Failure of Universality in Noncompact Lattice Field Theories

T. E. Gallivan* and Arie Kapulkin†

Center for Relativity
Department of Physics
University of Texas at Austin
Austin, Texas 78712-1081
(November 13, 2018)

Abstract

The nonuniversal behavior of two noncompact nonlinear sigma models is described. When these theories are defined on a lattice, the behavior of the order parameter (magnetization) near the critical point is sensitive to the details of the lattice definition. This is counter to experience and to expectations based on the ideas of universality.
PACS numbers: 11.10.Lm, 04.60.Nc, 11.10.Hi

*timg@landau.ph.utexas.edu
†arik@landau.ph.utexas.edu
I. INTRODUCTION

Functional integration is an extremely useful tool for the non-perturbative analysis of quantum field theories (QFT from now on). To ensure that calculations involving these integrals be well defined, it is convenient to regularize them by working on a spacetime lattice. A continuum QFT is then defined in terms of a sequence of lattice field theories. There may be many lattice representations for a single continuum theory, so it is not obvious that such a definition is unique. In the majority of lattice field theory simulations, the issue of uniqueness is addressed by appealing to the principle of universality. Universality is the property that, in the continuum limit, the system ceases to depend on the exact nature of the lattice theory, and many different lattice field theories lead to the same QFT. In the approach to the continuum limit, which is located at special—critical—values of the coupling parameters, various quantities are expected to have a universal dependence on the couplings. Two different lattice theories that display the same behavior at this critical point are said to belong to the same universality class.

In the present paper, lattice models that do not conform to this expectation of universality are described, and the extent of universal behavior is clarified somewhat. Four models are considered here: two lattice versions of the $SO(1, 1)$ nonlinear sigma model and two lattice versions of the $SO(2, 1)$ sigma model. In each case the two lattice models become the same classical theory in the classical continuum limit.

Numerical evidence strongly suggests [1–3] that both lattice $O(2, 1)$ models do belong to the same universality class in the sense that they both describe a pair of free uncoupled massless scalar fields in the continuum limit (they are trivial). The situation for the $SO(1, 1)$ models is not as clear. One lattice $SO(1, 1)$ model is known to become a free massless scalar field in the continuum limit, but the continuum limit of the other lattice model is unknown. The current numerical data does not rule out triviality.

All of these lattice theories, however, have an order parameter, called the magnetization, which is sensitive to the details of the lattice model near the critical point. This is unlike the
behavior observed in more familiar models, such as the Ising model or the compact nonlinear sigma models [1]. It is argued that the cause for this nonuniversal behavior is the radically different scaling of the fields with a parameter $\beta$ (analogous to an inverse temperature in statistical mechanics) that is made possible by the noncompact configuration spaces. This point will be further discussed in Section V.

The remainder of the paper is organized as follows. In Sec. II, the nature of the approximation involved in defining a QFT on a lattice is explained, and the freedom involved in this approximation is demonstrated. The relevance of universality to lattice field theory is then explained. The language of the renormalization group (RG from now on) is used to show why certain quantities (e.g. critical exponents) are expected to be the same for lattice field theories that are in the same universality class. In addition, the process through which two different lattice theories become the same QFT in the continuum limit is explained in terms of the concept of irrelevant operators. The general structure of nonlinear sigma models is given in Sec. III. Sec. IV introduces two possible lattice $SO(1,1)$ models and shows that they have different behavior in the critical region. Sec. V discusses the analogous situation for the $SO(2,1)$ model. In Sec. VI the reason for the nonuniversal behavior is discussed, and the conclusions are presented.

It should be noted that unlike the two and three dimensional models that are of interest in the related discipline of critical phenomena, this article deals with models in four spacetime dimensions, since the motivation is to study quantum field theory. To carry out numerical simulations of the lattice field theory, it is necessary to analytically continue the time into the Euclidean regime. The result of this Euclideanization is mathematically equivalent to a statistical mechanical partition function, with a Hamiltonian given by $\frac{H}{kT} = S_E$ where $S_E$ is the Euclidean action of the lattice field theory.
A quantum field theory is constructed from a lattice field theory as follows. The QFT is defined by the vacuum expectation values of time ordered products of operators built out of the local quantum fields. These time ordered products can be expressed in terms of functional integrals. The functional integrals are in turn defined by discretizing spacetime, which turns them into multiple integrals, albeit in a very high dimensional space. The multiple integrals are then amenable to numerical analysis through Monte Carlo techniques. The lattice field theory approximants to the expectation values of the continuum QFT can then be calculated\footnote{Of course, non-numerical techniques may also be used.}. The simulations are carried out on successively larger lattices, and the approximants are then extrapolated to the continuum limit to yield the desired continuum vacuum expectation values.

Thus, there are two identifiable stages in the process of quantization through functional integrals. First, a lattice field theory is defined as a large dimensional multiple integral with an action constructed from the classical action. Second, the continuum QFT is recovered as the limit of a sequence of lattice theories simulated on successively larger lattices, with the lattice spacing simultaneously considered to be taken to zero. If the continuum QFT contains any finite length scale, such as the inverse of the mass or the correlation length $\xi$, this length scale becomes infinite when measured in units of the lattice spacing $a$, which goes to zero in the continuum limit. Therefore, in the second stage, when the lattice spacing is taken to zero, it is necessary to identify a point in the parameter space of the theory where the lattice version of the characteristic length scale of the QFT diverges in units of the lattice spacing. The correlation length of the nonlinear sigma models discussed here is already infinite on an infinite lattice because the models are massless. Instead, the relevant length scale here is $\mu_R^{-1}$ which is associated with the interaction. The continuum limit is sought where the characteristic length scale measured in units of the lattice spacing, $\frac{1}{\sqrt{\beta R}}$,
diverges ($\beta R \equiv \mu^2 a^2$). This means that a continuum limit exists only where $\beta R \to 0$.

The lattice action constructed during the first stage is not unique, because the continuum action contains derivatives of the fields. The customary prescription for passing from the classical continuum action to the lattice action is to replace $\partial_\mu \phi(x) \to \frac{1}{a} \Delta_\mu \phi(x)$, where $\Delta_\mu \phi(x) = \phi(x + \mu) - \phi(x)$. Here $x = (ai, aj, ak, al)$ where $a$ is the lattice spacing, $i, j, k, l \in \{1, \ldots, N\}$ are integers, and $x + \mu$ ($\mu \in \{1, \ldots, 4\}$) denotes the lattice site which is the nearest neighbor of $x$ in the forward $\mu$ direction. In the classical theory, one can perform a point transformation $\phi(x) \to \Phi(\phi(x))$, and the equations of motion in terms of the new fields $\Phi(x)$ have the same content as the equations of motion in terms of the old fields. However, the lattice action obtained from the classical action expressed in terms of the fields $\Phi(x)$ will be different from the lattice action obtained from the classical action expressed in terms of the fields $\phi(x)$. The equivalence of the theories at the continuum classical level rests on the chain rule for differentiation, which is not valid on the lattice. It is not a priori clear that the continuum QFT obtained at the second stage is the same for the two different lattice actions.

In the great body of lattice field theory simulations, this possible ambiguity is circumvented by appealing to the principle of universality. The principle of universality maintains that, in the critical region, the large distance behavior of lattice systems is mostly independent of the small distance features, such as the detailed choice of the lattice action. In extrapolating to the continuum QFT, one reexpresses the observables of interest in units of appropriate physical lengths. In the continuum limit, as noted above, these finite physical units of length become infinite as measured in terms of the lattice spacing. Therefore, the specific features of the lattice action automatically become small distance features as compared to anything of interest in the continuum QFT. Universality requires that these small distance features be unimportant.

More precisely, all possible lattice models separate into classes. All lattice field theories within a single universality class become the same QFT in the continuum limit, but lattice models belonging to different universality classes yield different continuum QFT’s. From
the point of view of quantum field theory, the details of the lattice theory do affect a small set of quantities that are nonuniversal in the critical region, but which do not affect the physical content of the continuum QFT. On the other hand, in statistical mechanics, these nonuniversal quantities may have physical content. For example, in the models considered in this article, it is found that the magnetization, which is related to an expectation value of a field (Eq. 4.4) in one of the formulations of the $SO(2, 1)$ model, has nonuniversal behavior. The magnetization is relevant for the wavefunction renormalization [1], but apparently does not affect the $S$ matrix of the theory (which is trivial).

There are three criteria which determine a universality class. They are ([5] p. 80):

1. The spacetime dimension of the system;
2. The internal symmetry group of the system, and which representation of this symmetry group is furnished by the fields in the lattice action;
3. The nature of the critical point.

These three criteria alone still leave one with the freedom to use many different lattice actions to simulate a given theory. Although the above criteria are widely known, additional criteria may be needed to determine a universality class uniquely. cf [6] p. 18.

It is now seen how universality leads to a resolution of the ambiguity inherent in formulating a lattice field theory. To ensure that the lattice field theories under consideration go to the same continuum limit, the lattice actions must have the same spacetime dimension, have the same internal symmetry group, and be constructed out of fields belonging to the same group representation. It is not immediately clear that different lattice actions (such as those obtained from the fields $\phi(x)$ and $\Phi(x)$ in the discussion above), which satisfy criteria 1) and 2) will satisfy the third as well. It is commonly assumed that the first two criteria are sufficient. This is occasionally tested when numerical studies are carried out. This assumption is often made for scalar fields in four spacetime dimensions, and is buttressed by the strong evidence existing [7] for the triviality of $\lambda\Phi^4$ theory. However, for other scalar
field theories, as yet unknown fixed points may exist.

The principle of universality appears most natural in the framework of the renormalization group (RG). A good introduction to the subject may be found in [5], [8], [9] or [10] while a detailed review is given in [11] and [12]. The following is a brief overview of the key ideas. As noted previously, the continuum limit is extracted from the features of the lattice theory with length scales much larger than the lattice spacing. The RG provides a means of studying these features by gradually integrating the high momentum degrees of freedom out of the lattice version of the partition function.

A renormalization group transformation consists of the following steps:

1. Begin with an action defined at the scale of some cutoff $\Lambda$ (on the lattice $\Lambda \sim \frac{1}{a}$). This action should be the most general possible action consistent with the symmetries of the theory;

2. Integrate modes with momenta in the range $\frac{\Lambda}{s} \leq k \leq \Lambda$, out of the functional integral. Here $s$ is some scale factor greater than one;

3. Rescale all the remaining momenta by $k \rightarrow sk$ so as to restore the original cutoff $\Lambda$;

4. Redefine the dummy field variables of integration in the partition function by $\tilde{\phi}'(k) = \alpha(s)\tilde{\phi}(k)$, where $\alpha(s)$ is the function required to restore the kinetic term in the action $(\frac{1}{2}k^2\tilde{\phi}(k)^2)$ to its original normalization.

The effective action obtained after an RG transformation has the same form as the original action, but the coupling parameters are modified. RG transformations thus generate a flow in the space of coupling parameters. In the cases of interest, this flow terminates at a fixed point, at which the effective action is RG invariant.

If the parameters in the original action have their critical values and the correlation length $\xi$ is infinite, then $\xi$ remains infinite under RG transformations. The flows that leave $\xi$ infinite generate a subspace of the coupling parameter space known as the critical surface. It is assumed that every point on the critical surface flows to some fixed point. The theory
at this fixed point has the same long distance behavior as any critical theory connected to it by RG transformations. It should be emphasized that the RG transformation changes the natural unit of length (the lattice spacing). The long distance behavior is only the same in both cases when the same unit of length is used. Therefore, the long distance behavior of the original critical theory can be deduced by studying the theory at the fixed point.

The origin of universality now becomes clear. The critical surface breaks up into domains, such that all the points in a domain flow to the same fixed point. Each domain defines a universality class. Within a given universality class, the details of the lattice action of some starting critical theory are not important because all theories have the same long distance behavior as the fixed point theory.

As will be shown below, the long distance behavior for a theory which is slightly off the critical surface, is the same as the behavior of the theory at some point in the neighborhood of the fixed point. For simplicity, the analysis of RG transformations in the neighborhood of the fixed point will be restricted to the case of a single coupling parameter. Under a general RG transformation,

\[ K' = R(s, K) \]

the function \( R(s, K) \) is a smooth function of the coupling parameter \( K \) for a finite rescaling of the cutoff by \( s \). If \( K \) is sufficiently close to the fixed point, where the coupling parameter has its fixed point value \( K^\ast \), the general RG transformation is well approximated by the linear part of its Taylor expansion:

\[ K' = K^\ast + \frac{\partial R(s, K)}{\partial K} \bigg|_{K=K^\ast} (K - K^\ast) \equiv K^\ast + \Lambda(s)(K - K^\ast). \]

Considering two successive RG transformations with scale parameters \( s_1 \) and \( s_2 \), one finds

\[ K'' = K^\ast + \Lambda(s_1)\Lambda(s_2)(K - K^\ast) = K^\ast + \Lambda(s_1s_2)(K - K^\ast). \]

This is just the semi-group property of the RG. The only possible functional form for \( \Lambda(s) \) is
\[ \Lambda(s) = s^y. \] (2.2)

It is now seen that the behavior of the coupling constant \( K \) near the fixed point under RG transformations is determined by the exponent \( y \). If \( y > 0 \), the coupling parameter is enhanced under an RG transformation, and \( K \) is called a relevant coupling. If \( y < 0 \), it is suppressed, and it is called an irrelevant coupling. If \( y = 0 \), the coupling remains constant under RG transformations (at least in the linear approximation) and it is then called marginal. In the general case of an infinite set of couplings \( K_i \), \( \Lambda(s) \) becomes an infinite-dimensional matrix. The coupling parameters no longer transform as simply as in Eq. (2.2). Instead, the eigenvalues of \( \Lambda(s) \) transform in this simple manner. These eigenvalues are usually called scaling fields. The coupling parameters, however, are linear combinations of the scaling fields. The operators multiplying the scaling fields in the action are also called relevant, irrelevant, or marginal, according to the behavior of the associated scaling fields. Terms multiplying irrelevant scaling fields in the effective action become small as successive RG transformations drive the system to the fixed point.

If the system is initially slightly off the critical surface, the direction of the RG flow of the relevant scaling fields will be away from the fixed point (and hence the critical surface), while the direction of flow of the irrelevant scaling fields will be towards the fixed point. The relevant scaling fields must vanish on the critical surface. Otherwise, RG transformations will take the system out of the critical surface. Therefore, from a starting point sufficiently close to the critical surface, RG transformations first drive the system to the neighborhood of the fixed point, because the irrelevant operators dominate. The long distance behavior of the original critical region is thus seen to be the same as the long distance behavior in a neighborhood of the fixed point, since flow trajectories connect the two regions. Eventually, the relevant scaling fields will drive the system away from the neighborhood of the fixed point.

The RG also explains why the critical exponents of certain quantities are universal. Their critical behavior can be analyzed in terms of the behavior near the fixed point. Such
quantities themselves obey transformation laws similar to those of the coupling parameters:

\[ A' = s^{y_A} A. \]  

One such quantity is the correlation length, whose scaling behavior is

\[ \xi(K_1, 0, \ldots, L_1, L_2, \ldots) \to \xi'(K'_1, 0, \ldots, L'_1, L'_2, \ldots) = s^{-1} \xi(s^{y_1} K_1, 0, \ldots, s^{x_1} L_1, s^{x_2} L_2, \ldots), \]

where \( K_i \) are the relevant scaling fields and \( L_i \) are the irrelevant scaling fields. The system is considered initially to have all the scaling fields except \( K_1 \) set to zero, and \( K_1 \ll 1 \), since the initial point is very close to the critical surface. Performing a transformation with \( s = K_1^{-\frac{1}{y_1}} \), one gets

\[ \xi'(K'_i, L'_i) = K_1^{\frac{1}{y_1}} \xi(1, 0, \ldots, K_1^{-\frac{1}{y_1}} L_1, K_1^{-\frac{2}{y_1}} L_2, \ldots). \]

Since \( K_1^{-\frac{1}{y_1}} \gg 1 \), and \( x_i < 0 \) for all \( i \), it follows that

\[ \xi'(K'_i, L'_i) \approx K_1^{-\frac{1}{y_1}} \xi(1, 0, 0, \ldots, 0, 0, \ldots) \sim K_1^{-\frac{1}{y_1}}. \]

This means that the critical exponent for the correlation length with respect to the coupling \( K_1 \) is \( -\frac{1}{y_1} \). The critical exponents of \( \xi \) with respect to any other relevant coupling parameter can be obtained by starting with that coupling as the only non-zero relevant coupling parameter. The reason that the critical exponents are universal now emerges. The values of the critical exponents are generically determined by the eigenvalues of the linearized RG near the fixed point in the given universality class, and not by the values of the coupling parameters themselves. Any system in a given universality class will therefore have the same critical exponents with respect to given relevant coupling parameters (such as the temperature, the external field, etc.).

**III. NONLINEAR SIGMA MODELS**

Numerical results for the nonlinear sigma models with noncompact configuration spaces demonstrate the need for a closer look at universality. Nonlinear sigma models are a class
of scalar field theories that can be obtained by imposing a constraint among a set of free
massless scalar fields \[9\]. The models are constructed to be invariant under the global
transformations of some finite-dimensional Lie group, \(G\). The configuration space in which
the fields live is generally a coset space of \(G\). For the compact \(SO(2)\) and \(SO(3)\) sigma
models, the configuration spaces are the circle and the 2-sphere respectively. These sigma
models are analogous to spins systems in statistical mechanics. If \(G\) is a noncompact group,
then the configuration space may also be noncompact. This article will be concerned only
with Euclidean models, invariant under \(SO(n,m)\) groups.

Consider a set of \(k + 1 = m + n\) massless free scalar fields whose Euclidean action is
invariant under transformations of \(G\).

\[
S[\phi] = \frac{1}{2} \int d^4 x \, \eta_{ij} \partial_\mu \phi^i \partial_\mu \phi^j, \quad i, j \in \{1, 2, \ldots, k+1\}. \quad (3.1)
\]

The fields \(\phi^i\) are cartesian coordinates for the internal space associated with each space-time
point. In this coordinate system, \(\eta_{ij}\) is diag\((1, 1, \ldots, -1, -1, \ldots)\).

If the fields at every point are required to satisfy the constraint,

\[
- \eta_{ij} \phi^i \phi^j = \mu^2, \quad (3.2)
\]

then the theory described by (3.1) together with (3.2) is known as a nonlinear sigma model.
The constraint merely requires that the length of the field vector (in the metric of the
configuration space) be a constant. Since both \(S[\varphi]\) and the constraint are invariant under
the global action of \(G\), the sigma model clearly possesses the same symmetry. Different
groups describe different theories, and the symmetry is generally used to distinguish among
them.

The fields are constrained to a \(k\)-dimensional curved surface, \(C\), in the \((k+1)\)-dimensional
flat space. This surface is a space of constant curvature and is generally the manifold of
a coset space of \(G\). The space \(C\) may be noncompact, if \(G\) is a noncompact group. The
curvature of the configuration space introduces interactions into the theory. The nature
of these interactions is most easily seen if the constraint is used to eliminate one of the
fields from the theory. After eliminating $\varphi^{k+1}$, the action can be expressed in terms of the remaining fields and $G_{ab}(\varphi)$, the induced metric on the constraint surface.

$$S[\varphi] = \frac{1}{2} \int d^4x \ G_{ab}(\varphi) \partial_\mu \varphi^a \partial_\mu \varphi^b, \quad a, b \in \{1, 2, \ldots, k\},$$

(3.3)

$$G_{ab}(\varphi) = \eta_{ab} - \frac{\eta_{ac} \eta_{bd} \varphi^c \varphi^d}{\mu^2 + \eta_{rs} \varphi^r \varphi^s}, \quad a, b, c, d, r, s \in \{1, 2, \ldots, k\}.$$ 

Expanding $G_{ab}(\varphi)$ around $\varphi^a = 0$ gives

$$S[\varphi] = \frac{1}{2} \int d^4x \ \left[ \eta_{ab} \partial_\mu \varphi^a \partial_\mu \varphi^b - \mu^{-2} \eta_{ac} \eta_{bd} \varphi^c \varphi^d \partial_\mu \varphi^a \partial_\mu \varphi^b ight.$

$$+ \mu^{-4} \eta_{ac} \eta_{bd} \eta_{rs} \varphi^c \varphi^d \varphi^r \varphi^s \partial_\mu \varphi^a \partial_\mu \varphi^b - \ldots \right]$$

(3.4)

It is clear from (3.4) that $\mu^{-2}$ plays the role of a coupling constant. It is also seen that the interactions involve field derivatives and that the action is not polynomial in nature.

In four dimensions, $\mu$ has dimensions of mass and the coupling constant has dimensions of $[\text{mass}]^{-2}$. Well known power counting arguments indicate that theories with couplings having negative mass dimension are not perturbatively renormalizable [13]. In addition, naive scaling arguments indicate that that $1/\mu^2$ will be irrelevant under the renormalization group transformation described in Sec. [1]. For this reason, it is generally believed that the nonlinear sigma models in four dimensions are all in the same universality class as the free field theory (they are trivial).

For purposes of numerical simulation, it is convenient to render the fields dimensionless by the rescaling $\phi^a = \varphi^a / \mu$. In terms of these dimensionless fields, the sigma model action becomes

$$S[\phi] = \frac{\mu^2}{2} \int d^4x \ G_{ab}(\phi) \partial_\mu \phi^a \partial_\mu \phi^b, \quad a, b \in \{1, 2, \ldots, k\},$$

(3.5)

where

$$G_{ab}(\phi) = \eta_{ab} - \frac{\eta_{ac} \eta_{bd} \phi^c \phi^d}{1 + \eta_{rs} \phi^r \phi^s}, \quad a, b, c, d, r, s \in \{1, 2, \ldots, k\}.$$ 

When spacetime is taken to have periodic boundary conditions, any constant field is a solution of the classical dynamical equations which minimizes the Hamiltonian. The
existence of a continuous global symmetry dictates that sigma models have a k-parameter family of ground state solutions. Because of this, the models display the phenomenon of spontaneous symmetry breaking, in which only one of the possibilities is realized as the physical vacuum.

Euclideanized nonlinear sigma models with compact symmetry groups have been widely studied as statistical mechanical spin systems, and the universality classes of these models appear to be distinguishable by the standard rules [4]. As will be shown in this article, the situation is different for sigma models with noncompact symmetry groups. The SO(1, 1) and SO(2, 1) models are the simplest sigma models of this type. Both of these exhibit behavior that would normally be considered nonuniversal. The SO(1, 1) model is particularly illustrative, since it is classically related to the free theory by a transformation of field variables. Different discretizations, however, lead to different lattice theories, only one of which is simply related to the continuum free theory.

IV. THE SO(1, 1) MODEL

The SO(1, 1) Sigma Model is defined by (3.3) with \((k + 1) = 2\) and \(\eta_{ij} = \text{diag}(1, -1)\). In terms of dimensionless fields, the action is

\[
S = \frac{\mu^2}{2} \int d^4x \left( \partial_\mu \phi^1 \partial_\mu \phi^1 - \partial_\mu \phi^2 \partial_\mu \phi^2 \right), \tag{4.1}
\]

subject to the constraint that \((\phi^2)^2 - (\phi^1)^2 = 1\). This constraint surface is disconnected, so the theory will be restricted to the connected component defined by \(\phi^2 > 0\). The internal space shown in Fig. [4] is the coset space \(SO(1, 1)/(Z_2 \times Z_2)\). It is topologically equivalent to \(\mathbb{R}\) and is intrinsically flat. Group transformations leave the surface invariant.
The $SO(1,1)$ Sigma Model defined in this way is classically equivalent to the theory of a massless free scalar field. Let $\phi^1 = \sinh s$ and $\phi^2 = \cosh s$ in (4.1). The action in terms of the field $s$ is

$$S = \frac{\mu^2}{2} \int d^4 x \partial_\mu s \partial_\mu s,$$  \hspace{1cm} (4.2)$$

which is just the massless free theory. The variable $s$ measures the arc length from the $\phi^2$ axis along the hyperbola in Fig. 1.

For numerical simulations of the quantum theory, the $SO(1,1)$ Sigma Model must be defined on a four-dimensional hypercubical lattice with periodic boundary conditions. The lattice consists of $N^4$ points (or sites), $N$ along each spacetime direction. The sites may be connected by imaginary links, which join each site to its nearest neighbors. The distance between lattice sites is denoted by $a$, and the length of one side of the lattice is $L = Na$. Lattice approximations to the continuum action are generally obtained, as noted in Sec.14, by replacing derivatives with lattice differences and integrals by sums. When this procedure
is performed on the $SO(1, 1)$ model, the action takes the form

$$S = \frac{\beta}{2} \sum \mathcal{L}(\phi).$$  \hspace{1cm} (4.3)

Here $\beta$ is $\mu^2a^2$—a dimensionless lattice version of the coupling.

The exact form of $\mathcal{L}(\phi)$ depends on the details of the differencing procedure (two of which will be discussed shortly), but it generally contains interactions which tend to align the fields at neighboring sites. In this respect, the lattice sigma models are analogous to ferromagnetic spin systems in statistical mechanics. The field values at each site play the role of magnetic spins, and $\beta$ plays the role of an inverse temperature. This analogy is more easily visualized for the compact sigma models, but it is useful for the noncompact models as well.

If the direction of symmetry breaking is chosen so that the average value of $\phi^1$ is zero (this must be enforced by hand in finite-lattice numerical simulations), then the average value of $\phi^2$ is a natural order parameter for this model. When $\beta$ is large, $\langle \phi^2 \rangle$ is near unity. This indicates that the fields at most sites are nearly aligned (highly ordered) around $s = 0$. For small $\beta$, $\langle \phi^2 \rangle$ is large while $\langle \phi^1 \rangle$ remains zero. This indicates that the fields fluctuate widely over the configuration space and that the system is highly disordered.

Evidently, $\langle \phi^2 \rangle$ is analogous to the magnetization in ferromagnetic systems, but because the configuration space is not compact, it behaves quite differently. In order to define a magnetization that behaves in a familiar fashion, the quantity

$$M = \frac{1}{\langle \phi^2 \rangle}$$ \hspace{1cm} (4.4)

will be designated the “magnetization” for this model.

Ferromagnetic systems generally have some characteristic temperature, called the critical temperature, below which the system becomes spontaneously aligned. Sigma models generally have a characteristic value of $\beta$, denoted by $\beta_c$, above which the fields begin to become aligned. This is called the critical value of $\beta$, or the critical point of the theory. The value of $\beta_c$ depends on the lattice model and the dimension of spacetime, but for compact
models, it is generally nonzero. For noncompact sigma models, however, $\beta_c$ is always zero. This is because the fields are always aligned to some degree unless an infinite “temperature” ($\beta = 0$) induces fluctuations over the entirety of the noncompact configuration space. For a finite temperature (finite $\beta$), fluctuations will be finite and the magnetization will be nonzero. A plot of the magnetization as a function of $\beta$ for two lattice $SO(1,1)$ models in Fig. 2 illustrates this.

The normal lattice approximation to the integrand of (4.2) is simply

$$(\partial_\mu s)^2 \to \frac{1}{a^2}(\Delta_\mu s)^2, \quad \Delta_\mu s = s(x + \mu) - s(x),$$

where the 4-vectors $\mu$ are given by

$$1 = (a, 0, 0, 0), \quad 2 = (0, a, 0, 0),$$

$$3 = (0, 0, a, 0), \quad 4 = (0, 0, 0, a).$$

Application of the same prescription to the integrand of (4.1) gives

$$(\partial_\mu \phi)^2 - (\partial_\nu \phi)^2 \to a^{-2}\left\{(\Delta_\mu \phi)^2 - (\Delta_\nu \phi)^2\right\}$$

$$= a^{-2}\left\{\left[\sinh(s(x + \mu)) - \sinh(s(x))\right]^2\right.$$

$$\left.-\left[\cosh(s(x + \mu)) - \cosh(s(x))\right]^2\right\}$$

$$= 4a^{-2}\sinh^2\left(\frac{\Delta_\mu s}{2}\right)$$

$$= a^{-2}\left\{(\Delta_\mu s)^2 + \frac{1}{12}(\Delta_\mu s)^4 + \ldots\right\}$$

As $a \to 0$, the expressions (4.5) and (4.6) clearly become the same when the fields are smooth. In four dimensions, however, the fields are not smooth. One of the odd properties of functional integrals is that the field configurations that contribute significantly to the integral are generally neither continuous nor differentiable (except in one dimension, where they are continuous). This issue is discussed at length in [14], but qualitatively correct behavior can be surmised from the rule of thumb that configurations which contribute significantly to the functional integral are those for which, $\langle S \rangle \sim N^d$, where $d$ is the dimension of spacetime. Applying this rule of thumb to the lattice action implied by (4.3) gives
\[ \Delta \mu s \sim a^{1-d/2}. \]

The same rule applied to (4.6) gives

\[ \Delta \mu s \sim \ln(a) \]

as \( a \to 0 \). This causes the lattice theories defined by (4.5) and (4.6) to behave very differently, despite the fact that they were obtained from the same classical action.

The numerical results [15] shown in Fig. 2 clearly confirm that (4.5) and (4.6) produce significantly different results for the critical exponent of the magnetization. These numerical simulations were performed on a lattice of \( 8^4 \) sites; at this point, accurate error estimates are unavailable.

![Graph](image)

**FIG. 2.** Magnetizations for the \( SO(1,1) \) Sigma Model.

In fact, the \( \beta \) dependence of the magnetization corresponding to (4.5) can be calculated analytically and is found [2] to behave as \( \exp\left(-\frac{\text{const.}}{\beta}\right) \), which does not allow a definition
of a critical exponent at all. The numerical results for (4.6), however, indicate that the magnetization behaves approximately as $\beta^{1.4}$ near the critical point. There is a significant qualitative difference between these two lattice models.

The radical difference in the behavior of the magnetizations for these two theories raises the possibility that the two models may belong to different universality classes; the model defined by (4.6) may not become a free continuum field theory. The numerical simulations performed to date [15], do not settle this question definitively. The same simulations, however, do show some differences in the critical exponents for the specific heat, $c_L$, of the two models (this result is further discussed in [2]). For the model with the action (4.6), $c_L \sim \beta^{-1.88}$, while for (4.5), $c_L \sim \beta^{-2.00}$. This difference, while not so drastic as the one for the magnetizations, appears to be another example of nonuniversal behavior.

V. THE SO(2,1) MODEL

The $SO(2,1)$ Sigma Model is defined by (3.3) with $(k + 1) = 3$ and $\eta_{ij} = \text{diag}(1,1,-1)$. In terms of dimensionless field variables,

$$S = \frac{\mu^2}{2} \int d^4 x \left( \partial_\mu \phi^1 \partial^\mu \phi^1 + \partial_\mu \phi^2 \partial^\mu \phi^2 - \partial_\mu \phi^3 \partial^\mu \phi^3 \right),$$

(5.1)

with the constraint that $(\phi^3)^2 - (\phi^2)^2 - (\phi^1)^2 = 1$. Again, the connected sheet $\phi^3 > 0$ is chosen. The resulting configuration space, shown in Fig. 3, is $SO(2,1)/(SO(2) \times \mathbb{Z}_2)$. This is topologically equivalent to $\mathbb{R}^2$ but is not flat.

Substitution of finite differences into (5.1) produces one possible lattice action for the $SO(2,1)$ model. Another can be obtained by means of geodesic differencing. Let $l$ be the geodesic length between two field values on the hyperboloid shown in Fig. 3. Geodesic differencing prescribes$^2$ that the lattice Lagrangian for the $SO(2,1)$ model be $l^2/a^2$. (Note that geodesic differencing of the $SO(1,1)$ model yields (4.6).)

$^2$Reasons for using this prescription are discussed in [14].
FIG. 3. Configuration Space for the SO(2, 1) Sigma Model.

If $s$ is the geodesic length along the internal space from the point $(0, 0, 1)$ to the point $(\phi^1, \phi^2, \phi^3)$, and $\theta$ is the angle measured from the $\phi^1$ axis, then the geodesic lattice action for the SO(2, 1) model is

$$S = \beta \sum_{x,\mu} \left\{ \cosh^{-1} \left[ \cosh[s(x + \mu) - s(x)] \cos^2 \left( \frac{\theta(x + \mu) - \theta(x)}{2} \right) \right] \right. \\
+ \cosh[s(x + \mu) + s(x)] \sin^2 \left( \frac{\theta(x + \mu) - \theta(x)}{2} \right) \right\}^2. \quad (5.2)$$

Again, $\beta = \mu^2 a^2$ may be interpreted as the reciprocal of a bare lattice coupling, analogous to an inverse temperature in a statistical mechanical model. In a fashion similar to the SO(1, 1) model, the magnetization is defined to be $\langle \phi^3 \rangle^{-1}$ (with $\langle \phi^1 \rangle = \langle \phi^2 \rangle = 0$).

Plots of the magnetizations for the two lattice models described above are shown in Fig. 4. For the geodesic model, the $\beta$ dependence of the magnetization near the critical point is analytically soluble and has exponential behavior similar to that displayed by the
geodesically differenced $SO(1,1)$ model. The model obtained by differencing (5.1), however, has a magnetization that displays the familiar power-law behavior near $\beta = 0$. For a lattice of $10^4$ sites, the magnetization behaves approximately as $\beta^{0.5}$ (error estimates for this value are presently unavailable). Again, there is a striking qualitative difference between the two lattice models.

Extensive numerical simulations of the geodesically-differenced $SO(2,1)$ model indicate clearly that this theory is trivial in the continuum limit [1,2]. Somewhat less thorough, but still convincing numerical studies for the naively-differenced theory [3] suggest the that this theory is also trivial. In this respect, expectations from the ideas of universality appear to be fulfilled. Nevertheless, the difference between magnetizations of the two lattice models remains to be explained. If there were a noncompact statistical mechanical model, the difference would certainly be physically significant.
VI. CONCLUSIONS

The noncompact nonlinear sigma models discussed here clearly display behavior that would ordinarily be considered a violation of universality. One possible explanation for this behavior is that the different lattice theories are associated with different universality classes. This explanation, however, is difficult to reconcile with the renormalization group arguments about irrelevant operators, which suggest that the different lattice theories should have the same continuum limit. It was mentioned in Secs. IV and V that the numerical evidence supports (to varying degrees) the view that, as quantum field theories, these noncompact models are free (trivial). This view is tenuous for the $O(1,1)$ model (the data is also consistent with a nontrivial continuum limit), but is well supported by numerical results for the $O(2,1)$ theory. If it is true, it means that, under the action of the renormalization group, the different lattice theories flow to the Gaussian fixed point.

In such a case, it must be that the critical exponent for the magnetization is one of the quantities in the theory that does not behave in a universal fashion. This opposes a large body of evidence, but much of that has been obtained in the context of models with compact configuration spaces. To understand this nonuniversal behavior, examine the standard argument why the magnetization should have universal power law behavior near the critical point:

1. The critical lattice theory is connected by renormalization group transformations to another theory, which is at the fixed point. The theory at the fixed point has the same long-distance behavior as the original theory.

2. Near a fixed point of the renormalization group transformation, the coupling constant $\beta$ will scale by some power of $s$, $\beta \to \beta' = s^y \beta$.

3. The field will also scale by $s$ to some power, $\phi \to s^{-d_\phi} \phi$. Since the magnetization is generally just $\langle \phi \rangle$, it picks up an overall factor of $s^{-d_\phi}$. 

21
4. The above behavior leads to the relationship $M'(\beta') = s^{-d\phi}M(sy\beta)$. This relationship implies $M(\beta) \sim \beta^{d\phi}$. 

It is the third of the above statements that is false when applied to the magnetization of the lattice free field and the geodesic $O(2, 1)$ model. In those cases, the magnetization is not just the average value of the field; it does not scale in the same fashion as the field in the kinetic term of the action. This means that the factor of $s^{-d\phi}$ in front of the magnetization is not correct in this case, and the argument for a universal exponent fails. This should not affect the universality of other critical exponents in the theory, except perhaps those that are related to the exponent for the magnetization.

The fact that the configuration spaces for these models are not compact plays a major role in the observed differences in critical behavior. In the action (4.6), the hyperbolic sine differs greatly from $\Delta_{\mu}s$ over the range of field values that contribute significantly to the integral (recall that the fields which contribute to the functional integral are generally discontinuous [14]). This causes the fields to spread over the configuration space at an exponential rate as $\beta \to 0$, resulting in an average field which scales exponentially with $\beta$, instead of with the usual power law. This type of scaling would be unlikely if the configuration space were compact, simply because a compact configuration space places a limit on the size of the fields.

It may seem unusual that the local scaling of the fields can affect behavior that is normally considered to be due to long range correlations. While it is the long range correlations that cause the fields to align, the value of the order parameter is influenced by the local scaling (with $\beta$) properties of the field. For the sigma models discussed above, the local behavior is sufficiently different to cause qualitative differences in the behavior of the magnetization near the critical point.

In this sense, the behavior of the magnetization is like that of the critical coupling (or critical temperature). The value of the critical coupling is known to depend strongly on the local details of the lattice model; it is not a universal quantity. The conclusion of this article
is that, for some noncompact models, the same is true of the magnetization. In field theory, this magnetization is not an observable quantity, and its nonuniversal behavior should not affect the continuum theory. As noted in Sec. II, however, it is significant from the statistical mechanical viewpoint.

Regarding conclusions about the triviality of these models, there remains uncertainty about the continuum limit of the model defined by (4.6). More complete numerical simulations of this model are necessary before its continuum limit can be cited with conviction. In addition, it would be helpful to obtain an analytical approximation to this theory for small values of $\beta$, similar to the one that supports the numerical data for the $O(2,1)$ model [2]. These tasks remain for future research.

**ACKNOWLEDGMENTS**

The authors wish to thank Michael Mandelberg for helpful conversations, and See Kit Foong for providing comments and numerical data. They are grateful to Eric Myers for sharing his numerical results for the $SO(1,1)$ model. As always, Bryce DeWitt started this mess, and should be mentioned here also.
REFERENCES

[1] J. L. deLyra et al., Phys. Rev. D 46, 2527 (1992).

[2] J. L. deLyra et al., Phys. Rev. D 46, 2538 (1992).

[3] J. L. deLyra, B. S. DeWitt, S. K. Foong, and T. E. Gallivan, lattice simulation of the
naively differenced $SO(2, 1)$ model (unpublished).

[4] D. J. Amit, Field Theory, the Renormalization Group, and Critical Phenomena (World
Scientific, London, 1984).

[5] N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group (Addison-We
Wesley, New York, 1992).

[6] L. P. Kadanoff, in Phase Transitions and Critical Phenomena, edited by C. Domb and
M. S. Green (Academic Press, London, 1975), Vol. 5A, p. 1.

[7] D. E. Callaway, Phys. Rep. 167, 241 (1988).

[8] M. Le Bellac, Quantum and Statistical Field Theory (Oxford University Press, Oxford,
1991).

[9] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena, 1 ed. (Oxford University
Press, Oxford, 1989).

[10] S. K. Ma, Rev. Mod. Phys. 45, 589 (1973).

[11] K. G. Wilson and J. Kogut, Phys. Reports 12C, 75 (1974).

[12] K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975).

[13] C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980).

[14] J. L. deLyra, S. K. Foong, and T. E. Gallivan, Phys. Rev. D 43, 476 (1991).

[15] E. A. Myers, simulations of the free scalar field and the naively differenced $SO(1, 1)$
model on a four dimensional lattice (unpublished).