On the solvability of p states quantum Rabi Model with $Z_p$-graded parity

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 Abstract.
 In this paper the p-level Rabi model with $Z_p$-graded symmetry is discussed. The p-level Rabi Hamiltonian is constructed by introducing the generalized Pauli matrices. The energy and wave function for the p-level Rabi equation are obtained by using the standard perturbation method.

 1. Introduction
 The quantum Rabi model(QRM) [1] is very useful in studying the interaction between atom and light. This model arises in various physical models such as quantum computation, condensed matter, atomic and molecular physics [2], nuclear and particle physics [3,4]. This model is described by the following Schrödinger equation:

 \[ H_R \psi = E \psi \] (1)

 where

 \[ H_R = w^n + \Delta \sigma_z + g(a^\dagger + a)\sigma_x, \]

 and $g$ is coupling strength and $a^\dagger, a$ are the creation and annihilation operator for the quantized field. The atomic ground state is $(0\ 1)^T$ and excited state is $(1\ 0)^T$. The energy scale is defined by setting $w = 1$.

 The quantum Rabi model can be extended to Jaynes-Cummings model [6] by taking the interaction picture and using the rotating wave approximation which can be applied for the weak coupling case. For dimensionless coupling strength $\kappa = g/w \leq 10^{-2}$, The quantum Rabi model is well captured by JCM which can be analytically solvable. The dynamics of the JCM have been studied in Refs. [7, 8].

 Recently, solid-state semiconductor [7] and superconductor systems [8-10] have allowed the advent of the ultrastrong coupling regime, where the dimensionless coupling strength

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κ = g/w ≥ 0.1 [11]. In this regime, rotating wave approximation cannot be adopted, so the related physics should be explained by QRM.

There has been many trials to obtain the exact solution of the Rabi Hamiltonian until Braak [12] recently succeeded in doing it. He solved the quantum Rabi equation explicitly without the famous rotating wave approximation. The solution for ∆ = 0 is given by Charlier polynomial, but for ∆ ≠ 0 it is not shown to be related to any kind of orthogonal function. So for this case Braak used the numerical method to obtain the energy levels for this model.

In this paper we discuss the p-level Rabi model with \( Z_p \)-graded symmetry, which is a generalization of standard two-level Rabi model. This paper is organized as follows: In section II we obtain the p-level Rabi Hamiltonian by introducing the generalized Pauli matrices. We also diagonalize the p-level Rabi Hamiltonian with a help of two kinds of unitary transformations. In section III we obtain the energy and wave function for the p-level Rabi equation by using the standard perturbation method.

2. p-level Rabi model with \( Z_p \)-graded symmetry

Now let us start with the p-level Rabi model with \( Z_p \)-graded symmetry, whose Hamiltonian is given by

\[
H = \sum_{k=0}^{p-1} \Delta_k \sigma^k_{p-1} + w a^\dagger a + g(a\sigma_0 + a^\dagger \sigma_0^p) \tag{3}
\]

where

\[
\sum_{k=0}^{N-1} \Delta_k q^{ki} = E_i, \quad \text{or} \quad \Delta_i = \frac{1}{p} \sum_{k=0}^{p-1} q^{-ki} E_k \tag{4}
\]

and the generalized Pauli matrices are

\[
\sigma_i = \sum_{k=0}^{p-1} q^{ki} e_{k,k+p-1}, \quad (i = 0, 1, \ldots, p-2) \tag{5}
\]

\[
\sigma_{p-1} = \sum_{k=0}^{p-1} q^{k} e_{k,k} \tag{6}
\]

and \( e_{ij} = |i\rangle \langle j|, \ q = e^{2\pi i/p} \). These matrices are idempotent because they obey

\[
\sigma_i^p = I, \quad (i = 0, 1, \ldots, p-1) \tag{7}
\]

and

\[
\sigma_i^\dagger = \sigma_i^{p-1} \tag{8}
\]

These matrices are unitary matrices:

\[
\sigma_i \sigma_i^\dagger = \sigma_i^\dagger \sigma_i = I, \quad (i = 0, 1, \ldots, p-1) \tag{9}
\]

These \( Z_p \)-graded matrices satisfy

\[
\{\sigma_1, \sigma_2, \ldots, \sigma_p\} = p! \delta_{i_1i_2\ldots i_p} I, \tag{10}
\]

where

\[
\{\sigma_1, \sigma_2, \ldots, \sigma_p\} = \sum_{(j_1, \ldots, j_p) \in \text{perm}(i_1, \ldots, i_p)} \sigma_{j_1} \sigma_{j_2} \ldots \sigma_{j_p} \tag{11}
\]
\[ \delta_{i_1 i_2 \cdots i_p} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_p \\ 0 & \text{otherwise} \end{cases} \tag{12} \]

and \( \text{perm}\{i_1, \ldots, i_p\} \) implies set of all permutations of \( \{i_1, \ldots, i_p\} \). The proof of the eq.(11) is given in the Appendix.

Now let us introduce the \( \mathbb{Z}_p \)-graded Walsh-Hadamard matrix as follows:

\[
W = \frac{1}{\sqrt{p}} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & q & q^2 & \cdots & q^{p-1} \\
1 & q^2 & (q^2)^2 & \cdots & (q^2)^{p-1} \\
\vdots \\
1 & q^{p-1} & (q^{p-1})^2 & \cdots & (q^{p-1})^{p-1}
\end{pmatrix}
\tag{13} \]

Indeed this matrix is unitary:

\[
WW^\dagger = W^\dagger W = I \tag{14} \]

The \( \mathbb{Z}_p \)-graded Walsh-Hadamard matrix transforms one of \( \mathbb{Z}_p \)-graded Pauli matrix into another one:

\[
W \sigma_{p-1} W^\dagger = \sigma_0^{p-1} \tag{15} \]

\[
W \sigma_0 W^\dagger = \sigma_{p-1} \tag{16} \]

Thus, under this unitary transform, the \( p \)-level Rabi Hamiltonian is changed into

\[
H' = WHW^\dagger = \Delta_0 + \sum_{k=1}^{p-1} \Delta_{p-k} \sigma_0^k + wa^\dagger a + g(a\sigma_2 + a^\dagger \sigma_2^2) \tag{17} \]

Now let us consider the case that \( \Delta_0 = 0 \) and \( \Delta_1 = \Delta_2 = \cdots = \Delta_{p-1} = \Delta \) is real. The Hamiltonian can be written in a matrix form, acting in the space spanned by \( |0\rangle, |1\rangle, \ldots, |p-1\rangle \),

\[
H = \begin{pmatrix}
a^\dagger a + g(a + a^\dagger) & \Delta & \Delta & \cdots & \Delta \\
\Delta & a^\dagger a + g(qa + q^{-1}a^\dagger) & \Delta & \cdots & \Delta \\
\Delta & \Delta & a^\dagger a + g(q^2a + q^{-2}a^\dagger) & \cdots & \Delta \\
\vdots \\
\Delta & \Delta & \Delta & \cdots & a^\dagger a + g(q^{p-1}a + q^{-(p-1)}a^\dagger)
\end{pmatrix} \tag{18} \]

Now let us diagonalize the Hamiltonian \( H' \) by means of the unitary matrix

\[
U = \frac{1}{\sqrt{p}} \begin{pmatrix}
1 & R & R^2 & \cdots & R^{p-1} \\
1 & qR & q^2R^2 & \cdots & q^{p-1}R^{p-1} \\
1 & q^2R & (q^2)^2R^2 & \cdots & (q^2)^{p-1}R^{p-1} \\
\vdots \\
1 & q^{p-1}R & (q^{p-1})^2R^2 & \cdots & (q^{p-1})^{p-1}R^{p-1}
\end{pmatrix} \tag{19} \]

where \( R = q^a^\dagger a \). Using the eq.(19) we have

\[
H'' = UH'U^\dagger = wa^\dagger a + g(a + a^\dagger) + \Delta \sum_{k=1}^{p-1} R^k \sigma_0^k \tag{20} \]
The \( Z_p \)-symmetry of the chiral three states Rabi model can be used to eliminate the discrete degree of freedom from the problem. Each \( Z_p \)-parity-chain \( \mathcal{H}_r, \ r = 0,1,2, \cdots, p-1 \) is isomorphic to the Hilbert space of photon number states \( \mathcal{N} \). In each \( \mathcal{H}_r \), the Hamiltonian (20) reads

\[
H_r = wa^\dagger a + g (a + a^\dagger) + \Delta \sum_{k=1}^{p-1} q^k \sigma^k
\]

(21)

The complication of this reduced Hamiltonian comes from the last two terms. On the other hand, these terms is instrumental for the analytical solution of the model. To elucidate its meaning, it is convenient to represent the eq. (21) in Bargmann’s space of analytical functions which is isometrically isomorphic to the Bargmann space for photon number states \( \mathcal{B} \), where \( a, a^\dagger \) are realized by \( \partial_z, z \), respectively. The space \( \mathcal{B} \) is spanned by functions \( f(z) \) of a complex variable \( z \). Using \( Rf(z) = f(qz) \), in each \( Z_p \)-parity-chain \( \mathcal{H}_r \), we can write the Schrödinger equation as

\[
z \frac{d}{dz} \psi_r(z) + g \left( \frac{d}{dz} + z \right) \psi_r(z) = \epsilon_r \psi_r(z) - \sum_{k=1}^{p-1} q^k \psi_r(q^k z)
\]

(22)

where \( r = 0,1,2, \cdots, p-1 \).

Now let us consider the case \( r = 0 \). In this case the Hamiltonian \( H_0 \) reads in \( \mathcal{B} \),

\[
H_0 = z \frac{d}{dz} + g \left( \frac{d}{dz} + z \right) + \sum_{k=1}^{p-1} R^k \sigma^k
\]

(23)

The Schrödinger equation \( (H_0 - \epsilon_0)\psi(z) = 0 \) corresponds to a linear but non-local differential equation in the complex domain,

\[
z \frac{d}{dz} \psi(z) + g \left( \frac{d}{dz} + z \right) \psi(z) = \epsilon_0 \psi(z) - \sum_{k=1}^{p-1} \psi(q^k z)
\]

(24)

The theory of these equations initiated by Riemann and Fuchs [13] can now be applied to the above equation. First, if we set

\[
\psi(z) = \phi_0(z), \psi(q^{p-1} z) = \phi_1(z), \psi(q^{p-2} z) = \phi_2(z), \cdots, \psi(qz) = \phi_{p-1}(z)
\]

(25)

we obtain the following equations:

\[
(z + q^k g) \frac{d}{dz} \phi_r + (gg^{-r} z - \epsilon_0) \phi_r = -\sum_{s \neq r} \phi_s
\]

(26)

This system has \( p \) regular singular points at \( z = -gq^k, k = 0,1,\cdots, p-1 \) and an irregular singular point at \( z = \infty \). From the eq.(25), we can easily find that

\[
\phi_0(z) = \phi_1(qz) = \phi_2(q^2 z) = \cdots \phi_{p-1}(q^{p-1} z)
\]

(27)

If \( \epsilon_0 \) belongs to the spectrum of \( H_0 \), all \( \phi_r \)'s must be analytic at points \( z = -q^r g, r = 0,1,\cdots, p-1 \).

The eq.(26) is a system of \( p \) differential equations of Fuchsian type. In general it has \( p \) independent solutions. The required solution must be analytic in the whole complex plane, i.e. is entire, in order for \( E_r \) to belong to the spectrum of the system. The singular points of the
system are at $z = -gq^k$. When $\Delta = 0$, each differential equation behaves like $(z + gq^k)^\rho$ in the vicinity of each $z = -gq^k$, so we are lead to the following indicial equation:

$$\rho - E_r - g^2 = 0$$

So for all $p$ independent solutions $\phi_{r,k}(z)$ to be analytic at the singular points, we need

$$E_r = N - g^2, \quad (N = 0, 1, 2, \cdots)$$

This gives the exact isolated energies of the $p$ states Rabi model. These energies have the Rabi-like form but are $p$-fold degenerate.

3. Pertubative solution of $p$ states quantum Rabi equation

In this section we solve the $p$ states quantum Rabi Model by standard perturbation method. If we set $\phi_{r,k}(z) = \psi_r(q^kz)$ in the eq.(26), we obtain the following equations:

$$(z + gq^{-r})\frac{d}{dz}\phi_{r,s} + (gq^r z - E_r)\phi_{r,s} = -\Delta \sum_{k=1}^{p-1} q^{kr}\phi_{r,k+s}$$

Now let us apply the perturbation method for small value of $\Delta$ by putting

$$\phi_{r,k} = \sum_{i=0}^{\infty} \phi_{r,k}^{(i)} \Delta^i, \quad E_r = \sum_{i=0}^{\infty} E_r^{(i)} \Delta^i$$

Inserting eq.(29) into the eq.(28), we have

$$(z + q^{-s}g)\frac{d}{dz}\phi_{r,s}^{(i)} + (gq^s z - E_r^{(0)})\phi_{r,s}^{(i)} = \sum_{m=0}^{i-1} E_r^{(i-m)}\phi_{r,s}^{(m)} - \sum_{k=1}^{p-1} q^{kr}\phi_{r,s+k}^{(i-1)}$$

where we set $\phi_{r,k+pN}^{(i)} = \phi_{r,k}^{(i)}$, $k = 0, 1, 2, \cdots, p-1$ for any integer $N$. If we replace $\phi_{r,s}^{(i)} = e^{-q^s g y_s}\phi_{r,s}^{(i)}$ with $y_s = z + q^{-s}g$, we have

$$y_s \partial_{y_s} \bar{\phi}_{r,s}^{(i)} = \sum_{m=0}^{i-1} E_r^{(i-m)}\bar{\phi}_{r,s}^{(m)} - \sum_{k=1}^{p-1} q^{kr} e^{g^2(q^k-1)} e^{g (q^s - q^{k+s}) y_s} \bar{\phi}_{r,s+k}^{(i-1)}$$

Solving the eq.(30) for $i = 0$, we have

$$\bar{\phi}_{r,s}^{(0)} = 1$$

where we set $E_r^{(0)} = -g^2$. Solving the eq.(30) for $i = 1$, we have

$$\bar{\phi}_{r,s}^{(1)} = -\sum_{k=1}^{p-1} q^{kr} e^{g^2(q^k-1)} (q^s - q^{k+s}) y_s \cdot \mathcal{F}(1, 1; 2, 2; (q^s - q^{k+s}) y_s)$$

and

$$E_r^{(1)} = \sum_{k=1}^{p-1} q^{kr} e^{g^2(q^k-1)}$$

For $p = 2M$, we get

$$E_r^{(1)} = (-1)^p e^{-2g^2} + 2 \sum_{k=1}^{M-1} e^{g^2 \cos \frac{kr}{M} - 1} \cos \left( \frac{kr}{M} \pi + g^2 \frac{k}{M} \pi \right)$$
For \( p = 2M + 1 \), we get

\[
E^{(1)}_r = 2 \sum_{k=1}^{M} e^{g^2 \cos \left( \frac{2k}{2M+1} \pi \right)} \cos \left( \frac{2kr}{2M+1} - \sin \frac{2k}{2M+1} \pi \right)
\]

For \( i = 2 \), we have

\[
E^{(2)}_r = -g^2 \sum_{k=1}^{p-1} \sum_{m=1}^{p-1} q^{(k+m)} r e^{g^2 (q^k + q^m - 2)} (1 - q^k)(1 - q^m)
\]

\[\times 2F_2(1, 1; 2, 2; g^2(1 - q^k)(1 - q^m))\]  

\[\text{and}\]

\[
\tilde{\phi}^{(2)}_{r,s} = -E^{(1)}_r \sum_{k=1}^{p-1} q^k r e^{g^2 (q^k - 1)} (1 - q^k) g y_{s,3} F_2(1, 1; 2, 2; -q^k (1 - q^k) g y_{s})
\]

\[+ \sum_{k=1}^{p-1} \sum_{m=1}^{p-1} q^{(k+m)} r e^{g^2 (q^k + q^m - 2)} 2F_2(1, 1; 2, 2; g^2(1 - q^k)(1 - q^m)) \]

\[\times 2F_2(1, 1; 2, 2; q^k (1 - q^k) g y_{s})
\]

\[+ \sum_{n=1}^{\infty} \sum_{k=1}^{p-1} \sum_{m=1}^{p-1} \sum_{l=1}^{n} q^{(k+m)} r e^{g^2 (q^k + q^m - 2)} \left( \frac{n}{l} \right) \frac{1}{l^n!} (q^k (1 - q^k) g)^n l \]

\[\times 1F_1(l; l + 1; g^2(1 - q^k)(1 - q^m)) y_s^n \]  

4. Conclusion

In this paper we discuss the p-level Rabi model with \( Z_p \)-graded symmetry, which is a generalization of standard two-level Rabi model. We obtained the p-level Rabi Hamiltonian by introducing the generalized Pauli matrices. We investigated some properties of the generalized Pauli matrices. We diagonalized the p-level Rabi Hamiltonian with a help of two kinds of unitary transformations. For this Hamiltonian, we eliminated the discrete degree of freedom by introducing \( Z_p \)-parity-chain \( H_r, r = 0, 1, 2, \ldots, p-1 \) which is isomorphic to the Hilbert space of photon number states \( H \). In each \( H_r \), we rewrote the Hamiltonian in terms of the Bargmann variables. For this Hamiltonian, we obtained the energy and wave function by using the standard perturbation method. In this process, we obtained the explicit form of the energy and wave function up to a second order in \( \Delta \).

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Appendix

When \( i_1 = i_2 = \cdots = i_p \) dose not hold, we have

\[ \{ \sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_p} \} \]

\[= \sum_{m=1}^{p-1} \sum_{\text{perm} \{i_1, \ldots, i_{p-1} \}} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_{m-1}} \sigma_{p-1} \sigma_{i_m} \cdots \sigma_{i_{p-1}} \]
\[
\begin{align*}
&= \sum_{m=1}^{p-1} \sum_{\text{perm}\{i_1,\ldots,i_{p-1}\}} q^{i_1 t_1} e_{t_1, t_1+p-1} \cdots q^{i_{p-2} t_{p-2}} e_{t_{p-2}, t_{p-2}+p-1} q^{i_{p-1} t_{p-1}} e_{t_{p-1}, t_{p-1}+p-1} \\
&\quad \times d \times \sum_{l_{m+1}=0}^{p-1} q^{l_{m+1} t_{m+1}} e_{l_{m+1}, l_{m+1}+p-1} \cdots \sum_{l_{p-1}=0}^{p-1} q^{l_{p-1} t_{p-1}} e_{l_{p-1}, l_{p-1}+p-1} \\
&= \sum_{m=1}^{p-1} \sum_{\text{perm}\{i_1,\ldots,i_{p-1}\}} q^{\sum_{n=0}^{p-1} i_n (l_1 + (p-1)(n-1)) + t_1 + (m-1)(p-1)} e_{l_1 + (p-1)(m-1), l_1 + 1} \\
&= 0
\end{align*}
\]

because

\[
\sum_{m=1}^{p-1} q^{(m-1)(p-1)} = 0
\]

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