THE SINGULAR VALUES OF THE LOGARITHMIC POTENTIAL
TRANSFORM ON BOUND STATES SPACES OF LANDAU
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ABSTRACT. The singular values of the logarithmic potential transform on the generalized
Bergmann space is calculated explicitly, too behavior in infinity.

1. INTRODUCTION

Let $\mathbb{D}$ be the complex unit disk endowed with its Lebesgue measure $\mu$ and let $\partial \mathbb{D}$ be its
boundary. Denote by $L^2(\mathbb{D}, d\mu)$ the space of complex-valued measurable functions which are
$d\mu$ square integrable on $\mathbb{D}$. The logarithmic potential transform $: L^2(\mathbb{D}) \rightarrow L^2(\mathbb{D})$ is defined
by

$$\mathcal{L}[f](z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\xi)}{\xi - z} \log \left( \frac{1}{|z - \xi|} \right) d\mu(\xi)$$

(1.1)

This operator is very important as the transformed Cauchy and it often appears in Analysis [6].

The dimensional analysis [3, 6] and scaling arguments [5] form an integral part in theoretical
physics to solve some important problems without doing much calculation.

The logarithmic potential in physics forms an interesting one as it provides some unusual prediction about the system. Moreover, this potential can be used suitably to illustrate some of the important features of field theory such as dimensional regularization and renormalization. In most of our text books, this potential is not discussed at detail; although the calculations are quite simple to demonstrate some of its unique features. We have obtained
the bound state energy of this logarithmic potential through uncertainty principle, phase
space quantization and Hellmann-Feynman theorem.

In [2] the authors have been dealing with the restriction of $\mathcal{L}$ to the space $L^2_\alpha(\mathbb{D})$ of analytic $\mu$-square integrable on $\mathbb{D}$. They precisely have considered the projection operator $P_0: L^2(\mathbb{D}) \rightarrow L^2_\alpha(\mathbb{D})$ and they have proved that the singular values $\lambda_k$ of $\mathcal{L}P_0$, which turn out to be eigenvalues of the operator $\sqrt{(\mathcal{LP}_0)\star(\mathcal{LP}_0)}$ behave like $k^{-1}$ as $k$ goes to $\infty$. They also concluded that $\mathcal{LP}_0$ belongs to the Schatten class $S_{1,\infty}$.

Now, consider the following weighted logarithmic potential transform

$$(1.1) \quad \mathcal{L}_\sigma[f](z) = -\frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\xi)}{\xi - z} \log \left( \frac{1}{|z - \xi|} \right) \left(1 - \frac{1}{\xi \xi} \right)^{\sigma-2} d\mu(\xi)$$

(1.1)
defined on the space \( L^2, \sigma(\mathbb{D}) \) of complex-valued measurable functions which are \((1 - \xi \bar{\xi})^{-\sigma} d\mu(\xi)\)-square integrable on \( \mathbb{D} \) where \( \sigma > 1 \) is a fixed parameter. We observe that the subspace \( L^2, \sigma(\mathbb{D}) \) of analytic functions on \( \mathbb{D} \) and belonging to \( L^2, \sigma(\mathbb{D}) \) coincides with the eigenspace

\[
A^\sigma_0(\mathbb{D}) := \{ \psi \in L^2, \sigma(\mathbb{D}), \Delta_\sigma \psi = 0 \}
\]

of the second order differential operator

\[
\Delta_\sigma := -4 (1 - z \bar{z}) \left( (1 - z \bar{z}) \frac{\partial^2}{\partial z \partial \bar{z}} - \sigma \frac{\partial}{\partial \bar{z}} \right)
\]

known as the \( \sigma \)-weight Maass Laplacian and its discrete eigenvalues are given by

\[
\epsilon_m := 4m (\sigma - 1 - m), \ m = 0, 1, 2, \ldots, \lfloor (\sigma - 1)/2 \rfloor
\]

with their corresponding eigenspaces

\[
A^\sigma_m(\mathbb{D}) := \{ \psi \in L^2, \sigma(\mathbb{D}) \text{ and } \Delta_\sigma \psi = \epsilon_m^\sigma \psi \}
\]

are here called generalized Bergman spaces since...

After noticing that, we here deal with analogous questions as in [2] in the context of the weighted Cauchy transform (1.2) and for its restriction to the space \( A^\sigma_m(\mathbb{D}) \). That is, we are concerned with the operator \( \mathcal{C}_\sigma P^\sigma_m \) where \( P^\sigma_m \) is the projection \( L^2, \sigma(\mathbb{D}) \to A^\sigma_m(\mathbb{D}) \). The obtained results are as follows.

Firstly, we find that the singular values of \( \mathcal{L}_\sigma P^\sigma_m \).

For \( k \neq m \), can be expressed as

\[
\lambda_k = \sqrt{J_1 + J_2 + J_3}
\]

where

\[
J_1 = \left( \frac{(1 + k - m)_m}{m!(k - m + 1)} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n + 2k - 2m + 6 - 1)\Gamma(4\nu - 2m - 1)}{\Gamma(2n + 2k - 4m + 4\nu + 6)}
\]

\[
J_2 = \left( \frac{\alpha^\nu_m}{2\nu - m - 1} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu - 2m - 1)\Gamma(2n + 2)}{\Gamma(2n + 4\nu - 2m + 1)}
\]

and

\[
J_3 = \frac{(1 + k - m)_m \alpha^\nu_m}{m!(k - m + 1)(2\nu - m - 1)} \left( \sum_{n=0}^{\infty} A_n \frac{\Gamma(k - m + 2)\Gamma(4\nu - 2m - 1)}{\Gamma(4\nu - k - 3m)} \right)
\]

For \( k = m \) can be expressed as

\[
\lambda^2_k = \frac{\alpha^\nu_m}{8 (\pi(2\nu - m + 1))} \sum_{n=0}^{\infty} B_n \frac{B_n}{n + 2\nu - m}
\]
The singular values of the logarithmic potential transform

where

\[ B_n = \sum_{n=0}^{\infty} \frac{\Gamma(-m+1)\Gamma(2\nu-m)\Gamma(2(\nu-m)+1)}{n!\Gamma(2(\nu-m))\Gamma(2\nu-m+2)} \]

\[ \alpha_{\nu,m}^{k} = \frac{\Gamma(2)\Gamma(2(m-\nu)+1)}{\Gamma(m+1)\Gamma(2+m-2\nu)} \]

Secondly, we show that these singular values behave like

\[ \lambda_k \sim C\sqrt{k^{m-4\nu+1}}, \text{ as } k \to \infty \]

where \( C \) is a constant.

The paper is organized as follows. In section 2, we review the definition of the weighted logarithmic potential transform as well as some of its needed properties. Section 3 deals with some basic facts on the spectral theory of Maass Laplacians on the Poincaré disk. In Section 4, a precise description of the generalized Bergman spaces is reviewed. Section 5 is devoted to the computation of the singular values of the weighted logarithmic potential transform. The asymptotic behavior of these singular values is established in Section 6.

2. The weighted Logarithmic Potential transform \( \mathcal{L}_{\nu} \)

2.1. The case \( \nu = 1 \). Let \( \mathbb{D} \) the complex unit disk endowed with its Lebesgue measure \( \mu \) and let \( \partial \mathbb{D} \) its boundary denote by \( L^2(\mathbb{D}) \) the space of complex-valued measurable functions on \( \mathbb{D} \) with finite norm

\[ \|f\| = \int_{\mathbb{D}} |f(\xi)|^2 d\mu(\xi) \]

The Logarithmic Potential operator \( \mathcal{L} : L^2(\mathbb{D}) \to L^2(\mathbb{D}) \) is defined by

\[ \mathcal{L}[f](z) = \int_{\mathbb{D}} f(\xi) \log \left( \frac{1}{|\xi - z|} \right) d\mu(\xi) \]

2.2. The case of \( \nu \geq 1 \). We fix a real parameter \( \nu \) such that \( 2\nu > 1 \) and we consider the following weighted Logarithmic Potential transform

\[ \mathcal{L}_{\nu}[f](z) = \int_{\mathbb{D}} f(\xi) \log \left( \frac{1}{|\xi - z|} \right) (1-\xi \xi)^{2\nu-2} d\mu(\xi) \]

defined on the space \( L^{2,\nu}(\mathbb{D}) \) complex-valued measurable functions which are \((1-\xi \xi)^{2\nu-2} d\mu(\xi)-\)
square integrable on \( \mathbb{D} \). As a convolution of \( L^{2,\nu} \)-functions with the compactly supported measure \( \frac{1-\xi \xi}{\xi} \delta_{\mathbb{D}} d\mu(\xi) \) \( \mathcal{L}_{\nu} : L^{2,\nu}(\mathbb{D}) \to L^{2,\nu}(\mathbb{D}) \) is obviously bounded. Moreover, it is not hard to show that \( \mathcal{L}_{\nu} \) is in fact compact \([1]\). This raises a question concerning the spectral picture of \( \mathcal{L}_{\nu} \).
3. The Landau Hamiltonian $H_\nu$ on the Poincaré Disk $\mathbb{D}$

Let $\mathbb{D} = \{ z \in \mathbb{C}, \, z \bar{z} < 1 \}$ be the complex unit disk with the Poincaré metric $ds^2 = 4 \left(1 - \frac{1}{2} \bar{z} z \right)^{-2} \, dz d\bar{z}$. $\mathbb{D}$ is a complete Riemannian manifold with all sectional curvature equal $-1$. It has an ideal boundary $\partial \mathbb{D}$ identified with the circle $\{ \omega \in \mathbb{C}, \, \omega \bar{\omega} = 1 \}$. One refers to points $\omega \in \partial \mathbb{D}$ as points at infinity. The geodesic distance between two points $z$ and $w$ is given by

\begin{equation}
\cosh d(z, w) = 1 + \frac{2 (z - w)(\bar{z} - \bar{w})}{(1 - z \bar{z})(1 - w \bar{w})}
\end{equation}

By [10] the Schrödinger operator on $\mathbb{D}$ with constant magnetic field of strength proportional to $\nu > 0$ can be written as:

\begin{equation}
L_\nu := -(1 - |z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \nu z \left(1 - |z|^2\right) \frac{\partial}{\partial z} + \nu \bar{z} \left(1 - |z|^2\right) \frac{\partial}{\partial \bar{z}} + \nu^2 |z|^2.
\end{equation}

which is also called Maass Laplacian on the disk. A slight modification of $L_\nu$ is given by the operator

\begin{equation}
H_\nu := 4L_\nu - 4\nu^2
\end{equation}

acting in the Hilbert space

\begin{equation}
L^{2,0}(\mathbb{D}) := \left\{ \varphi : \mathbb{D} \to \mathbb{C}, \int_{\mathbb{D}} |\varphi(z)|^2 \left(1 - |z|^2\right)^{-2} \, d\mu(z) < +\infty \right\},
\end{equation}

For our purpose, we shall consider the unitary equivalent realization $\tilde{H}_\nu$ of the operator $H_\nu$ in the Hilbert space

\begin{equation}
L^{2,\nu}(\mathbb{D}) := \left\{ \varphi : \mathbb{D} \to \mathbb{C}, \int_{\mathbb{D}} |\varphi(z)|^2 \left(1 - |z|^2\right)^{2\nu-2} \, d\mu(z) < +\infty \right\},
\end{equation}

which is defined by

\begin{equation}
\tilde{H}_\nu := \Omega_\nu^{-1} H_\nu \Omega_\nu,
\end{equation}

where $\Omega_\nu : L^{2,\nu}(\mathbb{D}) \to L^{2,0}(\mathbb{D})$ is the unitary transformation defined by the map $\varphi \mapsto \Omega_\nu [\varphi] := (1 - |z|^2)^{-\nu} \varphi$. Different aspects of the spectral analysis of the operator $\tilde{H}_\nu$ have been studied by many authors. For instance, note that $\tilde{H}_\nu$ is an elliptic densely defined operator on the Hilbert space $L^{2,\nu}(\mathbb{D})$ and admits a unique self-adjoint realization that we denote also by $\tilde{H}_\nu$. The spectrum of $\tilde{H}_\nu$ in $L^{2,\nu}(\mathbb{D})$ consists of two parts: (i) a continuous part $[1, +\infty[$, which correspond to scattering states, (ii) a finite number of eigenvalues (hyperbolic Landau levels) of the form

\begin{equation}
\epsilon_m^\nu := 4(\nu - m)(1 - \nu + m), \quad m = 0, 1, 2, \ldots, \left\lfloor \nu - \frac{1}{2} \right\rfloor
\end{equation}

with infinite degeneracy, provided that $2\nu > 1$. To the eigenvalues in (2.6) correspond eigenfunctions which are called bound states since the particle in such a state cannot leave the system without additional energy. A concrete description of these bound states spaces will be the goal of the next section.
4. The Bound States Spaces $\mathcal{A}_{\nu,m}^2(\mathbb{D})$

Here, we consider the eigenspace

\[ (3.1) \quad \mathcal{A}_{\nu,m}^2(\mathbb{D}) := \left\{ \Phi : \mathbb{D} \to \mathbb{C}, \Phi \in L^{2,\nu}(\mathbb{D}) \text{ and } \hat{H}_\nu \Phi = \epsilon_m \Phi \right\} \]

See [9, 14], for the following proposition.

**Proposition 1.** Let $2\nu > 1$ and $m = 0,1,2,\cdots, \left[ \nu - \frac{1}{2} \right]$. Then, we have

(i) an orthogonal basis of $\mathcal{A}_{\nu,m}^2(\mathbb{D})$ is given by the functions

\[ \phi_{k}^{\nu,m}(z) := |z|^{m-k}(1-|z|^2)^{-m} e^{-i(m-k)\arg z} \times {}_2F_1\left(\begin{array}{c} -m-k+m-k \, 2m-k \end{array}\right)_{2\nu-m+m}^\nu, 1+|m-k|; |z|^2 \right) \]

$k = 0,1,2,\cdots$, in terms of a terminating $2F_1$ Gauss hypergeometric function.

(ii) the norm square of $\phi_{k}^{\nu,m}$ in $L^{2,\nu}(\mathbb{D})$ is given by

\[ \|\phi_{k}^{\nu,m}\|^2 = \frac{\pi (\Gamma(1+|m-k|))^2}{(2\nu-m-1)} \frac{(m-\frac{|m-k|+m-k}{2}+1)}{\Gamma\left(m+\frac{|m-k|-m+k}{2}\right)} \frac{(2\nu-m-\frac{|m-k|+m-k}{2})}{\Gamma\left(2\nu-m+\frac{|m-k|-m+k}{2}\right)} \]

**Corollary 1.** The functions $\{\Phi_{k}^{\nu,m}\}, k = 0,1,2,\ldots$ given by

\[ \Phi_{k}^{\nu,m}(z) := (-1)^k \left( \frac{2(\nu-m)-1}{\pi} \right)^{\frac{1}{2}} \left( \frac{k!\Gamma(2(\nu-m)+m)}{m!\Gamma(2(\nu-m)+k)} \right)^{\frac{1}{2}} \times \left(1-|z|^2\right)^{\frac{m}{2}} P_{k}^{(m-k,2(\nu-m)-1)}(1-2z\overline{z}) \]

in terms of Jacobi polynomials constitute an orthonormal basis of $\mathcal{A}_{\nu,m}^2(\mathbb{D})$.

**Proof.** Write the connection between the $2F_1$-sum and the Jacobi polynomial

\[ P_k^{\alpha,\beta}(u) = \frac{(1+\alpha)_k}{k!} {}_2F_1(-k; 1+\alpha+\beta+k, 1+\alpha; \frac{1-u}{2}) \]

then the functions

\[ \phi_{k}^{\nu,m}(z) = \left( \frac{-1}{|z|^2} \right)^{\min(m,k)} |z|^{m-k} e^{-i(m-k)\arg z} P_{\min(m,k)}^{(m-k,2(\nu-m)-1)}(1-2z\overline{z}) \]

constitute an orthonormal basis of $\mathcal{A}_{\nu,m}^2$. The norm square of $\phi_{k}^{\nu,m}$ in $L^{2,\nu}(\mathbb{D})$ is given by

\[ \|\phi_{k}^{\nu,m}\|^2 = \frac{\pi (m \wedge k)!\Gamma(2(\nu-m)+m \wedge k)}{(2(\nu-m)-1)(m \wedge k)!\Gamma(2(\nu-m)+m \vee k)} \]

Here, $m \wedge k := \min(m,k)$ and $m \vee k := \max(m,k)$. Thus, the set of functions

\[ \Phi_{k}^{\nu,m} := \frac{\phi_{k}^{\nu,m}}{\|\phi_{k}^{\nu,m}\|}, \quad k = 0,1,2,\ldots \]
is an orthonormal basis of $A_{\nu,m}^2 (\mathbb{D})$ and can be rewritten as
\[
(\ast) \quad \Phi_{\nu,m}^k (z) = (-1)^k \left( \frac{2(\nu - m) - 1}{\pi} \right)^{\frac{1}{2}} \left( \frac{k! \Gamma (2(\nu - m) + m)}{m! \Gamma (2(\nu - m) + k)} \right)^{\frac{1}{2}} \times \left( 1 - |z|^2 \right)^{-m} e^{-i(m-k) \arg z} P_{k}^{(m-k, 2(\nu-m) - 1)} (1 - 2z\overline{z})
\]
by making appeal to the identity \cite[p. 63]{12}
\[
\frac{\Gamma (m + 1)}{\Gamma (m - s + 1)} P_{m}^{(-s,\alpha)} (u) = \frac{\Gamma (m + \alpha + 1)}{\Gamma (m - s + \alpha + 1)} \left( \frac{u - 1}{2} \right)^{s} P_{m-s}^{(s,\alpha)} (u), 1 \leq s \leq m
\]
for $s = m - k$, $t = 1 - 2|z|^2$ and $\alpha = 2(\nu - m) - 1$.

**Corollary 2.** The $L^2$-eigenspace $A_{\nu,0}^2 (\mathbb{D})$, corresponding to $m = 0$ in (3.1) and associated to the bottom energy $\epsilon_{\nu} = 0$ in (2.6), reduces further to the weighted Bergman space consisting of holomorphic functions $\phi : \mathbb{D} \to \mathbb{C}$ such that
\[
\int_{\mathbb{D}} |\phi (z)|^2 \left( 1 - |z|^2 \right)^{2\nu - 2} d\mu (z) < +\infty.
\]

**5. Computation of the singular values $\lambda_k$**

Elements of this basis are given in terms of Jacobi polynomials as
\[
\Phi_{\nu,m}^k (z) = \left( -1 \right)^{\min(m,k)} \left( \frac{1}{2^m} \right) \frac{|m-k|}{m!} e^{-i(m-k) \arg z} P_{\min(m,k)}^{(|m-k|, 2(\nu-m) - 1)} (1 - 2z\overline{z})
\]
The norm square of $\Phi_{\nu,m}^k$ in $L^{2,\nu} (\mathbb{D})$ is given by
\[
\rho_{\nu,m}^k = \frac{\pi}{(2(\nu-m) - 1)!} \frac{(m \land k)! \Gamma (2(\nu-m) + m \land k)}{(m \land k)! \Gamma (2(\nu-m) + m \land k)}.
\]
Here, $m \land k := \min (m, k)$ and $m \lor k := \max (m, k)$. Let us introduce the notation
\[
\gamma_{\nu,m}^k := \left( -1 \right)^{m \land k} \sqrt{\rho_{\nu,m}^k}, k = 0, 1, 2, ...
\]
So that we consider the elements
\[
\Phi_{\nu,m}^k (z) := \gamma_{\nu,m}^k \frac{1}{(1 - z\overline{z})^m} |z|^{m-k} e^{-i(m-k) \arg z} P_{\min(m,k)}^{(|m-k|, 2(\nu-m) - 1)} (1 - 2z\overline{z})
\]
5.1. The action $\mathcal{L}_\nu$.

Lemma 1. We set $z = \rho e^{it}$, and $I = -\int_0^{2\pi} e^{i(k-m)t} \log \left(|z - re^{i\theta}|\right) \frac{d\theta}{2\pi}$, we have

$$\tag{5.1} I = \begin{cases} -\log (\rho \land r) & k = m \\ \frac{e^{i(k-m)t} (\frac{z}{\rho})^{m-k} \land (\frac{z}{\rho})^{m-k}}{2(m-k)} & k \neq m \end{cases}$$

Proof. By [2], it remains to prove that this lemma for $k < m$. We have

$$\int_0^{2\pi} e^{i(k-m)t} \log \left(|\rho \rho e^{i} - re^{i\theta}|\right) d\theta = -\int_0^{2\pi} e^{i(m-k)(-\theta)} \log \left(|\rho e^{i(-t)} - \rho e^{i(-\theta)}|\right) d(-\theta)$$

The function $\theta \rightarrow e^{i(m-k)(-\theta)} \log \left(|\rho e^{i(-t)} - \rho e^{i(-\theta)}|\right)$ is a periodic mapping with the period equal $2\pi$, then

$$\int_0^{2\pi} e^{i(k-m)t} \log \left(|\rho \rho e^{i} - re^{i\theta}|\right) d\theta = -\int_0^{2\pi} e^{i(m-k)(-\theta)} \log \left(|\rho e^{i(-t)} - \rho e^{i(-\theta)}|\right) d(-\theta)$$

$$= \frac{e^{i(k-m)t}}{2(m-k)} \times \left(\frac{z}{\rho}\right)^{m-k} \land \left(\frac{z}{\rho}\right)^{m-k} \tag*{□}$$

Lemma 2. For all $\lambda \in \partial D$. $\mathcal{L}_\nu$ commutes with the rotations $R_\lambda$, where $(R_\lambda f)(z) = f(\lambda z)$

Proof. We observe that

$$(R_\lambda \phi_k^{\nu,m})(z) = \lambda^{k-m} \phi_k^{\nu,m}(z), \forall k \neq m \tag*{□}$$

Corollary 3. $\{\mathcal{L}_\nu (\phi_k^{\nu,m})\}_{k=0}^{\infty}$ are orthonormal in $L^2(\mathbb{D})$

Proof. As $R_\lambda$ is an isometry of $L^2(\mathbb{D})$,

$$\langle \mathcal{L}_\nu (\phi_k^{\nu,m}), \mathcal{L}_\nu (\phi_j^{\nu,m}) \rangle = \langle R_\lambda \mathcal{L}_\nu (\phi_k^{\nu,m}), R_\lambda \mathcal{L}_\nu (\phi_j^{\nu,m}) \rangle = \langle \mathcal{L}_\nu R_\lambda (\phi_k^{\nu,m}), \mathcal{L}_\nu R_\lambda (\phi_j^{\nu,m}) \rangle$$

$$= \lambda^{j-k} \langle \mathcal{L}_\nu (\phi_k^{\nu,m}), \mathcal{L}_\nu (\phi_j^{\nu,m}) \rangle, \text{ if } m > k$$

or

$$= \lambda^{k-j} \langle \mathcal{L}_\nu (\phi_k^{\nu,m}), \mathcal{L}_\nu (\phi_j^{\nu,m}) \rangle, \text{ if } m < k$$

□
For all $\lambda \in \partial D$, since $\lambda \neq 0$, we have

$$ \left( \mathcal{L}_\nu (\phi_k^{\nu,m}), \mathcal{L}_\nu (\phi_j^{\nu,m}) \right) = 0 \text{ if } j \neq k $$

\[ \square \]

**Proposition 2.** The action of the operator $\mathcal{L}$ on a basis element $\phi_k^{\nu,m}$, is of the form:

If $k = m$, We put $z = \rho e^{i\theta}$ then

$$ \mathcal{L}_\nu (\phi_k^{\nu,m})(z) = \beta (\rho) \, \frac{\omega_k^{\nu,m} \omega^i(k-m) }{2(k-m)} (I_3 + I_4) $$

with

$$ \beta (\rho) = \frac{\omega_k^{\nu,m}}{2(2\nu - m + 1)} \sqrt{\frac{2(\nu - m) - 1}{\pi}} (1 - \rho^2)^{2m-m-1}. $$

If $k \neq m$ then

$$ \mathcal{L}_\nu (\phi_k^{\nu,m})(z) = \frac{\pi \omega_k^{\nu,m} \omega^i(k-m) }{2(k-m)} (I_3 + I_4) $$

where

$$ I_3 = \frac{(1 + k - m)m}{m!(k - m + 1)} \rho^{k-m+2}(1 - \rho^2)^{2\nu-m-1} \, _2F_1 \left( \begin{array}{c} -m + 1, 2\nu - m + 1 \\ 2(\nu - m), 2\nu - m + 2 \end{array} \right) | \rho^2 \right) $$

and

$$ I_4 = \frac{\omega_k^{\nu,m}}{2\nu - m - 1}(1 - \rho^2)^{2\nu-m-1} \, _2F_1 \left( \begin{array}{c} -m + 1, 2\nu - m - 1 \\ 2(\nu - m), \end{array} \right) | \rho^2 \right) $$

**Proof.** For $k = m$, we have

$$ \mathcal{L}_\nu (\phi_k^{\nu,m})(z) = \frac{(-1)^m}{\pi} \sqrt{\frac{2(\nu - m) - 1}{\pi}} $$

$$ \int_{\mathbb{E}} (1 - |\xi|^2)^{2\nu-m-2} P_{m,2(\nu-m)-1} (1 - 2|\xi|^2) \log(|z - \xi|) \, d\mu(\xi) $$

$$ = (-1)^m \sqrt{\frac{2(\nu - m) - 1}{\pi}} \int_0^1 (1 - r^2)^{2\nu-m-2} P_{m,2(\nu-m)-1} (1 - 2r^2) \log(r \land r) \, dr $$

$$ = \frac{(-1)^m}{2} \sqrt{\frac{2(\nu - m) - 1}{\pi}} \int_0^1 (1 - t)^{2\nu-m-2} P_{m,2(\nu-m)-1} (1 - 2t) \log(r^2 \lor t) \, dt $$

$$ = \frac{(-1)^m}{2} \sqrt{\frac{2(\nu - m) - 1}{\pi}} [I_1 + I_2] $$

Where

$$ I_1 = \int_0^{\rho^2} (1 - t)^{2\nu-m-2} P_{m,2(\nu-m)-1} (1 - 2t) \log(r^2 \lor t) \, dt $$

and

$$ I_2 = \int_{\rho^2}^1 (1 - t)^{2\nu-m-2} P_{m,2(\nu-m)-1} (1 - 2t) \log(t) \, dt $$
Calculus of $I_1$.

$$I_1 = \log(\rho^2) \int_{\rho^2}^1 (1-t)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2t) \, dt$$

We use the formula

$$P_k^{(\alpha,\beta)}(u) = \frac{(1+\alpha)_k}{k!} 2F_1 \left( \begin{array}{c} -k, 1+\alpha+\beta+k \cr 1+\alpha \end{array} \bigg| \frac{1-u}{2} \right)$$

We have

$$I_1 = \log(\rho^2) \int_{\rho^2}^1 (1-t)^{2\nu-m-2} 2F_1 \left( \begin{array}{c} -m, 2\nu-m \cr 1 \end{array} \bigg| t \right) \, dt$$

By [11], we have

$$\int x^{c-1}(1-x)^{b-c-1} 2F_1 \left( \begin{array}{c} a, b \cr c \end{array} \bigg| x \right) \, dx = \frac{1}{c} x^c (1-x)^{b-c} 2F_1 \left( \begin{array}{c} a+1, b \cr c+1 \end{array} \bigg| x \right)$$

implies that

$$I_1 = \log(\rho^2) \rho^2 (1-\rho^2)^{2\nu-m-1} 2F_1 \left( \begin{array}{c} -m+1, 2\nu-m \cr 2 \end{array} \bigg| \rho^2 \right)$$

Calculus of $I_2$.

$$I_2 = \int_{\rho^2}^1 (1-t)^{2\nu-m-2} P_m^{(0,2(\nu-m)-1)} (1-2t) \log(t) \, dt$$

Use the previous formula in [11] and the integration by part gives

$$I_2 = \left[ t(1-t)^{2\nu-m-1} 2F_1 \left( \begin{array}{c} -m+1, 2\nu-m \cr 2 \end{array} \bigg| t \right) \log(t) \right]_{\rho^2}^1$$

$$- \int_{\rho^2}^1 (1-t)^{2\nu-m} 2F_1 \left( \begin{array}{c} -m+1, 2\nu-m \cr 2 \end{array} \bigg| t \right) \, dt$$

$$= -\rho^2 \log(\rho^2) (1-\rho^2)^{2\nu-m-1} 2F_1 \left( \begin{array}{c} -m+1, 2\nu-m \cr 2 \end{array} \bigg| \rho^2 \right)$$

$$- \int_{\rho^2}^1 (1-t)^{2\nu-m} 2F_1 \left( \begin{array}{c} -m+1, 2\nu-m \cr 2 \end{array} \bigg| t \right) \, dt$$

Calculus of

$$\int_{\rho^2}^1 (1-t)^{2\nu-m} 2F_1 \left( \begin{array}{c} -m+1, 2\nu-m \cr 2 \end{array} \bigg| t \right) \, dt$$

Use the following formula which has place in [13]

$$2F_1 \left( \begin{array}{c} a, b \cr c \end{array} \bigg| t \right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} 2F_1 \left( \begin{array}{c} a, b \cr a+b-c+1 \end{array} \bigg| 1-t \right)$$

$$+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-t)^{c-a-b} 2F_1 \left( \begin{array}{c} a, b \cr a+b-c+1 \end{array} \bigg| 1-t \right)$$
We put \(a = 1 - m, b = 2\nu - m, c = 2\) and use the formula Boher-Mollerup, for \(z \in \mathbb{R}^+_c\),

\[
\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{-\frac{z}{n}}
\]

which implies \(\frac{1}{\Gamma(1-m)} = 0\), then

\[
_2F_1\left(\frac{-m + 1, 2\nu - m}{2} \mid t\right) = \frac{2\Gamma(2(m - \nu) + 1)}{m!\Gamma(2 + m - 2\nu)}_2F_1\left(\frac{-m + 1, 2\nu - m}{2(\nu - m)} \mid 1 - t\right)
\]

implies that

\[
\int_{\rho^2}^1 (1 - t)^{2\nu - m} \_2F_1\left(\frac{-m + 1, 2\nu - m}{2} \mid t\right) dt = \frac{2\Gamma(2(m - \nu) + 1)}{m!\Gamma(2 + m - 2\nu)} \int_{\rho^2}^1 (1 - t)^{2\nu - m} \_2F_1\left(\frac{-m + 1, 2\nu - m}{2(\nu - m)} \mid 1 - t\right) dt
\]

By the change \(1 - t = s\), we get

\[
\int_{\rho^2}^1 (1 - t)^{2\nu - m} \_2F_1\left(\frac{-m + 1, 2\nu - m}{2} \mid t\right) dt = \frac{2\Gamma(2(m - \nu) + 1)}{m!\Gamma(2 + m - 2\nu)} \int_{0}^{1-\rho^2} s^{2\nu - m} \_2F_1\left(\frac{-m + 1, 2\nu - m}{2(\nu - m)} \mid t\right) dt
\]

In [11] page 44,

\[
\int x^{a-1} \_2F_1\left(\frac{a, b}{c} \mid -t\right) dx = \frac{x^a}{\alpha} \_3F_2\left(\frac{a, b, \alpha}{c, \alpha + 1} \mid -t\right) + \frac{\Gamma(a)\Gamma(a - \alpha)\Gamma(b - \alpha)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c - \alpha)}
\]

Since \(a = 1 - m, b = 2\nu - m, c = 2(\nu - m), \) and \(\alpha = 2\nu - m + 1\) we have

\[
\frac{\Gamma(a)\Gamma(a - \alpha)\Gamma(b - \alpha)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c - \alpha)} = 0
\]

and by the change \(t = -s\)

\[
\int_{0}^{1-\rho^2} s^{2\nu - m + 1} \_2F_1\left(\frac{-m + 1, 2\nu - m}{2(\nu - m)} \mid t\right) dt = (-1)^m \int_{0}^{\rho^2} s^{2\nu - m} \_2F_1\left(\frac{-m + 1, 2\nu - m}{2(\nu - m)} \mid t\right) dt
\]

\[
= (-1)^m \frac{(\rho^2 - 1)^{2\nu - m + 1}}{2\nu - m + 1} \_3F_2\left(\frac{-m + 1, 2\nu - m, 2\nu - m + 1}{2(\nu - m), 2\nu - m + 2} \mid 1 - \rho^2\right)
\]

we set \(a_{\nu,m} = \frac{2\Gamma(2(m-\nu)+1)}{m!\Gamma(2+m-2\nu)}\) We get

\[
I_2 = -\rho^2 \log(\rho^2)(1 - \rho^2)^{2\nu - m - 1} \_2F_1\left(\frac{-m + 1, 2\nu - m}{2} \mid \rho^2\right)
\]
Finally

\[ \mathcal{L}_\nu(\phi_{k}^{\nu,m})(z) = \frac{\alpha_k^{\nu,m}}{2(2\nu - m + 1)} \sqrt{\frac{2(\nu - m) - 1}{\pi}} (1 - \rho^2)^{2\nu - m - 1} \times 3F_2 \left( \begin{array}{c} -m + 1, 2\nu - m, 2\nu - m + 1 \\ 2(\nu - m), 2\nu - m + 2 \end{array} \bigg| 1 - \rho^2 \right) \]

Now if \( k > m \), set \( z = \rho e^{it} \).

\[ \mathcal{L}_\nu(\phi_{k}^{\nu,m})(z) = \frac{\gamma_k^{\nu,m}}{2(2\nu - m + 1)} \int_0^1 (1 - |\xi|^2)^{2\nu - m - 2} \xi^{k - m} \log \left( \frac{1}{|z - \xi|} \right) P_m^{(k-m,2(\nu-m)-1)} (1 - 2|\xi|^2) d\mu(\xi) \]

\[ = \frac{\pi \gamma_k^{\nu,m} \rho^{(k-m)t}}{2(k-m)} \int_0^1 (1 - \rho^2)^{2\nu - m - 2} \rho^{k-m} P_m^{(k-m,2(\nu-m)-1)} (1 - 2\rho^2) \left( \frac{\rho}{\rho} \right)^{k-m} \left( \frac{\rho}{\rho} \right)^{k-m} dr^2 \]

Calculus of \( I_3 \).

\[ I_3 = \frac{\rho^{m-k}(1 + k - m)_m}{m!} \int_0^{\rho^2} t^{k-m}(1 - t)^{2\nu - m - 2} 2F_1 \left( \begin{array}{c} -m, 2(\nu - m) + k \\ 1 + k - m \end{array} \bigg| t \right) dt \]

By the formula

\[ \int x^{c-1}(1 - x)^{b-c-1} 2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| x \right) dx = \frac{1}{c} x^c(1 - x)^{b-c} 2F_1 \left( \begin{array}{c} a + 1, b \\ c + 1 \end{array} \bigg| x \right) \]

we have

\[ I_3 = \frac{(1 + k - m)_m}{m!(k - m + 1)} \rho^{k-m+2}(1 - \rho^2)^{2\nu - m - 1} 2F_1 \left( \begin{array}{c} -m + 1, 2(\nu - m) + k \\ 2 + k - m \end{array} \bigg| \rho^2 \right) \]
Calculus of $I_4$.

$$I_4 = \int_{\rho}^{1} (1 - \rho^2)^{2\nu - m - 2} \rho^{k-m} P_m^{(k-m,2(\nu-m)-1)}(1 - 2\rho^2) \left( \frac{\tau}{\rho} \right)^{k-m} \frac{(\beta)^{k-m}}{2m!} \int_{\rho^2}^{1} (1 - t)^{2\nu - m - 2} \, 2F_1 \left( -m, 2(\nu - m) + k \mid t \right) \, dt$$

As the previous

$$\int_{\rho^2}^{1} (1 - t)^{2\nu - m - 2} \, 2F_1 \left( -m, 2(\nu - m) + k \mid t \right) \, dt$$

$$= \alpha_k^{\nu,m} \int_{\rho^2}^{1} (1 - t)^{2\nu - m - 2} \, 2F_1 \left( -m + 1, 2\nu - m \mid \frac{1}{2} \right) \, dt$$

$$= (-1)^m \alpha_k^{\nu,m} \int_{\rho^2}^{1} t^{2\nu - m - 2} \, 2F_1 \left( -m + 1, 2\nu - m \mid t \right) \, dt$$

$$= \frac{\alpha_k^{\nu,m}}{2\nu - m - 1} (1 - \rho^2)^{2\nu - m - 1} \, 3F_2 \left( -m + 1, 2\nu - m, 2\nu - m - 1 \mid 1 - \rho^2 \right)$$

also

$$3F_2 \left( -m + 1, 2\nu - m, 2\nu - m - 1 \mid 1 - \rho^2 \right) = 2F_1 \left( -m + 1, 2\nu - m - 1 \mid 2\nu - m \right)$$

Now if $k < m$. We have

$$\phi_k^{\nu,m}(z) = (-1)^k \sqrt{\frac{2(\nu - m) - 1}{\pi}} \left( \frac{k! \Gamma(2(\nu - m) + m)}{m! \Gamma(2(\nu - m) + k)} \right) \left( 1 - |z|^2 \right)^{-m} \frac{z^{m-k} P_k^{(m-k,2(\nu-m)-1)} \left( 1 - 2|z|^2 \right)}{2}$$

By the formula

$$\frac{\Gamma(m + 1)}{\Gamma(m + s + 1)} P_m^{(-s,a)}(u) = \frac{\Gamma(m + \alpha + 1)}{\Gamma(m + s + \alpha + 1)} \left( \frac{u - 1}{2} \right)^s \frac{P_m^{(s,a)}(u), 1 \leq s \leq m}{2}$$

and put $s = m - k$ and $\alpha = 2(\nu - m) - 1$, we have

$$P_k^{(m-k,2(\nu-m)-1)} \left( 1 - 2|z|^2 \right) = m! \frac{k! \Gamma(k + \alpha + 1)}{k! \Gamma(m + \alpha + 1)} P_m^{(k-m,2(\nu-m)-1)} \left( 1 - 2|z|^2 \right)$$

substituting in the expression of $\phi_k^{\nu,m}(z)$, we get

$$\phi_k^{\nu,m}(z) = (-1)^k \sqrt{\frac{2(\nu - m) - 1}{\pi}} \frac{2m! \Gamma(2(\nu - m) + k)}{k! \Gamma(2(\nu - m) + m)} \left( 1 - |z|^2 \right)^{-m} \frac{z^{k-m} P_m^{(k-m,2(\nu-m)-1)} \left( 1 - 2|z|^2 \right)}{2}$$

it’s the same formula for $k > m$, which prove the same formula of $L_\nu(\phi_k^{\nu,m})(z)$ if $k > m$. □

**Remark 1.** By the previous formula in [13], we have

$$2F_1 \left( -m + 1, 2(\nu - m) + k \mid \rho^2 \right) = \frac{k! \Gamma(2 + k - m)}{\Gamma(1 - 2(\nu - m))} \frac{2F_1 \left( -m + 1, 2(\nu - m) + k \mid 1 - \rho^2 \right)}{2(\nu - m)}$$
5.2. The spectrum of $\mathcal{L}_\nu$.

**Proposition 3.** If $k \neq m$, then

$$\lambda_k = \sqrt{J_1 + J_2 + J_3}$$

where

$$J_1 = \left( \frac{(1 + k - m)_m}{m!(k - m + 1)} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n + 2k - 2m + 6 - 1)\Gamma(4\nu - 2m - 1)}{\Gamma(2n + 2k - 4m + 4\nu + 6)}$$

$$J_2 = \left( \frac{\alpha_k^{\nu,m}}{2\nu - m - 1} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu - 2m - 1)\Gamma(2n + 2)}{\Gamma(2n + 4\nu - 2m + 1)}$$

and

$$J_3 = \frac{(1 + k - m)_m \alpha_k^{\nu,m}}{m!(k - m + 1)(2\nu - m - 1)} \left( \sum_{n=0}^{\infty} A_n \frac{\Gamma(k - m + 2)\Gamma(4\nu - 2m - 1)}{\Gamma(4\nu - k - 3m)} \right)$$

If $k = m$ then

$$\lambda_k^2 = \frac{\alpha_k^{\nu,m} (2(2\nu - m) - 1)}{8 \pi (2\nu - m + 1)} \sum_{n=0}^{\infty} n + 2\nu - m$$

where

$$B_n = \sum_{n=0}^{\infty} \frac{\Gamma(-m + 1)\Gamma(2\nu - m)\Gamma(2\nu - m + 1)}{n!\Gamma(2\nu - m + 2)}$$

**Proof.** If $k \neq m$. We have

$$(\mathcal{L}_\nu(\phi_k^{\nu,m}))(z) = \pi\gamma_k^{\nu,m} (I_3 + I_4) e^{i(k-m)t}$$

We set $\mathcal{H} = (L^2(\mathbb{D}), (1 - |\xi|^2)^{2\nu-2}d\mu(\xi))$, $I_3 = I_3(\rho)$, and $I_4 = I_4(\rho)$ we have

$$\lambda_k^2 = (\mathcal{L}_\nu(\phi_k^{\nu,m}), \mathcal{L}_\nu(\phi_k^{\nu,m}))_{\mathcal{H}}$$

$$= \frac{\pi^2 \gamma_k^{\nu,m}}{(k-m)} \int_0^1 (I_3(\rho) + I_4(\rho))^2 \rho d\rho$$

Calculus of $\int_0^1 (I_3(\rho))^2 d\rho$.

$$I_3(\rho) = \frac{(1 + k - m)_m}{m!(k - m + 1)} \rho^{k-m+2} (1 - \rho^2)^{2\nu-2m-1} \binom{-m + 1, 2(\nu - m) + k}{2 + k - m} 2F_1$$

Since

$$2F_1 \left( \begin{array}{c} -m + 1, 2(\nu - m) + k \\ 2 + k - m \end{array} \bigg| \rho^2 \right) = \sum_{n=0}^{\infty} \frac{(-m + 1)_n(2(\nu - m) + k)_n \rho^{2n}}{(2 + k - m)_n n!}$$

then
\((I_3(\rho))^2 = \left(\frac{(1+k-m)m}{m!(k-m+1)}\right)^2 \sum_{n=0}^{\infty} A_n \rho^{2n} (1 - \rho^2)^{4\nu-2m-2}\)

where

\[ A_n = \frac{1}{n!} \sum_{i=0}^{n} \frac{(-m+1)_i(-m+1)_{n-i} (2(\nu-m) + k)_i (2(\nu-m) + k)_{n-i}}{(2(\nu-m))_i (2(\nu-m))_{n-i}} \]

Thus

\[ J_1 = \int_0^1 (I_3(\rho))^2 \rho d\rho = \left(\frac{(1+k-m)m}{m!(k-m+1)}\right)^2 \sum_{n=0}^{\infty} A_n \int_0^1 \rho^{2n+2k-2m+6-1}(1 - \rho^2)^{4\nu-2m-1-1} d\rho \]

Use the fact that

\[ \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} \]

implies

\[ (5.3) \]

\[ \int_0^1 (I_3(\rho))^2 \rho d\rho = \left(\frac{(1+k-m)m}{m!(k-m+1)}\right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n+2k-2m+6-1)\Gamma(4\nu-2m-1)}{\Gamma(2n+2k-4m+4\nu+6)} \]

Calculus of \( \int_0^1 (I_4(\rho))^2 \rho d\rho \).

In the same

\[ (5.4) \]

\[ J_2 = \int_0^1 (I_4(\rho))^2 \rho d\rho = \left(\frac{\alpha_k^{\nu,m}}{2\nu-m-1}\right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu-2m-1)\Gamma(2n+2)}{\Gamma(2n+4\nu-2m+1)} \]

Calculus of \( 2 \int_0^1 (I_3(\rho)) (I_4(\rho)) \rho d\rho \).

\[ (5.5) \]

\[ J_3 = 2 \int_0^1 (I_3(\rho)) (I_4(\rho)) \rho d\rho = \frac{(1+k-m)m\alpha_k^{\nu,m}}{m!(k-m+1)(2\nu-m-1)} \sum_{n=0}^{\infty} A_n \frac{\Gamma(k-m+2)\Gamma(4\nu-2m-1)}{\Gamma(4\nu-k-3m)} \]

If \( k = m \).

Since

\[ \left( _3F_2\left( \left[ \begin{array}{c} -m+1, 2\nu-m, 2(\nu-m) + 1 \\ 2(\nu-m), 2\nu-m+2 \end{array} \right] \right) | 1 - \rho^2 \right)^2 \]

\[ = \sum_{n=0}^{\infty} \frac{\Gamma(-m+1)\Gamma(2\nu-m)\Gamma(2\nu-m+1)}{n!\Gamma(2\nu-m)\Gamma(2\nu-m+2)} (1 - \rho^2)^n \]

\[ \lambda_k^2 = \frac{\alpha_k^{\nu,m}(2\nu-m-1)}{8(\pi(2\nu-m+1))} \sum_{n=0}^{\infty} B_n \int_0^1 (1 - \rho^2)^{n+2\nu-m-1} d\rho \]
\[ \lambda_k = \alpha_k^{\nu,m} \left( 2(2\nu - m) - 1 \right) \frac{B_n}{8(\pi(2\nu - m + 1))} \sum_{n=0}^{\infty} \frac{B_n}{n + 2\nu - m} \]

where

\[ B_n = \sum_{n=0}^{\infty} \frac{\Gamma(-m + 1)\Gamma(2\nu - m)\Gamma(2(\nu - m) + 1)}{n!\Gamma(2(\nu - m))\Gamma(2\nu - m + 2)} \]

\[ \square \]

6. ASYMPTOTIC BEHAVIOR OF SINGULAR VALUES \( \lambda_k \) AS \( k \to \infty \)

Proposition 4.

\[ \lambda_k \sim C \sqrt{k^{m-4\nu+1}}, \text{ as } k \to \infty \]

where \( C \) is a constant

**Proof.** If \( k > m \), then

\[ \lambda_k = \sqrt{J_1 + J_2 + J_3} \]

where

\[ J_1 = \left( \frac{(1 + k - m)_m}{m!(k - m + 1)} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(2n + 2k - 2m + 6 - 1)\Gamma(4\nu - 2m - 1)}{\Gamma(2n + 2k - 4m + 4\nu + 6)} \]

\[ J_2 = \left( \frac{\alpha_k^{\nu,m}}{2\nu - m - 1} \right)^2 \sum_{n=0}^{\infty} A_n \frac{\Gamma(4\nu - 2m - 1)\Gamma(2n + 2)}{\Gamma(2n + 4\nu - 2m + 1)} \]

and

\[ J_3 = \frac{(1 + k - m)_m\alpha_k^{\nu,m}}{m!(k - m + 1)(2\nu - m - 1)} \left( \sum_{n=0}^{\infty} A_n \frac{\Gamma(k - m + 2)\Gamma(4\nu - 2m - 1)}{\Gamma(4\nu - k - 3m)} \right) \]

The limit of \( \lambda_k \) as \( k \to \infty \).

We use the formula

\[ \frac{\Gamma(k + a)}{\Gamma(k + b)} \sim k^{a-b} \]

we have

(6.1)

\[ J_1 \sim \left( \frac{k^{-1-m}}{m!} \right)^2 \sum_{n=0}^{\infty} A_n \Gamma(4\nu - 2m - 1)(2k)^{2m-4\nu-1} \sim k^{-4\nu-1}2^{2m-4\nu-1}\Gamma(4\nu - 2m - 1) \sum_{n=0}^{\infty} \frac{A_n}{m!} \]

(6.2)

\[ J_2 = O_{k \to \infty}(1) \]

In the same

(6.3)

\[ J_3 \sim k^{m-4\nu+1} \frac{\alpha_k^{\nu,m}}{m!(2\nu - m - 1)} \sum_{n=0}^{\infty} A_n \]

Therefore

\[ \lambda_k \sim C \sqrt{k^{m-4\nu+1}} \]

where \( C \) is a constant \( \square \)


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