Treatment of Nonlinear Multi Freedom Boundary Constraints in Finite Element Analysis of Frame System using Lagrange Multiplier Method

Vu Thi Bich Quyen 1, Dao Ngoc Tien 1, Nguyen Thi Lan Huong 2

1 Faculty of Civil Engineering, Hanoi Architectural University, Km 10, Nguyentrai, Hanoi, Vietnam
2 International Research Institute MICA, Hanoi University of Science and Technology, N1, Daicoviet, Hanoi, Vietnam

bquyen1312@gmail.com

Abstract. This paper is concerned with the treatment of nonlinear multi-freedom and multi-point boundary condition in finite element analysis of frame system. The treatment of boundary constraints is required to produce modified system of equation based on master stiffness equations considering nonlinear multi freedom constraints. The nonlinear constraints considerably increases the difficulty in constructing and solving the modified system of equations. Generally, the operation of imposing multi-freedom constraints can be developed using master-slave elimination, penalty augmentation or Lagrange multiplier adjunction methods. The master-slave method is useful only for simple cases but exhibits serious shortcomings for treating arbitrary constraints. The penalty method has difficulty in selecting appropriate weight values that balance solution accuracy with the violation of constraint conditions. In present work the Lagrange multiplier adjunction methods is employed and endowed with possibility of substitution and works particularly well for nonlinear constraints. The incremental-iterative algorithm based on Crisfield arc-length method is proposed to solve the nonlinear modified system of equation. Based on the presented algorithm, the paper proposed calculation procedure and established programs for determining internal forces and displacements of frames having nonlinear multi-freedom constrains condition. The numerical test examples are presented to investigate load-displacement and load-internal relationship of system having nonlinear multi freedom constraints. The calculation results show the efficiency and convergence of proposed algorithm.

1. Introduction
The solution of any finite element method problem [1] is possible with defining the boundary conditions. A boundary condition is a set of constraints imposed on nodal coordinates located at the boundaries of a virtual domain. In analysing the finite element models with multi freedom and multi node constraints, the implementing boundary constraints is done by changing the assembled master stiffness equations to produce a modified system of equations based on the master stiffness equation. Generally, the operation of imposing multi freedom constraints develops by master-slave elimination method, penalty augmentation method or Lagrange multiplier adjunction methods [1,2]. The master-slave method is useful only for simple cases but exhibits serious shortcomings for treating arbitrary constraints. The penalty augmentation method and Lagrange multiplier adjunction method, employed
in the solution of equality constrained optimization problems, are two most common used in boundary treatment. These methods are better suited to general implementations of the finite element method, whether linear or nonlinear. The penalty method is earliest method, which was and still is employed in the solution of contact problems. However, the penalty method has difficulty of choice of weight values that balance solution accuracy with the violation of constraint conditions. The multiplier method is sensitive to the degree of linear independence of the constraints, and the bordered stiffness is singular in the case of the dependent constrains. In contrast to the penalty method, the method of Lagrange multipliers has the advantage of being exact (aside from computational errors due to finite precision arithmetic). It provides directly the constraint forces, which are of interest in many applications. It does not require guesses as regards weights. This method can be extended without difficulty to nonlinear constraints.

The Lagrange multiplier adjunction has proven to be highly and efficient and accurate for finite element solution of nonlinear contact problem. This research is intended to employ the Lagrange multiplier adjunction method for imposing nonlinear boundary constraints in finite element analysis of frame systems. The nonlinear boundary constraints increases the difficulty in solving the nonlinear modified system of equations and requests the method of solving a nonlinear equation. In recent years, a number of solution procedures for solving the nonlinear equilibrium equation have been discussed in many research papers. The arc length method [3], both load and displacement control, is a very efficient method in solving non-linear systems of equations. In solving the problems of structural analysis, the potential applications of the arc length method is more appropriate, where numerical solutions based on Newton’s method fail. This research proposes new approach to employ arc-length method to solve the nonlinear modified system of equation, which have been formulated by using arc length method. The incremental-iterative algorithm to solve the nonlinear modified system of equations is developed based on proposed method. The calculation programs for determining internal forces and displacements of frames having nonlinear multi freedom and multi point constrains boundary are established.

2. Lagrange multiplier method for imposing nonlinear multi freedom constraints
2.1. Boundary condition and nonlinear multi freedom constraints
In finite element structural analysis [1], separating known and unknown components is needed for developing equations for a linear solver. The necessary step is applying the physical support conditions as displacement boundary conditions to eliminate rigid body motions and render the system non-singular. The constraint conditions can be single freedom constraints or multi freedom constraint.

Single freedom constraints that are mathematically expressable as constraints on individual degrees of freedom. Multi freedom constraint is defined as type of the constraint where two or more displacement components are combined to equate to a prescribed value. The multi freedom constraint is described as the canonical form of the constraint, when a multi freedom constraint is defined such that all the interacting nodal displacement components can be assembled to the right side of the equation and the left-hand side is either a zero or a non-zero prescribed value. The multi freedom constraint is nonlinear when the interacting nodal displacements combine in a nonlinear manner. Unlike the single freedom constrain, using hand (or computer) oriented techniques can not incorporate multi freedom constraints into the master stiffness equations. Accounting for multi freedom constraints is done by changing the assembled master equations to produce a modified system of equations.

2.2. Lagrange multiplier method for incorporating nonlinear multi freedom constrains into the master stiffness equations
Lagrange multiplier method is indirect method for constrained optimization. The basic idea of Lagrange multiplier method is to convert a constrained problem into a form such that the derivative
test of an unconstrained problem can still be applied. The gradient of the function at any stationary point of the function, that also satisfies the equality constraints, can be expressed as a linear combination of the gradients of the constraints, with the Lagrange multipliers acting as coefficients. The great advantage of this method is that it allows the optimization to be solved without explicit parameterization in terms of the constraints. As a result, the method of Lagrange multipliers is widely used to solve challenging constrained optimization problem.

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For incorporating the nonlinear multi freedom constraint into the master stiffness matrix, consider the following, having

- the global nodal displacement vector of frame system is \( \mathbf{u} = [u_1, u_2, ..., u_n]^T, \mathbf{u} \in \mathbb{R}^n \);
- the augmented potential energy of the unconstrained finite element model is \( \Pi(\mathbf{u}) \)

Equation of nonlinear multi freedom constrains is expressed as

\[ g_k(\mathbf{u}) = 0, k = 1..m \tag{1} \]

The governing system of equations can be developed by minimization of the augmented potential energy function by incorporating the nonlinear multi freedom constraints (1) as

\[ \min \left\{ \Pi(\mathbf{u}) : g_k(\mathbf{u}) = 0, k = 1..m, \mathbf{u} \in \mathbb{R}^n \right\} \tag{2} \]

The potential energy of the unconstrained finite element model is

\[ \Pi(\mathbf{u}) = U(\mathbf{u}) - \mathbf{u}^T \mathbf{P} = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{P} \tag{3} \]

Where

- \( U(\mathbf{u}) \) is strain energy of system and \( -\mathbf{u}^T \mathbf{P} \) is external work;
- \( \mathbf{P} = [P_1, P_2, ..., P_n]^T \) is global nodal force vector ;
- \( \mathbf{K} \) is global stiffness matrix.

Converting a constrained problem into unconstrained problem using the Lagrange multipliers method [3,4]. To impose the constraint, adjoin additional unknowns - Lagrange multipliers \( \lambda_k, k = 1..m \). Lagrange multipliers collected in vector \( \lambda = [\lambda_1, \lambda_2, ..., \lambda_m]^T \), \( \lambda \in \mathbb{R}^m \).

Form the Lagrangian \( L(\mathbf{u}, \lambda) \), then extremize \( L \) with respect to \( \mathbf{u} \) and \( \lambda \) yields the multiplier-augmented form as below

\[ L(\mathbf{u}, \lambda) = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{P} + \sum_{k=1}^{m} \lambda_k g_k(\mathbf{u}) \tag{4} \]

\[ \min \left\{ L(\mathbf{u}, \lambda) : \mathbf{u} \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \right\} \tag{5} \]

Taking the derivative of function \( L(\mathbf{u}, \lambda) \) with respect to \( \mathbf{u} \) and \( \lambda \) yields, setting equal to zero, getting system of \((n+m)\) equations having \((n+m)\) unknowns \( \mathbf{u}, \lambda_k \) as
Modified system of equations (or modified stiffness equations) considering nonlinear multi freedom constrains, the Eq. (6) is compactly written as

$$\begin{align*}
\left\{ Ku - P + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u} \right\} &= 0, \quad i = 1..n \\
g_k(u) &= 0, \quad k = 1..m
\end{align*}$$  \tag{6}

Where

$$\frac{\partial g_k(u)}{\partial u} = \left[ \frac{\partial g_1(u)}{\partial u_1}, \frac{\partial g_2(u)}{\partial u_2}, \ldots, \frac{\partial g_n(u)}{\partial u_n} \right]^T ; P = [P_1, P_2, \ldots, P_n]^T ; g(u) = [g_1(u), g_2(u), \ldots, g_n(u)]^T$$

3. Problem solving and algorithm

3.1. Constructing incremental equations

The problem solving method for nonlinear analysis using finite element method based on dividing the total load into incremental load step. For constructing the incremental equation, utilizing Taylor series formula for a short \( \delta u, \delta \lambda \), to expand function of Eq. (7) around of variable point, keeping only linear term in \( \delta u, \delta \lambda \), getting

$$\begin{align*}
\left\{ Ku + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u} \right\} \delta u + \left\{ K + \sum_{k=1}^{m} \lambda_k \frac{\partial}{\partial u} \left( \frac{\partial g_k(u)}{\partial u} \right) \right\} \delta u + \sum_{k=1}^{m} \left( \frac{\partial g_k(u)}{\partial u} \delta \lambda_k \right) &= P + \Delta P \\
g(u) + \frac{\partial g(u)}{\partial u} \delta u &= 0
\end{align*}$$  \tag{8}

Where

$$\delta u = [\delta u_1, \delta u_2, \ldots, \delta u_n]^T$$

is vector consist of nodal incremental displacements

$$\frac{\partial}{\partial u} \left( \frac{\partial g_k(u)}{\partial u} \right) = \begin{bmatrix}
\frac{\partial^2 g_1(u)}{\partial u_1 \partial u_1} & \frac{\partial^2 g_1(u)}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 g_1(u)}{\partial u_1 \partial u_n} \\
\frac{\partial^2 g_2(u)}{\partial u_1 \partial u_1} & \frac{\partial^2 g_2(u)}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 g_2(u)}{\partial u_1 \partial u_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 g_n(u)}{\partial u_1 \partial u_1} & \frac{\partial^2 g_n(u)}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 g_n(u)}{\partial u_1 \partial u_n}
\end{bmatrix}$$

By moving several components from left side to the right side of Eq. (8), getting

$$\begin{bmatrix}
(K + K_d(u, \lambda)) & \frac{\partial g(u)}{\partial u} \\
\frac{\partial g(u)}{\partial u} & 0
\end{bmatrix} \begin{bmatrix}
\delta u \\
\delta \lambda
\end{bmatrix} = \begin{bmatrix}
P + \Delta P - \left( Ku + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u} \right) \\
g(u)
\end{bmatrix}$$  \tag{9}
Where

$$K_g(u, \lambda) = \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u} = \sum_{k=1}^{m} \lambda_k \left[ \frac{\partial^2 g_k(u)}{\partial u \partial u} \right]$$

is built from constrained condition $g(u) = 0$

System of Eq. (9) is written in compact form as

$$\vec{K}(\vec{u}, \delta \vec{u}) = \vec{P} + \Delta \vec{P} - q(u)$$

(9)

Where

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \\ \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix} \quad \delta \vec{u} = \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \vdots \\ \delta u_n \\ \delta \lambda_1 \\ \vdots \\ \delta \lambda_m \end{bmatrix} \quad \vec{K}(\vec{u}) = \begin{bmatrix} (K + K_g(u, \lambda)) \frac{\partial g(u)}{\partial u} \\ \frac{\partial g(u)}{\partial u} \\ 0 \end{bmatrix}$$

$$\vec{P} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad q(u) = \left\{ \begin{array}{c} Ku + \sum_{k=1}^{m} \lambda_k \frac{\partial g_k(u)}{\partial u} \\ g_1(u) \\ g_2(u) \\ \vdots \\ g_m(u) \end{array} \right\}$$

3.2. Algorithm for solving incremental equations of equilibrium based on spherical arc length control method.

The nonlinear response of structures requires iterative solution techniques for finding equilibrium path [5, 6, 7, 8]. The concept of iterative techniques is based on using the constraint surface for finding the equilibrium point at any loading step “i” depending equilibrium point of previous loading step (i-1)th (shown in Fig.). The iterative process will be needed for finding the next equilibrium point B.

Consider the jth iterative step, designating
- $\vec{P}_j$ is the input load (not equilibrium load) at the jth iterative step
- $\Delta \vec{P}_j$ is the incremental load at the jth iterative step
- $\delta \vec{P}_j$ is the iterative load at the two following steps jth and (j-1)th
- $\vec{u}_j$ is displacement at the jth iterative step
- $\Delta \mathbf{u}_j$ is the incremental displacement at the $j^{th}$ iterative step
- $\delta \mathbf{u}_j$ is the iterative displacement at the two following steps $j^{th}$ and $(j-1)^{th}$

The displacement and external load at the current $i^{th}$ load step are computed by adding increments from the displacement and external loads at the previous $(i-1)^{th}$ load step, thus

$$\mathbf{u}_j = \mathbf{u}_{i-1} + \Delta \mathbf{u}_j \quad \mathbf{P}_j = \mathbf{P}_{i-1} + \Delta \mathbf{P}_j$$  \hspace{1cm} (10&11)

Similarly for incremental displacement and load, thus

$$\Delta \mathbf{u}_j = \Delta \mathbf{u}_{j-1} + \delta \mathbf{u}_j \quad \Delta \mathbf{P}_j = \Delta \mathbf{P}_{j-1} + \delta \mathbf{P}_j$$ \hspace{1cm} (12&13)

The residual load is given by

$$r^i_j = \mathbf{P}^i_j - q(\mathbf{u}^i_j)$$  \hspace{1cm} (14)

The governing system of nonlinear equations to be solved at the $j^{th}$ iteration of the $i^{th}$ incremental load step is given by

$$\mathbf{R}_{j-1} \cdot \delta \mathbf{u}_j = (\mathbf{P}^{i-1}_j + \Delta \mathbf{P}_j) - q(\mathbf{u}^{i-1}_j) = \mathbf{P}^i_j - q(\mathbf{u}^{i-1}_j) = r^i_{j-1} + \delta \mathbf{P}^i_j$$ \hspace{1cm} (15)

Where: $\mathbf{R}_{j-1}$ is tangent stiffness matrix at the $(j-1)^{th}$ iteration.

*Figure 1.* Arc length technique

The formulation of dimensional based on proposed above incremental-iterative procedure. Designating that, $\alpha$ is load parameter, $\bar{P}$ and reference load, the input load is computed by $\mathbf{P} = \alpha \cdot \bar{P}$
Similarly, the incremental load at the \( j \)th iterative step is in relationship with incremental load parameter as

\[
\delta P_j = \delta \alpha_j . p
\]  

(16)

Inputting \( \delta P_j \) from Eq. (16) to Eq. (15), getting

\[
K_{j-1} \delta u_j = r_{j-1} + \delta \alpha_j . p
\]  

(17)

Eq. (17) is a system of \((N+1)\) unknowns, including \( N \) incremental displacement components \( \delta u_j \) and one incremental load parameter \( \delta \alpha_j \), but only \( N \) equations. Therefore, an additional constraint equation must be added to the system, given by [7]

\[
a_j \delta \bar{u} + b_j \delta \alpha_j = c_j
\]  

(18)

Where: \( a_j, b_j, c_j \) are constraint parameters, depending on the solving method.

From Eq. (17) and Eq. (18), getting

\[
\left[ \begin{array}{c}
K_{j-1} -p \\
[a_j^T, b_j]
\end{array} \right] \left[ \begin{array}{c}
\delta \alpha_j \\
\delta \bar{u}_j
\end{array} \right] = \left[ \begin{array}{c}
r_{j-1} \\
c_j
\end{array} \right]
\]  

(19)

Decompose the iterative displacement vector into two parts (results from input load and residual load) [8]:

\[
\delta u_j = \delta \alpha_j. \delta p_j + \delta \bar{u}_j
\]  

(20)

Than Eq. (17) becomes

\[
\left\{ \begin{array}{l}
R_{j-1} \delta \bar{u}_j = p \\
R_{j-1} \delta \bar{u}_j = r_{j-1}
\end{array} \right.
\]  

(21)

The load parameter is necessary to compute the total displacement for the \( j \)th iteration of the \( i \)th incremental step. Solving Eq. (18) for the load parameter, then combining with Eq. (20) yields

\[
\delta \alpha_j = \frac{c_j - a_j \delta \bar{u}_j}{a_j \delta u_p + b_j}
\]  

(22)

The arc length method, both load control and displacement control, is a very efficient method in solving non-linear systems of equations. The concept of method is to constrain the solution path to an arc length, which is calculated via a norm of the load and displacement increment. The general constraint equation can be represented by [3] :

\[
\delta \alpha_j. \delta \alpha_j + \eta. \delta \alpha_j. \delta \bar{u}_j = (\Delta \alpha_j)^2
\]  

(23)

Where \( \eta \) is a non-negative real parameter, which is unique for each version of the arc length method including cylindrical, spherical, and elliptical version.

In this work, the spherical arc length method is employed with \( \eta=1 \). From Eq. (18) and Eq. (23), the following constraint parameters are obtained

\[
a_j = \delta \bar{u}_j = \delta \alpha_j. \delta \bar{u}_p
\]  

(24)
\[ b_j' = \delta \alpha'_j, \quad \delta \alpha'_j = \begin{cases} (\bar{\Delta} s)^2 & \text{for } j = 1 \\ 0 & \text{for } j \geq 2 \end{cases} \] (25)

Where: \( \bar{\Delta} s \) is initial arc length (prescribed arc length at the first iteration).

From equation (23), the load parameter can be given by

\[ \delta \alpha'_j = \begin{cases} \frac{\bar{\Delta} s}{\sqrt{\delta u'_p \cdot \delta u'_p + 1}} & \text{for } j = 1 \\ - \frac{\delta u'_i \cdot \delta u'_j}{\delta u'_i \cdot \delta u'_i + \delta \alpha'_i} & \text{for } j \geq 2 \end{cases} \] (26)

4. Numerical Investigation

4.1. Example formulation

The system is composed of bars made of the same material and had the same geometrical properties (system is shown in Fig. 2), having nonlinear multi freedom and multi node constrain. The geometric parameters, material parameters and loading parameters are given

\[ E = 2.10^8 \text{ (kN/cm)}^2, A = 10 \text{ cm}^2, L = 300 \text{ cm}, H_1 = 300 \text{ cm}, H_2 = 100 \text{ cm}, \beta = 1. \]

Equation of nonlinear multi freedom and multi node constrains is expressed as

Multi node constraint \[ g_1(u) = (L + u_5 - u_3)^2 + (H_2 + u_6 - u_4)^2 - (L^2 + H_2^2) = 0 \]

Multi freedom constraint \[ g_2(u) = u_2 + \beta u_1^2 = 0 \]

For investigating the convergence speed of the proposed method, the problem was solve with different options of converged tolerance.

![Figure 2. Examined system having multi freedom and multi node constraint](image)

For solving nonlinear modified system, the calculation program for static finite element analysis of system is written in MathCAD software. The incremental-iterative algorithm employing for programming process is shown in Fig. 3.
Figure 3. Incremental-iterative procedure for solving nonlinear buckling problem based on arc length technique

4.2. Numerical results
The calculating results are nodal displacements and internal forces. The load-nodal displacement and load-internal force relationships are shown in Fig. 4&5. The calculation results show fast converged speed in all cases of selecting converged tolerance.

Figure 4. Relationship diagram P-u
5. Conclusions

The presented method of treating the nonlinear multi freedom constraints is effective in finite element analysis. The Lagrange multipliers method has the advantage of possibility of substitution and works particularly well for nonlinear constraints, giving exact results in contrast to the penalty method. The prososed method can be employed in extended field of nonlinear constraint problems.

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