ON INTRINSICALLY KNOTTED OR COMPLETELY 3-LINKED GRAPHS

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Abstract. We say that a graph is intrinsically knotted or completely 3-linked if every embedding of the graph into the 3-sphere contains a nontrivial knot or a 3-component link any of whose 2-component sublink is nonsplittable. We show that a graph obtained from the complete graph on seven vertices by a finite sequence of \( \Delta Y \)-exchanges and \( Y \Delta \)-exchanges is a minor-minimal intrinsically knotted or completely 3-linked graph.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let \( f \) be an embedding of a finite graph \( G \) into the 3-sphere. Then \( f \) is called a spatial embedding of \( G \) and \( f(G) \) is called a spatial graph. We denote the set of all spatial embeddings of \( G \) by \( \text{SE}(G) \). We call a subgraph \( \gamma \) of \( G \) which is homeomorphic to the circle a cycle of \( G \). For a positive integer \( n \), \( \Gamma^{(n)}(G) \) denotes the set of all cycles of \( G \) if \( n = 1 \) and the set of all unions of mutually disjoint \( n \) cycles of \( G \) if \( n \geq 2 \). In particular, we denote \( \Gamma^{(1)}(G) \) by \( \Gamma(G) \) simply.

A graph \( G \) is said to be intrinsically linked (IL) if for every spatial embedding \( f \) of \( G \), \( f(G) \) contains a nonsplittable 2-component link. Conway-Gordon [1] and Sachs [20] showed that \( K_6 \) is IL, where \( K_m \) denotes the complete graph on \( m \) vertices. Moreover, IL graphs have been completely characterized as follows. For a graph \( G \) and an edge \( e \) of \( G \), we denote the subgraph \( G \setminus e \) by \( G - e \). Let \( e = uw \) is an edge of \( G \) which is not a loop. We call the graph which is obtained from \( G - e \) by identifying the end vertices \( u \) and \( v \) the edge contraction of \( G \) along \( e \) and denote it by \( G/e \). A graph \( H \) is called a minor of a graph \( G \) if there exists a subgraph \( G' \) of \( G \) and the edges \( e_1, e_2, \ldots, e_m \) of \( G' \) such that \( H \) is obtained from \( G' \) by a sequence of edge contractions along \( e_1, e_2, \ldots, e_m \). A minor \( H \) of \( G \) is called a proper minor if \( H \) does not equal \( G \). Let \( \mathcal{P} \) be a property for graphs which is closed under minor reductions; that is, for any graph \( G \) which does not have \( \mathcal{P} \), all minors of \( G \) also do not have \( \mathcal{P} \). A graph \( G \) is said to be minor-minimal with respect to \( \mathcal{P} \) if \( G \) has \( \mathcal{P} \) but all proper minors of \( G \) do not have \( \mathcal{P} \). Note that \( G \) has \( \mathcal{P} \) if and only if \( G \) has a minor-minimal graph with respect to \( \mathcal{P} \) as a minor. By the famous theorem of Robertson-Seymour [13], there are finitely many minor-minimal graphs with respect
to $P$. Nešetřil-Thomas [16] showed that IL is closed under minor reductions, and Robertson-Seymour-Thomas [19] showed that the set of all minor-minimal graphs with respect to IL equals the Petersen family which is the set of all graphs obtained from $K_6$ by a finite sequence of $\triangle Y$-exchanges and $Y \triangle$-exchanges. Here a $\triangle Y$-exchange is an operation to obtain a new graph $G_Y$ from a graph $G_\triangle$ by removing all edges of a cycle $\triangle$ of $G_\triangle$ with exactly three edges $uv, vw$ and $wu$, and adding a new vertex $x$ and connecting it to each of the vertices $u, v$ and $w$ as illustrated in Fig. 1.1 (we often denote $ux \cup vx \cup wx$ by $Y$). A $Y \triangle$-exchange is the reverse of this operation. This family contains exactly seven graphs as illustrated in Fig. 1.2 where $G \rightarrow G'$ means that $G'$ can be obtained from $G$ by a single $\triangle Y$-exchange. Note that $P_{10}$ is isomorphic to the Petersen graph. We remark here that if $G_\triangle$ is IL then $G_Y$ is also IL [15], and if $G_Y$ is IL then $G_\triangle$ is also IL [19]. Namely $\triangle Y$ and $Y \triangle$-exchanges preserve IL.

On the other hand, a graph $G$ is said to be intrinsically knotted (IK) if for every spatial embedding $f$ of $G$, $f(G)$ contains a nontrivial knot. Conway-Gordon [1] showed that $K_7$ is IK. Fellows and Langston [2] showed that IK is closed under minor reductions, and Motwani-Raghunathan-Saran [15] showed that $K_7$ is a minor-minimal IK graph. Although additional minor-minimal IK graphs are known by Kohara-Suzuki [13] and Foisy [6], [7], IK graphs have not been completely characterized yet. We remark here that if $G_\triangle$ is IK then $G_Y$ is also IK [15], but if $G_Y$ is IK then $G_\triangle$ may not always be IK. Namely the $Y \triangle$-exchange does not preserve IK in general. Actually Flapan-Naimi [3] exhibited that there exists a graph $G_{F,N}$ which is obtained from $K_7$ by five times of $\triangle Y$-exchanges and twice $Y \triangle$-exchanges such that it is not IK. We call the set of all graphs obtained from $K_7$ by a finite

![Figure 1.1](image1.png)

![Figure 1.2](image2.png)
sequence of $\triangle Y$ and $Y \triangle$-exchanges the Heawood family. This family contains exactly twenty graphs as illustrated in Fig. 1.3 where $G \to G'$ means that $G'$ can be obtained from $G$ by a single $\triangle Y$-exchange. Note that $C_{14}$ is isomorphic to the Heawood graph, see Remark 4.7.

Kohara-Suzuki showed that a graph $G$ in the Heawood family is a minor-minimal IK graph if $G$ is obtained from $K_7$ by a finite sequence of $\triangle Y$-exchanges, namely $G$ is one of fourteen graphs $K_7$, $H_8$, $H_9$,..., $H_{12}$, $F_9$, $F_{10}$, $E_{10}$, $E_{11}$ and $C_{11}, C_{12}$,..., $C_{14}$. On the other hand, $N_{10}'$ is isomorphic to $G_{FN}$, namely $N_{10}'$ is not IK. Our first purpose in this paper is to determine completely when a graph in the Heawood family is IK as follows.

**Theorem 1.1.** Let $G$ be a graph in the Heawood family. Then the following are equivalent:

1. $G$ is IK,
2. $G$ is obtained from $K_7$ by a finite sequence of $\triangle Y$-exchanges,
3. $\Gamma(G)$ is the empty set.

Namely, each of the graphs $N_9, N_{10}, N_{11}, N_{10}', N_{11}'$ and $N_{12}'$ in the Heawood family is not IK, and only these graphs in the Heawood family contain a union of mutually disjoint three cycles. Our second purpose in this paper is to show that any of the graphs in the Heawood family is a minor-minimal graph with respect to a certain kind of intrinsic nontriviality even if it is not IK. We say that a graph $G$ is intrinsically knotted or completely 3-linked (I(K or C3L)) if for every spatial embedding $f$ of $G$, $f(G)$ contains a nontrivial knot or a 3-component link any of whose 2-component sublink is nonsplittable. Note that an IK graph is I(K or C3L).

As we will show in Proposition 2.2, I(K or C3L) is closed under minor reductions. Then we have the following.

**Theorem 1.2.** All of the graphs in the Heawood family are minor-minimal I(K or C3L) graphs.

Actually, each of the graphs $N_9, N_{10}, N_{11}, N_{10}', N_{11}'$ and $N_{12}'$ in the Heawood family is not IK but I(K or C3L), and they are minor-minimal with respect to I(K or C3L).

**Remark 1.3.**

1. A graph $G$ is said to be intrinsically $n$-linked (InL) if for every spatial embedding $f$ of $G$, $f(G)$ contains a nonsplittable $n$-component link. Note that In2L coincides with IL. Let $G$ be a graph in the Heawood family which is not IK. Then we will show in Example 4.6 that there exists a spatial embedding $f$ of $G$ such that $f(G)$ does not contain a nonsplittable 3-component link. Namely $G$ is neither IK nor I3L.

2. A graph $G$ is said to be intrinsically knotted or 3-linked (I(K or 3L)) if for every spatial embedding $f$ of $G$, $f(G)$ contains a nontrivial knot or a nonsplittable 3-component link. It is clear that I(K or C3L) implies I(K or 3L), but the converse is not true. Actually in [8], although Foisy discovered an I(K or 3L) graph $G$ and exhibit a spatial embedding $f$ of $G$ such that $f(G)$ contains a nonsplittable 3-component link but does not

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1. In [10], van der Holst call the set of all graphs obtained from $K_7$ or $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ and $Y \triangle$-exchanges the Heawood family, where $K_{3,3,1,1}$ is the complete 4-partite graph on $3 + 3 + 1 + 1$ vertices.

2. We remark that one edge of $F_{10}$ in [13] Fig. 5 is wanting.
Figure 1.3.

contain a nontrivial knot, each of the nonsplittable 3-component links in $f(G)$ contains a split 2-component sublink.

The rest of this paper is organized as follows. In the next section, we show the general results about graph minors, $\Delta Y$-exchanges and spatial graphs. We prove Theorems 1.1 and 1.2 in sections 3 and 4, respectively.
2. Graph minors, $\Delta Y$-exchanges and spatial graphs

Let $H$ be a minor of a graph $G$. Then there exists a natural injection

$$\Psi^{(n)} = \Psi^{(n)}_{H,G} : \Gamma^{(n)}(H) \longrightarrow \Gamma^{(n)}(G)$$

for any positive integer $n$. In particular, we denote $\Psi^{(1)}$ by $\Psi$ simply. Let $f$ be a spatial embedding of $G$ and $e$ an edge of $G$ which is not a loop. Then by contracting $f(e)$ into one point, we obtain a spatial embedding $\psi(f)$ of $G/e$. Similarly we also can obtain a spatial embedding $\psi(f)$ of $H$ from $f$. Thus we obtain a map

$$\psi = \psi_{G,H} : \text{SE}(G) \longrightarrow \text{SE}(H).$$

Then we immediately have the following.

**Proposition 2.1.** For a spatial embedding $f$ of $G$ and an element $\lambda$ in $\Gamma^{(n)}(H)$, $\psi(f)(\lambda)$ is ambient isotopic to $f\left(\Psi^{(n)}(\lambda)\right)$. □

Now we show the following.

**Proposition 2.2.** $I(K$ or $C3L)$ is closed under minor reductions.

**Proof.** Let $G$ be a graph which is not $I(K$ or $C3L)$ and $H$ a minor of $G$. Let $f$ be a spatial embedding of $G$ which contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Then by Proposition 2.1, $\psi(f)$ also contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. This implies that $H$ is not $I(K$ or $C3L)$. □

**Remark 2.3.** Proposition 2.1 also implies that $IK$, $InL$ and $I(K$ or $3L)$ are closed under minor reductions.

Let $G_\triangle$ and $G_Y$ be two graphs such that $G_Y$ is obtained from $G_\triangle$ by a single $\Delta Y$-exchange as we said in the previous section. Let $\lambda$ be an element in $\Gamma^{(n)}(G_\triangle)$ which does not contain $\triangle$. Then there exists an element $\Phi^{(n)}(\lambda)$ in $\Gamma^{(n)}(G_Y)$ such that $\lambda \setminus \triangle = \Phi^{(n)}(\lambda) \setminus Y$. Thus we obtain a map

$$\Phi^{(n)} = \Phi^{(n)}_{G_\triangle,G_Y} : \left\{ \lambda \in \Gamma^{(n)}(G_\triangle) \mid \lambda \not\supset \triangle \right\} \longrightarrow \Gamma^{(n)}(G_Y)$$

for any positive integer $n$. In particular, we denote $\Phi^{(1)}$ by $\Phi$ simply. Note that $\Phi^{(n)}$ is surjective and the inverse image of $\lambda$ by $\Phi^{(n)}$ contains at most two elements in $\Gamma^{(n)}(G_\triangle)$ for any element $\lambda$ in $\Gamma^{(n)}(G_Y)$. Note also that the surjectivity of $\Phi^{(n)}$ implies the following.

**Proposition 2.4.** For $n \geq 2$, if $\Gamma^{(n)}(G_\triangle)$ is the empty set, then $\Gamma^{(n)}(G_Y)$ is also the empty set. □

Let $f$ be a spatial embedding of $G_Y$ and $D$ a 2-disk in the 3-sphere such that $D \cap f(G_Y) = f(Y)$ and $\partial D \cap f(G_Y) = \{ f(u), f(v), f(w) \}$. Let $\varphi(f)$ a spatial embedding of $G_\triangle$ such that $\varphi(f)(x) = f(x)$ for $x \in G_Y \setminus Y$ and $\varphi(f)(G_\triangle) = (f(G_Y) \setminus f(Y)) \cup \partial D$. Thus we obtain a map

$$\varphi = \varphi_{G_Y,G_\triangle} : \text{SE}(G_Y) \longrightarrow \text{SE}(G_\triangle).$$

Then we immediately have the following.

**Proposition 2.5.** For a spatial embedding $f$ of $G_Y$ and an element $\lambda$ in $\Gamma^{(n)}(G_Y)$, $f(\lambda)$ is ambient isotopic to $\varphi(f)(\lambda')$ for each element $\lambda'$ in the inverse image of $\lambda$ by $\Phi^{(n)}$. □
Now we show the following lemmas.

**Lemma 2.6.** If $G_\Delta$ is $I(K$ or $C3L)$, then $G_Y$ is also $I(K$ or $C3L)$.

*Proof.* Assume that $G_Y$ is not $I(K$ or $C3L)$, namely there exists a spatial embedding $f$ of $G_Y$ which contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. In the following we show that $\varphi(f)(G_\Delta)$ also contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Let $\gamma$ be an element in $\Gamma(G_\Delta)$. If $\gamma$ is not $\Delta$, then $\varphi(f)(\gamma)$ is ambient isotopic to $f(\Phi(\gamma))$ by Proposition 2.3 and $f(\Phi(\gamma))$ is a trivial knot by the assumption. Since $\varphi(f)(\Delta)$ is also a trivial knot, it follows that $\varphi(f)(G_\Delta)$ does not contain a nontrivial knot. Let $\lambda$ be an element in $\Gamma^{(3)}(G_\Delta)$. If $\lambda$ does not contain $\Delta$, then $\varphi(f)(\lambda)$ is ambient isotopic to $f(\Phi^{(3)}(\lambda))$ by Proposition 2.5 and $f(\Phi^{(3)}(\lambda))$ is a 3-component link which contains a split 2-component sublink by the assumption. If $\lambda$ contains $\Delta$, then $\varphi(f)(\lambda)$ is a split 3-component link. Thus we see that $\varphi(f)(G_\Delta)$ does not contain a 3-component link any of whose 2-component sublink is nonsplittable. $\square$

**Lemma 2.7.** If $G_Y$ is minor-minimal for $I(K$ or $C3L)$, then $G_\Delta$ is also minor-minimal for $I(K$ or $C3L)$.

*Proof.* In the following we show that for any edge $e$ of $G_\Delta$ which is not a loop, there exists a spatial embedding $f$ of $G_\Delta - e$ and a spatial embedding $g$ of $G_\Delta/e$ such that each of $f(G_\Delta - e)$ and $g(G_\Delta/e)$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. If $e$ is not $\overline{uv}$, $\overline{uw}$ or $\overline{vw}$, then there exists a spatial embedding $f'$ of $G_Y - e$ and a spatial embedding $g'$ of $G_Y/e$ such that both $f'(G_Y - e)$ and $g'(G_Y/e)$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Note that $G_Y - e$ (resp. $G_Y/e$) is obtained from $G_\Delta - e$ (resp. $G_\Delta/e$) by a single $\Delta Y$-exchange at the same $\Delta$. Then we see that each of $\varphi(f')(G_\Delta - e)$ and $\varphi(g')(G_\Delta/e)$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable in the similar way as the proof of Lemma 2.6. If $e$ is one of $\overline{uv}$, $\overline{uw}$ and $\overline{vw}$, we may assume that $e = \overline{uv}$ without loss of generality. Now there exists a spatial embedding $f'$ of $G_Y/\overline{uv}$ such that $f'(G_Y/\overline{uv})$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Then we can see that $G_\Delta - \overline{uv} = G_Y/\overline{uv}$. On the other hand, there exists a spatial embedding $g'$ of $G_Y/\overline{uv}$ such that $g'(G_Y/\overline{uv})$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Take a 2-disk $D'$ in the 3-sphere such that $D' \cap g'(G_Y/\overline{uv}) = g'(\overline{vw})$ and $\partial D' \cap g'(G_Y/\overline{uv}) = \{g'(u), g'(w)\}$. Then $(g'(G_Y/\overline{uv}) \setminus \text{int} g'(\overline{vw})) \cup \partial D'$ may be regarded as the image of a spatial embedding of $G_\Delta/\overline{uv}$, which is denoted by $g$. It is clear that $g(G_\Delta/\overline{uv})$ contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. $\square$

**Remark 2.8.** Lemma 2.7 has already been proven by Ozawa-Tsutsumi in a more general form [17] Lemma 3.1, Exercise 3.2]. But we give a proof of Lemma 2.7 as described above for the reader’s convenience.
3. **Proof of Theorem 1.1**

**Lemma 3.1.** Each of the graphs $N_9, N_10, N_{11}, N'_10, N'_11$ and $N'_12$ in the Heawood family is not IK.

**Proof.** Since the case of $N'_10$ has been already shown by Flapan-Naimi [3], we show that each of the graphs $N_9, N_{10}, N_{11}, N'_11$, and $N'_12$ is not IK. Let $f_9$ be the spatial embedding of $N_9$ as illustrated in Fig. 3.1. Then it can be checked directly that $f_9(N_9)$ does not contain a nontrivial knot. Thus $N_9$ is not IK. Let $f_{10}$ be the spatial embedding of $N_{10}$ as illustrated in Fig. 3.1. Let $\varphi_{N_{10}, N_9}$ be the map from $SE(N_{10})$ to $SE(N_9)$ induced by the $\Delta Y$-exchange from $N_{10}$ to $N_9$ at the marked $Y$ as illustrated in Fig. 3.1. Then it is clear that $\varphi(f_{10}) = f_9$. Since $f_9(N_9)$ does not contain a nontrivial knot, by Proposition 2.5 it follows that $f_{10}(N_{10})$ also does not contain a nontrivial knot. Namely $N_{10}$ is not IK. By repeating this argument, we can see that each of the graphs $N_{11}, N'_11$, and $N'_12$ is also not IK, see Fig. 3.1.

![Figure 3.1.](image)

**Proof of Theorem 1.1.** First we show that (1) and (2) are equivalent. Since we have already known that (2) implies (1), we show that (1) implies (2). If $G$ is IK, then by Lemma 3.1 we see that $G$ is not one of $N_9, N_{10}, N_{11}, N'_10, N'_11$, or $N'_12$. Namely $G$ is obtained from $K_7$ by a finite sequence of $\Delta Y$-exchanges. Next we show that (2) and (3) are equivalent. Assume that $G$ is obtained from $K_7$ by a finite sequence of $\Delta Y$-exchanges. Note that $\Gamma^{(3)}(K_7)$ is the empty set. Thus by Proposition 2.4 we see that $\Gamma^{(3)}(G)$ is the empty set. Conversely, if $G$ is one of $N_9, N_{10}, N_{11}, N'_10, N'_11$, and $N'_12$, then $\Gamma^{(3)}(G)$ is not the empty set. This completes the proof.

**Remark 3.2.** Let $f'_11$ be the spatial embedding of $N'_11$ as illustrated in Fig. 3.1 and $f'_{10}$ the spatial embedding of $N'_10$ as illustrated in Fig. 3.2. Let $\varphi_{N'_11, N'_10}$ be the map from $SE(N'_11)$ to $SE(N'_10)$ induced by the $\Delta Y$-exchange from $N'_11$ to $N'_10$ at the double-marked $Y$ as illustrated in Fig. 3.2. Then it is clear that $\varphi(f'_{11}) = f'_{10}$. Moreover we can see that $f'_{10}$ coincides with Flapan-Naimi’s example of a spatial embedding of $N'_10$ whose image does not contain a nontrivial knot as illustrated in the left side of Fig. 3.2 [3].
4. Proof of Theorem 1.2

We need some lemmas which are needed to prove Theorem 1.2.

**Lemma 4.1.** (Conway-Gordon [1], Taniyama-Yasuhara [21]) Let $G$ be a graph in the Petersen family and $f$ a spatial embedding of $G$. Then there exists an element $\lambda$ in $\Gamma^{(2)}(G)$ such that $\text{lk}(f(\lambda))$ is odd, where $\text{lk}$ denotes the linking number in the 3-sphere.

Let $D_4$ be the graph as illustrated in Fig. 4.1. We denote the set of all cycles with exactly four edges of $D_4$ by $\Gamma_4(D_4)$. For a spatial embedding $f$ of $D_4$, we define

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\gamma)) \pmod{2},$$

where $a_2$ denotes the second coefficient of the Conway polynomial. Note that $a_2(K)$ of a knot $K$ is congruent to the Arf invariant modulo two [12]. Then the following is known.

**Lemma 4.2.** (Taniyama-Yasuhara [21]) Let $f$ be a spatial embedding of $D_4$ and $\lambda, \lambda'$ all elements in $\Gamma^{(2)}(D_4)$. If both $\text{lk}(f(\lambda))$ and $\text{lk}(f(\lambda'))$ are odd, then $\alpha(f) = 1$.

Let $G$ be a graph which contains $D_4$ as a minor and $f$ a spatial embedding of $G$. Then we define

$$\alpha(f) \equiv \sum_{\gamma \in \Gamma_4(D_4)} a_2(f(\lambda)) \pmod{2}.$$

**Lemma 4.3.** Let $G$ be a graph which contains $D_4$ as a minor and $f$ a spatial embedding of $G$. For two elements $\mu$ and $\mu'$ in $\Psi_{D_4,G}^{(2)}(\Gamma^{(2)}(D_4))$, if both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu'))$ are odd, then $\alpha(f) = 1$. 
Proof. For two elements \( \lambda \) and \( \lambda' \) in \( \Gamma^{(2)}(D_4) \), we see that both \( \text{lk}(f(\Psi^{(2)}_{D_4,G}(\lambda))) \) and \( \text{lk}(f(\Psi^{(2)}_{D_4,G}(\lambda'))) \) are odd by the assumption. Then by Proposition 2.1, it follows that \( \text{lk}(\psi_{G,D_4}(f)(\lambda)) \) and \( \text{lk}(\psi_{G,D_4}(f)(\lambda')) \) are also odd. Thus by Lemma 4.2, we have that

\[
\alpha(f) \equiv \sum_{\gamma \in \Gamma(D_4)} a_2(f(\Psi_{D_4,G}(\gamma))) \equiv 1 \pmod{2}.
\]

Now we show the following theorem, which is the most important part in the proof of Theorem 1.2.

**Theorem 4.4.** Let \( G \) be \( N_9 \) or \( N'_{10} \). For every spatial embedding \( f \) of \( G \), there exists an element \( \gamma \) in \( \Gamma(G) \) such that \( a_2(f(\gamma)) \) is odd, or there exists an element \( \lambda \) in \( \Gamma^{(3)}(G) \) such that each 2-component sublink of \( f(\lambda) \) has an odd linking number.

Proof. We give a label to each vertex of \( G \) as illustrated in Fig. 4.2. In the following we denote a \( k \)-cycle \( i_1i_2 \cup i_2i_3 \cup \cdots \cup i_{k-1}i_k \cup i_ki_1 \) of \( G \) by \( [i_1i_2 \cdots i_k] \).

![Figure 4.2](image)

First we show in the case of \( G = N_9 \). Let \( f \) be a spatial embedding of \( N_9 \). Note that \( N_9 \) contains \( K_6 \) as the proper minor

\[
(((N_9 - \overline{78}) - \overline{89}) - \overline{97}) / 47/58/69.
\]

Thus by Lemma 4.1, there exists an element \( \nu \) in \( \Gamma^{(2)}(K_6) \) such that \( \text{lk}(\psi_{N_9,K_6}(f)(\nu)) \) is odd. Hence by Proposition 2.1, there exists an element \( \mu \) in \( \Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6)) \) such that \( \text{lk}(f(\mu)) \) is odd. Note that \( \Psi^{(2)}_{K_6,N_9}(\Gamma^{(2)}(K_6)) \) consists of ten elements, and by the symmetry of \( N_9 \), we may assume that \( \mu = [1743] \cup [2658] \) or \( [123] \cup [456] \) without loss of generality.

**Case 1.** \( \mu = [1743] \cup [2658] \).

Note that \( N_9 \) contains \( P_7 \) as the proper minor

\[
(((N_9 - \overline{67}) - \overline{68}) - \overline{78}) / 37/48/59.
\]

Thus by Lemma 4.1, there exists an element \( \nu' \) in \( \Gamma^{(2)}(P_7) \) such that \( \text{lk}(\psi_{N_9,P_7}(f)(\nu')) \) is odd. Hence by Proposition 2.1, there exists an element \( \mu' \) in \( \Psi^{(2)}_{P_7,N_9}(\Gamma^{(2)}(P_7)) \)
such that \( \text{lk}(f(\mu')) \) is odd. Note that \( \Psi_{\mu_1,N_9}(\Gamma(\mu_1)(P_1)) \) consists of nine elements 
\[
\begin{align*}
\mu_1' &= [3 4 5] \cup [1 2 8 7], \quad \mu_2' = [1 5 4 7] \cup [2 3 9 8], \quad \mu_3' = [2 8 5 4] \cup [3 1 7 9], \\
\mu_4' &= [1 2 4 7] \cup [3 5 8 9], \quad \mu_5' = [1 2 3] \cup [4 7 8 5], \quad \mu_6' = [1 2 8 5] \cup [3 4 7 9], \\
\mu_7' &= [2 3 4] \cup [1 5 8 7], \quad \mu_8' = [7 8 9] \cup [1 2 4 5], \quad \mu_9' = [1 5 3] \cup [2 8 7 4].
\end{align*}
\]

For \( i = 1, 2, \ldots, 9 \), let \( J^{i} \) be the subgraph of \( N_9 \) which is \( \mu \cup \mu_i' \cup \overline{6} \overline{9} \) if \( i = 3, 6 \) and \( \mu \cup \mu_i' \) if \( i \neq 3, 6 \). Assume that \( \text{lk}(f(\mu_i')) \) is odd for some \( i \neq 8 \). Then it can be easily seen that \( J^{i} \) contains a graph \( D^i \) as a minor so that \( D^i \) is isomorphic to \( D_4 \) and \( \{\mu, \mu_i'\} = \Psi_{\mu_1,N_9}(\Gamma(\mu_1)(D^i)) \). Since both \( \text{lk}(f(\mu)) \) and \( \text{lk}(f(\mu_i')) \) are odd, by Lemma 4.3, there exists an element \( \gamma \) in \( \Gamma(\mu_1) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \text{lk}(f(\mu_i')) \) is odd. We denote two elements \([7 8 9] \cup [1 2 6 5]\) and \([7 8 9] \cup [4 2 6 5]\) in \( \Gamma^{(2)}(J^8) \) by \( \mu_{s,1}^j \) and \( \mu_{s,2}^j \), respectively. We denote the subgraph \( \mu \cup \mu_{s,j}^j \) of \( J^8 \) by \( J^{8,j} \) \((j = 1, 2)\). Then it can be easily seen that \( J^{8,j} \) contains a graph \( D^{8,j} \) as a minor so that \( D^{8,j} \) is isomorphic to \( D_4 \) and \( \{\mu, \mu_{s,j}^j\} = \Psi_{\mu_1,N_9}(\Gamma(\mu_1)(D^{8,j})) \) \((j = 1, 2)\). Note that \([1 2 4 5] = [1 2 6 5] + [4 2 6 5]\) in \( H_1(J^8; \mathbb{Z}_2) \), where \( H_1(:, \mathbb{Z}_2) \) denotes the homology group with \( \mathbb{Z}_2 \)-coefficients. Then, by the homological property of the linking number, we have that
\[
1 \equiv \text{lk}(f(\mu_{s,j}^j)) \equiv \text{lk}(f(\mu_{s,1}^j)) + \text{lk}(f(\mu_{s,2}^j)) \pmod{2}.
\]
Thus we see that \( \text{lk}(f(\mu_{s,1}^j)) \) is odd or \( \text{lk}(f(\mu_{s,2}^j)) \) is odd. In either case, by Lemma 4.3, there exists an element \( \gamma \) in \( \Gamma(\mu_{s,j}^j) \) such that \( a_2(f(\gamma)) \) is odd.

**Case 2.** \( \mu = [1 2 3] \cup [4 5 6] \).

Note that \( N_9 \) contains \( P_9 \) as the proper minor
\[
(\{(N_9 - 1 2) - 2 3 - 3 1 - 4 5 - 5 6\} - 6 4).
\]

Thus by Lemma 4.1, there exists an element \( \nu' \) in \( \Gamma^{(2)}(P_9) \) such that \( \text{lk}(\psi_{N_9,P_9}(f)(\nu')) \) is odd. Hence by Proposition 2.1, there exists an element \( \mu' \) in \( \Psi_{\mu_1,N_9}(\Gamma^{(2)}(P_9)) \) such that \( \text{lk}(f(\mu')) \) is odd. Note that \( \Psi_{\mu_1,N_9}(\Gamma^{(2)}(P_9)) \) consists of seven elements, and by the symmetry of \( N_9 \), we may assume that \( \mu' = [1 5 8 7] \cup [2 6 9 3 4] \) or \([7 8 9] \cup [1 5 3 4 2 6]\) without loss of generality. We denote the subgraph \( \mu \cup \mu_i' \) of \( J^i \)(\( i = 1, 2\)). Then \( J^i \) contains a graph \( D^i \) as a minor so that \( D^i \) is isomorphic to \( D_4 \) and \( \{\mu, \mu_i'\} = \Psi_{\mu_1,N_9}(\Gamma(\mu_1)(D^i)) \) \((i = 1, 2)\). Since \([2 6 9 3 4] = [4 3 2] + [6 9 3 2] \) in \( H_1(J^i; \mathbb{Z}_2) \), it follows that
\[
1 \equiv \text{lk}(f(\mu_i')) \equiv \text{lk}(f(\mu_1')) + \text{lk}(f(\mu_2')) \pmod{2}.
\]
This implies that \( \text{lk}(f(\mu_1')) \) is odd or \( \text{lk}(f(\mu_2')) \) is odd. In either case, by Lemma 4.3, there exists an element \( \gamma \) in \( \Gamma(\mu_1) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \mu' = [7 8 9] \cup [1 5 3 4 2 6] \). We denote four elements \([7 8 9] \cup [3 4 5], [7 8 9] \cup [4 5 6], [7 8 9] \cup [1 5 6]\) and \([7 8 9] \cup [2 4 6]\) in \( \Gamma^{(2)}(J) \) by \( \mu_1', \mu_2', \mu_3', \mu_4' \), respectively. Since \([1 5 3 4 2 6] = [3 4 5] + [4 5 6] + [1 5 6] + [2 4 6]\) in \( H_1(J; \mathbb{Z}_2) \), it follows that 
\[
1 \equiv \text{lk}(\mu') \equiv \text{lk}(\mu_1') + \text{lk}(\mu_2') + \text{lk}(\mu_3') + \text{lk}(\mu_4') \pmod{2}.
\]
This implies that \( \text{lk}(\mu_i') \) is odd for some \( i = 1, 2, 3 \) or 4. Moreover, by the symmetry of \( J \), we may assume that \( \text{lk}(\mu_i') \) is odd or \( \text{lk}(\mu_i') \) is odd without loss of generality. Assume that \( \text{lk}(\mu_i') \) is odd. We denote the subgraph \( \mu \cup \mu_i' \cup \overline{7} \overline{6} \overline{2} \cup \overline{6} \overline{9} \) of \( N_9 \) by...
Note that $N$ each of the 2-component sublinks of $\{4 5 6\} \cup \nu \mu$ is odd. Hence by Proposition 2.1, there exists an element $J$ is odd. Hence by Proposition 2.1, there exists an element $\gamma$ such that $lk(J) \equiv 0 \pmod{2}$. Thus we see that $lk(\mu_i)$ is odd for some $i = 5, 6, 7$ or 8. Moreover, by the symmetry of $J$, we may assume that $lk(\mu_5)$ is odd or $lk(\mu_6')$ is odd without loss of generality. Assume that $lk(\mu_5)$ is odd. We denote the subgraph $\mu \cup \mu_5 \cup \{7 \ 8 \ 9\}$ of $N_9$ by $J^5$. Then $J^5$ contains a graph $D^5$ as a minor so that $D^5$ is isomorphic to $D_4$ and $\{\mu, \mu_5\} = \Psi(D_4, J^5) (\Gamma(D^5))$. Since both $lk(\mu)$ and $lk(\mu_5')$ are odd, by Lemma 4.3 there exists an element $\gamma$ such that $lk(\mu_5')$ is odd. Finally, assume that $lk(\mu_6')$ is odd. Let us consider the 3-component link $L = f\{1 2 3\} \cup \{5 6\} \cup \{7 8\}$. Since all 2-component sublinks of $L$ are $f(\mu), f(\mu_5')$ and $f(\mu_6')$, each of the 2-component sublinks of $L$ has an odd linking number.

Next we show in the case of $G = N'_{10}$. Let $f$ be a spatial embedding of $N'_{10}$. Note that $N'_{10}$ contains $P_7$ as the proper minor

$$(((N'_{10} - \{7 8\}) \cup \{8 9\}) \cup \{9\}) \cup \{\{4, 7, 8, 9\} \cup \{6\} \cup \{5\} \cup \{7\})$$

Thus by Lemma 4.1 there exists an element $\nu$ in $\Gamma(P_7)$ such that $lk(\psi_{N'_{10}} P_7(f)(\nu))$ is odd. Hence by Proposition 2.1 there exists an element $\mu$ in $\Psi(\Gamma(P_7))$ such that $lk(\mu)$ is odd. Note that $\Psi(\Gamma(P_7))$ consists of nine elements, and by the symmetry of $N'_{10}$ we may assume that $\mu = \{1 7 4 5\} \cup \{2 10 3 9 6\}, \{2 4 5 8\} \cup \{1 10 3 9 6\}, \{3 10 8 9\} \cup \{1 6 2 4 7\}, \{3 4 5\} \cup \{1 10 2 6\}$ or $\{2 8 10\} \cup \{1 6 9 3 4 7\}$ without loss of generality.

**Case 1.** $\mu = \{1 7 4 5\} \cup \{2 10 3 9 6\}$.

Note that $N'_{10}$ contains $P_7$ as the proper minor

$$(((N'_{10} - \{5 1\} \cup \{5 3\} \cup \{5 4\} \cup \{5 6\} \cup \{5 8\})) \cup \{7\})$$

Thus by Lemma 4.1 there exists an element $\nu'$ in $\Gamma(P_7)$ such that $lk(\psi_{N'_{10}} P_7(f)(\nu'))$ is odd. Hence by Proposition 2.1 there exists an element $\mu'$ in $\Psi(\Gamma(P_9))$ such that $lk(\mu')$ is odd. Note that $\Psi(\Gamma(P_9))$ consists of seven elements

$$\mu_1' = \{3 10 8 9\} \cup \{1 6 2 4 7\}, \mu_2' = \{1 7 8 10\} \cup \{2 4 3 9 6\}, \mu_3' = \{1 10 2 6\} \cup \{3 4 7 8 9\}, \mu_4' = \{2 4 3 10\} \cup \{1 7 8 9 6\}, \mu_5' = \{2 4 7 8\} \cup \{1 10 3 9 6\}, \mu_6' = \{2 8 9 6\} \cup \{1 10 3 4 7\}, \mu_7' = \{2 8 10\} \cup \{1 6 9 3 4 7\}.$$

For $i = 1, 2, \ldots, 7$, let $J^i$ be the subgraph of $N'_{10}$ which is $\mu \cup \mu_i' \cup \{5 8\}$ if $i = 1, 6$ and $\mu \cup \mu_i'$ if $i = 2, 3, 4, 5$. Assume that $lk(\mu_i')$ is odd for some $i$. Then $J^i$ contains a graph $D^i$ as a minor so that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu_i'\} = \Psi(D_4, J^i) (\Gamma(D^i))$. Since both $lk(\mu)$ and $lk(\mu_i')$ are odd, by Lemma 4.3 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(\gamma)$ is odd.

**Case 2.** $\mu = \{2 4 5 8\} \cup \{1 10 3 9 6\}$. 


Note that $N_{10}'$ contains another $P_8$ as the proper minor 
\[
(((((N_{10}' - 8 2) - 8 5) - 8 7) - 8 9) - 8 10) - 3 4.
\]
Thus by Lemma 4.4 there exists an element $\nu'$ in $\Gamma(2)(P_9)$ such that $\text{lk}(\psi_{N_{10}'}, P_9(f)(\nu'))$ is odd. Hence by Proposition 2.1 there exists an element $\mu'$ in $\Psi_{P_9, N_{10}'}^{(2)}(\Gamma(2)(P_9))$ such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_9, N_{10}'}^{(2)}(\Gamma(2)(P_9))$ consists of seven elements 
\[
\begin{align*}
\mu_1' &= [1 6 9 7] \cup [2 4 5 3 10], \\
\mu_2' &= [1 7 4 5] \cup [2 10 3 9 6], \\
\mu_3' &= [3 5 6 9] \cup [1 10 2 4 7], \\
\mu_4' &= [1 5 3 10] \cup [2 4 7 9 6], \\
\mu_5' &= [1 10 2 6] \cup [3 9 7 4 5], \\
\mu_6' &= [1 5 6] \cup [2 4 7 9 3 10], \\
\mu_7' &= [2 4 5 6] \cup [1 10 3 9 7].
\end{align*}
\]

For $i = 1, 2, \ldots, 7$, let $J^i$ be the subgraph of $N_{10}'$ which is $\mu \cup \mu_i' \cup \overline{T_8}$ if $i = 1, 7$ and $\mu \cup \mu_i'$ if $i \neq 1, 7$. Assume that $\text{lk}(f(\mu_i'))$ is odd for some $i$. Then $J^i$ contains a graph $D^i$ as a minor so that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu_i'\} = \Psi_{D^i, J^i}^{(2)}(\Gamma(2)(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu_i'))$ are odd, by Lemma 4.4 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

**Case 3.** $\mu = [3 10 8 5] \cup [1 6 2 4 7]$.

Let $P_9$ be the proper minor of $N_{10}'$ and $\mu_i'$ the element in $\Psi_{P_9, N_{10}'}^{(2)}(\Gamma(2)(P_9))$ ($i = 1, 2, \ldots, 7$) as in Case 2. For $i = 1, 2, \ldots, 7$, let $J^i$ be the subgraph of $N_{10}'$ which is $\mu \cup \mu_i' \cup \overline{T_8}$ if $i = 1, 4$ and $\mu \cup \mu_i'$ if $i \neq 1, 4$. Assume that $\text{lk}(f(\mu_i'))$ is odd for some $i$. Then $J^i$ contains a graph $D^i$ as a minor so that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu_i'\} = \Psi_{D^i, J^i}^{(2)}(\Gamma(2)(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu_i'))$ are odd, by Lemma 4.4 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd.

**Case 4.** $\mu = [3 4 5] \cup [1 10 2 6]$.

Note that $N_{10}'$ contains another $P_7$ as the proper minor 
\[
(((N_{10}' - 3 4) - 4 5) - 5 3) / 3 9 / 4 7 / 5 8.
\]
Thus by Lemma 4.4 there exists an element $\nu'$ in $\Gamma(2)(P_7)$ such that $\text{lk}(\psi_{N_{10}'}, P_7(f)(\nu'))$ is odd. Hence by Proposition 2.1 there exists an element $\mu'$ in $\Psi_{P_7, N_{10}'}^{(2)}(\Gamma(2)(P_7))$ such that $\text{lk}(f(\mu'))$ is odd. Note that $\Psi_{P_7, N_{10}'}^{(2)}(\Gamma(2)(P_7))$ consists of nine elements 
\[
\begin{align*}
\mu_1' &= [5 6 9 8] \cup [1 10 2 4 7], \\
\mu_2' &= [3 10 8 9] \cup [1 6 2 4 7], \\
\mu_3' &= [1 5 8 10] \cup [2 4 7 9 6], \\
\mu_4' &= [7 8 9] \cup [1 10 2 6], \\
\mu_5' &= [2 8 10] \cup [1 6 9 7], \\
\mu_6' &= [2 8 5 6] \cup [1 10 3 9 7], \\
\mu_7' &= [1 7 8 5] \cup [2 10 3 9 6], \\
\mu_8' &= [1 5 6] \cup [2 4 7 9 3 10], \\
\mu_9' &= [2 4 7 8] \cup [1 10 3 9 6].
\end{align*}
\]

For $i = 1, 2, \ldots, 9$, let $J^i$ be the subgraph of $N_{10}'$ which is $\mu \cup \mu_i' \cup \overline{T_8}$ if $i = 5$ and $\mu \cup \mu_i'$ if $i \neq 5$. Assume that $\text{lk}(f(\mu_i'))$ is odd for some $i \neq 4, 8$. Then $J^i$ contains a graph $D^i$ as a minor so that $D^i$ is isomorphic to $D_4$ and $\{\mu, \mu_i'\} = \Psi_{D^i, J^i}^{(2)}(\Gamma(2)(D^i))$. Since both $\text{lk}(f(\mu))$ and $\text{lk}(f(\mu_i'))$ are odd, by Lemma 4.4 there exists an element $\gamma$ in $\Gamma(J^i)$ such that $a_2(f(\gamma))$ is odd. Next assume that $\text{lk}(f(\mu_i'))$ is odd. We denote two elements $[1 5 6] \cup [2 4 3 10]$ and $[1 5 6] \cup [3 4 7 9]$ in $\Gamma(2)(J^8)$ by $\mu_{8,1}'$ and $\mu_{8,2}'$, respectively. We denote the subgraph $\mu \cup \mu_{8,1}'$ of $J^8$ by $J^{8,1}$ and the subgraph
\[ \mu \cup \mu_{5,2} \cup \{8,9\} \cup \{8,10\} \text{ of } N'_{10} \text{ by } J^{8,2}. \] Then \( J^{8,j} \) contains a graph \( D^{8,j} \) as a minor so that \( D^{8,j} \) is isomorphic to \( D_4 \) and \( \{\mu, \mu_{5,2}'\} = \Psi^{(2)}_{D^{8,j}, P_{8,j}}(\Gamma^{(2)}(D^{8,j})) \) \((j = 1, 2)\). Since \([2 \ 4 \ 7 \ 9 \ 3 \ 10] = [2 \ 4 \ 3 \ 10] + [3 \ 4 \ 7 \ 9] \) in \( H_1(J^{8}; \mathbb{Z}_2) \), it follows that
\[
1 \equiv \text{lk}(f(\mu_{5,2})) \equiv \text{lk}(f(\mu_{5,1}')) + \text{lk}(f(\mu_{5,2}')) \pmod{2}.
\]
This implies that \( \text{lk}(f(\mu_{5,1}')) \) is odd or \( \text{lk}(f(\mu_{5,2}')) \) is odd. In either case, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^{8,j}) \) such that \( a_2(f(\gamma)) \) is odd. Finally assume that \( \text{lk}(f(\mu_4')) \) is odd. Note that \( N'_{10} \) contains another \( P_9 \) as the proper minor
\[
((\{\{N'_{10} - \{2,4\}, \{6,8\}, \{10,5\}\} - 3) - 3, 3, 3)\).
\]
Thus by Lemma 4.1 there exists an element \( \mu'' \) in \( \Gamma^{(2)}(P_9) \) such that \( \text{lk}(\Psi^{(2)}_{N'_{10}, P_9}(f)(\mu'')) \) is odd. Hence by Proposition 2.1 there exists an element \( \mu'' \) in \( \Psi^{(2)}_{P_{9}, N'_{10}}(\Gamma^{(2)}(P_9)) \) such that \( \text{lk}(f(\mu'')) \) is odd. Note that \( \Psi^{(2)}_{P_{9}, N'_{10}}(\Gamma^{(2)}(P_9)) \) consists of seven elements
\[
\begin{align*}
\mu''_1 &= [5 \ 6 \ 9 \ 8] \cup [1 \ 10 \ 3 \ 4 \ 7], \\
\mu''_2 &= [4 \ 5 \ 8 \ 7] \cup [1 \ 10 \ 3 \ 9 \ 6], \\
\mu''_3 &= [1 \ 7 \ 8 \ 10] \cup [3 \ 4 \ 5 \ 6 \ 9], \\
\mu''_4 &= [3 \ 10 \ 8 \ 9] \cup [1 \ 7 \ 4 \ 5 \ 6], \\
\mu''_5 &= [1 \ 6 \ 9 \ 7] \cup [3 \ 4 \ 5 \ 8 \ 10], \\
\mu''_6 &= [3 \ 9 \ 7 \ 4] \cup [1 \ 10 \ 8 \ 5 \ 6], \\
\mu''_7 &= [7 \ 8 \ 9] \cup [1 \ 10 \ 3 \ 4 \ 5 \ 6].
\end{align*}
\]
For \( j = 1, 2, \ldots, 7 \), let \( J^{4,j} \) be the subgraph of \( N'_{10} \) which is \( \mu_4' \cup \mu_5'' \cup 24 \) if \( j = 2, 6 \) and \( \mu_4' \cup \mu_5'' \) if \( j \neq 2, 6 \). Assume that \( \text{lk}(f(\mu_4'')) \) is odd for some \( j \neq 7 \). Then \( J^{4,j} \) contains a graph \( D^{4,j} \) as a minor so that \( D^{4,j} \) is isomorphic to \( D_4 \) and \( \{\mu_4', \mu''_j\} = \Psi^{(2)}_{D^{4,j}, J^{4,j}}(\Gamma^{(2)}(D^{4,j})) \). Since both \( \text{lk}(f(\mu_4')) \) and \( \text{lk}(f(\mu_4'')) \) are odd, by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^{4,j}) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \text{lk}(f(\mu_4'')) \) is odd. We denote three elements \([7 \ 8 \ 9] \cup [1 \ 5 \ 3 \ 10], [7 \ 8 \ 9] \cup [1 \ 5 \ 6] \) and \([7 \ 8 \ 9] \cup [3 \ 4 \ 5] \) in \( \Gamma^{(2)}(N'_{10}) \) by \( \mu''_{7,1}, \mu''_{7,2} \) and \( \mu''_{7,3} \), respectively. We denote the subgraph \( \mu \cup \mu''_{7,k} \cup [4 \ 7] \cup [2 \ 8] \) of \( N'_{10} \) by \( J^{4.7,k} \) \((k = 1, 2)\). Then \( J^{4.7,k} \) contains a graph \( D^{4.7,k} \) as a minor so that \( D^{4.7,k} \) is isomorphic to \( D_4 \) and \( \{\mu, \mu''_{7,k}\} = \Psi^{(2)}_{D^{4.7,k}, J^{4.7,k}}(\Gamma^{(2)}(D^{4.7,k})) \) \((k = 1, 2)\). Since \([1 \ 10 \ 3 \ 4 \ 5 \ 6] = [1 \ 5 \ 3 \ 10] + [1 \ 5 \ 6] + [3 \ 4 \ 5] \) in \( H_1(N'_{10}; \mathbb{Z}_2) \), it follows that
\[
1 \equiv \text{lk}(f(\mu_4')) \equiv \text{lk}(f(\mu''_7)) + \text{lk}(f(\mu''_{7,2})) + \text{lk}(f(\mu''_{7,3})) \pmod{2}.
\]
This implies that \( \text{lk}(f(\mu''_{7})) \) is odd for some \( k \). If \( \text{lk}(f(\mu''_{7,k})) \) is odd or \( \text{lk}(f(\mu''_{7,2})) \) is odd, then by Lemma 4.3 there exists an element \( \gamma \) in \( \Gamma(J^{4.7,k}) \) such that \( a_2(f(\gamma)) \) is odd. If \( \text{lk}(f(\mu''_{7,3})) \) is odd, let us consider the 3-component link \( L = f([3 \ 4 \ 5] \cup [7 \ 8 \ 9] \cup [1 \ 10 \ 2 \ 6]) \). Since all 2-component sublinks of \( L \) are \( f(\mu), f(\mu_4') \) and \( f(\mu''_{7,3}) \), each of the 2-component sublinks of \( L \) has an odd linking number.

**Case 5.** \( \mu = [2 \ 8 \ 10] \cup [1 \ 6 \ 9 \ 3 \ 4 \ 7] \).

We denote two elements \([2 \ 8 \ 10] \cup [1 \ 6 \ 9 \ 7] \) and \([2 \ 8 \ 10] \cup [3 \ 9 \ 7 \ 4] \) in \( \Gamma^{(2)}(N'_{10}) \) by \( \mu_1 \) and \( \mu_2 \), respectively. Since \([1 \ 6 \ 9 \ 3 \ 4 \ 7] = [1 \ 6 \ 9 \ 7] + [3 \ 9 \ 7 \ 4] \) in \( H_1(N'_{10}; \mathbb{Z}_2) \), it follows that
\[
1 \equiv \text{lk}(f(\mu)) \equiv \text{lk}(f(\mu_1)) + \text{lk}(f(\mu_2)) \pmod{2}.
\]
This implies that \( \text{lk}(f(\mu_1)) \) is odd or \( \text{lk}(f(\mu_2)) \) is odd. By the symmetry of \( N'_{10} \), we may assume that \( \text{lk}(f(\mu_1)) \) is odd. Note that \( N'_{10} \) contains another \( P_7 \) as the
proper minor
\[ (((N'_{10} - 2R) - 8T) - 10U)/2G/3U/5R. \]
Thus by Lemma 4.1, there exists an element \( \nu' \) in \( \Gamma^{(2)}(P_7) \) such that \( \text{lk}(\psi_{N'_{10}}, p_j(f)(\nu')) \) is odd. Hence by Proposition 2.1, there exists an element \( \mu' \) in \( \Psi^{(2)}_{P_7, N'_{10}}(\Gamma^{(2)}(P_7)) \) such that \( \text{lk}(f(\mu')) \) is odd. Note that \( \Psi^{(2)}_{P_7, N'_{10}}(\Gamma^{(2)}(P_7)) \) consists of nine elements
\[
\begin{align*}
\mu'_1 &= [3 5 8 9] \cup [1 6 2 4 7], \\
\mu'_2 &= [1 7 8 5] \cup [2 4 3 9 6], \\
\mu'_3 &= [1 5 6] \cup [3 9 7 4], \\
\mu'_4 &= [3 4 5] \cup [1 6 9 7], \\
\mu'_5 &= [5 6 9 8] \cup [1 10 3 4 7], \\
\mu'_6 &= [4 5 8 7] \cup [1 10 3 9 6], \\
\mu'_7 &= [1 5 3 10] \cup [2 4 7 9 6], \\
\mu'_8 &= [2 4 5 6] \cup [1 10 3 9 7], \\
\mu'_9 &= [7 8 9] \cup [1 10 3 4 2 6].
\end{align*}
\]
For \( i = 1, 2, \ldots, 9 \), let \( J^i \) be the subgraph of \( N'_{10} \) which is \( \mu_1 \cup \mu_2 \cup 310 \cup 5R \) if \( i = 3 \) and \( \mu_1 \cup \mu'_i \) if \( i \neq 3 \). Assume that \( \text{lk}(f(\mu'_i)) \) is odd for some \( i \neq 4, 9 \). Then \( J^i \) contains a graph \( D^i \) as a minor so that \( D^i \) is isomorphic to \( D_4 \) and \( \{\mu_1, \mu'_i\} = \Psi^{(2)}_{D^i, J^i}(\Gamma^{(2)}(D^i)) \). Since both \( \text{lk}(f(\mu_{10})) \) and \( \text{lk}(f(\mu'_i)) \) are odd, by Lemma 4.3, there exists an element \( \gamma \) in \( \Gamma(J^i) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \text{lk}(f(\mu'_i)) \) is odd. We denote two elements \([7 8 9] \cup [1 6 2 4 10] \) and \([7 8 9] \cup [2 4 3 10] \) in \( \Gamma^{(2)}(J^9) \) by \( \mu'_9 \) and \( \mu'_{9, 2} \), respectively. We denote the subgraph \( \mu_1 \cup \mu'_{9, 1} \) of \( J^9 \) by \( J'^{9, 1} \) and the subgraph \( \mu_1 \cup \mu'_{9, 2} \cup 310 \cup \Gamma \) of \( N'_{10} \) by \( J'^{9, 2} \). Then \( J'^{9, 2} \) contains a graph \( D^{9, 2} \) as a minor so that \( D^{9, 2} \) is isomorphic to \( D_4 \) and \( \{\mu_1, \mu'_{9, 1}, \mu'_{9, 2}\} = \Psi^{(2)}_{D^{9, 2}, J'^{9, 2}}(\Gamma^{(2)}(D^{9, 2})) \) \( (j = 1, 2) \). Since \([1 10 3 4 2 6] = [1 6 2 10] + [2 4 3 10] \) in \( H_1(J^{9, 2}; \mathbb{Z}_2) \), it follows that
\[ 1 \equiv \text{lk}(f(\mu'_9)) \equiv \text{lk}(f(\mu'_{9, 1})) + \text{lk}(f(\mu'_{9, 2})) \pmod{2}. \]
This implies that \( \text{lk}(f(\mu'_{9, 1})) \) is odd or \( \text{lk}(f(\mu'_{9, 2})) \) is odd. In either case, by Lemma 4.3, there exists an element \( \gamma \) in \( \Gamma(J^{9, 2}) \) such that \( a_2(f(\gamma)) \) is odd. Finally assume that \( \text{lk}(f(\mu'_i)) \) is odd. Note that \( N'_{10} \) contains another \( P_6 \) as the proper minor
\[ (((((N'_{10} - 6T) - 6U - 6S) - 8T) - 8R - 8U). \]
Thus by Lemma 4.1, there exists an element \( \nu'' \) in \( \Gamma^{(2)}(P_9) \) such that \( \text{lk}(\psi_{N'_{10}}, p_j(f)(\nu'')) \) is odd. Hence by Proposition 2.1, there exists an element \( \mu'' \) in \( \Psi^{(2)}_{P_9, N'_{10}}(\Gamma^{(2)}(P_9)) \) such that \( \text{lk}(f(\mu'')) \) is odd. Note that \( \Psi^{(2)}_{P_9, N'_{10}}(\Gamma^{(2)}(P_9)) \) consists of seven elements
\[
\begin{align*}
\mu''_1 &= [3 5 8 9] \cup [1 10 2 4 7], \\
\mu''_2 &= [3 9 7 4] \cup [1 5 8 2 10], \\
\mu''_3 &= [1 7 4 5] \cup [2 8 9 3 10], \\
\mu''_4 &= [2 4 5 8] \cup [1 10 3 9 7], \\
\mu''_5 &= [2 4 3 10] \cup [1 5 8 9 7], \\
\mu''_6 &= [1 5 3 10] \cup [2 4 7 9 8], \\
\mu''_7 &= [3 4 5] \cup [1 10 2 8 9 7].
\end{align*}
\]
For \( j = 1, 2, \ldots, 7 \), let \( J^{4, j} \) be the subgraph of \( N'_{10} \) which is \( \mu'_1 \cup \mu'_j \cup 2G \) if \( j = 4, 5 \) and \( \mu'_1 \cup \mu''_j \) if \( j \neq 4, 5 \). Assume that \( \text{lk}(f(\mu''_j)) \) is odd for some \( j \neq 7 \). Then \( J^{4, j} \) contains a graph \( D^{4, j} \) as a minor so that \( D^{4, j} \) is isomorphic to \( D_4 \) and \( \{\mu'_1, \mu''_j\} = \Psi^{(2)}_{D^{4, j}, J^{4, j}}(\Gamma^{(2)}(D^{4, j})) \). Since both \( \text{lk}(f(\mu'_1)) \) and \( \text{lk}(f(\mu''_j)) \) are odd, by Lemma 4.3, there exists an element \( \gamma \) in \( \Gamma(J^{4, j}) \) such that \( a_2(f(\gamma)) \) is odd. Next assume that \( \text{lk}(f(\mu''_7)) \) is odd. We denote two elements \([3 4 5] \cup [1 10 8 9 7] \) and \([3 4 5] \cup [2 8 10] \) in \( \Gamma^{(2)}(N'_{10}) \) by \( \mu''_{7, 1} \) and \( \mu''_{7, 2} \), respectively. We denote the subgraph \( \mu_1 \cup \mu''_{7, 1} \cup 2G \cup 5R \) of \( N'_{10} \) by \( J^{4, 7} \). Then \( J^{4, 7} \) contains a graph \( D^{4, 7} \) as a minor so that \( D^{4, 7} \)
is isomorphic to $D_4$ and $\{\mu_1, \mu_{12}''\} = \Psi^{(2)}_{D_{3,7}, J_{4,7}} (\Gamma^{(2)}(D^{4,7}))$. Since $[1\ 10\ 2\ 8\ 9\ 7] = [1\ 10\ 8\ 9\ 7]\ +\ [2\ 8\ 10]$ in $H_1(N_{10}; \mathbb{Z}_2)$, it follows that

$$1 \equiv \text{lk}(f(\mu_1)) \equiv \text{lk}(f(\mu_{12}'')) + \text{lk}(f(\mu_{7,2}'')) \pmod{2}.$$  

This implies that $\text{lk}(f(\mu_{12}''))$ is odd or $\text{lk}(f(\mu_{7,2}''))$ is odd. If $\text{lk}(f(\mu_{12}''))$ is odd, then by Lemma 4.3 there exists an element $\gamma$ in $\Gamma(J^{4,7})$ such that $a_2(f(\gamma))$ is odd. If $\text{lk}(f(\mu_{7,2}''))$ is odd, let us consider the 3-component link $L = f([3\ 4\ 5] \cup [2\ 8\ 10] \cup [1\ 6\ 9\ 7])$. Since all 2-component sublinks of $L$ are $f(\mu_1), f(\mu_4')$ and $f(\mu_{7,2}'')$, each of the 2-component sublinks of $L$ has an odd linking number. This completes the proof. \qed 

**Remark 4.5.** A graph is said to be 2-apex if it can be embedded in the 2-sphere after the deletion of at most two vertices and all of their incidental edges. It is not hard to see that any 2-apex graph may have a spatial embedding whose image contains neither a nontrivial knot nor a 3-component link any of whose 2-component sublink is nonsplittable. Thus any 2-apex graph is not I(K or C3L). It is known that every graph of at most twenty edges is 2-apex [14] (see also [11]). Since the number of all edges of every graph in the Heawood family is twenty one, we see that any proper minor of a graph in the Heawood family is 2-apex, namely not I(K or C3L). This also implies that any graph in the Heawood family is minor-minimal for I(K or C3L).

**Example 4.6.** Let $g_9$ be the spatial embedding of $N_9$ and $g_{10}$ the spatial embedding of $N_{10}$ as illustrated in Fig. 4.3. Then it can be checked directly that both $g_9(N_9)$ and $g_{10}(N_{10})$ do not contain a nonsplittable 3-component link. Thus neither $N_9$ nor $N_{10}$ is I3L. Moreover, we can see that $N_{10}, N_{11}, N_{12}$ and $N_{14}$ are not I3L in a similar way as the proof of Lemma 5.1 see Fig. 4.3.

**Remark 4.7.** We remark that the Heawood graph is IK. The Heawood graph is the dual graph of $K_7$ which is embedded in a torus. It is known that there exists a unique graph $C_{14}$ obtained from $K_7$ by seven times applications of $\Delta Y$-exchanges [13]. The seven triangles corresponds to the black triangles of black-and-white coloring of the torus by $K_7$. Then $C_{14}$ and $H$ are mapped to each other by parallel transformation of the torus, see Fig. 4.4. Thus they are isomorphic. Since $C_{14}$ is IK, we have the result.

**Remark 4.8.** It is known that all twenty six graphs obtained from the complete four-partite graph $K_{3,3,1,1}$ by a finite sequence of $\Delta Y$-exchanges are minor-minimal IK graphs [13]. [9]. There exist thirty two graphs which are obtained from $K_{3,3,1,1}$ by a finite sequence of $\Delta Y$ and $Y\Delta$-exchanges but cannot be obtained from $K_{3,3,1,1}$ by a finite sequence of $\Delta Y$-exchanges. Recently, Goldberg-Mattman-Naimi announces that these thirty two graphs are also minor-minimal IK graphs [9].
Figure 4.3.

Figure 4.4.

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