A KNOTTED MINIMAL TREE

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Abstract. There is a finite set of points on the boundary of the three-dimensional unit ball whose minimal tree is knotted. This example answers a problem posed by Michael Freedman.

1. Introduction

In [1], “Problems in Low-dimensional Topology” by Rob Kirby, one can find the following:

Problem 5.17 (Freedman) Given a finite set of points $X$ in $\partial B^3$, let $T$ be a tree in $B^3$ of minimal length with $\partial T = X$. Is $T$ unknotted, that is, is there a PL imbedded 2-ball in $B^3$ containing $T$?

It is shown here that there is a finite set $X$ on the boundary unit 3-ball in $\mathbb{R}^3$ whose minimal tree is knotted. The cardinality of $X$ is quite large, but the construction essentially depends on seven elements only: 6 points and an arc on the boundary of the ball. An outline of the example, provided by the author of this paper, is contained in [1] following the statement of Problem 5.17.

The arc lies close to the equator circumventing it one and a half times. In a plane perpendicular to the equator, consider a regular hexagon whose one pair of antipodal vertices is very close to the endpoints of the arc. The set $X$ contains one of the other pairs of antipodal vertices. The remaining two vertices of the hexagon are split into two points each so that the points are closer to the equator and the minimal tree connects these points to the endpoints of the arc. Finally, the arc is replaced by a sequence of points of small mesh.

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2. Preliminaries and notation

Let $A$ be a locally connected compact set in $\mathbb{R}^3$ with finitely many components. A connecting graph for $A$ is a pair $(\mathcal{E}, \mathcal{V})$, where $\mathcal{E}$ is a finite collection of straight line segments (edges) and $\mathcal{V}$ is a set of points (vertices) consisting of endpoints of edges of $\mathcal{E}$, such that the set $A \cup \bigcup_{E \in \mathcal{E}} E$ is connected. We informally say that the graph consists of these edges. The length of a connecting graph is the sum of the lengths of its edges. A minimal graph for $A$ is a connecting graph for $A$ whose length is a minimum. A minimal graph for $A$ is denoted by $G(A)$, possibly with a subscript if more than one of such graphs is considered. The length of $G(A)$ is denoted by $|G(A)|$. No additional notation is used for the union of the elements of $G$, i.e., if two edges meeting at a vertex are collinear, then the vertex belongs to $A$. The order of a vertex is the number of edges meeting at this vertex.

If $A$ is finite, then a minimal graph $G(A)$ is a tree, i.e., it is connected and acyclic. It is then called a minimal tree for $A$ and it is denoted by $T(A)$. Its length is denoted by $|T(A)|$. A simple triod is a tree consisting of three edges meeting at a vertex. For a three point set $A = \{a, b, c\}, T(A)$ is unique. If one of the angles of the triangle $\triangle(a, b, c)$ is greater than or equal to $\frac{2\pi}{3}$, then the minimal tree consists of two edges. If all angles of $\triangle(a, b, c)$ are less than $\frac{2\pi}{3}$, then the minimal tree is a simple triod whose edges form $\frac{2\pi}{3}$ angles. In general, a minimal tree is not unique. For example, the set of vertices of a square has two minimal trees.

If four or more half-lines in $\mathbb{R}^3$ have a common endpoint $p$, then at least one of the angles between the half-lines is less than $\frac{2\pi}{3}$. If $q_1$ and $q_2$ are two distinct points equidistant to $p$ that are on two half-lines meeting at $p$ at an angle less than $\frac{2\pi}{3}$, then the minimal tree $T(\{p, q_1, q_2\})$ is a triod. Therefore, a vertex of a minimal graph $G(A)$ is either of order 3 or it belongs to $A$. The angles between the edges meeting at a vertex not in $A$ equal $\frac{2\pi}{3}$ and the edges are coplanar.

The segment joining the points $p$ and $q$ is denoted by $[p, q]$. The Euclidean distance is denoted by $d(p, q)$. For distinct points $p$, $q$ and $r$, denote by $L(q, r)$ the line passing through $q$ and $r$, and by $d(p, L(q, r))$ the perpendicular distance between $p$ and $L(q, r)$. The Hausdorff distance between the sets $A$ and $B$ is denoted by $d_H(A, B)$. We say that two sets $A$ and $B$ with the same finite number of components, $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$, respectively, are Hausdorff $\varepsilon$-close, if there is a permutation $\tau : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that for $i = 1, \ldots, k$, $d_H(A_i, B_{\tau(i)}) < \varepsilon$. If $p$ and $q$ are non-antipodal points on a circle or a sphere $C$, then $arc_C(pq)$ denotes the shortest arc in $C$ joining the points $p$ and $q$.

In here, the 3-ball $B^3$ is exactly the unit ball in $\mathbb{R}^3$. A PL imbedded 2-ball $D$ in $B^3$ is properly imbedded, i.e., $\partial B^3 \cap D = \partial D$. A tree for a finite set $A \subset \partial B^3$ is unknotted if there is a PL 2-ball $D$ containing the tree, or equivalently, if there is an isotopy of $B^3$ onto itself such that the image of the tree under the final stage of the isotopy is contained in the $xy$-plane.

Throughout the paper we use the following notation:

\begin{align*}
S^2 & = \partial B^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}, \\
P & = \{(x, y, z) \in S^2 \mid y = 0\}, \\
Q & = \{(x, y, z) \in S^2 \mid z = 0\},
\end{align*}

where $(x, y, z)$ denotes the Cartesian coordinates of a point in $\mathbb{R}^3$. 
3. Some special graphs

Let $H \subset P$ be the regular hexagon with vertices:

\[
\begin{align*}
a_1 &= \left(-\frac{1}{2}, 0, \sqrt{3} \right), & b_1 &= (-1, 0, 0), & c_1 &= \left(-\frac{1}{2}, 0, -\sqrt{3} \right), \\
a_2 &= \left(\frac{1}{2}, 0, -\sqrt{3} \right), & b_2 &= (1, 0, 0), & c_2 &= \left(\frac{1}{2}, 0, \sqrt{3} \right).
\end{align*}
\]

An easy verification gives the following:

**Lemma 1.** Every minimal tree $T(A)$ for the set $A$ consisting of four consecutive vertices of $H$ consists of three edges of $H$.

**Lemma 2.** Every minimal tree $T(A)$ for the set $A$ consisting of five vertices of $H$ consists of four edges of $H$.

**Proof.** Let $A = \{a_1, b_1, c_1, a_2, c_2\}$.

Suppose that $T(A)$ contains at least one edge of $H$. If $b_1$ is an order 1 vertex of $T(A)$ and it is one of the endpoints of the only edge of $H$ belonging to $T(A)$, then $|T(A)| = 1 + 2\sqrt{3}$.

The cases when either $a_1$ or $c_1$ is of order 1 can also be easily eliminated. If either $a_2$ or $c_2$ is of order 1, then Lemma 1 can be used.

Now suppose that no edge of $H$ belongs to $T(A)$, i.e., $T(A)$ has three additional vertices. We may assume that every edge of $T(A)$ with an endpoint in $H$ has length less than one. The combinatorial scheme for the edges of $T(A)$ is as in Figure 1. The tree to the left shows that starting with any edge joining two interior vertices and with the four adjacent edges, by attaching the sixth edge to one of the legs, we obtain the tree to the right. The edge $K_1$ is different from the other edges that have a vertex of $H$ as one of the endpoints: $K_1$ does not have a common endpoint with any of the other edges $K_i$. Consider three cases: $K_1 = [p, b_1]$, $K_1 = [p, c_1]$, and $K_1 = [p, a_2]$ for some $p$ in the interior of the hexagon, see Figures 2 and 3. Assume that all angles at interior vertices equal $\frac{2\pi}{3}$.

Suppose that $K_1$ has $b_1$ as one of its endpoints. Then the pentagon whose sides are the segment $[a_2, c_2]$ and four edges of $T(A)$, as shown in Figure 2, would have three $\frac{2\pi}{3}$ angles and the remaining two angles each less than $\frac{\pi}{2}$.

Suppose that $K_1$ has $c_1$ as one of its endpoints. Then $T(A)$ contains two paths, one from $a_1$ to $c_1$ and the other from $a_2$ to $c_2$ not overlapping in a segment, see Figure 3. Hence $|T(A)| > d(a_1, a_2) + d(c_1, c_2) = 4$. 

![Figure 1](image-url)
Figure 2. Configuration $K_1 = [p, b_1]$ and configuration $K_1 = [p, c_1]$.

Suppose that $K_1$ has $a_2$ as one of its endpoints. Let $K_2 = [q, c_2]$. The angle $\angle(q, c_2, a_2)$ is greater than $\frac{\pi}{6}$, otherwise $\angle(c_2, a_2, p) \geq \frac{\pi}{6}$. The segment $[c_1, c_2]$ either intersects the interior of $[p, q]$ or it intersects the edge $K_1$. In either case denote the point of intersection by $r$.

If $r$ is in the interior of $[p, q]$, then by reflecting the part of $T(a)$ below $r$ in the line passing through $c_1$ and $c_2$ we obtain a graph of the same length as $T(a)$ and with an additional vertex at $r$, which is not possible.

Suppose now that $[c_1, c_2]$ intersects $K_1$. Let $x = d(p, L(c_1, a_2))$ and $y = d(p_1, p_2)$, where the segment $[p_1, p_2]$ is parallel to $L(c_1, a_2)$, $p \in [p_1, p_2]$, $p_1 \in L(c_1, a_1)$, and $p_2 \in L(c_1, c_2)$. Let $y_1 = d(p_1, p)$ and $y_2 = d(p_2, p)$, see Figure 3.

Note that

1. since $|K_1| < 1$, $p$ is on the same side of $L(c_1, a_1)$ as $a_2$,
2. $|K_1| = d(a_2, r) + d(r, p) \geq \frac{\sqrt{3}}{2} + d(p, L(c_1, c_2)) \geq \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} y_2$,
3. $d(p, L(b_1, c_1)) = \frac{x + \sqrt{3} y_1}{2}$,
4. $y = \frac{x}{\sqrt{3}}$,
5. $|T\{p, a_1, c_2\}| \geq |G\{a_1, c_2\} \cup L(p_1, p_2)| = \frac{3}{2} \sqrt{3} - x$,
6. $|T\{p, b_1, c_1\}| \geq d(p, L(b_1, c_1)) + \frac{\sqrt{3}}{2} = \frac{x}{2} + \sqrt{3} y_1 + \frac{\sqrt{3}}{2}$ (this holds true even if $d(p, L(b_1, c_1)) < \frac{1}{2 \sqrt{3}}$ and $T\{p, b_1, c_1\}$ is not a triod).
We have $|T(A)| = |K_1| + |T(\{p, a_1, c_2\})| + |T(\{p, b_1, c_1\})| \geq \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} y_2 + \frac{\sqrt{3} - x + \frac{\sqrt{3}}{2}}{2} y_1 + \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2} - \frac{x}{2} + \frac{\sqrt{3}}{2} y = \frac{5\sqrt{3}}{2} > 4$. 

\[ \square \]

**Figure 4.** $|T(A)|$ estimates.

**Lemma 3.** Every minimal graph $G(A)$ for the set $A = Q \cup \{a_1, c_1, a_2, c_2\}$ consists of four edges of $H$.

**Proof.** Let $b_4 = (0, 1, 0)$ and $b_5 = (0, -1, 0)$.

If $G(A) \cap Q \subset \{b_1, b_2\}$, then $G(A)$ is planar; otherwise the projection of $G(A)$ onto the $xz$-plane would be of shorter length. Then the vertices of a component of $G(A)$ that are in $H$ form a sequence of consecutive vertices of $H$. By the previous lemmas, $G(A)$ consists of 4 edges of $H$.

Suppose that one of the edges $[a_1, c_2]$ or $[c_1, a_2]$ belongs to $G(A)$. The only segments in the convex hull of $A$ forming with these edges an angle $\geq \frac{\sqrt{3}}{2}$ at one of the endpoints $a_1, a_2, c_1$, or $c_2$ are contained in another edge of $H$. So if $G(A)$ contains either of these two edges, then it contains an edge adjacent to $[a_1, c_2]$ or $[c_1, a_2]$. If $G(A)$ does not contain $[a_1, c_2]$ (resp. $[c_1, a_2]$) but it contains an edge of $H$ adjacent to it, then this edge can be replaced by $[a_1, c_2]$ (resp. $[c_1, a_2]$) to get a connecting graph of the same length. Hence we may assume that if $G(A)$ contains an edge of $H$, then it also contains another edge on the same side of the $xy$-plane.

Our consideration may be reduced to graphs $G(A)$ whose components contain either two or four of the vertices $a_1, a_2, c_1, c_2$. The case when one component contains $a_1$ and $a_2$, and the other $c_1$ and $c_2$ can be easily eliminated. If one of the components contains $a_1$ and $c_1$, and the other contains $a_2$ and $c_2$, then $G(A)$ is planar. Suppose that a non-planar component of $G(A)$ contains exactly two vertices of $H$, both on the same side of the $xy$-plane, say $a_1$ and $c_2$. Then $G(A)$ has a component that is a simple triod with one additional endpoint $a_3 \in Q$ different from $b_1$ and $b_2$, and a vertex $s$ of order 3. Since $a_3$ is the closest point to $s$ on $Q$, the line $L(s, a_3)$ intersects the $z$-axis. The only possible choices for $a_3$ so that $\angle(a_1, s, a_3) = \angle(c_2, s, a_3)$ are $b_4$ and $b_5$. But for $i = 4, 5$, $|T(\{a_i, c_2, b_i\})| = \sqrt{x^2 + y^2} > 2$.

Hence if $G(A)$ is non-planar, then $G(A)$ is connected and contains a point $b_3 \in Q$ different from $b_1$ and $b_2$. We may assume that $b_3 \in arc_Q(b_2, b_4)$. $G(A)$ is combinatorially equivalent to the graph pictured in Figure 3. Let $p$ be the endpoint of $K_1$ that does not belong to $H$. The possible types of configurations are:
1. $K_1 = [b_3, p]$ and the endpoints of $K_2$ and $K_3$ that belong to $H$ are on the same side of the $xy$-plane (see Figure 5),

2. $K_1 = [b_3, p]$ and the endpoints of $K_2$ and $K_3$ that belong to $H$ are on the opposite sides of the $xy$-plane,

3. $K_1 = [c_2, p]$, $K_2$ connects to $a_1$ and $K_3$ connects to $b_3$ (see Figure 3),

4. $K_1 = [a_1, p]$, $K_2$ connects to $c_2$ and $K_3$ connects to $b_3$ (see Figure 3),

5. $K_1 = [c_2, p]$, $K_2$ connects to $b_3$ and $K_3$ connects to either $c_1$ or $a_2$,

6. $K_1 = [a_1, p]$, $K_2$ connects to $b_3$ and $K_3$ connects to either $c_1$ or $a_2$.

**Figure 5.** $K_1$ connects to $b_3$.

The most interesting is Configuration 1. Let $q$ and $r$ be the remaining two interior vertices different from $p$, with $q$ above the $xy$-plane and $r$ below. Let $p', q', r', b_3'$ be the points obtained from $p, q, r, b_3$, respectively, by a rotation of the tree $T(\{p, r, b_3\})$ in the $z$-axis so that $d(b_3', b_2) < d(b_3, b_2)$. Let $E$ be the ellipsoid given by the equation $d(x, a_1) + d(x, c_2) = |K_2| + |K_3|$. Since $|K_2| = d(c_2, q) \leq |K_3| = d(a_1, q)$, the point $q'$ is inside the ellipsoid $E$. Therefore $d(a_1, q') + d(c_2, q') < |K_2| + |K_3|$. Similarly $d(c_1, r') + d(a_2, r') < |K_4| + |K_5|$ and we obtain a connecting graph for the set $A$ of shorter length. Hence $b_3 = b_2$.

It is easy to eliminate Configurations 2, 5, and 6. In each of these cases $G(A)$ contains two non-overlapping paths from the set $\{a_1, c_2\}$ to the set $\{a_2, c_1\}$, and since $G(A)$ is connected, $|G(A)| > 4$.

Now consider Configuration 3. We have,

$$|G(A)| > |T(\{a_1, b_3, p\})| + |T(\{c_1, a_2, c_2\})| >$$

$$d(a_1, b_4) + |T(\{c_1, a_2, (0, 0, \frac{\sqrt{3}}{2})\}) = \sqrt{3} + 3 \frac{\sqrt{3}}{2} > 4.$$ 

Finally consider Configuration 4. Let $q$ and $r$ be the two additional interior vertices with $K_2 = [q, c_2]$ and $r$ being the common vertex of $K_4$ and $K_5$ as in the right-hand picture in Figure 3. Note that $b_3$ is the point in $Q$ closest to $q$, hence the line $L(b_3, q)$ intersects the $z$-axis. If the segment $[p, r]$ intersects the $yz$-plane at $\tilde{r}$ different from $r$, then by reflecting the tree $T(\{\tilde{r}, a_2, c_1\})$ in the $yz$-plane, we obtain a connecting graph for the set $A$ of the same
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length as $G(A)$ with an additional vertex $\tilde{r}$. Therefore, we may assume that the points $p$ and $r$ are not on the opposite sides of the $yz$-plane.

Suppose that $r$ is either on the $yz$-plane or on the same side of the $yz$-plane as $c_2$. Let $b_3', p', q', r'$ be the new vertices corresponding to $b_3, p, q, r$ obtained by rotating the path $[b_3, q] \cup [q, p] \cup [p, r]$ about the $z$-axis. The points $\overline{a_1}, \overline{c_2}, \overline{p}, \overline{q}, \overline{q}'$ and $\overline{r}$ in Figure 6 are the projections of $a_1, c_2, p, p', q, q'$ and $r$ onto the $xy$-plane; $o$ is the origin. Since $p$ is in the convex hull of \{a_1, c_1, a_2, c_2, b_3\} and $q$ is in the triangle $\triangle(p, b_3, c_2)$, then $\overline{p} \in \triangle(\overline{a_1}, o, \overline{q})$.

If $b_3' \in \text{arc}_{Q}(b_2, b_3)$ and $b_3' \neq b_3$, then, arguing in a similar fashion as for Configuration 1, we get $d(c_1, r') + d(a_2, r') < |K_4| + |K_5|$ and $d(a_1, q') + d(c_2, q') < d(a_1, q) + d(c_2, q)$. Also note that $d(a_1, q') > d(a_1, q)$ and $d(c_2, q') < d(c_2, q)$. Suppose that $\overline{q}, \overline{p}$ and $o$ are not collinear, and $\overline{q}', \overline{p}'$ and $\overline{r}$ are collinear as in the right-hand picture in Figure 7. Then $\angle(q', a_1, p') < \angle(q, a_1, p)$ and $\angle(a_1, q', p') < \angle(a_1, q, p)$, and since $d(p', q') = d(p, q)$, then $d(a_1, q') - d(a_1, p') > d(a_1, q) - d(a_1, p)$, see Figure 8. Hence $d(a_1, q') - d(a_1, q) > d(a_1, p') - d(a_1, p)$, which combined with $d(a_1, q') + d(c_2, q') < d(a_1, q) + d(c_2, q)$ gives $d(a_1, p') + d(c_2, q') < d(a_1, p) + d(c_2, q) = |K_1| + |K_2|$. We obtain a new connecting graph for the set $A$ with interior vertices $p', q', r'$ whose length is less than $|G(A)|$.

Figure 6. $K_1$ connects to $c_2$; $K_1$ connects to $a_1$.

Figure 7. Rotating part of $G(A)$. 
If the points $\overline{x}_1$, $\overline{p}$ and $\overline{q}$ are collinear, then $a_1$, $p$ and $q$ are in a plane perpendicular to the $xy$-plane and so is the tree $T(\{a_1, q, r\})$, in particular, so is the segment $[p, r]$. Then $G(A)$ is planar.

If the points $\overline{x}_1$, $\overline{p}$ and $\overline{q}$ are collinear, then $b_3$, $q$, $p$ and $a_1$ are in a plane that is perpendicular to the $xy$-plane and passes through the $z$-axis. This plane also contains $c_2$ and $G(A)$ is planar.

If $p$ and $r$ are on the same side or on the $yz$-plane as $a_1$ (including the case when one or both of these points are on the $yz$-plane), then

$$|G(A)| > |T(\{a_1, c_1, a_2\})| + \min_{v \in yz}\text{-plane}|T(\{v, c_2, b_3\})|.$$

If $\min_{v \in yz}\text{-plane}|T(\{v, c_2, b_3\})|$ is attained at $v_0 \in yz$-plane, then either

1. $v_0 = b_3 = b_4$, $|T(\{v_0, c_2, b_3\})| = \sqrt{2}$ and $|G(A)| = \sqrt{2} + \frac{3\sqrt{3}}{2} > 4$, or
2. $v_0 = (0, 0, \frac{\sqrt{3}}{2})$, $|T(\{v_0, c_2, b_3\})| = \frac{3}{2}$ and $|G(A)| = \frac{3}{2} + \frac{3\sqrt{7}}{2} > 4$ , or
3. $|T(\{v_0, c_2, b_3\})|$ is a simple triod.

In the last case, let $t$ be the order 3 vertex of $T(\{v_0, c_2, b_3\})$. Note that the line $L(v_0, t)$ is perpendicular to the $yz$-plane and the line $L(b_3, t)$ intersects the $z$-axis. The plane $K$ containing $L(v_0, t)$ and $L(b_3, t)$ also contains the tree $T(\{v_0, c_2, b_3\})$, see Figure 4. If $b_3 = (x, y, 0)$, then $\sqrt{3}x = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + y^2} = \sqrt{\frac{5}{4} - x^2}$. Then $x = \frac{\sqrt{7}}{2}$, $|T(\{v_0, c_2, b_3\})| = \frac{1}{4} + \frac{\sqrt{7}}{2}$ and $|G(A)| > 2(x - \frac{1}{2}) + \frac{1}{4} + \frac{\sqrt{3}}{2} \left(\frac{\sqrt{5}}{4} - x^2 - \sqrt{3}(x - \frac{1}{2})\right) = \frac{3\sqrt{3}}{2} + \frac{1}{4} + \frac{\sqrt{7}}{2} > 4$. 

**Corollary 1.** For every $\epsilon > 0$ there is a $\delta > 0$ such that if a set $A$ is Hausdorff $\delta$-close to $Q \cup \{a_1, c_1, a_2, c_2\}$, then every minimal graph $G(A)$ is in an $\epsilon$-neighborhood of four edges of $H$.

In the following lemmas, the notation $Q_\delta$ is used for a circle in $B^3$ that is in a plane parallel to the $xy$-plane, with center on the $z$-axis, and such that $d_H(Q, Q_\delta) < \delta$. 
Lemma 4. There are an \( \epsilon > 0 \), a \( \delta > 0 \), and an \( \eta > 0 \) such that if
1. \( a'_1 \) and \( c'_2 \) are points in the \( xz \)-plane such that \( d(a_1, a'_1) < \eta \), \( d(c_2, c'_2) < \eta \), and
2. \( G(A) \) is a minimal graph for \( A = Q_{\delta} \cup \{a'_1, c'_2\} \) contained in an \( \epsilon \)-neighborhood of the set
   \[ [a_1, b_1] \cup [a_1, c_2], \]
then \( G(A) \) is contained in the \( xz \)-plane.

Proof. Let \( q \) be the point of \( G(A) \) that belongs to \( Q_{\delta} \). Thus \( G(A) \) is the tree \( T(\{q, a'_1, c'_2\}) \).
For small \( \epsilon \), \( \delta \), and \( \eta \), the tree \( T(\{q, a'_1, c'_2\}) \) consists of 2 segments in the \( xz \)-plane or it is a simple triod with an additional vertex \( p \) close to \( a'_1 \). Then the point of intersection \( r \) of the line \( L(p, q) \) and the edge \([a'_1, c'_2] \) is also close to \( a'_1 \). Since \( q \) is the point on \( Q_{\delta} \) that is closest to \( p \), the line \( L(p, q) \) intersects the \( z \)-axis at some point \( s \). Hence \( L(p, q) \) has two distinct points \( r \) and \( s \) in the \( xz \)-plane and \( T(\{q, a'_1, c'_2\}) \) is in the \( xz \)-plane.

For \( 0 < \gamma < 1 \), let \( c_1(\gamma) = (\frac{1}{2}, 0, -\frac{\sqrt{\gamma}}{2}(1 - \gamma)) \) and \( c_2(\gamma) = (\frac{1}{2}, 0, \frac{\sqrt{\gamma}}{2}(1 - \gamma)) \).

Lemma 5. There is a \( \gamma_0 > 0 \), such that the set \( A = Q \cup \{a_1, c_1(\gamma), a_2, c_2(\gamma)\}, \) \( 0 < \gamma < \gamma_0 \), has a unique minimal graph \( G(A) \) consisting of two simple triods
   \[ T(\{a_1, b_1, c_1(\gamma)\}) \text{ and } T(\{a_2, b_2, c_2(\gamma)\}). \]

Proof. By Corollary, there are six cases of a minimal graph \( G(A) \) to consider. \( G(A) \) may be close to one of the sets of the following fours edges of \( H \):

1. \([a_1, b_1] \cup [b_1, c_1] \cup [a_2, b_2] \cup [b_2, c_2], \)
2. \([a_1, b_1] \cup [a_1, c_2] \cup [c_1, a_2] \cup [a_2, b_2], \)
3. \([a_1, c_2] \cup [c_2, b_2] \cup [a_2, c_1] \cup [c_1, b_1], \)
4. \([a_1, b_1] \cup [b_1, c_1] \cup [a_1, c_2] \cup [a_2, b_2], \)
5. \([a_1, b_1] \cup [b_1, c_1] \cup [c_1, a_2] \cup [b_2, c_2], \)
6. \([a_1, b_1] \cup [b_1, c_1] \cup [a_1, c_2] \cup [c_1, a_2]. \)

The remaining cases of subsets of \( H \) consisting of four edges are three cases symmetric with respect to the origin to Cases 4, 5, or 6, and six possibilities of graphs that do not connect to at least one of the vertices of \( H \).

For \( i = 1, \ldots, 6 \), let \( G_i(A) \) be a minimal graph for \( A \) corresponding to Case \( i \), see Figure. Note that \( G_1(A) \) and \( G_5(A) \) are clearly contained in the \( xz \)-plane. The see that the remaining graphs are also in the \( xz \)-plane, denote by \( q \) the additional vertex of the graph \( G_i(A) \) so that there is an edge \([q, b_1]\) for \( i = 4, 5, 6 \), and \( q = b_1 \) for \( i = 2 \). The vertex \( q \) separates \( G_i(A) \) into subgraphs. The two subgraphs different from the edge \([q, b_1]\) connect one or two of the points \( a_1, c_1(\gamma), a_2, c_2(\gamma) \) to a circle \( Q_{\delta} \). By Lemma, we may assume that the subgraphs are subsets of the \( xz \)-plane. Since the distance between a point in \( Q_{\delta} \) and the circle \( Q \) is constant, it easily follows that each \( G_i(A) \) is in the \( xz \)-plane. We analyze the six cases as follows:

1. \( |G_1(A)| < 2(d(a_1, b_1) + d(b_1, c_1(\gamma))) < 2(1 - 1 - \gamma + \frac{\sqrt{2}}{2}) = 4 - \gamma. \)
2. \( |G_2(A)| = 2(d(a_1, p) + d(p, b_1) + d(c_2(\gamma), p)) > 2(1 + d(c_2(\gamma), p)). \) Since \( p \) is on the same side of the line \( L(a_1, a_2) \) as \( q \), and \( d(p, L(a_1, a_2)) < d(c_2(\gamma), L(a_1, a_2)) \), then \( |G_2(A)| > 2(1 - 1 - \frac{\sqrt{2}}{2}) \) if \( |G_1(A)| \).
3. \( |G_3(A)| > 2(d(a_1, c_2(\gamma)) + d(c_2(\gamma), b_2)) > 2(1 + d(c_2(\gamma), b_2)) > |G_1(A)| \).
4. Let \( c'_1 = (-\frac{1}{2} + \frac{3}{2}, 0, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}), \) i.e., \( c'_1 \) is the point symmetric to \( c_1(\gamma) \) with respect to the line \( L(c_1, c_2) \). Let \( C \) be the circle of radius \( \frac{\sqrt{2}}{2} \) centered at \( c'_1 \). Since \( c_1(\gamma) \in C, \)
Lemma 6. There are a $\gamma_0 > 0$ and a $\delta > 0$, such that for $0 < \gamma < \gamma_0$ and $i = 1, 2$,

1. the minimal graph for the set $Q_6 \cup \{e_i(\gamma), f_i(\gamma)\}$ is the triod

$$T(\{v_i, e_i(\gamma), f_i(\gamma)\}),$$
A KNOTTED MINIMAL TREE

2. the minimal graph for the set $Q_\delta \cup \{a_i, e_i(\gamma), f_i(\gamma)\}$ is the minimal tree

$T(\{a_i, v_i, e_i(\gamma), f_i(\gamma)\})$,

where $v_i \in Q_\delta$ is a point in the $xz$-plane close to $b_i$.

Proof. For a given $\delta$, let $e_i'$ and $f_i'$ be points on a $Q_\delta$ closest to $e_i(\gamma)$ and $f_i(\gamma)$ respectively. Let $p \in \text{arc}_{Q_\delta}(e_i', f_i')$. The minimal tree $T(\{q, e_i(\gamma), f_i(\gamma)\})$ has an edge containing $q$, whose extension intersects the $z$-axis, see Figure 11. For small $\gamma$ and $\delta$ this is possible only for $q \in xz$-plane. From this, both Conclusions 1 and 2 of Lemma 6 follow.

Similarly, one can prove the following:

**Corollary 2.** There are a $\gamma_0 > 0$ and an $\epsilon > 0$, such that for $0 < \gamma < \gamma_0$ and $i = 1, 2$, if the minimal graph $G(A)$ for the set

$A = Q \cup \{a_1, e_1(\gamma), f_1(\gamma), a_2, e_2(\gamma), f_2(\gamma)\}$

is in the $\epsilon$-neighborhood of the hexagon $H$, then $G(A)$ is symmetric with respect to the $xz$-plane.

Finally, in this sequence of lemmas we have:

**Lemma 7.** There is a $\gamma_0 > 0$, such that for $0 < \gamma < \gamma_0$ and $i = 1, 2$, the minimal graph $G(A)$ for the set

$A = Q \cup \{a_1, e_1(\gamma), f_1(\gamma), a_2, e_2(\gamma), f_2(\gamma)\}$

is unique and consists of the two minimal trees $T(\{a_i, b_i(\gamma), f_i(\gamma)\})$, $i = 1, 2$.

**Lemma 8.** There is a $\delta > 0$ such that if $b_2' \in P$ is a point below the $x$-axis and $d(b_2, b_2') < \delta$, then the length of the minimal tree $|T(\{a_2, v, c_2(\gamma)\})|$ is a strictly monotone function of $v \in \text{arc}_P(b_2, b_2')$ and attains its maximum at $b_2$.

Proof. $P$ is the circle circumscribed around the hexagon $H$. Let $a$ and $b$ be points in $\text{arc}_P(b_2, b_2')$ with $b \in \text{arc}_P(b_2, a)$. Note that $T(\{a_2, b, c_2(\gamma)\})$ is a simple triod; denote the vertex of order 3 by $q$. Let $E$ be the ellipse with foci $a_2$ and $c_2(\gamma)$, and passing through the point $q$. Let $p$ be the point on $E$ closest to $a$, see Figure 12. We have

$|T(\{a_2, a, c_2(\gamma)\})| < d(a, p) + d(p, a_2) + d(p, c_2(\gamma)) =$
\[
    d(a, p) + d(q, a_2) + d(q, c_2(\gamma)) < |T(\{a_2, b, c_2(\gamma)\})|.
\]

\[\blacklozenge\]

**Figure 12.** Perturbation.

**Corollary 3.** There are a \(\gamma > 0\) and a \(\delta > 0\) such that if \(b'_2 \in P\) is a point below the \(x\)-axis and \(d(b_2, b'_2) < \delta\), then the length of the minimal tree \(|T(\{a_2, v, e_2(\gamma), f_2(\gamma)\})|\) is a strictly monotone function of \(v \in \text{arc}_P(b_2, b'_2)\) and attains its maximum at \(b_2\).

4. **A knotted minimal tree**

In this section, we use the spherical coordinates, denoted by \((r, \theta, \phi)_s\), where \(x = r \sin \phi \cos \theta\), \(y = r \sin \phi \sin \theta\), and \(z = r \cos \phi\).

Let \(d_1(\delta) = (1, \pi, \frac{\pi}{2} - \delta)_s\), \(d_2(\delta) = (1, 0, \frac{\pi}{2} + \delta)_s\), and

\[
    \begin{align*}
    M_1(\delta) &= \text{arc}_{S^2}(d_1, (1, \frac{5\pi}{4}, \frac{\pi}{2} - \delta)_s), \\
    M_2(\delta) &= \text{arc}_{S^2}((1, \frac{5\pi}{4}, \frac{\pi}{2} - \delta)_s, (1, \frac{7\pi}{4}, \frac{\pi}{2})_s), \\
    M_3 &= Q - \text{arc}_{S^2}((1, \frac{5\pi}{4}, \frac{\pi}{2})_s, (1, \frac{7\pi}{4}, \frac{\pi}{2})_s), \\
    M_4(\delta) &= \text{arc}_{S^2}((1, \frac{5\pi}{4}, \frac{\pi}{2})_s, (1, \frac{7\pi}{4}, \frac{\pi}{2} + \delta)_s), \\
    M_5(\delta) &= \text{arc}_{S^2}((1, \frac{5\pi}{4}, \frac{\pi}{2} + \delta)_s, d_2), \\
    M(\delta) &= \bigcup_{i=1}^5 M_i(\delta).
    \end{align*}
\]

Thus \(M(\delta)\) is a path from \(d_1(\delta)\) to \(d_2(\delta)\), close to \(Q\), and of length approximately \(\frac{3}{2}\) times the length of \(Q\).

**Lemma 9.** There are a \(\gamma\) and a \(\delta\) such that the minimal graph \(G(B)\) for the set

\[
    B = M(\delta) \cup \{a_1, e_1(\gamma), f_1(\gamma), a_2, e_2(\gamma), f_2(\gamma)\}
\]

is unique and consists of the two minimal trees \(T(\{a_i, d_i(\delta), e_i(\gamma), f_i(\gamma)\})\), \(i = 1, 2\).

**Proof.** By Lemma 8, if \(\gamma\) is sufficiently small, then the minimal graph \(G(A)\) for the set

\[
    A = Q \cup \{a_1, e_1(\gamma), f_1(\gamma), a_2, e_2(\gamma), f_2(\gamma)\}
\]

is unique and isomorphic to the Euclidean plane. \(\Box\)
is unique and
\[ G(A) = T(\{a_1, b_1, e_1(\gamma), f_1(\gamma)\}) \cup T(\{a_2, b_2, e_2(\gamma), f_2(\gamma)\}). \]
Since the sets \( A \) and \( B \) are Hausdorff \( \delta \)-close, there is a \( \delta \) such that
\[ G(B) = T(\{a_1, m_1, e_1(\gamma), f_1(\gamma)\}) \cup T(\{a_2, m_2, e_2(\gamma), f_2(\gamma)\}) \]
for some \( m_1 \) and \( m_2 \) on \( M \). By Lemma 3, the points \( m_i \) are on the \( xz \)-plane. For sufficiently small \( \delta \), by Corollary 3, \( m_i = d(\delta) \).

For some positive constants \( \gamma \) and \( \delta \) chosen so that Lemma 3 is satisfied, let \( M = M(\delta) \), \( d_i = d_i(\gamma) \), \( e_i = e_i(\gamma) \), and \( f_i = f_i(\gamma) \). For \( \epsilon > 0 \), let \( t_2, \ldots, t_{n(\epsilon)-1} \) be points on \( M \) such that \( d(t_i, t_{i+1}) < \epsilon \) for \( i = 1, \ldots, n(\epsilon) - 2 \), \( t_2 = (1, \pi + \epsilon, \frac{\pi}{2} - \delta)_s \), and \( t_{n(\epsilon)-1} = (1, -\epsilon, \frac{\pi}{2} + \delta)_s \). In addition let \( t_1 = (1, \pi - \epsilon, \frac{\pi}{2} - \delta)_s \) and \( t_{n(\epsilon)} = (1, \epsilon, \frac{\pi}{2} + \delta)_s \).

**Theorem 1.** There is an \( \epsilon > 0 \) such that the minimal tree for the set
\[ X = \{a_1, e_1, f_1, a_2, e_2, f_2, t_1, \ldots, t_{n(\epsilon)}\} \]
is unique and knotted.

**Proof.** Let \( A_1 = \{a_1, t_1, t_2, e_1, f_1\} \) and \( A_2 = \{a_2, t_{n(\epsilon)-1}, t_{n(\epsilon)}, e_2, f_2\} \). For small \( \epsilon \), \( T(X) \) contains two subgraphs close to the minimal trees \( T(\{a_1, d_1, e_1, f_1\}) \) and \( T(\{a_2, d_2, e_2, f_2\}) \). Since \( d(t_1, t_2) = d(t_{n(\epsilon)-1}, t_{n(\epsilon)}) \) is approximately \( 2\epsilon \) and \( d(t_i, t_{i+1}) < \epsilon \) for the remaining points \( t_i \), for sufficiently small \( \epsilon \), the two subgraphs are \( T(A_1) \) and \( T(A_2) \). Thus there is an \( \epsilon > 0 \) such that \( T(X) \) is unique and
\[ T(X) = T(A_1) \cup \{t_2, t_3\} \cup \ldots \cup \{t_{n(\epsilon)-2}, t_{n(\epsilon)-1}\} \cup T(A_2). \]
Such \( T(X) \) is knotted, see Figure 13. \( \square \)

**Remark.** A slight change of the arc \( M \) and an appropriate choice of the sequence of the points \( t_i \) can yield a finite set in \( S^2 \) with two minimal trees, one knotted and the other unknotted.

The example of the knotted minimal tree raises the following questions:

1. (M. Freedman) What does the set of \( k \)-tuples in \( S^2 \) whose minimal tree is unknotted look like in the \( k \)-fold product \( S^2 \times \cdots \times S^2 \)? In particular, what is the measure of this set?
2. What is the minimum number $k$ for which there is a set of points in $S^2$ whose minimal tree is knotted?
3. (W. Kuperberg) What is the minimum number of vertices of order 3 in a knotted minimal tree for a finite subset of $S^2$? The described knotted tree has 6 vertices of order 3.
4. (G. Kuperberg) There is a finite set whose minimal tree is knotted on the surface of an ellipsoid with one of the axes much shorter than the other two axes. What are the strictly convex closed surfaces in $\mathbb{R}^3$ containing a finite set whose minimal tree is knotted? Can any knot be realized in a minimal tree of a finite set on some convex surface?

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