C IS NOT EQUIVALENT TO $C^-$ IN ITS JACOBIAN: A TROPICAL POINT OF VIEW

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ABSTRACT. We show that the Abel-Jacobi image of a tropical curve $C$ in its Jacobian $J(C)$ is not algebraically equivalent to its reflection by using a simple calculation in tropical homology.

1. Introduction

A classical result of Ceresa [Cer83] states that for a generic smooth projective complex curve $C$ of genus $g \geq 3$ the cycles $W_k$ and $W_{-k}$, the image of $W_k$ under the involution $(-1): x \rightarrow -x$, are not algebraically equivalent in the Jacobian $J(C)$. In this article we will give an argument for analogous tropical statement.

There is a folklore dictionary between complex objects existing in families and their tropical counterparts appearing as degenerations. For instance, a family of complex curves which degenerates to a tropical curve comes with the corresponding family of Jacobians, which in turn, degenerates to the corresponding tropical Jacobian. Moreover, families of cycles in the Jacobians should become tropical cycles (balanced weighted polyhedral complexes with rational slopes) in the tropical Jacobian. Studying tropical cycles, which are essentially linear objects, is a much simpler task than studying the classical cycle problems.

Several correspondences between classical and tropical objects have been established (cf., e.g. [Ale04], or more recent [Viv12] and references therein). However the author is unaware of a general assertion that tropical algebraic inequivalence of cycles implies classical inequivalence for generic members of the family. Nor is this the goal of this paper.

Rather our main objective is to introduce a new tropical tool – the determinantal form. This is a natural closed 2-form with coefficients in the tropical cohomology groups $F^1$ that vanishes on every cell of a tropical (1,2)-chain which provides an algebraic equivalence between two 1-cycles. We hope that its generalization can be used to algebraically distinguish also higher dimensional cycles in more complicated tropical varieties. Eventually it may lead to

The research is partially supported by the NSF FRG grant DMS-0854989.
a new (tropical) approach for proving algebraic inequivalences in the classical setting, given the dictionary established.

2. TROPICAL CURVES AND JACOBIANS

2.1. Tropical curves. Let \( \Gamma \) be a connected finite graph and \( \mathcal{V}_1(\Gamma) \) be the set of its 1-valent vertices. We say \( \Gamma \) is a metric graph if the topological space \( \Gamma \setminus \mathcal{V}_1(\Gamma) \) is given a complete metric structure and \( \Gamma \) is its compactification. In particular, all leaves must have infinite lengths. A new two-valent vertex inserted in the interior of any edge produces another metric graph which we set to be equivalent to the original one.

**Definition 1.** A tropical curve \( C \) is an equivalence class of such metric graphs. Its genus is \( g = b_1(\Gamma) \) for any representative \( \Gamma \).

For purposes needed in this paper it is enough to consider finite graphs. That is we can always remove all (infinite) leaves. That affects neither the construction of the Jacobian \( J(C) \) nor the Abel-Jacobi map and its image \( W_1 \subset J(C) \) of \( C \), which are the main objects of our interest.

A divisor \( D = \sum a_ip_i \) on the curve \( C \) is a formal linear combination of points in \( C \) with integral coefficients. We say \( D \) has degree \( d = \sum a_i \), and \( D \) is effective if all \( a_i \geq 0 \). Degree 0 divisors form an abelian group.

2.2. Tropical tori. Let \( V \) be a \( g \)-dimensional real vector space containing two lattices \( \Gamma_1, \Gamma_2 \) of maximal rank, that is \( V \cong \Gamma_{1,2} \otimes \mathbb{R} \). Suppose we are given an isomorphism \( Q: \Gamma_1 \rightarrow \Gamma_2^* \), which is symmetric if thought of as a bilinear form on \( V \).

**Definition 2.** The torus \( J = V/\Gamma_1 \) is the principally polarized tropical torus with \( Q \) being its polarization. The tropical structure on \( X \) is given by the lattice \( \Gamma_2 \). If, in addition \( Q \) is positive definite, we say that \( J \) is abelian variety.

**Remark.** The map \( Q: \Gamma_1 \rightarrow \Gamma_2^* \) provides an isomorphism of \( J = V/\Gamma_1 \) with the tropical torus \( V^*/\Gamma_2^* \). The tropical structure on the latter is provided by the lattice \( \Gamma_1^* \). If \( Q \) is very natural, e.g., for a Jacobian, it may be quite confusing to distinguish between \( V/\Gamma_1 \) and \( V^*/\Gamma_2^* \).

The data \( (V, \Gamma_1, \Gamma_2, Q) \) above is equivalent to a non-degenerate real-valued quadratic form \( Q \) on a free abelian group \( \Gamma_1 \cong \mathbb{Z}^g \). The second lattice \( \Gamma_2 \subset V := \Gamma_1 \otimes \mathbb{R} \) is then identified with the lattice dual to the image of \( \Gamma_1 \) under the isomorphism \( Q: V \rightarrow (V)^* \).
2.3. **Jacobian of a curve.** Let $C$ be a tropical curve of genus $g$. For $\Gamma_1$ we take the lattice of integral 1-cycles in any representative of $C$. That is $\Gamma_1 \cong \mathbb{Z}^g$. We define the symmetric bilinear form $Q$ on $\Gamma_1$ as $Q(\gamma, \gamma) = \text{length}(\gamma)$ on simple cycles and extend $Q$ bilinearly to arbitrary pairs of cycles. We call the resulting tropical torus $J(C)$ the *Jacobian* of the curve $C$.

There is a natural way to visualize the vector space $V$ and the second lattice $\Gamma_2 \subset V$ in geometric terms as follows. Let $\text{Aff}$ be the sheaf of $\mathbb{Z}$-affine functions (in some coordinate charts on $C$). Define the integral cotangent local system $T^*_{\mathbb{Z}}$ on $C$ by the following exact sequence of sheaves:

$$0 \to \mathbb{R} \to \text{Aff} \to T^*_{\mathbb{Z}} \to 0.$$  

The rank $g$ free abelian group of 1-forms $\Gamma_2^* = \Omega^1_{\mathbb{Z}}(C)$ on $C$ is formed by the global sections of $T^*_{\mathbb{Z}}$. Each such form can be thought of as an integral circuit on $C$ satisfying Kirchhoff’s law. Then $V = \Omega(C)^*$ is the vector space of $\mathbb{R}$-valued linear functionals on $\Omega^1_{\mathbb{Z}}(C)$. The integral cycles in the lattice $\Gamma_1 = H_1(C, \mathbb{Z})$ become linear functionals on $\Omega(C)$ by integration. The isomorphism $Q : \Gamma_1 \cong \Gamma_2^*$ is the tautological identification between integral cycles and integral circuits on $C$.

Then the definition of the tropical Jacobian

$$J(C) := V/\Gamma_1 = \Omega(C)^*/H_1(C, \mathbb{Z})$$

agrees with the classical one for Riemann surfaces.

2.4. **The Abel-Jacobi map.** Let us fix a reference point $p_0 \in C$. Then we can identify the set $\text{Div}^d(C)$ of degree $d$ divisors with $\text{Div}^0(C)$, thus giving a group structure on $\text{Div}^d(C)$ for any $d$. Given a divisor $D = \sum a_i p_i$ we choose paths from $p_0$ to $p_i$. Integration along these paths defines a linear functional on $\Omega^1_{\mathbb{Z}}(C)$:

$$\hat{\mu}(D)(\omega) = \sum a_i \int_{p_0}^{p_i} \omega.$$  

For another choice of paths the value of $\hat{\mu}(D)$ on $\omega$ will differ from the above by $\int_{\gamma} \omega$ for some $\gamma \in \Gamma_1$. Thus, we get a well-defined *Abel-Jacobi map* $\mu^d : \text{Div}^d(C) \to J(C)$.

We restrict our attention to the effective divisors of degree $d$. As was noted in [MZh07] the Abel-Jacobi map $\mu^1 : C \to J(C)$ is a tropical morphism, and so are the maps from the curve’s $k$-th powers $\mu^k : C^k \to J(C)$. Let $W_k \subset J(C)$ denote its image.

### 3. Algebraic cycles in $J(C)$

Algebraic $k$-cycles in $J(C)$ are weighed balanced $k$-dimensional polyhedral complexes with $\Gamma_2$-rational slopes. The $W_k$ provide examples of such cycles.
since they are images of tropical varieties (the powers of $C$) under tropical morphisms (cf. [Mik06]).

![Figure 1. A genus 3 curve and its image $W_1$ in the $J(C)$.](image)

On Fig. 1 we showed $W_1$ as a $\Gamma_1$-periodic 1-cycle in $V \cong \mathbb{R}^3$, the universal covering of $J(C)$. The space $V$ is filled by the maximal Voronoi cells, each is the zonotope, more precisely, the Minkowski sum of intervals corresponding to the edges of $C$. (cf. Section 4). We will use same zonotope as a fundamental domain for future pictures as well.

We say two cycles $Z_1$ and $Z_2$ in $J$ are *algebraically equivalent*: $Z_1 \sim Z_2$, if there is a tropical curve $S$ with two points $s_1, s_2 \in S$ and an algebraic cycle $W \subset J \times S$ (cf. [Mik06]) such that

$$
\pi_*[W \cap (J \times s_1) - W \cap (J \times s_2)] = Z_1 - Z_2.
$$

Here $\cap$ means tropical (or stable) intersection (cf. [Mik06] or [RST05]) and $\pi : J \times S \to S$ is the projection. Let $\pi' : J \times S \to J$ denote the other projection.

Recall that choice of the base point $p_0$ makes $J(C)$ into an abelian group. Notice, however, that another choice of $p_0$ will result in translation of the cycle $W_k$, which makes the choice of the base point irrelevant modulo algebraic equivalence. Given this group structure one can define action by integers $n : J \to J$ by $n \times x \mapsto nx$, which descends to an action on the Chow group. The main purpose of this article is to compare $W_k$ with $W_k^- := (-1)_*W_k$ modulo algebraic equivalence.

First we restrict our attention to the genus 3 curves. There are two combinatorial types of generic (trivalent) curves: $K_4$ and hyperelliptic (see Fig. 2).

Other genus 3 curves are degenerations of these two.
Figure 2. The two types of generic genus 3 curves.

**Theorem 3.** Let $C$ be a generic genus 3 curve of type $K_4$. Then $W_1$ is not algebraically equivalent to $W_1^− := (-1)_∗W_1$ in $J(C)$.

**Remark.** For curves of hyperelliptic type we do have $W_1 \sim W_1^-$, which reflects the fact that the Jacobians for these curves coincide with the Jacobian of the true hyperelliptic representative and the $W_1$ is a deformation of the Abel-Jacobi image of this hyperelliptic curve.

The proof of the theorem mimics Ceresa’s original proof [Cer83] in the complex case with some simplifications. The connecting 3-chains are replaced by tropical chains with coefficients in $Γ_2$ and tropical (co)homology (cf. [MZh13]) plays the rôle of the Hodge decomposition in $H^3(C)$. But we do not need to consider the full 6-dimensional family or use Griffiths transversality. Our argument works for any (generic) fixed tropical curve.

**3.1. Algebraic cycles and tropical homology.** Let $C$ be a curve of type $K_4$ and let $J$ be its Jacobian. Let $C_k(J, Γ_2)$ denote the group of polyhedral $k$-chains with coefficients in $Γ_2$, that is, an element $γ ∈ C_k(J, Γ_2)$ has the form $∑_σ β_σ σ$, where $β_σ ∈ Γ_2$ (we call $β_σ$ the framing vector at $σ$) and $σ : Δ → J$ is a linear map from a polytope $Δ ⊂ R^k$ to $J$. Sometimes we will abuse notation and identify $σ$ with its image in $J$. The usual boundary map $∂$ makes $C_*(J, Γ_2)$ into a chain complex. The universal coefficient theorem allows us canonically identify homology $H_k(J, Γ_2)$ of this complex with the groups $\bigwedge^k Γ_1 \otimes Γ_2$.

Given an algebraic 1-cycle $Y$ in $J$ one can associate to it a *tautological* tropical cycle $[Y] ∈ C_1(J, Γ_2)$ as follows (cf. [MZh13] for details). Every (oriented) edge $e ⊂ Y$ with weight $w_e$ defines a 1-cell $w_e σ_e$, and the primitive vector along this edge defines its framing $β_e$, an element in $Γ_2$. If the orientation of the edge is reversed, then so is the direction of the primitive vector. Both effects cancel in $C_1(J, Γ_2)$.

Let $Z_1$ and $Z_2$ be two 1-cycles which are algebraically equivalent. Then their classes $[Z_1] = [Z_2]$ in $H_1(J, Γ_2)$, since one can view the algebraic equivalence as a deformation family connecting $Z_1$ and $Z_2$ (cf. [Mik06]). Our next goal is to describe a particular element $γ ∈ C_2(J, Γ_2)$ with $∂γ = [Z_1] − [Z_2]$ on the chain level.
Suppose $W$, a cycle in $J \times S$, provides an algebraic equivalence between $Z_1$ and $Z_2$. Choose a path $P$ in $S$ between $s_1$ and $s_2$ and let $Z_s := \pi_s(W \cap (J \times s))$ for any $s \in P$. We also denote by $W_P$ the support of $\pi^{-1}(P) \cap W$ (set-theoretic intersection), which is a polyhedral complex in $J \times S$.

Let $\tau \subset W_P$ be a 2-cell with weight $w_{\tau}$. If $\pi' : J \times S \to S$ is transversal on $\tau$ we can define a vector field on $\tau$ by pulling back the tautological framing from the cycles $Z_s$, for $s \in \pi'(\tau) \subset S$. Because of the linearity of the maps $\pi$ and $\pi'$ on $\tau$ this vector field is constant on $\tau$, and thus defines the coefficient $\beta_{\tau}$. If $\pi'$ maps $\tau$ to a point in $S$ we set $\beta_{\tau} = 0$.

**Lemma 4.** Let $W$ be an algebraic equivalence between $Z_1$ and $Z_2$. Then the tropical 2-chain defined above

$$\gamma = \sum_{\tau \in W_P} w_{\tau} \beta_{\tau} \pi(\tau)$$

connects the corresponding tropical cycles: $\partial \gamma = [Z_1] - [Z_2]$.

**Proof.** By subdividing $S$ if necessary and using additivity it is enough to consider the case when $P$ is an edge and the map $\pi'$ is transversal on every $\tau \subset W_P$ in the preimage of the interior of $P$. Then the tropical (stable) intersection $W_P \cap J_s$ used to define $Z_s$ becomes the usual set-theoretic intersection. In particular we see that along any interior edge of $W_P$ the balancing condition for the cycle $W$ turns into the zero-boundary property of the chain $\gamma$.

Passing to the limit at the end points of $P$ and using continuity of the stable intersection we can identify boundary of $\gamma$ with the tautological cycle $[Z_1] - [Z_2]$. \qed

### 3.2. Determinantal 2-form and its periods

Let us fix a basis $e_1, e_2, e_3$ of $\Gamma_2$ and let $e^*_1, e^*_2, e^*_3$ be the dual basis of the dual lattice $\Gamma_2^*$. The choice of the basis defines canonically (up to translation) the linear coordinates $x_1, x_2, x_3$ on $J$. Let $dx_1, dx_2, dx_3$ be the corresponding constant 1-forms on $J$.

We will use notation $d\hat{x}_i = dx_{i+1} \wedge dx_{i+2}$, where we assume cyclic ordering of 1,2,3. Then we define the constant 2-from with coefficients in $\Gamma_2^*$ as

$$\Omega_0 := \sum_{i=1}^3 e^*_i d\hat{x}_i.$$

We can integrate any $\Gamma_2^*$-valued 2-form along any tropical 2-chain in $\gamma \in C_2(J, \Gamma_2)$ by evaluating the coefficients and then taking the ordinary integral.

**Lemma 5.** Let $Z_1$ and $Z_2$ be two algebraically equivalent 1-cycles in $J$ and let $\gamma_0$ be a connecting chain between $[Z_1]$ and $[Z_2]$ as in Lemma 4 above. Then $\int_{\gamma_0} \Omega_0 = 0$. 
Proof. For any triple of vectors $\beta, v_1, v_2 \in \Gamma_2$ evaluating $\Omega_0$ on the expression $v_1 \wedge v_2 \otimes \beta$ amounts to calculating the volume of the cube spanned by the triple. But note that on every $\sigma$ the corresponding framing vector $\beta_{\sigma}$ by construction of $\gamma_0$ lies in the linear span by $\langle \sigma \rangle$. Thus $\Omega_0|_{\beta_{\sigma}\sigma} \equiv 0$ for every $\sigma$ in the support of $\gamma_0$. □

We will need to calculate the periods of the form $\Omega_0$, that is the integrals over the elements in $H_2(J, \Gamma_2) = \bigwedge^2 \Gamma_1 \otimes \Gamma_2$. First let us choose a basis $e_1, e_2, e_3$ of $\Gamma_2$ as follows. Let $e_1$ be the evaluation functional of 1-forms in $\Omega(C)$ on the edge A (see Fig. 3). Similar $e_2$ and $e_3$ correspond to edges B and C respectively. Then we can write the three cycles (cf. Fig. 3) $\gamma_1, \gamma_2, \gamma_3$ which form a basis of $\Gamma_1$ into the matrix

\[
Q = \begin{pmatrix}
  a + e + f & -f & -e \\
  -f & b + d + f & -d \\
  -e & -d & c + d + e
\end{pmatrix},
\]

whose columns are the coordinates of the $\gamma$'s in terms of the $\{e_i\}$. Here $a, b, c, d, e, f \in \mathbb{R}_{\geq 0}$ are the lengths of the corresponding edges of the curve. One can identify $Q$ as the matrix of the polarization form on the abelian variety $J$ in the basis $\{\gamma_1, \gamma_2, \gamma_3\}$.

Then the periods of $\Omega_0$ are generated over $\mathbb{Z}$ by minors of $Q$:

\[
\int_{\hat{\gamma}_i \otimes e_j} \Omega_0 = M_{ij}.
\]

Here we use the notation $\hat{\gamma}_i$ similar to $d\hat{x}_i$ (e.g., $\hat{\gamma}_1 = \gamma_2 \wedge \gamma_3$). Explicitly, the 6 periods are:

\[
\begin{align*}
  ab + ad + af + be + de + ef + bf + df, \\
  ac + ad + ae + ce + de + cf + df + ef, \\
  bc + bd + be + cd + de + cf + df + ef,
\end{align*}
\]

\[
\begin{align*}
  ad + de + df + ef, \\
  be + de + df + ef, \\
  cf + df + ef + de.
\end{align*}
\]
More symmetric set of the generators would be

\[
\begin{align*}
&ad - be, \quad ad - cf, \\
&de + df + ef + ad, \\
&ab + af + bf + ad, \\
&ac + ae + ce + ad, \\
&bc + bd + cd + ad.
\end{align*}
\]

In words the periods are generated by the differences of the products of opposite pairs of edges and sums of products of all three pairs adjacent to a common vertex plus an opposite pair.

### 3.3. Proof of Theorem 3

Now we are ready to finish the proof of the theorem.

**Proof of Theorem 3.** We will choose a convenient representative of the cycle \(W_1 - W_1^-\) and a 2-chain \(\gamma_0 \in C_2(J, \Gamma_2)\) with \(\partial \gamma_0 = [W_1] - [W_1^-]\). Another choice \(\gamma'_0\) of the connecting chain will differ from \(\gamma_0\) by a cycle in \(C_2(J, \Gamma_2)\). Thus to prove the theorem it is sufficient to show that \(\int_{\gamma_0} \Omega_0 \neq 0\) modulo periods of \(\Omega_0\).

First we move \(W_1\) and \(W_1^-\) such that the image of one of the edges, say \(C\) (of blue color) in \(W_1\) coincides with that of \(W_1^-\), thus canceling in the \(W_1 - W_1^-\). For \(\gamma_0\) we choose the 2-chain supported on 5 parallelograms: MNPQ, NLKP, LMQK, KQSP and PSML (see Fig. 4). The orientation of all cells are counterclockwise if viewed from above (there are no vertical cells in \(\gamma_0\)).

To help a reader visualize the picture we give a view of the chain from...
above, where we indicated the coordinates of the framing vectors on all five parallelograms (see Fig. 5). It is an easy check that $\partial \gamma_0 = [W_1] - [W_1^-]$. We also see that the framing on all parallelograms except LMQK is parallel to the supporting cell (thus contributing 0 into the integral). The integration along LMQK contributes minus its area $-ad$ into $\int_{\gamma_0} \Omega_0$.

But the real number $ad$ is not in the lattice of the periods for generic choice of the edge lengths $a, b, c, d, e, f \in \mathbb{R}$. This completes the proof. □

**Remark.** Note that if one of the edges collapses, then $\int_{\gamma_0} \Omega_0 = -ad$ become zero modulo periods. This shows that the proof does not work for the hyperelliptic type curves. In fact one can easily construct an explicit deformation between $W_1$ and $W_1^-$ in $J$ in this case (see Fig. 6).

Figure 6. Deformation from $C$ to $C^-$ with hyperelliptic curve in the middle (all curves in the family have the same Jacobian).

4. Dicings, zonotopes and higher genus curves

By *lattice* we mean a positive definite quadratic form $Q$ on a free abelian group $\Lambda \cong \mathbb{Z}^g$. In our case the group $\Lambda$ will be $\Gamma_2^* \cong \mathbb{Z}^g$. Since $Q$ provides an isomorphism $\Gamma_2^* \cong \Gamma_1$ this is equivalent to define $Q$ on $\Gamma_1$ as in the definition of the tropical abelian varieties.

We say a lattice $(\Lambda, Q)$ is a *dicing* if the associated Delaunay decomposition of the real vector space $\Lambda \otimes \mathbb{R} \cong \mathbb{R}^g$ is given by families of parallel hyperplanes.
Figure 7. Faces of the zonotope.

intersecting at the lattice points. There are two other equivalent formulations of the dicing condition (cf., e.g. [Erd99]). First is that the defining quadratic form can be written as $Q = \sum \alpha_i (e_i)^2$, where the collection of linear functionals $\{e_i\}$ form a totally unimodular system in the dual lattice $\Lambda^*$. Second is that the maximal Voronoi cell is the zonotope, with zone vectors $\alpha_i e_i$.

Let us now return to the case of tropical Jacobian. To any edge one can associate an element $e_i$ of $\Gamma_2$ (as we did in the proof of Theorem 3) as the evaluation of 1-forms on a primitive vector at any interior point of the edge. Then we can write the polarization form as $Q = \sum \alpha_i (e_i)^2$ (the choice of signs for the $e_i$’s won’t matter). Here $\alpha_i$ is the length of the corresponding edge. Thus we have the following:

Proposition 6. The Jacobian lattice $(\Gamma_2^*, Q)$ is a dicing with zone vectors given by the edges of $C$.

It is convenient to take the Voronoi zonotope as the fundamental domain of $\Gamma_1$ action on $V = \Omega^*$. That what we did on all our pictures. One simple observation is that the codimension $k$ faces of the zonotope (which are zonotopes themselves) correspond to contracting genus $k$ subgraphs $C_k \subset C$ (see Fig. 7). This can be seen by setting the lengths of all edges in $C_k$ to zero.

More important for the main subject of this paper is what happens if we remove an edge from $C$. This will reduce the genus of $C$ by 1. Effectively this corresponds to consider only those 1-forms (currents) on $C$ which do not pass through the missing edge. Thus it results in the projection of the zonotope along the direction of this edge (see Fig. 8).

This simple observation combined with Theorem 3 leads us to the main result of the paper:

Theorem 7. Let $C$ be a tropical curve of genus $g$ whose underlying graph contains $K_4$ as a subgraph. Then $W_k$ is not algebraically equivalent to $W_k^-$ in $J(C)$ for $k = 1, \ldots, g - 2$. 
Figure 8. Removing an edge reduces genus and projects the zonotope.

Proof. Suppose \( W_1 \sim W_1^- \). This equivalence survives in the projection \( J(C) \to J(C \setminus e) \) (which is a tropical map), where \( C \setminus e \) is obtained from \( C \) by removing an edge. Finally we arrive at a genus 3 curve of type \( K_4 \) which provides the contradiction.

As for \( W_k \) with \( k > 1 \) we can formally follow Ceresa’s inductive argument in [Cer83]. Namely we can degenerate our curve to a (generic) curve \( C' \) of genus \( g - 1 \) connected to a loop \( E \) (elliptic curve) at a common vertex. Then \( W_k \) decomposes into a product \( W_{k-1}(C') \times E \) in \( J(C') \times E = J(C) \) and we use the induction hypothesis. \( \square \)

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