Generalized Yang-Baxter Equation

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Abstract
A generalization of the Yang-Baxter equation is proposed. It enables to construct integrable two-dimensional lattice models with commuting two-layer transfer matrices, while single-layer ones are not necessarily commutative. Explicit solutions to the generalized equations are found. They are related with Botzmann weights of the $sl(3)$ chiral Potts model.

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1. Introduction

As is known the Yang-Baxter equation (YBE) ensures commutativity of one-layer transfer matrices (TM’s) in two-dimensional vertex lattice models [1]. Obviously, two and more-layer TM’s commute among themselves in this case as well. Is it possible to have commuting family of e. g. two-layer TM’s while one-layer ones being non-commutative? Clearly, a positive answer to this question would provide us with a wider class of solvable lattice models. In this paper we propose a generalized YBE, which realizes such a possibility (provided the ”cross” matrices of Boltzmann weights are non-degenerate). The generalized equation has the following form:

\[ \sum_{k_1, k_2, k_3} S(p, q)_{i_1, i_2}^{k_1, k_2} S(p, r)_{j_1, j_3}^{k_1, k_3} S(q, r)_{j_2, k_3}^{i_2, i_3} = R_{p, q, r} \sum_{k_1, k_2, k_3} \overline{S}(q, r)_{i_2, i_3}^{k_2, k_3} S(p, r)_{i_1, k_3}^{k_1, j_3} \overline{S}(p, q)_{j_1, j_2}^{i_1, i_2}, \]  

where all indices run over \( N \geq 2 \) distinct values \( 0, 1, \ldots, N - 1 \), “rapidities” \( p, q, r \) represent some continuous variables, and \( S(p, q)_{i,j}^{k,l} \), \( \overline{S}(p, q)_{i,j}^{k,l} \), \( R_{p, q, r} \) are functions of their arguments. In the standard way one can write eqs. (1.1a) in a matrix form:

\[ S_{12}(p, q) \overline{S}_{13}(p, r) S_{23}(q, r) = R_{p, q, r} \overline{S}_{23}(q, r) S_{13}(p, r) \overline{S}_{12}(p, q), \]  

where, for example, \( N^3 \)-by-\( N^3 \) matrix \( S_{12}(p, q) \) acts on basis ket-vectors as follows:

\[ S_{12}(p, q)|i, j, k\rangle = \sum_{i', j'} |i', j', k\rangle S(p, q)_{i, j}^{i', j'}, \]  

the other matrices being defined similarly. If we put in (1.1b) \( R_{p, q, r} = 1 \) and \( \overline{S} = S \), then we recognize the well known “vertex” YBE [2], [1]. In this sense eqs. (1.1) are a generalization of the latter. Note, the function \( R_{p, q, r} \), up to some root of unity, can be absorbed by multiplicative redefinition of \( S; \overline{S} \) matrices (of course if they are non-degenerate). Indeed, calculating determinants of both sides of (1.1b), we see that

\[ (R_{p, q, r})^{N^3} = f_{p,q}f_{q,r}/f_{p,r} \]
for some \( f \)'s. Renormalizing now \( \overline{S}(p, q) \rightarrow \overline{S}(p, q)/(f_{p,q})^{1/N^3} \) we eliminate the \( R \) function from (1.1) up to an \( N^3 \)-th root of unity. Nevertheless, we will keep \( R \)-factor in an explicit form to avoid possible problems with branching.

Note that eqs. (1.1) resemble the star-triangle relations [1], which can be written as

\[
W_{p,q}(m,k)\overline{W}_{p,r}(k,l)W_{q,r}(m,l) = R_{p,q,r} \sum_n \overline{W}_{q,r}(k,n)W_{p,r}(m,n)\overline{W}_{p,q}(n,l),
\]

where \( W \)'s and \( \overline{W} \)'s correspond to \( S \)'s and \( \overline{S} \)'s respectively, the difference being in number of discrete arguments as well as summations.

We should also notice a resemblance of eqs. (1.1) with “twisted” YBE of refs.[3], [4]*. In the “twisted” YBE from these papers matrices, corresponding to our \( \overline{S} \)'s, act nontrivially in all three sub-spaces of the tensor product space. Particular solutions from [3], however, can not be specialized to non-trivial solutions of (1.1). If one demands the analogs of our \( \overline{S} \)'s to be trivial in third sub-space, then the “twisted” YBE reduces to the usual one. Most general “twisted” YBE of [4] appeared in the context of quasi-Hopf algebras. From this point of view, eqs. (1.1) seem to correspond to a particular quasi-Hopf algebra. Unfortunately, the latter is far from to be clear for us, so we will not discuss this point anymore in this paper.

In Section 2, assuming non-degenerateness conditions in the “cross channel” for the matrices \( S, \overline{S} \), satisfying (1.1), we show that \( N^2 \)-by-\( N^2 \) box-matrices, constructed in terms of matrices \( S \), satisfy the usual YBE. As a consequence, the corresponding TM’s, being in fact two-layer ones, commute among themselves. In Section 3 explicit solutions to (1.1) are presented. In Section 4 the results obtained are summarized with some discussion. In Appendix A the functions of Sect. 3 are proved to satisfy eqs. (1.1). Appendix B contains an explanation of why in the case of even number of local states our solutions are degenerate in the “cross channel”.

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2. “Box” Construction

In this section we assume the non-degenerateness condition in the “cross channel”:

$$\det S^{t_2}(p, q) \neq 0, \quad \det \overline{S}^{t_2}(p, q) \neq 0,$$

(2.1)

where $S^{t_2}$, $\overline{S}^{t_2}$ are matrices $S$, $\overline{S}$, transposed in the second space. Introduce inverse “cross” matrices $S'$, $\overline{S}'$, satisfying

$$\sum_{j,k} S(p, q)^{k,l}_i j S'(q, p)^i m = \delta_i n \delta_l m,$$

(2.2)

with similar definition for $\overline{S}'(q, p)$. With the help of these matrices we can write three more forms of (1.1).

First, multiply both sides of (1.1a) by $S'(r, p)^{s_3, i_1}_{j_3, l_1} \overline{S}'(r, p)^{i_3, s_1}_{l_3, j_1}$ and sum over $i_1, j_1, i_3, j_3$. Then, using (2.2), we obtain

$$\sum_{i_1, k_2, j_3} S(p, q)^{s_1, k_2}_{i_1, i_2} S'(r, p)^{s_3, i_1}_{j_3, l_1} S(q, r)^{j_2, j_3}_{k_2, l_3}$$

$$= R_{p, q, r} \sum_{j_1, k_2, i_3} \overline{S}(q, r)^{k_2, s_3}_{i_2, i_3} \overline{S}'(r, p)^{i_3, s_1}_{i_1, j_3} \overline{S}(p, q)^{j_1, j_2}_{i_1, k_2}.$$

(2.3)

Next, multiplying (2.3) by $S'(r, q)^{l_3, s_2}_{i_3, j_2} \overline{S}'(r, q)^{j_3, i_2}_{s_3, l_2}$, summing over $i_2, j_2, l_3, s_3$, and using again (2.2), we obtain another form of (1.1)

$$\sum_{i_1, i_2, s_3} S(p, q)^{s_1, s_2}_{i_1, i_2} S'(r, p)^{s_3, i_1}_{i_3, l_1} \overline{S}'(r, q)^{j_3, i_2}_{s_3, l_2}$$

$$= R_{p, q, r} \sum_{j_1, j_2, l_3} S'(r, q)^{l_3, s_2}_{i_3, j_2} \overline{S}'(r, p)^{j_3, s_1}_{i_1, j_3} \overline{S}(p, q)^{j_1, j_2}_{i_1, l_2}.$$

(2.4)

At last, multiplication of (2.3) by $S'(q, p)^{i_2, j_1}_{l_2, s_1} \overline{S}'(q, p)^{s_2, l_1}_{j_2, i_1}$, summation over $l_1, s_1, i_2, j_2$, and the use of (2.2) lead to

$$\sum_{l_1, j_2, j_3} \overline{S}'(q, p)^{s_2, l_1}_{l_2, i_1} S'(r, p)^{s_3, j_1}_{j_3, l_1} S(q, r)^{j_2, j_3}_{l_2, l_3}$$

$$= R_{p, q, r} \sum_{s_1, i_2, i_3} \overline{S}(q, r)^{s_2, s_3}_{i_2, i_3} \overline{S}'(r, p)^{i_3, s_1}_{l_3, i_1} S'(q, p)^{i_2, j_1}_{l_2, s_1}.$$

(2.5)
Now introduce an $N^4$-by-$N^4$ matrix $\mathcal{R}(p, p'; q, q')$ through a “box” construction known for solutions of the star-triangle relations [5]:

$$\langle \mathbf{k}, \mathbf{l} | \mathcal{R}(p, p'; q, q') | \mathbf{m}, \mathbf{n} \rangle = \sum_{i, i', j, j'} S(p, q)^{i, n_1}_{k_1, j} S(q', p)^{j', m_1}_{n_2, i} S(p', q')^{j', l_2}_{m_2, j'} S'(q, p')^{j, k_2}_{l_1, i'}, \quad (2.6)$$

where $\mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}$ are two-component multi-indices, taking $N^2$ values:

$$\mathbf{k} = (k_1, k_2), \quad k_1, k_2 = 0, \ldots, N - 1, \quad (2.7)$$

and similarly for $\mathbf{l}, \mathbf{m}, \mathbf{n}$. We want to show that $\mathcal{R}(p, p'; q, q')$ solves the following YBE:

$$\mathcal{R}_{12}(p, p'; q, q') \mathcal{R}_{13}(p, p'; r, r') \mathcal{R}_{23}(q, q'; r, r') = \mathcal{R}_{23}(q, q'; r, r') \mathcal{R}_{13}(p, p'; r, r') \mathcal{R}_{12}(p, p'; q, q'). \quad (2.8)$$

To do that, define auxiliary $N^2$-by-$N^2$ matrices $U_{i,j}(p; q, r)$, $V_{i,j}(p; q, r)$ by

$$\langle \mathbf{k} | U_{i,j}(p; q, r) | \mathbf{l} \rangle = \sum_{s} S(p, q)^{s, l_1}_{i, k_1} S(r, p)^{k_2, j}_{l_2, s}, \quad (2.9)$$

$$\langle \mathbf{k} | V_{i,j}(p; q, r) | \mathbf{l} \rangle = \sum_{s} S'(q, p)^{l_1, i}_{k_1, s} S(p, r)^{s, k_2}_{j, l_2}. \quad (2.10)$$

Using consequently eqs. (2.3), and three times (1.1), we easily come to the following relation:

$$\sum_{k} U_{i,k}(r; p, p') \otimes U_{k,j}(r; q, q') \mathcal{R}(p, p'; q, q') = \rho_{p, p', q, q', r} \overline{\mathcal{R}}(p, p'; q, q') \sum_{k} U_{k,j}(r; p, p') \otimes U_{i,k}(r; q, q'), \quad (2.11a)$$

where $\overline{\mathcal{R}}(p, p'; q, q')$ being defined as in (2.6) with all $S$’s replaced by $\overline{S}$’s, and

$$\rho_{p, p', q, q', r} = R_{p', r, q} R_{r, p, q} R_{p', q', r}/R_{q', r, p}. \quad (2.11b)$$
Similarly, using consequently eqs. (2.4), (2.5), (1.1), and (2.3), we obtain

\[ \rho'_p,p',q,q',r' \sum_k V_{i,k}(r';p,p') \otimes V_{k,j}(r';q,q') R(p,p';q,q') \]
\[ = R(p,p';q,q') \sum_k V_{i,k}(r';p,p') \otimes V_{k,j}(r';q,q'), \]

where

\[ \rho'_p,p',q,q',r' = R_{r',p',q'} R_{r',p,q} R_{r',q',p} / R_{r',p',q}. \]  (2.12b)

As a consequence of (1.3) one obtains

\( (\rho_{p,p',q,q',r}/\rho'_{p,p',q,q',r'})^{N^3} = 1, \)  (2.13)

so \( \rho' \)'s coincide up to some \( N^3 \)-th root of unity. On the other hand, at the particular choice \( p = p' = q = q' = r = r' \) they coincide themselves, so by continuity argument we conclude that

\[ \rho_{p,p',q,q',r} = \rho'_{p,p',q,q',r'}. \]  (2.14)

Using this fact and

\[ \langle \bar{i}, \bar{j} | R(p,p';q,q') | k, l \rangle = \langle \bar{i} | V_{i_2,k_2}(p';q,q') U_{i_1,k_1}(p;q,q') | \bar{l} \rangle, \]  (2.15)

we see that YBE (2.8) holds as a consequence of (2.11) and (2.12).

To construct a commuting family of transfer matrices, fix some positive integer \( L \) and introduce a two-layer transfer matrix \( T(p,p') \) by

\[ \langle \bar{i}_1, \ldots, \bar{i}_L | T(p,p') | j_1, \ldots, j_L \rangle = \sum_{k_1, \ldots, k_L} \prod_{s=1}^L \langle k_s, i_s | R(p,p';q,q') | k_{s+1}, j_s \rangle \]  (2.16)

with periodicity condition \( k_1 = k_{L+1} \). Using YBE (2.8) in the standard way [1], we obtain a commutativity condition

\[ T(p,p') T(q,q') = T(q,q') T(p,p'). \]  (2.17)

So, the lattice model, corresponding to \( R \)-matrix (2.6) is integrable. In the next section we write down solutions to (1.1) and thereby realize explicitly constructions of the present section.
3. Some Solutions

Here we present particular solutions to eqs.(1.1).

Denote

$$\omega = \exp(2\pi i/N).$$  \hspace{1cm} (3.1)

First, following [6], remind an explicit form of Boltzmann weights of the sl(3) chiral Potts model. Let $\Gamma$ be an algebraic curve, defined by the following equations in $\mathbb{P}^5$:

$$\Gamma: \left( \begin{array}{c} (h_i^+)^N \\ (h_i^-)^N \end{array} \right) = K_i K_j^{-1} \left( \begin{array}{c} (h_j^+)^N \\ (h_j^-)^N \end{array} \right), \quad i, j \in Z_3 = \{0, 1, 2\},$$  \hspace{1cm} (3.2)

where $h_i^\pm$ are homogeneous coordinates in $\mathbb{P}^5$, and $K_i$, moduli 2-by-2 matrices with one and the same determinant for all $i$’s:

$$\det K_i = \det K_j, \quad i, j \in Z_3.$$  \hspace{1cm} (3.3)

We impose additional constraints on moduli matrices. Let

$$K_i = \left( \begin{array}{cc} a_i & b_i \\ c_i & d_i \end{array} \right), \quad i \in Z_3.$$  \hspace{1cm} (3.4)

Introduce one more set of matrices:

$$K_i' = \left( \begin{array}{cc} a_{i-1} & b_{i-1} \\ c_i & d_i \end{array} \right), \quad i \in Z_3,$$  \hspace{1cm} (3.5)

and demand for them the same conditions as (3.3):

$$\det K_i' = \det K_j', \quad i, j \in Z_3.$$  \hspace{1cm} (3.6)

Besides, define an algebraic curve $\Gamma'$ by (3.2) with $K_i'$’s instead of $K_i$’s. Obviously, $\Gamma$ and $\Gamma'$ are birationally isomorphic:

$$\tau : \Gamma \rightarrow \Gamma'$$

$$\tau^* (h_i^+) = h_{i-1}^+, \quad \tau^* (h_i^-) = h_i^-, \quad i \in Z_3.$$  \hspace{1cm} (3.7)
For $p, q \in \Gamma$ or $\Gamma'$ and

$$\bar{m} = (m_1, m_2) \in (\mathbb{Z}_N)^2 : \ m_1, m_2 = 0, \ldots, N - 1 \pmod{N} \quad (3.8)$$

define a function $\overline{W}_{p,q}(\bar{m})$ by the following relations:

$$\frac{\overline{W}_{p,q}(\bar{m} + \delta_1)}{\overline{W}_{p,q}(\bar{m})} = \frac{h^+_0(p)h^-_0(q) - h^+_0(q)h^-_0(p)\omega^{-m_1}}{h^+_1(p)h^-_1(q) - h^+_1(q)h^-_1(p)\omega^{1+m_1-m_2}}, \quad (3.9a)$$

$$\frac{\overline{W}_{p,q}(\bar{m} + \delta_2)}{\overline{W}_{p,q}(\bar{m})} = \frac{h^+_1(p)h^-_1(q) - h^+_1(q)h^-_1(p)\omega^{m_1-m_2}}{h^+_2(p)h^-_2(q) - h^+_2(q)h^-_2(p)\omega^{1+m_2}}, \quad (3.9b)$$

where $\delta_1 = (1, 0)$ and $\delta_2 = (0, 1)$, and

$$\overline{W}_{p,q}(0) = 1. \quad (3.9c)$$

With these definitions, Boltzmann weights of the $sl(3)$ chiral Potts model have the following form:

$$\overline{W}_{p,q}(\bar{m}, \bar{n}) = \omega^{(m_2-n_2)(n_1-n_2)-(m_1-n_1)n_1} \overline{W}_{p,q}(\bar{m} - \bar{n}). \quad (3.10)$$

Now we formulate the main result of the paper.

**Theorem.** The following functions satisfy (1.1):

$$S(p, q)^{n_1, n_2}_{m_2, m_1} = \lambda(\bar{m})\overline{W}_{p,q}(\bar{m}, \bar{n})/\lambda(\bar{n}), \quad (3.11a)$$

$$\overline{S}(p, q)^{n_1, n_2}_{m_2, m_1} = \lambda'(\bar{m})\overline{W}_{\tau(p), \tau(q)}(\bar{m}, \bar{n})/\lambda'(\bar{n}), \quad (3.11b)$$

where $p, q \in \Gamma$, and

$$\lambda(\bar{m}) = \omega^{-2m_1m_2-m_2(m_2+N)/2}, \quad \lambda'(\bar{m}) = \lambda(\bar{m})\omega^{3m_1m_2}. \quad (3.12)$$

An explicit form of the function $R_{p,q,r}$ and the proof of the theorem is given in appendix A. Note, that, if all moduli matrices are lower-triangular, then $\Gamma' = \Gamma$ and $\tau = \text{id}$. In this case, and when $N = 3$, we have $R_{p,q,r} = 1$, $\overline{S}_{p,q} = S_{p,q}$. So, the theorem gives a solution to the usual “vertex” YBE.

All the constructions of section 2 can be performed with solutions, given by the theorem, only for the odd $N$, since for even $N$ the conditions (2.1) do not hold (see appendix B).
4. Summary

In this paper we have shown that the generalized YBE (1.1), being in fact a “vertex” counterpart of the star-triangle equations (1.4), admit non-trivial solutions, given by (3.11). The latter reduce to the usual $R$-matrix, i.e. the solution of the “vertex” YBE, if the moduli matrices (3.4) are lower-triangular, and the number of local states $N = 3$.

By the use of matrix elements of the $S$-matrix from (1.1) it is possible to build integrable models on square lattice with commuting two-layer transfer matrices, single-layer ones being not necessarily so. More precisely, solutions of eqs. (1.1) should satisfy also conditions (2.1) (this is the case for solutions (3.11) only for odd number of local states).

The family of two-layer transfer matrices (2.16) is commutative provided the “box” $R$-matrix (2.6) solves the YBE (2.8). The latter is a consequence of the generalized YBE (1.1).

In conclusion note that from the physical point of view the lattice model, constructed in this paper, is unsatisfactory, since there is no region in the parameter space where the Botzmann weights are non-negative. The same problem we have also in the $sl(n)$ chiral Potts model for $n \geq 3$. Whether one can find a similarity transformation, leading to a physical lattice model, is an open question.

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Appendix A

In this appendix we prove the theorem of section 3. First, following [7] and [8], introduce more auxiliary objects. For any complex \( x \) and integer \( l \) define

\[
w(x|0) = 1, \quad w(x|l) = \prod_{j=1}^{l} \frac{1}{(1 - x\omega^j)}. \tag{A.1}
\]

This function has specific properties

\[
w(x|l + m) = w(x|l)w(x\omega^l|m), \tag{A.2a}
\]

\[
w(x/\omega|l)w(1/x| - l) = \omega^{l(l+1)}(-x)^l. \tag{A.2b}
\]

With the aid of definition (A.1) introduce

\[
f(x, y|z) = \sum_{\sigma=0}^{N-1} \frac{w(x|\sigma)}{w(y|\sigma)} z^\sigma, \tag{A.3}
\]
where complex parameters $x, y, z$ are constrained by the relation

$$z^N = \frac{1 - x^N}{1 - y^N}, \quad (A.4)$$

providing periodicity of the summand in $(A.1)$ on variable $\sigma$ with period $N$. In [8] the following automorphism property of $f(x, y|z)$ is proved, see $(A.14)$ there:

$$\frac{f(x \omega^k, y \omega^l | z \omega^m)}{f(x, y|z)} = \frac{w(x/y \omega|k - l) w(y|l) w(1/z| - m)}{z^l y^m \omega^{m(l+1)} w(x|k) w(x/y \omega|k - l - m)}. \quad (A.5)$$

Now we turn to the proof of $(1.1)$. To simplify formulae below, we will use the following notations:

$$W_{pq}(m_1, m_2) \equiv \overline{W}_{p,q}(\overline{m}), \quad W'_{pq}(-m_2, -m_1) \equiv \overline{W}_{\tau(p), \tau(q)}(\overline{m}). \quad (A.6)$$

Substituting (3.11), (3.12) into $(1.1a)$ and cancelling common factors, we obtain:

$$\text{L.H.S.} = \sum_{k_1, k_2, k_3} \Phi(i_2) \Phi(k_1) \Phi(k_2)^2 \Phi(k_3)^3 \omega^{k_1 k_2 - k_1 k_3 - 2 k_2 k_3}$$

$$\times \omega^{k_1(j_1 + i_1 - i_2 + i_3) + k_2(j_2 - j_3 - i_1) - 3 k_3(2 j_1 + j_2) + j_2 j_3 - j_1 i_3 - 2 j_1 i_2}$$

$$\times W_{pq}(i_2 - k_1, i_1 - k_2) W'_{pr}(k_3 - k_1, j_1 - i_3) W_{qr}(k_3 - j_2, k_2 - j_3), \quad (A.7a)$$

and

$$\text{R.H.S.} = R_{p,q,r} \times (\text{L.H.S. \ with \ } i_s \leftrightarrow j_s, \ \text{and \ } W \leftrightarrow W'), \quad (A.7b)$$

where

$$\Phi(k) = \omega^{k(k+N)/2}. \quad (A.8)$$

Below we will transform explicitly only the L.H.S. up to factors, symmetrical with respect to interchange $i_s \leftrightarrow j_s$, and $W \leftrightarrow W'$, simultaneously assuming that the R.H.S. is transformed in accordance with $(A.7b)$.

Let us shift indices $i_2, j_2, k_1, k_3$ by one and the same quantity in $(A.7)$:

$$i_2 \rightarrow i_2 + s, \quad j_2 \rightarrow j_2 + s, \quad k_1 \rightarrow k_1 + s, \quad k_3 \rightarrow i_2 + s. \quad (A.9)$$
Up to symmetrical factors, the only change we have is

\[ \text{summand of L.H.S.} \sim \omega^{s(i_2 + k_3 - k_1)}. \quad (A.10) \]

Multiplying now the both sides by \( \omega^{st} \) and summing over \( s \), one gets instead of \((A.10)\)

\[ \text{summand of L.H.S.} \sim \delta(t + i_2 + k_3 - k_1), \quad (A.11) \]

so the summations over \( k_1 \) can be performed trivially. Changing variables

\[ k_2 \rightarrow k_2 + j_3, \quad k_3 \rightarrow k_3 + j_2 \quad (A.12) \]

and after that,

\[ i_2 \rightarrow -i_2 - t, \quad j_2 \rightarrow -j_2 - t, \quad i_3 \rightarrow -i_3 + j_1, \quad j_3 \rightarrow -j_3 + i_1, \quad (A.13) \]

rewrite \((A.7)\) as follows

\[
\text{L.H.S.} = W_{pr}^{r'}(i_2, i_3) \sum_{k_2, k_3} W_{pq}(j_2 - k_3, j_3 - k_2)W_{qr}(k_3, k_2) \\
\times \omega^{k_2^2 + k_3^2 - k_2 k_3 - k_2(i_2 + j_3) + k_3(i_2 - i_3 - j_2 + j_3)}. \quad (A.14)
\]

To proceed further, we need an explicit form of \( W_{pq}(k, l) \) in terms of functions defined in \((A.1)\). Solving recurrence relations \((3.9)\), one can write

\[
W_{pq}(k, l) = w(x_{pq} | k - l)w(y_{pq} | l)w(z_{pq} | -k)\frac{u_{pq}^l}{v_{pq}^k}, \quad (A.15)
\]

where parameters \( x_{pq}, \ldots, v_{pq} \) can be expressed in terms of original variables \( h_i^\pm(p), \quad h_i^\pm(q) \). Note only, that as a consequence of \((3.2)\), the parameters in \((A.15)\) satisfy the following periodicity conditions:

\[
u_{pq}^N = (1 - y_{pq}^N)/(1 - x_{pq}^N), \quad v_{pq}^N = (1 - z_{pq}^N)/(1 - x_{pq}^N). \quad (A.16)
\]

If we define the symbol \( \overline{pq} \) as an abstract notation for a new set of variables:

\[
x_{pq} = 1/\omega x_{pq}, \quad y_{pq} = 1/\omega y_{pq}, \quad z_{pq} = 1/\omega z_{pq}, \quad (A.17a)
\]

\[
u_{pq} = u_{pq} x_{pq} / y_{pq}, \quad v_{pq} = u_{pq} x_{pq} / z_{pq}, \quad (A.17b)
\]
which satisfy relations (A.16) as well, then, applying (A.2b) to (A.15), we obtain
\[
W_{pq}(k, l) = \omega^{-k^2-l^2+kl}/W_{pq}(-k, -l).
\] (A.18)
The corresponding expression for the \(W'_{pq}(k, l)\) has the same form (A.15) with the all parameters being primed.

Let us transform \(W_{qr}(k_3, k_2)\) in (A.14) with the aid of (A.18). Using of (A.15) and dividing by \(W_{pr}(j_2, j_3)W'_{pr}(i_2, i_3)\) we have

\[
\text{L.H.S.} = \frac{W_{pq}(j_2, j_3)}{W_{pr}(j_2, j_3)} \sum_{k_1, k_2, k_3} \delta(k_1 + k_2 + k_3) \prod_{s=1}^{3} \xi_s^{k_s} \frac{w(x_s | k_s)}{w(\overline{x}_s | k_s)},
\] (A.19)

where
\[
x_1 = x_{pq} \omega^{j_2-j_3}, \quad x_2 = y_{pq} \omega^{j_3}, \quad x_3 = z_{pq} \omega^{-j_2},
\]
\[
\overline{x}_1 = x_{qr}, \quad \overline{x}_2 = y_{qr}, \quad \overline{x}_3 = z_{qr},
\]
\[
\xi_1 = \xi \omega^{-i_2-j_3}, \quad \xi_2 = \xi u_{pq}/u_{qr}, \quad \xi_3 = \xi v_{pq} \omega^{-i_3-j_2}/v_{qr},
\] (A.20a)

and \(\xi\) is any root of the equation
\[
\xi^N = (1 - x_{pq}^N)/(1 - x_{qr}^N).
\] (A.20b)

In (A.19) we have negated \(k_2\), added one more summation over \(k_1\) together with delta-symbol \(\delta(k_1 + k_2 + k_3)\), and distributed the remained factors among \(\xi_1, \xi_2, \xi_3\). To make the last step, remind the following expression for the delta-symbol:
\[
N \delta(k) = \sum_{\sigma} \omega^{k \sigma}.
\] (A.21)

Substituting it into (A.19) and using definition (A.3) together with (A.4), we finally obtain

\[
\text{L.H.S.} = \frac{W_{pq}(j_2, j_3)}{W_{pr}(j_2, j_3)} \prod_{s=1}^{3} f(x_s, \overline{x}_s | \xi_s) \sum_{\sigma=1}^{3} \prod_{s=1}^{3} x_s^{\sigma} \frac{w(\overline{x}_s \xi_s / x_s | k_s)}{w(\xi_s / \omega | k_s)}. \] (A.22)

Restoring the R.H.S. through (A.7b) and expressing all parameters in terms of original ones, we convince that L.H.S.=R.H.S., the \(R\)-factor being given by
\[
R_{p, q, r} = \prod_{s=1}^{3} f(x_s, \overline{x}_s | \xi_s) / f(x'_{s}, \overline{x}'_{s} | \xi'_{s}) \bigg|_{i_2 = i_3 = j_2 = j_3 = 0}.
\] (A.23)
Appendix B

Here we examine validity of conditions (2.1) for solutions (3.11).

Consider the linear equations

\[ \sum_{m_1,n_1} S(p,q)^{n_1,n_2}_{m_2+n_2,m_1+n_1} \omega^{m_1n_1} \Psi_{m_1,n_1} = 0 \]  

(B.1)

on \( N^2 \) unknowns \( \Psi_{m_1,n_1} \). The number of linearly independent solutions of eqs. (B.1) is equal to \( N^2 - \text{rank} S(p,q)^{t_2} \), so the first condition in (2.1) is equivalent to absence of non-zero solutions. Substituting (3.11a), and omitting non-zero factors, we rewrite (B.1) as

\[ \sum_{m_1,n_1} \omega^{-(2m_2+2n_2)m_1-m_2n_1} W_{p,q}(\overline{m}) \Psi_{m_1,n_1} = 0. \]  

(B.2)

To solve these relations introduce a new set of \( N^2 \) unknowns:

\[ \tilde{\Psi}_{m_2,n_2} = \sum_{m_1,n_1} \omega^{-(2m_2+n_2)m_1-m_2n_1} W_{p,q}(\overline{m}) \Psi_{m_1,n_1}, \]  

(B.3)

the inverse transformation being given by

\[ \Psi_{m_1,n_1} = \sum_{m_2,n_2} \omega^{(2m_2+n_2)m_1+m_2n_1} \frac{1}{N^2 W_{p,q}(\overline{m})} \tilde{\Psi}_{m_2,n_2}. \]  

(B.4)

Comparing (B.2) and (B.3), we immediately conclude, that (B.2) have a very simple form in the new variables:

\[ \tilde{\Psi}_{m_2,2n_2} = 0. \]  

(B.5)

For odd \( N \) variable \( 2n_2 \) runs over all \( N \) values \( 0, 1, \ldots, N-1 \), while for even \( N \), only \( N/2 \) values \( 0, 2, \ldots, N-2 \). Thus, we have proved that

\[ \text{rank} S(p,q)^{t_2} = \begin{cases} N^2, & \text{if } N = 1 \pmod{2}; \\ \frac{N^2}{2}, & \text{if } N = 0 \pmod{2}. \end{cases} \]  

(B.6)

Analogous consideration leads to the same result for \( \overline{S}(p,q)^{t_2} \).