Bicomplexes and Conservation Laws in Non-Abelian Toda Models

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ABSTRACT

A bicomplex structure is associated to the Leznov-Saveliev equation of integrable models. The linear problem associated to the zero curvature condition is derived in terms of the bicomplex linear equation. The explicit example of a Non-Abelian Conformal Affine Toda model is discussed in detail and its conservation laws are derived from the zero curvature representation of its equation of motion.
1 Introduction

Two dimensional Toda field theories are examples of relativistic integrable non-linear systems underlined by an Lie algebraic structure. Finite dimensional Lie algebras are associated to the Conformal Toda Models (see [1] for a review), whose basic representative is the Liouville model. The Affine Toda models are associated to the loop algebra (centerless Kac-Moody) and are characterized by the broken conformal symmetry. Basic examples within this class, we find the sine-Gordon, Lund-Regge (complex sine-Gordon), general abelian affine Toda and homogeneous sine-Gordon [2]. Dyonic integrable models such as the singular non abelian Toda models are also within this class [3]. Conformal Affine Toda models [4] are related to infinite dimensional Affine Lie Algebras (full Kac-Moody algebras). Such models are classified according to a grading operator decomposing the Lie algebra into graded subspaces. The graded structure is an important ingredient to obtain such models when employing the hamiltonian reduction procedure to the WZNW [1] and 2-loop WZNW models [4]. Alternatively, the field equations of those models can be obtained from the Leznov-Saveliev equation [5]. An important fact about this equation is that it can be written as a zero curvature condition. As a consequence, under specific boundary conditions, there are infinite conserved charges. Also, if the fundamental Poisson bracket relation holds [6], the involution condition among the conserved charges is verified. An important ingredient in this construction is the classical r-matrix satisfying the classical Yang-Baxter equation.

In recent papers [7], a structure called bicomplex was used to derive some integrable field equations (e.g., sine-Gordon, non-linear Schroedinger). It was argued that the bicomplex linear equation could, in some cases, lead to chains of conserved charges.

In this paper we generalize the bicomplex structure to derive the Leznov-Saveliev equation corresponding to an infinite dimensional affine Lie algebra, which includes the non abelian Toda equations. The linear problem associated to the zero curvature condition is also derived in terms of the bicomplex linear equation. Explicit construction, following the arguments of [8], for the conserved charges of a specific $A_2^{(1)}$ non-abelian Toda model is obtained.

2 Bicomplexes and Leznov-Saveliev equation

Let $V = \oplus_{r \geq 0} V^r$ be an $N_0$-graded linear space over $C$ and $d, \delta : M^r \rightarrow M^{r+1}$ linear maps. If $d^2 = \delta^2 = \delta d + d\delta = 0$, then this structure is called a bicomplex [9].

It is important to emphasize that nothing is said about Leibnitz rules. Let $\xi^1, \xi^2$ be a basis for $V^1$ such that $\xi^1 \xi^1 = \xi^2 \xi^2 = \xi^1 \xi^2 + \xi^2 \xi^1 = 0$. In this case $V^2$ is one-dimensional and $V = V^0 \oplus V^1 \oplus V^2$. It is convenient to introduce light-cone variables in the two-dimensional space-time with coordinates $(t, x) : z = (t + x)/2$, $\bar{z} = (t - x)/2$, $\partial = \partial/\partial z = \partial_t + \partial_x$, $\bar{\partial} = \partial/\partial \bar{z} = \partial_t - \partial_x$. Consider a infinite dimensional affine Lie algebra $\hat{G}$ and constant generators $(\varepsilon^+, \varepsilon^-) \in \hat{G}$ such that

$$[\varepsilon^+, \varepsilon^-] = \mu_1 \mu_2 \hat{C},$$

where $\hat{C}$ is the central charge generator and $(\mu_1, \mu_2) \in C$. The meaning of this choice will be explained in the end of this section.
Let $v^1 = (v_1^1\xi^1 + v_2^1\xi^2) \in V^1$ arbitrary and define:
\[
\delta v^1 \equiv (\delta v_1^1)\xi^1 + (\delta v_2^1)\xi^2; \quad dv^1 \equiv (dv_1^1)\xi^1 + (dv_2^1)\xi^2.
\]  
(2.2)

Similarly, for $v^2 = v_2^2\xi^1\xi^2 \in V^2$ arbitrary, define:
\[
\delta v^2 \equiv (\delta v_{1,2}^2)\xi^1\xi^2 = 0; \quad dv^2 \equiv (dv_{1,2}^2)\xi^1\xi^2 = 0.
\]  
(2.3)

Let $v^0 \in V^0$ arbitrary and define the maps $\delta, d$:
\[
\delta v^0 \equiv \bar{\delta}v^0\xi^1 + \varepsilon v^0\xi^2; \quad dv^0 \equiv -\varepsilon^+v^0\xi^1 + \partial v^0\xi^2.
\]  
(2.4)

An explicit computation reveals that for $v^0 \in V^0$ arbitrary:
\[
\delta^2 v^0 = \delta(\bar{\delta}v^0)\xi^1 + \delta(-\varepsilon v^0)\xi^2 = \bar{\delta}(\delta v^0)\xi^1 - \delta v^0\xi^2\xi^1 + \bar{\delta}(v^0)\xi^2 + (\varepsilon^-)^2v^0(\xi^2)^2 = \varepsilon - \bar{\delta}v^0(\xi^2\xi^1 + \xi^1\xi^2) = 0.
\]
That is,
\[
\delta^2 = 0; \quad d^2 = 0; \quad (\delta d + d\delta)v^0 = -\mu_1\mu_2\hat{C}v^0\xi^1\xi^2,
\]  
(2.5)

where the last two equations are derived in a similar way. The last equality can be rewritten as
\[
P^2 + (\delta d + d\delta) = 0,
\]  
(2.6)

where the map $P : V^r \to V^{r+1}$ is defined by
\[
Pv^0 \equiv \varepsilon^+v^0\xi^1 + \varepsilon^-v^0\xi^2.
\]

The action of $P$ in $V^1$ and $V^2$ is defined in the same way as it was done for $d, \delta$. Notice that the maps $(\delta, d)$ do not define a bicomplex, unless the central charge is taken equal to zero, which imply to be working with the loop algebra. Alternatively, let $g$ be an exponential of the generators belonging to $\hat{G}$. Define a dressing \cite{4} for $d$, introducing $D : V^r \to V^{r+1}$ such that, for arbitrary $v^0 \in V^0$
\[
Dv^0 \equiv g^{-1}d(gv^0) = -g^{-1}\varepsilon^+gv^0\xi^1 + (\bar{\partial} + g^{-1}\partial g)v^0\xi^2.
\]  
(2.7)

Extending the action of $D$ in $V^1$ and $V^2$ in the same way as before,
\[
D^2v^0 = g^{-1}d(gDv^0) = g^{-1}d(gg^{-1}d(gv^0)) = 0 \quad \rightarrow \quad D^2 = 0,
\]  
(2.8)

using (2.3) and the fact that $v^0$ is arbitrary. Now,
\[
(\delta D + D\delta)v^0 = \{\bar{\partial}(g^{-1}\partial g) - [g^{-1}\varepsilon^+g, \varepsilon^-]\}v^0\xi^1\xi^2.
\]  
(2.9)

In order to get the Leznov-Saveliev equation there are two different options here. The first is to take
\[
g = B \exp(-\mu_1\mu_2\hat{z}\hat{\bar{z}}\hat{C}),
\]  
(2.10)
where $B$ is a group element and impose:

$$\delta D + D\delta = \delta d + d\delta = -P^2. \quad (2.11)$$

As a consequence,

$$\bar{\partial}(B^{-1}\partial B) = [B^{-1}\varepsilon^+ B, \varepsilon^-]; \quad \partial(\bar{\partial}BB^{-1}) = [\varepsilon^+, B\varepsilon^- B^{-1}]. \quad (2.12)$$

Equations (2.12) correspond to the Leznov-Saveliev equation \[5\] in its two different versions. Notice, however, that the maps $(\delta, D)$ defined in this way do not define a bicomplex.

Consider now the second option. Take

$$g = B \rightarrow Dv^0 \equiv B^{-1}d(Bv^0) = -B^{-1}\varepsilon^+ Bv^0\varepsilon^1 + (\partial + B^{-1}\partial B)v^0\xi^2, \quad (2.13)$$

for $v^0 \in V^0$ arbitrary. Extend the action in $V^1$ and $V^2$ as before and impose $\delta D + D\delta = 0$. This leads to the Leznov-Saveliev equation again and, in this case, defines a bicomplex:

$$D^2 = \delta^2 = \delta D + D\delta = 0. \quad (2.14)$$

An explanation about (2.1) is important. If $(\varepsilon^+, \varepsilon^-)$ are chosen in such a way that (2.1) holds, then $B_0 = \exp(\mu_1\mu_2 z\bar{z}\hat{C})$ is a particular solution of the Leznov-Saveliev equation (2.12). In fact, this is a vacuum solution required by the dressing method as an input for non-trivial one soliton solutions \[3\].

### 3 The Bicomplex Linear Equation

A linear problem is associated to a given bicomplex in the following way \[7\]: Suppose there is $T(0) \in V^0$ such that $DJ(0) = 0$, where $J(0) \equiv \delta T(0)$. Using (2.14), $\delta J(0) = 0$. Defining $J^{(1)} \equiv DT(0)$ and using (2.14), $\delta J^{(1)} = 0$, $DJ^{(1)} = 0$. Suppose that $J^{(1)}$ can also be written as $J^{(1)} = \delta T^{(1)}$, $T^{(1)} \in V^0$. Then, defining $J^{(2)} \equiv DT^{(1)}$, $\delta J^{(2)} = -D\delta T^{(1)} = -DJ^{(1)} = 0$ and $DJ^{(2)} = 0$. Continuing indefinitely such steps and defining a formal expansion $T \equiv \sum_{m=0}^{\infty} \rho^m T^{(m)}; \rho \in C$, the bicomplex linear equation is obtained:

$$\delta(T - T^{(0)}) = \rho DT \quad \rightarrow \quad \delta T = \rho DT, \quad (3.1)$$

if $\delta T^{(0)} = 0$. Using (2.3) and (2.13) in (3.1), result:

$$\bar{\partial}T = -\bar{A}T; \quad \partial T = -AT; \quad A = -\rho^{-1}\varepsilon^- + B^{-1}\partial B; \quad \bar{A} = \rho B^{-1}\varepsilon^+ B. \quad (3.2)$$

Defining

$$\varepsilon^\pm = \rho^{\pm 1}\varepsilon^\pm; \quad [\varepsilon^+, \varepsilon^-] = [\varepsilon^+, \varepsilon^-] = \mu_1\mu_2\hat{C}, \quad (3.3)$$

the commutation relation gets the same structure. In fact, even the Leznov-Saveliev equation is invariant under (3.3). Now,

$$A = -\varepsilon^- + B^{-1}\partial B; \quad \bar{A} = B^{-1}\varepsilon^+ B; \quad \partial\bar{A} - \partial A + [A, \bar{A}] = 0, \quad (3.4)$$

3
is a standard representation of the connections associated to the zero curvature equation from which the Leznov-Saveliev equation is derived. The solution of (3.2) is [6]:

\[ T(t, y) = T_0 \mathcal{P}[\exp(\int_{0}^{t} A_\mu dx^\mu)], \]  

(3.5)

where \( \mathcal{P} \) is the path ordered operator and \( T_0 \) is a constant. We should point out that the spectral parameter \( \rho \) naturally arises under such framework in (3.2). Given an affine Lie algebra \( \hat{G} \) and a grading operator \( \hat{Q} \in \hat{G} \) follows a decomposition [10]: \( \hat{G} = \oplus \hat{G}_i; \quad [\hat{Q}, \hat{G}_i] = i\hat{G}_i; \quad [\hat{G}_i, \hat{G}_j] \in \hat{G}_{i+j}. \) Here, \( i \in \mathbb{Z} \). The hamiltonian reduction procedure applied to the 2-loops WZNW model [3] leads to the Leznov-Saveliev equation, where the group element \( B \) is associated to zero-grade generators of \( \hat{G} \) and \( \pm \) are generators of grade \( \pm j \), \( j \in \mathbb{Z} \). That is, these integrable models are classified in terms of the grading operators [10]. In particular, the class of singular \( A_2^{(1)} \) non abelian Toda models have been constructed in [3] by choosing the zero grade subgroup \( \hat{G}_0 = SL(2) \otimes U(1)^{n-1} \). One can realize (3.3) as:

\[ \varepsilon^\pm = \exp(\frac{\hat{Q}\ln \rho}{j}) \varepsilon^\pm \exp(-\frac{\hat{Q}\ln \rho}{j}). \]  

(3.6)

### 4 Conformal Affine Non-Abelian Toda Model

In this section we consider the example of a \( A_2^{(1)} \) conformal affine non-abelian Toda model, whose singular version was constructed in [3]. The zero grade subgroup \( \hat{G}_0 = SL(2) \otimes U(1) \subset SL(3) \) is parametrized by

\[ B = \exp(\beta \overline{\lambda} E_{-\alpha_1}^{(0)}) \exp(\beta \varphi_1 H_{\lambda_1}^{(0)} + \beta \varphi_2 h_2^{(0)} + \beta \nu \hat{C} + \beta \eta \hat{D}) \exp(\beta \overline{\psi} E_{\alpha_1}^{(0)}), \]  

(4.1)

\[ \hat{Q} = 2\hat{D} + H_{\lambda_2}^{(0)}; \quad \varepsilon^+ = \mu_1(E^{(0)}_{\alpha_2} + E_{-\alpha_2}^{(1)}); \quad \varepsilon^- = \mu_2(E_{-\alpha_2} + E_{\alpha_2}^{(1)}), \]  

(4.2)

where \( \hat{D} \) is the homogeneous grading operator, \( h_i^{(0)} = 2\alpha_i, H^{(0)}/\alpha_i^2 \) are Chevalley generators, \( H_i^{(0)} \) define the \( A_2 \) Cartan subalgebra in the Weyl-Cartan basis, \( H_\lambda^{(0)} = 2\lambda_i H^{(0)}/\alpha_i^2 \), \( \lambda_i \) are the fundamental weights of \( A_2 \) satisfying \( 2\alpha_i, \lambda_i/\alpha_i^2 = \delta_{i,j}, \) that is, \( \lambda_i = \sum_{j=1}^{3}(K^{-1})_{i,j} \alpha_j \). \( K \) is the Cartan matrix of \( A_2 \), \( (i,j) = (1,2), \beta^2 = -\beta_0^2 = -(2\pi)/k, \) \( k \) is the WZNW coupling constant. Also, the normalization \( \alpha^2 = 2 \) for all the roots is adopted (See [9] for a description on affine Lie algebras).

The constant generators \( \varepsilon^\pm \) have grade \( \pm 1 \) with respect to the generalized grading operator \( \hat{Q} \) and \( B \in \hat{G}_0 \). This grading is an intermediate between the homogeneous grading \( \hat{Q} = \hat{D} \) and the principal grading \( \hat{Q} = 3\hat{D} + H_{\lambda_1}^{(0)} + H_{\lambda_2}^{(0)} \). The Leznov-Saveliev equation leads to the field equations corresponding to the lagrangean density

\[ \mathcal{L} = (1/3)\partial \varphi_1 \overline{\partial} \varphi_1 + \partial \varphi_2 \overline{\partial} \varphi_2 + (1/2)(\partial \nu \overline{\partial} \eta + \partial \eta \overline{\partial} \nu) + \exp(\beta(\varphi_1 - \varphi_2))\partial \overline{\chi} \overline{\partial} \psi + \]  

\[ - (\mu_1 \mu_2 / \beta^2)(\exp(-2\beta \varphi_2) + \exp(\beta(2\varphi_2 - \eta)) + \beta^2 \overline{\psi} \overline{\chi} \exp(\beta(\varphi_1 + \varphi_2 - \eta))). \]  

(4.3)

In order to construct the singular non abelian Toda model, one observes that \( \hat{G}_0^0 \equiv H_{\lambda_1}^{(0)} \in \hat{G}_0 \) is such that \( [\hat{G}_0^0, \varepsilon_\pm] = 0 \), implying, from the Leznov-Saveliev equation, the
chiral conservation laws \( \partial T[\hat{G}_0^0 \partial BB^{-1}] = \bar{\partial} T[\hat{G}_0^0 B^{-1} \partial B] = 0 \). Those, in turn allow the following subsidiary constraints \( T[\hat{G}_0^0 \partial BB^{-1}] = T[\hat{G}_0^0 B^{-1} \partial B] = 0 \), responsible for the elimination of the non local field \( \varphi_1 \), \( \partial \varphi_1 = \frac{3}{2} \beta \psi \exp(\beta \varphi_2) \); \( \bar{\partial} \varphi_1 = \frac{3}{2} \beta \chi \psi \exp(\beta \varphi_2) \), where \( \chi = \chi \exp(\beta \varphi_1/2) ; \psi = \bar{\psi} \exp(\beta \varphi_2/2) ; \Delta = 1 + (3/4) \beta^2 \chi \psi \exp(-\beta \varphi_2) \). The classical r-matrix associated to this model is discussed in [11]. The Singular Non-Abelian Affine Toda model, that is, without the field \( \eta \) ( the field \( \nu \) is only an auxiliary field ) was already discussed in the literature. In [3] a complete spectrum of 1 and 2 soliton solutions was obtained using the dressing transformations and the vertex operator construction. Also, the T-dual version of this model was analysed and some results about semiclassical quantization were shown.

5 Conservation Laws

In order to derive the conservation laws for the model defined in [4,3], it is convenient to define a new basis for \( A_{2}^{(1)} \):

\[
\hat{Q} = 2 \hat{D} + H_{\lambda_{2}}^{(0)} ; \quad \hat{C} ; \quad A_{(2n)} = \sqrt{3} (H_{1}^{(n)} - (1/6) \delta_{n,0} \hat{C}) ; \quad A_{(2n+1)} = E_{\alpha_2}^{(n)} + E_{-\alpha_2}^{(n+1)} ;
\]

\[
F_{(2n)} = h_{2}^{(n)} - (1/2) \delta_{n,0} \hat{C} ; \quad F_{(2n+1)} = E_{\alpha_2}^{(n)} - E_{-\alpha_2}^{(n+1)} ;
\]

\[
F_{(2n)}^+ = E_{\alpha_1}^{(n)} ; \quad F_{(2n+1)}^+ = E_{\alpha_1+\alpha_2}^{(n)} ; \quad F_{(2n)}^- = E_{-\alpha_1}^{(n)} ; \quad F_{(2n+1)}^- = E_{-\alpha_1-\alpha_2}^{(n)} ;
\]

(5.1)

where \( n \in \mathbb{Z} \). The generators \( A_{(2n)}, A_{(2n+1)} \) define a infinite dimensional Heisenberg subalgebra:

\[
[A_{(2n)}, A_{(2n+1)}] = 0; \quad [A_{(2n)}, A_{(2m)}] = 2n \delta_{n+m,0} ; \quad [A_{(2n+1)}, A_{(2m+1)}] = (2n + 1) \delta_{n+m+1,0} .
\]

(5.2)

Also, it is verified that

\[
\begin{align*}
[F_{(2n+1)}, A_{(2m)}] &= [F_{(2n)}, A_{(2m)}] = 0; \\
[F_{(2n+p)}^+, A_{(2m)}] &= -\sqrt{3} F_{[2(n+m)+p]}^+ ; \quad [F_{(2n-p)}^-, A_{(2m)}] = \sqrt{3} F_{[2(n+m)-p]}^- ; \\
[F_{(2n+p)}^+, A_{(2m+1)}] &= F_{[2(n+m)+p]+1}^+ ; \quad [F_{(2n-p)}^-, A_{(2m+1)}] = -F_{[2(n+m+1)-p]+1}^- ; \\
[F_{(2n)}, A_{(2m+1)}] &= 2F_{[2(n+m)+1]} ; \quad [F_{(2n+1)}, A_{(2m+1)}] = 2F_{[2(n+m+1)]} ,
\end{align*}
\]

(5.3)

where \( p = (0,1) \). Defining \( \mathcal{A} = \{ A_{(2n+p)}, \hat{Q}, \hat{C} \} \) and \( \mathcal{F} = \{ F_{(2n+p)}^+, F_{(2n+p)}^- \} \), we see that linear combinations of generators \( \in \mathcal{F} \) are used to construct the vertex operators [10], used in the dressing method [3].

The conservation laws follow from the zero-curvature equation by gauge transforming \( A \) and \( \bar{A} \) into \( A^R_{ab} \) and \( \bar{A}^R_{ab} \) such that \( [A^R_{ab}, \bar{A}^R_{cd}] = 0 \) [3,12]. It is convenient to define the notation: \( \mathcal{F} = \mathcal{F}^+ \oplus \mathcal{F}^- \oplus \mathcal{F}^0 ; \quad \mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^- \oplus \mathcal{A}^0 \), where the subspaces \( \mathcal{F}^\pm, \mathcal{F}^0 \) have a (positive/negative/zero) grade w.r.t. \( \hat{Q} \) and similarly to \( \mathcal{A}^\pm, \mathcal{A}^0 \). Consider

\[
A = B \varepsilon^{-} B^{-1} ; \quad \bar{A} = -\varepsilon^{+} - \bar{\partial} BB^{-1} ; \quad g_R = \prod_{m=1}^{\infty} \exp(S_{-m}) \exp(\xi \varepsilon^-) ,
\]

(5.4)

where the connections result in the Leznov-Saveliev equation under the zero-curvature equation and \( S_{-m} \) is a linear combination of generators in \( \mathcal{F}^- \) and \( \xi(z, \bar{z}) \) is a function of \( z \) and
We consider the gauge transformation:

\[
A^R = g_R A g_R^{-1} - \partial g_R g_R^{-1} = \sum_{m=-\infty}^{-1} (A^R_{A,m} + A^R_{F,m});
\]

\[
\bar{A}^R = g_R \bar{A} g_R^{-1} - \partial g_R g_R^{-1} = \sum_{m=-\infty}^{1} \bar{A}^R_{m} \tag{5.5}
\]

where \(\bar{A}^R_{m} \in \hat{C}_m\), \(A^R_{A,m}\) has grade \(m\) and is a linear combination of generators in \(A\) and similarly for \(A^R_{F,m}\). Explicitly,

\[
\bar{A}^R_{1} = -\varepsilon^+; \quad \bar{A}^R_{0} = -\bar{\partial}BB^{-1} + \mu_1 \mu_2 \xi \bar{C} - [S_{-1}, \varepsilon^+];
\]

\[
\bar{A}^R_{-1} = -\bar{\partial}(S_{-1} + \xi \varepsilon^-) - [S_{-2}, \varepsilon^+] - (1/2)[S_{-1}, [S_{-1}, \varepsilon^+]] - [\xi \varepsilon^- + S_{-1}, \bar{\partial}BB^{-1}]; \quad \ldots \tag{5.6}
\]

\[
\bar{\partial}BB^{-1} = \beta[\bar{\partial}\varphi_1 - (3/2)\beta \bar{\chi} \bar{\partial} \bar{\psi} e^{\beta(\varphi_1 - \varphi_2)}]H_{\lambda_1}^{(0)} + \beta[\bar{\partial}\varphi_2 + (1/2)\beta \bar{\chi} \bar{\partial} \bar{\psi} e^{\beta(\varphi_1 - \varphi_2)}]h_2^{(0)} +
\]

\[
+ \beta[\bar{\partial}v \bar{\dot{C}} + \bar{\partial} \bar{v} \bar{\dot{E}}_{\alpha_1}] + \beta \bar{\partial}v \bar{\dot{E}}_{\alpha_1} + \beta \bar{\delta} \bar{\dot{C}} + (\bar{\partial} \bar{\delta} v \bar{\dot{E}}_{\alpha_1}) - \beta^2 \bar{\chi}^2 \bar{\partial} \bar{\psi} e^{\beta(\varphi_1 - \varphi_2)} E_{\alpha_1}^{(0)}
\]

\[
\equiv \bar{J}^{(1)}H_{\lambda_1}^{(0)} + \bar{J} h_2^{(0)} + \bar{J} \bar{\dot{C}} + \bar{J} \bar{\dot{E}}_{\alpha_1} + \bar{J} \bar{\dot{E}}_{\alpha_1}^{(0)} + \bar{J} \bar{\dot{E}}_{\alpha_1}^{(0)}. \tag{5.7}
\]

Now, choose \(\{S_{-m}\}\) and \(\xi\) such that:

\[
\bar{A}^R = -\varepsilon^+ - \bar{J} \bar{\dot{D}}_A - \bar{J}^{(1)}H_{\lambda_1}^{(0)} + \sum_{m=-\infty}^{-1} \bar{a}^R_{A,m} A_{m};
\]

\[
\bar{D} \equiv \bar{D}_A + \bar{D}_F = [(1/2)\bar{Q} - \frac{\sqrt{3}}{12}A_{(0)}] - (1/6)\bar{C} + \frac{-F_{(0)}}{4}. \tag{5.8}
\]

The idea behind this structure is to solve for \(S_{-m}\) such that all terms in \(F\) are eliminated. Consider the zero-curvature equation and \((A^R, \bar{A}^R)\) as described. Note that \(A^R_{A,m} = a^R_{A,m} A_{(m)}\). In terms of the new basis \((\bar{A}^R)\), we find

\[
\partial \bar{J} = 0; \quad \partial \bar{J}^{(1)} = 0; \quad a^R_{A,1} = 0; \quad \partial a^R_{A,-1} = 0;
\]

\[
\partial a^R_{F,-m} - \bar{a} a^R_{-m} - \frac{m}{2} \bar{J} a^R_{A,-m} = 0, m \geq 2; \quad A^R_{F,m} = 0, \forall m. \tag{5.9}
\]

where \(\bar{J}\) and \(\bar{J}^{(1)}\) are defined in \((5.7)\). Under a gauge transformation defined by the group element \(g^R\), follows:

\[
g^R = \exp[-\varepsilon^+ \int_{L} \bar{J}^{\frac{1}{2}} J^{(\bar{v})} \bar{d} \bar{w}] \exp[-\int_{L} (\bar{J}^{(1)}(\bar{v})) H_{\lambda_1}^{(0)} + \bar{J} \bar{v} \bar{\dot{D}}_A \bar{d} \bar{w}] ; \tag{5.10}
\]

\[
\partial A^R_{ab} - \partial A^R_{ab} = 0; \tag{5.11}
\]

where

\[
A^R_{ab} \equiv \sum_{m=-\infty}^{-1} A^R_{A,m} e^{-\frac{\bar{d} \bar{w}}{2}} \int_{L} \bar{J}^{(\bar{v})} \bar{d} \bar{w} A_{(m)};
\]

\[
\bar{A}^R_{ab} \equiv \sum_{m=-\infty}^{-1} \bar{A}^R_{A,m} + a \bar{C} \bar{C}
\]

\[
= \sum_{m=-\infty}^{-1} \bar{a} R_{A,m} e^{-\frac{\bar{d} \bar{w}}{2}} \int_{L} \bar{J}^{(\bar{v})} \bar{d} \bar{w} A_{(m)} - \mu_1 \bar{a} R_{A,-1} e^{\frac{\bar{d} \bar{w}}{2}} \int_{L} \bar{J}^{(\bar{v})} \bar{d} \bar{w} \int_{L} \bar{J}^{(\bar{v})} \bar{d} \bar{w} \bar{C} \bar{C} ;. \tag{5.12}
\]
where $\bar{L} \in R$. Taking $\eta(z, \bar{z}) = \eta_1(z) + \eta_2(\bar{z})$ as solution for the equation of motion for $\eta$, results $\int_{L} \mathcal{P}(\bar{v}) d\bar{v} = \eta_2(\bar{z}) - \eta_2(\bar{L})$. This implies that all the terms in the abelian connections are local, except the term in $\hat{C}$. Under periodic boundary conditions [8],[12], the zero curvature equation for the abelianized connections implies an infinite set of conserved charges:

$$\partial_t Q^R_m = 0, m \leq -1; Q^R_m = \int_{-s}^s A_{x,ab}^R(t,y) dy; \quad A_{x,ab}^R = \frac{1}{2} (A_{ab}^R - \bar{A}_{ab}^R),$$

(5.13)

where $s \in R$. In order to verify the involution condition, one starts from the Fundamental Poisson Bracket relation (see for instance [11]):

$$\{ A^S_x(y, t) \otimes A^S_x(z, t) \}_P = [r, A^S_x(y, t) \otimes I + I \otimes A^S_x(z, t)] \delta(y - z);$$

$$A^S_x = \frac{1}{2} S(A - \bar{A}) S^{-1} - \partial_x S S^{-1}; \quad S = e^{-\frac{1}{2}(\beta \phi_1 H^{(0)} + \beta \phi_2 h_2^{(0)} + \beta \nu \hat{C} + \beta \eta \hat{D})} e^{-\beta \chi E^{(0)}};$$

$$r = \frac{\beta^2}{4} (C^+ - \sigma C^+); \quad \sigma(a \otimes b) = b \otimes a, \forall (a, b) \in \hat{G};$$

$$C^+ = \sum_{m=1,\alpha=0}^{\infty} \frac{\alpha_2}{2} (K^{-1})_{\alpha,b} \left(h_\alpha^m \otimes h_{\bar{b}}^{(-m)}\right) + \frac{1}{2} \sum_{\alpha>0} \alpha_2 \left(E^0_\alpha \otimes E^{-0}_{-\alpha}\right) + \sum_{m=1,\alpha>0} \alpha_2 \left[E^m_\alpha \otimes E^{(-m)}_{-\alpha} + E^m_{-\alpha} \otimes E^{(-m)}_\alpha\right],$$

(5.14)

where $(A, \bar{A})$ are defined in (5.4). As a consequence [8],[12],

$$\{ tr T^m, tr T^n \} = 0, \quad T = \mathcal{P}[exp(\int_{-s}^s A_x(t,y) dy)],$$

(5.15)

$(m, n) \in Z$. Since $tr T^n$ are gauge invariant quantities, the previous relation holds also for the abelian connections $A^R_{x,ab}$. It then follows that [8],[12]:

$$\{ Q^R_m, Q^R_n \} = 0, \forall (n, m).$$

(5.16)

Another set of conservation laws can be obtained in a completely analogous way by considering positive grade expansion in (5.4). The relevant equations are summarized in the appendix.

### 6 Conclusion

In this paper a bicomplex structure associated to the generalized Leznov-Saveliev equation is established. In this sense, the bicomplex structure is equivalent to the zero-curvature equation. Also, the linear problem associated to the zero curvature condition is derived in terms of the bicomplex linear equation. The conservation laws for a non-abelian Toda model were obtained, generalizing the standard procedure in the abelian models.

**Acknowledgements** The author thanks J. F. Gomes, G. M. Sotkov and A. H. Zimerman for discussions and FAPESP for financial support.
7 Appendix

Let $A = \varepsilon^- + B^{-1}\partial B; \quad \tilde{A} = -B^{-1}\varepsilon^+ B; \quad g_L = [\Pi_{m=1}^\infty \exp(S_m)] \exp(\tilde{\xi}\varepsilon^+),$ where $S_m$ is a linear combination of generators in $\mathcal{F}^+$ and $\tilde{\xi}(z, \bar{z})$ is a function.

Under a gauge transformation: $A^L = g_L^A g_L^{-1} - g_L^L g_L^{-1} = \sum_{m=-1}^\infty A^{L,m};$
\[
\tilde{A}^L = g_L \tilde{A} g_L^{-1} - \tilde{g}_L \tilde{g}_L^{-1} = \sum_{m=-1}^\infty (\tilde{A}_{L,m}^A + \tilde{A}_{L,m}^L),
\]
where $A_{L,m}^A$ has grade $m$ and is a linear combination of generators in $\mathcal{A}$. Similarly for $\tilde{A}_{L,m}^L$. Also,
\[
B^{-1}\partial B = \beta(\partial \varphi_1 - (3/2)\beta \psi \bar{\psi} \partial \bar{\chi} e^{\beta(\varphi_1-\varphi_2)} h_1^{(1)} + \beta[\partial \varphi_2 + (1/2)\beta \psi \bar{\psi} \partial \bar{\chi} e^{\beta(\varphi_1-\varphi_2)} h_2^{(0)} + \beta[\partial \bar{\psi} \bar{\partial} \bar{\hat{C}} + \partial \bar{\eta} \tilde{D} + \beta \partial \bar{\chi} e^{\beta(\varphi_1-\varphi_2)} e_{\alpha_1}^{(0)} + \beta[\partial \bar{\psi} + \beta \psi \partial (\varphi_1 - \varphi_2) - \beta^2 \bar{\psi} \bar{\psi} \partial \bar{\chi} e^{\beta(\varphi_1-\varphi_2)} e_{\alpha_1}^{(0)}]
\equiv J^{(1)} h_1^{(0)} + J^{(2)} h_2^{(0)} + J^\nu \hat{C} + J^n \tilde{D} + J^{-E_{\alpha_1}} + J^+ E_{\alpha_1}^{(0)}.
\]
Solving for $(S_m, \tilde{\xi})$ such that
\[
A^L = \varepsilon^- + J^n \tilde{D}_A + J^{(1)} h_1^{(0)} + \sum_{m=1}^\infty a^{L,m} A(m),
\]
the zero curvature equation leads to
\[
\partial J^a = 0; \quad \tilde{\partial} J^{(1)} = 0; \quad \partial a^{L,1} = 0, \tilde{A}_{L,m}^L = 0, \forall m.
\]

Under another gauge transformation defined by
\[
g_L^A = \exp[\varepsilon^- \int_L^1 e^{-\frac{\alpha}{2} \int_L^w J^{(0)}(w) dw}] \exp[\int_L^1 (J^{(1)}(w) h_1^{(0)} + J^n(w) \tilde{D}_A) dw];
\]
follows,
\[
\tilde{A}_{ab}^L \equiv \sum_{m=1}^\infty \tilde{A}_{ab}^{L,m} = \sum_{m=1}^\infty \tilde{a}_{ab}^{L,m} e^{\int_L^1 \int_L^w J^{(0)}(w) dw} A(m); \quad \partial \tilde{A}_{ab}^L - \partial A_{ab}^L = 0;
\]
\[
A_{ab}^L \equiv \sum_{m=1}^\infty a_{ab}^{L,m} + \hat{b} \hat{C}.
\]

The conserved charges are obtained: $\partial_t Q^L_m = 0, m \geq 1; Q^L_m = \int_{-s}^s \int_L^1 \int_L^w J^{(0)}(w) dw A(m)$, where $s \in R$.

References

[1] J. Balog, L. Feher, P.Forgacs, L. O’Raifeartaigh and A. Wipf, Phys. Lett. B227 (1989),214 ; J. F. Gomes, G. M. Sotkov and A. H. Zimerman, Ann. of Phys. 274 (1999), 289, hep-th/9803234.

[2] D. I. Olive, N. Turok and J. W. R. Underwood, Nucl. Phys. B401 (1993), 663. F. Lund and T. Regge, Phys. Rev. D14 (1976), 1524. O. A. Castro Alvaredo, J. L. Miramontes Nucl.Phys B581 (2000) 643, hep-th/0002213.

[3] J. F. Gomes, E. P. Guevoghlanian, G. M. Sotkov and A. H. Zimerman, Nucl. Phys. B598 (2001), 615, hep-th/0011187 see also J. F. Gomes, E. P. Guevoghlanian, G. M. Sotkov and A. H. Zimerman, hep-th/0007169 to appear in Nucl. Phys. B.

[4] H.Aratyn, L. A. Ferreira, J. F. Gomes and A. H. Zimerman, Phys. Lett. B254 (1991), 372. O. Babelon and L. Bonora, Phys. Lett. B244 (1990), 220.

[5] A. N. Leznov and M. V. Saveliev, Commun. Math. Phys. 89 (1983), 59; Theor. Math. Phys. 54 (1983), 209.
[6] L. D. Faddeev, Les Houches Lectures, Session XXXIX, Eds. J. B. Zuber and R. Stora, Elsevier (1984); L. D. Faddeev and L. A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer-Verlag, Berlin (1987).

[7] A. Dimakis and F. Muller-Hoissen, Int. J. Mod. Phys. B14 (2000), 2455, hep-th/0006005; J. Phys. A33, (2000), 6579, hep-th/0006023; hep-th/0007015; Phys. Lett. A278, (2000), 139, hep-th/0007074; J.Phys. A34 (2000), 2571, nlin.SI/0008016.

[8] D.I.Olive and N.Turok, Nucl.Phys. B257[FS14](1985),277.

[9] P. Goddard and D. I. Olive, Int. Jour. Mod. Phys. A1 (1986), 303. V. G. Kac, Infinite Dimensional Lie Algebras, Cambridge (1990).

[10] H.Aratyn,L.A.Ferreira,J.F.Gomes and A.H. Zimerman,Supersymmetry and Integrable Models,Lectures Notes in Physics 502,Ed. H.Aratyn et al.,Springer-Verlag,Berlin, (1998),197.

[11] J. F. Gomes, E. P. Gueuvoghlanian, G. M. Sotkov and A. H. Zimerman, hep-th/0010257, to appear in: Proc. XXIII Int. Colloquium on Group Theoretical Methods in Physics, G.Pogosyan et al.(Ed.), Dubna, 2000.

[12] H.Aratyn,L.A.Ferreira,J.F.Gomes and A.H. Zimerman,Mod.Phys.Lett.A9(1994),2783.