Spectral Properties of Finite Quantum Hall Systems*

Christian Ferrari and Nicolas Macris
Institut de Physique Théorique
Ecole Polytechnique Fédérale
CH - 1015 Lausanne, Switzerland

Abstract

In this note we review spectral properties of magnetic random Schrödinger operators $H_\omega = H_0 + V_\omega + U_\ell + U_r$ defined on $L^2(\mathbb{R} \times [-\frac{L}{2}, \frac{L}{2}], \, dx \, dy)$ with periodic boundary conditions along $y$. $U_\ell$ and $U_r$ are two confining potentials for $x \leq -\frac{L}{2}$ and $x \geq \frac{L}{2}$ respectively and vanish for $-\frac{L}{2} \leq x \leq \frac{L}{2}$. We describe the spectrum in two energy intervals and we classify it according to the quantum mechanical current of eigenstates along the periodic direction. The first interval lies in the first Landau band of the bulk Hamiltonian, and contains intermixed eigenvalues with a quantum mechanical current of $O(1)$ and $O\left(e^{-\gamma B(\log L)^2}\right)$ respectively. The second interval lies in the first spectral gap of the bulk Hamiltonian, and contains only eigenvalues with a quantum mechanical current of $O(1)$.

1 Introduction

In this note we review recent results on the spectrum of a magnetic random Schrödinger operator $H_\omega$ which describes the dynamics of an electron lying on a cylinder of circumference $L$ and which is confined along the cylinder axis by two smooth increasing potentials whose supports are separated by a distance $L$. We suppose our particle spinless, thus the Zeeman term in the Hamiltonian is neglected. The complete proofs of the theorems stated here can be found in [FM1] and [FM2].

First let us shortly recall previous results on random Schrödinger operators with magnetic field in the infinite two dimensional plane $\mathbb{R}^2$. We denote by $H_0$ the kinetic term $H_0 = (p - A)^2$, where $A$ is the vector potential associated to a constant magnetic field $B$. The spectrum of $H_0$ is given by the Landau levels \{(2n + 1)B : n \in \mathbb{N}\}. The

*Accepted for J. Oper. Theor.
bulk Hamiltonian is

$$H^b = H_0 + V_\omega$$  \hfill (1.1)

where $V_\omega$ is a Anderson-like random potential. The spectrum of (1.1) is contained in Landau bands around each Landau level, $\sigma(H^b) \subset \bigcup_{n \geq 0} [(2n + 1)B - V_0, (2n + 1)B + V_0]$ where $V_0 = \|V_\omega\|$, and if $V_0 < B$ there are open spectral gaps $G_n \supseteq ((2n + 1)B + V_0, (2n + 3)B - V_0)$ ($n \in \mathbb{N}$). It is proven that near the band edges the spectrum of $H^b_\omega$ is almost surely pure point with exponentially localized eigenfunctions [DMP1], [DMP2], [CH], [BCH], [W]. There are no rigorous results for energies at the band centers, except for a special model where the impurities are point scatterers [DMP3], [DMP4].

We now add a wall potential, translation invariant along the $y$-direction, such that $U_\ell(x)$ is confining for $x \leq -\frac{L}{2}$ and $U_\ell(x) = 0$ for $x \geq -\frac{L}{2}$. We have a semi-infinite system with a Hamiltonian

$$H^{si}_\omega = H_0 + V_\omega + U_\ell .$$ \hfill (1.2)

The spectrum contains the interval $[B, +\infty)$. For this system one can show that, for energies in intervals inside the gaps of the bulk Hamiltonian, the average velocity $(\psi, v_y \psi)$ in the $y$ direction, of an assumed eigenstate $\psi$ does not vanish. Since the velocity $v_y$ is the commutator between $-iy$ and the Hamiltonian, the Virial Theorem implies that an eigenstate cannot exist, and that therefore the spectrum is purely continuous inside the gaps of the bulk Hamiltonian [MMP], [F]. By Mourre theory one can show that the spectrum therein is purely absolutely continuous [FGW], [dBP].

Finally we can add a second wall potential $U_r$ such that $U_r(x) = 0$ for $x \leq \frac{L}{2}$ and which is confining for $x \geq \frac{L}{2}$. So the particle is confined between $x = -\frac{L}{2}$ and $x = \frac{L}{2}$. The Hamiltonian has the form

$$H_\omega = H_0 + V_\omega + U_\ell + U_r .$$ \hfill (1.3)

Additionally we make the $y-$direction periodic of length $L$ ($L$ a large parameter), this correspond to a finite, macroscopic cylindrical geometry.

Other models with confinement in the $x-$direction but without p.b.c. along $y$ have been studied. The first consists of a parabolic channel where $U_\ell + U_r$ is replaced by a parabolic confinement $\gamma x^2$, in this case it is shown that if the perturbation ($V_\omega$ in our case) is small enough in a suitable sense and satisfies a weak decay condition in the $y-$direction, there exists intervals of absolutely continuous spectrum [EJK]. The second model, more close to ours, consists to take two step walls of finite height for $U_\ell$ and $U_r$. In this case an initial state localized in energy in the spectral gap of the bulk Hamiltonian and near the left (resp. right) wall has a positive (resp. negative) velocity up to a finite time, limited by tunneling effect between the two walls [C].
The physical interest of our model is related to the integral quantum Hall effect [PG]. For the explanation of this effect Halperin [H] pointed out the importance of the boundary diamagnetic currents. Since many features of the integral quantum Hall effect can be described in the framework of one particle random magnetic Schrödinger operators it is important to understand their spectral properties for finite but macroscopic samples with boundaries.

2 The model

The family of random Schrödinger operators that we want to study is

$$H_\omega = p_x^2 + (p_y - Bx)^2 + V_\omega + U_r + U_l \quad \omega \in \Omega$$

(2.1)

these are densely defined self-adjoint operators acting in the Hilbert space $L^2(\mathbb{R} \times [-L/2, L/2], dx \, dy)$ with periodic boundary conditions along $y$: $\psi(x, -L/2) = \psi(x, L/2)$, $x \in \mathbb{R}$. $H_0 = p_x^2 + (p_y - Bx)^2$ is the kinetic Hamiltonian written in Landau gauge, its spectrum consists in infinitely degenerate eigenvalues

$$\sigma(H_0) = \sigma_{\text{ess}}(H_0) = \{(2n + 1)B; n \in \mathbb{N}\}.$$  

(2.2)

The two confining walls are assumed twice differentiable, strictly monotonic and satisfy

$$c_1|x + \frac{L}{2}|^{m_1} \leq U_l(x) \leq c_2|x + \frac{L}{2}|^{m_2} \quad \text{for } x \leq -\frac{L}{2}$$

(2.3)

$$c_1|x - \frac{L}{2}|^{m_1} \leq U_r(x) \leq c_2|x - \frac{L}{2}|^{m_2} \quad \text{for } x \geq \frac{L}{2}$$

(2.4)

for some constants $0 < c_1 < c_2 < \infty$ and $2 \leq m_1 < m_2 < \infty$. Moreover $U_l(x) = 0$ for $x \geq -\frac{L}{2}$ and $U_r(x) = 0$ for $x \leq \frac{L}{2}$. We could allow steeper confinements but the present polynomial conditions turn out to be technically convenient. The random potential $V_\omega$ consists of a sum of local perturbations located at the sites of a finite

Figure 1: The potentials along the x-axes.
lattice \( \Lambda = \mathbb{Z}^2 \cap [X \times [-\frac{L}{2}, \frac{L}{2}]] \) where \( X \) will be defined latter. Thus

\[
V_\omega(x,y) = \sum_{(n,m) \in \Lambda} X_{n,m}(\omega) V(x-n, y-m) \quad \omega \in \Omega
\]

where the coupling constants \( X_{n,m} \) are i.i.d. random variables with common bounded probability density \( h \in C^2((-1,1)) \). The local potential \( V \) satisfies \( V \in C^2 \), \( 0 \leq V(x,y) \leq V_0 < \infty \), \( \text{supp} \ V \subset \mathbb{B}(0, \frac{1}{4}) \) (the open ball centred at \( (0,0) \) of radius \( \frac{1}{4} \)). \( \Omega = [-1,1]^\Lambda \) is the set of all possibles realizations, we will denote by \( \mathbb{P}_\Lambda \) the product measure defined on \( \Omega \). Clearly for all \( \omega \in \Omega \) we have \( \|V_\omega\| \leq V_0 \). We will assume that \( V_0 \ll B \), that is, we work in a strong magnetic field regime or, equivalently, in a weak disorder regime.

Our first result concerns the study of \( \sigma(H_\omega) \) in the energy interval \( \Delta_\varepsilon = [B + \varepsilon, B + V_0] \) that lies inside the first Landau band of the infinite bulk system. In this case the interval \( X \), that defines the support of the random potential along the \( x- \)direction, is \([ -\frac{L}{2} + \log L, \frac{L}{2} - \log L ] \): we leave a thin strip of size \( \log L \) without random potential along each confining wall.

The second result is about \( \sigma(H_\omega) \) inside the first spectral gap of the infinite bulk system, more precisely in the energy interval \( \Delta = (2B - \delta, 2B + \delta) \subset (B + V_0 + \varepsilon, 3B - V_0 - \varepsilon) \). In this case the random potential fills the whole space in between the confining walls, that means \( X = [-\frac{L}{2}, \frac{L}{2}] \).

Since our system is confined the spectrum is made of discrete eigenvalues. There exists a natural classification of the eigenvalues via the quantum mechanical current along the periodic direction. If \( \psi \) satisfies the eigenvalue equation \( H_\omega \psi = E \psi \) the current is defined (here) as

\[
J_E \equiv (\psi, v_y \psi)
\]

where \( v_y = 2(p_y - Bx) \) is the velocity operator in the \( y- \)direction. Thanks to \( J_E \) we can classify the eigenvalues in two classes: the first consists on those which have \( |J_E| > C \) with \( C \) a positive constant uniform in \( L \), the second consists on those for which \( |J_E| < \epsilon(L) \) with \( \epsilon(L) \to 0 \) as \( L \to \infty \) (we stress that here \( L \) is finite but macroscopic, the limit means that \( \epsilon(L) \) is infinitesimally small with \( L \)). The physical meaning of this classification is briefly discussed at the end of section 3.

The main idea of our approach is to first look at some easier Hamiltonians and then link them together to get properties on the full Hamiltonian \( H_\omega \). In what follows we do not analyze these easier Hamiltonians but we just introduce the minimal notations and properties (see [FM1] and [FM2] for the details).
3 Main results

For the analysis of \( \sigma(H_\omega) \) in \( \Delta_\varepsilon \) we need to know some properties of the Hamiltonian

\[
H^0_\alpha = H_0 + U_\alpha \quad \alpha = \ell, r
\]

called pure edge Hamiltonian. Its spectrum is given by

\[
\sigma(H^0_\alpha) = \{ E^\alpha_{nk}; n \in \mathbb{N}, k \in \frac{2\pi}{T} \mathbb{Z} \}
\]  

\[
E^\ell_{0k} \quad E^\ell_{1k} 
\]

\[
E^r_{0k} \quad 3B \quad B \quad \frac{2\pi}{T} \mathbb{Z}
\]

Figure 2: The spectrum of \( H^0_\ell \) lies on monotonic decreasing branches. That of \( H^0_r \) lies on similar, but monotonic increasing, branches. The spectral branches are given by the dispersion relation for \( L = \infty \).

The quantum mechanical currents associated to the eigenfunctions \( \psi^\alpha_{nk} \) whose eigenvalue is in \( \Delta_\varepsilon \) (we have \( n = 0 \)) satisfy

\[
|J^\alpha_{0k}| = |(\psi^\alpha_{0k}, v_y \psi^\alpha_{0k})| > C
\]

with \( C > 0 \) a numerical constant independent of \( L \) [FM1]. Moreover we will assume the following

**Hypothesis 1.** Fix \( \varepsilon > 0 \). There exist \( L(\varepsilon) \) and \( d(\varepsilon) \) such that for all \( L > L(\varepsilon) \)

\[
\text{dist} \left( \sigma(H^0_\ell) \cap \Delta_\varepsilon, \sigma(H^0_r) \cap \Delta_\varepsilon \right) \geq \frac{d(\varepsilon)}{L}.
\]

This hypothesis is important because a minimal amount of non-degeneracy between the spectra of the two edge systems is needed in order to control backscattering effects induced by the random potential. Indeed in a system with two boundaries backscattering favors localization and has a tendency to destroy currents. Remark that this hypothesis can be verified by taking two symmetric confining potentials \( U_\ell(x) = U_r(-x) \) and adding a suitable flux line along the cylinder axes (see in particular Appendix C in [FM2]).
We also need to know some properties of the bulk Hamiltonian

\[ H_b = H_0 + V_\omega. \]  

(3.5)

Its essential spectrum is given by the Landau levels and the whole spectrum is contained in the Landau bands \( \sigma(H_b) \subset \bigcup_{n \geq 0} [(2n + 1)B - V_0, (2n + 1)B + V_0] \). We will suppose that the (discrete) spectrum in \( \Delta \) fulfills the

Hypothesis 2. Fix any \( \varepsilon > 0 \). There exist \( \mu(\varepsilon) \) a strictly positive constant and \( L(\varepsilon) \) such that for all \( L > L(\varepsilon) \) one can find a set of realizations of the random potential \( \Omega' \) with \( \mathbb{P}_\Lambda(\Omega') \geq 1 - L^{-\theta} \), \( \theta > 0 \), with the property that if \( \omega \in \Omega' \) the eigenstates corresponding to \( E_\beta \in \sigma(H_b) \cap \Delta \) satisfy

\[ |\psi_\beta^b(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L}, \quad |\partial_y \psi_\beta^b(x, \bar{y}_\beta)| \leq e^{-\mu(\varepsilon)L} \]  

(3.6)

for some \( \bar{y}_\beta \) depending on \( \omega \) and \( L \).

Since \( V_\omega \) is random we expect that eigenfunctions with energies in \( \Delta \) (not too close to the Landau levels where the localization length diverges) are exponentially localized on a scale \( O(1) \) with respect to \( L \). Inequalities (3.6) are a weaker version of this statement, and have been checked for the special case where the random potential is a sum of rank one perturbations \([FM3]\) using the methods of Aizenman and Molchanov \([AM]\). The main consequence of Hypothesis 2 is that a state satisfying (3.6) does not carry any appreciable current (contrary to the eigenstates of \( H_0^\alpha \)) in the sense that

\[ J_\beta^b = (\psi_\beta^b, v_y \psi_\beta^b) = \mathcal{O} \left( e^{-\mu(\varepsilon)L} \right). \]  

(3.7)

We are now ready to state our first result on the eigenvalues lying in \( \Delta \). We are ready to state our first result on the eigenvalues lying in \( \Delta \).

Theorem 1. Fix \( \varepsilon > 0 \) and assume that (H1) and (H2) are fulfilled. Assume \( B > 4V_0 \). Then there exists a numerical constant \( \gamma > 0 \) and an \( \bar{L} \geq L(\varepsilon) \) such that for all \( L > \bar{L} \) one can find a set \( \hat{\Omega} \subset \Omega \) of realizations of the random potential \( V_\omega \) with \( \mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-s} \) \( (s \gg 1) \) such that for any \( \omega \in \hat{\Omega} \), \( \sigma(H_\omega) \cap \Delta \) is the union of three sets \( \Sigma_\ell \cup \Sigma_b \cup \Sigma_r \), each depending on \( \omega \) and \( L \), and characterized by the following properties:

\( a) \) \( E_\alpha^k \in \Sigma_\alpha \) \( (\alpha = \ell, r) \) are a small perturbation of \( E_{0k}^\alpha \in \sigma(H_0^\alpha) \cap \Delta \) with

\[ |E_\alpha^k - E_{0k}^\alpha| \leq e^{-\gamma B(\log L)^2}, \quad \alpha = \ell, r. \]  

(3.8)

\( b) \) For \( E_\alpha^k \in \Sigma_\alpha \) the current \( J_\alpha^k \) of the associated eigenstate satisfies

\[ |J_\alpha^k - J_{0k}^\alpha| \leq e^{-\gamma B(\log L)^2}, \quad \alpha = \ell, r. \]  

(3.9)

\( c) \) \( \Sigma_b \) contains the same number of energy levels as \( \sigma(H_b) \cap \Delta \) and \( (p \gg 1) \)

\[ \text{dist}(\Sigma_b, \Sigma_\alpha) \geq L^{-p}, \quad \alpha = \ell, r. \]  

(3.10)
d) The current associated to each level $\mathcal{E}_\beta \in \Sigma_b$ satisfies
\[
|J_\beta| \leq e^{-\gamma B (\log L)^2}.
\] (3.11)

We now turn to the characterisation of eigenvalues lying in $\Delta = (2B - \delta, 2B + \delta)$. In this case the first easier Hamiltonian is
\[
H_\alpha = H_0 + U_\alpha + V_\omega^\alpha \quad \alpha = \ell, r
\] (3.12)
called random edge Hamiltonian. In (3.12) the random potential $V_\omega^\alpha$ is the restriction of $V_\omega$ to $\Lambda_\alpha \subset \Lambda$, where $\Lambda_\alpha$ is a strip of size $\sqrt{L}$ in the $x$–direction along the confining walls: $\Lambda_\ell = \mathbb{Z}^2 \cap \left[[-\frac{L}{2}, \frac{L}{2}] \times [-\frac{L}{2}, \frac{L}{2}]\right]$ and $\Lambda_r = \mathbb{Z}^2 \cap \left[[-\frac{L}{2}, \frac{L}{2}] \times [-\frac{L}{2}, \frac{L}{2}]\right]$. The spectrum of $H_\alpha$ is given by
\[
\sigma(H_\alpha) = \{E_\alpha^\kappa; \kappa \in \mathbb{Z}\}
\] (3.13)
and $E_\alpha^\kappa$ are isolated eigenvalues with accumulation points at the Landau levels. The quantum mechanical currents $J_\alpha^\kappa$ associated to the energies in $\Delta$ satisfy
\[
|J_\alpha^\kappa| = |(\psi_\alpha^{\kappa}, v_y \psi_\alpha^{\kappa})| > C'
\] (3.14)
with $C' > 0$ a numerical constant independent of $L$. By the way remark that, since the random variables in the Anderson potential are i.i.d., $H_\ell$ and $H_r$ are two independent random Hamiltonians. Here, as before we suppose that Hypothesis 1 is fulfilled.

We now state our second result.

**Theorem 2.** Let $V_0$ small enough, fix $\varepsilon > 0$ and let $0 < \delta \equiv \delta(V_0) < B - V_0 - \varepsilon$.

Suppose that (H1) holds. Then there exists $\mu > 0$, $\bar{L} \geq L(\varepsilon)$ such that if $L \geq \bar{L}$ one can find a set $\hat{\Omega} \subset \Omega$ of realizations of the random potential $V_\omega$ with $\mathbb{P}_\Lambda(\hat{\Omega}) \geq 1 - L^{-\nu}$ ($\nu \gg 1$) such that for all $\omega \in \hat{\Omega}$ the spectrum of $H_\omega$ in $\Delta = (B - \delta, B + \delta)$ is the unions of two sets $\Sigma_\ell'$ and $\Sigma_r'$, each depending on $\omega$ and $L$, with the following properties:

a) $\mathcal{E}_\alpha^\kappa \in \Sigma'_\alpha$ ($\alpha = \ell, r$) are a small perturbation of $E_\alpha^\kappa \in \sigma(H_\alpha) \cap \Delta$ with
\[
|\mathcal{E}_\alpha^\kappa - E_\alpha^\kappa| \leq e^{-\mu \sqrt{B \sqrt{L}}},
\] (3.15)

b) For $\mathcal{E}_\alpha^\kappa \in \Sigma'_\alpha$ the current $J_\alpha^\kappa$ of the associated eigenstate satisfies
\[
|J_\alpha^\kappa - J_\kappa^\alpha| \leq e^{-\mu \sqrt{B \sqrt{L}}},
\] (3.16)

The idea of the proofs of Theorems 1 and 2 is to link the resolvent of the full Hamiltonian $H_\omega$ to those of the easier Hamiltonians $R_\ell^0(z)$ (resp. $R_\ell(z)$), $R_r^0(z)$ (resp. $R_r(z)$) and $R_b(z)$. This is achieved via a decoupling formula for the resolvent [BCD], [BC]. Using it we can do deterministic estimates on the norm difference between the
projector \( P_{H_{\omega}}(\Gamma) \), associated to \( H_{\omega} \) into the disc with boundary \( \Gamma \), and the projector associated to one of the easier Hamiltonians. This is done for suitable circles \( \Gamma \) in the complex plane and a suitable set \( \hat{\Omega} \) of realizations of the random potential. Using Wegner estimates on \( H_{b} \) (resp. \( H_{a} \)) we control the probability of \( \hat{\Omega} \) and show that it can be made large.

Our classification of the spectrum via the quantum mechanical current leads to a well defined notion of extended edge states and localized bulk states. The former are those belonging to \( \Sigma_{\alpha} \) (resp. \( \Sigma'_{\alpha} \)), they are small perturbations of the eigenvalues of \( \sigma(H^0_{\alpha}) \) (resp. \( \sigma(H_{\alpha}) \)) and have a quantum mechanical current of order \( O(1) \) with respect to \( L \). The latter are those belonging to \( \Sigma_{b} \), and have an infinitesimal current with respect to \( L \) (of order \( O \left( e^{-\gamma B \log L^2} \right) \)), they “arise” from the spectrum of \( H_{b} \). It is interesting to note that our description leads, in the interval inside the first Landau band, to a spectrum in which extended edge and localized bulk states are intermixed and in some sense there is no “mobility edge”. On the other hand in the interval inside the spectral gap there exists only extended edge states.

**Acknowledgements**

C.F. is grateful to the organizers of the conference for the invitation to report on this work. The work of C.F. was supported by a grant from the Fonds National Suisse de la Recherche Scientifique.

**References**

[AM] M. Aizenman, S. Molchanov: Localization at large disorder and at extreme energies: an elementary derivation. Commun. Math. Phys. **157**, 245 (1993)

[BCD] P. Briet, J.M. Combes, P. Duclos: Spectral stability under tunneling. Commun. Math. Phys. **126**, 133 (1989)

[BCH] J.M. Barbaroux, J.M. Combes, P.D. Hislop: Localization near band edges for random Schrödinger operators. Helv. Phys. Acta **70**, 16 (1997)

[BG] F. Bentosela, V. Grecchi: Stark Wannier Ladders. Commun. Math. Phys. **142**, 169 (1991)

[dBP] S. de Bièvre, J.V. Pulé: Propagating edge states for magnetic Hamiltonian. Math. Phys. Electr. J. **5**, no. 3 (1999)

[CH] J.M. Combes, P.D. Hislop: Landau Hamiltonians with random potentials: localization and the density of states. Commun. Math. Phys. **177**, 603 (1996)

[C] J.M. Combes: private communication
[DMP1] T.C. Dorlas, N. Macris, J.V. Pulé: Localization in a single-band approximation to random Schrödinger operators in a magnetic field. Helv. Phys. Acta 68, 330 (1995)

[DMP2] T.C. Dorlas, N. Macris, J.V. Pulé: Localization in single Landau bands. J. Math. Phys. 37, 1574 (1996)

[DMP3] T.C. Dorlas, N. Macris, J.V. Pulé: The nature of the spectrum for a Landau Hamiltonian with delta impurities. J. Stat. Phys. 87, 847 (1997)

[DMP4] T.C. Dorlas, N. Macris, J.V. Pulé: Characterization of the spectrum of the Landau Hamiltonian with delta impurities. Commun. Math. Phys. 204, 367 (1999)

[EJK] P. Exner, A. Joye, H. Kovarik: Magnetic transport in a straight parabolic channel. J. Phys. A: Math. Gen. 34, 9733 (2001)

[F] C. Ferrari: Dynamique d’une particule quantique dans un champ magnétique inhomogène. Diploma work, EPFL (1999).

[FGW] J. Fröhlich, G.M. Graf, J. Walcher: On the extended nature of edge states of quantum Hall Hamiltonians. Ann. Henri Poincaré 1, 405 (2000)

[FM1] C. Ferrari, N. Macris: Intermixture of extended edge and localized bulk energy levels in macroscopic Hall systems. math-ph/0011013

[FM2] C. Ferrari, N. Macris: Extended energy levels in the gap for macroscopic Hall systems. Preprint

[FM3] C. Ferrari, N. Macris: Localized energy levels inside Landau bands. In preparation

[H] B.I. Halperin: Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential. Phys. Rev. B 25, 2185 (1982)

[MMP] N. Macris, P.A. Martin and J.V. Pulé: On Edge States In Semi-Infinite Quantum Hall Systems. J. Phys. A: Math. Gen. 32, 1985 (1999)

[PG] R.E. Prange and S.M. Girvin: The Quantum Hall Effect. New York: Graduate Texts in Contemporary Physics, Springer, 1987

[W] W.M. Wang: Microlocalization, percolation and Anderson localization for the magnetic Schrödinger operator with a random potential. J. of Funct. Anal. 146, 1 (1997)