Abstract

The main result of this article is the decomposition of tensor products of representations of $SL(2)$ in the sum of irreducible representations parametrized by outerplanar graphs. An outerplanar graph is a graph with the vertices $0, 1, 2, \ldots, m$, edges of which can be drawn in the upper half-plane without intersections. We allow for a graph to have multiple edges, but don’t allow loops. In more detail,

$$\rho_{d_1} \otimes \cdots \otimes \rho_{d_m} = \bigoplus_G T_G,$$

where $\rho_d$ denotes the irreducible representation of dimension $d + 1$, and the direct sum is taken over all the outerplanar graphs of degrees $d_0, d_1, \ldots, d_m$ with all possible values of $d_0$. $T_G$ is an irreducible subrepresentation of the type $\rho_{d_0}$, and we determine explicit formulas for the basis in the space of $T_G$ as well.

1 Introduction

In the classical quantum mechanics, each particle corresponds to an irreducible representations of $SL(2)$ of dimension $2s + 1$, where $s \in \frac{1}{2} \mathbb{Z}$ is the spin of the particle. For example, a photon having spin 0 corresponds to the trivial representation of dimension 1, and an electron having spin $1/2$ corresponds to the standard 2-dimensional representation of $SL(2)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} aA + bB \\ cA + dB \end{pmatrix}.$$  

(1)

The projections of the spin correspond to the elements of the fixed basis of the space of the representation, described below in section 2 just before (14). To study the systems of a few particles,
one needs to know explicit formulas for the bases of the isotypic components of the tensor product of corresponding representations, and especially the basis of the isotypic component of the type of the trivial representation, i.e. the basis of the subspace of invariants of the tensor product of representations. These bases are of interest in the valency theory and a few other branches of physics as well as mathematics.

The basis of the subspace of $SL(2)$-invariants of $V^\otimes m = V \otimes \cdots \otimes V$ ($m$ times), where $V$ is the two-dimensional linear space with the standard action (1) of $SL(2)$, was described in the terms of the 0-outerplanar graphs in the classic work [16] of Rumer, Teller and Weyl. This theory was developed and applied to the percolation theory by Temperley and Lieb [17], to the knots theory and invariants of 3-manifolds by Jones [5], Kauffman [6], Kauffman and Lins [7], Wenzl [19], Jaeger [1], Lickorish [1], Masbaum and Vogel [11], and others, to the quantum theory by Penrose [14] and Moussouris [2], to quantum groups and the quantum link theory by Reshetikhin and Turaev [15], Ohtsuki and Yamada [13], Carter, Flath and Saito [1] and others, especially to the theory of Lusztig’s canonical bases [10] by Khovanov and Frenkel [3], Varchenko [18] and Frenkel, Varchenko and Kirillov, Jr. [1]. I am sorry that I am not able to cite everybody who made a contribution, because the literature on this topic is very extensive. I became familiar with the description of the basis of the invariants of the tensor products of any irreducible representations of $SL(2)$ in the terms of the 0-outerplanar graphs from Kuperberg’s work [8]. Some people, and Frenkel and Khovanov [3] in particular, constructed bases not only in the invariants, but in the other components as well. The bases constructed in this work, are different. In all the previous works, the proofs used straightening algorithm. They showed that each invariant could be expressed as a linear combination of invariants corresponding to the 0-outerplanar graphs. That, after the calculation of the dimensions of an invariant space, and finding the number of 0-outerplanar graphs, confirmed that the given invariants form a basis.

In section 2 I give new proof of the result of Rumer, Teller and Weyl [16]. The method of my proof is the following. I show that the invariants corresponding to the 0-outerplanar graphs, are linearly independent. After the calculations of dimensions and numbers of 0-outerplanar graphs, it proves that the given invariants form a basis.

In section 3 I give new formulation for the extension of Rumer, Teller and Weyl’s work to the description of invariants of tensor products of any irreducible representations of $SL(2)$, using slightly different 0-outerplanar graphs, than in [3]. Then I give the new proof of the result, following the method described in the previous paragraph. Also, I give a few new explicit formulas of those
Invariants.

In section 4 I give a few examples and a few new formulas of the invariants.

In section 5 I introduce bases of all the isotypic components of tensor products of any irreducible representations of $SL(2)$, parametrized surprisingly by outerplanar graphs as well. Also, I describe the subdivisions of those bases, giving the decomposition of the isotypic component in the sum of the irreducible representations. The proofs are close to ones given in the previous sections.

Recall that all classes of equivalence of the irreducible polynomial finite dimensional representations of $SL(2)$ are parametrized by nonnegative integers, and we can choose the natural actions $\rho_k$ of $SL(2)$ in the symmetric powers $S^k V$, $k = 0, 1, 2, \ldots$ as their representatives.

**Definitions.** An outerplanar graph is a graph with the set of vertices $\{0, 1, 2, \ldots, m\} \subseteq \mathcal{H} = \{z \in \mathbb{C} \mid \Im z \geq 0\}$, edges of which can be drawn in the upper half-plane $\mathcal{H}$ without intersections. We allow for a graph to have multiple edges, but don’t allow loops. Denote $\mathcal{O}(G)$ the set of directed graphs, underlying undirected graph of which is $G$. Let $G$ be an outerplanar graph and $g \in \mathcal{O}(G)$. For each vertex $k$ of $g$ denote

$$x_k(g) = x_{d_k^{\text{out}}(g)} y_{d_k^{\text{in}}(g)} \in S^{d_k} V,$$

where $d_k^{\text{out}}(g)$ is the number of arrows in $g$, beginning in $k$, $d_k^{\text{in}}(g)$ is the number of arrows, ending in $k$, and

$$d_k = d_k^{\text{out}}(g) + d_k^{\text{in}}(g)$$

is the degree of $k$. In other words, we put $x$ at the beginning of each arrow of $g$, and $y$ — at the end, and multiply those $x$s and $y$s in each vertex. Denote

$$b_g = x_1(g) \otimes \cdots \otimes x_m(g) \in S^{d_1} V \otimes \cdots \otimes S^{d_m} V$$

and for any nonnegative integer $i \leq d_0(G)$ denote

$$t_{G,i} = \sum_{g \in \mathcal{O}(G)} (-1)^{\text{inv} g} b_g,$$

where $\text{inv} g$ is the number of inversions in $g$, i.e., the number of arrows $(k, l)$ in $g$ with $k > l$.

**Theorem 3.** For any fixed $d_0, d_1, \ldots, d_m$, tensors $t_{G,i}$ parametrized by all outerplanar graphs with degrees $d_0, d_1, \ldots, d_m$ and nonnegative integers $i \leq d_0$, form a basis in the isotypic component of the
type \( \rho_{d_0} \) in the representation \( \rho_{d_1} \otimes \cdots \otimes \rho_{d_m} \). For any fixed outerplanar graph \( G \), the subspace \( T_G \) spanned by the basis \( (t_G, 0, \ldots, t_G, d_0) \), are invariant; the linear homomorphism

\[
s_G : S^{d_0}V \rightarrow T_G, \quad x^iy^{d_0-i} \mapsto \frac{(-1)^i}{\binom{d_0}{i}} t_G,i
\]

defines the isomorphism of \( \rho_{d_0} \) and the subrepresentation of \( SL(2) \) in \( T_G \), and

\[
S^{d_1}V \otimes \cdots \otimes S^{d_m}V = \bigoplus_G T_G,
\]

where the direct sum in the right hand side is taken over all the outerplanar graphs of degrees \( d_0, d_1, \ldots, d_m \) with all possible values of \( d_0 \).

## 2 Invariants of tensor powers of the standard representation

Let \( f \) be a field of characteristic 0, and \( SL(2) \) —the group of \( 2 \times 2 \) \( f \)-matrices with determinant 1, acting on 2-dimensional linear \( f \)-space \( V \) with basis \( (x, y) \) by the standard way (8):

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Ax + By \\ \end{pmatrix} = \begin{pmatrix} aA + bB \\ cA + dB \end{pmatrix}x + \begin{pmatrix} aC + bD \\ cA + dB \end{pmatrix}y,
\]

i.e.

\[
x \mapsto ax + cy, \quad y \mapsto bx + dy.
\]

Then \( V \otimes V \) is 4-dimensional linear \( f \)-space with the basis \( x \otimes x, x \otimes y, y \otimes x, y \otimes y \), and \( SL(2) \) acts on \( V \otimes V \) through the tensor product of the standard actions (8), i.e.

\[
x \otimes x \mapsto (ax + cy) \otimes (ax + cy),
\]

and so on:

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} Ax \otimes x + Bx \otimes y + Cy \otimes x + Dy \otimes y \\ \end{pmatrix} = \begin{pmatrix} a^2A + ab(B + C) + b^2D \\ acA + adB + bcC + bdD \end{pmatrix}x \otimes x
\]

\[
+ \begin{pmatrix} acA + adB + bcC + bdD \\ (c^2A + cd(B + C) + d^2D \end{pmatrix}y \otimes x
\]

\[
+ \begin{pmatrix} acA + adB + bcC + bdD \\ (c^2A + cd(B + C) + d^2D \end{pmatrix}y \otimes x.
\]
Lemma 1. The subspace of $SL(2)$-invariants of $V \otimes V$ is one-dimensional, and we can choose
\[ x \wedge y = x \otimes y - y \otimes x \]
as a basis element in that space.

Proof. From (11),
\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left( x \otimes y - y \otimes x \right) = (ad - bc)(x \otimes y - y \otimes x) = x \otimes y - y \otimes x. \]
(13)
It means that $x \wedge y$ is invariant.

Recall some fundamental facts about the representations of $SL(2)$. The word representation will mean below a polynomial finite dimensional linear representation over $f$. Every representation of $SL(2)$ is equivalent to a sum of irreducible representations. All classes of equivalence of the irreducible representations are parametrized by nonnegative integers, and we can choose the natural actions $\rho_k$ of $SL(2)$ in the symmetric powers $S^k V$, $k = 0, 1, 2, \ldots$ as their representatives. In other words, $\rho_0$ is a trivial 1-dimensional representation, $\rho_1$ is the standard representation (8), and for $k > 0$ the representation $\rho_k$ acts in the $(k+1)$-dimensional linear $f$-space of the homogeneous polynomials of degree $k$ of two variables $x$ and $y$, which we can provide with the basis of monomials $(x^k, x^{k-1}y, \ldots, y^k)$, such that
\[ x^k \mapsto (ax + cy)^k, \quad x^{k-1}y \mapsto (ax + cy)^{k-1}(bx + dy), \quad \ldots, \quad y^k \mapsto (bx + dy)^k. \]
(14)
By definition,
\[ S^2 V = (V \otimes V)/(f x \wedge y). \]
(15)
It means that
\[ \rho_1 \otimes \rho_1 \simeq \rho_2 \oplus \rho_0. \]
(16)
As we see, the subspace of invariants is 1-dimensional.

To study decompositions of tensor products of representations, it is convenient to consider characters, i.e. traces of the representations. For a diagonal matrix $\begin{pmatrix} q^0 & 0 \\ 0 & q^{-1} \end{pmatrix}$ formulas (14) give us
\[ x^k \mapsto q^k x^k, \quad x^{k-1}y \mapsto q^{k-2}x^{k-1}y, \quad \ldots, \quad y^k \mapsto q^{-k}y^k. \]
(17)
It means that
\[
\text{ch } \rho_k = q^k + q^{k-2} + \cdots + q^{-k} = \frac{q^{k+1} - q^{-(k+1)}}{q - q^{-1}}.
\] (18)

**Characters** are elements of the \(\mathbb{Z}/2\)-graded ring
\[
K = \mathbb{Z}[q + q^{-1}] = K^{\text{odd}} \oplus K^{\text{even}}.
\] (19)

Two representations are equivalent iff they have the same characters. For any representations \(\sigma, \tau\) we have
\[
\text{ch } \sigma \oplus \tau = \text{ch } \sigma + \text{ch } \tau
\] (20)
and
\[
\text{ch } \sigma \otimes \tau = \text{ch } \sigma \cdot \text{ch } \tau.
\] (21)

Characters of the representations \(\rho_k\) with odd (even) \(k\) form a basis of \(\mathbb{Z}\)-module \(K^{\text{odd}}\) (or \(K^{\text{even}}\), correspondingly), and it is useful to know the exact formulas for the coefficients in that basis of arbitrary odd (or even) character:
\[
\sum_{\text{odd (even) } k} C_k q^k = \sum_{\text{odd (even) } k \geq 0} (C_k - C_{k+2}) \text{ch } \rho_k.
\] (22)

**Lemma 2.** The dimension of the subspace of \(\text{SL}(2)\)-invariants in \(V \otimes^m\) equals to the Catalan number
\[
c_n = \frac{(2n)!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n-1}
\] (23)
for \(m = 2n\), and 0 otherwise.

**Proof.** Using formula (22), we get
\[
(ch \rho_1)^m = (q + q^{-1})^m = \sum_{m \equiv k \mod 2} \binom{m}{(m-k)/2} q^k = \sum_{k=0}^{m} c_m^{(k)} \text{ch } \rho_k,
\] (24)
where
\[
c_m^{(k)} = \begin{cases} 
\binom{m}{(m-k)/2} - \binom{m}{(m-k)/2-1} & \text{if } m - k \text{ is even,} \\
0 & \text{otherwise.}
\end{cases}
\] (25)

In particular, the dimension of the subspace of invariants in \(V \otimes^m\) is equal to \(c_m^{(0)} = c_n\) for \(m = 2n\), and 0 otherwise.

\qed
Catalan numbers $c_n$ count the number of 0-outerplanar regular graphs of degree 1 with $m = 2n$ vertices.

**Definition 1.** A **0-outerplanar graph** is a graph with the set of vertices $\{1, 2, \ldots, m\} \subseteq \mathcal{H} = \{z \in \mathbb{C} \mid \Im z \geq 0\}$, edges of which can be drawn in the upper half-plane $\mathcal{H}$ without intersections.

For example, here are two of $c_3 = 5$ 0-outerplanar regular graphs of degree 1 with 6 vertices:

![Graphs](image)

The tensor products of invariants are invariants. Edges of each 0-outerplanar regular graph of degree 1 give us $n$ pairs of numbers from 1 to $m = 2n$. If we take the tensor product of the known invariants $x \land y$ in each of those pairs, we get an invariant in $V^{\otimes m}$. The count of such invariants is equal to the count of the considered 0-outerplanar graphs, i.e., with the Catalan number $c_n$. And we know that the dimension of the subspace of invariants in $V^{\otimes m}$ equals the same Catalan number $c_n$. It is natural to suppose that the constructed invariants form a basis of the space of invariants. To prove that, it is enough to show that they are linearly independent.

The standard basis $B$ of $V^{\otimes m}$ consists of $2^m$ tensor products $x_1 \otimes \cdots \otimes x_m$ with $x_1, \ldots, x_m \in \{x, y\}$. We suppose that $B$ is ordered lexicographically. Denote $\mathcal{O}(G)$ the set of directed graphs, underlying undirected graph of which is $G$. Let $G$ be an 0-outerplanar regular graph of degree 1, and $g \in \mathcal{O}(G)$. Denote

$$b_g = x_1(g) \otimes \cdots \otimes x_m(g) \in B$$

setting $x_i(g) = x$, $x_j(g) = y$ for each edge $(i, j)$ in $g$. In other words, we put $x$ at the beginning of each arrow of $g$, and $y$ — at its end. We can write the tensor product of $x \land y$ corresponding to $G$ that was introduced in the last paragraph as

$$t_G = \sum_{g \in \mathcal{O}(G)} (-1)^{\text{inv } g} b_g,$$

where $\text{inv } g$ is the number of inversions in $g$, i.e., the number of arrows $(i, j)$ in $g$ with $i > j$.

**Theorem 1.** Tensors $t_G$ parametrized by all regular 0-outerplanar graphs of degree 1, form a basis in the subspace of invariants in the representation $\rho_1^{\otimes m}$.  

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Proof. Notice that for each $G$ exists exactly one $g_0 \in \mathcal{O}(G)$ without inversions—with the orientation of each edge from the left to the right. Changing the orientation of the arrows of $g_0$ increases $b_g$ in the lexicographical order of $B$. It means that $b_{g_0}$ is the minimal element with a non-zero coefficient in the decomposition of $t_G$ in the basis $B$. Denote $b_G = b_{g_0}$. For the element of type $b_G$, we can reconstruct $G$, associating the left bracket with each $x$, the right bracket with each $y$, and connecting the corresponding left and right brackets. So, we have $c_n$ elements $b_G$—one for each $G$.

To prove the linear independence of the set of $t_G$, we can show that the rank of the $c_n \times 2^m$ matrix of the coefficients of $t_G$ in the basis $B$ is equal to $c_n$. To do that, we can find a non-zero $c_n \times c_n$ minor of that matrix. Consider the $c_n \times c_n$ submatrix with rows numbered by $G$ ordered the same way as $b_G$, and columns corresponding to $b_G$. As we noticed in the previous paragraph, $b_G$ is the first element with a nonzero coefficient in the row $G$, and this coefficient equals 1 by definition. So, this matrix is unipotent, its determinant equals 1, that completes the proof of the linear independence of $t_G$. 

3 General tensor invariants

Now we allow for a graph to have multiple edges, but don’t allow loops. Let $G$ be an $0$-outerplanar graph and $g \in \mathcal{O}(G)$. For each vertex $i$ of $g$ denote

$$x_i(g) = x^{d_i^{\text{out}}(g)} y^{d_i^{\text{in}}(g)} \in S^{d_i} V,$$  

where $d_i^{\text{out}}(g)$ is the number of arrows in $g$, beginning in $i$, $d_i^{\text{in}}(g)$ is the number of arrows, ending in $i$, and

$$d_i = d_i^{\text{out}}(g) + d_i^{\text{in}}(g)$$

is the degree of $i$. In other words, we put $x$ at the beginning of each arrow of $g$, and $y$—at the end, and multiply those $x$s and $y$s in each vertex. Denote

$$b_g = x_1(g) \otimes \cdots \otimes x_m(g) \in S^{d_1} V \otimes \cdots \otimes S^{d_m} V$$

and

$$t_G = \sum_{g \in \mathcal{O}(G)} (-1)^{\text{inv}} g b_g.$$
Theorem 2. For any fixed $d_1, \ldots, d_m$, tensors $t_G$ parametrized by all 0-outerplanar graphs with degrees $d_1, \ldots, d_m$, form a basis in the subspace of invariants in the representation $\rho_d \otimes \cdots \otimes \rho_{d_m}$.

Proof. First, we can compare the dimension of the space of invariants and the number of 0-outerplanar graphs with the given degrees. Using characters, we get

$$\text{ch} \rho_{d_1} \otimes \cdots \otimes \rho_{d_m} = \prod_{i=1}^{m} (q^{d_i} + q^{d_i-2} + \cdots + q^{-d_i}) = \sum_{k \equiv |d| \mod 2} \sum_{k=0}^{|d|} c_d^{(k)} \text{ch} \rho_k,$$

where $d = (d_1, \ldots, d_m)$, $|d| = d_1 + \cdots + d_m$ and

$$c_d^{(k)} = \begin{cases} C_d^{(k)} - C_d^{(k+2)} & \text{if } |d| - k \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Clebsh-Gordon formula to the last two items of the product, we get

$$\text{ch} \rho_{d_{m-1}} \otimes \text{ch} \rho_{d_m} = \text{ch} \rho_{d_{m-1}+d_{m}} + \text{ch} \rho_{d_{m-1}+d_{m-2}} + \cdots + \text{ch} \rho_{|d_m-d_{m-1}|}.$$  

(34)

It gives us the recursion relation

$$c_d^{(k)} = c_{(d_1, \ldots, d_{m-2}, d_{m-1}+d_{m})}^{(k)} + c_{(d_1, \ldots, d_{m-2}, d_{m-1}+d_{m}-2)}^{(k)} + \cdots + c_{(d_1, \ldots, d_{m-2}, |d_m-d_{m-1}|)}^{(k)}$$

(35)

for any $k$, and in particular, for $k = 0$ defining the dimension of the space of invariants. That relation makes it possible to decrease $m$, and for $m = 1$ we have the initial conditions

$$c_d^{(k)} = \delta_{dk},$$

(36)

because the representations $\rho_d$ and $\rho_k$ are irreducible.

Denote $c_d$ the number of 0-outerplanar graphs with degrees of vertices $d = (d_1, \ldots, d_m)$. For any 0-outerplanar graph with the given degrees, contracting $m$ to $m-1$, and deleting after that all the loops at the last vertex, we get an 0-outerplanar graph with the degrees $(d_1, \ldots, d_{m-2}, d_{m-1} + d_m - 2a_{m-1,m})$, where $a_{m-1,m}$ denotes the number of edges between $m-1$ and $m$ —each edge has two ends—it explains the coefficient 2. The number $a_{m-1,m}$ can be any integer between 0 and $\min\{d_{m-1}, d_m\}$. That means the last degree in the new 0-outerplanar graph can be $d_{m-1} + d_m, d_{m-1} + d_m - 2, \ldots, |d_m - d_{m-1}|$. This procedure is reversible: for any 0-outerplanar graph with admissible new degrees we can construct an 0-outerplanar graph with degrees $d$, moving the last $d_m$ ends of edges from $m-1$ to $m$ and adding $a_{m-1,m}$ edges between $m-1$ and $m$. I wrote 'the last' bearing in mind the natural order on the ends of edges of a vertex $i$ of an 0-outerplanar graph: we
can suppose that a small half-circle with the center in \(i\), is lying in the upper half-plane \(\mathcal{H}\), which intersects with each edge ending in \(i\), exactly in one point, and we can order these points on the half-circle clockwise, and these ends of edges—correspondingly. So, we have a bijection between the set of 0-outerplanar graphs with degrees \(d\), and the union of the sets of 0-outerplanar graphs of the mentioned above degrees. It gives us the same recursion relation as (35) for \(c^{(k)}_d\):

\[
c_d = c(d_1,\ldots,d_{m-2},d_{m-1}+d_m) + c(d_1,\ldots,d_{m-1},d_{m-2}+2) + \cdots + c(d_1,\ldots,d_{m-2},d_m-d_{m-1}).
\]

(37)

For \(m = 1\) we have only one 0-outerplanar graph—with one vertex 1 and without edges. So, we have the same initial conditions as (36) for \(c^{(0)}_d\):

\[
c_d = \delta_{d0}.
\]

(38)

Thus, we proved that the dimension \(c^{(0)}_d\) of the space of invariants is equal to the number \(c_d\) of the 0-outerplanar graphs with the given degrees \(d\).

The proof of the linear independence of the set of \(t_G\) is exactly the same as for the case of regular 0-outerplanar graphs of degree 1. Denote \(B\) the standard basis of \(S^{d_1}V \otimes \cdots \otimes S^{d_m}V\), consisting of \((d_1 + 1)\cdots(d_m + 1)\) tensor products \(x_1 \otimes \cdots \otimes x_m\) with \(x_i \in \{x^{d_i}, x^{d_i-1}y, \ldots, y^{d_i}\}\) for \(i = 1, \ldots, m\).

We suppose that \(B\) is ordered lexicographically. We have \(b_g \in B\) for all \(g \in \mathcal{O}(G)\). For each \(G\) exists exactly one \(g_0 \in \mathcal{O}(G)\) without inversions - with orientation each edge from the left to the right. Changing the orientation of the arrows of \(g_0\) increases \(b_g\) in the lexicographical order of \(B\). It means that \(b_{g_0}\) is the minimal element with a non-zero coefficient in the decomposition of \(t_G\) in the basis \(B\). Denote \(b_G = b_{g_0}\). In any vertex \(i\) of \(g_0\) we have, first of all the incoming arrows, and then - all the arrows coming out, in the order of the ends of the edges described above. Thus, for an element of type

\[
b_G = x^{d_{1_{\text{in}}}}y^{d_{1_{\text{out}}}} \otimes \cdots \otimes x^{d_{m_{\text{in}}}}y^{d_{m_{\text{out}}}},
\]

(39)

we can reconstruct \(G\), associating the sequence of \(d_{i_{\text{in}}}\) in the right brackets followed by \(d_{i_{\text{out}}}\) in the left brackets, with each vertex \(i\), and the connecting corresponding left and right brackets. So, we have \(c_d\) elements \(b_G\) —one for each \(G\). Consider the \(c_d \times c_d\) submatrix with rows numbered by \(G\) ordered the same way as \(b_G\), and columns corresponding to \(b_G\), of the \(c_d \times |B|\) matrix of the coefficients of \(t_G\) in the basis \(B\). As we noticed, \(b_G\) is the first element with a nonzero coefficient in the row \(G\), and this coefficient equals 1 by definition. So, this matrix is unipotent, its determinant equals 1, that completes the proof of the linear independence of \(t_G\).
To complete the proof of the theorem, we have to show that the tensors \( t_G \) are \( SL(2) \)-invariant. The representation \( \rho_{d_1} \otimes \cdots \otimes \rho_{d_m} \) is a subrepresentation of \( \rho^\otimes m \) in \( (SV)^\otimes m \), where \( SV = f[x, y] \) and \( \rho \) denotes the standard changing of variables. \( (SV)^\otimes m \) is an algebra, and \( \rho^\otimes m \) commutes with its multiplication

\[
(a_1 \otimes \cdots \otimes a_m)(b_1 \otimes \cdots \otimes b_m) = a_1 b_1 \otimes \cdots \otimes a_m b_m,
\]

because \( \rho \) commutes with the multiplication in \( SV \). In particular, a product of invariants is invariant. It follows from the definition that

\[
t_G = \prod_{(i,j) \in E(g_0)} \iota_{ij}(x \wedge y),
\]

where \( E(g_0) \) denotes the set of the arrows of \( g_0 \), and

\[
\iota_{ij} : V \otimes V \to (SV)^\otimes m, \quad v \otimes w \mapsto 1^\otimes(i-1) \otimes v \otimes 1^\otimes(j-i-1) \otimes w \otimes 1^\otimes(m-j),
\]

i. e. it puts \( v \) on the \( i \)-th place and \( w \) on the \( j \)-th place of the tensor product. Now, \( \iota_{ij} \) commutate with the actions of \( SL(2) \), \( x \wedge y \) is invariant, and \( t_G \), the product of invariants, is invariant. \( \square \)

**Proposition 1.** Let \( E(G) \) be the set of the edges of the 0-outerplanar graph \( G \) with \( m \) vertices. For any disjoint union

\[
E(G) = E_1 \amalg \cdots \amalg E_k
\]

we have

\[
t_G = t_{G_1} \cdots t_{G_k},
\]

where \( G_1, \ldots, G_k \) are the 0-outerplanar graphs with \( m \) vertices, with the sets of the edges \( E_1, \ldots, E_k \) correspondingly.

**Proof.** It follows from (41) immediately. \( \square \)

### 4 Examples

If \( m = 1 \), we have exactly one 0-outerplanar graph: with the vertex 1 and without edges. It means that the subspace of invariants of \( \rho_k \) is nontrivial iff \( k = 0 \). In that case, the corresponding invariant is \( 1 \in f \).
Corollary 1. The subspace of invariants of $S^{d_1}V \otimes S^{d_2}V$ is nontrivial iff $d_1 = d_2 = a$ for an integer $a \geq 0$. In that case, the space of invariants is one-dimensional with the basis element

$$(x \wedge y)^a = \sum_{i=0}^{a} (-1)^i \binom{a}{i} x^{a-i} y^i \otimes x^i y^{a-i} \in S^a V \otimes S^a V. \quad (45)$$

Proof. If $m = 2$, the degrees $d_1, d_2$ of any 0-outerplanar graph $G$ are equal. For any integer $a$ there is a unique 0-outerplanar graph with two vertices of degrees $d_1 = d_2 = a$. Formula (45) follows from (41).

Corollary 2. The subspace of invariants of $S^{d_1}V \otimes S^{d_2}V \otimes S^{d_3}V$ is nontrivial iff the integers $d_1, d_2$, and $d_3$ could be the lengths of the sides of a triangle (perhaps, degenerate, i.e., sides can have zero length, or a vertex can be situated in the opposite side) with an even perimeter. In that case, the space of invariants is one-dimensional with the basis element

$$t_G = ((x \wedge y)^a \otimes 1)(x \otimes 1 \otimes y - y \otimes 1 \otimes x)^b(1 \otimes (x \wedge y)^c), \quad (46)$$

where

$$a = \frac{d_1 + d_2 - d_3}{2}, \quad b = \frac{d_1 + d_3 - d_2}{2}, \quad c = \frac{d_2 + d_3 - d_1}{2}. \quad (47)$$

Proof. If $m = 3$, we have a 0-outerplanar graph with $a$ edges between 1 and 2, with $b$ edges between 1 and 3, and $c$ edges between 2 and 3 for any nonnegative integers $a$, $b$, and $c$. In this case

$$d_1 = a + b, \quad d_2 = a + c, \quad d_3 = b + c, \quad (48)$$

and conversely (47). Formula (46) follows from (41).

In the degenerate cases some of $a, b, c$ equal 0. If all of them equal 0, we have a trivial representation. If two of them equal 0, say $b = c = 0$, the situation is the same as for $m = 2$:

$$t_G = (x \wedge y)^a \otimes 1 = \sum_{i=0}^{a} (-1)^i \binom{a}{i} x^{a-i} y^i \otimes x^i y^{a-i} \otimes 1 \in S^a V \otimes S^a V \otimes f. \quad (49)$$
If one of them equals 0, say $b = 0$, it follows from Corollary 1 and Proposition 1, that
\[
t_G = ((x \land y)^a \otimes 1)(1 \otimes (x \land y)^c) = \sum_{i=0}^{a} \sum_{j=0}^{c} (-1)^{i+j} \binom{a}{i} \binom{c}{j} x^{a-i} y^i \otimes x^{c+i-j} y^{a-i-j} \otimes x^j y^{c-j} \in S^a V \otimes S^{a+c} V \otimes S^c V. \tag{50}
\]

It is a special case of the following

**Proposition 2.** Let $\Gamma_G$ denote the graph without multiple edges, with the same vertices as $G$, in which an edge between two vertices exists iff there are edges between these vertices in $G$. Then
\[
t_G = \sum_{i_1=0}^{a_1} \ldots \sum_{i_N=0}^{a_N} (-1)^{i_1+\ldots+i_N} \binom{a_1}{i_1} \ldots \binom{a_N}{i_N} b_{i_1,\ldots,i_N}, \tag{51}
\]
where $N$ is the number of edges of $\Gamma_G$, and for each edge $i_k$ of $\Gamma_G$ we denote $a_k$ the number of the edges of $G$ connecting the same vertices, and $b_{i_1,\ldots,i_N} = b_g$ for any directed graph $g \in \mathcal{O}(G)$ with the exactly $i_1$ inverted arrows connecting the vertices of the 1-st edge of $\Gamma_G, \ldots$, the exactly $i_N$ inverted arrows connecting the vertices of the $N$-th edge of $\Gamma_G$. If $\Gamma_G$ is a tree, i.e. a graph (not necessary connected) without cycles, then (51) is the decomposition of $t_G$ in the elements of the basis $B$.

**Proof.** It follows from Corollary 1 and Proposition 1 that formula (51) is true for any 0-outerplanar graph $G$. We have to prove that for the considered case in which $\Gamma(G)$ is a tree, tensors $b_{i_1,\ldots,i_N}$ for different $i_1, \ldots, i_N$ are not equal. We’ll use the induction on $N$. For $N = 0$ we have only one item in the right hand side of (51), so the statement is true. Let $N > 0$ and we know that the statement is true for $N - 1$. Choose the vertex $k$ of $\Gamma(G)$ with degree 1 (it exists because $\Gamma(G)$ is a tree). Then $k$-th component of $b_{i_1,\ldots,i_N}$ equals by formula (28) to
\[
x_k = \begin{cases} x^{a_K-i_K} y^{i_K} & \text{if } k < l, \\ x^{i_K} y^{a_K-i_K} & \text{if } k > l, \end{cases} \tag{52}
\]
where $K = \{k,l\}$ is the unique edge of $\Gamma(G)$ having an end $k$, and another its end is $l$. It means that $b_{i_1,\ldots,i_N}$ are different for different $i_K$. And if $i_K$ are the same, then by induction, considering $G$ without edges connecting $k$ and $l$, we see that $b_{i_1,\ldots,i_N}$ without $k$-th component, and without some factor on $l$-th place, are different for different $i_1, \ldots, i_N$. They will differ on the same place after adding equal $k$-th components and multiplying $l$-th component on equal factors.
If $\Gamma(G)$ is not a tree, some $b_{\ldots}$ in formula (51) can be the same. For instance, let $a = b = c = 1$. Then

$$(-1)^{\text{inv}} b_g = \begin{cases} +x^2 \otimes xy \otimes y^2 & \text{if } g = 1 \rightarrow 2 \rightarrow 3, \\ -xy \otimes xy \otimes xy & \text{if } g = 1 \rightarrow 2 \rightarrow 3, \\ -x^2 \otimes y^2 \otimes xy & \text{if } g = 1 \rightarrow 2 \rightarrow 3, \\ +xy \otimes y^2 \otimes x^2 & \text{if } g = 1 \rightarrow 2 \rightarrow 3, \\ -xy \otimes x^2 \otimes y^2 & \text{if } g = 1 \rightarrow 2 \rightarrow 3, \\ +y^2 \otimes x^2 \otimes xy & \text{if } g = 1 \rightarrow 2 \rightarrow 3, \\ +xy \otimes xy \otimes xy & \text{if } g = 1 \rightarrow 2 \rightarrow 3, \\ -y^2 \otimes xy \otimes x^2 & \text{if } g = 1 \rightarrow 2 \rightarrow 3. \end{cases}$$ (53)

The items corresponding to cycles, cancel, and $t_G$ is the sum of the other items. It is a special case of the following

**Corollary 3.** In the situation of the Corollary 2 for $d_1 = d_2 = d_3 = 2a$,

$$t_G = (x^2 \wedge xy \wedge y^2)^a.$$ (54)

**Proof.** For $a = 1$ see (53), and for the other $a$ it follows from this special case and Proposition 2. □

### 5 Tensor decompositions

In this section we use slightly different outerplanar graphs than before: with vertices starting from 0 instead of 1:

**Definition 2.** An **outerplanar graph** is a graph with the set of vertices $\{0, 1, 2, \ldots, m\} \subseteq \mathcal{H} = \{z \in \mathbb{C} \mid \Im z \geq 0\}$, edges of which can be drawn in the upper half-plane $\mathcal{H}$ without intersections.

We still allow for a graph to have multiple edges, but don’t allow loops. Also we’ll use the same formulas (28–30) to define $x_i(g)$ and $b_g$ for an directed graph $g \in \mathcal{O}(G)$. Note that formula (31) for $b_g$ doesn’t include $x_0(g)$. For any nonnegative integer $i \leq d_0(G)$ denote

$$t_{G,i} = \sum_{g \in \mathcal{O}(G) \atop d_{a_0}(g) = i} (-1)^{\text{inv}} b_g.$$ (55)

**Theorem 3.** For any fixed $d_0, d_1, \ldots, d_m$, tensors $t_{G,i}$ parametrized by all outerplanar graphs with degrees $d_0, d_1, \ldots, d_m$ and nonnegative integers $i \leq d_0$, form a basis in the isotypic component of the
type $\rho_{d_0}$ in the representation $\rho_{d_1} \otimes \cdots \otimes \rho_{d_m}$. For any fixed outerplanar graph $G$, the subspace $T_G$ spanned by the basis $(t_{G,0}, \ldots, t_{G,d_0})$, are invariant; the linear homomorphism

$$s_G : S^{d_0}V \to T_G, \quad x^iy^{d_0-i} \mapsto (-1)^i \frac{1}{d_0} t_{G,i}$$

(56)
defines the isomorphism of $\rho_{d_0}$ and the subrepresentation of $SL(2)$ in $T_G$, and

$$S^{d_1}V \otimes \cdots \otimes S^{d_m}V = \bigoplus_G T_G,$$

(57)
where the direct sum in the right hand side is taken over all the outerplanar graphs of degrees $d_0, d_1, \ldots, d_m$ with all possible values of $d_0$.

**Proof.** First we’ll compare the multiplicity of $\rho_{d_0}$ with the count of the outerplanar graphs. By formulas (32–33), this multiplicity equals

$$c_{d_1,\ldots,d_m}^{(d_0)} = \begin{cases} C_{d_1,\ldots,d_m}^{(d_0)} - C_{d_1,\ldots,d_m}^{(d_0+2)} & \text{if } d_0 + d_1 + \cdots + d_m \text{ is even,} \\ 0 & \text{otherwise,} \end{cases}$$

(58)
where the coefficients $C$: are defined by the generating function

$$\sum_k C_{d_1,\ldots,d_m}^{(k)} q^k = \prod_{i=1}^{m} (q^{d_i} + q^{d_i-2} + \cdots + q^{-d_i}).$$

(59)
We can easily transform an outerplanar graph to a 0-outerplanar graph and backwards, shifting it to the right, or to the left, correspondingly. Thus the number of the outerplanar graphs with degrees $d_0, \ldots, d_m$ is equal to the number of the 0-outerplanar graphs with the same degrees, but shifted: the degree of 1 must be $d_0$, ..., the degree of $m + 1$ must be $d_m$, i.e. equal to

$$c_{d_0,d_1,\ldots,d_m} = c_{d_0,0,d_1,\ldots,d_m}^{(0)}.$$ 

(60)
This equality was proved on the first step of the proof of Theorem 3. Now

$$c_{d_1,\ldots,d_m}^{(d_0)} = C_{d_1,\ldots,d_m}^{(d_0)} - C_{d_1,\ldots,d_m}^{(d_0+2)} = C_{d_1,\ldots,d_m}^{(d_0)} - C_{d_1,\ldots,d_m}^{(d_0-2)}$$

$$= \text{res}_0 (q^{-d_0-1} - q^{d_0+1}) \prod_{i=1}^{m} (q^{d_i} + q^{d_i-2} + \cdots + q^{-d_i})$$

$$= \text{res}_0 (q^{-1} - q) \prod_{i=0}^{m} (q^{d_i} + q^{d_i-2} + \cdots + q^{-d_i})$$

$$= c_{d_0,d_1,\ldots,d_m}^{(0)} - C_{d_0,d_1,\ldots,d_m}^{(2)} = \delta_{d_0,d_1,\ldots,d_m}^{(0)} = c_{d_0,d_1,\ldots,d_m},$$

(61)
where res$_0$ denotes a residue in 0. It means that the number of outerplanar graphs of degrees $d_0, d_1, \ldots, d_m$ coincides with the multiplicity of $\rho_{d_0}$ in $\rho_{d_1} \otimes \cdots \otimes \rho_{d_m}$. Notice, that using formula (60) we are able to obtain that result very easily from the following equality as well:

$$\text{mult}_{\rho} \tau = \dim \text{Inv}(\rho^* \otimes \tau)$$  \hspace{1cm} (62)$$

for any irreducible representation $\rho$ and representation $\tau$, where mult$_{\rho} \tau$ is the multiplicity of $\rho$ in $\tau$, $\rho^*$ denotes the representation conjugated to $\rho$, and dim Inv is the dimension of the subspace of invariants. In our case

$$c_{d_1, \ldots, d_m}^{(d_0)} = \text{mult}_{\rho_{d_0}} \rho_{d_1} \otimes \cdots \otimes \rho_{d_m} = \dim \text{Inv}(\rho_{d_0} \otimes \rho_{d_1} \otimes \cdots \otimes \rho_{d_m}) = c_{d_0, d_1, \ldots, d_m}^{(0)}.$$  \hspace{1cm} (63)$$

On the second step of the proof we’ll show that the set of $t_{G,0}$ is linearly independent. For each $G$ exists exactly one $g_0 \in O(G)$ without inversions - with orientation each edge from the left to the right. Changing the orientation of the arrows of $g_0$ with non-zero ends increases $b_g$ in the lexicographical order of $B$. It means that $b_{g_0}$ is the minimal element with a non-zero coefficient in the decomposition of $t_{G,0}$ in the basis $B$. Denote $b_G = b_{g_0}$. Note that for the 0-outerplanar graph $G'$, obtained from $G$ by shifting it to the right on 1, we have

$$b_{G'} = x^{d_0} \otimes b_G.$$  \hspace{1cm} (64)$$

In the proof of Theorem 4, we checked that all $c_{d_0, d_1, \ldots, d_m}$ elements $b_{G'}$ are different for all the 0-outerplanar graphs of degrees $d_0, d_1, \ldots, d_m$. Thus, by formula (64), all $c_{d_0, d_1, \ldots, d_m}$ elements $b_G$ are different for all the outerplanar graphs of degrees $d_0, d_1, \ldots, d_m$. Consider the $c_{d_0, d_1, \ldots, d_m} \times c_{d_0, d_1, \ldots, d_m}$ submatrix with rows numbered by $G$ ordered the same way as $b_G$, and columns corresponding to $b_G$, of the $c_{d_0, d_1, \ldots, d_m} \times |B|$ matrix of the coefficients of $t_{G,0}$ in the basis $B$. As we noticed, $b_G$ is the first element with a nonzero coefficient in the row $G$, and this coefficient equals 1 by definition. So, this matrix is unipotent, its determinant equals 1, that completes the proof of the linear independence of $t_{G,0}$.

Third step: $T_G$. First consider the case of a star with the center in 0, i.e. $G$ having degrees $d_0 = m$, $d_1 = \cdots = d_m = 1$:  

```
0 1 2 m-1 m
```

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In this case
\[ t_{G,i} = (-1)^i \binom{m}{i} \text{Sym}(x^\otimes i \otimes y^\otimes (m-i)), \] (65)
where \( x^\otimes i = x \otimes \cdots \otimes x \) \((m\text{ times})\) and \( y^\otimes (m-i) = y \otimes \cdots \otimes y \) \((m-i\text{ times})\) analogously, and
\[ \text{Sym}(x_1 \otimes \cdots \otimes x_m) = \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}, \] (66)
where \( S_m \) is the symmetric group of permutations of \(1, \ldots, m\). Thus
\[ T_G = S_mV = \text{Sym}(V^\otimes m). \] (67)

By the definition of the actions of \(SL(2)\) in \(S^mV\) and \(V^\otimes m\), the homomorphism
\[ s_G : S^mV \to S_mV, \quad x^i y^{m-i} \mapsto \text{Sym}(x^\otimes i \otimes y^\otimes (m-i)) \] (68)
is intertwining, therefore \(s_G\) is an isomorphism and \(T_G = S_mV\) is an invariant subspace.

Now let \( G \) be a star with the center in 0 and multiple edges, i. e.
\[ d_0 = d_1 + \cdots + d_m. \] (69)
Suppose for simplicity that all \(d_1, \ldots, d_m\) are not zero. Denote \( \tilde{G} \) the covering \(G\) star with the the same degree \(d_0\) of 0 but without multiple edges, i. e. with \(d_1 = \cdots = d_{d_0} = 1\). Consider a projection
\[ \text{Pr} = \text{pr}_{d_1} \otimes \cdots \otimes \text{pr}_{d_m} : V^{d_0} \to S^{d_1}V \otimes \cdots \otimes S^{d_m}, \] (70)
where
\[ \text{pr}_d : V^\otimes d \to S^dV \] (71)
is the projection defining \(S^d\). One can check that for any nonnegative integer \(i \leq d_0\),
\[ t_{G,i} = \text{Pr}(t_{\tilde{G},i}). \] (72)
Hense
\[ T_G = \text{Pr}(T_{\tilde{G}}). \] (73)
By definitions \((70, 71)\), the projection \(\text{Pr}\) is intertwining with the \(SL(2)\)-actions. Therefore, \(T_G\) is an invariant subspace since \(T_{\tilde{G}}\) is an invariant subspace. The representation of \(SL(2)\) in \(T_G\) is a
factor-representation of the irreducible representation of $SL(2)$ in $T_G$, and $T_G$ is non-zero space, because it contains $t_{G,0}$ that is not zero since the set of the tensors $t_{G,0}$ is linearly independent—it was stated on the first step of the proof of the theorem. It means that the representation of $SL(2)$ in $T_G$ is equivalent $\rho_{d_0}$, and

$$s_G = \Pr \circ s_{\tilde{G}} \quad (74)$$

is the corresponding isomorphism.

Adding a few vertices of degrees 0 to $G$ doesn’t change the situation, because of the canonical isomorphism:

$$\alpha : S^{d_1}V \otimes \cdots \otimes S^{d_m}V \to 1^ \otimes n_0 \otimes S^{d_1}V \otimes 1^ \otimes n_1 \otimes \cdots \otimes 1^ \otimes n_{m-1} \otimes S^{d_m} \otimes 1^ \otimes n_m, \quad (75)$$

adding 1’s to every place corresponding to a vertix of degree 0. In particular, we have

$$t_{G+,i} = \alpha(t_{G,i}), \quad T_{G+} = \alpha(T_G), \quad (76)$$

and

$$s_{G+} = \alpha \circ s_G, \quad (77)$$

where $G_+$ is an outerplanar graph obtained from $G$ by adding $n_0$ vertices of degree 0 between 0 and 1, $\ldots$, $n_k$ vertices of degree 0 between $k$ and $k+1$, $\ldots$, $d_m$ vertices of degree 0 after $m$.

Now, one can check that for an arbitrary outerplanar graph $G$, and for any nonnegative integer $i \leq d_0$,

$$t_{G,i} = t_{G+,i} \cdot t_{G_0}, \quad (78)$$

where $G_0$ is the 0-outerplanar graph obtained from $G$ by deleting the vertex 0 together with all the edges ending in 0, and $G_+$ is the star obtained from $G$ by deleting all the edges of $G_0$. Since we stated in Theorem 2 that $t_{G_0}$ is invariant, and the multiplication is intertwining, it follows from (78) that the linear homomorphism

$$\mu : T_{G_0} \to T_G, \quad t \mapsto t \cdot t_{G_0} \quad (79)$$

is intertwining and surjective. Further, $T_G$ is nonzero, because it contains $t_{G,0} \neq 0$, see above. Since the representation of $SL(2)$ in $T_G$ is irreducible, and equivalent $\rho_{d_0}$, the representation of $SL(2)$ in $T_G$, equivalent its factor-representation, is equivalent $\rho_{d_0}$ as well, and

$$s_G = \mu \circ s_{G_0} \quad (80)$$

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is the corresponding isomorphism.

Final step. We already know that all the subspaces $T_G$ are invariant, and the count of them is exactly the same as necessary for (\ref{eq:57}). Now we can use the following

**Lemma 3.** For any isotypic representation $T$ of $SL(2)$ and for any $q \in f$ that is not a root of unity, all the eigenspaces of $T\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ have the same dimension equal to the number of the irreducible components of $T$.

**Proof.** For $\rho_d$, the basis elements $x^d, \ldots, x^iy^{d-i}, \ldots, y^d$ are the eigenvectors of $\rho_d\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ with the eigenvalues $q^d, \ldots, q^{d-2i}, \ldots, q^{-d}$ that are all different if $q$ is not a root of unity. Therefore, for $T$ being a direct sum of $n$ representations equivalent to $\rho_d$, decomposing each component in the sum of the indicated above eigenlines (i.e. eigenspaces of dimension 1), we get the decomposition of the space of $T$ in the sum of $n$ eigenlines with the eigenvalue $q^d, \ldots, n$ eigenlines with the eigenvalue $q^{d-2i}, \ldots, n$ eigenlines with the eigenvalue $q^{-d}$. It means that the dimension of each eigenspace is $n$. \hfill \square

Consider the sum of $T_G$ for all $G$ with fixed degrees $d_0, d_1, \ldots, d_m$. It is a space of isotypic representation of the type $\rho_{d_0}$. It follows from formula (\ref{eq:54}) proved above, that each $t_{G,i}$ is an eigenvector with the eigenvalue $q^{d-2i}$ for the matrix in Lemma 3. In particular, the eigenspace with the eigenvalue $q^{-d}$ contains $c_{d_0,d_1,\ldots,d_m}$-dimensional space spanned by $t_{G,0}$. By Lemma 3, the number of irreducible components in the considered space is not less than $c_{d_0,d_1,\ldots,d_m}$. We know that the number of the irreducible components of the type $\rho_{d_0}$ in $\rho_{d_1} \otimes \cdots \otimes \rho_{d_m}$ is $c_{d_0,d_1,\ldots,d_m}$. It means that the considered sum of $c_{d_0,d_1,\ldots,d_m}$ subspaces $T_G$ is direct, and it is the space of the isotypic component of the type $\rho_{d_0}$ in $\rho_{d_1} \otimes \cdots \otimes \rho_{d_m}$. Taking the direct sum of the isotypic components, we get (\ref{eq:57}). \hfill \square

**Proposition 3.** Let $E(G)$ be the set of the edges of the outerplanar graph $G$ with $m$ vertices. For any disjoint union

$$E(G) = E_1 \amalg \cdots \amalg E_k$$

and for any nonnegative integer $i \leq d_0(G)$ we have

$$t_{G,i} = \sum_{i_1 + \cdots + i_k = i} t_{G_1,i_1} \cdots t_{G_k,i_k},$$

(82)
where $G_1, \ldots, G_k$ are the 0-outerplanar graphs with $m$ vertices, with the sets of the edges $E_1, \ldots, E_k$ correspondingly, and we suppose in the sum (82) that $i_1 \leq d_0(G_1), \ldots, i_k \leq d_0(G_k)$.

Proof. For a star without multiple edges, see (65), it follows directly from the definition (55) of $t_{G,i}$. Using formula (78) and Proposition 1 we obtain the general case.

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