Geometric modeling and study of properties of surfaces equidistant to two spheres

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Abstract. The paper considers the geometric locus of points equidistant to two spheres of different diameters. If these spheres are concentric, the sought multitude constitutes a single surface – a sphere of diameter equal to arithmetic mean of the diameters of the given spheres. In other cases the geometric locus of points equidistant to two spheres of different diameters constitutes two surfaces. In case the spheres intersect, are tangent or distant to each other, the first of these surfaces is a two-sheet hyperboloid of revolution that degenerates into a plane in case the spheres are equal. In case the spheres intersect, the second of the surfaces is an ellipsoid of revolution that degenerates into a straight line if the spheres are tangent to each other. In the case of distant spheres, the second of the surfaces is a two-sheet hyperboloid of revolution. In case the spheres contain one another, the sought geometric locus constitutes two co-axial co-focused ellipsoids of revolution. The equations defining the mentioned surfaces are presented. The regularities in shape and location of these surfaces were studied; the formulas for the major and the minor axes of the ellipsoids and the vertices of the two-sheet hyperboloids of revolution were derived.

1. Introduction

Both analytic and descriptive geometry feature a variety of problems with solutions that require knowing geometric locus of points (GLP), curves and other objects. Surfaces generated as geometric locus of points equidistant to various geometric shapes offer practical value in architecture, industrial construction and cosmonautics [1]. A frame designed of GLP surfaces can be optimized with methods documented in paper [2]. At the moment, various studies on modeling and application of complex surfaces are being held [3,4].

1.1. Introduction

As you can see from Table 1, the geometric locus of points equidistant to two spheres is designated 5.5.

Geometers Dmitry Ivanovich Kargin (1880-1949), A.D. Posvyanskiy (1909-1992), I.I. Aleksandrov (1856-1919), M. Ya. Vygodskiy (1898 – 1965) and N.A. Salkov dedicated their research to the problems of geometric locus of points [5]. V.V. Glogovskiy (1920 – 1998) is the author of papers on equidistant GLP. The present paper extends the series of articles on the topic of GLP [6,7,8].

The basic GLP classification is presented in Table 1 [6].

As you can see from Table 1, the geometric locus of points equidistant to two spheres is designated 5.5.
2. Problem definition
The present study is aimed at construction of geometric locus of points equidistant to two spheres of different radii. The problem is posed: to derive equations of the constructed surfaces, to investigate regularities in shape and location of the surfaces constituting the geometric locus of points equidistant to two spheres of different radii.

3. Theory
Let us consider the methods of geometric locus construction. A number of geometric objects are selected as given. Geometric locus can be constructed for two objects – for example, a point and a circle, a torus and a circle, a circle and a sphere – or, in fact, more than two objects, for example, three points or four straight lines. In order to find the geometric locus of points equidistant to the given objects, geometric loci of distance \( t \) – curves \( d \) are constructed. The multitude of these curves comprises the sought surface equidistant to the given objects. In order to acquire the multitude of curves \( d \), step-by-step incrementation of distance is applied. It is convenient to perform the construction if the incrementation step is fixed.

A geometric locus of points equidistant on \( t \) to a sphere of radius \( R \) constitutes two spheres of radius \( R \pm t \). Figure 1 depicts the given spheres \( \Delta \) and \( \Gamma \) of radius \( R \) and geometric locus of points equidistant to \( \Delta \) and \( \Gamma \) on distance \( t \) that is spheres \( R_1 \) and \( R_3 \) of radiuses \( R + t \) and \( R_2 \) and \( R_4 \) of radiuses \( R - t \).

GLP 3D modeling in CAD software involves construction of a curve \( d \) of intersection of spheres through step-by-step incrementation and subsequent creation of the sought 3D model of GLP surface through use of the “lofted surface” tool. Surface modeling can also be performed in CAD software through methods documented in paper [9].

There are two known approaches to GLP studies: the analytic approach and the graphic approach (see Figure 2).

It should be noted that:
1. The projectional method allows one to construct projections of GLP equidistant to the considered objects, determine focuses, directrices, vertices of the acquired curves and to estimate their shape.
2. The 3D models of GLP created by means of CAD software allow one to visualize their shape.
3. In order to determine the shape of the studied GLP surfaces, it is required to derive equations of these surfaces.
4. Computer algebra systems in general have the visualization feature and therefore this method can be classified as the middle ground between the graphic and the analytic methods. 3D models created in computer algebra systems are particularly interesting, since they allows one to observe variation in shape of the constructed GLP upon variation of mutual location of the given objects.

4. Results of the study
The geometric loci of points equidistant to two spheres can be classified into two cases: 5.5.1 – geometric loci of points equidistant to two spheres of the same diameter and 5.5.2 – geometric loci of points equidistant to two spheres of different diameters. In the present paper the case 5.5.2 of GLP is studied. There are the following known variants of geometric locus of points equidistant to spheres of different radiiuses 5.5.2:

5.5.2.0. Spheres are concentric.
Geometric locus \( \Gamma^{5.5.2.0} \) of points equidistant to two spheres of radiuses \( R \) and \( r \) is a sphere of radius \((R + r)/2\) presented on Figure 3.

5.5.2.1. Spheres are intersecting: distance between the centers of the spheres \( 2a < R + r \).
Geometric locus \( \Gamma^{5.5.2.1} \) of points is presented on Fig. 4. It consists of two co-axial shapes:
– a two-sheet hyperboloid of revolution;
– an ellipsoid of revolution.

5.5.2.2 Spheres are tangent: distance between the centers of the spheres $2a = R + r$.
Geometric locus $\Gamma^{5.5.2.2}$ of points is presented on Fig. 5. It consists of:
– a two-sheet hyperboloid of revolution, its major axis matching the line connecting the centers $A$ and $B$ of the given spheres and its focuses matching the centers of the given spheres;
– a straight line $AB$.

5.5.2.3 Spheres are distant: distance between the centers of the spheres exceeds $R + r$.
Geometric locus $\Gamma^{5.5.2.3}$ is presented on Figure 6. It consists of two co-focused co-axial hyperboloids of revolution.

5.5.2.4. Distance between spheres is lower than $r$ – the larger sphere contains the lesser one, yet they are not concentric.
Geometric locus $\Gamma^{5.5.2.4}$ consists of two co-focused co-axial ellipsoids of revolution.

5. Consideration of the results

GLP of concentric spheres (5.5.2.0)
Two spheres $\Delta$ and $\Gamma$ of different radiuses $R$ and $r$ and matching centers are given. Geometric loci of points equidistant to these surfaces on distance $t$ constitute spheres of the same center and radiuses $R \pm t$ and $r \pm t$. These spheres cannot intersect, however, they do match at $t = 0.5(R + r)$, as shown on Figure 3.

If in formula $0.5(R + r)$ we accept that $R = r = R$, then $0.5(R + r) = R$, in other words, the geometric locus of points equidistant to two equal concentric spheres is the exact same sphere.

![Figure 3. Geometric locus $\Gamma^{5.5.2.0}$ of concentric spheres](image)

GLP of intersecting spheres: distance between the centers of the spheres of different radiuses is lower than $R + r$ (5.5.2.1)
Spheres $\Delta_i$ and $\Gamma_i$ of radiuses $R + t_i$ and $r + t_i$ (both series of spheres are ascending) intersect generating parallel circles of increasing diameter $\Delta_1 \cap \Gamma_1 = 11'$, $\Delta_2 \cap \Gamma_2 = 22'$, $\Delta_3 \cap \Gamma_3 = 33'$. These parallel circles form the “positive” sheet of a two-sheet hyperboloid of revolution depicted on Figure 4 (a). One of the parallel circles is the result of intersection of the given spheres $\Delta$ and $\Gamma$. Since the diameters of the given spheres are different, these parallels do not belong to the same plane, as it happens in case of two spheres of equal diameter: here the plane (more precisely, one of its sheets) is transformed into one of the sheets of the two-sheet hyperboloid of revolution $\Omega$.

Spheres of radiuses $R - t_i$ and $r - t_i$ intersect generating the “negative” sheet of the two-sheet hyperboloid of revolution, for example, $\Delta_8 \cap \Gamma_8 = 88'$. The hyperboloid of revolution is focused at centers $A$, $B$ of the given spheres $\Delta$ and $\Gamma$, as depicted on Figure 4 (a). The directing plane is located midway between the focuses. Vertex $K$ of the hyperboloid is equidistant to spheres $\Delta_i$ and $\Gamma_i$ with a common step, for example, to spheres $\Delta$ and $\Gamma$.

Spheres $\Delta_i$ and $\Gamma_i$ with different values of radial step $R + t_i$ and $r - t_i$ (or vice versa) intersect generating decreasing parallels $\Delta^{11} \cap \Gamma^{11} = CD$, $\Delta^{12} \cap \Gamma^{12} = 66'$, $\Delta^{13} \cap \Gamma^{13} = 99'$ that generate the ellipsoid of revolution $\Sigma$. The ellipsoid $\Sigma$ is focused at centers $A$, $B$ of the given spheres $\Delta$ and $\Gamma$, i.e. the hyperboloid $\Omega$ and the ellipsoid $\Sigma$ are co-focused, therefore the minor ellipsoid axis $\Sigma_2$ belongs to the directing plane of the hyperboloid $\Omega$ and is formed upon intersection of equal spheres generating the largest parallel.

Knowing the focuses and the value of the minor ellipsoid axis $\Sigma_2$, it is easy to determine its major axis.

Conclusion: Geometric locus $\Gamma^{5.5.2.1}$ equidistant to two spheres of different diameters located at distance $2a < R + r$ consists of two co-focused figures:
– an ellipsoid,
– a two-sheet hyperboloid of revolution.
These figures are focused at centers A and B of the given spheres \( \Delta \) and \( \Gamma \). The vertices of the hyperboloid \( \Omega \) and the major axis of the ellipsoid \( \Sigma \) belong to the line \( AB \) connecting the centers of the given spheres. Directing plane of the hyperboloid and the minor axis plane of the ellipsoid match and are located midway between the focuses.

As distance \( AB \) between centers of the given spheres grows, the ellipsoid \( \Sigma \) depicted on Figure 4 gradually flattens. On the other hand, at \( AB = 2a = 0 \) the ellipsoid of revolution degrades into a sphere of positive radius at \( AB = 2a = 0 \), as in the previously considered case 5.5.2.0.

As the difference in diameters of the given spheres declines, the sheets of the two-sheet hyperboloid \( \Omega \) approach each other and flatten; eventually, at \( R = r \) the hyperboloid degrades into a plane. On the other hand, as the difference in radiuses \( R \) and \( r \) of the given spheres grows, radius of curvature of the generating parabolas grows.

**GLP of tangent spheres**: distance between the centers of the spheres of different radiuses equals \( R + r \) (5.5.2.2)

Spheres \( \Delta \) and \( \Gamma \) are tangent to each other at point \( N \), and this is the first point equidistant to both of them. Spheres \( \Delta_i \) and \( \Gamma_i \) of radiuses \( R + ti \) and \( r + ti \) have common parallels \( \Delta^1 \cap \Gamma^1 = 11' \), \( \Delta^2 \cap \Gamma^2 = 22' \), \( \Delta^3 \cap \Gamma^3 = 33' \) etc. generating the “positive” sheet of the two-sheet hyperboloid of revolution.

Spheres \( \Delta_i ' \) and \( \Gamma_i ' \) of radiuses \( R - ti \) and \( r - ti \) intersect at parallels \( \Delta^6 \cap \Gamma^6 = K \), \( \Delta^7 \cap \Gamma^7 = 44' \), \( \Delta^8 \cap \Gamma^8 = 55' \) etc. This is the way the “negative” sheet of the two-sheet hyperboloid of revolution is formed at negative values of sphere radiuses.

As a result of the difference in diameters of the given spheres, the parallels generated at the positive and the negative radiuses do not merge into a single plane, as in the case of equal spheres, but rather shift away from each other forming the hyperboloid of revolution.

The vertex of the “positive” sheet of the hyperboloid must match the point of tangency of the given spheres, \( N \) in the considered case. The vertex of the “negative” sheet \( K \) is located at distance \( \frac{R}{2} \) to the center of the larger of the given spheres. Both vertices belong to the straight line \( AB \) connecting centers of the given spheres. The focuses of hyperboloid sheets are located at the centers \( A \) and \( B \) of the given spheres, the directing planes pass through the vertex of the neighboring sheet: through \( K \) for sheet \( \Sigma \) and through \( N \) for sheet \( \Sigma ' \) (see Figure 5 (a)).

Spheres \( \Delta \), and \( \Gamma \), of radiuses \( R + ti \) and \( -r - ti \), i.e. a series of growing spheres and a series of declining (down to negative values) spheres, are always tangent to each other, for example, at points \( \Delta^3 \cap \Gamma^3 = 6 \), \( \Delta^4 \cap \Gamma^4 = 7 \), generating segment \( AB \) at positive radius values and two rays starting at points \( A \) and \( B \) at negative radius values.

Overall, intersection of growing, declining and negative spheres generates GLP in the form of a straight line \( AB \) passing through the centers of the given spheres.

As the distance \( AB \) between the centers of the given spheres grows, the ellipsoid gradually flattens. Eventually, as the spheres become tangent to each other \( (2a = R + r) \), the ellipsoid degenerates into a straight line.

Conclusion: Geometric locus of points \( \Gamma^5.5.2.2 \) equidistant to two tangent spheres of different diameters located at distance \( 2a = R + r \) constitutes the following figures depicted on Figure 5 (a), (b):
- a two-sheet hyperboloid of revolution $\Sigma$ with major axis belonging to the line connecting the centers $A$ and $B$ of the given spheres, and focused at the centers of the given spheres;
- a straight line $AB$.

![Figure 5](image)

**Figure 5.** Geometric locus $\Gamma^{5.5.2.2}$ of points equidistant to two spheres of different radii at $2a = R + r$ (tangent spheres): (a) front projection, (b) 3D model

**GLP of distant spheres:** distance between the centers of the spheres of different radii exceeds $R + r$ (5.5.2.3)

Spheres $\Delta_i$ and $\Gamma_i$ of radii $R + t_i$ and $r + t_i$ (both series of spheres have growing radii) intersect, generating parallel circles $\Delta^3 \cap \Gamma^3 = 22'$, $\Delta^4 \cap \Gamma^4 = 33'$ etc. depicted on Figure 6. Each subsequent parallel grows in diameter and is located further away from the directing plane located midway between the centers $A$ and $B$ of the given spheres.

These parallels, as in the cases 5.5.2.1 and 5.5.2.2, generate the “positive” shell of the two-shelled hyperboloid of revolution $\Sigma$. The vertex $K$ of the hyperboloid sheet $\Sigma$ depicted on Figure 6 (a) is equidistant to spheres $\Delta_i$ and $\Gamma_i$ of common step, e.g. spheres $\Delta$ and $\Gamma$; the vertex of the second sheet $P$ is symmetric to $K$ with respect to the plane passing midway between the focuses.

![Figure 6](image)

**Figure 6.** Geometric locus $\Gamma^{5.5.2.3}$ of points equidistant to two spheres of different radii at $2a > R + r$ (distant spheres): (a) front projection, (b) 3D model

Therefore, one of the sheets of the hyperboloid $\Sigma$ is generated at radii $R + t_i$ and $r + t_i$, while the other one is generated at radii $R - t_i$ and $r - t_i$. 
The second geometric locus $\Gamma^{5.5.2.3}$ of points equidistant to the two given spheres appears at the points of intersection of spheres $\Delta_i'$ and $\Gamma_i'$ of oppositely directed radiuses $R - t_i$ and $r + t_i$ (and vice versa) and constitutes a two-sheet hyperboloid of revolution $\Lambda$.

Conclusion: Geometric locus $\Gamma^{5.5.2.3}$ of points equidistant to two distant spheres of different diameters constitutes two co-focused co-axial hyperboloids of revolution focused at the centers of the given spheres. These hyperboloids are depicted on Figure 6 (a), (b).

**GLP of two spheres of different diameter, the larger sphere containing the lesser one (5.5.2.4)**

Spheres $\Delta_i$ and $\Gamma_i$ of radiuses $R + t_i$ and $r + t_i$ can never intersect, since the lesser spheres $\Gamma_i$ can never outgrow the larger spheres $\Delta_i$, as one can see from Figure 7 (a).

Spheres $\Delta_i$ and $\Gamma_i$ of radiuses $R - t_i$ and $r + t_i$ intersect at parallels $\Delta_i' \cap \Gamma_i' = 11'$, $\Delta_i' \cap \Gamma_i' = CC'$, $\Delta_i' \cap \Gamma_i' = 22'$ etc. generating the ellipsoid of revolution $\Lambda$ depicted on Figure 7 (a), (b).

**Figure 7.** Geometric locus $\Gamma^{5.5.2.4}$ of points equidistant to two spheres, the lesser sphere located inside the larger sphere: (a) front projection, (b) 3D model

As in the other cases of GLP equidistant to two spheres, the ellipsoid $\Lambda$ is focused at the centers of the two given spheres $A$ and $B$. The minor axis of the ellipse $\Lambda_2$ is, first, located midway between the focuses $A$ and $B$, and second, formed upon intersection of equal spheres generating the largest parallel. The major axis of the ellipse $\Lambda_2$ is the distance from the focus to the end of its minor axis.

Spheres $\Delta_i$ and $\Gamma_i$ of radiuses $R - t_i$ and $r - t_i$ also intersect at parallels $\Delta_i'' \cap \Gamma_i'' = 33'$, $\Delta_i'' \cap \Gamma_i'' = DD'$ etc. It is worth noting that radius $r$ of spheres $\Gamma_i$ here takes negative values. Parallels $33'$, $DD'$, etc. generate the second GLP – the lesser ellipsoid $\Psi$ depicted on Figure 7 (a), (b). The major and the minor axes of the ellipsoid $\Psi$ can be acquired the same way as for the ellipsoid $\Lambda$.

Given the radius $R$ of the larger sphere and the radius $r$ of the lesser sphere, one can find the minor axis of the ellipsoid that constitutes the GLP equidistant to two spheres of different diameters by constructing two spheres of radius $(R + r)/2$ depicted on Figure 8 and finding the points $C$ and $D$.

$CD$ constitutes the minor axis of the ellipse, while $KN = 2CF_i$ constitutes the major axis of the ellipse.

**Figure 8.** Construction of the axes of the ellipsoid of revolution that constitutes GLP equidistant to two spheres
However, $CF_j$ is in fact the value of the major half-axis of the ellipsoid, which brings us to the theorem.

Theorem. The value of the major axis of the larger ellipsoid forming the GLP equidistant to two spheres of radiuses $R$ and $r$ is equal to the sum of radiuses of the spheres $R + r$, while the value of the major axis of the lesser ellipsoid is equal to the difference between the radiuses $R - r$.

It is worth noting that if the centers of the two spheres match, the ellipsoid forming the GLP equidistant to those spheres transforms into a sphere of radius $(R + r)/2$ considered in the case 5.5.2.0.

The theorem is true for every case of mutual location of two spheres generating GLP in the form of ellipsoids, and within these constraints does not depend on the distance between the given spheres and the values of their radiuses.

Conclusion: Geometric locus of points Г5.5.2.4 equidistant to two spheres of different diameters containing one another constitutes two co-focused co-axial ellipsoids of revolution focused at the centers of the spheres, their major axes equal to the sum (for a larger ellipsoid) and to the difference (for the lesser one) of the radiuses of the given spheres.

Let us consider the analytic representation of GLP equidistant to two spheres of different radiuses using the canonical equations for second-degree surfaces. A drawing supplementing the derivation of equations for GLP surfaces is presented on Figure 9.

Figure 9. A drawing on derivation of equations for GLP surfaces

\[
(x - a)^2 + y^2 + z^2 = (R_1 \pm t)^2, \quad (1)
\]

\[
(x + a)^2 + y^2 + z^2 = (R_2 \pm t)^2. \quad (2)
\]

Let us assume that $R_1 > R_2$.

1) Let us consider the first case: both $R_1$ and $R_2$ grow.

Subtracting (2) from (1) results, after certain simplification, in the following expression:

\[
-4ax = R_1^2 - R_2^2 + 2t(R_1 - R_2). \quad (3)
\]

Let us express $t$ from (3):

\[
t = -2ax \frac{R_1 + R_2}{R_1 - R_2} - \frac{R_1^2 - R_2^2}{2}. \quad (4)
\]

Let us substitute the expression for $t$ (4) into the expression $(R_1 + t)$:

\[
R_1 - 2ax \frac{R_1 + R_2}{R_1 - R_2} = R_1^2 - R_2^2 - 2ax \frac{R_1 - R_2}{2}. \quad (5)
\]

Let us introduce designation $k = R_1 - R_2$ into expression (5):

\[
k - 2ax \frac{k}{2} = k^2 - 4ax \frac{k}{2k}. \quad (6)
\]

Substitution of (6) into (1) results in the following expression:

\[
(k^2 - 4a^2)x^2 + k^2y^2 + k^2z^2 = \frac{k^2(k^2 - 4a^2)}{4}. \quad (7)
\]

The expression (7) constitutes the general equation for a surface of GLP equidistant to two spheres for the case of growing radiuses. If $k = 0$, then $R_1 = R_2$ and

\[-4a^2x^2 = 0; \quad x = 0,
\]

which is the equation for profile level plane. Let us further consider the equation (7) and introduce the following designations:

\[k^2 - 4a^2 = A, \quad k^2 = B, \quad \frac{k^2(k^2 - 4a^2)}{4} = D.
\]

Let us substitute these designations into equation (7):
In case $A > 0$, $D > 0$, we have $Ax^2 + By^2 + Bz^2 = D$, which is the equation for an ellipsoid of revolution with major axis parallel to the $x$ axis.

In case $A = 0$, $D = 0$, we have $y^2 + Bz^2 = 0$, which is the equation for a profile projecting straight line, namely, the $x$ axis.

In case $A < 0$, $D < 0$, we have $-Ax^2 + By^2 + Bz^2 = -D$, which is the equation for a two-sheet hyperboloid of revolution with major axis parallel to the $x$ axis.

2) Let us consider the second case: one of the radiuses is growing, the other one is declining. As follows from (3),

$$-4ax = (R_1 + t)^2 - (R_2 - t)^2 = R_1^2 + 2R_1t + t^2 - R_2^2 + 2R_2t - t^2 =$$

$$= R_1^2 - R_2^2 + 2t(R_1 + R_2),$$

$$-4ax = (R_1 - R_2)(R_1 + R_2) + 2t(R_1 + R_2),$$

$$-4ax = R_1 - R_2 + 2t. #(24)$$

Let us express the parameter $t$ from the equation (8):

$$t = -\frac{2ax}{R_1 + R_2} - \frac{R_1 - R_2}{2}. #(9)$$

Let us substitute the expression for $t$ (9) into the expression $(R_1 + t)$ and designate the sum of radiuses of the given spheres as $d = R_1 + R_2$:

$$\frac{R_1 - R_2}{2} + \frac{d}{2} = \frac{R_1 + R_2}{2} - \frac{2ax}{R_1 + R_2} = \frac{d}{2} - \frac{2ax}{d}. #(10)$$

Substitution of (10) into (1) results in

$$x^2 - 2ax + a^2 + y^2 + z^2 = \left(\frac{d^2 - 4ax}{2d}\right)^2 = \frac{d^4 - 8ad^2x + 16a^2x^2}{4d^2},$$

$$(4d^2 - 16a^2)x^2 + 4d^2y^2 + 4d^2z^2 = d^2(d^2 - 4a^2).$$

Let us introduce designation $c = d^2 - 4a^2$ and simplify the equation:

$$c = \frac{1}{4}c. #(11)$$

The expression (11) constitutes the general equation for a surface of GLP equidistant to two spheres for the case of one growing radius and one declining radius. It is similar to the equation (7) and yields similar results.

Let us summarize the results of the study.

There are two geometric loci $\Gamma^{5.5}$ of points equidistant to two spheres:

- an ellipsoid of revolution;
- a two-sheet hyperboloid of revolution.

Particular cases may feature the following variants:

- if both spheres are of equal diameter, the focuses of the two-sheet hyperboloid of revolution tend to infinity, and the hyperboloid degenerates into a plane;
- if the spheres are tangent to each other, the ellipsoid degenerates into a straight line.

The following properties of geometric locus of points equidistant to two spheres are worth noting:

- focuses of the ellipsoids of revolution and the two-sheet hyperboloids of revolution are located at the centers of the given spheres. Therefore, the vertices and the major axes of the surfaces are located on a straight line connecting the centers of the given spheres;
- the acquired surfaces of revolution forming the GLP $\Gamma^{5.5}$ are co-focused, the minor axis of the ellipsoids belongs to the directing plane of the hyperboloids;
- the major axis $l$ of the ellipsoids forming the GLP $\Gamma^{5.5}$ is equal to the sum of radiuses of the given spheres:

$$l = R + r;$$

- the major axis $n$ of the lesser ellipsoid forming the GLP $\Gamma^{5.5.2.4}$ is equal to the difference of radiuses of the given spheres:

$$n = R - r;$$

- the minor axis of the ellipsoids of revolution forming the GLP $\Gamma^{5.5}$ is defined by intersection of spheres of radius $(R + r)/2$ centered at the focuses of the ellipsoids. A simple algorithm for construction of an ellipsoid as a geometric locus of points equidistant to two spheres follows from the above properties: first, construct the minor axis of the ellipsoid, second, construct the major axis of the ellipsoid, and finally, construct the ellipsoid outline;
– the first GLP equidistant to two equal spheres is a plane located perpendicular and in the middle of the major axis of the surface of revolution of the second GLP;
– one of the vertices of the two-sheet hyperboloid of revolution constituting GLP equidistant to two spheres is equally distant to any pair of spheres $\Delta_i$ and $\Gamma_1$, for example, to the initial ones.

Table 2 outlines the results of the study presented in the present paper.

Table 2. Dependence of GLP $\Gamma^{5.5}$ on the distance between centers of the given spheres

| No. | Designation                  | Distance $2a$ between the centers of the spheres | First GLP                             | Second GLP                                           | Figure |
|-----|------------------------------|--------------------------------------------------|---------------------------------------|-----------------------------------------------------|--------|
| 1   | 5.5.2.0                      | $2a = 0$                                          | Concentric spheres                    | Sphere $(R + r)/2$                                   | Figure 3|
| 2   | 5.5.2.1                      | $0 < 2a < R + r$                                  | Intersecting spheres                  | Ellipsoid of revolution co-focused and co-axial to the first GLP | Figure 4|
| 3   | 5.5.2.2                      | $2a = R + r$                                      | Tangent spheres                       | Two-sheet hyperboloid of revolution                 | Figure 5|
| 4   | 5.5.2.3                      | $2a > R + r$                                      | Distant spheres                       | Two-sheet hyperboloid of revolution co-focused and co-axial to the first GLP | Figure 6|
| 5   | 5.5.2.4                      | $0 < 2a \leq r$                                  | The larger sphere containing the lesser sphere | Ellipsoid of revolution                                           | Figure 7|

6. Conclusion
The present paper considers construction of the geometric loci of points equidistant to spheres of different diameters. The equations for ellipsoids of revolution and two-sheet hyperboloids of revolution constituting the geometric loci of points equidistant to spheres of different diameters were derived. The regularities in shape and location of these surfaces were investigated; the expressions for calculation of the major and the minor axes of the ellipsoids and vertices of the hyperboloids were derived.

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