On the geometry of the Titchmarsh counterexample

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Abstract

We study the lines of constant phase corresponding to the ratio formed by the building blocks of the Titchmarsh counterexample, that is by two Dirichlet \( L \)-functions whose characters are the complex conjugate of each other. This ratio on the critical line is sensitive to zeros off the critical line.

Keywords: Riemann zeta function, Titchmarsh counterexample, lines of constant phase, lines of constant height

(Some figures may appear in colour only in the online journal)

1. Introduction

The verification of the Riemann Hypothesis, that is of the claim that the so-called non-trivial zeros of the Riemann zeta function \( \zeta \) are all located on the critical line in the complex plane defined by the real part \( 1/2 \) is a famous still unsolved problem in mathematics. In his seminal
book [1] Edward Charles Titchmarsh introduced a function $\xi_T$ which shares many properties of the analytic continuation of $\zeta$, that is the Riemann function $\xi$ and is known to have [2] zeros off the critical line. The appearance of these zeros has been explained [3] as a consequence of the universality [4] of the underlying Dirichlet $L$-functions. In the present article we point out a crucial difference between $\xi$ and $\xi_T$ which manifests itself in the lines of constant phase [5] of the ratios formed by the elements of the corresponding analytic continuations.

1.1. Formulation of the problem

For this purpose, we study a class of complex-valued functions $F$ characterized by their representation as a superposition

$$F(s) = f(s) + f(1-s) \quad (1)$$

of a single complex-valued function $f$ evaluated at $s$ and at $1-s$ where $s \equiv \sigma + i\tau$.

Obviously, $F$ satisfies the elementary functional equation

$$F(s) = F(1-s). \quad (2)$$

A zero $s_0$ of $F$ appears when the ratio

$$g(s) \equiv \frac{f(s)}{f(1-s)} \quad (3)$$

assumes the value $-1$, that is

$$g(s_0) = -1. \quad (4)$$

This class includes but is not limited to an appropriately defined hyperbolic function $c$, the Riemann function $\xi$ and the Titchmarsh counterexample $\xi_T$. In the present article we show that for $c$ and $\xi$ the absolute values of the corresponding ratios $g_c$ and $g_R$ along the critical line $s = 1/2 + i\tau$ are unity.

In contrast, for the Titchmarsh counterexample the ratio $g_T$ on the critical line is real, and assumes all values from $-\infty$ to $+\infty$. This behavior originates from the fact that $f_T$ associated with $\xi_T$ displays zeros on the critical line, and the zeros of $f_T(s)$ and $f_T(1-s)$ are disjunct. As a result, poles and zeros appear in $g_T$.

In the case of two consecutive poles followed by two consecutive zeros, there must be two points, where the first derivative of $g_T$ vanishes. Here, two lines of constant phase leave symmetrically the critical line and unite again later. Provided the interval formed by the values of $g_T$ at the two points of vanishing derivative includes $-1$, there are two symmetrically located zeros of $\xi_T$.

We suspect that this dramatically different behavior of $g_T$ versus $g_R$ on the critical line may provide us with yet another perspective on the Riemann Hypothesis. Indeed, we have already followed [6–8] an approach based on the lines of constant phase of $\xi$ and $\xi_T$ rather than $g_R$ and $g_T$.

Moreover, we emphasize that the Riemann zeta function plays a central role not only in mathematics but also in physics. Three examples suffice to illustrate this point. (i) Indeed, the distribution of eigenvalues of Gaussian unitary ensembles of random matrices is similar [9] to that of the non-trivial zeros of the Riemann zeta function. (ii) There exists an intimate

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5 Throughout our article we refer to the product $\xi$ defined by equation (11) which is free of the pole and the trivial zeros of the Riemann zeta function $\zeta$ located at $s = 1$ and the negative even integers, respectively, as the Riemann function.
connection [10] between the inverted harmonic oscillator and the Mellin transform of $\xi$; and 
(iii) the long-standing Polya–Hilbert Hypothesis of finding a Hamiltonian whose eigenvalues 
are given by the zeros of $\xi$ has recently been verified [11].

1.2. Outline

Our article is organized as follows: In section 2 we show that $c$, $\xi$, and $\xi_T$ are elements of 
the class of functions defined by equation (1), and present explicit expressions for the cor-
responding building blocks $f_c$, $f_R$ and $f_T$. Then in section 3 we demonstrate that for $c$ and $\xi$ 
the critical line is a line of constant height of $g_c$ and $g_R$ with value unity. In sharp contrast, 
g_T corresponding to $\xi_T$ is real along the critical line and thus a line of constant phase. This 
distinct difference between $c$ and $\xi$ on one hand, and $\xi_T$ on the other, is the deeper reason for 
our ability of identifying zeros of $\xi_T$ located on and off the critical line by analyzing $g_T$ on it. 
Indeed, we demonstrate in section 4 that $g_T$ assumes zeros and poles on the critical line and 
the geometry of the lines of constant phase of $g_T$ allows us to determine the location of these 
zeros. We conclude in section 5 by providing a brief summary and an outlook.

In order to keep our article self-contained we first briefly review in appendix A properties of 
Dirichlet L-functions $\Lambda$ crucial for the main theme of our article. We then show that although 
in general their analytic continuation is not of the form, equation (1), the critical line is still 
a contour line of the corresponding ratio $g_\Lambda$ with value unity. Since the generalized Riemann 
Hypothesis states that all zeros of $\Lambda$ are also located on the critical line, this distinct property 
of $g_T$ compared to $g_c$, $g_R$ and $g_\Lambda$ supports the validity of the Riemann Hypothesis.

1.3. Dedication

It is with great pleasure that we dedicate this article to Prof. Sir Michael Victor Berry on the 
occasion of his 80th birthday. We have chosen for our contribution the possibility of identi-
fying zeros of the Titchmarsh counterexample off the critical line by the behavior of the ratio 
g_T of the building blocks of $\xi_T$ on the critical line. We are confident that this topic might find 
his interest since in a very stimulating discussion at the 600th Heraeus Seminar at Bad Hon-
nef in 2015 we have learned from him that Ernest Oliver Tuck [12, 13] has argued that his 
incompressibility function of $\xi$ on the critical axis line is sensitive to zeros off the critical line. 
Unfortunately, he was wrong as shown [14] by Michael Berry and Pragya Shukla.

‘Happy Birthday’ and many more happy and healthy years with fun in science!

2. Elements of class of functions

Throughout our article we consider functions that satisfy the elementary functional equation, 
equation (2), and result from the superposition, equation (1) of a single function $f$ evaluated at 
the two points $s$ and $1 - s$. In the present section we provide explicit expressions for $f$ giving 
rise to the hyperbolic cosine, the Riemann function and the Titchmarsh counterexample.

Indeed, the most elementary example

$$f_c(s) \equiv \frac{1}{2} e^{s-1/2}$$

(5)

leads us by equation (1) to the function

$$c(s) \equiv \cosh \left(s - \frac{1}{2}\right).$$

(6)
Moreover, for the choice
\[ f_R(s) \equiv \frac{1}{2} \left[ s(s-1) \gamma(s) + \frac{1}{2} \right] \]
with the integral transform
\[ \gamma(s) \equiv \int_{1}^{\infty} dx \, \omega(x) x^{s-1} \]
of the Jacobi theta function
\[ \omega(x) \equiv \sum_{n=1}^{\infty} e^{-\pi n^2 x} \]
we arrive at the familiar analytic continuation
\[ \xi(s) = \frac{1}{2} s(s-1) \left[ \gamma(s) + \gamma(1-s) \right] + \frac{1}{2} \]
of the Riemann function
\[ \xi(s) \equiv \pi^{-s/2} (s-1) \Gamma \left( \frac{s}{2} + 1 \right) \zeta(s), \]
where \( \Gamma \) and \( \zeta \) denote the Gamma and the Riemann zeta function, respectively.

Finally, we consider the case
\[ f_T(s) \equiv \frac{1}{2} \cos \theta e^{-i \theta} \Lambda(s, \chi_1) \]
with the Dirichlet L-function \( \Lambda = \Lambda(s, \chi_1) \) of complex-valued character \( \chi_1 \mod 5 \) where the parameter \( \theta \) is defined by the condition
\[ \tan(2 \theta) = \frac{\sqrt{5} - 1}{2}. \]

For a general introduction into and an overview over Dirichlet L-functions, we refer to [15, 16]. However, in appendix A we briefly summarize properties of \( \Lambda \) relevant to the present discussion.

Indeed, from equation (10) we recall the functional equation
\[ \Lambda(1-s, \chi_1) = e^{2i \theta} \Lambda(s, \chi_2) \]
and thus arrive at the expression
\[ f_T(1-s) = \frac{1}{2 \cos \theta} e^{i \theta} \Lambda(s, \chi_2), \]
leading us by the superposition, equation (1), of \( f_T(s) \) and \( f_T(1-s) \) to the Titchmarsh counter-example
\[ \xi_T(s) \equiv \frac{1}{2 \cos \theta} \left[ e^{-i \theta} \Lambda(s, \chi_1) + e^{i \theta} \Lambda(s, \chi_2) \right]. \]

Hence, the class of functions given by equation (1) includes all three examples.

3. Des Pudels Kern

The crucial difference between the functions \( c \) and \( \xi_R \) on the one hand, and \( \xi_T \) on the other, stands out most clearly when we consider for each of them the value \( f(s^*) \) where the star
indicates the complex conjugate. This operation tests the symmetry of $f$ with respect to the real axis.

Indeed, we find for $c$ and $\xi_R$ the symmetry relations

$$f_c(s^*) = f_c^*(s) \quad (16)$$

and

$$f_R(s^*) = f_R^*(s). \quad (17)$$

However, the dependence of $f_T$ on the complex-valued character $\chi_1$, and the presence of the phase factor $\exp(-i\theta)$, prevent a similar relation for the building block $f_T$ of the Titchmarsh counterexample, that is

$$f_T(s^*) \neq f_T^*(s). \quad (18)$$

In this section we first verify the identities, equations (16) and (17) as well as show the breakdown of this symmetry for $f_T$ as expressed by equation (18). We then demonstrate that this distinct difference implies that the critical line is a line of constant height for the ratios $g_c$ and $g_R$ formed by $f_c$ and $f_R$, but is a line of constant phase for $g_T$ defined by $f_T$.

3.1. Confirmation and breakdown of symmetry with respect to real axis

We start by noting that the definition, equation (5), of $f_c$ immediately implies the symmetry relation, equation (16).

A slightly more complicated argument verifies the corresponding property, equation (17), for $\xi_R$. Indeed, since the integration variable $x$ in $\gamma$ given by equation (8) is real the Jacobi theta function $\omega$ defined by equation (9) is real as well, and with the identity

$$x^{s^*/2} = \exp\left(\frac{x^*}{2} \ln x\right) = \left[\exp\left(\frac{x}{2} \ln x\right)\right]^* \quad (19)$$

we find

$$\gamma(s^*) = \gamma(s)^*. \quad (20)$$

Moreover, the polynomial $s(s-1)$ satisfies the relation $s^*(s^* - 1) = [s(s - 1)]^*$ which with the definition, equation (7), of $f_R$ leads us to equation (17).

Finally we address the case of $\xi_T$ where according to the definition, equation (12), we find

$$f_T(s^*) = \frac{1}{2} \cos\theta e^{-i\theta} \Lambda(s^*, \chi_1), \quad (21)$$

which due to the dependence of $\Lambda$ on $\chi_1$ takes the form

$$f_T(s^*) = \frac{1}{2} \cos\theta e^{-i\theta} [\Lambda(s, \chi_1^*)]^*. \quad (22)$$

When we recall the symmetry relation $\chi_1^* = \chi_2$, for the character $\chi_1$ we obtain the expression

$$f_T(s^*) = \frac{1}{2} \cos\theta e^{-i\theta} [\Lambda(s, \chi_2)]^* \quad (23)$$

which is obviously not identical to $f_T^*$. Two features of $f_T$ prevent this identity: (i) the presence of the phase factor $\exp(-i\theta)$ in front of the Dirichlet function $\Lambda$, and (ii) the dependence of $\Lambda$ on $\chi_1$ which leads to the emergence of $\chi_2$ rather than $\chi_1$.  

3.2. The critical line: line of constant height of ratios $g_c$ and $g_R$

The symmetry relations, equations (16) and (17) for $f_c$ and $f_R$ have an immediate consequence on the behavior of the corresponding ratios $g_c$ and $g_R$ on the critical line $s = 1/2 + i\tau$. Indeed, for $c$ and $\xi_R$ we obtain the representations

$$g_c \left( \frac{1}{2} + i\tau \right) = \frac{f_c \left( \frac{1}{2} + i\tau \right)}{f_c \left( \frac{1}{2} - i\tau \right)} = e^{i\varphi_c(\tau)}$$

and analogously

$$g_R \left( \frac{1}{2} + i\tau \right) = e^{i\varphi_R(\tau)}$$

where $\varphi_c = \varphi_c(\tau)$ and $\varphi_R = \varphi_R(\tau)$ are the phases of $f_c$ and $f_R$ along the critical line.

Hence, we arrive at the identity

$$\left| g_c \left( \frac{1}{2} + i\tau \right) \right| = \left| g_R \left( \frac{1}{2} + i\tau \right) \right| = 1$$

which indicates that along the critical line, $g_c$ and $g_R$ have lines of constant height with value unity.

According to equation (4) zeros of $c$ and $\xi_R$ occur on the critical line for the imaginary parts $\tau^{(k)}_c$ and $\tau^{(k)}_R$ where the phases $\varphi_c$ and $\varphi_R$ of $g_c$ and $g_R$ assume odd integer multiples of $\pi$, leading us to the condition

$$\varphi_c \left( \tau^{(k)}_c \right) = (2k + 1) \frac{\pi}{2}$$

and

$$\varphi_R \left( \tau^{(k)}_R \right) = (2k + 1) \frac{\pi}{2}.$$  

Here $k$ is an integer.

Since in the case of $c$ the phase $\varphi_c$ is just $\tau$, we obtain the familiar explicit formula

$$\tau^{(k)}_c = (2k + 1) \frac{\pi}{2}$$  

for the imaginary parts $\tau^{(k)}_c$ of the zeros of $c$ on the critical line.

Unfortunately, due to the more complicated expression for $f_R$ given by equations (7)–(9) no analytic expression for $\varphi_R(\tau)$ is known. Despite this complication, the zeros of $\xi_R$ on the critical line follow from a condition identical to that of $c$. Indeed, they are located where the phase lines of $g_R$ with an odd integer multiple of $\pi$ cross the critical line.

3.3. The critical line: line of constant phase of ratio $g_T$

We now turn to the case of $\xi_T$ and study the ratio

$$g_T \left( \frac{1}{2} + i\tau \right) = e^{-2i\theta} \frac{\Lambda \left( \frac{1}{2} + i\tau, \chi_1 \right)}{\Lambda \left( \frac{1}{2} + i\tau, \chi_2 \right)}$$

following from the definition, equation (3), of $g$ with equations (12) and (14).

We note that on the critical line the functional equation, equation (A9), of $\Lambda(s, \chi_1)$ reads

$$\Lambda \left( \frac{1}{2} + i\tau, \chi_1 \right) = e^{2i\theta} \Lambda \left( \frac{1}{2} + i\tau, \chi_1 \right) *$$

where $*$ denotes the complex conjugate.
and enforces the representation
\[
\Lambda \left( \frac{1}{2} + i \tau, \chi_1 \right) = e^{i \theta} \lambda_1(\tau), \tag{32}
\]
where \( \lambda_1 = \lambda_1(\tau) \) is a real function which is not necessarily positive.

Similarly, we find from the functional equation, equation (A10), the expression
\[
\Lambda \left( \frac{1}{2} + i \tau, \chi_2 \right) = e^{-i \theta} \lambda_2(\tau), \tag{33}
\]
where \( \lambda_2 \) is a real function which is not necessarily positive. Since \( \chi_1 \neq \chi_2 \), the two functions \( \lambda_1 \) and \( \lambda_2 \) are different.

When we substitute the representations equations (32) and (33) into the expression, equation (30), of \( g_T \), we arrive at the relation
\[
g_T \left( \frac{1}{2} + i \tau \right) = \frac{\lambda_1(\tau)}{\lambda_2(\tau)} \equiv \lambda(\tau). \tag{34}
\]

Hence, on the critical line \( g_T \) is real, and according to equation (4) a zero \( s_0 = \frac{1}{2} + i \tau_0 \) of \( \xi_T \) arises when
\[
g_T \left( \frac{1}{2} + i \tau_0 \right) = -1. \tag{35}
\]

A comparison between the behavior of the ratios \( g_e \), \( g_R \) and \( g_T \) on the critical line expressed by equations (24), (25) and (34), brings out the importance of the symmetry with respect to the real axis. Indeed, for functions such as \( e \) or \( \xi \) which enjoy this symmetry, the critical line is a line of constant height of \( g \) with \( |g| = 1 \). In this case, the phase of \( g \) can assume any value as \( \tau \) increases, and a zero of \( e \) or \( \xi \) arises for an odd integer multiple of \( \pi \).

In contrast, a violation of this symmetry as displayed by \( \xi_T \), leads to a situation where the ratio \( g_T \) is real along the critical line. When we allow for zeros or poles of \( g_T \), the critical line is a line of piecewise constant phase with \( \varphi_T \) being an integer multiple of \( \pi \). Zeros of \( \xi_T \) on the critical line arise at \( \tau \)–values where \( g_T \) is \( -1 \) as indicated by equation (35).

4. Zeros off the critical line

So far we have concentrated on the mechanism for the appearance of zeros of \( F \) on the critical line. Here the ratio \( g \) has played a central role. We now show that the behavior of \( g \) on the critical line, even allows us to identify zeros of \( F \) that are located off the critical line. We demonstrate this property using \( \xi_T \) which has such zeros.

For this purpose we first recall that the Dirichlet L-functions \( \Lambda_1 = \Lambda_1(s) \equiv \Lambda(s, \chi_1) \) and \( \Lambda_2 = \Lambda_2(s) \equiv \Lambda(s, \chi_2) \) have simple zeros [3, 6] on the critical line as exemplified by figure 1. Hence, at a zero of \( \Lambda_2 \) the ratio \( g_T \) has a simple pole, and at a zero of \( \Lambda_1 \) a zero. Since poles are sources and zeros are sinks of phase lines [6] we find a flow pattern of \( g_T \) illustrated in figure 2.

The influence of the locations of the zeros and poles of \( g_T \) on the zeros of \( \xi_T \) shown on the left of figure 2 stands out most clearly when we consider for increasing \( \tau \) the sequence of two zeros and two poles of \( g_T \) located on the critical line as depicted in the middle of figure 2. On the right we present \( g_T \) on the critical line where according to equation (34) \( g_T \) is real.

\[\text{[6] We emphasize that for our argument the still unverified generalized Riemann Hypothesis, that is the claim that all zeros of } \Lambda \text{ are on the critical line, is not of importance.}\]
Figure 1. Lines of constant phase (grey lines) for the Dirichlet L-functions $\Lambda_1(s) \equiv \Lambda(s, \chi_1)$ (left) and $\Lambda_2 \equiv \Lambda(s, \chi_2)$ (middle) corresponding to the characters $\chi_1$ and $\chi_2 \equiv \chi_1^*$ defined by equation (A6), compared and contrasted to the ones of the Titchmarsh counterexample $\xi_T$ (right) in identical domains of the complex plane $s \equiv \sigma + i\tau$. The phase shifts of $+2\theta$ and $-2\theta$ in $\Lambda_1$ and $\Lambda_2$ with respect to the critical line $\sigma = 1/2$ are a result of the corresponding phases of the normalized Gauss sums, equation (A5), and the functional equations, equations (A9) and (A10). In contrast, $\xi_T$ which is determined by a superposition, equation (15), of $\Lambda_1$ and $\Lambda_2$ such that it satisfies the functional equation, equation (2) does not display a phase shift but enjoys an antisymmetry of its phase.

Due to its simple pole, $g_T$ has to either increase from the second zero in the middle of figure 2 to plus infinity and increase from minus infinity after the pole, or decrease from the second zero to minus infinity and decay from plus infinity after the pole. Only in the second scenario, the condition, equation (35) for a zero of $\xi_T$ is satisfied for a value of $\tau$ before the pole as shown on the right of figure 2. Moreover, we note that after the second pole $g_T$ increases from minus infinity through the top zero of $g_T$. Hence, there must be another zero of $\xi_T$ between this pole and this zero of $g_T$.

So far we have explained the emergence of a zero of $\xi_T$ on the critical line. We now discuss the mechanism underlying the appearance of a zero off the critical line.

For this purpose we consider a situation depicted in the middle of figure 3 where $\Lambda_1$ has two consecutive zeros which are not separated by a zero of $\Lambda_2$. As a consequence, $g_T$ displays two adjacent simple zeros with a zero of the first derivative $g_T'$ in between. This point $\tau_0'$ on the critical line marked by a green triangle in the middle of figure 3 is the starting point of two lines of constant phase shown in green that move away symmetrically from their point of birth into the complex plane, and return to the critical line at another point $\tilde{\tau}_0'$ on the critical line,
Figure 2. Origin of the zeros of the Titchmarsh counterexample $\xi_T$ on the critical line explained by the behavior of the ratio $g_T$ formed by the two Dirichlet L-functions $\Lambda_1$ and $\Lambda_2$ together with the phase factor $\exp(-2i\theta)$ on the critical line. In the interval of $\tau-$values shown in the left picture the two zeros (red dots) of $\xi_T$ are located on the critical line, and the lines of constant phase (grey curves) terminating in them are separated by separatrices (green curves). Each separatrix merges with the critical line in a point (green triangles) where the first derivative of $\xi_T$ vanishes. The ratio $g_T$ displayed (middle) by its lines of constant phase (grey curves) shows for increasing values of $\tau$ on the critical line two zeros (red dots) of $g_T$ followed by two poles (black dots) and another zero (red dot). At $\tau$-values where the blue curve (right) representing $g_T$ on the critical axis, crosses the magenta solid line indicating the value $-1$, there are zeros of $\xi_T$. Indeed, the magenta solid lines (middle) which indicate the value $-1$ of $g_T$, pass through the critical line at points marked by crosses which are the zeros of $\xi_T$. Due to the pairing of two consecutive zeros and two consecutive poles, $g_T$ has to display vanishing first derivatives on the critical line as indicated for $g_T$ by green triangles. Here two phase lines first emerge and then terminate forming a loop. However, no zeros of $\xi_T$ exist on this loop since the values of $g_T$ in the points of vanishing first derivative are both positive, and the interval given by them does not include $-1$, as shown by the blue curve on the right.

now located between two adjacent zeros of $\Lambda_2$, that is two adjacent poles of $g_T$. In this case we have the possibility of zeros of $\xi_T$ located off the critical line since along this line the phase of the critical line is preserved.
Figure 3. Origin of the zeros of the Titchmarsh counterexample $\xi_T$ off the critical line explained by the behavior of $g_T$ on the critical line. In the interval of $\tau$—values shown on the left, the two zeros (red dots) of $\xi_T$ located off the critical line arise from three consecutive points (green triangles) where the first derivative vanishes. The ratio $g_T$ displayed in the middle for the same domain of the complex plane exhibits two consecutive poles and two consecutive zeros, which lead to a vanishing of the first derivative of $g_T$ (green triangles) between the poles, and between the zeros. Indeed, $g_T$ on the critical line shown on the right by blue curves displays at these points vanishing first derivatives as indicated by the green dashed horizontal lines. The inset enlarges the behavior of $g_T$ in one of these neighborhoods. In complete analogy to figure 2 a loop-shaped phase line appears, but now the interval formed by the values of $g_T$ at the points of the vanishing first derivative includes $-1$. As a result, there must be two symmetrically located zeros of $\xi_T$ off the critical line. This property is confirmed by the two crossings of the solid magenta line indicating the value $-1$ of $g_T$, and the phase loop marked by the two crosses.

However, we also need the ratio of $g_T$ to assume the value $-1$ and this requirement is not always satisfied. In figure 2 we show such an example of a loop-shaped phase line, but it does not lead to a zero off the axis.
Figure 4. Origin of the zeros of the Titchmarsh counterexample $\xi_T$ off the critical line, explained by the behavior of $g_T$ on the critical line. In the domain of the complex plane shown in the left picture $\xi_T$ displays three zeros (red dots), two of which are located off the critical line. In this case, two points of vanishing derivatives (green triangles) exist off the critical line as well. The flows of phase lines approaching the zero on the critical line from the right are divided by two separatrices collecting the flow toward the zero off the axis. Due to the antisymmetry of $\xi_T$ in the phase, the same behavior occurs for the flows left of the critical line. In this case $g_T$ (middle) displays two consecutive poles, a zero followed by a pole, and two consecutive zeros. Again a closed phase line appears which, due to the zero and the pole caught between the two pairs of poles and zeros, is in the shape of a pear. Indeed, two phase lines emerge from the point (green triangle) between the two poles where the first derivative of $g_T$ vanishes and meet again in the next point of a vanishing derivative (green triangle) caught between the two zeros. The interval formed by the values of $g_T$ on the critical line (right blue lines) at the points of the vanishing first derivative includes $-1$. As a result, there must be two symmetrically located zeros off the axis.

We can understand this result when we compare $g_T(1/2 + i\tau'_0)$ and $g_T(1/2 + i\tilde{\tau}'_0)$. Indeed only, if the interval $[g_T(1/2 + i\tau'_0), g_T(1/2 + i\tilde{\tau}'_0)]$ includes $-1$, we have two symmetrically located zeros off the axis.
The right panel of figure 2 shows that $-1$ is not included in the interval, whereas figures 3 and 4 depict situations where it is. Moreover, the zero of $\xi_T$ on the critical line in figure 4 is a consequence of the same scenario as in figure 2. Indeed, on the critical line $g_T$ decreases for increasing $\tau$ after the second pole from plus infinity to minus infinity at the second pole as shown by the blue line in the right panel of figure 4. Hence, $g_T$ must assume the value $-1$ leading to a zero of $\xi_T$ on the critical line.

5. Conclusions and outlook

In conclusion, the zeros of the Titchmarsh counterexample $\xi_T$ originate from the superposition of two Dirichlet $L$-functions. Since their zeros on the critical line are disjunct, the ratio of the two functions exhibit zeros and poles giving rise to closed lines of constant phase.

Moreover, this ratio is real along the critical line, which thus is a line of constant phase. We emphasize that this feature is in sharp contrast to the ratios corresponding to the hyperbolic cosine $c$ and the Riemann function $\xi$. Here the critical line is a line of constant height.

In appendix A we show that for a single Dirichlet $L$-function which can also be represented as a superposition of two contributions, and is expected to obey the generalized Riemann Hypothesis, the critical line is also a line of constant height of the corresponding ratio. We suspect that this property common to $c$, $\xi$ and $\Lambda$ could be intimately related to the validity of the Riemann Hypothesis.

It has often been argued that it is connected to the existence of an Euler product. Indeed, $\xi$ and $\Lambda$ enjoy such representations. However, due to $\xi_T$ being a superposition of two Dirichlet $L$-functions no Euler product can be found for $\xi_T$.

To establish a connection between the critical line being a line of constant height of the ratio of the building blocks of the analytic continuation, and the existence of an Euler product is a fascinating and challenging problem. However, this topic is only one of many that come to mind in the context of studying the lines of constant phase of this ratio.

Indeed, so far we have concentrated mainly on the behavior of these ratios on the critical line, but it is worthwhile to study the lines of constant phase of $g_R$ in the complete complex plane. Our analysis has clearly demonstrated that zeros of $\xi$ must be located on the critical line because there $|g_R| = 1$ and phase lines corresponding to an odd integer multiple of $\pi$ cross the critical line. But do zeros off the critical line as in the Titchmarsh counterexample exist also for $\xi$ ?

A violation of the Riemann Hypothesis would require an additional line of constant height of $g_R$ with $|g_R| = 1$ and at least one additional phase line that crosses it with a phase given by an odd integer multiple of $\pi$. Needless to say, we do not have a conclusive argument in favor or against it, but note the identity

$$g_R(s = 0) = g_R(s = 1) = 1$$

following from equations (3) and (7).

Hence, apart from the critical line there must be additional lines of constant height with $|g_R| = 1$. However, it is well-known that there are no zeros off the critical line close to the origin of the complex plane. Hence, there cannot be a phase line of integer multiple of $\pi$. Nevertheless, the question arises: how do the lines of constant phase of $g_R$ look in this domain? Moreover, there could be other contour lines with $|g_R| = 1$ away from the immediate neighborhood of the origin. However, the existence of such a line would require a pole of $g_R$ created either by a zero or a pole of $f_R$. Obviously $f_R$ is free of poles, but what about zeros?
Finally, we note from appendix A that the analytic continuation of $\Lambda$ given by equation (A2) is free of the off-set and the quadratic polynomial appearing in the corresponding expression, equation (10), for $\xi$ and giving rise to the complication of additional lines of constant height as expressed by equation (36). Hence, we expect the behavior of $g_\Lambda$ in the complete complex plane to be much simpler than $g_R$. Moreover, in this case $g_\Lambda$ is solely determined by the integral transform $\gamma = \gamma(s, \chi)$ which according to equation (A3) is free of a pole. Hence, an equivalent formulation of the generalized Riemann Hypothesis might be: The function $\gamma$ is free of zeros.

Unfortunately these questions go beyond the scope of the present article and have to be postponed to a future publication.

Data availability statement
No new data were created or analysed in this study.

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Appendix A. Dirichlet L-functions
Throughout our article, we focus on functions which satisfy the elementary representation, equation (1). For this class we show in the main body of our paper that in contrast to $\xi_T$ the functions $c$ and $\xi$ enjoy ratios $g_c$ and $g_\xi$ formed by the building blocks of the corresponding analytic continuations, where the critical line is a line of constant height with $|g| = 1$.

It is well-known [15, 16] that Dirichlet L-functions do not obey equation (1) but a slightly more complicated relation. Nevertheless, the critical line is still a line of constant height of the corresponding ratio $g_\Lambda$ as we verify now.

For this purpose, we first briefly summarize properties of Dirichlet L-functions central to our article. Here, for the sake of brevity we only state the results but do not derive them. We then turn to the discussion of $g_\Lambda$ on the critical line.

A.1. A brief review
The definition of a Dirichlet L-function $\Lambda$ of character $\chi \mod q$ reads [15, 16]

$$\Lambda(s, \chi) \equiv \left( \frac{d}{\pi} \right)^{s/2} \Gamma \left( \frac{s + \kappa}{2} \right) L(s, \chi), \quad (A1)$$

where

$$L(s, \chi) \equiv \sum_{n=1}^{\infty} \chi(n)n^{-s}$$
denotes the Dirichlet L-series and the parameter $\kappa$ assumes the values $\kappa = 0$ or $\kappa = 1$.

Moreover, $\Lambda$ enjoys the analytic continuation

$$\Lambda(s, \chi) = \gamma(s, \chi) + e^{i\beta(\chi)} \gamma(1 - s, \chi^*)$$

(A2)

where

$$\gamma(s, \chi) \equiv \left( \frac{q}{\pi} \right)^{-\kappa/2} \int_1^\infty dx \omega(x, \chi) \frac{q^x}{x^{s+1}}$$

(A3)

with

$$\omega(x, \chi) \equiv \sum_{n=1}^{\infty} \chi(n) n^\kappa e^{-\pi x^2/q}$$

(A4)

are the generalizations of the corresponding expressions equations (8) and (9) for the integral transform $\gamma$ of the Jacobi theta function $\omega$ in $\xi$ to $\Lambda$.

The phase factor $e^{i\beta(\chi)}$ in equation (A2) originates from the normalized Gauss sum

$$G(\chi) \equiv i^{-\kappa} \frac{1}{\sqrt{q}} \sum_{n=1}^{q} \chi(n) e^{2\pi i n/q} = e^{i\beta(\chi)}$$

(A5)

with $|G| = 1$ and makes equation (A2) different from the superposition, equation (1).

For the character $\chi_1 \mod 5$ with the values

$$\chi_1(1) = 1, \quad \chi_1(2) = i, \quad \chi_1(3) = -i, \quad \chi_1(4) = -1, \quad \chi_1(5) = 0$$

(A6)

we find from the definition, equation (A5), of the normalized Gauss sum the phase

$$\beta(\chi_1) = 2\theta,$$

where $\theta$ is given implicitly by the relation

$$\tan(2\theta) = \frac{\sqrt{5} - 1}{2}.$$  

(A7)

Since $\beta(\chi^*) = -\beta(\chi)$ we obtain for $\Lambda(s, \chi_2)$ with $\chi_2 \equiv \chi_1^*$ the phase

$$\beta(\chi_2) = -2\theta.$$

The analytic continuation of $\Lambda$ given by equation (A2) immediately leads us to the functional equation

$$\Lambda(s, \chi) = e^{i\beta(\chi)} \Lambda(1 - s, \chi^*),$$

(A8)

which for the two building blocks of the Titchmarsh counterexample defined by equation (15) takes the form

$$\Lambda(s, \chi_1) = e^{2i\theta} \Lambda(1 - s, \chi_2)$$

(A9)

and

$$\Lambda(s, \chi_2) = e^{-2i\theta} \Lambda(1 - s, \chi_1),$$

(A10)

where $\theta$ is given by equation (A7).
A.2. Critical line is line of constant height of $g_\Lambda$

Finally, we turn to the ratio

$$g_\Lambda(s, \chi) \equiv e^{-i\beta(\chi)} \frac{\gamma(s, \chi)}{\gamma(1-s, \chi^*)}$$

defined in analogy to equation (3) and following from the analytic continuation, equation (A2), of $\Lambda$ which on the critical line reads

$$g_\Lambda \left( \frac{1}{2} + i\tau, \chi \right) = e^{-i\beta(\chi)} \frac{\gamma \left( \frac{1}{2} + i\tau, \chi \right)}{\gamma \left( \frac{1}{2} + i\tau, \chi^* \right)}.$$

As a result, we arrive at the expression

$$g_\Lambda \left( \frac{1}{2} + i\tau, \chi \right) = e^{-i\beta(\chi)} e^{2i\delta_\Lambda(\tau; \chi)} \quad (A11)$$

where $\delta_\Lambda = \delta_\Lambda(\tau; \chi)$ is the phase of the integral transform $\gamma \left( \frac{1}{2} + i\tau, \chi \right)$ of the generalized theta function $\omega(x, \chi)$ on the critical line.

Hence, the critical line is indeed a contour line of $g_\Lambda$ with $|g_\Lambda \left( \frac{1}{2} + i\tau, \chi \right)| = 1.$

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