Geometric quantum computation and multi-qubit entanglement with superconducting qubits inside a cavity

Shi-Liang Zhu,1,2 Z. D. Wang,3,4 and Paolo Zanardi2

1School of Physics and Telecommunication Engineering, South China Normal University, Guangzhou, China
2Institute for Scientific Interchange Foundation, Viale Settimio Severo 65, I-10133 Torino, Italy
3Department of Physics, University of Hong Kong, Pokfulam Road, Hong Kong, China
4National Laboratory of Solid State Microstructures, Nanjing University, Nanjing, China

We analyze a new scheme for quantum information processing, with superconducting charge qubits coupled through a cavity mode, in which quantum manipulations are insensitive to the state of the cavity. We illustrate how to physically implement universal quantum computation as well as multi-qubit entanglement based on unconventional geometric phase shifts in this scalable solid-state system. Some quantum error-correcting codes can also be easily constructed using the same technique. In view of the gate dependence on just global geometric features and the insensitivity to the state of cavity modes, the proposed quantum operations may result in high-fidelity quantum information processing.

PACS numbers: 03.67.Lx, 03.65.Vf, 03.67.Pp, 85.25.Cp

Superconducting qubits have recently attracted significant interests because of their potential suitability for integrated devices in quantum information processing.1,2,3,4,5. So far, experimental research on quantum information processing using this kind of systems has mostly focused on the behavior of single isolated qubit as the decoherence time of this solid-state system is quite short. Only very recently, significant achievements on superconducting two-charge-qubit systems were reported, i.e., realization of an entangled state for two-qubits, and implementation of a conditional gate.5,6. Also note that multi-particle entanglement was experimentally reported only in photons and trapped ions. Implementation of a universal set of high-fidelity quantum gates and generation of multi-qubit entanglement will be the next significant and very challenging steps towards quantum information processing based on this scalable solid-state approach.

In this paper, by designing a device consisting of superconducting charge qubits coupled through a cavity, we propose a feasible scheme to implement a universal set of quantum gates and produce the Greenberger-Horne-Zeilinger state (GHZ)6 based on unconventional geometric phases. In addition, we find that this scheme is also applicable to construct quantum error-correcting codes (QECC).7. In particular, we shall show how to realize the geometric evolution operators $U_{x,y}(\gamma) = \exp(i\gamma J_{x,y}^2)$ with $J_{x,y}$ collective operators and $\gamma$ an unconventional geometric phase, which may have some inherent fault-tolerant features due to the fact that unconventional geometric phases depend only on some global geometric property.7,8,9,10,11,12,13. It is also shown that in terms of $U_{x,y}(\gamma)$ of the present system one is able to achieve a universal set of quantum gates, to generate even-N-qubit entanglement simultaneously by one operation,14 and to construct the QECC.15 All of these are essential ingredients in quantum information processing. Implementation of these tasks based on the same set of geometric quantum operators may simplify the experimental operations. Apart from the high-fidelity advantage, the geometric scheme proposed here has other distinctive merits: the operator is insensitive to the state of the cavity modes,14,16,17 is tolerant to device parameter nonuniformity in quantum computation, and the process can be fast.17,18 Moreover, a necessary condition for fault tolerant quantum computation, i.e., the two-qubit gate can act on any pair of qubits, can be realized using the cavity quantum electrodynamic technique. In addition, the system proposed is also a potential candidate for quantum communication and quantum network.

Superconducting nanocircuits coupling through cavities have been shown to be a promising solid-state system for implementation of quantum computation and quantum communication.21,22,23,24,25 One of advantages for qubits placed in a cavity is that the cavity can also protect the qubits from the environment, which is important for a useful operation of qubits especially in the scaling-up of the solid-state devices. Moreover, an architecture using one-dimensional transmission line resonators to reach the strong coupling limit between cavity and superconducting nanocircuits was theoretically proposed in Ref.23 and then experimentally achieved in Ref.24. In the present work, we address a system with N specially designed superconducting charge qubits coupled through a high-quality single mode cavity, as shown in Fig. 1. Clearly, as shown in Fig.1(a), the newly designed single qubit is significantly different from those studied before.1,2,10,21. The jth qubit consists of a small superconducting box with $n_j$ excess Cooper-pair charges, formed by two SQUIDs with Josephson coupling energies $E_{c}^{\text{ref}}$ ($c = a, b; l = 1,2$) and a $\pi$-phase junction, and each SQUID is penetrated by a magnetic flux $\varphi_j$. A control gate voltage $V_j^g$ is connected to the system via a
 capacitor $C_j$. The Hamiltonian of the $j$th qubit reads
\[ H = E_{ch}(n_j - \bar{n}_j)^2 - \sum_{\ell} E_{\ell}^{j} \cos \varphi_{\ell}^{j}, \]
where $E_{ch} = 2e^2/C_j$ is the charging energy with $C_j$ being the total capacitance of the box, $\bar{n}_j = C_j V_{0j}^2/2$ is the induced charge and can be controlled by changing $V_{0j}^2$. The gauge-invariant phase difference $\varphi_{\ell}^{j} = \Theta_j - (2\pi/\phi_0) \int A_{\ell}^{j} \cdot dl$ with $\Theta_j$ being the phase difference of the superconducting wave function across the junctions in a particular gauge and $A_{\ell}^{j}$ being the vector potential in the same gauge. Assuming that Josephson junctions are placed inside a single-mode resonant cavity, we have $(2\pi/\phi_0) \int A_{\ell}^{j} \cdot dl = (2\pi/\phi_0) \int A_{\ell}^{j} \cdot dl + g_j(a + a^\dagger)$, where $a$ and $a^\dagger$ are the creation and annihilation operators for the single-mode, $\phi_0 = \pi \hbar/e$ is the flux quantum, $g_j$ is the coupling constant between the junctions and the cavity, and $A_{\ell}^{j}$ is the vector potential determined by the magnetic flux $\phi_j^\ell$ penetrating the $\ell$th SQUID hole in the $j$th qubit.

![FIG. 1: Josephson charge qubits. (a) A single Josephson charge qubit. (b) A series of Josephson charge qubits coupled through a cavity](image)

At temperatures much lower than the charging energy and the gate voltage being close to a degeneracy ($\bar{n}_j \sim 1/2$), the relevant physics is captured by considering only the two charge eigenstates $n_j = 0, 1$, which constitute the basis $\{ |0\rangle, |1\rangle \}$ of the computational space of the qubit. If we have $N$ such qubits located inside a single-mode cavity [Fig.1(b)] with frequency $\omega_c/2\pi$, to a good approximation, the whole system can be considered as $N$ two-state systems coupled to a quantum harmonic oscillator. On the other hand, $\varphi_{\ell}^{j}$ are determined from the flux quantization for two independent loops and an effective $\pi$-phase junction between $a_2$ and $b_1$ for each qubit [see Fig 1(a)], that is, $\varphi_2^{a1} - \varphi_2^{a2} = \Phi_j^a$, $\varphi_1^{b1} - \varphi_2^{b2} = \Phi_j^b$, and $\varphi_2^{a2} - \varphi_2^{b1} = \pi/2$, where the flux $\Phi_j^a = \phi_j^a + g_j(a + a^\dagger)$ with $\phi_j^a = 2\pi \phi_j^a/\phi_0$. Moreover, the average phase drop $\sum_{\ell} \varphi_{\ell}^{j}$ is equal to $\Theta_j$, which conjugates to the Cooper pair number $n_j$. Here we assume that $E_{el}^{j} = E_{bd}^{j} = E_{lj}^{j}$ and $g_j^* = g_j$. In this case, the whole system can be described by the Hamiltonian $H = H_0 + H_{\text{int}}$, where
\[ H_0 = \hbar \omega_c (a^\dagger a + 1/2) + \sum_{j} N \bar{n}_j \sigma_j^z/2, \]
\[ H_{\text{int}} = -\frac{1}{2} \sum_{lj} [i(-1)^j E_{lj}^{j}(1-e^{i\pi/2})e^{-i\phi_j^a/2}\sigma_j^+ H.c.] \]

Here $E_{\bar{n}_j} = 2E_{ch}(\bar{n}_j - 1/2)$, $\Gamma_j^j = \phi_j + i(-1)^j [-\phi_j^a + g(a + a^\dagger)]$, and $\phi_j^\pm = (\phi_j^a \pm \phi_j^b)/2$. A spin notation is used for the qubit $j$ with Pauli matrices $\{ \sigma_j^x, \sigma_j^y, \sigma_j^z \}$, and $\sigma_j^\pm = (\sigma_j^x \pm i\sigma_j^y)/2$. It is interesting to note that this Hamiltonian is similar to that used in Ref. [21] when the external magnetic flux $\phi_j^a$ depends on the time $t$ with a constant rate $\omega_j^f$.

Comparing with the charge qubit made of one SQUID and coupled to the cavity [21], where the initial cavity mode should be in the ground state in performing quantum gates (the task is exceedingly difficult to implement experimentally), one more SQUID being penetrated through another magnetic flux is used here. Two magnetic fluxes for one qubit can be used to manipulate the qubit states. The function of these fluxes is similar to bichromatic lights in quantum computation based on trapped ions [14]: transition paths involving unpopulated cavity states interfere destructively to eliminate the influence of the cavity mode. This phenomenon plays a key role in the implementation of a quantum gate which is insensitive to the initial state of cavity mode and robust against changes in the cavity state occurring during operation. This scenario has been used in implementation of quantum computation with trapped ions [14, 13, 11, 19, 18, 17, 16, 15], and in generating experimentally the Dicke limit and under the rotating wave approximation [26].

In the following we illustrate how this strategy can be applied to the present system.

To eliminate the influence of the cavity mode, we now propose to apply two constantly growing external fluxes penetrating the designated SQUIDs in the $j$-th qubit: $\phi_j^a = \omega_j^0 t - \beta_j^0$ and $\phi_j^b = \omega_j^0 t - \beta_j^0 - 2k\pi$, where $\beta_j^0 (\beta_j^0 - 2k\pi)$ is the initial value of $\phi_j^a (\phi_j^b)$ and $k$ is an integer. Experimentally, one possible way is to place each SQUID just above an approximately-inductive circuit with a constant voltage $V_0 = \omega_j^0 \phi_0/2\pi$. Expanding the Hamiltonian [8] to the first order of $g_j$ in the Lamb-Dicke limit and under the rotating wave approximation as well as in the interaction picture $U_0 = \exp(-i H_0 t)$ with $E_{\bar{n}_j} = 0$ (i.e., the qubits are set at the degeneracy point) and $E_j^1 = E_j^2 = E_j^3$, yields
\[ H_{\text{int}} = \sum_{j} N g_j E_j(a^\dagger e^{i\delta_j t + i\delta_j^0} H.c.)\sigma_j^\eta, \quad (\eta = x, y) \]
where $\eta = x, y$ for $k = 0, 1$, and $\delta_j = \omega_c - \omega_j^0 << \omega_j^0$ [20].
We now show that the Hamiltonian described in Eq. (1) can be used to perform universal quantum computation as well as to produce the GHZ state. Let us first address a simpler case, in which all parameters for different qubits are the same, namely, $E_j^z = E, \delta_j = \delta, \text{ and } \beta_j^0 = \beta^0$. The corresponding Hamiltonian (4) can be rewritten as

$$H_{\text{int}}^g = gE(a^\dagger e^{i\delta t} + H.c.)J_\eta,$$

(5)

where $J_\eta = \sum_j^N \sigma_j^\eta$ is the collective spin operator. By using the Magnus' formula, the evolution operator $U(t)$ is found as

$$U_\eta(t) = e^{\gamma(t)J_\eta^2}e^{[\alpha(t,\beta^0)t - \alpha^*(t,\beta^0)J_\eta]}.$$

(6)

where $\alpha(t,\beta^0) = (gE/h\delta)(1 - e^{i\delta t})e^{i\beta^0 t}$ and $\gamma(t) = (gE/h\delta)^2(2\delta t - \sin \delta t)$. Here $\alpha(t,\beta^0)$ is a periodic function of time and vanishes at $t_m = 2m\pi/\delta$ for an integer $m$. At the time $t = t_m$, the evolution operator is explicitly expressed as

$$U_\eta(\gamma) = \exp[i\gamma J_\eta^2].$$

(7)

This operator is insensitive to heating by removing the influence of the cavity mode represented by the last exponent in Eq. (4).

More intriguingly, we can remove the influence of the cavity mode by using two operators in succession (17). Let us suppose that qubits first evolve with $\beta^0 = 0$ for a period $\tau/2$. Then the $\beta^0$ is shifted by $\pi$ for the other period $\tau/2$. Note that $\alpha(t,0) = -\alpha(t,\pi)$ and $\gamma(t)$ is independent on $\beta^0$, so we have $U_\eta = \exp[2\gamma(\tau/2)J_\eta^2]$. Comparing with the first approach, a distinctive merit lies in that the gate operation time is no longer restricted to be $2m\pi/\delta$ and it can then be shorter; of course a gate with fast speed is particularly important in solid state qubit systems since the decoherence time there is typically quite short.

Interestingly, the evolution operators (4) can provide a set of universal quantum gates. It is well known that for achieving universal quantum computation, we need to realize two noncommuting single-qubit gates and one nontrivial two-qubit gate. When only two qubits are considered, it is straightforward to check that $U_x$ (or $U_y$) is a nontrivial two-qubit gate. $U_x$ can be used to produce an entangled state from an unentangled state. For example, the maximally entangled state $(|00\rangle - i|11\rangle)/\sqrt{2}$ is derived when $U(\pi/2)$ is directly applied to the unentangled state $|00\rangle$. Moreover, a controlled-NOT gate is explicitly generated by $U_x$ plus single-qubit rotations in Ref. (14). We now work out how to use the operators $U_{x,y}$ to achieve a set of noncommuting single-qubit rotations if one of qubits is an auxiliary qubit. Denoting $|\psi_{\mu}^{x,y}\rangle_\alpha$ as the eigenstate of the operator $\sigma_x$ for the auxiliary qubit with eigenvalue $\pm 1$, and choosing the initial state as one of the eigenstates, then we have

$$U_{x,y}|\psi\rangle|\psi_{\mu}^{x,y}\rangle_\alpha = e^{2i\gamma}[e^{\pm 2i\gamma \sigma_x} |\psi\rangle]|\psi_{\mu}^{x,y}\rangle_\alpha.$$

(8)

It is clear from Eq. (8) that an effective single qubit gate $U^{(1)}_{x,y} = \exp(\pm 2i\gamma \sigma_x)$ is obtained (up to an irrelevant overall phase) by application of $U_{x,y}$ to two qubits, but one of them is an auxiliary qubit with a fixed initial state and then is disregarded after the gate operation. Since $|\psi_{\mu}^{\pm}\rangle_\alpha$ are the ground states of the Hamiltonian at $(\tilde{n}_a = 1/2, \phi^+ = 0)$ and $(\tilde{n}_a = 1/2, \phi^- = \pi)$, respectively, it is rather easy to experimentally realize the required initial state in the present system. Certainly, $U^{(1)}_{x,y}$ and $U^{(1)}_{y}$ are noncommuting and consist of the well-known single-qubit rotations. Although one more auxiliary qubit is required, all important operations required (all logical gates and construction of QECC addressed below) can be achieved with similar manipulations, which may simplify experimental operations. In addition, comparing with qubits coupled through capacitance, a nearest neighbor interaction, the distinctive merit is that the nontrivial two-qubit gate proposed here may act on any pair of qubits.

It is worth pointing out that a non-trivial two-qubit gate can also be implemented when the parameters $g_j$ and $E_j$ are dependent on $j$, thus the method is also insensitive to fabrication errors often appeared in realistic solid state experiments. We still consider $N$ charge qubits in a cavity, but only two of them are resonant with the cavity and thus are controlled by the Hamiltonian (4). So only the two resonant qubits are relevant and the evolution operator is given by

$$U(t) = e^{-i\sum_j j\alpha_j(t,\beta_j^0)\sigma_j^\gamma H.c.}.$$

(9)

where $\alpha_j(t,\beta_j^0) = (g_jE_j\delta_j)(1 - e^{i\delta_j t})e^{i\beta_j^0 t}$ and $\gamma_jl(t) = [g_jg_lE_jE_l/(h^2d_jd_l\delta_j\delta_l)][\delta_j \sin(d_jt + \beta_j^0) - d_l \sin(\delta_j t + \beta_l^0) - \delta_l \sin(\beta_j^0 t) + \delta_l \sin(\beta_j^0 t) - \beta_j^0]$, and $d_jl = \delta_j - \delta_l$ and $\beta_l = \beta_j^0 - \beta_j^0_l$ when $d_jl \neq 0$. Note the facts that $\sum_j \alpha_j(t,0) = -\sum_l \alpha_j(t,\pi)$, and $\gamma_jl(t)$ is the same for $\beta_j^0 = \beta_j^0_l$. Therefore, the influence of the cavity mode can also be eliminated by using two operators in succession: two qubits evolve with $\beta_j^0 = \beta_j^0_l = 0$ for a first period $\tau/2$ and then with $\beta_j^0 = \beta_j^0_l = \pi$ for the second period $\tau/2$. Moreover, the controlled phase gate diag$(1,1,1,-1)$ is obtained when $\gamma_jl = -\pi$ after performing one-qubit operations (17).

Remarkably, following the approach addressed in Ref. (4), it is straightforward to find that the phase $\gamma$ in the operators (7) satisfies the relation $\gamma = -\gamma_g$ ($\gamma_g = -2\gamma_g$), where $\gamma_g$ is the geometric phase and $\gamma_d$ is the dynamic phase accumulated in the evolution. Thus $\gamma$ is an unconventional geometric phase shift, which consists of both a geometric component and a nonzero dynamic component, but still depends only on global geometric features. Because of this, the high-fidelity of the gates in the present system may be experimentally achieved. Note that, a recent experiment on trapped ions demonstrated that the operation realized by unconventional geometric phases possesses the high-fidelity (15), which really benefits from the geometric features: the
phase is determined only by the path area, not on the exact starting state distributions, path shape, orientation in phase space, or the passage rate to traverse the closed path. Moreover, this operation is robust to small noncyclic perturbations.

At this stage, we illustrate how to produce the GHZ state in this system. Note that the operator (7) is independent on the number of qubits, by choosing $\gamma = \pi/2$ and an initial state $|\Psi\rangle = |00\cdots 0\rangle$, the final state $|\Psi_f\rangle = e^{i\pi J_2} |\Psi\rangle$ is found to be a GHZ state given by

$$|\Psi_f\rangle = \frac{1}{\sqrt{2}} \left[ e^{-i\pi} |00\cdots 0\rangle + e^{i\pi (\frac{1}{2} + \frac{1}{2})} |11\cdots 1\rangle \right],$$

when $N$ is even. Although the operator (7) does not directly produce the maximally entangled state described in Eq. (10) for odd $N$, it suffices to get GHZ state by applying the unitary operator $U = \exp(-i\pi J_2/2)$ in addition to application of Eq. (7). Starting from Eq (3), it is not difficult to see that the operation $U$ can be implemented in the system by just choosing ($\bar{n}_j = 1/2$, $\sigma_j^- = 0$, $\sigma_j^+ \neq 0$). To see the feasibility of the present scheme with current technology, we now use typical values of physical parameters in the system to estimate the operation time $\tau$. We have $\gamma \sim g^2 E^2 \tau / h^2 \delta$. Therefore, $\tau \sim \pi h^2 / 2 g^2 E^2$ for $\gamma = \pi / 2$, which is about 20 ns for $E = 40 \text{ meV}$, $\hbar \omega_c = 30 \text{ meV}$, $\delta = \omega_c/10$, and $g = 10^{-2}$. Thus the $\tau$ is significantly less than the decoherence time of qubit ($\sim 0.5 \mu s$) without the protection of the cavity, and is much less than the photon lifetime of the cavity mode $\tau_c = Q / \omega_c = 22 \mu s$, as the quality factor of the cavity $Q = 10^6$ was reported experimentally. In this case the voltage $V_0$ required to generate the external flux is $\omega_c \phi_0 / 2 \pi = 13 \mu V$, a typical voltage for manipulating Josephson junctions.

Another useful application of the present geometric approach is to construct QECC in this solid-state system. The operator (7) can also be used for QECC. For example, the Shor coding

$$\alpha \left( \frac{1}{\sqrt{2}} (|000\rangle - |111\rangle) \right)^{\otimes 3} + \beta \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)^{\otimes 3}$$

with $|\alpha|^2 + |\beta|^2 = 1$ is achieved by using the operator (7) plus single qubit measurement or using the operator (7) and a coupling $\sigma_x \sigma_z$, which arises naturally in charge qubits coupled through capacitors. Therefore, the approach proposed here provides a geometric way to efficiently construct this kind of essential codes in quantum information. A significant result here is that implementation of a universal set of quantum gates as well as construction of QECC can be based on the same set of geometric quantum operators with intrinsic fault tolerant features and then may simplify the experimental operations for quantum computation.

This work was supported by the European Union project TOPQIP (contract IST-2001-39215), the NSFC under Grant Nos. 10429401, 10204008 and 10334090, the NSF of Guangdong under Grant No.021088, Program for NCET, the RGC grant of Hong Kong (HKU7114/02P), and the URC fund of HKU.

[1] Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).
[2] Ya. A. Pashkin et al., Nature (London) 421, 823 (2003); T. Yamamoto et al. ibid., 425, 941 (2003).
[3] Yu. Yu et al., Science, 296, 889 (2002).
[4] D. Vion et al. 296, 886 (2002).
[5] D. M. Greenberger, M. A. Horne, and A. Zeilinger, in Bell’s Theorem, Quantum Theory and Conceptions of the Universe, edited by M. Kafatos (Kluwer Academic, Dordrecht, 1989), p. 69.
[6] P. W. Shor, Phys. Rev. A 52, R2493 (1995).
[7] S. L. Zhu and Z. D. Wang, Phys. Rev. Lett. 91, 187902 (2003).
[8] M. V. Berry, Proc. R. Soc. London A 392, 45 (1984).
[9] P. Zanardi and M. Rasetti, Phys. Lett. A 264, 94 (1999).
[10] G. Falci et al., Nature (London) 407, 355 (2000).
[11] L. M. Duan, J. I. Cirac, and P. Zoller, Science 292, 1695 (2001).
[12] S. L. Zhu and Z. D. Wang, Phys. Rev. Lett. 89, 097902 (2002); 89, 289901(E); Phys. Rev. A 66, 042322 (2002).
[13] D. Leibfried et al., Nature (London) 422, 412 (2003).
[14] K. Mølmer and A. Sørensen, Phys. Rev. Lett. 82, 1835 (1999); A. Sørensen and K. Mølmer, Phys. Rev. A 62, 022311 (2000).
[15] B. Zeng et al., quant-ph/0411122.
[16] G. J. Milburn, S. Schneider, and D. F. V. James, Fortschr. Phys. 48, 801 (2000).
[17] S. B. Zheng, Phys. Rev. Lett. 90, 217901 (2003).
[18] J. J. Garcia-Ripoll, P. Zoller, and J. I. Cirac, Phys. Rev. Lett. 91, 157901 (2003); L. M. Duan, Phys. Rev. Lett. 93, 105024 (2004).
[19] C. A. Sackett et al., Nature (London) 404, 256 (2000).
[20] X. Wang and P. Zanardi, Phys. Rev. A 65, 032327 (2002).
[21] S. L. Zhu, Z. D. Wang, and K. Yang, Phys. Rev. A 68, 034303 (2003).
[22] C. P. Yang, S. H. Chu, and S. Han, Phys. Rev. A 67, 042311 (2003); J. Q. You and F. Nori, Phys. Rev. B 68, 064509 (2003).
[23] S. M. Girvin et al., cond-mat/03010670.
[24] A. Blais et al., Phys. Rev. A 69, 062320 (2004).
[25] A. Wallraff et al., Nature (London) 431, 162 (2004).
[26] This $\pi$-phase junction is used to remove a dynamic phase shift because an additional term $-2E_j \delta_j^+ \delta_j^-$ will appear in Eq. (4) in the absence of it. Experimentally, this effective $\pi$-phase junction may be introduced by inserting an intrinsic $\pi$-junction between $a_2$ and $b_1$ (see, e.g., L. B. Ioffe et al., Nature (London) 398, 679 (1999) or a small $\pi$-flux SQUID with a negligible coupling to the cavity.
[27] All expanded terms oscillating with the frequencies $\omega_j^+$ and $\omega_c$ are disregarded under the rotating wave approximation, as done in Ref. [14].
[27] A. Friedenauer and E. Sjoqvist, Phys. Rev. A 67, 024303 (2003).

[28] P. K. Day et al., Nature (London) 425, 817 (2003).