HOMOGENIZATION OF A VISCOELASTIC MODEL FOR PLANT CELL WALL BIOMECHANICS

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Abstract. The microscopic structure of a plant cell wall is given by cellulose microfibrils embedded in a cell wall matrix. In this paper we consider a microscopic model for interactions between viscoelastic deformations of a plant cell wall and chemical processes in the cell wall matrix. We consider elastic deformations of the cell wall microfibrils and viscoelastic Kelvin–Voigt type deformations of the cell wall matrix. Using homogenization techniques (two-scale convergence and periodic unfolding methods) we derive macroscopic equations from the microscopic model for cell wall biomechanics consisting of strongly coupled equations of linear viscoelasticity and a system of reaction-diffusion and ordinary differential equations. As is typical for microscopic viscoelastic problems, the macroscopic equations for viscoelastic deformations of plant cell walls contain memory terms. The derivation of the macroscopic problem for degenerate viscoelastic equations is conducted using a perturbation argument.

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Key words: Homogenization; two-scale convergence; periodic unfolding method; viscoelasticity; plant modelling.

Introduction

To obtain a better understanding of the mechanical properties and development of plant tissues it is important to model and analyse the interactions between the chemical processes and mechanical deformations of plant cells. The main feature of plant cells are their walls, which must be strong to resist high internal hydrostatic pressure (turgor pressure) and flexible to permit growth. The biomechanics of plant cell walls is determined by the cell wall microstructure, given by microfibrils, and the physical properties of the cell wall matrix. The orientation of microfibrils, their length, high tensile strength, and interaction with wall matrix macromolecules strongly influences the wall’s stiffness. It is also supposed that calcium-pectin cross-linking chemistry is one of the main regulators of cell wall elasticity and extension \cite{30}. Pectin can be modified by the enzyme pectin methylesterase (PME), which removes methyl groups by breaking ester bonds. The de-esterified pectin is able to form calcium-pectin cross-links, and so stiffen the cell wall and reduce its expansion, see e.g. \cite{29}. It has been shown that the modification of pectin by PME and the control of the amount of calcium-pectin cross-links greatly influence the mechanical deformations of plant cell walls \cite{23, 24}, and the interference with PME activity causes dramatic changes in growth behavior of plant cells and tissues \cite{31}.

To address the interactions between chemistry and mechanics, in the microscopic model for plant cell wall biomechanics we consider the influence of the microstructure, associated with the cellulose microfibrils, and the calcium-pectin cross-links on the mechanical properties of plant cell walls. We model the cell wall as a three-dimensional continuum consisting of a polysaccharide matrix embedded with cellulose microfibrils. Within the matrix, we consider the dynamics of the enzyme PME, methylesterified pectin, demethylesterified pectin, calcium ions, and calcium-pectin cross-links. It was observed experimentally that plant cell wall microfibrils are anisotropic, see e.g. \cite{10}, and the cell wall matrix in addition to elastic deformations exhibits viscous behaviour, see e.g. \cite{14}. Hence we model the cell wall matrix as a linearly viscoelastic Kelvin–Voigt material, whereas microfibrils are modelled as an anisotropic linearly elastic material. The model for plant cell wall biomechanics in which the cell wall matrix was assumed to be a linearly elastic was derived and analysed in \cite{26}. The interplay between the mechanics and the cross-link dynamics comes in by assuming that the elastic and viscous properties of the cell wall matrix depend on the density of the cross-links and that stress within the cell wall can break calcium-pectin cross-links. The stress-dependent opening of calcium channels in the cell plasma membrane is addressed in the flux boundary conditions for calcium ions. The resulting microscopic model is a system of strongly coupled four diffusion-reaction equations, one ordinary differential equation, and the equations of linear viscoelasticity. Since only the cell wall matrix is viscoelastic we obtain degenerate elastic-viscoelastic equations.

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In our model we focus on the interactions between the chemical reactions within the cell wall and its deformation and, hence, do not consider the growth of the cell wall.

To analyse the macroscopic mechanical properties of the plant cell wall we rigorously derive macroscopic equations from the microscopic description of plant cell wall biomechanics. The two-scale convergence, e.g. [4, 21], and the periodic unfolding method, e.g. [7, 8], are applied to obtain the macroscopic equations. For the viscoelastic equations the macroscopic momentum balance equation contains a term that depends on the history of the strain represented by an integral term (fading memory effect). Due to the coupling between the viscoelastic properties and the biochemistry of a plant cell wall, the elastic and viscous tensors depend on space and time variables. This fact introduces additional complexity in the derivation and in the structure of the macroscopic equations, compared to classical viscoelastic equations.

The main novelty of this paper is the multiscale analysis and derivation of the macroscopic problem from a microscopic description of the mechanical and chemical processes. This approach allows us to take into account the complex microscopic structure of a plant cell wall and to analyze the impact of the heterogeneous distribution of cell wall structural elements on the mechanical properties and development of plants. The main mathematical difficulty arises from the strong coupling between the equations of linear viscoelasticity for cell wall mechanics and the system of reaction-diffusion and ordinary differential equations for the chemical processes in the wall matrix. Also the degeneracy of the viscoelastic equations, due to the fact that only the cell wall matrix is assumed to be viscoelastic and microfibrils are assumed to be elastic, induces additional technical difficulties.

A multiscale analysis of the viscoelastic equations with time-independent coefficients was considered previously in [12, 13, 15, 21]. Macroscopic equations for scalar elastic-viscoelastic equations with time-independent coefficients were derived in [11] by applying the H-convergence method [19]. A microscopic viscoelastic Kelvin–Voigt model with time-dependent coefficients in the context of thermo-viscoelasticity was analyzed in [1]. Macroscopic equations were derived by applying the method of asymptotic expansion.

The paper is organised as follows. In Section 1 we formulate a mathematical model for plant cell wall biomechanics in which the cell wall matrix is assumed to be viscoelastic. In Section 2 we summarise the main results of the paper. The well-posedness of the microscopic model is shown in Section 3. The multiscale analysis of the microscopic model is conducted in Section 4. Since we assume that only the cell wall matrix exhibits viscoelastic behaviour and microfibrils are elastic, the viscous tensor is zero in the domain occupied by the microfibrils. This fact causes technical difficulties in the multiscale analysis of the microscopic model. To derive the macroscopic equations for the elastic-viscoelastic model for cell wall biomechanics we first consider perturbed equations by introducing an inertial term. Then, letting the perturbation parameter in the macroscopic model tend to zero, we obtain the effective homogenized equations for the original elastic-viscoelastic model.

1. Microscopic model for viscoelastic deformations of plant cell walls

It was observed experimentally that in addition to elastic deformations the plant cell wall matrix exhibit viscoelastic behaviour [14]. Hence, in contrast to the problem considered in [26], here we assume that the deformation in the plant cell wall matrix is determined by the equations of linear viscoelasticity.

We consider a domain \( \Omega = (0,a_1) \times (0,a_2) \times (0,a_3) \) representing a part of a plant cell wall, where \( a_i, i = 1,2,3 \), are positive numbers and the microfibrils are oriented in the \( x_3 \)-direction, see Fig. 1(a). The part of \( \partial \Omega \) of the exterior of the cell wall is given by \( \Gamma_E = \{ a_1 \times (0,a_2) \times (0,a_3), \) and the interior boundary \( \Gamma_I \) of the cell wall is given by \( \Gamma_I = \{ 0 \times (0,a_2) \times (0,a_3). \) The top and bottom boundaries are defined by \( \Gamma_U = (0,a_1) \times \{ 0 \times (0,a_3) \cup (0,a_1) \times \{ a_2 \} \times (0,a_3). \)

To determine the microscopic structure of the cell wall, we consider \( Y = (0,1)^2 \times (0,a_3) \) and define \( \hat{Y} = (0,1)^2, \) and a subdomain \( \hat{Y}_F \) with \( \hat{Y}_F \subset \hat{Y} \) and \( \hat{Y}_M = \hat{Y} \setminus \hat{Y}_F, \) see Fig. 1(b). Then \( Y_F = \hat{Y}_F \times (0,a_3) \) and \( Y_M = \hat{Y}_M \times (0,a_3) \) represent the cell wall microfibrils and cell wall matrix. We also define \( \Gamma = \partial \hat{Y}_F \) and \( \hat{Y} = \partial \hat{Y}_F. \)

We assume that the microfibrils in the cell wall are distributed periodically and have a diameter on the order of \( \varepsilon \), where the small parameter \( \varepsilon \) characterise the size of the microstructure. The domains

\[
\Omega_F = \bigcup_{\xi \in \mathbb{Z}^2} \{ \varepsilon(\hat{Y}_F + \xi) \times (0,a_3) | \varepsilon(\hat{Y} + \xi) \subset (0,a_1) \times (0,a_2) \} \quad \text{and} \quad \Omega_M = \Omega \setminus \Omega_F
\]

denote the part of \( \Omega \) occupied by the microfibrils and by the cell wall matrix, respectively. The boundary between the matrix and the microfibrils is denoted by

\[
\Gamma = \partial \Omega_M \cap \partial \Omega_F.
\]

We adopt the following notation: \( \Omega_T = (0,T) \times \Omega, \Omega_{M,T} = (0,T) \times \Omega_M, \Gamma_{Z,T} = (0,T) \times \Gamma_Z, \Gamma_T = (0,T) \times \Gamma, \Gamma_{U,T} = (0,T) \times \Gamma_U, \Gamma_{E,T} = (0,T) \times \Gamma_E, \) and \( \Gamma_{EU,T} = (0,T) \times (\Gamma_E \cup \Gamma_U) \), and define

\[
W(\Omega) = \{ u \in H^1(\Omega; \mathbb{R}^3) | \int_\Omega u \, dx = 0, \int_\Omega [\nabla u]_{12} - (\nabla u)_{21} \, dx = 0 \text{ and } u \text{ is } a_3\text{-periodic } x_3 \},
\]

\[
V(\Omega_M) = \{ n \in H^1(\Omega_M) | n \text{ is } a_3\text{-periodic in } x_3 \}.
\]
By Korn’s second inequality, the $L^2$-norm of the strain
\[
\|u\|_{W(\Omega)} = \|e(u)\|_{L^2(\Omega)} \quad \text{for all } u \in W(\Omega)
\]
defines a norm on $W(\Omega)$, see [6, 17, 22]. For more details see also [26].

The microscopic model for elastic-viscoelastic deformations $u^\varepsilon$ of cell walls and the densities of enzyme and pectins: esterified pectin $p^1$, PME enzyme $p^2$, de-esterified pectin $n^1$, calcium ions $n^2$, and calcium-pectin cross-links $b^\varepsilon$ reads
\[
\begin{align*}
\text{div}(\mathbf{E}(n^\varepsilon, x)e(u^\varepsilon) + \mathbf{V}(n^\varepsilon, x)\partial_b u^\varepsilon)) &= 0 \quad \text{in } \Omega, \\
(\mathbf{E}(n^\varepsilon, x)e(u^\varepsilon) + \mathbf{V}(n^\varepsilon, x)\partial_b u^\varepsilon))\nu &= -p^\varepsilon\nu \quad \text{on } \Gamma_{\varepsilon,T}, \\
(\mathbf{E}(n^\varepsilon, x)e(u^\varepsilon) + \mathbf{V}(n^\varepsilon, x)\partial_b u^\varepsilon))\nu &= f \quad \text{on } \Gamma_{\varepsilon,T}, \\
\partial_t u^\varepsilon &= a_3\text{-periodic in } x_3, \\
u^\varepsilon(0, x) &= u_0(x) \quad \text{in } \Omega,
\end{align*}
\]
and in the cell wall matrix $\Omega_M$ we consider
\[
\begin{align*}
\partial_t p^\varepsilon &= \text{div}(D_p \nabla p^\varepsilon) - F_p(p^\varepsilon) \\
\partial_t n^\varepsilon &= \text{div}(D_n \nabla n^\varepsilon) + F_n(p^\varepsilon, n^\varepsilon) + R_n(n^\varepsilon, b^\varepsilon, N_\delta(e(u^\varepsilon)))) \\
\partial_t b^\varepsilon &= R_b(n^\varepsilon, b^\varepsilon, N_\delta(e(u^\varepsilon))),
\end{align*}
\]
where $p^\varepsilon = (p^1, p^2)^T$, $n^\varepsilon = (n^1, n^2)^T$, div($D_p \nabla p^\varepsilon$) = (div($D_p^1 \nabla p^1$), div($D_p^2 \nabla p^2$))$^T$, and div($D_n \nabla n^\varepsilon$) = (div($D_n^1 \nabla n^1$), div($D_n^2 \nabla n^2$))$^T$, together with the initial and boundary conditions
\[
\begin{align*}
D_p \nabla p^\varepsilon \nu &= J_p(p^\varepsilon) \quad \text{on } \Gamma_{\varepsilon,T}, \\
D_p \nabla p^\varepsilon \nu &= -\gamma_p p^\varepsilon \quad \text{on } \Gamma_{\varepsilon,T}, \\
D_n \nabla n^\varepsilon \nu &= N_\delta(e(u^\varepsilon))G(n^\varepsilon) \quad \text{on } \Gamma_{\varepsilon,T}, \\
D_n \nabla n^\varepsilon \nu &= J_n(n^\varepsilon) \quad \text{on } \Gamma_{\varepsilon,T}, \\
D_p \nabla p^\varepsilon \nu &= 0, \quad D_n \nabla n^\varepsilon \nu &= 0, \quad \text{on } \Gamma_T \text{ and } \Gamma_{\varepsilon,T}, \\
\partial_t p^\varepsilon, \ n^\varepsilon, \ b^\varepsilon &= a_3\text{-periodic in } x_3, \\
p^\varepsilon(0, x) &= p_0(x), \quad n^\varepsilon(0, x) = n_0(x), \quad b^\varepsilon(0, x) = b_0(x) \quad \text{in } \Omega_M.
\end{align*}
\]
Here $N_\delta(e(u^\varepsilon))$, defined as
\[
N_\delta(e(u^\varepsilon)) = \left(\int_{\Omega_T} \text{tr}(\mathbf{E}^\varepsilon(b^\varepsilon)e(u^\varepsilon))d\bar{x}\right)^+ \quad \text{in } \Omega_T, \quad \text{for } \delta > 0,
\]
represent the nonlocal impact of mechanical stresses on the calcium-pectin cross-links chemistry. From a biological point of view the non-local dependence of the chemical reactions on the displacement gradient is motivated by the fact that pectins are very long molecules and hence cell wall mechanics has a nonlocal impact on the
chemical processes. The positive part in the definition of $N_b(e(u^{r}))$ reflects the fact that extension rather than compression causes the breakage of cross-links. In the boundary conditions [3] we assumed that the flow of calcium ions between the interior of the cell and the cell wall depends on the displacement gradient, which corresponds to the stress-dependent opening of calcium channels in the plasma membrane [25].

The elasticity and viscosity tensors are defined as $E = E_{ijlk}(u) = \frac{1}{2}(\partial_i u^j + \partial_j u^i)$ and $\eta = \eta_{ij}(u) = \frac{1}{2}(\partial_i u^j - \partial_j u^i)$, for all $\xi, \eta \in \mathbb{R}^3$ and some $\gamma_0, \gamma_0 > 0$.

**Assumption 1.**

1. $D_1, D_2 \in \mathbb{R}^{3 \times 3}$ are symmetric, with $(D_1^T \xi, \xi) \geq d_1|\xi|^2$, $(D_2 \xi, \xi) \geq d_2|\xi|^2$ for all $\xi \in \mathbb{R}^3$ and some $d_1, d_2 > 0$, where $\alpha = p, n, j = 1, 2, \gamma_0, \gamma_0 > 0$.

2. $F_p : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $T_{\mu}^T$, with $F_p(0, 0) = 0$, $F_p(\xi, \eta) \geq 0$, and $|F_p(\xi, \eta)| \leq g_1(\xi)(1 + \eta)$ for all $\xi, \eta \in \mathbb{R}^2$ and some $g_1 \in C^1(\mathbb{R}^2: \mathbb{R}^2)$.

3. $J_p : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $T_{\mu}^T$, with $J_p(0, 0) \geq 0$, $J_p(\xi, \eta) \geq 0$, and $|J_p(\xi, \eta)| \leq g_2(\xi)(1 + \eta)$ for all $\xi, \eta \in \mathbb{R}^2$ and some $g_2 \in C^1(\mathbb{R}^2: \mathbb{R}^2)$.

4. $F_n : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $T_{\mu}^T$, with $F_n(0, 0) \geq 0$, $F_n(\xi, \eta, 0) \geq 0$, and $|F_n(\xi, \eta)| \leq g_3(\xi + |\eta|)$ for all $\xi, \eta \in \mathbb{R}^2$ and some $g_3 \in C^1(\mathbb{R}^2: \mathbb{R}^2)$.

5. $R_n : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ and $R_\mu : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable in $T_{\mu}^T \times \mathbb{R}^d$ and $R_\mu \in C^1(\mathbb{R}^d: \mathbb{R})$.

6. $J_n : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $T_{\mu}^T$, with $J_n(0, 0) \geq 0$, $J_n(\xi, \eta) \geq 0$, and $|J_n(\xi, \eta)| \leq g_4(\xi)(1 + \eta)$ for all $\xi, \eta \in \mathbb{R}^2$ and some $g_4 \in C^1(\mathbb{R}^2: \mathbb{R}^2)$.

7. $G(\xi, \eta) : \mathbb{R}^2 \to \mathbb{R}^2$, with $G(\xi, \eta) = (0, \gamma_0 - \gamma_0 \eta)^T$ for $\eta \in \mathbb{R}$ and some $\gamma_0, \gamma_0 > 0$.

8. $\nu_M \in C^1(\mathbb{R})$ possesses major and minor symmetries, i.e., $\nu_{M,ijkl} = \nu_{M,ikjl} = \nu_{M,jikl}$, and there exists $\nu > 0$ such that $\nu_M(\xi) A \cdot A \geq \omega_Q|A|^2$ for all symmetric $A \in \mathbb{R}^{3 \times 3}$ and $\xi \in \mathbb{R}^d$.

9. $\nu_F \in C^1(\mathbb{R})$, $\nu_F, \nu_M$ possess major and minor symmetries, i.e., $\nu_{F,ijkl} = \nu_{L,ijkl} = \nu_{L,jikl} = \nu_{L,jikl}$, for $L = F, M$, and there exists $\omega_P > 0$ such that $\nu_F A \cdot A \geq \omega_P|A|^2$ and $\nu_M(\xi) A \cdot A \geq \omega_Q|A|^2$ for all symmetric $A \in \mathbb{R}^{3 \times 3}$ and $\xi \in \mathbb{R}^d$.

10. The initial conditions $\nu_n, \nu_0 \in L^\infty(\Omega)^2, b_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ are non-negative, and $u_0 \in W(\Omega)$.

11. $f \in H^1(0, T; L^2(\Gamma_1 \cup \Gamma_2))$ and $p \in H^1(0, T; L^2(\Gamma_2))$.

**Remark.** Notice that Assumption 3 is not restrictive from a physical point of view, since every biological material will have a maximal possible stiffness. Also, in contrast to [25], we assume that $(R_\mu(\xi, \eta))^T = R_\mu(\xi, \eta)$ is bounded. This is required to show a priori estimates for solutions of equations of linear viscoelasticity independent of $\nu^r$.

**Definition 1.1.** A weak solution of the microscopic model (1)-(3) are functions $p^r, n^r$, and $b^r$ such that $b^r \in H^1(0, T; L^2(\Omega_M))$, $p^r, n^r \in L^2(0, T; V(\Omega_M))$, $\partial_t p^r, \partial_t n^r \in L^2(0, T; V(\Omega_M)^T)$, and satisfy the equations

$$\begin{align*}
\langle \partial_t p^r, \phi_p \rangle_{V, V^T} + \langle D_p \nabla p^r, \nabla \phi_p \rangle_{\Omega_M, T} & = -\langle F_p(p^r), \phi_p \rangle_{\Omega_M, T} + \langle J_p(p^r), \phi_p \rangle_{\Gamma_T} - \langle p^r, \phi_p \rangle_{\Gamma_T}, \\
\langle \partial_t n^r, \phi_n \rangle_{V, V^T} + \langle D_n \nabla n^r, \nabla \phi_n \rangle_{\Omega_M, T} & = \langle F_n(p^r, n^r) + R_n(n^r, b^r, N_b(u^{r})), \phi_n \rangle_{\Omega_M, T} \\
& + \langle \nu_\beta(u^{r}) G(n^r), \phi_n \rangle_{\Omega_T} + \langle J_n n^r, \phi_n \rangle_{\Gamma_T},
\end{align*}$$

for all $\phi_p, \phi_n \in L^2(0, T; V(\Omega_M))$.

$$\begin{align*}
\partial_t b^r & = R_b(n^r, b^r, N_b(u^{r})) \quad \text{a.e. in } \Omega_M, T,
\end{align*}$$

and $u^r \in L^2(0, T; W(\Omega))$, with $\partial_t u^r(b^r) \in L^2((0, T) \times \Omega_M)$, satisfying

$$\begin{align*}
\begin{align*}
\langle E^r(b^r, x) e(u^{r}) + \nu^r(b^r, x) \partial_t e(u^{r}), e(\psi) \rangle_{\Gamma_T} & = \langle f, \psi \rangle_{\Gamma_T} - \langle p^r e(u^{r}), \psi \rangle_{\Gamma_T},
\end{align*}
\end{align*}$$

for all $\psi \in L^2(0, T; W(\Omega))$. Furthermore, $p^r, n^r, b^r$ satisfy the initial conditions in $L^2(\Omega_M^T)$ and $u^r$ satisfies the initial condition in $W(\Omega)$, i.e., $u^r(t, -) \to u_0$ in $W(\Omega)$, $p^r(t, -) \to p_0$, $n^r(t, -) \to n_0$, $b^r(t, -) \to b_0$ in $L^2(\Omega_M^T)$ as $t \to 0$.
2. Main results

The main result of the paper is the derivation of the macroscopic equations for the microscopic viscoelastic model for plant cell wall biomechanics. The main difference between the homogenization results presented here and those in [20] is due to the presence of degenerate viscous term in the equation for mechanical deformations of a cell wall. The fact that only the cell wall matrix is viscoelastic and the dependence of the viscosity tensor on the time variable, via the dependence on the cross-links density $b^e$, make the multiscale analysis nonclassical and complex.

First we formulate the well-posedness result for the model (1)–(3).

**Theorem 2.1.** Under Assumptions 7 there exists a unique weak solution of (1)–(3) satisfying the a priori estimates

\[
\|b^e\|_{L^\infty(0,T;L^\infty(\Omega_M^e))} + \|\partial_t b^e\|_{L^\infty(0,T;L^\infty(\Omega_M^e))} \leq C_1,
\]

where the constant $C_1$ is independent of $\varepsilon$ and $\delta$,

\[
\|u^e\|_{L^\infty(0,T;W(\Omega^e))} + \|\partial_t u^e\|_{L^2(0,T;L^2(\Omega^e_M))} \leq C_2,
\]

where the constant $C_2$ is independent of $\varepsilon$ and $\delta$, and

\[
\|\partial_t b^e\|_{L^\infty(0,T;L^\infty(\Omega_M^e))} \leq C_3,
\]

\[
\|\nabla b^e\|_{L^\infty(0,T;L^\infty(\Omega_M^e))} + \|\nabla u^e\|_{L^\infty(0,T;L^\infty(\Omega_M^e))} + \|\nabla v^e\|_{L^2(\Omega_M^e)} \leq C_4,
\]

\[
\|\theta^e b^e - \theta^e\|_{L^2(\Omega_M^e)} \leq C_5\theta^{1/4}
\]

for any $h > 0$, where $\theta^e(t,x) = \theta(t + h, x)$ for $(t, x) \in \Omega_M^e = \Omega_M^e - \text{h}$ and the constant $C_5$ is independent of $\varepsilon$ and $h$.

The proof of Theorem 2.1 follows similar lines as the proof of the corresponding existence and uniqueness results in [20]. Thus here we will only sketch the main ideas of the proof and emphasise the steps that are different from those of the proof in [20].

To formulate the macroscopic equations for the microscopic model (1)–(3), first we define the macroscopic coefficients which will be obtained by the derivation of the limit equations. The macroscopic coefficients coming from the elasticity tensor are given by

\[
\hat{E}_{\text{hom},ijkl}(b) = \int_{\hat{Y}} \left[ E_{ijkl}(b, y) + (E(b, y)\hat{e}_b(w^{ij}))_{kl} \right] dy,
\]

\[
\hat{E}_{ijkl}(t, s, b) = \int_{\hat{Y}} \left[ E(b(t + s), y)\hat{e}_b(y^i(t, s))_{kl} \right] dy,
\]

and the macroscopic elasticity and viscosity tensors and memory kernel read:

\[
\hat{E}_{\text{hom},ijkl}(b) = \frac{1}{|Y|} \int_{Y_M} \left[ \mathcal{V}_M(b, y)\hat{e}_b(w^{ij})_{kl} \right] dy,
\]

\[
\hat{V}_{ijkl}(t, s, b) = \frac{1}{|Y|} \int_{Y_M} \left[ \mathcal{V}_M(b(t + s), y)\hat{e}_b(y^i(t, s))_{kl} \right] dy,
\]

where $w^{ij}$, $\chi^{ij}$, and $y^{ij}$ are solutions of the unit cell problems

\[
\text{div}_y \left[ E(b, y)\hat{e}_b(w^{ij}) + b_{ij} + \mathcal{V}_M(b, y)\hat{e}_b(w^{ij})_{XY} \right]_Y = 0 \quad \text{in} \ Y_T,
\]

\[
\text{div}_y \left[ \mathcal{V}_M(b, y)\hat{e}_b(w^{ij})_Y + b_{ij} \right] = 0 \quad \text{in} \ Y_M,
\]

\[
\text{div}_y \left[ \mathcal{V}_M(b(t + s), y)\hat{e}_b(y^{ij}(t, s)) + b_{ij} \right] = 0 \quad \text{on} \ \hat{\Gamma},
\]

\[
\int_{\hat{Y}} w^{ij} dy = 0, \quad \int_{Y_M} \chi^{ij} dy = 0, \quad w^{ij}, \chi^{ij} \quad \hat{Y}\text{-periodic}
\]

where $b_{jk} = \frac{1}{2}(b_1 \otimes b_k + b_k \otimes b_1)$, with $(b_j)_{1 \leq j \leq 3}$ being the canonical basis of $\mathbb{R}^3$, and

\[
\text{div}_y \left[ \mathcal{V}_M(b(t + s), y)\hat{e}_b(v^{ij}) + \mathcal{V}_M(b(t + s), y)\hat{e}_b(v^{ij})_{XY} \right] = 0 \quad \text{in} \ Y_{T-s},
\]

\[
v^{ij}(0, s, x, y) = \chi^{ij}_0(s, x, y) - w^{ij}(s, x, y) \quad \hat{Y}\text{-periodic}
\]

\[
\int_{\hat{Y}} v^{ij} dy = 0.
\]
Remark. Notice that the microfibrils do not intersect the boundaries \( \Gamma \) where the constant \( (20) \), we obtain the following lemma.

Theorem 2.2. A sequence of solutions of the microscopic model \([1] - [3]\) converges to a solution of the macroscopic equations

\[
\begin{align*}
\partial_t p &= \text{div}(D_p \nabla p) - F_p(p), \\
\partial_t n &= \text{div}(D_n \nabla n) + F_n(p, n) + R_n(n, b, \mathcal{N}_\delta^\text{eff}(e(u))), \\
\partial_t b &= R_b(n, b, \mathcal{N}_\delta^\text{eff}(e(u)))
\end{align*}
\]

in \( \Omega_T \) together with the initial and boundary conditions

\[
\begin{align*}
D_p \nabla p &= \theta_M^{-1} J_p(p), & D_n \nabla n &= \theta_M^{-1} G(n) \mathcal{N}_\delta^\text{eff}(e(u)) & \text{on } \Gamma_{\text{IT}}, \\
D_p \nabla p &= -\theta_M^{-1} \gamma_p p, & D_n \nabla n &= \theta_M^{-1} J_n n & \text{on } \Gamma_{\text{ET}}, \\
D_p \nabla p &= 0, & D_n \nabla n &= 0 & \text{on } \Gamma_{\text{IT}}, \\
p(0) = p_0(x), & n(0) = n_0, & b(0) = b_0 & \text{in } \Omega,
\end{align*}
\]

where \( \theta_M = |\hat{Y}_M|/|\hat{Y}| \), and the macroscopic equations of linear viscoelasticity

\[
\begin{align*}
\text{div} \left( \mathbb{E}_{\text{hom}}(e(u)) + \nabla \text{hOM}_\delta \partial_t e(u) + \int_0^t K(t-s, s) \partial_s e(u) \, ds \right) &= 0 & \text{in } \Omega_T, \\
(\mathbb{E}_{\text{hOM}}(e(u)) + \nabla \text{hOM}_\delta \partial_t e(u) + \int_0^t K(t-s, s) \partial_s e(u) \, ds) \nu &= f & \text{on } \Gamma_{\text{ET}},
\end{align*}
\]

(19)

\[
(\mathbb{E}_{\text{hOM}}(e(u)) + \nabla \text{hOM}_\delta \partial_t e(u) + \int_0^t K(t-s, s) \partial_s e(u) \, ds) \nu = -p_T \nu & \text{on } \Gamma_{\text{IT}}, \\
u(0, x) = u_0(x) & \text{in } \Omega.
\]

Here

\[
\mathcal{N}_\delta^\text{eff}(e(u)) = \left( \int_{B_j(x) \cap \Omega} \text{tr} \left[ \mathbb{E}_{\text{hOM}}(b) e(u) + \int_0^t \mathbb{K}(t-s, s, b) \partial_s e(u) \, ds \right] \, d\mathbf{x} \right)^+ \quad \text{for all } (t, x) \in (0, T) \times \overline{\Omega}.
\]

(20)

3. Existence of a unique weak solution of the microscopic problem \([1] - [3]\). A priori estimates.

In the derivation of a priori estimates for solutions of the microscopic problem \([1] - [3]\) we shall use an extension of a function defined on a connected perforated domain \( \Omega_M^\varepsilon \) to \( \Omega \). Applying classical extension results \([2, 3, 25]\), we obtain the following lemma.

Lemma 3.1. There exists an extension \( \hat{v}^\varepsilon \) of \( v^\varepsilon \) from \( W^{1,p}(\Omega_M^\varepsilon) \) into \( W^{1,p}(\Omega) \), with \( 1 \leq p < \infty \), such that

\[
|\hat{v}^\varepsilon|_{L^p(\Omega)} \leq \mu_1 |v^\varepsilon|_{L^p(\Omega_M^\varepsilon)} \quad \text{and} \quad \|\nabla \hat{v}^\varepsilon\|_{L^p(\Omega)} \leq \mu_1 \|\nabla v^\varepsilon\|_{L^p(\Omega_M^\varepsilon)},
\]

where the constant \( \mu_1 \) depends only on \( Y \) and \( Y_M^\varepsilon \), and \( Y_M^\varepsilon \subset Y \) is connected.

Remark. Notice that the microfibrils do not intersect the boundaries \( \Gamma_{\text{IT}}, \Gamma_{\text{ET}}, \) and \( \Gamma_{\text{ET}} \), and near the boundaries \( \partial \Omega \setminus (\Gamma_{\text{IT}} \cup \Gamma_{\text{ET}}) \) it is sufficient to extend \( v^\varepsilon \) by reflection in the direction normal to the microfibrils and parallel to the boundary. Thus, classical extension results \([2, 3, 13, 25]\) apply to \( \Omega_M^\varepsilon \). In the sequel, we identify \( p^\varepsilon \) and \( n^\varepsilon \) with their extensions.
First we show the well-posedness and a priori estimates for equations \(2\) and \(3\), for a given \(u^\varepsilon \in L^\infty(0, T; \mathbb{W}(\Omega))\). Next for a given \(b^\varepsilon\) we show the existence of a unique solution of the viscoelastic problem \(\text{(1)}\). Then using the fact that the estimates for \(b^\varepsilon\) can be obtained independently of \(u^\varepsilon\) and applying a fixed point argument we show the well-posedness of the coupled system.

**Lemma 3.2.** Under Assumption \(\text{(7)}\) and for \(u^\varepsilon \in L^\infty(0, T; \mathbb{W}(\Omega))\) such that
\[
\|u^\varepsilon\|_{L^\infty(0, T; \mathbb{W}(\Omega))} \leq C,
\]
where the constant \(C\) is independent of \(\varepsilon\), there exists a unique weak solution \((p^\varepsilon, n^\varepsilon, b^\varepsilon)\) of the microscopic model \(\text{(2)}\)–\(\text{(3)}\) satisfying
\[
p^\varepsilon(t, x) \geq 0, \quad n^\varepsilon(t, x) \geq 0, \quad b^\varepsilon(t, x) \geq 0 \quad \text{for a.a. } (t, x) \in (0, T) \times \Omega_M, \quad j = 1, 2,
\]
and the a priori estimates \(\text{(8)}\) and \(\text{(10)}\).

**Proof.** The proof of this lemma follows the same lines as the proof of Theorem 3.3 in \(\text{(20)}\). The only difference is in the derivation of the estimates for \(b^\varepsilon\). Using the non-negativity of \(n^\varepsilon_j, n^\varepsilon_3, b^\varepsilon\), and Assumptions \(\text{(14)}\) and \(\text{(15)}\) we obtain from the equation for \(b^\varepsilon\)
\[
0 \leq b^\varepsilon(t, x) \leq \|b_0\|_{L^\infty(\Omega)} + T\|(R_b(n^\varepsilon, b^\varepsilon, N_b(e(u^\varepsilon))))^\varepsilon\|_{L^\infty(\Omega_M, T)} \leq C \quad \text{for a.a. } (t, x) \in \Omega_M, \quad j = 1, 2,
\]
and the a priori estimates \(\text{(8)}\) and \(\text{(10)}\).

Hence, the bounds for \(b^\varepsilon\) and \((\partial_t b^\varepsilon)^+\) are independent of the bound for \(\|u^\varepsilon\|_{L^\infty(0, T; \mathbb{W}(\Omega))}\). This fact is important for the derivation of a priori estimates for \(u^\varepsilon\) and the fixed point argument for the proof of the existence of a solution for the coupled system.

Using the equation for \(b^\varepsilon\), the definition of \(N_b\) and the estimates for \(\|n^\varepsilon_j\|_{L^\infty(0, T; \mathbb{W}(\Omega))}, \|b^\varepsilon\|_{L^\infty(0, T; \mathbb{W}(\Omega))}, \) and \(\|u^\varepsilon\|_{L^\infty(0, T; \mathbb{W}(\Omega))}\), we obtain the estimate for \(\|\partial_t b^\varepsilon\|_{L^\infty(0, T; \mathbb{W}(\Omega))}\) uniformly in \(\varepsilon\).

Similar to \(\text{(20)}\), considering \(\phi_p = \int_0^{t+h}[\theta_0 p^\varepsilon(s, x) - p^\varepsilon(s, x)]ds\) and \(\phi_n = \int_0^{t+h}[\theta_1 n^\varepsilon(s, x) - n^\varepsilon(s, x)]ds\) as test functions in \(\text{(5)}\), respectively, we obtain the last estimate in \(\text{(10)}\).

Next we prove the existence, uniqueness and a priori estimates for a solution of viscoelastic equations for a given \(b^\varepsilon \in L^\infty(0, T; L^\infty(\Omega_M))\).

**Lemma 3.3.** Under Assumption \(\text{(7)}\) for a given \(b^\varepsilon \in L^\infty(0, T; \mathbb{W}(\Omega_M))\), satisfying
\[
\|b^\varepsilon\|_{L^\infty(0, T; L^\infty(\Omega_M))} + \|\partial_t b^\varepsilon\|^+_{L^\infty(0, T; L^\infty(\Omega_M))} \leq C;
\]
there exists a weak solution of the degenerate viscoelastic equations \(\text{(1)}\) satisfying the a priori estimates \(\text{(8)}\).

**Proof.** Using the estimates for \(u^\varepsilon\) and \(\partial_t u^\varepsilon\), similar to those in \(\text{(23)}\), along with the positive definiteness of \(E\) and \(V\), and applying the Galerkin method, yield the existence of a weak solution of the problem \(\text{(1)}\).

Considering \(\partial_t u^\varepsilon\) as a test function in \(\text{(7)}\) and using the non-negativity of \(b^\varepsilon\) and the assumptions on \(E\) and \(V\), we obtain
\[
\|\varepsilon \|_{L^2(\Omega_M, T)}^2 + \|\partial_t \varepsilon\|_{L^2(\Omega_M, T)}^2 \leq \|\partial_t b^\varepsilon\|_{L^\infty(\Omega_M, T)}^2 + \|\partial_t e(u^\varepsilon)\|_{L^2(\Omega_M, T)}^2 + C_1\|e(u_0)\|_{L^2(\Omega)}^2 + C_2\|f_\tau\|_{L^2(\Omega)}^2 + C_3\|p_\tau\|_{L^2(\Omega)}^2 + C_4\|p_\tau\|_{L^2(\Omega)}^2 + C_5\|p_\tau\|_{L^2(\Omega)}^2 + C_6\|p_\tau\|_{L^2(\Omega)}^2
\]
for \(\tau \in [0, T]\). Choosing \(\sigma\) sufficiently small, using the boundedness of \(b^\varepsilon\) and \((\partial_t b^\varepsilon)^+\) independent of \(\varepsilon\) and \(u^\varepsilon\), and applying Gronwall’s inequality imply
\[
\|\varepsilon \|_{L^2(\Omega_M, T)}^2 + \|\partial_t \varepsilon\|_{L^2(\Omega_M, T)}^2 \leq C,
\]
with a constant \(C\) independent of \(\varepsilon\). Then using the second Korn inequality yields \(\text{(9)}\).

Now applying a fixed point argument and using the results in Lemmas \(\text{(3.2)}\) and \(\text{(3.3)}\) we obtain the well-posedness result for the coupled system \(\text{(1)}\)–\(\text{(3)}\).

**Proof of Theorem \(\text{(2.1)}\)** We have that for a given \(\tilde{u}^\varepsilon \in L^\infty(0, T; \mathbb{W}(\Omega))\), with \(\|\tilde{u}^\varepsilon\|_{L^\infty(0, T; \mathbb{W}(\Omega))} \leq C\), Lemma \(\text{(3.2)}\) implies the existence of a non-negative weak solution \((p^\varepsilon, n^\varepsilon, b^\varepsilon)\) of the problem \(\text{(2)}\)–\(\text{(3)}\), where the estimates for \(\|b^\varepsilon\|_{L^\infty(0, T; L^\infty(\Omega_M))}\) and \(\|\partial_t b^\varepsilon\|^+_{L^\infty(0, T; L^\infty(\Omega_M))}\) are independent of \(\tilde{u}^\varepsilon\). Then for \(b^\varepsilon\) from Lemma \(\text{(3.3)}\) we have a solution \(u^\varepsilon\) of \(\text{(1)}\).

We define \(K : L^\infty(0, T; \mathbb{W}(\Omega)) \to L^\infty(0, T; \mathbb{W}(\Omega))\) by \(K(u^\varepsilon) = u^\varepsilon\), where \(u^\varepsilon\) is a solution of \(\text{(1)}\) for \(b^\varepsilon\) which is a solution of \(\text{(2)}\)–\(\text{(3)}\) with \(\tilde{u}^\varepsilon\) instead of \(u^\varepsilon\), and show that for sufficiently small \(T \in (0, T]\), the operator \(K : L^\infty(0, T; \mathbb{W}(\Omega)) \to L^\infty(0, T; \mathbb{W}(\Omega))\) is a contraction, i.e.
\[
\|K(u^\varepsilon) - K(\tilde{u}^\varepsilon)\|_{L^\infty(0, T; \mathbb{W}(\Omega))} \leq \gamma\|u^\varepsilon_2 - u^\varepsilon_1\|_{L^\infty(0, T; \mathbb{W}(\Omega))}, \quad \text{for some } 0 < \gamma < 1.
\]
Considering the difference of equation (7) for \( b^f \) and \( b^c \), and taking \( \partial_t (u^{f,1} - u^{c,2}) \) as a test function yield

\[
\left( \mathbb{E}^f (b^f, x) e(u^{f,1} - u^{c,2}), \partial_t (u^{f,1} - u^{c,2}) \right)_{\Omega} + \left( \mathbb{V}^f (b^f, x) \partial_t e(u^{f,1} - u^{c,2}), \partial_t e(u^{f,1} - u^{c,2}) \right)_{\Omega} = \left( (\mathbb{E}^f_M (b^f, x) - \mathbb{E}^c_M (b^f, x)) e(u_2^f), \partial_t e(u^{f,1} - u^{c,2}) \right)_{\Omega},
\]

for \( t \in (0, T) \). By the assumptions on \( \mathbb{E}^f (b^f, x) \) and \( \mathbb{V}^f (b^f, x) \), we have

\[
\| e(u^{f,1} - u^{c,2}) \|_{L^2(\Omega)}^2 \leq C_1 \left( \| \partial_t b^{f,1}^\circ \|_{L^\infty(0,T;L^\infty(\Omega_M))} \right)^{\alpha} \left( \int_0^T \| e(u^{f,1} - u^{c,2}) \|_{L^2(\Omega_M)}^2 \, dt \right)^{1/2} + C_2 \left( \| e(u^{c,2}) \|_{H^1(0,T;L^2(\Omega_M))} \right)^{\beta},
\]

for \( \widetilde{T} \in (0, T] \). Following the same calculations as in [20], by taking \( \phi_n = n^{f,1} - n^{c,2} \), where \( p = 2\kappa \) and \( \kappa = 1, 2, 3, \ldots \), as test functions in the differences of the equations for \( n^{f,1} \) and \( n^{c,2} \), and applying iterations in \( p \) similar to Lemma 3.2 in Alikakos [3], we obtain

\[
\| b^{f,1} - b^{c,2} \|_{L^\infty(0,\widetilde{T};L^\infty(\Omega_M))}^2 \leq C_3 \left( \| \partial_t u^{f,1} - \partial_t u^{c,2} \|_{L^\infty(0,\widetilde{T};L^2(\Omega))} \right)^2
\]

for \( \widetilde{T} \in (0, T] \). Thus, we have that the operator \( K : L^\infty(0, T; W(\Omega)) \to L^\infty(0, T; L^\infty(\Omega_M)) \), defined by \( K(u^c) = u^f \), where \( u^f \) is a weak solution of (1), is a contraction for sufficiently small \( \widetilde{T} \), where \( \widetilde{T} \) depends on the coefficients in the equations and is independent of \((p^f, n^f, b^f, u^f)\). Hence, using the Banach fixed point theorem and iterating over time intervals, we obtain the existence of a unique weak solution of the microscopic problem (1)-(3).

4. DERIVATION OF THE MACROSCOPIC EQUATIONS OF THE PROBLEM (1)-(3): PROOF OF THEOREM 2.2

Due to the fact that viscous term is defined only in the cell wall matrix and is zero for cell wall microfibrils, to conduct the multiscale analysis of the viscoelastic problem (1) we first consider a perturbed problem by adding the inertial term \( \vartheta \partial_t^2 u^f \), where \( \vartheta > 0 \) is a small perturbation parameter:

\[
\partial_t \chi_{\Omega_M} \partial_t^2 u^f = \text{div}(\mathbb{E}^f (b^f, x) e(u^f) + \mathbb{V}^f (b^f, x) \partial_t e(u^f)) \quad \text{on } \Omega_T,
\]

and the additional initial condition

\[
\partial_t u^f (0, x) = 0 \quad \text{in } \Omega.
\]

We split the proof of Theorem 2.2 into two steps. First we derive the macroscopic equations for the perturbed system. Then letting the perturbation parameter \( \vartheta \) go to zero we obtain the macroscopic equations (19) for the original degenerate viscoelastic problem.

**Lemma 4.1.** There exists a unique solution of the perturbed microscopic problem (2), (3) and (25), together with the initial and boundary conditions in (1) and (26), satisfying the a priori estimates

\[
\vartheta^2 \| \partial_t u^f \|_{L^\infty(0,T;L^2(\Omega_M))} + \| u^f \|_{L^\infty(0,T;W(\Omega))} + \| \partial_t e(u^f) \|_{L^2(\Omega_M,T)} \leq C,
\]

\[
\| p^f \|_{L^\infty(0,T;L^\infty(\Omega_M))} + \| \nabla p^f \|_{L^2(\Omega_M,T)} + \| n^f \|_{L^\infty(0,T;L^\infty(\Omega_M))} + \| \nabla n^f \|_{L^2(\Omega_M,T)} \leq C,
\]

with a constant \( C \) independent of \( \varepsilon \) and \( \vartheta \), and

\[
\| \theta_h p^f - p^f \|_{L^2(\Omega_M,T-h)} + \| \theta_h n^f - n^f \|_{L^2(\Omega_M,T-h)} \leq C h^{1/4},
\]

where \( \theta_h (t,x) = (t+h,x) \) for a.e. \( (t,x) \in \Omega_M, T-h \), and the constant \( C \) is independent of \( \varepsilon \) and \( \vartheta \).

**Proof.** For a given \( u^f \in L^\infty(0,T;W(\Omega)) \), with \( \| u^f \|_{L^\infty(0,T;W(\Omega))} \leq C \), in the same way as in Lemma 3.2 we obtain the existence of a unique solution of the problem (2), (3), satisfying the a priori estimates (28). Notice that the estimates for \( b^f \) and \( (\partial_t b^f)^+ \) are independent of \( u^f, \varepsilon, \) and \( \vartheta \).

Then for \( b^f \in L^\infty(0,T;L^\infty(\Omega_M)) \), with \( \| b^f \|_{L^\infty(0,T;L^\infty(\Omega_M))} \leq C \) and \( \| (\partial_t b^f)^+ \|_{L^\infty(0,T;L^\infty(\Omega_M))} \leq C \), similar to Lemma 3.3 we obtain the existence of a weak solution of the perturbed equations (25) with initial and boundary conditions in (1) and (26), satisfying the a priori estimates (27).
Similar to the proof of Theorem 2.1 considering the difference of the equations (25) for $b^j$, with $j = 1, 2$, and taking $\partial_t(u^1 - u^2)$ as a test function yield

$$
\frac{1}{2} \frac{d}{d\tau} \| \partial_t(u^1 - u^2)(\tau) \|^2_{L^2(\Omega_M)} + \langle \mathcal{E}(E^1, x) e(u^1 - u^2), \partial_t e(u^1 - u^2) \rangle_{\Omega_M} \\
+ \langle \mathcal{V}(E^1, x) \partial_t e(u^1 - u^2), \partial_t e(u^1 - u^2) \rangle_{\Omega_M} = \langle (\mathcal{E}_M(E^1, x) - \mathcal{E}_M(E^2, x)) e(u^2), \partial_t e(u^1 - u^2) \rangle_{\Omega_M, \tau}
$$

for $\tau \in (0, T]$. By the assumptions on $\mathcal{E}(E^1, x)$ and $\mathcal{V}(E^1, x)$, and applying the Gronwall inequality and the estimates for $\partial_t \hat{b}^j$ and $e(u^2)$ we obtain

$$
\| e(u^1) - e(u^2) \|^2_{L^\infty(0, T; L^2(\Omega))} \leq C_3 \| b^1 - b^2 \|^2_{L^\infty(0, T; L^2(\Omega_M))}
$$

for all $T \in (0, T]$. Then, using the estimates (24), (27), (30), and the a priori estimates for $\nu^\varepsilon$, $\hat{b}^\varepsilon$, and $b^\varepsilon$ in the same way as in the proof of Theorem 2.1, we obtain the existence of a unique weak solution of the perturbed problem (2), (3), and (25), with initial and boundary conditions in (1) and (26).

**Lemma 4.2.** There exist functions $p^\varepsilon, \nu^\varepsilon \in L^2(0, T; \mathcal{V}(\Omega) \cap L^\infty(0, T; L^\infty(\Omega))^2)$, $p^0, \nu^0 \in L^2(\Omega_M; H_{per}^1(\hat{Y}) / \mathbb{R})^2$ and $b^0 \in \mathcal{W}^{1, \infty}(0, T; L^\infty(\Omega))$, $u^0 \in H^1(0, T; \mathcal{W}(\Omega))$, $\nu^0 \in L^2(\Omega_M; H_{per}^1(\hat{Y}) / \mathbb{R})^3$, and $\partial_t \hat{b}^\varepsilon \in L^2(\Omega_M; H_{per}^1(\hat{Y}\mu^\varepsilon) / \mathbb{R})^3$ such that for a subsequence of solutions $(p^\varepsilon, \nu^\varepsilon, b^\varepsilon, u^\varepsilon)$ of the microscopic problem (2), (3), and (25), with initial and boundary conditions in (1) and (26), (denoted again by $(p^\varepsilon, \nu^\varepsilon, b^\varepsilon, u^\varepsilon)$) we have the following convergence results:

$$
p^\varepsilon \to p^0, \quad \nu^\varepsilon \to \nu^0 \quad \text{weakly in } L^2(0, T; H^1(\Omega)),
$$

$$
p^\varepsilon \to p^0, \quad \nu^\varepsilon \to \nu^0 \quad \text{strongly in } L^2(\Omega_M),
$$

$$\begin{align*}
\nabla p^\varepsilon &\to \nabla p^0 + \nabla_y p^0, \\
\nabla \nu^\varepsilon &\to \nabla \nu^0 + \nabla_y \nu^0
\end{align*} \quad \text{two-scale,}
$$

$$
\begin{align*}
b^\varepsilon &\to b^0, \\
\partial_t b^\varepsilon &\to \partial_t b^0
\end{align*} \quad \text{two-scale,}
$$

$$
T^\varepsilon_{\omega}(b^\varepsilon) \to b, \quad \text{strongly in } L^2(\Omega \times Y_M),
$$

$$
u^\varepsilon \to \nu^0, \quad \text{weakly in } L^2(0, T; \mathcal{W}(\Omega)),
$$

$$\nabla \nu^\varepsilon \to \nabla \nu^0 + \nabla_y \nu^0, \quad \text{two-scale,}
$$

$$\chi_{\Omega_M} \nabla \partial_t u^\varepsilon \to \chi_{\hat{Y}\mu^\varepsilon}(\nabla \partial_t u^0 + \nabla_y \partial_t u^0) \quad \text{two-scale.}
$$

Here $T^\varepsilon_{\omega}: L^p(\Omega_M, \mathbb{R}) \to L^p(\Omega_T \times \hat{Y}\mu^\varepsilon)$ is the unfolding operator defined as $T^\varepsilon_{\omega}(\phi)(t, x, y) = \phi(t, \hat{x}/\varepsilon, y)$ for $(t, x) \in \Omega_T$ and $y \in \hat{Y}\mu^\varepsilon$, where $\hat{x} = (x_1, x_2)$ and $[\hat{x}/\varepsilon]_{Y_M}$ is the unique integer combination of the periods such that $\hat{x}/\varepsilon - [\hat{x}/\varepsilon]_{Y_M} \in \hat{Y}\mu^\varepsilon$, see e.g. 8.

**Proof.** A priori estimates in (8) and (10) imply weak and two-scale convergences of $p^\varepsilon$, $\nu^\varepsilon$, $b^\varepsilon$, and $\partial_t b^\varepsilon$. Using the estimates for $p^\varepsilon(t + h, x) - p^\varepsilon(t, x)$ and $\nu^\varepsilon(t + h, x) - \nu^\varepsilon(t, x)$ together with the estimates for $\nabla \nu^\varepsilon$ and $\nabla p^\varepsilon$ in (10) and the properties of the extension of $\nu^\varepsilon$ and $p^\varepsilon$ from $\Omega_M$ to $\Omega$, see Lemma 3.1 and applying the Kolmogorov theorem [9, 20], we obtain the strong convergence of $\nu^\varepsilon$ and $p^\varepsilon$ in $L^2(\Omega_T)$.

In the same way as in (26), we show that, up to a subsequence,

$$
T^\varepsilon_{\omega}(b^\varepsilon) \to b \quad \text{strongly in } L^2(\Omega_T \times Y_M), \quad \text{as } \varepsilon \to 0.
$$

Here we present only the sketch of the calculations. Using the extension of $\nu^\varepsilon$ from $\Omega_M$ to $\Omega$, see Lemma 3.1 we define the extension of $b^\varepsilon$ from $\Omega_M$ to $\Omega$ as a solution of the ordinary differential equation

$$
\partial_t b^\varepsilon = R_0(\nu^\varepsilon, b^\varepsilon, N_0(\epsilon u^\varepsilon)) \quad \text{in } (0, T) \times \Omega,
$$

$$
b^\varepsilon(0, x) = b_0 \quad \text{in } \Omega.
$$

The construction of the extension for $\nu^\varepsilon$ and the uniform boundedness of $\nu_1^\varepsilon, \nu_2^\varepsilon$ in $\Omega_M, T$, see (10), ensure

$$
\| \nu^\varepsilon \|_{L^\infty(0, T; L^\infty(\Omega_M))} \leq C \| \nu^\varepsilon \|_{L^\infty(0, T; L^\infty(\Omega_M))},
$$

with the constant $C$ independent of $\varepsilon$. Hence from (32) we obtain also the boundedness of $b^\varepsilon$ and $\partial_t b^\varepsilon$. We show the strong convergence of $b^\varepsilon$ by applying the Kolmogorov theorem [9, 20]. Considering equation (32) at $(t, x + h)$ and $(t, x)$, where $h_j = h b_j$, with $(b_1, b_2, b_3)$ being the canonical basis in $\mathbb{R}^3$ and $h > 0$, taking
\[ b^\varepsilon(t, x + h_\varepsilon) - b^\varepsilon(t, x) \] as a test function and using the Lipschitz continuity of \( R_b \) yield
\[
\|b^\varepsilon(t, \cdot + h_\varepsilon) - b^\varepsilon(t, \cdot)\|_{L^2(\Omega_{2h})} \leq \|b_0(\cdot + h_\varepsilon) - b_0(\cdot)\|_{L^2(\Omega_{2h})} + C_1 \int_0^T \|b^\varepsilon(t, \cdot + h_\varepsilon) - b(t, \cdot)\|_{L^2(\Omega_{2h})} dt + C_2 \int_0^T (\|n^\varepsilon(t, \cdot + h_\varepsilon) - n(t, \cdot)\|_{L^2(\Omega_{2h})})^2 + \delta^{-6} \int B_{\varepsilon_2}(x) \cdot \Omega \text{ tr } E^\varepsilon(\beta^\varepsilon(t, \tilde{x})) d\tilde{x} \|_2^2 L^2(\Omega_{2h}) \) dt
\]
for \( \tau \in (0, T] \), where \( \Omega_{2h} = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \geq 2h \} \), \( B_{\varepsilon_2}(x) = [B_\varepsilon(x + h_\varepsilon) \setminus B_\varepsilon(x)] \cup [B_\varepsilon(x) \setminus B_\varepsilon(x + h_\varepsilon)] \), and the constants \( C_1, C_2 \) are independent of \( \varepsilon \) and \( h \). Using the regularity of the initial condition \( b_0 \in H^1(\Omega) \), the \textit{a priori} estimates for \( \varepsilon \) and \( \nabla n^\varepsilon \), along with the fact that \( B_{\varepsilon_2}(x) \cap \Omega \leq C\delta^2 h \) for all \( x \in \Omega \), and applying the Gronwall inequality we obtain
\[
\sup_{t \in (0, T)} \|b^\varepsilon(t, \cdot + h_\varepsilon) - b^\varepsilon(t, \cdot)\|_{L^2(\Omega_{2h})} \leq C_3 h.
\]
Extending \( b^\varepsilon \) by zero from \( \Omega_T \) into \( \mathbb{R}_+ \times \mathbb{R}^3 \) and using the uniform boundedness of \( b^\varepsilon \) in \( L^\infty(0, T; L^\infty(\Omega)) \) imply
\[
\|b^\varepsilon\|_{L^\infty(0, T, L^2(\Omega_{2h}))} + \|b^\varepsilon\|_{L^2(0, T-h, \Omega)} \leq C h,
\]
where \( \Omega_{2h} = \{ x \in \mathbb{R}^3 \mid \text{dist}(x, \partial \Omega) \leq 2h \} \) and the constant \( C \) is independent of \( \varepsilon \) and \( h \). The estimates for \( \partial_t b^\varepsilon \)
\[
\|b^\varepsilon(t, \cdot + h_\varepsilon) - b^\varepsilon(t, \cdot)\|_{L^2(\Omega_{2h})} \leq C_1 h^2 \|\partial_t b^\varepsilon\|_{L^2(\Omega_T)} \leq C_2 h^2,
\]
where \( C_1 \) and \( C_2 \) are independent of \( \varepsilon \) and \( h \). Combining (33) and (35) and applying the Kolmogorov theorem yield the strong convergence of \( b^\varepsilon \) to \( b^0 \) in \( L^2(\Omega_T) \). The definition of the two-scale convergence yields that \( \delta^\varepsilon = b^0 \) and hence the two-scale limit of \( b^\varepsilon \) is independent of \( y \). Then using the properties of the unfolding operator, see e.g. [22], we obtain the strong convergence of \( \mathcal{T}_\varepsilon(b^\varepsilon) \).

Considering an extension \( \partial_t w^\varepsilon \) of \( \partial_t u^\varepsilon \) from \( \Omega_{2h} \) into \( \Omega \) and applying the Korn inequality [22] yield
\[
\|\partial_t u^\varepsilon\|_{L^2(0, T, H^1(\Omega_{2h}))} \leq \|\partial_t w^\varepsilon\|_{L^2(0, T, H^1(\Omega_{2h}))} \leq C_1 \|\partial_t u^\varepsilon\|_{L^2(\Omega_T)} + \|e(\partial_t w^\varepsilon)\|_{L^2(\Omega_T)} \leq C_2 \|\partial_t u^\varepsilon\|_{L^2(\Omega_{2h}, T)} + \|e(\partial_t u^\varepsilon)\|_{L^2(\Omega_{2h}, T)} \leq C_3 \delta^{-\frac{1}{2}}\varepsilon,
\]
where the constant \( C_3 \) is independent of \( \varepsilon \) and \( h \).

Estimates (27) and (36) ensure the existence of \( u^0 \in L^2(0, T; W(\Omega)) \), \( u^1 \in L^2(\Omega_T; H^1_{per}(\hat{Y})) \), \( \xi^0 \in L^2(0, T; H^1(\Omega)) \) and \( \xi^1 \in L^2(\Omega_T; H^1_{per}(\hat{Y})) \) such that
\[
\begin{align*}
\text{two-scale,} & \quad u^\varepsilon \rightarrow u^0, \qquad \nabla u^\varepsilon \rightarrow \nabla u^0 + \hat{\nabla}_y u^1, \\
\text{two-scale,} & \quad \chi_{\Omega_M} \nabla \partial_t u^\varepsilon \rightarrow \chi_{\hat{Y}_M} \nabla \xi^0 \rightarrow \nabla \xi^0 + \hat{\nabla}_y \xi^1 \rightarrow \nabla \xi^0 \rightarrow \nabla \xi^0.
\end{align*}
\]

Considering the two-scale convergence of \( u^\varepsilon \) and \( \partial_t u^\varepsilon \), we obtain
\[
\frac{\mathcal{Y}_{\hat{M}}}{|Y|^2}(\xi^0, \phi)_{\Omega_T} = \lim_{\varepsilon \rightarrow 0} \langle \partial_t u^\varepsilon, \phi \rangle_{\Omega_{2h}, T} = \lim_{\varepsilon \rightarrow 0} \langle u^\varepsilon, \partial_t \phi \rangle_{\Omega_{2h}, T} = \frac{\mathcal{Y}_{\hat{M}}}{|Y|}(u^0, \partial_t \phi)_{\Omega_T}
\]
for all \( \phi \in C^\infty(\Omega_T) \). Hence, \( \partial_t u^0 \in L^2(\Omega_T) \), and \( \xi^0 = \partial_t u^0 \) a.e. in \( \Omega_T \times \hat{Y} \). The two-scale convergence of \( \nabla u^\varepsilon \) and \( \partial_t \nabla u^\varepsilon \) implies
\[
\begin{align*}
\langle \partial_t \nabla u^0 + \hat{\nabla}_y \xi^1, \phi \rangle_{\Omega_T \times \hat{Y}_M} = \lim_{\varepsilon \rightarrow 0} \langle \partial_t \nabla u^\varepsilon, \phi \rangle_{\Omega_{2h}, T} = \lim_{\varepsilon \rightarrow 0} \langle \nabla u^\varepsilon, \partial_t \phi \rangle_{\Omega_{2h}, T} = -\langle \hat{Y}^{-1}(\nabla u^0 + \hat{\nabla}_y u^1, \partial_t \phi)_{\Omega_T \times \hat{Y}_M} \rangle
\end{align*}
\]
for all \( \phi \in C^\infty(\Omega_T; C^\infty_{per}(\hat{Y})) \). Thus, \( \partial_t \nabla_y u^1 \in L^2(\Omega_T \times \hat{Y}_M) \) and \( \hat{\nabla}_y \xi^1 = \partial_t \nabla_y u^1 \) a.e. in \( \Omega_T \times \hat{Y}_M \). Therefore, \( u^0 \in H^1(0, T; W(\Omega)) \), \( \partial_t u^1 \in L^2(\Omega_T; H^1_{per}(\hat{Y})) \) and \( \chi_{\Omega_M} \partial_t e(u^\varepsilon) \rightarrow \chi_{\hat{Y}_M} \partial_t e(u^0) + \partial_t \hat{e}_y(u^1) \) two-scale. 

To derive macroscopic equations for the microscopic problem (1)–(3), we first derive the macroscopic equations for the perturbed system (2), (3), (25). Then letting the perturbation parameter to go to zero we obtain the macroscopic equations for (1)–(3).
Theorem 4.3. A sequence of solutions \((u^p, p^p, n^p, b^p)\), of the microscopic problem \([2], [3], [25]\), converges to a solution \((u^\delta, p^\delta, n^\delta, b^\delta)\) of the macroscopic perturbed equations

\[
\begin{align*}
\partial_t u^\delta_t - \text{div}(\mathbb{E}^\delta_{\text{hom}}(u^\delta) + \mathbb{V}^\delta_{\text{hom}}(u^\delta_t)) + \int_0^t \mathbb{K}^\delta(t-s,s)u^\delta_s ds &= 0 \quad \text{in } \Omega_T, \\
(\mathbb{E}^\delta_{\text{hom}}(u^\delta) + \mathbb{V}^\delta_{\text{hom}}(u^\delta_t)) + \int_0^t \mathbb{K}^\delta(t-s,s)u^\delta_s ds \nu &= f_\varepsilon \quad \text{on } \Gamma_{\varepsilon,T}, \\
(\mathbb{E}^\delta_{\text{hom}}(u^\delta) + \mathbb{V}^\delta_{\text{hom}}(u^\delta_t)) + \int_0^t \mathbb{K}^\delta(t-s,s)u^\delta_s ds \nu &= -p_\varepsilon \nu \quad \text{on } \Gamma_{\varepsilon,T}, \\
u^\delta_t(0,x) = u_0(x), \quad u^\delta_t(0,x) &= 0 \quad \text{in } \Omega,
\end{align*}
\]

and

\[
\begin{align*}
\partial_t p^\delta &= \text{div}(\nabla u^\delta) - F_p(p^\delta), \\
\partial_t n^\delta &= \text{div}(\nabla n^\delta) + F_n(p^\delta, n^\delta) + R_n(n^\delta, b^\delta, N^\text{eff}_\delta(u^\delta)), \\
\partial_t b^\delta &= R_b(n^\delta, b^\delta, N^\text{eff}_\delta(u^\delta)) \quad \text{in } \Omega_T \text{ together with the initial and boundary conditions}
\end{align*}
\]

\[
\begin{align*}
D_p \nabla p^\delta &= \theta^{1 - 1} J_p(p^\delta), & D_n \nabla n^\delta &= \theta^{1 - 1} G(n^\delta) N^\text{eff}_\delta(u^\delta) \quad \text{on } \Gamma_{\varepsilon,T}, \\
D_p \nabla p^\delta &= -\theta^{1 - 1} \gamma_p p^\delta, & D_n \nabla n^\delta &= \theta^{1 - 1} J_n(n^\delta) \quad \text{on } \Gamma_{\varepsilon,T}, \\
D_p \nabla p^\delta &= 0, & D_n \nabla n^\delta &= 0 \quad \text{on } \Gamma_{\varepsilon,T}, \\
p^\delta(0) &= p_0(x), \quad n^\delta(0) &= n_0, \quad b(0) = b_0 \quad \text{in } \Omega,
\end{align*}
\]

where \(\mathbb{E}^\delta_{\text{hom}} = \mathbb{E}^\delta_{\text{hom}}(t, b^\delta)\), \(\mathbb{V}^\delta_{\text{hom}} = \mathbb{V}^\delta_{\text{hom}}(t, b^\delta)\), and \(\mathbb{K}^\delta(t, s) = \mathbb{K}^\delta(t, s, b^\delta)\) are defined by

\[
\begin{align*}
\mathbb{E}^\delta_{\text{hom},ijkl}(b^\delta) &= \int_0^1 \left[ \mathbb{E}_{ijkl}(b^\delta, y) + (\mathbb{E}(b^\delta, y) \mathbb{e}_y(w^\delta_{ijkl}))_{kl} + (\mathbb{V}_M(b^\delta, y) \partial_s \mathbb{e}_y(w^\delta_{ijkl}))_{kl} \chi_{\varepsilon,T} \right] dy, \\
\mathbb{V}^\delta_{\text{hom},ijkl}(b^\delta) &= \frac{1}{|V|} \int_{Y_M} \left[ \mathbb{V}_{ijkl}(b^\delta, y) + (\mathbb{V}_M(b^\delta, y) \mathbb{e}_y(\chi^\delta_{ijkl})))_{kl} \right] dy, \\
\mathbb{K}^\delta_{ijkl}(t, s, b^\delta) &= \int_0^1 \left[ (\mathbb{E}(b^\delta(t + s), y) \mathbb{e}_y(\chi^\delta(t, s)))_{kl} + (\mathbb{V}_M(b^\delta(t + s), y) \partial_s \mathbb{e}_y(\chi^\delta(t, s)))_{kl} \chi_{\varepsilon,T} \right] dy,
\end{align*}
\]

and \(w^\delta_{ijkl}(t, x, y), \chi^\delta_{ijkl}(t, x, y), \text{ and } \mathbb{V}^\delta_{ijkl}\) are solutions of the unit cell problems \([13]\) and \([14]\) with \(b^\delta\) instead of \(b\). The macroscopic diffusion matrices \(D_p\), with \(\alpha = n, p\) and \(l = 1, 2\), are defined as in \([15]\) and \(N^\text{eff}_\delta\) is defined in \([20]\).

Proof. To pass to the limit in the equations for \(n^\delta\) and \(b^\delta\), we shall prove the strong convergence of \(\int_\Omega \mathbb{E}(u^\delta) dx\) in \(L^2(0,T)\) using the Kolmogorov compactness theorem \([5, 20]\). Considering the difference of \([25]\) for \(t\) and \(t + h\) and taking \(\delta_b u^\varepsilon(t, x) = u^\varepsilon(t + h, x) - u^\varepsilon(t, x)\) as a test function, we have

\[
\begin{align*}
&\int_0^{t+h} \left[ \mathbb{E}(b^\varepsilon(t + h)) u^\varepsilon(t + h) - \mathbb{E}(b^\varepsilon(t)) u^\varepsilon(t), e(\delta_b u^\varepsilon) \right]_\Omega \\
&\quad + \langle \mathbb{V}(b^\varepsilon(t + h)) \partial_t u^\varepsilon(t + h) - \mathbb{V}(b^\varepsilon(t)) \partial_t u^\varepsilon(t), e(\delta_b u^\varepsilon) \rangle_{\Omega_M} \right] dt \\
&\quad + \langle \delta_b \partial_t u^\varepsilon(T - h), \delta_b u^\varepsilon(T - h) \rangle_{\Omega_M} - \langle \delta_b \partial_t u^\varepsilon(0), \delta_b u^\varepsilon(0) \rangle_{\Omega_M} \\
&\quad = \int_0^{t+h} \left[ \partial \| \delta_b u^\varepsilon \|^2_{L^2(\Omega_M)} + \langle \delta_b \mathbb{f}_e, \delta_b u^\varepsilon \rangle_{L^2(\varepsilon)} - \langle \delta_b p_\varepsilon, \delta_b u^\varepsilon \rangle_{L^2(\varepsilon)} \right] dt.
\end{align*}
\]

To estimate the first term on the right-hand side we consider \(\delta_b u^\varepsilon(t, x) = u^\varepsilon(t + h, x) - u^\varepsilon(t, x) = \int_{t+h}^{t+h} \partial_t^2 u^\varepsilon(\tau, x) d\tau\), integrate \([25]\) over \((t, t + h)\) and take \(u^\varepsilon(t + h, x) - u^\varepsilon(t, x)\) as a test function, with \(\mathbb{r}^\varepsilon\) being an extension of \(u_t\) from \(\Omega_M\) to \(\Omega\) as in Lemma 3.1.

\[
\begin{align*}
\partial \| \delta_b u^\varepsilon \|^2_{L^2(\Omega_M)} &\leq h C_1 \left[ \| p_\varepsilon \|_{H^1(0,T;L^2(\varepsilon))} + \| \mathbb{f}_e \|_{H^1(0,T;L^2(\varepsilon))} \right] \\
&\quad + h \frac{1}{2} C_2 \left[ \| e(u^\varepsilon) \|^2_{L^2(0,T;L^2(\Omega_M))} + \| e(u^\varepsilon) \|^2_{L^2(\varepsilon)} + \| e(u^\varepsilon) \|^2_{L^2(\Omega_M)} \right] \leq C h^\frac{1}{2},
\end{align*}
\]

where the constant \(C\) is independent of \(\varepsilon, \vartheta, h,\) and \(h \in (0,T)\). Here we used estimates \([27]\) and the property of the extension, i.e. \(\| e(u^\varepsilon) \|^2_{L^2(\Omega_M)} \leq C_1 \| e(u^\varepsilon) \|^2_{L^2(\varepsilon)}\) with a constant \(C_1\) independent of \(\varepsilon,\) see e.g. \([22]\).
Using the estimate for $\vartheta^{1/2}\|\partial_t u^\vartheta\|_{L^\infty(0,T;L^2_2(\Omega^h_\vartheta))}$ in (27) we obtain
\[
\vartheta \langle \delta^h \partial_t u^\vartheta (T-h), \delta^h u^\vartheta (T-h) \rangle_{\Omega^h_\vartheta} \leq 2\|\partial_t u^\vartheta\|_{L^\infty(0,T;L^2_2(\Omega^h_\vartheta))}\|\delta^h u^\vartheta (T-h)\|_{L^2(\Omega^h_\vartheta)} \leq C \vartheta^{1/2}\|\partial_t u^\vartheta\|_{L^\infty(0,T;L^2_2(\Omega^h_\vartheta))} \leq C h.
\]
\[
(43)
\]
In the same way we also have
\[
\vartheta \langle \delta^h \partial_t u^\vartheta (0), \delta^h u^\vartheta (0) \rangle_{\Omega^h_\vartheta} \leq C h,
\]
where $C$ is independent of $\varepsilon$, $\vartheta$, and $h$. To estimate the first two terms on the left-hand side of (41) we use the uniform boundedness of $b^\vartheta$ and $\partial_t b^\vartheta$, the equality $\delta^h e(u^\vartheta (t, x)) = h \int_0^1 \partial_t e(u^\vartheta (t + hs, x)) ds$, and estimates (27):
\[
\int_0^{T-h} \langle (\mathbb{E}^\varepsilon (b^\vartheta (t + h)) - \mathbb{E}^\varepsilon (b^\vartheta (t))) e(u^\vartheta (t))\rangle_{\Omega} dt \leq h C_1 \|\delta^h b^\vartheta\|_{L^\infty(\Omega^h_{T-h}, T)} \|e(u^\vartheta)\|_{L^2(\Omega^h_{T-h}, T)} \leq C_2 h,
\]
\[
(45)
\]
\[
\int_0^{T-h} \langle \mathbb{V}^\varepsilon (b^\vartheta (t + h), x) \partial_t e(u^\vartheta (t + h)) - \mathbb{V}^\varepsilon (b^\vartheta (t), x) \partial_t e(u^\vartheta (t))\rangle_{\Omega^h_\vartheta} dt \leq h C_3 \|\delta^h b^\vartheta\|_{L^\infty(\Omega^h_{T-h}, T)} \|\partial_t e(u^\vartheta)\|_{L^2(\Omega^h_{T-h}, T)} \leq C_4 h,
\]
with the constants $C_j$, $j = 1, 2, 3, 4$, independent of $\varepsilon$, $\vartheta$, and $h$. Then, the assumptions on $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{P}$, estimates (27) and (42), and the boundedness of $b^\vartheta$ ensures
\[
|\mathbb{E}^\varepsilon (b^\vartheta (t + h)) - \mathbb{E}^\varepsilon (b^\vartheta (t))|_{L^2(\Omega^h_{T-h})} \leq C h^{1/2},
\]
\[
|\mathbb{V}^\varepsilon (b^\vartheta (t), x)\partial_t e(u^\vartheta (t))|_{L^2(\Omega^h_{T-h})} \leq h |\mathbb{V}^\varepsilon (b^\vartheta (t), x)|_{L^2(\Omega^h_{T-h})} \leq C h,
\]
with a constant $C$ independent of $\varepsilon$, $\vartheta$, and $h$.

Thus, the estimate (46), along with the estimate for $\partial_t b^\vartheta$, the Kolmogorov theorem, and the two-scale convergence of $u^\vartheta$, yields the strong convergence, up to a subsequence,
\[
\int_\Omega \mathbb{E}^\varepsilon (b^\vartheta, x) e(u^\vartheta) dx \rightarrow \int_\Omega \mathbb{E}^\varepsilon (b^\vartheta, x) e(\hat{u}^\vartheta) dx + \epsilon g(\hat{u}^\vartheta) dx \quad \text{in} \quad L^2(0, T),
\]
\[
\int_\Omega \mathbb{V}^\varepsilon (b^\vartheta, x) e(u^\vartheta) dx \rightarrow \int_\Omega \mathbb{V}^\varepsilon (b^\vartheta, x) e(\hat{u}^\vartheta) dx + \epsilon g(\hat{u}^\vartheta) dx \quad \text{in} \quad L^2(0, T), \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

Now we can pass to the limit as $\varepsilon \rightarrow 0$ in the microscopic equations (2), (3), and (25), with initial and boundary conditions in (1) and (26). Considering $\phi_\alpha (t, x) = \varphi_\alpha (t, x) + \varepsilon \psi_\alpha (t, x, \hat{x}/\varepsilon)$ as a test function in (5)–(6), where $\varphi_\alpha \in C^\infty (\Omega_T)$ and $\alpha_3$-periodic in $x_3$, and $\psi_\alpha \in C_0^\infty (\Omega_T; C_{per}^\infty (\hat{Y}))$, for $\alpha = 1, 2$, applying the two-scale convergence and using the strong convergence of $T^\varepsilon (b^\vartheta)$ and $\mathbb{P}^\varepsilon$, $\mathbb{N}^\varepsilon$, see Lemma 4.2, along with strong convergence of $\int_\Omega \mathbb{E}^\varepsilon (b^\vartheta, x) e(u^\vartheta) dx$ and $\int_\Omega \mathbb{V}^\varepsilon (b^\vartheta, x) e(u^\vartheta) dx$, we obtain macroscopic equations (38)–(39) for $\mathbb{P}^\varepsilon$, $\mathbb{N}^\varepsilon$, and $b^\vartheta$ in the same way as in (26).

Using the strong convergence of $T^\varepsilon (b^\vartheta)$ along with the two-scale convergence of $u^\vartheta$, $e(u^\vartheta)$, and $\partial_t e(u^\vartheta)$, as $\varepsilon \rightarrow 0$, yields the macroscopic equations
\[
\langle \mathbb{E}^\varepsilon (b^\vartheta, y) e(u^\vartheta) + \epsilon g(\hat{u}^\vartheta) \rangle_{\Omega^h_\vartheta} = \mathbb{V}^\varepsilon (b^\vartheta, y) \partial_t e(u^\vartheta) + \epsilon g(\hat{u}^\vartheta), e(\psi) + \epsilon g(\psi_1) \rangle_{\Omega^h_\vartheta} \quad \text{for} \quad \psi \in C_0^1 (0, T; C^1 (\Omega_T)) \cap C_0^\infty (\Omega_T; C_{per}^\infty (\hat{Y})) \neq 0.
\]
\[
(47)
\]
Taking $\psi \equiv 0$ we obtain
\[
\langle \mathbb{E}^\varepsilon (b^\vartheta, y) e(u^\vartheta) + \epsilon g(\hat{u}^\vartheta) \rangle_{\Omega^h_\vartheta} = \mathbb{V}^\varepsilon (b^\vartheta, y) \partial_t e(u^\vartheta) + \epsilon g(\hat{u}^\vartheta), e(\psi_1) \rangle_{\Omega^h_\vartheta} \quad \text{for} \quad \psi \in C_0^1 (0, T; C^1 (\Omega_T)) \cap C_0^\infty (\Omega_T; C_{per}^\infty (\hat{Y})) \neq 0.
\]
\[
(48)
\]
Considering the structure of (48) and taking into account the fact that $\mathbb{E}^\varepsilon (b^\vartheta, \cdot)$ and $\mathbb{V}^\varepsilon (b^\vartheta, \cdot)$ depend on $t$, we seek $u^\vartheta$ in the form
\[
u^\vartheta (t, x, y) = \sum_{i,j=1}^3 \left[ e(u^\vartheta (t, x))_{ij} w_{ij}^\vartheta (t, x, y) + \int_0^t \partial_t e(u^\vartheta (s, x))_{ij} v_{ij}^\vartheta (t - s, s, x, y) ds \right]
\]
and rewrite the equation (48) as
\[
\langle \mathbb{E}^\varepsilon (b^\vartheta, y) e(u^\vartheta) + \sum_{i,j=1}^3 \left[ e(u^\vartheta)_{ij} e_y (w_{ij}^\vartheta) + \int_0^t \partial_t e(u^\vartheta)_{ij} e_y (v_{ij}^\vartheta) ds \right] \rangle_{\Omega^h_\vartheta} \nu^\vartheta (t, x, y) + \langle \mathbb{V}^\varepsilon (b^\vartheta, y) \partial_t e(u^\vartheta) + \sum_{i,j=1}^3 \left[ \partial_t e(u^\vartheta)_{ij} e_y (w_{ij}^\vartheta) + e(u^\vartheta)_{ij} e_y (w_{ij}^\vartheta) \right] \rangle_{\Omega^h_\vartheta} \nu^\vartheta (t, x, y) = 0.
\]
\[
(49)
\]
Considering the terms with \( e(u^\theta) \) and \( \partial_t e(u^\theta) \), respectively, we obtain that \( v_{ij}^\theta(0, t, x, y) = \chi_{\psi, \vartheta}^{ij}(t, x, y) - w_{ij}^\theta(t, x, y) \) a.e. in \( \Omega_T \times \check{Y}_M \), where \( w_{ij}^\theta(t, x, y) \) and \( \chi_{\psi, \vartheta}^{ij}(t, x, y) \) are solutions of the unit cell problems \([13]\) with \( b^\theta \) instead of \( b \). Using this in \([19]\) implies that \( v_{ij}^\theta \) satisfies \([14]\) with \( b^\theta \) instead of \( b \). Taking \( \psi_1 \equiv 0 \) in \([17]\) yields the macroscopic equations \([37]\) for \( u^\theta \).

In the same way as for the macroscopic elasticity tensor for the equations of linear elasticity, see e.g. \([16, 22]\), we obtain that \( \nabla_{\text{hom}} \) is positive-definite and possesses major and minor symmetries, as in Assumption \([18]\). The assumptions on \( E \) and \( V_{\text{M}} \) and the uniform boundedness of \( b^\theta \) ensure the boundedness of \( E_{\text{hom}}^{\theta} \) and \( K^{\theta} \). Notice that the positive-definiteness and symmetry properties of \( \nabla_{\text{hom}}^{\theta} \) together with the boundedness of \( E_{\text{hom}}^{\theta} \) and \( K^{\theta} \) ensure the well-possedness of the viscoelastic equations \([37]\). \( \square \)

Now we can complete the proof of the main result of the paper.

**Proof of Theorem 2.2.** To complete the proof of Theorem 2.2 we have to show that \( \{p^\theta\}, \{n^\theta\}, \{b^\theta\}, \) and \( \{u^\theta\} \) converge to solutions of the macroscopic model \([17, 19, 20]\). Using the fact that the estimates \([27, 46]\) for \( \dot{u}^\theta \) are independent of \( \vartheta \) and \( \varepsilon \) and applying the weak and two-scale convergence of \( u^\theta \) together with the lower semicontinuity of a norm yield

\[
\|u^\theta\|_{L^\infty(0, T; W(\Omega))} + \|e(u^\theta)\|_{L^\infty(0, T; L^2(\Omega \times Y))} \leq C,
\]

\[
\|e(u^\theta(\cdot + h, \cdot)) - e(u^\theta)\|_{L^2((0, T-h) \times \Omega)} \leq Ch^{1/2},
\]

with a constant \( C \) independent of \( \vartheta \) and \( h \).

Similar to the proof of Lemma \([3, 2]\) using the estimates \([50, 51]\) we obtain the estimates for \( p^\theta \) and \( n^\theta \) in \( L^2(0, T; V(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)) \), and \( b^\theta \) in \( W^{1, \infty}(0, T; L^\infty(\Omega)) \). In a similar way as in the proof of Lemma \([4, 2]\) we show

\[
\|b^\theta(\cdot, h_1) - b^\theta\|_{L^\infty(0, T; L^2(\Omega))} \leq Ch,
\]

\[
\|b^\theta(\cdot + h, \cdot) - b^\theta\|_{L^2((0, T) \times \Omega)} \leq Ch,
\]

where \( b^\theta \) is extended by zero from \( \Omega_T \) into \( \mathbb{R}^3 \times \mathbb{R}_+ \) and \( h_1 = hb_j \), with \( h \in (0, T) \). Then, applying the Kolmogorov theorem we obtain the strong convergence of a subsequence of \( b^\theta \) in \( L^2(\Omega_T) \) as \( \vartheta \to 0 \).

In a similar way as in the proof of Lemma \([5, 3]\) considering the assumptions on \( E \) and \( V \), together with the boundedness of \( b^\theta \) and \( \partial_t b^\theta \), uniformly in \( \vartheta \), we obtain the existence of weak solutions of the unit cell problems \([13]\), with \( b^\theta \) instead of \( b \), satisfying

\[
\|w_{ij}^\theta\|_{L^\infty(0, T; L^2(Y))} + \|\partial_t \hat{e}_p(w_{ij}^\theta)\|_{L^2(0, T; L^2(Y_M))} \leq C \quad \text{for a.a. } x \in \Omega,
\]

\[
\|X_{ij}^\theta(\vartheta)\|_{H^{1/2}_0(Y_M)} \leq C \quad \text{for a.a. } (t, x) \in \Omega_T,
\]

where the constant \( C \) is independent of \( \vartheta \). The estimates \([52, 53]\) and boundedness of \( b^\theta \) and \( \partial_t b^\theta \) ensure the existence of a weak solution of the unit cell problem \([13]\) with \( b^\theta \) instead of \( b \) such that

\[
\|v_{ij}^\theta\|_{L^\infty(0, T; L^2(\Omega))} + \|\partial_t \hat{e}_p(v_{ij}^\theta)\|_{L^2(0, T; L^2(Y_M))} \leq C
\]

for a.a. \( x \in \Omega \) and \( s \in (0, T] \).

Using the assumptions on \( V_{\text{M}} \), we obtain the symmetry properties and strong ellipticity of \( V_{\text{hom}}^{\theta} \), see e.g. \([27, 22]\), with an ellipticity constant independent of \( \vartheta \). The assumptions on \( E \) and \( V_{\text{M}} \), the uniform boundedness of \( b^\theta \), and the estimates \([52, 53]\) ensure

\[
\|E_{\text{hom}}^{\theta}(b^\theta)\|_{L^2(0, T; L^\infty(\Omega))} + \|V_{\text{hom}}^{\theta}(b^\theta)\|_{L^\infty(0, T; L^\infty(\Omega))} + \|K^{\theta}(t-s,s,b^\theta)\|_{L^2(0, T; L^\infty(0, t; L^\infty(\Omega))}) \leq C,
\]

with a constant \( C \) independent of \( \vartheta \). Taking \( \dot{u}^\theta \) as a test function in the weak formulation of \([37]\), using the strong ellipticity of \( V_{\text{hom}}^{\theta} \) together with estimates \([50, 51]\) and \([54]\), and applying the second Korn inequality for \( u^\theta(t) \in W(\Omega) \) yield

\[
\|\partial_t u^\theta\|_{L^\infty(0, T; L^2(\Omega))} + \|u^\theta\|_{H^1(0, T; W(\Omega))} \leq C,
\]

with a constant \( C \) independent of \( \vartheta \). Hence we have the weak convergence, up to a subsequence, of \( u^\theta \) in \( H^1(0, T; W(\Omega)) \) and weak-* convergence of \( \partial_t u^\theta \) in \( L^\infty(0, T; L^2(\Omega)) \).

Hence, to pass to the limit as \( \vartheta \to 0 \) in the macroscopic equations \([37]\) we have to show the strong convergence of \( E_{\text{hom}}^{\theta}, V_{\text{hom}}^{\theta}, \) and \( K^{\theta} \) as \( \vartheta \to 0 \). First, we show the strong convergence of \( \int_Y \hat{e}_p(w_{ij}^\theta) \, dy \) and \( \int_Y \partial_t \hat{e}_p(w_{ij}^\theta) \, dy \) in \( L^2(\Omega_T) \). Considering the first equation in \([13]\), with \( b^\theta \) instead of \( b \), for \( t+h \) and \( t \), with \( h > 0 \), taking
\[ \delta^h w^{ij}_{\theta}(t, x, y) = w^{ij}_{\theta}(t + h, x, y) - w^{ij}_{\theta}(t, x, y) \] as a test function, and using \( \delta^h e_{\theta}(w^{ij}_{\theta}(t)) = h \int_0^1 \partial_t \hat{e}_{\theta}(w^{ij}_{\theta})(t + h \tau) d\tau, \) we obtain
\[
\| \delta^h e_{\theta}(w^{ij}_{\theta}) \|_{L^2((0,T-h) \times \hat{Y})} \leq C_{1,h} \| \partial_t \hat{e}_{\theta}(w^{ij}_{\theta}) \|_{L^2(\hat{Y}_{M,T})}^2 \\
+ \| \hat{e}_{\theta}(w^{ij}_{\theta}) \|_{L^2((0,T-h) \times \hat{Y})}^2 \leq C_2 h
\]
for a.a. \( x \in \Omega \) and the constants \( C_1 \) and \( C_2 \) are independent of \( \theta \). Taking an extension \( \delta^h \partial_{\theta} w^{ij}_{\theta} \) of \( \delta^h \partial_{\theta} w^{ij}_{\theta} \) from \( \hat{Y}_M \) to \( Y \) as a test function in the weak formulation of (13), with \( \theta \) instead of \( b \), yields
\[
\| \delta^h \partial_{\theta} w^{ij}_{\theta} \|_{L^2((0,T-h) \times Y_{M,T})} \leq C \| \theta \|_{L^2(0,T-L^\infty(\Omega))} \| \delta^h \partial_{\theta} w^{ij}_{\theta} \|_{L^2(\hat{Y}_{M,T})}^2 \\
+C_2 \{ 1 + \| \hat{e}_{\theta}(w^{ij}_{\theta}) \|_{L^2(\hat{Y}_{M,T})}^2 + \| \hat{e}_{\theta}(\partial_{\theta} w^{ij}_{\theta}) \|_{L^2(\hat{Y}_{M,T})} \} \| \delta^h \theta \|_{L^2(0,T-L^\infty(\Omega))}^2 \leq h C_3 \| \theta \|_{L^2(0,T-L^\infty(\Omega))}^2
\]
for a.a. \( x \in \Omega \) and the constants \( C_1, C_2, \) and \( C_3 \) are independent of \( \theta \) and \( h \). Here, we used the fact that due to the periodicity of \( w^{ij}_{\theta} \) and the Korn inequality we have
\[
\| \delta^h \partial_{\theta} w^{ij}_{\theta} \|_{L^2((0,T-h);H^1(\hat{Y}_{M,T}))} \leq C \| \delta^h \partial_{\theta} w^{ij}_{\theta} \|_{L^2((0,T-h) \times \hat{Y}_{M,T})},
\]
for a.a. \( x \in \Omega \), and \( \| \hat{e}_{\theta}(\delta^h \partial_{\theta} w^{ij}_{\theta}) \|_{L^2((0,T-h) \times Y_{M,T})} \leq C \| \hat{e}_{\theta}(\delta^h \partial_{\theta} w^{ij}_{\theta}) \|_{L^2((0,T-h) \times Y_{M,T})}, \) where the constant \( C \) is independent of \( \theta \).

Considering (13), with \( \theta \) instead of \( b \), for \( x + h_j \) and \( x \), where \( h_j = h b_j \), and using (51) imply
\[
\| \delta^h \partial_{\theta} w^{ij}_{\theta} \|_{L^2((0,T-L^2(\Omega';\hat{Y}))} \leq \| \partial_{\theta} w^{ij}_{\theta} \|_{L^2((0,T-L^2(\Omega';\hat{Y}))} \leq C h,
\]
where \( \delta^h w^{ij}_{\theta}(t, x, y) = w^{ij}_{\theta}(t, x + h_j, y) - w^{ij}_{\theta}(t, x, y) \), the function \( \theta \) is extended by zero from \( \Omega_{T} \) in \( \mathbb{R}^+ \times \mathbb{R}^3 \), and \( C \) is independent of \( \theta \). In the same manner we obtain
\[
\| \delta^h e_{\theta}(\chi_{ij,\theta}) \|_{L^2(\Omega_T \times \hat{Y}_{M,T})} \leq C h,
\]
where \( \theta \) and \( \chi_{ij,\theta} \) are extended by zero from \( \Omega_{T} \) into \( \mathbb{R}^+ \times \mathbb{R}^3 \), and
\[
\| \delta^h e_{\theta}(\chi_{ij,\theta}) \|_{L^2(0,T-L^2(\Omega';\hat{Y}))} \leq C h.
\]

Considering the difference of equations in (14), with \( \theta \) instead of \( b \), for \( s + h \) and \( s \), taking \( \hat{e}_{\theta}(\chi_{ij,\theta}^j(t + s, x, y)) \) as a test function, and using the estimates for \( \delta^h e_{\theta}(w^{ij}_{\theta}) \) and \( \delta^h e_{\theta}(\chi_{ij,\theta}) \) in (56) and (59), respectively, yield
\[
\| \hat{e}_{\theta}(\chi_{ij,\theta}^j(t - s, s)) - \hat{e}_{\theta}(\chi_{ij,\theta}^j(t - s, s)) \|_{L^2(0,T-L^2(\Omega';\hat{Y}))} \leq C h.
\]

Thus, (51) and (56)–(61) along with the Kolmogorov theorem and the strong convergence and boundedness of \( \theta \) ensure
\[
\begin{align*}
\int_{\hat{Y}} \hat{e}_{\theta}(w^{ij}_{\theta}) d\tau &\rightarrow \int_{\hat{Y}} \hat{e}_{\theta}(w^{ij}_{\theta}) d\tau, \\
\int_{\hat{Y}} \hat{e}_{\theta}((\partial \theta w^{ij}_{\theta}) d\tau &\rightarrow \int_{\hat{Y}} \hat{e}_{\theta}(w^{ij}_{\theta}) d\tau, \\
\int_{\hat{Y}} \hat{e}_{\theta}(\chi_{ij,\theta} d\tau &\rightarrow \int_{\hat{Y}} \hat{e}_{\theta}(\chi_{ij,\theta}) d\tau
\end{align*}
\]
in \( L^2(\Omega_T) \),
\[
\begin{align*}
\hat{E}_{\theta}^{\text{hom}}(b^\theta) &\rightarrow \hat{E}_{\theta}^{\text{hom}}(b), \\
\hat{V}_{\theta}^{\text{hom}}(b^\theta) &\rightarrow \hat{V}_{\theta}^{\text{hom}}(b)
\end{align*}
\]
in \( L^2(\Omega_T) \),
\[
\begin{align*}
\int_{\hat{Y}} \hat{e}_{\theta}(\chi_{ij,\theta}) d\tau &\rightarrow \int_{\hat{Y}} \hat{e}_{\theta}(\chi_{ij,\theta}) d\tau, \\
\int_{\hat{Y}} \hat{e}_{\theta}(\chi_{ij,\theta}) d\tau &\rightarrow \int_{\hat{Y}} \hat{e}_{\theta}(\chi_{ij,\theta}) d\tau
\end{align*}
\]
in \( L^2(0,T;L^2(\Omega^2)) \),
\[
\begin{align*}
\int_{\hat{Y}} \hat{e}_{\theta}(w^{ij}_{\theta}) d\tau &\rightarrow \int_{\hat{Y}} \hat{e}_{\theta}(w^{ij}_{\theta}) d\tau, \\
\int_{\hat{Y}} \hat{e}_{\theta}(w^{ij}_{\theta}) d\tau &\rightarrow \int_{\hat{Y}} \hat{e}_{\theta}(w^{ij}_{\theta}) d\tau
\end{align*}
\]
in \( L^2(0,T;L^2(\Omega^2)) \),
as \( \theta \rightarrow 0 \), where
\[
\begin{align*}
\hat{E}_{\theta}^{\text{hom},ijkl}(b^\theta) &\rightarrow \hat{E}_{ijkl}(b^\theta), \\
\hat{V}_{\theta}^{\text{hom},ijkl}(b^\theta) &\rightarrow \hat{V}_{ijkl}(b^\theta)
\end{align*}
\]
Using estimates (52) and (53), and the strong convergence of \( \theta \) yields
\[
\int_{0}^{T-S} \hat{K}_{ijkl}(t, x, b^\theta(t, x)) dt \rightarrow \int_{0}^{T-S} \hat{K}_{ijkl}(t, x, b(t, x)) dt
\]
as $\vartheta \to 0$, for a.a. $x \in \Omega_T$ and $s \in [0, T]$. Then, estimate [51] and the Lebesgue dominated convergence theorem, implies

$$\int_0^{T-s} \mathbb{K}_\vartheta(t, s, b^0)dt \to \int_0^{T-s} \mathbb{K}(t, s, b)dt \quad \text{in} \quad L^2(\Omega_T) \quad \text{as} \quad \vartheta \to 0.$$ 

The strong convergence of $\mathbb{E}_s^\vartheta$ and $\mathbb{K}_\vartheta$ and estimates [55] ensure the strong convergence $\mathcal{N}_s^\vartheta(\mathbf{u}^{s,3}) \to \mathcal{N}_s^{\text{hom}}(\mathbf{u})$ in $L^2(\Omega_T)$ as $\vartheta \to 0$.

Hence, taking the limit as $\vartheta \to 0$ in the weak formulation of (37) we obtain the macroscopic equations [19]. Notice that for the integral-term in (37) we have the macroscopic equations [19].

$$\int_0^{T} \mathbb{K}(t, s, b)\partial_t \mathbf{u}(t, s)dt \to \int_0^{T} \mathbb{K}(t, s, b)\partial_t \mathbf{u}(t, s)dt \quad \text{in} \quad L^2(\Omega_T) \quad \text{as} \quad \vartheta \to 0.$$ 

for all $\mathbf{u} \in C^\infty(\Omega_T)^3$, $\mathbf{u}$ is $a_3$-periodic in $x_3$. Thus, using the weak convergence of $\partial_t \mathbf{u}^{s,3}$ and the strong convergence of $\int_0^{T-s} \mathbb{K}_\vartheta(t, s, b^0)dt$ we can pass to the limit in the last term in (37).

The assumptions on the elastic $\mathbb{E}(b, y)$ and viscoelastic $\mathbb{V}_M(b, y)$ tensors together with the regularity and boundedness of $b$ ensure the existence of solutions of the unit cell problems [13] and [14]. As before, the assumptions on $\mathbb{E}$ and $\mathbb{V}_M$, the boundedness of $b$, and the estimates [52–53] yield the symmetry properties and strong ellipticity of $\mathbb{V}_M$, see e.g. [22], as well as the boundedness of the macroscopic tensors, i.e. $\mathbb{E}_{\text{hom}} \in L^\infty(0, T; L^\infty(\Omega))$, $\mathbb{V}_{\text{hom}} \in L^\infty(0, T; L^\infty(\Omega))$, $\mathbb{E}(t-s, s) \in L^\infty(0, T; L^\infty(0, t; L^\infty(\Omega)))$ and $\mathcal{K}(t-s, s) \in L^\infty(0, T; L^\infty(0, t; L^\infty(\Omega)))$. This together with the assumptions on the coefficients and nonlinear functions in the $\int_0^{T-s} \mathbb{K}_\vartheta(t, s, b^0)dt$ can pass to the limit in the last term in (37).

Using estimates [55] we obtain $\mathbf{u} \in H^1(0, T; W(\Omega))$. Hence, $\mathbf{u} \in C([0, T]; W(\Omega))$ and $\mathbf{u}$ satisfies the initial condition $\mathbf{u}(0, x) = \mathbf{u}_0(x)$ for $x \in \Omega$.

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