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THE EULER-MARUYAMA APPROXIMATIONS FOR THE CEV MODEL

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Abstract. The CEV model is given by the stochastic differential equation
\[ X_t = X_0 + \int_0^t \mu x_s \, ds + \int_0^t \sigma (x_s^+) p \, dW_s, \]
\[ \frac{1}{2} \leq p < 1, \]
where \( x^+ = 0 \lor x \), constants \( p \in \left( \frac{1}{2}, 1 \right) \), \( \mu \in \mathbb{R} \), \( \sigma > 0 \), and a positive deterministic initial condition \( X_0 \). Denote by \( \tau \) the first time of hitting zero \( \tau = \inf \{ t : X_t = 0 \} \).

It is known that zero is an absorbing state, e.g. \[24\], and that \( P(\tau < \infty) > 0 \).

The aim of this paper is to prove validity of approximations of expectations of some functionals of these diffusions by the Euler-Maruyama scheme. For example, in finance this model represents a price process and it is important to evaluate expectations of payoffs depending on past prices \( E g(X_{[0,T]}) \); as well as for evaluation of the ruin probability \( P(\tau \leq T) \) by simulations. In population modeling the diffusion with \( p = \frac{1}{2} \) represents the size of a population and is known as Feller’s branching diffusion. In this case \( \tau \) represents the time to extinction of the population.

1. Introduction and the main result. We consider the Constant Elasticity of Variance (CEV) model (e.g. \[5\], \[8\]) defined by the Itô equation with respect to a Brownian motion \( W_t \)
\[ X_t = X_0 + \int_0^t \mu x_s \, ds + \int_0^t \sigma (x_s^+) p \, dW_s, \]
where \( x^+ = 0 \lor x \), constants \( p \in \left( \frac{1}{2}, 1 \right) \), \( \mu \in \mathbb{R} \), \( \sigma > 0 \), and a positive deterministic initial condition \( X_0 \). Denote by \( \tau \) the first time of hitting zero \( \tau = \inf \{ t : X_t = 0 \} \).

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When exact theoretical expressions of such functionals exist then a comparison of simulated and exact results show how good the approximation is. However, in many cases exact expressions are not available. Then one needs to justify such approximations. Weak convergence established below assures convergence of expectations of bounded and continuous functionals of simulated values to the exact ones.

To allow a larger family of functionals to be approximated we show the weak convergence of approximations in the Skorokhod metric rather than uniform. We also consider approximation of the ruin probability \( P(\tau \leq t) \) which is the expectation of the past dependent and discontinuous (in this metric) functional \( I_{\{\tau \leq T\}} \).

The fact that the diffusion coefficient is singular and non-Lipschitz makes the analysis non-standard. Feller in [9] showed for the case \( p = 1/2 \) that the solution of the Fokker-Plank equation exists and is unique and gave its fundamental solution. This fact is used for evaluation of ruin probability by assuming that for any \( T \), \( X_T \) has a density on \((0, \infty)\), and in particular its distribution function is continuous at any positive point, moreover it is right-continuous (as a distribution function), \( \lim_{\varepsilon \to 0} P(X_T \leq \varepsilon) = P(X_T = 0) \).

Previous various results on the Euler-Maruyama algorithm for different type of diffusion models can be found in Kloeden and Platten [20] and Milstein and Tretyakov [22]. They are concerned mostly with the approximation of the final value \( X_T \), rather than the whole trajectory \( X_t \), \( 0 \leq t \leq T \). When both the drift and diffusion coefficients are Lipschitz and the diffusion coefficient is nonsingular, the standard theory applies. There is a body of literature on the topic of approximations, see e.g. Bally and Talay [2], Bossy and Diop [4], Gyöngy [10], Gyöngy and Krylov [11], Halidias and Kloeden [12], Higham, Mao and Stuart [15], Hutzenthaler and Jentzen [16], Jentzen, Kloeden and Neuenkirch [17]. In contrast to the above mentioned papers we always use continuous time setting, even for discrete time models as in [27] since the use of the Skorokhod metric allows to avoid some technical assumptions. Zähle [27] proved the weak convergence in terms of solutions to martingale problems in one dimension. He has shown that the solution of the martingale problem corresponding to \((1)\) can be approximated by solutions of the corresponding time-discrete martingale problems under some conditions. We propose a new method of proof by adapting (with a help of a specific approximation) a standard technique for weak convergence of semimartingales (e.g. [21]) that applies to cases of Hölder continuous diffusion and also when the the limiting process can be absorbed at zero. Our approach avoids the technical condition \((2.3)\) in [27], moreover, the suggested approximations can be used in high dimensions as well.

As a rule, the cases considered in the literature are such that the drift and diffusion coefficients exclude the absorption effect even when the diffusion coefficient is singular, such as in Bessel diffusion.

Our method applies to a more general situation of non-Lipschitz diffusion with absorption. We chose, however, to give the results for the particular model of CEV for the sake of transparency, and because this model is of significance in applications. The reader will note that the proofs are sufficiently involved already in this special case.

Recall next the Euler-Maruyama scheme for the diffusion process \( X_t \). Taking for simplicity equidistant partitions of \([0, T]\), \( 0 = t_0^0 < t_1^1 < \ldots < t_{n-1}^n < t_n^n \equiv T \).
for any bounded and continuous in the metric $d_0$ function $f(x)$. The aim of the paper is precisely to show this convergence. Note that such a function $f(x)$ need not be continuous in the uniform metric $d_0(x, y) = \sup_{t \in [0, T]} |x_t - y_t|$. 

We remark on the use of the function $x^+$ next. Since the process $X = (X_t)_{t \in [0, T]}$ is nonnegative, the notation $X_t^+$ in (1) is used formally. However, it must be used for the approximating process $X_t^{n+}$ in (2), because $X_t^n$ can (and does) become negative. Since $X_t^n$ has piece-wise constant paths the stopping time

$$
\tau_n = \inf\{t \leq T : X_t^n \leq 0\}
$$

much more likely corresponds to the negative value rather than the zero value of $X_t^n$. The plus notation $X_t^{n+}$ in (2) enables us to exclude negative values of $X_t^n$ up to the first time it becomes negative. Once the process becomes negative it stays at that value for all further times $t$. Thus, in simulations of $f(X)$ we must use $f(X^{n+})$ and stop at the first time it becomes negative. Next, the function $g(x) := f(x^+)$ inherits the properties of $f(x)$ and is bounded and continuous in the metric $d_0$, moreover, when applied to simulations it converges to the desired limit

$$
\lim_{n \to \infty} E[f(X^{n+})] = \lim_{n \to \infty} E[g(X^n)] = E[f(X^+)] = E[f(X)].
$$
We introduce now the continuous approximation \( \tilde{X}_t^n \) used in the proof of weak convergence

\[
\tilde{X}_t^n = X_0 + \sum_{k=1}^{n} \int_{t_{k-1}}^{t\wedge t_k} \mu \tilde{X}_s^n \, ds + \sum_{k=1}^{n} \int_{t_{k-1}}^{t\wedge t_k} \sigma(\tilde{X}_s^n) \, d\tilde{W}_s,
\]

where \( \tilde{W}_t \) is a Brownian motion such that \( \tilde{W}_{t_k} - \tilde{W}_{t_{k-1}} \equiv \sqrt{\frac{\xi_k}{n}} \). The above remark applies equally to the process \( X_t^n \).

Denote \( \tilde{Q}^n \) the distribution of \( (\tilde{X}_t^n)_{t\in[0,T]} \). The main result is that \( Q^n \overset{d_0}{\longrightarrow} Q \).

**Theorem 1.1.** For any \( T > 0 \), the Euler-Maruyama approximation for equation (2) converges weakly in the Skorokhod metric \( d_0 \) to the limit process \( (X_t)_{t\in[0,T]} \) defined in (1).

The proof is done in three steps: first we show that the distance in the uniform metric between the Euler-Maruyama and the continuous semimartingale approximations converges to zero in probability, secondly we show that the continuous approximation converges in the Skorokhod metric to the solution \( (X_t)_{t\in[0,T]} \), and finally we deduce convergence of the Euler-Maruyama approximation. Thus the three steps in the proof are

1. \( \rho(X^n, \tilde{X}^n) \overset{\text{prob}}{\longrightarrow} 0 \) as \( n \to \infty \)
2. \( \tilde{Q}^n \overset{d_0}{\longrightarrow} Q \)
3. \( \rho(X^n, \tilde{X}^n) \overset{\text{prob}}{\longrightarrow} 0 \) as \( n \to \infty \) \( \Rightarrow \) \( Q^n \overset{d_0}{\longrightarrow} Q \).

The proof of “step 1” uses standard stochastic calculus for semimartingales. The proof of “step 2” follows from the results on diffusion approximation for semimartingales (see [21, Ch.8]). The proof of “step 3” is done as a standard application of Billingsley [3, Theorem 4.4, Ch.1, §4].

We comment on the rate of weak convergence and why defer this important matter to another study. Our focus lies in computer simulation and Monte-Carlo technique. In computer realizations all numerical trajectories are independent objects. Therefore in practical Monte-Carlo simulations these trajectories can not be put on the same probability space. Stronger modes of convergence analyze the rate of decay to zero of the difference \( X^n - X \) between the pre-limit process \( X^n \) and the limit process \( X \) in various norms as a function of \( n \to \infty \) (see e.g. recent papers [17], [14], [26]). The rate of convergence in a weak mode, in particular in the Skorokhod metric, could not be analyzed in terms of the difference \( X^n - X \) since random processes are defined on different probability spaces. What we have is in fact convergence of pre-limit measures \( Q^n \) to the limit measure \( Q \), defined on the Skorokhod space, being distributions of \( X^n \) and \( X \) respectively. It is, of course, possible to compare \( Q \) and \( Q^n \) in the total variation \( \| Q - Q^n \|_{tv} \) or Prokhorov’s or Wasserstein’s metrics. The choice of any of these metrics requires a deep analysis (see e.g. [18] for the total variation metric). There are known results on approximation of \( Ef(X_T^n) \) at the final point \( T \) with a rate of convergence of the \( |Ef(X_T) - Ef(X_T^n)| \) termed as the weak error (see reviews of Kloeden and Platen [20] and Talay [25]). These rates depend on the smoothness of \( f \) and regularity of diffusion coefficients. Even for
the case of weak convergence at the final point the analysis of the rate is involved, for example Bally and Talay [2] use the Malliavin calculus (stochastic calculus of variation) to establish convergence rate of the distribution function. A recent paper of Alfonsi [1] is devoted entirely to weak error analysis for various approximation schemes for $f \in C^\infty_c(\mathbb{R})$ (compactly supported infinitely differentiable functions on $\mathbb{R}$), satisfying various conditions which applies to CIR model. Therefore due to complexity of the subject of the rate of convergence, and to preserve the clarity of ideas in the present paper, we postpone the study of this question.

The paper is organized as follows. In Section 2 we give preliminary results used in the proof, with some being of independent interest. The proof of the Theorem 1.1 is given in Section 3. Section 4 gives simulations for the ruin probability $P(\tau \leq T)$.

To be self contained, the existence and uniqueness of solutions of the stochastic differential equation for the CIR model (1) is shown in Section 5.

2. Preliminaries. We shall use the Doob maximal inequality for martingales (see e.g. [21, Ch.1, §9] and [19, p.201]) in the following context. Let $(\alpha_k)_{0 \leq k \leq n-1}$ and $(\xi_k)_{1 \leq k \leq n}$ be random variables such that

- $E\xi_k^2 < \infty$, $0 \leq k \leq n - 1$;
- $(\xi_k)_{1 \leq k \leq n}$ is i.i.d. with $E\xi_1 = 0$, $E\xi_1^2 = 1$;
- $\xi_k$ and $\{\alpha_0, \ldots, \alpha_{k-1}\}$ are independent for any $1 \leq k \leq n$.

Set $M_t = \sum_{j=1}^{\lfloor nt \rfloor} \alpha_{j-1} \xi_j$, where $t \in [0,1]$ and $\lfloor nt \rfloor = k$ if $t \in \left(\frac{k-1}{n}, \frac{k}{n}\right]$. The process $M_t$ is a square integrable martingale for a suitable filtration. So, the Doob maximal inequality $E(\sup_{t \in [0,1]} |M_t|^2) \leq 4EM_t^2$ is equivalent to

$$E \max_{1 \leq k \leq n} \left| \sum_{j=1}^{k} \alpha_{j-1} \xi_j \right|^2 \leq 4 \sum_{j=1}^{n} E\xi_j^2 = 4. \tag{4}$$

Next we give the result useful in the analysis of products of random variables.

Lemma 2.1. Let $X_n$, $Y_n$ be sequences of random variables such that $X_n$ converges to zero in probability and expectations of $Y_n$ are uniformly bounded, $\sup_n E|Y_n| = r < \infty$. Then $X_n Y_n$ converges to zero in probability.

Proof.

$$P(|X_n Y_n| > \varepsilon) = P(|Y_n| \leq C, |X_n Y_n| > \varepsilon) + P(|Y_n| > C, |X_n Y_n| > \varepsilon) \leq P(|Y_n| > \varepsilon) + P(|Y_n| > C).$$

By the Chebyshev inequality $P(|Y_n| > C) \leq \frac{r}{C}$. Hence

$$\lim_{n \to \infty} P(|X_n Y_n| > \varepsilon) \leq \frac{r}{C} \to 0.$$

Next we need the following result of convergence to zero the expectation of double-maximum the Brownian motion increments over partitions.

Lemma 2.2. Let $W_t$ be a Brownian motion and $\{t_i\}$, $i = 1, \ldots, n$ be an equidistant partition of $[0,1]$. Let $W^*_i = \sup_{t \in [t_{i-1}, t_i]} |W_t - W_{t_{i-1}}|$, and $M_n = \max_{1 \leq i \leq n} W^*_i$. Then $M_n$ converges to zero in $L^k$ for any $k \geq 1$. 

\qed
Proof. We calculate $k$-th moment of $M_n$, $k \geq 1$. Since $M_n \geq 0$
\[ \mathbb{E}M_n^k = \int_0^\infty P(M_n^k \geq x)dx = \int_0^\infty P(M_n \geq x^{1/k})dx \]
\[ \leq n \int_0^\infty P(W_1^* \geq x^{1/k})dx. \]

The inequality is obtained since $(W_i^*)_{1 \leq i \leq n}$ are i.i.d. random variables, therefore
for any $x \geq 0$
\[ P(M_n \geq x^{1/k}) = P\left( \max_{1 \leq i \leq n} W_i^* \geq x^{1/k} \right) = P\left( \bigcup_{i=1}^n \{W_i^* \geq x^{1/k}\}\right) \leq nP\left( W_1^* \geq x^{1/k}\right). \]

Next since $\sup_{t \in [0,1/n]} |W_t| \leq \sup_{t \in [0,1/n]} W_t + \sup_{t \in [0,1/n]} (-W_t)$, it follows
\[ P\left( W_1^* \geq x^{1/k}\right) \leq P\left( \sup_{t \in [0,1/n]} W_t + \sup_{t \in [0,1/n]} (-W_t) \geq x^{1/k}\right) \leq P\left( \sup_{t \in [0,1/n]} W_t \geq \frac{1}{2}x^{1/k}\right) + P\left( \sup_{t \in [0,1/n]} (-W_t) \geq \frac{1}{2}x^{1/k}\right) = 2P\left( W_{1/n} \geq \frac{1}{2}x^{1/k}\right), \]
where we have used the well-known law of the maximum of Brownian motion.
Thus
\[ \mathbb{E}M_n^k \leq 4n \int_0^\infty P\left( W_{1/n} \geq \frac{1}{2}x^{1/k}\right)dx \leq 4n \int_0^\infty P\left( |W_{1/n}| \geq \frac{1}{2}x^{1/k}\right)dx \]
\[ = 4n2^{k}\mathbb{E}|W_{1/n}|^k = C_k n^{1-k/2}, \]
where $C_k$ depends only on $k$, $(C_k = 2^{k+2}\mathbb{E}|\xi|^k$ with $\xi \sim N(0,1))$. Hence $M_n$ converges to zero in $L^k$ for any $k > 2$. But since convergence in $L^p$ for a $p > 1$
implies convergence in $L^k$ for any $k \in [1, p]$, the statement is proved. \hfill \Box

The next elementary inequality is new and is instrumental in the proof.

Lemma 2.3. For any $x, y \in \mathbb{R}$, and $p \in (\frac{1}{2}, 1)$,
\[ |(x^+)^{2p} - (y^+)^{2p}| \leq \begin{cases} |x - y|, & p = \frac{1}{2} \\ (2 + |x| + |y|)|x - y|^p, & p \in (\frac{1}{2}, 1). \end{cases} \]

Proof. For $p = \frac{1}{2}$ we have
\[ |x^+ - y^+| = |x - y|1_{\{x, y > 0\}} + |x|1_{\{x > 0, y \leq 0\}} + |y|1_{\{x \leq 0, y > 0\}} \]
\[ \leq |x - y|1_{\{x, y > 0\}} + |x - y|1_{\{x > 0, y \leq 0\}} + |y - x|1_{\{x \leq 0, y > 0\}} \leq |x - y|. \quad (5) \]

For $p \in (\frac{1}{2}, 1)$ we have
\[ |(x^+)^{2p} - (y^+)^{2p}| = |(x^+)^p - (y^+)^p||x|^p + (y^+)^p| \]
\[ \leq (2 + |x| + |y|)|(x^+)^p - (y^+)^p|. \]

Next we now show that
\[ |x^p - y^p| \leq |x - y|^p. \]

For $x = y = 0$ it is obvious.
Consider $x > 0$ and $y < 0$. Then, taking into account the proved inequality $|x^+ - y^+| \leq |x - y|$ (the statement of this lemma for $p = \frac{1}{2}$) and the fact that $y^+ = 0$, we obtain

$$|(x^+)p - (y^+)p| = |x^+ - (y^+)p| = |x^+ - y^+|p \leq |x - y|^p.$$ 

Clearly, the inequality remains true for $x < 0$ and $y > 0$.

If both $x$ and $y$ are positive and $x > y$, then

$$|x^+ - y^+|p^{1/p} = x\left|1 - \left(\frac{y}{x}\right)^p\right|^{1/p} \leq x\left|1 - \left(\frac{y}{x}\right)\right|^{1/p} \leq x\left|1 - \left(\frac{y}{x}\right)\right| = |x - y|$$

and, in turn, $|x^+ - y^+| \leq |x - y|^p$. It is easy to see the inequality remains true for $y > x$. □

3. **Proof of Theorem 1.1.** In the following result we show that the maximum of discrete approximations have uniformly bounded seconds moments. Henceforth $r$ denotes a generic positive constant independent of $n$ with different values at different appearances.

**Lemma 3.1.** Let $X_{t_k}^n$ be the Euler-Maruyama approximation defined in (2).

Then $E \max_{1 \leq k \leq n} |X_{t_k}^n|^2 \leq r$.

**Proof.** First we bound from above $\max_{1 \leq k \leq n} E|X_{t_k}^n|^2$. By (2)

$$E|X_{t_k}^n|^2 = E|X_{t_{k-1}}^n|^2 \left(1 + \frac{\mu T}{n}\right)^2 + \sigma^2 E(X_{t_{k-1}}^n)^{2p} T^n. \quad (6)$$

We use repeatedly the following bound of $(X_{t_{k-1}}^n)^{2p}$. Since $(x^+)^{2p} \leq |x|^{2p}$ and $2p < 2$, we have $(x^+)^{2p} \leq 1 + x^2$. Hence

$$E(X_{t_{k-1}}^n)^{2p} \leq 1 + E|X_{t_{k-1}}^n|^2. \quad (7)$$

Next, for sufficiently large $n$, there exists $r$ such that

$$(1 + \frac{\mu T}{n})^2 \leq 1 + \frac{r}{n}. \quad (8)$$

Hence for sufficiently large $n$, we obtain from (6) by using (8) the recurrent inequality

$$E|X_{t_k}^n|^2 \leq E|X_{t_{k-1}}^n|^2 \left(1 + \frac{r}{n}\right) + \sigma^2 E(X_{t_{k-1}}^n)^{2p} \left(1 + E|X_{t_{k-1}}^n|^2\right)$$

$$= E|X_{t_{k-1}}^n|^2 \left(1 + \frac{r + \sigma^2 T}{n}\right) + \frac{\sigma^2 T}{n} := E|X_{t_{k-1}}^n|^2 \left(1 + \frac{r}{n}\right) + \frac{r}{n}. \quad (9)$$

Iterating it, for $k \leq n$ we obtain

$$E|X_{t_k}^n|^2 \leq X_0^2 \left(1 + \frac{r}{n}\right)^k + \frac{r}{n} \sum_{j=1}^{k} \left(1 + \frac{r}{n}\right)^{k-j+1}$$

$$\leq (X_0^2 + r) \left(1 + \frac{r}{n}\right)^n = (X_0^2 + r)O(e^r).$$

Thus

$$\max_{1 \leq k \leq n} E|X_{t_k}^n|^2 \leq r. \quad (9)$$

From the definition of the scheme (2) we obtain by iterations
The distance between the two approximations in the sup norm converges to zero in probability and in \( L^2 \).

**Lemma 3.2.** \( \mathbb{E} \phi^2(X^n, \tilde{X}^n) \xrightarrow{n \to \infty} 0. \)

**Proof.** From the definitions of \( X^n_t \) and \( \tilde{X}^n_t \) it follows that at the points of the partitions both approximations coincide and at intermediate points the following holds

\[
X^n_{t_k} = \tilde{X}^n_{t_k}
\]

\[
\tilde{X}^n_t - X^n_{t_{k-1}} = \int_{t_{k-1}}^{t_k} \mu(X^n_{t_{k-1}}) + \int_{t_{k-1}}^{t_k} \sigma(X^n_{t_{k-1}}) \, d\tilde{W}_s, \quad t \in [t^n_{k-1}, t^{n}_{k}].
\]

By the formula (11),

\[
\sup_{t \in [t^n_{k-1}, t^n_{k}]} |\tilde{X}^n_t - X^n_{t_{k-1}}| \leq \frac{\mu|T|}{n} |X^n_{t_{k-1}}| + \sigma \sup_{t \in [t^n_{k-1}, t^n_{k}]} |\tilde{W}_t - \tilde{W}_{t^n_{k-1}}| |X^n_{t_{k-1}}|^p.
\]

Consequently,

\[
\phi(X^n, \tilde{X}^n) = \sup_{t \in [0, T]} |\tilde{X}^n_t - X^n_t| = \max_{1 \leq k \leq n} \sup_{t \in [t^n_{k-1}, t^n_{k}]} |\tilde{X}^n_t - X^n_{t_{k-1}}|.
\]

\[
\leq \frac{\mu|T|}{n} \max_{1 \leq k \leq n} |X^n_{t_{k-1}}| + \sigma \max_{1 \leq k \leq n} \sup_{t \in [t^n_{k-1}, t^n_{k}]} |\tilde{W}_t - \tilde{W}_{t^n_{k-1}}| \max_{1 \leq k \leq n} |X^n_{t_{k-1}}|^p.
\]
The first term converges to zero in $L^2$ using Lemma 3.1 as $E\max_{1 \leq k \leq n} |X^n_{t_{k-1}}| \leq r$. For the second term use Hölder’s inequality with parameters $\frac{2}{p}$ and $\frac{2}{2-p}$:

$$E \left( \max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} |\tilde{W}_t - \tilde{W}_{t_{k-1}}| \max_{1 \leq k \leq n} |X^n_{t_{k-1}}|^p \right)^{2/p} \leq \left( E \left( \max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} |\tilde{W}_t - \tilde{W}_{t_{k-1}}|^p \right) \right)^{2/p} \left( E \max_{1 \leq k \leq n} |X^n_{t_{k-1}}|^p \right)^{2/p}.$$

By Lemma 3.1 $\sup_n E \max_{1 \leq k \leq n} |X^n_{t_{k-1}}|^2 \leq r$. Since $\frac{2}{2-p} > 1$, by Lemma 2.2

$$\lim_{n \to \infty} E \left( \max_{1 \leq k \leq n} \sup_{t \in [t_{k-1}, t_k]} |\tilde{W}_t - \tilde{W}_{t_{k-1}}| \right)^{2(2-p)/p} = 0.$$

\[\square\]

3.2. **Step 2.** Weak convergence of continuous approximations.

**Lemma 3.3.** $\tilde{Q}^n \xrightarrow{d_0} Q$.

**Proof.** The proof rests on a general result on the weak convergence of semimartingales to a diffusion [21], Theorem 1, Ch. 8, §3. This theorem states that for weak convergence to a diffusion it is enough to check convergence of the drifts and quadratic variations evaluated at the pre-limit processes. The processes $X$ and $\tilde{X}^n$ are semimartingales with following decompositions

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma(X_t^+) dW_s,$$

$$\tilde{X}_t^n = X_0 + \sum_{k=1}^n \int_{t_{k-1}}^{t \wedge t_k} \mu \tilde{X}^n_{t_{k-1}} ds + \sum_{k=1}^n \int_{t_{k-1}}^{t \wedge t_k} \sigma(\tilde{X}^n_{t_{k-1}}^+) d\tilde{W}_s,$$

where we have denoted above

(B) $B_t(X)$ and $B^n_t(\tilde{X}^n)$ are drifts

(M) $M_t(X)$ and $M^n_t(\tilde{X}^n)$ are continuous martingales with predictable quadratic variations

$$\langle M \rangle_t(X) = \int_0^t \sigma^2(X_s^+) ds,$$

$$\langle M^n \rangle_t(\tilde{X}^n) = \sum_{k=1}^n \int_{t_{k-1}}^{t \wedge t_k} \sigma^2(\tilde{X}^n_{t_{k-1}}^+) ds.$$

The above mentioned Theorem 1 of [21], adapted to the present setting, states that the weak convergence takes place if the following three conditions hold:

\[
\begin{align*}
\text{(a)} & \text{ Equation (1) has a unique (at least weak) solution} \\
\text{(b)} & \text{ } \sup_{t \in [0,T]} |B_t(\tilde{X}^n) - B^n_t(\tilde{X}^n)| \xrightarrow{\text{prob.}} 0, \\
\text{(c)} & \text{ } \sup_{t \in [0,T]} |\langle M \rangle_t(X) - \langle M^n \rangle_t(\tilde{X}^n)| \xrightarrow{\text{prob.}} 0,
\end{align*}
\]

$$\Rightarrow \tilde{Q}^n \xrightarrow{d_0} Q.$$
We proceed to verify these conditions. The existence and uniqueness of (1) is known, e.g. [24], and is also given for completeness Proposition 1. Hence (a) holds. To show (b) write, taking into account (5),

\[
\sup_{t \in [0,T]} \left| B_t(\tilde{X}^n) - B^n_t(\tilde{X}^n) \right| \leq |\mu| \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t_k^n} |\tilde{X}_s^n - \tilde{X}_{t_{k-1}^n}| ds \leq T |\mu| \varrho(\tilde{X}^n, X^n).
\]

Hence (b) holds by applying Lemma 3.2.

To prove (c) write the bound

\[
\sup_{t \in [0,T]} \left| (M)_t(\tilde{X}^n) - (M^n)_t(\tilde{X}^n) \right| \leq \sigma^2 \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t_k^n} |(\tilde{X}_s^n)^{2p} - (X_{t_{k-1}^n}^n)^{2p}| ds
\]

\[
= \sigma^2 \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t_k^n} |(\tilde{X}_s^n)^{2p} - (X_{t_{k-1}^n}^n)^{2p}| ds,
\]

where we have used that the two approximations coincide on the grid. By applying Lemma 2.3 we have further bound on the expression under the integral

\[
|X_{t_{k-1}^n}^n|^{2p} - |X_{t_{k-1}^n}^n|^{2p} \leq \left\{ \begin{array}{ll}
|\tilde{X}_s^n - X_{t_{k-1}^n}^n|, & p = \frac{1}{2} \\

t + \sup_{t \in [0,T]} |\tilde{X}_t^n| + \sup_{t \in [0,T]} |X_t^n|, & p \in \left(\frac{1}{2}, 1\right) \end{array} \right.
\]

Hence for \( p = \frac{1}{2} \) the bound in (12) becomes

\[
\sigma^2 \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t_k^n} |(\tilde{X}_s^n)^{2p} - (X_{t_{k-1}^n}^n)^{2p}| ds \leq \sigma^2 T \varrho(\tilde{X}^n, X^n)
\]

and the statement follows by Lemma 3.2.

For \( p \in \left(\frac{1}{2}, 1\right) \) the bound in (12) becomes

\[
\sigma^2 \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t_k^n} |(X_{t_{k-1}^n}^n)^{2p} - (X_{t_{k-1}^n}^n)^{2p}| ds = \sigma^2 \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t_k^n} |(\tilde{X}_s^n)^{2p} - (X_{t_{k-1}^n}^n)^{2p}| ds
\]

\[
\leq \sigma^2 \left( 2 + \sup_{t \in [0,T]} |\tilde{X}_t^n| + \sup_{t \in [0,T]} |X_t^n| \right) \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t_k^n} \sup_{s \in [t_{k-1}^n, t_k^n]} |\tilde{X}_s^n - X_s^n|^p ds
\]

\[
\leq T \sigma^2 \left( 2 + \sup_{t \in [0,T]} |\tilde{X}_t^n| + \sup_{t \in [0,T]} |X_t^n| \right) \varrho(\tilde{X}^n, X^n).
\]

By Lemma 3.1, Lemma 3.2 we have the product of two terms, one of which has uniformly bounded expectations and the second converges in probability to zero. By Lemma 2.1 the product converges in probability to zero.

Thus the conditions of the Theorem 1 of [21] are verified and weak convergence is proved.

\[\square\]

3.3. Step 3. Weak convergence of the Euler-Maruyama approximations.

**Lemma 3.4.** For any bounded and continuous in the metric \( d_0 \) function \( f(x) \),

\[
\lim_{n \to \infty} |E f(X^n) - E f(X)| = \lim_{n \to \infty} |E f(\tilde{X}^n) - E f(X)| = 0.
\]
Proof. The result follows from the triangular inequality

\[ |E(f(X^n) - E(f(X))] \leq E(|f(X^n) - f(X^n)|) + |E(f(X^n) - E(f(X)]. \tag{13} \]

Since convergence in the uniform metric implies convergence in the Skorokhod metric, by Lemma 3.2 we have, in view of \( f \) is continuous in this metric,

\[ \lim_{n \to \infty} E|f(X^n) - f(X^n)| = 0. \]

Taking now the lim sup in (13) and using the previous result of weak convergence of \( X^n \) to \( X \) proves the step 3. \( \square \)

4. Evaluation of ruin probability by simulations. In this section we evaluate numerically a ruin probability \( P(\tau \leq T) \), for a finite positive \( T \), where \( \tau = \inf\{t : X_t = 0\} \) by Euler-Maruyama approximations.

The basis for analysis is an obvious formula \( P(\tau \leq T) = P(X_T = 0) \). It allows us to deal with the distribution function \( F(x) := P(X_T \leq x) \) of \( X_T \) instead of a harder to compute distribution function of \( \tau \), \( P(\tau \leq T) \). Notice that

\[ F(x) = \begin{cases} 0, & x < 0 \\ F(0) = P(X_T = 0) > 0, & x = 0, \end{cases} \]

that is, \( F(0) - F(0-) > 0 \) and so the distribution function \( F(x) \) has an atom at the point 0.

The measure \( Q \) is supported on the space of continuous functions. So the weak convergence of processes and measures \( Q_n \xrightarrow{d} Q \) implies weak convergence of finite dimensional distributions, and in particular marginals, \( X^n \xrightarrow{\text{law}} X_T \). That is if \( F_n(x) = P(X^n \leq x) \) then \( \lim_{n \to \infty} F_n(x) = F(x) \) at any point of continuity of \( F \). Unfortunately 0, our point of interest, is an atom of \( F \) and we cannot claim that

\[ \lim_{n \to \infty} F_n(0) = P(\tau \leq T) = F(0). \]

In view of this uncertainty, we give approximations for lower and upper bounds of \( F(0) \) by using the Lévy metric (see e.g. [13]):

\[ \mathcal{L}(F_n, F) = \inf\{h > 0 : F_n(x-h) - h \leq F(x) \leq F_n(x+h) + h; \forall x\}. \]

It is known that weak convergence of distributions implies convergence in the Lévy metric \( Q_n \xrightarrow{\text{d}} Q \Rightarrow \lim_{n \to \infty} \mathcal{L}(F_n, F) = 0. \) Although \( \lim_{n \to \infty} \mathcal{L}(F_n, F) = 0 \) does not catch the size of the atom it helps to localize its size. Namely we take \( x = 0 \) and a small suitable \( \varepsilon_n \), in each case determined experimentally, such that the interval \( [F_n(-\varepsilon_n) - \varepsilon_n, F_n(\varepsilon_n) + \varepsilon_n] \) is small enough and declare that an estimate of \( F(0) \) belongs to this interval. In practice, sometimes theoretical values might lie outside this interval. Such values of \( \varepsilon_n \) are pointed out below in each Table below.

It is important to compare a numerical method with theoretical results. Fortunately such results are available for \( p = \frac{1}{2} \). In this case the explicit formula for \( P_{X_0}(X_T = 0) \) for any \( \mu, \sigma \) is known (see e.g. Dawson [6])

\[ P_{X_0}(X_T = 0) = \begin{cases} \exp\left( -\frac{2X_0}{\sigma^2} \right), & \mu = 0 \\ \exp\left( -2X_0\frac{\mu e^{\mu T}}{\sigma^2[\exp(\mu T) - 1]} \right), & \mu \neq 0. \end{cases} \]

Our numerical results are obtained for the following values of parameters \( p, X_0, \mu, \sigma, T \):
- $p = 1/2, 3/4$
- $X_0 = 1/10, 1/4, 1$
- $\mu = -1, 0, 1$
- $T = 3, 6, 9$
- $\sigma = 1$.

We use Monte-Carlo simulations of $10^3$ independent copies of the process $(X^n_t)_{t \in [0,T]}$ generated by Euler-Maruyama algorithm with $t^n_k - t^n_{k-1} = 10^{-2}$. Numerical results are given below for $p = 1/2$ and $p = 3/4$. For $p = 1/2$, where theoretical formulae are available, they show good agreement for $\varepsilon_n = 10^{-7}, 10^{-6}$ and $X_0 = 1/4$ and $\sigma^2 = 1$. Here “lower bound” and “upper bound” denote respectively $P(X^n_T \leq -\varepsilon) - \varepsilon$ and $P(X^n_T \leq \varepsilon) + \varepsilon$.

| $p$ | $\mu$ | $T$ | $\varepsilon_n$ | lower bound | upper bound | theory |
|-----|-------|-----|-----------------|-------------|-------------|--------|
| 1/2 | -1    | 3   | $10^{-7}$       | 0.9730      | 0.9730      | 0.9830 |
| 1/2 | -1    | 9   | $10^{-6}$       | 0.9970      | 1.0000      | 0.9996 |
| 1/2 | 0     | 3   | $10^{-6}$       | 0.8477      | 0.8480      | 0.8465 |
| 1/2 | 0     | 6   | $10^{-6}$       | 0.9234      | 0.9253      | 0.9200 |
| 1/2 | 0     | 9   | $10^{-7}$       | 0.9384      | 0.9388      | 0.9460 |
| 1/2 | 1     | 3   | $10^{-7}$       | 0.5906      | 0.5912      | 0.5908 |
| 1/2 | 1     | 6   | $10^{-7}$       | 0.5950      | 0.5962      | 0.6058 |
| 1/2 | 1     | 9   | $10^{-7}$       | 0.5945      | 0.5958      | 0.6070 |

**Table 1.**

For $p = 3/4$, where no theoretical formulae are available, we give results of simulations for a few values of parameters.

| $p$ | $X_0$ | $\mu$ | $T$ | $\varepsilon_n$ | lower bound | upper bound | estimate |
|-----|-------|-------|-----|-----------------|-------------|-------------|----------|
| 3/4 | 1/4   | 1     | 9   | $10^{-9}$       | 0.3838      | 0.3864      | 0.3864 - 0.3864 |
| 3/4 | 1     | 1     | 9   | $10^{-9}$       | 0.0782      | 0.0790      | 0.0782 - 0.0790 |
| 3/4 | 1/4   | -1    | 3   | $10^{-9}$       | 0.8757      | 0.8803      | 0.8757 - 0.8803 |

**Table 2.**

A heuristic explanation why a larger $p$ leads to smaller ruin probabilities follows from the fact that the diffusion parameter $x^p$ for $x \in (0,1)$ decreases in $p$. The same argument clarifies why smaller $\varepsilon_n$ is needed to achieve the desired accuracy for larger $p$.

5. **Existence and Uniqueness of solution in the CEV model.** The proposition below is given only for reader convenience (see Deelstra and Delbaen [7] for more details).

**Proposition 1.** The Itô’s equation (1) possesses a unique strong nonnegative solution.

**Proof.** Define a sequence of processes indexed by integers $i, i > 1/X_0$, $(X_i^i)_{t \geq 0}$, such that $X_i^i$ is a strong solutions to the following stochastic differential equation

$$dX_i^i = \mu X_i^i dt + \sigma (i^{-1} \vee |X_i^i|)^p dW_t, \quad X_0^i = X_0.$$  \hspace{1cm} (14)
The diffusion coefficient $\sigma(i^{-1} \vee |x|)^p$ is Lipschitz continuous therefore $X_i^t$ is the unique strong solution of (14). Set $\vartheta_{i} = \inf\{t : X_{i}^{t} = i^{-1}\}$. Note that for $t \leq \vartheta_{i}$

$$X_{i}^{t+1} = X_{i}^{t},$$

and it follows that $\vartheta_{i+1} > \vartheta_{i}$. A strong solution to (1) is constructed by a natural prolongation

$$X_{t}^{\tau} := \sum_{i \geq n} X_{i}^{\vartheta_{i}} 1_{\{\vartheta_{i} \leq t < \vartheta_{i+1}\}}, \ \tau = \lim_{i \to \infty} \vartheta_{i}.$$

Finally, Yamada-Watanabe’s theorem (see, e.g. [23], p. 265) guarantees the uniqueness of the strong solution of equation (1), because the H"older parameter $p \geq \frac{1}{2}$.

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