TOP LOCAL COHOMOLOGY AND THE CATENARICITY OF THE UNMIXED SUPPORT OF A FINITELY GENERATED MODULE

NGUYEN TU CUONG, NGUYEN THI DUNG
Institute of Mathematics
18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam
E-mail adress: ntcuong@math.ac.vn

LE THANH NHAN
Department of Mathematics
Thai Nguyen Pedagogical University, Thai Nguyen, Vietnam
E-mail adress: trtrnhan@yahoo.com

Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M$ a finitely generated $R$–module with $\dim M = d$. This paper is concerned with the following property for the top local cohomology module $H^d_\mathfrak{m}(M)$:

$$\text{Ann}(0 :_{H^d_\mathfrak{m}(M)} \mathfrak{p}) = \mathfrak{p} \text{ for all prime ideals } \mathfrak{p} \supseteq \text{Ann } H^d_\mathfrak{m}(M).$$

In this paper we will show that this property is equivalent to the catenaricity of the unmixed support $\text{Usupp } M$ of $M$ which is defined by $\text{Usupp } M = \text{Supp } M / U_M(0)$, where $U_M(0)$ is the largest submodule of $M$ of dimension less than $d$. Some characterizations of this property in terms of system of parameters as well as the relation between the unmixed supports of $M$ and of the $\mathfrak{m}$-adic completion $\widehat{M}$ are given.

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring, $M$ a finitely generated $R$–module with $\dim M = d$, and $A$ an Artinian $R$–module. For each ideal $I$ of $R$, we denote by $V(I)$ the set of all prime ideals containing $I$.

An elementary property of finitely generated modules is that $\text{Ann}(M/\mathfrak{p}M) = \mathfrak{p}$ for all $\mathfrak{p} \in V(\text{Ann } M)$. The dual question for Artinian modules is to ask whether

$$(*) \quad \text{Ann}(0 : A \mathfrak{p}) = \mathfrak{p} \text{ for all } \mathfrak{p} \in V(\text{Ann } A).$$

\textbf{Keywords} Artinian module, top local cohomology, unmixed support, catenaricity.

\textbf{AMS Classification} 13D45, 13E10.

\textsuperscript{a}Senior Associate Member of ICTP, Trieste, Italy.
\textsuperscript{b}Junior Associate Member of ICTP, Trieste, Italy.
In case $R$ is complete with respect to the $m$–adic topology, the property (*) is satisfied for all Artinian $R$–modules $A$ because of the Matlis duality between the category of Noetherian $R$–modules and the category of Artinian $R$–modules. Unfortunately the property (*) is not satisfied in general. For example, let $R$ be the Noetherian local domain of dimension 2 constructed by D. Ferrand and M. Raynaud [FR] (see also [Na, App., Exam. 2]) such that its $m$–adic completion $\hat{R}$ has an associated prime $\hat{q}$ of dimension 1. Then the Artinian $R$–module $A = H^1_m(R)$ does not satisfy the property (*), cf [CN1]. However, it seems to us that the property (*) is an important property of Artinian modules. For example, the property (*) is closely related to some questions on dimension for Artinian modules. In [CN1], it is shown that $N\dim A = \dim R/\Ann A$ provided $A$ satisfies the property (*), where $N\dim A$ is the Noetherian dimension of $A$ defined by Roberts [R] (see also [K2]). Note that this equality does not hold in general. Concretely, with the Artinian $R$–module $A = H^1_m(R)$ as above, $N\dim A = 1 < 2 = \dim R/\Ann A$ although this ring $R$ is catenary and the top local cohomology module $H^2_m(R)$ satisfies the property (*).

The purpose of this paper is to study the property (*) for the top local cohomology Artinian module $H^d_m(M)$, $\dim M = d$, and its applications. We will show that, although $N\dim H^d_m(M)$ and $\dim R/\Ann H^d_m(M)$ are always equal to $d$, the property (*) is not necessarily satisfied for $H^d_m(M)$. Then we find conditions such that $H^d_m(M)$ satisfies the property (*). It seems surprising to us, that this property is equivalent to some important properties of $M$. In particular, the property (*) is satisfied for $H^d_m(M)$ if and only if the unmixed support $\Supp M = \Supp M/\hat{M}(0)$ is catenary, where $\hat{M}(0)$ is the largest submodule of $\hat{M}$ of dimension less than $d$.

From now on, we denote by $\hat{R}$ (resp. $\hat{M}$) the $m$–adic completion of $R$ (resp. $M$) and $\hat{\Supp R \hat{M}}$ the unmixed support of $\hat{M}$ as an $\hat{R}$–module. The main result of this paper is the following theorem.

**Theorem.** The following statements are equivalent:

(i) $\Ann(0 : H^d_m(M)) \subseteq \hat{p}$ for all $p \in V(\Ann H^d_m(M))$.

(ii) $\Supp M$ is catenary.

(iii) $\Supp M = \{ \hat{p} \cap R : \hat{p} \in \hat{\Supp R \hat{M}} \}$.

(iv) For every sequence $x_1, \ldots, x_d$ of elements in $m$, $(x_1, \ldots, x_d)$ is a system of parameters of $H^d_m(M)$ if and only if it is a system of parameters of $M/\hat{M}(0)$.

Here, the notion of system of parameters for Artinian modules is defined according to Section 2.

As an immediate consequence of the above main theorem, we have the following characterization for the catenaricity of a Noetherian local domain.

**Corollary.** Suppose that $(R, m)$ is a Noetherian local domain of dimension $d$. Then $R$ is catenary if and only if $H^d_m(R)$ satisfies the property (*).

This paper is divided into 4 sections. In Section 2 we introduce the property (*) for Artinian modules and recall some basic facts that we need in the sequel. In the last two sections we present the proof of the above main theorem. The characterizations of the property (*) for $H^d_m(M)$ in terms of system of parameters and the relation between two sets $\Supp M$ and $\hat{\Supp R \hat{M}}$ (the part (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)
of the main theorem) are proved in Section 3. In Section 4, we prove the equivalence between the property (*) for $H_m^d(M)$ and the catenaricity of $\text{Usupp} M$ (the part (i)$\iff$(ii) of the main theorem). We also examine some non-catenary domains to clarify the results.

2. When is $\text{Ann}(0 :_A p) = p$ for all prime ideals $p \supseteq \text{Ann} A$?

For each Noetherian $R$–module $M$, it is clear that $\text{Ann}(M/pM) = p$ for each prime ideal $p \in V(\text{Ann} M)$. Therefore, for each Artinian $R$–module $A$, we consider the following property:

\[ (*) \quad \text{Ann}(0 :_A p) = p \text{ for all prime ideals } p \in V(\text{Ann} A). \]

As we mentioned in the introduction, the property (*) is not satisfied for all Artinian modules. In this section, we give some conditions such that this property is satisfied.

First we have the following result which is proved in [CN1].

**Proposition 2.1.** A satisfies the property (*) if one of the following conditions holds.

(i) $R$ is complete with respect to the $m$–adic topology.

(ii) $A$ contains a submodule which is isomorphic to the injective hull of $R/m$.

It should be mentioned that $\text{Supp} M = \{ \widehat{p} \cap R : \widehat{p} \in \text{Supp} \hat{M} \}$ for each finitely generated $R$–module $M$. This means that $V(\text{Ann} M) = \{ \widehat{p} \cap R : \widehat{p} \in V(\text{Ann} \hat{M}) \}$.

We also note that, for each Artinian $R$–module $A$, $A$ has a natural structure as an Artinian $\hat{R}$–module (cf. [Sh]), and with this structure, a subset of $A$ is an $R$–submodule of $A$ if and only if it is an $\hat{R}$–submodule of $A$. Therefore it is natural to ask whether

\[ V(\text{Ann} A) = \{ \widehat{p} \cap R : \widehat{p} \in V(\text{Ann} \hat{R} A) \}. \]

Below we show that this equality holds if and only if $A$ satisfies the property (*).

Recall that $A$ has a minimal secondary representation $A = A_1 + \ldots + A_n$, where $A_i$ is $p_i$–secondary, and the set $\{ p_1, \ldots, p_n \}$ does not depend on the choice of the minimal secondary representation of $A$. This set is denoted by $\text{Att} A$ and called the set of attached prime ideals of $A$, cf. [Mac]. It follows by [Sh] that

\[ \text{Att} A = \{ \widehat{p} \cap R : \widehat{p} \in \text{Att} \hat{R} A \}. \]

**Proposition 2.2.** The following conditions are equivalent:

(i) $A$ satisfies the property (*).

(ii) $V(\text{Ann} A) = \{ \widehat{p} \cap R : \widehat{p} \in V(\text{Ann} \hat{R} A) \}$.

**Proof.** (i)$\Rightarrow$(ii). Let $\widehat{p} \in V(\text{Ann} \hat{R} A)$. Then $\widehat{p} \supseteq \widehat{q}$ for some minimal prime ideal $\widehat{q}$ containing $\text{Ann} \hat{R} A$. Note that any minimal prime ideal containing $\text{Ann} \hat{R} A$ is a minimal element of $\text{Att} \hat{R} A$, cf. [Mac]. Therefore $\widehat{q} \in \text{Att} \hat{R} A$. So, $\widehat{q} \cap R \in \text{Att} A$. Hence $\widehat{q} \cap R \in V(\text{Ann} A)$ and hence $\widehat{p} \cap R \in V(\text{Ann} A)$. Conversely, let $p \in V(\text{Ann} A)$. Then $\text{Ann}(0 :_A p) = p$ by the hypothesis (i). Since $p$ is a minimal prime ideal containing $\text{Ann}(0 :_A p)$, it follows that $p \in \text{Att}(0 :_A p)$. Therefore there exists a
prime ideal \( \hat{p} \in \text{Att}_R(0 :_A p) \) such that \( \hat{p} \cap R = p \). Since \( \hat{p} \in \text{Att}_R(0 :_A p) \), we have \( \hat{p} \supseteq \text{Ann}_R(0 :_A p) \), and hence \( \hat{p} \in V(\text{Ann}_R A) \) with \( \hat{p} \cap R = p \).

(ii) \( \Rightarrow \) (i). Let \( p \in V(\text{Ann} A) \). By the hypothesis (ii), there exists \( \hat{p} \in V(\text{Ann}_R A) \) such that \( \hat{p} \cap R = p \). It follows by Matlis duality that \( \text{Ann}_R(0 :_A \hat{p}) = \hat{p} \). Therefore

\[ p \subseteq \text{Ann}(0 :_A p) \subseteq \text{Ann}_R(0 :_A \hat{p}) \cap R = \hat{p} \cap R = p. \]

Thus, \( \text{Ann}(0 :_A p) = p. \)

Roberts [R] introduced the concept of Krull dimension for Artinian modules. Kirby [K2] changed the terminology of Roberts and used the terminology of Noetherian dimension to avoid confusion with Krull dimension defined for finitely generated modules. In this paper we use the terminology of Kirby [K2]. The Noetherian dimension of \( A \), denoted by \( \text{N-dim}_R A \), is defined inductively as follows:

when \( A = 0 \), put \( \text{N-dim}_R A = -1 \). Then by induction, for an integer \( d \geq 0 \), we put \( \text{N-dim}_R A = d \) if \( \text{N-dim}_R A < d \) is false and for every ascending sequence \( A_0 \subseteq A_1 \subseteq \ldots \) of submodules of \( A \), there exists \( n_0 \) such that \( \text{N-dim}_R(A_{n+1}/A_n) < d \) for all \( n > n_0 \).

The following result gives some good properties of Noetherian dimension for Artinian modules which are in some sense dual to that of Krull dimension for Noetherian modules, cf [R], [K1].

**Lemma 2.3.** \( \ell(0 :_A m^n) \) is a polynomial for \( n \gg 0 \), and

\[ \text{N-dim}_R A = \deg \ell(0 :_A m^n) = \inf\{t : \exists x_1, \ldots, x_t \in m : \ell(0 :_A (x_1, \ldots, x_t)R) < \infty\}. \]

It follows by Lemma 2.3 that there exists a sequence \( (x_1, \ldots, x_d) \), \( d = \text{N-dim}_R A \), such that \( \ell(0 :_A (x_1, \ldots, x_d)R) < \infty \). A such sequence is called a system of parameters of \( A \).

Note that, with the natural structure as an Artinian \( \widehat{R} \)-module, \( \text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A \). Therefore there is no confusion in writing \( \text{N-dim} A \) instead of \( \text{N-dim}_R A \) or \( \text{N-dim}_{\widehat{R}} A \). Moreover, it follows by Matlis duality that

\[ \text{N-dim} A = \dim \widehat{R}/\text{Ann} \widehat{R} A = \max\{\dim \widehat{R}/\widehat{p} : \widehat{p} \in \text{Att}_\widehat{R} A\}. \]

Note that the set of all minimal elements of \( \text{Att} A \) is exactly the set of all minimal prime ideals containing \( \text{Ann} A \). So, we have

\[ \dim R/\text{Ann} A = \max\{\dim R/p : p \in \text{Att} A\}. \]

The following result shows the relation between \( \text{N-dim} A \) and \( \dim R/\text{Ann} A \).
Proposition 2.4. [CN1]. The following statements are true.
(i) \( N \dim A \leq \dim R / \Ann A \).
(ii) If \( A \) satisfies the property (*) then \( N \dim A = \dim R / \Ann A \).

Remark 2.5.
(i) As we mentioned in the introduction, there exist Artinian modules \( A \) such that \( N \dim A < \dim R / \Ann A \).
(ii) The converse of Proposition 2.4, (ii) is not true. In the next sections, we will show that the top local cohomology module \( A = H^d_m(M) \) of a finitely generated \( R \)-module \( M \) of dimension \( d \) does not satisfy the property (*) in general, although it always satisfies the condition \( N \dim A = \dim R / \Ann A = d \).

3. The property (*) for the top local cohomology modules

From now on, let \( M \) be a finitely generated \( R \)-module with \( \dim M = d \). In this section, we examine the property (*) for the top local cohomology module \( H^d_m(M) \).

We first present a nice property of \( H^d_m(M) \), cf. [CN1, Corollary 3.6].

Lemma 3.1. \( N \dim H^d_m(M) = \dim R / \Ann H^d_m(M) = d \).

Let \( U_M(0) \) be the largest submodule of \( M \) of dimension less than \( d \). Note that if \( 0 = \bigcap_{p \in \Ass M} N(p) \) is a reduced primary decomposition of the zero submodule of \( M \) then \( U_M(0) = \bigcap_{\dim R/p = d} N(p) \), cf. [CN2]. Therefore we have
\[
\Ass M / U_M(0) = \{ p \in \Ass M : \dim R / p = d \}.
\]
Hence
\[
\Supp M / U_M(0) = \bigcup_{p \in \Ass M, \dim R/p = d} V(p).
\]
The set \( \Supp M / U_M(0) \) is called the unmixed support of \( M \) and denoted by \( \Usupp M \).

Lemma 3.2. Let \( p \in \Supp M \). Then \( p \in \Usupp M \) if and only if \( p \supseteq \Ann H^d_m(M) \). In particular, \( \Usupp M = V(\Ann H^d_m(M)) \).

Proof. We have by [BS] that
\[
\Att H^d_m(M) = \{ q \in \Ass M : \dim R / q = d \}.
\]
Moreover, the set of all minimal prime ideals containing \( \Ann H^d_m(M) \) and the set of all minimal elements of \( \Att H^d_m(M) \) are the same. Therefore
\[
V(\Ann H^d_m(M)) = \bigcup_{p \in \Ass M, \dim R/p = d} V(p) = \Usupp M.
\]
\[\square\]

There are some nice relations between associated primes and the supports of \( M \) and of its \( \widehat{m} \)-adic completion \( \widehat{M} \). For example, \( \Ass \hat{M} = \{ \hat{p} \cap R : \hat{p} \in \Ass \hat{R} \widehat{M} \} \) and \( \Supp \hat{M} = \{ \hat{p} \cap R : \hat{p} \in \Supp \hat{R} \widehat{M} \} \). Moreover,
\[
\{ p \in \Ass M : \dim R / p = d \} = \{ \hat{p} \cap R : \hat{p} \in \Ass \hat{R} M, \dim \hat{R} / \hat{p} = d \}.
\]
So, it is natural to ask about the relation between \( \Usupp \hat{M} \) and \( \Usupp \hat{R} \widehat{M} \). First we have the following lemma.
Lemma 3.3. \( \text{Usupp} M \supseteq \{ \hat{p} \cap R : \hat{p} \in \text{Usupp}_R \hat{M} \} \).

Proof. Let \( \hat{p} \in \text{Usupp} \hat{M} \). Then \( \hat{p} \supseteq \hat{q} \) for some \( \hat{q} \in \text{Ass}_R \hat{M} \) satisfying \( \dim \hat{R}/\hat{q} = d \). It follows that \( \hat{q} \cap R \in \text{Ass} M \) and \( \dim R/(\hat{q} \cap R) = d \). Since \( \hat{p} \cap R \supseteq \hat{q} \cap R \), we get \( \hat{p} \cap R \in \text{Usupp} M \). \( \Box \)

In general, the two sets \( \text{Usupp} M \) and \( \{ \hat{p} \cap R : \hat{p} \in \text{Usupp}_R \hat{M} \} \) are different (cf. Proposition 4.6). The following theorem shows that they are the same if and only if \( H^d_m(M) \) satisfies the property (*) for \( H^d_m(M) \) in term of systems of parameters is also given.

Theorem 3.4. The following statements are equivalent:

(i) \( H^d_m(M) \) satisfies the property (*).

(ii) \( \text{Usupp} M = \{ \hat{p} \cap R : \hat{p} \in \text{Usupp}_R \hat{M} \} \).

(iii) For every sequence \( x_1, \ldots, x_d \) of elements in \( m, (x_1, \ldots, x_d) \) is a system of parameters of \( H^d_m(M) \) if and only if it is a system of parameters of \( M/U_M(0) \).

Proof. (i) \( \Leftrightarrow \) (ii). We get by Lemma 3.2 that \( V(\text{Ann} H^d_m(M)) = \text{Usupp} M \) and \( V(\text{Ann}_R H^d_m(M)) = \text{Usupp}_R \hat{M} \). Therefore the condition (ii) is equivalent to the condition

\[
V(\text{Ann} H^d_m(M)) = \{ \hat{p} \cap R : \hat{p} \in V(\text{Ann}_R H^d_m(M)) \}.
\]

So, our claim follows by Proposition 2.2.

(i) \( \Rightarrow \) (iii). Let \( (x_1, \ldots, x_d) \) be a system of parameters of \( H^d_m(M) \). Let \( I \) be the ideal generated by \( x_1, \ldots, x_d \). For each prime ideal \( p \) of \( R \) containing \( I + \text{Ann} H^d_m(M) \), we have by (i) that

\[
p = \text{Ann}(0 : H^d_m(M), p) \supseteq \text{Ann}(0 : H^d_m(M), I).
\]

Therefore

\[
\text{rad} (I + \text{Ann} H^d_m(M)) = \bigcap_{p \supseteq I + \text{Ann} H^d_m(M)} p \supseteq \text{rad} (\text{Ann}(0 : H^d_m(M), I)).
\]

Hence \( \text{rad} (I + \text{Ann} H^d_m(M)) = \text{rad} (\text{Ann}(0 : H^d_m(M), I)) \). Since \( (x_1, \ldots, x_d) \) is a system of parameters of \( H^d_m(M) \), the length of \( (0 : H^d_m(M), I) \) is finite. So, we get by the last equality that \( I + \text{Ann} H^d_m(M) \) is a \( m \)-primary ideal. Since \( \text{rad} (\text{Ann}(M/U_M(0))) = \text{rad} (\text{Ann}(M/U_M(0))) \), (cf. Lemma 3.2), the ideal \( I + \text{Ann}(M/U_M(0)) \) is \( m \)-primary. Hence \( (x_1, \ldots, x_d) \) is a system of parameters of \( M/U_M(0) \). Conversely, assume that \( (x_1, \ldots, x_d) \) is a system of parameters of \( M/U_M(0) \). Then \( I + \text{Ann}(M/U_M(0)) \) is \( m \)-primary, and hence so is \( I + \text{Ann} H^d_m(M) \). Therefore \( I(0 : H^d_m(M), I) < \infty \), i.e. \( (x_1, \ldots, x_d) \) is a system of parameters of \( H^d_m(M) \).

(iii) \( \Rightarrow \) (i). Let \( p \in V(\text{Ann} H^d_m(M)) \). Assume that \( \text{N-dim} (0 : H^d_m(M), p) = d - r \). By [TZ, Proposition 2.10], there exist \( x_1, \ldots, x_r \in p \) which form a part of a system of parameters of \( H^d_m(M) \) in \( p \), and it is clear that this part of a system of parameters is maximal. Let

\[
0 : H^d_m(M) (x_1, \ldots, x_r)R = A_1 + \ldots + A_n
\]

be a minimal secondary representation of \( 0 : H^d_m(M) (x_1, \ldots, x_r)R \), where \( A_i \) is \( q_i \)-secondary. For each element \( y \in m \), note that \( y \) is a parameter element of
The following statements are equivalent:

(i) $\text{Supp } M$ is catenary.

(ii) $H^d_m(M)$ satisfies the property (*)

Before proving Theorem 4.1, we need the following lemmas.

Lemma 4.2. Assume that $R$ is complete with respect to the $m$-adic topology and $M$ a finitely generated $R$-module such that $\dim R/p = d$ for all $p \in \text{Ass } M$. Then $\dim R/p = d - r$ for any part of system of parameters $(x_1, \ldots, x_r)$ of $M$ and any minimal associated prime ideal $p$ of $M/(x_1, \ldots, x_r)M$.

Proof. As $(x_1, \ldots, x_r)$ is a part of a system of parameters of $M$, we have

$$\dim(R/\text{Ann } M + (x_1, \ldots, x_r)R) = \dim(M/(x_1, \ldots, x_r)M) = d - r.$$ 

Moreover $p$ is a minimal prime divisor of $\text{Ann } M + (x_1, \ldots, x_r)R$, so that $\dim R/p$ is at most $d - r$. There is a minimal prime divisor $q$ of $\text{Ann } M$ which is contained
in \( p \). As \( q \) belongs to \( \text{Ass}_R(M) \) it follows from our assumptions that \( \dim R/q = d \). Moreover \( p \) is a minimal prime divisor of \( q + (x_1, \ldots, x_r)R \), so that \( \text{ht}(p/q) \) does not exceed \( r \) (cf. [Mat, Theorem 18]). As \( R/q \) is catenary of dimension \( d \), it follows that \( \dim \hat{R}/p = d - \text{ht}(p/q) \) at least \( d - r \). Therefore \( \dim \hat{R}/p = d - r \). \( \square \)

**Lemma 4.3.** Let \( p \in V(\text{Ann} H^d_m(M)) \) such that \( \dim M_p + \dim R/p = d \). Then \( \text{Ann}(0 : H^d_m(M)) p = p \).

**Proof.** Let \( p \supseteq \text{Ann} H^d_m(M) \) be a prime ideal such that \( \dim M_p + \dim R/p = d \). Set \( \dim R/p = d - r \). It follows by the hypothesis that \( \dim M_p = r \). Therefore there exists a prime ideal \( q \in \text{Ass} M \) such that \( q \subseteq p \) and \( \text{ht}(p/q) = r \). Since

\[
\dim R/q \geq \dim R/p + \text{ht}(p/q) = d,
\]

it follows that \( \dim R/q = d \). It should be noted that \( \dim \hat{R}/p \hat{R} = \dim R/p = d - r \). So, there exists a prime ideal \( \hat{p} \subseteq \text{Ass}_R \hat{R}/p \hat{R} \) such that \( \dim \hat{R}/\hat{p} = d - r \). Since \( \hat{p} \subseteq \text{Ass}_R \hat{R}/p \hat{R} \), we get \( \hat{p} \cap p \subseteq R/p, \) i.e. \( \hat{p} \cap R = p \). Note that the natural map \( R \rightarrow \hat{R} \) is faithfully flat, and therefore the going down theorem holds (see [Mat, Theorem 4]). So, there exists a prime ideal \( \hat{q} \in \text{Spec} \hat{R} \) such that \( \hat{q} \cap R = q, \hat{q} \subseteq \hat{p} \) and \( \text{ht}(\hat{p}/\hat{q}) \geq r \). These facts imply that

\[
d = \dim R/q = \dim \hat{R}/\hat{q} \hat{R} \geq \dim \hat{R}/\hat{q} = \dim \hat{R}/\hat{p} + \text{ht}(\hat{p}/\hat{q}) \geq d - r + r = d.
\]

Hence \( \dim \hat{R}/\hat{q} = d \). Moreover, since the natural homomorphism \( R_q \rightarrow \hat{R}_q \) is faithfully flat and \( M_q \neq 0 \), we have

\[
M_q \otimes_{R_q} \hat{R}_q \cong \hat{M}_q \neq 0.
\]

Hence \( \hat{q} \in \text{Supp} \hat{R}.M \). Since \( \dim \hat{R}/\hat{q} = d \) and \( \hat{p} \supseteq \hat{q} \), we have \( \hat{p} \supseteq \text{Ann}_R H^d_m(M) \). Therefore we get by the Matlis duality that \( \text{Ann}_R(0 : H^d_m(M)) \hat{p} = \hat{p} \). Now we have

\[
p \subseteq \text{Ann}(0 : H^d_m(M)) p \subseteq \text{Ann}_R(0 : H^d_m(M)) \hat{p} \cap R = \hat{p} \cap R = p.
\]

Thus \( \text{Ann}(0 : H^d_m(M)) p = p \). \( \square \)

**Proof of Theorem 4.1.** (i) \( \Rightarrow \) (ii) follows by Lemma 4.3 and Lemma 3.2.

(ii) \( \Rightarrow \) (i). Let \( p \in \text{Usupp} M \). We need to show that \( \dim R/p + \dim M_p = d \). The case \( p = m \) is trivial. Assume that \( p \neq m \). Let \( \dim R/p = d - r \). Then it is enough to prove \( \dim M_p = r \). Since \( p \supseteq \text{Ann} M/U_M(0) \), we have

\[
\dim \left( M/U_M(0)/p(M/U_M(0)) \right) = \dim R/p = d - r.
\]

So, there exists a maximal part of a system of parameters \((x_1, \ldots, x_r)\) of \( M/U_M(0) \) in \( p \). Since \( p \in \text{Usupp} M \), there exists by Theorem 3.4, (i) \( \Leftrightarrow \) (ii) a prime ideal \( \hat{p} \in \text{Usupp} \hat{M} \) such that \( \hat{p} \cap R = p \). Set \( \hat{M}_1 = \hat{M}/U_{\hat{M}}(0) \). Since \((x_1, \ldots, x_r)\) is a part of a system of parameters of \( M/U_M(0) \), it is a part of a system of parameters of the \( m \)-adic completion \( \hat{M}/U_{\hat{M}}(0) \) of \( M/U_M(0) \). Because \( \hat{M}_1 \) is a quotient of \( \hat{M}/U_{\hat{M}}(0) \) and \( \dim \hat{M}_1 = \dim \hat{M}/U_M(0) \), it follows that \((x_1, \ldots, x_r)\) is a part of a system of parameters of \( \hat{M}_1 \). Note that \( \hat{p} \in \text{Supp} \hat{M}_1/x_1, \ldots, x_{r-1}\). Therefore
\(\tilde{p} \supseteq \tilde{p}_1\) for some minimal prime ideal \(\tilde{p}_1 \in \text{Supp}_R \tilde{M}_1/(x_1, \ldots, x_{r-1})\tilde{M}_1\). Since \(x_r\) is a parameter element of \(\tilde{M}_1/(x_1, \ldots, x_{r-1})\tilde{M}_1\), we get by Lemma 4.2 that \(x_r \notin \tilde{p}_1\). Set \(p_1 = \tilde{p}_1 \cap R\). Then \(x_r \notin p_1\). Therefore \(p \supseteq p_1\) and \(p \neq p_1\). By the same arguments, there exists a minimal prime ideal \(\tilde{p}_2 \in \text{Supp}_R \tilde{M}_1/(x_1, \ldots, x_{r-2})\tilde{M}_1\) such that \(\tilde{p}_1 \supseteq \tilde{p}_2\). Set \(p_2 = \tilde{p}_2 \cap R\). Then \(p \supseteq p_2\) and \(p \neq p_2\) since \(x_{r-1} \in p_1 \setminus p_2\). Continue the above process, after \(r\) steps, we get a chain \(p \supseteq p_1 \supseteq p_2 \ldots \supseteq p_r\) of prime ideals containing \(\text{Ann } M\) such that \(p_i \neq p_{i+1}\) for all \(i = 1, \ldots, r - 1\). Therefore \(\dim M_p = r\).

**Proof of the main theorem.** The equivalences between (i), (iii) and (iv) are proved by Theorem 3.4. The equivalence between (i) and (ii) is proved by Theorem 4.1. □

**Remark 4.4.** The catenaricity of \(\text{Usupp } M\) is equivalent to the property (*) for the top local cohomology module \(H_d^d(M)\) (see Theorem 4.1), but it is not related to the property (*) of other local cohomology modules of \(M\). In fact, let \(R\) be the Noetherian local domain constructed by Ferrand and M. Raynaud [FR] of dimension 2 such that the \(m\)-adic completion has an associated prime of dimension 1. It is clear that \(\text{Supp } R = \text{Usupp } R\) is catenary, but \(H^d_m(R)\) does not satisfy the property (*).

Let \(0 = M_0 \subset M_1 \subset \ldots \subset M_t = M\) be the filtration of submodules of \(M\), where \(M_{i-1}\) is the largest submodule of \(M_i\) of dimension less than \(\dim M_i\) for \(i = 1, \ldots, t\). Such a filtration always exists, and it is unique. We call this filtration to be the dimension filtration of \(M\) (cf. [CN2]). Let \(\dim M_i = d_i\) for \(i = 1, \ldots, t\). Then it is easy to check that

\[
\text{Supp } M = \bigcup_{i=1, \ldots, t} \text{Supp } M_i/M_{i-1}.
\]

For each \(i = 1, \ldots, t\), it should be noted that \(\dim R/p = d_i\) for all \(p \in \text{Ass } M_i/M_{i-1}\). Therefore we obtain by Theorem 4.1 the following result.

**Corollary 4.5.** \(\text{Supp } M\) is catenary if and only if \(H^d_m(M_i/M_{i-1})\) satisfies the property (*) for all \(i = 1, \ldots, t\).

Now we examine some non-catenary Noetherian local domains. Note that any domain of dimension 2 is catenary, but there exist non-catenary Noetherian local domains of dimension \(d\) for any \(d \geq 3\) (cf. [B, (8)]).

**Proposition 4.6.** Let \(R\) be a non-catenary Noetherian local domain of dimension 3. Set

\[
U = \{p \in \text{Spec } R : \dim R/p + \text{ht } p = 2\};
V = \{p \in \text{Spec } R : \dim R/p + \text{ht } p = 3\}.
\]

Then the following statements are true

(i) \(\text{Usupp } R = \text{Spec } R = U \cup V\) and \(U, V \neq \emptyset\).

(ii) \(\text{Ann}(0 : H^d_m(R) \ p) = p\) for all \(p \in V\). But \(\text{Ann}(0 : H^d_m(R) \ p) \neq p\) for all \(p \in U\).

(iii) For each \(p \in V\), there always exists \(\tilde{p} \in \text{Supp } R/U_R(0)\) such that \(\tilde{p} \cap R = p\). But for each \(p \in U\), there does not exist \(\tilde{p} \in \text{Supp } R/U_R(0)\) such that \(\tilde{p} \cap R = p\).
(iv) \( \text{N-dim} H^2_m(R) = 2 \) and \( \dim R / \text{Ann} H^2_m(R) = 3 \).

**Proof.** (i). This is clear since \( R \) is a non-catenary domain.

(ii). It follows by the proof of Theorem 4.1 that \( \text{Ann}(0 : H^3_m(R) p) \neq p \) for all \( p \in U \) and \( \text{Ann}(0 : H^3_m(R) p) = p \) for all \( p \in V \).

(iii). This follows by (ii) and by the proof of Theorem 3.4.

(iv). Let \( p \in U \). Then \( \dim R/p = 1 \). Let \( \hat{p} \in \text{Spec} \hat{R} \) such that \( \hat{p} \cap R = p \). Then \( \dim \hat{R}/\hat{p} = 1 \). It follows by (iii) that \( \hat{p} \not\supset \text{Ann}_R H^3_m(R) \). Moreover, \( \text{ht} \hat{p} \geq \text{ht} p = 1 \) by the going down theorem [Mat, Theorem 4]. Therefore there exists \( \hat{q} \in \text{Ass} \hat{R} \) such that \( \hat{q} \subset \hat{p} \) and \( \hat{q} \neq \hat{p} \). Hence \( \dim \hat{R}/\hat{q} \geq 2 \). Since \( \hat{p} \not\supset \text{Ann}_\hat{R} H^3_m(R) \), it follows that \( \dim \hat{R}/\hat{q} = 2 \). So we have by [BS, Corollary 11.3.3] that \( \hat{q} \in \text{Att}_\hat{R} H^2_m(R) \) and hence \( \hat{q} \supset \text{Ann}_\hat{R} H^2_m(R) \). Therefore \( \text{N-dim} H^2_m(R) = \dim \hat{R} / \text{Ann}_\hat{R} H^2_m(R) \geq 2 \). Note that \( \text{N-dim} H^2_m(R) \leq 2 \) by [CN1, Theorem 3.1]. So \( \text{N-dim} H^2_m(R) = 2 \). Since \( \hat{q} \in \text{Att}_\hat{R} H^2_m(R) \cap \text{Ass} \hat{R} \), we have

\[ \hat{q} \cap R \in \text{Att}_R H^2_m(R) \cap \text{Ass} R. \]

Since \( R \) is a domain, we have \( \hat{q} \cap R = 0 \). It follows that \( 0 = \text{Ann} H^2_m(R) \). Thus \( \dim R / \text{Ann} H^2_m(R) = 3 \). \( \square \)

**Acknowledgment.** We wish to express our gratitude to the referee for his/her useful suggestions and, especially, the shorter proof of Lemma 4.2.

**References**

[B] M. Brodmann, *A particular class of regular domains*, J. Algebra, 54, (1978), 366-373.

[BS] M. Brodmann and R. Y. Sharp, “Local cohomology: an algebraic introduction with geometric applications”, Cambridge University Press, 1998.

[CN1] N. T. Cuong and L. T. Nhan, *On the Noetherian dimension of Artinian modules*, Vietnam J. Maths., (2)30 (2002), 121-130.

[CN2] N. T. Cuong and L. T. Nhan, *On pseudo Cohen-Macaulay and pseudo generalized Cohen-Macaulay modules*, J. Algebra, 267 (2003), 156-177.

[FR] D. Ferrand and M. Raynaud, *Fibres formelles d’un anneau local Noetherian*, Ann. Sci. E’cole Norm. Sup., (4)3 (1970), 295-311.

[HIO] M. Herrmann, S. Ikeda and U. Orbanz, “Equimultiplicity and Blowing up”, Springer - Verlag, 1988.

[K1] D. Kirby, *Artinian modules and Hilbert polynomials*, Quart. J. Math. Oxford, (2)24 (1973), 47-57.

[K2] D. Kirby, *Dimension and length of Artinian modules*, Quart. J. Math. Oxford, (2)41 (1990), 419-429.

[Mac] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symposia Mathematica, 11 (1973), 23-43.

[Mat] H. Matsumura, “Commutative Algebra”, Second Edition (Benjamin, 1980).

[Na] M. Nagata, *Local rings*, Interscience, New York, 1962.

[R] R. N. Roberts, *Krull dimension for Artinian modules over quasi local commutative rings*, Quart. J. Math. Oxford, (2)26 (1975), 269-273.
[Sh] R. Y. Sharp, *A method for the study of Artinian modules with an application to asymptotic behaviour*, Commutative Algebra (Math. Sciences Research Inst. Publ. No. 15, Springer Verlag), (1989), 443-465.

[TZ] Z. Tang and H. Zakeri, *Co-Cohen-Macaulay modules and modules of generalized fractions*, Comm. Algebra., (6)22 (1994), 2173-2204.