A multiparameter Garsia-Rodemich-Rumsey inequality and some applications

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Abstract

We extend the classical Garsia-Rodemich-Rumsey inequality to the multiparameter situation. The new inequality is applied to obtain some joint Hölder continuity along the rectangles for fractional Brownian fields $W(t, x)$ and for the solution $u(t, y)$ of stochastic heat equation with additive white noise.

1 Introduction

Let the function $\Psi : [0, \infty) \to [0, \infty)$ be non decreasing with $\lim_{u \to \infty} \Psi(u) = \infty$ and let the function $p : [0, 1] \to [0, 1]$ be continuous and non decreasing with $p(0) = 0$. Set

$$
\begin{aligned}
\Psi^{-1}(u) &= \sup_{\Psi(v) \leq u} v & \text{if } \Psi(0) \leq u < \infty \\
p^{-1}(u) &= \max_{p(v) \leq u} v & \text{if } 0 \leq u \leq p(1)
\end{aligned}
$$

The celebrated Garsia-Rodemich-Rumsey inequality [6] takes the following form:

**Lemma 1.1** Let $f$ be a continuous function on $[0, 1]$ and suppose that

$$
\int_0^1 \int_0^1 \Psi \left( \frac{|f(x) - f(y)|}{p(x - y)} \right) \, dx \, dy \leq B < \infty.
$$

Then for all $s, t \in [0, 1]$ we have

$$
|f(s) - f(t)| \leq 8 \int_0^{|s-t|} \Psi^{-1} \left( \frac{4B}{u^2} \right) \, dp(u). \quad (1.1)
$$

This Garsia-Rodemich-Rumsey lemma 1.1 is very powerful in the study of the sample path Hölder continuity of a stochastic process and in other occasions. For example

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if \( \Psi(u) = |u|^p \) and \( p(u) = |u|^\alpha/\beta \), where \( p > 1 \), the inequality (1.1) implies the following Sobolev imbedding inequality

\[
|f(s) - f(t)| \leq C_{\alpha,p} |t - s|^\alpha \left( \int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^\alpha} \, dx \, dy \right)^{1/p} .
\] (1.2)

The Garsia-Rodemich-Rumsey lemma has been extended to several parameter or infinite many parameters. However the parameter space are assumed to have a distance (metric space) and the Garsia-Rodemich-Rumsey lemma is with respect to that distance. This method immediately yields the following result for a fractional Brownian field \( W^H(x) \) of Hurst parameter \( H = (H_1, \ldots, H_d) \), then for any \( \beta_i \) with \( \beta_i < H_i \), \( i = 1, \ldots, d \), one has

\[
|W(y) - W(x)| \leq L \sum_{i=1}^d |x_i - y_i|^\beta_i ,
\] (1.3)

where \( L \) is an integrable random variable. One can improve this result (Remark 4.4) by our version of multiparameter Garsia-Rodemich-Rumsey inequality. We do not seek for a suitable metric but rather deal directly with the multidimensional nature of the parameter space.

Let us explain our motivation by considering the two parameter fractional Brownian field \( \{W(x_1, x_2), (x_1, x_2) \in [0, 1]^2\} \) of Hurst parameter \( H = (H_1, H_2) \). Given two points \( x \) and \( y \) in \( \mathbb{R}^2 \), we consider the increment of \( W \) along with the rectangle determined by \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \):

\[
\square W := W(y_1, y_2) - W(x_1, y_2) - W(x_2, y_1) + W(x_1, x_2) .
\] (1.4)

In [10], using a two-parameter version (1.2), the author showed that for any \( \beta_1, \beta_2 \) with \( \beta_1 < H_1 \) and \( \beta_2 < H_2 \), there is an integrable random constant \( L_{\beta_1, \beta_2} \) such that

\[
|\square W| \leq L_{\beta_1, \beta_2} |y_1 - x_1|^{\beta_1} |y_2 - x_2|^{\beta_2} .
\] (1.5)

The above result was also obtained in [1] based on a two-parameter version of Kolmogorov continuity theorem. Along the paper (in Corollary 4.5), we shall see that the following sharper inequality than (1.5) holds

\[
|\square W| \leq L_{H_1, H_2} |y_1 - x_1|^{H_1} |y_2 - x_2|^{H_2} \sqrt{\log(|y_1 - x_1||y_2 - x_2|)} .
\] (1.6)

Consequently, this estimate implies

\[
|W(x_1, x_2) - W(y_1, y_2)| \leq L_{H_1, H_2} \left( |x_1 - y_1|^{H_1} |x_2|^{H_2} \sqrt{\log(|x_1 - y_1||x_2|)} \right)
+ |x_1|^{H_1} |x_2 - y_2|^{H_2} \sqrt{\log(|x_2 - y_2||x_1|)}
\]

which improves (1.3). We shall call such property as in (1.6) or (1.5) joint Hölder continuity. It turns out that a large class of Gaussian fields enjoys sample path joint Hölder continuity (Theorem 4.3.)

Our method is first formulate and prove a multiparameter version of the classical Garsia-Rodemich-Rumsey inequality (1.1). The generalized inequality is then applied to obtain sample path joint Hölder continuity for random fields. Our result generalizes
the results in [6], [10] and provides a different approach for sample path property problem of random fields (compare to the approach in [1], [2] and [11].)

The paper is structured as follows. In Section 2, we shall state and prove our multiparameter version of the Garsia-Rodemich-Rumsey lemma. The idea is to use induction on the dimension of the parameter space after some observations of the property of operator $\Box$ defined by (1.4). Some part of the proof is similar to the original proof of Garsia-Rodemich-Rumsey [6] with some modification. However, we feel it is more appropriate to give a detailed proof.

In Section 3, we introduce a multiparameter version of Kolmogorov continuity criteria (Theorem 3.1). To our best knowledge, a two-parameter of Theorem 3.1 first appeared in [1].

Section 4 is devoted for the study of sample path joint continuity for Gaussian fields. We give a sufficient condition for a Gaussian field to possess sample path joint continuity (Theorem 4.3). We also derive the estimate (1.6) for fractional Gaussian field. In Section 5, we shall study the joint Hölder continuity of solution of a stochastic heat equation with additive space-time white noise.

## 2 Multiparameter Garsia-Rodemich-Rumsey inequality

We state the following technical lemma which generalizes a crucial argument used in [6] in the proof of Lemma 1.1.

**Lemma 2.1** Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mu$ be a positive measure on $(\Omega, \mathcal{F})$. Let $g : \Omega \times [0, 1] \to \mathbb{R}^m$ be a measurable function such that

$$\int_0^1 \int_0^1 \int_{\Omega} \Psi \left( \frac{|g(z, t) - g(z, s)|}{p(|t - s|)} \right) \mu(dz) ds \leq B < \infty.$$ 

Then there exist two decreasing sequences $\{t_k, k = 0, 1, \cdots\}$ and $\{d_k, k = 0, 1, \cdots\}$ with

$$t_k \leq d_{k-1} = p^{-1} \left( \frac{1}{2} p(t_{k-1}) \right), \quad k = 1, 2, \cdots \quad (2.1)$$

such that the following inequality holds

$$\int_{\Omega} \Psi \left( \frac{|g(z, t_k) - g(z, t_{k-1})|}{p(|t_k - t_{k-1}|)} \right) \mu(dz) \leq \frac{4B}{d_{k-1}^2}. \quad (2.2)$$

**Proof** We follow the argument in [6]. Let

$$I(t) = \int_0^1 \int_{\Omega} \Psi \left( \frac{|g(z, t) - g(z, s)|}{p(|t - s|)} \right) \mu(dz) ds.$$

From the assumption $\int_0^1 I(t) dt \leq B$ it follows that there is some $t_0 \in (0, 1)$ such that

$$I(t_0) \leq B.$$

Now we can describe how to obtain the sequences $d_k$ and $t_k$ recursively for $k = 1, 2, \cdots$. Given $t_{k-1}$, define

$$d_{k-1} = p^{-1} \left( \frac{1}{2} p(t_{k-1}) \right).$$
Then we choose \( t_k \leq d_k - 1 \) such that
\[
I(t_k) \leq \frac{2B}{d_k - 1}
\] (2.3)
and
\[
\int_{\Omega} \Psi \left( \frac{|g(z, t_k) - g(z, t_k-1)|}{p(|t_k - t_{k-1}|)} \right) \mu(dz) \leq \frac{2I(t_{k-1})}{d_k - 1}.
\] (2.4)
It is always possible to find \( t_k \) such that the inequalities (2.3) and (2.4) hold simultaneously, since each of the two inequalities can be violated only on a set of \( t_k \)'s of measure strictly less than \( \frac{1}{2}d_{k-1} \). Now (2.3) and (2.4) gives
\[
\int_{\Omega} \Psi \left( \frac{|g(z, t_k) - g(z, t_k-1)|}{p(|t_k - t_{k-1}|)} \right) \mu(dz) \leq \frac{2I(t_{k-1})}{d_k - 1} \leq \frac{4B}{d_k - 1} \leq \frac{4B}{d_{k-1}^2}.
\]
This is (2.2). \( \square \)

Let \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) be in \( \mathbb{R}^n \). We denote \( x' = (x_1, \ldots, x_{n-1}) \) and \( y' = (y_1, \ldots, y_{n-1}) \). For each integer \( k = 1, 2, \ldots, n \), we define
\[
V_{k,y}x = (x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n).
\]
Let \( f \) be a function from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). We define the operator \( V_{k,y} \) acting on \( f \) in the following way:
\[
V_{k,y}f(x) = f(V_{k,y}x)
\]
or \( V_{k,y}f = f \circ V_{k,y} \) in short. It is straight forward to verify that
\[
V_{k,y}V_{k,y} = V_{k,y}
\]
and
\[
V_{k,y}V_{l,y} = V_{l,y}V_{k,y}
\]
for \( k \neq l \). Next, we define the joint increment of a function \( f \) on an \( n \)-dimensional rectangle, namely
\[
\Box^nf(x) = \prod_{k=1}^{n} (I - V_{k,y})f(x)
\]
where \( I \) denotes the identity operator.

**Example 2.2** If \( n = 2 \), then it is easy to see that \( \Box_y^2f(x) = f(y_1, y_2) - f(x_1, y_2) - f(y_1, x_2) + f(x_1, y_2) \), which is the increment of \( f \) over the rectangle containing the two points \( x \) and \( y \) with all sides parallel to the axis. In particular, if \( f(x_1, x_2) = x_1x_2 \), then \( \Box_y^2f(x) = (x_1 - y_1)(x_2 - y_2) \), which is the area of the rectangle. In a more general case, when \( f \) has the form \( f(x) = \prod_{j=1}^{n} f_j(x_j) \), then
\[
\Box^nf(x) = \prod_{j=1}^{n} [f_j(x_j) - f_j(y_j)].
\]
The following simple identity enable us to show our theorem by induction and plays an essential role in our approach:

\[
\Box^n_y f(x) = \prod_{k=1}^{n-1} (I - V_{k,y}) f(x) - \prod_{k=1}^{n-1} (I - V_{k,y}) f(x) = \Box^{n-1}_y f(x', x_n) - \Box^{n-1}_y f(x', y_n).
\] (2.5)

We are now in the position to state our general version of Lemma 1.1.

**Theorem 2.3** Let \( f(x) \) be a continuous function on \([0, 1]^n\) and suppose that

\[
\int_{[0,1]^n} \int_{[0,1]^n} \frac{|\Box^n_y f(x)|}{\prod_{k=1}^{n-1} p_k(|x_k - y_k|)} \, dx dy \leq B < \infty.
\] (2.6)

Then for all \( s, t \in [0, 1]^n \) we have

\[
|\Box^n_y f(t)| \leq 8^n \int_0^{|s_1-t_1|} \cdots \int_0^{|s_n-t_n|} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_n^2} \right) dp_1(u_1) \cdots dp_n(u_n). \] (2.7)

**Proof** We proceed by induction on \( n \). For \( n = 1 \), it coincides with the original Garsia-Rodemich-Rumsey inequality (1.1). Suppose (2.7) holds for \( n - 1 \). Let \( f \) be a continuous function on \([0, 1]^n\). For any \( x', y' \in \mathbb{R}^{n-1} \) and any \( s \in [0, 1] \), put

\[
g(x', y', s) = \frac{\Box^{n-1}_y f(x', s)}{\prod_{k=1}^{n-1} p_k(|x_k' - y_k'|)}.
\]

Let \( \Omega = [0, 1]^{n-1} \times [0, 1]^{n-1} \), \( z = (x', y') \). By (2.5) we can rewrite (2.6) as

\[
\int_0^1 \int_0^1 \int_{\Omega} \Psi \left( \frac{|g(z, s) - g(z, t)|}{p_n(|s-t|)} \right) d\omega ds dt \leq B < \infty.
\]

Applying Lemma 2.1, we can find sequences \( \{t_k\} \) and \( \{d_k\} \) such that

\[
t_k \leq d_{k-1} = p_n^{-1} \left( \frac{1}{2} p_n(t_{k-1}) \right)
\] (2.8)

and

\[
\int_{\Omega} \Psi \left( \frac{|g(z, t_k) - g(z, t_{k-1})|}{p_n(t_k - t_{k-1})} \right) d\omega \leq \frac{4B}{d_{k-1}^2}.
\] (2.9)

For each \( k \in \mathbb{N} \) and \( x' \in [0, 1]^{n-1} \), let

\[
h_k(x') = \frac{f(x', t_k) - f(x', t_{k-1})}{p_n(|t_k - t_{k-1}|)}.
\]

Again from (2.5) it follows

\[
\frac{\Box^{n-1}_y h_k(x')}{\prod_{i=1}^{n-1} p_i(|x_i' - y_i'|)} = \frac{\Box^{n-1}_y f(x', t_k) - \Box^{n-1}_y f(x', t_{k-1})}{\prod_{i=1}^{n-1} p_i(|x_i' - y_i'|) p_n(|t_k - t_{k-1}|)} = \frac{g(x', y', t_k) - g(x', y', t_{k-1})}{p_n(|t_k - t_{k-1}|)}.
\]
Thus, the inequality (2.9) becomes
\[
\int_{[0,1]^{n-1}} \int_{[0,1]^{n-1}} \Psi \left( \frac{\Box^{n-1} h_k(x')}{\prod_{i=1}^{n-1} p_i(|x_i - y'_i|)} \right) dx' dy' \leq \frac{4B}{d_{k-1}^2}.
\]

Now, by our induction hypothesis, for every \( k \geq 1 \), \( a, b \in [0,1]^{n-1} \), \( a = (a_1, \ldots, a_{n-1}) \) and \( b = (b_1, \ldots, b_{n-1}) \),
\[
|\Box^{n-1} h_k(b)| \leq 8^{n-1} \int_{0}^{|a_1-b_1|} \cdots \int_{0}^{|a_{n-1}-b_{n-1}|} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_{n-1}^2 d_{k-1}^2} \right) dp(u_1) \cdots dp_{n-1}(u_{n-1}).
\]

Denoting \( A = [0, |a_1-b_1|] \times \cdots \times [0, |a_{n-1}-b_{n-1}|] \) and \( dp(u_1, \ldots, u_{n-1}) = dp_1(u_1) \cdots dp_{n-1}(u_{n-1}) \), the above inequality can be rewritten as
\[
|\Box^{n-1} f(b, t_k) - \Box^{n-1} f(b, t_{k-1})| \leq 8^{n-1} \int_{A} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_{n-1}^2 d_{k-1}^2} \right) dp(u_1, \ldots, u_{n-1}) p_n(t_{k-1}-t_k).
\]

(2.10)

On the other hand, by (2.8), we have
\[
p_n(t_{k-1}-t_k) \leq p_n(t_{k-1}) = 2p(d_{k-1}) \leq 4[p(d_{k-1}) - p(d_k)].
\]

Combining this inequality with (2.10) yields
\[
|\Box^{n-1} f(b, t_0) - \Box^{n-1} f(b, 0)|
\leq \sum_{k=1}^{\infty} |\Box^{n-1} f(b, t_k) - \Box^{n-1} f(b, t_{k-1})|
\leq 8^{n-1} \sum_{k=1}^{\infty} 4[p(d_{k-1}) - p(d_k)] \int_{A} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_{n-1}^2 d_{k-1}^2} \right) dp(u_1, \ldots, u_{n-1})
\leq 8^{n-1} \int_{d_{k}}^{\infty} \int_{A} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_{n-1}^2 u_n^2} \right) dp(u_1, \ldots, u_{n-1} - dp_n(u_n))
\leq 8^{n-1} \int_{A} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_n^2} \right) dp(u_1, \ldots, u_{n-1}) dp_n(u_n).
\]

With \( f(x', 1-x_n) \) replaced \( f(x', x_n) \) we can obtain the same bound for
\[
|\Box^{n-1} f(b, t_0) - \Box^{n-1} f(b, 1)|.
\]

Hence, for every \( a, b \in [0,1]^{n-1} \),
\[
|\Box^{n-1} f(b, 1) - \Box^{n-1} f(b, 0)| \leq 8^n \int_{0}^{1} \int_{A} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_n^2} \right) dp(u_1, \ldots, u_{n-1}) dp_n(u_n).
\]

(2.11)

To obtain (2.7) for general \( s, t \) in \([0,1]^n\), we set
\[
\tilde{f}(t', \tau) = f(t', s_n + \tau(t_n - s_n)) \text{ for } \tau \in [0,1]
\]

in (2.11).
and
\[ \tilde{p}_n(u) = p_n(u|s_n - t_n|). \]

Upon restricting the range of the integration in (2.6) and carrying out a change of variables we get
\[
\int_{[0,1]^n} \int_{[0,1]^n} \Psi \left( \prod_{k=1}^{n-1} p_k(|x_k - y_k|) \tilde{p}_n(|x_n - y_n|) \right) dxdy \leq \frac{B}{|s_n - t_n|^2}.
\]

Thus, by (2.11), we deduce
\[
|\square^n f(t)| = |\square^n f(t', 1) - \square^n f(t', 0)| \\
\leq 8^n \int_0^1 \int_0^{s_1 - t_1} \cdots \int_0^{s_{n-1} - t_{n-1}} \Psi^{-1} \left( \frac{4^n B}{u_1^2 \cdots u_n |s_n - t_n|^2} \right) dp(u_1, \cdots, u_{n-1}) dp_n(u_n|s_n - t_n|).
\]

Another change of variables yields (2.7). \( \blacksquare \)

3 Sample path joint Hölder continuity of random fields

In this section, given a continuous random field \( W \), we study sample path joint continuity property. The first application of Theorem 2.3 is the following criteria for joint continuity of sample paths which is similar to Kolmogorov continuity theorem, which we shall call joint Kolmogorov continuity theorem.

Theorem 3.1 Let \( W \) be a continuous random field on \( \mathbb{R}^n \). Suppose there exist positive constants \( \alpha, \beta_k \) \((1 \leq k \leq n)\) and \( K \) such that for every \( x, y \) in \([0,1]^n\),
\[
\mathbb{E} \left[ |\square^n y W(x)|^\alpha \right] \leq K \prod_{k=1}^n |x_k - y_k|^{1+\beta_k}.
\]

Then, for every \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) with \( 0 < \epsilon_k \alpha < \beta_k \) \((1 \leq k \leq n)\), there exist a random variable \( \eta \) with \( \mathbb{E} \eta^\alpha \leq K \), such that the following inequality holds almost surely
\[
|\square^n W(s)| \leq C \eta(\omega) \prod_{k=1}^n |t_k - s_k|^{\beta_k \alpha^{-1} - \epsilon_k}
\]

for all \( s, t \) in \([0,1]^n\), where \( C \) is a constant defined by
\[
C = 8^n 4^n / \alpha \prod_{k=1}^n \left( 1 + \frac{2}{\beta_k - \alpha \epsilon_k} \right).
\]

Proof Let \( \Psi(u) = |u|^{\alpha}, p_k(u) = |u|^{\gamma_k} \) where \( \gamma_k \in (\frac{2}{\alpha}, \frac{2+\beta_k}{\alpha}) \), \( 1 \leq k \leq n \). A direct application of Theorem 2.3 gives that for all \( s, t \) in \([0,1]^n\)
\[
|\square^n W(t)| \leq 8^n \prod_{k=1}^n \frac{\gamma_k |t_k - s_k|^{\gamma_k - \frac{2}{\alpha}}}{\gamma_k^{\frac{2}{\alpha}} \prod_{k=1}^n |x_k - y_k|^{\gamma_k}} \left( 4^n \int_{[0,1]^2} \frac{|\square^n y W(x)|^\alpha}{\prod_{k=1}^n |x_k - y_k|^{\gamma_k}} dxdy \right)^{\frac{1}{n}}.
\]

(3.1)
Let
\[ B(\omega) = \int\int_{[0,1]^2n} \Psi \left( \left| \square^\omega W(x) \right| \prod_{k=1}^{n} p_k(x_k - y_k) \right) dxdy. \]

From our assumption and Fubini-Tonelli’s theorem,
\[ \mathbb{E}B = \int\int_{[0,1]^2n} \mathbb{E} \left| \square^\omega W(x) \right|^\alpha \prod_{k=1}^{n} |x_k - y_k|^{\alpha \gamma_k} dxdy \leq K \int\int_{[0,1]^2n} \prod_{k=1}^{n} |x_k - y_k|^{1+\beta_k - \alpha \gamma_k} dxdy < \infty. \]

Hence, the event \( \Omega^* = \{ \omega : B(\omega) < \infty \} \) has probability one. Therefore for each \( \omega \) in \( \Omega^* \), the inequality (3.1) gives
\[ |\square^\omega W(t, \omega)| \leq 8^n \prod_{k=1}^{n} \frac{\gamma_k |t_k - s_k|^{\gamma_k \frac{\gamma_k - \frac{2}{\alpha}}{\gamma_k - \frac{2}{\alpha}}} (4^n B(\omega))^{\frac{1}{\alpha}}} \]
for every \( s, t \) in \( [0,1]^n \). For each \( k \), the power \( \gamma_k - \frac{2}{\alpha} \) can be made arbitrarily close to \( \frac{\beta_k}{\alpha} \). This completes the proof with \( \eta = B^{1/\alpha} \).

**Remark 3.2** The result obtained by Ral’chenko [10] was the inequality (3.1) in the case \( n = 2 \).

### 4 Sample path joint continuity of Gaussian fields

We now focus on sample path joint continuity of Gaussian random fields. In case of Gaussian processes \( (n = 1) \), one of the first sufficient and necessary conditions for sample path continuity was given by Fernique [3] (see also [5]). Namely, let \( p(u) \) be an increasing positive function such that
\[ \mathbb{E}|W(x) - W(y)|^2 \leq p^2(|x - y|) \quad (4.1) \]
for any pair \((x, y)\) in \([0,1]^2\). Then Fernique [3] showed that a sufficient condition for almost sure continuity of the process \((W(x), 0 \leq x \leq 1)\) is
\[ \int_{0}^{1} \frac{p(u)}{u \sqrt{\log \frac{1}{u}}} du < \infty. \]

In the original paper of Garsia-Rodemich-Rumsey [6], the authors also observed that the above condition is equivalent to the condition (by integration by part)
\[ \int_{0}^{1} \sqrt{\log \frac{1}{u}} dp(u) < \infty. \]

Later, it was shown that the above condition is also necessary [4, 8]. In case of Gaussian fields, recent progress on modulus of continuity of Gaussian random fields has been reported in [11, 2, 7].
Let $W$ be a centered Gaussian random field with covariance function

$$E[W(x)W(y)] = Q(x, y).$$

(4.2)

We will always assume that $Q$ is a continuous function of $x$ and $y$. For any fixed $x, y$, the random variable $\Box^n_y W(x)$ is also Gaussian with mean zero. In the following proposition, we compute its variance.

**Proposition 4.1** Let $W$ be a centered Gaussian random field with covariance function given by (4.2). Then

$$E[|\Box^n_y W(x)|^2] = \Box^{2n}_{(y,y)} Q(x, x).$$

(4.3)

Furthermore, if the covariance function $Q$ has the following product form

$$Q(x, y) = \prod_{k=1}^n Q_k(x_k, y_k)$$

(4.4)

then (4.3) is simplified as

$$E[|\Box^n_y W(x)|^2] = \prod_{k=1}^n [Q_k(x_k, x_k) - Q_k(x_k, y_k) - Q_k(y_k, x_k) + Q_k(y_k, y_k)].$$

(4.5)

**Proof** We calculate the variance directly as follows

$$E[|\Box^n_y W(x)|^2] = E[\Box^n_y W(x) \cdot \Box^n_y W(x)]$$

$$= E[\Box^{2n}_{(y,y)} W(x)W(x)]$$

$$= \Box^{2n}_{(y,y)} E[W(x)W(x)]$$

$$= \Box^{2n}_{(y,y)} Q(x, x).$$

The identity (4.3) follows. To prove (4.5), we notice that the pair of operators $(I - V_{k,(y,y)})(I - V_{n+k,(y,y)})$ transforms the $k$-th factor of $Q$ in (4.4) to

$$Q_k(x_k, x_k) - Q_k(x_k, y_k) - Q_k(y_k, x_k) + Q_k(y_k, y_k).$$

Since the operators $I - V_{k,(y,y)}$, $(1 \leq k \leq 2n)$ are commutative, we can write

$$\Box^{2n}_{(y,y)} Q(x, x) = \prod_{k=1}^n (I - V_{k,(y,y)})(I - V_{n+k,(y,y)}) Q(x, x)$$

$$= \prod_{k=1}^n [Q_k(x_k, x_k) - Q_k(x_k, y_k) - Q_k(y_k, x_k) + Q_k(y_k, y_k)].$$

Hence, the identity (4.5) follows. ■

**Definition 4.2** Let $f$ be a continuous function on $\mathbb{R}^n$. We call a set of non-negative even functions $\{p_1, \ldots, p_n\}$ joint modulus of continuity of $f$ if

(i) For each $1 \leq k \leq n$, $p_k(0) = 0$, and $p_k$ is non-decreasing and continuous.
(ii) For every pair \((s, t)\) in \(\mathbb{R}^{2n}\), the following inequality holds

\[ |\square_s^n f(t)| \leq \prod_{k=1}^n p_k(|t_k - s_k|). \]

In view of Theorem 2.3 and Theorem 3.1, the continuity of sample paths is governed by the joint modulus of continuity of \(\square_{(y, y)}^{2n}Q(x, x)\). Such modulus of continuity always exists. For instance, we can define a joint modulus of continuity for \(\square_{(y, y)}^{2n}Q(x, x)\) as follows. We set

\[ p_1(u) = \sup_{x, y \in [0, 1]^n : |x - y| \leq u} \left[ \square_{(y, y)}^{2n}Q(x, x) \right]^{\frac{1}{2}}. \]

Given \(p_1, \ldots, p_{k-1}\), define

\[ p_k(u) = \sup_{x, y \in [0, 1]^n : |x - y| \leq u} \left[ \frac{\square_{(y, y)}^{2n}Q(x, x)}{\prod_{j=1}^{k-1} p_j(|x_j - y_j|)} \right]^{\frac{1}{2}}, \]

in which we have adopted the convention \(0/0 = 0\). It follows immediately that \(p_k\)'s are non-decreasing and continuous. Furthermore, we have \(p_k(0) = 0\) and

\[ \square_{(y, y)}^{2n}Q(x, x) \leq \prod_{k=1}^n p_k^2(|x_k - y_k|). \quad (4.6) \]

Namely, \(\{p_1, p_2, p_3, \ldots, p_n\}\) is a modulus of continuity for \(\square_{(y, y)}^{2n}Q(x, x)\). We also call \(\{p_1, \ldots, p_n\}\) a modulus of continuity for \(\square_{(y, y)}^{2n}Q(x, x)\).

In the following theorem, we give a sufficient condition for almost sure joint continuity of a Gaussian random field.

**Theorem 4.3** Let \(W\) be a continuous centered Gaussian random field with covariance function given by \((4.2)\), and \(p_k (1 \leq k \leq n)\) be a modulus of continuity for \(\square_{(y, y)}^{2n}Q(x, x)\), namely the inequality \((4.6)\) is satisfied. Suppose that

\[ \sum_{k=1}^n \int_0^1 \left( \log \frac{1}{u} \right)^{\frac{1}{2}} dp_k(u) < \infty. \quad (4.7) \]

Then, with probability one \(W\) has joint continuous sample path. Furthermore, we have almost surely that for any \(\delta > 0\),

\[ \sup_{0 \leq |x - y| \leq \delta} \frac{|\square_y^n W(x)|}{h(x, y)} \leq c_{n, \delta}, \quad (4.8) \]

where \(h(x, y)\) is the function

\[ h(x, y) = \prod_{k=1}^n p_k(|x_k - y_k|) \left[ \log \prod_{j=1}^n \frac{1}{|x_j - y_j|} \right]^{\frac{1}{2}}. \quad (4.9) \]
$c_{n, \delta}$ is a random variable, depending on $n$ and $\delta$, and $\mathbb{E}e^{c_{n, \delta}^2} < \infty$. Moreover, there exists a constant $\kappa_n$ such that

$$\lim_{\delta \downarrow 0} \sup_{|x-y| \leq \delta} \frac{\Box_y^n W(x)}{h(x, y)} \leq \kappa_n$$

(4.10) almost surely.

Proof. We set $\Psi(x) = e^{x^2/4}$ and

$$B(\omega) = \int_{[0,1]^{2n}} \exp \left[ \frac{\Box_y^n W(x)}{4 \prod_{k=1}^n p_k(|x_k - y_k|)} \right] \, dx \, dy.$$  

Theorem 2.3 gives

$$|\Box_y^n W(x)| \leq 2 \cdot 8^n \int_0^{|x_1-y_1|} \ldots \int_0^{|x_n-y_n|} \left( \log \frac{1}{u_1^2 \ldots u_n^2} \right)^{\frac{1}{2}} \, dp_1(u_1) \ldots dp_n(u_n)$$

$$+ \sqrt{\log(4^n B(\omega))} \prod_{k=1}^n p_k(|x_k - y_k|)$$

(4.11)

for $\omega$ such that $B(\omega)$ is finite.

It is elementary to see that

$$\lim_{|x-y| \to 0} \frac{1}{h(x, y)} \int_0^{|x_1-y_1|} \ldots \int_0^{|x_n-y_n|} \left( \log \frac{1}{u_1^2 \ldots u_n^2} \right)^{\frac{1}{2}} \, dp_1(u_1) \ldots dp_n(u_n) = c_n$$

for some constant $c_n$ and

$$\lim_{|x-y| \to 0} \frac{\prod_{k=1}^n p_k(|x_k - y_k|)}{h(x, y)} = 0.$$  

From (4.11) and the two facts above, the estimates (4.8) and (4.10) follow easily.

To see (4.11) indeed holds for almost every $\omega$ (and hence (4.8) and (4.10)), it is sufficient to show that $B$ has finite expectation. We notice that the random variable

$$N = \frac{\Box_y^n W(x)}{\prod_{k=1}^n p_k(|x_k - y_k|)}$$

is Gaussian, has mean zero and variance less than or equal to one. Thus, an application of Stirling’s formula gives

$$\mathbb{E} \exp \left( \frac{N^2}{4} \right) = \sum_{k=0}^{\infty} \frac{\mathbb{E}N^{2k}}{4^k k!} \leq 1 + \sum_{k=1}^{\infty} \frac{(2k)!}{8^k (k!)^2} (\mathbb{E}N^2)^k \leq 1 + \frac{1}{2} \sum_{k=1}^{\infty} 8^{-k} = \frac{15}{14}.$$  

Hence

$$\mathbb{E}B = \int_{[0,1]^{2n}} \mathbb{E} \exp \left( \frac{N^2}{4} \right) \, dx \, dy \leq \frac{15}{14}$$

and the proof is complete. •
Remark 4.4 Suppose that \( W(x) = 0 \) whenever \( x \) has at least one zero coordinate. Let \( \sigma(x, y) \) be the function defined below

\[
\sigma(x, y) = \sum_{k=1}^{n} \left( \prod_{j \neq k} p_j(|z_{j,k}|) \right) \left| \log \prod_{j \neq k} |z_{j,k}| \right|^{1/2} p_k(|x_k - y_k|) \log |x_k - y_k|^{1/2},
\]

(4.12)

where \( z_{j,k} = x_j \) if \( j < k \) and \( z_{j,k} = y_j \) if \( j > k \). The inequality (4.10) implies the following estimate which usually appears in literature

\[
\lim_{\delta \downarrow 0} \sup_{|x| \leq 1, |y| \leq 1, |x-y| \leq \delta} \frac{|W(x) - W(y)|}{\sigma(x, y)} \leq \kappa_n.
\]

(4.13)

Indeed, fix \( \omega \) such that (4.10) holds and \( \delta \) sufficiently small, for every \( x, y \) in \([0, \delta]^n\), with \( x \) and \((0, 0, \ldots, 0, y_n)\), the estimate (4.8) gives the following estimate for the increment along an edge of the \( n \)-dimensional rectangle \([x_1, y_1] \times \cdots \times [x_n, y_n]\)

\[
|W(x_1, \ldots, x_n) - W(x_1, \ldots, x_{n-1}, y_n)|
\leq c_{n,\delta} \left( \prod_{k=1}^{n-1} p_k(|x_k|) \right) \left| \log \prod_{k=1}^{n-1} |x_k| \right|^{1/2} p_n(|x_n - y_n|) \log |x_n - y_n|^{1/2}.
\]

(4.14)

Similarly, we can obtain analogue estimates along any edge of the \( n \)-dimensional rectangle \([x_1, y_1] \times \cdots \times [x_n, y_n]\). The increment along the diagonal is majorized by the total increments along all the edges connecting \( x \) and \( y \). Hence, this argument yields the following estimate

\[
|W(x) - W(y)| \leq c_{n,\delta} \sigma(x, y)
\]

(4.13)

which implies (4.13).

As an application of the above theorem, we obtain joint continuity for sample paths of fractional Brownian field, as mentioned in (1.6).

Corollary 4.5 Let \( W^H \) be a fractional Brownian field on \( \mathbb{R}^n \) with Hurst parameter \( H = (H_1, \ldots, H_n) \). Then, for any \( \delta > 0 \) the following inequality holds almost surely

\[
\sup_{|x-y| \leq \delta} \frac{\left| \square_{\delta} W^H(x) \right|}{h^H(x, y)} \leq c_{n,\delta}
\]

(4.15)

where \( h^H(x, y) \) is the function

\[
h^H(x, y) = \prod_{k=1}^{n} |x_k - y_k|^{H_k} \left| \log \prod_{j=1}^{n} |x_j - y_j| \right|^{1/2}
\]

(4.16)

for some finite random variable \( c_{n,\delta} \) depending on \( n \) and \( \delta \) such that \( \mathbb{E} c_{n,\delta}^2 < \infty \). Moreover, there is a constant \( \kappa_n \) such that

\[
\lim_{\delta \downarrow 0} \sup_{|x-y| \leq \delta} \frac{\left| \square_{\delta} W^H(x) \right|}{h^H(x, y)} \leq \kappa_n
\]

(4.17)

almost surely.
Proof The covariance function of a fractional Brownian field is given by

\[
\mathbb{E}[W^H(x)W^H(y)] = \prod_{k=1}^{n} R_k(x_k, y_k),
\]

where

\[
R_k(s, t) = \frac{1}{2} \left( |s|^{2H_k} + |t|^{2H_k} - |s-t|^{2H_k} \right), \quad \forall s, t \in \mathbb{R}.
\]

By Proposition 4.1, we obtain the second moment for \(\Box_y^n W^H(x)\)

\[
\mathbb{E}[\Box_y^n W^H(x)]^2 = \prod_{k=1}^{n} |x_k - y_k|^{2H_k}.
\]

This means that \(p_i(u) = u^{H_i}, i = 1, 2, \ldots, n\) are the modulus of continuity of \(\Box_{(y,y)}^n Q(x, x)\).

Now the corollary is a direct consequence of Theorem 4.3.

Remark 4.6 1. In case of \(n\)-parameter Wiener process, that is when \(H_1 = \cdots = H_n = 1/2\), the above corollary is comparable to a result of S. Orey and W. E. Pruitt in [9, Theorem 2.1].

2. As in Remark 4.4, let \(\sigma^H(x, y)\) be the function

\[
\sigma^H(x, y) = \sum_{k=1}^{n} \prod_{j \neq k} |z_{j,k}|^{H_j} \log \prod_{j \neq k} |z_{j,k}| |x_k - y_k|^{H_k} \log (|x_k - y_k|)^{1/2},
\]

where \(z_{j,k} = x_j\) if \(j < k\) and \(z_{j,k} = y_j\) if \(j > k\). Then the previous result implies the following

\[
\lim_{\delta \downarrow 0} \sup_{\|x\| \leq 1, \|y\| \leq 1} \frac{|W^H(x) - W^H(y)|}{\sigma^H(x, y)} \leq c_n
\]

where \(c_n\) is some constant.

5 Stochastic heat equations with additive space time white noise

In this section let us consider the following one dimensional stochastic differential equation

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + W & 0 < t \leq T, \quad y \in \mathbb{R} \\
u(0, y) = 0 & y \in \mathbb{R},
\end{cases}
\]

where \(\Delta u = \frac{\partial^2}{\partial y^2} u\), \(W\) is space time standard Brownian sheet, and \(\dot{W} = \frac{\partial^2}{\partial t \partial y} W\). Let \(p_t(y) = \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t}\). Then the (mild) solution of the above equation is given by

\[
u(t, y) = \int_0^t \int_{\mathbb{R}} p_{t-s}(y-z) W(dr, dz),
\]

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where the above integral is the usual (Itô) stochastic integral (however, the integrand is simple. It is a deterministic function). The solution \( u(t, y) \) is a Gaussian random field. It is known that \( u(t, y) \) is Hölder continuous of exponent \( \frac{1}{4} \) for time parameter and \( \frac{1}{2} \) for space parameter. Namely, for any \( \alpha < 1/4 \) and any \( \beta < 1/2 \), there is a random constant \( C_{\alpha, \beta} \) such that
\[
|u(t, y) - u(s, x)| \leq C_{\alpha, \beta} \left( |t - s|^\alpha + |x - y|^\beta \right).
\] (5.2)

We are interested in the joint Hölder continuity of the solution \( u(t, y) \). We need the following simple technical lemma.

**Lemma 5.1** Let \( a, b, \delta \) be some positive numbers, where \( a < b \), and let \( I, J \) be the integrations
\[
I = \int_a^b \frac{1}{\sqrt{r}} \left( 1 - e^{-\frac{4\delta^2}{4}} \right) dr,
\]
\[
J = \int_0^a \frac{1}{\sqrt{r}} \left( 1 - e^{-\frac{4\delta^2}{4}} \right) dr.
\]
Then for every \( \alpha \in [0, 1/2] \),
\[
2(\sqrt{b} - \sqrt{a}) \left( 1 - e^{-\frac{4\delta^2}{4}} \right) \leq I \leq 2(\sqrt{b} - \sqrt{a}) \left( 1 - e^{-\frac{4\delta^2}{4}} \right)
\]
and
\[
J \leq c_\alpha \delta^{2\alpha} a^{1/2 - \alpha}.
\]

**Proof** On the interval \( a \leq r \leq b \), we have \( 1 - e^{-\frac{4\delta^2}{4}} \leq 1 - e^{-\frac{4\delta^2}{4}} \leq 1 - e^{-\frac{4\delta^2}{4}} \). The estimate for \( I \) is then a straightforward consequence. To estimate \( J \), we first use integration by part to obtain
\[
J = 2\sqrt{\pi} \left( 1 - e^{-\frac{4\delta^2}{4}} \right) r_{=\alpha} + \int_0^a \left( \frac{d}{dr} e^{-\frac{4\delta^2}{4}} \right) 2\sqrt{\pi} dr
\]
\[
= 2\sqrt{\alpha} \left( 1 - e^{-\frac{4\delta^2}{4}} \right) + \delta^2 \int_0^a e^{-\frac{4\delta^2}{4}} r^{-3/2} dr.
\]
By a change of variable \( x = \frac{\delta}{\sqrt{2\pi}} \), we see that
\[
J = 2\sqrt{\alpha} \left( 1 - e^{-\frac{4\delta^2}{4}} \right) + 2\sqrt{2}\delta \int_0^{\sqrt{\pi}} e^{-x^2} dx.
\]
If \( \frac{\delta}{\sqrt{2\pi}} \geq 1 \), since \( \lim_{t \to \infty} \frac{\int_0^\infty e^{-x^2} dx}{(1 - e^{-t})^{1/2}} = 0 \), \( J \) is majorized by
\[
J \leq c\sqrt{\alpha} \left( 1 - e^{-\frac{4\delta^2}{4}} \right).
\]
If \( \frac{\delta}{\sqrt{2\pi}} \leq 1 \), the integration \( \int_0^\infty e^{-x^2} dx \) is bounded by \( \sqrt{\pi}/2 \), thus \( J \) is majorized by
\[
J \leq 2\sqrt{\alpha} \left( 1 - e^{-\frac{4\delta^2}{4}} \right) + c\delta.
\]
Therefore, for any \( 0 \leq \alpha \leq 1/2 \), employing the elementary inequality \( 1 - e^{-x} \leq c_\alpha x^\alpha \), we obtain
\[
J \leq c_\alpha \delta^{2\alpha} a^{1/2 - \alpha}
\]
and the lemma follows. \(\blacksquare\)
Theorem 5.2 For every $\alpha$ in $[0, 1/4]$ and $\delta > 0$, there is a finite random variable $c_{\alpha, \delta}$ depending on $\alpha$ and $\delta$ such that the following estimate holds almost surely

$$
\sup_{|t-s| \leq \delta, |x-y| \leq \delta} \frac{|u(t, y) - u(t, x) - u(s, y) + u(s, x)|}{|t-s|^\frac{1}{4} - \alpha |x-y|^{2\alpha} \log (|t-s||x-y|)} \leq c_{\alpha, \delta}.
$$

Moreover, for some absolute constant $\kappa_{\alpha}$, the following inequality holds

$$
\lim_{\delta \to 0} \sup_{|t-s| \leq \delta, |x-y| \leq \delta} \frac{|u(t, y) - u(t, x) - u(s, y) + u(s, x)|}{|t-s|^\frac{1}{4} - \alpha |x-y|^{2\alpha} \log (|t-s||x-y|)} \leq \kappa_{\alpha}.
$$

Proof $u(t, y)$ is a mean zero Gaussian field. The covariance of $u(t, y)$ and $u(s, x)$ is given by

$$
\mathbb{E}[u(s, x)u(t, y)] = \int_{\mathbb{R}^2} \chi_{[0,s]}(r)\chi_{[0,t]}(r)p_{s-r}(x-z)p_{t-r}(y-z)drdz
= \int_{\mathbb{R}^2} f(s, x)f(t, y)drdz,
$$

where $f(s, x) = \chi_{[0,s]}(r)p_{s-r}(x-z)$.

We calculate the second moment of $\Box_{(s,x)}^2 u(t, y)$ as follows

$$
\mathbb{E}
\left[\Box_{(s,x)}^2 u(t, y)\right]^2 = \mathbb{E}\Box_{(s,x)}^2 u(t, y)\Box_{(s,x)}^2 u(t, y)
= \mathbb{E}\Box_{(s,x)}^4 u(t, y)u(t, y)
= \Box_{(s,x)}^4 \mathbb{E}[u(t, y)u(t, y)]
= \Box_{(s,x)}^4 \int_{\mathbb{R}^2} f(t, y)^2 drdz
= \int_{\mathbb{R}^2} \Box_{(s,x)}^4 \Box_{(s,x)}^2 f(t, y) drdz
= \int_{\mathbb{R}^2} \Box_{(s,x)}^2 f(t, y) \left[\Box_{(s,x)}^2 f(t, y)\right] drdz
= \int_{\mathbb{R}^2} \left[\Box_{(s,x)}^2 f(t, y)\right]^2 drdz
$$

where

$$
\left[\Box_{(s,x)}^2 f(t, y)\right]^2 = [f(s, x) - f(t, x) - f(s, y) + f(t, y)]^2
= f(s, x)^2 + f(t, x)^2 + f(s, y)^2 + f(t, y)^2
- 2f(s, x)f(t, x) - 2f(s, y)f(s, y) + 2f(s, x)f(t, y)
+ 2f(t, x)f(s, y) - 2f(t, x)f(t, y) - 2f(s, y)f(t, y).
$$

Taking the integration with respect to $z$ and using the following identity

$$
\int_{\mathbb{R}} p_a(z - x)p_b(z - y)dz = p_{a+b}(x - y)
$$

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we obtain
\[
\mathbb{E} \left[ \mathbb{D}_{(s,x)}^2 u(t, y) \right]^2 = \int_\mathbb{R} \left[ 2\chi_{[0,s]}(r)p_{2s-2r}(0) + 2\chi_{[0,t]}(r)p_{2t-2r}(0) \right] dr \\
+ \int_\mathbb{R} \left[ -2\chi_{[0,s\wedge t]}p_{s+t-2r}(0) - 2\chi_{[0,s\wedge t]}p_{2s-2r}(x-y) + 2\chi_{[0,s\wedge t]}p_{s+t-2r}(x-y) \right] dr \\
+ \int_\mathbb{R} \left[ 2\chi_{[0,s\wedge t]}p_{s+t-2r}(x-y) - 2\chi_{[0,s\wedge t]}p_{2t-2r}(x-y) - 2\chi_{[0,s\wedge t]}p_{s+t-2r}(0) \right] dr \\
= 2 \int_0^s [p_{2s-2r}(0) - p_{2s-2r}(x-y)] dr + 2 \int_0^t [p_{2t-2r}(0) - p_{2t-2r}(x-y)] dr \\
-4 \int_0^{s\wedge t} [p_{s+t-2r}(0) - p_{s+t-2r}(x-y)] dr.
\]

By change of variables \(u = 2s - 2r, v = 2t - 2r\) and \(w = s + t - 2r\) in the above corresponding integrals respectively and noticing that \(s + t - 2(s \wedge t) = |t - s|\), we get
\[
\mathbb{E} \left[ \mathbb{D}_{(s,x)}^2 u(t, y) \right]^2 = \int_0^u [p_u(0) - p_u(x-y)] du + \int_0^v [p_v(0) - p_v(x-y)] dv \\
-2 \int_0^{s\wedge t} [p_w(0) - p_w(x-y)] dw \\
= \left( \int_{s+t}^{2(s\vee t)} - \int_{2(s\wedge t)}^{s+t} +2 \int_0^{s\wedge t} \right) [p_r(0) - p_r(x-y)] dr.
\]

By Lemma 5.1, we see that
\[
\left( \int_{s+t}^{2(s\vee t)} - \int_{2(s\wedge t)}^{s+t} \right) [p_r(0) - p_r(x-y)] dr \\
\leq \frac{1}{\sqrt{2\pi}} \left( 1 - e^{-\frac{(x-y)^2}{2|t-s|}} \right) \left( \sqrt{2s} + \sqrt{2t} - \sqrt{2s+2t} \right) \leq 0.
\]

and
\[
\int_0^{s\wedge t} [p_r(0) - p_r(x-y)] dr \leq c_\alpha |x-y|^{2\alpha} |s-t|^\frac{1}{2-\alpha}
\]
for every \(\alpha\) in \([0, 1/2]\). Thus
\[
\mathbb{E} \left[ \mathbb{D}_{(s,x)}^2 u(t, y) \right]^2 \leq 2 \int_0^{s\wedge t} [p_r(0) - p_r(x-y)] dr \leq c_\alpha |x-y|^{2\alpha} |s-t|^\frac{1}{2-\alpha}. \tag{5.5}
\]

An application of Theorem 4.3 immediately gives the desired result. ■

**Remark 5.3** Using the method in Remark 4.4, the above result implies there is a constant \(c\) such that
\[
\lim_{\delta \downarrow 0} \sup_{|t| \leq 1, |s| \leq 1, |t-s| \leq \delta} \frac{|u(s, x) - u(t, y)|}{|s-t|^{\frac{1}{4}} \sqrt{\log \frac{1}{|x-s-t|}} + |x-y|^{\frac{1}{2}} \sqrt{\log \frac{1}{|x-y| t}}} \leq c. \tag{5.6}
\]
which is sharper than (5.2).

Remark 5.4 After the completion of this paper it is communicated to us that recently, M. Meerschaert, W. Wang and Y. Xiao obtained (see [7], Theorem 4.1 and see also Theorem 6.1 for fractional multiparameter Brownian motion) the following result. Let \( W \) be an \( n \)-parameter Gaussian process with mean zero and

\[
\rho^2(x, y) = \mathbb{E} \left[ |W(x) - W(y)|^2 \right].
\]

Assume there are positive constants \( H_1, \cdots, H_n \in (0, 1) \) and positive constants \( C_1 < C_2 \) such that

\[
C_1 \left( \sum_{j=1}^{n} |x_j - y_j|^{H_j} \right)^2 \leq \rho^2(x, y) \leq C_2 \left( \sum_{j=1}^{n} |x_j - y_j|^{H_j} \right)^2.
\]

(5.7)

Assume further that \( W \) satisfies some conditions that we don’t repeat here and refer interested readers to [7]. Let \( I = [a, 1] \) where \( a \in (0, 1) \) is a constant. Then

\[
\lim_{\delta \downarrow 0} \sup_{x, y \in I, |x - y| \leq \delta} \frac{|W(x) - W(y)|}{\beta(x, y)} = \kappa
\]

for some positive constant \( \kappa \), where

\[
\beta(x, y) = \rho(x, y) \sqrt{\log(1 + \rho(x, y)^{-1})}.
\]

It is obvious that as \( |x - y| \to 0 \), we have

\[
\beta(x, y) \approx \rho(x, y) \sqrt{\log(|x - y|)}.
\]

Moreover, given (5.7), \( \beta(x, y) \) has the same order as \( \sigma^{11}(x, y) \) in (4.18) when \( x, y \) are bounded and \( |x - y| \to 0 \). Thus the identity (5.8) says that our inequality (4.13) is sharp. Besides, (4.13) does not require \( x, y \) to be bounded away from 0. We conjecture that the inequality (5.3) is also sharp. Moreover, it is interesting to know if an analogous identity to (5.8) holds for the increments over the rectangles of type (5.3) or not. Namely, for any \( \alpha \in [0, 1/4] \), is there a positive constant \( \kappa_\alpha \) such that

\[
\lim_{\delta \downarrow 0} \sup_{|t - s| \leq \delta, |x - y| \leq \delta} \frac{|u(t, y) - u(t, x) - u(s, y) + u(s, x)|}{|t - s|^{\frac{1}{4} - \alpha} |x - y|^{2\alpha} \log(|t - s||x - y|)}} = \kappa_\alpha?
\]

(5.9)

As it is well-known the Garsia-Rodemich-Rumsey inequality gives only the upper bound. It has not been powerful to obtain the lower bound. Therefore, one has to attack the above problem (5.9) using other means. As a confirmative example, we remark that in the case of Brownian sheet on \( \mathbb{R}^2 \), G. J. Zimmerman showed in [12] that

\[
\lim_{\delta_1, \delta_2 \downarrow 0} \sup_{|x_1 - y_1| = \delta_1, 0 < |x_2 - y_2| = \delta_2} \frac{\|\nabla^2 W(x)\|^2}{|2\delta_1 \delta_2 \log(1/(\delta_1 \delta_2))|^{\frac{2}{4}} = 1.}
\]

(5.10)
Remark 5.5 If one prefers to write one inequality rather than arbitrary $\alpha$ in Theorem 5.2, one can write the inequality (5.5) as

$$E \left[ \Box_{t,s,x}^2 u(t,y) \right]^2 \leq 2 \int_0^{||s-t||} \frac{1}{\sqrt{2\pi r}} \left[ 1 - e^{-\frac{||x-y||^2}{4r}} \right] dr$$

$$\leq \sqrt{\frac{2}{\pi}} ||x-y|| \int_0^{||t-s||} r^{-1/2}(1-e^{-1r}) dr$$

$$= ||x-y|| \rho \left( \frac{||t-s||}{||x-y||^2} \right),$$

where $\rho(u) = \sqrt{\frac{2}{\pi}} \int_0^u r^{-1/2}(1-e^{-1r}) dr$ which is of the order $\sqrt{u}$ as $u \to 0$ and bounded as $u \to \infty$. It is obvious that $||x-y|| \rho \left( \frac{||t-s||}{||x-y||^2} \right)$ goes to 0 if one of $|t-s|$ and $|x-y|$ goes to 0.

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References

[1] Ayache, A.; Leger, S. and Pontier, M. Drap brownien fractionnaire. Potential Anal. 17 (2002), no. 1, 31-43.

[2] Ayache, A. and Xiao, Y. Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets. J. Fourier Anal. Appl. 11 (2005), no. 4, 407-439.

[3] Fernique, X. Continuité des processus Gaussiens. C. R. Acad. Sci. Paris 258 (1964), 6058-6060.

[4] Fernique, X. Sérries de distributions aléatoires indépendantes. C. R. Acad. Sci. Paris Sér. A-B 263 (1966), A674-A677.

[5] Fernique, X. Fonctions aléatoires gaussiennes, vecteurs aléatoires gaussiens. Université de Montréal, Centre de Recherches Mathématiques, Montreal, QC, 1997.

[6] Garsia, A. M.; Rodemich, E.; and Rumsey, H., Jr. A real variable lemma and the continuity of paths of some Gaussian processes. Indiana Univ. Math. J. 20 (1970/1971), 565-578.

[7] Meerschaert, M. M.; Wang, W.; Xiao, Y. Fernique-type inequalities and moduli of continuity for anisotropic Gaussian random fields. Trans. Amer. Math. Soc. 365 (2013), no. 2, 1081-1107.

[8] Marcus, M. B.; Shepp, L. A. Continuity of Gaussian processes. Trans. Amer. Math. Soc. 151 (1970), 377-391.

[9] Orey, Steven; Pruitt, William E. Sample functions of the N-parameter Wiener process. Ann. Probability 1 (1973), no. 1, 138-163.
[10] Ral’chenko, K. V. The two-parameter Garsia-Rodemich-Rumsey inequality and its application to fractional Brownian fields. Theory Probab. Math. Statist. No. 75 (2007), 167-178.

[11] Xiao, Y. Sample path properties of anisotropic Gaussian random fields. A mini-course on stochastic partial differential equations, 145-212, Lecture Notes in Math., 1962, Springer, Berlin, 2009.

[12] Zimmerman, Grenith J. Some sample function properties of the two-parameter Gaussian process. Ann. Math. Statist. 43 (1972), 1235-1246.