An Optimal Multi-Unit Combinatorial Procurement Auction with Single Minded Bidders

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Abstract

The current art in optimal combinatorial auctions is limited to handling the case of single units of multiple items, with each bidder bidding on exactly one bundle (single minded bidders). This paper extends the current art by proposing an optimal auction for procuring multiple units of multiple items when the bidders are single minded. The auction minimizes the cost of procurement while satisfying Bayesian incentive compatibility and interim individual rationality. Under appropriate regularity conditions, this optimal auction also satisfies dominant strategy incentive compatibility. When bidders submit X-OR bids on two disjoint bundles, we show how the current state of the art can be used for developing optimal auctions.

Keywords: Bayesian Incentive Compatibility (BIC), Dominant Strategy Incentive Compatibility (DSIC), Individual Rationality (IR), Multi-Unit Combinatorial Procurement Auction, Optimal Mechanism

1 Introduction

1.1 Motivation and Background

Auction based mechanisms are extremely relevant in modern day electronic procurement systems \cite{2,15} since they enable a promising way of automating negotiations with suppliers and achieving the ideal goals of procurement efficiency and cost minimization. In many cases it may be beneficial to allow the suppliers to bid on combinations of items rather than on single items. Such auctions are called combinatorial auctions. Simply defined, a combinatorial auction is a mechanism where bidders can submit bids on combinations of items. The winner determination problem is to select a winning set of bids such that each item to be bought is included in at least one of the selected bids, and the total cost of procurement is minimized. In this paper, our interest is in multi-unit combinatorial procurement auctions, where a buyer is interested in procuring multiple units of multiple items.

In mechanism design literature, an optimal auction refers to an auction which optimizes a performance metric (for example maximizes revenue to a seller or minimizes cost to a buyer) subject to two critical game theoretic properties: (1) incentive compatibility and (2) individual rationality. Incentive compatibility comes in two forms: dominant strategy incentive compatibility (DSIC) and Bayesian incentive compatibility (BIC). DSIC property is a property that guarantees that reporting true valuations (or costs as the case may be) is a best response for each bidder, irrespective of the valuations (or costs) reported by the other bidders. BIC is a much weaker property which ensures that truth revelation is a best response for each bidder whenever the other bidders are also truthful. Individual rationality (IR) is a property which assures non-negative utility to each participant in the mechanism thus ensuring their voluntary participation. The IR property may be (1) ex-ante IR (if the bidders decide on participation even before knowing their exact types (valuations or costs)
or (2) interim IR (if the bidders decide on participation just after observing their types), or ex-post IR (if the bidders can withdraw even after the game is over). For more details on these concepts, the reader is referred to [5, 6, 12, 14].

1.2 Contributions and Outline

In his seminal work, Myerson [14] characterized an optimal auction for selling a single unit of a single item. Extending his work has been attempted by several researchers and there have been some generalizations of his work for multi-unit single item auctions [11, 9, 7]. Armstrong [1] characterized an optimal auction for two objects where type sets are binary. Malakhov and Vohra [11] studied an optimal auction for a single item multi-unit procurement auctions using a network interpretation. An implicit assumption in the above papers is that the sellers have limited capacity for the item. They also assume that the valuation sets are discrete. Kumar and Iyengar [9] and Gautam, Hemachandra, Narahari, Prakash [7] have proposed an optimal auction for multi-unit, single item procurement.

Recently, Ledyard [10] has looked at single unit combinatorial auctions in the presence of single minded bidders. A single minded bidder is one who only bids on a particular subset of the items. Ledyard’s auction, however, does not take into account multiple units of multiple items and this motivates our current work which extends Ledyard’s auction to the case of procuring multiple units of multiple items. The following are our specific contributions.

1. We characterize Bayesian incentive compatibility and interim individual rationality for procuring multiple units of multiple items when the bidders are single minded, by deriving a necessary and sufficient condition.

2. We design an optimal auction that minimizes the cost of procurement while satisfying Bayesian incentive compatibility and interim individual rationality.

3. We show under appropriate regularity conditions that the proposed optimal auction also satisfies dominant strategy incentive compatibility.

Some of the results presented here appeared in our paper [8].

The rest of the paper is organized as follows. First, we will explain our model in Section 2 and describe the notation that we use. We also outline certain essential technical details of optimal auctions from the literature. In Section 3, we present the three contributions listed above. Section 4 concludes the paper.

2 The Model

We consider a scenario in which there is a buyer and multiple sellers. The buyer is interested in procuring a set of distinct objects, $I$. She is interested in procuring multiple units of each object. She specifies her demand for each object. The sellers are single minded. That is each seller is interested in selling a specific bundle of the objects. We illustrate through an example below.

**Example 2.1.** Consider a buyer interested in buying 100 units of $A$, 150 units of $B$, and 200 units of $C$. Assume that there are three sellers. Seller 1 might be interested in providing 70 units of bundle $\{A, B\}$, that is, 70 units of $A$ and 70 units of $B$ as a bundle. Because he is single minded, he does not bid for any other bundles. We also assume that he would supply equal numbers of $A$ and $B$. Similarly, seller 2 may provide a bid for 100 units of the bundle $\{B, C\}$. The bid from seller 3 may be 125 units of the bundle $\{A, C\}$.

The sellers are capacitated i.e. there is a maximum quantity of the bundle of interest they could supply. The bid therefore specifies a unit cost of the bundle and the maximum quantity that can be supplied. After receiving these bids, the buyer will determine the allocation and payment as per auction rules.

We summarize below important assumptions in the model.

- The sellers are single minded.
- The sellers can collectively fulfill the demands specified by the buyer.
Table 1: Notation

| Symbol | Description |
|--------|-------------|
| I      | Set of items the buyer is interested in buying, \{1, 2, \ldots, m\} |
| D_j    | Demand for item j, j = \ldots, m |
| N      | Set of sellers, \{1, 2, \ldots, n\} |
| c_i    | True cost of production of one unit of bundle of interest to the seller i, \(c_i \in [c_i, \bar{c}_i]\) |
| q_i    | True capacity for bundle which seller i can supply, \(q_i \in [q_i, \bar{q}_i]\) |
| \hat{c}_i | Reported cost by the seller i |
| \hat{q}_i | Reported capacity by the seller i |
| \theta_i | True type i.e. cost and capacity of the seller i, \(\theta_i = (c_i, q_i)\) |
| b_i    | Bid of the seller i, \(b_i = (\hat{c}_i, \hat{q}_i)\) |
| b     | Bid vector, \((b_1, b_2, \ldots, b_n)\) |
| b_{-i} | Bid vector without the seller i, i.e. \((b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)\) |
| t_i(b) | Payment to the seller i when submitted bid vector is b |
| T_i(b_i) | Expected payment to the seller i when he submits bid \(b_i\). Expectation is taken over all possible values of \(b_{-i}\) |
| x_i = x_i(b) | Quantity of the bundle to be procured from the seller i when the bid vector is b |
| X_i(b_i) | Expected quantity of the bundle to be procured from the seller i when he submits bid \(b_i\). Expectation is taken over all possible values of \(b_{-i}\) |
| \(f_i(c_i, q_i)\) | Joint probability density function of \((c_i, q_i)\) |
| \(F_i(c_i, q_i)\) | Cumulative distribution function of \(f_i(c_i, q_i)\) |
| \(f_i(c_i | q_i)\) | Conditional probability density function of production cost when it is given that the capacity of the seller i is \(q_i\) |
| \(F_i(c_i | q_i)\) | Cumulative distribution function of \(f_i(c_i | q_i)\) |
| \(H_i(c_i, q_i)\) | Virtual cost function for seller i, \(H_i(c_i, q_i) = c_i + \frac{F_i(c_i | q_i)}{f_i(c_i | q_i)}\) |
| \(\rho_i(b_i)\) | Expected offered surplus to seller i, when his bid is \(b_i\) |
| \(u_i(b, \theta_i)\) | Utility to seller i, when bid vector is \(b\) and his type is \(\theta_i\) |
| \(U_i(b_i, \theta_i)\) | Expected utility to the seller i, when he submits bid \(b_i\) and his type is \(\theta_i\). Expectation is taken over all possible values of \(b_{-i}\) |

- The sellers are capacitated i.e. they can not supply beyond the capacity specified in the bids.
- The seller will never inflate his capacity, as it can be detected. If he fails to supply the quantity exceeding his capacity, he incurs a penalty which is deterrent on inflating his capacity. This is an important assumption.
- Whenever a buyer buys anything from the seller, she will procure the same number of units of each of the items from the seller’s bundle of interest.
- All the participants are rational and intelligent.

Table 1 shows the notation that will be used in the rest of the paper.

### 2.1 Some Preliminaries

The problem of designing an optimal mechanism was first studied by Myerson [14] and Riley and Samuelson [16]. Myerson’s work is more general and considers the setting of a seller trying to sell a single unit of a single object to one of several possible buyers. Note here that, unlike the rest of paper, the auctioneer is the seller
and his objective is to maximize the revenue. (In the rest of the paper, the auctioneer will be a buyer and her objective will be to minimize the cost of procurement.) So in this particular setting, as per notation defined in Table 1, \( m = 1, D_1 = 1 \). (So, \( q_i \) will be 1 for all the agents and no longer a private information). \( F_i, H_i \) defined in Table 1 will be function of single variable. The buyer’s private information will be the maximum cost he is willing to pay, which we will denote as \( \theta_i \). \( \theta_i \in \Theta_i = [\bar{\theta}_i, \bar{\theta}_i] \).

Myerson [14] characterizes all auction mechanisms that are Bayesian incentive compatible and interim individually rational in this setting. From this, he derives the allocation rule and the payment function for the optimal auction mechanism, using an interesting notion called the virtual cost function, defined as follows:

\[
H_i(\theta_i) = \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}
\]

He has shown that an optimal auction is one with allocation rule as:

\[
x_i(\theta) = \begin{cases} 
1 & \text{if } H_i(\theta_i) > \max \left\{ 0, \max_{j \neq i} H_j(\theta_j) \right\} \\
0 & \text{otherwise} \end{cases}
\]

\[
T_i(\theta_i) = E_i(\theta) - H_i(\theta_i) = U_i(\theta_i) - \theta_i x_i(\theta_i)
\]

\[
= \int_{\theta_i}^{\theta_i} X_i(s)ds - \theta_i X_i(\theta_i) \tag{2}
\]

One such payment rule is given by,

\[
t_i(\theta_i, \theta_{-i}) = \left( \int_{\theta_i}^{\theta_i} x_i(s, \theta_{-i})ds \right) - \left( \theta_i x_i(\theta) \right) \forall \theta
\]

Any auction for single unit of an single item which satisfies Equation (1) and Equation (2) is optimal i.e. maximizes seller’s revenue and is BIC and IIR.

**Regularity Assumption:** If \( H_i(\theta_i) \) is increasing with respect to \( \theta_i \), then we say, the virtual cost function is regular or regularity condition holds true. Under this assumption on e such optimal auction is,

1. Collect bids from the buyers
2. Sort them according to their virtual costs
3. If the highest virtual cost is positive, allocate the object to the corresponding bidder
4. The winner, say \( i \), will pay \( t_i(\theta_{-i}) = \inf \{ \theta_i | H_i(\theta_i) > 0 \text{ and } H_i(\theta_i) > H_j(\theta_j) \forall j \neq i \} \)

From the payment rule, it is a dominant strategy for each bidder to bid truthfully under the regularity assumption. When bidders are symmetric, i.e. \( F_i \) is same \( \forall i \), then the above optimal auction is Vickrey’s second price auction [17].

Myerson’s work can be easily extended to the case of multi-unit auctions with unit demand. But problems arise when the unit-demand assumption is relaxed. We move into a setting of multi-dimensional type information which makes truth elicitation non-trivial. Several attempts have addressed this problem, albeit under some restrictive assumptions [11, 9, 7]. It is assumed, for example, that even though the seller is selling multiple units (or even objects), the type information of the entities is still one dimensional [3, 4, 18].

Researchers have also worked on extending Myerson’s work for an optimal auction for multiple objects. The private information, in this setting may not be single dimensional. Armstrong [1] has solved this problem for two object case, when type sets are binary by enumerating all incentive compatibility conditions. Recently, Ledyard [10] has characterized an optimal multi-object single unit auction, when bidders are single minded.
### 3 Optimal Multi-Unit Combinatorial Procurement Auction

We will start this section with an example to illustrate that in a multi-unit, multi-item procurement auction, the suppliers may have an incentive to misreport their costs.

**Example 3.1.** Suppose the buyer has a requirement for 1000 units. Also, suppose that there are four suppliers with \((c_i, q_i)\) values of \(S_1 : (10, 500), S_2 : (8, 500), S_3 : (12, 800)\) and \(S_4 : (6, 500)\). Suppose the buyer conducts the classic \(k^{th}\) price auction, where the payment to a supplier is equal to the cost of the first losing supplier. In this case, the sellers will be able to do better by misreporting types. To see this, consider that all suppliers truthfully bid both the cost and the quantity bids. The allocation then would be \(S_1 : 0, S_2 : 500, S_3 : 0, S_4 : 500\) and this minimizes the total payment. Under this allocation the payment to \(S_4\) would be \(10 \times 500 = 5000\) currency units. However, if he bids his quantity to be 600 units at the cost of 10 per unit. Being his capacity 500, he would not be able to supply the remaining 1000 units. Suppose the buyer conducts the \(k^{th}\) price auction, where the payment to a supplier is equal to the cost of the first losing supplier. In this case, the sellers will be able to do better by misreporting types. To see this, consider that all suppliers truthfully bid both the cost and the quantity bids. The allocation then would be \(S_1 : 0, S_2 : 500, S_3 : 0, S_4 : 500\) and this minimizes the total payment. Under this allocation the payment to \(S_4\) would be \(10 \times 500 = 5000\) currency units.

The intuitive explanation for this could be that by under reporting their quantity are private information. The intuitive explanation for this could be that by under reporting their capacity values, the suppliers create an artificial scarcity of resources in the system. Such fictitious shortages force the buyer to pay overboard for use of the virtually limited resources.

We also make another observation here. Suppose, the seller \(4\) bids \((6,600)\). Then the buyer will order from him 600 units at the cost of 10 per unit. Being his capacity 500, he would not be able to supply the remaining 1000 units. If he bids \((6,1000)\), then he will be paid only 8 per unit and the buyer will be ordering him 1000 units. This clearly indicates our assumption that a seller will not inflate his capacity is quite natural.

We are interested in designing an optimal mechanism, for a buyer, that satisfies Bayesian incentive compatibility (BIC) and individual rationality (IR). BIC means that the best response of each seller is to bid truthfully with the above offered incentive, we now state and prove the following theorem.

**Theorem 3.1.** Any mechanism in the presence of single minded, capacitated sellers is BIC and IR iff

1. \(\rho_i(b_i) = \rho_i(\bar{c}_i, \bar{q}_i) + \int_{\bar{c}_i}^{c_i} X_i(t, \bar{q}_i)dt\)
2. \(\rho_i(b_i)\) non-negative, and non-decreasing in \(\bar{q}_i\) \(\forall \bar{c}_i \in [\bar{c}_i, \hat{c}_i]\)
3. The quantity which seller \(i\) is asked to supply, \(X_i(c_i, q_i)\) is non-increasing in \(c_i \forall q_i \in [\bar{q}_i, \hat{q}_i]\).

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\(\forall\) represents for all.
Proof. : A similar theorem is presented by Kumar and Iyengar [9] for the case of multi-unit single item procurement auctions. Using the notion of single minded bidder [10], we state and prove a result for a wider setting.

To prove the necessity part of the theorem, we first observe that,

\[ U_i(b_i, \theta_i) = U_i(\hat{c}_i, \hat{q}_i, c_i, q_i) = T_i(b_i) - c_i X_i(b_i) \]

and BIC \( \Rightarrow U_i(\hat{c}_i, \hat{q}_i, c_i, q_i) \leq U_i(c_i, q_i, c_i, q_i), \)

\( \forall (\hat{c}_i, \hat{q}_i) \) and \( (c_i, q_i) \in \Theta_i \)

In particular,

\[ U_i(\hat{c}_i, q_i, c_i, q_i) \leq U_i(c_i, q_i, c_i, q_i) \]

Without loss of generality, we assume \( \hat{c}_i > c_i \). Rearrangement of these terms yields,

\[ X_i(\hat{c}_i, q_i) - X_i(c_i, q_i) \leq -X_i(c_i, q_i) \]

Similarly using,

\[ X_i(c_i, q_i) \leq U_i(\hat{c}_i, q_i, c_i, q_i) - U_i(c_i, q_i, c_i, q_i) \]

\[ \frac{U_i(\hat{c}_i, q_i, c_i, q_i) - U_i(c_i, q_i, c_i, q_i)}{\hat{c}_i - c_i} \leq -X_i(c_i, q_i). \]

Taking limit \( \hat{c}_i \to c_i \), we get,

\[ \frac{\partial U_i(c_i, q_i, c_i, q_i)}{\partial c_i} = -X_i(c_i, q_i). \]

Equation (6) implies, \( X_i(c_i, q_i) \) is non-increasing in \( c_i \). This proves statement 3 of the theorem in the forward direction. When the seller bids truthfully, from Equation (5),

\[ \rho_i(c_i, q_i) = U_i(c_i, q_i, c_i, q_i). \]

For BIC, Equation (7) should be true. So,

\[ \rho_i(c_i, q_i) = \rho_i(c_i, q_i) + \int_{\hat{c}_i}^{c_i} X_i(t, q_i)dt \]

This proves claim 1 of the theorem. BIC also requires,

\[ q_i \in \arg \max_{\hat{q}_i \in [\hat{q}_i, q_i]} U_i(c_i, \hat{q}_i, c_i, q_i) \forall c_i \in [\hat{c}_i, c_i] \]

(Note that \( \hat{q}_i \in [\hat{q}_i, q_i] \) and not \( \in [\hat{q}_i, \hat{q}_i] \) as it is assumed that a bidder will not over report his capacity.) This implies, \( \forall c_i \rho_i(c_i, q_i) \) should be non-decreasing in \( q_i \). The IR conditions (Equations (4) and (8)) imply

\[ \rho_i(c_i, q_i) \geq 0. \]

This proves statement 2 of the theorem. Thus, these three conditions are necessary for BIC and IR properties.

We now prove that these are sufficient conditions for BIC and IR. Assume that all three conditions are true,

\[ \Rightarrow U_i(\theta_i, \theta_i) = \rho_i(c_i, q_i) \geq 0. \]
So the IR property is satisfied.

\[
U_i(b_i, \theta_i) = \rho_i(\hat{c}_i, \hat{q}_i) + \rho_i(\hat{c}_i, \hat{q}_i) + \int_{\hat{c}_i}^{c_i} X_i(t, \hat{q}_i) dt + (\hat{c}_i - c_i) X_i(c_i, \hat{q}_i)
\]

\[
= \rho_i(\hat{c}_i, \hat{q}_i) + \int_{\hat{c}_i}^{c_i} X_i(t, \hat{q}_i) dt
\]

\[
= \rho_i(\hat{c}_i, \hat{q}_i) + \int_{\hat{c}_i}^{c_i} X_i(t, \hat{q}_i) dt
\]

\[
\leq \rho_i(c_i, \hat{q}_i)
\]

This proves the sufficiency of the three conditions.

\[
\square
\]

### 3.2 Allocation and Payment Rules of Optimal Auction

The buyer’s problem is to solve,

\[
\min E_b \sum_{i=1}^n t_i(b) \quad \text{s.t.}
\]

1. \( t_i(b) = \rho_i(b) + \hat{c}_i x_i(b) \)

2. All three conditions in Theorem 3.1 hold true.

3. She procures at least \( D_j \) units of each item \( j \).

Expectation being a linear operator, the buyer’s problem is to minimize \( \sum_{i=1}^n E_b T_i(\hat{c}_i, \hat{q}_i) \). Condition 1 of the theorem has to hold true, which will imply the \( i \)th term in the summation is given by,

\[
\int_{\hat{c}_i}^{c_i} X_i(t, \hat{q}_i) dt f(c_i|q_i) f_i(q_i) dc_i dq_i
\]

However,

\[
\int_{\hat{c}_i}^{c_i} \left( \int_{\hat{c}_i}^{c_i} X_i(t, q_i) dt \right) f_i(c_i, q_i) dc_i = \int_{\hat{c}_i}^{c_i} X_i(c_i, q_i) f_i(c_i) dc_i
\]

Condition 2 of Theorem 3.1 requires \( \rho_i(\hat{c}_i, q_i) \geq 0 \) and the buyer wants to minimize the total payment to be made. So, she has to assign \( \rho_i(\hat{c}_i, q_i) = 0 \) \( \forall q_i, \forall i \). So her problem is to solve,

\[
\min \sum_{i=1}^n \int_{\hat{c}_i}^{c_i} \left( c_i + \frac{E(c_i|q_i)}{X_i(c_i, q_i)} \right) X_i(c_i, q_i) f_i(c_i, q_i) dc_i dq_i
\]

That is,
\[
\min \sum_{i=1}^{n} \int_{c_i}^{\bar{c}_i} \int_{q_i}^{\bar{q}_i} H_i(c_i, q_i) X_i(c_i, q_i) f_i(c_i, q_i) dc_i dq_i
\]

where, \( H_i(c_i, q_i) \) is the virtual cost function, defined in Table 1. Define,
\[
\bar{c} = (c_1, c_2, \ldots, c_n)
\]
\[
c = (c_1, c_2, \ldots, c_n)
\]
\[
\underline{c} = (\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_n).
\]

Similarly, define \( \bar{q}, q \) and \( q \). Let,
\[
dc = dc_1 dc_2 \ldots dc_n
\]
\[
dq = dq_1 dq_2 \ldots dq_n
\]
\[
f(c, q) = \prod_{i=1}^{n} f_i(c_i, q_i)
\]

Her problem now reduces to,
\[
\min \int_{c}^{\bar{c}} \int_{q}^{\bar{q}} \left( \sum_{i=1}^{n} H_i(c_i, q_i) x_i(c_i, q_i) \right) f(c, q) dc dq \quad \text{s.t.}
\]

1. \( \forall i, \quad X_i(c_i, q_i) \) is non-increasing in \( c_i, \forall q_i \).
2. The Buyer’s minimum requirement of each item is satisfied.

This is an optimal auction for the buyer in the presence of the single minded sellers.

In the next subsection, we will see an optimal auction under regularity conditions.

### 3.3 Optimal Auction under Regularity Assumption

First, we make the assumption that,
\[
H_i(c_i, q_i) = c_i + \frac{F_i(c_i | q_i)}{f_i(c_i | q_i)}
\]
is non-increasing in \( q_i \) and non-decreasing in \( c_i \). This regularity assumption is the same as regularity assumption made by Kumar and Iyengar [9]. With this assumption, the buyer’s optimal auction when bidder \( i \) submits bid as \((c_i, q_i)\) is,
\[
\min \sum_{i=1}^{n} x_i H_i(c_i, q_i) \quad \text{subject to}
\]

1. \( 0 \leq x_i \leq q_i \), where \( x_i \) denotes the quantity that seller \( i \) has to supply of bundle \( \tilde{x}_i \).
2. Buyer’s demands are satisfied.

The condition \( X_i(c_i, q_i) \) is non increasing in \( c_i, \forall q_i \) and \( \forall i \). After this problem has been solved, the buyer pays each seller \( i \) the amount
\[
l_i = c_i x_i^* + \int_{c_i}^{\bar{c}_i} x_i(t, q_i) dt
\]
where \( x_i^* \) is what agent \( i \) has to supply after solving the above problem.

We exemplify the optimal mechanism with one example.

**Example 3.2.** Suppose, the buyer is interested in buying 100 units of \( \{A, C, D\} \) and 250 units of \( \{B\} \). Seller 1 (S1) is interested in providing \( q_1 = 100 \) units of bundle \( \{A, B\} \), seller 2 (S2): \( q_2 = 100 \) units of \( \{B\} \), seller 3, (S3) \( q_3 = 150 \) units of \( \{B, C, D\} \) and seller 4 (S4) is interested in up to \( q_4 = 120 \) units of \( \{A, B, C, D\} \). The unit costs of the respective bundles are \( c_1 = 100, c_2 = 50, c_3 = 70 \) and \( c_4 = 110 \). Each seller will submit his bid as \((c_i, q_i)\). After receiving the bids, buyer will solve,
\[
\min x_1 H_1(100, 100) + x_2 H_2(50, 100)
\]
\[ +x_3H_3(70, 150) + x_4H_4(110, 120) \]

s.t.

\[
\begin{align*}
  x_i & \geq 0 \quad i = 1, 2, 3, 4. \\
  x_1 & \leq 100 \\
  x_2 & \leq 100 \\
  x_3 & \leq 150 \\
  x_4 & \leq 120 \\
  x_1 + x_2 & \geq 100 \\
  x_1 + x_2 + x_3 + x_4 & \geq 250 \\
  x_3 + x_4 & \geq 100 
\end{align*}
\]  

Equation (11) is required to be satisfied as at least 100 units of \( A \) has to be procured. Equation (12) is for procuring at least 250 units of \( B \), and Equation (13) is for procuring at least 100 units of \( C \) and \( D \). After solving this optimization problem, she will determine the payment according to Equation (10).

It can be seen that for the seller \( i \), the best response is to bid truthfully irrespective of whatever the others are bidding. Thus, this mechanism enjoys the stronger property, namely dominant strategy incentive compatibility. Note that this property is much stronger than BIC. The above property is a direct consequence of the result proved by Mookherjee, and Stefan [13]. They have given the monotonicity conditions for DSIC implementation of a BIC mechanism. Under these regularity assumptions, \( x_i \) satisfies these conditions. So we have a DSIC mechanism. In the next section we consider X-OR bidding with unit demand case.

4 An Optimal Auction when Bidders are XOR Minded

Consider the situation where a supplier can manufacture some of the items required by the buyer, say \( A, B, C, D \). However, with the machinery he has, at a time either he can manufacture \( A, D \) or \( B, C \) but not any other combination simultaneously. Thus he can either supply \( A, D \) as bundle or \( B, C \) as a bundle but not both. That is, he is interested in X-OR bidding.

**Definition 4.1 (XOR Minded Bidder).** We say a bidder is an XOR minded if he is interested in supplying either of two disjoint subsets of items auctioned for but not both.

To simplify the analysis, in this section, we restrict ourselves to the unit demand case. That is the buyer is interested in buying single unit of each of the items from \( I \). And hence there are no capacity constraints. We formally state assumptions.

- We assume that the bidders are XOR minded.
- For each bidder, his costs of the two bundles of his interest are independent.
- The two bundles for which each seller is going to submit an X-OR bid, are known.
- The sellers can collectively supply the items required by the buyer.
- The buyer and the sellers are strategic.
- Free disposal. That is, if the buyer procures more than one unit of an item, he can freely dispose it of.

With the above assumptions, we now discuss an extension of the current art of designing optimal auctions for combinatorial auctions in the presence of XOR minded bidders. Though we assume the bidders are XOR minded, the BIC characterization and the auction designed here work even though the bidders are either single minded or XOR minded.
4.1 Notation

As, $q_i = 1$ for each bidder, we drop capacity from the types and bids for all the agents. Each agent will be reporting the costs for each bundle of his interest, he will be bidding two real numbers. And we need to calculate virtual costs on both the bundles. Thus, we need appropriate modifications in some of the notation used in the paper. We summarize the new notation for this section in Table 2. Each agent is submitting tow different bids on two different bundles. We will use $j$ to refer to the bundle.

| $j$ | $j = 1 \text{ or } 2$. Bundle index. |
|-----|-------------------------------------|
| $B_{ij}$ | The $j^{th}$ bundle of items for which the agent $i$ is bidding. $j = 1, 2$ |
| $c_{ij}$ | True cost of production of $B_{ij}$ to the seller $i$. $c_{ij} \in [c_i, \bar{c}_i]$ |
| $\theta_i$ | True type i.e. costs for $i$, $\theta_i = (c_i, c_{ij})$ |
| $b_i$ | Bid of the seller $i$. $b_i = (\hat{c}_i, \hat{c}_{ij})$ |
| $x_{ij} = x_{ij}(b)$ | Indicator variable to indicate whether $B_{ij}$ is to be procured from the seller $i$ when the bid vector is $b$ |
| $X_{ij}(b_i)$ | Probability that $B_{ij}$ is procured from the seller $i$ when he submits bid $b_i$. Expectation is taken over all possible values of $b_i$ |
| $f_i(c_{ij})$ | Probability density function of ($c_{ij}$) |
| $F_i(c_{ij})$ | Cumulative distribution function of $c_{ij}$ |
| $H_i(c_{ij})$ | Virtual cost function for seller $i$, for bundle $B_{ij}$ |
| $H_i(c_{ij}) = c_{ij} + \frac{F_i(c_{ij})}{f_i(c_{ij})}$ |

4.2 Optimal Auctions When Bidders Are XOR Minded

First we characterize the BIC and IIR mechanisms for the settings under consideration in next subsection. We design an optimal auction in subsection 4.2.2.

4.2.1 BIC and IIR: Necessary and Sufficient Conditions

The utility for the agent $i$ is

$$U_i(b_i, \theta_i) = -c_i X_{i1} - c_{ij} X_{i2} + T_i(b_i, \theta_i)$$

Using similar arguments as in the proof of the Theorem 3.1 for any mechanism in the presence of XOR minded bidders, the necessary condition for BIC is,

$$\frac{\partial U(.)}{\partial c_{ij}} = X_{ij}(c_i, c_{ij})$$

and

$$\frac{\partial U(.)}{\partial c_{ij}} = X_{ij}(c_i, c_{ij})$$

and $X_{ij}(c_i, c_{ij})$ should be non-increasing in $c_{ij}, j = 1, 2$.

We make an assumption that,

$$\frac{\partial X_{i1}}{\partial c_{ij}} = \frac{\partial X_{i2}}{\partial c_{ij}}$$

In general, the above assumption is not necessary for the mechanism to be truthful. However, if we assume that Equation (15) is true, we can solve PDE (14) analytically. Now we can state the following theorem,

**Theorem 4.1.** With assumption (15), a necessary and sufficient condition for a mechanism to be BIC and IIR in the presence of XOR minded bidders is,
\[ T_i(\cdot) = c_{i1}X_{i1} + c_{i2}X_{i2} + \int_{(c_{i1},c_{i2})} U_i(\cdot) d\theta_i \]

1. \( T_i(\cdot) = c_{i1}X_{i1} + c_{i2}X_{i2} + \int_{(c_{i1},c_{i2})} U_i(\cdot) d\theta_i \)

2. \( U_i(\overline{c}_i,\overline{c}_i) \geq 0. \)

### 4.2.2 Optimal Auction with Regularity Assumption

Suppose we assume that, \( H_{ij} \) is non-decreasing in \( c_{ij} \) for each \( i,j \). This is the same regularity assumption as Myerson [14]. Now, following similar treatment for buyers problem as in Section 3.3 reduces the buyers problem to:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{2} x_{ij} H_{ij}(c_{ij}) \\
\text{subject to} \ \\
1. x_{ij} \in 0, 1, \text{ where } x_{ij} \text{ indicates whether supplier } i \text{ is supplying his } j^{th} \text{ bundle or no.} \ \\
2. x_{i1} + x_{i2} \leq 1. \text{ (XOR minded bidder).} \ \\
3. \text{All the items are procured.} \ (16)
\]

Now, we show that at optimal allocation, the assumption (15) holds true. For an agent \( i \), fix, \( \theta_i \) and consider the square of his types \([c_i, \overline{c}_i] \times [c_i, \overline{c}_i]\). When he bids, \( b_i = (\overline{c}_i, \overline{c}_i) \), he does not win any item. However, if he decreases his bid on \( c_{ij} \), he wins the bundle \( B_{ij} \) at some lower bid and at a lower bid for \( B_{ij} \), he continues to win. Also, he being XOR minded, he cannot win both the bundles. Thus, the type set’s square can be partitioned into three regions, \( R_1, R_2 \) and \( R_3 \) as shown in Figure 1. When his type is in region \( R_j \), he is asked to supply \( B_{ij} \), \( j = 1, 2 \) and when it is in \( R_3 \) he is not in the list of winning agents. Now, except on the boundary between \( R_1 \) and \( R_2 \), the assumption (15) holds true. Hence, though we are not using (15) as a necessary condition, it is getting satisfied in optimization problem (16). Thus OCAX is an optimal combinatorial auction for the buyer in the presence of XOR minded bidders.

Figure 1: X-OR Bidding
4.2.3 The Case when Regularity Assumption is not Satisfied

Though we do not solve the buyer’s problem of optimal mechanism design without the regularity assumption, we highlight some thoughts on this. If we can assume (15), then we can design an optimal auction very similar to the OCAS, in the presence of XOR minded bidders. The challenge is, we cannot use (15) as a necessary condition nor can we assume it. However, it may happen that in an optimal auction, the condition (15), will hold true. We are still working on this.

5 Conclusion

In this paper,

• we have stated and proved a necessary and sufficient condition for incentive compatible and individually rational multi-unit multi-item auctions in the presence of single minded, capacitated buyers.
• We have given a blueprint of an optimal mechanism, for a buyer seeking to procure multiple units of multiple items in the presence of single minded and capacitated sellers.
• We also have shown that the mechanism minimizes the cost subject to DSIC and IIR if the virtual cost functions satisfy the regularity assumptions.
• When bidders are XOR minded, under certain regularity conditions, we designed an optimal auction for the buyer which we call as OCAX.

There are many natural extensions to this work. First, we can study optimal auctions in which the sellers are willing to give volume discounts. We also plan to study a case where the sellers are interested in supplying multiple bundles.

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