Breather and interacting soliton and periodic waves for modified KdV equation

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We present the discovery of a class of exact spatially localized as well as periodic wave solutions within the framework of the modified Korteweg-de Vries equation. This class comprises breather and interacting soliton solutions as well as interacting periodic wave solutions. The functional forms of these solutions include a joint parameter which can take both positive and negative values of unity. It is found that the existence of those closed form solutions depend strongly on whether the cubic nonlinearity parameter should be considered positive or negative. The derived wave structures show interesting properties that may find practical applications.

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The modified Korteweg-de Vries (mKdV) equation is an important model which applies to the description of wave dynamics in different physical systems, such as soliton propagation in lattices \cite{1}, meandering ocean currents \cite{2}, the dynamics of traffic flow \cite{3,4}, nonlinear Alfvén waves propagating in plasma \cite{5}, and ion acoustic soliton experiments in plasmas \cite{6}. This model is also relevant for nonlinear waves in distributed Schottky barrier diode transmission lines \cite{7} and internal waves in stratified fluids \cite{8}. Recent results have also demonstrated that the propagation of optical pulses consisting of a few cycles in Kerr-type media can be described beyond the slowly varying envelope approximation by using the mKdV equation \cite{10,11,12}. Additionally, this equation has gained further importance recently, mainly because of its effectiveness in modeling supercontinuum generation in nonlinear optical fibers \cite{14}.

The mKdV equation has been successfully used to model the evolution of long waves in the critical case of vanishing quadratic nonlinearity \cite{13}. Due to the wide-ranging potential applications of this nonlinear wave evolution equation, various powerful methods have been employed to search for its explicit solutions. This because wave solutions are helpful for a better understanding of physical phenomena modeled by this equation such as the stability of nonlinear wave propagation. Having solutions in analytic form is relevant not only for determining certain important physical quantities and serving as diagnostics for simulations but also even for comparing experimental results with theory. Particularly, Wadati derived the exact N-soliton solutions of the mKdV equation by using the inverse-scattering transform scheme \cite{10}. In addition, Kevrekidis et al. \cite{17} obtained some classes of periodic solutions of this model. Another class of nonlinear wave solutions that conserve their energy during evolution -breathers (oscillatory wave packets)- has been also found for this model (see, e.g., \cite{18,19}). The focusing and defocusing mKdV equations with nonzero boundary conditions are studied for inverse scattering transforms with matrix Riemann-Hilbert problem in Ref. \cite{21}.

In this Letter, we predict the existence of three types of nonlinear wave structures through discovery of physically important exact solutions of the mKdV model that includes the cubic nonlinearity term with either positive or negative sign. Significant classes of breather and interacting soliton pairs and periodic waves are presented for the first time. Remarkably, the breather and interacting periodic waves formation are observed in the case of a negative coefficient of the cubic nonlinear term while the pair of interacting soliton solution is formed when this coefficient is positive.

We start by considering the mKdV equation in standard dimensionless form as

$$u_t + u_{xxx} + 6\mu u^2u_x = 0, \quad (1)$$

where $u(x,t)$ is the real function. The parameter $\mu = \pm 1$ denotes the type of nonlinearity, i.e., $+1$ for focusing type of nonlinearity and $-1$ for defocusing nonlinearity. The mKdV is a fully integrable equation which means that it has an infinite number of conserved invariants \cite{22}.

Two soliton branches exist for the mKdV equation (1) in the case of a positive coefficient of the cubic nonlinear term (i.e., $\mu = 1$), they are defined as

$$u(x, t) = a + \frac{b^2}{\Lambda \cosh(b(x - x_0 - vt) + \psi_0) + 2a}, \quad (2)$$

where $\Lambda = \pm \sqrt{4a^2 + b^2}$, $v = 6a^2 + b^2$, and $a, b, x_0$ are the arbitrary constants. In the limit when $a \to 0$ the solution in Eq. (2) tends to the familiar sech-shaped soliton family which is known to be stable with respect to small
perturbations (see, e.g., [23]). In the case with \( \mu = 1 \) also exists the algebraic solitary wave solution as

\[
\text{u}(x, t) = p - \frac{p}{p^2(x - x_0 - vt)^2 + 1/4},
\]

(3)

where \( v = 6p^2 \), and \( p, x_0 \) are the arbitrary constants. We note that the N-soliton solution of (1) can be obtained using the Darboux transform:

\[
\text{u}(x, t) = -i \frac{\partial}{\partial x} \ln \frac{W(\Psi_{1x}, \Psi_{2x}, ..., \Psi_{Nx})}{W(\Psi_1, \Psi_2, ..., \Psi_N)},
\]

(4)

where \( W \) is the Wronskian for \( N \) eigenfunctions \( \Psi_j \) in the denominator and for their spatial derivatives \( \Psi_{jx} \) in the numerator. This formal expression has inner symmetries and the properties of determinants which can be used for construction corresponding solutions [24].

The breather’s expression for \( \mu = 1 \) was obtained from inverse scattering transform by Pelinovsky and Grimshaw in [15] and also in Ref. [25]. Note that the breather has much more complicated dynamics than soliton. Recently Slunyaev and Pelinovsky [26] also presented an explicit breather solution for the mKdV equation (1) with \( \mu = 1 \) as

\[
\text{u}(x, t) = 2pq \left( \frac{p \sinh(\theta) \sin(\phi) - q \cosh(\theta) \cos(\phi)}{p^2 \sin^2(\phi) + sq^2 \cosh^2(\theta)} \right),
\]

(5)

when \( s = -1 \). Here \( \theta \) and \( \phi \) which control the wave envelope and the inner wave respectively, are given by

\[
\theta = p(x - x_0) + p(3q^2 - p^2)t + \theta_0,
\]

(6)

\[
\phi = q(x - x_0) + q(q^2 - 3p^2)t + \phi_0,
\]

(7)

where \( p, q \) and \( \theta_0, \phi_0, x_0 \) are the arbitrary real parameters. We have remarked that this breather solution still also existing for the case when \( s = +1 \) with the same relations of \( \theta \) and \( \phi \) as those given in Eqs. (6) and (7). Thus the mKdV equation (1) with the focusing type of nonlinearity possesses two exact breather solutions (5) corresponding to the values \( s = \pm 1 \).

FIG. 1: (a) Evolution of the interacting soliton solution of mKdV equation defined by Eq. (8) (\( \mu = 1 \)) for the values \( p = 1, q = 0.2 \) and \( s = 1 \). (b) The corresponding contour plot of the interacting soliton solution in (a).

In what follows, we report what is to our knowledge the analytical demonstration of a new class of breather, interacting periodic and interacting soliton solutions for the mKdV equation with either focusing or defocusing types of nonlinearity. More precisely, using the inverse scattering method (the details can be published somewhere) we have found three types of exact solutions for the modified KdV equation (1) with \( \mu = \pm 1 \). These solutions are given in Eq. (8) (for \( \mu = 1 \)), and Eqs. (13) and (18) (for \( \mu = -1 \)). A first class of localized waves in the form of a pair interacting solitons is obtained for Eq. (11) in the case of focusing cubic nonlinearity (i.e., \( \mu = 1 \)) as

\[
\text{u}(x, t) = 2pq \left( \frac{p \sinh(\theta) \sinh(\phi) - q \cosh(\theta) \cosh(\phi)}{p^2 \sinh^2(\phi) + sq^2 \cosh^2(\theta)} \right),
\]

(8)
where \( s = \pm 1 \) and \( \theta, \phi \) are given by

\[
\theta = p(x - x_0) - p(3q^2 + p^2)t + \theta_0, \\
\phi = q(x - x_0) - q(3p^2 + q^2)t + \phi_0.
\]

(9) \hspace{1cm} (10)

Here and below \( p, q \) and \( \theta_0, \phi_0, x_0 \) are the arbitrary real parameters. The wave solution (8) represents a pair of interacting bipolar solitons whose dynamic features are delineated in Fig. 1 (with \( \mu = 1 \)) for the case \( p = 1, q = 0.2 \) and \( s = 1 \). Here we choose the initial soliton position \( x_0 \) and phases \( \theta_0 \) and \( \phi_0 \) equal to zeros. Figure 1 shows the overtaking interaction between two solitons of opposite polarity for \( u(x, t) \) given in Eq. (8), from which we can see that the solitonic amplitudes and shapes have not changed after the interaction. Compared with the well-known sech-shaped solitary wave which is a single soliton structure, this novel mKdV soliton solution takes the form of a pair of interacting solitons, which can interact purely elastically and behave like independent solitons far from the meeting point. We notice that the wave solution (8) with \( s = -1 \) also takes the shape of a pair of interacting solitons with opposite polarity.

In the case \( q = -p \) and \( s = 1 \) the solution given in Eq. (8) has the form of modified soliton solution traveling with dimensionless velocity \( v \). This modified soliton solution is given by

\[
u(x, t) = \frac{4q \cosh(\theta_0 + \phi_0)}{\cosh(2q\xi + 2\phi_0) + \cosh(2q\xi - 2\theta_0)},
\]

where \( \xi = (x - x_0) - vt \) and \( v = 4q^2 \). In the particular case \( \theta_0 + \phi_0 = 0 \) the solution (11) reduces to soliton solution in (2) with \( a = 0, b = 2q \) and \( \psi_0 = 2\phi_0 \). In the case \( q = p \) and \( s = 1 \) the solution given in Eq. (8) also has the form of modified soliton solution traveling with dimensionless velocity \( v \). This modified soliton solution is given by

\[
u(x, t) = \frac{-4q \cosh(\theta_0 - \phi_0)}{\cosh(2q\xi + 2\phi_0) + \cosh(2q\xi - 2\theta_0)},
\]

where \( \xi = (x - x_0) - vt \) and \( v = 4q^2 \). In the particular case \( \theta_0 - \phi_0 = 0 \) the solution (12) reduces to soliton solution in (2) with \( a = 0, b = 2q \) and \( \psi_0 = 2\phi_0 \).

We have obtained another exact solution for the mKdV equation (1) with \( \mu = -1 \) in the form of an interacting periodic wave solution as

\[
u(x, t) = 2pq \left( \frac{p \sin(\theta) \sin(\phi) + q \cos(\theta) \cos(\phi)}{p^2 \sin^2(\phi) - sq^2 \cos^2(\theta)} \right),
\]

where \( s = \pm 1 \) and \( \theta, \phi \) are

\[
\theta = p(x - x_0) + p(3q^2 + p^2)t + \theta_0, \\
\phi = q(x - x_0) - q(3p^2 + q^2)t + \phi_0.
\]

(13) \hspace{1cm} (14)
\[ \phi = q(x - x_0) + q(3p^2 + q^2)t + \phi_0. \] (15)

In the case \( q = -p \) and \( s = -1 \) the solution given by Eq. (13) takes the shape of periodic traveling wave,

\[ u(x, t) = \frac{2p \cos(\theta_0 + \phi_0)}{\sin^2(p \xi - \phi_0) + \cos^2(p \xi + \theta_0)}, \] (16)

where \( \xi = (x - x_0) - vt \) and \( v = -4p^2 \). The denominator in (16) is zero only in the case when \( \theta_0 + \phi_0 = \pi/2 + \pi n \) with \( n = 0, \pm 1, \pm 2, \ldots \). If this denominator is zero the numerator in (16) is not singular when \( \theta_0 + \phi_0 = \pi/2 + \pi n \) with \( n = 0, \pm 1, \pm 2, \ldots \). In the particular case \( \theta_0 + \phi_0 = 0 \) this periodic wave solution reduces to constant solution \( u(x, t) = 2p \).

In the case \( q = p \) and \( s = -1 \) the solution given by (13) also takes the shape of periodic traveling wave,

\[ u(x, t) = \frac{2p \cos(\theta_0 - \phi_0)}{\sin^2(p \xi + \phi_0) + \cos^2(p \xi + \theta_0)}, \] (17)

where \( \xi = (x - x_0) - vt \) and \( v = -4p^2 \). The denominator in (17) is zero only in the case when \( \theta_0 - \phi_0 = \pi/2 + \pi n \) with \( n = 0, \pm 1, \pm 2, \ldots \). If this denominator is zero the numerator in (17) is not singular when \( \theta_0 - \phi_0 = \pi/2 + \pi n \) with \( n = 0, \pm 1, \pm 2, \ldots \). In the particular case \( \theta_0 - \phi_0 = 0 \) this periodic wave solution reduces to constant solution \( u(x, t) = 2p \).

![Figure 3](image-url)  
**FIG. 3:** Breather solution of mKdV equation defined by Eq. (18) (\( \mu = -1 \)) for the values \( p = 1 \), \( q = 0.98 \) and \( s = -1 \). (with \( \mu = -1 \)) for the case \( p = 0.5 \), \( \phi_0 = 0 \) and \( \theta_0 = \pi/4 \). Here the initial parameter \( x_0 \) is chosen equal to zeros. We observe that the nonlinear waveform (16) presents an oscillating behaviour during the process of wave evolution.

We have also found that the mKdV equation (1) with the defocusing type of nonlinearity (i.e., \( \mu = -1 \)) has an exact breather solution of the form,

\[ u(x, t) = 2pq \left( \frac{p \sin(\theta) \sinh(\phi) + q \cos(\theta) \cosh(\phi)}{p^2 \sinh^2(\phi) - sq^2 \cos^2(\theta)} \right), \] (18)

where \( s = \pm 1 \) and \( \theta, \phi \) are

\[ \theta = p(x - x_0) + p(p^2 - 3q^2)t + \theta_0, \] (19)
\[ \phi = q(x - x_0) + q(3p^2 - q^2)t + \phi_0. \] (20)

To our knowledge, exact solutions (8), (13) and (18) to the mKdV equation with the focusing and defocusing type of nonlinearity are firstly reported in this paper. From these important results, one may conclude that a physical system described by the mKdV equation could allow a breather evolution in either the focusing or the defocusing nonlinearity. A typical example of the singular breather profile for solution in Eq. (18) is shown Fig. 3 (with \( \mu = -1 \)) for the values \( p = 1, q = 0.98 \) and \( s = -1 \). For this case, the initial parameters \( x_0, \theta_0 \) and \( \phi_0 \) are chosen equal to zeros. It can be seen that the breather structure has a nontrivial periodical behaviour while evolving in \((x, t)\) plane. We emphasis that this breather solution has periodic singularities in \((x, t)\) plane for condition \( \phi(x, t) = 0 \). Hence this singular solution has not a direct application for physical problems. However, such solutions are important for the general theory of nonlinear systems [24].

In conclusion, we have presented several types of spatially localized and periodic wave solutions for the mKdV equation describing propagation of nonlinear waves in many physics areas when there is polarity symmetry. A novel class of exact soliton solutions given by Eq. (8) is firstly obtained for the model. Thise solutions describe the propagation of a pair interacting solitons with opposite polarity, which interact locally and behave like independent solitons far from the meeting point. We have also found a number of exact interacting periodic wave solutions for the mKdV equation. The results showed that the formation of those closed form solutions is determined by the sign of cubic nonlinearity parameter solely. Undoubtedly, these new solutions will be useful for recognizing physical phenomena and dynamical processes in various physical systems where the mKdV equation can provide a realistically accurate description of the waves.

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