Open constraint algebras

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Abstract. The standard BRST procedure for canonical quantization of a system with symmetry generated by a closed constraint algebra is extended to the case where the algebra is partly open. The method is illustrated by a specific topological model.

1. Introduction
It is well known that the symmetries of a classical action manifest themselves in the Hamiltonian setting as constraints on the canonical variables. In many cases the constraints form a closed algebra under the action of the Poisson brackets, but there are systems which possess symmetry but none the less have constraint algebras which do not close. This article is concerned with the modification to the BRST procedure for a class of such theories. The outline of the article is that section 2 below describes a particular symmetric model with these properties, section 3 reviews the standard BRST procedure in the canonical setting while section 4 describes the modification for the open algebras considered here. The final section describes the BRST quantization of the model constructed in section 2; the model is a topological model, and its partition function carries information about the equivariant cohomology of the space concerned under a circle action.

2. A symmetric model with open algebra
We begin by constructing a model whose fields are paths $x(t)$ in an $n$-dimensional Riemannian manifold $M$ with metric $g$. The manifold $M$ has the property that the group $U(1)$ acts on it isometrically, with corresponding Killing vector $X$. Expressed in terms of local coordinates $x^i, i = 1, \ldots, n$ the action of the theory is

$$S(x(\cdot)) = \int_0^T v X_i(x(t)) x^i(t) \, dt,$$

where $v$ is a constant. Working in Euclidean time, the canonical momentum conjugate to $x^i$ is

$$p_i = ivX_i,$$

so that there are $n$ first class constraints

$$T_i \equiv p_i - ivX_i = 0.$$
However a straightforward calculation shows that
\[ \{T_i, T_j\} = i\nu D_i X_j, \]  
so that the Poisson bracket algebra of these constraints is not closed. In the following sections it is shown how such a system may be quantized.

3. Review of the canonical BRST procedure for closed algebras
Before describing the modified BRST procedure appropriate to constraint algebras such as that in the preceding section, the standard BRST procedure is first reviewed. This concerns a system whose phase space \( \mathcal{N} \) is a \( 2n \)-dimensional symplectic manifold with local coordinates \( p_i, x^i : i = 1, \ldots, n \) and standard symplectic form
\[ \omega = dp_i \wedge dx^i \]  
leading to canonical Poisson brackets
\[ \{ x^i, p_j \} = \delta^i_j. \]  
The symmetry of the system involves a group \( G \) which acts on the phase space \( \mathcal{N} \) in a manner which leaves the Poisson brackets fixed. (A gauge invariant Hamiltonian is of course required for the dynamics, in this section only the kinematics is considered.)

In this situation there is a map \( J \), known as the momentum map, which takes the Lie algebra \( \mathfrak{g} \) of \( G \) into the space \( F(\mathcal{N}) \) of smooth functions on \( \mathcal{N} \). The momentum map satisfies two important conditions,
\[ L_{\xi} f = \{ J_{\xi}, f \} \quad \text{and} \quad J_{\xi}(gy) = J_{\text{Ad}_g\xi}(y), \]  
where \( g \in G, \; \xi \in \mathfrak{g}, \; J_{\xi} = J(\xi) \) and \( \xi \) denotes the vector field on \( \mathcal{N} \) corresponding to the infinitesimal action of \( \xi \). These conditions mean that the action of the group on \( \mathcal{N} \) is induced infinitesimally by the action of functions by Poisson bracket. If we choose a basis
\[ \{ \xi_a | a = 1, \ldots, m \} \]  
of \( \mathfrak{g} \), where \( m \) is the dimension of \( G \), and denote the functions \( J(\xi_a) \) by \( J_a : a = 1, \ldots, m \) we make contact with the more familiar language of constraint functions. In particular, using (7), it can be shown that
\[ \{ J_a, J_b \} = C_{ac}^b J_c, \]  
where \( C_{ac}^b \) are structure constants of \( \mathfrak{g} \) in the basis \( \xi_a \), so that in this case the algebra of the constraints is closed. The constraint submanifold is defined to be the submanifold \( C \) of \( \mathcal{N} \) on which the conditions
\[ J_a(x, p) = 0, m = 1, \ldots, m \]  
hold. There is a dual approach to \( C \): if the transpose \( \phi \) of the momentum map is defined by
\[ \phi : \mathcal{N} \to \mathfrak{g}^*, \quad \langle \phi(y), \xi \rangle = J_{\xi}(y) \]  
then
\[ C = \phi^{-1}(0). \]  

A key result due to Marsden and Weinstein [1] establishes that the action of the symmetry group \( G \) preserves the constraint surface \( C \), and that the quotient space \( C/G \) obtained by dividing out by this group action on \( C \) is a symplectic manifold (and thus is even-dimensional with a Poisson bracket). This space, which is known as the reduced phase space, is the true
phase space of the theory; it has dimension \(2n - 2m\). The Poisson bracket obtained by the
Marsden Weinstein procedure is the same as the Dirac bracket; however it is defined without
reference to any gauge-fixing procedure. In many cases the true phase space is very complicated,
and quantization becomes difficult, particularly because the polarization necessary to define the
split between momentum and position is hard to construct. This problem is avoided by the use
of BRST cohomology, as will be described following Henneaux [2] and Kostant and Sternberg
[3]. In this approach the physical observables of the system are realised as cohomology classes
of an operator \(Q\) (satisfying \(Q^2 = 0\)) which acts on a phase space which is an extension of the
classical phase space by anticommuting variables known as ghosts.

The BRST cohomology of a system is built in two stages. First, introducing the exterior
algebra \(\Lambda(\mathfrak{g})\) over the Lie algebra \(\mathfrak{g}\) of \(G\), a superderivation \(\delta\) is defined with
\[
\delta : \Lambda^q(\mathfrak{g}) \otimes F(\mathcal{N}) \to \Lambda^{q-1}(\mathfrak{g}) \otimes F(\mathcal{N})
\]  
and
\[
\delta(\pi \otimes 1) = 1 \otimes J_\pi, \quad \delta(1 \otimes f) = 0.
\]
The operator \(\delta\) satisfies
\[
\delta^2 = 0,
\]
so that it has well-defined cohomology which we now calculate at degree zero. It follows from
the definition of \(\delta\) that \(\text{Ker}_0\delta\), the kernel of \(\delta\) at degree zero, satisfies
\[
\text{Ker}_0\delta = F(\mathcal{N})
\]
while \(\text{Im}_0\delta\), the image of \(\delta\) at degree zero, satisfies
\[
\text{Im}_0\delta = \delta(\mathfrak{g})F(\mathcal{N}) = \{\text{Functions which vanish on } C\}.
\]
As a result \(H^0_\delta\), the cohomology of \(\delta\) at degree zero, satisfies
\[
H^0_\delta = F(\mathcal{N})/\{\text{Functions which vanish on } C\}.
\]
This space can be identified with \(F(C)\), the set of functions on the constraint submanifold \(C\),
so that we have established that that
\[
F(C) \cong H^0_\delta.
\]
Next it is observed that \(\mathfrak{g}\), the Lie algebra of \(G\), acts on \(\Lambda(\mathfrak{g}) \otimes F(\mathcal{N})\), since it acts on \(F(\mathcal{N})\) by
(7) and on itself by the Lie bracket, so it is possible to define a second superderivation
\[
d : \Lambda^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}) \otimes F(\mathcal{N}) \to \Lambda^{p+1}(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}) \otimes F(\mathcal{N})
\]
by setting \(dk = \eta^a \xi_a k\) for \(k \in \Lambda(\mathfrak{g}) \otimes F(\mathcal{N})\) and letting \(d\) acts as exterior derivative in the normal
manner on \(\Lambda(\mathfrak{g}^*)\). (Here \(\mathfrak{g}^*\) denotes the dual of \(\mathfrak{g}\), which consists of left-invariant differential
forms on \(G\), and \(\{\eta^a\}\) is the basis of \(\mathfrak{g}^*\) dual to the basis \(\{\xi_a\}\) of \(\mathfrak{g}\), so that \(d\eta^a = \frac{1}{2} C^a_{bc} \eta^b \eta^c\).)
The operator \(d\) satisfies
\[
d^2 = 0
\]
so that it too has well-defined cohomology. Further it satisfies
\[
d\delta + \delta d = 0
\]
so that $H^0_d(H^0_d)$ is defined. The definition of $d$ leads immediately to the result that

$$\text{Ker}_d(H^0_d) = \{ g \text{ invariant elements of } (H^0_d) \} \quad \text{and} \quad \text{Im}_d(H^0_d) = 0$$

(21)

so that $H^0_d(H^0_d)$ is equal to the $g$ invariant elements of $F(C)$, that is, to $F(C/G)$ which is the space of observables on the reduced phase space. This double cohomology can be achieved by taking the cohomology of the single operator

$$D = d + (-1)^p \delta$$

(22)

which satisfies

$$D^2 = 0.$$  

(23)

Thus we have defined an operator on $\Lambda(g^*) \otimes \Lambda(g) \otimes F(N)$ whose cohomology at degree zero in $g^*$ and $g$ is the space of observables on the true phase space of the system.

The key feature which makes the quantization possible is that we can implement this operator by Poisson bracket action on an extended super phase space. To do this, we regard $\Lambda(g^*) \otimes \Lambda(g) \otimes F(N)$ as the space of functions on an extended super phase space of dimension $(2n, 2m)$ where the odd coordinates are $\eta^a, a = 1, \ldots, m$ corresponding to a basis of $g^*$ and $\pi_a, a = 1, \ldots, m$ corresponding to a basis of $g$. The symplectic form on this super phase space is

$$\omega = dp_i \wedge dx^i + d\pi_a \wedge d\eta^a$$

(24)

so that $\pi_a$ the canonical dual of $\eta^a$ and the additional Poisson bracket is

$$\{\eta^a, \pi_b\} = \delta^a_b.$$  

(25)

The $\eta^a$ are known as ghosts and the $\pi_b$ as ghost momenta.

Equipped with this symplectic supermanifold, $D$ is implemented by taking Poisson brackets with

$$Q = \eta^a J_a - \frac{1}{2} C^{ce}_{ab} \eta^a \eta^b \pi_c.$$  

(26)

Quantization is carried out by taking wave functions $\Psi(x, \eta)$ which are functions of the position variables in the original phase space together with the ghost variable; an appropriate split between position and momentum variables is straightforward, in contrast to the situation on the reduced phase space. When calculating partition functions by path integral methods, a gauge fixing mechanism ensures projection onto the BRST cohomology or ‘physical states’ [4]. Further details of this method may be found in the book of Henneaux and Teitelboim [5].

4. **Modified BRST in open case: general formalism**

In this section we consider the case of open constraint algebra which can arise when the group $G$ has abelian subgroup $H$ such that

$$g = h \oplus k$$

(27)

with $[h, k] \subset k$ where $h$ is the Lie algebra of $H$. It will be convenient to select a basis

$$\{\zeta_\alpha, \xi_r | \alpha = 1, \ldots, l; r = 1, \ldots, m - l\}$$

of $g$ such that $\{\zeta_\alpha | \alpha = 1, \ldots, l\}$ is a basis of $h$ and $\{\xi_r | r = 1, \ldots, m - l\}$ is a basis of $k$.

Suppose that we now consider constraints

$$T_\alpha = J_\alpha - \langle v, \zeta_\alpha \rangle, \quad T_r = J_r$$

(28)
where $v$ is a non-zero element of $h^*$, the dual of $h$. This corresponds to the constraint surface

$$C = \phi^{-1}(v), v \in h^*. \quad (29)$$

The algebra of these constraints does not close, and moreover the group $G$ does not act on the constraint surface $C$. However the theory does have redundant degrees of freedom, and one expects some form of reduced phase space and BRST procedure to be required.

The correct approach to this situation is to treat $v_\alpha = \langle v, \zeta_\alpha \rangle$ as variables and introduce conjugate momenta $u_\alpha$. This leads to further constraints

$$u_\alpha = 0, \alpha = 1, \ldots, l. \quad (30)$$

Once these constraints are recognised, it is apparent that that the conditions

$$J_\alpha - v_\alpha = 0 \quad (31)$$

are in fact gauge-fixing conditions for the symmetry generated by $u_\alpha, \alpha = 1, \ldots, l$. Thus, temporarily ignoring the $J_r$ constraints, we have the reduced phase space for $H$ symmetry generated by $u_\alpha$. This space will be denoted $N_H$.

If we now consider the constraint functions $J_r$, together with the secondary constraints $J_\alpha$ arising from the commutators of the $J_r$, we find that they generate a symplectic $G$ action on $N_H$. The next step is to further reduce the space $N_H$ by applying the constraints $J_r = 0$ and $J_\alpha = 0$ and factoring out by the corresponding $G$ action. This leads to a reduced phase space of dimension $2n + 2l - 2l - 2m = 2n - 2m$, which is the true phase space of the system.

Once again we have a complicated reduced phase space, and so we seek an appropriate BRST cohomology. If we had separate constraints $J_\alpha = 0$ and $v_\alpha = 0$ we would simply have a $G \times H$ symmetry with the corresponding BRST operator

$$Q = \eta^a J_a - \frac{1}{2} C_{ab}^c \eta^a \eta^b \pi_c + u^a \rho_\alpha, \quad (32)$$

where we have used $\rho_\alpha$ to denote the ghost momenta corresponding to the constraints $u_\alpha$. However, since in fact our primary constraints are $v_\alpha = J_\alpha$ (together with $J_r = 0$ and $u^\alpha = 0$) we must tie the two symmetries together. This done by recognising that $\pi_\alpha$ corresponds to the group action generated by $J_\alpha$ while $\rho_\alpha$ corresponds to the gauge fixing $v_\alpha$, and thus we must set $\pi_\alpha = \rho_\alpha$ (to match the constraints) and $J_\alpha - \eta^a C_{\alpha r}^s \pi_s = 0$ (for consistency).

The conditions are more readily implemented if we make the coordinate transformation

$$\eta^\alpha \rightarrow \eta^\alpha + \theta^\alpha, \quad (33)$$

where $\theta^\alpha$ are ghosts for the constraints $u_\alpha$, giving

$$Q = \eta^a J_a - \frac{1}{2} C_{ab}^c \eta^a \eta^b \pi_c + u^\alpha \rho_\alpha + u^\alpha \pi_\alpha + \theta^\alpha (J_\alpha - \eta^r C_{\alpha r}^s \pi_s) \quad (34)$$

with the condition $\pi_\alpha = \rho_\alpha$ replaced by

$$\rho_\alpha = 0. \quad (35)$$

As with the standard case, projection onto physical states can be achieved by a gauge-fixing mechanism.
5. Modified BRST: quantizing the model of Section 2

In this case the group $G$ is the diffeomorphism group of $\mathcal{M}$, which has an induced action on the phase space of the system which is the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$. The subgroup $H$ is $U(1)$, generated by $X$. The Lie algebra of the diffeomorphism group of $\mathcal{M}$ consists of vector fields on $\mathcal{M}$ and, glossing over some details, the $\mathfrak{t}, \mathfrak{h}$ split comes from taking vectors parallel and orthogonal to $X$. The condition $[\mathfrak{t}, \mathfrak{h}] \subset \mathfrak{t}$ required by (7) is satisfied because if $\xi^i \frac{\partial}{\partial x^i}$ is orthogonal to $X$, then so is $[\xi^i \frac{\partial}{\partial x^i}, X]$.

Suppose that $\{X, \xi_r, r = 1, \ldots, n-1\}$ is a basis of the tangent space at some point in $\mathcal{M}$ with the $\xi_r$ each orthogonal to $X$. Then the corresponding constraint functions are

$$ T_r = \xi^i p_i $$

and

$$ T_\alpha = X^i p_i - i\nu X^i X_i. $$

(Here the index $\alpha$ takes the single value 1 since $H$ is one-dimensional.) In this case it is actually easiest to work in a coordinate basis, which leads to the BRST operator

$$ Q = \eta^i p_i + u^\alpha \rho_\alpha $$

or, after making change of coordinate $\eta^i \rightarrow \eta^i + X^i \theta^\alpha$, and suppressing the indices on $u^\alpha$, $\rho_\alpha$ and $\theta^\alpha$,

$$ Q = \eta^i p_i + u \theta + u X^i \pi_i - \theta (X^i p_i - \eta^i \frac{\partial X^j}{\partial x^i} \pi_j). $$

(37)

The construction of the BRST Hamiltonian which follows from this BRST operator, including the gauge-fixing mechanism, has been described in [6], and is shown to lead to the supersymmetric model constructed by Witten [7] in relation to the equivariant cohomology of $\mathcal{M}$.

References

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