Dynamics of Totally Constrained Systems
I. Classical Theory

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ABSTRACT

This is the first of a series of papers in which a new formulation of quantum theory is developed for totally constrained systems, that is, canonical systems in which the hamiltonian is written as a linear combination of constraints $h_\alpha$ with arbitrary coefficients. The main purpose of the present paper is to make clear that classical dynamics of a totally constrained system is nothing but the foliation of the constraint submanifold in phase space by the involutive system of infinitesimal canonical transformations $Y_\alpha$ generated by the constraint functions. From this point of view it is shown that statistical dynamics for an ensemble of a totally constrained system can be formulated in terms of a relative distribution function without gauge fixing or reduction. There the key role is played by the fact that the canonical measure in phase space and the vector fields $Y_\alpha$ induce natural conservative measures on acausal submanifolds, which are submanifolds transversal to the dynamical foliation. Further it is shown that the structure coefficients $c^\gamma_{\alpha\beta}$ defined by $\{h_\alpha, h_\beta\} = \sum_\gamma c^\gamma_{\alpha\beta} h_\gamma$ should weakly commute with $h_\alpha$, $\sum_\gamma \{h_\gamma, c^\gamma_{\alpha\beta}\} \approx 0$, in order that the description in terms of the relative distribution function is consistent. The overall picture on the classical dynamics given in this paper provides the basic motivation for the quantum formulation developed in the subsequent papers.
1 Introduction

In the canonical approach to quantum gravity the Dirac quantization prescription is widely adopted, in which the classical constraints $h_\alpha \approx 0$ are formulated as constraints on physical states of the form

$$h_\alpha \Phi = 0. \quad (1.1)$$

As is well-known, however, this leads to the frozen formalism in which the dynamical evolution equation is lost. This problem is closely related with the fact that operators corresponding to the physical quantities which play the role of time are excluded from observables in this formulation[^2]. Further the Dirac quantization of general relativity does not give a mathematically well-defined formulation apart from the regularization and the operator ordering problems because Eq. (1.1) does not have normalizable solutions in general even if the spatial diffeomorphism freedom is eliminated before quantization.

As discussed in Ref.[^3], this difficulty comes from the too formal application of the Dirac procedure. Since the hamiltonian is written as a linear combination of the constraint functions in general relativity, the hamiltonian constraints carry all the information on dynamics. Hence if we formulate the hamiltonian constraint as the condition on state vectors as above, each state vector becomes a dynamical object. This should be compared with the ordinary quantum mechanics. There state vectors are used to describe dynamics of a system, but each state is not a dynamical object. Dynamics is described by a one-parameter family of states $\Phi(t)$ satisfying the Schrödinger equation. Each state vector in this family merely carries information by possible measurements at each instant $t$.

This observation indicates that we should impose the hamiltonian constraints on the object which picks up all the possible state vectors allowed by dynamics. The most natural object of this nature will be the probability amplitude $\Psi(\Phi)$ which assigns the probability to each state vector, taking account of the probabilistic nature of quantum mechanics.

Here one may notice that a similar phenomenon occurs in the classical dynamics of a system with hamiltonian constraints. For simplicity let us consider a system with a single hamiltonian constraint $h \approx 0$ on a phase space $\Gamma$. If we reduce this system to a canonical system without constraint, dynamics is described by a curve $\gamma_0(t)$ in a reduced phase space $\Gamma_0$ which is a solution to a canonical equation of motion. This curve corresponds to the family of states $\Phi(t)$ in quantum mechanics.

On the other hand in the original phase space this curve corresponds to a curve $\gamma$ contained in the constraint hypersurface $\Sigma_H$, which corresponds to $\Psi$ above. This analogy becomes better if we consider an ensemble of systems instead of a single system, for which $\gamma_0(t)$ is replaced by a family of distribution functions $\rho_0(t)$ on $\Gamma_0$ and $\gamma$ by a distribution function $\rho$ on $\Gamma$. Clearly $\rho$ does not represent a state but is a dynamical object which picks up possible states allowed by dynamics. Hence $\rho$ should be constant along each curve in $\Gamma$ corresponding to a solution
to the equation of motion in $\Gamma_0$. This implies that it is not normalizable on $\Gamma$. This may be regarded as the essential reason why the solutions to Eq. (1.1) is unnormalizable. It is obviously absurd to postulate dynamics so that it picks up $\rho$ which is normalizable in $\Gamma$. Applying the quantum hamiltonian constraints on states just corresponds to such an approach.

This observation suggests that the investigation of the structure of classical statistical dynamics of totally constrained systems will shed a good light on how to find a consistent quantum formulation of them. This is the motivation of the present paper. Since a variety of forms exist for the canonical formulation of gravity and since the structure of the problem is common in all the theories with general covariance, we consider a generic totally constrained system in most part of the paper.

The organization of the paper is as follows. First in §2 we consider a simple totally constrained system obtained by embedding a canonical system without constraint to a larger phase space in order to find how to interpret the unnormalizable distribution function on the extended phase space. Then in §3 on the basis of the result obtained there we describe how to formulate the statistical dynamics of a generic totally constrained system with a single constraint without reduction and not referring to special time variables. Further the general structure of reduction and its freedom is examined because it is relevant to the time variable problem. In particular by applying it to the totally constraint system describing a relativistic particle in curved spacetime, it is shown that the background spacetime should have a Killing vector in order that there is a natural reduction of this system. In the subsequent two sections the formulation obtained for a single constraint system is extended to a multiple constraint system. First in §4 an overview on the canonical structure of general relativity in terms of the ADM variables is given in order to make clear that dynamics of a totally constrained system with multiple constraints is completely determined by the foliation of the constraint submanifold by the involutive system of the infinitesimal canonical transformations generated by the constraint functions. Then in §5 based on this viewpoint, statistical dynamics for multiple constraint systems is formulated in terms of the relative distribution function by proving the existence of natural conservative induced measures on acausal submanifolds. Section 6 is devoted to discussion.

2 Embedding of an unconstrained system into a constrained system

In this section we study the dynamics of a simple constrained system obtained by embedding an unconstrained canonical system into a larger phase space. The main purpose is to find the way how to formulate the dynamics of a constrained system and its ensemble without reducing it into an unconstrained one.
2.1 Canonical system

A canonical dynamical system with no constraint is specified by a triplet \((\Gamma, \omega, h)\) of a phase space, a symplectic form and a hamiltonian\(^4\). The phase space \(\Gamma\) is a \(2n\)-dimensional smooth manifold, the symplectic form \(\omega\) is a closed non-degenerate 2-form on \(\Gamma\),

\[
d\omega = 0, \quad \Omega := \frac{1}{n!} \wedge^n \omega = \frac{1}{n!} \omega^n \neq 0,
\]

and the hamiltonian \(h\) is a smooth function on \(\mathbb{R} \times \Gamma\).

Let \(\mathcal{F}(\Gamma)\) and \(\mathcal{A}(\Gamma)\) be the sets of all smooth functions and all smooth vector fields on \(\Gamma\), respectively. Then for any \(f \in \mathcal{F}(\Gamma)\), \(\omega\) uniquely determines a vector field \(X_f \in \mathcal{A}(\Gamma)\), which is called the infinitesimal canonical transformation generated by \(f\), through the equation

\[
df = -I_{X_f}\omega \quad \Leftrightarrow \quad Vf = \omega(V, X_f) \quad \forall V \in \mathcal{A}(\Gamma),
\]

where \(I_V\) is the inner product operator which maps a \(p\)-form \(\chi\) to a \((p - 1)\)-form defined by

\[
(I_{X_f}\chi)(V_1, \ldots, V_{p-1}) = \chi(X, V_1, \ldots, V_{p-1}).
\]

From the identity

\[
\mathcal{L}_X = I_X \circ d + d \circ I_X
\]

the infinitesimal transformation \(X_f\) preserves \(\omega\) and \(\Omega\);

\[
\mathcal{L}_{X_f}\omega = 0, \quad \mathcal{L}_{X_f}\Omega = 0.
\]

Conversely any vector field which satisfies this equation is an infinitesimal canonical transformation generated by some function on \(\Gamma\), at least locally.

In terms of this infinitesimal canonical transformation the Poisson bracket of two functions \(f\) and \(g\) are defined by

\[
\{f, g\} := -X_{fg} = -\omega(X_f, X_g).
\]

It follows from this definition that

\[
X_{\{f,g\}} = -[X_f, X_g].
\]

Thus the correspondence \(f \mapsto X_f\) gives a homomorphism from \(\mathcal{F}(\Gamma)\) into \(\mathcal{A}(\Gamma)\) as Lie algebras, whose kernel is constant functions.

In appropriate local coordinates \((q^j, p_j)\) \(\omega\) can be always written as

\[
\omega = dp_j \wedge dq^j,
\]
In this local coordinate system $X_f$ is expressed as

$$X_f = \frac{\partial f}{\partial p_j} \frac{\partial}{\partial q^j} - \frac{\partial f}{\partial q^j} \frac{\partial}{\partial p_j},$$

(2.10)

which leads to the familiar expression

$$\{f, g\} = \frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q^j}.$$  

(2.11)

Finally the dynamics in the phase space is determined by the hamiltonian $h$ in the following way. Let the canonical coordinate of $R$ in $R \times \Gamma$ be $t$, and for each value of $t$ let $Y := X_h$ be the infinitesimal canonical transformation generated by $h$ regarded as a function on $\Gamma$. Then for a single system, its possible histories are given by the integration curves of the vector field $\partial_t + Y$ on $R \times \Gamma$ when $t$ is regarded as the time variable, and the value of $f \in \mathcal{F}(\Gamma)$ along each curve, when regarded as a function of $t$, satisfies the equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, h\}. $$

(2.12)

In particular in a local coordinate system $(t, u^a)$ of $R \times \Gamma$ each integral curve follows the canonical equation of motion

$$\frac{du^a}{dt} = Y^a = \{u^a, h\}. $$

(2.13)

On the other hand the behavior of an ensemble of the system is described by a distribution function $\rho \in \mathcal{F}(R \times \Gamma)$, which satisfies the two conditions: i) $\int_\Gamma \rho \Omega = 1$ for each $t$, and ii) $\rho \Omega$ is preserved by the equation of motion. This second condition yields the equation of motion for $\rho$,

$$\mathcal{L}_{\partial_t + Y}(\rho \Omega) = 0 \iff (\partial_t + Y) \rho = 0,$$

(2.14)

which implies that $\rho$ is constant along each integration curves in $R \times \Gamma$. When we introduce $\rho_t$ defined by $\rho_t(u) = \rho(t, u)(u \in \Gamma)$, the expectation value of $f \in \mathcal{F}(R \times \Gamma)$ at a time $t$ is given by

$$< f >_t = \int_\Gamma f \rho_t \Omega,$$

(2.15)

and its time-derivative by

$$\frac{d}{dt} < f >_t = \int_\Gamma \frac{\partial}{\partial t} (f \rho_t) \Omega = \int_\Gamma \left( \frac{df}{dt} \rho_t \Omega + f \rho_t \mathcal{L}_Y \Omega \right)$$

(2.16)

$$= \int_\Gamma \frac{df}{dt} \rho_t \Omega.$$  

(2.17)

In particular, when $f$ is a constant of motion, i.e., $\frac{df}{dt} = 0$, $< f >_t$ becomes time-independent.
2.2 Embedding into a totally constrained system

The above canonical dynamical system can be embedded into totally constrained canonical systems with larger phase spaces by various ways. Here we consider the simplest one.

Let \((\tilde{\Gamma}, \tilde{\omega}, \tilde{h})\) be a canonical system defined by

\[
\tilde{\Gamma} := \mathbb{R}^2 \times \Gamma \ni (q^0, p_0, u),
\]

\[
\tilde{\omega} := dp_0 \wedge dq^0 + \omega,
\]

\[
\tilde{h}(q^0, p_0, u) := h(q^0, u) + p_0.
\]

Then the infinitesimal canonical transformation \(\tilde{Y}\) generated by \(\tilde{h}\) is expressed as

\[
\tilde{Y} = Y + \partial_{q^0} - \partial_{q^0} h \partial_{p_0}.
\]

Hence by the projection \(\pi\) defined by

\[
\pi : \tilde{\Gamma} \rightarrow \mathbb{R} \times \Gamma
\]

\[
\begin{array}{ccc}
\psi & \rightarrow & \mathbb{R} \times \psi \\
(q^0, p_0, u) & \rightarrow & (q^0, u)
\end{array}
\]

it is mapped to \(\pi_* \tilde{Y} = Y + \partial_t\), and each integral curve of \(\tilde{Y}\) to a solution to the equation of motion in \(\mathbb{R} \times \Gamma\). Therefore, noting that \(\tilde{h}\) is conserved and \(\pi\) is injective on each \(\tilde{h} = \text{const}\) surface, one sees that the original canonical system is equivalent to the extended canonical system with a constraint \(\tilde{h} = \text{const}\).

In this embedding only the integral curves of \(\tilde{Y}\) in the extended phase space, which we call the hamiltonian flow, have a physical significance, while the canonical time variable for the extended system, which is denoted by \(\tau\), just plays the role of a parameter of these curves. Hence for an arbitrary function \(N(\tau)\) the system with \(\tilde{h}\) replaced by \(H = N \tilde{h}\) is also equivalent to the original system under the constraint \(\tilde{h} = \text{const}\). In particular for the special choice of the constraint, \(\tilde{h} = 0\), this equivalence holds for an arbitrary function \(N \in F(\mathbb{R} \times \tilde{\Gamma})\) since

\[
X_{N \tilde{h}} = N \tilde{Y} + \tilde{h} X_N \approx N \tilde{Y}.
\]

where \(A \approx B\) means that \(A = B\) under the constraint. We express this last situation by saying that the original canonical system is embedded into the totally constraint system \((\tilde{\Gamma}, \tilde{\omega}, \tilde{h})\). In this expression \(\tilde{h}\) is understood to play the double roles, one as the generating function of the hamiltonian flow and the other giving the constraint \(\tilde{h} = 0\).

2.3 Distribution function on the extended phase space

The distribution function \(\rho\) for the unconstrained system, if it is regarded as a function on \(\tilde{\Gamma}\), is constant along the hamiltonian flow from Eq. (2.14) and Eq. (2.21).
This is quite natural since each pure dynamical state of the constrained system is not represented by a point but by a hamiltonian flow line in the extended phase space. Hence, taking account of the constraint, it is natural to consider the distribution function $\tilde{\rho}$ on the extended phase space defined by

$$\tilde{\rho} = \rho \delta(\tilde{h}),$$

(2.24)

where $\delta(*)$ is the delta function. From this definition it follows that $\tilde{\rho}$ is characterized as a distribution on $\tilde{\Gamma}$ satisfying the two equations,

$$\tilde{Y} \tilde{\rho} = 0,$$

(2.25)

$$\tilde{h} \tilde{\rho} = 0.$$  

(2.26)

Here by a distribution on $\tilde{\Gamma}$ we mean a functional $f$ on the space of smooth functions $\phi$ with compact supports in $\tilde{\Gamma}$, which is expressed as

$$< f, \phi > := \int_{\tilde{\Gamma}} f\phi \left| I_{\tilde{\Omega}} \right|$$

(2.27)

when it can be identified with a function on $\tilde{\Gamma}$.

Since the original phase space $\Gamma$ at time $t$ can be identified with the intersection of the $q^0 = t$ hypersurface $\Sigma_t$ and the constraint hypersurface $\Sigma_H$ in $\tilde{\Gamma}$, the expectation value of $f \in \mathcal{F}(\mathbb{R} \times \Gamma)$ at a time $t$ is expressed in terms of $\tilde{\rho}$ and $E_t := \delta(q^0 - t)$ as

$$< f >_{t=} = < E_t \tilde{\rho}, f >.$$  

(2.28)

This fixes the interpretation and the normalization of the distribution function $\tilde{\rho}$ in the extended phase space.

From the dynamical point of view these $q^0 =$const surfaces have no special significance in the extended phase space, apart from that they are ‘Cauchy surfaces’ for $\tilde{\rho}$. In fact we can easily extend the expression (2.28) to that for the expectation value on an arbitrary hypersurface $\Sigma$ which is transversal to all the hamiltonian flow lines. Let us call such a surface a \textit{maximal acausal hypersurface} and denote the corresponding expectation value by $< f >_\Sigma$. Then for two maximal acausal hypersurfaces $\Sigma_1$ and $\Sigma_2$, $< f >_{\Sigma_1}$ and $< f >_{\Sigma_2}$ should coincide for any constant of motion $f$. If we express $< f >_\Sigma$ in terms of a measure $\mu_\Sigma$ as

$$< f >_\Sigma = \int_{\Sigma} f d\mu_\Sigma,$$  

(2.29)

the above condition implies that $\theta_* \mu_{\Sigma_1} = \mu_{\Sigma_2}$ where $\theta$ is the diffeomorphism from $\Sigma_1$ onto $\Sigma_2$ determined by the hamiltonian flow. Hence the requirement that $\mu_{\Sigma_i}$ coincides with that given by Eq.(2.28) completely determines $\mu_{\Sigma}$ for any maximal acausal hypersurface.

The explicit form of the measure is given by

$$d\mu_{\Sigma} = \tilde{\rho} I_{\tilde{Y} \tilde{\Omega}} \left| \Sigma \right|,$$  

(2.30)
where \(|\chi|_{\Sigma}\) implies that the differential form \(\chi\) is regarded as a positive measure on \(\Sigma\). In order to see this, first note that, for any constant of motion \(f\), \(fI_\tilde{Y}\tilde{\Omega}\) is a closed form from

\[
d(fI_\tilde{Y}\tilde{\Omega}) = \mathcal{L}_\tilde{Y}(f\tilde{\Omega}) - I_\tilde{Y}d(f\tilde{\Omega}) = 0.
\]

Hence by applying the Stokes’ theorem to a region bounded by \(\Sigma_1\) and \(\Sigma_2\) one gets

\[
\int_{\Sigma_1} fd\mu_{\Sigma_1} = \int_{\Sigma_2} fd\mu_{\Sigma_2}.
\]

On the other hand from the identity

\[
d\phi \land I_\tilde{Y}\tilde{\Omega} = -I_\tilde{Y}(d\phi \land \tilde{\Omega}) + I_\tilde{Y}d\phi\tilde{\Omega} = -\{\tilde{h}, \phi\}\tilde{\Omega},
\]

one can rewrite the above expression for \(< f >_{\Sigma}\) as

\[
< f >_{\Sigma} = < E_{\Sigma}\tilde{\rho}, f >;
\]

\[
E_{\Sigma} = |\{\tilde{h}, \phi\}|\delta(\phi - \tau),
\]

if the maximal acausal surface \(\Sigma\) is specified by the condition \(\phi = \tau(=\text{const})\). It is easy to see that the right-hand-side of this equation coincides with Eq.(2.28) for \(\phi = q^0\) and \(\tau = t\).

2.4 Probability interpretation of the relative distribution function

Since \(\tilde{\rho}\) is a distribution on \(\tilde{\Gamma}\), we are tempted to interpret it simply as giving a probability measure for measurements of physical quantities defined on \(\tilde{\Gamma}\). Following this interpretation, the probability for a set of quantities \(f_1, \cdots \in \mathcal{F}(\tilde{\Gamma})\) to take value in a given set of ranges \(\Delta_1, \cdots \subseteq \mathbb{R}\) should be given by

\[
\Pr(f_1 \in \Delta_1, \cdots) = C < \tilde{\rho}, E_{f_1}(\Delta_1) \cdots >,
\]

where \(C\) is a normalization constant and \(E_{f}(\Delta)\) is the characteristic function of the region where the value of \(f\) is contained in \(\Delta\). However, this naive interpretation does not work by itself because the integration of \(\tilde{\rho}\) over the whole extended phase space diverges:

\[
< \tilde{\rho}, 1 > = \int dt < 1 >_t = +\infty.
\]

Hence \(\tilde{\rho}\) cannot give a finite measure on \(\tilde{\Gamma}\) by simple renormalization. Nevertheless it can be interpreted as giving a relative probability density under some limited situations.

To see this, let us define the conditional probability for physical quantities \(f_1, \cdots\) to take value in \(\Delta_1, \cdots\) under the condition \(\phi = \tau\), by

\[
\Pr(f_1 \in \Delta_1, \cdots | \phi = \tau) := \lim_{|\Delta_\phi| \to 0} \frac{\Pr(f_1 \in \Delta_1, \cdots, \phi \in \Delta_\phi)}{\Pr(\phi \in \Delta_\phi)}.
\]
Then the expectation value of any $f \in \mathcal{F}(\hat{\Gamma})$ determined from this probability coincides with the right-hand side of Eq. (2.33), if and only if $\{\tilde{h}, \phi\}$ is given by some function of $\phi$ on $\Sigma_{H}$. This condition requires that $\phi$ is expressed in terms of a function $g$ and some constant of motion $k$ as

$$\phi \approx g(q^0 + k).$$

Let us call a function satisfying this condition a good time variable.

When a maximal acausal hypersurface $\Sigma$ is given, we can always find a good time variable which is constant on $\Sigma$. However, its freedom is just a rescaling of the variable so that it is in general impossible to find a good time variable which is constant on each of more than two given acausal hypersurfaces. Though this fact has no significance in the classical framework, it seems to have a deep implication in the quantum framework in connection with the unitarity problem as will be discussed in the next paper.

3 Non-trivial system with a single hamiltonian constraint

In this section we show that by a simple generalization of the formulae in the previous section we can discuss the dynamics of a totally constrained system with a single hamiltonian constraint without reducing it to an unconstrained system.

3.1 Dynamics in the extended phase space

Let $(\Gamma, \omega, h)$ be a totally constrained system with a single hamiltonian constraint $h = 0$, and $Y \in \mathcal{A}(\Gamma)$ be the infinitesimal canonical transformation $X_{\tilde{h}}$. Then from the consideration in the previous section, it is natural to interpret that each integration curve of $Y$ on the constraint hypersurface $\Sigma_{H}$ yields a possible time evolution of the system. Hence if $Y$ has a zero-point on $\Sigma_{H}$, it represents a solution for which any physical quantity takes a fixed value. Since such a solution is quite unphysical, we assume that $Y$ does not vanish on $\Sigma_{H}$. On the other hand if there is a closed orbit in the hamiltonian flow, it represents a completely periodic world like the anti-de Sitter spacetime. We do not consider such causality violating cases in this paper either. We further assume that the hamiltonian flow is not ergodic. This is equivalent to requiring that the foliation of $\Sigma_{H}$ by the hamiltonian flow has a locally trivial bundle structure whose fiber is homeomorphic to $R$. Thus it has a global section. We call an extension of such a global section off $\Sigma_{H}$ as hypersurface, a maximal acausal hypersurface.

When we consider an ensemble of totally constrained systems with the same structure, they are represented by a set of hamiltonian flow lines in $\Gamma$, each of which intersects with a maximal acausal hypersurface at a single point. In the
limit that the ensemble consists of very large number of members, these intersection points determine a measure $\mu_\Sigma$ on each maximal acausal hypersurface $\Sigma$. From its definition this measure is preserved by the mapping among maximal acausal hypersurfaces determined by the Hamiltonian flow. Let $c$ be a positive definite constant of motion. Then, since $cI_\Omega$ is a closed form and yields a measure with the same property on each maximal acausal hypersurface as shown in the previous section, the Radon-Nykodim derivative of $\mu_\Sigma$ by $c|I_\Omega|$ yields a function which is constant along the Hamiltonian flow on the constraint surface $\Sigma_H$. Thus if we regard this function as a distribution whose support is contained in $\Sigma_H$, we are naturally led to the distribution function $\rho$ on $\Gamma$ which satisfies

\begin{align}
Y\rho &= 0, \\
h\rho &= 0 \tag{3.1}
\end{align}

in the distribution sense. From its definition the expectation value of a physical quantity $f \in \mathcal{F}(\Gamma)$ on a maximal acausal hypersurface $\Sigma$ for the ensemble is given by

\begin{equation}
\langle f \rangle_\Sigma = \frac{\int_\Sigma f\rho|I_\Omega|}{\int_\Sigma c\rho|I_\Omega|}. \tag{3.3}
\end{equation}

In practical situations each maximal acausal hypersurface is specified by the condition $\phi = \text{const}$ in terms of a physical quantity $\phi$ on the phase space $\Gamma$. Here the constant should be a special value and the other values may not give maximal acausal hypersurface in general. We call such a function instant function.

In terms of the instant function the dynamics of the totally constrained system is formulated in the following way. First one selects an appropriate instant function $\phi_1$ from measured quantities, and specifies a maximal acausal hypersurface, say, by $\phi_1 = 0$. Then measurements of various physical quantities determines a measure $\mu_{\Sigma_1}$, which in turn determines the value of the distribution function $\rho$ on the maximal acausal hypersurface $\Sigma_1$. $\rho$ is uniquely extended over the phase space by Eqs. (3.1)-(3.2), at least around the constraint hypersurface $\Sigma_H$. Once the distribution function is determined, one can calculate the expectation value of any physical quantity at an instant specified by any instant function. Here, though the distribution function depends on the choice of the constant of motion $c$ in the above procedure, this freedom does not affect the predictions on the expectation values since $c$ and $\rho$ come into the theory always in the combination $c\rho$.

The reason why we have introduced the apparently superfluous freedom of $c$ in the definition of $\rho$ is to widen the concept of good time variables introduced in the previous section. Let $\phi$ be an instant function such that $\phi = \tau(=\text{const})$ gives a maximally acausal hypersurface for any $\tau$ in some open interval of $\mathbb{R}$. If we require that the probability measures on these hypersurfaces derived from the natural measure $\rho\Omega$ as in Eq. (2.36) coincides with the conserved measure $\mu_\Sigma$, we obtain the condition

\begin{equation}
c\{h, \phi\} \approx f(\phi), \tag{3.4}
\end{equation}

\[10\]
where \( f(\phi) \) is an appropriate function of \( \phi \). This condition is equivalent to the condition that \( \phi \) is a function of \( \phi_0 \) which is a solution to the equation
\[
cY \phi_0 \approx \text{const.} \tag{3.5}
\]
We will show later that we must allow for a nontrivial choice of \( c \) for a natural time variable to satisfy this condition even in simple cases.

### 3.2 Reduction

As clarified in the previous section, the dynamics of a totally constrained system \((\Gamma, \omega, h)\) can be described with no reference to a special time variable. Now let us study the relation of this description with that in terms of a time variable in a canonical system with no constraint which is obtained by reduction.

In general a canonical dynamical system without constraint \((\Gamma_0, \omega_0, h_0)\) is a reduction of the totally constrained system \((\Gamma, \omega, h)\) if there exists a diffeomorphic embedding
\[
\Phi : \mathbb{R} \times \Gamma_0 \to \Sigma_H \tag{3.6}
\]
which satisfies the following two conditions:

i) \( \Phi^* (\partial_t + Y_0) = kY \) \( (k \in \mathcal{F}(\Gamma)) \),

ii) \( \Phi^* \omega(Z_1, Z_2) = \omega_0(Z_1, Z_2) \) \( Z_1, Z_2 \in \mathcal{X}(t \times \Gamma_0) \),

where \( Y_0 \) is the infinitesimal canonical transformation on \( \Gamma_0 \) generated by \( h_0 \). A convenient characterization of \( \Phi \) is given by the following well-known result.

**Proposition 3.1** *The necessary and sufficient condition for the mapping \( \Phi \) to give a reduction is that the following equation holds:*
\[
\Phi^* \omega = \omega_0 - dh_0 \wedge dt. \tag{3.7}
\]

**Proof**

1) Necessity.

From the condition ii) for the reduction mapping there exists a 1-form \( \xi \) such that \( \Phi^* \omega = \omega_0 - \xi \wedge dt \). Since \( \omega \) and \( \omega_0 \) are both closed, we obtain \( d\xi \wedge dt = 0 \). From this it follows that we can choose \( \xi \) so that it is closed. Hence \( \Phi^* \omega \) can be written as \( \Phi^* \omega = \omega_0 - dp \wedge dt \) in terms of some function \( p \). Applying \( I_{\partial_t + Y_0} \) on this expression leads to
\[
I_{\partial_t + Y_0} \Phi^* \omega = d(p - h_0) - (\partial_t p + Y_0 p)dt.
\]
On the other hand from the condition ii)
\[
I_{\partial_t + Y_0} \Phi^* \omega = \Phi^*(kY \omega) = -\Phi^*(kd\omega) = 0.
\]
Hence we obtain \( dp = dh_0 + (\partial_t p + Y_0 p)dt \), which leads to the equation in the proposition.

2) Sufficiency.

If the equation in the proposition holds, the condition ii) is obvious. Further it is easy to check that \( \Phi^* I_{(\partial_t + Y_0)} \omega = I_{\partial_t + Y_0} \Phi^* \omega = 0 \). Hence there is a function \( k \) on \( \Gamma \) such that
\[
I_{\Phi^*(\partial_t + Y_0)} \omega \approx -kdh = I_k \omega.
\]
This is equivalent to the condition i).

The reduction mapping \( \Phi \) induces a function \( \phi \) and a vector field \( V \) on \( \Sigma_H \) defined by
\[
t = \Phi^* \phi, \quad \Phi^* \partial_t = V,
\]
which are related by
\[
V \phi = 1.
\]
The function \( \phi \), when extended off the constraint surface \( \Sigma_H \), yields a time variable on \( \Gamma \), and foliates \( \Gamma \) into a family of acausal hypersurfaces \( \Sigma_t = \{ u \in \Gamma | \phi(u) = t \} \).

On the other hand the vector field \( V \), when extended off \( \Sigma_H \), generates a one-parameter family of transformations \( \nu_t \) on \( \Gamma \) such that \( \nu_t(\Sigma_t) = \Sigma_{t+\tau} \), and \( \mathcal{F}(\Gamma_0) \) can be identified with the restriction to \( \Sigma_H \) of the set of functions on \( \Gamma \) that are invariant by these transformations. Hence the reduced phase space \( \Gamma_0 \) can be naturally identified with \( \Sigma_0 \cup \Sigma_H \). Under this identification the reduction mapping \( \Phi \) can be written as
\[
\Phi: \mathbb{R} \times \Gamma_0 \to \Sigma_H.
\]
Hence, when an acausal hypersurface \( \Sigma_0 \) is given, the vector field \( V \) completely determines the reduction. For this reason we call \( V \) the reduction field.

The reduction field yields a reference dynamics in describing the dynamics of the totally constrained system by time evolution, and the hamiltonian in the reduced system is essentially the generator of the deviation of the hamiltonian flow from this reference dynamics. To see this, let \( \theta_t \) be the mapping from \( \Sigma_0 \) to \( \Sigma_t \) determined by the hamiltonian flow. Then \( \eta_t = \nu_t \theta_t \) yields a family of transformations on \( \Gamma_0 = \Sigma_0 \cap \Sigma_H \) describing the deviation of the hamiltonian flow and the reference dynamics. For a given point \( u \in \Gamma \) the tangent vector \( Z_{\eta_t(u)}(t) \) of the curve \( \eta_t(u) \) is given by
\[
Z_{\eta_t(u)}(t) = -V_{\eta_t(u)} + (\nu_t)_*(kY)_{\eta_t(u)} = (\nu_t)_*(-V + kY)_{\eta_t(u)},
\]
where \( k \) is a function determined by the condition \( kY \phi \approx 1 \). Hence from the condition ii) for the reduction mapping we see that \( Z(t) \) is nothing but the infinitesimal canonical transformation \( Y_0 \) generated by \( h_0 \).

An arbitrary vector cannot be a reduction field. It must be approximately a canonical transformation as the following proposition shows.
Proposition 3.2 A vector field \( V \) on \( \Gamma \) is a reduction field if and only if it satisfies the following condition:

i) \( V h \approx 0, \)

ii) there exists a function \( \phi \) on \( \Gamma \) such that \( V \phi \approx 1 \) and \( \mathcal{L}_V \omega \wedge d\phi|_{\Sigma_H} = 0. \)

In particular any infinitesimal canonical transformation satisfying i) is a reduction field. For this case the reduced hamiltonian \( h_0 \) can be chosen to be independent of \( t. \)

Proof

1) Necessity

From the definition of the reduction field i) is obvious, and for the time variable \( \phi \) in the reduction the former half of condition ii) is satisfied by definition. Further since \( \Phi^* \omega = \omega_0 - d h_0 \wedge dt \) from Proposition 3.1

\[ \Phi^*(dI_V \omega \wedge d\phi) = dI_{\partial_t} \Phi^* \omega \wedge dt = 0. \]

This is equivalent to the latter half of ii).

2) Sufficiency

By taking the function \( \phi \) in condition ii) as a time variable, let us construct the mapping \( \Phi : \mathbb{R} \times \Gamma_0 \to \Sigma_H \) from \( \phi \) and \( V \) exactly in the same way as described below Eq.(3.9). Then it obviously follows that \( \Phi_* \partial_t = V \) and \( \Phi^* \phi = t. \) Hence from ii) we obtain

\[ 0 = \Phi^*(dI_V \omega \wedge d\phi) = d(I_{\partial_t} \Phi^* \omega) \wedge dt. \]

This implies that there exist \( h_0, \alpha \in \mathcal{F}(\mathbb{R} \times \Gamma_0) \) such that \( I_{\partial_t} \Phi^* \omega = dh_0 + \alpha dt. \) Since \( I_X^2 = 0, \) \( \alpha \) is expressed as \( \alpha = -\partial_t h_0. \) Hence \( \Phi^* \omega \) is written in terms of 2-form \( \omega_0 \) such that

\[ \Phi^* \omega = \omega_0 - dh_0 \wedge dt. \]

Since \( \omega \) is closed, \( \omega_0 \) is also closed. Further from

\[ \mathcal{L}_{\partial_t} \omega_0 \wedge dt = \mathcal{L}_{\partial_t} \Phi^* \omega \wedge dt = \Phi^*(\mathcal{L}_V \omega \wedge d\phi) = 0, \]

we obtain \( \mathcal{L}_{\partial_t} \omega_0 = 0. \) Hence from Proposition 3.1 \( \Phi \) yields a reduction mapping.

Though the reduction fields are rather restricted, we can find a reduction \( \Phi \) for which \( h_0 = \Phi^* p, \) for an arbitrarily given function \( p \) on \( \Gamma. \) To see this, let \( \phi \) be a function on \( \Gamma \) such that \( Y \phi = -\{h, \phi\} \neq 0 \) on \( \Sigma_H, \) and put \( V = X_p + \alpha X_\phi \) where \( p \) and \( \alpha \) are functions to be determined so that \( V \) is a reduction field. First from condition i) in Proposition 3.2 \( \alpha \) is uniquely determined as \( \alpha = -\{h, p\}/\{h, \phi\}. \) Since \( \mathcal{L}_V \omega = -d\alpha \wedge d\phi, \) condition ii) simply reduces to \( V \phi = -X_\phi p \approx 1. \) As \( X_\phi \) is transversal to \( \Sigma_H, \) we can always find a solution \( p \) to this equation such that \( p \) coincides with an arbitrarily given function on \( \Sigma_H. \) Since \( h_0 = \Phi^* p|_{\Sigma_H} \) and \( V p \approx Y p/Y \phi, \) this implies that \( h_0 \) can be an arbitrary function on \( \mathbb{R} \times \Gamma_0. \)

This observation shows that reduction and the corresponding reduced hamiltonian has a physical significance only when the original system has some kind of
time-translation symmetry which induces a reduction field in the phase space. In
generic cases for which no such symmetry exits it is more natural to discuss the
dynamics in the original phase space with a constraint as described in the previous
section.

3.3 Example: a relativistic particle in curved background

We illustrate the argument so far by a simple totally constrained system describing
a relativistic particle moving on a curved background \((M, g)\) with a mass which
may be position-dependent.

The action for this system is given by

\[
S_2 = -2 \int d\tau \left[ -U(x)g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu \right]^{1/2}. 
\]

This action is equivalent to

\[
S_1 = \int d\tau \left\{ p_\mu (\dot{x}^\mu - v^\mu) - 2 \left[ -Ug_{\mu\nu}v^\mu v^\nu \right]^{1/2} \right\},
\]

where \(p_\mu\) is a Lagrange multiplier. The variation of this action with respect to \(v^\mu\)
yields

\[
p_\mu = \frac{1}{N} g_{\mu\nu} v^\nu; \quad N := \frac{1}{2U} \left[ -Ug_{\mu\nu}v^\mu v^\nu \right]^{1/2}.
\]

By eliminating \(v^\mu\) in \(S_1\) with the help of this equation we get

\[
S = \int d\tau \left[ p_\mu \dot{x}^\mu - \frac{1}{2} N (g^{\mu\nu} p_\mu p_\nu + U) \right],
\]

where \(N\) is regarded as an independent variable. Hence the original system is
equivalent to the totally constrained canonical system

\[
(\Gamma, \omega, h) : \quad \Gamma = T^*M,
\]

\[
\omega = d\theta = dp_\mu \wedge dx^\mu,
\]

\[
h = \frac{1}{2} \left[ g^{\mu\nu}(x)p_\mu p_\nu + U(x) \right],
\]

where \(T^*M\) is the cotangent bundle of \(M\) and \(\theta\) is its canonical 1-form.

This totally constrained system is not simply reducible as the hamiltonian con-
straint is quadratic in the momentum unlike the system considered in §2. However,
one can discuss its dynamics in the sense discussed in this section except for special
cases. In fact, the generating vector \(Y\) of the hamiltonian flow of this system is
given by

\[
Y = p^\mu \frac{\partial}{\partial x^\mu} - \frac{1}{2} \left( \frac{\partial g^{\lambda\sigma}}{\partial x^\mu} p_\lambda p_\sigma + \frac{\partial U}{\partial x^\mu} \right) \frac{\partial}{\partial p_\mu},
\]

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which vanishes at points where $p_\mu = 0$ and $\partial U/\partial x^\mu = 0$. Hence if there is no point such that $U = 0$ and $\partial U/\partial x^\mu = 0$, the hamiltonian flow has global acausal hypersurfaces.

As shown in the previous section, we can always find a reduction of this system into an unconstrained system. However, in order for the corresponding reduction field to be associated with some time translation of the system, the system must have a symmetry. To see this, let $K$ be a vector field on $M$. Then it induces a vector field $\tilde{K}$ on the phase space $T^*M$ which is expressed in the local coordinate system $(x^\mu, p_\nu)$ as

$$\tilde{K} = K^\mu \frac{\partial}{\partial x^\mu} - p_\nu \partial_\mu K^\nu \frac{\partial}{\partial p_\mu}. \quad (3.18)$$

It is easy to see that this is an infinitesimal canonical transformation generated by the function $p_\mu K^\mu$. The condition for this field to be a reduction field is given by the following proposition.

**Proposition 3.3** The vector field $\tilde{K}$ yields a reduction field of a totally constrained system $(T^*M, \omega, h)$ if and only if $\tilde{K} h \approx 0$. In particular for the constraint $h$ given by Eq.(3.16) this is equivalent to the condition that $K$ is a Killing vector of the metric $\tilde{g} = U g$.

**Proof**

Since $\tilde{K}$ is an infinitesimal canonical transformation the former half of the proposition is obvious from Proposition 3.2. For $h$ given by Eq.(3.16) the condition is expressed as

$$0 \approx \tilde{K}(U^{-1} h) = -\nabla_\mu (\tilde{g}_{\nu\alpha} K^\alpha) \tilde{g}^\mu\nu \tilde{g}^\sigma\lambda, \quad p_\lambda p_\sigma,$$

where $\nabla_\mu$ is a covariant derivative with respect to $\tilde{g}$. This equation is equivalent to

$$\nabla_\mu (\tilde{g}_{\nu\alpha} K^\alpha) + \nabla_\nu (\tilde{g}_{\mu\alpha} K^\alpha) = 0,$$

which implies that $K$ is a Killing vector of the metric $\tilde{g}$. 

This restriction on the system is closely related with the condition for the system to have a good time variable which is independent of $p_\mu$. To see this, let $\phi$ be a function on $M$. Then, since $Y \phi$ is now given by

$$Y \phi = g^{\mu\nu} p_\nu \frac{\partial \phi}{\partial x^\mu}, \quad (3.19)$$

the condition for $\phi$ to be a good time variable, Eq.(3.5), is written as

$$p^\mu p^\nu \nabla_\mu \nabla_\nu \phi - \frac{1}{2} \nabla_\mu \phi \nabla_\mu U \approx 0, \quad (3.20)$$

which is equivalent to

$$U \nabla_\mu \nabla_\nu \phi = -\frac{1}{2} g_{\mu\nu} \nabla_\lambda \phi \nabla_\lambda U. \quad (3.21)$$
This last equation is written in terms of the covariant derivative with respect to the metric \( \tilde{g}_{\mu\nu} = U g_{\mu\nu} \) as

\[
\tilde{\nabla}_\mu(U \tilde{\nabla}_\nu \phi) + \tilde{\nabla}_\nu(U \tilde{\nabla}_\mu \phi) = 0.
\] (3.22)

Hence the metric \( \tilde{g}_{\mu\nu} \) must have a Killing vector \( K \) such that

\[
K^\mu = \nabla^\mu \phi.
\] (3.23)

In particular \( \tilde{g}_{\mu\nu} \) must be static.

Putting these two requirements together, we find that there should exists a function \( \phi \) such that \( \nabla^\mu \phi \) is a Killing vector of \( \tilde{g} \) and \( \nabla_\mu \phi \nabla^\mu \phi = \text{const} \) in order that there is a good time variable which is a function on \( M \) and whose gradient field generates a reduction field. It is easy to see that these conditions are satisfied if and only if \( g \) has a static Killing vector along which \( U \) is constant (cf. Kuchar’s article in Ref. [2]. For such cases the reduced hamiltonian is given by \( p_\mu \nabla^\mu \phi \).

For example for a relativistic free particle in Minkowski spacetime for which \( U = m^2 = \text{const} \), the translation Killing vector \( a^\mu \partial_\mu \) satisfies these conditions when \( a^\mu \) is a constant time-like vector. The time function and the reduced hamiltonian are given by \( \phi = a_\mu x^\mu \) and \( h_0 = a^\mu p_\mu \). On the other hand the boost and rotation Killing vectors do not correspond to good time variables on \( M \) because they are not gradient vectors.

4 General relativity as a totally constrained system

Before extending the argument on the totally constrained system with a single constraint to a more generic case, we give an overview on the structure of the totally constrained system obtained from general relativity \([6]\). The main purpose is to make clear that the classical dynamics of general relativity as a totally constrained system is nothing but a foliation of the constraint submanifold such that each leaf is one-to-one correspondence with a 4-dimensional diffeomorphism class of solutions to the Einstein equations. This fact will be used to establish an interpretation of generic systems in the next section.

4.1 ADM canonical formulation on the 3-metric space

For simplicity we only consider globally hyperbolic vacuum spacetime \((M, g)\), and assume that it is spatially compact. Hence \( M \) is diffeomorphic to \( \mathbb{R} \times S \) where \( S \) is a compact space. Let \( q(t) \) be the induced 3-metric on \( \{t\} \times S \), \( K(t) \) the extrinsic curvature tensor of \( \{t\} \times S \), and \( n(t) = \frac{1}{N}(\partial_t - \nu) \) the unit normal to \( \{t\} \times S \) where \( \nu \in \mathcal{X}(\{t\} \times S) \). Then by regarding \( q(t) \), \( K(t) \), \( N(t) \) and \( \nu(t) \) as quantities on \( S \).
with the time parameter \( t \), \( K(t) \) is written as

\[
K_{jk} = \frac{1}{2N}(-\partial_t q_{jk} + D_j \nu_k + D_k \nu_j), \tag{4.1}
\]

where \( D_j \) is the covariant derivative with respect to \( q \). Further the Einstein-Hilbert action for the spacetime \((M, g)\),

\[
S_2 = \frac{1}{2\kappa^2} \int_M d^4x \sqrt{-|g|} R, \tag{4.2}
\]

is expressed in terms of these quantities as

\[
S_2 = \frac{1}{2\kappa^2} \int dt \int_S d^3x \sqrt{|q|} N(\beta R + q(K, K) - (\text{Tr}K)^2). \tag{4.3}
\]

By introducing the momentum variable \( p^{jk} \) conjugate to \( q_{jk} \) by

\[
p^{jk} = -\sqrt{|q|}N \left( \frac{1}{2\kappa^2} (K_{jk} - q_{jk}\text{Tr}K) \right), \tag{4.4}
\]

This action is put into the canonical form,

\[
S = \int dt \left[ < p, \dot{q} > - ( < h_{\perp}, N > + < h_D, \nu >) \right], \tag{4.5}
\]

where \( h_{\perp}, h_D \) and \( p \) are linear functionals of functions, vector fields and 2nd-rank covariant tensor fields on \( S \), respectively, defined by

\[
<h_{\perp}, f > := \int_S d^3x \frac{2\kappa^2}{\sqrt{|q|}} \left( q(p, p) - \frac{1}{2} (\text{Tr}p)^2 \right) - \frac{\sqrt{|q|}}{2\kappa^2} R, \tag{4.6}
\]

\[
<h_D, X > := \int_S d^3x (-2X^j D_k p^k_j) = < p, \mathcal{L}_X q >, \tag{4.7}
\]

\[
<p, v > := \int_S d^3x v_{jk} p^{jk}. \tag{4.8}
\]

Let \( \mathcal{T}_q^p(S) \) be the set of all smooth \((p, q)\)-type tensor field on \( S \), and \( \hat{q} \) be a reference Riemannian metric on \( S \). Then by taking the completion with respect to the inner product

\[
(\alpha, \beta) = \sum_{l=0}^n \int_S d^3x \sqrt{|\hat{q}|} \hat{q}(\hat{D}^l\alpha, \hat{D}^l\beta), \tag{4.9}
\]

we obtain the Sobolev space \( H^1_R(\mathcal{T}_q^p(S)) \) which is a real Hilbert space. In particular, if we define the space of 3-metrics on \( S \) by

\[
\mathcal{M}(S) := \{ q \in H^1_R(\mathcal{T}_q^p(S)) | q \text{ is positive definite on } S \}, \tag{4.10}
\]

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\( \mathcal{M}(S) \) becomes an open subset of \( H^2_\mathbb{R}(T^0_2(S)) \). Hence it has a natural Hilbert manifold structure, and its tangent space and cotangent space are both isomorphic to the original Hilbert space:

\[
T_q(\mathcal{M}(S)) \cong T^*_q(\mathcal{M}(S)) \cong H^2_\mathbb{R}(T^0_2(S)).
\]  

(4.11)

Further, since the operator \( L \) defined by

\[
(\alpha, \beta) = < \alpha, L\hat{\beta} >,
\]

(4.12)

where \( \hat{\beta}^{jk} = \hat{q}^j l \hat{\beta}_{lm} \), is given by \( L = \sum_{n=0}^{\infty} (-1)^n \hat{\Delta}^n \) and elliptical, it defines an injection from \( T^*_q(\mathcal{M}(S)) \) into the space of linear functionals on \( H^2_\mathbb{R}(T^0_2(S)) \). Under the identification by this mapping the momentum \( p \) can be regarded as an element of \( T^*_q(\mathcal{M}(S)) \).

From this observation the action Eq.(4.5) defines a canonical system \((\Gamma, \omega, H)\) with an infinite number of constraints:

\[
< h_D, X > = 0 \quad \forall X \in \mathcal{A}(S),
\]

(4.13)

\[
< h_\perp, f > = 0 \quad \forall f \in \mathcal{F}(S),
\]

(4.14)

where the total phase space \( \Gamma \) is given by the cotangent bundle \( T^*\mathcal{M}(S) \), the symplectic form formally by \( \omega = < \delta p \wedge \delta q > \), and the hamiltonian \( H \) by

\[
H = < h_\perp, N > + < h_D, \nu >.
\]

(4.15)

Though we can give an exact expression for \( \omega \) by introducing the basis of \( H^2_\mathbb{R}(T^0_2(S)) \), we will not do it because the argument in this section is formal. Since the hamiltonian is written as a linear combination of the constraint functionals, this canonical system is a totally constrained system.

### 4.2 Dynamical foliation of the phase space and \( \text{Diff}_0(M) \)-classes

Let \( \Sigma_D \) and \( \Sigma_H \) be the submanifolds of \( \Gamma \) defined by

\[
\Sigma_D := \{ u \in \Gamma | < h_D, X > (u) = 0 \forall X \in \mathcal{A}(S) \},
\]

(4.16)

\[
\Sigma_H := \{ u \in \Gamma | < h_\perp, f > (u) = 0 \forall f \in \mathcal{F}(S) \}.
\]

(4.17)

Then from the Poisson bracket structure among the constraints,

\[
\{ < h_D, X_1 >, < h_D, X_2 > \} = < h_D, [X_1, X_2] >,
\]

(4.18)

\[
\{ < h_D, X >, < h_\perp, f > \} = < h_\perp, \mathcal{L}_X f >,
\]

(4.19)

\[
\{ < h_\perp, f_1 >, < h_\perp, f_2 > \} = < h_D, f_1 D f_2 - f_2 D f_1 >,
\]

(4.20)

the infinitesimal canonical transformation \( X_H \) generated by \( H \) is tangential both to \( \Sigma_D \) and \( \Sigma_H \), and each integration curve of \( X_H \) on \( \Sigma_D \cap \Sigma_H \) yields a solution.
to the Einstein equations. However, this correspondence is not one-to-one because the same spacetime allows an infinite number of different slicings with the same $N$ and $\nu$. Further for a different choice of the lapse function $N$ and the shift vector $\nu$ yields a different curve in $\Gamma$ for the same spacetime.

This ill correspondence between the spacetime solutions to the Einstein equations and the curves in the phase space, which arises due to the general covariance of general relativity, can be made well-defined by considering the subspace spanned by the integration curves in stead of each curve. To see this, let us denote the set of constraints symbolically by $h_\alpha$ and the corresponding infinitesimal canonical transformations by $Y_\alpha = X_{h_\alpha}$. Then from the first class nature of $h_\alpha$ shown above,

\[ \{ Y_\alpha \} \text{ yields an involutive system on } \Sigma_D \cap \Sigma_H: \]

\[ [Y_\alpha, Y_\beta] \approx -\sum_\gamma c_{\alpha\beta}^\gamma Y_\gamma, \quad (4.21) \]

where $c_{\alpha\beta}^\gamma$ is a set of functions on $\Gamma$. Hence we obtain a foliation of the constraint submanifold $\Sigma_D \cap \Sigma_H = \bigcup \lambda C_\lambda$ where each leaf $C_\lambda$ is a connected component of the integration submanifolds.

For an arbitrary non-degenerate curve $\gamma$ contained in a leaf $C_\lambda$, its tangent vector $X$ is written in terms of some set of functions $N^\alpha$ as $X = \sum_\alpha N^\alpha Y_\alpha$ because $\{Y_\alpha\}$ spans the tangent space of $C_\lambda$ at any point. Hence it is an integration curve of the hamiltonian flow for the hamiltonian $H = \sum_\alpha N^\alpha h_\alpha$, and corresponds to some spacetime solution $(M, g)$ to the Einstein equations. Further, if two curves $\gamma_1$ and $\gamma_2$ are contained in the same leaf and intersect with each other at a point $u$, they correspond to solutions to the Einstein equations with the same initial data for different lapse functions and shift vectors. Hence from the uniqueness of the initial value problem for the Einstein equations the corresponding spacetime solutions $(M, g_1)$ and $(M, g_2)$ are isometric if they are maximally extended. The same conclusion holds even if these two curves do not intersect. For one can find another curve $\gamma_3$ in the same leaf which intersects both with $\gamma_1$ and $\gamma_2$, which implies that $(M, g_3) \cong (M, g_1)$ and $(M, g_3) \cong (M, g_2)$, hence $(M, g_1) \cong (M, g_2)$. Therefore all the curves contained in the same leaf corresponds to a unique 4-dimensional diffeomorphism class of the spacetime solutions to the Einstein equations.

We can further show that this correspondence is one-to-one. Take two curves $\gamma_1 \subset C_1$ and $\gamma_2 \subset C_2$ and suppose that the corresponding spacetime solutions $(M, g_1)$ and $(M, g_2)$ are isometric. Then there exist isometric diffeomorphisms to a spacetime $(\tilde{M}, \tilde{g})$, $\Phi_1 : (M, g_1) \rightarrow (\tilde{M}, \tilde{g})$ and $\Phi_2 : (M, g_2) \rightarrow (\tilde{M}, \tilde{g})$. Let $S_1 = \Phi_1(\{t_1\} \times S)$ and $S_2 = \Phi_2(\{t_2\} \times S)$ be two space-like constant-time hypersurfaces in $\tilde{M}$, and choose two families of time slicings in $\tilde{M}$ such that they contain a common time slice, one of them contains $S_1$ and the other $S_2$. Further let the two curves in the phase space determined by these slicings be $\gamma_3$ and $\gamma_4$. Then from the construction $\gamma_3 \cap \gamma_1 \neq \emptyset$, $\gamma_4 \cap \gamma_2 \neq \emptyset$ and $\gamma_3 \cap \gamma_4 \neq \emptyset$. This implies that there is a curve which connects a point in $C_1$ and a point $C_2$. Hence from the connectedness of each leaf $C_1$ and $C_2$ must coincide with each other.
Thus we have found that the connected components of the integration manifolds of the involutive system \( \{ Y_\alpha \} \) are in one-to-one correspondence with the 4-dimensional diffeomorphism classes of the spacetime solutions to the Einstein equations. In other words the classical dynamics of general relativity is completely determined by the foliation of the constraint submanifold in terms of the infinitesimal canonical transformations generated by the constraints. We can discard the lapse function and the shift vector, or the corresponding hamiltonian. For this reason we can simply say that the canonical theory of general relativity is given by a totally constrained system \( (T^* \mathcal{M}(S), \omega, \{ h_D, h_\perp \}) \). We will call each leaf of the foliation a causal submanifold.

### 4.3 Elimination of the diffeomorphism constraints

As the structure constants of the Poisson brackets among \( h_D \) are genuinely constant from Eq.\((\ref{eq:18})\), the corresponding infinitesimal canonical transformations are involutive on the whole phase space and generate the action of the diffeomorphism group of \( S \), \( \text{Diff}_0(S) \) where the subscript 0 denotes the connected component containing the unit element. Since all the measurable quantities are invariant under these transformations, it is desirable to eliminate this kinematical gauge symmetry from the canonical theory, especially when one consider the quantization of the theory. Now we will show that the classical dynamics has the same structure as above even after the elimination of this gauge freedom.

First of all note that for a \( \mathcal{F}(S) \)-valued functional \( \tilde{N} \) on \( \Gamma \) which transforms covariantly under \( \text{Diff}_0(S) \) as

\[
\tilde{N}(a_* u) = a_*(\tilde{N}(u)) \quad \forall a \in \text{Diff}_0(S), \quad \forall u \in \Gamma, \tag{4.22}
\]

\( < h_\perp, \tilde{N} > \) is invariant under \( \text{Diff}_0(S) \) as a functional on \( \Gamma \) from

\[
\left\{ < h_\perp, \tilde{N} >, < h_D, X > \right\} = < \mathcal{L}_X h_\perp, \tilde{N} > + < h_\perp, \mathcal{L}_X \tilde{N} > = 0. \tag{4.23}
\]

Similarly for a \( \mathcal{X}(S) \)-valued functional \( \tilde{\nu} \) on \( \Gamma \) which is \( \text{Diff}_0(S) \)-covariant, \( < h_D, \tilde{\nu} > \) is invariant under \( \text{Diff}_0(S) \). Further by inspecting the argument on the correspondence between a curve in a causal submanifold and the hamiltonian flow generated by the hamiltonian \( H = < h_\perp, N > + < h_D, \nu > \) one easily sees that \( \tilde{N} \) and \( \nu \) can be replaced by some appropriate \( \text{Diff}_0(S) \)-covariant functionals \( \tilde{N} \) and \( \tilde{\nu} \). Hence the connected integration surfaces of the involutive system generated by the \( \text{Diff}_0(S) \)-invariant functionals \( < h_\perp, \tilde{N} > \) and \( < h_D, \tilde{\nu} > \) give the same foliation as that given by \( < h_D, \nu > \) and \( < h_\perp, N > \).

Further if \( < h_D, \xi > \neq 0 \) at a point \( u \in \Gamma \) for some \( \xi \in \mathcal{X}(S) \), there exists a functional \( \tilde{\xi} : \Gamma \to \mathcal{X}(S) \) such that \( < h_D, \tilde{\xi} > \neq 0 \) at the same point \( u \). Thus \( \Sigma_D \) can be redefined as

\[
\Sigma_D = \{ u \in \Gamma \mid < h_D, \tilde{\nu} > (u) = 0 \forall \tilde{\nu} : \Gamma \to \mathcal{X}(S); \text{Diff}_0(S) \text{–covariant} \}. \tag{4.24}
\]
Similarly $\Sigma_H$ can be expressed as

$$\Sigma_H = \{ u \in \Gamma \mid < h_{\perp}, \tilde{N} > (u) = 0 \forall \tilde{N} : \Gamma \rightarrow \mathcal{F}(S); \text{Diff}_0(S)-\text{covariant} \}.$$  \hfill (4.25)

These arguments indicate that the original canonical system can be naturally projected on $\Gamma/\text{Diff}_0(S)$. To confirm this, let us denote all the functions on $\Gamma$ which are invariant under $\text{Diff}_0(S)$ by $\mathcal{F}_{\text{inv}}$:

$$\mathcal{F}_{\text{inv}} := \{ f \in \mathcal{F}(\Gamma) \mid \{ f, < h_D, \xi > \} = 0 \forall \xi \in \mathcal{X}(S) \}.$$  \hfill (4.26)

Then it is easily shown that $\mathcal{F}_{\text{inv}}$ is closed with respect to the Poisson algebra and $\mathcal{F}_{\text{inv}} F_D = F_D$ where

$$F_D := \{ f \in \mathcal{F}_{\text{inv}} \mid f|_{\Sigma_D} = 0 \}.$$  \hfill (4.27)

Further since $\{ f, < h_D, \xi > \} = 0$ implies that $X_f$ is tangential to $\Sigma_D$, $\{ f, g \} = -X_fg$ vanishes on $\Sigma_D$ for $f \in \mathcal{F}_{\text{inv}}$ and $g \in F_D$. Hence $\{ \mathcal{F}_{\text{inv}}, F_D \} = F_D$. This implies that $F_D$ forms an ideal of $\mathcal{F}_{\text{inv}}$ and the Poisson bracket in $\mathcal{F}_{\text{inv}}$ naturally induces a Poisson bracket in $\mathcal{A}_{\text{inv}} := \mathcal{F}_{\text{inv}}/F_D$. Each element of $\mathcal{A}_{\text{inv}}$ is just a set of functions in $\mathcal{F}_{\text{inv}}$ which coincide with each other on $\Sigma_D$.

Let $\pi : \Gamma \rightarrow \Gamma/\text{Diff}_0(S)$ be the natural projection and put $\Gamma_{\text{inv}} := \pi(\Sigma_D)$. Then from the arguments above $\Gamma_{\text{inv}}$ is characterized as

$$\Gamma_{\text{inv}} = \{ u \in \Gamma/\text{Diff}_0(S) \mid < h_D, \tilde{\nu} > (u) = 0 \forall \tilde{\nu} : \Gamma \rightarrow \mathcal{X}(S); \text{Diff}_0(S)-\text{covariant} \},$$  \hfill (4.28)

and $\mathcal{A}_{\text{inv}}$ is naturally identified with $\mathcal{F}(\Gamma_{\text{inv}})$. Further the constraint $h_D$ is trivialized on $\Gamma_{\text{inv}}$ and the causal submanifolds in $\Sigma_D \cap \Sigma_D$ is bijectively mapped to the causal submanifolds in $\pi(\Sigma_H) \cap \Gamma_{\text{inv}}$ determined from $< h_{\perp}, \tilde{N} > |_{\Sigma_D} \in \Gamma_{\text{inv}}$.

Let $h_\alpha \in \mathcal{A}_{\text{inv}}$ be a generating set of all the functions of the form $< h_{\perp}, \tilde{N} > |_{\Sigma_D}$ such that

1) For any Diff$_0(S)$-covariant functional $\tilde{N} : \Gamma \rightarrow \mathcal{F}(S)$ there exists a set of elements $\lambda^\alpha \in \mathcal{A}_{\text{inv}}$ such that $< h_{\perp}, \tilde{N} > = \sum_\alpha \lambda^\alpha h_\alpha$,

2) $\pi(\Sigma_H) \cap \Gamma_{\text{inv}} = \{ u \in \Gamma_{\text{inv}} \mid h_\alpha(u) = 0 \forall \alpha \}$,

3) $\{ h_\alpha, h_\beta \} = \sum_\gamma c^{\gamma}_{\alpha\beta} h_\gamma$.

Further let us denote $\pi(\Sigma_H) \cap \Gamma_{\text{inv}}$ by the same symbol $\Sigma_H$. Then the arguments so far shows that the canonical dynamics of general relativity is described by the totally constrained system $(\Gamma_{\text{inv}}, \omega_{\text{inv}}, \{ h_\alpha \})$ and the causal submanifolds in $\Sigma_H$ is one-to-one correspondence with the Diff$_0(M)$-class of the spacetime solutions to the Einstein equations.
4.4 Cotangent-bundle structure of $\Gamma_{\text{inv}}$

In the last statement of the previous subsection $\omega_{\text{inv}}$ is understood to be the symplectic form corresponding to the Poisson brackets in $A_{\text{inv}}$. Hence in order to make the statement rigorous it should be shown that $\Gamma_{\text{inv}}$ has a manifold structure and the required symplectic form exits. Now we prove these facts by showing that $\Gamma_{\text{inv}}$ can be identified with $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$ and $\omega_{\text{inv}}$ coincides with the canonical symplectic form corresponding to the cotangent bundle.

Let $\phi_1 : \Sigma_D \to \mathcal{M}(S)$ be the restriction of the natural projection from $T^*(\mathcal{M}(S))$ to $\mathcal{M}(S)$. Then, since the diffeomorphism constraint implies that $p$ vanishes on the subspace of $T_*(\mathcal{M}(S))$ spanned by the tangent vectors to the $\text{Diff}_0(S)$-orbits from Eq.(4.7), $\phi_1$ is surjective and induces a surjective mapping $\phi_2 : \Gamma_{\text{inv}} \to \mathcal{M}(S)/\text{Diff}_0(S)$ such that $\pi\phi_1 = \phi_2\pi$ (See the diagram in Fig.1). Let $\eta_q \in T_q(\mathcal{M}(S))$ be a vector tangent to a $\text{Diff}_0(S)$-orbit passing through $q \in \mathcal{M}(S)$. Then, since it is written in terms of a vector field $X \in \mathcal{X}(S)$ as $\eta_q = L_X q$, it follows from Eq.(4.7) that for $(p, q) \in \Sigma_D$,

\[(p, q)(\eta_q) = \langle p, \eta_q \rangle = \langle p, L_X q \rangle = 0. \tag{4.29}\]

Further for $a \in \text{Diff}_0(S)$ and $v \in T_q(\mathcal{M}(S))$

\[a_*(p, q)(a_*v) = \langle a_*p, a_*v \rangle = \langle p, v \rangle = (p, q)(v), \tag{4.30}\]

from the diffeomorphism invariance of $\langle p, v \rangle$. Hence there is an injection $j : \Gamma_{\text{inv}} \to T^*(\mathcal{M}(S)/\text{Diff}_0(S))$ such that for $v \in T_q(\mathcal{M}(S))$

\[j\pi(p, q)(\pi v) = \langle p, v \rangle. \tag{4.31}\]

It is easily checked that for the natural projection $\phi_3 : T^*(\mathcal{M}(S)/\text{Diff}_0(S)) \to \mathcal{M}(S)/\text{Diff}_0(S)$, $\phi_2 = \phi_3 j$ holds and that $j$ is surjective. Thus $\Gamma_{\text{inv}}$ can be identified with $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$.

Next let us show that the symplectic form $\omega_{\text{inv}}$ induced from the cotangent bundle structure of $\Gamma_{\text{inv}}$ is equivalent to the Poisson brackets in $A_{\text{inv}} \cong \mathcal{F}(\Gamma_{\text{inv}})$ derived from the symplectic form $\omega$ in $\Gamma$. From now on we identify $\Gamma_{\text{inv}}$ with $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$ and write $j\pi$ simply as $\pi$. 

\[\begin{array}{ccc}
T^*(\mathcal{M}(S)) & \ni & \Sigma_D \\
\pi & \downarrow & \pi \\
T^*(\mathcal{M}(S)/\text{Diff}_0(S)) & \ni & \Gamma_{\text{inv}} \\
& \downarrow_{j} & \downarrow_{\phi_3} \\
T^*(\mathcal{M}(S)/\text{Diff}_0(S)) & \ni & \mathcal{M}(S)/\text{Diff}_0(S) \\
\end{array}\]
We first show that each element of $\mathcal{A}_{\text{inv}}$ uniquely determines a vector field on $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$, which will turn out to be an infinitesimal canonical transformation on $\Gamma_{\text{inv}}$. Let $\mathcal{X}_{\text{inv}}$ be

$$\mathcal{X}_{\text{inv}} = \{ X \in \mathcal{X}(\Sigma_D) \mid a_* X = X \forall a \in \text{Diff}_0(S) \}, \quad (4.32)$$

and for $[f] \in \mathcal{A}_{\text{inv}}$, let $\mathcal{X}_{[f]}$ be

$$\mathcal{X}_{[f]} = \{ X \in \mathcal{X}_{\text{inv}} \mid \omega(X, Z) = -Z f \forall Z \in \mathcal{X}_{\text{inv}} \}. \quad (4.33)$$

Then for each $X \in \mathcal{X}_{\text{inv}}$, $\pi_* X$ obviously defines a unique vector field on $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$. Further let $\theta$ and $\theta_{\text{inv}}$ be the canonical 1-forms on $T^*(\mathcal{M}(S))$ and $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$, respectively. Then from the commutativity of the diagram in Fig[4] and Eq.(4.31) it follows that $(\theta_{\text{inv}})_p(\pi_* X) = \theta_p(X)$ for any $X \in \mathcal{X}_{\text{inv}}$, i.e.,

$$\theta(X) = \pi^*\theta_{\text{inv}}(X) \quad \forall X \in \mathcal{X}_{\text{inv}}. \quad (4.34)$$

Hence for $\Delta X = X_1 - X_2 (X_1, X_2 \in \mathcal{X}_{[f]}$) and $Z \in \mathcal{X}_{\text{inv}}$, from

$$0 = \omega(\Delta X, Z) = d\theta(\Delta X, Z) = \Delta X(\theta(Z)) - Z(\theta(\Delta X)) - \theta([\Delta X, Z]), \quad (4.35)$$

it follows that

$$0 = \pi_* (\Delta X)(\theta_{\text{inv}}(\pi_* Z)) - \pi_* Z(\theta_{\text{inv}}(\pi_* X)) - \theta_{\text{inv}}(\pi_* [\Delta X, Z])$$

$$= d\theta_{\text{inv}}(\pi_* \Delta X, \pi_* Z) = \omega_{\text{inv}}(\pi_* \Delta X, \pi_* Z). \quad (4.36)$$

As $\pi_* Z$ can be any vector field on $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$, this equation implies that $\pi_* \Delta X = 0$, i.e., $\pi_* X_1 = \pi_* X_2$. Hence $[f] \in \mathcal{A}_{\text{inv}}$ determines a unique vector field on $T^*(\mathcal{M}(S)/\text{Diff}_0(S))$. We denote this vector field by $X_{[f]}$.

Next we show that for $X \in \mathcal{X}_{[f]}$ and $Y \in \mathcal{X}_{[g]}$, $\omega(X, Y) = -\{f, g\}$. For $X_1, X_2 \in \mathcal{X}_{[f]}$ and $Y_1, Y_2 \in \mathcal{X}_{[g]}$ from the definition $[4.33]$ it follows that

$$\omega(X_2, Y_2) - \omega(X_1, Y_1) = \omega(X_2 - X_1, Y_2) + \omega(X_1, Y_2 - Y_1) = 0. \quad (4.37)$$

Hence $\omega(X, Y)$ depends only on $[f]$ and $[g]$. Obviously $X_f \in \mathcal{X}_{[f]}$ and $X_g \in \mathcal{X}_{[g]}$. Therefore

$$\{f, g\} = -\omega(X_f, X_g) = -\omega(X, Y). \quad (4.38)$$

With the help of the equations derived so far for $X \in \mathcal{X}_{[f]}$ and $Y \in \mathcal{X}_{[g]}$ we obtain

$$\{ [f, g] \} = -[\omega(X, Y)] = -[d\theta(X, Y)]$$

$$= [Y(\theta(X)) - Y(\theta(X)) - \theta([X, Y])]$$

$$= \pi_* Y(\theta_{\text{inv}}(\pi_* X)) - \pi_* Y(\theta_{\text{inv}}(\pi_* X)) - \theta_{\text{inv}}(\pi_* [X, Y])$$

$$= -d\theta_{\text{inv}}(\pi_* X, \pi_* Y) = -\omega_{\text{inv}}(X_{[f]}, Y_{[g]})$$

$$= \{[f], [g]\}_{\text{inv}} \quad (4.39)$$

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This shows that the Poisson brackets induced from $\omega$ coincides with that defined by $\omega_{\text{inv}}$.

Note here that the arguments so far are not mathematically rigorous because $\mathcal{M}(S)/\text{Diff}_0(S)$ has conical singularities at metrics with Killing vectors. Though these singularities may have physical importance in quantization, we will not go into this problem in this paper.

We can go further and eliminate all the hamiltonian constraints to get the fully reduced phase space with a symplectic structure which represents the true physical degrees of freedom as done by Fischer and Marsden. However, we shall not follow this line because we will then lose the dynamics.

## 5 General Totally Constrained Systems

Now we discuss the dynamics of a generic totally constrained system. Here a totally constrained system is defined as a triplet of a phase space, a symplectic form and a set of constraint functions, $(\Gamma, \omega, \{h_\alpha\})$. For a technical reason we assume that the phase space is $2n$-dimensional smooth manifold with finite $n$. Further we assume that the constraints are of first class with the Poisson brackets given by

$$\{h_\alpha, h_\beta\} = \sum_\gamma c_{\alpha\beta}^\gamma h_\gamma,$$

where $c_{\alpha\beta}^\gamma$ are functions on $\Gamma$.

On the basis of the arguments in the previous section we understand that the physical evolution of the system is one-to-one correspondence with each leaf of the foliation determined by the involutive system of the infinitesimal canonical transformations $Y_\alpha = X_{h_\alpha}$ on the constraint submanifold $\Sigma_H = \{u \in \Gamma \mid h_\alpha(u) = 0 \ \forall \alpha\}$. We call each leaf a causal submanifold as so far. As is clear from the arguments in the previous section, this interpretation is equivalent to regard that two solutions to the canonical equation of motion for the hamiltonian $H = \sum_\alpha \lambda^\alpha h_\alpha$ with arbitrary functions $\lambda^\alpha$ represent the same physical evolution if they intersect with each other in $\Gamma$.

This is a natural generalization of the argument on the dynamics of a single totally constrained system with one constraint in §3. Now we extend this generalization to the statistical dynamics of an ensemble.

### 5.1 Relative distribution function

From this interpretation of dynamics of a single system and the argument in §3 it is natural to introduce the relative distribution function $\rho$ on $\Gamma$ to describe an ensemble, which vanishes outside the constraint submanifold and is constant on each causal submanifold:

$$Y_\alpha \rho = 0 \ \forall \alpha,$$  

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\[ h_\alpha \rho = 0 \quad \forall \alpha. \quad (5.3) \]

Let us define an acausal submanifold as a submanifold of \( \Gamma \) which intersects with causal submanifolds transversally. Then for any acausal submanifold \( \Sigma \) and for any distribution \( \rho|_\Sigma \) on \( \Sigma \) a solution to these equations which coincides with \( \rho|_\Sigma \) on \( \Sigma \) is unique, if it exists, on the causal development of \( \Sigma \) defined by

\[ D(\Sigma) := \bigcup_{C \cap \Sigma \neq \emptyset} C \quad (5.4) \]

where \( C \) runs over causal submanifolds. However, such solution may not exist in general. In fact the following theorem holds.

**Theorem 5.1** In order that there exists a solution to Eqs. (5.2)-(5.3) for arbitrary initial data on any acausal submanifold, the following condition should be satisfied:

\[ \sum_\gamma \{h_\gamma, c^\gamma_{\alpha \beta}\} \approx 0. \]

This condition is satisfied if and only if there exists a function \( f \neq 0 \) such that for \( c'^\gamma_{\alpha \beta} \) corresponding to the constraints \( h'_\alpha = fh_\alpha \)

\[ c'_\alpha := \sum_\beta c'^\beta_{\alpha \beta} \approx 0 \]

holds.

**Proof**

Since \( \rho \) is a distribution, to be exact, Eqs. (5.2)-(5.3) are expressed as

\[ < h_\alpha \rho, \phi >= < \rho, h_\alpha \phi > = 0, \]
\[ < Y_\alpha \rho, \phi >= < \rho, -Y_\alpha \phi > = 0, \]

where \( \phi \) is an arbitrary smooth function with a compact support on \( \Gamma \). However, since the commutators among \( Y_\alpha \)'s are given by

\[ [Y_\alpha, Y_\beta] \approx -\sum_\gamma c^\gamma_{\alpha \beta} Y_\gamma, \]

the consistency condition yields

\[ 0 = < \rho, [Y_\alpha, Y_\beta] \phi >= < \rho, \sum_\gamma (Y_\gamma c^\gamma_{\alpha \beta}) \phi > . \]

Hence, noting the relation \( Y_\gamma c^\gamma_{\alpha \beta} = -\{h_\gamma, c_{\alpha \beta}\} \), we obtain the first condition in the theorem.
In order to show the latter half of the theorem, first note that the Jacobi identity for the Poisson brackets among $h_\alpha$ yields

$$Y_\alpha c_\beta - Y_\beta c_\alpha + \sum_\gamma c_\alpha^\gamma c_\beta = \sum_\gamma \{h_\gamma, c_\alpha^\gamma \},$$

where $c_\alpha := \sum_\beta c_\alpha^\beta$. From this it immediately follows that the first condition of the theorem holds if $c_\alpha \approx 0$.

On the other hand for $h'_\alpha = fh_\alpha c_\alpha$ changes as

$$c'_\alpha = c_\alpha - (m - 1)Y_\alpha \ln f,$$

where $m$ is the number of the constraints. Hence the second condition of the theorem is satisfied if $f$ is a solution to the equation

$$Y_\alpha f^{m-1} \approx c_\alpha f^{m-1}.$$

However, if the first condition of the theorem is satisfied, we obtain

$$Y_\alpha c_\beta - Y_\beta c_\alpha \approx -\sum_\gamma c_\alpha^\gamma.$$

This is nothing but the consistency condition for the first-order differential equation system for $f^{m-1}$ above. Hence the first condition of the theorem implies the second.

Note that for a matrix function $\Lambda = (\Lambda^\alpha_\beta)$ on $\Gamma$ with $\det \Lambda \neq 0$, the totally constrained system with the constraints $h'_\alpha = \sum_\beta \Lambda^\alpha_\beta h_\beta$ is equivalent to the original system. Hence the precise meaning of the requirement of the theorem is that $\sum_\gamma \{h_\gamma, c_\alpha^\gamma \}$ can be put to zero by such a transformation and that Eqs.$(5.2)-(5.3)$ are consistent only for such choice of the constraints. This result is interesting in relation to the quantization of the totally constrained system because this condition implies that the operators corresponding to $c_\alpha^\gamma$ and $h_\alpha$ should commute in a weak sense.

On the basis of this theorem we assume that $c_\alpha = 0$ from now on. Under this condition if we put

$$\rho = \rho_0 \prod_\alpha \delta(h_\alpha),$$

Eq.$(5.2)$ is automatically satisfied and Eq.$(5.3)$ reduces to the equation for the function $\rho_0$,

$$Y_\alpha \rho_0 \approx 0,$$

because

$$\mathcal{L} Y_\alpha \rho = Y_\alpha \prod_\beta \delta(h_\beta) + \sum_\beta \sum_\gamma c_\alpha^\beta h_\gamma \delta'(h_\beta) \prod_\mu \delta(h_\mu)$$

$$= (Y_\alpha \rho_0 - c_\alpha \rho_0) \prod_\mu \delta(h_\mu).$$

(5.7)
5.2 Statistical dynamics in terms of conservative measure on acausal submanifolds

Now we show that we can formulate the statistical dynamics for an ensemble of the totally constrained system without reducing it to a system without constraint. The basic idea is the same as that used in §2 and §3 for totally constrained systems with a single constraint.

First note that for any pair of acausal submanifolds $\Sigma_1$ and $\Sigma_2$ the foliation of the constraint submanifold by the constraints uniquely determines the causal mapping

$$\theta : \Sigma_1 \cap D(\Sigma_2) \cap \Sigma_H \to \Sigma_2 \cap D(\Sigma_1) \cap \Sigma_H,$$  \hspace{1cm} (5.8)

where $D(\Sigma_1)$ and $D(\Sigma_2)$ are causal developments of $\Sigma_1$ and $\Sigma_2$, respectively. We extend this causal mapping to a neighborhood of $\Sigma_H$ by considering a foliation of the tubular neighborhood such that the intersection of each leaf with $\Sigma_H$ coincides with the foliation of $\Sigma_H$ by $Y_\alpha$. If a measure $\mu_{\Sigma_0}$ on an acausal submanifold $\Sigma_0$ with its support contained in $\Sigma_0 \cap \Sigma_H$ is given, this causal mapping uniquely determines a measure $\mu_\Sigma$ on $\Sigma \cap D(\Sigma_0)$ with its support contained in $\Sigma_H$ such that for any constant of motion, i.e., a function $f \in \mathcal{F}(\Gamma)$ which is constant along each leaf,

$$\int \Sigma f d\mu_\Sigma = \int \Sigma_0 f d\mu_{\Sigma_0},$$  \hspace{1cm} (5.9)

if supp $f \cap \Sigma_0 \subset \text{dom } \theta$. As in the single constraint systems, this measure can be expressed locally in terms of $\rho$, $Y_\alpha$ and $\Omega$.

**Theorem 5.2** If $c_\alpha = 0$, for any distribution $\rho$ satisfying Eqs. (5.2)-(5.3) the measure

$$d\mu_\Sigma = \rho|I_{Y_1} \cdots I_{Y_m} \Omega|_{|\Sigma}$$

is conserved by causal mappings where $m$ is the number of independent constraints.

**Proof**

Let us denote $I_{Y_1} \cdots I_{Y_m}(\rho\Omega)$ simply as $\chi$. Then from the identities

$$d \circ I_X + I_X \circ d = \mathcal{L} X,$$

$$[\mathcal{L} X, I_Y] = I_{[X,Y]};$$

we obtain

$$d\chi = -I_{Y_1} dI_{Y_2} \cdots dI_{Y_m}(\rho\Omega) + \sum_{\alpha=1}^{m} c_{1\alpha}^1 (-1)^{\alpha} \prod_{\beta \neq \alpha} I_{Y_\beta}(\rho\Omega).$$

In a similar way we obtain

$$(-1)^{\gamma-1} I_{Y_1} \cdots dI_{Y_\gamma} \cdots I_{Y_m}(\rho\Omega) = (-1)^{\gamma} I_{Y_1} \cdots I_{Y_\gamma} dI_{Y_{\gamma+1}} \cdots (\rho\Omega)$$

$$+ (-1)^{\gamma-1} \sum_{\alpha=\gamma+1}^{m} \left[ c_{\beta \gamma}^{\alpha} \prod_{\beta \neq \gamma} I_{Y_\beta}(\rho\Omega) - (-1)^{\alpha-\gamma} c_{\gamma \alpha}^{\gamma} \prod_{\beta \neq \alpha} I_{Y_\beta}(\rho\Omega) \right].$$

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Since \(d(\rho\Omega) = 0\) and \(\mathcal{L}_{Y_\alpha}(\rho\Omega) = 0\), summation of these equations over \(\gamma = 1, \ldots, m\) yields
\[
d\chi = \sum_{\gamma=1}^{m} (-1)^{\gamma-1} c_\gamma \sum_{\beta \neq \gamma} I_{Y_\beta}(\rho\Omega) = 0.
\]

Hence for a pair of acausal submanifolds \(\Sigma_1\) and \(\Sigma_2\) such that \(D(\Sigma_1) = D(\Sigma_2)\), from Stokes’ theorem on \((2n - m + 1)\)-dimensional submanifold \(N\) such that \(\partial N = \Sigma_1 \cup \Sigma_2 \cup \Sigma'\) and \(\Sigma'\) is parallel to the leaves, we obtain
\[
\int_{\Sigma_2} |\chi| - \int_{\Sigma_1} |\chi| = \pm \int_{\partial N} \chi \mp \int_{\Sigma'} \chi = \pm \int_{N} d\chi = 0.
\]

In realistic situations each acausal submanifold is specified by a set of independent \(m\) functions \(\phi_\alpha\) such that \(Y_\alpha \phi_\beta = \{\phi_\beta, h_\alpha\}\) is a regular matrix as, say,
\[
\phi_1 = \cdots = \phi_m = 0. \tag{5.10}
\]

Let us call such a set of functions instant functions. Then the measure given in the previous theorem is expressed in terms of these instant functions as follows:

**Theorem 5.3** If \(\phi_1, \cdots, \phi_m\) are instant functions for an acausal submanifold \(\Sigma\), for any \(f \in \mathcal{F}(\Gamma)\) the following equality holds:
\[
\int_{\Sigma} f \rho |I_{Y_1} \cdots I_{Y_m} \Omega| = \int_{\Gamma} f \rho |\det\{h_\alpha, \phi_\beta\}| \prod_{\gamma} \delta(\phi_\gamma)|\Omega|.
\]

**Proof**

From
\[
d\phi_1 \wedge \cdots \wedge d\phi_m \wedge I_{Y_1} \cdots I_{Y_m} \Omega
\]
\[
= \frac{1}{m!} \sum_{\alpha_1 \cdots \alpha_m} \epsilon^{\alpha_1 \cdots \alpha_m} d\phi_{\alpha_1} \wedge \cdots \wedge d\phi_{\alpha_m} \wedge I_{Y_1} \cdots I_{Y_m} \Omega
\]
\[
= \frac{(-1)^m}{m!} \sum_{\alpha_1 \cdots \alpha_m} \epsilon^{\alpha_1 \cdots \alpha_m} I_{Y_1}(d\phi_{\alpha_1} \wedge \cdots \wedge d\phi_{\alpha_m} \wedge I_{Y_2} \cdots I_{Y_m} \Omega)
\]
\[
= \frac{(-1)^m}{(m-1)!} \sum_{\alpha_1 \cdots \alpha_m} \epsilon^{\alpha_1 \cdots \alpha_m} (I_{Y_1}d\phi_{\alpha_1})d\phi_{\alpha_2} \wedge \cdots \wedge d\phi_{\alpha_m} \wedge I_{Y_2} \cdots I_{Y_m} \Omega
\]
\[
= \frac{(-1)^m}{(m-1)!} \sum_{\alpha_1 \cdots \alpha_m} \epsilon^{\alpha_1 \cdots \alpha_m} \{h_1, \phi_{\alpha_1}\}d\phi_{\alpha_2} \wedge \cdots \wedge d\phi_{\alpha_m} \wedge I_{Y_2} \cdots I_{Y_m} \Omega
\]
\[
= \cdots
\]
\[
= (-1)^{m(m+1)/2} \sum_{\alpha_1 \cdots \alpha_m} \epsilon^{\alpha_1 \cdots \alpha_m} \{h_1, \phi_{\alpha_1}\} \cdots \{h_m, \phi_{\alpha_m}\} \Omega,
\]
we obtain
\[
\prod_{\alpha} \delta(\phi_\alpha) \prod_{\beta} d\phi_\beta |I_{Y_1} \cdots I_{Y_m} \Omega| = |\det\{h_\alpha, \phi_\beta\}| \prod_{\gamma} \delta(\phi_\gamma)|\Omega|.
\]
By multiplying $f\rho$ on the both sides of this equation and integrating over $\Gamma$, we obtain the equation in the theorem. 

From these theorems we can formulate the statistical dynamics of an ensemble of the totally constrained system with multiple constraints in the following way. First, from the data set obtained by measurements, pick up a set of instant functions $\phi_{(1)\alpha}$ which take a common set of values in the data set. For simplicity assume that these values are all zero, and let the corresponding acausal submanifold in $\Gamma$ be $\Sigma_1$, and define the measure $d\nu$ by

$$d\nu := c|I_{Y_1} \cdots I_{Y_m} \Omega| \quad (5.11)$$

where $c$ is some fixed positive constant of motion. Then the other data uniquely determines the distribution $\rho$ on $\Sigma_1$ through the formula

$$< f >_{\Sigma_1} = \int_{\Sigma_1} f\rho d\nu. \quad (5.12)$$

Extend this distribution $\rho$ over $D(\Sigma_1)$ by the evolution equations Eqs.(5.2)-(5.3). Then for another set of instant functions $\phi_{(2)\alpha}$ corresponding to an acausal submanifold $\Sigma_2$, the expectation value of a function $f \in F(\Gamma)$ on that submanifold is given by

$$< f >_{\Sigma_2} = \int_{\Sigma_2} f\rho d\nu, \quad (5.13)$$

if supp $f \in D(\Sigma_1)$. Of course we do not need the explicit knowledge on the acausal submanifolds, because from Theorem 5.3 the expectation values are written as an integration over $\Gamma$ in terms of measures expressed by the constraints and the instant functions.

Like the case of a single constraint system we can define that a set of functions $\phi_{\alpha}$ are good time variables if the natural measure $\prod_\alpha \delta(\phi_{\alpha} - \tau_\alpha)|\Omega|$ on a set of acausal submanifolds

$$\Sigma(\tau_\alpha) := \{ u \in \Gamma \mid \phi_{\alpha}(u) = \tau_\alpha \} \quad (5.14)$$

coincides with the conserved measure. This condition is expressed as

$$c|\det\{h_{\alpha}, \phi_{\beta}\}| \approx f(\phi_1, \cdots, \phi_m), \quad (5.15)$$

where $c$ is some positive constant of motion and $f$ is some function of $m$ variables.

6 Discussion

In this paper we have shown that the dynamics of a classical totally constrained system can be consistently formulated without reducing it to an unconstrained system by solving the constraints or referring to a special time variable. The basis idea has been to consider the relative distribution function which is constant on
each leaf of the foliation defined by the infinitesimal canonical transformations generated by the constraint functions, and to normalize it on an acausal submanifold which is transversal to the foliation in terms of the conservative measure.

The fact that we can formulate the classical statistical dynamics of a totally constrained system without referring a special time variable is very important for considering a quantum theory of the totally constrained system because a quantum theory has a similar structure to the classical statistical dynamics in general. In fact in the next paper we will show that by introducing a similar foliation structure into a state space of quantum theory and by considering a relative probability amplitude in stead of the relative distribution we can construct a consistent formulation of the quantum dynamics of a totally constrained system without referring to a special time variable under some restrictions.

Though the main purpose of the present paper has been to give a basic motivation for the quantum formulation developed in the subsequent papers, the results obtained in the paper may be interesting by themselves. In particular the fact that the conservative measure can be written only by the canonical volume form and the constraint functions even for multi-constrained systems seems to be useful in the arguments of the probability distribution of the initial condition of the universe in the classical framework and stochastic treatment of general relativity.

Of course the expression for the measure given in this paper cannot be applied to general relativity directly because we have only considered systems with finite degrees of freedom. However, it seems possible to extend the formulation to general relativity by taking an appropriate limit. To examine this limiting procedure explicitly in some simple situations such as the perturbation theory of general relativity on cosmological background spacetimes and spherical black hole spacetimes with scalar fields will be interesting.

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