Weighted Radon transforms for which Chang’s approximate inversion formula is exact

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We consider the weighted ray transforms $P_W$ defined by the formula

$$P_W f(s, \theta) = \int_{\mathbb{R}} W(s\theta^\perp + t\theta, \theta) f(s\theta^\perp + t\theta) \, dt, \quad s \in \mathbb{R}, \quad \theta = (\theta_1, \theta_2) \in \mathbb{S}^1,$$  \hspace{1cm} (1)

where $\theta^\perp = (-\theta_2, \theta_1)$, $W = W(x, \theta)$ is the weight, and $f = f(x)$ is a test function, $x \in \mathbb{R}^2$. Up to change of variables, $P_W$ is also known as the weighted Radon transform on the plane.

We recall that in definition (1) the product $\mathbb{R} \times \mathbb{S}^1$ is interpreted as the set of all oriented straight lines in $\mathbb{R}^2$. If $\gamma = (s, \theta) \in \mathbb{R} \times \mathbb{S}^1$, then $\gamma = \{ x \in \mathbb{R}^2 : x = s\theta^\perp + t\theta, \ t \in \mathbb{R} \}$ (modulo orientation) and $\theta$ gives the orientation of $\gamma$.

We assume that $W$ is complex-valued, $W \in C(\mathbb{R}^2 \times \mathbb{S}^1) \cap L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$,

$$w_0(x) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{S}^1} W(x, \theta) \, d\theta \neq 0, \quad x \in \mathbb{R}^2,$$  \hspace{1cm} (2)

where $d\theta$ is the standard element of arc length on $\mathbb{S}^1$.

If $W \equiv 1$, then $P_W$ is known as the classical ray (or Radon) transform on the plane. If $W(x, \theta) = \exp(-Da(x, \theta))$, $Da(x, \theta) = \int_0^{+\infty} a(x + t\theta) \, dt$, \hspace{1cm} (3)

where $a$ is a complex-valued sufficiently regular function on $\mathbb{R}^2$ which decays sufficiently quickly at infinity, then $P_W$ is known as the attenuated ray (or Radon) transform.

The classical Radon transform arises, in particular, in X-ray transmission tomography. The attenuated Radon transform (at least, with $a \geq 0$) arises, in particular, in single-photon emission computed tomography (SPECT). Certain other weights $W$ also arise in applications. For more information in this connection see, for example, [1] and [2].

Exact and simultaneously explicit inversion formulae for the classical and attenuated Radon transforms were given for the first time in [3] and [4], respectively. For certain other weights $W$, exact and simultaneously explicit inversion formulae were given in [5] and [6].

On the other hand, the following Chang approximate inversion formula for $P_W$, where $W$ is given by (3) with $a \geq 0$, has been used for a long time (see [7], [8], [2]):

$$f_{\text{appr}}(x) = \frac{1}{4\pi w_0(x)} \int_{\mathbb{S}^1} h'(x\theta^\perp, \theta) \, d\theta, \quad h'(s, \theta) = \frac{d}{ds} h(s, \theta),$$  \hspace{1cm} (4)

$$h(s, \theta) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{P_W f(t, \theta)}{s - t} \, dt, \quad s \in \mathbb{R}, \quad \theta \in \mathbb{S}^1, \quad x \in \mathbb{R}^2,$$

where $w_0$ is defined in (2). It is known that formula (4) is efficient as the first approximation in SPECT reconstructions and, in particular, is sufficiently stable with respect to strong Poisson noise in SPECT data.

The results of the present note consist of Theorem 1 and Remark 1.

Let $C_0(\mathbb{R}^2)$ denote the space of continuous compactly supported functions on $\mathbb{R}^2$. 

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Theorem 1. Let the assumptions (2) hold and let \( f_{\text{appr}}(x) \) be given by (4). Then

\[
f_{\text{appr}} = f \quad \text{(in the sense of distributions)} \quad \text{on } \mathbb{R}^2 \text{ for all } f \in C_0(\mathbb{R}^2),
\]

if and only if

\[
W(x, \theta) - w_0(x) \equiv w_0(x) - W(x, -\theta), \quad x \in \mathbb{R}^2, \quad \theta \in S^1. \tag{6}
\]

A scheme of the proof of Theorem 1 is given in [9] and makes use, in particular, of the following formula:

\[
\frac{1}{2} \left( P_W f(s, \theta) + P_W f(-s, -\theta) \right) = P_{W_{\text{sym}}} f(s, \theta), \quad (s, \theta) \in \mathbb{R} \times S^1,
\]

\[
W_{\text{sym}}(x, \theta) = \frac{1}{2} \left( W(x, \theta) + W(x, -\theta) \right), \quad x \in \mathbb{R}^2, \quad \theta \in S^1. \tag{7}
\]

The statement that, under the assumptions of Theorem 1, property (6) implies the identity (5) can also be deduced from considerations developed in [2].

Using the fact that \( W_{\text{sym}} \equiv w_0 \) under condition (6), we also obtain the following.

Remark 1. Let conditions (2) and (6) be fulfilled and let \( f \in C_0(\mathbb{R}^2) \). Then:

(A) \( P_W f \) on \( \Omega(D) \) uniquely determines \( f \) (or, more precisely, \( w_0 f \)) on \( \mathbb{R}^2 \setminus D \) via (7) and the Cormack inversion from \( P_{W_{\text{sym}}} f \) on \( \Omega(D) \), where \( D \) is a convex compact set in \( \mathbb{R}^2 \) and \( \Omega(D) \) denotes the set of all straight lines in \( \mathbb{R}^2 \) which do not intersect \( D \);

(B) \( P_W f \) on \( \mathbb{R} \times (S_+ \cup S_-) \) uniquely determines \( f \) on \( \mathbb{R}^2 \) via (7) and standard inversion from the limited angle data \( P_{W_{\text{sym}}} f \) on \( \mathbb{R} \times S_+ \), where \( S_+ \) is an arbitrary non-empty open connected subset of \( S^1 \), and \( S_- = -S_+ \).

For the case when \( W \) is given by formulae (3) under the additional conditions that \( a \geq 0 \) and \( \text{supp } w \subset D \), where \( D \) is some known bounded domain which is not too large, and for \( f \in C(\mathbb{R}^2) \), \( f \geq 0 \), \( \text{supp } f \subset D \), the transform \( P_W f \) is relatively well approximated by \( P_{W_{\text{appr}}} f \), where \( W_{\text{appr}}(x, \theta) = w_0(x) + (W(x, \theta) - W(x, -\theta))/2 \). In addition, this \( W_{\text{appr}} \) already satisfies (6). This explains the efficiency of formula (4) as the first approximation in SPECT reconstructions (at the level of integral geometry).

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