The classifications of certain kind of isoparametric submanifolds in non-compact symmetric spaces

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Abstract

In this paper, we prove that full irreducible curvature-adapted isoparametric submanifolds of codimension greater than one in a symmetric space of non-compact type are principal orbits of Hermann actions on the symmetric space under certain condition. In the proof, it is key to show the homogeneity of the submanifold. The proof of the homogeneity is performed by showing the homogeneity of the submanifold in an infinite dimensional anti-Kaehler space arising from the complexification of the original submanifold. Furthermore, by using this fact, we classify such submanifolds in all irreducible symmetric spaces of non-compact type.

Keywords: isoparametric submanifold, equifocal submanifold, Hermann action, focal radius, principal curvature

1 Introduction

In this introduction, we shall mainly explain the validity of imposing the condition \((\ast_{\mathbb{C}})\) in the main theorem (see Theorem A) of this paper. Let \(M\) be a submanifold in a complete Riemannian manifold \(N\), \(\psi : T^\perp M \to M\) the normal bundle of \(M\) and \(\exp^\perp\) the normal exponential map of \(M\). Denote by \(\mathcal{V}\) the vertical distribution on \(T^\perp M\) and \(\mathcal{H}\) the horizontal distribution on \(T^\perp M\) with respect to the normal connection of \(M\). Let \(v\) be a unit normal vector of \(M\) at \(x(\in M)\) and \(r\) a real number. Denote by \(\gamma_v\) the normal geodesic of \(M\) of direction \(v\) (i.e., \(\gamma_v(s) = \exp^\perp(sv)\)). If \(\psi_*(\text{Ker } \exp^\perp_{rv}) \neq \{0\}\), then \(\exp^\perp(rv)\) (resp. \(r\)) is called a focal point (resp. a focal radius) of \(M\) along \(\gamma_v\). For a focal radius \(r\) of \(M\) along \(\gamma_v\),
ψ∗(Ker exp∗) is called the nullity space for r and its dimension is called the multiplicity of r. Denote by \( \mathcal{FR}_{M,v}^R \) the set of all focal radii of M along \( \gamma_v \). Set

\[
\mathcal{FR}_{M,x} := \bigcup_{v \in T_x^* M \text{ s.t. } ||v|| = 1} \{ r v \mid r \in \mathcal{FR}_{M,v}^R \},
\]

which is called a tangential focal set of M at x. If, for any \( y \in M \), \( \Sigma_y := \exp_{T_y^* M} ((T_y^* M) || v || = 1 \{ r v \mid r \in \mathcal{FR}_{M,v}^R \} \), which is called a tangential focal set of M at \( x \). Assume that N is a symmetric space \( G/K \) and that M is a submanifold with flat section. Then we can show \( \ker \exp_r \subset H_r \) and

\[
(1.1) \quad \exp_r^L (X_r v) = P_{\gamma_r} \left( \cos(r \sqrt{R(v)}) - \frac{\sin(r \sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v \right)(X) \quad (X \in T_x^* M),
\]

where \( X_r^L \) is the horizontal lift of \( X \) to \( rv \), \( P_{\gamma_r} \) is the parallel translation along the normal geodesic \( \gamma_r \), \( R(v) \) is the normal Jacobi operator \( R(\bullet, v)v \) (\( R \) : the curvature tensor of \( G/K \)) and \( A \) is the shape tensor of \( M \). Hence \( \mathcal{FR}_{M,v}^R \) coincide with the set of all zero points of the real-valued function

\[
F_v(s) := \det \left( \cos(s \sqrt{R(v)}) - \frac{\sin(s \sqrt{R(v)})}{\sqrt{R(v)}} \circ A_v \right) \quad (s \in \mathbb{R}).
\]

In particular, in the case where \( G/K \) is a Euclidean space, we have \( F_v(s) = \det(id - s A_v) \) (id : the identity transformation of \( T_x^* M \)). Hence \( \mathcal{FR}_{M,v}^R \) is equal to the set of all the inverse numbers of the eigenvalues of \( A_v \) and the nullity space for \( r \in \mathcal{FR}_{M,v}^R \) is equal to \( \ker(A_v - \frac{1}{s} \mathbb{I}) \). Therefore the nullity spaces for focal radii of M along \( \gamma_v \) span \( T_x^* M \subset \ker A_v \). Also, in the case where \( G/K \) is a sphere of constant curvature \( c(>0) \), we have

\[
F_v(s) = \det \left( \cos(s \sqrt{c}) \mathbb{I} - \frac{\sin(s \sqrt{c})}{\sqrt{c}} A_v \right).
\]

Hence we have

\[
\mathcal{FR}_{M,v}^R = \left\{ \frac{1}{\sqrt{c}} \left( \arctan \frac{\sqrt{c}}{\lambda} + j\pi \right) \mid \lambda : \text{the eigenvalue of } A_v, \quad j \in \mathbb{Z} \right\}
\]

and the nullity space for \( \frac{1}{\sqrt{c}} \left( \arctan \frac{\sqrt{c}}{\lambda} + j\pi \right) \) is equal to \( \ker(A_v - \lambda \mathbb{I}) \), where we note that \( \arctan \frac{\sqrt{c}}{\lambda} \) means \( \pi/2 \) when \( \lambda = 0 \). Therefore the nullity spaces for focal radii of M along \( \gamma_v \) span \( T_x^* M \). Also, in the case where \( G/K \) is a hyperbolic space of constant curvature \( c(<0) \), we have

\[
F_v(s) = \det \left( \cosh(s \sqrt{-c}) \mathbb{I} - \frac{\sinh(s \sqrt{-c})}{\sqrt{-c}} A_v \right).
\]
Hence we have

\begin{equation}
\mathcal{FR}_{M,v}^R = \left\{ \frac{1}{\sqrt{-c}} \arctanh \frac{\sqrt{-c}}{\lambda} : \text{the eigenvalue of } A_v \text{ s.t. } |\lambda| > \sqrt{-c} \right\}
\end{equation}

and the nullity space for \( \frac{1}{\sqrt{-c}} \arctanh \frac{\sqrt{-c}}{\lambda} \) is equal to \( \text{Ker}(A_v - \lambda \text{id}) \). Therefore the nullity spaces for focal radii of \( M \) along \( \gamma_v \) span \( T_x M \) if and only if all the absolute values of eigenvalues of \( A_v \) is greater than \( \sqrt{-c} \). As a non-compact submanifold \( M \) with flat section in a symmetric space \( G/K \) of non-compact type deforms as its principal curvatures approach to zero, its focal set vanishes beyond the ideal boundary \((G/K)(\infty)\) of \( G/K \). This fact will be guessed from (1.2). From this fact, we \cite{Koi2} considered that a focal radius of \( M \) along the normal geodesic \( \gamma_v \) should be defined in the complex number field \( \mathbb{C} \). We \cite{Koi2} introduced the notion of a complex focal radius as the zero points of the complex-valued function \( F_c^v \) over \( \mathbb{C} \) defined by

\[ F_c^v(z) := \det \left( \cos(z \sqrt{R(v)^c}) - \frac{\sin(z \sqrt{R(v)^c})}{\sqrt{R(v)^c}} \circ A_c^v \right) \quad (s \in \mathbb{C}), \]

where \( A_c^v \) and \( \sqrt{R(v)^c} \) are the complexifications of \( A_v \) and \( \sqrt{R(v)} \), respectively.

**Terminology.** In this paper, we shall call a complex focal radius focal radius simply.

For a focal radius \( z \) of \( M \) along \( \gamma_v \), \( \text{Ker} \left( \cos(z \sqrt{R(v)^c}) - z A_c^v \right) \) (id : the identity transformation of \( (T_x M)^c \) is called the nullity space for \( z \) and its complex dimension is called the multiplicity of \( z \). Denote by \( \mathcal{FR}_{M,v}^C \) the set of all focal radii of \( M \) along \( \gamma_v \). In the case where \( G/K \) is a Euclidean space, we have \( F_c^v(z) = \det(\text{id} - z A_c^v) \) (id : the identity transformation of \( (T_x M)^c \)). Hence we have \( \mathcal{FR}_{M,v}^C = \mathcal{FR}_{M,v}^R \) and the nullity space for \( z \in \mathcal{FR}_{M,v}^C \) is equal to \( \text{Ker}(A_c^v - \frac{1}{z} \text{id}) \). Therefore the nullity spaces for focal radii of \( M \) along \( \gamma_v \) span \( (T_x M)^c \ominus \text{Ker} A_c^v \). Also, in the case where \( G/K \) is a sphere of constant curvature \( c(>0) \), we have \( F_c^v(z) = \det \left( \cos(z \sqrt{c}) - \frac{\sin(z \sqrt{c})}{\sqrt{c}} A_c^v \right) \).

Hence \( \mathcal{FR}_{M,v}^C = \mathcal{FR}_{M,v}^R \) and the nullity space for \( \frac{1}{\sqrt{c}} \arctanh \frac{\sqrt{c}}{\lambda} \) is equal to \( \text{Ker}(A_c^v - \lambda \text{id}) \). Therefore the nullity spaces for focal radii of \( M \) along \( \gamma_v \) span \( (T_x M)^c \).

Also, in the case where \( G/K \) is a hyperbolic space of constant curvature \( c(<0) \), we have \( F_c^v(z) = \det \left( \cos(iz \sqrt{-c}) - \frac{\sin(iz \sqrt{-c})}{i \sqrt{-c}} A_c^v \right) \).
Hence \( \mathcal{F} \mathcal{R}^G_{M,v} \) is equal to

\[
= \left\{ \frac{1}{\sqrt{-c}} \left( \text{arctanh} \frac{\sqrt{-c}}{\lambda} + j\pi i \right) \bigg| \lambda : \text{the eigenvalue of } A_v \text{ s.t. } |\lambda| > \sqrt{-c}, \ j \in \mathbb{Z} \right\} \cup \\
\left\{ \frac{1}{\sqrt{-c}} \left( \text{arctanh} \frac{\lambda}{\sqrt{-c}} + (j + \frac{1}{2})\pi i \right) \bigg| \lambda : \text{the eigenvalue of } A_v \text{ s.t. } |\lambda| < \sqrt{-c}, \ j \in \mathbb{Z} \right\},
\]

the nullity space for \( \frac{1}{\sqrt{-c}} \left( \text{arctanh} \frac{\sqrt{-c}}{\lambda} + j\pi i \right) \) (|\lambda| > \sqrt{-c}) is equal to Ker\((A_v^c - \lambda \text{id})\) and the nullity space for \( \frac{1}{\sqrt{-c}} \left( \text{arctanh} \frac{\lambda}{\sqrt{-c}} + (j + \frac{1}{2})\pi i \right) \) (|\lambda| > \sqrt{-c}) is equal to Ker\((A_v^c - \lambda \text{id})\). Therefore the nullity spaces for focal radii of \( M \) along \( \gamma_v \) span \( T_x M \) if and only if all the eigenvalues of \( A_v \) are not equal to \( \pm \sqrt{-c} \).

Let \( M \) be a \( C^\infty \)-submanifold with flat section in a symmetric space \( G/K \). Then we can define the complexification \( M^c \) of \( M \) as an anti-Kaehler submanifold in the anti-Kaehler symmetric space \( G^c/K^c \) associated with \( G/K \) (see [Koi3]). Denote by \( J \) and \( \hat{R} \) the complex structure and the curvature tensor of \( G^c/K^c \), respectively, and \( \hat{A} \) and \( \hat{\exp}^\perp \) the shape tensor and the normal exponential map of \( M^c \), respectively. Denote by \( \hat{\mathcal{R}} \) the horizontal distribution on the normal bundle \( T^\perp (M^c) \) of \( M^c \) with respect to the normal connection of \( M^c \). Take \( v \in T^\perp_x M(\subset T^\perp_x (M^c)) \) and \( z = s + ti \in \mathbb{C} \). Then we can show Ker\( \hat{\exp}_{sv+tJv}^\perp \subset \hat{\mathcal{R}}_{sv+tJv} \) and

\[
\hat{\exp}_{sv+tJv}^\perp(X_{sv+tJv}^L) = P_{\gamma_{sv+tJv}}(Q_{v,z}(X)) \quad (X \in T_x (M^c)),
\]

where \( X_{sv+tJv}^L \) is the horizontal lift of \( X \) to \( sv + tJv \), \( P_{\gamma_{sv+tJv}} \) is the parallel translation along the normal geodesic \( \gamma_{sv+tJv} \) of \( M^c \) and

\[
Q_{v,z} := \cos \left( s \sqrt{\hat{R}(v)} + t \left( J \circ \sqrt{\hat{R}(v)} \right) \right) - \sin \left( s \sqrt{\hat{R}(v)} + t \left( J \circ \sqrt{\hat{R}(v)} \right) \right) \circ \hat{A}_v
\]

(\( \hat{R}(v) := \hat{R}(\bullet, v)v \)). Hence \( \hat{\exp}^\perp(sv + tJv) \) is a focal point of \( M^c \) along the geodesic \( \gamma_{sv+tJv} \) if and only if \( z = s + ti \) is a zero point of the complex-valued function \( \hat{F}_v \) over \( \mathbb{C} \) defined by \( \hat{F}_v(z) := \det Q_{v,z} \), where \( Q_{v,z} \) is regarded as a \( \mathbb{C} \)-linear transformation of \( T_x (M^c) \) regarded as a complex linear space by \( J \). On the other hand, it is clear that the set of all zero points of \( \hat{F}_v \) is equal to that of \( F_v^c \). Therefore \( z = s + ti \) is a focal radius of \( M \) along \( \gamma_v \) if and only if \( \hat{\exp}^\perp(sv + tJv) \) is a focal point of \( M^c \) along \( \gamma_{sv+tJv} \) (see Figures 1 and 2). Thus we can catch the geometrical meaning of the focal radius (defined in \( \mathbb{C} \)).
In 1995, C.L. Terng and G. Thorbergsson [TT] introduced the notion of an equifocal submanifold in a symmetric space $G/K$. This notion is defined as a compact submanifold (which we denote by $M$) in $G/K$ with flat section, trivial normal holonomy group and parallel focal structure. Here the parallelity of the focal structure means that the tangential focal set $\mathcal{F}_{M,x}^R$’s ($x \in M$) move to one another under the parallel translations with respect to the normal connection of $M$. For a compact submanifold $M$ with flat section and trivial normal holonomy group, it is equifocal if and only if $\mathcal{F}_{M,x}^R$ is independent of the choice of $x \in M$ for any parallel normal vector field $\tilde{v}$ of $M$. Compact isoparametric submanifolds in a Euclidean space and compact isoparametric hypersurfaces in a sphere or a hyperbolic space are equifocal. On the other hand, Heintze-Liu-Olmos [HLO] defined the notion of isoparametric submanifold with flat section in a general complete Riemannian manifold as a (properly embedded) complete submanifold with flat section and trivial normal holonomy.
group whose sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction.

**Terminology.** In this paper, we shall call an isoparametric submanifold with flat section an *isoparametric submanifold* simply.

For a compact submanifold in a symmetric space of compact type, they [HLO] proved that the equifocality and the isoparametricness are equivalent. We [Koi2] introduced the notion of a complex equifocal submanifold as a (properly embedded) complete submanifold with flat section, trivial normal holonomy group and parallel complex focal structure, where the parallelity of complex focal structure means that \( \mathcal{F}_x \) is independent of the choice of \( x \in M \) for any parallel normal vector field \( \tilde{v} \) of \( M \). Here we note that, in the case where \( M \) is real analytic, the parallelity of the complex structure of \( M \) means the parallelity of the focal structure of \( M^c \) (see [Koi3]).

**Terminology.** In this paper, we shall call a complex equifocal submanifold an *equifocal submanifold* simply.

Let \( M \) be a submanifold in a symmetric space \( G/K \). If, for any unit normal vector \( v \) of \( M \), the normal Jacobi operator \( R(v) \) preserves \( T_x M \) (the base point \( v \)) invariantly and \( R(v) \) commutes with the shape operator \( A_v \), then \( M \) is said to be *curvature-adapted*. We [Koi3] proved that, for a (properly embedded) complete curvature-adapted submanifold in a symmetric space of non-compact type, the (complex) equifocality and the isoparametricness are equivalent (see Theorem 15 of [Koi3]). Assume that \( M \) is a curvature-adapted submanifold with flat section. If \( G/K \) is of compact type or Euclidean type, then it follows from (1.1) that the following fact \((\ast_R)\) holds:

\[
(\ast_R) \quad \text{For any unit normal vector } v \text{ of } M, \text{ the nullity spaces for real focal radii along the normal geodesic } \gamma_v \text{ span } T_x M \ominus (\text{Ker } A_v \cap \text{Ker } R(v)).
\]

See also the facts stated in the first paragraph in the case where \( G/K \) is a Euclidean space or a sphere. However, if \( G/K \) is of non-compact type, then it follows from (1.1) that this fact \((\ast_R)\) does not necessarily hold. For example, in the case where \( G/K \) is a hyperbolic space of constant curvature \( c(<0) \) and where \( M \) is a hypersurface, \((\ast_R)\) holds if and only if all the absolute values of the principal curvatures of \( M \) at each point are greater than \( \sqrt{-c} \) (see the fact stated in the first paragraph in the case where \( G/K \) is a hyperbolic space). So, in this paper, we considered the following condition:

\[
(\ast_C) \quad \text{For any unit normal vector } v \text{ of } M, \text{ the nullity spaces for focal radii (defined in } \mathbb{C} \text{) along the normal geodesic } \gamma_v \text{ span } (T_x M)^c \ominus (\text{Ker } A_v \cap \text{Ker } R(v))^c.
\]

This condition \((\ast_C)\) is the condition weaker than \((\ast_R)\). In the case where \( G/K \) is of non-
comact type, \((\ast_C)\) also does not necessarily hold. For example, in the case where \(G/K\) is a hyperbolic space of constant curvature \(c(<0)\) and where \(M\) is a hypersurface, \(M\) satisfies \((\ast_C)\) if and only if all the principal curvatures of \(M\) at each point of \(M\) are not equal to \(\pm \sqrt{-c}\) (see the fact stated in the first paragraph in the case where \(G/K\) is a hyperbolic space).

Let \(G/K\) be a symmetric space of non-compact type and \(H\) a closed subgroup of \(G\). If there exists an involution \(\sigma\) of \(G\) with \((\text{Fix } \sigma)_0 \subset H \subset \text{Fix } \sigma\), then we ([Koi4]) called the \(H\)-action on \(G/K\) a Hermann type action, where \(\text{Fix } \sigma\) is the fixed point group of \(\sigma\) and \((\text{Fix } \sigma)_0\) is the identity component of \(\text{Fix } \sigma\). In this paper, we call this action a Hermann action simply. According to the result in [Koi4], it follows that principal orbits of a Hermann action are curvature-adapted equifocal (hence isoparametric) \(C^\omega\)-submanifolds and that they satisfy the condition \((\ast_C)\).

In this paper, we prove that the following fact holds conversely.

Theorem A. Let \(M\) be a full irreducible curvature-adapted isoparametric \(C^\omega\)-submanifold of codimension greater than one in a symmetric space \(G/K\) of non-compact type. If \(M\) satisfies the above condition \((\ast_C)\), then \(M\) is a principal orbit of a Hermann action on \(G/K\).

Remark 1.1. (i) In this theorem, both the condition of the curvature-adaptedness and the condition \((\ast_C)\) are indispensable. In fact, we have the following examples. Let \(G/K\) be an irreducible symmetric space of non-compact type and rank greater than one such that the (restricted) root system of \(G/K\) is non-reduced. Let \(g = \mathfrak{t} + \mathfrak{p}\) \((g = \text{Lie } G, \mathfrak{t} = \text{Lie } K)\) be the Cartan decomposition associated with a symmetric pair \((G, K)\) and \(\mathfrak{a}\) a maximal abelian subspace of \(\mathfrak{p}\). Also, let \(\Delta_+\) be the positive root system of \(G/K\) with respect to \(\mathfrak{a}\) and \(\Pi = \{\lambda_1, \cdots, \lambda_r\}\) the simple root system of \(\Delta_+\), where we fix a lexicographic ordering of the dual space \(\mathfrak{a}^*\) of \(\mathfrak{a}\). Set \(n = \sum_{\lambda \in \Delta_+} g_{\lambda} \) and \(N := \exp n\), where \(g_{\lambda}\) is the root space for \(\lambda\) and \(\exp\) is the exponential map of \(G\). Any orbit of the \(N\)-action on \(G/K\) is a full irreducible curvature-adapted isoparametric \(C^\omega\)-submanifold of codimension one but it does not satisfy the condition \((\ast_C)\) (see [Koi9]). On the other hand, it is a principal orbit of no Hermann action. Thus the condition \((\ast_C)\) is indispensable in this theorem. Let \(H_{\lambda}\) be the element of \(\mathfrak{a}\) defined by \(\langle H_{\lambda}, \bullet \rangle = \lambda(\bullet)\). Take elements \(\lambda_i\) and \(\lambda_j\) of \(\Pi\) such that they are not connected in the Dynkin diagram and that \(2\lambda_1\) belongs to \(\Delta_+\), and one-dimensional subspaces \(l_k\) of \(\mathbb{R}H_{\lambda_k} + g_{\lambda_k}\) \((k = i, j)\). Set \(S := \exp((a + n) \odot (l_i + l_j))\). Then \(S\) is a subgroup of \(AN := \exp(a + n)\) and any orbit of the \(S\)-action on \(G/K\) is a full irreducible isoparametric \(C^\omega\)-submanifold of codimension two but it is not curvature-adapted (see [Koi9]). Furthermore, we can find an orbit satisfying the condition \((\ast_C)\) among orbits of the \(S\)-action. On the other hand, it is a principal orbit of no Hermann action. Thus the condition of the curvature-adaptedness is indispensable in this theorem.

(ii) In the proof of this theorem, the condition of the real analyticity of \(M\) is indis-
pensable. In fact, in the proof, it is key to prove the homogeneity of $M$ and the proof of the homogeneity is performed by showing the homogeneity of the lift $\tilde{\phi}^{-1}(M^c)$ of the complete complexification $M^c$ of $M$ to the path space $H^0([0,1],g^c)$ ($g^c := \text{Lie } G$) through a pseudo-Riemannian submersion $\phi : H^0([0,1],g^c) \to G^c/K^c$. However, $M^c$ cannot be defined unless $M$ is not real analytic.

Also, we prove the following fact.

**Theorem B.** Let $M$ be a full irreducible curvature-adapted isoparametric $C^\omega$-submanifold of codimension greater than one in an irreducible symmetric space $G/K$ of non-compact type. If $M$ satisfies the above condition $(\ast_R)$, then $M$ is a principal orbit of the isotropy action of $G/K$.

In the remained part of this introduction, we shall state the outline of the proof of the homogeneity of $M$ which will be given in Sections 3～5. Let $M$ be an isoparametric $C^\omega$-submanifold in a symmetric space $G/K$ of non-compact type as in Theorem A and $M^c$ the complexification of $M$, where we note that $M^c$ is not necessarily (geodesically) complete. Let $\pi$ be the natural projection of $G^c$ onto $G^c/K^c$ and $\phi : H^0([0,1],g^c) \to G^c$ the parallel transport map for $G^c$. See the next section about the definition of the parallel transport map for $G^c$. In Section 3, we shall first show that $\pi(\phi^{-1}(M^c))$ is a proper anti-Kaehler isoparametric submanifold in $H^0([0,1],g^c)$. See the next section about the definition of a proper anti-Kaehler isoparametric submanifold. Hence the complete complexification of $M$ is defined (see [Koi7] in detail). Denote by the same symbol $M^c$ this complete complexification under abuse of the notation, and $\tilde{M}^c$ the inverse image of this complete complexification by $\pi \circ \phi$. Next, in the section, we shall prove that $\tilde{M}^c$ is homogeneous by imitating the proof of the homogeneity of a full irreducible infinite dimensional isoparametric submanifold of codimension greater than one in a Hilbert space by Heintze-Liu [HL2]. In more detail, we shall construct the group of holomorphic isometries of $H^0([0,1],g^c)$ which preserves $\tilde{M}^c$ invariantly and act on $\tilde{M}^c$ transitively. In Section 4, by imitating the discussion in [GH], we shall prove that the holomorphic Killing fields associated with one-parameter subgroups of the group of holomorphic isometries of $H^0([0,1],g^c)$ constructed in Section 3 are defined on the whole of $H^0([0,1],g^c)$. In Section 5, by imitating the proof of the homogeneity of a full irreducible equifocal(=isoparametric) submanifold of codimension greater than one in a symmetric space of compact type by Christ [Ch], we shall construct the group of isometries of $G/K$ which preserves $M$ invariantly and acts on $M$ transitively from the group of holomorphic isometries of $H^0([0,1],g^c)$ (which preserves $\tilde{M}^c$ invariantly and act on $\tilde{M}^c$ transitively) constructed in Section 3, where we use the fact in Section 4.
2 Basic notions and facts

In this section, we first shall recall the notion of an infinite dimensional proper anti-Kaehler isoparametric submanifold introduced in [Koi3]. Let $M$ be an anti-Kaehler Fredholm submanifold in an infinite dimensional anti-Kaehler space $V$. See [Koi3] about the definitions of an infinite dimensional anti-Kaehler space and anti-Kaehler Fredholm submanifold in the space. Denote by $A$ the shape tensor of $M$ and the same symbol $J$ the complex structures of $M$ and $V$. Fix a unit normal vector $v$ of $M$. If there exists $X(\neq 0) \in TM$ with $A_vX = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a $J$-eigenvalue of $A_v$ (or a $J$-principal curvature of direction $v$) and call $X$ a $J$-eigenvector for $a + b\sqrt{-1}$. Also, we call the space of all $J$-eigenvectors for $a + b\sqrt{-1}$ a $J$-eigenspace for $a + b\sqrt{-1}$. The $J$-eigenspaces are orthogonal to one another and they are $J$-invariant, respectively. We call the set of all $J$-eigenvalues of $A_v$ the $J$-spectrum of $A_v$ and denote it by $\text{Spec}_J A_v$. Since $M$ is an anti-Kaehler Fredholm submanifold, the set $\text{Spec}_J A_v \setminus \{0\}$ is described as follows:

$$\text{Spec}_J A_v \setminus \{0\} = \{\mu_i | i = 1, 2, \ldots \}$$

$$\left(\begin{array}{c}
|\mu_i| > |\mu_{i+1}| \text{ or } |\mu_i| = |\mu_{i+1}| \text{ & } \text{Re}\, \mu_i > \text{Re}\, \mu_{i+1} \\
\text{or } |\mu_i| = |\mu_{i+1}| \text{ & } \text{Re}\, \mu_i = \text{Re}\, \mu_{i+1} \text{ & } \text{Im}\, \mu_i = -\text{Im}\, \mu_{i+1} > 0
\end{array}\right).$$

Also, the $J$-eigenspace for each $J$-eigenvalue of $A_v$ other than 0 is of finite dimension. We call the $J$-eigenvalue $\mu_i$ the $i$-th $J$-principal curvature of direction $v$. Assume that the normal holonomy group of $M$ is trivial. Fix a parallel normal vector field $\tilde{v}$ of $M$. Assume that the number (which may be $\infty$) of distinct $J$-principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$. Then we can define complex-valued functions $\tilde{\mu}_i$ $(i = 1, 2, \ldots)$ on $M$ by assigning the $i$-th $J$-principal curvature of direction $\tilde{v}_x$ to each $x \in M$. We call this function $\tilde{\mu}_i$ the $i$-th $J$-principal curvature function of direction $\tilde{v}$. The submanifold $M$ is called an anti-Kaehler isoparametric submanifold if it satisfies the following condition:

For each parallel normal vector field $\tilde{v}$ of $M$, the number of distinct $J$-principal curvatures of direction $\tilde{v}_x$ is independent of the choice of $x \in M$, each $J$-principal curvature function of direction $\tilde{v}$ is constant on $M$ and it has constant multiplicity.

Let $\{e_i\}_{i=1}^\infty$ be an orthonormal system of $T_x M$. If $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$ is an orthonormal basis of $T_x M$, then we call $\{e_i\}_{i=1}^\infty$ (rather than $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$) a $J$-orthonormal base. If there exists a $J$-orthonormal base consisting of $J$-eigenvectors of $A_v$, then $A_v$ is said to be diagonalized with respect to the $J$-orthonormal base. If $M$ is anti-Kaehler isoparametric and, for each $v \in T^\perp M$, the shape operator $A_v$ is diagonalized with respect to a $J$-orthonormal base, then we call $M$ a proper anti-Kaehler isoparametric submanifold. We
Proposition 3.7 of [Koi5], \( J \) contains all complex Coxeter group associated with the \( PW \) (up to group isomorphicness). Hence we simply denote it by reflection groups) if and only if there exist two \( J \)-decomposable (i.e., it is decomposed into a non-trivial product of two discrete complex curvature normals of \( M \), its \( f \)-principal curvatures. Set \( T_x^\perp M = \bigoplus_{i \in I} (E_i)_x \) (the closure of \( \bigoplus_{i \in I} (E_i)_x \)). We regard \( T_x^\perp M \) \((x \in M)\) as a complex vector space by \( J_x \vert_{T_x^\perp M} \) and denote the dual space of the complex vector space \( T_x^\perp M \) by \( (T_x^\perp M)^e \). Also, denote by \((T^\perp M)^e \) the complex vector bundle over \( M \) having \((T_x^\perp M)^e \) as the fibre over \( x \). Let \( \lambda_i \) \((i \in I)\) be the section of \((T^\perp M)^e \) such that \( A_v \equiv \text{Re}(\lambda_i)_x(v) \text{id} + \text{Im}(\lambda_i)_x(v)J_x \) on \((E_i)_x \) for any \( x \in M \) and any \( v \in T_x^\perp M \). We call \( \lambda_i \) \((i \in I)\) \( J \)-principal curvatures of \( M \) and \( E_i \) \((i \in I)\) \( J \)-curvature distributions of \( M \). The distribution \( E_i \) is integrable and each leaf of \( E_i \) is a complex sphere. Each leaf of \( E_i \) is called a complex curvature sphere. It is shown that there uniquely exists a normal vector field \( n_i \) of \( M \) with \( \lambda_i(\cdot) = \langle n_i, \cdot \rangle - \sqrt{-1} (Jn_i, \cdot) \) (see Lemma 5 of [Koi3]). We call \( n_i \) \((i \in I)\) the \( J \)-curvature normals of \( M \). Note that \( n_i \) is parallel with respect to the complexification of the normal connection of \( M \). Note that similarly are defined a (finite dimensional) proper anti-Kahler isoparametric submanifold in a finite dimensional anti-Kahler space, its \( J \)-principal curvatures, its \( J \)-curvature distributions and its \( J \)-curvature normals. Set \( l_i^\perp := (\lambda_i)_x^{-1}(1) \). According to (i) of Theorem 2 in [Koi3], the tangential focal set of \( M \) at \( x \) is equal to \( \bigcup_{i \in I} l_i^\perp \). We call each \( l_i^\perp \) a complex focal hyperplane of \( M \) at \( x \). Let \( \tilde{v} \) be a parallel normal vector field of \( M \). If \( \tilde{v}_x \) belongs to at least one \( l_i \), then it is called a focal normal vector field of \( M \). For a focal normal vector field \( \tilde{v} \), the focal map \( f_{\tilde{v}} \) is defined by \( f_{\tilde{v}}(x) := \exp_{\tilde{v}_x}^\perp(x) \) \((x \in M)\). The image \( f_{\tilde{v}}(M) \) is called a focal submanifold of \( M \), where we denote by \( F_{\tilde{v}} \). For each \( x \in F_{\tilde{v}} \), the inverse image \( f_{\tilde{v}}^{-1}(x) \) is called a focal leaf of \( M \). Denote by \( T_i^\perp \) the complex reflection of order 2 with respect to \( l_i^\perp \) (i.e., the rotation of angle \( \pi \) having \( l_i^\perp \) as the axis), which is an affine transformation of \( T_x^\perp M \). Let \( W_x \) be the group generated by \( T_i^\perp \)'s \((i \in I)\). According to Proposition 3.7 of [Koi5], \( W_x \) is discrete. Furthermore, it follows from this fact that \( W_x \) is isomorphic an affine Weyl group. This group \( W_x \) is independent of the choice of \( x \in M \) (up to group isomorphism). Hence we simply denote it by \( W \). We call this group \( W \) the complex Coxeter group associated with \( M \). According to Lemma 3.8 of [Koi5], \( W \) is decomposable (i.e., it is decomposed into a non-trivial product of two discrete complex reflection groups) if and only if there exist two \( J \)-invariant linear subspaces \( P_1 \) \((\neq \{0\}) \) and \( P_2 \) \((\neq \{0\}) \) of \( T_x^\perp M \) such that \( T_x^\perp M = P_1 \oplus P_2 \) (orthogonal direct sum), \( P_1 \cup P_2 \) contains all \( J \)-curvature normals of \( M \) at \( x \) and that \( P_i \) \((i = 1, 2)\) contains at least one \( J \)-curvature normal of \( M \) at \( x \). Also, according to Theorem 1 of [Koi5], \( M \) is irreducible if
and only if \( W \) is not decomposable.

Next we shall recall the notions of an anti-Kaehler symmetric space and the aks-representation introduced in [Koi3] and [Koi7]. Let \( J \) be a parallel complex structure on an even dimensional pseudo-Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) of half index. If \( \langle JX, JY \rangle = -\langle X, Y \rangle \) holds for every \( X, Y \in TM \), then \((M, \langle \cdot, \cdot \rangle, J)\) is called an anti-Kaehler manifold. Let \( G/K \) be a symmetric space of non-compact type, \((\mathfrak{g}, \sigma)\) its orthogonal symmetric Lie algebra and \( g = \mathfrak{k} + \mathfrak{p} \) the Cartan decomposition associated with a symmetric pair \((G, K)\), where \( \mathfrak{k} \) is the Lie algebra of \( K \). Note that \( \mathfrak{p} \) is identified with the tangent space \( T_eK(G/K) \), where \( e \) is the identity element of \( G \). Let \( \langle \cdot, \cdot \rangle \) be the \( \text{Ad}_G(G) \)-invariant non-degenerate inner product of \( \mathfrak{g} \) inducing the Riemannian metric of \( G/K \), where \( \text{Ad}_G \) is the adjoint representation of \( G \). Let \( G^c \) (resp. \( K^c \)) be the complexification of \( G \) (resp. \( K \)). Without loss of generality, we may assume that \( K^c \) is connected and that \( G^c \) is simply connected. The 2-multiple of the real part \( \text{Re}\langle \cdot, \cdot \rangle^c \) of \( \langle \cdot, \cdot \rangle^c \) is equal to the Killing form of \( \mathfrak{g}^c \) regarded as a real Lie algebra. The restriction \( 2\text{Re}\langle \cdot, \cdot \rangle^c|_{\mathfrak{p}^c \times \mathfrak{p}^c} \) is an \( \text{Ad}(K^c) \)-invariant non-degenerate inner product of \( \mathfrak{p}^c \) (\( = T_eK^c(G^c/K^c) \)). Denote by \( \langle \cdot, \cdot \rangle^A \) the \( G^c \)-invariant pseudo-Riemannian metric on \( G^c/K^c \) induced from \( 2\text{Re}\langle \cdot, \cdot \rangle^c|_{\mathfrak{p}^c \times \mathfrak{p}^c} \). Define an almost complex structure \( J_0 \) of \( \mathfrak{p}^c \) by \( J_0X = \sqrt{-1}X \) \( X \in \mathfrak{p}^c \). It is clear that \( J_0 \) is \( \text{Ad}(K^c) \)-invariant. Denote by \( J \) the \( G^c \)-invariant almost complex structure on \( G^c/K^c \) induced from \( J_0 \). It is shown that \((G^c/K^c, \langle \cdot, \cdot \rangle^A, J)\) is an anti-Kaehler manifold and a (semi-simple) pseudo-Riemannian symmetric space. We call this anti-Kaehler manifold an anti-Kaehler symmetric space associated with \( G/K \) and simply denote it by \( G^c/K^c \). The action \( \text{Ad}_{G^c}(K^c) \) on \( \mathfrak{g}^c \) preserves \( \mathfrak{p}^c \) invariantly, where \( \text{Ad}_{G^c} \) is the adjoint representation of \( G^c \). Define a representation \( \rho \) of \( K^c \) on \( \mathfrak{p}^c \) by \( \rho(k)(X) := \text{Ad}_{G^c}(k)(X) \) \( (k \in K^c, X \in \mathfrak{p}^c) \). If \( G^c/K^c \) is irreducible, then we call this representation \( \rho \) the aks-representation (associated with \( G^c/K^c \)). Let \( X_0 \) be a semi-simple element of \( \mathfrak{p}^c \), where the semi-simleness of \( X_0 \) means that the complexification of \( \text{ad}_{\mathfrak{p}^c}(X_0) \) is diagonalizable. If the orbit \( \rho(K^c) \cdot X_0 \) is principal, then it will be shown that the orbit is a (finite dimensional) proper anti-Kaehler isoparametric submanifold (see Lemma 3.7.3). As in the infinite dimensional case, we can define the complex Coxeter group associated with a finite dimensional proper anti-Kaehler isoparametric submanifold. This group is isomorphic to a Weyl group. For example, the complex Coxeter group associated with a principal orbit of the aks-representation associated with \( G^c/K^c \) is isomorphic to the Weyl group associated with the symmetric pair \((G, K)\).

Next we shall recall the notion of the parallel transport map for the complexification \( G^c \) of a semi-simple Lie group \( G \) introduced in [Koi3]. Let \( K \) be a maximal compact subgroup of \( G \), \( \mathfrak{g} \) (resp. \( \mathfrak{k} \)) the Lie algebra of \( G \) (resp. \( K \)) and \( g = \mathfrak{k} + \mathfrak{p} \) a Cartan decomposition of \( \mathfrak{g} \). Also, let \( \langle \cdot, \cdot \rangle \) be the \( \text{Ad}_G(G) \)-invariant non-degenerate inner product of \( \mathfrak{g} \). The Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) is an orthogonal time-space decomposition of \( \mathfrak{g} \) with respect to \( \langle \cdot, \cdot \rangle \), that is, \( \langle \cdot, \cdot \rangle|_{\mathfrak{k} \times \mathfrak{k}} \) is negative definite, \( \langle \cdot, \cdot \rangle|_{\mathfrak{p} \times \mathfrak{p}} \) is positive definite.
and $\langle \cdot, \cdot \rangle_{\mathbb{T}x\mathbb{P}}$ vanishes. Set $\langle \cdot, \cdot \rangle^A := 2\text{Re}\langle \cdot, \cdot \rangle^c$, where $\langle \cdot, \cdot \rangle^c$ is the complexification of $\langle \cdot, \cdot \rangle$ (which is a $\mathbb{C}$-bilinear form of $\mathfrak{g}^c$). The $\mathbb{R}$-bilinear form $\langle \cdot, \cdot \rangle^A$ on $\mathfrak{g}^c$ regarded as a real Lie algebra induces a bi-invariant pseudo-Riemannian metric on $G^c$ and furthermore a $G^c$-invariant anti-Kaehler metric on $G^c/K^c$. It is clear that $\mathfrak{g}^c = (\mathbb{T} + \sqrt{-1}\mathbb{P}) \oplus (\sqrt{-1}\mathbb{T} + \mathbb{P})$ is an orthogonal time-space decomposition of $\mathfrak{g}^c$ with respect to $\langle \cdot, \cdot \rangle^A$. For simplicity, set $\mathfrak{g}^c_{\pm} := \mathbb{T} + \sqrt{-1}\mathbb{P}$ and $\mathfrak{g}^c_+ := \sqrt{-1}\mathbb{T} + \mathbb{P}$. Note that $\mathfrak{g}^c_\pm$ is the compact real form of $\mathfrak{g}^c$. Set $\langle \cdot, \cdot \rangle^A_{\mathfrak{g}^c_{\pm}} := -\pi^*_c(\cdot, \cdot)^A + \pi^*_c(\cdot, \cdot)^A$, where $\pi^*_c$ (resp. $\pi^*_c$) is the projection of $\mathfrak{g}^c$ onto $\mathfrak{g}^c_\pm$ (resp. $\mathfrak{g}^c_+ \cap \mathfrak{g}^c_- \cap \mathfrak{g}^c_0$). Let $H^0([0, 1], \mathfrak{g}^c)$ be the space of all $L^2$-integrable paths $u : [0, 1] \to \mathfrak{g}^c$ with respect to $\langle \cdot, \cdot \rangle^A_{\mathfrak{g}^c_{\pm}}$ and $H^0([0, 1], \mathfrak{g}^c)$ (resp. $H^0([0, 1], \mathfrak{g}^c)$) the space of all $L^2$-integrable paths $u : [0, 1] \to \mathfrak{g}^c_\pm$ (resp. $u : [0, 1] \to \mathfrak{g}^c_+$) with respect to $-\langle \cdot, \cdot \rangle^A_{\mathfrak{g}^c_{\pm}} \times \mathfrak{g}^c_-$ (resp. $\langle \cdot, \cdot \rangle^A_{\mathfrak{g}^c_+ \cap \mathfrak{g}^c_- \cap \mathfrak{g}^c_0}$). Clearly we have $H^0([0, 1], \mathfrak{g}^c) = H^0([0, 1], \mathfrak{g}^c) \oplus H^0([0, 1], \mathfrak{g}^c)$. Define a non-degenerate inner product $\langle \cdot, \cdot \rangle^A_0$ of $H^0_0([0, 1], \mathfrak{g}^c)$ by $\langle u, v \rangle^A_0 := \int_0^1 \langle u(t), v(t) \rangle^A dt$. It is easy to show that the decomposition $H^0([0, 1], \mathfrak{g}^c) = H^0_0([0, 1], \mathfrak{g}^c) \oplus H^0([0, 1], \mathfrak{g}^c) \oplus H^0_0([0, 1], \mathfrak{g}^c)$ is an orthgonal time-space decomposition with respect to $\langle \cdot, \cdot \rangle^A_0$. For simplicity, set $\mathfrak{H}^0 := H^0([0, 1], \mathfrak{g}^c) (\varepsilon = - \text{ or } +)$ and $\langle \cdot, \cdot \rangle^A_{\mathfrak{H}^0} := -\pi^*_c(\cdot, \cdot)^A + \pi^*_c(\cdot, \cdot)^A$, where $\pi^*_c$ (resp. $\pi^*_c$) is the projection of $H^0([0, 1], \mathfrak{g}^c)$ onto $H^0_\varepsilon$ (resp. $H^0_{\varepsilon} \cap \mathfrak{g}^c_0$). It is clear that $\langle u, v \rangle^A_{\mathfrak{H}^0} = \int_0^1 \langle u(t), v(t) \rangle^A dt (u, v \in H^0([0, 1], \mathfrak{g}^c))$. Hence $(H^0([0, 1], \mathfrak{g}^c), \langle \cdot, \cdot \rangle^A_{\mathfrak{H}^0})$ is a Hilbert space, that is, $(H^0([0, 1], \mathfrak{g}^c), \langle \cdot, \cdot \rangle^A_{\mathfrak{H}^0})$ is a pseudo-Hilbert space in the sense of [Koi2]. Let $J$ be the endomorphism of $\mathfrak{g}^c$ defined by $JX = \sqrt{-1}X$ ($X \in \mathfrak{g}^c$). Denote by the same symbol $J$ the bi-invariant almost complex structure of $G^c$ induced from $J$. Define the endomorphism $\tilde{J}$ of $H^0([0, 1], \mathfrak{g}^c)$ by $\tilde{J}u = \sqrt{-1}u (u \in H^0([0, 1], \mathfrak{g}^c))$. From $JH^0_\varepsilon = H^0_\varepsilon$, $\tilde{J}H^0_\varepsilon = H^0_\varepsilon$ and $\langle Ju, Jv \rangle^A_0 = -\langle u, v \rangle^A_0 (u, v \in H^0([0, 1], \mathfrak{g}^c))$, the space $(H^0([0, 1], \mathfrak{g}^c), \langle \cdot, \cdot \rangle^A_{\mathfrak{H}^0}, J)$ is an anti-Kaehler space. Let $H^1([0, 1], \mathfrak{g}^c)$ be a pseudo-Hilbert subspace of $H^0([0, 1], \mathfrak{g}^c)$ consisting of all absolutely continuous paths $u : [0, 1] \to \mathfrak{g}^c$ such that the weak derivative $u'$ of $u$ is squared integrable (with respect to $\langle \cdot, \cdot \rangle^A_{\mathfrak{H}^0}$).

Also, let $H^1([0, 1], G^c)$ be the Hilbert Lie group of all absolutely continuous paths $g : [0, 1] \to G^c$ such that the weak derivative $g'$ of $g$ is squared integrable (with respect to $\langle \cdot, \cdot \rangle^A_{\mathfrak{H}^0}$), that is, $g^{-1}g' \in H^0([0, 1], \mathfrak{g}^c)$. Define a map $\phi : H^0([0, 1], \mathfrak{g}^c) \to G^c$ by $\phi(u) := g_u(1) (u \in H^0([0, 1], \mathfrak{g}^c))$, where $g_u$ is the element of $H^1([0, 1], G^c)$ with $g_u(0) = e$ and $g_u^{-1}g_u = u$. This map is called the parallel transport map for $G^c$. This map is an anti-Kaehler submersion. Set $P(G^c, e \times G^c) := \{ g \in H^1([0, 1], G^c) | g(0) = e \}$ and $\Omega^e(G^c) := \{ g \in H^1([0, 1], G^c) | g(0) = g(1) = e \}$. The group $H^1([0, 1], G^c)$ acts on $H^0([0, 1], \mathfrak{g}^c)$ by gauge transformations, that is,

$$g \ast u := \text{Ad}_{g^c}(g)u - g'g^{-1} \quad (g \in H^1([0, 1], G^c), \quad u \in H^0([0, 1], \mathfrak{g}^c)).$$

It is shown that the following facts hold:

(i) The above action of $H^1([0, 1], G^c)$ on $H^0([0, 1], \mathfrak{g}^c)$ is isometric,
(ii) The above action of $P(G^e, e \times G^e)$ in $H^0([0, 1], g^e)$ is transitive and free,
(iii) $\phi(g \ast u) = (Lg(0) \circ R^{-1}_{g(1)})(\phi(u))$ for $g \in H^1([0, 1], G^e)$ and $u \in H^0([0, 1], g^e)$,
(iv) $\phi : H^0([0, 1], g^e) \to G^e$ is regarded as a $\Omega_e(G^e)$-bundle.
(v) If $\phi(u) = (Lx_0 \circ R^{-1}_{x_1})(\phi(v))$ ($u, v \in H^0([0, 1], g^e)$, $x_0, x_1 \in G^e$), then there exists $g \in H^1([0, 1], G^e)$ such that $g(0) = x_0$, $g(1) = x_1$ and $u = g \ast v$. In particular, it follows that any $u \in H^0([0, 1], g^e)$ is described as $u = g \ast 0$ in terms of some $g \in P(G^e, G^e \times e)$.

3 Homogeneity of the lift of the complexification

Let $M$ be a full irreducible curvature-adapted isoparametric $C^\omega$-submanifold of codimension greater than one in a symmetric space $G/K$ of non-compact type. Assume that $M$ satisfies the condition $(\ast_C)$ stated in Introduction. Let $M^c(\to G^e/K^e)$ be the complexification of $M$, where we note that $M^c$ is not necessarily complete. See Section 4 of [Koi3] about the definition of the complexification $M^c(\to G^e/K^e)$. Also, see [Koi12] about the definition of the complexification of a general pseudo-Riemannian $C^\omega$-submanifold. Let $\pi$ be the natural projection of $G^e$ onto $G^e/K^e$ and $\phi : H^0([0, 1], g^e) \to G^e$ the parallel transport map for $G^e$. Set $\tilde{M}^c := (\pi \circ \phi)^{-1}(M^c)$. Without loss of generality, we may assume that $K^c$ is connected and that $G^e$ is simply connected. Hence $\tilde{M}^c$ is connected. Denote by $A$ the shape tensor of $M$ and $R$ the curvature tensor of $G/K$. First we shall show the following fact.

**Lemma 3.1.** The submanifold $\tilde{M}^c$ is a (not necessarily complete) full irreducible proper anti-Kaehler isoparametric submanifold.

**Proof.** Fix $x \in M$ and a unit normal vector $v$ of $M$ at $x$. Denote by $\Spec A_v$ and $\Spec R(v)$ the spectrum of $A_v$ and $R(v)$, respectively. For simplicity, set $D_{\lambda \mu} := \Ker(A_v - \lambda \id) \cap \Ker(R(v) - \mu \id)$, $\mathcal{S} := \{(\lambda, \mu) \in \Spec A_v \times \Spec R(v) \mid D_{\lambda \mu} \neq \{0\}\}$, $\mathcal{S}_+ := \{(\lambda, \mu) \in \mathcal{S} \mid |\lambda| > \sqrt{-\mu}\}$ and $\mathcal{S}_- := \{(\lambda, \mu) \in \mathcal{S} \mid |\lambda| < \sqrt{-\mu}\}$. Since $M$ is curvature-adapted, we have $T_xM = \oplus_{(\lambda, \mu) \in \mathcal{S}} D_{\lambda \mu}$. For simplicity, set

$$Q(z) := \cos(z \sqrt{R(v)^e}) - \frac{\sin(z \sqrt{R(v)^e})}{\sqrt{R(v)^e}} \circ A_v^e.$$  

Clearly we have

$$Q(z)|_{D_{\lambda \mu}} = \left(\cos(iz \sqrt{-\mu}) - \frac{\lambda \sin(iz \sqrt{-\mu})}{i \sqrt{-\mu}}\right) \id.$$  

Hence, if $(\lambda, \mu) \in \mathcal{S}_+$ and $\mu \neq 0$, then $\frac{1}{\sqrt{-\mu}} \left(\arctanh \frac{\sqrt{-\mu}}{\lambda} + k \pi i\right)$ ($k \in \mathbb{Z}$) are focal radii along $\gamma_v$ including $D_{\lambda \mu}$ as the nullity space. Also, if $(\lambda, \mu) \in \mathcal{S}_-$ and $\mu \neq 0$, then
\( \frac{1}{\sqrt{-\mu}} \left( \text{arctanh} \sqrt{-\frac{\mu}{\lambda}} + (k + \frac{1}{2})\pi i \right) \) (\( k \in \mathbb{Z} \)) are focal radii along \( \gamma_v \) including \( D_{\lambda\mu} \) as the nullity space. Also, if \( \lambda \in \text{Spec} A_v \setminus \{0\} \) satisfying \( (\lambda, 0) \in \mathcal{S} \), then \( \frac{1}{\lambda} \) is a focal radii along \( \gamma_v \) including \( D_{\lambda0} \) as the nullity space. Also, if \( |\lambda| = \sqrt{-\mu} \), then there exists no focal radius along \( \gamma_v \) such that the intersection of its nullity space and \( D_{\lambda\mu} \) is not equal to the zero space. Hence, since \( M \) satisfies the condition \( (*_C) \), there exists no \( (\lambda, \mu) \in \mathcal{S} \) satisfying \( |\lambda| = \sqrt{-\mu} \neq 0 \). Thus we have

\[ (3.1) \quad T_x M = D_{00} \oplus \left( \bigoplus_{(\lambda, \mu) \in \mathcal{S}_+ \cup \mathcal{S}_-} D_{\lambda\mu} \right). \]

Denote by \( \tilde{A} \) the shape tensor of \( \tilde{M}^c \). Fix \( u \in (\pi \circ \phi)^{-1}(x) \). Let \( v_u^L \) be the horizontal lift of \( v \) to \( u \). Then we have

\[ \text{Spec } \tilde{A}_{v_u^L} = \{ \lambda | \lambda \in \text{Spec} A_v \text{ s.t. } (\lambda, 0) \in \mathcal{S}_+ \} \cup \left\{ \frac{\sqrt{-\mu}}{\text{arctanh} \sqrt{-\frac{\mu}{\lambda}} + \pi ki} | (\lambda, \mu) \in \mathcal{S}_+, \mu \neq 0 \right\} \]

\[ \cup \left\{ \frac{\sqrt{-\mu}}{\text{arctanh} \sqrt{-\frac{\mu}{\lambda}} + (k + \frac{1}{2})\pi i} | (\lambda, \mu) \in \mathcal{S}_- \right\} \]

(see Proposition 4 of [Koi3] in detail). For simplicity, set

\[ \Lambda^+_{\lambda,\mu,k} := \frac{\sqrt{-\mu}}{\text{arctanh} \sqrt{-\frac{\mu}{\lambda}} + \pi ki} \quad ((\lambda, \mu) \in \mathcal{S}_+ \text{ s.t. } \mu 
eq 0, \ k \in \mathbb{Z}) \]

and

\[ \Lambda^-_{\lambda,\mu,k} := \frac{\sqrt{-\mu}}{\text{arctanh} \sqrt{-\frac{\mu}{\lambda}} + (k + \frac{1}{2})\pi i} \quad ((\lambda, \mu) \in \mathcal{S}_-, \ k \in \mathbb{Z}). \]

Also, we set \( \tilde{D}_\lambda := \text{Ker } (\tilde{A}_{v_u^L} - \lambda \text{id}) \ (\lambda \in \text{Spec} A_v \text{ s.t. } (\lambda, 0) \in \mathcal{S}_+) \), \( \tilde{D}^+_{\lambda,\mu,k} := \text{Ker } (\tilde{A}_{v_u^L} - \Lambda^+_{\lambda,\mu,k} \text{id}) \) \((\lambda, \mu) \in \mathcal{S}_+ \text{ s.t. } \mu 
eq 0, \ k \in \mathbb{Z})\) and \( \tilde{D}^-_{\lambda,\mu,k} := \text{Ker } (\tilde{A}_{v_u^L} - \Lambda^-_{\lambda,\mu,k} \text{id}) \) \((\lambda, \mu) \in \mathcal{S}_-, \ k \in \mathbb{Z})\). Then, by using (3.1), we can show that \( T_u \tilde{M}^c \) is equal to

\[ \left( \bigoplus_{\lambda \in \text{Spec} A_v \text{ s.t. } (\lambda, 0) \in \mathcal{S}_+} \tilde{D}_\lambda \right) \oplus \left( \bigoplus_{(\lambda, \mu) \in \mathcal{S}_+ \text{ s.t. } \mu 
eq 0} \tilde{D}^+_{\lambda,\mu,k} \right) \oplus \left( \bigoplus_{(\lambda, \mu) \in \mathcal{S}_-, \ k \in \mathbb{Z}} \tilde{D}^-_{\lambda,\mu,k} \right) \]

(see [Koi2,3] in detail). That is, \( \tilde{A}_{v_u^L} \) is diagonalized with respect to a \( J \)-orthonormal base of \( T_u \tilde{M}^c \). Therefore \( \tilde{M}^c \) is a proper anti-Kaehler isoparametric submanifold by the arbitrariness of \( x, v \) and \( u \). Hence \( M \) is a proper complex equifocal submanifold in the sense of [Koi4]. Since \( M \) is irreducible, it follows from Theorem 2 of [Koi5] that the complex Coxeter group associated with \( M \) is not decomposable, where we note that the complex
Coxeter groups associated with $M$ is equal to one associated with $\tilde{M}^c$ (see Introduction of [Koi5]). Hence, it follows from Theorem 1 of [Koi5] that $\tilde{M}^c$ is irreducible. Also, since $M$ is full, it is shown that the $J$-curvature normals of $\tilde{M}^c$ span the normal space of $\tilde{M}^c$ at each point of $\tilde{M}^c$ (see the discussion in the proof of Theorem 2 of [Koi5]). Furthermore, it follows from this fact that $\tilde{M}^c$ is full (see the discussion in the proof of Theorem 1 of [Koi5]). q.e.d.

Since $M$ is a proper complex equifocal submanifold in the sense of [Koi4] by this lemma, we can define the complete complexification of $M^c$ (see [Koi7] in detail). In the sequel, we denote this complete complexification by $\tilde{M}^c$. Also we denote the inverse image of this complete complexification by $\pi \circ \phi$ by $\tilde{M}^c$. E. Heintze-X.Liu [HL2] proved that a full irreducible isoparametric submanifold of codimension greater than one in a Hilbert space is homogeneous. In this section, we shall prove the following homogeneity theorem by imitating their proof.

**Theorem 3.2.** The submanifold $\tilde{M}^c$ is homogeneous.

Now we shall prepare some lemmas (or theorems) to prove this theorem. First we shall recall the generalized Chow’s theorem, which was proved in [HL2]. Let $N$ be a (connected) Hilbert manifold and $\mathcal{D}$ a set of local (smooth) vector fields which are defined over open sets of $N$. If two points $x$ and $y$ of $N$ can be connected by a piecwise smooth curve each of whose smooth segments is an integral curve of a local smooth vector field belonging to $\mathcal{D}$, then we say that $x$ and $y$ are $\mathcal{D}$-equivalent and we denote this fact by $x \sim_D y$. Let $\Omega_{\mathcal{D}}(x) := \{y \in N \mid y \sim_D x\}$. The set $\Omega_{\mathcal{D}}(x)$ is called the set of reachable points of $\mathcal{D}$ starting from $x$. Let $\mathcal{D}^*$ be the minimal set consisting of local smooth vector fields on open sets of $N$ which satisfies the following condition:

$$\mathcal{D} \subset \mathcal{D}^* \text{ and } \mathcal{D}^* \text{ contains the zero vector field and, for any } X, Y \in \mathcal{D}^* \text{ and any } a, b \in \mathbb{R},$$

$$aX + bY \text{ and } [X, Y] \text{ (which are defined on the intersection of the domains of } X \text{ and } Y) \text{ also belong to } \mathcal{D}^*.$$  

For each $x \in N$, set $\mathcal{D}^*(x) := \{X_x \mid X \in \mathcal{D}^* \text{ s.t. } x \in \text{Dom}(X)\}$. Then the following generalized Chow’s theorem holds.

**Theorem 3.3 ([HL2])** If $\overline{\mathcal{D}^*(x)} = T_xN$ for each $x \in N$, $\overline{\Omega_{\mathcal{D}}(x)} = N$ holds for each $x \in N$, where (·) implies the closure of (·).

For simplicity, we set $V := H^0([0, 1], g^c)$. Denote by $\tilde{A}$ the shape tensor of $\tilde{M}^c$. Let $\{E_i \mid i \in I\} \cup \{E_0\}$ be the set of all $J$-curvature distributions of $\tilde{M}^c$, where $E_0$ is defined by $(E_0)_u := \cap_{v \in T_u\tilde{M}^c} \text{Ker}A_v \ (u \in \tilde{M}^c)$. Also, let $\lambda_i$ and $n_i$ be the $J$-principal curvature and the $J$-curvature normal corresponding to $E_i$, respectively. Fix $u_0 \in \tilde{M}^c$. Denote by $l_i$ the
complex focal hyperplane $(\lambda_i)^{-1}(1)$ of $\tilde{M}^c$ at $u_0$. Let $Q(u_0)$ be the set of all points of $\tilde{M}^c$ connected with $u_0$ by a piecewise smooth curve in $\tilde{M}^c$ each of whose smooth segments is contained in some complex curvature sphere (which may depend on the smooth segment). By using the above generalized Chow’s theorem, we shall show the following result.

**Proposition 3.4.** The set $Q(u_0)$ is dense in $\tilde{M}^c$.

**Proof.** Let $D_E$ be the set of all local (smooth) tangent vector fields on open sets of $\tilde{M}^c$ which is tangent to some $E_i$ ($i \neq 0$) at each point of the domain. Define $\Omega_{D_E}(u_0), D^*_{E}$ and $D^*_E(u_0)$ as above. By imitating the proof of Proposition 5.8 of [HL1], it is shown that $\overline{D^*_E(u)} = T_u \tilde{M}^c$ for each $u \in \tilde{M}^c$. Hence, $\Omega_{D_E}(u_0) = \tilde{M}^c$ follows from Theorem 3.3. It is clear that $\Omega_{D_E}(u_0) = Q(u_0)$. Therefore we obtain $Q(u_0) = \tilde{M}^c$.

For each complex affine subspace $P$ of $T_u \tilde{M}^c$, define $I_P$ by

$$I_P := \left\{ i \in I \mid (n_i)_{u_0} \in P \right\} \cup \{0\} \ (0 \notin P).$$

Define a distribution $D_P$ on $\tilde{M}^c$ by $D_P := \bigoplus E_i$. If $0 \notin P$, then it is easy to show that $I_P$ is finite and that $(\bigcap_{i \in I_P \setminus \{0\}} l_i) \setminus (\bigcup_{i \notin I_P} l_i) \neq \emptyset$. Also, if $0 \in P$, then it is easy to show that $I_P$ is infinite or $I_P = \{0\}$, and that $(\bigcap_{i \in I_P \setminus \{0\}} (\lambda_i)^{-1}(0)) \setminus (\bigcup_{i \notin I_P} (\lambda_i)^{-1}(0))$ is dense in $\tilde{M}^c$. Assume that $0 \notin P$. Take $v \in \bigcap_{i \in I_P \setminus \{0\}} l_i \setminus (\bigcup_{i \notin I_P} l_i)$. Let $\tilde{v}$ be a parallel normal vector field on $\tilde{M}^c$ with $\tilde{v}_{u_0} = v$ and set $\pi := (\pi \circ \phi)_*(\tilde{v})$, which is well-defined because $\tilde{v}$ is projectable. These normal vector fields $\tilde{v}$ and $\pi$ are focal normal vector fields of $\tilde{M}^c$ and $M^c$, respectively. Let $f_{\tilde{v}}$ (resp. $f_{\pi}$) be the focal map (i.e., the end point map) for $\tilde{v}$ (resp. $\pi$) and $F_{\tilde{v}}$ (resp. $F_{\pi}$) the focal submanifold for $\tilde{v}$ (resp. $\pi$) (i.e., $F_{\tilde{v}} = f_{\tilde{v}}(\tilde{M}^c)$ (resp. $F_{\pi} = f_{\pi}(M^c)$)). Also, let $L^{D_P}_u$ be the leaf of $D_P$ through $u \in M^c$. We have $L^{D_P}_u = f_{\tilde{v}}^{-1}(f_{\tilde{v}}(u))$. By applying Theorem A of [Koi7] to $M^c$, we can show the following homogeneous slice theorem for $\tilde{M}^c$.

**Theorem 3.5.** If $0 \notin P$, then the leaf $L^{D_P}_u$ is a principal orbit of the direct sum representation (having $T_{F_{\tilde{v}}(u)} F_{\tilde{v}}$ as the representation space) of aks-representations.

**Proof.** Let $u_1 := f_{\tilde{v}}(u), F_{\tilde{v}} := (\pi \circ \phi)(F_{\tilde{v}})$ and $x_1 := (\pi \circ \phi)(u_1)$. Also, set $L := f_{\tilde{v}}^{-1}(u_1)$ and $L' := f_{\pi}^{-1}(x_1)$, which are leaves of the focal distributions corresponding to $\tilde{v}$ and $\pi$, respectively. According to Theorem A of [Koi7], $L'$ is the image of a principal orbit of the direct sum representation of aks-representations on $T_{x_1} F_{\tilde{v}}$ by the normal exponential map $\exp_{\tilde{v}}$. On the other hand, under the identification of $T_{u_1} F_{\tilde{v}}$ with $T_{x_1} F_{\tilde{v}}$, $L'$ is the image of $L$ by $\exp_{\tilde{v}}$. Hence it follows that $L$ is a principal orbit of the direct sum representation (having $T_{F_{\tilde{v}}(u)} F_{\tilde{v}}$ as the representation space) of aks-representations. Since
\[0 \notin P\] by the assumption, we have \(L_u^{D_P} = L\). Therefore the statement of this theorem follows.

Set \((W_P)_u := u + (D_P)_u \oplus \text{Span}_\mathbb{C}\{(n_i)_u | i \in I_P \setminus \{0\}\}\) \((u \in \tilde{M}^c)\). Let \(\gamma : [0,1] \to \tilde{M}^c\) be a piecewise smooth curve. In the sequel, we assume that the domains of all piecewise smooth curves are equal to \([0,1]\). If \(\gamma(t) \perp (D_P)_{\gamma(t)}\) for each \(t \in [0,1]\), then \(\gamma\) is said to be horizontal with respect to \(D_P\) or \(D_P\)-horizontal. Let \(\beta_i (i = 1,2)\) be curves in \(\tilde{M}^c\). If \(L_{\beta_i(t)}^{D_P} = L_{\beta_2(t)}^{D_P}\) for each \(t \in [0,1]\), then \(\beta_1\) and \(\beta_2\) are said to be parallel. By imitating the proof of Proposition 1.1 in [HL2], we can show the following fact.

**Lemma 3.6.** For each \(D_P\)-horizontal curve \(\gamma\), there exists an one-parameter family \(\{h_{\gamma,t}^{D_P} | 0 \leq t \leq 1\}\) of holomorphic isometries \(h_{\gamma,t}^{D_P} : (W_P)_{\gamma(0)} \to (W_P)_{\gamma(t)}\) satisfying the following conditions:

(i) \(h_{\gamma,t}^{D_P}(L_{\gamma(0)}^{D_P}) = L_{\gamma(t)}^{D_P}\) \((0 \leq t \leq 1)\),

(ii) for any \(u \in L_{\gamma(0)}^{D_P}\), \(t \mapsto h_{\gamma,t}^{D_P}(u)\) is a \(D_P\)-horizontal curve parallel to \(\gamma\),

(iii) for any \(u \in L_{\gamma(0)}^{D_P}\) and any \(i \in I_P\), \((h_{\gamma,t}^{D_P})_*((E_i)_u) = (E_i)_{h_{\gamma,t}^{D_P}(u)}\).

**Proof.** First we consider the case of \(0 \notin P\). Take \(v \in \bigcap_{i \in I_P} l^i \setminus \bigcup_{i \in I_P} l^i\). Let \(\tilde{v}\) be the parallel normal vector field of \(\tilde{M}^c\) with \(\tilde{v}_u = v\). Let \(\gamma : [0,1] \to \tilde{M}^c\) be a piecewise smooth curve. Denote by \(\tilde{F}_P\) the orthogonal complementary distribution of \(\tilde{F}_P\). Let \(U\) be a neighborhood of \(\gamma(0)\) in \(L_{\gamma(0)}^{D_P}\) such that there exists a family \(\psi_t : U \to U_t | 0 \leq t \leq 1\) of diffeomorphisms such that, for any \(u \in U\), the curve \(\gamma_t(u) := \psi_t(u)\) is a \(D_P\)-horizontal curve, where \(U_t\) is a neighborhood of \(\gamma(t)\) in \(L_{\gamma(t)}^{D_P}\). Note that such a family of diffeomorphisms is called an element of holonomy along \(\gamma\) (with respect to \(F_P\) and \(D_P\)) in [BH]. Let \(\Delta\) be a fundamental domain containing \(u_0\) of the complex Coxeter group of \(\tilde{M}^c\) at \(u_0\). Denote by \(\Delta_u\) a domain of \(T_u^{\\tilde{M}^c}\) given by parallel translating \(\Delta\) with respect to the normal connection of \(\tilde{M}^c\). Set \(\tilde{U} := \bigcup_{u \in U_t} (\text{Span}_\mathbb{C}\{(n_i)_u | i \in I_P \setminus \{0\}\} \cap \Delta_u)\), which is an open
subset of the affine subspace \((W_P)_{\gamma(0)}\). Define a map \(h_t : \bar{U} \to (W_P)_{\gamma(t)}\) \((0 \leq t \leq 1)\) by 
\[h_t(u + w) = \gamma_u(t) + \tau_{\gamma_0}^{-1}(w)\] \((u \in U, w \in \text{Span}\{(n_i)_{\gamma} \mid i \in P \setminus \{0\} \cap \Delta_u\} (\text{see Figure 4}). By imitating the proof of Lemma 1.2 in [HL2], it is shown that \(h_t\) is a holomorphic isometry into \((W_P)_{\gamma(t)}\). Hence \(h_t\) extends to a holomorphic isometry of \((W_P)_{\gamma(0)}\) onto \((W_P)_{\gamma(t)}\). Denote by \(\tilde{h}_t\) this holomorphic extension. It is shown that \(\tilde{h}_t\) gives the desired one-parameter family by imitating the discussion in Step 3 of the proof of Proposition 1.1 in [HL2]. q.e.d.

Fix \(i_0 \in I \cup \{0\}\) and take a complex affine subspace \(P_{i_0}\) of \(T^+_u\tilde{M}^c\) with \(P_{i_0} = \{i_0\}\). Note that \(D_{P_{i_0}}\) is equal to \(E_{i_0}\). Denote by \(\Phi_{i_0}(u_0)\) the group of holomorphic isometries of \((W_{P_{i_0}})_{u_0}\) generated by \(\{h^E_{\gamma,1} \mid \gamma : E_{i_0} \to \text{horizontal curve s.t. } \gamma(0), \gamma(1) \in L^E_{u_0}\}\), where \(L^E_{u_0}\) is the integral manifold of \(E_{i_0}\) through \(u_0\). Also, denote by \(\Phi^0_{i_0}(u_0)\) the identity component of \(\Phi_{i_0}(u_0)\) and \(\Phi^0_{i_0}(u_0)_{u_0}\) the isotropy subgroup of \(\Phi^0_{i_0}(u_0)\) at \(u_0\). Define a \(\text{Ad}_{\Phi^0_{i_0}(u_0)}(\Phi^0_{i_0}(u_0))\)-invariant non-degenerate inner product \(\langle \cdot, \cdot \rangle\) of the Lie algebra \(\text{Lie } \Phi^0_{i_0}(u_0)\) of \(\Phi^0_{i_0}(u_0)\) by 
\[\langle X, Y \rangle := B(X, Y) + \text{Tr}(X \circ Y)\] \((X, Y \in \text{Lie } \Phi^0_{i_0}(u_0))\),

where \(B\) is the Killing form of \(\text{Lie } \Phi^0_{i_0}(u_0)\) and, in \(\text{Tr}(X \circ Y)\), \(X\) and \(Y\) are regarded as linear transformations of \((W_{P_{i_0}})_{u_0}\). Take \(X \in \text{Lie } \Phi^0_{i_0}(u_0) \cap \text{Lie } \Phi^0_{i_0}(u_0)_{u_0}\). Set \(g(t) := \exp t X\) and \(\gamma(t) := g(t)u_0\), where \(\exp\) is the exponential map of \(\Phi^0_{i_0}(u_0)\). It is clear that \(\gamma\) is a geodesic in \(L^E_{u_0}\). Hence \(\gamma\) is an \(E_i\)-horizontal curve for \(i \in I\) with \(i \neq i_0\). Let \(F_\gamma\) be the holomorphic isometry of \(V\) satisfying \(F_\gamma(\gamma(0)) = \gamma(1)\) and 

\[(F_\gamma)_{*\gamma(0)} = \begin{cases} 
\frac{g(1)}{\gamma(0)} & \text{on } (E_{i_0})_{\gamma(0)} \\
(h^E_{\gamma,1})_{*\gamma(0)} & \text{on } (E_i)_{\gamma(0)} (i \neq i_0) \\
\tau_{\gamma}^{-1} & \text{on } T^+_\gamma(0)^c.
\end{cases}
\]

In similar to Theorem 4.1 of [HL2], we have the following fact.

**Proposition 3.7.** The holomorphic isometry \(F_\gamma\) preserves \(\tilde{M}^c\) invariantly (i.e., \(F_\gamma(\tilde{M}^c) = \tilde{M}^c\)). Furthermore, it preserves \(E_i\) (\(i \in I\)) invariantly (i.e., \(F_\gamma(E_i) = E_i\)).

To show this proposition, we prepare some lemmas. By imitating the proof (P163~166) of Proposition 3.1 in [HL2], we can show the following fact.
Figure 3.

Figure 4.
Lemma 3.7.1. Let \( N \) and \( \hat{N} \) be irreducible proper anti-Kaehler isoparametric submanifolds of complex codimension greater than one in an infinite dimensional anti-Kaehler space. If \( N \cap \hat{N} \neq \emptyset \) and, for some \( x_0 \in N \cap \hat{N}, T_{x_0}N = T_{x_0}\hat{N} \) and there exists a complex affine line \( l_0 \) of \( T_{x_0}N = T_{x_0}\hat{N} \) such that \( L_{x_0} = \hat{L}_{x_0} \) for any complex affine line \( l \) of \( T_{x_0}N \) with \( l \neq l_0 \), then \( N = \hat{N} \) holds, where \( D_l \) (resp. \( \hat{D}_l \)) is the integrable distribution on \( N \) (resp. \( \hat{N} \)) defined for \( l \) in similar to \( D_P \).

Proof. Let \( \{\lambda_i \mid i \in I\} \) (resp. \( \{\hat{\lambda}_i \mid i \in \hat{I}\} \)) be the set of all \( J \)-principal curvatures of \( N \) (resp. \( \hat{N} \)), \( n_i \) (resp. \( \hat{n}_i \)) the \( J \)-curvature normal corresponding to \( \lambda_i \) (resp. \( \hat{\lambda}_i \)) and \( E_i \) (resp. \( \hat{E}_i \)) the \( J \)-curvature distribution corresponding to \( \lambda_i \) (resp. \( \hat{\lambda}_i \)). Denote by \( A \) (resp. \( \hat{A} \)) the shape tensor of \( N \) (resp. \( \hat{N} \)). Let \( E_0 \) be the \( J \)-curvature distribution on \( N \) with \( (E_0)_x := \bigcap_{v \in T^*_xN} \text{Ker} A_v \) (\( x \in N \)) and \( \hat{E}_0 \) the \( J \)-curvature distribution on \( \hat{N} \) with \( (\hat{E}_0)_x := \bigcap_{v \in \hat{T}^*_x\hat{N}} \text{Ker} \hat{A}_v \) (\( x \in \hat{N} \)). Let \( Q_0(x_0) \) (resp. \( \hat{Q}_0(x_0) \)) be the set of all points of \( N \) (resp. \( \hat{N} \)) connected with \( x_0 \) by a piecewise smooth curve in \( N \) (resp. \( \hat{N} \)) each of whose smooth segments is contained in some complex curvature sphere or some integral manifold of \( E_0 \) (resp. \( \hat{E}_0 \)). Take any \( x \in Q_0(x_0) \). There exists a sequence \( \{x_0, x_1, \cdots, x_k (= x)\} \) such that, for each \( j \in \{1, \cdots, k\}, \) \( p_j \in (\bigcup_{i \in I} L_{x_{j-1}}^{E_i}) \cup L_{x_{j-1}}^{E_0} \). Assume that there exists \( j_0 \in \{1, \cdots, k\} \) such that \( x_{j_0} \in L_{x_{j_0-1}}^{E_{x_{j_0}}^0} \) for some \( i_0 \in I \) with \( (n_{i_0})_{x_0} \in \emptyset \). Since \( N \) is irreducible, the complex Coxeter group associated with \( N \) is not decomposable. Hence, according to Lemma 3.8 of \[\text{Koi5}], \] we can find a \( J \)-curvature normal \( n_{i_1} \) of \( N \) satisfying \( (n_{i_1})_{x_0} \notin \text{Span}_C\{(n_{i_0})_{x_0}\} \cup \text{Span}_C\{(n_{i_0})_{x_0}\} \) (see the final part of the first paragraph of the previous section), where we use also codim \( N \geq 2 \). Furthermore, since \( n_{i_1} \) is a \( J \)-curvature normal, then so are also infinitely many complex-constant-multiples of \( n_{i_1} \). Hence we may assume that \( (n_{i_1})_{x_0} \) does not belong to \( \emptyset \) by replacing \( n_{i_1} \) to a complex-constant-multiple of \( n_{i_1} \) if necessary. Denote by \( l_{i_0} \) the affine line in \( T_{x_{j_0}}N \) through \( (n_{i_0})_{x_0} \) and \( (n_{i_0})_{x_0} \), and set \( D_{i_0} := D_l \) for simplicity. According to Theorem 3.5, \( L_{x_{j_0-1}}^{D_{i_0}} \) is a proper anti-Kaehler isoparametric submanifold in \( (W_{l_{i_0}})_{x_{j_0-1}} \) of complex codimension two. Furthermore, since both \( (n_{i_0})_{x_0} \) and \( (n_{i_1})_{x_0} \) are \( J \)-curvature normals of \( L_{x_{j_0-1}}^{D_{i_0}} \subset (W_{l_{i_0}})_{x_{j_0-1}} \) and since they are not orthogonal, it follows from Lemma 3.8 of \[\text{Koi5}], \] that \( L_{x_{j_0-1}}^{D_{i_0}} \) is irreducible. Hence, by the anti-Kaehler version of Theorem D of \[\text{HOT}], \] \( x_{j_0-1} \) can be joined to \( x_{j_0} \) by a piecewise smooth curve each of whose smooth segments is tangent to one of \( E_i \)'s \( (i \in I \) s.t. \( (n_{i})_{x_0} \neq l_{i_0} \) and \( (n_{i})_{x_0} \neq (n_{i_0})_{x_0} \)). Therefore, we can find a sequence \( \{x_0, x'_1, \cdots, x'_k (= x)\} \) such that, for each \( j \in \{1, \cdots, k\}, \) \( x'_j \in \left( \bigcup_{i \in I \text{ s.t. } (n_{i})_{x_0} \neq l_{i_0}} L_{x_{j-1}}^{E_i} \right) \cup L_{x_{j-1}}^{E_0} \). Hence it follows from Lemma 3.7.2 (see below) that \( x'_1 \in \hat{Q}_0(x_0), x'_2 \in \hat{Q}_0(x'_1), \cdots, x'_{k-1} \in \hat{Q}_0(x'_{k-2}) \) and \( x \in \hat{Q}_0(x'_{k-1}) \) inductively. Therefore

20
we have \( x \in \tilde{Q}_0(x_0) \). From the arbitrariness of \( x \), it follows that \( Q_0(x_0) \subset \tilde{Q}_0(x_0) \). Similarly we can show \( \tilde{Q}_0(x_0) \subset Q_0(x_0) \). Thus we obtain \( Q_0(x_0) = Q_0(x_0) \) and hence \( Q_0(x_0) = \tilde{Q}_0(x_0) \).

Let \( D^0_E \) (resp. \( \tilde{D}^0_E \)) be the set of all local (smooth) vector fields of \( N \) (resp. \( \tilde{N} \)) which is tangent to some \( E_i \) (resp. \( \tilde{E}_i \)) (where \( i \) may be equal to 0) at each point of the domain. Since \( (D^0_E)^* = (E_0)_x \oplus (\oplus (E_i)_x) = T_NN \) for each \( x \in N \), it follows from Theorem 3.3 that \( \Omega_{D^0_E}(x_0) = N \). Similarly, we have \( \Omega_{\tilde{D}^0_E}(x_0) = \tilde{N} \). Also, it is clear that \( \Omega_{D^0_E}(x_0) = Q_0(x_0) \) and \( \Omega_{\tilde{D}^0_E}(x_0) = \tilde{Q}_0(x_0) \). Therefore we obtain \( N = \tilde{N} \). q.e.d.

**Lemma 3.7.2.** Let \( N, \tilde{N}, x_0 \) and \( l_0 \) be as in Lemma 3.7.1. Then, for any \( x \in L_{E_0}^0 \cup \left( \bigcup_{i \in I, \text{s.t. } (n_i)_x \notin l_0} L_{E_i} \right) \), we have \( T_xN = T_x\tilde{N} \) and \( L_x^{x_0} = \tilde{L}_x^{x_0} \) for any complex affine line \( l \) of \( T_{x_0}^N \) with \( l \neq l_0 \).

**Proof.** First we consider the case where \( x \in L_{E_i} \) for some \( i \) with \( (n_i)_x \notin l_0 \). Then, from the assumption, we have \( x \in L_{E_i} = L_{E_0}^{E_i} \) and hence \( L_x^{E_i} = L_{x_0}^{E_i} \). Let \( l \) be a complex affine line of \( T_{x_0}^N \) with \( l \neq l_0 \). Assume that \( (n_i)_x \in l \). Then we have \( x \in L_{E_0}^{E_i} \subset L_{x_0}^{E_i} \).

Since \( l \neq l_0 \), it follows from the assumption that \( L_x^{D_i} = L_{x_0}^{D_i} \). Hence we have \( L_x^{D_i} = L_{x_0}^{D_i} \). Therefore it follows from the arbitrariness of \( l \) that \( \sum_{l \neq l_0} T_x L_x^{D_i} = \sum_{l \neq l_0} T_x \tilde{L}_x^{D_i} \). On the other hand, we have \( T_xN = \sum_{l \neq l_0} T_x L_x^{D_i} \) and \( T_x\tilde{N} = \sum_{l \neq l_0} T_x \tilde{L}_x^{D_i} \). Therefore we have \( T_xN = T_x\tilde{N} \).

Assume that \( (n_i)_x \notin l \). Take a curve \( \gamma : [0, 1] \to L_{E_i}^{E_0} \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = x \). Since \( (n_i)_x \notin l \), \( \gamma \) is \( D_i \)-horizontal. For the holomorphic isometries \( h_i^{D_i} : (\tilde{W}_i)_x \to (W_i)_x \) and \( \tilde{h}_i^{D_i} : (\tilde{W}_i)_x \to (\tilde{W}_i)_x \) as in Lemma 3.6, we have \( h_{\gamma, 1}^{D_i}(L_{x_0}^{D_i}) = L_x^{D_i} \) and \( \tilde{h}_{\gamma, 1}^{D_i}(L_{x_0}^{D_i}) = \tilde{L}_x^{D_i} \).

On the other hand, by imitating the discussion from Line 7 from bottom of Page 164 to Line 4 of Page 165 in [HL2], we can show \( h_{\gamma, 1}^{D_i} = h_{\gamma, 1}^{\tilde{D}_i} \). Hence we obtain \( L_x^{D_i} = L_{x_0}^{D_i} \). Therefore it follows from the arbitrariness of \( l \) that \( T_xN = T_x\tilde{N} \).

Next we consider the case of \( x \in L_{x_0}^{E_0} \). Let \( l \) be a complex affine line of \( T_{x_0}^N \) with \( l \neq l_0 \). Assume that \( 0 \in l \). Then we have \( x \in L_{E_0}^{E_0} \subset L_{x_0}^{D_i} = L_{x_0}^{D_i} \) and hence \( L_x^{D_i} = L_{x_0}^{D_i} \). Therefore it follows from the arbitrariness of \( l \) that \( T_xN = T_x\tilde{N} \). Assume that \( 0 \notin l \), \( \gamma \) is \( D_i \)-horizontal. For the holomorphic isometries \( h_{\gamma, 1}^{D_i} : (W)_x \to (W)_x \) and \( \tilde{h}_{\gamma, 1}^{D_i} : (\tilde{W}_i)_x \to (\tilde{W}_i)_x \), we have \( h_{\gamma, 1}^{D_i}(L_{x_0}^{D_i}) = L_x^{D_i} \) and \( \tilde{h}_{\gamma, 1}^{D_i}(L_{x_0}^{D_i}) = \tilde{L}_x^{D_i} \). On the other hand, by imitating the discussion from Line 18 of Page 165 to Line 6 of Page 166 in [HL2], we have \( h_{\gamma, 1}^{D_i} = h_{\gamma, 1}^{\tilde{D}_i} \). Hence we
obtain $L^D_x = L^D_x$. Therefore it follows from the arbitrariness of $l$ that $T_x N = T_x \tilde{N}$. This completes the proof. q.e.d.

In similar to Lemma 4.2 in [HL2], we have the following fact.

**Lemma 3.7.3.** If $N$ is a principal orbit through a semi-simple element of an aks-representation, then $N$ is a full irreducible proper anti-Kaehler isoparametric submanifold and each holomorphic isometry of the ambient (finite dimensional) anti-Kaehler space defined for $N$ in similar to the above holomorphic isometry $F_\gamma$ preserves $N$ invariantly.

**Proof.** Let $L/H$ be an irreducible Riemannian symmetric space of non-compact type and $\rho$ the aks-representation associated with $L^c/H^c$. Denote by $l$ (resp. $h$) the Lie algebra of $L$ (resp. $H$). Let $\theta$ be the Cartan involution of $L$ with $(\text{Fix} \theta)_0 \subset H \subset \text{Fix} \theta$ and denote by the same symbol $\theta$ the involution of $l$ associated with $\theta$. Set $q := \text{Ker}(\theta + \text{id})$, which is identified with $T_{eH}(L/H)$. The complexification $q^c$ is identified with $T_{eH^c}(L^c/H^c)$. Let $N$ be a principal orbit through a semi-simple element $w \in q^c$ of $\rho$. Denote by $A$ the shape tensor of $N$. Let $\alpha$ be a Cartan subspace of $q^c$ containing $w$. See [Ha] about the definition of a Cartan subspace. The space $a$ contains the maximal split abelian subspace $a^c_v$ of vector-type and $a = a^c_v$ holds. See [OS] or [R] about the definition of a maximal split abelian subspace of vector-type. For each $\alpha \in a^c_v$, we set

$$q^c_\alpha := \{ X \in q^c | \text{ad}(a)^2(X) = \alpha(a)^2 X \ (\forall a \in a_v) \}.$$ 

Set $\Delta := \{ \alpha \in a^c_v | q^c_\alpha \neq \{0\} \}$, which is called the root system with respect to $a_v$. Then we have the root space decomposition

$$q^c = a + \sum_{\alpha \in \Delta_+} q^c_\alpha,$$

where $\Delta_+$ is the positive root system under some lexicographical ordering of $a^c_v$ and we note that $a$ is equal to the centralizer of $a_v$ in $q^c$. For each $\alpha \in \Delta_+$, the complexification $\alpha^c$ of $\alpha$ is regarded as a $\mathbb{C}$-linear function on $a$ and we have

$$q^c_\alpha = \{ X \in q^c | \text{ad}(a)^2(X) = \alpha^c(a)^2 X \ (\forall a \in a) \}.$$ 

Since $N$ is a principal orbit of $\rho$ and hence $w$ is a regular element, we have $\alpha^c(w) \neq 0$ for any $\alpha \in \Delta_+$. Under the identification of $T_w q^c$ and $q^c$, $T_w^+ N$ is identified with $a$. Under this identification, $\alpha^c$ is regarded as a $\mathbb{C}$-linear function on $T_w^+ N$. Denote by $\overline{\alpha^c}$ this $\mathbb{C}$-linear function on $T_w^+ N$. Easily we can show

$$A^c_v|_{\overline{q^c_\alpha}} = -\frac{\overline{\alpha^c}(w)}{\alpha^c(w)} \text{id} \ (\alpha \in \Delta_+).$$
for any $v \in T^\perp_w N$. Let $\lambda_{a^e}$ be the parallel section of the $C$-dual bundle $(T^\perp N)^*$ of $T^\perp N$ with $(\lambda_{a^e})_w = -\frac{1}{\alpha^e(w)}\alpha^e$. It is clear that $N$ is a proper anti-Kaehler isoparametric submanifold having $\{\lambda_{a^e} \mid \alpha \in \Delta_+\}$ as the set of all $J$-principal curvatures. Denote by $E_{\lambda^e}$ the $J$-curvature distribution for $\lambda_{a^e}$. Take $v_0 \in (\lambda_{a^e})_w^{-1}(1) \setminus \bigcup_{\alpha \in \Delta_+} (\lambda_{a^e})_w^{-1}(1)$ and set $F := \rho(H^c) \cdot (w + v_0)$, where $v_0$ is regarded as an element of $q^c$ under the identification of $T_w q^c$ and $q^c$. It is clear that $F$ is a focal submanifold of $N$ whose corresponding focal distribution is equal to $E_{\lambda^e}$. We have the relations $h^c = 3_{q^c}(a_w) + \sum_{\alpha \in \Delta_+} h^c_\alpha$, $T_{w+v_0} F = \sum_{\alpha \in \Delta_+} q^c_\alpha$ and $T_{w+v_0} F = a + q^c_\alpha$, where $3_{q^c}(a_w)$ is the centralizer of $a_w$ in $h^c$ and $h^c_\alpha := \{X \in h^c \mid \text{ad}(a)^2X = \alpha(a)X \, (\forall a \in a_w)\}$. Denote by $H^c_w$ (resp. $H^c_{w+v_0}$) the isotropy group of the $H^c$-action at $w$ (resp. $w + v_0$) and by $h^c_w$ (resp. $h^c_{w+v_0}$) the Lie algebra of $H^c_w$ (resp. $H^c_{w+v_0}$). Then we have $h^c_w = 3_{q^c}(a_w)$ and $h^c_{w+v_0} = 3_{q^c}(a_w) + h^c_{\alpha_0}$. For the restriction of the $\rho(H^c_{w+v_0})$-action on $q^c$ to $T_{w+v_0} F$ is called the slice representation of the action at $w+v_0$. It is shown that this slice representation coincides with the normal holonomy group action of $F$ at $w+v_0$ and that $\rho(H^c_{w+v_0}) \cdot w$ is equal to $L_{\Phi(w+v_0)}$. Set $\Phi(w+v_0) := \rho(H^c_{w+v_0})$ and $\Phi(w) := \rho(H^c_w)$. The leaf $L_{\Phi(w)}$ is identified with the quotient manifold $\Phi(w+v_0)/\Phi(w)$. Take $X(= \text{ad}_{q^c}(\overline{X})) \in \text{Lie } \Phi(w+v_0) \ominus \text{Lie } \Phi(w)$, where $\overline{X} \in h^c_{w+v_0}$, and set $g(t) := \exp_{\Phi(w+v_0)}(tX)$ and $\gamma(t) := g(t) \cdot w$, where $t \in [0,1]$. Let $F_\gamma$ be the holomorphic isometry of the ambient anti-Kaehler space satisfying $F_\gamma(w) = \gamma(1)$, $(F_\gamma)^{\ast w} | (E_{\alpha^e})_w = g(1)^{\ast w} | (E_{\alpha^e})_w$, $(F_\gamma)^{\ast w} | (E_{a^e})_w = h^c_{\alpha^e,\gamma} | (E_{a^e})_w$ (\alpha \in \Delta_+ \, \text{s.t.} \, \alpha \neq \alpha_0) and $(F_\gamma)^{\ast w} | T_w^\perp N = \tau^\perp_\gamma$, where $h^c_{\alpha^e,\gamma}$ is the holomorphic isometry as in the statement of Lemma 3.6 defined for this $\gamma$ and $\tau^\perp_\gamma$ is the parallel translation along $\gamma$ with respect to the normal connection of $N$. Easily we can show $h^c_{\alpha^e,\gamma} | (E_{a^e})_w = g(1)^{\ast w} | (E_{a^e})_w$ and $\tau^\perp_\gamma = g(1)^{\ast w} | T_w^\perp N$. Hence we have $(F_\gamma)^{\ast w} = g(1)^{\ast w}$. Furthermore, since both $F_\gamma$ and $g(1)$ are affine transformations of $q^c$, they coincide with each other. Therefore, we obtain $F_\gamma(N) = g(1)(\rho(H^c) \cdot w) = \rho(\exp_{\Phi(w)}(\overline{X}))(\rho(H^c) \cdot w) = N$. This completes the proof.

By using Lemmas 3.7.1 and 3.7.3, we shall prove Proposition 3.7.

Proof of Proposition 3.7. Since $\tilde{M}^c$ is a full irreducible proper anti-Kaehler isoparametric submanifold and $F_\gamma$ is a holomorphic isometry of $V$, $M^c := F_\gamma(M^c)$ also is a full irreducible proper anti-Kaehler isoparametric one. Denote by $\tilde{A}'$ the shape tensor of $\tilde{M}^c$. Let $\{E'_i \mid i \in I\} \cup \{E'_0\}$ be the set of all $J$-curvature distributions on $\tilde{M}^c$ and $n'_i$ the $J$-curvature normal corresponding to $E'_i$, where $E'_0$ is a distribution on $\tilde{M}^c$ defined by $(E'_0)_u := \bigcap_{v \in T^\perp_u \tilde{M}^c} \text{Ker } \tilde{A}'_v \, (u \in \tilde{M}^c)$. Clearly we have $\gamma(1) \in \tilde{M}^c \cap \tilde{M}^c$. Since $(F_\gamma)^{\ast w}((n_i)_{\gamma(0)}) = \tau^\perp_\gamma((n_i)_{\gamma(0)}) = (n_i)_{\gamma(1)} \, (i \in I)$, we have $(n'_i)_{\gamma(1)} = (n_i)_{\gamma(1)} \, (i \in I)$.
Also, since \((F_t)_{\gamma}((E_i)_{\gamma}(0)) = (h_{E_i}^F)_{\gamma}(0)((E_i)_{\gamma}(0)) = (E_i)_{\gamma}(1)\) (i.e., \(\gamma\) is not congruent to the \(n\text{-curvature normal of } L\)), we have
\[
(E'_i)_{\gamma}(1) = (E_i)_{\gamma}(1) \quad (i \in I \setminus \{i_0\}).
\]
From this fact and \((n'_i)_{\gamma}(1) = (n_i)_{\gamma}(1)\), we have
\[
L_{\gamma}(1) = L_{\gamma}(1) \quad (i \in I \setminus \{i_0\}).
\]
Also, since \((F_t)_{\gamma}((E_i)_{\gamma}(0)) = g(1)_{\gamma}(0)((E_i)_{\gamma}(0)) = (E_i)_{\gamma}(1)\), we have
\[
(E'_i)_{\gamma}(1) = (E_i)_{\gamma}(1).\]
From this fact and \((n'_i)_{\gamma}(1) = (n_i)_{\gamma}(1)\), we have
\[
L_{\gamma}(1) = L_{\gamma}(1).
\]
Also, from \((E'_i)_{\gamma}(1) = (E_i)_{\gamma}(1) \quad (i \in I)\), \(T_{\gamma}(1)\) follows. Let \(b_0\) be the complex affine line through 0 and \((n_{i_0})_{\gamma}(1)\). Let \(b\) be any complex affine line of \(T_{\gamma}(1)\) with \(b \neq b_0\). Now we shall show that \(L_{\gamma}(1) = L_{\gamma}(1)\), where \(D_t\) is the distribution on \(M\) (resp. \(M'\)) defined as above for \(l\). If \((n_{i_0})_{\gamma}(1) \notin b\), then \(\gamma\) is a \(D_t\)-horizontal curve and hence we have \(F_{\gamma}(L_{u_0}^{E_{i_0}}) = L_{\gamma}(1)\) and hence \(L_{\gamma}(1) = L_{\gamma}(1)\). Next we consider the case of \((n_{i_0})_{\gamma}(1) \in l\). Then we have \(\gamma \notin b\) because of \(b \neq b_0\). If there does not exist \(i_1(\neq i_0) \in I\) with \((n_{i_1})_{\gamma}(1) \in l\), then we have \(L_{\gamma}(1) = L_{\gamma}(1) = L_{\gamma}(1)\). Next we consider the case where there exists \(i_1(\neq i_0) \in I\) with \((n_{i_1})_{\gamma}(1) \in l\). Let \(\gamma\) be a focal normal vector field of \(M\) such that the corresponding focal distribution is equal to \(D_t\). Since \(\gamma \notin b\), it follows from Theorem 3.5 that \(L_{\gamma}(1)\) is a principal orbit of the direct sum representation (having \(T_{f_\gamma}^\perp(1))\) as the representation space) of \(\{\gamma, \ldots, \gamma\}\). Hence we have
\[
L_{\gamma}(1) = L_{\gamma}(1).
\]
On the other hand, we have \(F_{\gamma}(L_{u_0}^{E_{i_0}}) = L_{\gamma}(1)\). Therefore we obtain \(L_{\gamma}(1) = L_{\gamma}(1)\). Assume that \(L_{\gamma}(1)\) is irreducible. Then \(L_{\gamma}(1)\) is a principal orbit of an \(\gamma\)-representation on \(T_{\gamma}^\perp\). Then it follows from Lemma 3.7.3 that \(F_{\gamma}(L_{\gamma}(1)) = (F_{\gamma}(L_{\gamma}(1))) = L_{\gamma}(1)\). Hence we obtain \(L_{\gamma}(1) = L_{\gamma}(1)\) and \(L_{\gamma}(1) = L_{\gamma}(1)\) in general. Therefore, from Lemma 3.7.1, we obtain \(M = M\), that is, \(F_{\gamma}(M) = M\). q.e.d.

By using Proposition 3.7, we prove the following fact.
Proposition 3.8. For any \( u \in Q(u_0) \), there exists a holomorphic isometry \( f \) of \( V \) such that \( f(u_0) = u \), \( f(\tilde{M}^c) = \tilde{M}^c \), \( f_*(E_i) = E_i \) (\( i \in I \)), \( f(Q(u_0)) = Q(u_0) \) and that \( f|_{T^\perp_{u_0} \tilde{M}^c} \) coincides with the parallel translation along a curve in \( \tilde{M}^c \) starting from \( u_0 \) and terminating to \( u \) with respect to the normal connection of \( \tilde{M}^c \).

Proof. Take a sequence \( \{u_0, u_1, \ldots, u_k(= u)\} \) of \( Q(u_0) \) such that, for each \( i \in \{0, 1, \ldots, k - 1\} \), \( u_i \) and \( u_{i+1} \) belong to a complex curvature sphere \( S_i^c \) of \( \tilde{M}^c \). Furthermore, for each \( i \in \{0, 1, \ldots, k-1\} \), we take the geodesic \( \gamma_i : [0, 1] \to S_i^c \) with \( \gamma_i(0) = u_i \) and \( \gamma_i(1) = u_{i+1} \). Set \( f := F_{\gamma_{k-1}} \circ \cdots \circ F_{\gamma_1} \circ F_{\gamma_0} \), where \( F_{\gamma_i} \) (\( i = 0, 1, \ldots, k - 1 \)) are holomorphic isometries of \( \tilde{V} \) defined similarly to the above \( F_{\gamma} \). According to Proposition 3.7, \( f \) preserves \( \tilde{M}^c \) invariantly, \( f_*(E_i) = E_i \) (\( i \in I \)) and the restriction of \( f|_{u_0} \) to \( T^\perp_{u_0} \tilde{M}^c \) coincides with the parallel translation along a curve in \( \tilde{M}^c \) starting from \( u_0 \) and terminating to \( u \) with respect to the normal connection of \( \tilde{M}^c \). Also, since \( f \) preserves complex curvature spheres invariantly, it is shown that \( f \) preserves \( Q(u_0) \) invariantly. Thus \( f \) is the desired holomorphic isometry.

q.e.d.

By using Propositions 3.4 and 3.8, we shall prove the homogeneity of \( \tilde{M}^c \).

Proof of Theorem 3.2. Take any \( \tilde{u} \in \tilde{M}^c \). Since \( \overline{Q(u_0)} = \tilde{M}^c \) by Proposition 3.4, there exists a sequence \( \{u_k\}_{k=1}^\infty \) in \( Q(u_0) \) with \( \lim_{k \to \infty} u_k = \tilde{u} \). According to Proposition 3.8, for each \( k \in \mathbb{N} \), there exists a holomorphic isometry \( f_k \) of \( V \) with \( f_k(u_0) = u_k, f_k(\tilde{M}^c) = \tilde{M}^c, f_k(Q(u_0)) = Q(u_0) \) and \( f_k(L^E_{u_0}) = L^E_{u_k} \) (\( i \in I \)).

(Step I) In this step, we shall show that, for each \( i \in I \), there exists a subsequence \( \{f_{k_j}\}_{j=1}^\infty \) of \( \{f_k\}_{k=1}^\infty \) such that \( \{f_{k_j}|_{L^E_{u_0}}\}_{j=1}^\infty \) pointwisely converges to a holomorphic isometry of \( L^E_{u_0} \) onto \( L^E_{\tilde{u}} \). For any point \( u \) of \( \tilde{M}^c \), denote by \( (L^E_u)_{\tilde{R}} \) the compact real form through \( u \) of the complex sphere \( L^E_u \), satisfying \( (T_u(L^E_u))_{\tilde{R}}, JT_u(L^E_u))_{\tilde{R}}) = 0 \), where a real form of \( L^E_u \) means the fixed point set of an anti-holomorphic diffeomorphism of \( L^E_u \). Here we note that such a compact real form \( (L^E_u)_{\tilde{R}} \) of \( L^E_u \) is determined uniquely (see Figure 5) and that it is isometric to a \( m_i \)-dimensional sphere, where \( m_i := \dim E_i \). Clearly we have \( f_k((L^E_{u_0})_{\tilde{R}}) = (L^E_{u_k})_{\tilde{R}} \). Denote by \( \tilde{\mathcal{F}}_i \) the foliation on \( \tilde{M}^c \) whose leaf through \( u \in \tilde{M}^c \) is equal to \( (L^E_u)_{\tilde{R}} \), and \( \tilde{M}^c/\tilde{\mathcal{F}}_i \) the leaf space of \( \tilde{\mathcal{F}}_i \). Take a \( \tilde{\mathcal{F}}_i \)-saturated tubular neighborhood \( U \) of \( (L^E_u)_{\tilde{R}} \) in \( \tilde{M}^c \), where "\( \tilde{\mathcal{F}}_i \)-saturation" of \( U \) means that \( L^E_u \subseteq U \) for any \( u \in U \). Take a base \( \{e_1, \ldots, e_{m_i}\} \) of \( T_{u_0}((L^E_{u_0})_{\tilde{R}}) \) such that the norms \( ||e_1||, \ldots, ||e_{m_i}|| \) are sufficiently small and set \( \tilde{a}_a := \exp(e_a) \) (\( a = 1, \ldots, m_i \)), where \( \exp \) is the exponential map of \( (L^E_{u_0})_{\tilde{R}} \). Since \( (L^E_u)_{\tilde{R}} \)'s (\( u \in U \)) are compact, \( \tilde{\mathcal{F}}_i \) is a Hausdorff foliation. From this fact and the compactness of \( (L^E_{u_0})_{\tilde{R}} \), it follows that there exists a subsequence \( \{f_{k_j}\}_{j=1}^\infty \) of \( \{f_k\}_{k=1}^\infty \) such that \( \{f_{k_j}(\tilde{a}_a)\}_{j=1}^\infty \) (\( a = 1, \ldots, m_i \)) converge. Set
\( \tilde{u}_a := \lim_{j \to \infty} f_{k_j}(u_a) \) (\( a = 1, \ldots, m_i \)). Since \( \lim_{j \to \infty} f_{k_j}(u_0) = \tilde{u} \) and \( f_{k_j}(L^{E_i}_u) = R^{E_i}_u \) it follows from the Hausdorffness of \( \mathfrak{F} \) that \( \tilde{u}_a \) belongs to \( L^{E_i}_u \) (\( a = 1, \ldots, m_i \)). Denote by \( d_0, d_j \) \( (j \in \mathbb{N}) \) and \( \tilde{d} \) the (Riemannian) distance functions of \( L^{E_i}_u \) and \( L^{E_i}_u \), respectively. Since each \( f_{k_j} \) is an isometry onto \( L^{E_i}_u \), we have 
\[
d_j(f_{k_j}(u_0), f_{k_j}(\tilde{u}_a)) = d_0(u_0, \tilde{u}_a) \quad \text{and} \quad d_j(f_{k_j}(\tilde{u}_a), f_{k_j}(\tilde{u}_b)) = d_0(\tilde{u}_a, \tilde{u}_b), \quad (a, b = 1, \ldots, m_i).
\]
Hence we have \( \tilde{d}(\tilde{u}, \tilde{u}_a) = d_0(u_0, \tilde{u}_a) \) and \( \tilde{d}(\tilde{u}_a, \tilde{u}_b) = d_0(\tilde{u}_a, \tilde{u}_b) \) (\( a, b = 1, \ldots, m_i \)). Therefore, since \( L^{E_i}_u \) and \( L^{E_i}_u \) are spheres isometric to each other, there exists a unique isometry \( \tilde{f} \) of \( L^{E_i}_u \) onto \( L^{E_i}_u \) satisfying \( \tilde{f}(u_0) = \tilde{u} \) and \( \tilde{f}(\tilde{u}_a) = \tilde{u}_a \) (\( a = 1, \ldots, m_i \)). It is clear that \( \tilde{f} \) is uniquely extended to a holomorphic isometry of \( L^{E_i}_u \) onto \( L^{E_i}_u \). Denote by \( f \) this holomorphic extension. It is easy to show that \( \{f_{k_j}\}_{j=1}^{\infty} \) pointwisely converges to \( \tilde{f} \). Furthermore, it follows from this fact that \( \{f_{k_j}\}_{j=1}^{\infty} \) pointwisely converges to \( f \).

(Step II) Next we shall show that, for each fixed \( w \in Q(u_0) \), there exists a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) of \( \{f_{k_j}\}_{k=1}^{\infty} \) such that \( \{f_{k_j}(w)\}_{j=1}^{\infty} \) converges. There exists a sequence \( \{u_0, u_1, \ldots, u_m = w\} \) in \( Q(u_0) \) such that, for each \( j \in \{1, \ldots, m\} \), \( u_j \) is contained in a complex curvature sphere \( L^{E_i}_{u_j-1} \). For simplicity, we shall consider the case of \( m = 3 \). From the fact in Step I, there exists a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) of \( \{f_{k_j}\}_{k=1}^{\infty} \) such that \( \{f_{k_j}\}_{j=1}^{\infty} \) pointwisely converges to a holomorphic isometry \( f^{(1)} \) of \( L^{E_i}_{u_0} \) onto \( L^{E_i}_{u_1} \). Furthermore, by noticing \( \lim_{j \to \infty} f^{(1)}(u_j) = 1 \) and imitating the discussion in Step I, we can show that there exists a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) of \( \{f_{k_j}\}_{j=1}^{\infty} \) such that \( \{f_{k_j}\}_{j=1}^{\infty} \) pointwisely converges to a holomorphic isometry \( f^{(2)} \) of \( L^{E_i}_{u_1} \) onto \( L^{E_i}_{u_2} \). Furthermore, by noticing \( \lim_{j \to \infty} f^{(2)}(u_2) = 1 \) and imitating the discussion in Step I, we can show that there exists a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) of \( \{f_{k_j}\}_{j=1}^{\infty} \) such that \( \{f_{k_j}\}_{j=1}^{\infty} \) pointwisely converges to a holomorphic isometry \( f^{(3)} \) of \( L^{E_i}_{u_2} \) onto \( L^{E_i}_{u_3} \). In particular, we have \( \lim_{j \to \infty} f_{k_j}(w) = f^{(3)}(w) \). Thus \( \{f_{k_j}\}_{j=1}^{\infty} \) is the desired subsequence of \( \{f_{k_j}\}_{j=1}^{\infty} \).

(Step III) Let \( W \) be the complex affine span of \( M^c \). Next we shall show that there exists a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) such that \( \{f_{k_j}(w)\}_{j=1}^{\infty} \) pointwisely converges to some holomorphic isometry of \( W \). Take a countable subset \( B := \{w_j \mid j \in \mathbb{N}\} \) of \( Q(u_0) \) with \( \overline{B} = Q(u_0)(= M^c) \). According to the fact in Step II, there exists a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) of \( \{f_{k_j}\}_{j=1}^{\infty} \) such that \( \{f_{k_j}(w_j)\}_{j=1}^{\infty} \) converges. Again, according to the fact in Step II, there exists a subsequence \( \{f_{k_j}\}_{j=1}^{\infty} \) of \( \{f_{k_j}\}_{j=1}^{\infty} \) such that \( \{f_{k_j}(w_2)\}_{j=1}^{\infty} \) converges. In the sequel, we take subsequences \( \{f_{k_j}\}_{j=1}^{\infty} \) (\( l = 3, 4, 5, \ldots \)) inductively. It is clear that \( \{f_{k_j}(w_l)\}_{j=1}^{\infty} \) converges for each \( l \in \mathbb{N} \), that is, \( \{f_{k_j}(w_l)\}_{j=1}^{\infty} \) pointwisely converges to some map \( f \) of \( B \)
into $\tilde{M}$. Since each $f_{kj}$ is a holomorphic isometry, $f$ extends to a holomorphic isometry of $\tilde{M}$. Denote by $\tilde{f}$ this extension. It is clear that $\{f_{kj}|_{\tilde{M}}\}_{j=1}^{\infty}$ pointwisely converges to $\tilde{f}$. Furthermore, since each $f_{kj}$ is an affine transformation and hence the restriction $f_{kj}|_W$ of $f_{kj}$ to $W$ is a holomorphic isometry of $W$, $\tilde{f}$ extends to a holomorphic isometry of $W$. Denote by $(\tilde{f})$ this extension. It is clear that $\{f_{kj}|_W\}_{j=1}^{\infty}$ pointwisely converges to $(\tilde{f})$.

(Step IV) Denote by $H$ the group generated by all holomorphic isometries of $V$ preserving $\tilde{M}$ invariantly. Let $\tilde{f}$ and $(\tilde{f})$ be as in Step III. It is clear that $(\tilde{f})$ extends to a holomorphic isometry of $V$. Denote by $\hat{f}$ this extension. Then we have $\tilde{f}(\tilde{M}) = (\tilde{f})(\tilde{M}) = \tilde{M}$ and $\hat{f}(u_0) = f(u_0) = \lim_{j \to \infty} f_{kj}(u_0) = \lim_{j \to \infty} u_{kj} = \hat{u}$. Hence we have $\hat{u} \in H \cdot u_0$. From the arbitrariness of $\hat{u}$, we obtain $\tilde{M} \subset H \cdot u_0$. On the other hand, it follows from the definition of $H$ that $H \cdot u_0 \subset \tilde{M}$. Therefore we obtain $H \cdot u_0 = \tilde{M}$. q.e.d.

Figure 5.

4 On the holomorphic Killing field associated with the constructed one-parameter family of holomorphic isometries

Let $M(\rightarrow G/K)$ be as in Theorem A and $M^e$ the (complete extrinsic) complexification of $M$. Set $\tilde{M} := \pi^{-1}(M^e)$ and $\hat{M} := (\pi \circ \phi)^{-1}(M^e)$, where $\phi$ is the parallel transport.
map for $G^e$ and $\pi$ is the natural projection of $G^e$ onto $G^e/K^e$. Without loss of generality, we may assume that $K^e$ is connected and that $G^e$ is simply connected. Hence both $\tilde{M}^e$ and $\tilde{M}^e$ are connected. Also, without loss of generality, we may assume $\tilde{0} \in \tilde{M}^e$ and hence $e \in \tilde{M}^e$. For simplicity, set $V := H^0([0,1], g^e)$, $V_- := H^0([0,1], g^e_-)$ and $V_+ := H^0([0,1], g^e_+)$. 

**Notations.** For simplicity, denote by $(\cdot, \cdot)$ the non-degenerate inner product $\langle \cdot, \cdot \rangle_0^A$ of $V$ and $(\cdot, \cdot)_\pm$ the positive definite inner product $\langle \cdot, \cdot \rangle_0^A_{A, H^e_\pm}$ of $V$. See Section 2 about the definitions of $(\cdot, \cdot)_0^A$ and $(\cdot, \cdot)_0^A_{A, H^e_\pm}$. Also, denote by $|| \cdot ||$ the norm associated with $(\cdot, \cdot)_\pm$.

Denote by $\nabla$ and $\overline{\nabla}$ the Riemannian connection of $\tilde{M}^e$ and $V$, respectively, and $\widehat{A}$ and $\overline{h}$ the shape tensor and the second fundamental form of $\tilde{M}^e$, respectively. Set $p := T_{eK}(G/K)$ and $b := T^*_{eK}M$. Let $a$ be a maximal abelian subspace of $p (\subset g)$ containing $b$ and $p = a + \sum_{\alpha \in \triangle_+} p_\alpha$ be the root space decomposition with respect to $a$, that is, $p_\alpha := \{ X \in p \mid \text{ad}(a)^2(X) = \alpha(a)^2 X \ (\forall a \in a) \}$ and $\triangle_+$ is the positive root system of the root system $\Delta := \{ \alpha \in a^* \setminus \{0\} \mid p_\alpha \neq \{0\} \}$ under a lexicographic ordering of $a^*$. Set $\Delta_b := \{ \alpha|b| \mid \alpha \in \Delta \text { s.t. } \alpha|b| \neq 0 \}$ and let $p = p_0 + \sum_{\beta \in (\Delta_b)_+} p_\beta$ be the root space decomposition with respect to $b$, where $p_\beta$ is the centralizer of $b$ in $p$, $p_\beta := \sum_{\alpha \in \triangle_+ \text { s.t. } \alpha|b| = \pm \beta} p_\alpha$. The positive root system of the root system $\Delta_b$ under a lexicographic ordering of $b^*$. For convenience, we denote $p_0$ by $p_0$. Denote by $\widehat{A}$ (resp. $A$) the shape tensor of $M$ (resp. $\tilde{M}^e$). Also, denote by $R$ the curvature tensor of $G/K$. Let $m_A := \max_{v \in b \setminus \{0\}} \sharp \text{Spec } A_v$ and $m_R := \max_{v \in b \setminus \{0\}} \sharp \text{Spec } R(v)$, where $\sharp(\cdot)$ is the cardinal number of $(\cdot)$. Note that $m_R = \sharp((\Delta_b)_+)$. Let $U := \{ v \in b \setminus \{0\} \mid \sharp \text{Spec } A_v = m_A, \sharp \text{Spec } R(v) = m_R \}$, which is an open dense subset of $b \setminus \{0\}$. Fix $v \in U$. Note that $\text{Spec } R(v) = \{ -\beta(v)^{2} \mid \beta \in (\Delta_b)_+ \}$. From $v \in U$, $\beta(v)^{2}$'s $(\beta \in (\Delta_b)_+)$ are mutually distinct. Let $\text{Spec } A_v = \{ \lambda_1^v, \ldots, \lambda_{m_A}^v \} (\lambda_1^v > \cdots > \lambda_{m_A}^v)$. Set

$$I_0^v := \{ i \mid p_\beta \cap \text{Ker}(A_v - \lambda_i^v \text{id}) \neq \{0\} \},$$

$$I_i^v := \{ i \mid p_\beta \cap \text{Ker}(A_v - \lambda_i^v \text{id}) \neq \{0\} \},$$

$$I^+ := \{ i \mid I_i^v \mid \lambda_i^v > |\beta(v)| \},$$

$$I^- := \{ i \mid I_i^v \mid \lambda_i^v < |\beta(v)| \},$$

$$I_0^v := \{ i \mid I_i^v \mid \lambda_i^v = |\beta(v)| \}.$$

Let $\mathcal{F}$ be the sum of all complex focal hyperplanes of $\tilde{M}^e$ at $\tilde{0}$. Since the normal space $T^*_0(M^e)$ of $\tilde{M}^e$ at $\tilde{0}$ is identified with the complexification $b^e$ of $b$, each complex focal hyperplane of $\tilde{M}^e$ at $\tilde{0}$ is regarded as a complex hyperplanes of $b^e$. Denote by $\text{pr}_R$ the natural projection of $b^e$ onto $b$ and set $\mathcal{F}_R := \text{pr}_R(\mathcal{F})$. Then we can show the following fact.
Lemma 4.1. The set \((I_\beta^v)^0\) is empty and Spec \(A_v|_{\sum_{\beta \in (\Delta_b)_+} p_\beta}\) is equal to
\[
\left\{ \frac{\beta(v)}{\tanh \beta(Z)} \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I_\beta^v)^+ \neq \emptyset \right\}
\cup \left\{ \beta(v) \tanh \beta(Z) \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I_\beta^v)^- \neq \emptyset \right\}
\]
for some \(Z \in b\). Also, we have
\[
p_\beta = \begin{cases} 
\ker (A_v - \frac{\beta(v)}{\tanh \beta(Z)}) & ((I_\beta^v)^- = \emptyset) \\
\ker (A_v - \beta(v) \tanh \beta(Z)) & ((I_\beta^v)^+ = \emptyset) \\
\ker (A_v - \frac{\beta(v)}{\tanh \beta(Z)}) \oplus \ker (A_v - \beta(v) \tanh \beta(Z)) & ((I_\beta^v)^+ \neq \emptyset \& (I_\beta^v)^- \neq \emptyset) 
\end{cases}
\]
for any \(\beta \in (\Delta_b)_+\).

Proof. From \(v \in U\), we have \(\beta(v) \neq 0\) for any \(\beta \in (\Delta_b)_+\). Hence, according to the proof of Lemma 3.1, we have \((I_\beta^v)^0 = \emptyset\) because \(M\) satisfies the condition \((\ast C)\). Set \(c_{\beta,i,v}^+ := \frac{\beta(v)}{\lambda_i}\) \((i \in (I_\beta^v)^+ (\beta \in (\Delta_b)_+))\) and \(c_{\beta,i,v}^- := \frac{\lambda_i}{\beta(v)}\) \((i \in (I_\beta^v)^- (\beta \in (\Delta_b)_+))\). According to the proof of Theorems B and C in [Koi8], we have
\[
\mathcal{F} = \left( \bigcup_{\beta \in (\Delta_b)_+} \bigcup_{(i,j) \in (I_\beta^v)^+} (\beta^c)^{-1}(\arctanh c_{\beta,i,v}^+ + j\pi\sqrt{-1}) \right) 
\cup \left( \bigcup_{\beta \in (\Delta_b)_+} \bigcup_{(i,j) \in (I_\beta^v)^-} (\beta^c)^{-1}(\arctanh c_{\beta,i,v}^- + (j + \frac{1}{2})\pi\sqrt{-1}) \right)
\]
and
\[
\mathcal{F}_R = \left( \bigcup_{\beta \in (\Delta_b)_+} \bigcup_{i \in (I_\beta^v)^+} (\beta^c)^{-1}(\arctanh c_{\beta,i,v}^+) \right) 
\cup \left( \bigcup_{\beta \in (\Delta_b)_+} \bigcup_{i \in (I_\beta^v)^-} (\beta^c)^{-1}(\arctanh c_{\beta,i,v}^-) \right).
\]
Since \(M\) is of codimension greater than one and \(M\) has flat section, we have rank \(G/K \geq 2\). From this fact, it follows that \(\mathcal{F}_R\) is not empty. Let \(W_R\) be the group generated by reflections with respect to the members of \(\mathcal{F}_R\). Then, since \(\mathcal{F}\) is invariant with respect to the complex Coxeter group associated with \(M\), it is shown that \(\mathcal{F}_R\) is \(W_R\)-invariant. Therefore, since \(\mathcal{F}_R\) consists of finite pieces of hyperplanes in \(b\), it is shown that the intersection of all the members of \(\mathcal{F}_R\) is not empty. Take an element \(Z\) of their intersection (see Figure 9). Then we have
\[
\lambda_i^v = \begin{cases} 
\frac{\beta(v)}{\tanh \beta(Z)} & (i \in (I_\beta^v)^+) \\
\beta(v) \tanh \beta(Z) & (i \in (I_\beta^v)^-). 
\end{cases}
\]
Hence we obtain the statement of this lemma. q.e.d.

Denote by $\mathcal{P}_J(\tilde{M}^c)$ the set of all $J$-principal curvatures of $\tilde{M}^c$. For simplicity, set $E^+_\beta := \text{Ker} \left( A_v - \frac{\beta(v)}{\tanh \beta(Z)} \right)$ and $E^-_\beta := \text{Ker} \left( A_v - \beta(v) \tanh \beta(Z) \right)$. By using this lemma and Lemma 13 in [Koi3], we can show the following fact for $\mathcal{P}_J(\tilde{M}^c)$.

Lemma 4.2. The set $\mathcal{P}_J(\tilde{M}^c)$ is equal to

$$
\left\{ \frac{1}{\beta(Z) + j\pi i} \tilde{\beta}^c \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I^+_\beta)^+ \neq \emptyset, \ j \in \mathbb{Z} \right\} 
\cup 
\left\{ \frac{1}{\beta(Z) + (j + \frac{1}{2})\pi i} \tilde{\beta}^c \mid \beta \in (\Delta_b)_+ \text{ s.t. } (I^-_\beta)^- \neq \emptyset, \ j \in \mathbb{Z} \right\}
$$

for some $Z \in \mathfrak{b}$, where $\tilde{\beta}^c$ is the parallel section of the dual bundle $(T^\perp \tilde{M}^c)^*$ of $T^\perp \tilde{M}^c$ with $(\tilde{\beta}^c)_0 = \beta^c$.

Proof. Let $v$ be an element of $U$, where $U$ is as above. According to Lemma 12 in [Koi3], we have

$$
\Omega_\beta(a^c) \oplus H^0([0, 1], \mathfrak{f}^c(a^c)) \subset \text{Ker} \tilde{A}_v.
$$

Also, according to Lemma 13 in [Koi3], for $w \in \mathfrak{p}_\beta \cap \text{Ker}(A_v - \frac{\beta(v)}{\tanh \beta(Z)} \text{id})$ and $j \in \mathbb{Z}$, we have

$$
w^L + l^j_w \in \text{Ker} \left( \tilde{A}_v - \frac{\beta(v)}{\beta(Z) + j\pi i} \text{id} \right),
$$

where $l^j_w$ is a vertical vector with respect to the submersion $\pi \circ \phi$. Also, according to the lemma, for $w \in \mathfrak{p}_\beta \cap \text{Ker}(A_v - \beta(v) \tanh \beta(Z))$ and $j \in \mathbb{Z}$, we have

$$
w^L + l^j_w \in \text{Ker} \left( \tilde{A}_v - \frac{\beta(v)}{\beta(Z) + (j + \frac{1}{2})\pi i} \right),
$$

where $l^j_w$ is a vertical vector with respect to the submersion $\pi \circ \phi$. Note that the above vertical vectors $l^j_w$ and $l^j_w$ are unique for each $w$ and each $j$, respectively. Furthermore, we have

$$
\bigoplus_{j \in \mathbb{Z}} \left( \text{Ker} \left( \tilde{A}_v - \frac{\beta(v)}{\beta(Z) + j\pi i} \text{id} \right) \oplus \text{Ker} \left( \tilde{A}_v - \frac{\beta(v)}{\beta(Z) + (j + \frac{1}{2})\pi i} \right) \right) = H^0([0, 1], \mathfrak{f}^c \oplus \mathfrak{p}^c).
$$

Let $\lambda \in \text{Spec} \tilde{A}_v \setminus \{0\}$. According to the proof of Theorems B and C of [Koi8], there exists a complex linear function $\phi$ on $\mathfrak{b}^c$ with $\phi(v) = \lambda$ and $\phi_i^{-1}(1) \subset \mathcal{F}$. According to the above
facts, \( \phi^{-1}(1) \) coincides with one of \((\beta^c)^{-1}(\beta(Z) + j\pi i)^\ell\)'s \((\beta \in (\Delta_b)_+, \text{ s.t. } (I^c_\beta)^+ \neq \emptyset, \ j \in \mathbb{Z})\) and \((\beta^c)^{-1}(\beta(Z) + (j + \frac{1}{2})\pi \sqrt{-1})\)'s \((\beta \in (\Delta_b)_+, \text{ s.t. } (I^c_\beta)^- \neq \emptyset, \ j \in \mathbb{Z})\). Hence \( \phi \) coincides with one of \( \frac{1}{\beta(Z) + j\pi i} \beta^c \)'s \((\beta \in (\Delta_b)_+, \text{ s.t. } (I^c_\beta)^+ \neq \emptyset, \ j \in \mathbb{Z})\) and \( \frac{1}{\beta(Z) + (j + \frac{1}{2})\pi \sqrt{-1}} \beta^c \)'s \((\beta \in (\Delta_b)_+, \text{ s.t. } (I^c_\beta)^- \neq \emptyset, \ j \in \mathbb{Z})\). Therefore \( \mathcal{PC}_f(\tilde{M}^c) \) is given as in the statement of this lemma.

\( \text{q.e.d.} \)

Let \( v \in U \). Set

\[
\lambda_{(\beta,j)} := \begin{cases} 
\frac{1}{\beta(Z) + j\pi i} \tilde{\beta}^c & ((I^v_\beta)^- = \emptyset) \\
1 & ((I^v_\beta)^+ = \emptyset) \\
\frac{1}{\beta(Z) + (j + \frac{1}{2})\pi i} \tilde{\beta}^c & ((I^v_\beta)^+ \neq \emptyset \& (I^v_\beta)^- \neq \emptyset).
\end{cases}
\]

Denote by \( n_{(\beta,j)} \) the \( J \)-curvature normal corresponding to \( \lambda_{(\beta,j)} \) and \( E_{(\beta,j)} \) the \( J \)-curvature distribution corresponding to \( \lambda_{(\beta,j)} \). Also, define a distribution \( E_0 \) by \((E_0)_u := \bigcap_{v \in T_u \tilde{M}^c} \text{Ker} \tilde{A}_v \) \((u \in \tilde{M}^c)\). Also, let \((E_{(\beta,j)})_+^\ell\) be the half-dimensional subdistribution of \( E_{(\beta,j)} \) such that \( \langle \cdot, \cdot \rangle_{(E_{(\beta,j)})_+^\ell \times (E_{(\beta,j)})_+^\ell} \) is positive definite and that \((E_{(\beta,j)}^\ell, JE_{(\beta,j)}^\ell) = 0\), and set \((E_{(\beta,j)})_- := J(E_{(\beta,j)})_+^\ell\). Similarly we define subdistributions \((E_0)_+^\ell\) and \((E_0)_-^\ell\) of \( E_0 \).

Set \( I := (\Delta_b)_+ \times \mathbb{Z} \). Also, set

\[
(T\tilde{M}^c)^- := (E_0)_- + \bigoplus_{i \in I} (E_i)_-
\]

and

\[
(T\tilde{M}^c)_+ := (E_0)_+ + \bigoplus_{i \in I} (E_i)_+.
\]

For each complex affine subspace \( P \) of \( T^\perp_0 \tilde{M}^c \), \( D_P \) is a totally geodesic distribution on \( \tilde{M}^c \). We call the integral manifold \( L^D_u \) of \( D_P \) through \( u \) a slice of \( \tilde{M}^c \). Note that, if \( 0 \not\in P \), then \( L^D_u \) is a focal leaf, where \( 0 \) is the zero vector of \( T^\perp_0 \tilde{M}^c \). Denote by \( l_{(i,j)} \) the complex focal hyperplane \((\lambda_{(i,j)})_0^{-1}(1)\) of \( \tilde{M}^c \) at \( 0 \). Also, let \((l_{(\beta,j)})_+ \) be the half dimensional totally real affine subspace of \( l_{(\beta,j)} \) through the intersection point of \( l \) and \( \text{Span}_C\{(n_{(\beta,j)})_0\} \) such that \( \langle \cdot, \cdot \rangle_{(T(l_{(\beta,j)}))_+ \times (T(l_{(\beta,j)}))_+} \) is positive definite and that \((T(l_{(\beta,j)})_+, JT(l_{(\beta,j)})_+^\ell) = 0\), where \( T(l_{(\beta,j)})_+ \) is the tangent bundle of \((l_{(\beta,j)})_+ \). For example, in case of \((I^v_\beta)^+ \neq \emptyset\), \((n_{(\beta,j)})_0 \)'s and \( l_{(\beta,j)} \) \((j \in \mathbb{Z})\) are as in Figure 6. Also, in case of \((I^v_\beta)^+ = \emptyset\), they are as in Figure 7.
The case of \((I_\beta^+) \neq \emptyset\)

Figure 6
For each $\beta \in (\Delta_b)^+$, we define $n_\beta \in b$ by $\beta(\cdot) = \langle n_\beta, \cdot \rangle$. Also, for $i = (\beta, j) \in I$, we define a real affine hyperplane $l_i^R$ in $b$ by

$$l_i^R := \begin{cases} 
\beta^{-1}(j\pi) & ((I_\beta^+)^- = \emptyset) \\
\beta^{-1}((j + \frac{1}{2})\pi) & ((I_\beta^-)^- = \emptyset) \\
\beta^{-1}(\frac{j\pi}{2}) & ((I_\beta^+)^- \neq \emptyset \& (I_\beta^-)^- \neq \emptyset).
\end{cases}$$

Set $R := \{ (n_\beta, l_i^R) | \beta \in (\Delta_b)^+, j \in \mathbb{Z} \}$. Denote by $p_{R_1}$ the natural projection $b^c$ onto $i_b$

33
and set \( l_i := \text{pr}_1(l_i) \) (\( i \in I \)), which is a real hyperplane in \( i\mathfrak{b} \). Let \( \mathcal{W}' \) (resp. \( \mathcal{W}'' \)) be the group generated by the reflections with respect to \( \theta_i^{R} \)'s (resp. \( \theta_i^{L} \)'s) (\( i \in I \)). It is clear that \( \mathcal{W}' \) is isomorphic to \( \mathcal{W}'' \). Also, it is shown that \( \mathcal{W}'' \) is isomorphic to the complex Coxeter group \( \mathcal{W} \) associated with \( \tilde{M}^c \) through \( \text{pr}_1 \). Hence, since \( \mathcal{W} \) is an affine Weyl group, so is also \( \mathcal{W}' \). Also, \( \triangle_b \) is a root system and it is shown that \( \triangle_b \) is preserved invariantly by the differential of each element of \( \mathcal{W}' \). Thus \( \mathcal{R} \) is an affine root system.

**Definition.** We call \( \mathcal{R} \) (resp. \( \triangle_b \)) the affine root system (resp. the root system) associated with \( \tilde{M}^c \).

Since \( \tilde{M}^c \) is a full and irreducible proper anti-Kaehler isoparametric submanifold of codimension greater than one, \( \mathcal{R} \) is an irreducible affine root system of rank greater than one. Let \( P \) be a complex affine subspace in \( T^*_0 \tilde{M}^c \). If \( 0 \in P \), then the slice \( L^P_0 \) is an infinite dimensional proper anti-Kaehler isoparametric submanifold in \( (W_P)_{0} \). Hence the root system and the affine root system associated with \( L^P_0 \) are defined similarly. If \( 0 \notin P \), then \( L^P_0 \) is regarded as a principal orbit of the isotropy representation of an anti-Kaehler symmetric space \( G^c/K^c \) by the homogeneous slice theorem in [Koi7].

**Definition.** We call the root system of the symmetric pair \( (G', K') \) the root system associated with \( L^P_0 \).

Fix \( \hat{0} \in \tilde{M}^c \) and \( w_0 \in (E_i)_{\hat{0}} \) (\( i \in I \)). Let \( \gamma : [0,1] \to L^E_{\hat{0}} \) be the geodesic in \( L^E_{\hat{0}} \) with \( \gamma'(0) = w_0 \) and \( F_\gamma \) the holomorphic isometry satisfying \( F_\gamma(\gamma(0)) = \gamma(1) \) and the relation (3.2). In more general, let \( F_{\gamma_{[0,t]}} \) (\( t \in \mathbb{R} \)) be the holomorphic isometry satisfying \( F_{\gamma_{[0,t]}}(\gamma(0)) = \gamma(t) \) defined in similar to \( F_\gamma \). For simplicity, set \( F^w_{\gamma_{[0,t]}} := F_{\gamma_{[0,t]}} \). Let \( X^w_0 \) be the holomorphic Killing field associated with the one-parameter transformation group \( \{F^w_{\gamma_{[0,t]}} \}_{t \in \mathbb{R}} \), that is, \( X^w_0 := \frac{d}{dt} \bigg|_{t=0} F^w_{\gamma_{[0,t]}}(u) \), where \( u \) moves over the set (which we denote by \( U \)) of all elements \( u \)'s where the right-hand side exists. Set \( A^w_0 := \frac{d}{dt} \bigg|_{t=0} (F^w_{\gamma_{[0,t]}})_{\hat{0}} \) and \( b^w_0 := (X^w_0)_{\hat{0}} \) (i.e., \( (X^w_0)_{\hat{0}} = A^w_0 + b^w_0 \)). Clearly we have

\[
\left( \bigoplus_{i \in I \cup \{0\}} (E_i)_{\hat{0}} \right) \oplus T^0_0 \tilde{M}^c \subset U.
\]

However, \( U \) does not necessarily coincide with the whole of \( V \). For simplicity, we set \( V' := \left( \bigoplus_{i \in I \cup \{0\}} (E_i)_{\hat{0}} \right) \oplus T^0_0 \tilde{M}^c \) and \( V'_T := \bigoplus_{i \in I \cup \{0\}} (E_i)_{\hat{0}} \). Define a map \( \Gamma_{w_0} : V'_T \to V \) by \( \Gamma_{w_0}(w) := \frac{d}{dt} \bigg|_{t=0} (A^w_{t=0})_{\hat{0}}(w) = A^w_0 \) (\( w \in V'_T \)) and a map \( \Gamma_{w_0} : V'_T \to T^0_0 \tilde{M}^c \) by
Let $i_1 \in I$ and $i_2, i_3 \in I \cup \{0\}$.

(i) For any $w_k \in (E_{i_k})_0$ ($k = 1, 2, 3$), we have
\[
\langle \Gamma_{w_1} w_2, w_3 \rangle + \langle w_2, \Gamma_{w_1} w_3 \rangle = 0,
\]
(ii) For any $w_k \in (E_{i_k})_0$ ($k = 1, 2$) and any holomorphic isometry $f$ of $V$ preserving $\tilde{M}^c$ invariantly, we have
\[
f_* \Gamma_{w_1} w_2 = \Gamma_{f_* w_1} f_* w_2.
\]

In similar to Lemma 3.5 of [GH], we have the following fact.

**Lemma 4.5.** Let $L$ be a slice of $\tilde{M}^c$, $i_0$ an element of $I \cup \{0\}$ with $(E_{i_0})_0 \subset T_0 L$ and $W$ the complex affine span of $L$. If $w_0 \in (E_{i_0})_0$, then $F_{t w_0}^t (L) = L$ holds for all $t \in [0, 1]$ and $X_{w_0}^t$ is tangent to $W$ along $W$. Furthermore, if $L$ is irreducible and is of rank greater than one, then $F_t^{w_0} | W = L F_t^{w_0}$ holds for all $t \in [0, 1]$, where $L F_t^{w_0}$ is the one-parameter transformation group of $W$ associated with $X_{w_0}^t | W$, and hence the homogeneous structure of $L(\subset W)$ at $0$ is the restriction of $\Gamma$. 

35
Let \( \bar{v} \) be a (non-focal) parallel normal vector field of \( \tilde{M}^c, \eta_{\bar{v}} : \tilde{M}^c \to V \) the end-point map for \( \bar{v} \) (i.e., \( \eta_{\bar{v}}(u) := \exp^{-1}(\bar{v}_u) \) \((u \in \tilde{M}^c)\)) and \( \tilde{M}^c_{\bar{v}} \) the parallel submanifold for \( \bar{v} \) (i.e., the image of \( \eta_{\bar{v}} \)). Denote by \( \bar{v} \Gamma \) the homogeneous structure of \( \tilde{M}^c_{\bar{v}} \) at \( \eta_{\bar{v}}(\bar{v}) \). In similar to Lemma 3.6 of [GH], we have the following fact.

**Lemma 4.6.** For any \( w_1 \in (E_{i_1})_0 \) \((i_1 \in I)\) and any \( w_2 \in (E_{i_2})_0 \) \((i_2 \in I \cup \{0\})\), we have

\[
\bar{v} \Gamma_{(\eta_{\bar{v}}) \ast w_1} w_2 = (\eta_{\bar{v}}) \ast (\Gamma_{w_1} w_2),
\]

where we note that \( T_{\bar{v}} \tilde{M}^c = T_{\eta_{\bar{v}}(\bar{v})} \tilde{M}^c_{\bar{v}} \) under the parallel translation in \( V \). Also, we have \((\eta_{\bar{v}}) \ast w_1 = (1 - (\lambda_{i_1})_0(\bar{v}_0)) w_1\).

**Proof.** From \((\eta_{\bar{v}}) \ast 0 = \text{id} - \bar{A}_{\eta_{\bar{v}}}, \) the second relation follows directly. Since \((\eta_{\bar{v}}) \ast 0 \) maps the \( J \)-curvature distributions of \( \tilde{M}^c \) to those of \( \tilde{M}^c_{\bar{v}}, \) \( \eta_{\bar{v}} \) maps the complex curvature spheres of \( \tilde{M}^c \) through \( \bar{v} \) to those of \( \tilde{M}^c_{\bar{v}} \) through \( \eta_{\bar{v}}(\bar{v}) \). On the other hand, since \( F_t^{u_1} \) preserves \( \tilde{M}^c \) invariantly and its differential at a point of \( \tilde{M}^c \) induces the parallel translation with respect to the normal connection of \( \tilde{M}^c, \) we have \( \eta_{\bar{v}} \circ F_t^{u_1} |_{\tilde{M}^c_{\bar{v}}} = F_t^{u_1} \circ \eta_{\bar{v}}. \) By using these facts and the properties of \( F_t^{u_1}, \) we can show that \( F_t^{u_1} \) coincides with \( F_t^{(\eta_{\bar{v}}) \ast u_1}. \) From this fact, the first relation follows.

q.e.d.

In similar to Proposition 3.8 of [GH], we have the following fact for a principal orbit of an aks-representation of complex rank greater than one.

**Lemma 4.7.** Let \( N \) be a principal orbit of an aks-representation of complex rank greater than one, \( \{n_i \mid i \in I\} \) the set of all \( J \)-curvature normals of \( N, \) \( E_i \) the \( J \)-curvature distribution corresponding to \( n_i \) and \( \Gamma \) the homogeneous structure of \( N \) at \( x. \) If the 2-dimensional complex affine subspace \( P \) through \( n_{i_1}, n_{i_2} \) and \( n_{i_3} \) which does not pass through \( 0, \) then, for any \( w_k \in (E_{i_k})_x \) \((k = 1, 2, 3, \) \), we have

\[
\Gamma_{w_1} \Gamma_{w_2} w_3 - \Gamma_{w_2} \Gamma_{w_1} w_3 = \Gamma_{(\Gamma_{w_1} w_2 - \Gamma_{w_2} w_1) w_3}.
\]

**Proof.** Let \( N \) be a principal orbit of an aks-representation of an anti-Kaehler symmetric space \( L^c/H^c \) of complex rank greater than one. We use the notations in the proof of Lemma 3.7.3. According to the proof of Lemma 3.7.3, for any \( w \in T_x N, \) the holomorphic isometry \( F_t^{w} \) is equal to \( \rho(\exp_{x_c}(t\bar{w})), \) where \( \bar{w} \) is an element of \( h^c \ominus h^c \circ (a_v). \) Here we take \( \bar{a}_v, \) as \( x \in a_v. \) Hence we have

\[
(4.1) \quad \Gamma_{w} = \text{ad}(\bar{w}).
\]

36
By using this fact and imitating the proof of Proposition 3.8 in [GH], we can show the desired relation. q.e.d.

For each $i \in I$, denote by $W_i$ the complex affine subspace $\tilde{\mathcal{O}} + (\langle E_i \rangle_0 \oplus \text{Span}_\mathbb{C}\{n_i\})$ of $V$. Also, let $f_i$ be the focal map having $L^i_{E_i}$'s ($u \in \tilde{M}^c$) as fibres, $\Phi_i$ the normal holonomy group of the focal submanifold $f_i(\tilde{M}^c)$ at $f_i(0)$ and $(\Phi_i)_0$ the isotropy group of $\Phi_i$ at $0$. This group $(\Phi_i)_0$ acts on $T_0 \tilde{M}^c$ and has $(E_i)_0$ as an invariant subspace. The irreducible decomposition of the action $(\Phi_i)_0 \curvearrowright (E_i)_0$ is given by the form $(E_i)_0' = (E_i)_0' \oplus (E_i)_0''$, where $\dim_{\mathbb{C}}(E_i)_0'$ is even and $\dim_{\mathbb{C}}(E_i)_0'' = 0, 1$ or 3. Set $m_i := \dim_{\mathbb{C}}E_i$. Note that $\Phi_i$ is orbit equivalent to the aks-representation associated with one of the following irreducible complex rank one anti-Kaehler symmetric spaces:

$$SO(m_i + 2, \mathbb{C})/SO(m_i + 1, \mathbb{C}), \quad SL(\frac{m_i+1}{2} + 1, \mathbb{C})/SL(\frac{m_i+1}{2}, \mathbb{C}) \cdot C_*,$$

and that

$$\dim_{\mathbb{C}}(E_i)_0'' = \begin{cases}0 & (\Phi_i = SO(m_i + 1, \mathbb{C})) \\ 1 & (\Phi_i = SL(\frac{m_i+1}{2}, \mathbb{C}) \cdot C_*) \\ 3 & (\Phi_i = Sp(1, \mathbb{C}) \times Sp(\frac{m_i+1}{2}, \mathbb{C})).\end{cases}$$

In similar to Proposition 3.11 of [GH], we have the following fact.

**Lemma 4.8.** Let $i \in I$. Then we have

$$\Gamma_{\langle E_i \rangle_0'}(E_i)_0'' = 0, \quad \Gamma_{\langle E_i \rangle_0''}(E_i)_0' \subset (E_i)_0', \quad \Gamma_{\langle E_i \rangle_0'}(E_i)_0' \subset (E_i)_0' \quad \text{and} \quad \Gamma_{\langle E_i \rangle_0''}(E_i)_0'' \subset (E_i)_0''.$$

**Proof.** By using Lemma 4.5 and (4.1), and imitating the proof of Proposition 3.11 in [GH], we can show these relations. q.e.d.

Also, by imitating the proof of Proposition 3.12 in [GH], we have the following fact.

**Lemma 4.9.** For $i_1 \in I$ and $i_2 \in I \cup \{0\}$ with $i_2 \neq i_1$, we have $\langle \Gamma_{\langle E_{i_1} \rangle_0}(E_{i_2})_0, (E_{i_2})_0' \rangle = 0$.

Also, by imitating the proof of Proposition 3.13 in [GH], we have the following fact.

**Lemma 4.10.** Let $i_1 \in I$ and $i_2, i_3 \in I \cup \{0\}$. For $w_k \in (E_{i_k})_0$ ($k = 1, 2, 3$), we have

$$(\nabla_{w_1}h)(w_2, w_3) = (\Gamma_{w_1}w_2, w_3)((n_{i_2})_0 - (n_{i_3})_0) \quad \text{and} \quad \Gamma_{w_1}w_2 = \tilde{\nabla}_{w_1}w_2 \pmod{(E_{i_2})_0},$$

where $\nabla$ is the connection of the tensor bundle $T^* M^c \otimes T^* \tilde{M}^c \otimes T^\perp \tilde{M}^c$ induced from $\nabla$ and the normal connection $\nabla^\perp$ of $\tilde{M}^c$, and $\tilde{w}_2$ is a local section of $E_{i_2}$ with $(\tilde{w}_2)_0 = w_2$. 

37
Let $i_1, i_2, i_3 \in I \cup \{0\}$ with $i_2 \neq i_3$. Then we define $\frac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}}$ by

$$\frac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}} := \begin{cases} b & \text{(when } n_{i_1} - n_{i_3} = b(n_{i_2} - n_{i_3}) \text{ for some } b \in \mathbb{C}) \\ 0 & \text{(when } n_{i_1} - n_{i_3} \text{ is not the complex multiple of } n_{i_2} - n_{i_3}) \end{cases}$$

For this quantity, we have the following fact.

**Lemma 4.11.** Let $i_1, i_2$ and $i_3$ be elements of $I$. If there exists a complex affine subspace $P$ of $T^*_0M^e$ such that $P$ passes through $(n_{i_1})_0$, $(n_{i_2})_0$ and $(n_{i_3})_0$ but that does not pass through $0$, then $\frac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}}$ is a real number.

**Proof.** Assume that there exists a complex affine subspace $P$ of $T^*_0M^e$ as in the statement. Then $L^P_0(\subset (W_P)_0)$ is a finite dimensional proper anti-Kaehler isoparametric submanifold of complex codimension greater than one. Let $L^P_0 = L^P_0 \times \cdots \times L^P_{k_0}(\subset (W_P)_0 \oplus \cdots \oplus (W_P)_{0})$ be the irreducible decomposition of $L^P_0$. Each $L^P_{k_0}(\subset (W_P)_{0})$ is of complex codimension one or regarded as a principal orbit of the aks-representation associated with an irreducible anti-Kaehler symmetric space of complex rank greater than one. Denote by $L^c/H^c$ the irreducible anti-Kaehler symmetric space. We use the notations in the proof of Lemma 3.7.3. Let $L^P_{k_0} = \rho(H^c) \cdot w$, where $\rho$ is the aks-representation and $w$ is the element of $\in w$ identified with $0$. Let $a_v$ be the maximal split abelian subspace of vector-type of $q$ containing $w$ and $a$ the Cartan subspace of $q^e$ containing $a_v$. The space $a$ is identified with the normal space of $T^*_0M^e$ of $M^e$ at $0$. Let $\Delta_+$ be the positive root system under some lexicographic ordering of $\alpha_v$. Let $\alpha_v$ and $\lambda_{\alpha_v}$ be as in the proof of Lemma 3.7.3. The set of all $J$-principal curvatures of $L^P_{k_0}$ is equal to $\{\lambda_{\alpha_v} | \alpha \in \Delta_+\}$. Let $n_{\alpha} be the $J$-curvature normal corresponding to $\lambda_{\alpha_v}$. Since $(\lambda_{\alpha_v})_0 = -\frac{\alpha_v}{\alpha_v w}$, we have $(n_{\alpha})_w \in a_v$. Hence we have

$$n_{\alpha_1} - n_{\alpha_3} \in \mathbb{R} \quad (\forall \alpha_1, \alpha_2, \alpha_3 \in \Delta_+ \text{ s.t. } \alpha_2 \neq \alpha_3.)$$

On the other hand, $T^*_0L^P_0$ is decomposed orthogonally as $T^*_0L^P_0 = T^*_0L^P_1 \oplus \cdots \oplus T^*_0L^P_{k_0}$ and all $J$-curvature normals of $L^P_{k_0}$ belong to $T^*_0L^P_1 \cup \cdots \cup T^*_0L^P_{k_0}$. These facts together with (4.2) deduce the statement of this lemma.

q.e.d.
Denote by $w_k$ the $(E_k)_0$-component of $w \in T_0 \tilde{M}^c$. In similar to Proposition 3.15 in [GH], the following fact directly follows from the first relation in Lemma 4.10 and the Codazzi equation (i.e, the symmetricness of $\nabla h$).

**Lemma 4.12.** Let $i_1, i_2 \in I$ and $i_3 \in I \cup \{0\}$ with $i_3 \neq i_2$. For any $w_k \in (E_{i_k})_0$ ($k = 1, 2$), we have

$$(\Gamma_{w_1} w_2)_{i_3} = \frac{n_{i_1} - n_{i_3}}{n_{i_2} - n_{i_3}} (\Gamma_{w_2} w_1)_{i_3}.$$ 

Also, in similar to Lemma 3.16 in [GH], we have the following fact.

**Lemma 4.13.** (i) Let $i_1 \in I$ and $i_2, i_3 \in I \cup \{0\}$. If $(\Gamma_{w_1} w_2)_{i_3} \neq 0$ for some $w_1 \in (E_{i_1})_0$ and $w_2 \in (E_{i_2})_0$, then $(n_{i_1})_0, (n_{i_2})_0$ and $(n_{i_3})_0$ are contained in a complex affine line.

(ii) Let $i_1, i_2, i_3 \in I$. The condition $(\Gamma_{w_1})_0 (E_{i_2})_0)_{i_3} \neq 0$ is symmetric in $i_1, i_2, i_3$.

In similar to Theorem 4.1 in [GH], we have the following fact.

**Lemma 4.14.** \[ \sum_{i_1, i_2 \in I \text{ s.t. } i_1 \neq i_2} \Gamma_{E_{i_1})_0 (E_{i_2})_0} \text{ is dense in } T_0 \tilde{M}^c \text{ and includes } \sum_{i \in I} (E_i)_0. \]

**Proof.** Set $D_u := \sum_{i_1, i_2 \in I \text{ s.t. } i_1 \neq i_2} \Gamma_{E_{i_1}} (E_{i_2}) u$ for each $u \in \tilde{M}^c$. Then the correspondence $u \mapsto D_u$ ($u \in \tilde{M}^c$) is a possibly non-smooth distribution on $\tilde{M}^c$. According to (ii) of Lemma 4.4, $D$ is invariant under all holomorphic isometries of $V$ preserving $\tilde{M}^c$ invariantly. By using this fact and imitating the proof of Theorem 4.1 in [GH], we can show the statement of this lemma. q.e.d.

By using this lemma and imitating the proof of Corollary 4.2 in [GH], we can show the following fact in similar to Corollary 4.2 in [GH].

**Lemma 4.15.** (i) For each $i_1 \in I$, we have

$$\sum_{i_2, i_3 \in I \text{ s.t. } n_{i_2}, n_{i_3} \notin \text{Span}_C(n_{i_1})} (\Gamma_{E_{i_2})_0 (E_{i_3})_0})_{i_1} = (E_{i_1})_0.$$ 

(ii) \[ \sum_{i_1, i_2 \in I \text{ s.t. } n_{i_1}, n_{i_2} : \text{lin. dep.}} (\Gamma_{E_{i_1}} (E_{i_2})_0) \text{ is dense in } (E_0)_0, \] where "lin. dep." means "linearly dependent".

**Notation.** In the sequel, for $w \in (E_i)_0$ ($i \in I \cup \{0\}$), $\tilde{w}$ means a local section of $E_i$ with $\tilde{w}_0 = w$. 

39
For \( w_1 \in (E_i)_0 \) and \( w_2 \in (E_i)_0 \) \((i_1, i_2 \in I \cup \{0\})\), define \( \nabla'_{\tilde{w}_1} \tilde{w}_2 \) by \( (\nabla'_{\tilde{w}_1} \tilde{w}_2)_u := (\nabla_{\tilde{w}_1} \tilde{w}_2)_u - \Gamma^u_{(\tilde{w}_1)u} (\tilde{w}_2)_u \), where \( u \) moves over the domain of \( \tilde{w}_1 \) and \( \tilde{w}_2 \). Denote by \( R \) the curvature tensor of \( \tilde{M}^c \). Let \( i_1, i_2, i_3 \in I \), \( i_4 \in I \cup \{0\} \) and \( w_k \in (E_{i_k})_0 \) \((k = 1, \ldots, 4)\). According to the Gauss equation, we have

\[
\langle R(w_1, w_2)w_3, w_4 \rangle = \frac{\langle w_1, w_4 \rangle \langle w_2, w_3 \rangle - \langle w_1, w_3 \rangle \langle w_2, w_4 \rangle}{n_{i_1}, n_{i_2}}. \tag{4.3}
\]

Also, from the definition of \( \nabla' \), we have

\[
\langle R(w_1, w_2)w_3, w_4 \rangle = \langle \Gamma_{w_1}w_3, \Gamma_{w_2}w_4 \rangle - \langle \Gamma_{w_2}w_3, \Gamma_{w_1}w_4 \rangle - \langle (\nabla_{\tilde{w}_1} \tilde{w}_2)_0, w_3 \rangle - \langle (\nabla_{\tilde{w}_1} \tilde{w}_2)_0, w_4 \rangle - \langle (\nabla_{\tilde{w}_1} \tilde{w}_2)_1, w_3 \rangle + \langle (\nabla_{\tilde{w}_1} \tilde{w}_2)_1, w_4 \rangle. \tag{4.4}
\]

For \( \nabla' \) and \( \Gamma \), we have the following relations.

**Lemma 4.16.** Let \( i_1, i_2, i_3 \in I \) and \( i_4 \in I \cup \{0\} \).

(i) For any \( w_k \in (E_{i_k})_0 \) \((k = 1, 2, 3)\), we have

\[
w_1 \langle \tilde{w}_2, \tilde{w}_3 \rangle = \langle (\nabla'_{\tilde{w}_1} \tilde{w}_2)_0, \tilde{w}_3 \rangle + \langle \nabla_{\tilde{w}_1} \tilde{w}_3, (\nabla'_{\tilde{w}_1} \tilde{w}_2)_0 \rangle.
\]

(ii) If \( i_1 \neq i_2 \), then we have \( \nabla_{\tilde{w}_1} \tilde{w}_2 = (\nabla_{\tilde{w}_1} \tilde{w}_2)_{i_3} \) for any \( w_k \in (E_{i_k})_0 \) \((k = 1, 2)\).

(iii) For any \( w_k \in (E_{i_k})_0 \) \((k = 1, 2, 3)\), we have

\[
(\nabla_{\tilde{w}_1} ((\Gamma_{w_2}w_3)_{i_3}))_0 = (\Gamma_{(\nabla'_{\tilde{w}_1} \tilde{w}_2)_0}w_3)_{i_3} + (\Gamma_{w_2}((\nabla_{\tilde{w}_1} \tilde{w}_3)_0))_{i_3}.
\]

**Proof.** The relations in (i) and (ii) are trivial. By using (ii) of Lemma 4.4 and imitating the proof of Lemma 5.2 in [GH], we can show the relation in (iii).

Let \( i_1 \in I \) and \( i_2 \in I \cup \{0\} \). For \( w \in T_0 \tilde{M}^c \), \( w_1 \in (E_{i_1})_0 \) and \( w_2 \in (E_{i_2})_0 \), we define \( \langle \Gamma_{w_1}w_1, w_2 \rangle \) by

\[
\langle \Gamma_{w_1}w_1, w_2 \rangle := -\sum_{i \in I} \langle \Gamma_{w_1}w_2, \frac{n_i - n_{i_2}}{n_{i_1} - n_{i_2}} w_i \rangle. \tag{4.5}
\]

According to (i) of Lemma 4.4 and Lemma 4.12, this definition is valid. By using the relation in (iii) of Lemma 4.16, we can show the following fact in similar to Theorem 5.7 in [GH].
Lemma 4.17. Let \( i_1, i_2, i_3 \in I \) and \( i_4 \in I \cup \{0\} \) with \( i_4 \neq i_3 \). For any \( w_k \in (E_{ik})_0 \) (\( k = 1, \ldots, 4 \)), we have
\[
\langle \langle [\Gamma_{w_1}, \Gamma_{w_2}] - \Gamma_{w_1}w_2 - \Gamma_{w_2}w_1 \rangle \rangle w_3, w_4 \rangle = -\langle \langle w_1, w_4 \rangle \langle w_2, w_3 \rangle - \langle w_1, w_3 \rangle \langle w_2, w_4 \rangle \rangle \langle n_{i_1}, n_{i_2} \rangle.
\]

By using Lemmas 4.12 and 4.17, we can show the following fact in similar to Corollary 5.10 in [GH].

Lemma 4.18. Let \( (i_1, i_2, i_3) \) be an element of \( I^2 \times (I \cup \{0\}) \) such that there exits no complex affine line containing \((n_{i_1})_0, (n_{i_2})_0 \) and \((n_{i_3})_0 \) and \( i_4 \) an element of \( I \). For any \( w_k \in (E_{ik})_0 \) (\( k = 1, \ldots, 4 \)), we have
\[
\langle \langle \Gamma_{w_1}w_2, \Gamma_{w_4}w_3 \rangle \rangle = \langle \Gamma_{w_4}w_2, \Gamma_{w_1}w_3 \rangle + c\langle \Gamma_{w_1}w_4, \Gamma_{w_2}w_3 \rangle.
\]
Furthermore, if \( i_1 = i_4 \) or the intersection of the complex affine lines through \((n_{i_1})_0, (n_{i_4})_0 \) and \((n_{i_2})_0, (n_{i_3})_0 \) contains no J-curvature normal, then we have \( c = 0 \). On the other hand, if the intersection of the complex affine lines through \((n_{i_1})_0, (n_{i_4})_0 \) and \((n_{i_2})_0, (n_{i_3})_0 \) contains a J-curvature normal \((n_{i_5})_0 \), then we have
\[
c = \frac{n_{i_3} - n_{i_2}}{n_{i_3} - n_{i_4}} \times \frac{n_{i_1} - n_{i_4}}{n_{i_1} - n_{i_5}}.
\]

By imitating the proof of Corollary 5.11 in [GH], we can show the following fact.

Lemma 4.19. Let \( i_1, i_2, i_3 \in I \) satisfying \( i_3 \neq i_1, i_2 \) and \( \frac{n_{i_2}}{n_{i_3}} \neq -\frac{n_{i_4} - n_{i_2}}{n_{i_3}} \). Assume that \( \langle \langle \Gamma_{(E_{i_1})_0}(E_{i_2})_0 \rangle \rangle \langle E_{i_3} \rangle_0 = 0 \) for any \( i_4 \in I \) and \( \langle \langle \Gamma_{(E_{i_1})_0}(E_{i_2})_0 \rangle \rangle \langle E_{i_3} \rangle_0 = 0 \) (these conditions hold if \( \Gamma_{(E_{i_1})_0}(E_{i_2})_0 \subset (E_0)_0 \)). Then we have \( \langle \langle \Gamma_{(E_{i_1})_0}(E_{i_2})_0, \Gamma_{(E_{i_1})_0}(E_{i_3})_0 \rangle \rangle = 0 \).

In similar to Corollary 5.13 in [GH], we have the following fact.

Lemma 4.20. Let \( i_1, i_2 \in I \) with \( i_1 \neq i_2 \). For any \( w_k \in (E_{ik})_0 \) (\( k = 1, 2 \)), we have
\[
\sum_{i_3 \in (I \cup \{0\}) \setminus \{i_1\}} \Re \left( \frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}} \right) \| \langle \Gamma_{w_1}w_2 \rangle_{i_3} \|^2 = \frac{1}{2} \langle \langle n_{i_1}, n_{i_2} \rangle \rangle \langle w_1, w_1 \rangle \| w_2 \|^2.
\]
Proof. Let \( w_2 = (w_2)_- + (w_2)_+ \) \((w_2)_- \in (T_0\tilde{M}^c)_-, (w_2)_+ \in (T_0\tilde{M}^c)_+ \). In similar to Corollary 5.13 in [GH], we can show

\[
\sum_{i_3 \in (I \cup \{0\}) \setminus \{i_1\}} \langle (\Gamma_{w_1}(w_2)_\epsilon)_{i_3}, \frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}}(\Gamma_{w_1}(w_2)_\epsilon)_{i_3} \rangle = \frac{1}{2} \langle n_{i_1}, n_{i_2} \rangle \langle w_1, w_1 \rangle \langle (w_2)_\epsilon, (w_2)_\epsilon \rangle,
\]

where \( \epsilon = - \) or \(+\). On the other hand, since \( F_{w_1}^i \)'s are holomorphic isometries, \( \Gamma_{w_1} \) preserves \((T_0\tilde{M}^c)_-\) and \((T_0\tilde{M}^c)_+\) invariantly, respectively. Hence we have \( \Gamma_{w_1}(w_2)_\epsilon = (\Gamma_{w_1}w_2)_\epsilon \). Also, from the definition of \((T\tilde{M}^c)_\epsilon\), we have \( \langle (\Gamma_{w_1}w_2)_{i_3} \rangle = \langle (\Gamma_{w_1}w_2)_{i_3} \rangle \epsilon \).

From these relations, we have

\[
\langle (\Gamma_{w_1}(w_2)_\epsilon)_{i_3}, \frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}}(\Gamma_{w_1}(w_2)_\epsilon)_{i_3} \rangle = \text{Re} \left( \frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}} \right) \langle (\Gamma_{w_1}w_2)_{i_3} \rangle \langle (\Gamma_{w_1}w_2)_{i_3} \rangle \epsilon. \]

By summing the \((-1)\)-multiple of (4.6) for \( \epsilon = - \) and (4.6) for \( \epsilon = + \), and using this relation, we have

\[
\sum_{i_3 \in (I \cup \{0\}) \setminus \{i_1\}} \text{Re} \left( \frac{n_{i_2} - n_{i_3}}{n_{i_1} - n_{i_3}} \right) \langle (\Gamma_{w_1}w_2)_{i_3} \rangle \langle (\Gamma_{w_1}w_2)_{i_3} \rangle = \frac{1}{2} \langle n_{i_1}, n_{i_2} \rangle \langle w_1, w_1 \rangle \langle w_2 \rangle^2.
\]

q.e.d.

By imitating the proof of Proposition 6.1 in [GH], we can show the following fact.

**Lemma 4.21.** Assume that \( \mathcal{W} \) is of type \( \tilde{A}, \tilde{D} \) or \( \tilde{E} \). Let \( i_1 \) and \( i_2 \) be elements of \( I \) such that \( n_{i_1} \) and \( n_{i_2} \) are linearly independent.

(i) If \( n_{i_1} \) and \( n_{i_2} \) are orthogonal, then we have \( \Gamma_{w_1}w_2 = 0 \) for any \( w_k \in (E_{i_k})_0 \) \((k = 1, 2)\).

(ii) If \( n_{i_1} \) and \( n_{i_2} \) are not orthogonal, then we have \( ||\Gamma_{w_1}w_2|| \leq \frac{1}{2} ||w_1|| ||w_2|| ||n_{i_1}|| \) for any \( w_k \in (E_{i_k})_0 \) \((k = 1, 2)\).

Proof. Let \( P \) be the complex affine line in \( T_0^+\tilde{M}^c \) through \((n_{i_1})_0\) and \((n_{i_2})_0\). Since \( n_{i_1} \) and \( n_{i_2} \) are linearly independent, we have \( 0 \notin P \). Hence the slice \( L_0^\perp \) is a finite dimensional proper anti-Kaehler isoparametric submanifold \((\text{of codimension two in } (W_0)_0)\). Hence, since \( \mathcal{W} \) is isomorphic to an affine Weyl group of type \( \tilde{A}, \tilde{D} \) or \( \tilde{E} \), the root system \((\text{which we denote by } \Delta_P)\) of \( L_0^\perp \) is of type \( A_1 \times A_1 \) or \( A_2 \). First we shall show the statement (i). Assume that \((n_{i_1})_0\) and \((n_{i_2})_0\) are orthogonal. Then \( \Delta_P \) is of type \( A_1 \times A_1 \) and hence \( P \) contains no other \( J \)-curvature normal. By using this fact and Lemma 4.5, we can show
First we shall show \( \Gamma_{w_1, w_2} = t_0^P \Gamma_{w_1, w_2} = 0 \) for any \( w_k \in (E_{i_k})_0 \) (\( k = 1, 2 \)), where \( t_0^P \Gamma \) is the homogeneous structure of \( L_0^P \). Next we shall show the statement (ii). Assume that \( (n_{i_1})_0 \) and \( (n_{i_2})_0 \) are not orthogonal. Then \( \Delta_P \) is of type \( \Delta_2 \) and hence there exists \( i_3 \in I \setminus \{ i_1, i_2 \} \) with \( (n_{i_3})_0 \in P \). The set \( l_{i_1} \cap l_{i_2} \cap l_{i_3} \cap \text{Span}_C \{(n_{i_1})_0, (n_{i_2})_0\} \) consists of the only one point. Denote by \( p_0 \) this point. Let \( e_1, e_2 \) and \( e_3 \) be a unit normal vector of \( l_{i_1}, l_{i_2} \) and \( l_{i_3} \), respectively. We may assume that \( e_3 = e_1 + e_2 \) by replacing some of these vectors to the \((-1)\)-multiples of them if necessary. Since \( \frac{(n_{i_1})_0}{(n_{i_1})_0, (n_{i_3})_0} \in l_{i_1} \), we have \( (n_{i_1})_0 = \frac{e_1}{\langle p_0, e_1 \rangle} \), \( (n_{i_2})_0 = \frac{e_2}{\langle p_0, e_2 \rangle} \) and \( (n_{i_3})_0 = \frac{e_3}{\langle p_0, e_3 \rangle} \).

By using these facts, Lemmas 4.9, 4.11 and 4.20, we can show

\[
\| \Gamma_{w_1, w_2} \|^2 = \| (\Gamma_{w_1, w_2})_{i_3} \|^2 \leq \frac{1}{2} n_{i_1} - n_{i_3} || n_{i_1}, n_{i_2} || \| w_1 \|^2 \| w_2 \|^2 \\
\leq \frac{1}{4} || w_1 \|^2 \| w_2 \|^2 || n_{i_1} ||^2.
\]

Thus we obtain the desired relation. q.e.d.

In similar to Proposition 6.2 in [GH], we can show the following fact.

**Lemma 4.22.** We have

\[
\sup_{i \in I} \sup_{P \in \mathcal{H}_i} \sup_{(w_1, w_2) \in (E_i)_0 \times (D_P)_0} \frac{|| \Gamma_{w_1, w_2} ||}{|| w_1 || || w_2 || || (n_i)_0 ||} < \infty,
\]

where \( \mathcal{H}_i \) is the set of all complex affine subspaces \( P \) in \( T_0^\perp M_c \) with \( 0 \notin P \) and \( (n_i)_0 \in P \).

**Proof.** Let \( \mathcal{H}^{irr}_i \) be the set of all elements \( P \) of \( \mathcal{H}_i \) such that \( L_0^P (\subset (W_P)_0) \) is irreducible. First we shall show

\[
(4.8) \quad \sup_{i \in I} \sup_{P \in \mathcal{H}^{irr}_i} \sup_{(w_1, w_2) \in (E_i)_0 \times (D_P)_0} \frac{|| \Gamma_{w_1, w_2} ||}{|| w_1 || || w_2 || || (n_i)_0 ||} < \infty.
\]

Fix \( i_0 \in I \) and \( P_0 \in \mathcal{H}^{irr}_{i_0} \). If the complex codimension of \( L_0^P (\subset (W_{P_0})_0) \) is equal to one, then we take \( P'_0 \in \mathcal{H}^{irr}_{i_0} \) such that \( P_0 \subset P'_0 \) and that the complex codimension of \( L_0^{P'_0} (\subset (W_{P'_0})_0) \) is greater than one. Then we have

\[
\sup_{(w_1, w_2) \in (E_{i_0})_0 \times (D_{P'_0})_0} \frac{|| \Gamma_{w_1, w_2} ||}{|| w_1 || || w_2 || || (n_{i_0})_0 ||} \leq \sup_{(w_1, w_2) \in (E_{i_0})_0 \times (D_{P'_0})_0} \frac{|| \Gamma_{w_1, w_2} ||}{|| w_1 || || w_2 || || (n_{i_0})_0 ||}.
\]

\]

43
and hence

\[
\sup_{i \in I} \sup_{P \in \mathcal{H}_{i_1}^{\text{irr}}} \sup_{(w_1, w_2) \in (E_i)_{0} \times (D_P)_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_i)_0||} = \sup_{(i,P) \in I \times \mathcal{H}_{i_1}^{\text{irr}, \geq 2}} \sup_{(w_1, w_2) \in (E_i)_{0} \times (D_P)_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_i)_0||},
\]

where \( \mathcal{H}_{i_1}^{\text{irr}, \geq 2} \) is the set of all elements \( P \)'s of \( \mathcal{H}_{i_1}^{\text{irr}} \) such that the complex codimension of \( L_{0}^P (\subset (W_P)_0) \) is greater than one. Fix \( i_1 \in I \) and \( P_1 \in \mathcal{H}_{i_1}^{\text{irr}, \geq 2} \). Let \( i_1 = (\beta_1, j_1) \). Set \( I_{\beta_1} := \{(\beta_1, j) \mid j \in \mathbb{Z} \} \). By using Lemma 4.5, we can show

\[
\sup_{(w_1, w_2) \in (E_{i_1})_{0} \times (D_{P_1})_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_{i_1})_0||} < \infty.
\]

Take any \( j \in \mathbb{Z} \). For each \( P \in \mathcal{H}_{i_1}^{\text{irr}} \), there exists \( P' \in \mathcal{H}_{(\beta_1, j)}^{\text{irr}} \) such that \( (W_P)_0 \) and \( (W_{P'})_0 \) are identified and that \( L_{0}^{P} (\subset (W_{P'})_0) \) is regarded as a parallel submanifold of \( L_{0}^{P'} (\subset (W_{P'})_0) \) under the identification. Even if \( P \) and \( P' \) are interchanged, this statement holds. Hence, by using Lemma 4.6, we can show

\[
\sup_{P \in \mathcal{H}_{(\beta_1, j)}^{\text{irr}}} \sup_{(w_1, w_2) \in (E_{(\beta_1, j)})_{0} \times (D_P)_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_{(\beta_1, j)})_0||} = \sup_{P \in \mathcal{H}_{i_1}^{\text{irr}}} \sup_{(w_1, w_2) \in (E_{i_1})_{0} \times (D_P)_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_{i_1})_0||}
\]

and hence

\[
\sup_{i \in I} \sup_{P \in \mathcal{H}_{i_1}^{\text{irr}}} \sup_{(w_1, w_2) \in (E_i)_{0} \times (D_P)_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_i)_0||} = \sup_{\beta \in (\Delta_i)_0} \sup_{P \in \mathcal{H}_{(\beta, 0)}^{\text{irr}}} \sup_{(w_1, w_2) \in (E_{(\beta, 0)})_{0} \times (D_P)_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_{(\beta, 0)})_0||} < \infty.
\]

Thus we obtain (4.8).

For simplicity, set

\[
C := \sup_{i \in I} \sup_{P \in \mathcal{H}_{i_1}^{\text{irr}}} \sup_{(w_1, w_2) \in (E_i)_{0} \times (D_P)_{0}} \frac{||\Gamma_{w_1} w_2||}{||w_1|| ||w_2|| ||(n_i)_0||}.
\]

Fix \( i_0 \in I \) and \( P_0 \in \mathcal{H}_{i_0} \setminus \mathcal{H}_{i_0}^{\text{irr}} \). Let \( L_{0}^{D_{P_0}} = L_1 \times \cdots \times L_k \) be the irreducible decomposition of \( L_{0}^{D_{P_0}} \). Take any \( i_1, i_2 \in I \) with \( (n_{i_1})_0, (n_{i_2})_0 \in P_0 \). If \( (n_{i_1})_0 \) and \( (n_{i_2})_0 \) are not orthogonal,
then \((E_i)_0 \oplus (E_i)_0 \subset T_0 L_a\) for some \(a \in \{1, \cdots, k\}\). Hence we have

\[
\sup_{(w_1, w_2) \in (E_i)_0 \times (E_i)_0} \frac{\|\Gamma w_1 w_2\|}{\|w_1\| \|w_2\| \|(n_i)_0\|} \leq C.
\]

If \((n_i)_0\) and \((n_i)_0\) are orthogonal, then the complex affine line through \((n_i)_0\) and \((n_i)_0\) contain no other \(J\)-curvature normal. Hence we have \(\Gamma_{(E_i)_0}(E_i)_0 = 0\). Therefore, we obtain

\[
\sup_{i \in I} \sup_{P \in \mathcal{H}_i} \sup_{(w_1, w_2) \in (E_i)_0 \times (E_i)_0} \frac{\|\Gamma w_1 w_2\|}{\|w_1\| \|w_2\| \|(n_i)_0\|} = C.
\]

Thus we obtain the desired relation. \(\text{q.e.d.}\)

In similar to Proposition 6.3 in [GH], we can show the following fact by using this lemma.

**Lemma 4.23.** Let \(i_0 = (\beta_0, j_0) \in I\) and \(w \in (E_i)_0\). Then \(\Gamma w\) can be extended continuously to \(T_0 \tilde{\mathcal{M}}^c\) if and only if the restriction of \(\Gamma_w\) to \(\oplus_{j \in Z} (E_i)_0\) can be extended continuously to \(\oplus_{j \in Z} (E_i)_0\).

**Proof.** Set \(V_0 := (E_0)_0, V_1 := \bigoplus_{i \in I \setminus \{(\beta_0)\}} (E_i)_0\) and \(V_2 := \bigoplus_{j \in Z} (E_i)_0\). Clearly we have \(T_0 \tilde{\mathcal{M}}^c = V_0 \oplus V_1 \oplus V_2\). Since \(\Gamma_w\) is a closed operator and \((E_0)_0\) is contained in the domain of \(\Gamma_w\) and closed in the domain, \(\Gamma_w|_{(E_0)_0}\) is a closed operator. Hence, according to the closed graph theorem, \(\Gamma_w|_{(E_0)_0}\) is continuous. Easily we can show

\[
V_1 = \bigoplus_i \left( \bigoplus_{i \in I \setminus \{(\beta_0)\}} (E_i)_0 \right),
\]

where \(l\) runs over the set of all complex affine lines in \(T_0^1 \tilde{\mathcal{M}}^c \setminus \{0\}\) through \((n_{i_0})_0\). For simplicity, set

\[
V_{1, l} := \bigoplus_{i \in I \setminus \{(\beta_0)\}} \bigoplus_{(n_i)_0 \in l} (E_i)_0.
\]

For each \(l\), \(L_0^l(\subset (W_l)_0)\) is a proper anti-Kaehler isoparametric submanifold of complex codimension two whose tangent space at \(\hat{0}\) contains \((E_{i_0})_0\). Hence, according to Lemma 4.22, we have

\[
\sup_{w' \in V_{1, l}} \frac{\|\Gamma w' w'\|}{\|w'\|} \leq C\|n_{i_0}\| \|w\|,
\]

45
where $C$ is the positive constant as in the proof of Lemma 4.20, and hence

$$\sup_{w' \in V_1} \frac{||\Gamma_w w'||}{||w'||} \leq C ||n_0|| ||w||.$$

Therefore the restriction of $\Gamma_w$ to $V_1$ can be extended continuously to $\nabla_1$. From these facts, the statement of this lemma follows. q.e.d.

According to Lemma 6.4 in [GH], we have the following fact.

**Lemma 4.24.** Let $W$ be a Hilbert space, $W = \bigoplus_{i \in \mathbb{Z}} W_i$ the orthogonal decomposition of $W$ and $f$ a linear map from $\bigoplus_{i \in \mathbb{Z}} W_i$ to $W$. Assume that there exists a positive constant $C$ such that $||f(w)|| \leq C ||w||$ for all $w \in \bigcup_{i \in \mathbb{Z}} W_i$ and that there exist injective maps $\mu_i : \mathbb{Z} \to \mathbb{Z}$ ($i = 1, \cdots, r$) such that $\langle f(W_i), f(W_j) \rangle = 0$ for any $j \notin \{\mu_1(i), \cdots, \mu_r(i)\}$. Then we have $||f|| \leq \sqrt{rC}$ and hence $f$ can be extended continuously to $W$.

Since

$$l_{(\beta,j)} = (\lambda_{(\beta,j)})_0^{-1}(1) = \begin{cases} (\beta^c)^{-1}(\beta(Z) + j\pi i) & (I_{\beta}^v)^- = \emptyset \\ (\beta^c)^{-1}(\beta(Z) + (j + \frac{1}{2})\pi i) & (I_{\beta}^v)^+ = \emptyset \\ (\beta^c)^{-1}(\beta(Z) + \frac{j\pi i}{2}) & (I_{\beta}^v)^+ \neq \emptyset \& (I_{\beta}^v)^- \neq \emptyset, \end{cases}$$

we have

$$\frac{n_{(\beta,j_1)} - n_{(\beta,j_2)}}{n_{(\beta,j_2)} - n_{(\beta,j_3)}} = \frac{j_1 - j_3}{j_2 - j_3} \times \frac{1 + j_2 r_{\beta} i}{1 + j_1 r_{\beta} i},$$

where $r_{\beta}$ is given by

$$r_{\beta} := \begin{cases} \frac{\beta(Z)}{2\pi} & (I_{\beta}^v)^- = \emptyset \\ \frac{2\beta(Z) + \pi i}{2\pi} & (I_{\beta}^v)^+ = \emptyset \\ \frac{2\beta(Z)}{2\beta(Z) + \pi i} & (I_{\beta}^v)^+ \neq \emptyset \& (I_{\beta}^v)^- \neq \emptyset. \end{cases}$$

By using (4.10) and Lemma 4.20, we can show the following fact in similar to Proposition 6.5 in [GH].
Lemma 4.25. Let $\beta \in (\Delta_\beta)_+$ and $j_1, j_2 \in \mathbb{Z}$. For any $w_1 \in (E(\beta,j_1))_0$ and any $w_2 \in (E(\beta,j_2))_0$, we have

$$\sum_{j \in \mathbb{Z}\setminus\{j_1\}} \frac{j - j_2}{j - j_1} \| (\Gamma_{w_1} w_2)(\beta,j) \|^2 + \| (\Gamma_{w_1} w_2)_0 \|^2 = \frac{1}{2} \left( \text{Re} \left( \frac{1}{1 + j_1 r_\beta i (1 + j_2 r_\beta i)} \right) \right)^{-1} \left( \frac{1}{\beta(Z)^2} \right) \langle n_\beta, n_\beta \rangle \langle w_1, w_1 \rangle \| w_2 \|^2.$$

All of reduced irreducible affine root systems of rank greater than one are those of the following types:

- $(\tilde{A}_m)$ $(m \geq 2)$,
- $(\tilde{B}_m)$ $(m \geq 3)$,
- $(\tilde{C}_m^\vee)$ $(m \geq 3)$,
- $(\tilde{C}_m)$ $(m \geq 2)$,
- $(\tilde{C}_m^\vee)$ $(m \geq 2)$,
- $(\tilde{D}_m)$ $(m \geq 4)$,
- $(\tilde{E}_6)$,
- $(\tilde{E}_7)$,
- $(\tilde{E}_8)$,
- $(\tilde{F}_4)$,
- $(\tilde{G}_2)$,
- $(\tilde{G}_2)$.

See [GH] about the definition of the reduced irreducible affine root system of each type.

All of non-reduced irreducible affine root systems of rank greater than one are those of the following types:

- $(\tilde{B}_m, \tilde{B}_m^\vee)$ $(m \geq 3)$,
- $(\tilde{C}_m^\vee, \tilde{C}_m^\vee)$ $(m \geq 3)$,
- $(\tilde{C}_m, \tilde{C}_m^\vee)$ $(m \geq 3)$,
- $(\tilde{C}_m^\vee, \tilde{C}_m)$ $(m \geq 3)$,
- $(\tilde{C}_m, \tilde{C}_m)$ $(m \geq 3)$,
- $(\tilde{C}_2, \tilde{C}_2)$.

See [GH] about the definition of the non-reduced irreducible affine root system of each type. In similar to Lemma 8.1 in [GH], we can show the following fact.

Lemma 4.26. Let $P$ be a complex affine line through 0 such that $P = \text{Span}_c\{n_\beta_0\}$ for some $\beta_0 \in (\Delta_\beta)_+$.

(i) If the affine root system $\mathcal{R}$ is of type $(\tilde{A}_m)$ $(m \geq 2)$, $(\tilde{D}_m)$ $(m \geq 4)$, $(\tilde{E}_m)$ $(m = 6, 7, 8)$ or $(\tilde{F}_4)$, then there exists a (complex) 2-dimensional complex affine subspace $P'$ through 0 containing $P$ such that the affine root system associated with $L_0^P (\subset (W_P)_0)$ is of type $(\tilde{A}_2)$.

(ii) If the affine root system $\mathcal{R}$ is of type $(\tilde{B}_m)$, $(\tilde{B}_m^\vee)$ or $(\tilde{B}_m, \tilde{B}_m^\vee)$ $(m \geq 2)$, then there exists a (complex) 2-dimensional complex affine subspace $P'$ through 0 containing $P$ such that the affine root system associated with $L_0^P (\subset (W_P)_0)$ is of type ”$(\tilde{A}_2)$” or ”$(\tilde{A}_2)$” or ”$(\tilde{C}_2)$” or ”$(\tilde{C}_2, \tilde{C}_2)$”, respectively.

(iii) If the affine root system $\mathcal{R}$ is of type $(\tilde{C}_m)$, $(\tilde{C}_m^\vee)$, $(\tilde{C}_m^\vee)$, $(\tilde{C}_m, \tilde{C}_m^\vee)$, $(\tilde{C}_m^\vee, \tilde{C}_m)$, $(\tilde{C}_m, \tilde{C}_m)$, $(\tilde{C}_m, \tilde{C}_m^\vee)$ $(m \geq 2)$, then there exists a (complex) 2-dimensional complex affine subspace $P'$ through 0 containing $P$ such that the affine root system associated with $L_0^P (\subset (W_P)_0)$ is of type ”$(\tilde{A}_2)$” or ”$(\tilde{C}_2)$” or ”$(\tilde{A}_2)$” or ”$(\tilde{C}_2)$” or ”$(\tilde{A}_2)$” or ”$(\tilde{C}_2)$” or ”$(\tilde{A}_2)$” or ”$(\tilde{C}_2)$”, respectively.

47
Proof. We shall show the statement (i) by imitating the discussion in the proof of (i) of Lemma 8.1 in [GH]. Let \( \Pi(\Delta_0) \) be a simple root system of \( \Delta_0 \). Without loss of generality, we may assume that \( \beta_0 \) is one of the elements of \( \Pi \). Since \( \mathcal{R} \) is of \((\tilde{A}_m)\) (\( m \geq 2 \)), \((D_m)\) (\( m \geq 4 \)), \((E_m)\) (\( m = 6, 7, 8 \)) or \((F_4)\), it follows from their Dynkin diagrams that there exists \( \beta_1 \in \Pi \) such that the angle between \( n_{\beta_0} \) and \( n_{\beta_1} \) is equal to \( \frac{2\pi}{3} \). Let \( P' \subset \mathfrak{b}^* \) be the complex affine line through \( \langle n_{(\beta_0,0)0} \rangle \) and \( \langle n_{(\beta_1,0)0} \rangle \) and set \( \tilde{P}' := \text{Span}_{\mathbb{C}}\{n_{\beta_0}, n_{\beta_1}\} \). It is clear that \( P' \subset \tilde{P}' \). Then it is shown that the root system associated with \( L_0^{\tilde{P}'} \) is of type \((A_2)\) and hence the affine root system associated with \( L_0^{\tilde{P}'} \) is of type \((\tilde{A}_2)\).

Next we shall show the statement (ii) by imitating the discussion in the first-half part of the proof of (ii) of Lemma 8.1 in [GH]. Since \( \Delta_0 \) is of type \((B_m)\), the positive root system \((\Delta_0)_{+}\) is described as

\[
(\Delta_0)_{+} = \{\theta_a | 1 \leq a \leq m\} \cup \{\theta_a \pm \theta_b | 1 \leq a < b \leq m\}
\]

for an orthonormal base \( \theta_1, \ldots, \theta_m \) of the dual space \( \mathfrak{b}^* \) of \( \mathfrak{b} \), the simple root system \( \Pi \) is equal to \( \{\theta_i - \theta_{i+1} | 1 \leq i \leq n - 1\} \cup \{\theta_n\} \) and the highest root is equal to \( \theta_1 + \theta_2 \), where we need to replace the inner product \( \langle , \rangle_{\mathfrak{b} \times \mathfrak{b}} \) to its suitable constant-multiple. Without loss of generality, we may assume that \( \beta_0 \) is one of the elements of \( \Pi \). In the case where \( \beta_0 \) is other than \( \theta_n \), there exists \( \beta_1 \in \Pi \) such that the angle between \( n_{\beta_0} \) and \( n_{\beta_1} \) is equal to \( \frac{2\pi}{3} \). Set \( \tilde{P}' := \text{Span}_{\mathbb{C}}\{n_{\beta_0}, n_{\beta_1}\} \). Then it is shown that the affine root system associated with \( L_0^{\tilde{P}'} \) is of type \((\tilde{A}_2)\). In the case where \( \beta_0 \) is equal to \( \theta_n \), we can take \( \beta_1 \in \Pi \) such that the angle between \( n_{\beta_0} \) and \( n_{\beta_1} \) is equal to \( \frac{3\pi}{4} \). Let \( P' \subset \mathfrak{b}^* \) be the complex affine line through \( \langle n_{(\beta_0,0)0} \rangle \) and \( \langle n_{(\beta_1,0)0} \rangle \), and set \( \tilde{P}' := \text{Span}_{\mathbb{C}}\{n_{\beta_0}, n_{\beta_1}\} \). Then it is shown that, in correspondence to \( \mathcal{W} \) is of type \((\tilde{B}_m)\), \((\tilde{B}_m^n)\) or \((\tilde{B}_m, \tilde{B}_m^n)\) (\( m \geq 2 \)), the root system associated with \( L_0^{\tilde{P}'} \) is of type \((C_2)\), \((C_2^n)\) or \((C_2, C_2^n)\) (\( m \geq 2 \)) and hence the affine root system associated with \( L_0^{\tilde{P}'} \) is of type \((\tilde{C}_2)\), \((\tilde{C}_2^n)\) or \((\tilde{C}_2, \tilde{C}_2^n)\) (\( m \geq 2 \)).

Next we shall show the statement (iii) by imitating the discussion in the second-half part of the proof of (ii) of Lemma 8.1 in [GH]. Since \( \Delta_0 \) is of type \((C_m)\), the positive root system \((\Delta_0)_{+}\) is described as

\[
(\Delta_0)_{+} = \{2\theta_a | 1 \leq a \leq m\} \cup \{\theta_a \pm \theta_b | 1 \leq a < b \leq m\}
\]

for an orthonormal base \( \theta_1, \ldots, \theta_m \) of the dual space \( \mathfrak{b}^* \), the simple root system \( \Pi \) is equal to \( \{\theta_i - \theta_{i+1} | 1 \leq i \leq n - 1\} \cup \{2\theta_n\} \) and the highest root is equal to \( 2\theta_1 \), where we need to replace the inner product \( \langle , \rangle_{\mathfrak{b} \times \mathfrak{b}} \) to its suitable constant-multiple. Without loss of generality, we may assume that \( \beta_0 \) is one of the elements of \( \Pi \). In the case where \( \beta_0 \) is other than \( 2\theta_n \), there exists \( \beta_1 \in (\Delta_0)_{+} \) such that the angle between \( n_{\beta_0} \) and \( n_{\beta_1} \) is equal to \( \frac{2\pi}{3} \). Set \( \tilde{P}' := \text{Span}_{\mathbb{C}}\{n_{\beta_0}, n_{\beta_1}\} \). Then it is shown that the affine root system associated with \( L_0^{\tilde{P}'} \) is of type \((\tilde{A}_2)\). In the case where \( \beta_0 \) is equal
Lemma 4.28. We can take $\beta_1 \in (\Delta_b)_+$ such that the angle between $n_{\beta_0}$ and $n_{\beta_1}$ is equal to $\frac{3\pi}{4}$. Set $\tilde{P}' := \text{Span}_C \{n_{\beta_0}, n_{\beta_1}\}$. Then it is shown that, in correspondence to $\mathcal{W}$ is of type $(C_m), (\tilde{C}_m^u), (\tilde{C}_m^d), (\tilde{C}_m^a, \tilde{C}_m^a), (\tilde{C}_m^a, \tilde{C}_m^d)$ or $(\tilde{C}_m^a, \tilde{C}_m^a)$ ($m \geq 2)$, the affine root system associated with $L'_0$ is of type $(\tilde{C}_2), (\tilde{C}_2^2), (\tilde{C}_2', \tilde{C}_2), (\tilde{C}_2, C_2), (\tilde{C}_2, \tilde{C}_2)$ or $(\tilde{C}_2, \tilde{C}_2^2)$ ($m \geq 2$).

In similar to Lemma 8.2 in [GH], we can show the following fact.

**Lemma 4.27.** If the affine root system $\mathcal{R}$ is of type $(\tilde{G}_2)$, then $\Gamma_{(E_{i_1})_0}(E_{i_2})_0 = 0$ for any $i_1, i_2 \in I$ with $\langle n_{i_1}, n_{i_2} \rangle = 0$.

**Proof.** Let $i_k = (\beta_k, j_k)$ ($k = 1, 2$). Let $P$ be the complex affine line through $(n_{i_1})_0$ and $(n_{i_2})_0$. If there does not exist further $i_3 \in I$ with $(n_{i_3})_0 \in P$, then the root system associated with the slice $L'_0$ is of type $(A_1 \times A_1)$. Hence we have $\Gamma_{(E_{i_1})_0}(E_{i_2})_0 = 0$. Otherwise, it is shown that $\{i \in I \mid (n_i)_0 \in P\}$ consists of exactly six elements because $\Delta_b$ is of type $(G_2)$. The root system $\Delta_P$ associated with the slice $L'_0(\subset (W_P)_0)$ is of type $(G_2)$. The slice $L'_0$ is regarded as a principal orbit of the isotropy action of an anti-Kaehler symmetric space $G_P^\circ/K_P^\circ$ whose root system is of type $(G_2)$. Let $\mathfrak{g}_P^\circ = \mathfrak{r}_P^\circ + \mathfrak{p}_P^\circ$ be the canonical decomposition of the Lie algebra $\mathfrak{g}_P^\circ$ of $G_P^\circ$ associated with the symmetric pair $(G_P^\circ, K_P^\circ)$. The space $\mathfrak{p}_P^\circ$ is identified with $(W_P)_0$ and the normal space of $L'_0(\subset (W_P)_0)$ at 0 is identified with a maximal abelian subspace $\mathfrak{b}_P$ of $\mathfrak{p}_P^\circ$. Denote by $(\mathfrak{r}_P^\circ)_0(\subset \mathfrak{p}_P^\circ)$ and $(\mathfrak{r}_P^\circ)_0(\subset \mathfrak{p}_P^\circ)$ be the root spaces for $\alpha \in \Delta_P$. Let $\alpha_k$ ($k = 1, 2$) be the elements of $\Delta_P$ such that $(\mathfrak{p}_P^\circ)_{\alpha_k}$ is identified with $(E_{i_k})_0$. For any $w_k \in (E_{i_k})_0$ ($k = 1, 2$), we have

$$\Gamma_{w_1 w_2} \in [(\mathfrak{r}_P^\circ)_{\alpha_1}, (\mathfrak{p}_P^\circ)_{\alpha_2}] \subset (\mathfrak{p}_P^\circ)_{\alpha_1 + \alpha_2} + (\mathfrak{p}_P^\circ)_{\alpha_1 - \alpha_2}.$$ 

Since $\alpha_1$ and $\alpha_2$ are orthogonal and $\Delta_P$ is type $(G_2)$, we have $\alpha_1 \pm \alpha_2 \notin \Delta_P$. Hence we have $\Gamma_{w_1 w_2} = 0$. Therefore we have $\Gamma_{(E_{i_1})_0}(E_{i_2})_0 = 0$. This completes the proof.

q.e.d.

We need the following lemma in Euclidean plane geometry, which is stated in [GH] (Lemma 8.3).

**Lemma 4.28.** Let $l_k$ ($k = 1, 2, 3$) be mutually distinct parallel lines in the plane $\mathbb{R}^2$, $x_k \in l_k$ ($k = 1, 2, 3$) and $l_{k_1 k_2}$ the line through $x_{k_1}$ and $x_{k_2}$.

(i) If the angles between $l_1$ and $l_j$'s ($1 \leq i < j \leq 3$) are equal to multiples of $\frac{\pi}{6}$ other than $\frac{\pi}{2}$, then $x_1, x_2, x_3$ lie in a line.
(ii) If the angles between \(l_i\) and \(l_j\)'s \((1 \leq i < j \leq 3)\) are equal to multiples of \(\frac{\pi}{4}\), then \(x_1, x_2, x_3\) lie in a line or one of \(l_1, l_2, l_3\), say \(l_2\), lies in the half way distant between the other two. In the later case, \(l_{13}\) is orthogonal to \(l_1\).

By using Lemmas 4.8, 4.9, 4.13, 4.14, 4.17, 4.26, 4.27 and 4.28, and imitating the proof of Theorem 8.4 in [GH], we can show the following fact.

**Lemma 4.29.** If \(\mathcal{R}\) is of type \((\tilde{A}_m)\) \((m \geq 2)\), \((\tilde{D}_m)\) \((m \geq 4)\), \((\tilde{E}_6)\), \((\tilde{E}_7)\), \((\tilde{E}_8)\), \((\tilde{F}_4)\) or \((\tilde{G}_2)\), then we have \(\Gamma_{(E_{\beta,j_1})_0}(E_{\beta,j_2})_0 \subset (E_0)_0\) for any \(\beta \in (\Delta_+)\) and any \(j_1, j_2 \in \mathbb{Z}\).

**Proof.** According to Lemma 4.26, we may assume that \(\mathcal{R}\) is of type \((\tilde{A}_2)\) or \((\tilde{G}_2)\). Furthermore, according to Lemma 4.8, we may assume that \(j_1 \neq j_2\). Set \(i_k := (\beta, j_k)\) \((k = 1, 2)\). Suppose that \(\langle \Gamma_{(E_{i_1})_0}(E_{i_2})_0 \rangle_{i_3} = 0\) for some \(i_3 \in I\). Take \(w_k \in \langle (E_{i_k})_0 \rangle\) \((k = 1, 2)\) with \(\langle (\Gamma_{w_1}w_2)_i \rangle = 0\). Let \(P := \text{Span}_C\{(n_{i_1})_0\}\). Since \(P\) is totally geodesic, we have \((n_{i_3})_0 \in \mathbb{P}\). Hence \(i_3\) is expressed as \(i_3 = (\beta, j_3)\) in terms of some \(j_3 \in \mathbb{Z}\). According to Lemmas 4.8 and 4.9, we have \(j_3 \neq j_1, j_2\). According to Lemma 4.14, there exists \(i_4, i_5 \in I\) such that \((n_{i_4})_0\) and \((n_{i_5})_0\) are \(C\)-linearly independent and that \(\langle (\Gamma_{w_1}w_2)_i \rangle_{i_3} \neq 0\). Hence it follows from Lemma 14.3 that \((n_{i_4})_0\) and \((n_{i_5})_0\) are contained in a complex affine line. Since the intersection of \(\ell\) and \(P\) is at most one point, we have \(\langle (\Gamma_{w_1}w_2)_i \rangle_{i_3} \neq 0\). Hence \(\langle (\Gamma_{w_1}w_2)_i \rangle_{i_3} \neq 0\). Hence it follows from Lemma 4.3 that \(\langle (\Gamma_{w_1}w_2)_i \rangle_{i_3} \neq 0\) and no three of \((n_{i_1})_0, (n_{i_2})_0, (n_{i_3})_0\) are contained in a complex affine line. Hence, it follows from Lemma 4.18 that

\[
\langle (\Gamma_{w_1}w_2)_i \rangle_{i_3} = (\Gamma_{w_4}w_5, \Gamma_{w_1}w_4) = c(\Gamma_{w_5}w_2, \Gamma_{w_1}w_4),
\]

where \(c\) is as in Lemma 4.18. Hence we have \(\langle (\Gamma_{w_5}w_2, \Gamma_{w_1}w_4) \rangle = 0\) or \(\langle (\Gamma_{w_5}w_5, \Gamma_{w_5}w_4) \rangle = 0\). According to Lemma 4.13, this fact implies that the intersection of the complex affine lines through \((n_{i_1})_0\), \((n_{i_2})_0\) and \((n_{i_3})_0\) contains \((n_{i_6})_0\) for some \(i_6 \in I\) or the intersection of the complex affine lines through \((n_{i_1})_0\), \((n_{i_2})_0\) and \((n_{i_3})_0\) contains \((n_{i_6})_0\) for some \(i_6 \in I\). The analysis is completely similar in both cases. So we consider the first case. Then, since \((n_{i_1})_0\), \((n_{i_2})_0\) and \((n_{i_3})_0\) are \(C\)-linearly dependent pairwisely, the complex focal hyperplanes \(l_{14}, l_{24}, l_{34}\) are mutually parallel. Note that they are complex lines because we assume that \(\mathcal{R}\) is of type \((\tilde{A}_2)\) or \((\tilde{G}_2)\). Hence the lines \(l_{14}^R, l_{24}^R, l_{34}^R\) (in b) are mutually parallel. Also, since \((n_{i_3})_0\), \((n_{i_4})_0\) and \((n_{i_5})_0\) are contained in a complex line which does not pass \(0\), we have \(l_{14}, l_{24}\) and \(l_{34}\) have a common point. Hence the lines \(l_{14}^R, l_{24}^R, l_{34}^R\) have a common point. Denote by \(p_{345}\) this common point. Similarly, since \((n_{i_2})_0\), \((n_{i_5})_0\) and \((n_{i_6})_0\) are contained in a complex line which does not pass \(0\), we have \(l_{24}, l_{34}, l_{45}\) have a common point. Hence the lines \(l_{24}^R, l_{34}^R, l_{45}^R\) have a common point. Denote by \(p_{256}\) this common point. Also, since \((n_{i_1})_0\), \((n_{i_4})_0\) and \((n_{i_6})_0\) are contained in a complex line which does not pass \(0\), \(l_{14}, l_{45}\) and \(l_{46}\) have a common point. Hence the lines \(l_{14}^R, l_{45}^R, l_{46}^R\) and
have a common point. Denote by \( p_{146} \) this common point. These three intersection points \( p_{345}, p_{256} \) and \( p_{146} \) lie in no line in \( b \) because of \( i_4 \neq i_5 \). On the other hand, since \( R \) is of type \((A_2)\) or \((G_2)\), it follows from Lemmas 4.13 and 4.27 that the angle between any two of \( \eta_k \) \((k = 1, \ldots, 6)\) is equal to a multiple of \( \frac{\pi}{6} \) other than \( \frac{\pi}{2} \). Hence, it follows from (i) of Lemma 4.28 that \( p_{345}, p_{256} \) and \( p_{146} \) lie in a line in \( b \). Thus a contradiction arises. Therefore we obtain \( (\Gamma(E_{i_1})_{0}(E_{i_2})_{0})_i = \{0\} \) for all \( i \in I \), that is, \( \Gamma(E_{i_1})_{0}(E_{i_2})_{0} \subset (E_{0})_{0} \).

From Lemmas 4.20, 4.24 and 4.29, we have the following fact.

**Proposition 4.30.** If \( R \) is one of the following types:

\[
\begin{align*}
(A_m) & \ (m \geq 2), \quad (D_m) \ (m \geq 4), \quad (E_6), \quad (E_7), \quad (F_4), \quad (F_4^\vee), \quad (G_2), \quad (G_2^\vee),
\end{align*}
\]

then \( \Gamma_w \) is bounded for any \( w \in \bigcup_{i \in I} E_i \).

**Proof.** Let \( \beta \in (\Delta_b)_+ \) and \( j_1, j_2 \in \mathbb{Z} \). Set \( i_k := (\beta, j_k) \ (k = 1, 2) \). From Lemmas 4.20 and 4.29, we have

\[
||\Gamma_{w_1}w_2||^2 = \frac{1}{2} \text{Re} \left( \frac{n_{i_1} - 0}{n_{i_2} - 0} \right) \langle n_{i_1}, n_{i_2} \rangle \langle w_1, w_1 \rangle ||w_2||^2
\]

for any \( w_k \in (E_{i_k})_{0} (k = 1, 2) \). Clearly we have

\[
\sup_{j \in \mathbb{Z}} \left| \text{Re} \left( \frac{n_{i_1} - 0}{n_{(\beta,j)} - 0} \right) \langle n_{i_1}, n_{(\beta,j)} \rangle \right| < \infty.
\]

Denote by \( C \) this supremum. Then we have

\[
||\Gamma_{w_1}w_2|| \leq \sqrt{\frac{C}{2}} ||w_1|| ||w_2||.
\]

Hence, it follows from the arbitrariness of \( w_2 \) and \( j_2 \) that

\[
||\Gamma_{w_1}w|| \leq \sqrt{\frac{C}{2}} ||w_1|| ||w||
\]

for any \( w \in \bigcup_{j \in \mathbb{Z}} (E_{(\beta,j)})_{0} \). On the other hand, since \( \Gamma(E_{i_1})_{0}(E_{(\beta,j)})_{0} \subset (E_{0})_{0} \ (j \in \mathbb{Z}) \) by Lemma 4.29, it follows from Lemma 4.19 that

\[
\langle \Gamma(E_{i_1})_{0}(E_{(\beta,j)})_{0}, \Gamma(E_{i_1})_{0}(E_{(\beta,j')})_{0} \rangle = 0
\]

51
for any $j' \in \mathbb{Z}$ satisfying $j' \neq j_1, j, 2j_1 - j$. Therefore, by using Lemma 4.24, we can show that

$$||\Gamma_{w_1}w|| \leq \sqrt{\frac{3C}{2}}||w_1|| ||w||$$

for any $w \in \bigoplus_{j \in \mathbb{Z}} (E_{(\beta,j)})_0$. Thus the restriction $\Gamma_{w_1}$ to $\bigoplus_{j \in \mathbb{Z}} (E_{(\beta,j)})_0$ is bounded. Therefore, according to Lemma 4.23, $\Gamma_{w_1}$ is bounded.

From Lemmas 4.13, 4.14, 4.18, 4.24, 4.26, 4.28 and 4.29, we have the following fact in similar to Theorem 8.6 of [GH].

**Lemma 4.31.** For any $\beta \in (\triangle_b)_+$ and any $j_1, j_2 \in \mathbb{Z}$, we have

$$\Gamma(E_{(\beta,j_1)})_0(E_{(\beta,j_2)})_0 \subset (E_0)_0 \oplus (E_{(\beta,2j_1-j_2)})_0 \oplus (E_{(\beta,2j_2-j_1)})_0 \oplus (E_{(\beta,4j_2)})_0,$$

where the last term is omitted in the case where $j_1 + j_2$ is odd.

**Proof.** For simplicity, set $i_k := (\beta, j_k)$ $(k = 1, 2)$. According to (ii),(iii) of Lemma 4.26 and Proposition 4.29, we may assume that $(R)$ is of type $(\hat{C}_2), (\hat{C}_4), (\hat{C}_5, \hat{C}_6, \hat{C}_7, \hat{C}_8), (\hat{C}_9, \hat{C}_2)$ or $(\hat{C}_2, \hat{C}_2)$. Let $P := \text{Span}_C\{n_{i_3}\}$. Since $L^P_0$ is totally geodesic, we have

$$\Gamma(E_{(\beta,j_1)})_0(E_{(\beta,j_2)})_0 \subset (E_0)_0 \oplus \left(\bigoplus_{j \in \mathbb{Z}} (E_{(\beta,j)})_0\right).$$

Suppose that $(i_{w_1}w_2)(\beta,j_3) \neq 0$ for some $w_k \in (E_{i_{k}})_0$ $(k = 1, 2)$ and some $j_3 \in \mathbb{Z}$. Set $i_3 := (\beta, j_3)$. Then we have $j_3 \neq j_1, j_2$ automatically. According to Lemma 4.14, there exist $i_k = (\beta_k, j_k)$ $(k = 4, 5)$ such that $\langle (\Gamma_{w_1}w_2)i_3, \Gamma_{w_4}w_5 \rangle \neq 0$ for some $w_k \in (E_{i_{k}})_0$ $(k = 4, 5)$. As in the proof of Proposition 4.28, we can show $\langle \Gamma_{w_5}w_2, \Gamma_{w_1}w_4 \rangle \neq 0$ or $\langle \Gamma_{w_3}w_5, \Gamma_{w_2}w_4 \rangle \neq 0$ in terms of Lemmas 4.13 and 4.18. According to Lemma 4.13, this fact implies that the intersection of the complex lines through $(n_{i_2})_0, (n_{i_3})_0$ and $(n_{i_1})_0, (n_{i_4})_0$ contains $(n_{i_5})_0$ for some $i_6 \in I$ or the intersection of the complex lines through $(n_{i_1})_0, (n_{i_5})_0$ and $(n_{i_2})_0, (n_{i_4})_0$ contains $(n_{i_6})_0$ for some $i_6 \in I$. The analysis is completely similar in both cases. So we consider the first case. Then, as in the proof of Proposition 4.29, we can show that $i_{11}, i_{12}, i_{13}$ are mutually parallel, $i_{13}, i_{14}, i_{15}$ have the common point (which we denote by $p_{345}$), $i_{12}, i_{15}, i_{16}$ have the common point (which we denote by $p_{256}$) and that $i_{11}, i_{12}, i_{13}$ have the common point (which we denote by $p_{146}$). These three intersection points $p_{345}, p_{256}$ and $p_{146}$ are lie in no line in $b$ because of $i_4 \neq i_5$. Hence, it follows from (ii) of Lemma 4.28 that one of $i_{11}, i_{12}, i_{13}$ lies in the half way distant between the other two, that is, one of $j_1, j_2, j_3$ is equal to the half of the sum of the other two (i.e., $j_3 = \frac{j_1 + j_2}{2}, 2j_1 - j_2$ or $2j_2 - j_1$). Thus we obtain the desired relation.

q.e.d.
From this proposition and Lemmas 4.19 and 4.24, we can show the following fact in similar to Corollary 8.7 of [GH].

**Lemma 4.32.** Let \( \beta \in (\Delta_b)_+ \) and \( j_k \in \mathbb{Z} \ (k = 1, 2, 3) \) with \( j_1 \neq j_2 \). Then we have\( \langle \Gamma_{(E(\beta,j_1)_0)}(E(\beta,j_2)_0), \Gamma_{(E(\beta,j_1)_0)}(E(\beta,j_3)_0) \rangle = 0 \) if \( j_3 \) is not one of

\[
4j_2 - 3j_1, \ 2j_2 - j_1, \ j_2, \ \frac{j_1 + j_2}{2}, \ \frac{3j_1 + j_2}{4}, \ \frac{3j_1 - j_2}{2}, \ 2j_1 - j_2, \ 3j_1 - 2j_2.
\]

**Proof.** For simplicity, set \( i_k := (\beta,j_k) \ (k = 1, 2, 3) \). According to Lemmas 4.8, 4.9 and (i) of Lemma 4.4, we may assume that \( j_3 \neq j_1 \). The condition \( \frac{n_{i_2}}{n_{i_3}} \neq -\frac{n_{i_4} - n_{i_2}}{n_{i_4} - n_{i_3}} \) in Lemma 4.19 is equivalent to \( j_3 \neq 2j_1 - j_2 \). From the assumption for \( j_3 \), we have

\[
\{2j_1 - j_2, 2j_2 - j_1, \frac{j_1 + j_2}{2}\} \cap \{2j_1 - j_3, 2j_3 - j_1, \frac{j_1 + j_3}{2}\} = \emptyset.
\]

Hence it follows from Proposition 4.32 that \( \langle (\Gamma_{(E_{i_1})_0})(E_{i_2})_0), \Gamma_{(E_{i_1})_0}(E_{i_3})_0) \rangle = 0 \) for any \( i \in I \). Also, since \( j_3 \neq 2j_1 - j_2, 2j_2 - j_1, \frac{j_1 + j_2}{2} \), it follows from Lemma 4.31 that

\[
\langle (\Gamma_{(E_{i_1})_0})(E_{i_2})_0), \Gamma_{(E_{i_1})_0}(E_{i_3})_0) \rangle = 0.
\]

q.e.d.

Let \( P \) be a complex affine line in \( T_{\emptyset}^\perp \tilde{M}^c \) containing exactly four \( J \)-curvature normals \( (n_{i_k})_0 \ (k = 1, \ldots, 4) \) at \( \emptyset \). Then the root system (which we denote by \( \triangle_P \)) of the slice \( L_0^P \) is of type \((B_2)\) or \((BC_2)\). The root system \( \triangle_P \) is given by

\[
\{ \pm c_k \hat{\alpha}_k | k = 1, \ldots, 4 \} \ (\subset \text{Span}_\mathbb{R}\{c_k \hat{\alpha}_k | k = 1, \ldots, 4\})
\]

or

\[
\{ \pm c_k \hat{\alpha}_k | k = 1, \ldots, 4 \} \cup \{ \pm 2c_k \hat{\alpha}_k | k = 1, 2 \} \ (\subset \text{Span}_\mathbb{R}\{c_k \hat{\alpha}_k | k = 1, \ldots, 4\}),
\]

where \( c_k \)'s \( (k = 1, \ldots, 4) \) are complex constants and \( \hat{\alpha}_k \) is defined by \( \hat{\alpha}_k(\cdot) = \langle (n_{i_k})_0, \cdot \rangle \). For simplicity, set \( \alpha_k := c_k \hat{\alpha}_k \ (k = 1, \ldots, 4) \). Here we may assume that \( \alpha_1, \alpha_2 \) are short roots and that \( \alpha_3, \alpha_4 \) are long roots by permuting the indices if necessary. If \( \triangle_P \) is of type \((B_2)\), then \( E_{i_1}, \ldots, E_{i_4} \) are irreducible, and if it is of type \((BC_2)\), then \( E_{i_1}, E_{i_2} \) are reducible and \( E_{i_3}, E_{i_4} \) are irreducible. In similar to Lemma 8.8 in [GH], we have the following lemma.
Lemma 4.33. Let $P$ be as above.

(i) If $\Delta_P$ is of type $(B_2)$, then we have $\Gamma_{(E_{43})_0}(E_{43})_0 = 0$.

(ii) If $\Delta_P$ is of type $(BC_2)$, then we have $\Gamma_{(E_{43})_0}(E_{43})_0 = 0$ $(k = 1, 2)$.

(iii) If $\Delta_P$ is of type $(BC_2)$, then we have $\Gamma_{(E_{43})''_0}(E_{43})_0 = 0 = \Gamma_{(E_{43})''_0}(E_{43})_0$.

Proof. Since the slice $L^P_0$ is homogeneous, it is regarded as a principal orbit of the isotropy action of an anti-Kaehler symmetric space $G^c_P/K^c_P$ whose root system is of type $(B_2)$ or $(BD_2)$. We use the same notations as in the proof of Lemma 4.27. For the bracket product, we have

\[
[(\mathfrak{f}^c_P)_{\alpha,3}, (\mathfrak{p}^c_P)_{\alpha,4}] = [(\mathfrak{f}^c_P)_{\alpha,4}, (\mathfrak{p}^c_P)_{\alpha,3}] = 0 \quad ((B_2)\text{-case})
\]

\[
[(\mathfrak{f}^c_P)_{\alpha,3}, (\mathfrak{p}^c_P)_{\alpha,4}] = [(\mathfrak{f}^c_P)_{\alpha,4}, (\mathfrak{p}^c_P)_{\alpha,3}] \subset (\mathfrak{p}^c_P)_{2\alpha_1} + (\mathfrak{p}^c_P)_{2\alpha_2} \quad ((BC_2)\text{-case})
\]

\[
[(\mathfrak{f}^c_P)_{2\alpha_1}, (\mathfrak{p}^c_P)_{2\alpha_2} + (\mathfrak{p}^c_P)_{2\alpha_2}] = [(\mathfrak{f}^c_P)_{2\alpha_2}, (\mathfrak{p}^c_P)_{\alpha,1} + (\mathfrak{p}^c_P)_{2\alpha_2} = 0 \quad ((BC_2)\text{-case}).
\]

From (4.1) and these relations, we get the statements (i) ~ (iii).

For $\beta \in (\Delta_b)_+$, we set

\[ C_\beta := \sup_{j, j' \in \mathbb{Z}} \left| \text{Re} \left( \frac{1 + j' r_\beta i}{1 + j r_\beta i} \right)^{-1} \times \text{Re} \left( \frac{1}{(1 + j r_\beta i)(1 + j' r_\beta i)} \right)^{1/2} \right|. \]

Clearly we have $C_\beta < \infty$. By using Lemmas 4.25 and 4.31, we can show the following fact in similar to Proposition 9.1 in [GH].
Proposition 4.35. Let $i_k = (\beta, j_k) \ (k = 1, 2)$ and $w_k \in (E_{i_k})_0 \ (k = 1, 2)$. If $j_1 - j_2$ is not divisible by $2^m$ for some integer $m \geq 1$, then we have
\[
\|\Gamma_{w_1} w_2\| \leq 2^{m-1} C \frac{\|n_{\beta}\|}{\beta(Z)} \|w_1\| \|w_2\|.
\]

Proof. From Lemmas 4.25 and 4.31, we have
\[
\begin{aligned}
2 \|\Gamma_{w_1} w_2\|_{(\beta, 2j_1 - j_2)}^2 &+ \frac{1}{2} \|\Gamma_{w_1} w_2\|_{(\beta, 2j_2 - j_1)}^2 \\
- \|\Gamma_{w_1} w_2\|_{(\beta, i + j_2)}^2 &+ \|\Gamma_{w_1} w_2\|_0^2
\end{aligned}
\]
\[
\leq \frac{1}{2} \left| \Re \left( \frac{1 + j_2 r_{\beta 1}}{1 + j_1 r_{\beta 1}} \right)^{-1} \times \Re \left( \frac{1}{(1 + j_1 r_{\beta 1})(1 + j_2 r_{\beta 1})} \right) \right| \frac{\|n_{\beta}\|^2}{\beta(Z)^2} \|w_1\|^2 \|w_2\|^2.
\]
By multiplying 2 to both sides and adding 3\|\Gamma_{w_1} w_2\|_{(\beta, i + j_2)}^2 \text{ to both sides, we obtain}
\[
\|\Gamma_{w_1} w_2\|^2 \leq \left| \Re \left( \frac{1 + j_2 r_{\beta 1}}{1 + j_1 r_{\beta 1}} \right)^{-1} \times \Re \left( \frac{1}{(1 + j_1 r_{\beta 1})(1 + j_2 r_{\beta 1})} \right) \right| \frac{\|n_{\beta}\|^2}{\beta(Z)^2} \|w_1\|^2 \|w_2\|^2 + 3 \|\Gamma_{w_1} w_2\|_{(\beta, i + j_2)}^2
\]
\[
\leq C \frac{\|n_{\beta}\|^2}{\beta(Z)^2} \|w_1\|^2 \|w_2\|^2 + 3 \|\Gamma_{w_1} w_2\|_{(\beta, i + j_2)}^2.
\]
We use the induction on $m$. In case of $m = 1$, the statement of this proposition is derived from (4.11) directly. Now we assume that the statement of this proposition holds for $m(\geq 1)$ and that $j_1 - j_2$ is not divisible by $2^{m+1}$. Set $w := (\Gamma_{w_1} w_2)_{(\beta, i + j_2)}$. Since $F^w_{w_1}$'s are holomorphic isometries, $\Gamma_{w_1}$ preserves $(T_{0,0}^\epsilon)^-$ and $(T_{0,0}^\epsilon)^+$ invariantly, respectively. Hence we have $\Gamma_{w_1} ((w_2)_\epsilon) = (\Gamma_{w_1} w_2)_\epsilon$ ($\epsilon = - \text{ or } +$). Also, it follows from the definitions of $(T_{0,0}^\epsilon)_\epsilon$ ($\epsilon = - \text{ or } +$) that $(\Gamma_{w_1} w_2)_\epsilon = (\Gamma_{w_1} w_2)_{(\beta, i + j_2)}$ ($\epsilon = - \text{ or } +$).
From (i) of Lemma 4.4 and these relations, we have
\[
\langle (\Gamma_{w_1} w_2)_\epsilon, w_\epsilon \rangle = \langle \Gamma_{w_1} w_2, w_\epsilon \rangle = - \langle (w_2)_\epsilon, (\Gamma_{w_1} w)_\epsilon \rangle.
\]
Hence we have
\[
\langle \Gamma_{w_1} w_2, w \rangle = - \langle w_2, \Gamma_{w_1} w \rangle.
\]
From (4.11) and (4.12), we have

\[
\left\| \left( \Gamma_{w_1} w_2 \right)_{(\beta, \frac{j_1 + j_2}{2})} \right\|^2 = \langle \Gamma_{w_1} w_2, w \rangle_{\pm} = -\langle w_2, \Gamma_{w_1} w \rangle_{\pm} \leq \| w_2 \| \| \Gamma_{w_1} w \| \\
\leq 2^{m-1} \left| \text{Re} \left( \frac{1 + \frac{j_1 + j_2}{2} r_{\beta} i}{1 + j_1 r_{\beta} i} \right) \right|^{-1} \left. \frac{1}{(1 + j_1 r_{\beta} i)(1 + \frac{j_1 + j_2}{2} r_{\beta} i)} \right| \frac{\| n_{\beta} \| \| w_1 \| \| w_2 \| \| w \|}{\| \beta(Z) \|},
\]

that is,

\[
\left\| \left( \Gamma_{w_1} w_2 \right)_{(\beta, \frac{j_1 + j_2}{2})} \right\| \leq 2^{m-1} C_{\beta} \frac{\| n_{\beta} \|}{\| \beta(Z) \|} \| w_1 \| \| w_2 \|.
\]

From this inequality and (4.10), we obtain

\[
\| \Gamma_{w_1} w_2 \| \leq 2^m C_{\beta} \frac{\| n_{\beta} \|}{\| \beta(Z) \|} \| w_1 \| \| w_2 \|.
\]

Thus the statement of this proposition holds for \( m + 1 \). Therefore the statement of this proposition is true for all \( m \in \mathbb{Z} \). q.e.d.

By using Lemmas 4.12, 4.22, 4.24, 4.32, 4.34 and 4.35, we shall prove Theorem 4.3.

**Proof of Theorem 4.3.** Let \( i = (\beta, j) \in I \) and \( w \in (E_i)_{\hat{0}} \). We suffice to show that \( \Gamma_w \) is bounded in order to show that \( X^w \) is defined on the whole of \( V \). First we shall show that, in the case where \( j' \) is an integer with \( j' \neq j \) such that \( j' - j \) is divided by 4, there exists a positive constant \( \tilde{C}_{\beta} \) depending on only \( \beta \) such that

\[
\left\| \left( \Gamma_{w_1} w_2 \right)_{(\beta, \frac{j_1 + j_2}{2})} \right\| \leq 2^m \tilde{C}_{\beta} \frac{\| n_{\beta} \|}{\| \beta(Z) \|} \| w_1 \| \| w_2 \|.
\]

In similar to (4.12), we have

\[
\langle \Gamma_{w_1} w_2, \Gamma_{w_1} w_2 \rangle_{\pm} = -\langle \Gamma_{w_1} w_2, \Gamma_{w_1} w_2 \rangle_{\pm}.
\]
From this relation, Lemmas 4.12, (iii) of Proposition 4.34 and (4.14), we have

\[
\| (\Gamma w')_{(n')}^2 = (\Gamma w', u''')_{(n')} \leq \|w\| \| (\Gamma w')_{(n)} \| \leq 2 \text{Re} \left( \frac{n_i - n_{i'}}{n_{i'} - n_{i'}} \right) \cdot \text{Re} \left( \frac{1 + j r_{\beta}}{1 + j \beta i} \right) \cdot \text{Re} \left( \frac{1}{(1 + \frac{j r_{\beta}}{2}) (1 + j r_{\beta} i)} \right) \frac{1}{2} \cdot \left| \frac{n_{\beta}}{\beta(Z)} \|w\| \|w'\| \right|
\]

that is,

\[
\| (\Gamma w')_{(n')} \| \leq 2 \text{Re} \left( \frac{n_i - n_{i'}}{n_{i'} - n_{i'}} \right) \cdot \text{Re} \left( \frac{1 + j r_{\beta} i}{1 + j \beta i} \right) \cdot \text{Re} \left( \frac{1}{(1 + \frac{j r_{\beta}}{2}) (1 + j r_{\beta} i)} \right) \frac{1}{2} \cdot \left| \frac{n_{\beta}}{\beta(Z)} \|w\| \|w'\| \right|
\]

On the other hand, from (4.10), we have

\[
\text{Re} \left( \frac{n_i - n_{i'}}{n_{i'} - n_{i'}} \right) = \text{Re} \left( \frac{2 + (j + j') r_{\beta} i}{1 + j r_{\beta} i} \right)
\]

Hence we have

\[
\| (\Gamma w')_{(n')} \| \leq 2 \text{Re} \left( \frac{2 + (j + j') r_{\beta} i}{1 + j r_{\beta} i} \right) \cdot \text{Re} \left( \frac{1 + j r_{\beta} i}{1 + j \beta i} \right) \cdot \text{Re} \left( \frac{1}{(1 + \frac{j r_{\beta}}{2}) (1 + j r_{\beta} i)} \right) \frac{1}{2} \cdot \left| \frac{n_{\beta}}{\beta(Z)} \|w\| \|w'\| \right|
\]

Easily we can show

\[
\sup_{j, j' \in \mathbb{Z}} \left| \text{Re} \left( \frac{2 + (j + j') r_{\beta} i}{1 + j r_{\beta} i} \right) \cdot \text{Re} \left( \frac{1 + j r_{\beta} i}{1 + j \beta i} \right) \cdot \text{Re} \left( \frac{1}{(1 + \frac{j r_{\beta}}{2}) (1 + j r_{\beta} i)} \right) \frac{1}{2} < \infty
\]

Denote by \( \bar{C}_\beta \) this supremum. For this constant \( \bar{C}_\beta \), the inequality (4.13) holds. From this fact, Lemmas 4.22, 4.24, 4.32, 4.34 and 4.35, it follows that there exists a positive constant \( \bar{C}_\beta \) depending on only \( \beta \) such that

\[
\| \Gamma w' \| \leq \bar{C}_\beta \|w\| \|w'\|
\]

57
for any \( w' \in (E_0)^\perp_0 \). Assume that \( w' \in (E_0)_0 \). Then, since \( \Gamma_w w' \in (E_0)^\perp_0 \) by Lemma 4.9, we have we can find a sequence \( \{w''_k\} \) in \( \oplus (E_0)_0 \) with \( \lim_{k \to \infty} w''_k = \Gamma_w w' \) (with respect to \( || \cdot || \)). Then we have

\[
||\Gamma_w w'||^2 = \lim_{k \to \infty} \langle \Gamma_w w', w''_k \rangle = - \lim_{k \to \infty} \langle w', \Gamma_w w''_k \rangle \\
\leq \lim_{k \to \infty} ||w'|| ||\Gamma_w w''_k|| \leq C_\beta ||w'|| ||w'|| ||\Gamma_w w'||,
\]

that is,

\[
||\Gamma_w w'|| \leq \tilde{C}_\beta ||w'|| ||w'||,
\]

where \( \tilde{C}_\beta \) is as above. Thus \( \Gamma_w \) is bounded. Therefore, \( X^{\omega_0} \) is defined on the whole of \( V \).

q.e.d.

5 Proof of Theorem A

In this section, we shall prove Theorem A. Let \( M(\to G/K) \) be as in Theorem A and \( M^c \) the (complete extrinsic) complexification of \( M \). Set \( \tilde{M}^c := \pi^{-1}(M^c) \) and \( \tilde{M}^e := (\pi \circ \phi)^{-1}(M^c) \), where \( \phi \) is the parallel transport map for \( G^e \) and \( \pi \) is the natural projection of \( G^e \) onto \( G^c/K^c \). Without loss of generality, we may assume that \( K^c \) is connected and that \( G^c \) is simply connected. Hence both \( \tilde{M}^c \) and \( \tilde{M}^e \) are connected. Also, without loss of generality, we may assume \( \hat{0} \in \tilde{M}^e \) and hence \( e \in \tilde{M}^e \). In the previous section, set \( V := H^0([0, 1], g^c), \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle^A_0 \) and \( \langle \cdot, \cdot \rangle_\pm := \langle \cdot, \cdot \rangle^A_{0, H^0} \) for simplicity. Also, denote by \( || \cdot || \) the norm associated with \( \langle \cdot, \cdot \rangle_\pm \). U. Christ [Ch] proved that a full irreducible equifocal submanifold of codimension greater than one in a simply connected symmetric space of compact type is homogeneous in terms of the Heintze-Liu’s homogeneity theorem for full irreducible isoparametric submanifolds of codimension greater than one in a Hilbert space. By imitating his proof, we shall first prove the following homogeneity theorem in terms of Theorems 3.2 and 4.3.

**Theorem 5.1.** The submanifold \( M \) is homogeneous.

First we shall prepare some lemmas to prove this theorem. Denote by \( I_h(V) \) the group of all holomorphic isometries of \( V \), \( K^h \) the Lie algebra of all holomorphic Killing fields on \( V \) and \( K^h_{\tilde{M}^c} \) the Lie subalgebra of \( K^h \) consisting of elements of \( K^h \) which are tangent to \( \tilde{M}^c \) along \( \tilde{M}^e \). Also, denote by \( o_{AK}(V) \) the Lie algebra of all continuous skew-symmetric complex linear maps from \( V \) to oneself. Any \( X \in K^h \) is described as \( X_u = Au + b \) (\( u \in V \)) for some \( A \in o_{AK}(V) \) and some \( b \in V \). Hence \( K^h \) is identified with \( o_{AK}(V) \times V \). Give \( o_{AK}(V) \) the operator norm (which we denote by \( || \cdot ||_{op} \) associated with \( \langle \cdot, \cdot \rangle_\pm \) and \( K^h \).
the product norm of this norm \( \| \cdot \|_{\text{op}} \) of \( \mathfrak{o}_{AK}(V) \) and the norm \( \| \cdot \| \) of \( V \). The space \( \mathcal{K}^h \) is a Banach Lie algebra with respect to this norm. Denote by \( I^h_b(V) \) the Banach Lie group consisting of all holomorphic isometry \( f \)'s of \( V \) which admit a one-parameter transformation group \( \{ f_t \mid t \in \mathbb{R} \} \) of \( V \) such that each \( f_t \) is a holomorphic isometry of \( V \), \( f_1 = f \) and that \( \frac{d}{dt} \bigg|_{t=0} (f_t)_* \) is an element of \( \mathfrak{o}_{AK}(V) \). Note that, for a general holomorphic isometry \( f \) of \( V \), \( \frac{d}{dt} \bigg|_{t=0} (f_t)_* \) is not necessarily defined on the whole of \( V \) (but it can be defined on a dense linear subspace of \( V \)). It is clear that the Lie algebra of this Banach Lie group \( I^h_b(V) \) is equal to \( \mathcal{K}^h \).

\[ \mathcal{K}^h : \text{the space of all holomorphic Killing fields} \]
\[ \text{defined on dense linear subspaces of } V \]
\[ \exp : \text{the exponential map of } I_h(V) \]

**Figure 8.**

*Example.* We shall give an example of an element of \( I_h(V) \setminus I^h_b(V) \). Let \( V \) be a complex linear topological space consisting of all complex number sequences \( \{ z_k \}_{k=1}^{\infty} \)’s satisfying \( \sum_{k=1}^{\infty} |z_k|^2 < \infty \), and \( \langle \cdot, \cdot \rangle \) a non-degenerate inner product of \( V \) defined by

\[
\langle \{ z_k \}_{k=1}^{\infty}, \{ w_k \}_{k=1}^{\infty} \rangle := 2 \Re \left( \sum_{k=1}^{\infty} z_k w_k \right) \quad (\{ z_k \}_{k=1}^{\infty}, \{ w_k \}_{k=1}^{\infty} \in V).
\]
The pair \((V, \langle \cdot, \cdot \rangle)\) is an infinite dimensional anti-Kaehler space. Define a complex linear transformation \(A_t \ (t \in \mathbb{R})\) of \(V\) by assigning \(\{w_k\}_{k=1}^\infty\) defined by
\[
\begin{pmatrix}
    w_{2k-1} \\
    w_{2k}
\end{pmatrix} :=
\begin{pmatrix}
    \cos kt & -\sin kt \\
    \sin kt & \cos kt
\end{pmatrix}
\begin{pmatrix}
    z_{2k-1} \\
    z_{2k}
\end{pmatrix} \quad (k \in \mathbb{N})
\]
to each \(\{z_k\}_{k=1}^\infty \in V\). It is clear that each \(A_t\) is a holomorphic linear isometry of \(V\). Define \(f_t \in I_h(V)\) by \(f_t(u) := A_t u + b_t \ (u \in V)\), where \(b_t\) is an element of \(V\). Set
\[
B := \frac{d}{dt} \bigg|_{t=0} f_t = \frac{d}{dt} \bigg|_{t=0} A_t.
\]
It is easy to show that \(B\) is a skew-symmetric complex linear map from a dense linear subspace \(U\) of \(V\) to \(V\) assigning \(\{w_k\}_{k=1}^\infty\) defined by
\[
\begin{pmatrix}
    w_{2k-1} \\
    w_{2k}
\end{pmatrix} :=
\begin{pmatrix}
    0 & -k \\
    k & 0
\end{pmatrix}
\begin{pmatrix}
    z_{2k-1} \\
    z_{2k}
\end{pmatrix} \quad (k \in \mathbb{N}),
\]
to each \(\{z_k\}_{k=1}^\infty \in U\), where \(U\) is the set of all elements \(\{z_k\}_{k=1}^\infty\) of \(V\) satisfying \(B([z_k]) \in V\). Let \(\{a_k\}_{k=1}^\infty\) be an element of \(V\) defined by \(a_k := [\frac{1}{2k^2}] \ (k \in \mathbb{N})\), where \([\cdot]\) is the Gauss’s symbol of \(\cdot\). Then we can show \(B(\{a_k\}_{k=1}^\infty) \notin V\), that is, \(\{a_k\}_{k=1}^\infty \notin U\). Thus \(B\) is not an element of \(\mathfrak{o}_{AK}(V)\) and hence \(f_1\) is not an element of \(I_h(V)\).

Let \(H\) be the closed subgroup of \(I_h(V)\) of all elements of \(I_h(V)\) preserving \(\tilde{M}^c\) invariantly. According to Theorem 3.2 (and its proof) and Theorem 4.3, we have the following fact.

**Lemma 5.2.** We have \(H \cdot \hat{0} = \tilde{M}^c\).

**Proof.** Let \(\gamma : [0, 1] \to L_{\hat{0}}^{E_i}\) be the geodesic in \(L_{\hat{0}}^{E_i}\) and set \(w_0 := \gamma'(0)\). Also, let \(F_{\gamma}\) be the holomorphic isometry satisfying \(F_{\gamma}(\gamma(0)) = \gamma(1)\) and the relation (3.2). In more general, let \(F_{\gamma|_{[0,t]}} \ (t \in \mathbb{R})\) be the holomorphic isometry satisfying \(F_{\gamma|_{[0,t]}(\gamma(0)) = \gamma(t)}\) defined in similar to \(F_{\gamma}\). For simplicity, set \(F_{\gamma|_{[0,1]}} := F_{\gamma|_{[0,t]}}\). Let \(X_{w_0}\) be the holomorphic Killing field associated with the one-parameter transformation group \(\{F_{t_{w_0}}\}_{t \in \mathbb{R}}\), that is, \(X_{w_0}|_u := \frac{d}{dt} \bigg|_{t=0} F_{w_0} (u)\), where \(u\) moves over the set of all elements \(u\)’s where the right-hand side exists. According to Theorem 4.3, we have \(X_{w_0}\) is defined over the whole of \(V\), that is, \(X_{w_0} \in \mathfrak{K}^h\). Therefore we obtain \(F_{\gamma} \in I_h^h(V)\). On the other hand, \(F_{\gamma}\) preserves \(\tilde{M}^c\) invariantly. Hence we have \(F_{\gamma} \in H\). Since the holomorphic isometries \(f_k\’s\) in the proof of Theorem 3.2 are given as the composition of the holomorphic isometries of \(F_{\gamma}\)-type, it is then shown that \(f_k\’s\) are elements of \(H\) and hence the holomorphic isometry \(\tilde{f}\) in Step IV of the proof of Theorem 3.2 also is an element of \(H\) (see the construction of \(\tilde{f}\) in Step IV). Therefore we obtain \(H \cdot \hat{0} = \tilde{M}^c\). q.e.d.
In this section, the notation \( \rho \) denotes the homomorphism from \( H^1([0,1], G^e) \) to \( I_h(V) \) defined by assigning \( g \ast \cdot \) to each \( g \in H^1([0,1], G^e) \) (i.e., \( \rho(g)(u) := g \ast u \) \( g \in H^1([0,1], G^e), \ u \in V \)), where \( g \ast u \) is as stated in Section 2.

**Lemma 5.3.** The group \( \rho(H^1([0,1], G^e)) \) is a closed subgroup of \( I_h^b(V) \).

*Proof.* Take an arbitrary \( v \in H^1([0,1], g^e) \) and set \( \psi_s := \rho(\exp \circ sv) \), where \( \exp \) is the exponential map of \( G^e \). Note that \( \exp \circ sv \) is equal to the image of \( sv \in H^1([0,1], g^e) \) by the exponential map of \( H^1([0,1], G^e) \). The group \( \{ \psi_s \mid s \in \mathbb{R} \} \) is a one-parameter transformation group of \( V \). The holomorphic Killing field \( X \) associated with \( \{ \psi_s \mid s \in \mathbb{R} \} \) is given by

\[
X_u = \frac{d}{ds}\bigg|_{s=0} \psi_s(u) = \frac{d}{ds}\bigg|_{s=0} (\exp \circ sv) \ast u = \text{ad}(v)(u) - v'.
\]

Set \( I_c := \{ t \in [0,1] \mid \text{max Spec}(\text{ad}(v(t))^2) \geq c \} \) and \( c_0 := \min \{ c \mid I_c \text{ is of measure zero in } [0,1] \} \), where \( \text{ad} \) is the adjoint operator of \( g^e \). Then we have

\[
|\text{ad}(v)u|^2 = \int_0^1 \langle \text{ad}(v(t))u(t), \text{ad}(v(t))u(t) \rangle_{g^e}^A dt = -\int_0^1 \langle \text{ad}(v(t))^2 u(t), u(t) \rangle_{g^e}^A dt \leq c_0||u||^2,
\]

where \( \langle \ , \ \rangle_{g^e}^A \) is the inner product of \( g^e \) stated in Section 2. Thus \( \text{ad}(v) \) is bounded. Hence we have \( X \in K^h \), that is, \( \rho(\exp \circ v) \in I_h^b(V) \). Therefore, it follows from the arbitrariness of \( v \) that \( \rho(H^1([0,1], G^e)) \) is a subgroup of \( I_h^b(V) \). The closedness of \( \rho(H^1([0,1], G^e)) \) is trivial. \( \Box \)

In the proof of Theorem 5.1, it is key to show the following fact.

**Proposition 5.4.** The above group \( H \) is a subgroup of \( \rho(H^1([0,1], G^e)) \).

To prove this proposition, we prepare some lemmas. For \( X \in K^h \), we define a map \( F_X : \Omega_e(G^e) \to g^e \) by \( F_X(g) := \phi_{\hat{g}}((\rho(g), X)_{\hat{0}}) \). For simplicity, denote by \( \text{Ad} \) the adjoint operator \( \text{Ad}_{G^e} \) of \( G^e \). For this map \( F_X \), we have the following fact.

**Lemma 5.4.1.** (i) For \( g \in \Omega_e(G^e) \), \( F_X(g) = \int_0^1 \text{Ad}(g)(X_{\rho(g^{-1})(\hat{0})})dt \).

(ii) If \( X \in K^h_{\hat{M}^e} \), then the image of \( F_X \) is included by \( T_{e\hat{M}^e} \).

*Proof.* Let \( \{ \psi_s \}_{s \in \mathbb{R}} \) be the one-parameter transformation group associated with \( X \). For
each \( g \in \Omega_e(G^e) \), we have

\[
(r(g)_s X)_0 = \frac{d}{ds}\bigg|_{s=0} \rho(g)(\psi_s(g^{-1} \ast 0)) = \frac{d}{ds}\bigg|_{s=0} (\text{Ad}(g)(\psi_s(\rho(g^{-1})(\hat{0})))) - g'g^{-1}_s = \text{Ad}(g)(X_{\rho(g^{-1})(\hat{0}))}.
\]

Also we have \( \phi_{s\hat{0}}(u) = \int_0^1 u(t)dt \ (u \in T_{\hat{0}}V(= V)) \) (see Lemma 6 of [Koi3]). Hence we obtain the relation in (i). Since \( g \in \Omega_e(G^e) \), it maps each fibre of \( \phi \) to oneself. Hence, if \( X \in K^h_{M^e} \), then \( \rho(g)_s X \in K^h_{M^e} \). In particular, we have \( (\rho(g)_s X)_0 \in T_{\hat{0}}M^e \). Therefore we obtain \( F_X(g) = \phi_{s\hat{0}}(T_{\hat{0}}\tilde{M}_e) = T_{\hat{0}}\tilde{M}_e \).

q.e.d.

For \( v \in H^1([0, 1], g^e) \), we define a vector field \( X^v \) on \( V \) by \( (X^v)_u := [v, u] - v' \ (u \in V) \). Let \( \{ \exp s\rho v \mid s \in \mathbb{R} \} \) be the one-parameter subgroup of \( H^1([0, 1], G^e) \) associated with \( v \). Then the holomorphic Killing field associated with the one-parameter transformation \( \{ \rho(\exp s\rho v) \mid s \in \mathbb{R} \} \) of \( V \) is equal to \( X^v \). Furthermore, we can show \( X^v \in K^h_e \) by the discussion in the proof of Lemma 5.3. For \( X^v \), we have the following fact.

**Lemma 5.4.2.** The map \( F_X^v \) is a constant map.

**Proof.** Take elements \( g_1 \) and \( g_2 \) of \( \Omega_e(G^e) \). Since \( \rho(g_i) \) maps each fibre of \( \phi \) to oneself by the fact (iii) for \( \phi \) stated in Section 2, we have \( \phi \circ \rho(g_i) = \phi \ (i = 1, 2) \) and hence

\[
F_X^v(g_i) = \phi_{s\hat{0}}((\rho(g_i)_s(X^v))_0) = \phi_{s\rho(g_i^{-1})(\hat{0})}((X^v)_{\rho(g_i^{-1})(\hat{0})}) \quad (i = 1, 2).
\]

Since \( \rho(\exp s\rho v) \) maps the fibres of \( \phi \) to them by the fact (iii) for \( \phi \) stated in Introduction and \( \phi(\rho(g_1^{-1})(\hat{0})) = \phi(\rho(g_2^{-1})(\hat{0})) \), we have \( \phi(\rho(\exp s\rho v)(\rho(g_1^{-1})(\hat{0}))) = \phi(\rho(\exp s\rho v)(\rho(g_2^{-1})(\hat{0})))) \) and hence \( \phi_{\rho(g_1^{-1})(\hat{0})}((X^v)_{\rho(g_2^{-1})(\hat{0})}) = \phi_{\rho(g_2^{-1})(\hat{0})}((X^v)_{\rho(g_2^{-1})(\hat{0})}) \). From this relation and (5.1), we obtain \( F_X^v(g_1) = F_X^v(g_2) \). Therefore it follows from the arbitrariness of \( g_1 \) and \( g_2 \) that \( F_X^v \) is a constant map.

q.e.d.

For each \( u \in V \), denote by \( \tilde{u} \) the element \( t \mapsto \int_0^t u(t)dt \ (0 \leq t \leq 1) \) of \( H^1([0, 1], g^e) \). Also we have the following fact for \( F_X \).

**Lemma 5.4.3.** (i) The map \( X \mapsto F_X \) is linear.

(ii) \( F_X(g_1g_2) = F_{\rho(g_2)}X(g_1) \quad (g_1, g_2 \in \Omega_e(G^e)) \).

(iii) \( (dF_X)_g \circ (dR_g)_\hat{e} = (dF_{\rho(g)})_\hat{e} \quad (g \in \Omega_e(G^e)) \).

(iv) If \( X_u = Au + b \ (u \in V) \) for some linear transformation \( A \) of \( V \) and some \( b \in V \), then we have \( (dF_X)_e(u) = \int_0^1 (A + \text{ad}(\hat{b}))(u')dt \ (u \in \Omega(g^e)) \), where \( \text{ad} \) is the adjoint representation of \( g^e \) and \( \Omega(g^e) := \{ u \in H^1([0, 1], g^e) \mid u(0) = u(1) = 0 \} \).

(v) If \( X, X \in K^h \) and if \( \overline{X} = X = X^v \) for some \( v \in H^1([0, 1], g^e) \), then \( F_X - F_X \) is a constant map.
Proof. The statements (i) ∼ (iii) are trivial. The statement (iv) is shown by imitating the proof of Proposition 2.3 of [Ch]. The statement (v) follows from Lemma 5.4.2 and (i) directly. 

q.e.d.

By imitating the proof of Theorem 2.2 of [Ch], we can show the following fact in terms of Lemmas 5.4.1 ∼ 5.4.3.

**Lemma 5.4.4.** Let $X$ be an element of $\mathcal{K}^h$ given by $X_u := [v,u] - b$ ($u \in V$) for some $v, b \in V$. If $X \in \mathcal{K}^h_{\tilde{M}^c}$, then we have $v \in H^1([0,1], g^c)$ and $b = v'$ (i.e., $X = X^v$).

**Proof.** Set $\overline{X} := X - X^b$ and $w := v - \tilde{b}$. First we consider the case where $G^c$ is simple. From $\overline{X} = \text{ad}(w)$, we have

$$
(\rho(g)_{\overline{X}})_u = \rho(g)_{\overline{X}}(\rho(g^{-1})(u)) = \text{Ad}(g)([w, \rho(g^{-1})(u)]) = [\text{Ad}(g)w, u - g \hat{0}] \ (u \in V).
$$

From this relation and (i) of Lemma 5.4.1, we have

$$
(dF_{\rho(g)_{\overline{X}}})_{\hat{e}}(u) = \left. \frac{d}{ds} \right|_{s=0} F_{\rho(g)_{\overline{X}}} (\exp su)
$$

$$
= \frac{d}{ds} \left|_{s=0} \int_0^1 \text{Ad}(\exp su)(\rho(g)_{\overline{X}})_{\rho(\exp(-su))(\hat{0})} dt
$$

$$
= \int_0^1 \left( [u, (\rho(g)_{\overline{X}})_{\hat{0}}] + \left. \frac{d}{ds} \right|_{s=0} (\rho(g)_{\overline{X}})_{\rho(\exp(-su))(\hat{0})} \right) dt
$$

$$
= \int_0^1 \left( [u, [\text{Ad}(g)w, g \hat{0}]) + \left. \frac{d}{ds} \right|_{s=0} [\text{Ad}(g)w, \rho(\exp(-su))(\hat{0}) - g \hat{0}] \right) dt
$$

(5.2)

$$
= \int_0^1 \left( [u, [\text{Ad}(g)w, g'g_{u-1}^{-1}]] - [\text{Ad}(g)w, \left. \frac{d}{ds} \right|_{s=0} ((\exp(-su))' \exp(-su_{u-1}^{-1})) \right) dt
$$

$$
= \int_0^1 \left( [u, [\text{Ad}(g)w, g'g_{u-1}^{-1}]] + [\text{Ad}(g)w, u']) \right) dt
$$

$$
= [u(t), \text{Ad}(g)w, g'g_{u-1}^{-1}(t)]_{t=0} - [u(t), \text{Ad}(g)w, g'g_{u-1}^{-1}(t)]_{t=0}
$$

$$
- \int_0^1 [u', [\text{Ad}(g)w, g'g_{u-1}^{-1}]] dt + \int_0^1 [\text{Ad}(g)w, u'] dt
$$

$$
= \int_0^1 [\text{Ad}(g)w, g'g_{u-1}^{-1}] + \text{Ad}(g)w, u' dt
$$

for $u \in T_{\hat{e}} (\Omega_{\hat{e}}(G^c)) = \Omega_0(g^c)$, where each of the notation $'$ means the derivative with respect to $t$, $\hat{e}$ is the constant path at the identity element $e$ of $G^c$ and $\Omega_0(g^c) := \{ u \in H^1([0,1], g^c) \ | \ u(0) = u(1) = 0 \}$. According to (ii) of Lemma 5.4.1, we have $\text{Im} F_X \subset T_{\hat{e}} \tilde{M}^c$ and hence $\dim_{\mathbb{C}} (\text{Span}_\mathbb{C} \text{Im} F_X) \leq \dim_{\mathbb{C}} T_{\hat{e}} \tilde{M}^c \leq \dim_{\mathbb{C}} g^c - 2$, where $\text{Span}_\mathbb{C}(\cdot)$ means the complex linear span of $(\cdot)$ and $\dim_{\mathbb{C}}(\cdot)$ means the complex dimension of $(\cdot)$. Since $F_X - F_X$
is a constant map by (v) of Lemma 5.4.3, we have $\dim C(\text{Span}_C \text{Im} \frac{F_X}{g}) \leq \dim C g - 1$, that is, $\dim C(g^c \oplus \text{Span}_C \text{Im} \frac{F_X}{g}) \geq 1$. Take $Y (\neq 0) \in g^c \oplus \text{Span}_C \text{Im} \frac{F_X}{g}$. Also, take $g \in \Omega_\varepsilon(G^c)$ and $u \in T_\varepsilon(\Omega_\varepsilon(G^c))$. By using (iii) of Lemma 5.4.3 and (5.2), we have

$$
\langle (dF_X)_g((dR_g)_\varepsilon(u)), Y \rangle^A = \langle (dF_{\rho(g)}_X)_\varepsilon(u), Y \rangle^A
$$

$$
= \int_0^1 \langle [\text{Ad}(g)w, g'g_{-1}] + \text{Ad}(g)w, u' \rangle dt,
$$

$$
= \int_0^1 \langle [\text{Ad}(g)w, g'g_{-1}] + \text{Ad}(g)w, u' \rangle dt
$$

$$
= - \int_0^1 \langle u', [\text{Ad}(g)w, g'g_{-1}] + \text{Ad}(g)w, Y \rangle dt
$$

$$
= - \langle u', [\text{Ad}(g)w, g'g_{-1}] + \text{Ad}(g)w, Y \rangle,
$$

where $\langle \cdot, \cdot \rangle^A$ is the non-degenerate symmetric bilinear form of $g^c$ stated in Section 2. For simplicity, we set $\eta := [\text{Ad}(g)w, g'g_{-1}] + \text{Ad}(g)w$. On the other hand, from $(dF_X)_g((dR_g)_\varepsilon(u)) \in \text{Span}_C \text{Im} \frac{F_X}{g}$, we have $\langle (dF_X)_g((dR_g)_\varepsilon(u)), Y \rangle^A = 0$. Hence we have $\langle u', [\eta, Y] \rangle = 0$. The space $\Omega_0(g^c)$ is identified with the vertical space (which is denoted by $V_0$ at 0 of $\phi$ under the correspondence $u \mapsto u'$ ($u \in \Omega_0(g^c)$), where we note that $\phi_{_{\mathcal{A}}}(u') = \int_0^1 u'(t) dt = 0$ by Lemma 6 of [Koi3] (hence $u' \in V_0$). Hence, from the arbitrariness of $u$, it follows that $[\eta, Y]$ belongs to the horizontal space (which is denoted by $\mathcal{H}_0$ at 0 of $\phi$). Since $G^c$ has no center, there exists $Z \in g^c$ with $[Y, Z] \neq 0$. Set $W := [Y, Z]$. By using Lemma 6 of [Koi3], we can show that $\mathcal{H}_0$ is equal to the set of all constant paths in $g^c$. Hence it follows from $[\eta, Y] \in \mathcal{H}_0$ that $[\eta, Y]$ is a constant path. Furthermore it follows from $\langle \eta, W \rangle^A = \langle [\eta, Y], [\eta, Y] \rangle$ that $\langle \eta, W \rangle^A$ is constant, that is,

$$
\langle [\text{Ad}(g)w, g'g_{-1}], W \rangle^A + \langle \text{Ad}(g)w, W \rangle^A = \text{const.}
$$

Since $g^c$ has no center, there exists $W \in g^c$ with $[W, W] \neq 0$. Since $G^c$ is simple, $\text{Ad}(G^c)[W, W]$ is full in $g^c$. Hence there exist $h_1, \ldots, h_{2m} \in G^c$ such that $\{\text{Ad}(h_i)[W, W], \ldots, \text{Ad}(h_{2m})[W, W]\}$ is a base of $g^c$ (regarded as a real vector space), where $m := \dim g^c$. For a sufficiently small $\varepsilon > 0$, we take $g_i \in \Omega_\varepsilon(G^c)$ with $g_i|_{[\varepsilon, 1-\varepsilon]} = h_i$ ($i = 1, \ldots, 2m$). Since $g_i$ ($i = 1, \ldots, 2m$) are constant over $[\varepsilon, 1-\varepsilon]$, it follows form (5.3) ($g = g_i$-case) that $\langle w, \text{Ad}(h_i^{-1})W \rangle^A$ ($i = 1, \ldots, 2m$) are constant over $[\varepsilon, 1-\varepsilon]$. Hence $w$ is constant over $[\varepsilon, 1-\varepsilon]$. Hence it follows from the arbitrariness of $\varepsilon$ that $w$ is constant over $[0, 1]$. That is, we obtain $b = v'$ and hence $v \in H^1([0, 1], g^c)$.

Next we consider the case where $G^c$ is not simple. Let $G^c = G_1^c \times \cdots \times G_k^c$ be the irreducible decomposition of $G^c$ and $g_i^c$ be the Lie algebra of $G_i^c$ ($i = 1, \ldots, k$). Let $g_X^c$ be the maximal ideal of $g^c$ such that the orthogonal projection of $w = v - \bar{b}$ onto the ideal is a constant path, where we note that any ideal of $g^c$ is equal to the direct sum of some
$g_i^c$'s and hence it is a non-degenerate subspace with respect to $\langle \ , \ \rangle^A$. Now we shall show

$$(5.4) \quad (g^c_X)^\perp \subset T_c\tilde{M}^c,$$

where $(g^c_X)^\perp$ is the orthogonal complement of $g^c_X$ in $g^c$ with respect to $\langle \ , \ \rangle^A$. Let $V_i := H^0([0,1], g^c_i) \ (i = 1, \cdots, k)$. It is clear that $V = V_1 \oplus \cdots \oplus V_k$ (orthogonal direct sum).

The holomorphic Killing field $\overline{X}$ is described as $\overline{X} = \overline{X}_1 + \cdots + \overline{X}_k$ in terms of some holomorphic Killing field $\overline{X}_i$ on $V_i \ (i = 1, \cdots, k)$, where $\overline{X}_i^L$ is the holomorphic Killing field on $V$ defined by $(\overline{X}_i^L)_u = (\overline{X}_i)_u \ (u = (u_1, \cdots, u_k) \in V)$. For $g = (g_1, \cdots, g_k) \in \Omega_c(G^c) (= \Omega_c(G^c_1) \times \cdots \times \Omega_c(G^c_k))$, we have $\text{Ad}(g)(\overline{X}_{\rho(g^{-1})(\bar{0})}) = \sum_{i=1}^k \text{Ad}_i(g_i)(\overline{X}_i)_{\rho_i(g_i^{-1})(\bar{0})}$, where $\text{Ad}_i$ is the adjoint representation of $G_i^c$ and $\rho_i$ is the homomorphism from $H^1([0,1], G_i^c)$ to $I_{h_i}(V_i)$ defined in similar to $\rho$. Hence, from (i) of Lemma 5.4.1, we have $F^c_{\overline{X}}(g) = \sum_{i=1}^k F^c_{\overline{X}_i}(g_i)$, where $F^c_{\overline{X}_i}$ is the map from $\Omega_c(G^c_i)$ to $g^c_i$ defined in similar to $F^c_{\overline{X}}$. Therefore we obtain $\text{Span}_c \text{Im} F^c_{\overline{X}} = \bigoplus_{i=1}^k \text{Span}_c \text{Im} F^c_{\overline{X}_i}$. Let $v = \sum_{i=1}^k v_i$ and $\tilde{b} = \sum_{i=1}^k \tilde{b}_i$, where $v_i, \tilde{b}_i \in V_i \ (i = 1, \cdots, k)$. Since $\overline{g}^c_X$ is an ideal of $g^c$, it is described as $\overline{g}^c_X = \bigoplus_{i \in I} g^c_i \ (I \subset \{1, \cdots, k\})$.

Since $v_i - \tilde{b}_i \ (i \in I)$ are constant paths by the definition of $\overline{g}^c_X$, $\text{Ad}_i(g_i)[v_i - \tilde{b}_i, \rho(g_i^{-1})(\bar{0})]$ $(i \in I)$ are loops and hence

$$F^c_{\overline{X}_i}(g_i) = \int_0^1 \text{Ad}_i(g_i)[v_i - \tilde{b}_i, \rho(g_i^{-1})(\bar{0})]dt = 0 \quad (i \in I).$$

Hence we have

$$\text{Span}_c \text{Im} F^c_{\overline{X}} \subset (\overline{g}^c_X)^\perp (= \bigoplus_{i \in I} g^c_i).$$

Also we can show $\text{Span}_c \text{Im} F^c_{\overline{X}_i} = g^c_i \ (i \notin I)$. Therefore we obtain

$$(5.5) \quad \text{Span}_c \text{Im} F^c_{\overline{X}} = (\overline{g}^c_X)^\perp.$$

Also, since $F^c_{\overline{X}} - F_X$ is a constant map by (v) of Lemma 4.2.3 and $0 \in \text{Im} F^c_{\overline{X}}$, we have

$$(5.6) \quad \text{Span}_c \text{Im} F^c_{\overline{X}} \subset \text{Span}_c \text{Im} F_X.$$

From (5.5), (5.6) and (ii) of Lemma 5.4.1, we obtain $(\overline{g}^c_X)^\perp \subset T_c\tilde{M}^c$. Next we shall show that $(R_g)_* ((\overline{g}^c_X)^\perp) \subset T_g\tilde{M}^c$ for any $g \in \tilde{M}^c$. Fix $g \in \tilde{M}^c$. Define $\tilde{g} \in H^1([0,1], G^c)$ with $\tilde{g}(0) = e$ and $\tilde{g}(1) = g$ by $\tilde{g}(t) := \exp tY$ for some $Y \in g^c$. Since $\phi \circ \rho(\tilde{g}) = R_g^{-1} \circ \phi$, we
have $\phi^{-1}(R_g^{-1}(\hat{M}^c)) = \rho(\bar{g})(\hat{M}^c)$. Also we have $\rho(\bar{g})_sX \in K^h_{\rho(\bar{g})(\hat{M}^c)}$. Hence, by imitating the above discussion, we can show

\[ (5.7) \quad (g_{\rho(\bar{g})sX}^c)_\perp \subset T_c R_g^{-1}(\hat{M}^c) = (R_g)_s^{-1}(T_g \hat{M}^c). \]

Also, we have

\[ (5.8) \quad (\rho(\bar{g})_sX)_u = \rho(\bar{g})_s(X_{\rho(\bar{g})^{-1}(u)}) = [\text{Ad}(\bar{g})v, u] - [\text{Ad}(\bar{g})v, \rho(\bar{g})(\bar{0})] - \text{Ad}(\bar{g})b. \]

Set $\varpi := \text{Ad}(\bar{g})v$ and $\bar{b} := [\text{Ad}(\bar{g})v, \rho(\bar{g})(\bar{0})] + \text{Ad}(\bar{g})b$. Denote by $\text{pr}_{g_X^c}$ the orthogonal projection of $g_X^c$ onto $g_X^c$. Since $\text{Ad}(\bar{g})$ preserves each $g_X^c$ and $(g_X^c)_\perp$ invariantly, respectively. Hence we have $\text{pr}_{g_X^c} \circ \text{Ad}(\bar{g}) = \text{Ad}(\bar{g}) \circ \text{pr}_{g_X^c}$ and $\text{pr}_{g_X^c} \circ \text{Ad}(Y) = \text{Ad}(Y) \circ \text{pr}_{g_X^c}$. Also, we have $\rho(\bar{g})(\bar{0}) = -Y = -\text{Ad}(\bar{g})Y$. By using these facts and noticing that $\text{pr}_{g_X^c}(v - \bar{b})$ is a constant path, we have

\[
\begin{align*}
\frac{d}{dt} \text{pr}_{g_X^c}(\varpi - \bar{b}) &= \frac{d}{dt} \text{pr}_{g_X^c} \left( \text{Ad}(\bar{g})v + \text{Ad}(\bar{g})b - [\text{Ad}(\bar{g})v, \rho(\bar{g})(\bar{0})] - \text{Ad}(\bar{g})b \right) \\
&= \text{Ad}(\bar{g})[Y, \text{pr}_{g_X^c}(v - \bar{b})] + \text{Ad}(\bar{g})[Y, \text{pr}_{g_X^c}(\bar{b})] \\
&\quad + \text{Ad}(\bar{g})[\text{Ad}(\bar{g})v, Y] - \text{Ad}(\bar{g})\text{pr}_{g_X^c}(b) \\
&= (\text{pr}_{g_X^c} \circ \text{Ad}(\bar{g}))(\{Y, v - \bar{b}\} + [Y, \bar{b}] + [v, Y]) = 0.
\end{align*}
\]

Thus $\text{pr}_{g_X^c}(\varpi - \bar{b})$ is a constant path. This fact together with (5.8) implies $g_X^c \subset g_{\rho(\bar{g})sX}^c$. By exchanging the roles of $X$ and $\rho(\bar{g})_sX$, we have $g_X^c \subset g_{\rho(\bar{g})sX}^c$. Thus we obtain $g_X^c = g_{\rho(\bar{g})sX}^c$. Therefore the relation $(R_g)_s(h_{\rho(\bar{g})sX}) \subset T_g \hat{M}^c$ follows from (5.7). Since this relation holds for any $g \in \hat{M}^c$ and $g_X^c$ is an ideal of $g_X^c$, we have $\hat{M}^c = \hat{M}^c \times G_X^{c_\perp} \subset G_X^c \times G_X^{c_\perp} (= G_X^c)$ for some submanifold $\hat{M}^c$ in $G_X^c$, where $G_X^c := \exp(g_X^c)$ and $G_X^{c_\perp} := \exp((g_X^c)_\perp)$. Since $\hat{M}^c$ is irreducible and $\dim \hat{M}^c < \dim G_X^c$, we have $(g_X^c)_\perp = \{0\}$, that is, $g_X^c = G_X^c$. This implies that $v - \bar{b}$ is a constant path. Therefore we obtain $b = v'$ and hence $v \in H^1([0, 1], g_X^c)$. q.e.d.

Also we have the following fact.

**Lemma 5.4.5.** The set $K^h_{\hat{M}^c}$ is closed in $K^h$.

**Proof.** Denote by $\overline{K^h_{\hat{M}^c}}$ the closure of $K^h_{\hat{M}^c}$ in $K^h$. Take $X \in \overline{K^h_{\hat{M}^c}}$. Then there exists a sequence $\{X_n\}_{n=1}^\infty$ in $K^h_{\hat{M}^c}$ with $\lim_{n \to \infty} X_n = X$ (in $K^h$). Let $(X_n)_u = A_n u + b_n$ ($A_n \in$
for each \( b \in V \).

From \( \lim_{n \to \infty} X_n = X \) (in \( K^h \)), we have \( \lim_{n \to \infty} A_n = A \) (in \( \mathfrak{o}_{AK}(V) \)) and hence \( \lim_{n \to \infty} A_n u = A u \) (\( u \in V \)). Also, we have \( \lim_{n \to \infty} b_n = b \). Hence we have \( \lim_{n \to \infty} (X_n) u = X u \) (\( u \in V \)). For each \( u \in \tilde{M}^e \), denote by \( \text{pr}^u \) the orthogonal projection of \( V \) onto \( T_u \tilde{M}^e \). Since \( \dim T_u \tilde{M}^e < \infty \), \( \text{pr}^u \) is a compact operator. Hence, since \( \text{pr}^u(X_n) u = 0 \) for all \( n \), we obtain \( \text{pr}^u(X_u) = 0 \) and hence \( X \in K^h_{\tilde{M}^e} \).

Therefore we obtain \( K^h_{\tilde{M}^e} = K^h_{\tilde{M}^e} \).

Take \( v \in V \) and \( X \in K^h \). Define \( \tilde{v} \in H^1([0,1],\mathfrak{g}^e) \) by \( \tilde{v}(t) := \int_0^t v(t) dt \). Also, define \( g_n \in H^1([0,1],\mathfrak{g}^e) \) (\( n \in \mathbb{N} \)) by \( g_n(t) := \exp(n\tilde{v}(t)) \) and a vector field \( X_n^v \) (\( n \in \mathbb{N} \)) by \( X_n^v := \frac{1}{n}\rho(g_n),X \). We can show that \( X_n^v \) is defined on the whole of \( V \) (i.e., \( X_n^v \in K^h \)) as follows.

Let \( X_u = Au + b \) (\( A \in \mathfrak{o}_{AK}(V) \), \( b \in V \)), where \( u \in V \), and \( (X_n^v) u = A_n^v u + b_n^v \) (\( A_n^v \) : a skew-symmetric complex linear map from the domain of \( X_n^v \) to \( V \), \( b_n^v \in V \)), where \( u \) is an arbitrary point of the domain of \( X_n^v \). Then we have

\[
(X_n^v) u = \frac{1}{n} \text{Ad}(g_n)(X_{\rho(g^{-1})}(u)) = \frac{1}{n} \text{Ad}(g_n)(A\rho(g^{-1}))(u) + b
\]

and hence

\[
A_n^v = \frac{1}{n} \text{Ad}(g_n) A \circ \text{Ad}(g^{-1}) \quad \text{and} \quad b_n^v = \frac{1}{n} \text{Ad}(g_n)A \rho(g^{-1})(0) + b.
\]

From the first relation in (5.9), we have \( A_n^v \in \mathfrak{o}_{AK}(V) \) and hence \( X_n^v \in K^h \).

For \( \{X_n^v\}_{n=1}^\infty \), we have the following fact.

**Lemma 5.4.6.** If \( X \in K^h_{\tilde{M}^e} \) and \( v \) is an element of \( H^0_{0,e} \) with \( \int_0^1 v(t) dt = 0 \), then there exists a subsequence of \( \{X_n^v\}_{n=1}^\infty \) converging to the zero vector field.

**Proof.** Take \( u \in V \). Let \( u = u_- + u_+ \) (\( u_- \in H^0_{0,e}, u_+ \in H^0_{0,e} \)). Then we have

\[
(\text{Ad}(g_n)u_\varepsilon)(t) = \text{Ad}(\exp(n\tilde{v}(t)))u_\varepsilon(t) = \text{exp}(\text{ad}(n\tilde{v}(t)))u_\varepsilon(t) \in \mathfrak{g}^e_\varepsilon \quad (\varepsilon = - \text{ or } +)
\]

for each \( t \in [0,1] \) because \( \tilde{v}(t) \in \mathfrak{g}^e_\varepsilon \) (\( 0 \leq t \leq 1 \)) by the assumption and \( [\mathfrak{g}^e_\varepsilon, \mathfrak{g}^e_\varepsilon] \subseteq \mathfrak{g}^e_\varepsilon \) (\( \varepsilon = - \text{ or } + \)). Hence we have

\[
(\text{Ad}(g_n)u, \text{Ad}(g_n)u)_\varepsilon = -(\text{Ad}(g_n)u_- \text{Ad}(g_n)u_-) + (\text{Ad}(g_n)u_+ \text{Ad}(g_n)u_+)
\]

\[
= -(u_- u_-) + (u_+ u_+) = (u, u)_\varepsilon.
\]

(5.10)
Therefore, by using (5.9), we can show \[\|A_n^v\|_{op} = \frac{1}{n}\|A\|_{op} \to 0 \ (n \to \infty)\) and
\[
\|b^v_n\| \leq \frac{1}{n} (\|A\rho(g_n^{-1})(\hat{0})\| + \|b\|) = \|Av\| + \frac{1}{n}\|b\| \to \|Av\| \ (n \to \infty).
\]
Since the sequence \(\{X_n^v\}_{n \in \mathbb{N}}\) in \(\mathcal{K}^h\) is bounded, there exists its convergent subsequence \(\{X_{n_j}^v\}_{j=1}^{\infty}\). Set \(X_{\infty}^v := \lim_{j \to \infty} X_{n_j}^v\). From \(\lim_{n \to \infty} A_n^v = 0\), \(X_{\infty}^v\) is a parallel Killing field on \(V\). From \(\int_0^1 v(t)dt = 0\), we have \(g_n \in \Omega_e(G^c)\) and hence \(\rho(g_n)(\hat{M}^c) = \hat{M}^c\). This fact together with \(X \in \mathcal{K}^h_{\hat{M}^c}\) deduces \(X_{\infty}^v \in \mathcal{K}^h_{\hat{M}^c}\). Also, from \(\|A_n^v\|_{op} = \frac{1}{n}\|A\|_{op} < \infty\), we have \(X_{\infty}^v \in \mathcal{K}^h\). Hence we have \(X_{\infty}^v \in \mathcal{K}^h_{\hat{M}^c}\). Therefore we have \(X_{\infty}^v \in \mathcal{K}^h_{\hat{M}^c}\). Furthermore, from Lemma 5.4.5, we have \(X_{\infty}^v \in \mathcal{K}^h_{\hat{M}^c}\). Thus, since \(X_{\infty}^v\) is parallel and \(X_{\infty}^v \in \mathcal{K}^h_{\hat{M}^c}\), it follows from Lemma 5.4.4 that \(X_{\infty}^v = 0\). This completes the proof.
q.e.d.

On the other hand, we have the following fact.

**Lemma 5.4.7.** Let \(X\) be an element of \(\mathcal{K}^h_{\hat{M}^c}\) given by \(X_u = Au + b (u \in V)\) for some \(A \in \mathfrak{o}_{\mathcal{K}}(V)\) and some \(b \in V\), \(Y\) an element of \(H^0([0,1],\mathcal{C})(= H^0([0,1],\mathbb{R}^2))\) satisfying \(\int_0^1 f(t)dt = 0\) or \(f = \text{const}\). Then we have \(A(fY) = [Y, w]\) for some \(w \in V\).

**Proof.** Set \(v := fY\). Define \(\tilde{f} \in H^1([0,1],\mathcal{C})\) by \(\tilde{f}(t) := \int_0^t f(t)dt \ (0 \leq t \leq 1)\). Let \(A(fY)(t) = u_1(t) + u_2(t) \ (u_1(t) \in \text{Ker} \ ad(Y)\) and \(u_2(t) \in \text{Im} \ ad(Y))\), and \(u_i(t) = u_i^{-}(t) + u_i^{+}(t) \ (i = 1, 2)\) and \(b(t) = b^{-}(t) + b^{+}(t) \ (b^{-}(t) \in \mathfrak{g}^c, b^{+}(t) \in \mathfrak{g}^c_+)\). Let \(g_n(t) := \exp(n\tilde{f}(t)Y)\). From (5.9) and \(\text{Ad}(g_n)|_{\text{Ker} \ ad(Y)} = \text{id}\), we have
\[
b^v_n = \frac{1}{n} \text{Ad}(g_n)(\rho(g_n^{-1})(\hat{0}) + b) = \text{Ad}(g_n)(AfY + \frac{b}{n})
\]
\[
= u_1 + \text{Ad}(g_n)(u_2 + \frac{b}{n}).
\]
Since \(\text{Ad}(g_n)\) preserves \(\mathfrak{g}^c\) and \(\mathfrak{g}^c_+\) invariably, respectively, and \(\text{Ad}(g_n)|_{\text{Ker} \ ad(Y)} = \text{id}\), we have
\[
\langle b^v_n, u_1 \rangle \pm = \langle u_1, u_1 \rangle \pm + \langle \text{Ad}(g_n)(u_2 + \frac{b}{n}), \text{Ad}(g_n)u_1 \rangle \pm
\]
\[
= \langle u_1, u_1 \rangle \pm + \frac{1}{n} \langle b, u_1 \rangle \pm \to \langle u_1, u_1 \rangle \pm \ (n \to \infty).
\]
Assume that \(\int_0^1 f(t)dt = 0\) or \(f = \text{const}\) and \(Y\) is the initial vector of a closed geodesic in \(G^c_+\) of period one". Then we have \(\int_0^1 v(t)dt = 0\). Also we have \(v \in H^0, c\) because of \(Y \in \mathfrak{g}^c_+\). Hence, according to Lemma 5.4.6, there exists a subsequence \(\{X_{n_i}^v\}_{i=1}^{\infty}\) of
\{X_n\}_{n=1}^\infty converging to the zero vector field. Clearly we have \(\lim_{i \to \infty} b_{n_i}^\epsilon = 0\) and hence \(u_1 = 0\). Thus we see that \(A(fY)(t) \in \Im \mathrm{ad}(Y)\) holds for all \(t \in [0, 1]\). That is, we have \(A(fY) = [Y, w]\) for some \(w \in V\). Next we consider the case where \(f = \text{const}\) and \(Y\) is the initial vector of no closed geodesic \(G_n^\epsilon\) of period one. Set

\[ B := \{aZ \mid a \in \mathbb{R}, \ Z : \text{the initial vector of a closed geodesic in } G_n^\epsilon \text{ of period one}\}. \]

Since \(g_\mathbb{C}^\epsilon\) is the compact real of \(g_\mathbb{C}^\epsilon\), \(B\) is dense in \(g_\mathbb{C}^\epsilon\). Take a sequence \(\{a_iZ_i\}_{i=1}^\infty\) in \(B\) with \(\lim_{i \to \infty} a_iZ_i = fY\). For each \(i\), there exists \(w_i \in V\) with \(A(a_iZ_i) = [a_iZ_i, w_i]\). We can show that the sequence \(\{w_i\}_{i=1}^\infty\) is a convergent sequence and that

\[ A(fY) = \lim_{i \to \infty} [a_iZ_i, w_i] = [Y, f \lim_{i \to \infty} w_i]. \]

This completes the proof. q.e.d.

Since \(w\) in this lemma depends on \(X, f\) and \(Y\), we denote it by \(w_{X,f,Y}\). According to Lemma 2.10 of [Ch], we have the following fact.

**Lemma 5.4.8.** Let \(B\) be a map from \(g_\mathbb{C}^\epsilon\) to oneself defined by \(B(Y) = [\mu(Y), Y] \ (Y \in g_\mathbb{C}^\epsilon)\) in terms of a map \(\mu : g_\mathbb{C}^\epsilon \to g_\mathbb{C}^\epsilon\). If \(B\) is linear, then \(\mu\) is a constant map.

By using Lemmas 5.4.7 and 5.4.8, we can show the following fact.

**Lemma 5.4.9.** Fix \(X \in \mathcal{K}_{\mathbb{M}^\epsilon}^b\) and \(f \in \mathcal{H}^0([0, 1], \mathbb{C})\) satisfying \(\int_0^1 f(t)dt = 0\) or \(f = \text{const}\). Then \(w_{X,f,Y}\) is independent of the choice of \(Y \in g_\mathbb{C}^\epsilon\).

**Proof.** For simplicity, set \(w_Y := w_{X,f,Y}\). Define a linear map \(B_1^\epsilon : g_\mathbb{C}^\epsilon \to g_\mathbb{C}^\epsilon\) by \(B_1^\epsilon(Y) := A(fY)(t)_{g_\mathbb{C}^\epsilon} \ (Y \in g_\mathbb{C}^\epsilon)\) and a linear map \(B_2^\epsilon : g_\mathbb{C}^\epsilon \to g_\mathbb{C}^\epsilon\) by \(B_2^\epsilon(Y) := \sqrt{-1}A(fY)(t)_{g_\mathbb{C}^\epsilon} \ (Y \in g_\mathbb{C}^\epsilon)\), where \((\cdot)_{g_\mathbb{C}^\epsilon} (\varepsilon = - \text{ or } +)\) is the \(g_\mathbb{C}^\epsilon\)-component of \((\cdot)\). Since \(A(fY) = [Y, w_Y]\), we have \(B_1^\epsilon(Y) = [Y, w_Y(t)_{g_\mathbb{C}^\epsilon}]\) and \(B_2^\epsilon(Y) = [Y, \sqrt{-1}w_Y(t)_{g_\mathbb{C}^\epsilon}]\), it follows from Lemma 4.4.8 that, for each \(t \in [0, 1]\), \(w_Y(t)_{g_\mathbb{C}^\epsilon}\) and \(w_Y(t)_{g_\mathbb{C}^\epsilon}\) are independent of the choice of \(Y \in g_\mathbb{C}^\epsilon\). Hence \(w_Y\) is independent of the choice of \(Y \in g_\mathbb{C}^\epsilon\). q.e.d.

According to this lemma, \(w_{X,f,Y}\) is independent of the choice of \(Y \in g_\mathbb{C}^\epsilon\), we denote it by \(w_{X,f}\). Define \(\psi_n \in \mathcal{H}^0([0, 1], \mathbb{C})\) by \(\psi_n(t) = \exp(2n\pi\sqrt{-1}t) \ (0 \leq t \leq 1)\), where \(n \in \mathbb{Z}\).

**Lemma 5.4.10.** For each \(X \in \mathcal{K}_{\mathbb{M}^\epsilon}^b\) and each \(f \in \mathcal{H}^0([0, 1], \mathbb{C})\) satisfying \(\int_0^1 f(t)dt = 0\) or \(f = \text{const}\), we have \(w_{X,f} = f w_{X,1}\), where the subscript 1 in \(v_1\) means \(1 \in \mathcal{H}^0([0, 1], \mathbb{C})\).
Proof. Let \(( , )^c\) be the complexification of the \(\text{Ad}(G)\)-invariant non-degenerate symmetric bilinear form \(( , )\) of \(g\) inducing the metric of \(G/K\). Let \(a\) be a maximal abelian subspace of \(\sqrt{-1}p\) and \(g^c_\alpha = \delta g^c_\alpha(a) + \sum_{\alpha \in \Delta}(g^c_\alpha)_\alpha\) the root space decomposition of \(g^c\) with respect to \(a\), where \(\delta g^c_\alpha(a)\) is the centralizer of \(a\) in \(g^c\) and \(\Delta := \{\alpha \in a^* | (g^c_\alpha)_\alpha \neq \{0\}\}\). It follows from the arbitrariness of \(Y\) of \(\sqrt{c}\), of \(\sqrt{c}\). Also, from \(\text{Ad}(\psi_a)\) it follows from the arbitrariness of \(Y\) of \(\sqrt{c}\). Let \(\overline{\alpha}_a,\overline{\alpha}_n\) := \(\overline{\alpha}_a,\overline{\alpha}_n\). Then, from \(\overline{\alpha}_a,\overline{\alpha}_n\), we have

\[
[\text{Ad}(g_a, n)w_{\overline{\alpha}_a,\overline{\alpha}_n}, Y_0] = [\text{Ad}(g_a, n)w_{\overline{\alpha}_a,\overline{\alpha}_n}, \text{Ad}(g_a, n)Y_0] = -[\text{Ad}(g_a, n)\overline{\alpha}_a, Y_0] = -[\text{Ad}(g_a, n)\overline{\alpha}_a, Y_0] = -AY_0 = [w_{X,\overline{\alpha}_a}, Y_0].
\]

It follows from the arbitrariness of \(Y_0(\in \delta \alpha_\alpha(a))\) that

\[
(5.11) \quad \text{Im}(\text{Ad}(g_a, n)w_{\overline{\alpha}_a,\overline{\alpha}_n}, 1 - w_{X,1}) \subset a.
\]

Also, from \(\text{Ad}(g_a, n)Y_\alpha = \psi_n Y_\alpha\), we have

\[
[\text{Ad}(g_a, n)w_{\overline{\alpha}_a,\overline{\alpha}_n}, Y_\alpha] = \psi_n[\text{Ad}(g_a, n)w_{\overline{\alpha}_a,\overline{\alpha}_n}, \text{Ad}(g_a, n)Y_\alpha] = -\psi_n[\text{Ad}(g_a, n)\overline{\alpha}_a, Y_\alpha] = -\psi_n[\text{Ad}(g_a, n)\overline{\alpha}_a, Y_\alpha] = -\psi_n[\text{Ad}(g_a, n)\overline{\alpha}_a, Y_\alpha] = -\psi_n[w_{X,\overline{\alpha}_a}, Y_\alpha]
\]

and hence

\[
[\text{Ad}(g_a, n)w_{\overline{\alpha}_a,\overline{\alpha}_n}, 1 - \psi_n w_{X,\overline{\alpha}_a}, Y_\alpha] = 0.
\]

It follows from the arbitrariness of \(Y_\alpha(\in (\delta \alpha_\alpha)\alpha)\) that

\[
\text{Im}
\left(\text{Ad}(g_a, n)w_{\overline{\alpha}_a,\overline{\alpha}_n}, 1 - \psi_n w_{X,\overline{\alpha}_a}\right) \subset \delta \alpha_\alpha((\delta \alpha_\alpha)\alpha).
\]

This together with (5.11) implies

\[
\text{Im}(\psi_n w_{X,1} - w_{X,\overline{\alpha}_a}) \subset a + \delta \alpha_\alpha((\delta \alpha_\alpha)\alpha).
\]

From the arbitrariness of \(\alpha\), we obtain

\[
\text{Im}(\psi_n w_{X,1} - w_{X,\overline{\alpha}_a}) \subset a + \bigcap_{\alpha \in \Delta} \delta \alpha_\alpha((\delta \alpha_\alpha)\alpha) = a.
\]

Take another maximal abelian subspace \(a'\) of \(\sqrt{-1}p\) with \(a' \cap a = \{0\}\). Similarly we can show

\[
\text{Im}(\psi_n w_{X,1} - w_{X,\overline{\alpha}_a}) \subset a'
\]

70
and hence
\[(5.12) \quad w_{X, \psi_n} = \psi_n w_{X,1}.\]

Take any \(f \in H^0([0,1], \mathbb{C})\) satisfying \(\int_0^1 f(t) dt = 0\) or \(f = \text{const.}\). Let \(f = \sum_{n=-\infty}^{\infty} c_n \psi_n\) be the Fourier’s expansion of \(f\), where \(c_n\) is constant for each \(n\). Then, since \(A\) is continuous and linear, we have
\[(5.13) \quad A(fY) = \sum_{n=-\infty}^{\infty} c_n A(\psi_n Y) \quad (Y \in g^c).\]

From (5.12) and (5.13), we obtain
\[
[Y, w_{X,f}] = A(fY) = \sum_{n=-\infty}^{\infty} c_n [Y, w_{X,\psi_n}] = [Y, f w_{X,1}] \quad (Y \in g^c).
\]

Thus \(w_{X,f} - f w_{X,1}\) belongs to the center of \(g^c\). Therefore, since \(g^c\) has no center, we obtain \(w_{X,f} = f w_{X,1}\).

q.e.d.

From Lemmas 5.4.7 and 5.4.10, we have the following fact.

**Lemma 5.4.11.** Let \(X\) be an element of \(K^h_{\tilde{M}e}\) given by \(X_u = Au + b \ (u \in V)\) for some \(A \in \mathfrak{o}_AK(V)\) and \(b \in V\). Then we have \(A = \text{ad}(v)\) for some \(v \in V\).

**Proof.** Take any \(u \in V\) and a base \(\{e_1, \cdots, e_m\}\) of \(g^c\). Let \(u = \sum_{i=1}^{m} u_i(t) e_i\) and \(u_i(t) = \sum_{n=-\infty}^{\infty} c_{i,n} \psi_n(t)\) be the Fourier expansion of \(u_i\). Then, since \(A\) is continuous and linear, we have \(Au = \sum_{n=-\infty}^{\infty} \sum_{i=1}^{m} c_{i,n} A(\psi_n(t) e_i)\). According to Lemmas 5.4.7 and 5.4.10, we have \(A(fY) = [w_{X,1}, fY]\) for any \(Y \in g^c\) and any \(f \in H^0([0,1], \mathbb{C})\) satisfying \(\int_0^1 f(t) dt = 0\) or \(f = \text{const.}\). Hence we have
\[Au = \sum_{n=-\infty}^{\infty} \sum_{i=1}^{m} c_{i,n} [w_{X,1}, \psi_n(t) e_i] = [w_{X,1}, u].\]

Thus we obtain \(A = \text{ad}(w_{X,1})\).

q.e.d.

By using Lemmas 5.4.4 and 5.4.11, we shall prove Proposition 5.4.
Proof of Proposition 5.4. Let $H$ be as in the statement of Proposition 5.4. Take any $X \in \text{Lie } H$. Since $\text{Lie } H^0 \subset K_{\tilde{M}^c}$, it follows from Lemmas 5.4.4 and 5.4.11 that $X = X_v$ for some $v \in V$. Since $X_v$ is the holomorphic Killing field associated with an one-parameter subgroup $\{\rho(\exp \circ s v) \mid s \in \mathbb{R}\}$ of $\rho(H^1([0,1],G^c))$, we have $X \in \text{Lie } \rho(H^1([0,1],G^c))$. Hence we obtain $\text{Lie } H \subset \text{Lie } \rho(H^1([0,1],G^c))$, that is, $H \subset \rho(H^1([0,1],G^c))$. q.e.d.

By using Proposition 5.4, we shall prove Theorem 5.1.

Proof of Theorem 5.1. Since $H$ is a subgroup of $\rho(H^1([0,1],G^c))$ by Proposition 5.4, we have $H = \rho(Q)$ for some subgroup $Q$ of $H^1([0,1],G^c)$. Let $Q'$ be a closed connected subgroup of $G^c \times G^c$ generated by $\{(h(0), h(1)) \mid h \in Q\}$. Since $\phi \circ \rho(h) = (L_{h(0)} \circ R_{h(1)}^{-1}) \circ \phi$ for each $h \in H$, we have $\tilde{M}^c = Q' \cdot e$, where $e$ is the identity element of $G^c$. Here we note that $G^c \times G^c$ acts on $G^c$ by $(g_1, g_2) \cdot g := (L_{g_1} \circ R_{g_2}^{-1})(g)$ $(g_1, g_2, g \in G^c)$. Set $\tilde{M} := \pi_R^1(M)$, where $\pi_R$ is the natural projection of $G$ onto $G/K$. Since $\tilde{M}$ is a component of $\tilde{M}^c \cap G$ containing $e$ and $(Q' \cap (G \times G)) \cdot e$ is a complete open submanifold of $\tilde{M}^c \cap G$, $\tilde{M}$ is a component of $(Q' \cap (G \times G)) \cdot e$. Therefore we have $\tilde{M} = (Q' \cap (G \times G))_0 \cdot e$, where $(Q' \cap (G \times G))_0$ is the identity component of $Q' \cap (G \times G)$. Set $Q'_R := (Q' \cap (G \times G))_0$. Since $\tilde{M}$ consists of fibres of $\pi_R$, we have $(Q'_R \cup (e \times K)) \cdot e = \tilde{M}$, where $(Q'_R \cup (e \times K))$ is the group generated by $Q'_R \cup (e \times K)$. Denote by the same symbol $Q'_R$ the group $(Q'_R \cup (e \times K))$ under abuse of the notation. Set $(Q'_R)_1 := \{g_1 \in G \mid \exists g_2 \in G$ s.t. $(g_1, g_2) \in Q'_R\}$ and $(Q'_R)_2 := \{g_2 \in G \mid \exists g_1 \in G$ s.t. $(g_1, g_2) \in Q'_R\}$. Also, set $(Q'_R)_1^* := \{g \in G \mid (g, e) \in Q'_R\}$ and $(Q'_R)_2^* := \{g \in G \mid (e, g) \in Q'_R\}$. It is clear that $(Q'_R)_i^*$ is a normal subgroup of $(Q'_R)_i$ $(i = 1, 2)$. From $e \times K \subset Q'_R$, we have $K \subset (Q'_R)_2^*$. Since $K \subset (Q'_R)_2^* \subset (Q'_R)_2 \subset G$ and $K$ is a maximal subgroup of $G$, we have $(Q'_R)_2 = K$ or $G$ and $(Q'_R)_2^* = K$ or $G$. Suppose that $(Q'_R)_2^* = G$. Then we have $\tilde{M} = G$ and hence $M = G/K$. Thus a contradiction arises. Hence we have $(Q'_R)_2^* = K$. Since $K$ is not a normal subgroup of $G$ and it is a normal subgroup of $(Q'_R)_2$, we have $(Q'_R)_2 \neq G$. Therefore we have $(Q'_R)_2 = K$ and hence $Q'_R \subset G \times K$. Set $Q'_R^u := \{g \in G \mid \{(g) \times K\} \cap Q'_R \neq \emptyset\}$. Then, since $\tilde{M} = Q'_R \cdot e$ and $M = \pi(\tilde{M})$, we have $M = Q'_R^u(eK)$. Thus $M$ is homogeneous. q.e.d.

By using Theorem 5.1 and Lemma 4.1, we shall prove Theorem A.

Proof of Theorem A. We shall use the notations in Lemma 4.1. Let $Z$ be as in the proof of Lemma 4.1 and $\tilde{Z}$ the parallel normal vector field of $M$ with $\tilde{Z}_{eK} = Z$. Denote by $\eta_{\tilde{Z}}$ $(0 \leq t \leq 1)$ the end-point map for $t\tilde{Z}$ (i.e., $\eta_{t\tilde{Z}}(x) = \exp^t(t\tilde{Z}_x)$ $(x \in M)$), where $\exp^t$ is the normal exponential map of $M$. Set $M_t := \eta_{t\tilde{Z}}(M)$. If $0 < t < 1$, then $M_t$ is a a parallel submanifold of $M$ and $M_1$ is a focal submanifold of $M$ (see

72
Figures 9 and 10). Fix $v \in U$. Let $\widehat{v}$ be the parallel tangent vector field on the flat section $\Sigma_{eK} := \exp b$ with $\widehat{v}_{eK} = v$. Note that $\widehat{v}_{\eta_Z(eK)}$ is a normal vector of $M_t$ at $\eta_Z(eK)$. Denote by $A^t$ the shape tensors of $M_t$ ($0 < t < 1$). According to the proof of Theorems B and C of [Koi8], there exists a complex linear function $\phi_i$ on $b^c$ with $\phi_i(v) = \lambda_i^v$ and $\phi_i^{-1}(1) \subset F$ for each $i \in I_0^v$ with $\lambda_i^v \neq 0$. Fix $i_0 \in I_0^v$ with $\lambda_{i_0}^v \neq 0$. According to (5.14), $\phi_{i_0}^{-1}(1)$ coincides with one of $(\beta^c)^{-1}(\arctanh(\beta, i, v) + j\pi \sqrt{-1})$’s $(\beta \in (\Delta_b)_+, (i, j) \in (I_0^v)^+ \times \mathbb{Z})$ and $(\beta^c)^{-1}(\arctanh(\beta, i, v) + (j + \frac{1}{2})\pi \sqrt{-1})$’s $(\beta \in (\Delta_b)_+, (i, j) \in (I_0^v)^- \times \mathbb{Z})$, where $c_{\beta, i, v}$ and $c_{\beta, i, v}^-$ are as in the proof of Lemma 4.1. Since $(\beta^c)^{-1}(\arctanh(\beta, i, v) + j\pi \sqrt{-1}) = (\beta^c)^{-1}(\arctanh(\beta, i, v) + (\beta)^c|_{\sqrt{-1}})^{-1}((j + \frac{1}{2})\pi \sqrt{-1})$, we have $\phi_{i_0}^{-1}(1) = \beta_1^{-1}(\arctanh(\beta, i, v) + (\beta)^c|_{\sqrt{-1}})^{-1}(0)$ for some $\beta_1 \in (\Delta_b)_+$ and $i_1 \in (I_0^v)^+$. Hence, since $Z$ and $\beta_{i_0}$ belong to the complex hyperplane $\phi_{i_0}^{-1}(1)$, we have $\ker(\eta_Z)_{eK} = \ker(\eta_Z)_{\mathbb{Z}eK}$. On the other hand, we have $p_0 \cap \ker(A_v - \lambda_{i_0}^v id) \subset \ker(\eta_Z)_{\mathbb{Z}eK}$. Hence we have

(5.14) \hspace{1cm} p_0 \cap \ker(A_v - \lambda_{i_0}^v id) \subset \ker(\eta_Z)_{eK}.

Also, by using (1.1), we can show $\ker\left( (\eta_Z)_{eK} | p_0 \cap \ker A_v \right) = \{0\}$ and

(5.15) \hspace{1cm} A^1_{\eta_Z(eK)}(\eta_Z)_{eK} = \ker A_v.

By imitating the proof of Lemma 4.1, we can show

$$\text{Spec } A^t_{\eta_Z(eK)} = \sum_{\beta \in (\Delta_b)_+ \setminus \{\eta_Z\}_+} (\eta_Z)_+ \cdot (\beta_{i_0})$$

(5.16) \hspace{1cm} \left\{ \frac{\tan((1 - t)\beta(Z))}{\tan((1 - t)\beta(Z))} | \beta \in (\Delta_b)_+ \text{ s.t. } (I_0^v)^+ \neq \emptyset \right\} \cup \left\{ \beta \in (\Delta_b)_+ \text{ s.t. } (I_0^v)^+ \neq \emptyset \right\}

for each $t \in [0, 1)$. Set $m := \dim M_1$. Denote by $\text{Gr}_m(G/K)$ the Grassmann bundle of $G/K$ consisting of $m$-dimensional subspaces of the tangent spaces. Define $D_t$ ($0 \leq t < 1$) by

$$D_t := (\eta_Z)_+ (p_0 \cap \ker A_v) + \sum_{\beta \in (\Delta_b)_+ \text{ s.t. } (I_0^v)^+ \neq \emptyset} \left( p_{\beta} \cap \ker \left( A^t_{\eta_Z(eK)} - \beta(v) \tanh((1 - t)\beta(Z)) id \right) \right)$$

(see Figure 10). From (5.14), (5.15) and (5.16), we can show $\lim_{t \to 1^-} D_t = T_{\eta_Z(eK)} M_1$ (in $\text{Gr}_m(G/K)$) and

$$\text{Spec } A^1_{\eta_Z(eK)} = \{0\} \cup \left\{ \lim_{t \to 1^-} \beta(v) \tanh((1 - t)\beta(Z)) | \beta \in (\Delta_b)_+ \text{ s.t. } (I_0^v)^+ \neq \emptyset \right\} = \{0\}.$$
Thus we have \( A^1_{\hat{w}\eta_Z(eK)} = 0 \). Since this relation holds for any \( v \in U \) and \( U \) is open and dense in \( b(= T_{eK}^1 M) \), \( A^1_{\hat{w}\eta_Z(eK)} = 0 \) holds for any \( w \in T_{eK}^1 M \), where \( \hat{w} \) is the parallel tangent vector field on the section \( \Sigma_{eK} \) with \( \hat{w}_{eK} = w \). Set \( L := \eta_Z^{-1}(\eta_Z(eK)) \), which is the focal leaf through \( eK \) for the focal map \( \eta_Z \) of \( M \). Take any \( x \in L \). Similarly we can show \( A^1_{\hat{w}\eta_Z(x)} = 0 \) for any \( w \in T^1_x M \), where \( \hat{w} \) is the parallel tangent vector field on the section \( \Sigma_x \) of \( M \) through \( x \) with \( \hat{w}_x = w \). It is easy to show that \( \eta_Z(x) = \eta_Z(eK) \) and that \( \cup_{x \in L} \{ \hat{w}\eta_Z(x) \mid w \in T^1_x M \} = T^1_{\eta_Z(eK)}M_1 \). Hence we see that \( A^1 \) vanishes at \( \eta_Z(eK) \).

Similarly we can show that \( A^1 \) vanishes at any point of \( M_1 \). That is \( M_1 \) is totally geodesic in \( G/K \). On the other hand, since \( M \) is homogeneous by Theorem 5.1, it follows from Theorem A of [Koi6] that \( M \) is a principal orbit of a complex hyperpolar action on \( G/K \). See [Koi2] (or [Koi6]) about the definition of a complex hyperpolar action. Furthermore, since this action admits a totally geodesic singular orbit \( M_1 \) and since it is of cohomogeneity greater than one, it follows from Theorem C and Remark 1.1 of [Koi6] that this action is orbit equivalent to a Hermann action. Therefore we obtain the statement of Theorem A.

q.e.d.
Next we prove Theorem B.

**Proof of Theorem B.** Let $G/K$ be the ambient symmetric space and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ associated with the symmetric pair $(G, K)$, where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of $G$ (resp. $K$). Since $M$ satisfies the condition $(\ast_R)$, it satisfies the condition $(\ast_C)$ as stated in Introduction. Hence, according to Theorem A, $M$ is a principal orbit of a Hermann action $H \curvearrowright G/K$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ be the canonical decomposition associated with the symmetric pair $(G, H)$, where $\mathfrak{h}$ is the Lie algebra of $H$. Also, let $\theta$ be the Cartan involution of $G$ with $(\text{Fix} \, \theta)_0 \subset K \subset \text{Fix} \, \theta$ and $\sigma$ the involution of $G$ with $(\text{Fix} \, \sigma)_0 \subset H \subset \text{Fix} \, \sigma$. Denote by the same symbols the involutions of $\mathfrak{g}$ induced from $\theta$ and $\sigma$, respectively. We may assume that $\theta$ commutes with $\sigma$ by replacing $H$ to its suitable conjugate group if necessary (see Lemma 10.2 of [Be]). Hence we have $\mathfrak{p} = \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$.

Following to this replacement, we replace $M$ to a suitable congruent one. In the sequel we shall show that the $H$-action is orbit equivalent to the isotropy action of $G/K$. Suppose that the $H$-action is not orbit equivalent to the isotropy action of $G/K$. Then the orbit $H(eK)$ is a reflective submanifold (see [Koi4] for example). Denote by $F$ this orbit and set $F^\perp = \exp^\perp(T^\perp_{eK}F)$, where $\exp^\perp$ is the normal exponential map of $F$. Take $x \in M \cap F^\perp$ and $v(\neq 0) \in T^\perp_{eK}M$. Let $Z$ be the element of $\mathfrak{p} \cap \mathfrak{q}(= T^\perp_{eK}F)$ with $\text{Exp} \, Z = x$ and set $\bar{v} := (\exp Z)^{-1}(v)$, where $\text{Exp}$ is the exponential map of $G/K$ at $eK$ and $\exp$ is the exponential map of $G$. Also, set $\mathfrak{b} := (\exp Z)^{-1}(T^\perp_{eK}M)$, which is the maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$. Let $\mathfrak{p} = \mathfrak{z}_\mathfrak{p}(\mathfrak{b}) + \sum_{\beta \in \Delta^+_\mathfrak{p}} \mathfrak{p}_\beta$ be the root space decomposition with
respect to \(b\), where \(\mathfrak{z}_p(b)\) is the centralizer of \(b\) in \(p\), \(\Delta'_+\) is the positive root system of \(\Delta' := \{\beta \in b^* | \exists X(\neq 0) \in p \text{ s.t. } \text{ad}(b)^2(X) = \beta(b)^2X (\forall b \in b)\}\) under some lexicographic ordering of \(b^*\) and \(p_\beta := \{X \in p | \text{ad}(b)^2(X) = \beta(b)^2X (\forall b \in b)\} (\beta \in \Delta'_+)\). Also, let \(\Delta'_V := \{\beta \in \Delta'_+ | p_\beta \cap q \neq \{0\}\}\) and \(\Delta'_{VH} := \{\beta \in \Delta'_+ | p_\beta \cap h \neq \{0\}\}\). Then we have \(p \cap q = b + \sum_{\beta \in \Delta'_V} (p_\beta \cap q)\) and \(p \cap h = \mathfrak{z}_p (b) + \sum_{\beta \in \Delta'_{VH}} (p_\beta \cap h)\), where \(\mathfrak{z}_p (b)\) is the centralizer of \(b\) in \(p \cap h\). Assume that \(v\) is a regular element, that is, \(\beta(v) \neq 0\) for any \(\beta \in \Delta'_+\) (see Figure 11). In similar to (5.1) and (5.2) in [Koi11], we have

\begin{equation}
A_v |_{(\exp Z)_\ast(p_\beta \cap q)} = -\frac{\beta(v)}{\tanh (Z)} \text{id} \quad (\beta \in \Delta'_V)
\end{equation}

and

\begin{equation}
A_v |_{(\exp Z)_\ast(p_\beta \cap h)} = -\beta(v) \tanh (Z) \text{id} \quad (\beta \in \Delta'_{VH}),
\end{equation}

where we need to rescale the metric of \(G/K\) by a suitable positive constant. Set \(S_+ := \{(\lambda, \mu) \in \text{Spec } A_v \times \text{Spec } R(v) | |\lambda| > |\mu|\}\) and \(S_- := \{(\lambda, \mu) \in \text{Spec } A_v \times \text{Spec } R(v) | |\lambda| < |\mu|\}\). Also, we can show

\begin{equation}
A_v |_{(\exp Z)_\ast(\mathfrak{z}_p (b))} = 0.
\end{equation}

According to (5.17) \sim (5.19), we have

\begin{equation}
(\exp Z)_\ast \left( \sum_{\beta \in \Delta'_V} (p_\beta \cap q) \right) = \bigoplus_{(\lambda, \mu) \in S_+} (\text{Ker}(A_v - \lambda \text{id}) \cap \text{Ker}(R(v) - \mu \text{id})),
\end{equation}

and

\begin{equation}
(\exp Z)_\ast \left( \sum_{\beta \in \Delta'_{VH}} (p_\beta \cap h) \right) = \bigoplus_{(\lambda, \mu) \in S_-} (\text{Ker}(A_v - \lambda \text{id}) \cap \text{Ker}(R(v) - \mu \text{id})).
\end{equation}

Clearly we have

\begin{equation}
(\exp Z)_\ast \left( \sum_{\beta \in \Delta'_V} (p_\beta \cap q) \right) = T_x (M \cap F^\perp)
\end{equation}

and

\begin{equation}
(\exp Z)_\ast \left( \sum_{\beta \in \Delta'_{VH}} (p_\beta \cap h) \right) = (T_x M \oplus \text{Ker } R(v)) \oplus T_x (M \cap F^\perp).
\end{equation}
On the other hand, we can show that the nullity spaces for (real) focal radii of $M$ along $\gamma_v$ span $\oplus_{(\lambda,\mu) \in S^+}(\text{Ker}(A_v - \lambda \text{id}) \cap \text{Ker}(R(v) - \mu \text{id}))$ (see the fact (1.2) stated in the first paragraph of Introduction in the case where $G/K$ is a hyperbolic space). Hence, from (5.20) and (5.22) we have the following fact:

1. The nullity spaces for (real) focal radii of $M$ along $\gamma_v$ span $T_x(M \cap F^\perp)$.

According to Table 2 in [Koi11], in the case where $M$ is of codimension two, (i.e., the cohomogeneity of the $H$-action is equal to two), we have $\Delta^H \neq \emptyset$ because the $H$-action is not orbit equivalent to the isotropy action of $G/K$. Similarly, we can show $\Delta'^H \neq \emptyset$ in the case where $M$ is of codimension greater than two. Hence it follows from (5.23) that $(T_x M \ominus \text{Ker } R(v)) \ominus T_x(M \cap F^\perp) \neq \{0\}$. On the other hand, it is clear that $\text{Ker } R(v) \cap T_x M = (\exp Z)_*(\delta_{p\in\mathfrak{h}}(b))$. According to these facts, $T_x(M \cap F^\perp)$ is properly included by $T_x M \ominus \text{Ker } R(v)$. According to this fact and the above fact (1), the condition ($R$) does not hold. This contradicts that $M$ satisfies the condition ($\ast R$). Therefore the $H$-action is orbit equivalent to the isotropy action of $G/K$. Therefore we obtain the statement of Theorem B.

q.e.d.

Figure 11.
6 Classification

From Theorem A and the list of Hermann actions in [Koi6], we can classify isoparametric submanifolds in irreducible symmetric spaces of non-compact type as in Theorem A as follows.

**Theorem 6.1.** Let $M$ be an isoparametric submanifold in an irreducible symmetric space $G/K$ of non-compact type as in Theorem A. Then $M$ is congruent to a principal orbit of the action of one of symmetric subgroups $H$’s of $G$ as in Tables 1 ~ 3.
| $G/K$ | $H$ |
|-------|-----|
| $\text{SL}(n,\mathbb{R})/\text{SO}(n)$ | $\text{SO}(n)$, $\text{SO}_0(p, n-p)$ ($1 \leq p \leq n-1$), $\text{Sp}(\frac{n}{2}, \mathbb{R})$, $\text{SL}(\frac{n}{2}, \mathbb{C}) \cdot U(1)$ $\text{SL}(p, \mathbb{R}) \times \text{SL}(n-p, \mathbb{R}) \cdot \mathbb{R}^*$ ($2 \leq p \leq n-2$) |
| $(n \geq 6$, $n$ : even)$ | |
| $\text{SL}(4, \mathbb{R})/\text{SO}(4)$ | $\text{SO}(4)$, $\text{SO}_0(1, 3)$, $\text{SO}_0(2, 2)$, $\text{SL}(2, \mathbb{C}) \cdot U(1)$, $(\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})) \cdot \mathbb{R}^*$ |
| $\text{SL}(n,\mathbb{R})/\text{SO}(n)$ | $\text{SO}(n)$, $\text{SO}_0(p, n-p)$ ($1 \leq p \leq n-1$), $\text{SL}(\frac{n}{2}, \mathbb{C}) - \mathbb{R}^*$ $\text{SL}(p, \mathbb{R}) \times \text{SL}(n-p, \mathbb{R}) \cdot \mathbb{R}^*$ ($2 \leq p \leq n-2$) |
| $(n \geq 5$, $n$ : odd)$ | |
| $\text{SL}(3,\mathbb{R})/\text{SO}(3)$ | $\text{SO}(3)$, $\text{SO}_0(1, 2)$ |
| $\text{SU}^*(2n)/\text{Sp}(n)$ ($n \geq 4$) | $\text{Sp}(n)$, $\text{SO}^*(2n)$, $\text{Sp}(p, n-p)$ ($1 \leq p \leq n-1$), $\text{SL}(n, \mathbb{C}) \cdot U(1)$ $\text{SU}^*(2p) \times \text{SU}^*(2n-2p) \times U(1)$ ($2 \leq p \leq n-2$) |
| $\text{SU}^*(6)/\text{Sp}(3)$ | $\text{Sp}(3)$, $\text{SO}^*(6)$, $\text{Sp}(1, 2)$ |
| $\text{SU}(p,q)/\text{SU}(p) \times U(q)$ | $\text{SU}(p) \times U(q)$, $\text{SO}_0(p, q)$, $\text{Sp}(\frac{p}{2}, \frac{q}{2})$, $\text{SU}(i,j) \times U(p-i, q-j)$ ($1 \leq i \leq p-1, 1 \leq j \leq q-1$) |
| $(4 \leq p < q$, $p,q$ : even)$ | |
| $\text{SU}(p,q)/\text{SU}(p) \times U(q)$ | $\text{SU}(p) \times U(q)$, $\text{SO}_0(p, q)$, $\text{SU}(i,j) \times U(p-i, q-j)$ ($1 \leq i \leq p-1, 1 \leq j \leq q-1$) |
| $(3 \leq p < q$, $p$ or $q$ : odd)$ | |
| $\text{SU}(2,q)/\text{SU}(2) \times U(q)$ | $\text{SU}(2) \times U(q)$, $\text{SO}_0(2, q)$, $\text{SU}(1,j) \times U(1, q-j)$ ($1 \leq j \leq q-1$) |
| $(q \geq 3)$ | |
| $\text{SU}(p,p)/\text{SU}(p) \times U(p)$ | $\text{SU}(p) \times U(p)$, $\text{SO}_0(p, p)$, $\text{SU}^*(2p)$, $\text{Sp}(\frac{p}{2}, \frac{q}{2})$, $\text{Sp}(p, \mathbb{R})$, $\text{SL}(p, \mathbb{C}) \cdot U(1)$ $\text{SU}(i,j) \times U(p-i, p-j)$ ($1 \leq i \leq p-1, 1 \leq j \leq p-1$) |
| $(p \geq 4$, $p$ : even)$ | |
| $\text{SU}(2,2)/\text{SU}(2) \times U(2)$ | $\text{SU}(2) \times U(2)$, $\text{SO}_0(2, 2)$, $\text{SO}^*(4)$, $\text{SL}(2, \mathbb{C}) \cdot U(1)$, $\text{SU}(1,1) \times U(1,1)$ |
| $\text{SU}(p,p)/\text{SU}(p) \times U(p)$ | $\text{SU}(p) \times U(p)$, $\text{SO}_0(p, p)$, $\text{SU}^*(2p)$, $\text{Sp}(p, \mathbb{R})$, $\text{SL}(p, \mathbb{C}) \cdot U(1)$ $\text{SU}(i,j) \times U(p-i, p-j)$ ($1 \leq i \leq p-1, 1 \leq j \leq p-1$) |
| $(p \geq 5$, $p$ : odd)$ | |
| $\text{SU}(3,3)/\text{SU}(3) \times U(3)$ | $\text{SU}(3) \times U(3)$, $\text{SO}_0(3, 3)$, $\text{SO}^*(6)$, $\text{SL}(3, \mathbb{C}) \cdot U(1)$, $\text{SU}(1,1) \times U(2,2)$, $\text{SU}(1,2) \times U(2,1)$ |
| $\text{SL}(n,\mathbb{C})/\text{SU}(n)$ | $\text{SU}(n)$, $\text{SO}(n, \mathbb{C})$, $\text{SL}(n, \mathbb{R})$, $\text{SU}(i,n-i)$ ($1 \leq i \leq n-1$), $\text{Sp}(\frac{n}{2}, \mathbb{C})$, $\text{SU}^*(n)$ $\text{SL}(i, \mathbb{C}) \times \text{SL}(n-i, \mathbb{C}) \times U(1)$ ($2 \leq i \leq n-2$) |
| $(n \geq 6$, $n$ : even)$ | |
| $\text{SL}(4,\mathbb{C})/\text{SU}(4)$ | $\text{SU}(4)$, $\text{SO}(4, \mathbb{C})$, $\text{SL}(4, \mathbb{R})$, $\text{SU}(i,4-i)$ ($1 \leq i \leq 3$), $\text{SU}^*(4)$ $\text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \times U(1)$ |
| $\text{SL}(n,\mathbb{C})/\text{SU}(n)$ | $\text{SU}(n)$, $\text{SO}(n, \mathbb{C})$, $\text{SL}(n, \mathbb{R})$, $\text{SU}(i,n-i)$ ($1 \leq i \leq n-1$) $\text{SL}(i, \mathbb{C}) \times \text{SL}(n-i, \mathbb{C}) \times U(1)$ ($2 \leq i \leq n-2$) |
| $(n \geq 5$, $n$ : odd)$ | |
| $\text{SL}(3,\mathbb{C})/\text{SU}(3)$ | $\text{SU}(3)$, $\text{SO}(3, \mathbb{C})$ |

Table 1.
| $G/K$                  | $H$                                                                 |
|-----------------------|----------------------------------------------------------------------|
| $SO_0(p, q)/SO(p) \times SO(q)$ (4 ≤ $p < q$, $p, q$: even)   | $SO(p) \times SO(q)$, $SU(\frac{p}{2}, \frac{q}{2}) \cdot U(1)$, $SO_0(i, j) \times SO_0(p - i, p - j)$ (1 ≤ $i < p - 1$, 1 ≤ $j < p - 1$) |
| $SO_0(2, q)/SO(2) \times SO(q)$ (4 ≤ $q$, $q$: even)         | $SO(2) \times SO(q)$, $SO_0(1, j) \times SO_0(1, q - j)$ (1 ≤ $j < q - 1$) |
| $SO_0(p, q)/SO(p) \times SO(q)$ (2 ≤ $p < q$, $p$ or $q$: odd) | $SO(p) \times SO(q)$, $SO_0(i, j) \times SO_0(p - i, q - j)$ (1 ≤ $i < p - 1$, 1 ≤ $j < q - 1$) |
| $SO_0(p, p)/SO(p) \times SO(p)$ (p ≥ 4, $p$: even)            | $SO(p) \times SO(p)$, $SU(\frac{p}{2}, \frac{p}{2}) \cdot U(1)$, $SL(p, \mathbb{R}) \cdot U(1)$ |
| $SO_0(2, 2)/SO(2) \times SO(2)$                               | $SO(2) \times SO(2)$, $SO(p, \mathbb{C})$, $SO_0(1, 1) \times SO_0(1, 1)$ |
| $SO_0(p, p)/SO(p) \times SO(p)$ (p ≥ 5, $p$: odd)             | $SO(p) \times SO(p)$, $SO_0(p, \mathbb{C})$, $SL(p, \mathbb{R}) \cdot U(1)$ |
| $SO_0(3, 3)/SO(3) \times SO(3)$                                | $SO(3) \times SO(3)$, $SO(3, \mathbb{C})$, $SO_0(1, 1) \times SO_0(2, 2)$ |
| $SO^*(2n)/U(n)$ (n ≥ 6, $n$: even)                             | $U(n)$, $SO(n, \mathbb{C})$, $SU^*(n) \cdot U(1)$ |
| $SO^*(2n)/U(n)$ (n ≥ 5, $n$: odd)                              | $SO^*(2n) \times SO^*(2n - 2n)$ (2 ≤ $i < n - 2$), $SU(i, n - i) \cdot U(1)$ ($\frac{n}{2}$ + $\frac{n - i}{2}$ ≥ 2) |
| $SO(n, \mathbb{C})/SO(n)$ (n ≥ 8, $n$: even)                   | $SO(n)$, $SO(i, \mathbb{C}) \times SO(n - i, \mathbb{C})$ (2 ≤ $i < n - 2$), $SO_0(i, n - i)$ ($\frac{n}{2}$ + $\frac{n - i}{2}$ ≥ 2), $SL(\frac{n}{2}, \mathbb{C}) \cdot SO(2, \mathbb{C})$, $SO^*(n)$ |
| $SO(6, \mathbb{C})/SO(6)$                                     | $SO(6)$, $SO(i, \mathbb{C}) \times SO(6 - i, \mathbb{C})$ (2 ≤ $i < 4$), $SO_0(2, 4)$, $SO_0(3, 3)$, $SO^*(6)$ |
| $SO(4, \mathbb{C})/SO(4)$                                     | $SO(4)$, $SO(2, \mathbb{C}) \times SO(2, \mathbb{C})$, $SO_0(2, 2)$, $SO^*(4)$ |
| $SO(n, \mathbb{C})/SO(n)$ (n ≥ 5, $n$: odd)                   | $SO(n)$, $SO(i, \mathbb{C}) \times SO(n - i, \mathbb{C})$ (2 ≤ $i < n - 2$), $SO_0(i, n - i)$ ($\frac{n}{2}$ + $\frac{n - i}{2}$ ≥ 2) |
| $Sp(n, \mathbb{R})/U(n)$ (n ≥ 4, $n$: even)                    | $U(n)$, $SU(i, n - i) \cdot U(1)$ (1 ≤ $i < n - 1$), $SL(n, \mathbb{R}) \cdot U(1)$ |
| $Sp(2, \mathbb{R})/U(2)$                                       | $U(2)$, $SU(1, 1) \cdot U(1)$ |
| $Sp(n, \mathbb{R})/U(n)$ (n ≥ 5, $n$: odd)                    | $Sp(n, \mathbb{R}) \times Sp(n - i, \mathbb{R})$ (2 ≤ $i < n - 2$) |
| $Sp(3, \mathbb{R})/U(3)$                                       | $U(3)$, $SU(1, 2) \cdot U(1)$, $SL(3, \mathbb{R}) \cdot U(1)$ |

Table 2.
| $G/K$                                                                 | $H$                                                                 |
|---------------------------------------------------------------------|----------------------------------------------------------------------|
| $Sp(p,q)/Sp(p) \times Sp(q)$                                        | $Sp(p) \times Sp(q)$, $SU(p,q) \cdot U(1)$, $Sp(i,j) \times Sp(p-i,q-j)$ (1 ≤ i ≤ p − 1, 1 ≤ j ≤ q − 1) |
| ($2 \leq p < q$)                                                     |                                                                     |
| $Sp(p,p)/Sp(p) \times Sp(p)$                                        | $Sp(p) \times Sp(p)$, $SU(p,p) \cdot U(1)$, $SU^*(2p) \cdot U(1)$, $Sp(p,\mathbb{C})$ |
| ($p \geq 3$)                                                        |                                                                     |
| $Sp(2,2)/Sp(2) \times Sp(2)$                                        | $Sp(2) \times Sp(2)$, $SU(2,2) \cdot U(1)$, $SU^*(4) \cdot U(1)$, $Sp(1,1) \times Sp(1,1)$ |
| $Sp(n,\mathbb{C})/Sp(n)$                                            | $Sp(n)$, $SL(n,\mathbb{C}) \cdot SO(2,\mathbb{C})$, $Sp(n,\mathbb{R})$, $Sp(i,n-i)$ (1 ≤ i ≤ n − 1), $Sp(i,\mathbb{C}) \times Sp(n-i,\mathbb{C})$ (2 ≤ i ≤ n − 2) |
| ($n \geq 4$)                                                        |                                                                     |
| $E_6^4/(Sp(4)/(\pm 1))$                                             | $Sp(4)/(\pm 1)$, $Sp(4,\mathbb{R})$, $Sp(2,2)$, $SU^*(6) \cdot SU(2)$, $SL(6,\mathbb{R}) \times SL(2,\mathbb{R})$, $SO_0(5,5) \cdot \mathbb{R}$, $F_4$ |
| $E_6^2/SU(6) \cdot SU(2)$                                           | $SU(6) \cdot SU(2)$, $Sp(1,3)$, $Sp(4,\mathbb{R})$, $SU(2,4) \cdot SU(2)$, $SU(3,3) \cdot SL(2,\mathbb{R})$, $SO^*(10) \cdot U(1)$, $SO_0(4,6) \cdot U(1)$ |
| $E_6^{-14}/Sp(10) \cdot U(1)$                                      | $Sp(10) \cdot U(1)$, $Sp(2,2)$, $SU(2,4) \cdot SU(2)$, $SU(1,5) \cdot SL(2,\mathbb{R})$, $SO^*(10) \cdot U(1)$, $SO_0(2,8) \cdot U(1)$ |
| $E_6^{-26}/F_4$                                                     | $F_4$, $F_4^{-20}$, $Sp(1,3)$ |
| $E_6^0/E_6$                                                         | $E_6$, $E_6^0$, $E_6^2$, $E_6^{-14}$, $Sp(4,\mathbb{C})$, $SL(6,\mathbb{C}) \cdot SL(2,\mathbb{C})$, $SO(10,\mathbb{C}) \cdot Sp(1)$, $F_4^C$, $E_6^{-26}$ |
| $E_7^2/(SU(8)/(\pm 1))$                                            | $SU(8)/(\pm 1)$, $SL(8,\mathbb{R})$, $SU^*(8)$, $SU(4,4)$, $SO^*(12) \cdot SU(2)$, $SO_0(6,6) \cdot SL(2,\mathbb{R})$, $E_6^0 \cdot U(1)$, $E_6^0 \cdot U(1)$ |
| $E_7^{-5}/SO^*(12) \cdot SU(2)$                                    | $SO^*(12) \cdot SU(2)$, $SU(4,4)$, $SU(2,6)$, $SO^*(12) \cdot SL(2,\mathbb{R})$, $SO_0(4,8) \cdot SU(2)$, $E_6^0 \cdot U(1)$, $E_6^0 \cdot U(1)$ |
| $E_7^{-25}/E_6 \cdot U(1)$                                         | $E_6 \cdot U(1)$, $SU^*(8)$, $SU(2,6)$, $SO^*(12) \cdot SU(2)$, $SO_0(2,10) \cdot SL(2,\mathbb{R})$, $E_6^{-14} \cdot U(1)$, $E_6^{-26} \cdot U(1)$ |
| $E_7^2/E_7$                                                        | $E_7$, $E_7^2$, $E_7^{-5}$, $E_7^{-25}$, $SL(8,\mathbb{C})$, $SO(12,\mathbb{C}) \cdot SL(2,\mathbb{C})$, $E_8^0 \cdot C^*$ |
| $E_7^2/SO^*(16)$                                                   | $SO^*(16)$, $SO^*(16)$, $SO_0(8,8)$, $E_7^{-5} \cdot Sp(1)$, $E_7^2 \cdot SL(2,\mathbb{R})$ |
| $E_7^{-25}/E_7 \cdot Sp(1)$                                        | $E_7 \cdot Sp(1)$, $E_7^{-5} \cdot Sp(1)$, $E_7^{-25} \cdot SL(2,\mathbb{R})$, $SO^*(16)$, $SO_0(4,12)$ |
| $E_8^2/E_8$                                                        | $E_8$, $E_8^0$, $E_8^{-24}$, $SO(16,\mathbb{C})$, $E_8^2 \times SL(2,\mathbb{C})$ |
| $F_4^4/Sp(3) \cdot Sp(1)$                                          | $Sp(3) \cdot Sp(1)$, $Sp(1,2) \cdot Sp(1)$, $Sp(3,\mathbb{R}) \cdot SL(2,\mathbb{R})$ |
| $F_4^2/F_4$                                                        | $F_4$, $F_4^2$, $F_4^{-20}$, $Sp(3,\mathbb{C}) \cdot SL(2,\mathbb{C})$ |
| $G_2^2/\text{SO}(4)$                                               | $SO(4)$, $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, $\alpha(\text{SO}(4))$ (α : an outer automorphism of $G_2^2$) |
| $G_2^2/G_2$                                                        | $G_2$, $G_2^2$, $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ |

Table 3.
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