SELDER GROUP ASSOCIATED TO THE CHOW GROUP OF CERTAIN CODIMENSION TWO CYCLES

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Abstract. Let $X$ be a surface with geometric genus and irregularity zero which is defined over a number field $K$. Let $\mathfrak{X}$ denote a smooth spread of $X$ over $\mathcal{O}_K[1/f]$ for some element $f \in \mathcal{O}_K$ and $A^2$ stands for the group of algebraically trivial cycles on schemes modulo rational equivalence. If $j^*: A^2(\mathfrak{X}) \to A^2(X)$ be the flat pull-back corresponding to the embedding $j: X \hookrightarrow \mathfrak{X}$ then we prove that $\text{im}(j^*)(K)/A^2(\mathfrak{X})(K)$ is a torsion group. Here $\text{im}(j^*)(K), A^2(\mathfrak{X})(K)$ stand for the cycles fixed under the action of the absolute Galois group.

1. Introduction

Suppose $X$ be a smooth projective surface defined over a number field $K$ and assume that it can be spread out to a smooth projective scheme $\mathfrak{X}$ over an affine open subset of the spectrum of the number ring $\mathcal{O}_K$. Let $A^2(X)$ denote the group of algebraically trivial cycles of codimension 2 modulo rational equivalence on $X$.

We recall the definition of algebraic equivalence over $\mathcal{O}_K$, which will be used in the sequel. Let us consider the free abelian group of codimension two cycles on $\mathfrak{X}$. Two cycles $z_1, z_2$ are said to be algebraically equivalent if there exists a smooth projective curve $C$ defined over $\mathcal{O}_K$, two scheme theoretic points $x_0, x_1$ on $C$ and a relative correspondence $\Gamma$ on $C \times_{\mathcal{O}_K} \mathfrak{X}$, such that the intersection

$$\Gamma.(x_0 \times_{\mathcal{O}_K} \mathfrak{X}) - \Gamma.(x_1 \times_{\mathcal{O}_K} \mathfrak{X}) = z_1 - z_2.$$  

Here . denotes the relative intersection product in the sense of [Fu].

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Let $K$ denote the algebraic closure of $K$ and $G = \text{Gal}(\overline{K}/K)$ be the absolute Galois group. Also, $X_\overline{K}$ denotes the surface $X \times_K \overline{K}$ and

\[ X_\overline{K} := X \times_{\mathcal{O}_K} \overline{\mathcal{O}_K}. \]

Here $\mathcal{O}_K$ denotes the integral closure of $\mathcal{O}_K$ in $\overline{K}$.

Let $A^2(X_\overline{K})(K), A^2(X_\overline{K})(K)$ be the $G$-fixed part of the action of $G$ on $A^2(X_\overline{K}), A^2(X_\overline{K})$ respectively, and $j$ be the embedding of $X_\overline{K}$ into $X_\overline{K}$. If one considers the flat pullback $j^*: A^2(X_\overline{K}) \to A^2(X_\overline{K})$ of codimension 2-cycles, it gives a map $j^*: A^2(X_\overline{K})(K) \to A^2(X_\overline{K})(K)$. Then a general question is: what is the cokernel of this homomorphism?

Mildenhall [Mil] studied this flat pull-back for $X$ to be the self-product of an elliptic curve admitting a complex multiplication. He has shown that the kernel of this flat-pullback over any number field $K$ is finite. We use Mildenhall’s result to derive the information about the quotient $\text{im}(j^*)(K)[n]/A^2(X)(K)[n]$ at the level of $n$-torsions of this homomorphism for the case where $X$ is the self-product of a CM elliptic curve.

Let $E$ be an elliptic curve with complex multiplication by the ring of integers of a number field $K$ and $N$ be it’s discriminant (for this we fix an Weirstrass equation for $E$ once and for all). We denote $E \times E$ by $X$ and suppose that there exists a smooth spread of $X$, say $\mathcal{X}$ defined over $\overline{\mathcal{O}_K}[1/6N]$. Let $j^*$ be the flat pull-back at the level of $A^2$ induced by the embedding $j : X_\overline{K} \to X_\overline{K}$.

Then the main result is:

\textbf{Theorem 2.2} The group $\Sigma^6N(\Sigma_\overline{K})$ is a colimit of the unramified cohomology groups

\[ H^1(G, \Sigma_K[6N]) \to H^1(I_v, \Sigma_\Sigma[6N]) \]

here $G$ is the Galois group $\text{Gal}(\overline{K}/L)$ for a finite extension $L$ of $K$ and $I_v$ is the inertia subgroup of the Galois group $G$ corresponding to a
finite place $v$ and hence

$$\text{im}(j^*[6N])(K)/A^2(X_{\overline{K}})[6N](K)$$

is a colimit of unramified cohomology groups.

The main tools used here are the Galois module structure of the Chow group of $X_{\overline{K}}$ and that of $X_{\overline{K}}$ and the Galois cohomology of the groups $A^2(X_{\overline{K}})$ and that of $A^2(X_{\overline{K}})$. The proof involves similar techniques as to show that the Selmer group of an abelian variety defined over a number field is finite.

Coombes [Co] proved that for a surface $X$ with geometric genus and irregularity zero, $A^2(X)$ is finite under the assumption that $A^2(X_{\overline{K}}) = 0$.

Towards proving our main result Theorem 2.2, we start with a surface $X$ of geometric genus and irregularity zero defined over $K$ which satisfies the condition that the map:

$$\text{Pic}(X) \to \text{Pic}(X_{\overline{K}}) \to \text{NS}(X_{\overline{K}})$$

is surjective,

$$H^2(X_{\mathcal{O}_K[1/f]}, \mathcal{O}_{X_{\mathcal{O}_K[1/f]}}) = 0$$

for all $p$ in Spec($\mathcal{O}_K[1/f]$), here $f$ is some element in $\mathcal{O}_K$ and $X(K) \neq \emptyset$.

Then we prove by using the result of [CTR][lemma 3.3], that:

Theorem 2.6: The group

$$\text{im}(j^*)(K)/A^2(X_{\mathcal{O}_K[1/f]})(K)$$

is a torsion group.

Here we do not assume the vanishing of $A^2(X_{\overline{K}})$ as in [Co]. The above theorem is important to prove the finiteness or triviality of $A^2(X_{\overline{K}})(K)$. It says that, at least to prove that $\text{im}(j^*)(K) \otimes_\mathbb{Z} \mathbb{Q} = \{0\}$ it is enough to prove that

$$A^2(X_{\mathcal{O}_K[1/f]})(K) \otimes_\mathbb{Z} \mathbb{Q} = \{0\}.$$ 

This gives some information on how to prove $A^2(X)$ is finite.

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2. Proof of the theorems

Let $E$ be as before having complex multiplication by $\mathcal{O}_K$ and $X = E \times E$. Let us fix a Weirstrass equation for the elliptic curve once and for all. Let us consider a spread $\mathcal{E}_{\mathcal{O}_K[1/6N]}$ of $E$ over $\mathcal{O}_K[1/6N]$ which is smooth and denote $\mathcal{E}_{\mathcal{O}_K[1/6N]} \times \mathcal{E}_{\mathcal{O}_K[1/6N]}$ by $\mathfrak{X}$. Then we consider the restriction homomorphism from $A^2(\mathfrak{X}) \to A^2(X)$. It is known due to Mildenhall’s result that the kernel of this restriction map is finite and we name it $\Sigma_K$. Then we have the exact sequence

$$0 \to \Sigma_K \to A^2(\mathfrak{X}) \to A^2(X).$$

Now consider the sequence at the level of $\bar{K}$, namely

$$0 \to \Sigma_{\bar{K}} \to A^2(\mathfrak{X}_{\bar{K}}) \to A^2(X_{\bar{K}}).$$

Note that $G$ acts naturally on each member of the above short exact sequence. Also if the inclusion of $X_{\bar{K}} \hookrightarrow \mathfrak{X}_{\bar{K}}$ be denoted by $j$, then the pullback map $j^*$ is from $A^2(\mathfrak{X}_{\bar{K}})$ to $A^2(X_{\bar{K}})$. Therefore one has the natural long exact sequence on the group cohomology level of $G$ for these Galois modules,

$$0 \to \Sigma^G_K \to A^2(\mathfrak{X}_{\bar{K}})^G \to \text{im}(j^*)^G \to H^1(G, \Sigma_K) \to H^1(G, A^2(\mathfrak{X}_{\bar{K}})) \to H^1(G, \text{im}(j^*)).$$

Here $M^G$, for a $G$-module $M$, denotes the group of $G$-invariants in $M$, i.e.

$$\{m \in M | g.m = m, \forall g \in G\}.$$ 

Let us denote the groups

$$A^2(X_{\bar{K}})^G, A^2(\mathfrak{X}_{\bar{K}})^G, \text{im}(j^*)^G$$

as

$$A^2(X_{\bar{K}})(K), A^2(\mathfrak{X}_{\bar{K}})(K), \text{im}(j^*)(K)$$

respectively. Also for notational convenience we continue to denote $\Sigma^G_K$ as $\Sigma_K$. Then we have the following exact sequence

$$0 \to \text{im}(j^*)(K)/A^2(\mathfrak{X}_{\bar{K}})(K) \to H^1(G, \Sigma_K) \to H^1(G, A^2(\mathfrak{X}_{\bar{K}})) \to H^1(G, \text{im}(j^*)).$$

Let $v$ be a place of $K$ and $K_v$ be the completion of $K$ at $v$. Let $\bar{K}_v$ be the algebraic closure of $K_v$ and we embed $\bar{K}$ into $\bar{K}_v$. This embedding gives an injection of $\text{Gal}(\bar{K}_v/K_v) = G_v$ into $\text{Gal}(\bar{K}/K) = G$ and
consequently a homomorphism (considering the Galois cohomology)

\[ H^1(G, \Sigma_K) \to H^1(G_v, A^2(X_{\bar{\mathcal{O}}_{K_v}[1/6N]})) . \]

Again for notational convenience we write \( A^2(X_{\bar{\mathcal{O}}_{K_v}[1/6N]}) \) in the above as \( A^2(X_{\bar{K}_v}) \). Then we have the following commutative diagrams:

\[
\begin{array}{cccc}
\text{im}(j^*)(K)/A^2(X_{\bar{K}})(K) & \to & H^1(G, \Sigma_{\bar{K}}) & \to & H^1(G, A^2(X_{\bar{K}})) & \to & H^1(G, \text{im}(j^*)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{im}(j^*)(K_v)/A^2(X_{\bar{K}_v})(K_v) & \to & H^1(G_v, \Sigma_{\bar{K}_v}) & \to & H^1(G_v, A^2(X_{\bar{K}_v})) & \to & H^1(G_v, \text{im}(j^*_v))
\end{array}
\]

Let us now focus on

\[ H^1(G, \Sigma_K) \to \prod_v H^1(G_v, A^2(X_{\bar{K}_v})) \]

and consider the sequence of \( n \)-torsion subgroups of \( \Sigma_{\bar{K}}, A^2(X_{\bar{K}}), A^2(X_{\bar{K}}) \) given by:

\[ 0 \to \Sigma_K[n] \to A^2(X_{\bar{K}})[n] \to A^2(X_{\bar{K}})[n] . \]

Here \( A[n] \) for an abelian group \( A \), denotes the group of \( n \)-torsions of \( A \). Then considering the above groups as \( G \)-modules we have a homomorphism at the level of Galois cohomology given by:

\[ H^1(G, \Sigma_K[n]) \to \prod_v H^1(G_v, A^2(X_{\bar{K}_v}))[n] . \]

**Definition 2.1.** The kernel of this map is defined to be the \( n \)-Selmer group associated to the restriction homomorphism \( A^2(X_{\bar{K}}) \to A^2(X_{\bar{K}}) \), at the level of \( n \)-torsions in the group of algebraically trivial codimension 2-cycles and it is denoted by \( S^n(\Sigma_{\bar{K}}) \).

Let \( Alb(X_{\bar{K}}) \) be the Albanese variety such that there exists a natural (universal) homomorphism of abelian groups from \( A^2(X_{\bar{K}}) \) to \( Alb(X_{\bar{K}}) \). Now \( X_{\bar{K}} \) is an abelian variety \( Alb(X_{\bar{K}}) \cong X_{\bar{K}} \). Since the following argument is more general in nature, that is, it works for any smooth projective \( X_{\bar{K}} \) and for its albanese variety \( Alb(X_{\bar{K}}) \), provided the kernel of

\[ j^*: A^2(X_{\bar{K}})[n](K) \to A^2(X_{\bar{K}})[n](K) \]

is finite, we do not use the isomorphism \( Alb(X_{\bar{K}}) \cong X_{\bar{K}} \). Specifically this is required to prove the analogous result as stated in Remark 2.5.
Let's consider the commutative diagram:

\[
\begin{array}{ccc}
H^1(G, \text{Alb}(X_{\bar{K}})[n]) & \rightarrow & \prod_v H^1(G_v, \text{Alb}(X_{\bar{K}_v})[n]) \\
\downarrow & & \downarrow \\
H^1(G, A^2(X_{\bar{K}})[n]) & \rightarrow & \prod_v H^1(G_v, A^2(X_{\bar{K}_v})[n])
\end{array}
\]

Now by Roitman's theorem [R2], the groups $\text{Alb}(X_{\bar{K}})[n]$ and $A^2(X_{\bar{K}})[n]$ are isomorphic as Galois modules and therefore the group cohomologies are isomorphic. Thus the left vertical arrow in the above diagram is an isomorphism. Let

\[
S^n(\text{Alb}(X_{\bar{K}})/K) := \ker(H^1(G_K, \text{Alb}(X_{\bar{K}})[n]) \rightarrow \prod_v H^1(G_{K_v}, \text{Alb}(X_{\bar{K}_v})[n]))
\]

here $v$ varies over all finite places of $K$. Similarly

\[
S^n(A^2(X_{\bar{K}})/K) := \ker(H^1(G_K, A^2(X_{\bar{K}})[n]) \rightarrow \prod_v H^1(G_{K_v}, A^2(X_{\bar{K}_v})[n])).
\]

If we take an element in $S^n(\text{Alb}(X_{\bar{K}})/K)$, then by the commutativity of the above diagram, the image of the element under the left vertical homomorphism is in $S^n(A^2(X_{\bar{K}})/K)$. Now we prove our main result which has already been stated in the introduction.

**Theorem 2.2.** The group $S^{6N}(\Sigma_{\bar{K}})$ is a colimit of the unramified cohomology groups

\[
H^1(G, \Sigma_{\bar{K}}[6N]) \rightarrow H^1(I_v, \Sigma_v[6N])
\]

here $G$ is the Galois group $\text{Gal}(\bar{K}/L)$ for a finite extension $L$ of $K$ and $I_v$ is the inertia subgroup of the Galois group $G$ corresponding to a finite place $v$ and hence

\[
\text{im}(j^*[6N])(K)/A^2(X_{\bar{K}})[6N](K)
\]

is a colimit of unramified cohomology groups.
Proof. Let $n$ be a positive integer. Let us consider the diagram

$$
\begin{array}{ccc}
H^1(G, \Sigma \bar{K}[n]) & \longrightarrow & \prod_v H^1(G_v, \Sigma_{K_v}[n]) \\
\downarrow & & \downarrow \\
H^1(G, A^2(\mathcal{X}_\bar{K})[n]) & \longrightarrow & \prod_v H^1(G_v, A^2(\mathcal{X}_{K_v})[n])
\end{array}
$$

Suppose that some element is there in $S^n(\Sigma_{K/K})$.

Consider the following commutative squares:

$$
\begin{array}{ccc}
\Sigma_{K_v}/n\Sigma_{K_v} & \longrightarrow & H^1(G_v, \Sigma_{K_v}[n]) \\
\downarrow & & \downarrow \\
A^2(\mathcal{X}_{K_{v}})(K_{v})/nA^2(\mathcal{X}_{K_{v}})(K_{v}) & \longrightarrow & H^1(G_v, A^2(\mathcal{X}_{K_{v}})[n])
\end{array}
$$

Let $\eta$ be in the kernel of $H^1(G, \Sigma \bar{K}[n]) \to H^1(G_v, A^2(\mathcal{X}_{K_v}))[n]$.

Then it follows by the exactness of the sequence, induced by the long exact sequence corresponding to the short exact sequence of Galois modules given as:

$$
\begin{array}{c}
0 \to \Sigma_K[n] \to A^2(\mathcal{X}_{K_v}) \xrightarrow{n} A^2(\mathcal{X}_{K_v}) \to 0 \\
A^2(\mathcal{X}_{K_v})(K_v)/nA^2(\mathcal{X}_{K_v})(K_v) \to H^1(G_v, A^2(\mathcal{X}_{K_v})[n]) \to H^1(G_v, A^2(\mathcal{X}_{K_v}))[n]
\end{array}
$$

that there exists an element $z$ in $A^2(\mathcal{X}_{K_v})(K_v)$ such that

$$
\phi(\eta)(\sigma) = \sigma.z - z
$$

for all $\sigma$ in $G_v$. Here $\phi$ is from $Z^1(G, \Sigma \bar{K}[n])$ to $Z^1(G_v, A^2(\mathcal{X}_{K_v})[n])$ and $\eta$ is a co-cycle such that it’s cohomology class is in the kernel the homomorphism induced by $\phi$ at the level of cohomology. For simplicity as before we denote $\mathcal{X}_{K_v}, \Sigma_{K_v}$ by $\mathcal{X}_v, \Sigma_v$ respectively. In particular for all $\sigma$ in the inertia group $I_v$, we have

$$
\phi(\eta)(\sigma) = \sigma.z - z
$$

Let $v$ be a finite place such that $v$ does not divide $n$ and $Alb(X_{\bar{K}}), X_{\bar{K}}$ have good reduction at $v$. We consider the specialization homomorphism from $A^2(X_v)$ to $A^2(X'_v)$, where $X'_v$ is the reduction of $X_v$ at $v$. 
Then it follows that the image of
\[ \sigma.z - z \]
in \( A^2(X_v) \) goes to zero under the specialization homomorphism for all \( \sigma \) in \( I_v \). But on the other hand
\[ \sigma.z - z \]
is an \( n \)-torsion for each \( \sigma \) in \( \mathcal{G}_v \) (as \( \eta \) is an \( n \)-torsion), so by Roitman’s theorem on torsion, the image of the element \( \sigma.z - z \) in \( A^2(X_v) \) corresponds to an \( n \)-torsion on \( Alb(X_v) \). By the previous argument this \( n \)-torsion on \( Alb(X_v) \) is mapped to zero under \( Alb(X_v) \rightarrow Alb(X'_v) \).

But we know that the \( n \)-torsions of \( Alb(X_v) \) are embedded in \( Alb(X'_v) \) (for \( v \) which does not divide \( n \), this follows from the theory of formal groups over \( v \)-adic numbers). Therefore this \( n \)-torsion on \( A^2(X_v) \) is zero (this is because of the injectivity of the albanese map on \( n \)-torsions) and consequently
\[ \sigma.z - z \in \Sigma_v[n] \]
for all \( \sigma \in I_v \) and
\[ \sigma.z - z = 0 \]
in \( \Sigma_v[n] \), for all \( \sigma \in I_v \) (where \( v \) does not divide \( n \)).

This implies \( S^n(\Sigma_{\overline{K}}) \) consists of elements which are unramified for all but finitely many places \( v \), i.e., the image of the elements in \( S^n(\Sigma_{\overline{K}}) \) under the map
\[ H^1(G, \Sigma_{\overline{K}}[n]) \rightarrow H^1(I_v, \Sigma_v[n]) \]
is zero for all but finitely many places \( v \).

Hence the following variance of lemma [Sil][lemma 4.3, chapter X] tells us that \( S^{6N}(\Sigma_{\overline{K}}) \) is a colimit of this unramified cohomology groups.

**Lemma 2.3.** [Sil][Lemma 4.3, Chapter 10] Let \( L \) be a finite extension of the number field \( K \). Let \( M \) be the finite \( G = \text{Gal}(\overline{K}/L) \) module \( \Sigma_L \) and \( S \) be a set of finitely many places in \( L \). Consider
\[ H^1(G, M; S) \]
consisting of all elements \( \eta \) in \( H^1(G, M) \), which are unramified outside \( S \), that is in the kernel
\[ \ker(H^1(G, M) \rightarrow \prod_{v \notin S} H^1(I_v, M_v)) \]
here $M_v = \Sigma_v$, for a place in $L$. Then $H^1(G, M; S)$ is finite.

Thus this result, applied to $M = \Sigma_L[6N]$ for any finite extension $L$ over $K$ (is finite by Theorem 1.1 in [CTR]), we have that $S^{6N}(\Sigma_L)$ is finite. Also observe that

$$\Sigma_K = \cup_{K \subset L} \Sigma_L,$$

$$G = \cup_{K \subset L} \text{Gal}(\bar{K}/L).$$

Therefore $H^1(G, \Sigma_K[6N])$ is isomorphic to the colimit of

$$H^1(\text{Gal}(\bar{K}/L), \Sigma_L[6N]).$$

Hence $S^{6N}(\Sigma_K)$ is isomorphic to the colimit of $S^{6N}(\Sigma_L)$, all of which are finite unramified cohomology groups. Consequently

$$\text{im}(j^*[6N])(K)/A^2(X)[6N](K)$$

is a colimit of unramified cohomology groups. The proof actually follows from the finiteness of $S^{6N}(\Sigma_L)$.

Remark 2.4. In the previous theorem 2.2, it is interesting to see whether the groups $S^{6N}(\Sigma_L)$ are subgroups of the group $S^{6N}(\Sigma_K)$.

Remark 2.5. The analogue of Mildenhall’s result was proved for a Fermat quartic surface in [Ot]. Thus for the Fermat quartic surface too, Theorem 2.2 is true.

Now by lemma 3.3 in [CTR], if $\mathcal{O}_L[1/f]$ is such that $X_L := X_{\mathcal{O}_L[1/f]}$ is smooth and the following conditions are true:

$$H^2(X_{L,p}, \mathcal{O}_{X_{L,p}}) = 0 \text{ for all } p \in \text{Spec}(\mathcal{O}_L[1/f]),$$

$$X(K) \neq \emptyset,$$

$$\text{Pic}(X_L) \rightarrow \text{Pic}(X_{\bar{K}}) \rightarrow \text{NS}(X_{\bar{K}})$$

is surjective,

then for a finite extension $L$ of $K$, $A^2(X_L) \rightarrow A^2(X_L)$ has finite kernel. This leads us to prove:

Theorem 2.6. Under the above conditions

$$\text{im}(j^*)(K)/A^2(X_{\bar{K}})(K)$$

is torsion and is described by a colimit of unramified cohomology groups

$$H^1(G, \Sigma_K[n]) \rightarrow H^1(I_v, \Sigma_v[n]).$$
Proof. The proof goes verbatim as Theorem 2.2.

Remark 2.7. For a surface with geometric genus and irregularity zero, such that the above conditions as in Theorem 2.2 are satisfied, one has that

\[ \text{im}(j^*)(K) / A^2(X_K)(K) \]

is torsion. Therefore tensoring with \( \mathbb{Q} \) gives

\[ A^2(X_K)(K) \otimes \mathbb{Q} \rightarrow \text{im}(j^*)(K) \otimes \mathbb{Q} \]

is surjective. Therefore to prove the triviality for \( \text{im}(j^*)(K) \otimes \mathbb{Q} \), it is enough to prove that

\[ A^2(X_K)(K) \otimes \mathbb{Q} \]

is trivial. This gives some information about the structure of of \( A^2(X_K) \).

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