We study the problem of constructing $N=2$ superconformal algebras out of an $N=1$ affine Lie algebra. Following a recent independent observation of Getzler and the author, we derive a simplified set of $N=2$ master equations, which we then proceed to solve for the case of $sl(2)$. There is a unique construction for all noncritical values of the level, which can be identified as the Kazama–Suzuki coset associated to the hermitian symmetric space $SO(3)/SO(2)$. We also identify the construction with a generalized parafermionic construction or, after bosonization, with a bosonic construction of the type analyzed by Kazama and Suzuki. A mild generalization of this construction can be associated to any embedding $sl(2) \subset g$. 

† e-mail: jmf@avzw01.physik.uni-bonn.de
‡ † Address after September 1, 1993: Physics Department, Queen Mary and Westfield College, London, UK.
In the course of constructing $N=2$ superconformal algebras (SCAs) from Lie algebraic data, the following characterization of an $N=2$ SCA was recently found \cite{1}\cite{2}. Let $G^\pm(z)$ be fields satisfying the following operator product expansions:

$$G^\pm(z)G^\pm(w) = \text{reg.},$$

and

$$G^\pm(z)G^\mp(w) = \frac{2c/3}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2T(w) \pm \partial J(w)}{z-w} + \text{reg.},$$

for some fields $J(z)$, $T(z)$ and constant $c$ defined by the above OPE. If, in addition, $J(z)$ and $G^\pm(z)$ satisfy

$$J(z)G^\pm(w) = \pm \frac{G^\pm(w)}{z-w} + \text{reg.},$$

then it was proven independently in \cite{1} and \cite{2} that $J$, $G^\pm$, and $T$ obey an $N=2$ superconformal algebra with central charge $c$. The purpose of this paper is to use this result to analyze the construction of $N=2$ SCAs using $N=1$ affine Lie algebras.

The study of $N=2$ SCAs is of fundamental importance. Among the conformal field theories, those with $N=2$ superconformal symmetry ($N=2$ SCFTs) are arguably the most interesting both from the point of view of applications and from purely structural considerations. It was realized rather early in modern string theory that $N=2$ superconformal invariance is necessary for spacetime supersymmetry, whence $N=2$ SCFTs span the phenomenologically interesting classical vacua for string theory; for example, string compactifications on a Calabi-Yau space.

On the more mathematical side, $N=2$ SCFTs have a very rich algebraic structure which makes them omnipresent in string theory and topological field theory. It has been realized for some time now that to any $N=2$ SCFT one can associate a cohomology theory which is intimately linked to the topological conformal field theory (TCFT) one obtains after twisting \cite{3}. The cohomology of the TCFT—generically a topological invariant of some moduli space \cite{4}—inherits in the $N=2$ formulation a natural ring structure. This so-called chiral ring\cite{5} is a useful invariant of the theory and accounts for much of the recent popularity of these theories—lying, for instance, at the center of the mirror symmetry revolution\cite{6}. Through the Landau-Ginzburg description of $N=2$ SCFTs, the chiral ring lets one also make contact with singularity theory\cite{7}.

Conversely, there is a growing body of evidence that suggests that to any reasonable cohomology theory there is associated an $N=2$ SCA. This was shown explicitly for the first time in the literature in \cite{8} (see also \cite{9}) for the $c=1$ noncritical string. In \cite{10} this observation was generalized and argued to be a generic property of string theories: be it the “humble” string, the superstring, or the $W$-string. Explicit constructions of the $N=2$ algebra depend on the model under consideration and were given in \cite{10} for the bosonic string (critical and noncritical), the NSR string (critical and noncritical)—where one actually has an $N=3$ SCA—and for the noncritical $W_3$-string, where one finds an extension of $N=2$ by an
$N=2$ primary of weight 2 and charge 0; and in [11] for the $N=2$ string—where one actually has an $N=4$ SCA. For the critical $W_3$ string with matter representation given by the Romans realization of $W_3$ in terms of a free boson and an underlying Virasoro algebra [12] one can also construct such an $N=2$ extended algebra [13] whose central charge is fixed to either of the two values $c = -18$ or $c = -\frac{15}{2}$. It is expected that this continues to be the case for other $W$-string theories built on other $W$-algebras; but proving it by the current means requires knowing (at least the existence of) the BRST charge for the relevant $W$-algebra, for which no general construction is known and hence must be done case by case. The computational complexity soon becomes forbidding (even to a computer) and so far the only other algebras whose BRST charge is known are: $W_4$ [14] [15], $W(2, 4)$ [15] and some special quadratically nonlinear algebras [16]. One can certainly envision a few more algebras to be reached in the near future, but this is far from a general existence proof.

Since $W$-algebras are generically defined via the quantum Drinfel’d-Sokolov reduction [17], it is widely believed that the BRST charge and the underlying $N=2$ structure should also come induced from the analogous structures in the affine Lie algebra from which one reduces. A first step in this direction was given in [1], where the underlying $N=2$ structure for the BRST complex associated to the affinization of a semisimple Lie algebra was unveiled. In contrast with the BRST complexes in string theory, the construction in [1] is completely natural and does not depend on any details of the representation of the affine Lie algebra that one is considering.

Independently and at the same time, Getzler [2] found a very general class of $N=2$ constructions associated to Manin triples. In doing so he makes explicit an observation of [18] concerning the work in [19]. The Getzler constructions interpolate between the construction in [1] and (a deformation of) the Kazama–Suzuki coset construction [20].

Despite all of these constructions now in existence, much fewer constructions are known for $N=2$ SCFTs than for their $N=0$ and $N=1$ counterparts. For example, no natural Sugawara-type construction exists for $N=2$ SCFTs (although see [21]) and already the $N=2$ extension of the coset construction [22] imposes restrictions on the geometry of the coset manifold, requiring it to be Kähler [20]. This kind of rigidity is typical of $N=2$ constructions and we shall encounter it again in this paper.

Although in contrast to the $N=0$ and $N=1$ cases, there is no canonical construction which one is generalizing, we can place this work in the context of generalized $N=2$ superconformal constructions. Generalized Virasoro constructions, conceived originally by Halpern and Kiritsis [23], have so far had a short but lively history which is amply documented in [24]. The $N=1$ case was analyzed originally by [25] and [26] and then further in [27]. For the $N=2$ case much less is known. A set of master equations was derived in an appendix of [26], who verified that the Kazama–Suzuki constructions satisfied the equations, but did not look further. In this paper we will derive a set of master equations for “generalized” $N=2$ constructions which comprise a subset of the equations in [26] and generate the remaining ones. Although the equations have sufficient structure to expect that some results of a general nature can be obtained for all $g$, we will only solve the case of $g = sl(2)$. We find that there is a unique construction, which after some cosmetics can be identified with the Kazama–
Suzuki coset construction associated to $SO(3)/SO(2)$ or with a generalized parafermionic construction. We also identify it with a bosonic construction the type analyzed by Kazama and Suzuki in [28]. We also remark a similarity with the BRST complex of the quantum Drinfel’d–Sokolov reduction that suggests a conjecture described in the conclusions.

The paper is organized as follows. In Section 2 we derive the $N=2$ master equations associated to an $N=1$ affine Lie algebra at level $k$ and prove that there are no nontrivial solutions for $k = 0$. This means that we are free to decouple the fermions, which we do in Section 3 where we derive the simplified master equations. In Section 4 we solve the master equations for the simplest case $\mathfrak{g} = sl(2)$. The solution benefits from a geometrization of the master equation that is particular to $sl(2)$, but makes the underlying mechanism very transparent. A mild deformation of this construction then applies to an arbitrary simple Lie algebra and we discuss this in Section 5. In Section 6 we identify this construction. It turns out to be equivalent to the Kazama–Suzuki coset construction associated to $SO(3)/SO(2)$, to a (generalized) parafermionic construction and to a bosonic construction of the type analyzed by Kazama and Suzuki. Finally in section 7 we summarize our results.

§2 The $N=2$ Master Equations

Let $\mathfrak{g}$ be a simple complex Lie algebra with a fixed invariant metric $\langle , \rangle$. Let us fix a basis $\{ X_i \}$ for $\mathfrak{g}$. Relative to this basis, $\langle X_i , X_j \rangle = g_{ij}$ and $[X_i , X_j] = f_{ijk} X_k$. The $N=1$ affine Lie algebra (sometimes called a super Kač–Moody algebra, or a Kač–Todorov algebra, with some abuse of notation) associated to $\mathfrak{g}$ is the natural $N=1$ supersymmetric extension of the affine Lie algebra $\hat{\mathfrak{g}}$. It is generated by currents $I_i(z)$ and $\psi_i(z)$ obeying the following OPEs:

$$I_i(z)I_j(w) = \frac{kg_{ij}}{(z-w)^2} + \frac{f_{ijk}I_k(w)}{z-w} + \text{reg.}, \quad (2.1)$$

$$I_i(z)\psi_j(w) = \frac{f_{ijk}\psi_k(w)}{z-w} + \text{reg.}, \quad (2.2)$$

and

$$\psi_i(z)\psi_j(w) = \frac{kg_{ij}}{z-w} + \text{reg.}, \quad (2.3)$$

It is well-known that for $k \neq 0$ one can decouple the fermions by introducing currents

$$J_i(z) \equiv I_i(z) - \frac{1}{2k} f_{ijk} g^{jl}(\psi_k \psi_l)(z) \quad (2.4)$$

which have a regular OPE with the fermions $\psi_i(z)$. The $J_i(z)$ satisfy (2.1) with a shifted level $k - \frac{1}{2}c_\mathfrak{g}$, with $c_\mathfrak{g}$ being defined as the eigenvalue of the quadratic casimir $g^{ij}\text{ad}X_i\text{ad}X_j$ in the adjoint representation. One may worry that after decoupling, one may lose solutions at $k = 0$, but a closer examination shows that the only solution for $k = 0$ is the trivial one where all fields are zero. We will see this after we derive the master equations.
In order to consider the construction of an $N=2$ SCA with this data, and according to the result quoted in the introduction, one need only write the most general $G^\pm(z)$

$$G^\pm(z) = A^i_\pm (I_i \psi_j)(z) + B^{ijk}_\pm (\psi_i \psi_j \psi_k)(z) + C^i_\pm \partial \psi_i(z),$$

(2.5)

where $B^{ijk}_\pm$ is totally antisymmetric and $A^i_\pm$ is arbitrary, and then derive the equations on the $\frac{1}{3}n^2 + n^2 + \frac{8}{3}n + 1$ ($n = \dim g$) free parameters: $A^i_\pm$, $B^{ijk}_\pm$, $C^i_\pm$, and $k$ that come induced from (1.1) and (1.3).

Imposing (1.1) gives rise to the following equations. From the third order pole we find

$$0 = 3k + 4k A^{ij} A_{\pm i j} - 6k^2 B^{ijk}_\pm B_{\pm i j k}$$

$$- 6k A^{ij} B^{jkl}_\pm f_{k l}^i + (f_{i j m} f_{m l k} - f_{j i m} f_{m l k}) A^{i j}_\pm A^{k l}_\pm .$$

(2.6)

The equations from the double pole are much simpler:

$$0 = A^{i j}_\pm A^{k l}_\pm f_{i l}^m f_{j k}^n - (m \leftrightarrow n).$$

(2.7)

Finally let us consider the equations coming from the simple pole:

$$0 = k A^{i j}_\pm A^{k l}_\pm g_{j l} + (i \leftrightarrow j)$$

(2.8)

$$0 = k \left( A^{i j}_\pm A^{k l}_\pm f_{j k l} + \frac{1}{2} A^{m j}_\pm A^{k l}_\pm g_{j l} f_{m k}^i + k A^{i j}_\pm C^l_\pm \right)$$

(2.9)

$$0 = 2 A^{m k}_\pm B^{j m}_\pm f_{m n}^l + 3 k B^{j m}_\pm B^{k m}_\pm g_{m n} + \text{signed perms in } (i, j, k, l)$$

(2.10)

$$0 = A^{k l}_\pm C^j_\pm f_{j k l} + 6 k A^{k l}_\pm B^{i m n}_\pm f_{i n m} g_{k l} + 3 k B^{j m k}_\pm C^l_\pm g_{k l} + 3 k A^{k l}_\pm B^{i m n}_\pm f_{k l m}$$

$$+ 9 k^2 B^{i k l}_\pm B^{j m n}_\pm g_{k l m} + 3 k A^{k l}_\pm B^{i m n}_\pm f_{k l m} + (i \leftrightarrow j)$$

(2.11)

$$0 = 6 k A^{i l}_\pm B^{j m k}_\pm g_{l m} + A^{k l}_\pm A^{m k}_\pm f_{i m}^l - 2 A^{i l}_\pm A^{m j}_\pm f_{i m}^k + (j \leftrightarrow k)$$

(2.12)

$$0 = \frac{1}{2} A^{k l}_\pm A^{i m n}_\pm f_{j k l} f_{i m}^n - A^{k l}_\pm A^{i m n}_\pm f_{j k l}^m f_{i m}^n + \frac{k}{2} A^{k l}_\pm A^{i j}_\pm g_{k l} + (i \leftrightarrow j).$$

(2.13)

From (1.2) we can read off the central charge

$$c = \frac{9 k^2}{4} \left( -C^i_\pm C^i_\pm - \frac{k}{2} A^{i j}_\pm A_{- i j} - A^{i j}_\pm C^k_\pm f_{i j k} - 3 k^2 B^{i j k}_\pm B_{- i j k}$$

$$+ 3 k^2 A^{k l}_\pm B^{i m n}_\pm f_{i k l} g_{j m} - \frac{1}{2} A^{i j}_\pm A^{k l}_\pm f_{j k l}^m f_{i m} - A^{i j}_\pm A^{k l}_\pm f_{i j k}^m f_{j k l} + (\leftrightarrow -) \right),$$

(2.14)

and the expression for $J(z)$. We choose to write $J(z) = E^i I_i(z) + F^{i j}(\psi_i \psi_j)(z)$, where $E^i$ and $F^{i j}$ can be found from the double pole in (1.2):

$$E^i = \frac{k}{2} \left( A^{i j}_\pm C_{- j} + A^{j k}_\pm A^{i l}_\pm f_{j k l} - \frac{1}{2} A^{i j}_\pm A^{l m}_\pm g_{k l} f_{j i m} - \frac{1}{2} A^{i j}_\pm A^{l m}_\pm g_{k l} f_{j i m} - \frac{1}{2} A^{i j}_\pm A^{l m}_\pm g_{k l} f_{j i m} - (\leftrightarrow -) \right).$$

(2.15)

1 Needless to say, we have used the Mathematica™ package OPEdefs written by Kris Thielemans [29]. I would like to take this opportunity to thank him for his patient tutoring, and Steffen Mallwitz for access to the computer. Without their help, this would have taken much longer and not been as enjoyable.
\[
F^{ij} = \frac{1}{2} \left( A^k_i C^l_j f_{kl}^i + \frac{1}{2} A^k_i A^m_{+} f_{km}^i f_{lm}^j + A^{kl}_+ A^m_{-} f_{km}^n f_{ln}^j + \frac{b}{2} A^k_j A^l_i g_{kl} \right.
\]

\[
+3k B_{-}^{ij} C_{-} k + 3k A^{kl}_{-} B_{+}^{jm} f_{klm} + 9k^2 B^{jk}_{+} B^{imn}_{-} g_{kn9lm} + 3k A^{kl}_{+} B^{jlm}_{-} f_{klm}
\]

\[
+6k^2 A^{kl}_{-} B^{imn}_{+} f_{km}^j g_{lm} - ( + \leftrightarrow - ) \right)
\]

Finally from (1.3) one can obtain the final set of equations, which we leave in terms of \( E^i \) and \( F^{ij} \). The first set of equations comes from the double pole

\[
0 = 2k C_{-} + k F^{i} + 6k^2 B_{+}^{jk} C_{-} F_{kj} - 2k A^{jk}_{+} F^{lm} f_{jl}^i g_{km} + 2k A^{jk}_{-} F^{il} f_{jkl}
\]

\[
+ k A^{ij}_{+} F^{kl} f_{jkl} - C_{+}^j E^k f_{jk}^i + k A^{ij}_{-} E_j + A^{jk}_{-} E_l f_{jlm} f_{km}^i + 3k B^{ijk}_{+} E^l f_{jkl},
\]

whereas from the single pole come the following equations

\[
A^{ij}_{+} = \pm 2k A^{ij}_{+} F^{kl} g_{kl} \pm A^{jk}_{+} E^l f_{kl}^i \pm A^{jk}_{+} E^l f_{kl}^i \tag{2.18}
\]

\[
B^{ijk}_{+} = \pm \frac{1}{3} A^{ij}_{+} F^{km} f_{lm}^k \pm k B^{jk}_{+} E^m f_{lm} g_{tm}
\]

\[
\pm \frac{1}{2} B^{ij}_{+} E^m f_{lm}^k + \text{signed perms in } (i, j, k) \tag{2.19}
\]

\[
C^i_{+} = \pm 2k A^{ik}_{-} F^{jl} f_{jl}^i g_{km} \pm 2k A^{ik}_{-} F^{il} f_{jkl} - C^j_{+} E^k f_{jk}^i \tag{2.20}
\]

The form of this last set of equations suggests a deformation-type approach, in which by starting from a nontrivial solution of the master equations we can find other solutions depending formally on some parameter and agreeing with the original solution as the parameter tends to zero. This will undoubtedly miss many interesting “nonperturbative” solutions which only exist for a discrete set of values of the parameter, but it has proven in the past to be a useful method. A natural deformation parameter is the level \( k \). However we have to determine around which point to deform. The only natural points are \( k = 0 \) and \( k^{-1} = 0 \). It is easy to see that for \( k = 0 \), the only solution is the trivial one. In fact, from (2.15), we see that \( E^i = 0 \) at \( k = 0 \). Plugging this into (2.18) and (2.20) we see that \( A^{ij}_{+} = 0 \) and \( C^{i}_{+} = 0 \). Finally into (2.19), we find that \( B^{ijk}_{+} = 0 \), leading to a trivial solution. Therefore the natural point from which to expand is \( k^{-1} = 0 \). The high-level analysis of the Virasoro master equation has been studied in [30]. We will, however, not deal any further with the perturbative approach in this paper.

Having ruled out anything interesting at \( k = 0 \) we are free to decouple the fermions. We derive the master equations for this case in the next section.
We assume that $k \neq 0$ and we shift the currents as in (2.4). The shifted currents $J_i(z)$ satisfy the following algebra

$$J_i(z)J_j(w) = \frac{k'g_{ij}}{(z-w)^2} + \frac{f_{ij}^k J_k(w)}{z-w} + \text{reg.}, \quad (3.1)$$

where $k' = k - \frac{1}{2}c_g$, and

$$J_i(z)\psi_j(w) = \text{reg.}, \quad (3.2)$$

while the fermions still satisfy (2.3). It will prove useful in what follows to rescale the fermions so that they obey the following OPE:

$$\psi_i(z)\psi_j(w) = g_{ij} \frac{z-w}{z-w} + \text{reg.}, \quad (3.3)$$

The most general expression for $G^\pm(z)$ is given by

$$G^\pm(z) = A^{ij}_\pm(J_i\psi_j)(z) + B^{ijk}_\pm(\psi_i\psi_j\psi_k)(z) + C^i_\pm \partial \psi_i(z), \quad (3.4)$$

where the currents have been shifted and the fermions rescaled.

Imposing (1.1) we find that the double pole equation is trivially satisfied, whereas from the third and first order poles we obtain:

$$0 = k' A^{ij}_\pm A_{\pm i j} - 2C^i_\pm C_{\pm i} - 6B^{ijk}_\pm B_{\pm ijk} \quad (3.5)$$

$$0 = A^{ik}_\pm A^j_\pm g_{kl} \quad (3.6)$$

$$0 = A^{ij}_\pm C_{\pm j} \quad (3.7)$$

$$0 = B^{ijk}_\pm C_{\pm k} \quad (3.8)$$

$$0 = A^{ij}_\pm A^{mk}_\pm f_{lm}^i + 6A^{il}_\pm B^{jkm}_\pm g_{lm} \quad (3.9)$$

$$0 = B^{ijm}_\pm B^{klm}_\pm g_{mn} + \text{signed perms in } (i,j,k,l) \quad (3.10)$$

$$0 = k' A^{ki}_\pm A^{lj}_\pm g_{ik} + 18B^{ikl}_\pm B^{jmn}_\pm g_{km}g_{lm} \quad (3.11)$$

which are to be compared to (2.6) and (2.8)–(2.13).

Computing (1.2) we can read off the central charge

$$c = \frac{3k'}{2} A^{ij}_\pm A_{\pm i j} - 3C^i_\pm C_{\pm i} - 9B^{ijk}_\pm B_{\pm ijk}, \quad (3.12)$$

and the expression for $J(z)$. We again choose to write $J(z) = E^i J_i(z) + F^{ij}(\psi_i\psi_j)(z)$, where $E^i$ and $F^{ij}$ can be found from the double pole in (1.2):

$$E^i = \frac{1}{2} A^{ij} C_{\pm j} - \frac{1}{4} A^{ik} A^{lm}_\pm f_{jl}^i g_{km} - (+ \leftrightarrow -) \quad (3.13)$$

$$F^{ij} = \frac{k'}{4} A^{kj} A^{li}_\pm g_{kl} + \frac{3}{2} B^{ikj}_\pm C_{\pm k} - \frac{9}{2} B^{jkl}_\pm B^{imn}_\pm g_{km}g_{ln} - (+ \leftrightarrow -) \quad (3.14)$$

which are to be compared with (2.15) and (2.16).
Finally, we use (1.3) to obtain the final set of master equations. From the double pole we obtain

\[ 0 = 2F^{ij}C_{\pm j} - 6B_{\pm}^{jk}F_{j k} + k'A_{\pm}^{j}E_{j} , \]  

(3.15)

whereas from the first order pole we find

\[ A_{\pm}^{ij} = \pm \left( 2A_{\pm}^{ik}F_{j l} - A_{\pm}^{kj}E_{f l}^{i} \right) \]  

(3.16)

\[ B_{\pm}^{ijk} = \pm \left( B_{\pm}^{jkl}F_{lm}^{i}g_{lm} + \text{signed perms in } (i,j,k) \right) \]  

(3.17)

\[ C_{\pm}^{i} = \pm 2C_{\pm}^{j}F^{ij} \]  

(3.18)

which replace (2.17)–(2.20).

A few observations can already be made. Contracting (3.18) with \( C_{\pm i} \) and using the antisymmetry of \( F_{ij} \) we find that \( C_{\pm i}C_{\pm}^{i} = 0 \). Similarly contracting (3.6) with \( g_{ij} \) we get that \( A_{\pm}^{ij}A_{\pm}^{ij} = 0 \). Therefore, the sole content of (3.5) is simply \( B_{\pm}^{ijk}B_{\pm}^{ijk} = 0 \), which at the same time follows from (3.11) by contracting with \( g_{ij} \) and using the previous observation. Consequently, (3.5) is rendered redundant. In other words, only the equations from the first pole are relevant.

If we specialize to \( C_{\pm}^{i} = 0 \), we can compare our equations with those found in the appendix D of [26]. In fact, in [26] the authors do not consider \( N=1 \) affine Lie algebras, but rather an affine Lie algebra and free fermions in the fundamental representation of (a real form of) \( SO(n,C) \). Despite this small difference, the equations for one case can be read from the equations in the other; and the comparison is possible. A quick glance at the two sets of equations allows us to identify the equations (D.6a)-(D.6f) in [26] with our equations (3.6), (3.11), (3.9), (3.10), (3.16), and (3.17) respectively. Equation (D.5) in [26], which corresponds to the \( N=1 \) master equation, finds no analogue here, since as shown in [1] and [2], it is redundant.

These master equations seem to have sufficient structure to allow for a general analysis and perhaps their solution. Work on this is in progress and we hope to report on this in the near future. In the next section, however, we look for solutions in the simple case of \( g = sl(2) \). This will suggest an Ansatz for general \( g \) that we analyze in section 5.

§4  \( g = sl(2) \), A Geometrical Approach

We now study the solutions that can be constructed from \( g = sl(2) \). Without loss of generality we can rescale our generators such that \( g_{ij} = \delta_{ij} \) and such that \( f_{ijk} = \epsilon_{ijk} \). This allows us to identify \( sl(2) \) with \( C^{3} \) in such a way that the Lie bracket goes over to the cross product and the invariant metric goes over to the dot product. This mild geometrization will prove very useful in solving the master equations for \( sl(2) \).

Notice that \( B_{\pm}^{ijk} = b_{\pm}\epsilon^{ijk} \), and contracting (3.18) with \( \epsilon_{ijk} \) one finds that

\[ b_{\pm} = \pm 2F^{ij}\delta_{ij} = 0 , \]  

(4.1)

whence \( B_{\pm}^{ijk} = 0 \).
To analyze the rest of the equations, let us define a family of vectors \(\vec{V}_\pm \in \mathbb{C}^3\), by \((\vec{V}_\pm)^i = A_{ij}^\pm \). We can then rewrite the equations involving \(A_{ij}^\pm\) as geometric equations for these vectors. For instance, in this language (3.9) becomes

\[
0 = \vec{V}_\pm \times \vec{V}_\pm^j
\]

which immediately says that the vectors \(\vec{V}_\pm^1\) and \(\vec{V}_\pm^i\) are separately collinear; in other words, there are vectors \(\vec{V}_\pm\) and complex numbers \(\alpha_\pm, \beta_\pm\) and \(\gamma_\pm\) such that

\[
\begin{align*}
\vec{V}_\pm^1 & = \alpha_\pm \vec{V}_\pm , \\
\vec{V}_\pm^2 & = \beta_\pm \vec{V}_\pm , \\
\vec{V}_\pm^3 & = \gamma_\pm \vec{V}_\pm .
\end{align*}
\]

(4.3)

If we introduce two new vectors \(\vec{W}_\pm \in \mathbb{C}^3\), by \(\vec{W}_\pm = (\alpha_\pm, \beta_\pm, \gamma_\pm)^t\), then it follows that \(A_{ij}^\pm = \vec{V}_\pm^i \vec{W}_\pm^j\); in other words, \(A_\pm\) are decomposable.

In terms of these new variables, (3.6) becomes

\[
0 = (\vec{W}_\pm \cdot \vec{V}_\pm) \vec{V}_\pm \otimes \vec{V}_\pm ,
\]

(4.4)

which is easily seen to imply (3.5). Since we don’t want a trivial solution, (4.4) simply says that \(\vec{W}_\pm\) are null:

\[
\vec{W}_\pm \cdot \vec{W}_\pm = 0 .
\]

(4.5)

This does not imply that they are zero, however, since they are vectors in \(\mathbb{C}^3\) and the inner product is not hermitian, as it does not involve complex conjugation. Similarly, (3.11) also says that \(\vec{V}_\pm\) are null away from \(k' = 0\)

\[
\vec{V}_\pm \cdot \vec{V}_\pm = 0 .
\]

(4.6)

From now on, we assume that \(k' \neq 0\). We shall treat the case \(k' = 0\) separately. We will see, however, that nothing special happens there.

If we now use (3.15) and (3.18) we find

\[
\vec{C}_\pm = \mp k'(\vec{E} \cdot \vec{V}_\pm)\vec{W}_\pm .
\]

(4.7)

This equation, together with the fact that \(\vec{W}_\pm\) are null (4.5) immediately implies (3.7).

Now, on the one hand, we can rewrite (3.13) as

\[
\vec{E} = \frac{1}{2}(\vec{C}_+ \cdot \vec{W}_+)\vec{V}_+ - \frac{1}{2}(\vec{C}_- \cdot \vec{W}_-)\vec{V}_+ + \frac{1}{2}(\vec{W}_+ \cdot \vec{W}_-)\vec{V}_+ \times \vec{V}_- ,
\]

(4.8)

which dotted with \(\vec{V}_\pm\) becomes

\[
\vec{E} \cdot \vec{V}_\pm = \mp \frac{1}{2}(\vec{C}_\pm \cdot \vec{W}_\pm)(\vec{V}_+ \cdot \vec{V}_-) .
\]

(4.9)

On the other hand, (3.14) can be rewritten as

\[
F^{ij} = \frac{k'}{4}(\vec{V}_+ \cdot \vec{V}_-)^3 W_i^j W_j^i - W_i^j W_j^i .
\]

(4.10)

In this guise, it is easy to see that (3.18) is equivalent to (4.7) after using (4.9).
The last equation to be transcribed into this geometric language—(3.16)—becomes after some manipulation

\[
\left(1 - \frac{k'}{2}(\vec{V}_+ \cdot \vec{V}_-)(\vec{W}_+ \cdot \vec{W}_-)\right) \vec{V}_\pm = \pm \vec{E} \times \vec{V}_\pm .
\] (4.11)

The RHS of this equation can be read off of

\[
\vec{E} \times \vec{V}_\pm = \frac{1}{2}(\vec{C}_\pm \cdot \vec{W}_\mp) \vec{V}_+ \times \vec{V}_- \mp \frac{1}{2}(\vec{V}_+ \cdot \vec{V}_-)(\vec{W}_+ \cdot \vec{W}_-) \vec{V}_\pm .
\] (4.12)

Plugging (4.12) into (4.11) we find

\[
\left(1 - \frac{k'-1}{2}(\vec{V}_+ \cdot \vec{V}_-)(\vec{W}_+ \cdot \vec{W}_-)\right) \vec{V}_\pm = \pm \frac{1}{2}(\vec{C}_\pm \cdot \vec{W}_\mp) \vec{V}_+ \times \vec{V}_- .
\] (4.13)

We can now argue that $\vec{C}_\pm$ have to be zero. Indeed, dot (4.13) with $\vec{V}_\mp$. On the RHS we get zero, but on the LHS we find

\[
\left(1 - \frac{k'-1}{2}(\vec{V}_+ \cdot \vec{V}_-)(\vec{W}_+ \cdot \vec{W}_-)\right) (\vec{V}_+ \cdot \vec{V}_-) = 0 .
\] (4.14)

Now, this can have either of two solutions, either $(\vec{V}_+ \cdot \vec{V}_-) = 0$ or not. We argue that it cannot be zero. If it were, then $\vec{C}_\pm = 0$ because of (4.7) and (4.9). And feeding this back into (4.13) we see that $\vec{V}_\pm = 0$, yielding the trivial solution. Therefore, we must have that

\[
\frac{k'-1}{2}(\vec{V}_+ \cdot \vec{V}_-)(\vec{W}_+ \cdot \vec{W}_-) = 1 .
\] (4.15)

If this is the case then the RHS of (4.13) is also zero, whence crossing with $\vec{V}_+$, yields that $\vec{C}_\pm = 0$ after using (4.9) and (4.7).

Therefore $\vec{C}_\pm = 0$ and we are left with (4.15) as our only equation. Notice that for $k' = 1$ there are no solutions except the trivial one. This restriction is not new. In fact, in the normalization of the algebra that we have used, the eigenvalue of the quadratic casimir $\sum_i \text{ad}X_i \text{ad}X_i$ on the adjoint representation is $-2$. Thus, $k' = k + 1$, and $k' = 1$ corresponds to the case $k = 0$ which we had ruled out previously.

In summary, the initial system has been reduced to the following: Given four null vectors $\vec{W}_\pm$ and $\vec{V}_\pm$ satisfying (4.15), we have an $N=2$ SCA. A priori this would indicate that there is a continuum of solutions, even after we pass to the moduli space; that is, after we account for the freedom to change basis in the algebra. However, computing the central charge of the $N=2$ SCA, we find from (3.12) that

\[
c = \frac{3k'}{2}(\vec{W}_+ \cdot \vec{W}_-)(\vec{V}_+ \cdot \vec{V}_-) \tag{4.16}
\]

which upon using (4.15) becomes

\[
c = \frac{3k'}{k' - 1} = 3 + \frac{3}{k} . \tag{4.17}
\]

Remarkably it only depends on the level! This suggests that there is essentially only one construction.
Let us try to understand this construction. The decomposition of $A_{ij}^{\pm}$ suggests that we first define currents $V_{\pm}(z) \equiv V_{\pm}^{i} J_{i}(z)$ and fermions $\psi_{\pm}(z) = W_{\pm}^{i} \psi_{i}(z)$. The currents obey the following algebra

$$V_{+}(z) V_{-}(w) = \frac{k' \bar{V}_{+} \cdot \bar{V}_{-}}{(z - w)^2} + \frac{V_{0}(w)}{z - w} + \text{reg.}, \quad (4.18)$$

where $V_{0}(w) \equiv (\bar{V}_{+} \times \bar{V}_{-})^{i} J_{i}(z)$. Clearly then, the rescaled currents $\bar{V}_{\pm}$ and $\bar{V}_{0}$, where $\bar{V}_{0} = \lambda^{2} V_{0}$ and $\bar{V}_{\pm} = \lambda V_{\pm}$ and $\lambda^{2} = -2/(\bar{V}_{+} \cdot \bar{V}_{-})$, obey an affine $sl(2)$ at level $k^{*} \equiv -2k'$. On the other hand, the fermions $\psi_{\pm}$ obey

$$\psi_{+}(z) \psi_{-}(w) = \frac{\bar{W}_{+} \cdot \bar{W}_{-}}{z - w} + \text{reg.}, \quad (4.19)$$

which becomes the standard OPE of a fermionic $bc$-system after rescaling them by $(\bar{W}_{+} \cdot \bar{W}_{-})^{-1/2}$. Notice that both the rescaling of the fermions and of the currents can be made, since (4.15) forbids either $\bar{V}_{+} \cdot \bar{V}_{-}$ or $\bar{W}_{+} \cdot \bar{W}_{-}$ from being zero. Notice also that in terms of the level $k^{*} = -2k'$ of the affine $sl(2)$, (4.17) agrees (for positive integer values of $k^{*}$) with the formula for the unitary minimal series. This suggests that this construction may be related to the parafermionic constructions of the $N=2$ minimal models. We will see in Section 6 that this is indeed the case.

Thus the construction now becomes clear: given an $sl(2)$ at level $k^{*} \neq -2$ and a fermionic $bc$-system, we can construct an $N=2$ SCA, with central charge given by (4.17). Furthermore, as we will show in the next section, this construction admits an embedding in any $N=1$ affine Lie algebra and also a mild generalization. But before, and as advertised, we discuss the case $k' = 0$.

Equation (3.11) does not imply now that $\bar{V}_{\pm}$ are null. However, since $F^{ij} = 0$ now, then (3.18) says that $\bar{C}_{\pm} = 0$. This means that $\bar{E} = \frac{1}{2}(\bar{W}_{+} \cdot \bar{W}_{-}) \bar{V}_{+} \times \bar{V}_{-}$, and thus equation (3.16) becomes, after discarding the trivial solution,

$$\bar{V}_{\pm} = \pm \bar{E} \times \bar{V}_{\pm}, \quad (4.20)$$

which implies that $\bar{V}_{\pm}$ are indeed null. Therefore, the solution found before extends to $k' = 0$ and is moreover the only solution there.

§5 A Construction for general $g$

We work with decoupled fermions, so the notation is as in Section 3. We will take as our Ansatz

$$G_{\pm}(z) = A_{\pm}^{ij}(J_{i} \psi_{j})(z), \quad (5.1)$$

where $A_{\pm}$ is decomposable. That is, we consider elements $V_{\pm} = \sum_{i} V_{\pm}^{i} X_{i}$ and $W_{\pm} = \sum_{i} W_{\pm}^{i} X_{i}$ in $g$ and define $A_{\pm}^{ij} = V_{\pm}^{i} W_{\pm}^{j}$.
Equations (3.6) says that $W_\pm$ are null; whereas, away from $k' = 0$, (3.11) says that $V_\pm$ are null too. Similar considerations as in the previous section allow us to conclude that nothing special happens for $k' = 0$ and that $V_\pm$ are null there too. We shall omit the details this time.

If we define $E \in \mathfrak{g}$ by $E = \sum_i E^i X_i$ then (3.13) can be rewritten as

$$E = \frac{1}{2} \langle W_+, W_- \rangle [V_+, V_-] ,$$

(5.2)

whereas defining $F \in \wedge^2 \mathfrak{g}$ by $F = \frac{1}{2} \sum_{i,j} F^{ij} X_i \wedge X_j$ turns (3.14) into

$$F = \frac{k'}{4} \langle V_+, V_- \rangle W_+ \wedge W_- .$$

(5.3)

Now, using (5.3) we can rewrite (3.15) as $k'W_+ \langle E , V_\pm \rangle = 0$. Since $k' \neq 0$ and we are discarding the trivial solution, this says that $\langle E , V_\pm \rangle = 0$. Using (5.2) this becomes

$$\langle E , V_\pm \rangle = \frac{1}{2} \langle W_+, V_- \rangle \langle [V_+, V_-] , V_\pm \rangle ,$$

(5.4)

which is already zero by invariance of $\langle , \rangle$.

Finally (3.16) becomes—discarding the trivial solution—

$$\left( 1 - \frac{k'}{2} \langle V_+, V_- \rangle \langle W_+, W_- \rangle \right) V_\pm = \pm [E , V_\pm] ,$$

(5.5)

which using (5.2) turns into

$$\left( 1 - \frac{k'}{2} \langle V_+, V_- \rangle \langle W_+, W_- \rangle \right) V_\pm = \pm \frac{1}{2} \langle W_+, W_- \rangle [[V_+, V_-] , V_\pm] .$$

(5.6)

This equation may seem formidable, but it is actually easy to recognize. Let us define $V_0 \equiv [V_+, V_-]$. Then (5.6) becomes

$$[V_0 , V_\pm] = \pm 2 \frac{\left( 1 - \frac{k'}{2} \langle V_+, V_- \rangle \langle W_+, W_- \rangle \right)}{\langle W_+, W_- \rangle} V_\pm .$$

(5.7)

Now, we can always choose $W_\pm$ such that the RHS is not zero and also finite. In fact, we can always divide by $\langle W_+, W_- \rangle$, since if it were zero, then (5.6) would say that $V_\pm = 0$ rendering the solution trivial. Similarly, for any fixed value of the level, one can rescale $W_\pm$ in such a way that the RHS of (5.7) is not zero. Therefore for those generic\(^2\) $W_\pm$, (5.7) simply says that suitably rescaling $V_\pm$ and $V_0$ they span an $sl(2)$ subalgebra of $\mathfrak{g}$. Explicitly, if we let $\widetilde{V}_\pm = \lambda V_\pm$ and $\widetilde{V}_0 = \lambda^2 V_0$, where

$$\lambda^2 = \frac{2 \langle W_+, W_- \rangle}{2 - k' \langle W_+, W_- \rangle \langle V_+, V_- \rangle} ,$$

(5.8)

then $[\widetilde{V}_0 , \widetilde{V}_\pm] = \pm 2 \widetilde{V}_\pm$ and $[\widetilde{V}_+, \widetilde{V}_-] = \widetilde{V}_0$.

\(^2\) It would remain to study those $W_\pm$ for which $\langle W_+, W_- \rangle = 2/k' \langle V_+, V_- \rangle$. These correspond to embeddings in $\mathfrak{g}$ of the Lie algebra defined by $[V_+, V_-] = V_0$ and $[V_0 , V_\pm] = 0$. For $\mathfrak{g} = sl(2)$ such embeddings clearly do not exist; but they do exist for other simple $\mathfrak{g}$. It may be interesting to classify them.
Conversely, given any $sl(2) \subset g$, can one find null vectors $W_\pm$ such that we obtain an $N=2$ SCA? It follows from the invariance of $\langle,\rangle$ that if $\tilde{V}_\pm$ and $\tilde{V}_0 \equiv [\tilde{V}_+, \tilde{V}_-]$ satisfy an $sl(2)$ then $V_\pm$ are necessarily null. Thus the question reduces to whether we can find null vectors $W_\pm$ such that

$$1 - \frac{k'}{2} \langle V_+, V_- \rangle \langle W_+, W_- \rangle = \langle W_+, W_- \rangle,$$

(5.9)

where $V_\pm$ are suitable rescalings of $\tilde{V}_\pm$ which, together with a rescaling $V_0$ of $\tilde{V}_0$, still obey an $sl(2)$. The answer to this is clearly affirmative, since we can always rescale $V_\pm$ in such a way that $k' \langle V_+, V_- \rangle \neq -2$, whatever the value of $k'$.

Moreover, notice that from (3.12), we can obtain the central charge:

$$c = \frac{3k' \langle V_+, V_- \rangle}{2 + k' \langle V_+, V_- \rangle}.$$

(5.10)

Notice that $k' \langle V_+, V_- \rangle$ is the induced level $k^*$ of $sl(2) \subset g$ – the index of embedding need not be 1. As before, we recover the minimal unitary series for $k^*$ a positive integer.

Finally, let us remark that for $g \neq sl(2)$ the construction affords a slight generalization, since one can have that $C_\pm \neq 0$. It is easy to see that if we can find elements $C_\pm \in g$ obeying

$$\langle C_+, C_+ \rangle = \langle C_-, W_\pm \rangle = \langle C_\pm, W_\mp \rangle = 0,$$

(5.11)

then we obtain an $N=2$ SCA with central charge

$$c = \frac{3k' \langle V_+, V_- \rangle}{2 + k' \langle V_+, V_- \rangle} - 3 \langle C_+, C_- \rangle.$$

(5.12)

Notice that it is impossible to find nonzero vectors $C_\pm \in \mathbb{C}^3$ obeying (5.11), which is why this deformation of the construction does not exist for $g = sl(2)$.

§6 Relation to Kazama–Suzuki Models and Parafermions

In this section we will identify the unique construction for $sl(2)$, with the Kazama–Suzuki coset construction associated to $SO(3)/SO(2)$, with a (generalized) parafermionic construction and also with a bosonic construction of the Kazama–Suzuki type.

Kazama–Suzuki Coset Construction

As we made explicit in the previous two sections, the construction of the $N=2$ SCA derives from the following one. Let us start with an $sl(2)$ affine Lie algebra at level $k^*$ with currents $J_\pm(z), J_0(z)$ obeying the OPEs

$$J_0(z) J_\pm(w) = \frac{\pm 2J_\pm(w)}{z-w} + \text{reg.},$$

$$J_+(z) J_-(w) = \frac{k^*}{(z-w)^2} + \frac{J_0(w)}{z-w} + \text{reg.},$$

(6.1)
\[ J_0(z)J_0(w) = \frac{2k^*}{(z-w)^2} + \text{reg.} ; \]

and we then add a fermionic \( bc \)-system \( \psi_\pm(z) \) with OPE

\[ \psi_+(z)\psi_-(w) = \frac{1}{z-w} + \text{reg.} . \] (6.2)

Then the following generators

\[ J = \frac{1}{k^*+2} \left[ J_0 + k^*(\psi_+\psi_-) \right] , \] (6.3)

\[ G^\pm = \frac{1}{\sqrt{k^*+2}}(J_0\psi ) , \] (6.4)

and

\[ T = \frac{1}{k^*+2} \left[ (J_+J_-) + (J_0\psi_+\psi_-) - \frac{1}{2}\partial J_0 + \frac{k^*}{2} ((\partial \psi_+\psi_-) - (\psi_+\partial \psi_-)) \right] , \] (6.5)

satisfy an \( N=2 \) SCA with central charge given by

\[ c = \frac{3k^*}{k^*+2} . \] (6.6)

The form of \( G^\pm(z) \) agrees (modulo conventions) with the Cartan–Weyl basis description of the Kazama–Suzuki coset associated to the 2-sphere as the hermitian symmetric space \( SO(3)/SO(2) \)—see, for example, equation (4.4) in [20]. Since the \( G^\pm(z) \) determine the whole SCA, we conclude that the two constructions are identical.

Parafermionic Construction

One can gain some more insight into this construction by paying close attention to \( T(z) \). It is easy to see that it is made out of two commuting pieces: \( T(z) = T_1(z) + T_2(z) \). The first term is nothing but the coset construction \( sl(2)/gl(1) \):

\[ T_1(z) = T_{sl(2)}(z) - T_{gl(1)}(z) , \] (6.7)

with

\[ T_{sl(2)} = \frac{1}{2(k^*+2)} \left[ (J_+J_-) + (J_-J_+) + \frac{1}{2}(J_0J_0) \right] \] (6.8)

the Sugawara tensor, and

\[ T_{gl(1)} = \frac{1}{4k^*}(J_0J_0) \] (6.9)

the \( gl(1) \) piece. And the second term is another \( gl(1) \) coming from \( J(z) \):

\[ T_2(z) = \frac{k^*+2}{2k^*}(JJ)(z) . \] (6.10)

This split suggests that the realization is nothing but a generalized (that is, for not necessarily integer level) parafermionic construction [31] [32], where the \( N=2 \) theory is equivalent to the tensor product of a generalized parafermionic theory and a free boson. Indeed, as we now show, bosonizing the \( \psi_\pm \) system, one can easily see that the two theories are the same.
Let us first rewrite the $sl(2)$ currents in terms of generalized parafermions $\chi_i$, $\chi_i^\dagger$ and a free boson $\phi$ normalized to $\phi(z)\phi(w) = -\log(z-w) + \cdots$. The affine currents are then written as

$$
\begin{align*}
J_0 &= i\sqrt{2k^*}\partial\phi, \\
J_+ &= \sqrt{k^*}\chi_1 \exp(i\sqrt{\frac{2}{k^*}}\phi), \\
J_- &= \sqrt{k^*}\chi_1^\dagger \exp(-i\sqrt{\frac{2}{k^*}}\phi),
\end{align*}
$$

(6.11)

with the parafermion fields obeying

$$
\chi_1(z)\chi_1^\dagger(w) = \frac{1}{(z-w)^{2\Delta_1}} \left[ 1 + \frac{2\Delta_1}{c_\chi}(z-w)^2T_1(w) + \cdots \right]
$$

(6.12)

with $T_1$ the coset energy-momentum tensor given by (6.7), $\Delta_1 = \frac{k^*-1}{k^*}$, and $c_\chi = \frac{2k^*-2}{k^*+2}$ the central charge of the coset theory. Other parafermionic OPEs involve further parafermionic pairs $\chi_i$ and $\chi_i^\dagger$ of dimension $\Delta_i = \frac{i(k^*+i)}{k^*}$. For $k^*$ a positive integer we can truncate the spectrum getting rid of all fields of negative dimension by the constraint $\chi_i^\dagger = \chi_{k^*-i}$; but for arbitrary $k^*$ these currents are all there.

We now introduce a second scalar field $\xi$ with the same normalization as $\phi$, and we use it to bosonize the $bc$-system $\psi_\pm$ by

$$
\psi_\pm = \exp(\pm i\xi) \quad \text{and} \quad (\psi_+\psi_-) = i\partial\xi.
$$

(6.13)

In terms of the two bosons and the parafermions, the $N=2$ currents can now be written as

$$
\begin{align*}
J &= i\sqrt{\frac{k^*}{k^*+2}}\partial\pi, \\
G^+ &= \sqrt{\frac{k^*}{k^*+2}}\chi_1 \exp(i\sqrt{\frac{k^*+2}{k^*}}\pi), \\
G^- &= \sqrt{\frac{k^*}{k^*+2}}\chi_1^\dagger \exp(-i\sqrt{\frac{k^*+2}{k^*}}\pi),
\end{align*}
$$

(6.14) \quad (6.15) \quad (6.16)

and

$$
T = T_1 - \frac{1}{2}\partial\pi\partial\pi,
$$

(6.17)

where $\pi \equiv \sqrt{\frac{2}{k^*+2}}(\phi + \sqrt{\frac{k^*}{2}}\xi)$. Thus the expression of the $N=2$ currents agrees with the ones for the generalized parafermionic constructions as written, for example, in equation (4.3) of [32] – up to a rescaling of $\pi$. 

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Bosonic Construction

Alternatively, notice that both \( T_{\text{gl}(1)} \) and \( T_2 \) have central charge equal to 1. Hence they can be bosonized by a free boson without background charge. In fact, if we bosonize the \( sl(2) \) currents as well we see that the whole construction takes the form of a bosonic \( N=2 \) construction of the type analyzed by Kazama and Suzuki in [28]. There are at least two ways to bosonize the \( sl(2) \) currents: we could first introduce the Wakimoto representation in terms of a free boson and a bosonic \((\beta,\gamma)\)-system; and then bosonize the \((\beta,\gamma)\)-system in terms of another two free bosons. Alternatively, we can simply use (6.11) and bosonize the parafermion fields in terms of two extra bosons. The bosonization of generalized parafermions was considered in [32] and we will roughly follow their approach.

Let us introduce free bosons \( \varphi_i(z) \varphi_j(w) = - \delta_{ij} \log(z-w) + \cdots \). In [32] and references therein, the parafermion currents \( \chi_1 \) and \( \chi_1^\dagger \) are shown to admit the following bosonization:

\[
\chi_1 = \frac{1}{\sqrt{2}} \left( \partial \varphi_2 - i \sqrt{\frac{k^* + 2}{k^*}} \partial \varphi_1 \right) \exp \left( \sqrt{\frac{2}{k^*}} \varphi_2 \right) \quad (6.18)
\]

and

\[
\chi_1^\dagger = - \frac{1}{\sqrt{2}} \left( \partial \varphi_2 + i \sqrt{\frac{k^* + 2}{k^*}} \partial \varphi_1 \right) \exp \left( -\sqrt{\frac{2}{k^*}} \varphi_2 \right) . \quad (6.19)
\]

Plugging these formulas into (6.14)–(6.17), and letting \( \varphi_3 \equiv \pi \), we find that the \( N=2 \) currents are given by

\[
J = i \sqrt{\frac{k^*}{k^* + 2}} \partial \varphi_3 , \quad (6.20)
\]

\[
G^\pm = - \frac{i}{\sqrt{2}} \left[ \sqrt{\frac{k^* + 2}{k^*}} \partial \varphi_1 \pm i \partial \varphi_2 \right] \exp \left( \frac{2}{k^*} \varphi_2 \pm i \sqrt{\frac{k^* + 2}{k^*}} \varphi_3 \right) , \quad (6.21)
\]

and

\[
T = - \frac{1}{2} \partial \vec{\varphi} \cdot \partial \vec{\varphi} + i \sqrt{\frac{2}{k^* + 2}} \partial^2 \varphi_1 . \quad (6.22)
\]

§7 Conclusions

Let us recapitulate the highlights of this paper. Based on the observation that only a few of the OPEs in the \( N=2 \) SCA are sufficient to determine the rest, we have written down the minimal set of \( N=2 \) master equations starting from an \( N=1 \) affine Lie algebra with data \((g,k)\). We saw that for \( k = 0 \) there was no nontrivial solutions to the master equations, whence we could decouple the fermions without loss of generality. After decoupling, the equations simplify tremendously. The set of master equations is given by equations (3.6)–(3.11) coming from the first order pole in the OPE \( G^\pm(z)G^\pm(w) \); and by equations (3.15)–(3.18) coming from the OPE \( J(z)G^\pm(w) \).

After a mild but helpful geometrization of the master equations, we solved them in general for the simplest case of \( g = sl(2) \). The solution was shown to be equivalent to a construction out of an affine \( sl(2) \) at a shifted level, and a fermionic \( bc \)-system and corresponding to the Kazama–Suzuki coset construction on the hermitian symmetric space
SO(3)/SO(2). Bosonizing the bc-system we could also identify the construction with a
generalized parafermionic construction. We also remarked that after bosonizing the affine
currents as well, the construction could be understood also as a bosonic construction of the
ones analyzed by Kazama and Suzuki in [28].

Moreover, we saw that these results extend to some extent to general g and we showed
that all solutions that fit the Ansatz (5.1)—or even the more general Ansatz where we allow
for derivatives of the fermions—are related to embeddings sl(2) ⊂ g or to degenerations
thereof (see the footnote in Section 5).

Embeddings sl(2) ⊂ g are also the building blocks—via the (generalized) Drinfel’d–
Sokolov (DS) reduction—of W-algebras. Thus it behooves one to investigate the relation
that could exist between DS reduction and N=2 SCAs. Some N=2 W-superalgebras can
be obtained by a DS-type reduction of affine Lie superalgebras, but we have here something
else in mind. From equation (6.4) it follows that the charge associated to the current \( G^+(z) \)
can be interpreted as the BRST operator associated to the constraint \( J^+ = 0 \). This is to be
contrasted with the constraint imposed in the DS reduction \( J^+ = 1 \). These comments suggest
the following conjecture: that associated to the W-algebra coming from the Drinfel’d–Sokolov
reduction with data sl(2) ⊂ g there is an N=2 W-superalgebra whose spectrum (in terms of
N=2 supermultiplets) agrees with the spectrum of g as irreducible representations of sl(2).
For the special case of the principal embeddings sl(2) ⊂ A_n, these N=2 W-superalgebras do
exist (see, for example, [33]). The proof of this conjecture, should it hold true, is work in
progress.

Finally, the uniqueness of the generalized construction for \( g = sl(2) \) suggests that the
N=2 structures inside N=1 affine Lie algebra are much more rigid than the Virasoro struc-
tures inside affine Lie algebras, for already in the case of sl(2) one has a rich structure of
generalized Virasoro constructions. Should this rigidity persist, the generalized N=2 con-
structions on N=1 affine Lie algebras may allow for a classification.

ACKNOWLEDGEMENTS

It is a pleasure to thank Ezra Getzler, Marty Halpern, Noureddine Mohammedi, and
Niels Obers for their many comments and discussions; and Ralph Blumenhagen and Michael
Terhoeven for sharing some of their knowledge of the N=2 literature. I am also thankful to
the Department of Physics of Queen Mary and Westfield College for their hospitality during
the final stages of this work.

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