ON $K_1$ AND $K_2$ OF ALGEBRAIC SURFACES

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§0. Introduction.

Let $X$ be a projective smooth surface over a field $k$ of characteristic zero. In this paper we study the higher Chow groups $CH^2(X, 1)$ and $CH^3(X, 2)$ using Hodge theoretical methods. They are the most interesting graded pieces of the Quillen $K$-groups $K_1(X)$ and $K_2(X)$. We recall that these groups are generated by higher cycles that are curves together with sets of rational functions on them modulo certain relations arising from tuples of rational functions on $X$. More precisely, $CH^2(X, 1)$ is the cohomology of the complex

$$K_2(k(X)) \to \bigoplus_{Z \subseteq X} k(Z)^* \to \bigoplus_{x \in X} \mathbb{Z},$$

where $Z \subseteq X$ ranges over all irreducible curves on $X$ and $x \in X$ ranges over all the closed points of $X$. The two boundary maps are given respectively by tame symbols for $K_2$ and by divisors of rational functions. Similarly $CH^3(X, 2)$ is the cohomology of the complex

$$K_3(k(X)) \to \bigoplus_{Z \subseteq X} K_2(k(Z)) \to \bigoplus_{x \in X} k(x)^*,$$

where the second boundary map is given by tame symbols and the first by localization theory for algebraic $K$-theory. We note that, by a result of Merkurjev and Suslin, one is allowed to replace the Quillen $K_3$ by the Milnor $K_3^M$ and then the first boundary map is given also by tame symbols for $K_3^M$.

For the study of the above groups we fix $X \supset Z = \bigcup_{i \in I} Z_i \ (I = \{1, 2, \ldots, m\})$, a simple normal crossing divisor on $X$ and consider particularly higher cycles supported on $Z$: we write

$$CH^1(Z, 1) = \text{Ker}(\bigoplus_{i \in I} k(Z_i)^* \to \bigoplus_{x \in Z} \mathbb{Z}),$$

$$CH^2(Z, 2) = \text{Ker}(\bigoplus_{i \in I} K_2(k(Z_i)) \to \bigoplus_{x \in Z} k(x)^*).$$

These are the Bloch’s higher Chow groups of $Z$ (cf. [Bl]) and we have the exact sequence for $r = 1, 2$

$$CH^{r+1}(U, r+1) \to CH^r(Z, r) \to CH^{r+1}(X, r).$$
Our first question is if one can find any interesting elements in $CH^r(Z, r)$ whose images in $CH^{r+1}(X, r)$ or $CH^{r+1}(X, r)^{ind}$, the so-called indecomposable part of it, are non-torsion. Our main result Th.(0-1) suggests that this is impossible if $X \subset \mathbb{P}^3$ is a very general hypersurface of sufficiently high degree and the components of $Z$ are very general hypersurface sections of $X$ (see Def.(1-1) for the definition of $(X, Z)$ being very general). In order to state them, we need to introduce the indecomposable parts of $CH^r(Z, r)$:

$$CH^r(Z, r)^{ind} = \text{Coker}(\bigoplus_{i \in I} CH^r(Z_i, r) \to CH^r(Z, r)).$$

For an alternative description of $CH^r(Z, r)^{ind}$ see Pr.(1-1).

**Theorem(0-1).** Let $X \subset \mathbb{P}^3$ be a very general hypersurface of degree $d$ and let $Z = \cup_{i \in I} Z_i$ with $Z_i \subset X$; a very general hypersurface sections of degree $e_i$.

1. If $d \geq 5$, $CH^2(U, 2) \to CH^1(Z, 1)^{ind}$ is surjective.
2. Assume $d \geq 6$ and that $(e_i, e_j, e_l) \neq (1, 1, 2)$ for distinct $i, j, l \in I$. Then $CH^3(U, 3) \to CH^2(Z, 2)^{ind}$ is surjective.

The proof is easily reduced to the case where our base field $k = \mathbb{C}$. Then the essential idea of the proof goes back to Griffiths’ fundamental work on algebraic cycles (cf. [Gri]) and new improvements made later by Green [G1] and Voisin [V]. The assumption on the generality allows us to extend our varieties to a family $(X, Z)/S$ of varieties over a large parameter space $S$ with the fibers $(X_s, Z_s)$ over $s \in S$. Then, by using the theory of variation of Hodge structures, we construct the cycle class map for $CH^r(Z, r)$

$$\phi^r_{Z, S} : CH^r(Z, r) \to H^0(S, \Phi^r_{Z, S}),$$

where $\Phi^r_{Z, S}$ is a sheaf on $S_{an}$, the analytic site on $S(\mathbb{C})$. We have

$$\Phi^r_{Z, S} = \begin{cases} H^2_{\mathbb{Q}, Z}(X/S)(2) \cap F^2 H^2_{\mathbb{Q}, Z}(X/S), & \text{if } r = 1, \\ \text{Ker}(H^2_{\mathbb{Q}, Z}(X/S) / H^2_{\mathbb{Q}, Z}(X/S)(3) \to \Omega^1_S \otimes H^2_{\mathbb{Q}, Z}(X/S) / F^2 H^2_{\mathbb{Q}, Z}(X/S)), & \text{if } r = 2, \end{cases}$$

where $H^q_{\mathbb{Q}, Z}(X/S) = H^q_{\mathbb{Q}, Z}(X/S)(n) \otimes \mathbb{C}$ with the Gauss-Manin connection and $F^p H^2_{\mathbb{Q}, Z}(X/S) \subset H^2_{\mathbb{Q}, Z}(X/S)$ is the holomorphic subbundle whose fibers give the Hodge filtration on $H^2_{\mathbb{Q}, Z}(X_s, \mathbb{C})$ defined by Deligne [D2]. The above map is essentially given rise to by the regulator map from higher Chow groups to Deligne cohomology (cf. [EV]). For $r = 2$ it is an analogue of the Abel-Jacobi map defined by Griffiths [Gri] after which we call the sections of $\Phi^r_{Z, S}$ normal functions. Now the key to the proof of Th.(0-1) is that one can compute the space $H^0(S, \Phi^r_{Z, S})$ by using the theory of generalized Jacobian rings developed in [AS1]. A preliminary version of this method has been used in [SMS] to prove the vanishing of Deligne classes for $d \geq 5$ in the case $r = 1$. The key steps of the present computation have already been carried out in [AS2] and [AS3], where it has been applied to the so-called Beilinson’s Hodge and Tate conjecture for $U = X - Z$. The disturbing assumption on the $e_i$’s in Th.(0-1)(2) is caused by a technical obstruction in the Jacobian ring computation in [AS3].

The second objective of the paper is to apply our Hodge theoretic invariants for the purpose of detecting non-trivial elements in the indecomposable part $CH^{r+1}(X, r)^{ind}$ of $CH^{r+1}(X, r)$. As Th.(0-1) suggests, it is not hopeful for very general $(X, Z)$, while one may still hope for the possibility to find non-trivial examples among either special families or complete families of surfaces of low degree. This is done in §5, §6 and §7. After presenting the necessary formalism of infinitesimal invariants of normal functions in §4, we will show the following results:

**Theorem(0-2).** Consider the family

$$X_{u, v} = \{ F_{u, v} = x_0^5 + x_1 x_2^4 + x_2 x_1^4 + x_3^5 + u x_1^3 x_2^3 + v x_0 x_3 K(x_0, ..., x_3) = 0 \}, \quad u, v \in \mathbb{C}$$
of quintic surfaces over \( \text{Spec}(\mathbb{C}[u,v]) \) where \( K \) is a homogenous polynomial of degree 3 with coefficients in \( \mathbb{C}[u,v] \). Then there exist elements \( \alpha_{u,v} \in CH^3(X_{u,v},2) \) supported on \( Z = X \cap \{ x_0x_3 = 0 \} \) such that, for \( u,v \in \mathbb{C} \) and \( K \) very general, these elements are indecomposable modulo the image of \( \text{Pic}(X_{u,v}) \otimes K_2(\mathbb{C}) \).

**Theorem (0-3).** On the family

\[
X_u = \{(x_0: \ldots : x_3) \in \mathbb{P}^3 \mid F_u(x) = x_0x_1^4 + x_1x_2^4 + x_2x_0^4 + x_3^5 + ux_1^4 = 0\}, \quad u \in \mathbb{C}
\]

of quintic surfaces, there exist elements \( \alpha_u \in CH^2(X_u,1) \) supported on \( Z = X \cap \{ x_0x_3 = 0 \} \) such that, for \( u \) very general, these elements are indecomposable modulo the image of \( \text{Pic}(X_u) \otimes \mathbb{C}^* \).

The following examples were provided to us by Alberto Collino in a letter from September 19, 1999. We are very grateful to him for letting us reproduce the contents here. His result shows in particular that in case of surfaces of low degree, even a very general surface can carry indecomposable cycles in \( CH^3(S,2) \) which is supported on a smooth hyperplane section and which need not be rigid on the surface \( S \):

**Theorem (0-4).** On every very general quartic \( K3 \)-surface \( S \), there exists a 1-dimensional family of elements \( \alpha_t \in CH^3(S,2) \) supported on a smooth hyperplane section of \( X \) such that, for \( t \) very general, these elements are indecomposable modulo the image of \( \text{Pic}(S) \otimes K_2(\mathbb{C}) \).

\[
\]
the moduli space of \((X \supset Z = \cup_{i \in I} Z_i)\) where \(X \hookrightarrow \mathbb{P}^d\) is a smooth hypersurface of degree \(d\) and \(Z_i \hookrightarrow X\) for \(i \in I\) is a family of smooth hypersurface sections of degree \(e_i\) such that \(Z = \cup_{i \in I} Z_i \subset X\) is a normal crossing divisor. The algebraic group \(G = \text{PGL}_4\) acts naturally on \(\hat{M}\). By [GIT], there exists a dense open subset \(\hat{M}' \subset \hat{M}\) such that

1. \(\hat{M}'\) is stable under the action of \(G\),
2. the geometric quotient \(\hat{M} = \hat{M}'/G\) exists and it is smooth over \(\text{Spec}(\mathbb{Q})\).
3. the universal family over \(\hat{M}\) descends to the family \((X \hookrightarrow \tilde{Z} = \cup_{i \in I} Z_i)/\hat{M}\).

**Definition(1-1).** Let \(k\) be a field of characteristic zero.

1. A family \((X, Z)/S\) over a field \(k\), such as in the beginning of this section, is complete, if there exist a dominant, rational map \(\pi : S \to \hat{M}\) of schemes such that \((X \hookrightarrow Z)/\pi\) is the pullback of \((X \hookrightarrow \tilde{Z})\) via \(\pi\). By the same way, in the case \(I = 0\), we define a family \(X/S\) of hypersurfaces of degree \(d\) to be complete.
2. A pair \((X, Z)\) of a smooth hypersurface \(X \subset \mathbb{P}^3\) and a normal crossing divisor \(Z \subset X\) defined over \(k\) is very general, if it is complete as a family over \(S = \text{Spec}(k)\).

Now Th.(0-1) is a direct consequence of the following:

**Theorem(1-1).** If \((X, Z)/S\) is complete and \(d \geq 5\), \(CH^2(U, 2) \to CH^1(Z, 1)^{\text{ind}}\) is surjective.

**Theorem(1-2).** Assume \((X, Z)/S\) is complete and \(d \geq 6\) and that \((e_i, e_j, e_l) \neq (1, 1, 2)\) for distinct \(i, j, l \in I\). Then \(CH^3(U, 3) \to CH^2(Z, 2)^{\text{ind}}\) is surjective.

§2. Normal functions associated to higher cycles on \(Z\).

This section contains preliminary technical results for the proofs of Th.(1-1) and Th.(1-2) in the next section. First we note that, by a well-known argument, we can reduce our problem to the case where our base field \(k\) is equal to \(\mathbb{C}\), the fields of complex numbers. Let \(S, X, Z\) be as in the beginning of §1 and assume that \(S\) is a smooth affine variety over \(\mathbb{C}\). Let \(j : X = X - Z \hookrightarrow X\) and \(i : Z \to X\) be the inclusions. Consider the following complexes of sheaves on \(X_{an}\) introduced by Deligne (cf. [EV]):

\[
\begin{align*}
Q_D(n)_X &= \text{Cone}(Q(n) \oplus \Omega_X^\geq n \to \Omega_X)[-1], \\
Q_D(r)_U &= \text{Cone}(\mathbb{R}_j^* Q(r) \oplus \Omega_X^\geq n (\log Z) \to \mathbb{R}_j \Omega_U)[-1], \\
Q_D(n)_{(Z,X)} &= \text{Cone}(Q_D(n)_X \to Q_D(n)_U)[-1].
\end{align*}
\]

By [D1][Pr.3.1.8] we have the following distinguished triangles in \(D^b(X_{an})\), the derived category of complexes of sheaves of bounded constructible cohomologies,

\[
(2 - 1) \quad i_* \mathbb{R}^i Q(n)_X \to \Omega_X^\geq n (\log Z)/W_0 [-1] \to Q_D(n)_{(Z,X)} \xrightarrow{d},
\]

where \(\Omega_X^\geq n (\log Z)\) is the truncated de Rham complex of \(X\) with logarithmic poles along \(Z\) and \(W_p = W_p \Omega_X (\log Z) \subset \Omega_X (\log Z)\) denotes the weight filtration. We define the following sheaves on \(S_{an}\)

\[
\begin{align*}
\mathcal{H}^i_D(X/S)(n) &= R^i f_* Q_D(n)_X, \\
\mathcal{H}^i_D(U/S)(n) &= R^i f_* Q_D(n)_U, \\
\mathcal{H}^i_D(Z/X/S)(n) &= R^i f_* Q_D(n)_{(Z,X)}.
\end{align*}
\]

Our fundamental tool to study higher Chow groups is the following commutative diagram

\[
\begin{array}{ccc}
CH^n(U, r + 1) & \xrightarrow{\rho^u_{n, r + 1}} & H^0(S, \mathcal{H}^{2n-r-1}_D(U/S)(n)) \\
\downarrow & & \downarrow \\
CH^{n-1}(Z, r) & \xrightarrow{\rho^{n, r}_{Z, X}} & H^0(S, \mathcal{H}^{2n-r}_D(Z/X/S)(n)) \\
\downarrow & & \downarrow \\
CH^n(X, r) & \xrightarrow{\rho^{n, r}_X} & H^0(S, \mathcal{H}^{2n-r}_D(Z/X/S)(n))
\end{array}
\]
with the right vertical sequence arising from the following distinguished triangle in $D^b(X_{an})$

$$\mathbb{Q}_D(n)_Z \rightarrow \mathbb{Q}_D(n)_X \rightarrow \mathbb{Q}_D(n)_U \rightarrow$$

and the left vertical arrow arising from localization theory for higher Chow groups. The horizontal maps are defined by the theory of Chern classes in Deligne cohomology. The commutativity of the diagram is a consequence of the functoriality of Chern class maps into Deligne cohomology (cf. [D1]).

In order to go further, we must introduce some more notations. For integers $q,n$ put

$$H^q_{\mathbb{Q}}(X/S)(n) = R^nf_\ast\mathbb{Q}(n),$$

$$H^q_{\mathbb{Q}}(U/S)(n) = R^nf_\ast\mathbb{Q}(n),$$

$$H^q_{\mathbb{Q},Z}(X/S)(n) = R^nf_\ast\mathbb{Q}(n).$$

These are local systems on $S_{an}$ and we have the long exact sequence

$$\cdots \rightarrow H^{q-1}_{\mathbb{Q}}(U/S)(n) \rightarrow H^q_{\mathbb{Q},Z}(X/S)(n) \rightarrow H^q_{\mathbb{Q}}(X/S)(n) \rightarrow H^q_{\mathbb{Q}}(U/S)(n) \rightarrow \cdots.$$ 

We also consider the local systems

$$H^q_C(X/S) = H^q_C(X/S)(n) \otimes_{\mathbb{Q}} \mathbb{C}, \quad H^q_C(U/S) = H^q_C(U/S)(n) \otimes_{\mathbb{Q}} \mathbb{C}, \quad H^q_{C,Z}(X/S) = H^q_{C,Z}(X/S)(n) \otimes_{\mathbb{Q}} \mathbb{C}$$

and locally free $\mathcal{O}$-modules

$$H^q_C(X/S) = H^q_C(X/S) \otimes_{\mathcal{O}} \mathcal{O}, \quad H^q_C(U/S) = H^q_C(U/S) \otimes_{\mathcal{O}} \mathcal{O}, \quad H^q_{C,Z}(X/S) = H^q_{C,Z}(X/S) \otimes_{\mathcal{O}} \mathcal{O},$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions on $S$. By the relative Poincaré lemma we have the canonical isomorphisms

$$H^q_C(X/S) \xrightarrow{\sim} R^qf_\ast\Omega_{X/S},$$

$$H^q_C(U/S) \xrightarrow{\sim} R^qf_\ast\Omega_{X/S}(\log Z),$$

$$H^q_{C,Z}(X/S) \xrightarrow{\sim} R^{q-1}f_\ast(\Omega_{X/S}(\log Z)/W_0),$$

where $\Omega_{X/S}(\log Z)$ is the relative de Rham complex with logarithmic poles along $Z$ and $W_p \subset \Omega_{X/S}(\log Z)$ is the weight filtration (cf. [D1]). It follows from Deligne’s theory of mixed Hodge structures that they are locally free $\mathcal{O}$-modules and that the natural maps

$$F^nH^q_{\mathcal{O}}(X/S) := R^nf_\ast\Omega^n_{X/S} \rightarrow H^q_{\mathcal{O}}(X/S),$$

$$F^nH^q_{\mathcal{O}}(U/S) := R^nf_\ast\Omega^n_{X/S}(\log Z) \rightarrow H^q_{\mathcal{O}}(U/S),$$

$$F^nH^q_{\mathcal{O},Z}(X/S) := R^{n-1}f_\ast(\Omega^n_{X/S}(\log Z)/W_0) \rightarrow H^q_{\mathcal{O},Z}(X/S)$$

are injective and their images are locally free $\mathcal{O}$-submodules. We have a long exact sequence

$$(2-3) \quad \cdots \rightarrow F^nH^q_{\mathcal{O},Z}(X/S) \rightarrow F^nH^q_{\mathcal{O}}(X/S) \rightarrow F^nH^q_{\mathcal{O}}(U/S) \rightarrow F^nH^{q+1}_{\mathcal{O}}(X/S) \rightarrow \cdots$$

and we have integrable connections

$$\nabla_X : H^q_C(X/S) \rightarrow \Omega^1_S \otimes H^q_C(X/S),$$

$$\nabla_U : H^q_C(U/S) \rightarrow \Omega^1_S \otimes H^q_C(U/S),$$

$$\nabla_Z : H^q_{C,Z}(X/S) \rightarrow \Omega^1_S \otimes H^q_{C,Z}(X/S),$$

such that $\text{Ker}(\nabla_X) = H^q_C(X/S)$, $\text{Ker}(\nabla_U) = H^q_C(U/S)$ and $\text{Ker}(\nabla_Z) = H^q_{C,Z}(X/S)$. They satisfy

$$\nabla_X(F^nH^q_{\mathcal{O}}(X/S)) \subset \Omega^1_S \otimes F^{n-1}H^q_{\mathcal{O}}(X/S),$$

$$\nabla_U(F^nH^q_{\mathcal{O}}(U/S)) \subset \Omega^1_S \otimes F^{n-1}H^q_{\mathcal{O}}(U/S),$$

$$\nabla_Z(F^nH^q_{\mathcal{O},Z}(X/S)) \subset \Omega^1_S \otimes F^{n-1}H^q_{\mathcal{O},Z}(X/S).$$
We now define the sheaf of normal functions with support in $Z$:

**Definition (2-1).** Assume $n = r + 1$ and $r = 1$ or $r = 2$. We define

$$
\Phi^r_{Z/S} = \begin{cases} 
\ker(\Omega^3_{X/S}/H^3_{\mathbb{Q}Z}(X/S)(3) \to \Omega^1 \otimes H^3_{\mathbb{Q}Z}(X/S)/F^2 H^3_{\mathbb{Q}Z}(X/S)), & \text{if } r = 2, \\
H^3_{\mathbb{Q}Z}(X/S)(2) \cap F^2 H^3_{\mathbb{Q}Z}(X/S), & \text{if } r = 1,
\end{cases}
$$

$$
\Phi^r_{U/S} = \begin{cases} 
\ker(H^3_{\mathbb{Q}Z}(U/S)/H^3_{\mathbb{Q}Z}(U/S)(3) \to \Omega^1 \otimes H^3_{\mathbb{Q}Z}(U/S)/F^2 H^3_{\mathbb{Q}Z}(U/S)), & \text{if } r = 2, \\
H^3_{\mathbb{Q}Z}(U/S)(2) \cap F^2 H^3_{\mathbb{Q}Z}(U/S), & \text{if } r = 1,
\end{cases}
$$

**Proposition (2-1).** For $r = 1$ and $r = 2$, diagram (2-2) induces the commutative diagram

$$
CH^{r+1}(U, r + 1) \xrightarrow{\phi^r_{U/S}} H^0(S, \Phi^r_{U/S}) \xrightarrow{\phi^r_{Z/S}} H^0(S, \Phi^r_{Z/S}),
$$

where the right vertical arrow is induced by the localization map $H^q_{\mathbb{Q}Z}(U/S)(n) \to H^q_{\mathbb{Q}Z}(X/S)(n)$.

**Proof.** We filter $\Omega^n_X(\log Z)/W_0$ by the subcomplexes

$$
F_S^r(\Omega^n_X(\log Z)/W_0) = \text{Im}(f^* \Omega^p_S \otimes \Omega^{n-p}(\log Z) \to \Omega^n_X(\log Z)/W_0)
$$

so that we have

$$
\text{Gr}^F_S(\Omega_X^n(\log Z)/W_0) = f^* \Omega^p_S \otimes (\Omega^{n-p}_X(\log Z)/W_0)[-p].
$$

The projection formula now gives

$$
R^{p+q}f_* \text{Gr}^F_S(\Omega^n_X(\log Z)/W_0) = \Omega^p_S \otimes H^q_{\mathbb{Q}Z}(X/S)/F^{n-p}.
$$

In view of the exact sequence (2-1), this gives rise to the spectral sequence

$$
(2-5) \quad \text{I}^{p,q}(n) \Rightarrow H^{p+q}_{D, Z}(X/S)(n) \quad \text{with} \quad \text{I}^{p,q}(n) = \begin{cases} 
H^q_{\mathbb{Q}Z}(X/S)(n) & \text{if } p = 0, \\
\Omega^{p-1}_S \otimes H^q_{\mathbb{Q}Z}(X/S)/F^{n-p+1} & \text{if } p \geq 1.
\end{cases}
$$

Note that the $d_1$-differential of the spectral sequence is induced by the connection $\nabla_Z$. In view of Lem.(2-1) below we have the edge homomorphism $H^q_{\mathbb{Q}Z}(X/S)(n) \to \Phi^r_{Z/S}$ (Note $\Phi^r_{Z/S} = \text{I}^{p-1, q}(r + 1)$). Now $\phi^r_{Z/S}$ is defined to be the composite of this map and $\rho^{r+1, r}_{Z, X}$ (cf. (2-2)). The construction of $\phi^r_{U/S}$ is parallel to that of $\phi^r_{Z/S}$ by using the following spectral sequence

$$
\text{I}^{p,q}(n) \Rightarrow H^p_{D}(U/S)(n) \quad \text{with} \quad \text{I}^{p,q}(n) = \begin{cases} 
H^q_{\mathbb{Q}Z}(U/S)(n) & \text{if } p = 0, \\
\Omega^{p-1}_S \otimes H^q_{\mathbb{Q}Z}(U/S)/F^{n-p+1} & \text{if } p \geq 1.
\end{cases}
$$

The compatibility with localization sequences and with Chern class maps to Betti cohomology follows from basic properties of Chern class maps in Deligne cohomology (cf. [EV, J]). \qed

Here are some basic vanishing results we need later:

**Lemma (2-1).** $\text{I}^{p,q}(n) = 0$ either if $q \leq 1$ or $p \neq 1, p + q \leq n$ or $p \neq 1, q = n + p + 1 > 3$.

**Proof.** The lemma follows from Lem.(2-2) below in view of the exactness of the sequence

$$
0 \to H^1_{\mathbb{Q}Z}(X/S) \to H^1_{\mathbb{Q}Z}(X/S) \xrightarrow{\nabla_x} \Omega^1 \otimes H^1_{\mathbb{Q}Z}(X/S) \xrightarrow{\nabla_x} \Omega^1 \otimes H^1_{\mathbb{Q}Z}(X/S) \xrightarrow{\nabla_x} \cdots.
$$

**Lemma (2-2).**

1. $H^1_{\mathbb{Q}Z}(X/S) = 0$.
2. $F^m H^q_{\mathbb{Q}Z}(X/S) = 0$ for $m \geq q + 1$.
3. $F^q H^q_{\mathbb{Q}Z}(X/S) = 0$ for $q > 2 = \dim(X/S)$. 
Proof. For \( p \geq 0 \) we denote
\[
Z[p] = \begin{cases} 
X & \text{if } p = 0, \\
\prod_{1 \leq i \leq m} Z_i & \text{if } p = 1, \\
\prod_{1 \leq i < j \leq m} Z_i \cap Z_j & \text{if } p = 2, \\
\emptyset & \text{if } p \geq 3.
\end{cases}
\]

We have the spectral sequence
\[
E_1^{a,b} = F^{r+a}H^{b+2a}(Z^{-a}/S) \Rightarrow F^r H^{a+b+1}_{O,S,Z}(X/S),
\]
which is constructed from the weight filtration \( W_p \subset \Omega^r_{X/S}(\log Z) \) with the isomorphism
\[
Gr_p W \Omega^r_{X/S}(\log Z) \cong (i_p)_* \Omega^{r-p}Z[-p],
\]
where \( i_p : Z[p] \to X \) is the natural morphism. The lemma follows from this noting that
\[
F^{r+a}H^{a+b+1}_{O,S,Z}(Z^{-a}/S) = 0 \quad \text{if } r > \max\{ q, d \},
\]
since \( \dim(Z^{-a}/S) = \dim(X/S) + a \).

Proposition(2-2). Let \( x \in CH^r(Z, r) \) and assume \( \phi_{Z/S}(x) = 0 \). Then \( \pi_Z(x) = 0 \), if \( r \leq 2 \) and where \( \pi_Z \) is as in Pr.(1-1).

Proof. By localization theory we have the maps
\[
H^3_{0,Z}(X/S)(2) \to H^0_{0}(Z^{[2]}/S) = g_* \mathbb{Q},
\]
\[
H^3_{0,Z}(X/S)/H^3_{0,Z}(X/S)(3) \to H^0_{0}(Z^{[2]}/S)/H^0_{0}(Z^{[2]}/S)(1) = g_* (\mathcal{O}_{Z^{[2]}}/\mathbb{Q}(1)),
\]
where \( g : Z^{[2]} \to S \) is the natural morphisms. It induces the commutative diagrams
\[
\begin{array}{ccc}
CH^1(Z, 1) & \xrightarrow{\phi^1_{Z/S}} & H^0(S, \Phi^1_{Z/S}) \\
\downarrow \pi_Z & & \downarrow \pi_Z \\
CH^0(Z^{[2]}) & \xrightarrow{\alpha} & H^0(Z^{[2]}, \mathbb{Q})
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
CH^2(Z, 2) & \xrightarrow{\phi^2_{Z/S}} & H^0(S, \Phi^2_{Z/S}) \\
\downarrow \pi_Z & & \downarrow \pi_Z \\
CH^1(Z^{[2]}, 1) & \xrightarrow{\beta} & H^0(Z^{[2]}, \mathcal{O}_{Z^{[2]}}/\mathbb{Q}(1)),
\end{array}
\]
where \( \alpha \) is the obvious map and \( \beta \) is the isomorphism \( CH^1(Z^{[2]}, 1) \cong \mathcal{O}_{Zar}(Z^{[2]})^* \) followed by the logarithm. Either of the maps is injective and the desired assertion is proven in view of Pr.(1-1). □

§3. PROOF OF THE MAIN RESULTS.

In this section we prove Th.(1-1) and Th.(1-2). By Pr.(2-2) they follow from the following claims. Recall that \( \delta : CH^{r+1}(U, r + 1) \to CH^r(Z, r) \) denotes the boundary map.

Claim(3-1). Under the assumption of Th.(1-1), \( \text{Im}(\phi^1_{Z/S}) = \text{Im}(\phi^1_{Z/S} \cdot \delta) \).

Claim(3-2). Under the assumption of Th.(1-2), \( \text{Im}(\phi^2_{Z/S}) = \text{Im}(\phi^2_{Z/S} \cdot \delta) \).

First we prove Claim(3-1).

Definition(3-1). With the notation as in the beginning of §1, let \( M_\Omega(U/S) \subset H^2_{0}(U/S)(2) \) be the (constant) subsheaf generated by the sections \( \omega_{ij} := \phi^1_{U/S}([g_i, g_j]) \) for \( 1 \leq i < j \leq m - 1 \).

We need the following result from [AS2].
**Theorem (3-1).** If \((X, Z)\) is complete and \(d \geq 4\), \(H^0(S, H^3_{O,S}(X/S)(2)) = M_{Q}(U/S)\). In particular \(\phi_{U/S}^1\) is surjective.

By the above theorem, Claim (3-1) in the case \(r = 1\) follows from the following.

**Proposition (3-1).** Assuming \((X, Z)\) is complete and \(d \geq 5\), \(H^0(S, H^3_{Q,S}(U/S)(2)) \rightarrow H^0(S, H^3_{Q,Z}(X/S)(2))\) is surjective.

We note that it suffices to show the surjectivity of \(H^0(S, H^3_{Q,S}(U/S)) \rightarrow H^0(S, H^3_{Q,Z}(X/S))\).

**Lemma (3-1).** Under the assumption of Pr. (3-1), \(H^0(S, H^3_{Q,Z}(X/S)(2)) \subset H^0(S, H^3_{Q,Z}(X/S) \cap F^2 H^3_{O,Z}(X/S))\).

**Proof.** Localization theory provides the exact sequence \(0 \rightarrow K_Q \rightarrow H^3_{Q,Z}(X/S)(2) \rightarrow J_Q \rightarrow 0\) with
\[
K_Q = \bigoplus_{i \in I} H^1_{Q}(Z/I/S)(1), 
J_Q = \text{Ker}(H^0_{Q}(Z[2]/S) \rightarrow \bigoplus_{i \in I} H^2_{Q}(Z/I/S)(1))).
\]

We have \(H^0(S, K_Q) = 0\), which follows from [AS2, Th.(4-2)] or the monodromy argument of Green and Voisin (cf. [G1]). This proves the desired assertion. \(\square\)

By Lem. (3-1) the proof of Pr. (3-1) is reduced to the following.

**Lemma (3-2).** Under the assumption of Pr. (3-1), the following map is surjective
\[
H^0(S, H^2_{O}(U/S) \cap F^2 H^2_{O}(U/S)) \rightarrow H^0(S, H^3_{O,Z}(X/S) \cap F^2 H^3_{O,Z}(X/S)).
\]

**Proof.** Note that we have the exact sequence
\[
0 \rightarrow H^2_{Q}(X/S)_{pr}(2) \rightarrow H^2_{Q}(U/S)(2) \rightarrow H^3_{Q,Z}(X/S)(2) \rightarrow 0,
\]
where \(H^2_{Q}(X/S)_{pr}(2)\) is the primitive part of \(H^2_{Q}(X/S)(2)\). It induces the following commutative diagram
\[
\begin{array}{cccc}
0 & \rightarrow & H^2_{Q}(X/S)_{pr}(2) & \rightarrow H^2_{Q}(U/S)(2) & \rightarrow H^3_{Q,Z}(X/S)(2) & \rightarrow 0, \\
\downarrow & \searrow & \downarrow & \searrow & \downarrow & \\
F^2 H^2_{O}(X/S)_{pr} & \rightarrow & \Omega^1_S \otimes F^1 H^2_{O}(X/S)_{pr} & \rightarrow & \Omega^2_S \otimes F^0 H^2_{O}(X/S)_{pr} & \rightarrow \Omega^3_S \otimes F^{-1} H^2_{O}(X/S)_{pr} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
F^2 H^2_{O}(U/S) & \rightarrow & \Omega^1_S \otimes F^1 H^2_{O}(U/S) & \rightarrow & \Omega^2_S \otimes F^0 H^2_{O}(U/S) & \rightarrow \Omega^3_S \otimes F^{-1} H^2_{O}(U/S) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
F^2 H^3_{O,Z}(X/S) & \rightarrow & \Omega^1_S \otimes F^1 H^3_{O,Z}(X/S) & \rightarrow & \Omega^2_S \otimes F^0 H^3_{O,Z}(X/S) & \rightarrow \Omega^3_S \otimes F^{-1} H^3_{O,Z}(X/S) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \\
\end{array}
\]
where the vertical sequences are exact. Noting that \(S\) is affine, it implies that Lem. (3-2) follows from the exactness at the middle of the upper horizontal sequence, which in turn follows from the following result, a consequence of the theory of Jacobian rings (cf. [G2]).

**Proposition (3-2).** Consider the following complex
\[
\Omega^p_{S} \otimes H^{1+q-1-q}_O(X/S)_{pr} \xrightarrow{\nabla_X} \Omega^p_{S} \otimes H^{2+q-2-q}_O(X/S)_{pr} \xrightarrow{\nabla_X} \Omega^{p+1}_{S} \otimes H^{3+q-3-q}_O(X/S)_{pr},
\]
where \(H^{a,b}_O(X/S) = Gr^a_p H^{a,b}_O(X/S) = R^b f_* \Omega^a_{X/S}\) and \(H^{a,b}_O(X/S)_{pr}\) denotes the primitive part of \(H^{a,b}_O(X/S)\).

By convention \(\Omega^r_S = 0\) if \(r < 0\). Assuming \((X, Z)\) is complete, the complex is exact if \(q \geq 1\) and \(dq \geq p+4\).

We now show Claim (3-2).

**Definition (3-2).** We define
\[
\Psi^2_{Z/S} = \mathrm{Ker}(\Omega^1_S \otimes F^2 H^3_{O,Z}(X/S) \xrightarrow{\nabla} \Omega^2_S \otimes F^1 H^3_{O,Z}(X/S)),
\]
We have the following maps induced by $\phi$ be the $C$-linear subspace generated by $\psi$.

**Lemma (3-4).** Assuming Theorem (3-2).

By definition we have the exact sequences

$$0 \rightarrow H^0_C(X/S)/H^0_Q(Z(X/S)(3)) \rightarrow \Phi^2_{Z/S} \rightarrow \Psi^2_{Z/S},$$

$$0 \rightarrow H^0_C(U/S)/H^0_Q(U/S)(3) \rightarrow \Phi^2_{U/S} \rightarrow \Psi^2_{U/S}.$$  

We have the following maps induced by $\phi^2_{Z/S}$ and $\phi^2_{U/S}$ respectively

$$\psi^2_{Z/S} : CH^2(Z, 2) \otimes C \rightarrow H^0(S, \Psi^2_Z), \quad \psi^2_{U/S} : CH^3(U, 3) \otimes C \rightarrow H^0(S, \Psi^2_U).$$

**Definition (3-3).** With the notation as in the beginning of §1, let

$$N_C(U/S) \subset H^0(S, \Psi^2_U) \subset H^0(S, \Omega^1_S \otimes F^2H^0_C(U/S))$$

be the $C$-linear subspace generated by $\psi^2_U((g_1, g_2))$ with $1 \leq i < j < k \leq m - 1$.

We need the following result from [AS3].

**Theorem (3-2).** Under the assumption of Th. (1-2),

$$\Omega^1_{S,d=0} \otimes M_C(U/S) \otimes N_C(U/S) \rightarrow \Psi^2_U,$$

where $M_C(U/S) = M_Q(U/S) \otimes C \subset H^0_C(U/S) \cap F^2H^0_C(U/S)$.

Thanks to Th. (3-2), Claim (3-2) follows from the following three lemmas.

**Lemma (3-3).** Consider the composite map

$$\alpha : N_C(U/S) \otimes M_C(U/S) \otimes H^0(S, \Omega^1_{S,d=0}) \rightarrow H^0(S, \Psi^2_U) \rightarrow H^0(S, \Psi^2_Z).$$

Assume $\alpha(x + y) \in \text{Im}(\psi^2_Z)$ for $y \in N_C(U/S)$ and

$$x = \sum_{1 \leq i < j \leq m - 1} \omega_{ij} \otimes c_{ij} \in M_C(U/S) \otimes H^0(S, \Omega^1_{S,d=0}) \quad (\text{cf. Def. (3-1)}).$$

Then $c_{ij}$ is contained in the image of $dlog : CH^1(S, 1) \otimes C \rightarrow H^0(S, \Omega^1_{S,d=0})$.

**Lemma (3-4).** Assuming $(X, Z)$ is complete and $d \geq 5$, we have

$$\text{Ker}(H^0(S, \Psi^2_Z) \rightarrow H^0(S, \Psi^2_Z)) = \sum_{1 \leq i < j \leq m - 1} \phi^2_Z(\delta\{g_i, g_j, x\}).$$

**Lemma (3-5).** Assuming $(X, Z)$ is complete and $d \geq 6$, $H^0(S, \Psi^2_Z) \rightarrow H^0(S, \Psi^2_Z)$ is surjective.

**Proof of Lem. (3-3).** We have the commutative diagram

$$\begin{array}{ccc}
\text{CH}^3(U, 3) \otimes C & \xrightarrow{\delta} & \text{CH}^2(Z, 2) \otimes C \\
\downarrow \psi^2_U & & \downarrow \psi^2_Z \\
H^0(S, \Psi^2_U) & \xrightarrow{\beta} & H^0(S, \Psi^2_Z) \\
\downarrow \text{dlog} & & \downarrow \text{dlog}
\end{array}
$$

(3-3-1)

where $\beta$ is induced by $\text{Res}_Z : H^3_{Q,Z}(X/S)(2) \rightarrow H^0_{Q,Z}(Z^[2]/S) = g_* \mathbb{Q}_{Z^[2]}$ with $g : Z^[2] \rightarrow S$ the projection.

By a standard norm argument, we may prove Lem. (3-3) after a finite étale base change of $S$, so that
we may assume $Z^{[2]}$ is a disjoint union of copies of $S$. This implies that the horizontal maps $\iota$ in the following commutative diagram are isomorphisms

\[(3-3-2) \quad H^0(Z^{[2]}, \mathbb{Q}) \otimes CH^1(S, 1) \otimes \mathbb{C} \xrightarrow{\partial} CH^1(Z^{[2]}, 1) \otimes \mathbb{C} \]

\[\xrightarrow{d \log} \quad H^0(Z^{[2]}, \mathbb{Q}) \otimes H^0(S, \Omega^1_{S, d=0}) \xrightarrow{\partial} H^0(Z^{[2]}, \Omega^1_{Z^{[2]}, d=0}).\]

By (3-3-1), the assumption of Lem.(3-5) follows from the exactness at the middle of $\Omega^1_{S, d=0}$. From this Lem.(3-5) follows from the exactness at the middle notation in Proof of Lem.(3-5).

As in the proof of Lem.(3-2) we have the commutative diagram

\[
\begin{array}{ccc}
H^0(Z^{[2]}, \mathbb{Q}) & \xrightarrow{\partial} & CH^1(Z^{[2]}, 1) \\
\downarrow{d \log} & & \downarrow{d \log} \\
H^0(Z^{[2]}, \mathbb{Q}) \otimes H^0(S, \Omega^1_{S, d=0}) & \xrightarrow{\partial} & H^0(Z^{[2]}, \Omega^1_{Z^{[2]}, d=0})
\end{array}
\]

where the injectivity of the above map follows from the first exact sequence in Def.(3-2). This proves Lem.(3-4) since we have

\[
\sum_{1 \leq i < j \leq m-1} \iota(Res_U(\omega_{ij}) \otimes c_{ij}) \in \text{Im}(d \log : CH^1(Z^{[2]}, 1) \otimes \mathbb{C} \to H^0(Z^{[2]}, \Omega^1_{Z^{[2]}, d=0})),
\]

where $Res_U : M_Q(U/S) \to H^0(Z^{[2]}, \mathbb{Q})$ is the composite of $M_Q(U/S) \to H^3_{Q,Z}(X/S)(2)$ and $Res_Z$. This implies the desired assertion due to (3-3-2) and the linear independence of $Res_U(\omega_{ij})$ with $1 \leq i < j \leq m-1$, which is easily checked.

\[\square\]

**Proof of Lem.(3-4).** By Th.(3-1) and Pr.(3-1) we have

\[
M_Q(U/S) \otimes \mathbb{C}/\mathbb{Q}(1) \xrightarrow{\sim} H^0(S, H^3_{Q,Z}(X/S)/H^3_{Q,Z}(X/S)(3)) \xrightarrow{\sim} \text{Ker}(H^0(S, \Phi^2_{Z/S}) \to H^0(S, \Psi^2_{Z/S})),
\]

where the injectivity of the above map follows from Lem.(3-3) and the second isomorphism follows from the first exact sequence in Def.(3-2). This proves Lem.(3-4) since we have

\[
\phi^2_{Z/S}([g_i, g_j, x]) = \omega_{ij} \otimes \log x \in M_Q(U/S) \otimes \mathbb{C}/\mathbb{Q}(1) \quad \text{for} \quad x \in \mathbb{C}^*.
\]

\[\square\]

**Proof of Lem.(3-5).** As in the proof of Lem.(3-2) we have the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{0} & 0 \\
\downarrow & & \downarrow \\
\Omega^3_S \otimes F^2 H^3_{Q,Z}(X/S)_{pr} & \xrightarrow{\nabla} & \Omega^3_S \otimes F^1 H^3_{Q,Z}(X/S)_{pr} \\
\downarrow & & \downarrow \\
\Omega^1_S \otimes F^2 H^2_{Q,Z}(U/S) & \xrightarrow{\nabla} & \Omega^1_S \otimes F^1 H^2_{Q,Z}(U/S) \\
\downarrow & & \downarrow \\
\Omega^3_S \otimes F^2 H^3_{Q,Z}(X/S) & \xrightarrow{\nabla} & \Omega^3_S \otimes F^1 H^3_{Q,Z}(X/S) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{0} & 0
\end{array}
\]

where the vertical sequences are exact. From this Lem.(3-5) follows from the exactness at the middle of the upper horizontal sequence, which follows from Pr.(3-2).

\[\square\]

§4. **Indecomposable parts of infinitesimal invariants.**

Let the notation be as in the beginning of §1 and assume additionally the condition:

\[(4-1) \quad H^3_{Q}(X/S) = 0.\]

We want to capture elements of $CH^r(Z, r)$ whose image in $CH^{r+1}(X, r)$ are indecomposable in the sense defined below. For this purpose we use the infinitesimal invariants of normal functions. Recall the notation in §2. By definition we have the natural maps

\[
\begin{align*}
\mathcal{H}^0_n(X/S)(n) & \to R^q f_* \Omega^\geq n_X, \\
\mathcal{H}^0_n(U/S)(n) & \to R^q f_* \Omega^\geq n_X(\log Z), \\
\mathcal{H}^0_{D,Z}(X/S)(n) & \to R^{q-1} f_* \Omega^\geq n_X(\log Z)/W_0.
\end{align*}
\]
We denote the above maps by the same letter \( \delta \). The commutative diagram (2-2) gives rise to the following commutative diagram

\[
\begin{align*}
CH^n(U, r + 1) & \xrightarrow{\delta \rho_{r+1}} \Gamma(S, R^{2n-r-1}f_*\Omega_X^{\geq n}(\log Z)) \\
\downarrow & \\
CH^{n-1}(Z, r) & \xrightarrow{\delta \rho_{r}} \Gamma(S, R^{2n-r-1}f_*\Omega_X^{\geq n}(\log Z)/W_0) \\
\downarrow & \\
CH^n(X, r) & \xrightarrow{\delta \rho_{r}} \Gamma(S, R^{2n-r}f_*\Omega_X^{\geq n})
\end{align*}
\]

with the right vertical sequence arising from the exact sequence

\[
0 \to \Omega_X^{\geq n} \to \Omega_X^{\geq n}(\log Z) \to \Omega_X^{\geq n}(\log Z)/W_0 \to 0.
\]

We proceed in a similar manner for the construction of the spectral sequence (2-5). Filtering \( \Omega_X^{\geq n}(\log Z)/W_0 \) by the subcomplexes

\[
F_p^s\Omega_X^{\geq n}(\log Z)/W_0 = \text{Im}(f^*\Omega_S^p \otimes \Omega_X^{\geq n-p}(\log Z) \to \Omega_X^{\geq n}(\log Z)/W_0),
\]

we have

\[
Gr^p_r(\Omega_X^{\geq n}(\log Z)/W_0) = f^*\Omega_S^p \otimes (\Omega_X^{\geq n-p}(\log Z)/W_0).
\]

Thus we obtain the spectral sequence

\[
(4-3) \quad E^{p,q}_1 = \Omega_S^p \otimes F^{n-p}H^q_{D, Z}(X/S) \Rightarrow R^{p+q-1}f_*\Omega_X^{\geq n}(\log Z)/W_0.
\]

Using the same arguments, we also construct the spectral sequence

\[
(4-4) \quad E^{p,q}_1 = \Omega_S^p \otimes F^{n-p}H^q_{D, Z}(X/S) \Rightarrow R^{p+q}f_*\Omega_X^{\geq n}.
\]

The following construction provides a more intrinsic definition of \( \psi_{Z/S}^2 \) given in Def.(3-2).

**Definition (4-1).** Let \( n = r + 1 \geq 2 \) be an integer.

1. We define

\[
\Psi_{Z/S}^r = \text{Ker}(\Omega_S^{-1} \otimes F^2H^3_{D, Z}(X/S) \xrightarrow{\nabla} \Omega_S^r \otimes F^1H^3_{D, Z}(X/S)),
\]

where \( \nabla \) is induced by the connection (2-4) and coincides with \( d_1 \)-differential of (4-3). By definition \( \Psi_{Z/S}^r = \Omega_S^{-1} \otimes F^2H^3_{D, Z}(X/S) \xrightarrow{\nabla} \Omega_S^r \otimes F^1H^3_{D, Z}(X/S)) \),

where \( \nabla \) is induced by the connection (2-4) and coincides with \( d_1 \)-differential of (4-3). By definition \( \Psi_{Z/S}^r = \Omega_S^{-1} \otimes F^2H^3_{D, Z}(X/S) \xrightarrow{\nabla} \Omega_S^r \otimes F^1H^3_{D, Z}(X/S) \),

we define the map

\[
\psi_{Z/S}^r : CH^r(Z, r) \to \Gamma(S, \Psi_{Z/S}^r)
\]

to be the composite of \( \delta \rho_{Z,X}^{r+1} \) and the map induced by \( R^{r+1}f_*\Omega_X^{r+1}(\log Z)/W_0 \to \Psi_{Z/S}^r \) that is an edge homomorphism of the spectral sequence (4-3).

2. We define \( \Psi_{X/S}^r = \Omega_S^{-1} \otimes F^2H^3_{D, Z}(X/S) \xrightarrow{\nabla} \Omega_S^r \otimes F^1H^3_{D, Z}(X/S) \),

which is the homology of the complex

\[
\Omega_S^{-1} \otimes F^2H^3_{D, Z}(X/S) \xrightarrow{\nabla} \Omega_S^r \otimes F^1H^3_{D, Z}(X/S) \xrightarrow{\nabla} \Omega_S^{r+1} \otimes F^0H^3_{D, Z}(X/S)
\]

and define

\[
\psi_{X/S}^r : CH^{r+1}(X, r) \to \Gamma(S, \Psi_{X/S}^r)
\]

to be the composite of \( \delta \rho_{X}^{r+1} \) and the map induced by \( R^{r+1}f_*\Omega_X^{r+1} \to \Psi_{X/S}^r \) that is an edge homomorphism of (4-4).
Note that the edge homomorphisms exist due to the fact that on one hand \( H^p q(Z) = 0 \) if \( q \leq 1 \) or \( p + q \leq n \) or \( q = n - p + 1 > 3 \) by Lem.(2-2) and that on the other hand \( H^p q(X) = 0 \) unless \( q = 0, 2, 4 \) by the assumption (4-1) and that \( H^p q(X) = \Omega^p S \otimes F_{r-p+1} H^2 \mathbb{O}(X/S) = 0 \) if \( p \leq r - 2 \).

Next we consider the indecomposable part
\[
CH^{r+1}(X, r)^{ind} = \text{Coker}(CH^r(S, r) \otimes \mathbb{C} CH^1(X) \to CH^{r+1}(X, r)).
\]
Write
\[
F^p H^q \mathbb{O}(X/S) = \text{Ker}(F^p H^q \mathbb{O}(X/S) \xrightarrow{\nabla X} \Omega^1 S \otimes F^p-1 H^2 \mathbb{O}(X/S)),
\]
which is \( \mathbb{C} \)-vector space. Let
\[
cl_X : CH^1(X) \to \Gamma(S, F^1 H^2 \mathbb{O}(X/S))
\]
be the composite of the Chern class map \( \delta \rho^0 X : CH^1(X) \to \Gamma(S, R^2 f_* \Omega^{21} X) \) and the map induced by
\[
R^2 f_* \Omega^{21} X \to F^1 H^2 \mathbb{O}(X/S)
\]
that is an edge homomorphism of (4-4) with \( n = r + 1 = 1 \) \( (H^2 E^0_2 (X) = F^1 H^2 \mathbb{O}(X/S))) \). Let
\[
\psi^1 S : CH^r(S, r) \to \Gamma(S, \Omega^r S, d=0)
\]
be induced by \( \delta \rho^r X \) with \( X = S \) (cf. (4-2)). Define the map
\[
\alpha = \psi^r S \otimes cl_X : CH^r(S, r) \otimes \mathbb{C} CH^1(X) \to \Gamma(S, \Omega^r S, d=0 \otimes \mathbb{C} F^1 H^2 \mathbb{O}(X/S)).
\]

**Proposition(4-1).** We have the commutative diagram
\[
\begin{array}{ccc}
CH^r(S, r) \otimes \mathbb{C} CH^1(X) & \xrightarrow{\beta} & \Gamma(S, \Omega^r S, d=0 \otimes \mathbb{C} F^1 H^2 \mathbb{O}(X/S)) \\
\downarrow \beta & & \downarrow \gamma \\
CH^{r+1}(X, r) & \xrightarrow{\psi^r S} & \Gamma(S, \Psi^r X/S),
\end{array}
\]
where \( \beta \) is induced by the product structure on higher Chow groups and \( \gamma \) is induced by the natural map
\[
\Omega^r S, d=0 \otimes \mathbb{C} F^1 H^2 \mathbb{O}(X/S) \to \Omega^r S \otimes \mathbb{C} F^1 H^2 \mathbb{O}(X/S).
\]

**Proof.** We consider the following diagram
\[
\begin{array}{ccc}
CH^r(S, r) \otimes CH^1(X) & \xrightarrow{\beta} & CH^{r+1}(X, r) \\
\downarrow \delta \rho^r S \otimes \delta \rho^1 X & & \downarrow \delta \rho^r X \\
\Gamma(S, \Omega^r S, d=0) \otimes \mathbb{C} \Gamma(S, R^2 f_* \Omega^{21} X) & \xrightarrow{\epsilon} & \Gamma(S, R^+ f_* \Omega^{21+1} X) \\
\downarrow & & \downarrow \\
\Gamma(S, \Omega^r S, d=0) \otimes \mathbb{C} \Gamma(S, F^1 H^2 \mathbb{O}(X/S)) & \xrightarrow{\psi^r S} & \Gamma(S, \Psi^r X/S).
\end{array}
\]
Here \( \delta \rho^r S \) and \( \delta \rho^1 X \) are Chern class maps (cf. (2-2)) and \( \epsilon \) is induced by applying \( R^+ f_* \) to the product
\[
f^* \Omega^r S \otimes \Omega^{21} X \xrightarrow{r-1} \Omega^{2r+1} X.
\]
The upper diagram is commutative due to the compatibility of Chern class map with product. The commutativity of the lower diagram is easily seen. Now Pr.(4-1) follows from the fact that \( \rho^r S = \psi^r S \).

**Definition(4-2).** We define the indecomposable part of \( \Psi^r X/S \)
\[
\Psi^r X/S, ind = \text{Coker}(\Omega^r S, d=0 \otimes \mathbb{C} F^1 H^2 \mathbb{O}(X/S) \to \Psi^r X/S),
\]
and the map
\[ \psi^r_{X/S} : CH^{r+1}(X,r)^{ind} \to \Gamma(S,\Psi^r_{X,S,ind}), \]
which is induced by the commutative diagram of Pr.(4-1).

Now the commutative diagram (2-2) gives rise to the commutative diagram
\[ (4 - 5) \]
\[
\begin{array}{ccc}
CH^r(Z, r) & \xrightarrow{\psi^r_{Z/S}} & \Gamma(S, \Psi^r_{Z/S}) \\
\downarrow & & \downarrow \\
CH^{r+1}(X, r)^{ind} & \xrightarrow{\psi^r_{X/S}} & \Gamma(S, \Psi^{r+1}_{X, S, ind})
\end{array}
\]
where \( \lambda \) is obtained as follows. We consider the commutative diagram
\[ (4 - 6) \]
\[
\begin{array}{ccc}
\Omega^{-1}_S \otimes F^2 H^3_0(X/S) & \xrightarrow{\nabla_X} & \Omega^1_S \otimes F^1 H^3_0(X/S) \\
\downarrow & & \downarrow \\
\Omega^{-1}_S \otimes F^2 H^3_0(U/S) & \xrightarrow{\nabla_U} & \Omega^1_S \otimes F^1 H^3_0(U/S) \\
\downarrow & & \downarrow \\
\Omega^{-1}_S \otimes F^2 H^3_{0, Z}(X/S) & \xrightarrow{\nabla_Z} & \Omega^1_S \otimes F^1 H^3_{0, Z}(X/S) \\
\downarrow & & \\
& & 0.
\end{array}
\]
All vertical sequences are exact. The surjectivity of \( \beta \) is a consequence of the assumption (4-1). Putting
\[ H^q_{X,Z}(X/S) = \text{Ker}(H^3_{0, Z}(X/S) \xrightarrow{\nabla_Z} \Omega^1_S \otimes H^3_{0, Z}(X/S)), \]
the spectral sequence (2-6) gives rise to the isomorphism
\[ H^q_{X,Z}(X/S) \sim \bigoplus_{i \in I} \text{Ker}(\mathcal{O}_{Z_i} \xrightarrow{d} \Omega^1_{Z_i}) \sim \bigoplus_{i \in I} \mathbb{C}. \]
Thus we have
\[ \text{Ker}(\gamma) = \Omega^1_S \otimes_{\mathbb{C}} \text{Im}(H^3_{0, Z}(X/S) \to H^2_0(X/S)) = \Omega^1_S \otimes_{\mathbb{C}} \text{Im}(H^3_{X,Z}(X/S) \to H^2_{X,Z}(X/S)) \subset \Omega^1_S \otimes_{\mathbb{C}} F^1 H^2_{X,Z}(X/S), \]
so that \( \text{Ker}(\gamma) \cap \text{Ker}(\nabla_X') \subset \Omega^1_{S,d=0} \otimes_{\mathbb{C}} F^1 H^3_{X,Z}(X/S). \) Hence the commutative diagram (4-6) gives rise to the map
\[ \text{Ker}(\nabla_Z) \to \text{Coker}(\Omega^1_{S,d=0} \otimes_{\mathbb{C}} F^1 H^3_{X,Z}(X/S) \to \text{Ker}(\nabla_X')/\text{Im}(\nabla_X)), \]
which is defined to be \( \lambda. \)

For later use, we introduce the following linear version of the above construction.

**Definition (4-3).**

1. We define \( \Lambda^r_{X/S} \) to be the homology of the complex of \( \mathcal{O} \)-modules
\[ \Omega^{-1}_S \otimes H^3_{0, S}(X/S) \xrightarrow{\nabla_X} \Omega^1_S \otimes H^3_{1, S}(X/S)_{pr} \xrightarrow{\nabla_X} \Omega^1_S \otimes H^0_{0, S}(X/S), \]
where \( H^p_{0, S}(X/S) = \text{Gr}_p H^p_{0, S}(X/S) = R^p\tau^r_{X/S} \) and \( H^3_{0, S}(X/S)_{pr} \) is the primitive part of \( H^1_{0, S}(X/S) \). We have the canonical map \( \Psi^r_{X/S} \to \Lambda^r_{X/S} \) and we put
\[ \Lambda^r_{X, S, ind} = \text{Coker}(\Omega^1_{S,d=0} \otimes_{\mathbb{C}} F^1 H^3_{X,Z}(X/S) \to \Psi^r_{X/S} \to \Lambda^r_{X/S}). \]
The map \( \psi^r_{X/S} \)
\[ \tau^r_{X/S} : CH^{r+1}(X,r)^{ind} \to \Gamma(S, \Lambda^r_{X, S, ind}). \]
We define
\[ \Lambda_{Z/S}^r = \text{Ker}(\Omega_{Z/S}^r \otimes H^{2r}_{O,Z}(X/S) \rightarrow \Omega_{Z/S}^{r+1} \otimes H^{1,r}_{O,Z}(X/S)), \]
where \( H^{p,q}_{O,Z}(X/S) = Gr^p_F H^{p+q}_{O,Z}(X/S) = R^{p+q}f_* \Omega_{X/S}^p(\log Z)/W_0 \). By Lem.(2-2)(1) we have the natural injection \( \Psi_{Z/S} \hookrightarrow \Lambda_{Z/S}^r \) and the map \( \psi_{Z/S}^r \) induces
\[ \tau_{Z/S}^r : CH^r(Z, r) \rightarrow \Gamma(S, \Lambda_{Z/S}^r). \]

The commutative diagram (4-5) gives rise to the commutative diagram
\[ CH^r(Z, r) \quad \tau_{Z/S}^r \quad \Gamma(S, \Lambda_{Z/S}^r) \quad \Lambda \]
\[ CH^{r+1}(X, r)^{\text{ind}} \quad \tau_{X/S}^r \quad \Gamma(S, \Lambda_{X/S}^{r, \text{ind}}) \]
where the right vertical arrow is induced by the commutative diagram
\[ \begin{align*}
\Omega_{S}^{r-1} \otimes H^{2,0}_{O}(X/S) & \xrightarrow{\nabla_x} \Omega_{S} \otimes H^{1,1}_{O}(X/S)_{pr} \xrightarrow{\nabla_x} \Omega_{S}^{r+1} \otimes H^{0,2}_{O}(X/S) \\
\Omega_{S}^{r-1} \otimes H^{2,0}_{O}(U/S) & \xrightarrow{\nabla_u} \Omega_{S} \otimes H^{1,1}_{O}(U/S) \xrightarrow{\nabla_u} \Omega_{S}^{r+1} \otimes H^{0,2}_{O}(U/S) \\
\Omega_{S}^{r-1} \otimes H^{2,1}_{O,Z}(X/S) & \xrightarrow{\nabla_z} \Omega_{S} \otimes H^{1,2}_{O,Z}(X/S) \xrightarrow{0}
\end{align*} \]
with \( H^{p,q}_{O}(U/S) = Gr^p_F H^{p+q}_{O}(U/S) = R^{p+q}f_* \Omega_{X/S}^p(\log Z) \).

§5. An Example in \( CH^3(X, 2) \).

We mostly keep the notations of the previous paragraphs. In this section we prove Th.(0-2) which provides an example of classes in \( CH^3(X, 2) \) on a family of smooth projective complex surfaces \( X_{u,v} \), which are indecomposable modulo the image of \( K_2(\mathbb{C}) \otimes CH^3(X) \). We state the result here again.

**Proposition(5-1).** Consider the family
\[ X_{u,v} = \{ F_{u,v} = x_0^5 + x_1 x_2^4 + x_2 x_1^4 + x_3^5 + ux_1^2 x_2^3 + vx_0 x_3 | x_0, ..., x_3 = 0 \}, \quad u, v \in \mathbb{C} \]
of quintic surfaces over \( \text{Spec}(\mathbb{C}[u, v]) \) where \( K \) is a homogenous polynomial of degree 3 with coefficients in \( \mathbb{C}[u, v] \). Then there exist elements \( \alpha_{u,v} \) in \( CH^3(X_{u,v}, 2) \) supported on \( Z = X \cap \{ x_0 x_3 = 0 \} \) such that, for \( u, v \in \mathbb{C} \) and \( K \) very general, these elements are indecomposable modulo the image of \( \text{Pic}(X_{u,v}) \otimes K_2(\mathbb{C}) \).

Note that the parameter \( v \) is redundant since it is contained in the coefficients of the cubic form \( K \) already. However in the proof we fix \( K = x_0^5 x_1 + x_0 x_1 x_2 + x_0 x_2^2 + x_1^3 + x_0 x_1 x_3 \) and vary \( u \) and \( v \) in a 2-dimensional local parameter space to compute the infinitesimal data. We suppress \( K \) in the notation most of the time.

Our examples are deformations of the following quintic hypersurface in \( \mathbb{P}^3 \):
\[ X := \{ (x_0 : ... : x_3) \in \mathbb{P}^3 | F(x) = x_0^5 + x_1 x_2^4 + x_2 x_1^4 + x_3^5 = 0 \}. \]
The gradient of \( F \) is \((5x_0^4 x_1 : x_0^2 x_2^3 : x_1^5 : 4x_1^3 x_2 : 4x_2^3 x_1 : 5x_2^4)\). It is nowhere zero, therefore \( X \) is smooth. Now we cut out the hyperplane sections \( Z_1 := X \cap \{ x_3 = 0 \} \) and \( Z_2 := X \cap \{ x_0 = 0 \} \). Again \( Z_1 \) and \( Z_2 \) are smooth since the gradients of \( Z_1, Z_2 \) have no zeroes. The intersection
\[ Z_1 \cap Z_2 = \{ (x_1 : x_2) \in \mathbb{P}^1 | x_1 x_2 (x_2^3 + x_1^3) = 0 \} \]
consists of 5 distinct points $P_1 = (0 : 0 : 1 : 0), P_2 = (0 : 1 : 0 : 0), P_3 = (0 : -1 : 1 : 0), P_4 = (0 : -\zeta : 1 : 0)$ and $P_5 = (0 : -\zeta^2 : 1 : 0)$, where $\zeta^3 = 1$. The automorphism group of $X$ contains two copies of $\mathbb{Z}/5\mathbb{Z}$ which are generated by the diagonal matrices $\sigma_1 := (1 : \eta : \eta : \eta)$ and $\sigma_5 := (\eta : \eta : \eta : 1)$ respectively with $\eta^5 = 1$. $\sigma_i$ operates on $\mathbb{Z}_i$ and fixes $\mathbb{Z}_{2-i}$ pointwise. The fixed points of each operation of $\sigma_i$ on $\mathbb{Z}_i$ are exactly the points $P_1, ..., P_5$. By the Hurwitz formula the quotient of $Z_1$ by $\mathbb{Z}/5\mathbb{Z}$ is a rational curve and the quotient map is ramified in these five points. Hence on both curves we have the relations

$$5P_1 = 5P_2 = 5P_3 = 5P_4 = 5P_5 \in CH^1(Z_1),$$

in particular, on both curves, $5(P_1 - P_2)$ and $5(P_3 - P_4)$ are rationally equivalent. On $Z_1$ choose the function $f_1 = z$, where $z = x_1/x_2$ is the pullback of the standard coordinate function on $Z_1/\sigma_1 \cong \mathbb{P}^1$. Then $div(f_1) = 5(P_1 - P_2)$ and $f_1(P_3) = -1, f_1(P_4) = -\zeta$. Let $g_1 := \frac{\delta + 1}{z + \zeta^2}$. Then $div(g_1) = 5(P_3 - P_4)$, $g_1(P_1) = \zeta, g_1(P_2) = 1$ and the tame symbol is

$$T(f_1, g_1) = (\zeta^5, 1, -1, -\zeta^{-5}, 1) \in \oplus_{i=1}^5 \mathbb{C}^*.$$

In a similar way, we can construct $(f_2, g_2)$ on $Z_2$ with the property that $T(f_2, g_2) = T(f_1, g_1)^{-1}$. We see that this element is already quite nice because it is non-zero in $CH^2(Z, 2)^{ind}$ but it is 6-torsion. In order to get a non-torsion element, we have to deform $X$ slightly: look at the two-parameter family

$$X_{u,v} = \{x_0^5 + x_1x_2^4 + x_2x_3^4 + x_3^5 + uxx_1x_2^3 + vx_0x_3K(x_0, ..., x_3) = 0\} \quad u, v \in \mathbb{C},$$

where $K$ is a fixed cubic form. On all $X_{u,v}$ and their hyperplane sections $Z_1 = \{x_3 = 0\}$ and $Z_2 = \{x_0 = 0\}$, we have the same automorphisms $\sigma_1, \sigma_2$ of order 5, and again $Z_1 \cap Z_2$ consists of 5 fixed points $P_1 = (0 : 0 : 1 : 0), P_2 = (0 : 1 : 0 : 0), P_3 = (0 : \alpha : 1 : 0), P_4 = (0 : \beta : 1 : 0)$ and $P_5 = (0 : \gamma : 1 : 0)$, where $\alpha, \beta, \gamma$ are the 3 roots of the polynomial $f(z) = z^3 + uz + 1 \in \mathbb{C}[z]$. Define on $C_1$ the rational functions $f_1 = z$ and $g_1 = \frac{\delta - \alpha}{\alpha - \beta}$. Then $f_1(P_3) = \alpha, f_1(P_4) = \beta, f_1(P_5) = \gamma$ and $g_1(P_1) = \alpha, g_1(P_2) = g_1(\infty) = 1$. The tame symbol is given by

$$T(f_1, g_1) = (\left(\frac{\beta}{\alpha}\right)^5, 1, \alpha^5, \beta^{-5}, 1) \in \oplus_{i=1}^5 \mathbb{C}^*.$$

Now, if $u$ is sufficiently generic (any $v, K$), this tame symbol is not a torsion element in $\oplus_{i=1}^5 \mathbb{C}^*$. We have therefore obtained a non-torsion example in $CH^2(Z, 2)^{ind}$.

We now turn to the infinitesimal computation which shows that the cycles are indecomposable in our sense. Let $F_{u,v} = x_0^3 + x_1x_2^3 + x_2x_3^3 + uxx_1^2x_2^2 + vx_0x_3K(x_0, ..., x_3)$ be the quintic polynomial, where $K$ is a generic cubic form with coefficients in $\Lambda = \mathbb{C}[u, v]$. Let $B = B[x_0, x_1, x_2, x_3]$ and consider the Jacobian rings (cf. [AS1])

$$R_F = B/(\frac{\partial F_{u,v}}{\partial x_i} (0 \leq i \leq 3)),
R_F = B/(\frac{\partial F_{u,v}}{\partial x_i} (i = 1, 2), x_j \frac{\partial F_{u,v}}{\partial x_j} (j = 0, 3)).$$

Then we have isomorphisms (cf. [G2])

$$H^{2,0}(U/S) \xrightarrow{\cong} R_F = B^3, H^{1,1}(U/S) \xrightarrow{\cong} R_F^8,
H^{2,0}(X/S) \xrightarrow{\cong} R_F = B^3, H^{1,1}(X/S) \xrightarrow{\cong} R_F^8,$$

and the natural map $H^{p,2-p}(X/S) \rightarrow H^{p,2-p}(U/S)$ is given by multiplication with $x_0x_3$. The map

$$\nabla_U : \Omega_S \otimes H^{2,0}(U/S) \rightarrow \Omega^{2,0}_S \otimes H^{1,1}(U/S)$$
is given by

\[ Pdu + Qdv \to (P \frac{\partial F_{u,v}}{\partial v} - Q \frac{\partial F_{u,v}}{\partial u})du \land dv = (Px_0x_3K - Qx_1^2x_2^2)du \land dv, \quad (P, Q \in B^3). \]

Thus the element lies in \( \text{Ker}(\nabla) \) if and only if it satisfies

\[ (5-1) \quad x_1^2x_2^2(Px_1 - Qx_3) \in \left( \frac{\partial F_{u,v}}{\partial x_1}, \frac{\partial F_{u,v}}{\partial x_2}, x_j \frac{\partial F_{u,v}}{\partial x_j} = 5x_j^5 \quad (j = 0, 3) \right). \]

On the other hand we have the “residue map” (5-8)

\[ H^{2,0}(U/S) (= B^3) \to \bigoplus_{i=1}^5 \Lambda; P \to \left( \text{Res}_{P, \frac{\overline{P}}{F}} \overline{P} \right)_{1 \leq i \leq 5}, \]

where \( P = P \text{ mod } <x_0, x_3> \in \Lambda[x_1, x_2] \) and \( \Omega = x_1dx_2 + x_2dx_1 = x_2^3dz \). The “residue” of our element is given by

\[ (d \log \phi_i)_{1 \leq i \leq 5} = \frac{1}{\phi_i} \left( \frac{\partial \phi_i}{\partial u} du + \frac{\partial \phi_i}{\partial v} dv \right)_{1 \leq i \leq 5}, \]

where \( \phi_1 = \beta^5\alpha^5, \phi_2 = 1, \phi_3 = \alpha^5, \phi_4 = \beta^{-5}, \phi_5 = 1 \). Thus we want to show the following fact: there exist no \( P, Q \in B^3 \) satisfying (5 - 1) and for \( 1 \leq i \leq 5 \)

\[ \text{Res}_{P, \frac{\overline{P}}{F}} \overline{P} = \frac{1}{\phi_i} \frac{\partial \phi_i}{\partial u}, \quad \text{Res}_{P, \frac{\overline{Q}}{F}} \overline{Q} = \frac{1}{\phi_i} \frac{\partial \phi_i}{\partial v}. \]

Note that the map \( H^{2,0}(U) \to \bigoplus_{i=1}^5 \Lambda \) is surjective. Thus our equation

\[ \text{Res}_{P, \frac{\overline{P}}{F}} \overline{P} \Omega = \frac{1}{\phi_i} \frac{\partial \phi_i}{\partial u}, \quad \text{Res}_{P, \frac{\overline{Q}}{F}} \overline{Q} \Omega = \frac{1}{\phi_i} \frac{\partial \phi_i}{\partial v} \quad (1 \leq i \leq 5) \]

must have a solution. It is the condition (5 - 1) that destroys such solutions and we show this in the sequel. Let \( a, b, c \) (for simplicity instead of \( a, b, \gamma \)) be the 3 roots of the equation \( f(z) = z^3 + uz + 1 = 0 \). They satisfy \( abc = -1, u = ab + bc + ba \) and \( -a - b - c = 0 \). As above let \( \phi_1 = b^5a^{-5}, \phi_2 = 1, \phi_3 = a^5, \phi_4 = b^{-5}, \phi_5 = 1 \). Using implicit differentiation it is straightforward to prove that

\[ \frac{1}{a} \frac{\partial a}{\partial u} = \frac{-1}{f'(a)} = \frac{-1}{(a-b)(a-c)} \]

and

\[ \frac{1}{a} \frac{\partial a}{\partial v} = 0, \]

(since roots do not depend on \( v \)). Therefore we obtain the following set of equations:

\[
\begin{align*}
\frac{1}{\phi_1} \frac{\partial \phi_1}{\partial u} &= -5 \frac{b-c}{b-a} + 5 \frac{a-b}{a-c} \\
\frac{1}{\phi_2} \frac{\partial \phi_2}{\partial u} &= 0 \\
\frac{1}{\phi_3} \frac{\partial \phi_3}{\partial u} &= -5 \frac{(a-b)(a-c)}{(b-a)(b-c)} \\
\frac{1}{\phi_4} \frac{\partial \phi_4}{\partial u} &= 5 \frac{(b-c)(b-a)}{(b-a)(b-c)} \\
\frac{1}{\phi_5} \frac{\partial \phi_5}{\partial u} &= 0 \\
\frac{1}{\phi_1} \frac{\partial \phi_1}{\partial v} &= 0 \\
\frac{1}{\phi_2} \frac{\partial \phi_2}{\partial v} &= 0 \\
\frac{1}{\phi_3} \frac{\partial \phi_3}{\partial v} &= 0 \\
\frac{1}{\phi_4} \frac{\partial \phi_4}{\partial v} &= 0 \\
\frac{1}{\phi_5} \frac{\partial \phi_5}{\partial v} &= 0
\end{align*}
\]
Now we make the convention that the variable \( z \) is equal to \( \frac{u}{x_2} \) and the “Ansatz”

\[
\bar{P} = a_0 + a_1 z + a_2 z^2 + a_3 z^3, \quad \bar{Q} = b_0 + b_1 z + b_2 z^2 + b_3 z^3.
\]

Recall that \( \bar{F} = z + uz^2 + z^4 \) and \( \Omega = x_2^2 dz \). We then obtain

\[
\frac{\bar{P}}{\bar{F}} \Omega = \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3}{z + uz^2 + z^4} dz, \\
\frac{\bar{Q}}{\bar{F}} \Omega = \frac{b_0 + b_1 z + b_2 z^2 + b_3 z^3}{z + uz^2 + z^4} dz.
\]

Now compute residues at \( P_i \) i.e. \( z = 0, a, b, c, \infty \):

\[
\text{Res}_{z=0} \frac{\bar{P}}{\bar{F}} \Omega = a_0, \quad \text{Res}_{z=a} \frac{\bar{P}}{\bar{F}} \Omega = \frac{a_0 + a_1 a + a_2 a^2 + a_3 a^3}{a(a-b)(a-c)}, \\
\text{Res}_{z=b} \frac{\bar{P}}{\bar{F}} \Omega = \frac{a_0 + a_1 b + a_2 b^2 + a_3 b^3}{b(b-a)(b-c)}, \\
\text{Res}_{z=c} \frac{\bar{P}}{\bar{F}} \Omega = \frac{a_0 + a_1 c + a_2 c^2 + a_3 c^3}{c(c-a)(c-b)}, \\
\text{Res}_{z=\infty} \frac{\bar{P}}{\bar{F}} \Omega = -a_3.
\]

\[
\text{Res}_{z=0} \frac{\bar{Q}}{\bar{F}} \Omega = b_0, \quad \text{Res}_{z=a} \frac{\bar{Q}}{\bar{F}} \Omega = \frac{b_0 + b_1 a + b_2 a^2 + b_3 a^3}{a(a-b)(a-c)}, \\
\text{Res}_{z=b} \frac{\bar{Q}}{\bar{F}} \Omega = \frac{b_0 + b_1 b + b_2 b^2 + b_3 b^3}{b(b-a)(b-c)}, \\
\text{Res}_{z=c} \frac{\bar{Q}}{\bar{F}} \Omega = \frac{b_0 + b_1 c + b_2 c^2 + b_3 c^3}{c(c-a)(c-b)}, \\
\text{Res}_{z=\infty} \frac{\bar{Q}}{\bar{F}} \Omega = -b_3.
\]

This leads to the following 10 equations

\[
a_0 = \frac{-5}{(b-c)(b-a)} + \frac{5}{(a-b)(a-c)}, \\
a_0 + a_1 a + a_2 a^2 + a_3 a^3 = 0, \\
a_0 + a_1 b + a_2 b^2 + a_3 b^3 = \frac{-5}{(a-b)(a-c)}, \\
a_0 + a_1 c + a_2 c^2 + a_3 c^3 = \frac{5}{(b-c)(b-a)}, \\
a_0 + a_1 a + a_2 a^2 + a_3 a^3 = -a_3 = 0, \\
b_0 = 0, \\
b_0 + b_1 a + b_2 a^2 + b_3 a^3 = 0, \\
b_0 + b_1 b + b_2 b^2 + b_3 b^3 = 0, \\
b_0 + b_1 c + b_2 c^2 + b_3 c^3 = 0, \\
b_0 + b_1 c + b_2 c^2 + b_3 c^3 = -b_3 = 0.
\]

This system, though 8 variables and 10 equations, has always non-zero solutions for all \( a, b, c \), for example in the case \( u = 0 \) where \( a = \exp(i\pi/3) \), \( b = -1 \), \( c = a^2 = \exp(-i\pi/3) \), we get the solution

\[
(a_0, ..., a_3) = (\frac{5}{6} i \sqrt{3} - \frac{5}{2}, 0, \frac{5}{3} i \sqrt{3}, 0) = \frac{-5}{3} (a + 1, 0, a(a + 1), 0), \quad (b_0, ..., b_3) = (0, 0, 0, 0).
\]

We therefore need to use condition \((5-1)\) in order to destroy such solutions. Let us return to homogenous coordinates. Our solutions are

\[
\bar{P} = a_0 x_3^3 + a_1 x_1 x_2^2 + a_2 x_1^2 x_2, \quad \bar{Q} = 0.
\]
The partial derivatives of $F_{u,v}$ are given by
\[
\begin{align*}
\frac{\partial F}{\partial x_0} &= 5x_0^4 + x_3K + x_0x_3 \frac{\partial K}{\partial x_0}, \\
\frac{\partial F}{\partial x_1} &= x_2^4 + 4x_1^2x_2 + 2ux_1x_2^3 + vx_0x_3^3 \frac{\partial K}{\partial x_1}, \\
\frac{\partial F}{\partial x_2} &= x_1^4 + 4x_1^2x_2 + 3ux_1^2x_2 + vx_0x_3^3 \frac{\partial K}{\partial x_2}, \\
\frac{\partial F}{\partial x_3} &= 5x_3^4 + x_0K + x_0x_3 \frac{\partial K}{\partial x_3}.
\end{align*}
\]

We have to check that the polynomial $R = P x_0x_3K - x_1^2x_2^3Q$ is for any two liftings $P = \tilde{P} + x_0P_0 + x_3P_3$ and $Q = \tilde{Q} + x_0Q_0 + x_3Q_3$ with quadratic polynomials $P_i, Q_j$ not contained in the ideal
\[ I := \begin{pmatrix} x_0 \frac{\partial F}{\partial x_0}, & \frac{\partial F}{\partial x_1}, & \frac{\partial F}{\partial x_2}, & \frac{\partial F}{\partial x_3} \end{pmatrix} \]

considered as an ideal of $\mathbb{C}[x_0, ..., x_3]$.

We compute that
\[ R(x_0, ..., x_3) = P x_0x_3K + x_0^2x_3KP_0 + x_0x_3^2KP_3 - x_0x_1^2x_2^3Q_0 - x_1^2x_2^3x_3Q_3, \]

which is an element of the ideal
\[ J := (x_0^2x_3^3K, x_1x_2^2x_0x_3K, x_1^2x_2x_0x_3K, x_0^2x_3K, x_0x_3^2K, x_1^2x_2^2x_0, x_1^2x_2^3x_3). \]

It is sufficient to show that $I \cap J$ has no generators in degree $d \leq 8$ for a generic choice of $K$. It is possible to check this with the computer algebra program SINGULAR [S] by using the values $u = 1, v = 1$ and $K = x_0^2x_1 + x_0x_1x_2 + x_0x_2^3 + x_2 + x_0x_1x_3$. Below is a copy of the corresponding Singular session. All the generators of the intersection of the two ideals are computed to have degree $\geq 9$.

```singular
ring r=0,(w,x,y,z),dp; // here we have w = x_0, x = x_1, y = x_2, z = x_3.
poly K=w2x+wxy+wy2+y3+wxz; //sparsepoly(3).
int u = 1;
int v = 1;
poly f1=5w5+wz*K+w2*z*diff(K,w);
poly f2=y4+4x3y+2*u*x*y3+v*wz*diff(K,x);
poly f3=x4+4xy3+3*u*x2*y2+v*z*w*diff(K,y);
poly f4=5z5+zw*K+zw2*diff(K,z);
ideal i = f1,f2,f3,f4; //defining ideal of $\tilde{R}$.
ideal j = y3zw*K,xy2zw*K, x2ywy*K, w2*z*K, wz2*K, x2y3w, x2y3z; //ideal J
ideal I = intersect(i,j); //Intersection $I \cap J$.
dim(I); //the dimension of I (projective dimension +1)!
//Now compute minimal resolution with homogenous entries (ires)!
list T = ires(I,0);
print(betti(T),"betti"); //prints the table of the resolution.
int n;
for (n=ncols(T)); n >=-1; n=n-1)
deg(I[n]), homog(I[n]);
quit;
```

In order to finish the proof that our cycles are indecomposable, we will need that the element we obtained in $H_{pr}^{1,1}(X/S)$ is not in the Picard group for a general deformation, provided $u, v$ and $K$ are sufficiently general. This will be true, once the element is not mapped to zero under the map
\[ W \otimes H_{pr}^{1,1}(X/S) \rightarrow H^{0,2}(X/S), \]
where $W \subset H^1(X, T_X)$ is the 2-dimensional infinitesimal deformation space of our family parametrized by $u, v$ and fixed $K$. Since the element is essentially given by $PK$, we need to show that $NPK$ is not zero in $R^3_F$ for some quintic polynomial $N \in W \subset R^3_F$, which is generated by the two elements $x_0x_3K$ and $x_1^2x_2^2$. We do this in the point $u = 0$ with $K = w^2x + wxy + wy^2 + y^3 + wzx$ and $N = x_0x_3K$. The following little SINGULAR [S] program compares the dimension of $R^{11}_F$ with $R^{11}_F/(NPK)$ via Hilbert series and shows that $NPK \notin R^{11}_F$.

```plaintext
ring r=(0,a),(w,x,y,z),dp; minpoly=a^2 - a + 1;
// here we have w = x_0, x = x_1, y = x_2, z = x_3.
poly K= w2x+wxy+wy2+y3+wzx;/sparsepoly(3);
poly P = (a+1)*y3+a*(1+a)*x2y;
poly N = zw*K; // linear combination of x2y3 and zw*K;
poly f=w5+z5+xy4+x4y+zw*K;
ideal j = jacob(f);
ideal i=std(j);
print("Hilbert series of R^{11}_F:");
hilb(i,2);
print("Hilbert series of R^{11}_F/(NPK)");
ideal k = jacob(f),N*P*K;
ideal l=std(k);
hilb(l,2);
quit;
```

§6. AN EXAMPLE IN $CH^2(X, 1)$.

In this section we prove Th.(0-3) which provides an example of a class in $CH^2(X, 1)$ on a smooth projective complex surface $X$ which is indecomposable modulo the image of $CH^1(X) \otimes \mathbb{C}^*$. We state the result here again.

**Proposition (6-1).** On the family

$$X_u = \{(x_0 : \ldots : x_3) \in \mathbb{P}^3 \mid F_u(x) = x_0x_1^4 + x_1x_2^4 + x_2x_0^4 + x_3 + u_1x_3^4 = 0\}, \quad u \in \mathbb{C}$$

of quintic surfaces, there exist elements $\alpha_u$ in $CH^2(X_u, 1)$ such that, for $u$ very general, these elements are indecomposable modulo the image of $Pic(X_u) \otimes \mathbb{C}^*$.

In order to construct the examples, we consider the following quintic hypersurfaces in $\mathbb{P}^3$ from [SMS]:

$$X_u := \{(x_0 : \ldots : x_3) \in \mathbb{P}^3 \mid F_u(x) = x_0x_1^4 + x_1x_2^4 + x_2x_0^4 + x_3 + u_1x_3^4 = 0\}.$$ 

In [SMS] it was shown that on both curves $Z_1 = X_u \cap \{x_3 = 0\}$ and $Z_2 = X_u \cap \{x_0 = 0\}$ the points $P_1 = (0 : 0 : 1 : 0)$ and $P_2 = (0 : 1 : 0 : 0)$ satisfy $52(P_1 - P_2) = 0$ in $CH^1(Z_1)$. This defines an element $\alpha \in CH^2(X_u, 1)$ for all $u$. It is known by a result of Shioda (cf. [SMS]) that the Picard group of $X_u$ has rank one for almost all $u$. We use now the method of the previous subsection to deduce that these elements are indecomposable for very general $u$ modulo $CH^1(X_u) \otimes \mathbb{C}^*$.

We work over a parameter space $S = Spec(\Lambda) \subset Spec(\mathbb{C}[u])$. Let $B = \Lambda[x_0, x_1, x_2, x_3]$ and consider the Jacobian rings

$$R_F = B/\left(\frac{\partial F_u}{\partial x_i}\right) (0 \leq i \leq 3)$$

$$\tilde{R}_F = B/\left(\frac{\partial F_u}{\partial x_i} (i = 1, 2), x_j\frac{\partial F_u}{\partial x_j} (j = 0, 3)\right).$$

Then we have isomorphisms ([AS1], [G2])

$$H^{2,0}(U/S) \xrightarrow{\sim} \tilde{R}_F = B^3, \quad H^{1,1}(U/S) \xrightarrow{\sim} R_F^8,$$
H^{2,0}(X/S) \cong R^1_F = B^1, H^{1,1}(X/S) \cong R^0_F,

and the natural map \( H^{p-2,p}(X/S) \to H^{p-2,p}(U/S) \) is given by multiplication with \( x_0 x_3 \). The map \( \nabla_U : H^{2,0}(U/S) \to \Omega^2_S \otimes H^{1,1}(U/S) \) is given by

\[
G \xrightarrow{\partial F} G \frac{\partial F}{\partial u} du = G x_3 x_1^4 du, \quad (G \in B^3).
\]

Thus the element lies in \( \text{Ker}(\nabla_U) \) if and only if it satisfies

\[
(6-1) \quad G x_3 x_1^4 \in \left( \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, x_j \frac{\partial F}{\partial x_j} \right) (j = 0, 3)\]

On the other hand we have the “residue map” (5-8)

\[
H^{2,0}(U/S)(= B^3) \to \bigoplus_{i=1}^2 \Lambda_i G \to (\text{Res}_{P_i} \frac{G}{F})_{1 \leq i \leq 2},
\]

where \( \bar{G} = G \mod <x_0, x_3> \in \Lambda[x_1, x_2] \) and \( \Omega = x_1 dx_2 + x_2 dx_1 = x_2^2 dz \) \((z = \frac{x_4}{x_2})\) is the fundamental form on \( \mathbb{P}_x^1 = \text{Proj} \Lambda[x_1, x_2] \). The “residue” of our element is given by \( \text{Res}_{P_1} \frac{G}{F} \Omega = 52 \), \( \text{Res}_{P_2} \frac{G}{F} \Omega = -52 \).

It has to be taken also into account that \( Z_1 \cup Z_2 \) is not a normal crossing divisor, since \( P_2 \) is a point of multiplicity 4 on the intersection. This gives rise to 3 more condition which will be discussed below. Now we make the convention that the variable \( z \) is equal to \( \frac{x_4}{x_2} \) and the “Ansatz”

\[
\bar{G} = a_0 + a_1 z + a_2 z^2 + a_3 z^3.
\]

Recall that \( F_u = x_1 x_2^4 = z x_2^2 \) and \( \Omega = x_2^2 dz \). Therefore we obtain

\[
\frac{G}{F} \Omega = \frac{a_0 + a_1 z + a_2 z^2 + a_3 z^3}{z} dz.
\]

Now compute residues at \( P_1 \) (\( P_2 \) is a point of multiplicity 4):

\[
\text{Res}_{z=0} \frac{G}{F} \Omega = a_0, \\
\text{Res}_{z=\infty} \frac{G}{F} \Omega = -a_0.
\]

This leads to the solution

\[
a_0 = 52, a_1 = a_2 = a_3 = 0.
\]

We therefore need to use condition (6 - 1) in order to destroy such solutions. Let us return to homogenous coordinates. Our solutions are

\[
\bar{G} = a_0 x_2^3 = 52 x_2^3.
\]

The partial derivatives of \( F_u \) are given by

\[
\frac{\partial F}{\partial x_0} = x_1^4 + 4x_0 x_2, \quad \frac{\partial F}{\partial x_1} = 4x_0 x_1^3 + x_2^4 + 4x_1 x_3, \quad \frac{\partial F}{\partial x_2} = 4x_1 x_2^3 + x_0^4, \quad \frac{\partial F}{\partial x_3} = 5x_3^4 + 4x_1^4.
\]
We have to check that the polynomial \( R = Gx_1^4x_3 \) is – for any lifting \( G = 52x_3^2 + x_0G_0 + x_3G_3 \) and with quadratic polynomials \( G_i \) – not contained in the ideal

\[
I := \left( x_0 \frac{\partial F}{\partial x_0}, x_1 \frac{\partial F}{\partial x_1}, x_2 \frac{\partial F}{\partial x_2}, x_3 \frac{\partial F}{\partial x_3} \right)
\]

considered as an ideal of \( \mathbb{C}[x_0, ..., x_3] \).

We compute that

\[
R(x_0, ..., x_3) = (52x_3^2 + x_0G_0 + x_3G_3)x_1^4x_3,
\]

is an element of the ideal \( J := (x_1^4x_3) \).

It is sufficient to show that \( I \cap J \) for \( u = 0 \) has only one generator \( x_0x_1^4x_3 \) in degree 6, which does not divide \( G, b \) and otherwise has no generators in degree \( d \leq 8 \). It is possible to check this with the computer algebra program SINGULAR \( [S] \) by using the value \( u = 0 \). Here is the corresponding Singular session:

```plaintext
ring r=0,(w,x,y,z),dp; //here we have w = x_0, x = x_1, y = x_2, z = x_3.
int u = 0;
poly f1=wx4+4*w4*y;
poly f2=4*x3*w+y4+4*u*x3*z;
poly f3=4*x*y3+w4;
poly f4=5*z5+4*x4*z;
ideal i = f1,f2,f3,f4;
ideal j = w*x4y3z, w*x4z, w*x3z;
ideal I = intersect(i,j);
dim(I); //the dimension of I (projective dimension +1)!
//I; Now compute minimal resolution with homogenous entries (lres)!
list T = lres(I,0);
print(betti(T),"betti");
print(I[1]); print(I[2]);print(I[3]);
quit;
```

§7. Appendix (by Alberto Collino).

In this section we prove Th.(0-4) which was provided to us by Alberto Collino in a letter from September 19, 1999. We are very grateful to him for letting us reproduce the contents here. His result shows in particular that indecomposable cycles in \( CH^3(S, 2) \) need not be rigid on the surface \( S \):

**Proposition(7-1).** On every (very) general quartic \( K3 \)-surface \( S \), there exists a 1-dimensional family of elements \( Z_t \) in \( CH^3(S, 2) \) such that, for \( t \) very general, these elements are indecomposable modulo the image of Pic(S) \( \otimes K2(\mathbb{C}) \).

**Proof** All the cycles which are used originate from the existence of smooth bielliptic hyperplane sections \( C \) of genus 3 on \( S \), which means that there exists a double cover \( C \to E \) onto a smooth elliptic curve \( E \). Their existence is guaranteed by the following lemma:

**Lemma(7-2).** On every general quartic surface \( S \) there exists a 1-dimensional family of bielliptic curves \( C_t \) such that the underlying family \( E_t \) of elliptic curves has varying \( j \)-invariant.

**Proof** The family \( \mathcal{A} \) of plane quartic curves in \( \mathbb{P}^3 \) is a projective bundle over the dual projective space \( \mathbb{P}^3* \). Let \( B \subset \mathcal{A} \) be the closure of the locus where the plane quartics are bielliptic, thus \( B \) is a subvariety of codimension 2 in \( \mathcal{A} \). Let \( \mathcal{K} \) be the linear system of quartic surfaces in \( \mathbb{P}^3 \). Then the incidence family \( \mathcal{F} \subset \mathcal{A} \times \mathcal{K} \) parametrizes couples \( (Y, S) \) with \( Y \subset S \). Furthermore, if \( Y \subset \mathcal{F} \) denotes the restriction of \( \mathcal{F} \) over \( B \), then one has \( \dim(Y) = \dim(K) + 1 \). Our aim is to prove that the general quartic surface contains a 1-dimensional family of bielliptic curves with varying elliptic curve. To show this it is enough to check that the tangent space to a point \( (Y, S) \) in a nonempty fiber of \( Y \to \mathcal{K} \) is at most of dimension 1 in general and that there is variation of the elliptic curve. A smooth bielliptic curve \( Y \) has an involution acting on it which
also acts on the space of canonical forms on the curve. The eigenspace with eigenvalue 1 is of dimension 1 while the eigenspace with eigenvalue -1 is of dimension 2. In the associated coordinates the polynomial of the bielliptic curve is biquadratic in y, say it is of the form \( F(x, y, z) := y^2 + P(x, z)y^2 + Q(x, z) \). Now \( Y \) is the double cover of the elliptic curve \( E \) of weighted equation \( t^2 + P(x, z)t + Q(x, z) = 0 \). Note that \( Y \to E \) is ramified at \( y = Q(x, z) = 0 \), while \( E \to \mathbb{P}^1 \) is ramified over \( P^2 - 4Q = 0 \). We remark that a deformation of type \( F(x, y, z) + sG(x, z) = 0 \) changes the elliptic curve if \( G(x, z) \) is not in the ideal generated by \( \partial(\mathbb{P}^2 - 4Q)/\partial x \) and \( \partial(\mathbb{P}^2 - 4Q)/\partial z \). The tangent space \( TB \) at \( Y \) to the moduli space of bielliptic plane quartic up to projective equivalence is isomorphic to the quotient of the vector space of polynomials of type \( A(x, z)y^2 + B(x, z) \) modulo the relations generated by \( \partial F/\partial x = 0 \) and \( \partial F/\partial z = 0 \). \( TB \) is of dimension 4, if we take \( P := xz \) and \( Q := x^4 - z^4 \) then \( \{x^4, x^3z, x^2z^2, xz^3\} \) is a basis. The tangent space at \( Y \) to the moduli of quartic curves in the plane is isomorphic to \( \mathbb{P}^4 \), the degree 4 part of the jacobain ring of \( F \), with the basis \( \{x^2zy, xz^2y, x^4, x^3z, x^2z^2, xz^3\} \). Consider the quartic surface \( S \) with equation

\[
A_2(x, y, z, w)w^2 + A_1(x, y, z)w + F(x, y, z) = 0.
\]

The plane \( \pi \) with equation \( w = 0 \) cuts \( S \) along our bielliptic curve \( Y \). Let \( w = \epsilon L(x, y, z) \) be the equation of a first order deformation \( \pi(L) \) of \( \pi \). By projecting to \( w = 0 \), the section \( S \cap \pi(L) \) gives the curve \( Y(L) \) with equation \( F + \epsilon LA_1 \). Using the preceeding basis for \( \mathbb{P}^4 \), one can check that for a given \( A_1 \) (we have used \( A_1 := ax^3 + bz^3 + y^3 \) for our computation), the request that \( Y(L) \) is bielliptic imposes two independent conditions on \( L \) and thus the tangent space at \( Y \) to the family of bielliptic curves on \( S \) is of dimension 1 in general. Furthermore the conditions for the variation of the related elliptic curve are also satisfied. \( \square \)

To continue with the proof of \((7 - 1)\), let \( S \) be a quartic surface in \( \mathbb{P}^3 \) such that \( S \) admits a smooth bielliptic hyperplane section \( C \) of genus 3 with a double cover \( C \to E \) onto a smooth elliptic curve \( E \). Given any element in \( CH^2(E, 2) \), we can take its pull-back in \( CH^2(C, 2) \), which defines an element in \( CH^3(S, 2) \) by the Gysin map. In \([C]\) it has been shown that on a general elliptic curve there are cycles in \( CH^2(E, 2) \) which are associated to a pair of Weierstrass points and have a non-trivial invariant with regard to the regulator map

\[
CH^2(E, 2) \to H^1(E, \mathbb{C}^*)
\]

when \( E \) varies in moduli. For convenience let us recall the definition of those cycles in \( CH^2(E, 2) \): first choose a non-trivial divisor class \( e \) of degree zero and order 4 on \( E \). Set \( \sigma = 2e \) and fix a base point \( p \in E \). Then define \( q := p + \sigma \), \( a := p + e \) and \( b := p - e \). Then there is a unique rational function \( f \) on \( E \) such that \( div(f) = 2q - 2p \) and \( f(a) = f(b) = 1 \). Furthermore, by symmetry, there is a rational function \( g \) on \( E \) with the property that \( div(g) = 2b - 2a \) and \( g(p) = g(q) = 1 \). Then the resulting cycle \( \{f, g\} \in CH^2(E, 2) \) associated to the graph of \( (f, g) \); \( E \to \mathbb{P}^1 \times \mathbb{P}^1 \) defines a cubical cycle in the higher Chow group \( CH^2(E, 2) \).

Now we will do the infinitesimal computations necessary to show proposition \((7 - 1)\). Since we use essentially the same methods as in the previous paragraph, we do not have to introduce any new methods. Let \( S = \{F(x_0, \ldots, x_3) = 0\} \) be the equation of \( S \) and \( C = \{x_3 = 0\} \) the equation of the hyperplane section \( C \). Denote by \( U = X \setminus C \) the open complement. Denote by \( T \) a parameter space of pairs \( (S, C) \), where \( C \subset S \) is a bielliptic hyperplane section. Consider, over \( T \), the higher Chow group \( CH^3(S, 2) \) with support in the divisor \( C \) (the universal bielliptic section), and inside there the cycle \( Z \) which is the pullback of the elliptic cycle from \( [C] \). The infinitesimal invariant of \( Z \) in our sense is the lift of the infinitesimal invariant from the family of elliptic curves, hence it gives by lemma \((7 - 2)\) and \([C]\) a non zero element

\[
\gamma \in Ker(\Omega_{T,0,0} \otimes H^0(S, \Omega_S^2(\log C))/W_0 \to \Omega_{T,0}^2 \otimes H^1(S, \Omega_S^2(\log C))/W_0).
\]

Our goal is to prove that any lifting of \( \gamma \) into \( H^0(S, \Omega_S^2(\log C)) \) is not contained in the kernel of \( \nabla_U \). To this aim we must identify everything in terms of Jacobians rings with the previous notations, \( H^{2,0}(U) \cong \bar{R}_F^1 \), and \( H^{1,1}(U) \cong \bar{R}_F^3 \). After restricting to a 1-parameter family of surfaces, \( \Omega_{T,0}^2 \) has two generators, called \( d\sigma \) and \( d\tau \), dual to the tangent directions \( \frac{\partial}{\partial \sigma} \) and \( \frac{\partial}{\partial \tau} \); \( \bar{R}_F^1 \) is the space of tangent directions to the deformation space of couples \( (S, C) \), hence our vectors \( \frac{\partial}{\partial \sigma} \) and \( \frac{\partial}{\partial \tau} \) are associated with polynomials of degree 4. We now assume that \( \frac{\partial}{\partial \sigma} \) corresponds to a deformation direction where the plane section \( C \) is
...preserved, and then the polynomial associated to \( \frac{\partial}{\partial \sigma} \) is of type \( \tau := x_3W \), \( W \) a polynomial of degree 3. We assume furthermore that \( \frac{\partial}{\partial \sigma} \) corresponds to a deformation direction where \( S \) is preserved. Therefore the polynomial associated to \( \frac{\partial}{\partial \sigma} \) is of the type \( \sigma := L \frac{\partial F}{\partial x_3} \), with \( L \) a linear polynomial. The map \( \nabla_U \) acts on \( d\sigma \otimes P + d\sigma \otimes Q \) by sending it to \( (d\sigma \wedge d\tau)(P\tau - Q\sigma) \). Recall that \( P \) and \( Q \) are polynomials of degree 1 and that we are working in the jacobian ring \( \tilde{R}^*_F \).

Our element \( \gamma \), the infinitesimal invariant of elliptic origin, is by its nature of type \( d\sigma \otimes M \), \( M \) of degree 1 in \( x_0, \ldots, x_2 \). Indeed in direction \( \tau \) the curve is fixed. So we want to see that, by a wise choice, an element like \( d\sigma \otimes M \) is not in the image of the kernel of \( \nabla_U \). Elements which map to \( d\sigma \otimes M \) must be of type \( d\sigma \otimes (cx_3 + M) + d\sigma \otimes (kx_3) \). This element is in the kernel of \( \nabla_U \) iff in the jacobian ring \( \tilde{R}^*_F \) it is \((cx_3 + M)x_3W - (kx_3)L \frac{\partial F}{\partial x_3} = 0 \), or, equivalently, iff \((cx_3 + M)x_3W = 0 \) in \( \tilde{R}^*_F \).

Here comes the wise choice, (recall our freedom of choice of \( W \), this amounts to the choice of how the surface changes, keeping the curve fixed): look at \((cx_3 + M)x_3W \) in \( \tilde{R}^*_F \), this is clearly not 0, hence by duality there is a polynomial \( G \) of degree 7 so that \((cx_3 + M)x_3G \neq 0 \) in \( \tilde{R}^*_F \). This implies that there is a polynomial \( W \) of degree 3 so that \( W(cx_3 + M)x_3 \neq 0 \) in \( \tilde{R}^*_F \) and we are done. □

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