ROBIN PROBLEMS FOR THE $p$–LAPLACIAN
WITH GRADIENT DEPENDENCE

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Dedicated to Vicentiu, on the occasion of his 60th birthday,
with sincere friendship and esteem

Abstract. We consider a nonlinear elliptic equation with Robin boundary condition driven by the $p$–Laplacian and with a reaction term which depends also on the gradient. By using a topological approach based on the Leray-Schauder alternative principle, we show the existence of a smooth solution.

1. Introduction. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$–boundary $\partial \Omega$. In this paper, we study the following nonlinear Robin problem with dependence on the gradient

$$
\begin{cases}
-\Delta_p u(z) = f(z, u(z), Du(z)), & \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0, & \text{on } \partial \Omega.
\end{cases}
$$

In this problem, $p \in (1, \infty)$ and $\Delta_p$ denotes the usual $p$–Laplace differential operator defined by

$$
\Delta_p u = \text{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega).
$$

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The reaction term \( f(z,x,y) \) is a Carathéodory function (that is, for all \((x,y) \in \mathbb{R} \times \mathbb{R}^N \) the function \( z \in \mathbb{R} \mapsto f(z,x,y) \) is measurable, and for a.a. \( z \in \Omega \) the map \((x,y) \mapsto f(z,x,y) \) is continuous). Note that the reaction term depends also on the gradient of \( u \) (convection term). This fact precludes the use of variational methods directly on problem (1). Finally, in the boundary condition \( \frac{\partial u}{\partial n} \) denotes the conormal derivative defined by extension of the map \( u \in C^1(\Omega) \mapsto |Du|^{p-2} \frac{\partial u}{\partial n}, \) \( n(\cdot) \) being the outward unit normal on \( \partial \Omega \).

Using topological methods and, more precisely, the Leray-Schauder alternative principle, we show the existence of a smooth solution for problem (1), under general conditions on the convection term. The main hypothesis is that, asymptotically as \( x \to \pm \infty \), the quotient \( \frac{f(z,x,y)}{|x|^{p-2}x} \) stays below the principal eigenvalue \( \hat{\lambda}_1 \) uniformly in \( y \in \mathbb{R}^N \) in the measure sense, in the sense that we allow only partial interaction with \( \hat{\lambda}_1 \) (nonuniform nonresonance), see condition \( H(f,\text{ii}) \) below.

Elliptic problems with a reaction term depending on the gradient have been studied using a variety of methods. Most of the works deal with the Dirichlet problem. In this setting, we mention the papers of de Figueiredo - Girardi - Matzeu [2], Girardi - Matzeu [5] (semilinear equations) and Ruiz [13], Faraci - Motreanu - Puglisi [1], Huy - Quan - Khanh [6] (nonlinear equations driven by the \( p \)-Laplacian). For the Neumann problem, we have the recent works of Gasinski - Papageorgiou [4] and Papageorgiou - Radulescu - Repovs [12]. We remark that in these works the differential operator depends strongly on both \( u \) and \( Du \).

2. Mathematical background - hypotheses. In this section we recall the main mathematical tools which we will use in the analysis of problem (1) and we state our hypotheses on the reaction term \( f(z,x,y) \).

Let \( X \) be a reflexive Banach space and \( X^* \) its topological dual. By \( \langle \cdot, \cdot \rangle \) we denote the duality brackets for the pair \( (X,X^*) \). Given a map \( A : X \to 2^{X^*} \), the graph of \( A \) is the set
\[
GrA = \{(x,x^*) \in X \times X^* : x^* \in A(x)\}.
\]
The domain of \( A \) is the set
\[
D(A) = \{x \in X : A(x) \neq \emptyset\}.
\]
We say that \( A(\cdot) \) is monotone if
\[
\langle x^* - u^*, x - u \rangle \geq 0 \quad \text{for all } (x,x^*), (u,u^*) \in GrA.
\]
The map \( A(\cdot) \) is maximal monotone, if it is monotone and its graph is maximal with respect to inclusion among the graphs of monotone maps, that is,
\[
\langle u^* - x^*, u - x \rangle \geq 0 \quad \text{for all } (x,x^*) \in GrA \Rightarrow (u,u^*) \in GrA.
\]
The map \( A(\cdot) \) is coercive if
\[
\lim_{x \in D(A) \atop \|x\| \to \infty} \inf \left\{ \|x^*\| : x^* \in A(x) \right\} = +\infty.
\]
From Gasinski - Papageorgiou [3], p. 319, we have:

**Proposition 1.** If $A : X \to 2^{X^*}$ is maximal monotone and coercive, then $A$ is surjective.

Let $V$ and $Y$ be two Banach spaces. A map $h : V \to Y$ is said to be compact, if $h(\cdot)$ is continuous and maps bounded subsets of $V$ into relatively compact subsets of $Y$.

As we already mentioned, our approach being topological, we use the so called Leray - Schauder alternative principle (see Gasinski - Papageorgiou [3], p. 827).

**Theorem 2.1.** If $Y$ is a Banach space, $h : Y \to Y$ is a compact map and

$$K(h) = \{y \in Y : y = th(y) \text{ for some } t \in (0,1)\},$$

then either $K(h)$ is unbounded or $h(\cdot)$ has a fixed point.

The analysis of problem (1) uses the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^q(\partial \Omega)$, $1 \leq q \leq \infty$. By $\| \cdot \|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left\{ \|u\|_p^p + \|Du\|_p^p \right\}^{\frac{1}{p}}$$

for all $u \in W^{1,p}(\Omega)$.

Recall that the Banach space $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$. On $\partial \Omega$ we consider the $(N - 1)$–dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue spaces $L^q(\partial \Omega)$, $1 \leq q \leq \infty$. The theory of Sobolev spaces says that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial \Omega)$ known as the trace map such that

$$\gamma_0(u) = u|_{\partial \Omega} \quad \text{for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map assigns boundary values to every Sobolev function. We know that $\gamma_0(\cdot)$ is compact into $L^q(\partial \Omega)$ for all $q \in \left[1, \frac{p(N-1)}{N-p} \right]$ if $N > p$ and into $L^q(\partial \Omega)$ for all $q \geq 1$ if $N \leq p$. Moreover, we have

$$\text{Ker} \gamma_0 = W^{1,p}_0(\Omega) \quad \text{and} \quad \text{Im} \gamma_0 = W^{\frac{p}{p-1},p}(\partial \Omega)$$

with $\frac{1}{p} + \frac{1}{p'} = 1$.

In what follows for notational simplicity we drop the use of the trace map $\gamma_0$, and all restrictions of Sobolev functions on $\partial \Omega$ are understood in the sense of traces.

Our hypotheses on the boundary coefficient $\beta(\cdot)$ are the following:

$$H(\beta) : \beta \in C^{0,\alpha}(\partial \Omega) \text{ with } \alpha \in (0,1) \text{ and } \beta(z) \geq 0 \text{ for all } z \in \partial \Omega.$$

**Remark 1.** Of course, the case $\beta \equiv 0$ corresponds to the Neumann problem.

Before, considering problem (1), we start considering the following nonlinear eigenvalue problem

$$\begin{cases}
-\Delta_p u(z) = \lambda |u(z)|^{p-2}u(z), & \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0, & \text{on } \partial \Omega.
\end{cases}$$

(2)

We say that $\lambda \in \mathbb{R}$ is an *eigenvalue*, if problem (2) admits a nontrivial solution $\hat{u} \in W^{1,p}(\Omega)$, known as an *eigenfunction* corresponding to $\lambda$. The nonlinear regularity theory (see Lieberman [7, Theorem 2]) implies that $\hat{u} \in C^1(\Omega)$. We recall that
there is a smallest eigenvalue \( \lambda_1 \) with the following properties (see Papageorgiou - Radulescu [10]):

- \( \lambda_1 \geq 0 \). More precisely, if \( \beta \equiv 0 \) (Neumann eigenvalue problem), then \( \lambda_1 = 0 \), while if \( \beta \not\equiv 0 \), then \( \lambda_1 > 0 \).
- \( \lambda_1 \) is isolated in the spectrum \( \hat{\sigma}(p) \) of (2) (that is, there exists \( \epsilon > 0 \) such that \((\lambda_1, \lambda_1 + \epsilon) \cap \hat{\sigma}(p) = \emptyset\)).
- \( \lambda_1 \) is simple (that is, if \( \hat{u}, \hat{v} \) are eigenfunctions corresponding to \( \lambda_1 \), then \( \hat{u} = \xi \hat{v} \) for some \( \xi \in \mathbb{R} \setminus \{0\} \)).
- if \( \xi(u) = \|Du\|_p^p + \int_{\partial \Omega} \beta(z)|u|^p d\sigma \) for all \( u \in W^{1,p}(\Omega) \), then

\[
\hat{\lambda}_1 = \inf \left\{ \frac{\xi(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0 \right\}.
\]  

(3)

Hence, in (3) the infimum is realized on the one-dimensional eigenspace corresponding to \( \hat{\lambda}_1 \). The above properties imply that the elements of this eigenspace do not change sign and lead to the following lemma (see [9, Lemma 4.11]).

**Lemma 2.2.** If \( \Theta \in L^\infty(\Omega) \) satisfies \( \Theta(z) \leq \hat{\lambda}_1 \) for a.a. \( z \in \Omega \) and \( \Theta \not\equiv \hat{\lambda}_1 \), then there exists \( c_0 > 0 \) such that

\[
c_0\|u\|^p \leq \xi(u) - \int_{\Omega} \Theta(z)|u|^p dz \quad \text{for all } u \in W^{1,p}(\Omega).
\]

**Definition 2.3.** We say that a function \( f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is \( L^\infty \)-locally Hölder continuous, if for every \( r > 0 \), there exists \( \eta_r \in L^\infty(\Omega) \) and \( \lambda \in (0,1] \) such that

\[
|f(z, x, y) - f(z, x', y')| \leq \eta_r(z)||x - x'|^\lambda + ||y - y'|^\lambda
\]

for a.a. \( z \in \Omega \), all \( |x|, |y| \leq r \).

Now, we can introduce the hypotheses on the reaction term \( f(z, x, y) \).

**H(f) \** \( f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \) is an \( L^\infty \)-locally Hölder continuous function such that

1. \( |f(z, x, y)| \leq \gamma(z)[1 + |y|^{p-1} + |y|^{p-1}] \) for a.a. \( z \in \Omega \), all \( x \in \mathbb{R} \), all \( y \in \mathbb{R}^N \) with \( \gamma \in L^\infty(\Omega) \)
2. there exists a function \( \Theta \in L^\infty(\Omega) \) such that \( \Theta(z) \leq \hat{\lambda}_1 \) for a.a. \( z \in \Omega \), \( \Theta \not\equiv \hat{\lambda}_1 \),

and

\[
\limsup_{x \to \pm \infty} \frac{f(z, x, y)}{|x|^{p-2}x} \leq \Theta(z) \quad \text{uniformly for a.a. } z \in \Omega, \text{ all } y \in \mathbb{R}^N.
\]

**Example 1.** The following function satisfies hypotheses \( H(f) \). For the sake of simplicity we drop the \( z \)-dependence

\[
f(x) = \Theta|x|^{p-2}x + g(x)|y|^{p-1}
\]

with \( \Theta < \hat{\lambda}_1 \) and \( g \in C(\mathbb{R}) \) is such that \( \lim_{x \to \pm \infty} \frac{g(x)}{|x|^{p-2}x} = 0 \).
3. Existence of solutions. In this section we shall prove the existence of one solution to problem (1).

For this, let \( g \in L^\infty(\Omega) \) and \( \hat{\xi} > \|\Theta\|_\infty \). We start by considering the following auxiliary Robin problem

\[
\begin{align*}
-\Delta_p u(z) + \hat{\xi}|u(z)|^{p-2}u(z) &= g(z), & \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u &= 0, & \text{on } \partial \Omega.
\end{align*}
\] (4)

Proposition 2. If Hypothesis \( H(\beta) \) holds, then for every \( g \in L^\infty(\Omega) \) problem (4) admits a unique solution \( E(g) \in C^1(\overline{\Omega}) \), and the map \( E : L^\infty(\Omega) \to C^1(\overline{\Omega}) \) is compact.

Proof. First we show the existence and uniqueness of a solution for problem (4). To this end let \( V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^* \) be defined by

\[
\langle V(u), h \rangle = \int_\Omega |Du|^{p-2}(Du,Dh)_{\mathbb{R}^N} \, dz + \hat{\xi} \int_\Omega |u|^{p-2}uh \, dz + \int_{\partial \Omega} \beta(z)|u|^{p-2}uh \, d\sigma
\]

for all \( u, h \in W^{1,p}(\Omega) \).

Clearly \( V(\cdot) \) is continuous, monotone, hence maximal monotone, too (see Gasinski - Papageorgiou [3], p. 310). Moreover, for all \( u \in W^{1,p}(\Omega) \), we have

\[
\langle V(u), u \rangle \geq \|Du\|_p^p + \hat{\xi}\|u\|_p^p \geq c_1\|u\|^p
\]

with \( c_1 = \min\{1, \hat{\xi}\} \) (see hypothesis \( H(\beta) \)). Hence \( V(\cdot) \) is coercive.

By Proposition 1, we infer that \( V(\cdot) \) is surjective. So, given \( g \in L^\infty(\Omega) \subseteq W^{1,p}(\Omega)^* \), we can find \( \hat{u} \in W^{1,p}(\Omega) \) such that

\[
V(\hat{u}) = g \quad \text{in } W^{1,p}(\Omega)^*.
\]

Hence

\[
\int_\Omega |D\hat{u}|^{p-2}(D\hat{u},Dh)_{\mathbb{R}^N} \, dz + \hat{\xi} \int_\Omega |\hat{u}|^{p-2}\hat{u}h \, dz + \int_{\partial \Omega} \beta(z)|\hat{u}|^{p-2}\hat{u}h \, d\sigma = \int_\Omega gh \, dz
\]

for all \( h \in W^{1,p}(\Omega) \). Thus

\[
\begin{align*}
-\Delta_p \hat{u}(z) + \hat{\xi}|\hat{u}(z)|^{p-2}\hat{u}(z) &= g(z), & \text{for a.a. } z \in \Omega, \\
\frac{\partial \hat{u}}{\partial n_p} + \beta(z)|\hat{u}|^{p-2}\hat{u} &= 0, & \text{on } \partial \Omega,
\end{align*}
\] (5)

see Papageorgiou - Radulescu [10].

From (5) and Papageorgiou - Radulescu [11], we have that

\[
\hat{u} \in L^\infty(\Omega).
\]

Then [7, Theorem 2] implies that

\[
\hat{u} \in C^1(\overline{\Omega}).
\]

Now, we show the uniqueness of this solution. So, suppose that \( \tilde{u} \in W^{1,p}(\Omega) \) is another solution of (4). As above, we show that \( \tilde{u} \in C^1(\overline{\Omega}) \). We have

\[
\langle V(u) - V(\hat{u}), u - \tilde{u} \rangle = 0,
\]

thus

\[
\int_\Omega (|D\hat{u}|^{p-2}D\hat{u} - |D\hat{u}|^{p-2}D\tilde{u},D\hat{u} - D\tilde{u})_{\mathbb{R}^N} \, dz + \hat{\xi} \int_\Omega (|\hat{u}|^{p-2}\hat{u} - |\hat{u}|^{p-2}\tilde{u}) (\hat{u} - \tilde{u}) \, dz + \int_{\partial \Omega} \beta(z)(|\hat{u}|^{p-2}\hat{u} - |\hat{u}|^{p-2}\tilde{u}) (\hat{u} - \tilde{u}) \, d\sigma = 0.
\]
Since the map $x \mapsto |x|^{p-2}x$ is strictly increasing on $\mathbb{R}$ and we have
\[
\int_\Omega |(\tilde{u}|^{p-2} \tilde{u} - |\tilde{u}||^{p-2} \tilde{u})| \, dz \leq 0
\]
(see Hypothesis $H(\beta)$), it follows that $\tilde{u} = \tilde{u}$. This proves the uniqueness of the solution $\tilde{u} \in C^1(\overline{\Omega})$.

Now, consider the solution map $E : L^\infty(\Omega) \to C^1(\overline{\Omega})$. First, we show that this map is continuous. So, suppose that $g_n \to g$ in $L^\infty(\Omega)$. Let $u_n = E(g_n)$, $n \in \mathbb{N}$, and $u = E(g)$. Then we have
\[
\int_\Omega |Du_n|^{p-2}(Du_n, Dh)_{\mathbb{R}^N} \, dz + \xi \int_\Omega |u_n|^{p-2} u_n \, dz
\]
\[+ \int_{\partial \Omega} \beta(z)|u_n|^{p-2} u_n \, d\sigma = \int_\Omega g_n h \, dz
\]
for all $h \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$.

In (6) we choose $h = u_n \in C^1(\overline{\Omega}) \subseteq W^{1,p}(\Omega)$. Then
\[
||Du_n||^p + \xi ||u_n||^p \leq c_2 ||u_n||
\]
for some $c_2 > 0$, all $n \in \mathbb{N}$. Thus
\[
\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)\text{ is bounded.}
\]
We know that for every $n \in \mathbb{N}$ we have
\[
\begin{cases}
-\Delta_p u_n + \xi |u_n(z)|^{p-2} u_n(z) = g_n(z), & \text{for a.a. } z \in \Omega, \\
\frac{\partial u_n}{\partial \nu} + \beta(z)|u_n|^{p-2} u_n = 0, & \text{on } \partial \Omega.
\end{cases}
\]
From (7), (8) and [11, Proposition 7], we deduce that there exists $c_3 > 0$ such that
\[
\|u_n\|_{\infty} \leq c_3 \quad \text{for all } n \in \mathbb{N}.
\]
Then, by applying Lieberman’s regularity result [7, Theorem 2], we can find $\alpha \in (0, 1)$ and $c_4 > 0$ such that
\[
u_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } ||u_n||_{C^{1,\alpha}(\overline{\Omega})} \leq c_4
\]
for all $n \in \mathbb{N}$. Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, from (9) we see that we can find a subsequence $\{u_{n_k}\}_{k \geq 1}$ of $\{u_n\}_{n \geq 1}$ such that
\[
u_{n_k} \to \tilde{u} \text{ in } C^1(\overline{\Omega})
\]
as $k \to +\infty$. Hence, we have
\[
\int_\Omega |Du_{n_k}|^{p-2}(Du_{n_k}, Dh)_{\mathbb{R}^N} \, dz + \xi \int_\Omega |u_{n_k}|^{p-2} u_{n_k} \, dz + \\
\int_{\partial \Omega} \beta(z)|u_{n_k}|^{p-2} u_{n_k} \, d\sigma = \int_\Omega g_{n_k} h \, dz
\]
for all $h \in W^{1,p}(\Omega)$, all $k \in \mathbb{N}$. Passing to the limit as $k \to +\infty$ in (11) and using (10), we obtain
\[
\int_\Omega |\tilde{u}|^{p-2}(D\tilde{u}, Dh)_{\mathbb{R}^N} \, dz + \xi \int_\Omega |\tilde{u}|^{p-2} \tilde{u} \, dz + \\
\int_{\partial \Omega} \beta(z)|\tilde{u}|^{p-2} \tilde{u} \, d\sigma = \int_\Omega gh \, dz
\]
for all $h \in W^{1,p}(\Omega)$. Thus $\tilde{u} = E(g) = u$.

This being true for any fixed subsequence of the original sequence $\{u_n = E(g_n)\}_{n \geq 1}$, we have that
\[
E(g_n) = u_n \to u = E(g) \text{ in } C^1(\overline{\Omega}).
\]
This implies that $E : L^\infty(\Omega) \to C^1(\bar{\Omega})$ is continuous.

Next, let $B \subseteq L^\infty(\Omega)$ be bounded. As before, invoking [7, Theorem 2], we can find $\alpha \in (0, 1)$ and $c_5 > 0$ such that

$$u \in C^{1,\alpha}(\bar{\Omega}) \quad \text{and} \quad \|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq c_5$$

for all $u \in E(B)$.

The compact embedding of $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$ implies that $E(B) \subseteq C^1(\bar{\Omega})$ is relatively compact. So, we conclude that $E$ is compact.

Now, let $\hat{N} : C^1(\bar{\Omega}) \to L^\infty(\Omega)$ be defined by

$$\hat{N}(u)(\cdot) = f(\cdot, u(\cdot)) + \xi|u(\cdot)|^{p-2}u(\cdot)$$

for all $u \in C^1(\bar{\Omega})$.

**Proposition 3.** If Hypothesis $H(f).i)$ holds, then $\hat{N} : C^1(\bar{\Omega}) \to L^\infty(\Omega)$ is continuous and bounded.

**Proof.** Suppose that $u_n \to u$ in $C^1(\bar{\Omega})$. So, we can find $r > 0$ such that

$$\|u_n\|_{C^1(\bar{\Omega})} \leq r \quad \text{for all } n \in \mathbb{N}.$$

Recall that $f(z, \cdot, \cdot)$ is $L^\infty$–locally Hölder continuous. So, we have

$$|f(z, u_n(z), Du_n(z)) - f(z, u(z), Du(z))| \leq \eta_r|u_n(z) - u(z)|^\lambda + |Du_n(z) - Du(z)|^\lambda$$

for a.a. $z \in \Omega$, with $\eta_r \in L^\infty(\Omega)$, $0 < \lambda \leq 1$.

Thus $\hat{N}(u_n) \to \hat{N}(u)$ in $L^\infty(\Omega)$ as $n \to \infty$ and $\hat{N}(\cdot)$ is continuous.

Finally, hypothesis $H(f).i)$ implies that $\hat{N}(\cdot)$ is bounded (that is, maps bounded sets to bounded sets).

Propositions 2 and 3 imply that $E \circ \hat{N} : C^1(\bar{\Omega}) \to C^1(\bar{\Omega})$ is compact. Let

$$\mathcal{K} = \{u \in C^1(\bar{\Omega}) : u = t(E \circ \hat{N})(u), \text{ with } 0 < t < 1\}.$$

**Proposition 4.** If Hypotheses $H(\beta)$, $H(f)$ hold, then $\mathcal{K} \subseteq C^1(\bar{\Omega})$ is bounded.

**Proof.** Let $u \in \mathcal{K}$. Then

$$u = t(E \circ \hat{N})(u)$$

for some $t \in (0, 1)$. Hence $\frac{1}{t}u = E(\hat{N}(u))$. So, we have

$$\int_\Omega \left| D \left( \frac{1}{t}u \right) \right|^{p-2} D \left( \frac{1}{t}u \right), Dh \right|_{\mathbb{R}^N} dz + \frac{\xi}{t^{p-1}} \int_\Omega |u|^{p-2}uh dz + \frac{1}{t^{p-1}} \int_{\partial \Omega} \beta(z)|u|^{p-2}uh d\sigma = \int_\Omega [f(z, u, Du) + \hat{\xi}|u|^{p-2}u]h dz$$

for all $h \in W^{1,p}(\Omega)$. Thus

$$\int_\Omega |Du|^{p-2} (Du, Dh)_{\mathbb{R}^N} dz + \xi \int_\Omega |u|^{p-2}uh dz + \int_{\partial \Omega} \beta(z)|u|^{p-2}uh d\sigma = t^{p-1} \int_\Omega [f(z, u, Du) + \hat{\xi}|u|^{p-2}u]h dz$$

for all $h \in W^{1,p}(\Omega)$. By Hypothesis $H(f).ii)$, we see that given $\epsilon > 0$, we can find $M = M(\epsilon) > 0$ such that

$$f(z, x, y)x \leq (\Theta(z) + \epsilon)|x|^p$$

for a.a. $z \in \Omega$, all $|x| \geq M$, all $y \in \mathbb{R}^N$. (13)
On the other hand Hypothesis $H(f,i)$ implies that there exists $c_6 > 0$ such that
\[ f(z,x,y)x \leq c_6[1 + |y|^{p-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } |x| < M, \text{ all } y \in \mathbb{R}^N. \tag{14} \]
From (13), (14) and since $\Theta \in L^\infty(\Omega)$ it follows that
\[ f(z,x,y)x \leq (\Theta(z) + \epsilon) |x|^p + c_7 |y|^{p-1} + c_7 \tag{15} \]
for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $y \in \mathbb{R}^N$ and some $c_7 > 0$.

In (12) we choose $\hat{h} = u \in W^{1,p}(\Omega)$ and use (15). Then
\[
\|Du\|_p^p + \hat{\xi}\|u\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma
\leq t^{p-1} \int_{\Omega} \left[ (\Theta(z) + \epsilon + \hat{\xi}) |u|^p + c_7 |Du|^{p-1} + c_7 \right] dz
\leq \int_{\Omega} \left[ (\Theta(z) + \epsilon + \hat{\xi}) |u|^p + c_7 |Du|^{p-1} + c_7 \right] dz,
\]
since $\hat{\xi} > \|\Theta\|_\infty$, and so $\Theta(z) + \epsilon + \hat{\xi} > 0$ for a.a. $z \in \Omega$, and $t \in (0,1)$. Hence
\[
\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_{\partial\Omega} \Theta(z)|u|^p dz - \epsilon \|u\|_p^p \leq c_7 \|u\|^{p-1} + c_7 |\Omega|_N,
\]
where $| \cdot |_N$ denotes the Lebesgue measure on $\mathbb{R}^N$). Together with Lemma 2.2, this implies that
\[ (c_0 - \epsilon) \|u\|_p^p \leq c_7 \|u\|^{p-1} + c_7 |\Omega|_N. \]
Choosing $\epsilon \in (0,c_0)$, we conclude that
\[ \|u\|_p^p \leq c_8 [\|u\|^{p-1} + 1] \quad \text{for some } c_8 > 0, \]
so that
\[ \{u\}_{u \in K} \subseteq W^{1,p}(\Omega) \] is bounded. \tag{16}

For every $u \in K$, we have
\[
\begin{cases}
-\Delta_p u(z) + \hat{\xi} |u(z)|^{p-2} u(z) = t^{p-1}[f(z,u(z),Du(z)) + \hat{\xi}|u(z)|^{p-2} u(z)], & \text{for a.a. } z \in \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2} u = 0, & \text{on } \partial\Omega.
\end{cases} \tag{17}
\]
(see Papageorgiou - Radulescu [10]).

From (16), (17) and [11, Proposition 7], we know that there exists $c_9 > 0$ such that
\[ \|u\|_\infty \leq c_9 \quad \text{for all } u \in K. \]
Invoking [7, Theorem 2], we can find $\alpha \in (0,1)$ and $c_{10} > 0$ such that
\[ u \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and } \|u\|_{C^{1,\alpha}(\overline{\Omega})} \leq c_{10} \]
for all $u \in K$. Hence $K \subseteq C^{1}(\overline{\Omega})$ is bounded (in fact relatively compact). \hfill \Box

At this point, we are ready for our existence theorem

**Theorem 3.1.** If Hypotheses $H(\beta), H(f)$ hold, then problem (1) admits a solution $\hat{u} \in C^{1}(\overline{\Omega})$.

**Proof.** By using Proposition 4 and Theorem 2.1 (the Leray - Schauder alternative principle), we can find $\hat{u} \in C^{1}(\overline{\Omega})$ such that
\[ \hat{u} = (E \circ \hat{N})(\hat{u}). \]
Clearly $\hat{u} \in C^{1}(\overline{\Omega})$ is a solution of (1), and the theorem is proved. \hfill \Box
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