We propose a method for estimation and inference for bounds for heterogeneous causal effect parameters in general sample selection models where the treatment can affect whether an outcome is observed and no exclusion restrictions are available. The method provides conditional effect bounds as functions of policy relevant pre-treatment variables. It allows for conducting valid statistical inference on the unidentified conditional effects. We use a flexible debiased/double machine learning approach that can accommodate non-linear functional forms and high-dimensional confounders. Easily verifiable high-level conditions for estimation, misspecification robust confidence intervals, and uniform confidence bands are provided as well. Re-analyzing data from a large scale field experiment on Facebook, we find significant depolarization effects of counter-attitudinal news subscription nudges. The effect bounds are highly heterogeneous and suggest strong depolarization effects for moderates, conservatives, and younger users.

Keywords: Affective polarization; Debiased/double machine learning; Effect bounds; Facebook; Partial identification

JEL classification: C14, C21, D72, L82
1. Introduction

In this paper, we propose a novel method for estimation and inference for bounds of heterogeneous causal effects when outcome data is only selectively observed and no exclusion restrictions or instruments are available. In particular, we are concerned with the case when the treatment of interest itself can affect the selection process and when effects are heterogeneous along both observable and unobservable dimensions. The bounds are derived from a conditional monotonicity assumption in the selection equation. They can be used to study the effects of interventions on always-taker units, e.g. the effects of active labor market policies on earnings on the population that is working regardless of whether they were subject to the intervention or not.\footnote{Note that in contrast to the typical setup and nomenclature in the instrumental variables literature, the principal strata are defined with respect to potential selection state caused by treatment, not potential treatment states caused by an instrument. The always-taker stratum is also sometimes referred to as inframarginal or always-observed.} They can also be applied to obtain credible bounds in experimental studies where the original treatment can affect selection.

When applying established partial identification approaches for similar sample selection problems in practice, unconditional or subgroup specific effect bounds (Horowitz and Manski, 2000; Lee, 2009; Semenova, 2020) are often wide and thus too uninformative for assisting policy. Narrower bounds that also exploit covariate information can be used for a better targeting of interventions under weaker, i.e. more credible, conditions compared to restrictive point-identified methods that require exclusion restrictions and/or distributional assumptions. The nonparametric heterogeneity based approach in this paper helps to tighten bounds along policy relevant pre-treatment variables. This is due to the fact that the severity of the identification problem, i.e. the width of the identified set, can vary substantially along the confounding dimensions most associated with the heterogeneity variables of interest. In addition, the procedure has significant advantages over calculating bounds within discrete partitions of the data: It can accommodate continuous variables and, as it extracts signals for the bounds before conditioning on heterogeneity variables, exploits larger samples as well as potential group patterns/restrictions for modeling selection probabilities and other relevant nuisance functions.

The method can also incorporate a high-dimensional number of confounders building on
debased machine learning (DML) methodology (Chernozhukov et al., 2018a; Semenova and Chernozhukov, 2021). We derive explicit high-level conditions regarding the quality of the nuisance quantity estimators that can be verified in a variety of settings for popular non-parametric or machine learning estimators such as high-dimensional sparse regression, deep neural networks, or random forests. We provide analytical confidence intervals for heterogeneous effects that are robust against different types of model misspecification as well as uniform confidence bands using a multiplier bootstrap.

Figure 1.1.: Partially Identified Effect and Confidence Intervals

The black lines (left plot) are generalized Lee bounds (Semenova, 2020) with pointwise 95%-confidence intervals (dashed). The red area is the identification region using the method from this paper. The dashed blue lines are the pointwise 95%-confidence intervals for the heterogeneous effects. The black line (right plot) is the true conditional average treatment effect curve. Nuisance parameters are estimated via honest generalized random forests. Heterogeneous bounds use basis-splines with sample size $n = 5000$ and $p = 100$ regressors. For more details consider Section 5.

Figure 1.1 contains an illustration of the proposed method for a one-dimensional heterogeneity analysis. It plots the identified set and confidence intervals for the causal effect in dependence of a pre-treatment variable $z$. These could e.g. be bounds on the effect of job training on earnings as a function of pre-training earnings. We can see that unconditional analysis cannot rule out a zero effect while the confidence intervals of the heterogeneous bounds clearly suggest significant negative ($0.4 < z < 0.8$) and positive effects ($z > 0.9$) leading to different policy recommendations. Note that these conclusions are achieved not just by heterogeneous locations of the bounds but also by narrower widths for certain $z$ values. Thus, heterogeneous bounds can reduce uncertainty stemming from weaker identification assumptions for certain sub-populations. Monte Carlo simulations suggest, that the presented confidence intervals perform well in finite samples.

Our application is concerned with the effect of social media news consumption on political
polarization. In 2022, over 70% of US adults consumed news on social media. The consequences of social media and online news consumption on political polarization are of major importance: High partisan attitudes threaten the functioning of society and democracy as well as trust in public and private institutions (Phillips, 2022). Many democratic societies have been experiencing significant changes in partisan attitudes since the broad roll-out of social networks such as Facebook and Twitter. However, the exact contributions are under debate (Haidt and Bail, ongoing).

We study the effect of Facebook news subscription on affective polarization. Facebook is the most dominant social media site for news consumption among US adults (31% regularly get news on this site). Affective polarization measures relative attitudes toward opposing partisans in terms of (dis)like and (dis)trust. We re-examine data collected by Levy (2021) who employs a large scale field experiment on Facebook where units are nudged towards subscribing to popular media outlets with clear partisan ideology such as FoxNews or MSNBC. In particular, we measure the effects of the counter-attitudinal treatment in terms of political leaning on affective polarization after two months. The outcome measure suffers from large differential attrition rates between treatment and control groups and within political ideology (over 50% total). This is an ideal setup for the method developed in this paper. We find weak evidence against an overall null-effect after correcting for attrition. Our identified set using random forest-based methods of \([-0.156, -0.016]\) (standard deviations) suggest a decrease in affective polarization and covers the original point estimate of \(-0.055\) by Levy (2021) not corrected for attrition. We also detect significant heterogeneity: The counter-attitudinal treatment seems to significantly reduce affective polarization for moderates and conservatives. The corresponding identified sets cover parameters of 2-3 times the magnitude of the unconditional estimate by Levy (2021). The identified set for liberals also excludes zero but is statistically insignificant. Moreover, there are significant negative effects for Facebook users below the age of 44. The identified sets for younger users again suggest 2-3 times larger effects compared to the uncorrected unconditional point estimate.

The paper is structured as follows: Section 2 discusses the methodological literature. Section 3 contains model, estimator, and confidence intervals. Section 4 provides technical

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2Pew Research Center, Survey of U.S. adults, July 18-Aug 21, 2022.
assumptions, misspecification analysis, confidence bands, and large sample properties. Section 5 presents Monte Carlo simulations. Section 6 contains the empirical study. Section 7 concludes. All proofs are in the Appendix. The \texttt{R} package \texttt{HeterogeneousBounds} and replication notebook can be found on the author’s personal web-page.

2. Methodological Literature

Bounds for causal effects under weak assumptions have been considered in a series of papers by Charles Manski and others, see Molinari (2020) for a comprehensive overview. Horowitz and Manski (2000) develop nonparametric bounds for treatment effects in selected samples. Zhang and Rubin (2003) consider bounds for always-taker units under a monotonicity assumption regarding the effect of treatment on selection, commonly imposed in generic sample selection models, and/or stochastic dominance assumptions on the potential outcomes. Imai (2008) demonstrates sharpness of these bounds. Huber and Mellace (2015) consider similar sharp bounds for other principal strata. Lee (2009) provides asymptotic theory for Zhang and Rubin (2003) bounds using (conditional) monotonicity. Without further assumptions, these bounds are only applicable unconditionally or for low-dimensional discrete partitions of the covariate space and now commonly referred to as “Lee bounds”. Semenova (2020) provides “generalized Lee bounds” under a conditional monotonicity assumption. Our paper uses similar identification assumptions. Semenova (2020) also allows for high-dimensional and continuous confounders and generalizes the approach to multiple outcomes and endogenous treatment receipt but does neither consider heterogeneity analysis nor misspecification-robust inference. Bartalotti et al. (2021) also propose identification of bounds for always-takers within a marginal treatment effect framework. Using monotonicity and stochastic dominance, they tighten effect bounds based on underlying treatment propensities. However, they do not neither address heterogeneity beyond the propensity score, asymptotic properties, inference, nor potential misspecification. Moreover, their method is not suitable for many confounding variables without imposing additional parametric assumptions.

Our work is also directly related to the literature on robust or (Neyman-)orthogonal mo-
ment functions and DML. For point-identified parameters, there are now many approaches that use machine learning and orthogonal moments in both experimental and observational studies, see e.g. Belloni et al. (2014), Farrell (2015), Chernozhukov et al. (2018a), and Wager and Athey (2018). Chernozhukov et al. (2018a) develop a canonical framework that can be used for inference on low-dimensional target parameters such as the average treatment effect. In the context of heterogeneity analysis, orthogonal moments have been exploited in point-identified problems by using them as pseudo-outcomes in (nonparametric) regression models to obtain predictive causal summary parameters, see e.g. Lee et al. (2017), Fan et al. (2020), Semenova and Chernozhukov (2021), and Heiler and Knaus (2021) or Knaus (2022) for an overview. Our localization approach is closest to Semenova and Chernozhukov (2021) who estimate heterogeneity parameters via nonparametric projections using least squares series methods (Belloni et al., 2015). Employing a series approach is particularly useful for studying potential misspecification since the heterogeneity step can then be reformulated as a relaxed moment inequality problem with deviation parameters that are proportional to the difference between effect bounds and their best linear predictors. It also allows for construction of uniform confidence bands under correct specification.

Semenova (2020) constructs DML estimators for unconditional Lee-type bounds. Semenova (2023) considers partially identified parameters for linear moment functions with DML. A crucial point in both of these papers is that, while the effect of interest might not be point-identified, bounds themselves are characterized by well-understood convex moment problems or the corresponding support function. The same applies to the heterogeneous bounds considered in this paper. We derive the asymptotic distribution of the nonparametric heterogeneous DML based estimator for the identified set. This nests the univariate generalized Lee bounds by Semenova (2020) as a special case. In contrast to Semenova (2023), the moment functions are nonlinear in the outcome and non-differentiable with respect to the underlying nuisance functions. The use of machine learning (random forests) for Lee-type bounds has also been heuristically discussed by Cornelisz et al. (2020). They do, however, not provide any formal theory for estimation and inference. Olma (2021) also discusses nonparametric estimation of generic truncated conditional expectations based on similar conditional moments as the ones used in this paper. He suggest kernel estimation for both stages and
provides pointwise linearization and distribution results. In contrast, our approach can handle generic first-stage learners that also work in setups with high-dimensional confounding. Moreover, Olma (2021) does neither consider local power improvements, misspecification, nor uniform inference.

Inference for partially identified parameters in sample selection models is a non-trivial task. Relying on quantiles of large sample distributions of the bounds can be overly conservative for the actual effect of interest. The relevant uncertainty for the latter depends on the actual width of the identified set. If small, then deviations from a null value are likely to occur in both positive and negative direction, i.e. the problem is effectively two-sided. If large, then uncertainty in one direction dominates, rendering the testing problem close to one-sided. Imbens and Manski (2004) consider related confidence intervals for partially identified parameters. Their method is not uniformly valid with regards to width of the underlying identified set due to an implicit superefficiency assumption. Stoye (2009) suggests to artificially impose superefficiency via shrinkage. Andrews and Soares (2010) provide a more general approach within a moment inequality framework. Andrews and Shi (2013) and Andrews and Shi (2017) study inference based on (many) conditional moment inequalities in the presence of nuisance parameters. While the latter can be infinite dimensional as in this paper, their inference target is essentially parametric. Our approach is closest in spirit to Andrews and Shi (2014) who consider nonparametric estimation based on conditional moment inequalities. Their method can be adjusted to conduct pointwise inference on conditional effect bounds. However, as they study a relatively general setup, they do neither address local power improvements for effects, estimated nuisances, misspecification, nor uniform inference. Chernozhukov et al. (2019) also consider inference using generic moment inequalities in possibly high dimensions allowing for nonparametric sieve-type parameters. In contrast, we exploit the specific shape of the identified set as well as the restriction to low-dimensional heterogeneity variables to obtain stronger asymptotic results that can be used for power improvements.3

When analyzing heterogeneity, we are concerned with effect bounds at potentially many

3Chernozhukov et al. (2019), Appendix B also heuristically discusses approximate moment inequalities/mis-
specification arising from estimation using for parametric nuisance models.
points and thus there is an additional risk of local misspecification compared to unconditional effects. For instance, even when the conditional moments are correctly specified and ordered, their best linear predictors might intersect. In addition, there can be global misspecification, e.g. due to misspecified propensity scores, that can lead to a reversed ordering of the orthogonalized local bounds in both the population and in finite samples. Under such misspecification, aforementioned bounds and moment inequality methods can produce empty or very narrow confidence regions suggesting spuriously precise inference (Andrews and Kwon, 2019). Andrews and Kwon (2019) propose inference methods that extend the notion of coverage to pseudo-true parameter sets in general moment inequality frameworks, see also Stoye (2020) for an adaptation to generic regular parametric bound estimators. This paper adapts the inference method of Stoye (2020) to nonparametric heterogeneous bounds with machine learning in the first stage.

3. Methodology

3.1. Identification

In this section, we introduce the sample selection model and the main identification assumptions followed by a brief review of the construction of effect bounds. We then show how to exploit the latter to construct, estimate, and conduct inference on the heterogeneous partially identified effect parameters.

Assume for i = 1, ..., n we observe iid data \( W_i = (X_i', D_i, S_i, Y_i S_i) \) where \( X_i \) is a vector of predetermined covariates supported on \( X \subseteq \mathbb{R}^d \), \( D_i \) is a treatment indicator, \( S_i \) indicates whether the outcome is observed, i.e. we only observe \( Y_i S_i \) and not \( Y_i \). We would like to evaluate the average causal effect of \( D_i \) on outcome \( Y_i \) for the units that are selected under both treatment or control condition. We focus on the case of known treatment propensities \( e(x) = P(D_i = 1|X_i = x) \) during the exposition, but the methodology also extends to observational studies where they are unknown and have to be estimated. The prototypical setup is depicted by the graph in Figure 3.1 derived from the nonparametric structural equation model. The model nests the classic sample selection model (Heckman, 1979) where the
This is a graphical representation of a nonparametric sample selection model. Nodes denote variables and edges are structural relationships, i.e. a missing arrow from one node to another is an exclusion restriction. Unobserved independent components at each node are omitted. Unobserved variables are in dashed nodes. The Markov assumption for the graph holds, i.e. it is a directed acyclical graph where variables are independent of all their non-descendants given their direct parental nodes (Tian and Pearl, 2002). For inclusion of missing variables and their use in graphical models see Mohan and Pearl (2021).

Based on this model, we define unit potential outcomes $Y_i(1)$ and $Y_i(0)$ and potential selection indicators $S_i(1)$ and $S_i(0)$ for units $i = 1, \ldots, n$. They correspond to a unit’s value in outcome or selection when the treatment is exogeneously set to $D_i = 1$ or $D_i = 0$ respectively under the model in Figure 3.1. Note that potential selection and potential outcomes can still be dependent. This implies that, even conditional on $X_i$, causal effects can differ for different sub-types defined by their respective $S_i(1)$ and $S_i(0)$ variables. The target is to evaluate the causal effect of $D_i$ for the units that are selected in both control and treatment state, i.e. units for which $S_i(1) = S_i(0) = 1$. As $D$ affects $S$, a comparison of treated and control units for selected units does not yield a valid causal comparison for the sub-population $S_i(1) = S_i(0) = 1$. However, the model implies the following conditional
independence relationships in Rubin-Neyman potential outcome notation:

\[ Y_i(d), S_i(d) \perp\!\!\!\!\!\perp D_i \mid X_i \text{ for } d = 0, 1. \]

(3.1)

Conditions (3.1) cannot be exploited in a standard selection on observables strategy for point identification as, even conditional on \( X_i \), we only observe the selected subset of potential outcomes for which \( S_i = 1 \). However, we can use the fact that, conditional on \( X_i \), the treatment is exogenous with respect to selection to obtain bounds for the causal effect in the \( S_i(1) = S_i(0) = 1 \) population, i.e. the effect is partially identified under the model. For that, it is not necessary to impose any restrictions on the sign of the effect of \( D \) on \( S \). In particular, treatment can affect selection positively or negatively. The sign, however, must be uniquely determined by the vector of covariates \( X_i \). This is a weak or conditional monotonicity assumption based on ex-ante unknown partitions:

**Assumption 3.1** There exist partitions of the covariate space \( \mathcal{X} = \mathcal{X}_+ \cup \mathcal{X}_- \) such that

\[ P(S_i(1) \geq S_i(0) \mid X_i \in \mathcal{X}_+) = P(S_i(1) \leq S_i(0) \mid X_i \in \mathcal{X}_-) = 1 \]

We also assume that, with positive probability, there are comparable units between selected treated and selected non-treated units as well as between treated and non-treated units overall (multiple overlap):

**Assumption 3.2** For all \( x \in \mathcal{X} \) and \( d \in \{0, 1\} \) we have that \( 0 < P(S_i = 1 \mid X_i = x, D_i = d) < 1 \) and \( P(D_i = d \mid X_i = x) > 0 \).

Moreover, we rule out the case where treatment does not affect selection as in this case effects are point-identified:

**Assumption 3.3** There is a constant \( c > 0 \) and a set \( \tilde{\mathcal{X}} \subset \mathcal{X} \) with \( P(\mathcal{X}\setminus\tilde{\mathcal{X}}) = 0 \) such that

\[ \inf_{x \in \tilde{\mathcal{X}}} [P(S_i = 1 \mid X_i = x, D_i = 1) - P(S_i = 1 \mid X_i = x, D_i = 0)] > c. \]

For constructing the effect bounds, consider first a stronger version of Assumption 3.1: \( S_i(1) \geq S_i(0) \) with probability one, i.e. strong or unconditional monotonicity. In this case, selected units within the control group are always-takers or inframarginal in the sense of

\[^4\text{These are the marginal versions of the joint independence assumption } Y_i(0), Y_i(1), S_i(0), S_i(1) \perp\!\!\!\!\!\perp D_i \mid X_i \text{ considered in Lee (2009) that yield the same identification results.}\]
$S_i(1) = S_i(0) = 1$ while within the treated group there is a mixture of always-takers and compliers or marginal units who are induced to be selected by the treatment. This allows us to define sharp bounds on the conditional causal effect of treatment on the outcome for the population of always-takers

$$\theta(x) = E[Y_i(1) - Y_i(0)|S_i(1) = S_i(0) = 1, X_i = x]. \quad (3.2)$$

This is the expected causal effect for a unit with covariates $x$ whose outcome would be observed independently of its treatment status. Let $s(d, x) = P(S_i = 1|X_i = x, D_i = d)$ and denote $p_0(x) = s(0, x)/s(1, x)$ as the share of always-takers relative to always-takers and complier units conditional on $X_i = x$. For a given $x \in \mathcal{X}$, the worst case for $\theta(x)$ under monotonicity is then obtained when the smallest $1 - p_0(x)$ values of $Y_i$ are all complier units and the remaining $p_0(x)$ units are always-takers. Thus, trimming the outcome for the selected treated units from below by its corresponding $1 - p_0(x)$ quantile allows us to obtain an upper bound for the effect. Trimming the observed treated selected outcome by the $p_0(x)$ quantile from above yields the lower bound. Under the weaker conditional monotonicity Assumption 3.1, an equivalent argument holds separately for units with either $p_0(x) < 1$ or $p_0(x) > 1$ and inverted trimming shares. Denote the conditional quantile $q(u, x) = \inf\{q : u \leq P(Y_i \leq q|D_i = 1, S_i = 1, X_i = x)\}$. As $p_0(x)$ is identified from the observed joint distribution of $(X_i', D_i, S_i)$, we obtain the following sharp conditional bounds for any $x \in \mathcal{X}$

$$\theta_L(x) \leq \theta(x) \leq \theta_U(x) \quad (3.3)$$

where

$$\theta_L(x) = E[Y_i|D_i = 1, S_i = 1, Y_i \leq q(p_0(x), x), X_i = x] - E[Y_i|D_i = 0, S_i = 1, X_i = x]$$

$$\theta_U(x) = E[Y_i|D_i = 1, S_i = 1, Y_i \geq q(1 - p_0(x), x), X_i = x] - E[Y_i|D_i = 0, S_i = 1, X_i = x]$$

(3.4)

if $p_0(x) < 1$ and

$$\theta_L(x) = E[Y_i|D_i = 1, S_i = 1] - E[Y_i|D_i = 0, S_i = 1, Y_i \geq q(1 - 1/p_0(x), x), X_i = x]$$

$$\theta_U(x) = E[Y_i|D_i = 1, S_i = 1] - E[Y_i|D_i = 0, S_i = 1, Y_i \leq q(1/p_0(x), x), X_i = x]$$

(3.5)

if $p_0(x) > 1$. If $p_0(x) = 1$, the treatment would not affect the response and thus, conditional on observables, causal effects are point-identified by comparing the outcome of selected
treated to selected control units and $\theta_L(x) = \theta_U(x)$.

While the $x$-specific or “personalized bounds” (3.3) can provide some insights, they are not very useful in applications with continuous variables and/or many discrete cells to evaluate. Moreover, in such cases it is often no longer possible to consistently estimate the conditional bounds and to construct asymptotically valid confidence intervals without further restrictions. This is equivalent to estimation of personalized treatment effects under unconfoundedness in high dimensions (Chernozhukov et al., 2018b). Therefore, we propose to instead consider heterogeneous effect bounds conditional on a smaller, pre-specified (policy relevant) variables $Z_i$ supported on $Z$. These can be (functions of) any observable variable possibly affecting treatment, selection, and outcome. Thus, without loss of generality, we can treat $Z_i = f(X_i)$ where $f(\cdot)$ is a measurable, possibly multi-valued function. The true lower-dimensional heterogeneous effects are then defined as

$$\theta(z) = E[\theta(X_i)|Z_i = z].$$  \hspace{1cm} (3.6)

The goal is to estimate heterogeneous bounds

$$\theta_L(z) \leq \theta(z) \leq \theta_U(z).$$  \hspace{1cm} (3.7)

for any $z \in Z$. The bounds here are for the predictive causal effect $\theta(z)$. Such a functional parameter allows for evaluating effect heterogeneity along the pre-specified dimensions. This is conceptually equivalent to heterogeneity analysis of causal effects when dividing the sample into specific subgroups. However, it is more informative as it exploits the full variation in outcomes, selection, and treatment before conditioning on the subgroup variables. A special case are the unconditional effect bounds considered by Lee (2009) and Semenova (2020)

$$E[\theta_L(X_i)] \leq E[\theta(X_i)] \leq E[\theta_U(X_i)].$$  \hspace{1cm} (3.8)

Semenova (2020) derives Neyman-orthogonal moment functions $\psi_L(W_i, \eta)$ and $\psi_U(W_i, \eta)$ for the upper and lower effect bounds under the conditional monotonicity assumption, see Ap-

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5In principle, one could also be interested in bounds also conditional on the latent sub-type, i.e. $E[\theta(X_i)|S_i(1) = S_i(0) = 1, Z_i = z]$. These always-taker types, however, are usually not known or identified in practice and could therefore not be used e.g. for choosing an intervention population from a policy maker perspective. Thus, we focus explicitly on the expected bound given pre-treatment policy variables that could be used for treatment assignment on a population that potentially consists of all latent sub-types.
Appendix A.1 for the formal definitions. They depend on the observed data and a nuisance quantity vector \( \eta = \eta(W_i) \) that includes propensity scores, conditional selection probabilities, and conditional quantiles of the observed outcome distribution of the treated. These functions serve as approximately unbiased signals for the personalized bounds for the partially identified \( \theta(x) \). Thus, they can be used for (nonparametric) projections onto lower dimensional sub-spaces such as \( Z \) to obtain the upper and lower bounds (3.3). As \( Z \) is of low dimension, this allows for conducting asymptotically valid inference for the heterogeneous causal effect even if the original confounding dimension of \( X \) is large. In particular, Semenova (2020) shows that \( E[\psi_B(W_i, \eta_0)|X_i] = 0 \) for both \( B = L, U \) where \( \eta_0 \) denotes the true nuisance vector. Thus, as \( Z_i \) is known conditional on \( X_i \), the heterogeneous bounds are nonparametrically identified as

\[
\theta_B(z) = E[\theta_B(X_i)|Z_i = z] = E[E[\psi_B(W_i, \eta_0)|X_i]|Z_i = z] = E[\psi_B(W_i, \eta_0)|Z_i = z]. \tag{3.9}
\]

### 3.2. Estimation

The very right hand sides of (3.9) can be estimated using flexible nonparametric methods. In particular, the heterogeneous bounds can be obtained by two separate regressions of each score function onto the spaces spanned by their respective \( k_B \)-dimensional transformations of \( Z_i \), \( b_B(Z_i) \) for each \( B \in \{L, U\} \)

\[
\psi_B(W_i, \eta_0) = b_B(Z_i)' \beta_{B,0} + r_{\theta_B}(Z_i) + \varepsilon_{i,B} \tag{3.10}
\]

with conditional mean errors \( E[\varepsilon_{i,B}|Z_i] = 0 \) and approximation errors

\[
r_{\theta_B}(Z_i) = E[\psi_B(W_i, \eta_0)|Z_i] - b_B(Z_i)' \beta_{B,0}. \tag{3.11}
\]

Replacing the unobserved true score by its sample counterpart yields the two estimators

\[
\hat{\beta}_B = \left( \sum_{i=1}^{n} b_B(Z_i)b_B(Z_i)' \right)^{-1} \sum_{i=1}^{n} b_B(Z_i)\psi_B(W_i, \hat{\eta}) \tag{3.12}
\]
where the scores of the effect bounds with estimated nuisance quantities $\psi_B(W_i, \hat{\eta})$ serve as pseudo-outcomes in two separate least squares regression on their respective basis functions.\(^6\) The heterogeneous effect bounds at policy variable value $Z_i = z$ can then be calculated from the point predictions of the two models:

$$\hat{\theta}_B(z) = b_B(z)' \hat{\beta}_B$$  \quad (3.13)

When $b_B(Z_i)$ only contains a constant for $B = L, U$, the estimator collapses back to the one for the unconditional generalized Lee bounds in Semenova (2020). Estimation of $\hat{\eta}$ can be done via modern machine learning such as random forests, deep neural networks, high-dimensional sparse likelihood and regression models, or other non- and semiparametric estimation methods with good approximation qualities for the nuisance functions at hand. The influence of their learning bias/approximation on the functional bound estimator is limited by the Neyman-orthogonality of the chosen moment functions (Chernozhukov et al., 2018a; Semenova, 2020). In particular, under suitable conditions, the estimation does not affect the large sample distribution if the RMSE-approximation rates for the nuisance quantities $\hat{\eta} - \eta_0$ are of order $o((nk_B)^{-1/4})$. This is similar to standard DML estimation of conditional average treatment effects using Neyman-orthogonal moment functions for nonparametric projection (Semenova and Chernozhukov, 2021). When $z$ is one-dimensional, popular $k_B$ choices for many bases under weak smoothness assumptions are of rate $O(n^{1/5})$ leading to an overall RMSE convergence requirement for the nuisances of $o(n^{-3/10})$. This is a rate achievable by many nonparametric and machine learning estimators such as forests, deep neural networks, or high-dimensional sparse models under moderate complexity and/or dimensionality restrictions, see Semenova (2020), Semenova and Chernozhukov (2021) and Heiler and Knaus (2021) for examples. More details regarding the technical assumptions can be found in Section 4. We also require that all components in $\hat{\eta}$ are obtained via $K$-fold cross-fitting.

\(^6\)In principle, the two equations based on (3.10) are only seemingly unrelated and could thus also be estimated using a system based approach that takes into account the correlation structure of the conditional mean errors to increase efficiency. While this is straightforward in the case of a finite-dimensional parametric mean functions, it introduces additional dependencies in the two-step estimation that might offset potential gains in efficiency in the nonparametric case. Thus, we leave an extension along this line for future work.
Definition 1 **K-fold cross-fitting** (see Definition 3.1 in Chernozhukov et al. (2018a))

Take a K-fold random partition \((I_f)_{f=1}^{K}\) of observation indices \([K] = \{1, \ldots, n\}\) with each fold size \(n_f = n/K\). For each \(f \in [K] = \{1, \ldots, K\}\), define \(I_f^c := \{1, \ldots, n\} \setminus I_f\). Then for each \(f \in [K]\), the machine learning estimator of the nuisance function are given by

\[
\hat{\eta}_f = \hat{\eta}(W_{i \in I_f^c}).
\]

Thus for any observation \(i \in I_f\) the estimated score only uses the model for \(\eta\) learned from the complementary folds \(\psi_B(W_i, \hat{\eta}) = \psi_B(W_i, \hat{\eta}_f)\).

The use of cross-fitting controls potential bias arising from over-fitting using flexible machine learning methods without the need to evaluate complexity/entropy conditions for the function class that contains true and estimated nuisance quantities with high probability. If finite dimensional parametric models such as linear or logistic are assumed and estimated for the nuisance quantities, the proposed methodology can be applied without the need for cross-fitting. Under suitable assumptions, the heterogeneous bound estimators are jointly asymptotically normal

\[
\sqrt{n} \hat{\Omega}_n^{-\frac{1}{2}}(z) \left( \begin{array}{c} \hat{\theta}_L(z) - \theta_L(z) \\ \hat{\theta}_U(z) - \theta_U(z) \end{array} \right) \overset{d}{\rightarrow} \mathcal{N} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right),
\]

\[
\hat{\Omega}_n(z) = \begin{pmatrix} \hat{\sigma}_L^2(z) & \hat{\rho}(z)\hat{\sigma}_L(z)\hat{\sigma}_U(z) \\ \hat{\rho}(z)\hat{\sigma}_L(z)\hat{\sigma}_U(z) & \hat{\sigma}_U^2(z) \end{pmatrix}.
\]

at each \(z \in \mathcal{Z}\). For the complete definitions of the variance terms consider Appendix A.2. The variance can depend on the sample size and is generally increasing in norm if we allow the basis functions \(b_B(z)\) to grow with the sample size. Thus, convergence is slower than the parametric rate equivalently to conventional nonparametric series regression (Belloni et al., 2015). If the approximation errors \(r_{\theta_B}(z)\) are rather large, then this distributional result is still valid if centered around the best linear predictors of the true bounds \(b_B(z)'\beta_{B,0}\) similar to the best linear predictor in standard regression estimation.

### 3.3. Inference

Inverting the quantiles of the distribution in (3.14) could in principle be used to construct confidence intervals for the upper and lower bounds. However, this would be overly conser-
vative for the actual effect of interest $\theta(z)$. Inference on this partially identified parameter should be adaptive to the underlying true width of the interval. In addition, if we allow for (local) misspecification, corresponding confidence regions could be empty or very narrow suggesting overly precise inference (Andrews and Kwon, 2019). Robustness to misspecification is important in our setup as, in contrast to unconditional bounds, the heterogeneous bound functions are estimated at potentially many points with different variances and varying strength of identification in the sense of different widths of the underlying true identified set. A flexible estimator that is chosen e.g. by a global goodness-of-fit criterion for the effect bound curves could well be locally misspecified at some points. A limiting case of this type of misspecification would be to “overfit” the effect of a treatment that is fully independent of selection for some units instead of imposing local point identification. Corresponding confidence regions should be adaptive to such cases. To do so, we introduce the notion of a pseudo-true parameter $\theta^*(z)$ and its corresponding standard deviation $\sigma^*(z)$ that can be estimated as

$$\dot{\theta}^*(z) = \frac{\hat{\sigma}_U(z)\dot{\theta}_L(z) + \hat{\sigma}_L(z)\dot{\theta}_U(z)}{\hat{\sigma}_L(z) + \hat{\sigma}_U(z)}, \quad \dot{\sigma}^*(z) = \frac{\hat{\sigma}_L(z)\hat{\sigma}_U(z)\sqrt{2(1 + \hat{\rho}(z))}}{\hat{\sigma}_L(z) + \hat{\sigma}_U(z)}. \quad (3.15)$$

This is a variance weighted version of the upper and lower bound. The pointwise $(1 - \alpha)$%-confidence intervals for the true always taker effect $\theta(z)$ can then be obtained by the union of two intervals

$$CI_{\theta(z),1-\alpha} = [\hat{\theta}_L(z) - \frac{\hat{\sigma}_L(z)}{\sqrt{n}}\dot{c}(z), \hat{\theta}_U(z) + \frac{\hat{\sigma}_U(z)}{\sqrt{n}}\dot{c}(z)] \cup [\dot{\theta}^*(z) \pm \frac{\dot{\sigma}^*(z)}{\sqrt{n}}\Phi^{-1}(1 - \alpha/2)], \quad (3.16)$$

where the critical value $\dot{c}(z)$ uniquely solves

$$\inf_{\Delta \geq 0} P(u_1 - \Delta - c \leq 0 \leq u_2 \text{ or } |u_1 + u_2 - \Delta| \leq \sqrt{2(1 + \hat{\rho}(z))}\Phi^{-1}(1 - \alpha/2)) = 1 - \alpha, \quad (3.17)$$

with $(u_1, u_2)$ being jointly normal with unit variances and covariance $\hat{\rho}(z)$. This is an adaptation of the method by Stoye (2020) who considers simple parametric estimators for generic bounds without two-stage estimation, cross-fitting, or additional nuisance functions. Interval (3.16) is robust against misspecification that yields reverse ordering of the bounds as it is never empty and guarantees at least nominal coverage over an extended parameters
space. In particular, it has at least \((1 - \alpha)\%\) asymptotic coverage uniformly for all widths \(\theta_U(z) - \theta_L(z)\) pointwise at each \(z \in Z\). For more details as well as uniform confidence bands consider Section 4.3 and 4.4 respectively.

4. Large Sample Properties and Extensions

4.1. Assumptions and Pointwise Limiting Distribution

In this section we provide the assumptions for asymptotic normality, validity of the confidence intervals proposed in (3.16), and some more technical discussion. For simplicity, we present the case with known propensity scores.\(^7\) Denote \(|| \cdot ||_p\) as the \(L_p\) norm. Recall that 
\[
E[\psi_B(W_i, \eta_0)|Z_i = z] = \theta_B(z), \quad \theta_B \in \mathcal{G}_B
\]
where \(\mathcal{G}_B\) is a space of functions (possibly depending on \(n\)) that map from \(Z\) to the real line. Note that \(\theta_B(z) = b_B(z)'\beta_{B,0} + r\theta_B(z)\) with basis transformations \(b_B(z) \in \mathcal{S}_k^B := \{b \in \mathbb{R}^k_B : ||b|| = 1\}\) and \(\beta_{B,0}\) being the parameter of the best linear predictor defined as root of equation 
\[
E[\psi_B(W_i, \eta_0) - b_B(Z_i)'\beta_{B,0}] = 0.
\]
Define bound \(\xi_{k,B} = \sup_{z \in Z} ||b_B(z)||\). Let \(\eta \in T\) where \(T\) is a convex subset of some normed vector space. Denote the realization set 
\[
\mathcal{T}_n = (S_{0,n} \times S_{1,n} \times Q_n) \subset T
\]
as the set that with high probability contains estimators \(\hat{\eta} = \{\hat{s}(0,X_i),\hat{s}(1,X_i),\hat{q}(u,X_i)\}\) for nuisance quantities \(\eta_0 = \{s(0,X_i),s(1,X_i),q(u,X_i)\}\). Let their corresponding \(L_p\) error rates be 
\[
\lambda_{s,n,p} = \sup_{d \in \{0,1\}} \sup_{\hat{s}(d) \in S_{d,n}} E[(\hat{s}(d,X_i) - s(d,X_i))^p]^{1/p}
\]
\[
\lambda_{q,n,p} = \sup_{u \in \tilde{U}} \sup_{\hat{q}(u) \in Q_n} E[(\hat{q}(u,X_i) - q(u,X_i))^p]^{1/p}
\]
where \(\tilde{U}\) is a compact subset of \((0,1)\) containing the relevant quantile trimming threshold support unions \(([supp(p_0(X_i)) \cup supp(1-p_0(X_i))] \cap \mathcal{X}_+) \cup ([supp(1/p_0(X_i)) \cup supp(1-1/p_0(X_i))] \cap \mathcal{X}_-)\). All of the following assumptions are uniformly over \(n\) if not stated differently for both \(B = L, U\):

A.1) (Identification) \(Q_B = E[b_B(Z_i)b_B(Z_i)']\) has eigenvalues bounded above and away from zero.

\(^7\)With propensity scores estimated, one has to augment the bias-corrector in the moment functions and the nuisance parameter space as in Semenova (2020) as well as the Assumption A.5 by additional terms that equivalently depend on the (squared) \(L_p\) error rate of the propensity score estimator.
A.2) (Regular outcome) The outcome has bounded conditional moments $E[Y_i^m|X_i = x, D_i = d, S_i = s]$ for some $m > 2$ and a continuous density $f(y|X = x, D = d, S_i = s)$ that is uniformly bounded from above and away from zero with bounded first derivative for any $x \in \mathcal{X}$ and $s, d \in \{0, 1\}$.

A.3) (Strong multiple overlap) There exist constants $\epsilon, s \in (0, 1/2)$ such that

$$
\epsilon < \inf_{x \in \mathcal{X}} \epsilon(x) \leq \sup_{x \in \mathcal{X}} \epsilon(x) < 1 - \epsilon
$$

$$
s < \inf_{x \in \mathcal{X}, d \in \{0, 1\}} s(d, x) \leq \sup_{x \in \mathcal{X}, d \in \{0, 1\}} s(d, x) < 1 - s
$$

A.4) (Approximation) For any $n$ and $k_B$, there are finite constants $c_{k,B}$ and $l_{k,B}$ such that for each $\theta_B \in \mathcal{G}_B$

$$
||r_{\theta_B}||_{P, 2} := \sqrt{\int_{z \in Z} r_{\theta_B}^2(z)dP(z)} \leq c_{k,B},
$$

$$
||r_{\theta_B}||_{P, \infty} := \sup_{z \in Z} |r_{\theta_B}(z)| \leq l_{k,B}c_{k,B}.
$$

A.5) (Basis growth) Let $\sqrt{n}/\xi_{k,B} - l_{k,B}c_{k,B} \to \infty$ such that

$$
\sqrt{\frac{\xi_{k,B}^2 \log k_B}{n}} \left(1 + \sqrt{k_Bl_{k,B}c_{k,B}}\right) = o(1).
$$

A.6) (Machine learning bias) Let $e_n = o(1)$. For all $f \in \mathcal{K}$, $\hat{\eta}_f$ obtained via cross-fitting belongs to a shrinking neighborhood $T_n$ around $\eta_0$ with probability of at least $1 - e_n$, such that

$$
\xi_{k,B}(\lambda_{q,n,2} + \lambda_{s,n,2}) = o(1)
$$

and (i) either

$$
\sqrt{n k_B}(\lambda_{q,n,4}^2 + \lambda_{s,n,4}^2) = o(1)
$$

or (ii) the basis is bounded $\sup_{z \in Z} ||b(z)||_{\infty} < C$ and

$$
\sqrt{n k_B}(\lambda_{q,n,2}^2 + \lambda_{s,n,2}^2) = o(1).
$$

Assumption A.1 excludes collinearity of the basis transformations of the heterogeneity variables. A.2 puts restrictions on the tails and the smoothness of the distribution for the observed outcome distribution in different selection and treatment states. A.3 assures that there are comparable units between units of different selection and/or treatment status. A.2
and A.3 together imply a continuously differentiable conditional quantile function for the selected observed units that is almost surely bounded. This, together with the strong overlap for the treatment and selection probabilities assures that the effect bounds are regularly identified (Khan and Tamer, 2010; Heiler and Kazak, 2021).

A.4 defines $L_2$ and uniform approximation error bounds for function class $G_B$. This is a typical characterization in the literature on nonparametric series regression without nuisance functions (Belloni et al., 2015). We say the model is correctly specified if the basis is sufficiently rich to span $G_B$, i.e. $c_{k,B} \to 0$ as $k_B \to \infty$. However, the distributional theory also allows for the case of misspecification, i.e. $c_{k,B} \not\to 0$. A.5 controls the approximation error from linearization of the estimator with unknown design matrix $Q_B$. This is equivalent to the condition required for localization in general least squares series regression (Belloni et al., 2015). For more specific series methods such as splines (Huang, 2003) or local partitioning estimators (Cattaneo et al., 2020), this rate can be improved to $\sum \xi_{k,B}^2 \log k_B/n(1 + \sqrt{\log k_B} c_{k,B})$, see also Belloni et al. (2015), Section 4 and Cattaneo et al. (2020), Supplemental Appendix Remark SA-4 for a related discussion.

A.6 is crucial: It says that the machine learning estimators for the conditional quantiles and selection probabilities have sufficiently good approximation qualities in an $L_p$ sense. In the case of a bounded finite basis, the conditions reduce to the well-known requirement in the semiparametric/DML literature that the nuisance functions have root have mean squared error rates of order $o(n^{-1/4})$ (Chernozhukov et al., 2018a). In the more general case, they are comparable to the conditions in Semenova and Chernozhukov (2021) required for nonparametric estimation of conditional average treatment effects.

For the estimation of the asymptotic variance, we also assume that A.V holds:

A.V) (Asymptotic variance) The conditions in Appendix A.2, Assumption B.1 hold, i.e.

$$||\hat{\Omega}_n(z) - \Omega(z)|| = o_p(1) \text{ pointwise at each } z \in \mathcal{Z}.$$ 

A.V can require somewhat stronger outcome moment/tail and basis growth conditions. The corresponding primitive assumptions and discussion can be found in Appendix A.2 and are omitted for brevity. We obtain the following Theorem:
Theorem 4.1 Suppose Assumptions 3.1 - 3.3 and A.1 - A.6 hold and \( \hat{\theta}_B(z_0) \) for \( B = L, U \) and \( \hat{\Omega}_n(z_0) \) are estimators according to (3.13) and (A.1) respectively. Then, for any sequence \( z_0 = z_{0,n} \),

\[
\sqrt{n}\hat{\Omega}_n^{-\frac{1}{2}}(z_0) \left( \hat{\theta}_L(z_0) - b_L(z_0)'\beta_{L,0} \right) \xrightarrow{d} \mathcal{N} \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right).
\]

Moreover if \( \sup_B n^{1/2}k_B^{-1/2}l_{k,B}c_{k,B} = o(1) \), then

\[
\sqrt{n}\hat{\Omega}_n^{-\frac{1}{2}}(z_0) \left( \hat{\theta}_U(z_0) - \theta_U(z_0) \right) \xrightarrow{d} \mathcal{N} \left( \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right).
\]

Theorem 4.1 shows that nonparametric heterogeneous bounds using DML are jointly asymptotically normal. It allows for the case of misspecification when centered around the best linear predictor. It is most useful under the additional undersmoothing condition that makes any misspecification bias vanish sufficiently fast.\(^8\)

### 4.2. Pseudo Parameterization and Pointwise Inference

Theorem 4.1 would in principle be sufficient to construct confidence intervals for the heterogeneous effect bounds. They are, however, too wide for the actual effect parameter of interest \( \theta(z) \) depending on the width of \( \theta_U(z) - \theta_L(z) \) (Imbens and Manski, 2004). If the difference between upper and lower bound is large, the testing problem is essentially one-sided compared to the case of only having a small difference. This raises the question of how to conduct inference that is uniformly valid with respect to the underlying difference between upper and lower effect bound and has more power compared to using simple two-sided critical values. Moreover, there are multiple sources of misspecification that could invalidate or suggest spuriously precise inference when upper and lower bound estimates are close to each other or reverted. In addition, even when the population bounds do not intersect, their best linear predictors still might. Such misspecifications are particularly relevant for heterogeneous bounds as here we estimate two functions at potentially many points which

\(^8\)For example, when \( G_B \) is in a \( s \)-dimensional ball on \( Z \) of finite diameter, then the condition simplifies to \( n^{1/2}k_B^{-1/2+\hat{\xi}} \log(k_B) \to 0 \). See Belloni et al. (2015), Comment 4.3 for additional details. Note that undersmoothing does in general not admit mean-squared error optimal \( k_B \) choices, see e.g. Cattaneo et al. (2020) for multiple bias-correction alternatives methods for local partitioning estimators.
makes (local) potential misspecification and/or reordering a much larger concern compared to a single partially identified parameter as in Lee (2009) or Semenova (2020). We relax the notion of coverage following Andrews and Kwon (2019) to include an artificial pseudo-true target parameter that guarantees non-empty confidence intervals. For more details on misspecification and corresponding formalization consider Section 4.3.

Our method is based on Stoye (2020) who demonstrates that in the simple case of two regular, jointly asymptotically normally distributed parameters, uniformly valid intervals with good power properties can be obtained by concentrating out the unobserved true difference in effect bounds. We adapt his framework to the nonparametric heterogeneous effect bounds with estimated nuisances. For any $z \in Z$ let the identified set be $\Theta_z = [\theta_L(z), \theta_U(z)]$ Now denote the pseudo-true parameter $\theta^*(z)$ and its variance as

$$
\theta^*(z) = \frac{\sigma_U(z)\theta_L(z) + \sigma_L(z)\theta_U(z)}{\sigma_L(z) + \sigma_U(z)},
\sigma^*(z) = \frac{\sigma_L(z)\sigma_U(z)\sqrt{2(1 + \rho(z))}}{\sigma_L(z) + \sigma_U(z)}.
$$

The pseudo-true identified set is then given by $\Theta^*_z = \Theta_z \cup \{\theta^*(z)\}$. This set is implicitly defined by an estimand corresponding to setting a test statistic for the heterogeneous effect at $z$ to (i) zero (meaning no rejection of the null) if the interval is nonempty and the null value is inside the estimated intervals and to (ii) the larger of the two $t$-statistics for upper and lower bound for the null of the bound being equal to the hypothesized value. In particular, it chooses the test statistic under the null as

$$
\max\{\frac{(\theta(z) - \hat{\theta}_U(z))}{\hat{\sigma}_U(z)}, (\hat{\theta}_L(z) - \theta(z))/\hat{\sigma}_L(z), 0\}.
$$

Case (ii) collapses to a standard test for the parameter lying in $\Theta_z$ in the case of a well-defined interval with $\theta_U > \theta_L$ and to a test on the pseudo-true parameter under misspecification $\theta_U < \theta_L$. This inference procedure can also be interpreted as resulting from a moment inequality problem that allows for misspecification by adding slacks/deviations to equations defining upper and lower effect bounds, see Section 4.3. If such slacks are large, $\theta_U < \theta_L$, can be admitted as solution. In principle, alternative definitions for the pseudo-true set $\Theta^*_z$ that use different weighting compared to (4.1) could be considered as well. This would change the

\[ \text{Note also that we allow for the use of different basis functions across when estimating upper and lower effect bounds. This seems reasonable as it could well be that lower and upper effect bounds are curves of e.g. different smoothness and not equally difficult to approximate. Alternatively one could impose the largest complexity of any bound for both models, i.e. paying the price of a potentially higher estimation variance in the effect bound with a higher degree of smoothness. While this will generally yield a better approximation for each separate bound, it could also contribute to a reverse ordering in the sense that } \hat{\theta}_L(z) \geq \hat{\theta}_U(z) \text{ at some } z. \]
definition of the pseudo-true parameter. Not all choices of such pseudo-true parameter are of equal use. This is similar to e.g. GMM models under misspecification where the pseudo-true parameter is defined implicitly as the maximizer of a GMM population criterion using a particular weighting matrix. Therefore, the choice of the pseudo-true parameter and its usefulness should be balanced versus the robustness against spurious precision under misspecification (Andrews and Kwon, 2019). The particular choice in (4.1) leads to a convenient solution in terms of critical value adjustment due to the specific asymptotic bivariate normal distribution (Stoye, 2020). We obtain the following Theorem:

**Theorem 4.2 (Misspecification Robust Inference)** Suppose Assumptions 3.1 - 3.3 and A.1 - A.6 hold and \( \hat{\theta}_B(z_0) \) for \( B = L, U \) and \( \hat{\Omega}_n(z_0) \) are estimated according to (3.13) and (A.1) respectively. Let the confidence interval be constructed according to (3.16). Then, for any sequence \( z_0 = z_{0,n} \),

\[
\lim \inf_{n \to \infty} \inf_{\theta^{BLP}(z_0) \in \Theta^{*}_{z_0}} P(\theta^{BLP}(z_0) \in CI_{\theta(z_0),1-\alpha}) \geq 1 - \alpha
\]

where \( \Theta^{*}_{z_0} = [\theta_L(z_0), \theta_U(z_0)] \cup \theta^*(z_0) \) where is defined analogously to \( \theta^*(z) \) in (4.1) with best linear predictors instead of bounds. Furthermore if \( \sup_B n^{1/2} k_B^{-1/2} l_{k,Bc,k,B} = o(1) \), then

\[
\lim \inf_{n \to \infty} \inf_{\theta(z_0) \in \Theta^{*}_{z_0}} P(\theta(z_0) \in CI_{\theta(z_0),1-\alpha}) \geq 1 - \alpha
\]

where \( \Theta^{*}_{z_0} = [\theta_L(z_0), \theta_U(z_0)] \cup \theta^*(z_0) \) as defined in (4.1).

Theorem 4.2 demonstrates the asymptotic validity of the confidence intervals proposed in (3.16) for both the best linear predictor as well as the correctly specified case. In particular, we achieve at least nominal coverage independently of the actual width of the true identified region for the heterogeneous effect bound. Coverage is uniform with respect to the width of the true identified region \( \theta_U(z) - \theta_L(z) \) pointwise at each \( z \in Z \) or along sequences therein. The coverage notion over the augmented parameter set \( \Theta^{*}_z \) assures non-emptiness of the interval and avoids spuriously precise inference in regions where the estimated lower bound might be too large relative to the estimated upper bound. Coverage will be closer to the nominal level when correlations between conditional mean errors are small. In particular, near one-sided critical values apply for the usual levels of confidence when \( \rho(z) = 0 \). Note that the residual correlation \( \rho(z) \) can generally vary between different values of \( z \). Thus,
power and size properties of the corresponding tests depend on the location of the local effect bound. In particular, they are driven by the share of missing outcomes at given \( z \). In the case of point identification (no missing outcomes), upper and lower bounds and pseudo true parameter are equivalent and \( \rho(z) = 1 \). Thus, the confidence interval collapses to one using standard two-sided critical values.\(^{10}\)

### 4.3. Misspecification Framework and Examples

In this subsection we provide a more explicit discussion of misspecification using a moment inequality framework as well as two specific examples. In particular, we consider (i) global misspecification of the conditional moment or signal arising from misspecified nuisance functions and (ii) local misspecification of the heterogeneous effect bounds themselves. In practice, of course, these phenomena can occur simultaneously. First, we review the minimal relaxation framework by Andrews and Kwon (2019) and compare it to the heterogeneity setup considered in this paper. In essence, our formalization is a conditional version where the target parameter is nonparametric/semiparametric instead of a finite dimensional vector. Andrews and Kwon (2019) consider minimally relaxed moment inequalities such that the identified set is non-empty under correct specification and misspecification. For a generic \( k \)-dimensional moment function \( m(\cdot) \), they define

\[
\delta(\theta) := \inf \{ \delta \geq 0 : E[m(W,\theta)] \geq -\delta 1_k \}
\]

\[
\delta_{\inf} := \inf_{\theta \in \Theta} \delta(\theta).
\] (4.2)

The corresponding misspecification robust identified set is then given by

\[
\Theta_I := \{ \theta \in \Theta : E[m(W,\theta)] \geq -\delta_{\inf} 1_k \}.
\] (4.3)

In our framework, the moment condition behind auxiliary regression model (3.10), can be rephrased as conditional versions at \( Z_i = z \). In particular, for all \( z \in \mathcal{Z} \), we consider

\(^{10}\)Alternatively, one could employ an additional data splitting step for estimating the two bounds on different subsamples to assure that \( \rho(z) \) is equal to zero. We refrain from this approach to avoid inaccuracies in finite samples as the first estimation step already requires cross-fitting and potential data splitting for tuning of the machine learning methods within folds.
pointwise deviations

\[ \delta(\theta(z)) := \inf \{ \delta \geq 0 : E[m(W, \theta(Z)) | Z = z] \geq -\delta_1 k \} \]

\[ \delta^\inf_z := \inf_{\theta(z) \in \Theta(z)} \delta(\theta(z)) \quad (4.4) \]

and corresponding misspecification robust identified sets

\[ \Theta_I(z) := \{ \theta(z) \in \Theta(z) : E[m(W, \theta(Z)) | Z = z] \geq -\delta^\inf_z 1_k \}. \quad (4.5) \]

For simplicity, we restrict the basis function to be non-negative \( b_B(z) \geq 0 \) in what follows. This is without loss of generality and all derivations apply with reversed signs as well.

Consider case (i): Under correct specification in the heterogeneity step, \( \theta_B(z) = b_B(z)' \beta_B, 0 \) or \( \psi_B(W, \eta_0) = \theta_B(Z) + \varepsilon_B \). The moment function for the bound with estimated and potentially misspecified nuisance parameters then yields

\[ E[b_B(Z)(\psi_B(W, \hat{\eta}) - \theta_B(Z)) | Z = z] = 0 \quad (4.6) \]

for all \( z \in Z \). Neyman-orthogonality of the upper bound signal \( \psi_U(W, \eta) \) then yields

\[ E[b_U(Z)(\psi_U(W, \hat{\eta}) - \psi_U(W, \eta_0) + \varepsilon_U) | Z = z] \]

\[ = b_U(z) E[(\psi_U(W, \hat{\eta}) - \psi_U(W, \eta_0)) | Z = z] \]

\[ = b_U(z) \partial^2_r E[\psi_U(W, \eta_0 + r(\bar{\eta} - \eta_0)) | Z = z] \quad (4.7) \]

where \( \bar{\eta} \) is on the line-segment between \( \eta_0 \) and \( \hat{\eta} \). Thus, overall we have that

\[ E[b_U(Z)(\psi_U(W, \hat{\eta}) - \theta_U(Z)) | Z = z] = b_U(z) g_r(z) =: -\delta_U(\theta(z)) \quad (4.8) \]

where \( \sup_{r \in [0,1]} |g_r(z)| \lesssim_p \lambda^2_{q,n,2} + \lambda^2_{p,n,2} \); see Proof of Theorem 4.1. Now note that, for an upper bound \( \theta_U(z) \), inference is potentially invalid if there is underestimation. Under overestimation, confidence intervals based on misspecified moments are still valid albeit more conservative. Thus, \( \delta_U(\theta(z)) \) can be treated as a positive function in what follows. As \( \theta_U(z) \geq \theta(z) \) it also follows that

\[ E[b_U(Z)(\psi_U(W, \hat{\eta}) - \theta(U(Z)) | Z = z] \geq E[b_U(Z)(\psi_U(W, \hat{\eta}) - \theta_U(Z)) | Z = z]. \quad (4.9) \]

Applying the same with reverted signs and ordering for the bounds then yields the following
moment inequalities for the effect parameter

\[
E[b_U(Z)(\psi_U(W, \hat{\eta}) - \theta(Z))|Z = z] \geq -\delta_U(\theta(z))
\]

\[
E[b_L(Z)(\theta(Z) - \psi_L(W, \hat{\eta}))|Z = z] \geq -\delta_L(\theta(z))
\]

(4.10)

where \(\delta_B(\theta(z))\) are (vectors of) positive functions. This exactly corresponds to the conditional relaxed moment inequality framework outlined above with \(k = k_U + k_L\) and \(\delta_{\inf} = \sup_B \inf_{\theta(z) \in \Theta(z)} \delta_B(\theta(z))\). Under correct specification and convergence of the nuisance functions as in Assumption A.6, naturally \(\delta_B(\theta(z)) = o(1)\), so the uniform deviation would be \(\delta_{\inf} = 0\). However, without correct specification, \(\delta_{\inf} > 0\) can occur if we overestimate the lower bound and/or underestimate the upper bound. The suggested confidence interval with the pseudo-parameterization is then robust to such deviations from Assumption A.6.

This formalization also provides a bridge between a population misspecification and a finite sample misspecification. In particular, \(\delta_{\inf} > 0\) arguably provides a better approximation to cases where, asymptotically, there might be a correct specification but, for a given \(n\), the \(L_p\) errors are relatively large. In finite samples, point estimates after orthogonalization can also be reversed simply because they are not population means. This is also more likely to occur if nuisance parameters are estimated imprecisely.

Now consider case (ii): Local misspecification in the heterogeneity step but correct specification of the nuisance functions. This is the best linear predictor case. If one or both are misspecified, \(b_B(z)'\beta_{B,0} \neq \theta_B(z)\) for some \(B \in \{L, U\}\), then there is no guarantee that these best linear predictors do not intersect or reverse ordering for some \(z\). The relaxed moment inequality representation for such best linear predictor cases follows by definition: Recall that \(E[b_B(Z)(\psi_B(W, \eta_0) - b_B(Z)'\beta_{B,0})|Z = z] = 0\) and \(\theta_U(z) \geq \theta(z)\).

Thus \(E[b_U(Z)(\psi_U(W, \eta_0) - b_U(Z)'\beta_{B,0})|Z = z] + E[b_U(Z)(b_U(Z)'\beta_{B,0} - \theta_U(Z))|Z = z] + E[b_U(Z)(\theta_U(Z) - \theta(Z))|Z = z] \geq b_U(z)r_U(z)\) and therefore

\[
E[b_U(Z)(\psi_U(W, \eta_0) - \theta(Z))|Z_i = z] \geq b_U(z)r_U(z) =: -\delta_U(\theta(z))
\]

(4.11)

where again \(\delta_U(\theta(z))\) can be treated as positive as we are only concerned about underestimation of the upper bound. Applying the same with reverted sign and ordering for the lower
bound then again yields

\[ E[b_U(Z)\psi_U(W, \eta_0) - \theta(Z)|Z = z] \geq -\delta_U(\theta(z)) \]

\[ E[b_L(Z)(\theta(Z) - \psi_L(W, \eta_0))|Z = z] \geq -\delta_L(\theta(z)) \] (4.12)

where \( \delta_B(\theta(z)) = b_B(z)r_B(z) \). For the joint problem of upper and lower bound using two moment functions at a given \( z \), the minimal deviation \( \delta_{2}^{\text{inf}} \) is then chosen simultaneously such that the bounds are allowed to intersect \( b_L(z)^\prime \beta_{L,0} > b_U(z)^\prime \beta_{U,0} \). Thus, even when the conditional moment functions \( \psi_B(W, \eta_0) \) are correctly specified, their best linear predictors for \( \theta_B(z) \) can intersect for some \( z \in \mathcal{Z} \) if there is local misspecification \( r_B(z) \neq 0 \). Our confidence intervals are adaptive to these situations whenever such misspecification would lead to undercoverage and guarantee nominal coverage over the augmented parameter space.

### 4.4. Uniform Inference

We now consider inference for the whole process \( \theta(z) \) without misspecification. Under stronger approximation requirements, the series approach allows for a uniform linearization that can be used to obtain a process analogue of Theorem 4.1. We exploit this to construct a simple multiplier bootstrap procedure that guarantees uniform coverage over \( \mathcal{Z} \) and avoids retraining the first-stage estimators. Inference is then based on a joint process results for the bounds and does not exploit locally varying widths of the identified sets and convergence in distribution. As a result, inference will generally be more conservative. The bootstrap works as follows: For each \( b = 1, 2, \ldots \), generate an \( n \) independent standard exponential random variables \( h_1, \ldots, h_n \). Then, for both \( B = L, U \), use the same \( h_i \) in two weighted regressions

\[ \sqrt{h_i} \psi_B(W_i, \eta_0) = \sqrt{h_i} b_B(Z_i)^\prime \beta_{B,0} + \sqrt{h_i} r_{\theta_B}(Z_i) + \sqrt{h_i} \varepsilon_{i,B} \] (4.13)

with estimated nuisance parameters to obtain bootstrap estimates

\[ \hat{\beta}^b_B = \left( \sum_{i=1}^{n} h_i b_B(Z_i)b_B(Z_i)^\prime \right)^{-1} \sum_{i=1}^{n} h_i b_B(Z_i)\psi_B(W_i, \hat{\eta}). \] (4.14)

Then, for all \( z \), calculate the centered bootstrap \( t \)-statistic for both bounds as

\[ t^b_B(z) = \frac{b_B(z)^\prime (\hat{\beta}^b_B - \hat{\beta}_B)}{\hat{\sigma}^b_B(z)}, \] (4.15)
Now denote \( \bar{t}_B = \sup_{z \in Z} t_B^b(z) \) and \( l_B^b = \inf_{z \in Z} t_B^b(z) \) and, for any \( \alpha \), let \( c_{n,\alpha}(t^b) \) be the \( \alpha \)-quantile of \( t^b \) over the bootstrap samples \( b = 1, 2, \ldots \). The \((1 - \alpha)\) confidence band for \( \theta(z) \) for all \( z \in Z \) is then given by

\[
CB_{\theta(z), 1 - \alpha} = \left[ \hat{\theta}(z) - c_{n,\alpha/2}(t^b_L) \frac{\hat{\sigma}_L(z)}{\sqrt{n}}, \hat{\theta}(z) + c_{n,1 - \alpha/2}(\bar{t}_B) \frac{\hat{\sigma}_U(z)}{\sqrt{n}} \right] \quad (4.16)
\]

Theorem 4.3 contains the joint strong approximation result for both series and bootstrap processes. Theorem 4.4 contains the uniform inference result. The assumptions are generally stronger with regards to the basis function growth and existence of moments compared to the pointwise case (Belloni et al., 2015).

**Theorem 4.3 (Strong Approximation of Series and Bootstrap Process)** Under the assumptions of Theorem 4.1 with (i) \( m > 3 \) and (ii) \( \sup_B \sqrt{k_B \log^3 n/n(n^{1/m_B} \log^{1/2} n + \sqrt{k_B l_k c_{k,B}})} = o(a_n^{-1}) \), (iii) \( a_n^3 n^{-1/2} \sup_B k_B^2 \xi_{k,B}(\log n + \log^2 n/k_B) = o(1) \), and (iv) \( \sup_B n^{1/2} k_B l_k c_{k,B} \log^2 n = o(a_n^{-1}) \) there exists a random standard normal vector \( N_{[k_U + k_L]} \) of length \( k_U + k_L \) such that

\[
\sqrt{n} \hat{\Omega}^b_n(z)^{-1/2} (\hat{\theta}(z) - \theta(z)) \Rightarrow \Omega(z)^{1/2} N_{[k_U + k_L]} + o_P(a_n^{-1}) \text{ in } \ell^\infty(Z)
\]

and

\[
\sqrt{n} \hat{\Omega}_n^b(z)^{-1/2} (\hat{\theta}(z) - \theta(z)) \Rightarrow \Omega(z)^{1/2} N_{[k_U + k_L]} + o_{P^*}(a_n^{-1}) \text{ in } \ell^\infty(Z)
\]

where \( P^* \) is the conditional probability computed given the data \( \{W_i\}_{i=1}^n \).

**Theorem 4.4 (Uniform Inference)** Denote \( P \) as the set of probability measures obeying the assumptions of Theorem 4.3. The confidence bands (4.16) have uniform coverage

\[
\liminf_{n \to \infty} \inf_{P \in P} P(\theta(z) \in CB_{\theta(z), 1 - \alpha} \text{ for all } z \in Z) \geq 1 - \alpha.
\]

5. Monte Carlo Simulation

In this section we analyze the size and power properties of the proposed misspecification robust confidence intervals in finite samples. In particular we look at the size along a grid of \( z \)-values that vary with respect to the share of always-takers and thus the width of the identified set in the population. We consider a generalized Roy model with a random binary treatment and missing responses:
Table 5.1.: Monte Carlo Study: Generalized Roy Model

\[ S_i = 1(X_i'\gamma + D_i - v_i \geq 0) \]
\[ Y_i(0) = \varepsilon_i(0) \]
\[ Y_i(1) = \mu_1(X_i, 1) + \varepsilon_i(1) \]
\[ Y_i^* = D_i Y_i(1) + (1 - D_i) Y_i(0) \]
\[ Y_i = S_i Y_i^* \]
\[ D_i \sim \text{binomial}(0.5) \]
\[ X_{i,j} \sim \text{uniform}(0, 1), \quad j = 1, \ldots, p \]
\[ \varepsilon_i(1) \sim N\left( \begin{bmatrix} 0 \\ 0 \\ 0 \\ \rho \sigma_1 \\ 1 \end{bmatrix} \right) \]

with \( X_i = (X_{i,1}, \ldots, X_{i,p})' \), \( \gamma = (\Phi^{-1}(0.99) - 1, 0, 0, \ldots, 0)' \), \( \mu_1(x) = 0.35 - 4x^2 + 4x^3 \), \( \sigma_1 = \sigma_0 = 0.2 \), and \( \rho = 0.5 \). \( \Phi^{-1}(\cdot) \) denotes the quantile function of the standard normal distribution.

Similar designs have been considered for simulation for point-identified effects under exclusion restrictions, see e.g. Heckman and Vytlacil (2005) or Heiler (2022). Note that the true model here assumes strong monotonicity of the effect of the treatment on response. This knowledge, however, is not imposed onto the estimation procedures. We chose \( Z_i = X_{i,1} \). Thus, the true response function for the always-takers is given by

\[
\theta(z) = E[E[Y_i(1) - Y_i(0)|X_i, S_i(1) = S_i(0) = 1]|Z_i = z] = \mu_1(z) - \rho \frac{\phi(z\gamma)}{\Phi(z\gamma)}.
\]  

(5.1)

The parameters are chosen such that non-response rates vary from 84.13% to 99.00% and are monotonically increasing in \( z \). This will allow us to discover potential differences in coverage rates for varying setups including scenarios close to point identification. We consider both the continuous case, i.e. the size of confidence intervals for the continuous \( \theta(z) \) as well as power curves for a simple discretized version where \( z \) is integrated from 0 to 0.5 and 0.5 to 1 respectively. The nuisance quantities are estimated using honest probability random forests and honest quantile regression forests from the \texttt{grf} package (Athey et al., 2019) with default tuning parameters and two-fold cross-fitting. The design is sufficiently sparse for the forest to achieve the required convergence rates for estimating (5.1) according to Assumption A.6. For the heterogeneity analysis, we use basis splines with nodes and order selected via leave-one-out cross-validation for the continuous case and indicator functions for the discrete case. We analyze the size and power of the misspecification robust confidence intervals (3.16) at a 95% confidence level.
Figure 5.1.: Coverage Rates for $\hat{\theta}(z)$

(a) Design 1: $p = 10$

(b) Design 2: $p = 100$

Coverage rates for varying parameter and sample sizes with nominal level 95%. Results are based on 1000 replications.

Figure 5.1 depicts the simulated coverage rates in the case of $p = 10$ and $p = 100$ regressors for total sample sizes of $n = 400$ and $n = 2000$. The rates for $n = 400$ can sometimes drop to around 90% but overall coverage is still decent. For $n = 2000$, the confidence intervals have at least nominal size with coverage ranging from 95% to 100% depending on $z$. Thus, the theoretical large sample guarantee in Theorem 4.2 seems to approximate the finite sample behavior reasonably well in these designs.

Figure 5.2 depicts the power curves for two different heterogeneous effect parameters $\theta(z_0)$ and $\theta(z_1)$ for $p = 10$ and $p = 100$ at sample sizes $n = 400$ and $n = 2000$. $\theta(z_0)$ corresponds to an area with larger uncertainty regarding the partially identified parameter, i.e. it is integrated over the range of the heterogeneity with the largest share of unobserved outcomes while $\theta(z_1)$ integrates over the range with the largest share of observed outcomes. The difference in uncertainty can also be seen in Figure 1.1. The results show that the power curves are close to zero around the null for all sample size and designs reflecting the conservativeness of the inferential method. However, power converges quickly to 100% when moving away from the null. Power is lower for smaller sample sizes and a larger amount of possible confounding variables as expected. Moreover, in the $z = z_0$ case for which the share of missing outcome is larger, confidence intervals have lower power compared to the $z = z_1$ case that is closer to the case of point identification. Overall, intervals seem to perform reasonably well for different degrees of identification in the overall heterogeneous effect but
Designs with $z = z_0$ depict power curves as function of the deviation from the null for parameter $\theta(z_0) = \int_0^{1/2} \theta(z)dF(z)$. Designs with $z = z_1$ depict the power curves as a function of the deviation from the null for parameter $\theta(z_1) = \int_1^{1/2} \theta(z)dF(z)$. Results are based on 1000 replications.

being closer to point identification tends to yield more power in finite samples.

6. Empirical Study

6.1. Literature on Social Media and Political Polarization

There is a large developing literature regarding the effects of social media on political polarization. However, the evidence on the effects of social media and news exposure via social media on polarization is very mixed, see Haidt and Bail (ongoing) for a comprehensive collaborative review. Effects and channels are often context-dependent and can vary by time, country, platform, algorithm, research design, subgroup, outcome measure, and more. Here
we focus on research regarding affective polarization, Facebook, online news, and questions of selection and heterogeneity.

There are significant changes in measures of affective polarization in many countries since the 1980’s with steepest increase in the US (Boxell et al., 2020). The precise role of online news consumption and social media is still under debate: Suhay et al. (2018) show that online news that contains partisan criticism that derogates political opponents increases affective polarization. Munger et al. (2020) use different Amazon Turk and Facebook Ad samples with clickbait and conventional news headline treatments. They find that older non-democrat users read clickbait articles more often, but no effects on polarization. Cho et al. (2020) demonstrate that political videos selected via the YouTube algorithm increase affective polarization. Nordbrandt (2021) uses Dutch Survey data to argue that higher levels of affective polarization lead to increased social media consumption but finds no evidence for the reverse channel. However, there seems to be significant user-level heterogeneity. Beam et al. (2018) show that, over the course of the US 2016 presidential election campaign, attitudes towards political opponents remained relatively stable. Moreover, they find that Facebook leads to modest depolarization due to an increase in counter-attitudinal news exposure in this period. However, they look at partisan measures based on identity formation and not at standard affective polarization metrics. Bail et al. (2018) study the effects of following counter-attitudinal Twitter bots. Their findings suggest a heterogeneous increase in polarization from counter-attitudinal exposure on Twitter. In particular, Republicans adopted more conservative attitudes after being exposed to a liberal bot while for Democrats the increase in liberal attitudes after following a conservative bot is insignificant. Allcott et al. (2020) show that deactivation of Facebook for one month in Fall 2018 decreased exposure to polarizing news and polarization of political views. Their point estimate on affective polarization is negative but insignificant. However, the study is underpowered to detect small effects and they use a consumption based metric of polarization instead of attitudinal measures. Feezell et al. (2021) do not find evidence that algorithmic or non-algorithmic news sources contribute to higher levels of partisan polarization using explorative survey data. Di Tella et al. (2021) study varying social media status treatments for inside and outside echo chamber units. They find that Twitter, in particular when allowing for
interactions, increases polarization for groups which are already classified as being inside an echo chamber. Subjects in the outside echo chamber group show no significant increases. Yarchi et al. (2021) argue that there are important cross-platform differences. They provide evidence that Facebook is the least homophilic social media network in terms of interactions, positions, and expressed emotions.

Levy (2021) employs a large field experiment regarding news consumption, polarization, and algorithmic news selection on Facebook. He shows that a counter-attitudinal nudge towards subscribing to an outlet with an opposing political ideology decreases affective polarization but does not affect political opinions. He argues that the small nudge setup reflects a realistic user experience on Facebook. A channel seems to be that a shock to the selection of news consumption has lasting effects as units do not re-optimize their feed much afterwards. However, Levy (2021) also provides evidence that the Facebook algorithm limits exposure to precisely these counter-attitudinal news outlets and thus can increase polarization overall. This study has a clear stratified randomization design but suffers from large differential attrition rates in the endline survey, in particular when considering pro- and counter-attitudinal treatments separately. Conventional Lee bounds for the pro- and counter-attitudinal treatment effects include zero. Levy (2021) also does not find significant heterogeneity in treatment effects when using simple interacted linear regression models that ignore attrition. We re-analyze the specific question of the effect of a counter-attitudinal nudge on Facebook on affective polarization by Levy (2021) using the refined DML based heterogeneous bounding method developed in this paper.

6.2. Experiment, Data, and Attrition

In Levy (2021), users were recruited via Facebook ads and filled out a baseline survey between February–March 2018. Units were then stratified by self-reported political ideology and randomly allocated into one of three different treatment arms 1) Liberal, 2) Conservative, or 3) Control. The treatments in 1) and 2) consisted of a nudge to subscribe to (“like”) a selection of four potential (liberal or conservative) outlets. It was explained that a subscription could provide new perspectives, but there were no other incentives or rewards offered. Likes on Facebook make posts from the corresponding outlet more likely to appear
on the user’s feed, thus exposes them to potentially new information and opinions. The liberal outlets were HuffPost, MSNBC, The New York Times, and Slate. The conservative outlets were Fox News, The National Review, The Wall Street Journal, and The Washington Times. Based on this, the counter-attitudinal treatment is defined as nudge towards outlets with ideological leanings contrary to the leaning of the user.\footnote{Individual leanings are based on party affiliation. If units do not identify as Democrats or Republicans, it is according to self-reported ideology. If they neither identify as liberal nor conservative, support of the candidate in the 2016 elections is used. This excludes about 3\% of the total sample that provide no information on leaning.}

Around two months after the baseline survey, participants were asked to fill out an endline survey where political opinions and measures of affective polarization were recorded. Levy (2021) does not find effects from any treatment on political opinions. However, he detects significant effects for the counter-attitudinal treatment on affective polarization relative to the pro-attitudinal treatment. The counterfactual comparison for affective polarization there is made between treatment arms and not relative to the control group due to differential attrition rates. Our method can account for such differences and thus enables us to study whether the counter-attitudinal treatment itself drives the effect on polarization compared to no intervention. Thus, we consider the index for affective polarization constructed by Levy (2021) as outcome in what follows. In particular, we analyze the effect of nudging a user towards counter-attitudinal subscriptions on affective polarization. The parameters can also be interpreted as intent-to-treat effects of subscription. The outcome measure is standardized, such that all coefficients are measured in terms of standard deviations in what follows.

The covariates collected via the baseline survey and Facebook contain information on political ideology, party affiliation, voting behavior, approval of President Trump, baseline polarization, news consumption, and socio-demographic variables such as age and gender. For more information and descriptive statistics consider Levy (2021), Section II. The final sample (including missing endline survey units) consists of 24230 units of which 12126 are in the treatment and 12104 in the control group. Levy (2021) estimates the effects of the intervention on affective polarization using only the units which replied to the endline survey, i.e. for which the outcome variable is observed. He argues that the main estimates are likely
to generalize beyond the selected population.

The experiment, however, suffers from large differential attrition rates: Table 6.1 contains the differential attrition rates between treatment and control group stratified by political ideology. Attrition in the endline survey is large with rates between 46.23% to 65.35%. Moreover, there is significant heterogeneity when looking at the difference in attrition rates between treatment and control condition. In addition, the association between treatment and attrition is not homogeneous across all subgroups. This indicates that there are potential interactions between treatment and baseline characteristics that can lead to heterogeneous response rates. In this case, looking only at unconditional attrition rates between treatment and control group provides a distorted view on the potential bias introduced by ignoring the selection into response. We provide a more thorough analysis of the effect bounds for the counter-attitudinal treatment fully accounting for heterogeneity in treatment effects and attrition rates and potential effects of the treatment on selection into the endline survey. In particular, we re-analyze the unconditional effect bounds using the method suggested in this paper as well as heterogeneous bounds in terms of relevant pre-treatment characteristics.

### 6.3. Estimation and Inference Methods

Replication of unconditional point estimates and unconditional Lee bounds provided by Levy (2021), Table A.12(b) use the same methods. We employ two versions of the DML based generalized lee bounds that differ in terms of nuisance parameter models: The first
(DML parametric) uses logistic regression with all confounding variables interacted with
the treatment for the response selection probabilities and linear quantile regression with all
confounding variables for the conditional quantile of the selected treated. These models are
more likely to be misspecified. The second (DML forest) uses probability forests and quantile
forests with 1000 trees and honest splitting (Athey et al., 2019). For both specifications, all
categorical variables are coded as flexible dummy variables leaving us with 36 confounders in
total. Conditional quantile trimming levels are rounded towards the closest value on a grid
from (0.01, 0.02, ..., 0.99). Cross-fitting is based on 10 folds. For the heterogeneity analysis
we use the estimated signals \( \psi_B(W_i, \hat{o}) \) provided by the forest-based DML specification and
basis splines for the continuous variables with node and order selection via leave-one-out
cross-validation. Confidence intervals are based on the misspecification robust method (3.16),
confidence bands on the multiplier bootstrap (4.16). We report both at 90% due to the
conservativeness of the methods.

6.4. Results

6.4.1. Unconditional Effects

In this subsection, we provide the unconditional effect analysis of the counter-attitudinal
nudge on affective polarization. Table 6.2 contains the estimates by Levy (2021), the naive
Lee bounds assuming (strong) monotonicity as well as the two DML-based methods.\(^\text{12}\) The
conventional Lee bounds (3) as well as the DML bounds under parametric assumptions (4)
contain a null effect in the estimated identified set. The more flexible DML approach using
forests (5) does not contain zero and contains the point estimates provided by Levy (2021).
Correcting for the statistical uncertainty, however, the effect is not significantly different
from zero. From Theorem 4.2 we know that inference on the effect is conservative, thus we
interpret this as weak evidence against a null effect of the counter-attitudinal treatment on

\(^{12}\) Table A.12(b) by Levy (2021) contains an error as his bounds \([-0.172, 0.060]\) (no CI provided) are cal-
culated from \( \theta_L(x) = E[Y_i|D_i = 1, S_i = 1, Y_i \leq q(p_0(X_i)), X_i = x] - E[Y_i|D_i = 0, S_i = 1, Y_i \leq
q(p_0(X_i)), X_i = x] \) (and equivalently for \( \theta_U(x) \)) and not based on (3.4). Moreover, in the specifications
with controls, he assumes constant effect bounds and a linear form of the truncated mean which is heavily
restrictive and generally does not identify the true bounds. Columns (4) and (5) produce the correctly
calculated bounds under the more credible assumptions allowing for non-linearity and weak monotonicity.
Table 6.2.: Unconditional Counter-Attitudinal Treatment Effect on Affective Polarization

|                  | Levy (2021)       | Levy (2021)       | Lee Bounds        | DML Bounds        | DML Bounds        |
|------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
|                  | (unconditional)   | (conditional)     | (unconditional)   | (parametric)      | (forest)          |
|                  | (1)               | (2)               | (3)               | (4)               | (5)               |
| Estimate         | \(-0.055^{***}\) | \(-0.028^{**}\)  | \([-0.161, 0.060]\) | \([-0.238, 0.033]\) | \([-0.156, -0.016]\) |
| 90%-CI           | (-0.086, -0.024) | (-0.048, -0.008) | (-0.195, 0.094)  | (-0.279, 0.071)  | (-0.203, 0.024)  |

Point and interval estimates of the counter-attitudinal treatment on affective polarization using 16896 (columns 1 and 2) and 24230 (columns 3 to 5) observations. (Un)conditional refers to the case of (not) using X in the regression or bound analysis. Parametric and forest refers to the nuisance models for the DML bounds. \(^{***} = p < 0.01, ^{**} = p < 0.05, ^{*} = p < 0.10.\)

affective polarization in line with the original study. The applicability of the conventional Lee bounds (3) under strong monotonicity is questionable. In particular, our estimates for the conditional selection probabilities based on the specification for Column (5) suggest that the effect of the treatment on attrition is negative (\(\hat{p}_0(x) > 1\)) for 65.42% and positive (\(\hat{p}_0(x) < 1\)) for 34.58% of the sample indicating violation of strong monotonicity. This is in line with the heterogeneous attrition rates observed in Table 6.1.

### 6.4.2. Heterogeneous Effects

In this subsection, we analyze the effect bounds of the counter-attitudinal nudge on affective polarization as functions of heterogeneity variables. In particular, we look at political ideology (categorical) and age (continuous). We select political ideology from the baseline survey as it was used for block-randomization of the original experiment and could provide insight regarding potential asymmetries in the effect of counter-attitudinal nudges in terms of partisanship. Age can easily be used for targeting of such an intervention based on social media information only (no survey required) and has been shown to be an important determinant of aggregate affective polarization levels in the US (Phillips, 2022). These variables were also suggested by Levy (2021) for heterogeneity analysis.

Figure 6.1 contains the heterogeneous bounds sorted by ideology plus 90% confidence intervals. First note that the identified sets for liberal, moderate, slightly conservative, and conservative do not include zero. The effects are also significant for moderates with set estimate \([-0.159, -0.160]\) and for conservatives with \([-0.167, -0.091]\). This suggests that a
This Figure contains the effect bounds (in standard deviations) of the counter-attitudinal treatment on affective polarization stratified by ideology. The red intervals are the estimated identified sets. The dashed blue lines are the misspecification robust 90%-confidence intervals for the heterogeneous effects.

nudge on Facebook for these groups can significantly decrease their affective polarization. It is noteworthy that these estimated effect bounds are about 2-3 times as large as the unconditional estimates suggested by Levy (2021) that do not correct for attrition.

This Figure contains the effect bounds (in standard deviations) of the counter-attitudinal treatment on affective polarization as a function of age. The red band is the estimated identified set. The dashed blue lines are the misspecification robust 90%-confidence intervals. The black area is the 90% uniform confidence band using the multiplier bootstrap.

Figure 6.2 contains the estimated effect bounds and confidence intervals as functions of age. We can see that the estimated lower bound is constant and very close to the uncon-
ditional lower bound provided in Table 6.2, Column (5).\textsuperscript{13} The upper bound, however, is monotonically increasing with age. In particular, the identified set is very narrow at age 18, suggesting a significant effect with estimated set \([-0.160, -0.121]\). From there, the identified set gets progressively wider. The last age of significance is at 44 years with estimated set \([-0.166, -0.041]\). From 57 years on, the estimated set does no longer exclude zero. It is noteworthy that both effect bounds for the lowest age groups are much larger in magnitude compared to the unconditional point estimates that do not correct for attrition by, again, a rough factor of 2-3.

6.5. Discussion

Overall, the findings support the conclusion by Levy (2021) that a counter-attitudinal nudge can decrease affective polarization. Moreover, effects are potentially heterogeneous. With regards to political ideology, the fact that effects are only significant for conservatives and moderates could be driven by measurement/low power given the amount of attrition considering that the identified set is negative for liberals as well. Note that, as the treatment varies depending on political ideology, we cannot distinguish heterogeneity in user response from heterogeneity in news outlets. It could well be that conservative or liberal leaning outlets offer counter-attitudinal viewpoints at smaller or larger doses. Assessing the differences in partisan attitudes by outlets and the consequences on polarization is an interesting question. However, from a policy assignment perspective, the total effects presented here are most relevant, not hypothetical channels mediated through treatment intensity as the latter can reasonably be considered fixed. Similarly, the baseline of news consumption on Facebook could differ between users with different political ideology and thus being exposed to the same dose of nudges has a different effect on the relative amount of counter-attitudinal news which could potentially be more important for affective polarization than the absolute level.

Regarding age differences, first note that the data does not reject a constant effect across

\textsuperscript{13}Note that here we have excluded the units for which age information was missing \((n = 791)\). Thus, the estimate for the lower bound differs slightly from the unconditional estimate in Table 6.2, Column (5). Estimating the bounds separately for the omitted category yields interval \([-0.036, 0.345]\).
age levels. However, potential differences also in line with the data could well be dose-related as younger users spend more time online. Their overall activity is also likely contributing to lower attrition rates which leads to the tighter bounds reported. Alternatively or additionally, affective polarization is usually understood through the lens of social identity theory: People internalize partisan affiliation as part of their sense of self. The latter tends to be more malleable for younger people, in particular in their formative years (Phillips, 2022). The last age at which we detect significant effects (44 years) roughly corresponds to the median age of Facebook users (43 years). Thus, the evidence suggests that targeting users with counter-attitudinal subscription nudges on Facebook could decrease affective polarization for a substantial share of the population.

To put the size of effects into perspective, we compare the estimates to experimental estimate by Allcott et al. (2020). Our unconditional DML bounds are insignificant, but the point estimates suggest that the effect of nudging towards subscribing to 1 to 4 counter-attitudinal outlets after two months is approximately between 1/7 to 1.5 times the effect of deactivating Facebook for a whole month. The significant bounds for conservatives, moderates, and the younger age levels suggest that, for these groups, the impact is around 1.5 times as large as the effect of deactivation. As a limitation, note that we are comparing to an unconditional baseline from Allcott et al. (2020). The effect of deactivation could potentially be much larger (or smaller) for these groups as well.

All these results have to be taken with some caution given the finding by Levy (2021) that the Facebook algorithm itself does not favor counter-attitudinal outlets. However, they suggest that Facebook has the potential to decrease or at least mitigate its effects on affective polarization in line with Beam et al. (2018) and Yarchi et al. (2021). For further research it would be interesting to see whether typically encountered subscription advertisements run by news outlets themselves have comparable intent-to-treat effects as nudges provided by outside sources.
7. Concluding Remarks

This paper provides a method for estimation and inference for bounds for heterogeneous treatment effects under sample selection. We make the general point that heterogeneity in partially identified problems requires special attention as both effect parameters as well as identified sets can be subject to heterogeneity. Exploiting the latter can yield more precise inference in empirical applications compared to crude, unconditional approaches. This is also what we find in our evaluation study: There are highly heterogeneous bounds for the effects of subscription nudges on Facebook on affective polarization. Unconditional bounds mask effects that are up to three times as large in magnitude compared to point estimates that ignore missing outcomes completely. There are also multiple extensions possible: In many applications where the method could be useful, the iid assumption is overly restrictive. In particular, in social experiments units are often clustered within groups such as schools or regions. In this case, the cross-fitting requires adaptation to the dependence structure and standard errors should be based on cluster-robust or jackknife type estimators. It would also be interesting to see under which conditions the methodology in this paper can be extended to more general moment inequality problems.

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**Appendix A  Moment Functions and Variances**

**A.1 Moment Functions**

Assume propensity scores are known, i.e. \( P(D_i = 1|X_i = x) = e(x) \) known. For the correction terms for the unknown case consider Semenova (2020). The true nuisance vector \( \eta_0(X_i) = \eta_0 = \{s(0, X_i), s(1, X_i), q(u, X_i)\} \) where \( s(d, x) = P(S_i = 1|X_i, D_i = d) \) and \( q(u, x) = \inf\{q : u \leq P(Y_i \leq q|D_i = 1, S_i = 1, X_i = x)\} \) for all \( u \in [0, 1] \), \( x \in \mathcal{X} \), \( d \in \{0, 1\} \).

Moreover define \( p_0(x) = s(0, x)/s(1, x) \), \( 1_i^+ = 1(p_0(X_i) < 1) \), \( \mu_{i0} = E[s(0, X_i)|1_i = 1] \), \( 1_i^- = 1(p_0(X_i) > 1) \), and \( \mu_{i1} = E[s(1, X_i)|1_i = 1] \). The Neyman-orthogonal moment functions are given by

\[
\psi_{L}(W_i, \eta) = \psi_{L}^+(W_i, \eta) + \frac{1^+}{\mu_{i0}} \alpha_{L}^+(W_i, \eta) + \frac{1^-}{\mu_{i1}} \alpha_{L}^-(W_i, \eta)
\]

\[
\psi_{U}(W_i, \eta) = \psi_{U}^+(W_i, \eta) + \frac{1^+}{\mu_{i0}} \alpha_{U}^+(W_i, \eta) + \frac{1^-}{\mu_{i1}} \alpha_{U}^-(W_i, \eta)
\]

with non-robust moment functions

\[
\psi_{L}^+(W_i, \eta) = \frac{1^+}{\mu_{i0}} \left( \frac{D_i}{e(X_i)} S_i Y_i 1(Y_i \leq q(p_0(X_i), X_i)) - \frac{1 - D_i}{1 - e(X_i)} S_i Y_i \right)
\]

\[
+ \frac{1^-}{\mu_{i1}} \left( \frac{D_i}{1 - e(X_i)} S_i Y_i - \frac{1 - D_i}{1 - e(X_i)} S_i Y_i 1(Y_i \geq q(1 - 1/p_0(X_i), X_i)) \right)
\]

\[
\psi_{U}^+(W_i, \eta) = \frac{1^+}{\mu_{i0}} \left( \frac{D_i}{e(X_i)} S_i Y_i 1(Y_i \geq q(1 - p_0(X_i), X_i)) - \frac{1 - D_i}{1 - e(X_i)} S_i Y_i \right)
\]

\[
+ \frac{1^-}{\mu_{i1}} \left( \frac{D_i}{1 - e(X_i)} S_i Y_i - \frac{1 - D_i}{1 - e(X_i)} S_i Y_i 1(Y_i \leq q(1/p_0(X_i), X_i)) \right)
\]
and bias-corrections

\[
\begin{align*}
\alpha_L^+ (W_i, \eta) &= q(p_0(X_i), X_i) \left( \frac{(1 - D_i) S_i}{e(X_i)} - s(0, X_i) \right) - q(p_0(X_i), X_i) p_0(X_i) \left( \frac{D_i S_i}{e(X_i)} - s(1, X_i) \right) \\
&\quad - q(p_0(X_i), X_i) s(1, X_i) \left( \frac{D_i S_i \mathbb{I}(Y_i \leq q(p_0(X_i), X_i))}{s(1, X_i) e(X_i)} - p_0(X_i) \right) \\
\alpha_L^- (W_i, \eta) &= q(1 - 1/p_0(X_i), X_i) \frac{1}{p_0(X_i)} \left( \frac{(1 - D_i) S_i}{1 - e(X_i)} - s(0, X_i) \right) + q(1 - 1/p_0(X_i), X_i) \left( \frac{D_i S_i}{e(X_i)} - s(1, X_i) \right) \\
&\quad + q(1 - 1/p_0(X_i), X_i) s(0, X_i) \left( \frac{D_i S_i \mathbb{I}(Y_i \leq q(1 - 1/p_0(X_i), X_i))}{s(0, X_i) e(X_i)} + \left( 1 - \frac{1}{p_0(X_i)} \right) \right) \\
\alpha_U^+ (W_i, \eta) &= q(1 - p_0(X_i), X_i) \left( \frac{(1 - D_i) S_i}{e(X_i)} - s(0, X_i) \right) - q(1 - p_0(X_i), X_i) p_0(X_i) \left( \frac{D_i S_i}{e(X_i)} - s(1, X_i) \right) \\
&\quad + q(1 - p_0(X_i), X_i) s(1, X_i) \left( \frac{D_i S_i \mathbb{I}(Y_i \leq q(1 - p_0(X_i), X_i))}{s(1, X_i) e(X_i)} - (1 - p_0(X_i)) \right) \\
\alpha_U^- (W_i, \eta) &= q(1/p_0(X_i), X_i) \frac{1}{p_0(X_i)} \left( \frac{(1 - D_i) S_i}{e(X_i)} - s(0, X_i) \right) + q(1/p_0(X_i), X_i) \left( \frac{D_i S_i}{e(X_i)} - s(1, X_i) \right) \\
&\quad + q(1/p_0(X_i), X_i) s(0, X_i) \left( \frac{D_i S_i \mathbb{I}(Y_i \leq q(1/p_0(X_i), X_i))}{s(0, X_i) e(X_i)} - \frac{1}{p_0(X_i)} \right)
\end{align*}
\]

A.2 Definition of Variances

The true variance is given by \( \Omega(z) = B(z)' E[B(Z_i)(\varepsilon_i + r_i)(\varepsilon_i + r_i)'B(Z_i)']B(z) \). with definitions given in Appendix B.4.1. Its consistent estimator is

\[
\hat{\Omega}_n(z) = B(z)' \frac{1}{n} \sum_{i=1}^{n} [B(Z_i)e_i e_i' B(Z_i)'] B(z), \tag{A.1}
\]

where \( e_i = (e_i, L(Z_i), e_i, U(Z_i))' \) and \( e_i, B = \psi_B(W_i, \hat{\eta}) - b_B(Z_i)' \hat{\beta}_B \) being the residuals of the nonparametric regression for \( B = L, U \). The bootstrapped \( \hat{\Omega}^b_n(z) \) is defined analogously to (A.1) with bootstrap weights \( h_i \).

Appendix B Proof of Theorem 4.1 and 4.2

B.1 Preliminaries

First we introduce notation and establish auxiliary results. We then show that our assumptions imply small machine learning bias, provide a linearization result, and verify a sufficient condition for asymptotic normality. This, together with a matrix convergence, then serves as main input for the assumptions required for adaptation of the method by Stoye (2020) that
provides the coverage result. We use \( x \lesssim y \) whenever \( x = O(y) \) and \( x \lesssim_P y \) when \( x = O_p(y) \). Statements about random variables are almost surely if not stated differently. We refer to Semenova and Chernozhukov (2021) and Belloni et al. (2015) as SC and BCCK respectively.

### B.2 Auxiliary Results

**B.2.1 p and s rates** By definition \( \hat{p}(x) = s(0, x)/s(1, x) \) and equivalently for \( \hat{p}(x) \), thus

\[
\hat{p}(x) - p_0(x) = \frac{s(0, x)}{\hat{s}(1, x)} - \frac{s(0, x)}{s(1, x)} = \frac{s(0, x) - s(0, x) s(1, x)}{s(1, x)} = \frac{s(1, x) - s(1, x) s(0, x) s(1, x)}{s(1, x) \hat{s}(1, x)}
\]

almost surely in \( {\mathcal{X}} \) due to A.3.

**B.2.2 Quantile function and level bounds** First note that, for generic \( u, u' \in \tilde{U} \), we have that, almost surely on \( {\mathcal{X}} \),

\[
|\hat{q}(u, x) - \hat{q}(u', x)| \leq |q(u, x) - q(u', x)| + 2 \sup_{u \in \tilde{U}} |\hat{q}(u, x) - q(u, x)|
\]

Now, on the realization set \( {\mathcal{T}}_n \) with probability \( 1 - e_n \), the second term is \( o_p(1) \) by A.6. Using a mean-value expansion for the first term yields

\[
|q(u, x) - q(u', x)| \leq \frac{1}{\inf_{y} f(y|x, d = 1, s = 1)} |u - u'| \leq C |u - u'|
\]

by A.2. Equivalently, we have that on \( {\mathcal{T}}_n \) that

\[
|\hat{q}(\hat{p}(X), X) - q(p_0(X), X)| = \frac{1}{\hat{f}(q'(\hat{p}(X), X))} |\hat{p}(X) - p_0(X)| \lesssim_P |\hat{p}(X) - p_0(X)|
\]

as \( \hat{p} \) is almost surely bounded away from 0 and 1 on \( \tilde{U} \). This implies that \( E[|\hat{q}(\hat{p}(X), X) - \hat{q}(p_0(X), X)|] \lesssim E[|\hat{p}(X) - p_0(X)|^2]^{1/2} \). Now we bound the expectation of estimated versus true truncation thresholds

\[
E[|1_{Y < \hat{q}(\hat{p}(X), X)} - 1_{Y < \hat{q}(p_0(X), X)}| \mid D_i = 1, S_i = 1]
\]

\[
\lesssim E[|F_{y|x, d=1, s=1}(\hat{q}(\hat{p}(X), X)) - F_{y|x, d=1, s=1}(\hat{q}(p_0(X), X))| \mid D_i = 1, S_i = 1]
\]

\[
\lesssim \sup_{y} f_{y|x, d=1, s=1}(y) E[|\hat{q}(\hat{p}(X), X) - \hat{q}(p_0(X), X)|]
\]

\[
\lesssim E[|\hat{p}(X) - p_0(X)|^2]
\]

\[
\lesssim \lambda_{s,n,2}^2
\]
by a mean-value expansion and \((a,i)\) together with A.2. Similarly,

\[
E[\mathbb{I}_{Y<\hat{q}(p_0(X),X)}] - E[\mathbb{I}_{Y<q(p_0(X),X)}] | D_i = 1, S_i = 1 \\
\lesssim E[|F_{y|x,d=1,s=1}(\hat{q}(p_0(X),X)) - F_{y|x,d=1,s=1}(q(p_0(X),X))|] | D_i = 1, S_i = 1 \\
\lesssim \sup_y f_{y|x,d=1,s=1}(y) E[|\hat{q}(p_0(X),X) - q(p_0(X),X)|] \\
\lesssim \lambda^2_{q,n,i}.
\]

### B.3 Machine Learning Bias and Linearization

We know verify that our assumptions are sufficient for SC, A3.5 and provide a linearization result. For simplicity, we consider the lower bound under positive monotonicity. The full bounds under conditional monotonicity follows analogously. SC A3.5 requires

\[
B_n := \sqrt{n} \sup_{\eta \in T_n} ||E[b_i(W_i,\eta) - \psi(W_i,\eta_0)||] = o(1) \\
\Lambda_n := \sup_{\eta \in T_n} E[||b_i(\psi(W_i,\eta) - \psi(W_i,\eta_0)||^2]^{1/2} = o(1).
\]

Subscripts \(i\) of are omitted in the remainder of this subsection. Define the moment function for the lower bound with bias correction as \(\psi(W,\eta) = \sum_{j=1}^{4} \psi^{[j]}(W,\eta)\) with

\[
\psi^{[1]}(W,\eta) := \frac{D}{e(X)} SY \mathbb{I}_{Y<q(p_0(X),X)} - \frac{1-D}{1-e(X)} SY \\
\psi^{[2]}(W,\eta) := q(p_0(X),X) \left( \frac{1-D}{1-e(X)} S - s(0,X) \right) - q(p_0(X),X) p_0(X) \\
\psi^{[3]}(W,\eta) := -q(p_0(X),X) p_0(X) \left( \frac{D}{e(X)} S - s(1,X) \right) \\
\psi^{[4]}(W,\eta) := -q(p_0(X),X)s(1,X) \left( \frac{DS}{e(X)s(1,X)} \mathbb{I}_{Y<q(p_0(X),X)} - p_0(X) \right).
\]

For \(B_n\) note that, conditional on the cross-fitted model, the following expansion applies for any \(\eta \in T_n\)

\[
\frac{1}{\sqrt{k}} E[b_\psi(W,\eta)] = \frac{1}{\sqrt{k}} E[b_\psi(W,\eta_0)] + \frac{1}{\sqrt{k}} \partial_\eta E[b_\psi(W,\eta_0 + r(\eta - \eta_0))] + \frac{1}{\sqrt{k}} \partial^2_\eta E[b_\psi(W,\tilde{\eta})]
\]

where \(\partial_\eta\) denotes the Gateaux-derivative operator and \(\tilde{\eta}\) is on the line segment between \(\eta_0\) and \(\eta\). Define \(\tilde{Q} := \sup_{x \in X} \sup_{u \in \tilde{U}} Q(u,x)\) which is bounded as the boundaries of \(\tilde{U}\) are bounded away from 0 and 1 due to A.3. This, for any \(\eta \in T_n\), yields
\[ E[|\psi(W, \eta)| \mid X = x] \leq \frac{1}{\xi} |E[Y(1)| \mid X = x] + \frac{1}{1 - \xi} E[|Y(1)| \mid X = x] + |\tilde{Q}| + |\tilde{Q}| \]
\[ + \frac{|\tilde{Q}|}{s} \left( \frac{1}{\xi} + \frac{1}{s} \right) + \left( \frac{1}{\xi} + \frac{1}{s} \right) \]
\[ \lesssim E[|Y(1)| \mid X = x] \]
almost surely in \( \mathcal{X} \) due to A.3. Now note that \( b \) is \( \mathcal{X} \)-measurable and thus we can apply the following bound using the Cauchy-Schwarz inequality:
\[ \frac{1}{\sqrt{k}} E[|b\psi(W, \eta)|] \lesssim \frac{1}{\sqrt{k}} E[|b| \mid |Y| \mid X)] \leq \frac{E[|b|^1/2]}{\sqrt{k}} E[E[Y^2|X]]^{1/2} \lesssim 1 \]
where the last inequality comes from Assumptions A.1 and A.2. This bounds allows us to apply dominated convergence
\[ \frac{1}{\sqrt{k}} \partial, E[b\psi(W, \eta_0 + r(\eta - \eta_0))] = \frac{1}{\sqrt{k}} E[b\partial, E[\psi(W, \eta_0 + r(\eta - \eta_0))]|X] = 0 \]
where the last step follows from Neyman-orthogonality of \( \psi(W, \eta) \) around \( \eta_0 \) (Semenova, 2020). A uniform bound for the second derivative of \( E[\psi(W, \eta)|X = x] \) has been provided in Semenova (2020), Proof of Lemma C.1. Thus, we obtain
\[ \| \frac{1}{\sqrt{k}} E[b\psi(W, \eta)] - \frac{1}{\sqrt{k}} E[b\psi(W, \eta_0)] \| \lesssim \frac{1}{\sqrt{k}} E[|b|| \|\eta - \eta_0\|^2]. \]
Together this implies that, as \( E[|b(z)||] = k_B \)
\[ \sqrt{n} \sup_{\eta \in T_n} \|E[b_i(\psi(W_i, \eta) - \psi(W_i, \eta_0))] \| \lesssim \sqrt{n}k_L(\lambda_{s,n,4}^2 + \lambda_{q,n,4}^2) = o(1) \]
by the Cauchy-Schwarz and triangle inequality. In the case of a bounded basis similarly
\[ \sqrt{n} \sup_{\eta \in T_n} \|E[b_i(\psi(W_i, \eta) - \psi(W_i, \eta_0))] \| \lesssim \sqrt{n}k(\lambda_{s,n,2}^2 + \lambda_{q,n,2}^2) = o(1). \]
by A.6 and Lemma 6.1 by Chernozhukov et al. (2018a). Now consider the variance term \( \Lambda_n \). We first bound the differences for each component \( \psi^{[1]}, \ldots, \psi^{[4]} \) separately: For any \( \hat{\eta} \in T_n \), note that
\[ \psi^{[1]}(W, \hat{\eta}) - \psi^{[1]}(W, \eta_0) = \frac{D}{e(X)} SY \mathbb{I}_{\{Y < \hat{q}(\hat{p}(X), X)\}} - \frac{D}{e(X)} SY \mathbb{I}_{\{Y < q(p(X), X)\}} \]
\[ = \frac{D}{e(X)} SY \left( \mathbb{I}_{\{Y < \hat{q}(\hat{p}(X), X)\}} - \mathbb{I}_{\{Y < \hat{q}(\hat{p}(X), X)\}} \right) + \mathbb{I}_{\{Y < q(p(X), X)\}} - \mathbb{I}_{\{Y < q(p(X), X)\}} \]
and thus, by (a.ii) and A.2, we obtain

\[
\sup_{\eta \in \mathcal{T}_n} E[(\psi^{[1]}(W, \eta) - \psi^{[1]}(W, \eta_0))^2] \lesssim \lambda_{q,n,2}^2 + \lambda_{s,n,2}^2
\]

\[
\psi^{[2]}(W, \eta) - \psi^{[2]}(W, \eta_0) = \hat{q}(\hat{p}(X), X) \left( \frac{(1 - D)S}{1 - c(X)} - \hat{s}(0, X) \right) - q(p_0(X), X) \left( \frac{(1 - D)S}{1 - c(X)} - s(0, X) \right)
\]

\[
= [\hat{q}(\hat{p}(X), X) - q(p(X), X)] \left( \frac{(1 - D)S}{1 - c(X)} \right) + (\hat{q}(\hat{p}(X), X) - \hat{q}(p_0(X), X))\hat{s}(0, X)
\]

\[
+ \hat{q}(p_0(X), X)(\hat{s}(0, X) - s(0, X)) + (\hat{q}(p_0(X), X) - q(p_0(X), X))s(0, X)
\]

\[
\lesssim |\hat{q}(\hat{p}(X), X) - q(p_0(X), X)| + |\hat{q}(p_0(X), X) - q(p_0(X), X)| + |\hat{s}(0, X) - s(0, X)|
\]

almost surely by A.2 and (a.ii). (a.i) then implies that

\[
\sup_{\eta \in \mathcal{T}_n} E[(\psi^{[3]}(W, \eta) - \psi^{[3]}(W, \eta_0))^2] \lesssim \lambda_{q,n,2}^2 + \lambda_{s,n,2}^2.
\]

\[
\psi^{[3]}(W, \eta) - \psi^{[3]}(W, \eta_0) = -\hat{q}(\hat{p}(X), X)\hat{p}(X) \frac{DS}{e(X)} + \hat{q}(\hat{p}(X), X)\hat{s}(0, X)
\]

\[
+ q(p_0(X), X)p_0(X) \frac{DS}{e(X)} - q(p_0(X), X)s(0, X)
\]

\[
\lesssim |\hat{p}(X) - p_0(X)| + |\hat{q}(p_0(X), X) - q(p_0(X), X)| + |\hat{s}(0, X) - s(0, X)|
\]

almost surely by the same logic as for the previous components. Thus, we again obtain by (a.i) that

\[
\sup_{\eta \in \mathcal{T}_n} E[(\psi^{[4]}(W, \eta) - \psi^{[4]}(W, \eta_0))^2] \lesssim \lambda_{q,n,2}^2 + \lambda_{s,n,2}^2.
\]

\[
\psi^{[4]}(W, \eta) - \psi^{[4]}(W, \eta_0) = -\hat{q}(\hat{p}(X), X) \left( \frac{DS}{e(X)} \mathbb{1}_{\{Y < \hat{q}(\hat{p}(X), X)\}} \right) + \hat{q}(\hat{p}(X), X)\hat{s}(0, X)
\]

\[
+ q(p_0(X), X) \left( \frac{DS}{e(X)} \mathbb{1}_{\{Y < q(p_0(X), X)\}} \right) - q(p_0(X), X)s(0, X)
\]

\[
\lesssim |\hat{q}(\hat{p}(X), X)\mathbb{1}_{\{Y < \hat{q}(\hat{p}(X), X)\}} - q(p_0(X), X)\mathbb{1}_{\{Y < q(p_0(X), X)\}}|
\]

\[
+ |\hat{q}(p_0(X), X) - q(p_0(X), X)| + |\hat{s}(0, X) - s(0, X)|
\]

almost surely. Thus, using (a.i) and (a.ii), we obtain

\[
\sup_{\eta \in \mathcal{T}_n} E[(\psi^{[5]}(W, \eta) - \psi^{[5]}(W, \eta_0))^2] \lesssim \lambda_{q,n,2}^2 + \lambda_{s,n,2}^2.
\]

Putting everything together we then obtain the required

\[
\sup_{\eta \in \mathcal{T}_n} E[|b_i(\psi(W, \eta) - \psi(W, \eta_0))|^2]^{1/2} \lesssim \xi_{k,L} E[(\psi(W, \eta) - \psi(W, \eta_0))^2]^{1/2}
\]

\[
\lesssim \xi_{k,L}(\lambda_{q,n,2} + \lambda_{s,n,2})
\]

\[
= o(1)
\]
by the triangle inequality and A.6. Therefore Assumption 3.5 in SC applies. This together with A.1, A.2, A.4, and A.5 then covers all conditions required for SC, Lemma 3.1(b). Thus, for any \( z \in \mathcal{Z} \) and \( B \in \{L, U\} \), the estimators are asymptotically linear:

\[
\sqrt{n}b_B(z)'(\hat{\beta}_B - \beta_{B,0}) = b(z)'E[b_B(Z_i)b_B(Z_i)']^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} b_B(Z_i)(\varepsilon_{i,B} + r_{\theta_B}(Z_i)) + o_p(1),
\]

where the remainder convergence follows from A.5.

**B.4 Proof of Theorem 4.1 and Theorem 4.2**

**B.4.1 Notation**

Without loss of generality, instead of A.1, assume basis functions are orthogonalized, i.e. \( E[b_B(Z_i)b_B(Z_i)'] = I_{k_B} \). Stacking the linearizations from the previous section yields

\[
\sqrt{n}b_B(z)'(\hat{\beta}_L - \beta_{L,0}) = b_L(z)'E[b_L(Z_i)b_L(Z_i)']^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [b_L(Z_i)(\varepsilon_{i,L} + r_{\theta_L}(Z_i))] + o_p(1),
\]

\[
\sqrt{n}b_B(z)'(\hat{\beta}_U - \beta_{U,0}) = b_U(z)'E[b_U(Z_i)b_U(Z_i)']^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [b_U(Z_i)(\varepsilon_{i,U} + r_{\theta_U}(Z_i))] + o_p(1),
\]

Define

\[
\Omega(z) = B(z)'E[B(Z_i)(\varepsilon_i + r_i)(\varepsilon_i + r_i)'B(Z_i)']B(z)
\]

and

\[
\sqrt{n}\Omega(z)^{-1/2}B(z) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [B(Z_i)(\varepsilon_i + r_i)] = \sum_{i=1}^{n} w_{i,n}(z)(\varepsilon_i + r_i)
\]

with

\[
w_{i,n}(z) = \frac{\Omega(z)^{-1/2}B(z)B(Z_i)}{\sqrt{n}}.
\]

Note that invertibility of \( \Omega(z) \) is guaranteed by Assumption 3.3 as this excludes the case of perfectly correlated conditional mean errors between upper and lower bound.

**B.4.2 Auxiliary Results**

For simplification we use notation \( B = B(z) \) and \( B_i = B(Z_i) \) whenever it does not cause confusion. We make use of the fact that for block matrices the induced \( L_2 \) norm \( ||\cdot|| \) is given
by
\[ M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \Rightarrow ||M|| = \max\{||M_1||, ||M_2||\}. \]

\((an.i)\) **Expected norm of weights** \(w_{i,n}(z)\)  Note that Assumption 3.3 implies that the conditional mean have correlation bounded away from one. Thus, their conditional variance-covariance matrix is strictly p.d. with \(\inf_{z \in Z} ||E[(\varepsilon_i + r_i)(\varepsilon_i + r_i)'|Z_i = z]|| > \lambda_{\text{min}} > 0\) and hence \(||E[B_i(\varepsilon_i + r_i)(\varepsilon_i + r_i)'B_i']|| \geq ||\lambda_{\text{min}}^{-1}E[B_iB_i']||.\) This implies that
\[
\begin{align*}
nE[||w_{i,n}(z)||^2] &\leq ||(B'E[B_i(\varepsilon_i + r_i)(\varepsilon_i + r_i)'B')^{-1/2}E[B'B_i],||^2 E[||B'B_i||^2] \\
&= ||(B'E[B_iE[(\varepsilon_i + r_i)(\varepsilon_i + r_i)'|Z_iB_i']B)^{-1}|| E[||B'B_i||^2] \\
&\leq \frac{1}{\lambda_{\text{min}}} ||(B'E[B_iB_i']B)^{-1}|| \sup_B E[||b_B(z)'b_B(Z_i)||^2] \\
&= \frac{1}{\lambda_{\text{min}}} ||(B')^{-1}|| \sup_B ||b_B(z)'b_B(z)|| \\
&\leq 1
\end{align*}
\]
as \(||b_B(z)|| = 1\) and \(E[b_B(Z_i)b_B(Z_i)'] = I_{k_B}\) for \(B = L, U.\)

\((an.ii)\) **Norm of weights** \(w_{i,n}(z)\)
\[
\begin{align*}
||w_{i,n}|| &= \frac{1}{\sqrt{n}} ||(B'E[B_i(\varepsilon_i + r_i)(\varepsilon_i + r_i)'B')^{-1/2}B'B_i|| \\
&\leq \frac{1}{\sqrt{n}} ||(B'E[B_iE[(\varepsilon_i + r_i)(\varepsilon_i + r_i)'|Z_iB_i']B)^{-1/2}|| ||B_i|| ||B_i|| \\
&\leq \frac{1}{\sqrt{n}\lambda_{\text{min}}} ||(B'E[B_iB_i']B)^{-1/2}|| ||B|| \sup_{z \in Z} ||B(z)|| \\
&\leq \sup_B \sup_{z \in Z} \frac{||b_B(z)||}{\sqrt{n}} \\
&\leq \frac{\xi_{k,B}}{\sqrt{n}}
\end{align*}
\]
which is \(o(1)\) by A.5.

**B.4.3 Multivariate Lindeberg Condition**

In the following denote \(\bar{\xi} = \sup_B \xi_{k,B}\) and \(\bar{c} = \sup_B c_{k,B}(1 + \xi_{k,B})\) and note that by A.5 we have that for any constant \(C_1, C_2 > 0\)
\[
\frac{C_1\sqrt{n}}{\xi} - C_2\bar{c} \rightarrow \infty.
\]
Now consider the weighted sum of the asymptotically linear representation. First note that by definition of the variance we have that
\[ V\left[ \sum_{i=1}^{n} w_{i,n}(z)(\varepsilon_i + r_i) \right] = I_2. \]

Now we verify the Lindeberg condition. Note that for any \( \delta > 0 \),
\[ \sum_{i=1}^{n} E[||w_{i,n}(z)(\varepsilon_i + r_i)||^2 1(||w_{i,n}(\varepsilon_i + r_i)|| > \delta)] \]
\[ = nE[||w_{i,n}(z)||^2 ||\varepsilon_i + r_i||^2 1(||\varepsilon_i + r_i|| > \delta/||w_{i,n}(z)||)] \]
\[ \leq 2nE[||w_{i,n}(z)||^2 ||\varepsilon_i||^2 1(||\varepsilon_i|| + ||r_i|| > \delta/||w_{i,n}(z)||)] \]
\[ + 2nE[||w_{i,n}(z)||^2 ||r_i||^2 1(||r_i|| > \delta/||w_{i,n}(z)||)] \]
\[ = (E.1) + (E.2) \]

Now note that by the law of iterated expectations for some \( C > 0 \)
\[ (E.1) \leq 2nE[||w_{i,n}(z)||^2 E[||\varepsilon_i||^2 1(||\varepsilon_i|| + ||r_i|| > \delta/||w_{i,n}(z)||)|Z_i = z]] \]
\[ \leq 2nE[||w_{i,n}(z)||^2 \sup_{z \in Z} E[||\varepsilon_i||^2 1(||\varepsilon_i|| + ||r_i|| > \delta/||w_{i,n}(z)||)|Z_i = z]] \]
\[ \leq 2nE[||w_{i,n}(z)||^2 \sup_{z \in Z} E[||\varepsilon_i||^2 1(||\varepsilon_i|| + 2\xi > C\delta/\bar{c})|Z_i = z]] \]
\[ = o(1) \]

by (an.i) and A.5 implying \( C\delta\sqrt{n}/\bar{c}^2 - 2\bar{c} \to \infty \) together with the fact that the higher moment in A.2 implies uniform integrability of \( ||\varepsilon_i||^2 \) as the moment function error tails are driven only by the (conditional) distribution of \( Y_i \).

\[ (E.2) \leq E[||w_{i,n}(z)||^2 2\sup_{B \in \mathcal{Z}} ||r_B(z)||^2 E[1(||\varepsilon_i|| + ||r_i|| > \delta/||w_{i,n}(z)||)|Z_i]] \]
\[ \leq nE[||w_{i,n}(z)||^2 2\sup_{z \in Z} P(||\varepsilon_i|| > C\delta\sqrt{n}/\bar{c} - 2\bar{c})Z_i = z) \]
\[ \leq \bar{c}^2 2\sup_{B \in \mathcal{Z}} \sup_{z \in Z} E[\varepsilon_i^2 B|Z_i = z] \]
\[ \leq C\delta\sqrt{n}/\bar{c} - 2\bar{c}^2 \]
\[ = o(1) \]

where the second to last step follows from Chebyshev’s inequality and the divergence of the denominator in the last step from A.5. The numerator is again bounded due to A.2. Thus, overall we have that
\[ \lim_{n \to \infty} \sum_{i=1}^{n} E[||w_{i,n}(z)(\varepsilon_i + r_i)||^2 1(||w_{i,n}(\varepsilon_i + r_i)|| > \delta] = 0. \]
which implies asymptotic normality around the best linear predictor. For the case with small approximation error, note that
\[
\begin{align*}
\sqrt{n} \Omega(z)^{-1/2} & \left( b_L(z)'(\hat{\beta}_L - \beta_L, 0) \right) = \sqrt{n} \Omega(z)^{-1/2} \left( b_L(z)'(\hat{\beta}_L - \theta_L(z)) \right) + \sqrt{n} \Omega(z)^{-1/2} \left( r_{\theta_L}(z) \right) \\
& = \sqrt{n} \Omega(z)^{-1/2} \left( b_L(z)'(\hat{\beta}_L - \theta_L(z)) \right) + o(1)
\end{align*}
\]
where the last step follows from
\[
||\sqrt{n} \Omega(z)^{-1/2} (r_L(z) + r_u(z))'|| \leq \sqrt{n}||\Omega(z)^{-1/2}|| \sup_{B, z \in Z} |r_B(z)| \lesssim \sqrt{n} \sup_B k_B^{-1/2} l_{k,B} \]
and the additional assumption in Theorem 4.1, Part 2.

**B.4.4 Matrix Estimation**

Now we consider difference between \( \Omega(z) \) and \( \hat{\Omega}_n(z) \). Let \( a(z) := b(z)/||b(z)|| \) and denote \( \xi_{k,B} := \sup_{z, z' \in Z, z \neq z'} ||a(z) - a(z')||/||z - z'|| \). Assume the following conditions hold for \( B = L, U \):

**Assumption B.1** There exists an \( m_B > 2 \) such that (i) \( \sup_{x \in X} E[|Y|^{m_B}|X = x] \lesssim 1 \), (ii) \((\xi_{k,B}^L)^{2m/(m-2)} \log k_B/n \lesssim 1 \), and (iii) \( \xi_{k,B}^L \lesssim \log k \). Moreover assume that uniformly over \( T_n \)
\[
\begin{align*}
\kappa_1^n := \sup_{\eta \in T_n} E[\max_{1 \leq i \leq n} |\psi_B(W_i, \eta) - \psi_B(W_i, \eta_0)|] = o(n^{-\frac{1}{m_B}}) \\
\kappa_n := \sup_{\eta \in T_n} E[\max_{1 \leq i \leq n} (\psi_B(W_i, \eta) - \psi_B(W_i, \eta_0))^2]^{1/2} = o(1). 
\end{align*}
\]
These assumptions imply Assumptions 3.6 - 3.8 in SC and the additional conditions (i) and (ii) in their Theorem 3.3. Thus, pointwise consistency of \( \hat{\Omega}_n(z) \) follows directly for any \( z \in Z \). Note that in their Assumption 3.8 they first assume the weaker \( \kappa_1^n = o(1) \) for the asymptotic normality with true covariance. For estimating the latter consistently, however, they also require that \( n^{1/m_B} \kappa_1^n = o(1) \), see Assumption (ii) in their Theorem 3.3. Thus, there is no qualitative difference to the assumptions imposed here, see also Heiler and Knaus (2021) for a similar argument in a modified setup.
B.4.5 Proof of Theorem 4.2

For each \( z \), the asymptotic distribution in Theorem 4.1 matches the assumptions behind Stoye (2020), Theorem 1 (his Assumption 1) with adapted parameter spaces as in the Theorem 4.2. As all statements are pointwise in \( z \), they follow directly.

B.5 Proof of Theorem 4.3 and Theorem 4.4

First we introduce some auxiliary definitions. Then we provide a joint uniform linearization result for the series process for the bounds and its remainder. This is then used to derive a strong approximation result for the whole series process under correct specification. We then prove an equivalent result for the weighted bootstrap up to some logarithmic terms. We then show the uniform asymptotic size control of the suggested confidence bands for the treatment effect under the correct specification.

B.5.1 Preliminaries

Define the following quantities:

\[
\hat{Q} := \begin{pmatrix} \hat{Q}_L & 0 \\ 0 & \hat{Q}_U \end{pmatrix} := \begin{pmatrix} \frac{1}{n} \sum_{i=1}^{n} b_L(Z_i) b_L(Z_i)' & 0 \\ 0 & \frac{1}{n} \sum_{i=1}^{n} b_U(Z_i) b_U(Z_i)' \end{pmatrix} = \frac{1}{n} \sum_{i=1}^{n} B_i B_i'
\]

and

\[
\beta_0 = \begin{pmatrix} \beta_{L,0} \\ \beta_{U,0} \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} \hat{\beta}_L \\ \hat{\beta}_U \end{pmatrix}, \quad \Sigma(z) = E[B_i \varepsilon_i \varepsilon_i' B_i']
\]
B.5.2 Uniform Linearization and Remainder

The uniform linearization follows exactly the same steps as in BCCK, Proof of Lemma 4.2. Due to the two-dimensional problem, however, we have to use the following bounds:

\[
\max_{1 \leq i \leq n} |\varepsilon_i| \lesssim P \mathbb{E} \left[ \max_{1 \leq i \leq n} |\varepsilon_{i,L}| + |\varepsilon_{i,U}| \right] \lesssim n^{-1/m_L} + n^{-1/m_U}
\]

\[
||\hat{Q} - I_{[k_L + k_U]}|| \lesssim P \sup_B ||\hat{Q} - I_{k_B}|| \lesssim P \sup_B \sqrt{\frac{k_B^2 \log k_B}{n}}
\]

\[
||\hat{Q}|| \lesssim P \sup_B ||\hat{Q}_B|| \lesssim P 1
\]

\[
||B(z) - B(\tilde{z})|| \lesssim P \sup_B ||b_B(z) - b_B(\tilde{z})|| \leq \sup_B \xi_{k,B}^L ||z - \tilde{z}||
\]

\[
||G_n[B_i r_i]|| \lesssim P \sup_B ||G_n[b_B(Z_i) r_{\theta_B}(Z_i)]|| \lesssim P \sup_B l_{k,B} c_k B \sqrt{k_B}
\]

by the block-diagonality and definition of the underlying components, Assumption A.V and the proof of the previous theorems. We decompose

\[
\sqrt{n}B(z)'(\hat{\beta} - \beta_0) = B(z)'G_n[B_i \varepsilon_i] + B(z)'G_n[B_i r_i] + B(z)'[\hat{Q} - I]G_n[B_i(\varepsilon_i + r_i)]
\]

Using the modified bounds above within the Proof of Lemma 4.2 in BCCK, Step 1 and Step 2 yields:

\[
\sup_{z \in \mathbb{Z}} B(z)'[\hat{Q}^{-1} - I]G_n[B_i(\varepsilon_i + r_i)] \lesssim P \sup_B \sqrt{\frac{\xi_{k,B}^2 \log k}{n}} \left( n^{1/m_B} \log^{1/2} k_B \sqrt{k_B} l_{k,B} c_k B \right)
\]

while Step 3 implies

\[
\sup_{z \in \mathbb{Z}} ||B(z)'G_n[B_i r_i]|| \lesssim P \sup_B l_{k,B} c_k B \log^{1/2} k_B
\]

Plugging this back into the decomposition we obtain a uniform linearization

\[
\sqrt{n}B(z)'(\hat{\beta} - \beta_0) = B(z)'G_n[B_i(\varepsilon_i + r_i)] + R_{1n}(z) = B(z)'G_n[B_i \varepsilon_i] + R_{2n}(z) + R_{1n}(z)
\]

with remainder

\[
\sup_{z \in \mathbb{Z}} R_{1n}(z) \lesssim P \sup_B \sqrt{\frac{\xi_{k,B}^2 \log k}{n}} \left( n^{1/m_B} \log^{1/2} k_B \sqrt{k_B} l_{k,B} c_k B \right) = o(a_n^{-1})
\]

\[
\sup_{z \in \mathbb{Z}} R_{2n}(z) \lesssim P \sup_B l_{k,B} c_k B \log^{1/2} k_B = o(a_n^{-1})
\]

under the conditions of Theorem 4.3.
B.5.3 Strong Approximation

Now we apply Yurinskii’s Coupling to the first-order process from above under with sup_{z \in Z} R_{2n}(z) = o_p(a_n^{-1}). Consider the following copy of the first-order term above:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i \text{ with } \zeta_i = \Sigma(z)^{-1/2} B_i \varepsilon_i$$

Note that E[ζ_i] = 0 and V[ζ_i] = I_{k_L+k_U}, by definition. To apply the coupling, we need have to bound the (sum of the) third moments \( \sum_{i=1}^{n} E[|\zeta_i|^3] \). Note that

$$E[|\zeta_i|^3] \leq \frac{1}{\lambda_{\min}(\Sigma(z))^3/2} E[|B_i \varepsilon_i|^3]$$

\[\lesssim E[\sup_B |b_B(Z_i)|^3 \sup_{z \in Z} E[|\varepsilon_i|^3|Z_i = z]]\]

\[\lesssim \sup_B \varepsilon_k B E[|b_B(Z_i)|^2]\]

\[\lesssim \sup_B \varepsilon_k B k_B\]

where the bounded eigenvalues follows from Assumption A.1 and Assumption 3.3 that ensures invertibility. The third inequality follows from the conditional moment bound in Theorem 4.3. Thus, the Yurinskii’s coupling says implies that, for each \( \delta > 0 \), there exist a random standard normal vector of length \( k_L + k_B, N_{[k_L+k_u]} \), such that

$$P\left( \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta_i - N_{[k_L+k_u]} \right| \geq 3 \delta a_n^{-1} \right)$$

\[\lesssim \frac{n[k_L+k_U] \sup_B k_B \varepsilon_k B a_n^3}{(\delta \sqrt{n})^3} \left( 1 + \frac{\log(\delta^3/\varepsilon_k B k_B[k_L+k_U])}{[k_L+k_U]} \right)\]

\[\lesssim n^{-1/2} a_n^3 [k_L+k_U] \sup_B k_B \varepsilon_k B \left( 1 + \frac{\log n}{k_L+k_U} \right)\]

\[\lesssim n^{-1/2} a_n^3 \sup_B k_B^2 \varepsilon_k B \left( 1 + \frac{\log n}{k_B} \right)\]

where the second inequality follows from \( \delta \) being fixed and \( \sup_B \varepsilon_k B k_B^2/n = o(1) \). Now note that, under correct specification \( \sup_B n^{1/2} k_B l_{k,B} c_{k,B} \log^2 n = o(a_n^{-1}) \), we have that

$$\sqrt{n} \Omega(z)^{-1/2} B(z)^{1/2} (\hat{\theta}(z) - \beta_0) - \sqrt{n} \Omega(z)^{-1/2} (\hat{\theta}(z) - \theta(z)) = o_p(a_n^{-1})$$

where \( \Omega(z) = B(z)^{1/2} \Sigma(z) B(z) \). Thus, overall we have that, on \( \ell_{\infty}(Z) \),

$$\sqrt{n} \Omega(z)^{-1/2} (\hat{\theta}(z) - \theta(z)) = d \Omega(z)^{1/2} N_{[k_L+k_u]} + o_p(a_n^{-1}).$$
B.5.4 Bootstrap: Strong Approximation

Define the bootstrap estimator $\hat{\beta}_b^b = (\hat{\beta}_b^b L, \hat{\beta}_b^b U)'$ analogously to first least squares problem (3.10) but with random independent exponential weights for both $B = L, U$:

$$\hat{\beta}_b^b = \arg\min_{\beta} E_n [h_i(\psi_B(W_i, \hat{\eta}) - b_B(Z_i)' \beta)]$$

Note that the $h_1, \ldots, h_n$ have to be the same for both bounds. Else, it would affect the dependency between estimators. Note that $E[(h_i - 1)] = 0$, $E[h_i^{m_B/2}] \lesssim 1$, $\max_{1 \leq i \leq n} h_i \lesssim_p \log n$, and $E[\max_{1 \leq i \leq n} |\sqrt{h_i} \varepsilon_i, B|] \lesssim_p n^{1/m_B} \log n$ by the properties of the standard exponential. The least squares problem above can be considered as the regression (3.10) with both sides multiplied by $\sqrt{h_i}$. Thus, all results from the previous sections apply with $\xi_{k,B}^b = \xi_{k,B} \log^{1/2} n$ and $l_{k,B}^b c_{k,B}^b = l_{k,B} c_{k,B} \log^{1/2} n$, see also BCCK Proof of Theorem 4.5. Adding and subtracting then yields the following centered decomposition:

$$\sqrt{n} B(z)'(\hat{\beta}^b - \beta) = \sqrt{n} B(z)'(\hat{\beta}^b - \beta) - \sqrt{n} B(z)'(\hat{\beta}^b - \beta)$$

$$= B(z)G_n[B_i \varepsilon_i (h_i - 1)] + R_n^b(z) + R_n^{b2}(z)$$

with

$$\sup_{z \in Z} R_n^b(z) \lesssim_p B \sup_{z \in Z} \frac{\varepsilon^2}{n} (n^{1/m_B} \log^{1/2} n + \sqrt{k_B l_{k,B} c_{k,B}}) = o(n^{-1})$$

$$\sup_{z \in Z} R_n^{b2}(z) \lesssim_p B \sup_{z \in Z} l_{k,B} c_{k,B} \log n = o(n^{-1})$$

Then, due to existence of the higher moments of the (weighted) process, Yurinkii’s coupling can be applied analogously. Thus, on $\ell^\infty(Z)$,

$$\sqrt{n} \Omega(z)^{-1/2}(\hat{\theta}(z)^b - \theta(z)) =_d \Omega(z)^{1/2} N_{[k_L + k_u]} + o_p(n^{-1})$$

which gives the desired result in Theorem 4.3 by adding and subtracting the estimated parameter vector $\hat{\theta}(z)$ and using the triangle inequality.
B.5.5 Uniform Confidence Band Coverage

Under correct specification, uniformly over $\mathcal{Z}$, $\theta_L(z) \leq \theta(z) \leq \theta_U(z)$ and thus

$$P\left( \theta(z) \in CB_{\theta(z), 1-\alpha} \text{ for all } z \in \mathcal{Z} \right)$$

$$\geq P\left( \sup_{z \in \mathcal{Z}} \theta(z) \leq c_{n,1-\alpha/2}(\hat{t}_U^b)\hat{\sigma}_U(z) \right) - P\left( \inf_{z \in \mathcal{Z}} \theta(z) \leq c_{n,\alpha/2}(\hat{t}_L^b)\hat{\sigma}_L(z) \right)$$

$$\geq P\left( \sup_{z \in \mathcal{Z}} \theta_U(z) \leq c_{n,1-\alpha/2}(\hat{t}_U^b)\sigma_U(z) \right) - P\left( \inf_{z \in \mathcal{Z}} \theta_L(z) \leq c_{n,\alpha/2}(\hat{t}_L^b)\sigma_L(z) \right) + o(1)$$

$$\geq 1 - \alpha/2 - \alpha/2 + o(1)$$

where the second inequality is due to the correct specification and consistency of the asymptotic variance. The third inequality follows from the strong approximation result for the centered bootstrap process.