Analysis of a diffusive epidemic system with spatial heterogeneity and lag effect of media impact

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Abstract
We considered an SIS functional partial differential model cooperated with spatial heterogeneity and lag effect of media impact. The wellposedness including existence and uniqueness of the solution was proved. We defined the basic reproduction number and investigated the threshold dynamics of the model, and discussed the asymptotic behavior and monotonicity of the basic reproduction number associated with the diffusion rate. The local and global Hopf bifurcation at the endemic steady state was investigated theoretically and numerically. There exists numerical cases showing that the larger the number of basic reproduction number, the smaller the final epidemic size. The meaningful conclusion generalizes the previous conclusion of ordinary differential equation.

Keywords Media impact · Functional partial differential model · Spatial heterogeneity · Hopf bifurcation

1 Introduction

Infectious diseases can have a great impact on the development of human society as they can negatively bring morbidity, mortality, unemployment and inequality. Therefore, prevention and control of infectious diseases are of significance for public health and welfare. Recent outbreaks of infectious diseases, such as Ebola, severe acute respiratory syndrome (SARS), the 2009 novel influenza A(H1N1) pandemic, Covid-19 have highlighted an important role played by global public health systems of surveillance and response which help to quickly curb an emerging disease and reduce its...
influence on socioeconomic activities (Lau et al. 2004; Tang et al. 2012; Winters et al. 2018; Lai et al. 2020). However, the impact of infectious disease, the massive news coverage and fast information flow on the emerging diseases are all subject to behavioral changes of human trying to minimize the effect of the disease onto themselves. In recent years, several emerging infectious diseases confirmed the existence of a so-called behavioural immune system (Schaller 2011). For example, during the 2003 SARS and Covid-19 outbreak, people took precautionary actions such as wearing face masks, hand-washing, avoiding public contact (Lau et al. 2004; Beutels et al. 2009; Lai et al. 2020). Moreover, the 2009 A/H1N1 influenza pandemic and Covid-19 had induced a significant proportion of the population to adapt their behaviour and take preventive measures such as social distancing, home isolation or school closure (Tang et al. 2012; Lai et al. 2020).

The exact impact of media coverage can have on the infectious disease, however, is difficult to quantify and often subject to speculation in mathematical modelling. A number of mathematical models were developed to investigate the impact of media coverage on the spreading and control of infectious diseases (Verelst et al. 2016; Funk 2010; Cui et al. 2008; Xiao et al. 2013, 2015; Yan et al. 2016; Tang et al. 2010; Liu et al. 2007; Li and Cui 2009; Tchuenche et al. 2011; Sun et al. 2011; Song and Xiao 2018, 2019). In order to characterize the media impact on disease, a media function, decreasing in the number of infected individuals, was often included. For example, in Liu et al. (2007), a media function 
\[ \beta e^{-\alpha_1 E - \alpha_2 I - \alpha_3 H} \]
was introduced into the transmission coefficient, where \(E\), \(I\), and \(H\) are the numbers of reported exposed, infectious, and hospitalized individuals, respectively. Li and Cui (2009) used the media function \[ \beta_1 - \beta_2 \frac{I}{m+I} \] (or \[ \beta_2 \frac{I}{m+I} \]) to reflect the reduced amount of contact rate due to media coverage. Xiao et al. (2015) extended these media functions by assuming that the function depends on both the case number and its rate of change, and obtained that media impact switches on and off in a highly nonlinear fashion. Yan et al. (2016) further extended extended a class epidemic model of SEIR type by including extra compartment, i.e., the level of media coverage \(M\), and characterize the media impact by including the function \(e^{-\mu M}\) with \(\mu > 0\) in the incidence. It was found that although the media coverage itself is not a determined fact to eradicate the infection of the disease, media coverage can greatly delay the epidemic peak and decrease severity of outbreak (Liu et al. 2007; Song and Xiao 2019).

However, these models built on ordinary differential equations ignore two important factors: the lag effect of media impact and human mobility in heterogeneous environment. The lag of media impact are induced directly by the mass media’s response to the disease infection, and indirectly by the time from for individuals’ response to the media coverage such as symptom onset to hospitalization. In fact, by analyzing the case data and media coverage on A/H1N1 in Shaanxi province in 2009, Yan et al. (2016) obtained the correlation between the case number and media coverage, confirmed the existence of time lags and further identified the time lags. Hence, including time delay in the incidence rate is more reasonable. Recently, we (Song and Xiao 2018, 2019) initially included time delay in the media function and explored the lag effect of media impact on infectious diseases. However, human mobility in heterogeneous environment were ignored in Song and Xiao (2018, 2019). There is increasing evidence which
show that environmental heterogeneity and human mobility have significant impact on the spread of infectious diseases (Cantrell and Cosner 2003; Murray 2002; Riley 2007; y Piontti et al. 2018). In recent years, numerous reaction-diffusion models have been proposed to investigate the roles of diffusion and spatial heterogeneity on the transmission of diseases (Allen et al. 2008; Wang and Zhao 2012; Zhao 2017; Peng and Zhao 2012; Cui and Lou 2016; Cui et al. 2017; Deng and Wu 2016; Wu and Zou 2016; Li et al. 2017; Ge et al. 2015; Li et al. 2020). Among these works, Allen et al. (2008) proposed a susceptible-infected- susceptible (SIS) reaction-diffusion system cooperated with spatial heterogeneity as follows:

\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_S \Delta S - \frac{\beta(x)SI}{S + I} + \gamma(x)I, & x \in \Omega, t > 0, \\
\frac{\partial I}{\partial t} &= d_I \Delta I + \frac{\beta(x)SI}{S + I} - \gamma(x)I, & x \in \Omega, t > 0, \\
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{align*}
\]

(1)

The main results of Allen et al. (2008) concern with the definition, monotonicity and asymptotic properties of basic production number ($R_0$), threshold-type results on the global dynamics in terms of $R_0$ and particularly the existence, uniqueness and asymptotic behaviors of the endemic steady state ($EE$) as the diffusion rate of the susceptible individuals ($d_S$) approaches to zero. Peng and Zhao (Peng and Zhao 2012) recently considered the same SIS reaction-diffusion model, but the rates of disease transmission and recovery are assumed to be spatially heterogeneous and temporally periodic. In Deng and Wu (2016), Wu and Zou (2016), the authors investigated an SIS model with mass action infection mechanism. In Li et al. (2017), Li et al. provided qualitative analysis on an SIS reaction diffusion system with a linear source term. Ge et al. introduced a free boundary model for characterizing the spreading front of the disease in Ge et al. (2015). The effects of diffusion and advection for SIS epidemic reaction-diffusion model in heterogeneous environments were studied in Cui and Lou (2016), Cui et al. (2017). Dynamics and asymptotic profiles of endemic steady state for two frequency-dependent SIS epidemic models with cross-diffusion was studied in Li et al. (2020). Moreover, to grasp the impact of the media coverage and heterogeneous environment on preventing and controlling the transmission of infectious diseases, Ge et al. (2017) consider an SIS reaction-diffusion equation with media impact. However, these reaction diffusion models only consider human mobility in heterogeneous environment or the media impact without lag effect. The report delay and response time for individuals to the current infection were ignored in these models.

1.1 Model description

In order to investigate the lag effect of media impact and human mobility in heterogeneous environment on the transmission dynamics of infectious diseases, we devide the population into two groups: susceptible (S), infected (I), and consider the following system:
\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_S \Delta S + \Lambda(x) - \frac{\beta(x)e^{-m(x)I(x,t-r)}SI}{S+I} + \gamma(x)I - \alpha(x)S, \quad x \in \Omega, t > 0, \\
\frac{\partial I}{\partial t} &= d_I \Delta I + \frac{\beta(x)e^{-m(x)I(x,t-r)}SI}{S+I} - \gamma(x)I - \alpha(x)I, \quad x \in \Omega, t > 0, \\
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial \Omega, t > 0.
\end{align*}
\]

(2)

Here, \( \Omega \) is a bounded domain in \( \mathbb{R}^{n_0} \) with smooth boundary \( \partial \Omega \), where \( n_0 \) is a positive integer and the homogeneous Neumann boundary conditions assumed in model (2) mean that no population flux crosses the boundary \( \partial \Omega \). \( S(x, t) \) and \( I(x, t) \) denote the density of susceptible and infected individuals at location \( x \) and time \( t \), respectively. \( d_S \) and \( d_I \) represent the diffusion coefficients associated with susceptible and infected individuals, respectively. The positive Hölder continuous functions \( \Lambda(x), \alpha(x), \beta(x) \) and \( \gamma(x) \) on \( \overline{\Omega} \) represent the natural birth rate, the natural death rate and transmission rate and recovery rate at \( x \), respectively.

Media coverage and fast information flow induce a profound psychological impact on the public, a reduction in the incidence rate at the position \( x \), is represented by \( e^{-m(x)I(x,t-r)} \). Here time delay \( r \) denotes the report delay and response time for individuals to the current infection, and the positive Hölder continuous functions \( m(x) \) stands for the weight of media effect sensitive to number of infected population at the position \( x \). Here we assume that people care more about the prevalence of infectious disease in the place they stay, therefore we choose the media impact function \( e^{-m(x)I(x,t-r)} \) rather than \( e^{-\int_{\partial \Omega} m(x)I(x,t-r)dx} \). We also point out here that the disease-related death was not included in model (2), since, on one hand, we focus our model on the effect of spatial heterogeneity and media impact on prevalence, incidence or accumulated cases rather than case fatal rate of infectious disease, and the disease-induced death rate is far smaller than recovery rate. On the other hand, we compromise here to make further bifurcation analysis not too complicated.

It is easy to verify that \( \frac{\partial SI}{S+I} \) is a Lipschitz continuous function of \( S \) and \( I \), therefore we define it to be zero whenever \( S = 0 \) or \( I = 0 \). For notation convenience, we denote

\[
\underline{g} = \min_{x \in \Omega} \{ g(x) \}, \quad \overline{g} = \max_{x \in \Omega} \{ g(x) \}.
\]

where \( g \) can be \( \Lambda(x), \alpha(x), \beta(x), \gamma(x) \) and \( m(x) \). Moreover, throughout this paper, we assume

**A1:** \( S(x, 0) \geq 0, I(x, 0) \geq 0 \) for \( x \in \overline{\Omega} \) and \( \int_{\partial \Omega} I(x, 0)dx > 0 \);

**A2:** \( H^+ = \{ x \in \Omega | \beta(x) > \alpha(x) + \gamma(x) \} \) and \( H^- = \{ x \in \Omega | \beta(x) < \alpha(x) + \gamma(x) \} \) are nonempty.

For further purposes, we also define two conditions as follows:

**B1:** \( \int_{\Omega} \beta dx < \int_{\Omega} (\gamma(x) + \alpha(x))dx \);

**B2:** \( \int_{\Omega} \beta dx > \int_{\Omega} (\gamma(x) + \alpha(x))dx \).
1.2 Steady state problems

For further purposes, we define non-negative steady state solutions of model (2):

\[
\begin{aligned}
&d_S \Delta \tilde{S} + \Lambda(x) - \frac{\beta(x)e^{-m(x)}\tilde{I}\tilde{S}}{\tilde{S} + \tilde{I}} + \gamma(x)\tilde{I} - \alpha(x)\tilde{S} = 0, & x \in \Omega, \\
&d_I \Delta \tilde{I} + \frac{\beta(x)e^{-m(x)}\tilde{S}\tilde{I}}{\tilde{S} + \tilde{I}} - \gamma(x)\tilde{I} - \alpha(x)\tilde{I} = 0, & x \in \Omega, \\
&\frac{\partial \tilde{S}}{\partial n} = \frac{\partial \tilde{I}}{\partial n} = 0, & x \in \partial \Omega.
\end{aligned}
\]

Here, \( \tilde{S}(x) \) and \( \tilde{I}(x) \) denote the density of susceptible and infected individuals, respectively, at \( x \in \Omega \). A disease-free steady state (DFE) is a solution of (3) so that \( \tilde{I}(x) = 0 \) for every \( x \in \Omega \). An endemic steady state (EE) of (3) is a solution in which \( \tilde{I}(x) > 0 \) for some \( x \in \Omega \). By direct calculation, the disease free steady state (DFE) is

\[
E_0 := (\tilde{N}, 0)
\]

and it is unique, where \( \tilde{N} \) is the unique solution of

\[
d_S \Delta \tilde{N} + \Lambda(x) - \alpha(x)\tilde{N} = 0 \text{ in } \Omega; \quad \frac{\partial \tilde{N}}{\partial n}|_{\partial \Omega} = 0.
\]

Denote the endemic steady state (EE) by \( (\tilde{S}, \tilde{I}) \). By the strong maximum principle, any endemic steady state, \( (\tilde{S}, \tilde{I}) \) are positive for any \( x \in \Omega \).

The rest of this paper is organized as follows. In Sect. 2, we study the wellposedness, define the basic reproduction number and investigate the threshold dynamics of model (2). In Sect. 3, we assume that \( d_S = d_I \) and explore the local Hopf bifurcation at the endemic steady state. Section 4 is devoted to global existence of periodic solutions. Numerical simulations are presented in Sect. 5 to graphically illustrate the effect of delayed media impact and human mobility in heterogeneous environment on the transmission dynamics of infectious diseases. The paper ends with a conclusion section.

2 Wellposedness, Basic reproduction number and threshold dynamics

Let \( \mathcal{X} = C(\Omega, \mathbb{R}^2) \) be the Banach space of continuous functions with the supremum norm \( \| \cdot \|_\mathcal{X} \). Set \( C_r = C([-r, 0], \mathcal{X}) \). For any \( \phi \in C_r \), define \( \| \phi \| = \max_{\theta \in [-r, 0]} \| \phi(\theta) \|_\mathcal{X} \). Then \( C_r \) is an ordered Banach space with the cone \( C_r^+ \).

Denote \( \mathcal{Y} = C(\Omega, \mathbb{R}) \). Let \( T_1(t) \) and \( T_2(t) : \mathcal{Y} \to \mathcal{Y}, t \geq 0 \), be the semigroups associated with \( d_S \Delta \) and \( d_I \Delta \) with the homogeneous Neumann boundary conditions, respectively, and let \( A_i : D(A_i) \to \mathcal{Y}, i = 1, 2 \). Clearly, \( T(t) = (T_1(t), T_2(t)) : \mathcal{X} \to \mathcal{X}, t \geq 0 \), is a semigroup generated by the operator
\[ A = (A_1, A_2) \text{ defined on } D(A) = D(A_1) \times D(A_2). \text{ Then for each } t > 0, \ T(t) : X' \to X' \text{ is compact and positive (see, e.g., (Smith 1995, Section 7.1 and Corollary 7.2.3)). Define } F = (F_1, F_2) : C^r_+ \to C^r_+ \text{ by} \]

\[
F_1(\phi_1, \phi_2)(x) = \Lambda(x) - \frac{\beta(x)e^{-m(x)\phi_2(-r,x)}\phi_1(0,x)\phi_2(0,x)}{\phi_1(0,x) + \phi_2(0,x)} + \gamma(x)\phi_2(0,x) - \alpha(x)\phi_1(0,x),
\]

\[
F_2(\phi_1, \phi_2)(x) = \frac{\beta(x)e^{-m(x)\phi_2(-r,x)}\phi_1(0,x)\phi_2(0,x)}{\phi_1(0,x) + \phi_2(0,x)} - \gamma(x)\phi_2(0,x) - \alpha(x)\phi_2(0,x),
\]

for all \( \phi = (\phi_1, \phi_2) \in C^r_+, \ x \in \overline{\Omega} \). Given a function \( u : [-r, \sigma) \to X(\sigma > 0) \), define \( u_t \in C_r \) by \( u_t(\theta) = u(t + \theta) \) with \( \theta \in [-r, 0] \). Then we can rewrite system (2) as an abstract functional differential form:

\[
\frac{du(t)}{dt} = Au(t) + F(u_t), \ t > 0,
\]

with the initial condition \( u_0 \in C^r_+ \).

For further purposes to obtain the wellposedness of model (2), we give the following lemma:

**Lemma 1** Du and Peng (2016) Consider the parabolic system

\[
\begin{align*}
\frac{\partial u_i}{\partial t} - d_i \Delta u_i &= f_i(x, t, u), \quad x \in \Omega, \ t > 0, \ i = 1, \ldots, l, \\
\frac{\partial u_i}{\partial n} &= 0, \quad x \in \partial \Omega, \ t > 0, \\
u_i(x, 0) &= u^0_i(x), \quad x \in \Omega,
\end{align*}
\]

where \( u = (u_1, \ldots, u_l) \), \( u^0_i \in C(\overline{\Omega}) \) and \( d_i > 0 (i = 1, \ldots, l) \) are constants, and assume that, for each \( k = 1, \ldots, l \), the functions \( f_k \) satisfy the polynomial growth condition:

\[
|f_k(x, t, u)| \leq c_1 \sum_{i=1}^{l} |u_i|^q + c_2
\]

for some nonnegative constants \( c_1 \) and \( c_2 \), and positive constant \( q \). Let \( p_0 \) be a positive constant such that \( p_0 > \frac{n}{2} \max \{0, (q - 1)\} \) and \( r(u^0) \) be the maximal existence time of the solution \( u \) corresponding to the initial data \( u^0 \). Suppose that there exists a positive constant \( C_{p_0}(u^0) \) such that \( \| u(\cdot, t) \|_{L^{p_0}(\Omega)} \leq C_{p_0}(u^0), \forall t \in (0, r(u^0)) \), then the solution \( u \) exists for all time and there is a positive constant \( C_\infty \) such that \( \| u(\cdot, t) \|_{L^{p_0}(\Omega)} \leq C_\infty(u^0), \forall t \in (0, \infty) \). Moreover, if there exist finite numbers \( \rho \) and \( K_\rho \) independent of initial data such that \( \| u(\cdot, t) \|_{L^{p_0}(\Omega)} \leq K_\rho(\rho), \forall t \in \overline{\Omega} \).
Theorem 1
For any initial value \( \phi \in C^+_r \), system (2) admits a unique nonnegative uniformly bounded solution \( u(t, \phi) \) on \([0, \infty)\) with \( u_0 = \phi \), and there exists \( u_t(\phi) := (u_1(\phi), u_2(\phi)) \in C^+_r \) for all \( t \geq 0 \), and there exists a positive constant \( C_1 \) depending on initial values such that the solution \((S, I) \in X^+\) of system (2) satisfies
\[
\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1, \forall t \geq 0,  \tag{7}
\]
and there exists a positive constant \( C_2 \) independent of initial values such that for some large time \( T_0 > 0 \),
\[
\|S(\cdot, t)\|_{L^\infty(\Omega)} + \|I(\cdot, t)\|_{L^\infty(\Omega)} \leq C_2, \forall t \geq T_0. \tag{8}
\]
Moreover, the solution semiflow, denoted by \( \Phi(t) = u_t(\cdot) : C^+_r \to C^+_r, t \geq 0, \) has a strong global attractor.

Proof
It is easy to verify that
\[
\lim_{h \to 0^+} \frac{1}{h} \text{dist}(\phi(0) + hF(\phi), X^+) = 0, \quad \forall \phi \in C^+_r.
\]
By ( Martin and Smith 1990, Proposition 3 and Remark 2.4), it then follows that for every \( \phi \in C^+_r \), system (2) admits a unique noncontinuable mild solution \( u(t, \phi) \in X^+ \) in its maximal interval of existence \([0, \sigma_\phi)\) with \( u_0 = \phi \). Integrating the first and second equations of (2) and adding the resulting two identities yield
\[
\frac{d}{dt} \int_\Omega (S(x, t) + I(x, t)) dx \leq \int_\Omega \Lambda(x) dx - a \int_\Omega (S(x, t) + I(x, t)) dx.  \tag{9}
\]
Then the well-known Gronwall’s inequality applied to (9) asserts that there exists some constant \( C_0 > 0 \), such that
\[
\int_\Omega (S(x, t) + I(x, t)) dx \leq C_0, \quad \forall t \in (0, \sigma_\phi).
\]
Set \( h(x, t) = I(x, t - r), t < \sigma_\phi + r \), system (2) becomes
\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_S \Delta S + \Lambda(x) - \frac{\beta(x) e^{-m(x) h(x, t)} S I}{S + I} + \gamma(x) I - \alpha(x) S, \quad x \in \Omega, t > 0, \\
\frac{\partial I}{\partial t} &= d_I \Delta I + \frac{\beta(x) e^{-m(x) h(x, t)} S I}{S + I} - \gamma(x) I - \alpha(x) I, \quad x \in \Omega, t > 0, \\
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \partial \Omega, t > 0.
\end{align*}
\tag{10}
\]
By (9) and (Du and Peng 2016, Lemma 2.1) (due to Le (1997)) with \( \sigma = p_0 = 1 \),
along with the positiveness of \( \tilde{S}, \tilde{I} \), we have

\[
\| \tilde{S}(\cdot, t) \|_{L^\infty(\Omega)} + \| \tilde{I}(\cdot, t) \|_{L^\infty(\Omega)} \leq C_1, \; \forall 0 \leq t < \sigma \phi + r,
\]
and the maximal interval of existence extends to \([0, \sigma \phi + r)\). Reset \( h(x, t) = I(x, t - r), t < \sigma \phi + 2r \) and repeat the above procedure again, we can obtain

\[
\| \tilde{S}(\cdot, t) \|_{L^\infty(\Omega)} + \| \tilde{I}(\cdot, t) \|_{L^\infty(\Omega)} \leq C_1, \; \forall 0 \leq t < \sigma \phi + 2r,
\]
and the maximal interval of existence extends to \([0, \sigma \phi + 2r)\). Repeating the procedures yields (7) and (8).

Therefore, the solution semiflow \( \Phi(t) = u_t(\cdot) : C^+_r \to C^+_r \) is point dissipative. By
(Wu 1996, Theorem 2.2.6), \( \Phi(t) = u_t(\cdot) : C^+_r \to C^+_r \) is compact for \( t > r \). Thus, it follows from (Zhao 2017, Theorem 1.1.3) (see also (Hale 1988, Theorem 3.4.8) ) that \( \Phi(t) \) has a strong global attractor on \( C^+_r \).

\[ \square \]

2.1 Definition of basic reproduction number

For infectious disease models, the basic reproduction number, defined as the expected number of secondary cases produced in a completely susceptible population by an infective individual, is one of the most significant concepts in studying the transmission of infectious disease (Diekmann and Heesterbeek 2000; Anderson and May 1991). More importantly, it often determines the threshold behavior for many epidemic models. It is often the case that a disease dies out if the basic reproduction number is less than unity and the disease is established in the population if it is greater than unity. We refer to Diekmann et al. (1990) for the approach of next genenumber operators for the basic reproduction number and to Zhao (2017), Wang and Zhao (2012), Thieme (2009), Liang et al. (2019) for related works.

We now make use of the theory developed in Liang et al. (2019) to derive the basic reproduction number of system (2).

**Lemma 2** Let \( \mu_0 \) denote the unique positive eigenvalue with a positive eigenfunction corresponding to the following problem:

\[
d_1 \Delta \phi + \mu \beta(x) \phi - (\alpha(x) + \gamma(x)) \phi = 0 \; \text{in} \; \Omega; \quad \frac{\partial \phi}{\partial n} \bigg|_{\partial \Omega} = 0,
\]

then the basic reproduction number of system (2) satisfies

\[
R_0 = \frac{1}{\mu_0} = \sup_{\psi \neq 0} \frac{\int_{\Omega} \beta \psi^2 dx}{\int_{\Omega} (d_1 |\nabla \psi|^2 + (\gamma + \alpha) \psi^2) dx}.
\]

\[ \square \]
For further purposes to study the local stability of disease-free steady state, we consider the following eigenvalue problem:

\[-d_I \Delta \psi + (\alpha(x) + \gamma(x) - \beta(x))\psi = \lambda \psi \text{ in } \Omega; \quad \frac{\partial \psi}{\partial n} \bigg|_{\partial \Omega} = 0, \tag{13}\]

Let \( \lambda_1 \) be the principal eigenvalue of (13) with the positive eigenfunction \( \phi_I \). Then, we have the following properties of \( R_0 \), the proof of which resembles that of (Allen et al. 2008, Lemma 2.3) and hence is omitted.

**Lemma 3** Suppose that A1 – A2 hold.

(i) \( R_0 \) is a monotone decreasing function of \( d_I \) with \( R_0 \to \max \left\{ \frac{\beta(x)}{\gamma(x) + \alpha(x)}, x \in \Omega \right\} \) as \( d_I \to 0 \) and \( R_0 \to \int_\Omega \beta(x)dx \int_\Omega (\gamma(x) + \alpha(x))dx \) as \( d_I \to \infty \).

(ii) If \( \int_\Omega \beta(x)dx < \int_\Omega (\gamma(x) + \alpha(x))dx \), there exists a threshold value \( d^*_I \) in \( (0, \infty) \) such that \( R_0 > 1 \) for \( d_I < d^*_I \) and \( R_0 < 1 \) for \( d_I > d^*_I \), where

\[ d^*_I = \sup \left\{ \frac{\int_\Omega (\beta - \gamma - \alpha) \varphi^2 dx}{\int_\Omega |\nabla \varphi|^2 dx} \left| \varphi \in W^{1,2}(\Omega) \text{ and } \int_\Omega (\beta - \gamma - \alpha) \varphi^2 dx > 0 \right\}; \]

(iii) If \( \int_\Omega \beta(x)dx > \int_\Omega (\gamma(x) + \alpha(x))dx \), we have \( R_0 > 1 \) for all \( d_I > 0 \);

(iv) \( \text{sign}(1 - R_0) = \text{sign}(\lambda_1) \).

2.2 Threshold dynamics

The following lemma concerned with the local stability of DFE is a direct result of Lemma 3(iv) and we omit the proof.

**Lemma 4** The disease-free steady state \( E_0 \) in system (2) is locally asymptotically stable if \( R_0 < 1 \), unstable if \( R_0 > 1 \).

**Theorem 2** (i) If the basic reproduction number \( R_0 < 1 \), then DFE is globally asymptotically stable.

(ii) If \( R_0 > 1 \), there exists a small positive constant \( \epsilon_0 \) such that any positive solution of system (2) satisfies

\[ \liminf_{t \to \infty} \|(S(\cdot, t), I(\cdot, t)) - (\tilde{N}, 0)\| > \epsilon_0. \]

Besides, system (2) admits at least one endemic steady state.

**Proof** See Appendix.
2.3 Dynamics at the endemic steady state without delay

In this part, we assume \( r = 0 \) and \( d_S = d_I = d \), and system (2) becomes the following reaction-diffusion system:

\[
\begin{align*}
\frac{\partial S}{\partial t} &= d \Delta S + \Lambda(x) - \frac{\beta(x)e^{-m(x)}I(x)S I}{S + I} + \gamma(x)I - \alpha(x)S, & x \in \Omega, t > 0, \\
\frac{\partial I}{\partial t} &= d \Delta I + \frac{\beta(x)e^{-m(x)}I(x)S I}{S + I} - \gamma(x)I - \alpha(x)I, & x \in \Omega, t > 0, \\
\frac{\partial S}{\partial n} &= \frac{\partial I}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{align*}
\]

(14)

Denote \( N = S + I \), then system (14) is equivalent to the following system:

\[
\begin{align*}
\frac{\partial N}{\partial t} &= d \Delta N + \Lambda(x) - \alpha(x)N, & x \in \Omega, t > 0, \\
\frac{\partial I}{\partial t} &= d \Delta I + \frac{\beta(x)e^{-m(x)}I(x)(N - I)I}{N} - \gamma(x)I - \alpha(x)I, & x \in \Omega, t > 0, \\
\frac{\partial N}{\partial n} &= \frac{\partial I}{\partial n} = 0, & x \in \partial \Omega, t > 0.
\end{align*}
\]

(15)

**Theorem 3** Let \( r = 0 \) and suppose that \( R_0 > 1 \) and \( d_S = d_I \). Then system (2) admits a unique endemic steady state, denoted by \( E_1 = (\tilde{N} - I^*, I^*) \), which is globally asymptotically stable.

**Proof** By (15), we can regard \( N(t, x) \) as a fixed function on \( \mathbb{R}^+ \times \overline{\Omega} \) and \( \lim_{t \to \infty} N(t, \cdot) = \tilde{N} \). Then system is asymptotic to the following system:

\[
\begin{align*}
\frac{\partial I}{\partial t} &= d \Delta I + \frac{\beta(x)e^{-m(x)}I(x)(\tilde{N} - I)I}{N} - \gamma(x)I - \alpha(x)I, & x \in \Omega, t > 0, \\
\frac{\partial I}{\partial n} &= 0, & x \in \partial \Omega, t > 0.
\end{align*}
\]

(16)

If \( R_0 > 1 \), by (Zhao 2017, Theorem 3.1.5) (See also (Freedman and Zhao 1997, Colorrary 2.2)), system (16) admits a unique positive steady state, denoted by \( I^* \), which is globally asymptotically stable. Therefore, system (15) admits a unique endemic steady state, denoted by \( \hat{E}_1 = (\tilde{N}, I^*) \), and by (Zhao 2017, Theorem 1.2.1 with Remark 1.3.2) (see also (Thieme 1992, Theorem 4.1)), \( \hat{E}_1 \) is globally asymptotically stable. Thus Theorem 4 follows. \( \Box \)
3 Local hopf bifurcation at the endemic steady state

Throughout this section, we assume that $R_0 > 1$, $d_S = d_I = d$ and $\textbf{A1, A2, B1}$ hold, consider the limit system of (2)

\[
\begin{aligned}
\frac{\partial I}{\partial t} &= d \Delta I + \beta(x)e^{-m(x)I(x,t-\tau)} \left(1 - \frac{I}{N}\right) I - (\gamma(x) + \alpha(x))I, \quad x \in \Omega, t > 0, \\
\frac{\partial I}{\partial n} &= 0, \quad x \in \partial \Omega, t > 0.
\end{aligned}
\]  

(17)

Letting $\tilde{I} = I$, $\tilde{t} = td$, dropping the tilde since no confusion occurs, and denoting $\lambda = \frac{1}{d}$, $\tau = dr$, system (17) can be rewritten as

\[
\begin{aligned}
\frac{\partial I}{\partial t} &= \Delta I + \lambda \beta(x)e^{-m(x)I(x,t-\tau)} \left(1 - \frac{I}{N}\right) I - \lambda(\gamma(x) + \alpha(x))I, \quad x \in \Omega, t > 0, \\
\frac{\partial I}{\partial n} &= 0, \quad x \in \partial \Omega, t > 0.
\end{aligned}
\]  

(18)

The wellposedness (existence, uniqueness, and positivity) of solutions to systems (17) and (18) can be obtained by similar arguments as Theorem 1, so we omit the proof here. We refer to Su et al. (2012); Chen and Shi (2012); Chen et al. (2018); Faria et al. (2002) for some related works on Hopf bifurcation of functional partial differential equations.

Denote

\[
X = \left\{ u \in H^2(\Omega) : \frac{\partial u}{\partial n} = 0, x \in \partial \Omega \right\}, \quad Y = L^2(\Omega), C = C([-\tau, 0], Y).
\]

Moreover, we denote the the complexification of a linear real-valued vector space $Z$ to be $Z_C = Z \oplus iZ$, and the positive cone of $Z$ if it exists by $Z^+$, the domain of a linear operator $L$ by $\mathcal{D}(L)$, the kernel of $L$ by $\mathcal{N}(L)$ and the range of $L$ by $\mathcal{R}(L)$. For Hilbert space $Y_C$, we use the standard inner product $\langle u, v \rangle = \int_\Omega \bar{u}(x)v(x)dx$. For a nonlinear mapping $F$, we denote by $D_u F$ the Fréchet derivative with respect to variable $u$.

For further purposes, set

\[
L := \Delta + \lambda_*(\beta - \gamma - \alpha), \tag{19}
\]

where $\lambda_*$ is the unique positive principal eigenvalue of the following problem with positive eigenfunction $\phi$ under conditions $\textbf{A2, B1}$ (Cantrell and Cosner 2003, Theorem 2.4):

\[
-\Delta \phi = \lambda_*(\beta - \gamma - \alpha)\phi \quad \text{in} \quad \Omega; \quad \frac{\partial \phi}{\partial n} = 0. \tag{20}
\]
Then the endemic steady state, denoted by $I_\lambda$, of (18) can be written as

$$LI_\lambda + (\lambda - \lambda^*) (\beta - \alpha - \gamma) I_\lambda + \lambda \beta I_\lambda \left(e^{-mI_\lambda} \left(1 - \frac{I_\lambda}{\bar{N}}\right) - 1\right) = 0 \text{ in } \Omega; \quad \frac{\partial I_\lambda}{\partial n} \bigg|_{\partial \Omega} = 0.$$  

(21)

Note that $X = N(L) \oplus X_1$, $Y = N(L) \oplus Y_1$, where $N(L) = \text{span}\{\phi\}$,

$$X_1 = X \cap \mathcal{R}(L) = \left\{ \varphi \in X : \int_\Omega \varphi \phi dx = 0 \right\}, \quad Y_1 = \mathcal{R}(L) = \left\{ \varphi \in Y : \int_\Omega \varphi \phi dx = 0 \right\}.$$

By using the implicit function theorem, we can calculate the unique positive steady state $I_\lambda$ near $\lambda^*$, which will be used later.

**Lemma 5** Assumed that $A_2$, $B_1$ hold. There exist $\lambda^* > \lambda^*$ and a continuously differential mapping $\lambda \mapsto (\xi_\lambda, A_\lambda)$ from $[\lambda^*, \lambda^*]$ to $X_1 \times \mathbb{R}^+$ such that, for $\lambda \in [\lambda^*, \lambda^*]$, the unique positive steady state of (21) has the following form:

$$I_\lambda = A_\lambda (\lambda - \lambda^*) (\phi + (\lambda - \lambda^*) \xi_\lambda).$$  

(22)

Moreover, for $\lambda = \lambda^*$,

$$A_{\lambda^*} = \frac{\int_\Omega (\beta - \alpha - \gamma) \phi^2 dx}{\int_\Omega \lambda^* \beta (m + 1/\bar{N}) \phi^2 dx}$$  

(23)

and $\xi_{\lambda^*} \in X_1$ is the unique solution of the following equation:

$$L \xi_{\lambda^*} + (\beta - \alpha - \gamma) \phi - \lambda^* A_{\lambda^*} \beta (m + 1/\bar{N}) \phi^2 = 0,$$  

(24)

where $L$ is defined in (19).

**Proof** To start with, we show that $A_{\lambda^*}$ and $\xi_{\lambda^*}$ are well defined. Note that

$$\lambda^* \int_\Omega (\beta - \alpha - \gamma) \phi^2 dx = \int_\Omega |\nabla \phi|^2 dx,$$

then $A_{\lambda^*}$ is well defined and positive. Note that $L$ is bijective from $X_1$ to $\mathcal{R}(L)$ and

$$(\beta - \alpha - \gamma) \phi - \lambda^* A_{\lambda^*} \beta (m + 1/\bar{N}) \phi^2 \in \mathcal{R}(L),$$

hence $\xi_{\lambda^*}$ is well defined.
Substituting $I_\lambda = A_\lambda (\lambda - \lambda_*) (\phi + (\lambda - \lambda_*) \xi_\lambda)$ into (21), we see that $(A_\lambda, \xi_\lambda)$ satisfies

$$F(\xi_\lambda, A_\lambda, \lambda) := L \xi_\lambda + (\beta - \alpha - \gamma)(\phi + (\lambda - \lambda_*) \xi_\lambda) + \lambda \beta (\phi + (\lambda - \lambda_*) \xi_\lambda) M_\lambda = 0,$$

where

$$M_\lambda = \frac{e^{-mI_\lambda} (1 - I_\lambda/\tilde{N}) - 1}{\lambda - \lambda_*}.$$

It is easy to verify by standard Sobolev embedding theory that $F(\xi_\lambda, A_\lambda, \lambda)$ is a function from $X \times \mathbb{R}^2$ to $Y$. Note from (23) and (24) that $F(\xi_\lambda^*, A_\lambda^*, \lambda^*) = 0$ and the Fréchet derivative of $F$ with respect to $(\xi, A)$ yields $(\lambda^*, \xi_\lambda^*, A_\lambda^*)$

$$D_{(\xi, A)} F(\xi_\lambda^*, A_\lambda^*, \lambda^*) (\psi, \epsilon) = L \psi - \epsilon \lambda^* \beta (m + 1/\tilde{N}) \phi^2.$$

Note that $D_{(\xi, A)} F(\xi_\lambda^*, A_\lambda^*, \lambda^*)$ is a bijection from $X \times \mathbb{R}^2$ to $Y$, which together with the implicit function theorem imply that there exist $\lambda^* > \lambda_*$ and a continuously differential mapping $\lambda \mapsto (A_\lambda, \xi_\lambda)$ from $[\lambda_*, \lambda^*]$ to $X \times \mathbb{R}^+$ such that

$$F(\xi_\lambda, A_\lambda, \lambda) = 0, \lambda \in [\lambda_*, \lambda^*].$$

Therefore, $A_\lambda (\lambda - \lambda_*) (\phi + (\lambda - \lambda_*) \xi_\lambda)$ is a positive solution of (21). \qed

### 3.1 Eigenvalue problems

In this part, we assume $\lambda \in [\lambda_*, \lambda^*]$, derive the eigenvalue problem relative to the positive steady state $I_\lambda$ in system (18) and investigate the existence of purely imaginary roots. Linearizing system (18) at $I_\lambda$ yields

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta u + \lambda K_\lambda u - \lambda N_\lambda u(t - \tau), & x \in \Omega, t > 0, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, t > 0,
\end{cases}$$

(25)

where

$$K_\lambda = \beta e^{-mI_\lambda} (1 - 2I_\lambda/\tilde{N}) - (\alpha + \gamma), \quad N_\lambda = \beta me^{-mI_\lambda} I_\lambda (1 - I_\lambda/\tilde{N}).$$

It follows from (Wu 1996, Theorem 3.1.5) that the solution semigroup of (25) has the infinitesimal generator $A_\tau (\lambda)$ given by

$$A_\tau (\lambda) \Psi = \dot{\Psi},$$

where

$$D(A_\tau (\lambda)) = \{ \Psi \in C_C \cap C_C^1 : \dot{\Psi} (0) = \Delta \Psi (0) + \lambda K_\lambda \Psi (0) - \lambda N_\lambda \Psi (-\tau) \}.$$
and $C^1_C = C^1([-\tau, 0], Y_C)$. Note that $\mu \in C$ is an eigenvalue of $A_\tau$, if and only if there exists $\psi \in X_C \setminus \{0\}$ such that $\Delta(\lambda, \mu, \tau)\psi = 0$, where

$$\Delta(\lambda, \mu, \tau)\psi := \Delta\psi + \lambda K_\lambda \psi - \lambda N_\lambda e^{-\mu\tau} \psi - \mu\psi = 0. \quad (26)$$

Moreover, the eigenvalues of $A_\tau$ depend continuously on $\tau$ ((Chow and Hale 1982, Chapter 14)). It can be seen from (26) that $A_\tau(\lambda)$ has a pair of purely imaginary eigenvalue $\mu = \pm iw$ for some $w > 0$, if and only if

$$\Delta\psi + \lambda K_\lambda \psi - \lambda N_\lambda e^{-i\theta} \psi - iw\psi = 0 \quad (27)$$

is solvable for some value of $w > 0$, $\theta \in [0, 2\pi)$, and $\psi \in X_C \setminus \{0\}$.

Solving (27) for any $\lambda > \lambda_*$ is still a challenging problem. In what follows, we will solve (27) for $\lambda \in [\lambda_*, \lambda^*]$ by using the implicit function theorem. It follows from $X = N(L) + X_1$ that if $(w, \theta, \psi)$ solves (27), then ignoring a scalar factor, $\psi \in X_C \setminus \{0\}$ can be represented as

$$\psi = \kappa \phi + (\lambda - \lambda_*)z, \| \psi \|_{Y_C}^2 = \| \phi \|_{Y_C}^2, \quad (28)$$

where $z \in (X_1)_C$ and $\kappa \geq 0$. Now for $\lambda \in [\lambda_*, \lambda^*]$, substituting (22), (28) and $w = (\lambda - \lambda_*)h$ into (27), we obtain that $(w, \theta, \psi)$ with $w > 0$, $\theta \in [0, 2\pi)$, $\psi \in X_C \setminus \{0\}$ and $\| \psi \|_{Y_C}^2 = \| \phi \|_{Y_C}^2$ solves (27), if and only if the following system:

$$\begin{cases} 
  g_1 = Lz + \frac{\lambda K_\lambda - \lambda_* (\beta - \gamma - \alpha)}{\lambda - \lambda_*} (\kappa \phi + (\lambda - \lambda_*)z) \\
  -\frac{\lambda N_\lambda}{\lambda - \lambda_*} (\kappa \phi + (\lambda - \lambda_*)z) e^{-i\theta} - i h (\kappa \phi + (\lambda - \lambda_*)z) = 0, \\
  g_2 = (\kappa^2 - 1) \| \phi \|_{Y_C}^2 + (\lambda - \lambda_*)^2 \| z \|_{Y_C}^2 = 0, 
\end{cases} \quad (29)$$

is solvable for some value $z \in (X_1)_C$, $h > 0$, $\kappa \geq 0$, $\theta \in [0, 2\pi)$. Define $G : (X_1)_C \times \mathbb{R}^4 \to Y_C \times \mathbb{R}$ by $G = (g_1, g_2)$. Then we have the following results where the proof is in Appendix:

**Theorem 4** Assumed that A2, B1 hold. There exist $\tilde{\lambda}^* > \lambda_*$ and a continuously differentiable mapping $\lambda \to (z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)$ from $[\lambda_*, \tilde{\lambda}^*]$ to $(X_1)_C \times \mathbb{R}^3$ such that $G(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda, \lambda) = 0$. Moreover, for $\lambda \in [\lambda_*, \tilde{\lambda}^*]$,

$$G(z, \kappa, h, \theta, \lambda) = 0, \quad z \in (X_1)_C, \quad h \geq 0, \quad \kappa \geq 0, \quad \theta \in [0, 2\pi) \quad (30)$$

has a unique solution $(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)$.

If Theorem 4 holds true, then the following results can be directly derived.

**Corollary 1** Assumed that A2, B1 hold. For each $\lambda \in [\lambda_*, \tilde{\lambda}^*]$, the following equation

$$\Delta(\lambda, i w, \tau)\psi = 0, \quad w > 0, \quad \tau \geq 0, \quad \psi \in X_C \setminus \{0\}$$

has a unique solution $(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)$. 

$$\square$$ Springer
has a nontrivial solution \((w, \tau, \psi)\), if and only if

\[
    w = w_\lambda = (\lambda - \lambda_*) h_\lambda, \quad \psi = \psi_\lambda = c(\kappa_\lambda \phi + (\lambda - \lambda_*) z_\lambda), \quad \tau = \tau_n = \frac{\theta_\lambda + 2n\pi}{w_\lambda},
\]

where \(n = 0, 1, 2, \ldots\), \(c\) is a nonzero constant and \((z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda)\) is defined in Theorem 4.

### 3.2 Hopf bifurcation

In what follows, we will always assume \(\lambda \in [\lambda_*, \tilde{\lambda}^*]\) for simplicity and take the delay \(\tau\) as a bifurcation parameter to investigate the stability of the positive steady state \(I_\lambda\) in system (18). Particularly, we will explore the existence of local Hopf bifurcation at \(I_\lambda\) for system (18). By Theorem 3, we have

**Lemma 6** Assume that \(\lambda \in [\lambda_*, \tilde{\lambda}^*]\). If \(\tau = 0\), then all eigenvalues of \(A_\tau(\lambda)\) have negative real parts; if \(\tau > 0\), then 0 is not an eigenvalue of \(\lambda \in [\lambda_*, \tilde{\lambda}^*]\).

We show \(i w_\lambda\) is a simple eigenvalue of \(A_{\tau_n}(\lambda)\) for \(n = 0, 1, 2, \ldots\) in the subsequent lemma, where \(\tau_n\) is defined in (31). Thus by the implicit function theorem, there exists a neighborhood \(O_n \times P_n \times Q_n \subset \mathbb{R} \times \mathbb{C} \times X_C\) of \((\tau_n, i w_\lambda, \psi_\lambda)\) and a continuously differential function \((\mu(\tau), \psi(\tau)) : O_n \to P_n \times Q_n\) such that for any \(\tau \in O_n\), the only eigenvalue of \(A_{\tau}(\lambda)\) in \(P_n\) is \(\mu(\tau)\), i.e.,

\[
    \Delta(\lambda, \mu(\tau), \tau)\psi(\tau) = 0, \quad \tau \in O_n
\]

with \(\mu(\tau_n) = i w_\lambda, \psi(\tau_n) = \psi_\lambda\).

**Lemma 7** Assume that \(\lambda \in [\lambda_*, \tilde{\lambda}^*]\). Then \(\mu = i w_\lambda\) is a simple eigenvalue of \(A_{\tau_n}(\lambda)\) for \(n = 0, 1, 2, \ldots\), where \(i w_\lambda\) and \(\tau_n\) are defined as in Corollary 1.

**Proof** See Appendix.

Moreover, we have the following transversality condition:

**Lemma 8** Assume that \(\lambda \in [\lambda_*, \tilde{\lambda}^*]\). Then

\[
    \text{Re} \left( \frac{d\mu(\tau)}{d\tau} \bigg|_{\tau = \tau_n} \right) > 0, \quad n = 0, 1, 2, \ldots
\]

**Proof** See Appendix.

**Remark 1** Here \(0 < \tilde{\lambda}^* - \lambda_* \ll 1\) and the value of \(\tilde{\lambda}^*\) may be chosen smaller than the one in Theorem 4, since perturbation arguments are used in the proof of Lemma 8.

From Corollary 1, Lemmas 7 and 8, we have the results on the distribution of eigenvalues of \(A_\tau(\lambda)\) for \(\lambda \in [\lambda_*, \tilde{\lambda}^*]\).
**Theorem 5** For $\lambda \in [\lambda_*, \tilde{\lambda}^*]$, the infinitesimal generator $A_\tau(\lambda)$ has exactly $2(n+1)$ eigenvalues with positive real parts when $\tau \in (\tau_n, \tau_{n+1}]$, $n = 0, 1, 2, \cdots$.

Then by the local Hopf bifurcation theorem for partial functional differential equations ((Wu 1996, Theorem 2.1 in Chapter 6)), Lemmas 6, 7 and 8, we obtain the stability and associated local Hopf bifurcations of the positive steady state solution $I_\lambda$ in system (18).

**Theorem 6** Assumed that $R_0 > 1$, $d_S = d_I = d$, and $A_1, A_2, B_1$ hold, and $\lambda \in [\lambda_*, \tilde{\lambda}^*]$. Then the positive steady state $I_\lambda$ of system (18) is locally asymptotically stable when $\tau \in [0, \tau_0)$, and unstable when $\tau \in (\tau_0, \infty)$. Moreover, when $\tau = \tau_n$, system (18) occurs Hopf bifurcation at the positive steady state $I_\lambda$.

If $B_2(\int_{\Omega} \beta dx > \int_{\Omega} (\gamma(x) + \alpha(x)) dx)$ rather than $B_1(\int_{\Omega} \beta dx < \int_{\Omega} (\gamma(x) + \alpha(x)) dx)$ holds, we can similarly obtain stability and local Hopf bifurcation at the positive steady state $I_\lambda$ and the proof for $B_2$ is slightly different from that for $B_1$. For $B_2$, $\lambda_* = 0$ and $\phi$ become constant, and $L, X_1, Y_1$ should be made some adjustment, and other calculations are similar, so we omit the proof here.

**Theorem 7** Assumed that $R_0 > 1$, $d_S = d_I = d$, and $A_1, A_2, B_2$ hold, and $\lambda \in [0, \tilde{\lambda}^*]$. Then the positive steady state $I_\lambda$ of system (18) is locally asymptotically stable when $\tau \in [0, \tau_0)$, and unstable when $\tau \in (\tau_0, \infty)$. Moreover, when $\tau = \tau_n$, system (18) occurs Hopf bifurcation at the positive steady state $I_\lambda$.

### 4 Global existence of periodic solutions

Throughout this section, we assume that $R_0 > 1$, $d_S = d_I = d$, $\lambda \in [\lambda_*, \tilde{\lambda}^*]$ and study the global continuation of periodic solutions bifurcating from the point $(I_\lambda, \tau_n)$, $n = 1, 2, \cdots$ for system (18) by using global Hopf bifurcation theorem developed in (Wu 1996, Section 6.5). For convenience, we use the notations in (Wu 1996, Section 6.5). Let $\tilde{T}(t)$ be the semigroup on $Y$ associated with $\Delta$ under Neumann boundary condition and set $A_T : \mathcal{D}(A_T) \rightarrow Y$ to be the generator of $\tilde{T}(t)$. Denoting $u(\cdot, t) = I(\cdot, \tau t)$, we can rewrite system (18) as the following semilinear functional differential equation:

$$\dot{u} = \tau A_T u + \tau f(\tau, u_t), \quad (33)$$

where $u_t(\theta) = u(\cdot, t + \theta), \theta \in [-\tau, 0]$ and

$$f(\tau, u_t)(x) = \lambda \beta(x)e^{-m(x)u(x, t-1)} \left(1 - \frac{u(x, t)}{N(x)}\right) u(x, t) - \lambda(\gamma(x) + \alpha(x))u(x, t).$$

It follows from (Wu 1996, Theorem 3.1.5) that the solution semigroup of (33) has the infinitesimal generator $\tilde{A}_\tau(\lambda)$ given by

$$\tilde{A}_\tau(\lambda)\Psi = \dot{\Psi},$$
with
\[ D(\tilde{A}_\tau(\lambda)) = \{ \Psi \in C_C \cap C_C^1 : \dot{\Psi}(0) = \tau \Delta \Psi(0) + \tau \lambda K_\lambda \Psi(0) - \tau \lambda N_\lambda \Psi(-1) \} \]
and \( C_C^1 = C^1([-1, 0], Y_C) \).

To state the global Hopf bifurcation theorem, similar to (Wu 1996, Section 6.5), we define
(i) \( E = C(S_1; X) \) is a real isometric Banach representation of the group \( G = S_1 := \{ z \in C : |z| = 1 \} \);
(ii) Let \( E^G := \{ x \in E : gx = x \) for all \( g \in G \). Then \( E^G = X \), and \( E \) has an
isotypical direct sum decomposition \( E = E^G \bigoplus_{k=1}^{\infty} E_k \) where \( E_k = \{ e^{ikt}x : x \in X \} \)
for \( k \geq 1 \).

Then from (Wu 1996, Section 6.5), system (33) can be casted into an integral equation
which is continuously differentiable, completely continuous, and G-invariant. Now
we verify the three conditions \( H1-3 \) in (Wu 1996, Section 6.5).

**H1:** Note that \( I_\lambda \in D(A_T) \), \( \tau_\eta \in \mathbb{R} \) satisfies \( A_T I_\lambda + f(\tau_\eta, I_\lambda) = 0 \). From Lemma 6,
for any \( \tau \geq 0 \), \( 0 \) is not an eigenvalue of \( \tilde{A}_\tau(\lambda) \), hence the assumption (H1) in (Wu
1996, Section 6.5) is satisfied.

**H2:** When \( \tau = \tau_\eta \), \( \tilde{A}_\tau(\lambda) \) has a unique pair of purely imaginary eigenvalues \( i w_\lambda \tau_\eta \),
hence the assumption (H2) in (Wu 1996, Section 6.5) holds.

**H3:** We choose sufficiently small \( \epsilon_0, \eta_0 > 0 \), and define the local steady state manifold
\[ M_\lambda = \{ (I_\lambda, \tau, \zeta) : |\tau - \tau_\eta| < \epsilon_0, |\zeta - w_\lambda \tau_\eta| < \eta_0 \} \subset E^G \times \mathbb{R} \times \mathbb{R}_+ \]

Then for \( (\tau, \zeta) \in [\tau_\eta - \epsilon_0, \tau_\eta + \epsilon_0] \times [w_\lambda \tau_\eta - \eta_0, w_\lambda \tau_\eta + \eta_0] \), \( i \zeta \) is an eigenvalue of
\( \tilde{A}_\tau(\lambda) \) if and only if \( \tau = \tau_\eta \) and \( \zeta = w_\lambda \tau_\eta \) from Lemma 7. This verifies the assumption
(H3) in (Wu 1996, Section 6.5). Thus by (Wu 1996, Lemma 6.5.3), \( (I_\lambda, \tau_\eta, w_\lambda \tau_\eta) \) is
an isolated singular point in \( M_\lambda \).

Let \( \mu_k(I_\lambda, \tau_\eta, w_\lambda \tau_\eta)(k = 1, 2, \cdots) \) be the generalized crossing number defined
in (Wu 1996, Section 6.5). Then from Lemma 8, if \( \zeta(\tau) = \alpha(\tau) \pm i \beta(\tau) \) are the
eigenvalues of \( \tilde{A}_\tau(\lambda) \) satisfying \( \zeta(\tau_\eta) = i w_\lambda \tau_\eta \), then \( \alpha'(\tau_\eta) > 0 \). This implies that
\( \mu_1(I_\lambda, \tau_\eta, w_\lambda \tau_\eta) = 1 \).

Then by (Wu 1996, Theorem 6.5.4), we obtains the local topological Hopf bifurcation for system (18) at \( \tau = \tau_\eta \).

Next we consider the global continuation of the local Hopf bifurcation. Let
\[ S := \{(z, \tau, \zeta) \in E \times \mathbb{R} \times \mathbb{R}_+ : u(\cdot, t) = z(\cdot, \omega t) \ \text{is a nontrivial} \ 2\pi/\zeta \ \text{periodic solution of system (1.4)}\} \]

We also define the complete steady state manifold:
\[ M_\lambda^* = \{ (I_\lambda, \tau) \} \subset E^G \times \mathbb{R} \]

Let \( C_n = \mathbb{C}(I_\lambda, \tau_\eta, w_\lambda \tau_\eta) \) denote the connected component of \( S \) with respect to
the local bifurcation \( (I_\lambda, \tau_\eta, w_\lambda \tau_\eta) \). Then it follows from the global Hopf bifurcation theorem ((Wu 1996, Theorem 6.5.5)) that
Theorem 8 For each \( n = 1, 2, \ldots, \), \( C_n \) is unbounded, i.e.,

\[
\sup \{ \max_{t \in \mathbb{R}} |z(t)| + |\tau| + \zeta + \zeta^{-1} : (z, \tau, \zeta) \in C_n \} = \infty.
\]

Proof It follows from the global Hopf bifurcation theorem ((Wu 1996, Theorem 6.5.5)), one of the following assertions holds:
(i) \( C_n \) is unbounded; or
(ii) \( C_n \cap (M^*_\lambda \times \mathcal{R}^+) \) is finite and for all \( k \geq 1, \)

\[
\sum_{(z, \tau, \zeta) \in C_n \cap (M^*_\lambda \times \mathcal{R}^+)} \mu_k(z, \tau, \zeta) = 0,
\]

where \( \mu_k \) is the k-th generalized crossing number. By Lemma 8, if \( \zeta(\tau) = \alpha(\tau) \pm i \beta(\tau) \) are the eigenvalues of \( \bar{A}_\tau(\lambda) \) satisfying \( \zeta(\tau_n) = i w_\lambda \tau_n \), then \( \alpha'(\tau_n) > 0 \), which implies \( \mu_1(I_\lambda, \tau_n, w_\lambda \tau_n) = 1, n = 1, 2, \ldots \). Thus for \( k = 1 \), we have

\[
\sum_{(z, \tau, \zeta) \in C_n \cap (M^*_\lambda \times \mathcal{R}^+)} \mu_1(z, \tau, \zeta) > 0.
\]

Hence, (ii) fails and (i) holds. \( \square \)

Now we have the connected component \( C_n \) is unbounded. Therefore, if we verify the following three assertions:

Claim one: The projection of \( C(I_\lambda, \tau_n, w_\lambda \tau_n) \) onto \( T \)-space is bounded,
Claim two: The projection of \( C(I_\lambda, \tau_n, w_\lambda \tau_n) \) onto the \( \tau \)-space does not intersect with \( \tau = 0 \),
Claim three: The projection of \( C(I_\lambda, \tau_n, w_\lambda \tau_n) \) onto \( z \)-space is bounded,

then we can conclude that the projection of \( C(I_\lambda, \tau_n, w_\lambda \tau_n) \) onto the \( \tau \)-space can be extended to \( \infty \). Therefore, we reap the final results of global Hopf bifurcation branches.

To start with, we prove Claim one. (31) yields that

\[
\frac{1}{n + 1} < \frac{2\pi}{\tau_n w_\lambda} < 1.
\] (34)

If we exclude the existence of periodic solutions of period 1, then system (33) has no periodic solutions of period \( \frac{1}{n} \) for any positive integer \( n \). Then, we can obtain the projection of \( C(I_\lambda, \tau_n, w_\lambda \tau_n) \) onto \( T \)-space is bounded.

Lemma 9 If \( \mathcal{R}_0 > 1, \lambda \in [\lambda_*, \tilde{\lambda}^*], \) then the system (33) has no periodic solutions of period 1.
Proof Assume by contradiction that system (33) has periodic solutions of period 1. Then the following system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \tau \Delta u + \lambda \tau \beta(x)e^{-m(x)u} \left( \tilde{N} - u \right) u - \lambda \tau \gamma(x) u - \lambda \tau \alpha(x) u, \\
\frac{\partial u}{\partial n} &= 0,
\end{align*}
\] (35)
x \in \Omega, t > 0,

admits some periodic solution of period 1, which contradicts Theorem 3. This contradiction completes the proof.

From (Friesecke 1993, Theorem 2) (see also (Wu 1996, Section 10.2)), for small enough \( \tau > 0 \), the unique positive steady state solution \( I_\lambda \) is globally asymptotically stable for all positive initial values for system (33). Thus Claim two holds.

Lemma 10 If \( R_0 > 1, \lambda \in [\lambda_*, \tilde{\lambda}^*], \) then system has no positive nontrivial periodic orbit for small \( \tau > 0 \).

Now we proceed to prove Claim three. Theorem 1 yields the following lemma, implying that the projection of \( C(I_\lambda, \tau_n, w_\lambda, \tau_n) \) onto \( z \)-space has upper bound.

Lemma 11 For any initial value \( \phi > 0 \), the solution of system (33) are uniformly bounded.

Lemma 12 For any \((z, \tau, \zeta) \in C_n\), let \( u(x, t) \) be the \( \omega \)-periodic solution of system (33) with delay \( \tau \) and \( u \) is a representation of \( z \). Then we have \( u(x, t) > 0 \) for \( t \in \mathbb{R} \) and \( x \in \Omega \).

Proof If \((z, \tau, \zeta) \in C_n\), then we obtain from Lemma 10 that \( \tau > 0 \), and from (31) that \( \zeta > 0 \). Note that \((I_\lambda, \tau_n, w_\lambda, \tau_n) \in C_n\) then any \((z, \tau, \zeta) \in C_n\) near \((I_\lambda, \tau_n, w_\lambda, \tau_n)\) satisfies \( u(x, t) > 0 \) for \( t \in \mathbb{R} \) and \( x \in \Omega \), where \( u(x, t) \) is an \( \zeta \)-periodic solution of (33) with delay \( \tau \) and \( u \) is a representation of \( z \). Suppose by contradiction that Lemma 12 does not hold for all \((z, \tau, \zeta) \in C_n\). Then there exists a \((z^*, \tau^*, \zeta^*) \in C_n\) such that if \( u^*(x, t) \) is an \( \zeta^* \)-periodic solution of (1.4) with delay \( \tau^* \) and \( u^* \) is a representation of \( \zeta^* \), and \( u^*(x^*, t^*) = 0 \) for some \( x^* \in \Omega \) and \( t^* \in \mathbb{R} \), which by the strong maximum principle of parabolic equations (Protter and Weinberger 1984, Chapter 5, Theorem 5, P.173) implying that \( u^*(x, t) \equiv 0 \). Thus system (33) occurs Hopf bifurcation at \( u = 0 \). This contradiction completes the proof.

Finally we verify the conditions H1-3 in the global Hopf bifurcation theorem ((Wu 1996, Section 6.5)) and finish the proof of Claims one, two, three, thus we obtain

Theorem 9 Assume that \( R_0 > 1, \lambda \in [\lambda_*, \tilde{\lambda}^*], \) then for any \( \tau > \tau_1 \) system (33) has at least one nontrivial periodic solution.

5 Numerical simulations and discussions

In this section, we present some numerical simulations to demonstrate the analytic results in previous sections and investigate the effect of delayed media impact and
human motility in heterogeneous environment on the transmission dynamics of infectious diseases. Particularly, we will explore the effect of delayed media impact and human mobility on persistence and final epidemic size of infectious disease.

In epidemiology, disease persistence and epidemic size are mostly cared about. Disease persistence is directly related to the basic reproduction number $R_0$ (Theorem 2). It is often the case that a disease dies out if the basic reproduction number is less than unity and the disease is established in the population if it is greater than unity. Here we mention that if $R_0 > 1$ (endemic is established), we characterize the epidemic size by

$$H_I := \int_\Omega \tilde{I} \, dx;$$

if $R_0 < 1$ (disease dies out), and the epidemic size is characterized by

$$H_I := \int_0^\infty \int_\Omega \beta(x)e^{-m(x)I_S}I \frac{SI}{S + I} \, dxdt. \quad (36)$$

Note that if $R_0 < 1$, then $I(x, t)$ is exponentially decay to zero. Thus

$$H_I = \int_0^\infty \int_\Omega \beta(x)e^{-m(x)I_S}I \frac{SI}{S + I} \, dxdt < \infty$$

and well defined. Here we mention that for $R_0 > 1$, we can not calculate the total infection anymore since it will tend to infinity, and we use $H_I := \int_\Omega \tilde{I} \, dx$ to represent epidemic size. The definition is motivated by population size in ecology theory (Lou 2006).

Throughout this section, we fixed $\Omega = (0, 1)$ and the total population $\tilde{N}$ as 1, then $I(x, t)$ represents the fraction of infected individuals at the position $x$. Here we mention that some parameters fixed in this section is purposely for demonstrating our theoretical results, such as the occurrence of local Hopf bifurcation.

5.1 The effect of human mobility in heterogeneous environment

By Lemma 3, we obtain that $R_0$ is a monotone decreasing function of $d_I$ with $R_0 \to \frac{\int_\Omega \beta dx}{\int_\Omega (\gamma(x) + \alpha(x)) dx}$ as $d_I \to \infty$. On one hand, the basic reproduction number in spatial heterogeneous model is bigger than spatial homogeneous model, which implies spatial heterogeneity enhance the disease persistence. On the other hand, it seems nonintuitive that the bigger the diffusion rate, the smaller the basic reproduction number. An explanation is that the diffusion pattern is medical resources oriented, since diffusion has no effect on transmission rate ($\beta(x)$).

For spatial homogeneous ordinary differential models, it is often the case that the bigger the basic reproduction number $R_0$, the larger the epidemic size. Thus $R_0$ is also a commonly used measure of the effort needed to control an infectious disease. However, for model with human mobility in heterogeneous environment, the situation
may change. In what follows, we will show that the case that the larger the basic reproduction number $R_0$, the larger the epidemic size, may be not true.

The following two special case shows that the smaller the basic reproduction number, the larger the epidemic size under the condition $R_0 < 1$ and $R_0 > 1$, respectively.

Fix other parameters and let $d$ change

$$
r = 0, \omega = (0, 1), d_s = 0.2, \lambda = 2 + 0.01 \sin(2\pi x), \alpha = 0.2 + 0.01 \sin(2\pi x), \beta(x) = 2(3 + \sin(2\pi x)), \gamma = 2(4 + \sin(\pi x))(\text{day}^{-1}),$$

$$\alpha = 0.1(1 + \cos(2\pi x))(\text{day}^{-1}), m(x) = 0. \quad (37)$$

and

$$
r = 0, \omega = (0, 1), d_s = 0.2, \lambda = 2 + 0.01 \sin(2\pi x), \alpha = 0.2 + 0.01 \sin(2\pi x), \beta(x) = 2(5 + \sin(2\pi x)), \gamma = 1(\text{day}^{-1}),$$

$$\alpha = 0.1(1 + \cos(2\pi x))(\text{day}^{-1}), m(x) = 0. \quad (38)$$

By Lemma 3, the basic reproduction number is decreasing in $d_I$, however Fig. 1 (a) and (b) shows that the epidemic size is increasing in $d_I$ under $R_0 < 1$ and $R_0 > 1$, respectively, which to some extent shows that the epidemic size may be a decreasing function of the basic reproduction number. We mention here that two numerical examples are special cases and the results are not general. Actually, for the SIS reaction diffusion model (1) proposed in Allen et al. (2008), larger $R_0$ may not imply larger population size. One can see some theoretical results on this issue from epidemiology perspective in Gao (2020) and ecology perspective in Lou (2006). The epidemic model includes natural birth rate and the infection of newly increment population induces $H_I > 1$.

5.2 The effect of delayed media impact

In this part, we focus the simulations on system (17) for simplicity. We fixed the parameters as
Fig. 2 Solutions of system (17) showing that (A) the endemic steady state is asymptotically stable for $r = 6.4 < r_0 \approx 6.4$, and (B) the bifurcated periodic solution is feasible for $r = 6.8 > r_0 \approx 6.4$. Parameters are fixed as (39).

By direct calculation we obtain that $R_0 = 1.1$ and the bifurcation point $r_0 \approx 6.4$. It can be observed from Fig. 2(A) that the endemic steady state is asymptotically stable for $r = 6 < r_0 \approx 6.4$ and from Fig. 2(A) that the bifurcated periodic solution is feasible for $r = 6.8 > r_0 \approx 6.4$. Further, we plotted the bifurcation diagram by using the delay $r$ as the bifurcation parameter (shown in Fig. 3).

By direct calculation from (39), the local basic reproduction number of system (17) $R(x) := \beta(x)/(\gamma(x) + \alpha(x)) < 1$ at $0 < x < 0.65$ and the basic reproduction number
with respect to the spatial homogeneous environment \( \int_{\Omega} \beta(x)dx < \int_{\Omega}(\gamma(x) + \alpha(x))dx \) is smaller than one. However, Fig. 3 shows that for system (17), disease persists in any place and when the delay is large, disease oscillates at any place, which implies spatial heterogeneity enhances disease persistence and oscillation, making disease prediction and control much harder.

6 Conclusions

In this paper, we consider an SIS (suspected-infected-suspected) functional partial differential equation model cooperated with spatial heterogeneity and delayed media impact. The psychological impact of media coverage and rapid information flow on the public is depicted by the reduction in incidence rate at location \( x \), which is expressed as a media function \( e^{-m(x)I(x,t-r)} \) with \( x \) depending on location. \( r \) is the delay of mass media impact on infectious disease and it may directly means the mass media response time, or indirectly means individual response time to media reports, such as the time from symptom onset to hospitalization.

We first show the wellposedness of the model including the existence and uniqueness of the solution, and that the solution semiflow is point dissipative, so a global attractor follows. Then we define the basic reproduction number of the system, and prove that when the basic reproducing number \( R_0 < 1 \), the disease-free steady state is globally asymptotically stable; when \( R_0 > 1 \), the disease is uniformly persist. We find that the basic reproduction number has nothing to do with the media impact parameters \( (m(x)) \) and the lag effect \( (r) \), but only related to the diffusion rate. This shows that the delayed media impact does not affect the disease persistence, but the human mobility does. The asymptotic behaviors and monotonicity of the basic reproduction number with respect to the diffusion rate of the infected individuals are studied. The theoretical results show that the basic reproduction number in the spatial heterogeneous environment is larger than that in the spatial homogeneous environment, and when the diffusion rate of the infected individuals is small, even if the basic reproduction number in the space homogeneous environment is less than one (the disease is eliminated), the basic reproduction number is still greater than one under spatial heterogeneous environment.

We prove the existence of local Hopf bifurcation at the endemic steady state state with time delay as the bifurcation parameter and the global Hopf bifurcation theorem is used to prove the global continuation of periodic solutions. Here we mention that we can only obtain the existence of pure imaginary roots of eigenvalue equation in a small range of susceptibility diffusion coefficient by using implicit function theorem. Our theoretical and numerical results show that the lag effect of media impact may lead to the periodic oscillation of disease, which brings great challenges to the prevention and control of disease. Moreover, spatial heterogeneity not only makes the disease more likely to persist, but also makes the disease more prone to periodic oscillation and the oscillation places becomes larger, which makes disease harder to prevent and control. Human mobility in spatial heterogeneous environments makes the disease situation in different regions interact with each other. The prevention and control of this disease is no longer an independent matter of each region, but needs overall planning.
In epidemiology, for the spatially homogeneous ordinary differential epidemic model, a general conclusion is that the larger the basic reproduction number, the more people will eventually be infected. Therefore, the basic reproductive number can be used as an important indicator to control of infectious disease, assess the potential for disease invasion and persistence, to predict the extent of an epidemic, and to infer the impact of interventions and of relaxing control measures. However, the utility of \( R_0 \) may be overstated. One misconception is that the reproductive number is enough to tell us how large an epidemic will be. For spatial epidemic model, the relationship between basic reproduction number and epidemic size is more subtle and complex. Smaller basic reproduction number may induce larger epidemic size. It may not be effective to control the basic reproduction number of the disease alone. This poses a greater challenge for more effective prediction and control of infectious diseases.

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Appendix

Proof of Theorem 2

We first prove (i) by constructing a Lyapunov functional and applying LaSalle’s invariance principle ((Hale 1969, Theorem 1); (Zhao 2017, Theorem 1.1.1)) for infinite dimensional dynamical systems. Recall that system (2) defines a dynamical system on \( C_r^+ \), and \( \Phi(t) \) is the solution semiflow of system (2) on \( C_r^+ \), i.e., \( \Phi(t) \varphi = u_t(\varphi) \), \( t \geq 0 \), where \( u(t, \varphi) \) is the unique solution of system (2) with \( u_0 = \varphi \). By (Wu 1996, Theorem 2.2.6), \( \Phi(t) = u_t(\cdot) : C_r^+ \rightarrow C_r^+ \) is compact, and for each \( \varphi \in C_r^+ \), \( t > r \), the orbit of \( \varphi \) under \( \Phi(t) \) has compact closure in \( C_r^+ \).

For any \( \psi \in C_r^+ \), define the functional

\[
L(\psi) = \int_\Omega \psi(0)\phi_1 dx,
\]

where \( \phi_1 \) is the positive eigenfunction corresponding to the principal eigenvalue \( \lambda_1 \) of the problem (13). For an arbitrary solution \( u(t, \varphi) \) of (2), we obtain

\[
\frac{d}{dt} L(u_t(\varphi)) = \int_\Omega \left( d_I \Delta I + \frac{\beta(x)e^{-m(x)I(x,t-r)}S}{S+I} - \gamma(x)I - \alpha(x)I \right) \phi_1 dx
\]

\[
= \int_\Omega \left( \beta(x)I \left( \frac{e^{-m(x)I(x,t-r)}S}{S+I} - 1 \right) - \lambda_1 I \right) \phi_1 dx.
\]  

(A.1)
Therefore, $\frac{d}{dt} L(u_t(\varphi)) \leq 0$, which implies that $L(\varphi)$ is a Lyapunov functional on $C^+_r$ relative to the system (2).

Next define

$$\dot{L}(\varphi) := \frac{d}{dt} L(u_t(\varphi))\bigg|_{t=0} \quad \text{and} \quad S = \{\varphi \in C^+_r | \dot{L}(\varphi) = 0\},$$

where $u(t, \varphi)$ is the unique solution of (2) with initial condition $u_0 = \varphi \in C^+_r$. By (A.1), we have $S = \{\varphi \in C^+_r | \varphi_2 = 0\}$ and $S$ is invariant under $\Phi(t)$. Thus by the LaSalle invariant principle ((Hale 1969, Theorem 1)), we obtain $\lim_{t \to \infty} I(\cdot, t) = 0$. By (Zhao 2017, Theorem 1.2.1 with Remark 1.3.2) (see also (Thieme 1992, Theorem 4.1)), we have $\lim_{t \to \infty} S(\cdot, t) = \tilde{N}$.

For (ii). We appeal to the theory of uniform persistence theory developed in Zhao (2017); Magal and Zhao (2005). Denote

$$U_0 := \{(\varphi, \varphi_2) \in C^+_r | \varphi_2(0) \neq 0\}; \quad \partial U_0 := C^+_r \setminus U_0.$$

Then $C^+_r = U_0 \cup \partial U_0$, $U_0$ and $\partial U_0$ are relatively open and closed subsets of $U$, respectively, and $U_0$ is convex. Let $\Phi(t)(s_0, i_0) = (S(\cdot, t), I(\cdot, t))$ be the unique solution of system (2) with the initial value $(s_0, i_0) \in C^+_r$ for any $t > 0$. By Theorem 1, $\Phi(t)$ has a global attractor.

**Step 1.** We have $\Phi(t)U_0 \subseteq U_0$ for all $t > 0$. This is a direct result of the strong maximum principle for parabolic equations.

**Step 2.** Let $A_{\beta}$ be the maximal positively invariant set for $\Phi(t)$ in $\partial U_0$, i.e.

$$A_{\beta} := \{(s_0, i_0) \in C^+_r | \Phi(t)(s_0, i_0) \in \partial U_0, t \geq 0\}.$$

It is easy to verify that $A_{\beta} = \{u_0 = (s_0, i_0) \in C^+_r | i_0 = 0\}$. Denote $\omega((s_0, i_0))$ as the $\omega$-limit set of $(s_0, i_0)$ in $C^+_r$ (see Zhao (2017)) and

$$\hat{A}_{\beta} = \cup_{(s_0, i_0) \in A_{\beta}} \omega((s_0, i_0)).$$

It can be seen that $\hat{A}_{\beta} = \{E_0 = (\tilde{N}, 0)\}$. Thus, $\{E_0\}$ is a compact and isolated invariant set for $\Phi(t)$ restricted in $A_{\beta}$.

**Step 3.** We prove that there exists some constant $\epsilon_1 > 0$ independent of initial values such that

$$\limsup_{t \to \infty} \|\Phi(t)(s_0, i_0) - (\tilde{N}, 0)\| > \epsilon_1.$$
Given any small $\epsilon_3 > 0$ and let $\lambda_1(\epsilon_3)$ be the unique principal eigenvalue of the following eigenvalue problem with a positive eigenfunction $\psi_I$:

$$
-d_I \Delta \psi_I - \frac{\beta(x) e^{-m(x)\epsilon_3}(\tilde{N} - \epsilon_3)}{\tilde{N} + 2\epsilon_3} \psi_I + (\gamma(x) + \alpha(x))\psi_I = \lambda \psi_I \quad \text{in} \; \Omega; \quad \frac{\partial \psi_I}{\partial n} \big|_{\partial \Omega} = 0.
$$

Note that $\lim_{\epsilon_3 \to 0} \lambda_1(\epsilon_3) = \lambda_1 < 0$, where $\lambda_1$ is the principal eigenvalue of eigenvalue problem (13). Therefore, we can choose $\epsilon_3$ such that $\lambda_1(\epsilon_3) < 0$. Since $\epsilon_2$ is arbitrary, choose $\epsilon_2 = \epsilon_3$. By (A.2), there exists $T_0 > 0$ such that $|S^* - \tilde{N}|, I^* \leq \epsilon_3$ for all $t > T_0$. By the strong maximum principle of parabolic equations, $S^*(\cdot, t), I^*(\cdot, t) > 0$ for all $t > 0$. Then we can find a small positive constant $c_s$ such that $I^*(x, T) \geq c_s \psi_I$. It is easy to verify that $I^*(x, t)$ is a supersolution of the problem

$$
\begin{aligned}
\frac{\partial \hat{I}}{\partial t} &= d_I \Delta \hat{I} + \frac{\beta(x) e^{-m(x)\epsilon_3}(\tilde{N} - \epsilon_3)}{\tilde{N} + 2\epsilon_3} \hat{I} - (\gamma(x) + \alpha(x))\hat{I}, & x \in \Omega, t > T, \\
\frac{\partial \hat{I}}{\partial n} &= 0, & x \in \partial \Omega, t > T, \\
\hat{I}(x, T) &= c_s \psi_I,
\end{aligned}
$$

and $c_s e^{-\lambda_1(\epsilon_3)(t-T)} \phi_I$ is the unique solution to system (A.3). Note that $\lambda_1(\epsilon_3) < 0$, therefore $I^*(x, t) \geq c_s e^{-\lambda_1(\epsilon_3)(t-T)} \psi_I \to \infty$ uniformly in $\Omega$ as $t \to \infty$. This contradiction finishes the proof of Step 3.

The result of Step 3 implies that $\{E_0\}$ is an isolated invariant set for $\Phi(t)$ in $C^+_r$, and $W^S(\{E_0\}) \cap U_0$ is an empty set, where $W^S(\{E_0\})$ is the stable set of $\{E_0\}$ for $\Phi(t)$.

Finally, by Steps 1-3 and (Zhao 2017, Theorem 1.3.1), $\Phi_t$ is uniformly persistent with respect to $(U, \partial U_0)$. Moreover, by (Zhao 2017, Theorem 1.3.7), (2) admits at least one endemic steady state.

**Proof of Theorem 4**

Before proving Theorem 4, we give three lemmas which will be used to conclude our assertion. To start with, we give estimates for solutions of (27).

**Lemma A.1** If $(w_\lambda, \theta_\lambda, \psi_\lambda)$ is a solution to (27) with $w_\lambda > 0, \theta_\lambda \in [0, 2\pi)$ and $\psi_\lambda \in X_C \setminus \{0\}$, then

$$
\lambda \sin(\theta_\lambda) \int_{\Omega} N_\lambda |\psi_\lambda|^2 dx - w_\lambda \int_{\Omega} |\psi_\lambda|^2 dx = 0. \quad \text{(A.4)}
$$

Moreover, $\frac{w_\lambda}{\lambda - \lambda_*}$ is bounded for $\lambda \in [\lambda_*, \lambda^*]$.
**Proof** It follows from substituting \((w_\lambda, \theta_\lambda, \psi_\lambda)\) into (27), multiplying (27) by \(\bar{\psi}_\lambda\), and integrating the result by part over \(\Omega\) that

\[
- \int_\Omega |\nabla \psi_\lambda|^2 dx + \lambda \int_\Omega K_\lambda |\psi_\lambda|^2 dx - \lambda e^{-i\theta_\lambda} \int_\Omega N_\lambda |\psi_\lambda|^2 dx - iw_\lambda \int_\Omega |\psi_\lambda|^2 dx = 0,
\]

which implies (A.4). Moreover, we obtain

\[
\frac{w_\lambda}{\lambda - \lambda_*} = \frac{\lambda \sin(\theta_\lambda) \int_\Omega N_\lambda |\psi_\lambda|^2 dx}{(\lambda - \lambda_*) \int_\Omega |\psi_\lambda|^2 dx} \leq \lambda_* \max(\beta m) \|\phi\|_\infty + (\lambda - \lambda_*) \|\xi_\lambda\|_\infty.
\]

Thus \(\frac{w}{\lambda - \lambda_*}\) is bounded for \(\lambda \in [\lambda_*, \lambda^*]\).

By similar arguments as Lemma 2.3 in Busenberg and Huang (1996), we get the following result:

**Lemma A.2** If \(z \in X_C\) and \(\langle \phi, z \rangle = 0\), then \(|\langle Lz, z \rangle| \geq \mu^2 \|z\|_{Y_C}^2\), where \(\mu\) is the second eigenvalue of operator \(-L\).

We prove that \(G(z, \kappa, h, \theta, \lambda) = 0\) is uniquely solvable for \(\lambda = \lambda_*\).

**Lemma A.3** The following equation

\[
G(z, \kappa, h, \theta, \lambda) = 0, \quad z \in (X_1)_C, \quad h > 0, \kappa \geq 0, \theta \in [0, 2\pi)
\]

admits a unique solution \((z_{\lambda_*}, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})\), where

\[
\kappa_{\lambda_*} = 1, \quad \theta_{\lambda_*} = \frac{\pi}{2}, \quad h_{\lambda_*} = \frac{\int_\Omega \lambda_* m \beta A_{\lambda_*} \phi^3 dx}{\int_\Omega \phi^2 dx},
\]

and \(z_{\lambda_*} \in (X_1)_C\) is the unique solution of

\[
Lz_{\lambda_*} + (\beta - \gamma - \alpha - \lambda_* \beta A_{\lambda_*} (m + 2/\tilde{N}) \phi) \phi + i \lambda_* m \beta A_{\lambda_*} \phi^2 - ih_{\lambda_*} \phi = 0,
\]

where \(L\) is defined in (19).

**Proof** It follows from the second equation of (A.5) that \(\kappa = \kappa_{\lambda_*} = 1\). Substituting \(\kappa = 1\) and \(\lambda = \lambda_*\) into \(g_2 = 0\) yields

\[
g_2 = Lz + (\beta - \gamma - \alpha - \lambda_* \beta A_{\lambda_*} (m + 2/\tilde{N}) \phi) \phi - e^{-i\theta} \lambda_* m \beta A_{\lambda_*} \phi^2 - ih_{\lambda_*} \phi = 0,
\]

which implies

\[
\begin{align*}
\lambda_* \int_\Omega m \beta A_{\lambda_*} \phi^3 dx \sin(\theta) &= h \int_\Omega \phi^2 dx, \\
\lambda_* \int_\Omega m \beta A_{\lambda_*} \phi^3 dx \cos(\theta) &= 0.
\end{align*}
\]
Therefore, \( \theta = \theta_{\lambda_*} = \frac{\pi}{2} \), \( h = h_{\lambda_*} = \frac{\int_{\Omega} \lambda_* m \beta A_{\lambda_*} \phi^2 dx}{\int_{\Omega} \phi^2 dx} \). Moreover, it is easy to verify that

\[
(\beta - \gamma - \alpha - \lambda_* \beta A_{\lambda_*} (m + 2/N) \phi) \phi, -\lambda_* m \beta A_{\lambda_*} \phi^2 - h_{\lambda_*} \phi \in Y_1.
\]

Thus, \( g_1(z, \kappa, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*) = 0 \) has a unique solution \( z_{\lambda_*} \) satisfies (A.6).

\( \square \)

Now we proceed to prove Theorem 4.

**Proof** let \( T = (T_1, T_2) : (X_1)_C \times \mathbb{R}^3 \to Y_C \times \mathbb{R} \) be the Fréchet derivative of \( G \) with respect to \((z, \kappa, h, \theta)\) at \((z_{\lambda_*}, \kappa, h_{\lambda_*}, \theta_{\lambda_*}, \lambda_*))\), i.e.,

\[
T(z, \kappa, h, \theta) = Lz + \kappa \phi (\beta - \gamma - \alpha - \lambda_* \beta A_{\lambda_*} (m + 2/N) \phi + i \lambda_* m \beta A_{\lambda_*} \phi - ih_{\lambda_*})
\]

\[
- i h \phi + \theta \lambda_* m \beta A_{\lambda_*} \phi^2,
\]

\[
T_2(r) = 2r \| \phi \|_Y^2.
\]

Note that \( T \) is a bijection from \((X_1)_C \times \mathbb{R}^3\) to \(Y_C \times \mathbb{R}\). Thus it follows from the implicit function theorem that there exist \( \tilde{\lambda}_* > \lambda_* \) and a continuously differentiable mapping \( \lambda \to (z_\lambda, \kappa, h_\lambda, \theta_\lambda) \) from \([\lambda_*, \tilde{\lambda}_*]\) to \((X_1)_C \times \mathbb{R}^3\) such that \( G(z_\lambda, \kappa, h_\lambda, \theta_\lambda, \lambda) = 0 \).

Now it remains to prove the uniqueness. We only need to verify that if \( G(z, \kappa, h, \theta, \lambda) = 0 \) with \( z_\lambda \in (X_1)_C, \kappa, h_\lambda > 0, \theta_\lambda \in [0, 2\pi) \), then

\[
(z_\lambda, \kappa, h_\lambda, \theta_\lambda) \to (z_{\lambda_*}, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})
\]

as \( \lambda \to \lambda_* \) in the norm of \( X_C \times \mathbb{R}^3 \). To start with, we show \( z_\lambda \) is bounded in \((X_1)_C\).

Note that \( A_\lambda \) and \( \| \xi_\lambda \|_\infty \) are bounded for \( \lambda \in [\lambda_*, \tilde{\lambda}_*] \). By Lemma A.2 and similar arguments in (Busenberg and Huang 1996, Theorem 2.4), we can obtain that there exist \( M_1, M_2 > 0 \) such that

\[
\lambda_2 \| z_\lambda \|_{Y_C}^2 \leq | < Lz_\lambda, z_\lambda | \leq M_1 \| \phi \|_{Y_C} \| z_\lambda \|_{Y_C} + M_2 (\lambda - \lambda_*) \| z_\lambda \|_{Y_C}^2.
\]

Therefore, if \( \tilde{\lambda}_* \) is sufficiently small, \( z_\lambda \) is bounded in \( Y_C \) for \( \lambda \in [\lambda_*, \tilde{\lambda}_*] \). Then \( z_\lambda \) is also bounded in \((X_1)_C\). Moreover, it follows from Lemma A.1 and (29) that \( \kappa_\lambda, h_\lambda, \theta_\lambda \) are bounded for \( \lambda \in [\lambda_*, \tilde{\lambda}_*] \), which together with the boundedness of \( z_\lambda \) in \((X_1)_C\) implies that \( \{(z_\lambda, \kappa_\lambda, h_\lambda, \theta_\lambda) : \lambda \in [\lambda_*, \tilde{\lambda}_*]\} \) is precompact in \( Y_C \times \mathbb{R}^3 \). Then, there exists a subsequence \( \{(z_{\lambda_n}, \kappa_{\lambda_n}, h_{\lambda_n}, \theta_{\lambda_n})\} \) is convergent in \( Y_C \times \mathbb{R}^3 \) for \( \lambda_n \to \lambda_* \) as \( n \to \infty \). Taking the limit of the equation \( L^{-1} g_1(z_{\lambda_n}, \kappa_{\lambda_n}, h_{\lambda_n}, \theta_{\lambda_n}, \lambda_n) = 0 \) as \( n \to \infty \), we can see from Lemma A.3 that

\[
(z_{\lambda_n}, \kappa_{\lambda_n}, h_{\lambda_n}, \theta_{\lambda_n}) \to (z_{\lambda_*}, \kappa_{\lambda_*}, h_{\lambda_*}, \theta_{\lambda_*})
\]

in \((X_1)_C \times \mathbb{R}^3\) as \( n \to \infty \). Thus we obtain the uniqueness and complete the proof. \( \square \)
Proof of Lemma 7

It follows from Corollary 1 that \( \mathcal{N}(\mathcal{A}_{\tau_n}(\lambda) - iw_{\lambda}) = \text{Span}\{e^{iw_{\lambda} \theta} \psi_{\lambda}\} \). If \( \varphi \in \mathcal{N}(\mathcal{A}_{\tau_n}(\lambda) - iw_{\lambda}) \cap \mathcal{N}(\mathcal{A}_{\tau_n}(\lambda) - iw_{\lambda})^2 \), then

\[
(\mathcal{A}_{\tau_n} - iw_{\lambda})\varphi \in \mathcal{N}(\mathcal{A}_{\tau_n} - iw_{\lambda}) = \text{Span}\{e^{iw_{\lambda} \theta} \psi_{\lambda}\}
\]

Therefore, there exists a constant \( a \) such that

\[
(\mathcal{A}_{\tau_n} - iw_{\lambda})\varphi = ae^{iw_{\lambda} \theta} \psi_{\lambda},
\]

which implies

\[
\begin{cases}
\dot{\varphi}(\theta) = i w_{\lambda} \varphi(\theta) + ae^{iw_{\lambda} \theta} \psi_{\lambda}, \theta \in [-\tau_n, 0], \\
\varphi(0) = \Delta \varphi(0) + \lambda K_{\lambda} \varphi(0) - \lambda N_{\lambda} \varphi(-\tau_n).
\end{cases}
\] (A.7)

We can obtain from the first equation of (A.7) that

\[
\varphi(-\tau_n) = \varphi(0)e^{-i\omega_{\lambda} \tau_n} - a \tau_n e^{-i\omega_{\lambda} \tau_n} \psi_{\lambda}; \dot{\varphi}(0) = i w_{\lambda} \varphi(0) + a \psi_{\lambda},
\]

which together with the second equation of (A.7) yields

\[
\Delta \varphi(0) + \lambda K_{\lambda} \varphi(0) - \lambda N_{\lambda} \varphi(0)e^{-i\omega_{\lambda} \tau_n} - a \tau_n e^{-i\omega_{\lambda} \tau_n} \psi_{\lambda}) - iw_{\lambda} \varphi(0) - a \psi_{\lambda} = 0,
\]

i.e.,

\[
\Delta(\lambda, iw_{\lambda}, \tau_n) \varphi(0) = a \psi_{\lambda} (1 - \lambda N_{\lambda} \tau_n e^{-i\theta_{\lambda}}).
\] (A.8)

Multiplying (A.8) by \( \overline{\psi_{\lambda}} \) and integrating over \( \Omega \) by parts yield that

\[
a \int_{\Omega} |\psi_{\lambda}|^2 (1 - \lambda N_{\lambda} \tau_n e^{-i\theta_{\lambda}}) \, dx = 0.
\] (A.9)

For (A.9), taking the limit \( \lambda \to \lambda_* \), we have

\[
\begin{align*}
\theta_{\lambda} &\to \frac{\pi}{2}, \quad \tau_n(\lambda - \lambda_*) \to \frac{\pi/2 + 2n\pi}{h_{\lambda_*}}, \\
\psi_{\lambda} &\to \phi, \quad \frac{\lambda N_{\lambda}}{\lambda - \lambda_*} \to \lambda_* m \beta A_{\lambda_*} \phi \text{ in } X_{C}.
\end{align*}
\]

Thus \( a = 0 \) can obtained from

\[
\int_{\Omega} \psi_{\lambda}^2 (1 - \lambda N_{\lambda} \tau_n e^{-i\omega_{\lambda} \tau_n}) \, dx = \int_{\Omega} \phi^2 \left(1 + i \frac{\pi/2 + 2n\pi}{h_{\lambda_*}} \lambda_* m \beta A_{\lambda_*} \phi \right) \, dx + o(\lambda - \lambda_*) \neq 0.
\]
Therefore,

\[ \mathcal{N}(\mathcal{A}_n - i w_\lambda)^k = \mathcal{N}(\mathcal{A}_n - i w_\lambda), \quad k = 2, 3, \ldots, n = 0, 1, 2, \ldots. \]

Thus \( \mu = i w_\lambda \) is a simple eigenvalue of \( \mathcal{A}_n(\lambda) \) for \( n = 0, 1, 2, \ldots \).

**Proof of Lemma 8**

It follows from differentiating (32) with respect to \( \tau \) at \( \tau = \tau_n \) that

\[ \Delta(\lambda, \mu, \tau) \psi' + \lambda N_\lambda \psi e^{-\mu \tau} \mu = \frac{d\mu}{d\tau} (1 - \lambda \tau N_\lambda e^{-\mu \tau}) \psi. \]

Then multiplying by \( \overline{\psi} \) and integrating over \( \Omega \) by parts yield

\[ \text{Re} \left( \frac{d\mu}{d\tau} \right) = \text{Re} \left( \frac{\int_{\Omega} (1 - \lambda \tau N_\lambda e^{-\mu \tau}) \psi^2 dx}{\int_{\Omega} \lambda N_\lambda \psi^2 e^{-\mu \tau} v dx} \right) \]

\[ = \text{Re} \left( \frac{\int_{\Omega} \psi^2 dx}{\int_{\Omega} \lambda N_\lambda \psi^2 e^{-\mu \tau} \mu dx} - \frac{\tau}{\mu} \right). \]

Let \( \tau = \tau_n \). Recalling that \( \mu(\tau_n) = i w_\lambda \), \( \psi(\tau_n) = \psi_\lambda \), thus we obtain

\[ \text{Re} \left( \frac{d\mu}{d\tau} \bigg|_{\tau=\tau_n}^{-1} \right) = \text{Re} \left( \frac{\int_{\Omega} \psi^2_\lambda dx}{\int_{\Omega} i \lambda N_\lambda \psi^2_\lambda e^{-i w_\lambda \tau_n} w_\lambda dx} \right). \]

Moreover, taking the limit \( \lambda \to \lambda_* \), we have

\[ \frac{w_\lambda}{\lambda - \lambda_*} \to h_{\lambda_*}, \quad \tau_n (\lambda - \lambda_*) \to \frac{\pi/2 + 2n\pi}{h_{\lambda_*}}, \]

\[ \psi_\lambda \to \phi, \quad \frac{\lambda N_\lambda}{\lambda - \lambda_*} \to \lambda_* m \beta A_{\lambda_*} \phi \text{ in } X_C. \]

Therefore, we have

\[ (\lambda - \lambda_*^2) \text{Re} \left( \frac{d\mu}{d\tau} \bigg|_{\tau=\tau_n}^{-1} \right) = \frac{\int_{\Omega} \phi^2 dx}{\int_{\Omega} \lambda_* m \beta A_{\lambda_*} \phi^3 h_{\lambda_*} dx} + o(\lambda - \lambda_*) > 0. \]

This completes the proof.
References

Allen LJS, Bolker BM, Lou Y, Nevis AL (2008) Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model. Discrete Contin Dyn Syst Ser A 21(1):1–20

Anderson RM, May RM (1991) Infectious Diseases of Humans: Dynamics and Control. Cambridge University Press

Beutels P, Jia N, Zhou QY, Smith R, Cao WC, De Vlas SJ (2009) The economic impact of SARS in Beijing, China. Trop Med Int Health 14:85–91

Busenberg S, Huang W (1996) Stability and Hopf bifurcation for a population delay model with diffusion effects. J Differ Equ 124(1):80–107

Cantrell RS, Cosner C (2003) Spatial Ecology via Reaction-Diffusion Equations. John Wiley and Sons Ltd, Chichester

Chen S, Lou Y (2016) A spatial SIS model in advective heterogeneous environments. J Differ Equ 261(6):3305–3343

Cui R, Lou Y (2017) Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments. J Differ Equ 263(4):2343–2373

Du Z, Peng R (2016) A priori $L^\infty$ estimates for solutions of a class of reaction-diffusion systems. J Math Biol 72(6):1429–1439

Faria T, Huang W, Wu J (2002) Smoothness of center manifolds for maps and formal adjoints for semilinear fdes in general banach spaces. SIAM J Math Anal 34(1):173–203

Freedman HI, Zhao XQ (1997) Global asymptotics in some quasimonotone reaction-diffusion systems with delays. J Differ Equ 137(2):340–362

Friesecke G (1993) Convergence to equilibrium for delay-diffusion equations with small delay. J Dyn Differ Equ 5(1):89–103

Funk S (2010) Modelling the influence of human behaviour on the spread of infectious diseases: a review. J R Soc Interface 7(50):1247–1256

Gao D (2020) How does dispersal affect the infection size? SIAM Journal on Applied Mathematics 80(5):2144–2169

Ge J, Lin L, Zhang L (2017) A diffusive SIS epidemic model incorporating the media coverage impact in the heterogeneous environment. Discrete Contin Dyn Syst Ser B 22(7):2763–2776

Hale J (1988) Asymptotic Behavior of Dissipative Systems. American Mathematical Society

Hale JK (1969) Dynamical systems and stability. J Math Anal Appl 26(1):39–59

Lai S, Ruktanonchai N, Zhou L, Prosper O, Luo W, Floyd J, Wesołowski A, Santillana M, Zhang C, Du X et al (2020) Effect of non-pharmaceutical interventions to contain COVID-19 in China. Nature 585:410–413

Lau JT, Yang X, Tsui H, Pang E (2004) SARS related preventive and risk behaviours practised by Hong Kong-mainland China cross border travellers during the outbreak of the SARS epidemic in Hong Kong. J Epidemiol Community Health 58(12):988–996

Le D (1997) Dissipativity and global attractors for a class of quasilinear parabolic systems. Commun Partial Differ Equ 22(3–4):413–433
Li H, Peng R, Wang FB (2017) Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model. J Differ Equ 262(2):885–913  
Li H, Peng R, Xiang T (2020) Dynamics and asymptotic profiles of endemic equilibrium for two frequency-dependent SIS epidemic models with cross-diffusion. Eur J Appl Math 31(1):26–56  
Li Y, Cui JA (2009) The effect of constant and pulse vaccination on SIS epidemic models incorporating media coverage. Commun Nonlinear Sci Numer Simul 14(5):2353–2365  
Liang X, Zhang L, Zhao XQ (2019) Basic reproduction ratios for periodic abstract functional differential equations (with application to a spatial model for lyme disease). J Dyn Differ Equ 31(3):1247–1278  
Liu R, Wu J, Zhu H (2007) Media/psychological impact on multiple outbreaks of emerging infectious diseases. Comput Math Methods Med 8(3):153–164  
Lou Y (2006) On the effects of migration and spatial heterogeneity on single and multiple species. Journal of Differential Equations 223(2):400–426  
Magal P, Zhao XQ (2005) Global attractors and steady states for uniformly persistent dynamical systems. SIAM J Math Anal 37(1):1–44  
Martin RH Jr, Smith HL (1990) Abstract functional-differential equations and reaction-diffusion systems. Trans Amer Math Soc 321(1):1–44  
Murray JD (2002) Mathematical Biology, 2nd edn. Springer-Verlag, New York  
Peng R, Zhao XQ (2012) A reaction-diffusion SIS epidemic model in a time-periodic environment. Nonlinearity 25(5):1451–1471  
y Piontti AP, Perra N, Rossi L, Samay N, Vespignani A (2018) Charting the Next Pandemic: Modeling Infectious Disease Spreading in the Data Science Age. Springer  
Protter MH, Weinberger HF (1984) Maximum Principles in Differential Equations, 2nd edn. Springer-Verlag, Berlin  
Riley S (2007) Large-scale spatial-transmission models of infectious disease. Science 316(5829):1298–1301  
Schaller M (2011) The behavioural immune system and the psychology of human sociality. Philos Trans R Soc B-Biol Sci 366(1583):3418–3426  
Smith HL (1995) Monotone Dynamical Systems: an Introduction to the Theory of Competitive and Cooperative Systems. American Mathematical Soc  
Song P, Xiao Y (2018) Global Hopf bifurcation of a delayed equation describing the lag effect of media impact on the spread of infectious disease. J Math Biol 76(5):1249–1267  
Song P, Xiao Y (2019) Analysis of an epidemic system with two response delays in media impact function. Bull Math Biol 81(5):1582–1612  
Su Y, Wei J, Shi J (2012) Hopf bifurcation in a diffusive logistic equation with mixed delayed and instantaneous density dependence. J Dyn Differ Equ 24(4):897–925  
Sun C, Yang W, Arino J, Khan K (2011) Effect of media-induced social distancing on disease transmission in a two patch setting. Math Biosci 230(2):87–95  
Tang S, Xiao Y, Yang Y, Zhou Y, Wu J, Ma Z (2010) Community-based measures for mitigating the 2009 H1N1 pandemic in China. PLoS One 5(6):e10911  
Tang S, Xiao Y, Yuan L, Cheke RA, Wu J (2012) Campus quarantine (fengxiao) for curbing emergent infectious diseases: lessons from mitigating a/h1n1 in xi’an, china. J Theor Biol 295:47–58  
Tchuente JM, Dube N, Bhunu CP, Smith RJ, Bauch CT (2011) The impact of media coverage on the transmission dynamics of human influenza. BMC Public Health 11(1):55  
Thieme HR (1992) Convergence results and a Poincaré-Bendixon trichotomy for asymptotically autonomous differential equations. J Math Biol 30(7):755–763  
Thieme HR (2009) Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity. SIAM J Appl Math 70(1):188–211  
Verelst F, Willem L, Beutels P (2016) Behavioural change models for infectious disease transmission: a systematic review (2010–2015). J R Soc Interface 13(125):20160820  
Wang W, Zhao XQ (2012) Basic reproduction numbers for reaction-diffusion epidemic models. SIAM J Appl Dyn Syst 11(4):1652–1673  
Winters M, Julloho MF, Sengeh P, Julloho MB, Coniah L, Bunnell R, Li W, Zeebali Z, Nordenstedt H (2018) Risk communication and ebola-specific knowledge and behavior during 2014–2015 outbreak, sierra leone. Emerg Infect Dis 24(2):336  
Wu J (1996) Theory and Applications of Partial Functional-Differential Equations. Springer-Verlag, New York  
Wu Y, Zou X (2016) Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism. J Differ Equ 261(8):4424–4447
Xiao Y, Zhao T, Tang S (2013) Dynamics of an infectious diseases with media/psychology induced non-smooth incidence. Math Biosci Eng 10(2):445–461
Xiao Y, Tang S, Wu J (2015) Media impact switching surface during an infectious disease outbreak. Sci Rep 5:7838
Yan Q, Tang S, Gabriele S, Wu J (2016) Media coverage and hospital notifications: correlation analysis and optimal media impact duration to manage a pandemic. J Theoret Biol 390:1–13
Zhao XQ (2017) Dynamical Systems in Population Biology, 2nd edn. Springer, Cham

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