A Quantum Deformation of the Virasoro Algebra 
and 
the Macdonald Symmetric Functions

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Abstract

A quantum deformation of the Virasoro algebra is defined. The Kac determinants
at arbitrary levels are conjectured. We construct a bosonic realization of the quantum
deformed Virasoro algebra. Singular vectors are expressed by the Macdonald symmetric
functions. This is proved by constructing screening currents acting on the bosonic Fock
space.

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1 Introduction

The hidden correspondence between the Calogero-Sutherland model (CSM) and the conformal field theory (CFT) is very interesting and seems even a mysterious thing. This correspondence is studied by using the theory of the Jack symmetric polynomials and the Feigin-Fuchs bosonization of the Virasoro algebra. In CSM every excited state is written in terms of the Jack symmetric polynomial, and the singular vectors of CFT are also given by the Jack symmetric functions with rectangular diagrams. The Macdonald symmetric functions also can be written by using $q$-analogues of the screening currents. This is a strong evidence that the correspondence of CSM and CFT still survive after the $q$-deformation. Therefore an answer to the following question should be desired: what algebraic structure does emerge from a quantum deformation of CSM obtained by replacing the Jack polynomials with the Macdonald polynomials? A quite interesting answer to this question does exist, and we can regard the new algebra as an deformation of the Virasoro algebra. In this letter we study this quantum deformation of the Virasoro algebra. Highest weight modules for this quantum deformed Virasoro ($q$-Virasoro) algebra are defined. In order to have reducibility conditions, we calculate the Kac determinants for some lower levels and obtained a conjectural formula for general level $N$. We construct a free boson realization of the $q$-Virasoro algebra. This bosonization can be regarded as a $q$-deformation of the Feigin-Fuchs representation of the Virasoro algebra. We show that, after bosonization, singular vectors are written in terms of the Macdonald symmetric functions. This is analogous to the case of the ordinary Virasoro algebra whose singular vectors are given by the Jack symmetric functions. Surprisingly, our $q$-Virasoro algebra has a connection with the quantum affine algebra in the following sense. Quantum deformation of the Virasoro and $\mathcal{W}$-algebras associated with the quantum affine algebras at critical level were successfully constructed by Frenkel and Reshetikhin. One can show that our $q$-Virasoro algebra tends to theirs in a certain limit (see Discussions). Frenkel and Reshetikhin pointed out their bosonized $q$-Virasoro and $\mathcal{W}$-currents resemble the dressed vacuum form (DVF) in the algebraic Bethe ansatz. We can notice that the bosonization formula for our $q$-Virasoro algebra has DVF structure, too (see section 4). We present a heuristic derivation of our bosonized $q$-Virasoro current in Appendix.

Applying the collective field theory to the Calogero-Sutherland model, it can be shown that the Virasoro generators are related to the Calogero-Sutherland Hamiltonian $\hat{H}_\beta$ as follows,

$$\hat{H}_\beta = \beta \sum_{n=1}^{\infty} a_{-n} L_n + (\beta (N + 1 - 2a_0) - 1) \hat{P},$$

(1)

where $\beta$ is the coupling constant of the Calogero-Sutherland model and $N$ is the number of the particles, $\hat{P}$ is the momentum of CSM, $L_n$’s are the generators of the Virasoro algebra and $a_{-n}$’s are the creation operators of the free boson field and $a_0$ is its zero mode. This splitting $\hat{H}_\beta$ into $a_{-n}$ and $L_n$ means that the singular vectors of the Virasoro algebra are given by the Jack symmetric functions. It is natural to expect that we have an analogous split expression for the bosonized Macdonald operator $\hat{D}_{q,t}$

$$\hat{D}_{q,t} = \sum_{n=1}^{\infty} \psi_{-n} T_n + \cdots,$$

(2)

where $T_n$’s should be understood as the generators of the $q$-Virasoro algebra and $\psi_{-n}$’s do not
contain positive modes of the free boson. In this article, we will show that this splitting certainly can be done, and $T_n$’s generate an algebra having the Macdonald symmetric functions as its singular vectors. We can obtain screening currents $S^{\pm}_n(z)$ for our $q$-Virasoro algebra. Using this free boson realization scheme, we discuss Feigin-Fuchs like construction of the highest weight modules of the $q$-Virasoro algebra and study the singular vectors.

This letter is organized as follows. In section 2, we define the quantum deformation of the Virasoro algebra. In section 3, the Kac determinant of the $q$-Virasoro algebra is calculated at level 1 and level 2. In section 4, the free boson realization of the $q$-Virasoro algebra and the splitting of the Macdonald operator are given. Screening currents are constructed in section 5, and the singular vectors are written down by the screening currents. We show the singular vectors are the Macdonald symmetric functions with rectangular diagrams. Section 6 is devoted to discussions. In Appendix we show our heuristic derivation of the bosonized $q$-Virasoro current.

2 Definition of the $q$-Virasoro algebra $\mathcal{V}_{ir_{q,t}}$

Let $q, t$ be two generic complex parameters. For simplicity we will consider the case $|p| < 1$, where we have set $p = qt^{-1}$ and will use this notation frequently. We also write $t = q^\beta$ by introducing a complex parameter $\beta$, which plays the role of the coupling constant of the CSM in the limit of $q \to 1$. The $q$-Virasoro algebra $\mathcal{V}_{ir_{q,t}}$ is an associative algebra generated by \{\(T_n | n \in \mathbb{Z}\)\} with the following relations

\[
[T_n, T_m] = -\sum_{l=1}^{\infty} f_l (T_{n-l} T_{m+l} - T_{m-l} T_{n+l}) - \frac{(1 - q)(1 - t^{-1})}{1 - p} (p^n - p^{-n}) \delta_{m+n,0},
\]

where the coefficients $f_l$’s are given by the following generating function $f(z)$

\[
f(z) = \sum_{l=0}^{\infty} f_l z^l = \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})}{1 + p^n} z^n \right\}.
\]

Introducing $\mathcal{V}_{ir_{q,t}}$ current $T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}$, the defining relation (3) can be written as follows

\[
f(w/z) T(z) T(w) - T(w) T(z) f(z/w) = -\frac{(1 - q)(1 - t^{-1})}{1 - p} \left[ \delta\left(\frac{pw}{z}\right) - \delta\left(\frac{p^{-1}w}{z}\right) \right],
\]

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$. Note that the defining relation (3) is invariant under the transformation

\[T_n \to -T_n,
\]

and also invariant under

\[(q, t) \to (q^{-1}, t^{-1}).
\]

Here we give the relation between our quantum deformed Virasoro current $T(z)$ and the ordinary one $L(z)$. Let us study the limit $q \to 1$ by parameterizing $q = e^h$. Suppose that $T(z)$ has the expansion

\[T(z) = 2 + \beta \left( z^2 L(z) + \frac{(1 - \beta)^2}{4\beta} \right) h^2 + T^{(2)}(z) h^4 + \cdots.
\]
This expansion is consistent with the invariance under transformation (7). The defining relation (3) gives us the well known relations for \( L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \), namely

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c(n^3 - n)}{12}\delta_{n+m,0},
\]

where

\[
c = 1 - \frac{6(1 - \beta)^2}{\beta}.
\]

This relation between the central charge of the Virasoro algebra and the coupling constant of CSM is discussed by studying the Virasoro constraints of generalized matrix models[3].

3 Highest weight modules of \( \mathcal{V}ir_{q,t} \)

Let us define the Verma module of \( \mathcal{V}ir_{q,t} \). Let \( |\lambda\rangle \) be the highest weight vector having the properties

\[
T_0|\lambda\rangle = \lambda|\lambda\rangle, \quad T_n|\lambda\rangle = 0 \quad \text{for} \quad n \geq 1.
\]

The Verma module \( M(\lambda) \) is defined by

\[
M(\lambda) = \mathcal{V}_{q,t} |\lambda\rangle.
\]

The irreducible highest module \( V(\lambda) \) is obtained from \( M(\lambda) \) by removing all singular vectors and their descendants. Right modules are defined in a similar way from the lowest weight vector \( \langle \lambda | \) s.t. \( \langle \lambda | T_0 = \lambda \langle \lambda | \), \( \langle \lambda | T_n = 0 \) \( n \leq -1 \). A unique invariant paring is defined by setting \( \langle \lambda | \lambda \rangle = 1 \). The Verma module of the ordinary Virasoro algebra may have the singular vectors. The Verma module \( M(\lambda) \) may also have the singular vectors in the same way. Let us introduce the (outer) grading operator \( d \) which satisfies \( [d, T_n] = n T_n \). Set \( d|\lambda\rangle = 0 \). We call a vector \( |v\rangle \in M(\lambda) \) of level \( n \) if \( d|v\rangle = -n|v\rangle \).

Whether there exist the singular vectors or not is checked by calculating the Kac determinant. Here, we give some explicit forms of \( f_n \) which we will use for the calculations

\[
f_1 = \frac{(1 - q)(1 - t^{-1})}{1 + p},
\]

\[
f_2 = \frac{(1 - q^2)(1 - t^{-2})}{2(1 + p^2)} + \frac{(1 - q)^2(1 - t^{-1})^2}{2(1 + p)^2}.
\]

At level 1, the Kac determinant is the \( 1 \times 1 \) matrix as follows

\[
\langle \lambda | T_1 T_-1 |\lambda \rangle = \frac{(1 - q)(1 - t)}{q + t}(\lambda^2 - (p^{1/2} + p^{-1/2})^2).
\]

Therefore, there exist a singular vector at level 1 iff \( \lambda = \pm \left( p^{1/2} + p^{-1/2} \right) \), since \( q \) and \( t \) are generic. The signs \( \pm \) in the RHS are due to the symmetry (8).
At level 2, the Kac determinant is
\[
\begin{vmatrix}
\langle \lambda | T_1 T_1 T_{-1} T_{-1} | \lambda \rangle & \langle \lambda | T_1 T_{-1} | \lambda \rangle \\
\langle \lambda | T_2 T_{-1} T_{-1} | \lambda \rangle & \langle \lambda | T_2 T_{-2} | \lambda \rangle
\end{vmatrix}
\]
\[
= \frac{(1 - q^2)(1 - q)^2 q^{-4}(1 - t^2)(1 - t)^2 t^{-4}}{(q + t)^2 (q^2 + t^2)}
\times (\lambda^2 q t - (q + t)^2) (\lambda^2 q^2 t - (q^2 + t)^2) (\lambda^2 q t^2 - (q + t^2)^2).
\] (17)

The vanishing conditions of the Kac determinant are

(i) \( \lambda = \pm \left( p^{1/2} + p^{-1/2} \right) \),

(ii) \( \lambda = \pm \left( p^{1/2} q^{1/2} + p^{-1/2} t^{-1/2} \right) \),

(iii) \( \lambda = \pm \left( p^{1/2} t^{-1/2} + p^{-1/2} t^{1/2} \right) \).

(18) \( \text{(19)} \) \( \text{(20)} \)

In the case (i), there is a singular vector at level 1. In the cases (ii) and (iii), we have a singular vector at level 2. The singular vector for the case (ii) is

\[
\frac{q t^{-1/2} (q + t)}{(1 - q)^2 (1 + t)} T_{-1} T_{-1} | \lambda \rangle \mp T_{-2} | \lambda \rangle,
\] (21)

and for (iii) is

\[
\frac{q^{-1/2} t (q + t)}{(1 - t)^2 (1 + t)} T_{-1} T_{-1} | \lambda \rangle \mp T_{-2} | \lambda \rangle.
\] (22)

In discussion, we will state a conjecture of the Kac determinant for arbitrary level \( N \).

4 Free boson realization of \( \mathcal{V}ir_{q,t} \)

In this section we construct a free boson realization of \( \mathcal{V}ir_{q,t} \). One may expect that a free boson realization of \( \mathcal{V}ir_{q,t} \) enables us to investigate new aspects of the singular vectors. Let us introduce bosonic oscillators \( a_n \) \( n \in \mathbb{Z} \) and \( Q \) with the commutation relations

\[
[a_n, a_m] = \frac{n}{1 - t} \delta_{n+m,0},
\] (23)

\[
[a_n, Q] = \frac{1}{\beta} \delta_{n,0}.
\] (24)

Let us define the bosonic Fock space. Introduce the vacuum state \( |0\rangle \) which satisfy the following conditions

\( a_n |0\rangle = 0 \) for \( n \geq 0 \).

(25)

Furthermore, add the zero-mode momentum for \( r, s \in \mathbb{Z} \)

\( |r, s\rangle = e^{\alpha_{r,s} Q} |0\rangle, \)

(26)

where

\[ \alpha_{r,s} = \frac{1}{2} (1 + r) \beta - \frac{1}{2} (1 + s). \] (27)
We define the Fock space $\mathcal{F}_{r,s}$ by $C[a_{-1}, a_{-2}, \cdots | r, s]$. By studying the operator product expansion, we can check that $\text{Vir}_{q,t}$ current $T(z)$ is bosonized as follows:

$$
T(z) = p^{1/2} \exp \left\{ - \sum_{n=1}^{\infty} \frac{1 - t^n}{1 + p^n} a_n z^n t^{-n} p^{-n/2} \right\} \exp \left\{ - \sum_{n=1}^{\infty} \frac{1 - t^n}{1 + p^n} a_n z^n p^{-n/2} \right\} q^{\beta a_0} 
$$

$$
+ p^{-1/2} \exp \left\{ \sum_{n=1}^{\infty} \frac{1 - t^n}{1 + p^n} a_n z^n t^{-n} p^{-n/2} \right\} \exp \left\{ \sum_{n=1}^{\infty} \frac{1 - t^n}{1 + p^n} a_n z^n p^{-n/2} \right\} q^{-\beta a_0}. \quad (28)
$$

We can observe that this formula has strong resemblance to the dressed vacuum form in the algebraic Bethe ansatz. This profound $q$-Virasoro-DVF correspondence was discovered by Frenkel and Reshetikhin [12]. In Appendix we state our heuristic derivation of the formula (28) which does not rely on the defining relation (3).

We have the following highest weight conditions

$$
T_0|r, s\rangle = \left( p^{1/2} q^{\alpha_{r,s}} + p^{-1/2} q^{-\alpha_{r,s}} \right) |r, s\rangle,
$$

$$
T_n|r, s\rangle = 0 \quad \text{for} \quad n \geq 1. \quad (29)
$$

So we obtained the embedding $V(\lambda_{r,s}) \rightarrow \mathcal{F}_{r,s}$, where $\lambda_{r,s} = \left( p^{1/2} q^{\alpha_{r,s}} + p^{-1/2} q^{-\alpha_{r,s}} \right)$.

Next we will show how the Macdonald operator $D_{q,t}$ is factorized in terms of the $\text{Vir}_{q,t}$-current $T(z)$. The operator $D_{q,t}$ is defined by

$$
D_{q,t} = \sum_{i=1}^{N} \prod_{j=1, j \neq i}^{N} \frac{t x_i - x_j}{x_i - x_j} T_{q, x_i}, \quad (30)
$$

where $T_{q, x_i}$ is the shift operator defined by $T_{q, x_i} g(x_1, \cdots, x_i, \cdots, x_r) = g(x_1, \cdots, x_i, \cdots, x_r)$. $D_{q,t}$ acts on the ring of the symmetric polynomials with $N$-variables $\Lambda_N$. We have the projection $\pi_N : C[a_{-1}, a_{-2}, \cdots | 0] \rightarrow \Lambda_N$ given by the mapping $\pi_N : a_{-n_1} a_{-n_2} \cdots | 0 \mapsto p_{n_1} p_{n_2} \cdots$ where $p_n = \sum_{i=1}^{N} x_i^n$. We have the bosonized Macdonald operator $\widehat{D}_{q,t}$ which satisfies $D_{q,t} \circ \pi_N = \pi_N \circ \widehat{D}_{q,t}$ given by the following formula:

$$
\widehat{D}_{q,t} = \frac{t^N}{t - 1} \oint \frac{dz}{2\pi \sqrt{-1} z} \exp \left\{ \sum_{n=1}^{\infty} \frac{1 - t^n}{n} a_n z^n \right\} \exp \left\{ - \sum_{n=1}^{\infty} \frac{1 - t^n}{n} a_n z^n \right\} - \frac{1}{t - 1}. \quad (31)
$$

One may find that this bosonized operator $\widehat{D}_{q,t}$ can be decomposed by using the $q$-Virasoro current $T(z)$ and the operator $\psi(z)$ defined by

$$
\psi(z) = \sum_{n=0}^{\infty} \psi_n z^n = p^{-1/2} \exp \left\{ - \sum_{n=1}^{\infty} \frac{1 - t^n}{1 + p^n} a_n z^n p^{n/2} t^{-n} \right\} q^{-\beta a_0}, \quad (32)
$$

as follows,

$$
\widehat{D}_{q,t} = \frac{t^N}{t - 1} \left[ \oint \frac{dz}{2\pi \sqrt{-1} z} \frac{1}{\psi(z)} T(z) - p^{-1} q^{-2\beta a_0} \right] - \frac{1}{t - 1}
$$

$$
= \frac{t^N}{t - 1} \left[ \sum_{n=0}^{\infty} \psi_n T_n - p^{-1} q^{-2\beta a_0} \right] - \frac{1}{t - 1}. \quad (33)
$$
We study the singular vectors of $Vir_{q,t}$ in the bosonic Fock space. Define the screening currents $S_{\pm}(z)$ as follows:

$$S_+(z) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1 - t^n}{1 - q^n} a_{-n} z^n \right\} \exp \left\{ - \sum_{n=1}^{\infty} \left( 1 + p^n \right) \frac{1 - t^n}{1 - q^n} a_{n} z^{-n} \right\} e^{\beta Q z^{2\beta \alpha_0}},$$

$$S_-(z) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{a_{-n} z^n}{n} \right\} \exp \left\{ \sum_{n=1}^{\infty} \left( 1 + p^n \right) \frac{a_{n} z^{-n} p^{-n}}{n} \right\} e^{-Q z^{-2\alpha_0}}.$$  

The commutation relation between $T_n$ and the screening currents are

$$[T_n, S_+(w)] = -(1 - q)(1 - t^{-1}) \frac{d_q}{d_q w} \left( (p^{-\frac{1}{2}} w)^{n+1} A_+ (w) \right),$$

$$[T_n, S_- (w)] = -(1 - q^{-1})(1 - t) \frac{d_t}{d_t w} \left( (p^{\frac{1}{2}} w)^{n+1} A_- (w) \right),$$

where

$$A_+ (z) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1 + t^n}{1 + p^n} \frac{1 - t^n a_{-n} z^n t^{-n}}{1 - q^n} \right\} \exp \left\{ - \sum_{n=1}^{\infty} \left( 1 + t^n \right) \frac{1 - t^n a_{n} z^{-n} p^n}{1 - q^n} \right\} e^{\beta Q z^{2\beta \alpha_0} t^{-\alpha_0}},$$

$$A_- (z) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1 + q^n a_{-n} z^n t^{-n}}{1 + p^n} \right\} \exp \left\{ \sum_{n=1}^{\infty} \left( 1 + q^n \right) \frac{a_{n} z^{-n} p^{-n}}{n} \right\} e^{-Q z^{-2\alpha_0} t^{\alpha_0}},$$

and the difference operator with one parameter is defined by

$$\frac{d_{\xi}}{d_{\xi} z} g(z) = \frac{g(z) - g(\xi z)}{(1 - \xi) z}.$$  

We can construct the BRS charges in the same way discussed in the paper[11]. On the Fock space $\mathcal{F}_{-r,s}$, we have the single-valued BRS current

$$J^{(r)}_+ (w) = S_+ (w) \oint \prod_{i=2}^{r} \frac{dw_i}{2\pi \sqrt{-1}} S_+ (w_i),$$

where the integration cycle is chosen to be Felder’s one. Even though the commutator $[T_n, S_+(w)]$ is a “total difference”, we can check that $[T_n, \oint dw J^{(r)}_+ (w)] = 0$ on the Fock space $\mathcal{F}_{-r,s}$. This is because on the Fock space $\mathcal{F}_{-r,s}$ the expansion $[T_n, J^{(r)}_+ (w)] = \sum_{m \in \mathbb{Z}} \mathcal{O}_m w^{-m}$ is well defined and $\mathcal{O}_1 = 0$ since the commutator is a total difference.

We also have another BRS current

$$J^{(s)}_- (w) = S_- (w) \oint \prod_{i=2}^{s} \frac{dw_i}{2\pi \sqrt{-1}} S_- (w_i),$$

which is single-valued on the Fock space $\mathcal{F}_{r,-s}$.
The singular vector in $\mathcal{F}_{r,s}$ is written by using BRS current as

$$|\chi_{r,s}\rangle = \oint \frac{dw}{2\pi\sqrt{-1}} f_{+}^{(r)}(w)|r, s\rangle = \oint \frac{dw}{2\pi\sqrt{-1}} f_{-}^{(s)}(w)|r, -s\rangle,$$

up to normalization. Using equation (33), we obtain

$$\hat{D}_{q,t}|\chi_{r,s}\rangle = \left(\sum_{i=1}^{r} t^{N-i} q^{s} + \sum_{i=r+1}^{N} t^{N-i}\right)|\chi_{r,s}\rangle.$$  (44)

This means that the singular vector $|\chi_{r,s}\rangle$ is the eigen vector of the Macdonald operator $\hat{D}_{q,t}$. Namely $|\chi_{r,s}\rangle$ is proportional to the Macdonald symmetric function $P_{\lambda}(q,t)$ with the rectangular Young diagram $\lambda = (s^r)$.

It may be helpful to give another proof that $|\chi_{r,s}\rangle$ is the bosonized Macdonald symmetric function without using $q$-Virasoro algebra. We calculate $\hat{D}_{q,t}|\chi_{r,s}\rangle$ in the following way. We have

$$|\chi_{r,s}\rangle = \oint \prod_{i=1}^{r} \frac{dw_{i}}{2\pi\sqrt{-1}} F^{(+)}(w_{1}, w_{2}, \cdots, w_{r}) : S_{+}(w_{1})S_{+}(w_{2})\cdots S_{+}(w_{r})e^{a_{-r,s}Q} : |0\rangle,$$  (45)

where

$$F^{(+)}(w_{1}, \cdots, w_{r}) = \prod_{j=1}^{r} w_{j}^{2\alpha_{-r,s}+2(r-j)\beta} \prod_{1 \leq i < j \leq r} \frac{w_{j}/w_{i}; q_{\infty}}{1-\frac{w_{j}/w_{i}}{1-\frac{qw_{j}/w_{i}}{q}}},$$  (46)

here we used the standard notation $(z; q_{\infty}) \equiv \prod_{k=0}^{\infty} (1-q^{k}z)$. An explicit residue calculation gives us the following

$$t^{-N+1} \left(\hat{D}_{q,t} - \frac{1-t^{N-r}}{1-t}\right)|\chi_{r,s}\rangle = \oint \prod_{i=1}^{r} \frac{dw_{i}}{2\pi\sqrt{-1}} \sum_{i=1}^{r} q^{-1} \prod_{j=1}^{i-1} t^{-1} \frac{1-p^{-1}w_{i}/w_{j}}{1-q^{-1}w_{i}/w_{j}} \cdot \prod_{j=i+1}^{r} \frac{1-pw_{j}/w_{i}}{1-qw_{j}/w_{i}} \cdot (T_{q^{-1}, w_{i}} F^{(+)}(w_{1}, \cdots, w_{r}))$$

$$\times : S_{+}(w_{1})S_{+}(w_{2})\cdots S_{+}(w_{r})e^{a_{-r,s}Q} : |0\rangle = \oint \prod_{i=1}^{r} \frac{dw_{i}}{2\pi\sqrt{-1}} t^{1-r} q^{s}\prod_{i=1}^{r} \frac{1-tw_{j}/w_{i}}{1-w_{j}/w_{i}} F^{(+)}(w_{1}, \cdots, w_{r})$$

$$\times : S_{+}(w_{1})S_{+}(w_{2})\cdots S_{+}(w_{r})e^{a_{-r,s}Q} : |0\rangle = \sum_{i=1}^{r} q^{s-(r-1)\beta}|\chi_{r,s}\rangle.$$  (47)

In the calculation we used the following equations

$$T_{q^{-1}, w_{i}} F^{(+)}(w_{1}, \cdots, w_{r}) = \left[ \prod_{j=1}^{i-1} \frac{(1-q^{-1}w_{i}/w_{j})(1-t^{-1}w_{i}/w_{j})}{(1-p^{-1}w_{i}/w_{j})(1-w_{i}/w_{j})} \right] \left[ \prod_{j=i+1}^{r} \frac{(1-tw_{j}/w_{i})(1-qw_{j}/w_{i})}{(1-w_{j}/w_{i})(1-pw_{j}/w_{i})} \right]$$

$$\times q^{-2\alpha_{-r,s}} t^{-2(r-1)} F^{(+)}(w_{1}, \cdots, w_{r}).$$  (48)
and

\[ \sum_{i=1}^{r} \prod_{j \neq i} \frac{1 - tw_j/w_i}{1 - w_j/w_i} = \frac{1 - t^r}{1 - t^r}, \quad (49) \]

Then we get \( \hat{D}_{q,t}|\chi_{r,s}\rangle = \left( \sum_{i=1}^{r} t^{N-i}q^i + \sum_{i=r+1}^{N} t^{N-i} \right)|\chi_{r,s}\rangle \). A similar calculation can be done for the singular vectors composed of the screening currents \( S_-(w)'s \).

6 Discussions

In this section we look through some of our further results and unclarified problems.

- To calculate the Kac determinant becomes difficult task when \( N \) increases. We have calculated up to level 4, and write down the conjectural form at level \( N \) as follows

\[ \det_N = \det \left( \langle i|j \rangle \right)_{1 \leq i,j \leq p(N)} = \prod_{r,s \geq 1, rs \leq N} \left( \lambda^2 - \lambda_{r,s}^2 \right)^{p(N-rs)} \frac{(1 - q^r)(1 - t^r)}{q^r + t^r}^{p(N-rs)}, \]

where the basis at level \( N \) is defined \( |1\rangle = T_{-N}|\lambda\rangle, |2\rangle = T_{-N+1}T_{-1}|\lambda\rangle, \ldots, |p(N)\rangle = T_{-1}^N|\lambda\rangle \), and \( p(N) \) is the number of the partition of \( N \). We remark that the \( \lambda \) dependence has essentially the same structure as the case of the usual Virasoro algebra. Therefore, if \( q \) and \( t \) are generic, the character of the quantum Virasoro algebra \( \mathcal{V}_{irq,t} \), which counts the degeneracy at each level, exactly coincides with that of the usual Virasoro algebra. The \( \lambda \)-independent factor in the RHS will play an important role when we study the case that \( q \) is a root of unity.

- We can study the limit of \( \beta \to 0 \) and \( q \) be fixed. In this case \( T_n \)'s become commutative and the defining relations of the \( q \)-Virasoro algebra (3) reduces to the eq. (9.4) of Frenkel-Reshetikhin [12] if we replace the commutation bracket \( \frac{1}{\beta} [ , ] \) by the Poisson bracket \( \{ , \} \). The limit of \( q \to 0 \) and \( t \) be fixed is interesting in another sense. We obtain non-trivial algebra at \( q = 0 \) if we normalize the generators \( T_n \)'s as \( \tilde{T}_n = T_n q^n \). We can define the Verma module generated by \( \tilde{T}_n \)'s with the highest weight \( \lambda \). It may be seen that at \( q = 0 \) the Kac determinant dose not depend on \( \lambda \). Therefore, if \( t \) is generic, we have no singular vectors. The most remarkable thing at \( q = 0 \) is that after bosonization every weight vector can be written down by the Hall-Littlewood symmetric function[13].

- The Virasoro algebra plays a central role in the conformal field theory in two dimensions because it generates the conformal transformations of the local fields. If we want to understand the physical meaning of the quantum deformed Virasoro algebra \( \mathcal{V}_{irq,t} \) which was introduced to study the \( q \)-deformed CSM-CFT correspondence, we have to investigate the symmetry generated by the algebra \( \mathcal{V}_{irq,t} \). In other words, we have to understand \( \mathcal{V}_{irq,t} \) from a geometric point of view.

1. We have the representation of the Virasoro algebra for \( c = 0 \) given by \( L_n = -z^{n+1} \partial_z \), i.e. generators of the conformal transformations of holomorphic functions. It is interesting to find a similar representation for \( T_n \)'s.
2. The primary fields in CFT are characterized by the operator product expansion with the energy momentum tensor $L(z)$. In the $q$-deformed theory, operator product expansions do not work well. Therefore, in order to define the primary fields of $\text{Vir}_{q,t}$, an algebraic language for treating the intertwining property will be needed. In other words, we need the coproduct of $\text{Vir}_{q,t}$.

3. The correlation functions of the local fields of the minimal models satisfy some differential equations [2]. What equations do the correlation functions of $\text{Vir}_{q,t}$ satisfy? Do that coincide with the $q$-hypergeometric difference equation [16] for some cases?

4. The Jack symmetric functions with rectangular diagrams are the singular vectors of the Virasoro algebra and diagrams composed of $N$ rectangulars are that for the $\mathcal{W}_N$ algebra [3]. We proved in this article that the singular vectors of $\text{Vir}_{q,t}$ are expressed by the Macdonald symmetric functions with rectangular diagrams. Hence we expect that the Macdonald symmetric functions with diagrams made of $N$ rectangulars correspond to the singular vectors of a quantum deformed $\mathcal{W}_N$ algebra.

- A traditional way to introduce the quantum deformation of an infinite dimensional symmetry algebra such as quantum affine algebra $U_q(\hat{\mathfrak{g}})$ [14] [15] is based on the Yang-Baxter equation. The principle we have used, on the other hand, is just to “factor” the Macdonald operator as $\hat{D}_{q,t} = \oint dz \psi(z) T(z) + \cdots$. So far, we have not applied the idea of the Yang-Baxter equation directly. But, there are many evidences that the algebra $\text{Vir}_{q,t}$ has deep connection with the integrable systems. This point should be clarified.

1. The relation between $\text{Vir}_{q,t}$ and $U_q(\hat{\mathfrak{s\ell}}_2)$ must be clarified. If we proceed to this direction, a lattice theoretical interpretation of $\text{Vir}_{q,t}$ will be desired. Can we regard $\text{Vir}_{q,t}$ as an algebra acting on the Hilbert space of the XXZ spin chain model?

2. Frenkel and Reshetikhin defined the $q$-$\mathcal{W}$ algebra based on $U_q(\hat{\mathfrak{g}})$ and pointed out the remarkable formal resemblance between the bosonized $q$-$\mathcal{W}$ currents and the dressed vacuum form (DVF) in the algebraic Bethe ansatz [17]. This similar structural coincidence can be observed in the bosonized $\text{Vir}_{q,t}$ current $T(z)$ (eq. (28)).

3. The sine-Goldon theory is understood as the integrable massive deformation theory of CFT. Since this theory is integrable, there exist infinitely many conserved charges $\{P_1, P_3, \cdots\}$. It seems interesting to construct a $q$-deformation of this scenario. Does it have to do with the quantum KdV theory [18]?

4. In the previous paper [3] we studied the generalized matrix model and the Virasoro constraint. Construct a matrix model whose constraints are described by $\text{Vir}_{q,t}$. Is any integrable hierarchy associated with such matrix model?

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Appendix

In this appendix, we show our heuristic derivation of the bosonized \( \text{Vir}_{q,t} \) current \( T(z) \). We start from the known data obtained from the study of the integral representation of the Macdonald symmetric functions \([3][4]\). That are

- The bosonized Macdonald operator:
  \[
  \hat{D}_{q,t} = \frac{t^{N}}{t-1} \oint \frac{dz}{2\pi \sqrt{-1}z} \exp \left\{ \sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^{n} \right\} \exp \left\{ -\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} a_{n} z^{-n} \right\} - \frac{1}{t-1}. \tag{50}
  \]

- The creation operator and the zero-mode parts of the screening currents:
  \[
  S_{+}(z) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1-t^{n}}{1-q^{n}} a_{-n} z^{n} \right\} \exp \{ \text{annihilation part for } S_{+} \} e^{\delta Q z^{2} a_{0}}, \tag{51}
  \]
  \[
  S_{-}(z) = \exp \left\{ -\sum_{n=1}^{\infty} \frac{a_{-n}}{n} z^{n} \right\} \exp \{ \text{annihilation part for } S_{-} \} e^{-Q z^{-2} a_{0}}. \tag{52}
  \]

Note that the zero-mode parts are borrowed from the \( q = 1 \) free boson realization of the screening currents.

We want to “factorize” the bosonized Macdonald operator. First, let us choose the ansatz for factorization as follows

\[
\oint \frac{dz}{2\pi \sqrt{-1}z} \exp \left\{ \sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} a_{-n} z^{n} \right\} \exp \left\{ -\sum_{n=1}^{\infty} \frac{1-t^{n}}{n} a_{n} z^{-n} \right\} = \oint \frac{dz}{2\pi \sqrt{-1}z} \exp \left\{ \sum_{n=1}^{\infty} \frac{1-t^{-n}}{1+q^{n}} a_{-n} z^{n} q^{\epsilon n} \right\} q^{\delta a_{0}} \left( B_{+}(z) + \eta B_{-}(z) \right) - \eta q^{(\delta+\gamma)a_{0}},
\]

where

\[
B_{\pm}(z) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1-t^{-n}}{1+q^{n}} a_{-n} z^{n} q^{\epsilon n} \right\} \exp \left\{ -\sum_{n=1}^{\infty} (1-t^{n}) a_{n} z^{-n} \right\} q^{-\delta a_{0}}, \tag{53}
\]

\[
B_{-}(z) = \exp \left\{ -\sum_{n=1}^{\infty} \frac{1-t^{-n}}{1+q^{n}} a_{-n} z^{n} q^{\epsilon n} \right\} \exp \left\{ \sum_{n=1}^{\infty} (1-t^{n}) a_{n} z^{-n} q^{\alpha n} \right\} q^{\gamma a_{0}}, \tag{54}
\]

and \( \alpha, \delta, \gamma, \eta \) and \( \epsilon \) are parameters. Expand \( B_{\pm}(z) \) as \( B_{\pm}(z) = \sum B_{\pm,n} z^{-n} \).

The problem is to find suitable expressions for the annihilation parts of \( S_{\pm}(z) \) and fix the parameters \( \alpha, \delta, \gamma, \eta, \epsilon \) by imposing the following conditions.

The commutation relations between \( B_{+,n} + \eta B_{-,n} \) and \( S_{\pm}(w) \) are given by total differences of some fields \( O_{\pm}(w) \) with some parameters \( \chi, \xi \):

\[
[B_{+,n} + \eta B_{-,n}, S_{+}(w)] = \frac{d_{\chi}}{d_{\chi}w} O_{+}(w), \tag{55}
\]
\[
[B_{+,n} + \eta B_{-,n}, S_{-}(w)] = \frac{d_{\xi}}{d_{\xi}w} O_{-}(w). \tag{56}
\]
The reasons why we applied this ansatz i.e. to factorize the Macdonald operator by \( B_+ (z) + \eta B_- (z) \) are the following. i) Since the RHS’s of (55) and (56) are the total differences (that are given by sums of two operators), we have to introduce “two terms” \( B_+ \) and \( B_- \). This principle worked very well in the study of the screening currents of the quantum affine algebra \( U_q (\mathfrak{sl}_N) \) \([19]\). ii) Frenkel and Reshetikhin succeeded in finding the \( q \)-Virasoro algebra and its bosonization basing on the quantum affine algebra \( U_q (\mathfrak{sl}_2) \) at the critical level \( k = -2 \). Their bosonized formula of the \( q \)-Virasoro algebra also consists of two terms as \( S(z) = q^{-1} e^{-\lambda (q z)} + q e^{\lambda (q^{-1} z)} \) (see \([12]\) as for notation). Therefore, if we start from this ansatz and solve the problem, there may be a chance to observe close connection with their theory. However the factorization problem is quite complicated, it seems very difficult to find an ansatz having higher symmetry of the parameters at the beginning as Frenkel-Reshetikhin’s \( q \)-Virasoro current has.

Let us proceed to find the annihilation parts of \( S_\pm \). We have the following operator product expansions:

\[
B_+ (z) S_+ (w) = : B_+ (z) S_+ (w) : \frac{z - w}{z - tw} q^{-\delta}, \\
B_- (z) S_+ (w) = : B_- (z) S_+ (w) : \frac{z - q^2 tw}{z - q^2 w} q^\gamma. 
\]

So as to obtain (55), (56), it is desired that the commutation relations between \( B_\pm (z) \) and \( S_\pm (w) \) can be factorized by the delta functions \( \delta (q^\rho w / z) \) with some suitable parameters \( \rho_\pm \). To this end, let us impose the following operator product expansions:

\[
S_+ (w) B_+ (z) = : B_+ (z) S_+ (w) : \frac{w - z}{tw - z} q^{-\delta}, \\
S_+ (w) B_- (z) = : B_- (n) S_+ (w) : \frac{q^2 tw - z}{q^2 w - z} q^\gamma.
\]

Then we can find the annihilation part of \( S_+ (z) \) as follows

\[
S_+ (z) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1 - t^n}{1 - q^n} \frac{a_n z^n}{n} \right\} \exp \left\{ - \sum_{n=1}^{\infty} (1 + q^n) \frac{1 - t^n}{1 - q^n} \frac{a_n z^n}{n} \right\} e^{\beta Q z} 2^{\beta a_0},
\]

and we also have the relations \( \epsilon = -\alpha, \delta = -\beta \) and \( \gamma = -\beta \).

Repeat the same argument with \( B_\pm (z) \) and \( S_- (z) \). We have

\[
B_+ (z) S_- (w) = : B_+ (z) S_- (w) : \frac{z - q w}{z - w} q^{-1}, \\
B_- (z) S_- (w) = : B_- (z) S_- (w) : \frac{z - q^2 w}{z - q^2 w} q^\gamma.
\]

Postulating

\[
S_- (w) B_+ (z) = : B_+ (z) S_- (w) : \frac{q^w - z}{w - z} q^{-1}, \\
S_- (w) B_- (z) = : B_- (z) S_- (w) : \frac{q^2 w - z}{q^2 w - z} q^\gamma.
\]
we obtain

\[ S_-(z) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{a_n}{n} z^n \right\} \exp \left\{ \sum_{n=1}^{\infty} (1 + q^n) \frac{a_n}{n} q^{-n+1} w^n \right\} e^{-Q q z^2} \tag{66} \]

Summarizing the commutation relations, we have

\[
[B_{+}, S_{+}(w)] = : B_{+}(z) S_{+}(w) : (t - 1) \delta(t w / z), \tag{67}
\]

\[
[B_{-}, S_{+}(w)] = : B_{-}(z) S_{+}(w) : (t^{-1} - 1) \delta(q^a w / z), \tag{68}
\]

\[
[B_{+}, S_{-}(w)] = : B_{+}(z) S_{-}(w) : (q^{-1} - 1) \delta(w / z), \tag{69}
\]

\[
[B_{-}, S_{-}(w)] = : B_{-}(z) S_{-}(w) : (q - 1) \delta(q^{a+1} w / z). \tag{70}
\]

These equations are equivalent to the following:

\[
[B_{+}, S_{+}(w)] = \exp \left\{ \sum_{n=1}^{\infty} \frac{(1 - t^n)(q^n + q^n a_n - n) w^n}{n} \right\} \exp \left\{ - \sum_{n=1}^{\infty} \frac{(1 - t^n)(t^{-n} + 1 + q^n t^{-n} + q^n) a_n}{n} w^n \right\} e^{\beta Q q^{a} w^{2a}(t - 1)t^n w^n}, \tag{71}
\]

\[
[B_{-}, S_{+}(w)] = \exp \left\{ \sum_{n=1}^{\infty} \frac{(1 - t^n)(t^{-n} + 1 + q^n t^{-n} - q^n) a_n}{n} w^n \right\} \exp \left\{ - \sum_{n=1}^{\infty} \frac{(1 - t^n)(q^n + q^n a_n - n) w^n}{n} \right\} e^{\beta Q q^{-a} w^{2a}(t^{-1} - 1)p^{-n} w^n}, \tag{72}
\]

\[
[B_{+}, S_{-}(w)] = \exp \left\{ - \sum_{n=1}^{\infty} \frac{t^{-n} + q^n a_n - n}{1 + q^n} w^n \right\} \exp \left\{ \sum_{n=1}^{\infty} \frac{(-1 + t^n + q^{-n} t^n + q^{(a-1)n} a_n}{1 + q^n} w^n \right\} e^{-Q q^{a} w^{-2a}(q^{-1} - 1) w^n}, \tag{73}
\]

\[
[B_{-}, S_{-}(w)] = \exp \left\{ - \sum_{n=1}^{\infty} \frac{q^n - q^{n-1} t^{-n} + 1 + q^n a_n - n}{1 + q^n} w^n \right\} \exp \left\{ \sum_{n=1}^{\infty} \frac{(q^{-n} + q^{(a-1)n} a_n}{1 + q^n} w^{-n} \right\} e^{-Q q^{-a} w^{-2a}(q - 1) q^{(a+1)n} w^n}. \tag{74}
\]

Let us examine in what conditions can we have \( (\ref{55}) , (\ref{56}) \). Studying the zero-mode factor in the equations \((\ref{71})\) and \((\ref{72})\), namely \(q^{a} w^{2a} \) and \(q^{-a} w^{2a} \), it is found that we have to set \( \chi = q \). Next, examining the oscillator factors, we have

\[
(1 - t^n)(q^n + q^n) = \frac{(1 - t^n)(t^{-n} + 1 + q^n t^{-n} - q^n)}{(1 + q^n)(1 - q^n)} q^n \tag{75}
\]

\[
(1 - t^n)(t^{-n} + 1 + q^n t^{-n} + q^n) = \frac{(1 - t^n)(q^n + q^n)}{(1 - q^n)} q^{-n} \tag{76}
\]

These equations can be solved uniquely and we obtain \( q^\tau = q t^{-1} \equiv p \). In the same way, we have \( \xi = t \) from the zero-mode factors in the equations \((\ref{73}) , (\ref{74})\) and condition from the oscillator factors gives us also \( q^\tau = p \). We have obtained

\[
q^{-a} = q^\tau = p, \quad q^\delta = q^\gamma = t^{-1}, \quad \chi = q, \ \xi = t. \tag{77}
\]
Finally if we set $\eta = p^{-1}$ then we have the desired equations

\[
[B_{\pm,n} + p^{-1} B_{-n}, S_{\pm}(w)] = \frac{d_q}{d_q w} O_{\pm}(w),
\]

(78)

\[
[B_{\pm,n} + p^{-1} B_{-n}, S_{-}(w)] = \frac{d_t}{d_t w} O_{-}(w),
\]

(79)

where

\[
O_{\pm}(w) = -(1 - q)(1 - t^{-1})(p^{-1} w)^{n+1} \exp \left\{ \sum_{n=1}^{\infty} \frac{1 + t^n}{1 + p^n} \frac{1 - t^n}{1 - q^n} \frac{a_n}{n} w^n t^{-n} \right\}
\]

\[
\times \exp \left\{ -\sum_{n=1}^{\infty} \frac{1 + t^n}{1 - q^n} \frac{1 - t^n}{1 + p^n} \frac{a_n}{n} w^n p^n \right\} e^{\beta Q} w^{2\beta a_0 t^{-a_0}},
\]

(80)

\[
O_{-}(w) = -(1 - q^{-1})(1 - t) w^{n+1} \exp \left\{ -\sum_{n=1}^{\infty} \frac{1 + q^n}{1 + p^n} \frac{a_n}{n} w^n t^{-n} \right\}
\]

\[
\times \exp \left\{ \sum_{n=1}^{\infty} \frac{1 + q^n}{1 + p^n} \frac{a_n}{n} w^n p^n \right\} e^{-Q} w^{-2a_0 t^a_0}.
\]

(81)

\(\) \(\) \(\)

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