COHERENT STATES AS KÄHLER EMBEDDINGS IN $\mathbb{C}P^\infty$

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Abstract. In this semi-expository paper we study two examples of coherent states based on the Weyl-Heisenberg group and the group of $2 \times 2$ upper triangular matrices. It is known that sometimes the coherent states provide us with a Kähler embedding of a coadjoint orbit into the projective Hilbert space $\mathbb{C}P^n$ or $\mathbb{C}P^\infty$. We show an explicit computation of this in the above two examples. We also note the presence of other coadjoint orbits which only embed symplectically into the projective Hilbert space. These correspond to "squeezed states", which have several applications in physics. Our exposition includes a detailed study of the geometric quantisation of the coadjoint orbits of the Lie Algebra of upper triangular matrices. This reveals the presence of distinguished orbits which correspond to coherent states, as well as others corresponding to squeezed states. The coadjoint orbit of SUT we consider is intimately connected to the 2-dimensional Toda system.

Keywords: Coherent states, Squeezed States, Coadjoint orbits, Toda system

1. Introduction

Coherent states were first studied by Schrödinger in an effort to construct quantum states with classical-like behaviour [17, 6, 19, 7, 8, 9]. He searched for states which saturate the inequality expressed by the Heisenberg uncertainty principle

$$\Delta q \Delta p \geq \hbar / 2.$$ 

These states are called minimum uncertainty states. By minimising the quantum uncertainty, one finds quantum states that are the closest one can get to classical states. States of a classical system are described as points in phase space, which is a symplectic manifold. Coherent states can be thought of as localised around these points. They have equal uncertainty in position and momentum.

However, requiring minimum uncertainty is an incomplete characterisation of coherent states. There exist other states which also saturate the Heisenberg bound. These states are called squeezed states [16] and they play an important role in physics. Squeezed states have a larger uncertainty in one variable (either position or momentum) and a smaller uncertainty in the conjugate variable. From the viewpoint of minimising uncertainty, squeezed states are just as good as coherent states.

Squeezed states are extremely important in quantum optics and quantum metrology. Squeezed states of light are used, for example, in the LIGO detector to give an accurate determination of position, while sacrificing accuracy in the conjugate variable, momentum. Although, both saturate the uncertainty bound, squeezed states are physically quite different from coherent states.

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1Not only do squeezed and coherent states saturate the Heisenberg uncertainty relation, but in fact, they both saturate a slightly stronger version, the Robertson-Schrödinger inequality[15 15].
While coherent states have been studied mathematically from advanced points of view, using co-adjoint orbits of Lie Groups, squeezed states have not received the same attention in the mathematical literature. Our objective here is to place squeezed states in their proper place in the mathematical theory of coherent states. We will show below in two pertinent examples that coherent states give us a Kähler embedding of the classical phase space into the ray space of quantum mechanics (as is well known); while squeezed states give us only a symplectic embedding. This property sets apart coherent states from the other minimum uncertainty states.

Coherent and squeezed states pulled back from projective spaces have been considered in [4]. In this article authors define squeezed states mathematically, in analogy with coherent states.

In this semi-expository paper we study two examples of coherent states based on the Weyl-Heisenberg group and the group of $2 \times 2$ upper triangular matrices. In both cases it is known that the coherent states provide us with a Kähler embedding of a coadjoint orbit into the projective Hilbert space $\mathbb{C}P^\infty$, [5,11,13]. We show an explicit computation of this.

One of the examples dealt with (namely the coadjoint orbit of $\text{SUT}^+$) is connected to the 2-dimensional Toda system. In [11] Adler showed that a finite $n$-dimensional Toda system has a coadjoint orbit description of the group of lower triangular matrices of non-zero diagonal. In fact, one can restrict the action to that of lower triangular matrices of determinant 1 and positive diagonal elements. The orbit is homeomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, just described by $a_i > 0$, $i = 1, \ldots, n - 1$ and $b_i$, $i = 1, \ldots, n$ such that $b_1 + b_2 + \ldots + b_n = c$, $c$ a constant. In this paper we deal with the case $n = 2$. In [3], the authors geometrically quantized this system and studied the coherent states. But the viewpoint is different in this paper.

The paper is structured as follows: In section 2, we introduce squeezed and coherent states and describe how Schrödinger coherent states emerge from geometric quantisation of co-adjoint orbits of the Weyl-Heisenberg group. In section 3, we define the ray space of quantum mechanics and describe the Kähler structure of the ray space. In section 4 we show that the Schrödinger coherent states give us a Kähler embedding of the complex plane into the ray space. and we show a similar embedding of the unit disc. In section 5, we study the coadjoint orbits of the group of upper triangular matrices and note the presence of two kinds of orbits. The phase space is the right half plane. Section 6 shows that the two kinds of orbits yield coherent and squeezed states on quantisation. In section 7 we have an appendix that describes the Berezin quantisation of the upper half plane which is already known for the unit disc.

2. Squeezed and Coherent States

**Schrödinger coherent states:** Schrödinger coherent states are defined solutions to the eigenvalue equation

\[ (\hat{q} + i\hat{p}) |\alpha\rangle = \alpha |\alpha\rangle \]

where the eigenvalue $\alpha$ is a complex number. These states saturate the uncertainty bound eq.(1). The eigenvalue equation is easily solved to give an expression for
coherent states in terms of the oscillator eigenstates $|n\rangle$.

\begin{equation}
|\alpha\rangle = \left[\exp -|\alpha|^2/2\right] \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
\end{equation}

$|n\rangle$ are orthonormal eigenstates of the Hermitian operator $\hat{H} = (\hat{q}^2 + \hat{p}^2)/2$.

**Squeezed states:** As mentioned above, coherent states are not the only states which saturate the uncertainty inequality. Squeezed states, defined by

\begin{equation}
\left(\hat{q}\lambda + i\frac{\hat{p}}{\lambda}\right) |\alpha\rangle = \alpha |\alpha\rangle,
\end{equation}

also share this property. Squeezed states are anisotropic in phase space being “squeezed” in one direction and expanded in the other. This squeezing operation in phase space preserves the symplectic structure but alters the complex structure of the plane.

Mathematically, Schrödinger coherent states emerge naturally from the Weyl-Heisenberg group $\mathcal{W}$, which is the exponential of the nilpotent W-H Lie algebra, generated by $e_1, e_2, e_3$ with the only nonzero commutation relations being

\begin{equation}
[e_1, e_2] = e_3.
\end{equation}

The mathematical theory of coherent states, starts with an irreducible, unitary representation of the W-H group in a Hilbert space $[14]$. Taking a fiducial vector $|\psi_0\rangle$ satisfying

\begin{equation}
(\hat{q} + i\hat{p}) |\alpha\rangle = 0,
\end{equation}

we have

\begin{equation}
|\alpha\rangle = D(\alpha) |\psi_0\rangle
\end{equation}

where the complex number $\alpha$ determines $D(\alpha) = \exp [i\hat{q}\alpha_1 + i\hat{p}\alpha_2]$, and the group commutation relations ensure that (2) $[14]$ is satisfied. The complex number $\alpha$ (or equivalently the two real numbers $(\alpha_1, \alpha_2)$, where $\alpha = \alpha_1 + i\alpha_2$), parametrises the points of a two dimensional co-adjoint orbit of $\mathcal{W}$. The co-adjoint orbit $\Gamma$ has a natural symplectic structure from the KKS construction. $\Gamma$ is in fact the classical phase space of the Harmonic oscillator, identified with the complex plane. $\Gamma$ admits a Kähler structure, with the symplectic, Riemannian and complex structures coexisting compatibly.

### 3. Ray Space as a Kähler Manifold

The states of a quantum system are described projectively as rays in a Hilbert space $\mathcal{H}$. Let us consider the space of normalized states $\mathcal{N} = \{\psi \in \mathcal{H} : \|\psi\|^2 = 1\}$. We can regard the Ray space as elements of $\mathcal{N}$, modulo a overall phase

\begin{equation}
\mathcal{R} = \mathcal{N}/\sim
\end{equation}

where $|\psi\rangle \sim |\psi'\rangle$ if $|\psi\rangle = \exp i\gamma |\psi'\rangle$. An equivalent definition of $\mathcal{R}$ is to view it as the space of one dimensional projections on $\mathcal{H}$. These can be written as $\rho = |\psi\rangle\langle\psi|$. $\rho$ is Hermitean, $\rho^\dagger = \rho$, a projection operator $\rho^2 = \rho$ and normalised $\text{Tr}\rho = 1$. In finite dimensional quantum systems such as occur physically in spin systems, the
ray space is $CP^n$. In the systems we deal with in this paper, the classical phase space is non-compact and has infinite symplectic volume. As a result the Hilbert space $\mathcal{H}$ as well as the ray space $\mathcal{R}$ are infinite dimensional. We can describe the ray space as $\mathbb{C}P^n$ or $\mathbb{C}P^\infty$ depending on whether the dimension is finite or infinite.

The ray space $\mathcal{R}$ is naturally endowed with a metrical structure, (with distances $\delta$ determined by shortest geodesics of the Fubini-Study metric)

\[ \|\langle \psi_1 | \psi_2 \rangle\| = \cos \delta/2 \]

where $\delta$ ranges from 0 to $\pi$ and gives the distance between rays. $\delta$ is directly measureable in a laboratory as a transition probability.

The ray space also has two more structures inherited from $\mathcal{H}$: a symplectic structure and a complex structure. The symplectic structure on the ray space $\mathcal{R}$ is described by a closed, non-degenerate two form, the curvature of the universal $U(1)$ connection on the bundle: $\mathcal{N} \rightarrow \mathcal{R}$. The symplectic structure on $\mathcal{R}$ has been interpreted as a geometric phase [20], which also is amenable to experiments. The complex structure is less evident in laboratory terms but follows mathematically from the complex structure on $\mathcal{H}$. Physically, the complex structure is crucial to a discussion of time reversal symmetry, which acts by conjugation on the complex structure. The three structures - metrical, symplectic and complex- give us a Kähler structure on the ray space.

The equation (3) defining a coherent state gives us an embedding of the classical phase space into the ray space $\mathcal{R}$. Each complex number $\alpha \in \Gamma$, determines an unique nonzero element $|\alpha\rangle \in \mathcal{H}$, which projects down to the ray space $|\alpha\rangle$. (The square brackets refer to the equivalence class of $|\alpha\rangle$. This can also be expressed as a projection operator $\rho = |\alpha\rangle \langle \alpha|$)

\[ \phi : \Gamma \rightarrow \mathcal{R} \]
\[ \phi(\alpha) = \rho = |\alpha\rangle \langle \alpha| . \]

It is known that sometimes the coherent states provide us with a Kähler embedding of a coadjoint orbit into the projective Hilbert space $\mathbb{C}P^n$ or $\mathbb{C}P^\infty$, [5], [11, ?] [13]. In this paper, we investigate the nature of this embedding for coherent and squeezed states in two examples.

We have seen that the classical phase space $\Gamma$ of the oscillator is a Kähler manifold. As we remarked above, the Ray Space $\mathcal{R}$ has a Kähler structure which gives mutually compatible symplectic, complex and metrical structures. Given the embedding (10) it is reasonable to ask if the pullback of the structures on $\mathcal{R}$ agrees with the corresponding structure on $\Gamma$. We first discuss this question for the Schrödinger coherent states and then consider the more abstract case of Berezin quantisation of the upper half plane.

Consider two tangent vectors $\dot{\alpha}$ and $\alpha'$ on the complex plane $\Gamma$. The symplectic and metrical structures on the $\alpha$ plane are given respectively by

\[ \omega(\alpha', \dot{\alpha}) = \text{Im}(\alpha'^* \dot{\alpha}) \]

and

\[ g(\alpha', \dot{\alpha}) = \text{Re}(\alpha'^* \dot{\alpha}) \]
We can evaluate the push-forward of $\alpha'$ and $\dot{\alpha}$ to the Ray space and use the definition (3) of $|\alpha\rangle$, the metric and symplectic structure on $\mathcal{R}$, to find the real and imaginary parts of

\[(13) \quad \langle \alpha' | P | \dot{\alpha} \rangle \]

(where $P$ is the projector $1 - |\alpha\rangle \langle \alpha|}$ orthogonal to the rays of $\mathcal{H}$)

A calculation given in detail in section 4 with (3) shows that this works out to the symplectic (11) and metrical structure (12) on $\Gamma$. Thus Schrödinger coherent states have the property that the pull back of the Kähler structure on $\mathcal{R}$ to $\Gamma$ agrees with naturally occurring structures on $\Gamma$: $\phi$ gives us a Kähler embedding of $\Gamma$ into $\mathcal{R}$.

This property distinguishes coherent states from squeezed states. Squeezed states are got by composing the map (10) with the transformation $z \rightarrow \lambda(z + \bar{z})/2 + \lambda^{-1}(z - \bar{z})/2$. This transformation is not complex analytic for $\lambda \neq 1$ and does not respect the complex structure of $\Gamma$. However it does preserve the symplectic form. Squeezed states only give us a symplectic embedding but not a Kähler embedding. To see whether such a feature holds in general, we consider in detail another nilpotent Lie Algebra: the algebra of the group $G$ of nonsingular Upper triangular matrices in two real dimensions. This is done in a later section.

4. Kähler embedding of the classical phase space into $\mathbb{C}P^\infty$

In this section we explicitly show in two examples that sometimes the coherent states provide us with a Kähler embedding of a coadjoint orbit into the projective Hilbert space $\mathbb{C}P^\infty$, which illustrates a general result in [5, 11, 13].

4.1. Kähler embedding of the classical phase space: Schrödinger Coherent states. The coherent state on the complex plane is given by:

\[|\alpha\rangle = \exp \left( -\frac{\|\alpha\|^2}{2} \right) \sum_0^\infty \frac{\alpha^n}{\sqrt{n!}} |n\rangle.\]

Let $\alpha(s)$ be a curve in the complex plane. We have the following tangent vector on $\mathbb{C}$:

\[\alpha' = \frac{d\alpha}{ds}.\]

This defines a curve $|\alpha(s)\rangle \in \mathcal{N}$. We can project this down to the ray space to obtain a curve $[|\alpha(s)\rangle] \in \mathcal{R}$.

The push forward of $\alpha'$ to $\mathcal{N}$ is $\frac{d}{ds}|\alpha(s)\rangle$. Composing with $\pi : \mathcal{H} - \{0\} \rightarrow \mathcal{R}$

we have

\[\left( \frac{d}{ds}|\alpha(s)\rangle \right)^\perp = P \left( \frac{d}{ds}|\alpha(s)\rangle \right)\]
where \( P = \mathbb{1} - |\alpha(s)\rangle\langle \alpha(s) | \) is the projector perpendicular to \( |\alpha\rangle \).

\[
\frac{d}{ds} |\alpha(s)\rangle = \frac{d}{ds} \left( e^{-i\alpha^2 \frac{s}{2}} \sum_0^\infty \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right) = \frac{d}{ds} \left( e^{-i\alpha^2 \frac{s}{2}} |\alpha\rangle + e^{-i\alpha^2 \frac{s}{2}} \alpha' \sum_0^\infty \frac{n\alpha^{n-1}}{\sqrt{n!}} |n\rangle \right).
\]

On projection the first term, vanishes. Letting \( |b\rangle = \frac{d}{ds} |\alpha(s)\rangle \), we have

\[
(14) \quad P |b\rangle = P \left( \frac{d}{ds} |\alpha(s)\rangle \right) = P \left( e^{-i\alpha^2 \frac{s}{2}} \alpha' \sum_0^\infty \frac{n\alpha^{n-1}}{\sqrt{n!}} |n\rangle \right).
\]

Similarly, we set \( |b\rangle = \frac{d}{dt} |\alpha(t)\rangle \), to describe another tangent vector in the \( \alpha \) plane. This leads to a similar expression as (14) with \( \alpha' \) replaced by \( \dot{\alpha} \). We need to calculate:

\[
\langle a | P |b\rangle = \langle a | (\mathbb{1} - |\alpha\rangle\langle \alpha |)|b\rangle = \langle a |b\rangle - \langle a |\alpha\rangle\langle \alpha |b\rangle.
\]

Carrying out these calculations we obtain:

\[
\langle a |b\rangle = \overline{\alpha' \dot{\alpha} e^{-|\alpha|^2}} \sum_{m,n=0}^\infty \frac{mn\alpha^{m-1} \alpha^{n-1}}{\sqrt{m!}!} \langle m |n\rangle
\]

\[
= \overline{\alpha' \dot{\alpha} e^{-|\alpha|^2}} \sum_{n=0}^\infty \frac{n^2(\overline{\alpha})^{n-1}}{n!} = \overline{\alpha' \dot{\alpha}} \left[ \frac{1}{\alpha} |\alpha|^2 + e|\alpha|^2 \right]
\]

\[
= \overline{\alpha' \dot{\alpha}} (1 + \overline{\alpha} \alpha).
\]

\[
\langle a |b\rangle = \dot{\alpha} e^{-|\alpha|^2} \sum_{m,n=0}^\infty \frac{\overline{\alpha} \alpha^{n}}{\sqrt{m!}!} \langle m |n\rangle
\]

\[
= \dot{\alpha} e^{-|\alpha|^2} \sum_{n=0}^\infty \frac{n\alpha^{n-1}}{n!} = \dot{\alpha} \overline{\alpha'}.
\]

Similarly, \( \langle a |\alpha\rangle = \overline{\alpha' \dot{\alpha}} \).

Substituting these values in the equation above we obtain:

\[
\langle a | P |b\rangle = \overline{\alpha' \dot{\alpha}}
\]

which exactly agrees with the Hermitean metric on \( \mathbb{C} \).

**4.2. Kähler embedding of the unit disc.** This following calculation repeats for the upper half plane the calculation we did for Schrödinger coherent states. The argument is very similar. It is actually more convenient to work on the unit disc. The expression for coherent states are taken from section 2 of [2]. There is a parameter \( k \) which is continuous and \( k > \frac{1}{2} \).
Let us start with a unit hyperboloid in $2 + 1$ dimensions. This space is isomorphic to the unit disc via the stereographic projection, which in turn is biholomorphic to the upper half plane. Following [2], we denote:

$$\xi = -\frac{\tau}{2} e^{-i\phi},$$
$$\alpha = -\tanh \left( \frac{\tau}{2} \right) e^{-i\phi}.$$

The coherent states are given as:

$$|\alpha, k\rangle = (1 - \overline{\alpha}\alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} \right]^{\frac{1}{2}} \alpha^n |n, k\rangle$$

where $\langle m, k | n, k\rangle = \delta_{mn}$. $k$ is fixed from this point on.

Consider a curve $\alpha(s)$ in the unit disc. The tangent vector is given by:

$$\alpha'(s) = \frac{d}{ds} \alpha(s).$$

The corresponding tangent vector in $\mathcal{N}$ is the pushforward,

$$\frac{d}{ds} |\alpha(s), k\rangle = u |\alpha(s), k\rangle + (1 - \overline{\alpha}\alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} \right]^{\frac{1}{2}} n\alpha^{n-1} |n, k\rangle \alpha'.$$

Now, we have the projection operator, $P = 1 - |\alpha\rangle \langle \alpha|$. Denote:

$$|b\rangle = \frac{d}{ds} |\alpha(s), k\rangle$$
$$= \left\{ (1 - \overline{\alpha}\alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} \right]^{\frac{1}{2}} n\alpha^{n-1} |n, k\rangle \alpha' \right\}.$$

$$|a\rangle = \frac{d}{dt} |\alpha(t), k\rangle$$
$$= \left\{ (1 - \overline{\alpha}\alpha)^k \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} \right]^{\frac{1}{2}} n\alpha^{n-1} |n, k\rangle \dot{\alpha} \right\}.$$

We need to compute

$$\langle a|P|b\rangle = \langle a|b\rangle - \langle a|\alpha\rangle \langle \alpha|b\rangle.$$

The terms are calculated using binomial expansion.

$$\frac{1}{(1 - x)^{2k}} = \sum_{n=0}^{\infty} \left( \frac{2k}{n+1} \right) x^n = \sum_{n=0}^{\infty} \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} x^n.$$

Now,

$$\langle a|b\rangle = (1 - \overline{\alpha}\alpha)^{2k} \overline{\alpha}\dot{\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2k)}{n!\Gamma(2k)} n^2 (\overline{\alpha}\alpha)^{n-1}$$

$$= \frac{2k}{(1 - \overline{\alpha}\alpha)^2} \overline{\alpha}\dot{\alpha} (1 + 2k\overline{\alpha}\alpha).$$
\[ \langle a \vert \alpha \rangle = (1 - \overline{\alpha} \alpha)^{2k - \alpha' \alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2k)}{n! \Gamma(2k)} n(\overline{\alpha} \alpha)^{n-1} \]
\[ = \frac{2k}{1 - \overline{\alpha} \alpha} \alpha' \alpha. \]

\[ \langle a \vert b \rangle = \frac{2k}{1 - \overline{\alpha} \alpha} \overline{\alpha' \alpha}. \]

Putting it all together, we calculate:

\[ \langle a \vert b \rangle - \langle a \vert \alpha \rangle \langle \alpha \vert b \rangle = \overline{\alpha' \alpha} \left[ \frac{2k}{(1 - \overline{\alpha} \alpha)^2} (1 + 2k \overline{\alpha} \alpha) - \frac{(2k)^2 \overline{\alpha} \alpha}{(1 - \overline{\alpha} \alpha)^2} \right] \]
\[ = \frac{2k \overline{\alpha} \alpha' \alpha}{(1 - \overline{\alpha} \alpha)^2} \]

which is the Kähler metric on the unit disc apart from a constant depending on \( k \).

5. An Example: Upper Triangular Matrices

In this section we study the geometric quantisation of the coadjoint orbits of the group of \( 2 \times 2 \) upper triangular matrices with a different point of view from [3]. In the process we give elementary proofs of some known results in [1]. Let \( \text{SUT}(2, \mathbb{R}) \) denote the Lie group of \( 2 \times 2 \) upper triangular matrices over the reals with determinant 1.

\[ \text{SUT}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} : a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R} \right\}. \]

\( \text{SUT}(2, \mathbb{R}) \) is not a connected group. It has two disjoint components, matrices of positive diagonal and negative diagonal elements respectively. Let \( \text{sut} \) denoted the Lie algebra and \( \text{sut}^* \) denote it’s dual. Then

\[ \text{sut} = \left\{ \begin{pmatrix} u & v \\ 0 & -u \end{pmatrix} : u, v \in \mathbb{R} \right\}, \]
\[ \text{sut}^* = \left\{ \begin{pmatrix} u & 0 \\ v & -u \end{pmatrix} : u, v \in \mathbb{R} \right\}. \]

The group \( \text{SUT}(2, \mathbb{R}) \) acts on its Lie algebra \( \text{sut} \) by the adjoint action. This induces an action on the dual space, \( \text{sut}^* \), called the co-adjoint action. Take the following basis for \( \text{sut} \), \( \mathcal{B} = \{E_1, E_2\} \)

\[ E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]
\[ E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
The corresponding dual basis of $\text{sut}^*$ is given by $\mathcal{B}^* = \{E_1^*, E_2^*\}$

$$E_1^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$E_2^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

Let $g = \begin{bmatrix} g_1 & g_2 \\ 0 & 1 \end{bmatrix}$ be an element in $\text{SUT}(2, \mathbb{R})$ and $X = \begin{bmatrix} u_0 & 0 \\ v_0 & -u_0 \end{bmatrix}$ be an element of $\text{sut}^*$.

**Lemma 1.** The co-adjoint action of $g \in \text{SUT}(2, \mathbb{R})$ on $\text{sut}^*$, denoted by $\text{Ad}_g^*$, with respect to the basis $\{E_1^*, E_2^*\}$ is given by the following matrix:

$$\text{Ad}_g^* : \text{sut}^* \rightarrow \text{sut}^*$$

$$\text{Ad}_g^* = \begin{bmatrix} g_1^{-2} & 0 \\ -g_2 g_1^{-1} & 1 \end{bmatrix}.$$  

**Lemma 2.** The orbit of the co-adjoint action on $X \in \mathfrak{g}^*$, denoted by $O_X$ is given as follows.

$$O_X = \{\text{Ad}_g^* X : g \in \text{SUT}(2, \mathbb{R})\}$$

$$= \left\{ \begin{bmatrix} u_0 - \lambda v_0 & 0 \\ \mu v_0 & -u_0 + \lambda v_0 \end{bmatrix} : \lambda, \mu \in \mathbb{R}, \mu > 0 \right\}.$$ 

$O_X$ is a 2-plane when $v_0 \neq 0$ and a point when $v_0 = 0$.

From now on, we take $v_0 \neq 0$.

Fix a point $A = \begin{bmatrix} a_1 & 0 \\ a_2 & -a_1 \end{bmatrix}$ in $O_X$. $O_X$ is a plane and any point on this plane is given as $\begin{bmatrix} s & 0 \\ t & -s \end{bmatrix}$. This is diffeomorphic to $\mathbb{R}^2$ with global charts given as:

$$\begin{bmatrix} s & 0 \\ t & -s \end{bmatrix} \mapsto (s, t).$$

The tangent space at $A$ is denoted by $T_A O_X$. With the chart above, elements of the tangent space $T_A O_X$ are given as $\mu_1 \frac{\partial}{\partial s}|_A + \mu_2 \frac{\partial}{\partial t}|_A$.

**Lemma 3.** Every element of $\xi \in T_A O_X$ has a representation given as $\xi = \text{ad}_V^* A$ (where $V \in \text{sut}$ and $\text{ad}_V^*$ is the differential of $\text{Ad}^*$ at $V$)

**Proof.**

$$\text{Ad}^* : \text{SUT}(2, \mathbb{R}) \rightarrow \text{End}(\text{sut}^*).$$

$$d(\text{Ad}^*) = \text{ad}^* : \text{sut} \rightarrow \text{End}(\text{sut}^*).$$

Denote, $\text{ad}^*(V) = \text{ad}_V^*$. Now take $\mu \in \text{sut}^*$ and $Y \in \text{SUT}(2, \mathbb{R})$.

$$\langle \text{ad}_V^* \mu, Y \rangle = \langle \mu, -\text{ad}_V(Y) \rangle$$

$$= -\langle \mu, [V, Y] \rangle.$$
Consider the following curve:

\[ \gamma : \mathbb{R} \to \mathcal{O}_X \]
\[ t \mapsto \text{Ad}^*_t(V) \].

\[ \gamma(0) = \text{Ad}^*_{0,V}(A) = \text{Ad}^*_0(A) = A \]
\[ \gamma'(0) = \text{ad}^*_V A. \]

Since \( \gamma \) is a smooth curve in \( \mathcal{O}_X \) with \( \gamma(0) = A, \gamma'(0) \in T_A \mathcal{O}_X \), we deduce that \( \text{ad}^*_V A \in T_A \mathcal{O}_X \). Conversely, given \( Y \in T_A \mathcal{O}_X \), there exists \( V \in \mathfrak{g} \) s.t \( \text{ad}^*_V(A) = Y \).

We make the identification more precise. Let \( V = \begin{bmatrix} v_1 & v_2 \\ 0 & -v_1 \end{bmatrix} \).

\[ \langle \text{ad}^*_V(A), E_i \rangle = -\langle A, [V, E_i] \rangle = -\text{tr}(A[V, E_i]). \]

\[ \text{ad}^*_V(A) = \sum_{i=1}^2 \langle \text{ad}^*_V(A), E_i \rangle E_i^* \]
\[ = \begin{bmatrix} 2a_2v_2 & 0 \\ -2a_2v_1 & -2a_2v_2 \end{bmatrix}. \]

Hence the identification of the two definition of tangent spaces is given as:

\[ \begin{bmatrix} 2a_2v_2 & 0 \\ -2a_2v_1 & -2a_2v_2 \end{bmatrix} \mapsto 2a_2v_2 \frac{\partial}{\partial s} \bigg|_A + 2a_2v_1 \frac{\partial}{\partial t} \bigg|_A. \]

\( \square \)

**Lemma 4.** Let \( \xi_1, \xi_2 \in T_A \mathcal{O}_X \). Then from lemma 2 we know we can write,

\[ \xi_1 = \text{ad}^*_{V_1} A, \]
\[ \xi_2 = \text{ad}^*_{V_2} A \]

where \( V_1, V_2 \in \mathfrak{g} \). The Kirillov-Kostant-Souriau symplectic form, \( \omega \in \Lambda^2(\mathcal{O}_X) \) is defined as follows:

\[ \omega_A(\xi_1, \xi_2) : = \omega_A(\text{ad}^*_{V_1} A, \text{ad}^*_{V_2} A) \]
\[ = \langle A, [V_1, V_2] \rangle \]
\[ = A([V_1, V_2]). \]

Then we have the following well known theorem (see [1], for instance) for which we give a simple proof.

**Theorem 5.1.** \( \mathcal{O}_X \) is a symplectic manifold. For \( v_0 \neq 0, \mathcal{O}_X \) has a natural structure of a symplectic manifold given by the Kirillov-Kostant-Souriau symplectic form, \( \omega \).
Proof. Since $\omega$ is a symplectic form on $O_X$, $\omega_A \in \Lambda^2(T_A O_X)$.
Now using the definition of tangent space, since the manifold is two dimensional,
$$T_A(O_X) = \{(A, \mu_1, \mu_2) : \mu_1, \mu_2 \in \mathbb{R} \}$$
$$= \{\text{ad}^*_V(A) : V \in \text{sut}\}$$

Take two tangent vectors $\mu, \xi$.
$$\mu = \mu_1 \frac{\partial}{\partial s}|_A + \mu_2 \frac{\partial}{\partial t}|_A,$$
$$\xi = \xi_1 \frac{\partial}{\partial s}|_A + \xi_2 \frac{\partial}{\partial t}|_A.$$

We calculate $X_\mu, X_\xi$ such that
$$\mu = \text{ad}^*_{X_\mu}(A) = \begin{bmatrix} \mu_1 & 0 \\ -\mu_2 & -\mu_1 \end{bmatrix},$$
$$\xi = \text{ad}^*_{X_\xi}(A) = \begin{bmatrix} \xi_1 & 0 \\ -\xi_2 & -\xi_1 \end{bmatrix}.$$ 

Using the identification given above,
$$X_\mu = \begin{bmatrix} \frac{\mu_2}{2a_2} & \frac{\mu_1}{2a_2} \\ 0 & -\frac{\mu_2}{2a_2} \end{bmatrix},$$
$$X_\xi = \begin{bmatrix} \frac{\xi_2}{2a_2} & \frac{\xi_1}{2a_2} \\ 0 & -\frac{\xi_2}{2a_2} \end{bmatrix}.$$ 

Now we calculate the symplectic form:
$$\omega_A(\mu, \xi) = \omega_A(\text{ad}^*_{X_\mu}(A), \text{ad}^*_{X_\xi}(A))$$
$$= \langle A, [X_\mu, X_\xi] \rangle$$
$$= -\text{tr}(A[X_\mu, X_\xi])$$
$$= -\frac{1}{2a_2}(\xi_1\mu_2 - \mu_1\xi_2).$$ 

Hence,
$$\omega_A = \frac{1}{a_2} ds|_A \wedge dt|_A,$$
$$\omega = \frac{1}{t} ds \wedge dt.$$ 

Now, let $P = \begin{bmatrix} s & 0 \\ t & -s \end{bmatrix} \in O_X$. Define the functions $J_1, J_2$ on $O_X$ as follows:
$$J_i = \text{tr}(P E_i).$$
Lemma 5. The Hamiltonian vector fields corresponding to $J_i$’s are denoted by $X_{J_i}$, satisfying, $\omega(X_{J_i}, -) = dJ_i(-)$ are

$$X_{J_1} = 2t \frac{\partial}{\partial s},$$

$$X_{J_2} = -4t \frac{\partial}{\partial t}.$$
We then have direct proofs of the following (known) results in the next two propositions. The results can be found for instance in [1].

**Proposition 5.3.** $SUT^+$ is a symplectic manifold.

**Proof.** Using the identification of $SUT^+$ with $O_X$ (which depends on the vector $X$), we can pull back the symplectic form on $O_X$ to $SUT^+$. This identification is explicitly given below:

$$
\phi : O_X \rightarrow SUT^+
$$

$$
\begin{pmatrix}
 s & 0 \\
 t & -s
\end{pmatrix} \mapsto \begin{pmatrix}
 \sqrt{\frac{s}{t}} & \frac{u_0-s}{\sqrt{vu_0}} \\
 0 & \sqrt{\frac{t}{vu_0}}
\end{pmatrix}.
$$

$$
\phi' := \phi^{-1} : SUT^+ \rightarrow O_X
$$

$$
\begin{pmatrix}
 a & b \\
 0 & \frac{1}{a}
\end{pmatrix} \mapsto \begin{pmatrix}
 u_0 - \frac{ba_0}{a^2} & 0 \\
 \frac{vu_0}{a^2} & -u_0 + \frac{bu_0}{a}
\end{pmatrix}.
$$

Then calculating the pull-back,

$$
(\phi')^* \omega = (\phi')^* \left( -\frac{dv}{v} \wedge du \right)
$$

$$
= -\frac{2v_0}{a^2} da \wedge db
$$

$$
= 2v_0 d\left( \frac{1}{a} \right) \wedge db.
$$

\[\square\]

**Remark 1.** As such this form seems to dependent on $v_0$ and hence on the vector $X$ (whose orbit we are working with). However we can get rid of this dependence by looking at another global chart, $F_2$, for $SUT^+$ which is compatible with the original chart.

$$
F_2 : SUT^+ \rightarrow \mathbb{R}^{>0} \times \mathbb{R}
$$

$$
\begin{pmatrix}
 a & b \\
 0 & \frac{1}{a}
\end{pmatrix} \mapsto (a, bv_0) \equiv (a, y).
$$

On this chart, $\omega = 2d\left( \frac{1}{a} \right) \wedge dy$.

When we think of $SUT^+$ as a differential manifold, we start with only the chart $(SUT^+, F_1)$, but this chart is actually a part of the maximal atlas, which gives the differentiable structure on $SUT^+$ and $(SUT^+, F_2)$ is a part of this maximal atlas. Even though the pull-back $(\phi')^* \omega$ seemed to depend on $v_0$ on the chart $(SUT^+, F_1)$, this was just due to a bad choice of coordinates. On $(SUT^+, F_2)$, we see that the symplectic form has a identification independent of $v_0$.

Now let us try to analyse this in the complex case. Again we have a global chart, $(SUT^+, G_1)$, which maps onto $U = \{ z \in \mathbb{C} : Re(z) > 0 \}$

$$
G_1 : SUT^+ \rightarrow U
$$
\[
\begin{pmatrix}
a & b \\
0 & \frac{1}{a}
\end{pmatrix} \mapsto a + ib \equiv z.
\]

On \((\text{SUT}^+, G_1)\), the symplectic form can be written as \(\frac{v_0}{2i\text{Re}(z)}dz \wedge d\bar{z}\). We see that it depends on \(v_0\).

Let us take another suitable chart, \((\text{SUT}^+, G_2)\).

\[
G_2 : \text{SUT}^+ \to U
\]
\[
\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \mapsto a + ibv_0 \equiv z'.
\]

On this chart, the symplectic form has the representation, \(\frac{1}{2i\text{Re}(z')}dz' \wedge d\bar{z}'\). Hence we get a form independent of \(v_0\). However the two charts, \((\text{SUT}^+, G_1)\) and \((\text{SUT}^+, G_2)\) are not compatible as complex charts. Consider the transition functions:

\[
G_2 \circ G_1^{-1} : U \to U
\]
\[
G_2 \circ G_1^{-1}(z) = z \left( \frac{1 + v_0}{2} \right) + \bar{z} \left( \frac{1 - v_0}{2} \right).
\]

Clearly this map is not biholomorphism unless \(v_0 = 1\). The orbit containing \(v_0 = 1\) is thus a distinguished orbit in regard to its complex structure. Hence \(M_1 = (\text{SUT}^+, G_1)\) and \(M_2 = (\text{SUT}^+, G_2)\) are two distinct complex manifolds. The pull-back is independent of \(M_2\), but notice that \(M_2\) is defined by it’s chart \(G_2\), which depends on \(v_0\). Hence in the complex case, we can say that there exists a complex structure on \(\text{SUT}^+\) for which \((\phi')^* \omega\) doesn’t depend on \(v_0\). But the choice of this complex structure depends on \(v_0\).

**Proposition 5.4.** \(\text{SUT}^+\) is a Kähler manifold.

**Proof.** We have the complex structure given on \(\text{SUT}^+\) by the global chart \(G_2\).

\[
G_2 : \text{SUT}^+ \to U = \{z \in \mathbb{C} : \text{Re}(z) > 0\}
\]
\[
\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} \mapsto a + ibv_0 \equiv x' + iy' \equiv z'.
\]

The symplectic form is given by: \(\omega = \frac{1}{2i \text{Re}(z')}dz' \wedge d\bar{z}'\), where \(dz' = dx' + idy', d\bar{z}' = dx' - idy'\).

Now since we have a complex structure, this induces an almost complex structure, \(J\), on the tangent space.

\[
J \left( \frac{\partial}{\partial x'} \right) = \frac{\partial}{\partial y'},
\]
\[
J \left( \frac{\partial}{\partial y'} \right) = -\frac{\partial}{\partial x'}.
\]

The complex structure is compatible with the symplectic form if:

\[
g(u, v) = \omega(Ju, v)
\]
is a Riemannian metric. Let us calculate $g$.

$$g \left( \frac{\partial}{\partial x'}, \frac{\partial}{\partial x'} \right) = \omega \left( J \frac{\partial}{\partial x'}, \frac{\partial}{\partial x'} \right)$$

$$= \omega \left( \frac{\partial}{\partial y'}, \frac{\partial}{\partial x'} \right)$$

$$= \frac{1}{2i \text{Re}(z')^2} (d\bar{z}' \wedge dz') \left( \frac{\partial}{\partial y'}, \frac{\partial}{\partial x'} \right)$$

$$= \frac{1}{2i \text{Re}(z')^2} (-2idx' \wedge d\bar{y}') \left( \frac{\partial}{\partial y'}, \frac{\partial}{\partial x'} \right)$$

$$= \frac{1}{\text{Re}(z')^2} = \frac{1}{(x')^2}.$$  

Similar calculations will show that:

$$g \left( \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'} \right) = g \left( \frac{\partial}{\partial y'}, \frac{\partial}{\partial x'} \right) = 0$$

$$g \left( \frac{\partial}{\partial y'}, \frac{\partial}{\partial y'} \right) = \frac{1}{(x')^2}.$$  

Hence,

$$g = \begin{pmatrix} \frac{1}{(x')^2} & 0 \\ 0 & \frac{1}{(x')^2} \end{pmatrix},$$

which is clearly defines a Riemannian metric. \hfill \Box

Note: In this section we have described the phase space as the right half plane. It could very well have been described as the upper half plane. Both are biholomorphic to the unit disc.

6. Symplectic and Kähler Embeddings

A glance at proposition (5.3) reveals a one parameter ambiguity in the definition of the symplectic structure on $O_X$. This stems from the choice of the vector $X$ and appears as a $v_0$ dependence of the symplectic structure. However, over the reals $\mathbb{R}$, this ambiguity can be removed. By a change of chart on $O_{OX}$, we can absorb the positive, real parameter into the change of coordinates and arrive at a $v_0$ independent symplectic structure.

However if we work over the complex numbers $\mathbb{C}$, this change of chart is not admissible in general. As shown just above proposition (5.4), the transition functions are not biholomorphic except in the case $v_0=1$. There appears to be a genuine difference between the choice $v_0 = 1$ and other choices. Our objective here is to interpret this difference in terms of Kähler and symplectic embeddings in ray space.

Co-adjoint orbits naturally possess a symplectic structure (KKS). Do they naturally admit Kähler structures? In the case of compact groups, there is the positive definite Cartan-Killing metric on the Lie algebra. This metric permits us to identify $\mathfrak{g}$ with its dual and we get a Riemannian metric on $\mathfrak{g}^*$. The generic co-adjoint orbit is a submanifold of $\mathfrak{g}$ and we get an induced Riemannian metric on $O_X$. This leads
to a metric and then a Kähler structure on $O_X$. However, in our case, G is upper
triangular and non-compact. The Cartan-Killing form is degenerate. We can regard
G as a subgroup of SL(2, R) where the Cartan-Killing metric is nondegenerate, but
of Lorentzian signature. This metric does not pull back to a Riemannian metric on
$O_X$. We are led to ask what metrics naturally exist on $O_X$? Since $O_X$ is a homoge-
nous space $G/H$, we would expect that the metric must also be homogeneous and
therefore have constant curvature. In two dimensions, constant curvature metrics
can only be the plane, the sphere and the upper half plane. These have isometry
groups which are respectively $E(2)$, SO(3) and SO(2, 1). The Lie algebras of these
groups would generate symmetries of $O_X$. In our case the symmetry group of $O_X$
is two dimensional and nonabelian. The Lie algebras of $E(2)$ and SO(3) do not admit such a subalgebra. We conclude that the natural Riemannian metric on $O_X$
is the constant negative curvature metric. We can use such a metric to determine a
complex structure on $O_X$, thereby turning it into a Kähler manifold.

We note that the choice $v_0 = 1$ leads us to coherent states, while the choices
$v_0 \neq 1$ give us squeezed states, characterised by a single positive parameter $\lambda^2$.
$\lambda = 1$ describes coherent states, while the other values describe squeezed states.

7. Appendix: Coherent states and Berezin quantization of the phase space

The phase space is identified with the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$. In this section we show the Berezin Quantization of the upper half plane adapting
the one for the unit disc as described in Perelomov [14], chapter 16.

Recall there is a biholomorphism from $\mathbb{H}$ the unit disc $\mathbb{D}$ given by: $\epsilon : \mathbb{H} \rightarrow \mathbb{D}$

$$\epsilon(w) = \frac{w - i}{w + i}$$

Let $\chi$ be the inverse of $\epsilon$. The Kähler form of $\mathbb{D}$ is $d\mu(z, \bar{z}) = \frac{1}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2}$. Using $z = \epsilon(w)$ we get

$$d\mu(z, \bar{z}) = \frac{1}{2\pi i} \frac{dw \wedge d\bar{w}}{4(Imw)^2} = d\mu(w, \bar{w})$$

Also, $(1 - |z|^2)^{1/2} = \left(\frac{4Im(w)}{|w|^2 + 2Im(w) + 1}\right)^{1/2}$.

Let $f \in C^\infty(\mathbb{H})$. Then $f = \epsilon^*(\phi)$ where $\phi \in C^\infty(\mathbb{D})$ and $\chi^*(f)$.

Let $f, g \in C^\infty(\mathbb{H})$. Let $\psi = \chi^*(g)$. Then $(f, g)_\mathbb{H} = (\phi, \psi)_\mathbb{D}$ where

$$\langle \phi, \psi \rangle_\mathbb{D} = \int_\mathbb{D} \bar{\phi}(z) \psi(z) (1 - |z|^2)^{1/2} d\mu(z, \bar{z})$$

$$= \int_\mathbb{D} \bar{f}(w) \phi(w) \left(\frac{4Im(w)}{|w|^2 + 2Im(w) + 1}\right)^{1/2} d\mu(w, \bar{w})$$

Let $\mathcal{F}_h$ be the space of all smooth square integrable functions with respect to the
above inner product, i.e. $\|f\|_H < \infty$.

We proceed as in Perelomov [14], chapter 16 where the Berezin quantization on the
Lobachevsky plane is explained.

Let $\psi_l(z) = (l!)^{-1/2} \times \left[\left(l\frac{i}{\pi} - 1 + l\right)\right]^{1/2} \times z^l$ be the orthonormal basis for $\chi^*(\mathcal{F}_h)$
(from 16.3.3, [14]).
Let \( f_i(w) = ((l!)^{-1/2} \times \left(\frac{1}{n} \right)^{i} - 1 + l) \right)^{1/2} \times (\frac{-i}{n})^l \) be a basis for \( \mathcal{F}_h \).

\( (f_i, f_m)_H = (\psi_i, \psi_m) = \delta_{im} \).

Let \( K_p(q) = \sum f_i(p)f_i(q) \), \( p, q \in \mathbb{H} \).

Let \( \tau_{q} = \sum f_i(p)f_i(w) \), \( w \in \mathbb{H} \).

One can show that for any function \( f \in \mathcal{F}_h, (\tau_p, f) = f(p) \).

Let \( \hat{\mathcal{P}} \) be a bounded linear operator acting on \( \mathcal{F}_h \).

Then it is easy to check that there exists an operator \( \hat{A} \) acting on the square integrable smooth functions on \( \mathbb{D} \) such that:

\[
\hat{\mathcal{P}}(f) = \hat{\mathcal{P}}(e^*(\phi)) = e^*(\hat{A}(\phi)) = e^*\hat{A}(\chi^*(f))
\]

where \( \hat{A} = \chi^*\hat{\mathcal{P}}e^* \).

The symbol of \( \mathcal{P} \) is defined to be \( \mathcal{P}(p, q) = (\tau_p, \hat{\mathcal{P}}\tau_q)_\mathbb{H} \). An easy calculation shows that \( \chi^*(\tau_p) = \psi_\zeta \) is a coherent state on \( \mathbb{D} \) parametrized by \( \zeta = \epsilon(p) \). In fact all coherent states on \( \mathbb{D} \) are of this form. Let the symbol of \( \hat{A} \) be denoted by \( A(\zeta, \bar{\eta}) \) where \( \eta = \epsilon(q) \). One can show that the symbol \( \mathcal{P}(p, q) = A(\zeta, \bar{\eta}) \), i.e. the two symbols are in fact the same.

**The star product:** Let \( \hat{\mathcal{P}}_1, \hat{\mathcal{P}}_2 \) be two bounded linear operators acting on \( \mathcal{F}_h \) and \( \hat{A}_1, \hat{A}_2 \) be two bounded linear operators acting on \( \chi^*(\mathcal{F}_h) \). Then it is easy to show that the star product \( (\mathcal{P}_1 * \mathcal{P}_2)(p, \bar{p}) = (\hat{A}_1 * \hat{A}_2)(\zeta, \bar{\zeta}) \) is the symbol for the composition operator \( \hat{\mathcal{P}}_1 \circ \hat{\mathcal{P}}_2 \).

It has been proved in [14], chapter 16, that the correspondence principle holds for \( A_1 * A_2 \). Thus,

\[
\lim_{\hbar \to 0}(\mathcal{P}_1*\mathcal{P}_2)(p, \bar{p}) = \mathcal{P}_1(p, \bar{p})\mathcal{P}_2(p, \bar{p})
\]

Also,

\[
\lim_{\hbar \to 0}((\mathcal{P}_1*\mathcal{P}_2) - (\mathcal{P}_2*\mathcal{P}_1)) = \{A_1, A_2\}_\mathbb{D} = \{\mathcal{P}_1, \mathcal{P}_2\}_\mathbb{H}
\]

where the Poisson brackets are defined as follows:

\[
\{A_1, A_2\}_\mathbb{D} = (1 - |z|^2)^2 \left( \frac{\partial A_1}{\partial \zeta} \frac{\partial A_2}{\partial \zeta} - \frac{\partial A_2}{\partial \zeta} \frac{\partial A_1}{\partial \zeta} \right)
\]

\[
= 4(Im(w))^2 \left( \frac{\partial \mathcal{P}_1}{\partial w} \frac{\partial \mathcal{P}_2}{\partial w} - \frac{\partial \mathcal{P}_2}{\partial w} \frac{\partial \mathcal{P}_1}{\partial w} \right)
\]

\[
= \{\mathcal{P}_1, \mathcal{P}_2\}_\mathbb{H}
\]

The last equality follows from the fact that \((1 - |z|^2)^2 = \frac{16(Im(w))^2}{(|w|^2 + 2Im(w) + 1)^2} \) and \(|\frac{\partial}{\partial \zeta}|^2 = \frac{1}{4}(|w|^2 + 2Im(w) + 1)^2 \) where recall \( z = \frac{|w| - 1}{|w| + 1} \).

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