Critical exponent $\eta_0$ of the Lattice London Superconductor and vortex loops in the 3D XY model

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(November 21, 2018)

The anomalous dimension of the lattice London superconductor is determined from finite size scaling of the susceptibility to be $\eta_0 = -0.79(1)$. Indirect determinations of $\eta_0$ from properties of the vortex loops in the 3D XY model are also attempted, but it is found that the results are sensitive to details in the simulations related to vortex loop intersections. It is suggested that the same value of $\eta_0$ can at most be obtained from vortex loop properties in the limit of low vortex density.

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The properties of the Meissner transition is a classical problem in statistical physics. Whereas this transition was originally believed to always be of first order, the work by Dasgupta and Halperin [1] gave strong arguments that the transition should instead be continuous. The argument is based on a duality transformation of the lattice London Superconductor (LLS) and suggests that the transition should be 3D XY-like, but with the temperature scale inverted. A direct consequence of this relation is the expectation that the correlation length exponent $\nu$ should be the same in the superconductor as in the 3D XY (planar rotor) model. However, with fluctuations in both the phase angle and the gauge field, it becomes possible to define two characteristic lengths, and the behavior of the magnetic screening length $\lambda$ has recently been a subject of some controversy [2]. The current evidence points to a scenario where both characteristic lengths diverge with the same exponent. This is related to the presence of an anomalous dimension $\eta_A = 1$ for the gauge fluctuations [2].

What is needed beside $\nu$ and $\eta_A$ for the characterization of the universality class of the transition is a knowledge of the anomalous dimension $\eta_0$ associated with the phase correlations. This quantity was first determined to be $\eta_0 = -0.20$ from a renormalization group calculation to one loop order [3]. Determinations from simulations have so far only been indirect through the properties of vortex loops in the 3D XY model [3]. The main motivation for this analysis was the possible connection between the sign of $\eta_0$ and the existence of a vortex loop blowout transition in high-$T_c$ superconductors in applied magnetic fields [3].

In this Letter we report on a direct determination of the anomalous dimension $\eta_0$ from Monte Carlo (MC) simulations of the LLS that gives a surprisingly large negative value, $\eta_0 = -0.79 \pm 0.01$. We also consider in detail the approach to determine this exponent from the properties of the vortex loops in the 3D XY model. The main conclusion from that study is that the results depend strongly on the details in the simulation related to vortex loop intersections. We suggest that the theoretically expected value can at most be found in the limit of low vortex density.

The Hamiltonian of the LLS [1,3] is

$$H = \sum_{i\mu} \left\{ U(\phi_i + \mu - \phi_i - A_{i\mu}) + \frac{1}{2} J\lambda_0^2 |D \cdot A|_{i\mu}^2 \right\},$$

where $\phi_i$ is the phase of the superconducting wave function on site $i$, and $A_{i\mu}$ is the vector potential on the link starting at site $i$ and pointing along $\mu$. The sum is over all bonds of a 3D simple cubic lattice of unit grid spacing and $\mu = x, y, z$. In the first term, the kinetic energy of flowing supercurrents, $U(\phi)$ is the Villain function [4].

$$e^{-U(\phi)/T} = \sum_{p=-\infty}^{\infty} e^{-J(\phi - 2\pi p)^2/(2T)}.$$

In the second term, the magnetic field energy, $\lambda_0$ is the bare magnetic penetration length, and $D \cdot A$ is the discrete circulation of the vector potential.

In our simulations we fix $\lambda_0 = 0.3$ to be able to compare with Ref. [3] and fix the gauge through $D \cdot A = 0$ to facilitate a simple determination of the spin correlations. To fulfill this constraint we update the $A_{i\mu}$ by simultaneously adding $\delta A$ to four $A_{i\mu}$ around an elementary plaquette, a procedure which was also used in Ref. [3]. We perform our MC simulations with the standard Metropolis algorithm, first sweeping sequentially through the phase angles $\theta_i$ and then sweeping three times with attempts to change the circulation of $A$ in the three different directions. The length of the runs close to $T_c$ were at least $10^7$ sweeps through the lattice, but often much longer. The measured quantities discussed here are the second and fourth moment of the magnetization,

$$m^p = \frac{1}{L^3} \sum_i e^{i\phi_i} |^p,$$

which are used to obtain the dimensionless fraction

$$Q = \frac{\langle m^2 \rangle^2}{\langle m^4 \rangle}.$$  

(2)

similar to Binder’s cumulant, which is used to determine $T_c$. The susceptibility at the transition is obtained from the standard relation $\chi = L^3 \langle m^2 \rangle$. 

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The critical temperature for the LLS with the same parameter, $\lambda_0 = 0.3$, has already been determined with high precision. Our present analysis of $Q$ therefore mainly serves to confirm that the critical properties may be correctly determined from our simulations with $D \cdot A = 0$. Figure 1a shows $Q$ versus temperature for system sizes $L = 8, 12, 16, \text{ and } 24$. We find that the curves for $L > 8$ cross at $T \approx 0.80$. The $L = 8$ data does however not cross the other curves at the same temperature, a typical sign of corrections to scaling. This data is therefore not included in the scaling collapse.

The points do indeed to an excellent approximation fall on a straight line and our result is $\eta_0 = -0.79 \pm 0.01$. Panel (b) shows a determination of the exponent $\beta$ from the temperature dependence of $(m^2)$. This data is for temperatures at which we expect the finite size effects to be negligible. The result is $\beta = 0.069 \pm 0.003$ in good agreement with what we expect from $\eta_0$: $\beta = \frac{1}{2}(d - 2 + \eta_0) = 0.070 \pm 0.003$.

We now turn to the alternative and indirect approach to determine $\eta_0$ through the behavior of the dual model which is the ordinary 3D XY model. The idea is that the vanishing of the line tension of the vortex loops (see below) as $T \rightarrow T_c^-$ is governed by $\gamma_0$, $$\epsilon \sim (1 - T/T_c)^{\gamma_0},$$ which is related to $\eta_0$ through the Fisher scaling relation $\gamma = \eta(2 - \eta_0)$. The basis for Eq. (3) is the connection between $\phi$ correlations in the LLS and the vortex loops in the 3D XY model. Whereas this less direct determination of $\eta_0$ of course is more prone to errors, it is interesting to examine if the same value of $\eta_0$ can be obtained in that way. We will in the following examine the vanishing of $\epsilon$ for two different cases, examine the distribution of vortex loop diameters at $T_c$, and finally comment on the analysis in Ref. [4] that gave $\eta_0 \approx -0.18$.

The standard way to locate vortices in a XY model is from the angular difference $\varphi_{ij} = \theta_i - \theta_j$ between nearest neighbors, with $|\varphi_{ij}| < \pi$. For each plaquette the vorticity is then obtained from

$$n = \frac{1}{2\pi}(\varphi_{12} + \varphi_{23} + \varphi_{34} + \varphi_{41}).$$

With a Villain interaction it is also possible to define vortices in a different way, which is inspired by the wire model. For each link one first calculates $\varphi_{ij}^0 = \theta_i - \theta_j$ and then probabilistically sets the angular difference to $\varphi_{ij} = \varphi_{ij}^0 - 2\pi p_{ij}$, where $p_{ij}$ is an integer chosen with the relative weight $e^{-J(\varphi_{ij}^0 - 2\pi p_{ij})^2/(2T)}$. The vorticity is finally calculated from Eq. (4). The vortices obtained in that way are exactly equivalent to the dual vortex line model, a fact that follows from a straightforward generalization of the results in Ref. [1] to three dimensions. The density of these probabilistic vortices (PV) is always higher than the density of standard vortices (SV).

The simulations of the 3D XY model were performed with the Wolff cluster update method and Villain interaction on a cubic lattice with $L = 128$. The average for each data point is based on typically $10^5$ measurements. The simulations were performed with both SV and PV discussed above. The tracing out of vortex loops were done by always choosing the path randomly when two (or more) vortex loops meet at the same elementary cube.

After identifying the vortex loops we measure both the perimeter $p$ and the “diameter” $a$ of each such loop. Whereas $p$ is directly given by the number of vortex loop
segments the calculation of $a$ is somewhat more involved. With a loop made up by $p$ segments with each segment starting at $r_i$ and pointing along the unit vector $n_i$, the equivalent to the "magnetic moment" is

$$\mathcal{M} = \sum_{i=1}^{p} r_i \times n_i,$$

and the "diameter" becomes $a = 2\sqrt{|\mathcal{M}|/\pi}$. From these values we calculate the distributions $D(p)$ and $D(a)$.

The line tension $\epsilon$ is obtained by fitting \(2\)

$$D(p) \propto p^{-\alpha} \exp(-c\epsilon p/T), \quad (5)$$

with $\alpha$, $\epsilon$ and a prefactor as free parameters. To get good quality fits one has to restrict the fitting to large $p$ only. We found that good quality fits are obtained by only using data for $p > T/\epsilon$. Figure 3 shows the line tension $\epsilon$ determined with both SV and PV (open and solid symbols, respectively). Note that we are here examining a very narrow temperature region to be able to probe the behavior in the critical region; most data points are within a few percent below $T_c = 3.0024(1)$ \(12\). With logarithmic scales on both axes the slope as $T \to T_c$ is expected to give the exponent $\gamma_\phi$. To facilitate a comparison with the above obtained $\gamma_\phi = -0.79$ we draw a dashed line in Fig. 3, that corresponds to $\gamma_\phi = \nu(2 - \eta_\phi) = 1.87$. The slopes of the data seem to be in reasonable agreement with that prediction.

A more detailed analysis, however, casts doubt on this agreement. Panel (b) in Fig. 3 shows the temperature dependence of the local slope where each value is calculated from two consecutive points. The curvature in the original data is here seen as a trend towards larger values as $T_c$ is approached. There is, however, no evidence that

the local slope saturates at $\gamma_\phi = 1.87$ (horizontal dashed line); the last point lies significantly above the predicted value. To check for finite size effects the relevant points in panel (a) have also been analyzed with $L = 192$ but with no change in the result. For the PV data (solid circles) this deviation is even more significant.

We will below present more evidence that suggest that, generally speaking, the properties of vortex loops at or close to $T_c$ tend to not agree with theoretical predictions. We will also argue that this discrepancy is related to the existence of vortex line intersections which are normally not accounted for in theoretical considerations. The quantity we focus on is $D(a)$ – the distribution of vortex loop diameters – at $T_c$. There is a simple analytical prediction for this quantity based on the scale invariance of the system at $T_c$ \(13\).

$$D(a) \propto a^{-\lambda}, \quad \lambda = 4 \text{ at } T_c.$$  

Assume that there is on the average $N$ loops with diameter in the interval $(a/s, a/s)$ in a system with volume $L^3$, i.e. $D(a)da = N/L^3$. Scale invariance implies that the distribution should not be changed by a rescaling of the system. After rescaling with a factor $s$ the new system size is given by $L' = L/s$ and $N$ is now the number of loops with diameter in the interval $(a/s, a/s + da/s)$. We therefore get $D(a/s)da = sN/L'^3 = s^4D(a)da$ which in turn leads to $D(a) \propto a^{-4}$.

![FIG. 4. Distribution of loop diameters at $T = 3.002 \approx T_c$. The expected $\lambda = 4$ would give a horizontal line.](image)

As shown in Fig. 4 this expectation for $\lambda$ is not confirmed by the simulations \(4\). The values of $\lambda$ for SV and PV are instead found to be 4.13(1) and 4.18(1), respectively. The deviation from the expected $\lambda = 4$ increases with increasing vortex density and appears to be proportional to the vortex density, 0.156 and 0.226, respectively.

We now suggest that the reason for $\lambda \neq 4$ is related to the existence of vortex loop intersections. Consider the rescaling of a loop with a scale factor $s$. A non-trivial effect of the rescaling is related to the smaller resolution of a rescaled loop. This may lead to new vortex intersections that could change the outcome of the tracing. As an example, consider the vortex line loop in Fig. 3. The original loop (a) will always be treated as a single one, but since the rescaled loop (b) enters the same elementary cube twice the tracing can now give the loops in panel (c) or (d), each with a probability of 50%. Since a splitting will give an increased weight to smaller loops,
the effect is to make $\lambda$ larger. Note that a higher vortex density will make this effect more important which is in accordance with our results for $\lambda$.

![Image](image.png)

**FIG. 5.** The rescaling of a vortex loop (for simplicity shown as a coarsening of the vortex lattice) may give rise to new vortex line intersections. This is suggested as the reason why the analytically expected $\lambda = 4$ is not found in the simulations.

From the above it seems that the presence of vortex loop intersections is the reason for the failure to obtain the expected $\lambda = 4$ from $D(a)$. One could expect the vortex loop intersections to also affect other vortex loops properties. This is in accordance with our results from the analysis of the vortex loop line tension that the high density data (PV) is far off whereas the low density data (SV) is fairly close to the expected behavior. The correct behavior of $\epsilon$ should then at most be expected in the limit of low vortex density.

![Image](image.png)

**FIG. 6.** $D(p)$ for the 3D XY model with cosine interaction and splitting of self-intersecting loops. The fitting lines from Ref. [6] do not agree well with our more precise data (solid dots). The inset shows our values for $\epsilon$; the slope is clearly different from $\gamma_0 = 1.45$ (dashed line) obtained in Ref. [6].

The determination of the line tension for vortex loops in a 3D XY model was first made in Ref. [6]. Their simulations were with cosine spin interaction ($T_c \approx 2.2$) and a different method for tracing out the loops; self-intersecting vortex loops are always split into two. They found $\eta_0 = -0.18 \pm 0.07$, in agreement with the analytically obtained $\eta_0 \approx -0.2$. However, a comparison with our more precise MC data (now obtained with cosine interaction and splitting of intersecting loops) casts strong doubt on their analysis. Figure 6 shows our values for $D(p)$ (dots) together with both the data (open circles) and the fitting curves (solid lines) for $T = 2.0$ and 2.1 from Ref. [6]. We first note that the two different sets of MC data agree well. From the large-$p$ part of the data it is however clear that the fitting curves from Ref. [6] do not agree with our data. The same discrepancy can actually also be seen for $T = 2.0$ in Fig. 3 of Ref. [6]. The inset in Fig. 6 shows our values for $\epsilon$ obtained from a fit to Eq. (5) only including data for $p > 1.2T/\epsilon$. The slope differs clearly from $\gamma_0 = 1.45$ shown by the dashed line. We therefore conclude that the good agreement in Ref. [6] with the analytically obtained $\eta_0 \approx -0.2$ was only accidental. In our analysis the slope approaches $\gamma_0 \approx 1.15$ as $T \to T_c$. If one instead allows for vortex loop intersections $\gamma_0$ behaves much the same as in Fig. 6 (again with $\gamma_0$ significantly above 1.87). This again shows the central role of the intersections for the vortex loop properties.

To conclude, our main result is a direct determination of the anomalous dimension in the LLS, giving $\eta_0 = -0.79 \pm 0.01$. We have also attempted determinations of the same quantity from the vanishing of the line tension in the 3D XY model, but found that it was not possible to get a well-defined value for the exponent $\gamma_0$. From considerations involving the distribution of vortex loop diameters we suggest that it is the vortex loop intersections that are responsible for the failure to find the analytically expected values of some vortex-loop related exponents. This implies that the “correct” behavior at criticality can at most be expected in the limit of low vortex density.

The author thanks P. Minnhagen, A. Sudbø, S. Teitel, and Z. Tešanović for valuable discussions. This work was supported by the Swedish Natural Science Research Council through Contract No. E 5106-1643/1999.

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