Integrable cross-field generation based on imposed singularity configuration – the 2D manifold case –

Jovana Jezdimirović, Alexandre Chemin, Jean-François Remacle

Université catholique de Louvain, Louvain la Neuve, Belgium
jean-francois.remacl@uclouvain.be

Abstract

This work presents the mathematical foundations for the generation of integrable cross-field on 2D manifolds based on user-imposed singularity configuration. In this paper, we either use singularities that appear naturally, e.g., by solving a non-linear problem, or use as an input user-defined singularity pattern, possibly with high valence singularities that typically do not appear in cross-field computations. This singularity set is under the constraint of Abel-Jacobi’s conditions for valid singularity configurations. The main contribution of the paper is the development of a formulation that allows computing an integrable isotropic 2D cross-field from a given set of singularities through the resolution of only two linear PDEs. To address the issue of possible suboptimal singularities’ distribution, we also present the mathematical setting for the generation of an integrable anisotropic 2D cross-field based on a user-imposed singularity pattern. The developed formulations support both an isotropic and an anisotropic block-structured quad mesh generation.

Keywords: integrable 2D cross-field, valid singularity configuration, quad layout, quad meshing

1. Introduction and related work

Numerous methods for surface parametrization/representation have been developed for a large number of applications [1, 2, 3]. In cases when a shape exhibits complex topological or geometrical characteristics, it is necessary to split it into simple partitions to obtain a quad mesh. The special case of partitioning it into a simply connected network of conformal quadrilateral partitions is called the quad layout [4]. The latter manner of surface representation is a subject of great interest in meshing and computer graphics communities, due to providing a wide range of benefits [1, 2, 5]. Nevertheless, these advantages come with the high price of dealing with complex and time-consuming algorithms [5].

Among the developed methods for the quad layout generation, a general distinction can be made among the ones which are: computing a seamless global
parametrization of the domain where integer iso-values of the parameter fields form the sides $[6, 7, 8]$, using Riemann geometry $[9, 10, 11]$, or like in our case, constructing a cross-field structure that will guide the integral lines emanating from singularities $[12, 13, 14, 15, 16, 17]$.

Although leaning on heterogeneous approaches, all the above-mentioned methods share the common challenge: dealing with the inevitable singularity configuration. A singularity appears where a cross-field vanishes and it represents an irregular vertex of a quad layout/quad mesh $[18]$, i.e., a vertex which doesn’t have exactly four adjacent quadrilaterals. The singular configuration is constrained by the Euler characteristic $\chi$, which is a topological invariant of a surface. Moreover, a suboptimal number or location of singularities can have severe consequences: causing undesirable thin partitions, large distortion, not an adequate number and/or tangential crossings of separatrices as well as limit cycles $[5, 7, 17]$.

Cross-field guided methods can be very useful and flexible but they typically lack direct control over the positions of the singularities and the structures of the quad layout $[19]$. Our cross-field formulation, with mathematical foundations detailed in Section 2, offers a contribution to this issue through the concept of user-imposed singularity configuration in order to gain direct control over their number, location, and valence (number of adjacent quadrilaterals). The user is entitled to use either naturally appearing singularities, obtained by solving a non-linear problem $[14, 17, 20, 21]$, using globally optimal direction fields $[22]$, or to impose its own singularity configuration, possibly with high valences, as illustrated in Fig. 1. It is important to note that the choice of singularity pattern is not arbitrary, though. Moreover, it is under the direct constraint of Abel-Jacobi theory $[9, 10, 11]$ for valid singularity configurations. Here, the singularity configuration is taken as an input and an integrable isotropic cross-field is computed by solving only two linear systems, Section 3. Computation of the scalar field $H$ used for this cross-field generation bears some resemblance to the one developed for unstructured mesh generation on planar and curved surface domains in $[23]$. Finally, the preliminary results of the developed cross-field formulation for an isotropic block-structured quad mesh generation are outlined using the 3-step pipeline $[15]$ in Section 4.

Computing only one scalar field $H$ (a metric that is flat except at singularities) imposes a strict constraint on singularities’ placement, i.e., fulfilling all Abel-Jacobi conditions. In practice, imposing suboptimal distribution of singularities may lead to not obtaining boundary-aligned cross-field, disabling an isotropic quad mesh generation. Section 4.1 and 4.2. To bypass this issue, we develop a new cross-field formulation on the imposed singularity configuration, which considers the integrability, while relaxing the condition on isotropic scaling of crosses’ branches. Here, two independent metrics $H_1$ and $H_2$ are computed instead of only one as in the Abel-Jacobi framework, enabling an integrable 2D cross-field generation with anisotropic scaling, Section 5.

Lastly, final remarks and some of the potential applications are discussed in Section 6.
2. Cross-field computation on prescribed singularity configuration

We define a 2D cross \( \mathbf{c} \) as a set of 2 unit coplanar orthogonal vectors and their opposite, \( \mathbf{c} = \{ \mathbf{u}, \mathbf{v}, -\mathbf{u}, -\mathbf{v} \} \), with \( \mathbf{u}, \mathbf{v} \neq 0, |\mathbf{u}| = |\mathbf{v}| = 1 \) and \( \mathbf{u}, \mathbf{v} \) coplanar. These vectors are called cross’ branches.

A 2D cross-field \( C_M \) on a 2D manifold \( M \), now, is a map \( C_M : X \in M \rightarrow c(X) \), and the standard approach to compute a smooth boundary-aligned cross-field is to minimize the Dirichlet energy:

\[
\min_{C_M} \int_M \| \nabla C_M \|^2
\]

subject to the boundary condition \( c(X) = g(X) \) on \( \partial M \), where \( g \) is a given function.

The classical boundary condition for cross-field computation is that \( \forall P \in \partial M \), with \( T(P) \) a unit tangent vector to \( M \) at \( P \), one branch of \( c(P) \) has to be colinear to \( T(P) \). In the general case, there exists no smooth cross-field matching this boundary condition. The cross-field will present a finite number of singularities \( S_j \), located at \( X_j \), and of index \( k_j \), related to the concept of valence as \( k_j = 4 - \text{valence}(S_j) \).

We define a singularity configuration as the set

\[
S = \{ S_j, j \in [1, N], N \in \mathbb{Z} \}.
\]

In the upcoming section, a method to compute a cross-field \( C_M \) matching a given singularity configuration \( S \) is developed. In other words, we are looking for \( C_M \) such as:

\[
\begin{align*}
&\cdot \text{ if } X \in \partial M, \text{ at least one branch of } C_M(X) \text{ is tangent to } \partial M, \\
&\cdot \text{ singularities of } C_M \text{ are matching the given } S \\
&(\text{the same number, location, and indices}).
\end{align*}
\]

Before developing the method to compute such a cross-field, a few operators on the 2D manifold have to be defined.
2.1. Curvature and Levi-Civita connection on the 2D manifold

Let $E^3$ be the Euclidean space equipped with a Cartesian coordinates system $\{x^i, i = 1, 2, 3\}$, and $\mathcal{M}$ be an oriented two-dimensional manifold embedded in $E^3$. We note $n(X)$ the unit normal to $\mathcal{M}$ at $X \in \mathcal{M}$. It is assumed that the normal field $n$ is smooth and that the Gaussian curvature $K$ is defined and smoothed on $\mathcal{M}$.

If $\gamma(s)$ is a curve on $\mathcal{M}$ parametrized by arc length, the Darboux frame is the orthonormal frame defined by

\begin{align*}
T(s) &= \gamma'(s) \\
n(s) &= n(\gamma(s)) \\
t(s) &= n(s) \times T(s).
\end{align*}

One then has the differential structure

\begin{equation}
d \begin{pmatrix} T \\ t \\ n \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_r \\ -\kappa_n & -\tau_r & 0 \end{pmatrix} \begin{pmatrix} T \\ t \\ n \end{pmatrix} \, ds
\end{equation}

where $\kappa_g$ is the geodesic curvature of the curve, $\kappa_n$ the normal curvature of the curve, and $\tau_r$ the relative torsion of the curve. $T$ is the unit tangent, $t$ the tangent normal and $n$ the unit normal.

Arbitrary vector fields $V$ and $W \in E^3$ can be expressed as

\begin{align*}
V &= V^i E_i, \\
W &= W^i E_i
\end{align*}

in the natural basis vectors $\{E_i, i = 1, 2, 3\}$ of this coordinate system, and we shall note

\begin{align*}
<V, W> &= V^i W^j \delta_{ij}, \\
||V|| &= \sqrt{<V, V>}
\end{align*}

the Euclidean metric and the associated norm for vectors. The Levi-Civita connection on $E^3$ in Cartesian coordinates is trivial (all Christoffel symbols vanish), and one has

$$\nabla^E_V W = (\nabla_V W^i) E_i.$$  

The Levi-Civita connection on the Riemannian submanifold $\mathcal{M}$, now, is not a trivial one. It is the orthogonal projection of $\nabla^E_V$ in the tangent bundle $T\mathcal{M}$, so that one has

\begin{equation}
\nabla_V W = P_{T\mathcal{M}}[\nabla^E_V W] = (\nabla_V W^i) P_{T\mathcal{M}}[E_i]
\end{equation}

where $P_{T\mathcal{M}} : E^3 \to T\mathcal{M}$ is the orthogonal projection operator on $T\mathcal{M}$.

An arbitrary orthonormal local basis $(u_X, v_X, n)$ for every $X \in \mathcal{M}$, can be represented through the Euler angles $(\psi, \gamma, \phi)$ which are $C^1$ on $\mathcal{M}$, and with the
shorthands $s_\phi \equiv \sin \phi$ and $c_\phi \equiv \cos \phi$, as:

$$
\begin{align*}
\mathbf{u}_X &= \begin{pmatrix}
-s_\phi s_\psi c_\gamma + c_\phi c_\psi \\
s_\phi c_\psi c_\gamma + s_\psi c_\phi \\
s_\phi s_\gamma
\end{pmatrix}, \\
\mathbf{v}_X &= \begin{pmatrix}
-s_\phi s_\psi - s_\psi c_\phi c_\gamma \\
s_\psi c_\phi c_\gamma + s_\phi s_\gamma \\
s_\gamma
\end{pmatrix}, \\
\mathbf{n} &= \begin{pmatrix}
s_\psi s_\gamma \\
s_\gamma c_\phi \\
c_\gamma
\end{pmatrix}
\end{align*}
$$

in the vector basis of $E^3$.

### 2.2. Conformal mapping

We are looking for a conformal mapping

$$
\mathcal{F} : \mathcal{P} \rightarrow \mathcal{M} \subset E^3
$$

where $\mathcal{P}$ is a parametric space. As finding $\mathcal{F}$ right away is a difficult problem, one focuses instead on finding the $3 \times 2$ jacobian matrix of

$$
J(P) = (\partial_\xi \mathcal{F}(P), \partial_\eta \mathcal{F}(P)) \equiv (\mathbf{\hat{u}}(P), \mathbf{\hat{v}}(P)),
$$

where $\mathbf{\hat{u}}, \mathbf{\hat{v}} \in T\mathcal{M}$ are the columns vectors of $J$. The mapping $\mathcal{F}$ being conformal, the columns of $J(P)$ have the same norm $L(P) \equiv ||\mathbf{\hat{u}}(P)|| = ||\mathbf{\hat{v}}(P)||$ and are orthogonal to each other, $\mathbf{\hat{u}}(P) \cdot \mathbf{\hat{v}}(P) = 0$. We can also write:

$$
J = L(\mathbf{u}, \mathbf{v}), \quad \mathbf{n} = \mathbf{u} \wedge \mathbf{v}
$$

where

$$
\mathbf{u} = \frac{\mathbf{\hat{u}}}{||\mathbf{\hat{u}}||} \quad \mathbf{v} = \frac{\mathbf{\hat{v}}}{||\mathbf{\hat{v}}||}.
$$

Recalling that finding a conformal transformation $\mathcal{F}$ is challenging, we will from now on be looking for the jacobian $J$, i.e., the triplet $(\mathbf{u}, \mathbf{v}, L)$.

The triplet $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ forms a set of 3 orthonormal basis vectors and can be seen as a rotation of $(\mathbf{u}_X, \mathbf{v}_X, \mathbf{n})$ among the direction $\mathbf{n}$. Therefore, a 2D cross $c(\mathbf{X})$, $\mathbf{X} \in \mathcal{M}$ can be defined with the help of a scalar field $\theta$, where $\mathbf{u} = R_{\theta, \mathbf{n}}(\mathbf{u}_X)$ and $\mathbf{v} = R_{\theta, \mathbf{n}}(\mathbf{v}_X)$, and the local manifold basis $(\mathbf{u}_X, \mathbf{v}_X, \mathbf{n})$ as:

$$
\mathbf{u} = c_\theta \mathbf{u}_X + s_\theta \mathbf{v}_X, \quad \mathbf{v} = -s_\theta \mathbf{u}_X + c_\theta \mathbf{v}_X.
$$

By using the Euler angles $(\psi, \gamma, \phi)$ and $\theta$, the triplet $(\mathbf{u}, \mathbf{v}, \mathbf{n})$ can also be
expressed as:

\[
\begin{align*}
\mathbf{u} &= \begin{pmatrix}
-s\theta + \phi \psi C + c\theta + \phi C \\
 s\theta + \phi C \gamma + s\phi c\theta + \phi \\
 s\theta + \phi s\gamma
\end{pmatrix}, \\
\mathbf{v} &= \begin{pmatrix}
-s\theta + \phi \psi C - s\phi c\theta + \phi C \\
 -s\theta + \phi \psi + c\theta + \phi C \gamma \\
 s\gamma c\theta + \phi \\
 s\phi s\gamma \\
 c\gamma
\end{pmatrix}, \\
\mathbf{n} &= \begin{pmatrix}
 s\psi s\gamma \\
 -s\gamma C \psi \\
 c\gamma
\end{pmatrix},
\end{align*}
\]  

(13)

It is important to note that \(\mathbf{u}\) and \(\mathbf{v}\) are the two branches of the cross-field \(C_M\) we are looking for. The projection operator \(P_{T\mathcal{M}}\) introduced in Eq. (7) then simply amounts to disregarding the component along \(\mathbf{n}\) of vectors.

For a vector field \(\mathbf{w}\) defined on \(\mathcal{M}\), one can write by derivation of Eq. (13)

\[
\nabla^E_w \mathbf{u} = \mathbf{v} \nabla_w (\theta + \phi) + s\theta + \phi \mathbf{n} \nabla_w \gamma \\
+ (c_\gamma \mathbf{v} - s_\gamma c\theta + \phi \mathbf{n}) \nabla_w \psi
\]

\[
\nabla^E_w \mathbf{v} = -\mathbf{u} \nabla_w (\theta + \phi) + c\theta + \phi \mathbf{n} \nabla_w \gamma \\
+ (-c_\gamma \mathbf{u} + s_\gamma s\theta + \phi \mathbf{n}) \nabla_w \psi
\]

\[
\nabla^E_w \mathbf{n} = -(s\theta + \phi \mathbf{u} + c\theta + \phi \mathbf{v}) \nabla_w \gamma \\
+ s_\gamma (c\theta + \phi \mathbf{u} - s\theta + \phi \mathbf{v}) \nabla_w \psi
\]

and hence, using Eq. (7), the expression of the covariant derivatives on the submanifold \(\mathcal{M}\) is:

\[
\begin{align*}
\nabla_w \mathbf{u} &= \mathbf{v} \nabla_w (\theta + \phi) + c_\gamma \mathbf{v} \nabla_w \psi \\
\nabla_w \mathbf{v} &= -\mathbf{u} \nabla_w (\theta + \phi) - c_\gamma \mathbf{u} \nabla_w \psi.
\end{align*}
\]  

(15)

This allows writing the Lie bracket

\[
[\mathbf{u}, \mathbf{v}] = \nabla_\mathbf{u} \mathbf{v} - \nabla_\mathbf{v} \mathbf{u} \\
= -(\mathbf{u} \nabla_\mathbf{u} (\theta + \phi) + \mathbf{v} \nabla_\mathbf{v} (\theta + \phi) \\
- c_\gamma (\mathbf{u} \nabla_\mathbf{u} \psi + \mathbf{v} \nabla_\mathbf{v} \psi),
\]  

(16)

which will be used in the upcoming section.

3. Integrability condition with isotropic scaling

The mapping \(\mathcal{F}\), now, defines a conformal parametrization of \(\mathcal{M}\) if the columns of \(J\) commute as vector fields, \(i.e.,\) if the differential condition

\[
0 = [\tilde{\mathbf{u}}, \tilde{\mathbf{v}}] = \nabla_\tilde{\mathbf{u}} \tilde{\mathbf{v}} - \nabla_\tilde{\mathbf{v}} \tilde{\mathbf{u}} = [L \mathbf{u}, L \mathbf{v}]
\]  

(17)

is verified. Developing the latter expression and posing for convenience \(L = e^H\), it becomes

\[
0 = \mathbf{v} \nabla_\mathbf{u} H - \mathbf{u} \nabla_\mathbf{v} H + [\mathbf{u}, \mathbf{v}],
\]
and then
\[
\begin{align*}
\nabla_u H &= -\langle v, [u,v] \rangle \\
\nabla_v H &= \langle u, [u,v] \rangle
\end{align*}
\]

which after the substitution of Eq. (16) gives
\[
\begin{align*}
\nabla_u H &= \nabla_v \theta + \nabla_v \phi + c_\gamma \nabla_v \psi \\
-\nabla_v H &= \nabla_u \theta + \nabla_u \phi + c_\gamma \nabla_u \psi.
\end{align*}
\] (19)

In order to obtain the boundary value problem for \(H\), the partial differential equation (PDE) governing it will be expressed on \(\partial \mathcal{M}\) as well as on the interior of \(\mathcal{M}\).

### 3.1. \(H\) PDE on the boundary

As the boundary \(\partial \mathcal{M}\) is represented by curves on \(\mathcal{M}\), it is possible to parametrize them by arc length and thus associate for each \(X \in \partial \mathcal{M}\) a Darboux frame \((T(X), t(X), n(X))\). As we are looking for a cross-field \(C_\mathcal{M}\) fulfilling conditions (2), the triplet \((u, v, n)\) can be identified as \((T(X), t(X), n(X))\). One then has:
\[
\partial_s T = \kappa_t t + \kappa_n n \equiv \nabla_u u = \langle v, \nabla_u \phi + s_\gamma n \nabla_u \gamma + (c_\gamma v - s_\gamma c_\phi n) \nabla_u \psi \rangle
\]

where from follows
\[
\begin{align*}
\kappa_g &= \nabla_u \phi + c_\gamma \nabla_u \psi \\
\kappa_n &= s_\phi \nabla_u \gamma - s_\gamma c_\phi \nabla_u \psi.
\end{align*}
\] (20)

Using Eq. (19) it becomes:
\[
\nabla_t H = -\kappa_g,
\] (21)

the result that matches exactly the one found in the planar case [15].

### 3.2. \(H\) PDE in the smooth region on the interior of \(M\)

To find the PDE governing \(H\), let’s assume the jacobian \(J\) is smooth (and therefore \(H\)) in a vicinity \(V\) of \(X \in \mathcal{M}\).

We choose \(U \subset V\) such as \(X \in U\), \(\partial U\) such as unit tangent vector \(T_0\) to \(\partial U_0\) verifies \(T_0 = v\), \(T_1\) to \(\partial U_1\) verifies \(T_1 = u\), \(T_2\) to \(\partial U_2\) verifies \(T_2 = -v\), \(T_3\) to \(\partial U_3\) verifies \(T_3 = -u\).

Thus we have a submanifold \(U \subset \mathcal{M}\) on which \(H\) is smooth, and such as \(\partial U = \partial U_0 \cup \partial U_1 \cup \partial U_2 \cup \partial U_3\). Darboux frames of \(\partial U\) (Fig. 2) are:
\[
\begin{align*}
(T, t, n) &= (v, -u, n) \quad \text{on } \partial U_0 \\
(T, t, n) &= (-u, v, n) \quad \text{on } \partial U_1 \\
(T, t, n) &= (-v, u, n) \quad \text{on } \partial U_2 \\
(T, t, n) &= (u, v, n) \quad \text{on } \partial U_3
\end{align*}
\] (22)

For \((\mathbf{u}, \mathbf{v})\) to be a local coordinate system, we recall Eq. (21) demonstrated in Section 3.1:
\[
\kappa_g = -\nabla_t H, \text{ with } t = n \wedge T
\] (23)
and the divergence theorem stating that:

$$\int_{\partial U} \nabla_t H = -\int_U \Delta H.$$  \hfill (24)

Applying the Gauss-Bonnet theorem on $U$ leads to:

$$\int_U K \mathrm{d}U + \int_{\partial U} \kappa_g \mathrm{d}l + \frac{4\pi}{2} = 2\pi \chi(U)$$

where $K$ and $\chi(U)$ are respectively the Gaussian curvature and the Euler characteristic of $U$. As $\chi(U) = 1$ and using Eq. (23) and (24), it becomes:

$$\int_U K \mathrm{d}U = -\int_U \Delta H \mathrm{d}U$$  \hfill (25)

which holds for any chosen $U$. Hence, there is:

$$\Delta H = -K, \text{ if } J \text{ is smooth.}$$  \hfill (26)

In the general case, it is impossible for $J$ to be smooth everywhere. Indeed, let’s assume $M$ to be with smooth boundary $\partial M$ (i.e. with no corners) and of the Euler characteristic $\chi(M) = 1$. If we assume $J$ is smooth everywhere, it becomes:

$$\begin{cases}
\int_M K \mathrm{d}M + \int_{\partial M} \kappa_g \mathrm{d}l &= 0 \\
2\pi \chi(M) &= 2\pi
\end{cases}$$  \hfill (27)

which is not in accordance with the Gauss-Bonnet theorem. Therefore, $J$ has to be singular somewhere in $M$.

The goal is to build a usable parametrization of $M$, i.e., being able to use this parametrization to build a quad mesh of $M$. Therefore, we will allow $J$ to be singular on a finite number $N$ of points $S_j, j \in [0, N - 1]$ and show that this condition is sufficient for this problem to always have a unique solution.
3.3. H PDE at singular points

For now, we know boundary conditions for $H$, Eq. (21), and the local equation in smooth regions, Eq. (26). The only thing left is to determine a local PDE governing $H$ at singular points $\{S_j\}$. We define $k_j$ as the index of singularity $S_j$.

For this, we are making two reasonable assumptions:

$$
\begin{align*}
\Delta H(S_j) &= -K(S_j) + \alpha_j \delta(S_j) \\
k_i &= k_j \Rightarrow \alpha_i = \alpha_j,
\end{align*}
$$

(28)

where $\alpha_j$ is a constant, and $\delta$ is the Dirac distribution. We consider the disk $\mathcal{M}$ represented in Fig. 3 with 4 singularities $S_j, j \in [0, 3]$ of index $k_j = 1$.

The Gauss-Bonnet theorem states that:

$$
\int_{\mathcal{M}} K \, d\mathcal{M} + \int_{\partial \mathcal{M}} \kappa_g \, dl = 2\pi \chi(\mathcal{M}).
$$

Replacing $K$ and $\kappa_g$ by their values in Eq. (21) and (26), and using the hypothesis (28) we get $\alpha = 2\pi \frac{1}{4}$.

For the singularity of index 1 we have:

$$
\Delta H(S_j) = -K(S_j) + 2\pi \frac{1}{4} \delta(S_j).
$$

Using the same idea, we can generalize the following:

$$
\Delta H(S_j) = -K(S_j) + 2\pi k_j \frac{1}{4} \delta(S_j).
$$

(29)

3.4. Boundary value problem for $H$

To sum up, the equations governing $H$ on $\mathcal{M}$ are:

$$
\begin{align*}
\Delta H &= -K + 2\pi k_j \frac{1}{4} \delta(S_j) & \text{on } \mathcal{M} \\
\nabla_t H &= -\kappa_g & \text{on } \partial \mathcal{M}.
\end{align*}
$$

(30)

This problem is well-posed and admits a unique solution to an arbitrary additive constant. A triangulation $\mathcal{M}_T$ of the manifold $\mathcal{M}$ is generated and problem (30) is solved using a finite element formulation with order 1 Lagrange elements. Once $H$ is determined (illustrated in Fig. 4), the next step is to retrieve $J$ orientation, detailed in the next section. The fact that $H$ is only known up to an additive constant is not harmful as only $\nabla H$ will be needed to retrieve $J$ orientation.
3.5. Retrieving crosses orientation from $H$

In order to get an orientation at a given point $\mathbf{X} \in \mathcal{M}$, a local reference basis $(\mathbf{u}_{\mathbf{X}}, \mathbf{v}_{\mathbf{X}}, \mathbf{n})$ in $\mathbf{X}$ is recalled.

Equation (19) imposes that:

$$\begin{cases} \frac{\nabla u}{H} & = \nabla v (\phi + \theta) + c_{\gamma} \nabla v \psi \\ \frac{\nabla v}{H} & = -\nabla u (\phi + \theta) - c_{\gamma} \nabla u \psi \end{cases} \quad (31)$$

which is equivalent to:

$$\begin{cases} \frac{\nabla u_{\mathbf{X}}}{H} & = \nabla v_{\mathbf{X}} (\phi + \theta) + c_{\gamma} \nabla v_{\mathbf{X}} \psi \\ \frac{\nabla v_{\mathbf{X}}}{H} & = -\nabla u_{\mathbf{X}} (\phi + \theta) - c_{\gamma} \nabla u_{\mathbf{X}} \psi \end{cases} \quad (32)$$

and eventually gives:

$$\begin{cases} \nabla u_{\mathbf{X}} \phi = -\nabla v_{\mathbf{X}} H - \nabla u_{\mathbf{X}} \phi - c_{\gamma} \nabla u_{\mathbf{X}} \psi \\ \nabla v_{\mathbf{X}} \phi = \nabla u_{\mathbf{X}} H - \nabla v_{\mathbf{X}} \phi - c_{\gamma} \nabla v_{\mathbf{X}} \psi \end{cases} \quad (33)$$

which is linear in $\theta$.

We can show that there always exists a scalar field $\theta$ verifying Eq. (33). The $\theta$ exists if and only if we have:

$$\nabla u_{\mathbf{X}} \nabla v_{\mathbf{X}} \theta - \nabla v_{\mathbf{X}} \nabla u_{\mathbf{X}} \theta = 0 \quad (34)$$

Using Eq. (33) we obtain:

$$\nabla u_{\mathbf{X}} \nabla v_{\mathbf{X}} \theta - \nabla v_{\mathbf{X}} \nabla u_{\mathbf{X}} \theta = \Delta H + \nabla v_{\mathbf{X}} (c_{\gamma} \nabla u_{\mathbf{X}} \psi) - \nabla u_{\mathbf{X}} (c_{\gamma} \nabla v_{\mathbf{X}} \psi)$$

$$= -K + \nabla v_{\mathbf{X}} (c_{\gamma} \nabla u_{\mathbf{X}} \psi) - \nabla u_{\mathbf{X}} (c_{\gamma} \nabla v_{\mathbf{X}} \psi). \quad (35)$$

We know that, for 2D manifolds embedded in $\mathbb{R}^3$, the Gaussian curvature $K$ is equal to the jacobian of the Gauss map of the manifold \([24]\). We have:

$$\begin{align*}
\nabla u_{\mathbf{X}} \mathbf{n} & = s_{\gamma} \nabla u_{\mathbf{X}} \psi \begin{pmatrix} c_{\psi} \\ s_{\psi} \\ 0 \end{pmatrix} - \nabla u_{\mathbf{X}} \gamma \begin{pmatrix} -s_{\psi} c_{\gamma} \\ c_{\psi} c_{\gamma} \\ s_{\gamma} \end{pmatrix} \\
\nabla v_{\mathbf{X}} \mathbf{n} & = s_{\gamma} \nabla v_{\mathbf{X}} \psi \begin{pmatrix} c_{\psi} \\ s_{\psi} \\ 0 \end{pmatrix} - \nabla v_{\mathbf{X}} \gamma \begin{pmatrix} -s_{\psi} c_{\gamma} \\ c_{\psi} c_{\gamma} \\ s_{\gamma} \end{pmatrix} \end{align*} \quad (36)$$
Therefore we also have:

\[ K = s_{\gamma} \left( \nabla_{vX} \psi \nabla_{uX} \gamma - \nabla_{uX} \psi \nabla_{vX} \gamma \right). \quad (37) \]

Developing Eq. (35) and substituting \( K \) with the right-hand side of Eq. (37) we get:

\[
-K + \nabla_{vX} \left( c_{\gamma} \nabla_{uX} \psi \right) - \nabla_{uX} \left( c_{\gamma} \nabla_{vX} \psi \right) \\
- \nabla_{uX} \left( c_{\gamma} \nabla_{uX} \nabla_{vX} \psi \right) + s_{\gamma} \nabla_{uX} \gamma \nabla_{vX} \psi \\
= -K + c_{\gamma} \nabla_{vX} \nabla_{uX} \psi - s_{\gamma} \nabla_{vX} \nabla_{uX} \psi \\
- c_{\gamma} \nabla_{uX} \nabla_{vX} \psi + s_{\gamma} \nabla_{uX} \gamma \nabla_{vX} \psi \\
= 0. \quad (38)
\]

As Eq. (34) is verified, we know that there exists a scalar field \( \theta \) verifying Eq. (33), and therefore that our problem has a unique solution.

In order to solve Eq. (33), we first need to obtain a smooth global basis \((u_X, v_X, n)\) on \( M \). This is possible by generating a branch cut \( L \), as defined below, and computing a smooth global basis \((u_X, v_X, n)\) on \( M \) allowing discontinuities across \( L \).

A branch cut is a set \( L \) of curves of a domain \( M \) that do not form any closed loop and that cut the domain in such a way that it is impossible to find any closed loop in \( M \setminus L \) that encloses one or several singularities, or an internal boundary. As we already have a triangulation of \( M \), the branch cut \( L \) is in practice simply a set of edges of the triangulation.

The branch cut is generated with the method described in [15] which is based on [25]. An example of generated branch cut is presented in Fig. 5.

\[ \text{Figure 5: Edges of the branch cut } L \text{ are represented in blue. There exists no closed loop in } M \setminus L \text{ enclosing one or several singularities.} \]

Once a branch cut \( L \) is available, the field \( \theta \) can be computed by solving the linear equations (33). With equations (33), \( \theta \) is known up to an additive constant. For the problem to be well-posed, \( \theta \) value has to be imposed at one point of domain \( M \). The chosen boundary condition consists in fixing the angle \( \theta \) at one arbitrary point \( X_{BC} \in \partial M \) so that \( C_M(X_{BC}) \) has one of its branches collinear with \( T(X) \). The problem can be rewritten as the well-posed Eq. (39) and is solved using the finite element method on the triangulation \( M_T \) with order one Crouzeix-Raviart elements. This kind of elements has shown to be
more efficient for cross-field representation [14].

\[
\begin{align*}
P_{T,M}(\nabla \theta) &= P_{T,M}(\mathbf{n} \times \nabla H - \nabla \phi - c_\gamma \nabla \psi) \text{ in } \mathcal{M} \\
\theta(X_{BC}) &= \theta_{X_{BC}} \text{ for an arbitrary } X_{BC} \in \partial \mathcal{M} \\
\theta & \text{ discontinuous on } \mathcal{L}
\end{align*}
\] (39)

It is important to note that for Eq. (39) to be well-posed, the \( \theta \) value can only be imposed on a single point. A consequence is that if \( \mathcal{M} \) has more than one boundary (\( \partial \mathcal{M} = \partial \mathcal{M}_1 \cup \partial \mathcal{M}_2 \cup \cdots \cup \partial \mathcal{M}_n \)), the resulting cross-field is guaranteed to be tangent to the boundary \( \partial \mathcal{M}_i \) such as \( X_{BC} \in \partial \mathcal{M}_i \), which does not necessarily hold for all boundaries \( \partial \mathcal{M}_j \) for \( j \neq i \), as detailed in Section 4.1.

Once \( H \) and \( \theta \) scalar fields are computed on \( \mathcal{M} \) (illustrated respectively in Fig. 4 and Fig. 6), the cross-field \( \mathcal{C}_M \) can be retrieved for all \( X \in \mathcal{M} \):

\[
c(X) = \{ u_k = R_{\theta + k\frac{\pi}{2}} n(X), k \in [0, 3]\}.
\] (40)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Scalar field \( \theta \) obtained from scalar field \( H \) (represented in Fig. 4).}
\end{figure}

4. Preliminary results

As a proof of concept, the cross-field computation based on imposed singularity configuration is included in the 3-step quad meshing pipeline of [15] (illustrated in Fig. 7 and Fig. 8):

**Step 1:** impose a singularity configuration, i.e., position and valences of singularities (see [15]).

**Step 2:** compute a cross-field with the prescribed singularity configuration of Step 1 on an adapted mesh (singularities are placed in refined regions), by solving only two linear systems (Section 3).

**Step 3:** compute a quad layout on the accurate cross-field of Step 2, and generate a full block-structured isotropic quad mesh (see [14] [15]).

The presented pipeline includes the automatic check that singularity configuration obeys the Euler characteristic of the surface, but it does not inspect
all Abel-Jacobi conditions \[9, 10, 11\]. Further, the models of industrial complexity would require a more robust quad layout generation technique than the one followed here \([14, 15]\). The final quad mesh is isotropic, obtained from the quad layout via per-partition bijective parameterization aligned with the smooth cross-field (singularities can only be located on corners of the partitions) \([15]\), and following the size map implied by the \(H\), i.e., the element’s edge length is \(s = e^H\). In case when the application demands an anisotropic quad mesh, two sizing fields \((H_1, H_2)\) for the cross-field must be computed, more details in Section 5.

4.1. Valid singularity configurations for quad meshing

The singularity configuration, including both the positions and valences, plays a crucial role in the generation of conformal quad meshes \([19]\). It is essential to note that not all user-imposed singularity configurations matching the Euler’s characteristic of the surface will be valid for quad meshing, Fig. 9. The central cause for this lies in the fact that a combination of choices of valences and holonomy is not arbitrary \([26]\). Relevant findings on the non-existence of certain quadrangulations can be found in \([27, 28, 29]\).
The work of [30] presents the formula for determining the numbers of and indices of singularities, and [31] their possible combinations in conforming quad meshes. Latter authors also show that the presented formula is necessary but not sufficient for quad meshes, but neither of these works are proving the rules for the singularities’ placement.

Recently, the sufficient and necessary conditions for valid singularity configuration of the quad mesh are presented in the framework based on Abel-Jacobi’s theory [9, 10, 11]. The developed formulation here is under its direct constraint. In practice, imposing a singularity configuration fulfilling Euler’s characteristic constraint ensures that the flat metric, i.e., the $H$ field can be obtained. If this singularity configuration also verifies the holonomy condition, the cross-field will be aligned with all boundaries and consistent across the cut graph.

We recall here that our formulation entitles the user to impose its own singularity configuration, which in practice can contain a suboptimal distribution of singularities. As a consequence, computed cross-field may not be aligned with all boundaries, disabling the generation of the final conformal isotropic quad mesh. To bypass this issue, the following section develops an integrable cross-field formulation with two independent metrics (which are flat except at singularities), instead of only one as presented for Abel-Jacobi conditions.

Figure 9: Imposing a $3 - 5$ singularity configuration on a torus with the boundary marked in blue a), marking the boundary and the cut graph in black b), consistent cross-field across the cut graph c), and cross-field not aligned with the boundary d).

4.2. Dealing with suboptimal distribution of singularities

The issue of suboptimal distribution of singularities imposes the need for developing a new cross-field formulation on the imposed singularity configuration, which considers the integrability while relaxing the condition on isotropic scaling of crosses’ branches. More specifically, the integrability condition, along with computing only one scaling field $H$, $||\tilde{u}|| = ||\tilde{v}||$, imposes the strict constraint on
the valid singularity configurations, i.e., the need for fulfilling the Abel-Jacobi theorem. Therefore, two sizing fields $L_1 = ||\tilde{u}||$ and $L_2 = ||\tilde{v}||$ are introduced and the upcoming section presents the mathematical foundations for the generation of an integrable cross-field with anisotropic scaling on 2−D manifolds.

As it will be shown in the following, this setting presents promising results in generating an integrable and boundary-aligned cross-field on the imposed set of singularities, even when their distribution is not fulfilling all Abel-Jacobi conditions. Only for the sake of visual comprehensiveness, the presented motivational examples in Fig. 10 - Fig. 15 are planar.

![Figure 10: Obtained quad layouts on an imposed set of singularities that do not respect the location’s condition from the Abel-Jacobi theorem. Quad layouts obtained using the integrable cross-field with isotropic scaling are not aligned with boundaries (marked with "!"), and demonstrate the presence of t-junctions (marked with "T") a). Quad layouts obtained with imposing the $\theta$ value are boundary aligned but not integrable and demonstrate the presence of t-junctions b). Quad layouts obtained using the integrable cross-field with an anisotropic scaling c).](image)

5. Integrability condition with anisotropic scaling

As explained previously, a cross-field $\mathcal{C}_\mathcal{M}$ is integrable if and only if $\tilde{u}$ and $\tilde{v}$ commute under the Lie Bracket. In other words, the condition:

$$0 = [\tilde{u}, \tilde{v}] = \nabla_{\tilde{u}} \tilde{v} - \nabla_{\tilde{v}} \tilde{u} = [L_1 u, L_2 v] \quad (41)$$

where:

$$L_1 = ||\tilde{u}||, \quad L_2 = ||\tilde{v}|| \quad (42)$$
and
\[ u = \frac{\hat{u}}{||\hat{u}||}, \quad v = \frac{\hat{v}}{||\hat{v}||} \] (43)

has to be verified.

Developing the latter expression and posing for convenience \( L_1 = e^{H_1} \) and \( L_2 = e^{H_2} \), it becomes:

\[ 0 = v \nabla_u H_2 - u \nabla_v H_1 + [u, v], \]

and then
\[ \begin{cases} \nabla_u H_2 & = - <v, [u, v]> \\ \nabla_v H_1 & = <u, [u, v]> \end{cases} \] (44)

which after the substitution of Eq. (16) gives:
\[ \begin{cases} \nabla_u H_2 & = \nabla_v \theta + \nabla_v \phi + c_\gamma \nabla_v \psi \\ -\nabla_v H_1 & = \nabla_u \theta + \nabla_u \phi + c_\gamma \nabla_u \psi. \end{cases} \] (45)

It is important to note that the three scalar fields (\( \theta, H_1, H_2 \)) are completely defining the cross-field \( C_M \), as \( (\psi, \gamma, \phi) \) are known since they are defining the local manifold basis \( (t, T, n) \).

From Eq. (45), we can define the cross-field \( C_M \) integrability error \( E \) as:
\[
E^2(\theta, H_1, H_2) = \int_M (\nabla_u H_2 - \nabla_v \phi - c_\gamma \nabla_v \psi)^2
+ (\nabla_v H_1 + \nabla_u \theta + \nabla_u \phi + c_\gamma \nabla_u \psi)^2 \, dM. \] (46)

The problem of generating an integrable cross-field with anisotropic scaling can therefore be reduced at finding three scalar fields \( (\theta, H_1, H_2) \) verifying \( E(\theta, H_1, H_2) = 0 \).

The process of solving this problem presents several difficulties. First, the quadruple \( (\theta, \psi, \gamma, \phi) \) are multivalued functions. This kind of difficulty is commonly encountered in cross-field generation and is tackled here by cutting the domain \( M \) along a generated cut graph. Then, minimizing \( E \) regarding \( (\theta, H_1, H_2) \) is an ill-posed problem. Indeed, there are no constraints on \( \nabla_u H_1 \) and \( \nabla_v H_2 \). This is the main obstacle for generating an integrable 2D cross-field with an anisotropic scaling.

A simple approach to solve this problem is proposed here. In order to do so, it is needed to:

- be able to generate a boundary-aligned cross-field matching the imposed singularity configuration,
- compute \( (H_1, H_2) \) minimizing \( E \) for an imposed \( \tilde{\theta} \),
- compute \( \theta \) minimizing \( E \) for an imposed \( (H_1, H_2) \).

The final resolution solver (Algorithm 3), proposed in Section 5.4, allows finding a local minimum for \( E \) around an initialization \( (\theta^0, H_1^0, H_2^0) \).
5.1. Local manifold basis generation and θ initialization

As exposed earlier, in order to completely define a unitary cross-field \( C_M \) with a scalar field \( \theta \) it is needed to define a smooth global basis \((t, T, n)\) on \( M \).

This is possible by generating a branch cut \( L \) and computing a smooth global basis \((t, T, n)\) on \( M \) allowing discontinuities across \( L \).

The branch cut is generated using the method described in [25]. A local basis \((t, T, n)\) on \( M \) can be generated with any cross-field method. Such local basis will be smooth and will not show any singularities, as discontinuities are allowed across the cut graph \( L \) and no boundary alignment is required. Once the cut graph \( L \) and the local basis \((t, T, n)\) are generated, it is possible to compute \( \theta \) only if:

- \( \theta \) values on \( \partial M \) are known,
- \( \theta \) jump values across \( L \) are known.

These can be found using methods described in [25], or can be deduced from a low computational cost cross-field generation detailed in [15].

5.2. Computing \((H_1, H_2)\) from imposed \( \bar{\theta} \)

For a given \( \bar{\theta} \), it is possible to find \((H_1, H_2)\) minimizing \( E \). It is important to note that, in general, there does not exist a couple \((H_1, H_2)\) such as \( E = 0 \). Minimizing \( E \) with imposed \( \bar{\theta} \) is finding the couple \((H_1, H_2)\) for which the integrability error is minimal.

The problem to solve is the following:

\[
\text{Find } (H_1, H_2) \text{ such as } E(\bar{\theta}, H_1, H_2) = \min_{(H_1, H_2)\in(C^1(M))^2} E(\bar{\theta}, H_1, H_2). \quad (47)
\]

Let’s define \( S \) as:

\[ S = \{ (\bar{H}_1, \bar{H}_2) \mid (\bar{H}_1, \bar{H}_2) \text{ verifies Eq. (47)} \}. \]

For this problem to be well-posed, a necessary condition is to have 2 independent scalar equations involving \( \nabla H_1 \), and the same for \( \nabla H_2 \). We can note that in our case, there are no constraints on \( \nabla_u H_1 \) and \( \nabla_v H_2 \). Therefore, there is only 1 scalar equation involving \( \nabla H_1 \), and 1 scalar equation involving \( \nabla H_2 \). As a consequence, the problem we are looking to solve is ill-defined. As this problem is ill-defined, \( S \) will not be a singleton and, in the general case, there will be more than one solution to the problem (47).

To discuss this problem in detail, we will use the simple example of a planar domain \( \Omega \) illustrated in Fig. 11.

In this case, the unitary frame field \( C_\Omega \) obtained with common methods is:

\[
C_\Omega = \{ c(X) = \{x, y, -x, -y\}, X \in \Omega \} \quad (48)
\]

which is equivalent to:

\[
\bar{\theta} = 0. \quad (49)
\]
As in this case domain $\Omega$ is planar, we also have:

$$\psi = \gamma = \phi = 0.$$  \hfill (50)

Equation (45) becomes:

$$\\begin{align*}
\{ & \nabla_x H_2 = 0 \\
& -\nabla_y H_1 = 0
\}\end{align*}$$  \hfill (51)

which gives:

$$\\begin{align*}
H_1(x,y) &= f(x), \forall (x,y) \in \Omega, \forall f \in C^1(\mathbb{R}) \\
H_2(x,y) &= g(y), \forall (x,y) \in \Omega, \forall g \in C^1(\mathbb{R}).
\end{align*}$$  \hfill (52)

Knowing this, we finally have $S = (C^1(\mathbb{R}))^2$. There is an infinity of solutions, confirming the fact that problem (47) is ill-defined.

The solution we could expect to obtain for quad meshing purposes would be:

$$S = \{(H_1, H_2) = (0, 0)\},$$  \hfill (53)

which is equivalent to $(L_1, L_2) = (1, 1)$.

Based on this simple example, we can deduce that problem (47) has to be regularized in order to reduce the solution space. One way to achieve this goal is to add a constraint on the $(H_1, H_2)$ fields we are looking for. A natural one is to look for $(H_1, H_2)$ verifying Eq. (47) and being as smooth as possible.

With this constraint, the problem to solve becomes:

$$\text{Find } (\bar{H}_1, \bar{H}_2) \in S \text{ such that }$$

$$\int_M ||\nabla \bar{H}_1||^2 + ||\nabla \bar{H}_2||^2 dM$$

$$= \min_{(H_1, H_2) \in S} \int_M ||\nabla H_1||^2 + ||\nabla H_2||^2 dM.$$  \hfill (54)

Adding this constraint transforms the linear problem (47) into a non-linear one (54). Algorithm 1 is used to solve Eq. (54), leading to an $E$’s local minimum $(\bar{\theta}, \bar{H}_1, \bar{H}_2)$ close to $(\theta, H^0_1, H^0_2)$. 

18
$k = 0$

initial guess $H_1^0, H_2^0$

compute $\epsilon^0 = E(\bar{\theta}, H_1^0, H_2^0)$

while $\epsilon^k < \epsilon^{k-1}$ do
  
  $k = k + 1$
  find $(H_1^k, H_2^k)$ minimizing:
  
  $$
  E(\bar{\theta}, f_1, f_2) + \int_M \frac{1}{2} \left| \nabla f_1 - \nabla H_1^{k-1} \right|^2 + \frac{1}{2} \left| \nabla f_2 - \nabla H_2^{k-1} \right|^2 dM,
  $$
  
  $(f_1, f_2) \in (C^1(M))^2$
  compute $\epsilon^k = E(\theta, H_1^k, H_2^k)$

end

Algorithm 1: Regularized solver for $(H_1, H_2)$

5.3. Computing $\theta$ from $(\bar{H}_1, \bar{H}_2)$

For an imposed couple $(\bar{H}_1, \bar{H}_2)$, it is possible to find $\theta$ minimizing $E$. The problem to solve is formalized as:

Find $\bar{\theta} \in C^1(M)$ such as $E(\bar{\theta}, \bar{H}_1, \bar{H}_2) = \min_{\theta \in C^1(M)} E(\theta, \bar{H}_1, \bar{H}_2). \quad (55)$

This problem is non-linear too since $\nabla_v H_1$ and $\nabla_u H_2$ are showing a non-linear dependence regarding $\theta$. Algorithm 2 is used to solve Eq. (55), leading to an $E$’s local minimum $(\bar{\theta}, \bar{H}_1, \bar{H}_2)$ close to $(\theta^0, \bar{H}_1, \bar{H}_2)$.

$k = 0$

initial guess $\theta^0$

deduce $(u^0, v^0)$ from $\theta^0$

compute $\epsilon^0 = E(\theta^0, \bar{H}_1, \bar{H}_2)$

while $\epsilon^k < \epsilon^{k-1}$ do
  
  $k = k + 1$
  find $\theta^k$ minimizing:
  
  $$
  E^k(f, \bar{H}_1, \bar{H}_2) = \int_M \frac{1}{2} \left( \nabla_{u^{k-1}} H_2 - \nabla_{v^{k-1}} f - \nabla_{v^{k-1}} \phi - c_7 \nabla_{v^{k-1}} \psi \right)^2 + \frac{1}{2} \left( \nabla_{v^{k-1}} H_1 + \nabla_{u^{k-1}} f + \nabla_{u^{k-1}} \phi + c_7 \nabla_{u^{k-1}} \psi \right)^2 dM
  $$
  
  $f \in C^1(M)$
  deduce $(u^k, v^k)$ from $\theta^k$
  compute $\epsilon^k = E(\theta^k, \bar{H}_1, \bar{H}_2)$

end

Algorithm 2: Solver for $\theta$
5.4. Minimizing integrability error $E$ regarding $(\theta, H_1, H_2)$

Using the three steps exposed previously, it is possible to find a local minimum in the vicinity of an initialization $(\theta^0, H^0_1, H^0_2)$ following Algorithm 3.

\begin{verbatim}
k = 0
initial guess $\theta^0$ using method presented in Section 5.1
compute $(H^0_1, H^0_2)$ from $\theta^0$ using Alg. 1
compute $\epsilon^0 = E(\theta^0, H^0_1, H^0_2)$
while $\epsilon^k < \epsilon^{k-1}$ do
    $k = k + 1$
    compute $\theta^k$ from $(H^{k-1}_1, H^{k-1}_2)$ using Alg. 2
    compute $(H^k_1, H^k_2)$ from $\theta^k$ using Alg. 1
    compute $\epsilon^k = E(\theta^k, H^k_1, H^k_2)$
end
\end{verbatim}

Algorithm 3: Solver for $(\theta, H_1, H_2)$

For the sake of simplicity the motivational example, presented in Fig. 12, is planar and chosen to be topologically equivalent to a torus. A set of four of index 1 and four of index -1 singularities whose locations are not fulfilling the Abel-Jacobi condition is imposed. Consequently, a cross-field generated using the $H$ function will not be boundary aligned, and a cross-field generated using the method presented in Section 5.1 will not be integrable and therefore will generate limit cycles.

The method presented here is applied to compute an integrable boundary-aligned cross-field. Figure 13 represents the cross-field used as an initial guess and Fig. 14 is the one obtained at Algorithm 3 convergence.

Figure 13 demonstrates that integrability error density is not concentrated in certain regions, but rather quite uniformly spread over the domain. This suggests that addressing the integrability issue cannot be performed via local modifications but only via the global one, i.e., the convergence of the presented
non-linear problem. Figure 14 shows that generating a limit cycle-free 2D cross-field can indeed be done by solving Eq. (45). Nevertheless, this problem is highly non-linear and ill-defined, and solving it turns out to be difficult.

The method proposed here works well when initialization is not far from an integrable solution, i.e., when the imposed singularity set obeys Abel-Jacobi’s conditions. Otherwise, it does not converge up to the desired solution by reaching a local minimum \((\bar{\theta}, \bar{H}_1, \bar{H}_2)\) which does not satisfy \(E(\bar{\theta}, \bar{H}_1, \bar{H}_2) = 0\), as illustrated in Fig. 15. Although, it is interesting to note that, even without the presented method’s convergence, the number of T-junctions dramatically decreases and the valid solution, in the opinion of authors, can be “intuitively presumed”.

Figure 13: From left to right: quad layout obtained at initialization, and the integration error density on \(\Omega\). The total integration error is \(E = 0.307898\).

Figure 14: From left to right: quad layout obtained, and the integration error density on \(\Omega\). The total integration error at convergence is \(E = 1.45639e - 66\).
6. Conclusion and Future Work

We presented the mathematical foundations for the generation of integrable cross-field on 2D manifolds based on user-imposed singularity configuration with both isotropic and anisotropic scaling. Here, the mathematical setting is constrained by the Abel-Jacobi conditions for a valid singularity pattern. With the automatic algorithms to check and optimize singularity configuration (as recently presented in [9, 10, 11]), the developed framework can be used to effectively generate both an isotropic and an anisotropic block-structured quad mesh with preserved singularity distribution. When it comes to computational costs of our cross-field generation, the formulation with isotropic scaling $H$ takes solving only two linear systems, and the anisotropic one $(H_1, H_2)$ represents a non-linear problem.

An attractive direction for future work includes, although it is not limited to, working with the user-imposed size map. By using the integrable cross-field formulation relying on two sizing fields $H_1$ and $H_2$, it would be possible to take into account the anisotropic size field to guide the cross-field generation. The size field obtained from the generated cross-field would not precisely match the one prescribed by the user, but it would be as close as possible to the singularity configuration chosen for the cross-field generation.

It is important to note that employing the presented framework in the 3D volumetric domain would be possible only for a limited number of cases, in which the geometric and topological characteristics of the volume (more details in [31, 32]) allow the use of cross-field guided surface quad mesh for generating a hex mesh.
References

[1] D. Bommes, B. Lévy, N. Pietroni, E. Puppo, C. T. Silva, M. Tarini, D. Zorin, Quad meshing., in: Eurographics (STARs), pp. 159–182.

[2] M. Campen, Partitioning surfaces into quadrilateral patches: a survey, in: Computer Graphics Forum, volume 36, Wiley Online Library, pp. 567–588.

[3] M. S. Floater, K. Hormann, Surface parameterization: a tutorial and survey, in: Advances in multiresolution for geometric modelling, Springer, 2005, pp. 157–186.

[4] M. Campen, Quad Layouts–Generation and Optimization of Conforming Quadrilateral Surface Partitions, Ph.D. thesis, 2014.

[5] K. M. Shepherd, R. R. Hiemstra, T. J. Hughes, The quad layout immersion: A mathematically equivalent representation of a surface quadrilateral layout, arXiv preprint arXiv:2012.09368 (2020).

[6] M. Campen, D. Bommes, L. Kobbelt, Quantized global parametrization, Acm Transactions On Graphics (tog) 34 (2015) 1–12.

[7] D. Bommes, M. Campen, H.-C. Ebke, P. Alliez, L. Kobbelt, Integer-grid maps for reliable quad meshing, ACM Transactions on Graphics (TOG) 32 (2013) 1–12.

[8] N. Ray, W. C. Li, B. Lévy, A. Sheffer, P. Alliez, Periodic global parameterization, ACM Transactions on Graphics (TOG) 25 (2006) 1460–1485.

[9] W. Chen, X. Zheng, J. Ke, N. Lei, Z. Luo, X. Gu, Quadrilateral mesh generation i: Metric based method, Computer Methods in Applied Mechanics and Engineering 356 (2019) 652–668.

[10] N. Lei, X. Zheng, Z. Luo, F. Luo, X. Gu, Quadrilateral mesh generation ii: Meromorphic quartic differentials and abel–jacobi condition, Computer Methods in Applied Mechanics and Engineering 366 (2020) 112980.

[11] X. Zheng, Y. Zhu, W. Chen, N. Lei, Z. Luo, X. Gu, Quadrilateral mesh generation iii: Optimizing singularity configuration based on abel–jacobi theory, Computer Methods in Applied Mechanics and Engineering 387 (2021) 114146.

[12] N. Kowalski, F. Ledoux, P. Frey, A pde based approach to multidomain partitioning and quadrilateral meshing, in: Proceedings of the 21st international meshing roundtable, Springer, 2013, pp. 137–154.

[13] H. J. Fogg, C. G. Armstrong, T. T. Robinson, Automatic generation of multiblock decompositions of surfaces, International Journal for Numerical Methods in Engineering 101 (2015) 965–991.
[14] J. Jezdimirović, A. Chemin, J. F. Remacle, Multi-block decomposition and meshing of 2d domain using ginzburg-landau pde, Proceedings, 28th International Meshing Roundtable (2019).

[15] J. Jezdimirović, A. Chemin, M. Reberol, F. Henrotte, J. F. Remacle, Quad layouts with high valence singularities for flexible quad meshing, Proceedings of the 29th Meshing Roundtable (2021).

[16] N. Ray, B. Vallet, W. C. Li, B. Lévy, N-symmetry direction field design, ACM Transactions on Graphics (TOG) 27 (2008) 1–13.

[17] R. Viertel, B. Osting, An approach to quad meshing based on harmonic cross-valued maps and the ginzburg-landau theory, SIAM Journal on Scientific Computing 41 (2019) A452–A479.

[18] P.-A. Beaufort, J. Lambrechts, F. Henrotte, C. Geuzaine, J.-F. Remacle, Computing cross fields a pde approach based on the ginzburg-landau theory, Procedia engineering 203 (2017) 219–231.

[19] X. Gu, F. Luo, S. T. Yau, Computational conformal geometry behind modern technologies, Notices of the American Mathematical Society 67 (2020) 1509–1525.

[20] A. Vaxman, M. Campen, O. Diamanti, D. Panozzo, D. Bommes, K. Hildebrandt, M. Ben-Chen, Directional field synthesis, design, and processing, in: Computer Graphics Forum, volume 35, Wiley Online Library, pp. 545–572.

[21] A. Hertzmann, D. Zorin, Illustrating smooth surfaces, in: Proceedings of the 27th annual conference on Computer graphics and interactive techniques, pp. 517–526.

[22] F. Knöppel, K. Crane, U. Pinkall, P. Schröder, Globally optimal direction fields, ACM Transactions on Graphics (ToG) 32 (2013) 1–10.

[23] G. Bunin, A continuum theory for unstructured mesh generation in two dimensions, Computer Aided Geometric Design 25 (2008) 14–40.

[24] I. M. Singer, J. A. Thorpe, Lecture notes on elementary topology and geometry, Springer, 2015.

[25] D. Bommes, H. Zimmer, L. Kobbelt, Mixed-integer quadrangulation, ACM Transactions On Graphics (TOG) 28 (2009) 1–10.

[26] A. Myles, N. Pietroni, D. Zorin, Robust field-aligned global parametrization: Supplement 1, proofs and algorithmic details, Visual Computing Lab (2014).

[27] D. Barnette, E. Jucovič, M. Trenkler, Toroidal maps with prescribed types of vertices and faces, Mathematika 18 (1971) 82–90.

[28] E. Jucovič, M. Trenkler, A theorem on the structure of cell–decompositions of orientable 2–manifolds, Mathematika 20 (1973) 63–82.

[29] I. Izmestiev, R. B. Kusner, G. Rote, B. Springborn, J. M. Sullivan, There is no triangulation of the torus with vertex degrees 5, 6, ..., 6, 7 and related results: Geometric proofs for combinatorial theorems, Geometriae Dedicata 166 (2013) 15–29.

[30] P.-A. Beaufort, J. Lambrechts, F. Henrotte, C. Geuzaine, J.-F. Remacle, Computing cross fields a pde approach based on the ginzburg-landau theory, Procedia engineering 203 (2017) 219–231.

[31] H. J. Fogg, L. Sun, J. E. Makem, C. G. Armstrong, T. T. Robinson, Singularities in structured meshes and cross-fields, Computer-Aided Design 105 (2018) 11–25.

[32] D. R. White, T. J. Tautges, Automatic scheme selection for toolkit hex meshing, International Journal for Numerical Methods in Engineering 49 (2000) 127–144.