Accurate Study from
Adaptive Perturbation Method

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Abstract

The adaptive perturbation method decomposes a Hamiltonian by the diagonal elements and non-diagonal elements of the Fock state. The diagonal elements of the Fock state are solvable but can contain the information about interacting constants. The exact solution for each perturbed term can be obtained in the harmonic oscillator with the interacting potential, $\lambda_1 x^4/6 + \lambda_2 x^6/120$, where $\lambda_1$ and $\lambda_2$ are coupling constants, and $x$ is the position operator. We demonstrate the accurate study of the spectrum and $\langle x^2 \rangle$ up to the next leading-order correction. In particular, we study a similar problem of Higgs field from the inverted mass term to demonstrate the possible non-trivial application of particle physics.

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1 Introduction

Perturbation method is a known approximation for studying non-solvable systems [1]. The known procedure is to begin from the non-interacting system and then do a perturbation from the coupling terms. Therefore, the coupling constant cannot be large. Applying the perturbation method to Quantum Field Theory (QFT) builds a generic tool for a probe of a weakly coupled region. Although people still do not know how to study a strongly coupled region from a similar procedure [2], various physical phenomena and experiments were confirmed by the perturbation method. The most interesting problem in the strongly interacting system should be Quantum Chromodynamics (QCD). QCD describes the dynamics between quarks and gluons. The color confinement and asymptotic freedom are important problems, and it is necessary to extract physical information from the low-energy QCD. To study the low-energy physics, it is necessary to develop a new systematic-technology to study QCD because it is strongly coupled.

The standard model produces the Higgs boson by the excitation of the Higgs field, which is called the Higgs mechanism. The Higgs mechanism is to study the perturbation in a low-energy state (or true vacuum state), and it gives a natural way to interpret how to create particle’s mass. Therefore, the observation in the Large Hadron Collider (LHC) shows that a perturbation method is an important tool for studying the fundamental physics of our nature. However, the Higgs field has a non-zero value after the mechanism, and the value is inversely proportional to the square root of a coupling constant. Although redefining the Higgs field for obtaining the value is not problematic, doing the perturbation becomes problematic. Because the unperturbed state is the eigenstate of the vanishing coupling constant case in the perturbation method, and the unperturbed Higgs-field theory is not bounded from below, the unperturbed state is not a Fock state, labeled by the particle numbers. The current way can be seen as using a coupling constant with an extremely small value (but ignores the interacting terms) to work the perturbation. It is necessary to use the Fock state for an unperturbed state when applying the perturbation method to QFT. Hence the new skill is also necessary for the weakly coupled QFT.

To solve the above issues, we study the adaptive perturbation method [3]. The adaptive perturbation decomposes a Hamiltonian by the diagonal elements of a Fock space and the non-diagonal elements of a Fock space [3]. The perturbed term is the non-diagonal
sector. Therefore, the perturbation parameter is not coupling constant \[4\]. Because
the unperturbed part is controlled by the diagonal elements of a Fock space as in the
harmonic oscillator (but the adaptive perturbation method includes all diagonal ele-
ments), the unperturbed state is still a Fock state. To choose an unperturbed state
with a low energy, the adaptive perturbation method introduces an adaptive parameter
\(\gamma\). It is convenient to give such the state from the variation of the parameter \(3\). The
leading-order correction of the spectrum from the second-order perturbation provides
a practical study, and it successfully approaches to the numerical solution with a very
small error \[5\]. Hence the adaptive perturbation method seems to avoid the issue of
the weak-coupling perturbation. The central question that we would like to address in
this letter is the following: Whether the adaptive perturbation method can apply to the
Higgs field?

In this letter, we study the potential \(\omega^2 x^2/2 + \lambda_1 x^4/6 + \lambda_2 x^6/120\) because each perturbed
term has an exact solution. The \(\omega\) is frequency, \(\lambda_1\) and \(\lambda_2\) are coupling constants, and
\(x\) is the position operator. We show the analytical solution of the eigenenergy and \(\langle x^2 \rangle\)
up to the next leading-order calculation. The analytical formula shows a comparison
to the numerical solution with a small deviation. In particular, the inverted mass case
\((\omega^2 = -1)\) provides direct evidence of the possible application of the Higgs field.

2 Analytical Solution

We introduce the adaptive perturbation method \[3\] and show the analytical formula
for the spectrum and the \(\langle x^2 \rangle\) up to the next leading-order correction. In the final,
we demonstrate the accuracy of the analytical solution by comparing the perturbed
solution to the numerical solution.

2.1 Adaptive Perturbation Method

The main idea of the adaptive perturbation is to decide the decomposition of the Hamil-
tonian by whether the elements are in the diagonal places of the Fock state \[3\]. To choose
a suitable unperturbed state (with a low-energy), we introduce the adaptive parameter
\(\gamma\), which is allowed without changing the commutation relation \([p, x] = -i \[3\]. The
\(p\) is the momentum operator. The \(\gamma\) is introduced as \[3\]:

\[
x = (A^\dagger \gamma + A_\gamma)/\sqrt{2\gamma}
\]

and

\[
p = i\sqrt{\gamma/2}(A^\dagger_\gamma - A_\gamma),
\]

where \(A^\dagger_\gamma\) is the creation operator, and \(A_\gamma\) is the annihilation op-
erator. The operators have the same relation as in the harmonic oscillator case, except
for the dependence of the choice of $\gamma$, like the commutation relation $[A_\gamma, A_\gamma^\dagger] = 1$ \[3\]. Since the operators depend on the adaptive parameter, the vacuum state also depends $\langle A_\gamma | 0 \rangle = 0$ \[3\]. The adaptive parameter is just a scaling of the position operator. Then the unperturbed part $H_0(\gamma)$ is replaced by the diagonal elements of the Fock space, and the perturbed part $V(\gamma)$ is replaced by the non-diagonal elements of the Fock space. Here we study the Hamiltonian:

$$H = H_0 + V = \frac{p^2}{2} + \frac{\omega^2}{2} x^2 + \frac{\lambda_1}{6} x^4 + \frac{\lambda_2}{120} x^6$$

(1)
because each perturbed term has an exact solution \[4\].

### 2.2 Eigenenergy

Applying the time-independent perturbation to the adaptive perturbation method gives the same formula

$$E_n = E_n^{(0)} + \sum_{k \neq n} \frac{|\langle k^{(0)} | V | n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}} + \sum_{k \neq n} \sum_{m \neq n} \frac{\langle n^{(0)} | V | m^{(0)} \rangle \langle m^{(0)} | V | k^{(0)} \rangle \langle k^{(0)} | V | n^{(0)} \rangle}{(E_n^{(0)} - E_m^{(0)})(E_n^{(0)} - E_k^{(0)})} + \cdots ,$$

(2)

where $E_n^{(0)}$ is the $n$-th unperturbed eigenenergy, $|n^{(0)}\rangle$ is the $n$-th unperturbed eigenstate, and $E_{k,n}$ is the $k$-th unperturbed eigenenergy, calculated by the $n$-th unperturbed eigenstate’s $\gamma$. The first-order term $\langle n^{(0)} | V | n^{(0)} \rangle$ vanishes due to that $V$ is a non-diagonal element of the Fock space. The adaptive parameter $\gamma$ is determined by minimizing the unperturbed spectrum for the parameter \[4\]

$$E_n^{(0)} = \frac{\gamma}{4} (2n + 1) \frac{\omega^2}{4\gamma} (2n + 1) + \frac{\lambda_1}{4\gamma^2} \left( n^2 + n + \frac{1}{2} \right) + \frac{\lambda_2}{4\gamma^3} \left( \frac{1}{12} n^3 + \frac{29}{240} n^2 + \frac{1}{6} n + \frac{1}{16} \right),$$

(3)
in which the $\gamma$ is positive, and it satisfies the algebra equation \[4\]

$$\gamma^4 - \omega^2 \gamma^2 - \lambda_1 \frac{2n^2 + 2n_\gamma + 1}{2n_\gamma + 1} \gamma \gamma - \lambda_2 \frac{20n^3 + 29n^2 + 40n_\gamma + 15}{80} \frac{2n^2 + 2n_\gamma + 1}{2n_\gamma + 1} = 0.$$  

(4)

Here we use the value of the $\gamma$ in the higher-order calculation as in the solvable part. Because the adaptive parameter depends on the value of $n$, it is hard to guarantee whether
the adaptive perturbation method is practical (although the choice of the adaptive parameter should not affect the result if we do the perturbation to all-orders). However, the second-order perturbation is already enough to show a very small deviation to the numerical solution [5].

2.2.1 2nd-Order and 3rd-Order

To write the perturbed spectrum conveniently, we introduce the below parameters:

\[ T_1 \equiv \frac{\lambda_2}{960 \gamma^3}; \]
\[ T_2 \equiv \frac{\lambda_1}{24 \gamma^2} + \frac{\lambda_2}{320 \gamma^3}(2n + 5); \]
\[ T_3 \equiv -\frac{\gamma}{4} + \frac{\omega^2}{4 \gamma} + \frac{\lambda_1}{12 \gamma^2}(2n + 3) + \frac{\lambda_2}{64 \gamma^3}(n^2 + 3n + 3); \]
\[ T_4 \equiv -\frac{\gamma}{4} + \frac{\omega^2}{4 \gamma} + \frac{\lambda_1}{12 \gamma^2}(2n - 1) + \frac{\lambda_2}{64 \gamma^3}(n^2 - n + 1); \]
\[ T_5 \equiv \frac{\lambda_1}{24 \gamma^2} + \frac{\lambda_2}{320 \gamma^3}(2n - 3); \]
\[ T_6 \equiv T_1; \]
\[ T_7 \equiv -\frac{\gamma}{4} + \frac{\omega^2}{4 \gamma} + \frac{\lambda_1}{12 \gamma^2}(2n + 11) + \frac{\lambda_2}{64 \gamma^3}(n^2 + 11n + 31); \]
\[ T_8 \equiv \frac{\lambda_1}{24 \gamma^2} + \frac{\lambda_2}{320 \gamma^3}(2n + 9); \]
\[ T_9 \equiv -\frac{\gamma}{4} + \frac{\omega^2}{4 \gamma} + \frac{\lambda_1}{12 \gamma^2}(2n + 7) + \frac{\lambda_2}{64 \gamma^3}(n^2 + 7n + 13); \]
\[ T_{10} \equiv \frac{\lambda_1}{24 \gamma^2} + \frac{\lambda_2}{320 \gamma^3}(2n + 1); \]
\[ T_{11} \equiv -\frac{\gamma}{4} + \frac{\omega^2}{4 \gamma} + \frac{\lambda_1}{12 \gamma^2}(2n - 5) + \frac{\lambda_2}{64 \gamma^3}(n^2 - 5n + 7); \]
\[ T_{12} \equiv \frac{\lambda_1}{24 \gamma^2} + \frac{\lambda_2}{320 \gamma^3}(2n - 7); \]
\[ T_{13} \equiv -\frac{\gamma}{4} + \frac{\omega^2}{4 \gamma} + \frac{\lambda_1}{12 \gamma^2}(2n - 9) + \frac{\lambda_2}{64 \gamma^3}(n^2 - 9n + 21). \]

The necessary of the transition energy is given as in the following [4]:

\[
E_n^{(0)}(\gamma) - E_{n+6}^{(0)}(\gamma) = -3\gamma - \frac{3\omega^2}{\gamma} - \frac{3\lambda_1}{2\gamma^2}(2n + 7) + \frac{\lambda_2}{4 \gamma^3}\left( -\frac{3}{2}(n^2 + 6n + 12) - \frac{29}{20}(n + 3) - 1 \right); 
\]
\[ E_n^{(0)}(\gamma) - E_{n+2}^{(0)}(\gamma) \]
\[ = -2\gamma - \frac{2\omega^2}{\gamma} - \frac{\lambda_1}{\gamma^2}(2n + 5) - \frac{\lambda_2}{4\gamma^3}\left(\frac{1}{2}n^2 + n + \frac{2}{3}\right) + \frac{29}{30}(n + 2) + \frac{2}{3}; \]

\[ E_n^{(0)}(\gamma) - E_{n+2}^{(0)}(\gamma) \]
\[ = -\gamma - \frac{\omega^2}{\gamma} - \frac{\lambda_1}{2\gamma^2}(2n + 3) - \frac{\lambda_2}{4\gamma^3}\left(\frac{1}{2}n^2 + n + \frac{2}{3}\right) + \frac{29}{60}(n + 1) + \frac{1}{3}; \]

\[ E_n^{(0)}(\gamma) - E_{n-2}^{(0)}(\gamma) \]
\[ = \gamma + \frac{\omega^2}{\gamma} + \frac{\lambda_1}{2\gamma^2}(2n - 1) + \frac{\lambda_2}{4\gamma^3}\left(\frac{1}{2}n^2 - n + \frac{2}{3}\right) + \frac{29}{60}(n - 1) + \frac{1}{3}; \]

\[ E_n^{(0)}(\gamma) - E_{n-4}^{(0)}(\gamma) \]
\[ = 2\gamma + \frac{2\omega^2}{\gamma} + \frac{\lambda_1}{\gamma^2}(2n - 3) + \frac{\lambda_2}{4\gamma^3}\left(n^2 - 4n + \frac{16}{3}\right) + \frac{29}{30}(n - 2) + \frac{2}{3}; \]

\[ E_n^{(0)}(\gamma) - E_{n-6}^{(0)}(\gamma) \]
\[ = 3\gamma + \frac{3\omega^2}{\gamma} + \frac{3\lambda_1}{2\gamma^2}(2n - 5) + \frac{\lambda_2}{4\gamma^3}\left(\frac{3}{2}n^2 - 6n + 12\right) + \frac{29}{20}(n - 3) + 1. \] 

The second-order perturbation gives [5]:

\[ E_n(\gamma) \]
\[ = E_n^{(0)}(\gamma) + \sum_{k \neq n} \frac{\langle k^{(0)}|V(\gamma)|n^{(0)}\rangle|^2}{E_n^{(0)}(\gamma) - E_k^{(0)}(\gamma)} \]
\[ = E_n^{(0)} + \frac{T_1^2}{E_n^{(0)} - E_{n+6}^{(0)}} + \frac{T_2^2}{E_n^{(0)} - E_{n+4}^{(0)}} + \frac{T_3^2}{E_n^{(0)} - E_{n+2}^{(0)}} \]
\[ + \frac{T_4^2}{E_n^{(0)} - E_{n-2}^{(0)}} + \frac{T_5^2}{E_n^{(0)} - E_{n-4}^{(0)}} + \frac{T_6^2}{E_n^{(0)} - E_{n-6}^{(0)}}. \] 

5
The third-order perturbation gives:

\[
E_n(\gamma)^3 \\
= E_n(\gamma) + \sum_{k \neq n} \sum_{m \neq n} \frac{\langle n^{(0)} | V(\gamma) | m^{(0)} \rangle \langle m^{(0)} | V(\gamma) | k^{(0)} \rangle \langle k^{(0)} | V(\gamma) | n^{(0)} \rangle}{(E_n^{(0)}(\gamma) - E_m^{(0)}(\gamma))(E_n^{(0)}(\gamma) - E_k^{(0)}(\gamma))} \\
= E_n(\gamma) + (n + 1)(n + 2)(n + 3)(n + 4)(n + 5)(n + 6) \times \left( \frac{T_1T_2T_7}{(E_n^{(0)} - E_{n+6}^{(0)})(E_n^{(0)} - E_{n+4}^{(0)})} + \frac{T_1T_3T_8}{(E_n^{(0)} - E_{n+6}^{(0)})(E_n^{(0)} - E_{n+2}^{(0)})} \right) \\
+ (n + 1)(n + 2)(n + 3)(n + 4) \times \left( \frac{(n + 5)(n + 6)T_1T_2T_7}{(E_n^{(0)} - E_{n+4}^{(0)})(E_n^{(0)} - E_{n+6}^{(0)})} + \frac{T_2T_3T_9}{(E_n^{(0)} - E_{n+4}^{(0)})(E_n^{(0)} - E_{n+2}^{(0)})} \right) \\
+ (n + 1)(n + 2) \times \left( \frac{(n + 3)(n + 4)(n + 5)(n + 6)T_1T_3T_8}{(E_n^{(0)} - E_{n+2}^{(0)})(E_n^{(0)} - E_{n+6}^{(0)})} + \frac{(n + 3)(n + 4)T_2T_3T_9}{(E_n^{(0)} - E_{n+2}^{(0)})(E_n^{(0)} - E_{n+4}^{(0)})} \right) \\
+ (n + 1)(n + 2) \times \left( \frac{(n + 1)(n + 2)(n + 3)(n + 4)T_1T_2T_4}{(E_n^{(0)} - E_{n+2}^{(0)})(E_n^{(0)} - E_{n+4}^{(0)})} + \frac{(n + 1)(n + 2)T_3T_4T_{10}}{E_n^{(0)} - E_{n+2}^{(0)})(E_n^{(0)} - E_{n+4}^{(0)})} \right) \\
+ (n + 1)(n + 2)(n + 3)(n + 4) \times \left( \frac{(n + 5)(n + 4)(n + 3)(n + 2)T_4T_5T_{11}}{E_n^{(0)} - E_{n+2}^{(0)})(E_n^{(0)} - E_{n+4}^{(0)})} + \frac{(n + 5)(n + 4)T_5T_6T_{12}}{E_n^{(0)} - E_{n+2}^{(0)})(E_n^{(0)} - E_{n+4}^{(0)})} \right) \\
+ (n + 1)(n + 2) \times \left( \frac{(n + 1)(n + 2)(n + 3)(n + 4)T_1T_2T_5}{E_n^{(0)} - E_{n+4}^{(0)})(E_n^{(0)} - E_{n+2}^{(0)})} + \frac{T_4T_5T_{11}}{E_n^{(0)} - E_{n+4}^{(0)})(E_n^{(0)} - E_{n+2}^{(0)})} \right) \\
+ (n + 1)(n + 2)(n + 3)(n + 4) \times \left( \frac{(n + 5)(n + 4)(n + 3)(n + 2)T_4T_5T_{13}}{E_n^{(0)} - E_{n+4}^{(0)})(E_n^{(0)} - E_{n+2}^{(0)})} + \frac{T_5T_6T_{13}}{E_n^{(0)} - E_{n+4}^{(0)})(E_n^{(0)} - E_{n+2}^{(0)})} \right).
\]
2.2.2 Numerical Solution

We first demonstrate the accuracy of the analytical formula for the \( \omega^2 = 1 \) and \((\lambda_1, \lambda_2) = (16, 0); (16, 256)\) in Tables 1 and 2.

| \( n \) | \( E_n(\gamma)_2 \) | \( E_n(\gamma)_3 \) | Numerical Solution | Deviation 1 | Deviation 2 |
|---|---|---|---|---|---|
| 0 | 1.0292 | 1.0292 | 1.0268 | 0.2337\% | 0.2337\% |
| 1 | 3.5762 | 3.5762 | 3.5721 | 0.1147\% | 0.1147\% |
| 2 | 6.8789 | 6.8698 | 6.865 | 0.2024\% | 0.0699\% |
| 3 | 10.6461 | 10.6216 | 10.6141 | 0.3014\% | 0.0706\% |
| 4 | 14.7802 | 14.7385 | 14.7287 | 0.3496\% | 0.0665\% |
| 5 | 19.224 | 19.1638 | 19.1514 | 0.379\% | 0.0647\% |
| 6 | 23.9384 | 23.8587 | 23.8437 | 0.3971\% | 0.0629\% |
| 7 | 28.8951 | 28.7949 | 28.777 | 0.4103\% | 0.0622\% |

Table 1: The comparison between the perturbation and numerical solutions for the \( \omega^2 = 1, \lambda_1 = 16, \) and \( \lambda_2 = 0. \)

| \( n \) | \( E_n(\gamma)_2 \) | \( E_n(\gamma)_3 \) | Numerical Solution | Deviation 1 | Deviation 2 |
|---|---|---|---|---|---|
| 0 | 1.1681 | 1.172 | 1.1599 | 0.7069\% | 1.0431\% |
| 1 | 4.1655 | 4.1761 | 4.1545 | 0.2647\% | 0.5199\% |
| 2 | 8.2973 | 8.2828 | 8.2622 | 0.4248\% | 0.2493\% |
| 3 | 13.2538 | 13.19 | 13.1621 | 0.6966\% | 0.2119\% |
| 4 | 18.8898 | 18.7598 | 18.7216 | 0.8984\% | 0.204\% |
| 5 | 25.12 | 24.9111 | 24.8604 | 1.0442\% | 0.2039\% |
| 6 | 31.8858 | 31.5874 | 31.522 | 1.1541\% | 0.2074\% |
| 7 | 39.1433 | 38.7462 | 38.6639 | 1.2399\% | 0.2128\% |

Table 2: The comparison between the perturbation and numerical solutions for the \( \omega^2 = 1, \lambda_1 = 16, \) and \( \lambda_2 = 256. \)

The accuracy in the strong coupling region is lower than or around 1\% within the third-order perturbation. Therefore, the perturbation gives an accurate analytical-formula to the strong coupling region, and indeed, other coupling regions also shows so. The Deviation 1 is defined as the deviation of the leading-order correction from the numerical solution, and the Deviation 2 is defined as the deviation of the next leading-order correction from the numerical solution in all Tables.
The Hamiltonian in the numerical study is defined by the discretized kinetic energy \((p^2/2)\psi \rightarrow -(\psi_{j+1} - 2\psi_j + \psi_{j-1})/(2a^2)\), where \(\psi_j\) is the eigenfunction at the site \(x_j\) in the discrete theory, and \(a\) is the lattice spacing. The lattice index is labeled by \(j = 1, 2, \ldots, n\), where \(n\) is the number of lattice points. The discrete system has \(n + 1\) lattice points with a lattice size \(2L\) and the periodic boundary condition as the below:

\[-L \leq x_j \leq L; \quad x_0 = -L; \quad x_{j+1} \equiv x_j + a; \quad \psi_0 \equiv \psi_n; \quad 2L = na. (9)\]

The numerical parameters in this letter are chosen as that: lattice size \(L = 8\) and number of lattice points \(n = 16384\).

In the final, we show the accurate study from \(\omega^2 = 0\) in Tables 3 and 4. When coupling constants vanish, the unperturbed state is not a Fock state, and the region cannot be studied from the weak-coupling perturbation. Therefore, the accurate result should demonstrate the applicability of adaptive perturbation method to the similar problem of Higgs field before we give a more direct evidence from the inverted mass term.

| \(n\) | \(E_n(\gamma)_2\) | \(E_n(\gamma)_3\) | Numerical Solution | Deviation 1 | Deviation 2 |
|------|----------------|----------------|------------------|-------------|-------------|
| 0    | 0.9299         | 0.9299         | 0.9263           | 0.3886%     | 0.3886%     |
| 1    | 3.3249         | 3.3249         | 3.3193           | 0.1687%     | 0.1687%     |
| 2    | 6.5305         | 6.5194         | 6.5131           | 0.2671%     | 0.0967%     |
| 3    | 10.2112        | 10.1822        | 10.1726          | 0.3794%     | 0.0943%     |
| 4    | 14.2664        | 14.2181        | 14.206           | 0.4251%     | 0.0851%     |
| 5    | 18.6366        | 18.5683        | 18.5534          | 0.4484%     | 0.0803%     |
| 6    | 23.2818        | 23.1925        | 23.1747          | 0.4621%     | 0.0768%     |
| 7    | 28.1727        | 28.0616        | 28.0406          | 0.4711%     | 0.0748%     |

Table 3: The comparison between the perturbation and numerical solutions for the \(\omega^2 = 0, \lambda_1 = 16,\) and \(\lambda_2 = 0\).

### 2.3 \(\langle x^2 \rangle\)

The complete information of a quantum system contains the eigenenergy and also the eigenstates. Usually, it is harder to obtain the accurate eigenstate than the eigenvalue. When we choose the adaptive parameter by minimizing the energy, it possibly only guarantees that the suitable perturbed state for calculating eigenenergy, but the correlation functions may not be adaptive enough. In QFT, we are interested in the correlation
Table 4: The comparison between the perturbation and numerical solutions for the $\omega^2 = 0$, $\lambda_1 = 16$, and $\lambda_2 = 256$.

Table 5: The comparison between the perturbation and numerical solutions for the $\omega^2 = 1$, $\lambda_1 = 16$, and $\lambda_2 = 0$. 

Table 6: The comparison between the perturbation and numerical solutions for the $\omega^2 = 0$, $\lambda_1 = 16$, and $\lambda_2 = 256$. 
The perturbed eigenstate is

\[
|n\rangle = |n^{(0)}\rangle + \sum_{k \neq n} \frac{\langle k^{(0)}|V|n^{(0)}\rangle}{E_{n}^{(0)} - E_{k}^{(0)}} |k^{(0)}\rangle \\
+ \left( \sum_{k \neq n} \sum_{l \neq n} \frac{\langle k^{(0)}|V|l^{(0)}\rangle \langle l^{0}|V|n^{(0)}\rangle}{(E_{n}^{(0)} - E_{k}^{(0)})(E_{n}^{(0)} - E_{l}^{(0)})} |k^{(0)}\rangle \\
- \frac{1}{2} \sum_{k \neq n} \frac{\langle n^{(0)}|V|k^{(0)}\rangle \langle k^{(0)}|V|n^{(0)}\rangle}{(E_{n}^{(0)} - E_{k}^{(0)})^2} |n^{(0)}\rangle \right) + \cdots. \tag{10}
\]

The leading-order is \( |n^{(0)}\rangle \), the leading-order correction is

\[
|n^{(1)}\rangle = \sum_{k \neq n} \frac{\langle k^{(0)}|V|n^{(0)}\rangle}{E_{n}^{(0)} - E_{k}^{(0)}} |k^{(0)}\rangle, \tag{11}
\]

and the next leading-order correction is

\[
|n^{(2)}\rangle = \sum_{k \neq n} \sum_{l \neq n} \frac{\langle k^{(0)}|V|l^{(0)}\rangle \langle l^{0}|V|n^{(0)}\rangle}{(E_{n}^{(0)} - E_{k}^{(0)})(E_{n}^{(0)} - E_{l}^{(0)})} |k^{(0)}\rangle \\
- \frac{1}{2} \sum_{k \neq n} \frac{\langle n^{(0)}|V|k^{(0)}\rangle \langle k^{(0)}|V|n^{(0)}\rangle}{(E_{n}^{(0)} - E_{k}^{(0)})^2} |n^{(0)}\rangle. \tag{12}
\]

| \( n \) | \( S_{n}(\gamma)_{1} \) | \( S_{n}(\gamma)_{2} \) | Numerical Solution | Deviation 1 | Deviation 2 |
|---|---|---|---|---|---|
| 0 | 0.1984 | 0.2073 | 0.2072 | 4.2471% | 0.0482% |
| 1 | 0.502 | 0.5176 | 0.516 | 2.7131% | 0.31% |
| 2 | 0.703 | 0.7118 | 0.7124 | 1.3194% | 0.0842% |
| 3 | 0.8784 | 0.8864 | 0.8916 | 1.4804% | 0.5832% |
| 4 | 1.0378 | 1.0461 | 1.054 | 1.537% | 0.7495% |
| 5 | 1.1859 | 1.1947 | 1.2048 | 1.5687% | 0.8383% |
| 6 | 1.3253 | 1.3347 | 1.3466 | 1.5817% | 0.8837% |
| 7 | 1.4578 | 1.4679 | 1.4814 | 1.593% | 0.9113% |

Table 6: The comparison between the perturbation and numerical solutions for the \( \omega^2 = 0, \lambda_1 = 16, \) and \( \lambda_2 = 0. \)
The \( x^2 \) up to the first-order correction is given by:

\[
S_n(\gamma)_1
= \langle n^{(0)}|x^2|n^{(0)}\rangle + 2\langle n^{(1)}|x^2|n^{(0)}\rangle
= \frac{1}{2\gamma}(2n + 1) + \frac{1}{\gamma}\left((n + 1)(n + 2)\frac{T_3}{E_n(0) - E_{n+2}(0)} + (n - 1)n\frac{T_3}{E_n(0) - E_{n-2}(0)}\right).
\] (13)

The \( x^2 \) up to the second-order correction is given by:

\[
S_n(\gamma)_2
= S_n(\gamma)_1 + 2\langle n^{(2)}|x^2|n^{(0)}\rangle + \langle n^{(1)}|x^2|n^{(1)}\rangle
= S_n(\gamma)_1 + \frac{1}{\gamma}\left[(n + 1)(n + 2)\left(\frac{1}{(E_n(0) - E_{n+2}(0))(E_n(0) - E_{n-2}(0))} \langle n^{(0)}|\right)\right.
\]

\[
+ \frac{(n + 3)(n + 4)T_2T_3}{(E_n(0) - E_{n+2}(0))(E_n(0) - E_{n+4}(0))}
\]

\[
\left. + \frac{(n + 1)(n + 2)(n + 3)(n + 4)T_2T_3}{(E_n(0) - E_{n+4}(0))(E_n(0) - E_{n+2}(0))} \right] + \frac{(n - 1)(n + 1)_n}{(n - 1)nT_4T_6}
\]

\[
+ \frac{(n + 1)(n + 2)(n + 3)(n + 4)T_2T_3}{(E_n(0) - E_{n+4}(0))(E_n(0) - E_{n+2}(0))}
\]

\[
+ \frac{(n - 1)(n + 1)_n}{(n - 1)nT_4T_6}
\]

\[
+ \frac{(n - 1)(n + 1)_n}{(n - 1)nT_4T_6}
\]

\[
= \frac{6(n - 5)(n - 4)(n - 3)(n - 2)(n - 1)nT^2_6}{(E_n(0) - E_{n-6}(0))^2} + \frac{4(n - 3)(n - 2)(n - 1)nT^2_5}{(E_n(0) - E_{n-4}(0))^2}
\]

\[
- \frac{2(n - 1)nT^2_4}{(E_n(0) - E_{n-2}(0))^2} + \frac{2(n + 1)(n + 2)T^2_5}{(E_n(0) - E_{n+2}(0))^2} + \frac{4(n + 1)(n + 2)(n + 3)(n + 4)T^2_2}{(E_n(0) - E_{n+4}(0))^2}
\]

\[
+ \frac{6(n + 1)(n + 2)(n + 3)(n + 4)(n + 5)(n + 6)T^2_2}{(E_n(0) - E_{n+6}(0))^2}.
\] (14)
3 Inverted Mass Term

Now we discuss the analytical formula for the inverted mass term ($\omega^2 = -1$). For an oscillator, the eigenenergy is proportional to $\omega$. Therefore, the analytical continuation of the $\omega$ from the real-valued number to the imaginary number should be failed for the weak-coupling perturbation. The Hamiltonian is bounded from below for any positive value of the coupling constants. Therefore, the analytical continuation should not be problematic. Indeed, the real problem is the perturbation method. The perturbation relies on the existence of a discrete spectrum or a Fock state at the leading-order. The adaptive perturbation method introduces the coupling constants into the leading-order perturbation \[3\]. Hence the inverted mass case still has a discrete spectrum at the unperturbed level. For an inverted oscillator case, the eigenenergy can be negative. If the adaptive perturbation method is compatible with the analytical continuation, we can determine what is the critical value of the coupling constants for having a positive definite spectrum in the model. Here we turn off the $\lambda_2$ for the convenience. The third-order calculation shows the critical value lies on $0.8344 - 0.8345$ with a deviation of less than 10% from the true critical-value $0.9072 - 0.9073$ (numerical value). When the $\lambda_1 = 0$, the system loses a discrete spectrum. Hence the 10% deviation is not strange in the weak-coupling region, and one can find the perturbation result approaches to the numerical value by introducing the higher-order terms.

As we discussed above, the study of the inverted mass case is non-trivial because the weak-coupling perturbation is failed in a strong coupling region as in the usual situation and also in a weak-coupling region. To give a concrete evidence for the application of the adaptive perturbation to the inverted mass term, we show the eigenenergy and $\langle x^2 \rangle$ for the case $(\lambda_1, \lambda_2) = (16, 0)$ in Tables 7 and 8 respectively.

4 Outlook

The accurate study of $\langle x^2 \rangle$ demonstrated that the study of the adaptive perturbation method \[3\] can be extended to correlation functions. Confinement and asymptotic freedom are important open questions in theoretical physics. To explore the open questions, it is necessary to compute the correlation functions for all values of parameters. The information about the strongly coupled QFT was only given from the lattice method. It is hard to know whether the lattice method is correct due to the existence of the continuum limit. The adaptive perturbation method can be applied to lattice QFT and
Table 7: The comparison between the perturbation and numerical solutions for the $\omega^2 = -1$, $\lambda_1 = 16$, and $\lambda_2 = 0$.

| $n$ | $E_n(\gamma)_1$ | $E_n(\gamma)_2$ | Numerical Solution | Deviation 1 | Deviation 2 |
|-----|-----------------|-----------------|-------------------|-------------|-------------|
| 0   | 0.8247          | 0.8247          | 0.8193            | 0.7069%     | 1.0431%     |
| 1   | 3.0633          | 3.0633          | 3.05574           | 0.2768%     | 0.532%      |
| 2   | 6.1743          | 6.1608          | 6.1525            | 0.4479%     | 0.2723%     |
| 3   | 9.7691          | 9.7346          | 9.7223            | 0.7311%     | 0.2462%     |
| 4   | 13.7455         | 13.6896         | 13.6745           | 0.9474%     | 0.2527%     |
| 5   | 18.0423         | 17.9646         | 17.9466           | 1.1088%     | 0.268%      |
| 6   | 22.6183         | 22.5181         | 22.497            | 1.2366%     | 0.2892%     |
| 7   | 27.4434         | 27.32           | 27.2955           | 1.3413%     | 0.3132%     |

Table 8: The comparison between the perturbation and numerical solutions for the $\omega^2 = -1$, $\lambda_1 = 16$, and $\lambda_2 = 0$.

| $n$ | $S_n(\gamma)_1$ | $S_n(\gamma)_2$ | Numerical Solution | Deviation 1 | Deviation 2 |
|-----|-----------------|-----------------|-------------------|-------------|-------------|
| 0   | 0.2094          | 0.221           | 0.2211            | 5.2917%     | 0.0452%     |
| 1   | 0.5215          | 0.5405          | 0.5386            | 3.1748%     | 0.3527%     |
| 2   | 0.7193          | 0.7288          | 0.73              | 1.4657%     | 0.1643%     |
| 3   | 0.8939          | 0.9025          | 0.9095            | 1.7152%     | 0.7696%     |
| 4   | 1.053           | 1.0619          | 1.0718            | 1.754%      | 0.9236%     |
| 5   | 1.2009          | 1.2104          | 1.2225            | 1.7668%     | 0.9897%     |
| 6   | 1.3403          | 1.3503          | 1.3643            | 1.7591%     | 1.0261%     |
| 7   | 1.4727          | 1.4834          | 1.4991            | 1.761%      | 1.0472%     |

provides more clues to the open questions.

In the standard model of particle physics, the Higgs boson is produced by the excitation of the Higgs field. The Higgs boson is ill-defined when a coupling constant vanishes because the vacuum expectation value of the Higgs field is inversely proportional to the square root of a coupling constant. Hence the weak-coupling physics in the Higgs field possibly cannot use the weak-coupling perturbation to give complete information. Theoretical studies ignored the problem before. Although theoretical studies about the Higgs field was confirmed by LHC, the ill-defined problem possibly appears after reducing the statistical error. Our accurate study should demonstrate the applicability
for the inverted mass case and sheds light on the relevant problems of the Higgs field.

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