Sharpness of Muqattash-Yahdi problem

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Abstract. Let $\psi$ denote the psi (or digamma) function. We determine the values of the parameters $p$, $q$ and $r$ such that

$$
\psi(n) \approx \ln(n + p) - \frac{q}{n + r}
$$

is the best approximations. Also, we present closer bounds for psi function, which sharpens some known results due to Muqattash and Yahdi, Qi and Guo, and Mortici.

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The gamma function is usually defined for $x > 0$ by

$$
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt.
$$

The logarithmic derivative of the gamma function:

$$
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{or} \quad \ln \Gamma(x) = \int_1^x \psi(t) \, dt
$$
is known as the psi (or digamma) function. The successive derivatives of the psi function \( \psi(x) \):

\[
\psi^{(n)}(x) := \frac{d^n}{dx^n}\{\psi(x)\} \quad (n \in \mathbb{N})
\]

are called the polygamma functions.

The following asymptotic formula is well known for the psi function:

\[
\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nx^{2n}} - \frac{1}{252x^6} + \cdots \quad (x \to \infty)
\]

(see [1, p. 259]), where

\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \ldots, \quad B_{2n+1} = 0 \quad (n \in \mathbb{N})
\]

are the Bernoulli numbers.

Recently, the approximations of the following form:

\[
\psi(x) \approx \ln(x+a) - \frac{1}{x}, \quad a \in [0, 1], \quad x \in [2, \infty)
\]

were studied by Muqattash and Yahdi [6]. They computed the error

\[
\left| \psi(x) - \left( \ln(x+a) - \frac{1}{x} \right) \right| \leq \ln \left( 1 + \frac{1}{x} \right) \leq \ln \left( \frac{3}{2} \right) = 0.4054651081 \ldots ,
\]

and then the approximation (2) was compared with the approximation obtained by considering the first two terms of the series (1), that is

\[
\left| \psi(x) - \left( \ln(x+a) - \frac{1}{x} \right) \right| \leq \left| \psi(x) - \left( \ln(x) - \frac{1}{2x} \right) \right| .
\]

Very recently, the family (2) was also discussed by Qi and Guo [7]. One of their main results is the following inequality on \( x \in (0, \infty) \):

\[
\ln \left( x + \frac{1}{2} \right) - \frac{1}{x} < \psi(x) < \ln(x+e^{-\gamma}) - \frac{1}{x},
\]

where \( \gamma = 0.577215 \ldots \) is the Euler–Mascheroni constant.
In the final part of the paper [6], the authors wonder whether there are profitable constants \( a \in [0, 1] \) and \( b \in [1, 2] \) for which better approximations of the form
\[
\psi(x) \approx \ln(x + a) - \frac{1}{bx}
\]
can be obtained. Mortici [3] solved this open problem and proved that the best approximations (4) appear for
\[
a = \frac{1}{\sqrt{6}} , \quad b = 6 - 2\sqrt{6}
\]
and
\[
a = -\frac{1}{\sqrt{6}} , \quad b = 6 + 2\sqrt{6}.
\]
Moreover, the author derived from [3, Theorem 2.1] the following symmetric double inequality: For \( x > \frac{1}{\sqrt{6}} = 0.40824829 \ldots \),
\[
\ln\left( x - \frac{1}{\sqrt{6}} \right) - \frac{1}{(6 + 2\sqrt{6})x} \leq \psi(x) \leq \ln\left( x + \frac{1}{\sqrt{6}} \right) - \frac{1}{(6 - 2\sqrt{6})x}.
\]
This double inequality is more accurate than the estimations (3) of Qi and Guo.

We define the sequence \((v_n)_{n \in \mathbb{N}}\) by
\[
v_n = \psi(n) - \left( \ln(n + p) - \frac{q}{n + r} \right).
\]
We are interested in finding the values of the parameters \( p, q \) and \( r \) such that \((v_n)_{n \in \mathbb{N}}\) is the fastest sequence which would converge to zero. This provides the best approximations of the form:
\[
\psi(n) \approx \ln(n + p) - \frac{q}{n + r}.
\]
Our study is based on the following Lemma 1, which provides a method for measuring the speed of convergence.

**Lemma 1 (see [4] and [5]).** If the sequence \((\lambda_n)_{n \in \mathbb{N}}\) converges to zero and if there exists the following limit:
\[
\lim_{n \to \infty} n^k(\lambda_n - \lambda_{n+1}) = l \in \mathbb{R} \quad (k > 1),
\]
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then
\[ \lim_{n \to \infty} n^{k-1} \lambda_n = \frac{l}{k-1} \quad (k > 1). \]

**Theorem 1.** Let the sequence \((v_n)_{n \in \mathbb{N}}\) be defined by (6). Then for

\[
\begin{align*}
\begin{cases}
    p &= -\frac{1}{2} - \frac{1}{6} \sqrt{9 + 6 \sqrt{3}} \\
    q &= -\frac{1}{6} \sqrt{9 + 6 \sqrt{3}} \\
    r &= -\frac{1}{2} - \frac{1}{18} \sqrt{3} \sqrt{9 + 6 \sqrt{3}},
\end{cases}
\end{align*}
\]

or

\[
\begin{align*}
\begin{cases}
    p &= -\frac{1}{2} + \frac{1}{6} \sqrt{9 + 6 \sqrt{3}} \\
    q &= \frac{1}{6} \sqrt{9 + 6 \sqrt{3}} \\
    r &= -\frac{1}{2} + \frac{1}{18} \sqrt{3} \sqrt{9 + 6 \sqrt{3}},
\end{cases}
\end{align*}
\]

we have

\[ \lim_{n \to \infty} n^5 (v_n - v_{n+1}) = \frac{1}{180} + \frac{\sqrt{3}}{54} \quad \text{and} \quad \lim_{n \to \infty} n^4 v_n = \frac{1}{720} + \frac{\sqrt{3}}{216}. \]

The speed of convergence of the sequence \((v_n)_{n \in \mathbb{N}}\) is given by the order estimate \(O\left(n^{-4}\right)\) as \(n \to \infty\).

**Proof.** First of all, we write the difference \(v_n - v_{n+1}\) as the following power series in \(n^{-1}\):

\[
\begin{align*}
    v_n - v_{n+1} &= \frac{2q - 2p - 1}{2n^2} + \frac{-3q - 6qr + 3p + 3p^2 + 1}{3n^3} \\
    &\quad + \frac{4q + 12qr + 12qr^2 - 1 - 4p - 6p^2 - 4p^3}{4n^4} \\
    &\quad \quad + \frac{-5q - 20qr - 30qr^2 - 20qr^3 + 1 + 5p + 10p^2 + 10p^3 + 5p^4}{5n^5} \\
    &\quad \quad \quad + O\left(\frac{1}{n^5}\right) \quad (n \to \infty).
\end{align*}
\]
According to Lemma 1, the three parameters $p$, $q$ and $r$, which produce the fastest convergence of the sequence $(v_n)_{n \in \mathbb{N}}$ are given by (10)

$$
\begin{align*}
2q - 2p - 1 &= 0 \\
-3q - 6qr + 3p + 3p^2 + 1 &= 0 \\
4q + 12qr + 12r^2 - 1 - 4p - 6p^2 - 4p^3 &= 0,
\end{align*}
$$

that is, by (8) and (9). We thus find that

$$v_n - v_{n+1} = \left( \frac{1}{180} + \frac{\sqrt{3}}{54} \right) \frac{1}{n^3} + O \left( \frac{1}{n^6} \right) \quad (n \to \infty).$$

Finally, by using Lemma 1, we obtain the assertion (1) of Theorem 1. 

Solutions (8) and (9) provide the best approximations of type (7):

$$\psi(n) \approx \ln \left( n - \frac{1}{2} - \frac{1}{6\sqrt{9 + 6\sqrt{3}}} \right) + \frac{3\sqrt{9 + 6\sqrt{3}}}{18n - 9 - \sqrt{3\sqrt{9 + 6\sqrt{3}}}}. \quad (11)$$

and

$$\psi(n) \approx \ln \left( n - \frac{1}{2} + \frac{1}{6\sqrt{9 + 6\sqrt{3}}} \right) - \frac{3\sqrt{9 + 6\sqrt{3}}}{18n - 9 + \sqrt{3\sqrt{9 + 6\sqrt{3}}}}. \quad (12)$$

Theorem 2 below presents closer bounds for psi function.

**Theorem 2.** For $x > \left[ \frac{1}{2} + \frac{1}{6\sqrt{9 + 6\sqrt{3}}} \right] = 1.23394491 \ldots$, then

$$\ln \left( x - \frac{1}{2} - \frac{1}{6\sqrt{9 + 6\sqrt{3}}} \right) + \frac{3\sqrt{9 + 6\sqrt{3}}}{18x - 9 - \sqrt{3\sqrt{9 + 6\sqrt{3}}} + \left( \frac{1}{720} + \frac{\sqrt{3}}{216} \right) \frac{1}{x^4}} < \psi(x) < \ln \left( x - \frac{1}{2} + \frac{1}{6\sqrt{9 + 6\sqrt{3}}} \right) \quad (13)$$

$$- \frac{3\sqrt{9 + 6\sqrt{3}}}{18x - 9 + \sqrt{3\sqrt{9 + 6\sqrt{3}}} - \left( \frac{1}{720} + \frac{\sqrt{3}}{216} \right) \frac{1}{x^4}}.$$
Proof. The lower bound of (13) is obtained by considering the function $F$ defined by

$$F(x) = \psi(x) - \ln\left(x - \frac{1}{2} - \frac{1}{6}\sqrt{9 + 6\sqrt{3}}\right) - \frac{3\sqrt{9 + 6\sqrt{3}}}{18x - 9 - \sqrt{3}\sqrt{9 + 6\sqrt{3}}}$$

$$- \left(\frac{1}{720} + \frac{\sqrt{3}}{216}\right) \frac{1}{x^4}, \quad x > \frac{1}{2} + \frac{1}{6}\sqrt{9 + 6\sqrt{3}}.$$

We conclude from the asymptotic formula (1) that

$$\lim_{x \to \infty} F(x) = 0.$$

It follows from [2, Theorem 9] that

$$\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < \psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}$$

$$- \frac{1}{30x^5} + \frac{1}{42x^7}, \quad x > 0. \tag{14}$$

Differentiating $F(x)$ with respect to $x$ and applying the second inequality in (14) yields, for $x > \frac{1}{2} + \frac{1}{6}\sqrt{9 + 6\sqrt{3}},$

$$F'(x) = \psi'(x) - \frac{6}{6x - 3 - \sqrt{9 + 6\sqrt{3}}} + \frac{54\sqrt{9 + 6\sqrt{3}}}{(18x - 9 - \sqrt{3}\sqrt{9 + 6\sqrt{3}})^2}$$

$$+ \frac{10\sqrt{3} + 3}{540x^5} < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}$$

$$- \frac{6}{6x - 3 - \sqrt{9 + 6\sqrt{3}}} + \frac{54\sqrt{9 + 6\sqrt{3}}}{(18x - 9 - \sqrt{3}\sqrt{9 + 6\sqrt{3}})^2} + \frac{10\sqrt{3} + 3}{540x^5}$$

$$= - \frac{P(x)}{7x^7(6x - 3 - \sqrt{9 + 6\sqrt{3}})(18x - 9 - \sqrt{3}\sqrt{9 + 6\sqrt{3}})^2},$$

where

$$P(x) = 312 + 96\sqrt{9 + 6\sqrt{3}\sqrt{3}} + 426\sqrt{3} + 151\sqrt{9 + 6\sqrt{3}}$$

$$+ \left(593\sqrt{9 + 6\sqrt{3}} + 1902\sqrt{3} + 1113 + 372\sqrt{9 + 6\sqrt{3}\sqrt{3}}\right)(x - 1)$$

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\[ + \left( 540\sqrt{9} + 6\sqrt{3}\sqrt{3} + 880\sqrt{9} + 6\sqrt{3} + 1425 + 3192\sqrt{3} \right)(x - 1)^2 \]
\[ + \left( 567\sqrt{9} + 6\sqrt{3} + 336\sqrt{9} + 6\sqrt{3}\sqrt{3} + 705 + 2310\sqrt{3} \right)(x - 1)^3 \]
\[ + \left( 630\sqrt{3} + 84\sqrt{9} + 6\sqrt{3}\sqrt{3} + 189 + 147\sqrt{9} + 6\sqrt{3} \right)(x - 1)^4 \]
\[ > 0 \quad \text{for} \quad x > \frac{1}{2} + \frac{1}{6}\sqrt{9} + 6\sqrt{3}. \]

Therefore, \( F'(x) < 0 \) for \( x > \frac{1}{2} + \frac{1}{6}\sqrt{9} + 6\sqrt{3} \). This leads to
\[ F(x) > \lim_{x \to \infty} F(x) = 0. \]

This means that the first inequality in (13) holds for \( x > \frac{1}{2} + \frac{1}{6}\sqrt{9} + 6\sqrt{3} \).

The upper bound of (13) is obtained by considering the function \( G \) defined by
\[
G(x) = \psi(x) - \ln \left( x - \frac{1}{2} + \frac{1}{6}\sqrt{9} + 6\sqrt{3} \right) + \frac{3\sqrt{9} + 6\sqrt{3}}{18x - 9 + \sqrt{3}\sqrt{9} + 6\sqrt{3}}
- \left( \frac{1}{720} + \frac{\sqrt{3}}{216} \right) \frac{1}{x^4}, \quad x > 0.
\]

We conclude from the asymptotic formula (1) that
\[ \lim_{x \to \infty} G(x) = 0. \]

Differentiating \( G(x) \) with respect to \( x \) and applying the first inequality in (14) yields, for \( x > 0 \),
\[
G'(x) = \psi'(x) - \frac{6}{6x - 3 + \sqrt{9} + 6\sqrt{3}} - \frac{54\sqrt{9} + 6\sqrt{3}}{(18x - 9 + \sqrt{3}\sqrt{9} + 6\sqrt{3})^2}
+ \frac{10\sqrt{3} + 3}{540x^5} > \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} - \frac{6}{6x - 3 + \sqrt{9} + 6\sqrt{3}}
- \frac{54\sqrt{9} + 6\sqrt{3}}{(18x - 9 + \sqrt{3}\sqrt{9} + 6\sqrt{3})^2} + \frac{10\sqrt{3} + 3}{540x^5}
= \frac{Q(x)}{x^5(6x - 3 + \sqrt{9} + 6\sqrt{3})(18x - 9 + \sqrt{3}\sqrt{9} + 6\sqrt{3})^2}.
\]
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where

\[ Q(x) = (-90\sqrt{3} + 21\sqrt{9} + 6\sqrt{3} - 27 + 12\sqrt{3}\sqrt{9} + 6\sqrt{3})x^2 \]
\[ + \left(30\sqrt{3} - 39 - 3\sqrt{9} + 6\sqrt{3}\right)x + \sqrt{9} + 6\sqrt{3} - 6\sqrt{3} + 6 \]
\[ = \left(-90\sqrt{3} + 21\sqrt{9} + 6\sqrt{3} - 27 + 12\sqrt{3}\sqrt{9} + 6\sqrt{3}\right) \]
\[ \times (x - x_1)(x - x_2) \]

with

\[ x_1 = \frac{13 - 10\sqrt{3} + \sqrt{9} + 6\sqrt{3}}{2 \left(-30\sqrt{3} + 7\sqrt{9} + 6\sqrt{3} - 9 + 4\sqrt{3}\sqrt{9} + 6\sqrt{3}\right)} \]
\[ = 0.0638967475 \ldots , \]
\[ x_2 = \frac{13 - 10\sqrt{3} + \sqrt{9} + 6\sqrt{3}}{2 \left(-30\sqrt{3} + 7\sqrt{9} + 6\sqrt{3} - 9 + 4\sqrt{3}\sqrt{9} + 6\sqrt{3}\right)} \]
\[ = 0.158650823 \ldots . \]

Therefore, \( Q(x) > 0 \) and \( G'(x) > 0 \) for \( x > x_2 \). This leads to

\[ G(x) < \lim_{x \to \infty} G(x) = 0 \quad x > x_2 . \]

This means that the second inequality in (13) holds for \( x > 0.158650823 \ldots \). □

Some computer experiments indicate that for \( x > 2.30488055 \), the lower
bound in (13) is sharper than one in (5). For \( x > 0.5690291018 \), the upper
bound in (13) is sharper than one in (5).

The inequality (13) provides the best approximations:

\[ \psi(x) \approx \ln \left(x - \frac{1}{2} - \frac{3\sqrt{9} + 6\sqrt{3}}{18x - 9 - \sqrt{3}\sqrt{9} + 6\sqrt{3}} \right) \]
\[ + \frac{3\sqrt{9} + 6\sqrt{3}}{18x - 9 - \sqrt{3}\sqrt{9} + 6\sqrt{3}} \quad (15) \]
and

$$\psi(x) \approx \ln \left( x - \frac{1}{2} + \frac{1}{6} \sqrt{9 + 6\sqrt{3}} \right) - \frac{3\sqrt{9} + 6\sqrt{3}}{18x - 9 + \sqrt{3\sqrt{9} + 6\sqrt{3}}}.$$  (16)

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