GLOBAL STABILITY OF A MULTI-GROUP MODEL WITH GENERALIZED NONLINEAR INCIDENCE AND VACCINATION AGE

JINHU XU AND YICANG ZHOU *

School of Mathematics and Statistics
Xi’an Jiaotong University
Xi’an 710049, China

(Communicated by Pierre Magal)

ABSTRACT. A multi-group epidemic model with general nonlinear incidence and vaccination age structure has been formulated and studied. Mathematical analysis shows that the global stability of disease-free equilibrium and endemic equilibrium of the model are determined by the basic reproduction number $R_0$: the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$, the endemic equilibrium is globally asymptotically stable if $R_0 > 1$. The Lyapunov functionals for the global dynamics of the multi-group model are constructed by applying the theory of non-negative matrices and a novel grouping technique in estimating the derivative.

1. Introduction. Vaccination is one of the commonly used control measures to prevent and reduce the transmission of infectious diseases. The eradication of smallpox, which was last seen in a natural case in 1977, has been considered as the most spectacular success of vaccination. Some vaccines can offer lifelong immunity with only one dose, while others require boosters in order to maintain immunity since the acquired immunity varies with different vaccines and vaccination strategies. For example, the immunization period of hepatitis B vaccine is about five years.

It is natural to consider the vaccination and the waning immunity in modeling a disease dynamics. Li et al. [1] has investigated the global dynamics of an epidemic model with vaccination for newborns and susceptibles. Blower and McLean [2] have argued that a mass vaccination campaign may increase the severity of disease, if the vaccination is applied to only 50% of the population and the vaccine efficacy is 60%. Xiao et al. [3] assumed that the vaccinated individuals can be infected at a reduced rate compare to the susceptibles. Other mathematical models on vaccination have been studied in [4, 5, 6].

Although waning immunity has been included in several models [1, 3, 5], it was assumed that the rate of the immunity loss is a constant. A better assumption on the waning immunity is that the protection immunity depends on the vaccination.

2010 Mathematics Subject Classification. Primary: 92D25, 92D30; Secondary: 35B35, 37B25.

Key words and phrases. Multi-group, nonlinear incidence, vaccination age, Lyapunov functionals.

This work was supported by the National Natural Science Foundation of China (#11171267, #11301314, #11501443), and by a grant (# 104519-010) from the International Development Research Center, Ottawa, Canada.

* Corresponding author: Yicang Zhou, zhouyc@mail.xjtu.edu.cn.
age of an individual (the time from the vaccination). The epidemiological models with vaccination age structure can be a suitable choice to describe the dynamics of an infectious diseases with waning immunity after vaccination.

The mathematical models with the chronological age, disease age, and vaccination age have been widely used to describe the impact of the age on the disease evolution [7, 8, 9, 10, 11]. Iannelli et al. [9] have studied an epidemic model with vaccination age by assuming the immunity decreases with the time after vaccination. Li et al. [10] have proposed an epidemic model with vaccination age and treatment to show that backward bifurcation occurs due to a piecewise treatment function. Duan et al. [11] has simplified the model [10] by assuming no treatment and have obtained the global stabilities of the disease-free equilibrium and the endemic equilibrium. Motivated by [10, 11], we formulate and study a class multi-group epidemic models with the latent class and vaccination age. The total population is classified into five epidemiological compartments, the susceptible compartment, the latent compartment, the infected compartment, the removed compartment and the vaccinated compartment.

Let \( S(t), E(t), I(t), \) and \( R(t) \) be the numbers of the individuals in the susceptible, latent, infected and removed compartments at time \( t \), respectively. Let \( v(\theta, t) \) be the vaccination age density of the individuals in the vaccinated compartment at time \( t \), i.e., \( \int_{0}^{\infty} v(\theta, t) \, d\theta \) is the number of the individuals in the vaccinated compartment at time \( t \). In our model, it is assumed that the incidence rate is of the form \( \beta S f(I) \), where \( \beta \) is the transmission rate, and \( f(I) \) is the disease incidence function satisfying

\[
f(0) = 0, \quad f'(I) > 0, \quad f''(I) \leq 0. \tag{1}\]

It is easy to see that the incidence rate, \( \beta S f(I) \), includes the bilinear form \( (\beta SI) \) and the saturation form \( \frac{\beta S I}{1 + \rho I} \).

Following the compartment modeling approach, we construct the SVEIR model with vaccination age to describe the disease dynamics.

\[
\begin{align*}
S'(t) &= \Lambda - (\mu + \xi)S - \beta S f(I) + \int_{0}^{\infty} \alpha(\theta) v(\theta, t) \, d\theta, \\
\left( \frac{\partial}{\partial \theta} + \frac{\partial}{\partial t} \right) v(\theta, t) &= - (\mu + \alpha(\theta)) v(\theta, t), \\
E'(t) &= \beta S f(I) - (\mu + \delta) E, \\
I'(t) &= \delta E - (\mu + \gamma) I, \\
R'(t) &= (1 - p) \gamma I - \mu R, \\
v(0, t) &= \xi S(t), \quad v(\theta, 0) = v_{0}(\theta), \\
S(0) &= S_{0}, \quad E(0) = E_{0}, \quad I(0) = I_{0}, \quad R(0) = R_{0}.
\end{align*}
\]

The vaccinated compartment is structured by the vaccination age \( \theta \), and it is assumed that the newly vaccinated individuals enter the vaccinated class \( v(\theta, t) \) with vaccination age zero. \( \alpha(\theta) \) is the vaccine wane rate, and it is a nonnegative, bounded and continuous function of \( \theta \). For two given vaccination ages \( \theta_{1}, \theta_{2} \), \((0 \leq \theta_{1} \leq \theta_{2} \leq +\infty)\), the number of the vaccinated individuals with age of vaccination \( \theta \) between \( \theta_{1} \) and \( \theta_{2} \) at time \( t \) is \( \int_{\theta_{1}}^{\theta_{2}} v(\theta, t) \, d\theta \). The immunity lose rate (the number of individuals moving from the vaccinated class into the susceptible class due to the waning immunity) at time \( t \) is \( \int_{0}^{\infty} \alpha(\theta) v(\theta, t) \, d\theta \).

The parameters \( \Lambda, \beta, \mu, \xi \) and \( \delta \) are the recruitment rate of the susceptible class, the force of infection per contact per unit time, the mortality of individuals, the rate of vaccination of the susceptible individuals and the rate at which exposed individuals become infectious, respectively. \((1 - p) \gamma \) and \( p \gamma \) \((0 \leq p < 1)\) are the
per capital recovery rate and per capital death rate due to the disease, respectively. \( p = 0 \) implies that there is no disease induced death.

The main purpose of this paper is to study the global dynamics of the basic model (2) and the extended multi-group model (24). Organization of the paper is as follows: Preliminary results for the single group model (2) is presented in Section 2. The global stability of the equilibria for the single group model is proved in Section 3. The global stability of equilibria for the extended multi-group model is established in Section 4. A brief summary is given in the last Section.

2. Preliminaries. Let us define the state space

\[ X = \mathbb{R} \times \mathbb{R} \times L^1(0, \infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \]

equipped with norm

\[
\| (x, \varphi, y, z, u) \|_X := |x| + \int_0^\infty |\varphi(a)|da + |y| + |z| + |u|, \quad x, y, z, u \in \mathbb{R}, \varphi \in L^1(0, \infty). \quad (3)
\]

The initial condition for system (2) can be represented as

\[
X_0 := (S(0), v(0, \cdot), E(0), I(0), R(0)) = (S_0, v_0(\cdot), E_0, I_0, R_0) \in X_+, \quad (4)
\]

where

\[ X_+ = \mathbb{R}^+ \times \mathbb{R}^+ \times L^1(0, \infty) \times \mathbb{R}^+ \times \mathbb{R}^+. \]

Integrating the second equation in (2) along the characteristic line \( t - \theta = \text{constant} \) (see [12]) leads to the following formula

\[
v(\theta, t) = \begin{cases} 
\xi S(t - \theta) \Gamma_0(\theta), & t > \theta \geq 0, \\
v_0(\theta - t) \frac{\Gamma_0(\theta)}{\Gamma_0(\theta - t)}, & \theta \geq t \geq 0,
\end{cases} \quad (5)
\]

where \( \Gamma_0(\theta) = e^{-\int_0^\theta (\mu + \alpha(\tau))d\tau} \).

In order to investigate the global stability of system (2). We first define the continuous semi-flow associated with the system. Using standard methods, similar to [13, 14, 15], we can verify the existence and uniqueness of solutions to the system (2). Moreover, we can show that all solutions with nonnegative initial conditions \( X_0 \in X_+ \) will remain nonnegative for all \( t \geq 0 \). Thus, we can obtain a continuous semi-flow \( \Phi : \mathbb{R}^+ \times X_+ \rightarrow X_+ \) defined by system (2) such that

\[
\Phi(t, X_0) := (S(t), v(\cdot, t), E(t), I(t), R(t)), \quad t \geq 0, \quad X_0 \in X_+. \quad (6)
\]

Thus

\[
\| \Phi(t, X_0) \|_X = \| \Phi(S(t), v(\cdot, t), E(t), I(t), R(t)) \|_X = S(t) + \int_0^\infty v(\theta, t)d\theta + E(t) + I(t) + R(t). \quad (7)
\]

Therefore, from equations in (2), we know that

\[
d du dt \| \Phi(t, X_0) \|_X = \Lambda - \mu \| \Phi(t, X_0) \|_X - p\gamma I(t), \quad \| \Phi(0, X_0) \|_X = \| X_0 \|_X. \]

The comparison principle implies that the following inequality holds

\[
\| \Phi(t, X_0) \|_X \leq \| X_0 \|_X e^{-\mu t} + \frac{\Lambda}{\mu} (1 - e^{-\mu t}).
\]
From the fact that \(\|\Phi(t,X_0)\|_X < \frac{\Lambda}{\mu} \) (if \(\|X_0\|_X < \frac{\Lambda}{\mu}\)).

Define the state space for system (2) by
\[
\Omega := \left\{ (x, \varphi, y, z, u) \in \mathcal{X}_+ : \|(x, \varphi, y, z, u)\|_X \leq \frac{\lambda}{\mu} \right\}. \quad (8)
\]

Thus, \(\Phi(t,X_0)\) is point dissipative and \(\Omega\) attracts all points in \(\mathcal{X}_+\). Now, we are in a position to show the positive invariance of \(\Omega\).

**Lemma 2.1.** Let \(\Phi\) and \(\Omega\) be defined by (7) and (8), respectively. \(\Omega\) is positive invariant for \(\Phi\); that is,
\[
\Phi(t,X_0) \subset \Omega, \quad \forall t \geq 0, \quad X_0 \in \Omega.
\]

As we are now concerned with the infinite dimensional Banach space \(\mathcal{X}\) including \(L^1(0,\infty)\), the issue one first faces is to verify the relative compactness of the orbit \(\{\Phi(t,X_0) : t \geq 0\}\) in \(\mathcal{X}\) in order to make use of the invariance principle. To this end, we first decompose \(\Phi : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+\) into the following two operators \(\Theta, \Psi : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+\):

\[
\Theta(t,X_0) := (0,\tilde{\varphi}(\cdot,t),0,0,0), \quad (9)
\]
\[
\Psi(t,X_0) := (S(t),\tilde{v}(\cdot,t),E(t),I(t),R(t)), \quad (10)
\]

where
\[
\tilde{\varphi}(\theta,t) = \begin{cases} 0, & t > \theta \geq 0, \\
v(\theta,t), & \theta \geq t \geq 0, \end{cases}
\quad \text{and} \quad \tilde{v}(\theta,t) = \begin{cases} v(\theta,t), & t > \theta \geq 0, \\
0, & \theta \geq t \geq 0. \end{cases} \quad (11)
\]

It is easy to see that
\[
\Phi(t,X_0) = \Theta(t,X_0) + \Psi(t,X_0), \quad \forall t \geq 0,
\]
and from Proposition 3.13 in [12] and Lemma 2.1, we arrive at the following lemma.

**Lemma 2.2.** Let \(\Phi, \Omega, \Theta\) and \(\Psi\) be defined by (6), (8), (9) and (10), respectively. If the following two conditions hold, then \(\{\Phi(t,X_0) : t \geq 0\}\) for \(X_0 \in \Omega\) has compact closure in \(\mathcal{X}\).

(i) There exists a function \(\Delta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\) such that for any \(r > 0\),
\[
\lim_{t \to \infty} \Delta(t,r) = 0, \quad \text{and if} \quad X_0 \in \Omega \quad \text{with} \quad \|X_0\|_X \leq r, \quad \text{then} \quad \|\Theta(t,X_0)\|_X \leq \Delta(t,r)
\]
for \(t \geq 0\);

(ii) For \(t \geq 0\), \(\Psi(t,\cdot)\) maps any bounded sets of \(\Omega\) into sets with compact closure in \(\mathcal{X}\).

To show that conditions (i) and (ii) in Lemma 2.2 hold, we first prove the following lemma.

**Lemma 2.3.** Let \(\Omega\) and \(\Theta\) be defined by (8) and (9), respectively. For \(r > 0\), let \(\Delta(t,r) = e^{-\mu t} r\). Then,
\[
\lim_{t \to \infty} \Delta(t,r) = 0 \quad \text{and for} \quad t \geq 0, \quad \|\Theta(t,X_0)\|_X \leq \Delta(t,r)
\]
provided \(X_0 \in \Omega\) with \(\|X_0\|_X \leq r\).

**Proof.** \(\lim_{t \to \infty} \Delta(t,r) = 0\) is obvious. From (5), we have
\[
\tilde{\varphi}(\theta,t) = \begin{cases} 0, & t > \theta \geq 0, \\
v_0(\theta-t) - \frac{\Gamma(\theta)}{\Gamma(\theta-t)} \Gamma_0(\theta-t), & \theta \geq t \geq 0.
\end{cases}
\]
Then, for \(X_0 \in \Omega\) satisfying \(\|X_0\|_X \leq r\), we have
\[
\|\Theta(t, X_0)\|_X = |0| + \int_0^\infty |\hat{v}(\theta, t)|d\theta + |0| + |0| \\
= \int_t^\infty |v_0(\theta - t)| \frac{\Gamma_0(\theta)}{\Gamma_0(\theta - t)} |d\theta \\
\leq e^{-\mu t} \int_0^\infty |v_0(\theta)|d\theta \\
\leq e^{-\mu t} \|X_0\|_X \\
\leq e^{-\mu t} r = \Delta(t, r), \quad \forall t \geq 0,
\]
which completes the proof. \(\square\)

**Lemma 2.4.** Let \(\Omega\) and \(\Psi\) be defined by (8) and (10), respectively. Then, for \(t \geq 0\), \(\Psi(t, \cdot)\) maps any bounded sets of \(\Omega\) into sets with compact closure in \(X\).

**Proof.** It follows from Lemma 2.1 that \(S(t), E(t), I(t)\) and \(R(t)\) remain in the compact set \([0, \frac{1}{\mu}]\). Thus, we are in a position to show that \(\hat{v}(\theta, t)\) remains in a precompact subset of \(L^+ (0, \infty)\), which is independent of \(X_0 \in \Omega\). To this end, it suffices to verify the following conditions (see e.g., Theorem B.2 in [16]).

(i) The supremum of \(\int_0^\infty \hat{v}(\theta, t)d\theta\) with respect to \(X_0 \in \Omega\) is finite;
(ii) \(\lim_{h \to 0^+} \int_h^\infty \hat{v}(\theta, t)d\theta = 0\) uniformly with respect to \(X_0 \in \Omega\);
(iii) \(\lim_{h \to 0^+} \int_h^\infty |\hat{v}(\theta + h, t) - \hat{v}(\theta, t)|d\theta = 0\) uniformly with respect to \(X_0 \in \Omega\);
(iv) \(\lim_{h \to 0^+} \int_0^h \hat{v}(\theta, t)d\theta = 0\) uniformly with respect to \(X_0 \in \Omega\);

In fact, from (5) and (11) it follows that
\[
\hat{v}(\theta, t) := \begin{cases} 
\xi S(t - \theta)e^{-\int_0^\theta (\mu + \alpha(\tau))d\tau}, & t > \theta \geq 0, \\
0, & \theta \geq t \geq 0,
\end{cases} \tag{12}
\]
and hence, from Lemma 2.1 we obtain the inequality
\[
0 \leq \hat{v}(\theta, t) \leq \frac{\Lambda}{\mu} e^{-\int_0^\theta (\mu + \alpha(\tau))d\tau}. \tag{13}
\]
Thus, the aforementioned condition (i), (ii) and (iv) follow immediately from (13).

We claim that (iii) holds. In fact, for sufficiently small \(h \in (0, t)\), we have
\[
\int_0^\infty |\hat{v}(\theta + h, t) - \hat{v}(\theta, t)|d\theta \\
= \int_0^{t-h} |\xi S(t - \theta - h)\Gamma_0(\theta + h) - \xi S(t - \theta)\Gamma_0(\theta)|d\theta \\
+ \int_{t-h}^t |0 - \xi S(t - \theta)\Gamma_0(\theta)|d\theta \\
\leq \int_0^{t-h} \xi S(t - \theta - h)\Gamma_0(\theta + h) - \Gamma_0(\theta)|d\theta \\
+ \int_{t-h}^t \xi \Gamma_0(\theta)|S(t - \theta - h) - S(t - \theta)|d\theta + \frac{\Lambda}{\mu} h
\]
\[ \leq \frac{\xi}{\mu} \int_{0}^{t-h} |(\Gamma_0(\theta + h) - \Gamma_0(\theta))| d\theta \\
+ \int_{0}^{t-h} \xi \Gamma_0(\theta) |S(t - \theta - h) - S(t - \theta)| d\theta + \frac{\Lambda}{\mu} h. \]  
(14)

Note that
\[ \int_{0}^{t-h} |(\Gamma_0(\theta + h) - \Gamma_0(\theta))| d\theta = \int_{0}^{t-h} (\Gamma_0(\theta) - \Gamma_0(\theta + h)) d\theta \]
\[ = \int_{0}^{t-h} \Gamma_0(\theta) d\theta - \int_{h}^{t-h} \Gamma_0(\theta + h) d\theta \]
\[ = \int_{0}^{t-h} \Gamma_0(\theta) d\theta - \int_{h}^{t} \Gamma_0(\theta) d\theta - \int_{t-h}^{t} \Gamma_0(\theta) d\theta \]
\[ = \int_{0}^{h} \Gamma_0(\theta) d\theta - \int_{t-h}^{t} \Gamma_0(\theta) d\theta \]
\[ \leq h. \]  
(15)

Now, the Lipschitz continuity of \( S(\cdot) \) on \( \mathbb{R}_+ \) is easily verified from (2) and the boundedness of the solution (Lemma 2.1). Therefore, there exists a positive constant \( M_S > 0 \) such that
\[ |S(t - \theta - h) - S(t - \theta)| \leq M_S h. \]

From (14) and (15), we obtain
\[ \int_{0}^{\infty} |\tilde{v}(\theta + h, t) - \tilde{v}(\theta, t)| d\theta \]
\[ \leq \frac{\xi}{\mu} \int_{0}^{t-h} |\Gamma_0(\theta + h) - \Gamma_0(\theta)| d\theta \\
+ \int_{0}^{t-h} \xi \Gamma_0(\theta) |S(t - \theta - h) - S(t - \theta)| d\theta + \frac{\Lambda}{\mu} h \]
\[ \leq 2\frac{\xi}{\mu} h + \frac{\xi \Lambda}{\mu} h \int_{0}^{t-h} \Gamma_0(\theta) d\theta + \frac{\xi}{\mu} M_S h \]
\[ \leq \frac{\Lambda}{\mu} h + \frac{\xi}{\mu} M_S h. \]

Since this upper bound is independent of \( X_0 \in \Omega \) and converges to 0 as \( h \to 0_+ \), which implies that \( \tilde{v}(\theta, t) \) remains in a precompact subset of \( L^1_+ \) independent of \( X_0 \). Thus, condition (iii) holds. This completes the proof.

In summary, from Lemmas 2.2–2.4, we have proved the following result on the relative compactness of the orbit \( \{\Phi(t, X_0) : t \geq 0\} \).

**Theorem 2.5.** Let \( \Phi \) and \( \Omega \) be defined by (6) and (8), respectively. For \( X_0 \in \Omega \), \( \{\Phi(t, X_0) : t \geq 0\} \) has compact closure in \( \mathcal{X} \).

From (5) and applying the same method of [11], we obtain that the first equation of system (2) becomes
\[ S'(t) = \Lambda - (\mu + \xi) S - \beta S f(I) + \int_{0}^{\infty} S(t - \theta) \Gamma(\theta) d\theta, \]  
(16)
where \( \Gamma(\theta) = \xi \alpha(\theta) \Gamma_0(\theta) \).
In order to make the calculation simpler, we use the notations \( \Gamma \) equivalent to that of the following model explicitly in the first four equations. Thus, the dynamical behavior of model (2) is equivalent to that of the following model

\[
\begin{align*}
S'(t) &= \Lambda - (\mu + \xi)S - \beta Sf(I) + \int_0^\infty \Gamma(\theta)S(t - \theta)d\theta, \\
E'(t) &= \beta Sf(I) - (\mu + \delta)E, \\
I'(t) &= \delta E - (\mu + \gamma)I.
\end{align*}
\]  

(17)

For the reduced system (17), from [14], we consider the weighted space of continuous functions

\[
X := C_{\eta}((\infty, 0], \mathbb{R}^3) = \{ \phi \in C((\infty, 0], \mathbb{R}^3) : \sup_{\theta \leq 0} e^{\eta \theta}||\phi(\theta)|| < +\infty \}
\]

which is a Banach space endowed with the norm \( ||\phi||_{\eta} := \sup_{\theta \leq 0} e^{\eta \theta}||\phi(\theta)|| < +\infty \), and let \( \phi_1 \in C_{\eta} \) be such that \( \phi_1(s) = \phi(t + s), s \in (-\infty, 0] \) and \( \eta \) is a positive constant. Any initial condition \((S_0, E_0, I_0) \in X\) gives a solution \((S_t, E_t, I_t)\) that remains in the phase space for all time. Moreover, if \((S(t), E(t), I(t))\) is bounded for \( t \geq 0 \), then the positive orbit \( \Gamma_\ast = \{(S_t, E_t, I_t) : t \geq 0\}\) has compact closure in \( X \). According to the similar arguments as Lemma 2.1 in [17], it is not difficult to prove that if \((S(t), E(t), I(t))\) be the unique solution to model (17) with initial condition defined above and that \( S(t), E(t), I(t) \) an positive for all \( t \geq 0 \). Since the right hand side of system (17) is continuous differentiable and the system is positive invariant. Thus, the compactness of the orbit of system (17) holds. Furthermore, we can obtain the invariant region of the reduced system (17)

\[
\hat{\Omega} = \{(S, E, I) : S + E + I \leq \frac{\Lambda}{\mu}\}.
\]

In the rest of this Section we will focus on the dynamics of the reduced system (17). In order to make the calculation simpler, we use the notations \( \Gamma_0 \) and \( \Gamma \), where

\[
\Gamma_0 = \int_0^\infty \Gamma_0(\theta)d\theta, \quad \Gamma = \int_0^\infty \Gamma(\theta)d\theta = \xi(1 - \mu \Gamma_0).
\]

The biological interpretation of \( \Gamma_0 \) is the average time that an individual spends in the vaccinated compartment. Death and immunity loss are two ways for an individual to leave the vaccinated compartment. \( \mu \Gamma_0 \) or \( 1 - \mu \Gamma \) is the proportion that an individual moves out the vaccinated compartment because of death or the immunity loss, respectively. \( \xi(1 - \mu \Gamma_0) \) is the per capita rate at which individuals enter the susceptible compartment from the vaccinated compartment due to the immunity loss.

Obviously, model (17) always has a disease-free equilibrium \( P^0 = \left( \frac{\Lambda}{\mu(1 + \xi \Gamma_0)}, 0, 0 \right) \). Any positive equilibrium \((S^*, E^*, I^*)\) of model (17) satisfies the system of equations

\[
\begin{align*}
\Lambda - \beta S^*f(I^*) - (\mu + \xi)S^* + S^*\Gamma &= 0, \\
\beta S^*f(I^*) - (\mu + \delta)E^* &= 0, \\
\delta E^* - (\mu + \gamma)I^* &= 0.
\end{align*}
\]

(18)

The further calculation yields

\[
S^* = \frac{(\mu + \gamma)(\mu + \delta)I^*}{\delta \beta f(I^*)}, \quad E^* = \frac{(\mu + \gamma)I^*}{\delta}.
\]
where, $I^*$ is the positive root of the following equation
\[
g(I) = \Lambda - \frac{(\mu + \gamma)(\mu + \delta)I}{\delta} - \frac{\mu(1 + \xi \Gamma_0)(\mu + \gamma)(\mu + \delta)I}{\delta \beta f(I)}.
\]  
(19)

The derivative of $g(I)$ is
\[
g'(I) = -\frac{(\mu + \gamma)(\mu + \delta)}{\delta} - \frac{\mu(1 + \xi \Gamma_0)(\mu + \gamma)(\mu + \delta)(f(I) - I f'(I))}{\delta \beta f^2(I)}.
\]  
(20)

From the conditions in (1) we have $f(I) - I f'(I) \geq 0$, which leads to $g'(I) < 0$ for all $I > 0$. From equation (19) we have
\[
g(0) = \lim_{I \to 0} g(I) = \frac{\mu(1 + \xi \Gamma_0)(\mu + \gamma)(\mu + \delta)}{\delta \beta f'(0)} (R_0 - 1),
\]
where
\[
R_0 = \frac{\Lambda \delta \beta f'(0)}{\mu(1 + \xi \Gamma_0)(\mu + \gamma)(\mu + \delta)}.
\]

If $R_0 > 1$, then model (17) has a unique endemic equilibrium $P^* = (S^*, E^*, I^*)$. The steady state for the distribution of vaccinated individuals corresponding to $P^0$ and $P^*$ are $\psi^0(\theta) = \xi S^0 \Gamma_0(\theta)$ and $\psi^*(\theta) = \xi S^* \Gamma_0(\theta)$, respectively.

**Theorem 2.6.** Model (17) has the disease-free equilibrium $P^0$ if $R_0 < 1$. Model (17) has the disease-free equilibrium $P^0$ and an endemic equilibrium $P^*$ if $R_0 > 1$.

3. Stability analysis of equilibria. In this section, we investigate the stability of the disease-free equilibrium $P^0$ and the endemic equilibrium $P^*$ of model (17).

**Theorem 3.1.** The disease-free equilibrium $P^0$ of model (17) is stable if $R_0 < 1$ and is unstable if $R_0 > 1$.

**Proof.** The characteristic equation of the linearization of system (17) at $P^0$ is
\[
\left(\lambda + \mu + \xi - \int_0^\infty e^{-\lambda \theta} \Gamma(\theta) d\theta\right) \cdot H(\lambda) = 0,
\]  
(21)

where
\[
H(\lambda) = \lambda^2 + (\mu + \gamma + \mu + \delta)\lambda + (\mu + \gamma)(\mu + \delta)(1 - R_0).
\]  
(22)

It is easy to see that $H(\lambda) = 0$ has at least one positive root if $R_0 > 1$, and the disease-free equilibrium $P^0$ of model (17) is unstable. If $R_0 < 1$, then any root of $H(\lambda) = 0$ has a negative real part, and the disease-free equilibrium $P^0$ of model (17) is locally stable provided that all the roots of equation (23) have negative real parts, where
\[
\lambda + \mu + \xi - \int_0^\infty e^{-\lambda \theta} \Gamma(\theta) d\theta = 0.
\]  
(23)

If $\lambda$ is a root of (23) with $\text{Re} \lambda \geq 0$, then we have
\[
\mu + \xi \leq |\lambda + \mu + \xi| = \left|\int_0^\infty \xi \alpha(\theta) e^{-\lambda \theta} e^{-\int_0^\theta (\mu + \alpha(\tau)) d\tau} d\theta\right|
\]  
\[
< \left|\int_0^\infty \xi \alpha(\theta) e^{-\int_0^\theta \alpha(\tau) d\tau} d\theta\right| < \xi,
\]
which leads to a contradiction. We can conclude that all the roots of characteristic equation (21) have negative real parts, and the disease-free equilibrium $P^0$ of model (17) is stable.

\[\square\]
Theorem 3.2. If $R_0 < 1$, then the disease-free equilibrium $P^0$ of model (17) is globally asymptotically stable with non-negative initial data.

Proof. Let $(S(t), E(t), I(t))$ be any solution of system (17) with non-negative initial data. We consider

$$L_1 = S(t) - S^0 - S^0 \ln \frac{S(t)}{S^0} + E(t) + \frac{(\mu + \delta)I(t)}{\delta}.$$ 

Since the function $h(x) = x - 1 - \ln x$ ($x \in \mathbb{R}_+$), has the global minimum at $x = 1$ and $h(1) = 0$, we know that

$$S(t) - S^0 - S^0 \ln \frac{S(t)}{S^0} = S^0 \left( \frac{S(t)}{S^0} - 1 - \ln \frac{S(t)}{S^0} \right) \geq 0, \text{ for any } S(t) > 0$$

and $L_1(t)$ is nonnegative defined with respect to the disease-free equilibrium $P^0$, which is a global minimum.

Let

$$L = L_1 + \int_0^\infty \Gamma(\theta) \int_0^\theta \left( S(t - \tau) - S^0 - S^0 \ln \frac{S(t - \tau)}{S^0} \right) d\tau d\theta.$$ 

By using the following equality

$$\int_0^\theta \frac{\partial}{\partial \tau} \left( S(t - \tau) - S^0 - S^0 \ln \frac{S(t - \tau)}{S^0} \right) d\tau = -\int_0^\theta \frac{\partial}{\partial \tau} \left( S(t - \tau) - S^0 - S^0 \ln \frac{S(t - \tau)}{S^0} \right) d\tau,$$

we can get the derivative of $L$ along the solution of model (17)

$$\frac{dL}{dt} = \left( 1 - \frac{S^0}{S(t)} \right) \left( \Lambda - (\mu + \xi)S(t) - \beta S(t)f(I(t)) + \int_0^\infty \Gamma(\theta)S(t - \theta) d\theta \right)$$

$$+ \left( \beta S(t)f(I(t)) - (\mu + \delta)E(t) \right) + \frac{(\mu + \delta)}{\delta} \left( \delta E(t) - (\mu + \gamma)I(t) \right)$$

$$+ \Gamma \cdot S(t) - \int_0^\infty \Gamma(\theta)S(t - \theta) d\theta + \int_0^\infty \Gamma(\theta)S^0 \ln \left( \frac{S(t - \theta)}{S^0} \right) d\theta.$$ 

From the fact that $\Lambda = (\mu + \xi)S^0 - \int_0^\infty \Gamma(\theta)S^0 d\theta$ and conditions in (1), we get

$$\frac{dL}{dt} = -\frac{(\mu + \xi)(S(t) - S^0)^2}{S(t)} + \frac{\Gamma S^0(S^0 - S(t))}{S(t)} + \Gamma \cdot S(t)$$

$$+ \frac{(\mu + \delta)(\mu + \gamma)}{\delta} \left( \frac{\beta S^0 f(I(t))}{(\mu + \delta)(\mu + \gamma)I(t)} - 1 \right) I(t)$$

$$+ \int_0^\infty \Gamma(\theta)S^0 \left( \ln \frac{S(t - \theta)}{S(t)} - \frac{S(t - \theta)}{S(t)} \right) d\theta$$

$$= -\frac{(\mu + \xi)(S(t) - S^0)^2}{S(t)} + \frac{\Gamma S^0(S^0 - S(t))}{S(t)} + \Gamma \cdot (S(t) - S^0)$$

$$+ \frac{(\mu + \delta)(\mu + \gamma)}{\delta} \left( \frac{\beta S^0 f(I(t))}{(\mu + \delta)(\mu + \gamma)I(t)} - 1 \right) I(t)$$

$$+ \int_0^\infty \Gamma(\theta)S^0 \left( \ln \frac{S(t - \theta)}{S(t)} - \frac{S(t - \theta)}{S(t)} + 1 \right) d\theta$$

$$\leq -\frac{\mu(1 + \xi \Gamma_0)(S(t) - S^0)^2}{S(t)} + \frac{(\mu + \delta)(\mu + \gamma)}{\delta} \left( \frac{\beta S^0 f'(0)}{(\mu + \delta)(\mu + \gamma)} - 1 \right) I(t).$$
\[ + \int_0^\infty \Gamma(\theta)S^0 \left( \ln \frac{S(t-\theta)}{S(t)} - \frac{S(t-\theta)}{S(t)} + 1 \right) d\theta \]

\[ = - \frac{\mu(1 + \xi \Gamma_0)(S(t) - S^0)^2}{S(t)} + \frac{(\mu + \delta)(\mu + \gamma)}{\delta} (R_0 - 1) I(t) \]

\[ + \int_0^\infty \Gamma(\theta)S^0 \left( \ln \frac{S(t-\theta)}{S(t)} - \frac{S(t-\theta)}{S(t)} + 1 \right) d\theta. \]

From the inequality \( 1 + \ln \frac{S(t-\theta)}{S(t)} - \frac{S(t-\theta)}{S(t)} \leq 0 \), for \( S(t-\theta) > 0, S(t) > 0 \) with the equality holds if and only if \( S(t-\theta) = S(t) \). Thus, we know that \( \frac{dL}{dt} \leq 0 \) if \( R_0 < 1 \). The largest compact invariant set in \( \Omega = \left\{ (S, E, I) \mid \frac{dL}{dt} = 0 \right\} \) is the singleton \( \{P^0\} \). By LaSalle's invariance principle, the equilibrium \( P^0 \) of model (17) is globally asymptotically stable if \( R_0 < 1 \).

When \( R_0 > 1 \) we can obtain the global stability of the endemic equilibrium. The result is given in the following theorem.

**Theorem 3.3.** The unique endemic equilibrium \( P^* \) of model (17) is globally asymptotically stable if \( R_0 > 1 \).

**Proof.** Let \((S(t), E(t), I(t))\) be any solution of system (17) with non-negative initial data. Define the following Lyapunov functional

\[ L_2 = S(t) - S^* - S^* \ln \frac{S(t)}{S^*} + E(t) - E^* - E^* \ln \frac{E(t)}{E^*} \]

\[ + \frac{(\mu + \delta)}{\delta} \left( I(t) - I^* - \int_{I^*}^{I(t)} \frac{f(I^*)}{f(\tau)} d\tau \right) + \tilde{L}_2, \]

where

\[ \tilde{L}_2 = \int_0^\infty \Gamma(\theta) \int_0^\theta \left( S(t-\tau) - S^* - S^* \ln \frac{S(t-\tau)}{S^*} \right) d\tau d\theta. \]

By using the property of function \( h(x) = x - 1 - \ln x \ (x \in \mathbb{R}_+) \), we find that the function \( L_2(t) \) is nonnegative with its global minimum at \( P^0 \).

The similar argument as in the proof of Theorem 3.2 gives the the derivative of \( L_2 \) along the solution of model (17)

\[ \frac{dL_2}{dt} = \left( 1 - \frac{S^*}{S(t)} \right) \left( \Lambda - (\mu + \xi)S(t) - \beta S(t)f(I(t)) \right) \]

\[ + \int_0^\infty \Gamma(\theta)S(t-\theta)d\theta \right) + \left( 1 - \frac{E^*}{E(t)} \right) \left( \beta S(t)f(I(t)) - (\mu + \delta)E(t) \right) \]

\[ + \frac{(\mu + \delta)}{\delta} \left( 1 - \frac{f(I^*)}{f(I(t))} \right) \left( \delta E(t) - (\mu + \gamma)I(t) \right) \]

\[ + \Gamma S(t) - \int_0^\infty \Gamma(\theta)S(t-\theta)d\theta + \int_0^\infty \Gamma(\theta)S^* \ln \frac{S(t-\theta)}{S(t)} d\theta. \]
The equality $\Lambda = \beta S^* f(I^*) + (\mu + \xi)S^* - \int_0^\infty \Gamma(\theta)S^* d\theta$ leads to

$$\frac{dL_2}{dt} = -\frac{(\mu + \xi)(S(t) - S^*)^2}{S(t)} + \left(1 - \frac{S^*}{S(t)}\right) \int_0^\infty \Gamma(\theta)(S(t - \theta) - S^*) d\theta + (\mu + \delta)E^*$$

$$+ \frac{1}{S(t)} \left(3 - \frac{S^*}{S(t)} - \frac{E^* S f(I)}{E^* S f(I^*)} - \frac{E f(I^*)}{E^* f(I(t))}\right) - \frac{E^*}{E^*} \beta S f(I(t)) \frac{(\mu + \delta)(\mu + \gamma)I(t)}{\delta}$$

$$+ \frac{1}{S(t)} \int_0^\infty \Gamma(\theta)S^* \ln \frac{S(t - \theta)}{S(t)} d\theta.$$

From the equations

$$\beta S^* f(I^*) = (\mu + \delta)E^*, \quad \delta E^* = (\mu + \gamma)I^*,$$

we have

$$\frac{dL_2}{dt} = -\frac{(\mu + \xi)(S(t) - S^*)^2}{S(t)} + (\mu + \delta)E^* \left(3 - \frac{S^*}{S(t)} - \frac{E^* S f(I(t))}{E^* S f(I^*)}\right)$$

$$- \frac{E(t) f(I^*)}{E^* f(I(t))} + (\mu + \delta)E^* \left(\frac{f(I(t))}{f(I^*)} - \frac{I(t)}{I^*}\right) \left(1 - \frac{f(I^*)}{f(I(t))}\right)$$

$$+ \Gamma \cdot S(t) + \frac{\Gamma S^*}{S} (S^* - S(t)) + S^* \int_0^\infty \Gamma(\theta) \left(\ln \frac{S(t - \theta)}{S(t)} - \frac{S(t - \theta)}{S(t)}\right) d\theta$$

$$= \frac{\mu(1 + \xi_0)(S(t) - S^*)^2}{S(t)} + (\mu + \delta)E^* \left(3 - \frac{S^*}{S(t)} - \frac{E^* S f(I(t))}{E^* S f(I^*)}\right)$$

$$- \frac{E(t) f(I^*)}{E^* f(I(t))} + (\mu + \delta)E^* \left(\frac{f(I(t))}{f(I^*)} - \frac{I(t)}{I^*}\right) \left(1 - \frac{f(I^*)}{f(I(t))}\right)$$

$$+ S^* \int_0^\infty \Gamma(\theta) \left(1 + \ln \frac{S(t - \theta)}{S(t)} - \frac{S(t - \theta)}{S(t)}\right) d\theta.$$
From the conditions in (1), it is easy to obtain the following inequalities

\[
\frac{f(I)}{f(I^*)} \geq \frac{I}{I^*}, \quad \text{for } I \leq I^*; \quad \frac{f(I)}{f(I^*)} \leq \frac{I}{I^*}, \quad \text{for } I \geq I^*.
\]

Those two inequalities lead to

\[
\left( \frac{f(I)}{f(I^*)} - \frac{I}{I^*} \right) \left( 1 - \frac{f(I^*)}{f(I)} \right) \leq 0,
\]

with the equality holding true if and only if \( I = I^* \).

From the basic calculus we know that \( 1 + \ln x - x \leq 0 \) for \( x > 0 \), which leads to \( 1 + \ln \frac{S(t - \theta)}{S(t)} - \frac{S(t - \theta)}{S(t)} \leq 0 \), with equality holding if and only if \( S(t - \theta) = S(t) \). It follows from those inequalities that \( \frac{dL_2}{dt} \leq 0 \), and \( \frac{dL_2}{dt} = 0 \) holds if and only if \( S = S^*, E = E^*, I = I^* \). The largest compact invariant set in \( \Omega = \{(S, E, I) : \frac{dL_2}{dt} = 0\} \) is the singleton \( \{P^*\} \). By LaSalle’s invariance principle, we obtain that the equilibrium \( P^* \) of system (17) is globally asymptotically stable. This completes the proof of Theorem 3.3.

4. Multi-group model. One essential assumption in classical compartmental epidemic models is that the individuals are homogeneously mixed, and each individual has the same chance to get infected. More realistic models divide the host population into groups to consider the disease transmission in heterogeneous cases. The host population are classified into different groups according to their education levels, ethnic backgrounds, gender, age, professions, communities or geographic distributions for their diversities in disease transmission. The vital epidemic parameters varies among different population groups. For more information on multi-group epidemic models one can refer [5, 18, 19, 20, 21, 22, 23, 24, 25] and reference therein.

In this section, we formulate a multi-group epidemic model and study its dynamics. We assume that the disease can transmit within the same group and among different groups. The multi-group model has the following form,

\[
\begin{align*}
S_k'(t) &= \Lambda_k - \mu_k S_k - \sum_{j=1}^{n} \beta_{kj} S_k f_j(I_j) - \xi_k S_k + \int_0^\infty \alpha_k(\theta) v_k(\theta, t) d\theta, \\
E_k'(t) &= \sum_{j=1}^{n} \beta_{kj} S_j f_j(I_j) - (\mu_k + \delta_k) E_k, \\
I_k'(t) &= \delta_k E_k - (\mu_k + \gamma_k) I_k, \\
R_k'(t) &= (1 - p_k) \gamma_k I_k - \mu_k R_k, \quad k = 1, 2, ..., n.
\end{align*}
\]

(24)

In model (24), \( S_k, E_k, I_k, V_k \) and \( R_k \) \((k = 1, 2, ..., n)\) denote the numbers of susceptible, latent, infectious, vaccinated and recovered individuals at time \( t \) in the \( k \)-th group, respectively. \( v_k(\theta, t) \) is the age density of vaccinated individuals at time \( t \) in the \( k \)-th group. The non-negative constant \( \beta_{kj} \) is the transmission rate due to the contact of susceptible individuals in the \( k \)-th group with infectious individuals in the \( j \)-th group. The other non-negative constant parameters have the same meaning as those in model (2). The function \( f_k(I_k) \) satisfies

\[
f_k(0) = 0, \quad f_k'(I_k) > 0, \quad f_k''(I_k) \leq 0, \quad k = 1, 2, ..., n.
\]

(25)
We focus on the dynamical analysis of the reduced model (28). Once the solution of model (28) is determined, we can obtain the qualitative behavior of model (24). The relationship between the eigenvalues of the linearized matrix and the process is omitted here.

We replace the first equation in (24) by (27) and drop the variable \( R_k(t) \) does not appear explicitly in the first four equations in (24). The qualitative behavior of model (24) is equivalent to the following model which is a Banach space endowed with the norm \( \| \phi \| \). The biological interpretation of those quantities is the same as that given in Section 3.

The disease-free equilibrium of model (28) is given by the reduce system (28) in the phase space \( X = \prod_{k=1}^{n} (C_k \times C_k \times C_k) \).

Similarly, we define \( \Gamma_0 \) and \( \Gamma_k \) as \( \Gamma_0 = \int_0^\infty \Gamma_0k(\theta)d\theta \) and \( \Gamma_k = \int_0^\infty \Gamma_k(\theta)d\theta \). The biological interpretation of those quantities is the same as that given in Section 3.

The disease-free equilibrium of model (28) is

\[
P_0 = (S_1^0, 0, 0, ..., S_n^0, 0, 0)
\]

We also define the basic reproduction number

\[
R_0 = \rho(\mathcal{F} \mathcal{V}^{-1}), \quad \mathcal{F} \mathcal{V}^{-1} = \left( \frac{\delta_k \beta_{kj} S_k^0 f_j'(0)}{(\mu_k + \delta_k)(\mu_k + \gamma_k)} \right)_{n \times n}.
\]

Here \( R_0 \) is the spectrum radius of the matrix \( \mathcal{F} \mathcal{V}^{-1} \). From the biological interpretation of the basic reproduction number we have the following result.

**Theorem 4.1.** The disease-free equilibrium \( P_0 \) of model (28) is stable if \( R_0 < 1 \) while it is unstable if \( R_0 > 1 \).

The local stability of the disease-free equilibrium of model (28) comes from the relationship between the eigenvalues of the linearized matrix and \( R_0 \). The detailed process is omitted here.
Theorem 4.2. Assume that $B = (\beta_{kj})_{n \times n}$ is irreducible. If $R_0 < 1$, then the disease-free equilibrium $P_0$ of system (28) is globally asymptotically stable. If $R_0 > 1$, then system (28) is uniformly persistent.

Proof. Let $I = (I_1, ..., I_n)$, $S^0 = (S^0_1, ..., S^0_n)$, $F(I) = \left( \frac{\omega_k \kappa_k \beta_{kj} S_k f_j(I_j)}{(\mu_k + \delta_k)(\mu_k + \gamma_k)} \right)_{n \times n}$ for $k = 1, 2, ..., n$. Since $B$ is irreducible, we know that matrix $FV^{-1}$ is also irreducible, and has a positive left eigenvector $\omega = (\omega_1, ..., \omega_n)$ corresponding to the spectral radius $\rho(FV^{-1}) = R_0 > 0$. From the conditions given in (25), we have $F(I) \leq F(0) = FV^{-1}$. On the other hand, the irreducibility of $B$ implies that $F(I) + FV^{-1}$ is irreducible. It follows from the theory of nonnegative matrices (Corollary 1.5 of [26]) that $\rho(F(I)) < \rho(FV^{-1}) = R_0 \leq 1$. The vector equation $F(I)I = I$ has only the trivial solution $I = 0$. $P_0$ is the only equilibrium of model (28) in the positive orthant if $R_0 \leq 1$.

Let $c_k = \frac{\omega_k \delta_k}{(\mu_k + \delta_k)(\mu_k + \gamma_k)} > 0$, $(S_k(t), E_k(t), I_k(t))$ be any solution of system (28) with non-negative initial data and define

$$L_1 = \sum_{k=1}^{n} c_k \left\{ S_k - S^0_k - S^0_k \ln \frac{S_k}{S^0_k} + E_k + \frac{(\mu_k + \delta_k)}{\delta_k} I_k 
+ \int_{0}^{\infty} \Gamma_k(\theta) \int_{0}^{\theta} \left( S_k(t - \tau) - S^0_k - S^0_k \ln \frac{S_k(t - \tau)}{S^0_k} \right) d\tau d\theta \right\}.$$ 

Using the equation $\Lambda_k = (\mu_k + \xi_k)S^0_k - \int_{0}^{\infty} \Gamma_k(\theta)S^0_k d\theta$, we get the derivative of $L_1$ along the solution of (28)

$$\frac{dL_1}{dt} = \sum_{k=1}^{n} c_k \left\{ -\frac{(\mu_k + \xi_k)(S_k - S^0_k)^2}{S_k} + \sum_{j=1}^{n} \beta_{kj} S^0_k f_j(I_j) \n+ \int_{0}^{\infty} \Gamma_k(\theta)S^0_k \left( 1 + \ln \left( \frac{S_k(t - \theta)}{S_k} \right) - \frac{S_k(t - \theta)}{S_k} \right) d\theta \n- \frac{(\mu_k + \delta_k)(\mu_k + \gamma_k)}{\delta_k} I_k + \Gamma_k(S_k - S^0_k) + \frac{\Gamma_k S^0_k (S^0_k - S_k)}{S_k} \right\} \n= \sum_{k=1}^{n} c_k \left\{ -\frac{\mu_k (1 + \xi_k F_{ik})(S_k - S^0_k)^2}{S_k} \n+ \int_{0}^{\infty} \Gamma_k(\theta)S^0_k \left( 1 + \ln \left( \frac{S_k(t - \theta)}{S_k} \right) - \frac{S_k(t - \theta)}{S_k} \right) d\theta \n+ \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \frac{\omega_k \delta_k \beta_{kj} S^0_k f_j(I_j)}{(\mu_k + \delta_k)(\mu_k + \gamma_k)} I_j - \omega_k I_k \right) \n\leq \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \frac{\omega_k \delta_k \beta_{kj} S^0_k f_j(0)}{(\mu_k + \delta_k)(\mu_k + \gamma_k)} I_j - \omega_k I_k \right) \n= (\omega_1, ..., \omega_n)(F(0)I^T - I^T) \n= (\rho(F(0)) - 1)(\omega_1, ..., \omega_n)I^T \leq 0,$$ 

if $R_0 \leq 1$. 

990 JINHU XU AND YICANG ZHOU
Proof. For convenience of notations, define the endemic equilibrium $P_0$. By LaSalle’s invariance principle [27], the equilibrium $P_0$ of system (28) is globally asymptotically stable.

If $\mathcal{R}_0 = \rho(F(0)) = \rho(FV^{-1}) > 1$, and $I \neq 0$, we know that

$$\frac{dL_1}{dt} = 0 \text{ if and only if } L_1 = 0 \text{ and } S_k = S_k^0.$$ 

It follows that $\frac{dL_1}{dt} > 0$ in a small enough neighborhood of $P_0$. This implies that $P_0$ is unstable. With a uniform persistence result from [28] and a similar argument as in the proof of Proposition 3.3 of [29] we can show that the instability of $P_0$ of the system (28) implies the uniform persistence of system (28) when $\mathcal{R}_0 > 1$. This completes the proof of Theorem 4.2.

The uniform persistence of system (28), together with the uniform boundedness of solutions, implies the existence of an endemic equilibrium (see Theorem 2.8.6 in [30] or Theorem D.3 in [31]). Let $P_* = (S^*_1, E^*_1, I^*_1, ..., S^*_n, E^*_n, I^*_n)$ be the endemic equilibrium of model (28). $S^*_k$, $E^*_k$, and $I^*_k$ are positive and satisfy equations

$$\begin{align*}
\Lambda_k &= (\mu_k + \xi_k)S^*_k + \sum_{j=1}^n \beta_{kj}S^*_kE^*_j(I^*_j) - \int_0^\infty \Gamma_k(\theta)S^*_k d\theta, \\
\sum_{j=1}^n \beta_{kj}S^*_kE^*_j(I^*_j) &= (\mu_k + \delta_k)E^*_k, \\
\delta_kE^*_k &= (\mu_k + \gamma_k)I^*_k. 
\end{align*}$$

We can prove that the endemic equilibrium $P_*$ is globally asymptotically stable if it exists. The method is based on the graph-theoretical approach and Lyapunov functionals by Guo et al. [22, 23] and Li and Shuai [24].

**Theorem 4.3.** Assume that $B = (\beta_{kj})_{n \times n}$ is irreducible. If $\mathcal{R}_0 > 1$, then the endemic equilibrium $P_*$ of (28) is globally asymptotically stable and thus is the unique endemic equilibrium.

**Proof.** For convenience of notations, define

$$\overline{\beta}_{kj} = \beta_{kj}S^*_kE^*_j(I^*_j), \ 1 \leq k, j \leq n,$$

and

$$\overline{B} = \begin{pmatrix}
\sum_{l \neq 1} \overline{\beta}_{1l} & -\overline{\beta}_{12} & \cdots & -\overline{\beta}_{1n} \\
-\overline{\beta}_{12} & \sum_{l \neq 2} \overline{\beta}_{2l} & \cdots & -\overline{\beta}_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-\overline{\beta}_{1n} & -\overline{\beta}_{2n} & \cdots & \sum_{l \neq n} \overline{\beta}_{nl}
\end{pmatrix}.$$

$\overline{B}$ is also irreducible. By Lemma 2.1 in [22], the solution space of the linear system $\overline{B}v = 0$ has dimension 1 with a base

$$(v_1, ..., v_n) = (c_{11}, ..., c_{nn}),$$

where, $c_{kk} > 0$ is the co-factor of the $k$-th diagonal entry of $\overline{B}$. Let $(S_k(t), E_k(t), I_k(t))$ be any solution of system (28) with non-negative initial data. We construct
the following Lyapunov function:

\[ L_2 = \sum_{k=1}^{n} v_k \left\{ S_k(t) - S_k^* - S_k^* \ln \frac{S_k(t)}{S_k^*} + E_k(t) - E_k^* \ln \frac{E_k(t)}{E_k^*} \right\} \]

Using the equilibrium equations (29), we have

\[
\begin{align*}
\frac{dL_2}{dt} &= \sum_{k=1}^{n} v_k \left\{ \right. \\
&= \frac{1 - S_k^*}{S_k} \left( A_k - \mu_k S_k - \xi_k S_k \right) \\
&- \sum_{j=1}^{n} \beta_{kj} S_k(t) f_j(I_j) + \int_0^\infty \Gamma_k(\theta) S_k(t - \theta) d\theta \\
&+ \left( 1 - \frac{E_k^*}{E_k} \right) \left( \sum_{j=1}^{n} \beta_{kj} S_k f_j(I_j) - (\mu_k + \delta_k) E_k \right) \\
&+ \frac{\mu_k + \delta_k}{\delta_k} \left( 1 - \frac{f_k(I_k^*)}{f_k(I_k)} \right) \left( \delta_k E_k - (\mu_k + \gamma_k) I_k \right) \\
&+ \Gamma_k S_k - \int_0^\infty \Gamma_k(\theta) S_k(t - \theta) d\theta + \int_0^\infty \Gamma_k(\theta) S_k^* \ln \frac{S_k(t - \theta)}{S_k} d\theta \left. \right\}. \\
\end{align*}
\]

Computing the derivative of \( L_2 \) along the solution of model (28), we obtain that

\[
\frac{dL_2}{dt} = \sum_{k=1}^{n} v_k \left\{ \right. \\
&= \frac{1 - S_k^*}{S_k} \left( A_k - \mu_k S_k - \xi_k S_k \right) \\
&- \sum_{j=1}^{n} \beta_{kj} S_k(t) f_j(I_j) + \int_0^\infty \Gamma_k(\theta) S_k(t - \theta) d\theta \\
&+ \left( 1 - \frac{E_k^*}{E_k} \right) \left( \sum_{j=1}^{n} \beta_{kj} S_k f_j(I_j) - (\mu_k + \delta_k) E_k \right) \\
&+ \frac{\mu_k + \delta_k}{\delta_k} \left( 1 - \frac{f_k(I_k^*)}{f_k(I_k)} \right) \left( \delta_k E_k - (\mu_k + \gamma_k) I_k \right) \\
&+ \Gamma_k S_k - \int_0^\infty \Gamma_k(\theta) S_k(t - \theta) d\theta + \int_0^\infty \Gamma_k(\theta) S_k^* \ln \frac{S_k(t - \theta)}{S_k} d\theta \left. \right\}. \\
\end{align*}
\]

Using the equilibrium equations (29), we have

\[
\frac{dL_2}{dt} = \sum_{k=1}^{n} v_k \left\{ \right. \\
&= \frac{1 - S_k^*}{S_k} \left( A_k - \mu_k S_k - \xi_k S_k \right) \\
&- \sum_{j=1}^{n} \beta_{kj} S_k(t) f_j(I_j) + \int_0^\infty \Gamma_k(\theta) S_k(t - \theta) d\theta \\
&+ \left( 1 - \frac{E_k^*}{E_k} \right) \left( \sum_{j=1}^{n} \beta_{kj} S_k f_j(I_j) - (\mu_k + \delta_k) E_k \right) \\
&+ \frac{\mu_k + \delta_k}{\delta_k} \left( 1 - \frac{f_k(I_k^*)}{f_k(I_k)} \right) \left( \delta_k E_k - (\mu_k + \gamma_k) I_k \right) \\
&+ \Gamma_k S_k - \int_0^\infty \Gamma_k(\theta) S_k(t - \theta) d\theta + \int_0^\infty \Gamma_k(\theta) S_k^* \ln \frac{S_k(t - \theta)}{S_k} d\theta \left. \right\}. \\
\end{align*}
\]
From those expressions we have

\[ + \sum_{k,j=1}^{n} v_k \beta_{kj} \left( 3 - \frac{S^*_k}{S_k} - \frac{E_k f_k(I^*_k) \beta_{kj} S^*_k}{E_k f_k(I_k)} + \frac{E_k f_k(I^*_k) S^*_k f_j(I_j)}{E_k f_k(I_k) f_j(I^*_j)} \right) \]

\[ + \sum_{k,j=1}^{n} v_k \bar{\beta}_{kj} \left( \frac{f_j(I_j)}{f_j(I^*_j)} - \frac{f_k(I_k)}{f_k(I^*_k)} \right) \]

From the properties of function \( f_k(I_k) \), it is easy to obtain the following inequalities

\[ \frac{f_k(I_k)}{f_k(I^*_k)} \geq \frac{I_k}{I^*_k}, \quad \text{for } I_k \leq I^*_k, \quad \frac{f_k(I_k)}{f_k(I^*_k)} \leq \frac{I_k}{I^*_k}, \quad \text{for } I_k \geq I^*_k. \]

Those two inequalities imply that

\[ \left( \frac{f_k(I_k)}{f_k(I^*_k)} - \frac{I_k}{I^*_k} \right) \left( 1 - \frac{f_k(I_k)}{f_k(I^*_k)} \right) \leq 0, \]

with equality holding if and only if \( I_k = I^*_k \).

From \( S_k(t-\theta) > 0 \) and \( S_k(t) > 0 \) it follows that \( 1 + \ln \left( \frac{S_k(t-\theta)}{S_k(t)} \right) \leq 0, \)

with equality holding if and only if \( S_k(t-\theta) = S_k(t) \). It is obvious that \( \frac{S_k}{S^*_k} + \frac{S^*_k}{S_k} - 2 \geq 0 \) for all positive \( S^* \) and \( S_k \), with equality holding if and only if \( S^*_k = S_k(t) \).

The equality \( \sum_{j=1}^{n} v_j \beta_{kj} S^*_k f_k(I^*_k) v_j = \sum_{k=1}^{n} \beta_{kj} S^*_k f_k(I^*_k) v_k \), which implies

\[ \sum_{k=1}^{n} \sum_{j=1}^{n} v_k \beta_{kj} S^*_k f_j(I_j) = \sum_{k=1}^{n} \sum_{j=1}^{n} \beta_{kj} S^*_j v_j f_k(I_k) \]

\[ = \sum_{k=1}^{n} \sum_{j=1}^{n} \beta_{kj} S^*_j v_j f_k(I_k) \]

\[ = \sum_{k=1}^{n} \sum_{i=1}^{n} \beta_{ki} S^*_i v_k f_i(I^*_i) \]

\[ = \sum_{k=1}^{n} \sum_{j=1}^{n} \beta_{kj} S^*_k v_k f_j(I^*_j) \]

\[ = 0. \]

From those expressions we have

\[ \frac{dL_2}{dt} \leq \sum_{k,j=1}^{n} v_k \bar{\beta}_{kj} \left( 3 - \frac{S^*_k}{S_k} - \frac{E_k f_k(I^*_k) \beta_{kj} S^*_k}{E_k f_k(I_k)} + \frac{E_k f_k(I^*_k) S^*_k f_j(I_j)}{E_k f_k(I_k) f_j(I^*_j)} \right) =: H_n, \]

and the equality holds if and only if

\[ \frac{S_k}{S^*_k} + \frac{S^*_k}{S_k} - 2 = 0, \quad \left( \frac{f_k(I_k)}{f_k(I^*_k)} - \frac{I_k}{I^*_k} \right) \left( 1 - \frac{f_k(I_k)}{f_k(I^*_k)} \right) = 0, \]

\[ 1 + \ln \left( \frac{S_k(t-\theta)}{S_k(t)} \right) - \frac{S_k(t-\theta)}{S_k(t)} = 0, \quad k = 1, 2, ..., n. \]

A direct result of the inequality between the arithmetical mean and the geometrical mean gives that \( H_1 \leq 0, \) and \( H_1 = 0 \) if and only if \( S_1 = S^*_1, \ E_1 = E^*_1, \ I_1 = I^*_1 \).
We are going to show that $H_n \leq 0$ for all $n \geq 2$. Let $G$ denote the directed graph associated with matrix $(\overline{\beta}_{kj})$. $G$ has vertices $1, 2, \ldots, n$ with a directed arc $(k, j)$ from $k$ to $j$ if and only if $\overline{\beta}_{kj} \neq 0$. $E(G)$ denotes the set of all directed arcs of $G$. Using Kirchhoff’s Matrix-tree Theorem in graph theory, we know that $v_k = c_{kk}$ can be interpreted as a sum over all unicyclic subgraphs $Q$ of $G$, obtained from such a tree $T$ by adding a directed arc $(k, j)$ from vertex $k$ to vertex $j$. Note that the arc $(k, j)$ is part of the unique cycle $CQ$ of $Q$, and that the same unicyclic graph $Q$ can be formed when each arc of $CQ$ is added to a corresponding rooted tree $T$. Therefore, the double sum in $H_n$ can be reorganized as a sum over all unicyclic subgraphs $Q$ containing vertices $\{1, 2, \ldots, n\}$. That is

$$H_n = \sum_{Q} H_{n,Q},$$

where

$$H_{n,Q} = \prod_{(r,m) \in E(Q)} \overline{\beta}_{rm} \cdot \sum_{(k,j) \in E(CQ)} \left(3 - F_{k,j}\right),$$

$$= \prod_{(r,m) \in E(Q)} \overline{\beta}_{rm} \cdot (3l - \sum_{(k,j) \in E(CQ)} F_{k,j}),$$

with $l = l(Q)$ being the number of arc of $CQ$ and

$$F_{k,j} = \frac{S^*_k}{S_k} + \frac{E_k f_k(I^*_k)}{E_k f_k(I_k)} + \frac{E^*_k S_k f_j(I_j)}{E_k S^*_k f_j(I^*_j)}.$$

Taking $n = 2$ for example, the unique cycle $CQ$ has two vertices with the cycle $1 \rightarrow 2 \rightarrow 1$, and $E(CQ) = \{(1,2), (2,1)\}$, then we can obtain that $v_1 = \overline{\beta}_{21}$, $v_2 = \overline{\beta}_{12}$, and

$$H_2 = \sum_{k,j=1}^2 v_k \overline{\beta}_{kj} \left(3 - \frac{S^*_k}{S_k} - \frac{E_k f_k(I^*_k)}{E_k f_k(I_k)} - \frac{E^*_k S_k f_j(I_j)}{E_k S^*_k f_j(I^*_j)}\right)$$

$$= \overline{\beta}_{21} \overline{\beta}_{11} \left(3 - \frac{S^*_1}{S_1} - \frac{E_1 f_1(I^*_1)}{E_1 f_1(I_1)} - \frac{E^*_1 S_1 f_1(I_1)}{E_1 S^*_1 f_1(I^*_1)}\right) + \overline{\beta}_{12} \overline{\beta}_{22} \left(3 - \frac{S^*_2}{S_2} - \frac{E_2 f_2(I^*_2)}{E_2 f_2(I_2)} - \frac{E^*_2 S_2 f_2(I_2)}{E_2 S^*_2 f_2(I^*_2)}\right) + \overline{\beta}_{21} \overline{\beta}_{12} \left(6 - \frac{S^*_1}{S_1} - \frac{E_1 f_1(I^*_1)}{E_1 f_1(I_1)} - \frac{E^*_1 S_1 f_2(I_2)}{E_1 S^*_1 f_2(I^*_2)}\right) - \frac{S^*_2}{S_2} - \frac{E_2 f_2(I_2)}{E_2 f_2(I_2)} - \frac{E^*_2 S_2 f_1(I_1)}{E_2 S^*_2 f_1(I^*_1)}.$$

For each unicycle graph $Q$, it is not difficult to see that

$$\prod_{(k,j) \in E(CQ)} \frac{S^*_k}{S_k} \cdot \frac{E_k f_k(I_k^*)}{E_k f_k(I_k)} \cdot \frac{E^*_k S_k f_j(I_j)}{E_k S^*_k f_j(I^*_j)} = \prod_{(k,j) \in E(CQ)} f_j(I_j^*) f_k(I_k) = 1,$$

$$\sum_{(k,j) \in E(CQ)} F_{k,j} \geq 3l, \ H_{n,Q} \leq 0, \text{ and } H_{n,Q} = 0 \text{ if and only if }$$

$$\frac{S^*_k}{S_k} = \frac{E_k f_k(I_k^*)}{E_k f_k(I_k)} = \frac{E^*_k S_k f_j(I_j)}{E_k S^*_k f_j(I^*_j)}, \ (k,j) \in E(CQ). \quad (30)$$
Therefore, we have
\[ \frac{dL_2}{dt} \leq H_n = \sum_Q H_{n,Q} \leq 0, \]
and \( \frac{dL_2}{dt} = 0 \) if and only if \( S_k = S_k^* \), \( I_k = I_k^* \), and \( H_n = 0 \). We claim that if \( S_k = S_k^* \) and \( I_k = I_k^* \), then \( H_n = 0 \) if and only if \( E_k = E_k^* \), for \( k = 1, 2, \ldots, n \). Hence, we know that \( \frac{dL_2}{dt} = 0 \) if and only if \( S_k = S_k^* \), \( E_k = E_k^* \), \( I_k = I_k^* \). Thus, the only compact invariant subset of the set where \( \frac{dL_2}{dt} = 0 \) is the singleton \( \{P_*\} \). By LaSalle’s invariance principle [27], the unique endemic equilibrium \( P_* \) of system (28) is globally asymptotically stable if \( R_0 > 1 \). This completes the proof of Theorem 4.3.

5. Conclusions. One group and multi-group SVEIR epidemic models with general nonlinear incidence rate are proposed to describe heterogeneities in disease transmission. The global stability of those two models is established by using Lyapunov functions. The dynamical behavior is completely determined by the magnitude of the the basic reproduction number \( R_0 \).

We define \( \tilde{R}_0 = \rho \left( \frac{\beta_k S_k f(0)}{\mu_k + \gamma_k} \right)_{n \times n} = \lim_{\delta_k \to \infty} R_0 \) to investigate the influence of the latent period on \( R_0 \). It is obvious that \( \tilde{R}_0 > R_0 \) and \( \frac{\partial \tilde{R}_0}{\partial \xi_k} > 0 \), which shows that latent period has a positive role in disease control: a long latent period may lead to the extinction of the disease. Similarly, the fact that \( R_0^* = \rho \left( \frac{\Lambda_k \delta_k \beta_k f(0)}{\mu_k (\mu_k + \delta_k) (\mu_k + \gamma_k) (1 + \xi_k \Gamma_k)^2} \right)_{n \times n} \quad = \tilde{R}_0 |_{\xi_k=0} > R_0 \) and \( \frac{\partial \tilde{R}_0}{\partial \xi_k} = \rho \left( \frac{-\Lambda_k \delta_k \Gamma_k \beta_k f(0)}{\mu_k (\mu_k + \delta_k) (\mu_k + \gamma_k) (1 + \xi_k \Gamma_k)^2} \right) < 0 \) imply that the vaccination is helpful to eradicate the disease. If \( \tilde{R}_0 > 1 \), then there exists a unique \( \Theta \) such that \( R_0 < 1 \) for \( \xi_k > \Theta \) since \( \frac{\partial \tilde{R}_0}{\partial \xi_k} < 0 \) and \( \lim_{\xi_k \to -\infty} \tilde{R}_0 = 0 \). The immunity of a vaccine may not be permanent, a long immunity period of vaccines is still expected for diseases prevention. Our model with vaccination age may help to track the period of vaccination and the immunity wane.

The vaccination age is a prominent feature of our model, and the threshold result is the novel result of our paper. Of course, other factors, such as the discrete or continuous distributed delay for the latency, the population migration, can be integrated into the model to make it more realistic. Moreover, much attention should be paid to improve the results in this paper and make the results more completed.

Acknowledgments. The authors are grateful to the reviewers for their constructive comments and suggestions.

REFERENCES

[1] J. Q. Li, Y. L. Yang and Y. C. Zhou, Global stability of an epidemic model with latent stage and vaccination, Nonlinear Anal.: Real World Appl., 12 (2011), 2163–2173.
[2] S. M. Blower and A. R. McLean, Prophylactic vaccines, risk behavior change, and the probability of eradicating HIV in San Francisco, Science, 265 (1994), 1451–1454.
[3] Y. Xiao and S. Tang, Dynamics of infection with nonlinear incidence in a simple vaccination model, Nonlinear Anal.: Real World Appl., 11 (2010), 4154–4163.
[4] X. Y. Song, Y. Jiang and H. M. Wei, Analysis of a saturation incidence SVEIRS epidemic model with pulse and two time delays, Appl. Math. Comput., 214 (2009), 381–390.
D. Q. Ding and X. H. Ding, Global stability of multi-group vaccination epidemic models with delays, *Nonlinear Anal.: Real World Appl.*, 12 (2011), 1991–1997.

G. P. Sahu and J. Dhar, Analysis of an SVEIS epidemic model with partial temporary immunity and saturation incidence rate, *Appl. Math. Model.*, 36 (2012), 908–923.

F. Hoppensteadt, An age-dependent epidemic model, *J. Franklin Inst.*, 297 (1974), 325–333.

F. Hoppensteadt, *Mathematical Theories of Populations: Demographics, Genetics and Epiemics*, SIAM Publications, Philadelphia, 1975.

M. Iannelli, M. Martcheva and X. Z. Li, Strain replacement in an epidemic model with superinfection and perfect vaccination, *Math. Biosci.*, 195 (2005), 23–46.

X. Z. Li, J. Wang and M. Ghosh, Stability and bifurcation of an SIVS epidemic model with treatment and age of vaccination, *Appl. Math. Model.*, 34 (2010), 437–450.

X. C. Duan, S. L. Yuan and X. Z. Li, Global stability of an SVIR model with age of vaccination, *Appl. Math. Comput.*, 226 (2014), 528–540.

G. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics*, Marcel Dekker. New York, 1988.

P. Magal, Compact attractors for time periodic age-structured population models, *Electron. J. Diff. Eqns.*, 65 (2001), 1–35.

Z. Liu, P. Magal and S. Ruan, Center-unstable manifold theorem for non-densely defined Cauchy problem, and the stability of bifurcation periodic orbits by Hopf bifurcation, *Canadian Applied Mathematics Quarterly*, 20 (2012), 135–178.

P. Magal, C. C. McCluske and G. F. Webb, Liapunov functional and global asymptotic stability for an infection-age model, *Applicable Analysis*, 89 (2010), 1109–1140.

H. L. Smith and H. R. Thieme, *Dynamical Systems and Population Persistence*, Graduate Studies in Mathematics, 118. American Mathematical Society, Providence, RI, 2011.

G. S. Wolkowicz, H. Xia and S. Ruan, Competition in the chemostat: A distributed delay model and its global asymptotic behavior, *SIAM J. Appl. Math.*, 57 (1997), 1281–1310.

R. Y. Sun and J. P. Shi, Global stability of multigroup epidemic model with group mixing and nonlinear incidence rates, *Appl. Math. Comput.*, 218 (2011), 280–286.

T. Kuniya, Global stability of a multi-group SVIR epidemic model, *Nonlinear Anal.: Real World Appl.*, 14 (2013), 1135–1143.

H. Y. Shu, D. J. Fan and J. J. Wei, Global stability of multi-group SEIR epidemic models with distributed delays and nonlinear transmission, *Nonlinear Anal.: Real World Appl.*, 13 (2012), 1581–1592.

H. Guo, M. Y. Li and Z. Shuai, Global stability of the endemic equilibrium of multigroup SIR epidemic models, *Canad. Appl. Math. Quart.*, 14 (2006), 259–284.

H. Guo, M. Y. Li and Z. Shuai, A graph-theoretic approach to the method of global Lyapunov functions, *Proc. Amer. Math. Soc.*, 136 (2008), 2793–2802.

M. Y. Li and Z. Shuai, Global-stability problem for coupled systems of differential equations on networks, *J. Differential Equations*, 248 (2010), 1–20.

H. Guo and M. Y. Li, Impacts of migration and immigration on disease transmission dynamics in heterogeneous populations, *Discrete Contin. Dyn. Syst. Ser. B*, 17 (2012), 2413–2430.

A. Berman and R. J. Plemmons, *Nonnegative Matrices in Mathematical Science*, Academic Press, New York, 1979.

J. P. LaSalle, *The Stability of Dynamical Systems*, SIAM, Philadelphia, 1976.

H. I. Freedman, M. X. Tang and S. G. Ruan, Uniform persistence and flows near a closed positively invariant set, *J. Dyn. Differ. Equat.*, 6 (1994), 583–600.

M. Y. Li, J. R. Graef, L. Wang and J. Karsai, Global dynamics of a SEIR model with varying total population size, *Math. Biosci.*, 160 (1999), 191–213.

N. P. Bhatia and G. P. Szego, *Dynamical Systems: Stability Theory and Applications*, Lecture Notes in Mathematics, vol. 35, Springer, Berlin, 1967.

H. L. Smith and P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, Cambridge, 1995.