Pinning of sliding collective charge state in a 1D attractive fermion system

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We investigate an interacting fermion model with boundary potential by using Bethe ansatz method. The ground state properties of the system and the boundary effect are discussed. It is found that attractive boundary potential leads to the boundary bound state. An interesting phenomenon is that the sliding collective charges in a periodic system, which is formed due to the attractive interaction among the fermions, will be pinned around the boundary, as long as the negative boundary potential is strong enough.

I. INTRODUCTION

There has been extensive interest in the investigation of low-dimensional correlated fermion systems in the recent years. It is well known that the perturbation techniques are not very effective tools in dealing with the one-dimensional systems. The 1D systems, although they are somewhat artificial, can give some valuable information on the role of correlation effects in higher dimensions and thus are theoretically meaningful. Many approaches, such as bosonization and renormalization group techniques, have been successfully applied in this field and some fruitful results are obtained. Usually, studies on the exactly solvable models can provide an exactly theoretical understanding to the more complicated interacting systems. Thus the quantum integrable models, which have a long history in itself, again attract much attention.

Some recent works show that the behavior of a single impurity in the 1D quantum systems is rather different from that in a Fermi liquid. Kane and Fisher first investigated a 1D repulsive interacting system in the presence of a potential barrier and pointed out that it corresponds to a chain disconnected at the barrier site at low energy scales. This can be effectively described by the open boundary condition and is well investigated by the boundary conformal field theory. The open boundary problem is also studied by using the Bethe ansatz method. Attempting to understand the effect of boundary potential to an interacting fermion system, we study a spinless interacting model with boundary potential by the Bethe ansatz method. The results show that there exists a critical value for the attractive boundary potential. When it goes through the critical value, the strong attractive boundary potential leads to a boundary bound state. Moreover, the attractive interaction among the fermions also has a critical value. When the interaction is strong enough, a collective sliding charge state forms. The attractive boundary potential will make the whole collective state to be pinned down around the boundary barrier site. This phenomenon provides us a meaningful example how a local potential influences the global properties in a strongly correlated fermion system.

II. THE MODEL AND BETHE ANSATZ

Consider a 1D lattice with $L$ sites and $N$ particles. The Hamiltonian reads

$$H = \sum_{j=1}^{L-1} [-t(a_j^+ a_{j+1} + a_{j+1}^+ a_j) + U n_j n_{j+1}] + p_1 n_1 + p_L n_L,$$

where $a_j(a_j^+)$ are fermion annihilation (creation) operators and $t$, $U$, $p_1$, $p_L$ are hopping amplitude, coupling and boundary potential constants respectively. The hopping parameter is defined to be positive ($t > 0$). As a matter of fact, this model is equivalent to the XXZ Heisenburg model by a Jordan-Wigner transformation. For simplicity, we take $p_L = \infty$, which means that the particles arriving at the right boundary are completely reflected by an infinite high wall, and only consider the effect of the left boundary. To make the discussion clearer, we rederive some known results about this model in this section.

Taking the eigenstate as

$$|\Psi\rangle_N = \sum_{x_1 \cdots x_N} \varphi(x_1 \cdots x_N) a_{x_1}^+ \cdots a_{x_N}^+ |0\rangle,$$

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we get the eigenequation

\[ -t \sum_k \varphi(x_1...x_k \pm 1, ...x_N) + U \sum_k \delta_{x_1+x_k} \varphi(x_1 \cdots x_N) + |p_1 \sum_k \delta(x_k - 1) + p_N \sum_k \delta(x_k - L)|\varphi(x_1 \cdots x_N) = Eg(x_1 \cdots x_N). \] (2)

Considering the two-particle state, in the region of \( x_1 < x_2 \), we take the wavefunction as Bethe type

\[ \varphi(x_1, x_2) = \sum_{r_1, r_2} \{ A_{12}(r_1, r_2) \exp(i r_1 k_1 x_1 + i r_2 k_2 x_2) + A_{21}(r_2, r_1) \exp(i r_2 k_2 x_1 + i r_1 k_1 x_2) \}, \] (3)

where \( r_{1,2} = \pm 1 \) indicates that the particles move toward right or left. Far away from the boundary \( (2 < x_1, x_2 < L) \), when the two particles are not neighboring \( (x_1 \neq x_2 - 1) \), it is easy to get the eigenvalue from the eigenequation

\[ E = -2t \cos k_1 + \cos k_2. \] (4)

When the two particles are neighboring \( (x_1 = x_2 - 1) \), solving the eigenvalue equation, it is readily to obtain the following relation

\[ A_{21}(r_2, r_1) = S(r_1 k_1, r_2 k_2) A_{12}(r_1, r_2) \] (5)

with

\[ S(r_1 k_1, r_2 k_2) = -e^{i \theta(r_1 k_1, r_2 k_2)} = \frac{1 + \exp(i (r_1 k_1 + r_2 k_2) + U/t \exp i r_2 k_2)}{1 + \exp(i (r_1 k_1 + r_2 k_2) + U/t \exp i r_1 k_1)} \] (6)

Comparing to the periodic boundary case, the wavefunction with open boundary includes some coefficients corresponding to backscattering \( (r = -1) \) waves. In addition, the boundary conditions give some limitation to these coefficients. When a particle is located at the left boundary \( (x_1 = 1) \), the eigenvalue equation implies the relation

\[ A_{12}(-, r_2) = S_L(k_1) A_{12}(+, r_2) \] or \[ A_{21}(-, r_1) = S_L(k_2) A_{21}(+, r_1) \] with

\[ S_L(k_i) = \frac{e^{-ik_i} + p_1/t}{e^{ik_i} + p_1/t} e^{i 2k_i}. \] (7)

For a particle at the right boundary \( (x_2 = L) \), the eigenvalue equation also implies the relation \( A_{12}(r_1, -) = S_R(k_2) A_{12}(r_1, +) \) or \( A_{21}(r_2, -) = S_R(k_1) A_{21}(r_2, +) \) with

\[ S_R(k_i) = -e^{i 2k_i L}. \] (8)

In general, we suppose the wavefunction has the following form

\[ \varphi(x_1, \cdots, x_N) = \sum_{p_r, Q, r_p} (-1)^{I(Q)} A_P(r_p) \prod_{i=1}^N e^{i r_p k_r x^{Q_i}} \theta(x_{Q_1} < \cdots < x_{Q_N}), \] (9)

where \( I(Q) \) is the parity of the permutation of \( Q \). With the same procedure, a series of relations can be obtained by using the eigenequation (2). Similar to the two particle case, the eigenvalue is given by

\[ E = -2t \sum_{j=1}^N \cos k_j. \] (10)

When two particles are neighboring, we get

\[ A_{p_1 \cdots p_{j+1}, p_{j}, \cdots p_N} (r_1, \cdots, r_{p_{j+1}}, r_{p_j}, \cdots, r_{p_N}) = S(r_{p_j} k_{p_{j+1}}, r_{p_j} k_{p_{j}+1}) A_{p_1 \cdots p_{j+1}, p_{j}, \cdots p_N} (r_1, \cdots, r_{p_{j+1}}, r_{p_{j+1}}, \cdots, r_{p_N}), \] (11)

with \( S(k_{p_j}, k_{p_{j+1}}) \) given by (6). When the particle is at the boundaries, the eigen equation gives

\[ A_p (-, \cdots) = S_L(k_{p_1}) A_p (+, \cdots), \quad A_p (\cdots, -) = S_R(k_{p_N}) A_p (\cdots, +), \] (12)

with \( S_L(k_i), S_R(k_i) \) given by (7) and (8). From the relations (10-12) we conclude

\[ A_{p_1 \cdots p_N} = S_L^{-1}(k_{p_1}) S(k_{p_2}, k_{p_1}) \cdots S(k_{p_N}, k_{p_1}) S_R(k_{p_1}) S(k_{p_1}, k_{p_2}) \cdots S(k_{p_1}, k_{p_N}) A_{p_1, p_2, \cdots, p_N}. \]
For simplicity, we use $A_{p_1...p_j...p_N}$ to represent $A_{p_1...p_j...p_N} (+,\ldots,+)$ and $A_{p_1...p_j...p_N}$ to represent $A_{p_1...p_j...p_N} (+,\ldots,-,\ldots)$ in the following text. By taking $p_1 = j$, we get

$$S_L(-k_j)S_R(k_j) \prod_{i=1(\neq j)}^N e^{i\theta(k_i,-k_j)}e^{i\theta(k_j,k_i)} = 1. \quad (13)$$

Representing $S_L(k_i) = -e^{i\theta_L(k_i)}e^{i2k_i}$ and $S_R(k_i) = -e^{i2k_i}L$, we have

$$e^{-ik_j2(L-1)} = \prod_{i=1(\neq j)}^N e^{i\theta_L(-k_i)}e^{i\theta(k_i,k_j)}e^{i\theta(k_i,-k_j)}. \quad (14)$$

For convenience, we take $t = 1$ and put $\Delta = \frac{\eta}{2}$ in the following discussion. $\Delta > 0$ or $\Delta < 0$ corresponds to the repulsive or attractive interaction respectively. As is well known, $\Delta = \pm 1$ are two critical points. $|\Delta| > 1$ indicates the strong coupling regions while $|\Delta| < 1$ indicates weak coupling region. In the next section we discuss how the boundary potential influences the properties of the ground state of the whole system in different parameter regions.

### III. BOUNDARY BOUND STATES

We discuss first the gapless critical region. For $|\Delta| < 1$, it’s convenient to represent $\Delta = \cos \eta$ ($U = 2\cos \eta$) with $\eta \in (0, \pi)$. The repulsive or attractive interaction is decided by $\eta \in (0, \pi/2)$ or $(\pi/2, \pi)$ respectively. We parameterize $k_j$ as $k_j = \phi(\lambda_j, \eta)$ with $\phi(a, b) = 2\arctan(\tanh(a \cot b))$. It’s readily to get $\theta(k_j, k_i) = \phi(\lambda_i - \lambda_j, \eta)$. Put

$$p_1 = \frac{e^{i\Gamma+i\eta} - 1}{e^{i\Gamma} - e^{i\eta}} = \frac{\sin \frac{1}{2}(\Gamma + \eta)}{\sin \frac{1}{2}(\Gamma - \eta)}. \quad (15)$$

It follows that $e^{i\theta_L(k_j)+ik_j} = -\sinh(\lambda_j - \frac{i\eta}{2})/\sinh(\lambda_j + \frac{i\eta}{2})$. The Bethe ansatz equation is thus reduced to

$$\left(\frac{\sinh(\lambda_j + \frac{i\eta}{2})}{\sinh(\lambda_j - \frac{i\eta}{2})}\right)^{2L-1} \frac{\sinh(\lambda_j - \frac{i\eta}{2})}{\sinh(\lambda_j + \frac{i\eta}{2})} = \prod_{r=\pm 1(\neq j)}^N \frac{\sinh(\lambda_j - r\lambda_i + i\eta)}{\sinh(\lambda_j - r\lambda_i - i\eta)}, \quad (16)$$

and the eigenenergy is

$$E = 2t \sum_{j=1}^N [\cos \eta - \frac{\sin^2 \eta}{\cosh 2\lambda_j - \cos \eta}] \quad (17)$$

It should be noted that the Bethe ansatz equation with open boundary condition is reflecting invariant, which means $\lambda_j$ and $-\lambda_j$ corresponds to the same state, so we only need to choose one in dealing with concrete problem.

From equation (16), we observe that $\lambda_j = \frac{i\eta}{2}$ ($0 < \frac{i\eta}{2}$) is a possible solution of the Bethe ansatz equation in the condition of $|\frac{\sin \frac{1}{2}(\Gamma + \eta)}{\sin \frac{1}{2}(\Gamma - \eta)}| > 1$ ($|p_1| > 1$)when $L \to \infty$. It can also be seen from (17), when $\cos \Gamma > \cos \eta$, the energy contributed by the boundary mode is lower than that of a real mode. For example $\eta \in (0, \pi/2)$, taking $0 < \frac{1}{2}(\Gamma + \eta) < \frac{\pi}{2}$, and $-\frac{1}{2}(\Gamma - \eta) < 0$, the condition $|p_1| > 1$ is satisfied. By the limitation $0 < \Gamma < \frac{\pi}{2}$, the condition $\cos \Gamma > \cos \eta$ is satisfied if $\Gamma < \eta$. This implies $p_1 < 1$.

So far, we have learned that repulsive or the small attractive $(−1 < p_1 < 0)$ potential does not produce the boundary bound state. The BA equation has no imaginary solution in the ground state. The effect of boundary potential of this case has been extensively investigated by many authors [1]. The strong attractive $(p_1 < −1)$ potential produces a boundary bound state and the corresponding energy contributed by the boundary bound state is

$$e_b = 2t[\cos \eta - \frac{\sin^2 \eta}{\cos \Gamma - \cos \eta}],$$

with $0 < \Gamma < \eta < \frac{\pi}{2}$. In this case, the solutions of BA equation to the ground state consist of a boundary imaginary mode and $N−1$ real mode. This problem has also been discussed by many authors [11, 13] for XXZ and Hubbard model.
For $\Delta = -1$, the interaction term is attractive. In this case, the model corresponds to the well known XXX ferromagnetic spin chain \[14\] with fixed magnetization. Parameterizing $k_j = 2 \arccot 2 \lambda_j$, we have $\theta(k_j, k_i) = 2 \arctan(\lambda_j - \lambda_i)$. Putting

$$p_1 = \frac{1 + \Gamma}{1 - \Gamma},$$

the Bethe ansatz equation is given by

$$\left(\frac{\lambda_j + i}{\lambda_j - i}\right)^{2L-1} \frac{\lambda_j - i \Gamma}{\lambda_j + i \Gamma} = \prod_{r = \pm 1; r \neq j}^{N} \frac{\lambda_j - r \lambda_i + i}{\lambda_j - r \lambda_i - i} \quad \text{(19)}$$

The eigenvalue is given by

$$E = t \sum_{j=1}^{N} \frac{4}{4 \lambda_j^2 + 1} - 2. \quad \text{(20)}$$

With periodic boundary condition, all possible string solutions of the Bethe ansatz equation are $\lambda_{n,k}^\alpha = \lambda_{n}^\alpha - \frac{i}{2}(n + 1 - 2k)$ with $k = 1, \cdots, n$. Notice $\frac{1}{\lambda_{n+1/2}^\alpha} = i\left[\frac{1}{\lambda_{n+1/2}^\alpha} - \frac{1}{\lambda_{n-1/2}^\alpha}\right]$. Thus a $n$-string carries the energy

$$\sum_{k=1}^{n} \frac{4}{4 \lambda_{n,k}^\alpha + 1} = \frac{n}{\lambda_{n}^2 + n^2/4}. \quad \text{(21)}$$

Representing $M_n$ as the number of $n$-strings, the relation $N = \sum_{n=1}^{\infty} n M_n$ holds. It can be proved

$$\sum_{n=1}^{N} M_n \frac{n}{\lambda_{n}^2 + n^2/4} < \frac{N}{\lambda_{j}^2 + N^2/4}. \quad \text{(21)}$$

Thus the $N$ string state is favorable since it contributes much lower energy than the other states. The corresponding ground state wavefunction describes a bound state which decays exponentially as a function of coordinate differences $|x_j - x_i|$ with $j \neq i$.

It can be seen from (19), $\lambda_j = \frac{i}{2} \Gamma$ is a solution of Bethe equations in the limit of $L \to \infty$ if $|p_1| > 1$. Owing to $\lambda_j = \frac{i}{2} \Gamma$ is a solution of Bethe equation, the following type of strings

$$\lambda_{n,k}^\alpha = \frac{i}{2}(\Gamma + 2k - 2) \quad \text{(22)}$$

with $k = 1, \cdots, n$ are also possible solutions. The energy of an imaginary mode is

$$e_{n,k} = \frac{4}{4(\lambda_{n,k}^\alpha)^2 + 1} = \frac{4}{1 - (\Gamma + 2k - 2)^2}. \quad \text{(22)}$$

Now, it’s easy to see that when $\Gamma > 1$ the energy contributed by the boundary string is lower than that of a real mode.

As discussed, the attractive boundary potential produces a boundary bound state as soon as it is stronger than the hopping amplitude. In the case of attractive interaction, the ground state of systems will be a boundary string state with length $N$ in the form (23). In virtue of the relation

$$2 \sum_{k=1}^{n} \frac{1}{1 - (\Gamma + 2k - 2)^2} = \frac{1}{\Gamma + 2n - 1} - \frac{1}{\Gamma - 1},$$

the ground state energy is given by

$$E = t \sum_{k=1}^{N} \frac{4}{(4 \lambda_k^N)^2 + 1} - 2 = 2t \left(\frac{1}{\Gamma + 2N - 1} - \frac{1}{\Gamma - 1}\right) - 2Nt. \quad \text{(23)}$$
So far, we have shown all the particles are bounded by the attractive boundary potential in the ground state. The wavefunction decays exponentially away from the left boundary. In order to show it explicitly, we give an example of two particle case in the appendix. Generally, for a \( n \) particle system, the ground state is a boundary \( n \)-strings \((22)\), the corresponding wavefunction is given by

\[
\varphi(x_1, \cdots x_n) = c f(\lambda_1^n)^{-x_1} f(\lambda_2^n)^{-x_2} \cdots f(\lambda_n^n)^{-x_n},
\]

(24)

with \( f(\lambda_j) = e^{ik_j} = \frac{\lambda_j + \frac{1}{2}}{\lambda_j - \frac{1}{2}} \). Therefore, with the presence of the attractive impurity potential, the slidding collective charges will be pinned and loses its mobility. By the expression of energy \((24)\), we learn that the stronger the attractive potential is, the more quickly the wavefunction decays away from the boundary.

For the repulsive interaction \( U = 2 \) \((\Delta = 1)\), we re-parameterize \( k_j = 2 \arctan 2\lambda_j \), thus \( \theta(k_j, k_i) = 2 \arctan(\lambda_i - \lambda_j) \).

Putting \( p_1 = \frac{\Gamma + 1}{\Gamma + 2} \), the Bethe ansatz equation is also given by \((19)\), but the eigenvalue is given by

\[
E = t \sum_{j=1}^{N} \left( 2 - \frac{4}{4\lambda_j^2 + 1} \right).
\]

(25)

Similar to the case \( \Delta = -1 \), the Bethe ansatz equation has the imaginary solution \( \lambda_j = i\frac{\Gamma+1}{2} \) \((\text{or } -i\frac{\Gamma+1}{2})\) if \( |p_1| > 1 \). Thus \( \lambda_1^\prime = i\frac{\Gamma}{2}(\Gamma + 2k - 2) \) is also the possible solution of BA equation. From \((25)\), we see if \( \Gamma \in (0,1) \) the energy of the imaginary mode \( \lambda_1^1 = i\frac{\Gamma}{2} \) is lower than those of the real modes. However, the other imaginary solutions for \( n > 1 \) are not favorable in energy and thus corresponds to the highly excited state. In this case, the ground state is composed of an imaginary mode \( \lambda = i\frac{\Gamma}{2} \) with \( \Gamma \in (0,1) \) and \( N - 1 \) real mode.

In the following, we discuss the ground state properties of strong interaction cases, \( |\Delta| > 1 \). First, we consider the attractive interaction case. Owing to \( \Delta < -1 \), we can parameterize it as \( \Delta = -\cosh \eta \) with \( \eta > 0 \). By putting

\[
k_j = 2 \arccot(\tan \lambda_j \coth \frac{1}{2}\eta),
\]

(26)

\[
\theta(k_j, k_i) = 2 \arctan[\tan(\lambda_j - \lambda_i) \coth \eta],
\]

(27)

\[
p_1 = -\frac{e^{\Gamma + \eta} - 1}{e^{\Gamma} - e^{\eta}} = -\frac{\sinh \frac{1}{2}(\Gamma + \eta)}{\sinh \frac{1}{2}(\Gamma - \eta)},
\]

(28)

the Bethe ansatz equation can be reduced to

\[
\left( \frac{\sin(\lambda_j + \frac{1}{2}i\eta)}{\sin(\lambda_j - \frac{1}{2}i\eta)} \right)^{2L-1} \frac{\sin(\lambda_j - i\frac{\Gamma}{2})}{\sin(\lambda_j + i\frac{\Gamma}{2})} = \prod_{r=\pm 1}^{N} \prod_{j' \neq j}^{N} \frac{\sin(\lambda_j - r\lambda_j + i\eta)}{\sin(\lambda_j - r\lambda_j - i\eta)}.
\]

(29)

The eigenvalue is given by

\[
E = 2t \sum_{j=1}^{N} \left( \frac{\sinh^2 \eta}{\cosh \eta - \cos 2\lambda_j} - \cosh \eta \right)
\]

(30)

In the attractive interaction case \((\Delta = -\cosh \eta)\), the energy contributed by a string is lower than those of the real modes. In the period boundary condition, all the possible types of string solutions are

\[
\lambda_{n,k}^\alpha = \lambda_n^\alpha - \frac{i}{2}\eta(n + 1 - 2k)
\]

(31)

with \( k = 1, \cdots, n \). The energy of an \( n \)-string is given by

\[
e_n(\lambda) = \sinh \eta \frac{\sinh n\eta}{\cosh n\eta - \cos 2\lambda} - n \cosh \eta.
\]

(32)

It can be proved that
\[
\sinh N\eta \cosh N\eta - \cos 2\lambda < \sum_{n=1}^{N-1} M_n \sinh n\eta \cosh n\eta - \cos 2\lambda.
\]

Under the periodic boundary condition, the ground state in strong attractive case is given by an N-string \[15,16\]. Similar to the case \(\Delta = -1\), the ground state wavefunction decays exponentially as a function of coordinate differences \(|x_j - x_i|\) for all \(j \neq i\).

The Bethe ansatz equation (29) has an imaginary solution \(\lambda_j = i\frac{\Gamma}{2}\) for \(\Gamma > 0\) in the limit \(L \to \infty\). This solution corresponds to the boundary bound state. The imaginary mode contributes the energy

\[
e_k(i\frac{\Gamma}{2}) = \frac{\sinh^2 \eta}{\cosh \eta - \cosh \Gamma} - \cosh \eta.
\]

For \(\Gamma > \eta\), the imaginary mode \(i\frac{\Gamma}{2}\) contributes much lower energy than that of a real mode. Similar to the discussion for \(\Delta = -1\) case, if \(i\frac{\Gamma}{2}\) is a solution of BA equation, the following string

\[
\lambda_k^n = \frac{i}{2} \left[\Gamma + 2(k - 1)\eta\right]
\]

with \(k = 1, \cdots, n\), are also solutions. The energy of an imaginary mode is written as

\[
e_k^n = \frac{\sinh^2 \eta}{\cosh \eta - \cosh[\Gamma + 2(k - 1)\eta]} - \cosh \eta.
\]

Now, it’s easy to see that the energy of this state is lower than those of any other states. In this case, the ground state solution is not a sliding N-string as (31) but a boundary N-string given by (34). Put \(\Gamma = \Gamma' - (N - 1)\eta\). In virtue of

\[
\sum_{k=1}^{n} \frac{\sinh^2 \eta}{\cosh \eta - \cosh[\Gamma + (2k - n - 1)\eta]} = \sinh \eta \frac{\sinh n\eta}{\cosh n\eta - \cosh \Gamma'},
\]

we obtain the ground state energy as

\[
E = 2t \sum_{k=1}^{N} e_k^n = \frac{2t \sinh \eta \sinh N\eta}{\cosh N\eta - \cosh[\Gamma + (N - 1)\eta]} - 2tN \cosh \eta.
\]

Similar to the \(\Delta = -1\) case, the ground state wavefunction is given by

\[
\varphi(x_1, \cdots x_n) = cf(\lambda_1^n)^{-x_1} f(\lambda_2^n)^{-x_2} \cdots f(\lambda_n^n)^{-x_n},
\]

with

\[
f(\lambda_k^n) = \frac{\sinh \frac{1}{2}[\Gamma + (2k - 1)\eta]}{\sinh \frac{1}{2}[\Gamma + (2k - 3)\eta]}.
\]

For \(\Delta = \cosh \eta\) \((\eta > 0)\), we introduce the notations

\[
k_j = 2 \arctan(\tan \lambda_j \coth \frac{1}{2}\eta),
\]

\[
\theta(k_j, k_i) = 2 \arctan[\tan(\lambda_i - \lambda_j) \coth \eta],
\]

\[
p_1/t = e^{\Gamma + \eta} - 1 \left/ e^{\Gamma} - e^{\eta}\right. = \frac{\sinh \frac{1}{2}(\Gamma + \eta)}{\sinh \frac{1}{2}(\Gamma - \eta)}.
\]

The eigenvalue is expressed as

\[
E = 2t \sum_{j=1}^{N} \left[\cosh \eta - \frac{\sinh^2 \eta}{\cosh \eta - \cos 2\lambda_j}\right]
\]

In this case, the string solutions correspond to highly excited states. Now, we can see \(\lambda_j = i\frac{\Gamma}{2}\) is a possible solution of BA equation (29) if the condition \(|p_1| > 1\) is satisfied. It is easy to see that the energy of the imaginary mode \(\lambda_j = i\frac{\Gamma}{2}\) is smaller than those of real modes when \(0 < \Gamma < \eta\) \((p_1 < -1)\) for the repulsive interaction \(\Delta > 1\).
IV. CONCLUSION

In summary, we study an interacting spinless fermion system with boundary potential. By analyzing the Bethe ansatz equation and the eigenenergy, we discuss the boundary effect of the ground state. Our results show that, when the attractive potential increases to the value of the hopping amplitude, there will be a boundary bound state which leads to a fermion to be bounded on the first site. When the attractive potential is weaker comparing to the hopping amplitude, the attractive potential is not enough to resist the hopping charges. For the attractive interaction $U < 0$, there exists a critical value $U = -2t$ ($\Delta = -1$). When the attractive interaction exceeds the critical value, it makes all particles correlate with each other in a form similar to the “Cooper’s pair”. The corresponding ground state wavefunction decays exponentially as a function of coordinate differences $| x_j - x_i |$ for all $j \neq i$. In this case, the boundary bound state will influence the properties of the whole system drastically. Owing to the strong correlation effect among the fermions, the collective charge composite is pinned down in the neighborhood of the boundary barrier site when the attractive potential is strong. This is much different from the common Fermi liquid picture, in which a single potential barrier can not change the global properties of a system. This exactly solved model, although is very simple, reveals a meaningful picture of a strongly correlated fermion system.

Appendix

For the two particles system, the wavefunction is given by (3). We note $A_{21}(r_2, r_1) = S(r_1, k_1, r_2, k_2) A_{12}(r_1, r_2)$ with

$$S(r_1, k_1, r_2, k_2) = \frac{r_1 \lambda_1 - r_2 \lambda_2 - i}{r_1 \lambda_1 - r_2 \lambda_2 + i}.$$  

For convenience, we represent

$$f(\lambda) = e^{ikx} = \frac{\lambda_j + i \frac{t}{\lambda_j}}{\lambda_j - \frac{t}{\lambda_j}},$$

thus the wavefunction can be expressed as

$$\varphi(x_1, x_2) = \sum_{r_1, r_2} A_{12}(r_1, r_2) [f(\lambda_1)^{r_1x_1} f(\lambda_2)^{r_2x_2} + S(r_1, k_1, r_2, k_2) f(\lambda_2)^{r_2x_1} f(\lambda_1)^{r_1x_2}].$$  (40)

Using the relation (12) and (13), we have $A_{12} = S_L A_{12}, A_{12} = S_R A_{12}, A_{12} = S_L(\lambda_1) S_R(\lambda_2) A_{12}$. We note

$$S_L(\lambda_1) = -\frac{(\lambda_1 + i \frac{t}{\lambda_1})(\lambda_1 + i \frac{\Gamma}{\lambda_1})}{(\lambda_1 - \frac{t}{\lambda_1})(\lambda_1 - i \frac{\Gamma}{\lambda_1})}.$$  

Using the Bethe ansatz equation (14), we have.

$$S_R(\lambda_2) = -e^{ikx} = S_L(\lambda_2) S_{21} S_{21}.$$

Thus the wavefunction is written as

$$\varphi(x_1, x_2) = A_{12} [f(\lambda_1)^{x_1} f(\lambda_2)^{x_2} + S_{12} f(\lambda_2)^{x_1} f(\lambda_1)^{x_2}] + A_{12} [f(\lambda_1)^{-x_1} f(\lambda_2)^{-x_2} + S_{21} f(\lambda_2)^{-x_1} f(\lambda_1)^{-x_2}] + A_{12} [f(\lambda_1)^{-x_1} f(\lambda_2)^{-x_2} + S_{21} f(\lambda_2)^{-x_1} f(\lambda_1)^{-x_2}],$$  (41)

with $A_{12} = S_L(\lambda_1) A_{12}, A_{12} = S_R(\lambda_2) S_{21} A_{12}, A_{12} = S_L(\lambda_1) S_R(\lambda_2) S_{21} A_{12}$. If we take $\lambda_1 = \frac{\Gamma}{2}, \lambda_2 = \frac{\Gamma}{2}(\Gamma + 2)$, then $S_{12} = \infty, S_L(\lambda_1) = \infty$. That means only one term does not disappear for $\lambda_2 - \lambda_1 = i$. The wavefunction has the form

$$\varphi(x_1, x_2) = cf(\lambda_1)^{-x_1} f(\lambda_2)^{-x_2}.$$  

If we take $\lambda_1 = -\frac{\Gamma}{2}, \lambda_2 = -\frac{\Gamma}{2}(\Gamma + 2), S_{12} = 0, S_L(\lambda_1) = 0$, the wave function is written as

$$\varphi(x_1, x_2) = cf(\lambda_1)^{x_1} f(\lambda_2)^{x_2}.$$  

The wavefunction can also be expressed as

$$\varphi(x_1, x_2) = \sum_{r_1, r_2} A_{21}(r_2, r_1) [f(\lambda_2)^{r_2x_1} f(\lambda_1)^{r_1x_2} + S^{-1}(r_1, k_1, r_2, k_2) f(\lambda_1)^{r_1x_1} f(\lambda_2)^{r_2x_2}].$$
It can be written as the similar form of (41) with $A_{21} = S_L(\lambda_2)A_{21}$, $A_{2T} = S_T S_L(\lambda_1) S_{12}^{-1} A_{21}$, $A_{2T} = S_L(\lambda_1) S_L(\lambda_2) S_{21} A_{21}$. If we take $\lambda_2 = \frac{i}{2} \Gamma$, $\lambda_1 = \frac{i}{2} (\Gamma + 2)$, $S_{21} = \infty$, $S_L(\lambda_2) = \infty$, it is reduced to

$$\varphi(x_1, x_2) = cf(\lambda_2)^{-x_1} f(\lambda_1)^{-x_2}.$$ 

If we take $\lambda_2 = -\frac{i}{2} \Gamma$, $\lambda_1 = -\frac{i}{2} (\Gamma + 2)$, $S_{21} = 0$, $S_L(\lambda_2) = 0$, the wave function is written as

$$\varphi(x_1, x_2) = cf(\lambda_2)^{-x_1} f(\lambda_1)^{x_2}.$$ 

So far, we can see, the wavefunction can be always represented as the form

$$\varphi(x_1, x_2) = cf(\lambda_2^2)^{-x_1} f(\lambda_2^2)^{-x_2}$$

$$= c \left( \frac{\Gamma + 1}{\Gamma - 1} \right)^{-x_1} \left( \frac{\Gamma + 3}{\Gamma + 1} \right)^{-x_2}.$$ 

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