Long time convergence for a class of variational phase field models∗

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Dedicated to Professor Masayasu Mimura on the occasion of his 65th birthday

Abstract. In this paper we analyze a class of phase field models for the dynamics of phase transitions which extend the well-known Caginalp and Penrose-Fife models. Existence and uniqueness of the solution to the related initial boundary value problem are shown. Further regularity of the solution is deduced by exploiting the so-called regularizing effect. Then, the large time behavior of such a solution is studied and several convergence properties of the trajectory as time tends to infinity are discussed.

Key words: phase transition, gradient flow, \( \omega \)-limit set, Simon-Lojasiewicz inequality.

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1 Introduction

The present note is devoted to the analysis of the regularity and long-time behavior properties of the following class of PDE’s systems modeling phase change phenomena

\[
\begin{align*}
\varepsilon \vartheta_t + \lambda(\chi) - \Delta j'(\vartheta) &= f, \\
\delta \chi_t - \Delta \chi + W'(\chi) &= \lambda'(\chi) j'(\vartheta). 
\end{align*}
\]

Here, the unknowns are the relative temperature \( \vartheta \) (i.e., some critical freezing or melting temperature \( \tau_c \) has been normalized to 0) and the order parameter, or phase field, \( \chi \); both are functions of the spatial variable \( x \) (ranging in a bounded, connected, and sufficiently smooth domain \( \Omega \subset \mathbb{R}^d, 1 \leq d \leq 3 \)) and of the time \( t \in [0, \infty) \) (let us use the shorter notation \( \infty \) in place of \( +\infty \)). In the above system, \( \varepsilon, \delta > 0 \) are relaxation parameters; the function \( W : \text{dom} W \to \mathbb{R} \) represents a configuration potential in \( \chi \) and is assumed to be convex in its principal part; \( \lambda(\cdot) \) is a possibly non linear function with \( \varepsilon \vartheta + \lambda(\chi) \) yielding the internal energy of the system; finally, \( f \) stands for the heat source. However, the main novelty in (1.1)–(1.2) is given by the presence of the nonlinear, but convex function \( j \). This nonlinearity can be justified both on the physical and on the analytical side. Actually, several well-known models can be included in this general framework. For example, \( j(r) = r^2/2 \), corresponds to the Caginalp system [6], while \( j(r) = -\log(r + \tau_c) + r/\tau_c \) gives the so-called Penrose-Fife model [24, 25] (the complication of the expression, which is not the usual one, is compensated by the nice property \( j'(0) = 0 \)). Furthermore, also intermediate choices for \( j \) correspond to meaningful cases: for instance, a combination of the previous two expressions provides a variant of the Penrose-Fife model with special heat flux law introduced in [10,11].

Various systems concerned with the abovementioned phase field models, possibly including non-smooth potentials and also applying to martensitic phase transformations, have been intensively investigated in the last years. Among a number of recent contributions, let us quote [20] and [8] which treat the case of Neumann boundary conditions for the temperature; [22] devising a general (convex) framework for the study of Penrose-Fife systems; [12] involved with the analysis of the quasistationary (i.e., \( \delta = 0 \) in (1.2)) Penrose-Fife model; [19] that deals with the long-time behaviour and the study of inertial sets; [1] and [2] studying the long-time convergence of the Caginalp model with and without memory effects, and [30, 14] addressing the same questions for the Penrose-Fife model; [26, 27] showing the existence of a uniform attractor for Penrose-Fife systems; [9] and [28] in which a hyperbolic dynamics for \( \chi \), characterized by an extra inertial term \( \rho \chi_{tt} \) in the left hand side of equation (1.2), is considered.

In all this framework, the occurrence of a general convex function \( j \) seems to be an interesting issue, worth to be deepened. Recently, well-posedness of an initial and boundary value problem for (1.1)–(1.2) has been shown in [29] for completely arbitrary (convex) \( j \). More precisely, in [29] the homogeneous Neumann problem related to (1.1)–(1.2) is studied for a linear function \( \lambda \). We remark that, while the Neumann boundary conditions for the phase field \( \chi \) appear to be the most natural choice for phase field models, the case of no-flux conditions for \( j'(\vartheta) \) was motivated in [29] by the purpose of studying some singular limits of the system. In particular, in [29] it is shown that (1.1)–(1.2) gives rise to the Cahn-Hilliard equation in the viscous form if \( \varepsilon \) is sent to
0 and in the standard form if both $\varepsilon, \delta$ go to 0. On the other hand, the choice of the no-flux conditions for $j'(\vartheta)$ gives rise to some additional difficulties in the analysis that forced the author of [29] to restrict the range of the admissible potentials $W$ in order to get some a priori control of the spatial average of the unknowns (see [29], assumption (28)], see also [8]).

In this paper, we first extend the well-posedness result of [29], by adapting it to our slightly different setting. Then, we investigate here some further properties of the solution to suitable initial-boundary value problems related to (1.1)–(1.2). Namely, we shall concentrate our attention on the long-time behavior of the system from the point of view of $\omega$-limits of solution trajectories. Since we are not interested in singular limits, we shall take $\varepsilon = \delta = 1$ in the sequel.

In our approach, the basic observation is that the system, at least in the case of no external source, admits the Liapounov functional

$$E(\vartheta, \chi) := \int_\Omega \left( \frac{1}{2} |\nabla \chi|^2 + W(\chi) + j(\vartheta) \right),$$

(1.3)

which is obtained testing (1.1) by $j'(\vartheta)$ and (1.2) by $\chi_t$, then taking the sum. Hence, if we introduce the new variable $e := \vartheta + \lambda(\chi)$ (internal energy), (1.1)–(1.2) can be seen (at least in the case of no-flux conditions and no external source) as a gradient flow problem

$$e_t = -\partial_{e,e} E, \quad \chi_t = -\partial_{H,\chi} E.$$ (1.4)

Here, the symbol $\partial$ denotes here (sub)differentiation, and, more precisely, $\partial_{H,\chi}$ stands for the subdifferential w.r.t. $\chi$ in $H := L^2(\Omega)$ while $\partial_{e,e}$ indicates differentiation in $e$ in the space $H^1(\Omega)^*$ (which gives rise to the Laplacian in (1.1)).

In view of this variational structure, it is reasonable to expect good asymptotic properties of the solution as $t$ goes to infinity. This has already been noticed in some particular cases. For instance, for the Caginalp model corresponding to $j(r) = r^2/2$, the long time analysis has been performed in [2, 16] for various types of boundary conditions and assumptions on $\lambda, W$. Instead, the Penrose-Fife case ($j(r) = -\log r$) has been studied in [14], referring only to the case of (nonhomogeneous) Dirichlet boundary conditions for the temperature. Concerning [14], we point out that the problem addressed there presents a number of mathematical difficulties due the character of $j(\cdot) = -\log(\cdot)$, which is both singular at 0 and unbounded from below. Actually, the precompactness of trajectories, leading to the existence of a nonempty $\omega$-limit set, has been proved in [14] by strongly relying on ad hoc techniques to overcome, in particular, the noncoercive character of $j$ at $+\infty$.

Based on these considerations, it seems very difficult to perform a long time analysis of (1.1)–(1.2) by taking $j$ completely general as in [29]. Rather, we find there is room for a bit of compromise and ask at least that $j$ has some coercivity: see (hp $j$) below. On the other hand, $j$ can exhibit a singular character too. Still for the purpose of coercivity, we also need boundary conditions for $\vartheta$ which give some (uniform in time) control on its space average (differently from the no-flux case of [29]). Then, we shall consider two sets of boundary conditions:

$$i) \quad \vartheta = \vartheta_\infty, \text{ with } j'(\vartheta_\infty) = 0, \text{ and } \partial_n \chi = 0,$$

(1.5)
which we refer to as Dirichlet boundary conditions, and

\[
ii) \quad - \partial_n j'(\vartheta) = \eta (j'(\vartheta) - j'(\vartheta_\Gamma)) \quad \text{and} \quad \partial_n \chi = 0, \tag{1.6}
\]

which we refer to as Robin boundary conditions, where \( \vartheta_\Gamma \) represents the extremal boundary temperature and \( \eta \) is a positive constant. We point out that the latter case subsumes the presence of some (boundary) source term. We will see that this does not destroy the variational character of the system, nor does this the presence of a nonzero volumetric heat source \( f \) in (1.1), provided \( f \) and the boundary datum are globally \( L^2 \) in time. In our analysis, we will be able to consider nonlinear latent heat functions \( \lambda \) and, what is more important, rather general potentials \( W \) with the only restriction that they should not exhibit minima at the boundary of their domain (like instead it could happen in the case of a double-obstacle potential). These could neither be considered in [14], essentially for technical reasons, nor in [29] due to the quoted difficulties coming from the no-flux conditions for \( j'(\vartheta) \). We stress in particular that singular potentials \( W \) (i.e. which are \( +\infty \) outside an interval \( I \subset \mathbb{R} \)) are not completely easy to address even in the simpler case of the Caginalp model [16].

In the framework of the long time analysis, we prove the following results. First, we demonstrate that the \( \omega \)-limit of any solution trajectory is not empty and consists only of steady-state solutions \( (\vartheta_\infty, \chi_\infty) \). More in detail, the component \( \vartheta_\infty \) coincides with the (constant) temperature on the boundary (i.e., with the unique value such that \( j'(\vartheta_\infty) = 0 \), cf. assumption \( (\text{hp}_{j_2}) \) below); so that we also obtain the limit \( \vartheta(t) \to \vartheta_\infty \) holds in a suitable sense as \( t \to \infty \) and not only for a subsequence. Conversely, in general we only have that, as \( t \to \infty \), \( \chi(t) \) is precompact in a suitable topology and any of its limit points is a steady state solution of (1.2). In fact, (1.2) may have infinitely many stationary states due to nonconvexity of \( W \). We also point out that a careful use of parabolic regularization effects is a key step in the proof of the convergence result. Our second theorem, which follows the lines of some recent work devoted to the large time behavior of degenerate parabolic equations [15] and phase transition systems (see, among others, [2, 13, 14]), gives a sufficient condition under which the \( \omega \)-limit consists of only one point. Namely, we can prove that, in the case when \( W \) is analytic in the subdomain where the solution component \( \chi \) lives, then the \( \omega \)-limit set is a singleton and consequently also the entire trajectory of \( \chi(t) \) converges to \( \chi_\infty \). Here, the main ingredient of the proof is the so-called Simon-Lojasiewicz inequality [23, 31], that was originally [23] stated as a nontrivial (local) growth estimate for analytic functions of several complex variables in the neighbourhood of a critical point: Simon contributed by extending this inequality to the infinite dimensional setting, thus allowing to characterize the large time behavior of evolution systems with analytic nonlinearities. The last result we present is a convergence result which also establishes a (say, polynomial) rate of convergence for the \( L^2(\Omega) \)-norm of \( \chi(t) - \chi_\infty \). In proving such a result, we follow a method from [7] and argue partly as in [17].

Here is the plan of the paper. In Section 2 we introduce the functional framework, state precise hypotheses on the data and formulate our main results. Section 3 is devoted to the proof of the well-posedness of the system (Theorem 2.2) and of the basic uniform estimates (Theorem 2.6). Finally, Section 4 is concerned with all the properties of the \( \omega \)-limit set (proof of Theorem 2.7 and Theorem 2.13).
2 Main results

Let $\Omega$ be a $C^{1,1}$, bounded, and connected domain in $\mathbb{R}^d$, $1 \leq d \leq 3$, and let $\Gamma := \partial \Omega$. Set $H := L^2(\Omega)$, $V := H^1(\Omega)$, both endowed with their standard scalar products and norms. The norms in $H$ and in $H^d$ will be equally indicated by $|\cdot|$ and $(\cdot, \cdot)$ will denote the corresponding scalar products. Let also $V_0 := H^1_0(\Omega)$, endowed with the norm $|\cdot|_{V_0} := |\nabla \cdot|$. The symbol $|\cdot|_X$ will stand for the norm in the generic Banach space $X$ and $X^\ast (\cdot, \cdot)_X$ will denote the duality between $X$ and its topological dual space $X^\ast$. The space $H$ will be identified with its dual, so that we have the chains of continuous embeddings $V \subset H \subset V^\ast$ and $V_0 \subset H \subset V_0^\ast$. We introduce the elliptic operator

$$A : V \to V^\ast, \quad v^\ast(Av, z)_V := \int_\Omega \nabla v \cdot \nabla z, \quad \text{for } v, z \in V. \quad (2.1)$$

We also let $I, J$ be open intervals of $\mathbb{R}$, with $0 \in I$, and let $I_0$ be an open and bounded interval containing $0$ and whose closure is contained in $I$. Then, our basic hypotheses on the data are

$$\lambda \in C^{1,1}_{\text{loc}}(\mathbb{R}), \quad \exists \Lambda > 0 : |\lambda''(r)| \leq \Lambda \text{ for a.e. } r \in \mathbb{R}, \quad \text{(hp\lambda)}$$
$$W \in C^{1,1}_{\text{loc}}(I; [0, +\infty)), \quad \exists \kappa > 0 : W''(r) \geq -\kappa \text{ for a.e. } r \in I, \quad \text{(hpW1)}$$
$$\exists \mu > 0 : W'(r)/r \geq \mu \text{ for a.e. } r \in I \setminus I_0, \quad \text{(hpW2)}$$
$$j \in C^{1,1}_{\text{loc}}(J; [0, +\infty)), \quad \exists \sigma > 0 : j''(r) \geq \sigma \text{ for a.e. } r \in J, \quad \text{(hpj1)}$$
$$\exists \psi_\infty \in J : j'(\psi_\infty) = 0. \quad \text{(hpj2)}$$

Let us spend some words on the hypotheses on $W$ and $j$. Formula (hpW1) says that $W$ can be nonconvex, but just up to a quadratic perturbation (actually, $r \mapsto W(r) + \kappa r^2/2$ is convex). Instead, $j$ is uniformly strictly convex by (hpj1); moreover, it has some coercivity property in the sense that, by (hpj2), it attains its minimum value (which can be fixed at 0, for simplicity) at the point $\psi_\infty \in J$. The role of (hpW2) will be outlined later on.

In the sequel, both $W$ and $j$ will be extended to the whole real line by means of the following procedure. First, we prolong $j$ (resp. $W + \kappa \text{Id}^2/2$) giving it the value $+\infty$ outside $J$ (resp. $I$); then, we take the lower semicontinuous regularization, and, finally, only from $W$ we subtract $\sqrt{\kappa \text{Id}^2/2}$. This means, in the simpler (because convex) case of $j$ that, if $J$ is bounded and $j$ “explodes” (taking the value $+\infty$) at its boundary, we simply extend it at $+\infty$ outside $J$. If, instead, $J$ is bounded but $j$ does not explode at least on one side of $J$, then we first close the graph of $j$ and then extend it at $+\infty$. For $W$ the procedure is slightly more complicated due to its possibly non convex character. In any case, the extended $j$ and $W + \kappa \text{Id}^2/2$ are convex and lower semicontinuous functions from $\mathbb{R}$ to $[0, +\infty]$, so that their $\mathbb{R}$-subdifferentials are maximal monotone graphs coinciding, respectively on $J$ and on $I$, with the “original” functions $j'$ and $W' + \kappa \text{Id}$.

With all these conventions in mind, our assumptions on the initial data are

$$\vartheta_0 \in H, \quad j(\vartheta_0) \in L^1(\Omega), \quad \text{(hp\vartheta_0)}$$
$$\chi_0 \in V, \quad W(\chi_0) \in L^1(\Omega). \quad \text{(hp\chi_0)}$$
Remark 2.1. Recalling (1.3), let us notice that $(\text{hp}_0^{\vartheta})-(\text{hp}_0^{\lambda_0})$ are equivalent to asking $\mathcal{E}(\vartheta_0, \lambda_0) < +\infty,$ i.e., that the initial energy is finite. Indeed, the property $\vartheta_0 \in H$ follows from the quadratic growth of $j$ (cf. $(\text{hp}_1^{j})$). Note also that, by $(\text{hp}_W^{1})$ and $(\text{hp}_j^{2})$, $W$ and $j$ assume only non-negative values and consequently the functional $\mathcal{E}$ is non-negative.

Concerning the heat source, we assume in any case

$$f \in L^2(0, \infty; H),$$

(hpf)

but this hypothesis will need some refinement in the sequel.

In case we work with the Robin boundary condition

$$-\partial_n j'(\vartheta) = \eta(j'(\vartheta) - j'(\vartheta_\Gamma)) \quad \text{and} \quad \partial_n \chi = 0,$$  

(2.2)

where $\eta$ is some positive constant, we also suppose that the external boundary temperature $\vartheta_\Gamma$ satisfies

$$\vartheta_\Gamma : \Gamma \times (0, \infty) \to \mathbb{R} \quad \text{measurable,} \quad j'(\vartheta_\Gamma) \in L^2(0, \infty; L^2(\Gamma)).$$

(hp$\vartheta_\Gamma$)

We remark that, by $(\text{hp}_j^{1})$ (which ensures the Lipschitz continuity of $(j')^{-1}$) and $(\text{hp}_j^{2})$, $(\text{hp}$j$^{\vartheta_\Gamma}$) entails that, for $n \not\to \infty$ ($n$ denoting here the time variable),

$$j'(\vartheta_\Gamma(n+\cdot)) \to 0, \quad \vartheta_\Gamma(n+\cdot) \to \vartheta_\infty \quad \text{strongly in} \quad L^2(0,1; L^2(\Gamma)).$$

(2.3)

Next, we introduce the operator

$$R : V \to V^*, \quad v \cdot \langle Rv, z \rangle_V = \int_\Omega \nabla v \cdot \nabla z + \eta \int_\Gamma v z.$$  

(2.4)

Of course, $R$ turns out to be the Riesz isomorphism associated with the (equivalent) norm on $V$ given by

$$\|v\|_R^2 = v \cdot \langle Rv, v \rangle_V = \int_\Omega |\nabla v|^2 + \eta \int_\Gamma v^2.$$  

(2.5)

At this point, we are able to state our first result, related to existence and uniqueness of solutions (with $\varepsilon = 1$), which slightly extends [29, Thm. 1]:

**Theorem 2.2.** Let $(\text{hp}_A^{\lambda}), \ (\text{hp}_W^{1})-(\text{hp}_W^{2}), \ (\text{hp}_j^{1})-(\text{hp}_j^{2}), \ (\text{hp}_\vartheta_0^{\lambda}), \ (\text{hp}_\lambda_0^{\vartheta}), \ \text{(hp}_f^{\vartheta_\Gamma})$ hold. Moreover, assume either (Dirichlet conditions)

$$B := -\Delta : V_0 \to V_0^*, \quad \mathcal{V} := V_0, \quad \| \cdot \|_{\mathcal{V}} := \| \cdot \|_{V_0}, \quad g := f, \quad \text{(Dirichlet)}$$

or (hp$\vartheta_\Gamma$) and (Robin conditions)

$$B := R, \quad \mathcal{V} := V, \quad \| \cdot \|_{\mathcal{V}} := \| \cdot \|_{R}, \quad \mathcal{V} \cdot \langle g, v \rangle_{\mathcal{V}} := \langle f, v \rangle + \eta \int_\Gamma j'(\vartheta_\Gamma) v, \quad \text{(Robin)}$$

where $\vartheta_\Gamma : \Gamma \times (0, \infty) \to \mathbb{R}$ is measurable, $j'(\vartheta_\Gamma) \in L^2(0, \infty; L^2(\Gamma)).$
the last relation holding for all $v \in \mathcal{V} = \mathcal{V}$, a.e. in $(0, \infty)$. Then, there exist a constant $c > 0$, depending only on the data $\lambda, W, j, \psi_0, \chi_0$, and a unique pair $(\vartheta, \chi)$ such that if $u = j'(\vartheta)$

$$
\| \vartheta_t \|_{L^2(0, \infty; \mathcal{V}^*)} + \| \vartheta \|_{L^\infty(0, \infty; H)} \leq c,
$$

(2.6)

$$
\| u \|_{L^2(0, \infty; \mathcal{V})} \leq c,
$$

(2.7)

$$
\| \vartheta - \vartheta_\infty \|_{L^2(0, \infty; \mathcal{V})} \leq c,
$$

(2.8)

$$
\| \chi_t \|_{L^2(0, \infty; H)} + \| \chi \|_{L^\infty(0, \infty; \mathcal{V})} \leq c,
$$

(2.9)

$\chi \in C^0([0, t]; \mathcal{V}) \cap L^2(0, t; H^2(\Omega))$ for all $t > 0$, and $\vartheta, \chi, u$ satisfy

$$
\vartheta_t + \lambda(\chi)_t + Bu = g \quad \text{in } \mathcal{V}^*,
$$

(2.10)

$$
\chi_t + A\chi + W'(\chi) = \lambda'(\chi)u \quad \text{in } \mathcal{V}^*,
$$

(2.11)

a.e. in $(0, \infty)$, as well as

$$
\vartheta|_{t=0} = \vartheta_0, \quad \chi|_{t=0} = \chi_0.
$$

(2.12)

**Remark 2.3.** We point out that, in the (Robin) case, (hp $f$) and (hp $\vartheta_t$) entail

$$
g \in L^2(0, \infty; \mathcal{V}^*).$$

(2.13)

This is of course true also in the (Dirichlet) case, in which we even have by assumption the better relation $g \in L^2(0, \infty; H)$.

We shall not give the full proof of Theorem 2.2 since it is just a variant of the proof given in [29]. More precisely, the same argument of [29, Sec. 3] can be used to obtain existence of a solution to a suitable regularization of the problem. Concerning the a priori estimates which are required to remove the approximation, some points are technically different from [29, Sec. 3] especially due to our choice(s) of boundary conditions. Nevertheless, the required estimates will be easily obtainable from the uniform bounds we shall prove in the long-time analysis. Finally, the compactness argument necessary to pass to the limit and the proof of uniqueness will be briefly sketched in the next section.

Let us come now to our statement on further regularity properties of solutions. To this aim, we prepare an auxiliary result.

**Lemma 2.4.** Let $\mathcal{X}$ be a Banach space and let $\gamma \in L^2(0, \infty; \mathcal{X})$ satisfy, for some $p \in [1, \infty]$,

$$
\gamma \in W^{1,p}(0, t; \mathcal{X}) \quad \forall t \in (0, \infty), \quad \sup_{t \in [0, \infty)} \| \gamma_t \|_{L^p(t, t+1; \mathcal{X})} < \infty.
$$

(2.14)

Then, it turns out that $\gamma \in L^\infty(0, \infty; \mathcal{X})$. Moreover, if $p > 1$, the strong convergence $\gamma(t) \to 0$ holds in $\mathcal{X}$, as $t \nearrow \infty$.

**Proof.** First of all, the $L^\infty$ bound can be shown using the Fundamental Theorem of Calculus, by a simple contradiction argument. Concerning the convergence to 0, let $p > 1$ and $\{t_n\}$ be an arbitrary diverging sequence of times. Setting

$$
\gamma_n(t) := \gamma(t_n + t), \quad t \in [0, 1],
$$

...
it is clear that $\gamma_n \to 0$ strongly in $L^2(0,1; X)$. Thus, there is a subsequence, not relabelled, such that $\gamma_n(t) \to 0$ strongly in $X$ for a.e. $t \in (0,1)$. In particular, we can find a sequence $\{\delta_k\} \subset (0,1)$, with $\delta_k \to 0$ and such that, in $X$,
$$\lim_{n \to \infty} \gamma_n(\delta_k) = 0 \quad \forall k \in \mathbb{N}.$$ 

We then deduce from (2.14)
$$\|\gamma(t_n)\|_X \leq \|\gamma(t_n) - \gamma(t_n + \delta_k)\|_X + \|\gamma(t_n + \delta_k)\|_X \leq c\delta_k^{1/p^*} + \|\gamma_n(\delta_k)\|_X,$$
where $p^*$ is the conjugate exponent to $p$, and the latter quantity can be made arbitrarily small for $n$ large enough. Due to the arbitrariness of $\{t_n\}$, this shows that $\gamma(t) \to 0$ in $X$ as $t \nearrow \infty$. 

**Remark 2.5.** We point out that the second part of the statement above is false (even for $X = \mathbb{R}$) if one takes $p = 1$. Indeed, set, for $n \geq 2$, $v_n : \mathbb{R} \to \mathbb{R}$ given by
$$v_n(t) := n^2 \mathcal{X}_{(-1/n^2,0)} - n^2 \mathcal{X}_{(0,1/n^2)},$$
$\mathcal{X}$ denoting the characteristic function, and define
$$\gamma(t) := \sum_{n=2}^{\infty} v_n(t - n), \quad t \in \mathbb{R}, \quad g(t) := \int_0^t \gamma(s) \, ds.$$ 

Then, it is clear that $g$ stays in $L^2(0,\infty) \cap L^\infty(0,\infty)$ and satisfies (2.14) (with $X = \mathbb{R}$). However, $g(t)$ does not tend to 0 for $t \nearrow \infty$.

**Theorem 2.6.** Let (hp\lambda), (hpW 1)–(hpW 2), (hpj 1)–(hpj 2), (hp\vartheta 0), (hp\vartheta 0), (hpj) and either (Dirichlet), or (hp\vartheta f) and (Robin), hold. Let also (2.14) hold for $\gamma = g$, $X = \mathcal{V}^*$ and some $p \in [1,\infty]$. Then, for all $s > 0$ there exists a constant $c > 0$, depending on $\lambda$, $W$, $j$, $\vartheta_0$, $\lambda_0$, and $s$, such that
$$\sup_{t \geq s} \|\partial_t\|_{L^2(t,t+1; H)} + \|\partial^2_t\|_{L^\infty(s,\infty; \mathcal{V})} \leq c,$$  
(2.15)
$$\|u\|_{L^\infty(s,\infty; \mathcal{V})} \leq c, \quad \text{with} \quad u = j'(\vartheta),$$  
(2.16)
$$\|\chi_t\|_{L^\infty(s,\infty; \mathcal{V})} + \sup_{t \geq s} \|\chi_t\|_{L^2(t,t+1; \mathcal{V})} + \|\chi\|_{L^\infty(s,\infty; H^2(\Omega))} \leq c,$$  
(2.17)
$$\|W'(\chi)\|_{L^\infty(s,\infty; H)} \leq c.$$  
(2.18)

If, additionally,
$$g_t \in L^q(0,\infty; \mathcal{V}^*)$$  
(2.19)
for some $q \in [1,2]$, then we also have
$$\|\partial_t\|_{L^2(s,\infty; H)} + \|\chi_t\|_{L^2(s,\infty; \mathcal{V})} \leq c.$$  
(2.20)

In particular, the above theorem provides a priori estimates which have a uniform character for large times. Of course, this uniformity cannot be proved for the $L^2$ in time norm of $\chi$ in (2.19).

Let us now move to the study of long-time behavior, starting from existence of a nonempty $\omega$-limit set.
Theorem 2.7. Let \((hp\lambda), (hpW_1)-(hpW_2), (hpj_1)-(hpj_2), (hp\vartheta_0), (hp\lambda_0), (hp\varphi)\) and either \([\text{Dirichlet}]\), or \((hp\vartheta_1)\) and \([\text{Robin}]\), hold. Let also \((2.14)\) hold for \(\gamma = g\), \(X = \mathcal{V}^*\) and some \(p \in [1, \infty]\). Moreover, let us assume that, either

\[ g_t \in L^2(0, \infty; \mathcal{V}^*) \]  \hspace{1cm} (2.21)

or

\[ \exists c > 0, \, \alpha \leq 3 : \, |j''(r)| \leq c(1 + |j'(r)|^\alpha) \quad \forall r \in J. \]  \hspace{1cm} (2.22)

Then, we have that, as \(t \nearrow \infty\),

\[ u(t) \to 0, \quad \vartheta(t) \to \vartheta_\infty \quad \text{weakly in} \, \mathcal{V} \, \text{and strongly in} \, H. \]  \hspace{1cm} (2.23)

Moreover, any diverging sequence \(\{t_n\} \subset (0, \infty)\) admits a subsequence, not relabelled, such that

\[ \chi(t_n) \to \chi_\infty \quad \text{weakly in} \, H^2(\Omega) \, \text{and strongly in} \, V \cap C(\overline{\Omega}), \]  \hspace{1cm} (2.24)

where \(\chi_\infty\) is a solution of the stationary problem

\[ A\chi_\infty + W'(\chi_\infty) = 0 \quad \text{in} \, V^*. \]  \hspace{1cm} (2.25)

Remark 2.8. Assumption \((2.14)\) on \(g\) of course reinforces the convergence properties of \(g\) to 0 (cf. \((hp\varphi)\) and, in the \([\text{Robin}]\) case, \((hp\vartheta_1)\)).

Remark 2.9. The property \((2.22)\) is actually not very strong. For instance, in the situation when \(J = \mathbb{R}\), then \((2.22)\) is fulfilled provided \(j\) has a polynomial, or even exponential, growth at infinity. Moreover, \((2.22)\) also holds for the laws mentioned in the Introduction (corresponding to combinations of the Caginalp and Penrose-Fife models). A case in which \((2.22)\) does not hold (but \((hpj_1)\)–\((hpj_2)\) do hold) is given by \(j'(r) = r + \log(1 + r/\tau_c)\), due to the singular behavior of \(j'\) in proximity of \(\tau_c\).

Our last result characterizes the \(\omega\)-limit as a singleton in case the potential \(W\) is analytic. To introduce it we need some preliminaries. First of all, let us notice that \((hpW_1)-(hpW_2)\) entail by simple maximum principle arguments (see [1, Lemma 3.1]) that there exists a closed interval \(I_1 \subset I_0\) such that any solution \(\chi_\infty\) to \((2.25)\) satisfies

\[ \chi_\infty \in W^{2,q}(\Omega) \; \forall q \in [1, \infty), \quad \chi_\infty(x) \in I_1 \quad \text{for all} \; x \in \Omega. \]  \hspace{1cm} (2.26)

Moreover, if we set, for \(v \in V\),

\[ E(v) := \int_\Omega \frac{1}{2} |\nabla v(x)|^2 + W(v(x)) \, dx \]  \hspace{1cm} (2.27)

(which might be \(+\infty\) if \(W(v)\) is not summable), there holds the following form of the Simon-Lojasiewicz inequality, which is a reformulation of [1, Prop. 4.4] (see also [2, Prop. 4.2]):
Theorem 2.10. Let $\text{(hpW}_1\text{)--(hpW}_2\text{)}$ hold and let 

$$W \text{ be real analytic on } I_0.$$  

(2.28)

Let $\chi_\infty$ be a solution to (2.25). Then, there exist constants $c_\ell, \epsilon > 0, \zeta \in (0, 1/2)$, such that 

$$|E(v) - E(\chi_\infty)|^{1-\zeta} \leq c_\ell \|Av + W'(v)\|_{V^*}$$  

(2.29)

for all $v \in V$ such that 

$$\|v - \chi_\infty\|_{V \cap C(\overline{\Omega})} \leq \epsilon.$$  

(2.30)

Remark 2.11. The above statement is given in a slightly different fashion with respect to [1, Prop. 4.4] since this version seems to be more suitable for our specific problem. However, we point out that our hypotheses entail those of [1, Prop. 4.4]. Actually, by (2.26), it is clear that, as we possibly take a smaller $\epsilon$ in condition (2.30), then any $v$ fulfilling (2.30) also satisfies [1, (4.5)]. Indeed, $\epsilon$ can be taken so small that $v$ ranges into $I_0$, where Lipschitz continuity of $W$ holds (recall that $I_0 \subset I$).

Remark 2.12. Notice also that in this way we actually get rid of the possibly singular character of $W$ at the boundary of $I$. Indeed, it is not excluded that there is a transient dynamics where $W'(\lambda(t))$ may be unbounded. However, thanks to (2.26) and the precompactness of the trajectory in $C(\overline{\Omega})$, for sufficiently large times $\lambda(t)$ takes values into $I_0$, where $W$ is bounded and analytic. As in [16], the key condition ensuring this property is $\text{(hpW}_2\text{)},$ which essentially states that the leftmost and rightmost minima of $W$ are interior to its domain. The gradient flow structure of the system entails that the solution eventually moves away from these minima.

Here is, finally, our convergence result:

Theorem 2.13. Let the hypotheses of Theorem 2.7 hold. Furthermore, assume (2.28) and 

$$\sup_{t \geq 0} t^{1+\delta} \int_t^\infty \|g(s)\|_V^2, ds < \infty, \tag{2.31}$$

for some $\delta > 0.$ Then, as $(\vartheta_0, \chi_0)$ are initial data satisfying $\text{(hp}\vartheta_0\text{)--(hp}\chi_0\text{)},$ the $\omega$-limit of the corresponding trajectory $(\vartheta, \chi)$ of system (2.10)--(2.11) consists of a unique pair $(\vartheta_\infty, \chi_\infty),$ where $\vartheta_\infty$ is given by $\text{(hp}\vartheta_2\text{)}$ and $\chi_\infty \in V$ is a solution to (2.25). Moreover, as $t \nearrow \infty,$ (2.23) holds together with 

$$\chi(t) \rightarrow \chi_\infty \text{ weakly in } H^2(\Omega) \text{ and strongly in } V \cap C(\overline{\Omega}). \tag{2.32}$$

More precisely, if 

$$\delta > \frac{2\zeta}{1 - 2\zeta}, \tag{2.33}$$

where $\zeta$ is as in (2.29), then one can find $t^* > 0$ and a positive constant $c_\ast$ such that 

$$|\chi(t) - \chi_\infty| \leq c_\ast t^{-\frac{\zeta}{1 - 2\zeta}}, \quad \forall t \geq t^*. \tag{2.34}$$

Otherwise, one can find $\zeta_0 \in (0, \zeta)$ so that 

$$\delta > \frac{2\zeta_0}{1 - 2\zeta_0}, \tag{2.35}$$
a time $t^{**} > 0$ and a positive constant $c_{**}$ such that

$$|\chi(t) - \chi_\infty| \leq c_{**} t^{-\frac{\delta_0}{2\lambda_0}}, \quad \forall t \geq t^{**}. \quad (2.36)$$

**Remark 2.14.** Let us notice that (2.34), (2.36) give the convergence rate of $\chi(t)$ to $\chi_\infty$ with respect to the norm of $H$. Of course, an estimate of the rate of convergence in the norm of $V$ could be obtained from the uniform bound corresponding to the last of (2.17) and interpolation.

### 3  A priori estimates and well posedness

Let us first sketch an approximated version of system (2.10)–(2.11) along the lines of [29], [14, Sec. 3], to which we refer for more details. Namely, let us assume that $j$ and $W$ have been replaced in (2.10)–(2.11) by regularized functions $j_n$ and $W_n$ defined on the whole real line and such that

$$j_n, W_n + \frac{\kappa}{2} \text{Id}^2 \rightarrow j, W + \frac{\kappa}{2} \text{Id}^2 \quad \text{in the sense of Mosco} \quad (3.1)$$

(see, e.g., [3] for the definition of Mosco convergence and for the related notion of G-convergence of graphs). Moreover, we can assume [14, Sec. 3] that, for all $n \in \mathbb{N}$,

$$W_n''(r) \geq -\kappa \quad \forall r \in \mathbb{R}, \quad \frac{W_n'(r)}{r} \geq \mu/2 \quad \forall r \in \mathbb{R} \setminus I_0, \quad (3.2)$$

$$j_n''(r) \geq \sigma/2 \quad \forall r \in \mathbb{R}, \quad (3.3)$$

where $\mu$, $I_0$, and $\sigma$ are as in [[hpW1], [hpW2], [hpj1]] (cf. [14, Sec. 3] for an example of a possible regularizing sequence $\{W_n\}$). Then, we consider a family $\{(\vartheta_n, \chi_n)\}_{n \in \mathbb{N}}$ of (possibly local in time) solutions to the regularized problem specified by the subscript $n$. Existence of these solutions can be shown proceeding as in [29, Subsec. 3.1] and it turns out that $(\vartheta_n, \chi_n)$, as well as $u_n := j_n' (\vartheta_n)$ and $W_n' (\chi_n)$, are regular enough to give a rigorous meaning to the forthcoming computations. Actually, we shall now deduce some a priori estimates with the aim of taking the limit as $n \not\to \infty$. In this procedure, (2.10) and (2.11) will be implicitly considered in their $n$-approximated form. Moreover, $c$ will denote a positive constant, whose value is allowed to vary even inside one single row, but $c$ may depend only on $\lambda, j, W, \vartheta_0, \chi_0$ (and neither on $n$ nor on $t$). When we need to fix the value of some specific $c$, we shall use the notation $c_i, i \geq 0$. The symbols $C, C_i$ will denote constants which, instead, can explicitly depend on $t \in (0, \infty)$, but do not explode as $t \not\to 0$. For simplicity, we shall proceed as if the solutions were defined for all times $t \geq 0$. Indeed, although this might be not true at the approximating level, it will certainly hold at the limit in view of the uniform in time character of the estimates and of standard extension arguments.

**Energy estimate.** Test (2.10) by $u_n$ in the duality between $\mathcal{V}^*$ and $\mathcal{V}$ and sum the result to (2.11) tested by $\chi_{n,t}$ in the duality of $\mathcal{V}^*$ and $\mathcal{V}$. The smoothness properties assumed on the approximating solutions, standard integration by parts formulas, and the cancellation of a couple of opposite terms then give

$$\frac{d}{dt} \mathcal{E}_n(\vartheta_n, \chi_n) + |\chi_{n,t}|^2 + 1/2 \|u_n\|^2_{\mathcal{V}} \leq 1/2 \|g\|^2_{\mathcal{V}^*}, \quad (3.4)$$
where we also used the Young inequality to split the duality product of \( g \) and \( u_n \) resulting from the right hand side of (2.10). Also, accordingly with (1.3), we have set

\[
\mathcal{E}_n(\vartheta, \chi) := \int_\Omega \left( \frac{1}{2} |\nabla \chi|^2 + W_n(\chi) + j_n(\vartheta) \right).
\]  

(3.5)

Owing now to (hp0), (hp\( \chi_0 \)), and (2.11), we can integrate (3.4) over \((0, t)\) for arbitrary \( t > 0 \) and deduce

\[
\|u_n\|_{L^2(0,T;V)} + \|j_n(\vartheta_n)\|_{L^\infty(0,t;L^1(\Omega))} \leq c, \\
\|\chi_{n,t}\|_{L^2(0,t;H)} + \|\chi_{n}\|_{L^\infty(0,t;V)} + \|W_n(\chi_{n})\|_{L^\infty(0,t;L^1(\Omega))} \leq c.
\]

(3.6) \hspace{1cm} (3.7)

Let us note that, to obtain the second of (3.7), we used that, by (3.2),

\[
\exists c, c_0 > 0 : \int_\Omega \left( \frac{1}{2} |\nabla v|^2 + W_n(v) \right) \geq c \|v\|^2_{V^*} - c_0 \quad \forall v \in V.
\]

(3.8)

**Second estimate.** Let us now test (2.10) by \( \vartheta_n - \vartheta_\infty \). Owing to the monotonicity of \( j'_n \) and, more precisely, to (3.3), we can see that, both in the (Dirichlet) and in the (Robin) case,

\[
v^* (Bu_n, \vartheta_n - \vartheta_\infty) \geq 2c_1 \|\vartheta_n - \vartheta_\infty\|^2_{V^*}
\]

(3.9)

for some \( c_1 > 0 \) depending in particular on \( \sigma \). Thus, splitting the term depending on \( g \) by the Young inequality, we get

\[
\frac{1}{2} \frac{d}{dt} \|\vartheta_n - \vartheta_\infty\|^2 + c_1 \|\vartheta_n - \vartheta_\infty\|^2_{V^*} \leq c \|g\|^2_{V^*} - v^* (\lambda'(\chi_n)\chi_{n,t}, \vartheta_n - \vartheta_\infty)_{V^*}
\]

(3.10)

and the latter term is readily estimated as follows

\[
- v^* (\lambda'(\chi_n)\chi_{n,t}, \vartheta_n - \vartheta_\infty)_{V^*} = (\lambda'(\chi_n)\chi_{n,t}, \vartheta_n - \vartheta_\infty)_{V^*} \leq c(1 + \|\chi_n\|_{L^4(\Omega)}) \|\chi_{n,t}\| \|\vartheta_n - \vartheta_\infty\|_{L^4(\Omega)} \\
\leq c(1 + \|\chi_n\|^2_{V^*}) \|\chi_{n,t}\|^2 + \frac{c_1}{2} \|\vartheta_n - \vartheta_\infty\|^2_{V^*},
\]

(3.11)

where we used (hp\( \lambda \)) and the continuous embeddings \( V, V \subset L^4(\Omega) \). Hence, recalling (3.7) and using (2.13), we infer

\[
\|\vartheta_n\|_{L^\infty(0,t;H)} + \|\vartheta_n - \vartheta_\infty\|_{L^2(0,t;V)} \leq c.
\]

(3.12)

Moreover, arguing as in (3.11), we obtain that the term \( \lambda(\chi_n)_{t} = \lambda'_{(\chi_n)}\chi_{n,t} \) in (2.10) is uniformly controlled in \( L^2(0,T;V^*) \), so that (3.6), (2.13) and the equality \( \vartheta_{n,t} = -\lambda'(\chi_n)\chi_{n,t} - Bu_n + g \) yield

\[
\|\vartheta_{n,t}\|_{L^2(0,T;V^*)} \leq c.
\]

(3.13)

**Third estimate.** Thanks to (hp\( \lambda \)), the continuous embeddings \( V, V \subset L^4(\Omega) \), and (3.6)–(3.7), we infer that

\[
\|\lambda'(\chi_n)u_n\|_{L^2(0,t;H)} \leq c(1 + \|\chi_n\|_{L^\infty(0,t;L^4(\Omega))}) \|u_n\|_{L^2(0,t;L^4(\Omega))} \leq c.
\]

(3.14)
Then, we can test (2.11) by $A\chi_n$ and integrate over $\Omega \times (0, t)$. Using the first inequality of (3.2), we have that
\[
\int_0^t (A\chi_n, W'_n(\chi_n)) \geq -\kappa \|\chi_n\|_{L^2(0, t; V)}^2,
\]
which can be controlled thanks to (3.7), but only on bounded time intervals. Therefore, using also (hp$\chi_0$) it is not difficult to obtain
\[
\|\chi_n\|_{L^2(0, t; H^2(\Omega))} + \|W'_n(\chi_n)\|_{L^2(0, t; H)} \leq C,
\]
where the second bound comes from a further comparison of terms in (2.11).

**Limit as $n \to +\infty$ and existence.** Standard compactness tools enable us to pass to the limit in the $n$-approximated versions of (2.10)–(2.11). Indeed, thanks to estimates (3.6)–(3.7), (3.12)–(3.13), and (3.16), there exist four limit functions $\vartheta, u, \chi, v$, defined from $(0, \infty)$ to $H$ (at least), and a suitable subsequence of $n$ (not relabeled) such that the corresponding subsequences $\{\vartheta_n\}, \{u_n\}, \{\chi_n\}, \{W'_n(\chi_n)\}$ fulfill
\[
\vartheta_n \to \vartheta \text{ weakly star in } H^1(0, t; V^*) \cap L^\infty(0, t; H) \cap L^2(0, t; V),
\]
\[
u_n \to u \text{ weakly in } L^2(0, t; V),
\]
\[
\chi_n \to \chi \text{ weakly star in } H^1(0, t; H) \cap L^\infty(0, t; V) \cap L^2(0, t; H^2(\Omega)),
\]
\[
W'_n(\chi_n) \to v \text{ weakly in } L^2(0, t; H)
\]
as $n \not\to \infty$, for all $t > 0$. We note at once that the bounds in (2.6)–(2.9) are certainly satisfied by the limit functions: in fact, it suffices to take the lim inf in estimates (3.6)–(3.7), (3.12)–(3.13) which are uniform with respect to $t$. Next, (3.17), (3.19), the Ascoli theorem and the Aubin compactness lemma (see, e.g., [32, Cor. 4, Sec. 8]) enable us to deduce that
\[
\vartheta_n \to \vartheta \text{ strongly in } C^0([0; t]; V^*) \cap L^2(0, t; H),
\]
\[
\chi_n \to \chi \text{ strongly in } C^0([0; t]; H) \cap L^2(0, t; V),
\]
whence $\lambda'(\chi_n) \to \lambda'(\chi)$ strongly in $C^0([0; t]; H)$ due to the Lipschitz continuity of $\lambda'$. Then, (3.18)–(3.19) and the continuous embedding $V \subset L^6(\Omega)$ imply
\[
\lambda'(\chi_n) u_n \to \lambda'(\chi) u \text{ weakly in } L^2(0, t; L^{3/2}(\Omega)),
\]
\[
\lambda(\chi_n) t = \lambda'(\chi_n) \chi_n t \to \lambda(\chi) t = \lambda'(\chi) \chi t \text{ weakly in } L^2(0, t; L^1(\Omega)).
\]
At this point, we can pass to the limit in the $n$-approximated versions of (2.10)–(2.11) to obtain (2.10) (which a fortiori holds in $V^*$) and
\[
\chi_t + AX + v = \lambda'(\chi) u \text{ in } V^*,
\]
a.e. in $(0, \infty)$. Initial conditions (2.12) follow easily from (3.21)–(3.22). Then, in order to conclude the existence proof, it remains to identify functions $u$ and $v$, that is, to check that
\[
u = \vartheta', \quad v = W'(\chi) \quad \text{a.e. in } \Omega \times (0, \infty).
\]
However, due to the Mosco convergences in (3.1), it turns out that (cf., e.g., [3, Thm. 3.66]) the subdifferential operators $j'_n, W'_n + \kappa \text{Id}$ $G$-converge to $j', W' + \kappa \text{Id}$, as well as their extensions to $L^2(\Omega \times (0, t))$, for all $t > 0$. Then, we can apply the basic properties of $G$-convergence (see, e.g. [4, Prop. 1.1, p. 42]) stating that if $u_n = j'_n(\vartheta_n) \to u$ and $\vartheta_n \to \vartheta$ weakly in $L^2(\Omega \times (0, t))$ and limit supremum $\int_{\Omega \times (0, t)} u_n \vartheta_n \leq \int_{\Omega \times (0, t)} u \vartheta$, then $u = j'(\vartheta)$. But, in our case this follows easily from (3.18) and (3.21). The other identification in (3.20) is a bit longer, since we first check that $v + \kappa \chi \in (W' + \kappa \text{Id})(\lambda)$ with the help of (3.20) and (3.22) (recall that $W' + \kappa \text{Id}$ is monotone by (hpW1)), then extract the information $v = W'(\lambda)$. Hence, we conclude for the validity of (3.26).

**Uniqueness.** Assume, by contradiction, that there exist two solutions $(\vartheta_i, \chi_i), i = 1, 2$, to problem (2.10)–(2.12), let $u_i = j'(\vartheta_i), i = 1, 2$, and set temporarily $\vartheta := \vartheta_1 - \vartheta_2, \chi := \chi_1 - \chi_2, u := u_1 - u_2$. Now, we can take the difference of (2.10) written for the solutions corresponding to $i = 1, 2$, integrate it in time from $0$ to $t > 0$, and test by $u(t)$. At the same time, we test the difference of (2.11) by $\lambda(t)$ and sum the result to the previous relation. With the help of (hpj1) and (hpW1) it is straightforward to deduce

$$
\sigma|\vartheta(t)|^2 + \frac{1}{2} \frac{d}{dt} \left( \|\int_0^t u(s)ds\|^2_\nu + |\chi(t)|^2 \right) + |\nabla \chi(t)|^2 \\
\leq \kappa |\chi(t)|^2 + \int_\Omega \left( (\lambda'(\chi_1)u_1 - \lambda'(\chi_2)u_2)\chi - (\lambda(\chi_1) - \lambda(\chi_2))u \right)(t).
$$

(3.27)

To estimate the last term we use a remark from Kenmochi (see [18] and, e.g., [22, Lemma 3.2]): in fact, by the Taylor expansion and (hpLa) we have

$$
(\lambda'(\chi_1)u_1 - \lambda'(\chi_2)u_2)\chi - (\lambda(\chi_1) - \lambda(\chi_2))u \\
= u_2(\lambda(\chi_2) - \lambda(\chi_1) - \lambda'(\chi_1)(\chi_2 - \chi_1)) + u_2(\lambda(\chi_1) - \lambda(\chi_2) - \lambda'(\chi_2)(\chi_1 - \chi_2)) \\
\leq \Lambda \left(|u_1| + |u_2|\right)|\chi|^2.
$$

Therefore, the left hand side of (3.27) can be handled using the Hölder inequality and the continuous embeddings $V, \mathcal{V} \subset L^4(\Omega)$ in order to obtain

$$
\sigma \int_0^t |\vartheta(s)|^2 ds + \frac{1}{2} \left( \|\int_0^t u(s)ds\|^2_\nu + |\chi(t)|^2 \right) + \int_0^t |\nabla \chi(s)|^2 ds \\
\leq c \int_0^t \left( 1 + \|u_1(s)\|^2_\nu + \|u_2(s)\|^2_\nu \right)|\chi(s)|^2 ds + \frac{1}{2} \int_0^t \|\chi(s)\|^2_\nu ds.
$$

(3.28)

Finally, in view of the regularity (2.7) for $u_1, u_2$, the uniqueness property follows easily from the Gronwall lemma. This completes the proof of Theorem 2.2.

**Proof of Theorem 2.6** Let us test (2.10) by $u_t$ in the duality between $\mathcal{V}^*$ and $\mathcal{V}$. Then, differentiate in time (2.11), multiply the result by $\chi_t$, and integrate over $\Omega$. Summing together the obtained relations, noting that a couple of terms cancel out, and using (hpW1), we infer

$$
\frac{d}{dt} \left( \frac{1}{2} \|u\|^2_\nu + \frac{1}{2} |\chi_t|^2 - \nu\dot{\chi} \right) + \nu \nu' \langle g, u \rangle \nu + (u_t, \vartheta_t) + |\nabla \chi|^2 \\
\leq -\nu \langle g_t, u \rangle \nu + \kappa |\chi_t|^2 + \int_\Omega \lambda''(\chi) \chi^2_t u.
$$

(3.29)
Let us note that the computation above is just formal in the regularity setting of Theorem 2.2. However, the procedure might be made rigorous by working on the $n$-regularization sketched before and then passing to the limit. We omit the details just for brevity. By (hp1), we have

$$(u_t, \vartheta_t) \geq \sigma |\vartheta_t|^2.$$  \hfill (3.30)

Moreover, using (hp3) and once more the continuous embeddings $V, \mathcal{V} \subset L^4(\Omega)$, we obtain

$$\int_{\Omega} \chi''(\chi) \chi_t^2 u \leq \frac{1}{2} \|\chi_t\|_{\mathcal{V}}^2 + c \|u\|_{\mathcal{V}}^2 |\chi_t|^2.$$  \hfill (3.31)

Finally, we note that

$$- \nu \langle g_t, u \rangle \leq \|g_t\|_{\mathcal{V}} \|u\|_{\mathcal{V}}.$$  \hfill (3.32)

Next, we set

$$\mathcal{F} := \frac{1}{2} \|u\|_{\mathcal{V}}^2 + \frac{1}{2} |\chi_t|^2 - \nu \langle g, u \rangle.$$  \hfill (3.33)

and observe that, since $g \in L^\infty(0, \infty; \mathcal{V})$ thanks to Lemma 2.4, there exist constants $c_2, c_3 > 0$ depending only on $g$ and such that

$$\frac{1}{4} \|u\|_{\mathcal{V}}^2 + \frac{1}{2} |\chi_t|^2 \leq \mathcal{F} + c_2 \leq \|u\|_{\mathcal{V}}^2 + \frac{1}{2} |\chi_t|^2 + c_3$$  \hfill (3.34)

for all $t > 0$. Thus, setting $\mathcal{Y} := \mathcal{F} + c_2$ and using (3.30)–(3.32), (3.29) becomes

$$\frac{d}{dt} \mathcal{Y} + \sigma |\vartheta_t|^2 + \frac{1}{2} \|\nabla \chi_t\|^2 \leq c |\chi_t|^2 (1 + \|u\|_{\mathcal{V}}^2) + \|u\|_{\mathcal{V}} \|g_t\|_{\mathcal{V}}$$

$$\leq c(1 + \mathcal{Y}) \left(1 + \|u\|_{\mathcal{V}}^2 + \|g_t\|_{\mathcal{V}}^p \right).$$  \hfill (3.35)

Then, noting that, since by the estimates (2.7)–(2.9),

$$\sup_{t \geq 0} \int_t^{t+1} \mathcal{Y}(s) \, ds < \infty,$$  \hfill (3.36)

we can apply to $\mathcal{Y}$ the uniform Gronwall Lemma (see, e.g., [33, Lemma III.1.1]), which yields (2.16) and the first bound in (2.17). Next, integrating (3.35) in time over $(t, t + 1)$ for $t$ greater than or equal to a given $s > 0$, we get the first of (2.15) and the second of (2.17). Observing that, by (hp1)–(hp2) (and the Poincaré inequality in the Dirichlet case), there exists $c > 0$ such that

$$\|u\|_{\mathcal{V}} \geq c \|\vartheta - \vartheta_{\infty}\|_{\mathcal{V}},$$  \hfill (3.37)

we get the second of (2.15) from (2.16). Finally, by (hp3) we deduce

$$\|\lambda'(\chi) u\|_{L^\infty(s, T; H)} \leq c \left(1 + \|\chi\|_{L^\infty(s, T; \mathcal{V})}\right) \|u\|_{L^\infty(s, T; \mathcal{V})} \leq c.$$  \hfill (3.38)

Thus, using the first of (2.17) and (hp3) and viewing (2.11) as a time dependent family of elliptic equations with monotone (up to a linear perturbation) nonlinearities,
the last of (2.17) follows from the standard elliptic regularity theory. This, clearly, also gives (2.18). To conclude, we have to prove (2.20) in the case when (2.19) holds. Coming back to the first row of (3.35) it then suffices to note that, for a.e. \( t \geq s \), the right hand side is

\[
\leq c|\chi_t|^2(1 + \|u\|_{V}^2) + c\|u\|_{V}^2 + c\|g_t\|_{V^*}^2 \leq c|\chi_t|^2 + c\|g_t\|_{V^*}^2 + c\|u\|_{V}^2, \tag{3.39}
\]

where it is intended that, if \( q \in (1, 2] \), then \( q^* \in [2, \infty) \) denotes the conjugate exponent to \( q \); otherwise, i.e., if \( q = 1 \), the latter term on the right hand side has to be omitted. Relation (2.20) follows now by integrating (3.35) over \((s, \infty)\) and using (3.39). Indeed, the terms on the right hand side of (3.39) are controlled, respectively, by the first of (2.9), by (2.7), (2.16) and interpolation, and by (2.19). The proof of the Theorem is now complete. \( \blacksquare \)

4 Study of the \( \omega \)-limit

**Proof of Theorem 2.7.** Let us first notice that the convergence in (2.24) to some \( \chi_\infty \in H^2(\Omega) \) is an immediate consequence of (2.17) since \( \chi \) is weakly continuous from \([s, \infty)\) to \( H^2(\Omega) \), whence the estimate \( \|\chi(t)\|_{H^2(\Omega)} \leq c \) holds true for all \( t \in [s, \infty) \).

Analogously, the convergence \( \vartheta(t) \to \vartheta_\infty \) in (2.23) follows from (2.15) once one sees that \( \vartheta(t) \to \vartheta_\infty \) strongly in \( V^* \) by (2.6), (2.8), and Lemma 2.4; indeed, due to the identification of the limit \( \vartheta_\infty \), it turns out that the entire family \( \vartheta(t) \) converges.

Next, we show the convergence of \( u(t) \) in (2.23). With this purpose, let us observe that, if (2.22) holds, then

\[
\|u_t(t)\|_{L^1(\Omega)} \leq |j''(\vartheta(t))|\vartheta_t(t)| \leq c(1 + \|u(t)\|_{L^2(\Omega)}^2)\|\vartheta_t(t)\|
\leq c(1 + \|u\|_{L^\infty(\Omega, V)}^2)\|\vartheta_t(t)\| \leq c|\vartheta_t(t)|.
\]

for a.e. \( t \in (1, \infty) \), say. Thus, by (2.7) and (2.15) it is clear that \( \gamma = u \) fulfills the assumptions of Lemma 2.4 with \( p = 2 \) and \( X = L^1(\Omega) \) and consequently \( u(t) \to 0 \) in \( L^1(\Omega) \) as \( t \nearrow \infty \). Then, to show that (2.23) holds (i.e., \( u(t) \) weakly converges in \( V \)), it is now enough to point out the bound in (2.16).

Let us consider, instead, the case when (2.21) holds. Under this condition, using (2.7), (2.9) and Remark 2.3, we modify (3.34) as

\[
0 \leq \frac{1}{4}\|u\|_{V}^2 + \frac{1}{2}|\chi_t|^2 \leq \mathcal{Z} := \mathcal{F} + \|g\|_{V^*}^2 \leq \|u\|_{V}^2 + \frac{1}{2}|\chi_t|^2 + \frac{3}{2}\|g\|_{V^*}^2. \tag{4.1}
\]

Thus, in view of the first line of (3.35), it is not difficult to deduce

\[
\frac{d}{dt}\mathcal{Z} \leq c(\mathcal{Z}^2 + 1 + \|g\|_{V^*}^2 + \|g_t\|_{V^*}^2) \quad \text{a.e. in } (0, \infty), \tag{4.2}
\]

whence the convergence \( u(t) \to 0 \) with respect to the strong topology on \( V \) is a consequence of [34, Lemma 6.2.1, p. 225]. Then, recalling (3.37), in this case it happens that (2.23) is improved into

\[
u(t) \to 0, \quad \vartheta(t) \to \vartheta_\infty \quad \text{strongly in } V.
\]
To conclude, it remains to show that the limit value $\chi_\infty$ in (2.24) satisfies (2.25), and this can be done in a completely standard way. Namely, defining $(\chi_n, u_n) : (0, 1) \to V \times V$ as $(\chi_n, u_n)(\cdot) := (\chi, u)(t_n + \cdot)$, from (2.9) and (2.7) it is clear that

$$(\chi_n, u_n) \to (\chi_\infty, 0) \quad \text{strongly in } C^0([0, 1]; H) \times L^2(0, 1; V).$$

(4.3)
as $n \searrow \infty$. Here, of course we also used the strong convergence of $\chi_{n,t}$ to 0 in $L^2(0, 1; H)$. Then, by passing to the limit in

$$\chi_{n,t} + A\chi_n + W'(\chi_n) = \lambda'(\chi_n) u_n \quad \text{in } V^*, \; \text{a.e. in } (0, 1),$$

(4.4)it is not difficult to check that $\chi_\infty$ solves the stationary problem. Indeed, owing to (2.17), weak star compactness, and (4.3) it turns out that $\chi_n \to \chi_\infty$ weakly star in $L^\infty(0, 1; H^2(\Omega))$, whence $A\chi_n \to A\chi_\infty$ weakly star in $L^\infty(0, 1; H)$. Moreover, using (2.18) and exploiting the maximal monotonicity of $W' + \kappa \text{Id}$ (which is a continuous and increasing function thanks to (hpW)), one verifies that $W'(\chi_n)$ tends to $W'(\chi_\infty)$ weakly star in $L^\infty(0, 1; H)$ (and then weakly in $L^2(\Omega \times (0, 1))$) with the help of the strong convergence $\chi_n \to \chi_\infty$ in $L^2(\Omega \times (0, 1))$ and of [4, Prop. 1.1, p. 42]. Finally, thanks to (4.3), (hp\lambda) and the continuous embedding $V \subset L^6(\Omega)$, we infer that

$$\lambda'(\chi_n) u_n \to 0 \quad \text{strongly in } L^2(0, 1; L^{3/2}(\Omega)),$$

and consequently the right hand side of (4.4) tends to 0 in $L^2(0, 1; V^*)$. This concludes the proof of Theorem 2.10.  

Proof of Theorem 2.13. We proceed partly as in [7, Sec. 3], [17]. Let us assume, for simplicity, that (2.33) holds (otherwise, we can replace $\zeta$ with a value $\zeta_0$ such that (2.35) is satisfied and notice that Theorem 2.10 still holds with $\zeta_0$ in place of $\zeta$ in (2.29)). Letting $\chi_\infty$ be an element of the $\omega$-limit of $\chi(\cdot)$, we can set (cf. Theorem 2.10 for the notation)

$$\Sigma := \{ t > 0 : \| \chi(t) - \chi_\infty \|_{V' \cap C^0(\Omega)} \leq \epsilon/3 \}. \quad (4.5)$$

Clearly, $\Sigma$ is unbounded. Next, for $t \in \Sigma$, we put

$$\tau(t) := \sup \{ t' \geq t : \sup_{s \in [t, t']} \| \chi(s) - \chi_\infty \|_{V' \cap C^0(\Omega)} \leq \epsilon \} \quad (4.6)$$

and observe that, by continuity, $\tau(t) > t$ for all $t \in \Sigma$. Let us fix $t_0 \in \Sigma$ and divide $\mathcal{J} := [t_0, \tau(t_0))$ (where $\tau(t_0)$ might well be $+\infty$) into two subsets:

$$A_1 := \{ t \in \mathcal{J} : |\chi(t)| + \| u(t) \|_V > \left( \int_t^{\tau(t_0)} \| g(s) \|_{V^*}^2 \, ds \right)^{1-\zeta} \}, \quad (4.7)$$

$$A_2 := \mathcal{J} \setminus A_1. \quad (4.8)$$

Next, we define (cf. (2.27))

$$\Phi(t) := \int_{\Omega} j(\vartheta(x, t)) \, dx + \frac{1}{2} \int_t^{\tau(t_0)} \| g(s) \|_{V^*}^2 \, ds + E(\chi(t)) - E(\chi_\infty). \quad (4.9)$$
Then, it is not difficult to see that

$$\Phi'(t) \leq -\left(\frac{1}{2}\|u(t)\|^2 + |\chi_t(t)|^2\right).$$  \hspace{1cm} (4.10)

We remark that $\Phi$ is absolutely continuous thanks to \[2.15\]–\[2.18\] and \[5\] Lemme 3.3, p. 73. This justifies the above computation for a.e. $t \in J$. We then have (cf. also \[21\] (3.2))

$$\frac{d}{dt}(\Phi^\zeta \text{ sign } \Phi)(t) \leq -\zeta \Phi(t)^{\zeta-1} \left(\frac{1}{2}\|u(t)\|^2 + |\chi_t(t)|^2\right).$$  \hspace{1cm} (4.11)

Now, let us estimate $\Phi$ from above. If $t \in A_1$, thanks to \[2.27\] and Theorem \[2.10\] we obtain

$$|\Phi(t)|^{1-\zeta} \leq |E(\chi(t)) - E(\chi_\infty)|^{1-\zeta} + \int_{\Omega} j(\vartheta(t)) \left|1^{1-\zeta} + \left|\int_{\tau(t_0)}^{\tau(t)} \|g(s)\|_{V^*} \ ds\right|^{1-\zeta}\right)

\leq c_\ell - \chi_t(t) + \lambda(\chi(t)) u|_{V^*} + \int_{\Omega} j(\vartheta(t)) \left|1^{1-\zeta} + |\chi_t(t)| + \|u(t)\|_V, \right.$$

where we also used \(4.7\). Note now that, by \(hp\), the last of \(2.17\), and well-known continuous embeddings, we have

$$c_\ell - \chi_t(t) + \lambda(\chi(t)) u|_{V^*} \leq c(|\chi_t(t)| + \|u(t)\|_V).$$  \hspace{1cm} (4.13)

Moreover, by convexity of $j$, $j(\vartheta_\infty) = 0$, and Hölder’s inequality, we infer

$$0 \leq \int_{\Omega} j(\vartheta(x,t)) \ dx \leq \int_{\Omega} j'(\vartheta(x,t)) (\vartheta(x,t) - \vartheta_\infty) \ dx \leq |u(t)| |\vartheta(t) - \vartheta_\infty|.$$  \hspace{1cm} (4.14)

Using once more \(3.37\) together with \(2.16\) and recalling that $\zeta \in (0,1/2)$, one gets

$$\left|\int_{\Omega} j(\vartheta(t)) \right|^{1-\zeta} \leq c\|u(t)\|_{V^*}^{2(1-\zeta)} \leq c\|u(t)\|_V.$$  \hspace{1cm} (4.15)

Thus, collecting \(4.12\)–\(4.15\), we finally have

$$|\Phi(t)|^{1-\zeta} \leq c(|\chi_t(t)| + \|u(t)\|_V),$$  \hspace{1cm} (4.16)

whence from \(4.11\) we obtain

$$\|u(t)\|_V + |\chi_t(t)| \leq \frac{c}{4\zeta} \frac{d}{dt}(\Phi^\zeta \text{ sign } \Phi)(t).$$  \hspace{1cm} (4.17)

Since $\Phi$ is decreasing by \(4.10\), integration in time entails that $\|u\|_V$ and $|\chi_t|$ are summable over $A_1$. Of course, the same holds over $A_2$ by \[2.31\] and \[4.7\]–\[4.8\]. Thus, we conclude that $\chi_t \in L^1(J;H)$.

From this point on, the proof proceeds exactly as in \[17\] Sec. 3. Namely, a simple contradiction argument yields that $\tau(t_0) = \infty$ as $t_0 \in \Sigma$ is sufficiently large. This implies that $\chi_t \in L^1(t_0, +\infty; H)$, whence the convergence (in $H$) of the whole trajectory $\chi(t)$ to $\chi_\infty$ follows. More precisely, this convergence holds strongly in $V \cap C(\overline{\Omega})$ by precompactness of the trajectory (cf. \(2.17\)). Finally, the technical argument of \[17\] Sec. 3 leading to estimate \(2.34\) (or \(2.36\)) can be repeated just by adapting the notation.
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