TORIC CO-HIGGS BUNDLES ON TORIC VARIETIES

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Abstract. Starting from the data of a nonsingular complex projective toric variety, we define an associated notion of toric co-Higgs bundle. We provide a Lie-theoretic classification of these objects by studying the interaction between Klyachko’s fan filtration and the fiber of the co-Higgs bundle at a closed point in the open orbit of the torus action. This can be interpreted, under certain conditions, as the construction of a coarse moduli scheme of toric co-Higgs bundles of any rank and with any total equivariant Chern class.

1. Introduction

We begin with an algebraic or, equivalently, holomorphic vector bundle $V$ over a nonsingular complex projective variety $X$ with tangent bundle $TX$. Then, a co-Higgs field for $V$ is a holomorphic section $\phi$ of the twisted endomorphism bundle $\text{End}(V) \otimes TX$, subject to the integrability condition that the quadratic section $\phi \otimes \phi$ is symmetric — that is, that the section $\phi \wedge \phi$ of $\text{End}(V) \otimes \Lambda^2 TX$ vanishes identically. A pair $(V, \phi)$ satisfying the above conditions is referred to as a co-Higgs bundle. Co-Higgs bundles were introduced simultaneously by Hitchin [10] and the fourth-named author [14] in the context of generalized complex geometry. The name co-Higgs speaks to a duality with Higgs bundles in the sense of Hitchin [8, 9] and Simpson [17], where the Higgs fields are $T^* X$-valued.

Co-Higgs bundles have been classified and/or constructed on $\mathbb{P}^1$ [15, 1], $\mathbb{P}^2$ [16], $\mathbb{P}^1 \times \mathbb{P}^1$ [19], and logarithmic curves [1], for example. Over singular varieties, they have been used to some effect towards establishing inequalities related to vector-valued modular forms [5]. At the same time, there are “no-go” theorems for the existence of nontrivial co-Higgs bundles in some instances, such as over the moduli space of stable bundles on a nonsingular complex curve of genus at least 2 [3]. Recently, the first-named and fourth-named authors classified homogeneous co-Higgs bundles on Hermitian symmetric spaces [5]. The goal of the present work is to extend this to toric varieties. Accordingly, we define below a natural notion of toric co-Higgs bundle.

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We classify toric co-Higgs structures for a fixed toric bundle $V$ using Klyachko’s seminal work on classification of toric vector bundles [11]. We recall from [11, Theorem 2.2.1] that the category of toric bundles on $X$ with equivariant homomorphisms is equivalent to the category of compatible $\Sigma$-filtered vector spaces, where $\Sigma$ is the fan of $X$. Our strategy for classifying toric co-Higgs bundles $(V, \phi)$ is thus to reduce the data of $(V, \phi)$ to that of a tuple of commuting $\Sigma$-filtered endomorphisms of the fiber $V_{x_0}$, where $x_0$ is a closed point in the open orbit of $T$ in $X$. This is the content of Theorem 3.1, which is the main theorem in this note.

Similar to the symmetric space case [3], the resulting classification is Lie-theoretic in nature and admits an interpretation as a moduli construction. Subject to certain conditions (namely, the freeness of a certain group action), we identify a scheme that parametrizes toric co-Higgs bundles of fixed rank and total equivariant Euler characteristic in the sense of [12]. This scheme fibers over an associated moduli scheme of toric bundles constructed in [13].

2. Set-up and examples

2.1. Basic notions. Throughout, $X$ is a nonsingular complex projective variety. Assume that $X$ admits an algebraic (equivalently, holomorphic) action of a complex torus $T \cong (\mathbb{C}^*)^n$ so that it is a toric variety in the sense of [7]. Furthermore, fix a holomorphic vector bundle $V$ and suppose that it admits a lift of the action of $T$ from $X$ which is fiber-wise linear. In other words, $V$ is equipped with the structure of a $T$-equivariant vector bundle. We will refer to $V$ simply as a toric bundle.

There is subsequently an induced action of $T$ on the vector space of global holomorphic sections of $V$:

$$ (t \cdot s)(x) = ts(t^{-1}x) \quad (2.1) $$

for all $s \in H^0(X, V)$ and $t \in T$. A section $s \in H^0(X, V)$ is said to be semi-invariant if there exists a character $\chi$ of $T$ such that

$$ t \cdot s = \chi(t)s \quad (2.2) $$

for all $t \in T$. A semi-invariant section $s$ is said to be invariant if the associated character $\chi(t)$ is trivial, meaning

$$ t \cdot s = s \quad (2.3) $$

for all $t$. Combining (2.1) and (2.3), we have

$$ ts(t^{-1}x) = s(x) \quad (2.4) $$

for any invariant section $s$. 
2.2. $T$-equivariant structures and toric co-Higgs bundles. Now, a $T$-equivariant structure on $V$ induces one on $\text{End}(V)$ in the following way. Given an element $\psi$ in the fiber $(\text{End}(V))_x = \text{End}(V_x)$ over $x \in X$, we define $t\psi$ in $\text{End}(V_{tx})$ by
\[
(t\psi)(v) = (t \circ \psi \circ t^{-1})(v)
\]
for every $v \in V_{tx}$ and $t \in T$. Then, by (2.4), a holomorphic section $\phi$ of $\text{End}(V)$ is invariant if and only if
\[
t\phi(t^{-1}x) = \phi(x)
\]
for all $x \in X$ and $t \in T$. By (2.5), equation (2.6) is equivalent to
\[
t \circ \phi(t^{-1}x) = \phi(x) \circ t
\]
for all $x \in X$ and $t \in T$. In other words, $\phi$ is an invariant section of $\text{End}(V)$ if and only if $\phi$ is a $T$-equivariant endomorphism of $V$.

At the same time, the tangent bundle $TX$ has a natural $T$-equivariant structure that is simply the linearization of the $T$-action on $X$. Together with the above $T$-action on $\text{End}(V)$, we have an induced $T$-equivariant structure on the twisted bundle $\text{End}(V) \otimes TX$. This allows us to formulate the following:

**Definition 2.1.** A toric co-Higgs bundle on a toric variety $X$ is a pair $(V, \phi)$, where $V \rightarrow X$ is a toric bundle and $\phi \in H^0(X, \text{End}(V) \otimes TX)$ is an invariant co-Higgs field.

2.3. Examples. The tangent bundle $TX$ of a nonsingular projective toric variety $X$ always admits a nonzero invariant holomorphic section $s$. Let us take the tensor product of such a holomorphic vector field with the identity homomorphism $1$ of any toric bundle $V$. Call this product $\phi$. It follows immediately that
\[
\phi \wedge \phi = (s \otimes 1) \wedge (s \otimes 1) = [s, s] \otimes (1 \wedge 1) = 0.
\]
In other words, $\phi$ is a nontrivial invariant co-Higgs field on $V$, and hence every toric vector bundle on any toric variety is equipped with a family of invariant co-Higgs fields induced by the invariant holomorphic vector fields on $X$.

Another example, similar in spirit but for a specific bundle, is given by choosing $V = TX \oplus \mathcal{O}_X$ with its natural toric structure, where $\mathcal{O}_X$ is the structure sheaf of $X$. We can equip this $V$ with the co-Higgs field
\[
\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]
where $1$ is interpreted as the identity morphism on $TX$, which serves as the part of the co-Higgs structure that acts as $TX \rightarrow \mathcal{O}_X \otimes TX$. Because $\phi$ is built from just the identity map, the co-Higgs field automatically has the required invariance. It satisfies the vanishing condition $\phi \wedge \phi = 0$ since, as a matrix, $\phi$ is nilpotent. More generally, this example is present on any complex variety — it is the so-called
canonical co-Higgs bundle — and it is discussed in some detail from the point of view of slope stability and deformation theory in Chapter 6 of [14].

In the next section, we classify toric co-Higgs structures on a fixed toric bundle.

3. Toric co-Higgs bundles and the Klyachko fan filtration

3.1. $\Sigma$-filtrations. Let $X$ be a nonsingular complex projective toric variety equipped with an action of $T$. For any toric bundle $V$ on $X$, Klyachko [11] constructed a compatible family of full filtrations of decreasing subspaces of the fiber $E = V_{x_0}$, where $x_0$ is a closed point in the open orbit of $T$ in $X$. We will subsequently refer to $x_0$ simply as a “closed point for $T$”. The family is indexed by the $T$-invariant divisors or equivalently the generators $\{\rho\}$ of their stabilizing one-parameter subgroups. In other words, we have a family of filtrations

$$\{E^\rho(i) \mid \rho \in |\Sigma(1)|, i \in \mathbb{Z}\},$$

where $\Sigma$ denotes the fan of $X$, and $|\Sigma(1)|$ denotes the set of primitive integral generators of the 1-dimensional cones of $\Sigma$. Note that “decreasing” means

$$E^\rho(i + 1) \subseteq E^\rho(i)$$

for all $i$. For brevity, such a family of filtrations will be called a $\Sigma$-filtration. The compatibility condition mentioned above refers to the existence of cone-wise $T$-module structures on $E$ giving rise to the $\Sigma$-filtration. (We refer to [11] for the details.)

A morphism of compatible $\Sigma$-filtered vector spaces $\{E^\rho(i)\}$ and $\{F^\rho(i)\}$ is a vector space map $\phi : E \rightarrow F$ such that $\phi(E^\rho(i)) \subseteq F^\rho(i)$ for all $\rho$ and $i$. We call such a morphism a filtered linear map of $\Sigma$-filtered vector spaces.

**Theorem 3.1.** Let $X$ be a nonsingular complex projective toric variety equipped with an action of $T \cong (\mathbb{C}^*)^n$, let $V$ by any toric bundle on $X$, and let $x_0 \in X$ be a closed point for $T$. Then, there is a 1 : 1 correspondence between invariant co-Higgs fields $\phi$ and $n$-tuples of pairwise-commuting filtered linear maps of $E = V_{x_0}$ that respect the Klyachko $\Sigma$-filtration.

Before proving Theorem 3.1, we need to understand the integrability condition $\phi \wedge \phi = 0$ locally. In [14] [10] [16], a local criterion for the vanishing of $\phi \wedge \phi$ is identified. Suppose that $\{z_1, \cdots, z_n\}$ is a holomorphic coordinate system on an affine chart $U$ in a variety $X$. We can write

$$\phi|_U = \sum_{i=1}^n \phi_i \frac{\partial}{\partial z_i},$$

where each $\phi_i \in H^0(U, \text{End}(V))$. Then

$$\phi \wedge \phi = 0 \text{ on } U \iff [\phi_i, \phi_j] = 0 \text{ on } U \forall 1 \leq i, j \leq n. \quad (3.1)$$
With this observation, we can proceed with the proof of the main theorem.

**Proof of Theorem 3.1.** Let \((t_1, \ldots, t_n)\) be coordinates on \(T\) corresponding to an integral basis of \(\text{Lie}(T)\). We identify these with coordinates on the open dense \(T\)-orbit in \(X\); this should not cause any confusion. The corresponding vector fields \(\frac{\partial}{\partial t_i}\) are naturally \(T\)-invariant on the open orbit. By [2, Theorem 3.1], these vector fields admit \(T\)-invariant holomorphic extensions to the whole of \(X\). Now, let \(A_1, \ldots, A_n\) be pairwise-commuting linear endomorphisms of \(V_{x_0}\) that respect the \(\Sigma\)-filtration. Then by Klyachko’s theorem, these define \(T\)-equivariant endomorphisms \(\phi_1, \ldots, \phi_n\) of \(V\) such that \(\phi_j(x_0) = A_j\). Therefore, each \(\phi_j\) is an invariant section of \(\text{End}(V)\).

Applying equation (2.4) to \(\phi_j\) we see that
\[
\{t\phi_j(t^{-1}x_0) = \phi_j(x_0)\} \implies \{\phi_j(t^{-1}x_0) = t^{-1}\phi_j(x_0)\}
\]
for all \(t \in T\). It then follows from (2.5) that the \(\phi_j(t^{-1}x_0)\)'s commute with each other for every \(t \in T\). In other words, they commute mutually on the open dense \(T\)-orbit in \(X\). Therefore, by continuity, the \(\phi_j\)'s commute on entire \(X\).

Next, we define \(\phi \in H^0(X, \text{End}(V) \otimes TX)\) by
\[
\phi = \sum_j \phi_j \frac{\partial}{\partial t_j}. \tag{3.2}
\]
Consider any affine toric chart on \(X\) with coordinates \((z_1, \ldots, z_n)\). Then by [2, Lemma 3.1], we have
\[
\frac{\partial}{\partial t_j} = \sum_k c_{jk}(z_1, \ldots, z_n) \frac{\partial}{\partial z_k} \tag{3.3}
\]
where the \(c_{jk}\)'s are holomorphic functions. Substituting (3.3) in (3.2), we have the following representation of \(\phi\) in the \((z_1, \ldots, z_n)\) coordinates:
\[
\phi = \sum_k \psi_k \frac{\partial}{\partial z_k},
\]
where \(\psi_k = \sum_j c_{jk}(z_1, \ldots, z_n)\phi_j\). Since the \(\phi_j\)'s commute and the \(c_{jk}\)'s are scalars, the \(\psi_k\)'s also mutually commute. Thus by (3.1), \(\phi\) defines a co-Higgs structure on \(V\). Hence, given a tuple \((A_1, \ldots, A_n)\) of commuting filtered endomorphisms of \(V\), we obtain an equivariant co-Higgs structure on \(V\).

In the other direction, given any equivariant co-Higgs structure \(\phi\) on \(V\), we may write \(\phi\) on the open orbit, as in (3.2). We use the fact any torus-equivariant vector bundle is trivial over the open orbit. As the vector fields \(\frac{\partial}{\partial t_j}\) are \(T\)-invariant, the \(\phi_j\)'s are also \(T\)-invariant. Moreover, as the open orbit is contained in every affine toric chart, and \(\phi\) is a co-Higgs field, the \(\phi_j\)'s commute mutually by (3.1). Then we define \(A_j = \phi_j(x_0)\). As \(\phi_j\) is a \(T\)-equivariant endomorphism of \(V\), the endomorphisms \(A_j\) respect the Klyachko \(\Sigma\)-filtration.
It is now straightforward to check that the above association is a bijection. \(\square\)

4. Existence of a moduli scheme

We fix a toric bundle \(V\) on a toric variety \(X\) with fan \(\Delta\). Let \(x_0\) be a closed point of \(T\) and put \(E = V_{x_0}\) as in the preceding section. We use \(H^\rho\) to refer to the parabolic subgroup of \(GL(E)\) that preserves the filtration \(E^\rho\) on \(E\). Then, the group of endomorphisms of the \(\Sigma\)-filtration \(\{E^\rho(i)\}\) coincides with the group \(H_V := \bigcap \rho H^\rho\). Notice that the group \(H_V\) contains the center of \(GL(E)\). Denote by \(H_V[n]\) the set of \(n\)-tuples of pairwise-commuting elements of the group \(H_V\). Now, Theorem 3.1 can be recast as:

**Corollary 4.1.** If \(X\) is a nonsingular complex projective toric variety and \(V\) is any toric bundle on \(X\), then invariant co-Higgs fields \(\phi\) for \(V\) are in 1 : 1 correspondence with elements of \(H_V[n]\).

Now, we wish to consider all isomorphism classes \([V]\) of toric bundles on \(X\) having fixed rank \(r\) and fixed \(T\)-equivariant Chern classes. These classes can be defined explicitly within the equivariant Chow cohomology ring of \(X\) in terms of Klyachko’s filtration as per [12]. As per [13], let \(\mathcal{V}^{fr}_X(r, \psi)\) be the fine moduli space of rank-\(r\) toric vector bundles framed at \(x_0\) and with total equivariant Chern class \(\psi\). It is then a result of Payne [13, Corollary 3.11] that if \(PGL(r)\) acts freely on \(\mathcal{V}^{fr}_X(r, \psi)\), then there exists an associated coarse moduli scheme \(\mathcal{V}_X(r, \psi)\) of toric bundles on \(X\) with that total equivariant Chern class. In light of this, our result gives rise to:

**Corollary 4.2.** When the group \(PGL(r)\) acts freely on \(\mathcal{V}^{fr}_X(r, \psi)\), there exists a scheme

\[
\mathcal{C}_X(r, \psi) \overset{\pi}{\longrightarrow} \mathcal{V}_X(r, \psi),
\]

with fibers \(\pi^{-1}([V]) \cong H_V[n]\), that can be identified with a quasiprojective coarse moduli scheme of toric co-Higgs bundles on \(X\) of fixed rank \(r\) and total equivariant Chern class \(\psi\).

We mention a couple of natural questions for further study here. First, assuming the fibration \(\pi\) exists, when is it flat? Second, does \(\mathcal{C}_X(r, \psi)\) inherit arbitrarily bad singularities from \(\mathcal{V}_X(r, \psi)\), as per the “Murphy’s Law” for toric bundles in [13, Section 4]? Moreover, It would be desirable to understand the relationship of this construction to either Mumford-Takemoto or Gieseker stability for co-Higgs bundles in general. In particular, Simpson’s moduli space of \(\Lambda\)-modules [18], where \(\Lambda\) is a coherent sheaf of \(O_X\)-modules, produces a moduli space of Gieseker-stable coherent co-Higgs sheaves on \(X\) when \(\Lambda = \text{Sym}^*(T^*X)\) (cf. [16, Section 2] for further details on this correspondence in the co-Higgs setting). Interpreting the variation of toric
structures on $V$ with regards to the moduli problem for $\Lambda$-modules is an interesting direction for further exploration.

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