Tools for NLO automation: extension of the golem95C integral library

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Abstract

We present an extension of the program golem95C for the numerical evaluation of scalar integrals and tensor form factors entering the calculation of one-loop amplitudes, which supports tensor ranks exceeding the number of propagators. This extension allows various applications in Beyond the Standard Model physics and effective theories, for example higher ranks due to propagators of spin two particles, or due to effective vertices. Complex masses are also supported. The program is not restricted to the Feynman diagrammatic approach, as it also contains routines to interface to unitarity-inspired numerical reconstruction of the integrand at the tensorial level. Therefore it can serve as a general integral library in automated programs to calculate one-loop amplitudes.

PACS: 12.38.Bx

Key words: NLO computations, one-loop diagrams, tensor integrals, higher spin representations, effective theories
NEW VERSION PROGRAM SUMMARY

Manuscript Title: Tools for NLO automation: extension of the golem95C integral library
Authors: J. Ph. Guillet, G. Heinrich, J. F. von Soden-Fraunhofen
Program Title: golem95-1.3.0
Licensing provisions: none
Programming language: Fortran95
Computer: Any computer with a Fortran95 compiler
Operating system: Linux, Unix
RAM: RAM used per integral/form factor is insignificant
Keywords: NLO computations, one-loop diagrams, tensor integrals, higher spin representations, effective theories
PACS: 12.38.Bx
Classification: 4.4, 11.1
External routines/libraries: some finite scalar integrals are called from OneLOop [1,2], the option to call them from LoopTools [3,4] is also implemented.
Catalogue identifier of previous version: AEEO_v2_0
Journal reference of previous version: Comput. Phys. Commun. 182 (2011) 2276.
Does the new version supersede the previous version?: yes
Nature of problem: Evaluation of one-loop multi-leg integrals occurring in the calculation of next-to-leading order corrections to scattering amplitudes in particle physics. In the presence of particles with spin two in the loop, or effective vertices, or certain gauges, tensor integrals where the rank exceeds the number of propagators \(N\) are required.
Solution method: Extension of the reduction algorithm to rank \(r \leq 10\) for \(N \leq 4\) and \(r \leq N + 1\) for \(N \geq 5\), which is sufficient for most applications in Beyond the Standard Model Physics.
Reasons for the new version: The previous version was restricted to tensor ranks less or equal to the number of propagators.
Summary of revisions: Tensor ranks \(r > N\) are supported, an alternative reduction method for the case of small Gram determinants is implemented, numerical stability for the case of small mass differences has been improved.
Running time: Depends on the nature of the problem. A single call to a rank 6 five-point form factor at a randomly chosen kinematic point, using real masses, takes \(10^{-3}\) seconds on an Intel Core 4 i7-3770 CPU with a 3.4 GHz processor.
1 Introduction

Over the last years, enormous progress has been made to push the calculation of NLO corrections towards a higher number of particles in the final states, i.e. to “multi-leg” amplitudes, in QCD as well as in the electroweak sector.

Nowadays, the efforts are focused on the goals of automating multi-leg one-loop calculations and making them publicly available. Programs aiming at the complete automation of one-loop amplitude calculations, including the generation of the amplitude requested by the user “on the fly”, are e.g. FeynArts/FormCalc [3,5], HELAC-NLO [6], GoSam [7], aMCSNLO [8], BlackHat [9], NJET [10], OpenLoops [11], Recola [12]. Public programs which contain a collection of pre-generated processes are e.g. MCFM [13], VBFNLO [14].

An important ingredient for such programs is an integral library containing the one-loop integrals which are the basic building blocks of any one-loop amplitude unless it is calculated purely numerically. Several libraries are publicly available to date: FF [15], Looptools [4], QCDLoop [16], OneLOop [2], golem95C [17,18], PFRY [19]. A code for the calculation of one-loop four-point functions with complex masses (D0C) can be found in [20]. The latter has been integrated into the LoopTools library [4] where the complex versions of infrared finite integrals with less than four legs are already implemented. A complete set of scalar four-point integrals, both in dimensional and in mass regularisation and valid also for complex masses can be found in [21] in analytic form.

Public programs dedicated to the reduction of multi-leg one-loop amplitudes at integrand level are e.g. CutTools [22] and Samurai [23]. A novel reduction algorithm based on integrand reduction through Laurent series expansion also has been developed [24].

The calculation of scalar one-loop integrals has a long tradition of pioneering work, see e.g. [25,26,27,28,29,30,31]. For processes involving unstable particles, these integrals are also required for complex internal masses, in order to be able to work within the so-called “complex-mass scheme” developed in Refs. [32,33]. An extension of the golem95 library to complex masses has been presented in Ref. [18], called golem95C. The strategy in golem95C to avoid numerical instabilities due to small Gram determinants is to avoid the reduction to scalar basis integrals in such cases, in favour of turning to the numerical evaluation of a convenient one-dimensional integral representation of the respective tensor integral. Recent new developments in this direction can be found in Ref. [34].

In this article, we present an extension of the golem95C library to integrals with tensor ranks $r$ exceeding the number of propagators $N$. Such integrals occur for example in effective theories (a prominent example is the effective...
coupling of gluons to the Higgs boson), or in calculations within theories containing spin two particles beyond the leading order. Even though most of the modern methods to reduce one-loop amplitudes do not entirely rely on Feynman diagrams and tensor reduction anymore, the latter is still important in a number of cases. In particular, methods based on the reconstruction of tensor coefficients at integrand level \[35,36\] have proven very efficient recently \[11,12\]. Therefore, a tensor and scalar integral library represents an important tool for calculations within and beyond the Standard Model.

Further, tensor reduction, or tensorial reconstruction \[36\], can be an important “rescue system” for phase space points where cut-based techniques do not provide the desired accuracy. As such, the \texttt{golem95C} library is an important ingredient for the automated one-loop program \texttt{GoSam} \[7\]. The extension of cut based reduction methods at integrand level to higher ranks also has been tackled already \[24\]. The higher rank integrals (closely related to integrals in more than \(D = 4 - 2\epsilon\) space-time dimensions) can be written in terms of some “basis integrals” which can be called from the \texttt{golem95C} library presented here. However, the library contains the higher rank form factors in full generality. Therefore it can be used in various applications where the number of loop momenta in the integrand is large, in combination with integrand reduction methods ranging from traditional tensor reduction to cut-based methods.

This article is organized as follows. In Section 2, we review the theoretical background, while in Section 3 we focus on the new extension of the library to integrals of higher ranks. Section 4 contains installation instructions, while in Section 5 the user can find simple examples how to run the program. Section 6 contains our conclusions.

### 2 Theoretical background

The program is an update of the tensor and scalar integral library described in more detail in Ref. \[17\], based on the formalism developed in Refs. \[37,38\] to reduce tensor integrals to a convenient set of basis integrals. Similar reduction schemes can be found e.g. in Refs. \[30,31,39,40,41,42,43,44,45\]. Here we will describe the theoretical framework only briefly and focus on the new features of the program.

#### 2.1 Form Factors

Tensor integrals can be divided into a part containing the Lorentz structure and a part consisting of scalar quantities, which we call form factors.
We define an $N$-point tensor integral of rank $r$ in $D = 4 - 2\epsilon$ dimensions as

$$I_{D,m}^{\mu_1...\mu_r}(a_1,\ldots,a_r) = \int \frac{d^D q}{i \pi^{D/2}} \frac{q_{\mu_1}^{a_1} \cdots q_{\mu_r}^{a_r}}{(q_1^2 - m_1^2 + i\delta) \cdots (q_N^2 - m_N^2 + i\delta)}$$

(1)

where $q_a = q + r_a$, $q$ is the loop momentum, and $r_a$ is a combination of external momenta. Using the shift invariant vectors

$$\Delta_{ij}^\mu = r_i^\mu - r_j^\mu,$$

(2)

we can write down a general form factor decomposition of an arbitrary tensor integral

$$I_{D,m}^{\mu_1...\mu_r}(a_1,\ldots,a_r; S) = \sum_{j_1,\ldots,j_r \in S} \left[ \Delta_{j_1}^{\mu_1} \cdots \Delta_{j_r}^{\mu_r} \right]_{\{a_1\ldots a_r\}} A_{j_1...j_r}^{N,r}(S)$$

(3)

$$+ \sum_{j_1,\ldots,j_{r-2} \in S} \left[ g^{\mu} \Delta_{j_1}^{\mu_1} \cdots \Delta_{j_{r-2}}^{\mu_{r-2}} \right]_{\{a_1\ldots a_r\}} B_{j_1...j_{r-2}}^{N,r}(S)$$

$$+ \sum_{j_1,\ldots,j_{r-4} \in S} \left[ g^{\mu} g^{\nu} \Delta_{j_1}^{\mu_1} \cdots \Delta_{j_{r-4}}^{\mu_{r-4}} \right]_{\{a_1\ldots a_r\}} C_{j_1...j_{r-4}}^{N,r}(S)$$

$$+ \sum_{j_1,\ldots,j_{r-6} \in S} \left[ g^{\mu} g^{\nu} g^{\sigma} \Delta_{j_1}^{\mu_1} \cdots \Delta_{j_{r-6}}^{\mu_{r-6}} \right]_{\{a_1\ldots a_r\}} D_{j_1...j_{r-6}}^{N,r}(S)$$

$$+ \ldots .$$

The notation $[\cdots]_{\{a_1\ldots a_r\}}$ stands for the distribution of the $r$ Lorentz indices $\mu_i$, and the momentum labels $a_i$, to the vectors $\Delta_{j \ a_i}^\mu$ and metric tensors in all distinguishable ways. Note that the choice $r_N = 0$, $a_i = N \forall i$ leads to the well known representation in terms of external momenta where the labels $a_i$ are not necessary, but we prefer a completely shift invariant notation here.

$S$ denotes an ordered set of propagator labels, corresponding to the momenta forming the kinematic matrix $S$, defined by

$$S_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2, \quad i, j \in \{1,\ldots,N\}.$$  

(4)

The kinematic matrix $S$ is related to the Gram matrix $G_{ij}$ ($i, j = 1,\ldots,N - 1$ for $r_N = 0$) by

$$\det G = (-1)^{N+1} B \det S, \quad B = \sum_{i,j=1}^{N} S_{ij}^{-1}.$$  

(5)

We should point out that the form factors of type $D_{j_1...j_{r-6}}^{N,r}$ and beyond, i.e. form factors associated with three or more metric tensors, are not needed for integrals where the rank $r$ does not exceed the number $N$ of propagators, no
matter what the value of $N$ is. This is because integrals with $N \geq 6$ can be reduced algebraically to pentagons, without generating higher dimensional remainder terms, using the following formula recursively [37]

$$I^D,\mu_1...\mu_r(a_1,\ldots,a_r;S) = -\sum_{j\in S} C^\mu_{ja} I^D,\mu_1...\mu_{r-1}(a_1,\ldots,a_{r-1};S\{j\}) \quad (N \geq 6),$$

where $C^\mu_{ja}$ is the solution of the equation

$$\sum_{j\in S} S_{ij} C^\mu_{ja} = \Delta^\mu_{ia}, \quad a \in S. \tag{6}$$

If $N \geq 7$, or in the case of exceptional kinematics, $S$ is not invertible, so eq. (6) does not have a unique solution. However, an explicit solution can be constructed as shown in [37,46]. In this sense the tensor reduction of $N$-point integrals with $N \geq 6$ is trivial: integrals with $N \geq 6$ can be reduced iteratively to 5-point integrals. Therefore form factors for $N \geq 6$ are never needed. If the rank $r$ of an $N$-point integral does not exceed $N$, Lorentz structures carrying at most two factors of $g^{\mu\nu}$ are sufficient, as the maximal form factor needed is $C^5_5$, corresponding to a rank five 5-point integral. However, for $r > N$, we can also have e.g. rank six 5-point integrals, which contain Lorentz structures involving three factors of $g^{\mu\nu}$, and therefore form factors which go beyond the types $A^N_r, B^N_r, C^N_r$ are needed.

The form factors are linear combinations of algebraic reduction coefficients, derived from the matrix $S$, and $N$-point integrals with $N \leq 4$. Explicit expressions for $r \leq N$ are given in [37].

$2.2$ $Integrals$

The golem95c program uses the fact that tensor integrals are related to Feynman parameter integrals with Feynman parameters in the numerator. A scalar integral, after Feynman parametrisation, can be written as

$$I^D_N(S) = (-1)^N \Gamma(N - \frac{D}{2}) \int \prod_{i=1}^N dz_i \delta(1 - \sum_{l=1}^N z_l) \left( R^2 \right)^{\frac{D}{2} - N}$$

$$R^2 = -\frac{1}{2} \sum_{i,j=1}^N z_i S_{ij} z_j - i\delta. \tag{7}$$

The general relation between tensor integrals and parameter integrals with Feynman parameters in the numerator is well known [47,30,38]:

6
\[
I_N^{D, \mu_1 \ldots \mu_r} (a_1, \ldots, a_r; S) = (-1)^r \sum_{m=0}^{[r/2]} \left( -\frac{1}{2} \right)^m 
\sum_{j_1, \ldots, j_{r-2m}=1}^N \left[(g^{-}) \otimes^m \Delta_{j_1} \cdots \Delta_{j_r} \right]_{\{\mu_1 \ldots \mu_r\}} I_N^{D+2m} (j_1 \ldots, j_{r-2m}; S),
\]

where \(I_N^{D+2m} (j_1 \ldots, j_{r-2m}; S)\) is an integral with Feynman parameters in the numerator. \([r/2]\) stands for the nearest integer less or equal to \(r/2\) and the symbol \(\otimes m\) indicates that \(m\) powers of the metric tensor are present. Feynman parameter integrals corresponding to diagrams where propagators \(l_1, \ldots, l_m\) are omitted with respect to the “maximal” topology can be defined as

\[
I_N^D (j_1, \ldots, j_r; S \setminus \{l_1, \ldots, l_m\}) = (-1)^N \Gamma \left( N - \frac{D}{2} \right) 
\int \prod_{i=1}^N dz_i \delta \left( 1 - \sum_{k=1}^N z_k \right) \delta (z_{l_1}) \cdots \delta (z_{l_m}) z_{j_1} \cdots z_{j_r} \left( R^2 \right)^{D/2-N}.
\]

The program golem95C reduces the integrals internally to a set of basis integrals, i.e. the endpoints of the reduction (they do not form a basis in the mathematical sense, as some of them are not independent). The choice of the basis integrals can have important effects on the numerical stability in certain kinematic regions. Our reduction endpoints are 4-point functions in 6 dimensions \(I_6^4\), which are IR and UV finite, 4-point functions in \(D+4\) dimensions, and various 2-point and 3-point functions, some of the latter with Feynman parameters in the numerator. This provides us with a convenient separation of IR and UV divergences, as the IR poles are exclusively contained in the triangle functions.

Note that \(I_4^{D+2}\) and \(I_4^{D+4}\) are UV divergent, while \(I_4^D\) can be IR divergent. In the code, the integrals are represented as arrays containing the coefficients of their Laurent expansion in \(\epsilon = (4 - D)/2\).

If the endpoints of the reduction contain finite scalar four-point integrals, the latter have not all been coded explicitly, but some are automatically called from the library OneLOop [2].

3 Extension of the program to higher rank

We focus here on the new features, for more details on the software components which are the same as in version 1.2, we refer to [17,18] and to the documentation contained in the program.
The call syntax for the higher rank form factors is analogous to the one for \( r \leq N \), where the form factors defined in eq. (3) have been extended to include higher ranks. The code is built such that the reduction procedure is valid for arbitrary ranks. Form factors up to rank ten for \( N \leq 4 \) and up to rank six for \( N = 5 \) have been hardcoded, additional ones can be easily generated using the python script `gen_form_factors.py` in the subdirectory `tool/highrank`. Explicit examples how to call the form factor for a rank four triangle, a rank five box and a rank six pentagon are given in section 5.

### 4 Installation

The program can be downloaded as golem95-1.3.0.tar.gz from the following URL: [http://golem.hepforge.org/95/](http://golem.hepforge.org/95/). The installation instructions given below also can be found in the README file coming with the code.

The installation setup is based on autotools [48]. To install the `golem95C` library, type the following commands:

```
./configure [--prefix=mypath] [--precision=quadruple] [FC=compiler]
make
make install
```

The `--prefix` option denotes the installation prefix, under which the directories `lib/` and `include/` are generated. If no option is given, on a Linux system the configure script would choose `prefix=/usr/local`. The argument `--precision` selects double or quadruple precision to be used in the library; it should be noted that quadruple precision is not supported by all Fortran compilers and that `precision=double` is the default value. If the variable FC is not set the first Fortran compiler which is automatically detected will be used. Another variable commonly used is `FCFLAGS` which allows one to pass compiler flags to the Fortran compiler.

As an alternative to the call of finite box integrals from OneLOop [2], it is possible to call finite scalar box and triangle integrals with internal masses from LoopTools [3]. To use this option, the user should (a) install LoopTools, and
(b) use the option --with-looptools=path_to_libooptools.a for the configure script, i.e. type the following commands:

```bash
./configure [--prefix=mypath] [--with-looptools=path_to_libooptools.a]
[--precision=quadruple] [FC=compiler] [F77= fortran77compiler]
make
make install
```

5 Usage and examples

Examples for the usage of the program can be found in the subdirectory demos. The examples for the higher rank form factors are also described below.

The basic structure of a program using the calculation of \(N\)-point scalar integrals or form factors by golem95C is:

```fortran
!use precision_golem ! to get the type ki (for real and complex)
use matrix_s ! needed for initgolem95, s_mat, etc.
use constants ! contains useful constants
use form_factor_type
use parameter ! contains default parameter settings
use form_factor_3p ! module containing the three-point form factors
implicit none

program main
    type(form_factor) :: res
    real(ki) :: s1, s2, s3, m1sq, m2sq, m3sq
    s1 = 1.0_ki
    s2 = 0.0_ki
    s3 = 0.0_ki
    m1sq = 2.0_ki
    m2sq = 0.0_ki
    m3sq = 0.0_ki
    call initgolem95(3) ! initialisation of caching system and 3x3 matrix S
    ! definition of the kinematic matrix S
    s_mat(1,:) = (/ -m1sq+m2sq, s2-m1sq-m2sq, s1-m1sq-m3sq /)
    s_mat(2,:) = (/ s2-m1sq-m2sq, -m2sq+2.0_ki, s3-m2sq-m3sq /)
    s_mat(3,:) = (/ s1-m1sq-m3sq, s3-m2sq-m3sq, -m3sq+2.0_ki /)
    call preparesmatrix() ! call the scalar triangle
    res = a30(s_null)
end program main
```

A simple example for the calculation of a scalar three-point integral, which is evaluated by calling the form factor A30, is given in Figure 1.

Listing 1. Example of a program calling a scalar three-point integral.
5.1 Three-point rank 4 example

The executable for this example is created by make demo_3point. The program demos/demo_three_point.f90 contains as an option the call to a rank four three-point form factor, \( A_{34}(1, 1, 2, 3, S) \). The integers in the argument list denote the labels of the corresponding momenta in the form factor representation, i.e. \( A_{34}^{3,4} \) in the notation of eq. (3), denoting the coefficient of the term \( \Delta_{1a_1}^{\mu_1} \Delta_{2a_2}^{\mu_2} \Delta_{3a_3}^{\mu_3} \Delta_{4a_4}^{\mu_4} \) in the form factor decomposition of the Lorentz structure. Note that using \( a_j = 3 \) and \( r_3 = 0 \), \( \Delta_{ija} \) is simply replaced by \( r_i \). The call syntax is the same as for the lower rank form factors. The results are written to the file test3point.txt. As a check for the user, the results to be obtained when calling \( A_{34}(1, 1, 2, 3, S) \) are listed in table_of_results_3point_option_n.txt. The option \( n \) in table_of_results_3point_option_n.txt denotes different choices for the kinematics of the three-point function. Once the kinematics is chosen, the rank five three-point example corresponds to option (b) number 7.

5.2 Four-point rank 5 example

The executable for this example is created by make demo_4point. The program demos/demo_four_point.f90 contains as an option the call to a rank five four-point form factor, \( A_{45}(1, 2, 2, 3, 3, S) \). The integers in the argument list denote the labels of the corresponding momenta in the form factor representation, i.e. \( A_{45}^{4,5} \) in the notation of eq. (3), denoting the coefficient of the term \( \Delta_{1a_1}^{\mu_1} \Delta_{2a_2}^{\mu_2} \Delta_{3a_3}^{\mu_3} \Delta_{4a_4}^{\mu_4} \Delta_{5a_5}^{\mu_5} \) in the form factor decomposition of the Lorentz structure. The call syntax is the same as for the lower rank form factors. The results are written to the file test4point.txt. As a check for the user, the results to be obtained when calling \( A_{45}(1, 2, 2, 3, 3, S) \) are listed in table_of_results_4point_option_n.txt, where again \( n \) denotes different choices for the kinematics. Once the kinematics is chosen, the rank five 4-point example corresponds to option (b) number 9.
5.3 Five-point rank 6 example

The executable for this example is created by `make demo_5point`. The program `demos/demo_five_point.f90` contains as an option the call to a rank four three-point form factor, \( A_{56}(1, 2, 3, 4, 5, S) \). The integers in the argument list denote the labels of the corresponding momenta in the form factor representation, i.e. \( A_{112345}^{56} \) in the notation of eq. (3), denoting the coefficient of the term \( \Delta_{\mu_1}^{\mu_1} \Delta_{\mu_2}^{\mu_2} \Delta_{\mu_3}^{\mu_3} \Delta_{\mu_4}^{\mu_4} \Delta_{\mu_5}^{\mu_5} \Delta_{\mu_6}^{\mu_6} \) in the form factor decomposition of the Lorentz structure. The call syntax is the same as for the lower rank form factors. The results are written to the file `test5point.txt`. As a check for the user, the results to be obtained when calling \( A_{56}(1, 2, 3, 4, 5, S) \) (option 5 in `demo_five_point.f90`) are listed in `table_of_results_5point.txt`.

6 Conclusions

We have presented an extension of the program `golem95C` which provides a library of scalar and tensor integrals where the tensor rank \( r \), denoting the number of loop momenta in the numerator, can exceed the number of propagators \( N \). We have implemented ranks up to \( r = 10 \) for \( N \leq 4 \), and 5-point and 6-point functions up to rank \( r = N + 1 \). The use of both real or complex masses is possible within the same setup. The program, which is an extension of on an earlier version of the `golem95C` library, now also can be used in the presence of effective vertices and in models where the presence of spin two particles can require integrals with higher ranks than usually needed in renormalizable theories. Due to an appropriate interface, the program can be used both within a traditional tensor reduction approach as well as within unitarity-inspired numerical reconstruction of the integrand at the tensorial level. The program is publicly available at [http://golem.hepforge.org/95/](http://golem.hepforge.org/95/).

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A Appendix: \( \tilde{k} \) - integrals

Here we give a list of integrals which are needed if the momenta in the numerator are split into a 4-dimensional and a \((D - 4)\)-dimensional part, according to \( k^\mu_{(D)} = \tilde{k}^\mu_{(4)} + \tilde{k}^\mu_{(2s)} \), \( k^\mu_{(D)} = \tilde{k}^2 + \tilde{k}^2 \). Using \( \tilde{q}_a = \tilde{k} + r_a \), where \( \tilde{k} \) is the loop momentum in 4 dimensions, and \( r_a \) is a combination of external momenta, we define

\[
I_{N}^{D,\alpha;\mu_1 \ldots \mu_r}(a_1, \ldots, a_r; S) \equiv \int \frac{d^D k}{i\pi^{D/2}} \frac{(\tilde{k}^2)^\alpha}{\prod_{j=1}^N (q_j^2 - m_j^2 + i\delta)}. \tag{A.1}
\]

The integrals with additional powers of \((\tilde{k}^2)^\alpha\) in the numerator are related to integrals in higher dimensions by

\[
I_{N}^{D,\alpha;\mu_1 \ldots \mu_r}(a_1, \ldots, a_r; S) = (-1)^{r+\alpha} \frac{\Gamma(\alpha + D/2 - 2)}{\Gamma(D/2 - 2)} \sum_{m=0}^{[r/2]} \left( -\frac{1}{2} \right)^m \times \sum_{j_1, \ldots, j_{r-2m} = 1}^N \Delta_{j_1, \ldots, j_{r-2m}, \mu_1, \ldots, \mu_r} I_{N}^{D+2\alpha+2m}(j_1, \ldots, j_{r-2m}; S), \tag{A.2}
\]

where \( \Delta_{ij} = r_i - r_j \). Note that for \( D = 4 - 2\epsilon \), the prefactor in eq. (A.2) reads

\[
\frac{\Gamma(\alpha + D/2 - 2)}{\Gamma(D/2 - 2)} = \frac{\Gamma(\alpha - \epsilon)}{\Gamma(-\epsilon)} = -\epsilon (\alpha - 1)! + \mathcal{O}(\epsilon^2). \tag{A.3}
\]

Therefore, integrals with \( \alpha > 0 \) will only contribute if the integrals \( I_{N}^{D+2\alpha+2m} \) are UV divergent, since we can drop terms of \( \mathcal{O}(\epsilon) \) for one-loop applications. The coefficient of the UV pole of these integrals, which is projected out in this way, will contribute to the so-called “rational part” of an amplitude. For \( D = 4 - 2\epsilon \), the integral \( I_{N}^{D+2\alpha+2m} \) will be proportional to \( \Gamma(\epsilon - \eta) \), with \( \eta = 2 - N + \alpha + m \). Therefore it will contain an UV divergence if \( \eta \geq 0 \).

The results for those integrals which are relevant for the rational part can be given in a general form \[49\].

\[
\epsilon I_{N}^{D+2\alpha+2m}(l_1, \ldots, l_r; S) = \frac{(-1)^N}{2^\eta \eta!} \sum_{j_1, \ldots, j_{2\eta} = 1}^N \frac{S_{j_1 j_2} \cdots S_{j_{2\eta-1} j_{2\eta}} P_N(l_1, \ldots, l_r, j_1, \ldots, j_{2\eta})}{N - 1 + \sum_{t=1}^N t_t!}, \tag{A.4}
\]

where we define

\[
P_{t_1, t_2, \ldots, t_N} = \frac{\prod_{j=1}^N (t_j!)}{(N - 1 + \sum_{t=1}^N t_t)!} \tag{A.5}
\]

to arrive at \( P_N(j_1, \ldots, j_s) \), which counts the indices in an expression:

\[
P_N(j_1, \ldots, j_s) = P_{(\sum_{i=1}^s \delta_{i, j_1}), \ldots, (\sum_{i=1}^s \delta_{N, j_1})}. \tag{A.6}
\]
Further, $S_{j_1j_2}$ denotes an element of the kinematic matrix $S$.

The integrals $I_{N}^{D+2a+2m}$ will be UV divergent for $2a + 2m \geq 2N - 4$. Below we give the relevant explicit expressions, derived from eq. [A.4]. Terms which will be of order $\mathcal{O}(\epsilon)$ are dropped.

\[ \epsilon I_{N}^{D-4+2N}(S) = \frac{(-1)^N}{(N-1)!} \] (A.7a)

\[ \epsilon I_{N}^{D-4+2N}(l_1; S) = \frac{(-1)^N}{N!} \] (A.7b)

\[ \epsilon I_{N}^{D-4+2N}(l_1, l_2; S) = \frac{(-1)^N}{(N+1)!}(1 + \delta_{l_1l_2}) \] (A.7c)

\[ \epsilon I_{N}^{D-4+2N}(l_1, l_2, l_3; S) = \frac{(-1)^N}{(N+2)!} \times (1 + \delta_{l_1l_2} + \delta_{l_1l_3} + \delta_{l_2l_3} + 2\delta_{l_1l_2}\delta_{l_2l_3}) \] (A.7d)

\[ \epsilon I_{N}^{D-4+2N}(l_1, l_2, l_3, l_4; S) = \frac{(-1)^N}{(N+3)!} \times \left( \delta_{l_1l_2}(6\delta_{l_1l_3}\delta_{l_2l_4} + 2\delta_{l_1l_3} + 2\delta_{l_2l_4} + \delta_{l_3l_4}) \right) \] (A.7e)

\[ \epsilon I_{N}^{D-4+2(N+1)}(S) = \frac{(-1)^N}{2(N+1)!} \left( \sum_{j_1,j_2=1}^{N} S_{j_1j_2} + tr\{S\} \right) \] (A.7f)

\[ \epsilon I_{N}^{D-4+2(N+1)}(l_1; S) = \frac{(-1)^N}{2(N+2)!} \sum_{j_1,j_2=1}^{N} S_{j_1j_2} (1 + \delta_{j_1j_2}) \] (A.7g)

\[ \epsilon I_{N}^{D-4+2(N+1)}(l_1, l_2; S) = \frac{(-1)^N}{2(N+3)!} \sum_{j_1,j_2=1}^{N} S_{j_1j_2} \] (A.7h)

\[ \times (\delta_{j_1j_2}(6\delta_{j_1l_1}\delta_{j_2l_2} + 2\delta_{j_1l_1} + 2\delta_{j_2l_2} + \delta_{l_1l_2}) \right) \] \[ + 2\delta_{l_1l_2}(\delta_{j_1l_1} + \delta_{j_2l_2} + \delta_{j_1l_1}\delta_{j_2l_2} + \delta_{l_1l_2}\delta_{j_1l_1} \] \[ + \delta_{j_1l_2} + \delta_{j_1l_1} + \delta_{j_1l_2} + \delta_{j_2l_1} + \delta_{j_2l_2} + \delta_{l_1l_2} + 1) \right). \]

We also give here the explicit formulas for rank 6 pentagon integrals involving $\tilde{k}^2$ terms, because the latter are special to higher rank extensions. Again, terms of $\mathcal{O}(\epsilon)$ are dropped.
\[ I^{D,3}_{5}(S) = \int \frac{d^Dk}{i\pi^{D/2}} \frac{\left(\hat{k}^2\right)^3}{\prod_{j=1}^{5}(q_j^2 - m_j^2 + i\delta)} = -\frac{1}{12} \] (A.8)

\[ I^{D,2;\mu_1\mu_2}_{5}(a_1, a_2; S) = \int \frac{d^Dk}{i\pi^{D/2}} \frac{\left(\hat{k}^2\right)^2 \hat{q}_{a_1}^{\mu_1} \hat{q}_{a_2}^{\mu_2}}{\prod_{j=1}^{5}(q_j^2 - m_j^2 + i\delta)} = -\frac{1}{48} g^{\mu_1\mu_2} \] (A.9)

\[ I^{D,1;\mu_1\cdots\mu_4}_{5}(a_1, \ldots, a_4; S) = \int \frac{d^Dk}{i\pi^{D/2}} \frac{\hat{k}^2 \hat{q}_{a_1}^{\mu_1} \cdots \hat{q}_{a_4}^{\mu_4}}{\prod_{j=1}^{5}(q_j^2 - m_j^2 + i\delta)} \]

\[ = -\frac{1}{96} \left[g^{\mu_1\mu_2}g^{\mu_3\mu_4} + g^{\mu_1\mu_3}g^{\mu_2\mu_4} + g^{\mu_1\mu_4}g^{\mu_2\mu_3}\right]. \] (A.10)

As another example, we give the expressions for the rational part of box integrals in \(D+6\) dimensions, needed for rank six four-point functions, which can be obtained from (A.7f):

\[ \epsilon I^{D+6}_{4}(S) = \frac{1}{240} \left( \sum_{i,j=1}^{4} (\Delta_{ij}^2 - m_i^2 - m_j^2) - 2 \sum_{i=1}^{4} m_i^2 \right) + \mathcal{O}(\epsilon). \] (A.11)

References

[1] A. van Hameren, C. G. Papadopoulos, and R. Pittau. Automated one-loop calculations: a proof of concept. JHEP, 09:106, 2009.

[2] A. van Hameren. OneLOop: For the evaluation of one-loop scalar functions. Comput. Phys. Commun., 182:2427–2438, 2011.

[3] T. Hahn and M. Perez-Victoria. Automatized one-loop calculations in four and D dimensions. Comput. Phys. Commun., 118:153–165, 1999.

[4] T. Hahn. Feynman Diagram Calculations with FeynArts, FormCalc, and LoopTools. PoS, ACAT2010:078, 2010.

[5] S. Agrawal, T. Hahn, and E. Mirabella. FormCalc 7.5. PoS, LL2012:046, 2012.

[6] G. Bevilacqua, M. Czakon, M.V. Garzelli, A. van Hameren, A. Kardos, et al. HELAC-NLO. Comput. Phys. Commun., 184:986–997, 2013.

[7] G. Cullen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, et al. Automated One-Loop Calculations with GoSam. Eur.Phys.J., C72:1889, 2012.

[8] V. Hirschi, R. Frederix, S. Frixione, M. V. Garzelli, F. Maltoni, et al. Automation of one-loop QCD corrections. JHEP, 1105:044, 2011.

[9] C.F. Berger, Z. Bern, L.J. Dixon, F. Febres Cordero, D. Forde, et al. An Automated Implementation of On-Shell Methods for One-Loop Amplitudes. Phys.Rev., D78:036003, 2008.
[10] S. Badger, B. Biedermann, P. Uwer, and V. Yundin. NLO QCD corrections to multi-jet production at the LHC with a centre-of-mass energy of $\sqrt{s} = 8$ TeV. *Phys. Lett.*, B718:965–978, 2013.

[11] F. Cascioli, P. Maierhöfer, and S. Pozzorini. Scattering Amplitudes with Open Loops. *Phys. Rev. Lett.*, 108:111601, 2012.

[12] S. Actis, A. Denner, L. Hofer, A. Scharf, and S. Uccirati. Recursive generation of one-loop amplitudes in the Standard Model. *JHEP*, 1304:037, 2013.

[13] J. Campbell and R. K. Ellis. Next-to-leading order corrections to $W + 2$jet and $Z + 2$jet production at hadron colliders. *Phys. Rev.*, D65:113007, 2002.

[14] K. Arnold et al. VBFNLO: A parton level Monte Carlo for processes with electroweak bosons. *Comput. Phys. Commun.*, 180:1661–1670, 2009.

[15] G. J. van Oldenborgh. FF: A Package to evaluate one loop Feynman diagrams. *Comput. Phys. Commun.*, 66:1–15, 1991.

[16] R. K. Ellis and G. Zanderighi. Scalar one-loop integrals for QCD. *JHEP*, 02:002, 2008.

[17] T. Binoth, J. Ph. Guillet, G. Heinrich, E. Pilon, and T. Reiter. Golem95: a numerical program to calculate one-loop tensor integrals with up to six external legs. *Comput. Phys. Commun.*, 180:2317–2330, 2009.

[18] G. Cullen, J. Ph. Guillet, G. Heinrich, T. Kleinschmidt, E. Pilon, et al. Golem95C: A library for one-loop integrals with complex masses. *Comput. Phys. Commun.*, 182:2276–2284, 2011.

[19] J. Fleischer, T. Riemann, and V. Yundin. PJFry: A C++ package for tensor reduction of one-loop Feynman integrals. 2011.

[20] Dao Thi Nhung and Le Duc Ninh. D0C : A code to calculate scalar one-loop four-point integrals with complex masses. *Comput. Phys. Commun.*, 180:2258–2267, 2009.

[21] A. Denner and S. Dittmaier. Scalar one-loop 4-point integrals. *Nucl.Phys.*, B844:199–242, 2011.

[22] G. Ossola, C. G. Papadopoulos, and R. Pittau. CutTools: a program implementing the OPP reduction method to compute one-loop amplitudes. *JHEP*, 03:042, 2008.

[23] P. Mastrolia, G. Ossola, T. Reiter, and F. Tramontano. Scattering AMplitudes from Unitarity-based Reduction Algorithm at the Integrand-level. *JHEP*, 08:080, 2010.

[24] P. Mastrolia, E. Mirabella, and T. Peraro. Integrand reduction of one-loop scattering amplitudes through Laurent series expansion. *JHEP*, 1206:095, 2012.

[25] G. J. van Oldenborgh and J. A. M. Vermaseren. New Algorithms for One Loop Integrals. *Z. Phys.*, C46:425–438, 1990.
[26] G. ’t Hooft and M. J. G. Veltman. Scalar One Loop Integrals. *Nucl. Phys.*, B153:365–401, 1979.

[27] K. Fabricius and I. Schmitt. Calculation Of Dimensionally Regularized Box Graphs In The Zero Mass Case. *Z. Phys.*, C3:51–53, 1979.

[28] W. Beenakker and A. Denner. Infrared Divergent Scalar Box Integrals With Applications In The Electroweak Standard Model. *Nucl. Phys.*, B338:349–370, 1990.

[29] A. Denner, U. Nierste, and R. Scharf. A Compact expression for the scalar one loop four point function. *Nucl. Phys.*, B367:637–656, 1991.

[30] Z. Bern, L. J. Dixon, and D. A. Kosower. Dimensionally regulated one loop integrals. *Phys. Lett.*, B302:299–308, 1993.

[31] Z. Bern, L. J. Dixon, and D. A. Kosower. Dimensionally regulated pentagon integrals. *Nucl. Phys.*, B412:751–816, 1994.

[32] A. Denner, S. Dittmaier, M. Roth, and D. Wackeroth. Predictions for all processes $e^+e^- \rightarrow 4$ fermions + gamma. *Nucl. Phys.*, B560:33–65, 1999.

[33] A. Denner, S. Dittmaier, M. Roth, and L. H. Wieders. Electroweak corrections to charged-current $e^+e^- \rightarrow 4$ fermion processes: Technical details and further results. *Nucl. Phys.*, B724:247–294, 2005.

[34] J. Ph. Guillet, E. Pilon, M. Rodgers, and M.S. Zidi. Stable One-Dimensional Integral Representations of One-Loop N-Point Functions in the General Massive Case: I - Three Point Functions. *JHEP*, 1311:154, 2013.

[35] A. van Hameren. Multi-gluon one-loop amplitudes using tensor integrals. *JHEP*, 0907:088, 2009.

[36] G. Heinrich, G. Ossola, T. Reiter, and F. Tramontano. Tensorial Reconstruction at the Integrand Level. *JHEP*, 1010:105, 2010.

[37] T. Binoth, J. Ph. Guillet, G. Heinrich, E. Pilon, and C. Schubert. An algebraic/numerical formalism for one-loop multi-leg amplitudes. *JHEP*, 10:015, 2005, hep-ph/0504267 version 3.

[38] T. Binoth, J. P. Guillet, and G. Heinrich. Reduction formalism for dimensionally regulated one-loop N-point integrals. *Nucl. Phys.*, B572:361–386, 2000.

[39] G. Duplancic and B. Nizic. Reduction method for dimensionally regulated one-loop N-point Feynman integrals. *Eur. Phys. J.*, C35:105–118, 2004.

[40] W. T. Giele and E. W. Nigel Glover. A calculational formalism for one-loop integrals. *JHEP*, 04:029, 2004.

[41] F. del Aguila and R. Pittau. Recursive numerical calculus of one-loop tensor integrals. *JHEP*, 07:017, 2004.

[42] A. van Hameren, J. Volllinga, and S. Weinzierl. Automated computation of one-loop integrals in massless theories. *Eur. Phys. J.*, C41:361–375, 2005.
[43] A. Denner and S. Dittmaier. Reduction schemes for one-loop tensor integrals. *Nucl. Phys.*, B734:62–115, 2006.

[44] J. Fleischer and T. Riemann. A Complete algebraic reduction of one-loop tensor Feynman integrals. *Phys.Rev.*, D83:073004, 2011.

[45] J. Fleischer and T. Riemann. A solution for tensor reduction of one-loop N-point functions with $N \geq 6$. *Phys.Lett.*, B707:375–380, 2012.

[46] J. Fleischer, T. Riemann, and V. Yundin. New Results for Algebraic Tensor Reduction of Feynman Integrals. *PoS*, RADCOR2011:011, 2011.

[47] A. I. Davydychev. A Simple formula for reducing Feynman diagrams to scalar integrals. *Phys. Lett.*, B263:107–111, 1991.

[48] G. V. Vaughan, B. Elliston, T. Tromey, and I. L. Taylor. *GNU Autoconf, Automake, and Libtool: Expert insight into porting software and building large projects using GNU Autotools*. New Riders, Indianapolis, 2000.

[49] Thomas Reiter. Automated Evaluation of One-Loop Six-Point Processes for the LHC. *PhD Thesis*, arXiv:0903.0947, 2009.