LINEAR THEORY FOR A MIXED OPERATOR
WITH NEUMANN CONDITIONS

SERENA DIPIERRO, EDOARDO PROIETTI LIPPI, AND ENRICO VALDINOCI

ABSTRACT. We consider here a new type of mixed local and nonlocal equation under suitable Neumann conditions. We discuss the spectral properties associated to a weighted eigenvalue problem and present a global bound for subsolutions.

The Neumann condition that we take into account comprises, as a particular case, the one that has been recently introduced in [S. Dipierro, X. Ros-Oton, E. Valdinoci, Rev. Mat. Iberoam. (2017)].

Also, the results that we present here find a natural application to a logistic equation motivated by biological problems that has been recently considered in [S. Dipierro, E. Proietti Lippi, E. Valdinoci, preprint (2020)].

1. Introduction

The goal of this article is to discuss the spectral properties and the $L^\infty$-bounds associated to a mixed local and nonlocal problem, also in relation to some concrete motivations arising from population dynamics and mathematical biology. The methodology that we exploit here relies on functional analysis and methods from (classical and nonlocal) partial differential equations. Given the mixed character of the operator taken into account and the new set of external conditions, the standard mathematical framework to deal with partial and integro-differential equations needs to be conveniently modified to suit this new scenario.

More specifically, in [DPLV], we have introduced a new set of nonlocal Neumann conditions, extending those previously set forth in [DROV17], with the aim of dealing with a mathematical problem motivated by ethology and biology. More specifically, in [DPLV] a biological population was taken into consideration within an environment which could be partially hostile. The population competes for the resources via a logistic equation and diffuses by a possible combination of classical and nonlocal dispersal processes (a detailed derivation of the diffusion model is also presented in the appendix of [DPLV]).

The population can be also provided by an additional birth growth due to pollination, and the main question targeted in [DPLV] is whether or not it is possible to rearrange the given environmental resources (within given upper and lower constraints) to allow for the survival of the species.

The nonlinear mathematical analysis developed in [DPLV] also relies on some auxiliary results from the linear theory, such as spectral decompositions and uniform bounds for subsolutions.

2010 Mathematics Subject Classification. 35Q92, 35R11, 60G22, 92B05.

Key words and phrases. Long-range interactions, zero-flux condition, spectral theory, boundedness of subsolutions.

Serena Dipierro: Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Hwy, Crawley WA 6009, Australia. serena.dipierro@uwa.edu.au
Edoardo Proietti Lippi: Department of Mathematics and Computer Science, University of Florence, Viale Morgagni 67/A, 50134 Firenze, Italy. edoardo.proiettilippi@unifi.it
Enrico Valdinoci: Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Hwy, Crawley WA 6009, Australia. enrico.valdinoci@uwa.edu.au

The authors are members of INdAM. The first and third authors are members of AustMS and are supported by the Australian Research Council Discovery Project DP170104880 NEW “Nonlocal Equations at Work”. The first author is supported by the Australian Research Council DECRA DE180100957 “PDEs, free boundaries and applications”. Part of this work was carried out during a very pleasant and fruitful visit of the second author to the University of Western Australia, which we thank for the warm hospitality.
which have their independent interest. We collect here these results, providing full proofs in detail.

The setting in which we work is the following. We let \( s \in (0,1) \) and \( \alpha, \beta \in [0, +\infty) \) with \( \alpha + \beta > 0 \), and we consider the mixed operator

\[ (1) \quad -\alpha \Delta + \beta (-\Delta)^s. \]

As customary, the operator \((-\Delta)^s\) is the fractional Laplacian

\[ (-\Delta)^s u(x) := \frac{1}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x + \zeta) - u(x - \zeta)}{|\zeta|^{n+2s}} \, d\zeta, \]

where other normalization constants have been removed to ease the notation (in any case, additional normalizing constants do not affect our arguments, and they can also be comprised into the parameter \( \beta \) in (1) if one wishes to do so).

As a matter of fact, the theory that we develop here, as well as in [DPLV], works in greater generality (e.g., one can replace the fractional Laplacian with a more general integro-differential operator with only minor modifications in the main proofs), but we rather limit ourselves to the paradigmatic case of the fractional Laplacian for the sake of simplicity in the exposition. Moreover, the results obtained are new even in the case of “purely nonlocal diffusion”, i.e. when \( \alpha = 0 \) in (1).

In terms of theory and applications, we recall that operators with mixed classical and fractional orders have been studied under different points of views, see for instance [BK05, BK05b, JK05, JK06, BI08, dLLV09, BJK10, BCCI12, CKSV12, CKSV10, Cio12, CV13, BCCI14, CS16, DVV19, AV19, dTEJ17, DP18, AdTEJ19, BDVV20, AC20, CDV] and the references therein. Besides their clear mathematical interest, these operators find natural applications in biology, in view of the long-jump dispersal strategies followed by several species, as confirmed by a number of experimental data, see e.g. [VAB96], and theoretically studied under several perspectives, see e.g. [DHMP98, CCL07, CHL08, ABVV10, CDM12, KLS12, CLLR12, CDM13, MPV13, ACR13, Cov15, BCV16, CDV17, SV17, MV17, BCL17, PV18] (other concrete applications arise in plasma physics, see [BdCN13] and the references therein).

As usual, the mathematical framework in (1) is endowed by a spatial domain on which the corresponding equation takes place. For this, we take a bounded open set \( \Omega \subset \mathbb{R}^n \) of class \( C^1 \). When \( \beta = 0 \), we take the additional hypothesis that

\[ \Omega \text{ is connected.} \]

From the biological point of view, \( \Omega \) represents the natural environment inhabited by a given biological population, whose density is described by a function \( u : \mathbb{R}^n \to \mathbb{R} \) (as customary in nonlocal problems, one has to prescribe functions in all of the space to make sense of the fractional diffusive operators).

We prescribe external conditions to \( u \) in order to make \( \Omega \) an ecological niche. To this end, see [DPLV], we set a variational formulation related to the operator in (1) which endows the equation in the set \( \Omega \) with a suitable Neumann condition. The functional space that we consider is

\[ (3) \quad X_{\alpha,\beta} = X_{\alpha,\beta}(\Omega) := \begin{cases} H^1(\Omega) & \text{if } \beta = 0, \\ H^s_{\Omega} & \text{if } \alpha = 0, \\ H^1(\Omega) \cap H^s_{\Omega} & \text{if } \alpha \beta \neq 0, \end{cases} \]

where

\[ H^s_{\Omega} := \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } u \in L^2(\Omega) \text{ and } \int_{\Omega} \int \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy < +\infty \right\}, \]
and $Q$ is the cross-shaped set on $\Omega$ given by

$$Q := (\Omega \times \Omega) \cup (\Omega \times (\mathbb{R}^n \setminus \Omega)) \cup ((\mathbb{R}^n \setminus \Omega) \times \Omega).$$

We observe that $X_{\alpha,\beta}$ is a Hilbert space with respect to the scalar product

$$(u, v)_{X_{\alpha,\beta}} := \int_{\Omega} u(x)v(x) \, dx + \alpha \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx + \frac{\beta}{2} \int_{Q} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy,$$

for every $u, v \in X_{\alpha,\beta}$.

We also define the seminorm

$$(u)_{X_{\alpha,\beta}}^2 := \frac{\alpha}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{\beta}{4} \int_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.$$ 

Given $f \in L^2(\Omega)$, we say that $u \in X_{\alpha,\beta}$ is a solution of

$$-\alpha \Delta u + \beta (-\Delta)^s u = f \quad \text{in } \Omega$$

with $(\alpha, \beta)$-Neumann condition if

$$\alpha \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx + \frac{\beta}{2} \int_{Q} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy = \int_{\Omega} f(x) v(x) \, dx,$$

for every $v \in X_{\alpha,\beta}$.

We remark that, formally, the external condition in (7) can be detected by taking $v$ with $v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$ (which produces a normal derivative prescription along $\partial \Omega$) and then by taking $v = 0$ in $\overline{\Omega}$ (which produces a nonlocal prescription in $\mathbb{R}^n \setminus \Omega$): that is, formally, the external condition in (7) can be written in the form

$$\begin{cases} \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\nu$ is the exterior normal to $\Omega$, and we use the notation

$$\mathcal{N}_s u(x) := \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy$$

for every $x \in \mathbb{R}^n \setminus \overline{\Omega},$

and the first condition in (8) being dropped when $\alpha = 0$, the second condition in (8) being dropped when $\beta = 0$.

We recall that the nonlocal Neumann prescription in (9) is precisely the one introduced in [DROV17] in light of probabilistic consideration (i.e., a particle following a $\frac{s}{2}$-stable process is sent back to the original domain by following the same process). Also, as shown in [DROV17], the setting in (9) provides a coherent functional analysis setting.

In the situation treated in this paper, this setting is superimposed to a classical framework when $\alpha \neq 0$: in particular, we remark that, when $\alpha \neq 0$ and $\beta \neq 0$, both the prescriptions in (8) are in force, but they do not cause any overdetermined conditions, and indeed, as shown in [DPLV], the notion of solutions in this case is well-posed.

Moreover, we stress that the setting in (7) provides a “zero-flux” condition, in the sense that if (10) has a solution, then necessarily

$$\int_{\Omega} f(x) \, dx = 0,$$

as it can be seen by taking $v := 1$ in (7).

We now describe in detail the results stated and proved in this paper.
1.1. Eigenvalue and eigenfunctions for the \((\alpha, \beta)\)-Neumann condition. The first set of results that we discuss here is related to a generalized eigenvalue problem associated to equation (6) with \((\alpha, \beta)\)-Neumann condition.

Namely, we let \(m : \Omega \to \mathbb{R}\) and we consider the weighted eigenvalue equation

\[
\begin{cases}
-\alpha \Delta u + \beta (-\Delta)^s u = \lambda m u \\
\text{in } \Omega,
\end{cases}
\]

with \((\alpha, \beta)\)-Neumann condition.

According to (7) the notion of solution in (11) is in the weak sense in the space \(X_{\alpha,\beta}\): namely we say that \(u \in X_{\alpha,\beta}\) is a solution of (11) if

\[
\alpha \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx + \frac{\beta}{2} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy = \lambda \int_{\Omega} m(x) u(x) v(x) \, dx,
\]

for every \(v \in X_{\alpha,\beta}\).

To deal with the integrability condition of the weight \(m\), it is convenient to consider the following “critical” exponent:

\[
q := \begin{cases}
2^* - 2 & \text{if } \beta = 0 \text{ and } n > 2, \\
2^* - 2 & \text{if } \beta \neq 0 \text{ and } n > 2s, \\
1 & \text{if } \beta = 0 \text{ and } n \leq 2, \text{ or if } \beta \neq 0 \text{ and } n \leq 2s,
\end{cases}
\]

\[
= \begin{cases}
\frac{n}{2} & \text{if } \beta = 0 \text{ and } n > 2, \\
\frac{n}{2s} & \text{if } \beta \neq 0 \text{ and } n > 2s, \\
1 & \text{if } \beta = 0 \text{ and } n \leq 2, \text{ or if } \beta \neq 0 \text{ and } n \leq 2s.
\end{cases}
\]

As customary, the exponent \(2^*\) denotes the fractional Sobolev critical exponent for \(n > 2s\) and it is equal to \(\frac{2n}{n-2s}\). Similarly, the exponent \(2^s\) denotes the classical Sobolev critical exponent for \(n > 2\) and it is equal to \(\frac{2n}{n-2}\).

Furthermore, we suppose that

\[
m \in L^q(\Omega), \quad \text{for some } q \in \left( q, +\infty \right],
\]

where \(q\) is given in (13).

In this setting, problem (11) admits a spectral decomposition of classical flavor, according to the following result:

**Proposition 1.1.** Suppose that \(m^+, m^- \neq 0\) and\(^1\) that

\[
\int_{\Omega} m(x) \, dx \neq 0.
\]

Then, problem (11) admits two unbounded sequences of eigenvalues:

\[
\cdots \leq \lambda_{-2} \leq \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots.
\]

In particular, if

\[
\int_{\Omega} m(x) \, dx < 0,
\]

then

\[
\lambda_1 = \min_{u \in X_{\alpha,\beta}} \left\{ \|u\|^2_{X_{\alpha,\beta}} \text{ s.t. } \int_{\Omega} m(x) u^2(x) \, dx = 1 \right\}
\]

\(^1\)As customary, we use the standard notation

\[
m^+(x) := \max\{0, m(x)\} \quad \text{and} \quad m^-(x) := \max\{0, -m(x)\}.
\]
where we use the notation in (5). If instead
\[ \int_{\Omega} m(x) \, dx > 0, \]
then
\[ \lambda_{-1} = - \min_{u \in X_{\alpha,\beta}} \left\{ \|u\|_{X_{\alpha,\beta}}^2 \text{ s.t. } \int_{\Omega} m(x)u^2(x) \, dx = -1 \right\}. \]

The first positive eigenvalue \( \lambda_1 \), as given by Proposition 1.1, has the following structural properties:

**Proposition 1.2.** Suppose that \( m^+ \not\equiv 0 \) and
\[ \int_{\Omega} m(x) \, dx < 0. \]
Then, the first positive eigenvalue \( \lambda_1 \) of (11) is simple, and the first eigenfunction \( e \) can be taken such that \( e \geq 0 \).

A similar statement holds if \( m^- \not\equiv 0 \) and
\[ \int_{\Omega} m(x) \, dx > 0. \]

To deal with the eigenvalue problem in (11), it is convenient to recall the notation in (3) and to introduce the space
\[ V_m := \left\{ u \in X_{\alpha,\beta} \text{ s.t. } \int_{\Omega} m(x)u(x) \, dx = 0 \right\}. \]

To ease the notation, we will simply write \( V \) instead of \( V_m \) in what follows. We observe that, in view of (10),
\[ (18) \text{ all the eigenfunctions of problem (11) belong to } V. \]
As we will see in Corollary 1.4, a global bound holds true for these eigenfunctions. To obtain this bound, we develop a general theory, of independent interest, to bound globally from below the weak subsolutions that fulfill the \((\alpha, \beta)\)-Neumann conditions, as we now discuss in detail.

### 1.2. Global uniform bounds for subsolutions under \((\alpha, \beta)\)-Neumann condition.

We give here an \( L^\infty \)-result for solutions, and more general, subsolutions of equation (6) under \((\alpha, \beta)\)-Neumann condition. To apply this bound to the eigenfunctions of problem (11), it is also convenient to allow an additional linear term in the equation that we take into account. The result that we have is the following one:

**Theorem 1.3.** Let \( V \) be as in (17) and \( q \) be as in (13). Let \( q \in (q, +\infty) \) and \( c, f \in L^q(\Omega) \).

Let \( u \in V \) satisfy
\[ (19) \begin{align*}
\alpha \int_{\Omega} \nabla u \cdot \nabla v \, dx + \beta \frac{1}{2} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx dy & \leq \int_{\Omega} (c(x)u(x) + f(x)) \, v(x) \, dx
\end{align*} \]
for each \( v \in X_{\alpha,\beta} \) such that \( v \geq 0 \) in \( \Omega \).

Then, there exists \( C > 0 \), depending on \( n, \alpha, \beta, q, \Omega, \|c\|_{L^q(\Omega)} \) and \( m \) such that
\[ (20) \sup_{\Omega} u^+ \leq C \left( \|u^+\|_{L^2(\Omega)} + \|f\|_{L^q(\Omega)} \right). \]

In a forthcoming paper, we plan to use Theorem 1.3 as the cornerstone for a regularity theory for mixed equations under \((\alpha, \beta)\)-Neumann conditions.

As a consequence of (18) and Theorem 1.3 (applied with \( f := 0 \) and \( c := \lambda m \)), we easily obtain the following global bound for eigenfunctions:
Corollary 1.4. All the eigenfunctions of problem (11) belong to $L^\infty(\Omega)$.

In the rest of the paper, we provide full detailed proofs for Propositions 1.1 and 1.2 (in Section 2) and for Theorem 1.3 (in Section 3).

2. Eigenvalues and eigenfunctions and proof of Propositions 1.1 and 1.2

The proofs of Propositions 1.1 and 1.2 rely on classical functional analysis, revisited in a mixed local-nonlocal framework. We start these arguments by pointing out that a Poincaré-type inequality holds in the space $V$ introduced in (17):

Lemma 2.1. Let $m$ be such that
\begin{equation}
\int_\Omega m(x) \, dx \neq 0.
\end{equation}

Then, recalling the notation in (5), we have that
\begin{equation}
\int_\Omega u^2(x) \, dx \leq C |u|_{X_{\alpha,\beta}}^2,
\end{equation}

for every $u \in V$, where $C > 0$ depends only on $n$, $\Omega$, $s$ and $m$.

Proof. We argue by contradiction and we suppose that there exists a sequence of functions $u_k \in V$ such that
\begin{equation}
\int_\Omega u_k^2(x) \, dx = 1
\end{equation}

and
\begin{equation}
[u_k]_{X_{\alpha,\beta}}^2 < \frac{1}{k}.
\end{equation}

In particular, the sequence $(u_k)_k$ is bounded in $X_{\alpha,\beta}$ uniformly in $k$. As a consequence, from the compact embedding of $X_{\alpha,\beta}$ in $L^2(\Omega)$ (see e.g. Corollary 7.2 in [DNPV12] if $\alpha = 0$), we have that, up to a subsequence, $u_k$ converges to some function $u \in L^2(\Omega)$ as $k \to +\infty$. Moreover, $u_k$ converges to $u$ a.e. in $\Omega$ as $k \to +\infty$, and $|u_k| \leq h$ for some $h \in L^2(\Omega)$ for every $k \in \mathbb{N}$ (see e.g. Theorem IV.9 in [Bre83]).

As a result, since $u_k \in V$, we can apply the Dominated Convergence Theorem to conclude that
\begin{equation}
\int_\Omega m(x)u(x) \, dx = 0.
\end{equation}

In addition, we deduce from (23) that
\begin{equation}
\int_\Omega u^2(x) \, dx = 1.
\end{equation}

On the other hand, by the Fatou Lemma, the lower semicontinuity of the $L^2$-norm and (24) we have that
\begin{equation}
\frac{\alpha}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{\beta}{4} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy
\end{equation}

\begin{equation}
\leq \liminf_{k \to +\infty} \left( \frac{\alpha}{2} \int_\Omega |\nabla u_k|^2 \, dx + \frac{\beta}{2} \int_\Omega \int_\Omega \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right) \leq \lim_{k \to +\infty} \frac{1}{k} = 0.
\end{equation}

Now, if $\beta = 0$, this says that
\begin{equation}
\int_\Omega |\nabla u|^2 \, dx = 0,
\end{equation}

which implies that $u$ is constant in $\Omega$, thanks to (2). If instead $\beta \neq 0$, we have from (27) that
\begin{equation}
\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0,
\end{equation}

\begin{equation}
\alpha \int_\Omega |\nabla u|^2 \, dx + \frac{\beta}{4} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \leq \liminf_{k \to +\infty} \frac{1}{k} = 0.
\end{equation}

This implies that $u$ is constant in $\Omega$, thanks to (2).
which gives that \( u \) is constant in \( \Omega \). Hence in both case, we have that \( u \) is constant in \( \Omega \).

Moreover, we observe that \( u \) cannot vanish identically in \( \Omega \), in light of (26). Using these observations into (25) we conclude that
\[
\int_{\Omega} m(x) \, dx = 0,
\]
which is in contradiction with (21). This completes the proof of formula (22).

We notice that, thanks to (22), the seminorm in (5) is actually a norm on the space \( V \) and it is equivalent to the norm on \( X_{\alpha,\beta} \) given by (4). Moreover, the scalar product defined as
\[
(28) \quad \langle u, v \rangle_{X_{\alpha,\beta}} := \alpha \int_{\Omega} \nabla u \cdot \nabla v \, dx + \frac{\beta}{2} \iint_{Q} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy
\]
is equivalent to the one in \( X_{\alpha,\beta} \) given by (4). In this setting, we also denote
\[
\| u \|_V := \sqrt{\langle u, u \rangle_{X_{\alpha,\beta}}}.
\]

To complete the functional setting for the eigenvalue problem in (11), we also remark that \( V \) is closed with respect to the weak convergence:

**Lemma 2.2.** The space \( V \) introduced in (17) is closed with respect to the weak convergence in \( V \).

**Proof.** We take a sequence of functions \( u_j \in V \) weakly converging to some \( u \), and we claim that \( u \in V \). Indeed, we have that \( u_j \) weakly converges to \( u \) in \( X_{\alpha,\beta} \), and \( u \in X_{\alpha,\beta} \). Furthermore, by the compact embeddings (see e.g. Corollary 7.2 in [DNPV12] if \( \alpha = 0 \), \( u_j \to u \) in \( L^p(\Omega) \) for any \( p \in [1, 2^*_s) \) if \( \alpha = 0 \) and for any \( p \in [1, 2^*_s) \) if \( \alpha \neq 0 \). Moreover, \( u_j \) converges to \( u \) a.e. in \( \Omega \), and \( |u_j| \leq h \) for some \( h \in L^p(\Omega) \) (see e.g. Theorem IV.9 in [Bre83]). As a result, since \( u_j \in V \), recalling (14), we can apply the Dominated Convergence Theorem to conclude that
\[
\int_{\Omega} m(x) u(x) \, dx = 0,
\]
which proves that \( u \in V \), thus completing the proof of Lemma 2.2.

With this preliminary work, we can give the proofs of Propositions 1.1 and 1.2 by relying on functional analysis methods:

**Proof of Proposition 1.1.** We notice that
\[
(29) \quad \text{the simple eigenfunction } \lambda_0 = 0 \text{ has only constant functions as eigenfunctions.}
\]
Indeed, if \( u \) is an eigenfunction associated to \( \lambda_0 = 0 \), then, by (12),
\[
(30) \quad \alpha \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx + \frac{\beta}{2} \iint_{Q} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy = 0,
\]
for all functions \( v \in X_{\alpha,\beta} \). In particular, taking \( u \) as test function in (30), we obtain that
\[
(31) \quad \alpha \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{\beta}{2} \iint_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0.
\]
Now, if \( \beta = 0 \), formula (31) implies that
\[
\int_{\Omega} |\nabla u(x)|^2 \, dx = 0.
\]
This, together with (29), gives that \( u \) is constant in \( \Omega \), thus proving (29) in this case.

If instead \( \beta \neq 0 \), we deduce from (31) that
\[
\iint_{Q} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = 0,
\]
which implies (29).
Now, to obtain the other eigenvalues, we restrict to the space $V$ introduced in (17). We point out that the assumption in (15) guarantees that the Poincarè inequality in (22) holds true on the space $V$.

Also, we define the linear operator $T : V \to V$ by

$$\langle Tv, w \rangle_{X_{\alpha,\beta}} = \int_{\Omega} m(x)v(x)w(x)\,dx,$$

for every $v, w \in V$.

It is easy to see that $T$ is symmetric. Furthermore, we claim that

$$T \text{ is compact.}$$

To prove this, we let $(u_j)_j$ be a bounded sequence in $V$. Then, $(u_j)_j$ is a bounded sequence in $X_{\alpha,\beta}$, and therefore there exists $u \in X_{\alpha,\beta}$ such that $u_j$ weakly converges to $u$ in $X_{\alpha,\beta}$ as $j \to +\infty$. Moreover, from Lemma 2.2, we have that $u \in V$.

Now, by the compact embeddings,

$$u_j \to u \text{ in } L^p(\Omega) \text{ for any } p \in [1, 2^*) \text{ if } \alpha = 0 \text{ and for any } p \in [1, 2^*) \text{ if } \alpha \neq 0.$$

Using (32) with $v := u_j - u$ and $w := Tu_j - Tu$, we deduce that

$$\|Tu_j - Tu\|_V^2 = \langle (u_j - u), Tu_j - Tu \rangle_{X_{\alpha,\beta}} = \int_{\Omega} m(u_j - u)(Tu_j - Tu)\,dx.$$

Now we apply Hölder’s inequality with exponents $q$, as given in (14), $p$, as given by (34), and either $2^*$ if $\alpha = 0$ or $2^*$ if $\alpha \neq 0$. In this way, using also the continuous embedding of $V$ either in $L^{2^*}(\Omega)$ if $\alpha = 0$ or $L^{2^*}(\Omega)$ if $\alpha \neq 0$, we obtain from (35) that

$$\|Tu_j - Tu\|_V^2 \leq C\|m\|_{L^q(\Omega)}\|u_j - u\|_{L^p(\Omega)}\|Tu_j - Tu\|_V,$$

for some positive constant $C$ independent of $j$. This implies that

$$\|Tu_j - Tu\|_V \leq C\|m\|_{L^q(\Omega)}\|u_j - u\|_{L^p(\Omega)}.$$

Accordingly, recalling (34), we obtain that $Tu_j \to Tu$ in $V$ as $j \to +\infty$. This completes the proof of (33).

Now we observe that, in light of (12), and recalling (28) and (32), we can write the weak formulation of problem (11) as

$$\langle u, v \rangle_{X_{\alpha,\beta}} = \lambda \langle Tu, v \rangle_{X_{\alpha,\beta}} \quad \text{for all } v \in X_{\alpha,\beta}.$$

Therefore, we can apply standard results in spectral theory of self-adjoint and compact operators to obtain the existence and the variational characterization of eigenvalues (see e.g. [IF82, Proposition 1.10]; see also [BL80] and the references therein for related classical results).

**Proof of Proposition 1.2.** We first observe that if $\beta \neq 0$ and $w$ is an eigenfunction according to (11), then

$$w \equiv 0 \text{ in } \Omega \text{ entails that } w \equiv 0 \text{ in the whole of } \mathbb{R}^n.$$

To check this, suppose that $w \equiv 0$ in $\Omega$ and write (11) explicitly as in (12), namely

$$\alpha \int_{\Omega} \nabla w(x) \cdot \nabla v(x)\,dx + \frac{\beta}{2} \int_{\Omega} \frac{(w(x) - w(y))(v(x) - v(y))}{|x-y|^{n+2\alpha}}\,dx\,dy$$

$$= \lambda \int_{\Omega} m(x) w(x) v(x)\,dx$$

for all functions $v \in X_{\alpha,\beta}$. In particular, choosing $v := w$ in (38),

$$0 = \frac{\beta}{2} \int_{\Omega} \frac{(w(x) - w(y))^2}{|x-y|^{n+2\alpha}}\,dx\,dy = \beta \int_{\Omega \times (\mathbb{R}^n \setminus \Omega)} \frac{w^2(y)}{|x-y|^{n+2\alpha}}\,dx\,dy.$$

Whence, if $\beta \neq 0$, it follows that $w(y) = 0$ for each $y \in \Omega$, thus establishing (37).
Now, we prove that
\begin{equation}
\text{all the eigenfunctions corresponding to } \lambda_1 \text{ do not change sign.}
\end{equation}
For this, we let \( u \) be an eigenfunction corresponding to the first positive eigenvalue \( \lambda_1 \). In particular, recalling \((13)\), we have that \( u \in X_{\alpha,\beta} \) and
\begin{equation}
\int_{\Omega} m(x)u^2(x) \, dx = 1.
\end{equation}
If \( u \) is either nonnegative or nonpositive, then \((39)\) is established. Hence, we are left with the case in which \( u \) changes sign in \( \Omega \). In this case, we have that both \( u^+ \neq 0 \) and \( u^- \neq 0 \), and we claim that
\begin{equation}
\text{both } u^+ \text{ and } u^- \text{ are eigenfunctions corresponding to } \lambda_1.
\end{equation}
To this end, we notice that
\begin{equation}
\int_{\Omega} u^2(x) \, dx = \int_{\Omega} (u^+(x))^2 \, dx + \int_{\Omega} (u^-(x))^2 \, dx.
\end{equation}
Moreover, recalling \((5)\), by inspection one sees that
\begin{equation}
[u]_{X_{\alpha,\beta}}^2
= \alpha \int_{\Omega} |\nabla u|^2 \, dx + \beta \int_{\Omega} |u(x) - u(y)|^2 \, dx \, dy
= \alpha \int_{\Omega} (|\nabla u^+|^2 + |\nabla u^-|^2) \, dx + \beta \int_{\Omega} \frac{|u^+|^2 - |u^-|^2}{|x - y|^{n+2s}} \, dx \, dy
\end{equation}
\begin{equation}
\geq [u^+]_{X_{\alpha,\beta}}^2 + [u^-]_{X_{\alpha,\beta}}^2.
\end{equation}
This and \((12)\) imply that \( u^+, u^- \in X_{\alpha,\beta} \).

Also, in light of \((10)\), we have that
\begin{equation}
1 = \int_{\Omega} m(x)u^2(x) \, dx = \int_{\Omega} m(x)(u^+(x))^2 \, dx + \int_{\Omega} m(x)(u^-(x))^2 \, dx.
\end{equation}
Hence, using this and \((13)\), and recalling the characterization of \( \lambda_1 \) given in \((16)\),
\begin{equation}
\frac{1}{\lambda_1} = \frac{1}{[u]_{X_{\alpha,\beta}}^2} = \frac{\int_{\Omega} m(x)u^2(x) \, dx}{[u]_{X_{\alpha,\beta}}^2} \leq \frac{\int_{\Omega} m(x)(u^+(x))^2 \, dx + \int_{\Omega} m(x)(u^-(x))^2 \, dx}{[u^+]_{X_{\alpha,\beta}}^2 + [u^-]_{X_{\alpha,\beta}}^2}.
\end{equation}
Now we claim that, for any \( a_1, a_2, b_1, b_2 > 0, \) either
\begin{equation}
\frac{a_1 + a_2}{b_1 + b_2} = \frac{a_1}{b_1} = \frac{a_2}{b_2},
\end{equation}
or
\begin{equation}
\frac{a_1 + a_2}{b_1 + b_2} < \max \left\{ \frac{a_1}{b_1}, \frac{a_2}{b_2} \right\}.
\end{equation}
Indeed, if \( \frac{a_1}{b_1} = \frac{a_2}{b_2} \), then
\begin{equation}
\frac{a_1 + a_2}{b_1 + b_2} = \frac{a_2}{b_2} : \frac{a_2}{b_2} + 1 = \frac{a_2}{b_2} : \frac{a_2}{b_2} + 1 = \frac{a_2}{b_2},
\end{equation}
that is \((45)\). If instead we suppose that \( \frac{a_1}{b_1} > \frac{a_2}{b_2} \) (being the case in which \( \frac{a_1}{b_1} < \frac{a_2}{b_2} \) similar), then
\begin{equation}
\frac{a_1 + a_2}{b_1 + b_2} = \frac{b_1(a_1 + a_2)}{b_1(b_1 + b_2)} < \frac{a_1b_1 + a_1b_2}{b_1(b_1 + b_2)} = \frac{a_1(b_1 + b_2)}{b_1(b_1 + b_2)} = \frac{a_1}{b_1},
\end{equation}
which proves \((16)\).
Now, if we suppose that
\[
\int_{\Omega} m(x) (u^+(x))^2 \, dx > \int_{\Omega} m(x) (u^-(x))^2 \, dx
\]
then we deduce from (44) and (46), applied here with
\[
a_1 := \int_{\Omega} m(x) (u^+(x))^2 \, dx, \quad a_2 := \int_{\Omega} m(x) (u^-(x))^2 \, dx,
\]
\[
b_1 := [u^+]_{X_{\alpha,\beta}}^2, \quad \text{and} \quad b_2 := [u^-]_{X_{\alpha,\beta}}^2,
\]
that
\[
\frac{1}{\lambda_1} < \int_{\Omega} m(x) (u^+(x))^2 \, dx,
\]
which contradicts the minimality of \(\lambda_1\). Similarly, if
\[
\int_{\Omega} m(x) (u^+(x))^2 \, dx < \int_{\Omega} m(x) (u^-(x))^2 \, dx
\]
then
\[
\frac{1}{\lambda_1} < \int_{\Omega} m(x) (u^-(x))^2 \, dx,
\]
which is again a contradiction with the minimality of \(\lambda_1\).

As a consequence, we have that
\[
\int_{\Omega} m(x) (u^+(x))^2 \, dx = \int_{\Omega} m(x) (u^-(x))^2 \, dx
\]
In this case, we can apply (45) and we obtain from (47) that
\[
\frac{1}{\lambda_1} \leq \int_{\Omega} m(x) (u^+(x))^2 \, dx = \int_{\Omega} m(x) (u^+(x))^2 \, dx
\]
that is
\[
\lambda_1 \geq \frac{[u^+]_{X_{\alpha,\beta}}^2}{\int_{\Omega} m(x) (u^+(x))^2 \, dx} = \frac{[u^-]_{X_{\alpha,\beta}}^2}{\int_{\Omega} m(x) (u^+(x))^2 \, dx}
\]
Now, if the inequality in (47) is strict, we have a contradiction with the minimality of \(\lambda_1\). Accordingly,
\[
\lambda_1 = \frac{[u^+]_{X_{\alpha,\beta}}^2}{\int_{\Omega} m(x) (u^+(x))^2 \, dx} = \frac{[u^-]_{X_{\alpha,\beta}}^2}{\int_{\Omega} m(x) (u^+(x))^2 \, dx}
\]
This implies that \(u^+\) and \(u^-\) are both eigenfunctions corresponding to \(\lambda_1\) (unless they are trivial) thus establishing (11).

Our next claim is to prove that
\[
\text{either } u \equiv u^+ \text{ or } u \equiv u^-.
\]
We observe that, if \(\beta = 0\), then (48) follows from the standard maximum principle for the Laplace operator (see e.g. [Eva10]).
If instead $\beta \neq 0$, we use (11) and (14) to see that
\[
\frac{1}{\lambda_1} \leq \frac{\int_{\Omega} m(x)(u^+(x))^2 \, dx + \int_{\Omega} m(x)(u^-(x))^2 \, dx}{|u^+|_{X_\alpha,\beta}^2 + |u^-|_{X_\alpha,\beta}^2} = \frac{1}{\lambda_1}.
\]
In particular, equality holds in the latter formula, and accordingly, recalling (13), we have that
\[
0 = -\int_{\Omega} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x - y|^{n+2s}} \, dx \, dy = \int_{\Omega} \frac{2u^+(x)u^-(y)}{|x - y|^{n+2s}} \, dx \, dy.
\]
This gives that
\[
u^+(x)u^-(y) = 0 \quad \text{for all } (x, y) \in \Omega.
\]
We can also suppose that $u^+ \neq 0$ (in $\mathbb{R}^n$ if $\beta \neq 0$ and in $\Omega$ if $\beta = 0$), otherwise $u \equiv u^-$ and we are done. This and (37) give that $u^+ \neq 0$ in $\Omega$. Hence, we can take $\bar{x} \in \Omega$ such that $u^+(\bar{x}) \neq 0$. From this and (49), we obtain that
\[
u^+(\bar{x})u^-(y) = 0 \quad \text{for all } y \in \mathbb{R}^n.
\]
As a consequence, we find that $u^- \equiv 0$ in $\mathbb{R}^n$, which establishes (48).

In turn, the claim in (48) implies the one in (39), as desired.

We now prove that $\lambda_1$ is simple. First we show that
\[
\text{the geometric multiplicity of } \lambda_1 \text{ is } 1.
\]
For this, let $u_1$ and $u_2$ be eigenfunctions corresponding to $\lambda_1$. From (39) we know that $u_2$ does not change sign, hence (up to exchanging $u_2$ with $-u_2$), we can suppose that $u_2 \geq 0$ (in $\mathbb{R}^n$, if $\beta \neq 0$, and in $\Omega$, if $\beta = 0$).

From this and (37), it follows that
\[
\int_{\Omega} u_2(x) \, dx > 0.
\]
As a result, we can define
\[
a := \frac{\int_{\Omega} u_1(x) \, dx}{\int_{\Omega} u_2(x) \, dx},
\]
and we find that
\[
\int_{\Omega} \left( u_1(x) - au_2(x) \right) \, dx = 0.
\]
In addition, from (39), we know that the eigenfunction $u_1 - au_2$ does not change sign, and therefore (51) entails that $u_1 - au_2 \equiv 0$ in $\Omega$. This and (37) show that $u_1 - au_2 \equiv 0$ also in $\mathbb{R}^n$ when $\beta \neq 0$, and this proves that $u_1$ and $u_2$ are linearly dependent, giving (50), as desired.

Finally, we prove that
\[
\text{the algebraic multiplicity of } \lambda_1 \text{ is } 1.
\]
To this end, we recall the notation in (17) and (32), and we claim that
\[
Ker((I - \lambda_1 T)^2) = Ker(I - \lambda_1 T),
\]
where $I$ is the identity in $V$.

To prove (53), let $u \in Ker((I - \lambda_1 T)^2)$. Then, setting $U := u - \lambda_1 Tu$, we have that $U - \lambda_1 TU = 0$, and accordingly, by (36), $U$ is an eigenfunction corresponding to $\lambda_1$.

From this fact and (50), we conclude that $U = te_1$ for some $t \in \mathbb{R}$, where $e_1$ is a given eigenfunction corresponding to $\lambda_1$.

As a result,
\[
t\langle e_1, e_1 \rangle_{X_\alpha,\beta} = \langle U, e_1 \rangle_{X_\alpha,\beta} = \langle u - \lambda_1 Tu, e_1 \rangle_{X_\alpha,\beta} = \langle u, e_1 - \lambda_1 Te_1 \rangle_{X_\alpha,\beta} = \langle u, 0 \rangle_{X_\alpha,\beta} = 0,
\]
which implies that $t = 0$. This yields that $U = 0$ and therefore $u \in \text{Ker}(I - \lambda_1 T)$. This shows that \( \text{Ker}((I - \lambda_1 T)^2) \subseteq \text{Ker}(I - \lambda_1 T) \), and the other inclusion is obvious.

The proof of (53) is therefore complete. From (53), we obtain that for all $k \in \mathbb{N}$ with $k \geq 1$,

$$\text{Ker}((I - \lambda_1 T)^k) = \text{Ker}(I - \lambda_1 T),$$

and thus

$$\bigcup_{k=1}^{+\infty} \text{Ker}((I - \lambda_1 T)^k) = \text{Ker}(I - \lambda_1 T).$$

The latter has dimension 1, thanks to (50), and therefore the claim in (52) is established.  

\[ \square \]

3. BOUNDEDNESS OF WEAK SUBSOLUTIONS AND PROOF OF THEOREM 1.3

For the proof of Theorem 1.3, we give here a general Sobolev inequality for the functions in the space $V$ introduced in (17) which can be seen as a natural counterpart of the Poincaré inequality given in Lemma 2.1 (the proof is somewhat of classical flavor, but we provide full details for the sake of completeness):

**Lemma 3.1.** Let $m$ be such that

$$\int_{\Omega} m(x) \, dx \neq 0.$$

Let $\eta$ be the fractional Sobolev exponent $2^*_s := \frac{2n}{n - 2s}$ if $\beta \neq 0$ and $n > 2s$, the classical Sobolev exponent $2^* := \frac{2n}{n - 2}$ if $\beta = 0$ and $n > 2$ and $\eta \geq 1$ arbitrary in the other cases.

If $V$ is as in (17) and $u \in V$, then

$$\int_{\Omega} u^n(x) \, dx \leq C \left( \alpha \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{\beta}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n + 2s}} \, dx \, dy \right)^{\frac{\eta}{2}},$$

where $C > 0$ depends only on $n$, $\Omega$, $s$ and $m$.

**Proof.** As usual, in this proof we will freely rename $C > 0$ line after line. First of all, we observe that the following “generalized” Sobolev inequality for any function $f \in X_{\alpha, \beta}$ holds true:

$$\|f\|_{L^\eta(\Omega)} \leq C \|f\|_{H^\eta(\Omega)},$$

where $\eta_1 := 2^*$ if $n > 2$, and $\eta_1 \geq 1$ arbitrary if $n \leq 2$. Indeed, when $n > 2$, the claim in (55) is the standard Sobolev embedding (see e.g. Theorem 2 on page 279 of [Eva10]). If instead $n = 2$, we let $\sigma := \frac{m}{m + 1} \in (0, 1)$. By Proposition 2.2 in [DNPV12], we know that

$$\|f\|_{H^\eta(\Omega)} \leq C \|f\|_{H^\sigma(\Omega)}.$$

Also, we have that $2\sigma < 2 = n$ and

$$2^*_s = \frac{2n}{n - 2\sigma} = \frac{2}{1 - \sigma} = 2(\eta_1 + 1) \geq \eta_1.$$

Hence, by Theorem 6.7 in [DNPV12], we obtain that $\|f\|_{L^\eta(\Omega)} \leq C\|f\|_{H^\eta(\Omega)}$. From this and (55), we obtain (55) in this case.

Finally, when $n = 1$, we have that (55) is a consequence of Morrey embedding (see e.g. Theorem 5 on page 283 of [Eva10]). These considerations complete the proof of (55).

As a fractional counterpart of (55), we notice that

$$\|f\|_{L^\eta(\Omega)} \leq C \|f\|_{H^\eta(\Omega)},$$

where $\eta_s := 2^*_s$ if $n > 2s$, and $\eta_s \geq 1$ arbitrary if $n \leq 2s$. Indeed, when $n > 2s$, we can use Theorem 6.7 in [DNPV12] and obtain (57). If instead $n \leq 2s$, the claim in (57) is contained in Theorem 6.10 of [DNPV12].

Now we take $\eta$ as in the statement of Lemma 3.1 and we claim that

$$\|f\|_{L^n(\Omega)} \leq C \left( \alpha\|f\|_{H^\eta(\Omega)} + \beta\|f\|_{H^\sigma(\Omega)} \right).$$
Indeed, if $\beta \neq 0$, the claim in (58) follows from (57). If instead $\beta = 0$, then necessarily $\alpha > 0$ and thus the claim in (58) is a consequence of (55).

Having proved (55), we can now combine it with the Poincaré inequality in Lemma 2.1 in order to complete the proof of (54). To this end, since $u \in V$, Lemma 2.1 gives that

$$
\|u\|_{L^2(\Omega)} \leq C \left[ u \right]_{\alpha, \beta} = C \sqrt{\frac{\alpha}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{\beta}{4} \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx \, dy}.
$$

(59)

Moreover, by (58),

$$
\|u\|_{L^2(\Omega)} \leq C (\|u\|_{H^1(\Omega)} + \beta \|u\|_{H^s(\Omega)})
$$

(60)

$$
\leq C \sqrt{\frac{\alpha}{2} \int_{\Omega} |\nabla u(x)|^2 \, dx + \frac{\beta}{4} \int_{\Omega} \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} \, dx \, dy + C \|u\|_{L^2(\Omega)}}.
$$

Then, we insert (59) into (60), and we obtain (54), as desired. \hfill \square

Now, we dive into the details of the proof of Theorem 1.3, which is based on a suitable choice of test functions and an iteration argument.

**Proof of Theorem 1.3**
We combine for this proof some classical and nonlocal techniques, see e.g. [HL97, GM02, SV13, SV14, DCKP14, DCKP16, IMS16]. Differently from the previous literature, we focus here on the case of the $(\alpha, \beta)$-Neumann conditions. For the facility of the reader, we try to make our arguments as self-contained as possible.

Given $k \geq 0$, we let $v := (u - k)^+$. We claim that

$$
(u(x) - u(y))(v(x) - v(y)) \geq (v(x) - v(y))^2.
$$

(61)

To prove this, we can suppose that $u(x) \geq u(y)$, up to exchanging the roles of $x$ and $y$. Also, if both $u(x)$ and $u(y)$ are larger than $k$, we have that $v(x) = u(x) - k$ and $v(y) = u(y) - k$, and thus (61) follows in this case (in fact, with equality instead of inequality). Therefore, we can suppose that $u(x) \geq k \geq u(y)$, whence $v(x) = u(x) - k$ and $v(y) = 0$, and then

$$
(u(x) - u(y))(v(x) - v(y)) - (v(x) - v(y))^2 = (u(x) - u(y))(u(x) - k) - (u(x) - k)^2
$$

$$
= ((u(x) - u(y)) - (u(x) - k))(u(x) - k) = (k - u(y))(u(x) - k) \geq 0.
$$

This establishes (61).

By (61),

$$
\int\int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy \geq \int\int_{\Omega} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} \, dx \, dy.
$$

(62)

In addition,

$$
\int_{\Omega} \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} |\nabla v(x)|^2 \, dx.
$$

Consequently, by (19),

$$
J := \alpha \int_{\Omega} |\nabla v(x)|^2 \, dx + \frac{\beta}{2} \int_{\Omega} \frac{(v(x) - v(y))^2}{|x - y|^{n+2s}} \, dx \, dy
$$

$$
\leq \int_{\Omega} (c(x)u(x) + f(x)) v(x) \, dx
$$

$$
\leq \int_{\Omega} \left( |c(x)| |u(x)| v(x) + |f(x)| v(x) \right) \, dx.
$$

(63)

We also remark that

$$
|u(x)| v(x) \leq 4(v^2(x) + k^2).
$$

(64)
Indeed, if \( u(x) \leq k \), then \( v(x) = 0 \) and (64) plainly follows. If instead \( u(x) > k \), then

\[
|u(x)|v(x) - 4v^2(x) - 4k^2 = u(x)v(x) - 4v^2(x) - 4k^2 = (v(x) + k)v(x) - 4v^2(x) - 4k^2
\]

\[
= kv(x) - 3v^2(x) - 4k^2 \leq 0,
\]

thus establishing (64).

From (63) and (64), we conclude that

\[
I \leq C \int_{\Omega \cap \{v \neq 0\}} \left( |c(x)|v^2(x) + k^2|c(x)| + |f(x)|v(x) \right) dx,
\]

up to renaming \( C > 1 \).

Now, we denote by \( \mathcal{Z} \) the Lebesgue measure of the set \( \Omega \cap \{v \neq 0\} = \Omega \cap \{u > k\} \) and we let \( \eta \) be as in the statement of Lemma 3.1 with the additional requirement that \( \eta > \frac{2q}{q-1} \) if \( \beta \neq 0 \) and \( n \leq 2s \), and if \( \beta = 0 \) and \( n \leq 2 \) (these situations corresponding to “the other cases” mentioned in the statement of Lemma 3.1).

We claim that

\[
\frac{1}{q} + \frac{1}{\eta} < 1.
\]

Indeed, we use here (13) and we see that, if \( \beta = 0 \) and \( n > 2 \),

\[
\frac{1}{q} + \frac{1}{\eta} < \frac{1}{q} + \frac{n-2}{2n} = \frac{2}{n} + \frac{n-2}{2n} = \frac{n+2}{2n} < 1.
\]

If instead \( \beta \neq 0 \) and \( n > 2s \),

\[
\frac{1}{q} + \frac{1}{\eta} < \frac{1}{q} + \frac{n-2s}{2n} = \frac{2s}{n} + \frac{n-2s}{2n} = \frac{n+2s}{2n} < 1.
\]

In all the other cases,

\[
\frac{1}{q} + \frac{1}{\eta} < \frac{1}{q} + \frac{q-1}{q} = 1.
\]

These observations prove (66).

Now, from (66), we can define

\[
\eta' := \frac{1}{1 - \frac{1}{q} - \frac{1}{\eta}}
\]

and we can exploit the Hölder inequality with exponents \( q \) and \( \eta \) and \( \eta' \), thus finding that

\[
\int_{\Omega} |f(x)|v(x) dx \leq \|f\|_{L^q(\Omega)} \left( \int_{\mathbb{R}^n} (v(x))^\eta \right)^{\frac{1}{\eta'}} \mathcal{Z}^{\frac{1}{\eta'}}
\]

We fix now \( \delta \in (0, 1) \), to be taken conveniently small in what follows, and we claim that

\[
\int_{\Omega} |f(x)|v(x) dx \leq \delta J + C_\delta \|f\|_{L^q(\Omega)}^2 \mathcal{Z}^{\vartheta},
\]

with (recalling (64))

\[
\vartheta := \frac{2}{\eta'} = 2 \left( 1 - \frac{1}{q} - \frac{1}{\eta} \right),
\]

for a suitable \( C_\delta > 1 \).

Indeed, using (68) and Lemma 3.1

\[
\int_{\Omega} |f(x)|v(x) dx \leq C \|f\|_{L^q(\Omega)} \sqrt{J} \mathcal{Z}^{\frac{1}{\eta'}}
\]

\[
\leq \delta J + C_\delta \left( \|f\|_{L^q(\Omega)} \mathcal{Z}^{\vartheta} \right)^2,
\]
which gives (69). Then, combining (65) and (69), we find that
\[ J \leq C \int_{\Omega \setminus \{ \nu \neq 0 \}} \left( |c(x)| \nu^2(x) + k^2 |c(x)| \right) \, dx + C \delta J + C_{\delta} \| f \|_{L^q(\Omega)}^2 Z^0 \]
up to renaming constants.

Consequently, choosing \( \delta \) sufficiently small (and considering \( \delta \) fixed from now on), we obtain
\[ J \leq C \int_{\Omega \setminus \{ \nu \neq 0 \}} \left( |c(x)| \nu^2(x) + k^2 |c(x)| \right) \, dx + C \| f \|_{L^q(\Omega)}^2 Z^0 \]
up to renaming constants.

In this setting, formula (71) will play a role of a pivotal Caccioppoli-type inequality, according to the following argument. We claim that there exists \( c_* > 0 \) such that if \( |Z| < c_* \) then
\[ J \leq C \left( k^2 + \| f \|_{L^q(\Omega)}^2 \right) Z^{1 - \frac{1}{q}}. \]

To check this, we recall (70), and we use the Hölder inequality and Lemma 3.1 to see that
\[ \int_{\Omega} |c(x)| \nu^2(x) \, dx \leq \| c \|_{L^q(\Omega)} \| \nu \|_{L^q(\Omega)}^2 Z^{1 - \frac{1}{q} - \frac{2}{q}} \]
\[ = \| c \|_{L^q(\Omega)} \| \nu \|_{L^q(\Omega)}^2 Z^{\frac{2}{q} - \frac{1}{q}} \leq C J Z^{\frac{2}{q} - \frac{1}{q}} \]
and
\[ \int_{\Omega \setminus \{ \nu \neq 0 \}} |c(x)| \, dx \leq \| c \|_{L^q(\Omega)} Z^{1 - \frac{1}{q}} \leq C Z^{1 - \frac{1}{q}}. \]

We stress that here the constants denoted by \( C \) are allowed to depend also on \( \| c \|_{L^q(\Omega)} \). Plugging this information into (71), we obtain that
\[ J \leq C J Z^{\frac{2}{q} - \frac{1}{q}} + C k^2 Z^{1 - \frac{1}{q}} + C \| f \|_{L^q(\Omega)}^2 Z^0. \]

Noticing that \( \frac{2}{q} - \frac{1}{q} > 0 \) and \( 1 - \frac{1}{q} \leq \vartheta \), if \( |Z| \) is sufficiently small we obtain (72), as desired.

We also remark that, by Lemma 3.1
\[ \int_{\Omega} \nu^2(x) \, dx \leq \left( \int_{\Omega} \nu(x) \, dx \right)^{\frac{2}{q}} Z^{1 - \frac{2}{q}} \leq \frac{J}{2} Z^{1 - \frac{2}{q}}. \]

This and (72) yield that, if \( |Z| \leq c_* \),
\[ \int_{\Omega} \nu^2(x) \, dx \leq C \left( k^2 + \| f \|_{L^q(\Omega)}^2 \right) Z^{2 - \frac{1}{q} - \frac{2}{q}}. \]

We stress that \( 2 - \frac{1}{q} - \frac{2}{q} > 1 \), hence (73) gives that
\[ \int_{\Omega} \nu^2(x) \, dx \leq C \left( k^2 + \| f \|_{L^q(\Omega)}^2 \right) Z^{1 + \epsilon_0} \]
for some \( \epsilon_0 > 0 \).

That is, setting \( A(k) := \Omega \cap \{ u > k \} \) and
\[ \varphi(k) := \int_{A(k)} (u(x) - k)^2(x) \, dx := \int_{A(k)} v^2(x) \, dx, \]
in light of (74) we can write that, if \( |A(k)| \leq c_* \), then
\[ \varphi(k) \leq C \left( k^2 + \| f \|_{L^q(\Omega)}^2 \right) |A(k)|^{1 + \epsilon_0}. \]
We observe that if \( x \in A(k) \) then \( u(x) > k \) and thus \( u^+(x) > k \). Therefore,
\[ |A(k)| \leq \frac{1}{k} \int_{A(k)} u^+(x) \, dx \leq \frac{\sqrt{|A(k)|}}{k} \| u^+ \|_{L^2(\Omega)}. \]
Hence, it follows that

\begin{equation}
|A(k)| \leq \left( \frac{\|u^+\|_{L^2(\Omega)}}{k} \right)^2 \leq c_* ,
\end{equation}

as long as

\begin{equation}
k \geq \frac{\|u^+\|_{L^2(\Omega)}}{\sqrt{c_*}} =: \kappa .
\end{equation}

In particular, in view of (76), we know that (75) holds true for all \( k \) satisfying (77).

Now we define, for every \( \ell \in \mathbb{N} \),

\[
K := \kappa + \|f\|_{L^q(\Omega)}
\]

and

\[
k_\ell := \kappa + K \left( 1 - \frac{1}{2^\ell} \right).
\]

We point out that

\[
k_\ell - k_{\ell-1} = \frac{K}{2^\ell},
\]

and, as a result, if \( x \in A(k_\ell) \) then \( u(x) - k_{\ell-1} \geq k_\ell - k_{\ell-1} = \frac{K}{2^\ell} .\)

For this reason, we have that

\[
A(k_\ell) \leq \frac{2^{2\ell}}{K^2} \int_{A(k_\ell)} (u(x) - k_{\ell-1})^2 \, dx \leq \frac{2^{2\ell}}{K^2} \int_{A(k_{\ell-1})} (u(x) - k_{\ell-1})^2 \, dx = \frac{2^{2\ell}}{K^2} \varphi(k_{\ell-1}).
\]

Using this information together with (75) (exploited here with \( k := k_\ell \), and we remark that \( k_\ell \geq \kappa \), hence condition (77) is satisfied), we discover that

\begin{equation}
\varphi(k_\ell) \leq \frac{C^\ell (k_\ell^2 + \|f\|_{L^2(\Omega)})}{K^2} (\varphi(k_{\ell-1}))^{1+\epsilon_0} .
\end{equation}

Since \( k_\ell \leq \kappa + K \), up to renaming constants we obtain from (78) that

\[
\varphi(k_\ell) \leq \frac{C^\ell (\kappa^2 + K^2)}{K^2} (\varphi(k_{\ell-1}))^{1+\epsilon_0} ,
\]

and consequently, if \( c_* \) is sufficiently small,

\[
0 = \lim_{\ell \to +\infty} \varphi(k_\ell) = \varphi(\kappa + K).
\]

As a result, \( u^+(x) \leq \kappa + K \), whence the claim in (20) plainly follows. \( \square \)

REFERENCES

[AC20] Nicola Abatangelo and Matteo Cozzi, An elliptic boundary value problem with fractional nonlinearity, arXiv e-prints (2020), arXiv:2005.09515, available at 2005.09515.

[AV19] Elisa Affili and Enrico Valdinoci, Decay estimates for evolution equations with classical and fractional time-derivatives, J. Differential Equations 266 (2019), no. 7, 4027–4060, DOI 10.1016/j.jde.2018.09.031. MR3912710

[ACR13] Matthieu Alfaro, Jérôme Coville, and Gaël Raoul, Travelling waves in a nonlocal reaction-diffusion equation as a model for a population structured by a space variable and a phenotypic trait, Comm. Partial Differential Equations 38 (2013), no. 12, 2126–2154, DOI 10.1080/03605302.2013.828069. MR3169773

[AdTEJ19] Nathaël Alibaud, Félix del Teso, Jørgen Endal, and Espen R. Jakobsen, The Liouville theorem and linear operators satisfying the maximum principle, arXiv e-prints (2019), arXiv:1907.02495, available at 1907.02495.

[ABVV10] Narcisa Apreutesei, Nikolai Bessonov, Vitaly Volpert, and Vitali Vougalter, Spatial structures and generalized travelling waves for an integro-differential equation, Discrete Contin. Dyn. Syst. Ser. B 13 (2010), no. 3, 537–557, DOI 10.3934/dcdsb.2010.13.537. MR2601079

[BCCI12] Guy Barles, Emmanuel Chasseigne, Adina Ciomaga, and Cyril Imbert, Lipschitz regularity of solutions for mixed integro-differential equations, J. Differential Equations 252 (2012), no. 11, 6012–6060, DOI 10.1016/j.jde.2012.02.013. MR2911421
[BCCI14] __________. Large time behavior of periodic viscosity solutions for uniformly parabolic integro-differential equations, Calc. Var. Partial Differential Equations 50 (2014), no. 1-2, 283–304, DOI 10.1007/s00526-013-0636-2. MR3194684

[BI08] Guy Barles and Cyril Imbert, Second-order elliptic integro-differential equations: viscosity solutions’ theory revisited, Ann. Inst. H. Poincaré Anal. Non Linéaire 25 (2008), no. 3, 567–585, DOI 10.1016/j.anihpc.2007.02.007. MR2422079

[BK05a] Imran H. Biswas, Espen R. Jakobsen, and Kenneth H. Karlsen, Harnack inequalities for non-local operators of variable order, Comm. Partial Differential Equations 30 (2005), no. 7-9, 1249–1259, DOI 10.1080/03605300500257677. MR2180302

[BK05b] Richard F. Bass and Moritz Kassmann, Hölder continuity of harmonic functions with respect to operators of variable order, Comm. Partial Differential Equations 30 (2005), no. 7-9, 1249–1259, DOI 10.1080/03605300500257677. MR2180302

[BCV16] Henri Berestycki, Jérôme Coville, and Hoang-Hung Vo, Persistence criteria for populations with non-local dispersion, J. Math. Biol. 72 (2016), no. 7, 1693–1745, DOI 10.1007/s00285-015-0911-2. MR3498523

[BDVV20] Stefano Biagi, Serena Dipierro, Enrico Valdinoci, and Eugenio Vecchi, Mixed local and nonlocal elliptic operators: regularity and maximum principles, arXiv e-prints (2020), arXiv:2005.06907, available at 2005.06907.

[BJK10] Imran H. Biswas, Espen R. Jakobsen, and Kenneth H. Karlsen, Viscosity solutions for a system of integro-PDEs and connections to optimal switching and control of jump-diffusion processes, Appl. Math. Optim. 62 (2010), no. 1, 47–80, DOI 10.1007/s00245-009-9095-8. MR2653895

[BdCN13] Daniel Blazevski and Diego del-Castillo-Negrete, Local and nonlocal anisotropic transport in reversed shear magnetic fields: Shearless Cantori and nondiffusive transport, Phys. Rev. E 87 (2013), 063106, DOI 10.1103/PhysRevE.87.063106.

[BCL17] Olivier Bonnefon, Jérôme Coville, and Guillaume Legendre, Concentration phenomenon in some non-local equation, Discrete Contin. Dyn. Syst. Ser. B 22 (2017), no. 3, 763–781, DOI 10.3934/dcdsb.2017037. MR3639140

[Bre83] Haïm Brezis, Analyse fonctionnelle, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master’s Degree], Masson, Paris, 1983 (French). Théorie et applications. [Theory and applications]. MR697382

[BL80] K. J. Brown and S. S. Lin, On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function, J. Math. Anal. Appl. 75 (1980), no. 1, 112–120, DOI 10.1016/0022-247X(80)90309-1. MR576277

[CDV16] Xavier Cabré and Joaquim Serra, An extension problem for sums of fractional Laplacians and 1-D symplectic phase transitions, Nonlinear Anal. 137 (2016), 246–265, DOI 10.1016/j.na.2015.12.014. MR3485125

[CDV] Xavier Cabré, Serena Dipierro, and Enrico Valdinoci, The Bernstein technique for integro-differential equations, preprint.

[CDV17] Luis Caffarelli, Serena Dipierro, and Enrico Valdinoci, A logistic equation with nonlocal interactions, Kinet. Relat. Models 10 (2017), no. 1, 141–170, DOI 10.3934/krm.2017006. MR3579567

[CV13] Luis Caffarelli and Enrico Valdinoci, A priori bounds for solutions of a nonlocal evolution PDE, Analysis and numerics of partial differential equations, Springer INdAM Ser., vol. 4, Springer, Milan, 2013, pp. 141–163, DOI 10.1007/978-88-470-2592-9_10. MR3051400

[CCL07] Robert Stephen Cantrell, Chris Cosner, and Yuan Lou, Advection-mediated coexistence of competing species, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), no. 3, 497–518, DOI 10.1017/S0308210506000047. MR2332679

[CCLR12] Robert Stephen Cantrell, Chris Cosner, Yuan Lou, and Daniel Ryan, Evolutionary stability of ideal free dispersal strategies: a nonlocal dispersal model, Can. Appl. Math. Q. 20 (2012), no. 1, 15–38. MR3026598

[CHL08] Xinfu Chen, Richard Hambrock, and Yuan Lou, Evolution of conditional dispersal: a reaction-diffusion-advection model, J. Math. Biol. 57 (2008), no. 3, 361–386, DOI 10.1007/s00285-008-0166-2. MR2411225

[CKSV10] Zhen-Qing Chen, Panki Kim, Renming Song, and Zoran Vondraček, Sharp Green function estimates for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets and their applications, Illinois J. Math. 54 (2010), no. 3, 981–1024 (2012). MR2928344

[CKSV12] __________. Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$, Trans. Amer. Math. Soc. 364 (2012), no. 8, 4169–4205, DOI 10.1090/S0002-9947-2012-05542-5. MR2912450

[Cio12] Adina Ciomaga, On the strong maximum principle for second-order nonlinear parabolic integro-differential equations, Adv. Differential Equations 17 (2012), no. 7-8, 635–671. MR2963799

[CDM12] Chris Cosner, Juan Dávila, and Salome Martínez, Evolutionary stability of ideal free nonlocal dispersal, J. Biol. Dyn. 6 (2012), no. 2, 395–405, DOI 10.1080/17513758.2011.588341. MR2897881
[Cov15] Jérôme Coville, Nonlocal refuge model with a partial control, Discrete Contin. Dyn. Syst. 35 (2015), no. 4, 1421–1446, DOI 10.3934/dcds.2015.35.1421. MR3285831

[CDM13] Jérôme Coville, Juan Dávila, and Salomé Martínez, Pulsating fronts for nonlocal dispersion and KPP nonlinearity, Ann. Inst. H. Poincaré Anal. Non Linéaire 30 (2013), no. 2, 179–223, DOI 10.1016/j.anihpc.2012.07.005. MR3035974

[dF82] Djairo Guedes de Figueiredo, Positive solutions of semilinear elliptic problems, Differential equations (São Paulo, 1981), Lecture Notes in Math., vol. 957, Springer, Berlin-New York, 1982, pp. 34–87. MR679140

[dlLV09] Rafael de la Llave and Enrico Valdinoci, A generalization of Aubry-Mather theory to partial differential equations and pseudo-differential equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), no. 4, 1309–1344, DOI 10.1016/j.anihpc.2008.11.002. MR2542727

[DCKP14] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci, On distributional solutions of local and nonlocal problems of porous medium type, arXiv e-prints (2014), arXiv:1401.3672, available at 1401.3672.

[DCKP16] Agnese Di Castro, Tuomo Kuusi, and Giampiero Palatucci, Local behavior of fractional p-minimizers, Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2016), no. 5, 1279–1299, DOI 10.1016/j.anihpc.2015.04.003. MR3542614

[DNPV12] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), no. 5, 521–573, DOI 10.1016/j.bulsci.2011.12.004. MR2944369

[DPLV] Serena Dipierro, Edoardo Proietti Lippi, and Enrico Valdinoci, (Non)local logistic equations with Neumann conditions, preprint.

[DROV17] Serena Dipierro, Xavier Ros-Oton, and Enrico Valdinoci, Nonlocal problems with Neumann boundary conditions, Rev. Mat. Iberoam. 33 (2017), no. 2, 377–416, DOI 10.4171/RMI/942. MR3651008

[DVV19] Serena Dipierro, Enrico Valdinoci, and Vincenzo Vespri, Decay estimates for evolutionary equations with fractional time-diffusion, J. Evol. Equ. 19 (2019), no. 2, 435–462, DOI 10.1007/s00028-019-00482-z. MR3950697

[DHMP98] Jack Dockery, Vivian Hutson, Konstantin Mischaikow, and Mark Pernarowski, The evolution of slow dispersal rates: a reaction diffusion model, J. Math. Biol. 37 (1998), no. 1, 61–83, DOI 10.1007/s002850050120. MR1636644

[Eva10] Lawrence C. Evans, Partial differential equations, 2nd ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010. MR2597943

[GM02] Maria Giovanna Garroni and Jose Luis Menaldi, Second order elliptic integro-differential problems, Chapman & Hall/CRC Research Notes in Mathematics, vol. 430, Chapman & Hall/CRC, Boca Raton, FL, 2002. MR1911531

[HL97] Qing Han and Fanghua Lin, Elliptic partial differential equations, Courant Lecture Notes in Mathematics, vol. 1, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997. MR1669352

[IMS16] Antonio Iannizzotto, Sunra Mosconi, and Marco Squassina, Global Hölder regularity for the fractional p-Laplacian, Rev. Mat. Iberoam. 32 (2016), no. 4, 1353–1392, DOI 10.4171/RMI/921. MR3595328

[JK05] Espen R. Jakobsen and Kenneth H. Karlsen, Continuous dependence estimates for viscosity solutions of integro-PDEs, J. Differential Equations 212 (2005), no. 2, 278–318, DOI 10.1016/j.jde.2004.06.021. MR2129093

[JK06] _, A “maximum principle for semicontinuous functions” applicable to integro-partial differential equations, NoDEA Nonlinear Differential Equations Appl. 13 (2006), no. 2, 137–165, DOI 10.1007/s00030-005-0031-6. MR2243708

[KLS12] Chiu-Yen Kao, Yuan Lou, and Wenzian Shen, Evolution of mixed dispersal in periodic environments, Discrete Contin. Dyn. Syst. Ser. B 17 (2012), no. 6, 2047–2072, DOI 10.3934/dcdsb.2012.17.2047. MR2924452

[MPV13] Eugenio Montefusco, Benedetta Pellacci, and Giannmaria Verzini, Fractional diffusion with Neumann boundary conditions: the logistic equation, Discrete Contin. Dyn. Syst. Ser. B 18 (2013), no. 8, 2175–2202, DOI 10.3934/dcdsb.2013.18.2175. MR3082317
[PV18] Benedetta Pellacci and Gianmaria Verzini, *Best dispersal strategies in spatially heterogeneous environments: optimization of the principal eigenvalue for indefinite fractional Neumann problems*, J. Math. Biol. **76** (2018), no. 6, 1357–1386, DOI 10.1007/s00285-017-1180-z. MR3771424

[SV13] Raffaella Servadei and Enrico Valdinoci, *A Brezis-Nirenberg result for non-local critical equations in low dimension*, Commun. Pure Appl. Anal. **12** (2013), no. 6, 2445–2464, DOI 10.3934/cpaa.2013.12.2445. MR3060890

[SV14] ____, *Weak and viscosity solutions of the fractional Laplace equation*, Publ. Mat. **58** (2014), no. 1, 133–154. MR3161511

[SV17] Jürgen Sprekels and Enrico Valdinoci, *A new type of identification problems: optimizing the fractional order in a nonlocal evolution equation*, SIAM J. Control Optim. **55** (2017), no. 1, 70–93, DOI 10.1137/16M105575X. MR3590646

[VAB+96] G. M. Viswanathan, V. Afanasyev, S. V. Buldyrev, E. J. Murphy, P. A. Prince, and H. E. Stanley, *Lévy flight search patterns of wandering albatrosses*, Nature **381** (1996), 413–415, DOI 10.1038/381413a0.