Langevin theory of absorbing phase transitions with a conserved magnitude

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The recently proposed Langevin equation, aimed to capture the relevant critical features of stochastic sandpiles, and other self-organizing systems is studied numerically. This equation is similar to the Reggeon field theory, describing generic systems with absorbing states, but it is coupled linearly to a second conserved and static (non-diffusive) field. It has been claimed to represent a new universality class, including different discrete models: the Manna as well as other sandpiles, reaction-diffusion systems, etc. In order to integrate the equation, and surpass the difficulties associated with its singular noise, we follow a numerical technique introduced by Dickman. Our results coincide remarkably well with those of discrete models claimed to belong to this universality class, in one, two, and three dimensions. This provides a strong backing for the Langevin theory of stochastic sandpiles, and to the very existence of this new, yet meagerly understood, universality class.

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Aimed at shedding some light at the origin of “order” in Nature, some different routes to organization have been proposed in the last fifteen years or so. In particular, the concept of self-organization, as exemplified by sandpiles (for reviews see [1, 2, 3]) (one of the canonical instances of self-organizing systems) has generated a rather remarkable outburst of interest. In order to rationalize sandpiles in particular, and self-organized criticality (SOC) in general, and to understand their critical properties, it has been recently proposed to look at them as systems with many absorbing states [2, 7, 8, 10]. The underlying idea is that in the absence of external driving sandpile models get eventually trapped into stable configurations from which they cannot escape, i.e., absorbing states (AS) [11, 12, 13]. In order to make the aforementioned connection more transparent, the notion of fixed energy sandpiles (FES) was introduced. These modified sandpiles are defined in such a way that they share the microscopic rules with their standard (slowly driven) counterparts, but with no driving (no addition of sand-grains) nor dissipation; i.e. the total amount of sand (energy) becomes a conserved quantity that acts as a free tunable parameter. In this way, as noise crossed-correlations, whose role in the asymptotic properties is unclear.

Using this analogy with systems with AS, a field theoretical description of stochastic sandpiles has been proposed [2, 7, 8, 10], which includes the two more relevant features of stochastic sandpiles: i) the presence of infinitely many AS and, ii) the global conservation of the total energy. The phenomenological field theory (Langevin equation) aimed to capture the relevant critical features of this type of systems is similar to the well known Reggeon field theory (RFT) [12, 13, 14] (describing generic systems with AS) but it is coupled linearly to a conserved energy field, namely [7, 9]:

$$\frac{\partial \rho}{\partial t} = D_\rho \nabla^2 \rho - \mu \rho - \lambda \rho^2 + \omega \phi + \eta(x,t),$$

$$\frac{\partial \phi}{\partial t} = D_\phi \nabla^2 \phi,$$

where $D_\rho$, $D_\phi$, $\mu$, $\lambda$ and $\omega$ are constants, $\rho(x,t)$ and $\phi(x,t)$ are the activity and the energy field respectively, $\eta$ is a zero-mean Gaussian noise with:

$$\langle \eta(x,t) \eta(x',t') \rangle = 2 \sigma^2 \rho(x,t) \delta(x-x') \delta(t-t').$$

As usual in systems with AS, the variance of $\eta$ is proportional to $\rho$ [13, 14].

Soon after the introduction of the previous Langevin equation its range of applicability was extended, as it was conjectured to describe all systems with many AS and an auxiliary conserved and non-diffusive (or static) field [13]. In particular for a reaction-diffusion model in this family an equation similar to Eq. (i) was derived rigorously by using standard Fock-space formalism techniques [12, 13, 17]. To be more precise, we should mention that the derived set of equations, includes some higher order terms, as noise crossed-correlations, whose role in the asymptotic properties is unclear.

A priori, it is not straightforward to decide from a field theoretical point of view whether the extra conservation law induces a critical behavior different from that of RFT or if, on the contrary, it is an irrelevant perturbation at the RFT renormalization group fixed point [14].
From the theory side, it has been recently argued by van Wijland that the Langevin equation is renormalizable in $d_c = 6$, while other authors have previously claimed $d_c = 4$. Some mean field results and simulations in high dimensions of discrete models [19], and also a new method recently proposed by Lubeck to determine the upper critical dimension of systems with AS [24], leads rather convincingly to $d_c = 4$, but we are still far from a full clarification of these issues at a field theoretical level. In any case, it is commonly accepted, from numerical evidence, that this constitutes a new universality class, usually called Manna-class, or C-DP (in the spirit of Hohenberg and Halperin [22]). [6, 9, 15, 16, 17, 18, 19, 20].

Further details of the scheme can be found in [23]. It has been successfully applied to both the RFT [23] and to systems with many AS [25], leading to good estimations of phase diagrams and critical properties. In order to extend the algorithm to our problem, the second equation of [19] follows a subtle different procedure based on Dickman’s [26] usual Euler’s method. The usual scaling laws: 

\[ f(x, t + \Delta t) = f(x, t) + \Delta t \left[ D_\lambda \nabla_x^2 f(x, t) + \frac{Ds}{\nu_2} \phi(x, t) \right] + \sigma f \phi(x, t) \]

where $\Delta t = \Delta \rho$, $\eta'$ is a zero-mean Gaussian white noise, $\nabla_x^2$ is the discrete Laplacian operator and $f(x, t)$ is an auxiliary continuous field. Then, after each integration step, the number of quanta of $\rho$, $m(x, t)$, is updated according to:

\[ m(x, t + \Delta t) \rightarrow m(x, t) + \int [f(x, t + \Delta t)] \]

Initial conditions are taken as follows: i) $\phi(x, t = 0) = \phi_0 \times (1 + a \nabla_x^2 \epsilon(x, t))$, where $\epsilon$ is a normalized Gaussian noise with zero average, and $a$ is a constant establishing the range of relative variation allowed to $\phi$ with respect to its mean value, $\phi_0$. $\phi_0$, is the control parameter, and except for transient effects results should not depend on $a$. ii) The initial condition for $\rho$ is chosen by randomly distributing active-field quanta, in such a way that $\rho(x, t = 0) \leq \phi(x, t = 0)$ everywhere.

We have carried out extensive simulations of the coupled Eqs. [19] in one, two and three dimensional lattices. In all the cases, the time-mesh has been fixed to $\Delta t = 0.01$, and $\Delta x = 1$, (values below which we have verified that results are not further affected). We also fix $D_\lambda = D_\rho = 5$ and $\mu = \lambda = \omega = a = 1$. The noise amplitude $\sigma$ is taken different for the various dimensions in order to fix the transition in a reasonable (but arbitrary) value of $\phi_0$: $\sigma = 1$ in $1D$, $\sigma = 0.5$ in $2D$, and $\sigma = 0.35$ in $3D$. We have verified that the total energy is conserved within the considered precision, in all cases. The number of runs goes from $10^5$ up to $10^7$ depending on system size.

Results. – As we vary $\phi_0$, a continuous transition separating the absorbing from active phase, is observed at a critical threshold $\phi_c$. Below (above) this value the system is absorbing (active) states. The usual scaling laws: $\rho \sim (\phi_0 - \phi_c)^\beta$, $\xi \sim (\phi_0 - \phi_c)^{-\nu_x}$, $\tau \sim (\phi_0 - \phi_c)^{-\nu_t}$, where $\tau$ is the correlation length (time) are expected to hold [11, 12, 13]. This leads to the definition of the dynamic exponent as $\tau \sim \xi^z$, with $z = \nu_t / \nu_x$. It is also expected that at the critical point the density of activity presents a power law decay with time, $\rho \sim t^{-\eta}$. However, in some models, an anomalous critical time behavior of $\rho$ has been reported [2, 14, 15]. We shall later return to this issue. As usual the finite size of simulated systems induces the possibility of falling into the AS even for $\phi_0 > \phi_c$. This fact has two consequences. The density of activity $\rho$ does not reach a stationary state close to the critical point. Hence, we are forced to consider the density of surviving trials, $\rho_{\text{surv}}$, in order to realize the finite size analysis of $\rho$. On the other hand, this provides us with a method to measure the dynamic exponent $z$, by determining a characteristic decaying time as a function of system size. We have studied systems of linear size up to $L = 4000$ in $1D$, $L = 400$ in $2D$ and $L = 80$ in $3D$. The dependence of the stationary activity density on system size for several values of $\phi_0$ in a one dimensional system is shown in Fig. 1. From this picture, we deduce the position of the critical point in $1D$, $\phi_c(1D) = 1.6371(1)$.
FIG. 1: Stationary value of $\rho_{\text{surv}}$ for different system sizes in $1D$. Squares correspond to $\phi_0 = 1.6369$, circles to $\phi_0 = 1.6371$ and diamonds to $\phi_0 = 1.6373$. In the inset, the same graph is displayed but for $2D$ data; squares are for $\phi_0 = 0.631$, circles for $\phi_0 = 0.6325$ and diamonds for $\phi_0 = 0.635$.

FIG. 2: Evolution of $\rho$ (continuous line) and $\rho_{\text{surv}}$ (dashed lines) for several system sizes in $1D$. The curve of $\rho$ is for $L = 4000$ and those of $\rho_{\text{surv}}$ for from top to bottom $L = 20, 100$ and $500$. The slope of the straight line is $\theta = 0.14$. In the inset, the same data in $2D$ are represented; $\rho_{\text{surv}}$ curves correspond from top to bottom to $L = 10, 25, 70$ and $L = 280$ and $\rho$ to $L = 280$. The slope of the line is $\theta = -0.65$.  

FIG. 3: Anomalous time decay of the activity density in $2D$, for $L = 280$ and $\phi_0 = 0.711$. We find $\phi_s(2D) = 0.6325(5)$ and $\phi_s(3D) = 0.456(1)$, together with the critical exponents listed in Table I. In the table, we have also included the critical exponents of discrete models claimed to belong to the same universality, and also (for comparison) those of the Directed Percolation (DP) class. Observe the rather remarkable agreement (within errorbars) between all the measured exponents are their counterparts in discrete models. Let us remark that, for those exponents for which the differences with DP-values are larger, our values also deviate from DP. As we have already mentioned, some models in the Manna universality class may present an anomalous behavior in the time decay of the activity density at the critical point. $\phi_s(t, \phi_c)$. This anomaly implies that, apparently, the scaling relation $\beta = \theta \nu_\parallel$ fails and that $\rho(t)$ may decay in a nonmonotonous way at criticality. In our case, there is no anomalous decay in $d = 1$ (Fig.2). However, the anomaly is present both in $d = 2$ and in $d = 3$. As can be seen

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(numbers in parentheses correspond to the statistic uncertainty in the last digit). Also, from the slope of the log-log plot, we obtain $\beta/\nu_\perp(1D) = 0.213(6)$. The exponent $\beta$ may be estimated in an independent way from the scaling of the stationary value of $\rho_{\text{surv}}$ for large system sizes, as a function of $(\phi_0 - \phi_\perp)$ above the critical point. This gives $\beta(1D) = 0.28(1)$. By studying the time evolution of the characteristic time of the surviving probability $P(t)$ at criticality, we obtain $\zeta(1D) = 1.47(2)$. Finally, the exponent $\theta(1D) = 0.14(1)$ may be measured from the critical power law decay of $\rho$ in time, as may be seen in Fig. 2. Errors in these exponents mostly come from the uncertainty in the determination of the critical point. Repeating this process in higher dimensions,
in Fig. 3, the activity density decays initially faster than a power law, showing afterwards a non-monotonous behavior. The fast decay explains the anomalous values of the exponent $\theta$, not satisfying scaling relations, usually reported in the literature \([18]\). Later on, $\rho$ increases before reaching the steady state value, $\rho_{\text{stat}}(L)$ after a certain time $t_x(L)$ (calculated by fixing some arbitrary criterion). If the points $(t_x(L), \rho_{\text{stat}}(L))$ are represented in a log-log plot at the critical point, an alternative value for the exponent $\theta$ is found, which is related to the saturation time scale. This value in two dimensional space, $\theta \approx 0.50$ (with a large statistic uncertainty), \([18]\) is much closer to the more accurate measurements reported in the literature for models in this class ($\theta = 0.51$ for C-DP \([18]\)). The common presence of anomalous behavior in discrete systems \([1, 2, 11, 12]\) and in the continuous theory reinforces the claim that both belong to the same universality class: they share not only the critical behavior but also the dynamical anomalies. A deeper study of the physical origin of this anomaly is still missing but, essentially, it is related to the existence of different time scales.

Finally, let us mention that spreading experiments (performed using localized seeds of activity \([11, 13]\)) exhibit some numerical instabilities, due to the presence of large gradients, that merit a separate analysis and are, therefore, left aside for a future work.

Conclusions. – Strong numerical evidence confirms that there is a well defined new universality class of systems with absorbing states, different from DP and characterized by the presence of an extra static conserved field, linearly coupled to the activity field. More importantly, the numerical simulations of the phenomenological Langevin equation, proposed some time ago to capture the criticality of this universality class, Eq.(1), in one, two, and three dimensional systems, show that indeed it constitutes a sound minimal continuous representation of this class, sharing all the critical exponents as well as the dynamical anomalies with the discrete models. Therefore, no other higher order terms nor other noise correlations are needed to describe properly this class. Once the situation has been clarified from the numerical side, further theoretical analyses are highly desirable in order to put this puzzling universality class under a more firm basis.

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[27] Compare this with the value $\theta \approx 0.65$ obtained from the initial decay in $d = 2$. 