On d-Small Prime Modules

Adwia Jassim Abdul-Alkalik

Republic of Iraq, Ministry of Education, Directorate General of Education In Diyala
adwiaj@yahoo.com

Abstract: Let $R$ be a commutative ring with an identity, and $D$ be a unitary $R$-module. We say a non-zero submodule $E$ of $D$ is d-small prime if for each $c \in R$, $d \in D$, $(d) \ll_d E$ with $cd \in E$, then either $d \in E$ or $c \in [E:D]$ and an $R$-module $D$ is a d-small prime if $annD = annE$ for each non-zero submodule $E$ d-small in $D$. We have given and demonstrated some of the characterizations and features of these types of submodules (modules) in this paper.

Keywords: Prime submodules, Prime modules, d-Small submodules, d-Small prime submodule, d-Small prime modules.

1. Introduction

"A non-zero submodule $N$ of $M$ is called a prime if whenever $r \in R$ and $m \in M$ with $rm \in N$ implies that either $r \in [N:M]$ or $m \in N$ and $M$ is called a prime if $annM = annN$ for each proper submodule $N$ of $M$", [1]. These two concepts have been generalized by many researchers and have studied these generalizations, as examples: Adwia J. Abdul-Alkalik in [2] presented and studied the concept of I-Nearly Prime Submodules, where A non-zero submodule $N$ of $M$ is called a I-Nearly prime if whenever $r \in R$ and $m \in M$ with $rm \in N$ implies that either $r \in [N + J(M):M]$ or $m \in N + J(M)$. Layla S. Mahmood in [3] also presented and studied a new generalization for the two concepts, as follows "a submodule $N$ of an $R$-module $M$ is called small prime iff whenever $r \in R, x \in M$ with $(x) \ll M$ such that $rx \in N$ implies either $x \in N$ or $a \in [N:M]$ and $M$ is called a prime if $annM = annN$ for each non-zero small submodule $N$ of $M". Where "a submodule $N$ of $M$ is called small (notationally, $N \ll M$) if $N + K = M$ for all submodules $K$ of $M$ implies $K = M", [4]. And "an $R$-module $M$ is called a hollow module if every non-zero submodule of $M$ is small in $M" [5]. As for this research, we present and study a generalization of the concept a prime submodule and the concept prime module as follows: A non-zero submodule $E$ of a module $D$ is called a d-small prime submodule if whenever $c \in R, d \in D$, $(d)$ is d-small in $D$ and $cd \in E$, then either $d \in E$ or $c \in [E:D]$, and $D$ is a d-small prime module if $annD = annE$ for each non-zero submodule $E$ d-small in $D$. Where "a submodule $P$ of $X$ is called d-small (notationally, $P \ll_d X$) if $P + W = X$ where $W$ is a direct summand submodules of $X$, then $W = X"$, [6]. And "an $R$-module $D$ is called a d-hollow module if every non-zero submodule of $X$ is d-small in $X" [7].

2. d-small Prime Submodules

In this part, we introduce the concept of d-small prime submodules and demonstrated some of characterizations and features of this type of submodules.
**Definition (2.1):** A non-zero submodule \( E \) of \( R \)-module \( D \) is called d-small prime iff whenever \( c \in R \), \( d \in D \) and \( (d) \ll_d D \) such that \( cd \in E \), then either \( d \in E \) or \( c \in [E:D] \).

A non-zero ideal \( J \) of \( R \) is d-small prime if \( J \) is a d-small prime submodule of an \( R \)-module \( R \).

**Remarks (2.2):**

1. Every prime submodule is d-small prime.

**Proof:** Suppose that \( cd \in E \), where \( c \in R \), \( (d) \ll_d D \). Since \( E \) is a d-prime in \( D \), hence either \( c \in [E:D] \) or \( d \in E \). Therefore \( E \) is a d-small prime submodule in \( D \).

2. Suppose that \( D \) be \( R \)-module and let \( J \) be an ideal of \( R \) with \( J \subseteq \text{ann } D \). Then \( E \) is d-small prime \( R \)-submodule of \( D \) iff \( E \) is a d-small prime \( R/J \) submodule of \( D \).

**Proof:** Let \( \bar{c} \in R/J \), \( d \in D \) with \( (d) \ll_d D \) and \( \bar{c}d \in E \). But \( \bar{c}d = cd \). Therefore, we get the result.

3. Let \( D \) be a non-composable \( R \)-module, then every d-small prime submodule \( E \) of a module \( D \) is prime submodules.

**Proof:** Suppose that \( cd \in E \), where \( c \in R \), \( d \in D \). But \( D \) is a non-composable, so \( (d) \ll_d D \) by [6]. Since \( E \) is a d-small prime in \( D \), hence either \( c \in [E:D] \) or \( d \in E \). Therefore \( E \) is a prime submodule in \( D \).

4. Let \( D \) be a d-hollow \( R \)-module, then every d-small prime submodule \( E \) of a module \( D \) is prime submodule.

**Proof:** Suppose that \( cd \in E \), where \( c \in R \), \( d \in D \). But \( D \) is a d-hollow, so \( (d) \ll_d D \) by [7]. Since \( E \) is a d-small prime in \( D \), hence either \( c \in [E:D] \) or \( d \in E \). Therefore \( E \) is a prime submodule in \( D \).

5. Let \( D \) be a hollow \( R \)-module, then every d-small prime submodule \( E \) of a module \( D \) is prime submodule.

**Proof:** Suppose that \( cd \in E \), where \( c \in R \), \( d \in D \). But \( D \) is a hollow, so \( (d) \ll_d D \) and hence \( (d) \ll_d D \) by [6]. Since \( E \) is a d-small prime in \( D \), hence either \( c \in [E:D] \) or \( d \in E \). Therefore \( E \) is a prime submodule in \( D \).

6. If \( C \) is a d-small prime submodule of a module \( D \), then \( C \) is a small prime submodule in \( D \).

**Proof:** Let \( c \in R \), \( (d) \ll_d D \) with \( cd \in C \). Since \( (d) \ll_d D \), so \( (d) \ll_d D \). But \( C \) is a d-small prime. Hence either \( d \in C \) or \( c \in [C:X] \). Hence \( C \) is a small prime.

7. If \( W \subseteq E \subseteq D \) and \( E \) is a d-small prime of \( D \), then \( W \) need not be d-small prime in \( D \), the following example shows:

Consider \( D = Z_{24} \) as a \( Z \)-module, \( E = (\bar{3}) \) is d-small prime (since prime). However \( W = (\bar{12}) \) is not d-small prime submodule of \( D \), since \( (6) \ll_d Z_{24} \) and \( (12) = 2 \cdot 6 \in W \), but \( 2 \notin [W:D] = 12Z \) and \( 6 \notin W \).

8. The image of d-small prime submodule need not d-small prime in general for example: Let \( \theta: Z_{24} \to Z_{24} \) defined by \( \theta(\bar{d}) = 4\bar{d}, \forall \bar{d} \in Z_{24} \). The submodule \( (3) \) is d-small prime, but \( \theta(\bar{3}) = (12) \) is not d-small prime by (2.1.7).

9. Consider \( D = Z_{24} \) as a \( Z \)-module, \( E = (\bar{8}) \) is not d-small prime of \( D \), since \( (12) \ll_d Z_{24} \) and \( (0) = 2 \cdot 12 \in E \), but \( 2 \notin [E:D] = 8Z \) and \( 12 \notin E \).

10. If \( C \) is a direct summand of a d-small submodule \( E \), then it is not a general d-small prime submodule: For example, if \( D = Z_{24} \) as a \( Z \)-module and \( E = (\bar{2}) \) is d-small prime (since \( (2) \) is prime). \( (\bar{2}) = (\bar{8}) \oplus (\bar{6}) \). But \( C = (\bar{8}) \) is not d-small prime by (2.2, 9).

11. If \( D \) is a semisimple module, then \( (0) \) is d-small prime submodule.

**Proof:** Suppose that \( cd \in E \), where \( c \in R \), \( d \in D \), \( (d) \ll_d D \) and \( cd \in E \). But \( D \) is a semisimple, so is the only \( (0) \ll_d D \) by [6]. Hence \( 0 = c(0) \in E \). Therefore \( E \) is a d-small prime submodule in \( D \).

**Theorem (2.3):** Suppose that \( E \) be a non-zero submodule of a module \( D \). Then, the following are equivalent:

i. A submodule \( E \) is d-small prime.

ii. \( \forall \ c \in R \), \( W \ll_d D \) such that \( cW \nsubseteq E \), implies either \( W \nsubseteq E \) or \( c \in [E:D] \).

**Proof:** (i) \( \implies \) (ii): Let \( cw \nsubseteq E \), Suppose \( W \nsubseteq E \), then \( \exists, w \in W \) such that \( w \nsubseteq E \). Hence \( (w) \ll_d D \), since \( w \in W \) and \( W \ll_d D \) by [6]. Now \( cw \in E \). But \( E \) is a d-small prime submodule of \( D \) and \( w \nsubseteq E \), hence \( c \in [E:D] \).
(ii) \(\Rightarrow\text{(i)}\): Let \(c \in R, d \in X\) and \((d) \ll_d D\) such that \(cd \in E\). Then \(< c > < d > \subseteq E\). So either \(d > \subseteq E\) or \(c \in [E : D]\) by (ii). Thus, either \(d \in E\) or \(c \in [E : D]\). Hence \(E\) is a \(d\)-small prime.

**Theorem (2.4):** Let \(E\) be a non-zero submodule of a module \(D\). Then, the following are equivalent:

1. A submodule \(E\) is a \(d\)-small prime submodule.
2. A submodule \((E : X)\) is a \(d\)-small prime submodule.
3. A submodule \((E : (c))\) is a \(d\)-small prime submodule.

**Proof:** (1) \(\Rightarrow\) (2): Let \(d \in (E : X)\) and \((d) \ll_d D\); then \(cd \in E\) and \(so cd \in E, \forall b \in J\).

**Remark and Examples (3.2):**

**Proposition (2.6):** Suppose that \(D\) be \(R\)-module. Then \(E\) is a \(d\)-small prime submodule iff \(E\) is a \(d\)-small prime submodule of \(DS\), \forall \(E \neq DS\).

**Proof:** \(\Rightarrow\): Let \(E \ll_d D\). So there exists a direct summand submodule \(A\) of \(DS\) such that \(E^S + A = DS\), then \(E\) is a \(d\)-small submodule of \(DS\).

\(\Leftarrow\): Suppose there exists a direct summand submodule \(C\) of \(D\) such that \(E + C = D\). Hence \((E + C)S = DS\). But \(E \ll_d D\), therefore \(B = D\) and \(S = DS\) which is a contradiction.

**Proposition (2.7):** Let \(E\) be a non-zero submodule of a faithful finitely generated multiplication \(R\)-module \(D\). If \(P\) is a \(d\)-small prime submodule of \(D\), then \([E : D]\) is a small prime ideal of \(R\).

**Proof:** Suppose that \(E \subseteq D\) where \(c, b \in R\) such that \((c) \subseteq R\).

(ii) \(\Rightarrow\): Let \(E \subseteq D\). Then either \(rd \in E\) or \((c) \subseteq [E : D]\) Therefore either \(rd \subseteq E\) or \((c) \subseteq [E : D]\). Hence \((E : X)\) is a \(d\)-small prime submodule.

**Remarks and Examples (3.2):**

**Definition (3.1):** An \(R\)-module \(D\) is called a \(d\)-small prime ring iff \(annD = annE, \forall 0 \neq E \ll_d D\).

A ring \(R\) is a \(d\)-small prime ring iff \(annR = 0, \forall 0 \neq J \ll_d R\).

3. \(d\)-Small Primary Modules

In this part, we presented a definition of the \(d\)-small prime modules, studied and demonstrated some of their properties in detail.

**Definition (3.1):** An \(R\)-module \(D\) is called a \(d\)-small prime ring iff \(annD = annE, \forall 0 \neq E \ll_d D\).

A ring \(R\) is a \(d\)-small prime ring iff \(annR = 0, \forall 0 \neq J \ll_d R\).
1- If D is a prime R-module, then D is a d-small prime.
2- If D is a hollow (hollow) d-small prime R-module, then D is a prime.
3- If D be a indecomposable R-module d-small prime R-module, then D is a prime.
4- Since \(Z \triangleleft D\), but \(2Z = \text{ann}(Z) = \{0\}\), hence \(D\) is an integral domain. Let \(0 \neq d \in D\) and \(D\) is a torsion-free integral domain.
5- Since \(\bar{Z} \triangleleft D\), but \(2\bar{Z} = \text{ann}(\bar{Z}) = \{0\}\), hence \(D\) as a Z-module is an integral domain.
6- The module Z as a Z-module is a Z-module since each proper submodule E of Z is d-small by [7] and \(\text{ann}(E) = \{0\}\).

**Theorem (3.3):** Suppose that D is a module, then D is a d-small prime iff \(\text{ann}D = \text{ann}(d), \forall 0 \neq d \in D\) and \((d) \triangleleft D\).

**Proof:** Suppose that \(D\) is a module. Then \(D\) is a d-small prime iff there exists an R-isomorphism \(\theta : (d) \rightarrow \bar{D}\) such that \(\theta(d) = 1\), \(\forall 0 \neq d \in D\) and \((d) \triangleleft D\). It is clear that \(\theta\) is well defined and an R-isomorphism. To prove \(\theta\) is well defined, suppose that \(\text{cd} = \text{ed}\), \(c, e \in D\), \(c \cdot e \in \text{ann}(d)\). Since D is a d-small prime, \(c \cdot e \in \text{ann}(D)\) by theorem (3.3).

**Corollary (3.10):** Suppose that D is a faithful hollow, then D is a d-small prime module iff every submodule of D is isomorphic to the module R.
**Proposition (3.12):** Suppose that $D$ is a module. Then $D$ is a d-small prime iff $(0)$ is a d-small prime submodule of $D$.

**Proof:** Since $D$ is a faithful, so $annD = 0$ and hence $R/annD = R$. But $D$ is a d-hollow, so every non-zero cyclic is d-small submodule of $D$. Hence it is somorphic to the R-module $R$ by theorem (3.4).

**Theorem (3.11):** Suppose that $D$ is a module. Then $D$ is d-small prime iff $(0)$ is a d-small prime submodule of $D$.

**Proof:** $\Rightarrow$ Suppose that $c \in R, d \in D$ with $(d) \ll_d D$ such that $cd = 0$. If $d \neq 0$, so $c \in ann(d)$ and hence $c \in annD$ (since $D$ is d-small prime). So $c \in [0; D]$. If $d = 0$, so $d \in (0)$. Hence $(0)$ is a d-small prime submodule of $D$.

$\Leftarrow$ Suppose that $0 \neq E \ll_d D$ and let $a \in annE$. Then $cd = 0, \forall d \in E$. Hence $cd \in (0)$. Let $d \neq 0$, so $c \in [0; D] = annD$. Then $annE \subseteq annD$ therefore $annD = annE$. Thus $D$ is d-small prime.

**Corollary (3.12):** The following are equivalent of a module $D$:

- A module $D$ is d-small prime.
- $annD = Ann(d), \forall, 0 \neq d \in D$ and $(d) \ll_d D$.
- $(0)$ is a d-small prime.

**Examples (3.13):**

1. $(0)$ is not a d-small prime in $Z_{24}$, since $4 \cdot (6) = (0)$ and $(6) \ll_d Z_{24}$. But $(6) \notin (0)$ and $4 \notin ann Z_{24} = 24Z$. Hence $Z_{24}$ is not a d-small prime module.

2. $(0)$ is not a d-small prime in $Z_{32}$, since $4 \cdot (8) = (0)$ and $(8) \ll_d Z_{32}$. But $(8) \notin (0)$ and $4 \notin ann Z_{32} = 32Z$. Hence $Z_{32}$ is not a d-small prime module.

3. $Z_6$ as a $Z$-module is a semisimple module, then $0$ is a d-small prime submodule by (2.2, 11). Hence $Z_6$ is d-small prime module.

**Theorem (3.14):** Suppose that $D$ is a module. Then $D$ is d-small prime iff $R/annD$ is cogenerated R-module by each non-zero d-small submodule of $D$.

**Proof:** $\Rightarrow$: Suppose that $D$ is d-small prime. To prove $R/annD$ is cogenerated. Let $0 \neq E \ll_d D$ and suppose that $0 \neq d \in E$. So $0 \neq (d) \ll_d D$, [6]. Since $D$ is d-small prime, so $annD = Ann(d)$. Hence $R/annD = R/ann(d)$. But $R/ann(d) \cong Rd = (d)$ is a submodule of $E$. So, $\exists$ a monomorphism from $R/annD$ into $E$. Therefore $R/annD$ is cogenerated by $E$.

$\Leftarrow$: To prove $D$ is d-small prime. Suppose that $0 \neq E \ll_d D$. So $R/annD$ is cogenerated by $E$. Hence, $\exists$ a monomorphism $\theta: R/annD \rightarrow E^A$ for some index set $A$. Let $c \in ann E$, so $cE = 0$. Since $\tilde{1} \in R/ann D$, then $\theta(\tilde{1}) \in E^A$ and so $\theta(\tilde{1})(a) \in E_d = E, \forall a \in A$. But $\theta(c) = c \theta(\tilde{1})$. Hence $(\theta(c))(a) = (c \theta(\tilde{1}))(a) = c(\theta(\tilde{1}))(a) \in E$. Therefore $(\theta(c))(a) = 0, \forall a \in A$ and so $\theta(\tilde{c}) = 0$. Since $\theta$ is a monomorphism, so $\tilde{c} = 0$. Thus, $c + annD = 0$. Therefore $c \in ann D$ and so $ann E \subseteq ann D$. But $ann D \subseteq ann E$, so $ann E = ann D$. Hence $D$ is d-small prime.

**Corollary (3.15):** Suppose that $D$ is a faithful, then $D$ is a d-small prime module iff $R$ is cogenerated R-module by each non-zero d-small submodule of $D$.

**Proposition (3.16):** If $D$ is a d-small prime R-module, then $AnnE$ is a prime ideal of $R$, $\forall, 0 \neq E \ll_d D$.

**Proof:** Let $u, v \in R$ such that $uv \in AnnE$ and $0 \neq E \ll_d D$. Suppose that $v \notin AnnE$, so $vd \neq 0$ for some $d \in E$, and since $uv \in AnnE$ implies that $uvd = 0$. But $(vd)$ is a submodule of $E$ and $E \ll_d D$ implies that $(vd) \ll_d D$, [6]. On the other hand, $D$ is a d-small prime, so $(0)$ is a d-small prime of $D$. Then $u \in Ann D$. But $Ann D = Ann E$, hence $u \in Ann E$. Thus, $AnnE$ is a prime ideal in $R$.

**Proposition (3.17):** If $D$ is a d-small prime R-module, then a non-zero submodule is a d-small prime R-module.

**Proof:** Suppose that $E \neq 0$ be a submodule of $D$. Suppose that $0 \neq W \ll_d E$. So $W \ll_d D$, [6]. Hence $Ann D = Ann W$. But $Ann D \subseteq Ann E$, so $Ann W \subseteq Ann E$. Hence $Ann E = Ann W$ and therefore $E$ is a d-small prime.
The following example show the converse is not true: Let \( D = Z_6 \) as a \( Z \)-module, then \( Z_6 \) is a d-small prime \( Z \)-module by (3.13,3). While \( Z_{12} \) as a \( Z \)-module is not a d-small prime \( Z \)-module. Since (6) \( \ll_{d} Z_{12} \) but \( \text{ann}Z_{12} = 12Z = 2Z \).

**Proposition (3.18):** If \( \text{Rad}_d(D) \) is a non-zero direct summand d-small prime of a module \( D \) and \( \text{ann} D = \text{ann} \text{Rad}_d(D) \), then \( D \) is a d-small prime \( R \)-module, where \( \text{Rad}_d(D) = \sum \{ L \leq M \mid L \ll_{d} M \} \).

**Proof:** Suppose that \( 0 \neq d \in D \) and \( (d) \ll_{d} D \). Then \( d \in \text{Rad}_d(D) \), [6] and so \( (d) \ll_{d} \text{Rad}_d(D) \). Therefore \( \text{ann} \text{Rad}_d(D) = \text{ann} (d) \). But \( \text{ann} D = \text{ann} \text{Rad}_d(D) \), so \( \text{ann} D = \text{ann} (d) \) and therefore \( D \) is a d-small prime.

**Theorem (3.19):** Let \( D \) is a finitely generated \( R \)-module. Then \( D \) is a d-small prime \( R \)-module iff \( D_S \) is a d-small prime \( R_S \)-module, where \( S \) is a multiplicatively closed subset of \( R \).

**Proof:** \( \Rightarrow \) Let \( c/v \in R_S \), \( d/y \in D_S \) such that \( c/v \cdot d/y = 0_S \), and suppose \( 0_S \neq d/y \ll_{d} D_S \). So \( (d) \ll_{d} D \). Then for each \( s \in S, sd \neq 0 \). On the other hand, \( cd/vy = 0_S \), so \( \exists t \in S \) such that \( tc = c(td) = 0 \). But \( (td) \neq 0 \), is a submodule of \( (d) \) and \( (d) \ll_{d} D \) implies that \( 0 \neq (td) \ll_{d} D \), [6]. On the other hand, \( D \) is a d-small prime, \( 0 \) is a small primary of \( D \). Then \( c \in \text{ann} D \), therefore \( c/v \in (\text{ann}D)_S \). But \( D \) is finitely generated, so \((\text{ann}D)_S = \text{ann}D_S \), [11]. Hence \( c/v \in \text{ann}D_S \). Thus, \( \theta(S) \) is a d-small prime \( R_S \)-module.

\( \Leftarrow \) It follows similary.

**Proposition (3.20):** suppose that \( D \) is a multiplication \( R \)-module. If \( D \) is a d-small prime \( R \)-module, then \( D \) is a d-small prime \( S \)-module, where \( \text{End}(D) = S \).

**Proof:** Suppose that \( 0 \neq E \ll_{d} D \) and suppose \( \text{ann} E \nsubseteq \text{ann} D \), so \( \exists c \in \text{ann} E \) and \( c \notin \text{ann} D \). Thus, \( cD \neq 0 \). Define \( \theta : D \rightarrow D \) by \( \theta(d) = cd, \forall d \in D \). Clearly, \( 0 \neq \theta \) is \( R \)-homomorphism and well-defined. Since \( \theta(E) = cE = 0 \), so \( \theta \in \text{ann}E = \text{ann}S(D) \) (since \( D \) is a d-small prime \( S \)-module). Hence \( \theta(D) = 0 \), so \( \theta = 0 \) which is a contradiction. Thus, \( \text{ann}E = \text{ann}D \) and so \( D \) is a d-small prime \( R \)-module.

"Recall that an \( R \)-module \( X \) is called a scalar module if \( \forall, \phi \in \text{End}(X); \phi 
eq 0, \exists a \in R, a \neq 0 \) such that \( \phi(x) = ax \ \forall x \in X \", [12].

**Proposition (3.21):** suppose that \( D \) is a finitely generated multiplication \( R \)-module. If \( D \) is a d-small prime \( R \)-module, then \( D \) is a d-small prime \( S \)-module, where \( \text{End}(D) = S \).

**Proof:** Let \( 0 \neq E \ll_{d} D \) assume \( \exists \theta \in S, \theta \in \text{ann}E \) and \( \theta \notin \text{ann}D \). Since \( D \) is a multiplication finitely generated, hence \( D \) is a scalar \( R \)-module, [12]. Hence \( \theta(m) = cm, \forall d \in D \). Thus, \( \theta(E) = cE = 0 \) and so \( c \in \text{ann} E = \text{ann}D \). Hence \( cD = 0 \), so \( \theta(D) = 0 \), which is a contradiction. Hence \( \text{ann}E = \text{ann}D \). Thus, \( D \) is a d-small prime \( S \)-module.

**Corollary (3.22):** suppose that \( D \) is a finitely generated multiplication \( R \)-module. Then \( D \) is a d-small prime \( R \)-module iff \( D \) is a d-small prime \( S \)-module, where \( \text{End}(D) = S \).

References

[1] Lu., C.P., 1981, Prime Submodules of Modules, *Commutative mathematics, University Spatuls*, Vol. 33, pp.61-69.

[2] Abdul-Alkalik, A. J. 2019. I-Nearly Prime sumodules, *Iraqi Journal of Science*. Vol. 60. n0(11), pp. 2468-2472.

[3] Selman, M. L., 2012. Small Prime Modules and Small Prime Submodules, *Journal of Al-Nahrain University*. Vol. 15(4), pp. 191-199.

[4] Kash, F. 1982. *Modules and Rings*. Academic Press. London.

[5] Fleury, P. 1974, Hollow Modules and Local Endomorphism Rings, *Pac. J.Math.*. Vol. 53, No.2, pp.379-385.

[6] Mehdi, S. Abbas and Mohammad, F.Manhal., 2018, d-Small Submodules and d-Small Projective Modules, *International Journal of Algebra*, Vol. 12, no. 1, pp.25-30.

[7] Mehdi, S. Abbas and Mohammad, F.Manhal., 2018, d-Small M- Projective Modules and a Characterization of d-Small Submodules, *International Journal of Algebra*, Vol. 12, no. 1,
[8] Sharp, R. Y., 1990. *Steeps in Commutative Algebra*, Cambridge University Press.
[9] Smith, P.F., 1988, Some Remarks on Multiplication Modules, *Arch. Math.*, Vol.50, pp.223-235.
[10] Athab, I. A., 2004. Some Generalization of Projective Modules, Ph. D. Thesis, College of Science, University of Baghdad.
[11] Atiya, M. F., Macdonald, I. G., 1969, *Introduction to Commutative Algebras*, University of Oxford.
[12] Shihap, B. N., 2004, Scalar Reflexive Modules, Ph. D. Thesis, University of Baghdad.