Probing the statistical properties of Bose-Einstein condensates with light

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I. INTRODUCTION

The mean field approach to describe properties of Bose–Einstein condensates (BEC) in weakly interacting atomic gases [1–4] has proved to be immensely successful [5]. However, quantum statistical properties of the condensate, and in particular higher order correlations in BEC have been a subject of considerable interest in the recent years. In the series of papers the fluctuations of the number of condensed atoms of the ideal Bose gas were calculated in micro- and canonical ensembles [6]. The corrections due to the weak interatomic interactions were considered by several authors, but different models give different results, so that the impact of collisions on the statistics remains unclear [7]. The possible reason for these difficulties is the ambiguity in defining condensed fraction. Similar ambiguities lead to difficulties in determinations of the modification of the critical temperature for weakly interacting gases [8].

While the mean number of condensed atoms $N_0$ as a function of temperature has been measured in several laboratories [9], from the experimental point of view very little is known about fluctuations of BEC. The main source of information concerning higher order correlations comes from the studies of the depletion of BEC due to inelastic two-body and three-body collisions [10]. In this way the mean value of $N_0^2$ and $N_0^3$ have been estimated [11]. The experimental result have unequivocally ruled out the thermal fluctuations in the condensate. Precision of those measurements is, however, unable to distinguish between sub- and super-Poissonian, but normal fluctuations.

In fact different models of the interacting atoms condensate coexist in literature. In particular, the classic Bogolyubov approximation [12] tends to favour the Poisson statistics of the condensate. It would be desirable to discriminate between the models with the help of a clean experiment.

This paper describes such a proposal. We propose to use a scattering of a short weak non-resonant laser pulse as a means of probing the BEC statistics. The problem of light scattering on the condensate has been thoroughly studied in the recent years [13]. Most of these papers, however, concerned the signatures of BEC transition in scattering, rather statistics of the condensate. The references most closely related to the present paper are papers by You et al. [14,15] on scattering of short laser pulse on bosonic and fermionic clouds. These papers use very similar approximations to solve time-dependence of the system, but both use grand canonical ensemble, and do not address the problem of the condensate fluctuations.

The paper is organized as follows. In section II we present the theoretical model, and solve it for the photon operators of the scattered field. In Section III we analyze the mean number of scattered photons, whereas in Section IV its variance in function of the assumed statistical properties of the condensate, and physical parameters such as temperature and scattering angle. We compare the result for the thermally fluctuating gas, for the coherent state, for the ideal Bose gas described by the microcanonical ensemble and for the nonfluctuating gas. While the mean number of scattered photons can only discriminate between the thermal and the considered non-thermal condensate states, the variance can further distinguish the coherent state from the last two models. In Section V we present some conclusions.

II. THE MODEL

Our method of exploring the statistical properties of the condensate is based on the scattering of the series of short light pulses. The fluctuations in a given condensate occur at the time scale given by the interatomic collisions, thus to probe the fluctuations the time delay between consecutive pulses should be of the order of milliseconds. The distribution of the number of photons scattered into a given solid angle should be measured. Out of the distribution of the number of scattered photons we may compute the mean and its variance. We assume that the pulse of light is weak and far detuned in order to avoid heating the condensate during the interaction. The pulse of light should be also sufficiently short in time. It should satisfy two conditions: 1. $\omega_{\text{max}} t \ll 1$ where $\hbar \omega_{\text{max}}$ is the energy of the highest occupied level in the trap, since we are going to ignore factors $e^{i\omega t}$, 2. $t \ll \gamma^{-1}$ or the lifetime of the transition, since we are going to ignore the spontaneous emission. Note, that the conditions for the proposed experiment are complemen-
tary to the well known method of phase contrast used for nondestructive imaging of the condensate \[14\]. In fact we are interested in the regime in which the (small) condensate does not lead to significant refraction, while the scattered light carries then more direct information about fluctuations.

The total Hamiltonian consists of the following parts
\[
\hat{H} = \hat{H}_a + \hat{H}_{al} + \hat{H}_{af} + \hat{H}_f, \tag{1}
\]
where \(\hat{H}_a\) is the atomic Hamiltonian. In the second quantization formalism it reads
\[
\hat{H}_a = \sum_{\vec{n}} \hbar \omega_{\vec{n}} \hat{g}_{\vec{n}}^\dagger \hat{g}_{\vec{n}} + \sum_{\vec{m}} \hbar \omega_{\vec{m}}^e + \omega_0 \hat{e}_{\vec{m}}^\dagger \hat{e}_{\vec{m}}, \tag{2}
\]
where \(\hbar \omega_{\vec{n}}^g\) and \(\hbar \omega_{\vec{m}}^e\) are the energies of the CM motion of atoms, in the ground and the excited electronic state respectively. The \(\omega_0\) is the atomic resonance frequency. The operators \(\hat{g}_{\vec{n}}\) (\(\hat{e}_{\vec{m}}\)) are annihilation operators of atoms in ground (excited) electronic state respectively. They fulfill bosonic commutation relations \(\{\hat{g}_{\vec{n}}, \hat{g}_{\vec{n}'}\} = \delta_{\vec{n}, \vec{n}'}\). We use vector indices \(\vec{n}\) and \(\vec{m}\) because we consider a three dimensional trap.

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The term \(\hat{H}_f\) in the Hamiltonian is the energy of the electromagnetic (e-m) field:
\[
\hat{H}_f = \sum_{\lambda} \int d^3 k \ h \omega_{k} \hat{a}_{k\lambda}^\dagger \hat{a}_{k\lambda}, \tag{3}
\]
where \(\hat{a}_{k\lambda}\) \((\hat{a}_{k\lambda}^\dagger)\) are annihilation and creation operators of photons with wave-vector \(\vec{k}\) and frequency \(\omega_{\vec{k}} = kc\) and polarization \(\lambda\) respectively.

The interaction of the atoms with the laser light is described by the \(\hat{H}_{al}\) term:
\[
\hat{H}_{al} = \frac{\hbar \Omega}{2} \sum_{\vec{n}, \vec{m}} \langle \vec{n}, g | e^{-i\vec{k}_{L} \cdot \vec{r}} | \vec{m}, e \rangle e^{i\omega_{L} t} \hat{g}_{\vec{n}}^\dagger \hat{g}_{\vec{m}} + \h.c. \tag{4}
\]
where \(\omega_{L}\) is the laser frequency, \(\Omega = 2\vec{E}_0 \vec{d}/\hbar\) is the Rabi frequency with the transition dipole moment \(\vec{d}\) and amplitude of the electric field of the laser \(\vec{E}\). In the above equation Franck-Condon factors \(\langle \vec{n}, g | e^{-i\vec{k}_{L} \vec{r}} | \vec{m}, e \rangle\) appear. They are proportional to the amplitude of the transition between two states of the trap induced by absorption or emission of a photon. The trapping potential felt by excited and ground-state atoms may in general be different, and therefore their eigenstates may also be different. The labels \(e\) and \(g\) distinguish these two sets of states. However, in later derivations we will assume the same potential for all atoms, and we will use notation \(\vec{n}\) for the single atom trap states.

The atom-field interaction part of the Hamiltonian has the form:
\[
\hat{H}_{af} = i \sum_{\lambda} \int d^3 k \ \sqrt{\frac{\hbar \omega_{\vec{k}}}{2\epsilon_0 (2\pi)^3}} (\vec{d} \cdot \vec{e}_{k\lambda}) \hat{a}_{k\lambda}^\dagger \times \\
\times \sum_{\vec{n}, \vec{m}} \langle \vec{n}, g | e^{-i\vec{k}_{L} \vec{r}} | \vec{m}, e \rangle \hat{g}_{\vec{n}}^\dagger \hat{e}_{\vec{m}} + \h.c. \tag{5}
\]

The Heisenberg equations of motion for the atomic operators \(\hat{g}_{\vec{n}}, \hat{e}_{\vec{m}}\) can be solved in the following approximation. The atoms are driven only by laser field dominating over vacuum modes. In our approach we neglect the back action of atoms on the driving light and the influence of the vacuum modes. This last approximation is just a neglect of the spontaneous emission. It is justified if the duration of the pulse is short compared to the spontaneous emission life time. The laser light, as one may notice from \(\hat{H}_{al}\), is treated classically. The equation of motion for atomic operators in the interaction picture with respect to \(\hat{H}_a\) are given by
\[
\dot{\hat{g}}_{\vec{n}}(t) = -i \frac{\Omega}{2} e^{i\Delta t} \sum_{\vec{m}} \hat{e}_{\vec{m}}(t) e^{i(\omega_{\vec{n}} - \omega_0)t} \langle \vec{n}, g | e^{-i\vec{k}_{L} \vec{r}} | \vec{m}, e \rangle \hat{e}_{\vec{m}}, \tag{6}
\]
\[
\dot{\hat{e}}_{\vec{m}}(t) = -i \frac{\Omega}{2} e^{-i\Delta t} \sum_{\vec{n}} \hat{g}_{\vec{n}}(t) e^{i(\omega_{\vec{n}} - \omega_0)t} \langle \vec{n}, g | e^{-i\vec{k}_{L} \vec{r}} | \vec{m}, e \rangle \hat{g}_{\vec{n}}, \tag{7}
\]
where \(\Delta = \omega_{L} - \omega_0\) is the detuning of the laser. Now the assumption of the short pulse can be invoked to approximate the factors: \(e^{i(\omega_{\vec{n}} - \omega_0)t} \approx 1\). Then, the system of equations (6)-(7) is easy to solve analytically. To this end we introduce the operator \(\tilde{E}_{\vec{n}}(t) = \sum_{\vec{m}} \langle \vec{n}, e | e^{i\vec{k}_{L} \vec{r}} | \vec{m}, e \rangle \hat{e}_{\vec{m}}(t)\). The system of infinty many coupled equations then reduces to a set with only pairwise coupling. In the next step, we insert the found solution into the evolution equation for operator \(\tilde{a}_{k\lambda}\). In our approach the atom charges and currents distribution is a source for the vacuum modes of the e-m field, while the emmitted photons have no influence on the evolution of atoms. In other words we neglect the interaction of scattered photons with other atoms. The equation of motion is solved by simple integration in time, which leads to
\[
\tilde{a}_{k\lambda}(t) = e^{-i\omega_{\vec{k}} t} \left[ \tilde{a}_{k\lambda} + \sqrt{\frac{\omega_{\vec{k}}}{2\epsilon_0 (2\pi)^3}} \left( \frac{\Omega}{\Omega'} \right)^2 \times e^{(i\omega_{\vec{k}} - \omega_{L})t/2} \left( f_1(t) \sum_{\vec{n}, \vec{m}} \langle \vec{n}, g | e^{i\vec{k}_{L} \vec{r}} | \vec{m}, e \rangle \hat{g}_{\vec{n}} \hat{e}_{\vec{m}} + \ldots \right) \right. \\
+ \left. f_2(t) \sum_{\vec{n}, \vec{m}} \langle \vec{n}, g | e^{i(2\vec{k}_{L} - \vec{k}) \vec{r}} | \vec{m}, g \rangle \hat{g}_{\vec{n}} \hat{g}_{\vec{m}} + \ldots \right], \tag{8}
\]
where \(\Omega' = \sqrt{\Delta^2 + \Omega^2}\) and \(f_1(t), f_2(t)\) are functions of time given in Appendix A. All the operators on the RHS of Eq. (8) without specified dependence on time are taken at the time \(t = 0\). We will use this convention in the next sections. The dots in the brackets denote the remaining terms proportional to \(\hat{g}_{\vec{n}}^\dagger \hat{g}_{\vec{m}}\) and \(\hat{e}_{\vec{n}}^\dagger \hat{e}_{\vec{m}}\). They do not give any contribution to the mean number and variance of photons.
III. MEAN NUMBER OF SCATTERED PHOTONS

The basic statistical quantity we can construct from the solution is the mean number of photons. We would like to know if it depends on the statistical properties of the condensate. In particular we are interested in its relation to the condensate fluctuations $\delta^2 N_0$. In an experiment the angular distribution of scattered photons may be measured by scattering the series of short pulses, then calculating the mean from a number of detected photons in every pulse at the given angle. The calculated value will be approximately the mean number of photons of given mode, summed over polarization and integrated over the frequencies:

$$\frac{dN_{ph}(\theta, \phi, t)}{d\Omega} = \sum_{\lambda} \int dk \, k^2 \langle \hat{a}_E^\dagger(t) \hat{a}_{E\lambda}(t) \rangle.$$  \hspace{1cm} (9)

The $k$ under the mean value on the RHS of Eq (9) has a direction given by $(\theta, \phi)$. As we show later the spectrum has two spectral components, and we do the integration in Eq. (9) for each component separately. Hence, the integration leads to the total intensity of a given peak in the spectrum. The mean value under the integral should be calculated by tracing with the particular density matrix. Since we work in the Heisenberg picture, this density matrix can be well described by a density matrix of an ideal gas in microcanonical, or alternatively canonical ensemble. Experimental conditions favour the microcanonical description of the condensate (energy and the number of particles conserved), therefore the density matrix $\rho_g(0)$ is determined by the microcanonical ensemble. At the initial moment the only occupied modes of the e-m field are those corresponding to the laser light. But the occupation of these particular modes does not affect the results of the measurements that are done for the nonzero angles. Therefore we put $\rho_f(0) = |0, 0, \ldots \rangle \langle 0, 0, \ldots |$ at the initial moment.

Substituting the annihilation and creation operators in Eq. (9) by explicit formula, after some calculations (see details in Appendix A) we obtain

$$\frac{dN_{ph}}{d\Omega} (\theta, \phi) = \frac{d^2 \Omega^2}{32\pi^2 c_0 \hbar c \Delta^2} \left( 1 - (\vec{n}_k \vec{n}_d)^2 \right) \omega^3 f(q),$$  \hspace{1cm} (10)

and

$$f(q) = \sum_{\vec{n}, \vec{n}'} \langle \hat{g}^\dagger_{\vec{n}'} \hat{g}_{\vec{n}} \hat{g}^\dagger_{\vec{n}'} \hat{g}_{\vec{n}} \rangle \langle \vec{n}' | e^{i\vec{q}\vec{n}} | \vec{n} \rangle \langle \vec{m} | e^{-i\vec{q}\vec{m}} | \vec{m} \rangle.$$  \hspace{1cm} (11)

where $\vec{n}_k$ is the unit vector in the direction $(\theta, \phi)$ and $\vec{n}_d$ is the unit vector in the direction of the dipole moment $\vec{d}$. The frequency $\omega$ is the frequency of the given component of the spectrum. Following the calculation in Appendix A, the spectrum consists of two peaks with frequencies: $\omega_0$ (inelastic component) and $\omega_L$ (elastic component).

For the numerical purposes our expressions (11) were modified. After some calculations the mean value involving the operators $\hat{g}$ may be replaced by the statistical quantities for the microcanonical ensemble (see Eq. (34) in Appendix B). The statistical moments can be expressed by the microcanonical partition function, and then calculated by means of the recurrence algorithms. In order to check if the influence of the fluctuations may be noticed in the measurements of scattered photons, we compare the microcanonical results to the scattering on the nonfluctuating condensate. Such state may be theoretically realized in the following way: the mean occupation number for each level is the same as for the microcanonical condensate, while the higher statistical moments are decorrelated

$$\langle N_i N_j \rangle = \langle N_i \rangle \langle N_j \rangle,$$  \hspace{1cm} (12)

where indices $i$ and $j$ label the states in the trap. From this property it directly follows that the condensate does not fluctuate: $\delta N_0 = 0$ in this particular state.

There are other possible statistical properties of the condensate. The time-honored Bogolyubov approximation, taken at its face-value, assumes that the condensate is in a coherent state. We therefore consider also the Poissonian distribution:

$$\langle N_i^2 \rangle = \langle N_i \rangle^2 + \langle N_i \rangle, \quad \langle N_i N_j \rangle = \langle N_i \rangle \langle N_j \rangle, \quad i \neq j.$$  \hspace{1cm} (13) \hspace{1cm} (14)

The assumption that excited states are also coherent may be questionable, but we do the calculations for low temperatures, where the statistical properties of the condensate are dominant. For completeness of our review of the

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1 We omit the vector indices because the mean occupation number depends only on the energy of each state and not on its spatial properties. Therefore the energy is sufficient to enumerate the states.
different states we included also the results for the condensate described by thermal state. Of course this statistics, implicit in the grand canonical ensemble has already been ruled-out by the experiment [1]. The statistical properties of this last state are imposed by

\begin{align}
\langle N_i^2 \rangle &= 2 \langle N_i \rangle^2 + \langle N_i \rangle, \\
\langle N_i N_j \rangle &= \langle N_i \rangle \langle N_j \rangle, \quad i \neq j.
\end{align}

The comparison of the scattering of photons on these different condensates consisting of 1000 atoms is presented in Fig.1. We plot here the elastic part of the scattering. The conclusions for the inelastic component are very similar. The quantity proportional to the mean number of photons is plotted as a function of the angle \( \theta \). This mean photon falls-off exponentially for large angles due to the presence of the, so called, Debye factor (see Appendix A). The results were computed for realistic parameters. The oscillator length \( \xi \) was calculated for the sodium atoms in the spherically symmetric trap with \( \omega = 400 \text{s}^{-1} \). The resonance frequency of the transition \( \omega_0 \) corresponds to the wavelength \( \lambda_0 = 800 \text{nm} \), and detuning \( \Delta = 1 \text{GHz} \). Analysis of Fig.1 leads to the conclusion, that only scattering on the "thermal" condensate may be clearly distinguished. There are, of course, differences between the scattering on the remaining three models, but they are small, and become even smaller if the number of atoms increases. Our main conclusion in this section is that the mean values do not allow to detect and measure the condensate fluctuations.

![FIG. 1. Angular distribution of the mean number of the scattered photons measured with respect to the direction of the incident laser pulse in the plane perpendicular to the polarization axis of the incident light. The elastic component of the spectrum is shown. Of four different density matrices described in the text only the result for the thermally fluctuating source may be distinguished. The mean number of photons is scaled by the corresponding results for a single atom.](image)

**IV. THE VARIANCE**

As we have seen above, the mean number of scattered photons is fairly insensitive to the statistics of the condensed. We have to look at higher moments of the photon statistics. Its variance is a suitable quantity. It may be experimentally measured. It is sufficient only to scatter a series of pulses and after that to calculate the variance of the number of photons detected during every pulse at given angle. Repeating the series of measurements for different angles leads to the angular dependence of the variance. The formula which corresponds to this measured quantity is the following

\[
\delta \left( \frac{dN_{\text{ph}}}{d\Omega} \right)(\theta, \phi, t) = \left[ \left\langle \sum_{\lambda} \int d\mathbf{k} k^2 \hat{a}_{\mathbf{k}\lambda}^\dagger(t) \hat{a}_{\mathbf{k}\lambda}(t) \right\rangle^2 \right] - \left( \left\langle \sum_{\lambda} \int d\mathbf{k} k^2 \hat{a}_{\mathbf{k}\lambda}^\dagger(t) \hat{a}_{\mathbf{k}\lambda}(t) \right\rangle \right)^2.
\]

The mean values under the integrals should be calculated as in the previous case by tracing with density matrix of the initial state. The integration over the frequency is performed for each constituent of the spectrum separately, because we are interested in checking the variance of the photons both in elastic and inelastic scattering. Substituting the solution (3) for the creation and annihilation operators in (14), after a tedious but straightforward calculations we obtain the following result

\[
\delta \left( \frac{dN_{\text{ph}}}{d\Omega} \right)(\theta, \phi, t) = \frac{d^2 \Omega^2 \omega^3}{32 \pi^2 \epsilon_0 \hbar c^3 \Delta^2} \left( 1 - (\bar{n}_k \bar{n}_d)^2 \right) \sqrt{g(q)},
\]

and

\[
g(q) = \sum_{\bar{n}, \bar{n}', \bar{m}, \bar{m}', \ell, \ell', j, j'} \beta_{\bar{n}, \bar{n}'}(q) \beta_{\bar{m}, \bar{m}'}(-q) \beta_{\ell, \ell'}(q) \beta_{j, j'}(-q)
\times \left( \langle \hat{g}_{\bar{n}}^\dagger \hat{g}_{\bar{n}'} \hat{g}_{\bar{m}}^\dagger \hat{g}_{\bar{m}'} \rangle \langle \hat{g}_{\ell}^\dagger \hat{g}_{\ell'} \hat{g}_{j}^\dagger \hat{g}_{j'} \rangle \right)
\times \left( \langle \hat{g}_{\bar{n}}^\dagger \hat{g}_{\bar{n}'} \hat{g}_{\bar{m}}^\dagger \hat{g}_{\bar{m}'} \rangle \langle \hat{g}_{\ell}^\dagger \hat{g}_{\ell'} \hat{g}_{j}^\dagger \hat{g}_{j'} \rangle \right).
\]

where we denote Frank-Condon coefficients by \( \beta_{\bar{n}, \bar{m}}(q) \equiv \langle \bar{n} | e^{i q \hat{p}} | \bar{m} \rangle \). The Eq. (18) is very similar to Eq. (14) for the mean value, and we use the same notation. The frequency \( \omega \) is, as previously, the frequency of the considered component of the spectrum. In order to perform the numerical calculations the mean values of the atomic operators in formula (14) are expressed by means of the different statistical moments of the occupation number of the trap states. Unfortunately even when we couple the annihilation operators \( \hat{g} \) with the creation operators \( \hat{g}^\dagger \) into pairs, we still have to sum over four indices. These indices are additionally the vector ones. Furthermore the number of possible combinations of these pairs is much larger than in the case of the four operators in Eq. (14) for the mean number. We have to use some approximation to be able to perform the summation for reasonable size condensates (\( \sim 1000 \) atoms). We may utilise the fact
that for BEC the number of atoms in the ground state is of the order of the total number of atoms, so we expand our formula \( \hat{g}_0 \) in the powers of \( \hat{g}_0 \). As it turns out, for 1000 atoms a sufficient approximation requires inclusion of fourth-, third- and second order terms in this expansion. In this case summation is performed over two different excited states, what is possible to realize.

In Fig.2 we present the variance of the scattered photons calculated on the condensates with different statistical properties, described in the previous section. The numerical computations were realized for the same physical parameters. Again we only present the results for the elastic component. For vertical axis we also choose the same units as in the previous picture. Now we are able to distinguish the scattering on the coherent state, what was not possible on the basis of mean numbers. The larger difference occures for the elastic scattering for small angles. The difference between scattering on the microcanonical and uncorrelated condensates is still very small. However, for small angles these curves are different. In this regime our approximation is not valid for the microcanonical condensate. The expansion of the variance in the powers of \( q \), shows that the result behaves as \( \theta^2 \) and the curve is indistinguishable from the curve for uncorrelated condensate. As we expected, the scattering on "thermal" condensate leads to completely different results. In fact while microcanonical and uncorrelated states are even harder to distinguish for larger number of atoms, the distance to the results for the coherent states grows linearly with \( N \).

\[ V. CONCLUSIONS \]

The statistical properties of the Bose-Einstein condensate are of great interest. Its measurements would advance the theory of the coherence of existing atom lasers.

In the present paper we propose to use the scattering of short laser pulses for the study of fluctuations. One needs higher moments of the distribution of the number of the scattered photons to have a sufficiently subtle tool.

The main open question is the influence of weak interactions on the statistics. While the theory remains muddled, we know that the main impact of interactions is the emergence of two length-scales: that of the size of the condensate, broadened by the repulsive forces and that of the size of thermal cloud, expelled from the center of the trap by the condensate. It means that we would have two different Delye factors damping the scattering cross-section at large angles. This way by changing the angle one could, perhaps, gain access to the fluctuations of the condensate and fluctuations of the thermal cloud separately. They are lumped together for the ideal Bose gas studied in the present paper.

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\[ APPENDIX A: THE PHOTON SPECTRUM \]

The function of time which appears in the solution \( \hat{g}_0 \) are given by

\[
\begin{align*}
    f_1(t) &= \frac{\Delta}{\omega_k - \omega_L} \sin \left( \frac{\omega_k - \omega_L}{2} t \right) - \frac{\Omega' + \Delta}{4\omega_{1k}} e^{-\frac{\omega_1'}{2}} \sin(\omega_{1k}t) \\
    &\quad + \frac{\Omega' - \Delta}{4\omega_{2k}} e^{-\frac{\omega_4'}{2}} \sin(\omega_{2k}t), \\
    f_2(t) &= \frac{\Omega}{\omega_k - \omega_L} \sin \left( \frac{\omega_k - \omega_L}{2} t \right) - \frac{\Omega}{4\omega_{1k}} e^{-\frac{\omega_1'}{2}} \sin(\omega_{1k}t) \\
    &\quad - \frac{\Omega}{4\omega_{2k}} e^{-\frac{\omega_4'}{2}} \sin(\omega_{2k}t),
\end{align*}
\]

where we use the following notation: \( \Omega' = \sqrt{\Delta^2 + \Omega^2} \), \( \omega_{1k} = (\omega_k - \omega_L + \Omega')/2 \), \( \omega_{2k} = (\omega_k - \omega_L - \Omega')/2 \). Our laser pulse is weak and far detuned and we may simplify the equations \( A1 \)-\( A2 \). For these assumptions: \( \Delta \gg \Omega \) and \( \Omega' \approx \Delta \) for positive detuning. The functions of time \( f_1 \) and \( f_2 \) take the form
\[
f_1(t) = \frac{\Delta}{\omega_{kl}} \sin\left(\frac{\omega_{kl} t}{2}\right) - \frac{\Delta e^{\pm \Delta t/2}}{\omega_0} \sin\left(\frac{\omega_{k0} t}{2}\right),
\]
\[
f_2(t) = \frac{\Omega}{\omega_{kl}} \sin\left(\frac{\omega_{kl} t}{2}\right) - \frac{\Omega e^{\pm \Delta t/2}}{2 \omega_0} \sin\left(\frac{\omega_{k0} t}{2}\right)
- \frac{\Omega e^{-\Delta t/2}}{2 (\omega_k + \omega_0 - 2\omega_L t)} \sin\left(\frac{\omega_k + \omega_0 - 2\omega_L t}{2}\right),
\]
where \(\omega_{kl} = \omega_k - \omega_L\) and \(\omega_{k0} = \omega_k - \omega_0\). Calculation of the mean number of photon in a particular mode is done by taking solution \(f\) and then calculating the mean value. This mean value should be computed with the initial density matrix, whose properties are discussed in the text. After straightforward calculations we get
\[
\langle \hat{a}_{k\lambda}^\dagger (t) \hat{a}_{k\lambda}(t) \rangle = \frac{\omega \Omega^2 (\hat{d} \cdot \hat{e}_{k\lambda})^2}{16 \pi^3 \epsilon_0 \hbar \Delta^4} \times \left( f(\hat{q}) |f_1(t)|^2 + N |f_2(t)|^2 \right),
\]
where the function \(f(\hat{q})\) is defined in Eq. (11). This function is of the order of \(N^2\) what may be easily demonstrated. Additionally the modulus of the function \(|f_2|\) is much smaller than \(|f_1|\). Hence we are justified to neglect the contribution of function \(|f_2|\) to our formula. In order to compute mean number of photons we have to sum over polarizations and integrate over frequencies. The first task may be performed with the help of well-known identity: \(\sum_{\lambda} (\hat{n}_d \cdot \hat{e}_{k\lambda})^2 = (1 - \langle \hat{n}_k \hat{n}_d \rangle)^2\). The information about the spectrum is hidden in the function \(f_1(t)\). The integration over the frequencies may be easily performed if we assume the spectrum consist of very narrow lines. Since the pulse duration is long in comparison to the inverse of the optical frequencies: \(t \gg \omega_L^{-1}, \omega_0^{-1}\), we may approximate the spectrum to the form:
\[
|f_1(t)|^2 \approx \frac{\pi \Delta^2}{2} \left[ \delta(\omega_k - \omega_0) + \delta(\omega_k - \omega_L) \right].
\]
As one can see the spectrum consists of two peaks centered on the atomic and laser frequencies. The intensities of the two are the same, which is the result of the absence of the spontaneous emission during the short pulse. Now the result (14) is evident. The derivation of the Eq. (13) describing the variance is done in the similar way as for the mean.

**APPENDIX B: NUMERICAL STRATEGIES**

The basic numerical task is to compute the functions \(f(\hat{q})\) and \(g(\hat{q})\) which enter into equations for the mean value (10) and the variance (13). For this purpose we express the mean value in the functions \(f(\hat{q})\) and \(g(\hat{q})\) by mean occupation numbers and higher moments of trap energy levels. We explain how it should be done for the function \(f(\hat{q})\), the derivation in the case of variance is similar.

The summation in the function \(f(\hat{q})\)
\[
f(\hat{q}) = \sum_{\tilde{\alpha},\tilde{\alpha}',\tilde{\alpha},\tilde{\alpha}'} \langle \hat{g}\tilde{\alpha}^\dagger \hat{g}\tilde{\alpha}' \hat{g}\tilde{\alpha} \hat{g}\tilde{\alpha}' \rangle \beta_{\tilde{\alpha},\tilde{\alpha}'}(\hat{q}) \beta_{\tilde{\alpha},\tilde{\alpha}'}(-\hat{q}),
\]
is performed over four vector indices. We may exclude two of them because the mean of the four operators \(\langle \hat{g}\tilde{\alpha}^\dagger \hat{g}\tilde{\alpha}' \hat{g}\tilde{\alpha} \hat{g}\tilde{\alpha}' \rangle\) is nonzero only if the operators couple into pairs. For four operators there are two possibilities of pairing, and we obtain
\[
f(\hat{q}) = \sum_{\tilde{\alpha},\tilde{\alpha},\tilde{\alpha}',\tilde{\alpha}''} \langle \hat{N}_{\tilde{\alpha}} \hat{N}_{\tilde{\alpha}'} \rangle \beta_{\tilde{\alpha},\tilde{\alpha}'}(\hat{q}) \beta_{\tilde{\alpha},\tilde{\alpha}'}(-\hat{q}) + \beta_{\tilde{\alpha},\tilde{\alpha}'}(\hat{q})^2
+ \sum_{\tilde{\alpha}} \langle \hat{N}_{\tilde{\alpha}}^2 \rangle \beta_{\tilde{\alpha},\tilde{\alpha}}(\hat{q}) \beta_{\tilde{\alpha},\tilde{\alpha}}(-\hat{q}) + N,
\]
where \(\hat{N}_{\tilde{\alpha}} \equiv \hat{g}\tilde{\alpha}^\dagger \hat{g}\tilde{\alpha}\). In derivation of (B2) we use the identity \(\sum_{\tilde{\alpha}} \beta_{\tilde{\alpha},\tilde{\alpha}}(\hat{q}) \beta_{\tilde{\alpha},\tilde{\alpha}}(-\hat{q}) = 2 \delta_{\tilde{\alpha},\tilde{\alpha}'}\) and this results directly from the completeness of the states. The Franck-Condon coefficients for the harmonic trap may be calculated analytically. The result for three dimensional trap is simply the product of the 1D Franck-Condon coefficients
\[
\langle n|m e^{i\hat{q} \hat{r}} |m \rangle = \sqrt{n!} \frac{\pi \lambda}{m!} e^{-\frac{\xi^2}{4} \lambda^2} \left( \frac{q^2 \xi^2}{2} \right)^{m-n},
\]
where \(L^m_n(x)\) denotes Laguerre polynomial and \(\xi\) is the oscillator length. For simplicity we assume our trap to be spherically symmetric, therefore we may freely choose the orientation of the system of coordinates for the states in the trap. We choose the z-axis to be in the direction of the \(\hat{q}\) vector. For such system of coordinates the three dimensional Franck-Condon coefficients become \(\langle \hat{m}|e^{i\hat{q} \hat{r}} |\hat{m} \rangle = \langle n_x|m_x \rangle \delta_{n_x,m_x} \delta_{n_y,m_y} \delta_{n_z,m_z}\). Since the mean values of the different combinations of \(\hat{N}_{\tilde{\alpha}}\) operator depends only on the energies of the states, and not on the distribution of the quantum numbers, we may split the summation into two steps: the sum over energies and the sum over degeneracies. The last one may be performed for all terms, and the result is a combination of Laguerre polynomials. For the microcanonical ensemble
\[
f(\hat{q}) = e^{-\eta} \left[ \langle \hat{N}_E^2 \rangle - \langle \hat{N}_0 \rangle + 2 \sum_{E < \hat{E}'} \langle \hat{N}_E \hat{N}_{E'} \rangle (L^2_{E'}(\eta) L^{2}_{E}(\eta))
+ \sum_{E > \hat{E}} \langle \hat{N}_E \rangle (L^2_{E}(\eta)) \right]
+ \sum_{E > \hat{E}} \left( \langle \hat{N}_E^2 \rangle - \langle \hat{N}_E \hat{N}_{E'} \rangle \right) S^2_{E}(\eta) \right] + N,
\]
where \(\eta = q^2 \xi^2 / 2\) (the exponent in front of the square bracket is called the Debye factor) and the mean \(\langle \hat{N}_E \hat{N}_{E'} \rangle\) is the correlation between different levels with the same energies. In the formula (B4) \(S^2_{m}(x)\) denotes the sum including the Laguerre polynomials:
\[
S^2_{m}(x) = \sum_{n=0}^{m} \frac{n!}{(n+a)!} L^a_{n}(x)^2 (m+1-n).
\]
The sum \((B6)\) may be done analytically with use of the summation formula for Laguerre polynomials 8.974.1 [17]. The calculation leads to

\[
S_m^a(x) = \frac{(m+2)!}{(m+a)!} L_m^a L_{m-1}^a L_{m-1}^a - \frac{(m+1)!}{(m+a-1)!} \times \left[ \frac{1}{6} L_m^{a+2} L_m^a - \frac{1}{6} L_m^{a+2} L_m^a - \frac{1}{2} L_m^{a+2} L_m^a \right] + \frac{1}{2} L_m^{a+1} L_m^a \right] - \frac{(m+1)!}{(m+a)!} \left( L_m^a \right)^2,
\]

where we use shorthand notation \(L_m^a(x) = 0\) for \(m < 0\).

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[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995).

[2] K. B. Davis, M. -O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).

[3] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995); ibid 79, 1170 (1997).

[4] Extensive list of references can be found at the BEC Web page: [http://amo.phy.gasou.edu/bec.html/bibliography.html](http://amo.phy.gasou.edu/bec.html/bibliography.html).

[5] See, for instance, F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).

[6] H. D. Politzer, Phys. Rev. A 54, 5048 (1996); S. Grossman and M. Holthaus, Phys. Rev. E54, 3495 (1996); M. Gajda and K. Rzążewski, Phys. Rev. Lett. 78, 2686 (1997); M. Wilkens and C. Weiss, J. Mod. Opt. 44, 1801 (1997); P. Navez, D. Bitouk, M. Gajda, Z. Idziaszek, and K. Rzążewski, Phys. Rev. Lett. 79, 1789 (1997); S. Grossman and M. Holthaus, Opt. Express 1, 262 (1997); C. Weiss and M. Wilkens, Opt. Express 1, 272 (1997); M. Holthaus, E. Kalinowski, and K. Kirsten, Ann. Phys. (N. Y.) 270, 198 (1998).

[7] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 80, 5040 (1998); Z. Idziaszek, M. Gajda, P. Navez, M. Wilkens, and K. Rzążewski, Phys. Rev. Lett. 82, 4376 (1999).

[8] see for instance M. Holzmann and W. Krauth, Phys. Rev. Lett. 83, 2687 (1999); K. Huang, Phys. Rev. Lett. 83, 3770 (1999).

[9] J. R. Ensher, D. S. Jin, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 77, 4984 (1996); M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, and W. Ketterle, Phys. Rev. Lett. 77, 416 (1996).

[10] Yu. Kagan, B. V. Svistunov, and G. V. Shlyapnikov, JETP Lett. 42, 209 (1985).

[11] E. A. Burt, R. W. Ghrist, C. J. Myatt, M. J. Holland, E. A. Cornell, and C. E. Wieman, Phys. Rev. Lett. 79, 337 (1997); D. M. Stamper-Kurn, H.-J. Miesner, S. Inouye, M. R. Andrews, and W. Ketterle, Phys. Rev. Lett. 81, 500 (1998); similar methods has been used to detect the phase transition in 2D spin polarized hydrogen gas: A. I. Safonov, S. A. Vasilyev, I. S. Yasnikov, I. I. Lukashevich, and S. Jaakkola, Phys. Rev. Lett. 81, 4545 (1998).

[12] N. N. Bogolyubov, J. Phys. (Moscow) 11, 231 (1947).

[13] B. Svistunov and G. Shlyapnikov, Zh. Eksp. Teor. Fiz. 97, 821 (1990) [Sov. Phys. JETP 70, 460 (1990)]; ibid. 98, 129 (1990) [JETP 71, 71 (1990)]; H. D. Politzer, Phys. Rev. A 43, 6444 (1991); J. Javanainen, Phys. Rev. Lett. 72, 2375 (1994); O. Morice, Y. Castin and J. Dalibard, Phys. Rev. A 51, 3896 (1995); W. Zhang and D. F. Walls, ibid. 49, 3799 (1994); J. Javanainen and J. Ruostekoski, Phys. Rev. A 52, 3033 (1995).

[14] L. You, M. Lewenstein and J. Cooper, Phys. Rev. A 51, 4712 (1995).

[15] T. Wong, L. You, M. Lewenstein, quant-ph/9910054.

[16] M. Andrews, M.-O. Mewes, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science 273, 84 (1996).

[17] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals series and products, (Academic Press - New York and London - 1965).