A Virtual Element Method (VEM) for the quasilinear equation \(-\text{div}(\kappa(\nabla u)) = f\) using general polygonal and polyhedral meshes is presented and analysed. The nonlinear coefficient is evaluated with the piecewise polynomial projection of the virtual element ansatz. Well-posedness of the discrete problem and optimal order a priori error estimates in the $H^1$- and $L^2$-norm are proven. In addition, the convergence of fixed point iterations for the resulting nonlinear system is established. Numerical tests confirm the optimal convergence properties of the method on general meshes.

**Keywords:** virtual element method; quasilinear elliptic equations

1. Introduction

In this work we present an arbitrary-order conforming Virtual Element Method (VEM) for the numerical treatment of quasilinear diffusion problems. Both two and three dimensional problems are considered and the method is analysed under the same mesh regularity assumption used in the linear setting (Beirão da Veiga et al., 2013; Cangiani et al., 2017a), allowing for very general polygonal and polyhedral meshes.

Virtual element methods for general linear elliptic problems are now well-established, see e.g., (Beirão da Veiga et al., 2013; Beirão da Veiga & Manzini, 2014; Ahmad et al., 2013; Beirão da Veiga et al., 2016; Ayuso de Dios et al., 2016; Cangiani et al., 2017a; Brenner et al., 2017) and (Sutton, 2017b) for a simple implementation. See also (Beirão Da Veiga et al., 2017; Brenner & Sung, 2018) for an extension to meshes with arbitrarily small edges and (Cangiani et al., 2017b; Mora et al., 2017) where the mesh generality is exploited within an adaptive algorithm driven by rigorous a posteriori error estimates. The VEM framework has been concurrently extended to a number of different problems and applications, and, in particular, the literature on VEM for nonlinear problems is growing, the same being true for other approaches to polygonal and polyhedral meshes. Virtual Element methods are developed for semilinear parabolic problem in (Adak et al., 2019), Cahn-Hilliard in (Antonietti et al., 2016), stationary Navier-Stokes in (Beirão da Veiga et al., 2018; Gatica et al., 2018), nonlinear Birkman and quasi-Newtonian Stokes flow in (Gatica et al., 2018; Cáceres et al., 2018), computational mechanics in (Beirão da Veiga et al., 2015; Artioli et al., 2017; Wriggers & Hudobivnik, 2017; Hudobivnik et al., 2019; Wriggers et al., 2018; Taylor & Artioli, 2018; Artioli et al., 2018) and fracture problems in (Aldakheel et al., 2018). The related nodal Mimetic Finite Difference method is analysed in (Antonietti et al., 2015) for elliptic quasilinear problems whereby the nonlinear coefficient depends on the gradient of the solution, however only low-order discretisations are considered. We also mention the arbitrary order Hybrid High-Order method on polygonal meshes for the general class of Leray-Lions elliptic equations (Di Pietro & Droniou, 2017), including the problems considered here. The HHO method belongs to the class of nonconforming/discontinuous discretisations and is, in fact, related to the Hybrid Mixed Mimetic approach and to the nonconforming VEM (Droniou...
et al., 2010; Cockburn et al., 2016). In (Di Pietro & Droniou, 2017), the convergence of HHO is proven under minimal regularity assumptions, but the rate of convergence of the method is not analysed.

The VEM presented here is based on the $C^0$-conforming virtual element spaces of (Ahmad et al., 2013) whereby the local $L^2$-projection of virtual element functions onto polynomials is available and the VEM proposed in (Beirão da Veiga et al., 2016; Cangiani et al., 2017a) for the discretisation of linear elliptic problems with non-constant coefficients. In particular, to obtain a practical (computable) formulation, the nonlinear diffusion coefficient is evaluated with the element-wise polynomial projection of the virtual element ansatz. This results in nonlinear inconsistency errors which have to be additionally controlled.

We present an a priori analysis of the VEM which builds upon and extends the classical framework introduced by Douglas and Dupont (Douglas & Dupont, 1975) for standard conforming finite element methods. The analysis relies on the assumption that the nonlinear diffusion coefficient is bounded and Lipschitz continuous and is based on a bootstrapping argument: 1. existence of solutions for the numerical scheme is shown by a fixed point argument, 2. the $H^1$-norm error is bounded by optimal order terms plus the $L^2$-norm error, 3. using a standard duality argument and assuming that the discretisation parameter is small enough, the $L^2$-norm error is bounded by optimal order terms plus potentially higher-order terms, 4. based on the existence result, $L^2$-convergence is shown by a compactness argument, and now $H^1$-convergence follows from step 2. Within this approach, we also obtain optimal order a priori error estimates in the $H^1$- and $L^2$-norms, albeit under the (higher) regularity assumptions needed by the duality argument. To the best of our knowledge, this work provides the first optimal order error estimate for a conforming discretisation of quasilinear problems on general polygonal and polyhedral meshes.

To simplify the presentation, we consider homogeneous Dirichlet boundary value problems only. To this end, we introduce the model quasilinear elliptic problem

$$-\nabla \cdot (\kappa(u) \nabla u) = f(x) \quad \text{in} \quad \Omega, \quad \text{with} \quad u = 0 \quad \text{on} \quad \partial \Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a convex polygonal or polyhedral domain for $d = 2$ or $d = 3$, respectively. The diffusion coefficient is a twice differentiable function $\kappa : \mathbb{R} \to [\kappa_\ast, \kappa^\ast]$ such that $0 < \kappa_\ast \leq \kappa^\ast < +\infty$, and with bounded derivatives up to second order. Therefore $\kappa$ is Lipschitz continuous, namely there exists a positive constant $L$ such that

$$|\kappa(t) - \kappa(s)| \leq L|t - s|, \quad \text{for a.e } t, s \in \mathbb{R}.$$  

Writing (1.1) in variational form, we seek $u \in H^1_0(\Omega)$ such that

$$a(u; u, v) := (\kappa(u) \nabla u, \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega),$$

with $(\cdot, \cdot)$ denoting the standard $L^2$ inner-product. It is well known that for sufficiently smooth $f$, problem (1.1) possesses a unique solution $u$, see eg. (Douglas et al., 1971).

The remainder of this work is structured as follows. We introduce the virtual element method in Section 2. The method is then analysed in Section 3, where the well-posedness and a priori analysis are presented. In Section 4 we establish the convergence of fixed point iterations for the solution of the nonlinear system resulting from the VEM discretisation. We present a numerical test in Section 5 and, finally, we provide some conclusions in Section 6.

We use standard notation for the relevant function spaces. For a Lipschitz domain $\omega \subset \mathbb{R}^d$, $d = 2, 3$, we denote by $|\omega|$ its $d$-dimensional Hausdorff measure. Further, we denote by $H^s(\omega)$ the Hilbert space of index $s \geq 0$ of real–valued functions defined on $\omega$, endowed with the seminorm $|| \cdot ||_{s, \omega}$ and norm $||| \cdot |||_{s, \omega}$; further $(\cdot, \cdot)_\omega$ stands for the standard $L^2$-inner-product. The domain of definition will be omitted when this coincides with $\Omega$, eg. $|| \cdot ||_{\omega} := || \cdot ||_{1, \omega}$ and so on. Finally, for $\ell \in \mathbb{N} \cup \{0\}$, we denote by $P_\ell(\omega)$ the space of all polynomials of degree up to $\ell$.

2. The Virtual Element Method

We introduce the virtual element method for the discretisation of problem (1.3), using general polygonal and polyhedral decompositions of $\Omega$ in two and three dimensions, respectively. We start by recalling the definition of the virtual element spaces from (Ahmad et al., 2013; Cangiani et al., 2017a).
2.1 The Discrete Spaces

The definition of the virtual element method relies on the availability of certain local projector operators based on accessing the degrees of freedom. The choice of degrees of freedom for the virtual element spaces is thus important.

**Definition 2.1 (Degrees of freedom)** Let \( \omega \subset \mathbb{R}^d \), \( 1 \leq d \leq 3 \), be a \( d \)-dimensional polytope, that is, a line segment, polygon, or polyhedron, respectively. For any regular enough function \( v \) on \( \omega \), we define the following sets of degrees of freedom:

- **Nodal values.** For a vertex \( z \) of \( \omega \), \( \mathcal{N}_z^\omega(v) := v(z) \) and \( \mathcal{N}^\omega := \{ \mathcal{N}_z^\omega : z \text{ is a vertex} \} \);

- **Polynomial moments.** For \( l \geq 0 \),

\[
\mathcal{M}_l^\omega(v) = \frac{1}{|\omega|} \langle v, m_\alpha \rangle_\omega \quad \text{with} \quad m_\alpha := \left( \frac{x - x_\omega}{h_\omega} \right)^\alpha \quad \text{and} \quad |\alpha| \leq l,
\]

where \( \alpha \) is a multi-index with \( |\alpha| = \alpha_1 + \cdots + \alpha_d \) and \( x_\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d} \) in a local coordinate system, and \( x_\omega \) denoting the barycentre of \( \omega \). Further, \( \mathcal{M}_l^\omega := \{ \mathcal{M}_l^\omega : |\alpha| \leq l \} \). The definition is extended to \( l = -1 \) by setting \( \mathcal{M}_{-1}^\omega := \emptyset \).

Let \( \{ \mathcal{T}_h \}_h \) be a sequence of decompositions of \( \Omega \) into non-overlapping and not self-intersecting polygonal/polyhedral elements such that the diameter of any \( E \in \mathcal{T}_h \) is bounded by \( h \).

On \( \mathcal{T}_h \), we introduce element-wise projectors as follows. We denote by \( P_{\ell}^h \equiv P_{\ell}^{h,E} : L^2(E) \to \mathcal{P}_\ell(E) \), \( \ell \in \mathbb{N} \), the standard \( L^2(E) \)-orthogonal projection onto the polynomial space \( \mathcal{P}_\ell(E) \). With slight abuse of notation, the symbol \( P_{\ell}^h \) will also be used to denote the global operator obtained from the piecewise projections. Similarly, by \( P_{\ell}^h \equiv P_{\ell}^{h,E} \), \( \ell \in \mathbb{N} \), we denote the orthogonal projection of \( (L^2(E))^d \) onto the space \( \mathcal{P}_\ell(E) = (\mathcal{P}_\ell(E))^d \), obtained by applying \( P_{\ell}^{h,E} \) component-wise. Further, we consider the projection \( R_{\ell}^h \equiv R_{\ell}^{h,E} : H^1(E) \to \mathcal{P}_\ell(E) \), for \( \ell \in \mathbb{N} \), associating any \( v \in H^1(E) \) with the element in \( \mathcal{P}_\ell(E) \) such that

\[
(\nabla R_{\ell}^h v, \nabla p)_E = (\nabla v, \nabla p)_E, \quad \forall p \in \mathcal{P}_\ell(E),
\]

with, in order to uniquely determine \( R_{\ell}^h \), the addition of the following condition:

\[
\left\{ \begin{array}{ll}
\int_{\partial E} (v - R_{\ell}^h v) \, ds = 0 & \text{if } \ell = 1, \\
\int_E (v - R_{\ell}^h v) \, dx = 0 & \text{if } \ell \geq 2.
\end{array} \right.
\]

Let \( k \geq 1 \) be given, characterising the order of the method. We follow the construction of the corresponding \( C^0 \)-conforming VEM space presented in (Ahmad et al., 2013) to ensure that all of the above projectors, to be utilised in the definition of the method, are computable.

We first introduce the local spaces on each element \( E \) of \( \mathcal{T}_h \), for \( d = 2 \). Let \( B^2(E) \) be the space defined on the boundary of \( E \) as

\[
B^2_1(\partial E) := \{ v \in C^0(\partial E) : v|_e \in \mathcal{P}_k(e) \text{ for each edge } e \in \partial E \}.
\]

We define the local virtual element space \( V^E_h \) by

\[
V^E_h := \{ v_h \in H^1(E) : \forall e \in \partial E \in B^2_1(\partial E) ; \Delta v_h \in \mathcal{P}_k(E) \}
\]

and \( (v_h - R_{\ell}^h v_h, p)_E = 0, \forall p \in \mathcal{M}_{\ell}^\omega \setminus \mathcal{M}_{\ell-2}^\omega \). \]

In (Ahmad et al., 2013) it is shown that the following degrees of freedom (DoF) uniquely determine the elements of \( V^E_h \):

\[
\text{DoF}(V^E_h) := \mathcal{M}^E \cup \{ \mathcal{M}_{k-2}^\omega : \text{for each edge } e \in \partial E \} \cup \mathcal{M}_{k-2}^E.
\]
The global conforming space $V_h$ is obtained from the local spaces $V_h^E$ as

$$V_h := \{ v_h \in H^1_0(\Omega) : v_h|_E \in V_h^E, \quad \forall E \in \mathcal{T}_h \},$$

with degrees of freedom given in agreement with the local degrees of freedom (2.3).

The construction of the space for $d = 3$ is similar, although now we define the boundary space to be

$$B^3_k(\partial E) := \left\{ v \in C^0(\partial E) : v|_f \in V_h^f \text{ for each face } f \text{ of } \partial E \right\},$$

where $V_h^f$ is the two-dimensional conforming virtual element space of the same degree $k$ on the face $f$. The local virtual element space is defined to be

$$V_h^E := \{ v \in H^1(E) : v|_{\partial E} \in B^3_k(\partial E); \quad \Delta v \in \mathbb{P}_k(E); \quad \text{and } (v - R^k_h v, p)_E = 0, \forall p \in \mathcal{M}(E) \setminus \mathcal{M}_{k-2}(E) \}.$$  

with degrees of freedom

$$\text{DoF}(V_h^E) := \mathcal{N}^E \cup \{ \mathcal{M}^E_s \text{ for each edge and face } s \in \partial E \} \cup \mathcal{M}^E_{k-2}. \quad (2.4)$$

Finally, the global space and the set of global degrees of freedom for $d = 3$ are constructed from these in the obvious way, completely analogously to the case for $d = 2$.

The following are well established properties of the virtual element spaces introduced above (Beirão da Veiga et al., 2013; Ahmad et al., 2013; Cangiani et al., 2017a):

- For each $E \in \mathcal{T}_h$, we have $\mathbb{P}_k(E) \subset V_h^E$ as a subspace;
- For each $E \in \mathcal{T}_h$ and $v \in V_h^E$, the $H^1$-projector $R^k_{h,v}$ and $L^2$-projectors $P^k_{h,v}$ and $P^{k-1}_h \nabla v$ are computable just by accessing the local DoFs of $v$ given by (2.3) and (2.4) in the two and three dimensional case, respectively.
- The global virtual element space $V_h \subset H^1_0(\Omega)$ is a finite dimensional subspace.

2.2 Virtual element method

The virtual element method of order $k \geq 1$ for the discretisation of (1.1) reads: find $u_h \in V_h$ such that

$$a_h(u_h; u_h, v_h) = (P_{h}^{k-1} f, v_h), \quad \forall v_h \in V_h, \quad (2.5)$$

where $a_h(\cdot; \cdot; \cdot)$ is any bilinear form on $V_h$ defined as the sum of elementwise contributions $a^E_h(\cdot; \cdot; \cdot)$ satisfying the following assumption (Beirão da Veiga et al., 2013).

**Assumption 2.2.** For every $E \in \mathcal{T}_h$, the form $a^E_h(\cdot; \cdot; \cdot)$ is bilinear and symmetric in its second and third arguments and satisfies the following properties:

- **Polynomial consistency:** For all $p \in \mathbb{P}_k(E)$ and $v_h \in V_h^E$,

$$a^E_h(z; p, v_h) = \int_E \mathbf{k}(p) \nabla p \cdot (P_h \nabla v_h) \, dx, \quad \forall z \in L^2(E), \quad (2.6)$$

where $P_h = P^k_h$ and $P_{h}^{k-1} = P^{k-1}_h$.

- **Stability:** There exist positive constants $\alpha_*$, $\alpha^*$, independent of $h$ and the mesh element $E$, but may depend on the polynomial degree $k$, such that, for all $v_h, z_h \in V_h^E$,

$$\alpha_* a^E_h(z_h; v_h, v_h) \leq a^E_h(z_h; v_h, v_h) \leq \alpha^* a^E_h(z_h; v_h, v_h), \quad (2.7)$$

with $a^E(z; v, w) = (\mathbf{k}(z) \nabla v, \nabla w)_E$, for all $z \in L^\infty(\Omega)$ and $v, w \in H^1(\Omega)$. 
where $C$ is a positive constant which depends only on $k$. Then, using the fact that $w \in V_h$ of the form $a_h(z; \cdot, \cdot)$, for $z \in V_h$.

**Remark 2.2** The particular choice of local bilinear forms used in the numerical tests is given below in Section 5. We remark, however, that the following error analysis is valid whenever the assumption above is satisfied.

### 3. Error Analysis

We recall that $k \geq 1$ is a fixed natural number representing the order of accuracy of the method (2.5).

The convergence and a priori error analysis of the VEM relies on the availability of the following best approximation results.

#### 3.1 Approximation Properties

We recall the optimal approximation properties of the VEM space $V_h$ introduced above. These where established in a series of papers (Beirão da Veiga et al., 2013; Ahmad et al., 2013; Cangiani et al., 2017b) under the following assumption on the regularity of the decomposition $\mathcal{T}_h$.

**Assumption 3.1 (Mesh Regularity).** We assume the existence of a constant $\rho > 0$ such that

- for every element $E$ of $\mathcal{T}_h$ and every edge/face $e$ of $E$, $h_e \geq \rho h_E$
- every element $E$ of $\mathcal{T}_h$ is star-shaped with respect to a ball of radius $\rho h_E$
- for $d = 3$, every face $e \in \partial E$ is star-shaped with respect to a ball of radius $\rho h_e$

were $h_e$ is the diameter of the edge/face $e$ of $E$ and $h_E$ is the diameter of $E$.

The above star-shapedness assumption can be relaxed by including elements which are union of star-shaped domains (Beirão da Veiga et al., 2013). In particular, the following polynomial approximation result (Brenner & Scott, 2008) is extended to more general shaped elements in (Dupont & Scott, 1980) and the interpolation error bound below can be generalised by modifying the proof in (Cangiani et al., 2017b, see also (Sutton, 2017a).

**Theorem 3.2 (Approximation using polynomials)** Suppose that Assumption 3.1 is satisfied and let $s$ be a positive integer such that $1 \leq s \leq k + 1$. Then, for any $w \in H^s(E)$ there exists a polynomial $w_\pi \in P_k(E)$ such that

$$
\|w - w_\pi\|_{0,E} + h_E \|\nabla (w - w_\pi)\|_{0,E} \leq C h_E^s |w|_{s,E}.
$$

Moreover, we have

$$
\|\nabla (w - w_\pi)\|_{L^p(E)} \leq C |w|_{W^{1,p}(E)}.
$$

In the above bounds, $C$ are positive constants depending only on $k$ and on $\rho$.

The approximation properties of the virtual element space are characterised by the following interpolation error bound, whose proof can be found in (Cangiani et al., 2017b).

**Theorem 3.3 (Approximation using virtual element functions)** Suppose that Assumption 3.1 is satisfied and let $s$ be a positive integer such that $1 \leq s \leq k + 1$. Then, for any $w \in H^s(\Omega)$, there exists an element $w_I \in V_h$ such that

$$
\|w - w_I\| + h \|\nabla (w - w_I)\| \leq C h^s |w|_s
$$

where $C$ is a positive constant which depends only on $k$ and $\rho$.

Let $\mathcal{E}_h : L^2(\Omega) \times V_h \to \mathbb{R}$ denote the bilinear form

$$
\mathcal{E}_h(f,v_h) = \langle P_h^{k-1}f - f, v_h \rangle, \quad \forall v_h \in V_h.
$$

(3.1)

Then, using the fact that $P_h^{k-1}f$ is the $L^2$ projection on $P_{k-1}(E)$, we can show the following lemma.
**Lemma 3.1** For \( f \in H^s(\Omega), 0 \leq s \leq k \), there exists a positive constant \( C \), independent of \( h \) and of \( f \), such that
\[
|\varepsilon_h(f, \nabla v_h)| \leq C h^{s+1} \|f\|_{s} \|\nabla v_h\|, \quad \forall v_h \in V_h, \; j = 0, 1.
\] (3.2)

**Proof.** For \( j = 0 \), the desired estimate immediately follows from the Cauchy-Schwarz inequality and standard approximation estimates (Brenner & Scott, 2008). For \( j = 1 \), we employ the identity
\[
\int_{\Omega} (f - P_h^{k-1} f) v_h = \int_{\Omega} (f - P_h^{k-1} f) (v_h - P_h^{0} v_h),
\]
and the desired result follows similarly as before. \( \square \)

### 3.2 Existence

We first show the existence of a solution \( u_h \) of (2.5) using a fixed point argument. To this end, for \( M > 0 \), we let \( B_M = \{ v_h \in V_h : \| \nabla v_h \| \leq M \} \).

**Theorem 3.4** Let \( f \in L^2(\Omega) \) be given and assume that (1.2) holds. Choose \( M > 0 \) such that \( \|f\| \leq M c_* \), \( c_* = \kappa \alpha_* \), where \( \alpha_* \) is the lower bound constant in (2.7). Then, there exists a solution \( u_h \in B_M \subset V_h \) of (2.5).

**Proof.** We devise a fixed point iteration for (2.5): for a fixed \( f \in L^2(\Omega) \), consider an iteration map \( T_h : V_h \to V_h \) given by
\[
a_h(v_h; T_h v_h, w_h) = (P_h^{k-1} f, w_h), \quad \forall w_h \in V_h.
\] (3.3)

It is easy to see that there exists \( h_M > 0 \), such that for \( h < h_M \), \( T_h v_h \) is well defined, see for example (Cangiani et al., 2017a). For \( v_h \in B_M \) and \( w_h = T_h v_h \), in view of the stability assumption (2.7) and (3.3), we have
\[
c_* \| \nabla T_h v_h \|^{2} \leq \alpha_* a(h; T_h v_h, w_h) \leq a_h(v_h; T_h v_h, w_h) = (P_h^{k-1} f, w_h) \leq \|f\| \|w_h\|. \] (3.4)

Thus, choosing \( M \) sufficiently large, so that \( \|f\| \leq M c_* \), we get
\[
\| \nabla T_h v_h \| \leq c_*^{-1} \|f\| \leq M. \] (3.5)

Therefore, the operator \( T_h \) maps the ball \( v_h \in B_M \) into itself. By the Brouwer fixed point theorem, we know that \( T_h \) has a fixed point, which implies that (2.5) has a solution \( u_h \in B_M \). \( \square \)

### 3.3 Error bounds

In our a priori error analysis, we follow a similar-in-spirit approach to the classical work of Douglas and Dupont (Douglas & Dupont, 1975) where standard conforming finite element methods were analysed in the same context.

We start with the following preliminary \( H^1 \)-norm error bound.

**Theorem 3.5** Let \( u \in H^2(\Omega) \) be the solution of (1.1) and suppose that \( u \in H^{s}(\Omega) \cap W_{0}^{2}(\Omega), \; s \geq 2 \), assuming that \( f \in H^{s-2}(\Omega) \) and \( \kappa(u) \in W_{0}^{s-1}(\Omega) \). Then, for \( u_h \in V_h \) solution of (2.5) the following bound holds
\[
\| \nabla (u - u_h) \| \leq C (h^{-r} + \|u - u_h\|),
\] (3.6)

with \( r = \min\{s, k+1\} \) and \( C \) a positive constant independent of \( h \).

**Proof.** From Theorem 3.3, there exists a function \( u_t \in V_h \) such that \( u - u_t \) is bounded as desired. Thus, to show (3.6) it suffices to bound \( \| \nabla (u_h - u_t) \| \). Let \( \psi = u_h - u_t \), then using the stability Assumption 2.2 with \( c_* = \kappa \alpha_* \), we have
\[
c_* \| \nabla (u_h - u_t) \|^2 \leq \alpha_h(u_h; u_h - u_t, \psi)
= \varepsilon_h(f, \psi) + a(u; u, \psi) - a_h(u_h; u_h, \psi)
\]
= \varepsilon_h(f, \psi) + ((\mathbf{k}(u) - \mathbf{k}(P_h u_h)) \nabla u, \nabla \psi) + \sum_{E \in \mathcal{T}_h} a^E(P_h u_h; u - u_h, \psi) + \left\{ \sum_{E \in \mathcal{T}_h} a^E(P_h u_h; u_\pi, \psi) - a^E_h(u_h; u_\pi, \psi) \right\} + \sum_{E \in \mathcal{T}_h} a^E_h(u_h; u_\pi - u_h, \psi)

= I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.7}

where \( u_\pi \) is, on every element \( E \in \mathcal{T}_h \), the polynomial approximation of \( u \) given by Theorem 3.2. Next, we will bound the various terms \( I_i, i = 1, \ldots, 5 \). We start with \( I_1 \). Using Lemma 3.1, and the fact that \( r \leq s \), we have

\[ |I_1| \leq Ch^{r-1} \|f\|_{r-2} \|\nabla \psi\|. \tag{3.8} \]

To bound \( I_2 \), in view of (1.2), we get

\[ |I_2| \leq L \|\nabla u\|_{L_\infty} \|u - P_h u_h\| \|\nabla \psi\|. \tag{3.9} \]

Also, using the fact that \( \mathbf{k} \) is bounded along with Theorem 3.2, we obtain

\[ |I_3| \leq C \sum_E \|\nabla (u - u_\pi)\|_E \|\nabla \psi\|_E \leq Ch^{r-1} \|u\|_r \|\nabla \psi\|. \tag{3.10} \]

Using the fact that \( \nabla u_\pi \in \mathcal{P}_{k-1}(E) \) and Assumption 2.2, we have

\[ I_4 = \sum_{E \in \mathcal{T}_h} \int_E \mathbf{k}(P_h u_h) \nabla u_\pi \cdot (I - P_h) \nabla \psi \]

\[ = \sum_{E \in \mathcal{T}_h} \int_E \mathbf{k}(P_h u_h) \nabla (u_\pi - u) \cdot (I - P_h) \nabla \psi + \int_E \mathbf{k}(P_h u_h) \nabla u \cdot (I - P_h) \nabla \psi \]

\[ = \sum_{E \in \mathcal{T}_h} \int_E (\mathbf{k}(P_h u_h) - \mathbf{k}(u)) \nabla (u_\pi - u) \cdot (I - P_h) \nabla \psi + \int_E \mathbf{k}(u) \nabla (u_\pi - u) \cdot (I - P_h) \nabla \psi \]

\[ + \sum_{E \in \mathcal{T}_h} \int_E (\mathbf{k}(P_h u_h) - \mathbf{k}(u)) \nabla u \cdot (I - P_h) \nabla \psi + \int_E (I - P_h) \mathbf{k}(u) \nabla u \cdot \nabla \psi; \]

thus, in view of the stability of \( P_h \), the fact that \( \mathbf{k} \) is Lipschitz continuous, \( u \in W^1_\infty(\Omega) \), Theorem 3.2 and the hypothesis \( \mathbf{k}(u) \in W^{r-1}_\infty(\Omega) \), we deduce

\[ |I_4| \leq C \sum_{E \in \mathcal{T}_h} (\|\nabla (u - u_\pi)\|_E + \|P_h u_h - u\|_E) \|\nabla \psi\|_E + \|P_h u_h \mathbf{k}(u) \nabla u\|_E \|\nabla \psi\|_E \]

\[ \leq C (h^{r-1} \|u\|_r + \|P_h u_h - u\|) \|\nabla \psi\|. \tag{3.11} \]

Finally, we easily get

\[ |I_5| \leq C (\|\nabla (u - u_\pi)\| + \|\nabla (u - u_h)\|) \|\nabla \psi\| \leq Ch^{r-1} \|u\|_r \|\nabla \psi\|. \tag{3.12} \]

Therefore, combining the above estimates (3.8)-(3.12) with (3.7) we obtain

\[ c_\ast \|\nabla (u_h - u_1)\| \leq C (h^{r-1} + \|u - P_h u_h\|). \]

Then, in view of Theorem 3.2 and the stability of \( P_h \) in \( L^2 \)-norm, we obtain the estimate

\[ \|\nabla (u_h - u_1)\| \leq C (h^{r-1} + \|u - u_h\|). \]

Next we show two auxiliary lemmas in view of proving an \( L^2 \)-error bound.

**Lemma 3.2** Let \( u \in H^1_0(\Omega) \) be the solution of (1.1) and assume that \( u \in H^s(\Omega) \cap W^r_\infty(\Omega), s \geq 2, f \in H^{r-1}(\Omega), \mathbf{k}(u) \in W^{r-1_\infty}(\Omega) \) and \( \phi \in H^2 \cap H^1_0 \). Then, there exists a constant \( C \) independent of \( h \) such that

\[ \|u_h(u_h; u, \phi)_h^1 - a(u_h; u, \phi)_h^1\| \leq C (\|\nabla (u - u_h)\| + \|u - u_h\|^{1/2} \|\nabla (u - u_h)\|^{3/2} + h^r \|u\|_r) \|\phi\|_2, \]

where \( \phi_h^1 \in \mathcal{P}_1(E) \) for all \( E \in \mathcal{T}_h \), is given by Theorem 3.2, and \( r = \min\{s, k + 1\} \).
Proof. Let $\bar{\mathbf{x}}_\nu$ be such that

$$
\mathbf{k}(u) - \mathbf{k}(u_h) = (u - u_h) \int_0^1 \mathbf{k}_\nu(u - t(u - u_h)) \, dt = \bar{\mathbf{x}}_\nu(u - u_h). 
$$

Using polynomial consistency (2.6), the fact that $P_h \nabla u_\pi = \nabla u_\pi$, with $u_\pi \in P_h(E)$ given by Theorem 3.2 and the definition of $\bar{\mathbf{x}}_\nu$ given by (3.13), we have for all $E \in \mathcal{T}_h$

$$
a_E^E(u_h; u_h, \phi^1_E) - a^E(u_h; u_h, \phi^1_E) = \int_E \mathbf{k}(P_h u_h) (P_h - I) \nabla u_h \cdot \nabla \phi^1_E + (\mathbf{k}(P_h u_h) - \mathbf{k}(u_h)) \nabla u_h \cdot \nabla \phi^1_E \, dx
$$

$$
= \int_E \mathbf{k}(P_h u_h) (P_h - I) \nabla (u_h - u_\pi) \cdot \nabla \phi^1_E \, dx + \int_E \mathbf{h}(P_h u_h - u_h) \nabla u_h \cdot \nabla \phi^1_E \, dx
$$

$$
= \int_E \mathbf{k}(P_h u_h - u_h) (P_h - I) \nabla (u_h - u_\pi) \cdot \nabla \phi^1_E \, dx + \int_E \mathbf{k}(u) (P_h - I) \nabla (u_h - u_\pi) \cdot \nabla \phi^1_E \, dx
$$

$$
+ \int_E \mathbf{h}(P_h u_h - u_h) \nabla u_h \cdot \nabla \phi^1_E \, dx = I_1 + I_2 + I_3.
$$

Let $I = \sum I_E$, then we easily get

$$|I| \leq C \|P_h u_h - u\|_{L_2} \|\nabla \phi^1_E\|_{L_2} \|\nabla (u_h - u_\pi)\|.$$

Using Theorem 3.2, we have $\|\nabla \phi^1_E\|_{L_2} \leq C \|\phi\|_{W^{1,6}}$ and, hence, using a Sobolev imbedding,

$$\|\nabla \phi^1_E\|_{L_6} \leq C \|\phi\|_2. \quad (3.14)$$

Now, using Theorem 3.2 once again, we get

$$|I| \leq C (\|u_\pi - u_h\|^{1/2} \|\nabla (u_\pi - u_h)\|^{3/2} + h^{-1/2} \|\nabla (u_\pi - u_h)\|) \|\phi\|_2.$$

To bound $I_2$, we rewrite this term as

$$II = \int_E \mathbf{k}(u)(P_h - I) \nabla (u_h - u_\pi) \cdot \nabla \phi^1 - \phi \, dx + \int_E \mathbf{k}(u)(P_h - I) \nabla (u_h - u_\pi) \cdot \nabla \phi \, dx
$$

$$= \int_E \mathbf{k}(u)(P_h - I) \nabla (u_h - u_\pi) \cdot \nabla \phi^1 - \phi \, dx + \int_E (P_h - I)(\mathbf{k}(u) \nabla \phi) \nabla (u_h - u_\pi) \, dx
$$

Then for $II = \sum II_E$, using Theorem 3.2, it immediately follows that

$$|II| \leq Ch \|\nabla (u_h - u_\pi)\| \|\phi\|_2.$$

Next, we consider the term $III_E$, which can be rewritten as

$$III_E = \int_E (P_h u_h - u_h) \mathbf{h}\nabla (u_h - u_\pi) \cdot \nabla \phi^1_E + \nabla u_\pi \cdot \nabla \phi^1_E \, dx = III_{1,E} + III_{2,E}.$$

Then using the Hölder inequality

$$||vw|| \leq ||v||_{L_3} ||w||_{L_6}, \quad (3.15)$$

we obtain for $III_1 = \sum III_{1,E}$

$$|III_1| \leq C \|P_h u_h - u_h\|_{L_3} \|\nabla \phi^1_E\|_{L_6} \|\nabla (u_h - u_\pi)\|.$$
Further, using the stability property of $P_h$, namely $\|P_h \phi\|_{L^2(E)} \leq \tilde{C}\|\phi\|_{L^2(E)}$, with $\tilde{C} > 0$ independent of $E$ and the Gagliardo–Nirenberg–Sobolev inequality
\[
\|v\|_{L^3} \leq C\|v\|^{1/2}\|\nabla v\|^{1/2},
\]
we obtain
\[
\|P_h u_h - u_h\|_{L^3} \leq C\|u_h - u_h\|^{1/2}\|\nabla(u_h - u_h)\|^{1/2}.
\]
Then, in view of (3.14), we get
\[
|\text{III}_1| \leq C\|u_h - u_h\|^{1/2}\|\nabla(u_h - u_h)\|^{3/2}\|\phi\|_2.
\]

Next, in view of the fact that $\nabla u_h \cdot \nabla \phi_h \in \mathcal{P}_k(E)$, we have
\[
\text{III}_{E,2} = \int_E (P_h u_h - u_h)(\tilde{K}_u - c)\nabla u_h \cdot \nabla \phi_h \, dx, \quad \forall c \in \mathbb{R}.
\]
Thus, for $\text{III}_2 = \sum_k \text{III}_{E,2}$, we get
\[
|\text{III}_2| \leq Ch\|u_h - P_h u_h\|_{L^3}\|\nabla \phi_h\|_{L^6}\|\nabla u_h\|.
\]
Therefore, Theorem 3.2, and the Sobolev inequalities (3.16), (3.14), give
\[
|\text{III}_2| \leq Ch\|u_h - u_h\|^{1/2}\|\nabla(u_h - u_h)\|^{1/2}\|\phi\|_2.
\]
Collecting the above bounds, yields for $\text{III} = \text{III}_1 + \text{III}_2$
\[
|\text{III}| \leq C(h\|u_h - u_h\|^{1/2}\|\nabla(u_h - u_h)\|^{1/2} + \|u_h - u_h\|^{1/2}\|\nabla(u_h - u_h)\|^{3/2})\|\phi\|_2.
\]

Therefore
\[
|a_h(u_h; u_h, \phi_h) - a_h(u_h; u_h, \phi_h)| \leq C(h\|\nabla(u - u_h)\| + \|u - u_h\|^{1/2}\|\nabla(u - u_h)\|^{3/2} + h'|u|_r + h' ||f||_{r-1})\|\phi\|_2,
\]
from which the desired bound follows using once again Theorem 3.2. \hfill \Box

**Lemma 3.3** Let $u \in H^s_0(\Omega)$ be the solution of (1.1) and assume that $u \in H^s(\Omega) \cap W^s_{\text{loc}}(\Omega)$, $s \geq 2$, $f \in H^{s-1}(\Omega)$, $K(u) \in W^s_{\text{loc}}(\Omega)$ and $\phi \in H^s_0(\Omega) \cap H^2(\Omega)$. Then there exists a positive constant $C$ independent of $h$ such that
\[
|a(u; u, \phi) - a(u; u, \phi)| \leq C(h\|\nabla(u - u_h)\| + \|u - u_h\|^{1/2}\|\nabla(u - u_h)\|^{3/2} + h'|u|_r + h' ||f||_{r-1})\|\phi\|_2,
\]
where $r = \min\{s, k + 1\}$.

**Proof.** Let $\phi_l \in V_h$ be the approximation of $\phi$ given by Theorem 3.3 and using (1.3) and (2.5) we split the difference $a(u; u, \phi) - a(u; u, \phi)$ as
\[
a(u; u, \phi) - a(u; u, \phi) = \{a(u; u, \phi) - a(u; u, \phi_l)\} + \{a(u; u, \phi_l) - a(u; u, \phi_l)\} = I + II + III.
\]

Then, in view of (3.13), we rewrite term $I$ as
\[
I = (K(u)u)\nabla(u - u_h) + (K(u) - K(u_h))\nabla(u - u_h)\nabla(\phi - \phi_l)
\]
\[
= (K(u)u)\nabla(u - u_h) + K_u(u - u_h)\nabla(\phi - \phi_l).
\]

Employing Theorem 3.3 and (3.25), we obtain
\[
|I| \leq Ch\|\nabla(u - u_h)\| + \|u - u_h\|\|\nabla u_h\|\|\phi\|_2 \leq Ch\|\nabla(u - u_h)\|\|\phi\|_2.
\]
As for term II, using Lemma 3.1 we get

\[ |II| \leq C h^{r-1} \| \nabla \phi_I \| \leq C h^{r} \| f \|_{r-1} \| \phi \|_2. \quad (3.19) \]

In view of bounding term III, we write

\[ III = \{ a_h(u_h; u_h - u_h, \phi_I - \phi^1_h) - a(u_h; u_h - u_h, \phi_I - \phi^1_h) \} \]
\[ + \{ a_h(u_h; u_h, \phi_I - \phi^1_h) - a(u_h; u_h, \phi_I - \phi^1_h) \} + \{ a_h(u_h; u_h, \phi^1_h) - a(u_h; u_h, \phi^1_h) \} \]
\[ = III_1 + III_2 + III_3, \quad (3.20) \]

with \( \phi^1_h \in \mathbb{P}_1(E) \) and \( u_h \in \mathbb{P}_k(E) \), for any \( E \in \mathcal{T}_h \) given by Theorem 3.2. Using Theorems 3.2 and 3.3, we bound the term \( III_1 \) in (3.20) as

\[ |III_1| \leq C h \| \nabla (u_h - u_h) \| \| \phi_I \|_E \leq C (h^{r} \| \nabla (u_h - u_h) \| + h^{r-1} \| u \|_r) \| \phi_I \|_E. \]

Next, to estimate \( III_2 \), we split this term as a summation over each \( E \in \mathcal{T}_h \) and use the polynomial consistency (2.6) and the definition of \( k_n \), given by (3.13), to get

\[ a_h^E(u_h; u_h, \phi_I - \phi^1_h) - a_h^E(u_h; u_h, \phi_I - \phi^1_h) \]
\[ = \int_E (k(P_h u_h) \nabla \phi_I - \phi^1_h) - k(u_h) \nabla \phi_I - \phi^1_h) \nabla u_h \cdot \nabla (\phi_I - \phi^1_h) \]
\[ = \int_E (k(P_h u_h) \nabla \phi_I - \phi^1_h - (k(P_h u_h) - k(u_h)) \nabla \phi_I - \phi^1_h) \nabla u_h \cdot \nabla (\phi_I - \phi^1_h) \]
\[ = III_2^1 + III_2^2. \]

Then, following the steps used in the estimation of \( I_4 \) in (3.11) and using Theorems 3.2 and 3.3, we can see that

\[ |III_2^1| \leq C h^{r-1} \| u \|_{r,E} + \| P_h u_h - u \|_E \| \phi_I \|_2, \quad (3.21) \]

To bound \( III_2^2 \), we first note, in view of (3.15), that

\[ |III_2^2| \leq C \| P_h u_h - u_h \|_{L_1(E)} \| \nabla u_h \|_{L_1(E)} \| \nabla (\phi_I - \phi^1_h) \|_E. \quad (3.22) \]

Further, using the stability property of \( P_h \), namely \( \| P_h \phi_I \|_{L_1(E)} \leq C \| \phi_I \|_{L_1(E)} \), and the Gagliardo–Nirenberg–Sobolev inequality (3.16), we obtain

\[ \| P_h u_h - u_h \|_{L_1(E)} \leq C \| u_h - u_h \|_{L_1(E)}^{1/2} \| \nabla (u_h - u_h) \|_{L_1(E)}^{1/2}, \quad (3.23) \]

with \( C, C > 0 \) independent of \( E \). Using this in (3.22) and summing this new bound of (3.22) and (3.21) over all \( E \in \mathcal{T}_h \) and using Theorems 3.2 and 3.3, it follows that

\[ |II H_2| \leq C h \| \nabla (u_h - u_h) \| + \| P_h u_h - u \|_E \| \phi_I \|_E. \]

Finally, as a consequence of Lemma 3.2 below, we have

\[ |II H_3| \leq C \| \nabla (u_h - u_h) \|^{1/2} \| \nabla (u_h - u_h) \|^{3/2} + h^{r} \| u \|_r \| \phi_I \|_E. \]

Combining this with (3.19), the bounds for \( III_1 \) and \( II H_2 \), the desired bound follows.

We are now in a position to prove the following preliminary \( L^2 \)-norm, error bound.

**Theorem 3.6** Let \( u \in H^1_0(\Omega) \) be the solution of (1.1) and assume that \( u \in H^s(\Omega) \cap W^{k+1}_0(\Omega) \), \( s \geq 2, \)
\( f \in H^{r-1}(\Omega) \) and \( k(u) \in W^{r-1}_0(\Omega) \), with \( \Omega \) convex. Then, for \( h \) small enough and \( u_h \in V_h \) solution of (2.5) the following bound holds

\[ \| u - u_h \| \leq C (h^r + \| u - u_h \|^3), \quad (3.24) \]

where \( r = \min \{ s, k+1 \} \) and \( C \) is a positive constant independent of \( h \).
Proof. We use a duality argument. Consider the (linear) auxiliary problem: find \( \phi \in H^1_0(\Omega) \) such that
\[
-\text{div}(\kappa(u)\nabla\phi) + \kappa_u(u)\nabla u \cdot \nabla \phi = u - u_h.
\]
Noting that this equates to \( \kappa(u)\Delta\phi = u - u_h \) and since we have assumed that \( \Omega \) is convex, we have \( \phi \in H^2(\Omega) \) and
\[
\|\phi\|_2 \leq C\|u - u_h\|. \tag{3.25}
\]
In variational form, the above problem reads
\[
(\kappa(u)\nabla\phi, \nabla v) + (\kappa_u(u)\nabla u \cdot \nabla \phi, v) = (u - u_h, v), \quad \forall v \in H^1_0(\Omega), \tag{3.26}
\]
Then choosing \( v = u - u_h \) in (3.26)
\[
\|u - u_h\|_2^2 = (\kappa(u)\nabla u, \nabla (u - u_h)) + (\kappa_u(u)(u - u_h)\nabla u, \nabla \phi)
= (\kappa(u)\nabla u, \nabla \phi) - (\kappa_u(u)\nabla u, \nabla \phi) - ((\kappa(u) - \kappa_u(u))\nabla u_h, \nabla \phi)
+ (\kappa_u(u)(u - u_h)\nabla u, \nabla \phi)
= (\kappa(u)\nabla u, \nabla \phi) - (\kappa_u(u)\nabla u, \nabla \phi) + ((\kappa(u) - \kappa_u(u))\nabla u - \kappa_u(u)(u - u_h)\nabla u, \nabla \phi)
= (a(u; u, \phi) - a(u_h; u, \phi))
+ ((\kappa_u(u)(u - u_h)\nabla (u - u_h), \nabla \phi) - ((\kappa_{uu}(u - u_h)^2\nabla u, \nabla \phi) =: I + II, \tag{3.27}
\]
with \( \kappa_u \) given by (3.13) and \( \kappa_{uu} \) such that
\[
\kappa(u) - \kappa(u_h) - \kappa_u(u)(u - u_h) = (u - u_h)^2 \int_0^1 \kappa_{uu}(u - (u - u_h)) \, dt \equiv \kappa_{uu}(u - u_h)^2. \tag{3.28}
\]
In the sequel we will show Lemma 3.3, which in view of (3.25), gives
\[
|I| \leq C(h)\|\nabla (u - u_h)\| + \|u - u_h\|^{1/2}\|\nabla (u - u_h)\|^{3/2} + h^r\|\phi\|_r + h^{r-1}\|\phi\|_{r-1}\|u - u_h\|. \tag{3.29}
\]
For \( II \) in (3.27), using the Hölder inequality (3.15) and the fact that \( \kappa_u, \kappa_{uu} \) are bounded uniformly on \( \mathbb{R} \), we get
\[
|II| \leq C\|\nabla (u - u_h)\| \|u - u_h\|\|\nabla \phi\| + C\|\nabla (u - u_h)\| \|u - u_h\|\|\nabla \phi\|
\leq C\|\nabla (u - u_h)\| \|u - u_h\|_{L_3} \|\nabla \phi\|_{L_6} + C\|\nabla (u - u_h)\|_{L_3} \|\nabla \phi\|_{L_6}.
\]
Next, in view of the Gagliardo–Nirenberg–Sobolev inequality (3.16), the Sobolev Imbedding Theorem and the elliptic regularity (3.25), we have
\[
|II| \leq C\|\nabla (u - u_h)\|^{3/2} \|u - u_h\|^{1/2} \|u - u_h\| + C\|\nabla (u - u_h)\| \|u - u_h\| \|u - u_h\| \|u - u_h\|
\leq C\|\nabla (u - u_h)\|^{3/2} \|u - u_h\|^{1/2} \|u - u_h\|. \tag{3.30}
\]
Combining the previous estimates for terms \( I \) and \( II \), we obtain
\[
\|u - u_h\| \leq Ch\|\nabla (u - u_h)\| + Ch'\|\phi\|_r + Ch^r\|\phi\|_{r-1} + C\|\nabla (u - u_h)\|^3 + \frac{1}{2}\|u - u_h\|,
\]
from which, in view of Theorem 3.5, we conclude that
\[
\|u - u_h\| \leq Ch\|\nabla (u - u_h)\| + Ch'\|\phi\|_r + Ch^r\|\phi\|_{r-1} + C\|u - u_h\|^3.
\]
The desired bound now follows for \( h \) sufficiently small. \( \square \)

Having concluded the proof of Theorem 3.6, in order to show optimal convergence rate of the error in \( H^1 \) and \( L^2 \)-norms, it remains to demonstrate that \( u_h \) converge to \( u \).
Theorem 3.7 Under the same assumptions as in Theorems 3.5 and 3.6, the VEM solution \( u_h \) converges to the exact solution \( u \) in \( H^1_0(\Omega) \).

Proof. From Theorem 3.4 it follows that \( \| \nabla v_h \| \) is bounded from above. Therefore, we can choose a subsequence \( u_{h_k} \) such that for some \( z \in H^1_0(\Omega) \), \( u_{h_k} \to z \), weakly in \( H^1_0(\Omega) \), as \( h_k \to 0 \) and, thus, strongly in \( L^2(\Omega) \). Also, for arbitrary \( v \in C_0^\infty(\Omega) \) let \( v_{h_k} \) be a sequence in \( V_{h_k} \) such that

\[
\| \nabla (v - v_{h_k}) \| \to 0, \quad h_k \to 0. \tag{3.31}
\]

Then

\[
|a(z; v) - (f, v)| \leq |(\mathbf{k}(z) \nabla v, \nabla (v - v_{h_k}))| + |(\mathbf{k}(z) \nabla v, \nabla (v - v_{h_k}))| + |(P_h^{k-1} f, v - v_{h_k})| + \|v_{h_k}(f, v)\|
\]

\[
\leq C\|v - v_{h_k}\| + |(\mathbf{k}(z) \nabla v, \nabla (v - v_{h_k}))| - a_h(u_{h_k}; u_{h_k}, v_{h_k}) + Ch_h \|f\| \|v\|.
\]

Thus, if

\[
|(\mathbf{k}(z) \nabla v, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| \to 0, \quad h_k \to 0, \tag{3.32}
\]

then \( z \) is the weak solution of (1.1). To show (3.32), we rewrite its left-hand side as

\[
|(\mathbf{k}(z) \nabla v, \nabla v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| = |(\mathbf{k}(z) \nabla v, \nabla v_{h_k})| + |(\mathbf{k}(u_{h_k}) \nabla u_{h_k}, v_{h_k}) - a_h(u_{h_k}; u_{h_k}, v_{h_k})| + Ch_h \|f\| \|v\|.
\]

Using the fact that \( u_{h_k} \to z \) and \( v_{h_k} \to v \), we see that (3.32) holds. Hence \( a(z; v) = (f, v) \), and thus \( u = z \), since \( u \) is the unique solution of (1.1). Then, it follows that \( u_h \to u \) in \( L^2(\Omega) \). Hence, \( \|u - u_h\| \to 0 \) and the result follows from Theorems 3.6, and 3.5.

In view of Theorems 3.5, 3.6 and 3.7, the following a priori error estimates now readily follows.

Theorem 3.8 Let \( u \in H^1_0(\Omega) \) be the solution of (1.1) and suppose that \( u \in H^s(\Omega) \cap W^{1,s}_0(\Omega) \), \( s \geq 2 \), assuming that \( f \in H^{s-1}(\Omega) \) and \( \mathbf{k}(u) \in W_0^{s-1}(\Omega) \), with \( \Omega \) convex. Let also \( u_h \in V_h \) be the solution of (2.5). Then, there exists a constant \( C \) independent of \( h \) such that, for \( h \) sufficiently small,

\[
\|u - u_h\| + h \|\nabla (u - u_h)\| \leq Ch^r, \tag{3.33}
\]

where \( r = \min\{k + 1, s\} \).

4. Iteration method

In this section we show that, given a virtual element space \( V_h \), the sequence of solutions we obtain using fixed point iterations to solve the VEM problem (2.5) converges to the true solution \( u_h \in V_h \) of (2.5).

Starting with a given \( u_0 \in V_h \) we construct a sequence \( u_n \), \( n \geq 0 \), such that

\[
a_h(u_n; u_{n+1}, v_h) = (P_h^{k-1} f, v_h), \quad \forall v_h \in V_h. \tag{4.1}
\]

The convergence in \( H^1 \) of the sequence \( u_n \) as \( n \to \infty \) to a fixed point of (4.1), and hence a solution of (2.5), is an immediate consequence of the following result.

Theorem 4.1 Let \( \{u_n\} \subset V_h \) be the sequence produced in (4.1), then

\[
\|\nabla (u_n - u_{n+1})\| \to 0, \quad as \ n \to \infty. \tag{4.2}
\]

Proof. In view of Assumption 2.2 and the fact that \( a_h(u_n; \cdot, \cdot) \) is symmetric, we have

\[
c_\ast \|\nabla (u_n - u_{n+1})\|^2 \leq a_h(u_n; u_n - u_n^{n+1}, u_n - u_{n+1}^{n+1}) = a_h(u_n; u_n, u_n^{n+1}) - 2a_h(u_n, u_n^{n+1}, u_n) + a_h(u_n; u^{n+1}, u^{n+1}, u_{n+1}^{n+1}). \tag{4.3}
\]
with \( c = \mathbf{K} \cdot \alpha \). Then using (4.1), we obtain
\[
a_h(u^n_h; u^{n+1}_h, v_h) = (P^{k-1}_h f, u^n_h - u^{n+1}_h) + a_h(u^n_h; u^{n+1}_h, u^n_h),
\]
giving
\[
c \| \nabla(u^n_h - u^{n+1}_h) \|^2 \leq a_h(u^n_h; u^n_h, u^n_h) - 2(P^{k-1}_h f, u^n_h - u^{n+1}_h) - a_h(u^n_h; u^{n+1}_h, u^n_h) = \mathcal{F}(u^n_h) - \mathcal{F}(u^{n+1}_h),
\]
where \( \mathcal{F}(v) = a_h(u^n_h; v, v) - 2(P^{k-1}_h f, v) \). Therefore, \( \mathcal{F}(u^n_h) \) is a decreasing sequence and, in view of the fact that
\[
\mathcal{F}(v) = a_h(u^n_h; v, v) - 2(P^{k-1}_h f, v) \geq \mathbf{K} \cdot \| \nabla v \|^2 - 2 \| f \| \| \nabla v \| \geq -\| f \|^2 / \mathbf{K},
\]
\( \mathcal{F}(u^n_h) \) is bounded from below. Therefore \( \mathcal{F}(u^n_h) - \mathcal{F}(u^{n+1}_h) \to 0 \), as \( n \to \infty \), which completes the proof.

5. Numerical results

In order to test the VEM proposed in Section 2 we need to specify a bilinear form satisfying Assumption 2.2. We fix \( d^E_h \) as follows:
\[
a^E_h(z_h; v_h, w_h) = \int_E \mathbf{K}(P^0_h z_h)(P_h \nabla v_h) : (P_h \nabla u^n_h) \, d\mathbf{x} + S^E(z_h; (I - P_h) v_h, (I - P_h) w_h),
\]
with the VEM stabilising form \( S^E \) given by
\[
S^E(z_h; (I - P_h) v_h, (I - P_h) w_h) := \mathbf{K}^E(P^0_h z_h) [h^E_1(P_h \nabla v_h) - (I - P_h) v_h] \cdot (I - P_h) w_h.
\]
here, \( I \) denotes the identity operator, \( \nabla v_h \) is the vector with entries the degrees of freedom of \( v_h \in V^E_h \), and \( \overrightarrow{\nabla v_h} \) is the euclidean scalar product of the degrees of freedom of \( v_h, w_h \in V^E_h \).

The above definition of the local bilinear form extends to the nonlinear setting the one considered in (Cangiani et al., 2017a) and, similarly to the linear case, it is straightforward to show that it satisfies the stability condition (2.7). Following (Beirão da Veiga et al., 2013) instead, the projector \( R^E_h \) can be used in place of \( P_h \) in the stabilising term. The practical implementation of these projector operators and VEM assembly are discussed in (Beirão da Veiga et al., 2014; Cangiani et al., 2017a).

In the examples below, approximation errors are measured by comparing the piecewise polynomial quantities \( P^k_h u_h \) and \( P^{k-1}_h \nabla u_h \) with the exact solution \( u \) and solution’s gradient \( \nabla u \), respectively.

The tests are performed using the VEM implementation within the Distributed and Unified Numerics Environment (DUNE) library (Blatt et al., 2016), available from (Cangiani et al., 2019).

We use fixed point iterations analysed in Section 4 to solve the nonlinear system resulting from the VEM discretisation. This is compared below with Newton-Raphson iterations, defined as follows. Given an initial iterate \( u^n_h \in V_h \), we construct a sequence \( u^{n+1}_h = u^n_h + \delta^n, n \geq 0 \), by solving at each iteration the linearised problem: find \( \delta^n \in V_h \) such that
\[
a_h(u^n_h, \delta^n, v_h) + b_h(u^n_h, \delta^n, v_h) = (P^{k-1}_h f, v_h) - a_h(u^n_h; u^n_h, v_h), \quad \forall v_h \in V_h.
\]
Here, the extra terms stemming from the linearisation of both the consistency and stability terms in \( a_h \) are collected in the global form \( b_h := \sum_{E \in \mathcal{T}_h} b^E_h \), with the local form \( b^E_h \), \( E \in \mathcal{T}_h \), given by
\[
b^E_h(u^n_h, \delta^n, v_h) = \int_E \mathbf{K}_n(P^0_h u^n_h) P_h \delta^E \cdot (P_h \nabla u^n_h) \cdot (P_h \nabla v_h) \, d\mathbf{x} + b^{E-2}_h \mathbf{K}_n(P^0_h u^n_h) P_h \delta^E \cdot (P_h \nabla u^n_h - (I - P_h) v_h).
\]

**Numerical test 1.** We consider the following test problem from (Chatzipantelidis et al., 2005). We solve (1.1) on \( \Omega = (0, 1)^2 \) with \( \mathbf{K}(u) = 1/(1 + u)^2 \) and the function \( f \) chosen such that the exact solution is \( u = (x - x^2)(y - y^2) \). Note that, although the diffusion coefficient is not even bounded on the whole
of \( \mathbb{R} \), it is smooth in a neighbourhood of the range of \( u \). As initial guess for the nonlinear solve we use the constant zero function and the conjugate-gradient method is used to solve the linear system at each iteration. The relative errors for the approximation of \( u \) and its gradient as a function of the mesh size \( h \) are shown in Table 5 for \( k = 1 \) and a sequence of polygonal meshes generated using (Talischi et al., 2012), cf. the right-most plot in Figure 1. The numerical results confirm the theoretical rate of convergence. The relative errors for the approximation of \( u \) and its gradient as a function of the mesh size \( h \) are shown in Table 5 for \( k = 1 \) and a sequence of polygonal meshes generated using (Talischi et al., 2012), cf. the right-most plot in Figure 1. The numerical results confirm the theoretical rate of convergence.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{DOF} & \|u - P_h^k u_h\| & \text{EOC} & \|\nabla u - P_h^{k-1} \nabla u_h\| & \text{EOC} & \text{FP} & \text{NR} \\
\hline
9 & 1.30E-02 & \text{--} & 9.44E-02 & \text{--} & 6 & 4 \\
34 & 3.40E-03 & 2.018 & 4.96E-02 & 0.967 & 7 & 4 \\
129 & 8.16E-04 & 2.140 & 2.51E-02 & 1.022 & 6 & 4 \\
510 & 1.89E-04 & 2.131 & 1.25E-02 & 1.012 & 6 & 4 \\
2042 & 4.49E-05 & 2.070 & 6.26E-03 & 1.001 & 6 & 3 \\
8162 & 1.11E-05 & 2.011 & 3.12E-03 & 1.006 & 6 & 3 \\
\hline
\end{array}
\]

Table 1. Numerical test 1. Errors and empirical order of convergence (EOC) on a sequence of polygonal meshes. The Fixed Point (FP) and Newton-Raphson (NR) iterations needed to reach the tolerance \( 10^{-10} \) are reported in the right-most columns.

The convergence history with respect to all meshes in Figure 1 are reported in the loglog plots of Figure 2 showing that the performance is similar in all cases. Note that, as \( k = 1 \), in the case of the sequence of triangular meshes, the VEM coincides with the standard linear finite element method.
**Numerical test 2.** We consider a test problem with smooth diffusion coefficient proposed in (Bi & Ginting, 2007). Namely, we solve (1.1) on $\Omega = (0,1)^2$ with $\kappa(u) = 1 + 1/(1 + u^2)$ and the function $f$ chosen such that the exact solution is $u = \sin(3\pi x)\sin(3\pi y)$. We use the same initial guess and linear solver as in the first test, but only consider Newton-Raphson iterations this time. We test the VEM of order $k = 1$ up to 4 on a sequence of Voronoi meshes generated from random seeds exemplified in Figure 3. The convergence history reported in Figure 4 confirms the theoretical results. The slightly unsettled behaviour of some of the convergence curves is due to the uneven size of the mesh elements of Voronoi meshes. Another characteristic of Voronoi meshes is that mesh edges can be very small with respect to the element’s diameter. Hence this test confirms, in the quasilinear setting, the well-known robustness of the VEM with respect to mesh quality, even though we do not consider here the refined methods of (Beirão Da Veiga et al., 2017; Brenner & Sung, 2018).

![Fig. 3. Sample mesh from the Voronoi sequence used in numerical tests 2 and 3.](image)

**Numerical test 3.** The following test problem was proposed in (Chatzipantelidis et al., 2005). We solve (1.1) on $\Omega = (0,1)^2$ with $\kappa(u) = 1 + u$ and the forcing $f$ chosen such that the exact solution is $u = x^{1.6}$. This solution belongs to $H^2(\Omega)$ but not to $H^3(\Omega)$ and the source term is in $L^2(\Omega)$ only. We employ the same solution settings as for numerical test 2, including the same sequence of Voronoi meshes and, given the low regularity of the solution, we only consider $k = 1, 2$. In all cases, 3 Newton-Raphson iterations were needed to reach the tolerance $10^{-10}$. The respective convergence histories are reported in Figure 5. As expected, the rate of convergence does not increase for $k = 2$ for this non-smooth problem.

![Fig. 4. Numerical test 2. Convergence history for $k = 1, 2, 3, 4$ on a sequences of Voronoi meshes with random seeds.](image)
The results for $k = 1$ can be compared to those obtained with the similar order Finite Volume Element Method of (Chatzipantelidis et al., 2005) on structured triangular meshes. Although we have employed here the more irregular Voronoi meshes, the two methods give very similar results.

**Numerical test 4.** The following test problem is similar to a problem proposed in (Bi & Ginting, 2011). We solve (1.1) on $\Omega = (0,1)^2$ with $\kappa(u) = 1 - 0.9\sin(8\pi u)$ and the forcing $f$ chosen such that, as in numerical test 1, the exact solution is $u = (x - x^2)(y - y^2)$. Note that the diffusion coefficient is characterised by the oscillatory behaviour and may reach close to zero. We employ the same solution settings as for numerical test 2, including the same sequence of Voronoi meshes and $k = 1, 2, 3, 4$. In all computations, either 4 or 5 Newton-Raphson iterations were necessary to reach the tolerance of $10^{-10}$ starting from the initial guess $u = 0$. The convergence history is reported in Figure 6, once more confirming the theoretical rate of convergence. And we observe that, given that the solution is a simple polynomial, in the last iteration with $k = 4$ the $L^2$-norm error convergence is slowed down as the error has reached the Newton-Raphson tolerance.

6. Conclusions

With this paper, we propose a VEM for elliptic quasilinear problems with Lipschitz continuous diffusion in
two and three dimensions, showing that it suffices to evaluate the diffusion coefficient with the component of the VEM solution which is readily accessible. We prove optimal order a priori error estimates under the same mesh assumptions used in the linear setting.

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