PROOFS OF SOME RAMANUJAN SERIES FOR $1/\pi$
USING A ZEILBERGER’S PROGRAM

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Abstract. We show with some examples how to prove some Ramanujan-type series for $1/\pi$ in an elementary way by using terminating identities.

Introduction. Up till now, we know how to prove 11 Ramanujan-type series for $1/\pi$ by using the WZ (Wilf and Zeilberger) method [6]. Here we will show how to prove some more using a related Zeilberger’s algorithm.

1. The WZ Algorithm as a Black Box

Let $G(n, k)$ be hypergeometric in $n$ and $k$, that is such that $G(n + 1, k)/G(n, k)$ and $G(n, k + 1)/G(n, k)$ are rational functions. Then, we can use the Zeilberger’s Maple package SumTools[Hypergeometric]). The output of Zeilberger($G(n,k),k,n,K$)[1] is an operator $O(K)$ of the following form

$$O(K) = P_0(k) + P_1(k) K + P_2(k) K^2 + \cdots + P_m(k) K^m,$$

where $P_0(k), P_1(k), \ldots, P_m(k)$ are polynomials of $k$, and $K$ is an operator which increases $k$ in 1 unity, that is $KG(n, k) = G(n, k + 1)$. The output of Zeilberger($G(n,k),k,n,K$) [2] gives a function $F(n, k)$ such that

$$O(K)G(n, k) = F(n + 1, k) - F(n, k).$$

If we sum for $n \geq 0$, we get

$$O(K)r_k = \lim_{n \to \infty} F(n, k) - F(0, k), \quad r_k = \sum_{n=0}^{\infty} G(n, k).$$

If the above limit and $F(0, k)$ are equal to zero, we have

$$O(K)r_k = 0,$$

which is a recurrence of order $m$.

Example 1. Prove that:

$$\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \left( \frac{1}{4} \right)_n \left( \frac{3}{4} \right)_n (-1)^n \left( \frac{16}{63} \right)^n (65n + 8) = \frac{9\sqrt{7}}{\pi}. \quad (1)$$

We have not found a WZ-pair which proves this Ramanujan series. However our proof is closely related to the WZ-method.

Proof. Let

$$A(n, k) = 3 \left( \frac{64}{63} \right)^k \left( -\frac{1}{2} \right)_n \left( \frac{1}{4} \right)_n \frac{\left( \frac{1}{2} \right)_n}{\left( \frac{1}{2} - k \right)_n \left( \frac{1}{2} + k \right)_n} \left( \frac{1}{64} \right)^n (42n + 5),$$

$$B(n, k) = \left( -\frac{1}{2} \right)_n \left( \frac{1}{2} - \frac{1}{2} \right)_n \left( \frac{1}{2} + \frac{1}{2} \right)_n (-1)^n \left( \frac{16}{63} \right)^n (130n - 2k + 15),$$

and
We define the sequences
\[ r_k = \sum_{n=0}^{\infty} A(n, k), \quad s_k = \sum_{n=0}^{\infty} B(n, k). \]

Then we use a Zeilberger’s program which finds recurrences. Writing in a Maple session

```maple
with(SumTools[Hypergeometric]);
s := subs(n=0, Zeilberger(A(n,k),k,n,K)[2]);
t := subs(n=0, Zeilberger(B(n,k),k,n,K)[2]);
```

we see that \( s = t = 0 \). Then, writing

```maple
u := Zeilberger(A(n,k),k,n,K)[1];
v := Zeilberger(B(n,k),k,n,K)[1];
```

and executing it, we see that \( r_k \) and \( s_k \) satisfy a common recurrence of order 3. Then observe that the sums which define \( r_k \) and \( s_k \) are finite because the terms with \( n > k \) are equal to zero due to presence of \((-k)_n\). By direct evaluation, we check that \( r_0 = s_0\), \( r_1 = s_1 \) and \( r_2 = s_2 \). Hence, as the three first terms are equal, all of them are. Let

\[
\begin{align*}
    r(k) &= 3 \left( \frac{64}{63} \right)^k \sum_{n=0}^{\infty} \frac{(-k)_n}{\left( \frac{1}{2} \right)_n} \frac{2n}{3} \left( \frac{1}{64} \right)^n (42n + 5), \\
    s(k) &= \sum_{n=0}^{\infty} \frac{(-k)_n}{\left( \frac{1}{2} \right)_n} \frac{2n}{3} \left( \frac{1}{64} \right)^n (130n - 2k + 15),
\end{align*}
\]

Applying Carlson’s theorem [1, p. 39], we can deduce that for all complex values of \( k \) we have \( r(k) = s(k) \). Finally replacing \( k = -1/2 \), we get

\[
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n}{\left( \frac{1}{3} \right)_n} \frac{2n}{3} (-1)^n \left( \frac{16}{63} \right)^n (130n + 16) = \frac{9\sqrt{7}}{16} \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n}{\left( \frac{1}{3} \right)_n} \left( \frac{1}{64} \right)^n (42n + 5)
\]

But in 2002, we used the WZ-method to prove

\[
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n}{\left( \frac{1}{3} \right)_n} \left( \frac{1}{64} \right)^n (42n + 5) = \frac{16}{\pi},
\]

in an elementary way. Hence we are done. \( \square \)

**Example 2.** Prove that:

\[
\sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n}{\left( \frac{1}{3} \right)_n} \frac{2n}{3} \left( \frac{2}{11} \right)^n (126n + 10) = \frac{11\sqrt{33}}{2\pi}.
\] (2)

**Proof.** It is completely similar to that in our first example: Use Zeilberger to prove the identity

\[
11 \left( \frac{32}{33} \right)^k \sum_{n=0}^{\infty} \frac{(-3k)_n}{\left( \frac{2}{3} - 2k \right)_n} \frac{(-1)^n}{\left( \frac{1}{3} - 4k \right)_n} \left( \frac{2}{11} \right)^n (6n + 1)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-k)_n}{\left( \frac{2}{3} - k \right)_n} \frac{(-1)^n}{\left( \frac{2}{3} - 2k \right)_n} \left( \frac{2}{11} \right)^n (126n + 6k + 11),
\]

and take \( k = -1/6 \). \( \square \)
Example 3. Prove that:
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{4}{3}\right)_n \left(\frac{5}{3}\right)_n \left(\frac{8}{3}\right)_n}{(1)^3_n} \left(-\frac{4}{5}\right)^{3n} (63n + 8) = \frac{5\sqrt{15}}{\pi}.
\]

Proof. As in the preceding examples, first use Zeilberger to show that
\[
5 \sum_{n=0}^{\infty} (-3k)_n \left(\frac{3}{4} + k\right)_n \left(\frac{3}{4} - k\right)_n \frac{1}{64} (42n + 5) = (15)^{3k} \sum_{n=0}^{\infty} (-k)_n \left(\frac{1}{2} - k\right)_n \left(\frac{3}{4} - 2k\right)_n (1) - \frac{64}{125} (252n - 42k + 25).
\]
Then take \(k = -1/6\).

Example 4. With Zeilberger, we can also prove the following general identity:
\[
\sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{2}\right)^2}{(\frac{1}{2} - k)^2_n} z^n = (1 - z)^k \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{2} - k\right)_n \left(\frac{1}{2} - k\right)_n}{(\frac{1}{2} - k)^2_n} (\frac{-4z}{(1-z)^2})^n,
\]
which is a particular case of a multi-parameter formula due to Whipple [5]. Applying to it the operator \(5 + 42\theta\) at \(z = 1/64\), where \(\theta = zd/dz\) (Zudilin’s translation method [9]), we get an identity which we have proved directly in Example 1. In a similar way, If we apply the operator \(1 + 6\theta\) at \(z = -1/8\), we get an identity which we can reprove directly with Zeilberger. From this identity we immediately get an elementary proof of the formula
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{3}{3}\right)_n}{(1)^3_n} \left(\frac{32}{81}\right)^n (7n + 1) = \frac{9}{2\pi},
\]
as there is a WZ-method proof of the series in the other side of the identity [6].

Example 5. With Zeilberger, we can also prove the following general identity:
\[
\sum_{n=0}^{\infty} \frac{(-3k)_n \left(\frac{1}{3} - k\right)_n \left(\frac{1}{6} - 2k\right)_n}{(\frac{2}{3} - 2k)^2_n} z^n = 2 (4 - z)^k \sum_{n=0}^{\infty} \frac{(-k)_n \left(\frac{1}{3} - k\right)_n \left(\frac{3}{3} - 2k\right)_n}{(\frac{3}{3} - 2k)^2_n} (\frac{27z^2}{(4-z)^3})^n,
\]
which is a particular case of a multi-parameter formula due to Bailey [5]. Applying to it the operator \(1 + 6\theta\) at \(z = -1/8\), we get an identity which we have proved directly in Example 2. In a similar way, if we apply the operators: \(1 + 4\theta\) at \(z = -1/4\) and \(5 + 42\theta\) at \(z = 1/6\), we get identities which we can reprove directly with Zeilberger. From these identities we can derive respectively the formulas
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{(1)^3_n} \left(\frac{3}{5}\right)^{3n} (28n + 3) = \frac{5\sqrt{5}}{\pi},
\]
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{(1)^3_n} \left(\frac{4}{125}\right)^n (11n + 1) = \frac{5\sqrt{15}}{6\pi},
\]
\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{3}{2}\right)_n}{(1)^3_n} \left(\frac{4}{85}\right)^{3n} (133n + 8) = \frac{85\sqrt{255}}{54\pi},
\]
in an elementary way taking into account that we have shown that they are equal to series that we had already proved by the WZ-method [6].
Remarks.

(1) Our proofs are elementary (we do not use the modular theory).
(2) Formulas (6) and (7) are due to Ramanujan [8]. Formulas (1) and (4) are due to Berndt, Chan and Liaw [3]. Formulas (2) and (5) are due to the Borweins [4]. Formula (3) is due to Baruah and Berndt [2].

(3) For other elementary methods to prove these and other Ramanujan series see [9] and [7]. Those methods are based in the variable \( z \), while the proofs in this paper are based in the free parameter \( k \).

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