Loop Equation and Wilson line Correlators in Non-commutative Gauge Theories

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ABSTRACT

We investigate Schwinger-Dyson equations for correlators of Wilson line operators in non-commutative gauge theories. We point out that, unlike what happens for closed Wilson loops, the joining term survives in the planar equations. This fact may be used to relate the correlator of an arbitrary number of Wilson lines eventually to a set of closed Wilson loops, obtained by joining the individual Wilson lines together by a series of well-defined cutting and joining manipulations. For closed loops, we find that the non-planar contributions do not have a smooth limit in the limit of vanishing non-commutativity and hence the equations do not reduce to their commutative counterparts. We use the Schwinger-Dyson equations to derive loop equations for the correlators of Wilson observables. In the planar limit, this gives us a new loop equation which relates the correlators of Wilson lines to the expectation values of closed Wilson loops. We discuss perturbative verification of the loop equation for the 2-point function in some detail. We also suggest a possible connection between Wilson line based on an arbitrary contour and the string field of closed string.

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1 Introduction

Non-commutative gauge theories are realized on branes in the zero slope limit in the presence of a large NS-NS B-field \[1, 2, 3, 4, 5, 6, 7, 8\]. Recently these theories have attracted a lot of attention. Various aspects of these theories have been studied in \[10, 11, 9, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35, 36\].

In ordinary gauge theories, a generic gauge-invariant observable is provided by an arbitrary closed Wilson loop. Non-commutative gauge theories have more general gauge-invariant observables, defined on open contours. Such gauge-invariant observables in non-commutative gauge theories were constructed in \[16\]. Different aspects of these were studied in \[23, 24, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46\]. Roughly speaking, these gauge-invariant observables can be written as Fourier transforms of open Wilson lines. In the operator formalism they are given by the following expression

\[
W_C[y] = \text{Tr}(P \exp \{i \int_C d\sigma \, \partial_\sigma y_\mu(\sigma) A_\mu(\hat{x} + y(\sigma))\} \, e^{i k \cdot \hat{x}}),
\]

where

\[
[\hat{x}_\mu, \hat{x}_\nu] = i \theta_{\mu\nu},
\]

and the trace in (1.1) is over both the gauge group, taken here to be \(U(N)\), as well as the operator Hilbert space. These open Wilson lines are gauge-invariant in non-commutative gauge theories, unlike in ordinary gauge theories, provided the momentum \(k\) associated with the Wilson line is fixed in terms of the straight line joining the end points of the path \(C\), given by \(y_\mu(\sigma)\) where \(0 \leq \sigma \leq 1\), by the relation

\[
y_\mu(1) - y_\mu(0) = \theta_{\mu\nu}k_\nu.
\]

The path \(C\) is otherwise completely arbitrary. When \(k\) vanishes, the two ends of the path \(C\) must meet and we have a closed Wilson loop.

In an earlier work \[41\], based on a perturbative analysis of correlation functions of straight Wilson lines with generic momenta, we had suggested that at large momenta the Wilson lines are bound into the set of closed Wilson loops that can be formed by joining the Wilson lines together in all possible different ways. In the present work we will establish a more general connection between correlators of Wilson lines and expectation values of Wilson loops in a non-perturbative setting, for arbitrary Wilson lines. In this generic case, however, the closed Wilson loops to which the Wilson lines are related are not formed by simply joining the Wilson lines together, but by more complicated cutting and joining manipulations. Also, the statement is valid for arbitrary momenta, not necessarily large, carried by the Wilson lines. We will use the framework of Schwinger-Dyson equations and the closely related loop equations in the planar limit. In the context of non-commutative gauge theories similar equations have been studied earlier in \[17, 48, 49\].

This paper is organized as follows. In the next section we summarize some aspects of operator formulation of non-commutative gauge theories, which is used throughout this
paper, and in particular list some useful identities. In Sec.3 we derive Schwinger-Dyson equations for the correlators of open and closed Wilson observables and discuss these at finite $N$ as well as in the planar limit. As in commutative gauge theories, the splitting term disappears from the planar equations. However, unlike in the case of closed Wilson loops, the joining term survives in the planar equations for open Wilson lines. This has the consequence of eventually relating them to closed Wilson loops. We also find that at finite $N$, the splitting term does not reduce to the ordinary gauge theory result in the limit in which the non-commutativity is removed. We trace this result to the UV-IR mixing in non-commutative gauge theories. In Sec.4 we use the results of Sec.3 to write down loop equations for the correlators of open and closed Wilson observables. We consider the loop equation for the 2-point function of the open Wilson lines in the planar limit and discuss the verification of this equation in ’t Hooft perturbation theory in some detail. Sec.5 contains a discussion of a possible connection between a Wilson line based on an arbitrary contour and string field for closed string and the non-perturbative meaning of the new loop equations derived here. In the Appendix, we give details of the perturbative calculations.

2 Non-commutative gauge theories - operator formulation

We will be working in 4-dimensional Euclidean space with a generic non-commutativity parameter in (1.2). The non-commutative gauge theory action that we will consider is

$$S = \frac{1}{4g^2} \text{Tr}(F_{\mu\nu}(\hat{x}))^2 + \cdots$$

(2.1)

where

$$F_{\mu\nu}(\hat{x}) = \hat{\partial}_{\mu} A_{\nu}(\hat{x}) - \hat{\partial}_{\nu} A_{\mu}(\hat{x}) + i[A_{\mu}(\hat{x}), A_{\nu}(\hat{x})].$$

The dots stand for possible bosonic and fermionic matter coupled to the gauge field and the trace $\text{Tr} = \text{tr}_{U(N)} \text{tr}_{\mathcal{H}}$ is over the gauge group $U(N)$ as well as the operator Hilbert space $\mathcal{H}$. To define this latter trace more precisely, let us assume that $\theta_{\mu\nu}$ has the canonical form,

$$\theta_{01} = -\theta_{10} = \theta_a, \quad \theta_{23} = -\theta_{32} = \theta_b,$$

(2.2)

with all other components vanishing, and let us define the operators

$$a = \frac{\hat{x}_0 + \hat{x}_1}{\sqrt{2\theta_a}}, \quad b = \frac{\hat{x}_2 + \hat{x}_3}{\sqrt{2\theta_b}},$$

(2.3)

which satisfy the standard harmonic oscillator algebra,

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1.$$
The operator Hilbert space trace is then defined by
\[ \text{tr}_\mathcal{H} \hat{O}(\hat{x}) = (2\pi)^2 \theta_a \theta_b \sum_{n_a, n_b} < n_a, n_b | \hat{O}(\hat{x}) | n_a, n_b >. \] (2.5)

Note that with this definition of the operator Hilbert space trace, the coupling constant \( g \) appearing in the action (2.1) is dimensionless.

We use the standard Weyl operator ordering,
\[ \hat{O}(\hat{x}) = \int d^4 y \, O(y) \, \delta(4)(\hat{x} - y) \] (2.6)
where the \textit{operator} delta-function is defined in terms of the Heisenberg group elements by
\[ \delta(4)(\hat{x} - y) = \int \frac{d^4 k}{(2\pi)^4} \, e^{-ik \cdot y} \, e^{ik \cdot \hat{x}}. \] (2.7)

The use of this operator delta-function simplifies many calculations because it shares some properties of the usual delta-function. For example, (2.6) and
\[ \text{tr}_\mathcal{H} \delta(4)(\hat{x} - y) = 1. \] (2.8)

There are, of course, differences as in the following identity which encodes the star product:
\[ \delta(4)(\hat{x} - y) \, \delta(4)(\hat{x} - z) = e^{-\frac{i}{2} \theta_i \theta_0 \delta(4)(\hat{x} - y_+) \delta(4)(y_-)}. \] (2.9)

where \( y_+ = \frac{y + z}{2} \) and \( y_- = y - z \). Below we give two identities involving these operator delta-functions which will be used in deriving the Schwinger-Dyson equations in the next section. The first one “joins” together two operators which appear inside two different traces,
\[ \int d^4 z \sum_a \text{Tr}[\hat{O}_1(\hat{x}) t^a \delta(4)(\hat{x} - z)] \text{Tr}[\hat{O}_2(\hat{x}) t^a \delta(4)(\hat{x} - z)] = \text{Tr}[\hat{O}_1(\hat{x}) \hat{O}_2(\hat{x})], \] (2.10)

and the second one “splits” two operators which are inside the same trace,
\[ \int d^4 z \sum_a \text{Tr}[\hat{O}_1(\hat{x}) t^a \delta(4)(\hat{x} - z) \hat{O}_2(\hat{x}) t^a \delta(4)(\hat{x} - z)] = \frac{1}{(2\pi)^4 \det \theta} \text{Tr}[\hat{O}_1(\hat{x})] \text{Tr}[\hat{O}_2(\hat{x})]. \] (2.11)

Here the \( t^a \)'s are the generators for the gauge group, which we have taken to be \( U(N) \), with the normalization dictated by the completeness condition
\[ \sum_a t^a_{ij} t^a_{kl} = \delta_{il} \delta_{jk}. \] (2.12)
2.1 Wilson observables and cyclic symmetry

The generic gauge-invariant Wilson observable is given in (1.1). We will also need the Wilson operator
\[
\hat{\mathcal{W}}_{C[y]}^0 = P \exp \left\{ i \int_C d\sigma \, \partial_\sigma y_\mu(\sigma) A_\mu(\hat{x} + y(\sigma) - y(0)) \right\} e^{ik_s \cdot \hat{x}}. \tag{2.13}
\]

Here the subscripts ‘0’s’ indicate that the path-ordered phase factor runs from \(\sigma = 0\) to \(\sigma = s\) along the curve \(C\), and \(y(s) - y(0) = \theta k_s\). The Wilson operator \(\hat{\mathcal{W}}_{C[y]}^s\), which runs from \(\sigma = s\) to \(\sigma = 1\), is defined similarly:
\[
\hat{\mathcal{W}}_{C[y]}^s = P \exp \left\{ i \int_C d\sigma \, \partial_\sigma y_\mu(\sigma) A_\mu(\hat{x} + y(\sigma) - y(s)) \right\} e^{i\tilde{k}_s \cdot \hat{x}}. \tag{2.14}
\]

where \(y(1) - y(s) = \theta \tilde{k}_s\). These operators are related to the Wilson observable as follows:
\[
\text{Tr}(\hat{\mathcal{W}}_{C[y]}^0) = W_{C[y]} e^{ik_y(0)}. \tag{2.15}
\]

The Wilson observable \(W_{C[y]}\) possesses a “cyclic symmetry” because of the trace over both the gauge group and the operator Hilbert space. To arrive at a mathematical expression of this symmetry, we note that
\[
\hat{\mathcal{W}}_{C[y]}^s = e^{i\tilde{k}_s \cdot \hat{x}} W_{C[y]} e^{i\tilde{k}_s \cdot \hat{x}}. \tag{2.16}
\]

and, therefore,
\[
W_{C[y]} = e^{-ik_y(0)} e^{i\tilde{k}_s \cdot \hat{x}} \text{Tr}(\hat{\mathcal{W}}_{C[y]}^0 \hat{\mathcal{W}}_{C[y]}^s) = e^{-ik_y(s)} \text{Tr}(\hat{\mathcal{W}}_{C_s[y_s]}^s) \equiv W_{C_s[y_s]}, \tag{2.17}
\]

where in the second step we have used the cyclic property of the trace and recombined the two operators in the opposite order. The contour \(C_s\) is given by
\[
y_s(\sigma) = y(\sigma + s), \quad 0 \leq \sigma \leq (1 - s)
\]
\[= y(\sigma - 1 + s) + y(1) - y(0), \quad (1 - s) \leq \sigma \leq 1. \tag{2.18}
\]

It is obtained from the curve \(C\) by cutting it at a point \(\sigma = s\) and rejoining the two pieces in the opposite order, as shown in Fig. 1.

\footnote{In the following \(\hat{W}_{C[y]}^0\), which goes over the full parametric range, will be denoted by \(\hat{W}_{C[y]}\) for notational convenience.}
If the original curve is a straight line, then the transformed curve is also a straight line shifted by an amount \( s(\theta k) \). It is easy to see more directly that such shifts are a symmetry of the straight Wilson line. This symmetry was used very effectively in \([1]\) to simplify perturbative calculations. More generally, the cyclic symmetry relates Wilson observables defined on contours that are nontrivially different. As we shall see later, the quantity

\[
V_{C_{\mu}}^{(k)}[y] = i \int_{C} dy_{\mu}(s) \ e^{-ik \cdot y(s)}
\]

(2.19)

frequently appears in perturbative calculations of multipoint Wilson line correlation functions. It is easy to see that in fact this quantity is invariant under the cyclic symmetry (2.18), and that may be reason for its appearance. It is also noteworthy that the above quantity is very similar to the vector vertex operator of open string theory. It would be interesting to have a better understanding of these connections and the implications of the cyclic symmetry.

### 3 Schwinger-Dyson equation

In this section we will first derive the Schwinger-Dyson equation for multipoint correlators of Wilson observables and then analyse it at finite \( N \) as well as in the planar limit. The Schwinger-Dyson equation follows from the standard functional integral identity

\[
0 = \int [\mathcal{D} A^b_{\mu}(x)] \int d^4 z \sum_a \frac{\delta}{\delta A^a_{\mu}(z)} \left[ e^{-S} \ Tr(\hat{W} C_1[y_1] t^a \delta^{(4)}(\hat{x} - z)) Tr(\hat{W} C_2[y_2]) \cdots Tr(\hat{W} C_n[y_n]) \right] \]

(3.1)

Using

\[
\frac{\delta}{\delta A^a_{\mu}(z)} \hat{W}_C[y] = i \int_{C} dy_{\mu}(s) \ e^{i k_{\mu} \theta k} \hat{W}_C[y] t^a \delta^{(4)}(\hat{x} - z)) \hat{W}_C[y]_{s1}
\]

(3.2)
and the joining and splitting identities, (2.10) and (2.11), we get

\[
\frac{1}{g^2} < \text{Tr}(\hat{W}_{C_1}[y_1]\hat{D}_\nu \hat{F}_{\mu\nu}(\hat{x})) \text{Tr}(\hat{W}_{C_2}[y_2]) \cdots \text{Tr}(\hat{W}_{C_n}[y_n]) >
\]

\[
= i \sum_{i=2}^n \int_{C_i} dy_{\mu}(s) e^{\frac{i}{\theta} k_{i\mu}} < \text{Tr}(\hat{W}_{C_2}[y_2]) \cdots \text{Tr}(\hat{W}_{C_1}[y_1]_0 \ast \hat{W}_{C_1}[y_1] \hat{W}_{C_1}[y_1]_{s1}) \cdots \text{Tr}(\hat{W}_{C_n}[y_n]) >
\]

\[
+ \frac{i}{(2\pi)^4 \det \theta} \int_{C_1} dy_{\mu}(s) e^{\frac{i}{\theta} k_{1\mu}} < \text{Tr}(\hat{W}_{C_1}[y_1]_0) \text{Tr}(\hat{W}_{C_1}[y_1]_{s1}) \text{Tr}(\hat{W}_{C_2}[y_2]) \cdots \text{Tr}(\hat{W}_{C_n}[y_n]) > .
\]

(3.3)

In this equation \(y_l(s) - y_l(0) = \theta k_{ls}\) and each of the contours \(C_1, C_2, \ldots C_n\) may be open or closed.

### 3.1 Closed Wilson loop

Let us first consider a single closed Wilson loop. In this case equation (3.3) reduces to

\[
\frac{1}{g^2} < \text{Tr}(\hat{W}_C[y] \hat{D}_\nu \hat{F}_{\mu\nu}(\hat{x})) > = \frac{i}{(2\pi)^4 \det \theta} \int_{C} dy_{\mu}(s) < \text{Tr}(\hat{W}_C[y]_0) \text{Tr}(\hat{W}_C[y]_{s1}) >
\]

(3.4)

Here \(C\) is a closed curve. The right hand side of (3.4) has a disconnected piece. However, because of momentum conservation, the disconnected piece contributes only when \(y(s) = y(0)\). In fact, both \(< \text{Tr}(\hat{W}_C[y]_0) >\) and \(< \text{Tr}(\hat{W}_C[y]_{s1}) >\) are proportional to \((2\pi)^4 \delta^{(4)}(k_s)\). One of these gives rise to the total space-time volume \(V\), while the other factor can be rewritten as \((2\pi)^4 \det \theta \delta^{(4)}(y(s) - y(0))\). Thus, we may rewrite (3.4) as

\[
\frac{1}{g^2} < \text{Tr}(\hat{W}_C[y] \hat{D}_\nu \hat{F}_{\mu\nu}(\hat{x})) >
\]

\[
= \frac{i}{V} \int_{C} dy_{\mu}(s) \delta^{(4)}(y(s) - y(0)) < \text{Tr}(\hat{W}_C[y]_0) > < \text{Tr}(\hat{W}_C[y]_{s1}) >
\]

\[
+ \frac{i}{(2\pi)^4 \det \theta} \int_{C} dy_{\mu}(s) < \text{Tr}(\hat{W}_C[y]_0) \text{Tr}(\hat{W}_C[y]_{s1}) >_{\text{conn}}.
\]

(3.5)

The second term on the right hand side of (3.4) contains the connected part of the 2-point function of open Wilson lines. At finite \(N\), it is easy to see that this term is down by a factor of \(1/N^2\) relative to the other terms in the equation. In the planar limit, therefore, this term drops out and the planar equation looks formally like the corresponding equation in commutative gauge theory. This is consistent with the perturbative result that, except for in an overall phase, the dependence on the non-commutative parameter \(\theta\) drops out of planar diagrams. However, there are new gauge-invariant observables in non-commutative gauge theory, the open Wilson lines, and so there are new equations. As we shall see shortly, these new equations have a non-trivial planar limit. One might then say that it is these new equations that reflect the new physics of non-commutative gauge theory.
At finite $N$ the second term on the right hand side of (3.3) contributes. One might wonder whether this term reduces to its commutative counterpart in the limit of small non-commutative parameter. An argument has been presented in [49] suggesting that this is the case. However, we find that, in fact, the small $\theta$ limit of this term is not smooth, at least in perturbation theory, as we will now show.

At the lowest order in perturbation theory, the second term on the right hand side of (3.5) evaluates to

$$
-ig^2 N \frac{1}{(2\pi)^4 \det \theta} \int_C dy_{\mu} s \left( \int_0^s dy(\sigma) e^{-ik_s y(\sigma)} \right) \left( \int_0^1 dy(\sigma') e^{ik_s y(\sigma')} \right),
$$

(3.6)

For simplicity, let us specialize to a rectangular contour of sides $L_1$ and $L_2$. We will also take $\theta$ to be of the form in (2.2) with $\theta_a = \theta_b = \theta_0$. In this case the diagrams that contribute to (3.6) are shown in Fig. 2.

We can easily evaluate (3.6) in this case. The result is

$$
g^2 N \frac{1}{(2\pi)^4 \theta_0^2} \left( L_{1\mu} - L_{2\mu} \right) \left[ 1 - \phi^{-2} (1 - e^{-i\phi})^2 - f(\phi) + i\phi^{-1} e^{-i\phi} (f(\phi) - f^*(\phi)) \right]
$$

(3.7)

where $\phi = L_1 \theta^{-1} L_2$ is the magnetic flux passing through the rectangular contour and

$$
f(\phi) = \int \frac{ds}{s} \left( 1 - e^{-i\phi} \right).
$$

(3.8)

In the limit of small non-commutativity, $\phi$ is large, and then $f(\phi) \sim \ln \phi$. In this case the right hand side of (3.3) is divergent with the leading term going as $\sim \ln \phi / \theta_0^2$. So we see that if we take the limit of small non-commutativity first, keeping $N$ finite, we do not recover the commutative result. It is easy to see in perturbation theory that the origin of this problem lies in UV-IR mixing. It has been argued in [22] that this phenomenon renders loop diagrams finite in non-commutative field theory. Now, the diagrams in Fig. 2 that contribute to the right hand side of (3.3) at order $1/N^2$ in the lowest order in 't Hooft perturbation theory actually come from non-planar one-loop diagrams on the left hand side of this equation, as shown in Fig. 3.
In fact, the relevant amplitude is

\[ g^4 N \int dy_1 \cdot \int dy_3 \int dy_2 \cdot \int dy_4 \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} e^{ip(y_1-y_3)} e^{iq(y_2-y_4)} \frac{e^{ipq}}{p^2 q^2} \]  

(3.9)

We can estimate the above momentum integral as follows

\[
\int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} e^{ip(y_1-y_3)} e^{iq(y_2-y_4)} \frac{e^{ipq}}{p^2 q^2} = \left\{ \begin{array}{ll}
\frac{1}{4\pi^2} \frac{1}{y_1-y_3}^2 \frac{1}{4\pi^2} \frac{1}{y_2-y_4}^2 \frac{1}{2(2\pi)^4} \frac{1}{\theta^4} \ln\left(\frac{|y_1-y_3|}{|y_2-y_4|}\right) & \text{if } |y_1-y_3||y_2-y_4| > \theta_0 \\
\frac{1}{2(2\pi)^4} \frac{1}{\theta^4} \ln\left(\frac{|y_1-y_3|}{|y_2-y_4|}\right) & \text{if } |y_1-y_3||y_2-y_4| < \theta_0
\end{array} \right.
\]  

(3.10)

In commutative gauge theory these diagrams have short distance singularities which are linearly divergent. In the non-commutative theory they get regularized at the non-commutativity scale, as can be seen from the above expression. The singularities of the commutative theory reappear in the limit of small non-commutativity. This is what is reflected in the singular behaviour of the right hand side of (3.9) for small non-commutativity.

### 3.2 Open Wilson lines

The generic equation satisfied by the \( n \)-point function of Wilson lines is (3.3). The second term on the right hand side of this equation has a disconnected part which is given by

\[ < \text{Tr}(\hat{W}_{C_1}[y_1]_{0s}) > < \text{Tr}(\hat{W}_{C_1}[y_1]_{s1})\text{Tr}(\hat{W}_{C_2}[y_2]) \cdots \text{Tr}(\hat{W}_{C_n}[y_n]) > + < \text{Tr}(\hat{W}_{C_1}[y_1]_{s1}) > < \text{Tr}(\hat{W}_{C_1}[y_1]_{0s})\text{Tr}(\hat{W}_{C_2}[y_2]) \cdots \text{Tr}(\hat{W}_{C_n}[y_n]) > . \]  

(3.11)

Because of momentum conservation, the first term contributes only for \( y(s) = y(0) \), while the second term contributes only for \( y(s) = y(1) \). In either term we get back the original \( n \)-point function. This is just like for the closed Wilson loop discussed above. The connected part of the second term on the right hand side can be easily seen to be
down by a factor of $1/N^2$ compared to the other terms in the equation. In the planar limit, therefore, it drops out, leaving only the “joining” term (the first term on the right hand side), apart from the disconnected term mentioned above. We then have the result that the planar Schwinger-Dyson equation for Wilson lines expresses any $n$-point function entirely in terms of $(n-1)$-point functions. By iterating this procedure $(n-1)$ times we may, in principle, express any $n$-point function entirely in terms of closed Wilson loops.

The simplest example of the above phenomenon is provided by the 2-point function. In this case, the planar Schwinger-Dyson equation reads

$$\frac{1}{g^2} < \text{Tr}(\hat{W}_{C_1}[y_1]\hat{D}_\mu \hat{F}_{\mu\nu}(\hat{x}))\text{Tr}(\hat{W}_{C_2}[y_2]) >$$

$$= i \int_{C_2} dy_2(s) e^{\frac{i}{\theta}k_2,\theta k_2} < \text{Tr}(\hat{W}_{C_2}[y_2]_0\hat{W}_{C_1}[y_1]\hat{W}_{C_2}[y_2]_s) >$$

$$+ \frac{i}{(2\pi)^4\text{det}\theta} \int_{C_1} dy_1(s) e^{\frac{i}{\theta}k_1,\theta k_1} \left[ < \text{Tr}(\hat{W}_{C_1}[y_1]_0) > < \text{Tr}(\hat{W}_{C_1}[y_1]_s) \text{Tr}(\hat{W}_{C_2}[y_2]) >
+ < \text{Tr}(\hat{W}_{C_1}[y_1]_s) > < \text{Tr}(\hat{W}_{C_1}[y_1]_0) \text{Tr}(\hat{W}_{C_2}[y_2]) > \right]. \quad (3.12)$$

We see that the right hand side involves closed Wilson loops, apart from the the 2-point function itself. The closed curves involved are obtained by first traversing the curve $C_1$ given by $y_1(\sigma), \ 0 \leq \sigma \leq 1$ and then the curve given by

$$y(\sigma) = y_2(\sigma + s) - y_2(s) + y_1(1), \quad 0 \leq \sigma \leq (1-s)$$

$$= y_2(\sigma - 1 + s) - y_2(s) + y_1(0), \quad (1-s) \leq \sigma \leq 1 \quad (3.13)$$

for different values of $s, \ 0 \leq s \leq 1$. Note that the closed curves obtained in this way are continuous because of momentum conservation.

Similarly, the 3-point function involves two different 2-point functions,

$$< \text{Tr}(\hat{W}_{C_2}[y_2]_0\hat{W}_{C_1}[y_1]\hat{W}_{C_2}[y_2]_s)\text{Tr}(\hat{W}_{C_3}[y_3]) >,$$

$$< \text{Tr}(\hat{W}_{C_3}[y_3]_0\hat{W}_{C_1}[y_1]\hat{W}_{C_3}[y_3]_s)\text{Tr}(\hat{W}_{C_2}[y_2]) >,$$

depending on whether $C_1$ and $C_2$ or $C_1$ and $C_3$ combine into a single curve. These 2-point functions are themselves related to closed loops, as discussed above. Thus, the 3-point function can eventually be related to closed Wilson loops, there being two distinct sets of structures for the closed contours involved. These closed contours can be obtained explicitly, as we have done above for the case of the 2-point function. For $n$-point function one eventually gets $(n-1)!$ distinct structures for the closed loops.

The above discussion establishes a general link between the correlators of Wilson lines and the expectation value of closed Wilson loops. In a previous work [1] we have presented a perturbative proof for long straight Wilson lines to be bound together into closed loops. The connection we have found here is more general in that the Wilson lines are based on arbitrary contours and the momenta need not be large. The closed loops we now find are also more general. We believe that the planar Schwinger-Dyson equation
indeed supports our previous claim for the high energy behaviour of the Wilson lines. This is because, firstly, it can be argued that the planar approximation is always valid in the high energy limit (or large $\theta$ limit) since the splitting term is suppressed by $1/\det \theta$. Secondly, we need to repeatedly insert the equation of the motion operator into the Wilson line correlator in order to eventually relate it to Wilson loops. Such an operation picks up contact terms in the sense that it makes two Wilson lines touch each other. In high energy limit, we expect to rediscover such contact terms, although the real singularities are expected to be regulated by the noncommutativity. Finally, the set of relevant closed loops may collapse to that found in [41], which is the set of extreme configurations of Wilson loops obtained by simply joining the Wilson lines end-to-end.

4 Loop equation for Wilson lines

In this section we will first derive the loop equation for multipoint correlators of Wilson lines. We will then consider the case of the 2-point function in detail and verify the planar loop equation in this case up to second order in 't Hooft perturbation theory.

The loop equation is basically the Schwinger-Dyson equation (3.3) with the insertion of the equation of motion operator replaced by a geometric variation of the contour. This is done with the help of the identity

$$\frac{\delta^2 \hat{W}_C[y]}{\delta y_\mu(\tau) \delta y_\mu(\tau')} = e^{i\tilde{k}_\tau y(\tau)} \hat{W}_C[y]_{0\tau} \left( i \partial_\tau y_\mu(\tau') \hat{F}_{\mu\nu}(\hat{x}) \right) \hat{W}_C[y]_{\tau\tau'} \left( i \partial_{\tau'} y_\rho(\tau') \hat{F}_{\mu\rho}(\hat{x}) \right) \hat{W}_C[y]_{\tau'1}$$

$$-\delta(\tau - \tau') e^{i\tilde{k}_\tau y(\tau)} \hat{W}_C[y]_{0\tau} \left( i \partial_\tau y_\mu(\tau) \hat{D}_\nu \hat{F}_{\mu\nu}(\hat{x}) \right) \hat{W}_C[y]_{\tau'1}$$

(4.1)

where, as before, $\theta_\tau = y(\tau) - y(0)$ and $\tilde{k}_\tau = k - k_\tau$. Note that this identity is valid at interior points of the Wilson line. At the boundaries of the Wilson line one has to be more careful. However, if we vary both the ends keeping $k$ fixed, and assume that the tangents to the contour at the ends are identical, then a very similar identity is valid at the ends also.

We need to separate out the equation of motion piece on the right hand side of (4.1). This may formally be done by defining the “loop laplacian”

$$\frac{\partial^2}{\partial y^2(\tau)} \equiv \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} dt \frac{\delta^2}{\delta y_\mu(\tau + t/2) \delta y_\mu(\tau - t/2)}.$$

(4.2)

In principle, if the quantum theory is regularized then the first term in (4.1) does not have any singularities as $\tau \to \tau'$ and so the loop laplacian picks up only the delta-function

\[\text{under these conditions the variation of contour at the end points is effectively like at an interior point.}\]
term on the right hand side of this equation. \footnote{In practice the separation of the two terms on the right hand side of (4.3) is a nontrivial issue. For a discussion in the case of commutative gauge theory, see, for example, [50, 51, 52].} We then get

\[- \frac{\partial^2 \hat{W}_C[y]}{\partial y^2(\tau)} = e^{\frac{i}{2}k_r \theta_k} \hat{W}_C[y]_{\theta\tau} \left( i \partial_\tau y_\mu(\tau) \hat{D}_\mu \hat{F}_{\mu\nu}(\hat{x}) \right) \hat{W}_C[y]_{\tau 1}. \] (4.3)

Using this in (3.3) we get the loop equation for Wilson line correlators

\[- \frac{1}{g^2} \frac{\partial^2}{\partial y_1^2(\tau)} < W_{C_1}[y_1]W_{C_2}[y_2] \cdots W_{C_n}[y_n] > 
\quad = \sum_{l=2}^{n} \int_{C_l} ds \left( i \partial_\tau y_1(\tau), i \partial_\tau y_l(s) \right) e^{-ik_1 y_1(\tau) - ik_l y_l(s)} 
\quad \times < W_{C_2}[y_2] \cdots \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}] \hat{W}_{C_{ls}}[y_{ls}]) \cdots W_{C_n}[y_n] > 
\quad + \frac{1}{(2\pi)^4 \text{det} \theta} \int_{C_{1\tau}} ds \left( i \partial_\tau y_1(\tau), i \partial_\tau y_{1\tau}(s) \right) e^{-ik_1 y_1(\tau) + \frac{ik_1 \theta k_1}{2}} 
\quad \times < \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{0\tau}) \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{s1}) W_{C_2}[y_2] \cdots W_{C_n}[y_n] >, \] (4.4)

where the contours $C_{1\tau}$ and $C_{ls}$ are as defined in (2.18). Also, the operator $\hat{W}_C[y]$ has been defined in (2.13) and $W_C[y]$ is the gauge-invariant observable defined in (1.1).

4.1 Two-point function

We will now discuss the case of the 2-point function in some detail. For the 2-point function, the loop equation reduces to

\[- \frac{1}{g^2} \frac{\partial^2}{\partial y_1^2(\tau)} < W_{C_1}[y_1]W_{C_2}[y_2] > 
\quad = \int_{C_2} ds \left( i \partial_\tau y_1(\tau), i \partial_\tau y_2(s) \right) e^{-ik_1 y_1(\tau) - ik_2 y_2(s)} < \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}] \hat{W}_{C_{2s}}[y_{2s}] > 
\quad + \frac{1}{(2\pi)^4 \text{det} \theta} \int_{C_{1\tau}} ds \left( i \partial_\tau y_1(\tau), i \partial_\tau y_{1\tau}(s) \right) e^{-ik_1 y_1(\tau) + \frac{ik_1 \theta k_1}{2}} 
\quad \times < \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{0\tau}) \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{s1}) W_{C_2}[y_2] >. \] (4.5)

We are interested in the planar limit of this equation. In this limit only the disconnected part of the 3-point function, appearing on the right hand side of (4.3), survives. This disconnected part is

\[< \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{0\tau}) > < \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{s1}) W_{C_2}[y_2] > + < \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{s1}) > < \text{Tr}(\hat{W}_{C_{1\tau}}[y_{1\tau}]_{0\tau}) W_{C_2}[y_2] >. \]

Because of momentum conservation, the first term survives only when $y_{1\tau}(s) = y_{1\tau}(0)$, while the second term survives only when $y_{1\tau}(s) = y_{1\tau}(1)$. Assuming that the contour

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$C_1\tau$ has no self-intersection, the first condition is satisfied only for $s = 0$, while the second condition is satisfied only for $s = 1$. Thus the disconnected part of the 3-point function takes the form

\[ (\langle \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau])_{0s} \rangle + \langle \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau])_{s1} \rangle < \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau])W_{C_2}[y_2] >, \]

which, using (2.17), is equivalent to

\[ e^{ik_1\cdot y_1(\tau)} (\langle \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau])_{0s} \rangle + \langle \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau])_{s1} \rangle < W_{C_1}[y_1]W_{C_2}[y_2] >. \]

Using this in (4.5), together with the fact that $\partial_\tau y_\mu(\tau)$ gives tangents at the two ends of the contour $C_1\tau$, which are equal by construction, we get

\[ -\frac{1}{g^2} \frac{\partial^2}{\partial y^2_1(\tau)} < W_{C_1}[y_1]W_{C_2}[y_2] > = -\frac{(\partial_\tau y_1(\tau))^2}{(2\pi)^4 \det \theta} \int_{c_1\tau} ds (\langle \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau])_{0s} \rangle + \langle \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau])_{s1} \rangle < W_{C_1}[y_1]W_{C_2}[y_2] > + \int_{c_2} ds (i\partial_\tau y_1(\tau).i\partial_s y_2(s)) e^{-ik_1\cdot y_1(\tau)-ik_2\cdot y_2(s)} < \text{Tr}(\hat{W}_{C_1\tau}[y_1\tau]\hat{W}_{C_2}[y_2]) >. \quad (4.6) \]

This is the final form of the planar loop equation for the 2-point function. Notice that the first term on the right hand side of this equation is proportional to $(\partial_\tau y_1(\tau))^2$ and also it involves the original 2-point function. Taken together with the left hand side, the two terms have the form of string hamiltonian acting on the 2-point function. However, it is not clear that this term is really physically meaningful. In fact, a corresponding term in the loop equation for the commuting gauge theory is often ignored in the regularized theory. In the present case also, an evaluation of the coefficient of $(\partial_\tau y_1(\tau))^2$ cannot be done unambiguously. This is because such a calculation involves computation of amplitudes for splitting off of tiny bits at the two ends of the Wilson line defined on the contour $C_1\tau$. The computation of this is delicate and needs a regulator. So the physical significance of this term remains unclear.

We should mention here that equations (4.4) and (4.6) are new type of loop equations since there is no analogue of these in commutative gauge theory. Also, it is clear that the planar equation (4.6) relates the 2-point function of open Wilson lines to the expectation value of a closed Wilson loop. The latter may be obtained by solving the planar loop equation for a closed Wilson line. Thus one needs equations for both types of Wilson observables to form a closed system of equations.

### 4.2 Perturbative verification

Perhaps the most interesting aspect of the loop equation (4.6) is its stringy interpretation. Investigating this aspect of the loop equation is bound to be inherently non-perturbative. In fact, recently such an exercise has been successfully carried out in [51, 52] for the loop equation in commutative gauge theory, using the AdS/CFT correspondence. A similar
exercise for the present non-commutative case seems to require a better understanding of
the connection between non-commutative gauge theory and its conjectured gravity dual [10, 11],
and is beyond the scope of the present work. Here we will restrict ourselves to a
perturbative verification of (4.6). We will do the computations upto the second order in
the ’t Hooft coupling.

Separating out the momentum conserving delta-function, we may parametrize the
2-point function as

\[ <W_{C_1[y_1]}W_{C_2[y_2]}> = (2\pi)^4 \delta^{(4)}(k_1 + k_2) G_{C_1C_2[y_1,y_2]}, \]  

where, in perturbation theory,

\[ G_{C_1C_2[y_1,y_2]} = \lambda G_{C_1C_2}^{(1)}[y_1,y_2] + \lambda^2 G_{C_1C_2}^{(2)}[y_1,y_2] + \cdots, \]  

(4.8)

Here \( \lambda = g^2 N \) is the ’t Hooft coupling constant. As indicated in (4.8), in perturbation
theory the function \( G_{C_1C_2[y_1,y_2]} \) starts at first order in the ’t Hooft coupling and is order
one in \( N \). The left hand side of (4.6) is, therefore, of order \( N \), the same as the right hand side.

The lowest order diagram contributing to the 2-point function is shown in Fig. 4.

\[ \text{Fig. 4: Lowest order diagram contributing to the 2-point function.} \]

A simple calculation gives the result

\[ G_{C_1C_2}^{(1)}[y_1,y_2] = \frac{1}{k_1^4} V_{C_1}^{(k_1)}[y_1] V_{C_2}^{(k_2)}[y_2]. \]  

(4.9)

where \( V_{C}^{(k)}[y] \) has been defined in (2.19). Operating the loop laplacian on (4.9), we get
the lowest order expression for the left hand side of (4.6)

\[ \left( i \partial_\tau y_{1\mu}(\tau) e^{-ik_1.y_{1}(\tau)} \right) V_{C_2}^{(k_2)}[y_2]. \]  

(4.10)

In arriving at this expression we have used that \( k.V_{C}^{(k)}[y] = 0 \), which is true because of
the identity \( k.y(1) = k.y(0) \) which follows from the definition of \( k \), (1.3).

\[ \text{Here and in the following we have omitted the momentum conserving delta-function factor} \]
\[ (2\pi)^4 \delta^{(4)}(k_1 + k_2) \text{ which is present on both sides of (4.6).} \]
On the right hand side of (4.6), at the lowest order in $\lambda$, the first term does not contribute. The relevant contribution comes from the second term by setting the gauge field to zero in each of the two Wilson line operators involved in making the closed Wilson loop. Omitting the momentum conserving delta-function, we get precisely the expression in (4.10).

At the next order in $\lambda$, there are several different types of diagrams that contribute to the 2-point function. Fig. 5 shows a representative example from each type.

![Examples of diagrams contributing to the 2-point function at second order in perturbation theory.](image)

Fig. 5: Examples of diagrams contributing to the 2-point function at second order in perturbation theory.

We have done a calculation of the quantity $G_{C_1C_2}^{(2)}[y_1,y_2]$, which gives the second order contribution to the 2-point function. Some details of this calculation and the result are given in the Appendix.

At the second order in $\lambda$, both terms on the right hand side of (4.6) contribute. The contribution of the first term comes from the lowest order calculation of the 2-point function, while that of the second term comes from a one gauge boson exchange. In the Appendix we have discussed in detail how each of these contributions arises as a result of operating the loop laplacian on $G_{C_1C_2}^{(2)}[y_1,y_2]$. Here we only mention that some of the terms from different sets of diagrams that appear in the calculation of the left hand side of (4.6) have a structure that does not occur on the right hand side. However, there are non-trivial cancellations between the contributions of different sets of diagrams. We have checked that many such terms disappear from the overall result for the left hand side as well, but we have not attempted a complete verification of this.

It would be interesting to extend the present perturbative analysis to all orders in $\lambda$. To verify the new loop equation (4.4) non-perturbatively, one needs to understand what the multipoint correlators of Wilson lines based on arbitrary contours map on to in the string/supergravity dual. A better understanding of the non-commutative gauge theory/string theory duality than we have at present appears to be necessary for this.
5 Discussion

In this paper we have investigated Schwinger-Dyson and loop equations in non-commutative gauge theory. A major difference from the commutative case is the existence of gauge-invariant Wilson line observables based on open contours, in addition to those on closed contours. The Schwinger-Dyson and loop equations in non-commutative gauge theories, therefore, involve both types of gauge-invariant Wilson observables. In the planar limit, the equations for a closed Wilson loop simplify and, like in their commutative counterparts, involve only closed loops. There are, however, new equations, those for open Wilson lines. These involve closed Wilson loops as well, so both types of Wilson observables are needed for a closed set of equations in the planar limit. In fact, as we have seen, these latter equations determine correlators of open Wilson lines entirely in terms of closed Wilson loops.

Recently in several works it has been argued [39, 42, 43, 44, 45, 46] that local operators in non-commutative gauge theory with straight Wilson lines attached to them are dual to bulk supergravity modes. In this context it is relevant to ask what bulk observables are dual to Wilson lines based on arbitrary open contours. This question is also important for a non-perturbative study of the new loop equations derived here. Our proposal is to identify a Wilson line based on an arbitrary open contour with the operator dual to bulk closed string. This proposal is based on the following reasoning.

The momentum variable appearing in a Wilson line satisfies the condition (1.3). This is a constraint on the contour enforced by gauge invariance. The contour is otherwise arbitrary. This condition may be regarded as a boundary condition on the curves involved. A generic curve with this boundary condition may be parametrized as

\[ y(\sigma) = y_0(\sigma) + \delta y(\sigma), \]

\[ y_0(\sigma) = y_0(0) + \sigma(\theta k), \]

\[ \delta y(\sigma) = \sum_{n=1}^{\infty} \left( \alpha_n e^{-2\pi i n \sigma} + \tilde{\alpha}_n e^{2\pi i n \sigma} \right). \]  

(5.1)

As the above parametrization suggests, what we are going to do is to assume that deviations of the given curve from a straight line are small and expand the Wilson line, based on the given curve, around the corresponding straight Wilson line. This gives

\[ W_C[y] = W_{C_0}[y_0] + \int_0^1 d\sigma \, \delta y_\mu(\sigma) \left( \frac{\delta W_C[y]}{\delta y_\mu(\sigma)} \right)_{y=y_0} \]

\[ + \frac{1}{2!} \int_0^1 d\sigma \int_0^1 d\sigma' \, \delta y_\mu(\sigma) \, \delta y_\nu(\sigma') \left( \frac{\delta^2 W_C[y]}{\delta y_\mu(\sigma) \delta y_\nu(\sigma')} \right)_{y=y_0} + \cdots \]  

(5.2)
Here $C_0$ refers to the straight line contour.

The first term in the above equation is known to be the non-commutative gauge theory operator dual to the bulk closed string tachyon. The second term vanishes, since

$$\left(\frac{\delta W_{C_0}[y]}{\delta y_\mu(\sigma)}\right)_{y=y_0} \]$$

is independent of $\sigma$, which can be easily verified using the cyclic symmetry of a straight Wilson line, and since $\delta y(\sigma)$ has no zero mode. The first non-trivial contribution comes from the third term. Using a generalization of the identity in (4.1), we may rewrite this term as

$$\int_0^1 d\sigma \int_0^1 d\sigma' \delta y_\mu(\sigma) \delta y_\nu(\sigma') \left[ \text{Tr} \left( \hat{U}_{C_0}(0, \sigma) \left( il_\lambda \hat{F}_{\mu\lambda}(\hat{x} + y_0(\sigma)) \right) \hat{U}_{C_0}(\sigma, \sigma') \right) \right.$$  

$$\times (il_\rho \hat{D}_{\rho} \hat{F}_{\mu\rho}(\hat{x} + y_0(\sigma'))) \hat{U}_{C_0}(\sigma', 1) e^{ik.\hat{x}} \theta(\sigma' - \sigma) + (\sigma' \leftrightarrow \sigma, \mu \leftrightarrow \nu) \right]$$  

$$+ \int_0^1 d\sigma \delta y_\mu(\sigma) \delta y_\nu(\sigma) \text{Tr} \left( \hat{U}_{C_0}(0, \sigma) \left( il_\sigma \hat{D}_\sigma \hat{F}_{\mu\sigma}(\hat{x} + y_0(\sigma)) \right) \hat{U}_{C_0}(\sigma, 1) e^{ik.\hat{x}} \right)$$  

$$+ \int_0^1 d\sigma \delta y_\mu(\sigma) i\partial_\sigma \delta y_\nu(\sigma) \text{Tr} \left( \hat{U}_{C_0}(0, \sigma) \hat{F}_{\mu\nu}(\hat{x} + y_0(\sigma)) \hat{U}_{C_0}(\sigma, 1) e^{ik.\hat{x}} \right),$$

(5.3)

where $\hat{U}_{C_0}(\sigma_1, \sigma_2)$ is the path-ordered phase factor, running along the straight line contour $C_0$, from the point $\sigma_1$ to $\sigma_2$. Note that in this notation $W_{C_0}[y] = \text{Tr}(\hat{U}_{C_0}(0,1) e^{i k.\hat{x}}$).

Substituting for $\delta y(\sigma)$ from (5.1) in this expression and extracting the part of the $n=1$ term symmetric in the indices $\mu, \nu$, we get precisely the operator that has been identified in [44] as being dual to the bulk graviton (in the bosonic string), polarized along the brane directions, modulo factors that connect the open string metric with the closed string metric and terms involving the scalar fields. Note that the last term in (5.3) is purely antisymmetric in the indices $\mu, \nu$ and hence contributes only to the operator dual to the bulk antisymmetric tensor field.

It seems quite likely that the above procedure gives us all the gauge theory operators dual to bulk string modes. It is, therefore, tempting to identify the Wilson line based on generic curves of the type described by (5.1) as dual to the bulk closed string. An expansion of the Wilson line around the corresponding straight line contour would then be like the expansion of the closed string field in terms of the various string modes carrying a definite momentum. If this is true, then multipoint correlators of Wilson lines should be identified with closed string scattering amplitudes. In particular, the 2-point function would then have the interpretation of closed string propagator and (4.6) would be the equation of motion satisfied by the propagator. Such a non-perturbative interpretation of (4.6), or more generally (4.4), should further enhance our understanding of gauge theory/string theory duality. The above discussion applies to the bosonic string. It would be interesting to extend these ideas to the case of the superstring.
A Appendix

In this appendix we will give some details of the calculation of $G_{C_1C_2}^{(2)}[y_1, y_2]$. We will also describe how the action of the loop laplacian on it reproduces the right hand side of the loop equation \((4.6)\).

The diagrams that contribute to $G_{C_1C_2}^{(2)}[y_1, y_2]$ can be collected into four different types of groups. A representative from each of these has been shown in Fig. 5. There are six self-energy type of diagrams, Fig. 5(a). Their total contribution to $G_{C_1C_2}^{(2)}[y_1, y_2]$ is

$$
\frac{1}{k_1^2} \int_{\sigma_1} \int_{\sigma_2>\sigma_2'} \left[ \frac{1}{4\pi^2} \frac{e^{-ik_1(y_1(\sigma_1)-y_2(\sigma_2'))}}{y_2(\sigma_2') - y_2(\sigma_2')^2} - \frac{1}{4\pi^2} \frac{e^{-ik_1(y_1(\sigma_1)-y_2(\sigma_2'))}}{y_2(\sigma_2') - y_2(\sigma_2')^2} + \frac{1}{4\pi^2} \frac{e^{-ik_1(y_1(\sigma_1)-y_2(\sigma_2'))}}{y_2(\sigma_2') - y_2(\sigma_2')^2} \right]
$$

where a dot on $y$ stands for a derivative with respect to the argument and the last contribution is obtained by the $1 \leftrightarrow 2$ interchange of the subscripts on $k$, $y$ and $\sigma$.

There are two diagrams in the second set represented by Fig. 5(b). Their total contribution to $G_{C_1C_2}^{(2)}[y_1, y_2]$ is

$$
\int d^4z \ e^{ik_1z} \int_{\sigma_1>\sigma_1'} \int_{\sigma_2>\sigma_2'} \left[ \frac{1}{4\pi^2} \frac{e^{-ik_1(y_1(\sigma_1)-y_2(\sigma_2'))}}{z + y_1(\sigma_1) - y_2(\sigma_2')^2} - \frac{1}{4\pi^2} \frac{e^{-ik_1(y_1(\sigma_1)-y_2(\sigma_2'))}}{z + y_1(\sigma_1) - y_2(\sigma_2')^2} + \frac{1}{4\pi^2} \frac{e^{-ik_1(y_1(\sigma_1)-y_2(\sigma_2'))}}{z + y_1(\sigma_1) - y_2(\sigma_2')^2} \right].
$$

In the third set, represented by Fig. 5(c), also there are two diagrams. Their total contribution to $G_{C_1C_2}^{(2)}[y_1, y_2]$ is

$$
-\frac{1}{k_1^2} \int d^4z \int_{\sigma_1} \int_{\sigma_2>\sigma_2'} \left[ \frac{1}{4\pi^2} \frac{e^{ik_1(z-y_2(\sigma_2)-y_1(\sigma_1))}}{z - y_2(\sigma_2) + y_1(\sigma_1)} - \frac{1}{4\pi^2} \frac{e^{ik_1(z-y_2(\sigma_2)-y_1(\sigma_1))}}{z - y_2(\sigma_2) + y_1(\sigma_1)} + \frac{1}{4\pi^2} \frac{e^{ik_1(z-y_2(\sigma_2)-y_1(\sigma_1))}}{z - y_2(\sigma_2) + y_1(\sigma_1)} \right]
$$

Finally, we have the gauge boson self-energy diagrams like Fig. 5(d), including those with ghosts. Their total contribution to $G_{C_1C_2}^{(2)}[y_1, y_2]$ is

$$
-\frac{1}{(k_1^2)^2} \int d^4z \int_{\sigma_1} \int_{\sigma_2} \frac{e^{ik_1(z-y_2(\sigma_2))}}{4\pi^2 z^2} \frac{\delta_{1\mu}(\sigma_1)\delta_{2\nu}(\sigma_2)}{y_2(\sigma_2) - y_2(\sigma_2')} \left[ \frac{1}{4\pi^2} \frac{e^{ik_1(z-y_2(\sigma_2))}}{z - y_2(\sigma_2) + y_1(\sigma_1)} - \frac{1}{4\pi^2} \frac{e^{ik_1(z-y_2(\sigma_2))}}{z - y_2(\sigma_2) + y_1(\sigma_1)} + \frac{1}{4\pi^2} \frac{e^{ik_1(z-y_2(\sigma_2))}}{z - y_2(\sigma_2) + y_1(\sigma_1)} \right]
$$
\[ \times \left( \delta_{\mu\nu}(-\partial_z^2 + 2i\delta_{\mu\nu} k_1 \partial_z - 5k_1^2) + 8\frac{\partial_2 \mu \partial_2 \nu}{|z + \theta k_1|^2} \left( \frac{1}{4\pi^2} - \frac{1}{4\pi^2 z^2} \right) \right). \tag{A.4} \]

Let us now evaluate the action of the loop laplacian, \(-\partial^2 / \partial y_1^2(\tau)\), on the expression for the second order contribution to the 2-point function given above. In the first term in (A.1), the only dependence on \(y_1\) is in the form of \(V_{C_{1\mu}}^{(k_1)}[y_1]\), which has been defined in (2.19). Applying the loop laplacian on it gives the result

\[ -\frac{\partial^2}{\partial y_1^2} V_{C_{1\mu}}^{(k_1)}[y_1] = (k_2^2 \delta_{\mu\nu} - k_{1\mu} k_{1\nu}) \dot{y}_{1\nu}(\tau) e^{-i k_1 y_1(\tau)}. \tag{A.5} \]

The \(k_1^2\) in the first term above cancels the factor of \(1/k_1^2\) in front of the first term in (A.1). The rest of this factor can be seen to precisely reproduce that contribution of the second term on the right hand side of the loop equation (A.6) in which a self-energy insertion is present on the contour \(C_{2s}\). The three terms correspond to the three possibilities that the marked point \(s\) on the contour \(C_{2s}\) is entirely above, entirely below or in-between the points where the self-energy insertion takes place. The second term in (A.5) gives rise to the following contribution from the first term in (A.1):

\[ -\frac{k_1}{k_1^2} \dot{y}_1(\tau) e^{-i k_1 y_1(\tau)} \int_{\sigma_2 > \sigma_2' > \sigma_2''} \frac{(k_1 \dot{y}_2(\sigma_2''))(\dot{y}_2(\sigma_2') \dot{y}_2(\sigma_2'))}{4\pi^2 |y_2(\sigma_2') - y_2(\sigma_2''')|^2} e^{ik_1 \cdot y_2(\sigma_2')} + \frac{(k_1 \dot{y}_2(\sigma_2'))(\dot{y}_2(\sigma_2') \dot{y}_2(\sigma_2'''))}{4\pi^2 |y_2(\sigma_2') - y_2(\sigma_2''')|^2} e^{ik_1 \cdot y_2(\sigma_2')} \tag{A.6} \]

This can be simplified to

\[ \frac{ik_1}{k_1^2} \dot{y}_1(\tau) e^{-i k_1 y_1(\tau)} \int_{\sigma_2 > \sigma_2'} \left( e^{ik_1 \cdot y_2(\sigma_2)} - e^{ik_1 \cdot y_2(\sigma_2')} \right) \left( \frac{1}{4\pi^2 |y_2(\sigma_2) - y_2(\sigma_2') + \theta k_1|^2} - \frac{1}{4\pi^2 |y_2(\sigma_2) - y_2(\sigma_2')|^2} \right). \tag{A.7} \]

Now, it is easy to see that on the right hand side of (A.6) there are no terms at this order having the above structure. Therefore, for consistency of the loop equation, (A.7) must get cancelled by another term in the 2-point function. In fact, this does happen and the required term comes from the first term of (A.3). In this term also \(y_1\) is in the form of \(V_{C_{1\mu}}^{(k_1)}[y_1]\), so applying the loop laplacian results in two terms because of (A.3). Let us look at the second term. It is

\[ \frac{k_1}{k_1^2} \dot{y}_1(\tau) e^{-i k_1 y_1(\tau)} \int d^4z \int_{\sigma_2 > \sigma_2'} \frac{e^{ik_1 \cdot (z + y_2(\sigma_2))}}{4\pi^2 z^2} \left[ -ik_1 \dot{y}_2(\sigma_2') k_1 \dot{y}_2(\sigma_2) + k_{1\mu} \left( \dot{y}_{2\mu}(\sigma_2') \dot{y}_{2\nu}(\sigma_2) + \dot{y}_{2\mu}(\sigma_2) \dot{y}_{2\nu}(\sigma_2') - 2\dot{y}_2(\sigma_2) \dot{y}_2(\sigma_2') \delta_{\mu\nu} \right) \partial_2 \nu \right] \times \left( \frac{1}{4\pi^2 |z + y_2(\sigma_2) - y_2(\sigma_2') + \theta k_1|^2} - \frac{1}{4\pi^2 |z + y_2(\sigma_2) - y_2(\sigma_2')|^2} \right). \tag{A.8} \]
In the last term in the square brackets above, let us rewrite $2\tau_1\partial_\tau$ as $-i\{(k_1 + i\partial_\tau)^2 + \partial_\tau^2 - k_1^2\}$ and then use $(k_1 + i\partial_\tau)^2(e^{ik_1\cdot z}/z^2) = e^{ik_1\cdot z}\partial_\tau^2(1/z^2)$ and $\partial_\tau^2(1/z^2) = -4\pi^2\delta^{(4)}(z)$. As a result of this simplification, one of the terms we get is precisely (A.7), but with opposite sign. So this unwanted term cancels in a rather nontrivial way, since the cancellation involves terms which come from two entirely different diagrams.

Going back to (A.4), let us now look at the second term, which is obtained from the first by $1 \leftrightarrow 2$ interchange:

$$\frac{1}{k_1^2} \int_{\sigma_1} \int_{\sigma_1 > \sigma_1' > \sigma_1''} \left[ \frac{(\hat{y}_2(\sigma_2),\hat{y}_1(\sigma_1'))(\hat{y}_1(\sigma_1),\hat{y}_1(\sigma_1'))}{4\pi^2|y_1(\sigma_1) - y_1(\sigma_1')|^2} e^{ik_1,(y_{22}(\sigma_2) - y_1(\sigma_1'))} 
+ \frac{(\hat{y}_2(\sigma_2),\hat{y}_1(\sigma_1'))(\hat{y}_1(\sigma_1),\hat{y}_1(\sigma_1''))}{4\pi^2|y_1(\sigma_1) - y_1(\sigma_1'')|^2} e^{ik_1,(y_{22}(\sigma_2) - y_1(\sigma_1''))} 
+ \frac{(\hat{y}_2(\sigma_2),\hat{y}_1(\sigma_1'))(\hat{y}_1(\sigma_1'),\hat{y}_1(\sigma_1''))}{4\pi^2|y_1(\sigma_1') - y_1(\sigma_1'')|^2} e^{ik_1,(y_{22}(\sigma_2) - y_1(\sigma_1'))} \right] (A.9)$$

The $y_1$ structure of this term is much more complicated than that of the first term in (A.3). So the result of applying the loop laplacian on it is also more complicated. For example, let us consider the first term in the above expression. If the loop laplacian acts on $y_1(\sigma_1'')e^{-ik_1,y_1(\sigma_1'')}$, the result is simple and, in fact, just reproduces that contribution of the second term on the right hand side of the loop equation (4.6) in which a self-energy insertion is present on the contour $C_1\tau$ entirely above the marked point $\tau$. A similar operation of the loop laplacian on the other two terms in (A.3) reproduces the other two contributions in which the self-energy insertion is either entirely below the marked point $\tau$ or across it. On the other hand, if the loop laplacian acts on the propagator $1/4\pi^2|y_1(\sigma_1) - y_1(\sigma_1')|^2$, the result is a delta-function type of contribution. Together with a similar contribution from the last term in (A.9) (the middle term has no contribution of this type since the delta-function does not click), this precisely reproduces the entire contribution of the first term on the right hand side of the loop equation in this order.

Let us now go to the next term, (A.4). Here, the loop laplacian may act on any of the four propagators, resulting in a delta-function. This gives four different terms and these precisely reproduce that contribution of the second term on the right hand side of the loop equation (4.6) in which a gauge boson is exchanged between the two contours $C_1\tau$ and $C_2s$. For this it is essential to remember that the loop on the right hand side, $\text{Tr}(\hat{W}_{C_1\tau}[y_{1\tau}]\hat{W}_{C_2s}[y_{2s}])$, involves the hatted operators. As defined in (2.17), these differ from the unhatted ones in that the argument of the gauge field is shifted by the starting point of the contour. For the contours $C_1\tau$ and $C_2s$, the starting points are respectively $y_{1\tau}(0) = y_1(\tau)$ and $y_{2s}(0) = y_2(s)$. The four terms mentioned above correspond to the four different possibilities of the two ends of the gauge field propagator landing above or below the marked points on the two contours.

Thus, we see that the action of the loop laplacian on the second order contribution to the 2-point function reproduces all the terms expected on the right hand side of the loop equation (4.6). There are also other terms produced in the process of applying the loop laplacian on the contribution of individual diagrams to the 2-point function.
We have seen above an explicit example of one such term which, however, eventually disappeared because of a nontrivial cancellation with another term. We expect that all other similar terms (which are produced in the process of applying the loop laplacian on the contributions of different diagrams to the 2-point function, but are not present on the right hand side of the loop equation) will eventually disappear through similar cancellations, but we have not attempted a complete verification of this.
References

[1] A. Connes, M. Douglas and A. Schwarz, “Non-commutative Geometry and Matrix Theory: Compactification on Tori”, JHEP 9802 (1998) 002, hep-th/9711162.

[2] M.R. Douglas and C. Hull, “D-branes and the non-commutative torus,” JHEP 9802 (1998) 008, hep-th/9711165.

[3] F. Ardalan, H. Arfaei and M.M. Sheikh-Jabbari, “Mixed branes and matrix theory on non-commutative torus,” hep-th/9803067, “Non-commutative geometry from strings and branes”, JHEP 9902 (1999) 016, hep-th/9810072.

[4] Y. Cheung and M. Krogh, “Non-commutative geometry from $D_0$ branes in a background B-field”, Nucl. Phys. B528 185 (1999), hep-th/9803031.

[5] H. Garcia-Compean, “On the deformation quantization description of matrix compactifications”, Nucl. Phys. B541 651 (1999), hep-th/9804188.

[6] C. Chu and P. Ho, “Non-commutative Open String and D-brane”, Nucl. Phys. B550 151 (1999), hep-th/9812219.

[7] V. Schomerus, “D-branes and deformation quantization”, JHEP 9906 (1999) 030, hep-th/9903205.

[8] N. Seiberg and E. Witten, “String theory and non-commutative geometry”, JHEP 9909 (1999) 032, hep-th/9908142.

[9] N. Nekrasov and A.S. Schwarz, “Instantons on non-commutative $R^4$ and (2,0) superconformal six dimensional theory”, Comm. Math. Phys. 198 (1998) 689, hep-th/9802068.

[10] J. Maldacena and J. Russo, “Large N limit of non-commutative gauge theories”, hep-th/9908134.

[11] A. Hashimoto and N. Itzhaki, “Non-commutative Yang-Mills and the AdS/CFT correspondence”, Phys. Lett. B465 (1999) 142, hep-th/9907166.

[12] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, “Non-commutative Yang-Mills in IIB matrix model”, Nucl. Phys. 565 (2000) 176, hep-th/9908141.

[13] M. Li, “Strings from IIB matrices”, Nucl. Phys. 499 (1997) 149, hep-th/9612222.

[14] L. Cornalba, “D-brane physics and noncommutative Yang-Mills theory”, hep-th/9909081.

[15] N. Ishibashi, “A relation between commutative and non-commutative descriptions D-branes”, hep-th/9909176.
[16] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, “Wilson loops in non-commutative Yang-Mills”, Nucl. Phys. B573 (2000) 573, hep-th/9910004.

[17] N. Seiberg and E. Witten, “D1/D5 system and singular CFT”, JHEP 9904 (1999) 017, hep-th/9903224.

[18] A. Dhar, G. Mandal, S.R. Wadia and K.P. Yogendran, “D1/D5 system with B field, non-commutative geometry and the CFT of the Higgs branch”, Nucl. Phys. B575 (2000) 177, hep-th/9910194.

[19] S.R. Das, S. Kalyana Rama and S. Trivedi, “Supergravity with self-dual B fields and instantons in non-commutative gauge theory”, JHEP 0003 (2000) 004, hep-th/9911137.

[20] I. Bars and D. Minic, “Non-commutative geometry on a discrete periodic lattice and gauge theory”, Phys. Rev. D62 (2000) 1050, hep-th/9910091.

[21] J. Ambjorn, Y. Makeenko, J. Nishimura and R. Szabo, “Finite N matrix models of non-commutative gauge theory”, JHEP 9911 (1999) 029, hep-th/9911041; “Non-perturbative dynamics of non-commutative gauge theory”, Phys. Lett. B480 399, hep-th/0002158; “Lattice gauge fields and discrete non-commutative Yang-Mills theory”, JHEP 0007 (2000) 013, hep-th/0003208.

[22] S. Minwalla, M.V. Raamsdonk and N. Seiberg, “Non-commutative perturbative dynamics”, JHEP 0007 (2000) 013, hep-th/0003160.

[23] O. Andreev and H. Dorn, “On open string sigma-model and non-commutative gauge fields”, Phys. Lett. B476 (2000) 402, hep-th/9912070.

[24] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, “String scale in non-commutative Yang-Mills”, Nucl. Phys. B583 (2000) 159, hep-th/0004038.

[25] S.R. Das and B. Ghosh, “A note on supergravity duals of non-commutative Yang-Mills theory”, JHEP 0006 (2000) 043, hep-th/0005007.

[26] L. Alvarez-Gaume and S.R. Wadia, “Gauge theory on a quantum phase space”, hep-th/0006219.

[27] A.H. Fatollahi, “Gauge symmetry as symmetry of matrix coordinates”, hep-th/0007023.

[28] D. Jatkar, G. Mandal and S.R. Wadia, “Nielsen-Olesen vortices in noncommutative abelian higgs model”, JHEP 0009 (2000) 018, hep-th/0007078.

[29] J.A. Harvey, P. Kraus, F. Larsen and E. Martinec, “D-branes and strings as non-commutative solitons”, JHEP 0007 (2000) 042, hep-th/0005031.

[30] K. Dasgupta, S. Mukhi and G. Rajesh, “Non-commutative tachyons”, JHEP 0006 (2000) 022, hep-th/0005006.
[31] E. Witten, “Non-commutative tachyons and string field theory”, hep-th/0006074.

[32] R. Gopakumar, S. Minwalla and A. Strominger, “Symmetry restoration and tachyon condensation in open string theory”, hep-th/0007226; “Non-commutative solitons”, JHEP 0005 (2000) 020, hep-th/0003160.

[33] N. Seiberg, “A note on background independence in non-commutative gauge theories, matrix model and tachyon condensation”, hep-th/0008013.

[34] P. Kraus, A. Rajaraman and S. Shenker, “Tachyon condensation in non-commutative gauge theory”, hep-th/0010016.

[35] G. Mandal and S.R. Wadia, “Matrix model, noncommutative gauge theory and the tachyon potential”, hep-th/0011094.

[36] A. Dhar and Y. Kitazawa, ‘Wilson loops in strongly coupled noncommutative gauge theories’, hep-th/0010256, to appear in Phys. Rev. D.

[37] Soo-Jong Rey and R. von Unge, “S-duality, non-critical open string and non-commutative gauge theory”, Phys.Lett. B499 (2001) 215, hep-th/0007089.

[38] S.R. Das and Soo-Jong Rey, “Open Wilson lines in non-commutative gauge theory and tomography of holographic dual supergravity”, Nucl.Phys. B590 (2000) 453, hep-th/0008042.

[39] D.J. Gross, A. Hashimoto and N. Itzhaki, “Observables of non-commutative gauge theories”, hep-th/0008075.

[40] A. Dhar and S.R. Wadia, “A note on gauge invariant operators in non-commutative gauge theories and the matrix model”, Phys. Lett. B495 (2000) 413, hep-th/0008144.

[41] A. Dhar and Y. Kitazawa, “High energy behaviour of Wilson lines”, JHEP 0102 (2001) 004, hep-th/0012170.

[42] H. Liu, “*-Trek II”, hep-th/0011125.

[43] S.R. Das and S. Trivedi, “Supergravity couplings to noncommutative branes, open Wilson line and generalized star products”, hep-th/0011131.

[44] Y. Okawa and H. Ooguri, “How noncommutative gauge theories couple to gravity”, hep-th/0012218; “Energy momentum tensors in matrix theory and in noncommutative gauge theories”, hep-th/0103123.

[45] H. Liu and J. Michelson, “Supergravity couplings of noncommutative D-branes”, hep-th/0101016.

[46] K. Okuyama, “Comments on open Wilson lines and generalized star products”, hep-th/0101177.
[47] M. Fukuma, H. Kawai, Y. Kitazawa and A. Tsuchiya, “String field theory from IIB matrix model”, Nucl. Phys. B510 (1998)158, hep-th/9705128.

[48] A. Dhar and S.R. Wadia, unpublished.

[49] M. Abou-Zeid and H. Dorn, “Dynamics of Wilson observables in non-commutative gauge theory”, hep-th/0009231.

[50] A.M. Polyakov, “Gauge fields and strings”, Harwood Academic Publishers (1997), pp. 119.

[51] A.M. Polyakov and V.S. Rychkov, “Gauge fields-strings duality and the loop equation”, hep-th/0002106; “Loop dynamics and AdS/CFT correspondence”, hep-th/0005173.

[52] N. Drukker, D.J. Gross and H. Ooguri, “Wilson loops and minimal surfaces”, Phys. Rev. D60 (1999) 125006, hep-th/9904191.