W-representation of Rainbow tensor model

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ABSTRACT: We analyze the rainbow tensor model and present the Virasoro constraints, where the constraint operators obey the Witt algebra and null 3-algebra. We generalize the method of W-representation in matrix model to the rainbow tensor model, where the operators preserving and increasing the grading play a crucial role. It is shown that the rainbow tensor model can be realized by acting on elementary function with exponent of the operator increasing the grading. We derive the compact expression of correlators and apply it to several models, i.e., the red tensor model, Aristotelian tensor model and r = 4 rainbow tensor model. Furthermore, we discuss the case of the non-Gaussian red tensor model and present a dual expression for partition function through differentiation.

KEYWORDS: Conformal and W Symmetry, Matrix Models

ArXiv ePrint: 2104.01332

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1 Introduction

W-representation of matrix model which realizes partition function by acting on elementary functions with exponents of the given W-operator has attracted considerable attention. It indeed gives a dual expression for partition function through differentiation rather than integration. As the fundamental matrix models, it turned out that the Gaussian Hermitian and complex matrix models can be written as the form of the W-representations [1]–[4]. Since these W-representations can be expressed in terms of characters, the corresponding matrix models are reformulated as the sum of Schur functions over all Young diagrams. For the β-deformed Gaussian Hermitian and complex models, their W-representations still exist. The character expansions of these models can be given by the Jack polynomials. The studies of W-representations have also been devoted to the supersymmetric generalizations of matrix models, i.e., supereigenvalue models [5, 6].

As the generalizations of matrix models from matrices to tensor, tensor models become very useful in the deep study of higher dimensional quantum gravity [7]–[9]. Quite recently, the operators/Feynman diagrams correspondence in quantum field theory was provided [10]. For the number of Feynman diagrams with n propagators in the rank r – 1 complex tensor model, it is equal to the number of singlet operators with 2n vertices in the rank r complex tensor model. Tensor models are also very interesting in their own right [11]–[22]. Recently the tensorial generalization of characters [18, 19] and correlators in tensor models from character calculus [20]–[22] have been analyzed. The Gaussian tensor model is a model of complex r-tensors with the Gaussian action. It can be expressed as
the forms of the characters and $W$-representation [22],

$$Z_r\{p^{(i)}\} = \sum_{R_1,\cdots,R_r} \sum_{\Delta \vdash n} \prod_{i=1}^r \psi_R(\Delta) \prod_{m=1}^r \chi_{R_m}\{p^{(m)}\} \cdot \prod_{(i,j) \in R_m} (N_m + i - j)$$

$$= e^{\hat{W}(N_1,\cdots,N_r)} \cdot 1, \quad (1.1)$$

where $\psi_R(\Delta)$, $\Delta$ and $\chi_{R_m}\{p^{(m)}\}$ are respectively the character of symmetric group, the symmetry factor of Young diagram $\Delta$ and the Schur function as a function of time-variables $p_k$, the operator $\hat{W}(N_1,\cdots,N_r)$ is given by

$$\hat{W}(N_1,\cdots,N_r) = \hat{O}_1(N_1) \cdots \hat{O}_m(N_m) \circ \sum_k \prod_{m=1}^r \frac{p_k^{(m)}}{k} \circ \hat{O}_1^{-1}(N_1) \cdots \hat{O}_m^{-1}(N_m), \quad (1.2)$$

here the subscript $m$ of $\hat{O}_m(N)$ means that this operator acts on the variables $p_k^{(m)}$, the operator $\hat{O}(N)$ satisfies $\hat{O}(N)\chi_R = \frac{D_R(N)}{d_R} \chi_R$, $D_R(N) = \chi_R\{p_k = N\}$ is the dimension of the linear group and $d_R = \chi_R\{p_k = \delta_{1k}\}$. The generalized characters which generate the partition function (1.1) form an over-complete basis in the space of all gauge invariant operators with non-vanishing Gaussian averages. It should be noted that in the generic tensor model, there is no simple way to remove the redundancy.

The Aristotelian rainbow tensor model with a single complex tensor of rank 3 and the RGB (red-green-blue) symmetry is the simplest of the rainbow tensor models [23]–[25]. Recently, with the example of the Aristotelian tensor model, Itoyama et al. [23] introduced a few methods which allow one to connect calculations in the tensor models to those in the matrix models. Well known is that the partition functions of various matrix models can be realized by the $W$-representations, where the operators preserving and increasing the grading play a crucial role [1]. The goal of this paper is to make a step towards the $W$-representation of the rainbow tensor model. We present its $W$-representation and give the compact expression of correlators.

This paper is organized as follows. In section 2, we show that the rainbow tensor model can be realized by acting on elementary function with exponent of the given operator. The Virasoro constraints are also presented. Then we derive the compact expressions of the correlators. In sections 3 and 4, we focus on the correlators in the Aristotelian and $r = 4$ tensor models, respectively. In section 5, we consider the (Non-Gaussian) red tensor model. We end this paper with the conclusions in section 6.

## 2 $W$-representation of rainbow tensor model

For the rainbow model with the rank $r$ complex tensors and with the gauge symmetry $U = U(N_1) \otimes \cdots \otimes U(N_r)$, the gauge-invariant operators of level $n$ are given by [25]

$$k^{(n)} = k^{(n)}_{\sigma_1 \otimes \cdots \otimes \sigma_n} = \prod_{p=1}^n \hat{A}^{(p)}_{j_1^{(p)} \cdots j_r^{(p)}} \hat{A}^{(p)}_{j_1^{(p)} \cdots j_r^{(p)}}^{-1}, \quad (2.1)$$

where $\hat{A}^{j_1 \cdots j_{r-1}}$ is a tensor of rank $r$ with one covariant and $r - 1$ contravariant indices, its conjugate tensor is $\hat{A}^{j_1 \cdots j_{r-1}}$, $\sigma$ is an element of the double coset $S_n^r = S_n \backslash S_n^{\otimes r} / S_n$.
and $\deg \sigma = n$. Here the different types of indices in the fields and fields themselves are assigned with different color. We may choose some operators in (2.1) to generate a graded ring of gauge invariant operators with addition, multiplication, cut and join operations. These operators are called keystones. The connected operators in this ring can generate the renormalization group (RG) completed rainbow tensor model [25].

Let us introduce the RG-completed rainbow tensor model

$$Z_R = \int dA \exp \left( -\mu TAA \right) + \sum_{n=1}^{\infty} \sum_{\deg \sigma = n} t^{(n)}_\sigma K^{(n)}_\sigma$$

where $\mu$ is a constant, $t^{(n)}_\sigma$ are the time variables, the measure is induced by the norm $\| \delta A \|^2 = \delta A^{1 \ldots J_r - 1} \delta A^{1 \ldots J_r - 1}$,

$$Z_R^{(s)} = \int dA \exp \left( -\mu TAA \right) \sum_{l=0}^{\infty} \sum_{n_1 + \ldots + n_l = s} 1 \prod \{ K^{(n_1)}_{\sigma_1} K^{(n_2)}_{\sigma_2} \ldots K^{(n_l)}_{\sigma_l} \} t^{(n_1)}_{\sigma_1} t^{(n_2)}_{\sigma_2} \ldots t^{(n_l)}_{\sigma_l},$$

and the correlators $\{ K^{(n_1)}_{\sigma_1} K^{(n_2)}_{\sigma_2} \ldots K^{(n_l)}_{\sigma_l} \}$ are defined by

$$\{ K^{(n_1)}_{\sigma_1} K^{(n_2)}_{\sigma_2} \ldots K^{(n_l)}_{\sigma_l} \} = \frac{\int dA \exp \left( -\mu TAA \right) \sum_{l=0}^{\infty} \sum_{n_1 + \ldots + n_l = s} 1 \prod \{ K^{(n_1)}_{\sigma_1} K^{(n_2)}_{\sigma_2} \ldots K^{(n_l)}_{\sigma_l} \} t^{(n_1)}_{\sigma_1} t^{(n_2)}_{\sigma_2} \ldots t^{(n_l)}_{\sigma_l}}{\int dA \exp \left( -\mu TAA \right)}.$$

For any connected operator $K^{(a)}_{\alpha}$ in the exponent of (2.2), let us consider the deformation $\delta A = \sum_{a=1}^{\infty} \sum_{\deg \alpha = a} t^{(a)}_\alpha \frac{\partial K^{(a)}_{\alpha}}{\partial A}$ of the integration variable in the integral (2.2). It gives

$$\int dA \left[ \sum_{a=1}^{\infty} \sum_{\deg \alpha = a} t^{(a)}_\alpha \frac{\partial K^{(a)}_{\alpha}}{\partial A} + \sum_{a=1}^{\infty} \sum_{\deg \alpha = a \ deg \sigma = n} t^{(a)}_\alpha t^{(n)}_\sigma \{ K^{(n)}_{\sigma}, K^{(a)}_{\alpha} \} \right] - \mu \sum_{a=1}^{\infty} \sum_{\deg \alpha = a} a t^{(a)}_\alpha K^{(a)}_{\alpha} \exp \left( -\mu TAA \right) + \sum_{n=1}^{\infty} \sum_{\deg \sigma = n} t^{(n)}_\sigma K^{(n)}_{\sigma} = 0,$$

where $\Delta$ and $\{ \}$ are respectively the cut and join operations, the actions of the cut and join operations on the gauge-invariant operators are

$$\Delta K^{(a)}_{\alpha} = \sum_{i=1}^{N_1} \sum_{j_1=1}^{N_2} \ldots \sum_{j_{r-1}=1}^{N_r} \frac{\partial^2 K^{(a)}_{\alpha}}{\partial A^{i_1 \ldots i_{J_r-1} \partial A^{i_1 \ldots i_{J_r-1}}}$$

$$= \sum_{k=1}^{\rho} \sum_{b_1 + \ldots + b_k + 1 = a} \Delta b_1 \ldots \Delta b_k K^{(b_1)}_{\beta_1} \ldots K^{(b_k)}_{\beta_k}, \ a \geq 2,$$

and

$$\{ K^{(a)}_{\alpha}, K^{(n)}_{\sigma} \} = \sum_{i=1}^{N_1} \sum_{j_1=1}^{N_2} \ldots \sum_{j_{r-1}=1}^{N_r} \frac{\partial K^{(a)}_{\alpha}}{\partial A^{i_1 \ldots i_{J_r-1} \partial A^{i_1 \ldots i_{J_r-1}}}$$

$$= \sum_{\deg \beta = n + a - 1} \gamma^\beta_{\sigma, \alpha} K^{(n+a-1)}_{\beta},$$

$\Delta b_1 \ldots \Delta b_k$ and $\gamma^\beta_{\sigma, \alpha}$ are the coefficients.
From (2.6), we may deduce that the partition function (2.2) satisfies
\[
\mu \hat{D}_r Z_R = \hat{W}_r Z_R, \tag{2.8}
\]
where
\[
\hat{D}_r = \sum_{a=1}^{\infty} \sum_{\deg \alpha = a} a t^{(a)}_\alpha \frac{\partial}{\partial t^{(a)}_\alpha}, \tag{2.9}
\]
\[
\hat{W}_r = \sum_{a,n=1}^{\infty} \sum_{\deg \alpha = a} \sum_{\deg \beta = n} \sum_{\beta = n+1}^{a-1} \gamma^{\beta} \gamma^{(a)}_\alpha \partial^{(a)}_\alpha \partial^{(n)}_\beta \partial^{(n-a-1)}_\beta \partial^{(1)}_{id} \otimes \cdots \otimes id^{N_1 \cdots N_r} + \sum_{a=1}^{\infty} \sum_{\deg \alpha = a} \sum_{k=1}^{r} \sum_{\beta_1, \ldots, \beta_k} (1 - \delta_{a,1}) \Delta^{\beta_1, \ldots, \beta_k}_\alpha \partial^{(a)}_\alpha \partial^{(b_1)}_{\beta_1} \cdots \partial^{(b_k)}_{\beta_k} \tag{2.10}
\]
The commutation relation between \( \hat{D}_r \) and \( \hat{W}_r \) is
\[
[\hat{D}_r, \hat{W}_r] = \hat{W}_r. \tag{2.11}
\]
In terms of the operators \( \hat{D}_r \) and \( \hat{W}_r \), we may introduce the Virasoro constraints
\[
L_m Z_R = 0, \tag{2.12}
\]
where the constraint operators \( L_m \) are given by
\[
L_m = -\frac{1}{\mu} \hat{W}_r^m (\hat{W}_r - \mu \hat{D}_r), \quad m \in \mathbb{N}, \tag{2.13}
\]
which yield the Witt algebra
\[
[L_m, L_n] = (n - m)L_{m+n}, \tag{2.14}
\]
and null 3-algebra
\[
[L_k, L_m, L_n] = 0. \tag{2.15}
\]
Let us consider the operators \( \hat{D}_r \) and \( \hat{W}_r \) acting on \( Z_R^{(s)} \), respectively. We have
\[
\hat{D}_r Z_R^{(s)} = s Z_R^{(s)}, \tag{2.16}
\]
\[
\hat{W}_r Z_R^{(s)} = \mu (s + 1) Z_R^{(s+1)}. \tag{2.17}
\]
It is similar with the case of the Gaussian hermitian model [1]. We immediately recognize that the operators \( \hat{D}_r \) and \( \hat{W}_r \) are indeed the operators preserving and increasing the grading, respectively. Thus the partition function can be realized by acting on elementary function with exponents of the operator \( \hat{W}_r \)
\[
Z_R = \exp \left( \frac{1}{\mu} \hat{W}_r \right) \cdot 1. \tag{2.18}
\]
As done in the matrix models, we formally write the $m$-th power of the operator $\hat{W}_r$ as

$$
\hat{W}_r^m = \sum_{i=1}^{2m} \sum_{j=1}^{m} \sum_{a_1 + \cdots + a_i = b_1 + \cdots + b_j + m} (P_r)^{\alpha_1, \cdots, \alpha_i}_{\beta_1, \cdots, \beta_j} \sum_{\text{deg } \alpha_i = a_i, \text{deg } \beta_j = b_j} t^{(a_1)}_{\alpha_1} \cdots t^{(a_i)}_{\alpha_i} \frac{\partial}{\partial t^{(b_1)}_{\beta_1}} \cdots \frac{\partial}{\partial t^{(b_j)}_{\beta_j}}
$$

$$
+ \sum_{i=1}^{m} \sum_{a_1 + \cdots + a_i = m} \sum_{\text{deg } \alpha_i = a_i} P_r^{\alpha_1, \cdots, \alpha_i} t^{(a_1)}_{\alpha_1} \cdots t^{(a_i)}_{\alpha_i},
$$

(2.19)

where the coefficients $P_r^{\alpha_1, \cdots, \alpha_i}$ and $(P_r)^{\alpha_1, \cdots, \alpha_i}_{\beta_1, \cdots, \beta_j}$ are the polynomials of $N_1, \cdots, N_r$.

Substituting (2.19) into (2.18) and comparing the coefficients of $t^{(a_1)}_{\alpha_1} \cdots t^{(a_i)}_{\alpha_i}$ in the expansion of (2.18) with the corresponding terms in (2.2), we finally derive the compact expression of correlators

$$
\langle \mathcal{K}^{(a_1)}_{\alpha_1} \cdots \mathcal{K}^{(a_i)}_{\alpha_i} \rangle = \frac{i!}{\mu^m m! \lambda_{(a_1, \cdots, a_i)}} \sum_{\tau} P_r^{\tau(a_1), \cdots, \tau(a_i)},
$$

(2.20)

where $m = a_1 + \cdots + a_i$, $\tau$ denotes all distinct permutations of $(a_1, \cdots, a_i)$ and $\lambda_{(a_1, \cdots, a_i)}$ is the number of $\tau$ with respect to $a_1, \cdots, a_i$.

Let us turn to the Virasoro constraints (2.12). It can be rewritten as

$$
\hat{W}_r^m Z_R = \mu^m \prod_{j=0}^{m-1} (\hat{D}_r - j) Z_R, \ m \in \mathbb{N}^+.
$$

(2.21)

Since the coefficients of $t^{(a_1)}_{\alpha_1} \cdots t^{(a_i)}_{\alpha_i}$ on both sides in (2.21) with $\sum_{j=1}^i a_j = m$ are equal, we can not only derive the correlators (2.20), but also the exact correlators

$$
\langle \mathcal{K}^{(a_1)}_{\alpha_1} \rangle = \frac{1}{\mu^i} \prod_{j=0}^{i-1} (N_r + j),
$$

(2.22)

where

$$
P_r^{1, \cdots, 1} = (i - 1 + N_r) \prod_{j=0}^{i-1} (N_r + j),
$$

(2.23)

$N_r = \prod_{i=1}^r N_i$, and $\mathcal{K}_1 = A^{j_1, \cdots, j_r}_{\cdots, \cdots, 1} \tilde{A}^{j_1, \cdots, j_r}_{\cdots, \cdots, 1}$.

For the Gaussian average of the rank $r$ operator $O^{(r)}$ in the rank $r$ complex tensor model, there is a limit relation with $(O^{(r+1)})_{r+1}$ in the rank $r + 1$ model [10], i.e.,

$$
\langle \mathcal{K}^{(a_1)}_{\alpha_1} \cdots \mathcal{K}^{(a_i)}_{\alpha_i} \rangle_r = \lim_{N_r, r+1 \to \infty} \frac{1}{N_r^{m+1}} \sum_{\sigma_{r+1} \in S_m} \left( \prod_{p=1}^{m} W_{p, \sigma_{r+1}(p)}(\mathcal{K}^{(a_1)}_{\alpha_1} \cdots \mathcal{K}^{(a_i)}_{\alpha_i}) \right)_{r+1},
$$

(2.24)

where $a_1 + \cdots + a_i = m$, $S_m$ is the symmetric group that consists of permutations of $m$ elements, $W_{p,q}(\tilde{A}^{(p)}_{(q)}) = A^{p}_{(q)} \cdots \tilde{A}^{p}_{(q)} \cdots = \prod_{i=1}^{b_i} \delta_{a_i}^{b_i}$ is the Wick contractions of the $p$-th $A$ and $q$-th $\tilde{A}$ in $\mathcal{K}^{(a_1)}_{\alpha_1} \cdots \mathcal{K}^{(a_i)}_{\alpha_i}$.
When particularized to the special correlators $\langle (K_i)^r \rangle_r$, we have

$$
\lim_{N_{r+1} \to \infty} \frac{1}{N_{r+1}} \sum_{\sigma_{r+1} \in S_i} \left( \prod_{p=1}^{i} W_p,_{\sigma_{r+1}(p)}(K_{id}^{(1)} \cdots K_{id}^{(1)}) \right)_{r+1} = \frac{1}{\mu^i} \prod_{j=0}^{i+1} (N_r + j). \tag{2.25}
$$

Taking $i = 1, 2$ and $3$ in (2.25), respectively, it gives

$$
\lim_{N_{r+1} \to \infty} \frac{1}{N_{r+1}} N_r = N_r,
$$

$$
\lim_{N_{r+1} \to \infty} \frac{1}{N_{r+1}^2} \left( P_{r+1}^{id \otimes \cdots \otimes id \otimes (12)} \right) = 2N_r,
$$

$$
\lim_{N_{r+1} \to \infty} \frac{1}{N_{r+1}^3} \left[ 2P_{r+1}^{id \otimes \cdots \otimes id \otimes (123)} + 3P_{r+1}^{id \otimes \cdots \otimes id \otimes (12), id} \right] = 18N_r^2 + 12N_r. \tag{2.26}
$$

### 3 Correlators in the Aristotelian tensor model

In the Aristotelian model with the tensor $A_{ji}^{j_i,j_2}$ of rank $r = 3$, the ring is generated by keystone operators [23]

$$
K_2 = K_{(12) \otimes id \otimes id} = A_{ji}^{(1),j_2,(1)} A_{j_1,(1),j_2,(1),j_3,(1)} A_{j_3,(1),j_2,(2)} A_{j_2,(1),j_3,(2)} A_{j_1,(1),j_3,(2)} = 1
$$

$$
K_2 = K_{id \otimes (12) \otimes id} = A_{ji}^{j_1,(1),j_2,(1)} A_{j_1,(1),j_2,(1),j_3,(1)} A_{j_3,(1),j_2,(2)} A_{j_2,(1),j_3,(2)} A_{j_1,(1),j_3,(2)} = 1
$$

where 1 and $\bar{1}$ represent the first two fields $A$ and $\bar{A}$, 2 and $\bar{2}$ represent the last two fields $A$ and $\bar{A}$, respectively, the vertices are fields (tensors), the different color thin lines represent the contraction of indices in the operators, and the directions of arrows depend on the choice of covariant and contravariant indices. Note that the ring contains the tree and loop operators. If the operator belongs to the sub-ring generated only by the join operation (2.7), this operator is a tree operator, otherwise, it is a loop operator.

The tree operators made from $K_2$ or $K_2$ alone are constructed by merging two vertices in two thick circles (propagators) of the same color (figures 1 (a), (b)). When tree operators involve chains with both $K_2$ and $K_2$, they are constructed by merging two vertices of two thick circles (propagators) of different colors, two tree operators are drawn in figures 1 (c) and (d). It is known that all tree-operators are single planar cycles, and these operators which are depicted as one connected diagram are called the connected operators.
The loop operators made from $K_2$ or $\bar{K}_2$ alone are constructed by merging two vertices inside a thick circle (propagator). Two such loop operators are drawn in figures 2 (a) and (b), they are disconnected collections of red or green circles. These operators which are depicted as disconnected collection of some diagrams are called disconnected operators. When the loop operators involve both $K_2$ and $\bar{K}_2$, they are either the red-green cycles with the intersecting blue shortcuts or several such red-green cycles with the shortcuts connected by thin blue lines. Two loop operators are drawn in figures 2 (c) and (d), which are constructed by merging two vertices in two thick circles (propagators) of two different colors.
Here the thick black line represents the Feynman propagator, $K_m = K_{(12-\cdots-m)\otimes id\otimes id}$ and $K_m = K_{id\otimes(12-\cdots-m)\otimes id}$ are depicted as the red and green circles of length $m$.

and the thick red and green lines are

By means of the keystones operators, connected tree and loop operators $K^{(n)}$ in the ring, we introduce the RG-completed Aristotelian tensor model from (2.2)

$$Z_A = \int dAd\bar{A} \exp \left( -\mu TrA\bar{A} + t_1^{(1)} K_1^{(1)} + \sum_{k=2}^{\infty} t_k^{(k)} K_k^{(k)} + \sum_{k=2}^{\infty} t_k^{(k)} K_k^{(k)} + \sum_{k=2}^{\infty} t_k^{(k)} K_k^{(k)} \right)
+ \sum_{k_1, k_2=2}^{\infty} t_{k_1, k_2}^{(k_1+k_2-1)} K_{k_1, k_2}^{(k_1+k_2-1)} + \sum_{k_1, k_2=2}^{\infty} t_{k_1, k_2}^{(k_1+k_2-1)} K_{k_1, k_2}^{(k_1+k_2-1)}
+ \sum_{k_1, k_2=2}^{\infty} t_{k_1, k_2}^{(k_1+k_2-1)} K_{k_1, k_2}^{(k_1+k_2-1)} + \cdots)
= \exp \left( \frac{1}{\mu} \hat{W}_3 \right) \cdot 1, \quad (3.2)$$

where

$$\hat{W}_3 = \sum_{a=1}^{\infty} \sum_{\text{deg } \alpha = a}^{\infty} \sum_{k=1}^{3} \sum_{\beta_1, \ldots, \beta_k}^{\delta_2, \ldots, \delta_k} (1 - \delta_{\alpha_1, 1}) \Delta^{\beta_1, \ldots, \beta_k} t_{(a)}^{(k)} \frac{\partial}{\partial t_{(b_1)_{\beta_1}}} \cdots \frac{\partial}{\partial t_{(b_k)_{\beta_k}}} + t_{1}^{(1)} \otimes \otimes N_1 N_2 N_3
+ \sum_{a,n=1}^{\infty} \sum_{\text{deg } \alpha = a}^{\infty} \sum_{\text{deg } \sigma = n}^{\infty} \sum_{\text{deg } \beta = n+a-1}^{\infty} \gamma^{\alpha}_{\alpha, \sigma} t_{(a)}^{(n)} \frac{\partial}{\partial t_{(n+a-1)_{\beta}}}, \quad (3.3)$$

$\alpha, \sigma$ and $\beta$ are taken from the indices of connected operators in the ring.

For the Aristotelian tensor model (3.2), the Virasoro constraint operators in (2.12) become

$$L_m = \left( -\frac{1}{\mu} \right) \hat{W}_3^{m} (\hat{W}_3 - \mu \hat{D}_3), \quad (3.4)$$

where $\hat{D}_3$ is given by (2.9) with $r = 3$ in which the index $\alpha$ is an element of the double coset $S_3 = S_3 \backslash S_3^{\otimes 3} / S_3$. 

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Since the coefficients $P^{r(\alpha_1),\ldots,r(\alpha_i)}$ in (2.20) follow from the precise expression of $\hat{W}^m$, we may give the exact correlators from (2.20). In particular,

$$
((K_1)^i) = \frac{1}{\mu} (N_3 + i - 1) ((K_1)^i) = \frac{1}{\mu} \prod_{j=0}^{i-1} (N_3 + j),
$$

where $N_3 = N_1 N_2 N_3$.

Let us give the correlators $\langle K_1 K_1 \rangle$ and $\langle K_1 K_1 K_1 \rangle$ and represent these correlators graphically as follows:

$$
\langle K_1 K_1 \rangle = \frac{N_3}{\mu^2} + \frac{N_3}{\mu^2} = \left(\begin{array}{c}
\frac{1}{\mu} \otimes \frac{N_3}{\mu^2} \\
\frac{1}{\mu} \otimes \frac{N_3}{\mu^2} \\
\frac{1}{\mu} \otimes \frac{N_3}{\mu^2}
\end{array}\right) + \left(\begin{array}{c}
\frac{1}{\mu} \otimes \frac{N_3}{\mu^2} \\
\frac{1}{\mu} \otimes \frac{N_3}{\mu^2} \\
\frac{1}{\mu} \otimes \frac{N_3}{\mu^2}
\end{array}\right),
$$

$$
\langle K_1 K_1 K_1 \rangle = \frac{N_3}{\mu^2} + 3 \frac{N_3}{\mu^2} + 2 \frac{N_3}{\mu^2}
$$

which the thick line depicts the Feynman propagator and each propagator gives a factor $\frac{1}{\mu}$, the red, green and blue circles represent $N_1$, $N_2$ and $N_3$, respectively.

By calculating $\hat{W}_3^i$, $i = 1, \ldots, 4$, we obtain the correlators which have been derived in [25]. In the following, we give two correlators by calculating $\hat{W}_3^5$$

$$
\langle K_{id \otimes (12345) \otimes id} \rangle = \frac{1}{\mu^3} \left[ N_3 (15 N_1^2 N_4^2 + 15 N_2^2 + N_1^4 N_3^4 + N_2^4 + 8) + 10 N_3^2 (N_1^2 N_3^2 + N_2^2 + 4) + 20 N_3^3 \right],
$$

$$
\langle K_{id \otimes (12345) \otimes (12)} \rangle = \frac{1}{\mu^3} \left[ N_3^2 (N_1^2 N_3^2 + 9 N_1^2 N_3 + 14 N_1 N_2 + 6 N_2^2 N_3 + 20 N_3) + 6 N_3^3 N_3 + N_3 (N_1^4 N_3^3 + 3 N_1^2 N_3^3 + 12 N_1^2 N_3^3 + 20 N_1 N_2 + 4 N_1 N_3 + 15 N_2^2 N_3 + N_2^4 N_3 + 8 N_3) \right].
$$

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4 Correlators in the $r = 4$ rainbow tensor model

In the rainbow tensor model with the tensor $A_{j_1,j_2,j_3}^{i_1,i_2}$ of rank $r = 4$, the keystone operators are

\[ K_2 = K_{(12) \otimes id \otimes id \otimes id} = A_{j_1^{(1)},j_2^{(1)},j_3^{(1)}}^{i_1^{(1)}}, A_{j_1^{(2)},j_2^{(2)},j_3^{(2)}}^{i_1^{(2)}}, A_{j_1^{(1)},j_2^{(2)},j_3^{(2)}}^{i_1^{(2)}} - A_{j_1^{(2)},j_2^{(1)},j_3^{(2)}}^{i_1^{(1)}} - A_{j_1^{(2)},j_2^{(1)},j_3^{(2)}}^{i_1^{(2)}} - A_{j_1^{(1)},j_2^{(2)},j_3^{(2)}}^{i_1^{(1)}} \]

(4.1)

\[ K_2 = K_{id \otimes (12) \otimes id \otimes id} = A_{j_1^{(1)},j_2^{(1)},j_3^{(1)}}^{i_1^{(1)}}, A_{j_1^{(2)},j_2^{(2)},j_3^{(2)}}^{i_1^{(2)}}, A_{j_1^{(1)},j_2^{(2)},j_3^{(2)}}^{i_1^{(2)}}, A_{j_1^{(2)},j_2^{(1)},j_3^{(2)}}^{i_1^{(1)}} - A_{j_1^{(2)},j_2^{(1)},j_3^{(2)}}^{i_1^{(2)}} - A_{j_1^{(2)},j_2^{(1)},j_3^{(2)}}^{i_1^{(2)}} - A_{j_1^{(2)},j_2^{(1)},j_3^{(2)}}^{i_1^{(1)}} \]

(4.2)

The ring generated by the keystone operators contains the tree and loop operators, red circles $K_n = K_{(12..n) \otimes id \otimes id}$, green circles $K_n = K_{id \otimes (12..n) \otimes id}$ and disconnected collections of the red and green circles, respectively.

In similarity with the case of Aristotelian tensor model, the tree and loop operators can also be constructed. In figures 3 and 4, we draw some tree and loop operators involving chains with both $K_2$ and $K_2$, respectively.

Corresponding to figures 1(a), (b) and figures 2 (a), (b), we can draw the similar tree and loop operators made from $K_2$ or $K_2$ alone.
Note that here the thick red and green circles are

\[ \begin{array}{c}
\text{\textcircled{\text{Red}}} = \text{\textcircled{\text{Red}}} \\
\text{\textcircled{\text{Green}}} = \text{\textcircled{\text{Green}}}
\end{array} \]

and the thick red and green lines are

\[ \begin{array}{c}
\text{\textcircled{\text{Red}}} = \text{\textcircled{\text{Red}}} \\
\text{\textcircled{\text{Green}}} = \text{\textcircled{\text{Green}}}
\end{array} \]

In terms of the connected operators \( K^{(n)}_\sigma \), we write the rainbow tensor model (2.2) with \( r = 4 \) as

\[
Z_4 = \int dA d\bar{A} \exp \left( -\mu \text{Tr} A \bar{A} + t^{(1)}_1 K^{(1)}_1 + \sum_{k=2}^\infty t^{(k)}_k K^{(k)}_k + \sum_{k=2}^\infty t^{(k)}_k K^{(k)}_k + \cdots \right) \\
= \exp \left( \frac{1}{\mu} \hat{W}_4 \right) \cdot 1, \tag{4.3}
\]

where the operator \( \hat{W}_4 \) is given by

\[
\hat{W}_4 = \sum_{a,n=1}^{\infty} \sum_{\text{deg } \alpha = a} \sum_{\text{deg } \beta = n} \sum_{\delta = n+a-1} \gamma^{t(a)}_{\alpha} t^{(n)}_{\sigma} \frac{\partial}{\partial t^{(n+a-1)}} + t^{(1)}_{\text{id } \otimes \text{id } \otimes \text{id } \otimes \text{id}} N_1 N_2 N_3 N_4 \\
+ \sum_{a=1}^{\infty} \sum_{\text{deg } \alpha = a} \sum_{k=1}^{4} \sum_{\beta_1, \ldots, \beta_k = a} (1 - \delta_{a,1}) \Delta^{\beta_1, \ldots, \beta_k} t^{(a)}_{\alpha} \frac{\partial}{\partial t^{(b_1)}} \cdots \frac{\partial}{\partial t^{(b_k)}}. \tag{4.4}
\]

For the partition function (4.3), the Virasoro constraint operators in (2.12) are

\[
L_m = \left( -\frac{1}{\mu} \right) \hat{W}^m_4 (\hat{W}_4 - \mu \hat{D}_4), \tag{4.5}
\]

where \( \hat{D}_4 \) is given by (2.9) with \( r = 4 \) in which the index \( \alpha \) is an element of the double coset \( S_n^4 = S_n \setminus S_n^{\otimes 4} / S_n \).

We may give the exact correlators from (2.20), where the coefficients \( P^{\tau(\alpha_1), \ldots, \tau(\alpha_i)} \) in (2.20) follow from the power of \( \hat{W}_4 \) (4.4). The special correlators \( \langle (K_1)^i \rangle \) are given by (2.22) with \( \mathcal{N}_4 = N_1 N_2 N_3 N_4 \).

Let us list some correlators (2.20) by calculating \( \hat{W}_4^i, i = 1, 2, 3 \), and represent them graphically with the same rules as the case of Aristotelian tensor model. Noted that the thick (4-colored) lines depict the Feynman propagators, and the yellow circles represent \( N_4 \).
there is only one possible attachment of the Feynman propagators to the operator $\mathcal{K}_1$, giving the result $\frac{\mathcal{N}_4}{\mu^2}$.

(ii)

\[
\langle \mathcal{K}_2 \rangle = \langle \mathcal{K}^{(2)}_{id \otimes id \otimes id \otimes id} \rangle = \frac{N_1 \mathcal{N}_4}{\mu^2} + \frac{N_2 N_3 \mathcal{N}_4 \mathcal{N}_4}{\mu^2}
\]

\[
= \left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\]

\[
\langle \mathcal{K}^{(2)}_{id \otimes id \otimes id \otimes id} \rangle = \frac{N_1 N_2 \mathcal{N}_4}{\mu^2} + \frac{N_3 \mathcal{N}_4 \mathcal{N}_4}{\mu^2}
\]

\[
= \left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\]

\[
\langle \mathcal{K}_1 \mathcal{K}_1 \rangle = \langle \mathcal{K}^{(1)}_{id \otimes id \otimes id \otimes id} \mathcal{K}^{(1)}_{id \otimes id \otimes id \otimes id} \rangle = \frac{\mathcal{N}_4 (\mathcal{N}_4 + 1)}{\mu^2}
\]

\[
= \left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\]

\[
\langle \mathcal{K}_2 \rangle = \langle \mathcal{K}^{(2)}_{id \otimes id \otimes id} \rangle = \frac{N_2 \mathcal{N}_4}{\mu^2} + \frac{N_1 N_3 \mathcal{N}_4 \mathcal{N}_4}{\mu^2}
\]

\[
= \left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\left( \begin{array}{c}
\mathcal{N}_4
\end{array} \right)
\]

\[
(4.7)
\]
Due to too many Feynman diagrams for the correlators (4.9), we do not present them here.
5 (Non-Gaussian) red tensor model

5.1 Correlators in the red tensor model

Let’s take $\sigma$ in (2.2) to be the simplest element of the double coset $S_{n r}^r = S_{n r} \backslash S_n \otimes S_n / S_n$, i.e. 

$$(12 \cdots n) \otimes id \otimes \cdots \otimes id,$$

then

$$K_n \equiv K_{\sigma}^{(n)} = K^{(n)}_{(12-n) \otimes id \otimes \cdots \otimes id} = A_{i(1)}^{j_1(1), j_2(1) \cdots j_{r-1}(1)} A_{j_1(2), j_2(2) \cdots j_{r-1}(2)}^{j_1(2), j_2(2) \cdots j_{r-1}(2)} \cdots A_{n(n)}^{n(n)}.$$

(5.1)

Let us take the keystone operator

$$K_2 = K_{(12) \otimes id \otimes \cdots \otimes id} = A_{i(1)}^{j_1(1), j_2(1) \cdots j_{r-1}(1)} A_{j_1(2), j_2(2) \cdots j_{r-1}(2)}^{j_1(2), j_2(2) \cdots j_{r-1}(2)} \cdots A_{n(n)}^{n(n)}.$$

(5.2)

Then the tree and loop operators are $K_n$ and $\prod_n K_n$, respectively. Note that the tree operators are connected.

We introduce the red tensor model

$$Z = \int dAd\bar{A} \exp \left( -\mu TrA\bar{A} + \sum_{n=1}^{\infty} t_n K_n \right) = \exp \left( \frac{1}{\mu} \hat{W} \right) \cdot 1,$$

(5.3)

where we denote the index $i$ in $A_{i}^{j_1 \cdots j_{r-1}}$ with color red, the operator $\hat{W}$ is given by

$$\hat{W} = \sum_{b_1, b_2=1}^{\infty} (b_1 + b_2 + 1) t_{b_1 + b_2 + 1} \frac{\partial}{\partial t_{b_1}} \frac{\partial}{\partial t_{b_2}} + \sum_{b_1, b_2=1}^{\infty} b_1 b_2 t_{b_1} t_{b_2} \frac{\partial}{\partial t_{b_1 + b_2 - 1}}$$

$$+ \hat{N}_r \sum_{b=1}^{\infty} (b + 1) t_{b+1} \frac{\partial}{\partial t_{b}} + \mathcal{N}_r t_1,$$

(5.4)

$$\hat{N}_r = N_1 + N_2 N_3 \cdots N_r$$

and

$$\mathcal{N}_r = N_1 N_2 N_3 \cdots N_r.$$

When particularized to the rank 3 tensor case in the partition function (5.3), the operator $\hat{W}$ reduces to the result derived in ref. [23].

For the case of the red tensor model (5.3), the Virasoro constraint operators in (2.12) are

$$L_m = \left( -\frac{1}{\mu} \right) \hat{W}^m (\hat{W} - \mu \hat{D}),$$

(5.5)
and the correlators (2.20) become
\[
\langle K_{a_1} \cdots K_{a_i} \rangle = \frac{i!}{\mu^m m! \lambda_{\{a_1, \ldots, a_i\}}} \sum_\tau P_{\tau}^{(a_1), \ldots, (a_i)},
\]
(5.6)
where \( m = a_1 + \cdots + a_i \) and \( \hat{D} = \sum_{a=1}^{\infty} \alpha_t a \frac{\partial}{\partial a} \). Furthermore the correlators \( \langle (K_1)^i \rangle \) are given by (2.22) with \( N_r = N_1 N_2 N_3 \cdots N_r \).

By calculating \( \hat{W}^i, i = 1, \ldots, 4 \) to give \( P_{\tau}^{(a_1), \ldots, (a_i)} \) in (5.6), we obtain the exact correlators of degree no more than 4. Let us list some correlators as follows:

\[
\langle K_2 \rangle = \frac{N_1 N_r}{\mu^2} = \frac{N_1 N_r}{\mu^2} + \frac{N_2 N_3 \cdots N_r N_r}{\mu^2}
\]
(5.7)

\[
\langle K_1 K_2 \rangle = \frac{N_1 N_1^2}{\mu^3} + \frac{N_2 N_3 \cdots N_r N_r^2}{\mu^3} + 2 \frac{N_1 N_r}{\mu^3} + 2 \frac{N_2 N_3 \cdots N_r N_r}{\mu^3}
\]

\[
\langle K_3 \rangle = \frac{N_1^2 N_r}{\mu^3} + 2 \frac{N_1 \cdot N_2 N_3 \cdots N_r N_r}{\mu^3} + \left( \frac{N_2 N_3 \cdots N_r}{\mu^3} \right)^2 + \frac{N_r^2}{\mu^3} + \frac{N_r^2}{\mu^3}
\]

– 15 –
5.2 Non-Gaussian red tensor model

The correlators in the matrix models including supereigenvalue models have attracted considerable attention. Much interest has also been attributed to the non-Gaussian cases [26]–[30]. But so far, no investigation has been made for the non-Gaussian tensor models. The red tensor model is a simple rainbow tensor model. In the previous section, we have presented its $W$-representation and the correlators. Let us now focus on the non-Gaussian red tensor model

$$Z_{NG}\{t,T\} = \int dA d\bar{A} \exp \left( -\frac{\mu}{p} K_p - \mu \sum_{n=1}^{p-1} T_n K_n + \sum_{n=1}^{\infty} t_n K_n \right). \quad (5.8)$$

In this model, the keystone operator is $K_2$ (5.2), the tree and loop operators are the same with the case of the red tensor model.

From the deformation $\delta A = \frac{\partial K_{m+1}}{\partial A}$ of the integration variable, we derive the Virasoro constrains

$$\hat{L}_m Z_{NG}\{t,T\} = 0, \quad (5.9)$$

where

$$\hat{L}_m = \delta_{m,0} N_r + (1 - \delta_{m,0}) N_r \frac{\partial}{\partial t_m} + \sum_{b=1}^{m-1} \frac{\partial}{\partial t_b} \frac{\partial}{\partial t_{m-b}} - \mu \frac{\partial}{\partial t_{m+p}} - \mu \sum_{n=1}^{p-1} n T_n \frac{\partial}{\partial t_{m+n}} + \sum_{n=1}^{\infty} n t_n \frac{\partial}{\partial t_{m+n}}. \quad (5.10)$$

The operators yield the Witt algebra (2.14), but the null 3-algebra (2.15) does not hold.

Let us take the deformation $\delta A = \sum_{m=0}^{\infty} (m+p) t_{m+p} \frac{\partial K_{m+1}}{\partial A}$ of the integration variable in the integral (5.8), we obtain

$$\left( \mu \hat{D}_p - \hat{W}_{NG} \right) Z_{NG}\{t,T\} = 0, \quad (5.11)$$

where

$$\hat{D}_p = \sum_{m=p}^{\infty} m t_m \frac{\partial}{\partial t_m}, \quad (5.12)$$

In this model, the keystone operator is $K_2$ (5.2), the tree and loop operators are the same with the case of the red tensor model.
and
\[
\hat{W}_{NG} = \sum_{m=1}^{\infty} \sum_{b=1}^{m-1} (m+p)t_{m+p} \frac{\partial}{\partial t_{m-b}} + pN_t p + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n(m+p)t_{nt_{m+p}} \frac{\partial}{\partial t_{m+n}} + \sum_{m=1}^{\infty} (m+p)\hat{N}_t t_{m+p} \frac{\partial}{\partial t_m} - \mu \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n(m+p)T_{nt_{m+p}} \frac{\partial}{\partial t_{m+n}}. \tag{5.13}
\]

On the other hand, there are the additional constraints for the partition function (5.8)
\[
\left( \mu \frac{\partial}{\partial t_m} + \frac{\partial}{\partial T_m} \right) Z_{NG}\{t,T\}, \quad m = 1, \cdots, p - 1. \tag{5.14}
\]
Combining (5.14) and (5.11), we have
\[
\left( \mu \hat{D} - \hat{W} \right) Z_{NG}\{t,T\} = 0, \tag{5.15}
\]
where \( \hat{D} = \sum_{m=0}^{\infty} mt_m \frac{\partial}{\partial t_m} \) and \( \hat{W} = \hat{W}_{NG} - \sum_{m=0}^{p-1} mt_m \frac{\partial}{\partial T_m} \).

By expanding the exponential term in (5.8), we rewrite the partition function as
\[
Z_{NG}\{t,T\} = \sum_{s=0}^{\infty} Z^{(s)}_{NG}\{t,T\} \tag{5.16}
\]
\[
= Z_{NG}\{T\} \cdot \left( 1 + \sum_{n_1=1}^{\infty} t_{n_1} \langle K_{n_1} \rangle_{NG} + \sum_{n_1,n_2=1}^{\infty} t_{n_1} t_{n_2} \langle K_{n_1} K_{n_2} \rangle_{NG} + \cdots \right),
\]
where
\[
Z^{(s)}_{NG}\{t,T\} = Z_{NG}\{T\} \cdot \sum_{l=0}^{\infty} \sum_{n_1+\cdots+n_l=s} \frac{1}{l!} t_{n_1} t_{n_2} \cdots t_{n_l} \langle K_{n_1} K_{n_2} \cdots K_{n_l} \rangle_{NG}, \tag{5.17}
\]
\[
Z_{NG}\{T\} = \int dA d\bar{A} \exp \left[ -\frac{\mu}{p} K_p - \mu \sum_{n=1}^{p-1} T_n K_n \right], \tag{5.18}
\]
the correlators are defined by
\[
\langle K_{n_1} K_{n_2} \cdots K_{n_l} \rangle_{NG} = \frac{\int dA d\bar{A} K_{n_1} K_{n_2} \cdots K_{n_l} \exp \left( -\frac{\mu}{p} K_p - \mu \sum_{n=1}^{p-1} T_n K_n \right) \frac{\partial}{\partial K_{n_1}}}{\int dA d\bar{A} \exp \left( -\frac{\mu}{p} K_p - \mu \sum_{n=1}^{p-1} T_n K_n \right)}. \tag{5.19}
\]

The operators \( \hat{D} \) and \( \hat{W} \) acting on \( Z^{(s)}_{NG}\{t,T\} \) show that \( \hat{D} \) is the operator preserving the grading. However, \( \hat{W} \) is not the desired operator increasing the grading, since it does not satisfy the similar relation (2.17). Unlike the case of (2.18), the partition function can not be realized by acting on elementary function with exponents of the operator \( \hat{W} \).
Note that $\hat{D} Z_{NG}^{T} = 0$. From (5.15), it is not difficult to obtain that

$$
\sum_{s=1}^{\infty} Z_{NG}^{(s)}(t, T) = (\mu \hat{D} - \hat{W})^{-1} \hat{W} Z_{NG}^{T}
$$

$$
= \sum_{k=1}^{\infty} \mu^{-k} (\hat{D}^{-1} \hat{W})^{k} Z_{NG}^{T}.
$$

(5.20)

Thus the partition function (5.8) can be expressed as

$$
Z_{NG}^{T} = \sum_{k=0}^{\infty} \mu^{-k} (\hat{D}^{-1} \hat{W})^{k} Z_{NG}^{T}.
$$

(5.21)

We observe that (5.21) is similar with the case of non-Gaussian matrix model [30]. It shows that the dual expression for the non-Gaussian red tensor model (5.8) through differentiation can also be formulated.

Since the usual $W$-representation of (5.8) fails, we can not present the compact expression of correlators here. In principle, we can give the correlators from (5.21). Let us list some correlators as follows:

$$
\langle K_{i} \rangle_{NG} = -\frac{1}{\mu} \frac{1}{Z_{NG}^{T}} \frac{\partial}{\partial T_{i}} Z_{NG}^{T}, \quad (i=1,\ldots,p-1),
$$

$$
\langle K_{p} \rangle_{NG} = \frac{N_{r}}{\mu} + \frac{1}{\mu} \sum_{i=1}^{p-1} i T_{i} \frac{\partial}{\partial T_{i}} Z_{NG}^{T},
$$

$$
\langle K_{p+1} \rangle_{NG} = -\frac{N_{r}}{\mu} \frac{\partial}{\partial T_{1}} Z_{NG}^{T} + \frac{1}{\mu} \sum_{i=1}^{p-2} i T_{i} \frac{\partial}{\partial T_{i}} Z_{NG}^{T}
$$

$$
- \frac{N_{r}}{\mu} (p-1) T_{p-1} - (p-1) T_{p-1} \sum_{i=1}^{p-1} i T_{i} \frac{\partial}{\partial T_{i}} Z_{NG}^{T},
$$

$$
\langle K_{i} K_{j} \rangle_{NG} = \frac{1}{\mu^{2}} \frac{1}{Z_{NG}^{T}} \frac{\partial^{2}}{\partial T_{i} \partial T_{j}} Z_{NG}^{T}, \quad (i_{1}, i_{2} = 1, \ldots, p-1),
$$

$$
\langle K_{p} K_{q} \rangle_{NG} = -\frac{1}{\mu^{2}} \sum_{i=1}^{p-1} i T_{i} \frac{\partial^{2}}{\partial T_{i} \partial T_{q}} Z_{NG}^{T}
$$

$$
- \frac{N_{r}}{\mu^{2}} \frac{\partial}{\partial T_{q}} Z_{NG}^{T}, \quad (q = 1, \ldots, p-1).
$$

(5.22)

6 Conclusions

$W$-representation is important for the understanding of matrix model, since it provides a dual formula for partition function through differentiation. We have investigated the $W$-representation of the rainbow tensor model with the rank $r$ complex tensor in this paper. By the given deformation of the integration variable in the integral, we derived the desired operators $\hat{D}_{r}$ and $\hat{W}_{r}$ which preserve and increase the grading, respectively. It was shown that the rainbow tensor model can be realized by acting on elementary function with exponent of the operator $\hat{W}_{r}$. In terms of the operators preserving and increasing the
grading, we can construct the Virasoro constraints for the rainbow tensor model, where the constraint operators obey the Witt algebra and null 3-algebra. An interesting aspect of these Virasoro constraints is that the compact expression of correlators can be derived from them. As examples, we have applied above results to analyze the red tensor model, Aristotelian tensor model and $r = 4$ rainbow tensor model in detail and presented the corresponding correlators in these models.

We have also considered the non-Gaussian red tensor model and presented the Virasoro constraints, where the constraint operators obey the Witt algebra, however the null 3-algebra does not hold. We showed that the partition function can be expressed as the infinite sum of the operators $(\hat{D}^{-1}\hat{W})^k$ acting on the given function. Namely, a dual form for partition function through differentiation can be formulated. Since $\hat{W}$ is not the desired operator increasing the grading, it causes the usual W-representation of the non-Gaussian red tensor model to fail here. For this reason, the dual expression (5.21) through differentiation can be regarded as the generalized W-representation. In terms of the operators $\hat{D}$ and $\hat{W}$, we can not construct the Virasoro constraints such that the constraint operators obey the Witt algebra and null 3-algebra. It should be noted that we can calculate the correlators from (5.21). However, the compact expression of correlators can not be derived. We have presented some correlators. How to represent these correlators graphically still deserves further study. Furthermore, further study should be done to investigate the W-representations of the non-Gaussian and fermionic tensor models.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Nos. 11875194 and 11871350).

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