Left-Right Pairs and Complex Forests of Infinite Rooted Binary Trees

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October 11, 2018

Abstract
Let $D_0 := \{x + iy \mid x, y > 0\}$, and let $(L, R)$ be a pair of Möbius transformations corresponding to $SL_2(N_0)$ matrices such that $R(D_0)$ and $L(D_0)$ are disjoint. Given such a pair (called a left-right pair), we can construct a directed graph $F(L, R)$ with vertices $D_0$ and edges $\{(z, R(z))\}_{z \in D_0} \cup \{(z, L(z))\}_{z \in D_0}$, which is a collection of infinite binary trees. We answer two questions of Nathanson by classifying all the pairs of elements of $SL_2(N_0)$ whose corresponding Möbius transformations form left-right pairs and showing that trees in $F(L, R)$ are always rooted.

1 Introduction
An infinite rooted binary tree is a directed tree with infinitely many vertices such that every vertex has outdegree 2, there is one vertex with indegree 0, and every other vertex has indegree 1. In an infinite rootless binary tree every vertex has outdegree 2 and indegree 1.

Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$$SL_2(\mathbb{N}_0) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{N}_0, ad - bc = 1 \right\}.$$
For \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{N}_0) \), we let \( T(z) : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}} \) be the corresponding Möbius transformation:

\[
T(z) := \frac{az + b}{cz + d}.
\]

We let \( \mathcal{D}_0 := \{ x + iy \mid x, y > 0 \} \subseteq \overline{\mathbb{C}} \), and for \( L, R \in \text{SL}_2(\mathbb{N}_0) \), we call \((L, R)\) a left-right pair if \( L(\mathcal{D}_0) \cap R(\mathcal{D}_0) = \emptyset \). For any \( T \in \text{SL}_2(\mathbb{N}_0) \), \( T(\mathcal{D}_0) \subseteq \mathcal{D}_0 \), so for \( L, R \in \text{SL}_2(\mathbb{N}_0) \), the graph \( \mathcal{F}(L, R) \) with vertices \( \mathcal{D}_0 \) and edges \( \{(z, L(z)) \mid z \in \mathcal{D}_0\} \cup \{(z, R(z)) \mid z \in \mathcal{D}_0\} \) is well-defined; moreover, if \((L, R)\) is a left-right pair, \( \mathcal{F}(L, R) \) is a collection of infinite binary trees \([12]\).

Graphs constructed this way on various domains often have interesting numerical properties. A particularly well-studied example of this is the Calkin-Wilf tree \([3]\), which is a tree with vertices \( \mathbb{Q}_{>0} \) built on the left-right pair

\[
R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Motivated by the beautiful properties of the Calkin-Wilf tree (see \([1], [2], [4], [5], [6], [7], [8], [9], [10], [13], [11], [14]\)), Nathanson \([12]\) studied the more general construction \( \mathcal{F}(L, R) \) described above. He posed the following problems:

1. Classify left-right pairs in \( \text{SL}_2(\mathbb{N}_0) \);

2. Determine left-right pairs \((L, R)\) whose associated forests contain infinite binary rootless trees.

We answer these questions with the following theorems:

**Theorem 1.1.** Let

\[
A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \text{SL}_2(\mathbb{N}_0).
\]

Then \((A, B)\) is a left-right pair if and only if \( a_1d_2 \leq b_2c_1 \) or \( a_2d_1 \leq c_2b_1 \).

**Theorem 1.2.** Let \( L, R \in \text{SL}_2(\mathbb{N}_0) \) such that \((L, R)\) is a left-right pair. Then all the trees in \( \mathcal{F}(L, R) \) are rooted.
We let \( \overline{D_0} := D_0 \cup \partial D_0 = D_0 \cup \mathbb{R}_{\geq 0} \cup i \cdot \mathbb{R}_{\geq 0} \cup \{\infty\} \). We let \( \mathbb{H} := \{x + iy \mid y > 0\} \) be the Poincaré half-plane model for hyperbolic space with \( \partial \mathbb{H} = \mathbb{R} \cup \{\infty\} \) its boundary. We let \( \mathcal{G} \) be the set of geodesics of \( \mathbb{H} \), which are semicircles with end points on the real axis and vertical rays emitting from points on the real axis, endpoints included. Every element of \( \mathcal{G} \) intersects the boundary of \( \mathbb{H} \) in exactly two points, which we will refer to as the endpoint of \( g \). Möbius transformations map elements of \( \mathcal{G} \) into one another bijectively, with endpoints mapping to endpoints.

For \( g \in \mathcal{G} \) such that \( g \subseteq \overline{D_0} \), the open region contained in \( D_0 \) and bounded by \( g \) and \( \mathbb{R} \cup \{\infty\} \) is called a slice. We call the endpoints of \( g \) the endpoints of that slice. For a slice \( P \), we let

\[
\text{rad}(P) := \sup_{z \in P} \text{Im}(z), \quad \text{diam}(P) = 2\text{rad}(P).
\]

Note that \( \text{rad}(P) \) is the radius of the geodesic bounding \( P \) when the geodesic is a semicircle and infinity when it is a ray. We let \( g(P) \) be the geodesic bounding \( P \). If \( g(p) \) has endpoints \( x, y \) with \( x < y < \infty \), we let \( I(P) := [x, y] \subseteq \mathbb{R}_{\geq 0} \); if \( g(P) \) has endpoints \( x, \infty \), we let \( I(P) := [x, +\infty) \cup \{\infty\} \). Note that \( \partial P = g(P) \cup I(P) \).

We call \( w \) an ancestor of \( z \) if there is a way from \( w \) to \( z \) via the edges of \( \mathcal{F}(L, R) \).

We will abuse notation by writing \( x/0 := \infty \in \mathbb{C} \) when \( x \neq 0 \). We will also write \( \infty > x \) for all \( x \in \mathbb{R} \).

## 2 Left-right pairs

In this section we classify left-right pairs. We begin with a Lemma:

**Lemma 2.1.** Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{N}_0) \), and let \( P \) be a slice with endpoints \( x, y \), with \( x < y \). Then \( M(P) \) is a slice with endpoints \( M(x), M(y) \) and \( M(x) < M(y) \).

**Proof.** If \( c = 0 \), since \( ad - bc = 1 \), we can conclude that \( a = d = 1 \) and \( M \) is a translation by a non-negative integer \( b \), so \( M(P) \) is again a slice with endpoints \( x + b = M(x), y + b = M(y) \).

Suppose now \( c \neq 0 \). Let \( I = I(P), g = g(P) \). \( M \) maps reals into reals, is continuous and injective and preserves orientation, so \( M(I) = [M(x), M(y)] \).
Also, \( M(g) = g' \in \mathcal{G} \) is again a geodesic. By a result of Nathanson ([12, Theorem 1], or simply by computing the real and imaginary parts), \( M(D_0) \subseteq \overline{D_0} \), so by continuity of \( M \) on \( \mathbb{C} \), \( M(D_0) \subseteq \overline{D_0} \). Hence, \( g' \subseteq \overline{D_0} \). Thus, \( M(P) \) is an open region contained in \( \overline{D_0} \) which is bounded by \( M(I) \) and some geodesic \( g' \subseteq \overline{D_0} \), which means a slice with endpoints \( M(x), M(y), M(x) < M(y) \).

\[ \square \]

**Corollary 2.2.** Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{N}_0) \). Then: \( M(D_0) \) is a slice with endpoints \( b/d \) and \( a/c \) with \( b/d < a/c \); moreover, if \( c \neq 0 \), \( \text{diam}(M(P)) < \infty \).

**Proof.** Note \( D_0 \) is a slice with endpoints \( 0, \infty \). We have \( M(0) = b/d, M(\infty) = a/c \), so by Lemma 2.1 \( M(P) \) is a slice with those endpoints. Since \( ad - bc = 1 \), \( d \neq 0 \), so when \( c \neq 0 \), we have

\[ \text{diam}(M(P)) = a/c - b/d < \infty. \]

\[ \square \]

Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** It is clear from the geometric interpretation of slices that two slices \( A(D_0), B(D_0) \) do not intersect if and only if \( I(A(D_0)), I(B(D_0)) \) do not intersect on the interior. By Corollary 2.2 this happens if and only if either \( a_1/c_1 \leq b_2/d_2 \) or \( a_2/c_2 \leq b_1/d_1 \), which is equivalent to the statement of the theorem.

\[ \square \]

### 3 Trees in \( \mathcal{F}(L, R) \) are always rooted

In this section we show \( \mathcal{F}(L, R) \) contains only rooted trees. We let

**Definition 3.1.** Let \( \mathcal{L} : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \) via

\[ \mathcal{L}(t) := \frac{t}{t+1}. \]

Note that for \( n \in \mathbb{N} \),

\[ \mathcal{L}^n(t) = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right)^n(t) = \frac{t}{nt+1}, \]

which decreases goes to 0 as \( n \to \infty \) for all \( t > 0 \).
Lemma 3.2. Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{N}_0) \). Let \( P \) be a slice. Write \( I := I(P) = [x, x + t] \subseteq \mathbb{R}_{\geq 0} \), where \( t = \text{diam}(P) < \infty \). Then \( M(P) \) is also a slice, and:

1. If \( c \neq 0 \), \( \text{diam}(M(P)) \leq \mathcal{L}(t) \);
2. If \( c = 0 \), \( \text{diam}(M(P)) = t \).

Proof. If \( c = 0 \), then \( M \) is a translation, and \( \text{diam}(M(P)) = \text{diam}(P) \).

Suppose now \( c \neq 0 \). By Lemma 2.1, \( M(P) \) is a slice with endpoints \( M(x) \) and \( M(x+t) \). We have:

\[
M(x+t) - M(x) = \frac{a(x+t) + b}{c(x+t) + d} - \frac{ax + b}{cx + d} = \frac{(a(x+t) + b)(cx + d) - (ax + b)(c(x+t) + d)}{(cx + d)(c(x+t) + d)} = \frac{t(acx + ad - cax - cb)}{(cx + d)(c(x+t) + d)} = \frac{t}{(cx + d)(c(x+t) + d)}
\]

Since \( ad - bc = 1 \) and by assumption \( c \neq 0 \), we have \( c, d \geq 1 \), so

\[
\frac{t}{(cx + d)(c(x+t) + d)} \leq \frac{t}{cx + ct + d} \leq \frac{t}{t+1} = \mathcal{L}(t)
\]

as desired. \( \square \)

Theorem 3.3. Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{N}_0) \) with \( c \neq 0 \). Then

\[
\text{diam}(M^n(D_0)) \xrightarrow{n \to \infty} 0.
\]

Proof. By Corollary 2.2, \( M(D_0) \) is a slice with \( \text{diam}(M(D_0)) =: d < \infty \). By Lemma 3.2 it follows that \( M^n(D_0) \) is a slice with

\[
\text{diam}(M^n(D_0)) \leq \mathcal{L}^{n-1}(d),
\]

which goes to 0 as \( n \) goes to infinity. \( \square \)
Corollary 3.4. For all \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{N}_0) \) with \( M \neq \text{Id} \),

\[ \bigcap_{n \in \mathbb{N}_0} M^n(D_0) = \emptyset. \]

Proof. If \( c \neq 0 \), this follows immediately from Theorem 3.3 because for every \( z \in D_0 \), for sufficiently large \( n \), we will have \( \text{rad}(M^n(D_0)) < \text{Im}(z) \). If \( c = 0 \), \( M \) is a translation by a positive integer \( b \), so again we are done.

Lemma 3.5. Let \( A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \) such that \((L, R)\) is a left-right pair. Then at least one of \( c_1, c_2 \) is non-zero.

Proof. Suppose \( c_1 = c_2 = 0 \). Then \( A, B \) are both translations, and clearly \( A(D_0) \cap B(D_0) \neq \emptyset \), which is a contradiction.

We are now ready to prove the main theorem. It suffices to show that any vertex \( z \in D_0 \) cannot have infinitely many ancestors. We will prove this by contradiction. In particular, by assuming \( z \) has infinitely many ancestors, we will show that \( \text{Im}(z) \) has to be less than \( \varepsilon \) for any \( \varepsilon > 0 \).

Proof of Theorem 1.2

Write

\[ L =: \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, R =: \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \]

By Lemma 3.5, without loss of generality \( c_1 \neq 0 \). Moreover, \( R \neq \text{Id} \), because else \( L(D_0) \cap R(D_0) = L(D_0) \neq \emptyset \). Suppose \( z \in D_0 \) has infinitely many ancestors. By Corollary 3.4, any vertex can only have finitely many consecutive right ancestors. Hence, for every \( n \in \mathbb{N} \), we can find \( w \in D_0 \) such that

\[ z = R^{\alpha_1} \circ L^{\alpha_2} \circ \ldots \circ L^{\alpha_{2n}}(w), \]

where \( \alpha_{2k} \geq 1 \) and \( \alpha_{2k-1} \geq 0 \) for \( k \in \{1, \ldots, n\} \). By Corollary 2.2 \( L(D_0) \) is a slice with \( \text{diam}(L(D_0)) =: d < \infty \) and, by Lemma 3.2

\[ \text{diam}(R^{\alpha_1} \circ L^{\alpha_2} \circ \ldots \circ L^{\alpha_{2n-1}}(L(D_0)) \leq L^{\alpha_2 + \cdots + \alpha_{2n-2}}(d) \leq L^{n-1}(d). \]
Since \( z \) has to lie in \( R^{\alpha_1} \circ L^{\alpha_2} \circ \cdots \circ L^{\alpha_{2n-1}}(L(D_0)) \) for all \( n \), it follows that
\[
0 < \text{Im}(z) \leq \mathcal{L}^{n-1}(d) \xrightarrow{n \to \infty} 0,
\]
which is a contradiction. \( \square \)

\section{Acknowledgments}

The research was conducted during the Undergraduate Mathematics Research Program at University of Minnesota Duluth, and supported by grants NSF-1358659 and NSA H98230-16-1-0026. I would like to thank Joe Gallian very much for the wonderful Duluth REU, and for begin such an incredible and supportive mentor.

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