DYNAMICS OF CHEBYSHEV POLYNOMIALS ON $\mathbb{Z}_2$

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ABSTRACT. The dynamical structure of Chebyshev polynomials on $\mathbb{Z}_2$, the ring of 2-adic integers, is fully described by showing all its minimal subsystems and their attracting basins.

1. INTRODUCTION

For each integer $m \geq 0$, the $m$-th Chebyshev polynomial is defined as

$$T_m(x) = \sum_{k=0}^{[m/2]} (-1)^k \frac{m}{m-k} \binom{m-k}{k} 2^{m-2k-1} x^{m-2k}. \quad (1.1)$$

The Chebyshev polynomials are useful in many parts of analysis, especially in approximation theory. For the fundamental properties and applications of Chebyshev polynomials, we refer to the books [14, 16].

Let $p$ be a prime number. Recently, polynomials were studied as 1-Lipschitz dynamical systems on the ring $\mathbb{Z}_p$ of $p$-adic integers. See the book [4] for the development in this topic.

A dynamical system is a continuous transformation acting on a topological space. To understand a dynamical system, we want to know how a point moves under the iteration of the transformation. A (sub)-system is called minimal if every orbit is dense in the (sub)-space. The minimality of the polynomial (or 1-Lipschitz) dynamical systems on $\mathbb{Z}_p$ was widely studied [1, 2, 3, 6, 9, 10, 12, 13, 15].

In general, the following theorem shows that a polynomial with degree at least 2 has only finite number of periodic orbits and has at most countably many minimal subsystems.

**Theorem 1** ([11], Theorem 1). Let $f \in \mathbb{Z}_p[x]$ be a polynomial of integral coefficients with degree at least 2. We have the following decomposition

$$\mathbb{Z}_p = \mathcal{P} \bigcup \mathcal{M} \bigcup \mathcal{B}$$

where $\mathcal{P}$ is the finite set consisting of all periodic points of $f$, $\mathcal{M} = \bigcup_i \mathcal{M}_i$ is the union of all (at most countably many) clopen invariant sets such that each $\mathcal{M}_i$ is a finite union of balls and each subsystem $f : \mathcal{M}_i \to \mathcal{M}_i$ is minimal, and each point in $\mathcal{B}$ lies in the attracting basin of a periodic orbit or of a minimal subsystem.

The decomposition in Theorem 1 is usually referred to as a minimal decomposition and the invariant subsets $\mathcal{M}_i$ are called minimal components.

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Even though we have a general decomposition theorem, it is difficult to describe the structure of a concrete polynomial. The exact minimal decomposition of quadratic polynomials on $\mathbb{Z}_2$ [11] and square mapping on $\mathbb{Z}_p$ (for all primes $p$) [12] have been investigated. In this note, we obtain the detailed minimal decomposition of Chebyshev polynomials $T_m$ on $\mathbb{Z}_2$.

Since $T_1(x) = x$ is trivial, we only study $T_m$ for $m \geq 2$. As will be shown in Theorem 2, the dynamical structure of $T_m$ will be simpler when $m$ is even. Our main task is to study the dynamical system $T_m$ when $m$ is odd. Note that for any odd number $m \geq 3$, there exists a unique integer $s = s(m) \geq 2$ such that $m = 2^s q + 1$ or $m = 2^s q - 1$ with $q \geq 1$ being another odd number. In fact,

$$s(m) := \max\{n \geq 2 : 2^n|(m+1) \text{ or } 2^n|(m-1)\}. \tag{1.2}$$

Then we can decompose the set of odd positive integers at least 3 as

$$\{m = 2k + 1 : k \geq 1, k \in \mathbb{Z}\} = \bigcup_{s \geq 2} \left( \bigcup_{q \text{ is odd}} \{2^s q \pm 1\} \right).$$

The following main result shows an exact minimal decomposition of the Chebyshev polynomials. It turns out that in the case that $m \geq 3$ is an odd number, the decomposition depends only on the number $s(m)$.

**Theorem 2.** Let $T_m(m \geq 2)$ be a Chebyshev polynomial defined as in the formula (1.1).

(i) If $m$ is even, then $1$ is an attracting fixed point of $T_m$, and all other points lie in the attracting basin of $1$.

(ii) If $m$ is odd with $s = s(m) \geq 2$ being defined in (1.2), then $\mathbb{Z}_2$ is decomposed as

$$\mathbb{Z}_2 = \{0, 1, -1\} \bigcup \left( E_1 \bigcup E_2 \bigcup E_3 \right),$$

where

$$E_1 = \bigcup_{n \geq 1} \bigcup_{0 \leq i < 2^n-1} E_1(n, i),$$

$$E_2 = \bigcup_{n \geq 2} \bigcup_{0 \leq i < 2^n} E_2(n, i),$$

$$E_3 = \bigcup_{n \geq 2} \bigcup_{0 \leq i < 2^n} E_3(n, i),$$

with

$$E_1(n, i) := 2^n(1 + 2i) + 2^{n+s} \mathbb{Z}_2 \quad (n \geq 1, 0 \leq i < 2^{s-1}),$$

$$E_2(n, i) := 1 + 2^n(1 + 2i) + 2^{n+s+1} \mathbb{Z}_2 \quad (n \geq 2, 0 \leq i < 2^s),$$

$$E_3(n, i) := -1 + 2^n(1 + 2i) + 2^{n+s+1} \mathbb{Z}_2 \quad (n \geq 2, 0 \leq i < 2^s)$$

being the minimal components of $T_m$.

The Chebyshev polynomials are important examples in arithmetic dynamical systems. See for example, in pages 29-30, 41, 328-336, and 380-381 of the book [18]. Recently in [17] the authors studied the Chebyshev polynomials as dynamical system on the finite fields $\mathbb{Z}/p\mathbb{Z}$ for general prime $p \geq 2$. In [8] the periodic orbits of Chebyshev polynomials considered as dynamics on the field $\mathbb{C}_p$ of complex $p$-adic number were studied.

We also point out that the prime $p = 2$ behaves differently from other primes, and the minimal decomposition of the dynamics of Chebyshev polynomials on $\mathbb{Z}_p$ with $p \geq 3$ needs much more difficult calculations.
2. Induced Dynamics of Chebyshev Polynomials on $\mathbb{Z}/p^n\mathbb{Z}$

In this section we will recall the techniques used in [11] and [12] for the minimal decomposition. The idea of these techniques was originally from [7]. It was later fully developed in [11].

Let $p \geq 2$ be an arbitrary prime number. Denote by $\mathbb{Z}_p[x]$ the set of polynomials with coefficients in $\mathbb{Z}_p$. Let $f \in \mathbb{Z}_p[x]$ and let $n \geq 1$ be a positive integer. We denote by $f_n$ the induced map of $f$ on $\mathbb{Z}/p^n\mathbb{Z}$, i.e.,

$$f_n(x \mod p^n) = f(x) \pmod{p^n}.$$ 

Many properties of the dynamics $f$ are linked to those of $f_n$. For a subset $E$ of $\mathbb{Z}_p$, denote

$$E/p^n\mathbb{Z}_p := \{x \in \mathbb{Z}_p/p^n\mathbb{Z}_p : \exists y \in E \text{ such that } x \equiv y \pmod{p^n}\}.$$ 

The following lemma can be found in [1, p. 111], [2, Theorem 1.2] and [5, Corollary 4].

**Lemma 1 ([1, 2, 5]).** Let $f \in \mathbb{Z}_p[x]$ and $E \subset \mathbb{Z}_p$ be a compact $f$-invariant set. Then $f : E \to E$ is minimal if and only if $f_n : E/p^n\mathbb{Z}_p \to E/p^n\mathbb{Z}_p$ is minimal for each $n \geq 1$.

It is clear that if $f_n : E/p^n\mathbb{Z}_p \to E/p^n\mathbb{Z}_p$ is minimal, then so is $f_m : E/p^m\mathbb{Z}_p \to E/p^m\mathbb{Z}_p$ for each $1 \leq m < n$.

Therefore, Lemma 1 shows that to obtain the minimality of $E$, it is important to investigate under what condition, the minimality of $f_n$ implies that of $f_{n+1}$.

Let $\sigma = (x_1, \ldots, x_k) \subset \mathbb{Z}/p^n\mathbb{Z}$ be a cycle of $f_n$ of length $k$ (also called a $k$-cycle), i.e.,

$$f_n(x_1) = x_2, \ldots, f_n(x_k) = x_{k+1}, \ldots, f_n(x_k) = x_1.$$ 

In this case we also say $\sigma$ is at level $n$. Let

$$X_\sigma := \bigsqcup_{i=1}^k X_i \text{ where } X_i := \{x_i + p^n t + p^{n+1}\mathbb{Z} : t = 0, \ldots, p-1\} \subset \mathbb{Z}/p^{n+1}\mathbb{Z}.$$ 

Then

$$f_{n+1}(X_i) \subset X_{i+1} \quad (1 \leq i \leq k-1) \quad \text{and} \quad f_{n+1}(X_k) \subset X_1.$$ 

In the following we shall study the behavior of the finite dynamics $f_{n+1}$ on the $f_{n+1}$-invariant set $X$ and determine all the cycles in $X$ of $f_{n+1}$, which will be called lifts of $\sigma$ (from level $n$ to level $n + 1$). Remark that the length of a lift of $\sigma$ is a multiple of $k$.

Let $g := f^k$ be the $k$-th iterate of $f$. Then any point in $\sigma$ is fixed by $g_n$, the $n$-th induced map of $g$. Let

$$X_i := x_i + p^n\mathbb{Z}_p = \{x \in \mathbb{Z}_p : x \equiv x_i \pmod{p^n}\}$$

be the closed disk of radius $p^{-n}$ corresponding to $x_i \in \sigma$ and $X_\sigma := \bigsqcup_{i=1}^k X_i$ be the clopen set corresponding to the cycle $\sigma$. For $x \in X_\sigma$, denote

$$a_n(x) := g'(x) = \prod_{j=0}^{k-1} f'(f^j(x)) \quad (2.1)$$

and

$$b_n(x) := \frac{g(x) - x}{p^n} = \frac{f^k(x) - x}{p^n}. \quad (2.2)$$

The values of the functions $a_n$ and $b_n$ are important for our purpose. They define an affine map

$$\Phi(x, t) = b_n(x) + a_n(x) t \quad (x \in X_\sigma, t \in \{0, \ldots, p-1\}).$$
The 1-order Taylor expansion of \( g \) at \( x \) implies that for \( 0 \leq t \leq p - 1 \),
\[
g(x + p^nt) \equiv x + p^n b_n(x) + p^n a_n(x) t \equiv x + p^n \Phi(x, t) \pmod{p^{2n}}.
\] (2.3)
The function \( \Phi(x, \cdot) \) is usually considered as an induced function from \( \mathbb{Z}/p\mathbb{Z} \) to \( \mathbb{Z}/p\mathbb{Z} \) by taking \( \mod{p} \). We will keep the notation \( \Phi(x, \cdot) \) if there is no confusion. Then the formula (2.3) implies that \( g_{n+1} : X_i \rightarrow X_i \) is conjugate to the linear map
\[
\Phi(x_i, \cdot) : \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}.
\]
Thus \( \Phi(x_i, \cdot) \) is a linearization of \( g_{n+1} : X_i \rightarrow X_i \).

As proved in Lemma 1 of [11], the coefficient \( a_n(x) \pmod{p} \) is always constant on \( X_i \) and the coefficient \( b_n(x) \pmod{p} \) is also constant on \( X_i \) but under the condition \( a_n(x) \equiv 1 \pmod{p} \). For simplicity, sometimes we write \( a_n \) and \( b_n \) without mentioning \( x \).

In this note, we study the dynamical systems of Chebyshev Polynomials on \( \mathbb{Z}_2 \). So we assume \( p = 2 \) in the reminder of this section. For the general prime \( p \), we refer the readers to [11, 12].

From the values of \( a_n \) and \( b_n \), one can predict the behaviors of \( f_{n+1} \) on \( X_\sigma \), by the linearity of the map \( \Phi = \Phi(x, \cdot) \):

(a) If \( a_n \equiv 1 \pmod{2} \) and \( b_n \neq 0 \pmod{2} \), then \( \Phi \) preserves a single cycle of length 2. So \( f_{n+1} \) restricted to \( X_\sigma \) preserves a single cycle of length \( 2k \). In this case we say \( \sigma \) grows. Furthermore, we say \( \sigma \) strongly grows if \( a_n \equiv 1 \pmod{4} \) and \( b_n \equiv 1 \pmod{2} \), and \( \sigma \) weakly grows if \( a_n \equiv 3 \pmod{4} \) and \( b_n \equiv 1 \pmod{2} \).

(b) If \( a_n \equiv 1 \pmod{2} \) and \( b_n \equiv 0 \pmod{2} \), then \( \Phi \) is the identity. So \( f_{n+1} \) restricted to \( X_\sigma \) preserves \( p \) cycles of length \( k \). In this case we say \( \sigma \) splits. Furthermore, we say \( \sigma \) strongly splits if \( a_n \equiv 1 \pmod{4} \) and \( b_n \equiv 0 \pmod{2} \), and \( \sigma \) weakly splits if \( a_n \equiv 3 \pmod{4} \) and \( b_n \equiv 0 \pmod{2} \).

(c) If \( a_n \equiv 0 \pmod{2} \), then \( \Phi \) is constant. So \( f_{n+1} \) restricted to \( X_\sigma \) preserves one cycle of length \( k \) and the remaining points of \( X_\sigma \) are mapped into this cycle. In this case we say \( \sigma \) grows tails.

Sometimes, the behavior of cycles will be inherited. For example, we have the following lemma.

**Lemma 2 ([7], see also [11], Proposition 1).** If a cycle grows tails at a certain level, then its single lift also grows tails.

As corollary of Lemma 2, one has the following proposition.

**Proposition 1 ([11], p.2123).** If \( \sigma = (x_1, \cdots, x_k) \) is a growing tails cycle at level \( n \), then \( f \) has a \( k \)-periodic orbit in the clopen set \( X_\sigma = \bigsqcup_{i=1}^k x_i + 2^n \mathbb{Z} \) with \( X_\sigma \) as its attracting basin.

The case of growing tails has been checked in Proposition 1. For growing and splitting cases, we need do further investigations. In fact, both of the cases of growing and splitting will be divided into two sub-cases.

The following lemma shows that the strong growing properties will be inherited.

**Lemma 3 ([11], Proposition 5).** Let \( \sigma \) be a cycle of \( f_n \) \((n \geq 2)\). If \( \sigma \) strongly grows then the lift of \( \sigma \) strongly grows.

Applying Lemma 3 consecutively, if \( \tilde{\sigma} \) is the lift of \( \sigma \), then \( \tilde{\sigma} \) also strongly grows and again the lift of \( \tilde{\sigma} \) also strongly grows and so on. In this case, we also say that the cycle \( \sigma \) strongly grows forever.

Using Lemma 3, we deduce that a strong growing cycle produces a minimal component.
Proposition 2. If $\sigma = (x_1, \cdots, x_k)$ is a strongly growing cycle at level $n$, then $f$ restricted onto the invariant clopen set $X_{\sigma} = \bigsqcup_{i=1}^k x_i + p^i \mathbb{Z}_p$ is minimal.

Proof. By Lemma 3, the cycle $\sigma$ strongly grows forever. Hence, $f_m$ is a single cycle (thus minimal) on $X_{\sigma}/p^m \mathbb{Z}_p$ for each $m \geq n$. Therefore, by Lemma 1, $f$ is minimal on $X_{\sigma}$. \qed

3. Dynamics of Chebyshev Mapping on $\mathbb{Z}_2$

Let $v_2(\cdot)$ be the $2$-adic valuation on $\mathbb{Z}_2$. We will first calculate the $2$-adic valuations of the coefficients of the Chebyshev polynomials.

Lemma 4. Let $m = 2k + 1 \geq 3$ be an odd number and

$$T_m(x) = \sum_{i \geq 0} c_{2i+1} x^{2i+1}$$

be the Chebyshev polynomial of degree $m$. Let $s = \max\{v_2(m + 1), v_2(m - 1)\}$. Then $v_2(c_3) = s$ and $v_2(c_{2i+1}) \geq s + 1$ for $1 \leq i \leq k$.

Proof. We distinguish two cases:

Case 1). If $k$ is an even integer, then $v_2(m + 1) = v_2(2k + 2) = v_2(2) + v_2(k + 1) = 1$ and $v_2(m - 1) = v_2(2k) \geq 2$. Let $s = v_2(m - 1) \geq 2$. Write $m = 2^s q + 1$ for some odd integer $q$. Then

$$T_m(x) = \sum_{i=0}^{2^s q - 1} (-1)^i \frac{2^s q + 1}{2^{s-1}q + i + 1} \left(\frac{2^{s-1}q + i + 1}{2i + 1}\right) 2^{2i} x^{2i+1}.$$ 

Making a change of variables $i = 2^{s-1}q - j$, we have

$$T_m(x) = \sum_{j=0}^{2^s q - 1} (-1)^j \frac{2^s q + 1}{2^{s-1}q + j + 1} \left(\frac{2^{s-1}q + j + 1}{2j + 1}\right) 2^{2j} x^{2j+1}.$$ 

So

$$c_{2i+1} = (-1)^j \frac{2^s q + 1}{2^{s-1}q + j + 1} \left(\frac{2^{s-1}q + j + 1}{2j + 1}\right) 2^{2j}, \text{ for } 0 \leq i \leq k.$$ 

Thus

$$v_2(c_3) = v_2\left(\frac{(2^s q + 1)(2^{s-1}q + 1)2^{s-1}q}{3!} \cdot 2^2\right) = s - 1 - 1 + 2 = s.$$ 

Furthermore,

$$c_{2i+3} = (-1)^{i+1} \frac{2^s q + 1}{2^{s-1}q + i + 2} \left(\frac{2^{s-1}q + i + 2}{2i + 3}\right) 2^{2i+2}$$

$$= (-1)^{i+1} \frac{(2^s q + 1)(2^{s-1}q + i + 1)!}{(2i + 3)!(2^{s-1}q - i - 1)!} \cdot 2^{2i+2}$$

$$= \frac{(2^{s-1}q + i + 1)(2^{s-1}q - i)}{(2i + 2)(2i + 3)} \cdot 2^2 \cdot c_{2i+1}$$

$$= \frac{(2^{s-1}q + i + 1)(2^{s-1}q - i)}{(i + 1)(2i + 3)} \cdot 2 \cdot c_{2i+1}.$$
If $1 \leq i < \min\{2^s - 1, k - 1\}$, then

$$v_2(i + 1) \leq s - 1$$

and

$$v_2(2^{s-1}q + i + 1) \geq \min\{v_2(2^{s-1}q), v_2(i + 1)\} = v_2(i + 1).$$

Observing $v_2(2i + 3) = 0$, we obtain

$$v_2(c_{2i+3}) = v_2(c_{2i+1}) + v_2(2^{s-1}q + 1 + i) + v_2(2^{s-1}q - i) - v_2(i + 1) + 1\geq v_2(c_{2i+1}) + 1. \quad (3.1)$$

If $2^s - 1 \leq i \leq k$, we claim

$$v_2(c_{2i+1}) \geq 2i.$$

In fact, since

$$c_{2i+1} = (-1)^i \frac{2^s q + 1}{2^{s-1} q + 1 + i} \left(\frac{2^{s-1} q + 1 + i}{2i + 1}\right)^{2i} = (-1)^i \frac{2^s q + 1}{2i + 1} \left(\frac{2^{s-1} q + i}{2i}\right)^{2i},$$

we have

$$v_2(c_{2i+1}) = v_2\left(\left(\frac{2^{s-1} q + i}{2i}\right)^{2i}\right) + 2i \geq 2i.$$

Noting $i \geq 2^s - 1$ and $s \geq 2$, we get

$$v_2(c_{2i+1}) \geq 2i = 2^{s+1} - 2 > s + 2. \quad (3.2)$$

By combining inequalities (3.1) and (3.2), we have $v_2(c_{2i+1}) \geq s + 1$ for $1 < i \leq k$.

Case 2). If $k$ is an odd number, then $v_2(m + 1) = v_2(2k + 2) = v_2(2) + v_2(k + 1) \geq 2$ and $v_2(m - 1) = v_2(2k) = 1$. Let $s = v_2(m + 1) \geq 2$. Write $m = 2^sq - 1$ with $q$ being an odd integer. Then by change of variables

$$T_m(x) = \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \frac{m}{m - i} \binom{m - i}{i} 2^{m-2i-1} x^{m-2i}$$

$$= \sum_{i=0}^{2^{s-1}q-1} (-1)^{i+1} \frac{2^s q - 1}{2^{s-1} q + i} \binom{2^{s-1} q + i}{2i + 1} 2^{2i} x^{2i+1}.$$

Thus

$$c_{2i+1} = (-1)^{i+1} \frac{2^s q - 1}{2^{s-1} q + i} \binom{2^{s-1} q + i}{2i + 1} 2^{2i}, \text{ for } 0 \leq i \leq k.$$

As in Case 1), we obtain

$$v_2(c_3) = v_2\left(\frac{(2^s q - 1)(2^{s-1} q)(2^{s-1} q - 1)}{3!} \cdot 2^2\right) = s - 1 - 1 + 2 = s.$$
Furthermore, the relation between \( c_{2i+3} \) and \( c_{2i+1} \) is follows:
\[
c_{2i+3} = (-1)^{i+2} \frac{2^s q - 1}{2^{s-1} q + i + 1} \left( \frac{2^{s-1} q + i - 1}{2^i} \right) 2^{2i+2}
\]
\[
= (-1)^{i+2} \frac{(2^s q - 1)(2^{s-1} q + i)!}{(2i+3)(2^{s-1} q - i - 2)!} \cdot 2^{2i+2}
\]
\[
= \frac{(2^{s-1} q + i)(2^{s-1} q - i - 1)}{(2i+2)(2i+3)} \cdot 2^2 \cdot c_{2i+1}
\]
\[
= - \frac{(2^{s-1} q + i)(2^{s-1} q - i - 1)}{(i+1)(2i+3)} \cdot 2 \cdot c_{2i+1}.
\]

If \( 1 \leq i < \min\{2^s - 1, k - 1\} \), then
\[
v_2(i + 1) \leq s - 1
\]
and
\[
v_2(2^{s-1} q - i - 1) \geq \min\{v_2(2^{s-1} q), v_2(i + 1)\} = v_2(i + 1).
\]
Notice that \( v_2(2i + 3) = 0 \). Then
\[
v_2(c_{2i+3}) = v_2(c_{2i+1}) + v_2(2^{s-1} q + i) + v_2(2^{s-1} q - i - 1) - v_2(i + 1) + 1
\]
\[
\geq v_2(c_{2i+1}) + 1. \tag{3.3}
\]
Now, we claim that
\[
v_2(c_{2i+1}) \geq 2i, \quad \text{if} \ 2^s - 1 \leq i \leq k - 1.
\]
In fact, by observing
\[
c_{2i+1} = (-1)^i \frac{2^s q + i}{2^{s-1} q + i} \left( \frac{2^{s-1} q + i}{2i + 1} \right) 2^{2i} = (-1)^i \frac{2^s q + i}{2i + 1} \left( \frac{2^{s-1} q + i - 1}{2i} \right) 2^{2i},
\]
we obtain
\[
v_2(c_{2i+1}) = v_2 \left( \left( \frac{2^{s-1} q + i - 1}{2i} \right) \right) + 2i \geq 2i.
\]
Since \( i \geq 2^s - 1 \) and \( s \geq 2 \), we have
\[
v_2(c_{2i+1}) \geq 2i = 2^{s+1} - 2 > s + 2. \tag{3.4}
\]
Therefore, by inequalities (3.3) and (3.4), we conclude that \( v_2(2i + 1) \geq s + 1 \) for \( 1 < i \leq k \).

Now we can give the proof of our main theorem.

**Proof of Theorem 2.** (1) If \( m \) is even, then \( T_m \) induces the constant map 1 on \( Z_2/2Z_2 \).
Thus at level 1, the point 0 is sent to the point 1 and \( \sigma = (1) \subset Z_2/2Z_2 \) is a fixed point.
Furthermore, we can check
\[
a_1(1) = (T_m)'(1) \equiv 0 \mod 2.
\]
Hence \( \sigma \) grows tails. By Lemma 2, the single lift \( \tilde{\sigma} \) of \( \sigma \) grows tails.
Therefore, 1 is a fixed point, and all other points of \( 1 + 2Z_2 \) lie in the attracting basin of 1.
Since \( T_m(2Z_2) \subset 1 + 2Z_2 \), we have
\[
\lim_{k \to \infty} T_m^k(x) = \lim_{k \to \infty} T_m^k(x) = 1 \quad \forall x \in Z_2.
\]
(2) Assume that \( m \) is odd. It is easy to check that 0, 1, \(-1\) are fixed points of \( T_m \).
By Lemma 4, it can be checked that \( T_m'(x) \equiv 1 \mod 4 \) for any \( x \in Z_2 \).
Thus a cycle \( \sigma \) with length 1 either strongly grows or strongly splits. Let \( s = s(m) \geq 2 \) be defined in (1.2).
Then \( m = 2^s q + 1 \) or \( m = 2^s q - 1 \) for some odd number \( q \geq 1 \). Since the treatments for both of the two cases are the same, and the minimal decompositions are also the same, we will omit the proof of the case \( m = 2^s q - 1 \) and we suppose that \( m = 2^s q + 1 \), for some odd number \( q \geq 1 \). We will give the minimal decomposition on \( 2\mathbb{Z}_2 \) and on \( \pm 1 + 4\mathbb{Z}_2 \) separately. Then the minimal decomposition of the whole space \( \mathbb{Z}_2 \) will be obtained directly.

**Decomposition of** \( 2\mathbb{Z}_2 \): Since 0 is a fixed point, for \( n \geq 1 \), \( \sigma = (0) \subset \mathbb{Z}/2^n\mathbb{Z} \) is a cycle of length 1 of \( (T_m)_n \) at level \( n \). Hence \( \sigma \) strongly grows or strongly splits. However, by definition (2.2) of \( b_n \),

\[
b_n(0) \equiv 0 \pmod{2}.
\]

So \( \sigma \) strongly splits. Thus \( \sigma_1 = (0) \) and \( \sigma_2 = (2^n) \subset \mathbb{Z}/2^{n+1}\mathbb{Z} \) are two lifts of \( \sigma \) at level \( n + 1 \). It can be checked that \( \sigma_1 \) splits. Let \( x_0 = 2^n + t2^{n+1} \) for some \( t \in \mathbb{Z}_2 \). Then

\[
T_m(x_0) - x_0 = (c_1 - 1)x_0 + \sum_{i=1}^{2^{s-1}q} c_{2i+1}x_0^{2i+1}.
\]

(3.5)

By Lemma 4, we get that

\[
v_2(T_m(x_0) - x_0) = n + s.
\]

By Lemma 3, we deduce that \( \sigma_2 \) strongly splits \( s - 1 \) times then all lifts at level \( n + s - 1 \) grow forever. By Proposition 2, we have the following decomposition

\[
2\mathbb{Z}_2 = \{0\} \bigcup \left( \bigcup_{n \geq 1} \left( \bigcup_{0 \leq i < 2^{s-1}} 2^n(1 + 2i) + 2^{n+s}\mathbb{Z}_2 \right) \right),
\]

with fixed point 0 and the minimal components:

\[
2^n(1 + 2i) + 2^{n+s}\mathbb{Z}_2 \ (n \geq 1, 0 \leq i < 2^{s-1}).
\]

**Decomposition of** \( \pm 1 + 4\mathbb{Z}_2 \): Let \( x_0 = 1 + 2^n t \) for some \( n \geq 2 \) and \( t \in 1 + 2\mathbb{Z}_2 \). Then

\[
T_m(1 + 2^n t) - (1 + 2^n t) = \sum_{i=0}^{k} c_{2i+1}(1 + 2^n t)^{2i+1} - (1 + 2^n t)
\]

\[
= (1 + 2^n t) \left( c_1 - 1 + \sum_{i=1}^{k} c_{2i+1}(1 + 2^n t)^{2i} \right).
\]

Since 1 is a fixed point of \( T_m \), we have

\[
\sum_{i=0}^{k} c_{2i+1} = 1,
\]

(3.6)

and for \( 1 \leq i \leq k \),

\[
c_{2i+1}(1 + 2^n t)^{2i} - c_{2i+1} = \sum_{j=1}^{2i} c_{2i+1} \binom{2i}{j} 2^jn^j.
\]

Note that for \( n \geq 2 \), we have

\[
v_2 \left( \binom{2i}{j} 2^jn^j \right) > n + 1 \text{ for all } 1 < j \leq 2i.
\]
Thus for $1 < i \leq k$,
\[ v_2(c_{2i+1}(1 + 2^n t)^{2i} - c_{2i+1}) = v_2(c_{2i+1}) + n + 1. \]  
(3.7)

By Lemma 4 and equalities (3.6) and (3.7),
\[ v_2(T_m(1 + 2^n t) - (1 + 2^n t)) = v_2(2^{n+1}c_3) = s + n + 1. \]  
(3.8)

For all $n \geq 2$, the cycle (1) of $(T_m)_n$ always splits to be two cycles (1) and $(1 + 2^n)$ of $(T_m)_{n+1}$. Let us consider the cycle $(1 + 2^n)$ of $(T_m)_{n+1}$.

By (3.8) we have
\[ b_{n+1}(1 + 2^n) = T_m(1 + 2^n) - (1 + 2^n) \equiv 0 \, (\text{mod } 2). \]

Thus the cycle $(1 + 2^n)$ strongly splits.

The formula (3.8) also implies that
\[ b_{n+i}(1 + 2^n + 2^{n+i}h) \equiv 0 \, (\text{mod } 2) \quad \text{if } 0 \leq i \leq s - 1 \text{ and } 0 \leq h \leq 2^i \]

and
\[ \forall x \in 1 + 2^n \mathbb{Z}_2, \quad b_{n+i+s}(x) \not\equiv 0 \, (\text{mod } 2). \]

Thus all the lifts of $(1 + 2^n)$ split $s - 1$ times then all lifts at level $n + s + 1$ strongly grow. Therefore, for each $n \geq 2$, $1 + 2^n \mathbb{Z}_2$ consists of $2^s$ minimal components
\[ 1 + 2^n(1 + 2i) + 2^{n+s+1} \mathbb{Z}_2 \quad (0 \leq i < 2^s). \]

Similarly, for each $n \geq 2$, we can decompose $-1 + 2^n \mathbb{Z}_2$ as $2^s$ minimal components
\[ -1 + 2^n(1 + 2i) + 2^{n+s+1} \mathbb{Z}_2 \quad (0 \leq i < 2^s). \]

We can now give the minimal decomposition of $\pm 1 + 4 \mathbb{Z}_2$ as follows
\[ \pm 1 + 4\mathbb{Z}_2 = \{ \pm 1 \} \bigcup \left( \bigcup_{n \geq 2} \left( \bigcup_{0 \leq i < 2^s} \pm 1 + 2^n(1 + 2i) + 2^{n+s+1} \mathbb{Z}_2 \right) \right). \]

\[ \square \]

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