Equidistant Polarizing Transforms
Sinan Kahraman

Abstract

This paper presents a non-binary polar coding scheme that can reach the equidistant distant spectrum bound for discrete memoryless channels. This is the best that can be reached up to now and is tight for additive white Gaussian channels. To do this, we first define the equidistant property for the polarizing transforms. We have designed transforms that have better distance characteristics using a simple procedure for signal sets. Furthermore, we designed a new four-component signal set in two dimensions to define the equidistant transform for $q = 4$.

We follow Şaşoğlu’s work, which introduces polarizing transforms of all channels for any input alphabet size and defines some constraints to avoid a polarization anomaly. Alternatively, we removed the negative effect of the anomaly on polarization by using transforms with appropriate distance properties. Finally, the speed of polarization is significantly increased by using the proposed transforms with better distance characteristics. Moreover, we show improvement in error performance.

Index Terms

Channel polarization, equidistant channel, non-binary polar codes, phase shift keying, polarizing transform, signal set design.

I. INTRODUCTION

The goal of this paper is to present a method for improving the minimum distance of non-binary polar codes. Following the notation in [1], we consider a memoryless channel $W : X \rightarrow Y$ with input alphabet $X$, output alphabet $Y$, and transition probabilities $\{W(y|x) : x \in X, y \in Y\}$. We assume that $X$ is a finite set and label its elements such that $X = \{0, 1, \ldots, q-1\}$, where $q \geq 2$ is an arbitrary integer. We leave $Y$ arbitrary for the moment. We will consider polarization schemes based on a basic transform of the type depicted in Fig. 1.

![Fig. 1](image)

Fig. 1. A basic scheme with a polarizing transform $f$ for q-ary input alphabet.

The transform is defined by a kernel

$$f : X^2 \rightarrow X,$$

which we assume is a mapping with the following properties:

- for any fixed $u_1 \in X$, $u_2 \rightarrow f(u_1, u_2)$ is an invertable function of $u_2$;
- for any fixed $u_2 \in X$, $u_1 \rightarrow f(u_1, u_2)$ is an invertable function of $u_1$.

The standard polar coding kernel as defined in [1] is a mapping of this type with $f(u_1, u_2) = u_1 \oplus u_2$, where $\oplus$ denotes addition mod-$q$. The present paper shows that it is possible to construct polar codes with better distance properties (hence better performance) using alternative kernels of the above type.

A. Review of non-binary polar coding results

The results of polar coding in [1–2] were generalized to $q$-ary input alphabets in [3–4]. In one of the approaches proposed in these works, the polarization of the synthetic channels to good or bad channels is guaranteed for channels with prime input alphabets of cardinality $q$ [3] in a similar way to [1] by using the polar coding kernel with addition mod-$q$ instead of mod-2. To extend these results to channels with input alphabet cardinalities given by composite numbers $q$, a randomized construction was proposed in [3] without any change of complexity in the encoder and decoder. The randomization was first used in [1] as a process to simplify the analysis of the polarization property. When $q$ is a composite number, properties of polarizing transforms were investigated in [5] and the following result was presented:

(i) for any $2 \leq K \leq q - 1$ and distinct $a_0, \ldots, a_{K-1}$, the matrix $B_{ij} = f(a_i, a_j)$, $i, j = 0, \ldots, K - 1$ of a polarizing mapping has at least $K + 1$ distinct entries.

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Later, a particular transform

\[ f(u_1, u_2) = u_1 \oplus \pi(u_2), \]

as shown in Fig. 1 was provided by Şaşoğlu in [5]–[6] with the following definition:

\[ \pi(x) = \begin{cases} \lfloor q/2 \rfloor & \text{if } x = 0, \\ x - 1 & \text{if } 1 \leq x \leq \lfloor q/2 \rfloor, \\ x & \text{otherwise}, \end{cases} \]  

(1)

as a solution to polarize channels with arbitrary input alphabet sizes. However, this type of transforms were not discussed up to now to construct non-binary polar codes with better distance properties. In this paper, we provide a particular type of transforms for non-binary polar codes. In Fig. 2 the reliabilities of the synthetic channels obtained by Monte-Carlo simulation are shown for non-binary polar codes, where A with the proposed transform in this paper, B with transform defined by (1) and C with standard transform. Here, the block length is 1024, \( q = 8 \), and 8-PSK signalling is used for additive white Gaussian noise (AWGN) channel. The figure shows that the speed of polarization is increased by the use of the proposed transform in this paper.

As an alternative approach, a multi-level code construction technique was also proposed in [3] to avoid designing transforms. In this regard, \( q \)-ary multi-level polarization was investigated in [7]–[8] independently. In this case, the synthetic channels converge to good, bad, and partially good channels. In [9], it was shown that the sum capacity can be achieved by the multi-level construction. Then, it was shown in [10] that the number of partially polarized levels can be adjusted by using a particular polarizing transform.

Apart from these research directions, the code construction method proposed in [11], and which is based on the quantization of the synthetic channels via degrading and upgrading channels, was generalized in [12] for arbitrary input alphabets by using successive approximations. An alternative code construction method was proposed in [13] for arbitrary input alphabets by using one multi dimensional approximation instead of successive approximations. The construction for moderate size input alphabets was also discussed in [14]. Then, efficient algorithms were studied in [15] for \( q \)-ary construction that merge the output symbols of the synthetic channels.

The rest of the paper is organized as follows. Section II provides a system model of non-binary polar coding for AWGN channel with the encoding scheme and signal sets in 2-dimensions. Furthermore, we propose the minimum distance of the synthetic good channel as a function of the polarizing transform for a given signal set. Section III provides a procedure to design polarizing transforms with better distance properties. Moreover, we propose a definition of the equidistant polarizing transforms that achieve the optimal distance spectrum upper bound for a given signal set. Section IV describes polarizing transforms for \( q \)-ary PSK signal sets to improve minimum distances and\ or distance spectrums. Besides, we show that the new polarizing transforms are better than the transforms in (1) and the standard transform. We propose the equidistant polarizing transform for \( q = 5 \) and 5-PSK signal set. We reported that the equidistant property is not achievable for \( q = 4 \) by the use of PSK signal set, and hence, Section V designs a new signal set for \( q = 4 \) in 2-dimensions in order to define an equidistant polarizing transform. Section VI reviews the polarization speed of the equidistant polarizing transforms. Section VII propose asymptotic behaviour of the equidistant property. Section VIII provides some performance results to investigate the effect of better distance properties by using the equidistant polarizing transforms. Section IX concludes this study with some opinions.
II. System Model

A non-binary encoder scheme with a polarizing transform is depicted in Fig. 3 for the block length $N = 8$.

![Encoder Scheme](image)

Fig. 3. An encoder scheme of non-binary polar codes.

The encoding complexity is the same for any given transform due to the butterfly structure. In this study, we consider PSK signal sets for $q$-ary input alphabets. Let $\mathcal{S}$ be a signal set with size $q$, where $s_k = \sqrt{E_s}e^{2\pi k/q} \in \mathcal{S}$ for $k = 0, 1, \ldots, q - 1$. Here, the signal energy is $E_s$ joule/2-dimensions, and the $q$-ary PSK signal sets $\mathcal{S} : \{\sqrt{E_s}, \sqrt{E_s}e^{\frac{2\pi}{q}}, \ldots, \sqrt{E_s}e^{\frac{2\pi(q-1)}{q}}\}$ are depicted in Fig. 4 for different values of $q$.

![PSK Signal Sets](image)

Fig. 4. $q$-ary PSK signal sets: $s_k = \sqrt{E_s}e^{2\pi k/q} \in \mathcal{S}$ for $q = \{3, 4, 5\}$.

Here, a natural mapping $x_i \rightarrow s_{x_i}$ is considered for transmission on AWGN channel. The noisy observations from the channel are defined for $i = 1, \ldots, N$ as follows:

$$y_i = s_{x_i} + n_i, \quad (2)$$

where $n_i$ is a complex Gaussian random variable with $CN(0, \sigma^2)$. The power spectral density is $\sigma^2 = N_0$ joule/2-dimensions. Hence, the signal to noise ratio is $E_s/N_0$.

The transition probabilities of the synthetic channels obtained after one-step of polarization are defined as follows:

$$W(y_1, y_2 | u_1) = \frac{1}{q} \sum_{u_2=0}^{q-1} W(y_1 | f(u_1, u_2)) W(y_2 | u_2), \quad (3)$$

$$W(y_1, y_2, u_1 | u_2) = \frac{1}{q} W(y_1 | f(u_1, u_2)) W(y_2 | u_2), \quad (4)$$

where $W(y|x) = \frac{1}{\pi \sigma^2} e^{-\|y-s_x\|^2/\sigma^2}$. These transition probabilities describe a link between $f$ and error performances for polarized channels. Note that $\sigma^2$ is the variance for 2-dimensions. Squared Euclidean distance is denoted by $\|\cdot\|^2$.

For the synthetic good channel, the distance of a given transform $f$ is

$$d = \sqrt{\|s_{f(u_1, u_2)} - s_{f(u_1, u_2')}\|^2 + \|s_{u_2} - s_{u_2'}\|^2} \quad (5)$$

for any $u_1$ and $u_2 \neq u_2'$, and the minimum distance of the standard transform is

$$d_{min} = \sqrt{\|s_m - s_{m+1}\|^2 + \|s_n - s_{n+1}\|^2}, \quad (6)$$

and the corresponding minimum distance for the PSK signal set is given by the following equation that the equation is given for the standard transform.

$$d_{min} = 2\sqrt{2}\sin(\pi/q) \sqrt{E_s} \quad (7)$$

This result is the same for the class of polarizing transforms introduced in (1).

In the next section, we will describe equidistant transforms which improve distance properties.
III. DESIGN OF POLARIZING TRANSFORMS

For an AWGN channel, the distance spectrum upper bound for the symbol error probability $P_e$ is given by:

$$P_e \leq \sum_{d \geq d_{min}} N(d) \cdot Q\left(d/(2\sigma)\right),$$  

(8)

where $Q(d/(2\sigma))$ corresponds to the pairwise error probability between two points at a distance $d$ apart, and $N(d)$ denotes the distance spectrum defined as the number of points at distance $d$.

We investigate the distance properties by using a table where $u_1$ and $u_2$ are shown in a cell with coordinates $(x_1, x_2)$ corresponding to the outputs of the scheme in Fig. 1. For more clarity, the cells are marked with a grey face for $u_1 = 0$. We give a table for the standard transform (i.e. with the type of $f(u_1, u_2) = u_1 \oplus \pi_0(u_2)$, where $\pi_0$ is the identical permutation) for $q = 5$ as follows:

| $x_2$ | 0 | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|---|
| 0     | 00| 10| 20| 30| 40|
| 1     | 11| 21| 31| 41| 01|
| 2     | 32| 42| 02| 12| 22|
| 3     | 23| 33| 43| 03| 13|
| 4     | 14| 24| 34| 44| 04|

$\pi_0 = \{0, 1, 2, 3, 4\}.$

The minimum distance of the standard transform is

$$d_{min} = \sqrt{2}\|s_0 - s_1\| = 1.66\sqrt{E_s}$$  

(9)

for 5-PSK signalling. The distance spectrum $N(d)$ for the standard transform is given as follows:

$$N(d) = \begin{cases} 2 & d = 1.66\sqrt{E_s}, \\ 2 & d = \sqrt{2}\|s_0 - s_2\| = 2.69\sqrt{E_s}, \\ 0 & \text{otherwise}. \end{cases}$$

(10)

An analytical expression of the distance spectrum upper bound for the synthetic good channel with the standard transform is

$$P_e \leq 2Q\left(1.66\sqrt{SNR/2}\right) + 2Q\left(2.69\sqrt{SNR/2}\right).$$

(11)

As we can see from Fig. 5, the analytic result (11) is a tight upper bound for the symbol error probability.

In this paper, we first focus on designing polarizing transforms which increase the minimum distance for $q$-ary PSK signal sets. To design a table with the type of $f(u_1, u_2) = u_1 \oplus \pi(u_2)$, we define a simple procedure as follows:

(p.i) Each row has only 1 candidate of $u_2$ for $u_1 = 0$.
(p.ii) Each column has only 1 candidate of $u_2$ for $u_1 = 0$.
(p.iii) Place all candidates of $u_2$ as far as from each others for $u_1 = 0$.
(p.iv) Fill the empty cells by $q$ candidates of $u_2$ that are placed in the $k$th cyclic right-shift cell for $u_1 = k$, where $k = 1, \ldots, q - 1$.

Hence, the completed table provides the polarizing transform with the type of $f(u_1, u_2) = u_1 \oplus \pi(u_2)$. Here, (p.i) and (p.ii) are constraints of the procedure to guarantee $(x_1, x_2)$ that can take all possible $q$-ary pairs. To achieve an increased minimum distance, (p.iii) can be done by a computer search. Then, (p.iv) is to complete the definition of the polarizing transform.

**Theorem 1:** The distance is conserved as follows:

$$\sum_{u_2} \left(\|s_{f(u_1, u_2)} - s_{f(u_1, u_2)}\|^2 + \|s_{u_2} - s_{u_2}\|^2\right) = 2\sum_{k=1}^{q-1} \|s_k - s_0\|^2$$

for $q$-ary PSK signal set by using a polarizing transform $f$.

**Proof 1:** It is obtained by the following steps:

- PSK is a signal set that is matched to a group $[16]$ that $\|s_{t+k} - s_l\| = \|s_k - s_0\|$.
- $\sum_{u_2} \|s_{u_2} - s_{u_2}\|^2 = \sum_{k=1}^{q-1} \|s_k - s_0\|^2$.
- For a fixed $u_1, u_2 \rightarrow f(u_1, u_2)$ is an invertible function of $u_2$. Hence, $\sum_{u_2} \|s_{f(u_1, u_2)} - s_{f(u_1, u_2)}\|^2 = \sum_{k=1}^{q-1} \|s_k - s_0\|^2$.

\(^1\)Obviously, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ for the channel with $\sigma = 1$. Notice that $\sigma^2$ is for 1-dimension in 6, and $SNR = E_s/N_0$ where the signal power is $E_s$ joule/2-dimensions and $N_0$ is $\sigma^2$ joule/2-dimensions.
Theorem 2: The minimum distance of a \( q \)-ary synthetic good channel is lower bounded by

\[
d_{\text{min}} \leq \sqrt{\frac{2}{q-1} \sum_{k=1}^{q-1} ||s_k - s_0||^2},
\]

where \( s_k \in S \), and \( S \) is the \( q \)-ary PSK signal set.

Proof 2: The distance is conserved which is seen in the previous result, and it is easy to see that

\[
\min\{d^2\} \leq D/(q-1),
\]

where

\[
D = 2\sum_{k=1}^{q-1} ||s_k - s_0||^2.
\]

By this way, the proof of \( d_{\text{min}} \leq \sqrt{D/(q-1)} \) is obtained.

Definition 1 (Equidistant Polarizing Transforms): For a given \( q \)-ary signal set the polarizing transforms with the distance spectrum \( N(d_{\text{min}}) = q - 1 \) are the Equidistant polarizing transforms.

An equidistant transform for \( q \)-ary signal set has the distance spectrum upper bound as follows:

\[
P_e \leq (q-1)Q(d_{\text{min}}/(2\sigma)),
\]

where the minimum distance \( d_{\text{min}} = \sqrt{\frac{2}{q-1}\sum_{k=1}^{q-1} ||s_k - s_0||^2} \). This is the best achievable distance spectrum bound that is shown by Theorem 3.

In the next section, we provide some examples for the better distance characteristics.

IV. POLARIZING TRANSFORMS

Example 1 (Equidistant Polarizing Transforms for \( q = 5 \)): We applied the procedure for 5-PSK signal set to design polarizing transforms with the type of \( f(u_1, u_2) = u_1 \oplus \pi_i(u_2) \) for \( q = 5 \). Hence, the following tables are provided for \( q = 5 \) and 5-PSK signal set.

\[
\begin{array}{cccc|cccc|c}
\hline
x_2^5 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\
\hline
x_2^5 & 00 & 10 & 20 & 30 & 40 & 00 & 10 & 20 & 30 & 40 \\
0 & 01 & 11 & 21 & 31 & 41 & 11 & 21 & 31 & 41 \\
1 & 12 & 22 & 32 & 42 & 02 & 12 & 22 & 32 & 42 \\
2 & 23 & 33 & 03 & 13 & 23 & 33 & 03 & 13 & 23 \\
3 & 34 & 44 & 14 & 24 & 34 & 44 & 14 & 24 & \\
4 & & & & & & & & & \\
\pi_1 = (01234) & \pi_2 = (01234)
\end{array}
\]

We investigate the distance properties for the polarizing transforms \( f(u_1, u_2) = u_1 \oplus \pi_i(u_2) \) for \( i = 1, 2 \), where \( \pi_1 = (01234) \) and \( \pi_2 = (01234) \) by using the tables that the distance properties are the same for \( \pi_1 \) and \( \pi_2 \). Hence, the minimum distance is

\[
d_{\text{min}} = \sqrt{||s_0 - s_1||^2 + ||s_0 - s_2||^2} = 2.24\sqrt{E_s}
\]

for 5-PSK signal set, and the distance spectrum is as follows:

\[
N(d) = \left\{ \begin{array}{ll}
4 & d = 2.24\sqrt{E_s}, \\
0 & \text{otherwise}.
\end{array} \right.
\]

for the polarizing transforms \( f(u_1, u_2) = u_1 \oplus \pi_i(u_2) \) for \( i = 1, 2 \). It is clear to see that these polarizing transforms are equidistant (i.e. \( N(d_{\text{min}}) = q - 1 \)) for 5-PSK signal set, and the minimum distance \( (12) \) of the equidistant transforms is larger than the minimum distance \( (9) \) of the standard transform. The distance spectrum upper bound is given for the equidistant transform as follows:

\[
P_e \leq 4Q\left(2.24\sqrt{SNR/2}\right).
\]

Theorem 3: For \( a > 0, \ b > 0 \) and \( a \neq b \),

\[
2Q\left(\sqrt{\frac{a^2 + b^2}{2}}\right) < Q(a) + Q(b).
\]

\[\footnote{Notice that the minimum distances of the class of polarizing transforms introduced in \cite{1} and the standard transform are the same.} \]
The proof of the theorem is provided in the Appendix. Theorem 3 shows that the distance spectrum upper bound is minimized by the help of an equidistant transform for the $q$-ary signal set. Then, we can say that the upper bound of the error performance is improved for the synthetic good channel by using equidistant transforms for a given signal set. To support the claim, we provide simulation results in Fig. 5 that the error performance of the synthetic good channel is improved for $q = 5$ and 5-PSK signal set by using the equidistant transform $f(u_1, u_2) = u_1 \oplus \pi_1(u_2)$, where $\pi_1 = (0 \; 1 \; 2 \; 3 \; 4)$.

We follow the same way to investigate the distance properties of the bad synthetic channel by using the table. The analysis of the synthetic bad channel shows that the upper bounds are (almost) the same for any polarizing transform. As such, the minimum distances of the synthetic bad channel are the same,

$$d_{\min} = \|s_i - s_{i+1}\| = 1.176\sqrt{E_s},$$

for the standard transform and the equidistant transform for $q$-ary PSK signal set.

The distance spectrum of the standard transform is

$$N(d) = \begin{cases} 
4 & d = \|s_0 - s_1\| = 1.176\sqrt{E_s}, \\
2 & d = \sqrt{2}\|s_0 - s_1\| = 1.663\sqrt{E_s}, \\
4 & d = \|s_0 - s_2\| = 1.902\sqrt{E_s}, \\
8 & d = \sqrt{2}\|s_0 - s_1\|^2 + \|s_0 - s_2\|^2 = 2.236\sqrt{E_s}, \\
2 & d = \sqrt{2}\|s_0 - s_2\| = 2.690\sqrt{E_s}, \\
0 & otherwise. 
\end{cases}$$

for $q = 5$ and 5-PSK signal set.

The distance spectrum of the equidistant transform is

$$N(d) = \begin{cases} 
4 & d = \|s_0 - s_1\| = 1.176\sqrt{E_s}, \\
4 & d = \sqrt{2}\|s_0 - s_1\| = 1.663\sqrt{E_s}, \\
4 & d = \|s_0 - s_2\| = 1.902\sqrt{E_s}, \\
4 & d = \sqrt{2}\|s_0 - s_1\|^2 + \|s_0 - s_2\|^2 = 2.236\sqrt{E_s}, \\
4 & d = \sqrt{2}\|s_0 - s_2\| = 2.690\sqrt{E_s}, \\
0 & otherwise. 
\end{cases}$$

for $q = 5$ and 5-PSK signal set. The difference between the upper bounds of the synthetic bad channel is insignificant for the standard transform and the equidistant transform. To support the claim, we provide simulation results in Fig. 6 that the error performances of the synthetic bad channel are (almost) the same for $q = 5$ and 5-PSK signal set by using the standard transform and the equidistant transform.

Here, one of the main results in this work is: the equidistant transforms for a given $q$-ary signal set provide superior synthetic good channel and (almost) the same synthetic bad channel, and hence, the performance of the error correction capability is improved for the block length $N > 2$. We will provide some simulation results in Section VIII in order to strengthen this claim.
Example 2 (A Polarizing Transform for $q = 4$): We applied the procedure for $q = 4$ and 4-PSK signal set to design a polarizing transform. The minimum distance is $\sqrt{2}||s_0 - s_1||$, and it is the same for all possible polarization transforms of type $f(u_1, u_2) = u_1 \oplus \pi(u_2)$. In this case, the procedure optimize the distance spectrum $N(d)$ for 4-PSK signal set to improve the upper bound in (8). The first term of the distance spectrum, $N(d_{min})$ is known as the kissing number. For this purpose, it minimizes the kissing number by using a particular polarization transform. The distance spectrum of the standard transform is

$$N(d) = \begin{cases} 
2 & d = 2\sqrt{E_s}, \\
1 & d = \sqrt{||s_0 - s_1||^2 + ||s_0 - s_2||^2} = 2.45\sqrt{E_s}, \\
0 & \text{otherwise}. 
\end{cases}$$

for $q = 4$ and 4-PSK signal set.

The distance spectrum of the optimized transform by using $\pi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{pmatrix}$ is

$$N(d) = \begin{cases} 
1 & d = 2\sqrt{E_s}, \\
2 & d = \sqrt{||s_0 - s_1||^2 + ||s_0 - s_2||^2} = 2.45\sqrt{E_s}, \\
0 & \text{otherwise}. 
\end{cases}$$

for $q = 4$ and 4-PSK signal set. Hence, the distance spectrum upper bound of the standard transform is

$$P_e \leq 2Q\left(2\sqrt{\frac{SNR}{2}}\right) + Q\left(2.45\sqrt{\frac{SNR}{2}}\right),$$

and it is reduced to the following upper bound by using the optimized transform for 4-PSK signal set.

$$P_e \leq Q\left(2\sqrt{\frac{SNR}{2}}\right) + 2Q\left(2.45\sqrt{\frac{SNR}{2}}\right)$$

It is easy to notice that (20) is greater than (21). We also notice that the equidistant transform does not exist for $q = 4$ and 4-PSK signal set. In the next section, we will propose a new signal set for $q = 4$ in 2-dimensions to design the equidistant transform with the distance spectrum $N(d_{min}) = q - 1 = 3$, and the minimum distance $d_{min} = \sqrt{\frac{2}{3}} \sum_{k=1}^{3} ||s_k - s_0||^2 = 2.31$.

Example 3 (Equidistant Polarizing Transforms for $q = 3$): To design the equidistant transform, $q = 3$ and 3-PSK signal set is the special case by the help of its following property:

$$||s_j - s_{j'}|| = \sqrt{3E_s},$$

for all $j \neq j'$. The distance spectrum is $N(d_{min}) = 2$, and the minimum distance is $d_{min} = 2.449\sqrt{E_s}$ for all possible polarization transforms, and hence, all polarizing transforms are equidistant for 3-PSK signal set.

Notice that 3-PSK is only two-dimensional signal set for $q > 2$ with constant $||s_j - s_{j'}||$ for all $j \neq j'$. It is possible to find signal sets with constant distances for $q > 3$ in a higher dimension. For that case spectral efficiency is reduced for a given fixed $|K\log_2 q|$ number of information bit. In this study, we consider signal sets in 2-dimensions for the non-binary polar coding.
Example 4 (Almost-Equidistant Transform for $q = 8$): We applied the procedure for $q = 8$ and 8-PSK signal set to design a polarizing transform. There is not exist an equidistant transform of type $f(u_1, u_2) = u_1 \oplus \pi(u_2)$. Thanks to the following geometric property of 8-PSK signal set, we can design almost-equidistant transform of type $f(u_1, u_2) = u_1 \oplus \pi(u_2)$, where $\pi = (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7)$ by using the procedure for $q = 8$.

$$\|s_0 - s_1\|^2 + \|s_0 - s_3\|^2 = 2\|s_0 - s_2\|^2 = \|s_0 - s_4\|^2$$

This geometric property is depicted in Fig. 7.

![Fig. 7. The geometric property of 8-PSK signal set for almost-equidistant transform.](image)

The minimum distance of the almost-equidistant transform is $d_{min} = 2\sqrt{E_s}$. The distance spectrum of the almost-equidistant transform is

$$N(d) = \begin{cases} 
6 & d = 2\sqrt{E_s}, \\
1 & d = \sqrt{\|s_0 - s_4\|^2 + \|s_0 - s_4\|^2} = 2.83\sqrt{E_s}, \\
0 & \text{otherwise.}
\end{cases}$$ (22)

for $q = 8$ and 8-PSK signal set. Hence, the distance spectrum upper bound of the almost-equidistant transform is

$$P_e \leq 6Q\left(2\sqrt{\frac{\text{SNR}}{2}}\right) + Q\left(2.83\sqrt{\frac{\text{SNR}}{2}}\right).$$ (23)

If an equidistant transform existed for 8-PSK signal set, the minimum distance would be $d_{min} = 2.14\sqrt{E_s}$, and the upper bound would be

$$P_e \leq 7Q\left(2.14\sqrt{\frac{\text{SNR}}{2}}\right).$$ (24)

The proposed transform has also (almost) the same with the equidistant upper bound for 8-PSK signal set that can be seen in the Appendix.

Notice that the proposed almost equidistant transform with $\pi = (0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7)$ has the same anomaly which is defined by Şaşoğlu in [5] to describe failure of the polarization. The group $(\mathcal{X}, f)$ has a proper nontrivial subgroup. There exists a set $S = \{0, 4\}$ where $S \subset \mathcal{X}$ with $|S| > 1$ such that $(S, f)$ is a group.

It is interesting to notice that with the help of the nearly equidistant characteristic that the proposed transform has, the negative effect on the polarization of the mentioned anomaly can be removed. The two-level polarization provided by the almost equidistant transform and the distorted polarization obtained by the standard transform are shown in Fig. 2. Moreover, the speed of polarization obtained by the almost equidistant transform is higher than the polarization obtained by the the non-anomaly transform defined in (1). We investigate in more detail the effects of the equidistant property obtained by transform designs on polarization in the Section VI-A.

To summarize this section, we provide the distance properties of the proposed and standard transforms for $q$-ary PSK signalling in Table I.

In the next section, we will describe a new signal set which make possible to define an equidistant transform for $q = 4$. 

![Fig. 7. The geometric property of 8-PSK signal set for almost-equidistant transform.](image)
| $q$ | Distance Properties of $f : L_q$ | Distance Properties of $[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$ |
|-----|--------------------------------|--------------------------------|
|     | $d_{\min}$ | $N(d)$ | $d_{\min}$ | $N(d)$ |
| 3   | 2.449      | 2      | 2.449      | 2      |
| 4   | 2.000      | 1, 2   | 2.000      | 2, 1   |
| 5   | 2.236      | 4      | 1.663      | 2, 2   |
| 8   | 2.000      | 6, 1   | 1.082      | 2, 2, 1|

$L_3 : \pi = (\begin{smallmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{smallmatrix})$

$L_4 : \pi = (\begin{smallmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 \end{smallmatrix})$

$L_5 : \pi = (\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 4 & 2 \end{smallmatrix})$

$L_8 : \pi = (\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \end{smallmatrix})$

V. SIGNAL SET DESIGN FOR EQUIDISTANT TRANSFORM

In the previous section, we have shown that the equidistant transform does not exist for 4-PSK signal set. Here, we are designing a new signal set in order to make equidistant transform possible for $q = 4$.

For this purpose, we consider the following polarizing transform

$$f(u_1, u_2) = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 3 & 0 & 2 \\ 2 & 0 & 3 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix}$$

that is not of type $f(u_1, u_2) = u_1 \oplus \pi(u_2)$.

The new signal set is obtained by changing the geometry of the 4-PSK signal set. For this purpose, an equal rotation in clockwise direction was applied to the $s_1$ and $s_3$. The following relation is found between the new Euclidean distances, depending on the amount of rotation as a result of the rotation process.

$$\|s_0 - s_1\| = x \sqrt{E_s},$$

$$\|s_0 - s_3\| = \sqrt{4 - x^2} \sqrt{E_s}.$$
The distance properties are improved by the equidistant transform for the new signal set as follows

- The minimum distance is $d_{\text{min}} = 2.309$
- The distance spectrum is $N(d_{\text{min}}) = 3$.

Unfortunately, the synthetic bad channel is worse for this case due to its decreased minimum distance.

As a result of this section, we show that the synthetic good channel reaches the best possible distance spectrum upper bound with the new signal set for $q = 4$ in 2-dimensions. It can be easily seen that $W^{++...+}$ channel is also optimal. As a consequence, for any $N$ and $K = 1$ the best non-binary ($q = 4$) polar code is defined in this way for two-dimensional signalling. On the other hand the polar coding we have proposed for 4-PSK in the previous section may provide superior error correction performance for $K > 2$ because it has a better synthetic bad channel than the code we proposed here with the new signal set. For now, we leave this problem to the future work.
VI. SPEED OF POLARIZATION

By the help of Monte-Carlo simulation, we investigate the speed of polarization for non-binary polar codes under a symbol based $q$-ary successive cancellation decoder. Reliabilities of the sorted synthetic channels are depicted in Fig. 10.

![Graph showing symbol error rate against sorted indices for different $q$ values.]

As we can see from the figure, better distance properties lead to an improvement on the speed of polarization for non-binary polar codes.

A. Experiments on the almost-equidistant transform

We experimentally investigate the contribution of equidistant polarization transforms that are placed at only the channel stage as shown in Fig. 11.

![Diagram of an encoder scheme of non-binary polar codes with $f$ only at the channel stage. Other stages have the standard transform.]

It was depicted in Fig. 12 that contribution of any polarization transforms from the first stage to the channel state are insignificant when the equidistant transform is placed at the channel stage. This result is not true for other (not equidistant) polarization transforms.

The equidistant channels are first discussed for polar coding in the related work [3]. It was mentioned that the standard transform polarizes equidistant channels, regardless of the input alphabet size. By similar argument, the equidistant transform that is placed at the channel stage creates an equidistant synthetic good channel, and hence, the standard transforms that are placed at all of the previous stages provide the polarization, regardless of the input alphabet size. This assertion is provided by the computer simulation in Fig. 12.
VII. Asymptotic Behaviour of the Equidistant Property

The minimum distances of the equidistant transforms are plotted for $q$-ary PSK signal set in Fig. 13.

The minimum distance of the standard transform and the transform in (1) is

$$\lim_{q \to \infty} d_{\text{min}} = 0$$

for $q$-ary PSK signal set. Notice that the minimum distance of the equidistant transforms is

$$\lim_{q \to \infty} d_{\text{min}} = 2\sqrt{E_s}$$

for $q$-ary PSK signal set.

Furthermore, the equidistant distant spectrum upper bound for $q \to \infty$ is

$$P_e \leq (q - 1)Q\left(\sqrt{2SNR}\right)$$

for $q$-ary PSK signal set.

The asymptotic behavior of the equidistant property for the Polarization transforms mentioned here has shown that non-binary equidistant polar codes are also a promising way for very high order input alphabet sizes. However, we are leaving this issue to be discussed for future studies. In the next section, we provide some simulation results to investigate the error performance for $q$-ary PSK signal sets.
VIII. Simulation Results: Experiments to Investigate the Error Performance

We are investigating the error performance of non-binary polar codes for AWGN channels under the \( q \)-ary successive cancellation decoder. These codes were constructed using Monte-Carlo simulations for 2 dB SNR. Spectral efficiency is set to 1 bit/channel-use.

The frame error rates are depicted in Fig. 14 for \( q = 5 \), and 5-PSK signalling. The equidistant transform outperforms the standard transform and the transform in (1).

In the next experiment, the frame error rates are provided in Fig. 15 for \( q = 4 \), and 4-PSK signal set. Note that we compare our results with the set partitioned (SP) binary polar codes [17] which are state-of-the-art transmission schemes with multi-level coding (MLC) for polar codes for 4-PSK signalling. The optimized transform outperforms state-of-the-art MLC.

Furthermore, proposed non-binary codes in this study can also be adopted to the set partitioned MLC schemes to improve the error performance. This is out of the scope in this paper. We left this research direction to the future work.
The frame error rates are depicted in Fig. 16 for $q = 8$, and 8-PSK signalling. The almost-equidistant transform outperforms the transform in (1).

![Fig. 16. Frame error rates of non-binary polar codes for 8-PSK signal set.](image)

In the next experiment, the frame error rates are provided in Fig. 17 for the optimized transform for $q = 4$, and 4-PSK signal set, the equidistant transform for $q = 5$, and 5-PSK signal set, the almost-equidistant transform for $q = 8$, and 8-PSK signal set. The equidistant transform outperforms the optimized and almost equidistant transforms.

![Fig. 17. Frame error rates of the proposed non-binary polar codes for 4, 5 and 8-PSK signal sets.](image)

IX. CONCLUSION

We have shown how to design polarizing transforms of non-binary polar codes to improve the minimum distance. For a given signal set we have designed polarization transforms that reach the limit of the best known distance spectrum. This limit is tight for AWGN channels. Furthermore, we designed a new signal set to define an equidistant transform for $q = 4$. The proposed polar coding scheme has the same quasi-linear encoding and decoding complexity advantage. The increase in the polarization rate and even the improvement in the frame error rate are confirmed by the simulation results. Non-binary equidistant polar codes have been shown to be promising error correction methods due to their high performance in error correction.
APPENDIX A

PROOF OF THEOREM 3

Here we show that for all \( q > 2 \), a class of polarizing transforms that are equidistant provides the best achievable distance spectrum upper bound. To accomplish this claim, first we prove

\[
2Q \left( \sqrt{\frac{a^2 + b^2}{2}} \right) < Q(a) + Q(b)
\]

for \( a > 0, \ b > 0 \) and \( a \neq b \).

By the use of Jensen’s inequality [18] for a decreasing concave function \( \psi \), following result is obtained.

\[
\psi \left( \frac{a+b}{2} \right) < \frac{\psi(a) + \psi(b)}{2}
\]

A graphical interpretation of this fact is depicted in Fig. [18].

The proof is completed by the following step. For all \( a > 0, \ b > 0 \)

\[
\sqrt{\frac{a^2 + b^2}{2}} > \frac{a + b}{2}.
\]

Hence, it is obvious that

\[
Q \left( \sqrt{\frac{a^2 + b^2}{2}} \right) < Q \left( \frac{a + b}{2} \right) < Q(a) + Q(b).
\]

Finally, this result can be generalized for any \( q > 2 \) by using multiple \( q - 1 \) distance terms instead of \( a \) and \( b \).

Fig. 18. A graphical interpretation of Jensen’s inequality.

APPENDIX B

DISTANCE SPECTRUM UPPER BOUND OF THE ALMOST-EQUIDISTANT TRANSFORM

Hence, the distance spectrum upper bound of the almost-equidistant transform is

\[
P_e \leq 6Q \left( 2\sqrt{\frac{SNR}{2}} \right) + Q \left( 2.83\sqrt{\frac{SNR}{2}} \right).
\]

If an equidistant transform existed for 8-PSK signal set, the minimum distance would be \( 2.14\sqrt{E_{s}} \), and the upper bound would be

\[
P_e \leq 7Q \left( 2.14\sqrt{\frac{SNR}{2}} \right).
\]
Fig. 19. Performance of almost-equidistant transform.

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