Microscopic description of the elusive Hoyle state

Alison Dreyfuss¹, Kristina D Launey², Tomáš Dytrych², Jerry P Drayer², and Chairul Bahri³

¹Keene State College, Keene, NH 03435, USA
²Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA
³Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556-5670, USA

Abstract. Within a symmetry-guided shell-model framework and using a fraction of the model space extended beyond current no-core shell-model limits along with a schematic effective many-nucleon interaction, we gain new insights into the many-body dynamics that give rise to the ground state and low-lying 0⁺ states of ¹²C and ¹⁶O. In particular, we gain further understanding of the alpha-clustering nature of the challenging Hoyle state and its first 2⁺ excitation in ¹²C, as well as the corresponding states in ¹⁶O. This provides guidance for ab initio shell models by informing key features of the underlying nuclear structure and interaction.

1. Introduction

¹²C is produced in hot stars through a series of nuclear fusion reactions called the triple-alpha process. The triple-alpha process begins with the statistically unlikely collision and subsequent fusion of two alpha particles into a highly unstable ⁸Be nucleus, which, in another highly improbable event, then fuses with a third alpha particle through collision into a ¹²C nucleus. In order for the ¹²C nucleus formed during this process to reach a stable state, there must exist a low-lying, spin-zero, even-parity state of ¹²C, as postulated by Hoyle [1]. The Hoyle state and its first 2⁺ excitation remain a difficult challenge for no-core shell model studies of ¹²C.

2. NCSpM basis

The fully microscopic no-core symplectic shell model (NCSpM) uses a symplectic Sp(3, R) basis and Sp(3, R)-preserving interactions. When applied to the full model space, the model coincides exactly with the NCSM for a given $N_{\text{max}}$ cutoff, which is the maximum allowed harmonic oscillator (HO) excitations above the lowest energy configuration for a given nucleus. The symplectic irreps of NCSpM divide the basis space into ‘vertical slices’, each of which is comprised of basis states of a definite $(\lambda \mu)$ deformation. By considering only those symplectic configurations of all Sp(3, R) irreps in the $N_{\text{max}}$ model space that contribute significantly to the shape of the nucleus in question, we can greatly reduce the model space. With this reduction in model space, the NCSpM is able to provide shell model calculations beyond current NCSM limits; namely, up through $N_{\text{max}} = 20$ for ¹²C and $N_{\text{max}} = 16$ for ¹⁶O, the model spaces we found sufficient for the convergence of results.
3. NCSpM interaction

We use a very simple and elegant Hamiltonian with an effective interaction derived from the long-range expansion of the two-body central nuclear force:

$$H_{\text{eff}} = H_0 - \frac{\chi}{2 \gamma} \left( e^{\gamma(Q.Q-(Q.Q)_n)} - 1 \right).$$  \hspace{1cm} (1)

This includes both the spherical HO potential, which, together with the kinetic energy yields the HO Hamiltonian $H_0$, and the $Q.Q$ quadrupole-quadrupole interaction not restricted to a single shell. We remove the average contribution, $(Q.Q)_n$, of $Q.Q$ within a subspace of $n$ HO excitations [2], which helps eliminate the considerable renormalization of the zero-point energy, while retaining the $Q.Q$-driven behavior of the wavefunctions.

In its zeroth-order approximation (for parameter $\gamma \to 0$) for a valence shell, this Hamiltonian simplifies to the established Elliott model [3, 4, 5]. The coupling constant $\chi$ is taken to be proportional to $\hbar \Omega$, and to decrease, at leading order, with the total number of HO excitations, as shown by Rowe [6]. Hence, in the Hamiltonian (1), $\chi$ is fixed by the value of $\hbar \Omega$ (not varied) and the eigenstates are thus $\hbar \Omega$-independent.

Table 1. Experimental and theoretical NCSpM $^{12}$C and $^{16}$O energy spectra ($E_{\text{exp}}$ and $E_{\text{theo}}$ in MeV, respectively). The NCSpM calculations are performed for $^{12}$C for $\gamma = -1.71 \times 10^{-4}$ and three Sp(3, $\mathbb{R}$) irreps, $0\hbar \Omega \ 0p-0h \ (0 \ 4)$, $2\hbar \Omega \ 2p-2h \ (6 \ 2)$, and $4\hbar \Omega \ 4p-4h \ (12 \ 0)$, in the $N_{\text{max}} = 20$ model space; and for $^{16}$O for $\gamma = -0.50 \times 10^{-4}$ and three Sp(3, $\mathbb{R}$) irreps, $0\hbar \Omega \ 0p-0h \ (0 \ 0)$, $2\hbar \Omega \ (4 \ 2)$, and $4\hbar \Omega \ (8 \ 4)$, in the $N_{\text{max}} = 16$ model space. Experimental energies for all states are taken from Ref. [7] unless otherwise specified.

| Sp(3, $\mathbb{R}$) irrep | $J^\pi$ | $E_{\text{theo}}$, MeV | $E_{\text{exp}}$, MeV |
|---------------------------|---------|------------------------|------------------------|
| $^{12}$C                  |         |                        |                        |
| (0 4)                     | 0$^+$   | 0.0                    | 0.0                    |
|                           | 2$^+$   | 4.44                   | 4.44 5                 |
|                           | 4$^+$   | 14.74                  | 14.08                  |
| (12 0)                    | 0$^+$   | 7.49                   | 7.65                   |
|                           | 2$^+$   | 9.90                   | 9.93$^{[8]}$           |
| (6 2)                     | 0$^+$   | 9.00                   | 9.00$^{[8]}$           |
| $^{16}$O                  |         |                        |                        |
| (0 0)                     | 0$^+$   | 0.0                    | 0.0                    |
| (8 4)                     | 0$^+$   | 4.66                   | 6.05                   |
|                           | 2$^+$   | 6.38                   | 6.92                   |
|                           | 4$^+$   | 10.68                  | 10.36                  |
| (4 2)                     | 0$^+$   | 12.29                  | 12.05                  |

4. Results

Table 1 shows the NCSpM results in considerable agreement with the experimental values for low-lying states in both $^{12}$C and $^{16}$O. NCSpM results were calculated for $^{12}$C using $\hbar \Omega = 18$ MeV given by the empirical estimate $\approx 41/A^{1/3} = 17.9$ MeV and $N_{\text{max}} = 20$, which we found sufficient to yield convergence. The space was further reduced by selecting only the most relevant symplectic irreps: the spin-zero ($S = 0$) $0\hbar \Omega \ 0p-0h \ (0 \ 4)$, $2\hbar \Omega \ 2p-2h \ (6 \ 2)$, and $4\hbar \Omega \ 4p-4h \ (12 \ 0)$.
symplectic bandheads, and all multiples thereof up through $N_{\text{max}} = 20$. Note that the NCSpM energies for $^{12}$C given in Table 1 are rescaled by an overall factor of $\sim 2$, which was found by fixing the lowest $2^+$ by its experimental value. This scaling factor has no implications on the underlying physics, as an overall factor for $H$ does not change its properties and eigenstates, nor any associated observables.

The low-lying energy spectrum and eigenstates for $^{16}$O were also calculated using the NCSpM with $\hbar \Omega = 16$ MeV (following the $41/A^{1/3}$ estimation). The $N_{\text{max}} = 16$ model space used is further reduced by selecting the spin-zero ($S = 0$) $0\hbar \Omega$ (00), $2\hbar \Omega$ (42), and $4\hbar \Omega$ (84) symplectic bandheads, and all multiples thereof up through $N_{\text{max}} = 16$. The values for $^{16}$O in Table 1 are not rescaled.

These results indicate that the lowest $0^+$, $2^+$, and $4^+$ states of the $0\hbar \Omega$ 0p-0h (04) symplectic irrep calculated for $\gamma = -1.71 \times 10^{-4}$ in $^{12}$C closely reproduce the $g.s.t.$ rotational band, the lowest $0^+$ states of the $4\hbar \Omega$ 4p-4h (120) slice is found to lie close to the Hoyle state, and the lowest $0^+$ state of the $2\hbar \Omega$ 2p-2h (62) slice lies close to the 10-MeV $0^+$ resonance (third $0^+$ state). Results for $^{16}$O indicate that, relative to the lowest $0^+$ state of the $0\hbar \Omega$ (00), the $4\hbar \Omega$ (84) slice reproduces a $0^+\text{-}2^+\text{-}4^+$ rotational band over the second $0^+$ state in $^{16}$O, while the lowest $0^+$ state of the $2\hbar \Omega$ (42) is found to lie very close to the third $0^+$ state.

Table 2. Probability distribution for $^{12}$C as a function of the $n$ total excitations for the lowest $0^+$ and $2^+$ states as calculated by the NCSpM (independent of $\hbar \Omega$) and the SA-NCSM with the bare JISP16 interaction for $\hbar \Omega=20$ MeV and $N_{\text{max}} = 8$.

| $n$ | NCSpM | SA-NCSM (with bare JISP16) |
|-----|-------|-----------------------------|
|     |       |                             |
| $J = 0^+$ |       |                             |
| 0   | 64.49 | 69.25                       |
| 2   | 18.80 | 17.24                       |
| 4   | 11.11 | 9.37                        |
| 6   | 3.69  | 2.93                        |
| 8   | 1.31  | 0.84                        |
| $J = 2^+$ |       |                             |
| 0   | 64.82 | 68.17                       |
| 2   | 0.68  | 17.88                       |
| 4   | 10.90 | 9.59                        |
| 6   | 3.68  | 2.79                        |
| 8   | 1.09  | 0.76                        |

A comparison of the probability distributions for the $g.s.t.$ rotational band as calculated using the NCSpM with $ab\text{ initio}$ results when only configurations of zero proton and neutron spins ($S_{p,n} = 0$) are selected (for $0^+_g.s.t.$ and $2^+_g.s.t.$ for $^{12}$C, see Table 2) shows encouraging similarities. In particular, we compare NCSpM eigenstates, which are $\hbar \Omega$-independent, to SA-NCSM calculations [9, 10] with the bare JISP16 realistic interaction for $\hbar \Omega = 20$ MeV (around the minimum of the calculated binding energy for $^{12}$C) and a $N_{\text{max}} = 8$ model space, which appears to be sufficient for convergence for the $^{12}$C $g.s.t.$ rotational band for both models. This agreement indicates that the schematic interaction used in NCSpM captures most of the underlying physics in the realistic interaction for low-energy nuclear dynamics.

The model also successfully reproduces a number of other observables in both $^{12}$C and $^{16}$O, including, for example, mass rms radii, electric quadrupole moments and $B(E2)$ transition
strengths, all of which give additional information about the structure of the nuclei. For example, for $^{12}\text{C}$, the $B(E2; 2^+_1 \rightarrow 0^+_{g.s.})$ is calculated to be 5.12 W.u., close to the 4.65 W.u. experimental measurement.

While the model includes an adjustable parameter, $\gamma$, this parameter only controls the decrease rate of the $Q.Q$ interaction toward higher-lying shells. The many-body apparatus itself is fully microscopic and no adjustments are possible. Hence, as $\gamma$ varies, there is only a small window of possible $\gamma$ values that, for large enough $N_{\text{max}}$, closely reproduces the relative positions of the three lowest $0^+$ states (see Fig. 1 for $^{12}\text{C}$). $B(E2)$ transition strengths are also found to be sensitive to the $\gamma$ values, as shown in Fig. 1 for $B(E2; 2^+_1 \rightarrow 0^+_{g.s.})$, and the small $\gamma$ window indeed corresponds to $B(E2)$ values that are in agreement with experiment. In addition, Figure 1 shows the dependence of the energy (not rescaled) of the $2^+$ and $4^+$ states of $(0 4)$, the $0^+$ state of $(12 0)$ (labeled as “0$^+_2$”), and the $0^+$ state of $(6 2)$ (labeled as “0$^+_3””), as well as of the $B(E2; 2^+_1 \rightarrow 0^+_{g.s.})$ transition strength in W.u. (insert), on $\gamma$ for $^{12}\text{C}$.

5. Conclusion
Calculations using the symplectic no-core shell model within a reduced model space and employing a schematic effective many-nucleon interaction were presented. We showed that this symmetry-guided framework is not limited to a specific nucleus but effectively describes low-lying eigenstates, e.g., of $^{12}\text{C}$ and $^{16}\text{O}$. This model can be used as a guide to help $\textit{ab initio}$
models move toward heavier nuclear systems, and points to important underlying collective and cluster phenomena in nuclei.

Acknowledgments
This work was supported by the U.S. NSF (0904874), the U.S. DOE (DE-SC0005248 & FG02-95ER-40934), and SURA. ACD further acknowledges support by the U.S. NSF (1004822) through the REU Site in the Department of Physics & Astronomy at LSU. We also acknowledge DOE/NERSC and LSU/LONI for providing HPC resources.

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