Abstract: Parametric and nonparametric inference for stochastic processes driven by a fractional Brownian motion were investigated in Mishura (2008) and Prakasa Rao (2010) among others. Similar problems for processes driven by an infinite dimensional fractional Brownian motion were studied in Prakasa Rao (2004, 2013), Cialenco et al. (2009) and others. Parametric estimation for processes driven by an infinite dimensional mixed fractional Brownian motion is discussed in this article.

1 Introduction

Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes has been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999a). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion to model processes having long range dependence. Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process driven by a fractional Brownian motion. This is a fractional analogue of the Ornstein-Uhlenbeck process driven by a standard Wiener process. It is a continuous time first order auto-regressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W^H_t, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

\begin{equation}
X_t = \theta \int_0^t X_s ds + \sigma W^H_t, t \geq 0.
\end{equation}

They investigated the problem of estimation of the parameters $\theta$ and $\sigma^2$ based on the observation $\{X_s, 0 \leq s \leq T\}$ and proved that the maximum likelihood estima-
tor $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. More general classes of stochastic processes satisfying linear stochastic differential equations driven by a fractional Brownian motion were studied and the asymptotic properties of the maximum likelihood and the Bayes estimators for parameters involved in such processes is investigated in Prakasa Rao (2003). Prakasa Rao (2010) gives a comprehensive discussion on problems of estimation for processes driven by a fractional Brownian motion.

Geometric Brownian motion driven by a standard Brownian motion has been widely used for modeling fluctuations of share prices in a stock market using Black-Scholes model. However efforts to model fluctuations in financial markets with long range dependence through processes driven by a fractional Brownian motion were not successful as it was noted that such a modeling creates arbitrage opportunities contrary to the fundamental assumption of no arbitrage opportunity for modeling rational market behaviour. Cheridito (2001) proposed modeling through processes driven by a mixed fractional Brownian motion. It was shown by Cheridito (2001) that a mixed fractional Brownian motion is a semimartingale if and only if the Hurst index $H$ is either equal to $\frac{1}{2}$ reducing the process to a Wiener process or $H \in (3/4, 1)$. Furthermore the probability measure generated by such a process is absolutely continuous with respect to the probability measure generated by a Wiener process if $H = 1/2$ or $H \in (3/4, 1)$. This in turn will lead to no arbitrage opportunities for modeling financial market behaviour through processes driven by a mixed fractional Brownian motion. This discussion is to motivate the study of processes driven by a mixed fractional Brownian motion.

The problem of estimation of parameters for processes driven by processes which are mixtures of independent Brownian and fractional Brownian motions started from the works of Cheridito (2001), Rudomino-Dusyatska (2003) and more recently in Prakasa Rao (2015a,b;2017a,b; 2018a,b; 2019, 2020, 2021a,b) among others. Mixed fractional Brownian models were studied in Mishura (2008) and Prakasa Rao (2010). Cai et al. (2016) present a new approach via filtering for analysis of mixed processes of type $\{X_t = B_t + G_t, 0 \leq t \leq T\}$ where $\{B_t, 0 \leq t \leq T\}$ is a Brownian motion and $\{G_t, 0 \leq t \leq T\}$ is an independent Gaussian process. Statistical analysis of mixed fractional Ornstein-Uhlenbeck process was investigated in Chigansky and Kleptsyna (2019). Fractional Ornstein-Uhlenbeck type process driven a mixed fractional Brownian motion has also been termed as “mixed fractional Ornstein-Uhlenbeck process” in Marushkevych (2016). Large deviations for drift parameter estimator of a mixed
fractional Ornstein-Uhlenbeck process were studied by Marushkevych (2016).

Huebner et al. (1995) initiated the study of parametric estimation for a class of stochastic partial differential equations (SPDE) in the presence of white noise or the driving force is an infinite dimensional Wiener process. These results were extended to parabolic stochastic partial differential equations in Huebner and Rozovskii (1995). Prakasa Rao (2000) studied Bayes estimation for stochastic partial differential equations in the white noise case. For other results on parametric inference for SPDEs, see Prakasa Rao (2000, 2001, 2002a,b,2004, 2013). A comprehensive survey of results is given in Prakasa Rao (2001,2002). Parameter estimation for a two-dimensional stochastic Navier-Stokes equation driven by infinite dimensional fractional Brownian motion was studied in Prakasa Rao (2013). Lototsky and Rozovsky (2017) give an extensive survey of theory of SPDE and a discussion on parametric inference for such processes. Cialenco and his coworkers obtained several results dealing with parametric inference for SPDE based on continuous observation or discrete sampling of the processes. Cialenco (2019) gives a survey of their results.

Our aim in this paper is to study parametric inference for processes driven by infinite dimensional mixed fractional Brownian motion. As far as we are aware, this problem has not been investigated earlier.

2 Properties of processes driven by a mfBm

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions. The natural filtration of a stochastic process is understood as the $P$-completion of the filtration generated by this process. Let $\{W_t, t \geq 0\}$ be a standard Wiener process and $W^H = \{W_t^H, t \geq 0\}$ be an independent normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0$, $E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s - t|^{2H}], t \geq 0, s \geq 0. \tag{2.1}$$

Let

$$\tilde{W}_t^H = W_t + W_t^H, t \geq 0.$$

The process $\{\tilde{W}_t^H, t \geq 0\}$ is called the mixed fractional Brownian motion with Hurst index $H$. We assume here after that Hurst index $H$ is known and that $H \in (\frac{3}{4}, 1)$. 

Let us consider a stochastic process \(Y = \{Y_t, t \geq 0\}\) defined by the stochastic integral equation
\[
Y_t = \int_0^t C(s) ds + \tilde{W}_H^t, t \geq 0
\]
where the process \(C = \{C(t), t \geq 0\}\) is an \((\mathcal{F}_t)\)-adapted process. For convenience, we write the above integral equation in the form of a stochastic differential equation
\[
dY_t = C(t) dt + d\tilde{W}_H^t, t \geq 0
\]
driven by the mixed fractional Brownian motion \(\tilde{W}_H^t\). Following the recent works by Cai et al. (2016) and Chigansky and Kleptsyna (2019), one can construct an integral transformation that transforms the mixed fractional Brownian motion \(\tilde{W}_H^t\) into a martingale \(M_H^t\). Let \(g_H(s, t)\) be the solution of the integro-differential equation
\[
g_H(s, t) + H \frac{d}{ds} \int_0^t g_H(r, t)|s-r|^{2H-1} \text{sign}(s-r) dr = 1, 0 < s < t.
\]
Cai et al. (2016) proved that the process
\[
M_H^t = \int_0^t g_H(s, t) d\tilde{W}_H^s, t \geq 0
\]
is a Gaussian martingale with quadratic variation
\[
<M_H^t>_t = \int_0^t g_H(s, t) ds, t \geq 0
\]
Let \(w_H^t\) denote the quadratic variation \(<M_H^t>_t\) over the interval \([0, t]\). It is known that the natural filtration of the martingale \(M_H^t\) coincides with that of the mixed fractional Brownian motion \(\tilde{W}_H^t\). Suppose that, for the martingale \(M_H^t\) defined by the equation (2.5), the sample paths of the process \(\{C(t), t \geq 0\}\) are smooth enough in the sense that the process
\[
Q_H(t) = \frac{d}{d <M_H^t>_t} \int_0^t g_H(s, t) C(s) ds, t \geq 0
\]
is well defined. Define the process
\[
Z_t = \int_0^t g_H(s, t) dY_s, t \geq 0.
\]
As a consequence of the results in Cai et al. (2016), it follows that the process \(Z\) is a fundamental semimartingale associated with the process \(Y\) in the following sense.

**Theorem 2.1:** Let \(g_H(s, t)\) be the solution of the equation (2.4). Define the process \(Z\) as given in the equation (2.8). Then the following relations hold.
(i) The process $Z$ is a semimartingale with the decomposition

\begin{equation}
Z_t = \int_0^t Q_H(t) d < M^H>_s + M^H_t, \ t \geq 0
\end{equation}

where $M^H$ is the martingale defined by the equation (2.5).

(ii) The process $Y$ admits the representation

\begin{equation}
Y_t = \int_0^t \hat{g}_H(s,t) dZ_s, \ t \geq 0
\end{equation}

where

\begin{equation}
\hat{g}_H(s,t) = 1 - \frac{d}{d < M^H>_s} \int_0^t g_H(r,s) dr.
\end{equation}

(iii) The natural filtrations $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ of the processes $Y$ and $Z$ respectively coincide.

Applying Corollary 2.9 in Cai et al. (2016), it follows that the probability measures $\mu_Y$ and $\mu_{\tilde{W}^H}$ generated by the processes $Y$ and $\tilde{W}^H$ on an interval $[0,T]$ are absolutely continuous with respect to each other and the Radon-Nikodym derivative is given by

\begin{equation}
\frac{d\mu_Y}{d\mu_{\tilde{W}^H}}(Y) = \exp\left[\int_0^T Q_H(s)dZ_s - \frac{1}{2} \int_0^T [Q_H(s)]^2 d < M^H>_s\right]
\end{equation}

which is also the likelihood function based on the observation $\{Y_s, 0 \leq s \leq T\}$. Since the filtrations generated by the processes $Y$ and $Z$ are the same, the information contained in the families of $\sigma$-algebras $(\mathcal{Y}_t)$ and $(\mathcal{Z}_t)$ is the same and hence the problem of the estimation of the parameters involved based on the observation $\{Y_s, 0 \leq s \leq T\}$ and $\{Z_s, 0 \leq s \leq T\}$ are equivalent. Since the process $\{Z_s, 0 \leq s \leq T\}$ is driven by a martingale, it is convenient to discuss asymptotic behaviour of the estimators through limit theorems available for martingales. This explanation motivates the study of problem of estimation through the process $Z$ instead of the original process $Y$.

### 3 Parametric estimation for SPDE driven by infinite dimensional mfBm

Kallianpur and Xiong (1995) discussed the properties of solutions of stochastic partial differential equations (SPDE) driven by infinite dimensional fractional Brownian
motion. They indicate that SPDE’s are being used for stochastic modelling, for instance, for the study of neuronal behaviour in neurophysiology and in building stochastic models of turbulence. The theory of SPDE’s is investigated in Ito (1984), Rozovskii (1990) and Da Prato and Zabczyk (1992). Huebner et al. (1993) started the investigation of maximum likelihood estimation of parameters of two types of SPDE’s and extended their results for a class of parabolic SPDE’s in Huebner and Rozovskii (1995). Asymptotic properties of Bayes estimators for such problems were discussed in Prakasa Rao (2000). A short review and a comprehensive survey of these results are given in Prakasa Rao (2001,2002). Our aim in this section is to study the problems of parameter estimation for some SPDE driven by an infinite dimensional mixed fractional Brownian motion.

Stochastic PDE with linear drift (absolutely continuous case)

Let $U$ be a real separable Hilbert space and $Q$ be a self-adjoint positive operator. Further suppose that the operator $Q$ is nuclear. Then $Q$ admits a sequence of eigenvalues $\{q_n, n \geq 1\}$ with $0 < q_n$ decreasing to zero as $n \to \infty$ and $\sum_{n=0}^{\infty} q_n < \infty$. In addition the corresponding eigen vectors $\{e_n, n \geq 1\}$ form an orthonormal basis in $U$. We define the infinite dimensional mixed fractional Brownian motion on $U$ with covariance $Q$ as

$$\tilde{W}^H_Q(t) = \sum_{n=0}^{\infty} \sqrt{q_n} e_n \tilde{W}^H_n(t)$$

where $\tilde{W}^H_n, n \geq 1$ are real independent mfBm’s with Hurst index $H$. Formal definition is given in the next section.

Let $U = L_2[0,1]$ and $W^H_Q$ be the infinite dimensional mfBm on $U$ with the Hurst index $H$ and with the nuclear covariance operator $Q$.

Consider the process $u_\varepsilon(t, x), 0 \leq x \leq 1, 0 \leq t \leq T$ governed by the stochastic partial differential equation

$$du_\varepsilon(t, x) = (\Delta u_\varepsilon(t, x) + \theta u_\varepsilon(t, x))dt + \varepsilon d\tilde{W}^H_Q(t, x)$$

where $\Delta = \frac{\partial^2}{\partial x^2}$. Suppose that $\varepsilon \to 0$ and $\theta \in \Theta \subset \mathbb{R}$. Suppose the initial and the boundary conditions are given by

$$u_\varepsilon(0, x) = f(x), f \in L_2[0, 1]$$

$$u_\varepsilon(t, 0) = u_\varepsilon(t, 1) = 0, 0 \leq t \leq T.$$
Let us consider a special covariance operator $Q$ with $e_k = \sin(k\pi x), k \geq 1$ and $\lambda_k = (\pi k)^2, k \geq 1$. Then \{e_k\} is a complete orthonormal system with the eigenvalues $q_i = (1 + \lambda_i)^{-1}, i \geq 1$ for the operator $Q$ and $Q = (I - \triangle)^{-1}$.

Guerra and Nualart (2008) proved an existence and uniqueness theorem for solutions of multidimensional time dependent stochastic differential equations driven by a multidimensional fractional Brownian motion with Hurst index $H > \frac{1}{2}$ and a multidimensional standard Brownian motion. Similar results were obtained by Mishura and Shevchenko (2011) and da Silva and Erraoui (2018) under weaker conditions. Mishura et al. (2019) has given sufficient conditions for the existence and uniqueness of a mild solution $u_\varepsilon(t, x)$ for stochastic differential equation driven by an infinite dimensional mfBm.

We assume that sufficient conditions hold so that there exists a unique square integrable solution $u_\varepsilon(t, x)$ of (3.2) under the conditions (3.3)-(3.4) and consider it as a formal sum

\begin{equation}
  u_\varepsilon(t, x) = \sum_{i=1}^{\infty} u_{i\varepsilon}(t)e_i(x)
\end{equation}

It can be checked that the Fourier coefficient $u_{i\varepsilon}(t)$ satisfies the stochastic differential equation

\begin{equation}
  du_{i\varepsilon}(t) = (\theta - \lambda_i)u_{i\varepsilon}(t)dt + \frac{\varepsilon}{\sqrt{\lambda_i + 1}}d\tilde{W}^H_i(t), \ 0 \leq t \leq T
\end{equation}

with the initial condition

\begin{equation}
  u_{i\varepsilon}(0) = v_i, \ v_i = \int_0^1 f(x)e_i(x)dx.
\end{equation}

Let $P^{(c)}_\theta$ be the probability measure generated by $u_\varepsilon$ when $\theta$ is the true parameter. Suppose $\theta_0$ is the true parameter. Observe that the process $\{u_{i\varepsilon}(t), 0 \leq t \leq T\}$ is a mixed fractional Ornstein-Uhlenbeck type process (cf. Marushkevych (2016), Chigansky and Kleptsyna (2019), Cai et al. (2016)).

Following the notation given in the previous section, define

\begin{equation}
  M^H_i(t) = \int_0^t g_H(s, t)d\tilde{W}^H_i(s), 0 \leq t \leq T,
\end{equation}

\begin{equation}
  Q_{i\varepsilon}(t) = \frac{\sqrt{\lambda_i + 1}}{\varepsilon} \int_0^t g_H(s, t)u_{i\varepsilon}(s)ds, t \in [0, T],
\end{equation}
\[(3.10) \quad Z_{i\varepsilon}(t) = (\theta - \lambda_i) \int_0^t Q_{i\varepsilon}(s)dw_s^H + M_i^H(t), 0 \leq t \leq T.\]

Observe that \(M_i^H\) is a zero mean Gaussian martingale. Furthermore, it follow that the process \(\{Z_{i\varepsilon}(t)\}\) is a semimartingale and the natural filtrations \((Z_{i\varepsilon})\) and \((U_{i\varepsilon})\) of the processes \(Z_{i\varepsilon}\) and \(u_{i\varepsilon}\) respectively coincide. Let \(P_{i\varepsilon}^{T}\) be the probability measure generated by the process \(\{u_{i\varepsilon}(t), 0 \leq t \leq T\}\) when \(\theta\) is the true parameter. Let \(\theta_0\) be the true parameter. It follows, by the Girsanov type theorem, that

\[(3.11) \quad \log \frac{dP_{i\varepsilon}^{T}}{dP_{i\varepsilon}^0} = \sum_{i=1}^N \lambda_i + 1 \varepsilon^2 \int_0^T Q_{i\varepsilon}(t)dz_{i\varepsilon}(t) - \frac{1}{2} \left\{ (\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2 \right\} \int_0^T Q_{i\varepsilon}^2(t)dW_t^H.\]

Let \(u_{i\varepsilon}^N(t, x)\) be the projection of the solution \(u_{i\varepsilon}(t, x)\) onto the subspace spanned by the eigen vectors \(\{e_i, 1 \leq i \leq N\}\). Then

\[(3.12) \quad u_{i\varepsilon}^N(t, x) = \sum_{i=1}^N u_{i\varepsilon}(t)e_i(x)\]

From the independence of the processes \(\tilde{W}_i^H, 1 \leq i \leq N\) and hence of the processes \(u_{i\varepsilon}, 1 \leq i \leq N\), it follows that the Radon-Nikodym derivative, of the probability measure \(P_{\theta}^{N,T,\varepsilon}\) generated by the process \(u_{i\varepsilon}^N, 0 \leq t \leq T\) when \(\theta\) is the true parameter with respect to the probability measure \(P_{\theta_0}^{N,T,\varepsilon}\) generated by the process \(u_{i\varepsilon}^N, 0 \leq t \leq T\) when \(\theta_0\) is the true parameter, is given by

\[(3.13) \quad \log \frac{dP_{\theta}^{N,T,\varepsilon}}{dP_{\theta_0}^{N,T,\varepsilon}}(u_{i\varepsilon}^N) = \sum_{i=1}^N \lambda_i + 1 \varepsilon^2 \int_0^T Q_{i\varepsilon}(t)dz_{i\varepsilon}(t) - \frac{1}{2} \left\{ (\theta - \lambda_i)^2 - (\theta_0 - \lambda_i)^2 \right\} \int_0^T Q_{i\varepsilon}^2(t)dW_t^H.\]

Furthermore the Fisher information is given by

\[(3.14) \quad I_{N\varepsilon}(\theta) = E_\theta \left[ \frac{\partial \log dP_{\theta}^{N,T,\varepsilon}}{\partial \theta} \right]^2 = \sum_{i=1}^N \lambda_i + 1 \varepsilon^2 \int_0^T Q_{i\varepsilon}^2(t)dW_t^H.\]

It is easy to check that the maximum likelihood estimator \(\hat{\theta}_{N,\varepsilon}\) of the parameter \(\theta\) based on the projection \(u_{i\varepsilon}^N\) of \(u_{i\varepsilon}\) is given by

\[(3.15) \quad \hat{\theta}_{N,\varepsilon} = \frac{\sum_{i=1}^N (\lambda_i + 1) \int_0^T Q_{i\varepsilon}(t)dz_{i\varepsilon}(t)}{\sum_{i=1}^N (\lambda_i + 1) \int_0^T Q_{i\varepsilon}^2(t)dW_t^H}.\]
Suppose $\theta_0$ is the true parameter. It is easy to see that
\[
\varepsilon^{-1}(\hat{\theta}_{N,\varepsilon} - \theta_0) = \frac{\sum_{i=1}^{N} \sqrt{\lambda_i + 1} \int_0^T Q_{\varepsilon}(t) dM^H(t)}{\sum_{i=1}^{N} (\lambda_i + 1) \int_0^T Q_{\varepsilon}^2(t) dw^H_t}.
\]

Observe that $M_i, 1 \leq i \leq N$ are independent zero mean Gaussian martingales with $\langle M_i \rangle = w^H, 1 \leq i \leq N$.

**Theorem 3.1:** The maximum likelihood estimator $\hat{\theta}_{N,\varepsilon}$ is strongly consistent, that is,
\[
\hat{\theta}_{N,\varepsilon} \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } \varepsilon \to 0
\]
provided
\[
\sum_{i=1}^{N} \int_0^T (\lambda_i + 1) Q_{\varepsilon}^2(t) dw^H_t \to \infty \text{ a.s } [P_{\theta_0}] \text{ as } \varepsilon \to 0.
\]

**Proof:** This theorem follows by observing that the process
\[
R_{\varepsilon}^N = \sum_{i=1}^{N} \int_0^T \varepsilon \sqrt{\lambda_i + 1} Q_{\varepsilon}(t) dM^H(t), T \geq 0
\]
is a local martingale with the quadratic variation process
\[
\langle R_{\varepsilon}^N \rangle_T = \sum_{i=1}^{N} \int_0^T \varepsilon^2 (\lambda_i + 1) Q_{\varepsilon}^2(t) dw^H_t
\]
and applying the Strong law of large numbers (cf. Liptser (1980); Prakasa Rao (1999b), p. 61) under the condition (3.18) stated above.

**Limiting distribution:**

We now discuss the limiting distribution of the MLE $\hat{\theta}_{N,\varepsilon}$ as $\varepsilon \to 0$.

**Theorem 3.2:** Assume that the process $\{R_{\varepsilon}^N, \varepsilon \geq 0\}$ is a local continuous martingale and that there exists a norming function $I_{\varepsilon}^N, \varepsilon \geq 0$ such that
\[
(I_{\varepsilon}^N)^2 < R_{\varepsilon}^N \to_T (I_{\varepsilon}^N)^2 \sum_{i=1}^{N} \int_0^T \varepsilon^2 (\lambda_i + 1) Q_{\varepsilon}^2(t) dw^H_t \to \eta^2 \text{ in probability as } \varepsilon \to 0
\]
where $\eta$ is a random variable such that $P(\eta > 0) = 1$. Then
\[
(I_{\varepsilon}^N, R_{\varepsilon}^N, (I_{\varepsilon}^N)^2 < R_{\varepsilon}^N \to_T (\eta Z, \eta^2) \text{ in law as } \varepsilon \to 0
\]
where the random variable $Z$ has the standard Gaussian distribution and the random variables $Z$ and $\eta$ are independent.

**Proof :** This theorem follows as a consequence of the central limit theorem for local martingales (cf. Theorem 1.49; Remark 1.47, Prakasa Rao(1999b), p. 65).

Observe that
\[(I_\varepsilon^N)^{-1}(\hat{\theta}_N - \theta_0) = \frac{I_\varepsilon^N R_\varepsilon^N}{(I_\varepsilon^N)^2 < R_\varepsilon^N>}.\]

Applying the above theorem, we obtain the following result.

**Theorem 3.3 :** Suppose the conditions stated in Theorem 3.2 hold. Then
\[(I_\varepsilon^N)^{-1}(\hat{\theta}_N - \theta_0) \to \frac{Z}{\eta} \text{ in law as } \varepsilon \to 0\]  
where the random variable $Z$ has the standard Gaussian distribution and the random variables $Z$ and $\eta$ are independent.

**Remarks :** (i) If the random variable $\eta$ is a constant with probability one, then the limiting distribution of the maximum likelihood estimator is Gaussian with mean 0 and variance $\eta^{-2}$. Otherwise it is a mixture of the Gaussian distributions with mean zero and variance $\eta^{-2}$ with the mixing distribution as that of $\eta$.

(ii) Suppose that
\[\lim_{N \to \infty} \lim_{\varepsilon \to 0} \varepsilon^2 I_\varepsilon^N = I(\theta)\]
exists and is positive. Since the sequence of Radon-Nikodym derivatives
\[\left\{\frac{dP_{\theta}^{N,T,\varepsilon}}{dP_{\theta_0}}, n \geq 1\right\}\]
form a non-negative martingale with respect to the filtration generated by the sequence of random variables \(\{u_\varepsilon^N, N \geq 1\}\), it converges almost surely to a random variable $\nu_{\varepsilon,\theta,\theta_0}$ as $N \to \infty$ for every $\varepsilon > 0$. It is easy to see that the limiting random variable is given by
\[\nu_{\varepsilon,\theta,\theta_0}(u_\varepsilon) = \exp\left\{\sum_{i=1}^{\infty} \frac{\lambda_i}{\varepsilon^2} \left(\theta - \theta_0\right) \int_0^T Q_{i\varepsilon}(t) dZ_{i\varepsilon}(t)\right\}\]
Furthermore the sequence of random variables \( u_N^\varepsilon(t) \) converge in probability to the random variable \( u_\varepsilon(t) \) as \( N \to \infty \) for every \( \varepsilon > 0 \). Hence, by Lemma 4 in Skorokhod (1965, p. 100), it follows that the measures \( P_\theta^\varepsilon \) generated by the processes \( u_\varepsilon \) for different values of \( \theta \), are absolutely continuous with respect to each other and the Radon-Nikodym derivative of the probability measure \( P_\theta^\varepsilon \) with respect to the probability measure \( P_{\theta_0}^\varepsilon \) is given by

\[
\frac{dP_\theta^\varepsilon}{dP_{\theta_0}^\varepsilon}(u_\varepsilon) = \nu_{\varepsilon, \theta, \theta_0}(u_\varepsilon)
\]

It can be checked that the MLE \( \hat{\theta}_\varepsilon \) of \( \theta \) based on \( u_\varepsilon \) satisfies the likelihood equation

\[
\alpha_\varepsilon = \varepsilon^{-1}(\hat{\theta}_\varepsilon - \theta_0)\beta_\varepsilon
\]

when \( \theta_0 \) is the true parameter where

\[
\alpha_\varepsilon = \sum_{i=1}^{\infty} \sqrt{\lambda_i + 1} \int_0^T Q_{i\varepsilon}(t) dM_i^H(t)
\]

and

\[
\beta_\varepsilon = \sum_{i=1}^{\infty} \frac{1}{(\lambda_i + 1)} \int_0^T Q_{i\varepsilon}^2(t) dw_i^H.
\]

One can obtain sufficient conditions for studying the asymptotic behaviour of the estimator \( \hat{\theta}_\varepsilon \) as in the finite projection case discussed above. We omit the details.

**Stochastic PDE with linear drift (singular case):**

Let \( (\Omega, F, P) \) be a probability space and consider the process \( u_\varepsilon(t,x), 0 \leq x \leq 1, 0 \leq t \leq T \) governed by the stochastic partial differential equation

\[
du_\varepsilon(t,x) = \theta \triangle u_\varepsilon(t,x) dt + \varepsilon(I - \triangle)^{-1/2}d\tilde{W}(t,x)
\]
where $\theta > 0$ satisfying the initial and the boundary conditions

\begin{equation}
(3.32) \quad u_\varepsilon(0, x) = f(x), \ 0 < x < 1, \ f \in L_2[0, 1],
\end{equation}

\begin{equation}
(3.32) \quad u_\varepsilon(t, 0) = u_\varepsilon(t, 1) = 0, \ 0 \leq t \leq T.
\end{equation}

Here $I$ is the identity operator, $\Delta = \frac{\partial^2}{\partial x^2}$ as defined above and the process $\tilde{W}(t, x)$ is the cylindrical infinite dimensional mfBm with $H \in \left[\frac{1}{2}, 1\right)$.

Following the discussion in the previous section, we assume the existence of a square integrable solution $u_\varepsilon(t, x)$ for the equation (3.31) subject to the boundary conditions (3.32). Then the Fourier coefficients $u_{i\varepsilon}(t)$ satisfy the stochastic differential equations

\begin{equation}
(3.33) \quad du_{i\varepsilon}(t) = -\theta \lambda_i u_{i\varepsilon}(t) dt + \frac{\varepsilon}{\sqrt{\lambda_i + 1}} d\tilde{W}_i(t), \ 0 \leq t \leq T,
\end{equation}

with

\begin{equation}
(3.34) \quad u_{i\varepsilon}(0) = v_i, \ v_i = \int_0^1 f(x) e_i(x) dx.
\end{equation}

Let $u^{(N)}_\varepsilon(t, x)$ be the projection of $u_\varepsilon(t, x)$ onto the subspace spanned by $\{e_1, \cdots, e_N\}$ in $L_2[0, 1]$. In other words

\begin{equation}
(3.35) \quad u^{(N)}_\varepsilon(t, x) = \sum_{i=1}^{N} u_{i\varepsilon}(t) e_i(x).
\end{equation}

Let $P^{(\varepsilon, N)}_\theta$ be the probability measure generated by $u^{(N)}_\varepsilon$ on the subspace spanned by $\{e_1, \cdots, e_N\}$ in $L_2[0, 1]$. It can be shown that the measures $\{P^{(\varepsilon, N)}_\theta, \theta \in \Theta\}$ form an equivalent family and

\begin{equation}
(3.36) \quad \log \frac{dP^{(\varepsilon, N)}_\theta}{dP^{(\varepsilon, N)}_{\theta_0}}(u^{(N)}_\varepsilon) = -\frac{1}{\varepsilon^2} \sum_{i=1}^{N} \lambda_i (\lambda_i + 1) \int_0^T Q_{i\varepsilon}(t) dZ_{i\varepsilon}(t) - \frac{1}{2} (\theta - \theta_0)^2 \lambda_i \int_0^T Q_{i\varepsilon}^2(t) dM_t^H.
\end{equation}

It can be checked that the MLE $\hat{\theta}_{\varepsilon, N}$ of $\theta$ based on $u^{(N)}_\varepsilon$ satisfies the likelihood equation

\begin{equation}
(3.37) \quad \alpha_{\varepsilon, N} = -\varepsilon^{-1}(\hat{\theta}_{\varepsilon, N} - \theta_0) \beta_{\varepsilon, N}
\end{equation}

when $\theta_0$ is the true parameter where

\begin{equation}
(3.38) \quad \alpha_{\varepsilon, N} = \sum_{i=1}^{N} \lambda_i \sqrt{\lambda_i + 1} \int_0^T Q_{i\varepsilon}(t) dM_t^H(t)
\end{equation}
and
\[ \beta_{\varepsilon,N} = \sum_{i=1}^{N} (\lambda_i + 1)\lambda_i^2 \int_0^T Q_{1,\varepsilon}^2(t) dw_t^H. \]  

Asymptotic properties of these estimators can be investigated as in the previous example. We do not go into the details as the arguments are similar.

Remarks: One can study the local asymptotic mixed normality (LAMN) of the family of probability measures generated by the log-likelihood ratio processes by the standard arguments as in Prakasa Rao (1999b) and hence investigate the asymptotic efficiency of the MLE using Hajek-Lecam type bounds.

4 Parametric estimation for stochastic parabolic equations driven by infinite dimensional mfBm

We now extend some work of Cialenco et al. (2009) dealing with problems of estimation in models more general than those discussed in the previous section. We introduce some notation.

Let \( H \) be a separable Hilbert space with the inner product \((.,.)_0\) and with the corresponding norm \( ||.||_0 \). Let \( \Lambda \) be a densely defined linear operator on \( H \) with the property that there exists \( c > 0 \) such that

\[ ||\Lambda u||_0 \geq c||u||_0 \]

for every \( u \) in the domain of the operator \( \Lambda \). The operator powers \( \Lambda^\gamma, \gamma \in R \) are well defined and generate the spaces \( H^\gamma \) with the properties (i) for \( \gamma > 0, H^\gamma \) is the domain of \( \Lambda^\gamma \), (ii) \( H^0 = H \), and (iii) for \( \gamma < 0, H^\gamma \) is the completion of \( H \) with respect to the norm \( ||.||_\gamma \equiv ||\Lambda^\gamma ||_0 \) (cf. Krein et al. (1982)). The family of spaces \( \{H^\gamma, \gamma \in R\} \) has the following properties:

(i) \( \Lambda^\gamma (H^r) = H^{r-\gamma}, \gamma, r \in R; \)

(ii) For \( \gamma_1 < \gamma_2 \), the space \( H^{\gamma_2} \) is densely and continuously embedded into \( H^{\gamma_1} \), that is, \( H^{\gamma_2} \subset H^{\gamma_1} \) and there exists a constant \( c_{12} > 0 \) such that \( ||u||_{\gamma_1} \leq c_{12}||u||_{\gamma_2}; \)
(iii) for every $\gamma \in \mathbb{R}$ and $m > 0$, the space $H^{\gamma - m}$ is the dual of the space $H^{\gamma + m}$ with respect to the inner product in $H^{\gamma}$, with duality $<\cdot, \cdot>_{\gamma,m}$ given by

$$<u_1, u_2>_{\gamma,m} = (\Lambda^{\gamma - m} u_1, \Lambda^{\gamma + m} u_2)_0, \quad u_1 \in H^{\gamma - m}, u_2 \in H^{\gamma + m}.$$ 

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\tilde{W}_j^H, j \geq 1\}$ be a family of independent mixed fractional Brownian motions on this space with the same Hurst index $H$ in $(0,1)$.

Consider the SDE

$$du(t) + (A_0 + \theta A_1)u(t)dt = \sum_{j \geq 1} g_j(t) d\tilde{W}_j^H(t), 0 \leq t \leq T, u(0) = u_0$$

(4.1)

where $A_0, A_1$ are linear operators, $g_j, j \geq 1$ are non-random and $\theta \in \Theta \subset \mathbb{R}$.

The equation (4.1) is said to be diagonalizable if the operators $A_0, A_1$ have the same system of eigenfunctions $\{h_j, j \geq 1\}$ such that $\{h_j, j \geq 1\}$ is an orthonormal basis in $H$ and each $h_j$ belongs to $\cap_{\gamma \in \mathbb{R}} H^\gamma$. It is called $(m, \gamma)$-parabolic for some $m \geq 0, \gamma \in \mathbb{R}$, if

(i) the operator $A_0 + \theta A_1$ is uniformly bounded from $H^{\gamma + m}$ to $H^{\gamma - m}$ for every $\theta \in \Theta$, that is, there exists $C_1 > 0$ such that

$$||(A_0 + \theta A_1)v||_{\gamma - m} \leq C_1 ||v||_{\gamma + m}, \quad \theta \in \Theta, v \in H^{\gamma + m};$$

(4.2)

and

(ii) there exists a $\delta > 0$ and $C \in \mathbb{R}$ such that,

$$-2 < (A_0 + \theta A_1)v, v >_{\gamma,m} + \delta ||v||_{\gamma + m}^2 \leq C ||v||_{\gamma}^2, \quad v \in H^{\gamma + m}, \theta \in \Theta.$$ 

(4.3)

If the equation (4.1) is $(m, \gamma)$-parabolic, then the condition (ii) implies that

$$<(2A_0 + 2\theta A_1 + CI)v, v>_\gamma, m \geq \delta ||v||_{\gamma + m}^2$$

where $I$ is the identity operator. The Cauchy-Schwartz inequality and the continuous embedding of $H^{\gamma + m}$ into $H^\gamma$ will imply that

$$|||2A_0 + 2\theta A_1 + CI)v||_\gamma \geq \delta ||v||_\gamma$$
for some $\delta_1 > 0$ uniformly in $\theta \in \Theta$.

Let us choose $\Lambda = [2A_0 + 2\theta_0 A_1 + CI]^{1/(2m)}$ for some fixed $\theta_0 \in \Theta$. If the operator $A_0 + \theta A_1$ is unbounded, we say that $A_0 + \theta A_1$ has order $2m$ and $\Lambda$ has order 1. If the equation (4.1) is $(m, \gamma)$-parabolic and diagonalizable, we will assume that the operator $\Lambda$ has the same eigenfunctions as the operators $A_0$ and $A_1$. This is justified by the comments made above.

Suppose the equation (4.1) is diagonalizable and there exists eigenvalues $\{\rho_j, j \geq 1\}, \{\nu_j, j \geq 1\}$ such that

$$A_0 h_j = \rho_j h_j \text{ and } A_1 h_j = \nu_j h_j.$$ 

Without loss of generality, we can also assume that there exists $\{\lambda_j, j \geq 1\}$ such that

$$\Lambda h_j = \lambda_j h_j.$$ 

Following the arguments in Cialenco et al. (2009), it can be shown that the equation (4.1) is $(m, \gamma)$-parabolic if and only if there exists $\delta > 0, C_1 > 0$ and $C_2 \in \mathbb{R}$ such that, for all $j \geq 1, \theta, \in \Theta$,

$$|\rho_j + \theta \nu_j| \leq C_1 \lambda_j^{2m} \tag{4.4}$$

and

$$-2(\rho_j + \theta \nu_j) + \delta \lambda_j^{2m} \leq C_2. \tag{4.5}$$

As the conditions in (4.4) and (4.5) do not depend on $\gamma$, we conclude that a diagonalizable equation (4.1) is $(m, \gamma)$-parabolic for some $\gamma$ if and only if it is $(m, \gamma)$-parabolic for every $\gamma$. Here after we will say that the equation (4.1) is $m$-parabolic. We will assume that the equation (4.1) is diagonalizable and fix the basis $\{h_j, j \geq 1\}$ in $\mathbf{H}$ consisting of the eigenfunctions of $A_0$, $A_1$ and $\Lambda$. Recall that set of eigenfunctions is the same for all the three operators. Since $h_j$ belongs to every $\mathbf{H}^\gamma$, and since $\bigcap_{\gamma} \mathbf{H}^\gamma$ is dense in $\bigcup_{\gamma} \mathbf{H}^\gamma$, every element $f$ of $\bigcup_{\gamma} \mathbf{H}^\gamma$, has a unique expansion $\sum_{j \geq 1} f_j h_j$ where $f_j = < f, h_j >_{0,m}$ for suitable $m$.

The functional structure described above follows the work in Cialenko et al. (2009).

**Definition :** The infinite dimensional mixed fractional Brownian motion $\tilde{W}^H$ is an element of $\bigcup_{\gamma \in \mathbb{R}} \mathbf{H}^\gamma$ with the expansion

$$\tilde{W}^H(t) = \sum_{j \geq 1} \tilde{h}_j \tilde{W}^H_j(t). \tag{4.6}$$
Definition : The solution of the diagonalizable equation

\[(4.7) \quad du(t) + (A_0 + \theta A_1)u(t)dt = d\bar{W}^H(t), 0 \leq t \leq T, u(0) = u_0,\]

with \(u_0 \in \mathbb{H}\), is defined to be a random process \(u(t), 0 < t \leq T\), with values in \(\cup_{\gamma} \mathbb{H}^\gamma\) and has an expansion

\[(4.8) \quad u(t) = \sum_{j \geq 1} h_j u_j(t)\]

where

\[(4.9) \quad u_j(t) = (u_0, h_j) e^{-(\theta \nu_j + \rho_j)t} + \int_0^t e^{-(\theta \nu_j + \rho_j)(t-s)} d\bar{W}_j^H(s).\]

Let

\[(4.10) \quad \mu_j(\theta) = \theta \nu_j + \rho_j, j \geq 1.\]

In view of (4.5), we get that there exists a positive integer \(J\) such that

\[(4.11) \quad \mu_j(\theta) > 0 \quad \text{for} \quad j \geq J\]

if the equation (4.1) is \(m\)-parabolic and diagonalizable.

**Theorem 4.1 :** Suppose that \(H \geq \frac{1}{2}\) and the equation (4.1) is \(m\)-parabolic and diagonalizable. Further suppose that there exists a positive real number \(\gamma\) such that

\[(4.12) \quad \sum_{j \geq 1} (1 + |\mu_j(\theta)|)^{-\gamma} < \infty.\]

Then, for every \(t > 0\), \(\bar{W}^H(t) \in L_2(\Omega, \mathbb{H}^{-m\gamma})\) and \(u(t) \in L_2(\Omega, \mathbb{H}^{-m\gamma + m \min(2H,1)})\).

**Proof :** The condition (3) implies that \(\lim_{j \to \infty} |\mu_j| = \infty\), and hence the operators \(A_0 + \theta A_1\) and \(\Lambda\) are unbounded. The parabolicity assumption and the equations (4.4) and (4.5) imply that, for sufficiently large \(j\),

\[1 + |\mu_j(\theta)|^m \leq C_2 \lambda_j^{2m}\]

uniformly in \(\theta \in \Theta\). Furthermore

\[E||\bar{W}^H(t)||_{-m\gamma}^2 = (t^{2H} + t) \sum_{j \geq 1} \lambda_j^{-2m\gamma} \leq C_2 (t^{2H} + t) \sum_{j \geq 1} (1 + \mu_j(\theta))^{-\gamma} < \infty.\]

From the definition of the mixed fractional Brownian motion and the fact that the component fractional Brownian motion and the Brownian motion are independent
and centered, it follows that
\[
E[u_j^2(t)] = 4H(2H - 1)e^{-2\mu_j(\theta)t} \int_0^t \int_0^t e^{\mu_j(\theta)(s_1 + s_2)} |s_1 - s_2|^{2H-2} ds_1 ds_2 + e^{-2\mu_j(\theta)t} \int_0^t e^{2\mu_j(s)} ds
\]
\[= J_1(t) + J_2(t) \text{ (say)}.\]

by the properties of fractional Brownian motion (cf. Prakasa Rao (2010)) and Brownian motion. It can be checked that
\[
\lim_{j \to \infty} |\mu_j(\theta)|^{2H} J_1(t) = H(2H - 1) \int_0^\infty x^{2H-2} e^{-x} dx = H(2H - 1) \Gamma(2H - 1)
\]
and
\[
\lim_{j \to \infty} |\mu_j(\theta)| J_2(t) = \frac{1}{2},
\]
Hence
\[
\lim_{j \to \infty} |\mu_j(\theta)|^{\min(2H,1)} E(u_j^2(t)) < \infty.
\]
and
\[
\sum_{j=1}^\infty (1 + |\mu_j(\theta)|)^{-\gamma + \min(2H,1)} E(u_j^2(t)) < \infty.
\]

Maximum likelihood estimation : 

Consider the diagonalizable equation
\[
(4.14) \quad du(t) + (A_0 + \theta A_1)u(t)dt = d\tilde{W}^H(t), u(0) = 0, 0 \leq t \leq T
\]
with
\[
(4.15) \quad u(t) = \sum_{j \geq 1} h_j u_j(t)
\]
as given by (4.9). Suppose the processes \{u_i(t), 0 \leq t \leq T\}, i = 1, \ldots, N can be observed continuously over the interval [0, T]. The problem is to estimate the parameter \(\theta\) using these observed paths over the interval [0, T].

Note that \(\mu_j(\theta) = \rho_j + \nu_j \theta\), where \(\rho_j\) and \(\nu_j\) are the eigenvalues of \(A_0\) and \(A_1\) respectively. Furthermore each process \(u_j\) is a mixed fractional Ornstein-Uhlenbeck process satisfying the stochastic differential equation
\[
(4.16) \quad du_j(t) = -\mu_j(\theta) u_j(t) dt + d\tilde{W}_j^H(t), u_j(0) = 0, 0 \leq t \leq T.
\]
Since the processes $\{\tilde{W}_j^H, j \geq 1\}$ are independent, it follows that the processes $\{u_j, 1 \leq j \leq N\}$ are independent. Following the notation introduced above, let

$$
M_j^H(t) = \int_0^t g_H(s, t) d\tilde{W}_j^H(s), \quad Q_j(t) = \frac{d}{dw_j^H(t)} \int_0^t g_H(s, t) u_j(s) ds
$$

and

$$
Z_j(t) = \int_0^t g_H(s, t) du_j(s)
$$

for $j = 1, \ldots, N$. Applying the Girsanov-type formula, it can be shown that the measure generated by the process $(u_1, \ldots, u_N)$ is absolutely continuous with respect to the measure generated by the process $(\tilde{W}_1^H, \ldots, \tilde{W}_N^H)$ and their Radon-Nikodym derivative is given by

$$
\exp\left(-\sum_{j=1}^N \mu_j(\theta) \int_0^T Q_j(s) dZ_j(s) - \sum_{j=1}^N \frac{[\mu_j(\theta)]^2}{2} \int_0^T Q_j^2(s) dw_H(s)\right).
$$

Maximizing this function with respect to the parameter $\theta$, we get the maximum likelihood estimator

$$
\hat{\theta}_N = -\frac{\sum_{j=1}^N \int_0^T \nu_j Q_j(s)(dZ_j(s) + \rho_j Q_j(s) dw_H(s))}{\sum_{j=1}^N \int_0^T \nu_j^2 Q_j^2(s) dw_H(s)}.
$$

**Theorem 4.2:** Suppose the Hurst index $H > \frac{1}{2}$. Suppose that

$$
\sum_{j=1}^\infty \frac{\nu_j^2}{\mu_j(\theta)} = \infty.
$$

Then

$$
\lim_{N \to \infty} \hat{\theta}_N = \theta \text{ a.s.}
$$

where $\theta$ is the true parameter. Furthermore, if the condition (i) holds, then

$$
\lim_{N \to \infty} \left(\sum_{j=1}^N \frac{\nu_j^2}{\mu_j(\theta)}\right)^{1/2}(\hat{\theta}_N - \theta) \xrightarrow{d} N(0, 1) \text{ as } N \to \infty
$$

where $J = \min\{j : \mu_i(\theta) = 0 \text{ for all } i \geq j\}$.

**Proof:** Observe that

$$
\hat{\theta}_N - \theta = \frac{\sum_{j=1}^N \int_0^T \nu_j Q_j(s) dM_j^H(s)}{\sum_{j=1}^N \int_0^T \nu_j^2 Q_j^2(s) dw_H(s)}.
$$
The numerator of the expression on the right side (4.22) is a local martingale and the denominator is its quadratic variation. It is known that

\[ \lim_{j \to \infty} \mu_j(\theta) E[\int_0^T Q_j^2(s)dw_H(s)] = \frac{T}{2} > 0 \]  

from the equation (2.1) in Chigansky and Kleptsyna (2019) (cf. Maruskeevych (2016)) which implies that

\[ \sum_{j=1}^N \int_0^T \nu_j^2 Q_j^2(s)dw_H(s) \to \infty \quad \text{a.s. as } N \to \infty \]

under the condition (4.20). Hence, an application of the Strong law of large numbers for local martingales (cf. Prakasa Rao (1999), Liptser and Shiryaev (1989)) will imply that the left side of the equation (4.22) converges to zero almost surely and hence

\[ \hat{\theta}_N - \theta \to 0 \quad \text{almost surely} \]

as \( N \to \infty \).

**Theorem 4.3:** Suppose there exists a norming function \( I_N \) such that

\[ I_N^2 \sum_{j=1}^N \int_0^T \nu_j^2 Q_j^2(s)dw_H(s) \to \eta^2 \quad \text{in probability as } N \to \infty \]

where \( \eta \) is a random variable such that \( P(\eta > 0) = 1 \). Let

\[ R_N = \sum_{j=1}^N \int_0^T \nu_j Q_j(s)dM_j^H(s). \]

Then

\[ (I_N R_n, I_N^2 < R_N >) \to (\eta Z, \eta^2) \quad \text{in law as } N \to \infty \]

where the random variable \( Z \) has the standard normal distribution and the random variables \( Z \) and \( \eta \) are independent. Here \( < R_N >, N \geq 1 \) is the quadratic variation of the process \( \{R_N, N \geq 1\} \).

**Proof:** This theorem follows as a consequence of the central limit theorem for local martingales (cf. Theorem 1.49 and remark 1.47, Prakasa Rao (1999b)).

Observe that

\[ I_N^{-1}(\hat{\theta}_N - \theta) = \frac{I_N R_N}{I_N^2 < R_N >}, N \geq 1. \]
Applying Theorem 4.3, we obtain the following result.

**Theorem 4.4:** Suppose the conditions stated in Theorem 4.2 hold. Then

\[
I_N^{-1}(\hat{\theta}_N - \theta) \rightarrow \frac{Z}{\eta} \text{ in law as } N \rightarrow \infty
\]

(4.28)

where the random variable $Z$ has the standard normal distribution and the random variable $Z$ and $\eta$ are independent.

**Remarks:** If the random variable $\eta$ is a constant almost surely, then the limiting distribution of the maximum likelihood estimator $\hat{\theta}_N$ is Gaussian with mean zero and variance $\eta^{-2}$. Otherwise it is a mixture of the normal distribution with mean zero and variance $\eta^{-2}$ with the mixing distribution as that the random variable $\eta$. Applying Berry-Esseen bound for sums of independent random variables, it is possible to obtain the limiting distribution and rate of convergence of the distribution of the MLE if a bound on the variance of the random variable

\[
\int_0^T Q_j^2(s) \, dw^H(s)
\]

can be obtained as $j \rightarrow \infty$ after suitable norming as in Lemma A.2 in Cialenco et al. (2009) in the fractional Brownian case. It has not been possible to obtain a similar result in the case mixed fractional Brownian motion as the function $g_H(s, t)$ defining the martingale $M_j^H$ defined by (3.8) is not explicitly known. The problem of obtaining a closed form for the function $g_H(s, t)$ remains open.

**Acknowledgment** This work was supported under the scheme “INSA Senior Scientist” by the Indian National Science Academy (INSA) at the CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad, India.

**References:**

Cai, C., Chigansky., and Kleptsyna, M. (2016) Mixed Gaussian processes ; A filtering approach, *Ann. Probab.*, 44, 3032-3075.

Cheridito, C. (2001) Mixed fractional Brownian motion, *Bernoulli*, 7, 913-934.

Chigansky, P., and Kleptsyna, M. (2019) Statistical analysis of the mixed fractional Ornstein-Uhlenbeck process. *Theory Probab. Appl.*, 63, 408-425.
Cialenco, I. (2018) Statistical inference for SPDEs: an overview, *Statist. Infer. Stoch. Proc.*, https://doi.org/10.1007/s11203-018-9177-9.

Cialenco, I., Lototsky, S.G., and Pospisil, J. (2009) Asymptotic properties of the maximum likelihood estimator for stochastic parabolic equations with additive fractional Brownian motion, *Stoch. and Dynam.*, 9, 169-185.

da Silva, Jose Luis., Erroui, M and essaky, El Hassan (2018) Mixed stochastic differential equations: Existence and uniqueness result, *J. Theor. Probab.*, 31, 1119-1141.

Da Prato, G. and Zabczyk, J. (1992) *Stochastic Equations in Infinite Dimensions*, Cambridge University Press.

Guerra, Joao., and Nualart, D. (2008) Stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, *Stoch. Anal Appl.*, 26, 1053-1075.

Huebner, M., Khasminski, R. and Rozovskii. B.L. (1993) Two examples of parameter estimation for stochastic partial differential equations, In *Stochastic Processes : A Festschrift in Honour of Gopinath Kallianpur*, Springer, New York, pp. 149-160.

Huebner, M., and Rozovskii, B.L. (1995) On asymptotic properties of maximum likelihood estimators for parabolic stochastic SPDE’s, *Prob. Theory and Relat. Fields*, 103, 143-163.

Ito, K. (1984) Foundations of Stochastic Differential Equations in Infinite Dimensional Spaces, Vol. 47, CBMS Notes, SIAM, Baton Rouge.

Kallianpur, G., and Xiong, J. (1995) *Stochastic Differential Equations in Infinite Dimensions*, IMS Lecture Notes, Vol.26, Hayward, California.

Kleptsyna, M. L. and Le Breton, A. (2002) Statistical analysis of the fractional Ornstein-Uhlenbeck type process, *Statist. Infer. for Stoch. Proc.*, 5, 229–248.

Krein, S.G., Petunin, Yu.I., and Semenov, E.M. (1982) *Interpolation of linear operators*, Volume 54 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, Rhode Island.
Le Breton, A. (1998) Filtering and parameter estimation in a simple linear model driven by a fractional Brownian motion, *Statist. Probab. Lett.*, **38**, 263-274.

Liptser, R. (1980) A strong law of large numbers, *Stochastics*, **3**, 217-228.

Lototsky, S.V. and Rozovsky, B.L. (2017) *Stochastic Partial Differential Equations*, Springer, Switzerland.

Marushkevych, Dmytro (2016) Large deviations for drift parameter estimator of mixed fractional Ornstein-uhlenbeck process, *Modern Stochastics: Theory and applications*, **3**, 107-117.

Mishura, Y., Ralchenko, K., and Shevchenko, G. (2019) Existence and uniqueness of mild solutions to the stochastic heat equation with white and fractional noises, *Theory of Prob. and Math. stat.*, **98** 149-170.

Mishura, Y., and Shevchenko, G. (2011) Existence and uniqueness of the solution of stochastic differential equation involving wiener process and fractional Brownian motion with hurst index $H > \frac{1}{2}$. *Commun. Stat. Theory Methods*, **40**, 3492-3508.

Prakasa Rao, B.L.S. (1987) *Asymptotic Theory of Statistical Inference*, Wiley, New York.

Prakasa Rao, B.L.S. (1999a) *Statistical Inference for Diffusion Type Processes*, Arnold, London and Oxford University Press, New York.

Prakasa Rao, B.L.S. (1999b) *Semimartingales and Their Statistical Inference*, CRC Press, Boca Raton and Chapman and Hall, London.

Prakasa Rao, B.L.S. (2000) Bayes estimation for stochastic partial differential equations, *J. Statist. Plan. Inf.*, **91**, 511-524.

Prakasa Rao, B.L.S. (2001) Statistical inference for stochastic partial differential equations, In *Selected Proceedings of the Symposium on Inference for Stochastic Processes*, Ed. I.V.Basawa, C.C.Heyde and R.L.Taylor, IMS Monograph Series, Vol. **37**, pp. 47-70.
Prakasa Rao, B.L.S. (2002a) On some problems of estimation for some stochastic partial differential equations, In *Uncertainty and Optimality*, Ed. J.C.Mishra, World Scientific, Singapore, pp. 71-153.

Prakasa Rao, B.L.S. (2002b) Minimum distance estimation for some stochastic partial differential equations, *J. Korean Stat. Soc.* **31** 213-228.

Prakasa Rao, B.L.S. (2003) Parameter estimation for linear stochastic differential equations driven by fractional Brownian motion, *Random Operators and Stochastic Equations*, **11**, 229-242.

Prakasa Rao, B.L.S. (2004) Parameter estimation for some stochastic partial differential equations driven by infinite dimensional fractional Brownian motion, *Theory Stochastic. Process*, **10 (26)** 116-125.

Prakasa Rao, B.L.S. (2009) Estimation for stochastic differential equations driven by mixed fractional Brownian motion. *Calcutta Stat. Assoc. Bull.* 61: 143-153.

Prakasa Rao, B.L.S. (2010) *Statistical Inference for Fractional Diffusion Processes*, Wiley, London.

Prakasa Rao, B.L.S. (2013) Parameter estimation for a two-dimensional stochastic Navier-Stokes equation driven by infinite dimensional fractional Brownian motion, *Random Operators and Stochastic Equations*, **21** 37-52.

Prakasa Rao, B.L.S. (2015a) Option pricing for processes driven by mixed fractional Brownian motion with superimposed jumps. *Probability in the Engineering and Information Sciences*. 29: 589-596.

Prakasa Rao, B.L.S. (2015b) Pricing geometric Asian power options under mixed fractional Brownian motion environment, *Physica A*, **446**, 92-99.

Prakasa Rao, B.L.S. (2017a) Instrumental variable estimation for a linear stochastic differential equation driven by a mixed fractional Brownian motion. *Stochastic Anal. Appl.* 35: 943-953.

Prakasa Rao, B.L.S. (2017b) Optimal estimation of a signal perturbed by a mixed fractional Brownian motion, *Theory of Stochastic Processes*, **22 (38)**, 62-68.
Prakasa Rao, B.L.S. (2018a) Parametric estimation for linear stochastic differential equations driven by mixed fractional Brownian motion, *Stochastic Analysis and Applications*, 36, 767-781.

Prakasa Rao, B.L.S. (2018b) Pricing geometric Asian options under mixed fractional Brownian motion environment with superimposed jumps, *Calcutta Statistical Association Bulletin*, 70, 1-6.

Prakasa Rao, B.L.S. (2019) Nonparametric estimation of trend for stochastic differential equations driven by mixed fractional Brownian motion, *Stochastic Analysis and Applications*, 37, 271-280.

Prakasa Rao, B.L.S. (2020) Nonparametric estimation for stochastic differential equations driven by mixed fractional Brownian motion with random effects, In the Special Issue in honour of CR Rao Birth Centenary, *Sankhya, Series A* (to appear).

Prakasa Rao, B.L.S. (2021a) Maximum likelihood estimation in the mixed fractional Vasicek model, *Journal of Indian Society for Probability and Statistics*.

Prakasa Rao, B.L.S. (2021b) Nonparametric estimation of linear multiplier for processes driven by mixed fractional Brownian motion, In the Special Issue in memory of Aloke Dey, *Statistics and Applications*, 19, 1-12 (to appear).

Rozovskii, B.L. (1990) *Stochastic Evolution Systems*, Kluwer, Dordrecht.

Samko, S.G., Kilbas, A.A., and Marichev, O.I. (1993) *Fractional Integrals and derivatives*, Gordon and Breach Science.

Skorokhod, A.V. (1965) *Studies in the Theory of Random Processes*, Addison-Wesley, Reading, MA.