EXACT ASYMPTOTIC FOR THE TAIL OF MAXIMUM OF
SMOOTH RANDOM FIELD DISTRIBUTION.

OSTROVSKY E.

Department of Mathematics and Statistics, Bar - Ilan University, 59200, Ramat Gan, Israel.
e-mail: galo@list.ru

ABSTRACT

We obtain in this paper using the saddle - point method the expression for the exact asymptotic for the tail of maximum of smooth (twice continuous differentiable) random field (process) distribution.

Key words: Random field, exact asymptotic, saddle - point method, Banach spaces of random variables, generic chaining, Hessian, metric entropy, natural distance, natural space, Grand Lebesgue Spaces, Tauberian theorems.

Mathematics Subject Classification (2000): primary 60G17; secondary 60E07; 60G70.

1. Introduction. Notations. Statement of problem.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with expectation $\mathbf{E}$ and variance $\text{Var}$. Let also $D$ be a open bounded convex domain with compact closure $[D]$ in the space $\mathbb{R}^d$ with $(d - 1)$ - dimensional boundary $\partial D = [D] \setminus D$ and let $\xi = \xi(x) = \xi(x, \omega), \, \omega \in \Omega, \, x \in D$ be a twice continuous differentiable on the set $[D]$ with probability one random process (field in the case $d \geq 2$) with the values on the real line:

$$\xi : D \times \Omega \to \mathbb{R}^1.$$ 

Let us denote

$$M = M(\omega) = \max_{x \in D} \xi(x), \ T_M(u) = \mathbf{P}(M > u). \quad (1.1)$$

Our goal is the calculation of the exact asymptotic as $u \to \infty$ for the tail - function $T_M(u)$ of maximum distribution in the terms of some finite - dimensional distributions of the considered field $\xi(x)$.

Recall that by definition the asymptotical expression

$$T_M(u) \sim Y(u), \ u \to \infty$$

is said to be exact, iff

$$\lim_{u \to \infty} \frac{T_M(u)}{Y(u)} = 1.$$
The symbol \( \preceq \) will denote as usually the weak relation: we write \( f(\cdot) \preceq g(\cdot) \), \( \lambda \in \Lambda \) for two numerical functions \( f(\lambda) \), \( g(\lambda) \) defined on the arbitrary set \( \Lambda \) iff

\[
0 < \inf_{\lambda \in \Lambda} f(\lambda)/g(\lambda) \leq \sup_{\lambda \in \Lambda} f(\lambda)/g(\lambda) < \infty.
\]

It is easy to see that the case of maximum absolute value \( M_1 = \max |\xi(x)| \) or \( M_0 = \min \xi(x) \) and \( z \to -\infty \) may be considered analogously.

The estimations of the probability \( T_M(u) \) as \( u \to \infty \) are used in the probability theory [6], [14], [15], [9], theory of random fields [5], [6], [16], [7], statistics [7], [8], theory of Monte Carlo method [7], [8], reliability theory [7], theory of approximation [10] etc.

The exact asymptotic for the tail \( T_M(u) \) for the Gaussian fields \( \xi(x) \) was obtained in [13]; see also [1]. The upper and low bounds for \( T_M(u) \) was obtained in many publications ([4], [2], [12], [7], [8], [11], etc.

Another denotations. Let

\[
\eta_{i,j}(x) = \frac{\partial^2 \xi(x)}{\partial x_i \partial x_j}, \quad i, j = 1, 2, \ldots, d
\]

be the Hessain of the random field \( \xi = \xi(x) \),

\[
\zeta(x) = \zeta(x, \omega) = | \det(\eta) |^{1/2}, \quad K(d) = (2\pi)^{-d/2},
\]

\[
I(\lambda) = I(\lambda, \omega) = \int_D \zeta(x) \exp(\lambda \xi(x)) \, dx,
\]

where the great parameter \( \lambda \), \( \lambda \to \infty \) belongs to the sector \( S = S(\epsilon) \) in the complex plane:

\[
S(\epsilon) = \{ \lambda : \arg(\lambda) \leq \pi/2 - \epsilon \},
\]

where \( \epsilon \) be a fixed number in the interval \((0, 1/2)\).

Further, we denote for the values \( \lambda \in S(\epsilon) \):

\[
G(\lambda) = K(d) E[I(\lambda)] = K(d) \int_D E[\zeta(x) \exp(\lambda \xi(x))] \, dx
\]

(We used Fubini theorem).

2. Assumptions.

1. We assume that the considered random field \( \xi(x) \) is non-degenerate in the following sense. For arbitrary finite set of pair-wise different elements \( x_k \), \( k = 1, 2, \ldots, n + m : k \neq l \Rightarrow x_k \neq x_l \) of the set \( D : x_k \in D \) and for all the values

\[
(i, j) = \{i_r, j_r\}, \quad r = 1, 2, \ldots, m
\]
the random vector \( \tilde{\theta}_{(i,j)} = \tilde{\theta} = \{ \xi(x_1), \xi(x_2), \ldots, \xi(x_n); \eta_{i,j_1}(x_{n+1}), \eta_{i,j_2}(x_{n+2}), \ldots, \eta_{i,m,j}(x_{n+m}) \} \)

has a bounded density of distribution

\[
f_{\tilde{\theta}}(y_1, y_2, \ldots, y_{n+m}) = f_{\tilde{\theta}}(y_1, y_2, \ldots, y_{n+m}; x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m})
\]

with respect to the usually Lebesgue measure:

\[
V_{(i,j)}(x_1, x_2, \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+m}) \overset{\text{def}}{=} \sup_{y_1, y_2, \ldots, y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+m}} f_{\tilde{\theta}}(y_1, y_2, \ldots, y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+m}) < \infty. \tag{2.1}
\]

The condition (2.1) guarantee, by virtue of theorem of Ilvisaker, that the (random) point of maximum of the field \( \xi = \xi(x) : \)

\[
x_0 = \arg\max_{x \in [D]} \xi(x)
\]

there exists, is unique, belongs to the open set \( D \) and is non - degenerate:

\[
det(\eta_{i,j}(x_0)) = det \left( \frac{\partial^2 \xi(x)}{\partial x_i \partial x_j}(x_0) \right) \neq 0. \tag{2.3}
\]

The two last properties might be understood with probability one.

**2.** We will suppose again that the fields \( \xi(x) \) and its Hessian \( \eta_{i,j}(x) \) satisfy the so - called uniform Kramer’s condition. In detail, we write the expectation decomposition

\[
\xi(x) = a(x) + \xi^o(x), \quad E\xi^o(x) = 0,
\]

such that \( a(x) = E\xi(x); a(\cdot) \in C^2([D]) \) and

\[
\eta_{i,j}(x) = \frac{\partial^2 a(x)}{\partial x_i \partial x_j} + \eta^o_{i,j}(x), \quad E\eta^o_{i,j}(x) = 0.
\]

We assume that

\[
\forall \lambda \in R^1 \exists \exp(\phi(\lambda)) \overset{\text{def}}{=} \sup_{x \in [D]} E \exp(\lambda \xi^o(x)) < \infty, \tag{2.4}
\]

and analogously suppose

\[
\max_{i,j} \sup_{x \in [D]} E \exp \left( \lambda \eta^o_{i,j}(x) \right) < \infty. \tag{2.5}
\]

The conditions (2.4) and (2.5) imply, e.g., that the integral \( I(\lambda) \) there exists for all values \( \lambda \in R^1 \). Moreover, we can introduce the so - called \( B(\phi) \) space (see, for
instance, [7], [11] ) and define the natural semi - distance on the set $D, z_1, z_2 \in D$
by the formula

$$d(z_1, z_2) = d_\phi(z_1, z_2) = ||\xi^o(z_1) - \xi^o(z_2)||B(\phi).$$

We must recall briefly for readers convenience some facts about the theory of
$B(\phi)$ spaces.
Let $\phi = \phi(\lambda), \lambda \in (-\lambda_0, \lambda_0), \lambda_0 = const \in (0, \infty]$
be some even strong convex which takes positive values for positive arguments continuous function, such that
$\phi(\lambda) = 0 \iff \lambda = 0$;

$$|\lambda| \leq 1 \Rightarrow C_- \lambda^2 \leq \phi(\lambda) \leq C_+ \lambda^2;$$
$C_-, C_+ = const, 0 < C_- \leq C_+ < \infty$;

$$\lim_{\lambda \to \lambda_0} \phi(\lambda)/\lambda = \infty.$$

We denote the set of all these function as $\Phi; \Phi = \{\phi(\cdot)\}$.

We say that the centered random variable (r.v) $\zeta = \zeta(\omega)$ belongs to the space $B(\phi)$, if there exists some non - negative constant $\tau \geq 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow E \exp(\lambda \zeta) \leq \exp[\phi(\lambda \tau)].$$

The minimal value $\tau$ satisfying the last inequality for all values $\lambda \in (-\lambda_0, \lambda_0)$ is
called a $B(\phi)$ norm of the variable $\zeta$, write $||\zeta||B(\phi) = 

\inf\{\tau, \tau > 0 : \forall \lambda : |\lambda| < \lambda_0 \Rightarrow E \exp(\lambda \zeta) \leq \exp(\phi(\lambda \tau))\}.$

Notice that in the considered in this article case $\lambda_0 = \infty$.

This spaces are very convenient for the investigation of the r.v. having a exp-
ponential decreasing tail of distribution, for instance, for investigation of the limit
theorem, the exponential bounds of distribution for sums of random variables, non-
asymptotical properties, problem of continuous of random fields, study of Central
Limit Theorem in the Banach space etc.

The space $B(\phi)$ with respect to the norm $|| \cdot ||B(\phi)$ and ordinary operations is

a Banach space which is isomorphic to the subspace consisted on all the centered
variables of Orlichs space $(\Omega, F, P), N(\cdot)$ with $N - function

$$N(u) = \exp(\phi^*(u)) - 1, \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)).$$

The transform $\phi \rightarrow \phi^*$ is called Young - Fenchel transform. The proof of considered
assertion used the properties of saddle - point method and theorem of Fenchel -
Moraux:

$$\phi^{**} = \phi.$$

The next facts about the $B(\phi)$ spaces are proved in [7], p. 19 - 40:

A. $\zeta \in B(\phi) \iff E\zeta = 0, \text{ and } \exists C = const > 0,$
\[ T(|\zeta|, u) \leq \exp(-\phi^*(u/C)), u \geq 0, \]

where \( T(|\zeta|, u) \) denotes the tail of distribution of the r.v. \( \zeta \):

\[ T(|\zeta|, u) = \mathbf{P}(|\zeta| > u), u \geq 0, \]

and this estimation is in general case asymptotically exact.

Henceforth \( C, C_j \) will denote the non-essentially positive finite ”constructive” constants.

More exactly, if \( \lambda_0 = \infty \), then the following implication holds:

\[ \lim_{\lambda \to \infty} \phi^{-1}(\log \mathbf{E} \exp(\lambda \zeta))/\lambda = K \in (0, \infty) \]

if and only if

\[ \lim_{u \to \infty} (\phi^*)^{-1}(\| \log T(\zeta, u) \|)/u = 1/K. \]

Here and further \( f^{-1}(\cdot) \) denotes the inverse function to the function \( f \) on the left-side half-line \((C, \infty)\).  

**B.** The function \( \phi(\cdot) \) may be constructive introduced by the formula

\[ \phi(\lambda) = \phi_0(\lambda) \overset{def}{=} \log \sup_{x \in D} \mathbf{E} \exp(\lambda \xi^\circ(x)), \]

if obviously the family of the centered r.v. \( \{\xi^\circ(x), x \in D\} \) satisfies the uniform Kramers condition:

\[ \exists \mu \in (0, \infty), \sup_{x \in D} T(|\xi^\circ(x)|, u) \leq \exp(-\mu u), u \geq 0. \]

In this case we will call the function \( \phi(\lambda) = \phi_0(\lambda) \) a natural function.

**C.** We define

\[ \psi(r) = \psi_\phi(r) = r/\phi^{-1}(r), \quad r \geq 2. \]

Let us introduce a new norm (the so-called moment norm) on the set of r.v. defined in our probability space by the following way: the space \( G(\psi) \), or, in the other words, Grand Lebesgue Space (GLS) \( G(\psi) = G(\psi_\phi) \) consist, by definition, on all the centered r.v. \( \{\zeta\} \) with finite norm

\[ ||\zeta||_{G(\psi)} \overset{def}{=} \sup_{r \geq 2} |\zeta|_r/\psi(r), \quad |\zeta|_r \overset{def}{=} \mathbf{E}^{1/r}|\zeta|^r. \]

It is proved that the spaces \( B(\phi) \) and \( G(\psi) \) coincides: \( B(\phi) = G(\psi) \) (set equality) and both the norm \( || \cdot ||_{B(\phi)} \) and \( || \cdot || \) are equivalent: \( \exists C_1 = C_1(\phi), C_2 = C_2(\phi) = const \in (0, \infty), \forall \xi \in B(\phi) \)

\[ ||\zeta||_{G(\psi)} \leq C_1 ||\zeta||_{B(\phi)} \leq C_2 ||\zeta||_{G(\psi)}. \]

**D.** The definition of GLS \( G(\psi) \) spaces is correct still for the non-centered random variables \( \zeta \). If for some non-zero r.v. \( \zeta \) we have \( ||\zeta||_{G(\psi)} < \infty \), then for all positive values \( u \)
and conversely if a r.v. $\zeta$ satisfies Kramers condition, then $||\zeta||G(\psi) < \infty$.

Without loss of generality we can and will suppose

$$
\sup_{x \in D} \left[ ||\xi^\phi(x) ||B(\phi) \right] = 1,
$$

(this condition is satisfied automatically in the case of natural choosing of the function $\phi : \phi(\lambda) = \phi_0(\lambda)$ ) and that the metric space $(T, d)$ relatively the so called natural distance (more exactly, semi - distance)

$$
d(z_1, z_2) \stackrel{\text{def}}{=} ||\xi^\phi(z_1) - \xi^\phi(z_2)||B(\phi)
$$

is complete.

For example, if $\xi(x)$ is a centered Gaussian field: $E\xi(x) = 0$, $x \in [D]$ and is normed:

$$
\max_{x \in [D]} \text{Var}[\xi(x)] = 1
$$

with covariation function

$$
W(z_1, z_2) = E[\xi(z_1) \xi(z_2)], \text{ then } \phi_0(\lambda) = 0.5 \lambda^2, \lambda \in R, \text{ and}
$$

$$
d(z_1, z_2) = d_{\phi_0}(z_1, z_2) = ||\xi(z_1) - \xi(z_2)||B(\phi_0) =
$$

$$
\sqrt{\text{Var}[\xi(z_1) - \xi(z_2)]} = \sqrt{W(z_1, z_1) - 2W(z_1, z_2) + W(z_2, z_2)}.
$$

E. Let us introduce for any subset $V$, $V \subset D$ the so-called entropy $H(V, d, \epsilon) = H(V, \epsilon)$ as a logarithm (natural) of a minimal quantity $N(V, d, \epsilon) = N(V, \epsilon) = N$ of a balls $S(V, t, \epsilon)$, $t \in V$:

$$
S(V, t, \epsilon) \stackrel{\text{def}}{=} \{ s, s \in V, \ d(s, t) \leq \epsilon \},
$$

which cover the set $V$:

$$
N = \min\{M : \exists \{t_i\}, i = 1, 2, \ldots, M, \ t_i \in V, \ V \subset \bigcup_{i=1}^{M} S(V, t_i, \epsilon) \},
$$

and we denote also

$$
H(V, d, \epsilon) = \log N; \ S(t_0, \epsilon) \stackrel{\text{def}}{=} S(T, t_0, \epsilon), \ H(d, \epsilon) \stackrel{\text{def}}{=} H(T, d, \epsilon).
$$

It follows from Hausdorff’s theorem that $\forall \epsilon > 0 \Rightarrow H(V, d, \epsilon) < \infty$ iff the metric space $(V, d)$ is precompact set, i.e. is the bounded set with compact closure.

It is known (see, for example, [7], [11]) that if the following series converges:

$$
\sum_{n=1}^{\infty} 2^{-n} H \left(D, d, 2^{-n} \right) < \infty,
$$

(2.6)
then a (non-centered) r.v. \( \beta = \max_{x \in D} \xi(x) \) belongs to the space \( B^+(\phi) \):

\[
T_{|\beta|}(u) \leq 2 \exp(-\phi^*(u/C)), \ u \geq 1.
\]

The condition (2.6) holds if for example the so-called metric dimension of the set \( D \) relative the distance \( d = d_\phi \) is finite:

\[
\kappa \overset{\text{def}}{=} \lim_{\epsilon \to 0^+} \frac{H(D, d, \epsilon)}{|\log \epsilon|} < \infty.
\]

Henceforth we will suppose also the condition (2.6) (and following the conclusion (2.7)) is satisfied.

Note that more modern result in the terms of majorizing measures or equally in the terms of generic chaining see in [14], [15], [16], [17], [11].

3. Main result.

**Theorem 1.** We assert under formulated above conditions: as \( \lambda \to \infty \) uniformly in \( \lambda \in S(\epsilon) \)

\[
\mathbb{E} \ e^{\lambda M} \sim K(d) \lambda^{d/2} G(\lambda).
\]

**Proof.** Let us consider the integral \( I(\lambda) \). Using the classical saddle-point method (see, e.g., [3], chapter 2, section 4), we obtain that with probability one

\[
I(\lambda) \sim K(d) \lambda^{-d/2} e^{\lambda M}.
\]

The passing to the limit as \( \lambda \to \infty, \lambda \in S(\epsilon) \) here and further may be proved on the basis of equality (2.7) and theorem of dominated convergence.

We get taking the expectation of equality (3.2):

\[
G(\lambda)/K(d) \sim \lambda^{-d/2} \mathbb{E} e^{\lambda M}.
\]

The equality (3.3) is equivalent to (3.1).

**Corollary 1.** As long as

\[
\mathbb{E} \exp(\lambda M) \sim \lambda \int_0^\infty \exp(\lambda z) T_M(z) \ dz,
\]

we conclude: \( \lambda \to \infty \Rightarrow \)

\[
\int_0^\infty \exp(\lambda z) T_M(z) \ dz \sim R(\lambda),
\]

where

\[
R(\lambda) \overset{\text{def}}{=} \frac{\lambda^{-1+d/2}}{(2\pi)^{d/2}} \int_D Q(\lambda, x) \ dx.
\]

It is evident that as \( \lambda \to \infty \)

\[
\int_0^\infty \exp(\lambda z) T_M(z) \ dz \sim \int_{-\infty}^\infty \exp(\lambda z) T_M(z) \ dz.
\]
4. Examples.

It is possible to verify that for the smooth Gaussian fields $\xi(x)$ the asymptotical equality (3.4) coincides with the classical results belonging to Piterbarg [13] and Adler [1].

Thus, we consider further only the non-Gaussian case. Namely, suppose that as $\lambda \to \infty$, $\lambda \in S(\epsilon)$

$$R(\lambda) \sim C(R) \lambda^\alpha \exp(\lambda^q / q) \quad (4.1)$$

for some constants $\alpha, C(R), q$; $C(R) \in (0, \infty), q > 1, \alpha \in (-\infty, \infty)$.

Introduce the conjugate power $p = q/(q - 1)$ and a function $\phi_p(\lambda)$ as follows:

$$\phi_p(\lambda) = \lambda^2, |\lambda| \leq 1;$$

$$\phi_p(\lambda) = |\lambda|^p, |\lambda| > 1.$$ 

We assume in addition to the condition (4.1) that

$$\sup_{x \in [D]} ||\xi^0(x)||B(\phi_p) < \infty \quad (4.2)$$

and moreover that

$$a(x) = E \xi(x) \in C^2([D]), \xi(\cdot) \in C^2([D])(\text{mod } P),$$

$$\sum_{n=1}^{\infty} 2^{-n} H \left( D, d_p, 2^{-n} \right) < \infty, \quad (4.3)$$

where

$$d_p(z_1, z_2) = ||\xi^0(z_1) - \xi^0(z_2)||B(\phi_p)$$

is the natural semi-distance on the set $[D]$ between the points $z_1, z_2$ from the set $[D]$.

It follows from the main result of [5], [8] that

$$||\max_{x \in [D]} \xi(x) ||G(\psi_p) < \infty, \psi_p = \psi_p = \psi_{\phi_p}(\cdot),$$

or equally

$$T_M(z) \leq \exp \left( - (z/C)^p \right), z \geq 0,$$

as long as

$$\phi_p^*(\lambda) \asymp \phi_q(\lambda), \lambda \in (-\infty, \infty).$$

Taking into account the following asymptotical equality (see [3], chapter 2, section 2):
\[ \int_0^\infty y^\gamma \exp(\lambda y - y^p/p) \, dy \sim (2\pi)^{1/2} \lambda^\Delta \exp(\lambda^q/q), \] (4.4)

where \( \gamma = \text{const}, \)

\[ \Delta = \frac{2\gamma + 2 - p}{2(p - 1)}, \]

we conclude that under considered conditions and using Tauberian - Richter theorems

\[ P(M > u) \sim (2\pi)^{-1/2} C(R) \ u^{\alpha(p - 1) - 1+p/2} \ \exp(-u^p/p), \] (4.5)
as \( u \to \infty. \)

References

[1] **Adler J.** (2003) *The asymptotical behavior for the tail of maximum Homogeneous Gaussian random field distribution.* Annals of Probability, v. 174 B. 7 p. 1543 - 1558.

[2] **Dudley R.M.** (1967) *The sizes of compact of Hilbert space and continuity of Gaussian processes.* J. Functional Analysis. B. 1 pp. 290 - 330.

[3] **Fedorjuk M.V.** (1990) *The Saddle - Point Method.* Kluvner, Amsterdam, New York.

[4] **Fernique X.** (1975). *Regularite des trajectories des function aleatoires gaussiennes.* Ecole de Probablite de Saint-Flour, IV 1975, Lecture Notes in Mathemat- 480 1 96, Springer Verlag, Berlin.

[5] **Kozachenko Yu. V., Ostrovsky E.I.** (1985). *The Banach Spaces of random Variables of subgaussian Type.* Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43 - 57.

[6] **Ledoux M., Talagrand M.** (1991) *Probability in Banach Spaces.* Springer, Berlin, MR 1102015.

[7] **Ostrovsky E.I.** (1999). *Exponential estimations for Random Fields and its applications.* (in Russian). Russia, OINPE.
[8] Ostrovsky E.I. (2002). Exact exponential estimations for random field maximum distribution. Theory Probab. Appl. 45 v.3, 281 - 286.

[9] Ostrovsky E., Sirota L. Exponential Bounds in the Law of iterated Logarithm for Martingales. Electronic publications, arXiv:0801.2125v1 [math.PR] 14 Jan 2008.

[10] Ostrovsky E., Sirota L. Nikolskii - type inequalities in some rearrangement invariant spaces. Electronic publications, arXiv 0804.2311v1 [math.FA], 15 Apr. (2008).

[11] Ostrovsky E., Rogover E. Exact exponential Bounds for the random Field Maximum Distribution via the Majorizing Measures. Electronic Publications, arXiv:0802.0349v1 [math.PR] 4 Feb 2008.

[12] Pizier G. (1980) Condition d’entropic assupant la continuite de certain processus et applications a l’analyse harmonique. Seminaire d’analyse fonctionnalle. Exp. 13 p. 23 - 24.

[13] Piterbarg V.I. (1988) Asymptotical methods in the theory of Gaussian Processes and Fields. Moscow, MSU.

[14] Talagrand M. (1996). Majorizing measure: The generic chaining. Ann. Probab. 24 1049 - 1103. MR1825156

[15] Talagrand M. (2001). Majorizing Measures without Measures. Ann. Probab. 29, 411-417. MR1825156

[16] Talagrand M. (2005). The Generic Chaining. Upper and Lower Bounds of Stochastic Processes. Springer, Berlin. MR2133757.

[17] Talagrand M. (1990). Sample boundedness of stochastic processes under increment conditions. Ann. Probab. 18, 1 - 49.