Equidistribution of decompositions of 132-avoiding permutations and consequences

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Abstract. In this paper, we mainly examine four distinct ways of decomposing a 132-avoiding permutation into subsequences, two of them refining ascents and the rest refining descents of permutations. We show combinatorially that the subsequence length distributions of the four decompositions are mutually equivalent, and there is a way to group the four into two groups such that each group is symmetric and the joint length distribution of one group is the same as that of the other. As a consequence, we obtain enumerative results on various restricted 132-avoiding permutations. Notably, we discover the probably first new characteristic for vertices of plane trees that is equally distributed as the heights of vertices, which in addition refines the classical result that internal vertices and leaves are equidistributed.

Keywords: permutation pattern, plane tree, height of a vertex, increasing run, descent
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1 Introduction

Permutations with or without certain patterns have been extensively studied since Knuth’s work [14]. Let \([n] = \{1, 2, \ldots, n\}\) and \(\mathcal{S}_n\) be the symmetric group of permutations on \([n]\). Let \(\tau = \tau_1\tau_2\cdots\tau_m \in \mathcal{S}_m\) with \(m \leq n\). A permutation \(\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n\) is said to have a pattern \(\tau\) if there exists a subsequence \(\pi_{i_1}\pi_{i_2}\cdots\pi_{i_m}\) of \(\pi\) such that \(\pi_{i_j} < \pi_{i_k}\) if and only if \(\tau_j < \tau_k\). If \(\pi\) does not have the pattern \(\tau\), \(\pi\) is called \(\tau\)-avoiding. The permutation \(\pi\) is said to have a consecutive pattern \(\tau\) if there exists a subsequence \(\pi_{i_i}\pi_{i_{i+1}}\cdots\pi_{i_{i+m-1}}\) of \(\pi\) that provides a pattern \(\tau\).

It is well understood that the number of permutations on \([n]\) avoiding a pattern \(\tau\) of length three is given by the Catalan number \(C_n = \frac{1}{n+1} \binom{2n}{n}\) for any \(\tau \in \mathcal{S}_3\). This paper is mainly concerned with the set \(\mathcal{S}_n(132)\) of 132-avoiding permutations. It is also very well known that \(C_n\) counts plane trees of \(n\) edges and Dyck paths of semilength \(n\). A bijection between 132-avoiding permutations and plane trees was given in Jani and Rieper [12], while a bijection between 132-avoiding permutations and Dyck paths was
given in Krattenthaler [15]. We refer to Claesson and Kitaev [6] and references therein for more detailed discussion on bijections related to permutations avoiding a length three pattern.

In this work, we examine four types of decompositions of permutations into subsequences. When restricted to 132-avoiding permutations, two of the four decompositions refine ascents and others refine descents. In fact, two of them are respectively increasing run and decreasing run decompositions which have been studied, for instance, in Zhuang [19], and Elizalde and Noy [9]. Note that an ascent is essentially a consecutive length two pattern 12 while a descent is a consecutive pattern 21. One of our main results states that the length distributions of the subsequences of the four decompositions are mutually equivalent, and there is a way to group the four into two groups such that each group is symmetric and the joint length distribution of one group is the same as that of the other. We prove this combinatorially, connecting several bijections, some are well-known and some are recently discovered. As a consequence, we are able to enumerate 132-avoiding permutations according to a variety of filtrations.

Another main result is about plane trees. Degrees and outdegrees of vertices are the most studied statistics. Then, it is natural for one to distinguish the vertices of a plane tree into two classes: internal vertices and leaves. Consequently, internal vertices and leaves are usually somehow separately studied or behave differently. For example, a fundamental result in this regard is that internal vertices and leaves are equidistributed, i.e., the number of plane trees of $n$ edges and $k$ internal vertices is the same as the number of plane trees of $n$ edges and $k$ leaves. To give one more example, it was proved in Deutsch [8] that the number of plane trees with $k$ internal vertices of outdegree $q$ is the same as the number of plane trees with $k$ odd-level vertices of outdegree $q-1$, and the number of plane trees with $l$ leaves equals the number of plane trees with $l$ even-level vertices. It seems in the literature there is no uniform statistic $A$ of vertices (independent of being internal or not) that behaves the same as another uniform statistic $B$ of vertices. Notably, we discover such a pair of statistics: one being the natural heights of vertices, and the other what we call the right spanning widths of vertices. The correspondence also refines the fundamental result mentioned above. Height of a vertex in a plane tree is the distance of the vertex to the root of the tree. In particular, the height of a plane tree is the maximum height of leaves in the tree. Average height of various trees and a single leaf there were examined in a plethora of work, see for instance, de Bruijin, Knuth and Rice [14], Kemp [13], and Prodinger [16].

The paper is organized as follows. In Section 2, we introduce the four types of decompositions and present some basic properties. In Section 3, several relevant bijections are discussed. In particular, we introduce the right spanning width of a vertex and show it is equally distributed as the height of vertices. Finally, we prove the equidistribution result of the four decompositions and provide a number of enumerative results as applications in Section 4.
2 Decompositions of permutations

For a permutation treated as a sequence, we have many ways to decompose it into different subsequences. Here we are interested in four distinct decompositions which will be introduced in order. The four decompositions can be viewed as refinements of the well studied statistics ascents and descents on permutations as we shall see shortly. Let $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$. An ascent of $\pi$ is an index $1 \leq i < n$ such that $\pi_i < \pi_{i+1}$. The rest are called descents of $\pi$. Note that $n$ is always a descent.

2.1 Increasing and decreasing run decompositions

In a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$, a subsequence $\pi_i \pi_{i+1} \cdots \pi_{i+k-1}$ is called an increasing run (IR\(^1\)) if $\pi_i < \pi_{i+1} < \cdots < \pi_{i+k-1}$, and it is not contained in a longer such subsequence. Obviously, a permutation $\pi$ can be uniquely decomposed into IR. This decomposition is simply referred to as IRD. For example, we can decompose a 132-avoiding permutation $\pi = 5346127$ into 3 IR:

$\tau_1 = 5$, $\tau_2 = 346$, $\tau_3 = 127$. The lengths of the IR of $\pi$ give the length distribution of $\pi$ w.r.t. IRD which can be encoded by an integer partition of $n$. IR can be regarded as a refinement of descents, as we obviously have

Lemma 2.1. The number of descents of $\pi$ equals the number of segments from its IRD.

Decreasing runs (DR) and decreasing run decomposition (DRD) are defined similarly. It is also obvious that the number of ascents of $\pi$ plus one is the same as the number of segments from the DRD of $\pi$.

2.2 Value-consecutive increasing subsequences

An increasing run can be alternatively interpreted as a maximal position-consecutive increasing subsequence. As a counterpart, we can uniquely decompose a permutation into maximal value-consecutive increasing subsequences (v-CIS). A v-CIS is a subsequence of the form $\pi_i \pi_{i+1} \cdots \pi_{ik} = j(j+1) \cdots (j+k-1)$. Taking $\pi = 5346127$ as an example, its v-CIS (decomposition) gives subsequences: 567, 34 and 12.

Lemma 2.2. The number of descents of a 132-avoiding permutation $\pi$ equals the number of segments from its v-CIS.

Proof. Let $\pi = \pi_1\pi_2 \cdots \pi_n$. It suffices to show that there is a one-to-one correspondence between the descents $i \neq n$ of $\pi$ and the maximal v-CIS of $\pi$ which do not start with $\pi_1$. First, if $i \neq n$ is a descent of $\pi$, then $\pi_{i+1}$ starts a maximal v-CIS of $\pi$. Otherwise, $\pi_j = \pi_{i+1} - 1$ for some $j < i$. In this case, $\pi_j \pi_i \pi_{i+1}$ yields a 132 pattern, a contradiction. Conversely, suppose $\pi_{i+1}$ $(i > 0)$ starts a maximal v-CIS. If $\pi_{i+1} = 1$, then obviously $i$

\(^1\)We will write IR (DR and v-CIS defined later) in singular as well as plural form.
is a descent. If $\pi_{i+1} \neq 1$, then $\pi_j = \pi_{i+1} - 1$ for some $j > i + 1$ due to maximality. Consequently, $\pi_i < \pi_{i+1}$ implies $\pi_i < \pi_{i+1} - 1 = \pi_j$, which again yields a 132 pattern $\pi_i \pi_{i+1} \pi_j$. Thus, we have $\pi_i > \pi_{i+1}$ whence $i$ is a descent of $\pi$. In view that there is a maximal v-CIS starting with $\pi_1$ and $n$ is a descent of $\pi$, the lemma follows.

We remark that the above relation is not true for a general permutation. For instance, $\pi = 153642$ has four descents but three segments in its v-CIS.

**Lemma 2.3.** Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a 132-avoiding permutation. Then, there do not exist $1 \leq i < j < k < l \leq n$ such that $\pi_i$ and $\pi_k$ are from the same v-CIS while $\pi_j$ and $\pi_l$ are from a distinct v-CIS.

*Proof.* Suppose such $i, j, k, l$ exist. By construction, if $\pi_j > \pi_i$, then $\pi_j > \pi_k$ as well. Thus, $\pi_i \pi_j \pi_k$ provides a 132 pattern, a contradiction. If otherwise $\pi_j < \pi_i$, we then have $\pi_l < \pi_k$, which implies $\pi_j \pi_k \pi_l$ being a 132 pattern. Either way yields a contradiction and the lemma follows.

**Lemma 2.4.** Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a 132-avoiding permutation. Suppose $\tau$ and $\tau'$ are two distinct v-CIS of $\pi$. Then, either all elements in $\tau$ lie between two consecutive elements in $\tau'$, or all elements in $\tau$ are to the left of the starting element of $\tau'$. Moreover, in the former case, the maximal element in $\tau$ is smaller than the minimal element of $\tau'$, while in the latter case, the maximal element in $\tau'$ is smaller than the minimal element in $\tau$.

*Proof.* The first statement follows from Lemma 2.3. In the remaining part, the “former” case is true because the minimal (i.e., starting) element of $\tau$ is the image of an element determining a descent in view of Lemma 2.2; the “latter” case is true since otherwise an element from $\tau$, $\pi_j$ and an element from $\tau'$ form a 132 pattern, where $\pi_j$ determines the descent corresponding to $\tau'$ in the light of Lemma 2.2. This completes the proof.

### 2.3 Layered decreasing envelopes

Given a 132-avoiding permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, we imagine it to represent a row of poles of distinct height where the $i$-th pole is $\pi_i$ units high. Next, we group these poles such that the heights of the poles in the same group decrease from left to right, $\pi_i$ and $\pi_{i+1}$ belonging to distinct groups implies $\pi_{i+1}$ is larger than the maximal of the group containing $\pi_i$, and that if we connect the top ends of the poles in the same group by straight lines, these lines do not cross each other and do not cross the poles as well. See an example for $\pi = 10, 8, 7, 9, 11, 6, 4, 3, 5, 12, 1, 2$ in Figure 1. Clearly, if $i \neq n$ is an ascent of $\pi$, then $\pi_i$ is the rightmost element (or pole) of a group and vice versa. Taking account of the group ending with $\pi_n$, the number of groups is one greater than the number of ascents of $\pi$. With a little effort, we can see that there is only one such grouping.

Note that two (decreasing) paths induced by the groups are either in a left-right position or in a covering relation. An alternative way to characterize this kind of decomposition can be done iteratively as follows. Starting with $\pi_n$, we search all right-to-left maxima which will give us a decreasing path; Next starting with the position that precedes the leftmost pole of the last found group, we search all right-to-left maxima which
will give us a new decreasing path; Continuing doing this until we get a path starting with $\pi_1$. At this point, we have found the “outermost” layer of decreasing paths. Repeat this procedure with respect to the segment of poles covered by an edge in the existing path until all poles have been placed into a group. This decomposition of permutation elements into groups, i.e., decreasing subsequences, will be called layered decreasing envelop decomposition, simply referred to as LDE. We will be interested in the length distribution of these decreasing envolops.

3 Relevant bijections: old and new

In this section, we review several bijections involving plane trees, RNA secondary structures, and 132-avoiding permutations which will be used later.

3.1 Plane trees and RNA secondary structures

A plane tree $T$ can be recursively defined as an unlabeled tree with one distinguished vertex called the root of $T$, where the unlabeled trees resulting from the removal of the root as well as its adjacent edges from $T$ are linearly ordered, and they are plane trees with the vertices in $T$ adjacent to the root of $T$ as their respective roots. In a plane tree $T$, the number of edges in the unique path from a vertex $v$ to the root of $T$ is called the height (or level) of $v$, and the vertices adjacent to $v$ having greater heights are called the children of $v$. The vertices on level $2i$ (resp. $2i - 1$) for $i \geq 0$ are called even-level (resp. odd-level) vertices. A non-root vertex without any child is called a leaf, and an internal vertex otherwise. We will draw a plane tree with its root on the top level, i.e., level 0, and with the children of a level $i$ vertex arranged on level $i + 1$ left-to-right following their linear order.

A plane tree can be decomposed into a set of paths where each path has a leaf as a terminate vertex. There are two ways to do that: left path decomposition and right path decomposition. The left path decomposition works as follows: suppose all leaves are ordered by their relative order in the depth-first search from left to right. The first path is the path from the first leaf to the root, and for $t > 1$, the $t$-th path should go from
the $t$-th leaf up to the first vertex that is already in a path that has been obtained. See Figure 2 (left) for an illustration. We will call the multiset consisting of the lengths of the obtained paths the left path distribution of the given tree. The right path decomposition is analogous. As a counterpart, there is a natural horizontal decomposition of a plane tree into fibers. A fiber is basically an internal vertex and its children, and the size of the fiber is the outdegree of the internal vertex.

RNA plays an important role in various biological processes within a cell. In particular, RNA secondary structures have been extensively studied from a computational and combinatorial viewpoint, in order for structure and function prediction. We refer to Smith and Waterman [18], and Schmitt and Waterman [17] for the definition and discussion. Two bijections between RNA secondary structures and plane trees are relevant, one being the Schmitt-Waterman bijection [17] and the other being the new bijection recently discovered by Chen [4]. Clearly, with RNA secondary structures as intermediates, we have a bijection $\varphi$ from plane trees to plane trees. See an example in Figure 2. We refer to [17] and [4] for details about the bijections, and here we only present the most relevant result.

Even-level degrees
$2, 2, 1, 1, 3$

Figure 2: The bijection $\varphi$ is the composition of the Schmitt-Waterman bijection (left) and Chen’s bijection (right). Through $\varphi$, the left path distribution is mapped to the even-level degree distribution.

**Theorem 3.1** (Chen [4]). *The number of plane trees of $n > 0$ edges with $x$ internal vertices of outdegree $q$ and $y$ (left) paths of length $l$ associated to leaves is the same as the number of plane trees of $n > 0$ edges with $x$ odd-level vertices of outdegree $q - 1$ and $y$ even-level vertices of degree $l$.*

In a sense, the horizontal determines the odd-levels while the vertical determines the even-levels via the bijection $\varphi$ on plane trees. This is also an example that internal vertices and leaves behave differently.
3.2 The Jani-Rieper bijection

An explicit bijection between plane trees and 132-avoiding permutations was given by Jani and Rieper [12]. The following is how it works. Let $T$ be a plane tree on $n$ edges. We use a preorder traversal of $T$ (from left to right) to label the non-root vertices in decreasing order with the integers $n, n-1, \ldots, 1$. As such, the first vertex visited gets the label $n$ and the last receives 1. A permutation written as a word is next obtained by reading the labelled tree in postorder, that is, traverse the tree from left to right and record the label of a vertex when it is last visited.

The reverse from a 132-permutation to a plane tree was not explicitly presented in Jani and Rieper [12]. Here we present one procedure and we leave it to the reader to verify its effectiveness. Let $\pi$ be a 132-avoiding permutation. Suppose the IR of $\pi$ from left to right are $\tau_1, \tau_2, \ldots, \tau_k$. Starting with $\tau_k$ and making it into a path with the maximal element attaching to the root of the expected tree. For example, suppose $\pi = 10, 8, 7, 9, 11, 6, 4, 3, 5, 12, 1, 2$. Then, $\tau_k = 12$ and the path will be the path from vertex 1 to the root of the left tree in Figure 3. After $\tau_i$ is “placed” in the (partial) tree, we find the minimal element $u$ in the leftmost path in the current partial tree that is larger than the maximal element $x$ in $\tau_{i-1}$, and attach the path induced by $\tau_{i-1}$ to the tree such that $u$ and $x$ are adjacent; if no such a $u$ exists, we attach the path to the root of the current tree. Eventually, we obtain a plane tree. In the following, we will regard the corresponding plane trees of 132-avoiding permutations as plane trees with vertex labels, although the vertex labels are uniquely determined (by the underlying bijections) and can be omitted.

Lemma 3.2 (Jani-Rieper [12]). Let $JR(\pi)$ be the corresponding plane tree of a 132-avoiding permutation $\pi$ from the Jani-Rieper bijection. Then, the longest increasing subsequence in $\pi$ starting with a leaf is the height of the leaf in $JR(\pi)$.

Lemma 3.3. Given a 132-avoiding permutation $\pi$, let $JR(\pi)$ be its corresponding plane tree from the Jani-Rieper bijection. Then, the outdegree distribution of the internal vertices of $JR(\pi)$ is the same as the LDE length distribution of $\pi$ while the right path distribution of $JR(\pi)$ is the same as the IR length distribution of $\pi$.

Proof. By construction, the descendants (if any) of a non-root vertex $v$ is smaller than $v$ and appear before $v$ but after any siblings to the left of $v$ in the tree. Thus, the outermost and rightmost decreasing path consists of the children of the root of $JR(\pi)$. Analogously, the children of the root of a subtree of $JR(\pi)$ determine a decreasing path. Iteration of the reasoning gives rise to the correspondence between the outdegree distribution and LDE distribution. The other aspect follows from our reverse procedure from 132-avoiding permutations to plane trees. □

3.3 A (possibly new) bijection $\phi$

Here we present another bijection between 132-avoiding permutations and plane trees which seems new. Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a 132-avoiding permutation. Suppose there
are \( k \) segments in its v-CIS: \( \tau_1 = \pi_1^1 \pi_1^2 \cdots \pi_1^{i_1}, \ldots, \tau_k = \pi_k^1 \pi_k^2 \cdots \pi_k^{i_k} \), and suppose \( \pi_1^1 = \max\{\pi_1^1, \pi_1^{i_1}, \ldots, \pi_k^1\} \). We then construct a plane tree \( T \) recursively. First, we arrange \( \pi_1^1, \ldots, \pi_1^{i_1} \) from left to right as the children of the root of \( T \). We next let \( \pi_1^1, \ldots, \pi_j^1 \) be the left-to-right children of vertex \( \pi_t \) that has already appeared in the constructed partial tree if \( \pi_j^1 \) is immediately to the right of \( \pi_t \) in \( \pi \). Eventually all labels in \([n]\) appear, and we obtain a labelled plane tree. It is not difficult to notice that there are \( k \) internal vertices in the tree.

For example, \( \pi = 10, 8, 7, 9, 11, 6, 4, 3, 5, 12, 1, 2 \) is a 132-avoiding permutation, and \( \tau_1 = 10, 11, 12, \tau_2 = 89, \tau_3 = 7, \tau_4 = 6, \tau_5 = 45, \tau_6 = 3, \tau_7 = 12 \). Then, its corresponding labelled plane tree is shown in Figure 3 (right). In view of Lemma 2.4, it is not hard to see that on the path from any vertex to the root of \( T \), the minimal child of an internal vertex on the path and closer to the root is larger than the maximal child of an internal vertex on the path but further to the root (i.e., the “former case”), and the minimal child of an internal vertex \( u \) is larger than the maximal child of an internal vertex \( v \) if \( v \) is on the right-hand side of the path from \( u \) to the root of \( T \) (i.e., the “latter case”). As a result, the labels of the vertices are uniquely determined by the underlying unlabelled plane tree as follows: if we travel the internal vertices of the plane tree in the left-to-right depth-first manner, then the \( k \) left-to-right children of the current internal vertex carry the remaining largest \( k \) elements in increasing order.

As for the reverse, from a plane tree that is uniquely labelled, it may not be difficult to see as well that the left-to-right depth-first search (or preorder) gives us the desired 132-avoiding permutation. We leave the proof of the following lemma to the reader.

**Lemma 3.4.** Given a 132-avoiding permutation \( \pi \), let \( T = \phi(\pi) \). Then, the outdegree distribution of internal vertices of \( T \) is the same as the v-CIS length distribution of \( \pi \) while the left path distribution of \( T \) equals the DR length distribution of \( \pi \).

It is well known that the number of Dyck paths of semilength \( n \) with \( j \) peaks is
given by the Narayana number $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n-1}{k-1}$ with $N(0, 0) = 1$. In Callan [3], it was proved via a novel combinatorial argument that the number of Dyck paths of semilength $n$ with $i$ returns to ground level and $j$ peaks is the generalized Narayana number $N_i(n, j) = \frac{1}{n} \binom{n}{j} \binom{n-i-1}{j-i}$.

**Theorem 3.5.** The number of 132-avoiding permutations on $[n]$ starting with $i$ and having $k$ descents is given by $\frac{n+1-i}{n} \binom{n}{n+1-k} \binom{i-2}{i-k}$. Furthermore, the number of 132-avoiding permutations on $[n]$ starting with $i$, ending with $j$ and having $k$ descents is

$$
\begin{cases}
N_{n-i}(n-1, n-k), & \text{if } i < j, \\
\sum_{m=0}^{k-1} N_{n+1-i}(n-j, n-m+1)N(j-1, j+m-k+1), & \text{else}.
\end{cases}
$$

(1)

**Proof.** For a 132-avoiding permutation in question in the first part of the theorem, the v-CIS containing the first element $i$ is $i(i+1) \cdots n$. According to the bijection $\phi$, this implies the outdegree of the root of the corresponding plane tree equals $n + 1 - i$ and the number of leaves of the tree is $n + 1 - k$. It is well known that the set of plane trees of $n$ edges and these features has the same size as the set of Dyck paths of semilength $n$ with $n + 1 - i$ returns to ground level and $n + 1 - k$ peaks. As a result, the first statement follows.

As for the remaining part, we need the following claim: In a 132-avoiding permutation ending with $j$, the last $j$ entries are from the set $[j]$. Otherwise, there is no difficulty to show there exists a 132 pattern.

In order to prevent the appearance of 132-pattern, if $i < j$ then $j = n$. Then, the considered set is equivalent to the set of 132-avoiding permutations on $[n-1]$ starting with $i$ and having $k$ descents which is given by $N_{n-i}(n-1, n-k)$ as in the first statement. Similarly, if $i > j$ and $j = 1$, the considered set is equivalent to the set of 132-avoiding permutations on $[n-1]$ starting with $i$ and having $k-1$ descents which is given by $N_{n+1-i}(n-1, n-k+1)$.

If $i > j$ and $j \neq 1$, from the above claim, any 132-avoiding permutation in this case is a concatenation of two 132-avoiding permutations: one is on $[n] \setminus [j]$ which starts with $i$ and has $m$ descents, and the other is on $[j]$ which ends with $j$ and has $k - m - 1$ descents, for some $0 \leq m < k$. The 132-avoiding permutations on $[n] \setminus [j]$ are equivalent to 132-avoiding permutations on $[n-j]$ which start with $i - j$ and have $m$ descents. According to previous discussion, the desired number is

$$
\sum_{m=0}^{k-1} N_{n+1-i}(n-j, n-m+1)N(j-1, j+m-k+1).
$$

Note that $N(0, 0) = 1$ and $N(0, x) = 0$ for $x > 0$. The last quantity agrees to the number for the case $j = 1$, and the proof follows.

Next we introduce a new quantity associated to vertices. Let $v$ be a vertex in a plane tree $T$. The sum of the number of edges attached to other vertices (than $v$) on the path from $v$ to the root of $T$ from the right-hand side (of the path) and the number of children
of $v$ is called the right spanning width at $v$, denoted by $rsw(v)$. See an illustration in Figure 4. In addition, we define

$$rsw(T) = \max\{rsw(v) : v \text{ is an internal vertex in } T\}.$$  

Now we are ready to present the main theorem about plane trees which refines the well-known fact that internal vertices and leaves of plane trees are equally distributed.

![Figure 4: The right spanning width of vertex $v$ is the number of bold edges.](image)

**Theorem 3.6.** The number of plane trees of $n$ edges the heights of whose vertices constitute a multiset $\mathcal{M}$ of $n+1$ elements is the same as the number of plane trees of $n$ edges the right spanning widths of whose vertices constitute $\mathcal{M}$. Moreover, the number of plane trees of $n$ edges and $k$ leaves whose heights constitute a multiset $\mathcal{M}'$ is the same as the number of plane trees of $n$ edges and $k$ internal vertices whose right spanning widths constitute $\mathcal{M}'$.

**Proof.** Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a 132-avoiding permutation. Denote by $T$ its corresponding plane tree from the Jani-Rieper bijection and by $T'$ its corresponding plane tree from the new bijection $\phi$. First, it is not hard to observe that the label $\pi_i$ of a leaf in $T$ is the label of the left-most child of an internal vertex in $T'$ by construction. Now consider the longest increasing subsequence $\tau$ starting with $\pi_i$. In terms of $T$, the length of $\tau$ is the height of the leaf $\pi_i$. We next show that in terms of $T'$, the length of $\tau$ is $rsw(\pi_i)$. According to $\phi$, the elements larger than $\pi_i$ and to the right of $\pi_i$ must be on the right-hand side of the path from $\pi_i$ to the root of $T'$. Moreover, they must attach to some internal vertices on the path. Consequently, we can easily argue that the longest path starting with $\pi_i$ consists of vertices incident to the internal vertices on the path where there are $rsw(\pi_i)$ of them.

Next, for the label $\pi_i$ of an internal vertex in $T$, the longest increasing subsequence starting with $\pi_i$ in $\pi$ also equals the height of $\pi_i$ in $T$. By construction, $\pi_i$ being of an internal vertex in $T$ implies that $i \neq 1$ and $\pi_{i-1} < \pi_i$, i.e., $i - 1$ is an ascent of $\pi$. However, in $T'$, $\pi_{i-1}$ must be a leaf. Similar to the above discussion, the longest increasing subsequence starting with $\pi_{i-1}$ has length exactly $rsw(\pi_{i-1}) + 1$ in $T'$. Accordingly, the height of an internal vertex $\pi_i$ in $T$ is the same as $rsw(\pi_{i-1})$ where $\pi_{i-1}$ is a leaf in $T'$, and the theorem follows. \qed
Corollary 3.7. The number of plane trees of $n$ edges and height $k$ equals the number of plane trees $T$ of $n$ edges with $rsw(T) = k$.

### 3.4 Odd-even level switching of plane trees

Given a plane tree $T$, we obtain a new tree $T'$ by taking the leftmost child $v$ of the root of $T$ as the root of the new tree $T'$, i.e., lifting $v$ to the top level. It is obvious that the vertices that were on even-level in $T$ become odd-level vertices in $T'$ and the vertices that were on odd-level in $T$ become even-level vertices in $T'$. In addition, the degree distribution of the even-level vertices becomes the degree distribution of the odd-level vertices and vice versa.

### 4 Equidistribution and enumerative results

Theorem 4.1. Given two partitions $\lambda, \mu \vdash n$, the following four sets are of equal size:

1. $\pi \in S_n(132)$ whose IRD and LDE length distributions are resp. $\lambda$ and $\mu$.
2. $\pi \in S_n(132)$ whose IRD and LDE length distributions are resp. $\mu$ and $\lambda$.
3. $\pi \in S_n(132)$ whose v-CIS and DRD length distributions are resp. $\lambda$ and $\mu$.
4. $\pi \in S_n(132)$ whose v-CIS and DRD length distributions are resp. $\mu$ and $\lambda$.

Proof. We first prove the sets (1) and (2) have the same size. Let $\pi \in S_n(132)$ and suppose its IRD and LDE length distributions are resp. $\lambda$ and $\mu$. We shall show there is a unique $\pi' \in S_n(132)$ whose IRD and LDE length distributions are resp. $\mu$ and $\lambda$, and vice versa.

Let $T_1 = JR(\pi)$, i.e., the corresponding tree of $\pi$ under the Jani-Rieper bijection. Denote by $T_2$ the mirror image of $T_1$ (i.e., horizontal flipping). Then, clearly, the internal vertex outdegree distribution of $T_2$ is $\mu$ and the left path length distribution of $T_2$ is $\lambda$. Accordingly, the odd-level outdegree distribution of $T_3 = \varphi(T_2)$ is $\mu - 1$ (that is, $(\mu_1 - 1)(\mu_2 - 1)\cdots(\mu_k - 1)$ for $\mu = \mu_1 \mu_2 \cdots \mu_k$) while the even-level degree distribution is $\lambda$. Let $T_4$ be the resulting tree via the odd-even level switching transform from $T_3$. Then, it is easily seen that the odd-level outdegree distribution of $T_4$ is $\lambda - 1$ while the even-level degree distribution is $\mu$. According to Theorem 3.1, the internal vertex outdegree distribution of $T_5 = \varphi^{-1}(T_4)$ is thus $\lambda$ and the left path length distribution of $T_5$ is $\mu$. Consequently, the mirror image $T_6$ of $T_5$ has the internal vertex outdegree distribution $\lambda$ and the right path length distribution $\mu$. The corresponding permutation of $T_6$ under the Jani-Rieper bijection is the desired $\pi'$ whose IRD and LDE length distributions are resp. $\mu$ and $\lambda$. See Figure 5 for an illustration. Obviously, the correspondence is a bijection as it is essentially the composition of multiple bijections. Hence, the sets (1) and (2) are of equal size.

Next we prove the sets (1) and (3) contain the same number of 132-avoiding permutations. Again, let $\pi \in S_n(132)$ and suppose its IRD and LDE length distributions are
resp. $\lambda$ and $\mu$. Let $T_1 = JR(\pi)$, and denote by $T_2$ the mirror image of $T_1$. Then, the internal vertex outdegree distribution of $T_2$ is $\mu$ and the left path length distribution is $\lambda$. Suppose $\pi'$ is the corresponding permutation of $T_2$ under the bijection $\phi$. According to Lemma 3.4, the v-CIS and DRD length distributions of $\pi'$ are resp. $\lambda$ and $\mu$. The correspondence is clearly reversible, so the sets (1) and (3) have equal size. Other pairs from the four sets can be done analogously, completing the proof.

Consequently, from enumeration point of view, we only need to enumerate one set of 132-avoiding permutations, which is tantamount to enumeration of plane trees according to even-level and odd-level vertices. In the following, we present two primary approaches for this purpose: one is based on generating functions and the other is a bijective approach.

In both forthcoming approaches, we need to make use of the concept of set-alternating trees. A labelled (resp. unlabelled) set-alternating E-tree is a plane tree where the even-level vertices carry distinguishable (resp. indistinguishable) labels from a set $E$ and the odd-level vertices carry distinguishable (resp. indistinguishable) labels from a set $O$. $O$-trees are defined similarly.

Let $\kappa_t(n, m)$ denote the number of weak compositions of $n$ into $m$ parts each of which is no larger than $t$, i.e., $a_1 + a_2 + \cdots + a_m = n$ and $0 \leq a_i \leq t$.

Theorem 4.2. The number of 132-avoiding permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ on $[n]$ with $p$ IR of which the one ending with $\pi_n$ has a length at most $h$ while each of the rest is of length no longer than $h+1$ and with $q$ LDE each of which is no longer than $l$ is given by

$$\kappa_l(p-1, q)\kappa_h(q, p) - \sum_{i=0}^{l-1} (i+1)\kappa_l(p-i-2, q-1) \sum_{j=0}^{h-1} (j+1)\kappa_h(q-j-1, p-1).$$

(2)
Proof. According to Theorem 4.1, the set of 132-avoiding permutations under consideration has the same size as the set of plane trees of $n$ edges where there are $p$ even-level vertices, each with outdegree at most $h$ and $q$ odd-level vertices, each with outdegree at most $l$. Let

$$\omega_1(t_1, t_2) = \sum_{T \in \mathcal{F}_E} \# \text{vertices in } E \in T \# \text{vertices in } O \in T,$$

$$\omega_2(t_1, t_2) = \sum_{T \in \mathcal{F}_O} \# \text{vertices in } E \in T \# \text{vertices in } O \in T,$$

where $\mathcal{F}_E$ denotes the set of unlabelled set-alternating $E$-trees with every $E$-vertex having at most $h$ children, every $O$-vertex having at most $l$ children, while $\mathcal{F}_O$ denotes the set of unlabelled set-alternating $O$-trees with every $E$-vertex having at most $h$ children, every $O$-vertex having at most $l$ children. Clearly, the number in question is $[t_1^p t_2^q] \omega_1$.

Note that the following relation is obvious

$$\omega_1 = t_1 \frac{1 - \omega_2^{h+1}}{1 - \omega_2}, \quad \omega_2 = t_1 \frac{1 - \omega_1^{l+1}}{1 - \omega_1}.$$ 

Let

$$g(x_1, x_2) = x_1, \quad f_1(x_1, x_2) = \frac{1 - x_2^{h+1}}{1 - x_2}, \quad f_2(x_1, x_2) = \frac{1 - x_1^{l+1}}{1 - x_1}.$$ 

Making use of the bivariate Lagrange inversion formula [2, 11], we have

$$[t_1^p t_2^q] \omega_1 = [x_1^p x_2^q] g \cdot f_1 \cdot f_2 \left[ 1 \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} \right]$$

$$= [x_1^p x_2^q] \left[ 1 \frac{x_1 x_2}{(1 - x_1)^q (1 - x_2)^p} \left( 1 - \frac{x_1 x_2}{(1 - x_1)(1 - x_2)} \right) \right]$$

$$= \kappa_l(p - 1, q) \kappa_h(q, p) \sum_{i=0}^{l-1} (i + 1) \kappa_l(p - i - 2, q - 1) \sum_{j=0}^{h-1} (j + 1) \kappa_h(q - j - 1, p + 1),$$

where the last simplification follows from Lemma A.1 in the Appendix. \qed

Let $l$ go to infinity, we obtain

**Corollary 4.3.** The number of 132-avoiding permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ on $[n]$ with the IR ending with $\pi_n$ has a length at most $h$ while each of the rest is of length no longer than $h + 1$ is given by

$$\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \sum_{i \geq 0} (-1)^i \binom{k}{i} \binom{n - i(h + 1)}{k - 1}.$$
Our next enumeration is based on a bijection involving labelled set-alternating trees which is a variation of Chen’s bijection \[5\] on general labelled plane trees. In order to state our bijection, we need further notation as follows. A forest is a set of trees. A (labelled) small tree is a (labelled) tree with only two levels in total. A small set-alternating tree is a set-alternating tree with only two levels. The size of a small tree is the number of edges in the small tree. We shall call a small set-alternating tree with a root in \(E\) (resp. \(O\)) a small \(E\)-tree (resp. \(O\)-tree).

**Theorem 4.4.** Suppose \(E = [k] \subset E^* = [k] \cup \{(k + 1)^*, \ldots, (k + l_e - 1)^*\}\) and \(O = [b + 1] \subset O^* = [b + 1] \cup \{(b + 2)^*, \ldots, (b + m + l_o)^*\}\). There is a bijection between the set \(T\) of labelled set-alternating \(E\)-trees over \(E \cup O\) with root \(k\) of degree \(d_r\) and \(b + k\) edges, where there are \(l_e - 1\) non-root even-level internal vertices with outdegree distribution \(1^t_1 2^t_2 \cdots\) and \(l_o\) odd-level internal vertices with outdegree distribution \(1^s_1 2^s_2 \cdots\), and the set \(F\) of forests of small labelled set-alternating trees over \(E^* \cup O^*\) where in each forest

(a) there are \(l_e\) small \(E\)-trees with roots from \(E\) including \(k\), and the size distribution for the small \(E\)-trees not having \(k\) as the root is \(1^t_1 2^t_2 \cdots\);

(b) and the size of the small tree rooted on \(k\) is \(d_r\), and \((b + 1 + l_o)^*\) is a leaf there;

(c) and there are \(l_o\) small \(O\)-trees with roots from \(O\) and size distribution \(1^s_1 2^s_2 \cdots\).

**Theorem 4.5.** For a fixed \(k > 2\), the number of 132-avoiding permutations on \([n]\) with \(m\) occurrences of the consecutive pattern \(2 \cdots k\) is given by

\[
\frac{1}{n} \sum_s \sum_q \left( \begin{array}{c} n \\ s \end{array} \right) \left( \begin{array}{c} m + q \\ m \end{array} \right) \sum_j (-1)^j \left( \begin{array}{c} q \\ j \end{array} \right) \left( \begin{array}{c} n - s - (j + m)(k - 3) \\ m + q + 1 \end{array} \right).
\]

Theorem 4.4 is used to prove Theorem 4.5, and proofs for both theorems can be found in the Appendix. The above bijective approach can be also used to count 132-avoiding permutations satisfying a detailed exact specification on the subsequence length distribution of any decomposition, e.g., any set in Theorem 4.1, and we leave the computation to the interested reader.

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**A  Computation in Theorem 4.2**

**Lemma A.1.**

\[
[x_1^p]x_1 \cdot \frac{(1 - x_1^{i+1})^q}{(1 - x_1)^q} \cdot \frac{x_1}{1 - x_1^{i+1}} \left( \frac{1 - x_1^i}{1 - x_1} - lx_1^i \right) = \sum_{i=0}^{l-1} (i + 1) \kappa_i(p - i - 2, q - 1).
\]
Proof. First, it is easy to see

\[
\sum_{n \geq 0} \kappa_t(n, m)x^n = (1 + x + \cdots + x^t)^m = \left(\frac{1 - x^{t+1}}{1 - x}\right)^m.
\]

Then,

\[
[x_1^p]x_1 \cdot \frac{(1 - x_1^{l+1})^q}{(1 - x_1)^q} \cdot \frac{x_1}{(1 - x_1)} \left(\frac{1 - x_1^l}{1 - x_1} - lx_1^l\right)
\]

\[
= [x_1^{p-2}] \frac{(1 - x_1^{l+1})^q - 1}{(1 - x_1)^q} \cdot \frac{1}{(1 - x_1)} \left(\frac{1 - x_1^l}{1 - x_1} - lx_1^l\right)
\]

\[
= [x_1^{p-2}] \frac{(1 - x_1^{l+1})^{q-1}}{(1 - x_1)^{q+1}} - [x_1^{p-l-2}] \frac{(1 - x_1^{l+1})^{q-1}}{(1 - x_1)^{q+1}} - l[x_1^{p-l-2}] \frac{(1 - x_1^{l+1})^{q-1}}{(1 - x_1)^q}
\]

\[
= \sum_{i=0}^{p-2} (i+1)\kappa_i(p - i - 2, q - 1) - \sum_{i=0}^{p-l-2} (i+1)\kappa_i(p - l - i - 2, q - 1)
\]

\[
- l \sum_{i=0}^{p-l-2} \kappa_i(p - l - i - 2, q - 1)
\]

\[
= \sum_{i=0}^{p-2} (i+1)\kappa_i(p - i - 2, q - 1) - \sum_{i=0}^{p-l-2} (i+l+1)\kappa_i(p - l - i - 2, q - 1)
\]

\[
= \sum_{i=0}^{p-2} (i+1)\kappa_i(p - i - 2, q - 1) - \sum_{i=l}^{p-2} (i+1)\kappa_i(p - i - 2, q - 1)
\]

\[
= \sum_{i=0}^{l-1} (i+1)\kappa_i(p - i - 2, q - 1).
\]

\[\blacksquare\]

B Proof of Theorem 4.4

Proof. For each \(T \in \mathcal{T}\), we first decompose \(T\) into a forest of small trees according to the following procedure.

i. Set \(i_e = 0, i_o = 0\);

ii. We assume \(\bar{j} < \bar{j} + 1\) and any element \(i \in [k]\) is smaller than \(\bar{j} \in [b+1]\). In \(T\), find the minimal internal vertex (in terms of its label from \([k]\) and \([b+1]\)) whose children are all leaves; Remove the small tree determined by \(v\) and its children, with all labels carried over.

iii. If \(v\) has a label in \([k]\), relabel \(v\) with label \((d + i_e + 1)^*\) in the remaining tree (at the original position of \(v\) in \(T\)), and update \(T\) as the resulting tree, and set \(i_e = i_e + 1\);
If \( v \) has a label in \([b+1]\), place a vertex with label \((t+i_o+1)^*\) in the remaining tree at the original position of \( v \) in \( T \), and update \( T \) as the resulting tree, and set \( i_o = i_o + 1 \);

iv. If \( T \) is not a small E-tree, go to ii and continue, the procedure terminates otherwise.

In the end, it is obvious that we obtain \( l_e \) small E-trees and \( l_o \) small O-trees. Next, we describe how to get back to a tree \( T \in \mathbb{T} \) from a forest \( F \in \mathbb{F} \).

(i) Find a tree in \( F \) with the minimum root such that there is no vertex with a starred label in the tree. If the root \( v \) of the found tree is an \( E \)-vertex, then merge the root with the vertex having the minimum label in the set \( \{(k+1)^*, (k+2)^*, \ldots, (k+s-1)^*\} \) in \( F \), and label the newly generated vertex by the label of \( v \) (and discard the starred label involved). If the root \( v \) of the found tree is an \( O \)-vertex, then merge the root with the vertex having the minimum label in the set \( \{(b+2)^*, (b+3)^*, \ldots, (b+1+l_o)^*\} \), and label the newly generated vertex by the label of \( v \). Update \( F \) as the resulting forest of trees. Here we remark that a tree without any starred label always exists, because there are in total \( s+l_o \) trees and \( s-1+l_o \) starred labels at the beginning, and later every time we decrease the number of starred E-labels (resp. O-labels) by one, we decrease the number of E-trees (resp. O-trees) by one at the same time. In addition, since the number of E-trees is always one larger than the number of starred E-labels, we eventually obtain an E-tree.

(ii) iterate (i) until \( F \) becomes a single labelled plane tree \( T \).

The numbers of even-level internal vertices and odd-level internal vertices in \( T \) are determined by the number of small trees with unstarred roots and are obviously \( l_e \) and \( l_o \). This completes the proof.

\[ \square \]

C Proof of Theorem 4.5

\begin{proof}
Suppose there are in total \( s \) occurrences of the consecutive patterns of the form \( 2 \cdots j1 \) for all \( j > 1 \) that are maximal. In order to have exactly \( m \) occurrences of the consecutive pattern \( 2 \cdots k1 \), the \( j \)'s for \( m \) out of the \( s \) consecutive patterns are no less than \( k \) and that of the other \( s-m \) are less than \( k \). It suffices to enumerate plane trees of \( n \) edges with \( n-s \) odd-level vertices and \( s+1 \) even-level vertices where \( m \) of non-root ones have degree at least \( k-1 \) respectively while the rest of the \( s-m \) non-root even-level vertices have degree at most \( k-2 \) respectively. Note that the degree \( d_r \) of the root could be any number no less than one. Those even-level vertices with degree at least \( k-1 \) are certainly internal vertices for \( k > 2 \), and those with degree at most \( k-2 \) could be either internal vertices or leaves. Suppose \( q \) out of \( s-m \) the latter are internal vertices. We also suppose \( y \) out of the \( n-s \) odd-level vertices are internal. According to Theorem 4.4, we next enumerate the corresponding forests.

Label the roots of the \( y \) small O-trees. There are in total \( n-s \) unstared O-labels, which gives \( \binom{n-s}{y} \) different ways.
\end{proof}
Label the roots of the small E-trees. There are in total \( s + 1 \) unstarred E-labels. Since \( s + 1 \) will be a root, we need to pick \( m + q \) out of the set \([s]\) which gives \( \binom{s}{m+q} \) different ways.

Label the leaves of the small O-trees. There are \( \binom{s-1}{y-1} \) \( s! \) ways.

Label the leaves of the small E-trees. For the small tree rooted on \( s+1 \), we need to pick \( d_r - 1 \) remaining O-labels and arrange them together with \((n - s + y)\) which gives in total \( \binom{n-s-1}{d_r-1} d_r! \) ways. We next pick up \( m \) to be those of size at least \( k - 2 \) in \( \binom{m+q}{m} \). For those \( m \) with size at least \( k - 2 \), suppose the total size is \( w \). Then we need to pick \( w \) out of \( n - s - d_r \) and distribute them, in \( \binom{n-s-d_r}{w} \) \( (w-m(k-3)-1)w! \) different ways. We next need to distribute the remaining \( n - s - d_r - w \) guys into \( q \) small E-trees such that each has a size at most \( k - 3 \) in the following number of ways (Abramson [1]):

\[
(n - s - d_r - w)! \sum_{j=0}^{q} (-1)^j \binom{q}{j} \frac{n - s - d_r - w - j(k - 3) - 1}{q - 1}.
\]

Therefore, the total number is

\[
\sum_{s} \sum_{d_r} \sum_{q} \sum_{y} \sum_{w} \left( \frac{n - s}{y} \right) \left( \frac{s}{m + q} \right) \left( \frac{s - 1}{y - 1} \right) s! \left( \frac{n - s - 1}{d_r - 1} \right) d_r! \left( \frac{m + q}{m} \right) \left( \frac{n - s - d_r}{w} \right) \left( \frac{w - m(k - 3) - 1}{m - 1} \right) w! \sum_{j=0}^{q} (-1)^j \binom{q}{j} \left( \frac{n - s - d_r - w - j(k - 3) - 1}{q - 1} \right) (n - s - d_r - w)!.
\]

Dividing \( s!(n - s)! \) and simplifying the last formula, we obtain

\[
\sum_{s} \sum_{d_r} \sum_{q} \sum_{w} \left( \frac{n - 1}{s} \right) \left( \frac{s}{m + q} \right) \frac{d_r}{n - s} \left( \frac{m + q}{m} \right) \left( \frac{w - m(k - 3) - 1}{m - 1} \right) \sum_{j=0}^{q} (-1)^j \binom{q}{j} \left( \frac{n - s - d_r - w - j(k - 3) - 1}{q - 1} \right)
\]

\[
= \sum_{s} \sum_{d_r} \sum_{q} \left( \frac{n - 1}{s} \right) \left( \frac{s}{m + q} \right) \frac{d_r}{n - s} \left( \frac{m + q}{m} \right) \sum_{j=0}^{q} (-1)^j \binom{q}{j} \left( \frac{n - s - d_r - (j + m)(k - 3) - 1}{m + q - 1} \right)
\]

\[
= \frac{1}{n} \sum_{s} \sum_{q} \left( \frac{n}{s} \right) \left( \frac{s}{m + q} \right) \left( \frac{m + q}{m} \right) \sum_{j=0}^{q} (-1)^j \binom{q}{j} \left( \frac{n - s - (j + m)(k - 3)}{m + q + 1} \right).
\]

In the above computation, we used the facts that

\[
\sum_{k} \binom{k}{r} \binom{n - k}{s} = \binom{n + 1}{r + s + 1},
\]
and

Lemma C.1 (Abramson [1]). The number of integer compositions of $n$ into $k$ parts such that each part is at most $w$ is given by

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{n - jw - 1}{k - 1}.$$ 

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