The amplitude of the de Broglie Gravitational Waves

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We calculate the amplitude of the de Broglie gravitational waves using the standard Einstein General Relativity. We find that these waves disappear in the limit \( \hbar \to 0 \) and when their source has a large mass and volume. From the experimental point of view, the knowledge of the amplitude allows to estimate the magnitude of the effect of the wave on a sphere of test particles. We propose also to measure a very special shift angle that does not change with time.

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1. Introduction

In the standard interpretation of Quantum Mechanics the wave function is the solution of the Schrödinger equation and plays the role of a mathematical tool in calculating the probability for finding the particle in a given volume of space. In this framework the normalization constant of the wave function is determined by the condition

\[
\int_V |\psi|^2 \, dV = 1.
\]

On the contrary, de Broglie interpreted this wave \( \psi \) as a genuine field guiding the particle in spacetime. In 1924 he proposed \(^1\) the relation \( P^\nu = \hbar K^\nu \) between the classical conserved momentum \( P^\nu = mcu^\nu = (E/c, \vec{p}) \) of a free particle and the wave number \( K^\nu \equiv (\omega/c, \vec{k}) \) of the associated plane wave. Of course the normalization of four velocity \( u^\mu u_\mu = 1 \) implies \( P^\nu P_\nu = m^2 c^2 \) where \( m \) is the rest mass of the particle and

\[
u = \left( \frac{1}{\sqrt{1 - (v^2/c^2)}} - \frac{v}{c \sqrt{1 - (v^2/c^2)}} \right)
\]

for a particle moving along the x - axis with a constant velocity \( v \).

Generalizing this concept, in 1927, de Broglie formulated "the theory of the double solution" \(^3\) in which the particles are accompanied in spacetime by a real "pilot wave"

\[
\psi = R(t, x, y, z) \exp \left( \frac{i}{\hbar} S(t, x, y, z) \right) = R(t, x, y, z) \exp \left( \frac{i}{\hbar} P_\mu x^\mu \right)
\]

1
in such a way that the guiding formula: \( \partial \mu S = -P^\mu \) holds. Then this idea was improved in 1952 by Bohm’s hidden variable theory \cite{4,5,6} and by Vigier and his co-workers in the Stochastic Interpretation of Quantum Mechanics \cite{8,9,10} that is now an alternative to the standard point of view. Following this line of thinking, if we want to believe in the reality of de Broglie waves, we must give an answer to Bell’s famous question: “What is it that <<waves>> in wave mechanics?” \cite{11} specifying which genuine field is responsible for the de Broglie wave and simultaneously determining the amplitude \( R \) of these waves somehow.

We stress that in the whole paper we use the word "amplitude" in the framework of classical physics as "the half of a total oscillation between a maximum and a minimum of the wave” and not with the meaning of "probability amplitude" related in quantum mechanics to inner products between state vectors.

The hope is that one can find an amplitude that will vanish in the limit \( \hbar \to 0 \) and will be very small for macroscopic bodies with a large mass and extension. A recent proposal consists in considering the de Broglie waves of one-particle quantum mechanics as a special kind of Gravitational Waves. A solution of the linearized Einstein equations found in 1998 by Feoli and Scarpetta \cite{12} has in fact the right features to be interpreted as the wave associated to a quantum particle because the guiding formula is satisfied. A similar solution and interpretation was also found in 2005 by Chang \cite{13} for a special kind of Electromagnetic Waves.

Starting from a metric tensor \( g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \) (where \( g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \)) and neglecting any terms of order \( |h_{\mu\nu}|^2 \), the linearized Einstein field equations in vacuo are

\[
\partial_\alpha \partial^\alpha h_{\mu\nu} = 0 \quad (4)
\]

if the de Donder-Lanczos conditions

\[
\partial_\mu \left( h^\mu_\nu - \frac{1}{2} \delta^\mu_\nu h \right) = 0 \quad (5)
\]

are satisfied. The classical solution is a plane wave with \( K_\mu K^\nu = 0 \). On the contrary, we found \cite{12} a solution in the form of a wave packet of cylindrical symmetry around the propagation direction:

\[
h_{\mu\nu} = e_{\mu\nu}(K_0, K_1)A J_0(\sqrt{y^2 + z^2}/\lambda) \cos \left( K^0 ct - K^1 x \right) \quad (6)
\]

where \( A \) is an arbitrary constant that in this paper we want to calculate, \( J_0 \) is the Bessel function of the 0th-order, \( e_{\mu\nu} \) is the polarization tensor and \( K^\mu \), the constant wave number, is such that \( K^\mu K_\mu = \lambda^{-2} \), corresponding to a gravitational wave propagating along the \( x \)-axis with a phase velocity \( V_{ph} = c / (1 + (\lambda k)^{-2}) \). If we put \( \lambda = \hbar / mc \), this solution can be interpreted as the de Broglie wave associated with a particle of rest mass \( m \) moving with constant velocity \( v \) along the \( x \)-axis because the phase has the right behavior \( K^0 ct - K^1 x = (P^\mu x^\nu) / \hbar = (E t - px) / \hbar \).

As in the standard Quantum Mechanics, we can add waves with slightly different velocities around a central value \( v_0 \) to obtain \cite{14,15} a de Broglie wave packet with a group velocity \( V_G = v_0 \) such that \( V_{ph} \cdot V_G = c^2 \).
The amplitude of the de Broglie Gravitational Waves

From (5) we obtained two relations among the nonvanishing components of the polarization tensor that turn out to be depending on the wave number:

\[ e_{00} = e_{11} = \frac{e_{10}}{2} \left( \frac{K_1}{K_0} + \frac{K_0}{K_1} \right) \]  
\[ e_{22} = e_{33} = \frac{e_{10}}{2} \left( \frac{K_1}{K_0} - \frac{K_0}{K_1} \right). \]

(7) \hspace{1cm} (8)

An interesting limit case is when \( K_1 = 0 \), i.e. when we have an oscillation in time but not a propagation of the wave and we are in the rest frame of the associated particle. In that case we found \( e_{10} = 0 \) (hence \( e_{10} \) is proportional to \( K_1 \), that is to the velocity of the associated particle \( v \)) and \( e_{22} = e_{33} = -e_{00} = -e_{11} \). As in this limit, the only difference among the diagonal components of the constant polarization tensor is the sign, we can write \( |e_{\mu\mu}| = q \) where \( q \) is a constant to be determined. But, without changing the solution (6), we can also put \( |e_{\mu\mu}| = 1 \) and redefine \( A \) as \( A' = qA \). In this way we have at least fixed the magnitude of the diagonal components of \( e_{\mu\nu} \) in the rest frame of the associated particle.

The properties of the de Broglie Gravitational waves were studied in a series of interesting papers \[12,14,15,16\], but the normalization constant \( A \) of these waves has always been left undetermined. In this letter we propose a way to calculate this constant using the standard methods of Einstein General Relativity (section 2). Once the amplitude is completely known, a test to verify the existence of this kind of Gravitational Waves becomes possible. When the wave meets a sphere of test particles, one can hope to measure not only the magnitude of the shift but also its direction. To this aim we will calculate (section 3) a very special shift angle that remains constant in time.

2. Determination of the amplitude

Following Misner, Thorne and Wheeler\[17\], given the linearized field equations (4) and assuming the gauge conditions (5), we can calculate what they call ”the effective smeared-out stress–energy of gravitational waves”, starting from an energy-momentum tensor in the form:

\[ T_{\mu\nu}^{(GW)} = \frac{e^4}{32\pi G} \left( \langle \partial_\mu h^{\alpha\beta} \partial_\nu h_{\alpha\beta} - \frac{1}{2} \partial_\mu h \partial_\nu h \rangle \right) \]  

where \( \langle .... \rangle \) denotes an average over several periods in time and wavelengths in space. This tensor\[17\] contributes to the large-scale background curvature (which linearized theory ignores) just as any other stress–energy does:

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} \left( T_{\mu\nu}^{(GW)} + T_{\mu\nu}^{(\text{matter})} + T_{\mu\nu}^{(\text{otherfields})} \right). \]

(9) \hspace{1cm} (10)

In our simple case there are no ”other fields” and no other ”matter” except the particle source of our gravitational field (6) already included in the \( T_{\mu\nu}^{(GW)} \). If this particle, associated with the de Broglie wave, is placed in a small volume \( V \) around
4 Antonio Feoli

$r = 0$, i.e. $V << \lambda^3$, the tensor $T_{\mu\nu}^{(GW)}$ must coincide in the limit $r \to 0$ with $T_{\mu\nu}^{(\text{matter})} = \rho u_\mu u_\nu$ so the Einstein equations including terms of the order $|h_{\mu\nu}|^2$ can be written:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}^{(GW)} \to \frac{8\pi G}{c^4} T_{\mu\nu}^{(\text{matter})} \text{ for } r \to 0.$$  

(11)

This really happens constructing our energy–momentum tensor from the gravitational field (6). Actually, if we write the solution (6), that (fixing $\lambda = \hbar/mc$)

$$h_{\alpha\beta} = e_{\alpha\beta}\phi$$

(12)

where

$$\phi = AJ_0 \left( \sqrt{y^2 + z^2}/\lambda \right) \cos \left[ (u_0 x^0 + u_1 x^1) / \lambda \right]$$

(13)

and $u_\mu$ is given by the equation (2), we obtain from (9) the nonvanishing components of the tensor

$$T_{\mu\nu}^{(GW)} = \left( \frac{c^4 (e_{22})^2}{16\pi G} \right) \partial_\mu \phi \partial_\nu \phi$$

(14)

that are the following:

$$T_{00}^{(GW)} = \frac{c^4 (e_{22})^2 A^2 J_0^2}{32\pi G\lambda^2} u_0 u_0$$

(15)

$$T_{11}^{(GW)} = \frac{c^4 (e_{22})^2 A^2 J_0^2}{32\pi G\lambda^2} u_1 u_1$$

(16)

$$T_{01}^{(GW)} = \frac{c^4 (e_{22})^2 A^2 J_0^2}{32\pi G\lambda^2} u_0 u_1$$

(17)

$$T_{22}^{(GW)} = \frac{c^4 (e_{22})^2 A^2}{32\pi G\lambda^2} \frac{y^2 J_1^2}{(z^2 + y^2)}$$

(18)

$$T_{33}^{(GW)} = \frac{c^4 (e_{22})^2 A^2}{32\pi G\lambda^2} \frac{z^2 J_1^2}{(z^2 + y^2)}$$

(19)

$$T_{23}^{(GW)} = \frac{c^4 (e_{22})^2 A^2}{32\pi G\lambda^2} \frac{yz J_1^2}{(z^2 + y^2)}$$

(20)

In the limit $r \to 0$ we have $J_0 \to 1$ and $J_1 \to 0$ so this tensor reduces to the desired form $T_{\mu\nu} = \rho u_\mu u_\nu$ where

$$\rho = \frac{c^4 A^2 J_0^2(0)(e_{22})^2}{32\pi G\lambda^2}$$

(21)

that can be interpreted as the energy density of a dust–like source of our gravitational field.
In the rest frame of the particle \( \rho = mc^2/V \) and, as discussed in the previous section, \( |e_{\mu\mu}| = 1 \) hence \( (e_{22})^2 = 1 \) and
\[
\frac{mc^2}{V} = \frac{c^4 A^2}{32\pi G \lambda^2}
\]
where \( V \) is the proper volume of the particle. Finally we find:
\[
A = \frac{4\lambda}{c} \sqrt{\frac{2\pi G m}{V}}
\]
and, fixing \( \lambda = \hbar/mc \), we obtain:
\[
A = \frac{4\hbar}{c^2} \sqrt{\frac{2\pi G}{mV}}.
\]
From \( |h_{\mu\nu}|^2 \ll 1 \) we have that our model works if
\[
mV \gg \frac{32\pi G \hbar^2}{c^4} = 9.2 \times 10^{-111} \text{ kg m}^3.
\]
For example, for an electron the dust-like approximation of the energy-momentum tensor works if \( V_e = 4\pi R_e^3/3 \ll \lambda^3 \) \((3.86 \times 10^{-13} m)^3 \) and the linear approximation of Einstein equations does if \( V_e \gg 10^{-80} m^3 \). Of course the two relations are satisfied both using the ”classical radius” of the electron \( R_e = 2.8 \times 10^{-15} m \), and with the lower experimental value \( R_e \simeq 10^{-22} m \) proposed by Dehmelt\[18\]. In this last case the constant (24) would become \( A \simeq 4.9 \times 10^{-8} \) (more precisely the Dehmelt value computed from \( R_e = \lambda_e |g - 2| \) is \( R_e = 4.3 \times 10^{-23} m \) and the corresponding constant becomes \( A = 1.7 \times 10^{-7} \)) and it would be more convenient to measure its effects rather than the ones produced by the standard gravitational waves.

Note that, in general, due to the cylindrical symmetry of the solution (12), the constraint \( V = \pi L (y^2 + z^2) \ll \lambda^3 \) can be a little bit relaxed. In order to obtain the dust-like tensor, it is enough that the transverse radius of the source \( \sqrt{y^2 + z^2} \ll \lambda \), while the length \( L \) in the \( x \)-direction of the cylindrical tube can be such that \( L > \lambda \). In that case, \( \sqrt{y^2 + z^2} \ll L \) and our particle becomes similar to a string.

Furthermore we stress that the constant \( A \), that in the rest frame fixes the magnitude of the amplitude \( R = AJ_0 \), has just the right expected behavior because it vanishes in the limit \( \hbar \to 0 \) and is negligible when \( m \) and \( V \) are very large. In the standard quantum mechanics a fixed scale parameter is missing, so the border between classical bodies and quantum particles is not clear. In our approach, the result (24) allows at least to say that the linear approximation works when (25) holds and the quantum effects disappear just when \( \hbar \to 0 \) or when the product \( mV \) is very large compared to a fixed parameter: \( 9.2 \times 10^{-111} \text{ kg m}^3 \) that specifies the realm of quantum physics.

3. Experimental test: a very special ”shift angle”

In order to propose an experimental test we must refer to the tidal acceleration of geodesic deviation between two particles that does not depend on the particular
gauge chosen. Considering a central particle $m_0$ in the origin of the reference frame, we could estimate the shift of any peripheral particle at distance $\ell$ from the origin. When the de Broglie gravitational wave passes, moving along the x-axis, it deforms what was a sphere of radius $\ell$ of test particles, as measured in the proper frame of the central particle. We will calculate explicitly only the shift of the three particles placed on the three axes at the position $\ell$ and the opposite will occur for the three particles at $-\ell$. In some previous papers we have already calculated the tidal acceleration between two test particles using the relation:

$$\frac{D^2 \eta^\mu}{c^2 d\tau^2} = -R^\mu_{\nu\tau\delta} \beta^\nu \beta^\delta \eta^\gamma$$

(26)

where $\tau$ is the proper time along the worldline of the two masses, $\beta^\mu$ is their four-velocity and $\eta^\mu$ their relative distance, while the velocity of the source is $u^\mu$ in (2). We can use two approximations: the velocity of the test particles is very small compared with the speed of light so $\beta^\mu \simeq (1, 0, 0, 0)$ and the relative displacement $\delta^\mu$ between two of these masses caused by the wave is very small. So, if the initial position of the first particle $m_0$ is at the origin and of the second one at $\xi^\mu = (0, x, y, z)$, we have $\eta^\mu = (0, x + \delta^1, y + \delta^2, z + \delta^3)$ and we can consider $x^i \gg \delta^i$. In this framework the acceleration of geodesic deviation is

$$\ddot{\eta}^k \simeq -c^2 R^{k}_{0j0} \xi^j$$

(27)

where a dot indicates a differentiation with respect to time $t$ and $i, j = 1, 2, 3$.

Starting from (6) we calculate:

a) The relative distance between the mass $m_0$ at the origin and the mass $m_1$ initially at $\xi^1 = (\ell, 0, 0)$

$$\eta_1 = \delta_1 + \ell = \frac{A\ell}{2K_0^2} J_0(0) \left[ (K_0^2 + K_1^2)e_{00} - 2K_0K_1 e_{01} \right] \cos(K_0ct + K_1 \ell) + \ell$$

(28)

b) The relative distance between $m_0$ and $m_2$ initially at $\xi^2 = (0, \ell, 0)$

$$\eta_2 = \delta_2 + \ell = \frac{A\ell}{2K_0^2} \left( K_0 e_{10} - K_1 e_{00} \right) J_1(\ell/\lambda) \sin(K_0ct)$$

(29)

$$\eta_3 = 0$$

(30)

c) The relative distance between $m_0$ and $m_3$ initially at $\xi^3 = (0, 0, \ell)$

$$(\eta_1)_m = (\eta_2)_m = (\eta_3)_m = 0$$

(31)

A remarkable result is the existence of a longitudinal displacement $\delta_1$. So the sphere of particles surrounding the central mass $m_0$ is transformed into a triaxial ellipsoid, as we showed in the figures of a previous paper. This effect is absent in
the standard transverse gravitational waves, while it was predicted by Grishchuk and Sazhin for their standing gravitational waves produced by an electromagnetic generator. The determination of the amplitude in the previous section allows to estimate the magnitude of the shift of the test particles and opens the possibility to perform an experimental test. Furthermore it would be useful to measure also a quantity that depends neither on the constant $A$ (and hence on the way used to fix it), nor on the polarization tensor. To this aim we can simply compute the direction of the shift, for example, of $m_2$ and the resulting angle $\alpha$ is such that

$$\frac{\delta_1}{\delta_2} = \tan \alpha = \frac{4J_1K_1}{\lambda[2K_0^2J_0 + (K_0^2 + K_1^2)(J_2 - J_0)]} \tan(K_0ct)$$

and changes quickly with time, so it cannot be useful for a test. In the limit $K_1 \rightarrow 0$ (the source’s velocity $v \rightarrow 0$) the shift occurs in the direction of the line joining the two particles just like the standard gravitational waves. But if we consider the case:

d) The shift between $m_0$ and $m_4$ at $\xi^2 = (\ell, \ell, 0)$

![Fig. 1. The "shift angle" $\theta$ between the direction of the shift and the x-axis vs the initial position $\ell$ of the test particle $m_4$.](image)

we can compute, in the limit $K_1 \rightarrow 0$, the angle $\theta$ in the $xy$-plane between the direction of the shift and the x-axis. It depends on the initial position $\ell$ of the test particle:

$$\frac{\delta_2}{\delta_1} = \tan \theta = -\frac{1}{2} \left[ 1 + \frac{J_2(\ell/\lambda)}{J_0(\ell/\lambda)} \right]$$

The dependence of the shift angle $\theta$ only on $\ell$ is a typical feature of these waves and it is shown in Fig. 1. Of course the angle with the line joining the two particles is $\theta = \pi/4$ and it is different from what happens with the standard gravitational waves. Due to the fact that $\theta$ depends neither on the time, nor on $A$, it could be used as a constant measurable quantity in an experimental test even better than the shift depending on $A$.

4. Conclusions

In some previous papers, we studied a particular solution (6) of the linearized Einstein equations that can play the same role of the de Broglie wave of quantum
mechanics, but we left undetermined an arbitrary constant $A$. In this letter, we have finally found a method to fix the constant, and hence the amplitude of this special gravitational wave, starting from the estimation of the corresponding energy-momentum tensor. Now we can write down the complete solution in the form:

$$h_{\mu\nu} = e_{\mu\nu}(K_0, K_1) \frac{4\hbar}{c^2} \sqrt{\frac{2\pi G}{mV}} J_o \left( \frac{mc\sqrt{y^2 + z^2}}{\hbar} \right) \cos \left( \frac{Et - px}{\hbar} \right)$$  (35)

The result, not $a$ priori given for granted, is very interesting because the wave associated to the quantum particle disappears in the limit $\hbar \to 0$ and when the source has a large mass or volume just as it was expected. Furthermore, the knowledge of the magnitude of the wave opens the possibility to see experimental effects that in some cases could be more evident with respect to the standard gravitational waves. For example, a longitudinal shift or a direction of the shift not along the line joining the two test particles, are peculiar effects caused by our waves. Finally, a measurement of a particular "shift angle" that depends only on the relative initial position of the two test particles has been proposed.

References
1. L. de Broglie, *Philosophical Magazine* 47 446 (1924).
2. L. de Broglie, Ph.D thesis on *Recherches sur la theorie des quanta* also published in *Annales de Physique* 3 22 (1925).
3. L. de Broglie, *Jour. de Phys.* 8 225 (1927).
4. D. Bohm, *Phys. Rev.* 85 166 (1952).
5. D. Bohm *Phys. Rev.* 85 180 (1952).
6. L. de Broglie, *Nonlinear wave mechanics* (Elsevier, Amsterdam, 1960).
7. J. P. Vigier, *Found. Phys.* 21 125 (1991).
8. D. Bohm and J. P. Vigier, *Phys. Rev.* 96 208 (1954).
9. E. Nelson, *Phys. Rev.* 150 1079 (1966).
10. J. P. Vigier, *Lett. Nuovo Cim.* 24 258 and 265 (1979).
11. J. S. Bell, *Speakable and unspeakable in quantum mechanics* (Cambridge University Press, Cambridge, 1993) p.187
12. A. Feoli and G. Scarpetta, *Found. Phys. Lett.* 11 395 (1998).
13. D. C. Chang, *On the wave nature of matter* ArXiv: physics/0505010 Preprint 2005.
14. A. Feoli, *Europhys. Lett.* 58 169 (2002).
15. A. Feoli and S. Valluri, *Int. Jour. Mod. Phys. D* 13 907 (2004).
16. A. Feoli and S. Valluri in *Proceedings of 16th SIGRAV Conference on General Relativity and Gravitational Physics*, eds. G. Esposito, G. Lambiase, G. Marmo, G. Scarpetta and G. Vila, AIP Conference Proceedings vol. 751, issue 1, (2005) p.179
17. C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, (W. H. Freeman, New York, 1973) par. 35.7.
18. H. Dehmelt, *Rev. Mod. Phys* 62 525 (1990).
19. A. Einstein and N. Rosen, *J. Franklin Inst.* 223 43 (1937).
20. L. P. Grishchuk and M. V. Sazhin, *Sov. Phys. Dokl.* 20 486 (1976).