frac{2 a_{\alpha,m}}{|x|^n} \int_0^{|x|^2} \frac{d}{dr} r^{n/2} \frac{d}{dr} [(1 + r)^{(\alpha-n)/2-m}] \, dr
\]

\[
= - \frac{2 a_{\alpha,m}}{|x|^n} \left( r^{n/2} \frac{d}{dr} [(1 + r)^{(\alpha-n)/2-m}] \right)_{r=|x|^2}
\]

\[
= -a_{\alpha,m} (\alpha - n - 2m) \left( 1 + |x|^2 \right)^{(\alpha-n)/2-m-1}.
\]

This gives \( \lim_{\varepsilon \to 0} J_\varepsilon = d_{\alpha,m} f(x), \) where

\[
d_{\alpha,m} = \int_{\mathbb{R}^n} \psi(x) \, dx = \frac{2^{2m-\alpha+1} \pi^{n/2} \Gamma(m+1-\alpha/2)}{\Gamma((n+\alpha-2m)/2)} = (2m-\alpha) \gamma_n(2m-\alpha),
\]

and the limit is understood in the required sense.
4. **Inversion of the Radon-John Transform**

We recall basic definitions. More information can be found in [5, 7, 23]. Let $G_{n,k}$ and $G_{n,k}$ be the affine Grassmann manifold of all non-oriented $k$-dimensional planes $\tau$ in $\mathbb{R}^n$ and the ordinary Grassmann manifold of $k$-dimensional linear subspaces $\zeta$ of $\mathbb{R}^n$, respectively. Each $k$-plane $\tau \in G_{n,k}$ is parameterized as $\tau = (\zeta, u)$, where $\zeta \in G_{n,k}$ and $u \in \zeta^\perp$ (the orthogonal complement of $\zeta$ in $\mathbb{R}^n$). The $k$-plane Radon-John transform of a function $f$ on $\mathbb{R}^n$ is defined by

$$(R_k f)(\tau) \equiv (R_k f)(\zeta, u) = \int_\zeta f(y + u) \, dy,$$

(4.1)

where $dy$ is the induced Lebesgue measure on the subspace $\zeta \in G_{n,k}$. This transform assigns to a function $f$ a collection of integrals of $f$ over all $k$-planes in $\mathbb{R}^n$. The corresponding dual transform of a function $\varphi$ on $G_{n,k}$ is defined as the mean value of $\varphi(\tau)$ over all $k$-planes $\tau$ through $x \in \mathbb{R}^n$:

$$(R_k^* \varphi)(x) = \int_{O(n)} \varphi(\sigma \zeta_0 + x) \, d\sigma, \quad x \in \mathbb{R}^n.$$  

(4.2)

Here $\zeta_0 \in G_{n,k}$ is an arbitrary fixed $k$-plane through the origin. If $f \in L^p(\mathbb{R}^n)$, then $R_k f$ is finite a.e. on $G_{n,k}$ if and only if $1 \leq p < n/k$ [29, 23].

A variety of inversion procedures are known for $R_k f$; see [5, 7, 22, 23]. One of the most important algorithms relies on the Fuglede formula [4],

$$R_k^* R_k f = d_{k,n} I^k f, \quad d_{k,n} = (2\pi)^k \sigma_{n-k-1}/\sigma_{n-1},$$

(4.3)

with the Riesz potential $I^k f$ on the right-hand side. Hence, Theorem A yields the following result. We denote

$$w(x) = (-\Delta)^m[(1+|x|^2)^{m-(n+k)/2}], \quad m = [(k+2)/2], \quad \tilde{w}(x) = (-\Delta w)(x);$$

$$\lambda_k = 4^m \pi^{(n+k)/2} \Gamma(n/2) \Gamma(m-k/2) \Gamma((n-k)/2) \Gamma((n+k)/2-m), \quad \delta_k = (2m-k) \lambda_k.$$  

**Theorem B.** If $\varphi = R_k f$, $f \in L^p(\mathbb{R}^n)$, $1 \leq p < n/k$, then

$$\lambda_k f = \lim_{t \to 0} (R_k^* \varphi * t^{-k} w_t), \quad \delta_k f = \int_0^\infty \frac{R_k^* \varphi * \tilde{w}_t}{t^{1+k}} \, dt,$$

(4.4)

where $\int_0^\infty (\ldots) = \lim_{\epsilon \to 0} \int_0^\infty (\ldots)$. The limit in both formulae exists in the $L^p$-norm and in the a.e. sense. If, moreover, $f \in C_0(\mathbb{R}^n)$, then it exists in the sup-norm.
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