HADAMARD TYPE INEQUALITIES FOR $m$–CONVEX AND 
$(\alpha, m)$–CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstraction. In this paper, we established some new Hadamard-type integral inequalities for functions whose derivatives of absolute values are $m$–convex and $(\alpha, m)$–convex functions via Riemann-Liouville fractional integrals.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following inequality is well known in the literature as the Hermite–Hadamard inequality:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$  

In [1], G.Toader defined the concept of $m$-convexity as the following:

Definition 1. The function $f : [0, b] \to \mathbb{R}$ is said to be $m$–convex, where $m \in [0, 1]$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ we have:

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y).$$

Denote by $K_m(b)$ the set of the $m$–convex functions on $[0, b]$ for which $f(0) \leq 0$.

Several papers have been written on $m$–convex functions and we refer the papers [2]–[11].

In [11], the following inequality of Hermite-Hadamard type for $m$–convex functions holds:

Theorem 1. Let $f : [0, \infty) \to \mathbb{R}$ be a $m$–convex function with $m \in (0, 1]$. If $0 \leq a < b < \infty$ and $f \in L^1[a, b]$, then one has the inequality:

$$(1.1) \quad \frac{1}{b - a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf(b)}{2}, \frac{f(b) + mf(a)}{2} \right\}.$$  

In [19], S.S. Dragomir proved the following theorem.

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Theorem 2. Let \( f : [0, \infty) \to \mathbb{R} \) be a \( m \)-convex function with \( m \in (0, 1] \). If \( f \in L_1(a, b) \) where \( 0 \leq a < b \), then one has the inequality:

\[
\frac{1}{m+1} \left\{ \frac{1}{mb-a} \int_a^{mb} f(x)\,dx + \frac{1}{b-na} \int_{ma}^b f(x)\,dx \right\} \leq \frac{f(a) + f(b)}{2}. 
\]

In \[15\], Mihe\c{s}an gave definition of \((\alpha, m)\)-convexity as following:

Definition 2. The function \( f : [0, b] \to \mathbb{R}, b > 0 \) is said to be \((\alpha, m)\)-convex, where \((\alpha, m) \in [0, 1]^2\), if we have

\[
f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha) f(y)
\]

for all \( x, y \in [0, b] \) and \( t \in [0, 1] \).

Denote by \( K_m^\alpha(b) \) the class of all \((\alpha, m)\)-convex functions on \([0, b]\) for which \( f(0) \leq 0 \). If we choose \((\alpha, m) = (1, m)\), it can be easily seen that \((\alpha, m)\)-convexity reduces to \( m \)-convexity and for \((\alpha, m) = (1, 1)\), we have ordinary convex functions on \([0, b]\). For the recent results based on the above definition see the papers \[2\], \[9\], \[6\], \[7\], \[4\], and \[5\].

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 3. Let \( f \in L_1(a, b) \). The Riemann-Liouville integrals \( J_{a^+}^\alpha f \) and \( J_{b^-}^\alpha f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a
\]

and

\[
J_{b^-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b
\]

where \( \Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \,dt \), here is \( J_{a^+}^\alpha f(x) = J_{b^-}^\alpha f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral. Properties of this operator can be found in the references \[12\]-\[14\].

In \[15\], Sarikaya et al. proved a variant of the identity is established by Dragomir and Agarwal in \[16\] Lemma 2.1 for fractional integrals as the following.

Lemma 1. Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with \( a < b \). If \( f'' \in L[a, b] \), then the following equality for fractional integrals holds:

\[
\frac{f(a) + f(b)}{2} = \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right]
= \frac{b-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b)\,dt. 
\]

The aim of this paper is to establish Hadamard type inequalities for \( m \)-convex and \((\alpha, m)\)-convex functions via Riemann-Liouville fractional integrals.
2. Inequalities for \(m\)-convex functions

**Theorem 3.** Let \(f : [a, b] \to \mathbb{R}\) be a positive function with \(0 \leq a < b\) and \(f \in L^1[a, b]\). If \(f\) is a \(m\)-convex function on \([a, b]\), then the following inequalities for fractional integrals with \(\alpha > 0\) and \(m \in (0, 1]\) hold:

\[
\Gamma(\alpha) \left( b - a \right)^\alpha J_\alpha^a f(b) \leq \frac{f(a)}{\alpha + 1} + m f \left( \frac{b}{m} \right) \Gamma(\alpha) \Gamma(2) \Gamma(\alpha + 2),
\]

\[
\Gamma(\alpha) \left( b - a \right)^\alpha J_\alpha^b f(a) \leq \frac{f(b)}{\alpha + 1} + m f \left( \frac{a}{m} \right) \Gamma(\alpha) \Gamma(2) \Gamma(\alpha + 2),
\]

where \(\Gamma\) is Euler Gamma function.

**Proof.** Since \(f\) is a \(m\)-convex function on \([a, b]\), we know that for any \(t \in [0, 1]\)

\[
f(ta + (1 - t)b) \leq tf(a) + m(1 - t)f \left( \frac{b}{m} \right)
\]

and

\[
f(tb + (1 - t)a) \leq tf(b) + m(1 - t)f \left( \frac{a}{m} \right).
\]

By multiplying both sides of (2.2) by \(t^{\alpha - 1}\), then by integrating the resulting inequality with respect to \(t\) over \([0, 1]\), we obtain

\[
\int_0^1 t^{\alpha - 1} f(ta + (1 - t)b) dt \leq \int_0^1 t^{\alpha - 1} \left[ tf(a) + m(1 - t)f \left( \frac{b}{m} \right) \right] dt
\]

\[
= \frac{f(a)}{\alpha + 1} + m f \left( \frac{b}{m} \right) \beta(\alpha, 2).
\]

It is easy to see that \(\int_0^1 t^{\alpha - 1} f(ta + (1 - t)b) dt = \frac{\Gamma(\alpha)}{(b - a)^\alpha} J_\alpha^a f(b)\) and we note that, the Beta and the Gamma function (see [17, pp 908-910])

\[
\beta(x, y) = \int_0^1 t^{x - 1} (1 - t)^{y - 1} dt, \quad x, y > 0, \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x - 1} dt, \quad x > 0
\]

are used to evaluate the integral

\[
\int_0^1 t^{\alpha - 1} (1 - t) dt,
\]

where

\[
\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}
\]

thus we can obtain that

\[
\beta(\alpha, 2) = \frac{\Gamma(\alpha) \Gamma(2)}{\Gamma(\alpha + 2)}
\]

which completes the proof.

For the proof of the second inequality in (2.1) we multiply both sides of (2.3) by \(t^{\alpha - 1}\), then integrate the resulting inequality with respect to \(t\) over \([0, 1]\). □

**Remark 1.** If we choose \(\alpha = 1\) in Theorem 3, then the inequalities (2.1) become the inequality in (1.1).
Theorem 4. Let \( f : I^o \subset [0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \) such that \( f' \in L[a, b] \) where \( a, b \in I, a < b \). If \( |f'|^q \) is \( m \)-convex on \([a, b]\) for some fixed \( m \in (0, 1] \) and \( q \geq 1 \), then the following inequality for fractional integrals holds:

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \right| \\
\leq \frac{b-a}{2} 2^{1-\frac{1}{q}} \left[ \frac{2^\alpha - 1}{2^\alpha (\alpha + 1)} \right] \left[ |f'(a)|^q + m \left| f' \left( \frac{b}{m} \right) \right|^q \right]^{\frac{1}{q}}.
\]

Proof. Suppose that \( q = 1 \). From Lemma 1 and by using the properties of modulus, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \right| \\
\leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| |f'(ta + (1-t)b)| \, dt.
\]

Since \( |f'| \) is \( m \)-convex on \([a, b]\), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_a^\alpha f(b) + J_b^\alpha f(a)] \right| \\
\leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^\alpha| \left[ t |f'(a)| + m(1-t) \left| f' \left( \frac{b}{m} \right) \right| \right] \, dt \\
= \frac{b-a}{2} \left\{ \int_0^\frac{1}{2} |(1-t)^\alpha - t^\alpha| \left[ t |f'(a)| + m(1-t) \left| f' \left( \frac{b}{m} \right) \right| \right] \, dt + \int_\frac{1}{2}^1 |t^\alpha - (1-t)^\alpha| \left[ t |f'(a)| + m(1-t) \left| f' \left( \frac{b}{m} \right) \right| \right] \, dt \right\} \\
= \frac{b-a}{2} \left[ \frac{2^\alpha - 1}{2^\alpha (\alpha + 1)} \right] \left[ |f'(a)| + m \left| f' \left( \frac{b}{m} \right) \right| \right]
\]

where we use the facts that

\[
\int_0^\frac{1}{2} (1-t)^\alpha \, dt = \int_0^1 t^\alpha (1-t) \, dt = \frac{1}{(\alpha + 1) (\alpha + 2)} - \frac{\alpha + 3}{2^{\alpha+2} (\alpha + 1) (\alpha + 2)},
\]

\[
\int_0^\frac{1}{2} t^{\alpha+1} \, dt = \int_0^1 (1-t)^{\alpha+1} \, dt = \frac{1}{2^{\alpha+2} (\alpha + 2)},
\]

\[
\int_0^\frac{1}{2} (1-t)^{\alpha+1} \, dt = \int_\frac{1}{2}^1 t^{\alpha+1} \, dt = \frac{1}{(\alpha + 2)} - \frac{1}{2^{\alpha+2} (\alpha + 2)}
\]

and

\[
\int_0^\frac{1}{2} t^\alpha (1-t) \, dt = \int_\frac{1}{2}^1 (1-t)^\alpha t \, dt = \frac{\alpha + 3}{2^{\alpha+2} (\alpha + 1) (\alpha + 2)}
\]
which completes the proof for this case. Suppose now that $q > 1$. From Lemma 1 $m-$convexity of $|f|^q$ and using the well-known Hölder’s inequality we have successively

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right|$$

$$\leq \frac{b - a}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha| |f'(ta + (1 - t)b)| \, dt$$

$$= \frac{b - a}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha|^{1 - \frac{q}{p}} |(1 - t)^\alpha - t^\alpha|^{\frac{q}{p}} |f'(ta + (1 - t)b)| \, dt$$

$$\leq \frac{b - a}{2} \left( \int_0^1 |(1 - t)^\alpha - t^\alpha| \, dt \right)^{\frac{q}{p}} \left( \int_0^1 |(1 - t)^\alpha - t^\alpha|^{1 + \frac{q}{p}} |f'(ta + (1 - t)b)|^q \, dt \right)^{\frac{1}{q}}$$

$$\leq \frac{b - a}{2} 2^{1 - \frac{q}{p}} \left[ 2^\frac{q - 1}{p(\alpha + 1)} \right] \left[ |f'(a)|^q + m \left| f'( \frac{b}{m} ) \right|^q \right]^{\frac{1}{q}}$$

where we use the fact that

$$\int_0^1 |(1 - t)^\alpha - t^\alpha| \, dt = \int_0^{\frac{1}{2}} [(1 - t)^\alpha - t^\alpha] \, dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1 - t)^\alpha] \, dt$$

$$= \frac{2}{\alpha + 1} \left( 1 - \frac{1}{2^\alpha} \right).$$

\[ \square \]

**Corollary 1.** Under the assumptions of Theorem 4 with $\alpha \in (0, 1]$, the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right|$$

$$\leq \frac{b - a}{2} \left( \frac{1}{\alpha p + 1} \right)^{\frac{1}{q}} \left[ |f'(a)|^q + m \left| f'( \frac{b}{m} ) \right|^q \right]^{\frac{1}{q}}.$$

**Proof.** It is similar the proof of Theorem 4. In addition, we used the following inequality

$$|t_1^\alpha - t_2^\alpha| \leq |t_1 - t_2|^\alpha,$$

where $\alpha \in (0, 1]$ and $t_1, t_2 \in [0, 1]$.\[ \square \]

**Theorem 5.** Let $f : [0, \infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0, 1]$. If $f \in L_1[am, b]$, where $0 \leq a < b$, then the following inequality for fractional integrals with $\alpha > 0$ holds:

$$\left( 2.4 \right) \frac{\Gamma(\alpha)}{m + 1} \left\{ \frac{1}{(nb - a)} [J_{a+}^\alpha f(mb) + J_{b-}^\alpha f(mb)] + \frac{1}{(b - ma)} [J_{ma}^\alpha f(mb) + J_{b-}^\alpha f(mb)] \right\}$$

$$\leq \frac{f(a) + f(b)}{\alpha}.$$

**Proof.** By the $m-$convexity of $f$ we can write

$$f(ta + m(1 - t)b) \leq tf(a) + m(1 - t)f(b),$$

$$f((1 - t)a + mtb) \leq (1 - t)f(a) + mtf(b),$$

$$J_{a+}^\alpha f(ta + m(1 - t)b) \leq tf(a) + m(1 - t)f(b),$$

$$J_{b-}^\alpha f((1 - t)(a + mtb)) \leq (1 - t)f(a) + mtf(b).$$
\[ f(tb + m(1-t)a) \leq tf(b) + m(1-t)f(a) \]

and

\[ f((1-t)b + mta) \leq (1-t)f(b) + mtf(a) \]

for all \( t \in [0,1] \) and \( 0 \leq a < b \).

If we add the above inequalities we get

\[ f(tb + m(1-t)b) + f((1-t)a + mtb) + f(tb + m(1-t)a) + f((1-t)b + mta) \leq (m+1)(f(a) + f(b)). \]

By multiplying both sides of (2.5) with \( t^{\alpha-1} \), then integrating the resulting inequality with respect to \( t \) over \([0,1]\), we obtain

\[
\int_0^1 t^{\alpha-1} \left[ f(tb + m(1-t)b) + f((1-t)a + mtb) + f(tb + m(1-t)a) + f((1-t)b + mta) \right] dt \\
\leq (m+1)(f(a) + f(b)) \int_0^1 t^{\alpha-1} dt.
\]

It is easy to see that

\[
\int_0^1 t^{\alpha-1} f(tb + m(1-t)b) dt = \frac{\Gamma(\alpha)}{(mb-a)^\alpha} J_{mb}^a f(mb),
\]

\[
\int_0^1 t^{\alpha-1} f((1-t)a + mtb) dt = \frac{\Gamma(\alpha)}{(mb-a)^\alpha} J_{mb}^a f(mb),
\]

\[
\int_0^1 t^{\alpha-1} f(tb + m(1-t)a) dt = \frac{\Gamma(\alpha)}{(ma-b)^\alpha} J_{ma}^b f(mb)
\]

and

\[
\int_0^1 t^{\alpha-1} f((1-t)b + mta) dt = \frac{\Gamma(\alpha)}{(ma-b)^\alpha} J_{ma}^b f(mb).
\]

So the proof is completed. \( \Box \)

**Remark 2.** If we choose \( \alpha = 1 \) in Theorem 5 then the inequality (2.4) becomes the inequality in (1.4).

### 3. Inequalities for \((\alpha, m)\)-Convex Functions

**Theorem 6.** Let \( f : [a, b] \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in L_1[a, b] \). If \( f \) is an \((\alpha_1, m)\)-convex function on \([a, b]\), then the following inequalities hold for fractional integrals with \( \alpha > 0 \) and \((\alpha_1, m) \in (0,1]^2\):

\[ \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a}^b f(b) \leq \frac{1}{\alpha + \alpha_1} f(a) + \frac{ma_1}{\alpha(\alpha + \alpha_1)} f \left( \frac{b}{m} \right), \]

\[ \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a}^b f(a) \leq \frac{1}{\alpha + \alpha_1} f(b) + \frac{ma_1}{\alpha(\alpha + \alpha_1)} f \left( \frac{a}{m} \right). \]

**Proof.** Since \( f \) is \((\alpha_1, m)\)-convex function on \([a, b]\), we know that for any \( t \in [0,1] \)

\[ f(ta + (1-t)b) \leq t^{\alpha_1} f(a) + m(1-t^{\alpha_1}) f \left( \frac{b}{m} \right) \]
and

\begin{equation}
(3.3) \quad f(tb + (1-t)a) \leq t^{\alpha_1} f(b) + m(1-t^{\alpha_1}) f \left( \frac{a}{m} \right).
\end{equation}

By multiplying both sides of (3.3) by \( t^{\alpha-1} \), then by integrating the resulting inequality with respect to \( t \) over \([0,1]\), we get

\[
\int_0^1 t^{\alpha-1} f(ta + (1-t)b) \, dt \leq \int_0^1 t^{\alpha-1} \left[ t^{\alpha_1} f(a) + m(1-t^{\alpha_1}) f \left( \frac{b}{m} \right) \right] \, dt = \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a^+}^\alpha f(b) + \frac{m\alpha_1}{\alpha(\alpha + \alpha_1)} f \left( \frac{b}{m} \right).
\]

It is easy to see that \( \int_0^1 t^{\alpha-1} f(ta + (1-t)b) \, dt = \frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a^+}^\alpha f(b) \), by using this fact the proof of the first inequality is completed. Similarly, by multiplying both sides of (3.3) by \( t^{\alpha-1} \), then by integration, the proof of the second inequality is completed. \(\square\)

**Corollary 2.** If we choose \( \alpha = \alpha_1 \) with \( \alpha, \alpha_1 \in (0,1] \) in Theorem 6, we have the following inequalities:

\[
\frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{a^+}^\alpha f(b) \leq \frac{1}{2\alpha} \left[ f(a) + mf \left( \frac{b}{m} \right) \right],
\]

\[
\frac{\Gamma(\alpha)}{(b-a)^\alpha} J_{b^-}^\alpha f(a) \leq \frac{1}{2\alpha} \left[ f(b) + mf \left( \frac{a}{m} \right) \right].
\]

**Remark 3.** If we choose \( \alpha = \alpha_1 = 1 \) in Theorem 6 then the inequalities reduces to the inequality (1.4).

**Theorem 7.** Let \( f : I^2 \subset [0,\infty) \to \mathbb{R} \) be a differentiable function on \( I^2 \) such that \( f' \in L[a,b] \) where \( a, b \in I, a < b \). If \( |f'(t)|^q \) is a \((\alpha, m)\)-convex function and \( |f'| \) is decreasing on \([a, b]\), then the following inequalities hold for fractional integrals with \( \alpha > 0, (\alpha_1, m) \in (0,1]^2 \):

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \leq \frac{b-a}{2} \left[ \frac{2^\alpha - 1}{(\alpha + 1)} \right]^{\frac{1}{q+1}} \left[ \frac{2^{\alpha+\alpha_1} - 1}{2^{\alpha+\alpha_1} (\alpha + \alpha_1 + 1)} \left( |f'(a)|^q - m \left| f' \left( \frac{b}{m} \right) \right|^q \right) \right.
\]

\[
\left. + \frac{m}{\alpha + 1} \left| f' \left( \frac{b}{m} \right) \right|^q \left( 1 - \frac{1}{2^\alpha} \right) \right].
\]

**Proof.** Suppose that \( q = 1 \). From Lemma 4 and by using the properties of modulus, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a) \right] \right| \leq \frac{b-a}{2} \int_0^1 |(1-t)^\alpha - t^{\alpha_1}| \left| f'(ta + (1-t)b) \right| \, dt.
\]
Since $|f'|$ is $(\alpha_1, m)$-convex on $[a, b]$, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J^a_{\alpha^+} f(b) + J^\alpha_b f(a)] \right| 
\leq \frac{b - a}{2} \int_0^1 |(1 - t)^\alpha - t^\alpha| \left[ t^\alpha |f'(a)| + m(1 - t^\alpha) |f'\left(\frac{b}{m}\right)| \right] dt
\]

= \frac{b - a}{2} \left\{ \int_0^{\frac{1}{2}} [(1 - t)^\alpha - t^\alpha] \left[ t^\alpha |f'(a)| + m(1 - t^\alpha) |f'\left(\frac{b}{m}\right)| \right] dt 
+ \int_{\frac{1}{2}}^1 [(t^\alpha - (1 - t)^\alpha] \left[ t^\alpha |f'(a)| + m(1 - t^\alpha) |f'\left(\frac{b}{m}\right)| \right] dt \right\}.
\]

By computing the above integrals, we obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J^a_{\alpha^+} f(b) + J^\alpha_b f(a)] \right| 
\leq \frac{b - a}{2} \left\{ \left( \frac{2^\alpha - 1}{2^\alpha + \alpha_1 - 1} \right) \left( |f'(a)| - m \left| f'\left(\frac{b}{m}\right) \right| \right) 
+ \frac{m}{\alpha + 1} \left| f'\left(\frac{b}{m}\right) \right| \left( \frac{1}{2} \right)^\frac{1}{\alpha} \right\}
\]

where we used the fact that

\[
\int_0^{\frac{1}{2}} t^{\alpha_1}(1 - t)^\alpha dt = \int_{\frac{1}{2}}^1 t^{\alpha_1}(1 - t)^\alpha dt = \beta(\frac{1}{2}; \alpha_1 + 1, \alpha + 1).
\]

This completes the proof of this case. Suppose now that $q > 1$. Again, from Lemma 1 and by applying well-known Hölder’s inequality, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J^a_{\alpha^+} f(b) + J^\alpha_b f(a)] \right| 
\leq \frac{b - a}{2} \left\{ \left( \frac{1}{2} \right)^{\frac{\alpha + 1}{q}} \left( \int_0^1 |(1 - t)^\alpha - t^\alpha| |f'(ta + (1 - t)b)|^q dt \right)^{\frac{1}{q}} \right\}
\]

By computing the above integrals and by using $(\alpha_1, m)$-convexity of $|f'|^q$, we deduce

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J^a_{\alpha^+} f(b) + J^\alpha_b f(a)] \right| 
\leq \frac{b - a}{2} \left\{ \left( \frac{1}{2} \right)^{\frac{\alpha + 1}{q}} \left( \frac{2^\alpha - 1}{2^\alpha + \alpha_1 - 1} \right) \left( |f'(a)|^q - m \left| f'\left(\frac{b}{m}\right) \right|^q \right)^{\frac{1}{q}} \right\}
\]

+ \frac{m}{\alpha + 1} \left| f'\left(\frac{b}{m}\right) \right|^q \left( \frac{1}{2} \right)^{\frac{1}{\alpha}}.
\]

Which completes the proof.
Corollary 3. If we choose $\alpha = \alpha_1$ with $\alpha, \alpha_1 \in (0, 1]$ in Theorem 7 we have the inequality:

$$
\begin{align*}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)\alpha} \left( J^\alpha_a f(b) + J^\alpha_b f(a) \right) \right| \\
&\leq \frac{b-a}{2} \left[ \frac{2^{\alpha} - 1}{2^{\alpha-1}(\alpha + 1)} \right]^{\frac{2\alpha - 1}{\alpha}} \left[ \left( \frac{2^{\alpha} - 1}{2^{2\alpha}(2\alpha + 1)} \right) \left( |f'(a)|^q - m |f'\left( \frac{b}{m} \right)|^q \right) \right]^{\frac{1}{q}} + \frac{m}{\alpha + 1} \left[ f'\left( \frac{b}{m} \right) \right]^q \left( 1 - \frac{1}{2^{\alpha}} \right)^{\frac{1}{q}}.
\end{align*}
$$

Remark 4. Identical result of Theorem 8 can be stated for $(\alpha, m)$-convex functions, but we omit the details.

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