A brief survey on algebraic hyperstructures: Theory and applications

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Abstract

I am working on algebraic hyperstructures from 1995. During the last twenty years, I together with my students and co-authors studied and developed the theory of algebraic hyperstructures in many directions. In particular, we tried to find real examples of hyperstructures in nature. In this paper we review some parts of these works such as (1) Fundamental relations on hyperstructures; (2) Fuzzy sets and hyperstructures; (3) Rough sets and hyperstructures; (4) Topology and hyperstructures; (5) Number theory and hyperstructures; (6) n-ary hypergroups and there extension to hyperrings and hypermodules; (7) Applications of hyperstructures in biology, physics and chemistry.

1 Fundamental relations on hyperstructures

The main tools connecting the class of hyperstructures with the classical algebraic structures are the fundamental relations. The fundamental relation has an important role in the study of algebraic hyperstructures and especially of hypergroups. For all $n > 1$, the relation $\beta_n$ on a semihypergroup $(H, \circ)$ is defined as follows: $x \beta_n y$ if there exists $a_1, \ldots, a_n$ in $H$ such that $\{x, y\} \subseteq \prod_{i=1}^{n} a_i$, and we set $\beta = \bigcup_{n \geq 1} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on $H$. This relation was introduced by Koskas [53] and studied mainly by Corsini [19], Davvaz [23], Davvaz and Leoreanu-Fotea [36], Freni [44], Vougiouklis [71], and many others. Clearly, the relation $\beta$ is reflexive and symmetric. Denote by $\beta^*$ the transitive closure of $\beta$. If $(H, \circ)$ is a (semi)hypergroup, then the relation $\beta^*$ is the smallest equivalence relation on $H$ such that the quotient $H/\beta^*$ is a (semi)group. Freni in [44] proved that if $(H, \circ)$ is a hypergroup, then the relation $\beta$ is an equivalence relation.

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on \( H \). It is a natural question that how we can change the definition of \( \beta \) to obtain an abelian group, a cyclic group, a solvable group or a nilpotent group. In order to see the answers of these question we refer the readers to \([1, 5, 45, 52, 62]\).

Several kinds of hyperrings are introduced and analyzed in \([36]\) such as Krasner hyperrings, multiplicative hyperrings, general hyperrings. A multivalued system \((R, +, \cdot)\) is a (general) hyper-ring if \((1) (R, +)\) is a hypergroup; \((2) (R, \cdot)\) is a semihypergroup; \((3) (\cdot)\) is (strong) distributive with respect to \((+), \) i.e., for all \(x, y, z \in R\) we have \(x \cdot (y + z) = x \cdot y + x \cdot z\) and \((x + y) \cdot z = x \cdot z + y \cdot z\). The above definition contains the class of multiplicative hyperrings and additive hyperrings as well. In a hyperring, Vougiouklis introduced the equivalence relation \(\beta^*\), which is similar to the relation \(\beta\). Let \((R, +, \cdot)\) be a hypergroup. The relation \(\gamma\) is defined as follows: \(a \gamma b\) if and only if \(\{a, b\} \subseteq u\), where \(u\) is a finite sum of finite products of elements of \(R\). As usual, we denote the transitive closure of \(\gamma\) by \(\gamma^*\). Let \((R, +, \cdot)\) be a hypergroup. Then, the relation \(\gamma^*\) is the smallest equivalence relation in \(R\) such that the quotient \(R/\gamma^*\) is a ring. The structure \(R/\gamma^*\) is called the fundamental ring \([70]\). The commutativity, as well as the existence of the unit, it is not assumed in the fundamental ring. In \([11]\), Davvaz and Vougiouklis defined a new fundamental relation to obtain an ordinary commutative ring from a hyperring. They introduced the following definition.

If \(R\) is a hyperring, then we set \(\alpha_0 = \{(x, x) \mid x \in R\}\) and, for every integer \(n \geq 1\), \(\alpha_n\) is the relation defined as follows:

\[
x \alpha_n y \iff \exists (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n, \exists \sigma \in S_n \text{ and } \exists (x_{i_1}, \ldots, x_{i_k}) \in R^{k_i}, \exists \sigma_i \in S_{k_i}, \ (i = 1, \ldots, n)\text{ such that}
\]

\[
x \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \text{ and } y \in \sum_{i=1}^{n} A_{\sigma(i)},
\]

where \(A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}\). Obviously, for every \(n \geq 1\), the relation \(\alpha_n\) is symmetric, and the relation \(\alpha = \bigcup_{n \geq 0} \alpha_n\) is reflexive and symmetric. If \(\alpha^*\) is the transitive closure of \(\alpha\), then the quotient \(R/\alpha^*\) is a commutative ring \([11]\). Then, this relation is investigated in \([59]\). Also, a similar relation is defined on hypermodules to obtain an ordinary module \([13, 14, 58]\). The largest class of hyperstructures called \(H_v\)-structures. These structures introduced by Vougiouklis in 1990 in the 4th AHA congress held in Greece. In \(H_v\)-groups, \(H_v\)-rings and \(H_v\)-modules, the fundamental relations are defined and they connect the algebraic hyperstructure theory with the classical one. There is a rich monograph about \(H_v\)-structures that published by Davvaz and Vougiouklis in 2019 \([42]\).

## 2 Fuzzy sets and hyperstructures

In 1971, Rosenfeld introduced the fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. There is a considerable amount of work on the association between fuzzy sets and hyperstructures. This work can be classified into three groups. A first group of works studies crisp hyperoperations defined through fuzzy sets. This study was initiated by Corsini and others. A second group of works concerns the fuzzy hyperalgebras. This is a direct extension of the concept of fuzzy algebras. This idea was applied by Zahedi and his group on polygroups. A third group deals also with fuzzy hyperstructures, but with a completely different approach. This was studied by Corsini, Zahedi and others. The basic idea is the following one: a crisp hyperoperation assigns to every pair of elements a crisp set; a fuzzy hyperoperation assigns to every pair of elements a fuzzy set. In 1999, Davvaz introduced the notion of fuzzy subhypergroup \((H_v\text{-subgroup, resp.})\) of a hypergroup \((H_v\text{-group, resp.})\) \([20]\). Let \((H, \cdot)\) be a hypergroup \((H_v\text{-group})\) and let \(\mu\) be a fuzzy subset of \(H\). Then, \(\mu\) is said to be a fuzzy subhypergroup \((H_v\text{-subgroup,}...
resp.) of $H$ if the following axioms hold: (1) $\min\{\mu(x), \mu(y)\} \leq \inf_{a \in x \cdot y} \mu(a)$, for all $x, y \in H$; (2) For all $x, a \in H$ there exists $y \in H$ such that $x \in a \cdot y$ and $\min\{\mu(a), \mu(x)\} \leq \mu(y)$; (3) For all $x, a \in H$ there exists $z \in H$ such that $x \in z \cdot a$ and $\min\{\mu(a), \mu(x)\} \leq \mu(z)$. Then, he gave the relation between a fuzzy subhypergroup and level subhypergroup. This relation is expressed in terms of a necessary and sufficient condition. After this definitions many authors developed fuzzy hyperstructures based on this viewpoint. We refer the readers to [27], for a deep study on fuzzy hyperstructures.

3 Rough sets and hyperstructures

The concept of rough set was originally proposed by Pawlak in [63]. Since then the subject has been investigated in many papers. Some authors studied algebraic properties of rough sets. Let $U$ be a universe of objects and $\rho$ be an equivalence relation on $U$. Given an arbitrary set $A \subseteq U$, a concept in $U$, it may be impossible to describe $A$ precisely using the equivalence classes of $\rho$. That is, the available information is not sufficient to give a precise representation of $A$. In this case, one may characterize $A$ by a pair of lower and upper approximations $\text{app}(A) := \bigcup_{[a]_{\rho} \subseteq A} [a]_{\rho}$ and $\overline{\text{app}}(A) := \bigcup_{[a]_{\rho} \cap A \neq \emptyset} [a]_{\rho}$, where $[a]_{\rho} = \{b \mid a \rho b\}$ is the equivalence class containing $a$. The lower approximation $\text{app}(A)$ is the union of all the elementary sets which are subsets of $A$. The upper approximation $\overline{\text{app}}(A)$ is the union of all the elementary sets which have a non-empty intersection with $A$. An element in the lower approximation necessarily belongs to $A$, while an element in the upper approximation possibly belong to $A$. We can express lower and upper approximations as follows: $\text{app}(A) = \{a \in U \mid [a]_{\rho} \subseteq A\}$ and $\overline{\text{app}}(A) = \{a \in U \mid [a]_{\rho} \cap A \neq \emptyset\}$. If $X \subseteq U$ is given by a predicate $P$ and $x \in U$, then (1) $x \in \text{app}(X)$ means that $x$ certainly has property $P$; (2) $x \in \overline{\text{app}}(X)$ means that $x$ possibly has property $P$; (3) $x \in U \setminus \text{app}(X)$ means that $x$ definitely does not have property $P$.

In [21], the author applied the concept of rough sets to algebraic hyperstructures. Let $R$ be a hyperring ($H_v$-ring, resp.). For a subset $A \subseteq R$ we define two approximations of $A$ relative to the fundamental relation $\gamma^*$ as follows: $\gamma^*(A) = \{x \in R \mid \gamma^*(x) \subseteq A\}$ and $\overline{\gamma^*(A)} = \{x \in R \mid \gamma^*(x) \cap A \neq \emptyset\}$. The set $\gamma^*(A)$ is called the $\gamma^*$-lower approximation of $A$, and the set $\overline{\gamma^*(A)}$ is called the $\gamma^*$-upper approximation of $A$. It is easy to see that (1) $\gamma^*(A) \subseteq A \subseteq \overline{\gamma^*(A)}$, (2) $\gamma^*(\gamma^*(A)) = \gamma^*(A)$ and $\overline{\gamma^*(\overline{\gamma^*(A)})} = \overline{\gamma^*(A)}$. The difference $\overline{\gamma^*(A)} = \overline{\gamma^*(A)} - \gamma^*(A)$ is called the $\gamma^*$-boundary region of $A$. In the case when $\gamma^*(A) = \emptyset$ the set $A$ is said to be $\gamma^*$-exact; otherwise $A$ is $\gamma^*$-rough. If $A$ and $B$ are non-empty subsets of $R$, then (1) $\overline{\gamma^*(A)} + \overline{\gamma^*(B)} \subseteq \overline{\gamma^*(A + B)}$; (2) $\gamma^*(A) \cdot \gamma^*(B) \subseteq \gamma^*(A \cdot B)$. The upper and lower approximations can be represented in an equivalent form as shown below. Let $A$ be a non-empty subsets of $R$. Then $\gamma^*(A) = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \subseteq A\}$ and $\overline{\gamma^*(A)} = \{\gamma^*(x) \in R/\gamma^* \mid \gamma^*(x) \cap A \neq \emptyset\}$. Now, we consider these sets as subsets of the fundamental ring $R/\gamma^*$ of an $H_v$-ring $R$, and we recall some results from [21]. Let $A$ and $B$ are non-empty subsets of $R$, then the following hold:

1. $\gamma^*(A \cup B) = \gamma^*(A) \cup \gamma^*(B)$;
2. $\gamma^*(A \cap B) = \gamma^*(A) \cap \gamma^*(B)$;
3. $A \subseteq B$ implies $\gamma^*(A) \subseteq \gamma^*(B)$;
4. $A \subseteq B$ implies $\overline{\gamma^*(A)} \subseteq \gamma^*(B)$;
5. $\gamma^*(A) \cup \gamma^*(B) \subseteq \gamma^*(A \cup B)$;
6. $\gamma^*(A \cap B) \subseteq \gamma^*(A) \cap \gamma^*(B)$.

If $A$ is an $H_v$-subgroup of $(R, +)$, then $\overline{\gamma^*(A)}$ is a subgroup of $(R/\gamma^*, +)$. If $A$ and $B$ are non-empty subsets of $R$, then $\gamma^*(A) \oplus \gamma^*(B) \subseteq \gamma^*(A + B)$. 
If $A$ is a non-empty subset of $R$ and $B$ is an $H_\gamma$-ideal of $R$, then
\[
\gamma^*(A) \circ \gamma^*(B) \subseteq \gamma^*(B).
\]

If $A$ and $B$ are $H_\gamma$-ideals of $R$, then $\gamma^*(A) \circ \gamma^*(B) \subseteq \gamma^*(A) \cap \gamma^*(B)$.

If $A$ is an $H_\gamma$-ideal of $R$, then $\gamma^*(A)$ is an ideal of $R/\gamma^*$. Let $A$, $B$ and $C$ be $H_\gamma$-ideals of $R$.

The sequence of strong homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be exact if $g \circ f(x) \in \omega_R$, for all $x \in A$. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of $H_\gamma$-ideals of $R$. Then the sequence
\[
\gamma^*(A) \xrightarrow{f} \gamma^*(B) \xrightarrow{g} \gamma^*(C)
\]
is an exact sequence of ideals of $R/\gamma^*$, where $F(\gamma^*(a)) = \gamma^*(f(a))$ and $G(\gamma^*(b)) = \gamma^*(g(b))$, for all $a \in A$ and $b \in B$. For more study about this subject we refer to [22, 25, 68, 69].

4 Topology and hyperstructures

In [12], Ameri studied the concept of a (pseudo, strong pseudo) topological hypergroup and then he gave some related results. In [47], Heidari et al. introduce the concept of topological hypergroups as a generalization of topological groups. Let $(H, \tau)$ be a topological space. In order to construct a topological hypergroup we need a topology on $P^*(H)$. Let $(H, \tau)$ be a topological space. Then, the family $\mathcal{U}$ consisting of all sets $S_V = \{U \in P^*(H) \mid U \subseteq V, U \in \tau\}$ is a base for a topology on $P^*(H)$. This topology is denoted by $\tau^*$. Let $(H, \tau)$ be a topological space. Then, we consider the product topology on $H \times H$ and the topology $\tau^*$ on $P^*(H)$. Let $(H, \circ)$ be a hypergroup and $(H, \tau)$ be a topological space. Then, the system $(H, \circ, \tau)$ is called a topological hypergroup.

If (1) the mapping $(x, y) \mapsto x \circ y$, from $H \times H$ to $P^*(H)$ is continuous; (2) the mapping $(x, y) \mapsto x/y$, from $H \times H$ to $P^*(H)$ is continuous, where $x/y = \{z \in H \mid x \in z \circ y\}$; (3) the mapping $(x, y) \mapsto y/x$, from $H \times H$ to $P^*(H)$ is continuous, where $y/x = \{z \in H \mid x \in y \circ z\}$.

Now, we recall some results from [47]. Let $(H, \circ)$ be a hypergroup and $\tau$ be a topology on $H$.

Then, the following assertions hold: (1) The mapping $(x, y) \mapsto x \circ y$ is continuous if and only if for every $x, y \in H$ and $U \in \tau$ such that $x \circ y \subseteq U$, there exist $V, W \in \tau$ such that $x \in V$, $y \in W$ and $V \circ W \subseteq U$; (2) the mapping $(x, y) \mapsto x/y$ is continuous if and only if for every $x, y \in H$ and $U \in \tau$ such that $x/y \subseteq U$, there exist $V, W \in \tau$ such that $x \in V$, $y \in W$ and $V/W \subseteq U$.

Evidently, every topological group is a topological hypergroup. Suppose that $X$ is a topological space. Let $x$ and $y$ be points in $X$. We say that $x$ and $y$ can be separated by open subsets if there exist open subsets $U$ and $V$ of $X$ containing $x$ and $y$, respectively, such that $U$ and $V$ are disjoint. A Hausdorff space is a topological space in which points can be separated by open subsets. Note that some properties in topological groups do not hold in topological hypergroups. For instance, if $G$ is a topological group and $U$ is an open subset of $G$, then $aU$ is open in $G$ for all $a \in G$. Let $X$ be a topological space and $\sim$ be an equivalence relation on $X$. For every $x \in X$, denote by $[x]$ its equivalence class. The quotient space of $X$ modulo $\sim$ is given by the set $X/\sim = \{[x] \mid x \in X\}$. We have the projection map $p : X \to X/\sim, x \mapsto [x]$ and we equip $X/\sim$ by the topology: $U \subseteq X/\sim$ is open if and only if $p^{-1}(U)$ is an open subset of $X$. Let $A$ be a subset of the topological space $X$ and $\sim$ be an equivalence relation on $X$. Then, the saturation of $A$ with respect to $\sim$ is the set $\hat{A} = \{x \in X \mid \exists a \in A, x \sim a\}$. If $\hat{A} = A$, then $A$ is called saturated. Let $(H, \circ, \tau)$ be a topological hypergroup such that every open subset of $H$ is a complete part. Then, $(H/\beta^*, \circ, \tau)$ is a topological group. Let $(G, \cdot)$ be a topological group and $H$ be a non-normal subgroup of it. Let $\beta^*$ be the fundamental relation of the hypergroup $(G/H, \circ)$. Then, there exists a normal subgroup $N$ of $G$ such that the topological groups $(G/H)/\beta^*$ and $G/N$ are topological isomorphic.
Then, in [18], Heidari et al. introduced the concept of topological polygroups. By considering the relative topology on subpolygroups they proved some properties of them. Also, the topological isomorphism theorems of topological polygroups are proved. Salehi Shadkami et all in [65] presented some facts about complete parts in polygroups and they used these facts to obtain some new results in topological polygroups. They defined the concept of cp-resolvable topological polygroups. A non-empty subset $X$ of a topological polygroup is called cp-resolvable if there exist disjoint dense subsets $A$ and $B$ such that at least one of them is a complete part. Then, they investigated a few properties of cp-resolvable topological polygroups. Also, in [66], they established various relations between complete parts and open sets. They studied the properties of big subsets in a topological polygroup. Al Tahan et al. [11], showed that there is no relation (in general) between pseudotopological and strongly pseudotopological hypergroupoids. In particular, they presented a few results in topological polygroups. They defined the concept of cp-resolvable topological polygroups. We observe that $(G, \ast)$ is a commutative hypergroup and $(\mathcal{AF}(G), \ast)$ is a normal subhypergroup of $(G, \ast)$. Let $G$ be the set of all arithmetic functions and $\mathcal{AF}(G)$ the set of all additive functions of $G$ and by $\varphi^*(G)$ the set of all non-empty subsets of $G$. Now, we define a hyperoperation $\ast$ on $G$. Define a hyperoperation on $G$ as follows: $\ast : G \times G \rightarrow \varphi^*(G)$, $(\alpha, \beta) \mapsto \alpha \ast \beta$ such that 
\[
(\alpha \ast \beta)(n) = \left\{ \alpha(d) + \beta\left(\frac{n}{d}\right) : d \mid n \right\} = \bigcup_{d \mid n} \alpha(d) + \beta\left(\frac{n}{d}\right).
\]
Let $\alpha$ and $\beta$ be two elements in $G$. If $\alpha(n) = \beta(n)$ for all natural numbers $n$, then $\alpha = \beta$. We observe that $(G, \ast)$ is a commutative hypergroup and $(\mathcal{AF}(G), \ast)$ is a normal subhypergroup of $(G, \ast)$. Let $G$ be the set of all arithmetic functions. Define a map $\ast$ on $G \ast G$ as follows: 
\[
\ast : (G \ast G) \times (G \ast G) \rightarrow \varphi^*(G), ((\alpha_1, \beta_1), (\alpha_2, \beta_2)) \mapsto (\alpha_1 \ast \beta_1) \ast (\alpha_2 \ast \beta_2)
\]
such that for all natural numbers $m$ and $n$
\[
((\alpha_1 \ast \beta_1) \ast (\alpha_2 \ast \beta_2))(m, n) = \bigcup_{\alpha \in (\alpha_1 \ast \beta_1)(m), \beta \in (\alpha_2 \ast \beta_2)(n)} \alpha + \beta.
\]
Let $G$ be the set of all arithmetic functions and $m, n$ be natural numbers. Then
\begin{enumerate}
  
  \item $((\alpha_1 \ast \beta_1) \ast (\alpha_2 \ast \beta_2))(m, n) = ((\alpha_2 \ast \beta_2) \ast (\alpha_1 \ast \beta_1))(n, m)$ for all $\alpha_1, \alpha_2, \beta_1$ and $\beta_2 \in G$.
  
  \item $(G \ast G, \ast)$ is associative.
\end{enumerate}

Let $\alpha$ and $\beta \in G$. Then $\alpha \ast \beta$ is a multiplicative function in $G \ast G$ if for all coprime natural numbers $m$ and $n$ the following condition holds: $\alpha \ast \beta(mn) = (\alpha \ast \beta)(m) \ast (\alpha \ast \beta)(n)$. We denote

\section{Number theory and hyperstructures}

In [17], Asghari and Davvaz introduced a hyperoperation associated to the set of all arithmetic functions and analyzed the properties of this hyperoperation. In [6], Al Tahan and Davvaz defined a new hyperoperation associated to the set of all arithmetic functions and $\mathcal{AF}(G)$ the set of all additive functions of $G$ and by $\varphi^*(G)$ the set of all non-empty subsets of $G$. Now, we define a hyperoperation $\ast$ on $G$. Define a hyperoperation on $G$ as follows: $\ast : G \times G \rightarrow \varphi^*(G)$, $(\alpha, \beta) \mapsto \alpha \ast \beta$ such that 
\[
(\alpha \ast \beta)(n) = \left\{ \alpha(d) + \beta\left(\frac{n}{d}\right) : d \mid n \right\} = \bigcup_{d \mid n} \alpha(d) + \beta\left(\frac{n}{d}\right).
\]
Let $\alpha$ and $\beta$ be two elements in $G$. If $\alpha(n) = \beta(n)$ for all natural numbers $n$, then $\alpha = \beta$. We observe that $(G, \ast)$ is a commutative hypergroup and $(\mathcal{AF}(G), \ast)$ is a normal subhypergroup of $(G, \ast)$. Let $G$ be the set of all arithmetic functions. Define a map $\ast$ on $G \ast G$ as follows: 
\[
\ast : (G \ast G) \times (G \ast G) \rightarrow \varphi^*(G), ((\alpha_1, \beta_1), (\alpha_2, \beta_2)) \mapsto (\alpha_1 \ast \beta_1) \ast (\alpha_2 \ast \beta_2)
\]
such that for all natural numbers $m$ and $n$
\[
((\alpha_1 \ast \beta_1) \ast (\alpha_2 \ast \beta_2))(m, n) = \bigcup_{\alpha \in (\alpha_1 \ast \beta_1)(m), \beta \in (\alpha_2 \ast \beta_2)(n)} \alpha + \beta.
\]
Let $G$ be the set of all arithmetic functions and $m, n$ be natural numbers. Then
\begin{enumerate}
  
  \item $((\alpha_1 \ast \beta_1) \ast (\alpha_2 \ast \beta_2))(m, n) = ((\alpha_2 \ast \beta_2) \ast (\alpha_1 \ast \beta_1))(n, m)$ for all $\alpha_1, \alpha_2, \beta_1$ and $\beta_2 \in G$.
  
  \item $(G \ast G, \ast)$ is associative.
\end{enumerate}

Let $\alpha$ and $\beta \in G$. Then $\alpha \ast \beta$ is a multiplicative function in $G \ast G$ if for all coprime natural numbers $m$ and $n$ the following condition holds: $\alpha \ast \beta(mn) = (\alpha \ast \beta)(m) \ast (\alpha \ast \beta)(n)$. We denote
by $AF(G \ast G)$ the set of all additive functions in $G \ast G$. Let $\alpha, \beta \in G$ and $m, n$ be two natural numbers. Then, we have

$$\bigcup_{d|m} \alpha(d) + \beta(D) = \bigcup_{d|m} \alpha(d) \ast \bigcup_{D|n} \beta(D).$$

If $\alpha$ and $\beta \in AF(G)$ then $\alpha \ast \beta \in AF(G \ast G)$. We define a hyperstructure on $G$ as follows: $\circ : G \times G \rightarrow \varphi^* (G)$, with $(\alpha, \beta) \mapsto \alpha \circ \beta$ such that $(\alpha \ast \beta)(n) = \bigcup_{d|m} \alpha(d) \beta(\frac{n}{d})$. If $\alpha$ is a multiplicative function that admits an inverse in $(G, \circ)$, then $\alpha = 1$. If $\alpha \in (G, \circ)$ such that $\alpha^{-1} \in G$, then $\alpha(1) \neq 0$ and $\alpha^{-1}(1) = \frac{1}{\alpha(1)}$. If $\alpha \in (G, \circ)$ such that $\alpha(1) = a \neq 0$ and $\alpha^{-1} \in G$, then

$$\alpha(n) = \begin{cases} a, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $\alpha \in (G, \circ)$ with $\alpha(1) = a \neq 0$, then $\alpha^{-1} \in G$ if and only if

$$\alpha(n) = \begin{cases} a, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We define $O_* : \mathbb{N} \rightarrow \mathbb{C}$ as $O_*(n) = 0$ for all $n \in \mathbb{N}$. Note that $O_* \in AF(G)$. If $\alpha \in G$ then $\alpha \in \alpha \ast O_*$. An element $\lambda$ is said to be identity in $(G, \ast)$ if for all natural numbers $n$, $\alpha \ast \lambda(n) = \bigcup_{d|m} \alpha(d)$. An element $\alpha^{-1}$ is said to be an inverse of $\alpha$ in $(G, \ast)$ if $\alpha \ast \alpha^{-1} = O_*$. We see that $(G, \ast)$ has unique identity. If $\alpha$ is an element in $(G, \ast)$ that admits an inverse $\alpha^{-1}$ in $(G, \ast)$, then $\alpha^{-1}$ is unique. If $\alpha \in AF(G)$ such that its inverse $\alpha^{-1}$ exists, then $\alpha^{-1} \in AF(G)$.

If $\alpha \in (G, \ast)$, then $\alpha^{-1} \in G$ if and only if $\alpha$ is a constant function. A set $W$ associated to the hyperoperations $+$ and $\cdot$ is said to be weak distributive if $x \cdot (y \cdot z) \cap x \cdot y \cdot z \neq \emptyset$ and $(x + y) \cdot z \cap x \cdot z + y \cdot z \neq \emptyset$ whenever $x, y$ and $z$ are in $W$. A set $W$ associated to the hyperoperations $+$ and $\cdot$ is said to be weak hyperring if the following conditions are satisfied: $(W, +)$ is a hypergroup; $(W, \cdot)$ is a semi-hypergroup; $(W, \cdot)$ is weak distributive. We observe that $(G, +, \cdot)$ is a weak hyperring. Denote by $(M, +, \cdot)$ the set of all constant arithmetic functions under the hyperoperations of $G$ and by $(N, +, \cdot)$ the largest distributive set contained in $(G, +, \cdot)$. If $M$ is the set of all constant arithmetic functions in $G$, then $(M, +, \cdot)$ is a Krasner hyperring. Moreover, $(M, +)$ is a join space with scalar identity. If $\alpha \in N$ with $\alpha(1) \neq 0$, then $\alpha(n) = \alpha(1)$, for all $n \in \mathbb{N}$. If $\alpha \in N$ with $\alpha(1) = 0$, then $\alpha(n) = 0$, for all $n \in \mathbb{N}$. If $\alpha \in N$, then $\alpha$ is a constant function. The largest hyperring contained in $(G, +, \cdot)$ is $(M, +, \cdot)$. Finally, there is no hyperfield contained in $(G, +, \cdot)$. Also, in [7], Al Tahan and Davvaz determined fundamental groups and fundamental rings of hyperstructures of arithmetic functions. In addition, they investigated their complete parts and strongly regular regular relations.

6 $n$-ary hypergroups and there extension to hyperrings and hypermodules

The notion of an $n$-ary group was introduced by Dornte which is a natural generalization of the notion of a group. $n$-ary generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. Since then many papers concerning various $n$-ary algebra have appeared in the literature. In [40], Davvaz and Vougiouklis introduced the notion of $n$-ary hypergroups. Let $H$ be a non-empty set and $f$ be a mapping $f : H \times H \rightarrow \varphi^*(H)$, where $\varphi^*(H)$ is the set of all non-empty subsets of $H$. Then $f$ is called a binary hyperoperation on $H$. We denote by $H^n$ the cartesian product $H \times \ldots \times H$ where $H$ appears $n$ times.
An element of $H^n$ will be denoted by $(x_1, \ldots, x_n)$ where $x_i \in H$ for any $i$ with $1 \leq i \leq n$. In general, a mapping $f : H^n \to \varphi^*(H)$ is called an $n$-ary hyperoperation. Let $f$ be an $n$-ary hyperoperation on $H$ and $A_1, \ldots, A_n$ subsets of $H$. We define 
$f(A_1, \ldots, A_n) = \bigcup \{f(x_1, \ldots, x_n) | x_i \in A_i, i = 1, \ldots, n\}$.

We use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by $x_i^j$.

For $j < i$, $x_i^j$ is the empty set. In this convention $f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$ is written as $f(x_1^i, y_{i+1}, z_{j+1})$. A non-empty set $H$ with an $n$-ary hyperoperation $f : H^n \to \varphi^*(H)$ is called an $n$-ary hypergroupoid and is denoted by $(H, f)$. An $n$-ary hypergroupoid $(H, f)$ will be called an $n$-ary semihypergroup if and only if the following associative axiom holds:

\[
f(x_1^{i-1}, f(x_i^{n+i-1}, x_{n+i}^{2n-1})) = f(x_1^{j-1}, f(x_j^{n+j-1}, x_{n+j}^{2n-1}))
\]

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$. If for all $(a_1, a_2, \ldots, a_n) \in H^n$, the set $f(a_1, a_2, \ldots, a_n)$ is singleton, then $f$ is called an $n$-ary operation and $(H, f)$ is called an $n$-ary groupoid (resp. $n$-ary semigroup). If $m = k(n - 1) + 1$, then the $m$-ary hyperoperation $g$ by

\[
g(x_1^{k(n-1)+1}) = f(f(\ldots, f(f(\underbrace{x_1^{k-1}, x_i^k, a_{i+1}^n}_{k})\ldots)))
\]

has solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \leq i \leq n$, is called an $n$-ary hypergroup. If $f$ is $n$-ary operation then [1] is as follows: $b = f(a_1^{i-1}, x_1, a_{i+1}^n)$. In this case $(H, f)$ is an $n$-ary group. The important question is the solvability of [1]. Let $(H, f)$ be an $n$-ary semihypergroup. Then $(H, f)$ is an $n$-ary hypergroup if and only if [1] is solvable at the place $i = 1$ and $i = n$ or at least one place $1 < i < n$. The reproduction axiom can be formulated for $n$-ary hypergroups as follows: $f(H^{i-1}, x, H^{n-i}) = H$ for all $x \in H$ and $i = 1, \ldots, n$. Let $(H, f)$ be an $n$-ary hypergroup. If the value of $f(x_1, x_2, \ldots, x_n)$ is independent on the permutation of elements $x_1, x_2, \ldots, x_n$, then $(H, f)$ is called a commutative $n$-ary hypergroup. The element $a \in H$ is called a scalar if $|f(x_1^i, a, x_{i+2}^n)| = 1$, for all $x_1, \ldots, x_i, x_{i+2}, \ldots, x_n \in H$. Element $e$ of an $n$-ary hypergroup $(H, f)$ is called neutral (identity) element if $f(e, \ldots, e, x, e, \ldots, e)$ includes $x$, for all $x \in H$ and all

\[1 \leq i \leq n\]

If $(H, f)$ is a commutative $n$-ary hypergroup and $a \in H$ is a scalar element such that $f(a, e, \ldots, e) = a$ for some $e \in H$, then $e$ is a neutral element. If the set of all neutral elements of a given commutative $n$-ary hypergroup is non-empty, then it is an $n$-ary group. Let $(H, f)$ be an $n$-ary hypergroup and $B$ be a non-empty subset of $H$. Then $B$ is an $n$-ary subhypergroup of $H$ if the following conditions hold: (1) $B$ is closed under the $n$-ary hyperoperation $f$, i.e., for every $(x_1, \ldots, x_n) \in B^n$ implies that $f(x_1, \ldots, x_n) \subseteq B$; (2) Equation $b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$ has the solution $x_i \in B$ for every $b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n, b \in B$ and $1 \leq i \leq n$.

Let $(H, f)$ be an $n$-ary hypergroup. An equivalence relation $\theta$ on $H$ is called compatible if $a_1 \theta b_1, \ldots, a_n \theta b_n$, then for all $a \in f(a_1, \ldots, a_n)$ there exists $b \in f(b_1, \ldots, b_n)$ such that $a \theta b$. An equivalence relation $\theta$ is called strongly compatible if $a_1 \theta b_1, \ldots, a_n \theta b_n$ implies that $a \theta b$ for all
If \( a \in f(a_1, \ldots, a_n) \) and \( b \in f(b_1, \ldots, b_n) \). If \( (H, f) \) is an \( n \)-ary hypergroup and \( \theta \) a compatible relation on \( H \), then \( (H/\theta, f|_{\theta}) \) is an \( n \)-ary hypergroup where

\[
f|_{\theta}(\theta(a_1), \ldots, \theta(a_n)) = \{ \theta(a) \mid a \in f(a_1, \ldots, a_n) \}.
\]

The natural map \( \pi : H \to H/\theta \) where \( \pi(x) = \theta(x) \) is an onto homomorphism. Let \( (A, f) \) and \( (B, g) \) be two \( n \)-ary hypergroups and let \( f : A \to B \) be a homomorphism. Then the kernel of \( f \), written \( \ker f \), is defined by \( \ker f = \{(a, b) \in A^2 \mid f(a) = f(b)\} \). It is easy to see that \( \ker f \) is a compatible relation. Let \( (A, f) \) and \( (B, g) \) be two \( n \)-ary hypergroups and let \( \varphi : A \to B \) be a homomorphism. Then there exists a compatible relation \( \theta \) on \( A \) and a monomorphism \( \psi : A/\theta \to B \) such that \( \psi \circ \varphi = \varphi \). If \( \rho \) and \( \theta \) are compatible relations on an \( n \)-ary hypergroup \( (H, f) \) such that \( \rho \subseteq \theta \), then there exists a compatible relation \( \mu \) on \( (H/\rho, f/\rho) \) such that \( (H/\rho)/\mu \) is isomorphic to \( H/\theta \).

We refer the reader to \([35, 34, 54, 55, 56, 57, 64]\).

If \( a, b \in H \), then \( H/\beta \) is a pair of factor compatible relations on \( H \). A compatible relation \( \theta \) on \( H \) is called a transitive closure of the relation \( \theta \). The diagonal relation \( \Delta \) on \( H \) is the transitive closure of the relation \( \triangle \). Let \( \pi_1, \pi_2 \) be two \( n \)-ary hypergroups. Define the direct hyperproduct \( \langle A_1 \times A_2, f_1 \times f_2 \rangle \) to be the \( n \)-ary hypergroup whose universe is the set \( A_1 \times A_2 \) and such that for \( a_i \in A_i \), \( a_i \in A_2, 1 \leq i \leq n \), \( (f_1 \times f_2)((a_1, a_1'), \ldots, (a_n, a_n')) = \{ (a, a') \mid a \in f_1(a_1, \ldots, a_n), a' \in f_2(a_1', \ldots, a_n') \} \). The mapping \( \pi_i : A_1 \times A_2 \to A_i, i = 1, 2 \), defined by \( \pi_i((a_1, a_2)) = a_i \), is the projection map on the ith coordinate of \( A_1 \times A_2 \). For \( i = 1, 2 \), the mapping \( \pi_i : A_1 \times A_2 \to A_i \) is an onto homomorphism. Furthermore, we have (1) \( \ker \pi_1 \cap \ker \pi_2 = \Delta \); (2) \( \ker \pi_1 \) and \( \ker \pi_2 \) are permute; (3) \( \ker \pi_1 \cap \ker \pi_2 = \nabla \), where \( \ker \pi_i = \{ ((a_1, a_2), (b_1, b_2)) \mid \pi_i(a_1, a_2) = \pi_i(b_1, b_2) \} \). Note that \( ((a_1, a_2), (b_1, b_2)) \in \ker \pi_1 \iff \pi_i((a_1, a_2)) = \pi_i((b_1, b_2)) \iff a_i = b_i \). Thus \( \ker \pi_1 \cap \ker \pi_2 = \Delta \). Also, \( (a_1, a_2) \) and \( (b_1, b_2) \) are any two elements of \( A_1 \times A_2 \), then \( (a_1, a_2) \) \( \ker \pi_1 \) and \( (b_1, b_2) \) \( \ker \pi_2 \), and so \( \nabla = \ker \pi_1 \cap \ker \pi_2 \). But, then \( \ker \pi_1 \) and \( \ker \pi_2 \) are permute, and their join is \( \nabla \). Let \( (H, f) \) be an \( n \)-ary hypergroup. A compatible relation \( \theta \) on \( H \) is called a \( \theta \)-ary hypergroup if there is a compatible relation \( \theta^* \) on \( H \) such that \( \theta \cap \theta^* = \Delta, \theta \cap \theta^* = \nabla \) and \( \theta \) and \( \theta^* \) permute with \( \theta^* \). The pair \( \theta, \theta^* \) is called a \( \theta \)-ary hypergroup.

Mirkavili and Davvaz in \([60, 61]\) introduced the concept of \((m, n)\)-hyperrings. Here, we present some definitions and results. An \((m, n)\)-hyperring is an algebraic hyperstructure \((R, f,g)\), which satisfies the following axioms:

1. \((R,f)\) is an \(m\)-ary hypergroup,
2. \((R,g)\) is an \(n\)-ary hypersemigroup,
3. the \(n\)-ary hyperoperation \( g \) is distributive with respect to the \(m\)-ary hyperoperation \( f \), i.e., for every \( a_1^{-1}, a_1^n, x_m^n \in R \), we have

\[
g(a_1^{-1}, f(x_1^n, a_1^n, x_1^{i+1}), \ldots, g(a_1^{-1}, x_m, a_1^{n+1}))
\]
$(R, f, g)$ is called an $n$-ary hyperring if $n = m$. If $(R, f)$ is an $m$-ary semihypergroup, then $(R, f, g)$ is called an $(m, n)$-semihypergroup. In $(m, n)$-hyperring $(R, f, g)$, if $f$ is an $m$-ary operation then $(R, f, g)$ is called an $(m, n)$-multiplicative hyperring and if $g$ be an $n$-ary operation then $(R, f, g)$ is called an additive $(m, n)$-hyperring. A multiplicative and additive $(m, n)$-hyperring is called an $(m, n)$-ring. A non-empty subset $S \subseteq R$ is called an $(m, n)$-subhyperring if $(S, f, g, R)$ is an $(m, n)$-hyperring. Let $i \in \{1, \ldots, n\}$. A $i$-hyperideal $I$ of $R$ is an $(m, n)$-subhyperideal of $R$ such that for every $r_1^n \in R$, $g(r_1^{i-1}, I, r_{n+1}^n) \subseteq I$. If $I$ is an $i$-hyperideal and for every $r_1^n \in R$, $g(r_1^{i-1}, I, r_{n+1}^n) = I$, then $I$ called a strong $i$-hyperideal. A non-empty subset $I$ of $R$ is called (a) an $(m, n)$-hyperideal if $I$ is (a) an $(m, n)$-hyperideal of $R$ for every $i \in \{1, \ldots, n\}$.

For any $(m, n)$-hyperring $(R, f, g)$ and $I \subseteq R$ the following conditions are equivalent: (1) $I$ is a strong $(m, n)$-hyperideal of $R$; (2) $I$ is a strong $i$-hyperideal of $R$ for $i = 1$ and $i = n$; (3) $I$ is a strong $i$-hyperideal of $R$ for some $1 < i < n$. An element $o$ is called a (scalar) zero of $(R, f, g)$ if it is a (scalar) identity of $(R, f)$ and for every $x_2^n \in R$ we have $(o = f(o, x_2^n) = f(x_2, o, x_3^n) = \ldots = f(x_3^n, o))$. Let $R, S$ be an $(m, n)$-hyperoperation and $\phi : R \rightarrow S$ be an $(m, n)$-hyperoperation. Let $\phi(S_1) = \{\phi(s) : s \in S_1\}$ be an $(m, n)$-hyperoperation of $R$, $S_2$ be an $(m, n)$-hyperoperation of $R$ and $\phi : S_1 \rightarrow S_2$ be an $(m, n)$-hyperoperation. A mapping $\phi : R \rightarrow S_2$ is called an $i$-hyperzero for some $1 < i < n$, i.e., for every $x_1^n \in R$, $g(x_1^{i-1}, o, x_{n+1}^n) = o$; (3) $o$ is a scalar $i$-zero for $i = 1$ and $i = n$, i.e, $g(o, x_1^n) = o = g(x_1^{i-1}, o)$, for all $x_1^n \in R$. Let $(R_1, f, g)$ and $R_2, f, g)$ be two $(m, n)$-hyperoperations. An $(m, n)$-hyperoperation $\phi$ from $R_1$ to $R_2$ is a mapping $\phi : R_1 \rightarrow R_2$ such that $\phi(f_1(a_1^n)) = f_2(\phi(a_1), \ldots, \phi(a_n))$ and $\phi(f_1(b_1^n)) = g_2(\phi(b_1), \ldots, \phi(b_n))$ hold for all $a_1^n, b_1^n \in R_1$. If $\phi$ is injective, then it is called embedding. The map $\phi$ is an isomorphism if $\phi$ is injective and onto. We say that $R_1$ is isomorphic to $R_2$, denote $R_1 \cong R_2$, if there is an isomorphism from $R_1$ to $R_2$. Let $\phi : R_1 \rightarrow R_2$ be a homomorphism and $S_1$ be an $(m, n)$-hyperoperation of $R_1$ and $S_2$ be an $(m, n)$-hyperoperation of $R_2$, then $\phi(S_1)$ is an $(m, n)$-hyperoperation of $R_2$ and if $\phi^{-1}(S_2)$ is non-empty, then $\phi^{-1}(S_2)$ is an $(m, n)$-hyperoperation of $R_1$. Let $\phi : R_1 \rightarrow R_2$ be a homomorphism, then the kernel $\phi$, is defined by $\ker \phi = \{(a, b) \in R_1 \times R_1 \mid \phi(a) = \phi(b)\}$. If $b, c \in R$ then we say that an $(m, n)$-hyperringoid $(R, f, g)$ is a $(b, c)$-derived from a hyperringoid $(R, +, \cdot)$ and denote this fact by $(R, f, g) = der_b^c(R, +, \cdot)$ if two $m$-ary hyperoperation and $n$-ary hyperoperation $f$ and $g$ respectively, have the form

$$f(x_1^n) = \sum_{i=1}^m x_i + b, \text{ for all } x_1^n \in R,$$

and

$$g(x_1^n) = \prod_{j=1}^n y_j \cdot c, \text{ for all } y_1^n \in R.$$

In this case, when $b$ is a zero scalar of $(R, +)$ and $c$ is an identity scalar of $(R, \phi)$ we say that $(R, f, g)$ is derived from $(R, +, \cdot)$ and denote this fact by $(R, f, g) = der(R, +, \cdot)$. It is clear that if $b$ belongs to the center of a semihypergroup $(R, +)$ and $c$ belongs to the center of a semihypergroup $(R, \cdot)$ then two $m$-ary hyperoperation and $n$-ary hyperoperation $f$ and $g$ are associative and $(R, f)$ and $(R, g)$ are $m$-ary semihypergroup and $n$-ary semihypergroup. Now, if $b$ is zero scalar or $f$ define by $f(x_1^n) = \sum_{i=1}^m x_i$ then denote $(R, f, g) = der_b(R, +, \cdot)$ and say $(R, f, g)$ is $c$-derived from $(R, +, \cdot)$. Now, if $(R, +, \cdot)$ be a hyperring and $c \in Z(R, \cdot)$ Then, the $c$-derived $(R, f, g)$ is an $(m, n)$-hyperring.

If $(R, f, g)$ is an $(m, n)$-hyperring and the relation $\rho$ is a strongly compatible relation on both $m$-ary hypergroup $(R, f)$ and $n$-ary semihypergroup $(R, g)$, then the quotient $(R/\rho, f/\rho, g/\rho)$ is an $(m, n)$-ring.

Let $(R, f, g)$ be an $(m, n)$-hyperring. For every $k \in \mathbb{N}^*$ and $l_i^s \in \mathbb{N}$, where $s = k(m-1)+1$, we define the relation $\gamma_{k,l_i^s}$, as follows: $x \gamma_{k,l_i^s} y$ if and only if there exist $x_{i1}^{l_i^s} \in R$, where $t_i = l_i(n-1)+1$,
i = 1, \ldots, s such that \{x, y\} \subseteq f(k)(u_1, \ldots, u_s), where for every i = 1, \ldots, s, u_i = g(u)(x_iu_i). Now, set \( \gamma_k = \bigcup_{i \in \mathbb{N}} \gamma_{k;l_i^1} \) and \( \gamma = \bigcup_{k \in \mathbb{N}^*} \gamma_k \). Then, the relation \( \gamma \) is reflexive and symmetric. Let \( \gamma^* \) be the transitive closure of relation \( \gamma \). It easy to see that \( \beta_f \subseteq \gamma, \beta_f^* \subseteq \gamma^* \), \( \beta_g \subseteq \gamma \) and \( \beta_g^* \subseteq \gamma^* \). If \((R, f, g)\) is an \((m, n)\)-hyperring, then for every \( k \in \mathbb{N}^* \) we have \( \gamma_k \subseteq \gamma_{k+1} \). If \((R, f, g)\) is an \((m, n)\)-hyperring, then for every \( k \in \mathbb{N}^* \) we have \( \gamma_k^* \subseteq \gamma_{k+1}^* \). The relation \( \gamma^* \) is a strongly compatible relation on both \( m \)-ary hypergroup \((R, f)\) and \( n \)-ary semihypergroup \((R, g)\). The quotient \((R/\gamma^*, f/\gamma^*, g/\gamma^*)\) is an \((m, n)\)-ring. The relation \( \gamma^* \) is the smallest equivalence relation such that the quotient \((R/\gamma^*, f/\gamma^*, g/\gamma^*)\) is an \((m, n)\)-ring. For all additive \((m, n)\)-hyperrings, we have \( \gamma^* = \beta_f^* \). For every additive \((m, n)\)-hyperring, the relation \( \gamma \) is an equivalence relation, i.e. \( \gamma = \gamma^* \). If \((R, f, g)\) is an \((m, n)\)-hyperring, then

1. \((R/\beta_f, f/\beta_f, g/\beta_f)\) is an \((m, n)\)-multiplicative hyperring;
2. \((R/\beta_g, f/\beta_g, g/\beta_g)\) is an additive \((m, n)\)-hyperring.

If \((R, f, g)\) is an \((m, n)\)-hyperring, then \( R/\beta_f \cong (R/\beta_f^*)/\beta_f/\beta_f^* \). Let \((R_1, f, g)\) and \((R_2, f, g)\) be two \((m, n)\)-hypersemirings. We define \((f_1, f_2) : (A \times B)^m \rightarrow \varphi(A \times B)\) by \((f, g)((a_1, b_1), \ldots, (a_n, b_n)) = \{(a, b) \mid a \in f(a_1, \ldots, a_n), b \in g(b_1, \ldots, b_n)\}\). Clearly \( (R_1 \times R_2, (f_1, f_2), (g_1, g_2)) \) is an \((m, n)\)-semihypergroup and we call this \((m, n)\)-semihyperring the direct hyperproduct of \( R_1 \) and \( R_2 \). Let \((R_1, f, g)\) and \((R_2, f, g)\) be two \((m, n)\)-hypersemirings, \( a, c \in R_1 \) and \( b, d \in R_2 \). If \( \gamma_{R_1}, \gamma_{R_2} \) and \( \gamma^*_{R_1 \times R_2} \) are the \( \gamma^* \)-relations on \( R_1 \), \( R_2 \) and \( R_1 \times R_2 \) respectively. Then, \( (a, b) \gamma^*_{R_1 \times R_2} (c, d) \) implies \( a \gamma_{R_1} c \) and \( b \gamma^*_{R_2} d \). Let \( a, c \in R_1 \) and \( b, d \in R_2 \). If \( \gamma^*_{R_1}, \gamma^*_{R_2} \) and \( \gamma^*_{R_1 \times R_2} \) are \( \gamma^* \)-relations on \( R_1 \), \( R_2 \) and \( R_1 \times R_2 \) respectively: \( (a, b) \gamma^*_{R_1 \times R_2} (c, d) \) if and only if \( a \gamma_{R_1} c \) and \( b \gamma_{R_2} d \). Then, we have \((R_1 \times R_2)/\gamma^*_{R_1 \times R_2} \cong R_1/\gamma^*_{R_1} \times R_2/\gamma^*_{R_2} \). To extend the concept of \( n \)-ary hyperstructures to hypermodules, we refer to [15] [16].

7 Applications of hyperstructures in biology, physics and chemistry

Mendel, the father of genetics took the first steps in defining “contrasting characters, genotypes in \( F_1 \) and \( F_2 \) . . . and setting different laws”. The genotypes of \( F_2 \) is dependent on the type of its parents genotype and it follows certain roles. In [46], Ghadiri, Davvaz and Nekouian analyzed the second generation genotypes of monohybrid and a dihybrid with a mathematical structure. They used the concept of \( H_v \)-semigroup structure in the \( F_2 \)-genotypes with cross operation and proved that this is an \( H_v \)-semigroup. They determined the kinds of number of the \( H_v \)-subsemigroups of \( F_2 \)-genotypes. Isso, in [30], inheritance issue based on genetic information is looked at carefully via a new hyperalgebraic approach. Several examples are provided from different biology points of view, and it is shown that the theory of hyperstructures exactly fits the inheritance issue. In [8], Al Tahan and Davvaz presented examples of five different types of Non-Mendelian inheritance and studied their relation with hyperstructure theory. They made some hypothetical crosses for the \( n \)- hybrid case for both simple and incomplete inheritances and studied their relations with hyperstructures. In [9], the authors considered \( n \)-ary hyperstructures associated to the genotypes of the second generation \( F_2 \) for \( n = 2, 3, 4 \). They defined a hyperoperation \( (mating) \) on \( F_2 \) and proved that it is a cyclic \( H_v \)-semigroup under the defined hyperoperation. Then they defined a ternary hyperstructure \( f \) associated to the genotypes of \( F_2 \) and proved that \((F_2, f)\) is a ternary \( H_v \)-semigroup. Finally, they defined a 4-ary hyperstructure \( g \) associated to the genotypes of \( F_2 \) and proved that \((F_2, g)\) is a 4-ary \( H_v \)-semigroup.
In 1996, R. M. Santilli and T. Vougiouklis [67] point out that in physics the most interesting hyperstructures are the one called \( e \)-hyperstructures. \( e \)-hyperstructures are a special kind of hyperstructures and they can be interpreted as a generalization of two important concepts for physics: Isotopies and Genotopies. In [37], Davvaz, Santilli and Vougiouklis studied multi-valued hyperstructures following the apparent existence in nature of a realization of two-valued hyperstructures with hyperunits characterized by matter-antimatter systems and their extensions, where matter is represented with conventional mathematics and antimatter is represented with isodual mathematics. Also see [38]. In [39], the authors presented Ying’s twin universes, Santilli’s isodual theory of antimatter, and Davvaz-Santilli-Vougiouklis two-valued hyperstructures representing matter and antimatter in two distinct but co-existing space times. They identified a seemingly new map for both matter and antimatter providing a mathematical prediction of Ying’s twin universes, and introduced a four-fold hyperstructure representing matter-antimatter as well as Ying’s twin universes, all co-existing in distinct space times. Another motivation for the study of hyperstructures comes from physical phenomenon as the nuclear fission. This motivation and the results were presented by S. Hošková, J. Chvalina and P. Račková (see [50], HCR2). In [43], the authors provided, for the first time, a physical example of hyperstructures associated with the elementary particle physics, Leptons. They have considered this important group of the elementary particles and shown that this set along with the interactions between its members can be described by the algebraic hyperstructures. The Standard Model (SM) of particle physics is a gauge theory including the Higgs boson, which plays a unique role in the SM. In the SM, all the elementary particles are classified into three generations of matter, i.e., Hadrons, Leptons and Gauge Bosons. In [33], Davvaz et al. showed that the leptons and gauge bosons along with the interactions between their members construct a weak algebraic hyperstructure. This new sight to the elementary particles would make a new arrangement to the elementary particles.

Another motivation for the study of hyperstructures comes from chemical reactions. In [24], Davvaz presented an introduction to some of the results, methods and ideas about chemical examples of weak algebraic hyperstructures. Some of these examples include

1. Weak algebraic hyperstructures associated with chain reactions [28];
2. Weak algebraic hyperstructures associated with dismutation reactions [29];
3. Weak algebraic hyperstructures associated with redox reactions [31].

Also, see [10, 18, 25, 32].

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