THE INITIAL-BOUNDARY VALUE PROBLEMS FOR A CLASS OF SIXTH ORDER NONLINEAR WAVE EQUATION

RUNZHANG XU*

College of Science
College of Science, Harbin Engineering University
Heilongjiang, Harbin 150001, China
The Institute of Mathematical Sciences, The Chinese University of Hong Kong
Shatin, N.T., Hong Kong, China

MINGYOU ZHANG

College of Automation
Harbin Engineering University
Heilongjiang, Harbin 150001, China

SHAOHUA CHEN

Department of Mathematics
Cape Breton University
Sydney, NS, B1P 6L2, Canada

YANBING YANG AND JIHONG SHEN

College of Science
Harbin Engineering University
Heilongjiang, Harbin 150001, China

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ABSTRACT. This paper considers the initial boundary value problem of solutions for a class of sixth order 1-D nonlinear wave equations. We discuss the probabilities of the existence and nonexistence of global solutions and give some sufficient conditions for the global and non-global existence of solutions at three different initial energy levels, i.e., sub-critical level, critical level and sup-critical level.

1. Introduction. In this paper, we consider the initial boundary value problem (IBVP) for the following 1-D nonlinear wave equation of sixth order

\[ u_{tt} - au_{xx} + u_{xxxx} + u_{xxxxxt} + f(u_x)_x = 0, \quad (x, t) \in (0, 1) \times (0, \infty), \]  
\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in [0, 1], \]  
\[ u(0, t) = u(1, t) = u_{xx}(0, t) = u_{xx}(1, t) = 0, \quad t \in (0, \infty), \]  

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* Corresponding author: Runzhang Xu, xurunzh@163.com.
where \(f(u) = \pm |u|^p \) or \(|u|^{p-1}u\), \(p > 1\) is a constant, \(u_0\) and \(u_1\) are given initial data, \(a > 0\) is a given constant satisfying certain conditions to be specified later.

In the study of a weakly nonlinear analysis of elastoplastic-microstructure models for longitudinal motion of an elasto-plastic bar in \([1]\) there arose the model equation

\[
u_{tt} + \alpha u_{xxxx} = \beta (u_x^2),
\]

where \(u(x,t)\) is the longitudinal displacement, and \(\alpha > 0, \beta \neq 0\) are any real numbers. The author in \([1]\) proved the instability of the special solution and the instability of the ordinary strain solution.

In the study of model for the propagation of small amplitude long waves on the surface of shallow water, there arose the classical Boussinesq type equation

\[
u_{tt} - \nu_{xx} + \gamma u_{xxxx} = \beta (u_x^2), \quad (4)
\]

The equation \([4]\) was first introduced by Boussinesq \([2]\). Later, extensive research has been carried out to study the Boussinesq equation by different views. Cho and Ozawa \([3]\) established the global existence and the scattering of a small amplitude solution to the Cauchy problem of the equation \([4]\). Considering the effect of damping, Varlamov \([15, 16]\) considered the following damped Boussinesq equation

\[
u_{tt} - 2bu_{txx} + au_{xxxx} = u_{xx} + (u_x^2)_{xx}, \quad x \in \mathbb{R}, \ t > 0.
\]

The local existence and the long-time decay of the initial value and initial boundary value problems with small initial data for one-dimensional space was studied in \([15]\). For two dimensional space, the long-time asymptotics of global solution for the initial boundary value problem in a ball can be seen in \([16]\).

Subsequently when Rosenau \([13]\) was concerned with the problem of how to describe the dynamics of a dense lattice, he discovered equation \([1]\) by a continuum method. Meanwhile one-dimensional homogeneous lattice wave propagation phenomena can also be described by equation \([1]\). Wang and Xu in \([19]\) considered the following Rosenau equation

\[
u_{tt} - \gamma \nu_{xx} + u_{xxxx} + u_{xxxxxtt} = f(u_x)_{xx} \quad (5)
\]

and proved global existence and blow up of the solution for the initial value problem with \(E(0) \leq d\) (initial energy below the mountain pass level), where \(E(0)\) is the initial energy and \(d\) is the depth of the potential well defined. As for equation \([5]\), Wang and Wang in \([17]\) proved the decay and scattering of small-amplitude solution for \(\gamma = 1\). Liu and Xu in \([9]\) obtained the global existence and nonexistence of solution for the initial value problem of the equation \([5]\) with \(E(0) \leq d\) in the absence of \(u_{xxxxxtt}\) term.

Later on Han and Chen \([5]\) considered the initial boundary value problems for a class of nonlinear wave equations of the following form

\[
u_{tt} - a_1u_{xx} + a_2u_{xxxx} + a_3u_{xxxxxtt} = \phi(u_x)_{xx} + f(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}), \quad (6)
\]

where \(a_1, a_2, a_3 > 0\) are constants and \(\phi(s)\) is a given nonlinear function. By the extension theorem they proved the existence and uniqueness of classical global solutions. Moreover sufficient conditions for blow up of solutions were also obtained. Recently, for the Cauchy problem for equation \([6]\) with \(f(s) = 0\), Wang Yuzhu and Wang Yinxia \([18]\) discussed the existence and nonexistence of global solutions to this problem under the case \(E(0) \leq d\). Shen et al. in \([14]\) showed that the local solutions blow up in finite time under the high initial energy level \(E(0) > 0\).
As the depth of potential well is very small even near zero, the range \( 0 < E(0) \leq d \) is actually a very small range. Hence the properties of the initial data ensuring \( E(0) > 0 \) are of great interests. The task of the present paper is to consider almost all the possibilities of the initial data, i.e., \( 0 < E(0) < d \) (sub-critical case), \( E(0) = d \) (critical case) and \( E(0) > 0 \) (sup-critical case). For the sub-critical case and the critical case, we shall treat the problem for more general nonlinear terms such as \( f(u) = \pm |u|^p \) or \( |u|^{p-1}u \).

In this paper, for the sub-critical initial energy case \( E(0) < d \) (Section 3), we first state a global existence of solution in the framework of the potential well method \([12, 8, 21, 20]\), and then inspired by the so-called concavity method \([6, 7]\) we prove a blow up result. For the critical initial energy case \( (E(0) = d) \) (Section 4), by utilizing the method of \([10, 20]\), we prove the global existence, finite time blow up of solutions. For the sup-critical initial energy case \( (E(0) > 0) \) (Section 5), by the ideas in \([4, 23, 22]\) we derive some sufficient conditions on the initial data such that certain solutions exist globally or blow up in a finite time.

2. Some assumptions and preliminary lemmas. In this section we give some assumptions and preliminary results stating the main results of this paper. In what follows, we use the following abbreviations for simplicity of notation:

\[
L^p = L^p[0, 1], \quad H^s = H^s[0, 1], \quad H = H^1[0, 1] \cap H^2[0, 1],
\]

\[
\|\cdot\| \equiv \|\cdot\|_{L^2[0, 1]}, \quad \|\cdot\|_p = \|\cdot\|_{L^p[0, 1]}, \quad \|\cdot\|_H = \|\cdot\|_{H^1[0, 1]} \cap H^2[0, 1].
\]

For all \( u, v \in H^1 \), let \( (u, v) = \int_0^1 u v dx \) denote the \( L^2 \)-inner product and put

\[
(u, v)_{H^1} = (u, v) + (u_x, v_x).
\]

First, we give some preliminary lemmas, then by using them we introduce the potential well \( W \) and the corresponding set \( V \).

Lemma 2.1. Let \( f(u) = \pm |u|^p \) or \( |u|^{p-1}u, \ p > 1 \) and \( F(u) = \int_0^u f(v)dv \). Then

(i) \( uf(u) > 0, F(u) > 0 \) for \( u > 0 \) if \( f(u) = |u|^p \);

uf(u) > 0, F(u) > 0 \) for \( u < 0 \) if \( f(u) = -|u|^p \);

uf(u) > 0, F(u) > 0 \) for \( u \neq 0 \) if \( f(u) = |u|^{p-1}u \).

(ii) \( f(\lambda u) = \lambda^p f(u), F(\lambda u) = \lambda^{p+1} F(u), \ \forall \ u \in \mathbb{R}, \ \lambda > 0. \)

(iii) \( |uf(u)| = |u|^{p+1}, |F(u)| = \frac{1}{p+1} |u|^{p+1}, \ \forall \ u \in \mathbb{R}. \)

(iv) \( (p+1)F(u) = uf(u), \ \forall \ u \in \mathbb{R}. \)

For problem (1)-(3) we introduce the potential energy functional

\[
J(t) \equiv J(u) = \frac{1}{2} (a\|u_x\|^2 + \|u_{xx}\|^2) - \int_0^1 F(u_x)dx,
\]

the Nehari functional

\[
I(t) \equiv I(u) = a\|u_x\|^2 + \|u_{xx}\|^2 - \int_0^1 u_x f(u_x)dx
\]

and the energy functional

\[
E(t) = \frac{1}{2} (\|u_t\|^2 + a\|u_x\|^2 + \|u_{xx}\|^2 + \|u_{xxt}\|^2) - \int_0^1 F(u_x)dx.
\]
Then from (7)-(9) we have
\[
E(t) = \frac{1}{2} \left( \|u_t\|_2^2 + \|u_{xxt}\|_2^2 \right) + J(u)
\]
\[
= \frac{1}{2} \left( \|u_t\|_2^2 + \|u_{xxt}\|_2^2 \right) + \frac{p-1}{2(p+1)} \left( a\|u_x\|_2^2 + \|u_{xx}\|_2^2 \right)
\]
\[
+ \frac{1}{p+1} I(u).
\]

**Definition 2.2.** We define
\[
H^+ = \{ u \in H \mid u_x \geq 0, \|u\|_H \neq 0 \},
\]
\[
H^- = \{ u \in H \mid u_x \leq 0, \|u\|_H \neq 0 \},
\]
\[
\Omega^+(u) = \{ x \in [0,1] \mid u_x > 0 \},
\]
\[
\Omega^-(u) = \{ x \in [0,1] \mid u_x < 0 \}.
\]

**Lemma 2.3.** Let \( f(u) = \pm |u|^p \) or \( |u|^{p-1} u \), \( u \in H \) and
\[
\varphi(\lambda) = \frac{1}{\lambda} \int_0^1 u_x f(\lambda u_x) dx.
\]
Furthermore if \( \int_0^1 u_x f(u_x) dx > 0 \), then
i) \( \varphi(\lambda) \) is increasing on \( 0 < \lambda < \infty \).
ii) \( \lim_{\lambda \to 0} \varphi(\lambda) = 0 \), \( \lim_{\lambda \to +\infty} \varphi(\lambda) = +\infty \).

**Proof.** This lemma follows from
\[
\varphi(\lambda) = \frac{1}{\lambda} \int_0^1 u_x f(\lambda u_x) dx = \lambda^{p-1} \int_0^1 u_x f(u_x) dx.
\]

**Lemma 2.4.** Let \( f(u) = \pm |u|^p \) or \( |u|^{p-1} u \), \( u \in H \). Then
i) \( \lim_{\lambda \to 0} J(\lambda u) = 0 \).
ii) \( I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u), \forall \lambda > 0 \).
Furthermore if \( \int_0^1 u_x f(u_x) dx > 0 \), then
iii) \( \lim_{\lambda \to +\infty} J(\lambda u) = -\infty \).
iv) In the interval \( 0 < \lambda < \infty \) there exists a unique \( \lambda^* \) such that
\[
\frac{d}{d\lambda} J(\lambda u) \bigg|_{\lambda=\lambda^*} = 0.
\]
v) \( J(\lambda u) \) is increasing on \( 0 \leq \lambda \leq \lambda^* \), decreasing on \( \lambda^* \leq \lambda < \infty \) and takes the maximum at \( \lambda = \lambda^* \).
vi) \( I(\lambda u) > 0 \) for \( 0 \leq \lambda \leq \lambda^* \), \( I(\lambda u) < 0 \) for \( \lambda^* \leq \lambda < \infty \) and \( I(\lambda^* u) = 0 \).

**Proof.** Part(i)-Part(iii) are obvious. Note that \( \int_0^1 u_x f(u_x) dx \neq 0 \) implies \( a\|u_x\|_2^2 + \|u_{xx}\|_2^2 \neq 0 \) and
\[
\frac{d}{d\lambda} J(\lambda u) = \lambda \left( (a\|u_x\|_2^2 + \|u_{xx}\|_2^2) - \varphi(\lambda) \right),
\]
which gives Parts (iv) and Part (v).

Part (vi) follows from part (ii) and (11).
Corollary 1. If in Lemma 2.3 and Lemma 2.4 the assumption \( \int_0^1 u_x f(u_x) dx > 0 \) is replaced by any one of the following

(i) \( f(u) = |u|^p, \ u \in H^1 \),
(ii) \( f(u) = -|u|^p, \ u \in H^- \),
(iii) \( f(u) = |u|^{p-1}u, \ u \in H, \ a||u_x||^2 + ||u_{xx}||^2 \neq 0 \),

then the conclusions of Lemma 2.3 and Lemma 2.4 also hold.

Lemma 2.5. Let \( f(u) = \pm|u|^p \) or \( |u|^{p-1}u, \ u \in H \) and \( 0 < a||u_x||^2 + ||u_{xx}||^2 < r \), then \( I(u) > 0 \), where
\[
  r = \left( \frac{1}{C_2^{p+1}} \right)^{\frac{2}{p-1}},
\]
\( C_* \) is imbedding constant from \( H \) into \( W^{1,p+1} \).

Proof. From \( 0 < a||u_x||^2 + ||u_{xx}||^2 < r \) we obtain
\[
  \int_0^1 u_x f(u_x) dx \leq \int_0^1 |u_x f(u_x)| dx = ||u_x||_{p+1}^{p+1}
  \leq C_2^{p+1} (a||u_x||^2 + ||u_{xx}||^2)^{\frac{p+1}{2}}
  = C_2^{p+1} (a||u_x||^2 + ||u_{xx}||^2)^{\frac{p-1}{2}} (a||u_x||^2 + ||u_{xx}||^2)
  < a||u_x||^2 + ||u_{xx}||^2,
\]
which gives \( I(u) > 0 \). \( \square \)

Lemma 2.6. Let \( f(u) = \pm|u|^p \) or \( |u|^{p-1}u, \ u \in H \) and \( I(u) < 0 \), then \( a||u_x||^2 + ||u_{xx}||^2 > r \), where
\[
  r = \left( \frac{1}{C_2^{p+1}} \right)^{\frac{2}{p-1}}.
\]

Proof. Note that \( I(u) < 0 \) implies \( a||u_x||^2 + ||u_{xx}||^2 \neq 0 \). From this and
\[
  a||u_x||^2 + ||u_{xx}||^2 < \int_0^1 u_x f(u_x) dx
  \leq C_2^{p+1} (a||u_x||^2 + ||u_{xx}||^2)^{\frac{p+1}{2}} (a||u_x||^2 + ||u_{xx}||^2),
\]
we obtain \( a||u_x||^2 + ||u_{xx}||^2 > r \). \( \square \)

Lemma 2.7. Let \( f(u) = \pm|u|^p \) or \( |u|^{p-1}u, \ u \in H, \ I(u) = 0 \) and \( a||u_x||^2 + ||u_{xx}||^2 \neq 0 \), then \( a||u_x||^2 + ||u_{xx}||^2 \geq r \), where
\[
  r = \left( \frac{1}{C_2^{p+1}} \right)^{\frac{2}{p-1}}.
\]

Proof. From \( I(u) = 0 \) we have
\[
  a||u_x||^2 + ||u_{xx}||^2 = \int_0^1 u_x f(u_x) dx
  \leq C_2^{p+1} (a||u_x||^2 + ||u_{xx}||^2)^{\frac{p+1}{2}} (a||u_x||^2 + ||u_{xx}||^2),
\]
which together with \( a||u_x||^2 + ||u_{xx}||^2 \neq 0 \) gives \( a||u_x||^2 + ||u_{xx}||^2 \geq r \). \( \square \)
Definition 2.8. For \( f(u) = \pm |u|^p \) or \(|u|^{p-1}u\), now we can define
\[
d = \inf_{u \in H \setminus \{0\}} \sup_{\lambda \geq 0} J(\lambda u).
\]

(12)

All nontrivial stationary solutions belong to the so-called Nehari manifold (see [11] and also [4]) defined by
\[
\mathcal{N} = \{ u \in H \setminus \{0\} | I(u) = 0 \}.
\]

Then (12) equals to
\[
d = \inf_{u \in \mathcal{N}} J(u).
\]

(13)

(i) For \( f(u) = |u|^p \), we define
\[
d_1 = \inf_{u \in \mathcal{N} \cap H^+} J(u).
\]

(ii) For \( f(u) = -|u|^p \), we define
\[
d_2 = \inf_{u \in \mathcal{N} \cap H^-} J(u).
\]

Lemma 2.9. Let \( f(u) = \pm |u|^p \) or \(|u|^{p-1}u\), \( u \in H \) and \( I(u) < 0 \), then
\[
d < \frac{p - 1}{2(p + 1)} (a \|u_x\|^2 + \|u_{xx}\|^2).
\]

(14)

Proof. Note that \( I(u) < 0 \) gives
\[
\int_0^1 u_x f(u_x) dx > a \|u_x\|^2 + \|u_{xx}\|^2 > 0.
\]

Hence from Lemma [2.4] it follows that \( J(\lambda u) \) takes the maximum at \( \lambda^* \), which satisfies
\[
\left. \frac{d}{d\lambda} J(\lambda u) \right|_{\lambda = \lambda^*} = 0.
\]

(15)

Let
\[
a(u) = a \|u_x\|^2 + \|u_{xx}\|^2
\]

and
\[
b(u) = \int_0^1 u_x f(u_x) dx.
\]

Then
\[
J(\lambda u) = \frac{\lambda^2}{2} a(u) - \frac{\lambda^{p+1}}{p+1} b(u)
\]

and
\[
a(u) < b(u).
\]

Note (15) gives
\[
\lambda^* = \left( \frac{a(u)}{b(u)} \right)^{\frac{1}{p+1}} < 1.
\]
Therefore from Definition 2.8 we obtain
\[ d \leq \sup_{\lambda > 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2} \left( \frac{a(u)}{b(u)} \right)^{\frac{p}{p-1}} a(u) - \frac{1}{p+1} \left( \frac{a(u)}{b(u)} \right)^{\frac{p+1}{p}} b(u) \]
\[ = \frac{p-1}{2(p+1)} \left( \frac{a(u)}{b(u)} \right)^{\frac{p}{p+1}} a(u) \]
\[ < \frac{p-1}{2(p+1)} a(u) = \frac{p-1}{2(p+1)} (a\|u_x\|^2 + \|u_{xx}\|^2). \]

**Definition 2.10.** We can define the stable set (potential well)
\[ W = \{ u \in H \mid I(u) > 0, J(u) < d \} \cup \{0\}, \tag{16} \]
and the unstable set
\[ V = \{ u \in H \mid I(u) < 0, J(u) < d \}. \tag{17} \]

In order to prove the main theorems, we first establish the local existence and uniqueness for solutions of problem (1)-(3).

By weak solution of problem (1) over \([0, T_{\max}]\), where \(T_{\max}\) is the maximum existence time of \(u\), we mean a function \(u \in C([0, T_{\max}); H)\) with \(u_t \in C([0, T_{\max}); H)\), such that
\[ u(0) = u_0, \quad u_t(0) = u_1 \]
and
\[ (u_t, v) + (u_{xxt}, v_{xx}) + \int_0^t \left( a(u_x, v_x) + (u_{xx}, v_{xx}) - (f(u_x), v_x) \right) d\tau \]
\[ = (u_1, v) + (u_{1xx}, v_{xx}) \]
for all \(v \in H\) and a.e. \(t \in [0, T_{\max}]\).

Now we show a conservation law.

**Lemma 2.11 (Conservation of energy).** Suppose that \(u_0(x), u_1(x) \in H, u \in C^1([0, T_{\max}); H)\) is the weak solution of problem (1)-(3), where \(T_{\max}\) is the maximal existence time of \(u\). Then we get the energy conservation
\[ E(t) = E(0), \quad t \in (0, T_{\max}) \]
holds.

Next, we present the following local existence theorem that can be established by combining arguments of [5] and [18].

**Theorem 2.12.** Suppose that \(\frac{3}{2} < s < p + 1\), \(u_0(x), u_1(x) \in H^s\). Then problem (1)-(3) admits a unique local solution \(u(x, t)\) defined on a maximal time interval \([0, T_{\max})\) with \(u(x, t) \in C^1(H^s, [0, T_{\max})\). Moreover if
\[ \sup_{t \in [0, T_0]} (\|u(x, t)\|_{H^s} + \|u_t(x, t)\|_{H^s}) < \infty, \]
then \(T_{\max} = \infty\).

3. Existence and nonexistence of global solutions at sub-critical energy level \(E(0) < d\). In this section we discuss the invariance of the stable set (16) and the unstable set (17) under the flow of problem (1)-(3).

First we show the invariance of the sets \(W\) and \(V\) under the flow of problem (1)-(3).
Lemma 3.1 (Invariant sets for $E(0) < d$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u$, $u_0(x), u_1(x) \in H$ and $u$ be a unique solution of problem (1)-(3). Assume that $E(0) < d$. Then for all $t \in (0, T_{\text{max}})$, where $T_{\text{max}}$ is the maximal existence time of the solution, we have

\begin{itemize}
  \item[i)] $u \in W$ if $u_0 \in W$;
  \item[ii)] $u \in V$ if $u_0 \in V$.
\end{itemize}

Proof. We only prove the invariance of $W$, the proof for the invariance of $V$ is similar. It follows from (10) and $E(0) < d$ that $J(u) < d$. Then, if condition $I(u_0) > 0$ holds, we have $u \in W$ for all $t \in [0, T_{\text{max}})$. Indeed, if it was false, there would exist a first time $t_0 \in (0, T_{\text{max}})$ such that

$I(u(t_0)) = 0$.

By the relation of $C_*$ and $d$, we get

\begin{align*}
J(u(t_0)) &= \frac{1}{2} (a \|u_x(t_0)\|^2 + \|u_{xx}(t_0)\|^2) - \frac{1}{p+1} \int_0^1 u_x(t_0) f(u_x(t_0)) dx \\
&= \left( \frac{1}{2} - \frac{1}{p+1} \right) (a \|u_x(t_0)\|^2 + \|u_{xx}(t_0)\|^2) + \frac{1}{p+1} I(u(t_0)) \\
&\geq \frac{p-1}{2(p+1)} \left( \frac{1}{C_{p+1}^2} \right) = d.
\end{align*}

This contradicts $J(u) < d$. Thus, we have $u \in W$ for all $0 \leq t < T_{\text{max}}$. So the proof is completed. \hfill \Box

Now we prove the existence and nonexistence of global solution for problem (1)-(3). And we give some sharp conditions for global well-posedness.

Theorem 3.2 (Global existence for $E(0) < d$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u$, $u_0(x), u_1(x) \in H$, $E(0) < d$ and $I(u_0) > 0$ or $a \|u_0x\|^2 + \|u_0xx\|^2 = 0$. Then problem (1)-(3) has a unique global weak solution $u \in C([0, \infty); H)$ with $u_t \in C([0, \infty); H)$ and $u(t) \in W$ for $t \in [0, \infty)$.

Proof. From Theorem 2.12 and Lemma 2.11 it follows that problem (1)-(3) admits a unique local weak solution $u \in C([0, T_{\text{max}}); H)$ satisfying

$$
\frac{1}{2} \|u_t(t)\|^2 + \|u_{xxt}(t)\|^2 + J(u) \equiv E(0), \quad 0 \leq t < T_{\text{max}},
$$

where $T_{\text{max}}$ is the maximal existence time of $u$. From (8) and (18) we have

$$
\frac{1}{2} (\|u_t(t)\|^2 + \|u_{xxt}(t)\|^2) + \frac{p-1}{2(p+1)} (a \|u_x\|^2 + \|u_{xx}\|^2) + \frac{1}{p+1} I(u) \\
\equiv E(0), \quad 0 \leq t < T_{\text{max}}.
$$

From Lemma 3.1 we have $u \in W$ for $0 \leq t < T_{\text{max}}$, which says

$$
\|u_t(t)\|^2 + \|u_{xxt}(t)\|^2 + a \|u_x\|^2 + \|u_{xx}\|^2 \leq \frac{2(p+1)}{p-1} E(0).
$$

Again by Theorem 2.12 we obtain $T_{\text{max}} = +\infty$. \hfill \Box

Theorem 3.3 (Blow up for $E(0) < d$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u$, $u_0(x), u_1(x) \in H$. Assume that $E(0) < d$ and $I(u_0) < 0$, then the existence time of weak solution for problem (7)-(3) is finite.
Proof. Let \( u(t) \) be any solution of problem \((1) - (3)\) with \( E(0) < d \). Moreover, the solution satisfies energy functional \((10)\). Then, assume by contradiction that \( u \) is global, for any \( T > 0 \) we may consider \( M(t) : [0, T] \to \mathbb{R}_+ \) defined by

\[
M(t) := \|u(t)\|^2 + \|u_{xx}(t)\|^2.
\]  

(20)

As \( M(t) \) is continuous on \([0, T]\), there exist constants \( \delta_1, \delta_2 > 0 \) such that

\[
\delta_1 \leq M(t) \leq \delta_2,
\]

(21)

Furthermore,

\[
M'(t) = 2(u, u_t) + 2(u_{xx}, u_{xxt}),
\]

(22)

and

\[
M''(t) = 2\|u_t\|^2 + 2\|u_{xxt}\|^2 + 2(u, u_{tt}) + 2(u_{xx}, u_{xxtt}).
\]

(23)

Testing the equation in \((1)\) with \( u \) and plugging the result into the expression of \( M''(t) \) we obtain

\[
M''(t) = 2\|u_t\|^2 + 2\|u_{xxt}\|^2 - 2a\|u_x\|^2 - 2\|u_{xx}\|^2 + 2 \int_0^1 u_x f(u_x) dx.
\]

(24)

From the definition of \( I(u) \), \((24)\) becomes

\[
M''(t) = 2\|u_t\|^2 + 2\|u_{xxt}\|^2 - 2I(u).
\]

(25)

Therefore, we get

\[
M(t)M''(t) - \frac{p + 3}{4}(M'(t))^2 = 2M(t)(\|u_t\|^2 + \|u_{xxt}\|^2 - I(u)) - (p + 3)((u, u_t) + (u_{xx}, u_{xxt}))^2
\]

\[
= 2M(t)(\|u_t\|^2 + \|u_{xxt}\|^2 - I(u)) + (p + 3)(\eta(t) - M(t)(\|u_t\|^2 + \|u_{xxt}\|^2)),
\]

where \( \eta : [0, T] \to \mathbb{R}_+ \) is the function defined by

\[
\eta(t) = (\|u\|^2 + \|u_{xxt}\|^2)(\|u_t\|^2 + \|u_{xxt}\|^2) - ((u, u_t) + (u_{xx}, u_{xxt}))^2.
\]

(26)

Using Schwarz, Hölder and Young inequalities, we obtain

\[
(u, u_t)^2 \leq \|u\|^2\|u_t\|^2,
\]

\[
(u_{xx}, u_{xxt})^2 \leq \|u_x\|^2\|u_{xxt}\|^2
\]

and

\[
2(u, u_t)(u_{xx}, u_{xxt}) \leq 2\|u\|\|u_t\|\|u_{xx}\|\|u_{xxt}\|
\]

\[
\leq \|u\|^2\|u_{xxt}\|^2 + \|u_t\|^2\|u_{xx}\|^2.
\]

These three inequalities entail \( \eta(t) \geq 0 \) for every \( t \in [0, T] \). Therefore, we obtain

\[
M(t)M''(t) - \frac{p + 3}{4}(M'(t))^2 \geq 2M(t)(\|u_t\|^2 + \|u_{xxt}\|^2 - I(u)) - (p + 3)M(t)(\|u_t\|^2 + \|u_{xx}\|^2)
\]

\[
= M(t)\xi(t)
\]

(27)

for a.e. \( t \in [0, T] \), where \( \xi : [0, T] \to \mathbb{R}_+ \) is the map defined by

\[
\xi(t) = -(p + 1)(\|u_t\|^2 + \|u_{xx}\|^2) - 2I(u).
\]

(28)
Case I. when $a$ solution

It follows from Theorem 2.12 that the problem (1)-(3) admits a unique local
Proof.

I. exists $u$ a unique global weak solution $u$

Theorem 4.1

4. Existence and nonexistence of global solutions at critical energy level

when $u$

there exists a

$(\text{nonexistence of global weak solution for problem (1)-(3) as follows.}$

By (10) and (14), (28) becomes

$$\xi(t) = -2(p + 1)E(t) + (p - 1)(a\|u_x\|^2 + \|u_{xx}\|^2)$$
$$= -2(p + 1)E(0) + (p - 1)(a\|u_x\|^2 + \|u_{xx}\|^2)$$
$$> -2(p + 1)(d - E(0))$$
$$> 0$$

since $E(0) < d$. Hence, there exists $\rho > 0$ such that

$$\xi(t) > \rho \quad \text{for all} \quad t \geq 0. \quad (29)$$

By (21), (27) and (29) it follows that

$$M(t)M''(t) - \frac{b + 3}{4} (M'(t))^2 \geq M(t)\xi(t) > \delta_1 \rho \quad \text{for a.e.} \quad t \in [0, T],$$

thus,

$$\left( M(t)^{-\alpha} \right)'' = \frac{-\alpha}{(M(t))^{\alpha + 2}} \left( M(t)M''(t) - (1 + \alpha)(M'(t))^2 \right)$$
$$< \frac{-\alpha \delta_1 \rho}{\delta_2^{\alpha + 2}} < 0, \quad (30)$$

where $\alpha = \frac{p - 1}{2}$. Hence, this proves that $M(t)^{-\alpha}$ reaches zero in finite time, say there exists a $T_\delta \in (0, T)$ such that

$$\lim_{t \to T_\delta} M(t) = \infty,$$

which contradicts $T_{\text{max}} = \infty$. The proof is completed. \(\square\)

From Theorems 3.2 and 3.3 we can obtain a sharp condition for existence and nonexistence of global weak solution for problem (1)-(3) as follows.

Corollary 2 (Sharp condition for $E(0) < d$). Let $f(u) = \pm |u|^p$ or $u|u|^{p-1}$, $u_0$, $u_1 \in H$. Assume that $E(0) < d$. Then problem (2)-(3) admits a global weak solution when $I(u_0) \geq 0$ and the problem (4)-(3) does not admit any global weak solution when $I(u_0) < 0$.

4. Existence and nonexistence of global solutions at critical energy level $E(0) = d$. We shall show the global existence theorems for two cases: $E(0) = d$, $I(u_0) \geq 0$ and $E(0) = d$, $I(u_0) < 0$. We shall begin with the first case.

Theorem 4.1 (Global existence for $E(0) = d$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u$, $u_0(x), u_1(x) \in H$. Assume that $E(0) = d$, $I(u_0) \geq 0$. Then problem (1)-(3) admits a unique global weak solution $u \in C^1([0, \infty); H)$ and $u(t) \in W = W \cap \partial W$ for $t \in [0, \infty)$.

Proof. It follows from Theorem 2.12 that the problem (1)-(3) admits a unique local solution $u \in C^1([0, T_{\text{max}}); H)$. In what follows, we prove that $T_{\text{max}} = \infty$.

We prove this theorem by considering the following two cases.

Case I. when $a\|u_0_x\|^2 + \|u_0_{xx}\|^2 \neq 0$.

From $I(u_0) \geq 0$, (vi) and (v) in Lemma 2.4 we have $\lambda^* = \lambda^*(u_0) \geq 1$, $J(\lambda u_0)$ is increasing and $I(\lambda u_0) > 0$ for $0 < \lambda < 1$ (see FIGURE1.). Let $\lambda_m = 1 - \frac{1}{m}$ and $u_m = \lambda_m u_0$, $m = 2, 3, \ldots$. Consider the initial conditions

$$u(x, 0) = u_{0m}(x), \quad u_t(x, 0) = u_1(x) \quad (31)$$
and corresponding problem \([1], [31]\) and \([3]\), then

\[
I(u_{0m}) = I(\lambda_m u_0) > 0,
\]

\[
J(u_{0m}) = \frac{1}{2} \left(a\|u_{0mx}\|^2 + \|u_{0mxx}\|^2\right) - \frac{1}{p+1} \int_0^1 u_{0mx} f(u_{0mx}) dx
\]

\[
= \left(\frac{1}{2} - \frac{1}{p+1}\right) \left(a\|u_{0mx}\|^2 + \|u_{0mxx}\|^2\right) + \frac{1}{p+1} I(u_{0m})
\]

\[
> 0
\]

and

\[
J(u_{0m}) = J(\lambda_m u_0) < J(u_0).
\]

In addition

\[
0 < E_m(0) = \frac{1}{2} \left(\|u_1\|^2 + \|u_{1xx}\|^2\right) + J(u_{0m})
\]

\[
= \frac{1}{2} \left(\|u_1\|^2 + \|u_{1xx}\|^2\right) + J(\lambda_m u_0)
\]

\[
< \frac{1}{2} \left(\|u_1\|^2 + \|u_{1xx}\|^2\right) + J(u_0) = E(0) = d.
\]

Hence from Theorem 3.2 it follows that for each \(m\) problem \([1], [31]\) and \([3]\) admits a unique global weak solution \(u_m \in C([0, \infty), H)\) with \(u_{mt} \in C([0, \infty), H)\) and \(u_m \in W\) for \(0 \leq t < \infty\). Therefore, up to a subsequence, we may pass to the limit and obtain a weak solution \(u\) of problem \([1]-[3]\).

**Figure 1.**

Case II. when \(a\|u_{0x}\|^2 + \|u_{0xx}\|^2 = 0\).

In this case have \(J(u_0) = 0\). Hence from

\[
\frac{1}{2} \left(\|u_1\|^2 + \|u_{1xx}\|^2\right) + J(u_0) = E(0) = d,
\]
we obtain $\frac{1}{4} (\|u_1\|^2 + \|u_{1xx}\|^2) = d$. Take a sequence $\{\lambda_m\}$ such that $0 < \lambda_m < 1$ and $\lambda_m \to 1$ as $m \to \infty$. Let $\lambda_m = 1 - \frac{1}{m}$ and $u_{1m}(x) = \lambda_m u_1(x)$. For the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_{1m}(x), \quad m = 2, 3, \ldots \quad (34)$$

and corresponding problem [1], [5] and [3].

$$E_m(0) = \frac{1}{2} (\|u_{1m}\|^2 + \|u_{1xx}\|^2) + J(u_0)$$

$$= \frac{1}{2} (\|\lambda_m u_1\|^2 + \|\lambda_m u_{1xx}\|^2)$$

$$\leq \frac{1}{2} (\|u_1\|^2 + \|u_{1xx}\|^2) = d. \quad (35)$$

The remainder of the proof is the same as that in the proof of Case I of the proof of this theorem.

The following lemma shows the invariance of the set $V$ under the flow of problem [1]-[3].

Lemma 4.2 (Invariant sets for $E(0) = d$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u$, $u_0(x)$, $u_1(x) \in H$ and $u$ is a unique solution of problem [1]-[3]. Assume that $E(0) = d$ and $(u_0, u_1) + (u_{0xx}, u_{1xx}) \geq 0$. Then for all $t \in (0, T_{\max})$, $u \in V$ if $u_0 \in V$.

Proof. Let $u(t)$ be any weak solution of problem [1] with $E(0) = d$, $I(u_0) < 0$ and $(u_0, u_1) + (u_{0xx}, u_{1xx}) \geq 0$, $T_{\max}$ being the existence time of $u(t)$. Let us prove $I(u) < 0$ for $0 < t < T_{\max}$. Arguing by contradiction, then by the continuity of $I(u)$, we suppose that there exists a $t_0 \in (0, T_{\max})$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $0 < t < t_0$.

By variational characterization of $d$, we get $J(u(t_0)) \geq d$. From this and $J(u(t)) \leq E(t_0) = E(0) = d$, we have $J(u(t_0)) = d$ and

$$\|u_t(t_0)\|^2 + \|u_{xx}(t_0)\|^2 = 0.$$ 

Let $M(t)$ be defined as [20], then we have

$$M'(t) = 2\|u_t\|^2 + 2\|u_{xx}\|^2 - 2I(u) > 0, \quad 0 < t < t_0.$$ 

Hence $M'(t)$ is strictly increasing on $[0, t_0]$. Since

$$M'(0) = \langle u_t, u_1 \rangle + (u_{0xx}, u_{1xx}) \geq 0,$$

we obtain $M'(t_0) > 0$, which contradicts $\|u_t(t_0)\|^2 + \|u_{xx}(t_0)\|^2 = 0$. This proves Lemma 4.2.

Theorem 4.3 (Blow up for $E(0) = d$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u$, $u_0, u_1 \in H$, $E(0) = d$ and $I(u_0) < 0$, and $(u_0, u_1) + (u_{0xx}, u_{1xx}) \geq 0$, then the solution of problem [1]-[3] blows up in finite time.

Proof. From Theorem 2.12 it follows that problem [1]-[3] admits a unique local weak solution $u \in C(0, T_{\max}; H)$. We prove that $T_{\max} < \infty$. Arguing by contradiction, we suppose that $T_{\max} = \infty$. Again let $M(t)$ be defined as [20], we have [21]-[25]. From (19) we have

$$\frac{1}{2} (\|u_t\|^2 + \|u_{xx}\|^2) + \frac{p - 1}{2(p + 1)} \langle u_t, u_x \rangle^2 + \|u_{xx}\|^2 + \frac{1}{p + 1} I(u) = E(0) = d,$$
then (25) becomes

\[ M''(t) = \left( p + 3 \right) \left( \|u_t\|^2 + \|u_{xx}\|^2 \right) + \left( p - 1 \right) (a \|u_x\|^2 + \|u_{xxt}\|^2) - 2(p + 1)d, \]

for \( 0 \leq t < \infty \). By Lemma 4.2, we have \( I(u) < 0 \) for \( 0 \leq t < \infty \), then from Lemma 2.9 we obtain

\[ (p - 1)(a \|u_x\|^2 + \|u_{xx}\|^2) > 2(p + 1)d. \]

Hence we have

\[ M''(t) > (p + 3)(\|u_t\|^2 + \|u_{xxt}\|^2), \quad 0 \leq t < \infty, \]

which gives

\[
M(t)M''(t) - \frac{p + 3}{4} \left( M'(t) \right)^2 > (p + 3)M(t)(\|u_t\|^2 + \|u_{xxt}\|^2) - (p + 3) \left( M'(t) \right)^2 = (p + 3) \left( (\|u\|^2 + \|u_{xx}\|^2) \left( \|u_t\|^2 + \|u_{xxt}\|^2 \right) - ((u, u_t) + (u_{xx}, u_{xxt}))^2 \right) = (p + 3)\eta(t),
\]

where \( \eta(t) \) is defined in (26). Using Schwarz, Hölder and Young inequalities, we obtain

\[
(u, u_t)^2 \leq \|u\|^2 \|u_t\|^2,
\]

\[
(u_{xx}, u_{xxt})^2 \leq \|u_{xx}\|^2 \|u_{xxt}\|^2
\]

and

\[
2(u, u_t) (u_{xx}, u_{xxt}) \leq 2\|u\| \|u_t\| \|u_{xx}\| \|u_{xxt}\|
\]

\[
\leq \|u\|^2 \|u_{xxt}\|^2 + \|u_t\|^2 \|u_{xx}\|^2.
\]

These three inequalities entail \( \eta(t) \geq 0 \) for every \( 0 < t < \infty \). Hence, there exists \( \varepsilon > 0 \) such that

\[
M(t)M''(t) - \frac{p + 3}{4} M'(t)^2 > \varepsilon,
\]

then

\[
\left( M(t)^{-\alpha} \right)'' = \frac{-\alpha}{(M(t))^{\alpha+2}} \left( M(t)M''(t) - (1 + \alpha)M'(t)^2 \right) < \frac{-\alpha \varepsilon}{\delta^{\alpha+2}} < 0, \quad (36)
\]

\[
\alpha = \frac{p - 1}{4}, \quad 0 \leq t < \infty.
\]

From (36) it follows that there exists a \( T_1 > 0 \) such that

\[
\lim_{t \to T_1} M^{-\alpha}(t) = 0,
\]

and

\[
\lim_{t \to T_1} M(t) = +\infty,
\]

which contradicts \( T_{\text{max}} = +\infty \). The proof of Theorem 4.3 is completed.

From Theorem 4.1 and Theorem 4.3 we can obtain the following threshold results of global existence and nonexistence of solution for problems (1)-(3) with \( E(0) = d \).
Corollary 3 (Sharp condition for $E(0) = d$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1} u$, $u_0(x), u_1(x) \in H$. Assume that $E(0) = d$. Then when $I(u_0) \geq 0$, problem (1) admits a unique global weak solution; when $I(u_0) < 0$ and $(u_0, u_1) + (u_{0xx}, u_{1xx}) \geq 0$ the problem does not admit any global weak solution and the weak solution blows up in finite time.

5. Existence and nonexistence of global solutions at high initial energy level $E(0) > 0$. In this section, we consider the existence and nonexistence of the global solutions for the problems (1) at high initial energy level $E(0) > 0$.

Theorem 5.1 (Global existence for $E(0) > 0$). Let $f(u) = \pm |u|^p$ or $|u|^{p-1} u$, $u_0(x), u_1(x) \in H$. Suppose that $E(0) > 0$, the initial data satisfy

$$\|u_0\|^2 + \|u_{0xx}\|^2 + 2(u_0, u_1) + 2(u_{0xx}, u_{1xx}) + \frac{2(p+1)}{p+3} E(0) < 0$$

and

$$\|u_1\|^2 + \|u_{1xx}\|^2 - I(u_0) < 0,$$

then the existence time of global solution for problem (1) is infinite.

Proof. Let $M(t)$ be defined as (20), we have (21)-(25). Then we can reach that

$$M''(t) = 2\|u_t\|^2 + 2\|u_{xx}\|^2 - 2I(u) : = -2K(u(t)).$$

From (38), we have $K(u_0) > 0$. Now we prove $K(u(t)) > 0$ for all $t \in (0, T_{\max})$. Arguing by contradiction, by the continuity of $K(u(t))$ in $t$, we suppose that $t_0 \in (0, T_{\max})$ be the first time such that

$$K(u(t_0)) = 0$$

and

$$K(u(t)) > 0 \text{ for any } t \in [0, t_0).$$

From $E(0) > 0$, (37) and (22) we can get

$$M'(0) = 2(u_0, u_1) + 2(u_{0xx}, u_{1xx}) < 0.$$

By (41), we may also write $M''(t) < 0$ for all $t \in [0, t_0)$. It is easy to see that

$$M'(t) < M'(0) < 0 \text{ for all } t \in [0, t_0).$$

And then by (37), for all $t \in [0, t_0)$, we have

$$M(t) < M(0) = \|u_0\|^2 + \|u_{0xx}\|^2$$

$$< -2(u_0, u_1) - 2(u_{0xx}, u_{1xx}) - \frac{2(p+1)}{p+3} E(0).$$

Therefore from the continuity of $\|u(t)\|^2 + \|u_{xx}(t)\|^2$ in $t$, we get

$$M(t_0) < \|u_0\|^2 + \|u_{0xx}\|^2 < -2(u_0, u_1) - 2(u_{0xx}, u_{1xx}) - \frac{2(p+1)}{p+3} E(0).$$

On the other hand, from Lemma 2.11 (10) and (40), we can obtain

$$E(0) = E(t_0)$$

$$= \frac{1}{2} (\|u(t_0)\|^2 + \|u_{xx}(t_0)\|^2) + \frac{1}{p+1} I(u(t_0))$$

$$= \frac{1}{2} (\|u_0\|^2 + \|u_{0xx}\|^2) + \frac{1}{p+1} I(u(0)) + \frac{2(p+1)}{p+3} E(0).$$
with (42) and (44), we derive

\[
E(0) \geq A \|u_t(t_0) + u(t_0)\|^2 + A \|u_{xx}(t_0) + u_{xx}(t_0)\|^2 - A \|u(t_0)\|^2
\]

\[- A \|u_{xx}(t_0)\|^2 - 2A(u(t_0), u_t(t_0)) - 2A(u_{xx}(t_0), u_{xx}(t_0)) \]

\[
\geq - A (\|u(t_0)\|^2 + \|u_{xx}(t_0)\|^2) - 2A(u_0, u_1) + (u_{0xx}, u_{1xx})
\]

where \( A = \frac{p+3}{2(p+1)} \) or equivalently

\[
M(t_0) \geq - 2(\|u_0, u_1\| - 2(\|u_{0xx}, u_{1xx}\| - \frac{2(p+1)}{p+3})E(0),
\]

which contradicts the first inequality of (43). So we can get \( K(u(t)) > 0 \) for any \( t \in [0, T_{\max}) \). Therefore from (10), we can obtain

\[
E(0) = E(t) = \frac{1}{2} (\|u_t\|^2 + \|u_{xx}\|^2) + \frac{p-1}{2(p+1)} (a\|u_x\|^2 + \|u_{xx}\|^2) + \frac{1}{p+1} f(u)
\]

\[
> \frac{p+3}{2(p+1)} (\|u_t\|^2 + \|u_{xx}\|^2) + \frac{p-1}{2(p+1)} (a\|u_x\|^2 + \|u_{xx}\|^2),
\]

which implies

\[
u(x, t), u_t(x, t) \) are bounded in \( C^1(0, T_{\max}; H)\).
\]

Hence from Theorem 2.12, it follows that \( T_{\max} = \infty \) and the solution of problem (1)-(3) exists globally.

In what follows, we show a preliminary lemma about the monotonicity of the functional \( \|u(x, t)\|^2 + \|u_{xx}(x, t)\|^2 \), which will be used to prove the invariance of the unstable set \( V \) under the flow of problem (1)-(3) at high initial energy level \( E(0) > 0 \).

**Lemma 5.2.** Let \( f(u) = \pm |u|^p \) or \( |u|^{p-1} u, u_0(x), u_1(x) \in H \) be given. Assume the initial data satisfy

\[
\int_0^1 u_0 u_1 dx + \int_0^1 u_{0xx} u_{1xx} dx \geq 0.
\]

Let \( u(x, t) \) be the solution of equation (4) with initial data \( u_0, u_1 \). Then the map \( \{ t \mapsto \|u(t)\|^2 + \|u_{xx}(t)\|^2 \} \) is strictly increasing as long as \( u(x, t) \in V \).
Lemma 5.3 (Invariant sets for E(0) > 0). Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u$, $u_0(x)$, $u_1(x) \in H$ and \((45)\) hold. Assume that the initial data satisfy
\[
\begin{align*}
|u_0|^2 + |u_{0xx}|^2 > & \frac{2}{aC} \left(1 + \frac{2}{p-1}\right) E(0), \quad a \leq \frac{1}{C}, \\
|u_0|^2 + |u_{0xx}|^2 > & \left(1 + \frac{2}{p-1}\right) E(0), \quad a \geq \frac{1}{C},
\end{align*}
\]
where $C$ is the coefficient of Poincaré inequality $|u_x|^2 \geq C |u|^2$. Then all solutions of problem \((1)-(3)\) with $E(0) > 0$ belong to $V$, provided $u_0 \in V$.

Proof. We prove $u(t) \in V$. Arguing by contradiction, we suppose that there exists $t_0 \in (0, T_{\max})$ such that $u(t_0) \in V$, i.e. $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for $t \in [0, t_0)$. Let $M(t)$ be defined as \((20)\) above. Hence by Lemma 5.2 we can get that $M(t)$ and $M'(t)$ are strictly increasing on the interval $[0, t_0)$. Then by \((46)\) for $t \in [0, t_0)$ we have
\[
\begin{align*}
M(t) > & \frac{2}{aC} \left(1 + \frac{2}{p-1}\right) E(0), \quad a \leq \frac{1}{C}, \\
M(t) > & \left(1 + \frac{2}{p-1}\right) E(0), \quad a \geq \frac{1}{C}.
\end{align*}
\]
Therefore from the continuity of $|u(t)|^2 + |u_{xx}(t)|^2$ in $t$ we get
\[
\begin{align*}
M(t_0) > & \frac{2}{aC} \left(1 + \frac{2}{p-1}\right) E(0), \quad a \leq \frac{1}{C}, \\
M(t_0) > & \left(1 + \frac{2}{p-1}\right) E(0), \quad a \geq \frac{1}{C}.
\end{align*}
\]
On the other hand, by \((10)\) we can obtain
\[
E(0) = E(t_0) \geq \left(\frac{1}{2} - \frac{1}{p+1}\right) (a|u_x(t_0)|^2 + |u_{xx}(t_0)|^2) + \frac{1}{p+1} I(u(t_0)).
\]
Note that $I(u(t_0)) = 0$, we have
\[
2 \left(1 + \frac{2}{p-1}\right) E(0) \geq a|u_x(t_0)|^2 + |u_{xx}(t_0)|^2.
\]
Next we prove that this lemma depends on the relation of $a$ and $\frac{1}{C}$ by considering two cases, where $C$ is the coefficient of Poincaré inequality $|u_x|^2 \geq C |u|^2$.

Case I. $a \leq \frac{1}{C}$. In this case we have
\[
\begin{align*}
2 \left(1 + \frac{2}{p-1}\right) E(0) \geq & a|u_x(t_0)|^2 + |u_{xx}(t_0)|^2 \\
\geq & aC |u(t_0)|^2 + |u_{xx}(t_0)|^2 \\
\geq & aC (|u(t_0)|^2 + |u_{xx}(t_0)|^2) \\
= & aCM(t_0),
\end{align*}
\]
which contradicts the first inequality of (47).

**Case II.** $a \geq \frac{1}{C}$. At this point we can get

$$2 \left(1 + \frac{2}{p-1}\right) E(0) \geq a \|u_x(t_0)\|^2 + \|u_{xx}(t_0)\|^2$$

$$\geq a C \|u(t_0)\|^2 + \|u_{xx}(t_0)\|^2$$

$$\geq \|u(t_0)\|^2 + \|u_{xx}(t_0)\|^2$$

$$= M(t_0),$$

which contradicts the second inequality of (47).

Next, we present the main blow up theorem for the solution of problem (1)-(3) at high initial energy level $E(0) > 0$.

**Theorem 5.4 (Blow up for $E(0) > 0$).** Let $f(u) = \pm |u|^p$ or $|u|^{p-1}u, u_0(x), u_1(x) \in H$ be given and (45), (46) hold. Suppose that $E(0) > 0$ and $u_0 \in V$, then the solution of problem (1)-(3) blows up in finite time.

**Proof.** Let $u(t)$ be any solution of problem (1)-(3) with $E(0) > 0$ and $u_0 \in V$, then from Lemma 5.3 we have $u \in V$. Next, we prove the solution of problem (1)-(3) blows up in finite time. Suppose by contradiction that the solution $u(x, t)$ is global. Then for any $T_0 > 0$, we define the auxiliary function $M(t)$ as (20) above, then we have (21)-(28). Note (27), we get

$$M''(t)M(t) - \frac{p+3}{4} (M'(t))^2 \geq M(t)\xi(t),$$

where $\xi(t)$ is defined in (28). Then by (10), we have

$$\xi(t) = (p-1) \left( a \|u_x\|^2 + \|u_{xx}\|^2 \right) - 2(p+1)E(0)$$

In the following from Lemma 5.2 and Lemma 5.3 we estimate $\xi(t)$ by two cases depending on the relation of $a$ and $\frac{1}{C}$, where $C$ is the coefficient of Poincaré inequality $\|u_x\|^2 \geq C\|u\|^2$.

On the one hand when $a \leq \frac{1}{C}$ we have

$$\xi(t) = (p-1) \left( a \|u_x(t)\|^2 + \|u_{xx}(t)\|^2 \right) - 2(p+1)E(0)$$

$$\geq (p-1) \left( aC \|u(t)\|^2 + \|u_{xx}(t)\|^2 \right) - 2(p+1)E(0)$$

$$\geq aC(p-1) \left( \|u(t)\|^2 + \|u_{xx}(t)\|^2 \right) - 2(p+1)E(0)$$

$$\geq aC(p-1) \left( \|u_0\|^2 + \|u_{0xx}\|^2 \right) - 2(p+1)E(0)$$

$$= \sigma_1 > 0.$$

On the other hand when $a \geq \frac{1}{C}$ we have

$$\xi(t) = (p-1) \left( a \|u_x\|^2 + \|u_{xx}\|^2 \right) - 2(p+1)E(0)$$

$$\geq (p-1) \left( aC \|u\|^2 + \|u_{xx}\|^2 \right) - 2(p+1)E(0)$$

$$\geq (p-1) \left( \|u\|^2 + \|u_{xx}\|^2 \right) - 2(p+1)E(0)$$

$$\geq (p-1) \left( \|u_0\|^2 + \|u_{0xx}\|^2 \right) - 2(p+1)E(0)$$

$$= \sigma_2 > 0.$$
Therefore we can obtain
\[ M''(t)M(t) - \frac{p + 3}{4} M'(t)^2 \geq M(t)\xi(t) \geq \delta_1 \sigma > 0, \ t \in [0, T_0], \]
where \( \sigma = \min\{\sigma_1, \sigma_2\} \). Setting \( y(t) = M(t)^{-\frac{\alpha}{\alpha+2}} \), this inequality becomes
\[ y''(t) = \frac{-\alpha}{(M(t))^{\alpha+2}} \left( M(t)M''(t) - (1 + \alpha) \left( M'(t) \right)^2 \right) \leq \frac{-\alpha \delta_1 \sigma}{\delta_2^{\alpha+2}} < 0, \ t \in [0, T_0], \]
where \( \alpha = \frac{\sigma_1}{\sigma_2} \). This proves that \( y(t) \) reaches 0 in finite time, say \( t \to T_* \). Since \( T_* \) is independent of the initial choice of \( T_0 \), we may assume that \( T_* < T_0 \). This tells us that
\[ \lim_{t \to T_*} M(t) = +\infty. \]
This completes the proof.

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E-mail address: xurunzh@163.com
E-mail address: zhangmingyou@163.com
E-mail address: george_chen@cbu.ca
E-mail address: yanbing_yang@yeah.net
E-mail address: shenjihong@126.com