Perturbation Theory for the Thermal Hamiltonian: 1D Case

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Abstract
This work continues the study of the thermal Hamiltonian, initially proposed by J. M. Luttinger in 1964 as a model for the conduction of thermal currents in solids. The previous work (De Nittis and Lenz in Spectral theory of the thermal Hamiltonian, 1D case, 2020) contains a complete study of the “free” model in one spatial dimension along with a preliminary scattering result for convolution-type perturbations. This work complements the results obtained in De Nittis and Lenz (2020) by providing a detailed analysis of the perturbation theory for the one-dimensional thermal Hamiltonian. In more detail, the following results are established: the regularity and decay properties for elements in the domain of the unperturbed thermal Hamiltonian; the determination of a class of self-adjoint and relatively compact perturbations of the thermal Hamiltonian; the proof of the existence and completeness of wave operators for a subclass of such potentials.

Keywords Thermal Hamiltonian · Self-adjoint extensions · Spectral theory · Scattering theory

Mathematics Subject Classification Primary: 81Q10; Secondary: 81Q05 · 81Q15 · 33C10

1 Introduction

In order to study the thermal transport in the matter, Luttinger proposed in 1964 a model which allows a “mechanical” derivation of the thermal coefficients [4]. Such a
model has been eventually studied and generalized successfully in various later works such as [11,12]. The essential insight of the Luttinger’s model is to describe the effect of the thermal gradient in the matter by a 

*fictitious* gravitational field which affects the dynamics of a charged particle moving in a *background* material.

In the absence of thermal fields, and ignoring all physical constants, the dynamics of a one-dimensional quantum particle is described by the Hamiltonian

\[
h_V := p^2 + V
\]

(1.1)

where \( p := -i \frac{d}{dx} \) is the *momentum operator* and \( V \) is the *background* (or electro-static) *potential* which takes care of the interaction of the particle with the atomic structure of the matter. In the absence of interaction with matter (\( V = 0 \)), the dynamics is simply described by \( h_0 = p^2 \). The effect of the thermal field is introduced in the model by a *thermal potential* which is proportional to the local content of energy. Since the latter is given by the Hamiltonian (1.1) itself, one ends with the following (effective) model

\[
H_{T,V} := h_V + \lambda \{ h_V, x \}
\]

(1.2)

known as *thermal Hamiltonian* or Luttinger’s Hamiltonian. We will refer to [1, Section 1.1], and references therein, for more details on the physical justification of (1.2). Here, it is worth to point out that mathematically the thermal potential is introduced by the anti-commutator \( \{, \} \) between \( h_V \) and the *position operator* \( x \), and that the parameter \( \lambda > 0 \) describes the strength of the thermal field.

The Hamiltonian (1.2) can be rearranged in the form

\[
H_{T,V} := H_T + W_V
\]

(1.3)

where

\[
H_T := h_0 + \lambda \{ h_0, x \}
\]

(1.4)

is the *thermal Hamiltonian* in the absence of a background potential and

\[
W_V(x) := (1 + \lambda x) V(x), \quad x \in \mathbb{R}
\]

(1.5)

is the resulting potential that combines the effects of the thermal field and the electrostatic interaction with the matter. The study of the spectral properties of the Hamiltonian \( H_T \) has been the central argument of [1]. The main aim of this work is to provide a satisfactory description of the spectral theory for the perturbed Hamiltonian \( H_{T,V} \) and to derive the scattering theory [3,9,13] for the pair \((H_T, H_{T,V})\) for a sufficiently general class of background potentials \( V \).

Before describing the new results, let us recall some essential facts about the “unperturbed” operator \( H_T \). On sufficiently regular functions \( \psi : \mathbb{R} \to \mathbb{C} \), the operator \( H_T \) acts as follows:
\[(H_T \psi)(x) = -(1 + \lambda x)\psi''(x) - \lambda \psi'(x)\] (1.6)

where \(\psi'\) and \(\psi''\) are the first and the second derivatives of \(\psi\), respectively. However, when restricted to the Schwartz space \(S(\mathbb{R})\), \(H_T\) turns out to be symmetric but not essentially self-adjoint. In fact, the operator initially defined on \(S(\mathbb{R})\) by (1.6) admits a one-parameter family of self-adjoint extensions. However, it turns out that all these self-adjoint extensions are unitarily equivalent and, without loss of generality, one can focus on a specific "canonical" realization [1, Theorem 1.1]. Such a realization is obtained by considering the dense domain

\[D_{T,0} := S(\mathbb{R}) + \mathbb{C}[\tilde{\kappa}_0]\] (1.7)

and the prescription

\[(H_T(\psi + c\tilde{\kappa}_0))(x) = (H_T \psi)(x) + c\lambda \tilde{\kappa}_1(x), \quad \psi \in S(\mathbb{R})\] (1.8)

where the term \(H_T \psi\) is given by (1.6) and

\[
\tilde{\kappa}_0(x) := -\sqrt{\frac{8}{\pi}} \text{sgn} \left( x + \frac{1}{\lambda} \right) \text{kei} \left( 2\sqrt{\left| x + \frac{1}{\lambda} \right|} \right),
\]

\[
\tilde{\kappa}_1(x) := \sqrt{\frac{8}{\pi}} \text{ker} \left( 2\sqrt{\left| x + \frac{1}{\lambda} \right|} \right).
\] (1.9)

In (1.9) \(\text{kei}\) and \(\text{ker}\) denote the irregular Kelvin functions of 0-th order (cf. [5, Chap. 55] or [6, Sect. 10.61]) while the sign function is defined by \(\text{sgn}(x) := x/|x|\) if \(x \neq 0\) and \(\text{sgn}(0) := 0\). It turns out that the operator defined by (1.8) is essentially self-adjoint on the domain (1.7), and this fact provides a rigorous definition for the thermal Hamiltonian.

**Definition 1.1** (1-D unperturbed thermal Hamiltonian) The unperturbed thermal Hamiltonian, still denoted with \(H_T\), is the self-adjoint operator on \(L^2(\mathbb{R})\) defined by (1.7) and (1.8) on the domain

\[D_T := \overline{D_{T,0}} \| H_T\]

obtained by the closure of \(D_{T,0}\) with respect to the graph norm \(\| \psi \|_{H_T} := \| \psi \|_{L^2} + \| H_T \psi \|_{L^2}\).

In view of [1, Theorem 1.1], we know that \(H_T\) has a purely absolutely continuous spectrum given by

\[\sigma(H_T) = \sigma_{a.c.}(H_T) = \mathbb{R}\] (1.10)

independently of \(\lambda > 0\).
We are now in position to present the main results of this work. For that, let us recall the definition of the critical point

$$x_c \equiv x_c(\lambda) := -\lambda^{-1}$$  \hspace{1cm} (1.11)

which plays an important role in the singular behavior of the dynamics generated by the unperturbed operator $H_T$ [1]. The first result concerns the determination of a class of self-adjoint perturbation of $H_T$.

**Theorem 1.2** (Self-adjoint perturbations) Let $H_T$ be the unperturbed thermal Hamiltonian described in Definition 1.1. Let $V : \mathbb{R} \to \mathbb{R}$ be a background potential such that

$$V(x) = \frac{V_1(x)}{|x - x_c|^\frac{3}{4}} + \frac{V_2(x)}{|x - x_c|}$$

with $V_1 \in L^2(\mathbb{R})$ and $V_2 \in L^\infty(\mathbb{R})$. Then, the perturbed thermal Hamiltonian $H_{T,V}$ given by (1.3), with potential $W_V$ given by (1.5), is self-adjoint on the domain $\mathcal{D}_T$.

**Remark 1.3** Although Theorem 1.2 stipulates that $H_{T,V}$ is self-adjoint for a large class of background potentials $V$, from a physical point of view this result is not yet totally satisfactory. In fact, the standard model for the dynamics of a charged particle in a (semi-)metal is $hV_{\text{per}} := p^2 + V_{\text{per}}$ with $V_{\text{per}}$ a periodic background potential. However, every $V_{\text{per}} \neq 0$ does not meet the conditions of Theorem 1.2, and as a consequence the question of the self-adjointness of $H_{T,V_{\text{per}}}$ remains open. This is not an irrelevant fact since $H_{T,V_{\text{per}}}$ is the relevant model (tacitly) considered in [11,12] for the derivation of thermal conductivity in condensed matter systems. It is also worth noting that Theorem 1.2 allows background potentials which are singular around the critical point $x_c$. \Halmos

The second main result describes a class of relatively compact perturbations. For that we will need to introduce the family of resolvents

$$R_z(H_T) := (H_T - z\mathbf{1})^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$  

**Theorem 1.4** (Relatively compact perturbations) Let $H_T$ be the unperturbed thermal Hamiltonian described in Definition 1.1. Let $V : \mathbb{R} \to \mathbb{R}$ be a background potential such that

$$V(x) = \frac{V_1(x)}{|x - x_c|^\frac{3}{4}}$$

with $V_1 \in L^2(\mathbb{R})$. Then, $W_V R_z(H_T)$ is Hilbert–Schmidt (hence compact) for every $z \in \mathbb{C} \setminus \mathbb{R}$, where the potential $W_V$ is given by (1.5).

By combining Theorem 1.4 with the Weyl theorem about the stability of the essential spectrum [10, Theorem XIII.14], one obtains the following result.
Corollary 1.5 (Essential spectrum) Let $V$ be a background potential as in Theorem 1.4. Then, the essential spectrum of the perturbed thermal Hamiltonian $H_{T,V}$ is $\sigma_{\text{ess}}(H_{T,V}) = \sigma_{\text{ess}}(H_T) = \mathbb{R}$.

The question of the existence of embedded eigenvalues is not answered by Corollary 1.5 and is left open for future investigations.

The final result concerns the scattering theory for the pair $(H_{T,V}, H_{T,V})$. Let us recall the (formal) definition of the wave operators [9, Section XI.3]

$$\Omega_{\pm}(V) := s - \lim_{t \to \mp \infty} e^{i H_{T,V} t} e^{-i H_T t},$$

where the limits are meant in the strong operator topology. It is worth noting that in the definition (1.12) we have tacitly used the fact that the spectral projection on the absolutely continuous part of the spectrum of $H_T$ coincides with the identity in view of (1.10). The scattering matrix is defined by

$$\mathcal{S}(V) := \Omega_-(V)^* \Omega_+(V).$$

Theorem 1.6 (Scattering theory) Let $V : \mathbb{R} \to \mathbb{R}$ be a background potential such that both $V$ and $|V|^{1/2}$ satisfy the conditions of Theorem 1.4. Then, the wave operators $\Omega_{\pm}(V)$ exist and the scattering matrix $\mathcal{S}(V)$ is unitary.

Theorem 1.6 boils up the application of the celebrated Kuroda–Birman Theorem [9, Theorem XI.9] which guarantees the existence and the completeness of the wave operators. In particular, the unitarity of $\mathcal{S}(V)$ is a consequence of the completeness of the wave operators.

A special class of bounded background potentials that meet the conditions of Theorems 1.4 and 1.6 are described in Remarks 3.5 and 4.5, respectively.

It is worth to end this introductory section with few words about the strategy used for the proofs of the main results described above. Instead of working with the “physical” operator $H_T$, we found more convenient to work with the unitarily equivalent (up to a scale factor) operator

$$T := \lambda^{-1} S_{\lambda} H_T S_{\lambda}^*$$

obtained from $H_T$ via the unitary shift

$$(S_{\lambda} \psi)(x) := \psi \left( x - \frac{1}{\lambda} \right) = \psi(x + x_c), \quad \psi \in L^2(\mathbb{R}).$$

The advantage relies on the fact that $T$ has a simpler expression with respect to $H_T$. In fact, at least formally, one has that $T = pxp$. It turns out that Theorems 1.2, 1.4 and 1.6 are nothing more that the transposition via the conjugation by $S_{\lambda}$ of the equivalent results proved for $T$ in Propositions 3.1, 3.3 and 4.2, respectively. Anyway, the passage...
from the results concerning $T$ to the related results concerning $H_T$ is described in some
detail in Remarks 3.2, 3.5 and 4.5.

**Structure of the paper** In Sect. 2, we recall some basic result for the operator $T$
originally obtained in [1] and we provide new results about the regularity and the
decay of the elements of the domain of $T$. Section 3 contains the results about the
self-adjoint and relatively compact perturbations of the operator $T$. Section 4 provides
the results about the scattering theory. Appendix 4 elaborates on the construction of a
unitary operator used in the definition of $T$ and it’s domain.

## 2 Analysis of the domain

In this section, we will provide some result about regularity and decay properties for
element in the domain of the operator $T$. Such results can be immediately transported
to elements in the domain of $H_T$ in view of the unitary mapping (1.14).

### 2.1 Basic facts about the unperturbed operator

We will start by recalling some important result concerning the spectral theory of the
operator $T$ given by (1.14). All the information presented here is taken from [1].

An important role for the study of the operator $T$ is played by the bounded operator
$B$ initially defined on elements $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by the integral formula (see
Appendix 4)

\[
(B\psi)(x) := \int_{\mathbb{R}} dy \, B(x, y) \psi(y)
\]

with kernel

\[
B(x, y) := i \frac{\text{sgn}(x) - \text{sgn}(y)}{2} J_0 \left( 2\sqrt{|xy|} \right).
\]

where $J_0$ denotes the *Bessel function* of the first kind [2]. We will use the symbol $B$
to denote the unique linear, bounded extension the dense defined operator (2.1). It turns
out that $B$ is a unitary involution on $L^2(\mathbb{R})$, i.e., $B = B^* = B^{-1}$.

Let $x$ be the position operator on $L^2(\mathbb{R})$, acting as multiplication by $x$ on its natural
domain

\[
Q(\mathbb{R}) := \left\{ \psi \in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} dx \, x^2 |\psi(x)|^2 < +\infty \right\}.
\]

The involution $B$ introduced above intertwines between the operator $T$ and the position
operators $x$. In fact, it holds true that $T = -BxB$, and as a consequence the domain
of $T$ can be described a $\mathcal{D}(T) = B[Q(\mathbb{R})]$. The construction of $B$, and an overview
of its properties are given in Appendix 4.
The resolvent of $T$ admits an explicit integral expression when evaluated on element of the dense domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Let $z \in \mathbb{C} \setminus \mathbb{R}$ and consider the polar representation $z = |z| e^{\pm i \phi}$ with $0 < \phi < \pi$. Let $R_z(T) := (T - z \mathbf{1})^{-1}$ be the resolvent of $T$ at $z$. If $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, then it holds true that

\[
(R_z(T)\psi)(x) = \int_{\mathbb{R}} dy \ (\text{sgn}(x) + \text{sgn}(y)) \cdot F_z(x, y) \psi(y) \quad (2.3)
\]

where

\[
F_z(x, y) := I_0 \left(2 \sqrt{|z|} \min\{|x|, |y|\} e^{\pm i \left[\frac{\phi}{2} - \frac{\pi}{4} \text{sgn}(x) + 1\right]}\right) 
\times \ K_0 \left(2 \sqrt{|z|} \max\{|x|, |y|\} e^{\pm i \left[\frac{\phi}{2} - \frac{\pi}{4} \text{sgn}(x) + 1\right]}\right) \quad (2.4)
\]

with $I_0$ and $K_0$ are the modified Bessel functions of the first and second kind, respectively. In the special case $z = \pm i$, by using [6, eq. 10.61.1 & eq. 10.61.2], the formula above reduces to

\[
F_{\pm i}(x, y) := \left(\text{ber} \left(2 \sqrt{\min\{|x|, |y|\}}\right) \mp i \text{sgn}(x) \text{bei} \left(2 \sqrt{\min\{|x|, |y|\}}\right)\right) 
\times \left(\text{ker} \left(2 \sqrt{\max\{|x|, |y|\}}\right) \mp i \text{sgn}(x) \text{kei} \left(2 \sqrt{\max\{|x|, |y|\}}\right)\right),
\]

with ber, bei, ker and kei the Kelvin functions of 0-th order [5, Chapter 55] or [6, Section 10.61]. In accordance with [6, Section 10.68], let us introduce the following notation:

\[
M_0(s) := \sqrt{\text{ber}(s)^2 + \text{bei}(s)^2}, \quad s \geq 0.
\]
\[
N_0(s) := \sqrt{\text{ker}(s)^2 + \text{kei}(s)^2}, \quad s \geq 0.
\]

This allows to rewrite the modulus of the kernel $F_{\pm i}$ as follows:

\[
|F|(x, y) := |F_{\pm i}(x, y)| = M_0 \left(2 \sqrt{\min\{|x|, |y|\}}\right) \ N_0 \left(2 \sqrt{\max\{|x|, |y|\}}\right) \quad (2.5)
\]

From (2.5), one infers immediately that $|F|$ is invariant under the reflections $x \mapsto -x$ and $y \mapsto -y$ and is independent on the sign in $\pm i$. For the study of the integrability properties of (2.5), we will make use of the inequalities

\[
M_0(s)^2 \leq C_M \frac{e^{\sqrt{2}s}}{s}, \quad \forall \ s > 0
\]
\[
N_0(s)^2 \leq C_N \frac{e^{-\sqrt{2}s}}{s},
\]
which can be deduced by the asymptotic expansions [6, eq. 10.67.9 & eq. 10.67.13]. The exact value of the positive constants $C_M$ and $C_N$ is not important for the purposes of this work.¹

For small arguments ($s \sim 0$), one has that the functions ber and bei are continuous around the origin with $\text{ber}(0) = 1$ and $\text{bei}(0) = 0$. Consequently, also $M_0$ is continuous in 0 and

$$\lim_{s \to 0^+} M_0(s) = M_0(0) = 1.$$  

On the other hand, $\text{kei}(0) = -\frac{\pi}{4}$, but ker, and in turn $N_0$, have a logarithmic pole in 0. By using the series expansions [6, eq. 10.65.5], one can prove that

$$\lim_{s \to 0^+} \left( N_0(s) + \ln(s) \right) = \ln(2) - \gamma$$  

where $\gamma$ is the Euler–Mascheroni constant. For the next result that will play a crucial role in Sect. 3.2, we need to introduce the positive semi-axis $\mathbb{R}_+ := (0, +\infty)$ and the first quadrant $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$.

**Lemma 2.1** Let $w : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\}$ be a nonnegative function such that

$$\int_{\mathbb{R}_+} \frac{w(x)}{\sqrt{x}} \, dx < +\infty$$

and $|\mathcal{F}|$ the function defined by (2.5). Then, the integral

$$I_w := \iint_{\mathbb{R}_+^2} \, dx \, dy \, w(x) \, |\mathcal{F}|(x, y)^2 < +\infty$$

is finite.

**Proof** Let us split the integral $I_w$ as follows:

$$I_w = I_w(\Sigma_1) + I_w(\Sigma_2)$$

where we used the notation

$$I_w(\Sigma_i) := \iint_{\Sigma_i} \, dx \, dy \, w(x) \, |\mathcal{F}|(x, y)^2, \quad i = 1, 2$$

with $\Sigma_1 := \{(x, y) \in \mathbb{R}_+^2 | x < y\}$ and $\Sigma_2 := \{(x, y) \in \mathbb{R}_+^2 | y < x\}$. We start with the integral $I_w(\Sigma_1)$. In view of the definition of $|\mathcal{F}|$, one has that

$$I_w(\Sigma_1) := \iint_{\Sigma_1} \, dx \, dy \, w(x) \, M_0(2\sqrt{x})^2 \, N_0(2\sqrt{y})^2.$$  

¹ The best constant $C_M$ is fixed by the maximum of the function $g_M(s) := sM_0(s)^2e^{-\sqrt{2s}}$. A numerical inspection with Wolfram Mathematica (version 12.1) shows that one can choose $C_M > \frac{1.65}{2\pi}$. A similar argument also provides $C_N > \frac{\pi}{2}$. 

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From the inequalities (2.6), one infers that
\[ M_0 \left( 2\sqrt{x} \right)^2 N_0 \left( 2\sqrt{y} \right)^2 \leq C \frac{e^{2\sqrt{2x}} e^{-2\sqrt{2y}}}{\sqrt{xy}}, \quad \forall (x, y) \in \Sigma_1. \]

with \( C := \frac{CM}{4} > 0 \). As a consequence, one has that
\[ I_w(\Sigma_1) \leq C \int \int_{\Sigma_1} dx \, dy \, w(x) \frac{e^{2\sqrt{2x}} e^{-2\sqrt{2y}}}{\sqrt{xy}} < +\infty \quad (2.7) \]

where the first equality (second line) follows from the Tonelli’s theorem and the last inequality is guaranteed by hypothesis. The treatment of the integral \( I_w(\Sigma_2) \) is quite similar. Indeed, one has that
\[ I_w(\Sigma_2) \leq \int \int_{\Sigma_2} dx \, dy \, w(x) \frac{e^{2\sqrt{2y}} e^{-2\sqrt{2x}}}{\sqrt{xy}} < +\infty \quad (2.8) \]

This concludes the proof. \( \square \)

2.2 Regularity and decay

This section is devoted to the description of elements in \( D(T) \). Let us start from a preliminary result.

**Lemma 2.2** For every \( \psi \in D(T) \), it holds true that
\[ \| B\psi \|_{L^1}^2 \leq 2\pi \| T\psi \|_{L^2} \| \psi \|_{L^2}. \quad (2.9) \]

As a consequence, one has that \( B[D(T)] \subset L^1(\mathbb{R}) \).

**Proof** Let \( v > 0 \) be an arbitrary number and consider the identity
\[ \| B\psi \|_{L^1} = \int_{-\infty}^{+\infty} dx \, |(B\psi)(x)| \left( x^2 + v^2 \right)^{\frac{1}{2}} \left( x^2 + v^2 \right)^{-\frac{1}{2}}. \]

By the classical Cauchy–Bunyakovsky–Schwarz inequality, one gets
\[ \| B\psi \|_{L^1}^2 \leq \left( \int_{-\infty}^{+\infty} dx \, |(B\psi)(x)|^2 \left( x^2 + v^2 \right) \right) \left( \int_{-\infty}^{+\infty} dx \, \left( x^2 + v^2 \right)^{-1} \right). \]

By recalling the equivalence between \( T \) and the position operator described in Sect. 2.1, one can rewrite the first integral as follows:
\[ \int_{-\infty}^{+\infty} dx \, |(B\psi)(x)|^2 \left( x^2 + v^2 \right) = \| BT\psi \|_{L^2}^2 + v^2 \| B\psi \|_{L^2}^2. \]
By using the fact that $B$ is unitary and the known formula
\[ \int_{-\infty}^{+\infty} dx \left( x^2 + v^2 \right)^{-1} = \frac{\pi}{v}, \]
one gets
\[ \| B \psi \|_{L^1}^2 \leq \frac{\pi}{v} \left( \| T \psi \|_{L^2}^2 + v^2 \| \psi \|_{L^2}^2 \right) \]
independently of $v > 0$. By minimizing with respect to $v$, one obtains the bound (2.9).

The following lemma makes use of the explicit form of the kernel of $B$.

**Lemma 2.3 (Boundedness)** For every $\psi \in D(T)$, it holds true that
\[ \| \psi \|_{L^\infty} \leq \left( 2\pi \| T \psi \|_{L^2} \| \psi \|_{L^2} \right)^{\frac{1}{2}} \leq \sqrt{\pi} \left( \| T \psi \|_{L^2} + \| \psi \|_{L^2} \right). \quad (2.10) \]
As a consequence, one has that $D(T) \subset L^\infty(\mathbb{R})$.

**Proof** Since $B$ is an involution, one has that $\psi = B(B\psi)$, and in view of Lemma 2.2 one infers that $B\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Therefore, the action of $B$ on $B\psi$ can be computed via the integral formula (2.1) and (2.2). This provides
\[ |\psi(x)| = \left| \int_{\mathbb{R}} dy \mathcal{B}(x,y) (B\psi)(y) \right| \leq \int_{\mathbb{R}} dy \ |\mathcal{B}(x,y)| \ |(B\psi)(y)| \leq \int_{\mathbb{R}} dy \ |(B\psi)(y)| = \| B\psi \|_{L^1} \]
where we used the bound $|\mathcal{B}(x,y)| \leq 1$. Since the inequality above holds for every $x \in \mathbb{R}$, one gets $\| \psi \|_{L^\infty} \leq \| B\psi \|_{L^1}$. The rest of the proof follows from the inequality in Lemma 2.2. \qed

The next result describes the continuity properties of elements of $D(T)$.

**Lemma 2.4 (Hölder continuity)** Let $\psi \in D(T)$ and $x, y \in \mathbb{R} \setminus \{0\}$ with $\text{sgn}(x) = \text{sgn}(y)$. Then, it holds true that
\[ |\psi(x) - \psi(y)|^2 \leq G_k \ |x - y|^k \ \| T \psi \|_{L^2}^{1+k} \ |\psi|_{L^2}^{1-k} \quad (2.11) \]
for every $0 \leq k < 1$, with $G_k$ a constant depending only on $k$. As a consequence, the element of $D(T)$ is $\alpha$-Hölder continuous in $\mathbb{R} \setminus \{0\}$ with $0 \leq \alpha < \frac{1}{2}$. \qed
Proof By using the identity $\psi = B(B\psi)$ and the integral expression of $B$ as in the proof of Lemma 2.3, one obtains

$$|\psi(x) - \psi(y)| = \left| \int_{\mathbb{R}} ds \left[ B(x, s) - B(y, s) \right] (B\psi)(s) \right|$$

$$\leq \int_{\mathbb{R}} ds \left| J_0 \left( 2\sqrt{|sx|} \right) - J_0 \left( 2\sqrt{|sy|} \right) \right| |(B\psi)(s)|$$

where in the last inequality the hypothesis $\text{sgn}(x) = \text{sgn}(y)$ has been used. In view of the integral representation of the Bessel $J_0$

$$J_0(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\tau \ e^{iz\sin(\tau)} \quad (2.12)$$

one obtains

$$|\psi(x) - \psi(y)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} ds \int_{-\pi}^{\pi} d\tau \ g(x,y)(s, \tau) \ |(B\psi)(s)| \quad (2.13)$$

where

$$g(x,y)(s, \tau) := \left| e^{i2\sqrt{|sx|}\sin(\tau)} - e^{i2\sqrt{|sy|}\sin(\tau)} \right|.$$ 

For fixed values of $s$, and using the classical estimate $|1 - e^{is}| \leq |s|$, one gets

$$g(x,y)(s, \tau) \leq \min \left\{ 2, 2 \left| \sqrt{|sx|} - \sqrt{|sy|} \right| |\sin(\tau)| \right\}.$$ 

Since the minimum between two numbers is dominated by any weighted geometric mean of the same, one obtains that

$$g(x,y)(s, \tau) \leq 2^{1-k} \left( 2 \left| \sqrt{|sx|} - \sqrt{|sy|} \right| |\sin(\tau)| \right)^k, \quad \forall \ 0 \leq k \leq 1.$$ 

After inserting the last inequality in (2.13), one gets

$$|\psi(x) - \psi(y)| \leq C_k \int_{\mathbb{R}} ds \left| \sqrt{|sx|} - \sqrt{|sy|} \right|^k |(B\psi)(s)| \quad (2.14)$$

where

$$C_k := \frac{1}{\pi} \int_{-\pi}^{\pi} d\tau |\sin(\tau)|^k = \frac{4}{\pi} \int_{0}^{\frac{\pi}{2}} d\tau \ \sin(\tau)^k = \frac{2}{\sqrt{\pi}} \ \Gamma \left( \frac{(k+1)}{2} \right) \Gamma \left( \frac{(k+2)}{2} \right) \quad (2.15)$$
with $\Gamma$ denoting the gamma function. Observing that

$$
\left| \sqrt{|s x|} - \sqrt{|y|} \right|^k = |s|^\frac{k}{2} \left| \sqrt{|x|} - \sqrt{|y|} \right|^k
$$

one obtains

$$
|\psi(x) - \psi(y)| \leq C_k |x - y|^\frac{k}{2} \int_{\mathbb{R}} |s|^\frac{k}{2} \left| \frac{B\psi}{s} \right|.
$$

(2.16)

By inserting inside the integral the identity $(s^2 + v^2)^{\frac{1}{2}}(s^2 + v^2)^{-\frac{1}{2}}$ with $v > 0$, and using the same trick employed in the proof of Lemma 2.2, one gets

$$
|\psi(x) - \psi(y)|^2 \leq C_k^2 |x - y|^k q_k(v) \left( \|T\psi\|^2_{L^2} + v^2 \|\psi\|^2_{L^2} \right)
$$

(2.17)

where

$$
q_k(v) := \int_{\mathbb{R}} \frac{|s|^k}{s^2 + v^2} = \frac{1}{v^{1-k}} \frac{\pi}{\cos \left( \frac{\pi k}{2} \right)}, \quad 0 \leq k < 1.
$$

A minimization procedure on $v$ provides

$$
\min_{v > 0} \frac{\|T\psi\|^2_{L^2} + v^2 \|\psi\|^2_{L^2}}{v^{1-k}} = I_k \|T\psi\|^{1+k}_{L^2} \|\psi\|^{1-k}_{L^2}
$$

with

$$
I_k := \frac{2}{(1 + k) \frac{1+k}{2} (1 - k) \frac{1-k}{2}}.
$$

Since inequality (2.17) holds for every $v > 0$, it holds true also when $v$ coincides with the minimizer. This provides the inequality (2.11) with the constant given by $G_k := C_k^2 I_k \frac{\pi}{\cos \left( \frac{\pi k}{2} \right)}$. This concludes the proof.

The following result gives an upper bound for the decay rate at infinity.

**Lemma 2.5 (Global control of the behavior)** Let $\psi \in \mathcal{D}(T)$. Then, it holds true that for some positive constant $K$,

$$
|\psi(x)| \leq K |x|^{-\frac{1}{2}} \|T\psi\|^\frac{1}{2}_{L^2} \|\psi\|^\frac{3}{2}_{L^2}, \quad \forall x \in \mathbb{R} \setminus \{0\}.
$$

(2.18)
Proof Let us start by observing that from [6, eq. 10.7.8] one infers that there exists a positive constant $M > 0$ such that

$$|J_0(z)| \leq \frac{M}{\sqrt{z}}, \quad \forall \ z \geq 0.$$  

The exact value of the constants $M$ is not important for the purposes of this work.\footnote{The best constant $M$ is fixed by the maximum of the function $f(z) := |J_0(z)|\sqrt{z}$. A numerical inspection with Wolfram Mathematica (version 12.1) shows that one can choose $M > \sqrt{\frac{2}{\pi}}$.}

Using the same argument in the proof of Lemma 2.4, one gets

$$|\psi(x)| \leq \int_\mathbb{R} \, ds \, |J_0(2\sqrt{|sx|})| |(B\psi)(s)|$$

which shows that $|\psi(x)|$ is dominated by $|x|^{-\frac{1}{4}}$. For the determination of the constant, one can insert inside the integral the identity $(s^2 + v^2)^{\frac{1}{2}} (s^2 + v^2)^{-\frac{1}{2}}$ with $v > 0$ and, with the same trick employed in the proof of Lemma 2.2, one gets

$$\int_\mathbb{R} \, ds \, |s|^{-\frac{1}{4}} |(B\psi)(s)| \leq \left( \int_{-\infty}^{+\infty} \, ds \, |s|^{-\frac{1}{4}} \left( \frac{|s|^{-\frac{1}{4}}}{(s^2 + v^2)} \right)^2 \left( \|T\psi\|_{L^2}^2 + v^2 \|\psi\|_{L^2}^2 \right)^{\frac{1}{2}} \right)^\frac{1}{2}$$

$$= \frac{2^{\frac{1}{4}} \sqrt{\pi}}{v^{\frac{3}{4}}} \left( \|T\psi\|_{L^2}^2 + v^2 \|\psi\|_{L^2}^2 \right)^{\frac{1}{2}}.$$  

After minimizing the last inequality with respect $v > 0$, and grouping all constants in $K > 0$, one finally obtains

$$|\psi(x)| \leq K \, |x|^{-\frac{1}{4}} \, \|T\psi\|_{L^2}^{\frac{3}{2}} \|\psi\|_{L^2}^{\frac{3}{2}}.$$  

This concludes the proof. \qed

Remark 2.6 (Synopsis) Let us summarize the main properties of elements in the domain $\mathcal{D}(T)$ obtained in this section. Lemma 2.3 provides:

(i) $\mathcal{D}(T) \subset L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

By combining Lemmas 2.3 and 2.4, one infers that:

(ii) Every $\psi \in \mathcal{D}(T)$ is at least Hölder continuous in $\mathbb{R} \setminus \{0\}$ of order $\alpha < \frac{1}{2}$. Moreover, the possible discontinuity in the critical point $x = 0$ is of the first kind meaning that both limits $\lim_{x \to 0^\pm} \psi(x) = L_{\pm}$ exist and are finite.

Finally, Lemma 2.5 tells us that:
(iii) The global behavior of $\psi \in D(T)$ in $\mathbb{R} \setminus \{0\}$ is dominated by $|x|^{-\frac{1}{4}}$. In particular, the elements of $D(T)$ vanish at infinity.

Let us point out that the existence of singular discontinuous elements in $D(T)$, as stipulated by (ii) is unavoidable. Indeed, we know from [1] that smooth functions are not enough to provide a core for $T$, while a core is provided by $S(\mathbb{R}) + \mathbb{C}[\kappa_0]$. The function $\kappa_0(x) \propto \text{sgn}(x) \text{kei}(2\sqrt{|x|})$ is smooth in $\mathbb{R} \setminus \{0\}$ and has a discontinuity of the first kind in $x = 0$. The decay at infinity of $\kappa_0$ is quite rapid. Indeed, one has that $\kappa_0(x) = o(e^{-|x|^\frac{1}{2}})$. Finally, it should be noted that Lemma 2.5 provides a somehow weak information about the decay of elements of $D(T)$ at infinity. In fact a decay of type $|x|^{-\frac{1}{4}}$ it is not enough to guarantee that $\psi \in L^2(\mathbb{R})$. However, Lemma 2.5 gives an important information that will be crucial in Proposition 3.1. Let $\psi \in D(T)$ and define $\phi(x) := |x|^{\frac{1}{4}} \psi(x)$. Then, $\|\phi\|_{L^\infty} < \infty$. Moreover, from the proof of Lemma 2.5, one also infers that

$$\|\phi\|^2_{L^\infty} \leq \frac{C}{v^2} \left( \|T \psi\|^2_{L^2} + v^2 \|\psi\|^2_{L^2} \right)$$

with $C > 0$ a suitable constant and for every $v > 0$. Let us conclude this remark by observing that the properties (i), (ii) and (iii) remain valid for elements of the domain $\mathcal{D}_T$ of the thermal Hamiltonian $H_T$ with the only difference that the critical point has to be shifted from 0 to $x_c$ as effect of the translation $S_\lambda$.

3 Perturbations by potential

In this section, we will study the perturbations of the operator $T$ by multiplicative, real-valued potentials.

3.1 Self-adjoint perturbations

Let $W : \mathbb{R} \to \mathbb{R}$ be a real-valued function. With a slight abuse of notation, we will denote with the same symbol also the multiplication operator defined on $\psi \in L^2(\mathbb{R})$ by $(W \psi)(x) := W(x) \psi(x)$. This is a self-adjoint operator with natural domain given by

$$\mathcal{D}(W) := \{ \psi \in L^2(\mathbb{R}) \ | \ W \psi \in L^2(\mathbb{R}) \}.$$

The next result provides condition on the potential $W$ for the self-adjointness of $T + W$.

**Proposition 3.1** (Self-adjoint perturbations) Let $W : \mathbb{R} \to \mathbb{R}$ be a real-valued function such that

$$W(x) := |x|^{\frac{1}{4}} V_1(x) + V_2(x), \ x \in \mathbb{R}$$
with \( V_1 \in L^2(\mathbb{R}) \) and \( V_2 \in L^\infty(\mathbb{R}) \). Then, for every \( \epsilon > 0 \) there exists a positive constant \( B_\epsilon > 0 \) such that

\[
\| W\psi \|_{L^2}^2 \leq \epsilon \| T\psi \|_{L^2}^2 + B_\epsilon \| \psi \|_{L^2}^2.
\]  

(3.1)

As a consequence, \( T + W \) defines a self-adjoint operator with domain \( D(T) \).

**Proof** Let \( \psi \in D(T) \). Then, one has that

\[
\| W\psi \|_{L^2} \leq \| V_1\phi \|_{L^2} + \| V_2\psi \|_{L^2} \leq \| V_1 \|_{L^2} \| \phi \|_{L^\infty} + \| V_2 \|_{L^\infty} \| \psi \|_{L^2}.
\]

(3.2)

where \( \phi(x) := |x|^{\frac{1}{4}}\psi(x) \) with \( \| \phi \|_{L^\infty} < \infty \) in view of Lemma 2.5. This proves that \( D(T) \subseteq D(W) \). By combining (3.2) with the inequality (2.19) (also deduced from Lemma 2.5), one obtains

\[
\| W\psi \|_{L^2}^2 \leq \left( \| V_1 \|_{L^2} \| \phi \|_{L^\infty} + \| V_2 \|_{L^\infty} \| \psi \|_{L^2} \right)^2 \\
\leq 2 \left( \| V_1 \|_{L^2}^2 \| \phi \|_{L^\infty}^2 + \| V_2 \|_{L^\infty}^2 \| \psi \|_{L^2}^2 \right) \\
\leq \frac{2C \| V_1 \|_{L^2}^2}{v^2} \| T\psi \|_{L^2}^2 + \left( 2Cv^{\frac{1}{4}} + \| V_2 \|_{L^\infty}^2 \right) \| \psi \|_{L^2}^2.
\]

(3.3)

Since (3.3) holds for every \( v > 0 \), one can always find a \( v_\epsilon \) such that \( 2C \| V_1 \|_{L^2}^2v_\epsilon^{-\frac{3}{2}} = \epsilon \). By setting \( B_\epsilon := 2Cv_\epsilon^{\frac{1}{4}} + \| V_2 \|_{L^\infty}^2 \), one obtains the inequality (3.1). The latter implies that \( W \) is infinitesimally small with respect to \( T \) in the sense of Kato (cf. [8, eq. X.19a & eq. X.19b]). Consequently \( T + W \) defined a self-adjoint operator with domain \( D(T) \) in view of the Kato–Rellich theorem [8, Theorem X.12].

**Remark 3.2** (Self-adjoint perturbation of the thermal Hamiltonian) Let \( W \) be a potential which meets the conditions of Proposition 3.1. Then, \( T + W \) is a self-adjoint operator with domain \( D(T) \). As a consequence, \( H_T + \tilde{W} \) with

\[
\tilde{W} := \lambda S^*_{\lambda} W S_{\lambda}
\]

is a self-adjoint perturbation of the thermal Hamiltonian \( H_T \) in view of the equivalence (1.14). The splitting of \( W \) stipulated in Proposition 3.1 translates to

\[
\tilde{W}(x) = |x - x_c|^{\frac{1}{4}} \tilde{V}_1(x) + \tilde{V}_2(x)
\]

where \( \tilde{V}_i := \lambda S^*_{\lambda} V_i S_{\lambda} \) and \( i = 1, 2 \). Since the conjugation by \( S_{\lambda} \) is a translation, one obtains that \( \tilde{V}_1 \in L^2(\mathbb{R}) \) and \( \tilde{V}_2 \in L^\infty(\mathbb{R}) \). The considerations above provide the main argument to deduce Theorem 1.2 from Proposition 3.1.
3.2 Relatively compact perturbations

The task of this section is to find a suitable class of relatively compact (or $T$-compact) perturbations $W : \mathbb{R} \to \mathbb{R}$ of the free operator $T$. This means that $\mathcal{D}(T) \subseteq \mathcal{D}(W)$ and the product $WR_{\pm i}(T)$ must be a compact operator [10, Section XIII.4], with

$$R_z(T) := (T - z1)^{-1}, \quad z \in \mathbb{C} \setminus \mathbb{R}$$

the resolvent of $T$ evaluated at $z$. In the next result, we will provide conditions on $W : \mathbb{R} \to \mathbb{R}$ such that the product $WR_{\pm i}(T)$ is a Hilbert–Schmidt operator.

**Proposition 3.3** Let $W : \mathbb{R} \to \mathbb{R}$ be a real-valued function such that

$$W(x) = |x|^4 V(x)$$

with $V \in L^2(\mathbb{R})$. Then, $WR_{\pm i}(T)$ are Hilbert–Schmidt operators (hence compact).

**Proof** The resolvents $R_{\pm i}(T)$ are integral operators with explicit integral kernels given by (2.4) when evaluated on the dense domain $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Let us introduce the symbol

$$\mathcal{K}_\pm(x, y) := (\text{sgn}(x) + \text{sgn}(y)) W(x) \mathcal{F}_\pm(x, y) \quad (3.4)$$

for the integral kernel of $WR_{\pm i}(T)$. Since the term $\text{sgn}(x) + \text{sgn}(y)$ vanishes when $x$ and $y$ have different signs, one has that

$$\int_\mathbb{R}^2 dxdy |\mathcal{K}_\pm(x, y)|^2 = 4 (I_+ + I_-)$$

where

$$I_\pm := \int_\mathbb{R}_\pm dxdy |W(x)|^2 |\mathcal{F}|(x, y)^2$$

$$= \int_\mathbb{R}_\pm dxdy |W(\pm x)|^2 |\mathcal{F}|(x, y)^2.$$

In the definition of the integrals $I_\pm$, we used the following notation for the domains of integration: $\mathbb{R}_\pm := \{x \in \mathbb{R} | \pm x > 0\}$ are the positive and negative semi-axis and $\mathbb{R}_\pm^2 := \mathbb{R}_\pm \times \mathbb{R}_\pm$ are the first and third quadrants. The function $|\mathcal{F}|$ is defined by (2.5). In the second equality, the invariance property of $|\mathcal{F}|$ under the reflections $x \mapsto -x$ and $y \mapsto -y$ has been exploited. In view of the splitting above, and after a comparison with the notation introduced in Lemma 2.1, one obtains that

$$I_\pm = I_{w\pm} := \int_\mathbb{R}_\pm^2 dxdy w_{\pm}(x) |\mathcal{F}|(x, y)^2$$
where the functions \( w_{\pm} : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{0\} \) are defined as follows:

\[
w_+(x) := |W(x)|^2, \quad w_-(x) := |W(-x)|^2.
\]

Since

\[
\int_{\mathbb{R}_+} dx \frac{w_+(x)}{\sqrt{x}} = \int_{\mathbb{R}_+} dx |V(x)|^2 \leq \|V\|_{L^2}^2,
\]

one can apply Lemma 2.1 to conclude that \( I_{\pm} < +\infty \). This implies that the kernels (3.4) are square-integrable, i.e., \( \mathscr{K}_{\pm} \in L^2(\mathbb{R}^2) \). Therefore, the kernels \( \mathscr{K}_{\pm} \) define two Hilbert–Schmidt operators \( K_{\pm} \) on \( L^2(\mathbb{R}) \) [7, Theorem VI.23]. Moreover, one has that \( K_\pm \) coincides with \( WR_{\pm i}(T) \) on the dense domain \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). This is enough to conclude that \( K_\pm = WR_{\pm i}(T) \) and this completes the proof. \( \Box \)

The hypotheses on \( W \) stipulated in Proposition 3.3 are stronger than the hypotheses in Proposition 3.1. Consequently, it holds true that under the hypotheses of Proposition 3.3 \( T + W \) is a self-adjoint operator and \( W \) is a relatively compact perturbation of \( T \).

As a consequence of the celebrated Weyl theorem [10, Theorem XIII.14], one obtains the equality of the essential spectra, i.e.,

\[
\sigma_{\text{ess}}(T + W) = \sigma_{\text{ess}}(T) = \mathbb{R}.
\]

It is also worth noting that under the hypotheses of Proposition 3.3 it holds true that \( WR_z(T) \) is a Hilbert–Schmidt operator for every \( z \in \mathbb{C} \setminus \mathbb{R} \). This follows from the identity

\[
WR_z(T) = WR_{\pm i}(T) \left[ (T \mp i 1)R_z(T) \right],
\]

the boundedness of the operator inside the square brackets and the fact that the class of Hilbert–Schmidt operators is a two-sided ideals inside the bounded operators.

The next result provides a class of prototype bounded potentials which meet the conditions stipulated in Proposition 3.3. For that we need to introduce the Japanese brackets \( \langle x \rangle : \mathbb{R} \to \mathbb{R}_+ \) defined by

\[
\langle x \rangle := \left( 1 + x^2 \right)^{\frac{1}{2}}.
\]

**Corollary 3.4** Let \( W \) be a real-valued potential such that

\[
W\langle x \rangle^s \in L^\infty(\mathbb{R})
\]

for some \( s > \frac{1}{4} \). Then, \( W \) is a relatively compact perturbation of the operator \( T \) and \( WR_z(T) \) are Hilbert–Schmidt operators for every \( z \in \mathbb{C} \setminus \mathbb{R} \).
Proof In view of the identity
\[ \text{WR}_\zeta(T) = \left( W \langle x \rangle^s \right) \left( \langle x \rangle^{-s} R_\zeta(T) \right) \]
the proof just follows by showing that \( \langle x \rangle^{-s} R_\zeta(T) \) is Hilbert–Schmidt whenever \( s > \frac{1}{4} \). An explicit computation provides
\[
\int_{\mathbb{R}} \frac{|\langle x \rangle^{-s}(x)|^2}{|x|^{\frac{3}{2}}} = \int_{\mathbb{R}} \frac{dx}{(1 + x^2)^{s}|x|^{\frac{1}{2}}} = 4 \Gamma \left( \frac{5}{4} \right) \frac{\Gamma \left( s - \frac{1}{4} \right)}{\Gamma (s)}, \quad s > \frac{1}{4}
\]
where \( \Gamma \) denotes the gamma function. Therefore, \( \langle x \rangle^{-s} \), with \( s > \frac{1}{4} \), satisfies the hypotheses of Proposition 3.3.

\[ \square \]

Remark 3.5 (Relatively compact perturbations of the thermal Hamiltonian) In the spirit of Remark 3.2, one can translate the conditions for the relative compactness of a perturbation from the operator \( T \) to the thermal Hamiltonian \( H_T \) just by using the equivalence (1.14) implemented by the translation \( S_\lambda \). As a result, one gets that potentials of the type
\[ \tilde{W}(x) = |x - x_c|^{\frac{1}{4}} \tilde{V}(x) \]
with \( \tilde{V} \in L^2(\mathbb{R}) \) are relatively compact perturbations of \( H_T \) of Hilbert–Schmidt type. This is the key observation to deduce Theorem 1.4 from Proposition 3.3. It is worth to translate the result of Corollary 3.4 in terms of the background potential \( V \) which defines the perturbation of the thermal Hamiltonian via Eq. (1.5). By using the usual argument involving the shift operator \( S_\lambda \), one infers that the condition
\[ V \langle x - x_c \rangle^{s + 1} \in L^\infty(\mathbb{R}), \quad (3.5) \]
for some \( s > \frac{1}{4} \), implies that \( W_V \) is a relatively compact perturbation of \( H_T \) of Hilbert–Schmidt type. In the last equation, we introduced the shifted Japanese brackets
\[ \langle x - x_c \rangle(x) := \left( 1 + |x - x_c|^2 \right)^{\frac{1}{2}}. \]
Since the function \( \langle x - x_c \rangle^{-1} \) is bounded and invertible, one can reformulate the condition (3.5) as follows:
\[ V \langle x \rangle^s \in L^\infty(\mathbb{R}), \]
for some \( s > \frac{5}{4} \). \[ \blacklozenge \] Springer
4 Scattering theory

In this section, we will present a class of perturbations \( W \) of the operator \( T \) for which the wave operators

\[
\Omega_{\pm}(W) := s - \lim_{t \to \mp \infty} e^{i(T+W)t} e^{-iTt}.
\]

exist and are complete. It is worth noticing that in the definition (4.1) we have tacitly used the fact that the spectrum of \( T \) is purely absolutely continuous, and consequently the spectral projection on the absolutely continuous part of the spectrum of \( T \) coincides with the identity, i.e., \( P_{a.c.}(T) = 1 \). The proof of the existence and completeness of \( \Omega_{\pm}(W) \) rely on the results of Sect. 3.2 along with the celebrated Kuroda–Birman theorem [9, Theorem XI.9] which, in our specific case, can be stated as follows:

**Theorem 4.1** (Kuroda–Birman) Let \( W : \mathbb{R} \to \mathbb{R} \) be a potential such that \( T + W \) is self-adjoint. Consider the resolvents

\[
R_{-i}(T) := (T + i1)^{-1}, \quad R_{-i}(T + W) := (T + W + i1)^{-1}.
\]

If the difference \( R_{-i}(T) - R_{-i}(T + W) \) is trace-class, then the wave operators \( \Omega_{\pm}(W) \) exist and are complete.

We are now in position to present our main result concerning the scattering theory of the operator \( T \).

**Proposition 4.2** (Scattering theory) Let \( W : \mathbb{R} \to \mathbb{R} \) be a potential such that both \( W \) and \( W' := |W|^{\frac{1}{2}} \) satisfy the conditions of Proposition 3.3. Then, the wave operators \( \Omega_{\pm}(W) \) exist and are complete.

**Proof** An iterated use of the resolvent identity provides

\[
R_{-i}(T) - R_{-i}(T + W) = R_{-i}(T)WR_{-i}(T + W)
\]

\[
= R_{-i}(T)W[R_{-i}(T) - R_{-i}(T)WR_{-i}(T + W)]
\]

\[
= Z_1 + Z_2
\]

where the two terms in the last line are given by

\[
Z_1 := R_{-i}(T)WR_{-i}(T), \quad Z_2 := -[R_{-i}(T)W]^2R_{-i}(T + W).
\]

In view of the hypotheses and of Proposition 3.3, one has that

\[
R_{-i}(T)W = (WR_{i}(T))^*
\]

is a Hilbert–Schmidt operator. As a consequence, one has that its square, and consequently \( Z_2 \), are trace-class. The operator \( Z_1 \) can be rewritten as

\[
Z_1 = (W'R_{i}(T))^* \text{sgn}(W) W'R_{-i}(T)
\]
where sgn\((W)\) denotes the bounded multiplicative operator which multiplies elements of \(L^2(\mathbb{R})\) by the sign of the function \(W\). Again, from the hypotheses and of Proposition \(3.3\) one gets that \(W'R_{\pm i}(T)\) are Hilbert–Schmidt operators and consequently \(Z_1\) is trace-class. Since we proved that the difference \(R_{-i}(T) - R_{-i}(T + W)\) is trace-class, the result follows from Theorem \(4.1\).

\[\square\]  

**Remark 4.3** (Scattering matrix) The existence of the wave operators \(\Omega_{\pm}(W)\) allows to define the *scattering matrix*

\[
\mathcal{S}(W) := \Omega_{-}(W)^{\ast} \Omega_{+}(W).
\]  

(4.2)

The completeness of the wave operators, along with \(P_{a.e.}(T) = 1\), ensures that \(\mathcal{S}(W)\) is a unitary operator on \(L^2(\mathbb{R}^2)\). ▶

The next result provides a class of prototype bounded potentials which meet the condition stipulated in Proposition \(4.2\).

**Corollary 4.4** Let \(W\) be a real-valued potential such that

\[W \langle x \rangle^s \in L^\infty(\mathbb{R})\]

for some \(s > \frac{1}{2}\). Then, the related wave operators \(\Omega_{\pm}(W)\) exist and are complete.

**Proof** Consider the identity \(W = W_{\infty}W_s\) with \(W_{\infty} := W \langle x \rangle^s \in L^\infty(\mathbb{R})\) and \(W_s := \langle x \rangle^{-s}\). Observe that \(W_s\) meets the condition of Corollary \(3.4\) for every \(s > \frac{1}{2}\) and \(|W_s|^{\frac{1}{2}}\) meets the condition of Corollary \(3.4\) for every \(s > \frac{1}{2}\). This proves that \(W_s\) meets the condition of Proposition \(4.2\). Since \(W\) and \(W_s\) differ from each other by the multiplication by the essentially bounded function \(W_{\infty}\), they share the same integrability properties. As a consequence, also \(W\) meets the condition of Proposition \(4.2\). □

**Remark 4.5** (Scattering theory of the thermal Hamiltonian) The passage from Proposition \(4.2\) to Theorem \(1.6\) is justified by the same argument already discussed in Remarks \(3.2\) and \(3.5\). It is interesting to translate the content of Corollary \(4.4\) in terms of the background potential \(V\) which defines the perturbation of the thermal Hamiltonian via Eq. (1.5). An analysis similar to that in Remark \(3.5\) provides the following result: If the background potential \(V : \mathbb{R} \rightarrow \mathbb{R}\) meets the condition

\[V \langle x \rangle^s \in L^\infty(\mathbb{R}),\]

for some \(s > \frac{3}{2}\), then the wave operators defined by \((1.12)\) exist and are complete. ▶
Appendix A. Some remark about the operator $B$

In this appendix, we will provide some details for the derivation of Eq. (2.2) which provides the integral kernel of the unitary involution $B$. The complete proof can be found in [1].

Let us start by recalling that the operator $T$ has a core given by

$$D_0(T) := \mathcal{S}(\mathbb{R}) + \mathbb{C}[\kappa_0]$$

(A.1)

where it acts according to the prescription

$$T(\psi + c\kappa_0)(x) = -\left(\psi'(x) + x\psi''(x)\right) + c\kappa_1(x),$$

$$\psi \in \mathcal{S}(\mathbb{R}), \quad c \in \mathbb{C}$$

(A.2)



with

$$\kappa_0(x) := -\sqrt{\frac{8}{\pi}} \text{sgn}(x) \text{kei}(2\sqrt{|x|}), \quad \kappa_1(x) := \sqrt{\frac{8}{\pi}} \text{ker}(2\sqrt{|x|}).$$

The operator $T$ is a differential operator of order two, and a multiplication operator of order one. Since, in general, multiplication conditions are easier to handle than differential conditions, we introduce the operator $/Pi_1 := FTF^*$, where $F$ is the Fourier transform on $L^2(\mathbb{R})$. It is well known that $F$ is an unitary operator satisfying $FpF^* = x$ and $FxF^* = -p$, and leaving the space $\mathcal{S}(\mathbb{R})$ invariant. We then have, as an immediate consequence, that $/Pi_1$ has a core given by

$$D_0(/Pi_1) := \mathcal{S}(\mathbb{R}) + \mathbb{C}[\zeta_0]$$

(A.3)

on which it acts according to the prescription

$$(/Pi_1(\psi + c\zeta_0))(x) = i\left(x^2\psi'(x) + x\psi(x)\right) + c\zeta_1(x),$$

$$\psi \in \mathcal{S}(\mathbb{R}), \quad c \in \mathbb{C}$$

(A.4)

where the functions $\zeta_i := F\kappa_i$, $i = 0, 1$, are explicitly given by

$$\zeta_0(x) = \frac{1}{x} e^{-\frac{1}{|x|}}, \quad \zeta_1(x) = \frac{1}{|x|} e^{-\frac{1}{|x|}}.$$ 

The operator $\Pi$ admits a family of generalized eigenvectors given by

$$\phi_\lambda(x) := \frac{1}{x} e^{i\frac{\lambda}{x}}, \quad \lambda \in \mathbb{R}.$$ 

(A.5)
In fact, a direct computation shows that $\Pi \phi_\lambda = -(xp)x \phi_\lambda = \lambda \phi_\lambda$. The next consists in defining the unitary involution $I : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$(I \psi)(x) := \frac{1}{x} \psi \left( \frac{1}{x} \right).$$

(A.6)

With a small abuse of notation, when $I$ is applied to the family of generalized eigenvectors of $\Pi$, one gets

$$(I \phi_\lambda)(x) = e^{i\lambda x}.$$

Thus, $I$ maps every generalized $\lambda$-eigenvector of $\Pi$ to the corresponding generalized $\lambda$-eigenvector of $p$. This suggests the fact that $I$ intertwines between $\Pi$ and the (standard) momentum operator $p$. This fact can be proved rigorously and indeed one has that $I \Pi I = p$ with domain $H^1(\mathbb{R})$.

By using all the unitary equivalences above, one gets

$$T = \mathcal{F}^* \Pi \mathcal{F} = \mathcal{F}^* I(p) I \mathcal{F} = (\mathcal{F}^* I \mathcal{F}) (-x) (\mathcal{F}^* I \mathcal{F}).$$

Then, the unitary involution $B := \mathcal{F}^* I \mathcal{F}$ intertwines between $T$ and $-x$.

**Lemma A.1** On the dense domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ the operator $B = \mathcal{F}^* I \mathcal{F}$ acts as an integral operator with kernel given by (2.1) and (2.2).

This fact is proved in full detail in [1, Lemma 3.1]. However, the crucial ingredient of the proof is a direct computation on a suitable family of approximating functions which shows that $B$ acts as an integral operator with kernel

$$B(x, y) := \frac{1}{2\pi} \lim_{n \to +\infty} \int_{I_n} \frac{e^{ixu} e^{-i\frac{y}{u}}}{u},$$

where $I_n := [-n, -n^{-1}] \cup [n^{-1}, n]$. By using standard tools from complex analysis, one can finally show that the latter principal value integral coincides with (2.2). A density argument, along with the Lebesgue’s dominated convergence theorem, completes the proof.

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**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.
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