Asynchronous Decentralized Parallel Stochastic Gradient Descent

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Abstract

Recent work shows that decentralized parallel stochastic gradient decent (D-PSGD) can outperform its centralized counterpart both theoretically and practically. While asynchronous parallelism is a powerful technology to improve the efficiency of parallelism in distributed machine learning platforms and has been widely used in many popular machine learning softwares and solvers based on centralized parallel protocols such as Tensorflow, it still remains unclear how to apply the asynchronous parallelism to improve the efficiency of decentralized parallel algorithms. This paper proposes an asynchronous decentralize parallel stochastic gradient descent algorithm to apply the asynchronous parallelism technology to decentralized algorithms. Our theoretical analysis provides the convergence rate or equivalently the computational complexity, which is consistent with many special cases and indicates we can achieve nice linear speedup when we increase the number of nodes or the batchsize. Extensive experiments in deep learning validate the proposed algorithm.

1 Introduction

While the centralized optimization algorithms such as parallel stochastic gradient descent have been widely used to develop modern machine learning tools and softwares such as Tensorflow [Abadi et al., 2016] and MXNet [Chen et al., 2015], a recent work [Lian et al., 2017] has revealed that decentralized parallel stochastic gradient descent is comparable to its centralized counterparts and can be much more efficient for high latency computational networks.

The centralized computational framework consists of a central node and a bunch of surrounding child nodes: the central node maintains a shared model, splits the computational workload to child nodes, and updates the model parameter using the results from child nodes. Although the centralized framework is easy to extend and implement existing optimization or computational algorithms without major modification, it may suffer from the communication bottleneck at the central node since all nodes can only exchange information through the central node.

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In contrast, the decentralized computational framework does not use any central node, which avoids the communication bottleneck. It could be built on an arbitrary connected network. Figure 1 illustrates the difference between the centralized network and the decentralized network. Decentralized algorithms are mainly studied as consensus optimization problems by the control community [Ram et al., 2010, Yan et al., 2013, Yuan et al., 2016] for the application scenario where only a decentralized network is available. It was unclear if decentralized algorithms could have advantages over centralized algorithms in some scenarios where both types of networks are feasible, until a recent study [Lian et al., 2017] validates the advantage of decentralized parallel stochastic gradient decent (D-PSGD) algorithm over its centralized counterpart in both theory and empirical study.

However, the potential of decentralized algorithms has not been fully developed, since a clear drawback of D-PSGD lies on the heavy synchronization overhead. Specifically, in D-PSGD to step into the next iteration, all nodes are required to finish updating gradients and communicate with their neighbors, which means many nodes may wait for the slowest one. To avoid such synchronization overhead, in the spirit of asynchronous parallel algorithms for centralized networks Agarwal and Duchi [2011], Avron et al. [2014], Feyzmahdavian et al. [2015], Feyzmahdavian et al. [2016], Liu et al. [2014], Mania et al. [2015], Paine et al. [2013], Recht et al. [2011], Sridhar et al. [2013], Tran et al. [2015], Zhang et al. [2014], this paper proposes an asynchronous decentralized parallel stochastic gradient decent (AD-PSGD) algorithm, where all nodes can feel free to communicate with their individual neighbors. To avoid deadlock, a bipartite graph based implementation is also proposed. Our theoretical analysis shows that the proposed AD-PSGD algorithm can admit a convergence rate of \( O\left(\frac{1}{\sqrt{K}}\right) \) where \( K \) is the number of updates. Our result is consistent with many special cases such as SGD and D-PSGD. Extensive experiments validate the proposed algorithm and theory.

Definitions and notations   Throughout this paper, we use the following notation and definitions:

- \( \| \cdot \| \) denotes the vector \( \ell_2 \) norm or the matrix spectral norm depending on the argument.
- \( \| \cdot \|_F \) denotes the matrix Frobenius norm.
- \( \nabla f(\cdot) \) denotes the gradient of a function \( f \).
- \( 1_n \) denotes the column vector in \( \mathbb{R}^n \) with 1 for all elements.
- \( f^* \) denotes the optimal solution to (1).
- \( \lambda_i(\cdot) \) denotes the \( i \)-th largest eigenvalue of a matrix.
- \( e_i \) denotes the \( i \)-th element of the standard basis of \( \mathbb{R}^n \).

\(^{1}\)Of course, the centralized network can be also considered as a special case of the decentralized network.
2 Related work

We review related work in this section. In the following, \( K \) and \( n \) refer to the number of iterations and the number of nodes, respectively. A comparison of the algorithms can be found in Table 1.

The **Stochastic Gradient Descent (SGD)** [Ghadimi and Lan [2013], Moulines and Bach [2011], Nemirovski et al. [2009]] is a powerful approach to solving large scale machine learning problems, with the optimal convergence rate \( O(1/\sqrt{K}) \) on nonconvex problems.

For **Synchronous Parallel Stochastic Gradient Descent (S-PSGD)**, every node fetches the model saved in a parameter server and computes a minibatch of stochastic gradients. Then they push the stochastic gradients to the parameter server. The parameter server synchronizes all the stochastic gradients and update their average into the model saved in the parameter server, which completes one iteration. The convergence rate is proved to be \( O(1/\sqrt{nK}) \) on nonconvex problems [Ghadimi et al., 2016]. Results on convex objectives can be found in Dekel et al. [2012]. Due to the synchronization step, all other nodes have to stay idle to wait for the slowest one. In each iteration the parameter server needs to synchronize \( O(n) \) nodes, which causes high communication cost especially \( n \) is large.

The **Asynchronous Parallel Stochastic Gradient Descent (A-PSGD)** [Agarwal and Duchi, 2011, Feyzmahdavian et al., 2016, Paine et al., 2013, Recht et al., 2011] breaks the synchronization in S-PSGD. A node in A-PSGD blindly executes this loop: 1) pull the model from the parameter server, and 2) compute a minibatch of stochastic gradients and push it to the parameter server. The parameter server updates the model as long as it receives a stochastic gradient. In this way, asynchronous algorithms significantly reduce the communication overhead by avoiding idling any node. The main issue with asynchronous algorithms lies on using stale models to compute the stochastic gradients, since while node \( a \) is computing stochastic gradients, another node \( b \) updates the model in the parameter server, which renders the model node \( a \) is using a stale one. On nonconvex problems, when this staleness is bounded, A-PSGD is proved to admit the same \( O(1/\sqrt{Kn}) \) convergence rate as S-PSGD [Lian et al., 2015, 2016]. In practice, the asynchronous variations can usually improve the efficiency of their synchronous counterparts.

In **AllReduce Stochastic Gradient Descent implementation (AllReduce-SGD)** [Luehr, 2016, MPI contributors, 2015, Patarasuk and Yuan, 2009], the update rule per iteration is exactly the same as in S-PSGD, so they share the same convergence rate. However, there is no parameter server in AllReduce-SGD, where the nodes are connected with a ring network and each node keeps the same local copy of the model. In each iteration, each node calculates a minibatch of stochastic gradients. Then all the nodes use AllReduce to synchronize the stochastic gradients, after which each node will get the average of all stochastic gradients. In this procedure, only \( O(1) \) amount of gradient is sent/received per node, but each node needs \( O(n) \) handshakes due to the limitation of the AllReduce algorithm. At the end of the iteration the averaged gradient is updated into the local model of each node. Since we still have synchronization in each iteration, the idle time is still high as in S-PSGD.

We next briefly review decentralized algorithms. Decentralized algorithms were initially studied by the control community for solving the consensus problem where the goal is to compute the mean of all the data distributed on multiple nodes [Aysal et al., 2009, Boyd et al., 2005, Carli et al., 2010, Fagnani and Zampieri, 2008, Olfati-Saber et al., 2007, Schenato and Gamba, 2007]. For decentralized algorithms used for optimization problems, Lu et al. [2010] proposed two non-gradient-based algorithms for solving one-dimensional unconstrained convex optimization problems where the objective on each node is strictly convex, by calculating the inverse function of the derivative of the local objectives and transmitting the gradients or local objectives to neighbors, and the algorithms can be used over networks with time-varying topologies. A convergence rate was not shown but the authors did prove the algorithms will converge to the solution eventually. Mokhtari and Ribeiro [2016] proposed a fast decentralized variance reduced algorithm for strongly convex optimization problems. The algorithm is proved to have linear convergence rate and a nice stochastic saddle point method interpretation is given. However, the speedup property is unclear and a table of stochastic gradients need to be stored. Yuan et al. [2016] studied decentralized
gradient descent on convex and strongly convex objectives. The algorithm in each iteration averages the models of the nodes with their neighbors’ and then updates the full gradient of the local objective function on each node. The subgradient version was considered in Nedic and Ozdaglar [2009], Ram et al. [2009]. The algorithm is intuitive and easy to understand. However, the limitation of the algorithm is that it does not converge to the exact solution because the exact solution is not a fixed point of the algorithm’s update rule. This issue was fixed later by Shi et al. [2015a], Wu et al. [2016] by using the gradients of last two instead of one iterates in each iteration, which was later improved in Li et al. [2017], Shi et al. [2015b] by considering proximal gradients. Decentralized ADMM algorithms were analyzed in Aybat et al. [2015], Shi et al., Zhang and Kwok [2014]. Wang et al. [2016] develops a decentralized algorithm for recursive least-squares problems.

As for **Decentralized Parallel Stochastic Gradient Descent (D-PSGD) [Lian et al., 2017]**, all nodes are connected with a network that forms a connected graph G. Every node has its local copy of model. In each iteration, all nodes compute stochastic gradients locally and at the same time average its local model with its neighbors. Then the locally computed stochastic gradients are updated into the local models. In this procedure, the busiest node only send/receive $O(\deg(G))$ models (on a ring network, $\deg(G) = O(1)$, which is the same as in AllReduce SGD). Unlike $O(n)$ handshakes per iteration in AllReduce SGD, D-PSGD only needs $O(\deg(G))$ handshakes per iteration on the busiest node. Note that in D-PSGD the computation and communication can be done in parallel, which means, when communication time is smaller than the computation time, the communication can be completely hidden. The idle time is still high in D-PSGD, since in each iteration, all nodes still need to be synchronized. Before Lian et al. [2017] there are also previous studies on decentralized stochastic algorithms (both synchronous and asynchronous versions) though none of them is proved to have speedup when the number of nodes increases. For example, Lan et al. [2017] proposed a decentralized stochastic primal-dual type algorithm with a computational complexity of $O(n/\epsilon^2)$ for general convex objectives and $O(n/\epsilon)$ for strongly convex objectives. Sirb and Ye [2016] proposed an asynchronous decentralized stochastic algorithm with a $O(n/\epsilon^2)$ complexity for convex objectives. These bounds do not imply any speedup for decentralized algorithms. Bianchi et al. [2013] proposed a similar decentralized stochastic algorithm. The authors provided a convergence rate for the consensus of the local models when the local models are bounded. The convergence to a solution was provided by using central limit theorem. However, they did not provide the convergence rate to the solution. Ram et al. [2010] proposed an asynchronous subgradient variations of the decentralized stochastic optimization algorithm for convex problems. The asynchrony was modeled by viewing the update event as a Poisson process and the convergence to the solution was shown. Srivastava and Nedic [2011], Sundhar Ram et al. [2010] are similar. The main differences from this work are 1) we take the situation where a node calculates gradients...
based on old model into consideration, which is the case in the asynchronous setting; 2) we prove the our algorithm can achieve linear speedup when we increase the number of learners, which is important if we want to use the algorithm to accelerate training; 3) we solve the deadlock problem; and 4) in our algorithm the communication and the computation can be overlapped.

3 AD-PSGD algorithm

We introduce the proposed AD-PSGD algorithm in this section. The decentralized communication topology is represented as an undirected graph: \((V, E)\), where \(V := \{1, 2, \ldots, n\}\) denotes the set of \(n\) nodes and \(E \subseteq V \times V\) is the set of the edges in the graph. Each node represents a machine owning its local data (or a sensor collecting local data online) such that each node is associated with a local loss function

\[ f_i(x) := \mathbb{E}_{\xi \sim D_i} F_i(x; \xi), \]

where \(D_i\) is a distribution associated with the local data at node \(i\) and \(\xi\) is a data point sampled via \(D_i\). The edge means that the connected two nodes can exchange information.

For the AD-PSGD algorithm, the overall optimization problem it solves is

\[
\min_{x \in \mathbb{R}^N} f(x) := \mathbb{E}_{i \sim I} f_i(x) = \sum_{i=1}^{n} p_i f_i(x),
\]

(1)

where \(p_i\)'s define a distribution, that is, \(p_i \geq 0\) and \(\sum p_i = 1\), and \(p_i\) indicates the updating frequency of node \(i\) or the percentage of node \(i\) performing updates. The faster a node, the higher the corresponding \(p_i\).

The intuition is that if a node is faster than another node, then the faster node will run more epochs given the same amount of time.

To solve the common form of objectives in machine learning using AD-PSGD

\[
\min_{x \in \mathbb{R}^N} \mathbb{E}_{\xi \sim D} F(x; \xi),
\]

we can appropriately distribute data (which also defines functions \(F_i\)'s) such that Eq. (1) solves the target objective above:

**Strategy-1** Let \(D_i = D\) and \(D\), that is, all nodes can access a shared database, and and \(F_i(\cdot; \cdot) = F(\cdot; \cdot)\), that is, all \(f_i(\cdot)\)'s are the same;

**Strategy-2** Split the data into all nodes appropriately such that the portion of data is \(p_i\) on node \(i\) and define \(D_i\) to be the uniform distribution over the assigned data samples.

From the perspective of each node, the AD-PSGD algorithm can be described in the following: each node maintains a local model \(x\) in its local memory and (using node \(i\) as an example in the following) repeats the following steps:

- **Sample data**: Sample a mini-batch of training data denoted by \(\{\xi_i^m\}_{m=1}^M\), where \(M\) is the batch size.

- **Compute gradients**: Use the sampled data to compute the stochastic gradient \(\sum_{m=1}^{M} \nabla F(\hat{x}_i^m; \xi_i^m)\), where \(\hat{x}_i\) is read from the model in the local memory.

- **Gradient update**: Update the model in the local memory by \(x^i \leftarrow x^i - \gamma \sum_{m=1}^{M} \nabla F(\hat{x}_i^m; \xi_i^m)\). Note that \(\hat{x}_i\) may not be the same as \(x^i\) as it may be modified by other nodes in the **averaging** step.

- **Averaging**: Randomly select a neighbor (say \(i'\)) and average the local model with the selected neighbor’s model \(x^{i'}\) (both models on two nodes are changed to the averaged model). More specifically, \(x^i, x^{i'} \leftarrow \frac{x^i}{2} + \frac{x^{i'}}{2}\).
The averaging step can be generalized into the following update for all nodes:

\[
x_1, x_2, \ldots, x_n \leftarrow [x_1, x_2, \ldots, x_n]W
\]

where \(W\) can be an arbitrary doubly stochastic matrix. This generalization gives plenty flexibility to us in implementation without hurting our analysis.

All nodes run this procedure above concurrently. From the global viewpoint, the algorithm can be described in Algorithm 1. We use \(k\) to denote the iteration counter – every single gradient update happens no matter on which node will increase \(k\) by 1. \(i_k\) is the node performing the \(k\)th update.

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**Algorithm 1 AD-PSGD (global view)**

**Require:** Initialize local models \(\{x_i^0\}_{i=1}^n\) with the same initialization, learning rate \(\gamma^i\), batch size \(M\), and total number of iterations \(K\).

1. for \(k = 0, 1, \ldots, K-1\) do
   2. Randomly sample a node \(i_k\) of the graph \(G\) and randomly sample an averaging matrix \(W_k\) which can be dependent on \(i_k\).
   3. Randomly sample a batch \(\xi_{k,i_k} := (\xi_{k,1}, \xi_{k,2}, \ldots, \xi_{k,M})\) from local data of the \(i_k\)-th node.
   4. Compute the stochastic gradient locally \(g_k(x_k^i; \xi_{k,i}) := \sum_{j=1}^{M} \nabla F(x_{k,j}^i, \xi_{k,j})\).
   5. Average local modes by \(x_k^i \leftarrow x_{k+1/2}^i - \gamma g_k(x_k^i; \xi_{k,i})\) and \(x_{k+1}^i \leftarrow x_{k+1/2}^i\) for \(j \neq i_k\).
   6. output the average of the model of all nodes.

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\(^a\)We will see in the theoretical analysis that \(\gamma\) should scale with \(n\).

\(^b\)Note that Line 4 and Line 5 can run concurrently.

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### 3.1 A smarter implementation to avoid deadlock

Acute readers may notice that the implementation described above may cause deadlock, as the averaging step needs to be atomic and involves updating two nodes (the selected node and one of its neighbors).

For example, suppose that there are three fully connected nodes \(A\), \(B\), and \(C\). \(A\) sends its local model \(x_A\) to \(B\) and waits for \(x_B\) from \(B\); \(B\) has already sent out \(x_B\) to \(C\) and waits for \(C\)'s response; and \(C\) has sent out \(x_C\) to \(A\) and waits for \(x_A\) from \(A\).

The deadlock can be avoided by a smarter implementation in the following. The computational network needs to be designed to be a bipartite graph, that is, the node set \(V\) can be split into two disjoint sets \(A\) (active set) and \(P\) (passive set) such that any edge in the graph connects one node in \(A\) and one node in \(P\).

Due to the property of the bipartite graph, the neighbors of any active node can only be passive nodes and the neighbors of any passive node can only be active nodes.

Every node (including active and passive nodes) runs two threads in parallel without synchronization: a computation thread to compute stochastic gradients as shown in Algorithm 2 and a communication thread to average and update the local model using local stochastic gradient. Each node has a local buffer \(g\) for sharing gradients between the computation thread and communication thread. Every active node’s communication thread actively sends out its local model and averages with a randomly chosen passive node, as shown in Algorithm 3, while every passive node’s communication thread only passively waits for averaging requests from active nodes as shown in Algorithm 4. Note that this implementation avoids the deadlock, but still fits in the general algorithm Algorithm 1 we are analyzing.
Algorithm 2 Computation thread on active or passive node (node index is $i$)

**Require:** Batch size $M$

1. **while** not terminated **do**
2. Pull model $x^i$ from the communication thread.
3. Update locally in the thread $x^i \leftarrow x^i - \gamma g^a$.
4. Randomly sample a batch $\xi^i := (\xi_{i1}^i, \xi_{i2}^i, \ldots, \xi_{iM}^i)$ from local data of the $i$-th node and compute the stochastic gradient $g^i(x^i; \xi^i) := \sum_{m=1}^{M} \nabla F(x^i; \xi_{im}^i)$ locally.
5. **wait until** $g = 0$ **then**
6. Local buffer $g \leftarrow g^i(x^i; \xi^i)$.
7. **end wait until**
8. **end while**

\(^a\)At this time the communication thread may have not update $g$ into $x^i$ so the computation thread pulls an old model. We compensate this by doing local update in computation thread. We observe this helps the scaling.

\(^b\)We can also make a queue of gradients here to avoid the waiting. Note that doing this will make the effective batch-size different from $M$.

Algorithm 3 Communication thread on active node (node index is $i$)

**Require:** Initialize local model $x^i$, learning rate $\gamma$.

1. **while** not terminated **do**
2. **if** $g \neq 0$ **then**
3. $x^i \leftarrow x^i - \gamma g$, $g \leftarrow 0$.
4. **end if**
5. Randomly select a neighbor (namely node $j$). Send $x^i$ to node $j$ and fetch $x^j$ from it.
6. $x^i \leftarrow \frac{1}{2}(x^i + x^j)$.
7. **end while**

Algorithm 4 Communication thread on passive node (node index is $j$)

**Require:** Initialize local model $x^j$, learning rate $\gamma$.

1. **while** not terminated **do**
2. **if** $g \neq 0$ **then**
3. $x^j \leftarrow x^j - \gamma g$, $g \leftarrow 0$.
4. **end if**
5. **if** receive the request of reading local model (say from node $i$) **then**
6. Send $x^j$ to node $i$.
7. $x^j \leftarrow \frac{1}{2}(x^i + x^j)$.
8. **end if**
9. **end while**

4 Theoretical analysis

In this section we provide theoretical analysis for the AD-PSGD algorithm. We will show that the convergence rate of AD-PSGD is consistent with SGD and D-PSGD.

Note that by counting each update of stochastic gradients as one iteration, the update of each iteration in Algorithm 1 can be viewed as

$$X_{k+1} = X_k W_k - \gamma \partial g(\hat{X}_k; \xi_k^i, i_k),$$

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$$X_{k+1} = X_k W_k - \gamma \partial g(\hat{X}_k; \xi_k^i, i_k),$$
where \( k \) is the iteration number, \( x_i^k \) is the local model of the \( i \)th node at the \( k \)th iteration, and

\[
X_k = \begin{bmatrix} x_1^k & \cdots & x_n^k \end{bmatrix} \in \mathbb{R}^{N \times n},
\]

\[
\hat{X}_k = \begin{bmatrix} x_1^k & \cdots & x_n^k \end{bmatrix} \in \mathbb{R}^{N \times n},
\]

\[
\partial g(\hat{X}_k; \xi_{k,i_k}) = \begin{bmatrix} 0 & \cdots & 0 & \sum_{j=1}^{M} \nabla f(\hat{x}_j^k, \xi_{k,j}) & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{N \times n},
\]

and \( \hat{X}_k = X_{k-\bar{n}} \) for some nonnegative integer \( \bar{n} \).

**Assumption 1.** Throughout this paper, we make the following commonly used assumptions:

1. **Lipschitzian gradient:** All functions \( f_i(\cdot) \)'s are with L-Lipschitzian gradients.
2. **Doubly stochastic averaging:** \( W_k \) is doubly stochastic for all \( k \).
3. **Spectral gap:** There exists a \( \rho \in [0,1) \) such that
   \[
   \max\{ |\lambda_2(\mathbb{E}[W_k^\top W_k])|, |\lambda_n(\mathbb{E}[W_k^\top W_k])| \} \leq \rho, \forall k.
   \]
4. **Unbiased estimation:**
   \[
   \mathbb{E}_{\xi \sim D_i} \nabla f(x; \xi) = \nabla f_i(x),
   \]
   \[
   \mathbb{E}_{i \sim \mathcal{I}} \mathbb{E}_{\xi \sim D_i} \nabla f(x; \xi) = \nabla f(x).
   \]
   Note that this is easily satisfied when all nodes can access all data so that \( \mathbb{E}_{\xi \sim D_i} \nabla f(x; \xi) = \nabla f(x) \). When each node can only access part of the data, the sampling of \( i \) need to be sophisticated.
5. **Bounded variance:** Assume the variance of the stochastic gradient
   \[
   \mathbb{E}_{\xi \sim D_i} \| \nabla f(x; \xi) - \nabla f_i(x) \|^2 \leq \sigma^2, \forall i, \forall x.
   \]
   \[
   \mathbb{E}_{i \sim \mathcal{I}} \| \nabla f_i(x) - \nabla f(x) \|^2 \leq \zeta^2, \forall x.
   \]
   Note that if all nodes can access all data, then \( \zeta = 0 \).
6. **Dependence of random variables:** \( \xi_k, i_k, k \in \{0,1,2,\ldots\} \) are independent random variables. \( W_k \) is a random variable dependent on \( i_k \).
7. **Bounded staleness:** \( \hat{X}_k = X_{k-\bar{n}} \) and there exists a constant \( T \) such that \( \max_k \bar{n}_k \leq T \).

Throughout this paper, we define the following notations for simpler notation

\[
\bar{\rho} := \frac{n - 1}{n} \left( \frac{1}{1 - \rho} + \frac{2\sqrt{\bar{\rho}}}{(1 - \sqrt{\bar{\rho}})^2} \right),
\]

\[
C_1 := 1 - 24M^2L^2\gamma^2 \left( T \frac{n - 1}{n} + \bar{\rho} \right),
\]

\[
C_2 := \frac{\gamma M}{2n} - \frac{\gamma^2 LM^2}{n^2} - \frac{2M^3L^2T^2\gamma^3}{n^3} - \left( \frac{6\gamma^2L^3M^2}{n^2} + \frac{\gamma M L^2}{n} + \frac{12M^3L^4T^2\gamma^3}{n^3} \right) \frac{4M^2\gamma^2(T\frac{n - 1}{n} + \bar{\rho})}{C_1},
\]

\[
C_3 := \frac{1}{2} + \frac{2}{C_1} \left( \frac{6\gamma^2L^2M^2 + \gamma n ML + \frac{12M^3L^4T^2\gamma^3}{n^3}}{\gamma M L^2 + \frac{12M^3L^4T^2\gamma^3}{n^3}} \right) \frac{\bar{\rho}}{C_1} + \frac{LT^2\gamma M}{n}.
\]

Under Assumption 1 we have the following results:
Theorem 1 (Main theorem). While \( C_3 \leq 1 \) and \( C_2 \geq 0 \) and \( C_1 > 0 \) are satisfied we have

\[
\sum_{k=0}^{K-1} \mathbb{E} \left\| \nabla f \left( \frac{X_k}{n} \right) \right\|^2_K \leq 2(\mathbb{E}f(x_0) - \mathbb{E}f^*)n + \frac{2\gamma L (M\sigma^2 + 6M^2c^2)}{\gamma KM}.
\]

Noting that \( \frac{X_k}{n} = \frac{1}{n} \sum_{i=1}^{n} \chi_i \), this theorem characterizes the convergence of the average of all local models. By appropriately choosing the learning rate, we obtain the following corollary

Corollary 2. Let \( \gamma = \frac{n}{10ML + \sqrt{\sigma^2 + 6M^2c^2} \sqrt{KM}} \). We have the following convergence rate

\[
\sum_{k=0}^{K-1} \mathbb{E} \left\| \nabla f \left( \frac{X_k}{n} \right) \right\|^2_K \leq \frac{20(f(x_0) - f^*)L}{K} + \frac{2(f(x_0) - f^* + L)\sqrt{\sigma^2/M + 6c^2}}{\sqrt{K}}.
\]

if

\[
K \geq \frac{ML^2n^2}{\sigma^2 + 6M^2c^2} \max \left\{ 192 \left( \frac{n-1}{\pi} + \frac{\bar{\rho}}{\bar{\rho}} \right), \frac{64T^2}{n^2}, 1024n^2\bar{\rho}^2, \left( \frac{8\sqrt{\sigma^2/3} + 8}{\pi^2} \right)^{2/3} \gamma \right\}.
\]

This corollary basically indicates that if the iteration number is big enough, AP-PSGD converges with a sublinear rate \( O(1/\sqrt{K}) \). We compare the convergence rate of AD-PSGD with existing results for SGD and D-PSGD to show the tightness of the proved convergence rate. We will also show the efficiency and the linear speedup property for AD-PSGD w.r.t. batch size, number of nodes, and staleness respectively.

Remark 1 (Consistency with SGD). Note that if \( T = 0 \) and \( n = 1 \) the proposed AD-PSGD reduces to the vanilla SGD algorithm Ghadimi and Lan [2013], Moulines and Bach [2011], Nemirovski et al. [2009]. Since \( n = 1 \), we do not have the variance among nodes, that is, \( \varsigma = 0 \), the convergence rate becomes \( O(1/\sqrt{K}) \) which is consistent with the convergence rate with SGD.

Remark 2 (Linear speedup w.r.t. batch size). When \( K \) is large enough the second term on the RHS of (7) dominates the first term. Note that the second term converges at a rate \( O(1/\sqrt{MK}) \) if \( \varsigma = 0 \), which means the convergence efficiency gets boosted with a linear rate if increase the mini-batch size. This observation indicates the linear speedup w.r.t. the batch size and matches the results of mini-batch SGD.

Note that when \( \varsigma^2 \neq 0 \), AD-PSGD does not admit this linear speedup w.r.t. batch size. It is unavoidable because increasing the minibatch size only decreases the variance of the stochastic gradients within each node, while \( \varsigma^2 \) characterizes the variance of stochastic gradient among different nodes, independent of the batch size.

Remark 3 (Linear speedup w.r.t. number of nodes). Note that every single stochastic gradient update counts one iteration in our all analysis and our convergence rate in Corollary 2 is consistent with SGD / mini-batch SGD. It means that the number of required stochastic gradient update to achieve a certain precision is consistent with SGD / mini-batch SGD, as long as the total number of iterations is large enough. It further indicates the linear speedup with respect to the number of nodes \( n \) (\( n \) nodes will make the iteration number advance \( n \) times faster, which means we will converge \( n \) times faster.). To the best of our knowledge, the linear speedup property w.r.t. to the number of nodes for decentralized algorithms has not been recognized until the recent analysis for D-PSGD by Lian et al. [2017]. Therefore, our analysis reveals similar observation but our asynchronous algorithm is more generic and more efficient in practice.

Remark 4 (Linear speedup w.r.t. the staleness). From (8) we can also see that as long as the staleness \( T \) is bounded by \( O(K^{1/4}) \) (if other parameters are considered to be constants), the linear speedup is achievable.

5 Experiments

In this section we provides our experimental results.
Compared algorithms We compare the proposed AD-PSGD algorithm to D-PSGD Lian et al. [2017] and AllReduce-SGD MPI contributors [2015].

Configuration All experiments are run on IBM res92 HPC cluster. Each computing node has 4 P100 GPUs, 20 Power 8 cores, each clocked at 3.84GHz and 500GB memory, the communication speed between nodes is 100Gb/s mellonox EDR infiniband. We use CIFAR10 as the testing dataset and use VGG and ResNet-20 as the deep learning model. CIFAR10 contains 50K 32x32 color images that fall into 10 different classes. The VGG model is of 60MB size and RESNET-20 model is of 1MB size. We use Torch-7 as the underlying solver for stochastic gradient computation and multi-threaded MPI processes for communication. The network topology is a ring with an even number of nodes which forms a bipartite graph.

We will show that AD-PSGD/D-PSGD converges similar to AllReduce and sometimes better than AllReduce even w.r.t. epochs. That means the decentralized algorithms (AD-PSGD/D-PSGD) will generally converge faster when it comes to runtime if the communication cost is high since they reduce the communication time.

When the performance of the devices varies, the AD-PSGD will converge much faster in runtime than both D-PSGD and AllReduce. Interestingly, with AD-PSGD, the final accuracy is constantly better than AllReduce. This may indicate decentralized optimization can help the generalization.

Empirical study on fast networks (100Gb/s Infiniband) In Figure 2 we plot loss for 16 nodes on VGG/ResNet-20 and choose the batch size to be 128 for VGG and 32 for ResNet-20. For VGG, the initial learning rate is 1 and reduced by half for every 25 epochs. For RESNET-20, the initial learning rate is 0.1, and it decays by a factor of 10 at the 81st epoch and decays by another factor of 10 at the 122nd epoch. For the convergence w.r.t. epochs, all three algorithms look similar. On VGG, AD-PSGD is slightly better than the other two algorithms. On ResNet, AllReduce is slightly better than the other two algorithms.

However when it comes to the runtime performance shown in Figure 3, we observe the decentralized algorithms are faster than AllReduce when the communication is intensive (e.g., VGG with a 60x larger model than ResNet). For ResNet, the runtime performance for AllReduce is slightly better since the model size is only 1MB and the network is very fast, which means the communication cost is not an issue. The runtime performance of AD-PSGD is better than the D-PSGD in both cases, since it avoids the synchronization cost. In Figure 4 we plot the speedup for 16 nodes for the VGG and ResNet-20, which is calculated by comparing the time taken for one iteration in each algorithm.

Empirical study on networks with efficiency diverse nodes A major advantage of asynchronous algorithms over their synchronous counterparts is that it can dynamically load-balance the system and transparently handle the situation where the speed of each participant is heterogeneous. We adjust the 16 learners so that one learner is 2X, 10X, 100X, 1000X and 10000X slower respectively and compare the performance on those networks. Table 3 reports the speedup when there is a slower node, where we can observe AD-PSGD is robust to networks with some extremely slow nodes. In contrast, the synchronous protocol such like AllReduce is significantly lagged due to the slow node and the runtime performance advantage of AD-PSGD can be orders of magnitude faster. We plot the loss w.r.t. epoch and runtime when there is a slower learner in Figure 5. Figure 5a shows that despite a very slow learner in the system, AD-PSGD maintains a robust convergence rate that is comparable to AllReduce and D-PSGD. In addition, the test accuracy of the model trained by AD-PSGD, in the presence of a slow learner, stays above 91%, which is still better than the result produced by using AllReduce (90.72%). Figure 5b shows that the training loss decreases significantly faster in runtime by using AD-PSGD than AllReduce and D-PSGD when there is a slower learner.

The update rule for the AllReduce implementation is exactly the same as in the parameter server implementation, so the convergence w.r.t. epochs will be the same. However, since AllReduce has lower communication cost than the parameter server implementation, for runtime AllReduce will generally be faster. Thus we omit the comparison with parameter server here.
Table 2: Testing accuracy comparison for AllReduce, D-PSGD, and AD-PSGD. The VGG model has been trained for 100 epochs, and the ResNet model for 164 epochs.

| Model     | AllReduce | D-PSGD | AD-PSGD |
|-----------|-----------|--------|---------|
| VGG       | 84.04%    | 84.48% | 88.58%  |
| ResNet    | 90.72%    | 90.81% | 91.02%  |

Conjecture: AD-PSGD is easier to escape bad local optimum. Table 2 reports the testing accuracy of all algorithms. We can observe that decentralized algorithms tend to achieve a better accuracy and the asynchronous algorithm achieves the best performance. Our conjecture is that both decentralization and asynchronicity can bring additional noise to the stochastic gradient, which is helpful to improve the escape from some shallow local optimum.

6 Conclusion

This paper proposes an asynchronous decentralized stochastic gradient descent algorithm, combining two key parallelism techniques: asynchronization and decentralization. Although the asynchronous parallelism has been proven to be efficient by many work and decentralized algorithms was shown to be more efficient than centralized algorithms recently, it still remains unknown how to combine these techniques to make them boost each other. In this paper, the proposed algorithm is proven to admit a fast convergence rate and good speedup property with respect to the number of computation nodes, the staleness of asynchronity, and the mini-batch size. Extensive experiments validate the proposed algorithm.
Figure 3: Runtime comparison on CIFAR10 using VGG/Resnet on fast networks.

Table 3: Comparison on efficiency diverse networks for CIFAR-10 dataset with ResNet-20 model.

| Slowdown of one node | AD-PSGD  | AllReduce/D-PSGD |
|----------------------|----------|------------------|
|                      | Time/epoch (sec) | Speedup | Time/epoch (sec) | Speedup |
| no slowdown          | 1.22      | 14.78            | 1.47/1.45        | 12.27/12.44 |
| 2X                   | 1.28      | 14.09            | 2.6/2.36         | 6.93/7.64   |
| 10X                  | 1.33      | 13.56            | 11.51/11.24      | 1.56/1.60   |
| 100X                 | 1.33      | 13.56            | 100.4/100.4      | 0.18/0.18   |
| 1000X                | 1.33      | 13.56            | >1000            | <0.018      |
| 10000X               | 1.33      | 13.56            | >10000           | <0.0018     |

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Figure 4: Speedup for CIFAR10 VGG/Resnet

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Figure 5: Loss for CIFAR10 ResNet-20 w.r.t (a) epochs and (b) runtime when there is a slower learner.

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Appendix: Proofs

In the following analysis we define

\[ M_k := \sum_{i=1}^{n} p_i \left\| \frac{X_k 1_n}{n} - X_k e_i \right\|^2 , \]

and

\[ \bar{M}_k := M_k - \eta. \]

We also define

\[ \partial f(X_k) := n \left[ \begin{array}{c} p_1 \nabla f_1(x_k^1) \\ p_2 \nabla f_2(x_k^2) \\ \vdots \\ p_n \nabla f_n(x_k^n) \end{array} \right] \in \mathbb{R}^{N \times n}, \]
\[ \partial f(X_k, i) := \left[ \begin{array}{cc} \cdots & \nabla f_i(x_k^i) \end{array} \right] \in \mathbb{R}^{N \times n}, \]
\[ \partial g(\hat{X}_k, \hat{\xi}_k) := n \left[ \begin{array}{c} p_1 \sum_{j=1}^{M} \nabla F(\hat{x}_k^j \xi_k^j) \\ \vdots \\ p_n \sum_{j=1}^{M} \nabla F(\hat{x}_k^n \xi_k^n) \end{array} \right] \in \mathbb{R}^{N \times n}. \]

We start from

\[ \tilde{\rho} := \frac{n-1}{n} \left( 1 - \frac{2\sqrt{\rho}}{(1 - \sqrt{\rho})^2} \right), \]
\[ C_1 := 1 - 24M^2L^2\gamma^2 \left( \frac{T_n - 1}{n} + \tilde{\rho} \right), \]
\[ C_2 := \frac{\gamma M}{2n} - \frac{2\gamma LM^2}{n^2} - \frac{12M^2L^2T^2\gamma^3}{n^3} \left( \frac{6\gamma^2L^3M^2}{n^2} + \frac{\gamma M L^2}{n^2} + \frac{12M^3L^4T^2\gamma^3}{n^3} \right) \frac{4M^2L^2T^2 (T_n - 1) + \tilde{\rho}}{C_1} \]
\[ C_3 := \frac{1}{2} + \frac{2 \left( 6\gamma^2L^2M^2 + \gamma n ML + \frac{12M^3L^4T^2\gamma^3}{n^3} \right)}{C_1} \rho + \frac{LT^2\gamma M}{n}. \]

Proof to Theorem 1. We start from

\[ \mathbb{E} f \left( \frac{X_k 1_n}{n} \right) = \mathbb{E} f \left( \frac{X_k W_k 1_n}{n} - \gamma \partial g(\hat{X}_k, \hat{\xi}_k^i, i_k) 1_n \right) = \mathbb{E} f \left( \frac{X_k 1_n}{n} - \gamma \partial g(\hat{X}_k, \hat{\xi}_k^i, i_k) 1_n \right) \]
\[ \leq \mathbb{E} f \left( \frac{X_k 1_n}{n} \right) - \gamma \mathbb{E} \left\langle \nabla f \left( \frac{X_k 1_n}{n} \right), \partial g(\hat{X}_k, \hat{\xi}_k^i, i_k) 1_n \right\rangle + \frac{\gamma^2 L^2}{2n} \left\| \partial g(\hat{X}_k, \hat{\xi}_k^i, i_k) 1_n \right\|^2 \]
\[ \leq \mathbb{E} f \left( \frac{X_k 1_n}{n} \right) - \gamma \mathbb{E} \left\langle \nabla f \left( \frac{X_k 1_n}{n} \right), \nabla f(\frac{X_k 1_n}{n}) \right\rangle + \frac{\gamma^2 L^2}{2n} \left\| \nabla f(\frac{X_k 1_n}{n}) \right\|^2 \]
\[ + \frac{\gamma^2 L^2}{2n} \sum_{i=1}^{n} p_i \mathbb{E} \left\| \nabla f(\frac{X_k 1_n}{n}) \right\|^2 + \frac{\gamma^2 L^2}{2n} \sum_{i=1}^{n} p_i \mathbb{E} \left\| \nabla f(\frac{X_k 1_n}{n}) \right\|^2 \]

Using the upper bound of \( \sum_{i=1}^{n} p_i \mathbb{E} \left\| \nabla f(\frac{X_k 1_n}{n}) \right\|^2 \) in Lemma 5:

\[ \mathbb{E} f \left( \frac{X_k 1_n}{n} \right) \]
\[
\begin{align*}
\mathbb{E} f\left(\frac{X_k}{n}\right) + \frac{\gamma M}{2n} \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) - \frac{\partial f(\hat{X}_k)}{n} \right\|^2 &- \frac{\gamma M}{2n} \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) \right\|^2 - \frac{\gamma M}{2n} \mathbb{E} \left\| \partial f(\hat{X}_k)_{1n} \right\|^2 \\
+ \frac{\gamma^2LM^2}{2n^2} \left( 12L^2 \hat{M}_k + 6\sigma^2 + 2 \sum_{i=1}^n \sum_{j=1}^n p_i \mathbb{E} \left\| \frac{\partial f(\hat{X}_k)}{n} \right\|^2 \right) &+ \frac{\gamma^2L\sigma^2M^2}{2n^2} \\
= \mathbb{E} f\left(\frac{X_k}{n}\right) + \frac{\gamma M}{2n} \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) - \frac{\partial f(\hat{X}_k)}{n} \right\|^2 &- \frac{\gamma M}{2n} \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) \right\|^2 - \frac{\gamma M}{2n} \mathbb{E} \left\| \partial f(\hat{X}_k)_{1n} \right\|^2 \\
- \left( \frac{\gamma M}{2n} - \frac{\gamma^2LM^2}{n^2} \right) \mathbb{E} \left\| \partial f(\hat{X}_k)_{1n} \right\|^2 &+ \frac{\gamma^2L(\sigma^2M + 6\sigma^2M^2)}{2n^2} + \frac{6\gamma^2L^3M^2}{n^2} \hat{M}_k.
\end{align*}
\] (11)

For \( T_1 \) we have
\[
T_1 = \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) - \frac{\partial f(\hat{X}_k)}{n} \right\|^2 \\
\leq 2\mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) - \nabla f\left(\frac{\hat{X}_k}{n}\right) \right\|^2 + 2\mathbb{E} \left\| \nabla f\left(\frac{\hat{X}_k}{n}\right) - \frac{\partial f(\hat{X}_k)}{n} \right\|^2 \\
= 2\mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) - \nabla f\left(\frac{\hat{X}_k}{n}\right) \right\|^2 + 2\mathbb{E} \sum_i \left\| \nabla f_i\left(\frac{\hat{X}_k}{n}\right) - \nabla f_i(\hat{X}_k^i) \right\|^2 \\
\leq 2\mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) - \nabla f\left(\frac{\hat{X}_k}{n}\right) \right\|^2 + 2\mathbb{E} \sum_i \left\| \nabla f_i\left(\frac{\hat{X}_k}{n}\right) - \nabla f_i(\hat{X}_k^i) \right\|^2
\] (12)

Assumption 1:3
\[
\leq \frac{2L^2}{n} \left\| \frac{X_k - \hat{X}_k}{n} \right\|^2 + \frac{2L^2}{n} \mathbb{E} \hat{M}_k.
\]

From (11) and (12) we obtain
\[
\mathbb{E} f\left(\frac{X_{k+1}}{n}\right) \leq \mathbb{E} f\left(\frac{X_k}{n}\right) + \frac{\gamma M}{2n} \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) \right\|^2 - \left( \frac{\gamma M}{2n} - \frac{\gamma^2LM^2}{n^2} \right) \mathbb{E} \left\| \partial f(\hat{X}_k)_{1n} \right\|^2 \\
- \frac{\gamma^2LM^2}{n^2} \mathbb{E} \hat{M}_k + \frac{6\gamma^2L^3M^2}{n^2} \hat{M}_k + \frac{6\gamma^2L(\sigma^2M + 6\sigma^2M^2)}{2n^2} \\
= \mathbb{E} f\left(\frac{X_k}{n}\right) - \frac{\gamma M}{2n} \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) \right\|^2 - \left( \frac{\gamma M}{2n} - \frac{\gamma^2LM^2}{n^2} \right) \mathbb{E} \left\| \partial f(\hat{X}_k)_{1n} \right\|^2 \\
+ \left( \frac{6\gamma^2L^3M^2}{n^2} + \frac{\gamma M}{nL} \right) \mathbb{E} \hat{M}_k + \frac{\gamma M}{nL} \mathbb{E} \left\| \frac{X_k - \hat{X}_k}{n} \right\|^2 + \frac{\gamma^2L(\sigma^2M + 6\sigma^2M^2)}{2n^2}
\] (8)

\[
\leq \mathbb{E} f\left(\frac{X_k}{n}\right) - \frac{\gamma M}{2n} \mathbb{E} \left\| \nabla f\left(\frac{X_k}{n}\right) \right\|^2 - \left( \frac{\gamma M}{2n} - \frac{\gamma^2LM^2}{n^2} \right) \mathbb{E} \left\| \partial f(\hat{X}_k)_{1n} \right\|^2 \\
+ \left( \frac{6\gamma^2L^3M^2}{n^2} + \frac{\gamma M}{nL} \right) \mathbb{E} \hat{M}_k + \frac{\gamma M}{nL} \mathbb{E} \left\| \frac{X_k - \hat{X}_k}{n} \right\|^2 + \frac{\gamma^2L(\sigma^2M + 6\sigma^2M^2)}{2n^2}
\]
\[ \begin{aligned} &+ \left( \frac{6\gamma^2 L^3 M^2}{n^2} + \frac{\gamma M L^2}{n^2} \right) \mathbb{E} \mathcal{M}_k + \frac{\gamma \sigma^2 L (\sigma^2 M + 6\zeta^2 M^2)}{2n^2} + \frac{L^2 T^2 \gamma^3 \sigma^2 M^2}{n^3} \\
&+ \frac{M^3 L^2 T \gamma^3}{n^3} \sum_{k=1}^{n} \left( \sum_{i=1}^{n} p_i \mathbb{E} \| \nabla f_i(\hat{x}_{k-1}^i) \|^2 \right) \\
\end{aligned} \]

Lemma 5
\[ \begin{aligned} \mathbb{E} f \left( \frac{X_k 1_n}{n} \right) &\leq \mathbb{E} f \left( \frac{X_0 1_n}{n} \right) - \frac{\gamma M}{2n} \sum_{k=0}^{K-1} \mathbb{E} \| \nabla f \left( \frac{X_k 1_n}{n} \right) \|^2 - \left( \frac{\gamma M}{2n} - \frac{\gamma^2 L M^2}{n^2} \right) \sum_{k=0}^{K-1} \mathbb{E} \| \mathbb{E} \frac{\partial f(\hat{x}_k) 1_n}{n} \|^2 \\
&+ \left( \frac{6\gamma^2 L^3 M^2}{n^2} + \frac{\gamma M L^2}{n^2} \right) \mathbb{E} \mathcal{M}_k + \frac{\gamma \sigma^2 L (\sigma^2 M + 6\zeta^2 M^2)}{2n^2} + \frac{L^2 T^2 \gamma^3 \sigma^2 M^2}{n^3} \\
&+ \frac{M^3 L^2 T \gamma^3}{n^3} \sum_{k=1}^{n} \left( 12 L^2 \mathcal{M}_{k-1} + 6\zeta^2 + 2 \mathbb{E} \left( \sum_{j=1}^{n} p_j \| \nabla f_j(\hat{x}_{k-1}^j) \|^2 \right) \right) \\
&= \mathbb{E} f \left( \frac{X_k 1_n}{n} \right) - \frac{\gamma M}{2n} \sum_{k=0}^{K-1} \mathbb{E} \| \nabla f \left( \frac{X_k 1_n}{n} \right) \|^2 - \left( \frac{\gamma M}{2n} - \frac{\gamma^2 L M^2}{n^2} \right) \sum_{k=0}^{K-1} \mathbb{E} \| \mathbb{E} \frac{\partial f(\hat{x}_k) 1_n}{n} \|^2 \\
&+ \left( \frac{6\gamma^2 L^3 M^2}{n^2} + \frac{\gamma M L^2}{n^2} \right) \mathbb{E} \mathcal{M}_k + \frac{\gamma \sigma^2 L (\sigma^2 M + 6\zeta^2 M^2)}{2n^2} + \frac{L^2 T^2 \gamma^3 \sigma^2 M^2}{n^3} \\
&+ \frac{M^3 L^2 T \gamma^3}{n^3} \sum_{k=1}^{n} \left( 6 L^2 \mathbb{E} \mathcal{M}_{k-1} + \mathbb{E} \| \mathbb{E} \frac{\partial f(\hat{x}_{k-1}^i) 1_n}{n} \|^2 \right) \right). \\
\end{aligned} \]

Summing from \( k = 0 \) to \( k = K - 1 \) we obtain
\[ \mathbb{E} f \left( \frac{X_k 1_n}{n} \right) \leq \mathbb{E} f \left( \frac{X_0 1_n}{n} \right) - \frac{\gamma M}{2n} \sum_{k=0}^{K-1} \mathbb{E} \| \nabla f \left( \frac{X_k 1_n}{n} \right) \|^2 - \left( \frac{\gamma M}{2n} - \frac{\gamma^2 L M^2}{n^2} \right) \sum_{k=0}^{K-1} \mathbb{E} \| \mathbb{E} \frac{\partial f(\hat{x}_k) 1_n}{n} \|^2 \\
+ \left( \frac{6\gamma^2 L^3 M^2}{n^2} + \frac{\gamma M L^2}{n^2} \right) \sum_{k=0}^{K-1} \mathbb{E} \mathcal{M}_k + \frac{\gamma \sigma^2 L (\sigma^2 M + 6\zeta^2 M^2) K}{2n^2} + \frac{L^2 T^2 \gamma^3 \sigma^2 M^2 (\sigma^2 M + 6\zeta^2 M^2) K}{n^3} \\
+ \frac{M^3 L^2 T \gamma^3}{n^3} \sum_{k=0}^{K-1} \left( 6 L^2 \mathbb{E} \mathcal{M}_{k-1} + \mathbb{E} \| \mathbb{E} \frac{\partial f(\hat{x}_{k-1}^i) 1_n}{n} \|^2 \right) \right). \\
\]
\[ \begin{align*}
\leq & \mathbb{E} f \left( \frac{X_0 1_n}{n} \right) - \gamma M \sum_{k=0}^{K-1} \mathbb{E} \left\| \nabla f \left( \frac{X_k 1_n}{n} \right) \right\|^2 \\
& - \left( \frac{\gamma M}{2n} - \frac{\gamma^2 L \sigma^2}{n^2} - \frac{2M^2 \sigma^2}{n^3} \right) \sum_{k=0}^{K-1} \mathbb{E} \left\| \frac{\partial f(X_k) 1_n}{n} \right\|^2 \\
& + \left( \frac{6\gamma^2 L^3 M^2}{n^2} + \frac{\gamma M L^2}{n^2} + \frac{12M^3 \sigma^2 \gamma^2}{n^3} \right) \sum_{k=0}^{K-1} \mathbb{E} \bar{M}_k \\
& + \gamma^2 L (\sigma^2 M + 6\sigma^2 M^2) K
\end{align*} \]

\[ C_1 > 0, \text{Lemma 7} \]

\[\begin{align*}
\leq & \mathbb{E} f \left( \frac{X_0 1_n}{n} \right) - \gamma M \sum_{k=0}^{K-1} \mathbb{E} \left\| \nabla f \left( \frac{X_k 1_n}{n} \right) \right\|^2 \\
& - \left( \frac{\gamma M}{2n} - \frac{\gamma^2 L \sigma^2}{n^2} - \frac{2M^2 \sigma^2}{n^3} \right) \sum_{k=0}^{K-1} \mathbb{E} \left\| \frac{\partial f(X_k) 1_n}{n} \right\|^2 \\
& + \left( \frac{6\gamma^2 L^3 M^2}{n^2} + \frac{\gamma M L^2}{n^2} + \frac{12M^3 \sigma^2 \gamma^2}{n^3} \right) K^2 \gamma^2 (M \sigma^2 + 6M^2 \kappa^2) \rho \\
& + \frac{\gamma^2 L (\sigma^2 M + 6\sigma^2 M^2) K}{2n^2} + \frac{L^2 T^2 \gamma^3 M (\sigma^2 M + 6\sigma^2 M^2) K}{n^3}
\end{align*} \]

\[ \begin{align*}
= & \mathbb{E} f \left( \frac{X_0 1_n}{n} \right) - \gamma M \sum_{k=0}^{K-1} \mathbb{E} \left\| \nabla f \left( \frac{X_k 1_n}{n} \right) \right\|^2 \\
& - C_2 \frac{1}{K} \left\| \sum_{k=0}^{K-1} \frac{\partial f(X_k) 1_n}{n} \right\|^2 + C_3 \frac{\gamma^2 L \kappa^2}{n^2} (M \sigma^2 + 6M^2 \kappa^2).
\end{align*} \]

Thus while $C_3 \leq 1$ and $C_2 \geq 0$ we have

\[ \frac{\sum_{k=0}^{K-1} \mathbb{E} \left\| \frac{\nabla f(X_k 1_n)}{n} \right\|^2}{K} \leq \frac{2}{\gamma KM/n} \left( \mathbb{E} f \left( \frac{X_0 1_n}{n} \right) - \mathbb{E} f \left( \frac{X_1 1_n}{n} \right) \right) + \frac{2\gamma L (M \sigma^2 + 6M^2 \kappa^2)}{Mn}
\]

\[ \leq \frac{2}{\gamma KM/n} \left( \mathbb{E} f(x_0) - \mathbb{E} f^* \right) + \frac{2\gamma L (M \sigma^2 + 6M^2 \kappa^2)}{Mn}.
\]

It completes the proof. \[\square\]

**Lemma 3.** Define $\prod_{k=1}^{0} W_k = I$, where $I$ is the identity matrix. Then

\[ \mathbb{E} \left\| \frac{1_n}{n} - \prod_{k=1}^{K} W_k e_i \right\|^2 \leq \frac{n-1}{n} \rho^K, \quad \forall K \geq 0.
\]

**Proof.** Let $y_K = \frac{1_n}{n} - \prod_{k=1}^{K} W_k e_i$. Then noting that $y_{K+1} = W_{K+1} y_K$ we have

\[ \begin{align*}
\mathbb{E} \left\| y_{K+1} \right\|^2 \\
= & \mathbb{E} \left\| W_{K+1} y_K \right\|^2 \\
= & \mathbb{E} \langle W_{K+1} y_K, W_{K+1} y_K \rangle \\
= & \mathbb{E} \langle y_K, W_{K+1}^T W_{K+1} y_K \rangle
\end{align*} \]

\[ 20 \]
\[ = \mathbb{E} \langle y_K, \mathbb{E} y_{k+1} (W_{K+1}^TW_{K+1}) y_K \rangle \]
\[ = \mathbb{E} \langle y_K, \mathbb{E} (W_{K+1}^TW_{K+1}) y_K \rangle. \]

Note that \( \mathbb{E} (W_{K+1}^TW_{K+1}) \) is symmetric and doubly stochastic and \( 1_n \) is an eigenvector of \( \mathbb{E} (W_{K+1}^TW_{K+1}) \) with eigenvalue 1. Starting from \( 1_n \) we construct a basis of \( \mathbb{R}^n \) composed by the eigenvectors of \( \mathbb{E} (W_{K+1}^TW_{K+1}) \), which is guaranteed to exist by the spectral theorem of Hermitian matrices. From (2) the magnitude of all other eigenvectors’ associated eigenvalues should be smaller or equal to \( \rho \). Noting \( y_K \) is orthogonal to \( 1_n \), we decompose \( y_K \) using this constructed basis and it follows that

\[ \mathbb{E} \| y_{K+1} \|^2 \leq \rho \mathbb{E} \| y_K \|^2. \]

Noting that \( \| y_0 \|^2 = \| 1_n/n - e_i \|^2 = \frac{(n-1)^2}{n^2} + \sum_{i=1}^{n-1} \frac{1}{n^2} = \frac{n^2 - 2n + n - 1}{n^2} = \frac{n-1}{n} \), by induction, we complete the proof.

**Lemma 4.**

\[ \mathbb{E} \left\| \frac{\partial g(x_k, \xi_k^i, i_k)}{n} 1_n \right\|^2 \leq \frac{\sigma^2 M}{n^2} + \frac{M^2}{n^2} \sum_{i=1}^{n} p_i \mathbb{E} \| \nabla f_i(x_k^i) \|^2, \quad \forall k \geq 0. \]

**Proof.** The LHS can be bounded by

\[ \mathbb{E} \left\| \frac{\partial g(x_k, \xi_k^i, i_k)}{n} 1_n \right\|^2 = \sum_{i=1}^{n} p_i \mathbb{E} \left\| \sum_{j=1}^{M} \nabla F_i(x_k^i, \xi_k^i) \right\|^2 \]
\[ = \sum_{i=1}^{n} p_i \mathbb{E} \left\| \sum_{j=1}^{M} \left( \nabla F_i(x_k^i, \xi_k^i) - \nabla F_i(x_k^i) \right) \right\|^2 \]
\[ \leq \frac{\sigma^2 M}{n^2} + \frac{M^2}{n^2} \sum_{i=1}^{n} p_i \mathbb{E} \| \nabla f_i(x_k^i) \|^2. \]

**Lemma 5.**

\[ \sum_{i=1}^{n} p_i \mathbb{E} \| \nabla f_i(x_k^i) \|^2 \leq 12L^2 \mathbb{E} \tilde{M}_k + 6C^2 + 2 \mathbb{E} \left\| \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) \right\|^2, \quad \forall k \geq 0. \]

**Proof.** The LHS can be bounded by

\[ \sum_{i=1}^{n} p_i \mathbb{E} \| \nabla f_i(x_k^i) \|^2 \leq \sum_{i=1}^{n} p_i \mathbb{E} \| \nabla f_i(x_k^i) - \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) + \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) \|^2 \]
\[ \leq 2 \sum_{i=1}^{n} p_i \mathbb{E} \left\| \nabla f_i(x_k^i) - \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) \right\|^2 + 2 \sum_{i=1}^{n} p_i \mathbb{E} \left\| \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) \right\|^2 \]
\[ = 2 \sum_{i=1}^{n} p_i \mathbb{E} \left\| \nabla f_i(x_k^i) - \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) \right\|^2 + 2 \sum_{i=1}^{n} \mathbb{E} \left\| \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) \right\|^2. \] (13)

For the first term on the RHS we have

\[ \sum_{i=1}^{n} p_i \mathbb{E} \left\| \nabla f_i(x_k^i) - \sum_{j=1}^{n} p_j \nabla f_j(x_k^i) \right\|^2 \]
\[
\begin{align*}
&\leq 3 \sum_{i=1}^{n} p_i E \left\| \nabla f_i (\hat{x}_k^i) - \nabla f_i \left( \frac{\hat{X}_k 1_n}{n} \right) \right\|^2 + 3 \sum_{i=1}^{n} p_i E \left\| \nabla f_i \left( \frac{\hat{X}_k 1_n}{n} \right) - \sum_{j=1}^{n} p_j \nabla f_j \left( \frac{\hat{X}_k 1_n}{n} \right) \right\|^2 \\
&\quad + 3 \sum_{i=1}^{n} p_i E \left\| \sum_{j=1}^{n} p_j \nabla f_j (\hat{x}_k^j) - \sum_{j=1}^{n} p_j \nabla f_j \left( \frac{\hat{X}_k 1_n}{n} \right) \right\|^2
\end{align*}
\]

\[
\leq 3 L^2 \sum_{i=1}^{n} p_i E \left\| \hat{x}_k^i - \frac{\hat{X}_k 1_n}{n} \right\|^2 + 3 \sum_{i=1}^{n} p_i E \left\| \nabla f_i \left( \frac{\hat{X}_k 1_n}{n} \right) - \sum_{j=1}^{n} p_j \nabla f_j \left( \frac{\hat{X}_k 1_n}{n} \right) \right\|^2 \\
&\quad + 3 E \left\| \sum_{j=1}^{n} p_j \nabla f_j (\hat{x}_k^j) - \sum_{j=1}^{n} p_j \nabla f_j \left( \frac{\hat{X}_k 1_n}{n} \right) \right\|^2
\]

\[
\leq 3 L^2 E \bar{M}_k + 3 \sum_{i=1}^{n} p_i E \left\| \nabla f_i \left( \frac{\hat{X}_k 1_n}{n} \right) - \nabla f \left( \frac{\hat{X}_k 1_n}{n} \right) \right\|^2 + 3 \sum_{i=1}^{n} p_i E \left\| \nabla f_i (\hat{x}_k^i) - \nabla f_j \left( \frac{\hat{X}_k 1_n}{n} \right) \right\|^2
\]

\[
\leq 6 L^2 E \bar{M}_k + 3 \epsilon^2.
\]

Plugging this upper bound into (13) we complete the proof. \qed

**Lemma 6.** For any \( k \geq -1 \) we have

\[
E \left\| \frac{X_{k+1} 1_n}{n} - X_{k+1} e_i \right\|^2 \leq 2 \gamma^2 (M \sigma^2 + 6 M^2 \epsilon^2) \bar{p}
\]

\[
+ 2 \frac{n-1}{n} M^2 \gamma^2 E \sum_{j=0}^{k} \left( 12 L^2 \bar{M}_j + 2 E \left\| \sum_{i=1}^{n} p_i \nabla f_i (\hat{x}_k^i) \right\|^2 \right) \left( \rho^{k-j} + 2(k-j) \rho^{k-j} \epsilon^2 \right).
\]

**Proof.** Note that for \( k = -1 \), we have

\[
E \left\| \frac{X_{k+1} 1_n}{n} - X_{k+1} e_i \right\|^2 = 0.
\]

Also note that the columns of \( X_0 \) are the same (all nodes start with the same model), we have \( X_0 W_k = X_0 \) for all \( k \) and \( X_0 1_n / n - X_0 e_i = 0, \forall i \). It follows that

\[
E \left\| \frac{X_{k+1} 1_n}{n} - X_{k+1} e_i \right\|^2 =
\]

\[
= E \left\| X_0 1_n - \gamma \partial g(\hat{X}_k, \tilde{\xi}_k, i_k) 1_n \right\|^2
\]

\[
= E \left\| X_0 1_n - \sum_{j=0}^{k} \gamma \partial g(\hat{X}_j, \tilde{\xi}_j, i_j) 1_n \right\|^2
\]

\[
= \gamma^2 E \left\| \sum_{j=0}^{k} \partial g(\hat{X}_j, \tilde{\xi}_j, i_j) \frac{1_n}{n} + \sum_{j=0}^{k} \gamma \partial g(\hat{X}_j, \tilde{\xi}_j, i_j) \prod_{q=j+1}^{k} W_q e_i \right\|^2
\]

\[
= \gamma^2 E \left\| \sum_{j=0}^{k} \partial g(\hat{X}_j, \tilde{\xi}_j, i_j) \left( \frac{1_n}{n} - \sum_{q=j+1}^{k} W_q e_i \right) \right\|^2
\]

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$$\leq 2\gamma^2 \mathbb{E} \left[ \sum_{j=0}^{k} (\partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j)) \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \right]^2$$

$$+ 2M^2\gamma^2 \mathbb{E} \left[ \sum_{j=0}^{k} \partial f(\hat{X}_j, i_j) \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \right]^2 .$$

(14)

For $A_1$,

$$A_1 = \mathbb{E} \left[ \sum_{j=0}^{k} (\partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j)) \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \right]^2$$

$$= \sum_{j=0}^{k} \mathbb{E} \left[ (\partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j)) \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \right]^2$$

$$+ 2 \mathbb{E} \sum_{k \geq j, j' \geq 0} \left\langle (\partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j)) \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right), (\partial g(\hat{X}_{j'}, \xi_{j'}, i_{j'}) - M \partial f(\hat{X}_{j'}, i_{j'})) \left( \frac{1}{n} - \prod_{q=j'+1}^{k} W_q e_i \right) \right\rangle .$$

$A_3$ can be bounded by a constant:

$$A_3 = \sum_{j=0}^{k} \mathbb{E} \left\| \partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j) \right\|^2 \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right)^2$$

$$\leq \sum_{j=0}^{k} \mathbb{E} \left\| \partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j) \right\|^2 \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right)^2$$

Lemma \(3 n - 1 \leq n \sum_{j=0}^{k} \mathbb{E} \left\| \partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j) \right\|^2 \rho^{k-j} \)

Assumption \(1.5 \leq n - 1 \leq n - 1 \frac{M \sigma^2}{n} \frac{1}{1 - \rho} .\)

$A_4$ can be bounded by another constant:

$$A_4 = \sum_{k \geq j, j' \geq 0} \mathbb{E} \left\langle (\partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j)) \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right), (\partial g(\hat{X}_{j'}, \xi_{j'}, i_{j'}) - M \partial f(\hat{X}_{j'}, i_{j'})) \left( \frac{1}{n} - \prod_{q=j'+1}^{k} W_q e_i \right) \right\rangle$$

$$\leq \sum_{k \geq j, j' \geq 0} \mathbb{E} \left\| \partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j) \right\| \left( \frac{1}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \times$$

$$\left\| \partial g(\hat{X}_{j'}, \xi_{j'}, i_{j'}) - M \partial f(\hat{X}_{j'}, i_{j'}) \right\| \left( \frac{1}{n} - \prod_{q=j'+1}^{k} W_q e_i \right)$$

$$\leq \mathbb{E} \sum_{k \geq j, j' \geq 0} \left( \left\| \partial g(\hat{X}_j, \xi_j, i_j) - M \partial f(\hat{X}_j, i_j) \right\|^2 \left\| \partial g(\hat{X}_{j'}, \xi_{j'}, i_{j'}) - M \partial f(\hat{X}_{j'}, i_{j'}) \right\|^2 \right)^{\frac{1}{2}} , \forall \alpha_{j, j'} > 0$$

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\[
\begin{align*}
\langle 5 \rangle & \leq \mathbb{E} \sum_{k \geq j > j' \geq 0} \left( \frac{1_n - \Pi_{q=j+1} W_q e_i}{2 \alpha_{j,j'}} \right)^2 \mathbb{E} \sum_{k \geq j > j' \geq 0} \left( \frac{1_n - \Pi_{q=j'+1} W_q e_i}{2 \alpha_{j,j'}} \right)^2 + M^2 \alpha_{j,j'}^2, \forall \alpha_{j,j'} > 0 \\
\leq & \mathbb{E} \sum_{k \geq j > j' \geq 0} \left( \frac{n - 1}{n} \left( \frac{1_n - \Pi_{q=j+1} W_q e_i}{2 \alpha_{j,j'}} \right)^2 + M^2 \alpha_{j,j'}^2 \right), \forall \alpha_{j,j'} > 0 \\
\text{Lemma 3} & \leq \mathbb{E} \sum_{k \geq j > j' \geq 0} \left( \left( \frac{n - 1}{n} \right)^2 \frac{\rho^{k-j}}{2 \alpha_{j,j'}} + M^2 \alpha_{j,j'}^2 \right), \forall \alpha_{j,j'} > 0.
\end{align*}
\]

We can choose \( \alpha_{j,j'} > 0 \) to make the term in the last step become \( \frac{n-1}{n} \sum_{k \geq j > j' \geq 0} \rho^{k-j} \cdot M^2 \) (by applying inequality of arithmetic and geometric means). Thus

\[
A_4 \leq \frac{n-1}{n} \sum_{k \geq j > j' \geq 0} \rho^{k-j} \cdot M^2 \leq \frac{n-1}{n} \cdot M^2 \sum_{j' = 0}^{k} \sum_{j = 0}^{k} \rho^{k-j} = \frac{n-1}{n} \cdot M^2 \cdot \frac{\sqrt{\rho}}{(1-\sqrt{\rho})^2}.
\]

Putting \( A_3 \) and \( A_4 \) back into \( A_1 \) we obtain:

\[
A_1 \leq \frac{n-1}{n} \cdot M^2 \left( \frac{1}{1-\rho} + \frac{2 \sqrt{\rho}}{(1-\sqrt{\rho})^2} \right) = M^2 \rho. \quad (15)
\]

We then start bounding \( A_2 \):

\[
\begin{align*}
A_2 = & \mathbb{E} \left\| \sum_{j=0}^{k} \nabla f(\hat{X}, i_j) \left( \frac{1_n}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \right\|^2 \\
= & \mathbb{E} \sum_{j=0}^{k} \left\| \nabla f(\hat{X}, i_j) \left( \frac{1_n}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \right\|^2 + 2 \mathbb{E} \sum_{j=0}^{k} \sum_{j' = j+1}^{k} \nabla f(\hat{X}, i_j) \left( \frac{1_n}{n} - \prod_{q=j+1}^{k} W_q e_i \right) \nabla f(\hat{X}, i_{j'}) \left( \frac{1_n}{n} - \prod_{q=j'+1}^{k} W_q e_i \right) \\
\text{Lemma 3, (1)} \leq & \frac{n-1}{n} \sum_{j=0}^{k} \left( \sum_{i=1}^{n} p_i \left\| \nabla f(\hat{X}, i_j) \right\|^2 \right) \rho^{k-j} \\
& + 2 \mathbb{E} \sum_{j=0}^{k} \sum_{j' = j+1}^{k} \left\| \nabla f(\hat{X}, i_j) \right\| \left\| \nabla f(\hat{X}, i_{j'}) \right\| \left\| \nabla f(\hat{X}, i_{j'}) \right\| \left\| \nabla f(\hat{X}, i_{j'}) \right\|. \quad (16)
\end{align*}
\]

For the second term:

\[
\begin{align*}
\mathbb{E} \sum_{j=0}^{k} \sum_{j' = j+1}^{k} \left( \left\| \nabla f(\hat{X}, i_j) \right\| \left\| \nabla f(\hat{X}, i_{j'}) \right\| \left\| \nabla f(\hat{X}, i_{j'}) \right\| \left\| \nabla f(\hat{X}, i_{j'}) \right\| \\
& \leq \mathbb{E} \sum_{j=0}^{k} \sum_{j' = j+1}^{k} \left( \frac{\left\| \nabla f(\hat{X}, i_j) \right\|^2 \left\| \nabla f(\hat{X}, i_{j'}) \right\|^2}{2 \alpha_{j,j'}} + \frac{\left\| \nabla f(\hat{X}, i_j) \right\|^2 \left\| \nabla f(\hat{X}, i_{j'}) \right\|^2}{2 \alpha_{j,j'}} \right), \forall \alpha_{j,j'} > 0
\end{align*}
\]

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Lemma 3
\[ \frac{1}{2} \mathbb{E} \sum_{j \neq j'}^k \left( \frac{\|\partial f(\hat{X}_j, i_j)\|^2 \|\partial f(\hat{X}_{j'}, i_{j'})\|^2}{2 \alpha_{j,j'}} + \frac{\rho^{k-\min\{j,j'\}}}{2 \alpha_{j,j'}} \left( \frac{n-1}{n} \right)^2 \right), \quad \forall \alpha_{j,j'} > 0, \alpha_{j,j'} = \alpha_{j',j}. \]

By applying inequality of arithmetic and geometric means to the term in the last step we can choose \( \alpha_{j,j'} > 0 \) such that

\[
\begin{align*}
\mathbb{E} & \sum_{j=0}^k \sum_{j'=j+1}^k \|\partial f(\hat{X}_j, i_j)\| \left\| \frac{1}{n} - \prod_{q=j+1}^k W_{q,i_q} \right\| \|\partial f(\hat{X}_{j'}, i_{j'})\| \left\| \frac{1}{n} - \prod_{q=j'+1}^k W_{q,i_q} \right\| \\
& \leq \frac{n-1}{2n} \mathbb{E} \sum_{j \neq j'}^k \left( \frac{\|\partial f(\hat{X}_j, i_j)\| \|\partial f(\hat{X}_{j'}, i_{j'})\| \rho^{\frac{k-\min\{j,j'\}}{2}}}{2} \right) \\
& \leq \frac{n-1}{2n} \mathbb{E} \sum_{j \neq j'}^k \left( \frac{\|\partial f(\hat{X}_j, i_j)\|^2 + \|\partial f(\hat{X}_{j'}, i_{j'})\|^2}{\rho^{\frac{k-\min\{j,j'\}}{2}}} \right) \\
& = \frac{n-1}{2n} \sum_{j \neq j'}^k \left( \sum_{i=1}^n \rho_{j,i} \|\nabla f_{i}(\hat{x}_j)\|^2 \right) (k-j) \rho^{k-j}.
\end{align*}
\]

(17)

It follows from (17) and (16) that

\[
A_2 \leq \frac{n-1}{n} \mathbb{E} \left( \sum_{j=0}^k \left( \sum_{i=1}^n \rho_{j,i} \|\nabla f_{i}(\hat{x}_j)\|^2 \right) \left( \rho^{k-j} + 2(k-j) \rho^{k-j} \right) \right)
\]

Lemma 5
\[
\leq \frac{n-1}{n} \sum_{j=0}^k \left( 12L^2 \mathbb{E} \hat{M}_j + 6\zeta^2 + 2 \mathbb{E} \left( \sum_{j=1}^n \rho_{j} \|\nabla f_{j}(\hat{x}_j)\|^2 \right)^2 \right) \left( \rho^{k-j} + 2(k-j) \rho^{k-j} \right)
\]

\[
= \frac{n-1}{n} \sum_{j=0}^k \left( 12L^2 \mathbb{E} \hat{M}_j + 2 \mathbb{E} \left( \sum_{j=1}^n \rho_{j} \|\nabla f_{j}(\hat{x}_j)\|^2 \right)^2 \right) \left( \rho^{k-j} + 2(k-j) \rho^{k-j} \right)
\]

\[
+ 6\zeta^2 \frac{n-1}{n} \frac{1}{1-\rho} \left( 1 - \frac{2\sqrt{\rho}}{(1-\sqrt{\rho})^2} \right) \cdot
\]

(18)

Finally from (15), (18) and (14) we obtain

\[
\begin{align*}
\mathbb{E} \left\| \frac{X_{k+1}}{n} - X_{k+1} \right\|^2 & \leq 2\gamma^2 A_1 + 2M^2 \gamma^2 A_2 \\
& \leq 2\gamma^2 \tilde{M} \sigma^2 \tilde{\rho}
\end{align*}
\]

\[
+ 2\gamma^2 M^2 \mathbb{E} \sum_{j=0}^k \left( 12L^2 \hat{M}_j + 2 \mathbb{E} \left( \sum_{i=1}^n \rho_{j} \|\nabla f_{i}(\hat{x}_j)\|^2 \right)^2 \right) \left( \rho^{k-j} + 2(k-j) \rho^{k-j} \right) + 12\gamma^2 M^2 \zeta^2 \tilde{\rho}
\]

\[
= 2\gamma^2 (M\sigma^2 + 6M^2 \zeta^2) \tilde{\rho}
\]

\[
+ 2 \frac{n-1}{n} M^2 \gamma^2 \mathbb{E} \sum_{j=0}^k \left( 12L^2 \hat{M}_j + 2 \mathbb{E} \left( \sum_{i=1}^n \rho_{j} \|\nabla f_{i}(\hat{x}_j)\|^2 \right)^2 \right) \left( \rho^{k-j} + 2(k-j) \rho^{k-j} \right).
\]

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This completes the proof.

\[ \square \]

**Lemma 7.** While \( C_1 > 0 \), we have

\[
\sum_{k=0}^{K-1} \frac{E \hat{M}_k}{K} \leq \frac{2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2)\bar{\rho}}{C_1} + \frac{4\gamma^2 M^2}{K} \left( \frac{T n^{1-1/\tau}}{\bar{\rho}} + \bar{\rho} \right) \sum_{k=0}^{K-1} \frac{\| \sum_{i=1}^{n} p_i \nabla f_i(\hat{x}_i^k) \|^2}{K}.
\]

**Proof.** From Lemma 6 and noting that \( \hat{X}_k = X_{k-\tau} \), we have

\[
E \left\| \frac{X_1}{n} - \hat{X}_k \epsilon_i \right\|^2 \leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2)\bar{\rho}
+ 2 \frac{n - 1}{n} M^2\gamma^2 \sum_{j=0}^{k-\tau-1} \left( 12L^2 E \hat{M}_j + 2E \left\| \sum_{i=1}^{n} p_i \nabla f_i(\hat{x}_i^j) \right\|^2 \right) \left( \rho^{k-\tau-1-j} + 2(k - \tau_k - 1 - j) \rho^{\frac{k-\tau-1-j}{2}} \right).
\]

By averaging from \( i = 1 \) to \( n \) with distribution \( \mathcal{I} \) we obtain

\[
E \hat{M}_k = \sum_{i=1}^{n} p_i E \left\| \frac{X_1}{n} - \hat{X}_k \epsilon_i \right\|^2 \leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2)\bar{\rho}
+ 2 \frac{n - 1}{n} M^2\gamma^2 \sum_{j=0}^{k-\tau-1} \left( 12L^2 E \hat{M}_j + 2E \left\| \sum_{i=1}^{n} p_i \nabla f_i(\hat{x}_i^j) \right\|^2 \right) \left( \rho^{k-\tau-1-j} + 2(k - \tau_k - 1 - j) \rho^{\frac{k-\tau-1-j}{2}} \right).
\]

It follows that

\[
\sum_{k=0}^{K-1} \frac{E \hat{M}_k}{K} \leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2)\bar{\rho}
+ 2 \frac{\gamma^2 n - 1}{n} M^2 \sum_{k=0}^{K-1} \sum_{j=0}^{k-\tau-1} \left( 12L^2 E \hat{M}_j + 2E \left\| \sum_{i=1}^{n} p_i \nabla f_i(\hat{x}_i^j) \right\|^2 \right) \times
\left( \rho^{\max(k-\tau-1-j,0)} + 2(\max(k - \tau_k - 1 - j,0)) \rho^{\frac{\max(k-\tau-1-j,0)}{2}} \right)
\leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2)\bar{\rho}
+ 2 \frac{\gamma^2 n - 1}{n} M^2 \sum_{j=0}^{\infty} \sum_{k=0}^{K-1} \left( 12L^2 E \hat{M}_j + 2E \left\| \sum_{i=1}^{n} p_i \nabla f_i(\hat{x}_i^j) \right\|^2 \right) \times
\left( \rho^{\max(k-\tau-1-j,0)} + 2(\max(k - \tau_k - 1 - j,0)) \rho^{\frac{\max(k-\tau-1-j,0)}{2}} \right)
\leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2)\bar{\rho}
+ 2 \frac{\gamma^2 n - 1}{n} M^2 \sum_{j=0}^{\infty} \left( 12L^2 E \hat{M}_j + 2E \left\| \sum_{i=1}^{n} p_i \nabla f_i(\hat{x}_i^j) \right\|^2 \right) \left( T + \sum_{h=0}^{\infty} \left( \rho^h + 2h\rho^h \right) \right)
\]
\[ \leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2) \bar{\rho} \]
\[ + \frac{2\gamma^2}{K} M^2 \left( T \frac{n-1}{n} + \bar{\rho} \right) \sum_{j=0}^{K-1} \left( 12L^2 E \hat{M}_j + 2E \| p_i \nabla f_i (\hat{x}_{ij}) \|^2 \right) \]
\[ \leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2) \bar{\rho} \]
\[ + \frac{4\gamma^2 M^2}{K} \left( T \frac{n-1}{n} + \bar{\rho} \right) \sum_{k=0}^{K-1} \left( \sum_{i=1}^{n} p_i \nabla f_i (\hat{x}_k^i) \right)^2 \]
\[ + \frac{24L^2 \gamma^2 M^2}{K} \left( T \frac{n-1}{n} + \bar{\rho} \right) \sum_{k=0}^{K-1} \text{E} \hat{M}_k. \]

By rearranging the terms we obtain

\[ \left( 1 - 24L^2 M^2 \gamma^2 \left( T \frac{n-1}{n} + \bar{\rho} \right) \right) \sum_{k=0}^{K-1} \text{E} \hat{M}_k \]
\[ \leq 2\gamma^2 (M\sigma^2 + 6M^2\varsigma^2) \bar{\rho} + \frac{4\gamma^2 M^2}{K} \left( T \frac{n-1}{n} + \bar{\rho} \right) \sum_{k=0}^{K-1} \left( \sum_{i=1}^{n} p_i \nabla f_i (\hat{x}_k^i) \right)^2, \]

we complete the proof.

**Lemma 8.** For all \( k \geq 0 \) we have

\[ \text{E} \left\| X_k 1_n - \hat{X}_k 1_n \right\|_{\bar{\rho}}^2 \leq \frac{\tau^2 \gamma^2 \sigma^2 M}{n^2} + \tau_k \gamma^2 \sum_{t=1}^{\tau_k} \left( \frac{M^2}{n^2} \sum_{i=1}^{n} p_i \| \nabla f_i (\hat{x}_{k-1}^i) \|^2 \right). \]

**Proof.**

\[ \text{E} \left\| X_k 1_n - \hat{X}_k 1_n \right\|_{\bar{\rho}}^2 \overset{\text{Assumption 1-7}}{\leq} \sum_{t=1}^{\tau_k} \gamma^2 \text{E} \left\| \sum_{i=1}^{\tau_t} \gamma \partial g (\hat{X}_{k-1}^t; \hat{\xi}_{k-1}^t, \hat{\xi}_{k-1}^t) 1_n \right\|_{\bar{\rho}}^2 \]
\[ \overset{\text{Lemma 4}}{\leq} \tau_k \sum_{i=1}^{\tau_k} \gamma^2 \left( \frac{\sigma^2 M}{n^2} + \frac{M^2}{n^2} \sum_{i=1}^{n} p_i \| \nabla f_i (\hat{x}_{k-1}^i) \|^2 \right), \]

where the first step comes from any \( n \times n \) doubly stochastic matrix multiplied by \( 1_n \) equals \( 1_n \) and Assumption 1-7.

**Proof to Corollary 2.** To prove this result, we will apply Theorem 1. We first verify that all conditions can be satisfied in Theorem 1.

First \( C_1 > 0 \) can be satisfied by a stronger condition \( C_1 \geq 1/2 \) which can be satisfied by \( \gamma \leq \frac{1}{4\sqrt{6} ML} \left( T \frac{n-1}{n} + \bar{\rho} \right)^{-1/2} \). Second \( C_3 \leq 1 \) can be satisfied by:

\[ \gamma \leq \min \left\{ \frac{n}{8MT^2 L}, \frac{1}{8\sqrt{3}LM} \bar{p}^{-1/2}, \frac{1}{32nML} \bar{p}^{-1}, \frac{n^{1/3}}{8\sqrt{6} ML^{2/3}} \bar{p}^{-1/3} \right\} \]
and $C_1 \geq 1/2$, which can be seen from

$$C_3 = \frac{1}{2} + \frac{2 \left( 6\gamma^2 L^2 M^2 + \gamma nML + \frac{12M^3 L^3 T^2}{n} \right) \bar{\rho}}{C_1} + \frac{LT^2 \gamma M}{n},$$

$$C_1 \geq 1 \quad \Rightarrow \quad \frac{C_1}{2} \geq \frac{1}{2} + 24\gamma^2 L^2 M^2 + 4\gamma nML + \frac{48M^3 L^3 T^2}{n} \gamma^3 \bar{\rho} + \frac{LT^2 \gamma M}{n}.$$ 

The requirements on $\gamma$ are given by making each of the last four terms smaller than $1/8$:

$$\frac{LT^2 \gamma M}{n} \leq \frac{1}{8} \iff \gamma \leq \frac{n}{8MT^2 L'},$$

$$24\gamma^2 L^2 M^2 \bar{\rho} \leq \frac{1}{8} \iff \gamma \leq \frac{1}{8\sqrt{3LM} \bar{\rho}^{-1/2}},$$

$$4\gamma nML \bar{\rho} \leq \frac{1}{8} \iff \gamma \leq \frac{1}{32nML} \bar{\rho}^{-1},$$

and

$$\frac{48M^3 L^3 T^2}{n} \gamma^3 \bar{\rho} \leq \frac{1}{8} \iff \gamma \leq \frac{n^{1/3}}{8\sqrt{6MLT}^2 / 8 \bar{\rho}^{-1/3}}.$$ 

Third $C_2 \geq 0$ can be satisfied by

$$\gamma \leq \min \left\{ \frac{n}{10LM'} \frac{n}{2 \sqrt{5ML'} \frac{n^{1/3}}{8ML} \left( \frac{T n}{T n - 1 + \bar{\rho}} \right)^{-1/3}, \frac{1}{4\sqrt{5LM}} \left( \frac{T n}{T n - 1 + \bar{\rho}} \right)^{-1/2}, \frac{n^{1/2}}{6MLT^{1/2}} \left( \frac{T n}{T n - 1 + \bar{\rho}} \right)^{-1/4} \right\}$$

and $C_1 \geq 1/2$, which can be seen from

$$C_2 := \frac{\gamma M}{2n} - \frac{\gamma^2 L^2}{n^2} - \frac{2M^3 L^2 T^2}{n^3} - \frac{6\gamma^2 L^2 M^2}{n^2} + \frac{\gamma ML^2 L^2}{n^2} + \frac{12M^3 L^4 T^2}{n^3} \gamma^3 \bar{\rho} \leq 0$$

$$C_1 \geq \frac{1}{2} \iff 2\gamma LM \frac{M}{n} + \frac{2M^2 L^2 T^2}{n^2} \gamma^2 + \frac{96\gamma M^3}{n} + 16L^2 + \frac{192M^2 L^4 T^2}{n^2} \gamma^2 M^2 \gamma^2 \left( \frac{T n}{T n - 1 + \bar{\rho}} \right).$$

The last inequality is satisfied given the requirements on $\gamma$ because each term on the RHS is bounded by $1/5$:

$$\frac{2\gamma LM}{n} \leq \frac{1}{5} \iff \gamma \leq \frac{n}{10LM'},$$

$$\frac{4M^2 L^2 T^2}{n^2} \gamma^2 \leq \frac{1}{5} \iff \gamma \leq \frac{n}{2\sqrt{5ML'}},$$

$$\frac{96\gamma M^3}{n} M^2 \gamma^2 \left( \frac{T n}{T n - 1 + \bar{\rho}} \right) \leq \frac{1}{5} \iff \gamma \leq \frac{n^{1/3}}{8ML} \left( \frac{T n}{T n - 1 + \bar{\rho}} \right)^{-1/3},$$

$$16L^2 M^2 \gamma^2 \left( \frac{T n}{T n - 1 + \bar{\rho}} \right) \leq \frac{1}{5} \iff \gamma \leq \frac{n^{1/2}}{4\sqrt{5LM}} \left( \frac{T n}{T n - 1 + \bar{\rho}} \right)^{-1/2},$$

$$\frac{192M^2 L^4 T^2}{n^2} \gamma^2 \left( \frac{T n}{T n - 1 + \bar{\rho}} \right) \leq \frac{1}{5} \iff \gamma \leq \frac{n^{1/2}}{6MLT^{1/2}} \left( \frac{T n}{T n - 1 + \bar{\rho}} \right)^{-1/4}.$$ 

Combining all above the requirements on $\gamma$ to satisfy $C_1 \geq 1/2, C_2 \geq 0$ and $C_3 \leq 1$ are

$$\gamma \leq \frac{1}{ML} \min \left\{ \frac{1}{4\sqrt{6}} \left( \frac{T n - 1 + \bar{\rho}}{T n - 1 + \bar{\rho}} \right)^{-1/2}, \frac{1}{8T^{1/2}}, \frac{1}{8\sqrt{3}} \bar{\rho}^{-1/2}, \frac{1}{32} \rho^{-1}, \frac{n^{1/3}}{8\sqrt{6MT}^2 \bar{\rho}^{-1/3}}, \frac{n^{1/2}}{8\sqrt{5T}^2 \bar{\rho}^{-1/2}}, \frac{n^{1/2}}{8\sqrt{6T}^{1/2} \bar{\rho}^{-1/4}} \right\}.$$
Note that the RHS is larger than
\[ U := \frac{1}{ML} \min \left\{ \frac{1}{8\sqrt{3} \sqrt{T \frac{n-1}{n} + \bar{\rho}}}, \frac{n}{8T^2}, \frac{n}{32n}, \frac{n^{1/12}(n-1)^{-1/4}}{(8\sqrt{3}T^{2/3} + 8)(T + \bar{\rho} \frac{n}{n-1})^{1/3}} \right\}. \]

Let \( \gamma = \frac{n}{10ML + \sqrt{\sigma^2 + 6M_2^2\sqrt{KM}}} \) then if \( \gamma \leq U \) we will have \( C_1 \geq 1/2, C_2 \geq 0 \) and \( C_3 \leq 1 \). Further investigation gives us

\[ \gamma = \frac{n}{10ML + \sqrt{\sigma^2 + 6M_2^2\sqrt{KM}}} \leq \frac{1}{ML} \min \left\{ \frac{1}{8\sqrt{3} \sqrt{T \frac{n-1}{n} + \bar{\rho}}}, \frac{n}{8T^2}, \frac{n}{32n}, \frac{n^{1/12}(n-1)^{-1/4}}{(8\sqrt{3}T^{2/3} + 8)(T + \bar{\rho} \frac{n}{n-1})^{1/3}} \right\} \]

\[ \iff 10ML + \sqrt{\sigma^2 + 6M_2^2\sqrt{KM}} \geq nML \max \left\{ \frac{8\sqrt{3}}{T^{2/3} + 8}, \frac{32n\bar{\rho}}{(8\sqrt{3}T^{2/3} + 8)(T + \bar{\rho} \frac{n}{n-1})^{1/3}} \right\} \geq K \geq \frac{ML^2n^2}{\sigma^2 + 6M_2^2\sqrt{KM}} \max \left\{ \frac{192}{\sqrt{T^{2/3} + 8}}, \frac{64T^2}{8\sqrt{3}T^{2/3} + 8}, \frac{1024n^2\bar{\rho}^2}{(8\sqrt{3}T^{2/3} + 8)^2(T + \bar{\rho} \frac{n}{n-1})^{2/3}} \right\}. \]

It follows from Theorem 1 that if the last inequality is satisfied and \( \gamma = \frac{n}{10ML + \sqrt{\sigma^2 + 6M_2^2\sqrt{KM}}} \), we have

\[ \sum_{k=0}^{K-1} \mathbb{E} \left\| \nabla f \left( \frac{X_{1+k}}{n} \right) \right\|^2 \leq 2\left( \mathbb{E} f(x_0) - f^* \right) n \frac{2\gamma L}{\gamma KM} + 2\gamma L \left( \frac{M\sigma^2 + 6M_2^2}{\gamma KM} \right) \]

\[ \leq 2\frac{(\mathbb{E} f(x_0) - f^*)L}{K} + 2\frac{\mathbb{E} f(x_0) - f^*}{\sqrt{KM}} \sigma^2 + 6M_2^2 \]

\[ + \frac{2L}{M \left( 10ML + \sqrt{\sigma^2 + 6M_2^2\sqrt{KM}} \right)} \left( M\sigma^2 + 6M_2^2 \gamma^2 \right) \]

\[ \leq 2\frac{(\mathbb{E} f(x_0) - f^*)L}{K} + 2\frac{\mathbb{E} f(x_0) - f^* + L}{\sqrt{KM}} \sigma^2 + 6M_2^2. \]

This completes the proof. \( \square \)