Entangled coherent states for multiple bosonic modes, also referred to as multimode cat states, not only are of fundamental interest, but also have practical applications. The nonclassical correlation among these modes is well characterized by the violation of the Mermin-Klyshko inequality. We here study Mermin-Klyshko inequality violations for such multi-mode entangled states with rotated quantum-number parity operators. Our results show that the Mermin-Klyshko signal obtained with these operators can approach the maximal value even when the average quantum number in each mode is only 1, and the inequality violation exponentially increases with the number of entangled modes. The correlations among the rotated parities of the entangled bosonic modes are in distinct contrast with those among the displaced parities, with which a nearly maximal Mermin-Klyshko inequality violation requires the size of the cat state to be increased by about 15 times.

I. INTRODUCTION

Quantum mechanics predicts many counterintuitive phenomena in microscopic world, among which nonlocality is particularly appealing. According to quantum mechanics, when the wavefunctions of two objects are entangled, there exist nonlocal correlations between the features of these two objects, irrespective of their spatial distance. Einstein, Podolsky, and Rosen (EPR) rejected this point of view, and thought that the results of measurements performed on two particles should be mutually independent when their separation is sufficiently large [1]. Based on this locality condition, they concluded that quantum mechanical description of nature is incomplete. Despite the debate between Einstein and Bohr [2], it was not until Bell’s discovery in 1964 that it was realized that predictions of quantum mechanics conflict with local realism for some entangled states [3]. In the framework of local realism, Bell developed an inequality, which sets an upper bound on the statistical correlations of the results of measurements performed on two distant spin-1/2 particles. According to quantum mechanics, this bound is exceeded with proper settings of the measurement orientations when these two particles are prepared in a highly entangled state before the measurements.

Various forms of Bell inequality have been formulated, among which Clauser, Horne, Shimony and Holt (CHSH) version [4] is most famous and has been widely used for experimental test of quantum nonlocality [5-7]. The Bell-CHSH inequality has been generalized in different directions. On one hand, an inequality, referred to as the Mermin-Klyshko (MK) inequality, was derived for systems composed of multiple spin-1/2 particles [8,9], which shows that the conflict between local hidden theories and quantum mechanics becomes stronger as the number of entangled particles increases. Experimental violation of the MK inequality was reported with four-photon entanglement [10], and was analyzed in the context of ion traps [11]. On the other hand, Banaszek and Wödkiewicz constructed a Bell-CHSH-type operator for two harmonic oscillators with continuous variables, and formulated an inequality in phase space [12,13], which was later generalized to maximize the Bell signal [14,15]; however, with this formalism the inequality is not maximally violated by the original EPR state. In contrast, based on the so-called pseudospin operators introduced by Chen et al. [16], the original EPR state can maximally violate the inequality, but the experimental measurements remains challenging. We have constructed a new Bell operator [17], with which the Bell signal for a two-mode cat state of moderate size can approach the upper bound of quantum mechanics. The issue of quantum nonlocality associated with multimode continuous variable states has also been addressed [18-23]. In particular, Jeong et al. [18] have discussed the quantum nonlocality associated with entangled coherent states for three bosonic modes using joint quasiprobability distribution function in phase space. With this method, there are 12 variables to be optimized to find the global maximum value of the MK signal, which is a difficult task. In Ref. [18], only some local maximum values were numerically found. For the Greenberger-Horne-Zeilinger-type (GHZ-type) [24] entangled coherent states, also referred to as 3-mode Schrödinger cat states [25-28], the obtained MK signal increases with the amplitude $\alpha$ of the coherent state components, reaching a maximal value of 3.6 when $\alpha \rightarrow \infty$; this maximum is still lower than the quantum-mechanical upper bound of 4 for a three-partite system by 10%. With this method, the number of the variables to be optimized increases with the number of entangled
bosonic modes, which makes it extremely difficult to obtain the optimal MK signal when more bosonic modes are involved. We note that, multimode entangled coherent states are intriguing both from the fundamental viewpoint and for practical applications because the components forming these highly nonclassical states have classical analogs and can be used for fault-tolerant quantum computation [29].

In this manuscript, we analyzed the MK inequality violations for GHZ-type entangled coherent states of \( n \) bosonic modes based on effective rotated parity correlations proposed in Ref. [17]. Our results show that the MK signal quickly increases with \( n \), and approaches the quantum-mechanical upper bound for confirming true 3-partite entanglement. We demonstrate that a nearly maximal inequality violation can be obtained even when \( \alpha \sim 1 \). We find that, when using joint quasiprobability distribution function, the average quantum number in each mode needs to be increased by about 15 times to obtain the same degree of violation. In Sec. 2, we construct the MK operator for \( n \)-mode cat states with effective rotated parity operators. With this formalism, the correlations among the bosonic modes are in almost perfect analogy to those among \( n \) qubits prepared in a GHZ state even for \( \alpha \sim 1 \), in contrast with the results obtained in continuous-variable representation. In Sec. 3, we detailedly analyze the MK inequality violation for the 3-mode cat state based on effective rotated parity correlations. Analytical results qualitatively show that the size of the cat state does not need to be large for the MK signal to approach the quantum-mechanical upper bound. We confirm this prediction with numerical simulation, which demonstrates that a nearly maximal inequality violation can be obtained even when \( \alpha \) is only about 1. We find that, when using joint quasiprobability distribution function, the average quantum number in each mode needs to be increased by about 15 times to obtain the same degree of violation. In Sec. 4, we investigate the MK signals for the 4-mode and 5-mode cat states. Numerical simulations show that, for these cases the MK signals can also nearly approach the corresponding quantum-mechanical upper bounds even when the average quantum number in each mode is only 1. Conclusions are presented in Sec. 5.

\[
|\psi\rangle_{\text{cat}} = \mathcal{N}_n (|\alpha\rangle_1 |\alpha\rangle_2 |\alpha\rangle_3 \ldots |\alpha\rangle_n + |\alpha\rangle_1 |\alpha\rangle_2 |\alpha\rangle_3 \ldots |\alpha\rangle_n) 
\]

where \( \mathcal{N}_n = \left[ 2 + 2e^{-2n|\alpha|^2} \right]^{-1/2} \) and the subscripts \( j = 1 \) to \( n \) label these modes. When \( |\alpha|^2 \gg 1 \), the two coherent states \( |\alpha\rangle \) and \( |\alpha\rangle \) are approximately orthogonal and can be considered two logic states of a qubit. With this encoding, the \( n \)-mode Schrödinger cat state is in analogy with the GHZ state of \( n \) qubits

\[
|\psi\rangle_{\text{GHZ}} = \frac{1}{\sqrt{2}} (|\uparrow\rangle_1 |\uparrow\rangle_2 |\uparrow\rangle_3 \ldots |\uparrow\rangle_n + |\downarrow\rangle_1 |\downarrow\rangle_2 |\downarrow\rangle_3 \ldots |\downarrow\rangle_n) .
\]

According to quantum mechanics, when \( n \) qubits are prepared in a GHZ state, there are nonlocal correlations among the outcomes of measurements individually performed on them. The conflict between the predictions of quantum mechanics and the results allowed by local realism becomes stronger as the number of entangled qubits increases, as evidenced by the violations of the MK inequality. The MK operator for a multipartite entangled state is recursively defined as

\[
O_k = \frac{1}{\sqrt{2}} \left[ O_{k-1} (\sigma_{k,a_k} + \sigma_{k,a_k'}) + O'_{k-1} (\sigma_{k,a_k} - \sigma_{k,a_k'}) \right] ,
\]

starting with \( O_1 = \sigma_{1,a_1} \). Here \( \sigma_{k,a_k} \) and \( \sigma_{k,a_k'} \) represent two-valued observables for the \( k \)th party, with \( a_k \) and \( a_k' \) denoting the corresponding measurement settings, and \( O_k \) is obtained from \( O_k \) by exchanging all \( a_j \) and \( a_j' \) \( (j \leq k) \). For a multi-qubit system, \( a_k \) is a unit vector and \( \sigma_{k,a_k} \) corresponds to the Pauli operator along the direction \( a_k \). In the framework of local hidden variable (LHV) theories, one can assign a value of 1 or \( -1 \) to \( \sigma_{k,a_k} \) and \( \sigma_{k,a_k'} \), so that \( O_k = \pm \sqrt{2} O_{k-1} \) or \( O_k = \pm \sqrt{2} O'_{k-1} \). As a result, the outcome of each measurement on \( O_n \) is \( 2^{(n-1)/2} \) or \( -2^{(n-1)/2} \), which implies that the MK signal within a LHV model, defined as \( S_n^l = \langle O_n \rangle_{LHV} \), satisfies \( S_n^l \leq D_n = 2^{(n-1)/2} \). To show the violation of the MK inequality by the \( n \)-qubit GHZ state, we set all of \( a_k \) and \( a_k' \) are on the xy-plane, and for simplicity, use the notation \( \sigma_k(\phi_k) \) to replace \( \sigma_{k,a_k} \), with \( \phi_k \) being the angle between \( a_k \) and the \( x \)-axis. Set

\[
\phi_1 = 0, \phi_1' = \pi/2, \quad \phi_k = -\pi/4, \phi_k' = \pi/4 , \quad k \neq 1 .
\]

Then we have

\[
O_n = 2^{n-1} |\uparrow\rangle_1 (|\downarrow\rangle \otimes |\uparrow\rangle_2 \otimes |\downarrow\rangle \ldots \otimes |\uparrow\rangle_n + \text{H.c.} ,
\]

where H.c. denotes the Hermitian conjugate. According to quantum mechanics, the expectation value of \( O_n \) for the ideal \( n \)-qubit GHZ state is \( S_n = \langle O_n \rangle_{QM} = 2^{n-1} \), which exceeds the bound imposed by the local realism.

II. CONSTRUCTION OF THE MK OPERATOR FOR N-MODE CAT STATES

The \( n \)-mode Schrödinger cat state under consideration is defined as the equal superposition of all the \( n \) bosonic modes being in the coherent state \( |\alpha\rangle \) and all in \( |\alpha\rangle \),
by a factor of $2^{(n-1)/2}$, indicating the violation of the MK inequality exponentially grows with the number of entangled qubits.

We first analyze the MK inequality violation for the $n$-mode cat state based on the effective cat state qubit rotation operators. For the $k$th cat state qubit, the quantum-number parity operator $P_k = (-1)^{n/2}a_k$ acts as $\sigma_{k,x}$, where $a_k^\dagger$ and $a_k$ is the quantum-number rising and lowering operators. We note the other components of the Pauli operators in the xy-plane for the cat state qubits can be operationally constructed by combining the parity operator and the effective rotation operator, defined as [17]

$$R_{k,z}(\phi_k) = D_k^z(\alpha)G_k(\phi_k)D_k(\alpha),$$

where

$$G_k(\phi_k) = |0\rangle_k \langle 0| + \sum_{n=1}^{\infty} |n\rangle_k \langle n|$$

is the phase gate, and $D_k(\alpha) = e^{i\alpha a_k^\dagger - \alpha^* a_k}$ denotes the displacement operator. When $|\langle \alpha | - \alpha \rangle|^2 \ll 1$, $R_{k,z}(\phi_k) |\alpha\rangle_k \approx |\alpha\rangle_k$ and $R_{k,z}(\phi_k) |\alpha\rangle_k \approx e^{i\phi_k} |\alpha\rangle_k$. Therefore, $R_{k,z}(\phi_k)$ is effectively equivalent to a rotation of the $k$th cat state qubit around the z axis of the Bloch sphere by an angle $\phi_k$. The observables needed for test of the MK inequality can be expressed in terms of this kind of rotation operators and the parity operator

$$\sigma_k(\phi_k) = R_{k,z}(\phi_k)P_kR_{k,z}(\phi_k).$$

The effective rotated parity operator $\sigma_k(\phi_k)$ corresponds to the Pauli operator along the axis with an angle $\phi_k$ to the x axis of the cat state qubit. For a finite value of $\alpha$, the coherent states $|\alpha\rangle_j$ and $|\alpha\rangle_j$ are not strictly orthogonal, so that the $n$-mode cat state is not perfectly equivalent to the $n$-qubit GHZ state. As a consequence, the MK signal $S_n$ is smaller than $2^{(n-1)/2}$. However, the overlap between $|\alpha\rangle_j$ and $|\alpha\rangle_j$, $e^{-4|\alpha|^2}$, is quite small even for a moderate value of $\alpha$. For example, for $\alpha = \sqrt{2}$ this overlap is only about $2.5 \times 10^{-4}$. Therefore, the MK signal $S_n$ for the $n$-mode cat state of a moderate size, obtained with the effective cat state qubit rotation operators, can approximate the result for an $n$-qubit GHZ state.

The test of quantum nonlocality in phase space is based on the measurement of displaced quantum-number parity observables, defined as

$$P_k(\beta_k) = D_k^\dagger(\beta_k)P_kD_k(\beta_k).$$

In this formalism, $P_k(\beta_k)$ and $P_k(\beta_k)$ corresponds to $\sigma_{k,a_k}$ and $\sigma_{k,a_k^\dagger}$, and the recursive definition of the MK operator can be written as

$$Q_k = Q_{k-1} \left[ P_k(\beta_k) + P_k(\beta_k) \right] + Q'_{k-1} \left[ P_k(\beta_k) - P_k(\beta_k) \right].$$

The expectation value of each term of $Q_n$ is proportional to the joint Wigner function (quasiprobability distribution function) for these $n$ bosonic modes at the corresponding point in phase space. For a limited value of $\alpha$, the displacement operation $D_k(\beta_k)$ moves the corresponding bosonic mode out of the logic space of the corresponding cat state qubit, and is not equivalent to a qubit rotation. Therefore, the obtained MK signal is expected to be smaller than that based on the effective cat qubit rotation operators.

### III. Violations of the MK Inequality with the 3-Mode Cat State

To quantitatively show the violations of MK inequality based on the above-mentioned rotated parity correlations, we will derive analytic expressions for the quantum mechanical expectation values of the MK observable $O_n$ and then perform numerical simulations. We first consider the 3-mode cat state, for which the MK operator based on the effective rotated parity correlations among the three modes is given by

$$O_3 = \sigma_1(0)\sigma_2(-\pi/4)\sigma_3(\pi/4) + \sigma_1(0)\sigma_2(\pi/4)\sigma_3(-\pi/4) + \sigma_1(\pi/2)\sigma_2(-\pi/4)\sigma_3(-\pi/4) - \sigma_1(\pi/2)\sigma_2(\pi/4)\sigma_3(\pi/4).$$

The MK signal, defined as the absolute value of the expectation value of $O_3$, is

$$S_3 = |E_3(0, -\pi/4, \pi/4) + E_3(0, \pi/4, -\pi/4) + E_3(\pi/2, -\pi/4, -\pi/4) - E_3(\pi/2, \pi/4, \pi/4)|,$$

where

$$E_3(\phi_1, \phi_2, \phi_3) = N_3^2 \left\{ e^{-6|\alpha|^2} + \sum_{k=1}^{3} K_{\alpha,\alpha}(\phi_k) + \prod_{k=1}^{3} K_{\alpha,\alpha}(\phi_k) + c.c. \right\},$$

with

$$K_{\alpha,\alpha}(\phi_k) = -e^{-2|\alpha|^2} + 2e^{-2|\alpha|^2} \cos \phi_k + 2e^{-4|\alpha|^2} (1 - \cos \phi_k),$$

(14)

When $e^{-2|\alpha|^2} \ll 1$, the dominant term of $E_3(\phi_1, \phi_2, \phi_3)$ is $2N^2 \cos(\phi_1 + \phi_2 + \phi_3)$. The ratio of this term to the corresponding correlation for a GHZ state is $2N_3^2$, which approximates 1 even for a very moderate value of $\alpha$. For example, it is about 0.9975 for $\alpha = 1$. This allows a nearly maximal violation of the MK inequality with the average quantum number in each mode being only about 1.

To confirm the validity of the rotated parity operators for revealing the quantum nonlocality among the three
bosonic modes, we first perform numerical simulations of the four correlations $E_3(0, -\pi/4, \pi/4)$, $E_3(0, \pi/4, -\pi/4)$, $E_3(\pi/2, -\pi/4, -\pi/4)$, and $E_3(\pi/2, \pi/4, \pi/4)$ as functions of $\alpha$. As shown in Eqs. (13) and (14), these correlations are independent of the argument of the complex amplitude $\alpha$, which is taken to be a positive real number for simplicity. The results are respectively shown in Fig. 1(a), (b), (c), and (d), where the solid lines denote the correlations of the 3-mode cat states, while the dotted lines represent the corresponding correlations for the equally-weighted classical mixture of the two components $|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3$ and $|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3$, which are given by

$$E^m_{3}(\phi_1, \phi_2, \phi_3) = \frac{1}{2} \left[ e^{-|\alpha|^2} + \sum_{k=1}^{3} K_{n,\alpha}(\phi_k) \right].$$

For $\alpha = 0$, each mode is in the vacuum state $|0\rangle$, which is not affected by the rotation operator $R_{k,\alpha}(\phi_k)$. Consequently, the rotated parity operator $\sigma_k(\phi_k)$ is equivalent to the parity operator $R_k$, which has a definite value of 1, so that each correlation is 1. In this case, all correlations are completely classical as there is no entanglement. As $\alpha$ increases, for both $|\alpha\rangle$ and $|\alpha\rangle$ the difference between the occupations between the even and odd quantum-number states quickly decreases, and consequently the correlations $E^m_{3}(\phi_1, \phi_2, \phi_3)$ for the classical mixture drop fast, and approach 0 when $e^{-2|\alpha|^2} \ll 1$. In contrast, for this case the 3-mode cat state exhibits nearly perfect correlations, like the three-qubits GHZ state $|\psi\rangle_{GHZ}$, which is the eigenstate of each of the four terms in $\hat{O}_3$, with the eigenvalues corresponding to the first three terms being 1 and that associated with the last term being $-1$. This is in distinct contrast with the two-mode case, where the correlations used to construct the Bell signal are not perfect even when $\alpha \to \infty$; the absolute value of each correlation tends to $\sqrt{2}/2$ in this limit [17]. These results unambiguously demonstrate that, when $e^{-2|\alpha|^2} \ll 1$, the correlations associated with the 3-mode cat state almost completely arise from the quantum coherence between $|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3$ and $|\alpha\rangle_1|\alpha\rangle_2|\alpha\rangle_3$, which is responsible for the entanglement among the three modes.

The MK signal ($S_3$) for the 3-mode cat state as a function of $\alpha$ is shown in Fig. 2, with the blue solid line representing the simulated result while the black dotted line denoting the bound imposed by local hidden variable theories. The result shows that $S_3$ first drops below 2, and then quickly increases with $\alpha$. This non-monotonous behavior is similar to the two-mode case [17]. When $\alpha < 0.266$, $E_3(\pi/2, -\pi/4, -\pi/4)$ and $E_3(\pi/2, \pi/4, \pi/4)$ almost have the same value and thus cancel each other out, so that $S_3$ is approximately equal to the sum of the other two correlations, each of which slightly decreases as $\alpha$ increases in this regime. Consequently, $S_3$ drops as $\alpha$ increases in this regime. When $\alpha > 0.266$, the difference between $E_3(\pi/2, -\pi/4, -\pi/4)$ and $E_3(\pi/2, \pi/4, \pi/4)$ increases quickly, whose contribution makes $S_3$ rise quickly. $S_3$ reaches the classical upper bound 2 at $\alpha = 0.36$, and exceeds the lower bound $2\sqrt{2}$ for true 3-partite entanglement [30] when $\alpha > 0.581$. When $\alpha$ further increases, it quickly approaches the quantum-mechanical upper bound 4, which implies that a nearly maximal MK inequality violation can be obtained with a small cat state. For example, when $\alpha = 1$, $S_3 \approx 3.916$, very close to upper bound 4. This can be interpreted as follows. The two coherent states $|\alpha\rangle$ and $|\alpha\rangle$ of each bosonic mode are approximately orthogonal to each other and can act as the basis states of a logic qubit even for a moderate value of $\alpha$. The bosonic mode is well restricted in the logic space of the cat state qubit under the application of the effective cat state qubit rotation operator $R_{k,\alpha}(\phi_k)$. Consequently, the rotated parity operator $\sigma_k(\phi_k)$ is approximate to the corresponding Pauli operator, and the 3-mode cat state is approximately an eigenstate of each of the four correlation operators in Eq. (11).

As noted in Ref. [18], for the MK signal based on the joint Wigner function, there are 6 complex displacement parameters, $\beta_k$ and $\beta_k'$ with $k = 1$ to 3, corresponding to 12 variables to be optimized to obtain the maximal MK signal for each value of $\alpha$, which is a difficult task. Instead of the global maximum value, a local maximum value for each value of $\alpha$ was numerically obtained in Ref. [18], which monotonously increases with $\alpha$, and approaches 3.6, instead of 4, when $\alpha \to \infty$. We note that the upper bound 4 can be reached in this limit with the choice $\beta_1 = 0$, $\beta_1' = i\pi/8$, $\beta_2 = -i\pi/16$, $\beta_2' = i\pi/16$, with $k > 1$. This is due to the fact that the displaced parity operators are equivalent to the corresponding Pauli operators of the cat state qubits when $\alpha \to \infty$, so that the MK inequality can be maximally violated in this limit. Based on these displaced parity correlations, the MK signal at
where

\[ \alpha = 3.824 \]

is equal to the result of Fig. 2 at \( \alpha = 1 \), which implies that with these correlations the average quantum number in each mode, \( \bar{n} = |\alpha|^2 \), should be increased by about 15 times for approaching the maximal violation. According to the result of Ref. [18], the MK signal at \( \alpha = 1 \) is only about 2.5, which is significantly below the lower bound for confirming true 3-partite entanglement. This is in stark contrast with the result obtained with our framework, where the upper bound allowed by 3-partite entanglement can be approached for \( \alpha \approx 1 \).

IV. MK INEQUALITY VIOLATIONS WITH 4-AND 5-MODE CAT STATES

The MK signal for the 4-mode cat state obtained with the effective rotated parity operators is given by

\[
S_4 = \frac{1}{\sqrt{2}}[ -E_4(0, -\pi/4, -\pi/4, -\pi/4) + \\
3E_4(0, -\pi/4, -\pi/4, \pi/4) + 3E_4(0, -\pi/4, \pi/4, \pi/4) - \\
E_4(\pi/2, \pi/4, \pi/4, -\pi/4) - E_4(\pi/2, \pi/4, \pi/4, \pi/4) - \\
3E_4(\pi/2, \pi/4, \pi/4, -\pi/4) + 3E_4(\pi/2, \pi/4, -\pi/4, -\pi/4) + \\
E_4(\pi/2, -\pi/4, -\pi/4, -\pi/4)] .
\]

where

\[
E_4(\phi_1, \phi_2, \phi_3, \phi_4) = N_4^2 \left\{ e^{-8|\alpha|^2} + \prod_{j=1}^{4} K_{\alpha,\alpha}(\phi_j) \\
+ \left[ \prod_{j=1}^{4} K_{\alpha,-\alpha}(\phi_j) + c.c. \right] \right\} .
\]

The blue solid line in Fig. 3 represents the rescaled MK signal \( R_4 \), as a function of the amplitude \( \alpha \) of the coherent state components. Here \( R_4 \) is defined as the ratio of the MK signal to the upper bound \( 2\sqrt{2} \) allowed by local hidden variable models. The results show \( R_4 \) exceeds the classical upper bound 1 when \( \alpha > 0.311 \), and surpasses the lower bound 2 for true 4-partite entanglement when \( \alpha > 0.616 \). The maximal value \( 2\sqrt{3} \) can be approached for a moderate value of \( \alpha \). For example, \( R_4 \approx 2.758 \) when \( \alpha = 1 \). Based on displaced parity operators, the MK signal reaches this value only when \( \alpha \) is above 3.94. For the 5-mode cat state, the MK signal based on the effective rotation operators is given by

\[
S_5 = -E_5(0, -\pi/4, -\pi/4, -\pi/4, -\pi/4) + \\
6E_5(0, -\pi/4, -\pi/4, \pi/4, \pi/4) - \\
E_5(\pi/2, \pi/4, \pi/4, \pi/4, -\pi/4) - \\
4E_5(\pi/2, \pi/4, \pi/4, \pi/4, \pi/4) + \\
4E_5(\pi/2, \pi/4, -\pi/4, -\pi/4, -\pi/4),
\]

where

\[
E_5(\phi_1, \phi_2, \phi_3, \phi_4) = N_5^2 \left\{ e^{-10|\alpha|^2} + \prod_{j=1}^{5} K_{\alpha,\alpha}(\phi_j) \\
+ \left[ \prod_{j=1}^{5} K_{\alpha,-\alpha}(\phi_j) + c.c. \right] \right\} .
\]

The red dashed line in Fig. 3 denotes the rescaled 5-mode MK signal, defined as \( R_5 = S_5/4 \), versus \( \alpha \). The results show \( R_5 \) reaches the classical upper bound 1 and
the lower bound $2\sqrt{2}$ for true 5-partite entanglement at $\alpha = 0.281$ and 0.64, respectively, and quickly approaches the maximum of 4. For $\alpha = 1$, $R_5 \simeq 3.882$, close to the maximal value. In contrast, with displaced parity operators, the same value of the MK signal corresponds to $\alpha \simeq 3.927$.

V. CONCLUSIONS

In conclusion, we have analyzed quantum nonlocality for $n$ bosonic modes in an entangled coherent state using rotated parity operators. Our results show that the correlations obtained with this approach are close to those for the n-qubit GHZ state, and the MK inequality can be nearly maximally violated even when the average quantum number in each mode is only about 1. This is in distinct contrast with the previous study based on displaced parity operators, where the obtained MK signal for the 3-mode cat state even does not exceed the lower bound for genuine 3-partite entanglement for this size. With this framework, a nearly maximal inequality violation requires the quantum number in each mode to be improved by about 15 times compared to the present result. Since the decoherence rate of a cat state increases with the size, our results are useful for experimentally investigating quantum nonlocality for entanglement of quasiclassical states of multiple bosonic modes.

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