EMBEDDING ALGORITHMS AND APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract. Algorithms for embedding certain types of nilpotent subalgebras in maximal subalgebras of the same type are developed, using methods of real algebraic groups. These algorithms are applied to determine non-conjugate subalgebras of the symmetry algebra of the wave equation, which in turn are used to determine a large class of invariant solutions of the wave equation. The algorithms are also illustrated for the symmetry algebra of a classical system of differential equations considered by Cartan in the context of contact geometry.

1. Introduction

One of the main applications of Lie algebras is to find solutions of differential equations, by reduction of order, or by using conjugacy classes of its subalgebras to find invariant solutions. The method of invariant solutions goes back to [Lie2, Ch. X]. This method is also explained in detail in the books of Ibragimov [Ib1, Ch 9], Ibragimov [Ib2], Bluman [Bl, B2] and Olver [Ol, Ch 3].

The Lie theoretic input in this method is a list of conjugacy classes of subalgebras of dimension depending on the order of the equation. A detailed structure of the symmetry algebra is also useful in finding linearizing coordinates for linearizable equations.

It is our experience, based on [ADGM], that if the algebras are not chosen appropriately, they are practically useless, because a preliminary step is to find their invariants and there is no algorithmic procedure to do that. However, if the subalgebras are constructed from the geometry of the space on which one is studying a given equation, for example by embedding translations or scalings in maximal subalgebras, the characteristics of the subalgebras obtained are manageable.

The principal aim of this note is to give algorithms for embedding given abelian and solvable algebras of certain types in maximal subalgebras of the same type, using standard commands of Maple.

The precise types of the subalgebras are given in the algorithms constructed below.

Maple is able to find the Cartan decomposition as well as root space decompositions for semisimple algebras of fairly high dimensions. The algorithms it uses are based on the fundamental papers of Rand, Winternitz and Zassenhaus [RWZ], of de Graaf [dG], and of Dietrich, Faccin and de Graaf [DFG] and Ian Anderson [An]. The recent book of Šnobl-Winternitz [SW] gives a detailed account of some of these algorithms.

Derksen, Jeandel and Koiran [DJK] have also developed algorithms for computing the Zariski closures of linear solvable Lie groups and the algorithms in this paper reduce the computation of Zariski closures of linear groups to those of abelian subgroups. The algorithms given in this paper are based on results of Mostow [Mo] on real algebraic groups; see a recent account of the

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subject in [AB]. All the algebras considered in this paper are assumed to be real algebraic Lie algebras. We recall their definition and some basic facts about them.

1. Let $\mathfrak{g} \subset \text{gl}(n, \mathbb{R})$ be a Lie algebra, with $G$ the corresponding Lie group. The algebra $\mathfrak{g}$ is called an algebraic Lie algebra if the group $G^\mathbb{C} \subset \text{GL}(n, \mathbb{C})$ with Lie algebra $\mathfrak{g}^\mathbb{C} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$ is an algebraic group as defined in [Bo].

2. The group $G^\mathbb{C}$ is generated by complex 1-parameter subgroups

$$\{\exp(zX) \mid z \in \mathbb{C}, X \in \mathfrak{g}\}$$

and the connected component of the real points of $G^\mathbb{C}$ is the group $G$.

All real semisimple Lie algebras, all real linear Lie algebras generated by nilpotents, all abelian real linear algebras of semisimple elements defined by integral equations as well as all subalgebras generated by the types of algebras already listed are examples of algebraic Lie algebras.

If an abstract Lie algebra is given by its commutator table, then one is tacitly working in its adjoint representation and all the concepts regarding semisimplicity, nilpotency etc. are with reference to this representation.

As regards the reduced root space decompositions, Maple needs a maximal split abelian algebra of semisimple elements. If that is not specified, the commands will give, in general, roots taking complex eigen-values.

The algorithms constructed in this paper give both the relative and absolute root system of the Lie algebra more or less automatically.

We will illustrate the algorithms by working out in detail an embedding of a subalgebra, which is clearly abelian, of the symmetry algebra of the wave equation on flat 4 dimensional space, in a maximal solvable subalgebra: this gives at the same time detailed structure of the symmetry algebra and several non-conjugate subalgebras.

These algebras are non-conjugate by construction. A solvable linear algebraic algebra is itself a sum of a torus and its maximal nilpotent ideal. A linear algebraic torus has either all real eigen-values or all purely imaginary eigen-values or it is itself a sum of two such tori [Bo, AB Proposition 1]. Thus if given subalgebras of a semisimple algebra are abstractly isomorphic and solvable one looks at their semisimple and nilpotent parts in the adjoint representation of any semisimple subalgebra containing them to decide if they are non-conjugate. The uniqueness of the Jordan decomposition [Bo] ensures that it is immaterial which ambient subalgebra one chooses as long as it is semisimple.

If the subalgebras are semisimple and abstractly isomorphic, one looks at their centralizers or normalizers to decide if they are non-conjugate.

It may happen that their centralizers or normalizers are the same. In that case one has to use Bruhat decomposition or its variants [Kn] or methods similar to [GW, PWZ] to decide non-conjugacy under the adjoint group. This type of complete symmetry analysis has not been done in this paper and the examples were chosen to illustrate that while doing the algorithms interactively and computing normalizers or centralizers one gets several subalgebras for which one could check their non-conjugacy without using Bruhat or Iwasawa decomposition.

We do the same for the symmetry algebra of a classical nonlinear ODE, whose symmetry algebra was determined and identified as the exceptional algebra $G_2$ by Cartan — see [Ag] for further references, and also [AKO] and [Ke]. The group $G_2$ also has a very interesting relation to mechanics. A recent account is in Bor and Montgomery [BM]. One of the main points in [BM] is to identify a maximal compact subalgebra of $G_2$ and give its explicit decomposition. The identification and decomposition of the maximal compact subalgebra of $G_2$ follows from algorithm 1 in a straight forward way.
The structural information obtained in Section 4 gives several three and four dimensional subalgebras that are non-conjugate in the adjoint representation. This gives a more extensive and useable list of subalgebras than that given in [ADGM] and these subalgebras are used in the last section to give solutions of the wave equation in flat 4-d space.

In a follow up of this paper, a similar analysis of the wave equation on all static spherically symmetric spaces times will be given to obtain a more extensive list of solutions than that given in [ADGM] of the wave equation on certain spherically symmetric four dimensional spaces. In particular, such invariant solutions will be given for all types of 3 dimensional subalgebras that can arise as subalgebras of the symmetry algebra of the equation.

The reader is referred to [Ib2] for very general results on the wave equation on Riemannian manifolds and several other equations of relevance to physics.

As far as Lie algebras are concerned, we need the following results:

1. If $X$ is an element of a Lie algebra $L$ which is the Lie algebra of a real algebraic subgroup of $GL(n, \mathbb{R})$, and $X = X_s + X_n$ is the Jordan decomposition of $X$, then both $X_s$ and $X_n$ are in $L$ [Bo, p. 14, Section 3.7, Proposition 1, Lemma 1], [AB]. In fact, they are in the center of the centralizer of $X$ in $L$.

2. If $S$ is an abelian subalgebra consisting of semisimple elements of a real algebraic Lie algebra, then its centralizer $Z(S)$ has the Levi decomposition

$$Z(S) = Z(Z(S)) \oplus Z(Z(S)),$$

where $Z(Z(S))$ is the center of $Z(S)$ [BT] (a proof of this is also given below in Proposition 1).

3. The derived algebra of any solvable algebra is ad-nilpotent. In particular, if $H$ is any subalgebra of $L$ then the derived algebras of the radical of centralizer of $H$ and of the normalizer of $H$ are ad-nilpotent [HN, p. 105, Theorem 5.4.7], [AB, Proposition 1].

4. If $H$ is a semisimple subalgebra of $L$ and $X$ is an ad-semisimple or ad-nilpotent element of $H$ in the adjoint representation of $H$ on itself, then $X$ is also ad-semisimple or ad-nilpotent in the adjoint representation of $L$ [Bo, p. 14, 3.7].

These facts are very useful in verifying that a certain element is semisimple or nilpotent by reducing the computations to subalgebras of small dimensions.

2. Roots

2.1. Roots of a semisimple algebra. A few words regarding the section on roots are in order. Maple will give – for any Cartan algebra – an array of complex numbers. In the following section we explain how to extract a simple system of roots and the corresponding Dynkin diagram directly from such a list.

Let $L$ be a semisimple Lie subalgebra of $gl(n, \mathbb{R})$ and $C$ a Cartan subalgebra of $L$. The algebra $C$ is, by definition, a maximal abelian subalgebra of diagonalizable elements in the complexification of $L$. A nonzero vector $v$ in $L \oplus \sqrt{-1}L$ such that

$$[h, v] = \lambda(h) \cdot v$$

for all $h \in C$ is called a root vector and the corresponding linear functional $\lambda$ is called a root of the Cartan algebra $C$.

In general, the roots will be complex valued, so one needs to define what it means for a complex valued root to be positive – based only on the list of roots provided by the program. This is sufficient to describe the Dynkin diagram algorithmically, as detailed below.
A complex number \( z = a + \sqrt{-1}b \), where \( a, b \in \mathbb{R} \), is positive if either its real part \( a \) is positive or \( a = 0 \) but \( b > 0 \).

Fix a basis \( h_1, \cdots, h_r \) of \( C \). A non-zero root \( \lambda \) is positive if the first nonzero number \( \lambda(h_i) \) is a complex positive number. Otherwise, it is called a negative root.

Positive roots which are not a sum of two positive roots are called simple roots.

For sake of convenience, henceforth a root will mean a non-zero root.

2.2. **Restricted roots.** An abelian subalgebra of \( L \) consisting of semisimple elements in the adjoint representation on \( L \), is, by definition, a torus. If, moreover, all its elements in the adjoint representation of \( L \) have real eigen-values, then it is a real torus; if all eigen-values are purely imaginary, it is called a compact torus.

Any real algebraic torus is a sum of a real and a compact torus and the dimensions its real and compact parts are invariants of the torus [AB, Proposition 1].

Moreover, all maximal solvable subalgebras \( B \) of a real semisimple algebra with real eigen-values in the adjoint representation are conjugate [Mo], [AB]. In the context of the Iwasawa decomposition [Kn], [HN], this is the algebra \( A \oplus N \).

If \( A \) is a maximal torus of \( B \) then the full algebra is a sum of \( A \)-invariant subspaces — of dimension possibly greater than one — and the roots \( A \) in \( B \) which are not a sum of two roots in \( B \) are simple roots of a root system — in the sense of [HN], [Kn]. In case that the real semisimple algebra has a maximal torus with all real eigen-values we can define positive roots without going to the complexification of the algebra. In this case the positive root spaces together with the torus give a maximal solvable algebra whose eigen-values are all real and all such algebras are conjugate. Therefore, the restricted root system and the absolute root system coincide in this case.

For each positive simple root \( \alpha \) we can find a standard set of generators \( X_{\alpha}, Y_{\alpha}, H_{\alpha} \) with \( X_{\alpha}, Y_{\alpha} \) eigen-vectors of \( \text{ad}(H_{\alpha}) \) with opposite and nonzero eigen-values. This three dimensional subalgebra is therefore isomorphic to \( \text{sl}(2, \mathbb{R}) \). If, for each simple root \( \alpha \) we fix an isomorphism

\[
\varphi_\alpha : \text{sl}(2, \mathbb{R}) \rightarrow \langle X_{\alpha}, Y_{\alpha}, H_{\alpha} \rangle,
\]

where \( \langle X_{\alpha}, Y_{\alpha}, H_{\alpha} \rangle \) is the Lie subalgebra generated by \( X_{\alpha}, Y_{\alpha}, H_{\alpha} \), then the elements

\[
\varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

generate, as a Lie subalgebra, a maximal compact subalgebra of the given Lie algebra — only in the case that the Lie algebra has a maximal torus whose eigen-values are all real [St, p. 100, Lemma 43].

2.3. **Procedure for constructing Dynkin diagram.** Positive roots which are not a sum of two positive roots are called simple roots. Thus to obtain simple roots from a given set of positive roots, one adds pairs of positive roots and marks those that are sums of positive roots; at the end, one strikes out those roots that are sums of two positive roots and the remaining ones will be simple roots.

Let \( a, b \) be simple roots. The positive roots among the integral combinations of them determine the bond between \( a \) and \( b \). The simple roots \( a, b \) are not joined if \( a + b \) is not a root. They are joined by a single bond if \( a, b \) and \( a + b \) are the only positive roots among the integral combinations of \( a \) and \( b \). They are joined by a double bond with arrow pointing from \( a \) to \( b \) if \( a, b, a + b \) and \( a + 2b \) are the only positive roots among the integral combinations of \( a \) and \( b \). They are joined by a triple bond with arrow pointing from \( a \) to \( b \) if \( a, b, a + b, a + 2b, a + 3b \) and \( 2a + 3b \) are the only positive roots among the integral combinations of \( a \) and \( b \).
The diagram then identifies the complexification of the Lie algebra $L$. 

3. Algorithms

We need the following result on centralizers of semisimple elements to implement the algorithms.

**Proposition 1.** Let $S$ be a commuting algebra of diagonalizable elements in a real semisimple algebra $L$. Then the centralizer of $S$ has the Levi decomposition

$$Z_L(S) = [Z_L(S), Z_L(S)] \oplus Z(Z_L(S)),$$

where $Z_L(S)$ is the centralizer of $S$ in $L$, and $Z(Z_L(S)) \subset Z_L(S)$ is its center.

**Proof.** Recall that we defined a nonzero complex number to be positive if its real part is positive or if its real part is zero and the imaginary part is positive.

Include $S$ in a maximal torus $T$. The complexification of $T$ is a maximal torus of the complexification $L^C$ of $L$ \[1\] Corollary 7], and the centralizer $Z_L(S)$ of $S$ in $L$ is the same as the real points of the centralizer of $S^C$ in $L^C$. Now $Z_L^C(S^C)$ is generated by $T^C$ and the root vectors $X_\alpha$ such that $\alpha(S) = 0$. This is a closed set of roots and the Lie algebra $Z_L^C(S^C)$ contains the root vector $X_{-\alpha}$ for every $\alpha$ as above. By extending a basis of $S^C$ say $\{s_i\}_{i=1}^n$ to a basis of $T^C$, say $\{s_i\}_{i=1}^m$ and declaring a root $r$ to be positive if the first nonzero number $r(s_i)$ is positive, we see that the indecomposable positive roots of $Z_L^C(S^C)$ are simple roots of $T^C$. Thus these generate a semisimple subalgebra $L_1$ of $Z_L^C(S^C)$ and $Z_L^C(S^C) = (L_1, T^C)$. Since $T^C$ normalizes $L_1$ the commutator is $L_1$. Hence $Z_L^C(S^C)/L_1$, being an image of $T^C$, contains no nilpotents.

Therefore, the Levi decomposition of $Z_L^C(S^C)$ is

$$Z_L^C(S^C) = L_1 + R,$$

and $R$, being solvable with no nilpotents, is a torus. As $[L_1, R]$ is contained in both $L_1$ and $R$ it must be 0. Hence $R$ is a central torus and it is equal to $Z(Z_L^C(S^C))$ — the center of $Z_L^C(S^C)$. Therefore, the Levi decomposition of $Z_L^C(S^C)$ is $[Z_L^C(S^C), Z_L^C(S^C)] \oplus Z(Z_L^C(S^C))$. Taking real points gives the Levi decomposition $Z_L(S) = [Z_L(S), Z_L(S)] \oplus Z(Z_L(S))$.  

The algorithms given below are similar to each other. For the convenience of the user we have written down complete details—at the expense of repetition—of the most frequent types of algebras encountered in practice.

3.1. **Algorithm for embedding a given abelian subalgebra of semisimple elements with real eigen-values.** Here we give an algorithm for embedding a given abelian subalgebra of semisimple elements with real eigen-values in a maximal algebra of such elements and in a maximally real Cartan algebra: in this algorithm, the ambient algebra is assumed to be semisimple.

For a subalgebra $H$ of $L$, let $N_L(H)$, $Z_L(H)$, $Z(H)$ and $H'$ denote its normalizer in $L$, its centralizer in $L$, its center and its derived algebra respectively. For notational convenience, we will also write $N(H)$ for $N_L(H)$.

Let $A$ be real torus (defined in Section \[2\]).

**Step 1:** Compute $Z_L(A)$, the centralizer of $A$ in $L$, the derived algebra $Z_L(A)'$ of $Z_L(A)$ and the center $Z(Z_L(A))$ of $Z_L(A)$. Then one has the direct sum decomposition

$$Z_L(A) = Z_L(A) ' \oplus Z(Z_L(A)).$$
Step 2: Compute the Killing form of $Z_L(A)'$. If it is negative definite, then the real part of the subalgebra $Z(Z_L(A))$ is a maximal real torus.

Step 3: If the Killing form of $Z_L(A)'$ is indefinite, compute the Cartan decomposition of $Z_L(A)'$, and pick any nonzero element from the radial part of the decomposition and adjoin it to $A$.

Repeat Step 1 and Step 2 till an abelian algebra, which we again denote by $A$, is obtained which has all real eigen-values in the adjoint representation – and in the decomposition $Z_L(A) = Z_L(A)' \oplus Z(Z_L(A))$, the Killing form of $Z_L(A)'$ is negative definite.

At this stage, a maximal real torus containing the given algebra has been obtained. Denote it again by $A$.

The compact part of $Z(Z_L(A))$ together with a maximal torus of $Z_L(A)'$ is a compact torus. Adjoining it to $A$ gives a maximally real Cartan algebra.

Remark 2. By an entirely similar procedure, a compact torus can be embedded in a maximally compact Cartan subalgebra.

3.2. Algorithm for embedding a commutative subalgebra of ad-nilpotent elements to a maximal commutative subalgebra of such elements. Let $U$ be an abelian algebra of ad–nilpotent elements. Let

$$Z_L(U) = S \oplus R$$

be the Levi decomposition of $Z_L(U)$. Compute the derived subalgebra $R' \subset R$. If $\dim(R' + U) > \dim U$, adjoin any element of $R'$ complementary to $U$ to obtain a commutative subalgebra of ad–nilpotent elements. Repeat this procedure until an abelian algebra of ad–nilpotent elements is obtained – which we denote again by $U$ – so that in the Levi decomposition on $Z_L(U)$, the algebra $R'$ is contained in $U$.

At this stage if $R$ contains $U$ as a proper subalgebra, consider an element $x$ of $R$ complementary to $U$. Then the nilpotent and semisimple parts of $x$ belong to $R$. If $x$ has a nonzero nilpotent part $x_n$, then the subalgebra generated by $U$ and $x_n$ is commutative consisting of ad–nilpotent elements.

Repeating the above procedure, we may assume that $U$ is a commutative subalgebra of ad–nilpotent elements such that in the Levi decomposition

$$Z_L(U) = S \oplus R,$$

$R' \subset U$, and every element in a basis of $R$ complementary to $U$ consists of semisimple elements.

If the Killing form of $S$ is not negative definite, then $S$ will have a nontrivial Cartan decomposition. Take an element $\alpha$ in the radial part of the Cartan decomposition of $S$. As $S$ has no center, the endomorphism $\text{ad}(\alpha)$ of $S$ has a nonzero real eigen-value. In fact any element all of whose eigen-values are real will do. If $u$ is a nonzero eigen-vector of $\text{ad}(\alpha)$ for such a nonzero real eigen-value, then $\text{ad}(u)$ is nilpotent on $S$. The reason is that $S$ is a direct sum of eigen-spaces for $\text{ad}(\alpha)$ and if $S_a, S_b$ are two such eigen-spaces, then $[S_a, S_b]$ is contained in $S_{a+b}$; therefore a sufficiently high power of $\text{ad}(u)$ annihilates $S$. This implies that $\text{ad}(u)$ is also nilpotent on $L$, by uniqueness of Jordan decomposition. Adjoin $u$ to $U$ to obtain a higher dimensional commutative algebra of nilpotents.

Thus repeating the above procedure we ultimately have a commutative subalgebra consisting of ad–nilpotent elements, which we again denote by $U$, such that

$$Z_L(U) = S + R,$$

$R' \subset U$, every element in a basis of $R$ complementary to $U$ consists of semisimple elements and $S$ has a negative definite Killing form.
At this stage $U$ is a maximal abelian subalgebra of ad–nilpotent elements.

A very similar argument (-detailed below-) gives the embedding of a given ad–nilpotent subalgebra in a maximal ad–nilpotent subalgebra. Here one is assured of conjugacy of these subalgebras [AB]. Also, its normalizer will pick up a torus whose eigen-values are all real and which is maximal with these properties. Embedding this in a maximal torus – using Algorithm 3.1 – one will obtain a maximally split Cartan subalgebra of $L$.

3.3. **Algorithm for embedding a subalgebra $U$ of ad–nilpotent elements to a maximal subalgebra of such elements.**

**Step 1:** Find a normalizer of $U$ and compute its Levi decomposition

$$N(U) = S \oplus R,$$

where $R$ is the radical.

**Step 2:** Compute the derived algebra $R'$ of $R$. If $\dim(R' + U) > \dim U$, then this is again an ad–nilpotent algebra.

Repeat Step 1 and Step 2 so that ultimately $R' + U = U$. At this stage $R/U$ is abelian.

**Step 3:** We want to enlarge $U$ further so that $R/U$ consists entirely of semisimple elements. To do this, take a basis of $U$, say $u_1, \cdots, u_k$, and enlarge it to a basis of $R$ by adjoining $v_1, \cdots, v_\ell$. Find the Jordan decomposition of all $v_1, \cdots, v_\ell$. Adjoin to $U$ the nilpotent parts of all the $v_1, \cdots, v_\ell$. Denote by $\tilde{U}$ the algebra obtained this way.

Now repeat Step 1, Step 2 and Step 3 till we have the Levi decomposition

$$N(\tilde{U}) = S + R$$

such that $R' \subset \tilde{U}$ and $R/\tilde{U}$ consists only of semisimple elements.

**Step 4:** If the Killing form of $S$ is not negative definite, then it will have a nontrivial Cartan decomposition. Take an element $\alpha$ in the radial part of the Cartan decomposition of $S$. As the center of $S$ is trivial, the endomorphism $\text{ad}(\alpha)$ of $S$ has a nonzero real eigen-value. Let $u$ be a nonzero eigen-vector for such an eigen-value. Then $\text{ad}(u)$ is nilpotent on $S$ and therefore on the Lie algebra $L$ – by uniqueness of the Jordan decomposition. Adjoin $u$ to $U$ to obtain a higher dimensional algebra of nilpotents.

Iterating this procedure, we finally reach the situation that we have an ad–nilpotent subalgebra, which we denote again by $U$, that contains the original ad–nilpotent subalgebra, such that the Levi decomposition of $N(U)$ is

$$N(U) = S \oplus R,$$

where $S$ has a negative definite Killing form, $R' \subset U$ and $R/U$ consists only of semisimple elements in the sense that if we extend a basis of $U$ to a basis of $R$ and find the Jordan decomposition of the basis elements outside $U$, then the nilpotent parts all belong to $U$ [AB].

At this stage, $U$ is a maximal ad–nilpotent algebra containing the given ad–nilpotent algebra.

Finally, the abelian algebra representing $R/U$ is a torus and its real part is a maximal abelian algebra consisting of real semisimple elements [AB]. Denote this algebra by $A$. Then $A$ can be enlarged to a maximally split Cartan algebra of the whole algebra, using Algorithm 3.1. This algebra permutes the common eigen-spaces of $A$.

4. **Applications to structure of symmetry algebras related to certain equations of physics**

Before giving applications to symmetry algebras of higher dimensions, we illustrate the algorithms to obtain structural information for the algebras $\mathfrak{so}(4)$, $\mathfrak{so}(1,3)$ and $\mathfrak{so}(2,2)$. 
Recall that if \( A \) is an \( n \times n \) diagonal matrix with diagonal entries 1, \(-1\), then the Lie algebra of the corresponding orthogonal group has generators \( e_{ij} - e_{ji} \) if \( a_{ii}a_{jj} = 1 \), and \( e_{ij} + e_{ji} \) if \( a_{ii}a_{jj} = -1 \). Moreover, if \( A \) has the first \( p \) diagonal entries 1, and the next \( q \) diagonal entries \(-1\), where \( p + q = n \), then the Lie algebra of the corresponding orthogonal group is generated by
\[
e_{1,2} - e_{2,1}, \ldots, e_{p-1,p} - e_{p,p-1}, e_{p,p+1} + e_{p+1,p}, e_{p+1,p+2} - e_{p+2,p+1}, \ldots, e_{n-1,n} - e_{n,n-1}.
\]

4.1. \( \mathfrak{so}(4) \). We now derive the factorization of \( \mathfrak{so}(4) \) using roots of its maximal torus.

A basis of \( V = \mathfrak{so}(4) \) is
\[
e_1 = e_{12} - e_{21}, e_2 = e_{13} - e_{31}, e_3 = e_{14} - e_{41}, e_4 = e_{23} - e_{32}, e_5 = e_{24} - e_{42}, e_6 = e_{34} - e_{43}.
\]
Using Algorithm 3.1, a Cartan subalgebra \( C \) is generated by \( \{e_1, e_6\} \). The roots of \( C \) are
\[
a := (\sqrt{-1}, \sqrt{-1}), b := (\sqrt{-1}, -\sqrt{-1}), -a, -b.
\]

As \( a + b \) is not a root, this root system is of type \( A_1 \times A_1 \).

Also, conjugation maps a root to its negative. Thus the subalgebras generated by the eigenspaces \( V_r, V_-r, r = a, b \), contain a real form of \( \mathfrak{sl}(2, \mathbb{C}) \), which must be isomorphic to \( \mathfrak{so}(3) \).

In more detail, we need to compute only the eigen-vectors for the positive roots. Their real and imaginary parts will give the decomposition of the compact algebra \( V \). Now,
\[
V_a = \langle e_2 + \sqrt{-1} e_3 + \sqrt{-1} e_4 - e_5 \rangle, \ V_b = \langle e_2 + \sqrt{-1} e_3 - \sqrt{-1} e_4 + e_5 \rangle.
\]
The real and imaginary parts of the basis elements in \( V_a \) and \( V_b \) and generate \( \mathfrak{so}(4) \).

Let
\[
u_1 = e_2 - e_5, \ v_1 = e_3 + e_4, \ u_2 = e_2 + e_5, \ v_2 = e_3 - e_4.
\]
Then \( u_1, v_1, [u_1, v_1] = -2(e_1 + e_6) \) generate a copy of \( \mathfrak{so}(3) \). Note that \( u_2, v_2, [u_2, v_2] = 2(e_1 - e_6) \) also generate a copy of \( \mathfrak{so}(3) \). These two copies of \( \mathfrak{so}(3) \) commute because the root system is of type \( A_1 \times A_1 \). This gives the well known fact that \( \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \).

4.2. \( \mathfrak{so}(1,3) \). A basis of \( V = \mathfrak{so}(1,3) \) is
\[
e_1 = e_{12} + e_{21}, e_2 = e_{13} + e_{31}, e_3 = e_{14} + e_{41}, e_4 = e_{23} - e_{32}, e_5 = e_{24} - e_{42}, e_6 = e_{34} - e_{43}.
\]
Using Algorithm 3.1 a Cartan subalgebra \( C = \langle e_1, e_6 \rangle \) is obtained. Note that there is no real split or compact Cartan subalgebra.

The roots of \( C \) are
\[
a := (1, -\sqrt{-1}), b := (1, \sqrt{-1}), a, -b.
\]
The root system is of type \( A_1 \times A_1 \), with positive roots \( a, b \) and conjugation maps \( a \) to \( b \).

The real rank is one, and the eigen-values of \( \text{ad}(e_1) \) are \(-1, -1, 0, 0, 1, 1 \). The corresponding root spaces are
\[
V_1 = \langle e_3 + e_5, e_2 + e_4 \rangle, V_{-1} = \langle -e_3 + e_5, -e_2 + e_4 \rangle.
\]

Also, \( [e_3 + e_5, -e_3 + e_5] = 2e_1 \); the subalgebra generated by \( e_3 + e_5, -e_3 + e_5 \) is \( \mathfrak{sl}(2, \mathbb{R}) \), while the subalgebra generated by \( e_4, e_5, e_6 \) is \( \mathfrak{so}(3) \).

Thus a maximal solvable subalgebra consisting of elements with real eigen-values in the adjoint representation is
\[
\langle e_1, e_3 + e_5, e_2 + e_4 \rangle,
\]
and a maximal solvable subalgebra is
\[
\langle e_6, e_1, e_3 + e_5, e_2 + e_4 \rangle.
\]
4.3. $\mathfrak{so}(2,2)$. A basis of $V = \mathfrak{so}(2,2)$ is

\[
e_1 = e_{12} - e_{21}, e_2 = e_{13} + e_{31}, e_3 = e_{14} + e_{41}, e_4 = e_{23} + e_{32}, e_5 = e_{24} + e_{42}, e_6 = e_{34} - e_{43}.
\]

Using Algorithm 3.1, a real split Cartan subalgebra is $C = \langle e_2, e_5 \rangle$, while a compact Cartan subalgebra is $\langle e_1, e_6 \rangle$.

The roots of $C$ are

\[
a := (1, 1), \ b := (1, -1), \ -a, \ -b.
\]

The root spaces are

\[
V_a = \langle e_1 - e_3 + e_4 - e_6 \rangle, \ V_b = \langle e_1 + e_3 + e_4 + e_6 \rangle,
\]

\[
V_{-a} = \langle e_1 + e_3 - e_4 - e_6 \rangle, \ V_{-b} = \langle e_1 - e_3 - e_4 + e_6 \rangle.
\]

Conjugation fixes the roots. Consequently, the subalgebra generated by a root spaces of a root and its negative is isomorphic to $\mathfrak{sl}(2\mathbb{R})$.

Therefore, denoting the subalgebra generated by $V_r, V_{-r}$ by $\langle V_r, V_{-r} \rangle$ the decomposition $\langle V_a, V_{-a} \rangle \oplus \langle V_b, V_{-b} \rangle$ gives an isomorphism of $\mathfrak{sl}(2\mathbb{R}) \oplus \mathfrak{sl}(2\mathbb{R})$ with $\mathfrak{so}(2,2)$.

4.4. Lie symmetries of wave equations. The algebra of Lie point symmetries of the wave equation in a flat 4-d space is sixteen dimensional and determined by the vector fields (following the same order as given in [ADGM] and using the notation in which $X$ is represented with $e$):

\[
e_1 = yt\partial_t + xy\partial_x + \frac{(y^2 + t^2 - x^2 - z^2)}{2}\partial_y + yz\partial_z - uy\partial_u,
\]

\[
e_2 = y\partial_t + t\partial_y,
\]

\[
e_3 = xt\partial_t + \frac{(x^2 + t^2 - y^2 - z^2)}{2}\partial_x + xy\partial_y + xz\partial_z - ux\partial_u,
\]

\[
e_4 = x\partial_t + t\partial_x,
\]

\[
e_5 = zt\partial_t + xz\partial_x + yz\partial_y + \zeta_{\partial_z} - uz\partial_u,
\]

\[
e_6 = z\partial_t + t\partial_z,
\]

\[
e_7 = t\partial_t + x\partial_x + y\partial_y + z\partial_z,
\]

\[
e_8 = \partial_t,
\]

\[
e_9 = (t^2 + x^2 + y^2 + z^2)\partial_t + 2tx\partial_x + 2ty\partial_y + 2tz\partial_z - 2ut\partial_u,
\]

\[
e_{10} = \partial_y,
\]

\[
e_{11} = \partial_x,
\]

\[
e_{12} = \partial_z,
\]

\[
e_{13} = z\partial_y - y\partial_z,
\]

\[
e_{14} = z\partial_x - x\partial_z,
\]

\[
e_{15} = y\partial_x - x\partial_y,
\]

\[
e_{16} = u\partial_u.
\]

The commutator algebra of the finite dimensional part of the symmetry algebra of the wave equation on Minkowski space-time is 15 dimensional. By computing its radical or the Killing form one sees that this 15-dimensional algebra is semisimple.

Its basis is $e_1, \ldots, e_6, e_7 - e_{16}, e_8, \ldots, e_{15}$. For notational convenience we will denote-only in this section $e_7 - e_{16}$ by $e_7$. 

The commutator table is reproduced in Appendix 1. The translations parallel to the coordinate axes are
\[ \partial_x = e_{11}, \partial_y = e_{10}, \partial_z = e_{12}, \partial_t = e_8 \]
and they form an ad–nilpotent subalgebra.

Let \( U = \langle e_8, e_{10}, e_{11}, e_{12} \rangle \).

We will use standard Maple commands and Algorithm 3.3 to embed \( U \) in a maximal ad-nilpotent subalgebra \( \tilde{U} \) and also compute the normalizer of this subalgebra. It may be pointed out that the implementation of Algorithm 3.3 will give rise to the identification of several lower dimensional non-conjugate subalgebras. Since the standard symmetry reduction from translations yield trivial solutions, therefore our algorithm will provide translations embedded into those subalgebras that will provide non-trivial solutions of the wave equation. Consequently this approach provides a direct use of translational subalgebras.

As explained in Algorithm 3.3, in general, if \( N(\tilde{U}) = S + R \) is the Levi decomposition of \( N(\tilde{U}) \) then \( S \) has a negative definite Killing form, \( R' \) is contained in \( U \) and \( R/U \) consists only of semisimple elements in the sense that if we extend a basis of \( U \) to a basis of \( R \) and find the Jordan decomposition of the basis elements outside \( U \), then the nilpotent parts all belong to \( U \).

As the abelian algebra representing \( R/U \) is a torus and its real part \( A \) is a maximal abelian algebra consisting of real semisimple elements, \( A \) can be enlarged to a maximally split Cartan algebra of the whole algebra, using Algorithm 3.1. This algebra permutes the common eigen-spaces of \( A \).

Following Algorithm 3.3, we first compute \( N(U) \) and its Levi decomposition. We have
\[ N(U) = R \oplus S, \]
where \( R = \langle U, e_7 \rangle \) is the radical, and
\[ S = \langle e_{15}, e_{14}, e_{13}, e_6, e_5, e_4 \rangle \]
is semisimple with Cartan decomposition
\[ S = \langle e_{15}, e_{14}, e_{13} \rangle \oplus \langle e_6, e_5, e_4 \rangle \]
with compact part \( K = \langle e_{15}, e_{14}, e_{13} \rangle \) and radial part \( P = \langle e_6, e_5, e_4 \rangle \).

The element \( e_7 \) representing \( R/U \) is real semisimple and \( e_6 \) is maximal abelian in \( P \). The compact subalgebra \( K \) is the subalgebra of spatial rotations.

The eigen-values of \( \text{ad}(e_6) \) in \( S \), counting multiplicities, are \( 1, 1, -1, -1, 0, 0 \) and eigen-vectors for eigen-value 1 are \( -e_{15} + e_4, -e_{13} + e_6 \). Therefore, as the eigen-vectors for positive eigen-values of a real semisimple element of \( S \) form an ad-nilpotent subalgebra, following Algorithm 3.3, we adjoin \( -e_{15} + e_4 , -e_{13} + e_6 \) to \( U \) to get an ad-nilpotent algebra \( \tilde{U} \) and compute its normalizer. We find that
\[ N(\tilde{U}) = \langle \tilde{U}, e_2, e_7, e_{14} \rangle. \]
The subalgebra \( \langle e_2, e_7, e_{14} \rangle \) is abelian, and is a torus, whose real part is \( \langle e_2, e_7 \rangle \) and compact part is \( \langle e_{14} \rangle \).

Thus \( N(\tilde{U}) \) is self-normalizing and solvable. Therefore, by Algorithm 3.3, \( \tilde{U} \) is a maximal ad-nilpotent subalgebra containing \( U \). Using Algorithm 3.1, we find that \( \langle e_2, e_7, e_{14} \rangle \) is a Cartan subalgebra and \( A = \langle e_2, e_7 \rangle \) is a maximal abelian subalgebra of real semisimple elements.
The positive roots are 
\((-1, 0), (-1, -1), (-1, 1), (0, 1)\);
here, to say that \((r, s)\) is a root means that there is a common eigen-vector \(X\) for \(A\) which is not centralized by \(A\) and 
\([e_7, X] = rX, [e_2, X] = sX\).

Let 
\(a = (-1, 0), b = (-1, -1), c = (-1, 1), d = (0, 1)\).

This is a positive system of roots for \(A\) determined by \(N(\tilde{U})\). The only positive roots which are sums of positive roots are \(a + d = c\) and \(b + d = a\). Therefore, the simple roots are \(b, d\) and the roots as nonnegative integral combinations of the simple roots are \(b, d, b + d, b + 2d\).

Therefore, the Dynkin diagram of the reduced root system is of type \(B_2\) with \(b\) a long root.

Let \(\omega_7, \omega_2\) be linear functions on \(A\) dual to the ordered basis \(e_7, e_2\). With this notation, the roots are 
\(-\omega_7, -\omega_7 - \omega_2, -\omega_7 + \omega_2, \omega_2\).

Let \(L = N(\tilde{U})\). The corresponding eigen-spaces in \(L\) are 
\(L_{-\omega_7} = \langle e_{12}, e_{11} \rangle, L_{-\omega_7 - \omega_2} = \langle e_8 + e_{10} \rangle, L_{-\omega_7 + \omega_2} = \langle e_8 - e_{10} \rangle, L_{\omega_2} = \langle -e_13 + e_6, -e_15 + e_4 \rangle\).

Finally \(L_0 = \langle A, e_{14} \rangle\) and \(e_{14}\) operates on these eigen-spaces, as rotations on \(L_{-\omega_7}\) and \(L_{\omega_2}\), while it commutes with \(L_{-\omega_7 - \omega_2}\) and \(L_{-\omega_7 + \omega_2}\).

The absolute root system is determined by common eigen-vectors for the Cartan algebra 
\(C = \langle e_7, e_2, e_{14} \rangle\).

The positive roots are 
\(a = (0, 1, -\sqrt{-1}), b = (0, 1, \sqrt{-1}), c = (1, 0, \sqrt{-1})\)
\(d = (1, 1, 0), e = (1, -1, 0), f = (1, 0, -1)\).

Forming sums of pairs of positive roots and removing those roots that are sums of positive roots, we find that the simple roots are \(a, e, b\) with Dynkin diagram of type \(A_3\) with \(e\) the simple middle root.

Conjugation maps \(a\) to \(b\) and fixes \(e\). Thus the algebra is a real form of \(\mathfrak{sl}(4, \mathbb{C})\).

To find a maximally compact subalgebra – if any – we follow a procedure analogous to Algorithm 3.1 – starting with a compact element – namely one which generates a compact subgroup. For example, as \(\langle e_{15}, e_{14}, e_{13} \rangle\) generate \(\mathfrak{so}(3)\), because \(e_{15} = x\partial_y - y\partial_x\) and \(e_{14} = x\partial_x - z\partial_x\), we can with begin with \(e_{15}\), compute its centralizer, the center of its centralizer and its derived algebra. If the derived algebra is trivial, then the centralizer \(e_{15}\) of would be a maximal compact torus and its compact part will be a maximal compact subalgebra containing \(e_{15}\). If the derived algebra is nontrivial, it must have a compact element, say \(t\). Adjoining it to \(e_{15}\) and computing the centralizer of \(\langle e_{15}, t \rangle\) and its derived algebra and repeating the process, we will ultimately obtain a maximal compact subalgebra containing \(e_{15}\).

In this case we find that \(C_k = \langle t_1, t_2, e_{15} \rangle\) is a maximally compact Cartan subalgebra, where 
\(t_1 = 2e_{12} + e_5, t_2 = e_9 + 4e_8\).

The positive roots are 
\(a = (\sqrt{-1}, -4\sqrt{-1}, 0), b = (0, 4\sqrt{-1}, \sqrt{-1}), c = (\sqrt{-1}, 0, -\sqrt{-1})\),
\(d = (0, 4\sqrt{-1}, -\sqrt{-1}), e = (\sqrt{-1}, 4\sqrt{-1}, 0), f = (\sqrt{-1}, 0, \sqrt{-1})\).

The simple positive roots are \(d, a, b\) with Dynkin diagram of type \(A_3\) with \(a\) the middle simple root.
Conjugation maps every root to its negative. The root algebras generated by the real and imaginary parts of the root vectors are copies of \( \mathfrak{sl}(2, \mathbb{R}) \) except for roots \( d + a, a + b \), where they generate copies of \( \mathfrak{so}(3) \). Specifically, the subalgebras generated by the real and imaginary parts of root vectors for \( d + a \) and \( a + b \) are

\[
\langle e_1 + 2e_{10} - 2e_{14}, e_3 + 2e_{11} + 2e_{13}, -4e_5 - 8e_{12} + 8e_{15} \rangle,
\]

\[
\langle e_1 + 2e_{10} + 2e_{14}, -e_3 - 2e_{11} + 2e_{13}, -4e_5 - 8e_{12} - 8e_{15} \rangle.
\]

Both are isomorphic to \( \mathfrak{so}(3) \). Denoting these subalgebras by \( k_1 \) and \( k_2 \) respectively, we find that the centralizer of \( k_1 \) is

\[
k_2 \oplus \langle 4e_8 + e_9 \rangle.
\]

Moreover as the centralizer of the copy of \( \mathfrak{so}(3) \) given by \( k_0 = \langle e_{15}, e_{14}, e_{13} \rangle \) is \( \langle e_7, e_8, e_9 \rangle = \mathfrak{sl}(2, \mathbb{R}) \), the subalgebra \( k_1 \) is not conjugate to \( k_0 \).

Finally, as the Killing form of the full algebra has seven negative eigen-values, a maximal compact subalgebra is

\[
k_1 \oplus k_2 \oplus \langle 4e_8 + e_9 \rangle
\]

because \( 4e_8 + e_9 \) generates the maximal compact subalgebra of \( \langle e_7, e_8, e_9 \rangle = \mathfrak{sl}(2, \mathbb{R}) \).

5. Lie Symmetries of \( f_{xx} = \frac{4}{9} f_{yy}^3, f_{xy} = f_{yy}^2 \) and \( v' = (u'')^2 \)

These equations were considered by Cartan in the context of symmetries of a certain system of equations defined by differential forms \([\text{Ca}]\); he showed that their symmetry algebra was the 14 dimensional simple group \( G_2 \). Maple is able to compute both the algebras by using commands for contact symmetries and for generalized symmetries as well as its root space decomposition and its maximal compact subalgebra. The latter equation was also considered by Anderson, Kamran and Olver, \([\text{AKO}]\), in the context of generalized symmetries. To illustrate the algorithms of this paper, we will use the table given in \([\text{AKO}]\) – reproduced in Appendix 2 to identify the algebra and determine several interesting subalgebras. To streamline the calculations we will use repeatedly the following facts-already mentioned in the Introduction.

If \( H \) is a semisimple subalgebra of a semisimple algebra \( G \) then and element \( X \) of \( H \) is real semisimple, compact or nilpotent in the adjoint representation \( H \) of on itself, if and only if it is, respectively, real semisimple, compact or nilpotent in the adjoint representation of \( H \) in \( G \). Moreover, the derived algebras of the radical of normalizer or centralizer of any subalgebra are nilpotent.

The symmetry algebra has basis \( X_1, X_2, \cdots, X_{14} \) and is given by:
clearly commute and by Maple, we find that it is non-zero. Thus, the algebra is semisimple. The translations

\[ X_5 = -2u'' \partial_x + (v - 2u' u'') \partial_u - \frac{2}{3} u''' \partial_v, \]
\[ X_6 = \frac{1}{2} u \partial_u + v \partial_v, \]
\[ X_7 = -\frac{1}{2} x^2 \partial_x - \frac{3}{2} x u \partial_u - 2u^2 \partial_v, \]
\[ X_8 = -x \partial_x - \frac{3}{2} u \partial_u, \]
\[ X_9 = -\partial_x, \]
\[ X_{10} = \frac{1}{6} x^3 \partial_u + 2(xu' - u) \partial_v, \]
\[ X_{11} = \frac{1}{2} x^2 \partial_u + 2u' \partial_v, \]
\[ X_{12} = x \partial_u, \]
\[ X_{13} = \partial_u, \]
\[ X_{14} = \partial_v. \]

Using the commutator table in Appendix 2, and computing the determinant of the Killing form by Maple, we find that it is non-zero. Thus, the algebra is semisimple. The translations

\[ X_{14} = \partial_v, X_{13} = \partial_u \]

clearly commute and \( X_{12} = x \partial_u \) commutes with both. One can check that they are nilpotent: this also follows from computing the derived algebra of the radical of normalizer of \( U \). We want to embed

\[ U = \langle X_{14}, X_{13}, X_{12} \rangle \]
in a maximal subalgebra whose elements are all nilpotent. Following Algorithm 3.3 we compute the normalizer \( N(U) \) and find its Levi decomposition, using standard Maple commands:

\[ N(U) = \langle X_9, X_8, X_6, X_5, X_{14}, X_{13}, X_{12}, X_{11}, X_{10} \rangle \]

and its Levi decomposition is \( R(N(U)) \oplus S \), where the radical

\[ R(N(U)) = \langle X_9, X_8 - 3X_6, X_{14}, X_{13}, X_{12}, X_{11} \rangle \]

and the semisimple part \( S = \langle X_8 + X_6, X_5, X_{10} \rangle \). The commutators for the semisimple part are

\[ [X_8 + X_6, X_5] = 2X_5, \]
\[ [X_8 + X_6, X_{10}] = -2X_{10}, \]
\[ [X_5, X_{10}] = -2(X_8 + X_6). \]

This means that \( X_5 \) and \( X_{10} \) are nilpotent in the full algebra, \( X_8 + X_6 \) is real semisimple in the full algebra and \( X_5 + X_{10} \) is a compact element. Following Algorithm 3.3, we compute the derived algebra of the radical \( R(N(U)) \). It is

\[ \tilde{U} = \langle X_9, X_{14}, X_{13}, X_{12}, X_{11} \rangle. \]
The quotient $R(N(U))/\widetilde{U}$ is represented by $X_8 - 3X_6$, which is a real semisimple element. (This also follows by computing the centralizer of $X_8 + X_{10}$ and its derived algebra, which turns out to be $\langle X_8 - 3X_6, X_{12}, X_3 \rangle$. This is a standard $\mathfrak{sl}(2,\mathbb{R})$ with $X_8 - 3X_6$ as real semisimple element.)

Following Algorithm 3.3 we compute again $N(\widetilde{U})$ and its Levi decomposition. It turns out to be identical to the Levi decomposition of $N(U)$. We therefore adjoin a nilpotent element coming from the semisimple part of the decomposition, say $X_5$. Let

$$\widetilde{U} = \langle \widetilde{U}, X_5 \rangle.$$ 

Its normalizer is $\langle \widetilde{U}, X_8, X_6 \rangle$ and it is solvable, with commutator $\widetilde{U}$ and the quotient is represented by the real torus $\langle X_6, X_8 \rangle$. This also follows from noticing that $X_8 + X_6$ is also real semisimple and commutes with $X_8 - 3X_6$.

Thus a maximal nilpotent subalgebra containing

$$U = \langle X_{14}, X_{13}, X_{12} \rangle$$

is $\widetilde{U} = \langle X_5, X_{14}, X_{13}, X_{12}, X_{11}, X_9 \rangle$. Finally, $C = \langle X_6, X_8 \rangle$ is self-centralizing and it is a real split Cartan subalgebra of the full 14 dimensional algebra $L$.

Maple gives the following roots for $C$ in $\widetilde{U}$: in fact, the basis vectors for $\widetilde{U}$ listed above are common eigen-vectors for $C$ with eigen-values

$$a = (\frac{1}{2}, \frac{3}{2}), b = (-1, 0), c = (-\frac{1}{2}, \frac{3}{2}), d = (-\frac{1}{2}, 1), e = (-\frac{1}{2}, -\frac{1}{2}), f = (0, 1).$$

As explained in Section 2, this is a positive system of roots and a simple system of roots is given by adding pairs of positive roots and removing those that are a sum of positive roots. We have

$$a + b = c, a + e = f, d + e = b, e + f = d.$$ 

Thus the simple roots are $a, e$ and the positive roots written in terms of these roots are

$$a, e, a + e = f, a + 2e = d, a + 3e = b, 2a + 3e = c.$$ 

Therefore, the algebra $L$ is of type $G_2$ with a real split Cartan subalgebra. Any semisimple split real Lie algebra is generated by copies of $\mathfrak{sl}(2,\mathbb{R})$ corresponding to the simple roots, with relations

$$[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$$

and its maximal compact subalgebra is generated by copies of the compact element $X - Y$, which generates a circle, in these generating root $\mathfrak{sl}(2,\mathbb{R})$ copies; see \cite{st} pp. 99–100 for a global version of these results.

Here, the root vectors corresponding to $a, -a$ are $X_5, X_{10}$; the root vectors corresponding to $e, -e$ are $X_{11}, X_4$ and a maximal compact compact subalgebra $K$ is thus generated by

$$J_1 = X_5 + X_{10}, J_2 = X_4 - X_{11}.$$ 

The algebra is spanned by

$$J_1, J_2, J_3 = X_1 + \frac{3}{8}X_{14}, J_4 = X_2 - \frac{3}{4}X_{13}, J_5 = X_3 + \frac{3}{4}X_{12}, J_6 = X_7 - \frac{3}{2}X_9.$$ 

To identify the structure of $K$, we must choose a Cartan subalgebra of $K$ and compute its roots in the complexification of $K$ — exactly as for $\mathfrak{so}(4)$ in Section 11.11 Now the centralizer of $J_1$ is $\langle J_1, J_5 \rangle$, and it is therefore a Cartan subalgebra of $K$. Its positive roots are

$$((\sqrt{2}, -\frac{1}{\sqrt{2}}), (\sqrt{2}, \frac{3}{\sqrt{2}})).$$
Therefore the system is of type $A_1 \times A_1$. The real and imaginary parts for the root vectors of $(\sqrt{-2}, -\sqrt{-2})$ are

$$J_3 = \frac{1}{6} J_6, \quad \sqrt{\frac{3}{4}} J_4 - \sqrt{\frac{3}{8}} J_2$$

(1)

and for the root $(\sqrt{-2}, \frac{3}{\sqrt{2}})$ they are

$$J_3 + \frac{1}{2} J_6, \quad -\sqrt{\frac{2}{4}} J_4 + \frac{3}{\sqrt{2}} J_2.$$  

(2)

The vectors in (1) generate a copy of $\mathfrak{so}(3)$ and in (2) also a copy of $\mathfrak{so}(3)$ and these subalgebras commute.

This gives an explicit decomposition of a maximal compact subalgebra of $L$ as a sum of copies of $\mathfrak{so}(3)$.

6. Solutions of the Wave Equation

Section 4 gives several non-conjugate subalgebras of the symmetry algebra of the wave equation. The reason that they are non-conjugate is that the structure of low dimensional Lie algebras is well documented in literature [PBNL], [SW]. In this case, this can be described very briefly. If $L$ is a 3 dimensional algebra and its commutator $L' = [L, L]$ is 1–dimensional, then $L$ is completely determined by the dimension of the centralizer of $L'$ in $L$; if $L'$ is 2–dimensional, then $L'$ is abelian and the structure of $L$ is completely determined by the eigen-values of $L/L'$ in $L'$ and their multiplicities; in case $L'$ is of dimension 3, the eigen-values of a single element suffice to determine the structure of $L$ [ADMM, Corollary 2.2 and Section 4.3]. The algebras given below were of all possible Lie–Bianchi types. Their identification is facilitated by determining the reduced root system using the algorithms of Section 3, by enlarging a given subalgebra of commuting ad-nilpotent elements to a maximal solvable subalgebra that contains up to conjugacy all solvable subalgebras with real eigen-values. If the vector fields are in polynomial form and contain translations with respect to the independent variables, then these translations are ad-nilpotent. In case the Cartan algebra so obtained has a compact part, it must operate on the positive root spaces of its real part and this way one may obtain all 3 dimensional subalgebras of solvable subalgebras of all Lie–Bianchi types.

In the specific example of the 15 dimensional algebra considered in Section 4.4, we denoted — for simplicity of notation the element $e_7 - e_{16}$ by $e_7$. Taking this into account, the Cartan algebras obtained in Section 4.4 were $\langle e_2, e_7 - e_{16}, e_{14} \rangle$ and $\langle 2e_{12} + e_5, e_9 + 4e_8, e_{15} \rangle$. This first Cartan algebra is maximally real with its real part $A = \langle e_2, e_7 - e_{16} \rangle$. For this reason, the relative root system is different from the absolute root system. The roots of $A$ were determined in Section 4.4 as

$$b = -\omega_7 - \omega_2, \quad d = \omega_2, \quad b + d = -\omega_7, \quad b + 2d = \omega_2 - \omega_7,$$

where $\omega_2, \omega_7$ are dual to the ordered basis $e_2, e_7 - e_{16}$ of $A$.

The root spaces of $A$ in the maximal solvable algebra $L$ determined in Section 4.4 were

- $L_b = L_{-\omega_7 - \omega_2} = \langle e_8, e_{10} \rangle$,
- $L_d = L_{\omega_2} = \langle e_6 - e_{13}, e_4 - e_{15} \rangle$,
- $L_{b+d} = L_{-\omega_7} = \langle e_{12}, e_{11} \rangle$,
- $L_{b+2d} = L_{\omega_2 - \omega_7} = \langle e_8 - e_{10} \rangle$.

Thus, this displays the common eigen-vectors of $A$ and their multiplicities. Moreover as $[L_r, L_s] \subset L_{r+s}$, and $2r$ is not a root, the root spaces given above contain commuting eigen-vectors of $e_2$.
with different eigen-values and of $e_7 - e_{16}$ with repeated eigen-values. As any element centralizing $A$ operates on each root space of $A$, applying this to the compact part of the Cartan algebra $\langle A, e_{14}\rangle$ gives all possible solvable 3-dimensional solvable Lie–Bianchi types. Finally, using the compact Cartan subalgebra, we found a maximal compact subalgebra, whose derived algebra was $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. The centralizer for the spatial rotations was $\mathfrak{sl}(2, \mathbb{R})$. For this reason, the 3-dimensional simple algebras given below are non-conjugate. We now proceed to find the corresponding reductions and invariant solutions.

A preliminary step is to find the invariants of a given algebra of vector fields. The number of functionally independent invariants can be found from the row reduced echelon form of the operators. In the row reduced echelon form, the resulting operators always commute [ABGM].

The equation is

$$\frac{\partial^2}{\partial x^2} u(t, x, y, z) + \frac{\partial^2}{\partial y^2} u(t, x, y, z) + \frac{\partial^2}{\partial z^2} u(t, x, y, z) - \frac{\partial^2}{\partial t^2} u(t, x, y, z) = 0$$

(I) $\dim G' = 0$

(I.a) $L_{1,0} = \langle e_8, e_{10}, e_{11}\rangle$

The joint invariants of $L_{1,0}$ are $z, u$

so that the corresponding similarity transformations

$$p = z, w(p) = u$$

transform wave equation (3) to

$$\frac{d^2}{dp^2} w(p) = 0$$

which has the solution

$$w(p) = C_1 p + C_2.$$  

This leads to solution

$$u(t, x, y, z) = C_1 z + C_2$$

of wave equation (3).

(I.b) $L_{2,0} = \langle e_2, e_7 - e_{16}, e_{14}\rangle$

The joint invariants of $L_{2,0}$ are

$$-x^2 + z^2 \sqrt{y^2 - t^2}$$

so that the corresponding similarity transformations

$$p = \frac{-x^2 + z^2}{y^2 - t^2}, w(p) = u \sqrt{-y^2 + t^2}$$

transform wave equation (3) to Jacobi ODE

$$4 \left( \frac{d^2}{dp^2} w(p) \right) p^2 - 4 \left( \frac{d}{dp} w(p) \right) p + 8 \left( \frac{d}{dp} w(p) \right) p - 4 \frac{d}{dp} w(p) + w(p) = 0$$

which has the solution

$$w(p) = C_1 EllipticK(\sqrt{p}) + C_2 EllipticCK(\sqrt{p})$$

(10)
in terms of complete and complementary complete elliptic integrals of the first kind (ref: [http://www.maplesoft.com/support/help/Maple/view.aspx?path=EllipticF]). This leads to solution

$$u(t, x, y, z) = \frac{1}{\sqrt{-y^2 + t^2}} \left( C_1 \text{EllipticK} \left( \sqrt{-\frac{x^2 - z^2}{y^2 - t^2}} \right) + C_2 \text{EllipticCK} \left( \sqrt{-\frac{x^2 - z^2}{y^2 - t^2}} \right) \right)$$  \hspace{1cm} (11)

of wave equation (3).

(I.c) $\mathcal{L}_{3,0} = \langle e_{12} + \frac{1}{2}e_5, e_9 + 4e_8, e_{15} \rangle$

The joint invariants of $\mathcal{L}_{3,0}$ are

$$-t^4 + (2x^2 + 2y^2 + 2z^2 - 8) t^2 - x^4 + (-2y^2 - 2z^2) x^2 - y^4 - 2y^2z^2 - (z^2 + 4)^2$$

$$4x^2 + 4y^2$$

so that the corresponding similarity transformations

$$p = \frac{-t^4 + (2x^2 + 2y^2 + 2z^2 - 8) t^2 - x^4 + (-2y^2 - 2z^2) x^2 - y^4 - 2y^2z^2 - (z^2 + 4)^2}{4x^2 + 4y^2}, u\sqrt{x^2 + y^2}$$  \hspace{1cm} (12)

transform wave equation (3) to

$$4 \left( \frac{d^2}{dp^2} w(p) \right) p^2 + 8 \left( \frac{d}{dp} w(p) \right) p - 16 \frac{d^2}{dp^2} w(p) + w(p) = 0$$  \hspace{1cm} (13)

which has the solution

$$w(p) = C_1 \text{LegendreP} \left( -1/2, p/2 \right) + C_2 \text{LegendreQ} \left( -1/2, p/2 \right)$$  \hspace{1cm} (15)

in terms of Legendre functions of the first and second kind (ref: [http://www.maplesoft.com/support/help/Maple/view.aspx?path=Legendre]). This leads to solution

$$u(t, x, y, z) = \frac{1}{\sqrt{x^2 + y^2}} \left( C_1 \text{LegendreP} \left( -1/2, p/2 \right) + C_2 \text{LegendreQ} \left( -1/2, p/2 \right) \right)$$  \hspace{1cm} (16)

of wave equation (3) where $p$ is given by (12).

(II) dim $G' = 1$

(II.a) $\mathcal{L}_{1,1} = \langle e_2, e_7 - e_{16}, e_8 + e_{10} \rangle$

The joint invariants of $\mathcal{L}_{1,1}$ are

$$\frac{z}{x}, ux$$

so that the corresponding similarity transformations

$$p = \frac{z}{x}, w(p) = ux$$  \hspace{1cm} (17)

transform wave equation (3) to

$$\left( \frac{d^2}{dp^2} w(p) \right) p^2 + 4 \left( \frac{d}{dp} w(p) \right) p + 2w(p) + \frac{d^2}{dp^2} w(p) = 0$$  \hspace{1cm} (18)

which has the solution

$$w(p) = \frac{C_1 p + C_2}{p^2 + 1}.$$  \hspace{1cm} (19)
This leads to solution

\[ u(t, x, y, z) = \frac{C_1 z + C_2 x}{x^2 + z^2} \]  

(20)

of wave equation (3).

(II.b) \( L_{2,1} = \langle e_{12}, -e_6 + e_{13}, -e_8 + e_{10} \rangle \)

The joint invariants of \( L_{2,1} \) are

\[ x, t + y, u \]

which gives the similarity transformation

\[ p = x, q = t + y, w(p, q) = u \]  

(21)

that transforms the wave equation into

\[ w_{pp} = 0 \]  

(22)

which gives the solution

\[ u(t, x, y, z) = xF_1(y + t) + F_2(y + t). \]  

(23)

(III) \( \dim G' = 2 \)

(III.a) \( L_{1,2} = \langle e_7 - e_{16}, e_{11}, e_{12} \rangle \)

The joint invariants of \( L_{1,2} \) are

\[ y, t^2u \]

so that the corresponding similarity transformations

\[ p = \frac{y}{t}, w(p) = ut \]  

(24)

transform wave equation (3) to

\[ \left( \frac{d^2}{dp^2}w(p) \right) p^2 + 4 \left( \frac{d}{dp}w(p) \right) p + 2w(p) - \frac{d^2}{dp^2}w(p) = 0 \]  

(25)

which has the solution

\[ w(p) = \frac{C_1 p + C_2}{p^2 - 1}. \]  

(26)

This leads to solution

\[ u(t, x, y, z) = \frac{C_1 y + C_2 t}{y^2 - t^2} \]  

(27)

of wave equation (3).

(III.b) \( L_{2,2} = \langle e_2, e_8 + e_{10}, e_8 - e_{10} \rangle \)

\[ = \langle y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \frac{\partial}{\partial t} + \frac{\partial}{\partial y}, \frac{\partial}{\partial t} - \frac{\partial}{\partial y} \rangle. \]

Clearly, joint invariants the same as the invariants of \( \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial t} \rangle \), therefore the basic invariants are \( x, z, u \). Substituting \( u = u(x, z) \) in the wave equation shows that \( u \) must be a harmonic function, so they are real parts of holomorphic functions in the variable \( x + \sqrt{-1}z \).

(III.c) \( L_{3,2} = \langle e_{14}, e_{11}, e_{12} \rangle \).

The joint invariants of \( L_{3,2} \) are

\[ t, y, u \]

so that the corresponding similarity transformations

\[ p = t, q = y, w(p, q) = u \]  

(28)
transform wave equation (3) to
\[
\frac{\partial^2}{\partial q^2} w(p, q) - \frac{\partial^2}{\partial p^2} w(p, q) = 0 \quad (29)
\]
which has the solution
\[
w(p, q) = F_1(q + p) + F_2(q - p). \quad (30)
\]
This leads to solution
\[
u(t, x, y, z) = F_1(y + t) + F_2(y - t) \quad (31)
\]
of wave equation (3).

(III.d) \( L_{4,2} = \langle e_{14}, -e_6 + e_{13}, -e_4 + e_{15} \rangle \)
The joint invariants of \( L_{4,2} \) are
\[
y + t, x^2 - 2ty + z^2 - 2t^2, u
\]
so that the corresponding similarity transformations
\[
p = y + t, \quad q = x^2 - 2t \quad y + z^2 - 2t^2, \quad w(p, q) = u \quad (32)
\]
transform wave equation (3) to
\[
\left( -p^2 + q \right) \frac{\partial^2}{\partial q^2} w(p, q) + \left( \frac{\partial^2}{\partial q\partial p} w(p, q) \right) p + 2 \frac{\partial}{\partial q} w(p, q) = 0 \quad (33)
\]
which has the solution
\[
w(p, q) = \frac{1}{p} \left( F_2(p) p + F_1 \left( \frac{p^2 + q}{p} \right) \right). \quad (34)
\]
This leads to solution
\[
u(t, x, y, z) = F_2(y + t) + \frac{1}{y + t} F_1 \left( \frac{x^2 + y^2 + z^2 - t^2}{y + t} \right) \quad (35)
\]
of wave equation (3).

(IV) \( \text{dim } G' = 3 \)

(IV.a) \( L_{1,3} = \langle e_{15}, e_{14}, e_{13} \rangle \)
The joint invariants of \( L_{1,3} \) are
\[
t, x^2 + y^2 + z^2, u
\]
so that the corresponding similarity transformations
\[
p = t, \quad q = x^2 + y^2 + z^2, \quad w(p, q) = u \quad (36)
\]
transform wave equation (3) to
\[
4 \left( \frac{\partial^2}{\partial q^2} w(p, q) \right) q + 6 \frac{\partial}{\partial q} w(p, q) - \frac{\partial^2}{\partial p^2} w(p, q) = 0 \quad (37)
\]
which has the solution
\[
w(p, q) = \frac{F_1(\sqrt{q} + p) + F_2(-\sqrt{q} + p)}{\sqrt{q}}. \quad (38)
\]
This leads to solution
\[
u(t, x, y, z) = \frac{F_1 \left( \sqrt{x^2 + y^2 + z^2 + t} \right) + F_2 \left( -\sqrt{x^2 + y^2 + z^2 + t} \right)}{\sqrt{x^2 + y^2 + z^2}} \quad (39)
\]
of wave equation (3).
(IV.b) \( L_{2,3} = \{e_7 - e_{16}, e_8, e_9\} \)

The joint invariants of \( L_{2,3} \) are
\[
y, z, u x
\]
so that the corresponding similarity transformations
\[
p = \frac{y}{x}, q = \frac{z}{x}, w(p, q) = u x
\]
transform wave equation (3) to
\[
\left( \frac{\partial^2}{\partial p^2} w \right) p^2 + 2 pq \frac{\partial^2}{\partial q \partial p} w + \left( \frac{\partial^2}{\partial q^2} w \right) q^2 + 4 p \frac{\partial}{\partial p} w + 4 \left( \frac{\partial}{\partial q} w \right) q + \frac{\partial^2}{\partial p^2} w + 2 w + \frac{\partial^2}{\partial q^2} w = 0
\]
which has the solution
\[
w(p, q) = C_1 p + C_2 \frac{p^2 + 1}{q^2 + 1}
\]
This leads to solution
\[
u(t, x, y, z) = \frac{1}{x} \left( \frac{(C_1 y + C_2 x) x}{x^2 + y^2} + \frac{(C_3 z + C_4 x) x}{x^2 + z^2} \right)
\]
of wave equation (3).

(IV.c) \( L_{3,3} = \{e_1 + 2 e_{10} + 2 e_{14}, -e_3 - 2 e_{11} + 2 e_{13}, -4 e_5 - 8 e_{12} - 8 e_{15}\} \)

The joint invariants of \( L_{3,3} \) are
\[
x^2 + y^2 + z^2 - t^2 + 4
\]
so that the corresponding similarity transformations
\[
p = \frac{x^2 + y^2 + z^2 - t^2 + 4}{t}, w(p) = ut
\]
transform wave equation (3) to
\[
\left( \frac{d^2}{dp^2} w(p) \right) p^2 + 4 \left( \frac{d}{dp} w(p) \right) p + 2 w(p) + 16 \frac{d^2}{dp^2} w(p) = 0
\]
which has the solution
\[
w(p) = \frac{C_1 p + C_2}{p^2 + 16}
\]
This leads to solution
\[
u(t, x, y, z) = \frac{-C_1 t^2 + C_2 t + C_1 \left( x^2 + y^2 + z^2 + 4 \right)}{t^4 + (-2 x^2 - 2 y^2 - 2 z^2 + 8) t^2 + (x^2 + y^2 + z^2 + 4)^2}
\]
of wave equation (3).

APPENDICES

Appendix 1

Appendix 2
Table 1. Commutator table for symmetry algebra of wave equation on Minkowski spacetime

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{14}$ | $X_{15}$ | $X_{16}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|---------|
| 0    | $\frac{\partial}{\partial x_4}$ | 0    | 0    | 0    | $-X_2$ | 0    | $-X_3$ | 0    | $X_{10}$ | $X_{11}$ | 0    | 0    | 0    | 0    | 0    |
| 0    | 0    | $-X_{13}$ | 0    | $-X_{14}$ | 0    | $-X_{15}$ | 0    | $-X_{16}$ | $X_{12}$ | 0    | 0    | 0    | 0    | 0    | 0    |
| 0    | 0    | $-X_4$ | 0    | 0    | 0    | $-X_5$ | 0    | 0    | $-X_6$ | 0    | 0    | 0    | 0    | 0    | 0    |
| 0    | $\frac{\partial}{\partial x_4}$ | 0    | $-X_1$ | 0    | $-X_2$ | 0    | $-X_3$ | 0    | $-X_4$ | $X_{10}$ | 0    | 0    | 0    | 0    | 0    | 0    |
| 0    | 0    | $-X_{13}$ | 0    | $-X_{14}$ | 0    | $-X_{15}$ | 0    | $-X_{16}$ | $X_{12}$ | 0    | 0    | 0    | 0    | 0    | 0    |
| $X_5$ | $\frac{\partial}{\partial x_4}$ | 0    | $-X_1$ | 0    | $-X_2$ | 0    | $-X_3$ | 0    | $-X_4$ | $X_{10}$ | 0    | 0    | 0    | 0    | 0    | 0    |
| 0    | 0    | 0    | $-X_1$ | 0    | $-X_2$ | 0    | $-X_3$ | 0    | $-X_4$ | $X_{10}$ | 0    | 0    | 0    | 0    | 0    | 0    |
| 0    | 0    | $-X_1$ | 0    | $-X_2$ | 0    | $-X_3$ | 0    | $-X_4$ | $X_{10}$ | 0    | 0    | 0    | 0    | 0    | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |

where $[X_i, X_j] = -[X_j, X_i]$.

Table 2. Commutator table for $G_2$

| $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ | $X_{13}$ | $X_{14}$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|
| 0    | 0    | 0    | 0    | 0    | $-X_2$ | 0    | 0    | 0    | $-X_3$ | 0    | $-X_4$ | 0    | 0    |
| 0    | 0    | 0    | 0    | 4$X_1$ | $-\frac{1}{2}X_2$ | $\frac{1}{2}X_2$ | $-X_3$ | 0    | $X_7$ | 0    | $-X_8$ | 0    | 0    |
| 0    | 0    | 0    | 0    | 0    | $\frac{1}{2}X_1$ | $-\frac{1}{2}X_2$ | $\frac{1}{2}X_3$ | $-X_4$ | 0    | $-X_7$ | $\frac{1}{2}X_8$ | $-2X_9$ | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | $\frac{1}{2}X_1$ | $-\frac{1}{2}X_2$ | $\frac{1}{2}X_3$ | $-X_4$ | 0    | $-X_7$ | $\frac{1}{2}X_8$ | $-2X_9$ | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | $-X_1$ | 0    | $-X_2$ | $-X_3$ | 0    | $-X_4$ | 0    | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | $-2X_{10}$ | 0    | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | 0    | 0    | 0    | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | 0    | 0    | 0    | 0    |
| 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 0    |

where $[X_i, X_j] = -[X_j, X_i]$.

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