A new fractional operator of variable order: application in the description of anomalous diffusion

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Abstract

In this paper, a new fractional operator of variable order with the use of the monotonic increasing function is proposed in sense of Caputo type. The properties in term of the Laplace and Fourier transforms are analyzed and the results for the anomalous diffusion equations of variable order are discussed. The new formulation is efficient in modeling a class of concentrations in the complex transport process.

Key words: fractional derivative of variable-order, Laplace transform, Fourier transform, anomalous diffusion.

1 Introduction

Fractional-order derivatives (FOD) of the Riemann-Liouville and Caputo types with respect to the power-law-function kernel \cite{1,2,3,4} are important for developing mathematical models in the areas of the control, nuclear physics, electrical circuits, signal processing, economy and biology. FOD were used to describe anomalous diffusion (AD) problems \cite{5,6,7}. For example, the Riemann-Liouville-type model of the AD was considered in \cite{8,9}, the Caputo-type model

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of the AD was reported in [10], and the AD equation with a generalized FOD of Riemann–Liouville type was proposed in [11].

POD using the function kernel of the exponential type were reported in [12,13,14,15]. The applications of the FOD to model the surface of shallow water (SSW) [16], heat transfer [17] and linear viscoelasticity [18] were also studied. Moreover, FOD involving the generalized Mittag-Leffler function were proposed in [19,20] and their application in chaos was presented in [21].

More recently, a new FOD of the differentiable function with respect to the monotonic increasing function was proposed in the sense of the Caputo type [22]. The FOD of the differentiable function involving the monotonic increasing function was discussed in the sense of the Riemann-Liouville type [22]. To our best knowledge, FOD of the differentiable function of variable order with respect to the monotonic increasing function (MIF) in the sense of the Caputo type has not been reported. Motivated by this idea [27,28], in this paper the new FOD is applied in the modeling of anomalous diffusion problems.

The paper is organized as follows. In Section 2, the concepts of FODs of Caputo and Riemann-Liouville types with the use of the MIF are presented. In Section 3, the results for the anomalous diffusion models (ADMs) are discussed. Finally, in Section 4 the conclusion are drawn.

2 A new FOD with respect to the MIF in the sense of the Caputo type

In this section, the concept of the FODs of the differentiable function of variable order in the sense of the Caputo type is presented. We start by recalling the FODs of the differentiable function within the MIF in the sense of the Caputo and Riemann-Liouville types.

2.1 The Almeida FOD of constant order with respect to the MIF

Suppose that $T$ is the interval $\infty \leq a < b \leq +\infty$, $\varphi \in C^1 (T)$ with $\varphi^{(1)} (t) \neq 0$ and $\psi \in C^1 (T)$ for $\forall t \in T$.

The left and right Almeida FOD of the function $\psi (t)$ of order $\xi \ (0 < \xi < 1)$ in the sense of the Caputo type are given by (see [22]):

$$C D_a^{(\xi,\varphi)} \psi (t) = \frac{1}{\Gamma (1-\xi)} \int_a^t \frac{\psi^{(1)} (\tau)}{(\varphi (t) - \varphi (\tau))^{\xi}} d\tau,$$  (1)
\[ C D_{a_-}^{(\xi, \varphi)} \psi (t) = \frac{-1}{\Gamma (1 - \xi)} \int_{a^-}^t \frac{\psi^{(1)}_\varphi (\tau)}{(\varphi (\tau) - \varphi (t))^{1 + \xi}} d\tau, \] (2)

respectively, where \( \varphi (t) \) is the MIF and \( \Gamma (\cdot) \) is the Gamma function (GF).

The Mattag-Leffler function is defined as [2,4,28]:

\[ E_{\xi} (\eta^\xi) = \sum_{i=0}^{\infty} \frac{\eta^i \xi}{\Gamma (1 + i \xi)} . \] (3)

The properties of the Almeida FOD [22] are as listed in Table 1.

| Functions \((\varphi(t) - \varphi(a^+))^{\nu}\) | Almeida FOD \((\varphi(t) - \varphi(a^+))^{\nu-\xi}\) |
| \frac{(\varphi(t) - \varphi(a^+))^{\nu}}{\Gamma(1 + \nu)} | \frac{(\varphi(t) - \varphi(a^+))^{\nu-\xi}}{\Gamma(1 + \nu - \xi)} |
| \frac{(\varphi(a^-) - \varphi(t))^{\nu}}{\Gamma(1 + \nu)} | \frac{(\varphi(a^-) - \varphi(t))^{\nu-\xi}}{\Gamma(1 + \nu - \xi)} |
| \frac{E_{\xi} \left( (\varphi (t) - \varphi (a^+))^{\xi} \right)}{E_{\xi} \left( (\varphi (t) - \varphi (a^-))^{\xi} \right)} | \frac{E_{\xi} \left( (\varphi (t) - \varphi (a^+))^{\xi} \right)}{E_{\xi} \left( (\varphi (a^-) - \varphi (t))^{\xi} \right)} |

The left and right FOD of the function \( \psi (t) \) of order \( \xi (0 < \xi < 1) \) within the MIF in the sense of the Riemann-Liouville type are given by [22]:

\[ RL D_{a_+}^{(\xi, \varphi)} \psi (t) = \frac{1}{\Gamma (1 - \xi)} \int_{a^+}^t \frac{\varphi^{(1)}_\varphi (\tau)}{(\varphi (\tau) - \varphi (t))^{1 + \xi}} d\tau, \] (4)

\[ RL D_{a_-}^{(\xi, \varphi)} \psi (t) = \frac{-1}{\Gamma (1 - \xi)} \int_{a^-}^t \frac{\varphi^{(1)}_\varphi (\tau)}{(\varphi (\tau) - \varphi (t))^{1 + \xi}} d\tau, \] (5)

respectively.

The relationship between Eqs. (1) and (4) and further details about Eqs. (1) and (2) can be seen in [2,22,23,24].

2.2 The new FOD of variable order with respect to the MIF

Suppose that \( T \) is the interval \( \infty \leq a < b \leq +\infty, \varphi \in C^1 (T) \) with \( \varphi^{(1)} (t) \neq 0 \), \( \psi \in C^1 (T) \) and \( 0 < \xi (t) < 1 \) for \( \forall t \in T \).
The left and right ϕ-FOD of the function ψ (t) of order ξ (t) are defined by:

\[
C_D^{(\xi(t), \phi)}_a \psi (t) = \frac{1}{\Gamma (1 - \xi (t))} \int_{a}^{t} \frac{\psi_{\phi}' (\tau)}{(\phi (t) - \phi (\tau))^{\xi(t)} d\tau,}
\]

(6)

\[
C_D^{(\xi(t), \phi)}_a \psi (t) = \frac{-1}{\Gamma (1 - \xi (t))} \int_{a}^{t} \frac{\psi_{\phi}' (\tau)}{(\phi (\tau) - \phi (t))^{\xi(t)} d\tau,}
\]

(7)

respectively, where \( \phi (t) \) is the MIF.

In a similar manner, the properties of the new ϕ-FOD are shown in Table 2.

Table 2

| Functions | new ϕ-FOD |
|-----------|-----------|
| \( (\phi(t) - \phi(a^+))^{\xi(t)} \) | \( (\phi(t) - \phi(a^+))^{\nu-\xi(t)} \) |
| \( (\phi(a^-) - \phi(t))^{\xi(t)} \) | \( (\phi(a^-) - \phi(t))^{\nu-\xi(t)} \) |
| \( E_{\xi} \left( (\phi (t) - \phi (a^+))^{\xi} \right) \) | \( E_{\xi} \left( (\phi (t) - \phi (a^+))^{\xi} \right) \) |
| \( E_{\xi} \left( (\phi (a^-) - \phi (t))^{\xi} \right) \) | \( E_{\xi} \left( (\phi (a^-) - \phi (t))^{\xi} \right) \) |

Let \( 0 < \xi (\sigma, t) < 1 \). The left and right ϕ-FOD of the function \( \psi (\sigma, t) \) of two-variable order \( \xi (\sigma, t) \) are defined by:

\[
C_D^{(\xi(\sigma,t), \phi)}_a \psi (\sigma, t) = \frac{1}{\Gamma (1 - \xi (\sigma, t))} \int_{a}^{t} \frac{\psi_{\phi}' (\sigma, \tau)}{(\phi (\sigma, t) - \phi (\tau))^{\xi(\sigma,t)} d\tau,}
\]

(8)

\[
C_D^{(\xi(\sigma,t), \phi)}_a \psi (\sigma, t) = \frac{-1}{\Gamma (1 - \xi (\sigma, t))} \int_{a}^{t} \frac{\psi_{\phi}' (\sigma, \tau)}{(\phi (\tau) - \phi (\sigma, t))^{\xi(\sigma,t)} d\tau,}
\]

(9)

respectively, where \( \phi (t) \) is the MIF.

Similarly, the properties of the new ϕ-FOD of the functions of two-variable order \( \xi (\sigma, t) \) are listed in Table 3.

For \( \phi (t) = t \), Eqs. (6) and (7) are written as:

\[
C_D^{(\xi(t))}_a \psi (t) = \frac{1}{\Gamma (1 - \xi (t))} \int_{a}^{t} \frac{\psi_{\phi}' (\tau)}{(t - \tau)^{\xi(t)} d\tau,}
\]

(10)

\[
C_D^{(\xi(t))}_a \psi (t) = \frac{-1}{\Gamma (1 - \xi (t))} \int_{a}^{t} \frac{\psi_{\phi}' (\tau)}{(\tau - t)^{\xi(t)} d\tau,}
\]

(11)

respectively.
Table 3
The properties of the new $\varphi$-FOD of the functions of two-variable order $\xi(\sigma, t)$ where $\upsilon$ is real number.

| Functions                            | new $\varphi$-FOD |
|--------------------------------------|--------------------|
| $\frac{(\varphi(t) - \varphi(a^+))^{\xi(\sigma, t)}}{\Gamma(1 - \xi(\sigma, t))}$ | $\frac{(\varphi(t) - \varphi(a^+))^{\upsilon - \xi(\sigma, t)}}{\Gamma(1 + \upsilon - \xi(\sigma, t))}$ |
| $\frac{(\varphi(t) - \varphi(a^-))^{\xi(\sigma, t)}}{\Gamma(1 - \xi(\sigma, t))}$ | $\frac{(\varphi(t) - \varphi(a^-))^{\upsilon - \xi(\sigma, t)}}{\Gamma(1 + \upsilon - \xi(\sigma, t))}$ |
| $E_\xi\left(\left(\varphi(t) - \varphi(a^+\right)\xi(\sigma, t))\right)$ | $E_\xi\left(\left(\varphi(t) - \varphi(a^+\right)\xi(\sigma, t))\right)$ |
| $E_\xi\left(\left(\varphi(a^-) - \varphi(t)\right)\xi(\sigma, t))\right)$ | $E_\xi\left(\left(\varphi(a^-) - \varphi(t)\right)\xi(\sigma, t))\right)$ |

For $\xi(t) = \xi$, Eqs. (6) and (7) can be expressed as [2, 24]:

$$CD_a^{\xi}\psi(t) = \frac{1}{\Gamma(1 - \xi)} \int_{a^+}^t \frac{\psi^{(1)}(\tau)}{(t - \tau)^{\xi}} d\tau, \quad (12)$$

$$CD_a^{\xi}\psi(t) = -\frac{1}{\Gamma(1 - \xi)} \int_{a^-}^t \frac{\psi^{(1)}(\tau)}{(\tau - t)^{\xi}} d\tau, \quad (13)$$

respectively.

As a matter of fact, we notice that Eqs. (10) and (11) are the left and right Caputo FODs of variable order (see [28] and the references therein), respectively. Obviously, Eqs. (1) and (6) are expressed in the special forms of Eq. (8). Similarly, Eqs. (2) and (7) can be considered as the special forms of Eqs. (9) and (12) and both $\psi(\tau)$ and $\psi^{(1)}(\sigma, \tau)$ are continuous functions.

The Laplace transform (LT) of Eq. (12) for $a = 0$ is given as [2]:

$$\hat{L}\left[\frac{1}{\Gamma(1 - \xi)} \int_0^t \frac{\psi^{(1)}(\tau)}{(t - \tau)^{\xi}} d\tau\right] = s^{\xi}(\psi(s) - s^{-1}\psi(0)), \quad (14)$$

where $\hat{L}$ denotes the LT operator (LTO) with respect to $t$.

Following the steps for obtaining Eq. (14), the LT of $t^{-1-\xi(t)}/\Gamma(-\xi(t))$, given by Coimbra [28], takes the form:

$$\hat{L}\left[\frac{t^{-1-\xi(t)}}{\Gamma(-\xi(t))}\right] = s^{\xi(t)}, \quad (15)$$

With the aid of Eqs. (14) and (15), the LT of $t^{-1-\xi(\sigma,t)}/\Gamma(-\xi(\sigma,t))$ is given by:

$$\hat{L}\left[\frac{t^{-1-\xi(\sigma,t)}}{\Gamma(-\xi(\sigma,t))}\right] = s^{\xi(\sigma,t)}. \quad (16)$$

For $\sigma = 0$, Eq. (16) can be written as Eq. (15).
Similarly, the LT of $(\varphi (t))^{-1-\xi(s,t)}/\Gamma (-\xi(s,t))$ is as follows:

$$\hat{L} \left[ \frac{(\varphi (t))^{-1-\xi(s,t)}}{\Gamma (-\xi(s,t))} \right] = \left[ \varphi^{(1)}_\varphi (t) \right] (s)^{\xi(s,t)}, \quad (17)$$

where $\varphi^{(1)}_\varphi (t)$ is the operator with respect to $t$.

For $\varphi (t) = t$, we have

$$\varphi^{(1)}_\varphi (t) = 1 \quad (18)$$

so that Eq. (17) becomes Eq. (16), that is,

$$\left[ \varphi^{(1)}_\varphi (t) \right] (s)^{\xi(s,t)} = [1] (s)^{\xi(s,t)} = s^{\xi(s,t)}. \quad (19)$$

In this case, we have

$$\hat{L} \left[ \frac{(\varphi (t))^{-1-\xi(s,t)}}{\Gamma (-\xi(s,t))} \right] = \left[ \varphi^{(1)}_\varphi (t) \right] (s)^{\xi(s,t)}, \quad (20)$$

such that

$$\hat{L} \left[ C D_{0+}^{(\xi(s,t),\varphi)} \psi (s,t) \right] = \left[ \varphi^{(1)}_\varphi (t) \right] (s)^{\xi(s,t)} \psi (s) - \left[ \varphi^{(1)}_\varphi (t) \right] (s)^{-1} \psi (0), \quad (21)$$

where

$$\hat{L} \left[ \frac{(\varphi (t))^{-\xi(s,t)}}{\Gamma (1-\xi(s,t))} \right] = \left[ \varphi^{(1)}_\varphi (t) \right] (s)^{\xi(s,t)}. \quad (22)$$

Taking $s = j\omega$, we have the following Fourier transforms (FT):

$$\hat{F} \left[ \frac{(\varphi (t))^{-1-\xi(s,t)}}{\Gamma (-\xi(s,t))} \right] = \left[ \varphi^{(1)}_\varphi (t) \right] (j\omega)^{\xi(s,t)}, \quad (23)$$

$$\hat{F} \left[ \frac{(\varphi (t))^{-1-\xi(s,t)}}{\Gamma (-\xi(s,t))} \right] = \left[ \varphi^{(1)}_\varphi (t) \right] (j\omega)^{\xi(s,t)}, \quad (24)$$

$$\hat{F} \left[ C D_{0+}^{(\xi(s,t),\varphi)} \psi (s,t) \right] = \left[ \varphi^{(1)}_\varphi (t) \right] (j\omega)^{\xi(s,t)} \psi (j\omega), \quad (25)$$

where $\hat{F}$ denotes the FT operator (FTO) with respect to $t$.

Suppose that $T$ is the interval $\infty \leq a < b \leq +\infty$, $\varphi \in C^{n+1} (T)$ with $\varphi^{(1)}_\varphi (t) \neq 0$, $\psi \in C^{n+1} (T)$ and $n-1 < \xi (t) < n$ for $\forall t \in T$.

The left and right $\varphi$-FOD of the function $\psi (t)$ of order $\xi (t)$ are defined by:

$$C D_{a+}^{(\xi(t),\varphi)} \psi (t) = \frac{1}{\Gamma (n-\xi (t))} \int_{a+}^{t} \frac{\psi^{(n)}_\varphi (\tau)}{(\varphi (t)-\varphi (\tau))^{n-\xi (t)-1}} d\tau, \quad (26)$$
\[ C D_a^{(\xi(t)\varphi)} \psi(t) = \frac{-1}{\Gamma(n - \xi(t))} \int_{a^-}^{t} \frac{\psi^{(n)}(\tau)}{(\varphi(\tau) - \varphi(t))^{n-\xi(t)-1}} d\tau, \] (28)

respectively, where \( \varphi(t) \) is the MIF.

Let \( n - 1 < \xi(\sigma, t) < n \). The left and right \( \varphi \)-FOD of the function \( \psi(\sigma, t) \) of two-variable order \( \xi(\sigma, t) \) are defined by:

\[ C D_{a^+}^{(\xi(\sigma,t)\varphi)} \psi(\sigma, t) = \frac{1}{\Gamma(n - \xi(\sigma, t))} \int_{a^+}^{t} \frac{\psi^{(n)}(\sigma, \tau)}{(\varphi(\sigma, \tau) - \varphi(\tau))^{n-\xi(\sigma, t)-1}} d\tau, \] (29)

\[ C D_{a^-}^{(\xi(\sigma,t)\varphi)} \psi(\sigma, t) = \frac{-1}{\Gamma(n - \xi(\sigma, t))} \int_{a^-}^{t} \frac{\psi^{(n)}(\sigma, \tau)}{(\varphi(\sigma, \tau) - \varphi(\tau))^{n-\xi(\sigma, t)-1}} d\tau, \] (30)

respectively, where \( \varphi(t) \) is the MIF.

Similarly, the LT of Eq. (29) is given by:

\[ \hat{L} \left[ C D_{0^+}^{(\xi(\sigma,t)\varphi)} \psi(\sigma, t) \right] = \left[ \varphi^{(n)}(t) \right] (s)^{\xi(\sigma,t)} \left( \psi(\sigma, s) - \sum_{n=1}^{n-1} \left[ \varphi^{(n)}(t) \right] (s)^{-n} \psi^{(n-1)}(\sigma, 0) \right), \] (31)

and the FT of Eq. (29) by:

\[ \hat{F} \left[ C D_{0^+}^{(\xi(\sigma,t)\varphi)} \psi(\sigma, t) \right] = \left[ \varphi^{(n)}(t) \right] (j\omega)^{\xi(\sigma,t)}, \] (32)

where \( \psi^{(0)}(\sigma, 0) = \psi(\sigma, 0) \).

Similarly, we present the following ODE of variable order with respect to the MIF:

\[ C D_{0^+}^{(\xi(\sigma,t)\varphi)} \psi(\sigma, t) = \Lambda \psi(\sigma, t) \] (33)

with the solution involving the MIF

\[ \psi(\sigma, t) = \Lambda E_{\xi(\sigma,t)} \left( (\varphi(t))^{\xi(\sigma,t)} \right), \] (34)

where \( \Lambda \) is a constant and

\[ E_{\xi(\sigma,t)} \left( (\varphi(t))^{\xi(\sigma,t)} \right) = \sum_{i=0}^{\infty} (\varphi(t))^{i\xi(\sigma,t)} / \Gamma(1 + i\xi(\sigma,t)). \]

The chart of Eq. (34) for the different parameters is displayed in Figure 1.

Thus, from Eq. (34), we suggest the following ODE of variable order with respect to the MIF:

\[ C D_{0^+}^{(\xi(t)\varphi)} \psi(t) = \Lambda \psi(t) \] (35)
with the solution involving the MIF
\[
\psi(t) = \Lambda E_{\xi(t)} \left((\varphi(t))^\xi(t)\right),
\]  
(36)
where \(\Lambda\) is a constant and
\[
E_{\xi(t)} \left((\varphi(t))^\xi(t)\right) = \sum_{i=0}^{\infty} (\varphi(t))^{i\xi(t)} / \Gamma(1 + i\xi(t)).
\]

The chart of Eq. (36) for the different parameters is displayed in Figure 2.

For \(\xi(t) = \xi\), Eq. (36) is in line with the result in [22].
3 The ADM described by $\varphi$-FODs

The ADMs represent the concentrations of the density of particles in the physical systems. They play an important role in the description of the complex medium, such as heat, electric circuit, geophysics and biological systems.

For predicting the complex behavior in the inhomogeneous media, we can follow the ADM given by:

$$C_D^{(\xi(\sigma,t),\varphi)} \psi_\rho(\sigma,t) = \varsigma \frac{\partial^2 \psi_\rho(\sigma,t)}{\partial \sigma^2}, 0 < \xi(\sigma,t) < 1,$$

subjected to the initial-boundary value conditions

$$\psi_\rho(\sigma,0) = \delta(\sigma),$$

where $\psi_\rho(\sigma,t)$ is the concentrations, $\varsigma (\varsigma > 0)$ is the diffusion constant, and $\delta(\sigma)$ is the Dirac function.

In the above equation, the $\varphi$-FOD involving the function $\varphi(t)$ of two-variable order $\xi(\sigma,t)$ is adopted to model the different concentrations including the constant-order (CO), time-variable (TV), space-variable (SV) and concentration-dependent variable-order (CDVO).

Following the above consideration, we have the following:

3.1 The COADM with respect to the monotonic increasing time-variable function

The COADM used to describe the system of the unknown variable transport process were

$$C_D^{(\xi,\varphi)} \psi_\rho(\sigma,t) = \varsigma \frac{\partial^2 \psi_\rho(\sigma,t)}{\partial \sigma^2}, 0 < \xi < 1,$$

and

$$\int_0^1 o(\xi) C_D^{(\xi,\varphi)} \psi_\rho(\sigma,t) d\xi = \varsigma \int_0^1 o(\xi) d\xi = 0,$$

where $o(\xi)$ is the function considered to decelerate or accelerate the diffusion process.

We observe that the proposed models in [26] are the special cases of Eqs. (39) and (40). Furthermore, from mathematical point of view, the concentration of the particle in the transport process is also described by the COADM. Eq. (40) is considered to present the COADM with the use of the decelerating or accelerating processes.
3.2 The TVADM with respect to the monotonic increasing time-variable function

With the support of the complex behaviors of the anomalous transport systems, we can write the following equation:

\[ C D_{a^{(\xi(t),\phi)}}^\alpha \psi_\rho (\sigma, t) = \frac{\partial^2 \psi_\rho (\sigma, t)}{\partial \sigma^2}, 0 < \xi (t) < 1, 0 \leq \sigma \leq \varpi, \]  

where the initial-boundary conditions are as follows [26]:

\[ \psi_\rho (\sigma, 0) = \sin \left( \frac{\sigma \pi}{\varpi} \right), 0 \leq \sigma \leq \varpi, \]  

\[ \psi_\rho (0, t) = \psi_\rho (\varpi, t) = 0, \ t = 0. \]  

In fact, Eq. (41) is the TVADM of the Caputo-type applied to the transport process in the inhomogeneous media. When \( \phi = t \), Eq. (41) is the time dependent variable-order ADM proposed in [26]. When \( \phi \neq t \), the Sun-Chen-Chen model [26] is invalid for handling the generalized complex media. However, Eq. (41) can be successfully applied to model the above problems in the different transitional regimes.

3.3 The SVAMD with respect to the monotonic increasing time-variable function

The SVADM in terms of the monotonic increasing time-variable function is expressed by:

\[ C D_{a^{(\xi(\sigma),\phi)}}^\alpha \psi_\rho (\sigma, t) = \frac{\partial^2 \psi_\rho (\sigma, t)}{\partial \sigma^2}, 0 < \xi (\sigma) < 1, 0 \leq \sigma \leq \varpi, \]  

where \( \varpi \) is a constant, and the fractional-order-space differential operator is defined by:

\[ C D_{a_{-}^{(\xi(\sigma),\phi)}}^\alpha \psi (t) = \frac{1}{\Gamma (n - \xi (\sigma))} \int_a^t \frac{\psi_\rho^{(1)} (\tau)}{(\phi (t) - \phi (\tau))^{\xi(\sigma)}} d\tau. \]  

In the above result, Eq. (44) is the SVADM of the Caputo-type with respect to the monotonic increasing time-variable function and is used to predict the transport process in the special complex media. When \( \phi = t \), the space-dependent variable-order ADM presented in [26] is a special case of Eq. (41).
3.4 The CDVOADM with respect to the monotonic increasing time-variable function

The CDVOADM with respect to the monotonic increasing time-variable function is written as:

\[ C^D_{a+}^{(\psi_\rho(\sigma,t),\varphi)}\psi_\rho(\sigma,t) = \xi \frac{\partial^2 \psi_\rho(\sigma,t)}{\partial \sigma^2}, \quad 0 < \xi(\sigma) < 1, \quad 0 \leq \sigma \leq \varpi, \quad (46) \]

where \( \varpi \) is a constant, and the fractional-order concentration-dependent differential operator is defined by:

\[ C^D_{a+}^{(\psi_\rho(\sigma,t),\varphi)}\psi(\sigma,t) = \frac{1}{\Gamma(1-\xi(\sigma,t))} \int_a^t \frac{\psi_\varphi^{(1)}(\sigma,\tau)}{(\varphi(t)-\varphi(\tau))^{\psi_\rho(\sigma,t)}} d\tau. \quad (47) \]

We easily observe Eq. (47) is the extended version of the Sun-Chen-Chen model reported in [26], when \( \varphi(t) = t \).

4 Conclusion

In this work, we developed a new variable-order FOD with respect to the MIF in the sense of the Caputo type. Meanwhile, we also proposed a class of the special functions of Mattag-Leffler type and the FT and the LT of the proposed FOD. Finally, an ADM of \( \varphi \)-FOD type with respect to the monotonic increasing time-variable function was obtained and its analogies are also discussed. The anomalous diffusion models, as the generalized result reported in [26], are useful for describing the transport process in the special complex media.

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