Gravitational Collapse of Self-Similar Perfect Fluid in 2 + 1 Gravity

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Perfect fluid with kinematic self-similarity is studied in 2 + 1 dimensional spacetimes with circular symmetry, and various exact solutions to the Einstein field equations are given. In particular, these include all the solutions of dust and stiff perfect fluid with self-similarity of the first kind, and all the solutions of perfect fluid with a linear equation of state and self-similarity of the zeroth or second kind. It is found that some of these solutions represent gravitational collapse, and the final state of the collapse can be either black holes or naked singularities.

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I. INTRODUCTION

One of the most outstanding problems in gravitation theory is the final state of a collapsing massive star, after it has exhausted its nuclear fuel. In spite of numerous efforts over the last three decades or so, because of the (mathematical) complexity of the problem our understanding is still limited to several conjectures, such as, the cosmic censorship conjecture [1] and the hoop conjecture [2]. To the former, many counter-examples have been found [3], although it is still not clear whether those particular examples are generic. To the latter, no counter-example has been found so far in four-dimensional spacetimes, but it has been shown recently that this is no longer true in five dimensions [4].

Lately, Brandt et al. have studied gravitational collapse of perfect fluid with kinematic self-similarities in four-dimensional spacetimes [5], a subject that has been recently studied intensively (for example, see [6] and references therein.). In this paper, we shall investigate the same problem but in 2+1 gravity [7]. The main motivation of such a study comes from recent investigation of critical collapse of a scalar field in 3D gravity [8–11]. It was found that in the 3D case the corresponding problem is considerably simplified and can be studied analytically. In particular, Garfinkle first found a class, say, $S[n]$, of exact solutions to Einstein-massless-scalar field equations, and then Garfinkle and Gundlach studied their linear perturbations and found that the solution with $n = 2$ has only one unstable mode [9]. By definition this is a critical solution, and the corresponding exponent, $\gamma$, of the black hole mass,

$$M_{BH} \propto (p - p^*)^{\gamma},$$

is $\gamma = |k_1|^{-1} = 4/3$, where $k_1$ denotes the unstable mode. Although the exponent $\gamma$ is close to the one found numerically by Pretorius and Choptuik, which is $\gamma \sim 1.2 \pm 0.02$ (but not the one of Husain and Olivier, $\gamma \sim 0.81$), this solution is quite different from the numerical one [8]. Using different boundary conditions 1, Hirschmann, Wang and Wu found that the solution with $n = 4$ has only one unstable mode [11]. As first noted by Garfinkle [9], this $n = 4$ solution matches extremely well with the numerical critical solution found by Pretorius and Choptuik [8]. However, the corresponding exponent $\gamma$ now is given by $\gamma = |k_1|^{-1} = 4$, which is significantly different from the numerical ones. In this paper we do not intend to solve these problems, but study some analytical solutions that represent gravitational collapse of perfect fluid in 2 + 1 gravity.

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1 Comparing the two sets of boundary conditions one will find that the only difference between them is that, in addition to the ones imposed by Garfinkle and Gundlach [9], Hirschmann, Wang and Wu [11] further required that no matter field should come out of the already formed black holes. This additional condition seems physically quite reasonable, and has been widely used in the studies of black hole perturbations [12].
It can be shown that the conception of kinematic self-similarities given by Carter and Henriksen [13] in four-dimensional spacetimes can be easily generalized to D-dimensional spacetimes with the metric

\[ ds^2 = l^2 \left\{ \gamma_{ab} \left(x^r \right) dx^a dx^b - s^2 \left(x^r \right) H_{ij} \left(x^r \right) dx^i dx^j \right\}, \tag{1.1} \]

where \( l \) is an unit constant with the dimension of length, so that all the coordinates \( x^\mu \) and metric coefficients \( \gamma_{ab} \) and \( H_{ij} \) are dimensionless. Here we use lowercase Latin indices, such as, \( a, b, c, ..., g \), to run from 0 to 1, lowercase Latin indices, such as, \( i, j, k, ..., \) to run from 2 to \( D - 1 \), and Greek indices, such as, \( \mu, \nu, ..., \) to run from 0 to \( D - 1 \). Clearly, the above metric is invariant under the coordinate transformations

\[ x^a = x'^a \left(x'^c \right), \quad x^i = x^i \left(x'^j \right). \tag{1.2} \]

On the other hand, the energy-momentum tensor (EMT) for a perfect fluid is given by

\[ T_{\mu\nu} = \left( \rho + p \right) u_\mu u_\nu - pg_{\mu\nu}, \tag{1.3} \]

where \( u_\mu \) denotes the velocity of the fluid, \( \rho \) and \( p \) are, respectively, its energy density and pressure. Using the coordinate transformations (1.2) we shall choose the coordinates such that,

\[ u_\mu = \left(g_{00} \right)^{1/2} \delta^0_\mu, \quad g_{01} = 0. \tag{1.4} \]

This implies that the coordinates are comoving with the perfect fluid. Then, metric (1.1) can be cast in the form,

\[ ds^2 = l^2 \left\{ e^{2\Phi(t,r)} dt^2 - e^{2\Psi(t,r)} dr^2 - r^2 S^2(t,r) H_{ij} \left(x^r \right) dx^i dx^j \right\}. \tag{1.5} \]

Following Carter and Henriksen, we define kinematic self-similarity by

\[ \mathcal{L}_\xi h_{\mu\nu} = 2h_{\mu\nu}, \quad \mathcal{L}_\xi u^\mu = -\alpha u^\mu, \tag{1.6} \]

where \( \mathcal{L}_\xi \) denotes the Lie differentiation along the vector field \( \xi^\mu \), \( \alpha \) is a dimensionless constant, and \( h_{\mu\nu} \) is the project operator, defined by \( h_{\mu\nu} \equiv g_{\mu\nu} - u_\mu u_\nu \). When \( \alpha = 0 \), the corresponding solutions are said to have self-similarity of the zeroth kind; when \( \alpha = 1 \), they are said to have self-similarity of the first kind (or homothetic self-similarity); and when \( \alpha \neq 0, 1 \), they are said to have self-similarity of the second kind.

Applying the above definition to metric (1.5), we find that the metric coefficients must take the form,

\[ \Phi(t, r) = \Phi(x), \quad \Psi(t, r) = \Psi(x), \quad S(t, r) = S(x), \tag{1.7} \]

where the self-similar variable \( x \) and the vector field \( \xi^\mu \) are given by

\[ \xi^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}, \quad x = \ln(r) - t, \quad (\alpha = 0), \tag{1.8} \]

for the zeroth kind, and

\[ \xi^\mu \frac{\partial}{\partial x^\mu} = \alpha t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r}, \quad x = \ln(r) - \frac{1}{\alpha} \ln(-t), \quad (\alpha \neq 0), \tag{1.9} \]

for the first (\( \alpha = 1 \)) and second (\( \alpha \neq 1 \)) kind, respectively.

As mentioned above, in this paper we shall study perfect fluid with kinematic self-similarity in 2 + 1-dimensional spacetimes with circular symmetry, for which metric (1.5) becomes

\[ ds^2 = l^2 \left\{ e^{2\Phi(x)} dt^2 - e^{2\Psi(x)} dr^2 - r^2 S^2(x) d\theta^2 \right\}, \tag{1.10} \]

where the hypersurfaces \( \theta = 0, 2\pi \) are identified. It should be noted that the above metric is of invariance under the following transformations,

\[ t = At, \quad r = Br, \quad g_{\mu\nu} = C^2 \tilde{g}_{\mu\nu}, \tag{1.11} \]
for self-similar solutions of the first and second kinds, and

\[ t = \tilde{t} + A, \quad r = B\bar{r}, \quad g_{\mu\nu} = C^2\bar{g}_{\mu\nu}, \]

(1.12)

for self-similar solutions of the zeroth kind, where \(A, B\) and \(C\) are arbitrary constants.

On the other hand, to have circular symmetry, some physical and geometrical conditions needed to be imposed \[14\]. In general this is not trivial. As a matter of fact, only when the symmetry axis is free of spacetime singularity, do we know how to impose these conditions. Since in this paper we are mainly interested in gravitational collapse, we shall assume that the axis is regular at the beginning of the collapse, so that we are sure that the singularity to be formed later on the axis is due to the collapse of the fluid. Following \[15\] we impose the following conditions:

(i) There must exist a symmetry axis, which can be expressed as

\[ X \equiv \left| \xi^\mu_{(\theta)} \xi^\nu_{(\theta)} g_{\mu\nu} \right| = |g_{\theta\theta}| \to 0, \]

(1.13)
as \(r \to 0^+\), where we have chosen the radial coordinate such that the axis is located at \(r = 0\), and \(\xi^\mu_{(\theta)}\) is the Killing vector with a close orbit, given by \(\xi^\alpha_{(\theta)} \partial_\alpha = \partial_\theta\).

(ii) The spacetime near the symmetry axis is locally flat, which can be written as \[16\]

\[ \frac{X_{,\alpha}X_{,\beta}g^{\alpha\beta}}{4X} \to -1, \]

(1.14)
as \(r \to 0^+\), where \((\ ),,_{\alpha} \equiv \partial(\ )/\partial x^\alpha\). Note that solutions failing to satisfy this condition are sometimes acceptable. For example, when the left-hand side of the above equation approaches a finite constant, the singularity at \(r = 0\) can be related to a point-like particle \[17\]. However, since here we are mainly interested in gravitational collapse, in this paper we shall assume that this condition holds strictly at the beginning of the collapse.

(iii) No closed timelike curves (CTC’s). In spacetimes with circular symmetry, CTC’s can be easily introduced. To ensure their absence we assume that

\[ g_{\theta\theta} = \xi^\mu_{(\theta)} \xi^\nu_{(\theta)} g_{\mu\nu} < 0, \]

(1.15)
holds in the whole spacetime.

In addition to these conditions, it is usually also required that the spacetime be asymptotically flat in the radial direction. However, since we consider solutions with self-similarity, this condition cannot be satisfied by such solutions, unless we restrict the validity of them only up to a maximal radius, say, \(r = r_0(t)\). Then, we need to join the solutions with others in the region \(r > r_0(t)\), which are asymptotically flat as \(r \to \infty\). In this paper, we shall not consider such a possibility, and simply assume that the self-similar solutions are valid in the whole spacetime.

It should be noted that the regularity conditions (1.13)-(1.15) are invariant under the transformations of Eq.(1.11) or Eq.(1.12). Using these transformations, we shall further assume that

\[ \Phi(t, 0) = 0, \]

(1.16)
that is, the timelike coordinate \(t\) measures the proper time on the axis.

Moreover, in the analytical studies of critical collapse, one is usually first to find some particular solutions by imposing certain symmetries, such as, homothetic self-similarity, and then study their perturbations, because by definition critical solutions have one and only one unstable mode. In this paper we shall be mainly concerned with the first question, and leave the study of linear perturbations to another occasion. In particular, the rest of the paper is organized as follows: Exact solutions of the Einstein field equations with self-similarity of the first, second, and zeroth kinds will be given and studied, respectively, in Secs. II, III, and IV, while in Sec. V, our main conclusions are presented. There are also two appendices \(A\) and \(B\), where in Appendix \(A\) the general expression of the Einstein tensor, among other things, for spacetimes with self-similarities of the zeroth, first and second kinds is given, while in Appendix \(B\) the apparent horizon is defined in terms of the self-similar variables.

**II. SELF-SIMILAR SOLUTIONS OF THE FIRST KIND**

In this section, we shall study solutions with self-similarity of the first kind. Substituting Eqs.(1.3), (1.4) and (A.14) with \(\alpha = 1\) into the Einstein field equations \(G_{\mu\nu} = \kappa T_{\mu\nu}\), where \(\kappa\) is the Einstein coupling constant, we find that
\[ y_x - (1 + y)(\Psi_x - y) - y\Phi_x = 0, \tag{2.1} \]
\[
\Phi_{xx} + \Phi_x(\Phi_x - \Psi_x - y - 2)
- e^{2(x + \Psi - \Phi)}[\Psi_{xx} - \Psi_x(\Phi_x - \Psi_x + y)] = 0, \tag{2.2}
\]
and
\[
\rho = \frac{y e^{-2\Psi}}{r^2} \left( \Psi_x e^{2(x + \Psi - \Phi)} - \Phi_x \right),
\]
\[
p = -\left( \frac{y + 1}{e^{-2\Psi}} \right) \left( \Psi_x e^{2(x + \Psi - \Phi)} - \Phi_x \right), \tag{2.3}
\]
where \( y \equiv S_{,x}/S \). Note that in writing the above expressions we had set \( \kappa = 1 \). In the rest of this paper we shall continuously choose units so that this is true. Clearly, to determine the metric coefficients completely, we need to impose the equation of state for the perfect fluid. In general, it takes the form [18],
\[
\rho = \rho(T, \Sigma), \quad p = p(T, \Sigma), \tag{2.4}
\]
where \( T \) is the temperature of the system and \( \Sigma \) the entropy. However, in some cases the system is weakly dependent on \( T \), so the equation of state can be written approximately as \( p = p(\rho) \). In the following we shall show that in the latter case the only equation of state that is consistent with the symmetry of homothetic self-similarity is the one given by
\[
p = k\rho, \tag{2.5}
\]
where \( k \) is an arbitrary constant. To show this, let us first write Eq. (2.3) in the form
\[
\rho = \frac{f(x)}{r^2}, \tag{2.6}
\]
\[
p = \frac{g(x)}{r^2}. \tag{2.7}
\]
Then, from Eq. (2.6) we find that
\[
x = x(r^2 \rho), \quad \frac{dx}{d(r^2 \rho)} = \frac{1}{f'(x)}, \tag{2.8}
\]
where a prime denotes the ordinary differentiation with respect to \( x \). Inserting Eq. (2.8) into Eq. (2.7) we find that
\[
p = \frac{g \left( x \left( r^2 \rho \right) \right)}{r^2}, \tag{2.9}
\]
which shows that in general \( p \) is a function of \( r \) and \( \rho \). Taking partial derivative of the above equation with respect to \( r \), and then setting it to zero, we find that
\[
\frac{\partial p(r, \rho)}{\partial r} = -2 \frac{g(x)}{r^3} + \frac{1}{r^2} \frac{dg(x)}{dx} \frac{dx}{d(r^2 \rho)} \frac{\partial (r^2 \rho)}{\partial r}
= \frac{2fg}{r^3 f} \left( \frac{g'}{g} - \frac{f'}{f} \right) = 0, \tag{2.10}
\]
which gives \( g(x) = kf(x) \). Then, Eq. (2.5) follows. In the following we shall further assume \( 0 \leq k \leq 1 \) so that all the energy conditions hold [19] and that the pressure is non-negative.

The combination of Eqs. (2.3) and (2.5) immediately yields
\[
y = -\frac{1}{1 + k}. \tag{2.11}
\]
Then, Eqs. (2.1) and (2.11) have the solutions,
\[
\Phi = k \left( \frac{x}{1 + k} + \Psi \right) + \Phi_0,
\]
\[
S(x) = S_0 e^{-x/(1+k)}, \tag{2.12}
\]
4
while Eq.(2.2) becomes
\[k \Psi_{,xx} + k \left( \Psi_{,x} + \frac{1}{k} \right) \left[ (k - 1) \Psi_{,x} - 1 \right] - e^{2 \left[ \frac{1}{e^x} + (1-k) \Psi_{,x} - \Phi_0 \right]} \left[ \Psi_{,xx} + (1-k) \Psi_{,x} \left( \Psi_{,x} + \frac{1}{k} \right) \right] = 0, \tag{2.13}\]
where \( \Phi_0 \) and \( S_0 \) are integration constants.

**A. Stiff Fluid \( (k = 1) \)**

When \( k = 1 \), i.e., the stiff fluid, Eq.(2.13) has the general solution,
\[
\Psi(x) = q \ln \left( 1 - e^{x-x_0} \right) - \frac{1}{2} (x - x_0) + \Psi_0, \\
\Phi(x) = q \ln \left( 1 - e^{x-x_0} \right) + x_0 + \Psi_0, \\
S(x) = S_0 e^{-x^2/2}, \tag{2.14}\]
where \( \Psi_0, x_0, S_0 \) and \( q \) are the integration constants, and \( x_0 \equiv 2 \Phi_0 \). Imposing the regularity conditions of Eqs.(1.13)-(1.15) and the gauge condition (1.16), we find that
\[
S_0 = 2 e^{-x_0/2}, \quad \Psi_0 = -x_0, \tag{2.15}\]
while \( x_0 \) and \( q \) are arbitrary and cannot be removed by the transformations of Eq.(1.11). Then, the general solution is given by
\[
ds^2 = l^2 \left\{ \left( 1 - e^{x-x_0} \right)^{2q} \left[ dt^2 - e^{-(x+x_0)} dr^2 \right] - 4r^2 e^{-(x+x_0)} d\theta^2 \right\}, \tag{2.16}\]
and the corresponding energy density of the fluid reads,
\[
\rho = p = \frac{1 - 2q}{4l^2(-t)^2 (1 - e^{x-x_0})^{2q}}. \tag{2.17}\]

It should be noted that in writing Eq.(2.16), we had implicitly assumed that \( x - x_0 \leq 0 \). On the other hand, from Eq.(2.17) we can see that we must have \( q < 1/2 \) in order to have \( \rho \geq 0 \), a condition that we shall assume in the rest of the paper. The metric is singular on the hypersurface \( x = x_0 \), and depending on the values of \( q \), the nature of the singularity is different.

1. \( 0 < q < 1/2 \)

In this case the singularity is a curvature one and marginally naked. In fact, from Eq.(B.10) we find that
\[
\theta_l = \frac{f e^x}{2l^2 t (1 - e^{x-x_0}) q} \left[ e^{(x_0-x)/2} - 1 \right], \\
\theta_n = -\frac{g e^x}{2l^2 t (1 - e^{x-x_0}) q} \left[ e^{(x_0-x)/2} + 1 \right], \tag{2.18}\]
from which we can see that \( \theta_n < 0 \) for any given \( x \in [0, x_0] \), and \( \theta_l \) is positive for \( x < x_0 \), zero only on the surface \( x = x_0 \). Thus, now the hypersurface \( x = x_0 \) is a marginally trapped surface and the spacetime singularity located there is marginally naked. On the other hand, for any given hypersurface \( x = C, \text{ say, } C \), its normal vector is given by
\[
n_\alpha \equiv \frac{\partial(x - C)}{\partial x^\alpha} = \frac{1}{r} \left( e^x \delta^t_\alpha + \delta^r_\alpha \right), \tag{2.19}\]
from which we find
\[
n_\alpha n_\beta g^{\alpha\beta} = -\frac{e^{x+x_0}}{l^2 r^2} (1 - e^{-x-x_0})^{1-2q}.
\] (2.20)

Clearly, on the hypersurface \( x = x_0 \) the normal vector, \( n_\alpha \), becomes null. That is, the nature of the singularity is null. The corresponding Penrose diagram is given by Fig. 1.

FIG. 1. The Penrose diagram for the solutions given by Eq.(2.16) with 0 < \( q < 1/2 \). The spacetime is singular on the double line \( x = x_0 \), which is a null surface and on which the expansion of the outgoing radial null geodesics is zero, \( \theta_l(t,r)|_{x=x_0} = 0 \), but \( \theta_n(t,r) < 0 \) in the whole spacetime, including the hypersurface \( x = x_0 \).

2. \( q = 0 \)

In this case the metric is free of spacetime and coordinate singularities on the hypersurface \( x = x_0 \), and valid in the whole region \( t \leq 0, r \geq 0 \). However, now the spacetime becomes singular on the hypersurface \( t = 0 \) [cf. Eq.(2.17)]. From Eq.(2.18), which is also valid for \( q = 0 \), we can see that this singularity is not naked, and always covered by the apparent horizon formed on the hypersurface \( x = x_0 \). In the region \( x > x_0 \), which is denoted as Region I in Fig. 2, Eq.(2.18) shows that \( \theta_l \) becomes negative and \( \theta_l \theta_n > 0 \), that is, all the rings of constant \( t \) and \( r \) now become trapped. Thus, in this case Region I can be considered as the interior of a black hole, which is formed by the gravitational collapse of the stiff fluid in Region II. The corresponding Penrose diagram is given by Fig. 2.
It should be noted that apparent horizons and black holes are usually defined in asymptotically flat spacetimes [19]. As mentioned previously, the spacetimes considered here are not asymptotically flat, therefore, strictly speaking the above definitions are, in some sense, the generalization of those given in [19]. As a matter of fact, to be distinguishable with the asymptotical case, Hayward called such apparent horizons as trapping horizons and defined black holes by above definitions are, in some sense, the generalization of those given in [19]. As a matter of fact, to be distinguishable

w them as trapping horizons and defined black holes by following [9–11] we shall continuously use the notions, apparent horizons.

3. $q < 0$

In this case the spacetime is free of curvature singularity on the hypersurface $x = x_0$, although we do have a coordinate one there. Then, to have a geodesically maximal spacetime, we need to extend it beyond this surface. Introducing two null coordinates $u$ and $v$ via the relations

$$
t = -\frac{1}{2} \left[ (u)^n + (v)^n \right]^2,
$$

$$
r = \frac{1}{2} e^{x_0} \left[ (u)^n - (v)^n \right]^2,
$$

where

$$
n = \frac{1}{2q + 1},
$$

we find that the metric (2.16) becomes

$$
\text{ds}^2 = l^2 \left\{ 2e^{2\sigma(u,v)} du dv - R^2(u,v) dt^2 \right\},
$$

with

$$
\sigma(u, v) = (1 - 2q) \ln \left[ (u)^n + (v)^n \right] + \sigma_0,
$$

$$
R(u, v) = (u)^{2n} - (v)^{2n},
$$

$$
\sigma_0 = \frac{1}{2} \ln \left( 2n^2 q^{2q} \right),
$$

and the corresponding velocity and energy density of the stiff fluid are given, respectively, by

$$
u_{\mu} = \frac{le^{\sigma}}{\sqrt{2}} \left( u_0 \delta_{\mu}^u + \frac{1}{u_0} \delta_{\mu}^v \right), \quad u_0 \equiv \left( \frac{u}{v} \right)^{(n-1)/2},
$$

$$
\rho = \frac{\rho_0 (uv)^{n-1}}{\left[ (u)^n + (v)^n \right]^{n-2/n}}, \quad \rho_0 = \frac{2n(2n-1)}{l^2} e^{-2\sigma_0}.
$$

From Eq.(2.21) we can see that the region $t \leq 0$, $r \geq 0$, $x \leq x_0$ in the $(t, r)$-plane has been mapped into the region $u \leq 0$, $v \leq 0$ and $v \geq u$ in the $(u, v)$-plane, which will be referred as to region $I$, as shown in Fig. 3. The null hypersurface $x = x_0$, as can be seen from Eq. (2.20), is mapped to the one $v = 0$. Region $I$, where $u \leq 0$, $v \geq 0$, $|u| \geq v$, represents an extended region. In this extended region, the metric is real only for $n \neq (2m + 1)/(2j)$, where $j$ and $m$ are integers. When $n = (2m + 1)/(2j)$, a possible extension may be given by replacing $-v$ by $|v|$. However, this extension is not analytical. In fact, no analytical extension exists in this case. The only case where the extension is analytical is the one where $n$ is an integer. When the extension is not analytical, it is also not unique. Thus, to have an unique extension in the following we shall assume that $n$ is an integer. On the other hand, from Eq.(B.7) we find that

$$
\theta_t = \frac{2ne^{-2\sigma}}{l^2 R} (-v)^{2n-1}, \quad \theta_n = -\frac{2ne^{-2\sigma}}{l^2 R} (-u)^{2n-1},
$$

FIG. 2. The Penrose diagram for the solutions given by Eq.(2.16) with $q = 0$. The spacetime is singular on the double line $t = 0$. All the rings of constant $t$ and $r$ are trapped in Region $I$ but not in Region $II$, where $I = \{ x^n : x > x_0, t < 0, r \geq 0 \}$ and $II = \{ x^n : x < x_0, t < 0, r \geq 0 \}$. The hypersurface $x = x_0$ is a null surface and represents an apparent horizon.
where $R \equiv rS$. From the above expressions we can see that $\theta_n$ is always negative in both regions $I$ and $II$, while $\theta_l$ is positive in Region $II$, zero on the hypersurface $v = 0$, and negative in the extended region, $I$. Near the surface $v = 0$ the only non-vanishing component of the energy-momentum tensor is given by

$$T_{uu} = \frac{l^2 \rho_0 e^{2\sigma_0}}{(-u)^2},$$

which represents the energy flow moving from Region $I$ to Region $I$ along the hypersurfaces $u = Const$. To study the above solutions further, let us consider the two cases $n = 2m + 1$ and $n = 2m$, separately, where $m = 1, 2, 3, \ldots$.

When $n$ is an odd integer, i.e., $n = 2m + 1$, from Eq.(2.25) we can see that the spacetime is singular on the hypersurface $u = -v$ in the extended region $I$, on which we have $R(u, v) = 0$. On the other hand, all the rings of constant $t$ and $r$ are trapped in this region, as we can see from Eq.(2.26), now we have $\theta_l(v > 0) < 0$ and $\theta_l\theta_n(v > 0) > 0$. Thus, region $I$ can be considered as the interior of the black hole that is formed from the gravitational collapse of the fluid in region $II$. The corresponding Penrose diagram is that of Fig. 3.

![Figure 3](image-url)

**FIG. 3.** The Penrose diagram for the solutions given by Eqs. (2.23) and (2.24) with $n$ being an integer. The rings of constant $t$ and $r$ are closed trapped surfaces in Region $I$, but not in region $II$. The dashed line $v = 0$, or $x = x_0$, represents an apparent horizon. When $n$ is an odd integer, the spacetime is singular on the horizontal double line $R = 0$, while when $n$ is an even integer, the spacetime has an angular defect there.

When $n$ is an even integer, say, $n = 2m$, from Eq.(2.25) we can see that no spacetime curvature singularity is developed on the axis $R = 0$ in the extended region $I$, although all the rings of constant $t$ and $r$ are also trapped in this region, as now we still have $\theta_l(v > 0) < 0$ and $\theta_l(v > 0)\theta_n(v > 0) > 0$. However, the local flatness condition (1.14) is not satisfied there. In fact, it can be shown that now we have

$$\frac{X_{,\alpha}X_{,\beta}g^{\alpha\beta}}{4X} \to +1,$$

as $v \to -u$. Thus, unlike that on the axis $R = 0$ in Region $II$, where the local flatness condition is satisfied, now the spacetime on the axis $R = 0$ in region $I$ has angle defect. The corresponding Penrose diagram is also given by Fig. 3, but now the horizontal double line $R = 0$, instead of representing a curvature singularity, now represents an angle-defect-like singularity [17].

Moreover, in the extended region $I$, where $uv < 0$, the function $\rho$ becomes negative, and the three-velocity $u_\mu$ becomes imaginary, as we can see from Eq.(2.25). A close investigation shows that the energy-momentum tensor in this extended region actually takes the form

$$T_{\mu\nu} = \bar{\rho}(2r_\mu r_\nu + g_{\mu\nu}),$$

where

$$r_\mu = \frac{le^{\sigma}}{\sqrt{2}} \left( r_0 \delta_\mu - \frac{1}{r_0} \delta_\mu \right), \quad r_0 \equiv \left| \frac{u}{v} \right|^{(2m-1)/2},$$

$$\bar{\rho} = \frac{\rho_0 |uv|^{2m-1}}{(u^{2m} + v^{2m})^{(6m-1)/m}}, \quad (n = 2m),$$

as $v \to -u$. Thus, unlike that on the axis $R = 0$ in Region $II$, where the local flatness condition is satisfied, now the spacetime on the axis $R = 0$ in region $I$ has angle defect. The corresponding Penrose diagram is also given by Fig. 3, but now the horizontal double line $R = 0$, instead of representing a curvature singularity, now represents an angle-defect-like singularity [17].
from which we find that
\[ g^{\mu\nu} r_\mu r_\nu = -1. \] (2.31)

Thus, in the extended region the source is no more a stiff fluid. Introducing the unit vectors,
\[ u_\mu = \frac{le^\sigma}{\sqrt{2}} \left( r_0^\delta_\mu + \frac{1}{r_0^\delta_\mu} \right), \quad \theta_\mu = lR^\delta_\theta, \] (2.32)
where \( u_\nu u^\nu = 1 \) and \( \theta_\nu \theta^\nu = -1 \), we find that Eq.(2.29) reads
\[ T_\mu^\nu = \rho u_\mu u^\nu - p_r r_\mu r^\nu - p_\theta \theta_\mu \theta^\nu, \]
\[ \rho = -p_r = p_\theta = \frac{\rho_0 |uv|^{2m-1}}{(u^{2m} + v^{2m})(6m-1)/m}. \] (2.33)

Thus, the source in region \( I \) now becomes an anisotropic fluid with its energy density \( \rho \) and two principal pressures, \( p_r \) and \( p_\theta \), in the direction of \( r_\mu \) and \( \theta_\mu \). Note that although the pressure in the \( r_\mu \) direction is negative, the anisotropic fluid satisfies all the three energy conditions [19]. With this odd feature, it is not clear whether the corresponding solution can be interpreted as representing gravitational collapse of the stiff fluid. In fact, if we consider the fluid from its microscopic point of view, we might be able to rule out such a change in the equation of state across the hypersurface \( v = 0 \) [21], a problem that is under our current investigation.

Before turning to study other cases, we would like to note that it is well-known that a stiff perfect fluid is energetically equivalent to a massless scalar field when the velocity of the fluid is irrotational \( \nabla [u_\mu u_\nu] = 0 \) and the massless scalar field is timelike \( \phi, \alpha \phi, \beta g^{\alpha\beta} > 0 \). Clearly, these conditions are fulfilled in Region \( II \). In fact, comparing the solutions given by Eq.(2.24) with the corresponding massless scalar field ones found in [11], one can find that they are actually the same in Region \( II \). However, as shown lately in 4-dimensional case [22], the spacetime across the horizon \( v = 0 \) can be quite different. Our analysis given above and the one given in [9,11] show that this is also the case in \( 2 + 1 \) Gravity. In [22], it was also shown that the linear perturbations in these two cases are different. Thus, it would be very interesting to study the linear perturbations of the above solutions in terms of stiff fluid.

### B. Dust Fluid (k = 0)

When \( k = 0 \), the pressure of the perfect fluid vanishes, i.e., the dust fluid. Then, it can be shown that in this case the general solution is given by
\[ \Phi(x) = \Phi_0, \]
\[ \Psi(x) = \ln \left( 1 - e^{x-x_0} \right) - x + \Psi_0, \]
\[ S(x) = S_0 e^{-x}, \] (2.34)
for which we find
\[ \rho = \frac{e^{-2\Phi_0}}{(2(-t))^2 (1 - e^{x-x_0})}, \quad p = 0. \] (2.35)

Since now we have \( R(t,r) = rS(x) = -S_0 t \), we can see that these solutions are Kantowski-Sachs like [16], and may be interpreted as representing cosmological models. Because in this paper we are mainly interested in gravitational collapse, in the following we shall not consider this case in any more details.

### C. Perfect Fluid With \( k \neq 0, 1 \)

When \( k \neq 0, 1 \), introducing the function \( Z(x) \) by
\[ Z(x) = \exp \left\{ 2 \left[ \frac{x}{1 + k} + (1 - k) \Psi - \Phi_0 \right] \right\}, \] (2.36)
we find that Eq.(2.13) can be written in the form,
\[2Z(k - Z)Z_{,xx} + (Z - 3k)Z_{,x}^2 + 2Z^2Z_{,x} + \frac{4k}{(k + 1)^2}Z^2(k^2 - Z) = 0.\] (2.37)

We are not able to find the general solution of this equation, but a particular one given by

\[
\begin{align*}
\Phi(x) &= \Phi_0, \\
\Psi(x) &= -\frac{x}{k + 1} + \frac{\Phi_0}{1 - k}, \\
S(x) &= S_0 e^{-\frac{x}{k + 1}}.
\end{align*}
\] (2.38)

Then, it can be shown that the conditions of Eqs.(1.13)-(1.16) are fulfilled, provided that

\[
\Phi_0 = 0, \quad S_0 = \frac{1 + k}{k}, \quad k > 0.
\] (2.39)

Thus, the corresponding metric takes the form

\[
\begin{align*}
ds^2 &= l^2 \left\{ dt^2 - e^{-2\pi/(1+k)} \left[ dv^2 + \left(\frac{1 + k}{k}\right)^2 r^2 d\theta^2 \right] \right\},
\end{align*}
\] (2.40)

and the pressure and energy density of the fluid are given by

\[
p = k\rho = \frac{k}{l^2(1 + k)^2(-t)^2}.
\] (2.41)

From the above expression we can see that the spacetime is always singular at \(t = 0\). However, this singularity is not naked. In fact, from Eq.(B.10) we find that

\[
\begin{align*}
\theta_t &= \frac{fe^{x/(1+k)}}{l^2(1 + k)r} \left( k - e^{kz/(1+k)} \right), \\
\theta_n &= -\frac{ge^{x/(1+k)}}{l^2(1 + k)r} \left( k + e^{kz/(1+k)} \right),
\end{align*}
\] (2.42)

from which we can see that \(\theta_n\) is always negative, and

\[
\theta_t(t, r) = \begin{cases} > 0, & x < x_0, \\ = 0, & x = x_0, \\ < 0, & x > x_0, \end{cases}
\] (2.43)

where \(x_0\) is given by

\[
x_0 = \frac{1 + k}{k} \ln(k).
\] (2.44)

Thus, the singularity at \(t = 0\) is always covered by the apparent horizon localized on the hypersurface \(x = x_0\). In the region where \(t < 0\) and \(x > x_0\), the rings of constant \(t\) and \(r\) are all trapped, as now we have \(\theta_t < 0\) and \(\theta_t\theta_n > 0\). Therefore, in the present case the collapse of the fluid always forms black holes. Moreover, the normal vector \(n_\alpha\) to the hypersurface \(x = x_0\) is still given by Eq.(2.19) but now with

\[
n_\alpha n_\beta g^{\alpha\beta} = -\frac{e^{2x_0/(1+k)}}{l^2r^2} (1 - k^2) < 0,
\] (2.45)

since now we have \(0 < k < 1\). That is, the apparent horizon is always timelike and the corresponding Penrose diagram is given by Fig. 4.
III. SELF-SIMILAR SOLUTIONS OF THE SECOND KIND

When $\alpha \neq 1$, the term that is proportional to $r^{-2}$ has different power-dependence on $r$ from the term that is proportional to $t^{-2}$, when they are written out in terms of $r$ and $x$, since $t = -r^\alpha e^{-\alpha x}$. Then, it can be shown that the Einstein field equations in this case become

\begin{align}
y_{,x} - (1 + y) (\Psi_{,x} - y) - y \Phi_{,x} &= 0, \quad (3.1) \\
\Phi_{,xx} + \Phi_{,x} (\Phi_{,x} - \Psi_{,x} - y - 2) &= 0, \quad (3.2) \\
\Psi_{,xx} - \Psi_{,x} (\Phi_{,x} - \Psi_{,x} + y + 1 - \alpha) + (1 - \alpha) y &= 0, \quad (3.3)
\end{align}

and

\begin{align}
\rho &= \frac{y}{l^2} \left\{ \frac{1}{(\alpha t)^2} e^{2\Phi} \Psi_{,x} - \frac{1}{r^2} e^{-2\Psi} \Phi_{,x} \right\}, \\
p &= \frac{1}{l^2} \left\{ \frac{1 + y}{r^2} e^{-2\Psi} \Phi_{,x} - \frac{1}{(\alpha t)^2} e^{-2\Phi} [(1 + y) \Psi_{,x} - (1 - \alpha) y] \right\}, \quad (3.4)
\end{align}

where in writing Eqs.(3.2) - (3.4), we had used Eq.(3.1). From the above equations we can see that now the Einstein field equations are already sufficient to determinate completely the metric coefficients $\Phi(x)$, $\Psi(x)$ and $S(x)$. Following Maeda et al [24], it can be shown that the symmetry of the self-similarity of the second kind is inconsistent with a polytropic equation of the kind,

\begin{equation}
p = k \rho^\beta, \quad (3.5)
\end{equation}

or

\begin{equation}
p = k n^\beta, \quad \rho = m_b n + \frac{p}{\beta - 1}, \quad (3.6)
\end{equation}

unless $k = 1$, where the constant $m_b$ denotes the mean baryon mass, and $n(t,r)$ the baryon number density [18]. Thus, in the following we shall consider only the case $\beta = 1$. Combining it with Eq.(3.4), we find that

\begin{align}
[1 + (1 + k) y] \Phi_{,x} &= 0, \quad (3.7) \\
[1 + (1 + k) y] \Psi_{,x} - (1 - \alpha) y &= 0. \quad (3.8)
\end{align}

Since $\alpha \neq 1$, we must have $\Phi_{,x} = 0$. Therefore, for the perfect fluid with the equation of state $p = k \rho$ and self-similarity of the second kind, it must move along the radial timelike geodesics. To solve Eqs.(3.1)-(3.3) and (3.8), let us consider the two cases $y \neq -1$ and $y = -1$ separately.
**Case A)** $y \neq -1$: In this case, from Eq.(3.1) we find that
\[
\Psi_{,x} = \frac{y_{,x}}{1 + y} + y, \tag{3.9}
\]
while from Eq.(3.8) we obtain
\[
\Psi_{,xx} = \frac{y_{,x}}{1 + (1 + k)y} [(1 - \alpha) - (1 + k) \Psi_{,x}]. \tag{3.10}
\]
Inserting the above expressions into Eq.(3.3) we find that it has three different solutions,
\[
(a) \ k = 0, \\
(b) \ y_{,x} = 0, \\
(c) \ y_{,x} - (\alpha - 2)y(1 + y) = 0. \tag{3.11}
\]
When $k = 0$, it can be shown that Eqs.(3.1)-(3.3), (3.7) and (3.8) have the general solution,
\[
\Phi(x) = \Phi_0, \quad \Psi(x) = \ln \left| 1 + (\alpha - 1)e^{\alpha(x_0 - x)} \right| + \Psi_0, \\
S(x) = S_0 \left| 1 - e^{\alpha(x_0 - x)} \right|, \quad (k = 0). \tag{3.12}
\]
When $y_{,x} = 0$, we find that the general solution is given by
\[
\Phi(x) = \Phi_0, \quad S(x) = S_0 e^{\alpha x}, \\
\Psi(x) = ax + \Psi_0, \quad a \equiv -\frac{\alpha}{1 + k}. \tag{3.13}
\]
When $y_{,x} - (\alpha - 2)y(1 + y) = 0$, it can be shown that the corresponding solution is given by Eq.(3.13) with $\alpha = 2$. Thus, this case is a particular one of Eq.(3.13).

**Case B)** $y = -1$: In this case from Eq.(3.8) we find that
\[
\Psi_{,x} = \frac{1 - \alpha}{k}. \tag{3.14}
\]
Inserting it into Eq.(3.3) we find that there are only two solutions given, respectively, by
\[
(i) \ k = 1, \\
(ii) \ \alpha = 1 + k. \tag{3.15}
\]
When $k = 1$, the general solution is given by
\[
\Phi(x) = \Phi_0, \quad S(x) = S_0 e^{-x}, \\
\Psi(x) = (1 - \alpha)x + \Psi_0, \quad (k = 1), \tag{3.16}
\]
and when $\alpha = 1 + k$, we have
\[
\Phi(x) = \Phi_0, \quad S(x) = S_0 e^{-x}, \\
\Psi(x) = -x + \Psi_0, \quad \alpha = 1 + k. \tag{3.17}
\]
Thus, the most general solutions with self-similarity of the second kind ($\alpha \neq 0, 1$) and the equation of state $p = k\rho$ consist of four classes of solutions, given, respectively, by Eqs.(3.12), (3.13), (3.16) and (3.17). In the following, let us consider them separately.
A. \( y \neq -1, \ k = 0 \)

In this case, the solutions are given by Eq.(3.12). Applying the conditions (1.13)-(1.16) to them, we find that \( \Phi_0 = 0, \ S_0 = e^{\Psi_0} \) and \( \alpha < 1 \). Then, using the transformations (1.11) we can further set \( \Psi_0 = 0 \). Thus, the corresponding metric finally takes the form,

\[
d s^2 = l^2 \left\{ dt^2 - \left[ 1 - (1 - \alpha) e^{\alpha(x_0 - x)} \right]^2 dr^2 - r^2 \left[ 1 - e^{\alpha(x_0 - x)} \right]^2 d\theta^2 \right\}, \quad (\alpha < 1),
\]

and the energy density and pressure of the fluid are given by

\[
\rho = \frac{\alpha^2(1 - \alpha)e^{2\alpha(x_0 - x)}}{l^2 (\alpha t)^2 (1 - e^{\alpha(x_0 - x)})},
\]

\[
p = 0, \quad (\alpha < 1).
\]

From the above expressions we can see that the spacetime is singular on the hypersurfaces,

\[
(i) \ t = 0, \quad (ii) \ x = x_0, \quad (iii) \ x = x_1,
\]

where

\[
x_1 \equiv x_0 - \frac{1}{\alpha} \ln \left( \frac{1}{1 - \alpha} \right) < x_0.
\]

On the other hand, from Eq.(B.10) we find that

\[
\theta_l = \frac{f}{l^2 r (1 - e^{\alpha(x_0 - x)})} \left[ 1 - \frac{r}{r_A} \right]^{1 - \alpha},
\]

\[
\theta_n = -\frac{g}{l^2 r (1 - e^{\alpha(x_0 - x)})} \left[ 1 + \frac{r}{r_A} \right]^{1 - \alpha}, \quad (0 < \alpha < 1),
\]

for \( 0 < \alpha < 1 \), and

\[
\theta_l = \frac{f}{l^2 r (1 - e^{\alpha(x_0 - x)})} \left[ 1 + \frac{r}{r_A} \right]^{1 - \alpha},
\]

\[
\theta_n = -\frac{g}{l^2 r (1 - e^{\alpha(x_0 - x)})} \left[ 1 - \frac{r}{r_A} \right]^{1 - \alpha}, \quad (\alpha < 0),
\]

for \( \alpha < 0 \), where \( r_A \equiv e^{-\alpha x_0/(1 - \alpha)} \).

From Eq.(3.22) we can see that, in the case \( 0 < \alpha < 1 \), \( \theta_n \) is always negative for any \( r \in [0, \infty) \), but \( \theta_l \) is negative when \( r > r_A \), zero when \( r = r_A \) and positive when \( r < r_A \) [cf. Fig. 5]. That is, the spacetime is closed far away from the axis even at the very beginning \( (t = -\infty) \). This property makes the solution very difficult to be considered as representing gravitational collapse.

When \( \alpha < 0 \), from Eq.(3.23) we can see that \( \theta_l \) now is always positive, but \( \theta_n \) changes signs at \( r = r_A \). In particular, it shows that the ingoing radial null geodesics become expanding in the region \( r > r_A \). With this odd property, it is also very difficult to consider the corresponding solutions as representing gravitational collapse.
FIG. 5. The spacetime in the \((t, r)\)-plane for the solutions given by the metric (3.18). It is singular on the hypersurfaces \(t = 0, \ x = x_0\) and \(x = x_1\), where \(x_0 > x_1\).

**B. \(y = \text{Const.}(\neq -1)\)**

In this case the solutions are given by Eq.(3.13). It can be shown that the regularity conditions (1.13)-(1.15) and the gauge one (1.16) require \(\Phi_0 = 0, \ S_0 = e^{\Psi_0}/(1 + a)\) and \(\alpha < 1 + k\). On the other hand, using the transformations (1.11), we can further set \(\Psi_0 = 0\). Then, we find that

\[
\text{ds}^2 = l^2 \left\{ dt^2 - e^{2ax} \left[ dr^2 + \left( \frac{r}{1 + a} \right)^2 d\theta^2 \right] \right\}, \quad (\alpha < 1 + k),
\]

(3.24)

and the corresponding physical quantities are

\[
p = k\rho = \frac{k\alpha^2}{l^2(1 + k)^2(a t)^2},
\]

\[
\theta_t = \frac{f e^{-ax}}{l^2(1 + k)r} \left[ (1 + k)(1 + a) - \frac{r^{1+a}}{(-t)^{k/(1+k)}} \right],
\]

\[
\theta_n = -\frac{g e^{-ax}}{l^2(1 + k)r} \left[ (1 + k)(1 + a) + \frac{r^{1+a}}{(-t)^{k/(1+k)}} \right].
\]

(3.25)

From the above we can see that the spacetime is singular on the hypersurface \(t = 0\), and the singularity is not naked. In fact, it is covered by the apparent horizon located on the hypersurface,

\[
r_{AH}(t) = \left[ (1 + k)(1 + a)(-t)^{k/(1+k)} \right]^{1/(1+a)}.
\]

(3.26)

The normal vector to the hypersurface \(r - r_{AH}(t) = 0\) is given by \(n_\alpha \equiv \partial(r - r_{AH}(t))/\partial x^\alpha\). Then, we find that

\[
n_\alpha n^\alpha = -l^{-2} (1 + k - \alpha)^{2a/(1+k-\alpha)} (-t)^{-2(1-\alpha)/(1+k-\alpha)} (1 - k^2).
\]

(3.27)

Thus, the apparent horizon is timelike for \(0 \leq k < 1\), and the corresponding Penrose diagram is that of Fig. 4. When \(k = 1\), it is null, and the corresponding Penrose diagram is that of Fig. 2.

**C. \(y = -1, \ k = 1\)**

In this case the solutions are given by Eq.(3.16). The transformations (1.11) and the gauge condition (1.16) enable us to set \(\Phi_0 = \Psi_0 = 0, \ S_0 = 1\). Then, the metric takes the form
\[ ds^2 = l^2 \left\{ dt^2 - e^{2(1-\alpha)x} dr^2 - (-t)^{2/\alpha} d\theta^2 \right\}, \]  
(3.28)

from which we can see that the solutions are Kantowski-Sachs like and do not satisfy the regular conditions (1.13) and (1.14). The corresponding energy density is given by

\[ \rho = p = \frac{\alpha - 1}{l^2 (at)^2}. \]  
(3.29)

Thus, to have the energy density be positive, we must assume that \( \alpha > 1 \).

**D. \( y = -1, \ \alpha = 1 + k \)**

In this case, the solutions are given by Eq.(3.17). Using the transformations (1.11) and the gauge condition (1.16) we can set \( \Phi_0 = 0 \) and \( S_0 = 1 \). Then, the metric takes the form

\[ ds^2 = l^2 \left\{ dt^2 - e^{2(x_0-x)} dr^2 - (-t)^{2/\alpha} d\theta^2 \right\}, \]  
(3.30)

where \( \alpha = 1 + k \) and \( x_0 \equiv \Psi_0 \). Thus, in this case the solutions are also Kantowski-Sachs like, and the corresponding energy density and pressure of the fluid are given by

\[ p = k \rho = \frac{k}{l^2 (at)^2}. \]  
(3.31)

As we mentioned in the last section, when \( k = 1 \) the perfect fluid is energetically equivalent to a massless scalar field. Indeed, the solutions given by metric (3.28) and the ones given by Eqs.(3.24) and (3.30) with \( k = 1 \) are the solutions found lately for a massless scalar field in [23].

**IV. SELF-SIMILAR SOLUTIONS OF THE ZEROTH KIND**

In this case, combining Eqs.(1.3) and (A.8), we find that the Einstein field equations can be cast in the forms,

\[ y_x - (1 + y) (\Psi_x - y) - y \Phi_x = 0, \]  
(4.1)

\[ \Phi_{xx} + \Psi_x (\Phi_x - \Psi_x - y - 2) = 0, \]  
(4.2)

\[ \Psi_{xx} - \Psi_x (\Phi_x - \Psi_x + y + 1) + y = 0, \]  
(4.3)

and

\[ \rho = \frac{y}{l^2 r^2} \left\{ r^2 e^{-2\Phi} \Phi_x - e^{-2\Psi} \Phi_x \right\}, \]  
(4.4)

\[ p = \frac{1}{l^2 r^2} \left\{ (1 + y)e^{-2\Phi} \Phi_x - r^2 e^{-2\Psi} [ (1 + y) \Psi_x - y ] \right\}, \]  
(4.4)

where in writing Eqs.(4.2) - (4.4), we had used Eq.(4.1). Similar to the solutions with self-similarity of the second kind, now the Einstein field equations already determine completely the metric coefficients. It can be also shown that the self-similarity of the zeroth kind is inconsistent with the equation of state given by Eq.(3.5), unless \( \beta = 1 \).

Then, following the last section, let us consider solutions that satisfy the equation of state (2.5), which, together with Eq.(4.4), yields,

\[ [1 + (1 + k) y] \Phi_x = 0, \]  
(4.5)

\[ [1 + (1 + k) y] \Psi_x - y = 0. \]  
(4.6)

The above equations have the solution

\[ \Phi(x) = \Phi_0, \quad \Psi_x = \frac{y}{1 + (1 + k) y}. \]  
(4.7)

Thus, similar to the case of self-similarity of the second kind, now the fluid must also move along radial timelike geodesics. Inserting the above expressions into Eqs.(4.1)-(4.3), we find that solutions exist only for three special cases, \( k = 0, \pm 1 \).
A. $k = 0$

When $k = 0$, the general solution is given by,

$$
\Phi(x) = \Phi_0, \quad S(x) = S_0 (x_0 - x),
\Psi(x) = \ln (x_0 - x - 1) + \Psi_0, \quad (k = 0). \quad (4.8)
$$

Then, it can be shown that the conditions (1.13)-(1.16) require $\Phi_0 = 0$ and $S_0 = e^{\Psi_0}$. On the other hand, using the transformations (1.12), we can further set $\Psi_0 = 0$. Then, the solutions are finally given by

$$
ds^2 = l^2 \left\{ dt^2 - (x_0 - x - 1)^2 dr^2 - r^2 (x_0 - x)^2 d\theta^2 \right\},
$$

and the corresponding energy density is given by

$$
\rho = \frac{1}{l^2 (x_0 - x - 1)(x_0 - x)}, \quad p = 0.
$$

From this expression we can see that the spacetime is singular on the hypersurfaces $x = x_0 - 1$ and $x = x_0$ [cf. Fig. 6]. From Eq.(B.9) we also find that

$$
\theta_t = \frac{f}{l^2(x_0 - x)r} (1 + r),
\theta_n = -\frac{g}{l^2(x_0 - x)r} (1 - r).
$$

Thus, $\theta_t$ is positive for $x < x_0$ and negative for $x > x_0$, while the signs of $\theta_n$ get changed when across the hypersurfaces $r = 1$ and $x = x_0$. With this property, one can see that the solutions cannot be considered as representing gravitational collapse.

![FIG. 6. The spacetime in the $(t, r)$-plane for the solutions given by Eq.(4.9). It is singular on the hypersurfaces $x = x_0$ and $x = x_0 - 1$. The signs of $\theta_n$ change when across the hypersurfaces $r = 1$ and $x = x_0$.](image)

B. $k = -1$

When $k = -1$, the general solution is given by

$$
\Phi(x) = \Phi_0, \quad \Psi(x) = ax + \Psi_0, \quad S(x) = S_0 e^{\alpha x}, \quad (4.12)
$$

for which we have

$$
\rho = -p = \frac{a^2 e^{-2\Phi_0}}{l^2}.
$$

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This is a 3-dimensional de-Sitter solution. In fact, setting
\[ t = e^{-\Phi_0} \bar{t} + \frac{\Psi_0}{a}, \quad r = [(1 + a) \bar{\bar{r}}]^{1/(1+a)}, \]
\[ \theta = \frac{e^{\Psi_0}}{(1 + a) S_0} \bar{\bar{\theta}}, \quad \beta \equiv ae^{-\Phi_0}, \]
we find that the corresponding metric can be cast in the form
\[ ds^2 = l^2 \left\{ d\bar{\bar{t}}^2 - e^{-2\beta \bar{\bar{t}}} (d\bar{\bar{r}}^2 + \bar{\bar{r}}^2 d\bar{\bar{\theta}}^2) \right\}. \]

**C. \( k = 1 \)**

In this case, the general solution is given by
\[ \Phi(x) = \Phi_0, \quad \Psi(x) = x + \Psi_0, \quad S(x) = S_0 e^{-x}, \]
for which we have
\[ \rho = p = -\frac{e^{-2\Phi_0}}{l^2}. \]

Clearly, none of the three energy conditions is satisfied, and the physics of the solution is unclear (if there is any).

**V. CONCLUSIONS**

In this paper we have first generalized the notion of kinematic self-similarity of four-dimensional spacetimes to any dimensions for the metric given by Eq.(1.1), and then restricted ourselves to (2 + 1)-dimensional spacetimes with circular symmetry.

In Sec. II, we have studied solutions of the Einstein field equations with homothetic (the first kind) self-similarity for a perfect fluid. It has been shown that the only equation of state that takes the form \( p = p(\rho) \) and is consistent with the self-similarity is \( p = k \rho \), where \( k \) is a constant. In the latter case, a master equation has been found, Eq.(2.13). Then, the general solutions for the stiff fluid \( (k = 1) \) have been given in closed form. It has been found that some of these solutions represent formation of marginally naked singularities and the others represent the formation of black holes from gravitational collapse of the self-similar fluid. All the solutions of dust fluid \( (k = 0) \) have also been given, and found that they are all Kantowski-Sachs-like. When \( k \neq 0, 1 \), only particular solutions have been found and shown that the corresponding collapse always forms black holes.

In Secs. III and IV, all the solutions with self-similarity of the second or zeroth kind and the equation of state \( p = k \rho \) have been found and studied. In particular, it has been shown that the fluid must move along timelike radial geodesics. It has been also shown that some of those solutions represent gravitational collapse of the perfect fluid, while the others don’t. In the collapsing case, black holes are always formed.

In critical Type II collapse, all the critical solutions found so far have either discrete self-similarity or kinematic self-similarity of the first kind, and no critical solutions with kinematic self-similarity of the zeroth or second kind have been found, yet [25,26]. However, in Statistical Mechanics, critical solutions with kinematic self-similarity of the second kind seem more generic than those of the first kind [27]. Thus, it would be very interesting to study the linear perturbations of the solutions found in this paper with self-similarity of the zeroth or second kind.

Finally, we would like to note that Ida recently showed that a (2+1)-dimensional gravity theory which satisfies the dominant energy condition forbids the existence of black holes [28]. This result does not contradict with the ones obtained here, as in our case the surfaces of apparent horizons of the black holes are not compact, a precondition that was assumed in [28].
APPEND A: CIRCULARLY SYMMETRIC SPACETIMES WITH KINEMATIC SELF-SIMILARITY

The general metric of \((2 + 1)\)-dimensional spacetimes with circular symmetry can be cast in the form,

\[
ds^2 = l^2 \left\{ e^{2\Phi(t,r)} dt^2 - e^{2\Psi(t,r)} dr^2 - r^2 S^2 (t,r) d\theta^2 \right\},
\]

where \(l\) is a constant and has dimension of length. Then, it is easy to show that the coordinates \(\{x^\mu\} = \{t, r, \theta\}\), the Christoffel symbols, \(\Gamma^\lambda_{\mu\nu}\), the Riemann tensor, \(R^\sigma_{\mu\nu\lambda}\), the Ricci tensor, \(R_{\mu\nu}\), and the Einstein tensor, \(G_{\mu\nu}\), are all dimensionless, while the Ricci scalar, \(R\), has the dimension of \(l^{-2}\), and the Kretschmann scalar, \(R \equiv R^\sigma_{\mu\nu\lambda} R_{\sigma\mu\nu\lambda}\), has the dimension of \(l^{-4}\).

For the metric (A.1), we find that the non-vanishing Christoffel symbols are given by

\[
\Gamma^0_{00} = \Phi_t, \quad \Gamma^0_{01} = \Phi_r, \quad \Gamma^0_{11} = e^{2(\Psi-\Phi)} \Psi_t, \quad \Gamma^0_{22} = r^2 e^{-2\Phi} SS_t,
\]

\[
\Gamma^1_0 = e^{2(\Psi-\Phi)} \Phi_r, \quad \Gamma^1_{01} = \Psi_t, \quad \Gamma^1_{11} = \Psi_r, \quad \Gamma^1_{22} = r e^{-2\Phi} (S + r S_r),
\]

\[
\Gamma^2_{02} = \frac{S_t}{S}, \quad \Gamma^3_{12} = \frac{S + r S_r}{r S},
\]

and the Einstein tensor has the following non-zero components

\[
G_{tt} = \frac{e^{-2\Phi}}{r S} \left\{ r e^{2\Phi} S_t \Psi_x - e^{2\Phi} [r S_{,rr} + 2 S_r - (S + r S_r) \Psi_r] \right\},
\]

\[
G_{tr} = \frac{1}{r S} [r S_{,rr} - S_t (r \Phi_r - 1) - (r S_r + S) \Psi_t],
\]

\[
G_{rr} = \frac{e^{-2\Phi}}{r S} \left\{ e^{2\Phi} (r S_r + S) \Phi_r - r e^{-2\Phi} (S_{tt} - \Phi_t S_t) \right\},
\]

\[
G_{\theta\theta} = -r^2 S^2 \left\{ e^{-2\Phi} [\Psi_{,tt} - (\Psi_t - \Phi_t) \Psi_t] - e^{-2\Phi} [\Phi_{,rt} + (\Phi_r - \Psi_r) \Phi_r] \right\}.
\]

### A.1 Spacetimes with Self-Similarity of the Zeroth Kind

To study solutions with self-similarity of the zeroth kind, let us first introduce the self-similar variables, \(x\) and \(\tau\) via the relations,

\[
x = \ln(r) - t, \quad \tau = t,
\]

or inversely

\[
t = \tau, \quad r = e^{x+t}.
\]

Then, for any given function \(f(t,r)\) we have

\[
f_{,t} = f_{,\tau} - f_{,x}, \quad f_{,\tau} = \frac{1}{r} f_{,x},
\]

\[
f_{,tr} = -\frac{1}{r} (f_{,xx} - f_{,\tau x}), \quad f_{,rt} = \frac{1}{r^2} (f_{,xx} - f_{,x}),
\]

\[
f_{,tt} = f_{,\tau\tau} - 2 f_{,\tau x} + f_{,xx}.
\]

Substituting these equations into Eq.(A.3), we find that

\[
G_{tt} = \frac{e^{-2\Phi}}{r^4 S} \left\{ e^{2\Phi} [S_{xx} + S_x - \Psi_x (S_x + S)] - r^2 e^{-2\Phi} (S_{,\tau} - S_x) (\Psi_{,\tau} - \Psi_x) \right\},
\]

\[
G_{tr} = \frac{1}{r S} [S_{,xx} - S_{,\tau x} + (S_x + S) (\Psi_{,\tau} - \Psi_x)].
\]
\[ G_{\tau\tau} = \frac{e^{-2\Phi}}{r^2} \left\{ e^{2\Phi} (S_x + S) - r^2 e^{2\Psi} [S_{x\tau} - 2S_{x\tau} + S_{xx} + (S_x - S_x) (\Phi_x - \Phi_x)] \right\} ,
\]
\[ G_{\theta\theta} = S^2 e^{-2(\Phi + \Psi)} \left\{ e^{2\Phi} [\Phi_{xx} + \Phi_x (\Phi_x - \Psi_x - 1)] - r^2 e^{2\Psi} [\Psi_{xx} + \Psi_{x\tau} - 2\Psi_{x\tau} - (\Psi_x - \Psi_x) (\Phi_x - \Psi_x - \Phi_x)] \right\} . \]

(A.7)

For the self-similar solutions, the metric coefficients \( \Phi, \Psi \) and \( S \) are functions of \( x \) only. Then, Eq.(A.7) reduce to,

\[ G_{tt} = -\frac{e^{-2\Psi}}{r^2} \left\{ e^{2\Phi} [y + (y + 1) (y - \Psi_x)] - r^2 e^{2\Psi} \Psi_{xy} \right\} ,
\]
\[ G_{tr} = \frac{1}{r} [y + (y + 1) (y - \Psi_x) - y \Phi_x] ,
\]
\[ G_{rr} = \frac{e^{-2\Phi}}{r^2} \left\{ e^{2\Phi} (y + 1) - r^2 e^{2\Psi} [y_x + y (y - \Phi_x)] \right\} ,
\]
\[ G_{\theta\theta} = S^2 e^{-2(\Phi + \Psi)} \left\{ e^{2\Phi} [\Phi_{xx} + \Phi_x (\Phi_x - \Psi_x - 1)] - r^2 e^{2\Psi} [\Psi_{xx} - \Psi_x (\Phi_x - \Psi_x)] \right\} , \]

(A.8)

where
\[ y = \frac{S_x}{S} . \]  

(A.9)

### A.2 Spacetimes with Self-Similarity of the First and Second Kinds

To study these kinds of self-similar solutions, let us introduce the self-similar variables, \( x \) and \( \tau \) by

\[ x = \ln \left( \frac{r}{(-t)^{1/2}} \right) , \quad \tau = -\ln (-t) , \]

(A.10)

or inversely,

\[ r = e^{(\alpha x - \tau)/\alpha} , \quad t = -e^{-\tau} , \]

(A.11)

where \( \alpha \) is a dimensionless constant. When \( \alpha = 1 \), the corresponding spacetimes are said to have self-similarity of the first kind or homothetic self-similarity. Otherwise, they are said to have self-similarity of the second kind.

For any given function \( f(t, r) \), now we have

\[ f_{,\tau} = \frac{1}{\alpha t} (\alpha f_{,\tau} + f_{,x}) , \quad f_{,r} = \frac{1}{r} f_{,x} ,
\]
\[ f_{,rr} = \frac{1}{\alpha^2 r^2} (\alpha f_{,xx} + f_{,xx}) , \]
\[ f_{,tt} = \frac{1}{\alpha^2 t^2} (\alpha^2 f_{,\tau\tau} + 2\alpha f_{,\tau x} + f_{,xx} + \alpha^2 f_{,x} + \alpha f_{,x}) . \]

(A.12)

Substituting these equations into Eq.(A.3), we find that

\[ G_{tt} = -\frac{1}{\alpha^2 r^2 S_{x} e^{2\Psi}} \left\{ \alpha^2 e^{2\Phi} [S_{xx} + S_x - \Psi_x (S_x + S)] - \frac{r^2}{t^2} \Psi_{,x} S_x e^{2\Psi} - \frac{\alpha r^2}{t^2} e^{2\Psi} (\alpha \Psi_{,\tau} S_x + \Psi_{,x} S_{,\tau} + \Psi_{,\tau} S_{,x}) \right\} , \]
\[ G_{tt} = \frac{1}{\alpha^{2}r^{2}S} \{ \alpha^{2}e^{2\Phi} [\Phi, x (S, x + S)] \} \]

\[ G_{rr} = \frac{1}{\alpha^{2}r^{2}S} \{ \alpha^{2}e^{2\Phi} [\Phi, x (S, x + S)] \] \]

\[ -\frac{r^{2}}{t^{2}} e^{-2\Phi} (S, xx - S, x \Phi, x) \]

\[ -\frac{\alpha r^{2}}{t^{2}} e^{-2\Phi} \{ \alpha S, xx + 2S, x \Phi, x - S, \tau \} \]

\[ -S, \tau [\alpha (\Phi, \tau - 1) + \Phi, x] \} \),

\[ G_{\theta\theta} = \frac{S^{2}}{\alpha} \{ \alpha^{2}e^{-2\Phi} [\Phi, xx + \Phi, x (\Phi, x - \Psi, x - 1)] \] \]

\[ -\frac{r^{2}}{t^{2}} e^{-2\Phi} [\Psi, xx - \Psi, x (\Phi, x - \Psi, x - \alpha)] \]

\[ -\frac{\alpha r^{2}}{t^{2}} e^{-2\Phi} \{ \alpha (\Phi, \tau + 2\Psi, x - \Psi, \tau - 1 + \Phi, x - \Psi, x - \Psi, x) \}

\[ -S, \tau [\Phi, \tau - 1 + \Phi, x - \Psi, x] \} \). \] \] 

(A.13)

For the self-similar solutions, the metric coefficients are also functions of \( x \) only, but now with \( x \) being given by Eq.(1.9). Thus, for the self-similar solutions, Eq.(A.13) reduces to

\[ G_{tt} = \frac{1}{\alpha^{2}r^{2}} \Psi, xx - \Psi, x [y, xx + y (y + 1) (y - \Psi, x)], \]

\[ G_{tr} = \frac{1}{\alpha^{2}r^{2}} \Psi, x y - \frac{1}{\alpha^{2}r^{2}} e^{2(\Phi - \Psi)} [y, xx + y (y + 1) (y - \Psi, x)], \]

\[ G_{rr} = -\frac{1}{\alpha^{2}r^{2}} e^{2(\Phi - \Psi)} [y, xx + y (y - \Phi, x + \alpha)] + \frac{1}{r^{2}} \Phi, x (y + 1), \]

\[ G_{\theta\theta} = S^{2} \{ \alpha^{2}e^{-2\Phi} [\Phi, xx + \Phi, x (\Phi, x - \Psi, x - 1)] \] \]

\[ -\frac{r^{2}}{\alpha^{2}r^{2}} e^{-2\Phi} [\Psi, xx - \Psi, x (\Phi, x - \Psi, x - \alpha)] \} \). \] \] 

(A.14)

where \( y \) is given by Eq.(A.9).

**APPEND B: APPARENT HORIZONS OF CIRCULARLY SYMMETRIC SPACETIMES**

In [11], the ingoing and outgoing radial null geodesics were studied in double null coordinates, and apparent horizon was defined as the outmost hypersurface where the expansion of outgoing null geodesics vanishes. In this appendix, we shall write down the expansions in terms of the self-similar variables. To do this, let us first introduce two null coordinates \( u \) and \( v \) via the relations

\[ du = f (e^{\Phi} dt + e^{\Psi} dr), \quad dv = g (e^{\Phi} dt + e^{\Psi} dr), \] \] 

(B.1)

where \( f \) and \( g \) satisfy the integrability conditions for \( u \) and \( v \),

\[ \frac{\partial^{2} u}{\partial t \partial r} = \frac{\partial^{2} u}{\partial r \partial t}, \quad \frac{\partial^{2} v}{\partial t \partial r} = \frac{\partial^{2} v}{\partial r \partial t}. \] \] 

(B.2)

Without loss of generality, we shall assume that they are all strictly positive,

\[ f > 0, \quad g > 0. \] \] 

(B.3)

From the above we can see that the rays moving along the hypersurfaces \( u = \text{Const.} \) are outgoing, while the ones moving along the hypersurfaces \( v = \text{Const.} \) are ingoing. Then, it is easy to show that, in terms of \( u \) and \( v \), the metric (A.1) takes the form

\[ ds^{2} = l^{2} \left\{ 2e^{2\sigma(u,v)} du dv - R^{2}(u,v) d\theta^{2} \right\}, \] \] 

(B.4)

20
where
\[ \sigma(u, v) = -\frac{1}{2} \ln (2fg), \quad R(u, v) = rS. \] (B.5)

On the other hand, from Eq.(B.1) we find that
\[ \frac{\partial t}{\partial u} = \frac{1}{2f}e^{-\Phi}, \quad \frac{\partial t}{\partial v} = \frac{1}{2g}e^{-\Phi}, \]
\[ \frac{\partial r}{\partial u} = \frac{1}{2f}e^{-\Psi}, \quad \frac{\partial r}{\partial v} = \frac{1}{2g}e^{-\Psi}. \] (B.6)

Then, the expansions of the outgoing and ingoing null geodesics are defined as [11],
\[ \theta_l \equiv \nabla_{l^\lambda} = e^{-2\sigma} \left[ \frac{R_v}{l^2 R} \left( e^{-\Phi} R_t + e^{-\Psi} R_r \right) \right], \]
\[ \theta_n \equiv \nabla_{n^\lambda} = e^{-2\sigma} \left[ \frac{R_u}{l^2 R} \left( e^{-\Phi} R_t - e^{-\Psi} R_r \right) \right], \] (B.7)
where \( \nabla_{\lambda} \) denotes the covariant derivative, and \( l^\lambda \) (\( n^\lambda \)) is the null vector defined the outgoing (ingoing) null geodesics and given by
\[ l_\lambda \equiv \frac{\partial u}{\partial x^\lambda} = \delta_\lambda^u, \quad n_\lambda \equiv \frac{\partial v}{\partial x^\lambda} = \delta_\lambda^v. \] (B.8)

For the self-similar solutions of the zeroth kind, Eq.(B.7) takes the form,
\[ \theta_l = \frac{f}{l^2 R} \left\{ (1 + y) e^{-\Psi} - ye^{x+r-\Phi} \right\}, \]
\[ \theta_n = -\frac{g}{l^2 R} \left\{ (1 + y) e^{-\Psi} + ye^{x+r-\Phi} \right\}, \] (\( \alpha = 0 \)), (B.9)
while for the ones of the first or second kind, it becomes
\[ \theta_l = \frac{f}{\alpha l^2 R} \left\{ \alpha(1 + y) e^{-\Phi} + ye^{x+(\alpha-1)r/\alpha-\Phi} \right\}, \]
\[ \theta_n = -\frac{g}{\alpha l^2 R} \left\{ \alpha(1 + y) e^{-\Phi} - ye^{x+(\alpha-1)r/\alpha-\Phi} \right\}, \] (\( \alpha \neq 0 \)). (B.10)

Setting \( \theta_l = 0 \) in the above expressions, we shall find the location of the apparent horizons.

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