A critical nonlinear elliptic equation with nonlocal regional diffusion

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Abstract

In this article we are interested in the nonlocal regional Schrödinger equation with critical exponent

$$
\varepsilon^{2\alpha}(\Delta)^{\alpha}\rho u + u = \lambda u^q + u^{2^*_\alpha - 1}
$$
in $\mathbb{R}^N$, where $\varepsilon$ is a small positive parameter, $\alpha \in (0, 1)$, $q \in (1, 2^*_\alpha - 1)$, $2^*_\alpha = \frac{2N}{N-2\alpha}$ is the critical Sobolev exponent, $\lambda > 0$ is a parameter and $(\Delta)^{\alpha}\rho$ is a variational version of the non-local regional laplacian, whose range of scope is a ball with radius $\rho(x) > 0$. We study the existence of a ground state and we analyze the behavior of semi-classical solutions as $\varepsilon \to 0$.

1. Introduction

In the present paper, we consider the existence and concentration phenomena of solutions to the nonlinear Schrödinger equation with nonlocal regional diffusion

$$
\varepsilon^{2\alpha}(\Delta)^{\alpha}\rho u + u = \lambda u^q + u^{2^*_\alpha - 1}
$$
in $\mathbb{R}^N$, where $\varepsilon$ is a small positive parameter, $\alpha \in (0, 1)$, $q \in (1, 2^*_\alpha - 1)$, $2^*_\alpha = \frac{2N}{N-2\alpha}$ is the critical Sobolev exponent, $\lambda > 0$ is a parameter and the operator $(\Delta)^{\alpha}\rho$ is a variational version of the non-local regional laplacian, whose range of scope is a ball with radius $\rho(x) > 0$. We make precise assumptions on the scope function $\rho$ we assume $\rho \in C(\mathbb{R}^n, \mathbb{R}^+)$ and it satisfies the following hypotheses:

(\rho_1) There are numbers $0 < \rho_0 < \rho_\infty \leq \infty$ such that

$$
\rho_0 \leq \rho(x) < \rho_\infty, \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad \lim_{|x| \to \infty} \rho(x) = \rho_\infty.
$$
For any $x_0 \in \mathbb{R}^n$, the equation
$$|x| = \rho(x + x_0), \quad x \in \mathbb{R}^n,$$
defines an $(n - 1)$-dimensional surface of class $C^1$ in $\mathbb{R}^n$.

In case $\rho_\infty = \infty$ we further assume that there exists $a \in (0, 1)$ such that
$$\limsup_{|x| \to \infty} \frac{\rho(x)}{|x|} \leq a.$$

2. Preliminaries

For any $\alpha \in (0, 1)$, the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined by
$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(z)|}{|x - z|^{N/2 + 2\alpha}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$
endowed with the norm
$$\|u\|_\alpha = \left( \int_{\mathbb{R}^N} |u(x)|^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N/2 + 2\alpha}} \, dz \, dx \right)^{1/2}.$$

For the reader’s convenience, we review the main embedding result for $H^\alpha(\mathbb{R}^N)$.

**FSE Lemma 2.1.** [2] Let $\alpha \in (0, 1)$ such that $2\alpha < N$. Then there exist a constant $C = C(N, \alpha) > 0$, such that
$$\|u\|_{L^{2^*_\alpha}} \leq C \|u\|_\alpha$$
for every $u \in H^\alpha(\mathbb{R}^N)$. Moreover, the embedding $H^\alpha(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2^*_\alpha]$ and is locally compact whenever $p \in [2, 2^*_\alpha)$.

Furthermore, we introduce the homogeneous fractional Sobolev space
$$H^\alpha_0(\mathbb{R}^N) = \left\{ u \in L^{2^*_\alpha}(\mathbb{R}^N) : |\xi|^\alpha \hat{u} \in L^2(\mathbb{R}^N) \right\} = C^\infty_0(\mathbb{R}^N) \|\cdot\|_0,$$
where
$$\|u\|_0^2 = \int_{\mathbb{R}^N} |\xi|^{2\alpha} |\hat{u}(\xi)|^2 \, d\xi.$$

Now we consider the best Sobolev constant $S$ as follows:

**P01** (2.1) \[ S = \inf_{u \in H^\alpha_0(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(z)|^2}{|x - z|^{N/2 + 2\alpha}} \, dz \, dx}{\left( \int_{\mathbb{R}^N} |u(x)|^{2^*_\alpha} \, dx \right)^{2/2^*_\alpha}}. \]

According to [1], $S$ is attained by the function $u_0(x)$ given by

**P02** (2.2) \[ u_0(x) = \frac{c}{(\theta^2 + |x - x_0|^2)^{N/2 + 2\alpha}}, \quad x \in \mathbb{R}^N, \]
Remark 2.1. Let $c \in \mathbb{R} \setminus \{0\}$, $\theta > 0$ and $x_0 \in \mathbb{R}^N$ are fixed constants. For any $\varepsilon > 0$ and $x \in \mathbb{R}^N$, let

$$U_\varepsilon(x) = e^{-\frac{N-2\alpha}{2} \varepsilon^2} \tilde{u}\left(\frac{x}{\varepsilon \delta^{1/2\alpha}}\right), \quad \tilde{u}(x) = \frac{u_0(x)}{\|u_0\|_{L^2_\alpha}}$$

which is solution of the problem

$$(-\Delta)^\alpha u = |u|^{2^*_\alpha - 2} u, \quad x \in \mathbb{R}^N.$$

Given a function $\rho$ as above, we define

$$(2.3) \quad \|u\|^2_\rho = \int_{\mathbb{R}^n} \int_{B(0, \rho(x))} \frac{|u(x) - u(z)|^2}{|x - z|^{n + 2\alpha}} \, dz \, dx + \int_{\mathbb{R}^n} u(x)^2 \, dx$$

and the space

$$H^\alpha_\rho(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) / \|u\|^2_\rho < \infty \}.$$

This space is very natural for the study of our problem. Furthermore, we have the following result

**Proposition 2.1.** \cite{3} If $\rho$ satisfies $(H_1)$ there exists a constant $C = C(n, \alpha, \rho_0) > 0$ such that

$$\|u\|_\alpha \leq C \|u\|_\rho.$$

**Remark 2.1.** By Proposition \ref{Proposition 2.1} we have that $H^\alpha_\rho(\mathbb{R}^n) \hookrightarrow H^\alpha(\mathbb{R}^n)$ is continuous and then, by Theorem \ref{Theorem 2.1}, we have that $H^\alpha_\rho(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ is continuous for any $q \in [2, 2^*_\alpha]$, and there exists $C_q > 0$ such that

$$\|u\|_{L^q} \leq C_q \|u\|_\rho, \quad \text{for every } u \in H^\alpha_\rho(\mathbb{R}^n) \quad \text{and } q \in [2, 2^*_\alpha].$$

Furthermore $H^\alpha_\rho(\mathbb{R}^n) \hookrightarrow L^q_{loc}(\mathbb{R}^n)$ is compact for any $q \in [2, 2^*_\alpha]$.

**Remark 2.2.** Since $\|u\|_\rho \leq \|u\|_\alpha$, under the condition $(\rho_1)$ Proposition \ref{Proposition 2.1} implies $\| \cdot \|_\rho$ and $\| \cdot \|_\alpha$ are equivalent norms in $H^\alpha(\mathbb{R}^n)$.

**Lemma 2.2.** \cite{3} Let $n \geq 2$. Assume that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H^\alpha_\rho(\mathbb{R}^N)$ and it satisfies

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, R)} |u_n(x)|^2 \, dx = 0,$$

where $R > 0$. Then $u_n \rightharpoonup u$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2^*_\alpha]$.

Now, we consider the limit equations. namely

$$(-\Delta)^\alpha u + u = \lambda u^q + u^{2^*_\alpha - 1} \quad \text{in } \mathbb{R}^N,$$

where $u \in H^\alpha(\mathbb{R}^N)$. This equation was study by Shang, Zhang and Yang in \cite{5}. The solution of problem \ref{2.4} are the critical point of the functional

$$I(u) = \frac{1}{2} \|u\|^2_\alpha - \frac{\lambda}{q + 1} \int_{\mathbb{R}^n} u^{q+1}_+ \, dx - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^n} u^{2^*_\alpha}_+ \, dx.$$
Furthermore, they studied the existence of ground state solutions to (2.4), namely, function in $\mathcal{N} = \{ u \in H^{\alpha}(\mathbb{R}^n) \setminus \{0\} : I'(u)u = 0 \}$ such that

$$C^* = \inf_{u \in \mathcal{N}} I(u)$$

is achieved and they got the following characterization:

$$C^* = \inf_{u \in H^{\alpha}(\mathbb{R}^n) \setminus \{0\}} \sup_{t \geq 0} I(tu) = C,$$

where $C = \inf_{s \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)) > 0$ is the mountain pass critical value.

On the other hand, assuming that if $q \in (1, 2^*_\alpha - 1)$, they have proved that there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the critical value satisfies

$$0 < C < \frac{\alpha}{n} S^{n/2\alpha},$$

where $S$ is the best Sobolev constant given by (2.1) and problem (2.4) has a nontrivial ground state solution.

3. Ground state

Let $\epsilon = 1$ and consider the following problem

\begin{equation}
(-\Delta)^{\alpha}_\rho u + u = \lambda u^q + u^{2^*_\alpha - 1} \quad \text{in} \quad \mathbb{R}^N,
\end{equation}

$$u \in H^{\alpha}(\mathbb{R}^N).$$

We recall that $u \in H^{\alpha}_\rho(\mathbb{R}^n) \setminus \{0\}$ is a solution of (1.1) if $u(x) \geq 0$ and

$$\langle u, \varphi \rangle_{\rho} = \lambda \int_{\mathbb{R}^n} u^q \varphi dx + \int_{\mathbb{R}^n} u^{2^*_\alpha - 1} \varphi dx, \quad \forall \varphi \in H^{\alpha}_\rho(\mathbb{R}^n).$$

where

$$\langle u, \varphi \rangle_{\rho} = \int_{\mathbb{R}^n} \int_{B(0,\rho(x))} \frac{|u(x+z) - u(x)| [\varphi(x+z) - \varphi(x)]}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} u \varphi dx$$

and $u_+ = \max\{u, 0\}$. In order to find solution for problem (3.1), we consider the functional $I_\rho : H^{\alpha}_\rho(\mathbb{R}^n) \to \mathbb{R}$ defined as

$$I_\rho(u) = \frac{1}{2} \|u\|^2_{\rho} - \frac{\lambda}{q+1} \int_{\mathbb{R}^n} u_+^{q+1} dx - \frac{1}{2^*_\alpha} \int_{\mathbb{R}^n} u_+^{2^*_\alpha} dx,$$

which is well defined and belongs to $C^1(H^\alpha_\rho(\mathbb{R}^n), \mathbb{R})$ with Fréchet derivative

$$I'_\rho(u)v = \langle u, v \rangle_{\rho} - \lambda \int_{\mathbb{R}^n} u_+^q v dx - \int_{\mathbb{R}^n} u_+^{2^*_\alpha - 1} v dx.$$

Now, we start recalling that the functional $I_\rho$ satisfies the mountain pass geometry conditions

Lemma 3.1. The functional $I_\rho$ satisfies the following conditions:

1. There exist $\beta, \delta > 0$, such that $I_\rho(u) \geq \beta$ if $\|u\|_{\rho} = \delta$.
2. There exists an $e \in H^\alpha_\rho(\mathbb{R}^n)$ with $\|e\|_{\rho}$ such that $I_\rho(e) < 0$. 

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From the previous Lemma, by using the mountain pass theorem without \((PS)_c\) condition \([\text{6}]\) it follows that there exists a \((PS)_{C_\rho}\) sequence \(\{u_k\} \subset H^\alpha_{\rho}(\mathbb{R}^n)\) such that

\[
I_\rho(u_k) \to C_\rho \quad \text{and} \quad I'_\rho(u_k) \to 0,
\]

where

\[
C_\rho = \inf_{\gamma \in \Gamma_\rho} \sup_{t \in [0,1]} I_\rho(\gamma(t)) > 0,
\]

and \(\Gamma_\rho = \{ \gamma \in C([0,1], H^\alpha_{\rho}(\mathbb{R}^n)) : \gamma(0) = 0, I_\rho(\gamma(1)) < 0 \}. \)

Also, we define

\[
C^* = \inf_{u \in N_\rho} I_\rho(u)
\]

where

\[
N_\rho = \{ u \in H^\alpha_{\rho}(\mathbb{R}^n) \setminus \{0\} : I'_\rho(u)u = 0 \}.
\]

\textbf{Lemma 3.2.} \(C^* > 0\)

\textit{Proof.} Suppose by contradiction that \(C^* = 0\). Then there exists \(u_n \in N_\rho\). Then there exists \(u_k \in N_\rho\) such that

\[I_\rho(u_k) \to C^* = 0, \quad \text{as} \quad k \to \infty.\]

But, since \(u_k \in N_\rho\) we have

\[
I_\rho(u_k) = I_\rho(u_k) - \frac{1}{2} I'_\rho(u_k)u_k
\]

\[
= \lambda \left( \frac{1}{2} - \frac{1}{q+1} \right) \|u_{k+}\|_{L^{q+1}}^{q+1} + \frac{a}{n} \|u_{k+}\|_{L_{2n}^2}^{2n} \to 0, \quad \text{as} \quad k \to \infty.
\]

So

\[
\|u_{k+}\|_{L_{2n}^2} \to 0 \quad \text{and} \quad \|u_{k+}\|_{L^{q+1}} \to 0, \quad \text{as} \quad k \to \infty.
\]

Therefore

\[
\|u_k\|_{\rho} \to 0, \quad \text{as} \quad k \to \infty.
\]

On the other hand, since \(0 \neq u_k \in N_\rho\), then by Remark 2.1 we have

\[
1 \leq \lambda C_{q+1} \|u_k\|_{\rho}^{q-1} + C_{2n} \|u_k\|_{L_{2n}^2}^{2n-2}, \quad \forall k.
\]

Combining this inequality with \((3.4)\) we get a contradiction. This proves the Lemma. \(\square\)

\textbf{Lemma 3.3.} Let \(C_\rho\) given by \((3.2)\) and \((3.3)\). Then

\[
C^* = \inf_{u \in H^\alpha_{\rho}(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I_\rho(tu) = C_\rho.
\]
Proof. We note that our nonlinearity \( f(t) = \lambda t^q + t^{2\alpha - 1}, t > 0 \), is a \( C^1 \) function and \( \frac{I(t)}{t} \) is a strictly increasing function. Let \( u \in \mathcal{N}_\rho \), then we can show that the function \( h(t) = I_\rho(tu), t \neq 0 \) has a unique maximum point \( t_u \) such that

\[
I_\rho(t_u u) = \max_{t \geq 0} I_\rho(tu).
\]

Furthermore, we can show that \( t_u = 1 \). Now choose \( t_0 \in \mathbb{R} \) and \( \tilde{u} = t_0 u \) such that \( I_\rho(\tilde{u}) < 0 \). Then \( \gamma(t) = t\tilde{u} \in \Gamma_\rho \) then \( I_\rho(u) \geq C_\rho \), that is,

\[
C^* \geq C_\rho.
\]

On the other hand, let \( \{u_k\} \) be the \( (PS)_{C_\rho} \) sequence satisfying \((3.2)\) and \((3.3)\). Since \( \{u_k\} \) is bounded, then \( I'_\rho(u_k)u_k \rightarrow 0 \) as \( k \rightarrow \infty \), moreover, from \((3.5)\) for each \( k \), there is a unique \( t_k \in \mathbb{R}^+ \) such that

\[
I'_\rho(t_k u_k) t_k u_k = 0, \ \forall k.
\]

Hence \( t_k u_k \in \mathcal{N}_\rho \).

Now we note that by \((3.7)\), we have

\[
\|u_k\|^2 \rho = \lambda t_k^{q-1/2} \|u_k\| \Gamma_{\rho+1} + t_k^{2\alpha-2} \|u_k\| \Gamma_{\rho+1}, \ \forall k.
\]

So \( t_k \) does not converge to 0; otherwise, since \( \{u_k\} \) is bounded in \( H^1_\rho(\mathbb{R}^n) \), using \((3.8)\) we obtain

\[
\|u_k\| \rho \rightarrow 0, \ \text{as} \ k \rightarrow \infty,
\]

which is impossible since \( C_\rho > 0 \). Also, \( t_k \) does not go to infinity. In fact, by \((3.8)\) we get

\[
\frac{\|u_k\|^2 \rho}{t_k^{2\alpha-2}} = \lambda t_k^{q-1-2\alpha} \|u_k\| \Gamma_{\rho+1} + \|u_k\|^2 \rho \Gamma_{\rho+1}, \ \forall k.
\]

So, assuming that \( t_k \rightarrow \infty \), as \( k \rightarrow \infty \), by \((3.9)\) we get that \( u_k \rightarrow 0 \) in \( L^{2\alpha}(\mathbb{R}^n) \), as \( k \rightarrow \infty \).

Then using interpolation inequality it follows that

\[
\|u_k\|^2_{\rho} = \lambda \|u_k\| \Gamma_{\rho+1} + \|u_k\|^2 \rho \Gamma_{\rho+1} + o(1), \ \text{as} \ k \rightarrow \infty.
\]

Moreover, since \( I'_\rho(u_k)u_k \rightarrow 0 \), as \( k \rightarrow \infty \), we obtain,

\[
\|u_k\|^2_{\rho} = \lambda \|u_k\|^2_{\rho} + \|u_k\|^2 \rho \Gamma_{\rho+1} + o(1), \ \text{as} \ k \rightarrow \infty.
\]

So, by \((3.10)\) and \((3.11)\), we conclude that \( \|u_k\|^2 \rho \rightarrow 0 \), as \( k \rightarrow \infty \), contradicting \( C_\rho > 0 \). Hence, the sequence \( \{t_k\} \) is bounded and there exists \( t_0 \in (0, \infty) \) such that (up to subsequence) \( t_k \rightarrow t_0 \) as \( k \rightarrow \infty \).

Now, from \((3.8)\) and \((3.11)\) we obtain

\[
o(1) = \lambda (t_k^{q-1} - 1) \|u_k\|^2_{\rho} + (t_k^{2\alpha-2} - 1) \|u_k\|^2 \rho \Gamma_{\rho+1}, \ \text{as} \ k \rightarrow \infty.
\]

From where \( t_0 = 1 \), namely

\[
t_k \rightarrow 1, \ \text{as} \ k \rightarrow \infty.
\]
Therefore, by (3.13) and recalling that $t_k u_k \in N_\rho$, we get

$$C^* \leq I_\rho(t_k u_k)$$

$$= t_k^2 \left[ I_\rho(u_k) + \frac{\lambda}{q+1} (1 - t_k^{q-1}) \| u_{k+1} \|_{L^{q+1}}^{q+1} + \frac{1}{2^\alpha} (1 - t_k^{2^\alpha-2}) \| u_{k+1} \|_{L^{2^\alpha}}^{2^\alpha} \right]$$

$$= t_k^2 I_\rho(u_k) + o(1)$$

$$= (t_k^2 - 1) I_\rho(u_k) + I_\rho(u_k) + o(1).$$

Passing to the limit we obtain $C^* \leq C_\rho$.

On the other hand, By the previous comments, for any $u \in H^{\alpha}_\rho(\mathbb{R}^n) \setminus \{0\}$ there is a unique $t_u = t(u) > 0$ such that $t_u u \in N_\rho$, then

$$C^* \leq \inf_{u \in H^{\alpha}_\rho(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I_\rho(tu).$$

Moreover, for any $u \in N_\rho$, we have

$$I_\rho(u) = \max_{t \geq 0} I_\rho(tu) \geq \inf_{u \in H^{\alpha}_\rho(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I_\rho(tu)$$

so

$$C^* = \inf_{N_\rho} I_\rho(u) \geq \inf_{u \in H^{\alpha}_\rho(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I_\rho(tu).$$

\[\Box\]

**Remark 3.1.** Suppose that $(\rho_1)$ holds and without loss of generality take $\epsilon = 1$, then

$$C_\rho < C.$$ 

In fact, let $u$ be a critical point of $I$ with critical value $C$ and for any $y \in \mathbb{R}^n$, define $u_y(x) = u(x + y)$. Then for any $t > 0$ we have

$$C = I(u_y) \geq I(tu_y) > I_\rho(tu_y).$$

By Lemma 3.2, we can take $t^* > 0$ such that $t^* u_y \in N_\rho$ and

$$I_\rho(t^* u_y) = \sup_{t > 0} I_\rho(tu_y),$$

consequently $C > I_\rho(t^* u_y) \geq C_\rho$. In the same way we can show that

$$C_\rho < C_{\rho\infty} < C.$$ 

**Remark 3.2.** According to (2.5) and by Remark 3.1 we have

$$0 < C_\rho < \frac{\alpha}{n} S^{n/2\alpha}.$$ 

**Lemma 3.4.** Suppose that $C_\rho < C$. Then there are $\nu, R > 0$ such that

$$\int_{B(0,R)} u_k^2(x) dx \geq \nu, \text{ for all } k \in \mathbb{N}.$$
Proof. By Lemma \[^{3.3}\] for each \( k \in \mathcal{N} \), there exist \( t_k \subset \mathbb{R} \) such that \( t_k \to 1 \) and
\[
I_\rho(t_k u_k) = \max_{t \geq 0} I_\rho(t u_k).
\]
Furthermore
\[
I_\rho(u_k) = I_\rho(t_k u_k) + o(1) \geq I_\rho(t u_k) + o(1), \quad \text{for all} \ t > 0.
\]
Now we consider two cases, namely, when \( \rho_{\infty} = \infty \) and \( \rho_{\infty} < \infty \). In the first case, by \((\rho_1)\), for every \( \epsilon > 0 \) there exist \( R > 0 \) such that
\[
\mathbb{R}^n \setminus B(0, \rho(x)) \subset \mathbb{R}^n \setminus B(0, 1/\epsilon) \quad \text{whenever} \ |x| > R.
\]
Then
\[
I_\rho(t u_k) = \max_{t \geq 0} I_\rho(t u_k),
\]
we obtain
\[
I_\rho(t_k u_k) = I_\rho(t u_k) - \frac{1}{2} \int_{\mathbb{R}^n} \frac{|u_k(x + z) - u_k(x)|^2}{|z|^{n+2\alpha}} \, dz \, dx
\]
\[
\geq I_\rho(t u_k) - \frac{|S^{n-1} 1|}{\alpha \rho_0^{2\alpha}} \|u_k\|_{L^2(B(0, R))}^2 + \frac{|S^{n-1} 1| 2 \alpha}{\alpha \rho_0^{2\alpha}} \|u_k\|_{L^2(\mathbb{R}^n)}^2, \quad \text{for all} \ t > 0.
\]
If \( \{\tau_k\} \) is the bounded real sequence seven by
\[
I_\rho(u_k) = \max_{t \geq 0} I_\rho(t u_k),
\]
we obtain
\[
I_\rho(t_k u_k) = I_\rho(t u_k) - \frac{|S^{n-1} 1|}{\alpha \rho_0^{2\alpha}} \|u_k\|_{L^2(B(0, R))}^2 + \frac{|S^{n-1} 1| 2 \alpha}{\alpha \rho_0^{2\alpha}} \|u_k\|_{L^2(\mathbb{R}^n)}^2.
\]
If
\[
\int_{B(0, R)} u_k^2(x) \, dx \to 0 \quad \text{as} \ k \to \infty,
\]
from \((3.14), (3.15)\) and \((3.16)\) yield
\[
C_\rho \geq C,
\]
which is a contradiction with Remark \[^{3.1}\].

Now we analyze the case \( \rho_{\infty} < +\infty \). In this case we compare the functionals \( I_\rho \) and \( I_{\rho_{\infty}} \) writing
\[
I_\rho(u) = I_{\rho_{\infty}}(u) - \frac{1}{2} \int_{\mathbb{R}^n} \frac{|u(x + z) - u(x)|^2}{|z|^{n+2\alpha}} \, dz \, dx.
\]
By hypothesis \((\rho_1)\), for any \( \epsilon > 0 \) there is \( R > 0 \) such that
\[
0 < \rho_{\infty} - \rho(x) < \epsilon, \quad \text{whenever} \ |x| > R.
\]
Then, we obtain
\[
I_\rho(t u_k) = I_{\rho_{\infty}}(t u_k) - C(\epsilon) t^2 \|u_k\|_{L^2}^2 - C t^2 \|u_k\|_{L^2(B(0, R))}^2.
\]
where
\[ C(\varepsilon) = \frac{2|S^{n-1}|}{\alpha} \left( \frac{1}{\rho_0 - \varepsilon}^2 - \frac{1}{\rho_0^{2\alpha}} \right) \] and
\[ C = \frac{2|S^{n-1}|}{\alpha} \left( \frac{1}{\rho_0^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}} \right). \]

Proceeding as before, by (3.16) we get \( C_\rho \geq C_{\rho_\infty} \), which is a contradiction with Remark 3.1. \(\Box\)

The next result shows the existence of positive solution to (1.1) with \( \varepsilon = 1 \).

**Theorem 3.1.** Suppose that \( \lambda > 0 \), \( q > 1 \) and (\( \rho_1 \)) hold. Then, problem (1.1) with \( \varepsilon = 1 \) possesses a positive ground state solution.

**Proof.** Using (3.2), Lemma 3.4 and the Sobolev embedding we have
\[ \int_{B(0,R)} u^2(x) dx \geq \nu > 0, \]
which proves that \( u \neq 0 \). Furthermore, by standard arguments we have
\[ I'_\rho(u)\varphi = 0 \text{ for all } \varphi \in H_\rho^\alpha(\mathbb{R}^n), \]
so choosing \( \varphi = u_- (x) \max\{-u(x), 0\} \) and noting that for \( x, z \in \mathbb{R}^n \) we have
\[ (u(x+z) - u(x))(u_-(x+z) - u_-(x)) = -u_+(x+z)u_-(x) - u_+(x)u_-(x+z) \]
\[ -u_-(x+z) - u_-(x))^2 \leq 0, \]
we can conclude that \( \|u_-\|_\rho = 0 \), thus \( u(x) \geq 0 \) a.e. \( x \in \mathbb{R}^n \).

Moreover, from Lemma 3.3
\[ C_\rho \leq \max_{t \geq 0} I_\rho(tu) = I_\rho(u). \]

On the other hand, we have
\[ C_\rho = I_\rho(u_k) - \frac{1}{2} I'_\rho(u_k)u_k + o_k(1) \]
\[ = \lambda \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_{\mathbb{R}^n} u_{k+1}^{q+1} dx + \frac{1}{n} \int_{\mathbb{R}^n} u_k^{2\alpha} dx + o_k(1). \]

Applying Fatou’s Lemma to last inequality, we obtain
\[ C_\rho \geq \lambda \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_{\mathbb{R}^n} u^{q+1} dx + \frac{1}{n} \int_{\mathbb{R}^n} u^{2\alpha} dx = I_\rho(u) - \frac{1}{2} I'_\rho(u)u = I_\rho(u). \]

From (3.18) and (3.19) we obtain
\[ I_\rho(u) = C_\rho, \]
and hence \( u \) is a least energy solution and the proof is finished. \(\Box\)

**Proof of Theorem 3.2.** In what follows, we denote by \( \{u_k\} \subset H_\rho^\alpha(\mathbb{R}^n) \) a sequence satisfying
\[ I_{\rho_\varepsilon}(u_k) \to C_{\rho_\varepsilon} \quad \text{and} \quad I'_{\rho_\varepsilon}(u_k) \to 0. \]

If \( u_k \to 0 \) in \( H_\rho^\alpha(\mathbb{R}^n) \), then
\[ \lim_{k \to 0} \int_{\mathbb{R}^n} u_k^p dx = 0 \quad \text{for } p \in [2, 2\alpha). \]
By $(\rho_1)$, we obtain

\[
I_{\rho_k}(tu_k) = \frac{t^2}{2} \int_{\mathbb{R}^n} \int_{B(0, \rho_k^\infty) \setminus B(0, \frac{\rho(x)}{\epsilon})} \frac{|u_k(x + z) - u_k(x)|^2}{|z|^{n+2\alpha}} dz \, dx
\]

where

\[
I_{\rho_k}(u) = \frac{1}{2} \left( \int_{\mathbb{R}^n} \int_{B(0, \frac{\rho(x)}{\epsilon})} \frac{|u(x + z) - u(x)|^2}{|z|^{n+2\alpha}} dz \, dx + \int_{\mathbb{R}^n} u^2 \, dx \right)
\]

\[\quad - \frac{\lambda}{q + 1} \int_{\mathbb{R}^n} u^{q+1} \, dx - \frac{1}{2^{\alpha}} \int_{\mathbb{R}^n} u^{2\alpha} \, dx.
\]

Now we know that there exists a bounded sequence \(\{\tau_k\}\) such that

\[
I_{\rho_k}(\tau_k u_k) \geq C(\rho_\infty),
\]

where

\[
C(\rho_\infty) = \inf_{v \in H^\alpha(\mathbb{R}) \setminus \{0\}} \sup_{t \geq 0} I_{\rho_k}(tv)
\]

Thus,

\[
C_{\rho_k} \geq C(\rho_\infty) - \frac{|S^{n-1}|}{\alpha} \left( \frac{1}{\rho_0^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}} \right) \tau_k^2 \epsilon^{2\alpha} \|u_k\|^2_{L^2(B(0, \frac{\rho_\infty}{\epsilon}))}
\]

\[\quad - \frac{|S^{n-1}|}{\alpha} \left( \frac{1}{(\rho_\infty - \delta)^{2\alpha}} - \frac{1}{\rho_\infty^{2\alpha}} \right) \tau_k^2 \epsilon^{2\alpha} \|u_k\|^2_{L^2(\mathbb{R}^n)}
\]

Taking the limit as \(k \to \infty\), and after \(\delta \to 0\), we find

\[
\text{eq21} \quad C_{\rho_k} \geq C(\rho_\infty)
\]

A standard argument shows that

\[
\liminf_{\epsilon \to 0} C(\rho_\infty) \geq C.
\]

Therefore, if there is \(\epsilon_k \to 0\) such that the \((PS)_{C_{\rho_k}}\) sequence has weak limit equal to zero, we must have

\[
C_{\rho_k} \geq C(\rho_\infty), \quad \forall k \in \mathbb{N},
\]

leading to

\[
\liminf_{n \to +\infty} C_{\rho_{\epsilon_k}} \geq C,
\]

which contradicts Remark 3.1. This proves that the weak limit is non trivial for \(\epsilon > 0\) small enough and standard arguments show that its energy is equal to \(C_{\rho_k}\), showing the desired result. \(\square\)
4. Concentration Behaviour

In this section we make a preliminary analysis of the asymptotic behavior of the functional associated to equation (1.1) when \( \epsilon \to 0 \). As is point up in [3], the scope function \( \rho \), that describes the size of the ball of the influential region of the non-local operator, plays a key role in deciding the concentration point of ground states of the equation. Even though, at a first sight, the minimum point of \( \rho \) seems to be the concentration point, there is a non-local effect that needs to be taken in account. We define the concentration function

\[
\mathcal{H}(x) = -\frac{|S^{n-1}|}{2\alpha} \left( \frac{1}{\rho(x)^{2\alpha}} - \frac{1}{\rho_{\infty}^{2\alpha}} \right) + \frac{1}{2} \int_{\mathcal{C}^+(x)} \frac{dy}{|y|^{n+2\alpha}} - \frac{1}{2} \int_{\mathcal{C}^-} \frac{dy}{|y|^{n+2\alpha}},
\]

where the sets \( \mathcal{C}^+(x) \) and \( \mathcal{C}^-(x) \) are defined as follows

\[
\mathcal{C}^-(x) = \{ y \in \mathbb{R}^n : \rho(x + y) < |y| < \rho(x) \}
\]

and

\[
\mathcal{C}^+(x) = \{ y \in \mathbb{R}^n : \rho(x) < |y| < \rho(x + y) \}.
\]

We start with some basic properties of the function \( \mathcal{H} \).

Lemma 4.1. [3] Assuming \( \rho \) satisfies \((\rho_1) - (\rho_3)\), the function \( \mathcal{H} \) is continuous and

\[
\lim_{|x| \to \infty} \mathcal{H}(x) = 0.
\]

Moreover, there exists \( x_0 \in \mathbb{R}^n \) such that

\[
\inf_{x \in \mathbb{R}^n} \mathcal{H}(x) = \mathcal{H}(x_0) < 0.
\]

Along this section we will consider a sequence of functions \( \{w_m\} \subset H^\alpha(\mathbb{R}^n) \) such that \( \|w_m - w\|_{L^2(\mathbb{R}^n)} \to 0 \), where \( w \in H^\alpha(\mathbb{R}^n) \). We will also consider sequences \( \{z_m\} \subset \mathbb{R}^n \) and \( \{\epsilon_m\} \subset \mathbb{R} \) and assume that \( \epsilon_m \to 0 \) as \( m \to \infty \). We define \( \bar{\rho}_m \) as

\[
\bar{\rho}_m(x) = \frac{1}{\epsilon_m} \rho(\epsilon_m x + \epsilon_m z_m),
\]

and the functional \( I_{\bar{\rho}_m} \) defined as

\[
I_{\bar{\rho}_m}(u) = \frac{1}{2} \left( \int_{\mathbb{R}^n} \int_{B(0, \bar{\rho}_m(x))} \frac{|u(x + z) - u(x)|^2}{|z|^{n+2\alpha}} \, dz \, dx + \int_{\mathbb{R}^n} u^2 \, dx \right) - \frac{\lambda}{q + 1} \int_{\mathbb{R}^n} |u^{q+1}| \, dx - \frac{1}{2\alpha} \int_{\mathbb{R}^n} u^{2\alpha} \, dx.
\]

We will be considering the functionals

\[
I_{\bar{\rho}_m}, I_{\rho(x)/\epsilon_m}
\]

and the functional \( I \) in \( \mathbb{R}^n \) (with \( \rho \equiv \infty \)).

As in [3] we have the following key Theorem
Theorem 4.1. Under hypotheses \((\rho_1) - (\rho_3)\), we assume as above that \(w_m, w \in H^\alpha(\mathbb{R}^n)\) are such that \(\|w_m - w\|_{L^2(\mathbb{R}^2)} \to 0\) and \(\epsilon_m \to 0\), as \(m \to \infty\). Then we have:

i) If \(\epsilon_m z_m \to \bar{x}\) then
\[
\lim_{m \to \infty} \frac{I_{\rho_m}(w_m) - I_{\rho_m}(w_m)}{\epsilon_m^{2\alpha}} = \|w\|^2_{L^2(\mathbb{R}^2)} \quad \text{and}
\]

ii) If \(|\epsilon_m| z_m \to \infty\) then
\[
\lim_{m \to \infty} \frac{I_{\rho_m}(w_m) - I_{\rho_m}(w_m)}{\epsilon_m^{2\alpha}} = 0.
\]

Now, we rescaling equation (1.1), for this purpose we define \(\rho_\epsilon(x) = \frac{1}{\epsilon} \rho(\epsilon x)\) and consider the rescaled equation
\[
(-\Delta)^\alpha \rho_\epsilon v + v = \lambda v^q + u^{2s-1}, \quad \text{in} \quad \mathbb{R}^n
\]
and we see that \(u\) is a weak solution of (1.1) if and only if \(v_\epsilon(x) = u(\epsilon x)\) is a weak solution of (4.8).

In order to study equations (1.1) and (4.8), we consider the functional \(I_{\rho_\epsilon}\) on the \(\epsilon\)-dependent Hilbert space \(H_{\rho_\epsilon}^\alpha(\mathbb{R}^n)\) with inner product \(\langle \cdot, \cdot \rangle_{\rho_\epsilon}\). The functional \(I_{\rho_\epsilon}\) is of class \(C^1\) in \(H_{\rho_\epsilon}^\alpha(\mathbb{R}^n)\) and the critical points of \(I_{\rho_\epsilon}\) are the weak solutions of (4.8). We further introduce
\[
N_{\rho_\epsilon} = \{v \in H_{\rho_\epsilon}^\alpha(\mathbb{R}^n) \setminus \{0\} : I_{\rho_\epsilon}'(v)v = 0\},
\]
\[
\Gamma_{\rho_\epsilon} = \{\gamma \in C([0, 1], H_{\rho_\epsilon}^\alpha(\mathbb{R}^n)) : \gamma(0) = 0, I_{\rho_\epsilon}(\gamma(1)) < 0\}
\]
and the mountain pass minimax value
\[
C_{\rho_\epsilon} = \inf_{\gamma \in \Gamma_{\rho_\epsilon}} \max_{t \in [0, 1]} I_{\rho_\epsilon}(\gamma(t)).
\]

From Lemma 3.3 we also have
\[
0 < C_{\rho_\epsilon} = \inf_{v \in N_{\rho_\epsilon}} I_{\rho_\epsilon}(v) = \inf_{v \in H_{\rho_\epsilon}^\alpha(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I_{\rho_\epsilon}(tv).
\]

For comparison purposes we consider the functional \(I\), whose critical points are the solutions of (2.4). We also consider the critical value \(C\) that satisfies
\[
C = \inf_{u \in H^\alpha(\mathbb{R}^n) \setminus \{0\}} \max_{t \geq 0} I(tu).
\]

Now we start the proof of Theorem 4.1 with some preliminary lemmas.

Lemma 4.2. Suppose \((\rho_1)\) holds. Then
\[
\lim_{\epsilon \to 0^+} C_{\rho_\epsilon} = C.
\]

Proof. Since we obviously have
\[
\int_{\mathbb{R}^n} \int_{B(0, \rho_\epsilon(x))} \frac{|u(x + z) - u(x)|^2}{|z|^{n+2\alpha}} dzdx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x + z) - u(x)|^2}{|z|^{n+2\alpha}} dzdx,
\]
for all \( u \in H^\alpha_\rho(\mathbb{R}^n) \), then we have \( I_{\rho_\epsilon}(u) \leq I(u) \) and therefore

\[
\text{(4.11)} \quad \limsup_{\epsilon \to 0^+} C_{\rho_\epsilon} \leq C.
\]

On the other hand, by \((\rho_1)\) we have \( \rho(\epsilon x) \geq \rho_0 \) for all \( x \in \mathbb{R}^n \) then

\[
C_{\rho_\epsilon} \geq C_{\rho_0}/\epsilon.
\]

By standard arguments we can show that

\[
\lim_{\epsilon \to 0} C_{\rho_\epsilon} = C.
\]

Thus

\[
\text{(4.12)} \quad \liminf_{\epsilon \to 0} C_{\rho_\epsilon} \geq C.
\]

Therefore, by (4.11) and (4.12) we obtain (4.10).

\[\square\]

**Lemma 4.3.** There are \( \epsilon_0 > 0 \), a family \( y_\epsilon \subset \mathbb{R}^n \), constants \( \beta, R > 0 \) such that

\[
\text{(4.13)} \quad \int_{B(y_\epsilon, R)} v^2 \, dx \geq \beta, \text{ for all } \epsilon \in [0, \epsilon_0].
\]

**Proof.** By contradiction, there is a sequence \( \epsilon_m \to 0 \) such that for all \( R > 0 \)

\[
\limsup_{m \to \infty} \sup_{y \in \mathbb{R}^n} \int_{B(y, R)} v^2 \, dx = 0
\]

Using the following notation \( v_m = v_{\epsilon_m} \) and \( C_{\rho_m} = C_{\rho_{\epsilon_m}} \), by Lemma 2.2

\[
\int_{\mathbb{R}^n} v_m^{q+1} \, dx = o_m(1).
\]

Furthermore, since \( I'_{\rho_m}(v_m)v_m = 0 \) then

\[
\int_{\mathbb{R}^n} \int_{B(0, \rho_m(x))} \frac{|v_m(x + z) - v_m(x)|^2}{|z|^{n+2\alpha}} \, dz \, dx + \int_{\mathbb{R}^n} v_m^2 \, dx = \int_{\mathbb{R}^n} v_m^{2\alpha} \, dx + o_m(1).
\]

Let \( l \geq 0 \) be such that

\[
\int_{\mathbb{R}^n} \int_{B(0, \rho_m(x))} \frac{|v_m(x + z) - v_m(x)|^2}{|z|^{n+2\alpha}} \, dz \, dx + \int_{\mathbb{R}^n} v_m^2 \, dx \to l.
\]

Now, since \( I_{\rho_m}(v_m) = C_{\rho_m} \), we obtain

\[
C_{\rho_m} = \frac{\alpha}{n} \left( \int_{\mathbb{R}^n} \int_{B(0, \rho_m(x))} \frac{|v_m(x + z) - v_m(x)|^2}{|z|^{n+2\alpha}} \, dz \, dx + \int_{\mathbb{R}^n} v_m^2 \, dx \right) + o_m(1),
\]

then by Lemma 4.2

\[
\text{(4.14)} \quad l = \frac{n}{\alpha} C
\]

hence \( l > 0 \): Now, using the definition of the Sobolev constant \( S \) and Remark 2.1, we have

\[
\left( \int_{\mathbb{R}^n} v_m^{2\alpha} \, dx \right)^{2/2\alpha} S \leq \int_{\mathbb{R}^n} \int_{B(0, \rho_m(x))} \frac{|v_m(x + z) - v_m(x)|^2}{|z|^{n+2\alpha}} \, dz \, dx + \int_{\mathbb{R}^n} v_m^2 \, dx
\]
Therefore, by (4.14) and taking the limit in the above inequality as $m \to \infty$ we achieved that
\[ C \geq \frac{\alpha}{n} S^{n/2\alpha} \]
which is a contradiction with (2.5). \qed

Now let
\[ w_\epsilon(x) = v_\epsilon(x + y_\epsilon) = u_\epsilon(\epsilon x + \epsilon y_\epsilon) \]
then by (4.13),
\[ \liminf_{\epsilon \to 0^+} \int_{B(0,R)} w_\epsilon^2(x)dx \geq \beta > 0. \]

To continue, we consider the rescaled scope function $\bar{\rho}_\epsilon$, as defined in (4.4),
\[ \bar{\rho}_\epsilon(x) = \frac{1}{\epsilon} \rho(\epsilon x + \epsilon y_\epsilon) \]
and then $w_\epsilon$ satisfies the equation
\[ (-\Delta)^{\alpha/2}_\bar{\rho} w_\epsilon(x) + w_\epsilon(x) = w_\epsilon^p(x), \quad \text{in} \quad \mathbb{R}^n. \]

Now we prove the convergence of $w_\epsilon$ as $\epsilon \to 0$.

**Lemma 4.4.** For every sequence $\{\epsilon_m\}$ there is a subsequence, we keep calling the same, so that $w_{\epsilon_m} = w_m \to w$ in $H^{\alpha}(\mathbb{R}^n)$, when $m \to \infty$, where $w$ is a solution of (2.4).

**Proof.** Note that
\[
C_{\rho_m} = I_{\rho_m}(v_m) - \frac{1}{q + 1} I'_{\rho_m}(v_m)v_m
\]
\[ = \left( \frac{1}{2} - \frac{1}{q + 1} \right) \left( \int_{\mathbb{R}^n} \int_{B(0,1/\epsilon_m \rho(\epsilon_m x))} \frac{|v_m(x + z) - v_m(x)|^2}{|z|^{n+2\alpha}} dzdx + \int_{\mathbb{R}^n} v_m^2(x)dx \right) + \left( \frac{1}{q + 1} - \frac{1}{2\alpha} \right) \int_{\mathbb{R}^n} v_m^2(x)dx \]
\[ \geq \left( \frac{1}{2} - \frac{1}{q + 1} \right) \left( \int_{\mathbb{R}^n} \int_{B(0,1/\epsilon_m \rho(\epsilon_m x))} \frac{|v_m(x + z) - v_m(x)|^2}{|z|^{n+2\alpha}} dzdx + \int_{\mathbb{R}^n} v_m^2(x)dx \right) \]
\[ + \left( \frac{1}{q + 1} - \frac{1}{2\alpha} \right) \int_{\mathbb{R}^n} v_m^2(x)dx = \Lambda_m \]
By Lemma 4.2 we obtain that
\[ \limsup_{m \to \infty} \Lambda_m \leq C. \]
On the other hand, by Fatou’s Lemma and the weak convergence of \( \{w_m\} \), we get
\[
C \leq I(w)
\]
\[
= \left( \frac{1}{2} - \frac{1}{q + 1} \right) \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|w(x + z) - w(x)|^2}{|z|^{n+2\alpha}} dz dx + \int_{\mathbb{R}^n} w^2 dx \right) + \left( \frac{1}{q + 1} - \frac{1}{2\alpha} \right) \int_{\mathbb{R}^n} w^{2\alpha} dx
\]
\[
\leq \liminf_{m \to \infty} \Lambda_m + \liminf_{m \to \infty} \left( \frac{1}{2} - \frac{1}{q + 1} \right) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \setminus B(0, \frac{2m}{\epsilon_m})} \frac{|w_m(x + z) - w_m(x)|^2}{|z|^{n+2\alpha}} dz dx
\]
\[
= \liminf_{m \to \infty} \Lambda_m.
\]

So, by (4.18) and (4.19), \( \lim_{m \to \infty} \Lambda_m = C \), from where we get
\[
\lim_{m \to \infty} \|w_m - w\|_\alpha = 0.
\]

We are now in a position to complete the proof of our second main theorem.

**Proof of Theorem ??** We first obtain an upper bound for the critical values \( C_{\rho_m} = C_m \), for the sequence \( \{\epsilon_m\} \) given in Lemma 4.4. Next we consider the scope function
\[
\tilde{\rho}_m(x) = \frac{1}{\epsilon_m} \rho(\epsilon_m x + x_0),
\]
where \( x_0 \) is a global minimum point of \( H \), see Lemma 4.1. To continue, we consider the function \( w_m = w_{\epsilon_m} \) as given in (4.15) and let \( t_m > 0 \) such that \( t_m w_m \in N_{\tilde{\rho}_m} \). According to Lemma 4.4, \( \{w_m\} \) converges to \( w \in \mathcal{N} \), then \( t_m \to 1 \) and \( t_m w_m \to w \).

Now we apply Theorem 4.1 to obtain that
\[
C \leq \tilde{I}_{\tilde{\rho}_m}(t_m w_m) = \tilde{I}_{\epsilon_m}(t_m w_m) + \epsilon_m^{2\alpha} \left( \|w\|^2_{L^2} H(x_0) + o(1) \right).
\]

We have used part (i) of Theorem 4.1 with \( z_m = x_0/\epsilon_m \).

On the other hand, since \( w_m \in H^\alpha(\mathbb{R}^n) \) is a critical point of \( I_{\tilde{\rho}_m} \), we have that
\[
C_m = I_{\tilde{\rho}_m}(w_m) \geq \tilde{I}_{\tilde{\rho}_m}(t_m w_m).
\]

We write \( y_m = y_{\epsilon_m} \). If \( \epsilon_m |y_m| \to \infty \), then we may apply part (ii) of Theorem 4.1 with \( z_m = y_m \) in (4.21) and obtain that
\[
C_m \geq \tilde{I}_{\epsilon_m}(t_m w_m) + \epsilon_m^{2\alpha} o(1),
\]
which contradicts (4.20). We conclude then, that \( \{\epsilon_m y_m\} \) is bounded and that, for a subsequence, \( \epsilon_m y_m \to \bar{x} \), for some \( \bar{x} \in \mathbb{R}^n \). Now we apply Theorem 4.1 again, but now part (i) with \( z_m = y_m \) in (4.21), and we obtain that
\[
C_m \geq \tilde{I}_{\epsilon_m}(t_m w_m) + \epsilon_m^{2\alpha} \left( \|w\|^2_{L^2} H(\bar{x}) + o(1) \right).
\]
From (4.20) and (4.22) we finally get that
\[ \|w\|_{L^2}^2 \mathcal{H}(x) + o(1) \leq \|w\|_{L^2}^2 \mathcal{H}(x_0) + o(1) \]
and taking the limit as \( m \to \infty \), we get
\[ \mathcal{H}(x) \leq \mathcal{H}(x_0) \]
completing the proof of the theorem. □

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