Superconductivity phenomenon induced by external in-plane magnetic field in (2+1)-dimensional Gross–Neveu type model

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Phase structure of the (2+1)-dimensional model with four-fermion interaction of spin-1/2 quasiparticles (electrons) both in the fermion-antifermion (or chiral) and fermion-fermion (or superconducting) channels is considered at nonzero chemical potential $\mu$ and under the influence of an in-plane, i.e. parallel to a system sheet, external magnetic field $\vec{B}_\parallel$. It is shown that at sufficiently large values of $\mu$ and/or $\vec{B}_\parallel$ the Cooper pairing (or superconducting) phase appears in the system at arbitrary relation between coupling constants, provided that there is an (arbitrary small) attractive interaction in the superconducting channel. In particular, at sufficiently weak attractive interaction in the chiral channel, the Cooper pairing occurs even at infinitesimal values of $\mu$ and/or $\vec{B}_\parallel$. The superconducting phase of the model is always a paramagnetic one.

I. INTRODUCTION

Recently much attention has been paid to investigation of (2+1)-dimensional quantum field theories (QFT) and, in particular, to models with four-fermion interactions of the Gross–Neveu (GN) type. Partially, this interest is explained by more simple structure of QFT in two-, rather than in three spatial dimensions. As a result, it is much easier to investigate qualitatively such real physical phenomena as dynamical symmetry breaking [1] and color superconductivity [13,17] as well as to model phase diagrams of real quantum chromodynamics [18,19] etc. in the framework of (2+1)-dimensional QFT. Another example of this kind is the spontaneous chiral symmetry breaking induced by external magnetic fields. This effect was for the first time studied also in terms of (2+1)-dimensional GN models [20–24]. Moreover, these theories are very useful in developing new QFT techniques like the optimized perturbation theory [18,19,25,26], and so on.

However, there is yet another more serious motivation for studying (2+1)-dimensional QFT. It is supported by the fact that there are many condensed matter systems which, firstly, have a (quasi-)planar structure and, secondly, their excitation spectrum is described adequately by relativistic Dirac-like equation rather than by Schrödinger one. Among these systems are the high-Tc cuprate and iron superconductors [27,28], the one-atom thick layer of carbon atoms, or graphene, [29,30] etc. Thus, many properties of such condensed matter systems can be explained in the framework of various (2+1)-dimensional QFT, including the GN-type models (see, e.g., [31–44] and references therein).

In the recent papers [41–44] a competition between chiral symmetry breaking (excitonic pairing) and superconductivity phenomenon (Cooper pairing) was investigated in the framework of (2+1)-dimensional GN-type models. There the influence of such external factors, as temperature $T$, chemical potential $\mu$ and external magnetic field $\vec{B}_\perp$ perpendicular to the system plane, on the chiral and electromagnetic $U(1)$ symmetries was studied. In particular, it was shown in [43] that sufficiently strong perpendicular magnetic field $\vec{B}_\perp$ destroys the superconducting state of a planar system.

In the present paper the (2+1)-dimensional GN-type model, which describes four-fermion interaction of quasiparticles with spin 1/2 (electrons) both in the chiral (with coupling constant $G_1$) and Cooper pairing (with coupling constant $G_2$) channels, is considered at $T = 0$ and $\mu \neq 0$. In addition, we suppose that the planar system of electrons is subjected to an external magnetic field $\vec{B} = \vec{B}_\perp + \vec{B}_\parallel$ in such a way that $\vec{B}_\parallel = 0$ ($\vec{B}_\parallel$ is the in-plane, i.e. parallel to the system plane, magnetic field). As a result, in our consideration the external magnetic field $\vec{B}$ couples only to the spin of electrons and not to its orbital angular momentum. (The interaction between $\vec{B}$ and orbital angular momentum of electrons appears in planar systems only at $\vec{B}_\perp \neq 0$.) In this case the way to account for the influence of external parallel magnetic field on the phase structure of any planar system is to introduce the Zeeman terms into Lagrangian, which are really due to Zeeman effect. These terms modify effectively the chemical potentials corresponding to the electrons with different spin projections along the direction of an external magnetic field, $\vec{B}_\parallel$. In doing so we show that the external magnetic field $\vec{B}_\perp$ parallel to the system plane induces the superconductivity phenomenon in the initial model even at infinitesimal coupling $G_2$ of the Cooper pairing channel. If in addition the coupling constant $G_1$ is sufficiently small, then superconductivity appears at arbitrary weak in-plane magnetic field $\vec{B}_\parallel$ (of course, provided that $G_2 > 0$).

1 In the recent paper [35] the magnetization of planar systems exposed to an external in-plane magnetic field was investigated in the framework of the same (2+1)-dimensional GN model but in the particular case $G_1 \neq 0$, $G_2 = 0$. In particular, it was shown there that at sufficiently high values of $\vec{B}_\parallel$ there is a restoration of the chiral symmetry of the model.
The paper is organized as follows. In Sec. II the (2+1)-dimensional GN-type model with four-fermion interactions in the fermion-antifermion (or chiral) and fermion-fermion (or superconducting) channels is presented. Here the renormalized thermodynamic potential (TDP) of the model is obtained in the leading order of the large $N$ technique, taking into account the nonzero values of the chemical potential $\mu$ and the external in-plane magnetic field $\vec{B}_{||}$ (temperature is put equal to zero). In the next Sec. III a renormalization group invariant expression for the TDP in the leading order of $1/N$ expansion is obtained. The TDP global minimum point provides us with chiral and Cooper pairing condensates. In Sec. IV A phase structure of the model is described at $\mu = 0$ and $\vec{B}_{||} = 0$. Finally, in Sec. IV B the $(\mu, |\vec{B}_{||}|)$-phase diagrams as well as the behavior of gaps and different thermodynamic quantities of the system such as particle density, magnetization and magnetic susceptibility are presented for some representative values of coupling constants. We show in this section that for arbitrary relations between coupling constants superconductivity is induced in the system at sufficiently large values of $\mu$ and/or in-plane magnetic field $\vec{B}_{||}$. In particular, it is found out that infinitesimal values of $\vec{B}_{||}$ induce the superconductivity phenomenon in the case of a rather weak attractive interaction in the fermion-antifermion channel.

II. THE MODEL AND ITS THERMODYNAMIC POTENTIAL

We investigate the influence of an external magnetic field $\vec{B}$ on the phase structure of (2+1)-dimensional version of the Chodos et al. model [44, 45] which describes low-energy dynamics of quasiparticles (electrons) both in the fermion-antifermion (or chiral) and fermion-fermion (or Cooper pairing) channels. In addition, we take into account the fact that there are two spin projections, $\pm 1/2$, of electrons on the direction of the magnetic field $\vec{B}$. If external magnetic field is parallel to the system plane, i.e. $\vec{B} = \vec{B}_{||}$, then the Lagrangian has the following form

$$L = \sum_{k=1}^{2} \bar{\psi}_k \left[ \gamma^\rho i \partial_\rho + \mu \gamma^0 - \nu(-1)\gamma^0 \right] \psi_k + \frac{G_1}{N} \left( \sum_{k=1}^{2} \bar{\psi}_k \psi_k \right)^2 + \frac{G_2}{N} \left( \sum_{k=1}^{2} \bar{\psi}_k C\psi_k \right) \left( \sum_{j=1}^{2} \bar{\psi}_j C\psi_j^T \right),$$

(1)

where the summation over the repeated indices $a, b = 1, ..., N$ of the internal $O(N)$ group as well as repeated Lorentz indices $\rho = 0, 1, 2$ is implied. For each fixed values of $k = 1, 2$ and $a = 1, ..., N$ the quantity $\psi_{ka}(x)$ in (1) means the massless Dirac fermion field, transforming over a reducible 4-component spinor representation of the (2+1)-dimensional Lorentz group. Moreover, all these Dirac fields $\psi_{ka}(x)$ are composed into two fundamental multiplets, $\psi_{1(a)}(x)$ and $\psi_{2(a)}(x)$ ($a = 1, ..., N$), of the internal auxiliary $O(N)$ group, which is introduced here in order to make it possible to perform all the calculations in the framework of the nonperturbative large $N$ expansion method. We suppose that spinor fields $\psi_{1(a)}(x)$ and $\psi_{2(a)}(x)$ ($a = 1, ..., N$) correspond to electrons with spin projections $1/2$ and $-1/2$ on the direction of external magnetic field, respectively. In (1) the symbol $T$ denotes the transposition operation, $\mu$ is a fermion number chemical potential and the $\nu$-term is introduced in order to take into account the Zeeman interaction energy of electrons with external magnetic field $\vec{B}_{||}$. Hence, in our case $\nu = \mu_B B/2$, where $B = |\vec{B}_{||}|$. $g$ is the spectroscopic Lande factor (in what follows it is supposed throughout the paper that $g = 2$) and $\mu_B$ is the Bohr magneton. Moreover, $C \equiv \gamma^2$ is the charge conjugation matrix. The algebra of the $\gamma^\rho$-matrices as well as their particular representations are given, e.g., in [44]. The model (1) is invariant under the discrete chiral transformation, $\psi_{ka} \rightarrow \gamma^5 \psi_{ka}$ (the particular realization of the $\gamma^5$-matrix is also presented in [44]), as well as with respect to the transformations from the continuous $U(1)$ fermion number group, $\psi_{ka} \rightarrow \exp(i\alpha)\psi_{ka}$ $(k = 1, 2, a = 1, ..., N)$, responsible for the fermion number conservation or, equivalently, for the electric charge conservation law in the system under consideration. Certainly, there is $O(N)$ invariance of the Lagrangian (1).

The linearized version of Lagrangian (1) that contains auxiliary bosonic fields $\sigma(x)$, $\Delta(x)$, $\Delta^*(x)$ has the form

$$L = -\frac{N \sigma^2}{4G_1} - \frac{N \Delta^* \Delta}{4G_2} - \sum_{k=1}^{2} \bar{\psi}_k \left( \gamma^\rho i \partial_\rho + \mu_1 \gamma^0 - \sigma \right) \psi_k - \frac{\Delta^*}{2} \bar{\psi}_k C\psi_k - \frac{\Delta}{2} \bar{\psi}_k C\psi_k^T,$$

(2)

where $\mu_1 = \mu + \nu$, $\mu_2 = \mu - \nu$ and from now on $\nu = \mu_B B$ (in this formula and below the summation over repeated indices is implied). Clearly, the Lagrangians (1) and (2) are equivalent, as can be seen by using the Euler-Lagrange equations of motion for scalar bosonic fields which take the form

$$\sigma(x) = -2G_1 \sum_{k=1}^{2} \bar{\psi}_k \psi_k, \quad \Delta(x) = -2G_2 \sum_{k=1}^{2} \bar{\psi}_k C\psi_k, \quad \Delta^*(x) = -2G_2 \sum_{k=1}^{2} \bar{\psi}_k C\psi_k^T.$$

(3)

One can easily see from (3) that the neutral field $\sigma(x)$ is a real quantity, i.e. $(\sigma(x))^\dagger = \sigma(x)$ (the superscript symbol $\dagger$ denotes the Hermitian conjugation), but the (charged) dierfermion fields $\Delta(x)$ and $\Delta^*(x)$ are mutually Hermitian.

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2 It is a (2+1)-dimensional generalization, made in [44], of the initially (1+1)-dimensional well-known model by Chodos et al [43].
conjugated complex quantities, so \((\Delta(x))^\dagger = \Delta^*(x)\) and vice versa. If the diferfion field \(\Delta(x)\) has a nonzero ground state expectation value, i.e. \(\langle \Delta(x) \rangle \neq 0\), the Abelian fermion number \(U(1)\) symmetry of the model is spontaneou5 broken down and the superconducting phase is realized in the model. (Note, at \(T = 0\) a continuous symmetry breaking is allowed to occur in two spatial dimensions. The clarifying discussion is presented, e.g., in the papers [1, 44]). However, if \(\langle \sigma(x) \rangle \neq 0\) then the discrete chiral symmetry of the model is spontaneously broken.

Let us now study the phase structure of the four-fermion model (1) by starting from the equivalent semi-bosonized Lagrangian [2]. In the leading order of the large \(N\) approximation, the effective action \(S_{\text{eff}}(\sigma, \Delta, \Delta^*)\) of the considered model is expressed by means of the path integral over fermion fields

\[
\exp(iS_{\text{eff}}(\sigma, \Delta, \Delta^*)) = \int \prod_{k=1}^{2} \prod_{a=1}^{N} [d\bar{\psi}_ka][d\psi_ka] \exp\left(i \int \mathcal{L} d^3x \right),
\]

where

\[
S_{\text{eff}}(\sigma, \Delta, \Delta^*) = -\int d^3x \left[ \frac{N}{4G_1} \sigma^2(x) + \frac{N}{4G_2} \Delta(x)\Delta^*(x) \right] + \bar{S}_{\text{eff}}. \tag{4}
\]

The fermion contribution to the effective action, i.e. the term \(\bar{S}_{\text{eff}}\) in (4), is given by

\[
\exp(i\bar{S}_{\text{eff}}) = \int \prod_{l=1}^{2} \prod_{a=1}^{N} [d\bar{\psi}_la][d\psi_ka] \exp\left\{ i \sum_{k=1}^{2} \left[ \bar{\psi}_ka \left( \gamma^\nu i\partial_\nu + \mu_k \gamma^0 - \sigma \right) \psi_ka - \frac{\Delta^*}{2} \bar{\psi}_ka C\psi_ka - \frac{\Delta}{2} \bar{\psi}_ka C\psi_ka k^T \right] d^3x \right\}. \tag{5}
\]

The ground state expectation values \(\langle \sigma(x) \rangle, \langle \Delta(x) \rangle, \) and \(\langle \Delta^*(x) \rangle\) of the composite bosonic fields are determined by the saddle point equations,

\[
\frac{\delta S_{\text{eff}}}{\delta \sigma(x)} = 0, \quad \frac{\delta S_{\text{eff}}}{\delta \Delta(x)} = 0, \quad \frac{\delta S_{\text{eff}}}{\delta \Delta^*(x)} = 0. \tag{6}
\]

For simplicity, throughout the paper we suppose that the above mentioned ground state expectation values do not depend on spacetime coordinates, i.e.

\[
\langle \sigma(x) \rangle \equiv M, \quad \langle \Delta(x) \rangle \equiv \Delta, \quad \langle \Delta^*(x) \rangle \equiv \Delta^*, \quad \tag{7}
\]

where \(M, \Delta, \Delta^*\) are constant quantities. In fact, they are coordinates of the global minimum point of the thermodynamic potential (TDP) \(\Omega(M, \Delta, \Delta^*)\). In the leading order of the large \(N\) expansion this quantity is defined by the following expression:

\[
\int d^3x \Omega(M, \Delta, \Delta^*) = -\frac{1}{N} S_{\text{eff}}\{\sigma(x), \Delta(x), \Delta^*(x)\}\big|_{\sigma(x) = M, \Delta(x) = \Delta, \Delta^*(x) = \Delta^*},
\]

which gives

\[
\int d^3x \Omega(M, \Delta, \Delta^*) = \int d^3x \left( \frac{M^2}{4G_1} + \frac{\Delta \Delta^*}{4G_2} \right) + \frac{i}{N} \ln \left( \int \prod_{l=1}^{2} \prod_{b=1}^{N} [d\bar{\psi}_lb][d\psi_ka] \exp\left( i \sum_{k=1}^{2} \left[ \bar{\psi}_ka D_k \psi_ka - \frac{\Delta^*}{2} \bar{\psi}_ka C\psi_ka k - \frac{\Delta}{2} \bar{\psi}_ka C\psi_ka k^T \right] d^3x \right) \right). \tag{8}
\]

where \(D_k = \gamma^\nu i\partial_\nu + \mu_k \gamma^0 - M\). To proceed, let us first point out that without loss of generality the quantities \(\Delta, \Delta^*\) might be considered as real ones. \[3\] So, in the following we will suppose that \(\Delta = \Delta^* \equiv \Delta\), where \(\Delta\) is already a real quantity. Then, in order to find a convenient expression for the TDP it is necessary to evaluate the Gaussian path integral in (8) (see, e.g., the paper [44], where a similar path integral was calculated). As a result, we obtain the following expression for the TDP of the model (1) at zero temperature:

\[
\Omega(M, \Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} + \frac{i}{2} \sum_{k=1}^{2} \int \frac{d^3p}{(2\pi)^3} \ln \left[ \left( p_0^2 - (\xi^+_{\Delta,k})^2 \right) \left( p_0^2 - (\xi^-_{\Delta,k})^2 \right) \right]. \tag{9}
\]

\[3\] Otherwise, phases of the complex values \(\Delta, \Delta^*\) might be eliminated by an appropriate transformation of fermion fields in the path integral \([3]\).
where $(E_{\Delta,k}^\pm)^2 = E^2 + \mu_1^2 + \Delta^2 \pm 2\sqrt{M^2\Delta^2 + \mu_1^2E^2}$ and $E = \sqrt{M^2 + |\vec{p}|^2}$. Throughout the paper we suppose that $\mu \geq 0$, $\nu \geq 0$, $M \geq 0$ and $\Delta \geq 0$. Using in the expression $\Omega$ a rather general formula

$$\int_{-\infty}^{\infty} dp_0 \ln (p_0 - A) = i\pi |A|$$

(10)

(obtained rigorously, e.g., in Appendix B of [46] and true up to an infinite term independent on real quantity $A$), it is possible to reduce it to the following one:

$$\Omega(M, \Delta) \equiv \Omega^{\text{ren}}(M, \Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \sum_{k=1}^{2} \int_{0}^{\Lambda} dp_1 \int_{0}^{\Lambda} dp_2 \left( E_{\Delta,k}^+ + E_{\Delta,k}^- \right).$$

(11)

The integral term in (11) is an ultraviolet divergent, hence to obtain any information from this expression we have to renormalize it.

Note finally that the formulae from this section resemble the corresponding relations from [44]. However, there is an essential difference which is due to the fact that in the present model we deal with two $O(N)$-multiplets of Dirac fields and, correspondingly, with two different chemical potentials.

### III. THE RENORMALIZATION PROCEDURE

First of all, let us regularize the zero temperature TDP (11) by cutting momenta, i.e. we suppose that $|p_1| < \Lambda$, $|p_2| < \Lambda$ in (11). As a result we have the following regularized expression (which is finite at finite values of $\Lambda$):

$$\Omega^{\text{reg}}(M, \Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \sum_{k=1}^{2} \int_{0}^{\Lambda} dp_1 \int_{0}^{\Lambda} dp_2 \left( E_{\Delta,k}^+ + E_{\Delta,k}^- \right).$$

(12)

Let us use in (12) the following asymptotic expansion ($k = 1, 2$)

$$E_{\Delta,k}^+ + E_{\Delta,k}^- = 2|\vec{p}| + \frac{M^2 + \Delta^2}{|\vec{p}|} + O(1/|\vec{p}|^3),$$

(13)

where $|\vec{p}| = \sqrt{p_1^2 + p_2^2}$. (Note, the leading asymptotic terms in (13) do not depend on $\mu_{1,2}$.). Then, upon integration there term-by-term, it is possible to find

$$\Omega^{\text{reg}}(M, \Delta) = M^2 \left[ \frac{1}{4G_1} - \frac{4\Lambda \ln(1 + \sqrt{2})}{\pi^2} \right] + \Delta^2 \left[ \frac{1}{4G_2} - \frac{4\Lambda \ln(1 + \sqrt{2})}{\pi^2} \right] - \frac{4\Lambda^3(\sqrt{2} + \ln(1 + \sqrt{2}))}{3\pi^2} + O(\Lambda^0),$$

(14)

where $O(\Lambda^0)$ denotes an expression which is finite in the limit $\Lambda \to \infty$. Second, we suppose that the bare coupling constants $G_1$ and $G_2$ depend on the cutoff parameter $\Lambda$ in such a way that in the limit $\Lambda \to \infty$ one obtains a finite expression in the square brackets of (14). Clearly, to fulfil this requirement it is sufficient to require that

$$\frac{1}{4G_1} \equiv \frac{1}{4G_1(\Lambda)} = \frac{4\Lambda \ln(1 + \sqrt{2})}{\pi^2} + \frac{1}{\pi g_1},$$

$$\frac{1}{4G_2} \equiv \frac{1}{4G_2(\Lambda)} = \frac{4\Lambda \ln(1 + \sqrt{2})}{\pi^2} + \frac{1}{\pi g_2},$$

(15)

where $g_{1,2}$ are finite and $\Lambda$-independent model parameters with dimensionality of inverse mass. Moreover, since bare couplings $G_1$ and $G_2$ do not depend on a normalization point, the same property is also valid for $g_{1,2}$. Hence, taking into account in (12) and (14) the relations (15) and ignoring there an infinite $M$- and $\Delta$-independent constant, one obtains the following renormalized, i.e. finite, expression for the TDP

$$\Omega^{\text{ren}}(M, \Delta) = \lim_{\Lambda \to \infty} \left\{ \Omega^{\text{reg}}(M, \Delta) \right|_{G_1=G_1(\Lambda),G_2=G_2(\Lambda)} + \frac{4\Lambda^3(\sqrt{2} + \ln(1 + \sqrt{2}))}{3\pi^2} \right\}.$$ 

(16)

It should also be mentioned that the TDP (16) is a renormalization group invariant quantity.

Suppose that $\mu = 0$ and $\nu = \mu g_2 B = 0$. In this case $\mu_{1,2} = 0$, so the $O(\Lambda^0)$ term in (14) can be calculated explicitly. As a result, we have for the TDP in this particular case the following expression:

$$V(M, \Delta) \equiv \Omega^{\text{ren}}(M, \Delta) \mid_{\mu=0,\nu=0} = \frac{M^2}{\pi g_1} + \frac{\Delta^2}{\pi g_2} + \frac{(M + \Delta)^3}{3\pi} + \frac{|M - \Delta|^3}{3\pi}.$$ 

(17)
(The parameters $g_{1,2}$ are introduced in [15] in such a way that at $\mu = \nu = 0$ the TDP (17) differs by a factor 2 from the corresponding quantity of the paper [44].)

Now, let us obtain an alternative expression for the renormalized TDP (16) at $\mu \neq 0$ and $\nu \neq 0$, i.e. at $\mu_{1,2} \equiv \mu \pm \nu \neq 0$. For this purpose one can rewrite the unrenormalized TDP $\Omega^{un}(M, \Delta)$ (11) in the following way

$$\Omega^{un}(M, \Delta) = \frac{M^2}{4G_1} + \frac{\Delta^2}{4G_2} - \sum_{k=1}^{2} \int \frac{d^2p}{(2\pi)^2} \left( E_{\Delta,k}^{+} |_{\mu,\nu=0} + E_{\Delta,k}^{-} |_{\mu,\nu=0} \right)$$

$$- \sum_{k=1}^{2} \int \frac{d^2p}{(2\pi)^2} \left( E_{\Delta,k}^{+} - E_{\Delta,k}^{-} |_{\mu,\nu=0} - E_{\Delta,k}^{-} |_{\mu,\nu=0} \right),$$

where for each $k = 1, 2$

$$E_{\Delta,k}^{+} |_{\mu,\nu=0} + E_{\Delta,k}^{-} |_{\mu,\nu=0} = \sqrt{|p|^2 + (M + \Delta)^2} + \sqrt{|p|^2 + (M - \Delta)^2}.$$ 

Since the leading terms of the asymptotic expansion (13) do not depend on $\mu_{1,2}$, it is clear that the integrals in the last sum in (18) are convergent. Other terms in (18) form the unrenormalized TDP of the particular case with $\mu = 0$ and $\nu = 0$ which is reduced after renormalization procedure to the expression (17). Hence, after renormalization we obtain from (18) the following finite expression (evidently, it coincides with renormalized TDP (16)):

$$\Omega^{ren}(M, \Delta) = V(M, \Delta) - \int \frac{d^2p}{(2\pi)^2} \sum_{k=1}^{2} \left( E_{\Delta,k}^{+} + E_{\Delta,k}^{-} \right) - \sqrt{|p|^2 + (M + \Delta)^2} - \sqrt{|p|^2 + (M - \Delta)^2},$$

where $V(M, \Delta)$ is presented in (17). The integral terms in (19) can be explicitly calculated. As a result, we have

$$\Omega^{ren}(M, \Delta) = \frac{M^2}{\pi g_1} + \frac{\Delta^2}{\pi g_2} + \sum_{k=1}^{2} \left\{ \frac{1}{6\pi} \left( M + \mu_k^2 + \Delta^2 \right)^{\frac{3}{2}} + \frac{1}{6\pi} \left| M - \mu_k^2 + \Delta^2 \right| \right\}$$

$$- \frac{1}{4\pi} \left[ t_k^+ \left( M + \sqrt{\mu_k^2 + \Delta^2} \right) + t_k^- \left| M - \sqrt{\mu_k^2 + \Delta^2} \right| \right]$$

$$- \left( \mu_k^2 - M^2 \right) \Delta^2 \ln \left( t_k^+ |_{\mu_k^2 + \Delta^2} \left| M - \mu_k^2 + \Delta^2 \right| \right),$$

(20)

where $t_k^\pm = M \mu_k^2 + \Delta^2 \pm \mu_k^2$. It is not so evident, but at $\mu_k = 0 \ (k = 1, 2)$ the expression (20) for $\Omega^{ren}(M, \Delta)$ coincides with $V(M, \Delta)$ (17).

Note also that in the model under consideration we have two $O(N)$-multiplets of Dirac fields. Due to this reason we have introduced in (15) the parameters $g_{1,2}$ in such a way that at $\nu = 0$ the TDP (20) differs by a factor 2 from the corresponding quantity of the model [44] with single multiplet. However, at $\nu \neq 0$, i.e. when $\mu_1 \neq \mu_2$, there is an essential difference between the TDPs of two models.

IV. PHASE STRUCTURE OF THE MODEL

As was mentioned above, the coordinates of the global minimum point $(M_0, \Delta_0)$ of the TDP $\Omega^{ren}(M, \Delta)$ define the ground state expectation values of auxiliary fields $\sigma(x)$ and $\Delta(x)$. Namely, $M_0 = \langle \sigma(x) \rangle$ and $\Delta_0 = \langle \Delta(x) \rangle$. The quantities $M_0$ and $\Delta_0$ are usually called order parameters, or gaps, because they are responsible for the phase structure of the model or, in other words, for the properties of the model ground state (see also the comment after 3). Moreover, the gap $M_0$ is equal to the dynamical mass of one-fermionic excitations of the ground state. As a rule, gaps depend on model parameters as well as on various external factors. In our consideration the gaps $M_0$ and $\Delta_0$ are certain functions of the free model parameters $g_1$ and $g_2$ and such external factors as chemical potential $\mu$ and external in-plane magnetic field $B$.

A. The case $\mu = 0, B = 0$

First of all, let us discuss the phase structure of the model (1) in the simplest case when $\mu = 0$ and $B = 0$. The corresponding TDP is given in (17) by the function $V(M, \Delta)$. Since the global minimum of this function was already investigated in [44, 47], although in the framework of another (2+1)-dimensional GN model, we present at once the phase structure of the initial model (1) at $\mu = 0$ and $B = 0$ (see Fig. 1 which is taken from [44]).

In Fig. 1 the phase portrait of the model is depicted depending on the values of the free model parameters $g_1$ and $g_2$. There the plane $(g_1, g_2)$ is divided into several areas. In each area one of the phases I, II or III is implemented.
The second one, i.e. the point \(M_G\) has two equivalent global minima. The first one, the point \((M, \Delta) = (0, 0)\). In the phase II \(\langle \sigma \rangle = 1/g_1\). In the phase III \(\langle \sigma \rangle = 1/g_2\). On the curve \(L \equiv \{(g_1, g_2) : g_1 = g_2\}\), where \(g_{1,2} < 0\), the TDP minima corresponding to the phase II and III are equivalent.

In the phase I, i.e. at \(g_1 > 0\) and \(g_2 > 0\), the global minimum of the effective potential \(V(M, \Delta)\) is arranged at the origin. So in this case we have \(M_0 = \langle \sigma(x) \rangle = 0\) and \(\Delta_0 = \langle \Delta(x) \rangle = 0\). As a result, in the phase I both chiral and electromagnetic \(U(1)\) symmetries remain intact and fermions are massless. Due to this reason the phase I is called symmetric. In the phase II, which is allowed only for \(g_1 < 0\), at the global minimum point \((M_0, \Delta_0)\) the relations \(M_0 = -1/g_1\) and \(\Delta_0 = 0\) are valid. So in this phase chiral symmetry is spontaneously broken down and fermions acquire dynamically the mass \(M_0\). Finally, in the superconducting phase III, where \(g_2 < 0\), we have the following values for the gaps \(M_0 = 0\) and \(\Delta_0 = -1/g_2\).

Note also that if \(g_1 = g_2 = g\) and, in addition, \(g < 0\) (it is just the line \(L\) in Fig. 1), then the effective potential \(V\) has two equivalent global minima. The first one, the point \((M_0 = -1/g, \Delta_0 = 0)\), corresponds to a phase with chiral symmetry breaking. The second one, i.e. the point \((M_0 = 0, \Delta_0 = -1/g)\), corresponds to superconductivity.

Clearly, if the cutoff parameter \(\Lambda\) is fixed, then the phase structure of the model can be described in terms of bare coupling constants \(G_1, G_2\) instead of finite quantities \(g_1, g_2\). Indeed, let us first introduce a critical value of the bare couplings, \(G_c = \frac{\pi^2}{16\Lambda^2(1+\sqrt{2})}\). Then, as it follows from Fig. 1 and (15), at \(G_1 < G_c\) and \(G_2 < G_c\) the symmetric phase I of the model is located. If \(G_1 > G_c\) and \(G_2 < G_c\) \((G_1 < G_c, G_2 > G_c)\), then the chiral symmetry broken phase II (the superconducting phase III) is realized. Finally, let us suppose that both \(G_1 > G_c\) and \(G_2 > G_c\). In this case at \(G_1 > G_2\) \((G_1 < G_2)\) we have again the chiral symmetry broken phase II (the superconducting phase III).

### B. The case \(\mu \neq 0\) and/or \(B \neq 0\)

The starting point of our investigations in this case is the TDP (20). The behavior of the global minimum point \((M_0, \Delta_0)\) of this TDP vs \(\mu\) and \(B\) supplies us with the phase structure of the model. Moreover, we are interested in considering such thermodynamic quantities as particle density \(n\), magnetization \(m\) and magnetic susceptibility \(\chi\),

\[
\begin{align*}
n &= -\frac{\partial \Omega^{ren}(M_0, \Delta_0)}{\partial \mu}, \\
m &= -\frac{\partial \Omega^{ren}(M_0, \Delta_0)}{\partial B}, \\
\chi &= \frac{\partial m}{\partial B}.
\end{align*}
\]

In the framework of the model (1) these quantities can be presented in the following form

\[
\begin{align*}
n &= n_1 + n_2, \\
m &= \mu_B(n_1 - n_2),
\end{align*}
\]

where

\[
\begin{align*}
n_1 &= -\frac{\partial \Omega^{ren}(M_0, \Delta_0)}{\partial \mu_1}, \\
n_2 &= -\frac{\partial \Omega^{ren}(M_0, \Delta_0)}{\partial \mu_2} = -\text{sign}(\mu_2)\frac{\partial \Omega^{ren}(M_0, \Delta_0)}{\partial |\mu_2|}
\end{align*}
\]

are densities of particles with spin projection \(1/2\) and \(-1/2\), respectively, and \(\text{sign}(x)\) is the sign-function. Note, in the particular case with \(\mu \neq 0\) and \(B = 0\) we have \(n_1 = n_2\). Therefore, in this case \(n \neq 0, m = 0\). However, in the opposite particular case with \(\mu = 0\) and \(B \neq 0\) the relations \(n_1 = -n_2\) and, as a result, \(n = 0, m \neq 0\) are valid. It
B = 0, we can say that the chemical potential and/or in-plane magnetic field
superconducting phase is realized in the system. Since at $g = 0$ the system is in the symmetric phase if $\mu_1 = 0$ and $\Delta_0 \neq 0$ (the chiral symmetry breaking phase II) or $\mu_1 = 0, \Delta_0 \neq 0$ (the symmetric phase III), i.e. the symmetric phase is absent in the model, if $\mu_1 \neq 0$ and/or $B \neq 0$. Therefore, it is useful to present the analytical expressions for the thermodynamic quantities in each of the phases II and III. Namely, for the phase II we have

$$n\text{_{phase II}} = \frac{1}{2\pi} \left[ (\mu_1^2 - M_0^2)\theta(\mu_1 - M_0) + \text{sign}(\mu_2)(\mu_2^2 - M_0^2)\theta(\mu_2 - M_0) \right],$$

$$m\text{_{phase II}} = \frac{\mu_B}{2\pi} \left[ (\mu_1^2 - M_0^2)\theta(\mu_1 - M_0) - \text{sign}(\mu_2)(\mu_2^2 - M_0^2)\theta(\mu_2 - M_0) \right],$$

$$\chi\text{_{phase II}} = \frac{\mu_B}{\pi} \left[ |\mu_1|\theta(\mu_1 - M_0) + |\mu_2|\theta(\mu_2 - M_0) \right],$$

whereas for the phase III these quantities look like

$$n\text{_{phase III}} = \frac{1}{2\pi} \left[ \mu_1 \sqrt{\mu_1^2 + \Delta_0^2} + \Delta_0^2 \ln \frac{\mu_1 + \sqrt{\mu_1^2 + \Delta_0^2}}{\Delta_0} \right. \left. + \text{sign}(\mu_2) \left| \mu_2 \right| \sqrt{\mu_2^2 + \Delta_0^2} + \Delta_0^2 \ln \frac{|\mu_2| + \sqrt{\mu_2^2 + \Delta_0^2}}{\Delta_0} \right],$$

$$m\text{_{phase III}} = \frac{\mu_B}{2\pi} \left[ \mu_1 \sqrt{\mu_1^2 + \Delta_0^2} + \Delta_0^2 \ln \frac{\mu_1 + \sqrt{\mu_1^2 + \Delta_0^2}}{\Delta_0} \right. \left. - \text{sign}(\mu_2) \left| \mu_2 \right| \sqrt{\mu_2^2 + \Delta_0^2} + \Delta_0^2 \ln \frac{|\mu_2| + \sqrt{\mu_2^2 + \Delta_0^2}}{\Delta_0} \right],$$

$$\chi\text{_{phase III}} = \frac{\mu_B^2}{\pi} \left[ \sqrt{\mu_1^2 + \Delta_0^2} + \sqrt{\mu_2^2 + \Delta_0^2} \right].$$

**The case $g_1 > 0$.** Our investigations show that in this case at arbitrary nonzero values of $\mu$ and/or $B$ the superconducting phase is realized in the system. Since at $g_2 = 0$ the phenomenon takes place even at $\mu = 0$ and $B = 0$, we can say that the chemical potential and/or in-plane magnetic field enhance superconductivity, which was originally generated in this case by a rather strong interaction in the fermion-fermion channel ($G_2 > G_c$). In contrast, at $g_2 > 0$ the system is in the symmetric phase if $\mu = 0$ and $B = 0$. However, arbitrary small nonzero values of $\mu$ and/or $B$ induce in this case the superconductivity. In Fig. 2 and 3 the behavior of the superconducting gap $\Delta_0$ and

![FIG. 2. Superconducting gap $\Delta_0$ and particle density $n$ vs $B$ at arbitrary fixed $g_1 > 0$ as well as at $g_2 = 0.5g_1$, and $\mu = 0.5/g_1$. Curves 1 and 2 are the plots of the dimensionless quantities $g_1\Delta_0$ and $g_1^2n$, respectively.](image1)

![FIG. 3. Magnetization $m$ and magnetic susceptibility $\chi$ vs $B$ at arbitrary fixed $g_1 > 0$ as well as at $g_2 = 0.5g_1$, and $\mu = 0.5/g_1$. Curves 1 and 2 are the plots of the dimensionless quantities $g_1^2m/\mu_B$ and $g_1\chi/\mu_B^2$, respectively.](image2)
such thermodynamic parameters of the model, as particle density \( n \), magnetization \( m \) and magnetic susceptibility \( \chi \) vs \( B \) are presented at arbitrary fixed \( g_1 > 0 \) and \( g_2 = 0.5g_1 \) as well as at fixed chemical potential, \( \mu = 0.5/g_1 \).

The case \( g_1 < 0 \). We have found that in this case the \((\mu, B)\)-phase structure of the model is richer than in the case \( g_1 > 0 \). Indeed, supposing that \( g_2 = -1.5|g_1| \), where \( g_1 \) is arbitrary fixed and negative, it is easy to find the \((\mu, B)\)-phase portrait of the model drawn in Fig. 4. There in the chiral symmetry breaking phase both the particle density \( n \) and the magnetization \( m \) are equal to zero. Qualitatively, the similar \((\mu, B)\)-phase structure occurs for each coupling \( g_2 \) from the interval \( g_2 \in (-k|g_1|, -|g_1|) \), where \( k \approx 3.08 \). However, if \( g_2 > 0 \) (recall, \( g_1 < 0 \)) or \( g_2 < -k|g_1| \), then the situation is changed qualitatively. In this case for the representative choice of coupling constants, \( g_2 = 0.5|g_1| \) \((g_1 \text{ is arbitrary fixed and negative})\), the \((\mu, B)\)-phase portrait of the model is presented in Fig. 5. It is clear from this figure that chiral symmetry breaking phase II is divided into two regions denoted as \( \Pi_1 \) and \( \Pi_2 \). In the region \( \Pi_1 \) the quantities \( n \) and \( m \) are still equal to zero, whereas in the phase \( \Pi_2 \) both \( n \neq 0 \) and \( m \neq 0 \). On the boundary between chiral symmetry breaking phases \( \Pi_1 \) and \( \Pi_2 \) the relation \(|g_1|\mu + \mu_B|g_1|B = 1\) is valid. Moreover, we represent in Fig. 6 and 7 the plots of gaps \( M_0 \) and \( \Delta_0 \) as well as particle density \( n \), magnetization \( m \) and magnetic susceptibility \( \chi \) as functions of external in-plane magnetic field \( B \) at \( g_2 = 0.5|g_1| \) and \( \mu = 0.7/|g_1| \).

Finally, we would like to note that the TDP (20) is symmetric with respect to the transformation \( \mu \leftrightarrow \nu \). So, such physical quantities as \( M_0, \Delta_0, \chi \) remain intact, whereas \( n \leftrightarrow m \) if the transformation \( \mu \leftrightarrow \nu \) is performed. We remark that all figures 2, 3, 6, and 7 show these physical quantities only as functions of \( \nu \equiv \mu_B B \) at some fixed values of \( \mu \), i.e. its dependence on \( \mu \) at fixed \( \nu \) is not presented in an explicite form in our paper. However, there is no need for a special numerical calculations in this direction, taking into account the above-mentioned symmetry under the permutation \( \mu \leftrightarrow \nu \). Indeed, let us suppose that \( g_1 > 0 \), \( g_2 = 0.5g_1 \) and \( \mu_B B = 0.5/g_1 \). In this case, in order to consider behavior of the quantities \( \Delta_0, n, m \) and \( \chi \) vs \( \mu \), it is sufficient to imagine that the horizontal axis in Figs 2 and 3 corresponds to a variable \( \mu \). Then the lines 1 and 2 of Fig. 9 will mean the plots of \( \Delta_0 \) and \( m \) vs \( \mu \), whereas the lines 1 and 2 of Fig. 3 will correspond to the plots of \( n \) and \( \chi \) vs \( \mu \), respectively. In a similar way it is possible to extract from Figs 6 and 7 the information about behavior of \( M_0, \Delta_0, n, m \) and \( \chi \) vs \( \mu \) in the case of \( g_1 < 0 \), \( g_2 = 0.5|g_1| \) and \( \mu_B B = 0.7/|g_1| \).

Hence, as it follows from the above consideration (see Figs 4 and 5), at arbitrary values of \( \mu \geq 0 \) and sufficiently strong external in-plane magnetic field the superconducting (or Cooper pairing) phenomenon appears in the framework of the \((2+1)\)-dimensional GN-type model (1) (of course, if \( G_2 > 0 \)), i.e. an external in-plane magnetic field \( B \) promotes superconductivity for arbitrary relations between coupling constants \( g_{1,2} \) (or, equivalently, \( G_{1,2} \)). In particular, if \( G_{1,2} < G_c \), i.e. \( g_{1,2} > 0 \), then superconductivity is induced by infinitesimal values of \( \mu \) and/or \( \vec{B}_f \). Moreover, it is clear from our investigations that superconducting phase of the model is accompanied by a magnetization, or spin polarization, with positive valued susceptibility. It means that induced magnetic moment (which disappears at \( \vec{B}_f = 0 \)) of the system and external magnetic field \( \vec{B}_f \) have the same direction, i.e. the superconducting state is a paramagnetic one. It is not a diamagnetic, as in conventional superconductivity. Note, the magnetic superconductivity

\footnote{All the Figs. 2-7 are drawn in terms of dimensionless quantities which are obtained after multiplication of appropriate powers of \( |g_1| \) with corresponding dimensional quantities. For example, instead of \( \mu, \mu_B B, \Delta_0, g_2 \) we use their dimensionless analogies \(|g_1|\mu, \mu_B |g_1| B, |g_1|\Delta_0, g_2/|g_1| \). Instead of magnetization \( m \) the dimensionless quantity \( g_1^2 m/\mu_B \) is depicted there etc.}
the superconductivity phenomenon appears, i.e. the spontaneous breaking of the electromagnetic plane (in-plane) magnetic field $\vec{B}$ increasing function vs chiral symmetry breaking (if $G$ magnetization) in the superconducting state.

However, in our model, since $\nu G$ above the Fermi surface. So, the Fermi surface is unstable in favor of Bose–Einstein condensate of Cooper pairs, and phenomena (which includes both the paramagnetic and ferromagnetic superconductivity) has a long history of investigations in condensed matter physics (see, e.g., in [48]).

V. SUMMARY AND DISCUSSION

In this paper we considered the (2+1)-dimensional GN-type model (1) with chiral and superconducting interaction channels under the influence of an (in-plane) external magnetic field. It is shown that in-plane magnetic field $\vec{B}_||$ catalyzes a creation of the paramagnetic superconductivity in the system. It means that if at $\vec{B}_|| = 0$ the electromagnetic $U(1)$ symmetry group was not broken, then at $|\vec{B}_|| > B_c$, where a critical field $B_c$ can be even zero at some particular relations between coupling constants $G_{1,2}$, there is a spontaneous breaking of $U(1)$. In addition, if at $\vec{B}_|| = 0$ the $U(1)$ symmetry was broken spontaneously (due to a rather strong interaction in the fermion-fermion channel), then nonzero values of $\vec{B}_||$ enhance superconductivity, i.e. the superconducting order parameter $\Delta_0$ is an increasing function vs $|\vec{B}_||$. Moreover, in-plane magnetic field induces also a nonzero spin polarization (paramagnetic magnetization) in the superconducting state.

Various particular cases of the problem have been already considered earlier. Indeed, twenty years ago, it was found that at $G_2 = 0$ and $G_1 > 0$ an arbitrary weak perpendicular external magnetic field $\vec{B}_\perp$ can not only enhance the chiral symmetry breaking (if $G_1 > G_c$) but also induce spontaneous breaking of the chiral symmetry at $G_1 < G_c$ (see, e.g., [6, 20, 23]). In contrast, recently it was found [33] that an application of an external parallel to the system plane (in-plane) magnetic field $\vec{B}_||$ results in the restoration of chiral symmetry (recall, in this case $G_2 = 0$). For an explanation of such a different reaction of the chiral symmetry on perpendicular and parallel magnetic fields one should remember that $\vec{B}_\perp$ acts on the orbital angular momentum of electrons, whereas the in-plane field $\vec{B}_||$ couples only to their spin. In the last case, due to the Zeeman effect, increase of the electron chemical potential takes place which eventually leads to the restoration of the chiral symmetry [2, 8, 24].

On the contrary, the results of our present paper demonstrate that the response of the electromagnetic $U(1)$ symmetry of (2+1)-dimensional GN-type models to the applied external magnetic field is completely different. Indeed, it was shown, in the framework of the (2+1)-dimensional GN-type model with nonzero coupling $G_2$ of the superconducting channel and at $G_1 = 0$, that restoration of the $U(1)$ symmetry takes place at sufficiently strong external perpendicular field $\vec{B}_\perp$ [43]. Moreover, the main result of our paper is that if only an external in-plane magnetic field is included in the (2+1)-dimensional GN model (1) (at least with $G_2 > 0$), then at sufficiently strong values of $\vec{B}_||$ the superconductivity phenomenon appears, i.e. the spontaneous breaking of the electromagnetic $U(1)$ symmetry is originated. In particular, if an interaction in both channels of the GN model (1) is sufficiently weak, i.e. $G_{1,2} < G_c$ (or $g_{1,2} > 0$), then superconductivity is induced by infinitesimal values of $\vec{B}_||$.

To understand the last property, one should take into account that due to the Zeeman effect, i.e. due to the Zeeman $\nu$-terms in Lagrangian (1), an external in-plane magnetic field $\vec{B}_||$ induces (at $\mu = 0$) Fermi surface for electrons in any planar system. However, in our model, since $G_2 > 0$, there exists an attractive interaction between electrons just above the Fermi surface. So, the Fermi surface is unstable in favor of Bose–Einstein condensate of Cooper pairs, and
a new (superconducting) ground state is formed in the system.

Finally, we would like to recall some QFT examples in which similar effects, although in a quite different physical contexts, are also observed. First of all, it is important to note that an ability of strong external magnetic field to induce electromagnetic superconductivity of vacuum was recently established in the framework of quantum chromodynamics (see, e.g., the review [49]). Next, it is argued in some papers (see, e.g., [50]) that ferromagnetism and superconductivity might coexist in dense quark matter, thus explaining super strong magnetic fields of magnetars.\footnote{In ferromagnetic superconductors the spin polarization, or magnetisation, is originated spontaneously, i.e. in this case there is a spontaneous breaking of the rotational invariance. In contrast, in our model (1) the magnetization of the superconducting state exists only in the presence of an external magnetic field $\vec{B}_0$, which destroys the rotational symmetry of the model explicitly.}
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