ON THE CATENARITY OF VIRTUALLY NILPOTENT MOD-\(p\) IWASAWA ALGEBRAS

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ABSTRACT

Let \(p > 2\) be a prime, \(k\) a finite field of characteristic \(p\), and \(G\) a nilpotent-by-finite compact \(p\)-adic analytic group. Write \(kG\) for the completed group ring of \(G\) over \(k\). We show that \(kG\) is a catenary ring.

Introduction

Fix a prime \(p\), a commutative pseudocompact ring \(k\) (e.g., \(\mathbb{F}_p\) or \(\mathbb{Z}_p\)) and a compact \(p\)-adic analytic group \(G\). (Such groups are perhaps most accessibly characterised as those groups \(G\) which are isomorphic to a closed subgroup of \(GL_n(\mathbb{Z}_p)\) for some \(n\).) The completed group ring \(kG\) (sometimes written \(k[[G]]\)) is defined by

\[
kG := \lim_{\rightarrow} k[G/N],
\]

where the inverse limit ranges over all open normal subgroups \(N\) of \(G\), and \(k[G/N]\) denotes the ordinary group algebra of the (finite) group \(G/N\) over \(k\). This ring satisfies an obvious universal property [24 Lemma 2.2], and modules over it characterise continuous \(k\)-representations of \(G\) (which has the profinite topology). When \(k = \mathbb{F}_p, \mathbb{Z}_p\) or related rings, this is often called the Iwasawa algebra of \(G\).

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Iwasawa algebras (and related objects, such as locally analytic distribution algebras [21]) have recently become a very active research area due to their number-theoretic interest, for instance in the $p$-adic Langlands programme: see [20], for example. They are also interesting objects of study in their own right, as an interesting class of noetherian rings: see [3] for a 2006 survey of what is known about these rings.

Our main result is the following.

**Theorem A:** Take $p > 2$. Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group, and let $k$ be a finite field of characteristic $p$. Then $kG$ is a catenary ring.

Recall that a ring $R$ is said to be catenary if any two maximal chains of prime ideals with common endpoints have the same length, i.e., whenever

$$P = P_1 \prec P_2 \prec \cdots \prec P_r = P',$$
$$P = Q_1 \prec Q_2 \prec \cdots \prec Q_s = P'$$

are two chains of prime ideals of $R$ which cannot be refined further (i.e., by adding an extra prime ideal $P_i \prec I \prec P_{i+1}$ or $Q_i \prec I \prec Q_{i+1}$), we have that $r = s$. This is a “well-behavedness” condition on the classical Krull dimension of $kG$: it says that, whenever $P \preceq P'$ are adjacent prime ideals and the height $h(P)$ of $P$ is finite, then we have

$$h(P') = h(P) + 1.$$  

This result goes some way towards redressing the long-standing gap between Iwasawa algebras and similar algebraic objects; for instance, similar catenarity results had already been established for classical group rings of virtually polycyclic groups (in a special case in [19], in full generality in [14]), for universal enveloping algebras of finite-dimensional soluble Lie algebras over $\mathbb{C}$ [6]; for quantised coordinate rings over $\mathbb{C}$ [8], and over more general fields in [26]; for $q$-commutative power series rings [11]; and so on.

In proving this result, we crucially use the prime extension theorem, [23, Theorem A]. Before we can state this, we need to recall a few concepts.

Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group. Then [25, Theorem C] there exists a unique maximal subgroup $H$ of $G$ with the property that $H$ is an open normal subgroup of $G$, $H$ contains a finite normal subgroup $F$, and $H/F$ is nilpotent $p$-valuable. This $H$ is called the **finite-by-(nilpotent $p$-valuable)** radical of $G$, and is written $\text{FN}_p(G)$. 
Given a prime ideal $P$ of $kG$, as in [2, §1.3] and [23, Introduction], we will write $P^\dagger$ for the kernel of the natural composite map

$$G \to (kG)^\times \to (kG/P)^\times.$$ 

We say that $P$ is **faithful** if $P^\dagger = 1$, and $P$ is **almost faithful** if $P^\dagger$ is finite.

We now state the prime extension theorem:

**Theorem (23, Theorem A):** Take $p > 2$. Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group, and let $k$ be a finite field. Write $H = \text{FN}_p(G)$. If $P$ is an almost faithful prime ideal of $kH$, then the ideal $P^G$ is a prime ideal of $kG$.

We will use this result to generalise Ardakov’s analogue of Zalesskii’s theorem [2, Theorem 8.6] to our current context.

Recall [25, Definition 1.4, Lemma 1.10] that a group $G$ is **orbitally sound** if, whenever $H$ is a subgroup of $G$ with finitely many $G$-conjugates and $H^o$ is the largest normal subgroup of $G$ contained in $H$, we have $[H : H^o] < \infty$; and recall that nilpotent $p$-valuable groups are indeed orbitally sound [2, Proposition 5.9].

As in [25], we will write throughout this paper

$$\Delta(G) = \{x \in G \mid [G : C_G(x)] < \infty \},$$

$$\Delta^+(G) = \{x \in \Delta \mid o(x) < \infty \},$$

where $o(x)$ denotes the order of $x$. We will also often simply write $\Delta$ and $\Delta^+$ to denote $\Delta(G)$ and $\Delta^+(G)$. For the basic properties of these (closed, characteristic) subgroups, see [25, Lemma 1.3 and Theorem D].

Following Roseblade [19], we say that a prime ideal $P$ of $kG$ is **controlled** by the normal subgroup $H$ of $G$ if the right ideal $(P \cap kH)kG$ is equal to $P$.

**Theorem B:** Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group. Suppose $P$ is an almost faithful prime ideal of $kG$. Then $P$ is controlled by $\Delta$.

The analogous classical result, for group algebras of polycyclic-by-finite groups, was proved by Roseblade [19, Corollary H3]. Taken together with the results of [1] and [24], this also gives a precise partial answer to a question of Ardakov and Brown [3, Question G]: when $G$ is as in Theorem B, this completely describes the prime ideals of $kG$ in terms of closed normal subgroups, central elements and prime ideals of (classical) group algebras of finite groups over $k$. 
This is all we need to deduce that $kG$ is catenary when $G$ is nilpotent-by-finite and orbitally sound—see Theorem 3.12. In order to pass from orbitally sound to general nilpotent-by-finite groups, we partly develop the theory of vertices and sources along the lines of [12], [13].

**Theorem C:** Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group, $P$ a prime ideal of $kG$, $H$ an orbitally sound open normal subgroup of $G$, and $Q$ a minimal prime ideal of $kH$ above $P \cap kH$. Write $N$ for the $G$-isolator [25, Definition 1.6] of $Q^\dagger$, and write $\nabla$ for the subgroup of $G$ containing $N$ defined by

$$\nabla/N = \Delta(G/N).$$

Then $P$ is induced from an ideal $L$ of $k\nabla$.

For the precise meaning of induced here, see §1.2.

Theorem A then follows from Theorems B and C by adapting an argument from [14], as follows. Let $G$ be a (not necessarily orbitally sound) nilpotent-by-finite compact $p$-adic analytic group, $k$ a finite field of characteristic $p > 2$, and $P$ a faithful prime ideal of $kG$. We already know [25, Theorem A] that $G$ contains an open normal orbitally sound subgroup, which we denote $\text{nio}(G)$. From Theorem C, we may deduce Corollary 3.20 that $P$ is induced from some proper open subgroup $H$ of $G$ containing $\text{nio}(G)$. If $H = \text{nio}(G)$, then we can deduce from Theorem B (as above) that $kH$ is catenary, and now by Lemma 3.22 we are done. In general, $\text{nio}(G) \leq H < G$, and we may not have equality: but it is easy to see that

$$[H : \text{nio}(H)] < [G : \text{nio}(G)] < \infty,$$

and Corollary 3.23 establishes Theorem A by induction on the index $[G : \text{nio}(G)]$.

1. Heights of primes and Krull dimension

1.1. Prime and $G$-prime ideals.

**Definition 1.1:** Let $G$ be a compact $p$-adic analytic group [5, Definition 8.14]. Suppose the group $G$ acts (continuously) on the ring $R$, and that the ideal $I \triangleleft R$ is $G$-stable. Then, following [17, §14], we will say that $I$ is $G$-prime if, whenever $A, B \triangleleft R$ are $G$-stable ideals and $AB \subseteq I$, then either $A \subseteq I$ or $B \subseteq I$. 
Lemma 1.2: Let $G$ be a compact $p$-adic analytic group and $H$ a closed normal subgroup.

(i) If $P$ is a prime ideal of $kG$, then $P \cap kH$ is a $G$-prime ideal of $kH$. If $H$ is open in $G$, then $P$ is a minimal prime ideal above $(P \cap kH)kG$.

(ii) Let $Q$ be a $G$-prime ideal of $kH$, and $P$ any minimal prime of $kH$ above $Q$. Then

$$Q = \bigcap_{x \in G} P^x.$$ 

Furthermore, the set of minimal primes of $kH$ above $Q$ is

$$\{P^x | x \in G\}.$$

Proof.

(1) The former statement follows from [17, Lemma 14.1(i)], and the latter from [17, Theorem 16.2(i)].

(2) This follows from [17, Lemma 14.2(i)(ii)].

Definition 1.3: Let $P$ be a prime ideal of a ring $R$. Then we define the height of $P$ to be the greatest integer $h(P) := r$ for which there exists a (finite) chain

$$P_0 \leq P_1 \leq \cdots \leq P_r = P$$

of prime ideals in $R$ (or $\infty$ if no such longest finite chain exists).

Suppose instead that the group $G$ acts on $R$ by automorphisms, and $P$ is a $G$-prime ideal of $R$. Then the $G$-height of $P$ is the greatest integer $h_G(P) := r$ for which there exists a chain $\langle 1 \rangle$ of $G$-prime ideals in $R$ (or $\infty$).

Finally, suppose that the group $G$ acts on $R$ by automorphisms, and $P$ is a $G$-orbital prime ideal of $R$ (i.e., a prime ideal of $R$ with finite orbit under the action of $G$). Then the $G$-orbital height of $P$ is the greatest integer $h_G^{\text{orb}}(P) := r$ for which there exists a chain $\langle 1 \rangle$ of $G$-orbital prime ideals in $R$ (or $\infty$).

We note the following consequence of the correspondence of Lemma 1.2.

Corollary 1.4: Let $G$ be a compact $p$-adic analytic group and $H$ an open normal subgroup. Take $P$ a prime ideal of $kG$, and let $Q$ be a minimal prime of $kH$ above $P \cap kH$. Then

$$h(P) = h_G(P \cap kH) = h(Q).$$
Proof. As $H$ is open in $G$, we may regard $kG$ as a crossed product of $kH$ by the finite group $G/H$. Hence the claim that $h(P) = h(Q)$ follows from \cite[Corollary 16.8]{17}.

Next, given any two $G$-prime ideals $I_1 \leq I_2$ of $kH$, we may find prime ideals $K_1 \leq K_2$ of $kH$ such that $I_i = \bigcap_{x \in G} K_i^x$ (for $i = 1, 2$) by Lemma 1.2(ii). Now, by taking a longest chain of $G$-prime ideals of $kH$ whose greatest member is $P \cap kH$, we get a chain of prime ideals of $kH$, say with greatest member $K$, some minimal prime of $kH$ above $P \cap kH$. This shows that $h_G(P \cap kH) \leq h(K)$. But now Lemma 1.2(ii) tells us that $Q$ is also a minimal prime ideal above $P \cap kH$, and hence $Q$ and $K$ are $G$-conjugate, so that $h(K) = h(Q)$.

Finally, suppose we are given two prime ideals $J_1 \leq J_2$ of $kH$, whose $G$-orbits are $O_1$ and $O_2$ respectively. Then it is easy to see that the ideals $I_1 = \bigcap O_1$ and $I_2 = \bigcap O_2$ are $G$-prime and satisfy $I_1 \leq I_2$, and that $J_i$ is a minimal prime above $I_i$ for $i = 1, 2$. But we must have $I_1 \neq I_2$: otherwise, both $J_1$ and $J_2$ are minimal primes above $I_1$, a clear contradiction. Now note that $\bigcap_{x \in G} Q^x = P \cap kH$ by Lemma 1.2(ii): so by taking a longest chain of prime ideals of $kH$ whose greatest member is $Q$, and passing to a chain of $G$-prime ideals of $kH$ in this way, we have shown that $h(Q) \leq h_G(P \cap kH)$.

1.2. Inducing ideals.

Definition 1.5: Let $H$ be an open (not necessarily normal) subgroup of $G$, and let $L$ be an ideal of $kH$. We define the induced ideal $L^G \triangleleft kG$ to be the largest (two-sided) ideal contained in the right ideal $LkG \triangleleft kG$. In other words, by \cite[2.1]{14}, $L^G$ is the annihilator of $kG/LkG$ as a right $kG$-module, or by \cite[Lemma 14.4(ii)]{17},

$$L^G = \bigcap_{g \in G} Lg kG.$$ 

Lemma 1.6: Induction of ideals is transitive: if $H$ and $K$ are open subgroups of $G$ with $H \leq K \leq G$, and $L \triangleleft kH$, then

$$L^G = (L^K)^G.$$ 

Proof. Let $N$ be an open normal subgroup of $G$ contained in $H$, and write $\overline{\cdot}$ to denote the quotient by $N$, so that we have $kG = kN \star \overline{G}$ with $\overline{H} \leq K \leq \overline{G}$, and we may view $L$ as an ideal of $kN \star \overline{H}$. The result now follows from \cite[Lemma 1.2(iii)]{13}.
1.3. KRULL DIMENSION. We recall some facts about Krull dimension, used here in the sense of Gabriel and Rentschler: see [7, §15].

**Definition 1.7:** Let \( 0 \neq M \) be an \( R \)-module, and fix some ordinal \( \alpha \). We define the following notation inductively:

- \( \text{Kdim}(M) = 0 \) if \( M \) is an Artinian module,
- \( \text{Kdim}(M) \leq \alpha \) if, for every descending chain
  \[
  M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots
  \]
  of submodules of \( M \), we have \( \text{Kdim}(M_i/M_{i+1}) < \alpha \) for all but finitely many \( i \).

Of course, if there exists some \( \alpha \) such that \( \text{Kdim}(M) \leq \alpha \), but we do not have \( \text{Kdim}(M) \leq \beta \) for any \( \beta < \alpha \), then we write \( \text{Kdim}(M) = \alpha \).

**Remark:** \( \text{Kdim}(M) \) is a measure of complexity of the poset of submodules of \( M \).

\( \text{Kdim}(M) \) may not be defined for some modules \( M \), that is, we may not have \( \text{Kdim}(M) \leq \alpha \) for any ordinal \( \alpha \). However, if \( M \) is a noetherian module, then \( \text{Kdim}(M) \) is defined [7, Lemma 15.3].

**Definition 1.8:** Suppose that \( \text{Kdim}(M) = \alpha \). We say that \( M \) is \( \alpha \)-**homogeneous** if \( \text{Kdim}(N) = \alpha \) for all nonzero submodules \( N \) of \( M \).

**Examples 1.9:**

(i) Nonzero Artinian modules are \( 0 \)-homogeneous.
(ii) Prime rings \( R \), as modules over themselves, are \( \alpha \)-homogeneous (where we set \( \alpha \) equal to \( \text{Kdim}(R_R) \)) [7, Exercise 15E].
(iii) The property of being \( \alpha \)-homogeneous is inherited by products [7, Corollary 15.2] and (nonzero) submodules (by definition).

We now cite and adapt some standard results on Krull dimension.

**Lemma 1.10:**

(i) [14, 1.4(ii)] Let the ring \( R \) be \( \alpha \)-homogeneous as a right \( R \)-module. If \( x \in R \) satisfies \( \text{Kdim}(R/xR) < \text{Kdim}(R) \), then \( x \) is a regular element of \( R \).

(ii) [10, Théorème 5.3] Suppose \( B \) is a finite normalising extension of \( A \), and let \( M \) be a \( B \)-module. Then \( \text{Kdim}(M_B) \) exists if and only if \( \text{Kdim}(M_A) \) does, and if so, then they are equal.
(iii) [7] Exercise 15R] If $R$ is a right noetherian subring of a ring $S$ such that $S$ is finitely generated as an $R$-module, and $M$ is a finitely generated $S$-module, then $\Kdim(M_S) \leq \Kdim(M_R)$. □

**Corollary 1.11:** Suppose $A \subseteq C \subseteq B$ are right noetherian rings, and $B$ is a finite normalising extension of $A$. Let $M$ be a finitely generated $B$-module. Then, if $\Kdim(M_B)$ exists, we have

$$\Kdim(M_A) = \Kdim(M_C) = \Kdim(M_B).$$

**Proof.** This follows immediately from Lemma [1.10(ii) and two applications of Lemma [1.10(iii)]. □

**Lemma 1.12:** Let $G$ be a compact $p$-adic analytic group, $H$ an open subgroup of $G$, and $k$ a field of characteristic $p$. Let $M$ be a finitely generated $kG$-module.

(i) $\Kdim(M_{kG}) = \Kdim(M_{kH}).$

(ii) Suppose that $M = WkG$ for some submodule $W$ of $M_{kH}$. Then we have $\Kdim(M_{kG}) = \Kdim(W)$.

(iii) $M_{kG}$ is $\alpha$-homogeneous if and only if $M_{kH}$ is $\alpha$-homogeneous.

**Proof (Adapted from [14, 1.4(iii)–(v)]).**

(i) Let $N$ be the (open) largest normal subgroup of $G$ contained in $H$, so that $kG$ is a finite normalising extension of $kN$. Now apply Corollary [1.11].

(ii) Let $N$ be as in (i). Then, by (i), it suffices to prove that $\Kdim(M_{kN})$ is equal to $\Kdim(W_{kN})$. But, as a $kN$-module, $M$ is a finite sum of modules $Wg$ for various $g \in G$, and these are all isomorphic, so in particular have isomorphic submodule lattices and therefore the same $\Kdim$.

(iii) It is clear from the definition that, if $M_{kH}$ is $\alpha$-homogeneous, then $M_{kG}$ is $\alpha$-homogeneous. Conversely, suppose that $M_{kG}$ is $\alpha$-homogeneous, and let $W$ be a nonzero submodule of $M_{kH}$. Then $WkG$ is a nonzero submodule of $M_{kG}$, so has Krull dimension $\alpha$ by assumption, and hence also $\Kdim(W) = \alpha$ by (ii). □

**Lemma 1.13:** Let $G$ be a finite group, $H$ a subgroup, and $R \ast G$ a fixed crossed product. Fix a semiprime ideal $I$ of $R \ast G$. If $R \ast G/I$ is $\alpha$-homogeneous, then $R \ast H/(I \cap R \ast H)$ is $\alpha$-homogeneous.
Proof (Adapted from [4, Lemma 4.2(i)]). Let $M$ be a nonzero right ideal of the ring $R*H/(I \cap R*H)$, and write $\beta = \text{Kdim}(M_{R*H})$. We wish to show that $\beta = \alpha$.

Now $M$ is a right module over both $R*H$ and $R$; and $R*G/I$ is a right module over both $R*G$ and $R$. As $R*G$ and $R*H$ are both finite normalising extensions of $R$, we may apply Lemma 1.10(ii) to both of these situations to see that

$$\beta = \text{Kdim}(M_{R*H}) = \text{Kdim}(M_R)$$

and

$$\alpha = \text{Kdim}((R*G/I)_{R*G}) = \text{Kdim}((R*G/I)_R).$$

Now, as right $R$-modules, we have

$$R*H/(I \cap R*H) \cong (R*H + I)/I \leq R*G/I,$$

and so $M$ is isomorphic to some nonzero $R$-submodule of $R*G/I$. In particular, this means that

$$\beta = \text{Kdim}(M_R) \leq \text{Kdim}((R*G/I)_R) = \alpha.$$

But now $(R*G/I)_R$ is $\alpha$-homogeneous by Corollary 1.11 so we must have $\beta = \alpha$. □

**Corollary 1.14:** Let $G$ be a compact $p$-adic analytic group, $H$ be an open subgroup of $G$, and $N$ the largest open normal subgroup of $G$ contained in $H$. Take $k$ to be a field of characteristic $p$, and let $Q$ be a prime ideal of $kH$, $I = Q^G \cap kN$, and $\alpha = \text{Kdim}(kH/Q)$. Then $kH/Q$, $kG/Q^G$, $kG/IkG$ are all $\alpha$-homogeneous rings.

**Proof.** As we observed in Example 1.9(ii), $kH/Q$ is already $\alpha$-homogeneous, as it is prime of Krull dimension $\alpha$.

We know from Definition 1.5 that the ideal $Q^G$ can be written as $\bigcap_{g \in G} Q^g kG$, and that this intersection can be taken to be finite. Hence, as a right $kG$-module, $kG/Q^G$ is isomorphic to a (nonzero) submodule of the direct product of the various (finitely many) $kG/Q^g kG$; and each $kG/Q^g kG$ is generated as a $kG$-module by $kH^g/Q^g$, which is ring-isomorphic to $kH/Q$. Hence

$$\text{Kdim}(kG/Q^G) = \text{Kdim}(kH/Q)$$

by Lemma 1.10(ii).
Finally, as
\[ Q^{G} = \bigcap_{g \in G} (QkG)^g, \]
we see that
\[ I = \bigcap_{g \in G} (QkG)^g \cap kN = \bigcap_{g \in G} (QkG \cap kN)^g = \bigcap_{g \in G} (Q \cap kN)^g, \]
and so, as above, \( kN/I \) is a (nonzero) subdirect product of the various \( kN/(Q \cap kN)^g \), which are all ring-isomorphic to \( kN/Q \cap kN \); now Lemma [1.13] implies that \( kN/Q \cap kN \) is \( \alpha \)-homogeneous, so \( kN/I \) is also, and \( kG/IkG \) is generated as a \( kG \)-module by \( kN/I \), so finally \( kG/IkG \) also inherits this property.

We borrow a result from the standard proof of Goldie’s theorem.

**Lemma 1.15** ([22, Lemma 3.13]): Suppose \( R \) is a semiprime ring, satisfying the ascending chain condition on right annihilators of elements, and which does not contain an infinite direct sum of nonzero right ideals. If \( I \) is an essential right ideal of \( R \) (i.e., a right ideal that has nonzero intersection \( I \cap J \) with each nonzero right ideal \( J \) of \( R \)), then \( I \) contains a regular element.

These hypotheses are satisfied when \( R \) is \( G \)-prime and noetherian, for example.

**Proposition 1.16**: With notation as in Corollary [1.14], suppose \( P \) is a prime ideal of \( kG \) containing \( Q^{G} \). If \( P \) is minimal over \( Q^{G} \), then
\[ h(P) = h(Q). \]

**Proof.** First, set \( I = Q^{G} \cap kN. \) This is a \( G \)-prime ideal contained in \( P \cap kN \).

Suppose for contradiction that the inclusion \( I \subseteq P \cap kN \) is strict.

First, we will show that \( P \cap kN/I \) is essential as a right ideal inside \( kN/I \). Indeed, the left annihilator \( L \) in \( kN/I \) of \( P \cap kN/I \) is a \( G \)-invariant ideal which annihilates the nonzero \( G \)-invariant ideal \( P \cap kN/I \), so we must have \( L = 0 \); and so, given any right ideal \( T \) of \( kN/I \) having zero intersection with \( P \cap kN/I \), as we must have \( T \leq L \), we conclude that \( T = 0 \).

Hence, by Lemma [1.15] we may find an element \( c \in P \cap kN \subseteq kN \) which is regular modulo \( I \). As \( kG/IkG \) is a free \( kN/I \)-module, \( c \) may also be considered
as an element of $P \subseteq kG$ which is regular modulo $IkG$. Hence

$$K\dim(kG/(Q^G + ckG))_{kG}$$

\begin{align*}
&\leq K\dim(kG/(IkG + ckG))_{kG} \quad \text{as } IkG + ckG \subseteq Q^G + ckG \\
&< K\dim(kG/IkG)_{kG} \quad \text{by Lemma 1.10(i)} \\
&= K\dim(kG/Q^G)_{kG} \quad \text{by Corollary 1.14}
\end{align*}

which, again by Lemma 1.10(i), shows that $c \in P$ is regular modulo $Q^G$.

However, we may now deduce from a reduced rank argument that $P$ cannot be minimal over $Q^G$, as follows. Write $\rho$ for the reduced rank [7, §11, Definition] of a right module over the semiprime noetherian (hence Goldie) ring $R = kG/Q^G$, and write $(\cdot)$ for images under the map $kG \to R$. Now, $c \in P$ implies $\overline{\tau R} \subseteq \overline{P}$, and so by [7, Lemma 11.3] we have

$$\rho(R/\overline{\tau R}) \geq \rho(R/\overline{P}) \geq 0.$$ 

Further, if $\overline{\tau}$ is a regular element of $R$, then $\overline{\tau R} \cong R$ as right $R$-modules, so $\rho(R/\overline{\tau R}) = 0$, again by [7, Lemma 11.3]. But now [7, Exercise 11C] implies that $\overline{P}$ cannot be a minimal prime of $R$.

This contradicts the assumption we made at the start of the proof, and so we have shown that

$$P \cap kN = Q^G \cap kN.$$ 

We observed during the proof of Corollary 1.14 that

$$Q^G \cap kN = \bigcap_{g \in G} (Q \cap kN)^g.$$ 

But $Q$ is a prime ideal of $kH$, so $Q \cap kN$ is an $H$-prime ideal of $kN$, so may be written as

$$Q \cap kN = \bigcap_{h \in H} Q^h_0$$

for some prime ideal $Q_0$ of $kN$. Combining these two shows that

$$P \cap kN = Q^G \cap kN = \bigcap_{g \in G} Q^g_0.$$ 

Now, by applying [17, corollary 16.8] to both $P \cap kN$ and $Q \cap kN$, we have that

$$h(P) = h(Q^G) = h(Q)$$

as required.  ■
2. Control theorem

2.1. The abelian case. We will require some facts about prime ideals in power series rings.

**Lemma 2.1:** Let $A$ be a free abelian pro-$p$ group of finite rank and $B$ a closed isolated (normal) subgroup. Take $k$ to be a field of characteristic $p$. Write $\text{Spec}^B(kA)$ for the set of primes of $kA$ that are controlled by $B$. Then the maps

$$\text{Spec}^B(kA) \leftrightarrow \text{Spec}(kB)$$

$$P \mapsto P \cap kB$$

$$QkA \leftrightarrow Q$$

are well-defined and mutual inverses, and preserve faithfulness.

**Proof.** If $P$ is a prime ideal of $kA$, then $P \cap kB$ is an $A$-prime ideal (and hence a prime ideal) of $kB$ by Lemma 1.2(i).

Conversely, note that, as $B$ is isolated in $A$, the quotient $A/B$ is again free abelian pro-$p$; so we may write $A = B \oplus C$, where the natural quotient map $A \to A/B$ induces an isomorphism $A/B \cong C$. Now, if $Q$ is a prime ideal of $kB$, then $kA/QkA = (kB/Q)[[C]]$ is a power series ring with coefficients in the commutative domain $kB/Q$, and is hence itself a domain.

It follows from [1, Lemma 5.1] that $QkA \cap kB = Q$, and by assumption, if $P$ is controlled by $B$ then we already have $(P \cap kB)kA = P$.

Now suppose the prime ideals $P \preceq Q$ of $kA$, and suppose $B$ controls $P$. Then, again viewing $A$ as $B \oplus C$, we may similarly consider $kA/P$ as the completed tensor product $[24]$ Definition 2.3 $kB/Q \hat{\otimes}_k kC$. Then the map $A \to (kA/P)^\times$ can be written as

$$B \oplus C \to (kB/Q)^\times \oplus (kC)^\times \lesssim (kB/Q \hat{\otimes}_k kC)^\times$$

$$(b, c) \mapsto ((b + Q), c),$$

so it is clear that $P$ is faithful if and only if $Q$ is faithful.  

**Lemma 2.2:** Let $A$, $B$, $k$ be as in Lemma 2.1. Take two neighbouring prime ideals $P \preceq Q$ of $kA$, and suppose $B$ controls $P$. Then

(i) $h(P) + \dim(A/P) = r(A)$,

(ii) $h(Q) = h(P) + 1$,

(iii) $h(P) = h(P \cap kB)$. 


Proof.

(i) This follows from [27, Ch. VII, §10, Corollary 1].

(ii) This follows from [27, Ch. VII, §10, Corollary 2].

(iii) Under the correspondence of Lemma 2.1, any saturated chain of prime ideals \(0 = Q_0 \preceq Q_1 \preceq \cdots \preceq Q_n = P \cap kB\) of \(kB\) extends to a chain of prime ideals \(0 = P_0 \preceq P_1 \preceq \cdots \preceq P_n = P\) of \(kA\). As any two saturated chains of prime ideals in \(kA\) have the same length [27, Ch. VII, §10, Theorem 34 and Corollary 1], we need only check that this chain is saturated.

Take two adjacent prime ideals \(I_1 \preceq I_2\) of \(kB\), so that \(h(I_2) = h(I_1) + 1\) [27, Ch. VII, §10, Corollary 2] and \(I_1 kA \leq I_2 kA\) are prime. We will show that \(I_1 kA\) and \(I_2 kA\) are adjacent by showing that their heights also differ by 1. By performing induction on \(r(A/B)\), it will suffice to prove this for the case \(r(A/B) = 1\), i.e., \(kA = kB[[X]]\).

It is clear that, when \(R\) is a commutative ring,
\[
\dim(R[[X]]) \geq 1 + \dim(R)
\]
(where \(\dim\) denotes the classical Krull dimension). But, giving \(R[[X]]\) the \((X)\)-adic filtration, we see that
\[
\text{gr}(R[[X]]) \cong R[x].
\]

By [16, 6.5.6], we have
\[
\dim(R[[X]]) \leq \dim(\text{gr}(R[[X]])) = \dim(R[x]) = 1 + \dim(R),
\]
where this last equality follows from [16, 6.5.4(i)].

Hence, for any prime ideal \(I\), we have
\[
\dim(kA/IkA) - \dim(kB/I) = \dim((kB/I)[[X]]) - \dim(kB/I) = 1.
\]
But, from (i), we see that
\[
\dim(kA/IkA) = r(A) - h(IkA),
\]
\[
\dim(kB/I) = r(B) - h(I),
\]
and hence we conclude that \(h(I) = h(IkA)\). Setting \(I = I_1, I_2\) now shows that
\[
h(I_2 kA) = h(I_1 kA) + 1
\]
as required. \(\blacksquare\)
2.2. Faithful primes are controlled by $\Delta$. As in [25], if $P$ is a prime ideal of some completed group ring $kG$, we will write

$$P^\dagger := \ker(G \to (kG/P)^\times),$$

and say that $P$ is **faithful** if $P^\dagger = 1$ and $P$ is **almost faithful** if $P^\dagger$ is finite.

Fix a prime $p$, which will be arbitrary until otherwise stated.

Recall the control theorem of Ardakov [2, 8.6]:

**Theorem 2.3:** Let $G$ be a nilpotent $p$-valued group of finite rank with centre $Z$.

(i) If $p$ is a prime ideal of $kZ$, then $pkG$ is a prime ideal of $kG$.

(ii) If $P$ is a faithful prime ideal of $kG$, then $P$ is controlled by $Z$.

**Proof.** This is [2, 8.4, 8.6].

**Lemma 2.4:** Let $G$ be finite-by-(nilpotent $p$-valuable), i.e., $G = \text{FN}_p(G)$. Then

$$Z(G/\Delta^+) = \Delta/\Delta^+.$$  

**Proof.** Given $x \in G$, the two conditions $[G/\Delta^+ : C_{G/\Delta^+}(x\Delta^+)] < \infty$ and $[G : C_G(x)] < \infty$ are equivalent, as $\Delta^+$ is finite; this shows that we have $\Delta(G/\Delta^+) = \Delta/\Delta^+$. Take some $x \in \Delta$, so that $x$ satisfies this condition: then, given arbitrary $g \in G$, there exists some $k$ such that $g^k \Delta^+ \in C_{G/\Delta^+}(x\Delta^+)$, so that $(x^{-1}gx)^k \Delta^+ = g^k \Delta^+$, and it now follows from [9, III, 2.1.4] that $x^{-1}gx\Delta^+ = g\Delta^+$. This shows that $\Delta/\Delta^+ \leq Z(G/\Delta^+)$. Conversely, we must have $Z(G/\Delta^+) \leq \Delta(G/\Delta^+)$ by definition.

We extend Theorem 2.3 to:

**Proposition 2.5:** Let $G$ be a finite-by-(nilpotent $p$-valuable) group and $k$ a finite field of characteristic $p$.

(i) If $p$ is a $G$-prime ideal of $k\Delta$, then $pkG$ is a prime ideal of $kG$.

(ii) If $P$ is an almost faithful prime ideal of $kG$, then $P$ is controlled by $\Delta$.

**Proof.** Adopt the notation of [24, Lemma 1.1 and Notation 1.2]. Let $e \in \text{cpi}^{k\Delta^+}(p)$, and write $f = e|_G$. To prove (i), it suffices to prove that the ideal $f \cdot \overline{pkG} \triangleleft f \cdot \overline{kG}$ is prime. But, by the Matrix Units Lemma [24, Lemma 6.1], we have an isomorphism

$$f \cdot \overline{kG} \cong M_s(e \cdot \overline{kG_1}),$$
where $G_1$ is the stabiliser in $G$ of $e$, and under which

$$f \cdot p k G \mapsto M_s(e \cdot p_1 k G_1)$$

for some $G_1$-prime ideal $p_1$ of $k[[\Delta \cap G_1]]$. So, by Morita equivalence, it will suffice to show that the ideal $e \cdot p_1 k G_1 < e \cdot k G_1$ is prime.

Now recall from [24, Theorems A and C] that we have an isomorphism

$$\psi : e \cdot k G_1 \simeq M_t(k'[G_1/\Delta^+])$$

under which

$$e \cdot p_1 k G_1 \mapsto q k'[G_1/\Delta^+]$$

for a $(G_1/\Delta^+)$-prime ideal $q$ of $k'[\Delta \cap G_1/\Delta^+]$. Hence we need now only show that $q k' N < k' N$ is prime, where $N = G_1/\Delta^+$.

Note that, as $G_1$ is open in $G$, we have $\Delta(G_1) = \Delta \cap G_1$ [25, Lemma 1.3(ii)]; and from Lemma 2.4,

$$\Delta(G_1)/\Delta^+ = Z(G_1/\Delta^+).$$

Hence, still writing $N = G_1/\Delta^+$, we see that $q$ is an $N$-prime ideal of $k'[Z(N)]$, and hence a prime ideal. But now $q k' N$ is prime by Theorem 2.3(i). This establishes part (i) of the proposition.

To show part (ii), take an almost faithful prime ideal $P$ of $kG$. We would like to show that $P$ is a minimal prime ideal above $(P \cap k \Delta)kG$. But this is clearly true when $\Delta^+ = 1$ by Theorem 2.3 and in the general case, another application of the Matrix Units Lemma [24 Lemma 6.1] and [24 Theorems A and C], as above, reduces to the case $\Delta^+ = 1$.

Hence, finally, we need only show that $(P \cap k \Delta)kG$ is prime; but $P \cap k \Delta$ is a $G$-prime ideal of $k \Delta$ (again by Lemma 1.2(i)), so we are done by part (i) of the proposition.

Until the end of this section, we will write $(-)^\circ$ to mean $\bigcap_{g \in G} (-)^g$.

**Corollary 2.6:** Let $G$ be a finite-by-(nilpotent $p$-valuable) group, and $H$ an open normal subgroup of $G$ containing $\Delta$. Let $k$ be a finite field of characteristic $p$. If $P$ is an almost faithful $G$-prime ideal of $kH$, then $P k G$ is a prime ideal of $kG$.

**Proof.** Take a minimal prime $Q$ of $kH$ above $P$. Then we have $Q^\circ = P$, so $Q^\dagger$ is finite (as $G$ is orbitally sound [25, Definition 1.4, Corollary 2.4]). Hence $Q$ is controlled by $\Delta$, by Proposition 2.5(ii), and by applying $(-)^\circ$ to both sides.
of the equality $Q = (Q \cap k\Delta)kH$, we see that $P$ is also: $P = (P \cap k\Delta)kH$. In particular, $PkG = (P \cap k\Delta)kG$. But now Proposition 2.5(i) shows that $(P \cap k\Delta)kG$ is prime.  

For the following results, we need to assume that $p > 2$ in order to be able to invoke [23, Theorem A].

**Proposition 2.7:** Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, and $k$ a finite field of characteristic $p > 2$. Let $H = \mathbf{FN}_p(G)$. If $P$ is an almost faithful prime ideal of $kG$, then $P$ is controlled by $H$.

**Proof.** Let $Q$ be a minimal prime ideal of $kH$ above $P \cap kH$. Then

$$(Q^\dagger)^\circ = P^\dagger \cap H$$

is finite, so, as $G$ is orbitally sound, $Q^\dagger$ is also finite. By [17, Corollary 14.8], in order to prove that $(P \cap kH)kG$ is prime, it suffices to show that $QkS$ is prime, where $S$ is the stabiliser in $G$ of $Q$.

Let $T = \mathbf{FN}_p(S)$. As $H$ is a finite-by-(nilpotent $p$-valuable) open normal subgroup of $S$, we see that $H$ must be an open normal subgroup of $T$. It is also clear that $\Delta(H) = \Delta(T) = \Delta(S) = \Delta(G)$ [25, Lemma 1.3(ii) and Theorem C]. Now, by Corollary 2.6, $QkT$ must be prime; and we have that $(QkT)^\dagger$ is finite. Now, by the prime extension theorem [23, Theorem A], $(QkT)kS = QkS$ is prime.  

**Lemma 2.8:** Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group, and let $H \geq K$ be any two closed normal subgroups of $G$. Take $P$ to be a prime ideal of $kG$. Let $Q$ be a minimal prime ideal of $kH$ above $P \cap kH$. If $P$ is controlled by $H$ and $Q$ is controlled by $K$, then $P$ is controlled by $K$.

**Proof.** By Lemma 1.2(ii), we have $Q^\circ = P \cap kH$, and so

$$(P \cap kK)kG = ((P \cap kH) \cap kK)kG$$

$$(Q^\circ \cap kK)kG$$

$$(Q \cap kK)^\circ kG$$

$$(Q \cap kK)^\circ kG$$

$$(Q^\circ kG)$$

$$(Q^\circ kG)$$

$$(P \cap kH)kG = P$$

as $P$ is controlled by $H$.  

$$(Q^\circ kG)^\circ kG$$

as $Q$ is controlled by $K$.

$$(Q^\circ kG)^\circ kG$$

as $Q$ is controlled by $K$.

$$(Q^\circ kG)^\circ kG$$

as $Q$ is controlled by $K$.

$$(Q^\circ kG)^\circ kG$$

as $Q$ is controlled by $K$.

$$(Q^\circ kG)^\circ kG$$

as $Q$ is controlled by $K$.  

$$(Q^\circ kG)^\circ kG$$

as $Q$ is controlled by $K$.  

$$(Q^\circ kG)^\circ kG$$

as $Q$ is controlled by $K$.
Now back to:

**Theorem 2.9:** Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, $k$ a finite field of characteristic $p > 2$, and $P$ an almost faithful prime ideal of $kG$. Then $P$ is controlled by $\Delta$.

**Proof.** Proposition 2.7 shows that $P$ is controlled by $H$. Let $Q$ be a minimal prime of $kH$ above $P \cap kH$: then $Q^\circ = P \cap kH$ by Lemma 1.2(ii), so we see that $(Q^\dagger)^\circ = P^\dagger \cap H$ is finite, so (as $G$ is orbitally sound) $Q^\dagger$ must also be finite. Hence, as $Q$ is almost faithful, Proposition 2.5(ii) shows that it is controlled by $\Delta$. Now Lemma 2.8 applies.

2.3. Primes adjacent to faithful primes. We begin with a property of the “finite-by-(nilpotent $p$-valuable) radical” operator.

**Lemma 2.10:** Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, let $N$ be a normal subgroup of $G$ which is contained in $\Delta$, and let $F$ be a finite normal subgroup of $G$. Then the following three statements hold.

(i) $\text{FN}_p(G/F) = \text{FN}_p(G)/F$.

(ii) Suppose that $\text{FN}_p(G/i\Delta(N)) = \text{FN}_p(G)/i\Delta(N)$. Then $\text{FN}_p(G/N)$ is equal to $\text{FN}_p(G)/N$.

(iii) Suppose $N$ is $\Delta$-isolated. Then we have either $\text{FN}_p(G/N) = \text{FN}_p(G)/N$ or $N = \Delta = \text{FN}_p(G)$.

**Proof.**

(i) This is clear from the construction of $\text{FN}_p(G)$ (see [25, Definition 5.3]).

(ii) First, note that $\text{FN}_p(G)/N$ is a quotient of a finite-by-(nilpotent $p$-valuable) normal subgroup of $G$, and hence is still a finite-by-(nilpotent $p$-valuable) normal subgroup of $G/N$, i.e., $\text{FN}_p(G)/N \leq \text{FN}_p(G/N)$. As both of these are of finite index in $G$, it will suffice to show that these indices are equal.

Consider the natural surjection

$$\alpha : G/N \to G/i\Delta(N).$$

We can see that

$$\ker \alpha = i\Delta(N)/N = \Delta^+(\Delta/N) \leq \Delta^+(G/N) \leq \text{FN}_p(G/N)$$
is a finite normal subgroup of $G/N$, and hence from (i) we see that
\[
\frac{\mathbb{F}N_p(G/N)}{i_\Delta(N)/N} \cong \mathbb{F}N_p(G/i_\Delta(N)).
\]
That is, the restricted map
\[
\alpha|_{\mathbb{F}N_p(G/N)} : \mathbb{F}N_p(G/N) \to \mathbb{F}N_p(G/i_\Delta(N))
\]
is also surjective with kernel $i_\Delta(N)/N$. Hence we have the following commutative diagram, in which the first two rows are exact, all three columns are exact, and $C_1$ and $C_2$ are the cokernels of the vertical maps.

\[
\begin{array}{cccccc}
1 & 1 & 1 \\
\downarrow & \downarrow & \downarrow \\
1 \to i_\Delta(N)/N & \mathbb{F}N_p(G/N) & \mathbb{F}N_p(G/i_\Delta(N)) & \to 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 \to i_\Delta(N)/N & G/N & G/i_\Delta(N) & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 \to 1 & C_1 & C_2 & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 1 & 1 \\
\end{array}
\]

By the Nine Lemma [15, Chapter XII, Lemma 3.4], the third row is now also exact, so that $C_1 \cong C_2$. But by assumption, $C_2 \cong G/\mathbb{F}N_p(G)$, and hence
\[
[G/N : \mathbb{F}N_p(G/N)] = |C_1| = [G : \mathbb{F}N_p(G)] = [G/N, \mathbb{F}N_p(G)/N],
\]
as required.

(iii) **Case 1.** First, assume that $\Delta^+ = 1$.

Write $H = \mathbb{F}N_p(G)$, and $\hat{H}$ for the preimage of $\hat{H}/N = \mathbb{F}N_p(G/N)$.

If $G = \mathbb{F}N_p(G)$, then we clearly have $\mathbb{F}N_p(G/N) = \mathbb{F}N_p(G)/N$ for any closed normal subgroup $N$. So suppose that $H \leq \hat{H} \leq G$, and take some $z \in \hat{H} \setminus H$. Now conjugation by $z$ induces the automorphism $x \mapsto x^\zeta$ on $H/H'$ (where $H'$ denotes the isolated derived subgroup), and hence also on $H/H'N$, for some $\zeta \in t(\mathbb{Z}_p^\times)$ [25, Lemma 4.2] satisfying $\zeta \neq 1$ [23, Lemma 3.3].
If $H/H'N$ has nonzero rank, we may take an element $x \in H$ whose image in $H/H'N$ has infinite order; and now the image in $\hat{H}/H'N$ of $\langle x, z \rangle$ is not finite-by-nilpotent, contradicting the definition of $\hat{H}$. So we must have $H = i_H(H'N)$.

In particular, this implies that

$$H = i_H(H'Z),$$

where $Z = Z(H) = \Delta(G)$, and so, by Lemma 3.5, we see that $H$ is abelian, i.e., $H = \Delta$. Furthermore, this implies that $H' = 1$, and as $N$ is already $H$-isolated (because $\Delta$ is $H$-isolated), the equality $H = i_H(H'N)$ simplifies to give $H = N$. This is what we wanted to prove.

**Case 2.** Now suppose instead that $\Delta^+ \neq 1$. As $N$ is isolated in $G$, we see that

- $\Delta^+ \leq N$, and $N/\Delta^+$ is isolated normal inside $G/\Delta^+$, contained in $\Delta/\Delta^+$;
- $\Delta^+(G/\Delta^+) = 1$;
- $\Delta(G/\Delta^+) = \Delta/\Delta^+ = Z(FN_p(G)/\Delta^+);$
- $FN_p(G/\Delta^+) = FN_p(G)/\Delta^+$;

and so the result follows by applying Case 1 to $G/\Delta^+$.

**Remark:** If $G$ is a compact $p$-adic analytic group, $H$ is a closed normal subgroup, and $Q$ is a $G$-stable ideal of $kH$, then $Q^\dagger = (Q + 1) \cap H$ is normal in $G$.

**Lemma 2.11:** Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, and let $k$ be a finite field of characteristic $p > 2$. If $Q$ is a $G$-prime ideal of $k\Delta$, and $FN_p(G/Q^\dagger) = FN_p(G)/Q^\dagger$, then $QkG$ is a prime ideal of $kG$.

**Remark:** The hypothesis

$$(\dagger) \quad FN_p(G/Q^\dagger) = FN_p(G)/Q^\dagger$$

has the following consequence. Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, $k$ a finite field of characteristic $p > 2$, and let $P \preceq P'$ be adjacent prime ideals of $kG$, with $P$ almost faithful. Then $P$ is controlled by $\Delta$, by Theorem

Set

$$Q := P' \cap k\Delta.$$
Consider $i_\Delta(Q^\dagger)$: if this is not equal to $\Delta$, then by Lemma 2.10(ii), (iii), the hypothesis $i_\Delta(Q^\dagger)$ is satisfied. So suppose it is equal to $\Delta$. Now, as $Q$ contains the ideal ker$(kG \to k[[G/Q^\dagger]])$ (the augmentation ideal of $Q^\dagger$), if we further have that $\text{FN}_p(G) = \Delta$, then $kG/Q$ is a finite prime ring, which is therefore simple, and so $Q$ must be a maximal ideal of $kG$ of $i_G(\Delta) = G$; otherwise, we again have $i_\Delta(Q^\dagger)$ by Lemma 2.10(ii), (iii).

That is, under these conditions, we always have $i_\Delta(Q^\dagger)$ unless $Q$ is a maximal ideal of $kG$ and $G$ is virtually abelian, in which case $Q^\dagger$ is open in $G$.

**Proof.** Write $H = \text{FN}_p(G)$.

As $Q$ is a $G$-prime, we may write it as $\bigcap_{g \in G} I^g$ for some minimal prime ideal $I$ above $Q$. Suppose the $G$-orbit of $I$ splits into distinct $H$-orbits $O_1, \ldots, O_r$, and write

$$P_i := \bigcap_{A \in O_i} A.$$  

Then $P_i$ is an $H$-prime of $k\Delta$, and $\bigcap_{i=1}^r P_i = Q$. In particular, since $P_i$ is an $H$-prime of $k\Delta$, we have that $P_i kH$ is prime by Proposition 2.5(i).

It remains to show that

$$\left( \bigcap_{g \in G} (P_i kH)^g \right) kG$$

is prime. By [17, Corollary 14.8], it suffices to show that $P_i kS$ is prime, where $S = \text{Stab}_G(P_i)$.

Write

$$p = P_i kH,$$

and note that $p^\dagger = P_i^\dagger \leq \Delta$. Now, if $\text{FN}_p(G)/\Delta^+$ is non-abelian, we have $\text{FN}_p(S/p^\dagger) = \text{FN}_p(S)/p^\dagger$. If, on the other hand, $\text{FN}_p(G)/\Delta^+$ is abelian, then we must have $Q^\dagger \leq \Delta$, and as $Q^\dagger$ is $H$-isolated orbital, we have $[\Delta : Q^\dagger] = \infty$. But as $G$ is orbitally sound, and

$$Q^\dagger = \bigcap_{g \in G} (P_i^\dagger)^g,$$

we must have that $Q^\dagger$ is open in $P_i^\dagger$, so that in particular $[\Delta : p^\dagger] = \infty$. Hence again we have $\text{FN}_p(S/p^\dagger) = \text{FN}_p(S)/p^\dagger$.

Write $\overline{\cdot}$ for the quotient map $S \to S/p^\dagger$. Now, to show that $P_i kS = p kS$ is prime, we need only show that $\overline{p} k\overline{S}$ is prime. But $\overline{p}$ is a faithful prime ideal of $\overline{kH}$, and $\overline{H} = \text{FN}_p(\overline{S})$, so by [23, Theorem A], we are done.  

Lemma 2.12: Let $k$ be a finite field of characteristic $p > 2$. Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, and let $P \leq Q$ be adjacent prime ideals of $kG$, with $P$ almost faithful. Suppose that $Q$ is not a maximal ideal of $kG$. Then $Q$ is controlled by $\Delta$.

Proof. $Q \cap k\Delta$ is a $G$-prime of $k\Delta$, and so $(Q \cap k\Delta)kG$ is prime by Lemma 2.11 and the accompanying remark. But

$$P = (P \cap k\Delta)kG \leq (Q \cap k\Delta)kG \leq Q$$

(with the equality as a result of Theorem 2.9, and $P$ and $Q$ are adjacent, so $(Q \cap k\Delta)kG$ must equal either $P$ or $Q$).

Let us assume for contradiction that $(Q \cap k\Delta)kG = P$. Then we must have

$$P \cap k\Delta \leq Q \cap k\Delta \leq (Q \cap k\Delta)kG = P,$$

and by intersecting each of these with $k\Delta$, we see that $P \cap k\Delta = Q \cap k\Delta$. In particular, by taking $(\cdot)^\dagger$ of both sides of this equality, we see that $Q^\dagger \cap \Delta$ is finite (as $P$ is almost faithful).

Let $N$ be an open normal nilpotent $p$-valued subgroup of $G$, and let $Z = Z(N)$. By [25, Lemma 1.3(ii)], $Z = \Delta(N)$ is a finite-index torsion-free subgroup of $\Delta$, and so $Q^\dagger \cap Z = 1$. Now, as $N$ is nilpotent and the normal subgroup $Q^\dagger \cap N$ has trivial intersection with its centre, [18, 5.2.1] implies that $Q^\dagger \cap N = 1$, and hence $Q^\dagger$ must be a finite normal subgroup of $G$. So $Q^\dagger \leq \Delta^+$, and in particular $Q^\dagger = Q^\dagger \cap \Delta$, which we earlier determined is finite. Hence $Q$ is almost faithful, and must be controlled by $\Delta$ by Theorem 2.9. In particular, we must have $P \cap k\Delta \neq Q \cap k\Delta$. But this contradicts our assumption.

3. Catenarity

3.1. The orbitally sound case: plinths and a height function. Much of the material in this subsection is adapted from [19].

Unless stated otherwise, throughout this section, $G$ is an arbitrary compact $p$-adic analytic group, and $k$ is a finite field of characteristic $p$. We start by outlining our plan of attack:

Lemma 3.1: Let $R$ be a ring in which every prime ideal has finite height. Suppose we are given a function $h : \text{Spec}(R) \to \mathbb{N}$ satisfying

- $h(P) = 0$ whenever $P$ is a minimal prime of $R$,
- $h(P') = h(P) + 1$ for each pair of adjacent primes $P \leq P'$ of $R$.

Then $R$ is a catenary ring.

Proof. Obvious. ■
Lemma 3.2: $kG$ has finite classical Krull dimension, i.e., the maximal length of any chain of prime ideals is bounded.

Proof. The classical Krull dimension of $kG$ is bounded above by $\text{Kdim}(kG)$ by \cite{16} Lemma 6.4.5], which is equal to $\text{Kdim}(\mathbb{F}_pG)$ by \cite{16} Proposition 6.6.16(ii)], and this is bounded above by the dimension (in the sense of \cite{5} Theorem 8.36]) of $G$, which is finite by definition (see the remarks after \cite{5} Definition 3.12)].

Definition 3.3: Let $V$ be a $\mathbb{Q}_pG$-module, and suppose it has finite dimension as a vector space over $\mathbb{Q}_p$. Take a chain

$$0 = V_0 \leq V_1 \leq \cdots \leq V_r = V$$

of $G$-orbital subspaces, that is, $\mathbb{Q}_p$-vector subspaces of $V$ with finitely many $G$-conjugates, or equivalently $\mathbb{Q}_p$-vector subspaces that are $\mathbb{Q}_pN$-submodules for some open subgroup $N$ of $G$. Assume further that this chain is saturated, in the sense that it cannot be made longer by the addition of some $G$-orbital subspace

$$V_i \leq V' \leq V_{i+1}.$$ 

Such a chain is necessarily finite, as it is bounded above in length by $\dim_{\mathbb{Q}_p}(V) + 1$. We call the number $r$ the $G$-plinth length of $V$, written $p_G(V)$. If $p_G(V) = 1$, we say that $V$ is a plinth for $G$.

Remark: The number $r$ is independent of the $V_i$ chosen. Indeed, fix a longest possible chain

$$0 = V_0 \leq V_1 \leq \cdots \leq V_r = V$$

of $G$-orbital subspaces, and let $G_0$ be the intersection of the normalisers $N_G(V_i)$, i.e., the largest subgroup of $G$ such that each $V_i$ is a $\mathbb{Q}_pG_0$-module; $G_0$ is open in $G$. Now, given any chain

$$0 = W_0 \leq W_1 \leq \cdots \leq W_s = V$$

of $G$-orbital subspaces, take

$$H_0 = \bigcap_{j=1}^s N_G(W_j),$$

and note that $G_0 \cap H_0$ is a finite-index open subgroup of $G$ that normalises each $V_i$ and $W_j$. Hence, by the Jordan–Hölder theorem \cite{7} Theorem 4.11], the chain $W_j$ may be refined to a chain of length $r$; so if the chain $W_j$ is saturated, then $s = r$. 
Definition 3.4: A \textbf{G-group} is a topological group \(H\) endowed with a continuous action of \(G\). For example, closed subgroups of \(G\), and quotients of \(G\) by closed normal subgroups of \(G\), are \(G\)-groups under the action of conjugation.

Let \(H\) be a nilpotent-by-finite compact \(p\)-adic analytic group with a continuous action of \(G\). We aim to define \(p_G(H)\). In fact, as plinths are insensitive to finite factors, we may immediately replace \(H\) by the open subgroup formed by the intersection of the (finitely many) \(G\)-conjugates of any given open normal nilpotent uniform subgroup of \(H\). Then there is a series

\[
1 = H_0 < H_1 < \cdots < H_n = H
\]

of \(G\)-subgroups such that \(A_i = H_i/H_{i-1}\) is abelian for each \(i = 1, \ldots, n\). Let

\[
V_i = A_i \otimes \mathbb{Q}_p
\]

for each \(i = 1, \ldots, n\), with \(G\)-action given by conjugation. In this case, we define

\[
p_G(H) = \sum_{i=1}^{n} p_G(V_i).
\]

Lemma 3.5: \(p_G(H)\) is well-defined, and does not depend on the series \((1)\).

Proof. Apply the Jordan–Hölder theorem, as in the remark above. \(\blacksquare\)

For our purposes, the most important property of \(p_G\) is that it is additive on short exact sequences of \(G\)-groups, which also follows from the Jordan–Hölder theorem. We record this as:

Lemma 3.6: Suppose that 

\[
1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1
\]

is a short exact sequence of \(G\)-groups. Then \(p_G(A) + p_G(C) = p_G(B)\). \(\blacksquare\)

We now define Roseblade’s function \(\lambda\). (Later, we will show that, in the case when \(G\) is nilpotent-by-finite and orbitally sound, \(\lambda\) is actually equal to the height function on \(\text{Spec}(kG)\).)

Definition 3.7:

\[
\lambda(P) = \begin{cases} 
p_G(P^\dagger) + \lambda(P^\pi), & P^\dagger \neq 1, \\
h_G(P \cap k\Delta), & P^\dagger = 1,
\end{cases}
\]

where \(P^\pi\) is the image of \(P\) under the map

\[
\pi : kG \to kG/(P^\dagger - 1)kG \cong k[[G/P^\dagger]].
\]
This definition is recursive, in that if $P$ is an unfaithful prime ideal, then $\lambda(P)$ is defined with reference to $\lambda(P^\pi)$; but $P^\pi$ is then a faithful prime ideal of $kG^\pi$, so this process terminates after at most two steps.

We make the following remark on this definition immediately:

**Lemma 3.8:** Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, $k$ a finite field of characteristic $p > 2$, and $P$ a faithful prime ideal of $k\Delta$. Then $\lambda(P) = h(P)$.

**Proof.** $\lambda(P)$ is defined to be $h_G(P \cap k\Delta)$. But, by Theorem 2.9 and Lemma 2.11 we see that there is a one-to-one, inclusion-preserving correspondence between faithful prime ideals of $kG$ and faithful $G$-prime ideals of $k\Delta$, so that

$$h_G(P \cap k\Delta) = h(P).$$

We return to the general case of $G$ an arbitrary compact $p$-adic analytic group.

**Lemma 3.9:** Let $P \leq Q$ be neighbouring prime ideals of $kG$, and write

$$\pi : kG \to kG/(P^\dagger - 1)kG \cong k[[G/P^\dagger]].$$

Then

$$\lambda(Q) - \lambda(P) = \lambda(Q^\pi) - \lambda(P^\pi).$$

**Proof.** Firstly, as $P \leq Q$, we have $P^\dagger \leq Q^\dagger$, so the map

$$\rho : kG \to k[[G/Q^\dagger]]$$

factors as

$$kG \xrightarrow{\pi} k[[G/P^\dagger]] \xrightarrow{\sigma} k[[G/Q^\dagger]].$$

We now compute $\lambda(Q) - \lambda(P)$ using Definition 3.7

$$\lambda(Q) - \lambda(P) = p_G(Q^\dagger) - p_G(P^\dagger) + \lambda(Q^\rho) - \lambda(P^\pi)$$

$$= p_G((Q^\dagger)^\pi) + \lambda(Q^\rho) - \lambda(P^\pi)$$

by Lemma 3.6

$$= p_G((Q^\dagger)^\pi) + \lambda(Q^\pi^\sigma) - \lambda(P^\pi)$$

by definition of $\rho$

$$= p_G((Q^\pi)^\dagger) + \lambda((Q^\pi)^\sigma) - \lambda(P^\pi)$$

as $(Q^\dagger)^\pi = (Q^\pi)^\dagger$

$$= \lambda(Q^\pi) - \lambda(P^\pi)$$

by Definition 3.7.
Remark: Suppose $G$ is a nilpotent-by-finite compact $p$-adic analytic group, and suppose we are given a subquotient $A$ of $G$ which is a plinth, with $G$-action induced from the conjugation action of $G$ on itself. Then it is easy to see that

$$\dim_{Q_p}(A \otimes Z_p Q_p) = 1.$$  

(Roseblade calls such plinths centric.) Indeed, suppose $A = H/K$, where $H$ and $K$ are closed normal subgroups of $G$ with $K$ contained in $H$. Then we may replace $G$ by an open normal nilpotent uniform subgroup $G'$, and $A$ by $A' = H'/K'$, where $H' = H \cap G'$ and $K' = i_{H'}(K \cap G')$; after doing this, we still have that $A'$ is a plinth for $G'$, and that

$$\dim_{Q_p}(A \otimes Z_p Q_p) = \dim_{Q_p}(A' \otimes Z_p Q_p).$$

But, as $G'/K'$ is nilpotent, and $A'$ is a non-trivial normal subgroup, $A'$ must meet the centre $Z(G'/K')$ non-trivially; and as $A'$ is torsion-free, we must have that $A' \cap Z(G'/K')$ is a plinth for $G'$, and so must be equal to $A'$. Hence $G'$ centralises $A'$, and its plinth length is simply equal to its rank.

Again, we will write $(-)^\circ$ to mean $\bigcap_{g \in G} (-)^g$.

**Lemma 3.10:** Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group. Let $U$ be a $G$-prime of $k\Delta$, and write $\rho : k\Delta \to k[[\Delta/U^\dagger]]$. Then

$$h(U) = h_G(U^\rho) + p_G(U^\dagger).$$

**Proof.** Let $A = Z(\Delta)$, and let $U_1$ be a minimal prime of $kA$ above $U \cap kA$, so that $U \cap kA = U_1^\circ$. Then $h_G(U) = h(U_1)$ by Corollary 1.4, and so $h_G(U^\rho) = h(U_1^\rho)$. Now, from Lemma 2.2(i), we have that $h(U_1) + \dim(kA/U_1) = r(A)$ and $h(U_1^\rho) + \dim(kA/U_1) = r(A^\rho)$, from which we may deduce that

$$h(U_1) = h(U_1^\rho) + r(A) - r(A^\rho).$$

But $r(A) - r(A^\rho) = r(A \cap \ker \rho) = p_G(U^\dagger \cap A)$ by the above remark. Now this is just $p_G(U^\dagger)$, as $A$ is open in $\Delta$.

**Lemma 3.11:** Let $G$ be arbitrary compact $p$-adic analytic. Let $H$ be a closed normal subgroup of $G$, and let $K$ be an open subgroup of $H$ which is normal in $G$. If $P$ is a $G$-prime ideal of $kH$, then

$$h_G(P) = h_G(P \cap kK).$$
Proof (Adapted from [19, Lemma 29]). We know that $P = Q^\circ$ for some prime $Q$ of $kH$, and $Q \cap kK = \bigcap_{h \in H} V^h$ for some prime $V$ of $kK$. Hence $P \cap kK = V^\circ$. Then, writing $h_G^{\text{orb}}$ for the height function on $G$-orbital primes,

$$h_G(P) = h_G^{\text{orb}}(Q) \quad \text{by Lemma 1.2(ii)}$$

$$= h_G^{\text{orb}}(V) \quad \text{by Lemma 1.2(i)}$$

$$= h_G(P \cap kK) \quad \text{by Lemma 1.2(ii).}$$

Here, we deduce from Theorem 2.9 and [19, proof of theorem H2] the following corollary:

**Theorem 3.12:** Let $G$ be a nilpotent-by-finite, orbitally sound compact $p$-adic analytic group, and $k$ a finite field of characteristic $p > 2$. Then $kG$ is a catenary ring.

**Proof.** Let $P \preceq Q$ be neighbouring prime ideals of $kG$. We will first show that $\lambda(Q) = \lambda(P) + 1$.

By passing to $k[[G/P]]$, we may assume that $P$ is a faithful prime ideal, by Lemma 3.9. Hence, by Theorem 2.9 (and as $p > 2$), we have that

$$(P \cap k\Delta)kG = P.$$ 

We also have either that $(Q \cap k\Delta)kG = Q$, by Lemma 2.12 or $Q^\dagger \geq \Delta$ by the remark of Lemma 2.11 and so, in either case, we have $P \cap k\Delta \preceq Q \cap k\Delta$.

We will now show that $P \cap k\Delta$ and $Q \cap k\Delta$ are neighbouring $G$-primes of $k\Delta$. Suppose that they are not: then there must be a $G$-prime $J$ strictly between them, i.e., $P \cap k\Delta \preceq J \preceq Q \cap k\Delta$. Then, again by Lemma 2.11 and the remark made there, we see that $JkG$ is a prime ideal of $kG$. Now it is clear that

$$P \preceq JkG \preceq Q$$

by the previous paragraph; and if $JkG = Q$, then intersecting both sides with $k\Delta$ shows that $J = Q \cap k\Delta$, and likewise if $JkG = P$. Hence we must have $P \preceq JkG \preceq Q$, so that $P$ and $Q$ are not neighbouring primes. But this contradicts our initial assumptions.

So we conclude that

$$h_G(Q \cap k\Delta) = h_G(P \cap k\Delta) + 1.$$ 

The right-hand side is, by definition, just equal to $\lambda(P) + 1$; and we have $\lambda(Q) = \lambda(Q^\rho) + p_G(Q^\dagger)$, where $\rho : G \to G/Q^\dagger$. It remains to show that this is equal to $h_G(Q \cap k\Delta)$. 
Case 1. $Q^\dagger$ is not open in $G$. Then $Q$ is controlled by $\Delta$ by Lemma 2.12 and the remark of Lemma 2.11, and so $Q^\rho$ is controlled by $\Delta^\rho$, and in particular by $i_{G^\rho}(\Delta^\rho) \leq i_{G^\rho}(\Delta(G^\rho))$. Write $A = Z(\Delta(G^\rho))$ and $B = A \cap i_{G^\rho}(\Delta^\rho)$: as $Q^\rho$ is controlled by $i_{G^\rho}(\Delta^\rho)$, we have that $Q^\rho \cap kA$ is controlled by $B$. Furthermore, we can write $Q^\rho \cap kA = q^\circ$ for some prime $q$ of $kA$, so that $q$ is also controlled by $B$, and hence

$$\lambda(Q^\rho) = h_G(Q^\rho \cap k[\Delta(G^\rho)]) = h_G(Q^\rho \cap kA) = h(q) = h(q \cap kB) = h_G(Q^\rho \cap kB) = h_G(Q^\rho \cap k[\Delta^\rho]) = h_G(Q^\rho \cap k\Delta^\rho)$$

We also have $p_G(Q^\dagger) = p_G((Q \cap k\Delta)^\dagger)$. Hence

$$\lambda(Q) = h_G((Q \cap k\Delta)^\rho) + p_G((Q \cap k\Delta)^\dagger).$$

Now we are done by Lemma 3.10.

Case 2. $Q^\dagger$ is open in $G$. We have already seen that this case only occurs when $G = i_G(\Delta)$, and so $\lambda(Q^\rho) = \lambda(0) = 0$, and $p_G(Q^\dagger) = p_G(G)$, and $h_G(Q \cap k\Delta) = h_G(Q \cap kA) = r(A)$. These are clearly equal, as $A$ is open in $G$.

In order to invoke Lemma 3.11 it remains only to show that $\lambda(P) = 0$ when $P$ is a minimal prime. But as all minimal primes are induced from $\Delta^\dagger$, this follows immediately from the definition of $\lambda$: we will have $P^\dagger \leq \Delta^\dagger$ (and hence $p_G(P^\dagger) = 0$), and $P^\pi \cap k\Delta^\pi$ will be a minimal $G$-prime of $k\Delta^\pi$ (and hence $h_G(P^\pi \cap k\Delta^\pi) = 0$).

3.2. Vertices and sources. We now study a more general setting. Let $G$ be an arbitrary compact $p$-adic analytic group, and $P$ an arbitrary prime ideal of $kG$.

Remark: Suppose $G$ is orbitally sound and nilpotent-by-finite, $N$ is a closed normal subgroup of $G$, and $I$ is a prime ideal of $kG$ with $N \leq I^\dagger$ and $[I^\dagger : N] < \infty$. Writing $\overline{(-)}$ for the natural map $kG \to k[[G/N]]$, it is clear that the prime ideal $\overline{I} \subset k[[G/N]]$ is almost faithful, and so, by Theorem 2.9, is controlled by $\Delta(G/N)$, and that $I$ is the complete preimage in $kG$ of $\overline{I}$, and is therefore controlled by the preimage in $G$ of $\Delta(G/N)$.
This motivates the following definition:

**Definition 3.13:** Let $I$ be an ideal of $kG$, and $N$ a closed subgroup of $G$. We say that $I$ is **almost faithful** mod $N$ if $I^\dagger$ contains $N$ as a subgroup of finite index. We also write $\nabla_G(N)$ for the subgroup of $N_G(N)$ defined by

$$\nabla_G(N)/N = \Delta(N_G(N)/N).$$

Diagrammatically:

```
G
|   |   |
|---|---|
N_G(N) --- N_G(N)/N
|   |   |
|---|---|
\nabla_G(N) --- \Delta(N_G(N)/N)
|   |   |
|---|---|
N --- N/N
|   |   |
|---|---|
1
```

We will extend this notion to ideals $I$ with $I^\dagger$ contained in $N$ as a subgroup of finite index.

**Lemma 3.14:** Let $H$ be an open subgroup of $N$. Then there exists an open characteristic subgroup $M$ of $N$ contained in $H$.

**Proof (Adapted from [17, 19.2]).** Let $[N : H] = n < \infty$. Now, as $N$ is topologically finitely generated, there are only finitely many continuous homomorphisms $N \to S_n$, where $S_n$ is the symmetric group. Take $M$ to be the intersection of the kernels of these homomorphisms. $\blacksquare$

**Lemma 3.15:** Let $N$ be a closed subgroup of $G$, and $A = N_G(N)$. Suppose $I$ is an ideal of $kA$, and $I^\dagger \leq N$ with $[N : I^\dagger] < \infty$. Then there is a closed normal subgroup $M$ of $A$ such that $I$ is almost faithful mod $M$. Furthermore, this $M$ can be chosen so that

$$\nabla_G(N) = \nabla_A(M).$$
Proof. Set $H = I^\dagger$ in Lemma 3.14, then the subgroup $M$ is characteristic in $N$, hence normal in $A$; $M$ contains $I^\dagger$; and $M$ is open in $N$, so we must have

$$[I^\dagger : M] < \infty.$$ 

By definition, we have $\nabla_G(N) = \nabla_A(N)$. Now, $N/M$ is a finite normal subgroup of $A/M$, so is contained in $\Delta^+(A/M)$. Hence the preimage under the natural quotient map $A/M \rightarrow A/N$ of $\Delta(A/N)$ is $\Delta(A/M)$. But this is the same as saying that

$$\nabla_A(N) = \nabla_A(M).$$

When $G$ is a general compact $p$-adic analytic group, we will use the following lemma to translate between prime ideals of $kG$ and prime ideals of $kA$ for certain open subgroups $A$ of $G$.

Lemma 3.16: Let $H$ be an open normal subgroup of $G$. Suppose $P$ is a prime of $kG$, and write $Q$ for a minimal prime of $kH$ above $P \cap kH$. Let $B$ be the stabiliser in $G$ of $Q$, and let $A$ be any open subgroup of $G$ containing $B$, so that

$$H \leq B \leq A \leq G.$$ 

Then there is a prime ideal $T$ of $kA$ with $P = T^G$, and furthermore this $T$ satisfies

$$T \cap kH = \bigcap_{a \in A} Q^a.$$ 

Proof. This follows from [17, 14.10(i)].

Definition 3.17: A prime $P \triangleleft kG$ is standard if it is controlled by $\Delta$ and we have

$$P \cap k\Delta = \bigcap_{x \in G} L^x$$

for some almost faithful prime $L \triangleleft k\Delta$.

Lemma 3.18: Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group and $H$ an open normal subgroup. Let $P$ be a prime ideal of $kG$, and $Q$ a minimal prime of $kH$ above $P$, so that

$$P \cap kH = \bigcap_{x \in G} Q^x.$$ 

If $Q$ is a standard prime, then $P$ is a standard prime.
Proof (Adapted from [17, 20.4(i)]). Write $\Delta = \Delta(G), \Delta_H = \Delta(H)$, and

$$P \cap k\Delta = \bigcap_{x \in G} S^x \quad \text{and} \quad Q \cap k\Delta_H = \bigcap_{y \in H} T^y,$$

for prime ideals $S \triangleleft k\Delta$ and $T \triangleleft k\Delta_H$. On the one hand,

$$P \cap k\Delta_H = (P \cap k\Delta) \cap k\Delta_H$$

$$= \left( \bigcap_{x \in G} S^x \right) \cap k\Delta_H$$

$$= \bigcap_{x \in G} (S \cap k\Delta_H)^x,$$

but on the other hand,

$$P \cap k\Delta_H = (P \cap kH) \cap k\Delta_H$$

$$= \left( \bigcap_{x \in G} Q^x \right) \cap k\Delta_H$$

$$= \bigcap_{x \in G} (Q \cap k\Delta_H)^x$$

$$= \bigcap_{x \in G} T^x.$$

Now, the conjugation action of $G$ on $\Delta_H$ has kernel $C_G(\Delta_H)$, which contains $C_G(\Delta)$ by [25, Lemma 1.3(ii)]. But

$$C_G(\Delta) = \bigcap C_G(a),$$

where the intersection runs over a set of topological generators $a$ for $\Delta$, and each $C_G(a)$ is open in $G$ by definition of $\Delta$. Now, as $\Delta$ is topologically finally generated, we see that $C_G(\Delta)$ and hence $C_G(\Delta_H)$ are also open in $G$.

That is, the conjugation action of $G$ on $\Delta_H$ factors through the finite group $G/C_G(\Delta_H)$, and hence the intersections above are finite, so that (by the primality of $T$) we have

$$S \cap k\Delta_H \subseteq T^x$$

for some $x \in G$.

Now, by assumption, $Q$ is standard, so $T$ is almost faithful. This means that

$$S^\dagger \cap \Delta_H \subseteq (T^\dagger)^x$$

is a finite group, and so, since $[\Delta : \Delta_H] < \infty$, we have that $S^\dagger$ is also finite, so $S$ is almost faithful.
It remains to show that $S^\circ kG = P$. By Lemma 2.11, we see that $S^\circ kG = P'$ is a prime ideal of $kG$ contained in $P$. Now,

$$
(P \cap kH)kG = \left( \bigcap_{g \in G} Q^g \right)kG
= \left( \bigcap_{g \in G} \left( \bigcap_{h \in H} T^h kH \right)^g \right)kG
= \left( \bigcap_{g \in G} T^g \right)kG
= (P \cap k\Delta_H)kG \quad \text{by calculation above}
\subseteq (P \cap k\Delta)kG = P' \subseteq P,
$$

and as $H$ is open and normal in $G$, we know from Lemma 1.2(i) that $P$ is a minimal prime above $(P \cap kH)kG$, so that $P = P'$. ■

Finally, the main theorem of this subsection:

**Theorem 3.19:** Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group, $P$ a prime ideal of $kG$, $H$ an orbitally sound open normal subgroup of $G$, $Q$ a minimal prime ideal above $P \cap kH$, and $N = i_G(Q^\dagger)$. Then there exists an ideal $L \subseteq k[[\nabla_G(N)]]$ with $P = L^G$.

**Remark:** The subgroup $N$ is a vertex of the prime ideal $P$, and the ideal $L$ is a source of $P$ corresponding to the vertex $N$.

**Proof.** We follow the proof of [13, 2.3], as reproduced in [17, 20.5].

Trivially, $H$ stabilises $Q$, i.e.,

$$
H \leq B := \text{Stab}_G(Q);
$$

and $B$ normalises $Q^\dagger$. Set

$$
N := i_G(Q^\dagger).
$$

Now we must have $N_G(Q^\dagger) \leq A := N_G(N)$: indeed, if $x \in G$ normalises $Q^\dagger$, then it permutes the (finitely many) closed orbital subgroups $K$ of $G$ containing $Q^\dagger$ as an open subgroup, and hence it normalises $N$, which is generated by those $K$. [25, Definition 1.6].
We are in the following situation:

Now, Lemma 3.16 shows that there is a prime ideal $T$ of $kA$ with $P = T^G$ and $T \cap kH = \bigcap_{a \in A} Q^a$. It will suffice to show the existence of a prime ideal $L$ of $k[[\nabla_G(N)]]$ with $T = L^A$, by Lemma 1.6.

Let $M$ be an open characteristic subgroup of $N$ contained in $Q^\dagger$, whose existence is guaranteed by Lemma 3.14. Write $\nabla = \nabla_G(N)$, which we know is equal to $\nabla_A(M)$ by Lemma 3.15 and denote by $\langle \cdot \rangle$ images under the natural map $kA \to k[[A/M]]$.

Now $Q$ is a prime ideal of $kH$ with $M \leq Q^\dagger$ an open subgroup, so $\overline{Q}$ is an almost faithful prime ideal of $k\overline{H}$; hence, as $\overline{H}$ is orbitally sound [25, Lemma 1.5(ii)], we see that $\overline{Q}$ is a standard prime of $k\overline{H}$.

But $T \cap kH = \bigcap_{a \in A} Q^a$ clearly implies

$$\overline{T} \cap k\overline{H} = \bigcap_{\pi \in \overline{A}} \overline{Q}^\pi,$$

by the modular law. Now Lemma 3.18 implies that $\overline{T}$ is also a standard prime ideal of $k\overline{A}$: that is, there is an almost faithful prime ideal $\overline{L}$ of $k[[\Delta(\overline{A})]]$ with $\overline{T} = \overline{L}$. Lifting this back to $kA$, we see that we have an almost faithful mod $M$ prime ideal $L$ of $k\nabla$ with $T = L^A$ as required. \[\square\]
We end this subsection with an important application of this theorem. Recall the definition of $\text{nio}(G)$ from \cite[Definition 2.5]{25}.

**Corollary 3.20:** Suppose $G$ is a nilpotent-by-finite compact $p$-adic analytic group which is not orbitally sound. Let $P$ be a faithful prime ideal of $kG$. Then $P$ is induced from some proper open subgroup of $G$ containing $\text{nio}(G)$.

**Proof.** Write $H = \text{nio}(G)$. $H$ is orbitally sound by \cite[Theorem 2.6(ii)]{25}.

Let $Q$ be a minimal prime ideal above $P \cap kH$, so that $N = i_G(Q^\dagger)$ is a vertex for $P$ by Theorem 3.19. Then $P$ is induced from $\nabla_G(N)$, which is contained in $N_G(N)$, and so $P$ is induced from $N_G(N)$ itself by Lemma 1.6 But, as $\text{nio}(G)$ is orbitally sound, in particular it must normalise $N$ \cite[Theorem 2.6(i)]{25}. Hence, if $N_G(N)$ is a proper subgroup of $G$, we are done.

Suppose instead that $N_G(N) = G$, i.e., that $i_G(Q^\dagger)$ is a normal subgroup of $G$. Then, for each $g \in G$, $(Q^\dagger)^g$ is a finite-index subgroup of $i_G(Q^\dagger)$ \cite[Proposition 1.7]{25}; and $Q^\dagger$ is orbital in $G$, so there are only finitely many $(Q^\dagger)^g$, and their intersection $(Q^\dagger)^o$ must also have finite index in $i_G(Q^\dagger)$. But $(Q^\dagger)^o = P^\dagger = 1$, so in particular we have $i_G(Q^\dagger) = \Delta^+$, and hence $P$ is induced from $\nabla_G(N) = \Delta$, again by Theorem 3.19. Hence, as $\text{nio}(G)$ contains $\Delta$, $P$ must be induced from $\text{nio}(G)$ itself. $\blacksquare$

### 3.3. The General Case: Inducing from Open Subgroups

Now we will proceed to show that $kG$ is catenary.

**Lemma 3.21:** Let $H$ be an open subgroup of $G$, and $P$ a prime ideal of $kG$. Suppose $Q$ is an ideal of $kH$ maximal amongst those ideals $A$ of $kH$ with $A^G \subseteq P$. Then $Q$ is prime, and $P$ is a minimal prime ideal above $Q^G$.

**Proof.** Suppose $I$ and $J$ are ideals strictly containing $Q$: then, by the maximality of $Q$, we see that $I^G$ and $J^G$ must strictly contain $P$. Hence $I^G J^G \subseteq (IJ)^G$ \cite[Lemma 14.5]{17} strictly contains $P$, and so $IJ$ strictly contains $Q$. Hence $Q$ is prime.

Now $P$ is clearly a prime ideal containing $Q^G$, so to show it is minimal it suffices to find any ideal $A$ of $kH$ with $P$ a minimal prime above $A^G$. Let $N$ be the normal core of $H$ in $G$, and take $A = (P \cap kN)^H$: then by Lemma 1.6 we have

$$A^G = (P \cap kN)^G = (P \cap kN)kG,$$

and by Lemma 1.2(i), $P$ is a minimal prime above this. $\blacksquare$
Lemma 3.22: Let $H$ be an open subgroup of $G$ with $kH$ catenary. If $P \lhd P'$ are adjacent primes of $kG$, and $P$ is induced from $kH$, then

$$h(P') = h(P) + 1.$$ 

Proof (Adapted from [14, 3.3]). Choose an ideal $Q$ (resp. $Q'$) of $kH$ which is maximal amongst those ideals $A$ of $kH$ with $A^G \subseteq P$ (resp. $A^G \subseteq P'$). Then $Q$ and $Q'$ are prime, and $P$ (resp. $P'$) is a minimal prime ideal over $Q^G$ (resp. $Q'^G$), by Lemma 3.21. Hence, by Proposition 1.16 we see that it suffices to show that $h(Q') = h(Q) + 1$.

Suppose not. Then there exists some prime ideal $I$ of $kH$ with $Q \lhd I \lhd Q'$; and we may choose a prime ideal $J$ of $kG$ which is minimal over $I^G$. Then $P \leq J \leq P'$. But $h(Q) < h(I) < h(Q')$ implies (by another application of Proposition 1.16) that $h(P) < h(J) < h(P')$, contradicting our assumption that $P$ and $P'$ were adjacent primes.

Corollary 3.23: Let $G$ be a nilpotent-by-finite compact $p$-adic analytic group, and $k$ a finite field of characteristic $p > 2$. Then $kG$ is a catenary ring.

Proof (Adapted from [14, 3.3]). Take two adjacent prime ideals $P \lhd Q$ of $kG$, and assume without loss of generality that $P$ is faithful. We proceed by induction on the index $[G : \text{nio}(G)]$. When this index equals 1, we are already done by Theorem 3.12; so suppose not. Then Corollary 3.21 implies that $P$ is induced from some proper open subgroup $H$ of $G$ containing $\text{nio}(G)$. As $\text{nio}(G)$ is an orbitally sound open normal subgroup of $H$, it must be contained in $\text{nio}(H)$ (by the maximality of $\text{nio}(H)$), and so we have

$$[H : \text{nio}(H)] < [G : \text{nio}(G)].$$

By induction, $kH$ is catenary, so we may now invoke Lemma 3.22 to show that $h(Q) = h(P) + 1$.

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