Quantization and Greed are Good:
One bit Phase Retrieval, Robustness and Greedy Refinements

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Abstract

In this paper, we study the problem of robust phase recovery. We investigate a novel approach based on extremely quantized (one-bit) measurements and a corresponding recovery scheme. The proposed approach has surprising robustness properties and, unlike currently available methods, allows to efficiently perform phase recovery from measurements affected by severe (possibly unknown) non linear perturbations, such as distortions (e.g. clipping). Beyond robustness, we show how our approach can be used within greedy approaches based on alternating minimization. In particular, we propose novel initialization schemes for the alternating minimization achieving favorable convergence properties with improved sample complexity.

1 Introduction

The phase recovery problem can be modeled as the problem of reconstructing a $n$-dimensional complex vector $x_0$ given only the magnitude of $m$ phase-less linear measurements. Such a problem arises for example in X-ray crystallography [Har93, Lea08], diffraction imaging [BDP+07, Rod08] or microscopy [MISE08], where one can only measure the intensities of the incoming waves, and wishes to recover the lost phase in order to be able to reconstruct the desired object.

In practice, phase recovery is often tackled via greedy algorithms [GS72, Fie82, GL84] which typically lack convergence guarantees. Recently, approaches based on convex relaxations, namely Phase Lift in [CSV11, DH12], and Phase cut in [WDM12], have been proposed and analyzed. These latter methods can be solved by Semi Definite Programing (SDP), and allow the exact and stable recovery of the signal (up to a global phase) from $O(n)$ measurements. A different approach have been recently considered in [NJS13], where it is shown that a greedy alternating minimization, akin to those in [GS72, Fie82, GL84], can be shown to geometrically converge to the true vector $x_0$ if $O(n \log^3(n))$ measurements are given. Indeed, alternating minimization algorithms are known to be extremely sensitive to the initialization and a suitable initialization is the key of the analysis in [NJS13]. While alternating minimization approaches provide a solution only up-to a given accuracy, they often have very good practical performances when compared to convex methods [NJS13], with dramatic computational advantages [NJS13]. The solution of the SDP in convex approaches is computationally expensive and needs to be close to a rank one matrix for tight recovery (which is rarely encountered in practice [WDM12]). Indeed, some greedy refinement of the SDP solution is often considered [WDM12].
In this paper, we propose and investigate a phase recovery approach based on extremely quantized measurements and a corresponding recovery procedure. In particular, we study the properties of the proposed method towards the following questions:

1. **Robustness**: *Is it possible to efficiently perform phase recovery, when the measurements are corrupted by severe perturbations such as non linear distortions or stochastic noise?*

2. **Refinements of Alternating Minimization**: *Are there initialization strategies for the alternating minimization approach that allow better sample complexity?*

Robustness to noise and distortions, such as clipping of the intensities or imperfections in Fourier optics such as multiple scattering [MK83] is a desirable property for a phase retrieval algorithm. At first this task might seem hopeless since current approaches to phase recovery are based on measurements magnitudes which might be completely altered by distortions or if the signal-to-noise ratio is very poor. In fact, we prove the somewhat surprising fact that phase retrieval is still possible, as long as the perturbations preserve (on average) the ranking of the measurements intensities. Indeed, key to our approach is considering a suitable quantization scheme based on comparing pairs of phase-less measurements: only the ranking of each measurement pairs becomes important, rather than the intensity values themselves. Using these extremely quantized (one-bit) measurements, we show that recovery is possible as soon as $O(n \log n)$ pairs of measurements are available. The corresponding recovery procedure reduces to a maximum eigenvalue problem ($1bitPhase$) which can be efficiently solved, for example using the power method. Our approach is inspired by the growing field of one-bit compressive sensing [BB08, PV13b, PV13a, JLBB13].

Beyond robustness, we show how the nature of the one-bit phase less measurements can be used to obtain better results for alternating minimization. We show that the solution of one-bit phase retrieval can be used to initialize alternating minimization to obtain the same convergence results in [NJS13] from only $O(n \log n)$ measurements. Finally, we study a further initialization (weighted one-bit phase retrieval), which is a hybrid between the one in [NJS13] and the one provided by the one-bit approach.

The rest of the paper is organized as follows. In Section 2, we discuss some background and previous work. In Section 3, we sketch our main results and techniques. In Section 4, we introduce and analyze the One-Bit Phase Retrieval approach. In Section 4.5 we introduce weighted one bit phase retrieval that uses both quantized and un-quantized measurements. In Section 5 we show how one-bit phase retrieval algorithms can be used to initialize the alternating minimization approach to get a better sample complexity. We provide a theoretical analysis of our approach in Section 6. Finally, in Section 7 we discuss some computational aspects and present some numerical results.

**Notations:** For $z \in \mathbb{C}$, $|z|^2$ is squared complex modulus of $z$. For $a, a' \in \mathbb{C}^n$, $\langle a, a' \rangle$ is the complex dot product in $\mathbb{C}^n$. For $a \in \mathbb{C}^n$, $a^*$ is the complex conjugate and $||a||_2$ or simply $||a||$ is the norm 2 of $a$. Let $A$ a complex hermitian matrix in $\mathbb{C}^n$, $||A||_F$ denotes the Frobenius norm of $A$, $||A||$ denotes the operator norm of $A$, $Tr(A)$ denotes the trace of $A$. Throughout the paper, we denote by $c, C$ positive absolute constants whose values may change from instance to instance.

### 2 Background and Previous Work

In this section, we formalize the problem of recovering a signal from phase-less measurements and discuss previous results. Throughout this section, and the rest of the paper, we consider measure-
ments defined by independent and identically distributed Complex Gaussian sensing vectors,

\[ a_i \in \mathbb{C}^n, \quad a_i \sim \mathcal{N}(0, \frac{1}{2}I_n) + i\mathcal{N}(0, \frac{1}{2}I_n), \quad i = 1 \ldots m. \]  

(1)

The (noiseless) phase recovery problem is defined as follows.

**Definition 1** (Phase-less Sensing and Phase Recovery). Suppose phase-less sensing measurements

\[ b_i = |\langle a_i, x_0 \rangle|^2 \in \mathbb{R}_+, \quad i = 1 \ldots m, \]  

(2)

are given for \( x_0 \in \mathbb{C}^n \), where \( a_i, i = 1 \ldots m \) are random vectors as in (1). The phase recovery problem is

\[ \text{find } x, \text{ subject to } |\langle a_i, x \rangle|^2 = b_i, \quad i = 1 \ldots m. \]  

(3)

The above problem is non convex and in the following we recall recent approaches to provably and efficiently recover \( x_0 \) from a finite number of measurements.

**SDP (Convex) Relaxation and PhaseLift.** The PhaseLift approach [CSV11] stems from the observation that

\[ |\langle a_i, x \rangle|^2 = \text{Tr}(a_ia_i^*xx^*), \]  

so that if we let \( X = xx^* \), Problem 3 can be written as,

\[ \text{find } X, \text{ subject to } \text{Tr}(a_ia_i^*X) = b_i, \quad i = 1 \ldots m, \quad X \succeq 0, \quad \text{rank}(X) = 1. \]  

(4)

While the above formulation is still non convex (and in fact combinatorially hard because of the rank constraint), a convex relaxation can be obtained noting that Problem 4 can be written as a rank minimization problem over the positive semidefinite cone,

\[ \min_X \text{rank}(X), \quad \text{subject to } \text{Tr}(a_ia_i^*X) = b_i, \quad i = 1 \ldots m, \quad X \succeq 0, \]  

(5)

and then considering the trace as a surrogate for the rank [CSV11],

\[ \min_X \text{Tr}(X), \quad \text{subject to } \text{Tr}(a_ia_i^*X) = b_i, \quad i = 1 \ldots m, \quad X \succeq 0. \]  

(6)

Indeed, the above problem is convex and can be solved via semidefinite programming (SDP). Intestingly, a different relaxation is obtained in [DH12] by ignoring the rank constraint in Problem 4. The results in [CSV11, DH12] show that, with high probability, the solution \( \hat{X}_m \) obtained via either one of the above relaxations can recover \( x_0 \) exactly, i.e. \( \hat{X}_m = x_0x_0^* \), as soon as \( m \geq cn \log n \). In fact, the latter requirement can be further improved to \( m \geq cn \) [CL12]. If the measurements are corrupted by noise, namely

\[ b_i = |\langle a_i, x_0 \rangle|^2 + w_i, \quad i = 1 \ldots m, \]  

(7)

the PhaseLift approach can be adapted [CL12] by considering

\[ \min_X \sum_{i=1}^m |\text{Tr}(a_ia_i^*X) - b_i|, \quad \text{subject to } X \succeq 0. \]  

(8)

The above problem is convex and can again be solved via an SDP approach. The properties of its solution have been studied in [CL12] for deterministic noise

\[ ||w||_1 \leq \delta, \quad w = (w_1, \ldots, w_n) \in \mathbb{R}_+^m, \]

where it is shown that the solution \( \hat{X}_m \) of (8) satisfies \( ||\hat{X}_m - x_0x_0^*||_F \leq c\delta/m, \) as soon as \( m \geq cn \). Moreover, the leading eigenvector \( \hat{x}_m \) of \( \hat{X}_m \) satisfies \( ||\hat{x}_m - e^{i\phi}x_0||_2 \leq c \min(||x_0||_2, \frac{\delta}{m||x_0||_2}) \), where
\( \phi \) is a global phase in \([0, 2\pi]\). Most importantly, the latter results suggests that \( x_0 \) can be recovered considering the leading eigenvector of \( \hat{X}_m \).

As mentioned in the introduction, while powerful, the convex relaxation approach incur in cumbersome computations—see Table 1, and in practice non convex approaches based on greedy alternating minimization (AM) [GS72, Fie82, GL84] are often used. The convergence properties of the latter methods depend heavily on the initialization and only recently [NJS13] they have been shown to globally converge (with high probability) if provided with a suitable initialization. We next briefly review these latter results, which we further discuss and extend in Section 5.

**Phase Retrieval via Suitably Initialized Alternating Minimization.** Let \( A \) be the matrix defined by \( m \) sensing vectors as in (1) and \( B = \text{Diag}(\sqrt{b}) \), where \( b \) is the vector of measurements as in (2). Then,

\[
Ax_0 = Bu_0,
\]

for \( u_0 = Ph(Ax_0) \) with \( Ph(z) = \left( \frac{z_1}{|z_1|}, \ldots, \frac{z_m}{|z_m|} \right) \), \( z \in \mathbb{C}^n \). The above equality suggests the following natural approach to recover \((x_0, u_0)\),

\[
\min_{x,u} \| Ax - Bu \|^2 \quad \text{subject to} \quad |u_i| = 1, \ i = 1 \ldots m,
\]

The above problem is non-convex because of the constraint on \( u \) and the AM approach (Algorithm 1) consists in optimizing \( u \), for a given \( x \), and then optimizing \( x \) for a given \( u \). It is easy to see that for a given \( x \), the optimal \( u \) is simply \( u = Ph(Ax) \), and for a given \( u \), the optimal \( x \) is the solution of a least squares problem.

The key result in [NJS13] shows that if such an iteration is initialized with maximum eigenvector of the matrix

\[
\hat{C}_m = \frac{1}{m} \sum_{i=1}^{m} b_i a_i a_i^* \tag{10}
\]

and \( m \geq Cn(\log n)^3 \), then the solution \( \hat{x}_m \) of the alternating minimization globally converge (with high probability) to the true vector \( x_0 \). Moreover for a given accuracy \( \epsilon \in [0,1] \), if

\[
m \geq c(n(\log^3(n) + \log\left(\frac{1}{\epsilon}\right) \log(\log(\frac{1}{\epsilon})))) \tag{11}
\]

then \( \| \hat{x}_m - e^{i\phi}x_0 \|_2 \leq \epsilon \).

The following two remarks will be useful in the following. First, a key observation, motivating the above initialization (called SubExpPhase in the following), is the fact that the expectation of \( \hat{C}_m \) can be shown to satisfy \( \mathbb{E}(\hat{C}_m) = x_0 x_0^* + I \). Indeed, the proof in [NJS13] (see Section 5) relies on the concentration properties of the random matrix \( \hat{C}_m \) around its expectation [Ver11]. Second, it is useful to note that these latter results crucially depend on a bound on the norm of \( b_i a_i a_i^* \) for \( i = 1, \ldots, m \). Indeed, it is this latter bound the main cause of the poly-logarithmic term in the sample complexity (11), since the \( b_i \)'s are sub-exponential random variables.

| Algorithm 1 AltMinPhase |
|--------------------------|
| 1: procedure ALTMINPHASE(A, b) |
| 2: \hspace{1cm} Initialize x . |
| 3: \hspace{1cm} for k=1\ldots do |
| 4: \hspace{2cm} u ← Ph(Ax) |
| 5: \hspace{2cm} x ← arg min \( \| Ax - Bu \|^2 \) |
| 6: \hspace{1cm} end for |
| 7: \hspace{1cm} return x |
| 8: end procedure |
3 Summary of Our Main Results and Techniques

In this paper, we propose and study a quantization scheme and a corresponding recovery procedure. In particular, we investigate the properties of our approach towards: 1) the phase recovery problem from severely perturbed measurements, and 2) the improvement of the AM approach discussed in the previous section.

3.1 Phase Recovery from Severely Perturbed Measurements

We investigate the phase recovery problem in the case in which we have at disposal measurements of the form

\[\theta(|\langle a_i, x_0 \rangle|^2), \quad i = 1, \ldots, 2m,\]

where \(a_i\) are sensing vectors as in (1) and \(\theta\) is a possibly unknown rank preserving transformation. In particular, we are interested in situations where \(\theta\) models a distortion, e.g., \(\theta(s) = \tanh(\alpha s)\), \(\alpha \in \mathbb{R}^+\), or an additive noise \(\theta(s) = s + \nu\), where \(\nu\) is a stochastic noise, such as an exponential or Poisson noise. As we noted before, the recovery problem from severely perturbed intensity values seems hopeless, and indeed, the key in our approach is a quantization scheme based on comparing pairs of phase-less measurements. More precisely, for each pair \(b^1, b^2\) of measurements of the form (12) we define \(y \in \{-1, 1\}\) as \(y = \text{sign}(b^1 - b^2)\). This one-bit quantization scheme draws inspiration from ideas in one-bit compressive sensing [BB08, PV13b, PV13a, JLBB13], but the fundamental difference is that our approach crucially depends on the comparison of two measurements: one-bit measurements involve the spacing between the order statistics of exponentially distributed random variables \(b^1\) and \(b^2\). Indeed, this will be a key fact in our analysis. While phase-recovery from one-bit phase-less measurements is in general a hard problem (see Section 4), we propose to consider a relaxation which reduces to a maximum eigenvalue problem induced by the matrix

\[\hat{C}_m = \frac{1}{m} \sum_{i=1}^{m} y_i (a_{1i}^1 a_{2i}^1 \ast - a_{1i}^2 a_{2i}^2 \ast).\]

(13)

When compared to (10) we see that the \(m\) phase-less measurements \(b_i\) are replaced by their quantized counterpart \(y_i\) (obtained from \(2m\) phase-less measurements), and the term given by sensing vectors is now given by pairs of sensing vectors. Indeed, we prove in Section 6 that the expectation of \(\hat{C}_m\) satisfies \(\mathbb{E}\hat{C}_m = \lambda x_0 x_0^*\), where \(\lambda\) is a suitable constant which depends on \(\theta\) and plays the role of a signal-to-noise ratio. Indeed, by studying the concentration properties of the matrix \(\hat{C}_m\), we show that, for a given accuracy \(\epsilon \in [0, 1]\), if \(O(\frac{n \log n}{\epsilon^2 \lambda})\) pairs of measurements are available, then the solution of the above maximum eigenvalue problem satisfies

\[||\hat{x}_m - x_0 e^{i\phi}||^2 \leq \epsilon,\]

where \(\phi \in [0, 2\pi]\) is a global phase. It is worth noting here that the signal can be recovered up to a scaling factor from one-bit measurements, but this is not a problem since our goal is to recover the missing phase.

3.2 One-Bit Phase Retrieval and Alternating Minimization

A key difference between the matrix in Eq. (10) and the one in Eq. (13) is that one-bit measurements are bounded and lead to improved concentration results. This motivates considering the effect of using the solution of the one-bit phase retrieval to initialize the alternating minimization procedure considered in [NJS13]. Indeed, leveraging results from [NJS13], we prove in Section
that, provided with the one-bit retrieval initialization, the alternating minimization algorithm globally converges (with high probability) to the true vector $x_0$, and if

$$m \geq c(n(\log n + \log(\frac{1}{\epsilon})\log(\frac{1}{\epsilon})))$$

then $||\hat{x}_m - e^{i\phi}x_0||_2 \leq \epsilon$. Comparing to (11), we see that the sample complexity depends now only on a logarithmic term. Quantization can be seen as playing the role of a preconditioning that enhances the sample complexity of the alternating minimization. Further, we note that it is possible to achieve similar results, see Table 1, considering a different initialization obtained via a weighted one-bit approach which combines quantized and un-quantized measurements, see Section 4.5.

| Sample complexity | Comp. complexity |
|-------------------|------------------|
| PhaseLift         | $O(n)$           | $O(n^3/\epsilon^2)$ |
| PhaseCut          | $O(n)$           | $O(n^3/\sqrt{\epsilon})$ |
| SubExpPhase+AM    | $O(n(\log^2 n + \log \frac{1}{\epsilon}\log \frac{1}{\epsilon}))$ | $O(n^2(\log^2 n + \log^2 \frac{1}{\epsilon}\log \frac{1}{\epsilon}))$ |
| 1bitPhase+AM      | $O(2n(\log(n) + \log \frac{1}{\epsilon}\log \frac{1}{\epsilon}))$ | $O(n^2(\log n + \log^2 \frac{1}{\epsilon}\log \frac{1}{\epsilon}))$ |
| Weighted1bitPhase+AM | $O(2n(\log(n) + \log \frac{1}{\epsilon}\log \frac{1}{\epsilon}))$ | $O(n^2(\log n + \log^2 \frac{1}{\epsilon}\log \frac{1}{\epsilon}))$ |

Table 1: Comparison of the sample and computational complexity of different phase retrieval schemes.

4 One-Bit Phase Retrieval

In this section, we set up the one-bit approach to phase-retrieval and state our main results on robust phase recovery.

4.1 Quantization and Recovery

Unlike in compressive sensing, in our context the intensities are non negative, and thus we cannot rely on only one measurement to build a quantizer.

**Definition 2** (One-bit quantizer). Let $A = (a^1, a^2)$, where $a^1, a^2$ are i.i.d. complex Gaussian vectors $\mathcal{N}(0, \frac{1}{2}I_n) + i\mathcal{N}(0, \frac{1}{2}I_n)$. For $x_0 \in \mathbb{C}^n$, a one bit quantizer is given by

$$Q_A : \mathbb{C}^n \rightarrow \{-1, 1\}, \quad Q_A(x_0) = \text{sign} (|\langle a^1, x_0 \rangle|^2 - |\langle a^2, x_0 \rangle|^2).$$

**Definition 3** (Quantized phase-less measurements). Let $\{A_i = (a^1_i, a^2_i)\}_{1 \leq i \leq m}$ be $2m$ i.i.d. gaussian complex vectors in $\mathbb{C}^n$, and $Q_{A_i}(x_0)$ as in Def 2. The Quantized Phase-less sensing is given by $Q : \mathbb{C}^n \rightarrow \{-1, 1\}^m$, $Q(x_0) = (Q_{A_1}(x_0), \ldots, Q_{A_m}(x_0))$.

In this paper, we are interested in recovering $x_0$ from its quantized phase-less measurements $y = (y_1 \ldots y_m) = Q(x_0) = (Q_{A_1}(x_0), \ldots, Q_{A_m}(x_0))$. It is easy to see that the recovery problem has the form,

$$\text{find } x, \quad \text{subject to } y_i \left( |\langle a^1_i, x \rangle|^2 - |\langle a^2_i, x \rangle|^2 \right) \geq 0, \quad i = 1 \ldots m, \quad ||x||^2 = 1.$$
Indeed, as in one-bit compressive sensing, we cannot hope to recover the norm of the vector from inequality constraints, hence the norm one constraint. Problem (15) can be equivalently written as the following quadratically constrained problem,

\[
\text{find } x, \text{ subject to } x^* y_i (a_i^1 a_i^{1,*} - a_i^2 a_i^{2,*}) x \geq 0, \quad i = 1 \ldots m, \quad ||x||_2^2 = 1. \tag{16}
\]

The above problem is a non-convex Quadratically Constrained Quadratic Program (QCQP) and can be shown to be NP-hard in general [DB03]. We propose to consider the following relaxation,

\[
\max x^* \left( \frac{1}{m} \sum_{i=1}^{m} y_i (a_i^1 a_i^{1,*} - a_i^2 a_i^{2,*}) \right) x, \text{ subject to } ||x||_2^2 = 1. \tag{17}
\]

The 1BitPhase problem is obtained noting that the above problem can be rewritten as the the maximum eigenvalue problem,

\[
\max_{x \text{ s.t. } ||x||_2 = 1} x^* \tilde{C}_m x, \tag{18}
\]

defined by the matrix

\[
\tilde{C}_m = \frac{1}{m} \sum_{i=1}^{m} y_i (a_i^1 a_i^{1,*} - a_i^2 a_i^{2,*}).
\]

As we comment in the following remark the 1BitPhase approach is inspired by one bit-compressed sensing.

**Remark 1** (Quantization in Compressive Sensing: One bit CS). *Non-linear, quantized measurements have been recently considered in the context of one-bit compressive sensing\(^1\). Here, binary (one-bit) measurements are obtained applying, for example, the “sign” function to linear measurements. More precisely, given \(x_0 \in \mathbb{R}^n\), a measurement vector is given by \(y = (y_1, \ldots, y_m)\), where \(y_i = \text{sign}(\langle a_i, x \rangle)\) with \(a_i \sim \mathcal{N}(0, I_n)\) independent Gaussian random vectors, for \(i = 1, \ldots, m\). It is possible to prove, see e.g. [PV13b], that, for a signal \(x_0 \in K \cap \mathbb{B}^n\) (\(\mathbb{B}^n\) is the unit ball in \(\mathbb{R}^n\)), the solution \(\hat{x}_m\) to the problem

\[
\max_{x \in K \cap \mathbb{B}^n} \sum_{i=1}^{m} y_i \langle a_i, x \rangle, \tag{19}
\]

satisfies \(\|\hat{x}_m - x_0\|^2 \leq \frac{\delta}{\sqrt{2}}, \quad \delta > 0\), with high probability, as long as \(m \geq C \delta^{-2} \omega(K)^2\) [PV13b].

Here \(\omega(K) = \mathbb{E} \sup_{x \in K-K} \langle w, x \rangle\) denotes the Gaussian mean width of \(K\). The 1BitPhase approach shows that a relaxation similar to problem 19 allows to perform phase recovery for a suitably defined quantization of phase-less linear measurements.

Before studying the recovery guarantees for the solution of 1BitPhase we briefly discuss a geometric intuition underlying the method.

### 4.2 Geometric Intuition

To understand the geometric intuition of one bit phase retrieval, we first consider the feasibility problem (15). Each measurements pair \((a_i^1, a_i^2)\) defines a hyperbolic paraboloid \(z = x^* (a_i^1 a_i^{1,*} - a_i^2 a_i^{2,*}) x\). The feasible zone is defined by the constraints \(y_i (x^* (a_i^1 a_i^{1,*} - a_i^2 a_i^{2,*}) x) \geq 0\), which enforce the geometric consistency with the bit \(y_i\). In other words each constraint says that \(x_0\) is in the

\(^1\)See [http://dsp.rice.edu/1bitCS/](http://dsp.rice.edu/1bitCS/) for an exhaustive list of references.
region of the space where the sign of the corresponding hyperbolic paraboloids is \( y_i \). Note that each hyperbolic paraboloid is symmetric with respect to the origin, thus the feasible region is also symmetric with respect to the origin. When we add more constraints we have that \( x_0 \) lies in the intersection of such symmetric feasible regions and the unit sphere. This intersection is also symmetric, thus we can solve the phase retrieval up to global sign flip (in the real valued case). As we mentioned before, the feasibility problem is a non convex QCQP which is NP hard in general and our relaxation (18) can be seen as requiring the geometric consistency with one bit measurements on average, rather than individually as in the feasibility problem. In figure 1 we plot the level sets of the objective function of the 1bitPhase Problem (18) for \( x_0 = (1,0) \). We see that the objective function achieves its maximum values (in red) in a symmetric region. This region intersects with the sphere in two regions close to the points \( x_0 \) and \(-x_0\). Thus we are able to recover the phase up to global sign and scaling from single bit measurements.

### 4.3 One bit Phase Retrieval from Distorted and Noisy measurements

In the following we assume that the intensities are undergoing an unknown non linearity \( \theta \), that is we observe,

\[
(b_i^1, b_i^2) = (\theta(|\langle a_i^1, x_0 \rangle|^2), \theta(|\langle a_i^2, x_0 \rangle|^2), \quad i = 1, \ldots, m.
\]

Thus the quantized measurements are,

\[
y_i = Q^\theta_{A_i}(x_0) = \text{sign} \left( \theta(|\langle a_i^1, x_0 \rangle|^2) - \theta(|\langle a_i^2, x_0 \rangle|^2) \right), \quad i = 1 \ldots m.
\] (20)

For instance clipping can be modeled by a sigmoid,

\[
\theta(z) = \tanh(\alpha z), \quad z > 0.
\]
where the parameter $\alpha$ controls how severe is the distortion. An additive noise (before quantization) can be modeled by,

$$\theta(z) = z + \nu,$$

where $\nu \sim Exp(\gamma)$ is a stochastic exponential noise with mean $\mu = \frac{1}{\gamma}$ and variance $\sigma = \frac{1}{\gamma^2}$.

When the intensities are contaminated with poisson noise we have,

$$\theta(|\langle a, x_0 \rangle|^2) = \mathcal{P}_\eta(|\langle a, x_0 \rangle|^2),$$

where $\mathcal{P}_\eta$ is a poisson noise, such that :

For $z, \eta > 0$, $\theta(z) = \mathcal{P}_\eta(z) = p$, where $p \sim \text{Poisson}\left(\frac{z}{\eta}\right)$.

We shall make one assumption on the non linearity $\theta$,

$$\lambda = \mathbb{E}(\text{sign}(\theta(E_1) - \theta(E_2))(E_1 - E_2)) > 0,$$

where $E_1, E_2$ are two independently distributed exponential random variables. To see why this assumption is natural, notice that $|\langle a, x_0 \rangle|^2 \sim Exp(1)$ if $a \sim \mathcal{CN}(0, I_n)$ and $||x_0|| = 1$, thus

$$\mathbb{E}(y_i(|\langle a_1, x_0 \rangle|^2 - |\langle a_1^2, x_0 \rangle|^2)) = \mathbb{E}(\text{sign}(\theta(E_1) - \theta(E_2))(E_1 - E_2)) = \lambda > 0.$$

Then the above assumption simply means that the one bit measurements preserve robustly the ranking of the intensities. For example, such an assumption is trivially satisfied whenever $\theta$ is increasing. In case $\theta(z) = z$, $\lambda$ achieves its maximal value,

$$\lambda = \mathbb{E}(\text{sign}(E_1 - E_2)(E_1 - E_2)) = \mathbb{E}|E_1 - E_2| = 1,$$

since $|E_1 - E_2| \sim Exp(1)$. For different models of observation, the value of $\lambda$ is given in Lemma 1, which shows that that $\lambda$ plays the role of a signal to noise ratio.

![Figure 2: $\lambda$ as a signal to noise ratio: $\lambda$ versus different parameters of various observation model $\theta$. $\lambda$ decreases as noise and distortion levels increase.](image)

We see in Figure (4.3) that $\lambda$ indeed decreases as the level of noise and distortion increases.
4.4 Main Results for One Bit Phase Retrieval

The following theorem describes the recovery guarantees for the solution \( \hat{x}_m \) of problem 1bitPhase (18).

**Theorem 1** (One bit Recovery). For \( x_0 \in \mathbb{C}^n, \|x_0\| = 1 \). Assume \( y_1 \ldots y_m \), follows the model given in (20). Then for any \( \epsilon \in [0, 1] \), we have with a probability at least \( 1 - O(n^{-2}) \),

\[
for \ m \geq \frac{C}{\epsilon^2 \lambda} n \log(n), \quad \|\hat{x}_m - x_0 e^{i\phi}\|^2 \leq \epsilon,
\]

where \( \phi \in [0, 2\pi] \) is a global phase and \( \lambda \) is given in (23).

For the simple model model where \( \theta(z) = z \), \( \lambda = 1 \). Theorem 1 implies that if \( m = O(n \log(n)) \) (so that the total number of measurements is \( 2m \)), then \( \hat{x}_m \) is an \( \epsilon \)-estimate of \( x_0 \), up to a global phase \( \phi \). In Corollary 1 we specify the above theorem to the noisy model (21).

**Corollary 1** (One bit Recovery/ Noise). For \( x_0 \in \mathbb{C}^n, \|x_0\| = 1 \), and \( \epsilon > 0 \). Assume \( y_1 \ldots y_m \), follows the noisy model given in (20), for \( \theta(z) = z + \nu, \nu \sim \text{Exp}(\gamma) \). Where \( \nu \) is an exponential noise with variance \( \sigma = \frac{1}{\gamma^2} \). Then for any \( \epsilon \in [0, 1] \), we have with a probability at least \( 1 - O(n^{-2}) \),

\[
for \ m \geq \frac{C}{\epsilon^2} (1 + \sqrt{\sigma})^2 n \log(n), \quad \|\hat{x}_m - x_0 e^{i\phi}\|^2 \leq \epsilon,
\]

where \( \phi \in [0, 2\pi] \) is a global phase.

In other words, under an exponential noise we have:

\[
\|\hat{x}_m - x_0 e^{i\phi}\|^2 \leq C \sqrt{\frac{n \log(n)}{m} \frac{(1 + \sqrt{\sigma})^2}{1 + 2\sqrt{\sigma}}},
\]

A similar result holds for Poisson noise for a different value of \( \lambda \) given in Lemma 1. Beyond robustness to noise, another desirable feature for phase retrieval from phase-less measurements, is the robustness to distortions of the values of intensities. Is it possible to retrieve the phase from intensities values that are undergoing clipping for instance?

**Corollary 2** (One bit Recovery/ Distortion). For \( x_0 \in \mathbb{C}^n, \|x_0\| = 1 \), and \( \epsilon > 0 \). Assume \( y_1 \ldots y_m \), follows the noisy model given in (20), for \( \theta(z) = \tanh(\alpha z), \alpha > 0 \). Then for any \( \epsilon \in [0, 1] \), we have with a probability at least \( 1 - O(n^{-2}) \),

\[
for \ m \geq \frac{C n \log(n)}{\epsilon^2 \lambda(\alpha)}, \quad \|\hat{x}_m - x_0 e^{i\phi}\|^2 \leq \epsilon,
\]

where \( \phi \in [0, 2\pi] \) is a global phase. \( \lambda(\alpha) = \mathbb{E}(|E_1 - E_2| \text{sign}(1 - \tanh(\alpha E_1) \tanh(\alpha E_2))) \) is a decreasing function in \( \alpha \).

The proof of the above results follow from a simple combination of Propositions 1,2, and 3 given in Section 6.
4.5 Weighted One Bit Phase Retrieval

The boundedness of one bit phase measurement \( y \) is appealing as it ensures better sample complexity. In this section, we ask the question of whether similar results can be obtained combining the un-quantized measurements \( b_1^i = |\langle a_1^i, x_0 \rangle|^2, b_2^i = |\langle a_2^i, x_0 \rangle|^2, i = 1 \ldots m \), and the quantized measurements \( y_i = \text{sign}(b_1^i - b_2^i) \).

We have therefore to keep in mind that we need to formulate the problem in such way the random variables depending on \((b_1^i, b_2^i)\) are bounded. Then we consider

\[
R_1^i = \frac{b_1^i}{b_1^i + b_2^i} \quad \text{and} \quad R_2^i = \frac{b_2^i}{b_1^i + b_2^i} \quad i = 1 \ldots m.
\]

\((R_1^i, R_2^i)\) take values in \([0, 1]^2\) (hence bounded), moreover they are Beta distributed \(\text{Beta}(1, 1)\), that is the uniform distribution \(\text{unif}[0, 1]\) (Lemma 6). Then, we can consider the following problem,

\[
\text{find } x \quad \text{subject to} \quad y_i \left( R_1^i |\langle a_1^i, x \rangle|^2 - R_2^i |\langle a_2^i, x \rangle|^2 \right) \geq 0, \quad i = 1 \ldots m.
\]

\[
||x||^2 = 1.
\]

We relax this problem to the following maximum eigen value problem that we call weighted one bit Phase retrieval (Weighted1bitPhase).

\[
\max_{x, ||x|| = 1} x^* \frac{1}{m} \sum_{i=1}^{m} y_i \left( R_1^i a_1^i a_1^{i,*} - R_2^i a_2^i a_2^{i,*} \right) x
\]

Thanks to the boundedness of \((R_1^i, R_2^i)\), one can carry the same analysis done in 6, and get correctness and sample complexity for this formulation. Indeed \(O(2n \log(n))\) measurements are also sufficient for phase retrieval from that weighted scheme. Nevertheless this scheme is more sensitive to noise and distortion than the original formulation.

**Theorem 2** (Weighted One bit Recovery). For \(x_0 \in \mathbb{C}^n, ||x_0|| = 1\), and \(\epsilon > 0\). Let \(\hat{x}_m\) be the solution of problem Weighted1Bit (25). Then for any \(\epsilon \in [0, 1]\), we have with we have with a probability at least \(1 - O(n^{-2})\),

\[
\text{for } m \geq \frac{C}{\epsilon^2} n \log(n), \quad ||\hat{x}_m - x_0 e^{i\phi}||^2 \leq \epsilon.
\]

where \(C\) is universal constant, and \(\phi \in [0, 2\pi]\) is a global phase.

The proof of this Theorem is given in the Section 6.

**Remark 2.** For simplicity of the exposure we limit the analysis to \(\theta(z) = z\).

5 Greedy Refinements via Alternating Minimization

We have now defined 2 variants of one bit phase retrieval: 1BitPhase and Weighted1BitPhase. Both formulation allows phase recovery via a spectral maximum Eigen value problem. In this section we start by analyzing the alternating minimization approach and the virtues of the initialization step proposed in [NJS13] that we call SubExpPhase. We then show that 1BitPhase
and Weighted1BitPhase offer a new way to initialize the alternating minimization problem. Note that we have now 3 randomized strategies (SubExpPhase, 1BitPhase and Weighted1BitPhase) to initialize the alternating minimization problem of phase retrieval. Each one succeeds with high probability, a multiple initialization strategy allows to choose the corresponding solution with lowest MSE.

5.1 Phase Recovery via Alternating Minimization

The alternating minimization algorithm proposed in [NJS13] has 2 main ingredients:

1. For an accuracy $\epsilon$, given $O(\frac{1}{\epsilon^2} n \log^3(n))$ measurements, the authors propose an initialization $x^0$ that is an $\epsilon$ estimate of $x_0$.

2. A resampling procedure that ensures a reduction in the error in each step of the alternating minimization provided with the above initialization.

The resulting algorithm has a sample complexity of $O(n |\log^3(n) + \log(\frac{1}{\epsilon}) \log(\log(\frac{1}{\epsilon})))$, and a computational complexity of $O(n^2 |\log^3(n) + \log^2(\frac{1}{\epsilon}) \log(\log(\frac{1}{\epsilon})))$. Thus in order to get a better sample complexity of the resulting algorithm and hence a better computational complexity, the challenge is to propose a better initialization. We will show that one bit phase retrieval offers a good strategy for initializing the alternating minimization.

5.1.1 Sub-Exponential Initialization

We comment in this section on the initialization and the alternating minimization procedure of [NJS13].

Let $b_i = |\langle a_i, x_0 \rangle|^2, i = 1 \ldots m$, where $a_i \sim \mathcal{CN}(0, I_n)$. The initialization proposed amounts to taking the maximum eigen vector $\hat{x}_m$ of

$$\hat{C}_m = \frac{1}{m} \sum_{i=1}^{m} b_i a_i a_i^*,$$

(26)

In [NJS13] authors show correctness and concentration of this approach. We restate here their main result, and give for completeness a sketch of the proof in C:

**Theorem 3** (Correctness and Concentration). Let $x, x_0 \in \mathbb{C}^n$. Assume that $x$ and $x_0$ are unitary, and $a \sim \mathcal{CN}(0, I_n)$. Let

$$\mathcal{E}^{x_0}(x) = \mathbb{E} |\langle a, x_0 \rangle|^2 |\langle a, x \rangle|^2, \quad \mathcal{E}^{x_0}(x) = \frac{1}{m} \sum_{i=1}^{m} b_i |\langle a_i, x \rangle|^2, \quad \hat{x}_m = \arg \max_{x, ||x||=1} \mathcal{E}^{x_0}(x).$$

We have the following claims:

1. $\mathcal{E}^{x_0}(x) = x^* C x$, where $C = \mathbb{E} (b a a^*)$, where $b = |\langle a, x_0 \rangle|^2$.

2. For all $x$, such that $||x|| = 1$, $\mathcal{E}^{x_0}(x) = |\langle x_0, x \rangle|^2 + 1$.

3. For all $x$, such that $||x|| = 1$, $\frac{1}{2} ||x x^* - x_0 x_0^*||_F^2 = (\mathcal{E}^{x_0}(x_0) - \mathcal{E}^{x_0}(x)) = (1 - |\langle x_0, x \rangle|^2)$.

4. $\frac{1}{2} ||\hat{x}_m - x_0 x_0^*||_F^2 \leq 2 \left|\left| \hat{C}_m - C \right|\right|_F$.

5. Let $\epsilon \in [0, 1]$ then For $m \geq c \frac{n \log^3(n)}{\epsilon^2}$, $\left|\left| \hat{C}_m - C \right|\right| \leq 2 \epsilon$ with probability at least $1 - O(n^{-2})$.

6. Let $\epsilon \in [0, 1]$ then For $m \geq c \frac{n \log^3(n)}{\epsilon^2}, ||\hat{x}_m - x_0 e^{i\phi}||^2 \leq \epsilon$ with probability at least $1 - O(n^{-2})$, where $c$ is a universal constant.
5.2 Discussion: One Bit Phase Retrieval as an Initialization to the Alternating Minimization

Note that $b_i, i = 1 \ldots m$ are exponential random variable thus we call that initialization Sub-exponential Initialization.

The concentration of $\hat{C}_m$ around $C$, depends upon the boundedness of $b_i$ and $a_i$ by the non commutative matrix Bernstein inequality (Theorem 5). We have with high probability that

$$b_i||a_i||^2 \leq 4 \log(m)n,$$

thus we have a sample complexity of $O(n \log^3(n))$ due to the extra contribution of $b_i$ with a $\log(n)$ term. Recall that the solution of one bit phase retrieval and weighted one bit phase retrieval is the maximum eigen vector of

$$\hat{C}_m = \frac{1}{m} \sum_{i=1}^{m} y_i(a_i^1a_i^{1,*} - a_i^2a_i^{2,*}) \quad \text{and} \quad \hat{C}_m = \frac{1}{m} \sum_{i=1}^{m} y_i(R_i^1a_i^1a_i^{1,*} - R_i^2a_i^2a_i^{2,*}),$$

respectively. The measurements $y_i, R_i^1, R_i^2$ are bounded by 1 and do not affect the bound . Thus, the sample complexity reduces to only $O(n \log(n))$ pairs of measurements for phase retrieval via one-bit measurements. Thus we can initialize the alternating minimization with the solution of One Bit Phase and Weighted One Bit Phase and get a better sample complexity especially in high dimensions.

**Geometric intuition.** We see in Figure 2 that the levels sets of the objective of SubExpPhase initialization consists of a paraboloid, that is symmetric, hence it intersects the unity sphere in a symmetric zone thus phase retrieval is possible up to a global phase. Compared to one bit initialization, the levels sets are hyperbolic paraboloids. Hyperbolic paraboloid are more "pointy" than paraboloid thus the surface of intersection with the sphere is smaller. The above discussion gives an intuition of the reasons behind the better sample complexity of one-bit phase retrieval.

We conclude this section with a surprising fact:

*Quantization and Greed are good. Quantization plays the role of a preconditioning that enhances the sample complexity of the initialization step of the greedy alternating minimization in phase retrieval.*

5.2.1 Resampling Procedure and Error Reduction

Now given one of the three initialization strategies, namely the Sub-Exponential initialization of [NJS13], One Bit Phase Retrieval and Weighted One Bit Phase Retrieval. The following algorithm proposed in [NJS13], proceeds in alternating the estimating of the phase and the signal. For technical reasons - mainly ensuring independence - the algorithm proceeds in a stage-wise alternating minimization. At each stage we use a new re-sampled sensing matrix and the corresponding measurements.
Figure 3: The level sets of the objective of the initialization: $x^* \sum_{i=1}^{m} b_i a_i a_i^* x$, for $x_0 = (1, 0)$. (in red maximal values, in blue minimal values).

Algorithm 2 AltMinPhase with Resampling

1: procedure ALTMINPHASERESAMPLING($A, b, \epsilon$)
2: $t_0 \leftarrow c \log(\frac{1}{\epsilon}) n$
3: Partition $b$ and the corresponding rows of $A$ into $t_0 + 1$ disjoint sets: $(b_0, A_0), \ldots, (b_{t_0}, A_{t_0})$.
4: Initialize $x$ using SUBEXPONENTIALPHASE or 1BITPHASE, or WEIGHTED1BIT PHASE.
5: for $t = 0 \ldots t_0 - 1$ do
6: $u_{t+1} \leftarrow Ph(A_{t+1} x_t)$
7: $x_t \leftarrow \arg\min ||A_{t+1} x - B_{t+1} u_{t+1}||^2_2$
8: end for
9: return $x_{t_0}$
10: end procedure

Combining results from [NJS13] with Theorem 1 and Theorem 2 we have:

**Theorem 4.** For every $\epsilon > 0$ Algorithm 2 outputs $x_{t_0}$ such that $||x_{t_0} - x_0 e^{i\theta}||_2 \leq \epsilon$ with high probability. The sample complexity depends upon the initialization step.

1. **Sub-Exponential Initialization:** the sample complexity is $O(n \log^3(n) + \log(\frac{1}{\epsilon}) \log(\log(\frac{1}{\epsilon})))$.

2. **Weighted/One Bit Phase Retrieval:** the sample complexity is $O(2n(\log(n) + \log(\frac{1}{\epsilon}) \log(\log(\frac{1}{\epsilon})))$.

This theorem is a consequence of the work of [NJS13] that does not depend on the initialization step. The greedy refinements of one bit solution, ensures convergence to the optimum with high probability and lower sample complexity than the one obtained in [NJS13].

**Remark 3 (Multiple Initialization).** Fix the total number of measurements. Let $x_s$ the solution of Algorithm 2 initialized with the Sub-exponential initialization. $x_{16}$ the solution of Algorithm 2
initialized with the One Bit Phase initialization. \(x_{w1b}\) the solution of Algorithm 2 initialized with the Weighted One Bit Phase initialization. Define

\[ x_* = \arg \min_{x \in \{ x_s, x_{1b}, x_{w1b} \}} MSE(x, u = Ph(Ax)) = ||Ax - Bu||^2_2, \]

The multiple initialization strategy produces \(x_*\) that has the lower MSE for a given accuracy.

6 Theoretical Analysis

In this section we give the main steps of the proof of Theorems 1 and 2 for one bit phase Retrieval and Weighted One Bit phase Retrieval respectively.

6.1 One Bit Phase Retrieval: Correctness and Concentration

In this section we state Propositions 1, 2, and 3 which form the core of our analysis for one bit Phase Retrieval. The proofs are given in Appendix A.1. We need the following preliminary definition.

**Definition 4 (Risk and Empirical risk).** Let \(x_0 \in \mathbb{C}^n, ||x_0|| = 1\). For \(x \in \mathbb{C}^n\) such that \(||x|| = 1\), and \(A = \{a^1, a^2\}\) i.i.d. complex Gaussians, let

\[ \mathcal{E}^{x_0}(x) = x^*Cx, \]

where \(C = \mathbb{E}(y(a^1a^{1,*} - a^2a^{2,*}))\) and \(y = \text{sign}(\theta(|\langle a^1, x_0 \rangle|^2) - \theta(|\langle a^2, x_0 \rangle|^2))\). Moreover, let

\[ \hat{\mathcal{E}}^{x_0}(x) = x^*\hat{C}_m x, \]

where \(\hat{C}_m = \frac{1}{m} \sum_{i=1}^{m} y_i(a^{1i}a^{1,*}_i - a^{2i}a^{2,*}_i), y_i = Q^\theta_{A_i}(x_0)\) and \(A_i = \{(a^{1i}, a^{2i})\}, i = 1 \ldots m\) are i.i.d. complex Gaussians.

We first state Proposition 1, that provides a theoretical justification to the relaxation introduced in the formulation 1bitPhase in (18).

**Proposition 1 (Correctness in Expectation).** The following statements hold:

1. For all \(x \in \mathbb{C}^n, ||x|| = 1\), we have the following equality,

\[ \mathcal{E}^{x_0}(x) = x^*Cx = \lambda |\langle x_0, x \rangle|^2. \] (27)

2. Let \(y = Q^\theta_{A}(x_0)\), \(C\) is a rank one matrix,

\[ C = \mathbb{E}(y(a^{1,*}a^{1,*} - a^{2,*}a^{2,*})) = \lambda x_0 x_0^*. \] (28)

3. \(x_0\) is an eigen vector of \(C\) with eigen value \(\lambda\),

\[ Cx_0 = \lambda x_0. \] (29)

4. The maximum eigenvector of \(C\) is of the form \(x_0 e^{i\phi}\), where \(\phi \in [0, 2\pi]\). The maximum eigen value is given by \(\lambda\).
Proposition 1 suggests that \( x_0 \) can be recovered up to global phase shift as the maximum eigen vector of the matrix \( C \). The Quality of the recovery of one bit phase recovery, and its sample complexity is therefore driven by how well the empirical Hermitian matrix \( \hat{C}_m \), concentrates around its mean \( C \). A key quantity in the analysis is \( \lambda \), which can be seen as a form of signal to noise ratio. Recall that:

\[
\lambda = \mathbb{E}(\text{sign}(\theta(E_1) - \theta(E_2))(E_1 - E_2)), \quad E_1, E_2 \sim \text{Exp}(1) (\text{iid}).
\]  

(30)

The following Lemma shows how the value of \( \lambda \) depends on the observation model \( \theta \), and how \( \lambda \) relates to noise and distortion levels.

**Lemma 1.** The values of \( \lambda \) for different observation models \( \theta \) are given in the following:

1. **Noiseless setup:** \( \theta(z) = z \), \( \lambda = 1 \).

2. **Exponential Noise:** \( \theta(z) = z + \nu, \nu \) is an exponential random variable with variance \( \sigma \), \( \lambda = \frac{1+2\sqrt{\sigma}}{(1+\sqrt{\sigma})^2} \).

3. **Poisson Noise:** \( \theta(z) = P_\eta(z), \quad P_\eta(z) = p \sim \text{Poisson}(\frac{z}{\eta}) \) \( \lambda = \mathbb{E}(\text{sign}(S(E_1, E_2))(E_1 - E_2)) \)

\( S(E_1, E_2) \sim \text{Skellam}(\frac{E_1}{\eta}, \frac{E_2}{\eta}) \), \( E_1, E_2 \sim \text{Exp}(1) \). \( \lambda \) is a decreasing function in \( \eta \).

4. **Distortion setup:** \( \theta(z) = \tanh(\alpha z) \), \( \lambda = \mathbb{E}(|E_1 - E_2|\text{sign}(1 - \tanh(\alpha E_1) \tanh(\alpha E_2))) \), \( E_1, E_2 \sim \text{Exp}(1) \). \( \lambda \) is a decreasing function in \( \alpha \).

From Lemma 1, we see that \( \lambda \) achieves its maximum value 1, in the noiseless case. \( \lambda \) interestingly captures the SNR as it decreases with noise and distortion levels.

**Lemma 2.** For any \( x \in \mathbb{C}^n, \|x\| = 1 \), the following equality holds:

\[
\mathcal{E}^{x_0}(x_0) - \mathcal{E}^{x_0}(x) = \frac{\lambda}{2} \|xx^* - x_0x_0^*\|_F^2.
\]  

(31)

Lemma 2 provides a *comparison equality* relating \( \|xx^* - x_0x_0^*\|_F^2 \) to the *excess risk* \( \mathcal{E}^{x_0}(x_0) - \mathcal{E}^{x_0}(x) \). Then using results from empirical processes we bound the excess risk with the operator norm of \( \hat{C}_m - C \). The rest of the proof uses results from matrix concentration inequalities [Ver11] in order to bound \( \|\hat{C}_m - C\| \).

**Proposition 2.** The following inequalities hold for the solution \( \hat{x}_m \) of problem (18),

\[
\mathcal{E}^{x_0}(x_0) - \mathcal{E}^{x_0}(\hat{x}_m) \leq 2 \sup_{x, \|x\|=1} (\hat{\mathcal{E}}^{x_0}(x) - \mathcal{E}^{x_0}(x))
\sup_{x, \|x\|=1} (\hat{\mathcal{E}}^{x_0}(x) - \mathcal{E}^{x_0}(x)) = \left\| \hat{C}_m - C \right\|.
\]

Finally we bound \( \|\hat{C}_m - C\| \) using Matrix Bernstein inequality [Ver11]:

**Proposition 3.** For \( \epsilon \in [0, 1] \), there exists a constant \( c \) such that:

\[
\text{For } m \geq \frac{c n \log(n)}{\lambda \epsilon^2}, \quad \|\hat{C}_m - C\| \leq \epsilon \lambda \text{ with probability at least } 1 - O(n^{-2}).
\]

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6.2 Weighted One Bit Phase Retrieval: Correctness and Concentration

In this section we sketch the proof architecture for results of Weighted one bit phase retrieval. We start first by a preliminary definition:

**Definition 5** (Risk and Empirical risk). Let \( x_0 \in \mathbb{C}^n, ||x_0|| = 1 \). For \( x \in \mathbb{C}^n \) such that \( ||x|| = 1 \), and \( A = \{a^1, a^2\} \) i.i.d. complex Gaussians, let

\[
\mathcal{E}_{x_0}^x(x) = x^*Cx,
\]

where \( C = \mathbb{E}(y(R^1a^1a^1* - R^2a^2a^2*)) \) and \( y = \text{sign}(b^1 - b^2) \), \( R^1 = \frac{b^1}{b^1 + b^2}, R^2 = \frac{b^2}{b^1 + b^2} \), and \( b^j = |\langle a^j, x_0 \rangle|^2, j = 1, 2 \). Moreover, let

\[
\hat{\mathcal{E}}_{x_0}^x(x) = x^*\hat{C}_m x,
\]

where \( \hat{C}_m = \frac{1}{m}\sum_{i=1}^{m} y_i(R^1_{i}a^1_{i}a_{i}^1* - R^2_{i}a^2_{i}a_{i}^2*), y_i = \text{sign}(b^1_{i} - b^2_{i}) \), \( R^j_{i} = \frac{b^j_{i}}{b^1_{i} + b^2_{i}}, j = 1, 2 \) and \( A_i = \{(a^1_i, a^2_i)\}, i = 1 \ldots m \) are i.i.d. complex Gaussians.

We first state Proposition 4, that provides a theoretical justification to the relaxation introduced in the formulation Weighted1bitPhase in (25). The proof of Proposition 4 is given in the appendix B.

**Proposition 4** (Correctness in Expectation). The following statements hold:

1. For all \( x \in \mathbb{C}^n, ||x|| = 1 \), we have the following equality,

\[
\mathcal{E}_{x_0}^x(x) = x^*C x = \frac{1}{2} |\langle x_0, x \rangle|^2 + \frac{1}{2}.
\]  

2. The maximum eigenvector of \( C \) is of the form \( x_0 e^{i\phi} \), where \( \phi \in [0, 2\pi] \).

Proposition 4 suggests that \( x_0 \) can be recovered up to a global phase as a maximum eigen value of the matrix \( C \). The rest of the proof consists in proving the concentration of the empirical matrix \( \hat{C}_m \) around its mean \( C \). The proof architecture is the same presented in Section 6.1.

The following lemma states a comparison equality that relates the excess risk to the distance to the optimum.

**Lemma 3** (Excess Risk). For any \( x \in \mathbb{C}^n, ||x|| = 1 \), the following equality holds:

\[
\mathcal{E}_{x_0}^x(x_0) - \mathcal{E}_{x_0}^x(x) = \frac{1}{4} ||xx^* - x_0x_0^*||_F^2.
\]  

The Rest of the proof consists in bounding the excess risk \( \mathcal{E}_{x_0}^x(x_0) - \mathcal{E}_{x_0}^x(x) \) using empirical processes tools and Non commutative Matrix Bernstein inequality. Note that the boundedness of \( (R^1, R^2) \) simplifies at that point the analysis and the proofs are a straightforward adaptation of the one presented in Section 6.1.
7 Computational Aspects

7.1 One bit Phase Retrieval Algorithms

A straightforward computation of the maximum eigenvector of the matrix $\hat{C}_m$ is expensive. One needs $O(n^2m)$ operations to compute the matrix $\hat{C}_m$, that is $O(n^3 \log(n))$, and then the computation of the first eigenvector requires $O(n^3)$ operations. The total computational cost is therefore $O(n^3 \log(n)) + n^2)$, and is dominated by the cost of computing $\hat{C}_m$.

An elegant method to avoid that overhead is the power method. The power method allows for the computation of the maximum eigenvector without having to compute the matrix $\hat{C}_m$, this reduce drastically the computational cost to $O(nm)$ at each iteration of the power method, that is $O(n^2 \log(n))$.

In the following we discuss 2 algorithms:

1. 1bitPhasePower: One bit Phase retrieval via the Power Method given in Algorithm 3.

2. Weighted1bitPhasePower: Weighted One bit Phase retrieval via the Power Method given in Algorithm 4.

Algorithm 3 1bitPhasePower

1: procedure 1BITPHASEPOWER$(A, y, \epsilon)$
2: Initialize $r_0$ at random, $j = 1$.
3: while $||r_j - r_{j-1}|| > \epsilon$ or $j = 1$ do
4: \[ r_j \leftarrow \frac{1}{m} \sum_{i=1}^{m} y_i \left( \langle a_1^i, r_{j-1} \rangle a_1^i - \langle a_2^i, r_{j-1} \rangle a_2^i \right) \]
5: \[ \hat{\lambda} \leftarrow ||r_j|| \]
6: \[ r_j \leftarrow \frac{r_j}{\hat{\lambda}} \]
7: \[ j \leftarrow j + 1 \]
8: end while
9: return $(\hat{\lambda}, r)$ \hspace{1cm} \triangleright $(\hat{\lambda}, r)$ is an estimate of $(\lambda, x_0)$.
10: end procedure

Algorithm 4 Weighted 1bitPhasePower

1: procedure WEIGHTED1BITPHASEPOWER$(A, y, R, \epsilon)$
2: Initialize $r_0$ at random, $j = 1$.
3: while $||r_j - r_{j-1}|| > \epsilon$ or $j = 1$ do
4: \[ r_j \leftarrow \frac{1}{m} \sum_{i=1}^{m} (R_1^i \langle a_1^i, r_{j-1} \rangle a_1^i - R_2^i \langle a_2^i, r_{j-1} \rangle a_2^i) \]
5: \[ \hat{\lambda} \leftarrow ||r_j|| \]
6: \[ r_j \leftarrow \frac{r_j}{\hat{\lambda}} \]
7: \[ j \leftarrow j + 1 \]
8: end while
9: return $(\hat{\lambda}, r)$ \hspace{1cm} \triangleright $(\hat{\lambda}, r)$ is an estimate of $(\lambda, x_0)$.
10: end procedure

7.2 Alternating Minimization Algorithms

Given an initialization the algorithm amount to simply solving a Least Squares that can be solved using conjugated gradient method that needs $O(mn)$ iterations.
8 Numerical Experiments

8.1 Robustness to distortion

We consider a signal $x_0 \in \mathbb{C}^n$ which is a a random complex Gaussian vector with i.i.d. entries of the form $x_0[j] = X + iY$, where $X, Y \sim \mathcal{N}(0, \frac{1}{2})$. Let $n = 128$, $\epsilon = 0.25$. We consider $r = 1/(\epsilon^2)\lfloor \log(n) \rfloor = 64$ and set $m = rn$. So that the total number of measurements is $2m$. We assume that we measure distorted (clipped) measurements according to the model:

$$b_1^i = \tanh(\alpha |\langle a_1^i, x_0 \rangle|^2), \quad b_2^i = \tanh(\alpha |\langle a_2^i, x_0 \rangle|^2) \quad a_1^i, a_2^i \sim \mathcal{CN}(0, I_n), i = 1 \ldots m.$$ 

$\alpha$ corresponds to the level of distortion. The distortion is more severe as $\alpha$ increases. In figure 3 we plot the error of recovery $1 - \text{dist}(\hat{x}_m, x_0) := 1 - |\langle \hat{x}_m, x_0 \rangle|^2$. Where $\hat{x}_m$ is either the solution of 1bitPhase or SubExpPhase. We see that one bit phase retrieval robustly recovers the signal while traditional approaches (SubExpPhase for instance) fail under sever distortions.

8.2 One Bit Phase Retrieval and Alternating Minimization

We consider the problem of recovering the phase from coded diffractions patterns [CLM13] or so the called Fourier masks [EJCV13]. Let $F$ be the discrete Fourier Matrix, In this setting we measure :

$$b = |FDiag(w)x_0|^2 \in \mathbb{R}_n^+,$$

where $w \sim \mathcal{CN}(0, I_n)$ is a Gaussian complex random mask. We generate a Gaussian random $x_0$ signal of dimension $n = 8000$. Let $m = rn$ we set $r = 4$. We measure :

$$b_1^i = |\langle FDiag(w_1^i), x_0 \rangle|^2 + \sigma. \max(\epsilon_i, 0), b_2^i = |\langle FDiag(w_2^i), x_0 \rangle|^2 + \sigma. \max(\epsilon_i, 0) \quad i = 1 \ldots r.$$ 

where $w_1^i, w_2^i \sim \mathcal{CN}(0, I_n)$, and $\epsilon_i \sim \mathcal{N}(0, I_n)$ is a random noise. We split the measurements in 2 sets of size $m, \{b_1^i, b_2^i, i = 1 \ldots m\}$ each and compute: $y_i = \text{sign}(b_1^i - b_2^i) \in \{-1, 1\}^n, i = 1 \ldots r$, and $R_1^i = \frac{b_1^i}{b_1^i + b_2^i}, R_2^i = \frac{b_2^i}{b_1^i + b_2^i}$ (ratio by coordinate). We then compute the solution of SubExpPhase,1bitPhase and Weigthed1bitPhase. We then run the alternating minimization initialized with one of those solutions as well as a random initialization. Note that all the algorithms can be now much faster.
thanks to the Fast Fourier transform. In figure 5(a) we see that in the noiseless setting all approaches converge, the convergence is faster for one bit variants in high dimension. In figure 5(b, 5(c), 5(d) we see that one bit variants are more robust in the noisy setting.

![Graph](image)

(a) Error $1 - |\langle x, x_0 \rangle|^2$ versus Iterations of AltMinPhase, (b) Error $1 - |\langle x, x_0 \rangle|^2$ versus Iterations of AltMinPhase, for $n = 8000$, and a total measurements $8n$ in the noiseless $n = 8000$ and a total measurements $8n$ in the noisy setting $\sigma = 0.4$.

![Graph](image)

(c) Error $1 - |\langle x, x_0 \rangle|^2$ versus Iterations of AltMinPhase, for (d) Error $1 - |\langle x, x_0 \rangle|^2$ versus Iterations of AltMinPhase, for $n = 8000$ and a total measurements $8n$ in the noisy setting $n = 8000$ and a total measurements $8n$ in the noisy setting $\sigma = 0.8$.

Figure 5: Alternating minimization convergence with different initializations: Random Initialization, 1bitPhase, Weighted1bitPhase, and SubExpPhase, in the noisy and noiseless setting.

9 Acknowledgements

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A One Bit Phase Retrieval

A.1 Proofs of Propositions 1, 2, 3

Proof of Proposition 1. i- For \( x \in \mathbb{C}^n, ||x|| = 1 \),

\[
\mathcal{E}^{x_0}(x) = \mathbb{E} \left( y \left( |\langle a^1, x \rangle|^2 - |\langle a^2, x \rangle|^2 \right) \right),
\]

(34)

where \( y = \text{sign}(|\langle a^1, x_0 \rangle|^2 - |\langle a^2, x_0 \rangle|^2) \). Recall \( a^1, a^2 \sim \mathcal{CN}(0, I_n) \) are complex Gaussian vectors, there exists \( g, h \sim \mathcal{N}(0, \frac{1}{2}) + i \mathcal{N}(0, \frac{1}{2}) \) i.i.d. and \( G, H \sim \mathcal{N}(0, \frac{1}{2}) + i \mathcal{N}(0, \frac{1}{2}) \) i.i.d.,

\[
\langle a^1, x_0 \rangle = g, \quad \langle a^2, x_0 \rangle = h.
\]

\[
\langle a^1, x \rangle = \langle a^2, x \rangle = \langle x_0, x \rangle g + \sqrt{1 - |\langle x_0, x \rangle|^2} h.
\]

\[
|\langle a^1, x \rangle|^2 - |\langle a^2, x \rangle|^2 = |\langle x_0, x \rangle g + \sqrt{1 - |\langle x_0, x \rangle|^2} h|^2 - |\langle x_0, x \rangle G + \sqrt{1 - |\langle x_0, x \rangle|^2} H|^2
\]

\[
= |\langle x_0, x \rangle|^2 (|g|^2 - |G|^2) + (1 - |\langle x_0, x \rangle|^2) (|h|^2 - |H|^2) + 2 \Re \left( \langle x_0, x \rangle \sqrt{1 - |\langle x_0, x \rangle|^2} (g^* h - G^* H) \right)
\]

Recall that \( y = \text{sign}(\theta(|g|^2) - \theta(|G|^2)) \). From Lemma 4, we know that \( E_1 = |g|^2 \), and \( E_2 = |G|^2 \) are two exponential independent random variables \( \text{Exp}(1) \).

\[
\mathcal{E}^{x_0}(x) = \mathbb{E} \left( y \left( |\langle a^1, x \rangle|^2 - |\langle a^2, x \rangle|^2 \right) \right)
\]

\[
= |\langle x_0, x \rangle|^2 \mathbb{E} \left( \text{sign}(\theta(E_1) - \theta(E_2)) (E_1 - E_2) \right)
\]

\[
= \lambda |\langle x_0, x \rangle|^2.
\]

ii- Let \( y = Q^\theta_A(x_0) \), by (i) we have that,

\[
\mathbb{E}(y(a^1 a^{1,*} - a^2 a^{2,*}) x x^*)_F = \lambda \langle xx^*, x_0 x_0^* \rangle_F, \forall x, ||x|| = 1.
\]

This means that \( C = \mathbb{E}(y(a^1 a^{1,*} - a^2 a^{2,*})) = \lambda x_0 x_0^* \). Hence \( C \) is a rank one matrix.

iii- By (ii) we have, \( Cx_0 = \lambda x_0 x_0^* x_0 = \lambda x_0, \) since \( ||x_0|| = 1 \).

iv- By (i) maximizers \( ||x|| = 1 x^* Cx = \lambda \max_{||x|| = 1} |\langle x_0, x \rangle|^2 \). It is easy to see that \( x = e^{i\phi} x_0, \phi \in [0, 2\pi] \) are maximizers of the right hand side of the equation.

\[\square\]

Proof of Lemma 1. i. Noiseless:

\( \lambda = \mathbb{E}(\text{sign}(E_1 - E_2)(E_1 - E_2)) = \mathbb{E}(|E_1 - E_2|) = 1 \), since \( E_1 - E_2 \sim \text{Exp}(1) \).

ii. Noisy:

Exponential Noise:

Let \( y = \text{sign}((E_1 + \nu_1) - (E_2 + \nu_2)) \). Let \( L = E_1 - E_2 \), \( L \) follows a Laplace distribution with mean 0 and scale parameter 1:

\( L \sim \text{Laplace}(0, 1) \).
Let \( N = \nu_1 - \nu_2 \), \( N \) follows a Laplace distribution, \( N \sim \text{Laplace}(0, \frac{1}{\gamma}) \). It follows that:

\[
\lambda = \mathbb{E}_{L,N}(\text{sign}(L + N)L) \\
= \mathbb{E}_L((1 - 2\mathbb{P}_N(N \leq -L))L) \\
= \mathbb{E}_L((1 - 2F_N(-L))L) \\
= \mathbb{E}_L\left\{ \left(1 - 2\left(\frac{1}{2} + \frac{1}{2}\text{sign}(-L)(1 - \exp(-\gamma|L|))\right)\right)L \right\} \\
= \mathbb{E}_L(\text{sign}(L)(1 - \exp(-\gamma|L|))L) \\
= \mathbb{E}_L[L(1 - \exp(-\gamma|L|))L] \\
= 1 - \int_0^{+\infty} z \exp(-\gamma z) \exp(-z)dz \\
= 1 - \frac{1}{(1 + \gamma)^2} > 0.
\]

Let \( \sigma = \frac{1}{\sqrt{\gamma}} \) be the variance of the exponential noise. We conclude that:

\[
\lambda = \frac{1 + 2\sqrt{\sigma}}{(1 + \sqrt{\sigma})^2}.
\]

**Poisson Noise:**

\[
\lambda = \mathbb{E}(\text{sign}(p_1 - p_2)(E_1 - E_2)), \quad p_1|E_1 \sim \text{Poisson}\left(\frac{E_1}{\eta}\right), \quad p_2|E_2 \sim \text{Poisson}\left(\frac{E_2}{\eta}\right).
\]

We know that:

\[
S(E_1, E_2) = p_1 - p_2| (E_1, E_2) \sim \text{Skellam}\left(\frac{E_1}{\eta}, \frac{E_2}{\eta}\right).
\]

Hence:

\[
\lambda = \mathbb{E}(\text{sign}(S(E_1, E_2))(E_1 - E_2)).
\]

### iii. Distortion:

\[
y = \text{sign}(\tanh(\alpha E_1) - \tanh(\alpha E_2)) \\
= \text{sign}(\tanh(\alpha(E_1 - E_2))) \cdot (1 - \tanh(\alpha E_1) \tanh(\alpha E_2)) \\
= \text{sign}(\tanh(E_1 - E_2)) \cdot \text{sign}(1 - \tanh(\alpha E_1) \tanh(\alpha E_2)) \\
= \text{sign}(E_1 - E_2) \cdot \text{sign}(1 - \tanh(\alpha E_1) \tanh(\alpha E_2))
\]

\[
\lambda = \mathbb{E}(y(E_1 - E_2)) = \mathbb{E}(\text{sign}(1 - \tanh(\alpha E_1) \tanh(\alpha E_2)) | E_1 - E_2|) > 0.
\]

**Proof of Lemma 2.** \( \mathcal{E}^{x_0}(x_0) - \mathcal{E}^{x_0}(x) = \lambda(1 - |\langle x_0, x \rangle|^2) = \frac{\lambda}{2} \|xx^* - x_0x_0^*\|_F^2 \), since \( x_0 \) and \( x \) are unitary.

**Proof of Proposition 2.** Following the classical approach to study empirical risk minimization in statistical learning theory we have,

\[
\mathcal{E}^{x_0}(x_0) - \mathcal{E}^{x_0}(\hat{x}_m) \leq \mathbb{E} \sup_{x, \|x\|=1} \left| \mathcal{E}^{x_0}(x) - \mathcal{E}^{x_0}(x) \right|.
\]
\[ \hat{\mathcal{E}}^{x_0}(x) - \mathcal{E}^{x_0}(x) = \left( x^*(\hat{C}_m - C)x \right) \]

Hence:
\[
\frac{\lambda}{2} \| \hat{x}_m \hat{x}_n - x_0 x_0^* \|^2_F \leq 2 \sup_{x, \|x\|=1} x^*(\hat{C}_m - C)x = 2 \left\| \hat{C}_m - C \right\|, \tag{35}
\]

where \( \hat{C}_m = \frac{1}{m} \sum_{i=1}^m C_i, C_i = y_i(\Delta_i), \Delta_i = a_i^1 a_i^{1*} - a_i^2 a_i^{2*} \) and \( C = E(y(a^1 a^{1*} - a^2 a^{2*})) = \lambda x_0 x_0^* \).

**Proof of Proposition 3.** Let

\[ X_i = \frac{1}{m} (y_i(a_i^1 a_i^{1*} - a_i^2 a_i^{2*}) - \lambda x_0 x_0^*). \]

We would like to get a bound on \( \| \sum_{i=1}^m X_i \| \), the main technical issue goes to the fact that \( \| X_i \| \) are not bounded almost surely. We will address that issue by rejecting samples outside the ball of radius \( \sqrt{M} \), where \( M \) is defined in the following.

Let \( M = 2n(1 + \beta)^2 \). Let \( E = \{ (a^1, a^2), \| a^1 \|^2 \leq M \text{ and } \| a^2 \|^2 \leq M \} \). Let

\[ (\hat{a}_i^1, \hat{a}_i^2) = (a_i^1, a_i^2) \text{ if } (a_i^1, a_i^2) \in E \text{ and 0 otherwise.} \]

Let

\[ \bar{y}_i = \text{sign} \left( \| \langle a_i^1, x_0 \rangle \|^2 - \| \langle a_i^2, x_0 \rangle \|^2 \right) \text{ if } (a_i^1, a_i^2) \in E \text{ and 0 otherwise.} \]

Let

\[ \bar{C}_m = \frac{1}{m} \sum_{i=1}^m \bar{y}_i (a_i^1 a_i^{1*} - a_i^2 a_i^{2*}) \quad \bar{C} = E(\bar{C}_m). \]

Note that \( \bar{C}_m \) is the sum of bounded random variable, so that we can use non commutative matrix Bernstein inequality given in Theorem 5, in order to bound \( \| \bar{C}_m - \bar{C} \| \). On the other hand by the triangular inequality we have:

\[
\| C_m - C \| \leq \| C_m - \bar{C}_m \| + \| \bar{C}_m - \bar{C} \| + \| \bar{C} - C \| \tag{36}
\]

**Bounding \( \| C_m - \bar{C}_m \| :**

Note that \( \| a \|^2 \sim \chi_{2n}^2 \), \( \| a \| \) is a Lipchitz function of Gaussian with constant one. A Gaussian concentration bound implies:

\[
P(\| a_i \|^2 \geq (\sqrt{2n} + t)^2) \leq e^{-\frac{t^2}{2}}. \tag{37}
\]

Setting \( t = \beta \sqrt{2n} \), it follows that:

\[
P(\| a_i \|^2 \geq 2n(1 + \beta)^2) \leq e^{-\beta^2 n}. \tag{38}
\]

\[
P(\max_{i=1, \ldots, m, j=1,2} \| a_i^j \|^2 > M) \leq 2mP(\| a \|^2 > M) \leq 2me^{-\beta^2 n}.
\]
It follows that:
\[ |C_m - \tilde{C}_m| = 0 \text{ with probability at least } 1 - 2me^{-\beta^2 n}. \]

**Bounding** \[||\tilde{C} - C||\]:

Let
\[ \tilde{X}_i = \frac{1}{m} \left( \tilde{y}_i (\tilde{a}_i^1 \tilde{a}_i^{1,*} - \tilde{a}_i^2 \tilde{a}_i^{2,*}) - \tilde{C} \right). \]

It is easy to see that \[||\tilde{C}|| \leq ||C|| = \lambda.\]

\[ ||\tilde{X}_i|| \leq \frac{1}{m} (||\tilde{y}_i (\tilde{a}_i^1 \tilde{a}_i^{1,*})|| + ||\tilde{y}_i (\tilde{a}_i^2 \tilde{a}_i^{2,*})|| + ||\tilde{C}||) = \frac{1}{m} (||\tilde{a}_i^1||^2 + ||\tilde{a}_i^2||^2 + \lambda) \leq \frac{2M + \lambda}{m} \leq \frac{4M + \lambda}{m} = K. \]

\[ \tilde{X}_i^2 = \frac{1}{m^2} \left( \tilde{y}_i (\tilde{a}_i^1 \tilde{a}_i^{1,*} - \tilde{a}_i^2 \tilde{a}_i^{2,*} - \tilde{C}) \right) \left( \tilde{y}_i (\tilde{a}_i^1 \tilde{a}_i^{1,*} - \tilde{a}_i^2 \tilde{a}_i^{2,*} - \tilde{C}) \right) \]
\[ = \frac{1}{m^2} \left( ||\tilde{a}_i^1||^2 ||\tilde{a}_i^1||^* + ||\tilde{a}_i^2||^2 ||\tilde{a}_i^2||^* - \langle \tilde{a}_i^1, \tilde{a}_i^2 \rangle \tilde{a}_i^1 \tilde{a}_i^2 - \langle \tilde{a}_i^1, \tilde{a}_i^2 \rangle \tilde{a}_i^2 \tilde{a}_i^1 + \tilde{C}_i^2 - \tilde{C}_i \right) \]

Note that \( \mathbb{E}(\langle a_i^1, a_i^2 \rangle a_i^1 a_i^2) = \mathbb{E}(a_i^1 a_i^1) = 1 \) by independence, \( \mathbb{E}(a_i^1 a_i^1) = 1 \), and \( \mathbb{E}(\tilde{C}_i \tilde{C}_i) = \tilde{C}_i^2 \), by definition.

On the other hand \( \mathbb{E}(\langle \tilde{a}_i^1, \tilde{a}_i^2 \rangle \tilde{a}_i^1 \tilde{a}_i^2) \) is zero on the off diagonal and less than one on the diagonal. Hence
\[ ||\mathbb{E}(\langle \tilde{a}_i^1, \tilde{a}_i^2 \rangle \tilde{a}_i^1 \tilde{a}_i^2) || \leq 1. \]

Also \( \mathbb{E}(\tilde{a}_i^1 a_i^1 a_i^1) \) is zero on the off diagonal and less than one on the diagonal, hence:
\[ ||\mathbb{E}(\tilde{a}_i^1 a_i^1 a_i^1) || \leq 1. \]

It follows that
\[ ||\mathbb{E}(||\tilde{a}_i^1||^2 a_i^1 a_i^1)|| \leq M ||\mathbb{E}(\tilde{a}_i^1 a_i^1)|| \leq M. \]

Taking the operator norm we have:
\[ ||\mathbb{E}(\tilde{X}_i^2)|| \leq \frac{1}{m^2} (2M + \lambda^2 + 2) \leq \frac{1}{m^2} (4M + \lambda^2). \]

Finally:
\[ \left| \sum_{i=1}^{m} \mathbb{E}(\tilde{X}_i^2) \right| \leq m \max_i ||\tilde{X}_i^2|| \leq \frac{4M + \lambda^2}{m} = \sigma^2. \]

The rest of the proof of this part is an adaptation of the proof of Theorem 6 in [Ver11] on covariance estimation of heavy tailed matrices.

We are now ready to apply the non commutative Bernstein’s inequality:
\[ \mathbb{P} \left( \left| \sum_{i=1}^{m} \tilde{X}_i \right| \geq \epsilon \right) \leq 2n \exp \left( -c \min \left( \frac{\epsilon^2}{\sigma}, \frac{\epsilon}{K} \right) \right) \leq 2n \exp \left( -c \min \left( \frac{\epsilon^2}{4M + \lambda^2}, \frac{\epsilon}{4M + \lambda} \right) m \right) \]

Clearly \( \lambda \leq 1 \), by definition. Hence \( \lambda \leq 4M \).

\[ \min \left( \frac{\epsilon^2}{4M + \lambda^2}, \frac{\epsilon}{4M + \lambda} \right) = \frac{1}{4M} \min \left( \frac{\epsilon^2}{\lambda (\frac{\lambda}{4M} + \frac{1}{\lambda})}, \frac{\epsilon}{1 + \frac{\lambda}{4M}} \right) \geq \frac{1}{4M} \min \left( \frac{\epsilon^2}{\lambda}, \epsilon \right) \]
Let \( \epsilon = \max(\sqrt{\lambda} \delta, \delta^2) \), \( \delta = s \sqrt{\frac{4M}{m}} \). It follows that:

\[
P \left( \left\| \sum_{i=1}^{m} \bar{X}_i \right\| \geq \epsilon \right) \leq 2n \exp \left( -c\delta^2 \frac{m}{4M} \right) = 2n \exp(-cs^2).
\]

Therefore we have with a probability at least \( 1 - 2n \exp(-cs^2) \):

\[
||\tilde{C}_m - \bar{C}|| \leq \max(\sqrt{\lambda} \delta, \delta^2) \quad \delta = s \sqrt{\frac{4M}{m}}.
\]

Setting \( s = \sqrt{\log(n)} \), we have finally:

\[
||\tilde{C}_m - \bar{C}|| \leq \max(\sqrt{\lambda} \delta, \delta^2) \quad \delta = t \sqrt{\frac{4M \log(n)}{m}} \text{ with probability at least } 1 - n^{\frac{1}{2}}.
\]

**Bounding \( ||\tilde{C} - C|| \):**

By the rotation invariance of Gaussian we can assume \( x_0 = (1, 0, \ldots, 0) \).

The off diagonal terms of \( \mathbb{E}(\bar{y}(\bar{a} \cdot \tilde{a}) - a^2 \bar{a} \cdot \tilde{a}^{*}) \) are zero. The same holds for \( \mathbb{E}(\bar{y}(a \cdot a^{*} - a^2 a^{*}) \).

The only term that is non zero on the diagonal is first one.

\[
||\tilde{C} - C|| = \mathbb{E}(\bar{y}(|a_1|^2 - |a_1^2|^2)1_{(a^1, a^2) \notin E})
\]

\[
\leq \left( \mathbb{E}(y^2(|a_1|^2 - |a_1^2|^2)) \right)^\frac{1}{2} \left( \mathbb{E}(1_{E^c}) \right)^\frac{1}{2}
\]

\[
= \left( \mathbb{E}(|a_1|^4 + |a_2|^4 - 2|a_1|^2|a_2|^2) \right)^\frac{1}{2} \sqrt{\mathbb{P}(E^c)}
\]

\[
\leq \sqrt{2} + 2 - 2\sqrt{2e^{-\beta^2n}}
\]

\[
= 2e^{\beta^2n/2}.
\]

**Putting all together:**

Setting \( \beta = t = \sqrt{2} \). For \( 2m = \hat{c}n \), \( 1 < \hat{c} < n \). We have with probability at least \( 1 - O(\frac{1}{n^2}) \), since \( (1 - 2me^{-2n}) \sim (1 - O(1/n^2)) \), for sufficiently large \( n \):

\[
||\tilde{C}_m - C|| \leq c \sqrt{\frac{4M \log(n)}{m}} + 2e^{-n}
\]

where \( M = 2n(1 + \sqrt{2})^2 \). There exists a constant \( c', \epsilon \in [0, 1] \) such that:

For \( m \geq \frac{c' n \log(n)}{\lambda^2} \) \( \|\tilde{C}_m - C\| \leq \epsilon \lambda \) with probability at least \( 1 - O(n^{-2}) \).

By equation (35) we conclude:

For \( m \geq \frac{c'' n \log(n)}{\lambda^2} \) \( \|\dot{x}_m - x_0 e^{i\phi}\|^2 \leq \|\dot{x}_m - x_0 x_0^{*}\|^2 \leq \epsilon \) with probability at least \( 1 - O(n^{-2}) \).

where \( \phi \) is a global phase.
A.2 Technical Tools

Here, we collect a few technical results needed in the proofs.

**Lemma 4** (Spacing of Exponentials). For \( x_0 \in \mathbb{C}^n, \) and \( a \sim \mathcal{N}(0, \frac{1}{2} I_n) + i \mathcal{N}(0, \frac{1}{2} I_n), \) \( E = |\langle a, x_0 \rangle|^2 \) follows an exponential distribution with parameter one (see [SW] for a proof). Moreover [SW], if we let and \( a^1, a^2 \) i.i.d. \( \sim \mathcal{N}(0, \frac{1}{2} I_n) + i \mathcal{N}(0, \frac{1}{2} I_n), \) let \( E_1(x_0) = |\langle a^1, x_0 \rangle|^2, \) and \( E_2(x_0) = |\langle a^2, x_0 \rangle|^2, \) and \( E^{(1)}(x_0) \) and \( E^{(2)}(x_0), \) then the corresponding order statistics i.e \( E^{(2)}(x_0) \geq E^{(1)}(x_0). \) \( \Delta(x_0, x_0) = |\langle a^{(2)}, x_0 \rangle|^2 - |\langle a^{(1)}, x_0 \rangle|^2 \) is also exponentially distributed with parameter one. \( \Delta \) is called spacing of order statistics of exponentials.

**Lemma 5** (Laplace Exponential). \( U \sim \exp(\gamma), \) \( V \sim \exp(\gamma), \) \( U \) and \( V \) are independent then \( U - V \sim \text{Laplace}(0, \frac{1}{\gamma}). \) The CDF of \( U - V \) is:

\[
F_{U-V}(z) = \frac{1}{2} + \frac{1}{2} \text{sign}(z)(1 - \exp(-\gamma |z|)).
\]

**Lemma 6** (Uniform Ratio [Mro10]). If \( X \) and \( Y \) are two independent chi-square variables with 2\( a \) and 2\( b \) degrees of freedom respectively, then \( Z = \frac{X}{X+Y} \) has the beta distribution with parameter \( a \) and \( b, \beta(a,b). \)

For \( a = b = 1: \) If \( X \) and \( Y \) are two independent Exponential random variable with mean one then \( Z = \frac{X}{X+Y} \) is uniformly distributed \( \text{Unif}[0,1]. \)

**Theorem 5** (Non commutative Bernstein Inequality [Ver11]). Consider a finite sequence \( X_i \) of independent centered self adjoint random \( n \times n \) matrices. Assume we have for some numbers \( K \) and \( \sigma \) that:

\[
\|X_i\| \leq K \text{ almost surely} \quad \|\sum X_i^2\| \leq \sigma^2.
\]

Then, for every \( t > 0, \) we have:

\[
P\{\|\sum X_i\| > t\} \leq 2n \exp \left(\frac{-t^2/2}{\sigma^2 + K t/3}\right).
\]

**Theorem 6** (Covariance Estimation for Arbitrary distributions [Ver11]). Consider a distribution with covariance matrix \( \Sigma \) supposed in some centered ball whose radius we denote \( \sqrt{M}. \) Let \( \Sigma_m \) be the empirical covariance. Let \( \epsilon \in [0,1], \) and \( t \geq 1. \) Then the following holds with probability at least \( 1 - n^{-t^2}:
\]

If \( m \geq C(t/\epsilon)^2\|\Sigma\|^{-1} M \log(n) \) then \( \|\Sigma_m - \Sigma\| \leq \epsilon\|\Sigma\|.
\]

B Weighted one Bit Phase Retrieval

**Proof of Proposition 4.** Recall that \( E^{x_0}(x) = x^* C x,\) where \( C = \mathbb{E}(y(R^1 a^1 a^{1,*} - R^2 a^2 a^{2,*})) \). where \( y = \text{sign}(b^1 - b^2), \) \( R^1 = \frac{b^1}{\sqrt{b^1+b^2}}, R^2 = \frac{b^2}{\sqrt{b^1+b^2}}, \) and \( b^j = |\langle a^j, x_0 \rangle|^2, j = 1, 2. \) Recall \( a^1, a^2 \sim \mathcal{CN}(0, I_n) \) are complex Gaussian vectors, there exists \( g, h \sim \mathcal{N}(0, \frac{1}{2}) + i \mathcal{N}(0, \frac{1}{2}) \) i.i.d. and \( G, H \sim \mathcal{N}(0, \frac{1}{2}) + \)
\(i \mathcal{N}(0, \frac{1}{2})\) i.i.d.,

\[
\begin{align*}
\langle a^1, x_0 \rangle &= g, \quad \langle a^1, x \rangle = \langle x_0, x \rangle g + \sqrt{1 - |\langle x_0, x \rangle|^2} h. \\
\langle a^2, x_0 \rangle &= G, \quad \langle a^2, x \rangle = \langle x_0, x \rangle G + \sqrt{1 - |\langle x_0, x \rangle|^2} H.
\end{align*}
\]

\[
R^1|\langle a^1, x \rangle|^2 - R^2|\langle a^2, x \rangle|^2 = R^1|\langle x_0, x \rangle g + \sqrt{1 - |\langle x_0, x \rangle|^2} h|^2 - R^2|\langle x_0, x \rangle G + \sqrt{1 - |\langle x_0, x \rangle|^2} H|^2
\]

\[
= |\langle x_0, x \rangle|^2 (R^1|g|^2 - R^2|G|^2) + (1 - |\langle x_0, x \rangle|^2) (R^1|h|^2 - R^2|H|^2)
\]

\[
+ 2 \Re \left( \langle x_0, x \rangle \sqrt{1 - |\langle x_0, x \rangle|^2} (R^1 g^* h - R^2 G^* H) \right)
\]

\[
= |\langle x_0, x \rangle|^2 (b^1 - b^2) + (1 - |\langle x_0, x \rangle|^2) (R^1|h|^2 - R^2|H|^2)
\]

\[
+ 2 \Re \left( \langle x_0, x \rangle \sqrt{1 - |\langle x_0, x \rangle|^2} (R^1 g^* h - R^2 G^* H) \right).
\]

Taking the expectation we get finally:

\[
\mathcal{E}^{x_0}(x) = \mathbb{E}(g(R^1|\langle a^1, x \rangle|^2 - R^2|\langle a^2, x \rangle|^2))
\]

\[
= |\langle x_0, x \rangle|^2 \mathbb{E}(g(b^1 - b^2)) + (1 - |\langle x_0, x \rangle|^2) \mathbb{E}(g(R^1 - R^2))
\]

\[
= \mathbb{E}(|b^1 - b^2| - |R^1 - R^2|) |\langle x_0, x \rangle|^2 + \mathbb{E}(|R^1 - R^2|)
\]

\[
= \frac{1}{2} (|\langle x_0, x \rangle|^2 + 1).
\]

Where the first equality follows from independence and that \(h\) and \(H\) have variance one. The last equality holds since \(|b^1 - b^2|\) is exponentially distributed with mean one. Note also that \(R^1 = U\), where \(U \sim \text{Unif}[0, 1]\), and \(R^2 = 1 - U\). Hence \(|R^1 - R^2| = |2U - 1|\), and \(\mathbb{E}(|R^1 - R^2|) = \mathbb{E}|2U - 1| = \frac{1}{2}\). \(\Box\)

**C** Sub-Exponential Initialization

**Proof of Theorem 3.**

\[
\mathcal{E}^{x_0}(x) = \mathbb{E}(\|g\|^2 |\langle x_0, x \rangle|^2 |g|^2 + (1 - |\langle x_0, x \rangle|^2) |h|^2 + 2 \sqrt{1 - |\langle x_0, x \rangle|^2} \Re(\langle x_0, x \rangle)^* g^* h)
\]

\[
= |\langle x_0, x \rangle|^2 \mathbb{E}(\|g\|^4) + (1 - |\langle x_0, x \rangle|^2) \mathbb{E}(|g|^2) \mathbb{E}(|h|^2)
\]

\[
= 2 |\langle x_0, x \rangle|^2 + (1 - |\langle x_0, x \rangle|^2)
\]

\[
= |\langle x_0, x \rangle|^2 + 1.
\]

The rest of the proof concerns the concentration of \(\hat{C}_m\) around \(C\) is a simple application of Theorem 6. Note that \(C = \mathbb{E}(baa^*)\) we have \(||C|| = 2\). \(b\) is an exponential random variable.

\[
|b||a||^2 \leq 4 \log(m)n,
\]

with high probability, thus we can apply Theorem 6, and get a sample complexity bound. \(\Box\)
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