Perturbative analysis of multiple-field cosmological inflation

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We develop a general formalism for analyzing linear perturbations in multiple-field cosmological inflation based on the gauge-ready approach. Our inflationary model consists of an arbitrary number of scalar fields with non-minimal kinetic terms. We solve the equations for scalar- and tensor-type perturbations during inflation to the first order in slow-roll, and then obtain the super-horizon solutions for adiabatic and isocurvature perturbations after inflation. Analytic expressions for power-spectra and spectral indices arising from multiple-field inflation are presented.

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1. INTRODUCTION

Observations suggest that the early Universe underwent a period of accelerated expansion called cosmological inflation. In addition to providing a causal mechanism for the generation and evolution of large-scale structure formation, inflation also leads to elegant resolutions of a number of puzzles of the Big-Bang theory, such as the isotropy, horizon and the flatness problems \cite{1, 2, 3}. The simplest implementation of inflation is achieved by assuming that the matter is described by a single scalar field, the \textit{inflaton} \cite{1, 5}. The quantum fluctuations of the scalar field generated during inflation become classical after crossing the event horizon, seeding the observed density perturbations. In addition, inflation also generates metric perturbation in the tensor sector, leading to a stochastic gravitational wave background. Indeed, the recent high accuracy CMBR data from the WMAP satellite \cite{4} do indeed support the general predictions of inflationary cosmology.

It was, however, realized, that with a single scalar field the inflationary scenario suffers from what is called the \textit{graceful-exit} problem \cite{1, 6}, namely, achieving sufficient inflation consistent with the observed density perturbations, before the Universe exits from the inflationary epoch. Kofman et al. \cite{7} suggested the possibility of using a first short stage of double inflation in order to generate a large value of a scalar field required for a second, longer stage with a graceful exit. Linde \cite{8} showed that one requires at least two scalar fields to overcome the graceful exit issue, without modifying Einstein gravity, and without sacrificing natural initial conditions. There are other motivations for incorporating multiple fields in the dynamics of cosmological inflation. For example, when constructing models of inflation inspired by particle physics theories such as low energy effective supergravity derived from superstrings, one obtains many scalar fields (see \cite{9} for a recent review). This calls for a general framework for handling cosmological perturbations in a situation where the matter sector consists of an arbitrary number of scalar fields.

Cosmological perturbations in a single field inflation has been thoroughly investigated in the past \cite{3, 10, 11}, following the seminal paper of Bardeen \cite{12}. In the context of multiple-field inflation, Starobinsky \cite{13} obtained an expression for density perturbations with an arbitrary number of scalar fields interacting between themselves through gravity. In a consistent treatment of cosmological perturbations with more than one field, one should consider the role of isocurvature, or entropy modes in addition to the adiabatic, or curvature perturbations \cite{14, 15, 16, 17}. Indeed, it is quite possible for the two to be correlated, leading to distinct observational results \cite{18}. Recently Gordon et al. \cite{19} analyzed the evolution of adiabatic and isocurvature perturbations in multicomponent inflation, where they performed a local rotation in the field space to separate out the adiabatic and entropy modes, while Wands et al. \cite{20} studied possible observational aspects of adiabatic and isocurvature spectra produced by two-field inflation. For a somewhat different approach, see Malik et al. \cite{21}. A method for treating density perturbations in multicomponent inflation was proposed in \cite{22}, but see also \cite{23, 24, 25}.

In a recent paper \cite{26} we presented a general formalism to analyze cosmological perturbations in inflation driven by multicomponent scalar fields using the \textit{gauge-ready} method developed by Hwang and colleagues \cite{27, 28, 29, 30}. This approach follows from a suggestion by Bardeen \cite{31}, that rather than imposing a particular gauge condition while dealing with cosmological perturbations right from the beginning, it is often advantageous to express the
perturbations without specifying any gauge. Thus one has the flexibility of adopting different gauge conditions at a much later stage, depending upon the nature of each problem. Moreover, it becomes easy to relate results between various gauge-dependent and gauge-invariant techniques.

This paper is organized as follows. In Section 2 we present the equations describing multicomponent scalar fields with a non-trivial field metric and having non-minimal kinetic terms coupled to Einstein gravity. By introducing a set of basis vectors based on the Gram-Schmidt orthonormalization technique, we can discriminate multiple-field effects from single-field ones. Metric perturbations are discussed in Section 3. These are conveniently decomposed into scalar, vector and tensor modes. Here we introduce the gauge-ready approach to cosmological perturbations and present the equations governing density perturbations in the gauge-ready form, as well as in terms of gauge-invariant variables. Next, we introduce the slow-roll variables in Section 4 and proceed to obtain the solutions to scalar and tensor equations governing density perturbations in the gauge-ready form, as well as in terms of gauge-invariant variables. In Section 5 we discuss adiabatic and isocurvature perturbations. We calculate the power-spectra and spectral indices during inflation in the first order in slow-roll. We then proceed to study perturbations after inflation. In Section 6 we conclude with adiabatic, isocurvature, correlated, and tensor modes. These should be useful in comparing theoretical predictions of various inflationary models with observations. We conclude in Section 6.

2. THE INFLATIONARY MODEL

2.1. Scalar fields

In this Section, we explain our notation and set up the basic equations needed for our analysis. We consider Einstein gravity coupled to an arbitrary number of real scalar fields. The Lagrangian then is

$$\mathcal{L} = \sqrt{-g} \left( \frac{1}{2\kappa_0^2} R - \frac{1}{2} \partial^\mu \phi \cdot \partial_\mu \phi - V(\phi) \right)$$

$$= \sqrt{-g} \left( \frac{1}{2\kappa_0^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi^T G \partial_\nu \phi - V(\phi) \right).$$

(2.1)

were $R$ is the scalar curvature, $\kappa_0^2 \equiv 8\pi G$, and we set $c = 1$. For the scalar fields we use a vector notation, $\phi \equiv (\phi^a)$, with the indices $a, b, c, \ldots = 1, 2, 3, \ldots, N$ labelling the $N$-components in field space. Further, $g \equiv \det(g_{\mu\nu})$, and $\mu, \nu, \ldots$ denote the spacetime indices. For repeated indices, the summation convention applies. The second quantity within the parentheses of Eq. (2.1) represents a non-minimal kinetic term. Such a kinetic term appears in various models of high-energy physics [9]. Also $V(\phi)$ is an arbitrary scalar potential.

The scalars $\phi$ may be interpreted as coordinates $(\phi^a)$ on a real manifold $\mathcal{M}$ induced with a symmetric Riemannian metric $G$ having components $G_{ab}$ in the field space [22]. The field metric is chosen to be positive-definite so that the Hamiltonian is bounded from below. The special case of minimally-coupled fields corresponds to the situation $G_{ab} \equiv \delta_{ab}$. From the components $G_{ab}$ we can define the connection coefficients $\Gamma^a_{bc}$ in the usual manner,

$$\Gamma^a_{bc} = \frac{1}{2} G^{nd} (G_{bd,c} + G_{cd,b} - G_{bc,d}).$$

(2.2)

The curvature tensor on $\mathcal{M}$ is introduced in terms of the tangent vectors $B, C, D$:

$$[R(B, C)D]^a \equiv R_{bcd}^a B^b C^c D^d \equiv (\Gamma^a_{bd,c} - \Gamma^a_{bd,c} + \Gamma^c_{cd} \Gamma^a_{ce} - \Gamma^c_{cd} \Gamma^a_{de}) B^b C^c D^d.$$  

(2.3)

For any two vectors $A$ and $B$, we define the inner product and the norm as

$$A \cdot B = A^T B \equiv \langle A, B \rangle = A^a G_{ab} B^b,$$

$$|A| \equiv \sqrt{(A \cdot A)},$$

(2.4)

respectively. Here $A^T$ is the cotangent vector such that $(A^T)_a \equiv A^b G_{ba}$. We also introduce the covariant derivative $\nabla_a$ on $\mathcal{M}$ acting upon a vector $A$ as

$$\nabla_a A^b \equiv A^b_{;a} + \Gamma^b_{ac} A^c,$$

(2.5)

while, the covariant derivative on $A$ with respect to the spacetime $x^\mu$ is

$$\nabla_a A^a \equiv \partial_\mu + \Gamma^a_{bc} \partial_\mu \phi^b A^c.$$

(2.6)
It should be noted that the covariant derivative reduces to the ordinary derivative when it acts upon a scalar.

By varying the action \[ \mathcal{L} \] with respect to \( g_{\mu \nu} \) and \( \phi \), we obtain the gravitational field equation,

\[
\frac{1}{k^2} G^\mu_\nu = T^\mu_\nu = \partial^\mu \phi \cdot \partial_\nu \phi - \delta^\mu_\nu \left( \frac{1}{2} \partial^\lambda \phi \cdot \partial_\lambda \phi + V \right),
\]

and the equation of motion for the scalar fields,

\[
g^{\mu \nu} \left( \partial_\mu \delta^\lambda_\nu - \Gamma^\lambda_\mu_\nu \right) \partial_\lambda \phi - G^{-1} \nabla^T V = 0,
\]

where \( G^\mu_\nu \) and \( T^\mu_\nu \) are Einstein and energy-momentum tensors.

It is often convenient to represent the scalar fields as effective fluid quantities. The energy-momentum tensor can be covariantly decomposed into fluid quantities using a time-like four-vector \( u^\mu \) normalized as \( u^\mu u_\mu = -1 \):

\[
T_{\alpha \beta} = \mu u_\alpha u_\beta + ph_{\alpha \beta} + q_\alpha u_\beta + q_\beta u_\alpha + \pi_{\alpha \beta},
\]

\[
\mu \equiv T_{\alpha \beta} u^\alpha u^\beta, \quad p \equiv \frac{1}{3} T_{\alpha \beta} h^\alpha_\beta, \quad q_\alpha \equiv -T_{\beta \gamma} u^\beta h^\gamma_\alpha,
\]

\[
\pi_{\alpha \beta} \equiv T_{\gamma \delta} h^\gamma_\alpha h^\delta_\beta - ph_{\alpha \beta}.
\]

Here \( \mu, p, q_\alpha, \) and \( \pi_{\alpha \beta} \) are the energy density, pressure, energy flux, and anisotropic pressure, respectively; \( h_{\alpha \beta} \equiv g_{\alpha \beta} + u_\alpha u_\beta \) is a projection tensor based on \( u_\alpha \) vector, \( q_\alpha u^\alpha = 0 = \pi_{\alpha \beta} \), and \( \pi^\alpha_\alpha = 0 \). Thus, for a multicomponent scalar field, the above decomposition gives

\[
\mu = \frac{1}{2} |\dot{\phi}|^2 + V, \quad p = \frac{1}{2} |\dot{\phi}|^2 - V, \quad q_\alpha = 0 = \pi_{\alpha \beta}.
\]

Equations \[2.7\] with \[2.8\] provide the fundamental expressions required for describing cosmological inflation.

### 2.2. Basis vectors

It will now prove useful to introduce a set of basis vectors generated using Gram-Schmidt orthonormalization \[22, 34\]. From the vector \( \phi \) we can construct a set of \( N \) linearly independent vectors \( \{ \phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(N)} \} \), where,

\[
\phi^{(1)} \equiv \phi, \quad \phi^{(n)} \equiv D_{n-1}^{(n-1)} \phi \quad (n \geq 2).
\]

Let \( e_1 = \phi^{(1)} / |\phi^{(1)}| \) be the first unit vector along the direction of the field velocity \( \dot{\phi} \). Define the second unit vector \( e_2 \) to be along that part of the direction of the field acceleration \( D_t \dot{\phi} \) which is normal to \( e_1 \) so that \( e_1 \cdot e_2 = 0 \). Repetitively applying this Gram-Schmidt procedure generates a set of mutually orthonormal vectors \( \{ e_n \} \), spanning the same subspace as the vectors \( \{ \phi^{(n)} \} \).

Introducing the projection operators \( P_n \) and \( \hat{P}_n \), which project on \( e_n \) and on the subspace perpendicular to \( \{ e_1, \ldots, e_n \} \) respectively, we may then write a general unit vector \( e_n \) as,

\[
e_n = \frac{P_{n-1}^{\perp} \phi^{(n)}}{|P_{n-1}^{\perp} \phi^{(n)}|},
\]

where,

\[
P_n = e_n e_n^\dagger, \quad P_n^{\perp} = \mathbb{1} - \sum_{q=1}^{n} P_q, \quad P_0^{\perp} = \mathbb{1},
\]

and we define,

\[
P^{\parallel} \equiv P_1 = e_1 e_1^\dagger, \quad P^{\perp} \equiv P_1^{\perp} = \mathbb{1} - P^{\parallel}.
\]

Note that when the denominator in Eq. \[2.12\] vanishes, the corresponding basis vector does not exist.

Since \( P^{\parallel} + P^{\perp} \equiv 1 \), we can decompose any vector \( A \) in directions parallel and perpendicular to the field velocity:

\[
A = A^{\parallel} + A^{\perp} \equiv (P^{\parallel} + P^{\perp}) A = e_1 (e_1 \cdot A) + e_2 (e_2 \cdot A).
\]

When there is just one field, \( e_1 \) by definition simply reduces to the normalized scalar \( \phi^{(1)}/|\phi^{(1)}| \) while \( e_2 \) vanishes identically, and so do all other basis vectors. Thus the decomposition \[2.15\] enables us to distinguish between single-field contributions, where only \( e_1 \) survives, from multiple-field ones.
3. THE PERTURBED UNIVERSE

3.1. Metric perturbations

The observed Universe is not perfectly homogeneous and isotropic. Assuming that the inhomogeneities are small enough, we can then treat the deviations by considering linear perturbations of the homogeneous and isotropic cosmological space-time described by the Friedmann-Robertson-Walker (FRW) model,

\[ ds^2 = -a^2 (1 + 2A) dt^2 - 2a^2 B_i dx^i + a^2 \left( g^{(3)}_{ij} + 2C_{ij} \right) dx^i dx^j, \]

where \( a(t) \) is the scale factor, \( dt \equiv ad\eta \), and indices \( i, j, \ldots \), run from 1 to 3 labelling the spatial components. The perturbed order variables \( A(t, x) \), \( B_i(t, x) \), and \( C_{ij}(t, x) \) are based on the metric \( g_{ij}^{(3)} \) of the 3-surfaces of constant curvature \( K = 0, \pm 1 \). Here \( t \) and \( \eta \) are the comoving and conformal times respectively. We denote a derivative with respect to comoving time by \( \dot{\cdot} \equiv \partial_t \) and one with respect to conformal time by \( \dot{\cdot} \equiv \partial_\eta \). The Hubble parameters in terms of comoving and conformal times are defined as \( H = \dot{a}/a \) and \( \mathcal{H} = \dot{a}/a = aH \).

Similar to the metric decomposition Eq. (3.1), we can decompose the scalar field as

\[ \phi(t, x) = \bar{\phi}(t) + \delta \phi(t, x), \]

where the perturbation \( \delta \phi \equiv (\delta \phi^a) \) is a tangent vector on \( \mathcal{M} \), while the energy-momentum tensor is decomposed as

\[
\begin{align*}
T_0^0 &= -\mu - (\bar{\mu} + \delta \mu), \\
T_0^i &= \frac{1}{a} [q_i + (\mu + p)u_i] \equiv (\mu + p)v_i, \\
T_{ij} &= \rho \delta_{ij} + \pi_{ij} \equiv (\rho + \delta \rho) \delta_{ij} + \pi^{(3)}_{ij},
\end{align*}
\]

The barred entities denote background variables. For notational simplicity we shall ignore the overbars unless required. In Eq. (3.1), \( v_i \) is the frame-independent flux variable, and \( \pi^{(3)}_{ij} \) are based on \( g_{ij}^{(3)} \).

From Eqs. (3.1) and (3.2), the equations for the background can be written as

\[
\begin{align*}
H^2 &= \frac{1}{3} \kappa_0^2 \mu - \frac{K}{a^2} = \frac{1}{3} \kappa_0^2 \left( \frac{1}{2} |\dot{\phi}|^2 + V \right) - \frac{K}{a^2}, \\
\dot{H} &= -\frac{1}{2} \kappa_0^2 (\mu + p) + \frac{K}{a^2} = -\frac{1}{2} \kappa_0^2 |\dot{\phi}|^2 + \frac{K}{a^2}, \\
R &= 6 \left( 2H^2 + \dot{H} + \frac{K}{a^2} \right), \\
\mathcal{D}_i \dot{\phi} + 3H \dot{\phi} + G^{-1} \nabla^T V &= 0, \\
\ddot{\mu} + 3H (\mu + p) &= 0.
\end{align*}
\]

We shall ignore the cosmological constant \( \Lambda \) in our work; nevertheless it can be easily included by making the replacements \( \mu \rightarrow \mu + \Lambda/\kappa_0^2 \) and \( p \rightarrow p - \Lambda/\kappa_0^2 \). Note that we have explicitly retained \( K(= 0, \pm 1) \), and only at a later stage shall we set \( K = 0 \).

3.2. Scalar, vector and tensor decompositions

To make further progress, we decompose the perturbed order variables into scalar-, vector-, and tensor-type perturbations. To the linear order, they decouple from one another and evolve independently. Accordingly, the metric perturbation variables \( A(t, x) \), \( B_i(t, x) \), and \( C_{ij}(t, x) \) may be decomposed as

\[
\begin{align*}
A &\equiv \alpha, \\
B_i &\equiv \beta_i + \bar{B}_i^{(v)}, \\
C_{ij} &\equiv \bar{g}_{ij}^{(3)} \varphi + \gamma_{ij}^{(v)} + C_{(ij)}^{(v)} + C_{ij}^{(t)}.
\end{align*}
\]
The superscripts \((s), (v)\) and \((t)\) indicate the scalar-, vector- and tensor-type perturbed order variables. The vertical bar represents a covariant derivative with respect to \(\partial_{(s)}^{(3)}\) and the round brackets in the subscript imply symmetrization of the indices. The scalar metric perturbations are then given by \(\alpha, \beta, \gamma\) and \(\varphi\). The transverse-type vector perturbations \(B_i^{(v)}\) and \(C_i^{(v)}\) satisfy \(B_i^{(v)}|_{t} = 0 = C_i^{(v)}|_{t}\) while the tensor-type perturbation \(C_{ij}^{(t)}\) is transverse-traceless \((C_{ij}^{(t)} = 0 = C_{ij}^{(t)}|_{ij})\). Both the vector and tensor perturbed order variables are based on \(B_i^{(3)}\). We define \(\Delta\) as a comoving three-space Laplacian, and introduce the following combinations of the metric variables,

\[
\chi \equiv a(\beta + \alpha \gamma), \quad \kappa \equiv 3(H\alpha - \varphi - \frac{\Delta}{a^2}\chi),
\]

\[
\psi^{(v)} \equiv B^{(v)} + a\dot{C}^{(v)}.
\]  

(3.10)

It is convenient to separate the temporal and spatial aspects of the perturbed order variables by expanding them in terms of harmonic eigenfunctions \(Q^{(s,v,t)}(k; x)\) of the generalized Helmholtz equation \[\square Q^{(s,v,t)} = 0\], with \(k\) the wave vector in Fourier space and \(k = |k|\). We can then write the scalar-type perturbed order variables as \(\alpha(t, x) \equiv \alpha(t, k)Q^{(s)}(k; x)\), with similar expressions for \(\beta, \gamma\) and \(\varphi\). The vector- and tensor-type perturbations are expanded as \(B_i^{(v)} \equiv B^{(v)}Q_i^{(v)}, C_i^{(v)} \equiv C^{(v)}Q_i^{(v)},\) \(C_{ij}^{(v)} \equiv C^{(v)}Q_{ij}^{(v)},\) and \(C_{ij}^{(t)} \equiv C^{(t)}Q_{ij}^{(t)}\). In each of these harmonic expansions, a summation over the modes of the eigenfunctions is implied. Thus, the perturbed scalar fields have the expansion

\[
\delta \phi(t, x) \equiv \delta \phi(t, k)Q^{(s)}(k; x),
\]

(3.11)

Similarly, the fluid variables \(v_i\) and \(\pi^{(3)}_{ij}\) can be expanded in terms of the harmonics as

\[
v_i \equiv v^{(s)}Q_i^{(s)} + v^{(v)}Q_i^{(v)}, \quad \pi^{(3)}_{ij} \equiv \pi^{(s)}Q_{ij}^{(s)} + \pi^{(v)}Q_{ij}^{(v)} + \pi^{(t)}Q_{ij}^{(t)},
\]

(3.12)

while the energy-momentum tensor in Eq. \[\ref{eq:3.13}\] has the expansion

\[
T_0^0 = -\mu \equiv -(\tilde{\mu} + \delta \mu),
\]

\[
T^i_0 = -\f{1}{k}(\mu + p)v^{(s)}_i + (\mu + p)v^{(v)}Q_i^{(v)},
\]

\[
T^i_j = (\tilde{\rho} + \delta \rho)\delta^i_j + \pi^{(s)}Q^{(s)}_{ij} + \pi^{(v)}Q^{(v)}_{ij} + \pi^{(t)}Q^{(t)}_{ij}.
\]

(3.13)

For a Universe having the matter sector composed exclusively of scalar fields, the quantity \(\pi^{(3)}_{ij}\) in Eq. \[\ref{eq:3.13}\] vanishes identically. We then have to the perturbed order,

\[
\delta \mu = \dot{\phi} \cdot D\phi - \alpha|\phi|^2 + \nabla V \cdot \delta \phi, \quad (3.14)
\]

\[
\delta \rho = \dot{\phi} \cdot D\phi - \alpha|\phi|^2 - \nabla V \cdot \delta \phi, \quad (3.15)
\]

\[
(\mu + p)v^a_k = \dot{\phi} \cdot \delta \phi, \quad (3.16)
\]

where we have written \(v \equiv v^{(s)}\) for simplicity. It is also convenient to decompose \(\delta \rho\) into an adiabatic part \(c_s^2\delta \mu\), and an entropy perturbation \(e\):

\[
\delta \rho = c_s^2\delta \mu + e, \quad (3.17)
\]

where \(c_s^2 \equiv \dot{\rho}/\dot{\mu}\) may be interpreted as an effective sound velocity. We shall also use the notation \(w \equiv p/\mu\).

### 3.3. Gauge-Ready formalism

We now briefly summarize the gauge-ready approach discussed in \[\ref{ref:26, 27, 28, 29, 30}\]. Under a gauge transformation, or coordinate shift, \(\tilde{x}^\mu = x^\mu + \xi^\mu\) with \((\xi^0, \xi^i) \equiv (a^{-1}\xi^t, a^{-1}\xi^i + \xi^{(v)i})\), and \(\xi^{(v)i}|_t = 0\), the metric and matter variables transform to linear order as

\[
\tilde{\alpha} = \alpha - \xi^t, \quad \tilde{\beta} = \beta - \frac{1}{a}\xi^t + a\left(\frac{\xi}{a}\right),
\]

\[
\tilde{\gamma} = \gamma - \xi^i, \quad \tilde{\varphi} = \varphi - \xi^t + a\left(\frac{\xi}{a}\right),
\]

\[
\tilde{\rho} = \rho - \xi^t + a\left(\frac{\xi}{a}\right), \quad \tilde{\mu} = \mu - \xi^t + a\left(\frac{\xi}{a}\right),
\]

\[
\tilde{\Delta} = \Delta + \xi^t + a\left(\frac{\xi}{a}\right),
\]

\[
\tilde{\Psi}^{(v)} = \Psi^{(v)} + a\xi^{(v)}.
\]
\[ \tilde{\gamma} = \gamma - \frac{1}{\alpha} \xi, \quad \tilde{\varphi} = \varphi - H \xi, \quad \tilde{\chi} = \chi - \xi, \]
\[ \tilde{k} = k + \left( 3H + \frac{\Delta}{a^2} \right), \quad \tilde{v} = v - \frac{1}{\alpha} \xi, \]
\[ \delta \tilde{\mu} = \delta \mu - \tilde{\mu} \xi, \quad \delta \tilde{p} = \delta p - \tilde{p} \xi, \quad \delta \tilde{\Phi} = \delta \Phi - \tilde{\Phi} \xi, \]
\[ \tilde{B}_i^{(v)} = B_i^{(v)} + a \xi_i^{(v)}, \quad \tilde{C}_i^{(v)} = C_i^{(v)} - \xi_i^{(v)}, \]
\[ \tilde{\psi}^{(s,v,t)} = \pi^{(s,v,t)}, \quad \tilde{C}_{ij}^{(t)} = C_{ij}^{(t)}. \]  

(3.18)

It is immediately obvious from Eq. (3.18) that the tensor-type perturbations are gauge-invariant. For the special case of scalar-type perturbations to the linear order, fixing the temporal part \( \xi^t \) of the gauge transformation leads to different gauge conditions: \( \alpha \equiv 0 \) (synchronous gauge), \( \chi \equiv 0 \) (zero-shear gauge), \( v/k \equiv 0 \) (comoving gauge), \( \varphi \equiv 0 \) (uniform-curvature gauge), and so on. Except for the synchronous gauge, the temporal gauge mode in the other gauges are completely fixed. Consequently, there is a unique correspondence between a variable in a gauge condition, and a gauge-invariant combination of the variable concerned and the variable used in the gauge condition. Using the perturbed order variables together with the variables used in the gauge condition, one can systematically construct various gauge-invariant variables, for example,

\[ \varphi_{\chi} \equiv \varphi - H \chi, \quad \alpha_{\chi} \equiv \alpha - \dot{\chi}, \quad v_{\chi} \equiv v - \frac{k}{a} \chi, \]
\[ \delta \mu_{\chi} \equiv \delta \mu - \dot{\mu} \chi, \quad \delta \rho_{\chi} \equiv \delta \rho - \dot{\rho} \chi, \]
\[ \delta \Phi_{\chi} \equiv \delta \Phi - \dot{\Phi} \chi, \quad \delta \varphi_{\varphi} \equiv \delta \varphi - \frac{\dot{\Phi}}{H} \varphi \equiv - \frac{\dot{\Phi}}{H} \delta \varphi, \]
\[ \varphi_v \equiv \varphi - \frac{aH}{k} v, \quad \delta \mu_v \equiv \delta \mu - \frac{a}{k} \dot{\mu} v. \]  

(3.19)

Thus, in the zero-shear gauge, also known as the longitudinal, or conformal Newtonian gauge, we have from Eq. (3.19), \( \varphi_{\chi} \equiv \varphi, \) \( \alpha_{\chi} \equiv \alpha, \) and \( \delta \Phi_{\chi} \equiv \delta \Phi. \) Similarly, in the uniform-curvature gauge, it follows that \( \delta \Phi_{\varphi} \equiv \delta \Phi \) which in turn is equivalent to \( -(\dot{\Phi}/H) \delta \Phi \) in the uniform-field gauge. In the notation of [10], our \( \alpha_{\chi} \) and \( \varphi_{\chi} \) correspond to their \( \Phi \) and \( -\Psi \) respectively.

Now, as is well known in the theory of cosmological perturbations, a judicious choice of gauge conditions often simplifies the mathematical structure of a particular problem. For example, density perturbations with hydrodynamical fluids are most conveniently treated using the comoving gauge, while gravitational potential and velocity perturbations are best handled in the zero-shear gauge. In the same spirit, the uniform-curvature gauge simplifies the analysis of perturbations due to minimally coupled scalar fields. Since, in general, we do not know the optimal gauge condition beforehand, it becomes advantageous to express the perturbations without imposing a specific temporal gauge condition. In other words, we write the governing equations in the gauge-ready form, which would give us the freedom to choose different gauge conditions, as adapted to the problem, at a later stage in the calculations. The equations are spatially gauge-invariant, but the temporal gauge condition remains unspecified. Once the temporal gauge mode is completely fixed so that no further gauge degrees of freedom are left, the resulting variables would then be gauge-invariant. Moreover, when a solution in a particular gauge is known, we can then easily derive the corresponding solution in other gauges, as well as in gauge-invariant forms. This is the basic concept of the gauge-ready method. The method is most useful when one considers relativistic hydrodynamic perturbations with mutually interacting imperfect fluids as well as kinetic components. The gauge-ready method then not only simplifies the analysis by enabling us to choose different gauge conditions for different aspects of the system on the fly, but also allows us to check the numerical accuracy by comparing solutions in different gauges.

To implement this gauge-ready strategy, it is most convenient to derive the perturbed set of equations from the (3+1) ADM [32], and the (1+3) covariant [33] formulations of Einstein gravity. A complete set of these equations may be found in the Appendix of Ref. [27]. In this Section we write the equations for scalar-type perturbations in the gauge-ready form:

\[ \dot{\varphi} = H \alpha - \frac{1}{3} \kappa + \frac{1}{3} \frac{k^2}{a^2} \chi. \]  

(3.20)

\[ -\frac{k^2 - 3K}{a^2} \varphi + H \kappa = -\frac{1}{2} \kappa_0^2 \delta \mu. \]  

(3.21)
\[
\kappa - \frac{k^2 - 3K}{a^2}\chi = \frac{3}{2} \kappa_0^2 (\mu + p) \frac{a}{k} v. \tag{3.22}
\]

\[
\dot{\chi} + H\chi - \alpha - \varphi = \kappa_0^2 \frac{a^2}{k^2} \pi^{(s)}. \tag{3.23}
\]

\[
\dot{k} + 2H\kappa + \left(3H - \frac{k^2}{a^2}\right) \alpha = \frac{1}{2} \kappa_0^2 (\delta \mu + 3\delta p). \tag{3.24}
\]

\[
\left(D_t^2 + 3HD_t - \frac{\Delta}{a^2} + M^2\right) \delta \phi = \left(\dot{\alpha} - 3\dot{\varphi} - \frac{\Delta}{a^2} \chi\right) \phi - 2\alpha G^{-1} \nabla^T V. \tag{3.25}
\]

\[
\delta \dot{\mu} + 3H (\delta \mu + \delta p) = (\mu + p) \left(\kappa - 3H \alpha - \frac{k}{a} v\right). \tag{3.26}
\]

\[
\frac{[a^4 (\mu + p) v]}{a^4 (\mu + p)} = \frac{k}{a} \left[\alpha + \frac{1}{\mu + p} \left(\delta p - \frac{2}{3} k^2 - 3K \frac{a}{k} \pi^{(s)}\right)\right]. \tag{3.27}
\]

Equations (3.20)-(3.27) are the definition of \(\kappa\), the ADM energy constraint (\(G^0_0\) component of the field equation), the ADM momentum constraint (\(G^0_i\) component), the ADM propagation (\(G^i_j - \frac{1}{2} \delta^i_j \delta^k_i\) component), the Raychaudhuri equation (\(G^i_i - G^0_0\) component), the equation of motion for scalar fields, energy conservation, and the momentum conservation, respectively. Here \(\delta \mu\) and \(\delta p\) are given by Eqs. (3.14) and (3.15) respectively, while

\[M^2 = G^{-1} \nabla^T \nabla V - R(\dot{\phi}, \dot{\phi}).\tag{3.28}\]

Note that these equations are valid for any \(K\), and for a scalar field, \(\pi^{(s)} = 0\).

Equations (3.20)-(3.27), together with the background equations (3.4)-(3.8), and the perturbed order variables for the scalar fields (3.14)-(3.16), provide a complete set of equations for analyzing scalar-type cosmological perturbations with multicomponent scalar fields. As we have not chosen a specific gauge so far, Eqs. (3.20)-(3.27) are therefore in the gauge-ready form. This allows us to impose any one of the available temporal gauge conditions, which would then fix the temporal gauge mode completely, leading to gauge-invariant variables.

### 3.4. Gauge-Invariant perturbation equations

As an illustration of the gauge-ready method, we derive some useful expressions using the gauge-invariant variables of Eq. (3.19) introduced in Section 3.3. These may be obtained by making judicious combinations of Eqs. (3.20)-(3.27).

Thus, from Eqs. (3.21) and (3.22) we obtain

\[
\frac{k^2 - 3K}{a^2} \varphi\chi = \frac{1}{2} \kappa_0^2 \delta \mu_v. \tag{3.29}
\]

Eq. (3.20) can be re-expressed as

\[
\alpha \chi + \varphi\chi = -\kappa_0^2 \frac{a^2}{k^2} \pi^{(s)}. \tag{3.30}
\]

Eqs. (3.22), (3.23) and (3.20) lead to

\[
\dot{\varphi}\chi - H\alpha\chi = -\frac{1}{2} \kappa_0^2 (\mu + p) \frac{a}{k} v\chi. \tag{3.31}
\]

Eqs. (3.20), (3.21) with (3.22) yield

\[
\delta \dot{\mu}_v + 3H \delta \mu_v = -\frac{k^2 - 3K}{a^2} \left[(\mu + p) \frac{a}{k} v\chi + 2H \frac{a^2}{k^2} \pi^{(s)}\right], \tag{3.32}
\]

\[\delta \dot{\mu}_v + 3H \delta \mu_v = -\frac{k^2 - 3K}{a^2} \left[(\mu + p) \frac{a}{k} v\chi + 2H \frac{a^2}{k^2} \pi^{(s)}\right]. \tag{3.32}
\]
while Eqs. (3.23) and (3.27) give
\[
\dot{v}_\chi + Hv_\chi = \frac{k}{a} \left[ \alpha_\chi + \frac{\delta p_v}{\mu + p} - \frac{2}{3} \frac{k^2 - 3K}{a^2} \frac{\pi^{(s)}}{\mu + p} \right].
\] (3.33)
Combining Eqs. (3.29) - (3.33) we can derive
\[
\ddot{\varphi}_\chi + (4 + 3c_s^2)H \varphi_x - c_s^2 \frac{\Delta}{a^2} \varphi_x + \left[ (\mu c_s^2 - p) - 2(1 + 3c_s^2) \frac{K}{a^2} \right] \varphi_x = - \frac{1}{2} \kappa_0^2 \left( e - \frac{2}{3} \pi^{(s)} \right) - \frac{1}{2} \kappa_0^2 \frac{\mu + p}{H} \left( \frac{2H^2 a^2}{\mu + p} \kappa_0^2 \pi^{(s)} \right),
\] (3.34)
where we used Eq. (3.17). From Eq. (3.30) we can draw the important conclusion that, for scalar-fields, \(\alpha_\chi = - \varphi_x\), since \(\pi^{(s)} = 0\). Using this result the equation of motion for scalar fields becomes
\[
(D^2_\eta + 2HD_\eta - \Delta + a^2 M^2) \delta \phi_x = - 4 \varphi_x \varphi' + 2a^2 \varphi_x G^{-1} \nabla^T V,
\] (3.35)
while Eqs. (3.31) and (3.34) become
\[
\varphi'_x + H \varphi_x = - \frac{1}{2} \kappa_0^2 \phi' \cdot \delta \phi_x,
\] (3.36)
\[
\varphi''_x + 6H \varphi'_x - \Delta \varphi_x + 2 \left[ H' + 2(H^2 - K) \right] \varphi_x = \kappa_0^2 a^2 \nabla \cdot \delta \phi_x,
\] (3.37)
where we used Eq. (3.16), and the relations
\[
e = \delta p - c_s^2 \delta \mu = \delta p_x - c_s^2 \delta \mu_x,
\]
\[
(1 - c_s^2) \delta \mu_x - e = \delta \mu_x - \delta p_x = \nabla \cdot \delta \phi_x.
\] (3.38)
Eq. (3.37) is often called the constraint equation. These equations contain most of the physics related to inflationary cosmological perturbations. They are expressed in terms of gauge-invariant forms of the variables, and from Section 3.3 we see that they retain the same algebraic forms in the zero-shear gauge.

We note that Eq. (3.37) may be recast in a different way. According to Eq. (2.15), \(\delta \phi_x\) may be decomposed into components parallel and perpendicular to the field velocity, \(\delta \phi_x = \delta \phi_x^\parallel + \delta \phi_x^\perp\). Using the background equation (3.7), the constraint equation (3.37), and the fact that \(|\varphi'| = (D_\eta \varphi') \cdot \phi'\), we can write Eq. (3.37) as
\[
\varphi''_x + 2 \left( H - \frac{|\varphi'|}{|\phi'|} \right) \varphi'_x
+ 2 \left[ H' - H \frac{|\phi'|}{|\phi'|} - 2K \right] \varphi_x - \Delta \varphi_x
= - \kappa_0^2 (D_\eta \varphi') \cdot \delta \phi_x.
\] (3.39)
Following our discussion in Section 2.2, we know that the perpendicular component of field perturbation vanishes when there is only one field. In this case, the right hand side of Eq. (3.39) vanishes, and the resulting equation is well known in the theory of single field inflationary perturbations [10].
4. PERTURBATIONS DURING INFLATION

4.1. Slow-roll variables

Continuing with our analysis, we now assume that the Universe has undergone inflation to complete flatness, so that henceforth we can set \( K = 0 \). This allows us to introduce a set of functions known as the slow-roll variables:

\[
\epsilon(\phi) \equiv -\frac{\dot{H}}{H^2}, \quad \eta(\phi) \equiv \frac{\phi'(2)}{H|\phi'|}, \quad (4.1)
\]

It is also convenient to decompose \( \eta \) into parallel and perpendicular components using Eq. (2.15),

\[
\eta^\parallel = e_1 \cdot \eta = \frac{D \dot{\phi} \cdot \dot{\phi}}{H|\phi'|^2}, \quad \eta^\perp = e_2 \cdot \eta = \frac{|(D \dot{\phi})^\perp|}{H|\phi'|}. \quad (4.2)
\]

The standard slow-roll assumptions are

\[
\epsilon = O(\zeta), \quad \eta^\parallel = O(\zeta), \quad \eta^\perp = O(\zeta), \quad (4.3)
\]

for some small parameter \( \zeta \), with \( \epsilon, \sqrt{\epsilon} \eta^\parallel \), and \( \sqrt{\epsilon} \eta^\perp \) much smaller than unity. If in an expansion in slow-roll variables we neglect terms of order \( O(\zeta^2) \), we claim that expansion to be of first order in slow-roll. Thus terms with \( \epsilon^2, \epsilon \eta^\parallel \), etc. are of second order. We list some useful relations involving the slow-roll variables:

\[
\mathcal{H}' = \mathcal{H}(1 - \epsilon), \quad \frac{|\phi'|}{|\phi|} = \mathcal{H}(1 + \eta^\parallel),
\]

\[
D_{\eta} \phi' = \mathcal{H}|\phi'|(\eta + e_1) = \kappa_0^{-1} \sqrt{2\mathcal{H}^2} \sqrt{\epsilon}(\eta + e_1),
\]

\[
\mathcal{H}^2 \epsilon = \frac{1}{2} \kappa_0^2 |\phi'|^2, \quad \epsilon' = 2\mathcal{H} \epsilon(\epsilon + \eta^\parallel). \quad (4.4)
\]

4.2. Analysis using gauge-invariant variables

In order to solve the system of perturbation equations (3.35), (3.36) and (3.39), we shall find it convenient to introduce the variables,

\[
q = a \left( \delta \phi \chi - \frac{\phi'}{H} \varphi \chi \right) = a \left( \delta \phi - \frac{\phi'}{H} \varphi \right), \quad (4.5)
\]

\[
u = -\frac{a}{\kappa_0^2 |\phi'|^2} \chi = \frac{1}{\kappa_0 \sqrt{2\mathcal{H}^2} \epsilon} \varphi \chi. \quad (4.6)
\]

Here \( q \) is a gauge-invariant quantity, and is a natural generalization of the single field Sasaki-Mukhanov variable [10]. We first express the constraint equation (3.36) in terms of the slow-roll variables (4.1) as

\[
\varphi' + \mathcal{H}(1 + \epsilon) \varphi = -\frac{1}{2} \kappa_0^2 \phi' \cdot \frac{q}{a}. \quad (4.7)
\]

The equation for the scalar field perturbations follows from (3.35),

\[
D^2_\eta q - (\Delta - \mathcal{H}^2 \Omega)q = 0, \quad (4.8)
\]

where

\[
\Omega = \frac{a^2 M^2}{\mathcal{H}^2} - (2 - \epsilon) \mathbb{I} - 2\epsilon \left( (3 + \epsilon) P^\parallel + \epsilon_1 \eta^\parallel + \eta e_1^\perp \right), \quad (4.9)
\]

and we also used (177). The corresponding Lagrangian \( \mathcal{L}' \) follows from Eq. (4.8):

\[
S = \int \mathcal{L}' \sqrt{g(3)} d\eta d^3 x = \int \frac{1}{2} (D_{\eta} q^\dagger D_{\eta} q + q^\dagger (\Delta - \mathcal{H}^2 \Omega)q) \sqrt{g(3)} d\eta d^3 x. \quad (4.10)
\]
Here \(g^{(3)}\) is the determinant of the metric \(g_{ij}^{(3)}\) of the 3-surfaces of constant curvature \(K = 0\), see below Eq. (3). The equation of motion for \(u\) is obtained by substituting its definition into Eq. (3.39):

\[
\dddot{u} - \Delta u - \frac{\theta''}{\theta} u = \mathcal{H} \eta^\dagger q_2, \quad q_2 \equiv e_2 \cdot q, 
\]

\[
\theta = \frac{\mathcal{H}}{a|\phi|} = \frac{\kappa_0}{\sqrt{2}} \frac{1}{a \sqrt{\epsilon}}. \tag{4.11}
\]

Further, from Eq. (4.7) it follows that

\[
\ddot{u} + \left(\frac{1}{\theta}\right)' \frac{1}{\theta} u = \frac{1}{2} q_1, \quad q_1 \equiv e_1 \cdot q. \tag{4.12}
\]

Differentiating Eq. (4.12) once with respect to the conformal time and using Eq. (4.11), we obtain the relation

\[
\frac{1}{2} \left( q_1' - \frac{(1/\theta)'}{1/\theta} q_1 \right) - \mathcal{H} \eta^\dagger q_2 = \Delta u. \tag{4.13}
\]

We pause to note that though the equations (4.7), (4.8) and (4.11) have been expressed in terms of the slow-roll variables, they are exact, and no slow-roll approximation has yet been made. Observe that, to the leading order in slow-roll, the perturbation variables \(q\) and \(u\) decouple, whereas at first order, mixing between these occur.

### 4.3. Quantization of density perturbations

We now briefly discuss the quantization of the density perturbations described by the Lagrangian in Eq. (4.10). Introduce the matrix \(Z_{mn}\) defined as

\[
(Z)_{mn} = -(Z^T)_{mn} = \frac{1}{\mathcal{H}} e_m \cdot D_{\eta} e_n. \tag{4.14}
\]

Thus \(Z\) is antisymmetric and traceless. Expanding \(q = q_m e_m\) using the basis \(\{e_m\}\), Eq. (4.10) may be expressed as,

\[
\mathcal{L} = \frac{1}{2} (q' + \mathcal{H} Z q)^T (q' + \mathcal{H} Z q) + \frac{1}{2} q^T (\Delta - \mathcal{H}^2 \Omega) q, \tag{4.15}
\]

where \((\Omega)_{mn} = e_m^\dagger \Omega e_n\), and for notational ease, we have suppressed the indices \(m, n\). Now redefine \(q\) using a new matrix \(R\) as

\[
q(\eta) = R(\eta) Q(\eta), \quad R' + \mathcal{H} Z R = 0, \quad \tilde{\Omega} = R^T \Omega R. \tag{4.16}
\]

From the equation of motion (4.16) for \(R\), it follows that \(R^T R\) and det\(R\) are constants, so that \(R\) represents a rotation. Without any loss of generality, the initial value of \(R\) may be chosen as \(R(\eta_0) = 1\). Substituting the variables defined in Eq. (4.16) into Eq. (4.15) reduces the Lagrangian to the canonical form:

\[
\mathcal{L} = \frac{1}{2} Q'^T Q' + \frac{1}{2} Q^T (\Delta - \mathcal{H}^2 \tilde{\Omega}) Q. \tag{4.17}
\]

The corresponding Hamiltonian is then given by

\[
\mathcal{H} = \frac{1}{2} \Pi'^T \Pi - \frac{1}{2} Q^T (\Delta - \mathcal{H}^2 \tilde{\Omega}) Q, \tag{4.18}
\]

where the momentum \(\Pi\) canonically conjugate to \(Q\) is

\[
\Pi(\eta, \mathbf{x}) = \partial \mathcal{L}/\partial \dot{Q} = Q'(\eta, \mathbf{x}). \tag{4.19}
\]

To implement the canonical quantization procedure, the variables \((Q, \Pi)\) are promoted to quantum operators \((\hat{Q}, \hat{\Pi})\) satisfying the commutation relations

\[
[a^T \hat{Q}(\eta, \mathbf{x}), \beta \hat{Q}(\eta, \mathbf{x}')] = [a^T \hat{\Pi}(\eta, \mathbf{x}), \beta \hat{\Pi}(\eta, \mathbf{x}')] = 0,
\]

\[
[a^T \hat{Q}(\eta, \mathbf{x}), \beta \hat{\Pi}(\eta, \mathbf{x}')] = i a^T \beta \delta(\mathbf{x} - \mathbf{x}'), \tag{4.20}
\]

\[
[a^T \hat{Q}(\eta, \mathbf{x}), \beta \hat{Q}(\eta, \mathbf{x}')] = [a^T \hat{\Pi}(\eta, \mathbf{x}), \beta \hat{\Pi}(\eta, \mathbf{x}')] = 0,
\]

\[
[a^T \hat{Q}(\eta, \mathbf{x}), \beta \hat{\Pi}(\eta, \mathbf{x}')] = i a^T \beta \delta(\mathbf{x} - \mathbf{x}'), \tag{4.20}
\]
where the delta function is normalized as

\[ \int \delta(x-x') \sqrt{g^{(3)} d^3x} = 1. \tag{4.21} \]

Here we have introduced the vectors \( \alpha, \beta \) with components \( \alpha_m, \beta_m \) in the basis \( \{ e_m \} \) to avoid writing the indices \( m, n \) in the commutators. Since we are considering spatially flat hypersurfaces \( (K = 0) \), the operator \( \hat{Q} \) may be expanded in a plane wave basis as

\[ \hat{Q} = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ Q_k(\eta) \hat{a}_k^\dagger e^{-i k \cdot x} + \text{h.c.} \right], \tag{4.22} \]

with a similar expansion for \( \hat{\Pi} \). It immediately follows from Eq. (4.16) that \( q \) must now be interpreted as the operator \( \hat{q} \) satisfying a mode expansion identical to Eq. (4.22). The creation and annihilation operators \( \hat{a}_k^\dagger \) and \( \hat{a}_k \) satisfy

\[ [\hat{\alpha}^T \hat{a}_k, \beta \hat{a}_k^\dagger] = [\hat{\alpha}^T \hat{a}_k^\dagger, \beta \hat{a}_k] = 0, \]
\[ [\hat{\alpha}^T \hat{a}_k, \beta \hat{a}_k^\dagger] = \alpha^T \beta \delta(k-k'). \tag{4.23} \]

To maintain consistency of the commutation relations (4.20) and (4.23), the Wronskian condition

\[ W\{Q_k, Q_k^*\} \equiv Q_k'(\eta)Q_k^*(\eta) - Q_k^*(\eta)Q_k(\eta) = i \tag{4.24} \]

must be satisfied. From the mode expansion (4.22) and the Hamiltonian (4.18), it follows that the equation of motion for \( Q_k \) is

\[ Q_k'' + \left( k^2 + \frac{H^2 \bar{\Omega}}{2} \right) Q_k = 0. \tag{4.25} \]

It may be easily verified using Eq. (4.25) that the Wronskian satisfies \( dW\{Q_k, Q_k^*\}/d\eta = 0 \).

We also interpret the variable \( u \) introduced in Eq. (4.6) as an operator \( \hat{u} \), and after performing a mode expansion identical to that of \( \hat{Q} \) in Eq. (4.22), it follows from Eq. (4.11) that the modes \( u_k \) satisfy

\[ u_k'' + \left( k^2 - \frac{g''}{\vartheta} \right) u_k = H q_2'^k \eta q_{2k}, \quad q_{2k} \equiv (e_2 \cdot e_m) q_k, \tag{4.26} \]

or, equivalently, from Eq. (4.13),

\[ H q_2'^k \eta q_{2k} - \frac{1}{2} \left( q_{1k} - \frac{(1/\vartheta)}{1/\vartheta} q_1 k \right) = k^2 u_k, \]
\[ q_{1k} \equiv (e_1 \cdot e_m) q_k. \tag{4.27} \]

4.4. First order solution

In order to present the solution of the scalar perturbation equations, it is convenient to introduce the time \( \eta_H \) when the mode with wave number \( k \) crosses the Hubble radius during inflation, so that the relation

\[ H(\eta_H) = k \tag{4.28} \]

is satisfied for each \( k \). Consequently, the inflationary epoch can be separated into three regions: the sub-horizon region \( (H \ll k) \), the transition region \( (H \sim k) \), and the super-horizon region \( (H \gg k) \). We now discuss each of these in turn.

In the sub-horizon region, we solve Eq. (4.26) with the \( H^2 \bar{\Omega} \) term subdominant compared to \( k^2 \). The solution is obtained in the limit \( k/H \to \infty \) for fixed \( k \) as

\[ Q_k(\eta) = \frac{1}{\sqrt{2k}} e^{ik(\eta-\eta_0)}, \quad R(\eta_0) = 1. \tag{4.29} \]

Since one is usually interested in calculating quantities at the end of inflation, this region is therefore irrelevant.
Solving Eq. (4.16) for the rotation matrix $R$ and integrating the relation for $H$ so that

$\frac{\dot{H}}{H} = -\frac{1}{3} \left[ 1 + \frac{3M^2}{H^2} \right] = \frac{1}{3} \frac{c}{H^2}$.

Here the second equality is valid to the first order in slow-roll, and we have used the notation $\Omega$ for the matrix valued order quantity.

Since the time-dependent terms in the matrix $\Omega$ in Eq. (4.9) are of first order, we can take $\Omega = \Omega(\eta H) \equiv \Omega_H$ in the transition region. Then the matrix $\bar{\Omega}$ is given by

$$\bar{\Omega} = R^{-1}(z)\Omega_H R(z) = \Omega_H + 3[\delta_H, Z_H] \left( \ln \frac{z}{z_H} + \frac{3}{4} \ln \epsilon_H \right),$$

using the above results in Eq. (4.25) to be a first order quantity. We also made the assumption that those components of $a^2M^2/H^2$ which cannot be expressed in terms of the slow-roll variables are of first order. Because $\delta_H$ and $Z_H$ are both of first order, we can take $\bar{\Omega} = \Omega_H$ in Eq. (4.32) to be a first order quantity.

In order to write the equation for the mode $Q_k$ in the transition region, we will find it convenient to define $Q_k \equiv R_H Q_k(z)$ and $\bar{\Omega} = R_H \Omega_R^{-1}$, with $R_H \equiv R(z_H)$. From Eq. (4.33), we have to the first order, $Q_k(z) = \tilde{Q}_k(z)$, while from Eq. (4.34) we conclude that $\bar{\Omega} = \Omega_H$ within a small region around $z_H$. Using the above results in Eq. (4.30), the mode equation for $Q_k$ may be written in terms of $\tilde{Q}_k$ as

$$\tilde{Q}_{k,zz} + \left( 1 - \frac{\nu_H^2 - 1}{2z^2} \right) \tilde{Q}_k = 0, \quad \nu_H^2 = \frac{9}{4} + 3 \delta_H.$$

This equation is similar to the one obtained for the single-field inflation, except that this is a matrix equation. The solution is then given in terms of the Hankel functions of matrix valued order $\nu_H$,

$$\tilde{Q}_k(z) = \sqrt{z} [c_1(k) H^{(1)}_{\nu_H}(z) + c_2(k) H^{(2)}_{\nu_H}(z)], \quad \nu_H = \frac{3}{2} + \delta_H.$$
We wish to match the solution in Eq. (4.38) so that in the limit \( k/\mathcal{H} \to \infty \), the modes approach plane waves, \( \tilde{Q}_k(z) = e^{iz/\sqrt{2k}} \), see (4.29). For \( |z| \gg 1 \), the Hankel functions have the asymptotic forms,

\[
H^{(1)}_{\nu}(z) \sim \sqrt{2/\pi z} e^{i(z-(\nu+1/2)\pi/2)}, \\
H^{(2)}_{\nu}(z) \sim \sqrt{2/\pi z} e^{-i(z-(\nu+1/2)\pi/2}).
\]  

(4.39)

We set \( c_1(k) = \sqrt{\pi/(4k)} e^{i(\nu+1/2)\pi/2} \), and \( c_2(k) = 0 \). The phase factor of \( c_1(k) \) is chosen in order to match with Eq. (4.29) at short scales, while the factor of \( \sqrt{\pi/(4k)} \) ensures conformity with the Wronskian in Eq. (4.29). Therefore the final solution with the appropriate normalization is

\[
\tilde{Q}_k(z) = \sqrt{\pi/(4k)} e^{i(\nu+1/2)\pi/2} \sqrt{z} H^{(1)}_{\nu}(z).
\]  

(4.40)

It is worth mentioning that that the matrix valued Hankel functions are to be interpreted as series expansions, just like the usual Hankel functions.

We finally discuss the solution in the super-horizon region. On super-horizon scales we have \( |z| \ll 1 \), for which the asymptotic form of the Hankel function is

\[
H^{(1)}_{\nu}(z) \sim \sqrt{2/\pi} e^{-i\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu+1/2)}{\Gamma(3/2)} z^{-\nu},
\]  

(4.41)

so that the asymptotic solution for \( \tilde{Q}_k(z) \) in the super-horizon region is given by

\[
\tilde{Q}_k(z) \sim \frac{1}{\sqrt{2k}} e^{i(\nu-1/2)\pi/2} 2^{\nu-3/2} \frac{\Gamma(\nu+1/2)}{\Gamma(3/2)} \frac{1}{\sqrt{z}} \mathbb{I} - \nu, \\
\sim -\frac{1}{\sqrt{2k}} e^{i(\nu-1/2+2\delta)\pi/2} E_{\mathcal{H}}(z/\mathcal{H})^{-\nu} \mathbb{I} - \delta,
\]  

(4.42)

where

\[
E_{\mathcal{H}} \equiv (1 - \epsilon_{\mathcal{H}}) \mathbb{I} + (2 - \gamma_{\mathcal{E}} - \ln 2) \delta_{\mathcal{H}},
\]  

(4.43)

and \( \gamma_{\mathcal{E}} \approx 0.5772 \) is the Euler constant.

In this region since \( k/\mathcal{H} \to 0 \), we can also solve Eq. (4.26) ignoring the \( k^2 \) dependent term, leading to

\[
u_k(\eta) = u_p \kappa + D_k \theta \int_{\eta_0}^\eta \frac{d\eta'}{\theta^2(\eta')}, \\
u_p \kappa = \theta \int_{\eta_0}^\eta \frac{d\eta'}{\theta^2} \int_{\eta_0}^{\eta'} d\eta'' \delta \eta' \phi q_{2k},
\]  

(4.44)

where \( C_k \) and \( D_k \) are constants of integration, and \( u_p \kappa \) is a particular solution. Note that since \( \theta \) is a rapidly decaying function, we can ignore \( C_k \) compared to \( D_k \). In the same approximation, the solution of Eq. (4.27) is

\[
q_{1k} = d_k (1/\theta) + 2(1/\theta) \int_{\eta_0}^\eta d\eta' \theta \delta \eta' q_{2k}.
\]  

(4.45)

From Eq. (4.12) we see that the integration constants \( D_k \) and \( d_k \) are related by \( \widetilde{D_k} = \frac{1}{2} d_k \). Considering the region where \( \eta \) is sufficiently close to \( \eta_{\mathcal{H}} \), the integral in Eq. (4.45) may then be neglected, so that using Eq. (4.32), we can write \( q_{1k} = 2D_k (1/\theta_{\mathcal{H}})(z/\mathcal{H})^{-1} \). Taking into account the asymptotic solution (4.29), and the fact that \( q_k = (e_1 \cdot e_m)^T q_{1k} \), we finally obtain,

\[
D_k = -\frac{1}{2(\sqrt{2k})} e^{i(\nu-1/2+2\delta)\pi/2} \theta_{\mathcal{H}} (e_1 \cdot e_m)^T E_{\mathcal{H}}.
\]  

(4.46)

Thus the integration constant in Eq. (4.43) is completely determined to first order in slow-roll. Inserting the result (4.40) for \( D_k \) in (4.6) together with (4.43), and using the relation \( a_{\mathcal{H}} H_{\mathcal{H}} = k \), we finally arrive at

\[
\varphi_{\mathcal{H}} \equiv e^{i(\nu-1/2+2\delta)\pi/2} \frac{k_0}{2k^{3/2}} \frac{H_{\mathcal{H}}}{\sqrt{c_{\mathcal{H}}}} \left[ f(t_{\mathcal{H}}, t)(e_1 \cdot e_m)^T + f(t_{\mathcal{H}}, t) \right] E_{\mathcal{H}},
\]  

(4.47)
where we ignored $C_k$, and

$$\mathcal{F}(t, t) = \frac{H}{a} \int_{t_H}^{t} dt' a \left( \frac{1}{H} \right)^3, \quad \mathcal{F}(t_H, t) = \frac{H}{a} \int_{t_H}^{t} dt' a \left( \frac{1}{H} \right)^3 \mathcal{W}(t_H, t),$$

$$\mathcal{W}(t, t) = \frac{2}{a} \int_{t_H}^{t} dt' H e^{\eta} \sqrt{\frac{\epsilon}{\epsilon - \alpha}} (e_2 - e_m)^T R \frac{Q_k}{Q_{k_H}}.$$ (4.48)

Here $Q_{kH}$ is the value of the asymptotic solution $e_{ij}$ for $Q_k$ evaluated at $\eta = \eta_H$. Observe that the solution $\varphi_{\lambda k}$ for $\varphi_{\lambda k}$ is expressed entirely in terms of background quantities and comoving time.

### 4.5. Vector and tensor perturbations

For the sake of completeness, we now present a brief discussion of vector- and tensor-type perturbations. From the $G_{ij}$ component of Eq. (2.7), together with Eq. (3.13), we have

$$\frac{1}{2} k^2 \Psi^{(v)} = \kappa_0^2 a^2 (\mu + \pi) \nu^{(v)},$$ (4.49)

while the condition $T_{i\mu}^\mu = 0$ yields

$$\frac{1}{a^4} \left[ a^4 (\mu + \pi) \nu^{(v)} \right]' = - \frac{1}{2} k \pi^{(v)}.$$ (4.50)

Equations (4.49) and (4.50) describe the vector-type, or rotational perturbations. Observe that $\Psi^{(s)}$, $\nu^{(s)}$, and $\pi^{(s)}$ appearing in these equations are gauge-invariant, see Eq. (3.18). Since vector sources are absent when the matter sector is composed entirely of scalar fields, the vector-type perturbations are therefore irrelevant in the inflationary scenario.

The equation for the tensor-type, or gravitational wave perturbations follows from the $G^{tr}_{ij}$ component of (2.7):

$$C_{ij}^{(t)\mu} + 2 HC_{ij}^{(t)} + k^2 C_{ij}^{(t)} = \kappa_0^2 a^2 \pi_{ij}^{(t)}.$$ (4.51)

For scalar fields we have $\pi^{(t)} = 0$. Here $C_{ij}^{(t)}$ is symmetric, transverse-traceless, and gauge-invariant. We quantize the tensor perturbations by interpreting $C^{(t)}_{ij}$ as the operator $\hat{C}_{ij}^{(t)}$ with the mode expansion

$$\hat{C}_{ij}^{(t)} = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ C_{ij}^{(t)}(k) \hat{a}_{\Delta_{k}}, e^{-ik \cdot x} + \text{h.c.} \right],$$ (4.52)

$$C_{ij}^{(t)}(k) = \sum_{\lambda = +, -} \frac{2\kappa_0}{a} \epsilon_{ij}(k; \lambda) v_{\lambda k}(k \cdot x).$$ (4.53)

The quantity $\epsilon_{ij}(k; \lambda)$ is the polarization tensor satisfying the conditions $\epsilon_{ij} = \epsilon_{ji}$, $\epsilon_{ii} = 0$, $k^i \epsilon_{ij} = 0$, and $\epsilon^{(i)}(k; \lambda) \epsilon^{(i)}(k; \lambda') = \delta_{\lambda \lambda'}$. The summation is over the two independent polarization states $+$ and $\times$. As before, the creation and annihilation operators satisfy the commutation relation,

$$[\hat{a}_{\lambda k}, \hat{a}^+_{\lambda' k'}] = \delta_{\lambda \lambda'} \delta^{(3)}(k - k').$$ (4.54)

The mode $v_{\lambda k}(\eta)$ satisfies

$$v''_{\lambda k} + \left( k^2 - \frac{a''}{a} \right) v_{\lambda k} = 0,$$ (4.55)

for each $\lambda$, and $a''/a = H^2 (2 - \epsilon)$. Proceeding similarly as in the case of scalar perturbations, the solution of Eq. (4.55) in a sufficiently small interval of time around $\eta_H$ to the first order in slow-roll may be written in terms of Hankel functions ($z = k\eta$):

$$v_{\lambda k}(z) = \sqrt{\pi/(4k)} \sqrt{z H^{(1)}_{\frac{1}{2} + \epsilon_\lambda}} (z).$$ (4.56)
Performing the asymptotic expansion of Hankel function in the region $|z| \ll 1$, the super-horizon solution for $v_k(z)$ becomes

$$v_k(z) \sim (1/\sqrt{2k}) e^{i(2c + 1)\pi^2/2} C_H(z/z_H)^{-1} e^{j},$$

$$C_H = 1 + (1 - \gamma_E - \ln 2) e^{j}.$$  \hspace{1cm} (4.57)

On the other hand, solving Eq. (4.55) in the super-horizon region yields

$$v_k(z) = A_k a + B_k a \int \frac{z}{a^2} dz'.$$ \hspace{1cm} (4.58)

We ignore the rapidly decaying term $B_k$, and matching the $A_k$ term with the solution (4.57), we finally obtain the solution of tensor perturbation in the super-horizon region:

$$C_{ij}^{(t)}(\eta) = e^{i(2c + 1)\pi^2/2} C_H \sum_{\lambda = +, c} e_{ij}(k; \lambda).$$ \hspace{1cm} (4.59)

5. POST-INFLATIONARY PERTURBATIONS

5.1. Adiabatic and Isocurvature perturbations

In the previous Section we presented the super-horizon solutions to scalar perturbations during inflation. We now extend the analysis to density perturbations after inflation, in the regime of radiation- / matter-domination and recombination.

It is well known that the most general density perturbation is a linear combination of adiabatic and isocurvature, or entropy perturbations. Adiabatic perturbations are just the total energy density perturbations. They perturb the solution along the same trajectory in phase-space as the background solution. On the other hand, isocurvature or entropy perturbations. Adiabatic perturbations are just the total energy density perturbations. They perturb the solution off the background solution. When there is just one scalar field driving the inflation, isocurvature perturbations among them are absent, while the remaining $N - 1$ isocurvature perturbations will be along directions orthonormal to $e_1$, while the $N - 1$ isocurvature perturbations will be along directions orthonormal to $e_1$.

In order to simplify our analysis, we assume that, of the $N$ scalar fields, one of them has decayed to Standard Model particles so that isocurvature perturbations among them are absent, while the remaining $N - 1$ components have decayed to non-interacting Cold Dark Matter (CDM). Then the CDM components may be considered as ideal fluids without mutual interactions.

We start with Eq. (3.34) in conformal time for $K = 0 = \pi^2$, with the gauge-invariant entropy perturbation $e$ defined in Eq. (3.41):

$$\varphi' + 3H(1 + c^2)\varphi' + c^2 k^2 \varphi + [2H' + (1 + 3c^2)H^2] \varphi = 2k_0^2 a^2 (\mu c_s^2 - p) S,$$

where

$$S = -\frac{1}{4} \frac{e}{\mu c_s^2 - p} = \sum_{i=1}^{N-1} \frac{\mu_i S_i}{\sum_{i=1}^{N-1} \mu_i},$$

is the total entropy perturbation, and

$$S_i = \frac{\delta \mu_i}{\mu_i + p_i} - \frac{\delta \mu_i}{\mu_i + p_i}.$$ \hspace{1cm} (5.3)

In Eq. (5.3) we used the notation that the Standard Model particles are represented as photons (with subscript $\gamma$) and $p_\gamma = 0$ for CDM components, while $p_\gamma = \mu_\gamma / 3$ for photons. Since we are considering the CDM components as mutually non-interacting ideal fluids, it can be shown that in flat space $S_i$ are constant in the super-Hubble region. Further, $\mu_i$ have the same time-dependence. Therefore, within our approximation, $S$ is constant for super-horizon scales.
5.2. Super-Horizon solutions

The solution of Eq. 5.1 in the super-horizon region, where we neglect the \( c_s^2 k^2 \) term, may be obtained in the same way that we derived the inflationary solution. Introducing the variables \( u \) and \( \theta \) defined in Eqs. 4.6 and 4.11 respectively, Eq. 5.1 becomes,

\[
\frac{u''}{\theta'} \mu = -2\sqrt{3} \frac{aH}{\kappa_0} \frac{c_s^2 - w}{\sqrt{1 + w}} S,
\]

with \( w \equiv p/\mu \). This equation admits the solution \( \varphi = \varphi^{(0)} + \varphi^{(P)} \), where

\[
\varphi^{(0)} = -\kappa_0^2 C \frac{H}{a} - 3D \frac{H}{a} \int_{t_0}^{t} dt' a(1 + w)
\]

\[
= - \left( \kappa_0^2 C + 2D \frac{a(t_0)}{H(t_0)} \right) \frac{H}{a} - 2D \mathcal{I}(t_0, t)
\]

is the homogeneous solution (with \( S = 0 \)), and

\[
\varphi^{(P)} = 6 \frac{H}{a} \int_{t_0}^{t} dt' a(1 + w) \int_{t_0}^{t'} dt'' H \frac{c_s^2 - w}{1 + w} S
\]

\[
= 2 \frac{H}{a} \int_{t_0}^{t} dt' \frac{1 + w}{\frac{3}{2} + \frac{1}{2} w} \left( \frac{a}{H} \right) \int_{t_0}^{t'} dt'' S \left( \frac{5}{2} + \frac{3}{2} w \right)
\]

is the particular solution. The function \( \mathcal{I} \) is defined in Eq. 4.48. It immediately follows from Eq. 5.1 that when \( S \) is constant, \( \varphi^{(P)} = 2S \) is a particular solution.

Returning back to our discussion in Section 5.1, we can claim that the adiabatic perturbation corresponds to the homogeneous solution of Eq. 5.1 with the source term \( S \) absent, while the isocurvature perturbation pertains to the particular solution due to the entropic source \( S \). The initial conditions that one imposes on the isocurvature perturbation are that both \( \varphi \) and \( \varphi' \) vanish at the beginning of the radiation-dominated era. If we further make the simplifying assumption that the end of inflation at time \( t_e \) marks the immediate beginning of the radiation epoch, we can then write the solution to isocurvature perturbation with a constant \( S \) as

\[
\varphi^{(iso)}(t) = 2S \left( 1 - \frac{3}{2} \mathcal{J}(t_e, t) \right)
\]

\[
= 2S \left( 1 - \frac{3}{2} \mathcal{J}(t_e, t) \right).
\]

In the radiation-dominated era we have \( a(t) \propto t^{1/2} \) and \( w = 1/3 \), so that the non-decaying part of \( \mathcal{J}(t_e, t) = 2/3 \). Thus, from Eq. 5.7, \( \varphi^{(iso)} \) vanishes for radiation domination. On the other hand, at the time of recombination during matter-domination, we have \( a(t) \propto t^{2/3} \), \( w = 0 \), and hence \( \mathcal{J}(t_e, t) = 3/5 \). In this case we have the simple relation

\[
\varphi^{(iso)} = \frac{1}{5} S.
\]

We now derive the expression for \( S \) in the super-horizon region during inflation. Using the slow-roll functions introduced in Section 4.1, we have

\[
\mu = \frac{1}{2} |\dot{\phi}|^2 + V = \frac{3H^2}{\kappa_0^2},
\]

\[
p = \frac{1}{2} |\dot{\phi}|^2 - V = \frac{3H^2}{\kappa_0^2} \left( 1 - \frac{2}{3} \epsilon \right),
\]

\[
w = \frac{p}{\mu} = -1 + \frac{2}{3} \epsilon,
\]

\[
c_s^2 = \frac{\dot{\phi}}{\mu} = -1 - \frac{2}{3} \eta,
\]

\[
\delta p - c_s^2 \delta \mu = \frac{2\sqrt{2}}{\kappa_0} H^2 \sqrt{c_s^2 \frac{\delta \phi}{a}}
\]

(5.9)
Substituting these into the definition \( S = \frac{1}{4}(\delta p - c_s^2 \delta \mu)/(p - c_s^2 \mu) \) yields
\[
S = \frac{\kappa_0}{2\sqrt{2} \epsilon + \eta} \eta^+ \frac{q_2}{a}. 
\] (5.10)

From Eq. (5.10), we see that the total entropy perturbation is generated along the directions orthonormal to the adiabatic perturbation, in agreement with [19]. It is worth mentioning here that when \( \dot{\epsilon} = 2H\epsilon(\epsilon + \eta^+) = 0 \), the expression for \( S \) as given by (5.10) develops a singularity. Hence \( S \) is not a convenient variable to use during inflation.

It remains to obtain the solution for the adiabatic perturbation. This is derived by matching the solution for \( \dot{\varphi}_k \) in Eq. (4.47) at the end of inflation \( t_e \) with the homogeneous solution (5.5), while maintaining continuity and differentiability. Ignoring the rapidly decaying \( \phi \) and spectral indices from inflation. The power spectra are conventionally defined as
\[
\Delta_A^2(k) = \frac{k^3}{2\pi^2} \langle |\varphi_{\chi k}^{(ad)}|^2 \rangle, \quad \Delta_S^2(k) = \frac{k^3}{2\pi^2} \langle |\varphi_{\chi k}^{(iso)}|^2 \rangle, 
\]
\[
\Delta_C^2(k) = \frac{k^3}{2\pi^2} \langle |\varphi_{\chi k}^{(t)}|^2 \rangle, 
\]
\[
\Delta_T^2(k) = \frac{k^3}{2\pi^2} \langle |\varphi_{\chi k}^{(iso)} \varphi_{\chi k}^{(ad)}|^2 \rangle. 
\] (5.11)

The functions \( \mathcal{I} \) and \( \mathcal{W} \) are defined in Eq. (5.5).

### 5.3. Power spectra and spectral indices from inflation

Having obtained the solutions for adiabatic and isocurvature perturbations given by Eqs. (5.11) and (5.8) respectively, and the solution for the tensor perturbation (4.59), we are now in a position to calculate the power spectra and spectral indices from inflation. The power spectra are conventionally defined as
\[
\Delta_A^2(k) = \frac{9}{25 \pi M_P^2 \epsilon H} \left[ (1 - 2\epsilon_H)(1 + \mathcal{W}_e^{T} \mathcal{W}_c) - 2C_0 \left( (2\epsilon_H + \eta^+_H) + 2\eta^+_{\mathcal{W}e} (e_2 \cdot e_m)^T \mathcal{W}_c + \mathcal{W}_c^{T} \delta_H \mathcal{W}_c \right) \right], 
\] (5.13)
\[
\Delta_S^2(k) = \frac{1}{25 \pi M_P^2 \epsilon H} \left[ (1 - 2\epsilon_H)\mathcal{W}_e^{T} \mathcal{W}_c - 2C_0 \mathcal{W}_e^{T} \delta_H \mathcal{W}_c \right], 
\] (5.14)
\[
\Delta_C^2(k) = \frac{3}{25 \pi M_P^2 \epsilon H} \left[ (1 - 2\epsilon_H)\mathcal{W}_e^{T} \mathcal{W}_c - 2C_0 (\eta^+_{\mathcal{W}e} e_2 \cdot e_m)^T \mathcal{W}_c + \mathcal{W}_c^{T} \delta_H \mathcal{W}_c \right], 
\] (5.15)
\[
\Delta_T^2(k) = \frac{16}{25 \pi M_P^2 \epsilon H} \left[ (1 - 2(C_0 + 1)\epsilon_H) \right], 
\] (5.16)
where \( M_P = G^{-1/2} \) is the Planck Mass, and
\[
\mathcal{W} = \frac{1}{2\sqrt{\epsilon H}} \sqrt{\epsilon H} \mathcal{Q}_H^+ \mathcal{Q}_H (e_2 \cdot e_m)^T, \quad C_0 = \gamma_E + \ln 2 - 2. 
\] (5.17)
The subscript \( e \) reminds us that \( \mathcal{W} \) and \( \mathcal{V} \) are evaluated at \( t = t_e \).
If we now assume that the power spectra $\Delta^2_X(k)$ depend weakly on $k$, where $X$ denotes adiabatic, isocurvature or cross-correlated spectra, we can parametrize them as

$$
\Delta^2_X(k) = \Delta^2_X(k_0) \left( \frac{k}{k_0} \right)^{n_X(k_0)-1},
$$

while the tensor spectrum is conventionally parametrized as

$$
\Delta^2_T(k) = \Delta^2_T(k_0) \left( \frac{k}{k_0} \right)^{n_T(k_0)}.
$$

The normalization factor $\Delta^2(k_0)$ is called the amplitude, and $n$ the spectral index. Here $k_0$ is a suitable pivot wavenumber. These parametrizations are valid for a range of $k$ when $n_X - 1$ and $n_T$ are close to zero, that is, when the power spectra are near scale-invariant. They lead to the definition of the spectral indices,

$$
n_X(k_0) - 1 = \frac{d \ln \Delta^2_X(k)}{d \ln k},
$$

for the scalar modes and

$$
n_T(k_0) = \frac{d \ln \Delta^2_T(k)}{d \ln k},
$$

for the tensor modes, with the right hand side of Eqs. (5.20) and (5.21) to be evaluated at $k = k_0$. Substituting the values of $\Delta^2(k)$ from above leads to the expressions for the spectral indices valid to first order in slow-roll:

$$
n_A - 1 = -4\epsilon_H - 2n_H^\parallel + \frac{2\eta_H^T (2\epsilon_H + \eta_H^\parallel - \delta_H) \eta_e - 4\eta_H^\parallel (e_2 \cdot e_m)^T \eta_e}{1 + \eta_e^T \eta_e},
$$

$$
n_S - 1 = -2 \frac{\eta_e^T \delta_H \eta_e}{\eta_e^T \eta_e},
$$

$$
n_C - 1 = -2 \frac{\eta_e^T \delta_H \eta_e + \eta_H^\parallel (e_2 \cdot e_m)^T \eta_e}{\eta_e^T \eta_e},
$$

$$
n_T = -2\epsilon_H + \left[ -2\epsilon_H^2 - 4(C_0 + 1) \epsilon_H (\epsilon_H + \eta_H^\parallel) \right].
$$

The contribution due to multicomponent scalar fields come from $\mathcal{U}$ and $\mathcal{V}$. In the case of a single scalar field, both these terms vanish, and we recover the single-field results. Observe that in this case $n_S$ and $n_C$ are irrelevant.

We would like to point out here that one often comes across the adiabatic power spectrum defined in terms of the curvature perturbation $\mathcal{R} = -\dot{\phi} + (H/\dot{H}) (\dot{\phi} + H\dot{\phi})$ and denoted by $\Delta^2_{\mathcal{R}}$ [19, 20, 38]. This definition simply removes the time-dependent factor $\mathcal{J}$ in Eq. (5.18), so that $\Delta^2_A = (9/25) \Delta^2_{\mathcal{R}}$, and $n_A = n_{\mathcal{R}}$.

Using the above results we obtain the consistency relation, or the tensor to scalar ratio [39],

$$
r = \frac{\Delta^2_T(k_0)}{\Delta^2_{\mathcal{R}}(k_0)}
\approx 16\epsilon_H \left( 1 + \mathcal{U}_e^T \mathcal{U}_e \right)^{-1} \approx -8n_T \left( 1 + \mathcal{U}_e^T \mathcal{U}_e \right)^{-1}.
$$

The single-field result $r \approx 16\epsilon_H \approx -8n_T$ follows immediately from above, while in the case of two fields, Eq. (5.26) may be written as [20, 35],

$$
r \approx -8n_T \sin^2 \Delta, \quad \cos \Delta = \Delta^2_C / \sqrt{\Delta^2_A \Delta^2_S}.
$$

For single-field inflation, WMAP limits the tensor to scalar ratio as $r(k_0 = 0.002$ Mpc$^{-1}) < 1.28$ (95% CL) [38]. An independent estimation of $n_T$ and $r$ would provide a decisive test for discriminating multiple-field inflation from single-field slow-roll models [41]. The results obtained in this Section may be applied towards identifying classes of inflation models differing by their observational signatures [41]. They may also be used to obtain information about the slow-roll potential by an inverse analysis of the observational results [35].
6. CONCLUSION

The foregoing Sections contain a general framework for analyzing the dynamics of cosmological perturbations in multiple-field inflation using the gauge-ready approach. The model comprises arbitrary number of scalar fields induced with a general field metric coupled to Einstein gravity. The complete set of equations describing scalar perturbations were presented in the gauge-ready form as well as in terms of gauge-invariant variables. Solutions for density perturbations were derived to the first order in slow roll during inflation, and for adiabatic and isocurvature modes after inflation. Tensor perturbations were also discussed. Expressions of the power-spectra and spectral indices for adiabatic, isocurvature, cross-correlated and tensor modes were obtained within the first order slow roll approximation. These results are of direct relevance when comparing theoretical predictions of various inflation models with observations. It would be of interest to interface our approach with the CMBFAST [42] or CAMB [43] computer codes and compare with the WMAP results.

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