Stratonovich representation of semimartingale rank processes

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Abstract

Suppose that $X_1, \ldots, X_n$ are continuous semimartingales that are reversible and have nondegenerate crossings. Then the corresponding rank processes can be represented by generalized Stratonovich integrals, and this representation can be used to decompose the relative log-return of portfolios generated by functions of ranked market weights.

Introduction

For $n \geq 2$, consider a family of continuous semimartingales $X_1, \ldots, X_n$ defined on $[0, T]$ under the usual filtration $\mathcal{F}_t^X$, with quadratic variation processes $\langle X_i \rangle_t$. Let $r_i(t)$ be the rank of $X_i(t)$, with $r_i(i) < r_i(j)$ if $X_i(t) > X_j(t)$ or if $X_i(t) = X_j(t)$ and $i < j$. The corresponding rank processes $X_{(1)}, \ldots, X_{(n)}$ are defined by $X_{(r_i(t))}(t) = X_i(t)$. We shall show that if the $X_i$ are reversible and have nondegenerate crossings, then the rank processes can be represented by

$$dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t) = X_{(k)}(t)\}} \circ dX_i(t), \quad \text{a.s.,} \quad (1)$$

where $\circ d$ is the generalized Stratonovich integral developed by Russo and Vallois (2007).

An Atlas model is a family of positive continuous semimartingales $X_1, \ldots, X_n$ defined as an Itô integral on $[0, T]$ by

$$d \log X_i(t) = (-g + ng \mathbb{1}_{\{r_i(t) = n\}}) dt + \sigma dW_i(t), \quad (2)$$

where $g$ and $\sigma$ are positive constants and $(W_1, \ldots, W_n)$ is a Brownian motion (see Fernholz (2001)). The Stratonovich representation (1) allows us to extend this decomposition to portfolios generated by $C^2$ functions of the ranked market weight processes in Atlas models.

Itô integrals and Stratonovich integrals

Let $X$ and $Y$ be continuous semimartingales on $[0, T]$ with the filtration $\mathcal{F}_t^{X,Y}$. Then the Fisk-Stratonovich integral is defined by

$$\int_0^t Y(s) \circ dX(s) \triangleq \int_0^t Y(s) dX(s) + \frac{1}{2} \langle Y, X \rangle_t, \quad (3)$$

for $t \in [0, T]$, where the integral on the right hand side is the Itô integral and $\langle X, Y \rangle_t$ is the cross variation of $X$ and $Y$ over $[0, t]$ (see Karatzas and Shreve (1991)). The Fisk-Stratonovich integral is defined only for semimartingales, but in some cases can be extended to more general integrands. Following Russo and Vallois (2007) Definition 1, for a continuous semimartingale $X$ and a locally integrable process $Y$, both defined on $[0, T]$, we define the forward integral, backward integral, and covariation process by

$$\int_0^t Y(s) d^- X(s) \triangleq \lim_{\varepsilon \downarrow 0} \int_0^t Y(s) \frac{X(s + \varepsilon) - X(s)}{\varepsilon} ds \quad (4)$$

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\[\int_0^t Y(s) d^r X(s) \triangleq \lim_{\varepsilon \downarrow 0} \int_0^t Y(s) \frac{X(s) - X(s-\varepsilon)}{\varepsilon} ds\]  
(5)

\[[X,Y]_t \triangleq \lim_{\varepsilon \downarrow 0} \int_0^t \frac{(X(s+\varepsilon) - X(s))(Y(s+\varepsilon) - Y(s))}{\varepsilon} ds,\]
(6)

for \(t \in [0,T]\), where the limits are uniform in probability on \([0,T]\). We shall use the convention of Russo and Vallois (2007) that for the evaluation of these limits a continuous function \(X\) defined on \([0,T]\) is implicitly extended to \(\mathbb{R}\) by setting \(X(t) = X(0)\) for \(t < 0\) and \(X(t) = X(T)\) for \(t > T\). Then, by Russo and Vallois (2007) Definition 10, the Stratonovich integral is given by

\[\int_0^t Y(s) \circ dX(s) \triangleq \int_0^t Y(s) dX(s) + \frac{1}{2} [Y,X]_t,\]  
(7)

where the integral on the right hand side is the Itô integral. If both \(X\) and \(Y\) are continuous semimartingales, then

\[(X,Y)_t = [X,Y]_t, \ a.s.,\]

and the Stratonovich integral is equivalent to the Fisk-Stratonovich integral.

For a continuous semimartingale \(X\) and \(C^2\) function \(F\) defined on the range of \(X\), Itô’s rule establishes that

\[F(X(t)) - F(X(0)) = \int_0^t F'(X(s)) dX(s) + \frac{1}{2} \int_0^t F''(X(s)) d\langle X, X \rangle_s, \ a.s.,\]

and with the Fisk-Stratonovich integral, this becomes

\[F(X(t)) - F(X(0)) = \int_0^t F'(X(s)) \circ dX(s), \ a.s.,\]
(8)

as in ordinary calculus (see Karatzas and Shreve (1991)). The relationship \(\Box\) can be extended to a wider class of functions in some cases. For example, for an absolutely continuous function \(F\) and Brownian motion \(W\), it was shown in Föllmer et al. (1995) Corollary 4.2, that \(\Box\) holds, so for the absolute-value function we have

\[|W(t)| = \int_0^t \text{sgn}(W(s)) \circ dW(s), \ a.s.,\]

where \(\text{sgn}(x) \triangleq 1_{\{x>0\}} - 1_{\{x<0\}}\). The Russo and Vallois (2007) results allow us to extend this relationship to a class of continuous semimartingales.

**Definition 1.** Let \(X\) be a continuous semimartingale defined on \([0,T]\) under the filtration \(\mathcal{F}_t^X\). Then \(X\) is reversible if the time-reversed process \(\bar{X}\) defined by \(\bar{X}(t) = X(T-t)\) is also a continuous semimartingale on \([0,T]\) under the time-reversed filtration \(\mathcal{F}_t^{\bar{X}}\).

**Definition 2.** The continuous semimartingales \(X_1, \ldots, X_n\) have nondegenerate crossings if for any \(i \neq j\) the set \(\{t : X_i(t) = X_j(t)\}\) almost surely has measure zero with respect to \(d\langle X_k, X_k \rangle_t\), for \(1 \leq k \leq n\).

**Lemma 1.** Suppose that the continuous semimartingales \(X_1, \ldots, X_n\) have nondegenerate crossings. Then the same is true for the rank processes \(X_{(1)}, \ldots, X_{(n)}\).

**Proof.** Let \(p_i \in \Sigma_n\) be the inverse permutation to the rank function \(r_i\). If \(X_{(k)}(t) = X_{(\ell)}(t)\) for \(k \neq \ell\), then \(X_i(t) = X_j(t)\) for \(i = p_i(k) \neq p_i(\ell) = j\), so

\[\bigcup_{i \neq j} \{t : X_i(t) = X_j(t)\} = \bigcup_{k \neq \ell} \{t : X_{(k)}(t) = X_{(\ell)}(t)\}.\]
(10)
From Banner and Ghomrasni (2008) we have the representation
\[ dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t) = X_{(k)}(t)\}} dX_i(t) + \text{finite variation terms}, \text{ a.s.,} \]

for \( k = 1, \ldots, n \), so that \( d(X_{(k)})_t \ll d(X_1)_t + \cdots + d(X_n)_t \), for \( k = 1, \ldots, n \). In the same manner we can show that \( d(X_i)_t \ll d(X_{(1)})_t + \cdots + d(X_{(n)})_t \), for \( i = 1, \ldots, n \), so sets of the form \( \{ t : X_i(t) = X_j(t) \} \), for \( i \neq j \), or \( \{ t : X_{(k)}(t) = X_{(\ell)}(t) \} \), for \( k \neq \ell \), will almost surely have measure zero with respect to \( d(X_i)_t \), for \( i = 1, \ldots, n \), and with respect to \( d(X_{(k)})_t \), for \( k = 1, \ldots, n \), and the same holds for finite unions of such sets, as in (10).

\[ \square \]

Lemma 2. Let \( X \) be a reversible continuous semimartingale defined on \([0,T]\), and suppose that the set \( \{ t : X(t) = 0 \} \) almost surely has measure zero with respect to \( d(X)_t \). Then
\[ |X(t)| - |X(0)| = \int_0^t \text{sgn}(X(s)) \circ dX(s), \text{ a.s.} \tag{11} \]

Proof. References in this proof denoted by R&V are from Russo and Vallois (2007).

The Tanaka-Meyer formula states that for the \( \text{Itô integral} \)
\[ \int_0^t \text{sgn}(X(s)) dX(s) = |X(t)| - |X(0)| + 2\Lambda_X(t), \text{ a.s.,} \tag{12} \]

where \( \Lambda_X \) is the local time at zero for \( X \) (see Karatzas and Shreve (1991)). From R&V, Proposition 1,
\[ \int_0^t \text{sgn}(X(s)) \circ dX(s) = \frac{1}{2} \left( \int_0^t \text{sgn}(X(s)) d^- X(s) + \int_0^t \text{sgn}(X(s)) d^+ X(s) \right), \tag{13} \]

with the forward and backward integrals defined by (14) and (15). Since \( \text{sgn}(X) \) is continuous outside the set \( \{ t : X(t) = 0 \} \), and this set almost surely has measure zero with respect to \( d(X)_t \), R&V Proposition 6 implies that
\[ \int_0^t \text{sgn}(X(s)) d^- X(s) = \int_0^t \text{sgn}(X(s)) dX(s) = |X(t)| - |X(0)| + 2\Lambda_X(t), \text{ a.s.,} \tag{14} \]

by equation (12).

By R&V, Proposition 1,
\[ \int_0^t \text{sgn}(X(s)) d^+ X(s) = -\int_{T-t}^T \text{sgn}(\widehat{X}(s)) d^- \widehat{X}(s), \tag{15} \]

where \( \widehat{X} \) is the time-reversed version of \( X \). By hypothesis, \( \widehat{X} \) is a continuous semimartingale on \([0,T]\) with respect to the reverse filtration, so as in (14) we have
\[ \int_{T-t}^T \text{sgn}(\widehat{X}(s)) d^- \widehat{X}(s) = |\widehat{X}(T)| - |\widehat{X}(T-t)| + 2(\Lambda_{\widehat{X}}(T) - \Lambda_{\widehat{X}}(T-t)) \]
\[ = |X(0)| - |X(t)| + 2\Lambda_X(t), \text{ a.s.} \tag{16} \]

If we combine (13), (14), (15), and (16), then (11) follows. \( \square \)
Lemma 3. Let \( X \) and \( Y \) be reversible continuous semimartingales defined on \([0, T]\) under a common filtration and suppose that they have nondegenerate crossings. Then

\[
|X(t) - Y(t)| - |X(0) - Y(0)| = \int_0^t \text{sgn}(X(s) - Y(s)) \circ dX(s) - \int_0^t \text{sgn}(X(s) - Y(s)) \circ dY(s), \quad \text{a.s.} \quad (17)
\]

Proof. References in this proof denoted by R&V are from Russo and Vallois (2007).

Since \( X - Y \) is a reversible continuous semimartingale, it follows from Lemma 2 that

\[
|X(t) - Y(t)| - |X(0) - Y(0)| = \int_0^t \text{sgn}(X(s) - Y(s)) \circ d(X(s) - Y(s)), \quad \text{a.s.,}
\]

so, due to linearity of the integral with respect to the differentials, it suffices to show that the integrals in (17) are defined. Let us first consider the integral with respect to \( dX \).

By the definition of the Stratonovich integral in (17),

\[
\int_0^t \text{sgn}(X(s) - Y(s)) \circ dX(s) = \int_0^t \text{sgn}(X(s) - Y(s))dX(s) + \left[ \text{sgn}(X - Y), X \right]_t, \quad (18)
\]

if the terms on the right hand side are defined. The Itô integral in (18) is defined, so we need only consider the covariation term. From R&V Proposition 1,

\[
\left[ \text{sgn}(X - Y), X \right]_t = \int_0^t \text{sgn}(X(s) - Y(s))d^+X(s) - \int_0^t \text{sgn}(X(s) - Y(s))d^-X(s), \quad (19)
\]

and will be defined if the two integrals are. Since \( \text{sgn}(X - Y) \) is continuous outside \( \{ t : X(t) = Y(t) \} \), which almost surely has measure zero with respect to \( d\langle X \rangle_t \), R&V Proposition 6 implies that

\[
\int_0^t \text{sgn}(X(s) - Y(s))d^-X(s) = \int_0^t \text{sgn}(X(s) - Y(s))dX(s), \quad (20)
\]

and since this Itô integral is defined, so is the forward integral. By R&V Proposition 1,

\[
\int_0^t \text{sgn}(X(s) - Y(s))d^+X(s) = -\int_{T-t}^T \text{sgn}(\hat{X}(s) - \hat{Y}(s))d^-\hat{X}(s), \quad (21)
\]

where \( \hat{X} \) and \( \hat{Y} \) are the time-reversed versions of \( X \) and \( Y \) on \([0, T]\). By hypothesis, the time-reversed process \( \hat{X} \) is a continuous semimartingale, so as in (20) its forward integral is defined, and this defines the backward integral in (21). Hence, the covariation in (19) is defined, so both terms on the right hand side of (18) are defined, and this defines the Stratonovich integral with respect to \( dX \) in (17).

The same reasoning holds for the integral with respect to \( dY \).

A Stratonovich representation for rank processes

We would like to prove (11), and we shall start with a lemma that establishes this result for \( n = 2 \) and then apply the lemma to prove the general case with \( n \geq 2 \).

Lemma 4. Let \( X_1 \) and \( X_2 \) be reversible continuous semimartingales defined on \([0, T]\) under a common filtration, and suppose that they have nondegenerate crossings. Then

\[
X_1(t) \lor X_2(t) - X_1(0) \lor X_2(0) = \int_0^t \mathbb{1}_{\{X_1(s) \geq X_2(s)\}} \circ dX_1(s) + \int_0^t \mathbb{1}_{\{X_1(s) < X_2(s)\}} \circ dX_2(s), \quad \text{a.s.,} \quad (22)
\]
and
\[ X_1(t) \land X_2(t) - X_1(0) \land X_2(0) = \int_0^t \mathbb{1}_{\{X_i(s) < X_2(s)\}} \circ dX_1(s) + \int_0^t \mathbb{1}_{\{X_1(s) \geq X_2(s)\}} \circ dX_2(s), \quad \text{a.s.} \quad (23) \]

Proof. For \( t \in [0, T] \) we have
\[ X_1(t) \lor X_2(t) = \frac{1}{2} \left( X_1(t) + X_2(t) + |X_2(t) - X_1(t)| \right), \quad \text{a.s.,} \]
so by Lemma 3,
\[ X_1(t) \lor X_2(t) - X_1(0) \lor X_2(0) = \frac{1}{2} \left( X_1(t) + X_2(t) - X_1(0) - X_2(0) + \int_0^t \text{sgn}(X_2(s) - X_1(s)) \circ dX_2(s) \right. \]
\[ \left. - \int_0^t \text{sgn}(X_2(s) - X_1(s)) \circ dX_1(s) \right) \]
\[ = \frac{1}{2} \int_0^t \left( 1 + \text{sgn}(X_2(s) - X_1(s)) \right) \circ dX_2(s) \]
\[ + \frac{1}{2} \int_0^t \left( 1 - \text{sgn}(X_2(s) - X_1(s)) \right) \circ dX_1(s) \]
\[ = \int_0^t \mathbb{1}_{\{X_1(s) < X_2(s)\}} \circ dX_2(s) + \int_0^t \mathbb{1}_{\{X_1(s) \geq X_2(s)\}} \circ dX_1(s), \quad \text{a.s.,} \]
which proves (22). Equation (23) follows from this and the fact that
\[ X_1(t) \land X_2(t) = X_1(t) + X_2(t) - X_1(t) \lor X_2(t), \quad \text{a.s.} \]

Proposition 1. Let \( X_1, \ldots, X_n \) be continuous semimartingales defined on \([0, T]\) that are reversible and have nondegenerate crossings. Then the rank processes \( X_1, \ldots, X_n \) satisfy
\[ dX_{(k)}(t) = \sum_{i=1}^n \mathbb{1}_{\{X_i(t) = X_{(k)}(t)\}} \circ dX_i(t), \quad \text{a.s.} \quad (24) \]

Proof. It follows from Lemma 4 that (24) holds for \( n = 2 \), so let us assume that it holds for \( X_1, \ldots, X_{n-1} \), and prove that it then holds for \( X_1, \ldots, X_n \). Let \( \tilde{X}_{(1)}, \ldots, \tilde{X}_{(n-1)} \) be the ranked processes \( X_1, \ldots, X_{n-1} \), so by our inductive hypothesis we have
\[ d\tilde{X}_{(k)}(t) = \sum_{i=1}^{n-1} \mathbb{1}_{\{X_i(t) = \tilde{X}_{(k)}(t)\}} \circ dX_i(t), \quad \text{a.s.,} \quad (25) \]
for \( k = 1, \ldots, n-1 \). By Lemma 1, the processes \( \tilde{X}_{(1)}, \ldots, \tilde{X}_{(n-1)} \) have nondegenerate crossings as do \( X_1, \ldots, X_n \), so the same holds for holds for \( \tilde{X}_{(1)}, \ldots, \tilde{X}_{(n-1)}, X_n \).

Now,
\[ X_{(1)}(t) = \tilde{X}_{(1)}(t) \lor X_n(t), \quad \text{a.s.,} \]
and we can apply Lemma 4, so by our inductive hypotheses,
\[ dX_{(1)} = \mathbb{1}_{\{X_{(1)}(t) \geq X_n(t)\}} \circ d\tilde{X}_{(1)}(t) + \mathbb{1}_{\{X_{(1)}(t) < X_n(t)\}} \circ dX_n(t) \]
\[ = \mathbb{1}_{\{X_{(1)}(t) \geq X_n(t)\}} \sum_{i=1}^{n-1} \mathbb{1}_{\{X_i(t) = X_{(1)}(t)\}} \circ dX_i(t) + \mathbb{1}_{\{X_n(t) = X_{(1)}(t)\}} \circ dX_n(t) \]
\[ = \sum_{i=1}^{n-1} \mathbb{1}_{\{X_i(t) = X_{(1)}(t)\}} \circ dX_i(t) + \mathbb{1}_{\{X_n(t) = X_{(1)}(t)\}} \circ dX_n(t) \]
Hence, by Lemma 4,

\[ \sum_{i=1}^{n} I_{\{X_i(t) = X_{(i)}(t)\}} \circ dX_i(t), \quad \text{a.s.} \]

For \( 2 \leq k \leq n-1 \), we have

\[ X_{(k)}(t) = \tilde{X}_{(k-1)}(t) \land (\tilde{X}_{(k)}(t) \lor X_n(t)), \quad \text{a.s.} \]

Since

\[ \{t : \tilde{X}_{(k-1)}(t) = (\tilde{X}_{(k)}(t) \lor X_n(t))\} \subset \{t : \tilde{X}_{(k-1)}(t) = \tilde{X}_{(k)}(t)\} \cup \{t : \tilde{X}_{(k-1)}(t) = X_n(t)\} \]

and \( d(\tilde{X}_{(k)} \lor X_n)_t \leq d(\tilde{X}_{(k)})_t + d(X_n)_t \), it follows that \( \tilde{X}_{(k-1)} \) and \( \tilde{X}_{(k)} \lor X_n \) have nondegenerate crossings. Hence, by Lemma 4,

\[
dX_{(k)}(t) = \sum_{i=1}^{n} I_{\{X_{(k)}(t) < X_{(k)}(t) \lor X_n(t)\}} \circ d\tilde{X}_{(k-1)}(t) + \sum_{i=1}^{n} I_{\{X_{(k)}(t) \geq X_{(k)}(t) \lor X_n(t)\}} \circ d\tilde{X}_{(k)}(t) + \sum_{i=1}^{n} I_{\{X_{(k)}(t) < X_{(k)}(t) \lor X_n(t)\}} \circ dX_n(t)
\]

Finally, for \( k = n \), we have

\[ dX_{(n)}(t) = \sum_{i=1}^{n} dX_i(t) - \sum_{k=1}^{n-1} dX_{(k)}(t), \quad \text{a.s.} \]

**Stratonovich representation for Atlas rank processes**

We would like to apply Proposition 1 to the Atlas model \([\text{2}]\). To do so we must show that the log-capitalization processes for an Atlas model are reversible and have nondegenerate crossings.

**Proposition 2.** For the Atlas model \([\text{2}]\), the processes \( \log X_1, \ldots, \log X_n \) are reversible and have nondegenerate crossings.

**Proof.** Girsanov’s theorem and the properties of multidimensional Brownian motion imply that the processes \( \log X_i \) of \([\text{2}]\) have nondegenerate crossings and that there are no *triple points*, i.e., for \( i < j < k \), \( \{t : \log X_i(t) = \log X_j(t) = \log X_k(t)\} = \), a.s. (see Karatzas and Shreve (1991)). It remains to show that the log \( X_i \) are reversible.
Choose \( k \in \{1, \ldots, n\} \) and \( t_0 \in [0, T] \), and suppose that \( \log X_j(t_0) = \log X_{(k)}(t_0) \). If for all \( i \neq j \) we have \( \log X_i(t_0) \neq \log X_j(t_0) \), then there is a neighborhood \( U \) of \( t_0 \) in \([0, T]\) such that for \( t \in U \), if \( i \neq j \) then \( \log X_i(t) \neq \log X_j(t) \). In this case, within \( U \), the process \( \log X_j \) is Brownian motion with drift, which is reversible. Now suppose that \( \log X_i(t_0) = \log X_j(t_0) \) for some \( i \neq j \). No-triple-points implies that there is a neighborhood \( U \) of \( t_0 \) such that for \( t \in U \), if \( \ell \neq i, j \) then \( \log X_i(t) \neq \log X_\ell(t) \neq \log X_j(t) \). Hence, within \( U \) we can confine our attention to the two processes \( \log X_i \) and \( \log X_j \), in which case it was shown in Fernholz, Ichiba, Karatzas, and Prokaj (2013) or Fernholz, Ichiba, and Karatzas (2013) that the time-reversed versions of these processes are continuous semimartingales.

For each \( k \), the compactness of \([0, T]\) ensures that a finite subfamily of the neighborhoods \( U \) will include all values of \( t \), so the \( \log X_i \) are reversible on \([0, T]\).

**Corollary 1.** For the Atlas model \([2]\), the rank processes \( \log X_{(1)}, \ldots, \log X_{(n)} \) satisfy

\[
d\log X_{(k)}(t) = \sum_{i=1}^{n} 1_{\{X_i(t) = X_{(k)}(t)\}} \circ d\log X_i(t), \quad \text{a.s.,} \tag{26}
\]

**Proof.** Follows immediately from Propositions 1 and 2. \( \square \)

**An application to portfolio return decomposition**

For \( n \geq 2 \), consider a stock market of stocks with capitalizations represented by the positive continuous semimartingales \( X_1, \ldots, X_n \) defined on \([0, T]\). The **market weight processes** \( \mu_1, \ldots, \mu_n \) are defined by

\[
\mu_i(t) \triangleq \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)},
\]

and the **ranked market weight processes** \( \mu_{(k)} \) are defined accordingly. If the processes \( \log X_1, \ldots, \log X_n \) of a market are reversible and have nondegenerate crossings, the same will hold for the log-weight processes \( \log \mu_1, \ldots, \log \mu_n \).

The **market portfolio** is the portfolio with weights \( \mu_i \) and portfolio value process

\[
Z_{\mu}(t) = X_1(t) + \cdots + X_n(t), \quad \text{a.s.}
\]

In Fernholz (2016) it was shown that for a portfolio \( \pi \), the relative log-return \( d\log(Z_{\pi}/Z_{\mu}) \) can be decomposed into

\[
d\log \left( Z_{\pi}(t)/Z_{\mu}(t) \right) = d\log S_{\pi}(t) + d\mathcal{J}_{\pi}(t), \quad \text{a.s.,}
\]

where \( S_{\pi} \) is a **structural process** defined by

\[
d\log S_{\pi}(t) \triangleq \sum_{i=1}^{n} \pi_i(t) \circ d\log \mu_i(t), \tag{27}
\]

and \( \mathcal{J}_{\pi} \) is a **trading process** with

\[
d\mathcal{J}_{\pi}(t) \triangleq d\log \left( Z_{\pi}(t)/Z_{\mu}(t) \right) - d\log S_{\pi}(t),
\]

at least when the Stratonovich integrals in \(27\) are all defined.

Let \( S \) be a real-valued \( C^2 \) function defined on a neighborhood of the unit simplex \( \Delta^n \subset \mathbb{R}^n \). Then we shall say that the portfolio \( \pi \) is **generated by** the function \( S \) of the ranked market weights if \( S(\mu(t)) = S(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)) \) and the portfolio weight processes \( \pi_i \) are given by

\[
\pi_{\mu_{(k)}}(t) = \left( D_k \log S(\mu_{(1)}(t)) + 1 - \sum_{j=1}^{n} \mu_{(j)}(t) D_j \log S(\mu_{(j)}(t)) \right) \mu_{(k)}(t), \tag{28}
\]
where $p_t$ is the inverse of $r_t \in \Sigma_n$. In this case, the relative log-return of $\pi$ will satisfy

$$d \log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = d \log S(\mu(t)) + d\Theta(t), \quad \text{a.s.,}$$

where $\Theta$ is a function of locally bounded variation (see Fernholz (2001) or Fernholz (2002) Theorem 4.2.1).

**Proposition 3.** Suppose that the market log-weight processes $\log \mu_1, \ldots, \log \mu_n$ are reversible and have nondegenerate crossings. Let $\pi$ be the portfolio generated by the function $S$ of the ranked market weights. Then

$$d \log S_\pi(t) = d \log S(\mu(t)), \quad \text{a.s.,} \quad (29)$$

and

$$d T_\pi(t) = d \Theta(t), \quad \text{a.s.} \quad (30)$$

**Proof.** By hypothesis, $S(\mu(t)) = S(\mu(\cdot)(t))$, where $S$ is a real-valued $C^2$ function defined on a neighborhood of the unit simplex $\Delta^n \subset \mathbb{R}^n$. Then

$$d \log S(\mu(t)) = d \log S(\mu(\cdot)(t))$$

$$= \sum_{k=1}^n D_k \log S(\mu(\cdot)(t)) \circ d \mu(k)(t)$$

$$= \sum_{k=1}^n D_k \log S(\mu(\cdot)(t)) \mu(k)(t) \circ d \log \mu(k)(t)$$

$$= \sum_{k=1}^n \pi_{p(k)}(t) \circ d \log \mu(k)(t)$$

$$= \sum_{k=1}^n \pi_{p(k)}(t) \sum_{i=1}^n \mathbb{1}\{X_i(t) = X_{k}(t)\} \circ d \log \mu_i(t)$$

$$= \sum_{i=1}^n \pi_i(t) \circ d \log \mu_i(t)$$

$$= d \log S_\pi(t) \quad \text{a.s.,}$$

where (31) is due to (28) and the fact that

$$\sum_{k=1}^n \mu(k)(t) \circ d \log \mu(k)(t) = \sum_{k=1}^n d \mu(k)(t) = d \sum_{k=1}^n \mu(k)(t) = 0, \quad \text{a.s.,}$$

and (32) follows from Proposition 1.

The representation for $T_\pi$ follows by construction. \qed

**Corollary 2.** For the Atlas model (2), a portfolio generated by a function of the ranked market weights satisfies the decomposition (29) and (30).

**Proof.** Follows immediately from Propositions 2 and 3. \qed

**Remark.** The lemmata leading to Proposition 1 depend on the reversibility of the semimartingales $X_i$. The localization argument in Proposition 2 to establish this reversibility for Atlas models depends on no-triple-points along with the $n = 2$ results of Fernholz et al. (2013) and Fernholz, Ichiba, and Karatzas (2013). However, triple points may exist in first-order models and hybrid Atlas models, so for these more general models localization to two dimensions fails and reversibility cannot immediately be established (see
Hence, it appears that other methods may be needed to extend Corollary 2 to more general rank-based models.

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