GENERALIZED PRIMITIVE ELEMENTS
OF A FREE GROUP

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ABSTRACT. We study endomorphisms of a free group of finite rank by means of their action on specific sets of elements. In particular, we prove that every endomorphism of the free group of rank 2 which preserves an automorphic orbit (i.e., acts “like an automorphism” on one particular orbit), is itself an automorphism. Then, we consider elements of a different nature, defined by means of homological properties of the corresponding one-relator group. These elements (“generalized primitive elements”), interesting in their own right, can also be used for distinguishing automorphisms among arbitrary endomorphisms.

1. Introduction

Let $F = F_n$ be the free group of a finite rank $n \geq 2$ with a set $X = \{x_i\}, 1 \leq i \leq n$, of free generators. An element $g \in F$ is called primitive if it is a member of some free basis of $F$. Or, equivalently: there is an automorphism $\phi \in \text{Aut} F$ that takes $g$ to $x_1$.

We start here by recalling Problem 2 from [12]. It is clear that the group $F$ is a disjoint union of different orbits under the action of the group $\text{Aut} F$. Denote an orbit $\{\phi(u), \phi \in \text{Aut} F\}$ by $O_{\text{Aut} F}(u)$. It seems plausible to assume (see [12] for a general set-up) that if an endomorphism preserves some orbit $O_{\text{Aut} F}(u)$, i.e., if it acts “like an automorphism” on one particular orbit, then it acts like an automorphism everywhere, i.e., is itself an automorphism.

This general conjecture appears to be quite difficult to prove; even the case of the “simplest” orbit (the one consisting of primitive elements) is still unsettled. Here we are able to settle the conjecture in the case when $F$ has rank 2:

**Theorem 1.1.** If an endomorphism $\phi$ of the group $F_2$ preserves an orbit $O_{\text{Aut} F_2}(u)$, then $\phi$ is actually an automorphism.

Our proof of this theorem uses the following interesting fact: there are elements of the group $F_2$ that cannot be subwords of any cyclically reduced primitive element of $F_2$; we call those elements primitivity-blocking words. More formally:

**Definition.** An element $g \in F$ is called a primitivity-blocking word if there is no cyclically reduced primitive element $w \in F$ such that $w = gh$ for some $h \in F$ (assuming, of course, there is no cancellation between $g$ and $h$).

In the group $F_2$, primitivity-blocking words are easy to find: for example, $[x_1, x_2]$ and $x_1^k x_2^l$ with $k, l \geq 2$, are primitivity-blocking words - this follows from a result of

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(see also [1]). On the other hand, in a free group of bigger rank, the situation becomes really intriguing. We ask:

**Problem.** Are there primitivity-blocking words in a free group $F_n$ if $n \geq 3$?

The attempts to extend the result of Theorem 1.1 have led me to the following generalization of the primitivity concept:

**Definition.** Let $J$ be a right ideal of the free group ring $ZF$. An element $u \in F$ is called $J$–primitive if Fox derivatives [8] of the element $u$ generate $J$ as a right ideal of $ZF$.

Now “usual” primitive elements are the same as $ZF$-primitive ones - this non-trivial fact follows from a matrix characterization of primitive elements given by Umirbaev [16] (cf. also [6], Corollary IV.5.3).

Another motivation for this definition comes from commutative algebra. We recall some facts here very briefly. A vector $(p_1, ..., p_m)$ of (Laurent) polynomials is called unimodular if $p_1, ..., p_m$ generate the whole (Laurent) polynomial algebra $P$ as an ideal. Then, a vector $(p_1, ..., p_m)$ of Laurent polynomials is called $\Delta$–modular (see [1]) if $p_1, ..., p_m$ generate the augmentation ideal $\Delta$ as an ideal of the whole algebra. Suslin [14] and Artamonov [1] have proved that the group $GL_m(P)$, $m \geq 3$, acts transitively on the set of all unimodular and all $\Delta$-modular vectors, respectively. See [1] for new interesting applications of these results to the study of metabelian groups.

Now the above cited result of Umirbaev can be considered a free group analog of Suslin’s result: it says that the group $AutF$ acts transitively on the set of all $ZF$-primitive elements (i.e., those with unimodular vector of Fox derivatives).

The desire to get a free group analog of Artamonov’s result draws our attention to $\Delta$-primitive elements. First we note that one can give an equivalent definition of $\Delta$-primitivity using a different language: an element $u \in F$ is $\Delta$-primitive if and only if the cohomology group $H^2(G, ZF)$ is infinite cyclic, where $G$ is one-relator group $\langle F \mid u \rangle$.

If the group $F$ has even rank $n = 2m$, then the element $u_n = [x_1, x_2][x_3, x_4]...[x_{n-1}, x_n]$ is $\Delta$-primitive (and hence so is every element from $O_{AutF}(u_n)$) - see Example 3.1. On the other hand, free group of an odd rank has no $\Delta$-primitive elements whatsoever - see Corollary 1.5 below. R.Bieri has pointed out to me that a combination of results of [3], [5] and [8] yields the following

**Theorem 1.2.** In a group $F_{2m}$, $m \geq 1$, any $\Delta$-primitive element is an automorphic image of $u_{2m} = [x_1, x_2][x_3, x_4]...[x_{2m-1}, x_{2m}]$.

This, together with Corollary 1.5 below, gives a free group analog of Artamonov’s result:

- in a free group $F$ of finite rank, the group $AutF$ acts transitively on the set of all $\Delta$-primitive elements.
In the case of $F_2$, Theorem 1.2 yields an explicit combinatorial description of $\Delta$-primitive elements since every automorphic image of $[x_1, x_2]$ in the group $F_2$ has the form $[x_1, x_2]^g$ or $[x_2, x_1]^g$ for some $g \in F_2$. Note that there is also an explicit combinatorial description of $ZF_2$-primitive elements due to [4] (see also [11]).

There is a nice matrix characterization of $\Delta$-primitive elements; based on that, we can use $\Delta$-primitive elements for recognizing automorphisms:

**Proposition 1.3.**
(i) An element $u \in F'_n$ is $\Delta$-primitive if and only if the matrix $D_u = (d'_j(d_i(u)))_{1 \leq i,j \leq n}$ is invertible over the ring $ZF_n$.
(ii) If $\phi$ takes a $\Delta$-primitive element of the group $F_n$ to another $\Delta$-primitive element, then $\phi$ is an automorphism.

The matrix $D_u$ (the “double Jacobian” matrix) has been introduced in [13]. Here $d'_j$ denotes right Fox derivation whereas $d_i$ is the “usual”, left Fox derivation - see Section 2.

We also treat here the case of a free metabelian group $M_n = F_n/F''_n$. (In the definition of a $\Delta$-primitive element in $M_n$, we consider abelianized Fox derivatives):

**Theorem 1.4.**
(i) The group $AutM_2$ acts transitively on the set of all $\Delta$-primitive elements of $M_2$. In other words, every $\Delta$-primitive element of $M_2$ is an automorphic image of the element $[x_1, x_2]$ and therefore has the form $[x_1, x_2]^g$ or $[x_2, x_1]^g$ for some $g \in M_2$;
(ii) There are no $\Delta$-primitive elements in the group $M_n$ if $n$ is odd.

Part (i) of this theorem answers a question from [3] in rank 2 case.

**Corollary 1.5.** There are no $\Delta$-primitive elements in the free group $F_n$ if $n$ is odd.

Talking about general case of $J$-primitive elements for an arbitrary right ideal $J$ of the group ring $ZF$, we note first of all that $J$-primitive elements do not always exist. For example, if $J$ cannot be generated by less than $(n+1)$ elements, then there are obviously no $J$-primitive elements in the group $F_n$. In fact, if $g$ is a $J$-primitive element, and the ideal $J$ is $k$-generated, then $g$ has outer rank $k$ in the sense defined in [12] (the minimal number of free generators on which an automorphic image of $g$ can depend). This follows from a result of Umirbaev [7].

Deciding for which (right) ideals $J$ of the group ring $ZF$ the group $AutF$ acts transitively on the set of all $J$-primitive elements of $F$, seems to be a difficult problem. Actually, it makes sense only for characteristic ideals, i.e., those invariant under free group automorphisms. Note that if an element $g \in F$ is $J$-primitive for a characteristic right ideal $J$, then every automorphic image of $g$ is $J$-primitive, too - this follows from the equality (1) in the next section.

Also, it is easy to show that for any $m$-generator right ideal $J$ of $ZF$, the group $GL_m(ZF)$ acts transitively on the set of all $m$-tuples of elements generating $J$ (Lemma 2.4).
Of particular interest are \( J \)-primitive elements of maximal outer rank \( n = \text{rank} F \) in the case when \( J \) is the right ideal generated by Fox derivatives of a group element:

**Proposition 1.6.** Let \( g \in F_n \) be an element of outer rank \( n \), and let \( J \) be the right ideal of \( ZF_n \) generated by Fox derivatives of \( g \). If a monomorphism (i.e., injective endomorphism) \( \phi \) of the group \( F_n \) takes \( g \) to another \( J \)-primitive element, then \( \phi \) is actually an automorphism.

In particular:

**Corollary 1.7.** (cf. [15]). Let \( g \in F_n \) be an element of outer rank \( n \). If \( \phi(g) = g \) for some monomorphism \( \phi \) of the group \( F_n \), then \( \phi \) is actually an automorphism.

### 2. Preliminaries

Let \(ZF\) be the integral group ring of the group \( F\) and \( \Delta \) its augmentation ideal, that is, the kernel of the natural homomorphism \( \varepsilon : ZF \to Z\). More generally, when \( R \subseteq F \) is a normal subgroup of \( F \), we denote by \( \Delta_R \) the ideal of \( ZF \) generated by all elements of the form \((r-1), r \in R\). It is the kernel of the natural homomorphism \( \varepsilon_R : ZF \to Z(F/R)\).

The ideal \( \Delta \) is a free left \( ZF \)-module with a free basis \( \{(x_i - 1)\}, 1 \leq i \leq n \), and left Fox derivations \( d_i \) are projections to the corresponding free cyclic direct summands. Thus any element \( u \in \Delta \) can be uniquely written in the form \( u = \sum_{i=1}^{n} d_i(u)(x_i - 1) \).

Since the ideal \( \Delta \) is a free right \( ZF \)-module as well, one can define right Fox derivatives \( d'_i(u) \) accordingly, so that \( u = \sum_{i=1}^{n} (x_i - 1)d'_i(u) \).

One can extend these derivations linearly to the whole \(ZF\) by setting \( d'_i(1) = d_i(1) = 0 \).

The next lemma is an immediate consequence of the definitions.

**Lemma 2.1.** Let \( J \) be an arbitrary left (right) ideal of \( ZF \) and let \( u \in \Delta \). Then \( u \in \Delta J \) \((u \in J \Delta) \) if and only if \( d_i'(u) \in J \) \((d_i(u) \in J) \) for each \( i, 1 \leq i \leq n \).

Proof of the next lemma can be found in [3].

**Lemma 2.2.** Let \( R \) be a normal subgroup of \( F \), and let \( y \in F \). Then \( y - 1 \in \Delta_R \Delta \) if and only if \( y \in R' \).

We also need the “chain rule” for Fox derivations (see [4]):

**Lemma 2.3.** Let \( \phi \) be an endomorphism of \( F \) (it can be linearly extended to \( ZF \)) defined by \( \phi(x_k) = y_k, 1 \leq k \leq n \), and let \( v = \phi(u) \) for some \( u, v \in ZF \). Then:

\[
d_j(v) = \sum_{k=1}^{n} \phi(d_k(u))d_j(y_k).
\]

For an endomorphism \( \phi : x_i \to y_i, 1 \leq i \leq n \), of the group \( F_n \), let \( J_\phi = (d_j(y_i))_{1 \leq i,j \leq n} \) be the Jacobian matrix of \( \phi \). We are going to need the following application of Lemma 2.3: if \( g, h \in F_n \) and \( h = \phi(g) \), then
\[(d_1(h)), ..., d_n(h)) = (\phi(d_1(g)), ..., \phi(d_n(g)))J_\phi. \quad (1)\]

Now comes the key lemma:

**Lemma 2.4.** Let \( J \) be a right ideal of \(ZF\) (hence a free right module over \(ZF\) - see \([3]\)) generated as a free module over \(ZF\) by \(u_1, ..., u_m\). Then the following conditions are equivalent:

(a) A matrix \(M = (a_{ij})_{1 \leq i, j \leq m}\) is invertible over \(ZF\) (i.e., \(M \in GL_m(ZF)\));

(b) The elements \(y_j = \sum_{k=1}^{m} (u_k - 1)a_{kj}, 1 \leq j \leq m\), generate the ideal \(J\) as a right ideal of \(ZF\).

**Proof.** Suppose \(M\) is invertible over \(ZF\); denote by \(U\) the row matrix \((u_1, ..., u_m)\), and by \(Y\) - the row matrix \((y_1, ..., y_m)\). Then \(UMM^{-1} = YM^{-1} = U\) which means that \(u_1, ..., u_m\) belong to the right ideal of \(ZF\) generated by \(y_1, ..., y_m\). Conversely, suppose we have \(YB = U\) for some matrix \(B\) over \(ZF\). Then \(UMB = U\), hence \(MB = I\), the identity matrix, because \((u_1, ..., u_m)\) form a free basis of a free right \(ZF\)-module \(J\). This implies \(M \in GL_m(ZF)\) - see e.g. \([5]\).

3. Proofs

**Proof of Theorem 1.1.** We consider several possibilities for an orbit \(O_{AutF_2}(u)\):

(1) \(u\) is a primitive element. In this case, the image of every primitive element of \(F_2\) under the endomorphism \(\phi\) is primitive. By composing \(\phi\) with some automorphism of the group \(F_2\) if necessary, we may assume that \(\phi(x_1) = x_1\).

Now write \(\phi(x_2) = x_1^k g\), where \(k\) is an integer, and \(g\) is an element of the normal closure of \(x_2\), so that \(g\) is a product of elements of the form \(h_i x_2^{\pm 1} h_i^{-1}\).

Since every element of the form \(x_1^n x_2\) is primitive, its image \(s = \phi(x_1^n x_2) = x_1^{n+k} g\) is primitive, too.

Recall now a result of \([4]\) which says that if \(w\) is a primitive element of \(F_2\), then some conjugate of \(w\) can be written in the form \(x_1^{k_1} x_2^{l_1} ... x_1^{k_m} x_2^{l_m}\), so that some of \(x_i\) occurs either solely in the exponent 1 or solely in the exponent \(-1\).

We see that for sufficiently large \(n\), generator \(x_1\) would not occur in any conjugate of \(s\) neither solely in the exponent 1 nor solely in the exponent \(-1\). Therefore, \(x_2\) should be the one with this property.

It follows that no \(h_i\) in the decomposition of \(g\) mentioned above, has entries of \(x_2\), so that every \(h_i\) is a power of \(x_1\).

Then, since the element \(x_1^{-k} x_2\) is primitive, its image \(\phi(x_1^{-k} x_2) = g\) is primitive as well. This implies that the sum of powers of \(x_2\) in the decomposition of \(g\) should be equal to \(\pm 1\); otherwise \(g\) would not be primitive even modulo \(F_2'\). It follows that our \(g\) is actually conjugate to \(x_2^{\pm 1}\).
Summing up, we see that $g$ has the form $hx_{2}^{\pm 1}h^{-1}$ for some $h \in F_{2}$, $h$ a power of $x_{1}$. Therefore, $\phi(x_{1}) = x_{1}$ and $\phi(x_{2}) = x_{1}^{k}g$ generate the group $F_{2}$, i.e., $\phi$ is an automorphism.

(2) $u = v^{k}$ is a power of a primitive element $v$. In this case, every image of $u$ under the endomorphism $\phi$ has the form $w^{k}$, $w$ a primitive element. It follows that the image of every primitive element of $F_{2}$ is primitive, hence we may apply the argument from the previous case.

(3) $u$ has outer rank 2, i.e., $u$ does not belong to a proper free factor of $F_{2}$. If $u$ does not belong to a proper retract of $F_{2}$, then $\phi$ is an automorphism by a result of Turner [15].

Suppose now that $u$ belongs to a proper retract $R$ of $F_{2}$, and $\phi$ is the corresponding retraction, i.e., $\phi(F_{2}) = R$ (otherwise $\phi$ would already be an automorphism by [15]). Since $R$ is a proper retract of $F_{2}$, it should have rank smaller than 2, i.e., $R$ is cyclic, and $R \subseteq O_{AutF}(u)$.

We show now that no automorphic orbit $O_{AutF}(u)$ can contain a non-trivial cyclic group. By means of contradiction, suppose some automorphic image of some $s \in O_{AutF}(u)$ is of the form $s^{k}$, $k \geq 2$, and let $s$ have minimal length among all the elements of $O_{AutF}(u)$ with this property. Let $\alpha(s) = s^{k}$ with $k \geq 2$, and let $\alpha^{-1}(s) = r$. Then $\alpha(s) = \alpha(r)^{k} = \alpha(r^{k})$, hence $s = r^{k}$. Thus, every automorphic image of $r$ is of the form $r^{m}$; furthermore, $r \in O_{AutF}(u)$ (since $r = \alpha^{-1}(s)$), and $r$ has length smaller than that of $s$ (see [10], Proposition I.2.15). This contradiction completes the proof of Theorem 1.1.

Before getting to a proof of Theorem 1.2, we consider a couple of examples.

Example 3.1. The element $u = [x_{1}, x_{2}][x_{3}, x_{4}]...[x_{m-1}, x_{m}]$ of the group $F_{2m}$ is $\Delta$-primitive since the corresponding double Jacobian matrix $D_{u}$ is invertible - see [12], Proposition 4.1.

Example 3.2. The element $v = [x_{1}, x_{2}][x_{2}, x_{3}][x_{3}, x_{4}]$ of the group $F_{4}$ is $\Delta$-primitive. It is not quite obvious that $v$ is an automorphic image of $u = [x_{1}, x_{2}][x_{3}, x_{4}]$. However, this is the case: $u$ is taken to $v$ by the following automorphism: $x_{1} \rightarrow x_{1}x_{3}^{-1}; x_{2} \rightarrow x_{2}x_{3}x_{1}^{-1}; x_{3} \rightarrow x_{3}; x_{4} \rightarrow x_{4}$.

Now we get to Theorem 1.2:

**Proof of Theorem 1.2.** We give a proof here without introducing background material, just referring to [4] and [7] for details.

First of all, it is an immediate consequence of the definition that $g \in F$ is a $\Delta$-primitive element if and only if for the right ideal $J_{g}$ of the group ring $ZF$ generated by Fox derivatives of $u$, one has factor-module $ZF/J_{g}$ isomorphic to the trivial $F$-module $Z$. In other words, as we have mentioned in the Introduction, the cohomology group $H^{2}(G, ZF)$ of the one-relator group $G = < F \mid g >$ is infinite cyclic. It is also clear that the group $G$ is torsion-free.
Then, Theorem 9.3 of [2] implies that the group $< F \mid g >$ is a Poincaré duality group of dimension 2 (PD$^2$-group). It has been proved later in [7] that every one-relator (torsion-free) PD$^2$-group is a surface group.

Applying now a result of Zieschang [18], we see that, in case $F = F_{2m}$, $g$ must be an automorphic image of $u_{2m} = [x_1, x_2][x_3, x_4]...[x_{2m-1}, x_{2m}]$. This completes the proof.

**Proof of Proposition 1.3.**

(i) First suppose the matrix $D_u = (d_i'(d_j(u)))_{1 \leq i, j \leq n}$ is invertible. Then, by Lemma 2.4, the elements $y_i = \sum_{k=1}^{n} (x_k - 1)d_k'(d_i(u)) = d_i(u) - \epsilon(d_i(u))$, 1 $\leq i \leq n$, generate the ideal $\Delta$ as a right ideal of $ZF$. Since $u \in F'$, we have $\epsilon(d_i(u)) = 0$, 1 $\leq i \leq n$, by Lemmas 2.1, 2.2. Therefore, $u$ is $\Delta$-primitive.

Conversely, if $u$ is $\Delta$-primitive, then the elements $d_i(u) = \sum_{k=1}^{n} (x_k - 1)d_k'(d_i(u))$, generate $\Delta$ as a right ideal of $ZF$. Again by Lemma 2.4, the matrix $D_u$ is invertible.

(ii) We use an argument from [13] implying that if a matrix $D_{\phi(u)}$ is invertible and $u \in F'$, then $\phi$ is an automorphism. Applying part (i) of this proposition completes the proof.

**Proof of Theorem 1.4.** First of all, note that if $u$ is $\Delta$-primitive, then $u \in M'$; otherwise some $d_i(u)$ wouldn’t belong to $\Delta$ by Lemmas 2.1, 2.2.

(i) Let $h \in M'_{2}$; then we can write $h$ as $[x_1, x_2]^w$ for some $w \in ZA_2 = Z(M_2/M'_2)$. Then for abelianized Fox derivatives (we denote them the same way as the ones in a free group ring when there is no ambiguity) we have: $d_i(h) = w \cdot d_i([x_1, x_2])$. Hence if $d_i(h)$, $i = 1, 2$, generate the same ideal of $ZA_2$ as $d_i([x_1, x_2])$, the element $w$ should be invertible in $ZA_2$, which means it has the form $\pm g$ for some $g \in M_2/M'_2$.

Thus $h = [x_1, x_2]^\pm g$; in particular, it is an automorphic image of $[x_1, x_2]$.

(ii) Consider basic commutators of weight 2 in the group $M_n$: $c_1 = [x_1, x_2]$; $c_2 = [x_1, x_3]$; ...; $c_N = [x_{n-1}, x_n]$, where $N = n(n - 1)/2$, and consider a product of the form $w = c_1^{k_1}c_2^{k_2}...c_N^{k_N}$. Evaluate abelianized Fox derivatives of the element $w$:

$d_1(w) = k_1(x_2 - 1) + ... + k_{n-1}(x_n - 1)$;
$d_2(w) = -k_1(x_1 - 1) + k_n(x_3 - 1) + ... + k_{2n-3}(x_n - 1)$;
$d_3(w) = -k_2(x_1 - 1) - k_n(x_2 - 1) + k_{2n-2}(x_4 - 1) + ... + k_{3n-6}(x_n - 1)$.

... .

$d_n(w) = -k_{n-1}(x_1 - 1) - ... - k_{n(n-1)/2}(x_{n-1} - 1)$.

We are now going to show that these derivatives do not generate $\Delta$ even modulo $\Delta^2$. To do that, it suffices to show that they are linearly dependent, i.e., that the $nxn$ matrix of coefficients (its $(i, j)$th entry is the coefficient at $(x_j - 1)$ in the decomposition of $d_i(w)$ above) has determinant 0.

It is easy to see that this matrix (denote it by $A = (a_{ij})$) is antisymmetric, with zeroes on the diagonal. The determinant of a matrix like that must be 0.

7
if $n$ is odd. Indeed, consider a summand $a_{1,i_1}a_{2,i_2}...a_{n,i_n}$ in the decomposition of the determinant. If there is at least one diagonal element among these $a_{k,i_k}$, then the product is 0. If all $a_{k,i_k}$ are off-diagonal elements, consider the “reflection” $a_{i_1,1}a_{i_2,2}...a_{i_n,n}$. These 2 summands go with different signs since $a_{ij} = -a_{ji}$, and $n$ is odd. Therefore, they cancel out which proves that the determinant of $A$ equals 0. This completes the proof of part (ii).

**Proof of Corollary 1.5.** If there were a $\Delta$-primitive element in $F_n$, its image in $M_n$ would be a $\Delta$-primitive element of $M_n$ which contradicts Theorem 1.4 (ii).

**Proof of Proposition 1.6.** Let $h = \phi(g)$ be a $J$-primitive element. Since $g$ has outer rank $n$, the right ideal $J$ is $n$-generated by $[17]$. It follows that the elements $d_1(g), ..., d_n(g)$ freely generate $J$ as a right ideal of $ZF_n$. Indeed, if there were a (right) $ZF_n$-dependence between these elements, then one of them would belong to the right ideal generated by the others - this follows from a general theory of $[3]$. Therefore, $J$ could be generated by less than $n$ elements.

Thus by Lemma 2.4, for some matrix $M \in GL_n(ZF_n)$, we have:

$$ (d_1(h), ..., d_n(h)) = (d_1(g), ..., d_n(g))M. \quad (2) $$

On the other hand, by the equation (1) in the Preliminaries, we have:

$$ (d_1(h), ..., d_n(h)) = (\phi(d_1(g)), ..., \phi(d_n(g)))J_\phi. $$

This together with (2) gives

$$ (d_1(g), ..., d_n(g)) = (\phi(d_1(g)), ..., \phi(d_n(g)))J_\phi M^{-1}. \quad (3) $$

This means $J \subseteq \phi(J)$. Since $\phi$ is a monomorphism, this yields $J = \phi(J)$, in which case the matrix $J_\phi M^{-1}$ on the right-hand side of (3) must be invertible by Lemma 2.4. Therefore, $J_\phi$ is invertible, too, hence $\phi \in Aut F_n$ by $[3]$. **Corollary 1.7** follows immediately.

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