Lattice Boltzmann Magnetohydrodynamics

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Abstract

Lattice gas and lattice Boltzmann methods are recently developed numerical schemes for simulating a variety of physical systems. In this paper a new lattice Boltzmann model for modeling two-dimensional incompressible magnetohydrodynamics (MHD) is presented. The current model fully utilizes the flexibility of the lattice Boltzmann method in comparison with previous lattice gas and lattice Boltzmann MHD models, reducing the number of moving directions from 36 in other models to 12 only. To increase computational efficiency, a simple single time relaxation rule is used for collisions, which directly controls the transport coefficients. The bi-directional streaming process of the particle distribution function in this paper is similar to the original model [H. Chen and W. H. Matthaeus, Phys. Rev. Lett., 58, 1845(1987), S. Chen, H. Chen, D. Martínez and W. H. Matthaeus, Phys. Rev. Lett. 67, 3776 (1991)], but has been greatly simplified, affording simpler implementation of boundary conditions and increasing the feasibility of extension into a workable three-dimensional model. Analytical expressions for the transport coefficients are presented. Also, as example cases, numerical calculation for the Hartmann flow is
performed, showing a good agreement between the theoretical prediction and numerical simulation, and a sheet-pincho simulation is performed and compared with the results obtained with a spectral method.

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1 Introduction

Lattice gas automata (LGA) methods [1-5], based upon dynamics of cellular automata (CA) have attracted considerable attention during the last several years for both modeling physical phenomena and simulating linear and nonlinear partial differential equations. The lattice gas method is similar to traditional molecular dynamics in that a particle representation is employed for microscopic processes such as particle collision and streaming, but the dynamics is much simpler. The fundamental idea underlying the lattice gas approach is that simple microscopic dynamics may lead to macroscopic complexity.

The first lattice gas automata model was introduced by Frisch, Hasslacher and Pomeau (FHP) [2] in a hexagonal lattice for simulating two-dimensional (2D) hydrodynamics. The basic dynamical model comprises particles scattering and moving in discretized space and time. The approach of this system is inspired by classical statistical mechanics treatments of systems such as the Ising model and simple cellular automata models. Although intuitively appealing, the lattice automata method for fluids requires an averaging process in order to obtain the macroscopic fluid variables and their dynamics, due to the high levels of noise naturally present in the discretized particle representation. More recently there has been a trend towards using the lattice Boltzmann (LB) scheme instead of the lattice gas automata method. Unlike the lattice gas method in which one keeps track of each individual particle, in the lattice Boltzmann approach we are interested only in the one-point distribution function. While retaining the advantages associated with parallel implementation of lattice automata, the LB method is more efficient and accurate computationally, and essentially noise-free.

The history of lattice gas and lattice Boltzmann models for magnetohydrodynam-
ics (MHD) can be very briefly summarized as follows. The first attempt to model 2D MHD with a lattice gas automata scheme was carried out by Montgomery and Doolen \[5, 6\]. In their model the basic FHP model is extended to include additional degrees of freedom to account for the vector potential. To update the dynamics, some space average quantities need to be evaluated. By doing so, the essential feature of locality, that characterizes lattice gas systems, is lost. In addition, because of the vector potential representation of magnetic field the model is intrinsically two dimensional. It is noted that the recent lattice Boltzmann MHD model by Succi et al. \[7\] has a similar limitation. Another MHD lattice gas automata model with pure local operations was proposed by H. Chen et al. \[8, 9\]. To account for the Lorentz force, the model introduced a tensor (i.e., two-indexed) particle representation, and a random walk (or, “bi-directional streaming”) mechanism. For each one of these particles, there are two vectors attached, representing the momentum and magnetic field vectors. During the streaming procedure, particles move along one of the two possible vector directions with a probability deduced by requiring that MHD behavior is obtained macroscopically. Later, S. Chen et al. \[10\] extended the lattice gas automata model into a lattice Boltzmann model. The simulation of the two dimensional LGA and LB models for problems with free boundary and simple wall boundaries, including the two dimensional Hartmann flow and two dimensional magnetic reconnection, has achieved reasonable success \[10, 11\] in test problems.

The random walk MHD LGA model \[8, 9\], and its extension to the LB scheme \[10\], however, have two major problems. First, because of the random streaming, implementation of wall boundary conditions becomes complicated, requiring increased computational memory and computational work \[10\]. Second and most important, although both the lattice gas automata model \[8, 9\] and the lattice Boltzmann model \[10\] can be formally extended into three dimensional (3D) space, real 3D implementa-
ition is impractical due to memory requirements. In order to include a correct Lorentz force, for a lattice with $N$ moving directions, $N \times N$ particle states are needed. In 3D LGA and LB models, a face-centered-hyper-cubic (FCHC) lattice of $N = 24$ is usually employed. For this case, the random walk model needs at least 576 states, requiring about 1.2 gigabytes for a system of $64^3$. Thus, to some degree, its actual value as a computational tool is diminished because of the requirement of vast amounts of memory.

In the present paper we introduce a new lattice Boltzmann model for MHD that requires considerably less memory than previous models, while continuing to offer the computational efficiency of the LB approach. The new model is, in essence, a reduction of the previous MHD LBE model [10], including a smaller number of allowed states, while maintaining the symmetries and most of the desirable analytical and computational properties of the earlier method. The new model utilizes a “13 bit” representation on a 2D hexagonal lattice, in contrast to the earlier requirement of a 37-bit 2D model.

The paper is organized as follows: In Section II we will describe the model and show how ideal MHD is obtained in the fluid limit. In Section III the next order terms in the Chapman-Enskog expansion are included, adding dissipative effects (viscosity and resistivity) to the model. Following this, two sections are devoted to numerical tests. Section IV discusses the linear Hartmann flow problem as a first test of the model, for several Hartmann numbers $H$, which parameterizes the solutions. Section V describes use of the model to solve numerically for the evolution of the MHD sheet pinch configuration, i.e. the dynamics of a highly sheared planar magnetic field. The results obtained with the LB run are compared with those obtained with a spectral method run, using the same initial conditions. Discussions of the results and of the model are presented in Section VI. Finally, in Appendix A some useful
tensorial relations for the derivation of the model are presented for completeness; and in Appendix B we show how the same principle that is applied to obtain MHD behavior on the hexagonal lattice, can be extended to the square lattice.

2 Description of the Model

The model described here is inspired by the previous random walk MHD LGA (or, CA) model [8, 9] and is motivated by the need for an MHD LB scheme that is computationally feasible. This requires overcoming the two problems mentioned above. The model in this section, for simplicity, concerns two dimensional problems, but its principle also applies to three-dimensional models.

For our two-dimensional system, we use the standard hexagonal grid [2]. In the vicinity of given lattice point at \( \mathbf{x} \), six nearest neighbors are located at positions \( \mathbf{x} + \mathbf{e}_a \), with \( \mathbf{e}_a = (\cos(2\pi(a-1)/6), \sin(2\pi(a-1)/6)) \), \( a = 1, \ldots, 6 \). Instead of using \( 6 \times 6 \) particle states as in previous models, we only consider a subset of them. Each state is labeled by the pair of indices \( (a, \sigma) \). The positive particle distribution function is represented by \( f^\sigma_a \) with \( a = 1, \ldots, 6 \) and \( \sigma = 1, 2 \), where \( \sigma \) is defined relative to \( \mathbf{e}_a \) in the following manner; \( \sigma = 1 \) corresponds to the direction \( a + 1 \) (mod 6), and \( \sigma = 2 \) to \( a - 1 \) (mod 6).

The evolution of the system consists of a sequence of a streaming stage in which the distribution \( f^\sigma_a \) is propagated from each cell to its neighbour cells, followed by a collision stage in which the distribution at each cell is redistributed according to some conservation laws, as we will see below. The propagation part of the evolution for our particular model consists of partitioning the particle distribution into the two directions associated with the state \( (a, \sigma) \),

\[
 f^\sigma_a(\mathbf{x}, T) \rightarrow (1 - p)f^\sigma_a(\mathbf{x} + \mathbf{e}_a, T + 1) + pf^\sigma_a(\mathbf{x} + \mathbf{e}_\sigma, T + 1), \quad (1)
\]
where $T$ corresponds to the discrete microscopic time and $p$ is a given parameter which represents the fraction of the distribution function $f_a^\sigma$ that propagates along the $\sigma$ direction. Notice that this streaming procedure improves in two ways the random walk used in [8, 9, 10]. First, the motion of the “magnetic” portion of the distribution function $f_a^\sigma$ is always “forward” because the streaming parameter $p$ is greater than zero. (For the 36-bit model [10], the distribution function is represented by $f_a^b$, $(a = 1, \ldots, 6, b = 1, \ldots, 6)$ and a fragment of $f_a^b$ moves in the direction $\text{sign}(p_{ab})e_b$ while the remainder of the distribution moves in the direction $e_a$. Therefore, there are states $(a, b)$ for which the distribution streams in the direction $-e_b$). In addition, in the present model the angle between the two directions $e_a$ and $e_\sigma$ is $\pi/3$ for all states $(a, \sigma)$. These properties are important for imposing boundary conditions. In addition to these twelve states, the model includes a $13^{th}$ state denoted by $f_0$ that represents the fraction of the distribution function at a cell that does not advect at all. This “stopped” distribution introduces additional freedom in the model that allows to get rid of undesirable dependence of the pressure on the velocity that had plagued earlier LGA and LB models [12, 13].

Associated with each state $(a, \sigma)$ we define the local microscopic velocity $v_a^\sigma$, which is equal to the mean velocity at each cell, and the microscopic magnetic field $B_a^\sigma$

$$v_a^\sigma \equiv (1 - p)e_a + pe_\sigma,$$  \hspace{1cm} (2)

$$B_a^\sigma \equiv re_a + qe_\sigma.$$ \hspace{1cm} (3)

Although in principle the parameters $r$, $q$ and $p$ are unrelated, we will see later that a connection between $r$ and $q$ is set by the dynamical requirements. Notice that unlike the velocity, the microscopic magnetic field does not, on the surface, appear to play an active role in the evolution of the system. (However, later we will see that this is not the case; see the discussion after Eq. (19).)
The density $\rho$ is defined as the summation of all particle distribution functions,

$$\rho(x, T) \equiv f_0(x, T) + \sum_{a=1}^{6} \sum_{\sigma=1}^{2} f^\sigma_a(x, T).$$

(4)

In (4) the limits of summation on $a$ and $\sigma$ are given explicitly; subsequently we will sometimes suppress them for convenience when no ambiguity is introduced.

The macroscopic velocity and magnetic field are defined as averages of the microscopic fields $v^\sigma_a$ and $B^\sigma_a$,

$$\rho v \equiv \sum_{a, \sigma} v^\sigma_a f^\sigma_a,$$

(5)

$$\rho B \equiv \sum_{a, \sigma} B^\sigma_a f^\sigma_a.$$  

(6)

The kinetic equation obeyed by $f^\sigma_a$ can be written by combining the effect of streaming, represented by (1) with the collisional effects, denoted by the symbol $\Omega^\sigma_a$, arriving at,

$$f^\sigma_a(x, T) = (1 - p)[f^\sigma_a(x - e_a, T - 1) + \Omega^\sigma_a(x - e_a, T - 1)] +$$

$$p[f^\sigma_a(x - e_\sigma, T - 1) + \Omega^\sigma_a(x - e_\sigma, T - 1)].$$

(7)

In the present model, for simplicity, we assume that the collision operator $\Omega^\sigma_a$ has a single time relaxation form [10],

$$\Omega^\sigma_a = -(f^\sigma_a - f^{\sigma(eq)}_a)/\tau,$$

where $\tau$ is the relaxation time and $f^{\sigma(eq)}_a$ is the local equilibrium distribution function depending on the local particle density, velocity and magnetic field.

A crucial step in the development of a LB method is the selection of an appropriate single particle equilibrium distribution function, associated with vanishing of the collision operator. This equilibrium distribution function has to be consistent with
definitions (4), (5) and (6), and in addition has to give rise to the MHD equations. A suitable equilibrium distribution function fulfilling all these conditions is given by

\[ f_{a(eq)} = \left( \frac{\rho}{12} \right) \left\{ \frac{12}{\alpha + 12} + \frac{4}{C} \left[ \nu_a^\sigma \cdot \nu + \frac{(2 - p)^2}{3q^2} B_a^\sigma \cdot B \right. \
+ \frac{4(2p - 1)}{C} [(e_\sigma \cdot \nu)^2 - (e_\sigma \cdot B)^2] \
+ \frac{4(1 - p^2)}{C} [(e_a \cdot \nu)(e_\sigma \cdot \nu) - (e_a \cdot B)(e_\sigma \cdot B)] \
+ \frac{2(2 - p)}{3q} [(e_a \cdot \nu)(e_\sigma \cdot B) - (e_a \cdot B)(e_\sigma \cdot \nu)] \
- \left. \frac{(2p - 1)(2 - p)}{C} \nu^2 - \frac{p^2 - 4p + 1}{C} B^2 \right\}, \] (8)

and

\[ f_0^{(eq)} = \rho \left[ \frac{\alpha}{12 + \alpha} - \frac{2}{C} \nu^2 \right], \] (9)

where \( C = 2(p^2 - p + 1) \). Positivity of the distribution function is guaranteed if \( \nu \) and \( B \) are sufficiently small.

To derive the MHD fluid model, we next form the continuum kinetic equation by Taylor expanding (7) in the limit of low frequencies and long wavelengths [2]. The result to lowest order is

\[ \frac{\partial f_a^\sigma}{\partial t} + \nu_a^\sigma \cdot \nabla f_a^\sigma = \Omega_a^\sigma. \] (10)

Equations for the density, momentum transport and magnetic momentum transport can now be found by taking moments of (10):

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nu) = 0, \]

\[ \frac{\partial (\rho \nu)}{\partial t} + \nabla \cdot \Pi^{(0)} = 0, \] (11)

\[ \frac{\partial (\rho B)}{\partial t} + \nabla \cdot \Lambda^{(0)} = 0, \]

where the momentum flux tensor \( \Pi^{(0)} = \sum_{a,\sigma} \nu_a^\sigma \nu_a^\sigma f_a^{\sigma(eq)} \), and the magnetic momentum flux tensor \( \Lambda^{(0)} = \sum_{a,\sigma} B_a^\sigma \nu_a^\sigma f_a^{\sigma(eq)} \) are defined in a similar fashion as in the
36-bit model $[3,10]$. Plugging the expression for $f_a^{\sigma(eq)}$, Eqs. (8) and (9), in the above definitions, we may compute the fluxes of density, momentum and magnetic field.

In computing the flux tensors, we need to make use of some of the freedom contained in our definition of the equilibrium distribution (8) and (9), which depends on parameters $p$, $q$, $r$, and $\alpha$. This flexibility in selecting the parameters will be used to eliminate certain unphysical terms that would prevent the appearance of MHD equations at leading order, and also to obtain other desirable properties.

First, we note that the presence of terms that mix the directions $e_a$ and $e_\sigma$ appears to be necessary to obtain a correctly structured induction equation. Next, the structure of the equilibrium permits appearance of an unphysical pressure-like term in the induction equation. This can be eliminated by choice of a relationship between $r$ and $q$, namely,

$$r = -q\frac{1+p}{2-p}. \quad (12)$$

Now we turn to some considerations with regard to selection of the value of the parameter $p$. Two interesting properties of this MHD system should be mentioned. First, in the limit case of $p = 0$, for which the streaming is only along the $a$ direction in an $(a, \sigma)$ state, incompressible hydrodynamics is recovered, as expected. Second, if the streaming parameter $p$ is changed to $1-p$, the roles played by $a$ and $\sigma$ are interchanged. To clarify this point, let us insert the relation between $r$ and $q$ into $B_\sigma^a$ to obtain

$$B_\sigma^a = \frac{q}{2-p}[-(1+p)e_a + (2-p)e_\sigma], \quad (13)$$

and recall that $v_\sigma^a = (1-p)e_a + p e_\sigma$. If $p$ is replaced by $1-p$, then $2-p$ and $1+p$ (and thus $q$ and $r$) also interchange their values. This property holds for the distribution function $f_a^{\sigma}$ as well; in other words, the same macroscopic MHD properties are obtained under this exchange, including sound speed, and transport
coefficients. The streaming parameter \( p \) is constrained to be between zero and one for mass conservation. However, we will see in the next section that choosing \( p = 1/2 \) eliminates spurious terms that would otherwise appear in the momentum and induction equations at second order in the spatial expansion of the kinetic equation.

Turning to the parameter \( q \), we note that, unlike \( p \), \( q \) can in principle take any desired real value with the exception of zero. For simplicity, we chose \( q = (2 - p)/\sqrt{3} \). For this particular value of \( q \), \( |v^{\alpha}_a| = |B^{\alpha}_a| \) and the microscopic velocity and magnetic field have the same intensity. Magnetohydrodynamics behavior is obtained for either \( q \) positive or negative. Notice that changing the sign of \( q \) will reverse the direction of \( B \). Therefore, this feature is associated to the fact that if the magnetic field is reversed everywhere the fluid flow is unchanged. This property was already present in the 36-bit MHD CA model [9] and in its LB version [10].

The equilibrium distribution function for these values of \( p \) and \( q \) becomes,

\[
f_a^{\sigma(eq)} = \left( \frac{\rho}{12} \right) \left\{ 12 \frac{1}{\alpha + 12} + \frac{8}{3} \left[ v^{\sigma} \cdot v + B^{\sigma} \cdot B \right. \right.
\]
\[
+ 2 \left\{ (e^{\alpha}_a \cdot v)(e^{\sigma} \cdot v) - (e^{\alpha}_a \cdot B)(e^{\sigma} \cdot B) \right\}
\]
\[
+ \frac{2}{\sqrt{3}} \left\{ (e^{\alpha}_a \cdot v)(e^{\sigma} \cdot B) - (e^{\alpha}_a \cdot B)(e^{\sigma} \cdot v) \right\} + \left. \frac{B^2}{2} \right\},
\]

and

\[
f_0^{\sigma(eq)} = \rho \left[ \frac{\alpha}{12 + \alpha} - \frac{4}{3} v^2 \right].
\]

Using this form of the equilibrium, after some straightforward algebra explicit forms for the flux tensors are obtained as,

\[
\Pi_{ij}^{(0)} = \frac{\rho}{2} \left\{ \frac{9}{12 + \alpha} \delta_{ij} + u_k u_l [\Delta_{ijkl} - \delta_{ij} \delta_{kl}] - B_k B_l [\Delta_{ijkl} - 2 \delta_{ij} \delta_{kl}] \right\},
\]

and

\[
\Lambda_{ij}^{(0)} = \rho (\delta_{il} \delta_{kj} - \delta_{ik} \delta_{jl}) u_k B_l.
\]

The ideal MHD equations emerge from this procedure,

\[
\rho \frac{\partial v}{\partial t} + \rho (v \cdot \nabla) v = -\nabla (P + B^2/2) + (B \cdot \nabla) B
\]

\[
(18)
\]
\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v},
\]

where the pressure \( P = \rho C_s^2 \). \( C_s \) is the sound speed of the system and has a simple form related to \( C \) and \( \alpha \), \( C_s = \sqrt{3C/(12+\alpha)} \), thus being controllable by the parameters \( p \) and \( \alpha \).

Now we make a small digression to clarify some points about the microscopic properties of the 13-bit model. For the 36-bit MHD model (LGA or LBE), we recall that the macroscopic velocity and magnetic field are defined as follows [8, 9, 10],

\[
\begin{align*}
\rho \mathbf{v} &= \sum_{a,b}^6 u_{ab} f_{ab} = \sum_{a,b}^6 [(1 - |p_{ab}|) \mathbf{e}_a + p_{ab} \mathbf{e}_b] f_{ab} \\
\rho \mathbf{B} &= \sum_{a,b}^6 B_{ab} f_{ab} = \sum_{a,b}^6 [r_{ab} \mathbf{e}_a + q_{ab} \mathbf{e}_b] f_{ab}.
\end{align*}
\]

The parameter matrices \( \mathbf{P}, \mathbf{Q}, \) and \( \mathbf{R} \) (of elements \( p_{ab}, q_{ab}, \) and \( r_{ab} \), respectively) involve, in principle, 108 independent scalars. Arguing that some conditions must be placed on these matrices to obtain the right MHD behavior, it was possible to reduce the number of independent scalars to only six [9]. Although we have not explicitly imposed such conditions for the case of the 13-bit model, we want to show that these properties are already present in the model as it is.

For the case of the 36-bit model, because the microphysics should be isotropic, we expect that rotating \( \mathbf{e}_a \) and \( \mathbf{e}_b \) together by a multiple of \( \pi/3 \), \( u_{ab} \) and \( B_{ab} \) should rotate by the same amount. This condition implies that the matrices should be circulant. Recall that a tensor \( \Xi \) is circulant if for \( c = 1, \ldots, 6 \), \( \Xi_{ab} = \Xi_{a+c,b+c} \mod 6 \), \( a, b = 1, \ldots, 6 \). In addition, the microscopic physics should be mirror symmetric: if \( \mathbf{e}_a \) and \( \mathbf{e}_b \) are interchanged, then the new values for \( u_{ab} \) and \( B_{ab} \), should be the mirror images with respect to the line that bisects the angle between \( \mathbf{e}_a \) and \( \mathbf{e}_b \). This implies that the matrices are symmetric, \( \Xi_{ab} = \Xi_{ba} \). These two conditions are trivially obeyed by our 13-bit model because the matrices \( \mathbf{P}, \mathbf{Q}, \) and \( \mathbf{R} \) in the 36-bit model are the
counterpart of the scalars $p$, $q$, and $r$.

There is another constraint to be enforced on the coefficients that is more subtle, and is associated with the vector nature of the velocity, and pseudovector nature of the magnetic field. There exist microscopic transformations that reverse the direction of one of the fields ($\mathbf{v}$ or $\mathbf{B}$) everywhere, while leaving the other one unchanged. Imposing such a property guarantees, for example that if $\mathbf{B} \rightarrow -\mathbf{B}$ the evolution of the velocity field will remain unchanged, while $-\mathbf{B}$ becomes the solution for the induction equation. This property was also included in the 36-bit model, by imposing some constraints on the parameter matrices, namely, $p_{ab} = -p_{ab+3}$, $q_{ab} = q_{ab+3}$, and $r_{ab} = -r_{ab+3}$, where all the sums are $modulo(6)$. It can be easily seen from the definitions $\mathbf{u}_{ab} = (1 - |p_{ab}|) \mathbf{e}_a + p_{ab} \mathbf{e}_b$ and $\mathbf{B}_{ab} = r_{ab} \mathbf{e}_a + q_{ab} \mathbf{e}_b$ for the 36-bit model, that by changing every $b$ by $b + 3$ $\mathbf{u}_{ab}$ is reversed, whereas $\mathbf{B}_{ab}$ is unchanged. The opposite is true if every $a$ is replaced by $a + 3$. In summary,

$$\mathbf{B}_{ab+3} = \mathbf{B}_{ab}, \quad \mathbf{u}_{ab+3} = -\mathbf{u}_{ab}$$
$$\mathbf{B}_{a+3b} = -\mathbf{B}_{ab}, \quad \mathbf{u}_{a+3b} = \mathbf{u}_{ab}.$$ 

This property should also be obeyed by the 13-bit model we are dealing with in this section. There is no straightforward “translation” from the $a + 3$ or $b + 3$ operation in the 36-bit scheme to our model here, because now for every $a$ there are only two $b$’s: $a + 1$ and $a - 1$ $modulo(6)$. Nevertheless, we find that the 13-bit model obeys the following relationships,

$$\mathbf{B}_{a+4,2} = \mathbf{B}_{a1}, \quad \mathbf{u}_{a+4,2} = -\mathbf{u}_{a1}$$
$$\mathbf{B}_{a+1,2} = -\mathbf{B}_{a1}, \quad \mathbf{u}_{a+1,2} = \mathbf{u}_{a1},$$

and therefore, such transformations are already embedded in the model as it is.

As a final comment, we note that this scheme would not exhibit MHD behavior in the context of a CA-type lattice gas model. If the idea of the splitting of the
distribution in two parts of the present model is “translated” to the CA realm, we would, most likely, end up with the 36-bit lattice gas scheme of Refs. [8, 9] that inspired our model, in the first place. Although this final statement would be hard to rigorously prove, we suspect that the present lattice Boltzmann scheme allowed us to get rid of all the “degeneracies” (or most of them) present in the 36 bit model.

3 Transport Coefficients

In this section we examine in detail the structure of the present model from the perspective of a Chapman-Enskog expansion procedure. This renders the long wavelength, low frequency behavior of the system, including corrections to the ideal equations in the form of dissipative transport effects. We start from the discrete kinetic equation (7)

$$f^\sigma_a(x, T) = (1 - p)[f^\sigma_a(x - e_a, T - 1) + \Omega^\sigma_a(x - e_a, T - 1)] + p[f^\sigma_a(x - e_\sigma, T - 1) + \Omega^\sigma_a(x - e_\sigma, T - 1)],$$

and expand up to second order in time and space variables, to obtain

$$\frac{\partial f^\sigma_a}{\partial t} + v^\sigma_a \cdot \nabla (f^\sigma_a + \Omega^\sigma_a) - v^\sigma_a \cdot \nabla \frac{\partial (f^\sigma_a + \Omega^\sigma_a)}{\partial t} - \Omega^\sigma_a - \frac{1}{2} \frac{\partial^2 \Omega^\sigma_a}{\partial t^2} - \frac{1}{2} [(1 - p)e_a e_a + pe_\sigma e_\sigma] : \nabla \nabla (f^\sigma_a + \Omega^\sigma_a) + \frac{\partial \Omega^\sigma_a}{\partial t} - \frac{1}{2} \frac{\partial^2 f^\sigma_a}{\partial t^2} = 0. \quad (20)$$

Now we adopt the following multiple scale expansion [4, 15]. The time derivative is expanded as

$$\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \cdots, \quad (21)$$

where $\epsilon$ is the expansion parameter assumed small, implying that $t_2$ is a slower time scale than $t_1$, and will be associated with diffusion effects. Likewise, the one-particle distribution function is expanded, assuming small departures from equilibrium,

$$f^\sigma_a = f^{\sigma(0)}_a + \epsilon f^{\sigma(1)}_a + \epsilon^2 f^{\sigma(2)}_a, \quad (22)$$
where \( f^{(0)}_a = f^{eq}_a \). Finally, for the collision operator we write

\[
\Omega^\sigma_a = -\frac{1}{\tau}(\epsilon f^{(1)}_a + \epsilon^2 f^{(2)}_a). \tag{23}
\]

Replacing all these expansions into (20), we find the following relation to order \( \epsilon \),

\[
\frac{\partial f^{(0)}_a}{\partial t} + \mathbf{v}_a \cdot \nabla f^{(0)}_a = -\frac{f^{(1)}_a}{\tau}, \tag{24}
\]

and

\[
\frac{\partial f^{(1)}_a}{\partial t_1} + \frac{\partial f^{(0)}_a}{\partial t_2} + \mathbf{v}_a \cdot \nabla f^{(1)}_a - \mathbf{v}_a \cdot \frac{\partial}{\partial t_1} f^{(0)}_a \\
-\frac{1}{2}[(1-p)e_a + pe_{\sigma}e_{\sigma}] : \nabla \nabla f^{(0)}_a \\
-\frac{1}{\tau} \frac{\partial}{\partial t_1} f^{(1)}_a - \frac{1}{2} \frac{\partial^2}{\partial t_1^2} f^{(0)}_a = -\frac{1}{\tau} f^{(2)}_a \tag{25}
\]

to order \( \epsilon^2 \). From (24) we can obtain the auxiliary relationship

\[
\frac{1}{2\tau}\left(\frac{\partial}{\partial t_1} + \mathbf{v}_a \cdot \nabla\right) f^{(1)}_a = -\frac{1}{2} \left[ \frac{\partial^2 f^{(0)}_a}{\partial t_1^2} + 2\mathbf{v}_a \cdot \nabla \frac{\partial f^{(0)}_a}{\partial t_1} + \mathbf{v}_a \cdot \mathbf{v}_a : \nabla \nabla f^{(0)}_a \right]
\]

that can be combined with (23), and after some algebraic manipulations the following equation is obtained:

\[
\frac{\partial f^{(0)}_a}{\partial t} + (1 - \frac{1}{2\tau}) \left[ \frac{\partial f^{(1)}_a}{\partial t_1} + (\mathbf{v}_a \cdot \nabla) f^{(1)}_a \right] = \\
p(1-p)(\mathbf{e}_a - \mathbf{e}_\sigma)(\mathbf{e}_a - \mathbf{e}_\sigma) : \nabla \nabla f^{(0)}_a = -\frac{1}{\tau} f^{(2)}_a. \tag{26}
\]

Summing equations (24) and (26) over all velocities, and using that
\[\sum_{a,\sigma} f^{(1)}_a = 0, \quad \text{and} \quad \sum_{a,\sigma} f^{(2)}_a = 0,\]
the following continuity equation up to second order is obtained,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \epsilon^2 \frac{p(1-p)}{2} \sum_{a,\sigma}(\mathbf{e}_a - \mathbf{e}_\sigma)(\mathbf{e}_a - \mathbf{e}_\sigma) : \nabla \nabla f^{(0)}_a. \tag{27}
\]
The term on the right hand side can be calculated using the tensorial relationships (71) in Appendix A, and the expression for \( f_a^{\sigma(eq)} \),

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \epsilon^2 \frac{p(1-p)}{2} \frac{\partial}{\partial x_i} \left\{ \left( \frac{6}{12 + \alpha} \right) \frac{\partial \rho}{\partial x_i} + 3 \frac{\partial (\rho v^2)}{\partial x_i} + \frac{2 \rho^2 - 2p - 1}{C^2} \left[ 2 \frac{\partial (\rho u_i u_j)}{\partial x_j} + \frac{\partial (\rho B_i B_j)}{\partial x_i} - 2 \frac{\partial (\rho B_i u_j)}{\partial x_j} \right] \right\}.
\] (28)

Therefore, there are second order corrections to the continuity equation. The most important term is apparently the one associated with the density diffusion, compared to the other terms that are quadratic in the fields \( u \) and \( B \). In the limit of hydrodynamics, i.e. \( p = 0 \) these additional terms vanish. No value of \( 0 < p < 1 \) (necessary for mass conservation) will make these spurious terms vanish. However, notice that the r.h.s. vanishes when integrated over the whole domain, and mass conservation is restored.

Similarly, adding moments of equations (24) and (26) with respect to \( \mathbf{v}_a^\sigma \) and \( \mathbf{B}_a^\sigma \), the following momentum equation and induction equation are obtained:

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot \mathbf{\Pi} = \frac{p(1-p)}{2} \sum_{a\sigma} \mathbf{v}_a^\sigma (\mathbf{e}_a - \mathbf{e}_\sigma) : \nabla \nabla f_a^{\sigma(0)} ,
\] (29)

\[
\frac{\partial (\rho \mathbf{B})}{\partial t} + \nabla \cdot \mathbf{\Lambda} = \frac{p(1-p)}{2} \sum_{a\sigma} \mathbf{B}_a^\sigma (\mathbf{e}_a - \mathbf{e}_\sigma) : \nabla \nabla f_a^{\sigma(0)} ,
\] (30)

where

\[
\mathbf{\Pi} = \sum_{a\sigma} \mathbf{v}_a^\sigma \mathbf{v}_a^\sigma \left[ f_a^{\sigma(0)} + \epsilon (1 - \frac{1}{2\tau}) f_a^{\sigma(1)} \right] ,
\]

and

\[
\mathbf{\Lambda} = \sum_{a\sigma} \mathbf{B}_a^\sigma \mathbf{v}_a^\sigma \left[ f_a^{\sigma(0)} + \epsilon (1 - \frac{1}{2\tau}) f_a^{\sigma(1)} \right] .
\]

We can see that there will be several contributions to the transport coefficients coming from (29) and (30). Let us first examine the contribution coming from the right hand side of the equations. For both the momentum and the induction equations only the terms linear in the fields in \( f_a^{\sigma(0)} \) will be different from zero, due to the microscopic
symmetries already embedded in the model at microscopic level. From the previous section we can write

\[ f_a^{(0)} \sim \frac{\rho}{12 + \alpha} + \frac{\rho}{3C} [v_a^{\sigma} \cdot v + B_a^{\sigma} \cdot B] + O(v^2, B^2). \]  

(31)

The following tensorial relationships are needed first,

\[
\sum_{a\sigma}(v_a^{\sigma})_i(e_a - e_{\sigma})_j(e_a - e_{\sigma})_k(v_a^{\sigma})_l = \frac{3}{4}(C - 3)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + \frac{3}{4}(C + 3)\delta_{il}\delta_{jk} \]

(32)

\[
\sum_{a\sigma}(v_a^{\sigma})_i(e_a - e_{\sigma})_j(e_a - e_{\sigma})_k(B_a^{\sigma})_l = \frac{3\sqrt{3}}{4}(2p - 1)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \]

\[
\sum_{a\sigma}(B_a^{\sigma})_i(e_a - e_{\sigma})_j(e_a - e_{\sigma})_k(B_a^{\sigma})_l = -\frac{3}{4}(C - 3)(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \]

Using the above relationships and after some algebra, we obtain

\[
\frac{p(1 - p)}{2} \sum_{a\sigma} v_a^{\sigma}(e_a - e_{\sigma})(e_a - e_{\sigma}) : \nabla \nabla f_a^{(0)} =
\frac{p(1 - p)}{8C} \left[ 2(C - 3)\nabla(\nabla \cdot (\rho v)) + (C + 3)\nabla^2(\rho v) \right]
+ \sqrt{3}(2p - 1) \left[ 2\nabla(\nabla \cdot (\rho B)) - \nabla^2(\rho B) \right],
\]

(33)

and

\[
\frac{p(1 - p)}{2} \sum_{a\sigma} B_a^{\sigma}(e_a - e_{\sigma})(e_a - e_{\sigma}) : \nabla \nabla f_a^{(0)} =
\frac{\sqrt{3}p(1 - p)}{8C}(2p - 1) \left[ 2\nabla(\nabla \cdot (\rho v)) - \nabla^2(\rho v) \right]
+ \frac{1}{\sqrt{3}} \left[ -2(C - 3)\nabla(\nabla \cdot (\rho B)) + 3(C - 1)\nabla^2(\rho B) \right].
\]

(34)

The other contribution, that is controllable through \(\tau\), comes from

\[
\Pi_{ij}^{(1)} = (1 - \frac{1}{2\tau}) \sum_{a\sigma}(v_a^{\sigma})_i(v_a^{\sigma})_jf_a^{\sigma(1)},
\]

(35)

\[
\Lambda_{ij}^{(1)} = (1 - \frac{1}{2\tau}) \sum_{a\sigma}(B_a^{\sigma})_i(v_a^{\sigma})_jf_a^{\sigma(1)},
\]

(36)
with \( f_a^{(1)} = -\tau [\partial/\partial t + \mathbf{v}^a \cdot \nabla] f_a^{(0)} \). We obtain as a contribution to the momentum equation,
\[
\nabla \cdot \Pi^{(1)} = \left( \tau - \frac{1}{2} \right) \left\{ \left( \frac{C}{2} + C_s^2 \right) \nabla [\nabla \cdot (\rho \mathbf{v})] + \frac{C}{8} \nabla^2 (\rho \mathbf{v}) \right\},
\]
whereas the contribution to the induction equation will be
\[
\nabla \cdot \Lambda^{(1)} = \left( \tau - \frac{1}{2} \right) \frac{C}{4} \left\{ -\nabla [\nabla \cdot (\rho \mathbf{B})] + \frac{3}{2} \nabla^2 (\rho \mathbf{B}) \right\}.
\]

We can write the macroscopic equations obtained, including all the above contributions to the transport coefficients and in the limit of low Mach number:
\[
\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla (P + \rho B^2/2) + \rho (\mathbf{B} \cdot \nabla) \mathbf{B} + \mathbf{B} \nabla \cdot (\rho \mathbf{B}) + \frac{\sqrt{3} p(1-p)(2p-1)}{C} \left[ 2 \nabla (\nabla \cdot (\rho \mathbf{B})) - \nabla^2 (\rho \mathbf{B}) \right],
\]
and
\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{v} + \frac{\mathbf{v}}{\rho} \nabla \cdot (\rho \mathbf{B}) + \frac{\sqrt{3} p(1-p)(2p-1)}{C} \left[ 2 \nabla (\nabla \cdot (\rho \mathbf{v})) - \nabla^2 (\rho \mathbf{v}) \right],
\]
where \( P \) corresponds to the mechanical pressure. We can readily notice that the unphysical terms \( \nabla (\nabla \cdot (\rho \mathbf{B})) \) and \( \nabla^2 (\rho \mathbf{B}) \) in (39) and \( \nabla (\nabla \cdot (\rho \mathbf{v})) \) and \( \nabla^2 (\rho \mathbf{v}) \) in (40) can be eliminated in the hydrodynamics limit \( (p = 0) \), but more importantly they can be removed for \( p = 1/2 \), so that the MHD macroscopic behavior can be maintained.
The extra terms in the macroscopic equations for $\rho$, $v$ and $B$ are also present in the 36-bit model described earlier in this work. The origin of these terms lies on the bidirectional streaming used in these models. The most seriously offending terms in (39) and (40) can be eliminated with the symmetrically appealing choice of splitting the distribution in halves. As mentioned in the previous section, the model should be symmetric about $p = 1/2$ and it is noticed that all coefficients appearing in the macroscopic equations are indeed invariant under the exchange $p \to 1 - p$.

For this choice of $p = 1/2$ (implying $C = 3/2$) the values for the transport coefficients are:

\begin{align}
\nu &= \frac{3}{16} \tau, \\
\nu_b &= \frac{3}{2} \left( \frac{1}{4} + \frac{3}{12 + \alpha} \right) - \frac{1}{4} \left( 1 + \frac{9}{12 + \alpha} \right), \\
\mu &= \frac{9}{16} \left( \tau - \frac{4}{9} \right), \\
\mu_b &= -\frac{1}{8} (3\tau - 2),
\end{align}

where $\nu_b$ and $\mu_b$ are the bulk viscosity and the bulk resistivity, respectively, and $\alpha$ is a free parameter introduced in the previous section that is used to set the sound speed. Notice that the ratio $\nu/\mu$ can be arbitrarily chosen by conveniently adjusting the parameters $\tau$ and $\alpha$. By inspecting these expressions we can realize that the model, unlike the hydrodynamics model, displays positive $\nu$ and $\mu$ beyond the threshold for stability ($\tau = 1/2$). Simple stability arguments indicate that the parameter $\tau$ should not be less than $1/2$, therefore imposing a lower bound on the transport coefficients.

4 Hartmann Flow

We now turn to the application of the lattice Boltzmann model we just described to a linear magnetohydrodynamics problem, namely Hartmann flow [16, 17]. This problem
represents one of the few MHD configurations that can be analytically solved without the need of linearizations (the equations are linear), with the additional assumption of constant density and constant transport coefficients.

The Hartmann configuration consists of a conducting liquid along a uniform channel, in steady regime and under the action of a transverse magnetic field. These flows can be used as flowmeters, by measurement of the potential induced in the fluids as it streams exposed to the external magnetic field [16, 18]. The fluid, assumed incompressible, is constrained to flow horizontally in a very long, ideally infinite channel alongside the x-direction. All relevant quantities, except the pressure, are a function of only the transverse coordinate (to the channel) \( y \), \( \mathbf{v} = (v_x(y), 0, 0) \), \( \mathbf{B} = (B_x(y), B_0, 0) \). A uniform and time independent pressure gradient is maintained along the channel direction to drive the fluid, so that \( p = p(x) \). The walls are located at \( y = -L \) and \( y = L \). Opposing to the propelling pressure gradient is the viscosity of the fluid and the tension in the magnetic field lines that resist the bending effect of the flow.

For this case, the incompressible MHD equations

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= \frac{1}{\rho} \nabla (p + \frac{B^2}{2}) + (\mathbf{B} \cdot \nabla) \mathbf{B} + \nu \nabla^2 \mathbf{v}, \\
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{B} &= (\mathbf{B} \cdot \nabla) \mathbf{v} + \mu \nabla^2 \mathbf{B},
\end{align*}
\]

can be reduced to the following linear system

\[
\begin{align*}
\nu \frac{d^2 v_x}{dy^2} + B_y \frac{dB_x}{dy} - \frac{1}{\rho} \frac{dp}{dx} &= 0, \\
\mu \frac{d^2 B_x}{dy^2} + B_y \frac{dv_x}{dy} &= 0,
\end{align*}
\]

(45)

(46)

where we are assuming that the system has reached a steady state; the density \( \rho \) is assumed uniform, and \( B_y(\equiv B_0 \text{ from now on}) \) corresponds to the known externally applied constant magnetic field transverse to the channel. If non-slip (static plane
walls) boundary conditions for the velocity, and \( B_x(-L) = B_x(L) = 0 \) for the magnetic field are applied, the following analytic solutions can be found for the system (45) and (46),

\[
v_x(y) = \frac{fL}{B_0 \rho} \sqrt{\frac{\mu}{\nu}} \coth(H) \left[ 1 - \frac{\cosh(Hy/L)}{\cosh(H)} \right], \tag{47}
\]

\[
B_x(y) = \frac{fL}{B_0 \rho} \left[ \frac{\sinh(Hy/L)}{\sinh(H)} - \frac{y}{L} \right], \tag{48}
\]

where the solutions depend on the dimensionless Hartmann number \( H \equiv B_0 L / \sqrt{\mu \nu} \), that measures the relative importance of viscous and magnetic forces. In the above expressions \( f = dp/dx \), and represents the force driving the fluid down the channel. It is easy to prove that in the limit of no external magnetic field \( B_0 \) (that corresponds to \( H = 0 \) if all other parameters are maintained constant), the solution for \( B_x \) is identically zero, and \( v_x = -(fL^2/2\nu)[(y/L)^2 - 1] \). This is the well-known solution for a simple Poiseuille flow.

The boundary conditions imposed for the magnetic field imply that the walls have a finite conductivity (they are neither perfectly conducting nor perfectly insulating). In this configuration (with these boundary conditions), the Hartmann flow is operating as an electromagnetic flowmeter as we will immediately show [17].

The current density \( j \) can be obtained from Ohm’s law as \( j = \sigma(E + \mathbf{v} \times \mathbf{B}) \), where the conductivity \( \sigma \) is the inverse of the resistivity \( \mu \) in our units, and \( E \) is the electric field. The only surviving component of \( j \) is the component normal to the plane of the flow, and it can be computed as

\[
j_z = \sigma(E_z + v_z B_0). \tag{49}
\]

On the other hand, from Maxwell’s induction equation we can get that \( j_z(y) = -\partial B_x / \partial y \). This expression for the current density can be immediately integrated to obtain the total current across the channel \( J_z \). Noting that \( B_x(y) \) is an odd function
of $y$, or simply by using (48) we get

$$J_z = -\int_{-L}^{L} \frac{\partial B_x}{\partial y} dy = B_x(-L) - B_x(L) = 0.$$ 

Combining with (49) we obtain that

$$\int_{-L}^{L} E_z = -B_0 \int_{-L}^{L} v_x(y) dy,$$

from which it follows that

$$E_z = -B_0 v_M,$$  \hspace{1cm} (50)

where $v_M$ represents the mean velocity of the flow across the channel. Consequently, $v_M$ can be computed by measuring $E_z$. ($B_0$ is assumed to be known since it is externally applied.)

Now we turn to the numerical simulation of Hartmann flow, making use of the lattice Boltzmann model with 12 moving states described in previous sections. The simulation domain was for all cases a lattice of only 4 cells in the $x$ direction of the problem $\times$ 60 cells in the transverse direction $y$. The $x$ direction coincides with the direction $e_1$ of the hexagons. The system is initialized by setting the distribution functions $f_a^\sigma$ and $f_0$ to their equilibrium values given by a uniform density, and a transverse magnetic field $B_y \neq 0$. $B_x$, $v_x$, and $v_y$ are initially zero.

The system is evolved by using the standard sequence of collisions and streaming processes, with the addition of an intermediate step that acts to generate an effective pressure gradient. To achieve this, the distribution of moving particles at each cell $f_a^\sigma$ is redistributed so that the total density and the magnetic field at the cell are unaltered, and the velocity receives a “kick” in the direction of the flow. Care must be taken that the redistribution of mass density along the different states does not push the distribution function too far out of its equilibrium value, nor makes it negative. That is, for the problem under consideration, a forcing function must be constructed.
that tries to increase \( \rho v_x \) while \textit{not} changing \( \rho v_y, \rho B \), or \( \rho \). To explicate the dynamics of this procedure, let us recall the definitions of the macroscopic quantities,

\[
\rho = f_0 + \sum f^\sigma_a
\]

\[
\rho v = \sum [(1-p)e_a + p_{\sigma}f^\sigma_a]
\]

\[
\rho B = \sum [r e_a + q_{\sigma}f^\sigma_a].
\]

Therefore, if at a certain microscopic time \( T \) the distribution function at a specific cell is \( f^\sigma_a \) before this “forcing” scheme, and \( f^\sigma_a' = f^\sigma_a + F^*C_{a\sigma} \) after the distribution is kicked, we wish to find a quantity \( F^*C_{a\sigma} \) with the constraints

\[
\rho = \sum_{a\sigma} f^{\sigma'}_a = \sum_{a\sigma} f^\sigma_a + \Delta \rho
\]

\[
\sum_{a\sigma} f^{\sigma'}_a[(1-p)e_a + p_{\sigma}] = \sum_{a\sigma} f^\sigma_a[(1-p)e_a + p_{\sigma}] + \rho(\Delta v)
\]

\[
\sum_{a\sigma} f^{\sigma'}_a[r e_a + q_{\sigma}] = \sum_{a\sigma} f^\sigma_a[r e_a + q_{\sigma}] + \Delta(\rho B),
\]

where \( \Delta \) represents a “small” change. As stated above, we are seeking \( \Delta(\rho) = 0, \Delta(\rho B) = 0 \). Using that \( f^{\sigma'}_a = f^\sigma_a + F^*C_{a\sigma} \) and adding over \( \sigma \), Eqs. 51-53 become

\[
\Delta(\rho) = 0 = \sum_a (C^1_a + C^2_a) = 0,
\]

\[
\Delta(\rho v) = F^*(1-p) \sum_a (C^1_a + C^2_a)e_a + F^*p \sum_a (C^1_{a-1} + C^2_{a+1})e_{\sigma},
\]

and

\[
\Delta(\rho B) = 0 = r \sum_a (C^1_a + C^2_a)e_a + q \sum_a (C^1_{a-1} + C^2_{a+1})e_{\sigma}.
\]

Combining \([54]\) and \([56]\) we obtain

\[
\Delta(\rho v) = (1-p - p^r \frac{r}{q}) \sum_a (C^1_a + C^2_a)e_a.
\]

Choosing \( \Delta(\rho v_x) = 1 \) and \( \Delta(\rho v_y) = 0 \) the following set of equations can be found

\[
\sum_a (C^1_a + C^2_a) = 0
\]
\[ C_1^1 + C_2^2 + C_3^1 + C_3^2 - C_5^1 - C_5^2 - C_6^1 - C_6^2 = 0 \]
\[ (1 - p - p \frac{r}{q}) \left[ C_1^1 + C_1^2 + \frac{C_2^1}{2} + \frac{C_2^2}{2} \right. \]
\[ - \frac{C_3^1}{2} - \frac{C_3^2}{2} - C_4^1 - C_4^2 - \frac{C_5^1}{2} - \frac{C_5^2}{2} + \frac{C_6^1}{2} + \frac{C_6^2}{2} \right] = 1, \quad (58) \]

for which a particular solution is

\[ C_1^1 = C_2^2 = -C_4^1 = -C_4^2 = 1 \]
\[ C_5^1 = -C_6^2 = \frac{3 - p}{2 - p} \]
\[ C_5^2 = -C_6^1 = \frac{3 - 2p}{2 - p}, \quad (60) \]

where we used that \( r/q = -(1 + p)/(2 - p) \). By appropriately choosing \( F^* \) we are able to control the strength of the forcing scheme. We note that the pressure gradient produced in this way is uniform, and that it follows the spirit of the CA interpretation of cell populations as “particles.” When this process is put into action, the total momentum in the direction that is being forced is seen to increase, until the driving force is balanced by the action of the viscosity of the fluid and the reluctance of the magnetic field lines to be bent, and the system reaches a stationary regime.

The boundary conditions were implemented by setting \( v_x, v_y \) and \( B_x \) equal to zero in the first and last layers (i.e., \( y = 0 \) and \( y = L \)), combined with a periodic streaming. An alternative way to achieve the boundary conditions would be to cancel the periodic streaming of the populations, and to make the “particles” bounce off the walls reversing the velocity at those cells while keeping unchanged the magnetic field.

Simulations of the Hartmann flow were carried out for Hartmann number \( H = 0 \) (zero magnetic field case), 1, 3, 5, 8, and 13. The results are presented in Fig. 1 and Fig. 2. Figure 1 displays all the velocity profiles \( v_x(y) \) plotted versus \( y \), for all six values of \( H \), and compared with the analytical solution \( (17) \). A very good agreement between the latter and the computational results is obtained for this range of values.
of H.

The flattening of the velocity profile can be understood in several ways \[16, 17, 18\]. From the linearized MHD equations (45) and (46), it can be seen that the transverse magnetic field tries to eliminate vorticity. If the magnetic forces dominate (i.e., for large Hartmann number $H$), then the velocity profile tends to flatten.\[19\] However, the velocity profile cannot be flat all across the channel, because the velocity at the walls vanishes, so there must be a region within a certain distance from the walls on which the gradients of the velocity are very large, i.e., the region where the vorticity (produced by the boundaries) is confined to.

Alternatively, the flattening of the velocity profile could be understood as follows. Combining (49) and (50) we obtain

$$j_z = \sigma B_0 (v_x - v_M).$$

The magnetic force on the fluid is given by the Lorentz force $\mathbf{F}_L = \mathbf{J} \times \mathbf{B}$, that in our case reduces to $(F_L)_x = -\sigma B_0^2 (v_x(y) - v_M)$, and therefore the flow tends to slow down where $v_x > v_M$ and tries to speed up where $v_x < v_M$, producing a flattening of the velocity profile.

The high degree of agreement seen for the velocity profile across the channel is also observed for Figure 2, that shows the same comparison for $B_x(y)$, for $H = 1, 3$ and 13 only. For this figure, only results for three different Hartmann numbers were included for the sake of clarity, since the effect of $H$ on $B_x$, is not as marked as it is for $v_x$. We note that the fit for the cases not shown is as good as those displayed in the figure. There is a point we would like to stress about the material presented in this figures. These are not “fits” in the usual sense. The Hartmann number $H$ is here constructed from $\nu$ and $\mu$ which come from the Chapman-Enskog theory. The analytic solutions (47) and (48) depend only upon $f, \mu, \nu, B_0, L$, and $\rho$. The forcing
$f$ is fixed; $B_0$ is fixed initially as are $\rho$ and the simulation size $L$. With this subtlety in mind, the solutions have no free parameters to adjust.

Some caution must be exerted when calculating the Hartmann number corresponding to the simulation, since the exponential character of the solutions would make them very sensitive to a small departure from the actual value of $H$. In particular, the width of the channel $2L$ should be evaluated as $(N_y - 1)\sqrt{3}/2$, taking into account the $x-y$ ratio for the hexagons, and the position of the boundary in the simulation is at $y = 1$ instead of $y = 0$.

The Hartmann number $H = B_0L/\sqrt{\mu\nu}$ was varied by changing the strength of the magnetic field $B_0$. The size (60 cells) was kept constant for convenience in the manipulation of data. The kinematic viscosity $\nu$ and resistivity $\mu$ can be obtained as a function of the relaxation time $\tau$. From Section III we recover the expressions $\nu = 3\tau/16$ and $\mu = 9\tau/16 - 1/4$. Only $\tau = 1$ was used for all the simulations in this section, therefore the system is forced to equilibrium in each iteration. The “forcing” coefficient was set to $F^* = 2 \times 10^{-5}$, thus being sufficiently small (comparing $F^*C_\sigma^a$ with the mean value of the density, $\rho_0 = 3.9$) to ensure that the distribution function $f_\sigma^a$ will be only slightly departed from equilibrium during the “forcing” step.

The limitations for the range of $H$ that the model will be able to accurately reproduce are as follows. 1) for the upper bound, we can see from (15) and (16) that we can estimate the width of the boundary layers, in which the Lorentz force is comparable to the viscous forces, as $\delta \sim \sqrt{\nu\mu}/B_0$, from which we gather that $\delta/L \sim H^{-1}$. When $H \gg 1$, the thickness $\delta$ becomes very small (the region of nonzero velocity gradients is confined to a very narrow layer away from the walls), and chances are that we need to increase the width of the domain to resolve $\delta$. For our simulations, for which 60 cells across the channel were used for all cases, we observed that for $H > 30$ the boundary layer thickness $\delta$ is of the order of one cell. Therefore,
if simulations with higher values for $H$ are desired, $L$ should be increased. We note that this is not a limitation of the lattice Boltzmann simulation scheme, but rather a resolution constraint: the computation of the analytical solution presents the same flaws. 2) for the lower bound, the limitation is given by the roundoff error of the machine since for $H \ll 1$ we are forced to use very weak external fields $B_0$.

The one dimensional nature of the problem is highlighted by the fact that the exact behavior of the solutions was reproduced for a domain that was 4 cells long in the fluid direction $x$. As a matter of fact, we observed that the same results can be obtained with a length of only one cell, making the domain truly one-dimensional, and supporting the hypothesis made for the derivation of the lattice Boltzmann approach, that the population of the cells correspond to ensemble averages of the discrete populations used for the Cellular Automaton approach.

5 2D Magnetic Reconnection

In the previous section, we argue in favor of our lattice Boltzmann MHD model, by comparing its solutions for the Hartmann flow problem, with the analytically obtained solutions. The results are encouraging to the extent that is very difficult to observe with the naked eye in Figures 1 and 2 departures of the LBE solution from the theoretical ones. This was the case for a wide range of values of the Hartmann number, the only parameter in the problem. Optimistic as we might be, we recognize that Hartmann flow is essentially a one-dimensional and linear problem. Thus, it is our intention in this section, to test the validity of the 13-bit LBE model for a situation that is both two-dimensional and nonlinear. The configuration we chose is the 2D MHD “sheet pinch”. Before presenting results for the present 13-bit model, we recall that the reconnection configuration had been chosen previously as a test problem
for 36-bit MHD model of Ref. [10]. The authors found that the LBE solutions were qualitatively correct, and they showed features of the evolution associated with nonlinear effects of the sheet pinch dynamics. However, no claim of comparison with the “real” solutions, or solutions from other numerical methods was made.

The “sheet pinch” is a magnetohydrodynamic configuration characterized by an inhomogeneous magnetic field that changes markedly in a very narrow region, thus producing very strong currents. This arrangement can be encountered in a variety of important physical phenomena as solar flares, and the earth’s magnetic field, reversing its sign embedded in the solar wind. It is not the goal of this section to discuss in detail the physical processes in a magnetofluid undergoing a reconnection process, a vast literature is available. Central are the theoretical efforts of Dungey [20], Parker [21], and Sweet [22]. For a discussion of the reconnection process, including the role played by the fluctuations from the point of view of turbulence, see Matthaeus and Montgomery [23], and Matthaeus and Lamkin [24].

Our approach to the “sheet pinch” problem will be similar to the procedure followed in another recent test of the LBE method [25], in which LBE and spectral method solutions for a 2D hydrodynamic shear layer were compared in detail. Here, we will briefly describe the reconnection runs from a technical point of view, and then we will move on to present and contrast the results obtained with both LBE and spectral methods.

5.1 The Sheet Pinch Simulations

The idealized sheet pinch consists of a uniform magnetic field reversing sign in a very thin zone, much in the same way as the velocity swaps its direction in the idealized nonlinear Kelvin-Helmholtz instability. This configuration gives rise to a current sheet because \( j = (\nabla \times B)_z \), with \( j \) the current density in the \( z \) direction. Therefore, the
initialization of the sheet pinch was done, in a $2\pi \times 2\pi$ simulation box, with a spectral truncated representation of delta functions located at $y = \pi/2$ and $y = 3\pi/2$ (with opposite signs) for the current, including wavevectors $k = 1$ through 31.

An uninteresting evolution follows unless some non-sheet pinch modes are excited, to initiate the nonlinear couplings. The current density and the vorticity Fourier modes with $1 \leq k \leq 15$ where excited with random phases and with spectra of $k^{-3}$ for high $k$ for both, kinetic and magnetic energy. The “noise” spectrum was peaked at $k = 3$ for both kinetic and magnetic energy, and added about 1% of the energy already present in the ideal sheet pinch.

For the spectral run the $z$ components of the vorticity $\omega = (0, 0, \omega)$ and the vector potential $a = (0, 0, a)$, are evolved according to the equations

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = \mathbf{B} \cdot \nabla j + \nu \nabla^2 \omega \quad (62)$$

$$\frac{\partial a}{\partial t} + \mathbf{v} \cdot \nabla a = \mu \nabla^2 a, \quad (63)$$

where $\mu$ is the resistivity and $\nu$ is the kinematic viscosity. The fields $\mathbf{v}$ and $\mathbf{B}$ are in the plane $x, y$ depending solely on those coordinates. The scalar functions $\omega$ and $a$ are related to $\mathbf{v}$ and $\mathbf{B}$ by $\omega \mathbf{z} = \nabla \times \mathbf{v}$, and $\mathbf{B} = \nabla \times a \mathbf{z} = \nabla a \times \mathbf{z}$, where $\mathbf{z}$ is the unit vector in the direction normal to the $x, y$ plane. The current density is $\nabla^2 a = -j$, and the vorticity is related to the stream function by $\nabla^2 \psi = -\omega$. The spectral run is of the Fourier-Galerkin type and has a resolution of $128 \times 128$.

For the LBE run we have to specify the initial density, velocity field, and magnetic field. The velocity field is obtained from the relation $\mathbf{v} = \nabla \psi \times \mathbf{z}$, the stream function $\psi$ being obtained from the solution to $\nabla^2 \psi = -\omega$, with $\omega$ being the initial vorticity used in the spectral run. Similarly, to initialize $\mathbf{B}$, we numerically evaluate $\mathbf{B} = \nabla a \times \mathbf{z}$ in Fourier space, $a$ being the initial vector potential used for the spectral run. The initial density is set to a uniform value $\rho = \rho_0 = 3.9$. The initial fields are used
to evaluate the equilibrium distribution function $f^\sigma_a$ for this model, given by Eqs. (8) and (9), for these initially specified fields. From then on, the LBE sequence of streaming and collisions is performed to evolve $f^\sigma_a$.

A short digression is needed at this point to justify the choice of $\nu$ and $\mu$ for the spectral run (the inverse of the Reynolds number $R$, and the magnetic Reynolds number $R_m$, respectively, in the set of units we are using), and the relaxation parameter $\tau$ and the simulation size, for the LBE simulation. Since it is our goal to test the MHD LBE model in a situation that involves strong nonlinear interactions, it is desirable to perform a simulation with the Reynolds numbers $R$ and $R_m$ as large as possible. The limitation is imposed by the LBE scheme. In Section III, explicit expression for the resistivity and viscosity of the LBE model were found:

$$\nu_{\text{LBE}} = \frac{3}{16} \tau$$

$$\mu_{\text{LBE}} = \frac{9}{16} \tau - \frac{1}{4}.$$  \hspace{1cm} (64)

(65)

For stability reasons, the parameter $\tau$ is constrained to be $\tau > 1/2$. On the other hand, for attaining small $\nu_{\text{LBE}}$ and $\mu_{\text{LBE}}$, and thus high $R$ and $R_m$, the relaxation parameter $\tau$ should be as small as possible. Choosing $\tau$ slightly larger than 1/2, to ensure stability, the values of $R$ and $R_m$ will be dictated essentially, by the simulation size. To strike a compromise between a computationally reasonable simulation size, and the degree of nonlinear activity, we use a $512 \times 512$ resolution domain, for the LBE simulation. The characteristic speed in our problem is the Alfvén speed, coinciding with $\sqrt{\langle B^2 \rangle}$ in our units. Consequently, the magnetic Reynolds number is given by

$$R_m = \frac{BL}{\mu} = 0.1 \times \frac{512}{2\pi} \times \frac{1}{\mu},$$

(66)

where $B = \sqrt{\langle B^2 \rangle} = 0.1$ initially. Similarly, for the mechanical Reynolds number we
obtain,
\[ R = \frac{BL}{\nu} = 0.1 \times \frac{512}{2\pi} \times \frac{1}{\nu}. \] (67)

For \( \tau = 0.5001 \) we get \( R = 86.9 \) and \( R_m = 260.3 \). The reciprocal of these numbers are those used for \( \nu \) and \( \mu \), respectively, for the spectral run. The factors of \( 2\pi \) are necessary because the length \( L \) used to define \( R \) and \( R_m \) for the spectral simulation is the physical length of the simulation box, divided by \( 2\pi \), i.e., \( L = 2\pi/2\pi = 1 \). Thus, for the LBE simulation simulation we need to use 512 (the physical size) divided by \( 2\pi \). Note that for the LBE run, because we are using the hexagonal lattice, the typical length used for the above evaluations, is somewhat ambiguous due to the \( \sqrt{3}/2 \) ratio between the \( y \) and \( x \) directions. Moreover, the system represented by both methods are physically different.

Last, before turning to the comparison of the results obtained for the two runs, and to be able to relate physical processes observed in both simulations, we need to find a relationship between the spectral and LBE characteristic times. Let us write,
\[ \frac{T_{LBE}}{T_{SP}} = \frac{L_{LBE}/L_{SP}}{U_{LBE}/U_{SP}} = \frac{512}{2\pi} \frac{1}{0.1}, \] (68)

where \( U_{LBE} = 0.1 \) is the characteristic speed of the LBE model, given by the Alfvén speed, and similarly \( U_{SP} = 1 \) for the spectral method run, that tells us that \( T_{LBE} = 814.87 \ T_{SP} \).

The runs were carried out up to about ten spectral-method characteristic times. Periodic boundary conditions were imposed for the simulations.

### 5.2 Comparison and Discussion of the Sheet Pinch Runs

The evolution of the sheet pinch dynamics, being an MHD system, is much more complex than its hydrodynamics counterpart. There are more dynamical variables evolving coupled to each other, and there is a larger parameter space. For example,
the relative values of $R$ and $R_m$ may influence the dynamics. Nevertheless, there are some global features thought to be similar for most decaying, 2D, incompressible MHD flows in periodic geometry. For example, there are three rugged invariants [26], that is constants of the nondissipative evolution that survive the truncation in $k$ space (Galerkin approximation). Those are [26], the total energy $\langle E \rangle = \sum_k \omega^2(k)/k^2 + a^2(k)k^2$, the cross helicity $H_c = \langle \mathbf{v} \cdot \mathbf{B} \rangle = \sum_k \omega(k)a(k)$, and the mean square vector potential $A = \langle a^2 \rangle = \sum_k a^2(k)$, where $\langle \cdots \rangle$ denotes a volume average.

In Fig. 3 we present time histories of bulk quantities that characterize the turbulence for both the spectral method and the LBE run. The continuous line corresponds to the spectral method simulation. Panels a) and b) of Fig. 3 display the evolution of $A$ and $E$, respectively. Both quantities, which are ideally conserved, decay monotonically due to the presence of nonzero dissipation coefficients $\mu$ and $\nu$. The slow decay of $A$ suggests that there is a dynamic redistribution of this quantity in favor of larger scales. In Fig. 3c) we present the evolution of the kinetic energy. The kinetic energy $E_k$, initially 1% of the total energy, decreases throughout the run, displaying some small bursts of activity that would be more pronounced for larger $R$ and $R_m$ [24].

In panels d) and e) of Fig. 3, we show the evolution of the enstrophy $\Omega$ and the mean square current $J = \langle j^2 \rangle$. These two quantities highlight the activity in small spatial scales, and for these runs they rapidly decay due to the relatively high viscosity and ohmic dissipations. Again, both quantities present a more “bursty” shape for more turbulent systems [24].

We now turn our attention to comparison of contour plots. In Fig. 4 we exhibit plots of constant magnetic field lines (constant $a$), for times approximately equal to one, seven, and ten. Emergence and subsequently growth of magnetic islands can be seen. For the same times, in Fig. 5, contours of constant vorticity are displayed.
Although the distribution of vorticity consists of small, nonlocalized structures, it can be seen that this quantity rapidly organizes in the region of the $X$-points to form quadrupole-like structures. The vorticity plots, combined with the stream function contour plots, shown in Fig. 6, provide a consistent picture of part of the activity of the magnetofluid in the reconnection zone. Jets of fluid are seen to come into the “hot” area (higher speeds are represented by denser $\psi$ lines) from the strong-field sides of the $X$ (up and down in our plots), and out through weak-field corners (sides of the $X$). These jets are responsible for the bursts that suggest themselves in the evolution of the kinetic energy. This is essentially a pressure-driven effect due to the steep gradients of the magnetic field present near the neutral sheet; the fluid finds itself pushed by the magnetic pressure into the neutral zone, and, being incompressible, it has no “choice” but to turn into the weak-field region producing the four vortices seen in the plots, at the $X$ points.

The last set of plots from the spectral and LBE runs (Fig. 7), shows contours of constant current density. Initially the current is concentrated, more or less uniformly, along both current sheets. As the system evolves, and the magnetic field lines start to reconnect, we see a tendency for filaments of current density to form. The regions with current filaments will participate in a significant part of the energy dissipation due to finite resistivity.

We conclude from the examination of these plots, that this hexagonal, two-dimensional MHD LBE model, is capturing the basic mechanisms of MHD dynamics. From the contour plots, we see that once a structure of one of the quantities is identified in one of the plots corresponding to the spectral run, a very similar structure can be also seen in the corresponding LBE plot. Similarly, the LBE tracks the evolution of relevant bulk quantities very closely. We would like to point out that a perfect agreement is not, in fact expected. Additional refinements could be introduced in
the LBE simulation to still further reduce the gap between the results from both simulations. For example, it was mentioned above that the spectral simulation box, and the LBE domain are physically different systems, due to the effects coming from the hexagonal lattice. This will certainly affect the magnitude of the mechanical and magnetic Reynolds numbers. There are, at least, two alternative ways of getting around this difficulty. One of them would be to make use of the similar MHD model derived based on the square lattice, which is described in Appendix B. (The earlier comparison of the LBE and spectral method hydrodynamic shear layer dynamics \[25\] employed an LBE on a square lattice.) Although the hexagonal LBE requires less memory, this is not a decisive advantage. Another possibility would be to abandon the use of the same number of cells in both dimensions \(N_x = N_y\), and to choose, for example \(N_y = N_x\sqrt{3}/2\). This choice would have introduced complications in diagnostics currently based upon Fast Fourier Transforms. Instead, considerable larger amounts of data would be required to be kept for later analysis.

An examination of the divergence of the magnetic field is mandatory, to make sure that monopoles are not being created by the model, thus casting doubt on the results. To this end, we decompose in Fourier space the magnetic field \(\mathbf{B}(\mathbf{k})\) in its longitudinal component \(B_L\), and its transverse component \(B_\perp\). A similar examination was carried out for the velocity field in the shear layer LBE study \[25\], as a way to quantitatively measure the compressibility of the flow. We calculate \(B_L\) and \(B_\perp\) as

\[
B_L^2 = \sum_k \frac{|\mathbf{k} \cdot \mathbf{B}(\mathbf{k})|^2}{k^2}, \quad (69)
\]

\[
B_\perp^2 = \sum_k \frac{|\mathbf{k} \times \mathbf{B}(\mathbf{k})|^2}{k^2}. \quad (70)
\]

This ratio is a good measure of the amount of “monopolar” (longitudinal) activity as compared with the transverse component, containing most of the energy. In Fig. 8 we show the evolution of the ratio \(B_L/B_\perp\). We readily notice that the overall tendency
of this quantity is to decrease, and that by the end of the run the “amount” of \( B_L \) is about one part in one thousand.

The nonzero initial value of \( B_L \) is attributed to the non-square nature of our LBE simulation box. Exactly the same field that produces \( \nabla \cdot \mathbf{B} = 0 \) for the square spectral simulation run, produces nonzero divergence on the hexagonal lattice. What is encouraging about this picture is that the LBE dynamics seems to possess self-adjusting mechanisms that reduce the amount of monopoles, much in the same way as Chen et al. [9] discuss in the context of their 36-bit MHD CA model.

Finally, we turn to a brief discussion of the efficiency of the method. At the end of the previous section, we noted that although the transport coefficients are directly controllable via the relaxation parameter \( \tau \), the threshold of stability with respect to \( \tau \), is higher than the value of \( \tau \) needed to make \( \nu \) and \( \mu \) zero. This technical problem limits the Reynolds numbers that can be attained in the MHD LBE for fixed grid size. In the case of the hydrodynamic shear layer LBE [25], the simulation domain was chosen with the objective of resolving the spatial structure of the turbulent activity, much in the same way as it is done for a spectral method simulation. Thus, at the same Reynolds number, an LBE and spectral method hydrodynamics simulation can have about the same grid size. For the MHD LBE, the size was also determined by the requirement of matching the Reynolds numbers with the spectral code. This \( 512 \times 512 \) LBE simulation was run on a CRAY-YMP computer, in the San Diego Supercomputer Center, and needed about 12 minutes per characteristic time (about 800 LBE microscopic times). The spectral run, 16 times smaller (\( 128 \times 128 \)), took about ten times less CPU time. Nevertheless, it should be noted that the numerical efficiency of LBE-type computations is greatly enhanced in massively parallel computers, due to its local dynamics.
6 Discussion and Conclusions

In this paper we have introduced a model for simulation of 2D MHD with the lattice Boltzmann equation technique. The idea of propagating the distribution at a given state into two directions associated with the velocity and magnetic fields, had been previously used for obtaining 2D MHD using CA dynamics [8, 9], and later extended to LBE [10]. In the present scheme, by utilization of the same idea combined with the flexibility of the LBE scheme a significantly more efficient and simpler method is obtained for simulation of 2D MHD. The improvement is two-fold; first, the number of discrete velocities is reduced in our model from 37 to 13, only. Second, the algorithm for the evolution of the present model is simplified by requiring a “forward” streaming, as explained in the text. These models possess the same microscopic symmetries necessary for guaranteeing the correct long wavelength, low frequency behavior. The theory for the model was presented, including the Chapman-Enskog expansion procedure to obtain second-order effects. In passing, we note that the simplicity of the model is apparent when evaluating all the second-order contributions displayed in Section III.

Evidence of correct MHD behavior was introduced in Sections IV and V, where the model is applied to reproduce a linear problem (steady Hartmann flow), and a nonlinear problem (evolution of the 2D sheet-pinch). For the former, the performance of our model is extremely good, for a reasonably wide range of the Hartmann number, that parameterizes the problem. The second numerical test of the model behaves reasonably well. Nevertheless, it should be mentioned that this more stringent test exposes what might be the most serious deficiency of the model, namely the inability to achieve relatively low transport coefficients. This disadvantage hurts the potential use of this model for highly turbulent simulations. Clearly further investigation into
this matter is required. At the moment we ignore the fundamental reason of this
crossover (in terms of the relaxation time $\tau$) between the stability threshold ($\tau > 1/2$),
and the region of low values of viscosity and resistivity that occurs for $\tau < 1/2$ (see
Eqs. (41) and (43)), and whether this effect is induced by the special streaming used
for the model, or the deficiency could be cured by choosing a more flexible collision
operator. The LBE model for hydrodynamics (with single-time-relaxation collision
operator) does not share this inconvenient feature since $\tau > 1/2$ is both a condition
for numerical stability and for positivity of the viscosity. Another two desirable
properties that an improved MHD LBE model could have are as follows; first, it
would be convenient to have independent control on the viscosity and resistivity (in
the present model, the choice of the relaxation time $\tau$ determines both $\mu$ and $\nu$).
Second, and more important, in the present model, and in other MHD models we
made reference to in this report, the divergenceless property of the magnetic field is
not imposed. It is seen, however (see Ref [9]) that the model possesses some self-
adjusting mechanisms that diffuse away the solenoidal component of the magnetic
field, as displayed in Section V, for the sheet-pinch simulation.

In spite of these improvable aspects, we believe that the 13-speed model is cap-
turing the essential features of the equations for MHD with a minimum number of
degrees of freedom, and that it is a valuable tool for non-turbulent regimes. More-
over, although the model was formulated explicitly for 2D in the present paper, its
extension to 3D with a reasonably low number of degrees of freedom does not seem
to pose any serious difficulty.
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Appendix A: Useful Tensorial Relationships

The relevant tensorial relationships used extensively for the derivation of the fluxes of density, momentum, and magnetic field in Section II are:

\[ \sum_{a,\sigma} (e_a)_i (e_a)_j (e_a)_k (e_\sigma)_l = 2 \times \frac{3}{4} \Delta_{ijkl} \]

\[ \sum_{a,\sigma} (e_a)_i (e_a)_j (e_\sigma)_k (e_\sigma)_l = \frac{3}{4} \Delta_{ijkl} \]

\[ \sum_{a,\sigma} (e_a)_i (e_\sigma)_j (e_a)_k (e_a)_l = \frac{3}{4} \Delta_{ijkl} + \frac{9}{2} \delta_{ij} \delta_{kl} \]

\[ \sum_{a,\sigma} (e_a)_i (e_a)_j (e_\sigma)_k = \sum_{a,\sigma} (e_\sigma)_i (e_\sigma)_j (e_\sigma)_k = 0 \]

\[ \sum_{a,\sigma} (e_a)_i (e_\sigma)_j - \sum_{a,\sigma} (e_\sigma)_i (e_\sigma)_j = 6 \delta_{ij} \]

\[ \sum_{a,\sigma} (e_a)_i (e_\sigma)_j = 3 \delta_{ij} \]

\[ \sum_{a,\sigma} (e_a)_i = \sum_{a,\sigma} (e_\sigma)_j = 0, \]

where \( \delta_{ij} \) is the Kronecker delta, and \( \Delta_{ijkl} = \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \).

Appendix B: Simulating 2D MHD on the “Square” Lattice

Moving from modeling hydrodynamics to modeling MHD requires the inclusion of new force terms, and more importantly a new whole equation to follow the evolution
of the magnetic field. This equation involves terms like \((v \cdot \nabla)B - (B \cdot \nabla)v\), and the technique that has worked in the 36-bit and in the 13-bit models, has been splitting the distribution in different directions during the streaming part of the evolution. This parting of the distribution produces a mixture of directions that allows to include terms mixing \(v\) and \(B\) and with different signs. Once this fact is recognized, it is not hard to generalize the same idea to the square lattice.

In this section we would like to simply display a method for modeling 2D MHD on the square lattice. This would have a slight advantage and a slight disadvantage over the hexagonal lattice (13-bit model). The latter is that the “square” model requires more memory than the hexagonal model: we need to keep track of 13 real numbers per cell, for the hexagons, whereas for the squares the requirements increase to 17 real numbers per cell. On the other hand, the square lattice provides a “better” simulation domain, in a geometrical sense, for some systems, unlike the hexagonal case, for which the basic cell has different physical lengths in the two directions.

We now turn to a brief description of this alternative approach. If one of the square cells is located at \(x\), its nearest neighbors are located at the face-centers \(x + c^I_a\), for \(a = 1, 2, 3, 4\), with \(c^I_a \equiv (\cos ((a-1)\pi/2), \sin ((a-1)\pi/2))\), and the vertices of the square centered about \(x\), i.e., \(x + c^{II}_a\), for \(a = 1, 2, 3, 4\), with \(c^{II}_a \equiv \sqrt{2}(\cos ((a-1/2)\pi/2), \sin ((a-1)\pi/2))\).

There will be three distribution functions: one that streams in the lattice indicated by the superscript \(I\), a second one that is moved in the lattice \(II\), and a “stopped” distribution. Therefore, the streaming part of the evolution can be represented by the expression,

\[
f^K_{ab}(x, T) \rightarrow \frac{1}{2}f^K_{ab}(x + c^K_a, T + 1) + \frac{1}{2}f^K_{ab}(x + c^K_b, T + 1), \tag{72}
\]

where \(K = I\) or \(II\), \(a = 1, 2, 3, 4\), \(b = a + 1\) or \(a - 1\) (modulo 4), and \(T\) is the discrete
lattice time.

The macroscopic quantities we are interested in following, the density, velocity, and magnetic field, are defined below,

\[ \rho = f_0 + \sum_{a,b,K} f_{ab}^K \]  \hspace{1cm} (73)

\[ \rho v = \frac{1}{2} \sum_{a,b,K} (c_a^K + c_b^K) f_{ab}^K \]  \hspace{1cm} (74)

\[ \rho B = \sum_{a,b,K} q_K (-c_a^K + c_b^K) f_{ab}^K, \]  \hspace{1cm} (75)

where \( q_1 = 1/2 \), and \( q_2 = 1 \) and \( f_0 \) represents the stopped distribution.

During the collisional part of the evolution is when the three types of distributions “see” each other. We use the single time relaxation approximation with parameter \( \tau \), so that the discrete kinetic equation obeyed by the system is,

\[ f_{ab}^K(x, T) = \frac{1}{2} \left[ f_{ab}^K(x - c_a^K, T - 1) + \Omega_{ab}(x - c_a^K, T - 1) \right] + \frac{1}{2} \left[ f_{ab}^K(x - c_b^K, T - 1) + \Omega_{ab}(x - c_b^K, T - 1) \right], \]  \hspace{1cm} (76)

where

\[ \Omega_{ab} = -(f_{ab}^K - f_{ab}^{K(eq)}) / \tau. \]  \hspace{1cm} (77)

The procedure to get the macroscopic MHD equations is familiar to us at this point. The only thing that is left to complete the definition of the model is to specify the distribution functions,

\[ f_{ab}^K = \rho d_K \left[ 1 + \frac{1}{24d_2} (c_a^K + c_b^K) \cdot v + \frac{\alpha K}{24d_2} (c_a^K + c_b^K) \cdot B + \frac{1}{32d_2} (c_a^K \cdot v)^2 \right. \]

\[ + \frac{1}{32d_2} (c_b^K \cdot v)^2 + \frac{1}{24d_2} (c_a^K \cdot v)(c_b^K \cdot B) - \frac{1}{24d_2} (c_a^K \cdot B)(c_b^K \cdot v) \]

\[ + \frac{1}{8d_2} (c_a^K \cdot v)(c_b^K \cdot v) - \frac{1}{8d_2} (c_a^K \cdot B)(c_b^K \cdot B) \]  \hspace{1cm} (78)

\[ f_{ab}^0 = \rho d_0 \left[ 1 - \frac{3}{2d_0} v^2 \right], \]  \hspace{1cm} (79)
where
\[
d_0 = 1 - 40d_2 \quad \quad d_1 = 4d_2
\]
\[
\alpha_1 = 1 \quad \quad \alpha_2 = \frac{1}{2},
\]

\( K = I \) or \( II \), and \( 0 < d_2 < 0.025 \) for positivity of \( d_0 \).

For the sake of completeness, we document the tensorial identities relevant to the derivation of the model

\[
\sum_{ab} c_a^K c_a^K = \sum_{ab} c_b^K c_b^K = 2A_K I,
\]
\[
\sum_{ab} c_a^K c_b^K = 0,
\]
\[
\sum_{ab} (c_a^K)_i (c_a^K)_j (c_a^K)_k (c_a^K)_l = 2Z_K \Delta_{ijkl} + 2Y_K \delta_{ijkl}, \tag{80}
\]
\[
\sum_{ab} (c_a^K)_i (c_a^K)_j (c_a^K)_k (c_b^K)_l = 0,
\]
\[
\sum_{ab} (c_a^K)_i (c_b^K)_j (c_b^K)_k (c_b^K)_l = 0,
\]
\[
\sum_{ab} (c_a^K)_i (c_b^K)_j (c_b^K)_k (c_b^K)_l = A_2^K \delta_{ij} \delta_{kl} - 2Z_K \Delta_{ijkl} - 2Y_K \delta_{ijkl}.
\]

where \( \Delta_{ijkl} = \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}, \) \( \delta_{ijkl} = 1 \) only if \( i = j = k = l \), otherwise is 0; \( A_1 = 2, A_2 = 4, Z_1 = 0, Z_2 = 4, Y_1 = 2, \) and \( Y_2 = -8 \). \( I \) represents the identity matrix.

The viscosity and resistivity can be calculated using the Chapman-Enskog expansion, in a similar fashion as it was done in Section III,

\[
\nu = \frac{\tau + 1}{6} \tag{81}
\]
\[
\mu = \frac{3\tau - 1}{6}. \tag{82}
\]

The transport coefficients have been numerically measured for the case of decaying shear flows, and the values found were in agreement better than 0.1\% with the predictions of (81) and (82).
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Figure Captions

Fig. 1 Profiles of $v_x$ vs. $y/L$ for Hartmann number $H = 0, 1, 3, 5, 8,$ and $13,$ from top to bottom. The $H = 0$ (unmagnetized) case corresponds to the Poiseuille flow. The lines indicate analytical results, and the symbols are the solutions provided by the LBE scheme.

Fig. 2 Profile of $B_x$ vs. $y/L$ for $H = 3$ (+ symbol), $H = 1$ ($\diamond$ symbol), and $H = 13$ (△ symbol). The lines indicate analytical solutions, and the symbols represent the LBE solutions. Simulations with $H = 5$ and $8$ were carried out with similar results, and they are not presented here for clarity.

Fig. 3 Time histories of bulk quantities for the LBE run and the spectral run. Evolution of a) the mean square vector potential ($A$); b) the total (magnetic plus kinetic) energy $E$; c) kinetic energy ($E_k$); d) mean square vorticity (the enstrophy $\Omega$), and e) the mean square current $J$. The LBE run is indicated by the solid line and the spectral run by the dashed line.

Fig. 4 Contours of constant magnetic field (constant $a$), for the spectral (SP) and LBE runs, at times approximately equal to one, seven, and ten. Growth of magnetic islands can be observed for both runs.

Fig. 5 Lines of constant vorticity at times approximately equal to one, seven, and ten, for both runs. Quadrupole-like structures can be noticed in the region of the $X$-points.

Fig. 6 Contours of constant stream function ($\psi$) at times approximately equals to one, seven, and ten. Similar features can be observed for both spectral (SP) and LBE runs.
Fig. 7 Lines of constant current density at times approximately one, seven, and ten, for the LBE run and the spectral run. Diffusion from the sheets area can be observed, as well as filamentation in the $X$-point regions.

Fig. 8 Evolution of $B_L/B_\perp$ for the LBE run, where $B_L$ is the longitudinal component of the magnetic field. The higher initial value is due to the unequal physical lengths in the $x$- and $y$-directions, associated to the use of the hexagonal lattice.