Gott time machines in the Anti-de Sitter space

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Abstract

In 1991 Gott presented a solution of Einstein's field equations in 2+1 dimensions with \( \Lambda = 0 \) that contained closed timelike curves (CTC's). This solution was remarkable because at first it did not seem to be unphysical in any other respect. Later, however, it was shown that Gott's solution is tachyonic in a certain sense. Here the case \( \Lambda < 0 \) is discussed. We show that it is possible to construct CTC's also in this case, in a way analogous to that used by Gott. We also show that this construction still is tachyonic.

\( \Lambda < 0 \) means that we are dealing with Anti-de Sitter space, and since the CTC-construction necessitates some understanding of its structure, a few pages are devoted to this subject.

1 Introduction

General relativity in 2+1 dimensional spacetimes is trivial in the sense that there is no gravitation, that is masses in such spacetimes do not attract each other [1]. This is due to the fact that in 2+1 dimensions Einstein’s equations (with \( \Lambda = 0 \)) imply that spacetime is flat in sourceless regions. This, however, does not mean that 2+1 dimensional spacetimes are uninteresting—in fact, despite the absence of gravitation the subject is very rich, and yet it is comparably easy to handle. One example of its richness was provided 1991 when Gott [2] showed how to construct closed timelike curves (CTC’s) by means of 2+1 dimensional masses. Since a CTC allows an observer to travel backwards in time, and constitutes what is generally meant by a time-machine, it was important to investigate Gott’s CTC-solution thoroughly (see e.g. [3]). Soon it was revealed that his construction was tachyonic—the system of 2+1 masses which give rise to the CTC’s corresponds to an effective mass travelling faster than light [4,5]. A stronger result, that every open 2+1 dimensional universe with \( \Lambda = 0 \) containing Gott-CTC’s has spacelike total momentum, also was shown [6,7]. The latter reference generalizes the results of the former and also makes some statements about the case \( \Lambda < 0 \). Furthermore, for closed universes ‘t Hooft [8,9] showed that if one starts without CTC’s and tries to build them according to Gott’s construction, then the universe always will collapse before the CTC’s come into existence.

In short, nature succeeds very well to prevent the creation of time-machines. But these results concern the case with vanishing cosmological constant \( \Lambda \), and it is interesting to investigate the situation when \( \Lambda \) is non-vanishing. Is it perhaps not even possible to construct a counterpart to Gott’s CTC’s in this case? This and similar questions were the

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motivation for this work, in which we repeat Gott’s construction in the case $\Lambda < 0$ (that is, in the Anti-de Sitter space) and then see whether the resulting CTC’s spring from a tachyonic system.

In the next section Gott’s construction for the case $\Lambda = 0$ is reviewed, and we also show that this construction leads to a tachyonic effective particle. Section 3 is devoted to Anti-de Sitter space, and in section 4 we perform the analogue of Gott’s construction when $\Lambda < 0$. After a lot of algebraic labour CTC’s are found, and it is shown that the condition for them to exist is essentially the same as in Gott’s case. In the subsequent section we go on to show that the CTC’s only exist for tachyonic systems of masses. This is done by a method analogous to that used in the flat case. The last section summarizes the results and contains a short discussion.

As will be seen, the results in the case $\Lambda < 0$ parallel those for $\Lambda = 0$ very closely. Since Gott’s construction is local, and since locally there is no essential difference between the cases, this is perhaps only what one would suspect. But the CTC’s generated also exist infinitely far away from the masses involved and on a global scale Anti-de Sitter space and Minkowski space differ very much. There is for example no obvious reason for why the condition for CTC’s to exist in AdS-space is independent of their size.

2 Closed timelike curves—Review of the situation with $\Lambda = 0$

In 2+1 dimensions, as mentioned before, spacetime is flat in sourceless regions, and the metric exterior to a mass $M$ at the origin can be written \[ ds^2 = -dt^2 + d\rho^2 + \rho^2 d\theta^2, \quad 0 \leq \theta \leq 2\pi - \alpha \] where $\alpha = 8\pi GM$ ($G$ is the gravitational constant in $c = 1$ units). This means that a mass can be represented simply by cutting a wedge out of ordinary Minkowski space, and identifying its sides. From now on the angle of this wedge will be used to characterize masses.

We assume that the reader is already familiar with Gott’s CTC construction—here we only make a short description of it, and give the condition for CTC’s to arise. The construction simply involves two masses $\alpha$ which are passing each other. If each of them travels with a velocity greater than $\cos(\alpha/2)$ they will give rise to CTC’s. The intuitive reason for this is that the identification of points at different sides of the cut out wedges, will, for a mass in motion, involve a time jump. For $v > \cos(\alpha/2)$ this jump in time will be large enough to permit a rocket travelling around both of the masses to arrive at the starting point at the same time that it left—that is, the time jumps will permit the rocket to perform a CTC. See figure 1. There are different methods to show that the spacetime thus obtained is tachyonic. To make the analogy with the Anti-de Sitter space as clear as possible we will do it by using a matrix formalism introduced by Deser, Jackiw and ’t Hooft in 1984 [1], in which any system of masses can be represented. The idea is to represent a mass or a system of masses by the Poincaré transformation that is the result of a parallell transport of a vector around the masses.\footnote{In [1] the Poincaré transformation representing a system of masses is regarded as describing the “matching conditions” for points on different sides of the cut out wedges. Then the formalism also contains information about the relative position of the masses, but since this information will be superfluous in this context we choose the parallell transport viewpoint.}

To begin with, a mass $\alpha$ at rest is represented by an ordinary rotation $R(\alpha)$ of the Lorentz group in 2+1 dimensions, since a vector parallell transported around the mass
rotated by an angle $\alpha$. A moving mass $\alpha$ is represented by $L(\xi)R(\alpha)L^{-1}(\xi)$ where $L(\xi)$ with $\cosh \xi = \gamma_v = (1 - v^2)^{-1/2}$ is the boost of the mass with respect to the laboratory. Namely, if we want to parallel transport a vector around a moving mass, then we can always first make the appropriate coordinate transformation to its restframe.

Now we want to represent the situation with the two masses in Gott’s construction in this way. In order to get as few matrices as possible we choose to work in the frame of mass I (in figure 1). There mass II is seen to pass by with velocity $u = 2v(1 + v^2)^{-1}$ and with matrices the situation is described as $R_I L_{II} R_{II}^{-1}$, where $R_I = R_{II} = R(\alpha)$ and $L_{II} = L(\xi)$ with $\cosh \xi = \gamma_u$. Suppose now that we try to describe the situation as only one effective mass instead of the two masses $\alpha$, that is $L(\xi_{eff}, \theta)R(\alpha_{eff})L^{-1}(\xi_{eff}, \theta)$, where $L(\xi_{eff}, \theta)$ denotes a general boost in direction $\theta$. Since we want the result of a parallel transport to be the same, independently of the viewpoint taken, we get the equation

$$R(\alpha) L(\xi) R(\alpha) L^{-1}(\xi) = L(\xi_{eff}, \theta) R(\alpha_{eff}) L^{-1}(\xi_{eff}, \theta)$$ (2)

If we use the trace of this equation together with its time-time-component we can get the velocity (in the I-frame) of the effective particle as an expression of $\alpha$ (each particle’s mass) and $u$ (the velocity of II in the I-frame). If we translate the velocities to the laboratory (that is, the frame in which both masses are seen to travel with velocity $v$ but in opposite directions) this expression reads

$$v_{eff} = v \gamma_v \tan(\alpha/2)$$ (3)

It is now easy to see that the Gott-condition $v > \cos(\alpha/2)$ is equivalent to $v_{eff} > 1$. Hence we see that this type of CTC’s only exists when the system of two masses is tachyonic. As was mentioned in the introduction stronger results than this have been shown, but this is the one that will be generalized here to the case with $\Lambda < 0$.

### 3 The Anti-de Sitter space

Just as Minkowski space is the vacuum solution to Einstein’s equations in 2+1 dimensions without cosmological constant $\Lambda$, Anti-de Sitter space (or ‘AdS-space’) is the
corresponding solution when \( \Lambda < 0 \). Here we discuss the general properties of this spacetime to be able to redo the Gott-construction. We will provide it with a special choice of coordinates in which the geodesics are nicely represented, and when discussing the isometries we show how to get explicit expressions for them in these coordinates.

As definition of the AdS-space we will take the 3-surface of the hyperboloid

\[
X^2 + Y^2 - Z^2 - T^2 = -1
\]  

embedded in a 4-dimensional space with metric

\[
ds^2 = dX^2 + dY^2 - dZ^2 - dT^2
\]  

This surface can be parameterized in coordinates \( r, \varphi \) och \( t \) according to

\[
\begin{align*}
X &= \sinh r \cos \varphi \\
Y &= \sinh r \sin \varphi \\
Z &= \cosh r \cos t \\
T &= \cosh r \sin t
\end{align*}
\]  

which yields the metric

\[
ds^2 = -\cosh^2 r dt^2 + dr^2 + \sinh^2 r d\varphi^2
\]  

We see that \( r \) and \( \varphi \) plays the same role as ordinary polar coordinates, while \( t \) is the timecoordinate. But \( t \) is periodic with \( 2\pi \) and the topology is \( S^1 \times \mathbb{R}^2 \). Such a spacetime, of course, trivially contains CTC’s and is therefore not what we are interested in. Instead we will work with the universal covering of the AdS-space, that is, the space where the time \( S^1 \) is “unwinded” to \( R^1 \) and \( t \) is regarded as a normal non-periodic variable. When we discuss the isometries, the space, strictly speaking, is the ordinary AdS-space (with topology \( S^1 \times \mathbb{R}^2 \)), but the result obtained is identical for its covering space.

If we transform the radial coordinate according to

\[
\rho(r) = \frac{\cosh r - 1}{\sinh r}
\]  

we obtain the metric

\[
ds^2 = -\left(1 + \frac{\rho^2}{1 - \rho^2}\right)^2 dt^2 + \frac{4}{(1 - \rho^2)^2} d\rho^2 + \frac{4\rho^2}{(1 - \rho^2)^2} d\varphi^2
\]  

Observe that \( \rho \to 1 \) when \( r \to \infty \), and in the coordinates \( (\rho, \varphi, t) \) the AdS-space is confined to a cylinder, with the time lengthwise. The cross-section of this cylinder is a \( (\rho, \varphi) \)-surface, and the point in using these special coordinates is that a spacelike geodesic in such a cross-section then always is an arc of a circle meeting the circle \( \rho = 1 \) at right angle. This means that a surface \( t=\text{CONSTANT} \) looks like the the Poincaré model of a space with constant negative curvature, and therefore we will call the coordinates \( (\rho, \varphi, t) \) the “Poincaré coordinates”. If we look at a \( (\rho, t) \)-plane of the spacetime cylinder we get something like a Penrose-diagram, but with non-straight lightlike geodesics—not with a constant slope of \( 45^\circ \). (This is the price we have to pay for letting the \( t=\text{CONSTANT} \) surfaces be Poincaré models.) Instead, lightlike geodesics are described by the formula

\[
\rho(t) = \frac{1 - \cos(t + E)}{\sin(t + E)}
\]
where \( E \) is a constant. Note that they traverse the whole space in a finite time, namely \( \pi \).

On the other hand, a timelike geodesic in a \((\rho, t)\)-plane never reaches infinity, but oscillates to-and-fro through the origin with a period of \( 2\pi \). Two such geodesics that at time \( t_0 \) each is a worldline of a resting object but at different places, a time \( \pi/2 \) later will focus at the origin, and now represent the worldlines of two objects both at the origin, having different velocities. This indicates a fundamental property of the isometries in this spacetime, namely that an isometry which at time \( t_0 \) looks like a translation, at time \( t_0 + \pi/2 \) has the effect of a boost. We will come back to this in a moment. Geodesics in Anti-de Sitter space: To the left in a \((\rho, t)\)-plane and to the right in a \((\rho, \varphi)\)-plane.

Geodesics in \((\rho, t)\)-planes and \((\rho, \varphi)\)-planes are illustrated in figure 2. A general timelike geodesic in the AdS-space, spirals around the time axis \( \rho = 0 \), but not necessarily, of course, in a circular motion.

To be able to find and express the isometries it is now convenient to parameterize the AdS-space by means of the group SU(1,1), in the sense that every spacetime-point will correspond to a SU(1,1) matrix, and conversely. Let a point \((X, Y, Z, T)\) on the hyperboloid (4) correspond to the matrix

\[
K = \begin{pmatrix}
Z + iT & Y + iX \\
Y - iX & Z - iT
\end{pmatrix}
\] (11)

which belongs to SU(1,1) if \( \det(K) = 1 \). Since \( \det(K) = Z^2 + T^2 - Y^2 - X^2 \) this is in accordance with (4), and this relation between points in the AdS-space and SU(1,1) matrices is 1-1.

The isometry group of 2+1 dimensional AdS-space is SO(2,2). But this group is locally isomorphic to SU(1,1) \( \times \) SU(1,1), and we now claim that a general isometry can be written

\[
K' = T_1KT_2
\] (12)

where \( K \) is given by (11) and \( T_1, T_2 \) are two arbitrary SU(1,1) matrices. To see this, write

\[
\begin{pmatrix}
dZ + idT & dY + idX \\
dY - idX & dZ - idT
\end{pmatrix} = dK
\]

Now the metric can be written \( ds^2 = - \det dK \), and we see that \( \det(dK') = \det(T_1dKT_2) = \det(dK) \) since \( T_1 \) and \( T_2 \) are SU(1,1). So the metric is preserved and (12) is indeed an isometry.
To begin with, let’s find all isometries leaving the time axis invariant, which means $t’ = t$. Using (12) together with the coordinate transformations (1) and (3) gives that $t’ = t$ if and only if the transformation is of the form:

$$K’ = RKR^{-1} \quad \text{where} \quad R = \pm \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

(13)

The lower sign in $R$ only correspond to a mirroring of space, and if we stick to the upper sign this transformation simply accounts to a rotation around the time axis: $\rho’ = \rho$, $\varphi’ = \varphi + \theta$ and $t’ = t$.

Note that (13) is the only isometry leaving the time axis invariant. Hence, in AdS-space we do not have any “pure” translations, but of course it is possible to perform a translation at one specified time. Before we see how such an isometry is written in form (12) we will try to understand its properties. For this purpose, look at figure 2 and remember that an isometry maps geodesics on geodesics. For example, by performing a translation in the boost translation (14) with $\tau$ the new coordinates ($\rho, \varphi, t$) gives that this transformation simply accounts to a rotation around the time axis: $\rho’ = \rho$, $\varphi’ = \varphi + \theta$ and $t’ = t$.

Lengthy calculations show that the boost-translation which is uniquely specified by mapping $\rho = 0$ on $\rho’ = \rho_0$, $\varphi’ = \theta$, in all planes $t = 2\pi n + \tau$ is written as

$$K’ = \frac{1}{1-\rho_0^2} \begin{pmatrix} 1 & i\rho_0 e^{i(\theta+\tau)} \\ -i\rho_0 e^{-i(\theta+\tau)} & 1 \end{pmatrix} K \begin{pmatrix} 1 & i\rho_0 e^{i(\theta-\tau)} \\ -i\rho_0 e^{-i(\theta-\tau)} & 1 \end{pmatrix}$$

(14)

For the sake of completeness we also write down the isometry corresponding to a translation in time, that is $\rho’ = \rho$, $\varphi’ = \varphi$ and $t’ = t + \tau$:

$$K’ = TKT \quad \text{where} \quad T = \begin{pmatrix} e^{i\tau/2} & 0 \\ 0 & e^{-i\tau/2} \end{pmatrix}$$

(15)

By means of the rotations (13), the boost-translations (14) and the time translations (15) we can express an arbitrary isometry of AdS-space, or in other words, we can map every two geodesics (both either timelike or spacelike) into each other. In the following pages we will need an explicit expression for a boost-translation in the Poincaré coordinates. Since both rotations and time translations are trivial in these, it will suffice to find such an expression with $\theta$ and $\tau$ in (14) already fixed. The following nasty expressions of the new coordinates $(\rho’, \varphi’, t’)$ in the old ones $(\rho, \varphi, t)$ and the boost parameter $\rho_0$ gives the boost translation (14) with $\tau = \pi/2$ and $\theta = 0$.

$$\rho’ = \sqrt{\frac{A(\rho_0; \rho, \varphi, t) - (1-\rho^2)}{A(\rho_0; \rho, \varphi, t) + (1-\rho^2)}}$$

with $A(\rho_0; \rho, \varphi, t) = \sqrt{(1+\rho^2)^2 \cos^2 t + \frac{1}{(1-\rho_0^2)^2} ([1+\rho_0^2](1+\rho^2) \sin t + 4\rho_0 \rho \cos \varphi)^2}$$

$$\varphi’ = \begin{cases} \arctan \left[ \frac{(1-\rho_0^2) \rho \sin \varphi}{(1+\rho_0^2) \rho \cos \varphi + \rho_0 (1+\rho^2) \sin t} \right] & \text{for } \cos \varphi > -\frac{\rho_0}{1+\rho_0^2} \frac{1+\rho^2}{\rho} \sin t \\ \pi + \arctan \left[ \frac{(1-\rho_0^2) \rho \sin \varphi}{(1+\rho_0^2) \rho \cos \varphi + \rho_0 (1+\rho^2) \sin t} \right] & \text{for } \cos \varphi < -\frac{\rho_0}{1+\rho_0^2} \frac{1+\rho^2}{\rho} \sin t \end{cases}$$

(17)
Figure 3: a) To represent a mass $\alpha$ at rest at $\rho = 0$ in AdS-space we simply cut a wedge out from the full spacetime. b) This shows the result from applying the boost-translation mapping point $A$ on $B$ in figure (a), and hence what a more general mass will look like.

Figure 4: A mass $\alpha$ at rest at $\rho = 0$, and the space projection of the lightlike geodesic between $A$ and $B$ through the wedge. This also is the path connecting $A$ and $B$ in shortest time.

\[
t' = \begin{cases} 
\arctan \left[ \frac{(1+\rho_0^2)(1+\rho^2) \sin t + 4\rho_0 \rho \cos \varphi}{(1-\rho_0^2)(1+\rho^2) \cos t} \right] & \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2} \\
\pi + \arctan \left[ \frac{(1+\rho_0^2)(1+\rho^2) \sin t + 4\rho_0 \rho \cos \varphi}{(1-\rho_0^2)(1+\rho^2) \cos t} \right] & \text{for } \frac{\pi}{2} < t < \frac{3\pi}{2}
\end{cases}
\]  

(18)

4 Construction of CTC’s

Before we can try to construct any CTC’s we must understand what a mass looks like in 2+1 dimensional AdS-space. This is carefully investigated in [10], but here it will suffice to use a wedge representation corresponding to that used for $\lambda = 0$. As in that case the spacetime exterior to sources is locally unaffected, and a mass at rest at the origin can be represented in the Poincaré coordinates simply as a wedge cut out according to figure 3a, with its sides identified at equal times. To find a more general representation we just apply an isometry to this figure. For example, the result of the isometry mapping point $A$ on point $B$ is shown in figure 3b, and from this picture two things become apparent: 1) At times when the mass is not situated at the origin the sides of the wedges are not straight lines, but arcs of circles—the sides of the wedges always follow spacelike geodesics. 2) The plane $t = 0$ in 3a is after the transformation not a plane $t' = C$ for any constant $C$, but the surface indicated in 3b. Since the identification is done between points in this surface, it will involve a time jump. This is in close analogy with the situation in figure 1, where the identification also involves a jump in time. The only difference is that in 3b the surfaces of identification have different slopes at different times, and, for example, we get no time jump at all time $t' = -\pi/2$.

We will now investigate whether these time jumps can be made big enough to give rise to CTC’s, and in that case find a CTC-condition analogous to $v > \cos(\alpha/2)$ in Gott’s situation. The first step is to find the path which gives the shortest time between $A$ and $B$ in figure 4 (The mass $\alpha$ is here at rest at the origin). This path is a lightlike geodesic passing through the wedge, and its space projection is shown in the picture. By applying the boost-translation $(16) - (18)$ on $(14)$ (with $E = -\pi/2$), which is a lightlike geodesic in the $(\rho, \varphi)$-plane, we get a more general lightlike geodesic, passing the origin at distance
\( \rho_0 = \rho_c \). Since the geodesic must meet the wedge in a straight angle, \( \rho_c \) is uniquely determined from \( \rho_A \), and it is straightforward, but algebraically tiresome, to find the time along this geodesic between \( A \) and \( B \) expressed in \( \alpha \) and \( \rho_A \):

\[
\begin{align*}
t_{AB} &= \pi - 2 \arctan \left[ \frac{\sqrt{(1 + \rho^2)^2 - 4 \rho^2 \cos^2(\alpha/2)}}{2 \rho \cos(\alpha/2)} \right] 
\end{align*}
\] (19)

Now, put \( t_A = -t_{AB}/2 \) and \( t_B = t_{AB}/2 \). We want to find an isometry mapping \( t_A \) and \( t_B \) on \( t'_A \) and \( t'_B \), such that \( t'_A = t'_B \). If such an isometry exists it is possible to travel from \( A \) to \( B \) in no time (by applying the isometry to figure 4), and we can repeat the argument in the lower part of the figure with another mass subject to an opposite isometry, which then will enable us to travel back again from \( B \) to \( A \) in no time, and the CTC will be completed. Bearing Gott’s case in mind, the required isometry ought to be of the form (16) – (18) since that will have the effect of a boost in the \( t = 0 \)-plane and, as in figure 3b, therefore cause a time jump in the identification.

Using the time transformation, that is (18), with \( \varphi_A = 0 \), \( \varphi_B = \pi \) we get the new time coordinates \( t'_A \) and \( t'_B \) expressed in the old ones:

\[
\begin{align*}
t'_A &= \arctan \left[ \frac{(1 + \rho^2_0)(1 + \rho^2_A) \sin t_A + 4 \rho_0 \rho_A}{(1 - \rho^2_0)(1 + \rho^2_A) \cos t_A} \right] \\
t'_B &= \arctan \left[ \frac{(1 + \rho^2_0)(1 + \rho^2_A) \sin t_B - 4 \rho_0 \rho_A}{(1 - \rho^2_0)(1 + \rho^2_A) \cos t_B} \right] 
\end{align*}
\] (20) (21)

Demanding \( t'_B < t'_A \) gives a condition on the boost parameter \( \rho_0 \), and when inserting (19) into this one finds that it is independent of \( \rho_A \):

\[
\rho_0 > \frac{1 - \sin(\alpha/2)}{\cos(\alpha/2)} \] (22)

If this condition is fulfilled we clearly can get CTC’s, because if \( t'_B < t'_A \) we can, as already mentioned, repeat the argument with another mass \( \alpha \) in the lower halfplane of figure 4, for the journey back from \( B \) to \( A \).

In (22) \( \rho_0 \) is the boost parameter of each mass, and as mentioned before this means that their velocity as they pass each other at the origin is \( v|_{\rho=0} = 2\rho_0(1 + \rho^2_0)^{-1} \). So (22) is equivalent to

\[
v|_{\rho=0} = \frac{2\rho_0}{1 + \rho^2_0} > \cos \frac{\alpha}{2} \] (23)

which is the same condition as in Gott’s construction. However, in AdS-space it is only true at \( \rho = 0 \) because here the geodesic velocity is non-constant.

### 5 The tachyonic nature of the CTC-spacetime

As in the Minkowskian case we can associate an isometry to every configuration of masses. Let a mass \( \alpha \) at rest (at \( \rho = 0 \)) correspond to a rotation with angle \( \alpha \), that is, to an isometry (13) with \( \theta = \alpha \). In analogy with LRL\(^{-1} \) for the flat case we then represent a mass that is translated for example in the \( t = 0 \)-plane, as \( K' = BRB^{-1}K^{-1}B^{-1} \) where \( B \) is the matrix in (14) with \( \tau = 0 \). When \( \Lambda = 0 \) this representation of masses with isometries has an immediate interpretation as the transformation resulting from a parallel transport.
around the system. This interpretation is not valid in the AdS-space, since in a curved space a parallel transport along a closed loop not containing any masses yields a rotation anyway. But actually, even though this interpretation does not work when $\Lambda < 0$, the representation as such is still meaningful.

To get an equation analogous to (2) we work in the frame of mass I, where mass II is seen to pass by in direction $\varphi = \pi$ at time $t = 0$. According to (14) the relative boost-translation between the masses then is described by

$$K' = SKS^{-1}, S = \frac{1}{\sqrt{1 - \rho_1^2}} \begin{pmatrix} 1 & -\rho_1 \\ -\rho_1 & 1 \end{pmatrix}$$  \hspace{1cm} (24)$$

where $\rho_1$ is the boost parameter of mass II in the I-frame. The isometry corresponding to Gott’s construction in AdS-space, seen from mass I, then is

$$K' = RSRS^{-1}KSR^{-1}S^{-1}R^{-1}$$ \hspace{1cm} (25)$$

The three pairs of matrices nearest to the $K$ account for mass II, and the remaining rotation takes care of mass I.

Now we want to express the situation in terms of only one effective particle. We do not know in which direction this effective particle is travelling, so we have to use the boost-translation $S_\theta$ that maps $\rho = 0$ on $\rho' = \rho_0$, $\varphi' = \theta$ in every plane $t = 2\pi n + \pi/2$, that is (14) with $\tau = \pi/2$. The isometry associated with one effective particle then is given by

$$K' = S_{eff}R_{eff}S_{eff}^{-1}KS_{eff}R_{eff}^{-1}S_{eff}^{-1}$$ \hspace{1cm} (26)$$

where $R_{eff}$ is a rotation by some “effective angle” and where $S_{eff} = S_\theta$ with boost parameter $\rho_{eff,1}$. (The index 1 denotes that this effective boost parameter is expressed in the I-frame.) Since (25) and (26) describe the same situation we get the analogue of (2) as

$$RSRS^{-1} = S_{eff}R_{eff}S_{eff}^{-1}$$ \hspace{1cm} (27)$$

Some algebra yields, after translation of $\rho_1$ and $\rho_{eff,1}$ to the corresponding velocities in the laboratory (the frame in which the two masses pass each other at $t = 0$ with equal velocities)

$$v_{eff}\big|_{\rho=0} = \frac{v\big|_{\rho=0} \tan(\alpha/2)}{\sqrt{1 - v_{\rho=0}^2}}$$ \hspace{1cm} (28)$$

This is the same expression as (3), and since the CTC-condition $v\big|_{\rho=0} > \cos(\alpha/2)$ is the same too, we have also in this case that $v_{eff}\big|_{\rho=0} > 1$ exactly when the two masses generate CTC’s: In AdS-space, as well as in Minkowski space, we cannot have CTC’s without permitting tachyonic effective particles.

6 Discussion

We have shown explicitly that it is possible to perform Gott’s construction also in the Anti-de Sitter space, and that it gives rise to CTC’s when $v\big|_{\rho=0} > \cos(\alpha/2)$. This condition is the same as that shown by Gott for $\Lambda = 0$, except for the fact that when $\Lambda < 0$ it concerns the velocity of each particle only when they pass each other at the origin—the velocities corresponding to Anti-de Sitter geodesics are non-constant, but always largest
at $\rho = 0$. We also showed that the CTC-construction is tachyonic in the same sense as in the case $\Lambda = 0$: A situation with this type of CTC’s corresponds to an effective particle with tachyonic momentum.

When $\Lambda = 0$ it is obvious that the CTC-condition must be independent of how far away the CTC’s are from the masses themselves, since the construction with two wedges cut out from Minkowski space is scale-invariant: Large CTC’s look exactly as magnified small ones. This is certainly not true in a curved spacetime, and particularly not in AdS-space: The constant negative curvature has more effect on a large scale than on a small one. Therefore it is a non-trivial fact that the CTC-condition here also is independent on the largeness of the generated CTC’s.

The methods used here are highly specialized to AdS-space and hard to generalize to other spacetimes. That the results are so similar to those for $\Lambda = 0$ indicates that it should be possible to give a more general treatment of the subject, and perhaps find a whole class of spacetimes in which these or similar results hold.

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