Imperative Process Algebra and Models of Parallel Computation

C.A. Middelburg
Informatics Institute, Faculty of Science, University of Amsterdam
Science Park 904, 1098 XH Amsterdam, the Netherlands
C.A.Middelburg@uva.nl

Abstract. In the theory of computation, a model of computation is used to study issues related to computability and computational complexity. Central in such a model are the computational processes considered. Processes of this kind can be described using an existing imperative process algebra based on ACP (Algebra of Communicating Processes). This paper studies whether this process algebra can play a role in the field of models of computation, in particular in the field of models of parallel computation.

Keywords: imperative process algebra, computational process, abstract machine, parallel random access machine, parallel time complexity parallel computation thesis

1998 ACM Computing Classification: D.1.3, F.1.1, F.1.2, F.1.3.

1 Introduction

Central in a model of computation are the computational processes considered. Computational processes are processes that solve some computational problem. A computational process is applied to a data environment that consists of data organized and accessible in a specific way. Well-known examples of data environments are the tapes of Turing machines and the memories of random access machines. The application of a computational process to a data environment effects a transformation of the data environment. The initial data environment represents an instance of the computational problem that is solved by the process and the final data environment represents the solution of that instance. A computational process is divided into simple steps, each of which depends on and affects only a small portion of the data environment to which the process is applied.

A basic assumption in this paper is that a model of computation is fully characterized by: (a) a set of possible computational processes, (b) for each possible computational process, a set of possible data environments, and (c) the effect of applying such processes to such environments. The set of possible computational processes is usually given indirectly, mostly by means of abstract machines that have a built-in program that belongs to a set of possible programs which is such that the possible computational processes are exactly the processes that are
produced by those machines when they execute their built-in program. Clearly, the abstract machines with their built-in programs emphasize the mechanical nature of the possible computational processes. However, in this way, details of how possible computational processes are produced become part of the model of computation. To the best of my knowledge, all definitions that have been given with respect to a model of computation and all results that have been proved with respect to a model of computation can do without reference to such details.

In [24], an extension of ACP [4] is presented whose additional features include assignment actions to change data in the course of a process, guarded commands to proceed at certain stages of a process in a way that depends on changing data, and data parameterized actions to communicate data between processes. The extension concerned is called ACP\(\tau\)-I. The term imperative process algebra was coined in [25] for process algebras like ACP\(\tau\)-I. In [24], it is discussed what qualities of ACP\(\tau\)-I distinguish it from other imperative process algebras, how its distinguishing qualities are achieved, and what its relevance is to the verification of properties of processes carried out by contemporary computer-based systems. Moreover, that paper goes into one of the application areas of ACP\(\tau\)-I, namely the area of information-flow security analysis.

The current paper studies whether ACP\(\tau\)-I can play a role in the field of models of computation. The idea of this study originates from the experience that definitions of models of computation and results about them in the scientific literature tend to lack preciseness, in particular if it concerns models of parallel computation. The study takes for granted the basic assumption about the characterization of models of computation mentioned above. Moreover, it focuses on models of computation that are intended for investigation of issues related to computability and computational complexity. It does not consider models of computation geared to computation as it takes place on concrete computers or computer networks of a certain kind. Such models are left for follow-up studies. Outcomes of this study are among other things mathematically precise definitions of models of computation corresponding to models based on (sequential) random access machines, asynchronous parallel random access machines, and synchronous parallel random access machines.

This paper is organized as follows. First, a survey of the imperative process algebra ACP\(\tau\)-I and its extension with recursion is given (Section 2). Next, its explained in this process algebra what it means that a given process computes a given function (Section 3). After that, a version of the RAM model of computation is described in the setting introduced in the previous two sections (Section 4). Following this, an asynchronous parallel RAM model of computation and a synchronous parallel RAM model of computation are described in that setting as well (Section 5 and Section 6, respectively). Then, complexity measures for the models of computation presented in the previous three sections are introduced (Section 7). Thereafter, the question whether the presented synchronous parallel RAM model of computation is a reasonable model of computation is treated (Section 8). Finally, some concluding remarks are made (Section 9).
Section 2 mainly summarizes the material presented in Sections 2–7 of [6]. Portions of that material have been copied near verbatim or slightly modified.

2 The Imperative Process Algebra ACP$^\tau$-I

The imperative process algebra ACP$^\tau$-I is an extension of ACP$^\tau$, the version of ACP with empty process and silent step constants that was first presented in [3, Section 5.3]. In this section, first a short survey of ACP$^\tau$ is given and then ACP$^\tau$-I is introduced as an extension of ACP$^\tau$. Moreover, recursion in the setting of ACP$^\tau$-I is treated and soundness and (semi-)completeness results for the axioms of ACP$^\tau$-I with recursion are presented.

2.1 ACP with Empty Process and Silent Step

In this section, a short survey of ACP$^\tau$ is given. A more comprehensive treatment of this process algebra can be found in [3]. In ACP$^\tau$, it is assumed that a fixed but arbitrary finite set $A$ of basic actions, with $\tau, \delta, \epsilon \not\in A$, and a fixed but arbitrary commutative and associative communication function $\gamma: (A \cup \{\tau, \delta\}) \times (A \cup \{\tau, \delta\}) \rightarrow (A \cup \{\tau, \delta\})$, such that $\gamma(\tau, a) = \delta$ and $\gamma(\delta, a) = \delta$ for all $a \in A \cup \{\tau, \delta\}$, have been given. Basic actions are taken as atomic processes. The function $\gamma$ is regarded to give the result of simultaneously performing any two basic actions for which this is possible, and to be $\delta$ otherwise. Henceforth, we write $A_\tau$ for $A \cup \{\tau\}$.

The algebraic theory ACP$^\tau$ has one sort: the sort $P$ of processes. This sort is made explicit to anticipate the need for many-sortedness later on. The algebraic theory ACP$^\tau$ has the following constants and operators to build terms of sort $P$:

- a basic action constant $a : P$ for each $a \in A$;
- a silent step constant $\tau : P$;
- a inaction constant $\delta : P$;
- an empty process constant $\epsilon : P$;
- a binary alternative composition or choice operator $+: P \times P \rightarrow P$;
- a binary sequential composition operator $\cdot: P \times P \rightarrow P$;
- a binary parallel composition or merge operator $\parallel: P \times P \rightarrow P$;
- a binary left merge operator $\lfloor\lfloor: P \times P \rightarrow P$;
- a binary communication merge operator $\mid: P \times P \rightarrow P$;
- a unary encapsulation operator $\partial_H: P \rightarrow P$ for each $H \subseteq A$ and for $H = A_\tau$;
- a unary abstraction operator $\tau_I: P \rightarrow P$ for each $I \subseteq A$.

It is assumed that there is a countably infinite set $X$ of variables of sort $P$, which contains $x$, $y$ and $z$. Terms are built as usual. Infix notation is used for the binary operators. The following precedence conventions are used to reduce the need for parentheses: the operator $\cdot$ binds stronger than all other binary operators and the operator $+$ binds weaker than all other binary operators.

The operators $\parallel$ and $\mid$ are of an auxiliary nature. They make a finite axiomatization of ACP$^\tau$ possible. Let $t$ and $t'$ be closed ACP$^\tau$ terms. Then $t \parallel t'$ denotes
the same process as \( t \parallel t' \), except that it starts with performing an action of the process denoted by \( t \); and \( t \mid t' \) denotes the same process as \( t \parallel t' \), except that it starts with performing an action of the process denoted by \( t \) and an action of the process denoted by \( t' \) simultaneously. In \( \partial_H(t \mid t') \), \( \partial_H \) enforces actions in \( H \) to be performed simultaneously.

The axioms of \( ACP^{\tau}_\epsilon \) are presented in Table 1. In these equations, \( a \), \( b \), and \( \alpha \) stand for arbitrary constants of \( ACP^{\tau}_\epsilon \) other than \( \epsilon \), \( H \) stands for an arbitrary subset of \( A \) or the set \( A_{\tau} \), and \( I \) stands for an arbitrary subset of \( A \). So, CM3, CM7, D0–D4, T0–T4, and BE are actually axiom schemas. In this paper, axiom schemas will usually be referred to as axioms.

The term \( \partial a(x) \cdot \partial a(y) \) occurring in axiom CM1E is needed to handle successful termination in the presence of \( \epsilon \): it stands for the process that behaves the same as \( \epsilon \) if both \( x \) and \( y \) stand for a process that has the option to behave the same as \( \epsilon \) and it stands for the process that behaves the same as \( \delta \) otherwise.

In [2], Section 5.3, the symbol \( \sqrt{\text{a}} \) is used instead of \( \partial a \).

Notice that the equation \( \alpha \cdot \delta = \alpha \) would be derivable from the axioms of \( ACP^{\tau}_\epsilon \) if operators \( \partial_H \) where \( H \neq A_{\tau} \) and \( \tau \in H \) were added to \( ACP^{\tau}_\epsilon \).

The notation \( \sum_{i=1}^{n} t_i \), where \( n \geq 1 \), will be used for right-nested alternative compositions. For each \( n \in \mathbb{N}^+ \) the term \( \sum_{i=1}^{n} t_i \) is defined by induction on \( n \) as follows: \( \sum_{i=1}^{1} t_i = t_1 \) and \( \sum_{i=1}^{n+1} t_i = t_1 + \sum_{i=1}^{n} t_{i+1} \). In addition, the convention will be used that \( \sum_{i=1}^{0} t_i = \delta \). Moreover, we write \( \partial a \) and \( \tau a \), where \( a \in A \), for \( \partial \{ a \} \) and \( \tau \{ a \} \), respectively.

### 2.2 Imperative \( ACP^{\tau}_\epsilon \)

In this section, \( ACP^{\tau}_\epsilon \)-I, imperative \( ACP^{\tau}_\epsilon \), is introduced as an extension of \( ACP^{\tau}_\epsilon \). In [24], the paper in which \( ACP^{\tau}_\epsilon \)-I was first presented, a comprehensive treatment of this imperative process algebra can be found. \( ACP^{\tau}_\epsilon \)-I extends \( ACP^{\tau}_\epsilon \) with features to communicate data between processes, to change data involved in a process in the course of the process, and to proceed at certain stages of a process in a way that depends on the changing data.

In \( ACP^{\tau}_\epsilon \)-I, it is assumed that the following has been given with respect to data:

- a many-sorted signature \( \Sigma_D \) that includes:
  - a sort \( D \) of data and a sort \( B \) of bits;
  - constants of sort \( D \) and/or operators with result sort \( D \);
  - constants 0 and 1 of sort \( B \) and operators with result sort \( B \);
- a minimal algebra \( \mathcal{D} \) of the signature \( \Sigma_D \) in which the carrier of sort \( B \) has cardinality 2 and the equation \( 0 = 1 \) does not hold.

We write \( \mathcal{D} \) for the set of all closed terms over the signature \( \Sigma_D \) that are of sort \( D \). The sort \( B \) is assumed to be given in order to make it possible for operators to serve as predicates.

\footnote{We write \( \mathbb{N}^+ \) for the set \( \{ n \in \mathbb{N} \mid n \geq 1 \} \) of positive natural numbers.}
Table 1. Axioms of $\text{ACP}^*$

| Axiom | Description |
|-------|-------------|
| A1    | $x + y = y + x$ |
| A2    | $(x + y) + z = x + (y + z)$ |
| A3    | $x + x = x$ |
| A4    | $(x + y) \cdot z = x \cdot z + y \cdot z$ |
| A5    | $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ |
| A6    | $x + \delta = x$ |
| A7    | $\delta \cdot x = \delta$ |
| A8    | $\epsilon \cdot x = x$ |
| A9    | $x \parallel \epsilon = x$ |
| CM1E  | $\epsilon \parallel (y + x) + x + y = \epsilon + \partial \alpha, \epsilon \cdot \partial \alpha, \epsilon (x)$ |
| CM2E  | $\epsilon \parallel \delta = \epsilon$ |
| CM3   | $\alpha \cdot x \parallel y = \alpha \cdot (x \parallel y)$ |
| CM4   | $(x + y) \parallel z = x \parallel z + y \parallel z$ |
| CM5E  | $\epsilon \mid x = \delta$ |
| CM6E  | $x \mid \epsilon = \delta$ |
| CM7   | $\alpha \cdot b \cdot y = \gamma(a, b) \cdot (x \parallel y)$ |
| CM8   | $(x + y) \mid z = x \mid z + y \mid z$ |
| CM9   | $x \mid (y + z) = x \mid y + x \mid z$ |
| D0    | $\partial H(\epsilon) = \epsilon$ |
| D1    | $\partial H(\alpha) = \alpha$ if $\alpha \notin H$ |
| D2    | $\partial H(\alpha) = \delta$ if $\alpha \in H$ |
| D3    | $\partial H(x + y) = \partial H(x) + \partial H(y)$ |
| D4    | $\partial H(x \cdot y) = \partial H(x) \cdot \partial H(y)$ |
| T0    | $\tau(\epsilon) = \epsilon$ |
| T1    | $\tau(\alpha) = \alpha$ if $\alpha \notin I$ |
| T2    | $\tau(\alpha) = \tau$ if $\alpha \in I$ |
| T3    | $\tau(x + y) = \tau(x) + \tau(y)$ |
| T4    | $\tau(x \cdot y) = \tau(x) \cdot \tau(y)$ |
| BE    | $\alpha \cdot (\tau \cdot (x + y) + x) = \alpha \cdot (x + y)$ |

It is also assumed that a finite or countably infinite set $\mathcal{V}$ of flexible variables has been given. A flexible variable is a variable whose value may change in the course of a process. An flexible variable valuation is a function from $\mathcal{V}$ to $\mathbb{D}$.

\footnote{The term flexible variable is used for this kind of variables in e.g. [31,21].}
The algebraic theory $\text{ACP}_\tau^\varepsilon$-I has the following sorts: the sorts from $\Sigma_D$, the sort $C$ of conditions, and the sort $P$ of processes.

It is assumed that there are countably infinite sets of variables of sort $C$ and $D$ and that the sets of variables of sort $P$, $C$, and $D$ are mutually disjoint and disjoint from $\mathcal{V}$.

Below, the constants and operators of $\text{ACP}_\tau^\varepsilon$-I are introduced. The operators of $\text{ACP}_\tau^\varepsilon$-I include two variable-binding operators. The formation rules for $\text{ACP}_\tau^\varepsilon$-I terms are the usual ones for the many-sorted case (see e.g. [30,34]) and in addition the following rule:

- if $O$ is a variable-binding operator $O : S_1 \times \ldots \times S_n \to S$ that binds a variable of sort $S'$, $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$, respectively, and $X$ is a variable of sort $S'$, then $OX(t_1, \ldots, t_n)$ is a term of sort $S$.

An extensive formal treatment of the phenomenon of variable-binding operators can be found in [29].

$\text{ACP}_\tau^\varepsilon$-I has the constants and operators from $\Sigma_D$ to build terms of the sorts from $\Sigma_D$ — which include the sort $B$ and the sort $D$ — and in addition the following constants to build terms of sort $D$:

- for each $v \in \mathcal{V}$, the flexible variable constant $v : D$.

We write $D$ for the set of all closed $\text{ACP}_\tau^\varepsilon$-I terms of sort $D$.

Flexible variable valuations are intended to provide the data values assigned to flexible variables when a $\text{ACP}_\tau^\varepsilon$-I term of sort $D$ is evaluated. In order to fit better in an algebraic setting, they provide closed terms from $D$ that denote those data values instead. The requirement that $D$ is a minimal algebra guarantees that each member of the carrier of a sort from $\Sigma_D$ can be represented by a closed term of the sort concerned.

In the rest of this paper, for each sort $S$ from $\Sigma_D$, let $ct_S$ be a function from the carrier of $S$ in $D$ to the set of all closed terms over the signature that are of sort $S$ such that, for each element $d$ of the carrier of sort $S$ in $D$, the term $ct_S(d)$ represents $d$. We write $d$, where $d$ is an element of the carrier of a sort $S$ from $\Sigma_D$, for $ct_S(d)$ if it is clear from the context that $d$ stands for a closed term representing $d$.

Let $V \subseteq \mathcal{V}$. Then a $V$-indexed data environment is a function from $V$ to the carrier of sort $D$ in $\mathcal{D}$. Let $\mu$ be a $V$-indexed data environment and $\rho$ be a flexible variable valuation. Then $\rho$ represents $\mu$ if $\rho(v) = ct_D(\mu(v))$ for all $v \in V$.

$\text{ACP}_\tau^\varepsilon$-I has the following constants and operators to build terms of sort $C$:

- a binary equality operator $= : B \times B \to C$;
- a binary equality operator $= : D \times D \to C$;
- a truth constant $t : C$;
- a falsity constant $f : C$;
- a unary negation operator $\neg : C \to C$;
- a binary conjunction operator $\land : C \times C \to C$;

\footnote{The overloading of $=$ can be trivially resolved if $\Sigma_D$ is without overloaded symbols.}
– a binary disjunction operator $\lor : C \times C \to C$;
– a binary implication operator $\Rightarrow : C \times C \to C$;
– a unary variable-binding universal quantification operator $\forall : C \to C$ that binds a variable of sort $D$;
– a unary variable-binding existential quantification operator $\exists : C \to C$ that binds a variable of sort $D$.

We write $C$ for the set of all closed $\tau_\epsilon$-I terms of sort $C$.

Each term from $C$ can be taken as a formula of a first-order language with equality of $D$ by taking the flexible variable constants as additional variables of sort $D$. The flexible variable constants are implicitly taken as additional variables of sort $D$ wherever the context asks for a formula. In this way, each term from $C$ can be interpreted as a formula in $\mathcal{D}$.

$\tau_\epsilon$-I has the constants and operators of $\tau_\epsilon$ and in addition the following operators to build terms of sort $P$:

– an $n$-ary data parameterized action operator $a : D \times \cdots \times D \to P$ for each $a \in A$, for each $n \in \mathbb{N}$;
– a unary assignment action operator $v := : D \to P$ for each $v \in V$;
– a binary guarded command operator $: : C \times P \to P$;
– a unary evaluation operator $V : P \to P$ for each $\rho \in V \to D$.

We write $P$ for the set of all closed $\tau_\epsilon$-I terms of sort $P$.

The same notational conventions are used as before. Infix notation is also used for the additional binary operators. Moreover, the notation $[v := e]$, where $v \in V$ and $e$ is a $\tau_\epsilon$-I term of sort $D$, is used for the term $v := (e)$.

The notation $\phi \iff \psi$, where $\phi$ and $\psi$ are $\tau_\epsilon$-I terms of sort $C$, is used for the term $(\phi \Rightarrow \psi) \land (\psi \Rightarrow \phi)$. The axioms of $\tau_\epsilon$-I (given below) include an equation $\phi \iff \psi$ for each two terms $\phi$ and $\psi$ from $C$ for which the formula $\phi \iff \psi$ holds in $\mathcal{D}$.

$\tau_\epsilon$-I concerns processes in which data are involved. Data parameterized actions make it possible that data is communicated between processes. Assignment actions make it possible to change data in the course of a process. Guarded commands make it possible that a process proceeds at certain stages in a way that depends on the changing data.

Let $t \in P$ and $\rho \in V \to D$. Then the term $V(\rho)(t)$ denotes the process that behaves the same as the process denoted by $t$ except that each subterm of $t$ that belongs to $D$ is evaluated using the flexible variable valuation $\rho$ updated according to the assignment actions that have taken place at the point where the subterm is encountered.

The following closed $\tau_\epsilon$-I term is reminiscent of a program that computes the difference between two integers by subtracting the smaller one from the larger one ($i, j, d \in V$):

$$[d := i] \cdot ((d \geq j = 1) \Rightarrow [d := d - j] + (d \geq j = 0) \Rightarrow [d := j - d])$$
That is, the final value of $d$ is the absolute value of the result of subtracting the initial value of $i$ from the initial value of $j$. Let $\rho$ be an flexible variable valuation such that $\rho(i) = 11$ and $\rho(j) = 3$. Then the following equation can be derived from the axioms of $\text{ACP}_\tau^\omega$ given below:

$$V_\rho([d := i] \cdot ((d \geq j = 1) \rightarrow [d := d - j] + (d \geq j = 0) \rightarrow [d := j - d])) = [d := 11] \cdot [d := 8].$$

This equation shows that in the case where the initial values of $i$ and $j$ are 11 and 3 the final value of $d$ is 8, which is the absolute value of the result of subtracting 11 from 3.

An flexible variable valuation $\rho$ can be extended homomorphically from flexible variables to $\text{ACP}_\tau^\omega$ terms of sort $D$ and $\text{ACP}_\tau^\omega$ terms of sort $C$. These extensions are denoted by $\rho$ as well. Below, we write $\rho[e \mapsto v]$ for the flexible variable valuation $\rho'$ defined by $\rho'(v') = \rho(v')$ if $v' \neq v$ and $\rho'(v) = e$.

Three subsets of $\mathcal{P}$ are defined:

$\mathcal{A}^{dp} = \bigcup_{n \in \mathbb{N}^+} \{a(e_1, \ldots, e_n) \mid a \in A \wedge e_1, \ldots, e_n \in D\}$,

$\mathcal{A}^{ass} = \{[v := e] \mid v \in V \wedge e \in D\}$,

$\mathcal{A} = \{a \mid a \in A\} \cup \mathcal{A}^{dp} \cup \mathcal{A}^{ass}$.

In $\text{ACP}_\tau^\omega$, the elements of $\mathcal{A}$ are the terms from $\mathcal{P}$ that denote the processes that are considered to be atomic. Henceforth, we write $\mathcal{A}_\tau$ for $\mathcal{A} \cup \{\tau\}$, $\mathcal{A}_\delta$ for $\mathcal{A} \cup \{\delta\}$, and $\mathcal{A}_{\tau,\delta}$ for $\mathcal{A} \cup \{\tau, \delta\}$.

The axioms of $\text{ACP}_\tau^\omega$ are the axioms presented in Table 1, on the understanding that $\alpha$ now stands for an arbitrary term from $\mathcal{A}_{\tau,\delta}$, $H$ now stands for an arbitrary subset of $\mathcal{A}$ or the set $\mathcal{A}_\tau$, and $I$ now stands for an arbitrary subset of $\mathcal{A}$, and in addition the axioms presented in Table 2 in the latter table, $\phi$ and $\psi$ stand for arbitrary terms from $\mathcal{C}$, $e$, $e_1, e_2, \ldots$, and $e', e'_1, e'_2, \ldots$ stand for arbitrary terms from $\mathcal{D}$, $v$ stands for an arbitrary flexible variable from $\mathcal{V}$, $\rho$ stands for an arbitrary flexible variable valuation from $\mathcal{V} \rightarrow \mathcal{D}$, $a, b,$ and $c$ stand for arbitrary basic actions from $\mathcal{A}$, and $\alpha$ stands for an arbitrary term from $\mathcal{A}_{\tau,\delta}$.

### 2.3 $\text{ACP}_\tau^\omega$ with Recursion

In this section, recursion in the setting of $\text{ACP}_\tau^\omega$ is treated. A closed $\text{ACP}_\tau^\omega$ term of sort $P$ denotes a process with a finite upper bound to the number of actions that it can perform. Recursion allows the description of processes without a finite upper bound to the number of actions that it can perform.

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4 Here and in subsequent examples in Section 2, the carrier of $D$ is assumed to be the set of all integers. Moreover, the usual integer constants, operators on integers, and predicates on integers are assumed (where operators with result sort $B$ serve as predicates).
A recursive specification over $\text{ACP}^*_\tau$-I is a set $\{X_i = t_i \mid i \in I\}$, where $I$ is a finite set, each $X_i$ is a variable from $\mathcal{X}$, each $t_i$ is a $\text{ACP}^*_\tau$-I term of sort $P$ in which only variables from $\{X_i \mid i \in I\}$ occur, and $X_i \neq X_j$ for all $i, j \in I$ with $i \neq j$. We write $\text{vars}(E)$, where $E$ is a recursive specification over $\text{ACP}^*_\tau$-I, for the set of all variables that occur in $E$. Let $E$ be a recursive specification and

\begin{table}[h]
\centering
\caption{Additional axioms of $\text{ACP}^*_\tau$-I}
\begin{tabular}{ll}
\hline
$e = e'$ & if $\mathcal{D} \models e = e'$ \hspace{1cm} IMP1 \\
$\phi = \psi$ & if $\mathcal{D} \models \phi \leftrightarrow \psi$ \hspace{1cm} IMP2 \\
$t : \rightarrow x = x$ & GC1 \\
$f : \rightarrow x = \delta$ & GC2 \\
$\phi : \rightarrow \delta = \delta$ & GC3 \\
$\phi : \rightarrow (x + y) = \phi : \rightarrow x + \phi : \rightarrow y$ & GC4 \\
$\phi : \rightarrow x \cdot y = (\phi : \rightarrow x) \cdot y$ & GC5 \\
$\phi : \rightarrow (\psi : \rightarrow x) = (\phi \land \psi) : \rightarrow x$ & GC6 \\
$(\phi \land \psi) : \rightarrow x = \phi : \rightarrow x + \psi : \rightarrow x$ & GC7 \\
$(\phi : \rightarrow x) \parallel y = \phi : \rightarrow (x \parallel y)$ & GC8 \\
$(\phi : \rightarrow x) \parallel y = \phi : \rightarrow (x \parallel y)$ & GC9 \\
$x \parallel (\phi : \rightarrow y) = \phi : \rightarrow (x \parallel y)$ & GC10 \\
$\partial_H(\phi : \rightarrow x) = \phi : \rightarrow \partial_H(x)$ & GC11 \\
$\tau_1(\phi : \rightarrow x) = \phi : \rightarrow \tau_1(x)$ & GC12 \\
$\mathcal{V}_\rho(e) = e$ & V0 \\
$\mathcal{V}_\rho(\tau \cdot x) = \tau \cdot \mathcal{V}_\rho(x)$ & V1 \\
$\mathcal{V}_\rho(a \cdot x) = a \cdot \mathcal{V}_\rho(x)$ & V2 \\
$\mathcal{V}_\rho(a(e_1, e_2, \ldots, e_n) \cdot x) = a(\rho(e_1), \ldots, \rho(e_n)) \cdot \mathcal{V}_\rho(x)$ & V3 \\
$\mathcal{V}_\rho([v := c] \cdot x) = [v := \rho(c)] \cdot \mathcal{V}_{\rho(\rho(e) \cdot v)}(x)$ & V4 \\
$\mathcal{V}_\rho(x + y) = \mathcal{V}_\rho(x) + \mathcal{V}_\rho(y)$ & V5 \\
$\mathcal{V}_\rho(\phi : \rightarrow y) = \rho(\phi) : \rightarrow \mathcal{V}_\rho(x)$ & V6 \\
$\alpha(e_1, e_2, \ldots, e_n \cdot x \mid b(e_1, e_2, \ldots, e_n) \cdot y = (e_1 = e'_1 \land \ldots \land e_n = e'_n) : \rightarrow (c(e_1, e_2, \ldots, e_n) \cdot (x \parallel y) \text{ if } \gamma(a, b) = c)$ & CM7Da \\
$\alpha(e_1, e_2, \ldots, e_n \cdot x \mid b(e_1, e_2, \ldots, e_n) \cdot y = \delta \text{ if } \gamma(a, b) = \delta \text{ or } n \neq m)$ & CM7Db \\
$\alpha(e_1, e_2, \ldots, e_n \cdot x \mid a \cdot y = \delta \text{ if } \alpha \notin A^\text{op})$ & CM7Dc \\
$\alpha \cdot x \mid a(e_1, e_2, \ldots, e_n) \cdot y = \delta \text{ if } \alpha \notin A^\text{op})$ & CM7Dd \\
$[v := c] \cdot x \mid a \cdot y = \delta$ & CM7De \\
$\alpha \cdot x \mid [v := c] \cdot y = \delta$ & CM7Df \\
$\alpha \cdot (\phi : \rightarrow \tau \cdot (x + y) + \phi : \rightarrow x) = \alpha \cdot (\phi : \rightarrow (x + y))$ & BED \\
\hline
\end{tabular}
\end{table}
let $X \in \text{vars}(E)$. Then the unique equation $X = t \in E$ is called the *recursion equation* for $X$ in $E$.

Below, recursive specifications over $\text{ACP}_\tau^\tau$-I are introduced in which the right-hand sides of the recursion equations are linear $\text{ACP}_\tau^\tau$-I terms. The set $\mathcal{L}$ of *linear $\text{ACP}_\tau^\tau$-I terms* is inductively defined by the following rules:

- $\delta \in \mathcal{L}$;
- if $\phi \in \mathcal{L}$, then $\phi : \rightarrow \epsilon \in \mathcal{L}$;
- if $\phi \in \mathcal{L}$, $\alpha \in \mathcal{A}_\tau$, and $X \in \mathcal{X}$, then $\phi : \rightarrow \alpha \cdot X \in \mathcal{L}$;
- if $t, t' \in \mathcal{L}$, then $t + t' \in \mathcal{L}$.

Let $t \in \mathcal{L}$. Then we refer to the subterms of $t$ that have the form $\phi : \rightarrow \epsilon$ or the form $\phi : \rightarrow \alpha \cdot X$ as the *summands* of $t$.

Let $X$ be a variable from $\mathcal{X}$ and let $t$ be an $\text{ACP}_\tau^\tau$-I term in which $X$ occurs. Then an occurrence of $X$ in $t$ is *guarded* if $t$ has a subterm of the form $\alpha \cdot t'$ where $\alpha \in \mathcal{A}$ and $t'$ contains this occurrence of $X$. Notice that an occurrence of a variable in a linear $\text{ACP}_\tau^\tau$-I term may be not guarded.

A *guarded linear recursive specification* over $\text{ACP}_\tau^\tau$-I is a recursive specification $\{X_i = t_i \mid i \in I\}$ over $\text{ACP}_\tau^\tau$-I where each $t_i$ is a linear $\text{ACP}_\tau^\tau$-I term, and there does not exist an infinite sequence $i_0 i_1 \ldots$ over $I$ such that, for each $k \in \mathbb{N}$, there is an occurrence of $X_{i_{k+1}}$ in $t_{i_k}$ that is not guarded.

A solution of a guarded linear recursive specification $E$ over $\text{ACP}_\tau^\tau$-I in some model of $\text{ACP}_\tau^\tau$-I is a set $\{p_X \mid X \in \text{vars}(E)\}$ of elements of the carrier of sort $\mathcal{P}$ in that model such that each equation in $E$ holds if, for all $X \in \text{vars}(E)$, $X$ is assigned $p_X$. A guarded linear recursive specification has a unique solution under the equivalence defined in [24] for $\text{ACP}_\tau^\tau$-I extended with guarded linear recursion. If $\{p_X \mid X \in \text{vars}(E)\}$ is the unique solution of a guarded linear recursive specification $E$, then, for each $X \in \text{vars}(E)$, $p_X$ is called the $X$-component of the unique solution of $E$.

$\text{ACP}_\tau^\tau$-I is extended with guarded linear recursion by adding constants for solutions of guarded linear recursive specifications over $\text{ACP}_\tau^\tau$-I and axioms concerning these additional constants. For each guarded linear recursive specification $E$ over $\text{ACP}_\tau^\tau$-I and each $X \in \text{vars}(E)$, a constant $\langle X \mid E \rangle$ of sort $\mathcal{P}$, that stands for the $X$-component of the unique solution of $E$, is added to the constants of $\text{ACP}_\tau^\tau$-I. The equation RDP and the conditional equation RSP given in Table 3 are added to the axioms of $\text{ACP}_\tau^\tau$-I. In RDP and RSP, $X$ stands for an arbitrary variable from $\mathcal{X}$, $t$ stands for an arbitrary $\text{ACP}_\tau^\tau$-I term of sort $\mathcal{P}$, $E$ stands for an arbitrary guarded linear recursive specification over $\text{ACP}_\tau^\tau$-I, and the notation $\langle t \mid E \rangle$ is used for $t$ with, for all $X \in \text{vars}(E)$, all occurrences of $X$ in $t$ replaced by $\langle X \mid E \rangle$. Side conditions restrict what $X$, $t$ and $E$ stand for.

| $\langle X \mid E \rangle$ | $\langle t \mid E \rangle$ if $X = t \in E$ | RDP |
|--------------------------|---------------------------------|-----|
| $E \Rightarrow X = \langle X \mid E \rangle$ if $X \in \text{vars}(E)$ | RSP |
We write $\text{ACP}_\tau^\varepsilon$-I+REC for the resulting theory. Furthermore, we write $\mathcal{P}_{\text{rec}}$ for the set of all closed $\text{ACP}_\tau^\varepsilon$-I+REC terms of sort $P$.

RDP and RSP together postulate that guarded linear recursive specifications over $\text{ACP}_\tau^\varepsilon$-I have unique solutions.

Because conditional equational formulas must be dealt with in $\text{ACP}_\tau^\varepsilon$-I+REC, it is understood that conditional equational logic is used in deriving equations from the axioms of $\text{ACP}_\tau^\varepsilon$-I+REC. A complete inference system for conditional equational logic can for example be found in [3,13].

We write $T \vdash t = t'$, where $T$ is $\text{ACP}_\tau^\varepsilon$-I+REC or $\text{ACP}_\tau^\varepsilon$-I+REC+CFAR (see below), to indicate that the equation $t = t'$ is derivable from the axioms of $T$ using a complete inference system for conditional equational logic.

The following closed $\text{ACP}_\tau^\varepsilon$-I+REC term is reminiscent of a program that computes by repeated subtraction the quotient and remainder of dividing a non-negative integer by a positive integer ($i,j,q,r \in \mathcal{V}$):

\[
[q := 0] \cdot [r := i] \cdot \langle Q \mid E \rangle,
\]

where $E$ is the guarded linear recursive specification that consists of the following two equations ($Q,R \in X$):

\[
Q = (r \geq j = 1) \rightarrow [q := q + 1] \cdot R + (r \geq j = 0) \rightarrow \varepsilon,
\]

\[
R = t \rightarrow [r := r - j] \cdot Q.
\]

Let $\rho$ be an flexible variable valuation such that $\rho(i) = 11$ and $\rho(j) = 3$. Then the following equation can be derived from the axioms of $\text{ACP}_\tau^\varepsilon$-I+REC:

\[
\forall \rho ([q := 0] \cdot [r := i] \cdot \langle Q \mid E \rangle)
\]

\[
= [q := 0] \cdot [r := 11] \cdot [q := 1] \cdot [r := 8] \cdot [q := 2] \cdot [r := 5] \cdot [q := 3] \cdot [r := 2].
\]

This equation shows that in the case where the initial values of $i$ and $j$ are 11 and 3 the final values of $q$ and $r$ are 3 and 2, which are the quotient and remainder of dividing 11 by 3.

In [24], an equational axiom schema CFAR (Cluster Fair Abstraction Rule) is added to $\text{ACP}_\tau^\varepsilon$-I+REC. CFAR expresses that every cluster of $\tau$ actions will be exited sooner or later. This is a fairness assumption made in the verification of many properties concerning the external behaviour of systems. We will write $\text{ACP}_\tau^\varepsilon$-I+REC+CFAR for the theory $\text{ACP}_\tau^\varepsilon$-I+REC extended with CFAR.

### 2.4 Soundness and Completeness Results

In [24], a structural operational semantics of $\text{ACP}_\tau^\varepsilon$-I+REC is presented and an equivalence relation $\equiv_{\text{ob}}$ on $\mathcal{P}_{\text{rec}}$ based on this structural operational semantics is defined. This equivalence relation reflects the idea that two processes are equivalent if they can simulate each other insofar as their observable potentials to make transitions by performing actions and to terminate successfully are
concerned, taking into account the assignments of data values to flexible variables under which the potentials are available.

In this section, soundness and (semi-)completeness results for the axioms of $\text{ACP}^\tau - I + \text{REC} + \text{CFAR}$ with respect to $\leftrightarrow_{rb}$ are presented. The proofs can be found in [24].

The axiom system of $\text{ACP}^\tau - I + \text{REC} + \text{CFAR}$ is sound with respect to $\leftrightarrow_{rb}$ for equations between terms from $\mathcal{P}_{\text{rec}}$.

**Theorem 1 (Soundness).** For all terms $t, t' \in \mathcal{P}_{\text{rec}}$, $t = t'$ is derivable from the axioms of $\text{ACP}^\tau - I + \text{REC} + \text{CFAR}$ only if $t \leftrightarrow_{rb} t'$.

The axioms of $\text{ACP}^\tau - I + \text{REC} + \text{CFAR}$ are incomplete with respect to $\leftrightarrow_{rb}$ for equations between terms from $\mathcal{P}_{\text{rec}}$ and there is no straightforward way to rectify this. Below two semi-completeness results are presented. The next two lemmas are used in the proofs of those results.

A term $t \in \mathcal{P}_{\text{rec}}$ is called abstraction-free if no abstraction operator occurs in $t$. A term $t \in \mathcal{P}_{\text{rec}}$ is called bool-conditional if, for each $\phi \in \mathcal{C}$ that occurs in $t$, $D \models \phi \iff t$ or $D \models \phi \iff f$.

**Lemma 1.** For all abstraction-free $t \in \mathcal{P}_{\text{rec}}$, there exists a guarded linear recursive specification $E$ and $X \in \text{vars}(E)$ such that $\text{ACP}^\tau - I + \text{REC} \vdash t = \langle X | E \rangle$.

**Lemma 2.** For all bool-conditional $t \in \mathcal{P}_{\text{rec}}$, there exists a guarded linear recursive specification $E$ and $X \in \text{vars}(E)$ such that $\text{ACP}^\tau - I + \text{REC} + \text{CFAR} \vdash t = \langle X | E \rangle$.

The following two theorems are the semi-completeness results referred to above.

**Theorem 2 (Semi-completeness I).** For all abstraction-free $t, t' \in \mathcal{P}_{\text{rec}}$, $\text{ACP}^\tau - I + \text{REC} \vdash t = t'$ if $t \leftrightarrow_{rb} t'$.

**Theorem 3 (Semi-completeness II).** For all bool-conditional $t, t' \in \mathcal{P}_{\text{rec}}$, $\text{ACP}^\tau - I + \text{REC} + \text{CFAR} \vdash t = t'$ if $t \leftrightarrow_{rb} t'$.

The following is a corollary of Theorems 1 and 2.

**Corollary 1.** For all $t, t' \in \mathcal{P}_{\text{rec}}$, for all $\rho \in \mathcal{V} \to \mathcal{D}$, $\text{ACP}^\tau - I + \text{REC} + \text{CFAR} \vdash V_\rho(t) = V_\rho(t')$ iff $V_\rho(t) \leftrightarrow_{rb} V_\rho(t')$.

Below, an elimination theorem is presented which can easily be proved using Theorem 3. That elimination theorem refers the set $\mathcal{B}$ of basic $\text{ACP}^\tau - I$ terms inductively defined by the following rules:

- $\delta \in \mathcal{B}$;
- if $\phi \in \mathcal{C}$, then $\phi :\to \epsilon \in \mathcal{B}$;
- if $\phi \in \mathcal{C}$, $\alpha \in \mathcal{A}_\tau$, and $t \in \mathcal{B}$, then $\phi :\to \alpha \cdot t \in \mathcal{B}$;
- if $t, t' \in \mathcal{B}$, then $t + t' \in \mathcal{B}$.

**Theorem 4 (Elimination).** For all bool-conditional $t \in \mathcal{P}$, there exists a bool-conditional $t' \in \mathcal{B}$ such that $\text{ACP}^\tau - I \vdash t = t'$. 

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2.5 Extensions

In this section, two extensions of ACP\textsuperscript{τ,I} are treated, namely projection and action renaming. It is not unusual to come across these extensions in applications of ACP-style process algebras. The first extension is treated here because projections can be used to determine the maximum number of actions that a finite process can perform. The second extension is treated here because action renaming enables to easily define the synchronous variant of the parallel composition operator of ACP\textsuperscript{τ,I} needed later in this paper.

Let $T$ be either ACP\textsuperscript{τ,I} or one of its extensions introduced before. $T$ can be extended with projection by adding, for each $n \in \mathbb{N}$, a unary projection operator $\pi_n : P \to P$ to the operators of $T$ and adding the axioms given in Table 4 to the axioms of $T$. In this table, $n$ stands for an arbitrary natural number, $\alpha$ stands for an arbitrary term from $A^{\text{act}}$, and $\phi$ stands for an arbitrary term from $C$.

Let $t$ be a closed term of the extended theory. Then the projection operator $\pi_n$ can be explained as follows: $\pi_n(t)$ denotes the process that behaves the same as the process denoted by $t$ except that it terminates successfully after $n$ actions have been performed.

Let $T$ be either ACP\textsuperscript{τ,I} or one of its extensions introduced before. Then we will write $T+\text{PR}$ for $T$ extended with the projection operators $\pi_n$ and the axioms PR1–PR6 from Table 4.

Let $T$ be either ACP\textsuperscript{τ,I} or one of its extensions introduced before. $T$ can be extended with action renaming by adding, for each function $f : A \to A$ such that $f(\alpha) = \alpha$ for all $\alpha \in A^{\text{act}}$, a unary action renaming operator $\rho_f : P \to P$ to the operators of $T$ and adding the axioms given in Table 5 to the axioms of $T$. In this table, $f$ stands for an arbitrary function $f : A \to A$ such that $f(\alpha) = \alpha$ for all $\alpha \in A^{\text{act}}$, $\alpha$ stands for an arbitrary term from $A$, and $\phi$ stands for an arbitrary term from $C$.

Let $t$ be a closed term of the extended theory. Then the action renaming operator $\rho_f$ can be explained as follows: $\rho_f(t)$ denotes the process that behaves the same as the process denoted by $t$ except that, where the latter process performs an action $\alpha$, the former process performs the action $f(\alpha)$.

Let $T$ be either ACP\textsuperscript{τ,I} or one of its extensions introduced before. Then we will write $T+\text{RN}$ for $T$ extended with the action renaming operators $\rho_f$ and the axioms RN1–RN7 from Table 5.

Table 4. Axioms for the projection operators

| Axiom | Description |
|-------|-------------|
| $\pi_n(\epsilon) = \epsilon$ | PR1 |
| $\pi_0(\alpha \cdot x) = \epsilon$ | PR2 |
| $\pi_{n+1}(\alpha \cdot x) = \alpha \cdot \pi_n(x)$ | PR3 |
| $\pi_n(x + y) = \pi_n(x) + \pi_n(y)$ | PR4 |
| $\pi_n(\phi :\to x) = \phi :\to \pi_n(x)$ | PR5 |
| $\pi_n(\tau \cdot x) = \tau \cdot \pi_n(x)$ | PR6 |
Table 5. Axioms for the action renaming operators

| Axiom | Description |
|-------|-------------|
| \( \rho f(\epsilon) = \epsilon \) | \( \text{RN1} \) |
| \( \rho f(\delta) = \delta \) | \( \text{RN2} \) |
| \( \rho f(\alpha) = f(\alpha) \) | \( \text{RN3} \) |
| \( \rho f(x + y) = \rho f(x) + \rho f(y) \) | \( \text{RN4} \) |
| \( \rho f(x \cdot y) = \rho f(x) \cdot \rho f(y) \) | \( \text{RN5} \) |
| \( \rho f(\phi \rightarrow x) = \phi \rightarrow \rho f(y) \) | \( \text{RN6} \) |
| \( \rho f(\tau) = \tau \) | \( \text{RN7} \) |

3 Computation and the RAM Conditions

In order to investigate whether ACP\(\tau\)-I+REC can play a role in the field of models of computation, it has to be explained in the setting of ACP\(\tau\)-I+REC what it means that a given process computes a given function. This requires that assumptions about \( \mathcal{D} \) have to be made. In this section, such assumptions are given. They are based on the idea that the data environment of a computational process consists of one or more RAM (Random Access Machine) memories. Because the assumptions amount to conditions to be satisfied by \( \mathcal{D} \), they are called the RAM conditions on \( \mathcal{D} \). In this section, it is moreover made precise what it means, in the setting of ACP\(\tau\)-I+REC where \( \mathcal{D} \) satisfies the RAM conditions, that a given process computes a given partial function from \( (\{0, 1\}^*)^n \) to \( \{0, 1\}^* \) \( (n \in \mathbb{N}) \).

3.1 The RAM Conditions

The memory of a RAM consists of a countably infinite number of registers which are numbered by natural numbers. Each register is capable of containing a bit string of arbitrary length. The contents of the registers constitute the state of the memory of the RAM. The execution of an instruction by the RAM amounts to carrying out an operation on its memory state that changes the content of at most one register or to testing a property of its memory state. The RAM conditions are presented in this section using the notions of a RAM memory state, a RAM operation, and a RAM property.

A RAM memory state is a function \( \sigma : \mathbb{N} \rightarrow \{0, 1\}^* \) that satisfies the condition that there exists an \( i \in \mathbb{N} \) such that, for all \( j \in \mathbb{N} \), \( \sigma(i + j) = \lambda^5 \). We write \( \Sigma_{\text{ram}} \) for the set of all RAM memory states.

Let \( \sigma \) be a RAM memory state. Then, for all \( i \in \mathbb{N} \), \( \sigma(i) \) is the content of the register with number \( i \) in memory state \( \sigma \). The condition on \( \sigma \) expresses that the part of the memory that is actually in use remains finite.

The input region and output region of a function \( o : \Sigma_{\text{ram}} \rightarrow \Sigma_{\text{ram}} \), written \( IR(o) \) and \( OR(o) \), respectively, are the subsets of \( \mathbb{N} \) defined as follows:

\[ 5 \) We write \( \lambda \) for the empty bit string.\]
The following theorem states that each basic RAM operation transforms states of a RAM memory that coincide on its input region to states that coincide on its output region.

**Theorem 5.** Let $\sigma_1, \sigma_2 \in \Sigma_{\text{ram}}$ and $o \in O_{\text{ram}}$. Then $\sigma_1 \models IR(o) = \sigma_2 \models IR(o)$ implies $o(\sigma_1) \models OR(o) = o(\sigma_2) \models OR(o)$.

*Proof.* It is easy to see that the 4-tuple $(\mathbb{N}, \{0,1\}^*, \Sigma_{\text{ram}}, O_{\text{ram}})$ is a computer according to Definition 3.1 from [23]. From this and Theorem 3.1 from [23], the theorem follows immediately. \ 

The input region of a function $p : \Sigma_{\text{ram}} \to \{0,1\}$, written $IR(p)$ is the subset of $\mathbb{N}$ defined as follows:

$$IR(p) = \{ i \in \mathbb{N} \mid \exists \sigma_1, \sigma_2 \in \Sigma_{\text{ram}} \cdot (\forall j \in \mathbb{N} \setminus \{i\}) \cdot \sigma_1(j) = \sigma_2(j) \land p(\sigma_1) \neq p(\sigma_2)\}.$$ 

Let $p : \Sigma_{\text{ram}} \to \{0,1\}$. Then $IR(p)$ consists of the numbers of all registers that can affect what the value of $p$ is.

A basic RAM property is a function $p : \Sigma_{\text{ram}} \to \{0,1\}$ that satisfies the condition that $IR(p)$ is finite. We write $P_{\text{ram}}$ for the set of all basic RAM properties.

Let $p$ be a basic RAM property and $\sigma$ be a RAM memory state. Then testing the property $p$ on a RAM memory in state $\sigma$ yields the value $p(\sigma)$ and does not change the state of the RAM memory. The condition on $p$ expresses that only a finite number of registers can affect what this value is. We say that $p$ holds in $\sigma$ if $p(\sigma) = 1$.

The following theorem states that each basic RAM property holds in some state of a RAM memory if and only if it holds in all states of the RAM memory that coincide with that state on its input region.

---

*Footnote:* We use the notation $f \upharpoonright D$, where $f$ is a function and $D \subseteq \text{dom}(f)$, for the function $g$ with $\text{dom}(g) = D$ such that for all $d \in \text{dom}(g)$, $g(d) = f(d)$.
Theorem 6. Let $\sigma_1, \sigma_2 \in \Sigma_{\text{ram}}$ and $p \in P_{\text{ram}}$. Then $\sigma_1 \upharpoonright IR(p) = \sigma_2 \upharpoonright IR(p)$ implies $p(\sigma_1) = p(\sigma_2)$.

Proof. Let $\Sigma'_{\text{ram}}$ be the set of all functions $\sigma : \mathbb{N} \cup \{-1\} \rightarrow \{0, 1\}$, let $O'_{\text{ram}}$ be the set of all functions $o : \Sigma'_{\text{ram}} \rightarrow \Sigma'_{\text{ram}}$, let $\sigma_1', \sigma_2' \in \Sigma'_{\text{ram}}$ be such that $\sigma_1' \upharpoonright \mathbb{N} = \sigma_1$ and $\sigma_2' \upharpoonright \mathbb{N} = \sigma_2$, and let $o : \Sigma'_{\text{ram}} \rightarrow \Sigma'_{\text{ram}}$ be such that, for all $\sigma \in \Sigma'_{\text{ram}}$, $o(\sigma) \upharpoonright \mathbb{N} = \sigma \upharpoonright \mathbb{N}$ and $o(\sigma)(-1) = p(\sigma)$. Then $\sigma_1 \upharpoonright IR(p) = \sigma_2 \upharpoonright IR(p)$ implies $p(\sigma_1) = p(\sigma_2)$ iff $\sigma_1' \upharpoonright IR(o) = \sigma_2' \upharpoonright IR(o)$ implies $o(\sigma_1') = o(\sigma_2')$. Because of this and the fact that $(\mathbb{N} \cup \{-1\}, \{0, 1\}^*, \Sigma'_{\text{ram}}, O'_{\text{ram}})$ is also a computer according to Definition 3.1 from [23], this theorem now follows immediately from Theorem 3.1 from [23]. □

With basic RAM operations only computational processes can be considered whose data environment consists of one RAM memory. Below, $n$-RAM operations are introduced to remove this restriction. They are defined such that the basic RAM operations are exactly the 1-RAM operations.

An $n$-RAM operation $(n \in \mathbb{N}^+)$ is a function $o : \Sigma_{\text{ram}} \rightarrow \Sigma_{\text{ram}}$ that satisfies the condition that there exist a basic RAM operation $o'$ and a $k \in \mathbb{N}$ with $1 \leq k \leq n$ such that, for all $\sigma_1, \ldots, \sigma_n \in \Sigma_{\text{ram}}$, $o'((\beta(\sigma_1, \ldots, \sigma_n))) = \beta(\sigma_1, \ldots, \sigma_{k-1}, o(\sigma_1, \ldots, \sigma_n), \sigma_{k+1}, \ldots, \sigma_n)$, where $\beta : \Sigma_{\text{ram}}^n \rightarrow \Sigma_{\text{ram}}$ is the unique function such that $\beta(\sigma_1, \ldots, \sigma_n)(n \cdot i + k - 1) = \sigma_k(i)$ for all $i \in \mathbb{N}$ and $k \in \mathbb{N}$ with $1 \leq k \leq n$. We write $O_{n,\text{ram}}$, where $n \in \mathbb{N}^+$, for the set of all $n$-RAM operations.

The function $\gamma : \{k \in \mathbb{N}^+ | k \leq n\} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $\gamma(k, i) = n \cdot i + k - 1$ is a bijection. From this it follows that the basic RAM operation $o'$ and the $k \in \mathbb{N}$ referred to in the above definition are unique if they exist.

The operations from $\bigcup_{n \geq 1} O_{n,\text{ram}}$ will be referred to as RAM operations.

In a similar way as $n$-RAM operations, $n$-RAM properties are defined. The basic RAM properties are exactly the 1-RAM properties.

An $n$-RAM property $(n \in \mathbb{N}^+)$ is a function $p : \Sigma_{\text{ram}}^n \rightarrow \{0, 1\}$ that satisfies the condition that there exists a basic RAM property $p'$ such that, for all $\sigma_1, \ldots, \sigma_n \in \Sigma_{\text{ram}}$, $p'((\beta(\sigma_1, \ldots, \sigma_n)) = p(\sigma_1, \ldots, \sigma_n)$, where $\beta : \Sigma_{\text{ram}}^n \rightarrow \Sigma_{\text{ram}}$ is defined as above. We write $P_{n,\text{ram}}$, where $n \in \mathbb{N}^+$, for the set of all $n$-RAM properties.

The properties from $\bigcup_{n \geq 1} P_{n,\text{ram}}$ will be referred to as RAM properties.

The RAM conditions on $\mathfrak{D}$ are:

1. the signature $\Sigma_{\mathfrak{D}}$ of $\mathfrak{D}$ includes:
   - a sort $\mathbf{BS}$ of bit strings and a sort $\mathbf{N}$ of natural numbers;
   - constants $\lambda, 0, 1 : \mathbf{BS}$ and a binary operator $\odot : \mathbf{BS} \times \mathbf{BS} \rightarrow \mathbf{BS}$;
   - constants $0, 1 : \mathbf{N}$ and a binary operator $+: \mathbf{N} \times \mathbf{N} \rightarrow \mathbf{N}$;
   - a constant $\lambda_0 : \mathbf{D}$ and a ternary operator $\oplus : \mathbf{D} \times \mathbf{N} \times \mathbf{BS} \rightarrow \mathbf{D}$;
2. the sorts, constants, and operators mentioned under 1 are interpreted in $\mathfrak{D}$ as follows:
   - the sort $\mathbf{BS}$ is interpreted as the set $\{0, 1\}^*$, the sort $\mathbf{N}$ is interpreted as the set $\mathbb{N}$, and the sort $\mathbf{D}$ is interpreted as the set $\Sigma_{\text{ram}}$;
   - the constant $\lambda : \mathbf{BS}$ is interpreted as the empty bit string, the constants $0, 1 : \mathbf{BS}$ are interpreted as the bit strings with the bit 0 and 1, respectively, as sole element, and the operator $\odot : \mathbf{BS} \times \mathbf{BS} \rightarrow \mathbf{BS}$ is interpreted as the concatenation operation on $\{0, 1\}^*$;
– the constants 0, 1 : N are interpreted as the natural numbers 0 and 1, respectively, and the operator + : N × N → N is interpreted as the addition operation on N;
– the constant σλ : D is interpreted as the unique σ ∈ Σram such that σ(i) = λ for all i ∈ N and the operator ⊕ : D × N × BS → D is interpreted as the override operation defined by ⊕(σ, i, w)(i) = w and, for all j ∈ N with i ≠ j, ⊕(σ, i, w)(j) = σ(j);
3. the signature ΣD of D is restricted as follows:
   – for each operator from ΣD, the sort of its result is D only if the sort of each of its arguments is D or the operator is ⊕;
   – for each operator from ΣD, the sort of its result is B only if the sort of each of its arguments is D;
4. the interpretation of the operators mentioned under 3 is restricted as follows:
   – each operator with result sort D other than ⊕ is interpreted as a RAM operation;
   – each operator with result sort B is interpreted as a RAM property.

The notation σ{i → w}, where σ ∈ Σram, i ∈ N, and w ∈ {0, 1}*, is used for the term ⊕(σ, i, w).

The RAM conditions make it possible to explain what it means that a given process computes a given partial function from (0, 1)n to 0, 1 (n ∈ N).

Moreover, the RAM conditions are nonrestrictive: presumably they allow to deal with all proposed versions of the RAM model of computation as well as all proposed models of parallel computation that are based on a version of the RAM model and the idea that the data environment of a computational process consists of one or more RAM memories.

3.2 Computing Partial Functions from (∀, 1)n to 0, 1

In this section, we make precise in the setting of ACP∥-I+REC+CFAR, where D is assumed to satisfy the RAM conditions, what it means that a given process computes a given partial function from (∀, 1)n to 0, 1 (n ∈ N).

In the rest of this paper, D is assumed to satisfy the RAM conditions. Moreover, it is assumed that m ∈ \mathcal{V}.

Let t ∈ P_{rec}. Then t eventually halts if there exists a t' ∈ B in which δ does not occur such that ACP∥-I+REC+CFAR ⊢ t = t'. This means that t eventually halts if the process denoted by t always terminates successfully after performing a finite number of actions.

The depth of a term t ∈ P, written depth(t), is defined as follows: for all t ∈ P, depth(t) = min{n ∈ N | π_n(t) = t}. This means that depth(t) is the maximum number of actions other than τ that the process denoted by t can perform.

Let t ∈ P_{rec}, let n ∈ N, let F : (∀, 1)n → 0, 1 and let W : N → N. Then t computes F in W steps if:

7 We write f : A → B, where A and B are sets, to indicate that f is a partial function from A to B.
for all \(w_1, \ldots, w_n \in \{0, 1\}^*\) such that \(F(w_1, \ldots, w_n)\) is defined, for all \(\rho \in V \rightarrow \mathbb{D}\) with \(\rho(m) = \sigma \Lambda \{1 \mapsto w_1\} \ldots \{n \mapsto w_n\}\), there exists a \(\sigma \in \Sigma_{\text{ram}}\) such that:

\[
V_\rho(t) \text{ eventually halts},
\]

\[
\text{ACP}_I^\tau + \text{REC} + \text{CFAR} \vdash V_\rho(t) = V_\rho(t \cdot (m = \sigma \Lambda \{0 \mapsto F(w_1, \ldots, w_n)\} : \epsilon) ) ,
\]

\[
\text{depth}(V_\rho(t)) \leq W(\ell(w_1) + \ldots + \ell(w_n));^8
\]

for all \(w_1, \ldots, w_n \in \{0, 1\}^*\) such that \(F(w_1, \ldots, w_n)\) is undefined, for all \(\rho \in V \rightarrow \mathbb{D}\) with \(\rho(m) = \sigma \Lambda \{1 \mapsto w_1\} \ldots \{n \mapsto w_n\}\):

\[
V_\rho(t) \text{ does not eventually halt}.
\]

We say that \(t\) computes \(F\) if there exists a \(W : \mathbb{N} \rightarrow \mathbb{N}\) such that \(t\) computes \(F\) in \(W\) steps, we say that \(F\) is a computable function if there exists a \(t \in P\) such that \(t\) computes \(F\), and we say that \(t\) is a computational process if there exists a \(F : (\{0, 1\}^*)^n \rightarrow \{0, 1\}^*\) such that \(t\) computes \(F\).

We write \(CP_{\text{rec}}\) for the set \(\{t \in P_{\text{rec}} | t\) is a computational process\}\).

With the above definition, we can establish whether a process of the kind considered in the current setting computes a given partial function from \((\{0, 1\}^*)^n\) to \(\{0, 1\}^*\) \((n \in \mathbb{N})\) by equational reasoning using the axioms of \(\text{ACP}_I^\tau + \text{REC} + \text{CFAR}\). This setting is more general than the setting provided by any known version of the RAM model of computation. It is not suitable as a model of computation itself. However, various known models of computation can be defined by fixing which RAM operations and which RAM properties belong to \(\mathbb{D}\) and by restricting the computational processes to the ones of a certain form. To the best of my knowledge, the models of computation that can be dealt with in this way include all proposed versions of the RAM model as well as all proposed models of parallel computation that are based on a version of the RAM model and the idea that the data environment of a computational process consists of one or more RAM memories.

Whatever model of computation is obtained by fixing the RAM operations and the RAM properties and by restricting the computational processes to the ones of a certain form, it is an idealization of a real computer because it offers an unbounded number of registers that can contain a bit string of arbitrary length instead of a bounded number of registers that can only contain a bit string of a fixed length.

4 The RAMP Model of Computation

In this section, a version of the RAM model of computation is described in the setting introduced in the previous sections. Because it focuses on the processes

^8 We write \(\ell(u)\), where \(u\) is a sequence, for the length of \(u\).
that are produces by RAMs when they execute their built-in program, the version of the RAM model of computation described in this section is called the RAMP (Random Access Machine Process) model of computation.

First, the operators are introduced that represent the RAM operations and the RAM properties that belong to $D$ in the case of the RAMP model of computation. Next, the interpretation of those operators as a RAM operation or a RAM property is given. Finally, the RAMP model of computation is described.

4.1 Operators for the RAMP Model

In this section, the operators that are relevant to the RAMP model of computation are introduced.

In the case of the RAMP model of computation, the set of operators from $\Sigma_D$ that are interpreted in $D$ as RAM operations or RAM properties is the set $O_{RAMP}$ defined as follows:

$$O_{RAMP} = \{ \text{binop}: s_1:s_2:d \mid \text{binop} \in \text{Binop} \land s_1, s_2 \in \text{Src} \land d \in \text{Dst} \} \cup \{ \text{unop}: s_1:d \mid \text{unop} \in \text{Unop} \land s_1 \in \text{Src} \land d \in \text{Dst} \} \cup \{ \text{cmpop}: s_1:s_2 \mid \text{cmpop} \in \text{Cmpop} \land s_1, s_2 \in \text{Src} \},$$

where

- $\text{Binop} = \{ \text{add}, \text{sub}, \text{and}, \text{or} \}$,
- $\text{Unop} = \{ \text{not}, \text{shl}, \text{shr}, \text{mov} \}$,
- $\text{Cmpop} = \{ \text{eq}, \text{gt}, \text{beq} \}$

and

- $\text{Src} = \{ \# i \mid i \in \mathbb{N} \} \cup \mathbb{N} \cup \{ @i \mid i \in \mathbb{N} \}$,
- $\text{Dst} = \mathbb{N} \cup \{ @i \mid i \in \mathbb{N} \}$.

We write $C_{RAMP}$ for the set $\{ \text{cmpop}: s_1:s_2 \mid \text{cmpop} \in \text{Cmpop} \land s_1, s_2 \in \text{Src} \}$.

The following is a preliminary explanation of the operators from $O_{RAMP}$:

- carrying out the operation denoted by an operator of the form $\text{binop}: s_1:s_2:d$ on a RAM memory in some state boils down to carrying out the binary operation named $\text{binop}$ on the values that $s_1$ and $s_2$ stand for in that state and then changing the content of the register that $d$ stands for into the result of this;
- carrying out the operation denoted by an operator of the form $\text{unop}: s_1:d$ on a RAM memory in some state boils down to carrying out the unary operation named $\text{unop}$ on the value that $s$ stands for in that state and then changing the content of the register that $d$ stands for into the result of this;
– carrying out the operation denoted by an operator of the form \textit{cmpop}:s_1:s_2 on a RAM memory in some state boils down to carrying out the binary operation named \textit{cmpop} on the values that \(s_1\) and \(s_2\) stand for in that state.

The value that \(s_i\) \((i = 1, 2)\) stands for is as follows:

– \textit{immediate}: it stands for the shortest bit string representing the natural number \(i\) if it is of the form \#\(i\);
– \textit{direct addressing}: it stands for the content of the register with number \(i\) if it is of the form \(i\);
– \textit{indirect addressing}: it stands for the content of the register whose number is represented by the content of the register with number \(i\) if it is of the form \(@i\);

and the register that \(d\) stands for is as follows:

– \textit{direct addressing}: it stands for the register with number \(i\) if it is of the form \(i\);
– \textit{indirect addressing}: it stands for the register whose number is represented by the content of the register with number \(i\).

The following kinds of operations and relations on bit strings are covered by the operators from \(O_{\text{RAMP}}\): arithmetic operations (\texttt{add, sub}), logical operations (\texttt{and, or, not}), bit-shift operations (\texttt{shl, shr}), data-transfer operations (\texttt{mov}), arithmetic relations (\texttt{eq, gt}), and the bit-wise equality relation (\texttt{beq}). The arithmetic operations on bit strings are operations that model arithmetic operations on natural numbers with respect to their binary representation by bit strings, the logical operations on bit strings are bitwise logical operations, and the data transfer operation on bit strings is the identity operation on bit strings (which is carried out when copying bit strings). The arithmetic relations on bit strings are relations that model arithmetic relations on natural numbers with respect to their binary representation by bit strings.

### 4.2 Interpretation of the Operators for the RAMP Model

In this section, the interpretation of the operators from \(O_{\text{RAMP}}\) in \(\mathcal{D}\) is defined.

We start with defining auxiliary functions for conversion between natural numbers and bit strings and evaluation of the elements of \(\text{Src}\) and \(\text{Dst}\).

We write \(\cdot\) for proper subtraction of natural numbers. We write \(\div\) for zero-totalized Euclidean division of natural numbers, i.e. Euclidean division made total by imposing that division by zero yields zero (like in meadows, see e.g. [7, 5]). We use juxtaposition for concatenation of bit strings.

The \textit{natural to bit string} function \(b : \mathbb{N} \rightarrow \{0, 1\}^*\) is recursively defined as follows:

\[ b(n) = n \text{ if } n \leq 1 \quad \text{and} \quad b(n) = (n \mod 2)b(n \div 2) \text{ if } n > 1 \]

and the \textit{bit string to natural} function \(n : \{0, 1\}^* \rightarrow \mathbb{N}\) is recursively defined as follows:
These definitions tell us that, when viewed as the binary representation of a natural number, the first bit of a bit string is considered the least significant bit. Results of applying $b$ have no leading zeros, but the operand of $n$ may have leading zeros. Thus, we have that $n(b(n)) = n$ and $b(n(w)) = w'$, where $w'$ is $w$ without leading zeros.

For each $\sigma \in \Sigma_{ram}$, the src-valuation in $\sigma$ function $v_\sigma : Src \to \{0,1\}^*$ is defined as follows:

$$v_\sigma(#i) = b(i), \quad v_\sigma(i) = \sigma(i), \quad \text{and} \quad v_\sigma(@i) = \sigma(n(\sigma(i)))$$

and, for each $\sigma \in \Sigma_{ram}$, the dst-valuation in $\sigma$ function $r_\sigma : Dst \to \mathbb{N}$ is defined as follows:

$$r_\sigma(i) = i \quad \text{and} \quad r_\sigma(@i) = n(\sigma(i)).$$

We continue with defining the operations on bit strings that the operation names from $Binop \cup Unop$ refer to.

We define the operations on bit strings that the operation names $\text{add}$ and $\text{sub}$ refer to as follows:

$$+: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*: \quad w_1 + w_2 = b(n(w_1) + n(w_2));$$

$$\div: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*: \quad w_1 \div w_2 = b(n(w_1) \div n(w_2)).$$

These definitions tell us that, although the operands of the operations $+$ and $\div$ may have leading zeros, results of applying these operations have no leading zeros.

We define the operations on bit strings that the operation names $\text{and}$, $\text{or}$, and $\text{not}$ refer to recursively as follows:

$$\land: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*: \quad \lambda \land \lambda = \lambda, \quad \lambda \land (bw) = 0(\lambda \land w),$$

$$(bw) \land \lambda = 0(w \land \lambda), \quad (b_1 w_1) \land (b_2 w_2) = (b_1 \land b_2) (w_1 \land w_2);$$

$$\lor: \{0,1\}^* \times \{0,1\}^* \to \{0,1\}^*: \quad \lambda \lor \lambda = \lambda, \quad \lambda \lor (bw) = b(\lambda \lor w),$$

$$(bw) \lor \lambda = b(w \lor \lambda), \quad (b_1 w_1) \lor (b_2 w_2) = (b_1 \lor b_2) (w_1 \lor w_2);$$

$$\neg: \{0,1\}^* \to \{0,1\}^*: \quad \neg \lambda = \lambda, \quad \neg (bw) = (-b)(\neg w).$$

These definitions tell us that, if the operands of the operations $\land$ and $\lor$ do not have the same length, sufficient leading zeros are assumed to exist. Moreover, results of applying these operations and results of applying $\neg$ can have leading zeros.

We define the operations on bit strings that the operation names $\text{shl}$ and $\text{shr}$ refer to as follows:

$$\ll: \{0,1\}^* \to \{0,1\}^*: \quad \ll \lambda = \lambda, \quad \ll (bw) = 0bw;$$

$$\gg: \{0,1\}^* \to \{0,1\}^*: \quad \gg \lambda = \lambda, \quad \gg (bw) = w.$$
These definitions tell us that results of applying the operations $\ll$ and $\gg$ can have leading zeros. We have that $n(\ll w) = n(w) \cdot 2$ and $n(\gg w) = n(w) \div 2$.

Now, we are ready to define the interpretation of the operators from $O_{RAMP}$ in $\mathcal{D}$. For each $o \in O_{RAMP}$, the interpretation of $o$ in $\mathcal{D}$, written $[o]$, is defined as follows:

\[
\begin{align*}
[\text{add}:s_1:s_2:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto v_\sigma(s_1) + v_\sigma(s_2) \}; \\
[\text{sub}:s_1:s_2:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto v_\sigma(s_1) - v_\sigma(s_2) \}; \\
[\text{and}:s_1:s_2:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto v_\sigma(s_1) \land v_\sigma(s_2) \}; \\
[\text{or}:s_1:s_2:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto v_\sigma(s_1) \lor v_\sigma(s_2) \}; \\
[\text{not}:s_1:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto \neg v_\sigma(s_1) \}; \\
[\text{shl}:s_1:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto \ll v_\sigma(s_1) \}; \\
[\text{shr}:s_1:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto \gg v_\sigma(s_1) \}; \\
[\text{mov}:s_1:d](\sigma) &= \sigma \{ r_\sigma(d) \mapsto v_\sigma(s_1) \}; \\
[\text{eq}:s_1:s_2](\sigma) &= 1 \text{ iff } n(v_\sigma(s_1)) = n(v_\sigma(s_2)); \\
[\text{gt}:s_1:s_2](\sigma) &= 1 \text{ iff } n(v_\sigma(s_1)) > n(v_\sigma(s_2)); \\
[\text{beq}:s_1:s_2](\sigma) &= 1 \text{ iff } v_\sigma(s_1) = v_\sigma(s_2). 
\end{align*}
\]

Clearly, the interpretation of each operator from $O_{RAMP} \setminus C_{RAMP}$ is a basic RAM operation and the interpretation of each operator from $C_{RAMP}$ is a basic RAM property.

### 4.3 RAMP Terms and the RAMP Model of Computation

In this section, the RAMP model of computation is characterized in the setting introduced in Sections 2 and 3. However, first the notion of a RAMP term is defined. This notion is introduced to make precise what the set of possible computational processes is in the case of the RAMP model of computation.

In this section, $\mathcal{D}$ is fixed as follows:

- $\Sigma_\mathcal{D}$ is the smallest signature including (a) all sorts, constants, and operators required by the assumptions made about $\mathcal{D}$ in $ACP^*_I$ or the RAM conditions on $\mathcal{D}$ and (b) all operators from $O_{RAMP}$;
- all sorts, constants, and operators mentioned under (a) are interpreted in $\mathcal{D}$ as required by the assumptions made about $\mathcal{D}$ in $ACP^*_I$ or the RAM conditions on $\mathcal{D}$;
- all operators from $O_{RAMP}$ are interpreted in $\mathcal{D}$ as defined at the end of Section 4.2.

Moreover, it is assumed that $m \in \mathcal{V}$.
A RAM process term, called a RAMP term for short, is a term from $P_{\text{rec}}$ that is of the form $\langle X | E \rangle$, where, for each $Y \in \text{vars}(E)$, the recursion equation for $Y$ in $E$ has one of the following forms:

$$Y = t : \rightarrow [m := o(m)] \cdot Z,$$

$$Y = (p(m) = 1) : \rightarrow [m := m] \cdot Z + (p(m) = 0) : \rightarrow [m := m] \cdot Z',$$

$$Y = t : \rightarrow \epsilon,$$

where $o \in O_{\text{RAMP}} \setminus C_{\text{RAMP}}$, $p \in C_{\text{RAMP}}$, and $Z, Z' \in \text{vars}(E)$. We write $P_{\text{RAMP}}$ for the set of all RAMP terms, and we write $CP_{\text{RAMP}}$ for $P_{\text{RAMP}} \cap CP_{\text{rec}}$.

A process that can be denoted by a RAMP term is called a RAMP process or a RAMP for short. So, a RAMP is a process that is definable by a guarded linear recursive specification over $ACP_{\tau \epsilon}$-I of the kind described above.

As mentioned in Section I, a basic assumption in this paper is that a model of computation is fully characterized by: (a) a set of possible computational processes, (b) for each possible computational process, a set of possible data environments, and (c) the effect of applying such processes to such environments.

$O$ as fixed above and $CP_{\text{RAMP}}$ induce the RAMP model of computation:

- the set of possible computational processes is the set of all processes that can be denoted by a term from $CP_{\text{RAMP}}$;
- for each possible computational process, the set of possible data environments is the set of all $\{m\}$-indexed data environments;
- the effect of applying the process denoted by a $t \in CP_{\text{RAMP}}$ to a $\{m\}$-indexed data environment $\mu$ is $V_{\mu}(t)$, where $\rho$ is a flexible variable valuation that represents $\mu$.

The RAMP model of computation described above is intended to be essentially the same as the standard RAM model of computation extended with logical instructions and bit-shift instructions. The RAMs from that model will be referred to as the BBRAMs (Basic Binary RAMs). There is a strong resemblance between $O_{\text{RAMP}}$ and the set $I_{\text{BBRAM}}$ of instructions from which the built-in programs of the BBRAMs can be constructed. Because the concrete syntax of the instructions does not matter, $I_{\text{BBRAM}}$ can be defined as follows:

$$I_{\text{BBRAM}} = (O_{\text{RAMP}} \setminus C_{\text{RAMP}}) \cup \{\text{jmp}:p:i \mid p \in C_{\text{RAMP}} \land i \in \mathbb{N}^+\} \cup \{\text{halt}\}.$$

A BBRAM program is a non-empty sequence $C$ from $I_{\text{BBRAM}}^*$ in which instructions of the form $\text{jmp}:p:i$ with $i > \ell(C)$ do not occur. We write $IS_{\text{BBRAM}}$ for the set of all BBRAM programs.

The execution of an instruction $o$ from $O_{\text{RAMP}} \setminus C_{\text{RAMP}}$ by a BBRAM causes the state of its memory to change according to $[o]$. The execution of an instruction of the form $\text{jmp}:p:i$ or the instruction $\text{halt}$ by a BBRAM has no effect on the state of its memory. After execution of an instruction by a BBRAM, the BBRAM proceeds to the execution of the next instruction from its built-in program except when the instruction is of the form $\text{jmp}:p:i$ and $[p] = 1$ or when
the instruction is \textit{halt}. In the case that the instruction is of the form \texttt{jmp:p:i} and 

\[[p] = 1\), the execution proceeds to the \textit{i}th instruction of the program. In the 

case that the instruction is \textit{halt}, the execution terminates successfully.

The processes that are produced by the BBRAMs when they execute their built-in program are given by a function \(M : IS_{BBRAM} \rightarrow P_{RAMP}\) that is defined up to consistent renaming of variables as follows: \(M(c_1 \ldots c_n) = \langle X_1|E \rangle\), where 

\(E\) consists of, for each \(i \in \mathbb{N}\) with \(1 \leq i \leq n\), an equation \(X_i = t : \rightarrow [m := c_i(m)] \cdot X_{i+1}\) if \(c_i \in O_{RAMP} \setminus C_{RAMP}\), \(X_i = (p(m) = 1) : \rightarrow [m := m] \cdot X_j + (p(m) = 0) : \rightarrow [m := m] \cdot X_{i+1}\) if \(c_i \equiv \texttt{jmp:p:j}\), \(X_i = t : \rightarrow \epsilon\) if \(c_i \equiv \textit{halt}\),

where \(X_1, \ldots, X_n\) are different variable from \(X\).

Let \(C \in IS_{BBRAM}\). Then \(M(C)\) denotes the process that is produced by the 

BBRAM whose built-in program is \(C\) when it executes its built-in program.

The definition of \(M\) is in accordance with the descriptions of various ver-

sions of the RAM model of computation in the literature on this sub-
ject (see e.g. [9,17,1]). However, to the best of my knowledge, none of these descriptions 

is precise and complete enough to allow of a proof of this.

The RAMPs are exactly the processes that can be produced by the BBRAMs 

when they execute their built-in program.

\textbf{Theorem 7.} \textit{For each constant} \(\langle X|E \rangle \in P_{rec}\), \(\langle X|E \rangle \in P_{RAMP}\) \textit{iff there exists} 

\(a C \in IS_{BBRAM}\) \textit{such that} \(\langle X|E \rangle\) \textit{and} \(M(C)\) \textit{are identical up to consistent re-

naming of variables.}

\textit{Proof.} It is easy to see that (a) for all \(C \in IS_{BBRAM}\), \(M(C) \in P_{RAMP}\) and 

(b) \(M\) is an bijection up to consistent renaming of variables. From this, the 

theorem follows immediately. \(\square\)

Notice that, if \(\langle X|E \rangle\) and \(\langle X'|E' \rangle\) are identical up to consistent renaming of 

variables, then the equation \(\langle X|E \rangle = \langle X'|E' \rangle\) is derivable from RDP and RSP 

(and \(\langle X|E \rangle \leftrightarrow_{rb} \langle X'|E' \rangle\)).

The following theorem is a result concerning the computational power of 

RAMPs.

\textbf{Theorem 8.} \textit{For each} \(F : \{0,1\}^* \rightarrow \{0,1\}^*\), \textit{there exists a} \(t \in P_{RAMP}\) \textit{such that} \(t\) \textit{computes} \(F\) \textit{iff} \(F\) \textit{is Turing-computable.}

\textit{Proof.} By Theorem 7 it is sufficient to show that each BBRAM is Turing equiv-

alent to a Turing machine. The BBRAM model of computation is essentially the 

same as the BRAM model of computation from [10] extended with bit-shift in-

structions. It follows directly from simulation results mentioned in [10] (part (3) 

of Theorem 2.4, part (3) of Theorem 2.5, and part (2) of Theorem 2.6) that 

each BRAM can be simulated by a Turing machine and vice versa. Because each
Turing machine can be simulated by a BRAM, we immediately have that each Turing machine can be simulated by a BBRAM. It is easy to see that the bit-shift instructions can be simulated by a Turing machine. From this and the fact that each BRAM can be simulated by a Turing machine, it follows that each BBRAM can be simulated by a Turing machine as well. Hence, each BBRAM is Turing equivalent to a Turing machine.

Henceforth, we write $\text{POLY}$ for $\{f | f : \mathbb{N} \rightarrow \mathbb{N} \wedge f \text{ is a polynomial function}\}$.

The following theorem tells us that the decision problems that belong to $\textbf{P}$ are exactly the decision problems that can be solved by means of a RAMP in polynomially many steps.

**Theorem 9.** For each $F : \{0,1\}^* \rightarrow \{0,1\}$, there exist a $t \in \mathcal{P}_{\text{RAMP}}$ and a $W \in \text{POLY}$ such that $t$ computes $F$ in $W$ steps iff $F \in \textbf{P}$.

**Proof.** By Theorem 7, it is sufficient to show that time complexity on BBRAMs under the uniform time measure, i.e. the number of steps, and time complexity on multi-tape Turing machines are polynomially related. The BBRAM is essentially the same as the BRAM model of computation from [10] extended with bit-shift instructions. It follows directly from simulation results mentioned in [10] (part (3) of Theorem 2.4, part (3) of Theorem 2.5, and part (2) of Theorem 2.6) that time complexity on BRAMs under the uniform time measure and time complexity on multi-tape Turing machines are polynomially related. It is easy to see that the bit-shift instructions can be simulated by a multi-tape Turing machine in linear time. Hence, the time complexities remain polynomially related if the BRAM model is extended with the bit-shift instructions.

5 The APRAMP Model of Computation

In this section, an asynchronous parallel RAM model of computation is described in the setting introduced in Sections 2 and 3. Because it focuses on the processes that are produced by asynchronous parallel RAMs when they execute their built-in programs, the parallel RAM model of computation described in this section is called the APRAMP (Asynchronous Parallel Random Access Machine Process) model of computation. In this model of computation, a computational process is the parallel composition of a number of processes that each has its own private RAM memory. However, together they also have a shared RAM memory for synchronization and communication.

First, the operators are introduced that represent the RAM operations and the RAM properties that belong to $\mathcal{D}$ in the case of the APRAMP model of computation. Next, the interpretation of those operators as a RAM operation or a RAM property is given. Finally, the APRAMP model of computation is described.

In the case of the APRAMP model of computation, the set of operators from $\Sigma_{\mathcal{D}}$ that are interpreted in $\mathcal{D}$ as RAM operations or RAM properties is the set
\( \mathcal{O}_\text{PRAMP} \) defined as follows:

\[
\mathcal{O}_\text{PRAMP} = \mathcal{O}_\text{RAMP} \cup \{ \text{ini} : \#i \mid i \in \mathbb{N}^+ \} \\
\quad \cup \{ \text{loa} : @i : d \mid i \in \mathbb{N} \land d \in \text{Dst} \} \cup \{ \text{sto} : s : @i \mid s \in \text{Src} \land i \in \mathbb{N} \},
\]

where \( \text{Src} \) and \( \text{Dst} \) are as defined in Section 4.1.

In operators of the forms \( \text{binop} : s_1 : s_2 : d, \text{unop} : s_1 : d, \) and \( \text{cmpop} : s_1 : s_2 : d \) from \( \mathcal{O}_\text{RAMP} \), \( s_1, s_2, \) and \( d \) refer to the private RAM memory. In operators of the form \( \text{loa} : @i : d \) and \( \text{sto} : s : @i \) from \( \mathcal{O}_\text{PRAMP} \setminus \mathcal{O}_\text{RAMP} \), \( s \) and \( d \) refer to the private RAM memory too. The operators of the form \( \text{loa} : @i : d \) and \( \text{sto} : s : @i \) differ from the operators of the form \( \text{mov} : @i : d \) and \( \text{mov} : s : @i \), respectively, in that \( @i \) stands for the content of the register and the register, respectively, from the shared RAM memory whose number is represented by the content of the register with number \( i \) from the private memory. The operators of the form \( \text{ini} : \#i \) initialize the registers from the private memory as follows: the content of the register with number 0 becomes the shortest bit string that represents the natural number \( i \) and the content of all other registers becomes the empty bit string.

Now, we are ready to define the interpretation of the operators from \( \mathcal{O}_\text{PRAMP} \) in \( \mathcal{D} \). For each \( o \in \mathcal{O}_\text{PRAMP} \), the interpretation of \( o \) in \( \mathcal{D} \), written \([ [ o ] ] \), is as defined in Section 4.2 for operators from \( \mathcal{O}_\text{RAMP} \) and as defined below for the additional operators:

\[
[[\text{ini} : \#i]\sigma_p] = \sigma_\land\{ 0 \mapsto b(i) \};
[[\text{loa} : @i : d]\sigma_p, \sigma_s] = \sigma_p\{ r_{\sigma_p}(d) \mapsto \sigma_s(\text{n}(\sigma_p(i))) \};
[[\text{sto} : s : @i]\sigma_p, \sigma_s] = \sigma_s\{ \text{n}(\sigma_p(i)) \mapsto \nu_{\sigma_p}(s) \}.
\]

Clearly, the interpretation of each operator of the form \( \text{ini} : \#i \) is a 1-RAM operation and the interpretation of each operator of the form \( \text{loa} : @i : d \) or \( \text{sto} : s : @i \) is a 2-RAM operation.

Below, the APRAMP model of computation is characterized in the setting introduced in Sections 2 and 3. However, first the notion of an APRAMP term is defined. This notion is introduced to make precise what the set of possible computational processes is in the case of the APRAMP model of computation.

In this section, \( \mathcal{D} \) is fixed as follows:

- \( \Sigma_{\mathcal{D}} \) is the smallest signature including (a) all sorts, constants, and operators required by the assumptions made about \( \mathcal{D} \) in ACP\(^\tau\)-I or the RAM conditions on \( \mathcal{D} \) and (b) all operators from \( \mathcal{O}_\text{PRAMP} \);
- all sorts, constants, and operators mentioned under (a) are interpreted in \( \mathcal{D} \) as required by the assumptions made about \( \mathcal{D} \) in ACP\(^\tau\)-I or the RAM conditions on \( \mathcal{D} \);
- all operators from \( \mathcal{O}_\text{PRAMP} \) are interpreted in \( \mathcal{D} \) as defined above.

Moreover, it is assumed that \( m \in \mathcal{V} \) and, for all \( i \in \mathbb{N}^+ \), \( m_i \in \mathcal{V} \). We write \( \mathcal{V}^m_n \), where \( n \in \mathbb{N}^+ \), for the set \( \{ m \} \cup \{ m_i \mid i \in \mathbb{N}^+ \land i \leq n \} \).
An \( n \)-APRAM process term \( (n \in \mathbb{N}^+) \), called an \( n \)-APRAMP term for short, is a term from \( \mathcal{P}_{\text{rec}} \) that is of the form \( \langle X_1|E_1 \rangle \parallel \ldots \parallel \langle X_n|E_n \rangle \), where, for each \( i \in \mathbb{N}^+ \) with \( i \leq n \):

- for each \( X \in \text{vars}(E_i) \), the recursion equation for \( X \) in \( E_i \) has one of the following forms:

  1. \( X = t \implies [m_i := \text{ini} : \#i(m_i)] \cdot Y \),
  2. \( X = t \implies [m_i := \text{loa} : \alpha : \delta : d : (m_i, m)] \cdot Y \),
  3. \( X = t \implies [m := \text{sto} : \delta : s : \alpha : i : (m_i, m)] \cdot Y \),
  4. \( X = t \implies [m_i := \text{o} : (m_i)] \cdot Y \),
  5. \( X = (p(m_i) = 1) \implies [m_i := m_i] \cdot Y + (p(m_i) = 0) \implies [m_i := m_i] \cdot Y' \),
  6. \( X = t \implies \epsilon \),

where \( o \in \mathcal{O}_{\text{RAMP}} \setminus \mathcal{C}_{\text{RAMP}} \), \( p \in \mathcal{C}_{\text{RAMP}} \), and \( Y, Y' \in \text{vars}(E_i) \);

- for each \( X \in \text{vars}(E_i) \), the recursion equation for \( X \) in \( E_i \) is of the form (1) iff \( X \equiv X_i \).

We write \( \mathcal{P}_{\text{APRAMP}} \) for the set of all terms \( t \in \mathcal{P}_{\text{rec}} \) such that \( t \) is an \( n \)-APRAMP terms for some \( n \in \mathbb{N}^+ \), and we write \( CP_{\text{APRAMP}} \) for \( \mathcal{P}_{\text{APRAMP}} \cap \mathcal{C}_{\text{rec}} \). Moreover, we write \( \text{deg}(t) \), where \( t \in \mathcal{P}_{\text{APRAMP}} \), for the unique \( n \in \mathbb{N}^+ \) such that \( t \) is an \( n \)-APRAMP term.

The terms from \( \mathcal{P}_{\text{APRAMP}} \) will be referred to as APRAMP terms.

A process that can be denoted by an APRAMP term is called an APRAM process or an APRAMP for short. So, an APRAMP is a parallel composition of processes that are definable by a guarded linear recursive specification over \( \text{ACP}_{\tau}^\text{I} \) of the kind described above. Each of those parallel processes starts with an initialization step in which the number of its private memory is made available in the register with number 0 from its private memory.

Notice that by Lemma 1 and Theorem 2, for all \( t \in \mathcal{P}_{\text{APRAMP}} \), there exists a guarded linear recursive specification \( E \) and \( X \in \text{vars}(E) \) such that \( t \xrightarrow{\text{rb}} \langle X|E \rangle \).

As mentioned before, a basic assumption in this paper is that a model of computation is fully characterized by: (a) a set of possible computational processes, (b) for each possible computational process, a set of possible data environments, and (c) the effect of applying such processes to such environments.

\( \mathcal{D} \) as fixed above and \( CP_{\text{APRAMP}} \) induce the APRAMP model of computation:

- the set of possible computational processes is the set of all processes that can be denoted by a term from \( CP_{\text{APRAMP}} \);
- for each possible computational process \( p \), the set of possible data environments is the set of all \( V_{\text{deg}(t)}^m \)-indexed data environments, where \( t \) is a term from \( CP_{\text{APRAMP}} \) denoting \( p \);
- the effect of applying the process denoted by a \( t \in CP_{\text{APRAMP}} \) to a \( V_{\text{deg}(t)}^m \)-indexed data environment \( \mu \) is \( V_\rho(t) \), where \( \rho \) is a flexible variable valuation that represents \( \mu \).
The APRAMP model of computation described above is intended to be close to the asynchronous parallel RAM model of computation sketched in [SP2020]9. However, the time complexity measure for this model introduced in Section 7 is quite different from the ones proposed in those papers.

The APRAMPs can be looked upon as the processes that can be produced by a collection of BBRAMs with an extended instruction set when they execute their built-in program asynchronously in parallel.

The BBRAMs with the extended instruction set will be referred to as the SMBRAMs (Shared Memory Binary RAMs). There is a strong resemblance between $\mathcal{O}_\text{PRAMP}$ and the set $\mathcal{I}_\text{SMBRAM}$ of instructions from which the built-in programs of the SMBRAMs can be constructed. Because the concrete syntax of the instructions does not matter, $\mathcal{I}_\text{SMBRAM}$ can be defined as follows:

$$\mathcal{I}_\text{SMBRAM} = (\mathcal{O}_\text{PRAMP} \setminus \mathcal{C}_\text{RAMP}) \cup \{\text{jmp}:p:i \mid p \in \mathcal{C}_\text{RAMP} \land i \in \mathbb{N}^+\} \cup \{\text{halt}\}.$$  

An SMBRAM program is a non-empty sequence $C$ from $\mathcal{I}_\text{SMBRAM}^*$ in which instructions of the form $\text{jmp}:p:i$ with $i > \ell(C)$ do not occur. We write $\mathcal{I}_\text{SMBRAM}$ for the set of all SMBRAM programs.

For the SMBRAMs whose private memory has number $i$ ($i \in \mathbb{N}^+$), the processes that are produced when they execute their built-in program are given by a function $\mathcal{M}_i : \mathcal{I}_\text{SMBRAM} \to \mathcal{P}_\text{APRAMP}$ that is defined up to consistent renaming of variables as follows: $\mathcal{M}_i(c_1 \ldots c_n) = \langle X_i | E_i \rangle$, where $E_i$ consists of the equation

$$X_i = t : \rightarrow [m_i := \text{ini} : #i(m_i)] \cdot Y_1$$

and, for each $j \in \mathbb{N}$ with $1 \leq j \leq n$, an equation

$$Y_j = t : \rightarrow [m_i := c_j(m_i, m)] \cdot Y_{j+1} \quad \text{if } c_j \in \text{Load},$$

$$Y_j = t : \rightarrow [m_i := c_j(m_i, m)] \cdot Y_{j+1} \quad \text{if } c_j \in \text{Store},$$

$$Y_j = t : \rightarrow [m_i := c_j(m_i)] \cdot Y_{j+1} \quad \text{if } c_j \in \mathcal{O}_\text{RAMP} \setminus \mathcal{C}_\text{RAMP},$$

$$Y_j = (p(m_i) = 1) : \rightarrow [m_i := m_i] \cdot Y_j + (p(m_i) = 0) : \rightarrow [m_i := m_i] \cdot Y_{j+1} \quad \text{if } c_j \equiv \text{jmp}:p:j',$$

$$Y_j = t : \rightarrow \epsilon \quad \text{if } c_j \equiv \text{halt},$$

where $\text{Load} = \{\text{loa}:@i:d \mid i \in \mathbb{N} \land d \in \text{Dst}\}$, $\text{Store} = \{\text{sto}:s:@i \mid s \in \text{Src} \land i \in \mathbb{N}\}$, and $Y_1, \ldots, Y_n$ are different variables from $\mathcal{X} \setminus \{X_i\}$.

The APRAMPs are exactly the processes that can be produced by a collection of SMBRAMs when they execute their built-in program asynchronously in parallel.

9 The APRAMP model is considered less close to the asynchronous parallel RAM model sketched in [12] because the latter provides special instructions for synchronization.
Theorem 10. Let $n \in \mathbb{N}^+$. For all constants $\langle X_1|E_1 \rangle, \ldots, \langle X_n|E_n \rangle \in \mathcal{P}_{\text{rec}}$, $\langle X_1|E_1 \rangle \| \ldots \| \langle X_n|E_n \rangle \in \mathcal{P}_{\text{APRAMP}}$ iff there exist $C_1, \ldots, C_n \in \mathcal{IS}_{\text{SMBRAM}}$ such that $\langle X_1|E_1 \rangle \| \ldots \| \langle X_n|E_n \rangle$ and $M_1(C_1) \| \ldots \| M_n(C_n)$ are identical up to consistent renaming of variables.

Proof. Let $i \in \mathbb{N}^+$ be such that $i \leq n$. It is easy to see that (a) for all $C \in \mathcal{IS}_{\text{SMBRAM}}$, $M_i(C) \in \mathcal{P}_{\text{APRAMP}}$ and (b) $M_i$ is an bijection up to consistent renaming of variables. From this, it follows immediately that there exists a $C \in \mathcal{IS}_{\text{SMBRAM}}$ such that $\langle X_i|E_i \rangle$ and $M_i(C)$ are identical up to consistent renaming of variables. From this, the theorem follows immediately. \hfill \Box

6 The SPRAMP Model of Computation

In the asynchronous parallel RAM model of computation presented in Section 5, the parallel processes that make up a computational process do not automatically synchronize after each computational step. In this section, we describe a parallel RAM model of computation where the parallel processes that make up a computational process automatically synchronize after each computational step.

For that purpose, a special instance of the synchronization merge operator of CSP [18] is defined in term of the operators of ACP$^\tau_{\text{I+RN}}$. It is assumed that $\text{sync}$, $\text{sync} \in A$ and $\gamma$ is such that $\gamma(\text{sync}, \text{sync}) = \text{sync}$, $\gamma(\text{sync}, a) = \delta$ for all $a \in A \setminus \{\text{sync}\}$, and $\gamma(\text{sync}, a) = \delta$ for all $a \in A$. The special instance of the synchronization merge operator, $\|_{\text{sync}}$, is defined as follows:

$$t \|_{\text{sync}} t' = \rho_f(\partial(\text{sync}) \rho_f(t) \| \rho_f(t')),$$

where the renaming function $f$ is defined by $f(\text{sync}) = \text{sync}$ and $f(\alpha) = \alpha$ if $\alpha \in A \setminus \{\text{sync}\}$.

The parallel RAM model of computation described in this section is called the SPRAMP (Synchronous Parallel Random Access Machine Process) model of computation.

The operators that represent the RAM operations and the RAM properties that belong to $\mathcal{D}$ in the case of the SPRAMP model of computation are the same as in the case of the APRAMP model of computation. The interpretation of those operators as a RAM operation or a RAM property is also the same as in the case of the APRAMP model of computation. Moreover, $\mathcal{D}$ is fixed as in Section 5.

Below, the SPRAMP model of computation is characterized. However, first the notion of an $n$-SPRAMP term is defined.

Like in Section 5, it is assumed that $m \in \mathcal{V}$ and, for all $i \in \mathbb{N}^+$, $m_i \in \mathcal{V}$. Again, we write $\mathcal{V}^m_n$, where $n \in \mathbb{N}^+$, for the set $\{m\} \cup \{m_i \mid i \in \mathbb{N}^+ \wedge i \leq n\}$.

An $n$-SPRAM process term ($n \in \mathbb{N}^+$), called an $n$-SPRAMP term for short, is a term from $\mathcal{P}_{\text{rec}}$ that is of the form $\langle X_1|E_1 \rangle \|_{\text{sync}} \ldots \|_{\text{sync}} \langle X_n|E_n \rangle$, where, for each $i \in \mathbb{N}^+$ with $i \leq n$: 29
for each $X \in \text{vars}(E_i)$, the recursion equation for $X$ in $E_i$ has one of the following forms:

1. $X = t \rightarrow \text{sync} \cdot Y$,
2. $X = t \rightarrow [m_i := \text{ini} \cdot i (m_i)] \cdot Y$,
3. $X = t \rightarrow [m_i := \text{loa} \cdot i : d (m_i, m)] \cdot Y$,
4. $X = t \rightarrow [m_i := \text{sto} \cdot i : s (m_i, m)] \cdot Y$,
5. $X = t \rightarrow [m_i := o (m_i)] \cdot Y$,
6. $X = (p(m_i) = 1) \rightarrow [m_i := m_i] \cdot Y + (p(m_i) = 0) \rightarrow [m_i := m_i] \cdot Y'$,
7. $X = t \rightarrow \epsilon$,

where $o \in O \cap CRAMP$, $p \in CRAMP$, and $Y, Y' \in \text{vars}(E_i)$;

– for each $X, Y \in \text{vars}(E_i)$ with $Y$ occurring in the right-hand side of the recursion equation for $X$ in $E_i$, the recursion equation for $X$ in $E_i$ is of the form (1) iff the recursion equation for $Y$ in $E_i$ is not of the form (1);

– for each $X \in \text{vars}(E_i)$, the recursion equation for $X$ in $E_i$ is of the form (2) iff $X \equiv X_i$.

We write $\mathcal{P}_{\text{SPRAMP}}$ for the set of all terms $t \in \mathcal{P}_{\text{rec}}$ such that $t$ is an $n$-SPRAMP term for some $n \in \mathbb{N}^+$, and we write $\mathcal{CP}_{\text{SPRAMP}}$ for $\mathcal{P}_{\text{SPRAMP}} \cap \mathcal{CP}_{\text{rec}}$. Moreover, we write $\deg(t)$, where $t \in \mathcal{P}_{\text{SPRAMP}}$, for the unique $n \in \mathbb{N}^+$ such that $t$ is an $n$-SPRAMP term.

The terms from $\mathcal{P}_{\text{SPRAMP}}$ will be referred to as SPRAMP terms.

A process that can be denoted by an SPRAMP term is called an SPRAM process or an SPRAMP for short. So, an SPRAMP is a synchronous parallel composition of processes that are definable by a guarded linear recursive specification over ACP$^\llbracket \epsilon \rrbracket$-I of the kind described above. Each of those parallel processes starts with an initialization step in which the number of its private memory is made available in the register with number 0 from its private memory.

Notice that by Lemma 1 and Theorem 2, for all $t \in \mathcal{P}_{\text{SPRAMP}}$, there exists a guarded linear recursive specification $E$ and $X \in \text{vars}(E)$ such that $t \leftrightarrow_{rb} (X \mid E)$.

$\mathcal{D}$ as fixed above and $\mathcal{CP}_{\text{SPRAMP}}$ induce the SPRAMP model of computation:

– the set of possible computational processes is the set of all processes that can be denoted by a term from $\mathcal{CP}_{\text{SPRAMP}}$;

– for each possible computational process $p$, the set of possible data environments is the set of all $V_{\deg(t)}$-indexed data environments, where $t$ is a term from $\mathcal{CP}_{\text{SPRAMP}}$ denoting $p$;

– the effect of applying the process denoted by a $t \in \mathcal{CP}_{\text{SPRAMP}}$ to a $V_{\deg(t)}$-indexed data environment $\mu$ is $V_{\rho}(t)$, where $\rho$ is a flexible variable valuation that represents $\mu$.

The SPRAMP model of computation described above is intended to be close to the synchronous parallel RAM model of computation sketched in.
ever, that model is a PRIORITY CRCW model whereas the SPRAMP model is essentially an ARBITRARY CRCW model (see [19][16]). This means basically that, in the case that two or more parallel processes attempt to change the content of the same register at the same time, the process that succeeds in its attempt is chosen arbitrarily. Moreover, in the model sketched in [32], the built-in programs of the RAMs that make up a PRAM must be the same whereas the parallel processes that make up an SPRAMP may be different.

The SPRAMPs can be looked upon as the processes that can be produced by a collection of SMBRAMs when they execute their built-in program synchronously in parallel.

For the SMBRAMs whose private memory has number \( i \) \((i \in \mathbb{N}^+)\), the processes that are produced when they execute their built-in program are now given by a function \( \mathcal{M}_i^{\text{sync}}: \mathcal{L}_S^{\text{SMBRAM}} \rightarrow \mathcal{P}_{\text{SPRAMP}} \) that is defined up to consistent renaming of variables as follows: \( \mathcal{M}_i^{\text{sync}}(c_1 \ldots c_n) = \langle X_i | E_i \rangle \), where \( E_i \) consists of the equation

\[
X_i = t \rightarrow [m_i := \text{ini} \# i \{m_i\}] \cdot Y_1
\]

and, for each \( j \in \mathbb{N} \) with \( 1 \leq j \leq n \), an equation

\[
Y_{2j-1} = t \rightarrow \text{sync} \cdot Y_{2j}
\]

\[
Y_{2j} = t \rightarrow [m_i := c_j(m_i, m)] \cdot Y_{2j+1}
\]

if \( c_j \in \text{Load} \),

\[
Y_{2j} = t \rightarrow [m := c_j(m, m)] \cdot Y_{2j+1}
\]

if \( c_j \in \text{Store} \),

\[
Y_{2j} = t \rightarrow [m_i := c_j(m_i)] \cdot Y_{2j+1}
\]

if \( c_j \in \text{Oramp} \setminus \text{CRAMP} \),

\[
Y_{2j} = (p(m_i) = 1) \rightarrow [m_i := m_i] \cdot Y_{2j' - 1} + (p(m_i) = 0) \rightarrow [m_i := m_i] \cdot Y_{2j+1}
\]

if \( c_j \equiv \text{jmp}:p';j' \),

\[
Y_{2j} = t \rightarrow \epsilon
\]

if \( c_j \equiv \text{halt} \),

where \( \text{Load} = \{ \text{loa}:@i:d \mid i \in \mathbb{N} \land d \in \text{Dst} \} \), \( \text{Store} = \{ \text{sto}:s:@i \mid s \in \text{Src} \land i \in \mathbb{N} \} \), and \( Y_1, \ldots, Y_{2n} \) are different variable from \( X \setminus \{ X_i \} \).

The SPRAMPs are exactly the processes that can be produced by a collection of SMBRAMs when they execute their built-in program synchronously in parallel.

**Theorem 11.** Let \( n \in \mathbb{N}^+ \). For all constants \( \langle X_1|E_1 \rangle, \ldots, \langle X_n|E_n \rangle \in \mathcal{P}_{\text{rec}} \), \( \langle X_1|E_1 \rangle \parallel_{\text{sync}} \cdots \parallel_{\text{sync}} \langle X_n|E_n \rangle \in \mathcal{P}_{\text{SPRAMP}} \) iff there exist \( C_1, \ldots, C_n \in \mathcal{L}_S^{\text{SMBRAM}} \) such that \( \langle X_1|E_1 \rangle \parallel_{\text{sync}} \cdots \parallel_{\text{sync}} \langle X_n|E_n \rangle \) and \( \mathcal{M}_i^{\text{sync}}(C_1) \parallel_{\text{sync}} \cdots \parallel_{\text{sync}} \mathcal{M}_i^{\text{sync}}(C_n) \) are identical up to consistent renaming of variables.

**Proof.** Let \( i \in \mathbb{N}^+ \) be such that \( i \leq n \). It is easy to see that (a) for all \( C \in \mathcal{L}_S^{\text{SMBRAM}} \), \( \mathcal{M}_i^{\text{sync}}(C) \in \mathcal{P}_{\text{SPRAMP}} \) and (b) \( \mathcal{M}_i^{\text{sync}} \) is an bijection up to consistent renaming of variables. From this, it follows immediately that there exists

\[10\] The model sketched in [32] is similar to, among others, the models sketched in [11][12][20][33][22].
a $C \in \mathcal{IS}_{\text{SMBRAM}}$ such that $\langle X_i | E_i \rangle$ and $\mathcal{M}_i^{\text{sync}}(C)$ are identical up to consistent renaming of variables. From this, the theorem follows immediately. \qed

The first synchronous parallel RAM models of computation, e.g. the models proposed in [11,14,32], are older than the first asynchronous parallel RAM models of computation, e.g. the models proposed in [8,20,26]. It appears that the synchronous parallel RAM models have been primarily devised to be used in the area of computational complexity and that the asynchronous parallel RAM models have been primarily devised because the synchronous models were considered of restricted value in the area of algorithm efficiency.

7 Time and Work Complexity Measures

This section concerns complexity measures for the models of computation presented in Sections 4–6. Before the complexity measures in question are introduced, it is made precise in the current setting what a complexity measure is and what the complexity of a computable function from $\{0,1\}^n$ to $\{0,1\}^*$ under a given complexity measure is.

Let $CP \subseteq CP_{\text{rec}}$. Then a complexity measure for $CP$ is a partial function $M : CP \times (\mathcal{V} \rightarrow \mathbb{D}) \rightarrow \mathbb{N}$ such that, for all $t \in CP$ and $\rho \in \mathcal{V} \rightarrow \mathbb{D}$, $M(t, \rho)$ is defined iff $V_\rho(t)$ eventually halts.

Let $CP \subseteq CP_{\text{rec}}$ and let $M : CP \times (\mathcal{V} \rightarrow \mathbb{D}) \rightarrow \mathbb{N}$ be a complexity measure for $CP$. Let $n \in \mathbb{N}$ and let $F : \{0,1\}^n \rightarrow \{0,1\}^*$ be a computable function. Let $V : \mathbb{N} \rightarrow \mathbb{N}$. Then $F$ is of complexity $V$ under the complexity measure $M$ if there exists a $t \in CP$ such that:

- $t$ computes $F$;
- for all $w_1, \ldots, w_n \in \{0,1\}^*$ such that $F(w_1, \ldots, w_n)$ is defined, for all $\rho \in \mathcal{V} \rightarrow \mathbb{D}$ with $\rho(m) = \sigma_{\lambda}\{1 \mapsto w_1 \} \ldots \{n \mapsto w_n\}$:

$$M(t, \rho) \leq V(\ell(w_1) + \ldots + \ell(w_n)).$$

The RAMP Model of Computation

Below, a time complexity measure and a work complexity measure for the RAMP model of computation are introduced.

The sequential uniform time measure is the complexity measure $M_{\text{SUT}} : CP_{\text{RAMP}} \times (\mathcal{V} \rightarrow \mathbb{D}) \rightarrow \mathbb{N}$ for $CP_{\text{RAMP}}$ defined by

$$M_{\text{SUT}}(t, \rho) = \text{depth}(V_\rho(t))$$

for all $t \in CP_{\text{RAMP}}$ and $\rho \in \mathcal{V} \rightarrow \mathbb{D}$ such that $V_\rho(t)$ eventually halts.

The sequential uniform time measure is essentially the same as the uniform time complexity measure for the standard RAM model of computation (see e.g [1]). It is an idealized time measure: the simplifying assumption is made
that a RAMP performs one step per time unit. That is, this measure actually yields, for a given RAMP and a given data environment, the maximum number of steps that can be performed by the given RAMP before eventually halting in the case where the initial data environment is the given data environment. However, the maximum number of steps can also be looked upon as the maximum amount of work. This makes the sequential uniform time measure a very plausible work measure as well.

The sequential work measure is the complexity measure \( M_{SW} : CP_{RAMP} \times (V \rightarrow \mathbb{D}) \rightarrow \mathbb{N} \) for \( CP_{RAMP} \) defined by

\[
M_{SW}(t, \rho) = M_{SUT}(t, \rho)
\]

for all \( t \in CP_{RAMP} \) and \( \rho \in V \rightarrow \mathbb{D} \) such that \( V_\rho(t) \) eventually halts.

In the sequential case, it is in accordance with our intuition that the uniform time complexity measure coincides with the work complexity measure. In the parallel case, this is not in accordance with our intuition: it is to be expected that the introduction of parallelism results in a reduction of the amount of time needed but not in a reduction of the amount of work needed.

### The APRAMP Model of Computation

Below, a time complexity measure and a work complexity measure for the APRAMP model of computation are introduced.

The asynchronous parallel uniform time measure is the complexity measure \( M_{APUT} : CP_{APRAMP} \times (V \rightarrow \mathbb{D}) \rightarrow \mathbb{N} \) for \( CP_{APRAMP} \) defined by

\[
M_{APUT}(t, \rho) = \max \{ \text{depth}(\tau H_i(V_\rho(t))) \mid 1 \leq i \leq \deg(t) \},
\]

where \( H_i \) is the set of all \( \alpha \in A \) in which \( m_i \) does not occur, for all \( t \in CP_{APRAMP} \) and \( \rho \in V \rightarrow \mathbb{D} \) such that \( V_\rho(t) \) eventually halts.

In the above definition, \( \tau H_i \) turns steps of the process denoted by \( V_\rho(t) \) that are not performed by the parallel process whose private memory is referred to by \( m_i \) into silent steps. Because \( \text{depth} \) does not count silent steps, \( \text{depth}(\tau H_i(V_\rho(t))) \) is the maximum number of steps that the parallel process whose private memory is referred to by \( m_i \) can perform.

Hence, the asynchronous parallel uniform time measure yields, for a given APRAMP and a given data environment, the maximum over all parallel processes that make up the given APRAMP of the maximum number of steps that can be performed before eventually halting in the case where the initial data environment is the given data environment. Because it yields the maximum number of steps of one of the parallel processes that make up the given APRAMP, the asynchronous parallel uniform time measure differs from the asynchronous parallel work measure.

The asynchronous parallel work measure is the complexity measure \( M_{APW} : CP_{APRAMP} \times (V \rightarrow \mathbb{D}) \rightarrow \mathbb{N} \) for \( CP_{APRAMP} \) defined by

\[
M_{APW}(t, \rho) = \text{depth}(V_\rho(t))
\]
for all \( t \in \mathcal{CP}_{\text{APRAMP}} \) and \( \rho \in \mathcal{V} \rightarrow \mathbb{D} \) such that \( \mathcal{V}_\rho(t) \) eventually halts.

The sequential work measure and the asynchronous parallel work measure are such that comparison of complexities under these measures have some meaning: both concern the maximum number of steps that can be performed by a computational process.

Like all complexity measures introduced in this section, the asynchronous parallel uniform time measure introduced above is a worst-case complexity measure. It is quite different from the parallel time complexity measures that have been proposed for the asynchronous parallel RAM model of computation sketched in [8,20]. The round complexity measure is proposed as parallel time complexity measure in [8] and an expected time complexity measure is proposed as parallel time complexity measure in [26]. Neither of those measures is a worst-case complexity measure: the round complexity measure removes certain cases from consideration and the expected time complexity measure is an average-case complexity measure.

It appears that the round complexity measure and the expected time complexity measure are more important to analysis of the efficiency of parallel algorithms whereas the asynchronous parallel time complexity measure introduced above is more important to analysis of the complexity of computational problems that are amenable to solution by a parallel algorithm. After all, the area of computational complexity is mostly concerned with worst-case complexity.

In [26], the asynchronous parallel uniform time measure introduced above is explicitly rejected. Consider the case where there exists an interleaving of the parallel processes that make up an APRAMP that is close to performing all steps of each of the processes uninterrupted by steps of the others. Then the interleaving concerned is not ruled out by synchronization (through the shared memory) and may even be enforced by synchronization. So it may be likely or unlikely to occur. Seen in that light, it is surprising why it is stated in [26] that such an interleaving has “very low probability, yielding a sequential measure”.

The SPRAMP Model of Computation

Below, a time complexity measure and a work complexity measure for the SPRAMP model of computation are introduced.

The time complexity measure introduced below is essentially the same as the uniform time complexity measure that goes with the synchronous parallel RAM model of computation sketched in [32] and similar models.

The \textit{synchronous parallel uniform time measure} is the complexity measure \( M_{\text{SPUT}} : \mathcal{CP}_{\text{SPRAMP}} \times (\mathcal{V} \rightarrow \mathbb{D}) \rightarrow \mathbb{N} \) for \( \mathcal{C P}_{\text{SPRAMP}} \) defined by

\[
M_{\text{SPUT}}(t, \rho) = \text{depth}(\text{sync}(\mathcal{V}_\rho(t))),
\]

where \( \text{sync} = A \setminus \{ \text{sync} \} \), for all \( t \in \mathcal{CP}_{\text{SPRAMP}} \) and \( \rho \in \mathcal{V} \rightarrow \mathbb{D} \) such that \( \mathcal{V}_\rho(t) \) eventually halts.

In the above definition, \( \text{sync} \) turns all steps of the process denoted by \( \mathcal{V}_\rho(t) \) other than synchronization steps, i.e. all computational steps, into silent steps.
Because \textit{depth} does not count silent steps, \( \text{depth}(\tau_{\text{sync}}(V_{\rho}(t))) \) is the maximum number of synchronization steps that can be performed by the process denoted by \( V_{\rho}(t) \) before eventually halting.

Hence, the synchronous parallel uniform time measure yields, for a given SPRAMP and a given data environment, the maximum number of synchronization steps that can be performed by the given SPRAMP before eventually halting in the case where the initial data environment is the given data environment. Because the parallel processes that make up the given SPRAMP synchronize after each computational step, the time between two consecutive synchronization steps can be considered one time unit. Therefore, this measure is a plausible time measure. Clearly, the maximum number of synchronization steps that can be performed by the given SPRAMP and the maximum number of computational steps that can be performed by the given SPRAMP are separate numbers. So the synchronous parallel uniform time measure differs from the synchronous parallel work measure.

The \textit{synchronous parallel work measure} is the complexity measure \( M_{\text{SPW}} : \mathcal{CP}_{\text{SPRAMP}} \times (\mathcal{V} \rightarrow \mathcal{D}) \rightarrow \mathbb{N} \) for \( \mathcal{CP}_{\text{SPRAMP}} \) defined by

\[
M_{\text{SPW}}(t, \rho) = \text{depth}(\tau_{\text{sync}}(V_{\rho}(t)))
\]

for all \( t \in \mathcal{CP}_{\text{SPRAMP}} \) and \( \rho \in \mathcal{V} \rightarrow \mathcal{D} \) such that \( V_{\rho}(t) \) eventually halts.

The sequential work measure and the synchronous parallel work measure are such that comparison of complexities under these measures have some meaning: both concern the maximum number of computational steps that can be performed by a computational process.

Take an SPRAMP and the APRAMP which is the SPRAMP without the automatic synchronization after each computational step. Assume that at any stage the next step to be taken by any of the parallel processes that make up the APRAMP does not depend on the steps that have been taken by the other parallel processes. Then the synchronous parallel time measure \( M_{\text{SPUT}} \) yields for the SPRAMP the same results as the asynchronous parallel time measure \( M_{\text{APUT}} \) yields for the APRAMP.

8 SPRAMPS and the Parallel Computation Thesis

The SPRAMP model of computation is a simple model based on an idealization of existing shared memory parallel machines that abstracts from synchronization overhead. The synchronous parallel uniform time measure introduced for this model is a simple, hardware independent, and worst-case complexity measure.

The question is whether the SPRAMP model of computation is a reasonable model of parallel computation. A model of parallel computation is generally considered reasonable if the parallel computation thesis holds. In the current setting, this thesis can be phrased as follows: the \textit{parallel computation thesis} holds for a model of computation if, for each computable partial function from
$((\{0,1\}^*)^n \to \{0,1\}^*$ \(n \in \mathbb{N}\)), its complexity under the time complexity measure for that model is polynomially related to its complexity under the space complexity measure for the multi-tape Turing machine model of computation.

Before we answer the question whether the SPRAMP model of computation is a reasonable model of parallel computation, we go into a classification of synchronous parallel RAMs. This classification is used later on in answering the question. Below, synchronous parallel RAMs will be called PRAMs for short.

First of all, PRAMs can be classified as PRAMs whose constituent RAMs may execute different programs or PRAMs whose constituent RAMs must execute the same program. The former PRAMs are called MIMD PRAMs and the latter PRAMs are called SIMD PRAMs.

In [19, Section 2.1], PRAMs are classified according to their restrictions on shared memory access as EREW (Exclusive-Read Exclusive-Write), CREW (Concurrent-Read Exclusive-Write) or CRCW (Concurrent-Read Concurrent-Write). CRCW PRAMs are further classified according to their way of resolving write conflicts as COMMON, where all values attempted to be written concurrently into the same shared register must be identical, ARBITRARY, where one of the values attempted to be written concurrently into the same shared register is chosen arbitrarily, or PRIORITY, where the RAMs making up the PRAM are numbered and, from all values attempted to be written concurrently into the same shared register, the one attempted to be written by the RAM with the lowest number is chosen.

Below, the next two lemmas about the above classifications of PRAMs will be used to show that the parallel computation thesis holds for the SPRAMP model of computation.

**Lemma 3.** Assuming a fixed instruction set:

1. MIMD PRIORITY CRCW PRAMs can be simulated by MIMD ARBITRARY CRCW PRAMs with the same number of RAMs and with the parallel time increased by a factor of $O(\log(p))$, where $p$ is the number of RAMs;
2. MIMD ARBITRARY CRCW PRAMs can be simulated by MIMD PRIORITY CRCW PRAMs with the same number of RAMs and the same parallel time.

**Proof.** Assume a fixed instruction set.

Part 1. It is shown in [19, Section 3.1] that MIMD PRIORITY CRCW PRAMs can be simulated by MIMD EREW PRAMs with the same number of RAMs and with the parallel time increased by only a factor of $O(\log(p))$, where $p$ is the number of RAMs. It follows directly from the definitions concerned that MIMD EREW PRAMs can be simulated by MIMD ARBITRARY CRCW PRAMs with the same number of RAMs and the same parallel time (the programs involved can be executed directly). Hence, MIMD PRIORITY CRCW PRAMs can be simulated by MIMD ARBITRARY CRCW PRAMs with the same number of RAMs and with the parallel time increased by a factor of $O(\log(p))$, where $p$ is the number of RAMs.
Part 2. It follows directly from the definitions concerned that MIMD ARBITRARY CRCW PRAMs can be simulated by MIMD PRIORITY CRCW PRAMs with the same number of RAMs and the same parallel time (the programs involved can be executed directly).

Lemma 4. Assuming a fixed instruction set:

1. SIMD PRIORITY CRCW PRAMs can be simulated by MIMD PRIORITY CRCW PRAMs with the same number of RAMs and with the same parallel time;
2. MIMD PRIORITY CRCW PRAMs can be simulated by SIMD PRIORITY CRCW PRAMs with the same number of RAMs and with the parallel time increased by a constant factor.

Proof. Assume a fixed instruction set.

Part 1. This follows directly from the definitions concerned (the programs involved can be executed directly).

Part 2. This is a special case of Theorem 3 from [35].

The next theorem expresses that the parallel computation thesis holds for the SPRAMP model of computation.

Theorem 12. Let $F : (\{0,1\}^*)^m \to \{0,1\}^*$ for some $m \in \mathbb{N}$ be a computable function and let $T, S : \mathbb{N} \to \mathbb{N}$. Then:

- if $F$ is of complexity $T(n)$ under the synchronous parallel time complexity measure $M_{\text{SPUT}}$ for the SPRAMP model of computation, then there exists a $k \in \mathbb{N}$ such that $F$ is of complexity $O(T(n)^k)$ under the space complexity measure for the multi-tape Turing machine model of computation;
- if $F$ is of complexity $S(n)$ under the space complexity measure for the multi-tape Turing machine model of computation, then there exists a $k \in \mathbb{N}$ such that $F$ is of complexity $O(S(n)^k)$ under the synchronous parallel time complexity measure $M_{\text{SPUT}}$ for the SPRAMP model of computation provided that $S(n) \geq \log(n)$ for all $n \in \mathbb{N}$.

Proof. In [14], SIMDAGs are introduced. SIMDAGs are SIMD PRIORITY CRCW PRAMs with a subset of the instruction set of SMBRAMs as instruction set. Because $\text{DSPACE}(S(n)) \subseteq \text{NSPACE}(S(n)) \subseteq \text{DSPACE}(S(n)^2)$, the variant of the current theorem for the SIMDAG model of computation follows immediately from Theorems 2.1 and 2.2 from [14] under a constructibility assumption for $S(n)$. However, the proofs of those theorems go through with the instruction set of SMBRAMs because none of the SMBRAM instructions builds bit strings that are more than $O(T(n))$ bits long in $T(n)$ time. Moreover, if we take forking variants of SIMDAGs with the instruction set of SMBRAMs (resembling the P-RAMs from [11]), the constructibility assumption for $S(n)$ is not needed. This can be shown in the same way as in the proof of Lemma 1a from [11].

In the rest of this proof, we write E-SIMDAG for a SIMDAG with the instruction set of SMBRAMs and forking E-SIMDAG for a forking variant of an E-SIMDAG.
The variant of the current theorem for the forking E-SIMDAG model of computation follows directly from the above-mentioned facts.

Now forking E-SIMDAGs can be simulated by E-SIMDAGs with \(O(p)\) number of SMBRAMs and with the parallel time increased by a factor of \(O(\log(p))\), where \(p\) is the number of SMBRAMs used by the forking E-SIMDAG concerned. This is proved as in the proof of Lemma 2.1 from [15]. The other way round, E-SIMDAGs can be simulated by forking E-SIMDAGs with eventually the same number of SMBRAMs and with the parallel time increased by \(O(\log(p))\), where \(p\) is the number of SMBRAMs of the E-SIMDAG concerned. This is easy to see: before the programs of the \(p\) SMBRAMs involved can be executed directly, the \(p\) SMBRAMs must be created by forking and this can be done in \(O(\log(p))\) time. It follows immediately from these simulation results that time complexities on forking E-SIMDAGs are polynomially related to time complexities on E-SIMDAGs.

The variant of the current theorem for the forking E-SIMDAG model of computation follows directly from the variant of the current theorem for the forking E-SIMDAG model of computation and the above-mentioned polynomial relationship. From this, the fact that E-SIMDAGs are actually SIMD PRIORITY CRCW PRAMs that are composed of SMBRAMs, Lemma 4, Lemma 3, and Theorem 11, the current theorem now follows directly.

\[ \Box \]

9 Concluding Remarks

In this paper, it has been studied whether the imperative process algebra ACP\( \tau \)-I can play a role in the field of models of computation. Models of computation corresponding to models based on (sequential) random access machines, asynchronous parallel random access machines, synchronous parallel random access machines, and complexity measures for those models have been described in a mathematically precise way in the setting of ACP\( \tau \)-I. Central in the models described are the computational processes considered instead of the abstract machines that produce those processes when they execute their built-in program.

The work presented in this paper pertains to formalizing models of computation. Little work has been done in this area. Three notable exceptions are [27,36,2]. Those papers are concerned with formalization in a theorem prover (HOL4, Isabelle/HOL, Matita) and focus on some version of the Turing machine model of computation. This makes it impracticable to compare the work presented in those papers with the work presented in this paper.

Whereas it is usual in versions of the RAM model of computation that bit strings are represented by natural numbers, here natural numbers are represented by bit strings. Moreover, the choice has been made to represent the natural number 0 by the bit string 0 and to adopt the empty bit string as the register content that indicates that a register is (as yet) unused.
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