The engine that powers quantum cryptography is the principle that there are no physical means for gathering information about the identity of a quantum system’s state (when it is known to be prepared in one of a set of nonorthogonal states) without disturbing the system in a statistically detectable way. This situation is often mistakenly described as a consequence of the “Heisenberg uncertainty principle.” A more accurate account is that it is a unique feature of quantum phenomena that rests ultimately on the Hilbert space structure of the theory along with the fact that time evolutions for isolated systems are unitary. In this paper we shall explore several aspects of the information/disturbance principle in an attempt to make it firmly quantitative and flesh out its significance for quantum theory as a whole.

1 Introduction and Model

Suppose an observer obtains a quantum system secretly prepared in one of two nonorthogonal pure quantum states. Quantum theory dictates that there is no measurement he can use to certify which of the two states was actually prepared. This is well known [1, and refs]. A simple, but less recognized, corollary is that no interaction used for performing such an information-gathering measurement can leave both states unchanged in the process [2]. If the observer could completely regenerate the unknown quantum state after measurement, then—by making further nondisturbing information-gathering measurements on it—he would be able eventually to infer the state’s identity after all.

This consistency argument is enough to establish a tension between information gain and disturbance in quantum theory. What it does not capture, however, is the extent of the tradeoff between these two quantities. In this talk, we shall lay the groundwork for a quantitative study that goes beyond the qualitative nature of this tension. Namely, we will show how to capture in a formal way the idea that, depending upon the particular measurement interaction, there can be a tradeoff between the disturbance of the quantum states and the acquired ability to make inferences about their identity. We shall also explore the extent to which the very existence of this tradeoff can be taken as a fundamental principle of quantum theory—one from which unitary time evolution itself can be derived.

The model we shall base our considerations on is most easily described in terms borrowed from quantum cryptography, from whence it takes its origin. Alice randomly prepares a quantum system to be in either a state $\hat{\rho}_0$ or a state $\hat{\rho}_1$. These states will most generally be described by $N \times N$ density operators on an $N$-dimensional Hilbert space, $N$ arbitrary; there is no restriction that they be pure states, orthogonal, or commuting for that matter. After the preparation, the quantum system is passed into a “black box” where it may be probed by an eavesdropper Eve in any way allowed by the laws of quantum mechanics. That is to say, Eve may first allow the system to interact with an auxiliary system, or ancilla, and then perform quantum mechanical measurements on the ancilla itself [3]. The outcome of such a measurement may provide Eve with some information about the quantum state and may even provide her a basis on which to make an inference as to the state’s identity. Upon this man-handling by Eve, the original quantum system is passed out of the “black box” and into the possession of a third person Bob.

A crucial aspect of this model is that even if Bob knows the state actually prepared by Alice and, furthermore, the manner in which Eve operates, because the system will have become entangled with Eve’s ancilla, he will still have to resort to a new description of the quantum system after it emerges from the “black box”—say by some $\hat{\rho}_0'$ or $\hat{\rho}_1'$. This is where the detail of our work begins. Eve now has the potential to gather information about the quantum state; meanwhile that state is no longer a valid description of Bob’s system because it has become entangled with Eve’s ancilla.

The ingredients required to formally pose an information/disturbance tradeoff principle follow from the details of the model. We shall need:

A) a concise account of all ancillas and interactions that

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Eve may use to obtain evidence about the identity of the state

B) a convenient description of the most general kind of quantum measurement she may then perform on her ancilla

C) a measure of the information or inference power provided by any given measurement,

D) a good notion by which to measure the distinguishability of mixed quantum states and a measure of disturbance based on it, and finally

E) a “figure of merit” by which to compare the disturbance with the inference.

In the next section, we shall carry this program out in some detail for two pure quantum states to give a flavor of how it should go in the general case. However, before pursuing this, perhaps we should say a word or two about the significance of these ideas. It is often said that it is the Heisenberg uncertainty relations that dictate that quantum measurements necessarily disturb the measured system. To the extent that this statement can be given a precise meaning, it is rather beside the point in this context. The Heisenberg relations concern the inability to get hold of two classical observables simultaneously, such as spin in the $x$ direction and spin in the $y$ direction. Thus it concerns our inability to ascribe classical states of motion to quantum mechanical systems. This, in and of itself, generally has very little to do with the limits of what can happen to a quantum state when information is gathered about its identity. The theme of this approach differs from that used in formulating the Heisenberg relations in that it makes no reference to conjugate or complementary variables. The only elements entering these considerations are related to the quantum states themselves. In this way one can get at a notion of state disturbance that is purely quantum mechanical, making no reference to the sort of semi-classical considerations that drove the early conceptual work on the theory’s interpretation.

What does it really mean to say that the states are disturbed in and of themselves without reference to classical variables? It means quite literally that Alice faces a loss of predictability about the outcomes of Bob’s measurements whenever an information gathering eavesdropper intervenes. Take as an example the case where $\rho_0$ and $\rho_1$ are nonorthogonal pure states. Then, for each of these, there exists one nontrivial observable for which Alice can predict the outcome with complete certainty—namely the projectors parallel to $\rho_0$ and $\rho_1$, respectively. However, after Alice’s quantum states pass into the “black box” occupied by Eve, neither Alice nor Bob will any longer be able to predict with complete certainty the outcomes of both these measurements. This is the real content of these ideas.

### 2 Pure States

Let us now carry out the last example in greater detail. Consider the case where Alice prepares, with equal probability, one of the following nonorthogonal states on a two-dimensional Hilbert space $\mathcal{H}_A$:

$$
|0\rangle = \cos \alpha |a_0\rangle + \sin \alpha |a_1\rangle \tag{1}
$$

$$
|1\rangle = \sin \alpha |a_0\rangle + \cos \alpha |a_1\rangle \tag{2}
$$

where $|a_0\rangle$ and $|a_1\rangle$ form an orthonormal basis on the space. A good measure of the nonorthogonality of these states is given by their inner product:

$$
S = \langle 0 | 1 \rangle = \sin 2\alpha \tag{3}
$$

Eve, in an attempt to gather information about the identity of the state Alice prepared, will interact Alice’s system with an ancilla described by the states in a Hilbert space $\mathcal{H}_E$. This interaction takes its formal description as a unitary operator $U$ on $\mathcal{H}_A \otimes \mathcal{H}_E$. Supposing the ancilla starts off in some standard pure state $|\psi\rangle$, we can describe the outcome of this interaction as follows:

$$
|s\rangle|\psi\rangle \rightarrow |\Psi_s^{AE}\rangle = \sum_{n=0}^{1} \sqrt{\lambda_n} |A_n^s\rangle |E_n^s\rangle \tag{4}
$$

$s = 0, 1$, and where $|\Psi_s^{AE}\rangle$ is written in Schmidt polar form for orthonormal bases (parameterized by Alice’s state $s$) $|A_n^s\rangle$ and $|E_n^s\rangle$ on $\mathcal{H}_A$ and $\mathcal{H}_E$ respectively. The bases and the constants $\lambda_n^s$ will of course depend on the particular unitary operation $U$ used. The state of the system finally left in Eve’s possession is given by tracing $\mathcal{H}_A$ out of the picture, i.e., the mixed state

$$
\rho_s^E = \text{tr}_A (|\Psi_s^{AE}\rangle \langle \Psi_s^{AE}|) = \lambda_0^s |E_0^s\rangle \langle E_0^s| + \lambda_1^s |E_1^s\rangle \langle E_1^s|. \tag{5}
$$

Similarly the states passed on to Bob are

$$
\rho_s^A = \text{tr}_E (|\Psi_s^{AE}\rangle \langle \Psi_s^{AE}|) = \lambda_0^s |A_0^s\rangle \langle A_0^s| + \lambda_1^s |A_1^s\rangle \langle A_1^s|. \tag{6}
$$

Eq. (5) tells us immediately that we never need consider ancillas for Eve with Hilbert space dimension greater than four $\mathbb{F}$. For, at most all four of the $|E_n^s\rangle$ can be linearly independent vectors.

With this much as set up, how can we best gauge the tradeoff between the disturbance Eve has caused in Alice’s system and the information or inference power she has gained? There are several ways to go about this $\mathbb{F}$. For specificity here, we assume that disturbance is gauged by the average probability that Bob will detect a discrepancy if Alice announces the state she sent and he checks on the
D(\hat{U}) = 1 - \frac{1}{2} \langle 0 | \rho_0^E | 0 \rangle - \frac{1}{2} \langle 1 | \rho_1^E | 1 \rangle. \quad (7)

(The operator \( \hat{U} \) is listed in this expression to remind us that \( D \) is explicitly a function of whatever unitary operation Eve uses to entangle her ancilla with Alice’s system.)

For a measure of inference power gained by Eve, we take the best possible average probability that she will make an error in guessing of the state’s identity after performing a measurement on the ancilla.

In order to get at this expression, one needs the formalism of Positive Operator Valued Measures (POVM). However, we record it here for reference:

\[
P_e(\hat{U}) = \frac{1}{2} - \frac{1}{4} \text{tr} \left| \rho_1^E - \rho_0^E \right|, \quad (8)
\]

where \( | \cdot | \) signifies an operator diagonal in the same basis as its argument, but with eigenvalues that are the absolute values of those of the argument. It is the tradeoff between \( D(\hat{U}) \) and \( P_e(\hat{U}) \) that we should like to explore as a function of \( \hat{U} \).

Rather than considering the full-blown problem of evaluating the tradeoff between these quantities for four-dimensional \( \mathcal{H}_E \), here we stave off some of the mathematical difficulties encountered there by restricting ourselves to three-dimensional \( \mathcal{H}_E \). This allows for an easier path to the essential physics. (Besides, in the problem of Ref. \(^3\) it was found by numerical simulation that two-dimensional \( \mathcal{H}_E \) always sufficed for the optimal tradeoff.)

Let us consider the set of unitary operators that take \( |s\rangle\langle \psi| \) to real superpositions of the basis states \( |a_m\rangle\langle e_\beta| \), \( m = 0, 1 \) and \( \beta = x, y, z \), on \( \mathcal{H}_A \otimes \mathcal{H}_E \). Also, let us make the reasonable assumption of only considering \( \hat{U}s \) that obtain the same symmetry on the ancilla’s relative states as that of Alice’s states in the sense that: if

\[
\hat{U}|a_m\rangle|\psi\rangle = \sum_n |a_n\rangle|R_{mn}\rangle,
\]

then all inner products \( \langle R_{m'n'}|R_{mn}\rangle \) remain invariant under an interchange of \( 0 \leftrightarrow 1 \) in the indices.\(^1\)

It will be shown in the full paper to be submitted to these proceedings that the set of all such \( \hat{U}s \) can be parameterized by three real numbers \( \lambda, \phi, \) and \( \theta \). Moreover,

\[
D(\hat{U}) \text{ and } P_e(\hat{U}) \text{ work out explicitly to be:}
\]

\[
D(\hat{U}) = \cos^2 \lambda \left( \sin^2 \theta - \frac{1}{2} S \cos 2\phi \sin 2\theta \right. \\
+ \frac{1}{2} S^2 \left( 1 - \sin 2\phi \right) \cos 2\theta \left. \right) \quad (10)
\]

and

\[
P_e(\hat{U}) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - S^2} \sqrt{G(\hat{U})}, \quad (11)
\]

where

\[
G(\hat{U}) = \cos^4 \lambda \cos^2 2\phi + \frac{1}{2} \sin^2 2\lambda \left( 1 - \sin 2\phi \right) \cos^2 \theta. \quad (12)
\]

Ideally, what one would like to do with these quantities is form from them a curve of the smallest possible \( D \) as a function of \( P_e \). However that leads to a pretty hefty constrained-variation problem. Suffice it for the present to consider a much simpler problem where the “figure of merit” is the ratio \( G/D \)—the larger this quantity, the more effective the given \( \hat{U} \) will have been at leading to a good inference while minimizing the disturbance. (More details of the full constrained variation problem will appear in the full paper.)

For fixed \( \theta \) and \( \phi \) with \( \sin 2\phi \geq 0 \), \( G/D \) is maximized when \( \lambda = 0 \)—that is to say, a two-dimensional ancilla again suffices. Then it is straightforward to calculate the best possible \( D \) for a fixed \( P_e \). It is given by

\[
D = \frac{1}{2} - \frac{1}{2} \left\{ S^2 G + \left[ 1 - S^2 \left( 1 - \sqrt{1 - G} \right) \right]^2 \right\}^{1/2}, \quad (13)
\]

where

\[
G = \frac{1}{1 - S^2} \left( 1 - 2P_e \right)^2. \quad (14)
\]

Notice that \( D \to 0 \) as \( P_e \to 1/2 \), just as one would expect. \( D \) reaches its maximum value when \( P_e \) reaches its minimal value. Eqs. \((13)\) and \((14)\) completely define the best possible inference/disturbance tradeoff in this context.

## 3 Mixed States

The deepest understanding of what the information/disturbance principle has to say about quantum theory as a whole will necessarily come from the mixed-state analog of these considerations. For only then can a direct comparison be made to classical probability distributions on phase space—the object of study in classical information theory.\(^4\) Unfortunately, in moving to the mixed state version of this principle, the mathematical difficulties become even more acute. Some progress in the face of these troubles will be reported in the full paper.

However, in the mean time, there is one restricted result about information gain vs. disturbance for mixed states.
that brings out an interesting mystery. This result is known as the “no-broadcasting theorem” 5.

Suppose a quantum system, secretly prepared in one state from the set \( \mathcal{A} = \{ \hat{\rho}_0, \hat{\rho}_1 \} \), is dropped into a “black box” whose purpose is to broadcast or replicate that quantum state onto two separate quantum systems. That is to say, a state identical to the original should appear in each system when it is considered without regard to the other (though there may be correlation or quantum entanglement between the systems). Can such a black box be built? If so, then that will certainly provide a way to gain information about the mixed state without causing a detectable disturbance in the system.

The “no-cloning theorem” 6,7 insures that the answer to this question is no when the states in \( \mathcal{A} \) are pure and nonorthogonal: for the only way to have each of the broadcast systems described separately by a pure state \( \ket{\psi} \) is for their joint state to be \( \ket{\psi} \otimes \ket{\psi} \). When the states are mixed, however, things are not so clear. There are many ways each broadcast system can be described by \( \hat{\rho} \) without the joint state being \( \hat{\rho} \otimes \hat{\rho} \), the mixed state analog of cloning. The systems may also be entangled or correlated in such a way as to give the correct marginal density operators.

For instance, consider the limiting case in which \( \hat{\rho}_0 \) and \( \hat{\rho}_1 \) commute and so may be thought of as probability distributions \( p_0(b) \) and \( p_1(b) \) for the eigenvectors in a common diagonalizing basis. In this case, one easily sees that the states can be broadcast; the broadcasting device need merely perform a measurement of the eigenbasis and prepare two systems, each in the state corresponding to the outcome it finds. The resulting joint state is not of the form \( \hat{\rho} \otimes \hat{\rho} \) but still reproduces the correct marginal probability distributions and thus, in this case, the correct marginal density operators.

It turns out that the case just described is indeed a rather special one, for two states \( \hat{\rho}_0 \) and \( \hat{\rho}_1 \) can be broadcast if and only if they commute 8. That is, if and only if they can be simultaneously thought of as classical probability distributions for some underlying reality. This is the content of the no-broadcasting theorem.

The enticing mystery that arises from the no-broadcasting theorem is the following. A detailed study of Eqs. (4) and (4) and their analogs in Ref. 9 reveals that information can be gained about the identity of two pure quantum states without disturbance if and only if they are orthogonal. (Also see Ref. 9.) It follows that there can be information gain without disturbance if and only if the two pure states can be cloned.

One might expect the same sort of behavior to hold for mixed states: that information can be gained without disturbance if and only if the two states can be broadcast. This thought, however, is misguided. Life becomes interestingly more complex when it comes to mixed states.

A simple example to consider is that of two density operators on a four-dimensional Hilbert space. In matrix representation, suppose they do not commute on a \( 2 \times 2 \) block but do on the orthogonal complement to it. The states will thus have two common eigenvectors. Suppose the eigenvalues associated with the common eigenvectors are all distinct. Then it follows immediately that there are information gathering measurements that will cause no disturbance to these states—it consists of checking whether the system is in the \( 2 \times 2 \) block or either of the other two eigenvectors. This measurement can be performed as a quantum nondemolition measurement. Since the eigenvalues are all distinct, there are distinct probabilities for all these outcomes and thus information to be gained. However, because these two states are noncommuting overall, they cannot be broadcast.

An interesting open question is the necessary and sufficient mathematical criteria required of two density operators to insure that information gathering measurements necessarily disturb the quantum states. A conjecture is that this will be the case when they commute on no nontrivial subspace.

4 Foundations

Finally we briefly sketch the statement of a theorem that indicates that the information/disturbance tradeoff principle may be a fundamental aspect of quantum theory. 10 Details of its proof will be given in the full conference paper.

Let us go back to the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_E \) and suppose we consider all of quantum mechanics intact and beyond question except for the general group of time evolutions that this space may be submitted. We shall suppose that this group \( T \) consists of all maps (continuous in time) that are bijections of \( \mathcal{H}_A \otimes \mathcal{H}_E \) onto itself, and that it contains at least all the unitary operations. A priori, however, we may not wish to tie down the set of evolutions any further than this—it might contain other linear maps, nonlinear maps, maps discontinuous with respect to the topology of the vector space, or what have you.

As it stands, not much can be said about the group \( T \). So let us now consider what would be required of a mapping \( \Phi \in T \) if it were to be capable of breaking the principles espoused in this paper. A mapping \( \Phi \) is said to allow illegal eavesdropping (i.e., information gain without disturbance) if there are two nonorthogonal states \( \ket{s}, s = 0, 1, \) in \( \mathcal{H}_A \) and a standard state \( \ket{\sigma} \) in \( \mathcal{H}_E \) such that

\[ \Phi(\ket{s}) = \alpha \ket{s} + \beta \ket{\sigma}, \]

where \( |\alpha|^2 + |\beta|^2 = 1 \).

5I thank Michel Boyer, Gilles Brassard, Nicolas Gisin, Asher Peres, and Bill Wootters for listening to me patiently on these points.
that
\[ \Phi(|s\rangle|\sigma\rangle) = |s\rangle|\sigma_s\rangle, \] (15)

where \( s = 0, 1 \), for which \( 0 \leq |\langle \sigma_0 | \sigma_1 \rangle| < 1 \). (All vectors in this description are assumed normalized.) If such a map existed, then—it since we are assuming all the other principles of quantum mechanics are still intact—a measurement on \( \mathcal{H}_E \) alone will certainly reveal information about the state \(|s\rangle\) without disturbing it.

Suppose now, we find the existence of such maps too unbearable, and we take it as a principle that such time evolutions cannot exist. The question is, can we still be left with time evolutions that are more general than those provided by the unitary group? For instance, one can easily imagine maps on \( \mathcal{H}_A \otimes \mathcal{H}_E \) that are perfectly well behaved on product states, doing just what we expect, even though their behavior goes completely awry on the set of entangled states.

The answer to the question is “no.” If there is a map in \( \mathcal{T} \) that allows the increase or decrease of the modulus of any inner product of two states in \( \mathcal{H}_A \otimes \mathcal{H}_E \), then we can construct another map that will use that effect for illegal eavesdropping. This will be shown in the full paper. Thus, if the information/disturbance tradeoff principle is to be upheld, \( \mathcal{T} \) can only contain inner-product modulus preserving maps. And this, it turns out, is the premise for Wigner’s famous theorem [4] stating that, allowing the possible redefinition of phase, all such maps must be unitary or anti-unitary. If the maps are to be continuous in time, then they must be unitary.

This simple point demonstrates that there are things to be learned about quantum theory itself by observing how it can be used for communication and computation. The result is not completely satisfactory in that we had to require that \( \mathcal{T} \) contain at least the unitary maps before we could make any progress. Nevertheless, it does provide food for thought.

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