Structure of the Spherical Black Hole Interior

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Abstract

The internal structure of a charged spherical black hole is still a topic of debate. In a nonrotating but aspherical gravitational collapse to form a spherical charged black hole, the backscattered gravitational wave tails enter the black hole and are blueshifted at the Cauchy horizon. This has a catastrophic effect if combined with an outflux crossing the Cauchy horizon: a singularity develops at the Cauchy horizon and the effective mass inflates. Recently a numerical study of a massless scalar field in the Reissner-Nordström background suggested that a spacelike singularity may form before the Cauchy horizon forms. We will show that there exists an approximate analytic solution of the scalar field equations which allows the mass inflation singularity at the Cauchy horizon to exist. In particular, we see no evidence that the Cauchy horizon is preceded by a spacelike singularity.

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1 Introduction

The final state of a star that undergoes gravitational collapse into a black hole is described by the uniqueness theorems of general relativity [1]. At late times, after the star’s irregularities have been radiated away [2], the external geometry is described by the stationary Kerr-Newman solution. But the structure of the realistic black hole’s interior has not been definitively determined and is still the subject of much debate.

The unproven, yet plausible strong cosmic censorship principle leads one to suspect that the singularity in a physical black hole ought to be spacelike, and described by the general mixmaster type solution [3]. But the Kerr-Newman singularity at $r = 0$ is timelike. How could the inclusion of the details of collapse alter this picture so drastically? Penrose has described the basic mechanism [4]. In the Kerr-Newman solution (and the Reissner-Nordström solution), the timelike singularity is preceded by a Cauchy horizon. The Cauchy horizon is the boundary of predictability, making the existence of a timelike singularity in a region beyond it physically irrelevant. The Cauchy horizon is also a surface of infinite blueshift. Any freefalling observer crossing the Cauchy horizon will measure infalling radiation to have infinite energy density. When only ingoing radiation is present a weak nonscalar singularity forms, which is classified as a whimper singularity [5]. Whimper singularities are known to be unstable to perturbations which can transform them into stronger singularities [5].

This scenario can be examined more closely by assuming spherical symmetry. When an outflow crossing the Cauchy horizon is included in the analysis of a charged spherical black hole [7, 8], the mass function is found to diverge exponentially. In spherical symmetry, the only non-vanishing component of the Weyl tensor, the invariant $\Psi_2$, is proportional to the mass function. The net result of including an outflux is that a scalar curvature singularity forms along the Cauchy horizon. This phenomenon has been dubbed “mass inflation” [7]. The Reissner-Nordström black hole has the same causal structure as the Kerr-Newman black hole, so we expect that results for spherical symmetry should be qualitatively the same for the more general black hole. Preliminary calculations suggest that the general picture derived for spherical symmetry does not change dramatically in a non-spherical black hole [9].

The important issue, which will be addressed in this paper, is to ask how generic is the mass inflation picture. When an outflow crossing the Cauchy horizon is included in the analysis of a charged spherical black hole [7, 8], the mass function is found to diverge exponentially. In spherical symmetry, the only non-vanishing component of the Weyl tensor, the invariant $\Psi_2$, is proportional to the mass function. The net result of including an outflux is that a scalar curvature singularity forms along the Cauchy horizon. This phenomenon has been dubbed “mass inflation” [7]. The Reissner-Nordström black hole has the same causal structure as the Kerr-Newman black hole, so we expect that results for spherical symmetry should be qualitatively the same for the more general black hole. Preliminary calculations suggest that the general picture derived for spherical symmetry does not change dramatically in a non-spherical black hole [9].

The important issue, which will be addressed in this paper, is to ask how generic is the mass inflation picture. General stability arguments [10] and the numerical study of a scalar field [11] suggest that the Cauchy horizon is preceded by a spacelike $r = 0$ singularity. Hence evolution of the interior should end before the Cauchy horizon forms. But while these studies are suggestive, a conclusive analysis of the internal structure is still lacking even for spherical holes.

Fig 1. Penrose conformal diagram of a collapsing star. Note that the point H is not part of the manifold, but a singular point of this mapping.

Our main objective is to examine the “corner” region, (“point” H in Fig. 1), bounded by the Cauchy and event horizons, and to decide whether it is possible for an early portion of the Cauchy horizon to survive. We have found, using analytic approximations and physical arguments, that there is a general solution of the scalar field equations which is consistent with the mass inflation picture [12].

The description of the black hole interior is simplified by causality: behind the event horizon the coordinate $r$ is timelike, so a descent into a black hole is an evolution in time. The solution down to any particular radius is only influenced by the initial data at larger radii. Most importantly, our lack of understanding of the true quantum description of gravity (applicable in the innermost layers) will not affect the description of the outer classical and semi-classical layers. The classical evolution problem is thus on firm ground.

In an aspherical collapse with zero angular momentum, the asphericities must be radiated away in order to form a static spherically symmetric black hole, as the no hair theorems demand. Price [2] has shown that the star settles into an asymptotically Reissner-Nordström state with perturbations that die off with advanced time as an inverse power law. These perturbations are scattered by the black hole’s external potential barrier so that part of the radiation falls into
the black hole. Near the event horizon, the backscattered energy flux takes the form \( v \rightarrow \infty \) where \( v \rightarrow \infty \) corresponds to \( Z^+ \) and \( p = 4l + 4 \), where \( l \) is the spherical harmonic multipole of the perturbation \([16]\). This result was recently verified by numerical studies of the coupled Einstein-scalar field equations \([13]\). We will use the Price tail \( v \rightarrow \infty \) as the initial data on the event horizon, the natural choice for the initial data surface for the black hole interior evolutionary problem.

Our paper is organised as follows. In §2 and §3 we will review the mass inflation model of the black hole interior and discuss its limitations. The boundary conditions will be discussed in section §4. These results will be applied to a null crossflow model in section §5.

In §6, we present an approximate analytic solution to the scalar field equations inside a spherical black hole.

## 2 Schematic Mass Inflation Models

The essential physics of mass inflation – a blueshifted influx combined with an outflux – can be illustrated using simple solvable models. The simplest model is that of two concentric spherical null shells, one ingoing, the other outgoing, which cross without interaction.

Imagine the spacetime split into four regions \( A, B, C \) and \( D \) by the two crossing shells (see Fig. 1 of \([16]\)). Each region \( i \), is described by a Reissner-Nordström solution with metric function \( f_i = 1 - 2m_i/r + \epsilon^2/r^2 \).

The masses of the ingoing and outgoing shells (before collision) are \( m_{in} = m_B - m_D \) and \( m_{out} = m_C - m_B \). The final result of the analysis is that the metric functions are related by the generalised DTR relation \([13, 14]\)

\[
f_{AB} = f_C f_D, \tag{1}
\]

which holds at \( r_0 \), the radius of crossing. Mass inflation occurs when the ingoing shell is very close (within distance \( \epsilon \)) of the Cauchy horizon corresponding to the extension of sector \( B \), the space between them before the shells cross. Then \( f_B \sim \epsilon \), while the product \( f_C f_D \) stays finite. This forces the metric function in region \( A \), after the shells cross, to diverge as \( f_A \sim 1/\epsilon \).

The crucial role of the outflux is illustrated by writing the expression for the mass after the crossing, \( m_A \) \([16]\)

\[
m_A = \frac{m_C m_D}{m_B} + \frac{m_{in} m_{out}}{m_B (1/f_B)}. \tag{2}
\]

The first term is finite. When the ingoing shell is close to the Cauchy horizon, the second term becomes very large. But note that the second term is proportional to the product \( m_{in} m_{out} \). The absence of the outgoing shell would render the second term harmless. Thus the outflow, however weak, is crucial for the divergence of the mass function after the shells cross.

Ori has introduced a generalization of this simple model \([8]\). He considers a continuous influx and treats the outflux as a thin shell. This allows the matching of two ingoing Vaidya solutions along the outgoing lightlike shell \( \Sigma \), a finite Kruskal time after the event horizon (see Fig. 1 of \([17]\)):

\[
ds^2 = dv_+^2 (f_+ dv_+ - 2dr) + r^2 d\Omega^2, \tag{3}
\]

\[
f_+ = 1 - 2m_+/(v_+) + e^2/r^2,
\]

where the subscript \(+ (-)\) refers to the region after (before) the shell. The Einstein equations link the mass with the influx

\[
L_\pm = dm_\pm/dv_\pm, \quad T^\pm_{ab} = \frac{L^\pm_+(v_+)}{4\pi r^2} \partial_a v_\pm \partial_b v_\pm. \tag{4}
\]

Continuity of the line element and the radial coordinate, \( r \), yields the equations

\[
f_+ dv_+ = f_- dv_- = 2dr \tag{5}
\]

along the shell. Continuity of the influx across the shell gives the equation

\[
\frac{1}{f_+^2} \frac{dm_+}{dv_+} = \frac{1}{f_-^2} \frac{dm_-}{dv_-}. \tag{6}
\]

These two equations yield the simple equation \([17]\)

\[
\frac{dm_+}{f_+} = \frac{dm_-}{f_-}, \tag{7}
\]

in which mass inflation is evident. The metric function \( f_- \) goes to zero as the Cauchy horizon is approached, causing the right hand side of the equation to diverge. The presence of the outgoing shell displaces the apparent horizon to smaller radii so that \( f_+ \neq 0 \) at the Cauchy horizon. This equation implies that beyond the shell, the mass will diverge at CH.

This model can be solved asymptotically close to the Cauchy horizon. The mass function prior to the shell must reproduce the Price power law tail:

\[
m_- = m_0 - \frac{\beta}{\kappa_0 (p-1)} (\kappa_0 v_-)^{- (p-1)}. \tag{8}
\]

Here \( v_- \) is the usual Eddington-Finkelstein advanced time coordinate which is infinite on the Cauchy horizon, \( \beta \) is a dimensionless constant and \( \kappa_0 \) is the surface gravity of the inner horizon. Equation \([10]\) is then integrated for \( r \) along \( \Sigma \) as \( v_- \rightarrow \infty \)

\[
r_\Sigma(v_-) = r_0 + \beta \frac{1}{r_0^2 (p-1)} (\kappa_0 v_-)^{-(p-1)} \left( 1 + \frac{(p-1)}{\kappa_0 v_-} + \ldots \right).
\]
Equation (4) can now be integrated using (5) to show that the mass function diverges exponentially

\[ m_+(v_-) \sim e^{\kappa_0 v_-} (\kappa_0 v_-)^{-p}, \quad v_- \to \infty. \]  

(9)

This phenomenon has been dubbed mass inflation [6]. Indeed this is a scalar curvature singularity since the Weyl curvature invariant diverges exponentially, \( \Psi_2 \sim e^{\kappa_0 v_-}/r_0^2 \) as the Cauchy horizon is approached.

It is worth noting that there are coordinates in which the metric is finite. Near the CH the line element is given to a very good approximation by

\[ ds^2 = 2\frac{dv_+}{r} (rdr + m_+(v_+)dv_+) + r^2 d\Omega^2 \]  

(10)

where \( v_+ \) is the standard advanced time coordinate behind \( \Sigma \). It is easily checked, that the coordinate \( u \), defined by

\[ du = rdr + m_+(v_+)dv_+ \]

is regular at the CH. (3) now becomes

\[ ds^2 = 2\frac{dv_+ du}{r} + r^2 d\Omega^2. \]

The mass inflation singularity, though much stronger than a whimper singularity is still very weak in this sense. It is exactly this property of the mass inflation singularity, that will allow us to construct an approximate solution to the full cross flow and scalar field equations.

3 The Continuous Crossflow Model

The previous idealized models show that when the outflux is concentrated as a shell, the mass function (and the Weyl curvature invariant) inflates exponentially. Originally, this was shown for a continuous, arbitrary outflux which starts a finite time after the event horizon. In this section we shall introduce the notation and Einstein equations for spherical symmetry, review the standard mass inflation solution and discuss its limitations.

In a general spherical spacetime it is convenient to introduce a coordinate system \( x^a = (x^a, \theta, \phi), \quad (a = 1, 2) \), where \( x^a \) are the coordinates of the radial two-spaces \( (\theta, \phi) = \text{constant} \). Introducing a function \( r(x^a) \) which measures the area of the two-spheres, the metric can be written:

\[ ds^2 = g_{ab} dx^a dx^b + r^2 d\Omega^2, \]  

(11)

where \( d\Omega^2 \) is the line element of the unit two-sphere and \( g_{ab} \) is the metric of the radial two-spaces.

Scalar fields \( m(x^a), f(x^a) \) and \( \kappa(x^a) \) can be defined by

\[ f = g^{ab} \partial_a r \partial_b r = 1 - \frac{2m}{r} + \frac{e^2}{r^2}, \]

\[ \kappa = -\frac{1}{2} \partial_a f = -\frac{1}{r^2} (m - \frac{e^2}{r}). \]

(12)

The Einstein field equations are then written as

\[ G_{\alpha\beta} = 8\pi (E_{\alpha\beta} + T_{\alpha\beta}), \]

(13)

where \( E_{\alpha\beta} \) is the Maxwellian contribution to the stress tensor. For a point charge of strength \( e \) located at the origin, this is

\[ E_\alpha^\beta = \frac{e^2}{8\pi r^4} \text{diag}(-1, -1, 1, 1). \]

(14)

(We shall assume that \( e \) is fixed.) The non-Maxwellian contribution will be decomposed as

\[ \mathfrak{d} T^a_b = T^a_b, \quad T^a_\theta = T^\phi_\theta = P \]

(15)

where \( P \) is the tangential pressure. The energy-momentum conservation equation is

\[ (r^2 T^{ab})_{;b} = (r^2)^{\alpha\beta} P. \]

(16)

The field equations can then be written as

\[ r_{;ab} + \kappa g_{ab} = -4\pi r (T^a_a - g_{ab} T), \quad \bar{T}^a_a = T^a_a \]

\[ R - 2\partial_\alpha \kappa = 8\pi (T - 2P). \]

(17)

Equations (12) and (17) yield the equations

\[ m_{;a} = 4\pi r^2 (T^b_a - 2^{\Theta b} T^a b) r_{;b} \]

(18)

which can be used to derive a wave equation for the mass function

\[ \Box m = 4\pi (2r f(P - T) + \kappa r^2 T) - (4\pi)^2 r^3 T_{ab} T^{ab}. \]

(19)

In order to be more specific, we will write the two-metric using null coordinates \( U, V \)

\[ g_{ab} dx^a dx^b = -2e^{2\sigma} dU dV. \]

(20)

We will choose \( V, U \) to be Kruskal coordinates in which the asymptotic Reissner-Nordström metric (mass \( m_0 \) and charge \( e \)) is regular on the inner horizon. They are related to the usual Eddington-Finkelstein advanced and retarded times \( v, u \) by

\[ U = e^{-\kappa_0 u}, \quad V = e^{-\kappa_0 v} \]

(21)

where \( \kappa_0 = \kappa(r_0) \) is the surface gravity of the Cauchy horizon and \( r_0 \) is its radius. The wave operator acting on a scalar \( \psi \) takes the form

\[ \Box \psi = -2e^{-2\sigma} \psi_{;UV}. \]

(22)
In this coordinate system, the Ricci scalar is \( R = -2 \sigma \).

Null radial vectors pointing inwards and outwards can be defined
\[
l_a = -\partial_a V, \quad n_a = -\partial_a U. \tag{23}
\]
In the original mass inflation analysis \([6]\), a null crossflow stress tensor was used to model the gravitational radiation. This effective stress-energy model \([3]\) is justified by the high blueshift near the Cauchy horizon. The stress tensor for null crossflowing radiation can then be written as
\[
T_{ab} = \frac{L_{in}(V)}{4\pi r^2} l_a l_b + \frac{L_{out}(U)}{4\pi r^2} n_a n_b \tag{24}
\]
which satisfies the conservation equations \([10]\) and has \( P = T = 0 \). The conservation equations force \( L_{in} \) (\( L_{out} \)) to be a function only of \( V (U) \).

In the Kruskal coordinate, \( V \), the Price power-law tail has the form
\[
L_{in}(V) = \frac{dm_{in}}{dv} \left( \frac{dv}{dV} \right)^2 = \frac{\beta}{(-\kappa_0 V)^p} (-\ln(-V))^{-p}. \tag{25}
\]
As the Cauchy horizon is approached, in the limit, \( V \to 0 \ldots \), \( L_{in} \) diverges and the source term in the wave equation for \( m \) diverges as well. The integral solution for the mass function is \([11]\)
\[
m(U, V) = \int_{U_1}^{V_1} \int_{V_1}^{U_1} r^{-1} e^{-2\sigma'} L_{in}(V') L_{out}(U') dU' dV' + m_{in}(V) + m_{out}(U) - m_1. \tag{26}
\]

The gravitational wave tail influx is turned on at advanced time \( V_1 \) and the outflux is assumed to be switched on at the advanced time \( U_1 \) which is behind the event horizon. The divergence of \( L_{in}(V') dV' \) leads to mass inflation with the mass function behaving as \( m \sim 1/V \). Of course, this is only true if the combination \( r^{-1} e^{-2\sigma} \) does not go to zero, but this was proved by Poisson and Israel \([3]\).

The previous mass inflation analyses suffer some limitations. In the picture presented \([1, 3]\) it is always assumed that the outflux is turned on abruptly after some finite time behind the event horizon. Essentially, this amounts to the assumption that a null portion of the Cauchy horizon exists because the solution before the outflux begins is the Vaidya solution. This segment’s existence depends on the form of the outflux crossing it. If the outflux at early retarded times \( U \to -\infty \) is too strong, a spacelike singularity will form. The effect of an outflux crossing a null ray is described by Raychaudhuri’s equation
\[
\frac{d^2 r}{d\lambda^2} = -4\pi r T_{\lambda\lambda} \tag{27}
\]
where \( \lambda \) is an affine parameter on the null hypersurface and \( T_{\lambda\lambda} = T_{ab} \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} \) is the transverse flux and \( dx^b/d\lambda \) is tangent to the generators of the null hypersurface. In the case of interest, the null hypersurface is the Cauchy horizon and \( \lambda \to -\infty \) corresponds to its “meeting” with the event horizon at \( H \) in Fig. 1.

Examination of (27) shows that in order for \( r \) to be finite as \( \lambda \to -\infty \), \( T_{\lambda\lambda} \) must satisfy
\[
\lambda^2 T_{\lambda\lambda} \to 0 \quad \text{as} \quad \lambda \to -\infty. \tag{28}
\]
To test this condition we need to relate the affine parameter \( \lambda \) to the null coordinate \( U \) by
\[
\frac{dU}{d\lambda} = -g^{VV} = e^{-2\sigma}. \tag{29}
\]
If \( e^{-2\sigma} \) diverges as \( V \to 0 \) and \( U \to -\infty \), then depending on \( T_{UU} \), the Cauchy horizon may not survive.

The earlier models of mass inflation do not address this issue since the outflux is turned on after the event horizon. The corner region, \( V \to 0, -\infty < U < U_1 \) is described by a Vaidya solution. (The metric function \( e^{-2\sigma} \) is finite in this region for Vaidya.) For more general models which include the corner region, the behaviour of \( \sigma \) must be found. To do this, we need to specify the appropriate initial conditions for \( T_{UU} \) which are physically reasonable.

4 The Outflux

A star collapsing through its event horizon provides two sources of outflux. First, the star shines as it collapses and will irradiate the Cauchy horizon after the event horizon is passed. While we will not attempt to describe the actual form of the star’s radiation, we do know that in a freely falling frame at the event horizon, the radiation must be measured to be bounded. Kruskal coordinates for the event horizon, \( U_+ = e^{\kappa_+ u} \) are appropriate for freely falling observers. These observers measure \( T_{UU} < \text{constant} \). Transforming this to the Kruskal coordinate \( U \) appropriate to the inner horizon the outflux across \( CH \) is
\[
(T_{UU})_{\text{star}} = T_{U+U+} \left( \frac{dU_+}{dU} \right)^2 \sim (-U)^{-2(1+\kappa_+/\kappa_0)}, \quad U \to -\infty. \tag{30}
\]
As we shall see, the outflux due to the star has a negligible effect compared to the backscattering of the incoming radiation.

Consider the evolution of a massless spherically symmetric scalar field in the black hole interior. The characteristic initial value problem is completely specified.
by data given on the event horizon. The physical initial data are determined by the Price power law wave tail \( v^{-p} \).

For reference purposes, consider the evolution of a scalar field in a fixed Reissner-Nordström background with mass \( m_0 \). The far right hand side of Fig. 2 (with all fluxes turned off), describes the static Reissner-Nordström solution. It is distinguished by an outer barrier. Since the actual scattering potential will be strongly blueshifted. It is important to note that the radiation is scattered long before it reaches the large blueshift zone. This motivates our treatment of the evolution of fields as a scattering problem on a static Reissner-Nordström background.

Mathematically, scattering of a massless field is given by \( \Psi = 0 \), which using the usual advanced/retarded coordinates is

\[
\phi_{uv} = V(r)\phi, \quad \Psi = \phi/r
\]

\[V(r) = \frac{f_s(r)}{4r} \frac{d}{dr} f_s(r)\]  \hfill (31)

where \( f \) has been defined in (22) and the subscript \( s \) denotes the static Reissner-Nordström case with mass \( m_0 \).

This is a one-dimensional scattering problem. It is greatly simplified by the fact that the potential \( V(r) \) is highly localized near \( e^2/m_0 \). It falls off exponentially in the tortoise coordinate defined by \( dx = dr/f_s(r) \) near the event horizon \( x = -\infty \) and the Cauchy horizon \( x = \infty \). A scalar field will propagate freely near the event and Cauchy horizons and will only strongly interact with the curvature in the thin belt around \( e^2/m_0 \).

At the horizons the scalar field solutions will be of the form of ingoing and outgoing waves \( e^{-i\omega v} \) and \( e^{-i\omega u} \). The effect of the potential will be to alter the amplitudes by the reflection and transmission coefficients, \( R(\omega) \) and \( T(\omega) \).

If the initial value on the event horizon is \( \phi_0(v) \) then its Fourier transform

\[
\tilde{\phi}_0(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_0(v)e^{i\omega v}dv
\]  \hfill (32)

allows us to write the form of the scattered waves as \( X(v) + Y(u) \) where

\[
X(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}_0(\omega)t(\omega)e^{-i\omega v}d\omega
\]

\[
Y(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}_0(\omega)r(\omega)e^{i\omega u}d\omega. \]  \hfill (33)

The initial conditions are \( \phi_0(v) = (\kappa_0 v)^{-p/2}\Theta(v - v_1) \) where the influx is assumed to start after \( v_1 \). The Fourier transform behaves as

\[
\tilde{\phi}_0(\omega) \sim \omega^{p/2-1}. \]  \hfill (34)

This can be used to calculate the transmitted and reflected flux. The stationary phase approximation can be used to evaluate the integrals (33). For large \( v \) the transmitted flux has the form

\[
X(v) \sim t(\omega_0)(k_0 v)^{-p/2}, \]

\[
\omega_0 = -ip/2v
\]

\[
t(\omega_0) \sim \frac{1}{v} \]  \hfill (35)

and the reflected outflux is for large negative \( u \)

\[
Y(u) \sim r(\omega_0)(-\kappa_0 u)^{-p/2}, \]

\[
\omega_0 = -ip/2u \]

\[
r(\omega_0) \sim \text{constant}. \]  \hfill (36)

We calculated the reflection and transmission coefficients shown in (33) and (34) using a simple model for the scattering potential: a rectangular well adjacent to a rectangular barrier. Since the actual scattering potential looks approximately like a well adjacent to a thin barrier, this model captures the essential features of the potential. For low energy scattering it is expected that the perturbations will be strongly influenced by the potential so that there will be an almost total reflection. Actually, it is more appropriate to use the term refraction here since the reflected beam continues on to smaller radii.
The general effect of the Reissner-Nordström curvature is to scatter the influx $T_{uv} \sim (\kappa_0 v)^{-p}$ into a reflected flux $T_{uu} \sim \alpha (-\kappa_0 u)^{-p}$ and a transmitted flux $T_{vv} \sim \beta (\kappa_0 v)^{-p-2}$ near the Cauchy horizon, where $\alpha$ and $\beta$ are the reflection and transmission coefficients and are $O(1)$. Is this linearized scattering theory useful for our problem?

Consider the initial layers just beneath the event horizon $r = r_+ - \epsilon$. This region is far above the strongly blueshifted region, so the flux of energy is only

$$\beta r^2$$

our problem?

The null crossflow stress tensor is of the form of equation (37) with the luminosity functions given by scattering

$$L_{in}(V) = \beta (-\kappa_0 V)^{-2} (-\ln(-V))^{-q}$$
$$L_{out}(U) = \alpha (-\kappa_0 U)^{-2} (\ln(-U))^{-p}$$

where $\alpha$ and $\beta$ are dimensionless positive numbers, corresponding to the reflection and transmission coefficients respectively and $q = p + 2$.

Introduce the functions $A(U), B(V)$ defined by

$$L_{out}(U) = A''(U)$$
$$L_{in}(V) = B''(V)$$

where $' \cdot$ denotes ordinary differentiation. For $V \to 0$

$$B(V) = -\frac{\beta}{\kappa_0^2 q} (-\ln(-V))^{-q+1}$$
$$B'(V) = \frac{\beta}{\kappa_0^2} (-\ln(-V))^{-q}$$

and for $U \to -\infty$

$$A(U) = \frac{\alpha}{\kappa_0^2 (p-1)} (-\ln(-U))^{-p+1}$$
$$A'(U) = \frac{\alpha}{\kappa_0^2 (-U)} (-\ln(-U))^{-p}$$

In the corner region $V \to 0, U \to -\infty$, the functions $A$ and $B$ are small, but derivatives of $B$ with respect to $V$ diverge.

We wish to concentrate on the region after the potential barrier at early times. Before the potential barrier we expect Reissner-Nordström to be a good model. In the innermost regions we must model the effect of the infinite blueshift of the inflowing radiation.

Using the metric (20) we note that the Einstein equations allow us to write wave equations for two combinations of the metric functions, which do not include the mass function as a source term:

$$(\ln(r^{-1} e^{-2\sigma}),_{UV} = -\frac{e^{2\sigma}}{2r^2} \left( 1 - \frac{3e^2}{r^2} \right)$$
$$(r^2)_{,UV} = -e^{2\sigma} \left( 1 - \frac{e^2}{r^2} \right).$$

In order to solve the evolutionary problem, we need to find the solution in the intermediate region after the potential barrier and before the region where $r$ goes to zero. This region will be defined by $r \neq 0$. When this stipulation is made it is impossible for $r^{-1} e^{-2\sigma}$ to go to zero [41]. This means that both wave equations (41) do not have any potentially diverging source terms

5 Analytic Approximation for lightlike Crossflow

Our aim is to construct an analytic model for the black hole interior after the potential barrier. We shall first start with a null crossflow stress tensor and introduce some approximations. This can then be used as a model for what we expect to happen in the scalar field evolution.

The null crossflow stress tensor is of the form of equation (24) with the luminosity functions given by scattering

$$L_{in}(V) = \beta (-\kappa_0 V)^{-2} (-\ln(-V))^{-q}$$
$$L_{out}(U) = \alpha (-\kappa_0 U)^{-2} (\ln(-U))^{-p}$$

where $\alpha$ and $\beta$ are dimensionless positive numbers, corresponding to the reflection and transmission coefficients respectively and $q = p + 2$.

Introduce the functions $A(U), B(V)$ defined by

$$L_{out}(U) = A''(U)$$
$$L_{in}(V) = B''(V)$$

where $' \cdot$ denotes ordinary differentiation. For $V \to 0$

$$B(V) = -\frac{\beta}{\kappa_0^2 q} (-\ln(-V))^{-q+1}$$
$$B'(V) = \frac{\beta}{\kappa_0^2} (-\ln(-V))^{-q}$$

and for $U \to -\infty$

$$A(U) = \frac{\alpha}{\kappa_0^2 (p-1)} (-\ln(-U))^{-p+1}$$
$$A'(U) = \frac{\alpha}{\kappa_0^2 (-U)} (-\ln(-U))^{-p}$$

In the corner region $V \to 0, U \to -\infty$, the functions $A$ and $B$ are small, but derivatives of $B$ with respect to $V$ diverge.

We wish to concentrate on the region after the potential barrier at early times. Before the potential barrier we expect Reissner-Nordström to be a good model. In the innermost regions we must model the effect of the infinite blueshift of the inflowing radiation.

Using the metric (20) we note that the Einstein equations allow us to write wave equations for two combinations of the metric functions, which do not include the mass function as a source term:

$$(\ln(r^{-1} e^{-2\sigma}),_{UV} = -\frac{e^{2\sigma}}{2r^2} \left( 1 - \frac{3e^2}{r^2} \right)$$
$$(r^2)_{,UV} = -e^{2\sigma} \left( 1 - \frac{e^2}{r^2} \right).$$

In order to solve the evolutionary problem, we need to find the solution in the intermediate region after the potential barrier and before the region where $r$ goes to zero. This region will be defined by $r \neq 0$. When this stipulation is made it is impossible for $r^{-1} e^{-2\sigma}$ to go to zero [41]. This means that both wave equations (41) do not have any potentially diverging source terms
and both will have finite solutions. For conciseness, introduce the bounded and non-zero variables

$$\chi = r^{-1}e^{-2\sigma}, \rho = \frac{1}{2}r^2.$$ \hfill (42)

The Einstein equations can then be written as equations (44) and the null hypersurface constraint equations:

$$\partial_U(\chi \rho, U) = -\chi A''$$
$$\partial_U(\chi \rho, V) = -\chi B''.$$ \hfill (43)

The mass function obeys the wave equation

$$m_{,UV} = \chi A''B''.$$ \hfill (44)

As long as \( r \neq 0 \), we can write a solution with \( \chi \) and \( \rho \) being close to their Reissner-Nordström values plus perturbations which are small in this region. The metric functions for static Reissner-Nordström with a mass \( m_0 \) will be denoted with a subscript “s”, so that \( f_s(r_s) \) and \( \kappa_s(r_s) \) are defined by equation (42). The functions \( \rho_s \) and \( \chi_s \), and their derivatives are

$$\rho_s = \frac{1}{2} r_s^2$$
$$\rho_{s, U} = -\frac{1}{2} r_s f_s$$
$$\rho_{s, V} = -\frac{1}{2} r_s f_s$$
$$\chi_s = -2UV \frac{k_0^2}{f_s}$$

$$\chi_{s, U} = \frac{-k_0 V}{r_s^2} \left( 1 + \frac{2k_0 r_s}{f_s} (\kappa_0 - \kappa_s) \right)$$
$$\chi_{s, V} = \frac{-k_0 U}{r_s^2} \left( 1 + \frac{2k_0 r_s}{f_s} (\kappa_0 - \kappa_s) \right).$$ \hfill (45)

In the limit of the Cauchy horizon, \((UV \to 0, r_s \to r_0)\) these functions take on the limiting value

$$f_s \to -2UV$$
$$\rho_s \to \frac{1}{2} r_0^2, \rho_{s, U} \to \frac{r_0 V}{\kappa_0}, \rho_{s, V} \to \frac{r_0 U}{\kappa_0}$$
$$\chi_s \to \frac{k_0^2}{r_0}, \chi_{s, U} \to \frac{\kappa_0 V}{r_0}, \chi_{s, V} \to -\frac{\kappa_0 U}{r_0}.$$ \hfill (46)

We can now construct a solution to the Einstein equations using an iterative approach, taking the static Reissner-Nordström solution as the zeroth order solution \((\chi^{(0)} = \chi_s, \rho^{(0)} = \rho_s)\) and substituting back into the Einstein equations to find the first order correction terms. Equations (45) can be integrated to solve for \( \rho \):

$$\rho = \rho_s$$
$$-\int_U \frac{dV'}{\chi(U', V')} \int_U^{V'} dV' \chi(U', V') B''(V')$$
$$-\int_U \frac{dU'}{\chi(U', V')} \int_U^{U'} dU' \chi(U', V') A''(U').$$ \hfill (47)

It is clear in our approximation scheme that (50) is the leading order contribution to the solution of the Einstein equations. The contribution from (41) will be of lower order.

Integration of (50) by parts gives the solution

$$\rho = \rho_s - (A + B) + \epsilon,$$ \hfill (51)

where \( \epsilon \) is

$$\epsilon = \int_U \frac{dV'}{\chi(U', V')} \int_U^{V'} dV' \chi_s A'(U', V') B'(V')$$
$$+ \int_U \frac{dU'}{\chi(U', V')} \int_U^{U'} dU' \chi_s B'(U', V') A'(U')$$
$$\sim U \int BdV + V \int AdU.$$ \hfill (52)

which is much smaller than \( A + B \) in the remote past of CH. Using this approximation in the second equation of (41) and expanding to first order in \( A \) and \( B \), allows the estimation

$$\chi = \chi_s + O\left(\frac{1}{r_0^2}(A + B)\right).$$ \hfill (53)

Substitution of this order of correction back into (50) yields a second order approximation

$$\rho = \rho_s - A(1 + O(\ln(-U)^{-p}))$$
$$- B(1 + O(\ln(-V)^{-q})).$$ \hfill (54)

Clearly, in the corner region where \((\ln(-U))^{-1} \sim (\ln(-V))^{-1} \sim 0\), \( \rho \) is well approximated by the leading order solution (51).

To linear order in \( A \) and \( B \), the mass function can be integrated from (44)

$$m(U, V) = \chi_s A'B'(1 + O(A + B)) + m_{in}' + m_{out}' - m_0$$ \hfill (55)

which in the limit \( V \to 0 \) is

$$m \sim \frac{\alpha \beta}{r_0^3 UV} (\ln(-U))^{-p} (\ln(-V))^{-q}$$ \hfill (56)

showing the usual \( 1/V \) inflation found in earlier work (6).
The solutions for the original metric functions $r$ and $\sigma$ are
\[
\begin{align*}
  r &= r_s - \frac{(A + B)}{rs} + \frac{2AB}{rs^3} + O(A^2 + B^2) \\
  \sigma &= \sigma_s + \frac{(A + B)}{2\nu^2} + \frac{AB}{r_s^4} + O(A^2 + B^2).
\end{align*}
\]

This approximation is not so good as the scattering surface is approached ($UV \to 1$, so that correction terms \[2\] are comparable to the first order terms in \[51\]). We already know that the solution near the scattering surface should be approximately described by the Reissner-Nordström solution. It is only after this region, deep into the blueshift region that an approximate solution is needed and this is where it is important that the solution be accurate. The solution that we have found is accurate where it matters, close to the Cauchy horizon.

**6 The Scalar Field Solution**

Using approximations similar to those just discussed for lightlike crossflow, we can develop an approximate analytic solution for the scalar field equations. As before the physics tells us that the interior solution can be approximated well by the static Reissner-Nordström solution from the event horizon, down until the scattering surface.

The Einstein equations for coupling to a massless scalar field are
\[
\begin{align*}
  r\phi_{,UV} &= -r_{,U}\phi_{,V} - r_{,V}\phi_{,U} \\
  \rho_{,UV} &= -\frac{2}{r\chi}(1 - \frac{\epsilon^2}{r^2}) \\
  (\ln(\chi))_{,UV} &= 8\pi\phi_{,U}\phi_{,V} - \frac{1}{r^3\chi}(1 - \frac{3\epsilon^2}{r^2}) \\
  (\chi\rho)_{,U} &= -8\pi\rho\chi\phi_{,U} \\
  (\chi\rho)_{,V} &= -8\pi\rho\chi\phi_{,V} \\
  m_{,UV} &= 16\pi^2r^4\phi_{,U}\phi_{,V} - 4\pi rf\phi_{,U}\phi_{,V}.
\end{align*}
\]

Define functions $a(U)$ and $b(V)$ by setting their derivatives equal to
\[
\begin{align*}
  a'(U) &\equiv \phi_{,U}|_b, \quad b'(V) \equiv \phi_{,V}|_b
\end{align*}
\]
where the subscript $b$ refers to the value of the scalar field given by scattering at the underside of the potential barrier.

As before, the wave equations for $\rho$ and $\chi$ have solutions which are finite and non-zero in the corner region as long as $\phi$ does not diverge. The initial conditions given by scattering \[35\, 36\] are that the scalar field is initially regular. Near the scattering surface the radius will be close to its Reissner-Nordström value, so using the scalar wave equation and the Reissner-Nordström radius \[45\], it can be seen that the $UV$ mixed derivative of the scalar field, near the initial surface is
\[
\phi_{,UV}|_b = \frac{1}{2r_s\kappa_0}\left(\phi_{,V} + \phi_{,U}\right)|_b \\
\sim (-\ln(-V))^{-\eta/2} + (\ln(-U))^{-\eta/2}.
\]

This derivative is small in the corner ($V \to 0, U \to -\infty$), so in the earliest regions the scalar field will not be changing rapidly from its initial value. This motivates us to make the ansatz that the leading order behaviour of the scalar field, near the Cauchy horizon should be
\[
\begin{align*}
  \phi^{(0)}_{,U} &= a' = \sqrt{\frac{A'}{4\pi r_0^4}} \\
  \phi^{(0)}_{,V} &= b' = \sqrt{\frac{B'}{4\pi r_0^4}} \\
  &\sim \frac{1}{\sqrt{4\pi r_0^4\kappa_0^2}}(-U)(\ln(-U))^{-\eta/2}.
\end{align*}
\]
With this ansatz, we can see that the scalar field will be small everywhere in the corner region, but that derivatives with respect to $V$ will diverge near the Cauchy horizon.

As before we can calculate the first order correction terms by iterating the Einstein equations, again taking the zeroth order solutions for $\rho$ and $\chi$ to be the same as the Reissner-Nordström solution. The solution for $\rho$ is the same as the lightlike cross flow solution \[51\]. Substitution of \[51\] and \[51\] into the scalar wave equation yields the first order equation
\[
\phi_{,UV} = -\frac{1}{2\rho_s}((\rho_s - A')b' + (\rho_s - B')a')
\]

which can be integrated asymptotically in the corner region, making use of \[13\]
\[
\phi = a + b + \frac{1}{r_0^2}(Ab + Ba) + O(UV(a + b)).
\]

Using the first order solutions for $\phi$ and $\rho$ the wave equation for $\chi$ can be integrated
\[
\ln\chi = \ln\chi_s + 8\pi ab + O(A + B).
\]
To leading order the mass function is integrated to be
\[
m(U, V) \sim \kappa_0^2/r_0A'B'
\]
and the metric functions \( r \) and \( \sigma \) are
\[
\begin{align*}
  r &= r_s - \frac{(A + B)}{r_s} - \frac{AB}{r_s^3} + O(A^2 + B^2) \\
  \sigma &= \sigma_s - 4\pi ab + \frac{(A + B)}{2r_s^2} + \frac{AB}{r_s^4} + O(A^2 + B^2).
\end{align*}
\]

The existence of the Cauchy horizon in this solution can now be tested. Substitution of the solution for \( \sigma \) into (29) gives the following asymptotic ordinate, \( U \),
\[
\lambda = \frac{U}{\kappa_0^2} (1 + O(a)), U \to -\infty. \tag{67}
\]
Condition (28) for the scalar field solution now reads
\[
\lim_{\lambda \to -\infty} \lambda^2 T_{\lambda\lambda} = \lim_{\nu \to -\infty} \frac{(\ln(-U))^{-p+2}}{4\pi^2 r_0^6} \left( 1 + O(\ln(-U))^{-p+1} \right) = 0. \tag{68}
\]
Since condition (28) is satisfied, the Cauchy horizon exists in our approximate solution to the Einstein-scalar field equations. This is of course evident directly from the asymptotic form of the metric given by (20) and (66).

### 7 Conclusions

We have found an approximate solution to the scalar field equations in the black hole interior which includes a complete past segment of the Cauchy horizon and contains the requisite number of arbitrary functions to be considered “general”. In our picture (see Fig. 2) the Cauchy horizon will be focussed by the outgoing flux, forcing it to eventually contract to \( r = 0 \), forming a singularity which is possibly spacelike. But we find that the \( r = 0 \) singularity does not precede the Cauchy horizon in the earliest region of the interior.

This is not in agreement with a previous numerical study \[11\]. That work consisted of two simulations, in the first of which \[11\] initial conditions were placed on the event horizon. The result of this simulation was that the Cauchy horizon survived the introduction of a scalar field. In the second study, initial conditions were set outside the event horizon, which in effect allows tails to form. These results (see their Fig. 4) show that any null portion is abbreviated or perhaps completely absent. It is difficult to judge from their evidence, but the authors state that the Cauchy horizon is destroyed and replaced by a spacelike \( r = 0 \) singularity. Further numerical work on this problem is in progress \[20\], which should help clear up the discrepancy in results.

### 8 Acknowledgments

We would like to thank Pat Brady and Eric Poisson for stimulating discussions. This work was supported by the Natural Sciences and Engineering Research Council of Canada. S.M.M. wishes to acknowledge an Avadh Bhatia Fellowship.

### References

[1] R.M. Wald, *General Relativity*, (University of Chicago Press, Chicago, 1984), pp. 322-324.
[2] R.H. Price, Phys. Rev. D 5, 2419 (1972); D 5, 2439 (1972).
[3] V.A. Belinskii, E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. 19, 525 (1970); C.W. Misner, Phys. Rev. Lett. 22, 1071 (1969).
[4] R. Penrose, in *Battelle Rencontres*, ed.s C.M. De Witt and J.A. Wheeler, (W. A. Benjamin, New York, 1968), p. 222.
[5] G.F.R. Ellis and A.R. King, Commun. Math. Phys. 38, 119 (1974). A.R. King, Phys. Rev. D 11, 763 (1975).
[6] S. Chandrasekhar and J.B. Hartle, Proc. R. Soc. London A284, 301 (1982); Y. Gursel, V.D. Sandberg, I.D. Novikov, and A.A. Starobinsky, Phys. Rev. D19, 413 (1979); Y. Gursel, I.D. Novikov, V.D. Sandberg, and A.A. Starobinsky, Phys. Rev. D 20, 1260 (1979).
[7] E. Poisson and W. Israel, Phys. Rev. D41, 1796, (1990).
[8] A. Ori, Phys. Rev. Lett. 67, 781 (1991).
[9] A. Ori, Phys. Rev. Lett. 68, 2117 (1992); S. Droz, *Lake Louise Winter Institute Proceedings, 1994*, to be published; A. Bonanno, S. Droz, W. Israel and S.M. Morsink, Can. J. Phys., to be published; P.R. Brady, S. Droz, W. Israel and S.M. Morsink, paper in preparation.
[10] U. Yurtsever, Class. Quantum Grav. 10, L17 (1993).
[11] M.L. Gnedin and N.Y. Gnedin, Class. Quantum Grav. 10, 1083 (1993).
[12] A. Bonanno, S. Droz, W. Israel and S.M. Morsink, Phys. Rev. D, to be published.

[13] C. Gundlach, R.H. Price and J. Pullin, Phys. Rev. D49, 883 (1994).

[14] C. Gundlach, R.H. Price and J. Pullin, Phys. Rev. D49, 890 (1994).

[15] T. Dray and G. ’t Hooft, Commun. Math. Phys. 99, 613 (1985).

[16] C. Barrabès, W. Israel and E. Poisson, Class. Quantum Grav. 7, L273 (1990).

[17] W.G. Anderson, P.R. Brady, W. Israel, and S.M. Morsink, Phys. Rev. Lett. 70, 1041 (1993).

[18] R.A. Isaacson, Phys. Rev. 160, 1263 (1968); 160, 1272 (1968).

[19] N.Y. Gnedin and M.L. Gnedin, Sov. Astron. 36, 216, (1992).

[20] M.W. Choptuik, personal communication; C. Gundlach, personal communication.