On the exterior Dirichlet problem for Hessian quotient equations*

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Abstract

In this paper, we establish the existence and uniqueness theorem for solutions of the exterior Dirichlet problem for Hessian quotient equations with prescribed asymptotic behavior at infinity. This extends the previous related results on the Monge-Ampère equations and on the Hessian equations, and rearranges them in a systematic way. Based on the Perron’s method, the main ingredient of this paper is to construct some appropriate subsolutions of the Hessian quotient equation, which is realized by introducing some new quantities about the elementary symmetric functions and using them to analyze the corresponding ordinary differential equation related to the generalized radially symmetric subsolutions of the original equation.

Keywords: Dirichlet problem, existence and uniqueness, exterior domain, Hessian quotient equation, Perron’s method, prescribed asymptotic behavior, viscosity solution

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1 Introduction

In this paper, we consider the Dirichlet problem for the Hessian quotient equation

\[
\frac{\sigma_k(\lambda(D^2u))}{\sigma_1(\lambda(D^2u))} = 1
\]

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in the exterior domain $\mathbb{R}^n \setminus \overline{D}$, where $D$ is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, $0 \leq l < k \leq n$, $\lambda(D^2u)$ denotes the eigenvalue vector $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of the Hessian matrix $D^2u$ of the function $u$, and

$$\sigma_0(\lambda) \equiv 1 \quad \text{and} \quad \sigma_j(\lambda) := \sum_{1 \leq s_1 < s_2 < \ldots < s_j \leq n} \lambda_{s_1}\lambda_{s_2} \ldots \lambda_{s_j} \quad (\forall 1 \leq j \leq n)$$

are the elementary symmetric functions of the $n$-vector $\lambda$. Note that when $l = 0$, (1.1) is the Hessian equation $\sigma_k(\lambda(D^2u)) = 1$; when $l = 0$, $k = 1$, it is the famous Monge-Ampère equation $\det(D^2u) = 1$; and when $l = 1$, $k = 3$, $n = 3$ or $4$, it is the special Lagrangian equation $\sigma_1(\lambda(D^2u)) = \sigma_3(\lambda(D^2u))$ in three or four dimension (in three dimension, this is $\det(D^2u) = \Delta u$ indeed) which arises from the special Lagrangian geometry [HL82].

For linear elliptic equations of second order, there have been much extensive studies on the exterior Dirichlet problem, see [MS60] and the references therein. For the Monge-Ampère equation, a classical theorem of Jörgens [Jor54], Calabi [Cal58] and Pogorelov [Pog72] states that any convex classical solution of $\det(D^2u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. Related results was also given by [CY86], [Ca95], [TW00] and [X01]. Caffarelli and Li [CL03] extended the Jörgens-Calabi-Pogorelov theorem to exterior domains. They proved that if $u$ is a convex viscosity solution of $\det(D^2u) = 1$ in the exterior domain $\mathbb{R}^n \setminus \overline{D}$, where $D$ is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, then there exist $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that

$$\limsup_{|x| \to +\infty} |x|^{n-2} \left| u(x) - \left( \frac{1}{2} x^T Ax + b^T x + c \right) \right| < \infty. \quad (1.2)$$

With such prescribed asymptotic behavior at infinity, they also established an existence and uniqueness theorem for solutions of the Dirichlet problem of the Monge-Ampère equation in the exterior domain of $\mathbb{R}^n$, $n \geq 3$. See [FMM99], [FMM00] or [Del92] for similar problems in two dimension. Recently, J.-G. Bao, H.-G. Li and Y.-Y. Li [BLL14] extended the above existence and uniqueness theorem of the exterior Dirichlet problem in [CL03] for the Monge-Ampère equation to the Hessian equation $\sigma_k(\lambda(D^2u)) = 1$ with $2 \leq k \leq n$ and with some appropriate prescribed asymptotic behavior at infinity which is modified from (1.2). Before them, for the special case that $A = c_0 I$ with $c_0 := (C_n^k)^{-1/k}$ and $C_n^k := n!/(k!(n-k)!)$, the exterior Dirichlet problem for the Hessian equation has been investigated by Dai and Bao in [DB11]. At the
same time, Dai [Dai11] proved the existence theorem of the exterior Dirichlet problem for the Hessian quotient equation (1.1) with \( k - l \geq 3 \), and with the prescribed asymptotic behavior at infinity of the special case that \( A = c_\ast I \), that is,
\[
\limsup_{|x|\to+\infty} |x|^{k-l-2} \left| u(x) - \left( \frac{c_\ast}{2} |x|^2 + c \right) \right| < \infty,
\]
where
\[
c_\ast := \left( \frac{C_n^k}{C_n^l} \right)^{\frac{1}{k-l}} \text{ with } C_n^i := \frac{n!}{i!(n-i)!} \text{ for } i = k, l.
\]

As they pointed out in [LB14] that the restriction \( k - l \geq 3 \) rules out an important example, the special Lagrangian equation \( \det(D^2 u) = \Delta u \) in three dimension. Later, [LD12] improve the result in [Dai11] for (1.1) with \( k - l \geq 3 \) to that for (1.1) with \( 0 \leq l < k \leq \left( \frac{n+1}{2} \right) \). More recently, Li and Bao [LB14] established the existence theorem of the exterior Dirichlet problem for a class of fully nonlinear elliptic equations related to the eigenvalues of the Hessian which include the Monge-Ampère equations, Hessian equations, Hessian quotient equations and the special Lagrangian equations in dimension equal and larger than three, but with the prescribed asymptotic behavior at infinity only in the special case of (1.2) that \( A = c_\ast I \) with \( c_\ast \) some appropriate constant, like (1.3) and (1.4).

In this paper, we focus our attention on the Hessian quotient equation (1.1) and establish the existence and uniqueness theorem for the exterior Dirichlet problem of it with prescribed asymptotic behavior at infinity of the type similar to (1.2). This extends the previous corresponding results on the Monge-Ampère equations [CL03] and on the Hessian equations [BLL14] to Hessian quotient equations, and also extends those results on the Hessian quotient equations in [Dai11], [LD12] and [LB14] to be valid for general prescribed asymptotic behavior condition at infinity. Since we do not restrict ourselves to the case \( k - l \geq 3 \) or \( 0 \leq l < k \leq \left( \frac{n+1}{2} \right) \) only, our theorems also apply to the special Lagrangian equations \( \det(D^2 u) = \Delta u \) in three dimension and \( \sigma_1(\lambda(D^2 u)) = \sigma_3(\lambda(D^2 u)) \) in four dimension. Indeed, we will show in our forthcoming paper [LL16] that our method still works very well for the special Lagrangian equations with higher dimension and with general phase.

We would like to remark that, for the interior Dirichlet problems there have been much extensive studies, see for example [CIL92], [CNS85], [Ivo85], [Kry83], [Urb90], [Tru90] and [Tru95]; see [BCGJ03] and the references given.
there for more on the Hessian quotient equations; and for more on the special Lagrangian equations, we refer the reader to [HL82], [Fu98], [Yuan02], [CWY09] and the references therein.

For the reader’s convenience, we give the following definitions related to Hessian quotient equation (see also [CIL92], [CC95], [CNS85], [Tru90], [Tru95] and the references therein).

We say that a function \( u \in C^2(\mathbb{R}^n \setminus \overline{D}) \) is \( k \)-convex, if \( \lambda(D^2u) \in \Gamma_k \) in \( \mathbb{R}^n \setminus \overline{D} \), where \( \Gamma_k \) is the connected component of \( \{ \lambda \in \mathbb{R}^n | \sigma_k(\lambda) > 0 \} \) containing the positive cone

\[
\Gamma^+ := \{ \lambda \in \mathbb{R}^n | \lambda_i > 0, \forall i = 1, 2, ..., n \} .
\]

It is well known that \( \Gamma_k \) is an open convex symmetric cone with its vertex at the origin and that

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n | \sigma_j(\lambda) > 0, \forall j = 1, 2, ..., k \} ,
\]

which implies

\[
\{ \lambda \in \mathbb{R}^n | \lambda_1 \lambda_2 + ... + \lambda_n > 0 \} = \Gamma_1 \supset ... \supset \Gamma_k \supset \Gamma_{k+1} \supset ... \supset \Gamma_n = \Gamma^+
\]
with the first term \( \Gamma_1 \) the half space and with the last term \( \Gamma_n \) the positive cone \( \Gamma^+ \). Furthermore, we also know that

\[
\partial_{\lambda_i} \sigma_j(\lambda) > 0, \forall 1 \leq i \leq n, \forall 1 \leq j \leq k, \forall \lambda \in \Gamma_k, \forall 1 \leq k \leq n \quad (1.5)
\]

(see [CNS85] or [Urb90] for more details).

Let \( \Omega \) be an open domain in \( \mathbb{R}^n \) and let \( f \in C^0(\Omega) \) be nonnegative. Suppose \( 0 \leq l < k \leq n \). A function \( u \in C^0(\Omega) \) is said to be a viscosity subsolution of

\[
\frac{\sigma_k(\lambda(D^2u))}{\sigma_l(\lambda(D^2u))} = f \quad \text{in} \ \Omega \quad (1.6)
\]

(or say that \( u \) satisfies

\[
\frac{\sigma_k(\lambda(D^2u))}{\sigma_l(\lambda(D^2u))} \geq f \quad \text{in} \ \Omega
\]

in the viscosity sense, similarly hereinafter), if for any function \( v \in C^2(\Omega) \) and any point \( x^* \in \Omega \) satisfying

\[
v(x) \geq u(x), \ \forall x \in \Omega \quad \text{and} \quad v(x^*) = u(x^*),
\]

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we have
\[
\frac{\sigma_k(\lambda(D^2v))}{\sigma_l(\lambda(D^2v))} \geq f \text{ in } \Omega.
\]

A function \( u \in C^0(\Omega) \) is said to be a \textit{viscosity supersolution} of (1.6) if for any \( k \)-convex function \( v \in C^2(\Omega) \) and any point \( x^* \in \Omega \) satisfying
\[
v(x) \leq u(x), \forall x \in \Omega \quad \text{and} \quad v(x^*) = u(x^*),
\]
we have
\[
\frac{\sigma_k(\lambda(D^2v))}{\sigma_l(\lambda(D^2v))} \leq f \text{ in } \Omega.
\]

A function \( u \in C^0(\Omega) \) is said to be a \textit{viscosity solution} of (1.6), if it is both a viscosity subsolution and a viscosity supersolution of (1.6). A function \( u \in C^0(\Omega) \) is said to be a \textit{viscosity subsolution} (respectively, supersolution, solution) of (1.6) and \( u = \varphi \) on \( \partial \Omega \) with some \( \varphi \in C^0(\partial \Omega) \), if \( u \) is a viscosity subsolution (respectively, supersolution, solution) of (1.6) and \( u \leq (\text{respectively}, \geq, =) \varphi \) on \( \partial \Omega \).

Note that in the definitions of viscosity solution above, we have used the ellipticity of the Hessian quotient equations indeed. For completeness and convenience, this will be proved in the end of Subsection 2.3. See also [CC95], [CIL92], [CNS85], [Urb90] and the references therein.

Define
\[
\mathcal{A}_{k,l} := \{ A \in S(n)|\lambda(A) \in \Gamma^+, \sigma_k(\lambda(A)) = \sigma_l(\lambda(A)) \}.
\]

Note that there are plenty of elements in \( \mathcal{A}_{k,l} \). In fact, for any \( A \in S(n) \) with \( \lambda(A) \in \Gamma^+ \), if we set
\[
\varphi := \left( \frac{\sigma_k(\lambda(A))}{\sigma_l(\lambda(A))} \right)^{-\frac{1}{k-l}},
\]
we then have \( \varphi A \in \mathcal{A}_{k,l} \). Let
\[
\widetilde{\mathcal{A}}_{k,l} := \{ A \in \mathcal{A}_{k,l}|m_{k,l}(\lambda(A)) > 2 \},
\]
where \( m_{k,l}(\lambda) \) is a quantity which plays an important role in this paper. We will give the specific definition of \( m_{k,l}(\lambda) \) in (2.11) in Subsection 2.2 and verify there that \( \widetilde{\mathcal{A}}_{k,l} \) possesses the following fine properties.

**Proposition 1.1.** Suppose \( 0 \leq l < k \leq n \) and \( n \geq 3 \).
(1) If \( k - l \geq 2 \), then \( \tilde{A}_{k,l} = A_{k,l} \).

(2) \( \tilde{A}_{n,0} = A_{n,0} \) and \( m_{n,0} \equiv n \).

(3) \( c_* I \in \tilde{A}_{k,l} \) and \( m_{k,l}(c_*(1,1,\ldots,1)) = n \), where \( c_* \) is the one defined in (1.4).

The main result of this paper now can be stated as below.

**Theorem 1.1.** Let \( D \) be a bounded strictly convex domain in \( \mathbb{R}^n \), \( n \geq 3 \), \( \partial D \in C^2 \) and let \( \varphi \in C^2(\partial D) \). Then for any given \( A \in \tilde{A}_{k,l} \) with \( 0 \leq l < k \leq n \), and any given \( b \in \mathbb{R}^n \), there exists a constant \( \tilde{c} \) depending only on \( n, D, k, l, A, b \) and \( \|\varphi\|_{C^2(\partial D)} \), such that for every \( c \geq \tilde{c} \), there exists a unique viscosity solution \( u \in C^{0}(\mathbb{R}^n \setminus D) \) of

\[
\begin{aligned}
\sigma_k(\lambda(D^2u)) &= 1 \quad \text{in } \mathbb{R}^n \setminus \overline{D}, \\
u &= \varphi \quad \text{on } \partial D, \\
\limsup_{|x| \to +\infty} |x|^{m-2} \left| u(x) - \left( \frac{1}{2} x^T Ax + b^T x + c \right) \right| &< \infty,
\end{aligned}
\]

where \( m \in (2,n] \) is a constant depending only on \( n, k, l \) and \( \lambda(A) \), which actually can be taken as \( m_{k,l}(\lambda(A)) \).

**Remark 1.1.** (1) One can easily see that Theorem 1.1 still holds with \( A \in \tilde{A}_{k,l} \) replaced by \( A \in \tilde{A}_{k,l}^* \) and \( \lambda(A) \) replaced by \( \lambda(A^*) \), where

\[
\tilde{A}_{k,l}^* := \left\{ A \in \mathbb{R}^{n \times n} \mid \lambda(A^*) \in \Gamma^+, \sigma_k(\lambda(A^*)) = \sigma_l(\lambda(A^*)), m_{k,l}(\lambda(A^*)) > 2 \right\}
\]

and \( A^* := (A + A^T)/2 \). This is to say that the above theorem can be adapted to a slightly more general form by modifying the meaning of \( \tilde{A}_{k,l} \).

(2) For the special cases that \( l = 0 \) (i.e., the Hessian equation \( \sigma_k(\lambda(D^2u)) = 1 \)) and that \( l = 0 \) and \( k = n \) (i.e., the Monge-Ampère equation \( \det(D^2u) = 1 \)), in view of Proposition 1.1-(2), our Theorem 1.1 recovers the corresponding results [BLL14, Theorem 1.1] and [CL03, Theorem 1.5], respectively.
(3) For $A = c_* I$ with $c_*$ defined in (1.4), by Proposition 1.1-(1),(3), our results improve those in [Dai11] and [LD12]. Indeed, by Proposition 1.1-(3), the main results in [Dai11], [LD12] and those parts concerning the Hessian quotient equations in [LB14] can all be recovered by Theorem 1.1 as special cases. Furthermore, our results also apply to the special Lagrangian equation $\det(D^2 u) = \Delta u$ in three dimension (respectively, $\sigma_1(\lambda(D^2 u)) = \sigma_3(\lambda(D^2 u))$ in four dimension), not only for $A = \sqrt{3}I$ (respectively, $A = I$), but also for any $A \in \mathcal{A}_{3,1}$. \hfill \square

The paper is organized as follows. In Section 2, after giving some basic notations in Subsection 2.1, we introduce the definitions of $\Xi_k, \xi_k, \xi_k$ and $m_{k,l}$, and investigate their properties in Subsection 2.2. Then we collect in Subsection 2.3 some preliminary lemmas which will be used in this paper. Section 3 is devoted to the proof of the main theorem (Theorem 1.1). To do this, we start in Subsection 3.1 to construct some appropriate subsolutions of the Hessian quotient equation (1.1), by taking advantages of the properties of $\Xi_k, \xi_k, \xi_k$ and $m_{k,l}$ explored in Subsection 2.2. Then in Subsection 3.2 after reducing Theorem 1.1 to Lemma 3.3 by simplification and normalization, we prove Lemma 3.3 by applying the Perron’s method to the subsolutions we constructed in Subsection 3.1.

2 Preliminary

2.1 Notation

In this paper, $S(n)$ denotes the linear space of symmetric $n \times n$ real matrices, and $I$ denotes the identity matrix.

For any $M \in S(n)$, if $m_1, m_2, ..., m_n$ are the eigenvalues of $M$ (usually, the assumption $m_1 \leq m_2 \leq ... \leq m_n$ is added for convenience), we will denote this fact briefly by $\lambda(M) = (m_1, m_2, ..., m_n)$ and call $\lambda(M)$ the eigenvalue vector of $M$.

For $A \in S(n)$ and $\rho > 0$, we denote by

$$E_\rho := \{ x \in \mathbb{R}^n \mid x^T Ax < \rho^2 \} = \{ x \in \mathbb{R}^n \mid r_A(x) < \rho \}$$

the ellipsoid of size $\rho$ with respect to $A$, where we set $r_A(x) := \sqrt{x^T Ax}$. 

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For any \( p \in \mathbb{R}^n \), we write

\[
\sigma_k(p) := \sum_{1 \leq s_1 < s_2 < \ldots < s_k \leq n} p_{s_1}p_{s_2} \ldots p_{s_k} \quad (\forall 1 \leq k \leq n)
\]
as the \( k \)-th elementary symmetric function of \( p \). Meanwhile, we will adopt the conventions that \( \sigma_{-1}(p) \equiv 0 \), \( \sigma_0(p) \equiv 1 \) and \( \sigma_k(p) \equiv 0 \), \( \forall k \geq n + 1 \); and we will also define

\[
\sigma_{k;i}(p) := \left. \left( \sigma_k(\lambda) \right|_{\lambda_i = 0} \right) \left|_{\lambda = p} \right. = \sigma_k(p_1, p_2, \ldots, \hat{p_i}, \ldots, p_n)
\]
for any \(-1 \leq k \leq n\) and any \( 1 \leq i \leq n \), and similarly

\[
\sigma_{k;i,j}(p) := \left. \left( \sigma_k(\lambda) \right|_{\lambda_i = \lambda_j = 0} \right) \left|_{\lambda = p} \right. = \sigma_k(p_1, p_2, \ldots, \hat{p_i}, \ldots, \hat{p_j}, \ldots, p_n)
\]
for any \(-1 \leq k \leq n\) and any \( 1 \leq i, j \leq n \), \( i \neq j \), for convenience.

### 2.2 Definitions and properties of \( \Xi_k, \xi_k, \xi_k \) and \( m_{k,l} \)

To establish the existence of the solution of (1.1), by the Perron’s method, the key point is to find some appropriate subsolutions of the equation. Since the Hessian quotient equation (1.1) is a highly fully nonlinear equation which including polynomials of the eigenvalues of the Hessian matrix \( D^2u \), \( \sigma_k(\lambda) \) and \( \sigma_l(\lambda) \), of different order of homogeneities, to solve it we need to strike a balance between them. It will turn out to be clear that the quantities \( \Xi_k, \xi_k, \overline{\xi}_k \) and \( m_{k,l} \), which we shall introduce below, are very natural and perfectly fit for this purpose.

**Definition 2.1.** For any \( 0 \leq k \leq n \) and any \( a \in \mathbb{R}^n \setminus \{0\} \), let

\[
\Xi_k := \Xi_k(a, x) := \frac{\sum_{i=1}^n \sigma_{k-1;i}(a) a_i^2 x_i^2}{\sigma_k(a) \sum_{i=1}^n a_i x_i^2}, \quad \forall x \in \mathbb{R}^n \setminus \{0\},
\]

and define

\[
\overline{\xi}_k := \overline{\xi}_k(a) := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \Xi_k(a, x)
\]

and

\[
\underline{\xi}_k := \underline{\xi}_k(a) := \inf_{x \in \mathbb{R}^n \setminus \{0\}} \Xi_k(a, x).
\]
Definition 2.2. For any \(0 \leq l < k \leq n\) and any \(a \in \mathbb{R}^n \setminus \{0\}\), let

\[
m_{k,l} := m_{k,l}(a) := \frac{k - l}{\xi_k(a) - \xi_l(a)}.
\] (2.1)

We remark, for the reader’s convenience, that \(\Xi_k\) originates from the computation of \(\sigma_k(D^2\Phi(x))\) where \(\Phi(x)\) is a generalized radially symmetric function (see Lemma 2.2 and the proof of Lemma 3.2), that \(\xi_k\) and \(\xi_k\) result from the comparison between \(\sigma_k(\lambda)\) and \(\sigma_l(\lambda)\) in the attempt to derive an ordinary differential equation from the original equation (see the last part of the proof of Lemma 3.2), and that \(m_{k,l}\) arises in the process of solving this ordinary differential equation (see (3.6) in the proof of Lemma 3.1). By \(\Xi_k, \xi_k\) and \(\xi_k\), we get a good balance between \(\sigma_k(\lambda)\) and \(\sigma_l(\lambda)\), which can be measured by \(m_{k,l}\). Furthermore, we will find that \(m_{k,l}\) has also some special meaning related to the decay and asymptotic behavior of the solution (see Lemma 3.1-(iii), Corollary 3.1 and Theorem 1.1).

It is easy to see that

\[
\xi_k(\lambda a) = \xi_k(a), \quad \xi_k(\lambda a) = \xi_k(a), \quad \forall \lambda \neq 0, \quad \forall a \in \mathbb{R}^n \setminus \{0\}, \quad \forall 0 \leq k \leq n,
\]

and

\[
\xi_k(C(1,1,...,1)) = \frac{k}{n} = \xi_k(C(1,1,...,1)), \quad \forall C > 0, \quad \forall 0 \leq k \leq n.
\]

Furthermore, we have the following lemma.

Lemma 2.1. Suppose \(a = (a_1, a_2,..., a_n)\) with \(0 < a_1 \leq a_2 \leq ... \leq a_n\). Then

\[
0 < \frac{a_1 \sigma_{k-1;1}(a)}{\sigma_k(a)} = \xi_{k}(a) \leq \frac{k}{n} \leq \frac{a_n \sigma_{k-n}(a)}{\sigma_k(a)} \leq 1, \quad \forall 1 \leq k \leq n;
\] (2.2)

\[
0 = \xi_0(a) < \frac{1}{n} \leq \frac{a_n}{\sigma_1(a)} = \xi_1(a) \leq \xi_2(a) \leq ... \leq \xi_{n-1}(a) < \xi_n(a) = 1; \quad (2.3)
\]

and

\[
0 = \xi_0(a) < \frac{a_1}{\sigma_1(a)} = \xi_1(a) \leq \xi_2(a) \leq ... \leq \xi_{n-1}(a) < \xi_n(a) = 1. \quad (2.4)
\]

Moreover,

\[
\xi_k(a) = \frac{k}{n} = \xi_k(a) \quad (2.5)
\]

for some \(1 \leq k \leq n - 1\), if and only if \(a = C(1,1,...,1)\) for some \(C > 0\).
Proof. (1°) By the definitions of $\sigma_k(a)$ and $\sigma_{k;i}(a)$, we see that

$$\sigma_k(a) = \sigma_{k;i}(a) + a_i \sigma_{k-1;i}(a), \quad \forall 1 \leq i \leq n; \quad (2.6)$$

and

$$\sum_{i=1}^{n} \sigma_{k;i}(a) = \frac{n C_{n-1}^{k}}{C_n^{k}} \sigma_k(a) = (n - k)\sigma_k(a).$$

Hence we obtain

$$\sum_{i=1}^{n} a_i \sigma_{k-1;i}(a) = k \sigma_k(a). \quad (2.7)$$

Now we show that

$$a_1 \sigma_{k-1;1}(a) \leq a_2 \sigma_{k-1;2}(a) \leq \ldots \leq a_n \sigma_{k-1;n}(a). \quad (2.8)$$

In fact, for any $i \neq j$, similar to (2.6), we have

$$a_i \sigma_{k-1;i}(a) = a_i (\sigma_{k-1;i,j}(a) + a_j \sigma_{k-2;i,j}(a))$$

and

$$a_j \sigma_{k-1;j}(a) = a_j (\sigma_{k-1;i,j}(a) + a_i \sigma_{k-2;i,j}(a)),$$

thus

$$a_i \sigma_{k-1;i}(a) - a_j \sigma_{k-1;j}(a) = (a_i - a_j) \sigma_{k-1;i,j}(a).$$

Hence if $a_i \leq a_j$, then

$$a_i \sigma_{k-1;i}(a) \leq a_j \sigma_{k-1;j}(a). \quad (2.9)$$

By the definition of $\bar{\xi}_k$, we have

$$\bar{\xi}_k(a) = \sup_{x \neq 0} \frac{\sum_{i=1}^{n} \sigma_{k-1;i}(a) a_i^2 x_i^2}{\sigma_k(a) \sum_{i=1}^{n} a_i x_i^2}$$

$$\geq \sup_{x_1 = \ldots = x_{n-1} = 0, \ x_n \neq 0} \frac{\sum_{i=1}^{n} \sigma_{k-1;i}(a) a_i^2 x_i^2}{\sigma_k(a) \sum_{i=1}^{n} a_i x_i^2}$$

$$= \sup_{x_n \neq 0} \frac{\sigma_{k-1;n}(a) a_n^2 x_n^2}{\sigma_k(a) a_n x_n^2}$$

$$= \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)}$$

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and
\[
\overline{\xi}_k(a) = \sup_{x \neq 0} \frac{\sum_{i=1}^n \sigma_{k-1;1}(a) a_i^2 x_i^2}{\sigma_k(a) \sum_{i=1}^n a_i x_i^2}
\]
\[
\leq \sup_{x \neq 0} \frac{a_n \sigma_{k-1;n}(a) \sum_{i=1}^n a_i x_i^2}{\sigma_k(a) \sum_{i=1}^n a_i x_i^2}
\]
by (2.8)
\[
= \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)}.
\]

Hence we obtain
\[
\overline{\xi}_k(a) = \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)}.
\]
(2.10)

Similarly
\[
\underline{\xi}_k(a) = \frac{a_1 \sigma_{k-1;1}(a)}{\sigma_k(a)}.
\]
(2.11)

From (2.7), we have
\[
\sum_{i=1}^n \frac{a_i \sigma_{k-1;i}(a)}{\sigma_k(a)} = k.
\]

Combining this with (2.8), (2.10) and (2.11), we deduce that
\[
\underline{\xi}_k(a) \leq \frac{k}{n} \leq \overline{\xi}_k(a).
\]

Thus the proof of (2.2) is complete, and (2.5) is also clear in view of (2.9).

(2°) Since it follows from (2.6) that
\[
a_i \sigma_{k-1;i}(a) < \sigma_k(a), \ \forall 1 \leq i \leq n, \ \forall 1 \leq k \leq n - 1,
\]
we obtain
\[
\underline{\xi}_k(a) \leq \overline{\xi}_k(a) < 1, \ \forall 0 \leq k \leq n - 1.
\]

On the other hand, we have \(\overline{\xi}_n(a) = \underline{\xi}_n(a) = 1\) which follows from
\[
a_i \sigma_{n-1;i}(a) = \sigma_n(a), \ \forall 1 \leq i \leq n.
\]

Combining (2.10) and (2.6), we discover that
\[
\overline{\xi}_k(a) = \frac{a_n \sigma_{k-1;n}(a)}{\sigma_k(a)} = \frac{a_n \sigma_{k-1;n}(a)}{\sigma_{k;n}(a) + a_n \sigma_{k-1;n}(a)}
\]
\begin{align*}
\leq \frac{a_n \sigma_{k:n}(a)}{\sigma_{k+1:n}(a)} + a_n \sigma_{k:n}(a) = \frac{a_n \sigma_{k:n}(a)}{\sigma_{k+1}(a)} = \xi_{k+1}(a),
\end{align*}
where we used the inequality
\begin{align*}
\frac{\sigma_{k-1:n}(a)}{\sigma_{k:n}(a)} \leq \frac{\sigma_{k:n}(a)}{\sigma_{k+1:n}(a)}
\end{align*}
which is a variation of the famous Newton inequality (see [HLP34])
\begin{align*}
\sigma_{k-1}(\lambda) \sigma_{k+1}(\lambda) \leq (\sigma_k(\lambda))^2, \quad \forall \lambda \in \mathbb{R}^n.
\end{align*}
Thus the proof of (2.3), and similarly of (2.4), is complete. \hfill \Box

Since it follows from (2.2) that
\begin{align*}
\frac{k - l}{n} \leq \bar{\xi}_k(a) - \xi_l(a) < \bar{\xi}_k(a) \leq 1,
\end{align*}
we obtain

**Corollary 2.1.** If $0 \leq l < k \leq n$ and $a \in \Gamma^+$, then
\begin{align*}
1 \leq k - l < m_{k,l}(a) \xi_k(a) \leq m_{k,l}(a) \leq n.
\end{align*}

As an application of Corollary 2.1 and Lemma 2.1, we now verify Proposition 1.1.

**Proof of Proposition 1.1.** (1) and (2) are clear. For (3), we only need to note that $c_s I \in \mathcal{A}_{k,l}$ and $m_{k,l}(c_s(1, 1, \ldots, 1)) = n > 2$. \hfill \Box

To help the reader to become familiar with these new quantities, it is worth to give the following examples which are also the applications of the above lemma.

**Example 2.1.** Note that, for $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ with $0 < a_1 \leq a_2 \leq a_3$, by Lemma 2.1, we have
\begin{align*}
\bar{\xi}_3(a) &\equiv 1 \equiv \xi_3(a), \\
\bar{\xi}_2(a) &\equiv \frac{a_3(a_1 + a_2)}{a_1 a_2 + a_1 a_3 + a_2 a_3}, \quad \xi_2(a) = \frac{a_1 (a_2 + a_3)}{a_1 a_2 + a_1 a_3 + a_2 a_3},
\end{align*}
\[ \xi_1(a) = \frac{a_3}{a_1 + a_2 + a_3}, \quad \xi_2(a) = \frac{a_1}{a_1 + a_2 + a_3}, \]

and

\[ \xi_0(a) \equiv 0 \equiv \xi_0(a). \]

Thus we can compute, for \( a = (1, 2, 3) \), that

\[ \xi_2 = \frac{9}{11}, \quad \xi_2 = \frac{5}{11}, \quad \xi_1 = \frac{1}{2}, \quad \xi_1 = \frac{1}{6}, \]

\[ m_{3, 2} = \frac{11}{6} < 2, \quad m_{3, 1} = \frac{12}{5} > 2, \quad m_{3, 0} \equiv 3 > 2, \]

\[ m_{2, 1} = \frac{66}{43} < 2, \quad m_{2, 0} = \frac{22}{9} > 2 \text{ and } m_{1, 0} = 2, \]

and, for \( a = (11, 12, 13) \), that

\[ \xi_2 = \frac{299}{431}, \quad \xi_2 = \frac{275}{431}, \quad \xi_1 = \frac{13}{36}, \quad \xi_1 = \frac{11}{36}, \]

\[ m_{3, 2} = \frac{431}{156} > 2, \quad m_{3, 1} = \frac{72}{25} > 2, \quad m_{3, 0} \equiv 3 > 2, \]

\[ m_{2, 1} = \frac{15516}{6023} > 2, \quad m_{2, 0} = \frac{862}{299} > 2 \text{ and } m_{1, 0} = \frac{36}{13} > 2, \]

Remark 2.1. (1) By definition of \( m_{k, l} \), we can easily check that for any \( 1 < k \leq n \), \( m_{k, k-1}(a) > 2 \) if and only if \( \xi_{k-1}(a) \leq \xi_k(a) \leq \xi_{k-1}(a) + 1/2 \). This will show us how \( m_{k, l} \) plays a role in the making of a balance between different order of homogeneities as we stated in the beginning of this subsection.

(2) Proposition 1.1 states that \( \tilde{\mathcal{A}}_{k, l} = \mathcal{A}_{k, l} \) provided \( k - l \geq 2 \). Note that this is the best case we can expect, since in general \( \tilde{\mathcal{A}}_{k, k-1} \nsubseteq \mathcal{A}_{k, k-1} \), which is evident by the fact stated in the first item of this remark (and also by the above examples). For example, in \( \mathbb{R}^3 \) we have

\[ m_{3, 2}(a) > 2 \iff \xi_3(a) \leq \xi_2(a) + 1/2 \iff a_1 > \frac{a_2 a_3}{a_2 + a_3}, \]

where the last inequality is not always true. \( \square \)
2.3 Some preliminary lemmas

In this subsection, we collect some preliminary lemmas which will be mainly used in Section 3.

We first give a lemma to compute $\sigma_k(\lambda(M))$ with $M$ of certain type. If $\Phi(x) := \phi(r)$ with $\phi \in C^2$, $r = \sqrt{x^T Ax}$, $A \in S(n) \cap \Gamma^+$ and $a = \lambda(A)$ (we may call $\Phi$ a generalized radially symmetric function with respect to $A$, according to [BLL14]), one can conclude that

$$\partial_{ij} \Phi(x) = \frac{\phi'(r)}{r} a_i \delta_{ij} + \frac{\phi''(r) - \phi'(r)}{r^2} (a_i x_i)(a_j x_j), \forall 1 \leq i, j \leq n,$$

provided $A$ is normalized to a diagonal matrix (see the first part of Subsection 3.2 and the proof of Lemma 3.2 for details). As far as we know, generally there is no explicit formula for $\lambda(D^2 \Phi(x))$ of this type, but luckily we have a method to calculate $\sigma_k(\lambda(D^2 \Phi(x)))$ for each $1 \leq k \leq n$, which can be presented as follows.

**Lemma 2.2.** If $M = (p_i \delta_{ij} + sq_i q_j)_{n \times n}$ with $p, q \in \mathbb{R}^n$ and $s \in \mathbb{R}$, then

$$\sigma_k(\lambda(M)) = \sigma_k(p) + s \sum_{i=1}^n \sigma_{k-1; i}(p) q_i^2, \forall 1 \leq k \leq n.$$

**Proof.** See [BLL14].

To process information on the boundary we need the following lemma.

**Lemma 2.3.** Let $D$ be a bounded strictly convex domain of $\mathbb{R}^n$, $n \geq 2$, $\partial D \in C^2$, $\varphi \in C^0(D) \cap C^2(\partial D)$ and let $A \in S(n)$, det $A \neq 0$. Then there exists a constant $K > 0$ depending only on $n$, diam $D$, the convexity of $D$, $\|\varphi\|_{C^2(D)}$, the $C^2$ norm of $\partial D$ and the upper bound of $A$, such that for any $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying

$$|\bar{x}(\xi)| \leq K \quad \text{and} \quad Q_\xi(x) < \varphi(x), \forall x \in \overline{D} \setminus \{\xi\},$$

where

$$Q_\xi(x) := \frac{1}{2} (x - \bar{x}(\xi))^T A (x - \bar{x}(\xi)) - \frac{1}{2} (\xi - \bar{x}(\xi))^T A (\xi - \bar{x}(\xi)) + \varphi(\xi), \forall x \in \mathbb{R}^n.$$
Proof. See [CL03] or [BLL14].

Remark 2.2. It is easy to check that $Q_\xi$ satisfy the following properties.

1. $Q_\xi \leq \varphi$ on $\overline{D}$ and $Q_\xi(\xi) = \varphi(\xi)$.

2. If $A \in \mathcal{A}_{k,l}$, then
   \[ \frac{\sigma_k(\lambda(D^2 Q_\xi))}{\sigma_l(\lambda(D^2 Q_\xi))} = 1 \quad \text{in} \quad \mathbb{R}^n. \]

3. There exists $\bar{c} = \bar{c}(D, A, K) > 0$ such that
   \[ Q_\xi(x) \leq \frac{1}{2} x^T A x + \bar{c}, \quad \forall x \in \partial D, \forall \xi \in \partial D. \]

Now we introduce the following well known lemmas about the comparison principle and Perron’s method which will be applied to the Hessian quotient equations but stated in a slightly more general setting. These lemmas are adaptations of those appeared in [CNS85] [Jen88] [Ish89] [Urb90] and [CIL92]. For specific proof of them one may also consult [BLL14] and [LB14].

Lemma 2.4 (Comparison principle). Assume $\Gamma^+ \subset \Gamma \subset \mathbb{R}^n$ is an open convex symmetric cone with its vertex at the origin, and suppose $f \in C^1(\Gamma)$ and $f_{\lambda_i}(\lambda) > 0$, $\forall \lambda \in \Gamma$, $\forall i = 1, 2, ..., n$. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $u, \bar{u} \in C^0(\overline{\Omega})$ satisfying

\[ f(\lambda(D^2 u)) \geq 1 \geq f(\lambda(D^2 \bar{u})) \]

in $\Omega$ in the viscosity sense. Suppose $u \leq \bar{u}$ on $\partial \Omega$ (and additionally

\[ \lim_{|x| \to +\infty} (u - \bar{u})(x) = 0 \]

provided $\Omega$ is unbounded). Then $u \leq \bar{u}$ in $\Omega$.

Lemma 2.5 (Perron’s method). Assume that $\Gamma^+ \subset \Gamma \subset \mathbb{R}^n$ is an open convex symmetric cone with its vertex at the origin, and suppose $f \in C^1(\Gamma)$ and $f_{\lambda_i}(\lambda) > 0$, $\forall \lambda \in \Gamma$, $\forall i = 1, 2, ..., n$. Let $\Omega \subset \mathbb{R}^n$ be a domain, $\varphi \in C^0(\partial \Omega)$ and let $u, \bar{u} \in C^0(\overline{\Omega})$ satisfying

\[ f(\lambda(D^2 u)) \geq 1 \geq f(\lambda(D^2 \bar{u})) \]
in \( \Omega \) in the viscosity sense. Suppose \( u \leq \bar{u} \) in \( \Omega \), \( u = \varphi \) on \( \partial \Omega \) (and additionally \( \lim_{|x| \to +\infty} (u - \bar{u})(x) = 0 \) provided \( \Omega \) is unbounded). Then

\[
u(x) := \sup \left\{ v(x) \mid v \in C^0(\Omega), \ u \leq v \leq \bar{u} \text{ in } \Omega, \ f(\lambda(D^2v)) \geq 1 \text{ in } \Omega \right\}
\]

is the unique viscosity solution of the Dirichlet problem

\[
\begin{cases}
f(\lambda(D^2u)) = 1 & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega.
\end{cases}
\]

**Remark 2.3.** In order to apply the above lemmas to the Hessian quotient operator

\[
f(\lambda) := \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)}
\]

in the cone \( \Gamma := \Gamma_k \), we need to show that

\[
\partial_{\lambda_i} \left( \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} \right) > 0, \ \forall 1 \leq i \leq n, \ \forall 0 \leq l < k \leq n, \ \forall \lambda \in \Gamma_k,
\]

which indeed indicates that the Hessian quotient equations \(\text{(1.1)}\) are elliptic equations with respect to its \(k\)-convex solution \(u\).

Indeed, for \( l = 0 \), \(\text{(2.12)}\) is clear in light of \(\text{(1.5)}\). For \( 1 \leq l < k \leq n \), since

\[
\partial_{\lambda_i} \sigma_k(\lambda) = \frac{\sigma_k(\lambda) - \sigma_{k;i}(\lambda)}{\lambda_i} = \sigma_{k-1;i}(\lambda)
\]

according to \(\text{(2.6)}\), we have

\[
\partial_{\lambda_i} \left( \frac{\sigma_k(\lambda)}{\sigma_l(\lambda)} \right) = \frac{\sigma_{k-1;i}(\lambda)\sigma_l(\lambda) - \sigma_k(\lambda)\sigma_{l-1;i}(\lambda)}{(\sigma_l(\lambda))^2}.
\]

Thus to prove \(\text{(2.12)}\), it remains to verify

\[
\sigma_{k-1;i}(\lambda)\sigma_l(\lambda) \geq \sigma_k(\lambda)\sigma_{l-1;i}(\lambda).
\]
In view of (2.6), this is equivalent to
\[
\sigma_{k-1;i}(\lambda)\sigma_{l;i}(\lambda) \geq \sigma_{k;i}(\lambda)\sigma_{l-1;i}(\lambda),
\]
which in turn is equivalent to
\[
\frac{\sigma_{l;i}(\lambda)}{\sigma_{l-1;i}(\lambda)} \geq \frac{\sigma_{k;i}(\lambda)}{\sigma_{k-1;i}(\lambda)},
\]
since \(\sigma_{j;i}(\lambda) = \partial_{\lambda} \lambda \sigma_{j+1}(\lambda) > 0\), \(\forall 1 \leq i \leq n\), \(\forall 0 \leq j \leq k - 1\), \(\forall \lambda \in \Gamma_k\), according to (1.5). For the proof of the latter, we only need to note that
\[
\frac{\sigma_{j;i}(\lambda)}{\sigma_{j-1;i}(\lambda)} \geq \frac{\sigma_{j+1;i}(\lambda)}{\sigma_{j;i}(\lambda)},
\]
which is the variation of the Newton inequality (see [HLP34])
\[
\sigma_{j-1}(\lambda)\sigma_{j+1}(\lambda) \leq (\sigma_{j}(\lambda))^2, \text{ } \forall \lambda \in \mathbb{R}^n,
\]
as we met in the proof of Lemma 2.1.

\[\Box\]

3 Proof of the main theorem

3.1 Construction of the subsolutions

The purpose of this subsection is to prove the following key lemma and then use it to construct subsolutions of (1.1). We remark that for the generalized radially symmetric subsolution \(\Phi(x) = \phi(r)\) that we intend to construct, the solution \(\psi(r)\) discussed in the following lemma actually is equivalent to \(\phi'(r)/r\) (see the proof of Lemma 3.2).

\textbf{Lemma 3.1.} Let \(0 \leq l < k \leq n\), \(n \geq 3\), \(A \in \tilde{A}_{k,l}\), \(a := (a_1, a_2, \ldots, a_n) := \lambda(A)\), \(0 < a_1 \leq a_2 \leq \ldots \leq a_n\) and \(\beta \geq 1\). Then the problem

\[
\begin{aligned}
\psi(r)^k + \xi_k(a)r\psi(r)^{k-1}\psi'(r) &- \psi(r)^l - \xi_l(a)r\psi(r)^{l-1}\psi'(r) = 0, \text{ } r > 1, \\
\psi(1) &= \beta,
\end{aligned}
\]

has a unique smooth solution \(\psi(r) = \psi(r, \beta)\) on \([1, +\infty)\), which satisfies
(i) $1 \leq \psi(r, \beta) \leq \beta$, $\partial_r \psi(r, \beta) \leq 0$, $\forall r \geq 1$, $\forall \beta \geq 1$. More specifically, $\psi(r, 1) \equiv 1$, $\psi(1, \beta) \equiv \beta$; and $1 < \psi(r, \beta) < \beta$, $\forall r > 1$, $\forall \beta > 1$.

(ii) $\psi(r, \beta)$ is continuous and strictly increasing with respect to $\beta$ and

$$\lim_{\beta \to +\infty} \psi(r, \beta) = +\infty, \forall r \geq 1.$$

(iii) $\psi(r, \beta) = 1 + O(r^{-m}) (r \to +\infty)$, where $m = m_{k,l}(a) \in (2, n]$ and the $O(\cdot)$ depends only on $k$, $l$, $\lambda(A)$ and $\beta$.

**Proof.** For brevity, we will often write $\psi(r)$ or $\psi(r, \beta)$ (respectively, $\xi(a)$, $\bar{\xi}(a)$) simply as $\psi$ (respectively, $\xi$, $\bar{\xi}$), when there is no confusion. The proof of this lemma now will be divided into three steps.

**Step 1.** We deduce from (3.1) that

$$\psi^k - \psi^l = -\frac{r}{dr} \left( \xi_k \psi^{k-1} - \xi \psi^{l-1} \right) d\psi$$

and

$$\frac{d\psi}{dr} = -\frac{1}{r} \cdot \frac{\psi^k - \psi^l}{\xi_k \psi^{k-1} - \xi \psi^{l-1}} = -\frac{1}{r} \cdot \frac{\psi^{k-l} - 1}{\psi^k - \xi_k} \psi^k =: g(\psi),$$

where we set

$$g(\nu) := -\frac{\nu}{\xi_k} \cdot \frac{\nu^{k-l} - 1}{\nu^k - \xi_k}.$$

Hence the problem (3.1) is equivalent to the following problem

$$\begin{cases}
\psi'(r) = \frac{g(\psi(r))}{r}, & r > 1, \\
\psi(1) = \beta.
\end{cases}$$

If $\beta = 1$, then $\psi(r) \equiv 1$ is a solution of the problem (3.4) since $g(1) = 0$. Thus, by the uniqueness theorem for the solution of the ordinary differential equation, we know that $\psi(r, 1) \equiv 1$ is the unique solution satisfies the problem (3.4).

Now if $\beta > 1$, since

$$h(r, \nu) := \frac{g(\nu)}{r} \in C^\infty((1, +\infty) \times (\nu_0, +\infty)),$$
where \(\xi_l < \nu_0 < 1\) (note that \(\nu_0\) exists, since we have \(\xi_l \leq l/n < k/n \leq \xi_k\) by Lemma 2.1), by the existence theorem (the Picard-Lindelöf theorem) and the theorem of the maximal interval of existence for the solution of the initial value problem of the ordinary differential equation, we know that the problem (3.4) has a unique smooth solution \(\psi(r) = \psi(r, \beta)\) locally around the initial point and can be extended to a maximal interval \([1, \zeta]\) in which \(\zeta\) can only be one of the following cases:

1. \(\zeta = +\infty\);
2. \(\zeta < +\infty\), \(\psi(r)\) is unbounded on \([1, \zeta]\);
3. \(\zeta < +\infty\), \(\psi(r)\) converges to some point on \(\{\nu = \nu_0\}\) as \(r \to \zeta^-\).

Since

\[
\frac{g(\psi(r))}{r} < 0, \quad \forall \psi(r) > 1,
\]

we see that \(\psi(r) = \psi(r, \beta)\) is strictly decreasing with respect to \(r\) which exclude the case (2o) above. We claim now that the case (3o) can also be excluded. Otherwise, the solution curve must intersect with \(\{\nu = 1\}\) at some point \((r_0, \psi(r_0))\) on it and then tends to \(\{\nu = \nu_0\}\) after crossing it. But \(\psi(r) \equiv 1\) is also a solution through \((r_0, \psi(r_0))\) which contradicts the uniqueness theorem for the solution of the initial value problem of the ordinary differential equation. Thus we complete the proof of the existence and uniqueness of the solution \(\psi(r) = \psi(r, \beta)\) of the problem (3.1) on \([1, +\infty]\).

Due to the same reason, i.e., \(\psi(r, \beta)\) is strictly decreasing with respect to \(r\) and the solution curve can not cross \(\{\nu = 1\}\) provided \(\beta > 1\), assertion (i) of the lemma is also clear now, that is, \(1 < \psi(r, \beta) < \beta\), \(\forall r > 1, \forall \beta > 1\).

Step 2. By the theorem of the differentiability of the solution with respect to the initial value, we can differentiate \(\psi(r, \beta)\) with respect to \(\beta\) as blew:

\[
\begin{cases}
\frac{\partial \psi(r, \beta)}{\partial r} = \frac{g(\psi(r, \beta))}{r}, \\
\psi(1, \beta) = \beta;
\end{cases}
\]
\[
\begin{aligned}
\Rightarrow \quad & \quad \left\{ \begin{array}{l}
\frac{\partial^2 \psi(r, \beta)}{\partial \beta \partial r} = \frac{g'(\psi(r, \beta))}{r} \cdot \frac{\partial \psi(r, \beta)}{\partial \beta}, \\
\frac{\partial \psi(1, \beta)}{\partial \beta} = 1.
\end{array} \right.
\end{aligned}
\]

Let

\[v(r) := \frac{\partial \psi(r, \beta)}{\partial \beta}.\]

We have

\[
\begin{aligned}
\frac{dv}{dr} &= \frac{g'(\psi(r, \beta))}{r} \cdot v, \\
v(1) &= 1.
\end{aligned}
\]

Therefore we can deduce that

\[
\frac{dv}{v} = \frac{g'(\psi(r, \beta))}{r} dr,
\]

and hence

\[
\frac{\partial \psi(r, \beta)}{\partial \beta} = v(r) = \exp \int_1^r \frac{g'(\psi(\tau, \beta))}{\tau} d\tau.
\]

Since

\[
g'(\nu) = -\frac{\nu^{k-l}-1}{\xi_k \nu^{k-l} - \xi_l} - \frac{\nu}{\xi_k} \cdot \frac{-(\xi_k - 1)(k-l)\nu^{k-l-1}}{(\nu^{k-1} - \xi_k \xi_k)^2}
\]

and

\[0 < \frac{\xi_k}{\xi_k} < 1 \leq \psi(r, \beta) \leq \beta = \psi(1), \ \forall r \geq 1, \quad (3.5)\]

we have

\[
g'(\psi(r, \beta)) \leq -\frac{(k-l)\left(1 - \frac{\xi_k}{\xi_k}\right)}{\xi_k \left(\nu^{k-1} - \xi_k \xi_k\right)^2} = -C(k, l, \lambda(A), \beta) < 0,
\]

and hence

\[0 < \frac{\partial \psi(r, \beta)}{\partial \beta} \leq r^{-C} \leq 1, \ \forall r \geq 1.
\]

Thus \(\psi(r, \beta)\) is strictly increasing with respect to \(\beta\).
Step 3. By (3.2), we have

\[-d \ln r = - \frac{dr}{r} = \frac{\xi_k \psi^{k-1} - \xi_l \psi^{l-1}}{\psi^k - \psi^l} d\psi = \frac{\xi_k}{\psi} \cdot \frac{\psi^{k-l} - \xi_l}{\psi^{k-l} - 1} d\psi \]

\[= \frac{\xi_k}{\psi} \left(1 + \frac{1 - \xi_l}{\psi^{k-l} - 1}\right) d\psi = \left(\frac{\xi_k}{\psi} + \frac{\xi_k - \xi_l}{\psi^{k-l} - 1}\right) d\psi \]

\[= \xi_k d\ln \psi - \frac{\xi_k - \xi_l}{k - l} d\ln \psi^{k-l} - 1 \]

\[= d\ln \left(\psi^{\xi_k} (1 - \psi^{k-l}) \frac{\xi_k - \xi_l}{k - l}\right).\]

Hence

\[-md \ln r = d\ln \left(\psi(r)^m \xi_k (1 - \psi(r)^{-k+l})\right), \quad (3.6)\]

where

\[m := m_{k,l}(a) := \frac{k - l}{\xi_k - \xi_l},\]

which has been already defined in (2.1) in Subsection 2.2. Note that, by the assumptions on \(A\) and Corollary 2.1, we have \(2 < m \leq n\) and \(m \xi_k > k - l\).

Integrating (3.6) from 1 to \(r\) and recalling \(\psi(1) = \beta \geq 1\), we get

\[\ln \left(\psi(r)^m \xi_k (1 - \psi(r)^{-k+l})\right) = \ln \left(\beta^m \xi_k (1 - \beta^{-k+l})\right) + \ln r^{-m},\]

and hence

\[\psi(r)^m \xi_k (1 - \psi(r)^{-k+l}) = \beta^m \xi_k (1 - \beta^{-k+l}) r^{-m} := B(\beta) r^{-m},\]

where we set

\[B(\beta) := \beta^m \xi_k (1 - \beta^{-k+l}) = \xi_k^m \xi_{k-l} \left(\beta^{k-l} - 1\right).\]

Since

\[\psi(r)^m \xi_k (1 - \psi(r)^{-k+l}) = \psi(r)^m \xi_k^{-k+l} \left(\psi(r)^{k-l} - 1\right)\]

\[= \psi(r)^m \xi_k^{-k+l} (\psi(r) - 1) \left(\psi(r)^{k-l-1} + \psi(r)^{k-l-2} + ... + \psi(r) + 1\right),\]
we thus conclude that
\[
\frac{\psi(r) - 1}{r^{-m}} = \left( \psi(r)^{m\xi_k - k + l} (\psi(r)^{k-l-1} + \psi(r)^{k-l-2} + ... + \psi(r) + 1) \right)^{-1} B(\beta). 
\]
(3.7)

Note that \( m \xi_k - k + l > 0 \) and
\[
\beta - 1 = \left( \beta^{m\xi_k - k + l} (\beta^{k-l-1} + \beta^{k-l-2} + ... + b + 1) \right)^{-1} B(\beta). 
\]
Recalling (3.5), we obtain
\[
\beta - 1 \leq \frac{\psi(r, \beta) - 1}{r^{-m}} \leq \frac{B(\beta)}{k-l}, \quad \forall \beta \geq 1. 
\]
(3.8)

Thus we have
\[
\lim_{\beta \to +\infty} \psi(r, \beta) = +\infty, \quad \forall r \geq 1, 
\]
and
\[
\psi(r, \beta) \to 1 \quad (r \to +\infty), \quad \forall \beta \geq 1. 
\]
Substituting the latter to (3.7), we get
\[
\frac{\psi(r, \beta) - 1}{r^{-m}} \to \frac{B(\beta)}{k-l}, \quad r \to +\infty, \quad \forall \beta \geq 1. 
\]
Therefore
\[
\psi(r, \beta) = 1 + \frac{B(\beta)}{k-l} r^{-m} + O(r^{-m}) = 1 + O(r^{-m}) \quad (r \to +\infty), 
\]
where \( o(\cdot) \) and \( O(\cdot) \) depend only on \( k, l, \lambda(A) \) and \( \beta \). This completes the proof of the lemma.

\[ \Box \]

**Remark 3.1.** For \( l = 0 \), i.e., the Hessian equation \( \sigma_k(\lambda) = 1 \), we have an easy proof. Consider the problem
\[
\left\{ \begin{aligned}
\psi(r)^k + \xi_k(a) r \psi(r)^{k-1} \psi'(r) &= 1, \quad r > 1, \\
\psi(1) &= \beta.
\end{aligned} \right. 
\]
(3.9)

Set \( m := m_{k,0}(a) = k/\xi_k \). We have
\[
\psi^k - 1 = -r\xi_k \psi^{k-1} \frac{d\psi}{dr} = -\frac{1}{m} r \cdot \frac{d(\psi^k - 1)}{dr}, 
\]
\[
\frac{d (\psi^k - 1)}{\psi^k - 1} = -m \frac{dr}{r}
\]

and

\[
d \ln (\psi(r)^k - 1) = -md \ln r = d \ln r^{-m}.
\]

Integrating it from 1 to \(r\) and recalling \(\psi(1) = \beta \geq 1\), we get

\[
\psi(r)^k - 1 = (\psi(1)^k - 1) r^{-m} = (\beta^k - 1) r^{-m}
\]

and

\[
\psi(r) = \left(1 + (\beta^k - 1) r^{-m}\right)^{\frac{1}{k}}
\]

\[
= \left(1 + (\beta^k - 1) r^{-\frac{k}{k^2}}\right)^{\frac{1}{k}}
\]

\[
= 1 + \frac{\beta^k - 1}{k} r^{-m} + o(r^{-m}) = 1 + O(r^{-m}) \quad (r \to +\infty).
\]

It is obvious that the \(\psi(r)\) that we here solved from (3.9) for \(l = 0\) satisfies all the conclusions of Lemma 3.1. Moreover, comparing (3.10) with the corresponding ones in [BLL14] and in [CL03], we observe that our method actually provides a systematic way for construction of the subsolutions, which gives results containing the previous ones as special cases.

Set

\[
\mu_R(\beta) := \int_R^{+\infty} \tau (\psi(\tau, \beta) - 1) d\tau, \quad \forall R \geq 1, \ \forall \beta \geq 1.
\]

Note that the integral on the right hand side is convergent in view of Lemma 3.1 (iii). Moreover, as an application of Lemma 3.1 we have the following.

**Corollary 3.1.** \(\mu_R(\beta)\) is nonnegative, continuous and strictly increasing with respect to \(\beta\). Furthermore,

\[
\mu_R(\beta) \geq \int_R^{+\infty} (\beta - 1) r^{-m+1} d\tau \to +\infty \quad (\beta \to +\infty), \quad \forall R \geq 1;
\]

and

\[
\mu_R(\beta) = O(R^{-m+2}) \quad (R \to +\infty), \quad \forall \beta \geq 1.
\]

**Proof.** By Lemma 3.1(ii),(iii) and the above property (3.8) of \(\psi(r, \beta)\).  

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For any $\alpha, \beta, \gamma \in \mathbb{R}$, $\beta, \gamma \geq 1$ and for any diagonal matrix $A \in \tilde{A}_{k,l}$, let
\[
\phi(r) := \phi_{\alpha,\beta,\gamma}(r) := \alpha + \int_{r}^{\infty} \tau \psi(\tau, \beta) d\tau, \quad \forall r \geq \gamma,
\]
and
\[
\Phi(x) := \Phi_{\alpha,\beta,\gamma,A}(x) := \phi(r) := \phi_{\alpha,\beta,\gamma}(r_A(x)), \quad \forall x \in \mathbb{R}^n \setminus E_{\gamma},
\]
where $r = r_A(x) = \sqrt{x^T A x}$. Then we have
\[
\phi_{\alpha,\beta,\gamma}(r) = \int_{r}^{\infty} \tau \left(\psi(\tau, \beta) - 1 \right) d\tau + \frac{1}{2} r^2 - \frac{1}{2} \gamma^2 + \alpha
\]
\[
= \frac{1}{2} r^2 + \left( \mu_{\gamma}(\beta) + \alpha - \frac{1}{2} \gamma^2 \right) - \mu_r(\beta) \quad (3.11)
\]
\[
= \frac{1}{2} r^2 + \left( \mu_{\gamma}(\beta) + \alpha - \frac{1}{2} \gamma^2 \right) + O(r^{-m+2}) \quad (r \to +\infty), \quad (3.12)
\]
according to Corollary 3.1 and now we can assert that

**Lemma 3.2.** $\Phi$ is a smooth $k$-convex subsolution of (1.1) in $\mathbb{R}^n \setminus \overline{E_{\gamma}}$, that is,
\[
\sigma_j \left( \lambda \left( D^2 \Phi(x) \right) \right) \geq 0, \quad \forall 1 \leq j \leq k, \quad \forall x \in \mathbb{R}^n \setminus \overline{E_{\gamma}},
\]
and
\[
\frac{\sigma_k \left( \lambda \left( D^2 \Phi(x) \right) \right)}{\sigma_1 \left( \lambda \left( D^2 \Phi(x) \right) \right)} \geq 1, \quad \forall x \in \mathbb{R}^n \setminus \overline{E_{\gamma}}.
\]

**Proof.** By definition we have $\phi'(r) = r\psi(r)$ and $\phi''(r) = \psi(r) + r\psi'(r)$. Since
\[
r^2 = x^T A x = \sum_{i=1}^{n} a_i x_i^2,
\]
we deduce that
\[
2r \partial_x r = \partial_x \left( r^2 \right) = 2a_i x_i \quad \text{and} \quad \partial_x r = \frac{a_i x_i}{r}.
\]
Consequently
\[
\partial_x \Phi(x) = \phi'(r) \partial_x r = \frac{\phi'(r)}{r} a_i x_i,
\]
\[
\partial_{x_i x_j} \Phi(x) = \frac{\phi'(r)}{r} a_i \delta_{ij} + \frac{\phi''(r) - \frac{\phi'(r)}{r}}{r^2} (a_i x_i)(a_j x_j)
\]

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\[ \psi(r) a_i \delta_{ij} + \frac{\psi'(r)}{r^2} (a_i x_i)(a_j x_j), \]

and therefore

\[ D^2 \Phi = \left( \psi(r) a_i \delta_{ij} + \frac{\psi'(r)}{r^2} (a_i x_i)(a_j x_j) \right)_{n \times n}. \]

So we can conclude from Lemma 2.2 that

\[
\sigma_j \left( \lambda \left( D^2 \Phi \right) \right) = \sigma_j (a) \psi (r) \frac{j}{j} + \frac{\psi'(r)}{r} \psi (r) \frac{j-1}{j-1} \sum_{i=1}^{n} \sigma_{j-1; i} (a) a_i^2 x_i^2 \\
= \sigma_j (a) \psi^{j} + \Xi_j (a, x) \sigma_j (a) r \psi^{j-1} \psi' \\
\geq \sigma_j (a) \psi^{j} + \tilde{\Xi}_j (a) \sigma_j (a) r \psi^{j-1} \psi' \\
= \sigma_j (a) \psi^{j-1} \left( \psi + \tilde{\Xi}_j (a) r \psi' \right), \forall 1 \leq j \leq n,
\]

where we have used the facts that \( \psi (r) \geq 1 > 0 \) and \( \psi'(r) \leq 0 \) for all \( r \geq 1 \), according to Lemma 3.1 (i).

For any fixed \( 1 \leq j \leq k \), in view of Lemma 2.1 and Lemma 3.1 (i), we have

\[ 0 \leq \frac{\psi^{k-1} - 1}{\psi^{k-1} - \frac{\xi_j (a)}{\xi_j (a)}} < 1 \leq \frac{\xi_k (a)}{\xi_j (a)}. \]

Hence it follows from (3.3) that

\[ \psi' = \frac{1}{r} \frac{\psi}{\xi_k (a)} \cdot \frac{\psi^{k-1} - 1}{\psi^{k-1} - \frac{\xi_j (a)}{\xi_j (a)}} > \frac{1}{r} \frac{\psi}{\xi_j (a)}, \]

which yields \( \psi + \tilde{\Xi}_j (a) r \psi' > 0 \).

Since \( A \in \mathcal{A}_{k, l} \) implies \( a \in \Gamma^+ \), that is, \( \sigma_i (a) > 0 \) for all \( 1 \leq i \leq n \), we thus conclude that

\[ \sigma_j \left( \lambda \left( D^2 \Phi \right) \right) > 0, \forall 1 \leq j \leq k. \]

In particular, we have

\[ \sigma_k \left( \lambda \left( D^2 \Phi \right) \right) > 0 \quad \text{and} \quad \sigma_l \left( \lambda \left( D^2 \Phi \right) \right) > 0. \]

On the other hand,

\[ \sigma_k \left( \lambda \left( D^2 \Phi \right) \right) - \sigma_l \left( \lambda \left( D^2 \Phi \right) \right) \]
\[
\sigma_k(a)\psi^k + \Xi_k(a,x)\sigma_k(a)r\psi^{k-1}\psi' - \sigma_l(a)\psi^l - \Xi_l(a,x)\sigma_l(a)r\psi^{l-1}\psi' \\
\geq \sigma_k(a)\psi^k + \bar{\xi}_k(a)\sigma_k(a)r\psi^{k-1}\psi' - \sigma_l(a)\psi^l - \bar{\xi}_l(a)\sigma_l(a)r\psi^{l-1}\psi' \\
= \sigma_k(a)\left(\psi^k + \bar{\xi}_k(a)r\psi^{k-1}\psi' - \psi^l - \bar{\xi}_l(a)r\psi^{l-1}\psi'\right) \\
= 0.
\]

Therefore
\[
\frac{\sigma_k(\lambda(D^2\Phi))}{\sigma_l(\lambda(D^2\Phi))} \geq 1.
\]

This completes the proof of Lemma 3.2.

\section*{3.2 Proof of Theorem 1.1}

We first introduce the following lemma which is a special and simple case of Theorem 1.1 with the additional condition that the matrix $A$ is diagonal and the vector $b$ vanishes.

**Lemma 3.3.** Let $D$ be a bounded strictly convex domain in $\mathbb{R}^n$, $n \geq 3$, $\partial D \in C^2$ and let $\varphi \in C^2(\partial D)$. Then for any given diagonal matrix $A \in \mathbb{R}^{n \times n}$ with $0 \leq l < k \leq n$, there exists a constant $\tilde{c}$ depending only on $n, D, k, l, A$ and $\|\varphi\|_{C^2(\partial D)}$, such that for every $c \geq \tilde{c}$, there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ of

\[
\begin{cases}
\frac{\sigma_k(\lambda(D^2u))}{\sigma_l(\lambda(D^2u))} = 1 & \text{in } \mathbb{R}^n \setminus D, \\
\quad u = \varphi & \text{on } \partial D, \\
\limsup_{|x| \to +\infty} |x|^{m-2} \left| u(x) - \left( \frac{1}{2} x^T Ax + c \right) \right| < \infty,
\end{cases}
\]

where $m = m_{k,l}(\lambda(A)) \in (2, n]$.

To prove Theorem 1.1 it suffices to prove Lemma 3.3. Indeed, suppose that $D, \varphi, A$ and $b$ satisfy the hypothesis of Theorem 1.1. Consider the decomposition $A = Q^T N Q$, where $Q$ is an orthogonal matrix and $N$ is a diagonal matrix which satisfies $\lambda(N) = \lambda(A)$. Let

\[
\tilde{x} := Qx, \quad \tilde{D} := \{Qx | x \in D\}
\]
and
\[ \tilde{\varphi}(\tilde{x}) := \varphi(x) - b^T x = \varphi(Q^T \tilde{x}) - b^T Q^T \tilde{x}. \]

By Lemma 3.3, we conclude that there exists a constant \( \tilde{c} \) depending only on \( n, \tilde{D}, k, l, N \) and \( \| \tilde{\varphi} \|_{C^2(\partial \tilde{D})} \), such that for every \( c \geq \tilde{c} \), there exists a unique viscosity solution \( \tilde{u} \in C^0(\mathbb{R}^n \setminus \tilde{D}) \) of

\[
\begin{cases}
\frac{\sigma_k(\lambda(D^2 \tilde{u}))}{\sigma_l(\lambda(D^2 \tilde{u}))} = 1 & \text{in } \mathbb{R}^n \setminus \tilde{D}, \\
\tilde{u} = \tilde{\varphi} & \text{on } \partial \tilde{D}, \\
\limsup_{|\tilde{x}| \to +\infty} |\tilde{x}|^{m-2} \left| \tilde{u}(\tilde{x}) - \left( \frac{1}{2} \tilde{x}^T N \tilde{x} + c \right) \right| < \infty,
\end{cases}
\]

where \( m = m_{k,l}(\lambda(N)) = m_{k,l}(\lambda(A)) \in (2, n] \). Let
\[ u(x) := \tilde{u}(\tilde{x}) + b^T x = \tilde{u}(Qx) + b^T x = \tilde{u}(\tilde{x}) + b^T Q^T \tilde{x}. \]

We claim that \( u \) is the solution of (1.7) in Theorem 1.1. To show this, we only need to note that
\[ D^2 u(x) = Q^T D^2 \tilde{u}(\tilde{x}) Q, \quad \lambda(D^2 u(x)) = \lambda(D^2 \tilde{u}(\tilde{x})); \]
\[ u = \varphi \quad \text{on } \partial D; \]

and
\[
|\tilde{x}|^{m-2} \left| \tilde{u}(\tilde{x}) - \left( \frac{1}{2} \tilde{x}^T N \tilde{x} + c \right) \right| = \left( x^T Q^T Q x \right)^{(m-2)/2} \left| u(x) - b^T x - \left( \frac{1}{2} x^T Q^T N Q x + c \right) \right| = |x|^{m-2} \left| u(x) - \left( \frac{1}{2} x^T A x + b^T x + c \right) \right|.
\]

Thus we have proved that Theorem 1.1 can be established by Lemma 3.3.

**Remark 3.2.** (1) We may see from the above demonstration that the lower bound \( \tilde{c} \) of \( c \) in Theorem 1.1 can not be discarded generally. Indeed, for the radial solutions of the Hessian equation \( \sigma_k(\lambda(D^2 u)) = 1 \) in \( \mathbb{R}^n \setminus \overline{B}_1 \), \([WB13, \text{Theorem 2}]\) states that there is no solution when \( c \) is too small.
Unlike the Poisson equation and the Monge-Ampère equation, generally, for the Hessian quotient equation, the matrix $A$ in Theorem 1.1 can only be normalized to a diagonal matrix, and cannot be normalized to $I$ multiplied by some constant. This is the reason why we study the generalized radially symmetric solutions, rather than the radial solutions, of the original equation (1.1). See also [BLL14].

Now we use the Perron’s method to prove Lemma 3.3.

**Proof of Lemma 3.3.** We may assume without loss of generality that $E_1 \subset \subset D \subset \subset E_r \subset \subset E_{\bar{r}}$ and $a := (a_1, a_2, ..., a_n) := \lambda(A)$ with $0 < a_1 \leq a_2 \leq ... \leq a_n$. The proof now will be divided into three steps.

**Step 1.**

Let

$$\eta := \inf_{x \in \partial D} Q_\xi(x), \quad Q(x) := \sup_{\xi \in \partial D} Q_\xi(x)$$

and

$$\Phi_\beta(x) := \eta + \int_{r_A(x)}^{r_A(x)} \tau \psi(\tau, \beta) d\tau, \quad \forall r_A(x) \geq 1, \forall \beta \geq 1,$$

where $Q_\xi(x)$ and $\psi(r, \beta)$ are given by Lemma 2.3 and Lemma 3.1 respectively. Then we have

(1) Since $Q$ is the supremum of a collection of smooth solutions $\{Q_\xi\}$ of (1.1), it is a continuous subsolution of (1.1), i.e.,

$$\frac{\sigma_r(\lambda(D^2 Q))}{\sigma_I(\lambda(D^2 Q))} \geq 1$$

in $\mathbb{R}^n \setminus \overline{D}$ in the viscosity sense (see [Ish89, Proposition 2.2]).

(2) $Q = \varphi$ on $\partial D$. To prove this we only need to show that for any $\xi \in \partial D$, $Q(\xi) = \varphi(\xi)$. This is obvious since $Q_\xi \leq \varphi$ on $\overline{D}$ and $Q_\xi(\xi) = \varphi(\xi)$, according to Remark 2.2(1).

(3) By Lemma 3.2, $\Phi_\beta$ is a smooth subsolution of (1.1) in $\mathbb{R}^n \setminus \overline{D}$. 

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(4) $\Phi_\beta \leq \varphi$ on $\partial D$ and $\Phi_\beta \leq Q$ on $\overline{E_\hat{r}} \setminus D$. To show them we first note that $\Phi_\beta(x)$ is strictly increasing with respect to $r_A(x)$ since $\psi(r,\beta) \geq 1 > 0$ by Lemma 3.1(i). Invoking $\Phi_\beta = \eta$ on $\partial E_\hat{r}$ and $\eta \leq Q$ on $\overline{E_\hat{r}} \setminus D$ by their definitions, we have $\Phi_\beta \leq \eta \leq Q$ on $\overline{E_\hat{r}} \setminus D$. On the other hand, according to Remark 2.2(1), we have $Q \leq \varphi$ on $\overline{D}$ which implies that $\eta \leq \varphi$ on $\overline{D}$. Combining these two aspects we deduce that $\Phi_\beta \leq \eta \leq \varphi$ on $\partial D$.

(5) $\Phi_\beta(x)$ is strictly increasing with respect to $\beta$ and
\[
\lim_{\beta \to +\infty} \Phi_\beta(x) = +\infty, \quad \forall r_A(x) \geq 1, \tag{3.15}
\]
by the definition of $\Phi_\beta(x)$ and Lemma 3.1(ii).

(6) As we showed in (3.11) and (3.12), for any $\beta \geq 1$, we have
\[
\Phi_\beta(x) = \eta + \int_{\hat{r}}^{r_A(x)} \tau \psi(\tau, \beta) d\tau
= \eta + \frac{1}{2}(r_A(x)^2 - \hat{r}^2) + \int_{\hat{r}}^{r_A(x)} \tau \left(\psi(\tau, \beta) - 1\right) d\tau
= \frac{1}{2} r_A(x)^2 + \left(\eta - \frac{1}{2} \hat{r}^2 + \mu_r(\beta)\right) - \mu_{r_A(x)}(\beta)
= \frac{1}{2} r_A(x)^2 + \mu(\beta) - \mu_{r_A(x)}(\beta)
= \frac{1}{2} x^T A x + \mu(\beta) + O\left(|x|^{-m+2}\right) \quad (|x| \to +\infty),
\]
where we set
\[
\mu(\beta) := \eta - \frac{1}{2} \hat{r}^2 + \mu_r(\beta),
\]
and used the fact that $x^T A x = O(|x|^2) \quad (|x| \to +\infty)$ since $\lambda(A) \in \Gamma^+$.

Step 2. For fixed $\hat{r} > \bar{r}$, there exists $\hat{\beta} > 1$ such that
\[
\min_{\partial E_{\hat{r}}} \Phi_{\hat{\beta}} > \max_{\partial E_{\bar{r}}} Q,
\]
in light of (3.15). Thus we obtain
\[
\Phi_{\hat{\beta}} > Q \quad \text{on} \, \partial E_{\bar{r}}. \tag{3.16}
\]
Let
\[ \tilde{c} := \max \left\{ \eta, \mu(\hat{\beta}), \bar{c} \right\}, \]
where the \( \bar{c} \) comes from Remark 2.2 (3), and hereafter fix \( c \geq \tilde{c} \).

By Lemma 3.1 and Corollary 3.1 we deduce that
\[ \psi(r, 1) \equiv 1 \Rightarrow \mu(r)(1) = 0 \Rightarrow \mu(1) = \eta - \frac{1}{2}r^2 < \eta \leq \tilde{c} \leq c, \]
and
\[ \lim_{\beta \to +\infty} \mu(\beta) = +\infty \Rightarrow \lim_{\beta \to +\infty} \mu(\beta) = +\infty. \]

On the other hand, it follows from Corollary 3.1 that \( \mu(\beta) \) is continuous and strictly increasing with respect to \( \beta \) (which indicates that the inverse of \( \mu(\beta) \) exists and \( \mu^{-1} \) is strictly increasing). Thus there exists a unique \( \beta(c) \) such that \( \mu(\beta(c)) = c \). Then we have
\[ \Phi(\beta(c))(x) = \frac{1}{2}r_A(x)^2 + c - \mu r_A(x)(\beta(c)) = \frac{1}{2}x^T Ax + c + O\left(|x|^{-m+2}\right) \quad (|x| \to +\infty), \]
and
\[ \beta(c) = \mu^{-1}(c) \geq \mu^{-1}(\bar{c}) \geq \hat{\beta}. \]

Invoking the monotonicity of \( \Phi(\beta) \) with respect to \( \beta \) and (3.16), we obtain
\[ \Phi(\beta(c)) \geq \Phi(\hat{\beta}) > Q \quad \text{on } \partial E_{\hat{r}}. \]

Note that we already know
\[ \Phi(\beta(c)) \leq Q \quad \text{on } \overline{E_{\hat{r}}} \setminus D, \]
from (4) of Step 1.

Let
\[ u(x) := \begin{cases} \max \left\{ \Phi(\beta(c))(x), Q(x) \right\}, & x \in E_{\tilde{r}} \setminus D, \\ \Phi(\beta(c))(x), & x \in \mathbb{R}^n \setminus E_{\tilde{r}}. \end{cases} \]

Then we have
(1) \( u \) is continuous and satisfies
\[ \frac{\sigma_k(\lambda(D^2u))}{\sigma_l(\lambda(D^2u))} \geq 1 \]
in \( \mathbb{R}^n \setminus \overline{D} \) in the viscosity sense, by (1) and (3) of Step 1.
(2) \( u = Q = \varphi \) on \( \partial D \), by (2) of Step 1.

(3) If \( r_A(x) \) is large enough, then
\[
\Phi_{\beta(c)}(x) = \frac{1}{2} x^T A x + c + O \left( |x|^{-m+2} \right) \quad (|x| \to +\infty).
\]

**Step 3.** Let
\[
\overline{u}(x) := \frac{1}{2} x^T A x + c, \; \forall x \in \mathbb{R}^n.
\]
Then \( \overline{u} \) is obviously a supersolution and
\[
\lim_{|x| \to +\infty} (\overline{u} - \overline{u})(x) = 0.
\]

To use the Perron’s method to establish Lemma 3.3, we now only need to prove that
\[
u \leq \overline{u} \in \mathbb{R}^n \setminus D.
\]
In fact, since
\[
\mu_{r_A(x)}(\beta) \geq 0, \quad \forall x \in \mathbb{R}^n \setminus E_1, \; \forall \beta \geq 1,
\]
according to Corollary 3.1, we have
\[
\Phi_{\beta(c)}(x) = \frac{1}{2} x^T A x + c - \mu_{r_A(x)}(\beta(c)) \leq \frac{1}{2} x^T A x + c = \overline{u}(x), \; \forall x \in \mathbb{R}^n \setminus D.
\]
(3.18)

We remark that this (3.18) can also be proved by using the comparison principle, in view of
\[
\Phi_{\beta(c)} \leq \eta \leq c \leq \overline{u} \quad \text{on } \partial D,
\]
and
\[
\lim_{|x| \to +\infty} (\Phi_{\beta(c)} - \overline{u})(x) = 0.
\]

On the other hand, for every \( \xi \in \partial D \), since
\[
Q_\xi(x) \leq \frac{1}{2} x^T A x + \tilde{c} \leq \frac{1}{2} x^T A x + c \leq \frac{1}{2} x^T A x + c = \overline{u}(x), \; \forall x \in \partial D,
\]
and
\[
Q_\xi \leq Q < \Phi_{\beta(c)} \leq \overline{u} \quad \text{on } \partial E_\delta,
\]

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follows from (3.17) and (3.18), we obtain
\[ Q_\xi \leq \overline{u} \quad \text{on } \partial (E_r \setminus D). \]

In view of
\[ \frac{\sigma_k(\lambda(D^2Q_\xi))}{\sigma_l(\lambda(D^2\overline{u}))} = 1 = \frac{\sigma_k(\lambda(D^2\overline{u}))}{\sigma_l(\lambda(D^2\overline{u}))} \quad \text{in } E_r \setminus D, \]
we deduce from the comparison principle that
\[ Q_\xi \leq \overline{u} \quad \text{in } E_r \setminus D. \]

Hence
\[ Q \leq \overline{u} \quad \text{in } E_r \setminus D. \tag{3.19} \]

Combining (3.18) and (3.19), by the definition of \( \underline{u} \), we get
\[ \underline{u} \leq \overline{u} \quad \text{in } \mathbb{R}^n \setminus D. \]

This finishes the proof of Lemma 3.3.

**Remark 3.3.** To prove Lemma 3.3 we have used above Lemma 2.4 and Lemma 2.5 presented in Subsection 2.3. In fact, one can follow the techniques in [CL03] (see also [DB11] [Dai11] and [LD12]) instead of Lemma 2.5 to rewrite the whole proof. These two kinds of presentation look a little different but are essentially the same.

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