Choosing with unknown causal information: Action-outcome probabilities for decision making can be grounded in causal models

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Abstract

Decision-making under uncertainty and causal thinking are fundamental aspects of intelligent reasoning. Decision-making has been well studied when the available information is considered at the associative (probabilistic) level. The classical Theorems of von Neumann-Morgenstern and Savage provide a formal criterion for rational choice using associative information: maximize expected utility. There is an ongoing debate around the origin of probabilities involved in such calculation. In this work, we will show how the probabilities for decision-making can be grounded in causal models by considering decision problems in which the available actions and consequences are causally connected. In this setting, actions are regarded as an intervention over a causal model. Then, we extend a previous causal decision-making result, which relies on a known causal model, to the case in which the causal mechanism that controls some environment is unknown to a rational decision-maker. In this way, action-outcome probabilities can be grounded in causal models in known and unknown cases. Finally, as an application, we extend the well-known concept of Nash Equilibrium to the case in which the players of a strategic game consider causal information.

1 Introduction

An important aspect of acting in the world is being able to make decisions under uncertain conditions: Which route should I use to get to work? Could there be traffic? If one wishes to avoid certain incoherent, inconsistent behavior, then the natural and most well-known idea is to balance between how desired an object is, and how available is it. Such balancing is to be done via the calculation of the expected utility (Bernoulli, 1954; Von Neumann and Morgenstern, 1944; Savage, 1954), where how desired an object is, is measured in terms of the utility it produces; and how available is it is encoded in terms of probabilities. Such probabilities are required to either be objectively known or subjectively (internally) defined by a decision maker. There is no standard agreement in the decision-making literature on where such probabilities must come from.

Acting in the world is conceived by human beings as causally intervening on it, and it is known that humans are also able to learn and use causal relations while making choices (Lagnado et al., 2007; Hagmayer et al., 2007; Hagmayer and Meder, 2008; Hagmayer and Sloman, 2009; Hagmayer and Meder, 2013; Hagmayer and Fernbach, 2017). Thinking in terms of causes and effects is an everyday task and in fact, causal reasoning is to be found at the very core
of our minds since we constantly ask why: Why do we get sick? Why does a drug work? (Tversky and Kahneman, 1977; Spirtes et al., 2000; Waldmann and Hagmayer, 2013; Danks, 2014; Lake et al., 2017; Pearl and Mackenzie, 2018; Neil et al., 2019).

If, as argued, another important aspect of acting in the world is making choices, then taking causal information as a basis is fundamental for decision-making. Therefore, it is a natural question how to formalize decision making when causal information (known or unknown) is present? Answering the such question is relevant given the importance of causal relations as well as reasoning in everyday life and in science (Spirtes et al., 2000; Pearl and Mackenzie, 2018). Given that human beings actually use causal information while making choices (Tversky and Kahneman, 1980), and the importance of decision-making results based on associative information, it is desirable to have an explicit and computationally implementable criterion for decision-making, which also addresses the question of the origin of probabilities for decision making in a sensitive, reasonable way.

The previous question has been already considered by Nozick (1969); Lewis (1981); Joyce (1999); Eells (1982); Stern (2017) as well as by Pearl (2009) who provides an optimality criterion for decision making under causal-controlled uncertainty when the causal mechanism which controls the environment is known by the decision maker (Pearl, 2009).

In this paper, we will provide a decision-making criterion for unknown causal information, and such criterion has the form of maximization of expected utility; thus showing that action-outcome probabilities for decision-making can be grounded in causal models.

The remainder of the paper goes as follows: In Section 2 we review the basics of Classical Decision Making: first we give some basic definitions; then we review the decision making results from von Neumann-Morgenstern and Savage, where we discuss the General Expected Utility Criterion, and finally we discuss about the problem around the origin of the action-outcome probabilities. In Section 3, the basics of Causation and Causal Graphical Models, as well as the basic definitions required for Causal Decision Making. In Section 4 we describe some fundamental previous results for Causal Decision Making, and in particular our main result in Section 4.2, finally in Section 5 we describe an application into the domain of Game Theory, where we are able to define a Causal Nash Equilibrium.

## 2 Classical Decision Making

A Decision Problem under Uncertainty is the mathematical model of a situation in which an agent must choose one out of many available actions with uncertain consequences which depend on different, possibly unknown, factors (Bernardo and Smith, 1994; Robert, 2007; Gilboa, 2009). Such consequences are assumed to produce satisfaction in the decision maker, and this satisfaction is represented by a preference relation, defined over actions, and denoted by $\succeq$, where $a_1 \succeq a_2$ is read as $a_1$ being preferred to $a_2$, in terms of the consequences that would be obtained by choosing action $a_1$ instead of action $a_2$, if these were the only two available options and given the current knowledge the agent has. The agent is assumed to be rational; that is, it is assumed that the preference relation satisfies the so-called rationality or coherence axioms described in Appendix ??.

**Von Neumann and Morgenstern** gave an answer for how to make choices if rational preferences are assumed, utilities are unknown, and the stochastic relation (i.e., probabilities of events) between actions and outcomes are considered as objective and given to the decision maker: maximize expected utility with respect to a utility function whose existence is guaranteed (Von Neumann and Morgenstern, 1944).

If probabilities are not known, then **Savage** showed that a rational decision maker must choose as if she is maximizing the expected utility using a subjective probability distribution (Savage, 1954). Such theorems provide a formal criterion for associative, also called evidential, decision-making if rationality is assumed: maximize expected utility.

Other decision-making theories exist, such as Kahneman and Tversky’s Prospect Theory (Kahneman and Tversky, 1979), Gilboa’s Case-Based Decision Theory (Gilboa and Schmeidler, 1995), among others that are out of the scope of this work, since such theories defy the classical
notion of rationality, which is the one we are adopting in this work.

In Section 4.1 we make reference to another theory, Joyce’s Causal Decision Making (Joyce, 1999) as well as Pearl’s criterion for known causal information. For further details on classical (non-causal) decision making, see Bernardo and Smith (1994) and Gilboa (2009). For further applications of Decision Making in other fields, such as Quantum Mechanics, see Wallace (2012); Berkovitz (2012).

2.1 Preliminary Definitions

In this Section, we formally state the definitions required for a precise formulation of what follows. The notation and setting we use here is the one used by Bernardo and Smith (1994). In Definitions 2.1 and 2.2, we consider a non-empty set Ω and a countable set A of available actions; for each action a_i ∈ A, a partition (E_j)_{j ∈ J(i)} of Ω and a set of consequences (C_j)_{j ∈ J(i)} over i. Bernardo and Smith (1994) derives the existence of a subjective probability measure from a set of coherence axioms, according to which a decision maker has some mechanism of quantifying uncertainty in terms of real numbers within the [0,1] interval.

Note: In the context of Chapter 2 of Bernardo and Smith (1994), from a preference relation ≥ defined over actions of the form \{c_{ij} \mid E_{ij}, j ∈ J(i)\}, and the coherence axioms stated in the Appendix, further relationships ≥, > as well as ∼ can be defined both over consequences, as well as over events, where E ≥ F is to be read as considering E more likely than F, as well as E ≥_C F, which is the conditional likelihood relation, but such development is beyond the scope of this work. Having said this, we can reduce the notation a_i = \{c_{ij} \mid E_{ij}, j ∈ J(i)\} to simply \{c_j \mid j ∈ J\}. But it must be kept in mind that the index set J depends on i as well as c_j and each E_j. We note that each a_i = \{c_{ij} \mid E_{ij}, j ∈ J(i)\} links, or maps, a partition of uncertain events \{E_j : j ∈ J\} to a corresponding set of consequences \{c_j : j ∈ J\}. Thus bridging this notation to the original formulation of Savage’s Theorem, where the objects of choice are functions from states, which are defined by Gilboa (2009) as ...an exhaustive list of all scenarios that might unfold..., to outcomes (Bernardo and Smith, 1994; Gilboa, 2009).

We will now first consider the general setting for a decision problem: an uncertain environment.

Definition 2.1. Let Ω be a non-empty set. An uncertain environment is a tuple (Ω, A, C, E). Where A is a non-empty set of available actions, C a set of consequences, and E is a σ-algebra of events over Ω.

If we consider the preferences of some decision maker over the set of actions of some uncertain environment, then we have a Decision Problem under Uncertainty (Bernardo and Smith, 1994).

Definition 2.2. A Decision Problem under Uncertainty is an uncertain environment (Ω, A, C, E) endowed with a preference relation ≥ over A.

As previously mentioned, we can identify, as done in Bernardo and Smith (1994), consequences as actions by simply writing c = \{c \mid Ω\} and say that consequence c_1 is preferred over consequence c_2, denoted as c_1 ≥ c_2, if action \{c_1 \mid Ω\} is preferred over action \{c_2 \mid Ω\}.

Definition 2.3. A Decision Problem under Uncertainty is said to be bounded if there exists a pair of consequences c_∗ and c^* such that for every c ∈ C, c^* ≥ c ≥ c_∗.

Definition 2.4. A Decision Problem under Uncertainty is said to be finite if the set A of available actions is finite.

2.2 Objective Probabilities: von Neumann-Morgenstern Theory

Consider the following scenario: throwing a die. In this scenario, using DeFinetti’s Representation Theorem (De Finetti, 1937; Schervish, 1995), we can model such throws as independent realizations of a random variable that has a known fixed law. We can think of such a law as
being objective. The Von Neumann and Morgenstern (vNM) Theorem considers a scenario of decision under risk with rational preferences; this is, rationally choosing between alternatives with uncertain outcomes with known, objective, probabilities.

Formally, using the notation by Gilboa (2009), we consider a set \( X \) of available options. Let \( L \) be the set of lotteries with finite support over \( X \). The objects of choice are elements \( l \in L \), which are known to the decision maker; we represent the decision maker’s preferences by a preference relation \( \succeq \subseteq L \times L \) that satisfies being complete, transitive, continuous, and a condition called independence. This family of conditions is called von Neumann-Morgenstern rationality axioms as described by Gilboa (2009) and Schervish (1995).

**Theorem 2.5** (von Neumann-Morgenstern). A preference relation \( \succeq \subseteq L \times L \) where \( L \) is a set of lotteries with finite support over a set \( X \) satisfies the von Neumann-Morgenstern rationality axioms if and only if there exists a function \( u : X \rightarrow \mathbb{R} \) such that for every \( P, Q \in L \) we have that

\[
P \succeq Q \text{ if and only if } \sum_{x \in X} P(x)u(x) \geq \sum_{x \in X} Q(x)u(x).
\]

The theorem states that if a rational decision maker knows the probabilities of obtaining a certain outcome, then she must choose as if maximizing the expected value of some function \( u \) whose existence is guaranteed by Theorem 2.5. See Gilboa (2009) for details on the proof.

### 2.3 Subjective Probabilities: Savage

Thanks to DeFinetti’s Theorem, we can think of objective probabilities as long-term frequencies, but the question remains open regarding inquiries of the type: what is the probability that John Doe was born on May 1st? It is not the case that John Doe sometimes is born one day and sometimes born again some other day. In this context, we may talk of probabilities as specifying beliefs an agent has over some the occurrence or not of some event. Savage’s Theorem studies decision-making in this context; this is, if a rational decision-maker is uncertain about the probabilities of obtaining certain outcomes and does not have a precise, objective quantification of her preferences (utility function), then it is Savage’s Theorem (Savage, 1954), which gives a formal decision criterion.

Savage’s result extends von Neumann-Morgenstern Theorem since it considers the case in which a rational decision maker knows neither her utility function nor the probabilities to be used in order to obtain the expected values required for making choices according to the von Neumann-Morgenstern Theorem (Gilboa, 2009).

#### 2.3.1 General Expected Utility Principle (Savage)

We now state the general Expected Utility Principle as found in (Bernardo and Smith, 1994). Further details can be found in (Savage, 1954; Anscombe et al., 1963; Kreps, 1988; Schervish, 1995; Robert, 2007; Gilboa, 2009).

**Theorem 2.6** (Expected Utility Principle, Savage (1954); Bernardo and Smith (1994)). In a finite, bounded Decision Problem under Uncertainty \((\mathcal{A}, \mathcal{C}, \mathcal{E}, \succeq)\), the preference relation \( \succeq \) satisfies the coherence rationality axioms if and only if there exists: A probability measure \( P \), called a subjective probability, that associates with each uncertain event \( E \in \mathcal{E} \) a real number \( P(E) \) and a utility function \( u : \mathcal{C} \rightarrow \mathbb{R} \) such that it associates each consequence with a real number \( u(c) \). Such that for \( a_1 \) and \( a_2 \) actions in \( \mathcal{A} \), and any \( G \neq \emptyset \)

\[
a_1 \succeq_G a_2 \text{ if and only if } \sum_{j \in J(a_1)} u(c_j)P(E_j) \geq \sum_{j \in J(a_2)} u(c_j)P(E_j)
\]

This theorem states that if a rational decision maker does not know the precise probabilities of outcomes given that a certain action has been taken, then she must choose as if having in
mind a probability assignment to the uncertainties in her environment and use such probabilities to calculate the expected utility with respect to a subjective utility function that represents her preferences. This result also gives a precise definition of subjective probability as a quantification of uncertainty which is used to make good decisions (Gilboa, 2009). See (Hens, 1992; Bernardo and Smith, 1994), and (Gilboa, 2009) for further details. See (Ellsberg, 1961; Tversky, 1975; Tversky and Kahneman, 1989; Binmore, 2008; Gilboa et al., 2009) and (Gilboa et al., 2012) for critiques of the coherence axioms.

2.4 Origins of outcome-action probabilities

The previous theorems, both vNM and Savage’s, have the limitation of being stated only in terms of associative information (probabilities), which leaves open the question of the origin of such probabilities. We know that vNM provides utilities given probabilities, and Savage provides both utilities and probabilities given the coherence axioms, but what do such probabilities mean and what is their origin?

Interpreting the action-outcome conditional probabilities, either as causal or observational, is exactly the issue at the heart of the debate between causal and evidential decision theorists, see Joyce (1999). It is recognized in the literature that the source of these conditional probabilities should be explained (Peterson, 2017), but there is substantial debate about their source, see for example Binmore (2008); Gilboa (2009); Eells (1982). We need a further justification to show that the right action-outcome conditional probabilities can be grounded in causal models. The objective of this paper is to provide that basis. In the next section we state some basic facts on causality and why \( P(\cdot | x) \) is different from \( P(\cdot | do(x)) \) for some value \( x \).

3 Causation

The concept of Causality deals with regularities found in a given environment (context) that are stronger than probabilistic (or associative) relations in the sense that a causal relation allows for evaluating a change in the consequence given a change in the cause. This is known as an intervention, which is different from observing and consists of a change in the joint distribution of the variables, which is performed by forcing the value of some variable to a specific value. Causal reasoning is able to deal with changes in the data-generating distributions, while observational reasoning does not. In a non-causal world, patients would avoid going to the doctor in order to avoid being sick (Spirtes et al., 2000; Pearl, 2009; Koller and Friedman, 2009; Pearl and Mackenzie, 2018).

In this work, the manipulationist interpretation of Causality is adopted (Woodward, 2003). The main paradigm is clearly expressed by Campbell and Cook as manipulation of a cause will result in a manipulation of the effect (Campbell and Cook, 1979). Consider the following example from Woodward: manually forcing a barometer to go down won’t cause a storm, whereas the occurrence of a storm will cause the barometer to go down (Woodward, 2003).

Here, we take the formal definition of Probabilistic Causality given by Spirtes et al. as a working definition for the notion of Causation. Similar descriptions of the manipulationist approach were described by Holland (1986). Causal inference tools, such as Pearl’s do-calculus allow finding the effect of an intervention in terms of probabilistic information when certain conditions are met (Pearl, 2009).

3.1 A definition of Causality

Causality is a stochastic relation between events within a probability space; this is, some event (or events) causes another event to occur, (Spirtes et al., 2000).

**Definition 3.1.** Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, and consider a binary relation \( \rightarrow \subseteq \mathcal{F} \times \mathcal{F} \) which is: Transitive: If \( A \rightarrow B \) and \( B \rightarrow C \) for any \( A, B, C \in \mathcal{F} \) then \( A \rightarrow C \). Irreflexive: For all \( A \in \mathcal{F} \) it doesn’t hold that \( A \rightarrow A \). Antisymmetric: For \( A, B \in \mathcal{F} \) such that \( A \neq B \) if \( A \rightarrow B \) then it doesn’t hold that \( B \rightarrow A \).

We consider an extra pair of conditions. The first one, known as Causal Sufficiency, is about the nature of the model: for any variables \( X, Y \) in the model \( \mathcal{G} \), there are no common causes of
X, Y outside of the model $\mathcal{G}$ (Spirtes et al., 2000; Pearl, 2009; Sucar, 2015). Common causes that are uncaused by other factors in $\mathcal{G}$ are the usual emphasis with Causal Sufficiency, but it also excludes some unobserved intermediate events (e.g., if $A \rightarrow B \rightarrow X$ and $B \rightarrow Y$, then $B$ must be observed). Causal Sufficiency thus implies that the causal connections in $\mathcal{G}$ do not share unmodeled mechanisms (Spirtes et al., 2000).

The second required condition is that an intervention on a particular variable or event $T$ will change the value of $T$ (and so can “break” the causal influences on $T$), but do not otherwise affect the causal mechanisms in $\mathcal{G}$. This assumption, called Invariance by Woodward (2003), more specifically implies that $T$ still has the same effects as before, even though the joint distribution of the variables is reconfigured (Woodward, 2003).

These conditions are required in an axiomatic fashion, so we do not discuss them further here.

3.2 Representation into a Directed Acyclic Graph

The causal relations, defined between events, contained in $\rightarrow$ can be summarized in a graph $G = (V, E)$ in the following way: If $A \rightarrow B$ then the graph must contain a node $A \in V$ representing A, a node $B \in V$ representing $B$ and a directed edge $e \in E$ connecting the respective nodes in the direction of the causal relation.

Notice that since the graph is finite, there exist some nodes that do not have causes, which are called exogenous. If an event $A$ is caused by some other event, then we say it is endogenous and we denote the set of its causes as $Pa(A)$. It is proven by Kiiveri et al. that at least one exogenous node exists in a causal graph (Kiiveri et al., 1984).

3.3 Causal Graphical Models

A Causal Graphical Model (CGM) consists of a set of random variables $X = \{X_1, ..., X_n\}$, and a Directed Acyclic Graph (DAG) $\mathcal{G}$ whose nodes are in correspondence with the variables in $X$ and whose edges represent relations of cause-effect in the sense that their realizations correspond to the events encoded by the causal relation (Koller and Friedman, 2009; Sucar, 2015). Also, the model is enriched with an operator called $do\()$ which is a functional defined over graphs, and whose action is described as follows: given $X \subseteq X$ and $x = \{x_{i_1}, ..., x_{i_k}\}$ an element of the set of all possible values of the variables belonging to $X$, $Val(X)$ the action $do(X = x)$ corresponds to assigning to each $X_j \in X$ the value $x_{i_j}$ and to delete every incoming edge into the node corresponding to each $X_j$ in the graph $\mathcal{G}$. In the context of CGMs, to apply the $do()$ operator over a variable (or set of variables) is called as an intervention over the variable. It is this interventional operator which separates associative models from causal models (Pearl, 2009; Koller and Friedman, 2009; Sucar, 2015).

It is required that the probability distribution that results from an intervention over a variable is Markov compatible with the graph; this is, the resulting interventional distribution is equivalent to the product of the conditional probability of every variable given its parents in the intervened graph (Sucar, 2015).

3.4 Causal Environments and Causal Decision Problems

We consider decision-making using causal information. In this section we define a Causal Environment to be an uncertain environment, as defined in Section 2.1, with the extra condition that there exists a CGM $\mathcal{G}$ which controls it in the following sense: there exists a causal relation between available actions and consequences in the sense that any chosen action will stochastically cause a consequence. The role of the CGM is to encode all of the causal relations present in the environment, not only between actions and consequences but also between any other variables in the environment.

**Definition 3.2.** A Causal Environment is a tuple $(\Omega, A, G, C, E)$ where $(\Omega, A, C, E)$ is an uncertain environment and $\mathcal{G}$ is a CGM such that the set of uncertain events $E$
correspond to the different realizations of the variables in $G$ as well as the possible ways that the variables are related one with each other.

Definition 3.3. We define a Causal Decision Problem (CDP) as a tuple $(\Omega, \mathcal{A}, G, \mathcal{E}, \mathcal{C}, \succeq)$, where $(\Omega, \mathcal{A}, G, \mathcal{E}, \mathcal{C})$ is a Causal Environment and $\succeq$ is a preference relation defined over actions.

For the CGM in a CDP, we distinguish two particular variables: one corresponding to the available actions, and one corresponding to the produced (caused) outcome. We are considering that only one variable can be intervened upon and that the values of such variable represent the actions available to the decision maker; i.e., the value forced upon such variable under an intervention represents the action taken by the decision maker.

The intuition behind the definition of a Causal Decision Problem is this: a decision maker chooses an action $a \in \mathcal{A}$, which is automatically fed into the model $G$, which outputs the causal outcome $c \in \mathcal{C}$. The definitions of a Bounded and Finite Causal Decision Problem extend in an analogous way from the classical definition.

4 Causal Decision Theory

Now that we have addressed basic notions of Classical Decision Making and Causation, we move on to Causal Decision Theory, where actions and outcomes are causally related. In this Section, we will recall some related and previous work done in Causal Decision Theory, with particular emphasis on decision-making results by Judea Pearl, in which causal information is assumed to be known to a decision-maker. Then, we further generalize this result to the case of unknown causal information.

4.1 Related Work

According to Joyce’s formulation of Causal Decision Theory, a decision maker must choose whatever action is more likely to (causally) produce the desired outcome while keeping any beliefs about causal relations fixed (Joyce, 1999). This is resumed in Stalnaker’s equation (Stalnaker, 1968):

$$u(a) = \sum_x P(a \Box \rightarrow x)u(x),$$

(2)

where $a \Box \rightarrow x$ is to be read as if the decision maker does $a$ then $x$ would be the case (Gibbard and Harper, 1978; Kleinberg, 2013). Lewis’ and Joyce’s work captured the intuition that causal relations may be used to control the environment and to predict what is caused by the actions of a decision-maker. We will refine the $\Box \rightarrow$ operator by an explicit way of calculating the probability of causing an outcome by doing a certain action in terms of Pearl’s do-calculus. Heckerman and Shachter provides a framework for defining the notions of cause and effect in terms of decision theoretical concepts and gives a theoretical basis for a graphical description of causes and effects, such as the causal influence diagrams introduced by Dawid (Dawid, 2002). Heckerman and Shachter gave an elegant definition of causality but did not address how to actually make choices using causal information (Heckerman and Shachter, 1995). Dawid presents a decision-theoretic approach to causal inference in which a decision maker must take into account how alternatives compare against each other in terms of the average causal effect, the such approach uses the well-known influence diagrams (Dawid, 2002, 2003) in order to derive formulas that allow an explicit calculation of the average causal effect (Dawid, 2012). Influence diagrams have the ability to express both intervention variables and chance variables in a single graphical structure. An optimality criterion for sequential interventions is obtained by Dawid and Didelez by maximizing the expectation of outcomes (Dawid and Didelez, 2008).

On the other hand, J. Pearl proposes the following criterion for decision-making: Consider a rational decision-maker who faces a causal environment in which she knows the causal model controlling the relation between her actions and outcomes. She can use the known causal model in order to find the probabilities of causing the desired outcome given she takes a certain action. The following theorem is found in Section 4.1 of Pearl (2009), but the intuitions that lie behind can be traced back to Lewis (1981) and Joyce (1999).
Theorem 4.1 (Pearl (2009), Section 4.1). Let \( G \) be a Causal Graphical Model, and its associated distribution \( P_G \). Let \( C \) be a set of consequences of interest for a decision-maker. If the decision maker faces a causal environment and if the causal graphical model \( G \) is known, then the preference relation \( \succeq \) satisfies the von Neumann-Morgenstern rationality axioms if and only if:

\[
a \succeq b \quad \text{if and only if} \quad \sum_{c \in C} P_G(c \mid \text{do}(a))u(c) \geq \sum_{c \in C} P_G(c \mid \text{do}(b))u(c).
\]

Equivalently, the action that must be chosen is

\[
a^* = \arg\max_{a \in A} \sum_{c \in C} P_G(c \mid \text{do}(a))u(c).
\]

Further related work Lewis (1981); Heckerman and Shachter (1995); Joyce (1999); Dawid (2002, 2012); Eells (1982); Stern (2017)

4.2 Main Result

We now consider the case in which a rational decision maker does not know the causal model which controls her environment. We must add a new axiom to the rationality axioms in order to keep simple our proof: Choosing within a Causal Decision Problem corresponds to intervening a variable of the true causal model which controls the environment.

Using Pearl’s result, we saw how a rational decision maker can use a known causal model in order to make a choice. Now, since the decision maker does not know the causal model, we argue that she must use beliefs about which causal relations hold in her environment, and use them in order to make a choice. In this case, in which a Causal Graphical Model controls the relation between actions and outcomes, any subjective information about the environment must consider causal structures. For this reason, we assert that the probability distribution that the decision maker has in mind is in fact a distribution over causal structures, where the decision maker uses each structure as if it were the true one in order to choose the best action within each structure by using Theorem 4.1. We assume a finite set of actions and outcomes.

Theorem 4.2 (Main Result). In a finite, bounded, Causal Decision Problem \((A, G, E, C, \succeq)\), where \( G \) is a Causal Graphical Model, we have that the preferences \( \succeq \) of a decision maker are rational if and only if there exists a utility function, a probability distribution \( P_C \) over a non-empty family \( F \) of causal graphical models such that for each \( a, b \in A \):

\[
a \succeq b \quad \text{if and only if} \quad \sum_{c \in C} u(c) \left( \sum_{g \in F} P_g(c \mid \text{do}(a))P_C(g) \right) \geq \sum_{c \in C} u(c) \left( \sum_{g \in F} P_g(c \mid \text{do}(b))P_C(g) \right)
\]

where \( P_g \) is the probability distribution associated with the causal structure \( g \).

Proof. Following Bernardo and Smith proof of the Expected Utility Principle (Proposition 2.22, p. 52, Bernardo and Smith (1994)), let \( a_i = \{c_{ij} \mid E_{ij}, j = 1, \ldots, n\} \) be an option.

The existence of a probability \( P(\cdot) \), associated with uncertain events \( E \), is given by Axiom 5(2). Also, the utility \( u(c) \), is the real number \( \mu(S) \), associated with any standard event \( S \), such that

\[
c \sim \{c^* \mid S, c_* \mid S^c\}.
\]

From the coherence Axioms 4(3) and 5(2), and Proposition 2.13 in Bernardo and Smith (1994), there exists events \( S_{ij} \) and \( S'_{ij} \), such that

\[
c \sim \{S'_{ij}, c_* \mid S'_{ij}, S_{ij} \perp \perp E, P(S'_{ij}) = P(S_{ij})\}.
\]
Then, by Proposition 2.10 in Bernardo and Smith (1994), $c_{ij} \sim \{c^* \mid S_{ij}, c^* \mid S_{ij}' \}$, where $P(S_{ij}) = u(c_{ij})$.

Now, for any other option $a$ and $i = 1, 2$,

$$\{[c_{ij} \mid E_{ij}, j = 1, \ldots, n, a] \sim \{(c^* \mid S_{ij}, c^* \mid S_{ij}') \mid E_{ij}, j = 1, \ldots, n, a\}.$$

According to Definitions 3.2 and 3.3, and the new axiom added, we can split each event $E_{ij}$ into $E'_{ij}, g_{ijk}, I$, where $E'_{ij} = \{X_1 = x_1, \ldots, X_n = x_n\}$, where $X_1, \ldots, X_n$, are the variables of the CGM that controls the environment $G$; and $g_{ijk}$, specifies a graphical structure over the set of variables $\{X_1, \ldots, X_n\}$; and $I$ is a pair of indices which indicates which variable has been intervened in order to affect which other variable.

We can define $A_i$ as

$$A_i = \cup_j (E_{ij} \cap S_{ij}),$$

Then, it is shown in Bernardo and Smith (1994) that events with this structure hold that:

$$a_1 \succeq a_2 \iff A_1 \succeq A_2 \iff P(A_1) \geq P(A_2).$$

And, we can express $P(E_{ij} \cap S_{ij})$ as:

$$P(E_{ij} \cap S_{ij}) = P(E_{ij})P(S_{ij}).$$

This means that,

$$P(A_i) = P(\cup_j (E_{ij} \cap S_{ij}))$$

$$= P(E_{ij})P(S_{ij})$$

$$= \sum_j u(c_{ij})P(E_{ij}).$$

We now use the fact that $E_{ij} = \cup_k \{E'_{ijk}, g_{ijk}, I\}$, to see that

$$P(E_{ij}) = P(E'_{ij} \cap (g_{ij1} \cup \ldots \cup g_{ijn_k}) \cap I)$$

$$= \sum_k P(E' \cap I \mid g_{ijk})P(g_{ijk}).$$

Where the last term $P(g_{ijk})$, is supported within a family of causal models. Now, this last expression can be further expanded as:

$$\sum_k P(E'_{ij} \cap I \mid g_{ijk})P(g_{ijk}) = \sum_k P(\{X_1, \ldots, X_n\} \cap I \mid g_{ijk})P(g_{ijk})$$

$$= \sum_k P_g(c_{ij} \mid do(x_i))P(g_{ijk}).$$

Therefore,

$$P(A_i) = \sum_j u(c_{ij})P(E_{ij})$$

$$= \sum_j u(c_{ij}) \left(\sum_k P_g(c_{ij} \mid do(x_i))P(g_{ijk})\right).$$

And the result follows, since $a_1 \succeq a_2 \iff P(A_1) \geq P(A_2)$. \qed
4.3 Interpretation

Theorem 4.2 asserts that a rational decision maker who faces a Causal Decision Problem with unknown causal information must use a probability distribution $P_C$ over a family $\mathcal{F}$ of causal structures, and, within each structure, $g \in \mathcal{F}$, use the term $P_g(c \mid do(a))$ in order to find the probability of obtaining a certain consequence given that the intervention $do(\cdot)$ is performed; in this way, the optimal action $a^*$ is given by:

$$
a^* = \arg\max_{a \in A} \sum_{c \in C} u(c) \left( \sum_{g \in \mathcal{F}} P_g(c \mid do(a)) P_C(g) \right).
$$

We note that $a^*$ is obtained by taking into account the utility obtained by every possible consequence weighted using both the probability of causing such action within a specific causal model $g$ and the probability that the decision-maker assign to such $g \in \mathcal{F}$.

We are considering a normative interpretation for Theorem 4.2 according to which a decision maker must use any causal information in order to obtain the best possible action. Such action must be obtained by considering the beliefs of the decision maker about the causal relations that hold in her environment (the distribution $P_C$), how such relations could produce the best action when considered as if they were true (distribution $P_g$), and the satisfaction (utility $u$) produced by the consequences of actions.

It is known that humans tend to ignore pure probabilistic information over causal information (Tversky and Kahneman, 1980), and are in fact able to learn, and use, causal models in sequential decision-making processes (Lagnado et al., 2007; Sloman and Hagnayer, 2006; Nichols and Danks, 2007; Meder et al., 2010; Hagnayer and Meder, 2013; Wellen and Danks, 2012), even though such learning is not perfect (Rottman and Hastie, 2014). Therefore, this theorem provides the basis for a much stronger, and computationally implementable, framework for decision-making in which causal information is used over associative information, even though complete causal information may not be available to the decision-maker.

5 Application: Causal Games and Nash Equilibrium

In this section we study and develop an application of the previous result in the domain of Game Theory: we consider a strategic game between $N$ rational players who are situated in a causal environment. A game is a model of a situation in which several players must take an action and afterward they will be affected both by the outcome of their own action as well as the actions of the other players (Osborne and Rubinstein, 1994).

In a strategic game, it is assumed that no player knows the action taken by any other players; we also assume that the causal mechanism, which is represented by a Causal Graphical Model $G$, remains fixed and it is unknown for each player. In this game, players ignore the actions taken by any other player, and since the causal model which controls the environment is unknown to every player, the players also ignore the information that players will use in order to take their respective actions: strategic games of this type are called Bayesian Games, introduced by Harsanyi (Harsanyi, 1967, 1968b,a).

With this contribution, we expect to show that standard notions of game theory such as Nash Equilibrium can be extended to the case in which causal information is considered over associative information. Therefore, provide motivation to further extend classical results to use causal information as a basis.

In the games we will consider, the uncertainty of every player consists of two levels: on the first level, the true causal model $G$; on the second level, what an action $do(a)$ causes if a certain Causal Graphical Model $\omega$ is considered to be the causal model.

**Definition 5.1.** A Bayesian strategic game (Osborne and Rubinstein, 1994) consists of: A finite set $N$ of players. A finite set $\Omega$ of states of nature. For each player, a nonempty set $A_i$ of actions. For each player, a finite set $T_i$ and a function $\tau_i : \Omega \rightarrow T_i$ is the signal function of the player. For each player, a probability measure $p_i$ over $\Omega$ such that $p_i(\tau_i^{-1}(t_i)) > 0$ for all $t_i \in T_i$. A preference relation $\succeq_i$ defined over the set of probability measures over $A \times \Omega$ where $A = A_1 \times \cdots A_n$. 

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We consider $\Omega$ to be a family of admissible causal models; in this way, $\omega \in \Omega$ being the true state of Nature fixes a causal model which controls the environment in which the players make their choices. In classical Bayesian games, once $\omega \in \Omega$ is realized as the true state, then each player receives a signal $t_i = \tau_i(\omega)$ and the posterior belief $p_i(\omega | \tau_i^{-1}(t_i))$ given by $p_i(\omega)/p_i(\tau_i^{-1}(t_i))$ if $\omega \in \tau_i^{-1}(t_i)$. In the case of causal Bayesian games, we must consider both the probability $p_i$ of $\omega$ being the true state as well as the probability $p_i^\omega$ of observing a certain consequence when doing some action $a_i$ if $\omega$ is the true model. Following Osborne and Rubinstein (1994), we define a new game $G^*$ in which its players are all of the possible combinations $(i, t_i) \in N \times T_i$, where the possible actions for $(i, T_i)$ is $A_i$. Osborne and Rubinstein (1994) show that for a fixed player $i \in N$, the posterior probability $p(\omega | \tau_i^{-1}(t_i))$ induces a lottery over the pairs $(a^*(j, \tau_j(\omega)))_{j, \omega}$ for some other $j \in N$. This lottery assigns to $(a^*(j, \tau_j(\omega))), \omega$ the probability $p_i(\omega)/p_i(\tau_i^{-1}(t_i))$ if $\omega \in \tau_i^{-1}(t_i)$. The classical Bayesian game's Nash Equilibrium is the Nash equilibrium of $G^*$ (Osborne and Rubinstein, 1994). Now, we consider the second level of uncertainty: the consequences caused by some action $a$ through a causal model $\omega \in \Omega$. We notice that the posterior probability itself induces a probability distribution defined over actions for each player once a desired consequence is fixed, this distribution, according to Theorem 4.2 is given by $p_i^r(c | do(a^*_i), a_i^\omega)p_i(\omega | \tau_i^{-1}(t_i))$. This motivates the following definition of a Causal Nash equilibrium.

5.1 Causal Nash Equilibrium

For each player $i \in N$ in the strategic game, we define the following probability distribution over consequences:

$$p_i^C(c) = p_i^r(c | do(a_i), a_{-i})p_i(\omega) \text{ for } a \in A = A_1 \times \cdots \times A_N,$$

where $p_i^C$ is the probability of causing a certain consequence within a causal structure $\omega$, and $p_i$ is the player’s posterior beliefs about the causal structure that controls the environment, and $do()$ is the well-known intervention operator from (Pearl, 2009). We now define:

$$u_i^C(a) = \sum_{c \in C} u_i(c)p_i^C(c) \text{ for } a \in A = A_1 \times \cdots \times A_N.$$

Notice that $u_i^C$ evaluates an action profile $a \in A$ in terms of: The knowledge each player has about the causal structure represented by $p_i$, which allows each player to evaluate the probability of causing outcomes in terms of actions by using the do operator, as well as the observed actions that are taken by the other players, given by $a_{-i}$, and the preferences of each player $u_i$. Using this new function, we define the equilibrium for a strategic game with causal information and Bayesian players as:

**Definition 5.2.** A Nash equilibrium for this causal strategic game is an action profile $a^* \in A$ if and only if

$$u_i^C(a^*) \geq u_i^C(a_i, a_{-i}) \text{ for any other } a_i \in A_i.$$

This is, an action profile is a Nash equilibrium if and only if each player uses her current knowledge about the causal structure of the environment in order to (causally) produce the best possible outcome given the actions taken by the other players. The existence of the Causal Nash Equilibrium is guaranteed if every $A_i$ is a nonempty compact convex set in some $\mathbb{R}^n$ and if the preference relation induced by $u_i^C$ is continuous and quasi-concave as proved by Osborne and Rubinstein (1994).

6 Limitations

We are working within the classical rationality assumption. Rationality can be ultimately thought of as a consistent or coherent way of making choices, but the precise definition has been a subject of debate. See Ellsberg (1961); Gilboa (2009) and Machina and Siniscalchi (2014) for critiques of the Savage Rationality Axioms. We have favored causal graphical models over other alternatives since it has been argued that several cognitive processes, such as causal reasoning, can be best represented as graphical models (Danks, 2014; Sloman and Lagnado, 2015; Hagemayer, 2016).
7 Summary

We have defined a Causal Decision Problem in terms of a classical Decision Problem under Uncertainty provided by a causal mechanism that mediates between actions and outcomes; this causal mechanism is assumed to be represented as a causal graphical model. In the case in which a rational decision maker knows such causal relations, we have seen that Pearl has provided a decision-making result Pearl (2009).

On the other hand, when a decision maker does not know the causal mechanism, in Theorem 4.2 we have provided preference representation result for causal decision making; our result explicitly states how a rational decision maker should use subjective beliefs, encoded as a probability distribution over causal models, as well as the causal inference machinery within the considered causal structures in order to find an optimal action.

With these two cases being covered, we have shown how causal models can provide the action-outcome probabilities for decision-making.

As an application, by using Theorem 4.2 and taking as a basis Harsanyi’s model of a Bayesian Game in which every player has incomplete information about both the actions taken by other players as well as the information that made each player take his action, we have been able to provide a definition of a Causal Nash Equilibrium (Equation 7) in which every player is aware that there exists a causal mechanism that will produce some consequence once he takes an action.

Finally, by extending a known result from Pearl to the case of unknown causal information, we have shown how action-outcome probabilities for decision-making can be grounded in causal models.

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A Coherence axioms (Bernardo and Smith, 1994)

• Axiom 1: Comparability of consequences and dichotomized options:
  1. There exists consequences \( c_1, c_2 \) such that \( c_2 \succ c_1 \).
  2. For any consequences \( c_1, c_2 \in \mathcal{C} \) and any events \( E, F \in \mathcal{E} \), then either \( \{ c_2 \mid E, c_1 \mid E^c \} \succeq \{ c_2 \mid F, c_1 \mid F^c \} \) or \( \{ c_2 \mid F, c_1 \mid F^c \} \succeq \{ c_2 \mid E, c_1 \mid E^c \} \).

• Axiom 2: Transitivity of preferences:
  1. \( a \succeq a \).
  2. \( a_2 \succeq a_1, a_3 \succeq a_2 \Rightarrow a_3 \succeq a_1 \).

• Axiom 3: Consistency of preferences:
  1. If \( c_2 \succeq c_1 \), then for any event \( G \) more likely than \( \emptyset \), \( c_2 \succeq_G c_1 \).
  2. If \( c_2 \succ c_1 \) and \( \{ c_2 \mid F, c_1 \mid F^c \} \succeq \{ c_2 \mid E, c_1 \mid E^c \} \), then \( F \succeq E \).

• Axiom 4: Existence of standard events: There exists a sub-algebra \( \mathcal{S} \) of \( \mathcal{E} \) and a function \( \mu : \mathcal{S} \to [0, 1] \) such that
  1. \( S_2 \succeq S_1 \) if and only if \( \mu(S_2) \geq \mu(S_1) \).
  2. Disjoint \( S_1 \) and \( S_2 \) imply that \( \mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2) \).
  3. For any \( \alpha \in [0, 1] \), and events \( E, F \in \mathcal{E} \), there exists a standard event \( S \in \mathcal{S} \) such that \( \mu(S) = \alpha \) and \( E \perp \perp S \) and \( F \perp \perp S \).
  4. For \( S_1 \) and \( S_2 \), independent standard events, we have that \( \mu(S_1 \cap S_2) = \mu(S_1)\mu(S_2) \).
  5. If \( E \perp \perp S, F \perp \perp S, \) and \( E \perp \perp F \), then \( E \sim S \Rightarrow E \sim_F S \).

• Axiom 5: Precise measurement:
  1. If \( c_2 \succeq c \succeq c_1 \), there exists a standard event \( S \in \mathcal{S} \) such that \( c \sim \{ c_2 \mid S, c_1 \mid S^c \} \).
  2. For each event \( E \in \mathcal{E} \), there exists a standard event \( S \in \mathcal{S} \) such that \( E \sim S \).

References

Anscombe, F. J., Aumann, R. J., et al. (1963). A definition of subjective probability. *Annals of mathematical statistics*, 34(1):199–205.

Berkovitz, J. (2012). The world according to de finetti: On de finetti’s theory of probability and its application to quantum mechanics. In *Probability in physics*, pages 249–280. Springer.

Bernardo, J. M. and Smith, A. F. M. (1994). Bayesian theory. Wiley Series in Probability and Statistics.

Bernoulli, D. (1738/1954). Exposition of a new theory on the measurement of risk. *Econometrica*, 22(1):23–36.

Binmore, K. (2008). *Rational Decisions*. The Gorman Lectures in Economics. Princeton University Press.

Campbell, D. T. and Cook, T. D. (1979). *Quasi-experimentation: Design & analysis issues for field settings*. Rand McNally College Publishing Company Chicago.

Danks, D. (2014). *Unifying the mind: Cognitive representations as graphical models*. MIT Press.

Dawid, P. (2002). Influence diagrams for causal modelling and inference. *International Statistical Review*, 70(2):161–189.
Dawid, P. (2003). *Causal inference using influence diagrams: the problem of partial compliance*, pages 45–81. Oxford University Press.

Dawid, P. (2012). *The Decision-Theoretic Approach to Causal Inference*, chapter 4, pages 25–42. John Wiley & Sons, Ltd.

Dawid, P. and Didelez, V. (2008). Identifying optimal sequential decisions. In *Proceedings of the Twenty-Fourth Conference on Uncertainty in Artificial Intelligence*, pages 113–120. AUAI Press.

De Finetti, B. (1937). Foresight: Its logical laws, its subjective sources. In *Breakthroughs in statistics*, pages 134–174. Springer.

Eells, E. (1982). *Rational decision and causality*. Cambridge University Press.

Ellsberg, D. (1961). Risk, ambiguity, and the savage axioms. *The quarterly journal of economics*, pages 643–669.

Gilboa, I. (2009). *Theory of Decision under Uncertainty*. Cambridge University Press.

Gilboa, I., Postlewaite, A., and Schmeidler, D. (2009). Is it always rational to satisfy savage’s axioms? *Economics & Philosophy*, 25(3):285–296.

Gilboa, I., Postlewaite, A., and Schmeidler, D. (2012). Rationality of belief or: why savage’s axioms are neither necessary nor sufficient for rationality. *Synthese*, 187(1):11–31.

Hagmayer, Y. and Meder, B. (2008). Causal learning through repeated decision making. In *Proceedings of the Annual Meeting of the Cognitive Science Society*, volume 30.

Hagmayer, Y. and Meder, B. (2013). Repeated causal decision making. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 39(1):33.

Harsanyi, J. C. (1967). Games with incomplete information played by “bayesian” players, i–iii part i. the basic model. *Management science*, 14(3):159–182.

Harsanyi, J. C. (1968a). Games with incomplete information played by ‘bayesian’ players, part iii. the basic probability distribution of the game. *Management Science*, 14(7):486–502.

Harsanyi, J. C. (1968b). Games with incomplete information played by “bayesian” players part ii. bayesian equilibrium points. *Management Science*, 14(5):320–334.

Heckerman, D. and Shachter, R. (1995). Decision-theoretic foundations for causal reasoning. *Journal of Artificial Intelligence Research*, 3:405–430.
Hens, T. (1992). A note on savage’s theorem with a finite number of states. *Journal of Risk and Uncertainty*, 5(1):63–71.

Holland, P. W. (1986). Statistics and causal inference. *Journal of the American Statistical Association*, 81(396):945–960.

Joyce, J. M. (1999). *The Foundations of Causal Decision Theory*. Cambridge University Press.

Kahneman, D. and Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–292.

Kiiveri, H., Speed, T. P., and Carlin, J. B. (1984). Recursive causal models. *Journal of the Australian Mathematical Society*, 36(1):30–52.

Kleinberg, S. (2013). *Causality, probability, and time*. Cambridge University Press.

Koller, D. and Friedman, N. (2009). *Probabilistic graphical models: principles and techniques*. MIT press.

Kreps, D. M. (1988). *Notes On The Theory Of Choice*. Routledge New York.

Lagnado, D. A., Waldmann, M. R., Hagmayer, Y., and Sloman, S. A. (2007). Beyond covariation. *Causal learning: Psychology, philosophy, and computation*, pages 154–172.

Lake, B. M., Ullman, T. D., Tenenbaum, J. B., and Gershman, S. J. (2017). Building machines that learn and think like people. *Behavioral and Brain Sciences*, 40.

Lewis, D. (1981). Causal decision theory. *Australasian Journal of Philosophy*, 59(1):5–30.

Machina, M. J. and Siniscalchi, M. (2014). Ambiguity and ambiguity aversion. In *Handbook of the Economics of Risk and Uncertainty*, volume 1, pages 729–807. Elsevier.

Meder, B., Gerstenberg, T., Hagmayer, Y., and Waldmann, M. R. (2010). Observing and intervening: Rational and heuristic models of causal decision making. *The Open Psychology Journal*, 3:119–135.

Neil, M., Fenton, N., Osman, M., and Lagnado, D. (2019). Causality, the critical but often ignored component guiding us through a world of uncertainties in risk assessment. *Journal of Risk Research*, 0(0):1–5.

Nichols, W. and Danks, D. (2007). Decision making using learned causal structures. In *Proceedings of the Annual Meeting of the Cognitive Science Society*, volume 29.

Nozick, R. (1969). Newcomb’s problem and two principles of choice. In *Essays in honor of Carl G. Hempel*, pages 114–146. Springer.

Osborne, M. J. and Rubinstein, A. (1994). *A course in game theory*. MIT press.

Pearl, J. (2009). *Causality: Models, Reasoning and Inference*. Cambridge University Press, New York, NY, USA, 2nd edition.

Pearl, J. and Mackenzie, D. (2018). *The Book of Why: The New Science of Cause and Effect*. Basic Books.

Peterson, M. (2017). *An Introduction to Decision Theory*. Cambridge University Press.

Robert, C. (2007). *The Bayesian Choice: From Decision-Theoretic Foundations to Computational Implementation*. Springer Science & Business Media.

Rottman, B. M. and Hastie, R. (2014). Reasoning about causal relationships: Inferences on causal networks. *Psychological bulletin*, 140(1):109.

Savage, L. J. (1954). *The Foundations of Statistics*. New York: John Wiley & Sons.
Schervish, M. J. (1995). *Theory of statistics*. Springer Series in Statistics. Springer-Verlag New York, Inc.

Sloman, S. A. and Hagmayer, Y. (2006). The causal psycho-logic of choice. *Trends in Cognitive Sciences*, 10(9):407–412.

Sloman, S. A. and Lagnado, D. (2015). Causality in thought. *Annual Review of Psychology*, 66:223–247.

Spirtes, P., Glymour, C. N., and Scheines, R. (2000). *Causation, prediction and search*. MIT Press.

Stalnaker, R. (1968). A theory of conditionals. In Rescher, N., editor, *Studies in Logical Theory (American Philosophical Quarterly Monographs 2)*, pages 98–112. Oxford: Blackwell.

Stern, R. (2017). Interventionist decision theory. *Synthese*, 194(10):4133–4153.

Sucar, L. E. (2015). *Probabilistic Graphical Models*. Advances in Computer Vision and Pattern Recognition. Springer London.

Tversky, A. (1975). A critique of expected utility theory: Descriptive and normative considerations. *Erkenntnis*, pages 163–173.

Tversky, A. and Kahneman, D. (1977). Causal thinking in judgment under uncertainty. In *Basic problems in methodology and linguistics*, pages 167–190. Springer.

Tversky, A. and Kahneman, D. (1980). Causal schemas in judgments under uncertainty. *Progress in social psychology*, 1:49–72.

Tversky, A. and Kahneman, D. (1989). Rational choice and the framing of decisions. In *Multiple criteria decision making and risk analysis using microcomputers*, pages 81–126. Springer.

Von Neumann, J. and Morgenstern, O. (1944). *Theory of games and economic behavior*. Princeton University Press.

Waldmann, M. R. and Hagmayer, Y. (2013). Causal reasoning. In Reisberg, D., editor, *The Oxford Handbook of Cognitive Psychology*. Oxford University Press.

Wallace, D. (2012). *The emergent multiverse: Quantum theory according to the Everett interpretation*. Oxford University Press.

Wellen, S. and Danks, D. (2012). Learning causal structure through local prediction-error learning. In *Proceedings of the Annual Meeting of the Cognitive Science Society*, volume 34.

Woodward, J. (2003). *Making things happen: A theory of causal explanation*. Oxford Studies in Philosophy of Science. Oxford University Press.