Curvature without connection

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Abstract

We show that the alternative theory of Lie groups and geometric structures proposed in the recent book [Or1] can be developed independently of connections. We show the details of this connection-free approach in the cases of absolute parallelism, affine and Riemannian structures and outline the method in the general case.

1 Introduction

A nonlinear/linear prehomogeneous geometry (PHG) as defined in [Or1] is a special transitive Lie equation in finite and infinitesimal forms whose theory is developed by D.C.Spencer and his coworkers around 1970. Lie equations, whose origin traces back to the original works of Sophus Lie, are later generalized to groupoids (nonlinear or finite in the old terminology) and algebroids (linear or infinitesimal). In [Or1] we defined the curvature of a PHG under a technical assumption which is equivalent to this curvature being the curvature of a connection. This turns out to be a very strong condition as explained in [Or1] (see pages 188-192). As a solution, we proposed another definition based on an idea communicated to us by A. Blaom. With time, however, we came to realize that this second definition that we advocated as the ultimate solution throughout [Or1] has the same weakness as the classical definition of a Cartan geometry: It is difficult to construct nonflat examples (see [CS] for parabolic geometries). The purpose of this paper is to give a detailed analysis of an affine and Riemannian PHG showing that the above mentioned technical assumption is redundant, a fact stated also in [Or1] without proof. This paper can be regarded also as a revision and extension of [Or2].

A few words about the general philosophy of [Or1] and this paper are in order here: Principal bundles, vector bundles and connections on these bundles are essentially topological objects. It is a standard practice today to do geometry using these topological tools. By geometry we mean here the study of transitive actions of Lie groups in the spirit of Klein’s Erlangen Program complemented with a concept of curvature, or briefly, the study of ”symmetry deformed by curvature” in the words of Blaom ([Bl]). In the approach commonly accepted today, which regards connection as the primary object, the search for some ”special
connection” that suits the geometric structure "best" becomes an unavoidable and sometimes an arduous task ([CS]). Naturally the question arises whether it is possible to start directly with geometry building on some simple and intuitive principles based on "group action" and recover the above topological framework as a generalization. Such a geometric framework is proposed in [Or1]. In this alternative approach certain "splittings" replace "special connections". These splittings are incorporated into the definition of the geometric structures and they do not always define connections, but when they do, these connections turn out to be "very special".

This alternative approach has many advantages over the classical one. For instance,

1) As mentioned above, it frees us from the search for special connections, and consequently from one of the most intriguing concepts in differential geometry: torsion.

2) It is much easier to construct nonflat examples.

3) It is modelfree, that is, the flat model (which is some homogeneous space) is not fixed beforehand. In particular, it avoids the Maurer-Cartan form which fixes the model. This fact gives an immense conceptual depth to the theory. For instance, Poincare Conjecture becomes tantamount to proving the existence of a flat absolute parallelism on a compact and simply connected 3-manifold.

4) A PHG is by definition a first order mixed system of PDE and curvature is by definition the integrability conditions of this system. Therefore, this definition of curvature is both elementary and intuitive and avoids the subtle aspects of Spencer cohomology, like involution, acyclicity, formal integrability...etc.

5) All geometric structures are defined on equal logical footing.

6) Nevertheless, there is an hierarchy determined by the order of jets measuring the inner complexity of the structure. In this hierarchy, the simplest one is absolute parallelism which renders in the flat case the theory of simply transitive actions of Lie groups and Lie algebras.

7) The study of all PHG’s reduces to the case of parallelism and the principle of this reduction is the moving frame method introduced by Fels and Olver in [FO1], [FO2] as a correction and perfection of Cartan’s repere mobile. This method equips the present approach with powerful computational algorithms.

It is our hope that the differential geometers will find this alternative approach worthwhile.

2 Review of absolute parallelism

We started Part 1 of [Or1] with a quote from Einstein: Everything should be made as simple as possible, but not simpler. With time, however, we came to realize that Part 1 of [Or1] is in need of further simplification, especially the unnecessarily long and obscure proof of the crucial Proposition 6.8. This simplification has a striking consequence: The connections \( \nabla, \bar{\nabla} \), which seem to play a fundamental role in Parts 1, 2 of [Or1], emerge as consequences of more fundamental concepts and one can develop the whole theory without even
mentioning them! More importantly, this new approach applies word by word to all geometric structures. The key fact is that the theory in Part 1 of [Or1] can be based on the “structure object” which is passed over in silence in [Or1] except in Chapter 7. Our purpose here is to recast Part 1 of [Or1] in this simplified form, refering to [Or1] for some further technical details.

Let \( M \) be a smooth manifold and \( \varepsilon \) a splitting of the groupoid projection \( \pi : \mathcal{U}_1 \to \mathcal{U}_0 = M \times M \). It is easy to show that such \( \varepsilon \) exists if and only if \( M \) is parallelizable ([Or1], Proposition 1.2). Since \( \varepsilon \) is a homomorphism of groupoids, it satisfies the following identities:

\[
\varepsilon_a^i(y, z)\varepsilon_a^a(x, y) = \varepsilon_j^j(x, z) \tag{1}
\]

\[
\varepsilon_j^j(x, x) = \delta_j^j \tag{2}
\]

\[
\varepsilon_a^i(y, x)\varepsilon_a^a(x, y) = \varepsilon_j^j(x, x) = \delta_j^j \tag{3}
\]

for all \( x, y, z \in M \). We fix a base point \( e \in M \) and define a geometric object \( w \) with components \( (w^i_j(x)) \) on \((U, x)\) by

\[
w^i_j(x) \overset{def}{=} \varepsilon_j^j(e, x) \tag{4}
\]

Now (1), (3) and (4) give

\[
\varepsilon_j^i(x, y) = \varepsilon_a^i(e, y)\varepsilon_a^a(x, e) = w_a^i(y)\bar{w}_a^a(x) \quad \bar{w} = w^{-1} \tag{5}
\]

Since \( \varepsilon(x, y) \circ \varepsilon(e, x) = \varepsilon(e, y) \) by (1), the components \( (w^i_j(x)) \) are subject to the transformation law

\[
\frac{\partial y^i}{\partial x^a} w^a_j(x) = w^i_j(y) \tag{6}
\]

upon a coordinate change \((x) \to (y)\) in view of (4). Therefore, if we are given an object \( w \) on \( M \) whose components \( (w^i_j(x)) \) transform according to (6), we can define \( \varepsilon \) by (5) which is easily seen to be a splitting and therefore satisfies (1)-(3). Conversely, given a splitting \( \varepsilon \) satisfying (1)-(3), we can fix a base point \( e \) and define \( w \) by (4) whose components transform according to (5). We observe that \( w \) determines \( \varepsilon \) canonically whereas \( \varepsilon \) determines \( w \) modulo the choice of a base point \( e \) and some coordinates around \( e \) satisfying \( w^i_j(e) = \delta^i_j \). It follows that \( w \) is more intrinsic than \( \varepsilon \) and henceforth, we will concentrate on \( w \) rather than \( \varepsilon \).

**Definition 1** The geometric object \( w = (w^i_j(x)) \) with the transformation law (6) is the structure object of the parallelizable manifold \((M, w)\).

There is a more conceptual derivation of \( w \): Let \( G \) be an abstract Lie group and \( L_g : G \to G \) be the left translation \( L_g(x) = gx \). Let \( \mathcal{U}^e_1 \) the set of 1-jets of all local diffeomorphisms (which we call 1-arrows) that fix \( e \). Then \( \mathcal{U}^e_1 \) is a Lie group and a choice of coordinates around \( e \) identifies \( \mathcal{U}^e_1 \) with \( GL(n, \mathbb{R}) \). The only \( L_g \) that fixes \( e \) is \( L_e \) which we identify with its 1-jet Id in \( \mathcal{U}^e_1 \cong GL(n, \mathbb{R}) \).
Now $GL(n, \mathbb{R})$ acts on the left coset space $GL(n, \mathbb{R})/\{Id\} = GL(n, \mathbb{R})$ by matrix multiplications as $(w^i_j) \rightarrow g^a_i w^a_j$ and this group action can be used to define the structure object $w$. This derivation applies to all homogeneous spaces $G/H$ and can be used to define the "structure object of that particular geometry" as we will see below.

We now define

$$
\Gamma_{ij}^k(x) \overset{\text{def}}{=} \left[ \frac{\partial \varepsilon^i_k(x, y)}{\partial y^j} \right]_{y=x} = \frac{\partial \varepsilon^i_k(y, x)}{\partial x^j} \varepsilon^a_k(x, y) = \frac{\partial w^a_i(x)}{\partial x^j} \tilde{w}^a_k(x) \quad (7)
$$

The second equality in (7) follows by differentiating (1) with respect to $z$ at $z = x$ and the third equality follows from (5). In particular, note that the third expression in (7) is independent of $y$. Similarly, we obtain

$$
- \Gamma_{jk}^i(x) = \left[ \frac{\partial \varepsilon^i_k(x, y)}{\partial x^j} \right]_{y=x} = \varepsilon^a_i(y, x) \frac{\partial \varepsilon^a_k(y, x)}{\partial x^j} = w^a_i(x) \frac{\partial \tilde{w}^a_j(x)}{\partial x^j} \quad (8)
$$

Using (6), we now define a subgroupoid $N_1^1(w) \subset \mathcal{U}_1$ as follows: The fiber $N_1^1(w)^{\mathcal{U}_1}_w$ of $N_1^1(w)$ over $(\mathfrak{a}, \mathfrak{g}) \in \mathcal{U}_0 = \mathbb{M} \times \mathbb{M}$ consists of those 1-arrows $(\mathfrak{a}, \mathfrak{g}, \tilde{T}_1) = (\mathfrak{a}, \mathfrak{g}, \tilde{T}_j)$ of $\mathcal{U}_1$ from $\mathfrak{a}$ to $\mathfrak{g}$ that preserve the geometric object $w$, that is

$$
\tilde{T}_a w^a_j(\mathfrak{a}) = w^a_j(\mathfrak{g}) \quad (9)
$$

**Definition 2** The subgroupoid $N_1^1(w) \subset \mathcal{U}_1$ is the invariance groupoid of the structure object $w$.

$N_1^1(w)$ is the most important example of a prehomogeneous geometry (PHG) and historically it is called an absolute parallelism. We observe that if $(\mathfrak{a}, \mathfrak{g}, \tilde{T}_1) = (\mathfrak{a}, \mathfrak{g}, \tilde{T}_j)$ is in $N_1^1(w)$, then $\tilde{T}_j = w^a_i(\mathfrak{g}) \tilde{w}^a_j(\mathfrak{a})$ by (9) and therefore $\tilde{T}_j$ is determined by $\mathfrak{a}$ and $\mathfrak{g}$. Therefore, above any 0-arrow $(\mathfrak{a}, \mathfrak{g}) \in \mathcal{U}_0 = \mathbb{M} \times \mathbb{M}$, there is a unique 1-arrow $(\mathfrak{a}, \mathfrak{g}, \tilde{T}_1)$ of $N_1^1(w)$ which we write as $\varepsilon(\mathfrak{a}, \mathfrak{g}) = (\mathfrak{a}, \mathfrak{g}, \tilde{T}_1)$. We define $N_0^1(w) \overset{\text{def}}{=} \mathcal{U}_0 = \text{the pair groupoid } \mathbb{M} \times \mathbb{M}$ so that $N_0^1(w) \cong N_1^1(w)$ where the isomorphism $\cong$ of groupoids is given by (omitting the indices) $(\mathfrak{a}, \mathfrak{g}) \rightarrow (\mathfrak{a}, \mathfrak{g}, w(\mathfrak{g})\tilde{w}(\mathfrak{a})) = (\mathfrak{a}, \tilde{\mathfrak{g}}, \varepsilon(\mathfrak{a}, \mathfrak{g}))$. Therefore $N_1^1(w) = \varepsilon(\mathcal{U}_0) \subset \mathcal{U}_1$.

**Definition 3** A local bisection of $\mathcal{U}_1$ consists of

1) An arbitrary local diffeomorphism $f: U \rightarrow V = f(U)$
2) A smooth choice of 1-arrows of $\mathcal{U}_1$ from $x \in U$ to $f(x) \in V$

In coordinates, a local bisection of $\mathcal{U}_1$ is of the form

$$
(x^i, f^i(x), f^j_i(x)) \quad (10)
$$

If we fix $x = \mathfrak{a}$ in (10), then (10) becomes the 1-arrow denoted by $(\mathfrak{a}, \mathfrak{g}, \tilde{T}_1)$ above. Similarly we define a local bisection of $N_1^1(w)$ by requiring the 1-arrows
in Definition 3 to belong to $\mathcal{N}_1(w)$. In this case, note that there is no "choice" of 1-arrows since from $x$ to $f(x)$ there is a unique 1-arrow of $\mathcal{N}_1(w)$.

If (10) is a local bisection of $\mathcal{N}_1(w)$, then it satisfies

$$f_i^j(x)w_j^a(x) = w_j^i(f(x))$$ (11)

according to (9).

**Definition 4** The local bisection (10) is prolonged (or holonomic) if

$$f_i^j(x) = \frac{\partial f_i^j(x)}{\partial x^j}$$ (12)

Now $\mathcal{N}_1(w)$ defines a first order nonlinear system of PDE’s on the pseudogroup $Diff_{fl}(M)$ of local diffeomorphisms of $M$ as follows: Some local diffeomorphism $y = f(x)$ is a local solution of $\mathcal{N}_1(w)$ if it satisfies

$$\frac{\partial f_i^j(x)}{\partial x^a}w_j^a(x) = w_j^i(f(x))$$ (13)

or equivalently

$$\frac{\partial f_i^j(x)}{\partial x^j} = \epsilon_j^i(x, f(x))$$ (14)

Therefore, a local solution of $\mathcal{N}_1(w)$ is a prolonged bisection of $\mathcal{N}_1(w)$ and now the question is whether $\mathcal{N}_1(w)$, which clearly admits many local bisections (as we can choose $y = f(x)$ arbitrarily), admits any prolonged bisections. The fibers $\mathcal{N}_1(w)^{\mathcal{F}, \mathcal{F}}$ of $\mathcal{N}_1(w)$ now serve as the initial conditions for the PDE (13): For any initial condition $(y, f(x), f_1)$ satisfying (9), can we find a prolonged bisection $(x, f(x), \frac{\partial f}{\partial x})$ defined around $f$ and satisfying $f'(f) = f, \frac{\partial f}{\partial x}(f) = f_1$? This is a particular instance of a general framework of PDE’s: Jets (1-arrows in our case) are pointwise but not always locally derivatives. A PDE (for instance (11)) is a condition on jets. Solving the PDE is equivalent to replacing jets by derivatives (or replacing (11) by (13)).

Now the well known existence and uniqueness theorem for first order systems of PDE’s with initial conditions asserts that the answer to the above question is affirmative if and only if the integrability conditions of (13) are satisfied. The check these integrability conditions, we differentiate (13) with respect to $x^k$, substitute back from (13) and alternate $j,k$. After some computation, this condition turns out to be

$$\left[ \frac{\partial w_i^a(f(x))}{\partial x^j}w_k^a(f(x)) \right]_{jk} - f'_i^j(x)\left[ \frac{\partial w_i^a(x)}{\partial x^a}w_k^a(x) \right]_{ab}g_j^a(x)g(x)_k^b = 0$$ (15)

for all bisections $(x,y,f_1) = (x^i,f^i(x),f_j^j(x))$ of $\mathcal{N}_1(w)$ where $g_1 = f_1^{-1}$. We now define the geometric object $I(w)$ by defining its components $I_{jk}(w;x)$ on $(U,x)$ by
\[ I^i_{jk}(w; x) \stackrel{\text{def}}{=} \left[ \frac{\partial w^i(x)}{\partial x^j} - \omega^a_{jk}(x) \right]_{[ij]} = \left[ \Gamma^i_{jk}(x) \right]_{[jk]} \]  

(16)

**Definition 5** \( I(w) \) is the integrability object of \( w \).

The name for \( I(w) \) will be justified below. In [Or1], \( I(w) \) is called the torsion of \( \mathcal{N}_1(w) \) because it turns out to be the torsion of a linear connection on the tangent bundle \( T(M) \to M \). However, this is a very misleading terminology as it holds only in the case of absolute parallelism as we will see below.

Using the LHS of (15) and (16), we define \( R^i_{jk}(x, y) \) on \( M \times M \) by

\[ R^i_{jk}(x, y) \stackrel{\text{def}}{=} I^i_{jk}(w; y) - I^i_{ab}(w; x) f^a_1 g^b_1 \]  

(17)

**Definition 6** \( R^i_{jk}(x, y) \) is the nonlinear curvature of \( \mathcal{N}_1(w) \).

\( R \) is denoted by \( \overline{R} \) in [Or1] (2.12, pg.20). Note that \( f_1 = g_1^{-1} \) on the RHS of (17) is not an independent variable since it is determined by \( x, y \). Now if (13) admits local solutions with arbitrary initial conditions, then clearly \( R = 0 \) on \( M \times M \) by (15). Conversely, if \( R = 0 \) on \( M \times M \), then according to our theorem, any initial condition (= any 1-arrow) of \( \mathcal{N}_1(w) \) integrates uniquely to a local solution of (13) satisfying this initial condition. In this case we call \( \mathcal{N}_1(w) \) uniquely locally integrable. Using a symbolic notation without the indices, we rewrite (17) as

\[ R(x, y) = I(w; y) - (x, y, f_1).I(w; x) \]  

(18)

According to (18), we translate \( I(w; x) \) from \( (U, x) \) to \( (V, y) \) using the 1-arrow \( (x, y, f_1) \) of \( \mathcal{N}_1(w) \) and compare the value \( (x, y, f_1).I(w; x) \) on \( (V, y) \) with \( I(w; y) \) by subtracting and \( R(x, y) \) measures the difference. Consequently, \( R = 0 \) if and only if \( I(w) \) is invariant with respect to \( \mathcal{N}_1(w) \). Therefore we can state

**Proposition 7** The following are equivalent for the subgroupoid \( \mathcal{N}_1(w) \subset \mathcal{U}_1 \).

1) \( R = 0 \) on \( M \times M \)

2) \( \mathcal{N}_1(w) \) is uniquely locally integrable, i.e., any 1-arrow of \( \mathcal{N}_1(w) \) integrates uniquely to a local solution of (13).

3) \( \mathcal{N}_1(w) \), which leaves \( w \) invariant by its definition, leaves also \( I(w) \) invariant.

It is worthwhile here to take a closer look at \( R \) defined by (17), (18). Clearly \( R(x, x) = 0 \), i.e., \( R \) vanishes on the diagonal of \( \mathcal{U}_0 = M \times M \). Now \( R \) is a 2-form on \( M \) and assigns to any \( (\overline{x}, \overline{y}) \in \mathcal{U}_0 \) a vector in the fiber of \( T \to M \) over \( \overline{y} \), i.e., a tangent vector at \( \overline{y} \). In the language of bisections, the 2-form \( R \) maps the bisection \( (x, f(x), f_1(x)) \) of \( \mathcal{N}_1(w) \) to a section of the vector bundle \( T \to M \) over \( f(x) \). Therefore,
\[ \mathcal{R} : \mathcal{U}_0 = M \times M \longrightarrow \wedge^2(T^*) \otimes T \]  

(19)

In (19), we use the same notation for the bundles and the (bi)sections of these bundles. Note that \( \mathcal{R} \) lifts the bisection \( (x, f(x)) \in \mathcal{U}_0 \) (same abuse of notation) to \( (x, f(x), f_1(x)) \in \mathcal{N}_1(w) \) by \( \varepsilon \) and then maps it to a section of \( \wedge^2(T^*) \otimes T \) over \( f(x) \).

Now our purpose is to linearize the above nonlinear picture. Consider the vector bundle \( J_1T \rightarrow M \) of 1-jets of vector fields and let \( \xi = (\xi^i(x)) \) be a vector field integrating to the 1-parameter subgroup \( f^i(t, x) \) of local diffeomorphisms. Therefore

\[ f^i(0, x) = x^i \quad \left[ \frac{\partial f^i(t, x)}{\partial t} \right]_{t=0} = \xi^i(x) \]  

(20)

and

\[ \frac{\partial}{\partial x^j} \left[ \frac{\partial f^i(t, x)}{\partial t} \right]_{t=0} = \frac{\partial \xi^i(x)}{\partial x^j} = \frac{\partial}{\partial t} \left[ \frac{\partial f^i(t, x)}{\partial x^j} \right]_{t=0} \]  

(21)

**Definition 8** A (smooth) path in \( \mathcal{U}_1 \) through the identity at \( \overline{x} \in M \) consists of

1) A path \( x(t) \) in \( M \), \( t \in (-\epsilon, \epsilon) \), \( x(0) = \overline{x} \)

2) A (smooth) choice of 1-arrows of \( \mathcal{U}_1 \) from \( \overline{x} \) to \( x(t) \) which is identity for \( t = 0 \)

In coordinates, such a path is of the form

\[ (\overline{x}^i, x^i(t), f^i_j(t)) \quad t \in (-\epsilon, \epsilon) \]  

(22)

\[ (\overline{x}^i, x^i(0), f^i_j(0)) = (\overline{x}^i, \overline{x}^j, \delta^j_i) \]

Given such a path in \( \mathcal{U}_1 \) (which we will simply call a path), we define its tangent at \( t = 0 \) by

\[ \frac{d}{dt} \left[ (\overline{x}^i, x^i(t), f^i_j(t)) \right]_{t=0} \overset{\text{def}}{=} (\overline{x}^i, \xi^i(\overline{x}), \xi^j_j(\overline{x})) \]  

(23)

and (20), (21) show that \( (\overline{x}^i, \xi^i(\overline{x}), \xi^j_j(\overline{x})) \) (briefly \( (\xi^i(\overline{x}), \xi^j_j(\overline{x})) \)) is an element in the fiber of \( J_1T \rightarrow M \) over \( \overline{x} \). Using (21), we see that all vectors in the fiber over \( \overline{x} \) are obtained in this way. Indeed, given \( (\xi^i(\overline{x}), \xi^j_j(\overline{x})) \), we choose any vector field \( \xi = (\xi^i(x)) \) around \( \overline{x} \) satisfying \( \frac{\partial \xi^i(\overline{x})}{\partial x^j} = \xi^j_j(\overline{x}) \), and \( (\overline{x}^i, f^i(t, \overline{x}), \frac{\partial f^i(t, \overline{x})}{\partial x^j}) \) has the tangent \((\xi^i(\overline{x}), \xi^j_j(\overline{x}))\) by (20), (21). Henceforth we will use the notations

\[ \xi_0(\overline{x}) = (\xi^i(\overline{x})) = \text{a tangent vector at } \overline{x} \]  

(24)

\[ \xi_1(\overline{x}) = (\xi^i(\overline{x}), \xi^j_j(\overline{x})) = \text{a 1-jet of a vector field at } \overline{x} \]

So \( \xi_0(\overline{x}) \) is a vector in the fiber of \( J_0T = T \rightarrow M \) over \( \overline{x} \), and \( \xi_1(\overline{x}) \) is a vector in the fiber of \( J_1T \rightarrow M \) over \( \overline{x} \), that projects to \( \xi_0(\overline{x}) \). We call this
process of passing from the groupoid $\mathcal{U}_1$ to the vector bundle $J_1T \to M$ (which is actually the algebroid of $\mathcal{U}_1$) linearization. Now our purpose is to linearize the subgroupoid $N_1(w) \subset \mathcal{U}_1$ in the same way whose linearization $N_1(w) \to M$ will be a subbundle of $J_1T \to M$.

A path in $N_1(w) \subset \mathcal{U}_1$ is of the form $(\overline{x}(t), x(t), \varepsilon(\overline{x}(t), x(t)))$ with the tangent

$$(\xi^i(\overline{x}), \xi^j(\overline{x})) = \frac{d}{dt} \left[ x^i(t), \varepsilon^j(\overline{x}(t), x(t)) \right]_{t=0} = (\xi^i(\overline{x}), \Gamma^i_{aj}(\overline{x})\xi^a(\overline{x}))$$

$$= (\xi^i(\overline{x}), \frac{\partial w_i^b(\overline{x})}{\partial x^a} \tilde{w}_j^b(\overline{x})\xi^a(\overline{x}))$$

(25)

Using (25) we define the fiber $N_1(w) \overline{x} \subset (J_1T)\overline{x}$ as those tangents $(\xi^i(\overline{x}), \xi^j(\overline{x}))$ satisfying

$$\xi^j(\overline{x}) = \Gamma^j_{ai}(\overline{x})\xi^i(\overline{x})$$

(26)

which is clearly a subspace. We define the bundle of vectors $N_1(w) \defeq \cup_{\overline{x} \in M} N_1(w)\overline{x}$ with the obvious projection $N_1(w) \to M$ which is easily seen to be a vector subbundle. Note that (26) gives a splitting $\varepsilon$ (using the same notation) of the projection

$$0 \longrightarrow T \otimes T^* \longrightarrow J_1(T) \longrightarrow T \longrightarrow 0$$

(27)

defined by

$$\varepsilon : (\xi^i) \longrightarrow (\xi^i, \Gamma^i_{aj}\xi^a)$$

(28)

The process of linearization is defined for all PHG’s (more generally for all Lie groupoids) yielding their algebroids. Since the linear connection (28) (denoted by $\nabla$ in [Or1]) drops out of this process, it will not be of primary importance for us. Put more succinctly, the splitting (28) is the linearization of the above nonlinear splitting $\varepsilon$ which is not a connection for otherwise the curvature of $\varepsilon$ would always vanish since $\varepsilon$ is a trivialization of the principle bundle $\mathcal{U}_1 \to M$ whereas just the opposite is true for $\mathcal{R}$: It is surely not always zero and a very subtle object!

To summarize, a section of $N_1(w) \to M$ is of the form $(\xi^i(x), \Gamma^i_{aj}(x)\xi^a(x))$.

**Definition 9** The section $(\xi^i(x), \xi^j(x))$ of $J_1(T) \to M$ is prolonged (or holonomic) if

$$\xi^j(x) = \frac{\partial \xi^i(x)}{\partial x^j}$$

(29)

So a vector field $\xi_0 = \xi = (\xi^i)$ defines a section of $J_1(T) \to M$ by prolonging as $pr(\xi) \defeq (\xi^i, \frac{\partial \xi^i}{\partial \overline{x}^j})$. Therefore, the section $(\xi^i(x), \Gamma^i_{aj}(x)\xi^a(x))$ of $N_1(w) \to M$ is prolonged if and only if
\[
\frac{\partial \xi^i(x)}{\partial x^j} = \Gamma^i_{aj}(x)\xi^a(x) = w^i_b(y)w^b_a(x)\xi^a(x)
\]  

(30)

Now (30) is a first order linear system and has unique local solutions with arbitrary initial conditions if and only if its integrability conditions are identically satisfied. Some computation shows that these integrability conditions are given by

\[
\Re^i_{kj,a} \overset{\text{def}}{=} \left[ \frac{\partial \Gamma^i_{aj}}{\partial x^k} + \Gamma^i_{bj} \Gamma^b_{ak} \right] \xi^a = 0
\]  

(31)

for all \( \xi = (\xi^i) \). Substituting (7) into (31) expresses \( \Re \) in terms of the structure object \( w \). Now if \( \Re = 0 \) near \( \mathcal{F} \), then for any tangent in the fiber \( N_1(w)\mathcal{F} \) as initial condition, there is a unique vector field \( \xi \) near \( \mathcal{F} \) which solves (31) and satisfies the given initial condition. If \( \Re = 0 \) on \( M \), we call \( N_1(w) \) uniquely locally integrable. Therefore

\[
\Re = 0 \iff N_1(w) \text{ is uniquely locally integrable}
\]  

(32)

**Definition 10** \( \Re \) is the linear curvature of the subgroupoid \( N_1(w) \subset \mathcal{U}_1 \).

Note that \( \Re(x) \) is a 2-form at \( x \) which assigns to the tangent vector \( (\xi^i(x)) \) the tangent vector \( \Re^i_{\xi,a}(x)\xi(x)^a \) at \( x \). Equivalently, we have

\[
\Re : T \longrightarrow \wedge^2(T^*) \otimes T
\]  

(33)

Yet another interpretation is that \( \Re \) is a 2-form on \( M \) with values in the vector bundle \( Hom(T,T) \rightarrow M \). At this point, inspecting (19) and (33) carefully and noting that \( T \rightarrow M \) is the linearization (the algebra) of \( \mathcal{U}_0 = M \times M \) (the pair groupoid), it is natural to expect that \( \Re \) will be in some sense the "linearization" of \( \Re \).

(32) is the linear analog of the equivalence 1) \( \iff \) 2) in Proposition 7 and it remains to find the linearization of 3) of Proposition 7. So let \( (\mathcal{F}, x^i(t), f^j_j(t)) \) be a path in \( \mathcal{U}_1 \) with the tangent \( \xi^i_1(\mathcal{F}) \) and \( \alpha(t) \) a first order geometric object, for instance a tensor field, defined on this path. Our purpose is to define the change of \( \alpha \) at \( \mathcal{F} \) in the direction of \( \xi^i_1(\mathcal{F}) \) which we will denote by \( (L_{\xi^i_1(\mathcal{F})})\alpha(\mathcal{F}) \). Note that \( (L_{\xi^i(\mathcal{F})})\alpha(\mathcal{F}) \) should not depend on the path but only on its tangent at \( \mathcal{F} \). When doing this, we should keep in mind the definition of the ordinary Lie derivative \( L_{\xi^i_0} \alpha \) of \( \alpha \) with respect to some vector field \( \xi_0 \) where both \( \xi_0 \) and \( \alpha \) are defined in some neighborhood of \( \mathcal{F} \). Now the path \( (\mathcal{F}, x^i(t), f^j_j(t)) \) maps the value \( \alpha(\mathcal{F}) = \alpha(x(0)) \) to \( (\mathcal{F}, x^i(t), f^j_j(t)),(\alpha(\mathcal{F})) \) and the idea is to compare \( (\mathcal{F}, x^i(t), f^j_j(t)),(\alpha(\mathcal{F})) \) with \( (\alpha(x(t)) \) by dividing their difference by \( t \) and letting \( t \rightarrow 0 \). The quantity \( (\mathcal{F}, x^i(t), f^j_j(t)),(\alpha(\mathcal{F})) \) is computed using the transformation rule of the tensor \( \alpha \). For instance, let \( \alpha = (\alpha^i_j) \) be a (1,1)-tensor field which transforms according to
Now (34) states the invariance of $\alpha$ with respect to $f$, i.e., the local diffeomorphism $f : (U, x) \to (V, y)$, $y = f(x)$, maps $\alpha(x)$ to $f_\ast \alpha(x)$ defined by the RHS of (34) which need not be equal to $\alpha(y)$ on the LHS of (34) unless $\alpha$ is left invariant by $f$. Now let $\xi = (\xi^i(x))$ be any vector field defined near $x$ satisfying $j_1(\xi)(x) = \xi_1(x)$, i.e., $\frac{\partial \xi^i(\xi)}{\partial x^j} = \xi^j_1$ and let $f^i(x, t)$ be the 1-parameter local diffeomorphisms defined by $\xi$ with $f^{-1}(t, x) = g(t, y)$. As we observed above, $(\bar{x}^i, f^i(t, \bar{x}), \frac{\partial f^i(t, \bar{x})}{\partial x^j})$ is a path (possibly different from the above one) having the same tangent $\xi_1(\bar{x}) = (\xi^i(\bar{x}), \xi^j_1(\bar{x}))$ at $\bar{x}$. We define

$$\langle \mathcal{L}_{(\xi^1(\bar{x}))} \rangle_{\bar{x}} \equiv_{def}$$

$$\frac{d}{dt} \left[ \alpha^i_1(\bar{x}) - \alpha^a_b(\bar{x}) \frac{\partial f^a(t, \bar{x})}{\partial x^b} \frac{\partial g^b(t, \bar{x})}{\partial y^i} \right] \bigg|_{t=0}$$

$$= \left[ \frac{\partial \alpha^i_1(\bar{x})}{\partial x^a} \frac{\partial f^a(t, \bar{x})}{\partial x^b} \frac{\partial g^b(t, \bar{x})}{\partial y^i} \right] \bigg|_{t=0} - \alpha^a_b(\bar{x}) \frac{d}{dt} \left[ \frac{\partial f^a(t, \bar{x})}{\partial x^b} \right] \bigg|_{t=0}$$

$$- \frac{d}{dt} \left[ \frac{\partial g^a(t, \bar{x})}{\partial y^i} \right] \bigg|_{t=0}$$

$$= \frac{\partial \alpha^i_1(\bar{x})}{\partial x^a} \xi^a(\bar{x}) - \alpha^a_b(\bar{x}) \frac{\partial \xi^i_b(\bar{x})}{\partial x^a} (\delta^b_j - \alpha^a_b(\bar{x}) (\delta^a_i)(- \frac{\partial \xi^b_1(\bar{x})}{\partial x^j}))$$

$$= \frac{\partial \alpha^i_1(\bar{x})}{\partial x^a} \xi^a(\bar{x}) - \alpha^a_b(\bar{x}) \xi^i_1(\bar{x}) + \alpha^a_b(\bar{x}) \xi^a(\bar{x})$$

(36)

(37)

We make six important observations.

1) To compute $\mathcal{L}_{(\xi^1(\bar{x}))} \alpha(\bar{x})$, (35) shows that we need only the values of $\alpha$ on the path and (37) shows that $\mathcal{L}_{(\xi^1(\bar{x}))} \alpha(\bar{x})$ depends only on the tangent of the path at $\bar{x}$ as required. Furthermore, $\mathcal{L}_{(\xi^1(\bar{x}))} \alpha$ is linear in the argument $\xi_1$.

2) We gave above all the details in the derivation of (37). For computational purposes however, all we have to do is to formally substitute $y^i = x^i + t \xi^i(x)$, $\frac{\partial \alpha^i}{\partial x^j} = \delta^i_j + t \frac{\partial \xi^i}{\partial x^j}$, $x^i = y^i - t \xi^i(y)$, $\frac{\partial \alpha^i}{\partial y^j} = \delta^i_j - t \frac{\partial \xi^i}{\partial y^j}$ into (34), collect all the terms on the LHS of (34) and differentiate the resulting expression with respect to $t$ at $t = 0$.

3) $\mathcal{L}_{(f^i(\bar{x}))} \alpha(x)$ is the ordinary Lie derivative $\mathcal{L}_\xi \alpha$ of $\alpha$ with respect to $\xi$ as expected. Therefore, (36) is the ordinary Lie derivative of $\alpha$ and the passage from (36) to (37) shows that $\mathcal{L}_{(\xi^1(\bar{x}))} \alpha$ is computed by first computing the ordinary Lie derivative of $\alpha$ and replacing the derivatives of the vector field $\xi$ with jet variables.

4) If $\alpha$ is a global $(1,1)$-tensor field on $M$ and $\xi_1$ is a global section of $J^1T \to M$, then $\mathcal{L}_{(\xi^1)} \alpha$ is defined pointwise by (37) and is another tensor of the same type as $\alpha$. 

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5) $\mathcal{L}_{\xi_1}$ is a derivation of the tensor algebra and commutes with contractions.

6) $\mathcal{L}_{\xi_1}\alpha$ is defined in the same way for all first order geometric objects $\alpha$, in particular, for all tensor fields.

**Definition 11** $\mathcal{L}_{\xi_1}\alpha$ is the formal Lie derivative of $\alpha$ with respect to the section $\xi_1$ of $J_1T \to M$.

Now let us choose our path in the definition of $\mathcal{L}_{\xi_1}\alpha$ in $N_1(w) \subset \mathcal{U}_1$ keeping in mind that $\xi_1 = \varepsilon\xi_0 = \varepsilon\xi$ in this case. Substituting (28) into (37), we find

$$
(\mathcal{L}_\varepsilon\alpha)^i_j(x) = \frac{\partial\alpha^i_j(x)}{\partial x^a} \xi^a(x) - \alpha^b_j(\tau) \Gamma^i_{ab}(\tau) \xi^a(x) + \alpha^i_b(\tau) \Gamma^b_{aj}(\tau) \xi^a(x)
$$

We observe that another linear connection pops up from (38) (denoted by $\tilde{\nabla}$ in [Or1]) and the remarkable fact is that this connection differs from the above one by the integrability object $I(w)^i_{jk}$ called torsion in [Or1]! Furthermore, this new connection has vanishing curvature!

On the hand, the formal Lie derivative $\mathcal{L}_{\xi+1}\alpha_k$ is defined in the obvious way for all $k$th order geometric object $\alpha_k$ and for a section $\xi_{k+1}$ of $J_{k+1}(T) \to M$ by computing the ordinary Lie derivative $\mathcal{L}_\xi\alpha_k$ and replacing the derivatives of $\xi$ with jet variables. For a PHG of order $k$, $\mathcal{L}_\xi\alpha_k$ is defined by $\mathcal{L}_{\varepsilon\xi}\alpha_k$. We observe the crucial fact that $\mathcal{L}_{\varepsilon\xi}\alpha_k$ is not a connection any more for $k \geq 2$ but forms the basis of an abstraction called "algebroid connection" in the modern theory of algebroids.

It is easy to check that $\mathcal{L}_{\varepsilon\xi}w = 0$ for all $\xi$. Indeed, $\mathcal{L}_{\varepsilon\xi}\alpha$ measures how $\alpha$ changes along the paths of $N_1(w)$ and $w$ is constant along all these paths by the definition of $N_1(w)$. Now we claim that $\mathcal{L}_{\varepsilon\xi}I(w) = \mathfrak{R}(\xi)$, or in more detail

$$
\left[\frac{d}{dt}\mathcal{R}^i_{kj}(x, x+t\xi)\right]_{t=0} = \mathcal{L}_{\varepsilon\xi} \left(I(w; x)^i_{kj}\right) = \mathfrak{R}^i_{kj,a}(x) \xi^a
$$

The first equality in (39) holds by the definition of $\mathcal{R}$ by (17) and it justifies, in view of the second equality in (39), that $\mathfrak{R}$ is indeed the linearization of $\mathcal{R}$ as forseen above. To prove the second equality, all we need to observe is that $\mathcal{L}_{\varepsilon}\xi = \tilde{\nabla}\xi$ and $I(w)^i_{kj} = T^i_{kj}$ in [Or1] and (39) is in fact the definition of $\mathfrak{R}$ in [Or1] (see Definition 6.1). Therefore, the linearization of 3) of Proposition 7 that we search for is given by the middle term in (39)! Combining (39) with (32), we now state

**Proposition 12** The following are equivalent.

1) $\mathfrak{R} = 0$

2) $N_1(w) \to M$ is uniquely locally integrable

3) $\mathcal{L}_{\varepsilon\xi}I(w) = 0$ for all $\xi \in T$
At this stage, it is easy to guess that 3) of Proposition 7 and 3) of Proposition 12 are equivalent. Indeed, in the language of [Or1], 3) of Proposition 7 asserts that $I(w)$ is $\varepsilon$-invariant and 3) of Proposition 12 asserts $I(w)$ is $\nabla$-parallel and these two concepts are equivalent according to Proposition 5.5 in [Or1]. A more intuitive and amusing argument goes as follows: Clearly $\mathcal{R} = 0$ implies $\mathcal{R} = 0$ by (39). Conversely, if $\mathcal{R} = 0$, then the "derivative of $\mathcal{R}(x, y)$ with respect to $y$" vanishes identically according to (39) and therefore $\mathcal{R}(x, y)$ is "constant in $y$". Therefore $\mathcal{R} = 0$ on $M \times M$ since $\mathcal{R}(x, x) = 0$.

Thus we state

**Proposition 13 (Lie’s 3’rd Theorem)** The conditions of Proposition 5 and Proposition 12 are equivalent.

It is explained in [Or1] why Proposition 13 is called Lie’s 3’rd Theorem.

There is another fundamental operator lurking in the above picture and this is a good place to pinpoint it. Consider the Spencer operator

$$D : J_1(T) \rightarrow \bigwedge (T^*) \otimes T$$

$$: (\xi^i, \xi^j) \rightarrow (\frac{\partial \xi^i}{\partial x^j} - \xi^i_j)$$

which restricts to

$$D : N_1(w) \rightarrow \bigwedge (T^*) \otimes T$$

$$: (\xi^i, \Gamma^i_{aj} \xi^a) \rightarrow (\frac{\partial \xi^i}{\partial x^j} - \Gamma^i_{aj} \xi^a)$$

Therefore $D_{\eta^k} \xi^j = \left( \frac{\partial \xi^i}{\partial x^j} - \Gamma^i_{aj} \xi^a \right) \eta^k = \nabla_{\eta^k} \xi^j$, i.e., $D_\eta \xi = \nabla_\eta$ (see (28)). As we remarked above, $\mathcal{L}_{[\eta]}$ will not be a connection on the PHG for $k \geq 1$ and we will see below that $D_{\eta_k}$ will sometimes become a "very special connection" but under a strong assumption!

### 3 Affine PHG’s

In this section we will imitate our above arguments word by word and therefore will not give all the details but elaborate when a new phenomenon occurs.

Consider the transformation group $\text{Aff}(\mathbb{R}^n) = GL(n, \mathbb{R}) \times \mathbb{R}^n$ of $\mathbb{R}^n$. Fixing, for instance, the origin $o \in \mathbb{R}^n$ and its stabilizer $GL(n, \mathbb{R})$, we define the map $j_k : GL(n, \mathbb{R}) = G_1(n) \rightarrow G_k(n)$ by $g \rightarrow j_k(g)$ which is injective for $k \geq 1$, where $G_k(n)$ is the $k$’th order jet group in $n$ variables. Identifying $G_1(n)$ with its image $j_2(G_1(n)) \subset G_2(n)$, the left coset space $G_2(n)/G_1(n)$ is parametrized by functions $(\Gamma^i_{jk})$ (see [Or1], 175-178 for details). Therefore, the left action of $G_2(n)$ on $G_2(n)/G_1(n)$ defines a geometric object $\Gamma$ with components $(\Gamma^i_{jk}(x))$ subject to the transformation rule.
\[ \Gamma_{ab}^{i}(f(x)) \frac{\partial f^a(x)}{\partial x^b} \frac{\partial f(x)^b}{\partial x^c} = \Gamma_{jk}^a(x) \frac{\partial f^i(x)}{\partial x^a} + \frac{\partial^2 f^i(x)}{\partial x^a \partial x^c} \]  

(42)

upon a coordinate change \((x) \rightarrow (f(x))\). It is standard to interpret \(\Gamma = (\Gamma_{jk}^a(x))\) as a "torsionfree affine connection" but we will carefully avoid this interpretation and regard \(\Gamma\) merely as a geometric object on \(M\) like \(w\). Using (42), we consider those 2-arrows \((\varPi, \vartheta, f_1, f_2) = (\varPi, \vartheta, f_j, f_{jk})\) of \(\mathcal{U}_2\) which preserve \(\Gamma\), that is

\[
\Gamma_{ab}^{i}(\vartheta) f_{jk} = f_j^a(x) \Gamma_{jk}^a(\vartheta) f_k + f_{jk}^i
\]

(43)

Definition 14 The subgroupoid \(\mathcal{H}_2(\Gamma) \subset \mathcal{U}_2\) defined by (43) is an affine PHG on \(M\) and \(\Gamma\) is its structure object.

We observe that \((\varPi, \vartheta, f_j) \in \mathcal{U}_1\) is arbitrary in (43) and \(f_{jk}\) is uniquely determined by \((\varPi, \vartheta, f_j)\), which gives the splitting

\[
\varepsilon : \mathcal{H}_1(\Gamma) = \mathcal{U}_1 \rightarrow \mathcal{U}_2
\]

(44)

If we set \(\varPi = \vartheta\) in (43), we get the the vertex groups \(\mathcal{H}_2(\Gamma)\vartheta \cong \mathcal{H}_1(\Gamma)\vartheta\) at \(\varPi\). We can always find coordinates \((x)\) around \(\varPi\) with the property \(\Gamma_{jk}^a(\vartheta) = 0\) as can be seen from (43). We call \((x)\) regular coordinates at \(\varPi\). In such coordinates, the vertex groups are identified with \(G_1(n) = GL(n, \mathbb{R})\).

We can clearly replace the arrows in (43) and (44) by bisections as before. A bisection \((x^i, f^i(x), f_j^i(x), f_{jk}^i(x))\) of \(\mathcal{U}_2\) is prolonged if \(\frac{\partial f^i(x)}{\partial x^j} = f_j^i(x)\) and \(\frac{\partial f_j^i(x)}{\partial x^k} = f_{jk}^i(x)\). Therefore, a bisection of \(\mathcal{H}_2(\Gamma)\) is prolonged if and only if

\[
\frac{\partial f^i(x)}{\partial x^j} = f_j^i(x)
\]

(45)

Clearly (45) has a solution if and only if (42) has a solution. However, (42) is a second order PDE whereas (45) is a first order system of PDE’s. Note that (45) is a closed system in the sense that it expresses the derivatives of the unknown functions \(f^i(x), f_j^i(x)\) in terms of themselves and \(x^*\)’s due to the splitting \(\varepsilon\). Therefore, we see that a prolonged bisection corresponds to the well known trick of introducing jet variables to reduce a second order PDE to a first order system. Now (44) serves as the initial conditions for (45). The integrability conditions of (45) are given by

\[
\frac{\partial f_j^i}{\partial x^k} - \frac{\partial f_k^i}{\partial x^j} = 0
\]

(46)

\[
\frac{\partial f_j^i}{\partial x^*} - \frac{\partial f_j^i}{\partial x^j} = 0
\]
Since the second expression of (45) is symmetric in $j$, $k$, the first condition of (46) is identically satisfied. To check the second, we differentiate the second expression of (45) with respect to $x^r$, substitute back from (45) and alternate $r$, $j$. After some straightforward computation, we find

\[
R_{i,rj,k}(x, f(x), f_1(x)) \overset{\text{def}}{=} \left[ \frac{\partial\Gamma^i_{jk}(f(x))}{\partial y^r} + \Gamma^i_{jb}(f(x))g^h_{rk}(f(x)) \right]_{[rj]} - g^a_j g^c_r \left[ \frac{\partial\Gamma^e_{ad}(x)}{\partial x^e} + \Gamma^e_{ab}(x)\Gamma^b_{cd}(x) \right]_{[ca]} f_i^j(x) = 0
\]

where $(x, f(x), f_1(x)) = (x^i, f^i(x), f^j_1(x))$ is a bisection of $H_1(\Gamma) = \mathcal{U}_1$, $g_1 = f_1^{-1}$. Note that $R_{i,rj,k}$ is a 2-form in the indices $r, j$ and a $(1, 1)$-tensor in the indices $i, k$

**Definition 15** $R$ is the nonlinear curvature of $H_2(\Gamma)$.

We define

\[
I_{i,rj,k}(\Gamma; x) \overset{\text{def}}{=} \left[ \frac{\partial\Gamma^i_{jk}(x)}{\partial x^r} + \Gamma^i_{ja}(x)\Gamma^a_{rk}(x) \right]_{[rj]}
\]

and call $I(\Gamma) = (I_{i,rj,k}(\Gamma; x))$ the integrability object of $\Gamma = (\Gamma^i_{jk}(x))$ (which is the curvature of the affine connection $\Gamma$). Like (18), we write (47) symbolically as

\[
R(x, y, f_1) = I(\Gamma; y) - (x, y, f_1)_* I(\Gamma; x) = 0
\]

and (49) expresses the invariance of $I(\Gamma)$ by the bisections (or equivalently 1-arrows) of $H_1(\Gamma)$. Note that $f_1 = (f^j_1)$ is a dependent variable in (18) whereas an independent variable in (49). We view $R$ as a map

\[
R : H_1(\Gamma) = \mathcal{U}_1 \rightarrow \wedge^2(T^*) \otimes T^* \otimes T
\]

We will see in the next section that $R$ is actually a map

\[
R : H_1(\Gamma) = \mathcal{U}_1 \rightarrow \wedge^2(T^*) \otimes J_1(T)
\]

but its projection on $T$ vanishes (see (27)) since $H_1(\Gamma) = \mathcal{U}_1$!

Therefore, (49) holds for all bisections if and only if all initial conditions (44) integrate locally and uniquely to local solutions of (42). Therefore we state

**Proposition 16** The following are equivalent.

1) $R = 0$ on $\mathcal{U}_1$

2) $H_2(\Gamma)$ is uniquely locally integrable

3) $H_2(\Gamma)$, which leaves $\Gamma$ invariant by definition, leaves also $I(\Gamma)$ invariant
Now a new phenomenon occurs due to do nontriviality of the stabilizers: Choosing \( x = y \) in \((49)\) and noting that \( f \) is arbitrary, it follows that the tensor \( I(\Gamma(x)) \) is fixed by \( \mathcal{H}_2(\Gamma)^\varepsilon \cong \mathcal{H}_1(\Gamma)^\varepsilon \cong GL(n, \mathbb{R}) \). This is possible if and only if \( I(\Gamma) = 0 \) on \( M \) and we obtain

**Proposition 17** The conditions of Proposition 16 are equivalent to \( I(\Gamma) = 0 \).

Therefore, an affine PHG is flat if and only if it is flat in the classical sense, i.e., "the torsionfree affine connection \( \Gamma \) has vanishing curvature". As a very intriguing fact, however, the linearization of \((50)\) will be the curvature of a connection on \( J_1(T) \rightarrow M \) and not \( T \rightarrow M \) where the affine connection \( \Gamma \) is defined!!

The linearization of \( \mathcal{H}_2(\Gamma) \) is now straightforward. A path in \( \mathcal{U}_2 \) (through the identity at \( \mathfrak{p} \in M \)) consists of a path \( x(t) \) in \( M \), \( t \in (-\epsilon, \epsilon) \), \( x(0) = \mathfrak{p} \) and a smooth choice of 2-arrows of \( \mathcal{U}_2 \) from \( \mathfrak{p} = x(0) \) to \( x(t) \) which is identity for \( t = 0 \). In coordinates, such a path is of the form \((\mathfrak{p}, x^i(t), f_j^i(t), f_{jk}^i(t))\), \( t \in (-\epsilon, \epsilon) \), \((x^i, x^i(0), f_j^i(0), f_{jk}^i(0)) = (\mathfrak{p}, \mathfrak{p}, \delta^i_j, 0)\) and the tangent of this path is defined by

\[
\frac{d}{dt}[(\mathfrak{p}, x^i(t), f_j^i(t), f_{jk}^i(t))]_{t=0} = (\xi^i(\mathfrak{p}), \xi_j^i(\mathfrak{p}), \xi_{jk}^i(\mathfrak{p}))
\]

(52)

giving all vectors in the fiber of \( J_2T \rightarrow M \) over \( \mathfrak{p} \). Choosing our paths in \( \mathcal{H}_2(\Gamma) \) and using \((44)\), we define the linearization \( H_2(\Gamma) \rightarrow M \) of the subgroupoid \( \mathcal{H}_2(\Gamma) \subset \mathcal{U}_2 \) which is a vector subbundle of \( J_2(T) \rightarrow M \) and its fiber over \( \mathfrak{p} \) is defined by (omitting \( \mathfrak{p} \) from our notation)

\[
\frac{\partial \xi^a_j}{\partial x^a_k}\xi^a_j + \Gamma^a_{ka}\xi^a_j + \Gamma^a_{ja}\xi^a_k - \xi^a_i\Gamma^a_{jk} = \xi^a_{jk}
\]

(53)

Clearly, \( H_0(\Gamma) = T \) is the tangent bundle of \( M \), \( H_1(\Gamma) = J_1(T) \) and \((53)\) defines a splitting \( \varepsilon : H_1(\Gamma) \rightarrow H_2(\Gamma) \) so that \( H_1(\Gamma) \cong H_2(\Gamma) \). Now \((53)\) defines a second order linear PDE on the vector fields \( \xi = (\xi^a_i(x)) \) on \( M \) which can be reduced to a first order system by introducing sections. The integrability conditions of this first order system are easily obtained from \((53)\) and are of the form

\[
\mathcal{R}(\xi_1)^i_{rjk} = 0
\]

(54)

We will not bother here to give the explicit form of \((54)\). Note that \( \mathcal{R}(\xi_1) \) depends linearly on the section \( \xi_1 \) of \( J_1(T) \rightarrow M \). Now \( \mathcal{R} \) is a 2-form on \( M \) which maps sections of \( J_1(T) \rightarrow M \) linearly to sections of \( T^* \otimes T \rightarrow M \) (actually to sections of \( J_1(T) \rightarrow M !) \), i.e.,

\[
\mathcal{R} : J_1(T) \longrightarrow \wedge^2(T^*) \otimes T^* \subset \wedge^2(T^*) \otimes J_1(T)
\]

(55)

and can be interpreted also as a 2-form on \( M \) with values in \( \text{Hom}(J_1, T^* \otimes T) \subset \text{Hom}(J_1, J_1) \). After some computation (which turns out to be redundant, see below) we deduce \( \mathcal{L}_{\varepsilon \xi_1} I(\Gamma) = \mathcal{R}(\xi_1) \), i.e.,
Therefore we deduce

\[ \mathcal{L}_{\xi_1} (I(\Gamma)^{i}_{rj,k}) = \mathcal{R}(\xi_1)^{i}_{rj,k} \]  \hspace{1cm} (56) \]

**Proposition 18** The following are equivalent.

1) \( \mathcal{R} = 0 \)
2) \( H_2(\Gamma) \to M \) is uniquely locally integrable
3) \( \mathcal{L}_{\xi_1} I(\Gamma) = 0 \) for all \( \xi_1 \in H_1(\Gamma) = J_1(T) \)

Now sit back a moment and look at what we did so far: We differentiate the nonlinear equations (45) with respect to \( t \) to get the linear system (53) and we differentiate (53) with respect to \( x \) to deduce the linear integrability conditions (54). However, these two operations commute: We can differentiate first (45) with respect to \( x \) and deduce the nonlinear integrability conditions (50) and then linearize (50) by differentiating with respect to \( t \). Hence we conclude that (55) must be the linearization of (50), that is (omitting the indices) we have

\[ \frac{d}{dt} \left[ \mathcal{R}(x, x + t\xi(0), \delta + t\xi(1)) \right]_{t=0} = \mathcal{R}(\xi_1) \]  \hspace{1cm} (57) \]

where we used the notation \( \xi_1 = (\xi(0), \xi(1)) \) (see (24)) for the section \( \xi_1 \) of \( J_1(T) = H_2(\Gamma) \to M \). Like (39), it follows from (57) that \( \mathcal{R} = 0 \) implies \( \mathcal{R} = 0 \) and the calculus fact suggests that the converse holds too (this can be proved rigorously by reducing the problem to absolute parallelism, see the next section).

Thus we state

**Proposition 19** (*Lie’s 3’rd Theorem*) The conditions of Propositions 16, 17 and 18 are equivalent.

It is worthwhile to emphasize again that our constructions are completely independent of connections (let alone torsionfree ones), but incorporate only splittings which are built into the definitions of the geometric structures and serve the purpose of reducing certain PDE’s to first order systems.

A last remark: As we have seen, what we did for affine PHG’s in this section follows word by word the prescription for absolute parallelism and the same will be true in the next section for Riemannian PHG’s. This fact suggests the existence of a unique principle that handles all cases at one stroke. This is indeed true and this process of reduction to parallelism will be explained in Section 6 below.

### 4 Riemannian PHG’s

Replacing \( Aff(\mathbb{R}^n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n \) with \( Iso(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n \), the same argument (see [Or1] for details) defines a geometric object \( g = (g_{ij}(x), \Gamma_{jk}^i(x)) \) where \( g = (g_{ij}(x)) \) is a Riemannian metric and \( \Gamma = (\Gamma_{jk}^i(x)) \) is a "torsionfree affine connection not necessarily the Levi-Civita connection". It is standard in
modern differential geometry to regard the components of \( g \) as separate objects whereas from the present standpoint \( g \) is a single object whose components are subject to the transformation rule

\[
g_{ab}(f(x)) \frac{\partial f^a(x)}{\partial x^j} \frac{\partial f^b(x)}{\partial x^k} = g_{jk}(x) \tag{58}
\]

\[
\Gamma^i_{ab}(f(x)) \frac{\partial f^a(x)}{\partial x^j} \frac{\partial f^b(x)}{\partial x^k} = \Gamma^i_{jk}(x) \frac{\partial f^i(x)}{\partial x^a} + \frac{\partial^2 f^a(x)}{\partial x^j \partial x^k} \tag{59}
\]

upon a coordinate change \((x) \rightarrow (f(x))\). Clearly (59) is identical with (42).

Using (58)+(59) we consider those 2-arrows \((\pi, \overline{g}, \overline{J}_j, \overline{J}_{jk})\) of \(U_2\) which preserve \(g = (g, \Gamma)\), that is

\[
g_{ab}(\overline{g}) \overline{J}_j \overline{J}_k = g_{jk}(\pi) \tag{60}
\]

\[
\Gamma^i_{ab}(\overline{g}) \overline{J}_j \overline{J}_k = \Gamma^i_{jk}(x) \overline{J}_a + \overline{J}_{jk} \tag{61}
\]

**Definition 20** The subgroupoid \(K_2(g) = K_2(g, \Gamma) \subset U_2\) defined by (60)+(61) is a Riemannian PHG on \(M\).

Now (61) defines a splitting

\[
\varepsilon : K_1(g) \longrightarrow K_2(g) \tag{62}
\]

which is the restriction of (44) to \(K_1(g)\). We can always find coordinates \((x)\) around \(\pi\) with the property \(g_{ij}(\pi) = \delta_{ij}\) and \(\Gamma^i_{jk}(\pi) = 0\). We call \((x)\) regular coordinates at \(\pi\). In such coordinates, the vertex group \(K_2(g)\pi \cong K_1(g)\pi\) is identified with the orthogonal group \(O(n)\). Clearly we can write (60)+(61) also in terms of bisections.

Substituting \(y^i = x^i + t\xi^i, f^i_j = \delta^i_j + t\xi^i_j, f^i_{jk} = t\xi^i_{jk}\) into (60)+(61) and differentiating with respect to \(t = 0\), we get the defining equations of the linearization \(K_2(g) \rightarrow M\) as a subbundle of \(J_2(T) \rightarrow M\) given by

\[
\frac{\partial g_{jk}}{\partial x^a} \xi^a_k + g_{ka} \xi^a_j + g_{ja} \xi^a_k = 0 \tag{63}
\]

\[
\frac{\partial \Gamma^i_{jk}}{\partial x^a} \xi^a_k + \Gamma^i_{ka} \xi^a_j + \Gamma^i_{ja} \xi^a_k - \xi^i_a \Gamma^a_{jk} = \xi^i_{jk} \tag{64}
\]

and (64) is clearly identical with (53). Note that for \(\xi^i = 0\), (63)+(64) become \(\xi^k_j = -\xi^k_j\) and \(\xi^i_{jk} = 0\) in regular coordinates.

Now (63)+(64) define a splitting

\[
\varepsilon : K_1(g) \longrightarrow K_2(g)
\]

and \(\varepsilon\) is the restriction of (44).
As it is clear by now, by introducing bisections, we can reduce the 2nd order nonlinear PDE (60)+(61) to an equivalent first order nonlinear mixed system and then compute the integrability conditions $\mathcal{R} = 0$. Similarly, by introducing sections, we can reduce the 2nd order linear PDE (63)+(64) to a first order linear mixed system and compute the integrability conditions $\mathcal{R} = 0$ keeping in mind that differentiations with respect to $t$ and $x$ commute so that $\mathcal{R}$ will be the linearization of $\mathcal{R}$.

To pinpoint the new phenomenon, we differentiate \( g_{ab}(f(x)) f^a_j(x) f^b_k(x) = g_{jk}(x) \) with respect to $x^r$ and substitute bisections. This gives

\[
\frac{\partial g_{ab}(f(x))}{\partial y^c} f^c_r(x) f^a_j(x) + g_{ab}(f(x)) f^a_j(x) f^b_k(x) + g_{ab}(f(x)) f^a_j(x) f^b_k(x) = \frac{\partial g_{jk}(x)}{\partial x^r} \tag{66}
\]

Now we substitute \( f^b_k(x) \) from (61) into (66) and alternate $r, j$ in (66). After some straightforward computation, we arrive at

\[
\left[ g^{ak} \left( \frac{\partial g_{ja}(y)}{\partial y^c} + g_{ja}(y) \Gamma^b_{ra}(y) \right) \right]_{[rj]} - \left[ g^{be} \left( \frac{\partial g_{ab}(x)}{\partial x^e} + g_{da}(x) \Gamma^a_{ja}(x) \right) \right]_{[cd]} f^e_r(x) f^d_j(x) f^k_i(x) = 0 \tag{67}
\]

We define $I_1(g; x)$ by

\[
I_k(r; g; x) \overset{def}{=} I_{rj}^k(g; x) \overset{def}{=} \left[ g^{ak} \left( \frac{\partial g_{ja}(x)}{\partial x^r} + g_{ja}(x) \Gamma^b_{ra}(x) \right) \right]_{[rj]} \tag{68}
\]

as the first component of $\mathcal{R}$ and rewrite (67) in the form

\[
I_1(g; x) - (x, f(x), f_1(x)) I_1(g; x) = 0 \tag{69}
\]

As a very crucial fact, if $\Gamma = (\Gamma^i_{jk})$ are the Christoffel symbols of the Levi-Civita connection $\nabla$, then

\[
\nabla_r g_{jk} = \frac{\partial g_{jk}}{\partial x^r} + g_{ja}(x) \Gamma^a_{jk}(x) + g_{ka}(x) \Gamma^a_{rj}(x) \tag{70}
\]

and using (68) and (70) we easily check that

\[
I^k(r; g; x) = \left[ g^{ab} \nabla_r g_{ja} \right]_{[rj]} \tag{71}
\]

Therefore $I_1(g; x) = 0$ if $\Gamma = (\Gamma^i_{jk})$ are the Christoffel symbols, i.e., this assumption makes the first component of $\mathcal{R}$ vanish!!
Therefore we obtain the map \((K(x, y, f) and it remains to clarify the meaning of \((x, f) section \((J the vector bundle \(\land 2 in the fiber of \(K 1 x is a 2-form at \((x, y, f) which we will rewrite here:

\[
\mathcal{R}^i_{r,j,k}(x, f(x), f_1(x)) \overset{\text{def}}{=} \left[ \frac{\partial \Gamma^i_{j,k}(f(x))}{\partial y^r} + \Gamma^i_{j,b}(f(x)) g^b_{i,k}(f(x)) \right]_{[rj]}
\]

\[
- g^a_j g^c_k \left[ \frac{\partial \Gamma^e_{a,d}(x)}{\partial x^a} + \Gamma^e_{a,b}(x) \Gamma^b_{c,d}(x) \right] f^i_{a}(x) = 0
\]

\[
I_2(g; x) \overset{\text{def}}{=} I^i_{r,j,k}(g; x) \overset{\text{def}}{=} \left[ \frac{\partial \Gamma^i_{j,k}(x)}{\partial x^r} + \Gamma^i_{j,a}(x) \Gamma^a_{c,k}(x) \right]_{[rj]}
\]

\[
\mathcal{R}(x, y, f_1) = I_2(g; y) - (x, y, f_1), I_2(g; y)) = 0
\]

Now (45) is uniquely locally integrable if and only if

\[
I_1(g; y) - (x, f(x), f_1(x)), I_1(g; x) \overset{\text{def}}{=} \mathcal{R}_1(x, f(x), f_1(x)) = 0
\]

\[
I_2(g; y) - (x, f(x), f_1(x)), I_2(g; x) \overset{\text{def}}{=} \mathcal{R}_2(x, f(x), f_1(x)) = 0
\]

We define

\[
\mathcal{R} \overset{\text{def}}{=} (\mathcal{R}_1, \mathcal{R}_2)
\]

and

\[
I(g; x) \overset{\text{def}}{=} (I_1(g), I_2(g)) = (I^i_{r,j,k}(g; x))
\]

To clarify the meaning of \(I(g; x)\), a section of \(J_1(T) \to M\) is of the form \((\xi^i, \xi^j)\) and using regular coordinates at \(x\), it is not difficult to check that \(I(g; x)\) is a 2-form at \(x\) with values in the fiber of \(J_1(T) \to M\) over \(x\) \((\text{but not necessarily in the fiber of } K_1(g) \to M\) over \(x\))! i.e., \(I(g; x)\) is a section of the vector bundle \(\land 2(T^*) \otimes J_1(T) \to M\).

Now we rewrite (75) in compact form as

\[
\mathcal{R}(x, y, f_1) \overset{\text{def}}{=} I(g; y) - (x, y, f_1), I(g; x)
\]

and it remains to clarify the meaning of \((x, y, f_1)\) in (78). We recall that the vector bundle \(J_1(T) \to M\) is associated with \(\mathcal{U}_2\) in the sense that a bisection \((x, f(x), f_1(x), f_2(x))\) of \(\mathcal{U}_2\) induces an isomorphism between the fibers \((x, f(x), f_1(x), f_2(x)), J_1(T)^x \to J_1(T)^y\). Now the bisection \((x, f(x), f_1(x))\) of \(K_1(g)\) lifts by (66) to a bisection of \(K_2(g)\) and induces the isomorphism \((x, y, f_1) : J_1(T)^x \to J_1(T)^y\) which restricts to \((x, y, f_1) : K_1(g) \to J_1(T)^y\). Therefore we obtain the map

\[
\mathcal{R} : K_1(g) \to \land 2(T^*) \otimes J_1(T)
\]
which gives the full integrability conditions of (58)+(59). We emphasize again that \( R \) does not necessarily take values in the smaller bundle \( \wedge^2(T^*) \otimes K_1(g) \) (therefore our last statement in the second paragraph on page 192 of [Or1] is incorrect). We observe the following remarkable fact: \( I_1(g) \) is a tensor in affine geometry and has an invariant meaning whereas \( I_2(g) \) is a component of the second order object \( R \) in coordinates and has no invariant meaning alone unless \( I_1(g) = 0 \)!!

Now the full integrability conditions of (63)+(64) are obtained by the linearization of (58)+(59) and is given by

\[
\mathfrak{R} : K_1(g) \to \wedge^2(T^*) \otimes J_1(T) \tag{80}
\]

where \( \mathfrak{R} \) is a section of the vector bundle \( \wedge^2(T^*) \otimes \text{Hom}(K_1(g), J_1(T)) \).

Therefore we state

**Proposition 21** (Lie’s 3’rd Theorem) The following are equivalent

1) \( R = 0 \) on \( K_1(g) \)
2) \( K_2(g) \), which leaves \( g \) invariant by definition, leaves also \( I(g) \) invariant
3) \( K_2(g) \) is uniquely locally integrable
4) \( R = 0 \)
5) \( L_{\xi_1} I(g) = 0 \) for all sections \( \xi_1 \) of \( K_1(g) \to M \)

By Proposition 21, we know that an affine PHG is flat if and only if the torsionfree affine connection \( \Gamma \) is flat. The answer to the following question will point at a new phenomenon for a Riemannian PHG.

**Q :** What is the meaning of the conditions of Proposition 21 in terms of the metric \( g = (g_{ij}) \)?

The very neat answer is given by

**Proposition 22** For a Riemannian PHG \( K_2(g) \), the following are equivalent

1) One of the conditions of Proposition 21 holds
2) \( \Gamma \) is the Levi-Civita connection and the metric \( g \) has constant curvature.

To prove Proposition 22, we first claim \( R = 0 \) implies that \( \Gamma = (\Gamma^i_{jk}) \) are the Christoffel symbols. Let \( y = f(x) \) be local diffeomorphism satisfying (58)+(59) with the initial conditions (60)+(61). Differentiating (58) with respect to \( x^r \) and evaluating at \( x = \mathfrak{r} \), we get

\[
\frac{\partial g_{ab}(y)}{\partial y^c} f^f_r f^j_s f^k_t \frac{\partial}{\partial x^r} + g_{ab}(y) f^a_r f^b_s f^c_t - g_{ab}(y) f^b_j f^a_k f^c_r = \frac{\partial g_{jk}(\mathfrak{r})}{\partial x^r} \tag{81}
\]

Shifting the indices \( r, j, k \) in (81) using the Gauss trick and defining

\[
\bar{\Gamma}^k_{ij}(\mathfrak{r}) \overset{def}{=} \frac{1}{2} g^{ka} \left( \frac{\partial g_{ja}(\mathfrak{r})}{\partial x^r} - \frac{\partial g_{jr}(\mathfrak{r})}{\partial x^a} + \frac{\partial g_{ar}(\mathfrak{r})}{\partial x^j} \right) \tag{82}
\]

= the Christoffel symbols of \( g \)
(81) becomes after some computation

\[ \tilde{\Gamma}_{ab}^c \bar{f}_j J_k = \tilde{\Gamma}_{jk}^a (\bar{\tau}) \bar{f}_j^i + \bar{f}_{jk} \]  

(83)

Therefore (60)+(61) and (60)+(83) have the same solutions for all initial conditions (60)+(61). Subtracting (83) from (61), setting \( y = x \) and using regular coordinates around \( \bar{\tau} \), we conclude that \( \mathcal{O} (n) \) fixes the \((1,2)\)-tensor \( \Gamma_{ijk} (x) \). Since \( 1 + 2 = 3 \) is odd, we conclude \( \Gamma_{ijk} (\bar{\tau}) + \tilde{\Gamma}_{ijk} (\bar{\tau}) = 0 \) as claimed (This derivation actually uses a much weaker assumption than \( R = 0 \), see the next section). Therefore \( I_1 (g) = 0 \) by (71) and we deduce from (73) that

\[ I_2 (g) = R = \text{the Riemann curvature tensor of } g \]  

(84)

Now it follows from Proposition 21 that \( \mathcal{R}_{kj,lm}^i \) and therefore \( \mathcal{R}_{kj,lm} \) is fixed pointwise by \( \mathcal{O} (n) \) and therefore can be expressed in terms of the components of \( g = (g_{ij}) \) since all tensor invariants of \( \mathcal{O} (n) \) are of this form. We define

\[ \mathcal{R}_{kj,lm} \overset{def}{=} g_{lk} g_{jm} - g_{lj} g_{km} \]  

(85)

and check the curvature identities

\[ \mathcal{R}_{kj,lm} = -\mathcal{R}_{jk,lm} \]
\[ \mathcal{R}_{kj,lm} = -\mathcal{R}_{kj,ml} \]  

(86)

Since the vector space of the \( \mathcal{O} (n) \)-invariant tensors \( \mathcal{R}_{kj,lm} \) satisfying (86) is spanned by \( \mathcal{R}_{kj,lm} \), we conclude

\[ I_{kj,lm} (g, \Gamma, \bar{\tau}) = \mathcal{R}_{kj,lm} (\bar{\tau}) = c(\bar{\tau}) \mathcal{R}_{kj,lm} = c(\bar{\tau}) (g_{lk} g_{jm} - g_{lj} g_{km}) \]  

(87)

for some scalar \( c(\bar{\tau}) \). Since \( \mathcal{R} \) and \( R \) are both \( \mathcal{K}_1 (g) \)-invariant on \( M \), it follows that \( c(\bar{\tau}) \) does not depend on \( \bar{\tau} \) and (87) becomes the constant curvature condition. Hence 1) implies 2). Conversely, if \( \Gamma = \tilde{\Gamma} \), then \( I_1 (g) = 0 \) by (71) and the above argument shows that 1) follows from 2), finishing the proof of Proposition 15.

We note here that the equivalence of 4) of Proposition 21 and 2) of Proposition 22 is shown also in [Po], 254-255 making heavy use of Spencer cohomology.

## 5 Connections

The equations (81) are obtained differentiating (60) and substituting jet variables. The equations (60)+(81) are called the (first) prolongation of (60) and denoted by \( pr (\mathcal{K}_1 (g)) \). The proof of the first part of Proposition 22 shows actually
\[ K_2(g) \subset pr(K_1(g)) \iff K_2(g) = pr(K_1(g)) \quad (88) \]

\[ \iff \Gamma = \text{Christoffel symbols} \]

\[ K_2(g) \] is called 1-flat (called 1-torsionfree in [Or1], unfortunately another misleading terminology) if (88) is satisfied. Of course, \( R = 0 \) implies 1-flatness of \( K_2(g) \). For an affine PHG \( H_2(\Gamma) \), we have

\[ \mathcal{H}_2(\Gamma) \subset pr(\mathcal{H}_1(\Gamma)) = pr(\mathcal{U}_1) = \mathcal{U}_2 \quad (89) \]

and therefore \( \mathcal{H}_2(\Gamma) \) is always 1-flat. Similarly, for an absolute parallelism \( \mathcal{N}_1(w) \), we have

\[ \mathcal{N}_1(w) \subset pr(\mathcal{N}_0(w)) = pr(M \times M) = \mathcal{U}_1 \quad (90) \]

and \( \mathcal{N}_1(w) \) is always 0-flat. Even though 1-flatness of \( K_2(g) \) is equivalent to the condition \( \Gamma = \text{Christoffel symbols} \) (the reason why (88) is called 1-torsionfreeness in [Or1]!!), we observe that 0-flatness of \( \mathcal{N}_1(w) \) has nothing to do with the torsions of the affine connections \( \nabla \) and \( \tilde{\nabla} \)!! Similarly, 1-flatness of \( \mathcal{H}_2(\Gamma) \) has nothing to do with the torsionfreeness of the affine connection \( \Gamma \).

\( i \)-flatness guarantees that the first operators in Spencer sequences take values in the "right spaces" as follows: Starting with \( \mathcal{N}_1(w) \), we have the first nonlinear Spencer sequence

\[ \mathcal{N}_0(w) = M \times M \xrightarrow{D_1} \Lambda^1(T^*) \otimes T \xrightarrow{D_2} \Lambda^2(T^*) \otimes T \quad (91) \]

and the first three terms of the linear Spencer sequence

\[ T \xrightarrow{D_1} \Lambda^1(T^*) \otimes T \xrightarrow{D_2} \Lambda^2(T^*) \otimes T \longrightarrow .... \quad (92) \]

Coordinate description of the operators in (91) are given in [Or1] and \( D_1 = \nabla \) defined by (41). Note that \( D_1 \neq \tilde{\nabla} = \mathcal{L} \). Now \( R \) and \( \mathfrak{R} \) drop out of the compositions \( D_2 \circ D_1 \) and \( D_2 \circ D_1 \) respectively and Proposition 21 asserts that (91) is locally exact if and only if (92) is locally exact. From our standpoint, the reason why \( D_1 \) turns out to be a connection is that \( \mathcal{N}_1(w) \) is 0-flat as will become clear below. Similar remarks apply to an affine PHG \( H_2(\Gamma) \) as follows: We have the nonlinear and linear Spencer sequences

\[ \mathcal{H}_2(\Gamma) = \mathcal{U}_1 \xrightarrow{D_1} \Lambda^1(T^*) \otimes H_1(\Gamma) \xrightarrow{D_2} \Lambda^2(T^*) \otimes H_1(\Gamma) \quad (93) \]

\[ H_1(\Gamma) \xrightarrow{D_1} \Lambda^1(T^*) \otimes H_1(\Gamma) \xrightarrow{D_2} \Lambda^2(T^*) \otimes H_1(\Gamma) \longrightarrow .... \quad (94) \]

and \( \mathfrak{R} \) is the curvature of the connection \( D_1 \) and is a section of \( \Lambda^2(T^*) \otimes Hom(H_1(\Gamma), H_1(\Gamma)) = \Lambda^2(T^*) \otimes Hom(J_1(T), J_1(T)) \). Note again that \( \mathfrak{R} \) is by no means the curvature of the torsionfree affine connection \( \Gamma \) defined on the tangent bundle \( T \rightarrow M \).
Something very interesting happens for the Riemannian PHG $K_2(g)$. Lifting a section of $K_1(g)$ to a section of $K_2(g)$ and then mapping by the Spencer operator $D_1$ (see the Appendix of [Or1] and [P1], [P2] for the definition of $D_1$), we get the map

$$K_1(g) \xrightarrow{D_1} \wedge^1(T^*) \otimes J_1(T)$$

but not necessarily a map

$$K_1(g) \xrightarrow{D_1} \wedge^1(T^*) \otimes K_1(g) \subset \wedge^1(T^*) \otimes J_1(T)$$

Linearizations of (95) and (96) are given by

$$K_1(g) \xrightarrow{D_1} \wedge^1(T^*) \otimes J_1(T)$$

$$K_1(g) \xrightarrow{D_1} \wedge^1(T^*) \otimes K_1(g)$$

We observe that $D_1$ in (97) is not a connection on $K_1(g)$!! Now it is easy to check that (88) holds if and only if (98) holds. With this assumption, $D_1$ becomes a connection and we get the first three terms of the linear Spencer sequence

$$K_1(g) \xrightarrow{D_1} \wedge^1(T^*) \otimes K_1(g) \xrightarrow{D_2} \wedge^2(T^*) \otimes K_1(g) \rightarrow ....$$

Now $R$ becomes the curvature of the connection $D_1$ and is a section of $\wedge^2(T^*) \otimes \text{Hom}(K_1(g), K_1(g))$. However, recall that (88) forces $I_1(g) = 0$ which makes $R$ a section of $\wedge^2(T^*) \otimes \text{Hom}(K_1(g), T^* \otimes T)$!

We will conclude with three remarks.

1) Since $R$ is a section of $\wedge^2(T^*) \otimes \text{Hom}(K_1(g), J_1(T))$ in general, one may object that $R$ is not intrinsic to $K_2(g)$ and (88) is necessary to make it intrinsic. However, $K_2(g)$ is by definition a subgoupid of $U_2$ and it is not possible to separate it from $U_2$ and define it as an abstract structure, i.e., $R$ is intrinsic to jets. Indeed, an abstract $G$-principal bundle is not always a $G$-structure defined as a reduction of the principal bundle $U_2^{\bullet} \rightarrow M$ (like $K_2(g)$). For instance, the concept of torsion emerges from connections on $G$-reductions and makes no sense (unless we introduce further structure) for connections on abstract $G$-principal bundles.

2) The above linear analysis becomes more intriguing (even catastrophic) in the nonlinear case when we attempt to interpret $R$ as the curvature of a torsion-free connection on the principal bundles $U_2^{\bullet} \rightarrow M$ (affine) and $K_2(g)^{\bullet} \rightarrow M$ (riemannian) and will be omitted here.

3) We recall that $L_{\xi_k}$ is not a connection for $k \geq 1$. This operator is used to localize the global sequences (92), (94), (99) and more generally the linear Spencer sequences arising from PHG’s. This localization generalizes the well known process of passing from de Rham cohomology to Lie algebra cohomology by localizing the forms in the sequence. The details of this construction for absolute parallelism are given in Chapter 11 of [Or1].
6 Reduction to parallelism

We start with an affine PHG $\mathcal{H}_2(\Gamma)$. We fix a basepoint $e \in M$ and consider the right principal bundle $\pi : \mathcal{H}_1(\Gamma)^{\bullet \bullet} = \mathcal{U}_1^{\bullet \bullet} \rightarrow M$ with the structure group $\mathcal{U}_1^{\bullet \bullet} \cong GL(n, \mathbb{R})$ whose fiber $\pi^{-1}(x)$ over $x$ is the set $\mathcal{U}_1^{x \cdot \bullet}$ of all 1-arrows from $e$ to $x$. Let $\bar{x} \in \mathcal{U}_1^{e \cdot \bullet}$, $\pi(\bar{x}) = x$ and consider the tangent space $T_{\bar{x}}(\mathcal{U}_1^{\bullet \bullet})$ of $\mathcal{U}_1^{\bullet \bullet}$ at $\bar{x}$ which is easily seen to be canonically isomorphic to the fiber $\mathcal{U}_1(x, \pi)$ over $x$, i.e.,

$$T_{\bar{x}}(\mathcal{U}_1^{\bullet \bullet}) \cong J_1(T)^x \quad \pi(\bar{x}) = x \quad (100)$$

The isomorphism in (100) is obtained by lifting the 1-parameter subgroup defined by $\xi_1 \in J_1(T)^x$ to $\mathcal{U}_1^{e \cdot \bullet}$ by composition at the target and differentiating at $t = 0$. In coordinates, it is given by

$$\xi^a(x) \frac{\partial}{\partial x^a} + \xi^b(x) f^b_c(x) \frac{\partial}{\partial f^c} \leftarrow (\xi^i(x), \xi^j(x)) \quad (101)$$

where $\bar{x} = (e^i, x^i, f^i_j) \in \mathcal{U}_1^{e \cdot \bullet}$. Note that the action of $\mathcal{U}_1^{e \cdot \bullet} \cong GL(n, \mathbb{R})$ on $\mathcal{U}_1^{\bullet \bullet}$ by composition at the source commutes with the above isomorphism. Therefore, some $\xi_1 \in J_1(T)^x$ defines tangent vectors at all points in the fiber $\pi^{-1}(x)$ which are invariant by the action of $\mathcal{U}_1^{e \cdot \bullet}$.

Now let $\bar{x}, \bar{y} \in \mathcal{U}_1^{\bullet \bullet}$, $\pi(\bar{x}) = x$, $\pi(\bar{y}) = y$ and consider the 1-arrow $\bar{y} \circ \bar{x}^{-1}$ on $M$ from $x$ to $y$. Therefore $\varepsilon(\bar{y} \circ \bar{x}^{-1})$ is a 2-arrow of $\mathcal{H}_2(\Gamma) \cong \mathcal{H}_1(\Gamma) = \mathcal{U}_1$ on $M$ from $x$ to $y$. Recalling that $H_1(T) = J_1(T)$ is associated with $\mathcal{U}_2$, the 2-arrow $\varepsilon(\bar{y} \circ \bar{x}^{-1})$ induces an isomorphism

$$\varepsilon(\bar{y} \circ \bar{x}^{-1}) : J_1(T)^x \rightarrow J_1(T)^y \quad (102)$$

In view of (100), (102) becomes

$$\varepsilon(\bar{y} \circ \bar{x}^{-1}) : T_{\bar{x}}(\mathcal{U}_1^{\bullet \bullet}) \rightarrow T_{\bar{y}}(\mathcal{U}_1^{\bullet \bullet}) \quad (103)$$

We conclude from (103) that $\varepsilon(\bar{y} \circ \bar{x}^{-1})$, which is a 2-arrow on $M$ from $x$ to $y$, is at the same time a 1-arrow on $\mathcal{U}_1^{\bullet \bullet}$ from $\bar{x}$ to $\bar{y}$!! We easily check that these 1-arrows are closed under composition and inversion and we conclude

**Proposition 23** The splitting (44) defines an absolute parallelism on $\mathcal{H}_1^{\bullet \bullet}(\Gamma) = \mathcal{U}_1^{\bullet \bullet}$.

Using the notation of Section 1, we will write the absolute parallelism in Proposition (23) in the form

$$\varepsilon^\alpha_\beta(\bar{x}, \bar{y}) \quad (104)$$

where the indices $\alpha, \beta$ refer to the components of the tangent vectors at $\bar{x}, \bar{y}$. It follows from (44) and (103) that $\varepsilon^\alpha_\beta(\bar{x}, \bar{y})$ depends on $\Gamma(x), \Gamma(y)$ but we will not bother here to write down the explicit form of (104). Now (104) explains
the reason for the strong analogy between our computations in Sections 2, 3 but much more importantly, it allows us to carry everything done for absolute parallelism in Part 2 of [Or1] over to the affine case (like Chern-Simons classes, homogeneous flow...etc).

When we attempt to generalize (104) to a Riemannian PHG, a problem arises: Now $K}_1(g)^{••} \not\subseteq \mathcal{U}_1^{••}$ and $J_1(T) \to M$, which is associated with $\mathcal{U}_2$, is clearly associated also with $K_2(g) \subseteq \mathcal{U}_2$, i.e., a 2-arrow $\varepsilon(f_x,y)$ of $\mathcal{K}_2(g)$ from $x$ to $y$ induces an isomorphism $\varepsilon(f_x,y) : J_1(T)^x \to J_1(T)^y$. However, this isomorphism may not restrict to an isomorphism $K_1(g)^x \to K_1(g)^y$, i.e., $K_1(g) \to M$ need not be associated with $K_2(g)$: All we can conclude is the injection $\varepsilon(f_x,y) : K_1(g)^x \to J_1(T)^y$. In fact, using the definitions, it is not difficult to show the following

**Proposition 24** For a Riemannian PHG the following are equivalent

1) $K_1(g) \to M$ is associated with $K_2(g)$

2) $K_2(g)$ is 1-flat, i.e., $K_2(g) \subseteq prK_1(g)$ or equivalently $\Gamma = (\Gamma_{jk})$ are the Christoffel symbols.

Fortunately, there is an easy way out of this difficulty.

**Definition 25** Let $L \subseteq S$ be a submanifold. Then $L$ is parallelizable relative to $S$ if for any $x, y \in L$, there is a unique 1-arrow $\varepsilon(x, y)$ of $S$ (not necessarily of $L$) such that these 1-arrows of $S$ are closed under composition and inversion.

We call $\varepsilon$ an $S$-splitting of $L$. If the 1-arrows $\varepsilon(x, y)$ of $S$ restrict to the 1-arrows of $L$, i.e., if the maps $\varepsilon(x, y) : T(S)^x \to T(S)^y$ restrict to $\varepsilon(x, y) : T(L)^x \to T(L)^y$ (our fiber notation $A^x$ forces us to denote the tangent space $T_x(S)$ by $T(S)^x$), then the $S$-splitting $\varepsilon$ on $L$ becomes a true splitting and $L$ becomes absolutely parallelizable.

Now the key fact is that if $L$ is parallelizable relative to $S$, then both sides of (14) are well defined and are contained in the tangent space $T(S)^{f(x)}$. Therefore, the equality in (14) makes perfect sense with the only difference that the index $i$ on the LHS of (14) refers to the tangent space of $L$ whereas on the RHS of (14) it refers to the tangent space of $S$. Now all the computations of Section 2 work through if we replace absolute parallelism with relative parallelism. For instance, the linear curvature $\mathcal{H}^{(i)}_{jk,l}$ becomes a 2-form on $L$ with values in $\text{Hom}(T(L), T(S))$...etc.

Our standard example of relative parallelism is $L = K_1(g)^{••} \subseteq \mathcal{U}_1^{••} = S$ and $\varepsilon$ is as in (104) which we may write now as $\varepsilon_0^{(β)}(\bar{x}, \bar{y})$. Note that our choice of $S$ is canonical. As in the affine case, we can now reduce the study of a Riemannian PHG to that of relative parallelism and furthermore carry all constructions of Part 2 of [Or1] over to the Riemannian case.

# 7 General PHG’s
When we attempt to define and study a general PHG, the first problem we face is the construction of the structure object. Once this is done, the rest follows as in the case of parallelizable, affine and Riemannian PHG’s along the same lines.

Let $G$ be a Lie group acting transitively on a smooth manifold $M$. Fixing a point $e \in M$, we can identify this action with the (say) left action of $G$ on $G/H$ where $H$ is the stabilizer at $e$. We fix some $p, q \in M$, $g \in G$ with $g(p) = q$ and define

$$G_k(p, q; g) \overset{\text{def}}{=} \{ f \in G \mid f(p) = q, \; j_k(f)^{p,q} = j_k(g)^{p,q}\} \quad (105)$$

Obviously $g \in G_k(p, q; g)$ for all $k \geq 0$ and $G_{k+1}(p, q; g) \subset G_k(p, q; g)$.

With some mild assumptions (like $G$ is connected and acts effectively) we can show (see [Or1]) the existence of a smallest integer $m$ with the property that $G_{m+1}(p, q; g) = \{g\}$ and furthermore $m$ is independent of $p, q$ and $g$. In particular, we may choose $p = q$ arbitrarily. In short, any transformation of $G$ is globally determined on $M$ by any of its $m$-arrows. It follows that above any $m$-arrow, there is a unique $(m+1)$-arrow. Indeed, since $g \in G$ is determined by $j_m(g)^{x,g(x)}$ for any $x \in M$, $j_{m+1}(g)^{x,g(x)} \overset{\text{def}}{=} \varepsilon(j_m(g)^{x,g(x)})$ is the unique $(m+1)$-arrow above $j_m(g)^{x,g(x)}$. In this way, we obtain a transitive subgroupoid $\mathcal{P}_{m+1} \subset U_{m+1}$ with the property $\mathcal{P}_{m+1} = \varepsilon(\mathcal{P}_m) \cong \mathcal{P}_m$. It turns out that $\mathcal{P}_{m+1}$ defines a first order nonlinear system on $M$ whose unique solutions are the restrictions of actions of the elements of $G$. Therefore, any $(m+1)$-arrow of $\mathcal{P}_{m+1}$ integrates uniquely to a global transformation of $G$. The PHG $\mathcal{P}_{m+1}$ is a flat model with $\mathcal{R} = 0$. The idea is now to forget the solution space $G$ and define $\mathcal{P}_{m+1}$ as an independent structure on a smooth $N$ with $\dim N = \dim M$ with the ”same stabilizer $H$” and with curvature $\mathcal{R}$ which will be the obstruction to the existence of (unique) local solutions. In order to do this, we need first to construct the structure object of our flat model $\mathcal{P}_{m+1}$ on $M$.

The above construction of the integer $m$ gives now the injective map

$$j_{m+1} : H \rightarrow j_{m+1}(h)^{e,e} \in U_{m+1} \cong G_{m+1}(n) \quad (106)$$

for $h \in H$ and $j_m(H) \cong j_{m+1}(H) = \varepsilon(j_m(H))$. It is easy to construct examples of action pairs $(G, M)$ with arbitrarily large $m$ using graded Lie algebras of vector fields (see [D] and the references therein) but it seems to us that the structure of such pairs is far from being well understood. Clearly, $m = 0$ for parallelism, $m = 1$ for affine and Riemannian PHG’s and $m = 2$ for projective and conformal PHG’s.

Now suppose that $G$ is an algebraic group and $H \subset G$ an algebraic subgroup. Recalling that $G_{m+1}(n)$ is an affine algebraic group, the first problem is the following question

**Q1:** Is $j_{m+1}(H) = \varepsilon(j_m(H)) \subset G_{m+1}(n)$ an algebraic subgroup?

Now consider the left coset space $G_{m+1}(n)/j_{m+1}(H)$ and the action of $G_{m+1}(n)$ on $G_{m+1}(n)/j_{m+1}(H)$. Assuming that the answer to **Q1** is affirmative, we now ask
Q2: Does there exist polynomial functions on $G_{m+1}(n)$ separating the left cosets of $j_{m+1}(H)$, i.e., a polynomial injective map

$$\Omega : G_{m+1}(n)/j_{m+1}(H) \longrightarrow \mathbb{R}^s \quad (107)$$

for some $s$.

The answers to Q1, Q2 are affirmative for parallelizable, affine and Riemannian PHG’s. In fact, the components of the structure objects $w = (w^i_j)$, $\Gamma = (\Gamma^i_{jk})$, $g = (g_{ij}, \Gamma^i_{jk})$ give the required imbedding (107) and $s$ is minimal in all these cases (see [Or1] for more details). We believe that the answers to Q1, Q2 are affirmative if $H \subset G$ are algebraic groups and therefore $G_{m+1}(n)/j_{m+1}(H)$ is always an affine variety.

Now assuming we found the polynomial function $\Omega = (\Omega^a)$ whose components parametrize the left coset space $G_{m+1}(n)/j_{m+1}(H)$, the action of $G_{m+1}(n)$ on $G_{m+1}(n)/j_{m+1}(H)$ gives the transformation rule of the components $(\Omega^a)$. Now it is easy to check that the $(m+1)$-arrows of our flat $\mathcal{P}_{m+1}$ on $M$ are those $(m+1)$-arrows in $\mathcal{U}_{m+1}$ that preserve the geometric object $\Omega$. Since $\mathcal{P}_{m+1}$ is flat, $\Omega$ is subject to some “integrability conditions”. Equivalently, the transformations of $G$ preserve both $\Omega$ and its integrability object $I(\Omega)$ and as a remarkable fact, $\Omega$ and $I(\Omega)$ now drop out of the all-important recursion formulas in the Fels-Olver theory of moving frames ([FO2]) that we will briefly mention in the next section.

Having the geometric object $\Omega = (\Omega^a)$ at our disposal, we now start anew on a smooth manifold $N$ with $\dim N = \dim M$ by postulating the existence of some $\Omega = (\hat{\Omega}^a(x))$ on $N$ whose components are subject to the above transformation rule and define $S_{m+1} \subset \mathcal{U}_{m+1}$ as the invariance subgroupoid of $\Omega$. It is now a straightforward matter to linearize $S_{m+1}$ and prove Lie’s 3’rd Theorem. Many highly nontrivial questions arise, but it seems pointless at this stage to elaborate further on the theory before making a detailed study of projective and conformal structures and understanding the advantages and disadvantages of the present theory of geometric structures.

# 8 Moving frames

In [FO1], [FO2] Olver and Fels introduced a new theoretical foundation of the moving frame method most closely associated with Elie Cartan which is very simple to apply and amazingly powerful. The wide range of new applications of this new approach (see the survey article [Ol2] and the references therein) underscores its significance. There is a remarkable relation between the present theory of geometric structures and the moving frame method. It turns out that the first is both a generalization and a specialization of the second: It is a generalization because it incorporates the concept of curvature which is not present in the second. It is a specialization because when $\mathcal{R} = 0$, the first considers only pseudogroups arising from transitive Lie group actions whereas the second applies to much more general pseudogroups, not even transitive.
To clarify this relation, we fix some integer $1 \leq k \leq n = \dim M$, $p \in M$ and define the fiber bundle $J_k(r, M) \to M$ whose fiber over $p \in M$ is the set consisting of the $k$-jets of (locally defined) maps $\mathbb{R}^r \to M$ with the source at the origin $o \in \mathbb{R}^r$, target at $p$ and have maximal rank $r$ at $o$ (hence near $o$). So we have the obvious projections of fiber bundles

\[ \ldots \to J_{k+1}(r, M) \to J_k(r, M) \to \ldots \to J_1(r, M) \to M \]  

(108)

We observe that the fiber of $J_k(r, M) \to M$ over $p$ can be identified with the $r$-dimensional (local) submanifolds of $M$ passing through $p$ modulo the equivalence relation defined by $k$-th order contact at $p$. Now the action of a transitive Lie group $G$ on $M$ lifts in the obvious way to $J_k(r, M)$ for all $k$ and "stabilizes" at some order $m$, i.e., $G$ acts freely on $J_k(r, M)$ for $k \geq m$ (For more general Lie group and pseudogroup actions the stabilization is a more delicate problem, see [AO], [OP]). This stabilization phenomenon which is first observed in [Ov] and corrected and generalized in [Ol1], is a fundamental fact and is the key to the theory of differential invariants by the moving frame method.

Once the action of $G$ becomes free on $J_m(r, M)$ with the orbit space $J_m(r, M)/\sim$ and the quotient map $\pi : J_m(r, M) \to J_m(r, M)/\sim$, the moving frame method proceeds by choosing a crosssection

\[ c : J_m(r, M)/\sim \to J_m(r, M) \]  

(109)

to the orbits of $G$, i.e., $c$ chooses from each orbit a single element in a smooth way. Since each orbit is in 1-1 correspondence with $G$ by freeness and $J_m(r, M)$ is disjoint union of orbits, (109) gives a map

\[ \tilde{c} : J_m(r, M) \to G \]  

(110)

defined as follows: If $x \in J_m(r, M)$, then $x$ belongs to the orbit $\pi(x)$ and $\tilde{c}(x)$ is the unique element in $G$ which maps $x$ to $c(\pi(x))$. It is easy to check that $\tilde{c}$ commutes with the action of $G$ (the choice of left/right moving frames arises if we work with an abstract Lie group and disappears for transformation groups). It is crucial to observe that all orbits are equal in the moving frame method, i.e., there is no canonical orbit and also there is no canonical crosssection (109). We also remark here that moving frame method is a local theory and all spaces, maps...etc above have only local meaning with the assumption that locally everything is "nice". However, a global version is proposed in [KL] for algebraic groups and their actions on algebraic manifolds. Once the moving frame (110) is constructed, the moving frame method allows us to compute everything in sight (and not in sight!) algorithmically and constructively and we refer to the survey article [Ol2] to give the reader an idea about the power and scope of this method.

Now we specialize to the case $r = \dim M = n$. Fixing some $e \in M$ arbitrarily, the fiber of $J_k(n, M) \to M$ over $p$ can now be identified with the fiber $U_{k}^{e,p}$ of the principal bundle $U_{k}^{e} \to M$ over $p$, i.e., $J_k(n, M) \cong U_{k}^{e,p}$ for all $k \geq 0$. 

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With this identification, the prolonged action of $G$ on $J_k(n, M)$ becomes simply the above mentioned composition with the arrows of $U_m^e$ at the target, i.e.,
$g \in G$ maps $j_k(f)^{e,p} \in U_k^e$ to $j_k(g)^{p,g(p)} \circ j_k(f)^{e,p} = j_k(g \circ f)^{p,g(p)}$. Now suppose we choose $m$ as in the first paragraph of the previous section. Then $g$ fixes $j_k(f)^{e,p} \iff j_k(f)^{e,p} = j_k(g)^{p,g(p)} \circ j_k(f)^{e,p} \iff g(p) = p$ and $j_k(g)^{p,p} = \text{Id} \iff g = \text{Id}$. Therefore $G$ acts freely on $U_m^e$ and the $m$ is the stabilization number in the moving frame method. We observe that in this special case $k = \dim M$, there is also a canonical orbit which is $P_m^e \subset U_m^e$ where $P_m \subset U_m$ is defined by the action of $G$ on $M$ as in the previous section. Therefore, we have the "moving frame" on this orbit

$$P_m^e \rightarrow G$$

since any element of $g \in G$ is uniquely determined by its $m$-arrow from $e$ to $g(e)$. Observe that we already have the "restriction of the moving frame $U_m^e \rightarrow G$ to the canonical orbit $P_m^e$" whereas the moving frame $U_m^e \rightarrow G$ itself is not in sight as we have not chosen any crossection yet !! Now let us take a closer look at the orbit space $U_m^e/\sim$. The stabilizer $H \subset G$ at $e$ acts on the fiber $U_m^e \cong G_m(n)$ on the left and the orbit space is the left coset space $U_m^e / j_m(H)$. However, since $G$ acts transitively on $M$, there is an obvious bijection

$$U_m^e / \sim \iff U_m^e / j_m(H) \cong G_m(n)/j_m(H) \quad (112)$$

because any equivalence class in $U_m^e$ has a representative in $U_m^e$ and two arrows are related in $U_m^e$ if and only if "their projections" are related in $U_m^e$. Therefore, the local crossection (109) amounts to choosing a local crossection

$$G_m(n)/j_m(H) \rightarrow G_m(n) \quad (113)$$

around the coset defined by $j_m(H)$. In particular, if $\dim G_m(n)/j_m(H) = t$ and $U \subset G_m(n)/j_m(H)$ is a neighborhood of the coset $j_m(H)$ in $G_m(n)/j_m(H)$, then a coordinate crossection amounts to introducing a coordinate patch

$$G_m(n)/j_m(H) \supseteq U \rightarrow \mathbb{R}^t \quad (114)$$

Now we compare (107) and (114). There are two main differences: 1) The order is $m + 1$ in (107) because our approach to geometric structures incorporates splittings and therefore the concept of curvature. 2) Since $G_m(n)/j_m(H)$ is a smooth manifold, we can always separate the orbits locally by smooth functions as in (114). However, to separate the orbits globally by polynomial functions as in (107), we need an affirmative answer to Q1, Q2.

A last remark: As we have seen, our approach to geometric structures arises from the special case of (108) for $r = \dim M$. The key property of the tower (108) is that any transitive Lie group action on $M$ prolongs to this tower and stabilizes at some order. There are many other such (nonlinear and linear) towers. Calling such a tower a jet-representation, the theory of PHG's can be developed in the framework of jet-representations, the PHG's themselves arising from the above canonical jet-representation with $r = \dim M ([Or3])$. 

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