Asymptotic Theory of the Sparse Group LASSO

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Abstract

This paper proposes a general framework for penalized convex empirical criteria and a new version of the Sparse-Group LASSO (SGL, Simon and al., 2013), called the adaptive SGL, where both penalties of the SGL are weighted by preliminary random coefficients. We explore extensively its asymptotic properties and prove that this estimator satisfies the so-called oracle property (Fan and Li, 2001), that is the sparsity based estimator recovers the true underlying sparse model and is asymptotically normally distributed. Then we study its asymptotic properties in a double-asymptotic framework, where the number of parameters diverges with the sample size. We show by simulations that the adaptive SGL outperforms other oracle-like methods in terms of estimation precision and variable selection.

Keywords: Asymptotic Normality, Consistency, Model Selection, Oracle Property.

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1 Introduction

Model complexity is an obstacle when one models richly parameterized dynamics such as multivariate nonlinear dynamic systems. For instance, dynamic variance correlation processes of size $N$ have an $O(N^2)$ complexity as in the dynamic conditional correlation parametrization (DCC, Ding and Engle, 2001). Another issue arises when the sample size, say $T$, is comparable to $N$, which may reduce the estimation performances. This is typically a high-dimensional statistical framework.

A significant literature developed on model penalization, which consists of reducing the number of parameters and performing variable selection. For instance, the Akaike’s or Bayesian information criteria aim at selecting the size of a model. However, these methods are unstable, computationally complex and their sampling properties are difficult to study as Fan and Li (2001) pointed out mainly because they are stepwise and subset selection procedures.

The LASSO procedure of Tibshirani (1996) overcomes these drawbacks as it simultaneously performs variable selection and model estimation. It then fosters sparsity and allows for continuity of the selected models. Other penalties were proposed such as the smoothly clipped absolute deviation (SCAD) of Fan, which modifies the LASSO to shrink large coefficients less severely. The elastic net regularization procedure of Zou and Hastie (2005) was developed to overcome the collinearity between the variables, which hampers the LASSO to perform well. Their idea consists of mixing a $l^1$ penalty, which performs variable selection, with a $l^2$ penalty, which stabilizes the solution paths. The Group LASSO of Yuan and Lin (2006) fosters sparsity and variable selection in a group of variables. Simon and al. (2013) designed the Sparse-Group LASSO (SGL) to foster sparsity both at a a group level and within a group. Their penalization involves a $l^1$ LASSO type penalty and a mixed $l^1/l^2$ penalty for group selection. All these procedures, together with the algorithms designed for performing selection and estimation, were developed within a linear framework. The penalized Ordinary Least Square (OLS) loss function is typically used for linear models as it is convex, which makes the computation easier, and allows for closed form solutions, such as the soft-thresholding operator for the LASSO penalty. Furthermore, linear modeling allows for deriving non asymptotic oracle inequalities straightforwardly: see Bühlmann and van de Geer (2011) on this non-asymptotic framework.

Knight and Fu (2000) explored the asymptotic properties of the LASSO penalty for OLS loss functions. Fan and Li (2001) proposed a penalization framework
for general convex functions and studied the asymptotic properties of the SCAD penalty. They proved that the SCAD estimator satisfies the oracle property, that is the sparsity based estimator recovers the true underlying sparse model and is asymptotically normally distributed. The LASSO as proposed by Tibshirani cannot enjoy the oracle property. To fix this drawback, Zou (2006) proposed the adaptive LASSO within an OLS framework, where adaptive weights are used to penalize different coefficients in the penalty. Nardi and Rinaldo (2008) applied the same methodology for the Group LASSO estimator within an OLS framework and studied its oracle property.

These theoretical studies were developed for fixed dimensional models with i.i.d. data, a case where $N$ does not depend on the sample size, and for least square type loss functions, except Fan and Li (2001). Fan and Peng (2004) considered the general penalized convex likelihood framework when the number of parameters grows with the sample size and focused on the oracle property for general penalties. Zou and Zhang (2009) also focused on the oracle property of the adaptive elastic-net within the double-asymptotic framework. Their work highlights that adaptive weights penalizing different coefficients are key quantities to enjoy the oracle property as one can modify the convergence rate of the penalty terms. Nardi and Rinaldo (2008) also proposed within the double-asymptotic setting selection consistency results, which states that asymptotically the right set of relevant variables is selected.

In this paper, we develop the asymptotic theory of penalized M-estimators for convex criteria and dependent variables and consider the asymptotic properties of the Sparse-Group LASSO estimator. This penalty is relevant for problems where one would like to foster sparsity for selecting active groups, that is a group for which some of the corresponding coefficients are non zero, and active coefficients within an active group, a situation where a coefficient is non zero within an active group. Hence this is somehow a two step approach as first the active groups are selected, and then the active variables within an active group are selected. We prove that the SGL as proposed by Simon and al. (2013) does not enjoy the oracle property. Then we propose a new version of the SGL, the adaptive SGL using the same methodology of Zou (2006), which consists of penalizing different coefficients and group of coefficients using random weights that are positive functions of a first step estimator. This enables to alter the rate of convergence of the penalties such that the adaptive SGL satisfies the oracle property. Our work is influenced by Fan and Peng (2004) concerning the oracle property for general penalized convex loss functions and by Zou and Zhang (2009) regarding the modeling of random weights
penalizing the coefficients differently. They both considered a double-asymptotic framework. We also prove that the adaptive SGL enjoys the oracle property in a double-asymptotic framework, a situation where the model complexity grows with the sample size.

The rest of the paper is organized as follows. In section 2, we describe our general framework for penalized convex empirical criteria and the SGL penalty. In section 3, we derive the optimality conditions of the statistical criterion. In section 4, we derive the asymptotic properties of both the SGL and adaptive SGL when the number of parameters is fixed. In section 5, we prove the oracle property of the adaptive SGL in a double-asymptotic setting.

2 Framework and notations

We consider a dynamic system in which the criterion is written as an empirical criterion, that is

\[ \theta \mapsto \mathcal{G}_T l(\theta) = \frac{1}{T} \sum_{t=1}^{T} l(\epsilon_t; \theta), \]

such that \( l(.) \) is "a general" known loss function on the sample space such that for any process \((\epsilon_t), \theta \mapsto l(\epsilon_t; \theta)\) is convex. This framework encompasses for instance the maximum likelihood method, where the \( l(.) \) function corresponds to \( l(\epsilon_t; \theta) = -\log f(\epsilon_t; \theta) \), where \( f(\epsilon_t; \theta) \) is the density of the observation \((\epsilon_t)\) under \( \mathbb{P}_\theta \). Alternatively, a linear model would imply \( l(\epsilon_t; \theta) = \| \epsilon_t^{(1)} - \theta' \epsilon_t^{(2)} \|_p \), where \((\epsilon_t^{(1)}, \epsilon_t^{(2)}) = \epsilon_t\). We denote the empirical score and Hessian of the empirical criterion respectively as

\[ \hat{\mathcal{G}}_T l(\theta) = \frac{1}{T} \sum_{t=1}^{T} \nabla \theta l(\epsilon_t; \theta), \quad \hat{\mathcal{G}}^2_T l(\theta) = \frac{1}{T} \sum_{t=1}^{T} \nabla^2 \theta \theta l(\epsilon_t; \theta). \]

The dependent nature of our framework requires the use of particular probabilistic tools to study the asymptotic properties of M-estimators. We extensively use the ergodic theorem and central limit theorem (Billingsley, 1961, 1995) to obtain convergence in probability of empirical quantities to their theoretical counterparts and central limit theorems. To do so, we assume the stationarity and the ergodicity of the underlying process \((\epsilon_t)\): see assumption in section 4.

In this setting, \( \epsilon_t \in \mathbb{R}^N \) and \( \theta \in \mathbb{R}^d \), a vector that can be split into \( m \) groups \( \mathcal{G}_k, k = 1, \cdots, m \), such that \( \text{card}(\mathcal{G}_k) = c_k \) and \( \sum_{k=1}^{m} c_k = d \). We suppose no overlap
between these groups. We use the notation \( \theta^{(i)} \) as the subvector of \( \theta \), that is the set \( \{ \theta_k : k \in \mathcal{G}_l \} \). Hence the vector \( \theta = (\theta_j, j = 1, \ldots, d) \) can be written as \( \theta = (\theta^{(k)}_i, k \in \{1, \ldots, m\}, i = 1, \ldots, c_k) \). We denote by \( \theta_0 \) the true parameter vector of interest. Moreover, \( \theta \to \mathbb{E}[(\epsilon; \theta)] \) is supposed to be a one-to-one mapping and is minimized uniquely at \( \theta = \theta_0 \).

We denote by \( \mathcal{S} := \{ k : \theta^{(k)}_i \neq 0 \} \) the set of indices for which the groups are active. Let \( \mathcal{A} := \{ j : \theta_{0,j} \neq 0 \} \) be the true subset model, which can be decomposed into sub-groups of active sets as \( l \in \mathcal{S}, \mathcal{A}_l = \{(l,i) : \theta^{(l)}_{0,i} \neq 0 \} \). Besides, there are inactive indices \( \mathcal{G}_l \setminus \mathcal{A}_l = \mathcal{A}_l^c = \{(l,i) : \theta^{(l)}_{0,i} = 0 \} \). We have \( \{ l \notin \mathcal{S} \} \Leftrightarrow \{ \forall i = 1, \ldots, c_l, \theta^{(l)}_{0,i} = 0 \} \). In this setting, \( \mathcal{A} = \bigcup_{l \in \mathcal{S}} \mathcal{A}_l \) such that for \( k \neq l, \mathcal{A}_k \cap \mathcal{A}_l = \emptyset \).

Furthermore, \( \mathcal{A}_c = \bigcup_{l=1}^m \mathcal{A}_l^c \) such that for \( k \neq l, \mathcal{A}_k^c \cap \mathcal{A}_l^c = \emptyset \).

Finally, we need the following notations: \( \mathcal{G}_T(\theta^{(k)}) \in \mathbb{R}^{c_k} \) is the ”score” vector of the empirical criterion taken over group \( k \) of size \( c_k \), \( \mathcal{G}_T(\theta^{(k)})(i) \in \mathbb{R} \) is the \( i \)-th component of this score, and \( \mathcal{G}_T(\theta)_{\mathcal{A}} \in \mathbb{R}^{\text{card}(\mathcal{A})} \) is the score over the set of active indices. \( \mathcal{G}_T(\theta)(k)(k) \in \mathcal{M}_{c_k \times c_k}(\mathbb{R}) \) (resp. \( \mathcal{H}(k)(k) \)) is the empirical (resp. theoretical) Hessian taken over the block representing group \( k \), and \( \mathcal{G}_T(\theta)_{\mathcal{A},\mathcal{A}} \in \mathcal{M}_{\text{card}(\mathcal{A}) \times \text{card}(\mathcal{A})}(\mathbb{R}) \) is the Hessian over the set of active indices.

The statistical problem consists of minimizing over the parameter space \( \Theta \) a penalized criterion of the form

\[
\hat{\theta} = \arg \min_{\theta \in \Theta} \{ \mathcal{G}_T \varphi(\theta) \}, \tag{2.2}
\]

where

\[
\theta \mapsto \mathcal{G}_T \varphi(\theta) = \frac{1}{T} \sum_{t=1}^T \{ l(\epsilon_t; \theta) + p_1(\lambda_T, \theta) + p_2(\gamma_T, \theta) \} = \mathcal{G}_T l(\theta) + p_1(\lambda_T, \theta) + p_2(\gamma_T, \theta).
\]

and both penalties are specified as

\[
\begin{align*}
\{ p_1 : \mathbb{R}_+ \times \mathbb{R}_+^m \times \Theta &\to \mathbb{R}_+ , \quad p_2 : \mathbb{R}_+ \times \mathbb{R}_+^m \times \Theta \to \mathbb{R}_+ , \\
(\lambda_T, \alpha, \theta) &\mapsto p_1(\lambda_T, \theta) = \lambda_T T^{-1} \sum_{k=1}^m \alpha_k \| \theta^{(k)} \|_1 , \quad (\gamma_T, \xi, \theta) &\mapsto p_2(\gamma_T, \theta) = \gamma_T T^{-1} \sum_{l=1}^m \xi_l \| \theta^{(l)} \|_2 .
\end{align*}
\]

\(^1\)Formally, there is a one-to-one mapping between two ways for writing \( \theta \):

\[
\psi : \{1, \ldots, d\} \to \{(k,i), k = 1, \ldots, m; i = 1, \ldots, c_k \}, \quad j \mapsto \psi(j) = (k_j, i_j).
\]

In the rest of this paper, this mapping is implicit such that we allow such writings as \( j = (k,i) \) or \( j = i_k \) where \( k \) is clear.
Both $\alpha_k$ and $\xi_l$ are non negative scalar quantities for each group and the tuning parameters $\lambda_T$ and $\gamma_T$ vary with $T$.

The estimator $\hat{\theta}$ obtained in (2.2) is not the minimum of the empirical unpenalized criterion $G_T l(.)$. Our main interest is to analyze the bias generated by the penalties and how the oracle property can be achieved in the sense of Fan and Li (2001). More precisely, the sparsity based estimator must satisfy

(i) $\hat{A} = \{i : \hat{\theta}_i \neq 0\} = A$ asymptotically, that is "model selection consistency".
(ii) $\sqrt{T}(\hat{\theta}_A - \theta_{0,A}) \xrightarrow{d} \mathcal{N}(0, V_0)$ with $V_0$ a covariance matrix related to the criterion of interest.

We highlight in Proposition 4.9 section 4 that actually the SGL as proposed by Simon and al. (2013) cannot perform the oracle property. Hence in section 4 we propose a new estimator based on the same idea as Zou (2006), the adaptive Sparse Group LASSO, for which the oracle property is obtained when the weights are randomized, as proved in Theorem 4.10.

This framework can be adapted to a broad range of problem. For instance, one can penalize a subset of groups with a $l^1$ penalty only, and the other groups with a $l^1/l^2$ penalty only. This framework encompasses the SGL, the LASSO and the group LASSO for proper choices of $\alpha$’s and $\xi$’s.

Let us motivate the interests of the SGL approach and illustrate our notations through a simple linear example. In finance, finding the right set of explanatory variables to predict future asset returns is a significant issue. For instance, one may use Japanese companies indices, the Japanese GDP or the Japanese aggregated dividend-price ratio to explain the Nikkei index return through a linear projection. But one should also consider some foreign variables, such as the S&P 500 index or the US yield curve. Consequently, some groups of variables naturally arise: group of financial companies, tech companies, and the like; group of foreign components such as American financial companies, and the like. Hence the set $G_k$ may represent the $k$-th ($k \leq m$) group of Japanese financial companies, composed (as a shortcoming) with Nomura (index 1), MUFG-Bank of Tokyo (index 2) and Sumitomo (index 3) represented by the parameter vector $\theta^{(k)} = (\theta_1^{(k)}, \theta_2^{(k)}, \theta_3^{(k)})$; then $k \in S$ if the whole group has a statistically significant effect on the Nikkei index. Suppose the $l^1/l^2$ penalty selects this group as active. Then $A_k$ represents the set of active components in $G_k$ such that $c_{A_k} = \text{card}(A_k) \leq \text{card}(G_k) = c_k$. The $l^1$ penalty fosters sparsity within this selected group. If Nomura is the only variable that is expelled, then $1 \in A_k^c = G_k \setminus A_k$, whereas $\{2, 3\} \in A_k$ and $c_{A_k} = 2$. 

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3 Optimality conditions

The statistical problem consists of solving (2.2). Both \( G_T(l(.), p_1(\lambda_T, \alpha, .) \) and \( p_2(\gamma_T, \xi, .) \) are convex functions and there are no inequality constraints. Consequently, by the Karush-Kuhn-Tucker optimality conditions, which are necessary and sufficient, the estimator \( \hat{\theta} \) satisfies for a group \( k \)

\[
\hat{G}_T l(\hat{\theta})_{(k)} + \lambda_T T^{-1} \alpha_k \hat{w}^{(k)} + \gamma_T T^{-1} \xi_k \hat{z}^{(k)} = 0,
\]

for some vectors \( w^{(k)} \) and \( z^{(k)} \) satisfying

\[
\hat{w}^{(k)} = \begin{cases} 
\text{sgn}(\hat{\theta}^{(k)}_i) & \text{if } \hat{\theta}^{(k)}_i \neq 0, i = 1, \ldots, c_k; \\
\{ \hat{w}^{(k)}_i : |\hat{w}^{(k)}_i| \leq 1 \} & \text{if } \hat{\theta}^{(k)}_i = 0, i = 1, \ldots, c_k.
\end{cases}
\]

\[
\hat{z}^{(k)} = \begin{cases} 
\left(\hat{\theta}^{(k)}/\|\hat{\theta}^{(k)}\|_2\right) & \text{if } \hat{\theta}^{(k)} \neq 0, \\
\{ z^{(k)} : \|z^{(k)}\|_2 \leq 1 \} & \text{if } \hat{\theta}^{(k)} = 0.
\end{cases}
\]

If \( \hat{\theta}^{(k)} = 0 \), we have \( \|z^{(k)}\|_2 \leq 1 \). Then, from (3.1), we obtain for such a \( k \notin S \)

\[
\sum_{i=1}^{c_k} (\hat{G}_T l(\hat{\theta})_{(k),i} + \lambda_T T^{-1} \alpha_k \hat{w}^{(k)}_i)^2 = \sum_{i=1}^{c_k} (\gamma_T T^{-1} \xi_k \hat{z}^{(k)}_i)^2 \leq \gamma_T^2 T^{-2} \xi_k^2 \|z^{(k)}\|_2^2.
\]

Consequently, if the subgradient equations are satisfied for \( \hat{\theta}^{(k)} \), then \( \hat{\theta}^{(k)} = 0 \) if

\[
\|\hat{G}_T l(\hat{\theta})_{(k)} + \lambda_T T^{-1} \alpha_k \hat{w}^{(k)}\|_2 \leq \gamma_T T^{-1} \xi_k.
\]

On the contrary, if this condition is not satisfied, then \( \hat{\theta}^{(k)} \neq 0 \). In this case, sparsity is fostered by the \( l^1 \) penalty as follows: using the optimality condition of (3.1), we have for \( \hat{\theta}^{(k)} \neq 0 \)

\[
\forall i = 1, \ldots, c_k, \ -\hat{G}_T l(\hat{\theta})_{(k),i} = \lambda_T T^{-1} \alpha_k \hat{w}^{(k)}_i + \gamma_T T^{-1} \xi_k \frac{\hat{\theta}^{(k)}_i}{\|\hat{\theta}^{(k)}\|_2}.
\]

If \( \hat{\theta}^{(k)}_i = 0 \), then \( |\hat{w}^{(k)}_i| \leq 1 \) and we obtain straightforwardly

\[
|\hat{G}_T l(\hat{\theta})_{(k),i}| \leq \lambda_T T^{-1} \alpha_k.
\]

Bertsekas (1995) proposed the use of subdifferential calculus to characterize necessary and sufficient solutions for problems such as (2.2). The conditions we derived are close to those of Simon and al. (2013) (obtained for a least square loss function). They will be extensively used in the rest of the paper.
4 Asymptotic properties

To prove the asymptotic results, we make the following assumptions.

**Assumption 1.** \((\epsilon_t)\) is a strictly stationary, ergodic and nonanticipative process.

**Assumption 2.** The parameter set \(\Theta \subset \mathbb{R}^d\) is convex and not necessarily compact.

**Assumption 3.** For any \((\epsilon_t)\), the function \(\theta \mapsto l(\epsilon_t; \theta)\) is strictly convex and \(C^\infty(\mathbb{R}, \Theta)\).

**Assumption 4.** The function \(\theta \mapsto \mathbb{E}[l(\epsilon_t; \theta)]\) is uniquely minimized in \(\theta_0\). Moreover, \((\nabla l(\epsilon_t; \theta_0))\) is a square integrable martingale difference.

**Assumption 5.** \(H := \mathbb{E}[\nabla^2_{\theta \theta} l(\epsilon_t; \theta_0)]\) and \(M := \mathbb{E}[\nabla_{\theta} l(\epsilon_t; \theta_0) \nabla_{\theta} l(\epsilon_t; \theta_0)]\) exist and are positive definite.

**Assumption 6.** Let \(\upsilon_t(C) = \sup_{k,l,m=1, \ldots, d} \{ \sup_{\theta: \|\theta - \theta_0\|_2 \leq \nu_T C} |\partial_{\theta_k \theta_l \theta_m}^3 l(\epsilon_t; \theta_0)|\}, \) where \(C > 0\) is a fixed constant and \(\nu_T \to 0\) as \(T \to \infty\), a quantity that will be made explicit. Then

\[
\eta(C) := \frac{1}{T^2} \sum_{t,t'=1}^{T} \mathbb{E}[\upsilon_t(C) \upsilon_{t'}(C)] < \infty.
\]

**Remark.** Assumptions 1 and 4 allows for using the central limit theorem of Billingsley (1961). We remind this result stated as a corollary in Billingsley (1961).

**Corollary 4.1.** (Billingsley, 1982)

If \((x_t, \mathcal{F}_t)\) is a stationary and ergodic sequence of square integrable martingal increments such that \(\sigma_x^2 = \text{Var}(x_t) \neq 0\), then

\[
\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_t \xrightarrow{d} \mathcal{N}(0, \sigma_x^2).
\]

Note that the square martingale difference condition can be relaxed by \(\alpha\)-mixing and moment conditions. For instance, Rio (2013) provides a central limit theorem for strongly mixing and stationary sequences.

**Theorem 4.2.** Under assumptions 1, 3, if \(\lambda_T / T \to \lambda_0 \geq 0\) and \(\gamma_T / T \to \gamma_0 \geq 0\), then for any compact set \(B \subset \Theta\) such that \(\theta_0 \in B\),

\[
\hat{\theta} \xrightarrow{P} \arg \min_{x \in B} \{G_\infty \varphi(x)\}.
\]
with

$$G_{\infty}\varphi(x) = G_{\infty}l(x) + \lambda_0 \sum_{k=1}^{m} \alpha_k \|x^{(k)}\|_1 + \gamma_0 \sum_{l=1}^{m} \xi_l \|x^{(l)}\|_2,$$

where $$\theta^*_0 = \arg \min_{x \in B} \{G_{\infty}\varphi(x)\}$$ is supposed to be a unique minimum, and $$G_{\infty}l(.)$$ is the limit in probability of $$G_Tl(.)$$.

To prove this theorem, we remind of Theorem II.1 of Andersen and Gill (1982) which proves that pointwise convergence in probability of random concave functions implies uniform convergence on compact subspaces.

**Lemma 4.3.** (Andersen and Gill, 1982)

Let $$E$$ be an open convex subset of $$\mathbb{R}^p$$, and let $$F_1, F_2, \ldots$$, be a sequence of random concave functions on $$E$$ such that $$F_n(x) \xrightarrow{p} f(x)$$ for every $$x \in E$$ where $$f$$ is some real function on $$E$$. Then $$f$$ is also concave, and for all compact $$A \subset E$$,

$$\sup_{x \in A} |F_n(x) - f(x)| \xrightarrow{n \to \infty} 0.$$

The proof of this theorem is based on a diagonal argument and theorem 10.8 of Rockafeller (1970), that is the pointwise convergence of concave random functions on a dense and countable subset of an open set implies uniform convergence on any compact subset of the open set. Then the following corollary is stated.

**Corollary 4.4.** (Andersen and Gill, 1982)

Assume $$F_n(x) \xrightarrow{p} f(x)$$, for every $$x \in E$$, an open convex subset of $$\mathbb{R}^p$$. Suppose $$f$$ has a unique maximum at $$x_0 \in E$$. Let $$\hat{X}_n$$ maximize $$F_n$$. Then $$\hat{X}_n \xrightarrow{p} x_0$$.

Newey and Powell (1987) use a similar theorem to prove the consistency of asymmetric least squares estimators without any compactness assumption on $$\Theta$$. We apply these results in our framework, where the parameter set $$\Theta$$ is supposed to be convex.

**Proof.** of Theorem 4.2.

By definition, $$\theta = \arg \min_{\theta \in \Theta} \{G_T\varphi(\theta)\}$$. In a first step, we prove the uniform convergence of $$G_T\varphi(.)$$ to the limit quantity $$G_{\infty}\varphi(.)$$ on any compact set $$B \subset \Theta$$, idest

$$\sup_{x \in B} |G_T\varphi(x) - G_{\infty}\varphi(x)| \xrightarrow{T \to \infty} 0. \quad (4.1)$$
We define $C \subset \Theta$ an open convex set and pick $x \in C$. Then by Assumption 1, the law of large number implies

$$G_T l(x) \xrightarrow{P} G_\infty l(x).$$

Consequently, if $\lambda_T/T \to \lambda_0 \geq 0$ and $\gamma_T/T \to \gamma_0 \geq 0$, we obtain the pointwise convergence

$$|G_T \varphi(x) - G_\infty \varphi(x)| \xrightarrow{P} 0.$$ 

By Lemma 4.3 of Andersen and Gill, $G_\infty \varphi(\cdot)$ is a convex function and we deduce the desired uniform convergence over any compact subset of $\Theta$, that is \(4.1\).

Now we would like that $\arg \min \left\{ G_T \varphi(x) \right\} \xrightarrow{P} \arg \min \left\{ G_\infty \varphi(x) \right\}$. By assumption 3, $\varphi(\cdot)$ is strictly convex, which implies

$$\|G_T \varphi(\theta)\|_{\|\|_{T \to \infty} \rightarrow \infty}.$$

Consequently, $\arg \min \left\{ G_T \varphi(x) \right\} = O(1)$, such that $\hat{\theta} \in B(\theta_0, C)$ with probability approaching one for $C$ large enough, with $B(\theta_0, C)$ an open ball centered at $\theta_0$ and of radius $C$. Furthermore, as $G_\infty \varphi(\cdot)$ is strictly convex, continuous, then $\arg \min \left\{ G_\infty \varphi(x) \right\}$ exists and is unique. Then by Corollary 4.4 of Andersen and Gill, we obtain

$$\arg \min \left\{ G_T \varphi(x) \right\} \equiv \arg \min \left\{ G_\infty \varphi(x) \right\},$$

that is $\hat{\theta} \xrightarrow{P} \theta_0^*$. \hfill $\square$

**Theorem 4.5.** Under Assumptions 2, 3 and 6, the sequence of penalized estimators $\hat{\theta}$ satisfies

$$\|\hat{\theta} - \theta_0\| = O_p(T^{-1/2} + \lambda_T^{-1}a_T + \gamma_T^{-1}b_T),$$

when $\lambda_T = o(T)$ and $\gamma_T = o(T)$, and $a_T := \text{card}(A).\left\{ \max_k \alpha_k \right\}$ and $b_T := \text{card}(A).\left\{ \max_l \xi_l \right\}$ satisfy $\lambda_T T^{-1}a_T \to 0$ and $\gamma_T T^{-1}b_T \to 0$.

**Proof.** We denote $\nu_T = T^{-1/2} + \lambda_T^{-1}a_T + \gamma_T^{-1}b_T$, with $a_T = \text{card}(A).\left\{ \max_k \alpha_k \right\}$ and $b_T = \text{card}(A).\left\{ \max_l \xi_l \right\}$. We would like to prove that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\mathbb{P}\left( \frac{1}{\nu_T} \|\hat{\theta} - \theta_0\| > C_\epsilon \right) < \epsilon. \quad (4.2)$$

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We have

\[ \mathbb{P}(\frac{1}{\nu_T}||\hat{\theta} - \theta_0|| > C_\epsilon) \leq \mathbb{P}(\exists \mathbf{u} \in \mathbb{R}^d, ||\mathbf{u}||_2 \geq C_\epsilon : \mathcal{G}_T \varphi(\theta_0 + \nu_T \mathbf{u}) \leq \mathcal{G}_T \varphi(\theta_0)). \]

Furthermore, \( ||\mathbf{u}||_2 \) can potentially be large as it represents the discrepancy \( \hat{\theta} - \theta_0 \) normalized by \( \nu_T \). Now based on the convexity of the objective function, we have

\[ \{ \exists \mathbf{u}^*, ||\mathbf{u}^*||_2 \geq C_\epsilon, \mathcal{G}_T \varphi(\theta_0 + \nu_T \mathbf{u}^*) \leq \mathcal{G}_T \varphi(\theta_0) \} \subset \{ \exists \bar{\mathbf{u}}, ||\bar{\mathbf{u}}||_2 = C_\epsilon, \mathcal{G}_T \varphi(\theta_0 + \nu_T \bar{\mathbf{u}}) \leq \mathcal{G}_T \varphi(\theta_0) \}, \tag{4.3} \]

a relationship that allows us to work with a fixed \( ||\mathbf{u}||_2 \). Let us define \( \theta_1 = \theta_0 + \nu_T \mathbf{u}^* \) such that \( \mathcal{G}_T \varphi(\theta_1) \leq \mathcal{G}_T \varphi(\theta_0) \). Let \( \alpha \in (0, 1) \) and \( \theta = \alpha \theta_1 + (1 - \alpha) \theta_0 \). Then by convexity of \( \mathcal{G}_T \varphi(.) \), we obtain

\[ \mathcal{G}_T \varphi(\theta) \leq \alpha \mathcal{G}_T \varphi(\theta_1) + (1 - \alpha) \mathcal{G}_T \varphi(\theta_0) \leq \mathcal{G}_T \varphi(\theta_0). \]

We pick \( \alpha \) such that \( ||\bar{\mathbf{u}}|| = C_\epsilon \) with \( \bar{\mathbf{u}} := \alpha \theta_1 + (1 - \alpha) \theta_0 \). Hence (4.3) holds, which implies

\[ \mathbb{P}(||\hat{\theta} - \theta_0|| > C_\epsilon \nu_T) \leq \mathbb{P}(\exists \mathbf{u} \in \mathbb{R}^d, ||\mathbf{u}||_2 \geq C_\epsilon : \mathcal{G}_T \varphi(\theta_0 + \nu_T \mathbf{u}) \leq \mathcal{G}_T \varphi(\theta_0)) \leq \mathbb{P}(\exists \bar{\mathbf{u}}, ||\bar{\mathbf{u}}||_2 = C_\epsilon : \mathcal{G}_T \varphi(\theta_0 + \nu_T \bar{\mathbf{u}}) \leq \mathcal{G}_T \varphi(\theta_0)). \]

Hence, we pick a \( \mathbf{u} \) such that \( ||\mathbf{u}||_2 = C_\epsilon \). Using \( p_1(\lambda_T, \alpha, 0) = 0 \) and \( p_2(\gamma_T, \xi, 0) = 0 \), by a Taylor expansion to \( \mathcal{G}_T l(\theta_0 + \nu_T \mathbf{u}) \), under assumption [4] we obtain

\[ \mathcal{G}_T \varphi(\theta_0 + \nu_T \mathbf{u}) - \mathcal{G}_T \varphi(\theta_0) = \nu_T \mathcal{G}_T l(\mathbf{0}) \mathbf{u} + \frac{\nu_T^2}{2} (\mathbf{u}' \mathcal{G}_T l(\theta_0) \mathbf{u}) + \frac{\nu_T^3}{6} \nabla' \{ \mathbf{u}' \mathcal{G}_T l(\mathbf{0}) \mathbf{u} \} \mathbf{u} + p_1(\lambda_T, \alpha, \theta_T) - p_1(\lambda_T, \alpha, \theta_0) + p_2(\gamma_T, \xi, \theta_T) - p_2(\gamma_T, \xi, \theta_0), \]

where \( \bar{\theta} \) is defined as \( ||\bar{\theta} - \theta_0|| \leq ||\theta_T - \theta_0|| \). We want to prove

\[ \mathbb{P}(\exists \mathbf{u}, ||\mathbf{u}||_2 = C_\epsilon : \mathcal{G}_T l(\mathbf{0}) \mathbf{u} + \frac{\nu_T}{2} \mathbb{E}[\mathbf{u}' \mathcal{G}_T l(\theta_0) \mathbf{u}] + \frac{\nu_T^2}{2} \mathcal{R}_T(\theta_0) + \frac{\nu_T^3}{6} \nabla' \{ \mathbf{u}' \mathcal{G}_T l(\mathbf{0}) \mathbf{u} \} \mathbf{u} + \nu_T^{-1} \{ p_1(\lambda_T, \alpha, \theta_T) - p_1(\lambda_T, \alpha, \theta_0) + p_2(\gamma_T, \xi, \theta_T) - p_2(\gamma_T, \xi, \theta_0) \} \leq 0 < \epsilon, \tag{4.4} \]

where \( \mathcal{R}_T(\theta_0) = \sum_{k,l=1}^d u_k u_l \{ \partial_{\theta_k \theta_l} \mathcal{G}_T l(\theta_0) - \mathbb{E}[\partial_{\theta_k \theta_l} \mathcal{G}_T l(\theta_0)] \} \). By assumption [4] (\( \epsilon_1 \)) is a non anticipative stationary solution and is ergodic. As a square integrable martingale difference,

\[ \sqrt{T} \mathcal{G}_T l(\theta_0) \mathbf{u} \overset{d}{\rightarrow} \mathcal{N}(0, \mathbf{u}' \mathbb{M} \mathbf{u}), \]
by the central limit theorem of Billingsley (1961), which implies \( \hat{G}_T l(\theta_0) u = O_p(T^{-1/2} u'Mu) \).

By the ergodic theorem of Billingsley (1995), we have

\[
\hat{G}_T l(\theta_0) \xrightarrow{\mathbb{P}} \mathbb{H}.
\]

This implies \( \mathcal{R}_T(\theta_0) = o_p(1) \).

Furthermore, we have by the Markov inequality and for \( b > 0 \) that

\[
\mathbb{P}(\exists u, \|u\|_2 = C_\varepsilon : \sup_{\tilde{\theta} : \|\theta - \theta_0\|_2 \leq \nu_T C_\varepsilon} \frac{\nu_T^2}{6} \nabla' \{ u' \hat{G}_T l(\tilde{\theta}) u \} > b) \leq \frac{\nu_T^4 C_\varepsilon^6}{36b^2 \eta(C_\varepsilon),}
\]

where \( \eta(C_\varepsilon) \) is defined in assumption 6. We now focus on the penalty terms. As \( p_1(\lambda_T, \alpha, 0) = 0 \), for the \( l^1 \) norm penalty, we have

\[
p_1(\lambda_T, \alpha, \theta_T) - p_1(\lambda_T, \alpha, \theta_0) = \lambda_T T^{-1} \sum_{k \in S} \alpha_k \{ \| \theta_0^{(k)} + \nu_T u^{(k)} \|_1 - \| \theta_0^{(k)} \|_1 \},
\]

and \( |p_1(\lambda_T, \alpha, \theta_T) - p_1(\lambda_T, \alpha, \theta_0)| \leq \text{card}(S) \{ \max_{k \in S} \alpha_k \} \lambda_T T^{-1} \nu_T \|u\|_1. \quad (4.5)
\]

As for the \( l^1/l^2 \) norm, we obtain

\[
p_2(\gamma_T, \xi, \theta_T) - p_2(\gamma_T, \xi, \theta_0) = \gamma_T T^{-1} \sum_{l \in S} \xi_l \{ \| \theta_0^{(l)} \|_2 - \| \theta_0^{(l)} \|_2 \},
\]

and \( |p_2(\gamma_T, \xi, \theta_T) - p_2(\gamma_T, \xi, \theta_0)| \leq \gamma_T T^{-1} \sum_{l \in S} \xi_l \nu_T \|u^{(l)}\|_2
\]

\[
\leq \text{card}(S) \{ \max_{l \in S} \xi_l \} \gamma_T T^{-1} \nu_T \|u\|_2. \quad (4.6)
\]

Then denoting by \( \delta_T = \lambda_{\text{min}}(\mathbb{H}) C_\varepsilon^2 \nu_T \), and using \( \frac{\nu_T}{2} \mathbb{E}[u' \hat{G}_T l(\theta_0) u] \geq \delta_T \), we deduce that [4.4] can be bounded as

\[
\mathbb{P}(\exists u, \|u\|_2 = C_\varepsilon : \hat{G}_T l(\theta_0) u + \frac{\nu_T}{2} u' \hat{G}_T l(\theta_0) u + \frac{\nu_T^2}{6} \nabla' \{ u' \hat{G}_T l(\tilde{\theta}) u \} u
\]

\[
+ \nu_T^{-1} \{ p_1(\lambda_T, \alpha, \theta_T) - p_1(\lambda_T, \alpha, \theta_0) + p_2(\gamma_T, \xi, \theta_T) - p_2(\gamma_T, \xi, \theta_0) \} \leq 0)
\]

\[
\leq \mathbb{P}(\exists u, \|u\|_2 = C_\varepsilon : |\hat{G}_T l(\theta_0) u| > \delta_T/8) + \mathbb{P}(\exists u, \|u\|_2 = C_\varepsilon : \frac{\nu_T}{2} |\mathcal{R}_T(\theta_0)| > \delta_T/8)
\]

\[
+ \mathbb{P}(\exists u, \|u\|_2 = C_\varepsilon : |p_1(\lambda_T, \alpha, \theta_T) - p_1(\lambda_T, \alpha, \theta_0)| > \nu_T \delta_T/8)
\]

\[
+ \mathbb{P}(\exists u, \|u\|_2 = C_\varepsilon : |p_2(\gamma_T, \xi, \theta_T) - p_2(\gamma_T, \xi, \theta_0)| > \nu_T \delta_T/8).
\]
We also have for $C_\epsilon$ and $T$ large enough, and using norm equivalences that

$$\mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : |\mathbf{p}_1(\lambda_T, \alpha, \theta_T) - \mathbf{p}_1(\lambda_T, \alpha, \theta_0)| > \nu T \delta_T / 8) \leq \mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : \text{card}(\mathcal{S}) \{ \max_{\xi_k} \alpha_k \} \lambda T^{-1} \nu T \|\mathbf{u}\|_1 > \nu T \delta_T / 8) < \epsilon / 5,$$

$$\mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : |\mathbf{p}_2(\gamma_T, \xi, \theta_T) - \mathbf{p}_2(\gamma_T, \xi, \theta_0)| > \nu T \delta_T / 8) \leq \mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : \text{card}(\mathcal{S}) \{ \max_{\xi_k} \xi_k \} \gamma T^{-1} \nu T \|\mathbf{u}\|_2 > \nu T \delta_T / 8) < \epsilon / 5.$$

Moreover, if $\nu T = T^{-1/2} + \lambda T^{-1} a_T + \gamma T^{-1} b_T$, then for $C_\epsilon$ large enough

$$\mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : |\hat{\mathbf{G}} T l(\theta_0) \mathbf{u}| > \delta_T / 8) \leq \frac{C_\epsilon^2 \nu_T \eta(C_\epsilon)}{T \delta_T^2} \leq \frac{C_{st} \nu_T^2 \eta(C_\epsilon)}{C_\epsilon^2} < \epsilon / 5.$$  

Moreover

$$\mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : \sup_{\theta : \|\theta - \theta_0\|_2 < \nu T C_\epsilon} \frac{\nu_T^2}{6} \nabla' \{ \mathbf{u}' \hat{\mathbf{G}} T l(\theta) \mathbf{u} \} > \delta_T / 8) \leq \frac{C_{st} \nu_T^2 \eta(C_\epsilon)}{\delta_T^2} \leq C_{st} \nu_T^2 C_\epsilon^2 \eta(C_\epsilon)$$

where $C_{st} > 0$ is a generic constant. Consequently, we obtain, for $T$ and $C_\epsilon$ large enough, we obtain

$$\mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : |\hat{\mathbf{G}} T l(\theta_0) \mathbf{u}| > \delta_T / 8) + \mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : \frac{\nu_T}{2} |\mathcal{R}_T(\theta_0)| > \delta_T / 8) + \mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : |\mathbf{p}_1(\lambda_T, \alpha, \theta_0) - \mathbf{p}_1(\lambda_T, \alpha, \theta_0)| > \nu T \delta_T / 8) + \mathbb{P}(\exists \mathbf{u}, \|\mathbf{u}\|_2 = C_\epsilon : |\mathbf{p}_2(\gamma_T, \xi, \theta_0) - \mathbf{p}_2(\gamma_T, \xi, \theta_0)| > \nu T \delta_T / 8) + 0 \leq \frac{C_{st}}{C_\epsilon^2} + \nu_T^2 C_\epsilon^2 \eta(C_\epsilon) C_{st} + 3 \epsilon / 5 \leq \epsilon,$$

for $C_\epsilon$ sufficiently large, and $T$ large enough. We then deduce

$$\|\hat{\theta} - \theta_0\| = O_T(\nu_T) = O_T(\lambda T^{-1} a_T + \gamma T^{-1} b_T + T^{-1/2}).$$

\[\square\]

**Remark.** We would like to highlight the use of the convexity property of $\mathcal{G}_T \varphi(.)$. It allowed us to obtain the upper bound \[\text{(a)}\]. Otherwise, the inequality would have
been uniform over $\|u\|_2 \geq C_\varepsilon$. A consequence is that $\|u\|_2$ can take significantly large values, which would have made the control of the random part in the Taylor expansion hard. This issue is overcome thanks to the convexity that allows for working with fixed $\|u\|_2$, as Fan and Li (2001), Fan and Peng (2004) or Nardi and Rinaldo (2008) do.

We now focus on the distribution of the SGL estimator. Deriving the asymptotic distribution for M-estimators is standard in the case the objective function is differentiable. It consists of characterizing the estimator by the orthogonality conditions and derive a linear representation by Taylor expansions of the estimator. But these techniques do not apply when the objective function is not differentiable. In our case, $\varphi(.)$ is not differentiable at 0 due to the penalty terms. In some specific context, it may be possible to treat the non-differentiability of $G_T \varphi(.)$ by applying the expectation operator $E[.]$ to $\varphi(.)$, which then becomes differentiable in $\theta_0$. Then Taylor expansions are feasible and one obtains the distribution, provided some regularity conditions of the empirical criterion, such as stochastic equi-continuity: see Andrews (1994, a,b). This approach works for specific loss functions, such as the LAD. But in our setting, the expectation operator fails at regularizing $\varphi(.)$ due to the penalty functionals.

Another approach to obtain the asymptotic distribution relies on the convexity property of $\varphi(.)$, and hence of $G_T \varphi(.)$, without assuming strong regularity conditions on $\varphi(.)$. The intuition behind this rather strong statement is as follows. Let $F_T(u)$ and $F_\infty(u)$, $u \in \mathbb{R}^d$, be random convex functions such that their minimum are respectively $u_T$ and $u_\infty$. Then if $F_T(.)$ converges in finite distribution to $F_\infty(.)$, and $u_\infty$ is the unique minimum of $F_\infty$ with probability one, then $u_T$ converges weakly to $u_\infty$. This method to prove the convergence of arg min processes is called the convexity argument. It was developed by Davis and al. (1992), Hjort, Pollard (1993), Geyer (1996a, 1996b) or Kato (2009). Chernozhukov and Huong (2004), Chernozhukhov (2005) use this convexity argument to obtain the asymptotic distribution of quantile regression type estimators. The convexity argument only requires the lower-semicontinuity and convexity of the empirical criterion. The convexity Lemma, as in Chernozhukov (2005), can be stated as follows.

**Lemma 4.6.** (Chernozhukov, 2005)

Suppose

(i) a sequence of convex lower-semicontinuous $F_T : \mathbb{R}^d \to \mathbb{R}$ marginally converges to $F_\infty : \mathbb{R}^d \to \mathbb{R}$ over a dense subset of $\mathbb{R}^d$;
(ii) $F_\infty$ is finite over a nonempty open set $E \subset \mathbb{R}^d$;
(iii) $F_\infty$ is uniquely minimized at a random vector $u_\infty$.
Then
$$\arg \min_{z \in \mathbb{R}^d} F_T(z) \xrightarrow{d} \arg \min_{z \in \mathbb{R}^d} F_\infty(z), \text{ that is } u_T \xrightarrow{d} u_\infty.$$ 

**Theorem 4.7.** Under Assumptions 1-6, if $T \xi T^{-1/2} \to \lambda_0$ and $\gamma T^{-1/2} \to \gamma_0$, then
$$\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{d} \arg \min_{\theta \in \mathbb{R}^d} \{F_\infty(\theta)\},$$
provided $F_\infty$ is the random function in $\mathbb{R}^d$, where
$$F_\infty(\theta) = \frac{1}{2} u' \mathbb{H} u + u' Z + \lambda_0 \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{c_k} \{||u^{(k)}_i||1_{\theta^{(k)}_{0,i}=0} + u^{(k)}_i \text{sgn}(\theta^{(k)}_{0,i})1_{\theta^{(k)}_{0,i} \neq 0}\}$$
$$+ \gamma_0 \sum_{l=1}^{m} \xi_l \{||u^{(l)}||2_{1_{\theta^{(l)}_0=0} + u^{(l)}/||\theta^{(l)}_0||2_{1_{\theta^{(l)}_0 \neq 0}}\},$$
with $\mathbb{H} = \mathbb{H}(\theta_0) := \mathbb{E}[\nabla^2 \theta \ell(\epsilon; \theta_0)]$ and some random vector $Z \sim \mathcal{N}(0, M)$, $M = M(\theta_0) := \mathbb{E}[\nabla \theta \ell(\epsilon; \theta_0) \nabla^\theta \ell(\epsilon; \theta_0)].$

**Proof.** Let $u \in \mathbb{R}^d$ such that $\theta = \theta_0 + u/T^{1/2}$ and we define the empirical criterion $F_T(u) = TG_T(\varphi(\theta_0 + u/T^{1/2}) - \varphi(\theta_0))$. First, we are going to prove the finite distributional convergence of $F_T$ to $F_\infty$. Then we use the convexity of $F_T(.)$ to obtain the convergence in distribution of the arg min empirical criterion to the arg min process limit. To do so, let $u = \sqrt{T}(\theta - \theta_0)$. We have
$$F_T(u) = T \{G_T(l(\theta) - l(\theta_0)) + p_1(\lambda T, \alpha, \theta) - p_1(\lambda T, \alpha, \theta_0) + p_2(\gamma T, \xi, \theta) - p_2(\gamma T, \xi, \theta_0)\}$$
$$= TG_T(l(\theta_0 + u/T^{1/2}) - l(\theta_0)) + \lambda T \sum_{k=1}^{m} \alpha_k \{||\theta^{(k)}_0 + u^{(k)}/\sqrt{T}||1 - ||\theta^{(k)}_0||1\}$$
$$+ \gamma T \sum_{l=1}^{m} \xi_l \{||\theta^{(l)}_0 + u^{(l)}/\sqrt{T}||2 - ||\theta^{(l)}_0||2\},$$
where $F_T(.)$ is convex and $C^0(\mathbb{R}^d)$. We now prove the finite dimensional distribution of $F_T$ to $F_\infty$ to apply Lemma 4.6. For the $l^1$ penalty, for any group $k$, we have for $T$ sufficiently large
$$||\theta^{(k)}_0 + u^{(k)}/\sqrt{T}||1 - ||\theta^{(k)}_0||1 = T^{-1/2} \sum_{i=1}^{c_k} \{||u^{(k)}_i||1_{\theta^{(k)}_{0,i}=0} + u^{(k)}_i \text{sgn}(\theta^{(k)}_{0,i})1_{\theta^{(k)}_{0,i} \neq 0}\},$$
which implies that

\[ \lambda T \sum_{k=1}^{m} \alpha_k [\| \theta_0^{(k)} + u^{(k)} / \sqrt{T} \|_1 - \| \theta_0^{(k)} \|_1] \xrightarrow{T \to \infty} \lambda_0 \sum_{k=1}^{m} \alpha_k \sum_{i=1}^{c_k} \{ u_i^{(k)} 1_{\theta_0^{(k)}} = 0 + u_i^{(k)} sgn(\theta_0^{(k)}) 1_{\theta_0^{(k)} \neq 0} \}, \]

under the condition that \( \lambda T / \sqrt{T} \to \lambda_0 \).

As for the \( l^1/l^2 \) quantity, for any group \( l \), we have

\[ \| \theta_0^{(l)} + u^{(l)} / \sqrt{T} \|_2 - \| \theta_0^{(l)} \|_2 = T^{-1/2} \{ \| u^{(l)} \|_2 1_{\theta_0^{(l)} - 0} + \frac{u^{(l)'} \theta_0^{(l)}}{\| \theta_0^{(l)} \|_2} 1_{\theta_0^{(l)} \neq 0} \} + o(T^{-1}). \]

Consequently, if \( \gamma_T T^{-1/2} \to \gamma_0 \geq 0 \), we obtain

\[ \gamma_T \sum_{l=1}^{m} \xi_l [\| \theta_0^{(l)} + u^{(l)} / \sqrt{T} \|_2 - \| \theta_0^{(l)} \|_2] = \gamma_0 \sum_{l=1}^{m} \xi_l \{ \| u^{(l)} \|_2 1_{\theta_0^{(l)} = 0} + \frac{u^{(l)'} \theta_0^{(l)}}{\| \theta_0^{(l)} \|_2} 1_{\theta_0^{(l)} \neq 0} \} + o(T^{-1}) \gamma_T. \]

Now for the unpenalized criterion \( G_T l(\cdot) \), by a Taylor expansion, we have

\[ T G_T (l(\theta_0 + u / T^{1/2}) - l(\theta_0)) = u T^{1/2} G_T l(\theta_0) + \frac{1}{2} u' \tilde{G}_T l(\theta_0) u + \frac{1}{6 T^{1/3}} \nabla' \{ u' \tilde{G}_T l(\tilde{\theta}) u \}, \]

where \( \tilde{\theta} \) is defined as \( \| \tilde{\theta} - \theta_0 \| \leq \| u \| / \sqrt{T} \). Then by Assumption 4 we have the central limit theorem of Billingsley (1961)

\[ \sqrt{T} G_T l(\theta_0) \xrightarrow{d} N(0, \mathbb{M}), \]

and by the ergodic theorem

\[ \tilde{G}_T l(\theta_0) \xrightarrow{p} \mathbb{H}. \]

Furthermore, we have by assumption 6

\[ | \nabla' \{ u' \tilde{G}_T l(\tilde{\theta}) u \} | \leq \frac{1}{T^2} \sum_{l,t'=1}^{T} \sum_{l,m=1}^{d} \sum_{l,m=1}^{d} u_{k_1} u_{l_1} u_{m_1} u_{k_2} u_{l_2} u_{m_2} | \tilde{\theta}_{k_1} \tilde{\theta}_{l_1} \tilde{\theta}_{m_1} l(\epsilon; \tilde{\theta}), \tilde{\theta}_{k_2} \tilde{\theta}_{l_2} \tilde{\theta}_{m_2} l(\epsilon; \tilde{\theta}) \mid \]

\[ \leq \frac{1}{T^2} \sum_{l,t'=1}^{T} \sum_{l,m=1}^{d} \sum_{l,m=1}^{d} u_{k_1} u_{l_1} u_{m_1} u_{k_2} u_{l_2} u_{m_2} v_l(C) v_{l'}(C), \]

for \( C \) large enough, such that the \( v_l(C) = \sup_{k,l,m=1,\ldots,d} \sup_{\theta : \| \theta - \theta_0 \|_2 \leq \nu T} \{ \tilde{\theta}_{k_1} \tilde{\theta}_{l_1} \tilde{\theta}_{m_1} l(\epsilon; \tilde{\theta}) \} \)

with \( \nu_T = T^{-1/2} + \lambda T^{-1} \alpha_T + \gamma T^{-1} b_T \). We deduce

\[ \nabla' \{ u' \tilde{G}_T l(\tilde{\theta}) u \} = O_p(\| u \|^3(C)). \]
Consequently, we obtain
\[
\frac{1}{6T^{1/3}} \nabla' \{ u' \mathcal{G}_T(\hat{\theta}) u \} u \xrightarrow{p} 0.
\]

Then we proved that \( F_T(u) \xrightarrow{d} F_\infty(u) \), for a fixed \( u \). Let us observe that
\[
u^*_T = \arg \min_u \{ F_T(u) \},
\]
and \( F_T(.) \) admits as a minimizer \( u^*_T = \sqrt{T}(\hat{\theta} - \theta_0) \). As \( F_T \) is convex and \( F_\infty \) is continuous, convex and has a unique minimum, then by the convexity Lemma 4.6, we obtain
\[
\sqrt{T}(\hat{\theta} - \theta_0) = \arg \min_u \{ F_T \} \xrightarrow{d} \arg \min_u \{ F_\infty \}.
\]

\[\square\]

**Theorem 4.8.** Under assumptions \( \square \square \) if \( \gamma T^{-1} \to 0 \) and \( \gamma T^{-1/2} \to \infty \) such that \( \lambda T^{-1} \to \mu_0 \), with \( \mu_0 \geq 0 \), then
\[
\frac{T}{\gamma T}(\hat{\theta} - \theta_0) \xrightarrow{d} \arg \min_u \{ \mathbb{K}_\infty(u) \},
\]
provided \( \mathbb{K}_\infty \) is a uniquely defined random function in \( \mathbb{R}^d \), where
\[
\mathbb{K}_\infty(u) = \frac{1}{2} u' \mathbb{H} u + \mu_0 \sum_{k=1}^m \alpha_k \{ \| u^{(k)} \|_1 1_{\theta_0^{(k)} = 0} + u^{(k)}' \text{sgn}(\theta_0^{(k)}) 1_{\theta_0^{(k)} \neq 0} \} \\
+ \sum_{l=1}^m \xi_l \{ \| u^{(l)} \|_2 1_{\theta_0^{(l)} = 0} + \frac{u^{(l)}' \theta_0^{(l)}}{\| \theta_0^{(l)} \|_2} 1_{\theta_0^{(l)} \neq 0} \}.
\]

The limit quantity \( \mathbb{K}_\infty(.) \) is non-random, which implies that the convergence in distribution implies the convergence in probability \( \frac{T}{\gamma T}(\hat{\theta} - \theta_0) \xrightarrow{p} \arg \min_u \{ \mathbb{K}_\infty(u) \} \) by Shiryaev (ex 7, p 259, 1995).

**Proof.** To prove this convergence result, we proceed as in Theorem 4.7. To do so, we define \( \theta = \theta_0 + u \gamma_T / T \) and we prove that \( \tilde{F}_T(u) = \mathbb{G}_T(\varphi(\theta_0 + u T / \gamma_T) - \varphi(\theta_0)) \) converges in finite distribution to \( \mathbb{K}_\infty(.) \). We have
\[
\tilde{F}_T(u) = T \{ \mathbb{G}_T(l(\theta) - l(\theta_0)) + \mathbb{P}_1(\lambda_T, \alpha, \theta) - \mathbb{P}_1(\lambda_T, \alpha, \theta_0) + \mathbb{P}_2(\gamma_T, \xi, \theta) - \mathbb{P}_2(\gamma_T, \xi, \theta_0) \} \\
= TG_T(l(\theta_0 + u \gamma_T / T) - l(\theta_0)) + \lambda_T \sum_{k=1}^m \alpha_k \| \theta_0^{(k)} + u^{(k)} \gamma_T / T \| - \| \theta_0^{(k)} \|_1 \\
+ \gamma_T \sum_{l=1}^m \xi_l [\| \theta_0^{(l)} + u^{(l)} \gamma_T / T \|_2 - \| \theta_0^{(l)} \|_2].
\]
For the unpenalized empirical criterion, we have the expansion

$$ T\gamma_T(l(\theta_0 + u\gamma_T/T) - l(\theta_0)) = \gamma_T\tilde{\gamma}_T l(\theta_0) u + \frac{\gamma^2_T}{2T} u'\tilde{\gamma}_T l(\theta_0) u + \frac{\gamma^3_T}{6T^2} \nabla \{ u'\tilde{\gamma}_T l(\bar{\theta}) u \}, $$

where $\bar{\theta}$ lies between $\theta_0$ and $\theta_0 + u\gamma_T/T$. This implies $\tilde{\gamma}_T(u) = \frac{\gamma^2_T}{\gamma_T} K_T(u)$, where

$$ K_T(u) = \frac{\sqrt{T}}{\gamma_T} (\sqrt{T}\tilde{\gamma}_T l(\theta_0) u) + \frac{1}{2} u'\tilde{\gamma}_T l(\bar{\theta}) u + \frac{\gamma_T}{\gamma_T} \nabla \{ u'\tilde{\gamma}_T l(\bar{\theta}) u \}. $$

Furthermore, by the ergodic theorem of Billingsley (1961), we have

$$ \sqrt{T}\tilde{\gamma}_T l(\theta_0) \xrightarrow{p} 0 \quad \text{as } T \to \infty, $$

and the ergodic theorem of Billingsley (1961), we have

$$ \gamma_T l(\theta_0) \xrightarrow{p} H. $$

As for the third order term, by assumption $[\mathbf{B}]$ and using the same reasoning as the
proof of Theorem 4.7 we have
\[
\frac{\gamma T}{6T} \nabla' \{u' \tilde{G}_T(l(\hat{\theta}))u\} \xrightarrow{P} 0,
\]
using \(\gamma_T = o(T)\). Then we proved that \(K_T(u) \xrightarrow{d} K_{\infty}(u)\), for a fixed \(u \in \mathbb{R}^d\). We have
\[
u_T^* = \arg \min_u \{K_T(u)\},
\]
and \(K_T(\cdot)\) admits as a minimizer \(u_T^* = T \gamma_T(\hat{\theta} - \theta_0)\). \(K_T(\cdot)\) is convex and \(K_{\infty}(\cdot)\) is continuous, then by the convexity Lemma, we deduce
\[
\frac{T}{\gamma_T}(\hat{\theta} - \theta_0) = \arg \min_u \{K_T\} \xrightarrow{d} \arg \min_u \{K_{\infty}\}.
\]

\(\square\)

**Remark.** The convergence rate of \(\hat{\theta}\) is slower than \(\sqrt{T}\). Furthermore, the limiting function is not random.

We now turn to the oracle property of the SGL. Model selection consistency consists of evaluating the probability that \(\{\hat{A} = A\}\), for \(T\) large enough. That means we check that the regularization asymptotically allows for identifying the right model.

**Proposition 4.9.** Under assumption 4.3, if \(\lambda_T T^{-1/2} \rightarrow \lambda_0\) and \(\gamma_T T^{-1/2} \rightarrow \gamma_0\), then
\[
\limsup_{T \rightarrow \infty} \mathbb{P}(\hat{A} = A) \leq c < 1,
\]
where \(c\) is a constant depending on the true model.

**Proof.** In Theorem 4.7, we proved
\[
\sqrt{T}(\hat{\theta} - \theta_0) := \arg \min_{u \in \mathbb{R}^d} \{F_T\} \xrightarrow{d} \arg \min_{u \in \mathbb{R}^d} \{F_{\infty}\},
\]
under the assumption \(\lambda_T/\sqrt{T} \rightarrow \lambda_0\) and \(\gamma_T/\sqrt{T} \rightarrow \gamma_0\). The limit random function is
\[
F_{\infty}(u) = \frac{1}{2} u' \mathbb{H} u + u' Z + \lambda_0 \sum_{k=1}^m \alpha_k \sum_{i=1}^c \{u_i^{(k)} \mathbf{1}_{\theta_0^{(i)} = 0} + u_i^{(k)} \text{sgn}(\theta_0^{(i)}) \mathbf{1}_{\theta_0^{(i)} \neq 0}\} + \gamma_0 \sum_{l=1}^m \xi_l \{\|u(l')\|_2 \mathbf{1}_{\theta_0^{(l')} = 0} + \frac{u(l')^{(l')} \theta_0^{(l')}}{\|\theta_0^{(l')}\|_2} \mathbf{1}_{\theta_0^{(l')} \neq 0}\}.
\]
First, let us observe that

\[ \{ \hat{A} = A \} = \{ \forall k = 1, \ldots, m, i \in A_k^c, \hat{\theta}_i^{(k)} = 0 \} \cap \{ \forall k = 1, \ldots, m, i \in \hat{A}_k^c, \theta_{0,i}^{(k)} = 0 \}. \]

Both sets describing \( \{ \hat{A} = A \} \) are symmetric, and thus we can focus on

\[ \{ \hat{A} = A \} \Rightarrow \{ \forall k = 1, \ldots, m, i \in A_k^c, T^{1/2} \hat{\theta}_i^{(k)} = 0 \}. \]

Hence

\[ \mathbb{P}(\hat{A} = A) \leq \mathbb{P}(\forall k = 1, \ldots, m, \forall i \in A_k^c, T^{1/2} \hat{\theta}_i^{(k)} = 0). \]

Denoting by \( u^* := \arg \min_{u \in \mathbb{R}^d} \{ \mathbb{P}_\infty (u) \} \), Theorem 4.7 corresponds to \( \sqrt{T}(\hat{\theta}_A - \theta_{0,A}) \xrightarrow{d} u_A^* \). By the Portmanteau Theorem (see Wellner and van der Vaart, 1996), we have

\[ \limsup_{T \to \infty} \mathbb{P}(\forall k = 1, \ldots, m, \forall i \in A_k^c, T^{1/2} \hat{\theta}_i^{(k)} = 0) \leq \mathbb{P}(\forall k = 1, \ldots, m, \forall i \in A_k^c, u_i^{(k)*} = 0), \]

as \( \theta_{0,A^c} = 0 \). Consequently, we need to prove that the probability of the right hand side is strictly inferior to 1, which is upper-bounded by

\[ \mathbb{P}(\forall k = 1, \ldots, m, \forall i \in A_k^c, u_i^{(k)*} = 0) \leq \min(\mathbb{P}(k \notin S, u_i^{(k)*} = 0), \mathbb{P}(k \in S, \forall i \in A_k^c, u_i^{(k)*} = 0)). \]

(4.7)

If \( \lambda_0 = \gamma_0 = 0 \), then \( u^* = -\mathbb{H}^{-1} Z \), such that \( \mathbb{P}_{u^*} = \mathcal{N}(0, \mathbb{H}^{-1} \mathbb{M} \mathbb{H}^{-1}) \). Hence, \( c = 0 \).

If \( \lambda_0 \neq 0 \) or \( \gamma_0 \neq 0 \), the necessary and sufficient optimality conditions for a group \( k \) tell us that \( u^* \) satisfies

\[ \begin{cases} (\mathbb{H} u^* + Z)(k) + \lambda_0 \alpha_k p(k) + \gamma_0 \xi_k \frac{\theta_{0,i}^{(k)}}{\| \theta_{0,i}^{(k)} \|_2} = 0, & k \in S, \\ (\mathbb{H} u^* + Z)(k) + \lambda_0 \alpha_k w(k) + \gamma_0 \xi_k z(k) = 0, & \text{otherwise}, \end{cases} \]

(4.8)

where \( w(k) \) and \( z(k) \) are the subgradients of \( \| u(k) \|_1 \) and \( \| u(k) \|_2 \) given by

\[ u_i^{(k)} = \begin{cases} \text{sgn}(u_i^{(k)}) \text{ if } u_i^{(k)} \neq 0, \\ \in \{ w_i^{(k)} : |w_i^{(k)}| \leq 1 \} \text{ if } u_i^{(k)} = 0, \end{cases} \]

\[ z^{(k)} = \begin{cases} u_i^{(k)} \text{ if } u_i^{(k)} \neq 0, \\ \in \{ z^{(k)} : \| z^{(k)} \|_2 \leq 1 \} \text{ if } u_i^{(k)} = 0, \end{cases} \]

and \( p_i^{(k)} = \partial u_i \{ u_i^{(k)} 1_{\tilde{\theta}_{0,i}^{(k)} = 0} + u_i^{(k)} \text{sgn}(\tilde{\theta}_{0,i}^{(k)}) 1_{\tilde{\theta}_{0,i}^{(k)} \neq 0} \} \).
If $u^{(m)*} = 0, \forall m \notin \mathcal{S}$, then the optimality conditions (4.8) become

$$
\begin{cases}
\mathbb{H}_{SS} u_S^* + Z_S + \lambda_0 \tau_S + \gamma_0 \zeta_S = 0, \\
\| - \mathbb{H}_{(l)S} u_S^* - Z_{(l)} - \lambda_0 \alpha_l w^{(l)} \|_2 \leq \gamma_0 \xi_l, \text{ as } \| z^{(l)} \|_2 \leq 1, l \in \mathcal{S}^c,
\end{cases}
$$

(4.9)

with $\tau_S = \text{vec}(k \in \mathcal{S}, \alpha_k p^{(k)})$ and $\zeta_S = \text{vec}(k \in \mathcal{S}, \xi_k \frac{\theta_0^{(k)}}{\| \theta_0^{(k)} \|_2})$, which are vectors of $\mathbb{R}^{\text{card}(\mathcal{S})}$.

For $k \in \mathcal{S}$, that is the vector $\theta_0^{(k)}$ is at least non-zero, then

$$
\begin{cases}
(\mathbb{H} u^* + Z)_i + \lambda_0 \alpha_k \text{sgn}(\theta_0^{(k)}) + \gamma_0 \xi_k \frac{\theta_0^{(k)}}{\| \theta_0^{(k)} \|_2} = 0, \text{ if } k \in \mathcal{S}, i \in \mathcal{A}_k, \\
(\mathbb{H} u^* + Z)_i + \lambda_0 \alpha_k w^{(k)} = 0, i \in \mathcal{A}_k^c.
\end{cases}
$$

(4.10)

Consequently, if $u_i^{(k)*} = 0, \forall i \in \mathcal{A}_k^c$, with $k \in \mathcal{S}$, then the conditions (4.10) become

$$
\begin{cases}
(\mathbb{H}_{A_k^c A_k} u^*_{A_k} + Z_{A_k} + \lambda_0 \alpha_k \text{sgn}(\theta_{0,A_k}) + \gamma_0 \xi_k \frac{\theta_{0,A_k}}{\| \theta_{0,A_k} \|_2} = 0, \\
\| - (\mathbb{H}_{A_k^c A_k} u^*_{A_k} + Z_{A_k^c}) \|_i \leq \lambda_0 \alpha_k.
\end{cases}
$$

Combining relationships in (4.9), we obtain

$$
\| \mathbb{H}_{(l)S} \mathbb{H}_{SS}^{-1} (Z_S + \lambda_0 \tau_S + \gamma_0 \zeta_S) - Z_{(l)} - \lambda_0 \alpha_l w^{(l)} \|_2 \leq \gamma_0 \xi_l, l \in \mathcal{S}^c.
$$

The same reasoning applies for active groups with inactive components, such that combining relationships in (4.10), we obtain

$$
\| (\mathbb{H}_{A_k^c A_k} \mathbb{H}_{A_k^c A_k}^{-1} (Z_{A_k} + \lambda_0 \alpha_k \text{sgn}(\theta_{0,A_k}) + \gamma_0 \xi_k \frac{\theta_{0,A_k}}{\| \theta_{0,A_k} \|_2}) - Z_{A_k^c} \|_i \leq \lambda_0 \alpha_k.
$$

Hence we deduce

$$
\mathbb{P}(\forall k = 1, \cdots, m, \forall \in \mathcal{A}_k^c, u_i^{(k)*} = 0) \leq \min(\mathbb{P}(k \notin \mathcal{S}, u^{(k)*} = 0), \mathbb{P}(k \in \mathcal{S}, \forall \in \mathcal{A}_k^c, u_i^{(k)*} = 0)) := \min(a_1, a_2).
$$

Under the assumption that $\lambda_0 < \infty$ and $\gamma_0 < \infty$, we obtain

$$
a_1 = \mathbb{P}(l \in \mathcal{S}^c, \| \mathbb{H}_{(l)S} \mathbb{H}_{SS}^{-1} (Z_S + \lambda_0 \tau_S + \gamma_0 \zeta_S) - Z_{(l)} - \lambda_0 \alpha_l w^{(l)} \|_2 \leq \gamma_0 \xi_l) < 1,
$$

$$
a_2 = \mathbb{P}(k \in \mathcal{S}, i \in \mathcal{A}_k^c, \| (\mathbb{H}_{A_k^c A_k} \mathbb{H}_{A_k^c A_k}^{-1} (Z_{A_k} + \lambda_0 \alpha_k \text{sgn}(\theta_{0,A_k}) + \gamma_0 \xi_k \frac{\theta_{0,A_k}}{\| \theta_{0,A_k} \|_2}) - Z_{A_k^c} \|_i \leq \lambda_0 \alpha_k) < 1.
$$
Thus $c < 1$, which proves (4.7), that is proposition 4.9.

**Remark.** The result in Proposition 4.9 highlights that the SGL as proposed by Simon and al. (2013) cannot achieve the oracle property since the penalties cannot recover the unknown set of active indices $A$, which is called model selection consistency. To fix this drawback in an ordinary least square framework, Zou (2006) proposes the adaptive LASSO, where random weights are used to penalize different coefficients and proves that the adaptive LASSO estimator satisfies the oracle property in the sense of Fan and Li (2001), that is asymptotic normality and selection consistency for a proper choice of $\lambda_T$ and $\alpha_i^{(k)}$. That is also the case for the adaptive group LASSO model proposed by Nardi and Rinaldo (2008), where adaptive weights are used to penalize grouped coefficients differently. We propose the same approach than Zou (2006) and use adaptive weights in the penalties such that the adaptive SGL enjoys the oracle property in the sense of Fan and Li (2001) as proved in Theorem 4.10.

The adaptive specification of the proposed estimator now becomes

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \{ G_T \psi(\theta) \},$$

(4.11)

where

$$\theta \mapsto G_T \psi(\theta) = \frac{1}{T} \sum_{t=1}^T l(\epsilon_t; \theta) + p_1(\lambda_T, \tilde{\theta}, \theta) + p_2(\gamma_T, \tilde{\theta}, \theta)$$

$$= G_T l(\theta) + p_1(\lambda_T, \tilde{\theta}, \theta) + p_2(\gamma_T, \tilde{\theta}, \theta),$$

such that both penalties are specified as

$$p_1(\lambda_T, \tilde{\theta}, \theta) = \lambda_T T^{-1} \sum_{k=1}^m \sum_{i=1}^c \alpha(\tilde{\theta}_i^{(k)}) |\theta_i^{(k)}|, p_2(\gamma_T, \tilde{\theta}, \theta) = \gamma_T T^{-1} \sum_{l=1}^m \xi(l) \|\theta(l)\|_2.$$ 

These penalties are now randomized through the $\tilde{\theta}$ argument in the weights $\alpha$’s and $\xi$’s. This first step estimator $\tilde{\theta}$ is supposed to be a $T^{1/2}$-consistent estimator of $\theta_0$. For instance, it can be defined as an M-estimator of the unpenalized empirical criterion $G_T l(\cdot)$, that is

$$\tilde{\theta} = \arg\min_{\theta \in \Theta} G_T l(\theta).$$

Adaptive weights are also used by Zou and Zhang (2009), who plug the elastic-net estimator in the adaptive weight and then estimate a new elastic net model using these weights, that is the adaptive elastic net.
The weights we use are now random and for any group \( k \) or \( l \), \( \alpha(\tilde{\theta}^{(k)}) \in \mathbb{R}_{+}^{c_k}, \xi(\tilde{\theta}^{(l)}) \in \mathbb{R}_{+} \) are specified as

\[
\alpha^{(k)}_{T} := \alpha(\tilde{\theta}^{(k)}) = (|\tilde{\theta}^{(k)}_i| - \eta, i = 1, \cdots, c_k), \xi_{T,l} := \xi(\tilde{\theta}^{(l)}) = \|\tilde{\theta}^{(l)}\|_2^{-\mu},
\]

for some constants \( \eta > 0 \) and \( \mu > 0 \) (to be specified).

**Remark.** Theorem 4.10 can be adapted to the adaptive specification of the SGL in (4.11), where the bound in probability of the error would be

\[
\|\hat{\theta} - \theta_0\| = O_p(T^{-1/2} + \lambda_{T}T^{-1}a_{T} + \gamma_{T}T^{-1}b_{T}),
\]

with \( a_T = \text{card}(A)\{\max_{k \in S} (\max_{i \in A_k} \alpha^{(k)}_{T,i})\}, b_T = \text{card}(A)\{\max_{l \in S} \xi_{T,l}\}, \) such that \( \lambda_{T}T^{-1}a_{T} \xrightarrow{P} 0 \) and \( \gamma_{T}T^{-1}b_{T} \xrightarrow{P} 0 \). The proof follows exactly the same steps as for Theorem (4.5), except \( a_T \) and \( b_T \) are random quantities.

**Theorem 4.10.** Under assumptions 4.10 if \( \lambda_{T}T^{-1/2} \rightarrow 0, \gamma_{T}T^{-1/2} \rightarrow 0, T^{(\eta-1)/2} \lambda_{T} \rightarrow \infty \) and \( T^{(\mu-1)/2} \gamma_{T} \rightarrow \infty \), then \( \hat{\theta} \) obtained in (4.11) satisfies

\[
\lim_{T \rightarrow \infty} \mathbb{P}(\hat{A} = A) = 1, \text{ and } \sqrt{T}(\hat{\theta}_{A} - \theta_{0,A}) \xrightarrow{d} \mathcal{N}(0, \mathbb{H}_{A\lambda}^{-1}M_{A\lambda}\mathbb{H}_{A\lambda}^{-1}).
\]

**Proof.** We start with the asymptotic distribution and proceed as in the proof of Theorem 4.7 where we used Lemma 4.6. To do so, we prove the finite dimensional convergence in distribution of the empirical criterion \( F_T(u) \) to \( F_{\infty}(u) \) with \( u \in \mathbb{R}^d \), where these quantities are respectively defined as

\[
F_T(u) = TG_T(\psi(\theta_0 + u/\sqrt{T}) - \psi(\theta_0)) = TG_T(l(\theta_0 + u/\sqrt{T}) - l(\theta_0)) + \lambda_T \sum_{k=1}^{m} \sum_{i=1}^{c_k} \alpha^{(k)}_{T,i}(\theta_{0,i}^{(k)} + u_i^{(k)})/\sqrt{T} - |\theta_{0,i}^{(k)}|)
+ \gamma_T \sum_{l=1}^{m} \xi_{T,l}(\theta_0^{(l)} + u_l^{(l)})/\sqrt{T} - \|\theta_0^{(l)}\|_2 - \|\theta_0^{(l)}\|_2,
\]

and

\[
F_{\infty}(u) = \begin{cases} 
1/2 u_A'\mathbb{H}_{A\lambda}u_A + u_A'Z_A & \text{if } u_i = 0, \text{ when } i \notin A, \text{ and } \\
\infty & \text{otherwise,}
\end{cases}
\]

with \( Z_A \sim \mathcal{N}(0, M_{A\lambda}). \) By Lemma 4.6, the finite dimensional convergence in distribution implies \( \arg \min_{u \in \mathbb{R}^d}\{F_T(u)\} \xrightarrow{d} \arg \min_{u \in \mathbb{R}^d}\{F_{\infty}(u)\}. \) We first consider the
unpenalized empirical criterion of $\mathbb{F}_T(\cdot)$, which can be expanded as

$$T\mathbb{G}_T(\psi(\theta_0 + u/\sqrt{T}) - \psi(\theta_0)) = T^{1/2}\mathbb{G}_T(\theta_0)u + \frac{1}{2}u'\mathbb{G}_T(\bar{\theta})u + \frac{1}{6T^{1/3}}\nabla'\{u'\mathbb{G}_T(\bar{\theta})\}u,$$

where $\bar{\theta}$ lies between $\theta_0$ and $\theta_0 + u/\sqrt{T}$. First, using the same reasoning on the third order term, we obtain

$$\frac{1}{6T^{1/3}}\nabla'\{u'\mathbb{G}_T(\bar{\theta})\}u \xrightarrow{P} 0.$$

By the ergodic theorem, we deduce $\mathbb{G}_T(\theta_0) \xrightarrow{P} \mathbb{I}$ and by assumption $\sqrt{T}\mathbb{G}_T(\theta_0) \xrightarrow{d} \mathcal{N}(0, \mathbb{M})$.

We now focus on the penalty terms of (1.11), we remind that $\alpha_{T,i}^{(k)} = |\tilde{\theta}_i^{(k)}|^{-\eta}$, such that for $i \in A_k$, $k \in \mathcal{S}$, $\tilde{\theta}_i^{(k)} \xrightarrow{P} \theta_{0,i}^{(k)} \neq 0$. Note that

$$\sqrt{T}(|\theta_0^{(k)} + u^{(k)}/\sqrt{T}| - |\theta_0^{(k)}|) \xrightarrow{P} u_i^{(k)} \text{sgn}(\theta_{0,i}^{(k)})1_{\theta_{0,i}^{(k)} \neq 0}.$$

This implies that, for $i \in A_k$, $k \in \mathcal{S}$, we have

$$\lambda T^{-1/2} \sum_{i=1}^{c_k} \alpha_{T,i}^{(k)} \sqrt{T}(|\theta_0^{(k)} + u_i^{(k)}/\sqrt{T}| - |\theta_0^{(k)}|) \xrightarrow{P} 0,$$

under the condition $\lambda T^{-1/2} \to 0$. For $i \in A_k$, $\theta_{0,i}^{(k)} = 0$, then $T^{\eta/2}(|\tilde{\theta}_i^{(k)}|)^{\eta} = O_p(1)$. Hence under the assumption $\lambda T^{(\eta-1)/2} \to \infty$, we obtain

$$\lambda T^{-1/2} \alpha_{T,i}^{(k)} \sqrt{T}(\theta_0^{(k)} + u_i^{(k)}/\sqrt{T}) - |\theta_0^{(k)}| = \lambda T^{-1/2} u_i^{(k)} \frac{T^{\eta/2}}{(T^{1/2}|\tilde{\theta}_i^{(k)}|)^{\eta}} \xrightarrow{P} \infty.$$

As for the $l^1/l^2$ quantity, we remind that $\xi_{T,l} = \|\tilde{\theta}^{(l)}\|^{-\mu}_2$, such that for $l \in \mathcal{S}$, $\tilde{\theta}^{(l)} \xrightarrow{P} \theta_0^{(l)}$, and in this case

$$\sqrt{T}\{\|\theta_0^{(l)} + u^{(l)}/\sqrt{T}\|_2 - \|\theta_0^{(l)}\|_2\} = \frac{u_i^{(l)}/\theta_0^{(l)}}{\|\theta_0^{(l)}\|_2} + o(T^{-1/2}).$$

Consequently, using $\gamma T^{-1/2} \to 0$, and for $l \in \mathcal{S}$, we obtain

$$\gamma T^{-1/2} \sqrt{T}\xi_{T,l}(\|\theta_0^{(l)} + u^{(l)}/\sqrt{T}\|_2 - \|\theta_0^{(l)}\|_2) \xrightarrow{P} 0.$$

Combining the fact $k \in \mathcal{S}$ and $\theta_0^{(k)}$ is partially zero, that is $i \in A_k$, we obtain the
divergence given in (4.13). Furthermore, if \( l \notin S \), that is \( \theta_0^{(l)} = 0 \), then

\[
\sqrt{T} \{\|\theta_0^{(l)} + u^{(l)}/\sqrt{T}\|_2 - \|\theta_0^{(l)}\|_2\} = \|u^{(l)}\|_2,
\]

and \( T^{\mu/2}(\|\hat{\theta}^{(l)}\|_2)^\mu = O_p(1) \), then under the assumption \( \gamma T^{-\mu/2} \rightarrow \infty \), we obtain

\[
\gamma T^{-1/2} \xi T^{1/2} \sqrt{T}[\|\theta_0^{(l)} + u^{(l)}/\sqrt{T}\|_2 - \|\theta_0^{(l)}\|_2] = \gamma T^{-1/2} ||u^{(l)}||_2 \frac{T^{\mu/2}}{(T^{1/2} \|\hat{\theta}^{(l)}\|_2)^\mu} \xrightarrow{p} \infty.
\]

We deduce the pointwise convergence \( F_T(u) \xrightarrow{d} F_\infty(u) \), where \( F_\infty(.) \) is given in (4.12). As \( F_T(.) \) is convex and \( F_\infty(.) \) is convex and has a unique minimum \( (\mathbb{H}^{-1} A_A Z_A, 0_{A^c}) \), by Lemma 4.6 we obtain

\[
\sqrt{T}(\hat{\theta} - \theta_0) = \arg\min_{u \in \mathbb{R}^d} \{F_T(u)\} \xrightarrow{d} \arg\min_{u \in \mathbb{R}^d} \{F_\infty(u)\},
\]

that is to say

\[
\sqrt{T}(\hat{\theta}_A - \theta_{0,A}) \xrightarrow{d} \mathbb{H}^{-1} A_A Z_A, \text{ and } \sqrt{T}(\hat{\theta}_{A^c} - \theta_{0,A^c}) \xrightarrow{d} 0_{A^c}.
\]

We now prove the model selection consistency. Let \( i \in A_k \), then by the asymptotic normality result, \( \hat{\theta}_i^{(k)} \xrightarrow{p} \theta_0^{(k)} \), which implies \( \mathbb{P}(i \in \hat{A}_k) \rightarrow 1 \). Thus the proof consists of proving

\[
\forall k = 1, \cdots, m, \forall i \in A_k^c, \mathbb{P}(i \in \hat{A}_k) \rightarrow 0.
\]

This problem can be split into two parts as

\[
\forall k \notin S, \mathbb{P}(k \in \hat{S}) \rightarrow 0, \text{ and } \forall k \in S, \forall i \in A_k^c, \mathbb{P}(i \in \hat{A}_k) \rightarrow 0. \quad (4.14)
\]

Let us start with the case \( k \notin S \). If \( k \in \hat{S} \), by the optimality conditions given by the Karush-Kuhn-Tucker theorem applied on \( G_T \psi(\hat{\theta}) \), we have

\[
\hat{G}_T l(\hat{\theta})^{(k)} + \frac{\lambda_T^{(k)}}{T^{1/2}} \hat{\omega}_T^{(k)} \odot \omega_T^{(k)} + \frac{\gamma T^{1/2}}{T^{1/2} \|\hat{\theta}^{(k)}\|_2} \hat{\omega}_T^{(k)} = 0,
\]

\( \odot \) is the Hadamard product and

\[
\omega_T^{(k)} = \begin{cases} 
\text{sgn}(\hat{\theta}_i^{(k)}) & \text{if } \hat{\theta}_i^{(k)} \neq 0, \\
\{\omega_T^{(k)} : |\omega_T^{(k)}| \leq 1 \} & \text{if } \hat{\theta}_i^{(k)} = 0.
\end{cases}
\]
Multiplying the unpenalized part by $T^{1/2}$, we have the expansion
\[
T^{1/2}G_{T,l}(\hat{\theta}(k)) = T^{1/2}G_{T,l}(\theta_0(k)) + T^{1/2}G_{T,l}(\theta_0(k)) (\hat{\theta} - \theta_0(k)) + T^{1/2}\nabla' \{(\hat{\theta} - \theta_0)^T(\hat{\theta}(k) - \theta_0(k))\},
\]
which is asymptotically normal by consistency, assumption \[ \text{regarding the bound on the third order term, the Slutsky theorem and the central limit theorem of Billingsley (1961).} \] Furthermore, we have
\[
\gamma_T T^{-1/2} \xi_{T,k} \frac{\hat{\theta}(k)}{\|\hat{\theta}(k)\|_2} = \gamma_T T^{(\mu-1)/2}(T^{1/2}\|\hat{\theta}(k)\|_2)^{-\mu} \frac{\hat{\theta}(k)}{\|\hat{\theta}(k)\|_2} \xrightarrow{P} \infty, T \to \infty.
\]
We obtain the same when adding $\lambda_T T^{-1/2} \alpha_T^{(k)}(\hat{\theta}(k))$. Therefore, we have
\[
\forall k \notin S, \mathbb{P}(k \in \hat{S}) \leq \mathbb{P}(-G_{T,l}(\hat{\theta}(k)) = \frac{\lambda_T}{T} \alpha_T^{(k)}(\hat{\theta}(k)) + \frac{\gamma_T}{T} \xi_{T,k} \frac{\hat{\theta}(k)}{\|\hat{\theta}(k)\|_2} \to 0.
\]

We now pick $k \in S$ and consider the event \{\[i \in \hat{A}_k\]. Then the Karush-Kuhn-Tucker conditions for $G_{T,l}(\hat{\theta})$ are given by
\[
(G_{T,l}(\hat{\theta}))(k,i) + \frac{\lambda_T}{T} \alpha_T^{(k)}(\hat{\theta}(k)) + \frac{\gamma_T}{T} \xi_{T,k} \frac{\hat{\theta}(k)}{\|\hat{\theta}(k)\|_2} = 0.
\]
Using the same reasoning as previously, $T^{1/2}(G_{T,l}(\hat{\theta}))(k,i)$ is also asymptotically normal, and $\hat{\theta}(k) \xrightarrow{P} \theta_0(k)$ for $k \in S$, and besides
\[
\lambda_T T^{-1/2} \alpha_T^{(k)}(\hat{\theta}(k)) \xrightarrow{P} \theta_0(k), T \to \infty,
\]
such that we obtain the same when adding $\gamma_T T^{-1/2} \xi_{T,k} \frac{\hat{\theta}(k)}{\|\hat{\theta}(k)\|_2}$. Therefore, we have
\[
\forall k \in S, \forall i \notin A_k, \mathbb{P}(i \in \hat{A}_k) \leq \mathbb{P}(-G_{T,l}(\hat{\theta}))(k,i) = \frac{\lambda_T}{T} \alpha_T^{(k)}(\hat{\theta}(k)) + \frac{\gamma_T}{T} \xi_{T,k} \frac{\hat{\theta}(k)}{\|\hat{\theta}(k)\|_2} \to 0.
\]
We have proved (4.14).
5 Double asymptotic

In the previous sections, we worked with a fixed dimension $d$, where $d = \sum_{k=1}^{m} c_k$. From now on, let us consider the case where $d = d_T$, such that $d_T \to \infty$ as $T \to \infty$. Note that $\text{card}(S) = O(\text{card}(A)) = O(d_T)$. The speed of growth of the dimension is supposed to be $d_T = O(T^c)$ for some $q_2 < c < q_1$. In this section, we prove that the adaptive SGL enjoys the oracle property, that is model selection consistency and optimal rate of convergence for proper choices of $0 \leq q_1 < q_2 < 1$. We highlight that our general framework unfortunately hampers a high degree of flexibility on the behavior of $d_T$, that is $c$ cannot be set in $(0, 1)$. This issue was encountered by Fan and Peng (2004) in an i.i.d. and non-adaptive framework. This lack of flexibility is a necessary cost to cope with the random remainder of the Taylor expansions as we should take the third order term into account. This problem is moved aside when considering the simple linear model, where the third order derivative is zero. For instance, Zou and Zhang (2009) proved the oracle property of the adaptive elastic-net in a double-asymptotic framework for linear models where $0 \leq c < 1$.

For the asymptotic normality, we use the method of Fan and Peng (2004) and Zou and Zhang (2009), where we derive the asymptotic distribution of the discrepancy $\sqrt{T}(\hat{\theta} - \theta_0)_A$ times a matrix sequence $(Q_T)$ of size $r \times \text{card}(A)$, $r$ being arbitrary but finite. This allows for switching from infinite dimensional distribution to finite dimensional distribution, where we can apply usual tools of asymptotic analysis.

In this section, we provide the conditions to achieve the oracle property as in Fan and Peng (2004) or Zou and Zhang (2009). In this double asymptotic framework, the quantities depend on $d_T$, hence on $T$. They should be indexed by $T$, which expresses that the dimension depend on the sample size. In the rest of the paper, we denote $H_T := E[\nabla^2 \theta l(\epsilon_t; \theta_0)]$ and $M_T := E[\nabla \theta l(\epsilon_t; \theta_0)\nabla \theta l(\epsilon_t; \theta_0)]$. To make the reading easier, we do not index other quantities by $T$, which will be implicit. We remind that the criterion is

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \{G_T l(\theta) + p_1(\lambda_T, \hat{\theta}, \theta) + p_2(\gamma_T, \hat{\theta}, \theta)\}$$

$$= \arg\min_{\theta \in \Theta} \left\{\frac{1}{T} \sum_{t=1}^{T} l(\epsilon_t; \theta) + \frac{\lambda_T}{T^c} m \sum_{k=1}^{m} c_k |\hat{\theta}_i^{(k)}| + \frac{\gamma_T}{T^c} m \sum_{l=1}^{m} \xi_{T,l} \|\theta^{(l)}\|_2\right\},$$

with $\alpha^{(k)}_{T,i} = |\hat{\theta}_i^{(k)}|^{-\eta}$ and $\xi_{T,l} = \|\hat{\theta}^{(l)}\|_2^{-\mu}$, where $\eta > 0, \mu > 0$, and $\hat{\theta}$ is a first step estimator satisfying

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \{G_T l(\theta)\}. $$

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The double asymptotic framework implies that the empirical criterion can be viewed as a sequence of dependent random variables for which we need refined asymptotic theorems for dependent sequence of arrays. Shiryaev (1991) proposes a version of the central limit theorem for dependent sequence of arrays, provided this sequence is a square integrable martingale difference satisfying the so-called Lindeberg condition. A similar theorem can be found in Billingsley (1995, theorem 35.12, p.476). We provide here the theorem of Shiryaev (see Theorem 4, p.543 of Shiryaev, 1991) that we will use to derive the asymptotic distribution of the adaptive SGL estimator.

**Theorem 5.1.** (Shiryaev, 1991)

Let a sequence of square-integrable martingale differences $\xi^T = (\xi_t, F_t^T), T \geq 0$, with $F_t^T = \sigma(\xi_s, s \leq t)$, satisfy the Lindeberg condition, for $\epsilon > 0$, given by

$$
\sum_{t=0}^{T} E[\xi_t^2 1_{|\xi_t| > \epsilon |F_{t-1}^T|}] \overset{p}{\rightarrow} 0,
$$

then if $\sum_{t=0}^{T} E[\xi_t^2 |F_{t-1}^T|] \overset{p}{\rightarrow} \sigma_t^2$, or $\sum_{t=0}^{T} \xi_t^2 \overset{p}{\rightarrow} \sigma_t^2$, then $\sum_{t=0}^{T} \xi_t^2 \overset{d}{\rightarrow} N(0, \sigma_t^2)$.

**Remark.** Note that central limit theorems relaxing the stationarity and martingale difference assumptions for sequences of arrays exist. Neumann (2013) proposes such a central limit theorem for weakly dependent sequences of arrays. Such sequences should also satisfy a Lindeberg condition and conditions on covariances. In the rest of the paper, we use Shiryaev’s result.

We consider problem (5.1), which is the adaptive SGL estimator. In the first step, we study the convergence rate of the first step unpenalized estimator, which is plugged in the adaptive specification. The convergence rate of a classic M-estimator is $T^{1/2}$, for $d$ fixed. For $d$ diverging, we need some additional assumptions.

The two next assumptions are similar to condition (F) of Fan and Peng (2004) and allows for controlling the minimum and maximum eigenvalues of the limits of the empirical Hessian and the score cross-product. We denote by $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ the minimum and maximum eigenvalues of any positive definite square matrix $M$.

**Assumption 7.** $H_T$ and $M_T$ exist. $H_T$ is nonsingular, and there exist $b_1, b_2$ with $0 < b_1 < b_2 < \infty$ and $c_1, c_2$ with $0 < c_1 < c_2 < \infty$ such that, for all $T$,

$$
b_1 < \lambda_{\min}(M_T) < \lambda_{\max}(M_T) < b_2, c_1 < \lambda_{\min}(H_T) < \lambda_{\max}(H_T) < c_2.
$$
Let $V_T = H_T^{-1}M_T H_T^{-1}$, we deduce there exist $a_1, a_2$ with $0 < a_1 < a_2 < \infty$ such that, for all $T$,
\[ a_1 < \lambda_{\min}(V_T) < \lambda_{\max}(V_T) < a_2. \]

**Assumption 8.** $\mathbb{E}[\{\nabla g_l(\epsilon_t; \theta_0) \nabla g_{l'}(\epsilon_{t'}; \theta_0)\}^2] < \infty$, for every $d_T$ (and then of $T$).

**Assumption 9.** There exist some functions $\Psi(.)$ such that, for all $T$,
\[
\sup_{k=1,\cdots,d_T} \mathbb{E}[\partial_{\theta_k} l(\epsilon_t; \theta) \partial_{\theta_k} l(\epsilon_t'; \theta)] \leq \Psi(|t - t'|),
\]
and
\[
\sup_T \frac{1}{T} \sum_{t, t'=1}^T \Psi(|t - t'|) < \infty.
\]

**Assumption 10.** Let $\zeta_{kl,t} := \partial^2_{\theta_k \theta_l} l(\epsilon_t; \theta_0) - \mathbb{E}[\partial^2_{\theta_k \theta_l} l(\epsilon_t; \theta_0)]$. There exist some functions $\chi(.)$ such that
\[
|\mathbb{E}[\zeta_{kl,t} \zeta_{kl,t'}]| \leq \chi(|t - t'|),
\]
and
\[
\sup_T \frac{1}{T} \sum_{t, t'=1}^T \chi(|t - t'|) < \infty.
\]

**Assumption 11.** Let $\nu_t(C) := \sup_{k,l,m=1,\cdots,d_T} \{ \sup_{\theta_0: \|\theta - \theta_0\|_2 \leq \nu_T C} |\partial^3_{\theta_k \theta_l \theta_m} l(\epsilon_t; \theta)| \}$, where $C > 0$ is a fixed constant and $\nu_T = (d_T/T)^{1/2}$. Then
\[
\eta(C) := \frac{1}{T^2} \sum_{t, t'=1}^T \mathbb{E}[\nu_t(C) \nu_{t'}(C)] < \infty.
\]

**Theorem 5.2.** Under Assumptions 1-3, 7-11 and if $d_T^4 = o(T)$, the sequence of unpenalized M-estimators solving \( \tilde{\theta} = \arg \min_{\theta \in \Theta} \{ g_T l(\theta) \} \) satisfies
\[
\| \tilde{\theta} - \theta_0 \|_2 = O_p\left(\frac{d_T}{T}\right)^{1/2}.
\]

Both vectors $\tilde{\theta}$ and $\theta_0$ depend on $T$ such that $\tilde{\theta} = \tilde{\theta}_T$ and $\theta_0 = \theta_{0,T} := \theta_{0,\infty} e_T$.

**Remark.** Note that this consistency result requires at most $d_T^4 = o(T)$, as Theorem 1 of Fan and Peng (2004).

**Proof.** We proceed as in the proof of Theorem 4.5. We denote $\nu_T = (d_T/T)^{1/2}$ and we would like to prove that, for any $\epsilon > 0$, there exists $C_{\epsilon} > 0$ such that
\[
\mathbb{P}(\| \tilde{\theta} - \theta_0 \|_2/\nu_T > C_{\epsilon}) < \epsilon.
\]

(5.2)
To prove (5.2), it is sufficient to show that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that
\[
P(\|\tilde{\theta} - \theta_0\|_2 > C_\epsilon \nu_T) \leq P(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 \geq C_\epsilon : \mathcal{G}_T l(\theta_0 + \nu_T u) \leq \mathcal{G}_T l(\theta_0))
\]
\[
= P(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \mathcal{G}_T l(\theta_0 + \nu_T u) \leq \mathcal{G}_T l(\theta_0)),
\]
by convexity. By a Taylor expansion of $\mathcal{G}_T l(\theta_0 + \nu_T u)$, we obtain
\[
\mathcal{G}_T l(\theta_0 + \nu_T u) = \mathcal{G}_T l(\theta_0) + \nu_T \mathcal{G}_T \mathcal{T} l(\theta_0) u + \frac{\nu_T^2}{2} u' \mathcal{G}_T \mathcal{T} l(\theta_0) u + \frac{\nu_T^3}{6} \nabla' \{ u' \mathcal{G}_T \mathcal{L} l(\tilde{\theta}) u \} u,
\]
where $\tilde{\theta} \in \Theta$ such that $\|\tilde{\theta} - \theta_0\|_2 \leq C_\epsilon \nu_T$. We would like to prove
\[
P(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \nu_T \mathcal{G}_T \mathcal{T} l(\theta_0) u + \frac{\nu_T^2}{2} u' \mathcal{G}_T \mathcal{T} l(\theta_0) u + \frac{\nu_T^3}{6} \nabla' \{ u' \mathcal{G}_T \mathcal{L} l(\tilde{\theta}) u \} u \leq 0) < \epsilon.
\]
(5.3)
To do so, we focus on each quantity of the Taylor expansion to extract the dominant term. First, for $a > 0$ and the Markov inequality, we have for the score term
\[
P(\sup_{\|u\|_2 = C_\epsilon} |\mathcal{G}_T l(\theta_0) u| > a) \leq P(\sup_{\|u\|_2 = C_\epsilon} \|\mathcal{G}_T l(\theta_0)\|_2 \|u\|_2 > a)
\]
\[
\leq P(\|\mathcal{G}_T l(\theta_0)\|_2 > \frac{a}{C_\epsilon})
\]
\[
\leq \left(\frac{C_\epsilon}{a}\right)^2 \mathbb{E}[\|\mathcal{G}_T l(\theta_0)\|_2^2]
\]
\[
\leq \left(\frac{C_\epsilon}{a}\right)^2 \sum_{k=1}^{d_T} \mathbb{E}[(\partial_{\theta_k} \mathcal{G}_T l(\theta_0))^2]
\]
\[
= \left(\frac{C_\epsilon}{a}\right)^2 \frac{1}{T^2} \sum_{t,t'=1}^{T} \sum_{k=1}^{d_T} \mathbb{E}[(\partial_{\theta_k} \mathcal{G}_T l(\theta_0)) (\partial_{\theta_k} \mathcal{G}_T l(\theta_0))]
\]
\[
\leq \left(\frac{C_\epsilon}{a}\right)^2 \{ \frac{1}{T^2} \sum_{t,t'=1}^{T} \Psi(|t - t'|) \}.d_T.
\]
By assumption \( \sup_{k=1, \ldots, d_T} \mathbb{E}[\partial_{\theta_k} l(\epsilon_t; \theta_0) \partial_{\theta_k} l(\epsilon_t'; \theta_0)] \leq \Psi(|t - t'|) \) and \( \frac{1}{T} \sum_{t,t'=1}^{T} \Psi(|t - t'|) < \infty \). This implies
\[
P(\sup_{\|u\|_2 = C_\epsilon} |\mathcal{G}_T l(\theta_0) u| > a) \leq \frac{C_\epsilon^2 d_T}{Ta^2} K_1,
\]
for some constant $K_1 > 0$.

We now focus on the hessian quantity that can be rewritten as
\[
u_T \mathcal{G}_T \mathcal{T} l(\theta_0) u = u' \mathbb{E}[\mathcal{G}_T \mathcal{L} l(\theta_0)] u + R_T(\theta_0),
\]

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where $\mathcal{R}_T(\theta_0) = \sum_{k,l=1}^{d_T} u_k u_l \{ \partial^2_{\theta_k \theta_l} \mathcal{G}_T I(\theta_0) - \mathbb{E}[\partial^2_{\theta_k \theta_l} \mathcal{G}_T I(\theta_0)] \}$. We have

$$
\mathbb{E}[\mathcal{R}_T(\theta_0)] = 0, \quad \text{Var}(\mathcal{R}_T(\theta_0)) = \frac{1}{T^2} \sum_{t,t'=1}^{T} \sum_{k,k',l,l'=1}^{d_T} u_k u_{k'} u_l u_{l'} \mathbb{E}[\zeta_{k,l,t} \zeta_{k',l',t'}],
$$

where $\zeta_{k,l,t} = \partial^2_{\theta_k \theta_l} I(\varepsilon_t; \theta_0) - \mathbb{E}[\partial^2_{\theta_k \theta_l} I(\varepsilon_t; \theta_0)]$. Let $b > 0$, we deduce by the Markov inequality and assumption 10,

$$
\mathbb{P}(|\mathcal{R}_T(\theta_0)| > b) \leq \frac{1}{b^2} \mathbb{E}[\mathcal{R}_T^2(\theta_0)] \leq \frac{K_2 \|u\|_2^2 d_T^2}{b^2} \leq \frac{K_2 C^4_\varepsilon d_T^2}{b^2 T},
$$

where $K_2 > 0$. Furthermore, by assumption 7,

$$
u' \mathbb{E}[\mathcal{G}_T I(\theta_0)] u \geq \lambda_{\min}(\mathbb{H}_T) u' u.
$$

As for the third order term, we have

$$
|\nabla \{ u' \mathcal{G}_T l(\theta) u \}| \leq \frac{1}{T^2} \sum_{t,t'=1}^{T} \sum_{k_1,k_2,k_3,l_2,l_3} |u_{k_1} u_{k_2} u_{k_3} u_l u_{l_2} u_{l_3} | |\partial^3_{\theta_{k_1} \theta_{k_2} \theta_{k_3}} l(\varepsilon_t; \bar{\theta}) | \leq \|u\|_2^6 \frac{1}{T^2} \sum_{t,t'=1}^{T} v_t(C) v'_t(C),
$$

where

$$
v_t(C) = \sup_{k_1,k_2,k_3} \sup_{\|\bar{\theta} - \theta_0\|_2 \leq \nu_T C_0} |\partial^3_{\theta_{k_1} \theta_{k_2} \theta_{k_3}} l(\varepsilon_t; \bar{\theta})|.
$$

Note that $v_t(C_0)$ depends on $d_T$ and $C_0$. By assumption 11 we have

$$
\eta(C_0) := \frac{1}{T^2} \sum_{t,t'=1}^{T} \mathbb{E}[v_t(C_0) v'_t(C_0)] < \infty.
$$

By the Markov inequality, for $c > 0$, we conclude that

$$
\mathbb{P}(\exists u, \|u\|_2 = C_\varepsilon : \frac{\nu^2_T}{6 \|\bar{\theta} - \theta_0\|_2 \leq \nu_T} |\nabla \{ u' \mathcal{G}_T l(\theta) u \}| > c) \leq \frac{\nu^4_T d_T^2 C^6_\varepsilon}{36 c^2} \eta(C_\varepsilon).
$$

We can now bound (5.3) thanks to proper choices of $a, b, c$ and $C_\varepsilon$. We denote
by \( \delta_T = \lambda_{\min}(\mathbb{H}_T)C_T^2 \nu_T \), and using \( \frac{\nu_T}{2} \mathbb{E}[u^T \tilde{G}_T l(\theta_0) u] \geq \delta_T \), we have

\[
\begin{align*}
\mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \tilde{G}_T l(\theta_0) u + \frac{\nu_T}{2} u^T \tilde{G}_T l(\theta_0) u + \frac{\nu_T^2}{6} \nabla \{u^T \tilde{G}_T l(\tilde{\theta}) u\} u \leq 0) \\
\leq \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : |\tilde{G}_T l(\theta_0) u| > \delta_T / 4) + \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \frac{\nu_T^2}{2} |R_T(\theta_0)| > \delta_T / 4) \\
\quad + \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \frac{\nu_T^2}{6} \sup_{\theta : \|\theta - \theta_0\|_2 < \nu_T C_\epsilon} |\nabla \{u^T \tilde{G}_T l(\tilde{\theta}) u\} u| > \delta_T / 4) \\
\leq \frac{16C_\epsilon^2 d_T K_1}{T \delta_T^2} + \frac{4\nu_T^2 d_T^2 C_\epsilon^4}{T \delta_T^2} + \frac{16\nu_T^4 d_T^4 C_\epsilon^6}{36\delta_T^2} \eta(C_\epsilon) \\
\leq C_1 \frac{d_T}{TC^2 \nu_T^2} + C_2 \frac{d_T^2}{T} + C_3 \nu_T^2 d_T^4 C_\epsilon^2 \eta(C_\epsilon),
\end{align*}
\]

where \( C_1, C_2, C_3 \) are strictly positive constants. We chose \( \nu_T = \left( \frac{d_T}{T} \right)^\frac{\kappa}{2} \), we then deduce

\[
\begin{align*}
\mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \nu_T \tilde{G}_T l(\theta_0) u + \frac{\nu_T^2}{2} u^T \tilde{G}_T l(\theta_0) u + \frac{\nu_T^3}{6} \nabla \{u^T \tilde{G}_T l(\tilde{\theta}) u\} u \leq 0) \\
\leq \frac{C_1}{C_\epsilon^2} + \frac{C_2 d_T^2}{T} + \frac{C_3 \nu_T^2 d_T^4 C_\epsilon^2}{T} \eta(C_\epsilon).
\end{align*}
\]

Now we fix \( C_\epsilon \) sufficiently large enough, such that \( C_1/C_\epsilon^2 < \epsilon/3 \). Once this constant is fixed, there exists a \( T_0 \) such that for \( T > T_0 \) we have \( C_2 \frac{d_T^2}{T} < \epsilon/3 \) and \( C_3 \frac{d_T^4 C_\epsilon^2}{T} \eta(C_\epsilon) < \epsilon/3 \) under the assumption that \( d_T = o(T) \). Consequently, we obtain

\[
\begin{align*}
\mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \tilde{G}_T l(\theta_0) u + \nu_T \tilde{G}_T l(\theta_0) u + \frac{\nu_T^2}{2} u^T \tilde{G}_T l(\theta_0) u + \frac{\nu_T^3}{6} \nabla \{u^T \tilde{G}_T l(\tilde{\theta}) u\} u \leq 0) < \epsilon.
\end{align*}
\]

This proves (5.2), that is \( \|\tilde{\theta} - \theta_0\|_2 = O_p\left( \left( \frac{d_T}{T} \right)^{\frac{\kappa}{2}} \right) \). \( \square \)

The first step estimator used for the adaptive weights is \( (T/d_T)^{1/2} \)-consistent. However, the estimated quantities on \( A^c \) converge to zero by consistency. We then propose a slight modification of the first step estimator, denoted \( \tilde{\theta} \), which disappears asymptotically as follows

\[
\tilde{\theta} = \hat{\theta} + e_T,
\]

such that \( e_T \to 0 \) is a strictly positive quantity. We choose \( e_T = T^{-\kappa} \) with \( \kappa > 0 \). This means we add in the adaptive weights a power of \( T \) to the first step estimator, that is

\[
\alpha_{T,i}^{(k)} = |\tilde{\theta}_{i}^{(k)}| = |\hat{\theta} + T^{-\kappa}|^{-\eta}, \xi_{T,l} = \|\hat{\theta}^{(l)}\|_2 = \|\tilde{\theta}^{(l)} + T^{-\kappa}\|_{2}^{-\mu}.
\]

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Theorem 5.3. Under assumptions [1],[2],[3],[4],[7],[11], if \( d_T = o(T) \), and if \( \frac{\gamma T^{\frac{3}{2} + \kappa \mu}}{T} \to \infty \) as \( T \to \infty \), and if \( \gamma T^{\frac{3}{2}} \to 0 \), then the sequence of penalized estimators \( \hat{\theta} \) solving (5.1) satisfies

\[
\|\hat{\theta} - \theta_0\|_2 = O_p((\frac{d_T}{T})^{\frac{1}{2}}).
\]

Remark. Note that \( d_T = o(T) \) is as in Fan and Peng (2004, Theorem 1). The asymptotic behaviors of the regularization terms provide a condition such that the penalty terms are negligible with respect to \( (d_T/T)^{1/2} \).

Proof. We proceed as we did for proving Theorem 5.2. Let \( \nu_T = (\frac{d_T}{T})^{1/2} \). We would like to prove that for any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that

\[
\mathbb{P}(\|\hat{\theta} - \theta_0\|_2 / \nu_T > C_\epsilon) < \epsilon.
\]

(5.4)

To prove (5.4), we show

\[
\mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \hat{G}_Tl(\tilde{\theta})u + \nu_T \tilde{G}_Tl(\tilde{\theta})u + \frac{\nu_T^2}{6} \nabla' \{u' \hat{G}_Tl(\tilde{\theta})u\}u + \nu_T^{-1} \{p_1(\lambda_T, \tilde{\theta}, \theta_0 + \nu_T u) - p_1(\lambda_T, \tilde{\theta}, \theta_0) + p_2(\gamma_T, \tilde{\theta}, \theta_0 + \nu_T u) - p_2(\gamma_T, \tilde{\theta}, \theta_0)\} \leq 0) < \epsilon.
\]

(5.5)

a relationship obtained by convexity and a Taylor expansion.

The score quantity can be upper bounded as

\[
|\hat{G}_Tl(\theta_0)u| \leq \|\hat{G}_Tl(\theta_0)\|_2 \|u\|_2 = O_p((\frac{d_T}{T})^{\frac{1}{2}}) \|u\|_2 = O_p(\nu_T) \|u\|_2,
\]

where we used assumption [3] to obtain the bound in probability of the score.

As for the third order term, we have by the Cauchy-Schwartz inequality

\[
|\nabla' \{u' \hat{G}_Tl(\tilde{\theta})u\}u|^2 \leq \|u\|_2^6 d_T^3 \frac{1}{T^2} \sum_{t, t'} \{ \sum_{k_1, l_1, m_1=1}^{d_T} \sum_{k_2, l_2, m_2=1}^{d_T} \partial^3_{\theta_{k_1} \theta_{l_1} \theta_{m_1}} l(\epsilon_{t}; \tilde{\theta}) \partial^3_{\theta_{k_2} \theta_{l_2} \theta_{m_2}} l(\epsilon_{t'}; \tilde{\theta}) \} = \|u\|_2^6 d_T^3 \eta(C_\epsilon).
\]

This implies

\[
\nabla' \{u' \hat{G}_Tl(\tilde{\theta})u\}u = O_p(d_T^{3/2} \|u\|_2^2).
\]

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Hence by the Markov inequality

\[ \mathbb{P}(\exists \mathbf{u} \in \mathbb{R}^d, \|\mathbf{u}\|_2 = C_\epsilon : |\nu_T^2 \nabla' \{ \mathbf{u}' \tilde{G}_T l(\tilde{\theta}) \mathbf{u} \}| > a) \leq \frac{\nu_T^4 C_\epsilon^6 d_T^3}{a^2} \eta(C_\epsilon). \]

where we used assumption 11.

Finally, the hessian quantity can be treated as in the proof of Theorem 5.2. We denote by \( R_T(\theta_0) = \sum_{k,l=1}^{d_T} u_k u_l \{ \partial^2_{\theta_k \theta_l} G_T l(\theta_0) - \mathbb{E}[\partial^2_{\theta_k \theta_l} G_T l(\theta)] \} \). We have

\[ u' \tilde{G}_T l(\theta_0) \mathbf{u} = u' \mathbb{E}[\tilde{G}_T l(\theta_0)] \mathbf{u} + R_T(\theta_0). \]

By assumption 10 and the Markov inequality, for any \( \kappa > 0 \), we obtain

\[ \mathbb{P}(\|R_T(\theta_0)\| > \kappa) \leq \frac{1}{\kappa^2} \mathbb{E}[R_T^2(\theta_0)] \leq \frac{K_2 \|\mathbf{u}\|_2^4 d_T^2}{\kappa^2} \leq \frac{K_2 C_\epsilon^4 d_T^2}{\kappa^2 T}, \]

with \( K_2 > 0 \). This relationship holds for any \( \kappa > 0 \). Then for \( T \) large enough, we deduce that \( |R_T(\theta_0)| = o_p(1) \). Consequently

\[ \frac{\nu_T^2}{2} u' \tilde{G}_T l(\theta_0) \mathbf{u} \geq \frac{\nu_T^2}{2} \lambda_{\min}(\Xi_T) \|\mathbf{u}\|_2^2 + o_p(1) \nu_T^2 \|\mathbf{u}\|_2^2. \]

We focus on the penalty terms. We have

\[ p_1(\lambda_T, \tilde{\theta}, \theta_0 + \nu_T \mathbf{u}) - p_1(\lambda_T, \tilde{\theta}, \theta_0) = \lambda_T T^{-1} \sum_{k \in S \in A_k} \sum_{i \in A_k} \alpha^{(k)}_{T,i} \{ |\theta^{(k)}_{0,i} + \nu_T u^{(k)}_i| - |\theta^{(k)}_{0,i}| \}, \]

and

\[ |p_1(\lambda_T, \tilde{\theta}, \theta_0 + \nu_T \mathbf{u}) - p_1(\lambda_T, \tilde{\theta}, \theta_0)| \leq \lambda_T T^{-1} \sum_{k \in S \in A_k} \sum_{i \in A_k} \alpha^{(k)}_{T,i} \nu_T \|u^{(k)}_i\|. \]

As for the \( l^1/l^2 \) norm, we obtain

\[ p_2(\gamma_T, \tilde{\theta}, \theta_0 + \nu_T \mathbf{u}) - p_2(\gamma_T, \tilde{\theta}, \theta_0) = \gamma_T T^{-1} \sum_{l \in S_1} \xi_{T,l} \{ \|\theta^{(l)}_0 + \nu_T \mathbf{u}\|_2 - \|\theta^{(l)}_0\|_2 \} \]

and

\[ |p_2(\gamma_T, \tilde{\theta}, \theta_0 + \nu_T \mathbf{u}) - p_2(\gamma_T, \tilde{\theta}, \theta_0)| \leq \gamma_T T^{-1} \sum_{l \in S} \xi_{T,l} \nu_T \|u^{(l)}\|_2. \]
For the $l^1$ norm penalty, using $\{\min_{k \in S, i \in A_k} |\tilde{\theta}^{(k)}_i|\}^{-\eta} \leq T^{\kappa \eta}$, then

$$\lambda_T T^{-1} \sum_{k \in S, i \in A_k} \alpha^{(k)}_{T,i} \nu_T |u^{(k)}_i| \leq \lambda_T T^{-1} \nu_T \{\sum_{k \in S, i \in A_k} |\tilde{\theta}^{(k)}_i|^{2-2\eta}\}^{1/2} \|u\|_2$$

$$\leq \lambda_T T^{-1} \nu_T \sqrt{d_T} \{\min_{k \in S, i \in A_k} |\tilde{\theta}^{(k)}_i|\}^{\eta} \|u\|_2$$

$$\leq \lambda_T T^{-1} \nu_T \sqrt{d_T T^{\kappa \eta}} \|u\|_2,$$

by the Cauchy-Schwartz inequality. Then if $\lambda_T T^{2-1+\kappa \eta}$ is bounded, we obtain

$$p_1(\lambda_T, \tilde{\theta}, \theta_0 + \nu_T u) - p_1(\lambda_T, \tilde{\theta}, \theta_0) = O(\nu_T^2) \|u\|_2.$$  

As for the $l^1/l^2$ term, using $\{\min_{i \in S} |\tilde{\theta}^{(i)}_i|\}^{-\mu} \leq T^{\kappa \mu}$, we obtain

$$\gamma_T T^{-1} \sum_{i=1}^m \xi_T \nu_T \|u^{(i)}\|_2 \leq \gamma_T T^{-1} \nu_T \{\sum_{i \in S} |\tilde{\theta}^{(i)}_i|^{2-2\mu}\}^{1/2} \|u\|_2$$

$$\leq \gamma_T T^{-1} \nu_T \sqrt{d_T} \{\min_{i \in S} |\tilde{\theta}^{(i)}|\}^{\mu} \|u\|_2$$

$$\leq \gamma_T T^{-1} \nu_T \sqrt{d_T T^{\kappa \mu}} \|u\|_2,$$

by the Cauchy-Schwartz inequality. Then if $\gamma_T T^{2-1+\kappa \mu}$ is bounded, we obtain

$$p_2(\gamma_T, \tilde{\theta}, \theta_0 + \nu_T u) - p_2(\gamma_T, \tilde{\theta}, \theta_0) = O(\nu_T^2) \|u\|_2.$$  

We now can prove (5.13). Let $\delta_T = \lambda_{\min}(\mathbb{H}_T)C_\epsilon^2 \nu_T$ and using $\frac{\nu_T^2}{2} \mathbb{E}[u^T \hat{G}_T l(\theta_0) u] \geq \delta_T$, we have

$$\mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \hat{G}_T l(\theta_0) u + \nu_T u^T \hat{G}_T l(\theta_0) u / 2 + \nu_T^2 \nabla \{u^T \hat{G}_T l(\theta) u\} u / 6$$

$$+ \nu_T^{-1} \{p_1(\lambda_T, \tilde{\theta}, \theta_0 + \nu_T u) - p_1(\lambda_T, \tilde{\theta}, \theta_0) + p_2(\gamma_T, \tilde{\theta}, \theta_0 + \nu_T u) - p_2(\gamma_T, \tilde{\theta}, \theta_0)\} \leq 0)$$

$$\leq \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : |\nu_T u^T \hat{G}_T l(\theta_0) u| / 2 \leq |\hat{G}_T l(\theta_0) u| + |\nu_T^2 \nabla \{u^T \hat{G}_T l(\theta) u\} u| / 6$$

$$+ \nu_T^{-1} \{p_1(\lambda_T, \tilde{\theta}, \theta_0) - p_1(\lambda_T, \tilde{\theta}, \theta_0 + \nu_T u) + p_2(\gamma_T, \tilde{\theta}, \theta_0) - p_2(\gamma_T, \tilde{\theta}, \theta_0 + \nu_T u)\} \}$$

$$\leq \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : |\hat{G}_T l(\theta_0) u| > \delta_T / 8) + \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : \frac{\nu_T^2}{2} |\mathcal{R}_T(\theta_0)| > \delta_T / 8$$

$$+ \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : |p_1(\lambda_T, \tilde{\theta}, \theta_0) - p_1(\lambda_T, \tilde{\theta}, \theta_0 + \nu_T u)| > \nu_T \delta_T / 8$$

$$+ \mathbb{P}(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_\epsilon : |p_2(\gamma_T, \tilde{\theta}, \theta_0) - p_2(\gamma_T, \tilde{\theta}, \theta_0 + \nu_T u)| > \nu_T \delta_T / 8$$

$$\leq \frac{C_{st}}{C_\epsilon^2} + C_{st} \{\nu_T^2 C_\epsilon^2 \nu_T^3 \eta(C_\epsilon)\} + C_{st} \nu_T^2 d_T^2 C_\epsilon^4 \frac{T \delta_T^2}{\nu_T^3} + \epsilon / 5 + \epsilon / 5$$

$$< \epsilon,$
with \( C_{st} > 0 \) a generic constant. We used \( d_T^2 = o(T) \) and for \( C_e \), large enough
\[
\Pr(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_e \leq \epsilon),
\]
\[
\Pr(\exists u \in \mathbb{R}^{d_T}, \|u\|_2 = C_e \leq \epsilon/5).
\]

Thus we obtain for \( C_e \) and \( T \) large enough, with the conditions \( \gamma T T^{-\frac{1}{2}+\kappa\mu} \rightarrow 0 \) and \( \lambda T T^{-\frac{1}{2}+\kappa\eta} \rightarrow 0 \) that
\[
\|\hat{\theta} - \theta_0\|_2 = O_p(\nu T) = O_p\left((\frac{d_T}{T})^{\frac{1}{2}}\right).
\]

\( \square \)

To achieve the oracle property, we need some additional assumptions regarding the adaptive penalty components.

**Assumption 12.** For any \( T \), there exists \( \beta \) such that \( 0 < \beta < \min_{i \in A_k} \theta_{0,i,k}, k \in S \).
Moreover,
\[
\beta^{-1}T^{-1}\{\lambda T T^{-\frac{1}{2}}E\left[\max_{k \in S, i \in A_k} \alpha_{T,i,k}\right] + \gamma T E\left[\max_{k \in S} \xi_{T,k}\right]\} \longrightarrow 0.
\]

**Assumption 13.** The model complexity is assumed to behave as \( d_T^\delta = o(T) \), which implies that \( 0 < c < \frac{1}{5} \). The regularization parameters are chosen such that they satisfy
\[
\gamma T T^{-\frac{1}{2}(1+\kappa\mu)} \longrightarrow 0,
\]
\[
\lambda T T^{-\frac{1}{2}(1+\kappa\eta)} \longrightarrow 0,
\]
\[
\gamma T T^{-\frac{1}{2}(1+\kappa\mu)(1-c)} \longrightarrow \infty,
\]
\[
\lambda T T^{-\frac{1}{2}(1+\kappa\eta)(1-c)} \longrightarrow \infty.
\]

**Remark.** The main condition is \( d_T^\delta = o(T) \), which is the same as Fan and Peng (2004). This condition comes from the control for the third order derivative of the empirical criterion. Note that simple cases allow for a framework where \( 0 \leq c < 1 \). Moreover, these asymptotic behaviors are closely related to condition (A5) of Zou and Zhang (2009). In section 6, we provide further details about the calibration of the adaptive weights and \( \kappa \).

**Assumption 14.** Let \( F_T^T = \sigma(X_{T,s}, s \leq t) \) with \( X_{T,t} = \sqrt{T}Q_T^{-1/2}H_{T,A}^{-1}G_{T,t}(\theta_0)A \), \( (Q_T) \) is a sequence of \( r \times \text{card}(A) \) matrices such that \( Q_T \times Q_T^T \cong \mathbb{C} \), for some...
r \times r \text{ nonnegative symmetric matrix } C, \forall T, AA = (H_{-1}^{-1} M_T H_{-1}^{-1})_{AA} \text{ and } \hat{\mathcal{C}}_{T,i}(\theta_0)_A = \frac{1}{T} \nabla \mathcal{A} l(\epsilon_i; \theta_0). \text{ Then } X_{T,i} \text{ is a martingale difference and we have}

\[\mathbb{E}[\sup_{i,j=1,\ldots,d_T} \mathbb{E}[(\partial_{\theta_i} l(\epsilon_i; \theta_0) \partial_{\theta_j} l(\epsilon_i; \theta_0))^2 | F_{t-1}^T] \lambda_{\max, t-1}(H_{t-1}^T)] \leq B < \infty,\]

with

\[H_{t-1}^T := \mathbb{E}[\nabla l(\epsilon_i; \theta_0) \nabla^T l(\epsilon_i; \theta_0) | F_{t-1}^T] \leq \lambda_{\max}(H_{t-1}^T) < \infty.\]

**Theorem 5.4.** Under assumptions 7-13 and assumptions 7-14 the sequence of adaptive estimator \(\hat{\theta}\) solving (5.4) satisfies

\[
\lim_{T \to \infty} P(\hat{A} = A) = 1, \text{ and } \quad \sqrt{T} Q_T \nabla^{-1/2}_{T, AA}(\hat{\theta}_A - \theta_0, A) \overset{d}{\to} \mathcal{N}(0, C),
\]

where \((Q_T)\) is a sequence of \(r \times \text{card}(A)\) matrices such that \(Q_T \times Q_T^T \overset{P}{\to} C, \text{ for some } r \times r \text{ nonnegative symmetric matrix } C \text{ and } \forall T, AA = (H_{-1}^{-1} M_T H_{-1}^{-1})_{AA}.\)

**Proof.** Model selection consistency consists of proving that the probability of the event \(\{\hat{A} = A\}\) tends to one asymptotically. This event is

\[\{\hat{A} = A\} = \{\forall k \in S, \forall i \in A_k, |\hat{\theta}_i^{(k)}| > 0\} \cap \{\forall k = 1, \ldots, m, \forall i \in A_k^c, \hat{\theta}_i^{(k)} = 0\}.\]

Hence we prove

\[P(\{\forall k \in S, \forall i \in A_k, |\hat{\theta}_i^{(k)}| > 0\} \cap \{\forall k = 1, \ldots, m, \forall i \in A_k^c, \hat{\theta}_i^{(k)} = 0\}) \overset{T \to \infty}{\longrightarrow} 1. \quad (5.6)\]

Model selection consistency can be decomposed into two parts: recovering the active indices by estimating nonzero coefficients; discarding the inactive indices by shrinking to zero the related coefficients. Now (5.6) can be proved by first showing that for any \(T\), there exists \(\beta\) such that \(0 < \beta < \min_{i \in A_k} \theta_{0,i,N_k}\), with \(k \in S\) and

\[P(\|\hat{\theta}_A - \theta_0, A\|_2 < \beta) \overset{T \to \infty}{\longrightarrow} 1. \quad (5.7)\]

The second part regarding nonactive indices can be proved as

\[
\left\{\begin{array}{l}
P(\cap_{k \in S} \{\|\hat{z}^{(k)}\|_2 < 1\}) \overset{T \to \infty}{\longrightarrow} 1, \\
P(\cap_{k \in S} \cap_{i \in A_k^c} \{\|\hat{w}_i^{(k)}\| < 1\}) \overset{T \to \infty}{\longrightarrow} 1,
\end{array}\right. \quad (5.8)
\]

where \(\hat{z}^{(k)}\) (resp. \(\hat{w}^{(k)}\)) is the subgradient of \(\|\hat{\theta}^{(k)}\|_2\) (resp. \(\|\hat{\theta}^{(k)}\|_1\)) given in (3.1).
Hence (5.7) and (5.8) prove (5.6).

We first focus on (5.7), which is equivalent to

\[ \Pr(||\hat{\theta}_{Ak} - \theta_{0,Ak}||_2 > \beta) \xrightarrow{T \to \infty} 0. \]

By the Karush-Kuhn-Tucker optimality conditions, we have

\[
\dot{G}_{T,l}(\hat{\theta})_A + \lambda_T T^{-1} \alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A) + \gamma_T T^{-1} \varsigma_T = 0,
\]

where \( \varsigma_T = \text{vec}(\xi_{T,k} \frac{\hat{\theta}_{Ak}}{||\theta_{Ak}||_2}, k \in S) \). We denote by \( \alpha_{T,A_k} = (\alpha_{T,i}, i \in A_k) \), a vector of size \( \mathbb{R}^{C_{Ak}} \). By a Taylor expansion of the gradient component around \( \theta_{0,A} \), we have

\[
\dot{G}_{T,l}(\theta)_A + H_{T,A,A}(\hat{\theta}_A - \theta_{0,A}) + P_T(\theta) (\hat{\theta}_A - \theta_{0,A}) + \frac{1}{2} \nabla_t' \{ (\hat{\theta}_A - \theta_{0,A})' \dot{G}_{T,l}(\hat{\theta})_A (\hat{\theta}_A - \theta_{0,A}) \} + \lambda_T T^{-1} \alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A) + \gamma_T T^{-1} \varsigma_T = 0
\]

\[
\iff \hat{\theta}_A = \theta_{0,A} - H_{T,A,A}^{-1}(\dot{G}_{T,l}(\theta)_A + \lambda_T T^{-1} \alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A) + \gamma_T T^{-1} \varsigma_T
\]

\[
- H_{T,A,A}^{-1} \frac{1}{2} \nabla_t' \{ (\hat{\theta}_A - \theta_{0,A})' \dot{G}_{T,l}(\hat{\theta})_A (\hat{\theta}_A - \theta_{0,A}) \} - H_{T,A,A}^{-1} P_T(\theta) (\hat{\theta}_A - \theta_{0,A}),
\]

where \( ||\bar{\theta} - \theta_0||_2 \leq ||\hat{\theta} - \theta_0||_2 \), \( P_T(\theta) = \dot{G}_{T,l}(\theta)_A - H_{T,A,A} \) and \( H_{T,A,A} = \mathbb{E}[\nabla^2_{\theta \theta} l(\epsilon_1; \theta_0)]_{A_4} \).

Then using \( ||\hat{\theta}_A - \theta_{0,A}||_2 = O_p((\frac{d_T}{T})^{1/2}) \), we obtain

\[
\Pr(||\hat{\theta}_A - \theta_{0,A}||_2 > \beta) \leq \Pr(||H_{T,A,A}^{-1} \dot{G}_{T,l}(\theta)_A||_2 + ||H_{T,A,A}^{-1}||_2 \lambda_T T^{-1} \alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A)||_2
\]

\[
+ ||H_{T,A,A}^{-1}||_2 ||\nabla_t' \{ (\hat{\theta}_A - \theta_{0,A})' \dot{G}_{T,l}(\hat{\theta})_A (\hat{\theta}_A - \theta_{0,A}) \} /2||_2
\]

\[
+ ||H_{T,A,A}^{-1}||_2 ||P_T(\theta) (\hat{\theta}_A - \theta_{0,A})||_2 > \beta
\]

\[
\leq \Pr(\lambda_{\text{min}}^{-1}(H_T) ||\dot{G}_{T,l}(\theta)_A||_2 + \lambda_{\text{min}}^{-1}(H_T) \lambda_T T^{-1} ||\alpha_{T,A}||_2
\]

\[
+ \lambda_{\text{min}}^{-1}(H_T) \gamma T^{-1} ||\varsigma_T||_2 + \lambda_{\text{min}}^{-1}(H_T) C_0^2 (d_T/2T) ||\nabla_t' \{ \dot{G}_{T,l}(\hat{\theta})_A \}||_2
\]

\[
+ \lambda_{\text{min}}^{-1}(H_T) C_0 (d_T/T)^{1/2} ||P_T(\theta_0)||_2 > \beta + \Pr(||\hat{\theta}_A - \theta_{0,A}||_2 > (d_T/T)^{1/2} C_0),
\]

for \( C_0 > 0 \) large enough, and we used \( ||H_T^{-1} x||_2 \leq \lambda_{\text{min}}^{-1}(H_T) ||x||_2 \) for any vector \( x \in \mathbb{R}^{d_T} \).

Let us proceed element-by-element. We have by the Markov inequality

\[
\Pr(\lambda_{\text{min}}^{-1}(H_T) C_0 \sqrt{\frac{d_T}{T}} ||P_T(\theta_0)||_2 > \frac{\beta}{6}) \leq \frac{36 \lambda_{\text{min}}^{-2}(H_T) C_0^2 d_T}{T \beta^2} \mathbb{E}||P_T(\theta_0)||_2^2
\]

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We have
\[ \mathbb{E}[\|P_T\|^2] = \frac{1}{T^2} \sum_{t,t'=1}^T \sum_{k,k' \in A_t,l,l' \in A} \mathbb{E}[\zeta_{kt,t} \zeta_{k't',t'}], \]
where \( \zeta_{kt,t} = \partial_{\theta_t}^2 l(\epsilon_t; \theta_0) - \mathbb{E}[\partial_{\theta_t}^2 l(\epsilon_t; \theta_0)] \). By assumption \([10]\) we obtain
\[ \mathbb{P}(\lambda_{\min}^{-1}(H_T)C_0 \sqrt{\frac{dT}{T}} \|P_T(\theta_0)\|_2 > \frac{\beta}{6}) \leq \frac{36 \lambda_{\min}^{-2}(H_T)C_0^2 d_T^2}{\beta^2 \mathbb{E}[\|\mathbb{V}_A(\hat{\mathbb{G}}_T l(\bar{\theta}))\|_2^2]}. \]

As for the third order term, by the Markov inequality
\[ \mathbb{P}(\frac{1}{2} \lambda_{\min}^{-1}(H_T)C_0^2 \frac{dT}{T} \|\mathbb{V}_A(\hat{\mathbb{G}}_T l(\bar{\theta}))\|_2 > \frac{\beta}{6}) \leq \frac{9 \lambda_{\min}^{-2}(H_T)C_0^4 d_T^2}{\beta^2 \mathbb{E}[\|\mathbb{V}_A(\hat{\mathbb{G}}_T l(\bar{\theta}))\|_2^2]}, \]
We obtain
\[ \mathbb{E}[\|\mathbb{V}_A(\hat{\mathbb{G}}_T l(\bar{\theta}))\|_2^2] \leq \frac{1}{T^2} \sum_{t,t'=1}^T \sum_{k,k_2,k_3 \in A_t,l_1,l_2,l_3 \in A} \mathbb{E}[\partial_{\theta_{k_1} \theta_{k_2} \theta_{k_3}}^2 l(\epsilon_t; \theta_0) \partial_{\theta_{l_1} \theta_{l_2} \theta_{l_3}}^2 l(\epsilon_t; \theta_0) l(\epsilon_{t'}; \theta_0)] \]
\[ \leq \frac{1}{T^2} \mathbb{E}[v_t(C_0)v_{t'}(C_0)] = \eta(C_0)d_T^2, \]
by assumption \([11]\) where \( v_t(C_0) = \sup_{k,k_2,k_3 \in A_t} \sup_{\theta: \|\theta - \theta_0\|_2 \leq \sqrt{T/C_0}} |\partial_{\theta_{k_1} \theta_{k_2} \theta_{k_3}}^3 l(\epsilon_t; \theta_0)| \). We deduce that
\[ \mathbb{P}(\frac{1}{2} \lambda_{\min}^{-1}(H_T)C_0^2 \frac{dT}{T} \|\mathbb{V}_A(\hat{\mathbb{G}}_T l(\bar{\theta}))\|_2 > \frac{\beta}{6}) \leq \frac{9 \lambda_{\min}^{-2}(H_T)C_0^4 d_T^2}{4 \beta^2 \eta(C_0)}. \]

We now turn to the score quantity. By the Markov inequality and assumption \([9]\) we have
\[ \mathbb{P}(\lambda_{\min}^{-1}(H_T)\|\hat{\mathbb{G}}_T l(\bar{\theta})\|_2 > \beta/6) \leq \frac{\lambda_{\min}^{-2}(H_T)^36}{\beta^2 \mathbb{E}[\|\hat{\mathbb{G}}_T l(\bar{\theta})\|_2]}. \]
\[ \leq \frac{\lambda_{\min}^{-2}(H_T)^36}{\beta^2} \frac{1}{T^2} \sum_{t,t'=1}^T \mathbb{E}[\partial_{\theta_{t'}} l(\epsilon_t; \theta_0) \partial_{\theta_{t'}} l(\epsilon_{t'}; \theta_0)] \]
\[ \leq \frac{\lambda_{\min}^{-2}(H_T)^36}{\beta^2} \frac{1}{T} \left( \frac{1}{T} \sum_{t,t'=1}^T \Psi(|t - t'|) \right) d_T \]
\[ \leq \frac{\lambda_{\min}^{-2}(H_T)^36 K d_T}{T \beta^2}, \]

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with $K > 0$. Hence we deduce

$$\mathbb{P} (\|\hat{\theta}_A - \theta_{0,A}\|_2 > \beta) \leq \mathbb{P} (\lambda_{\min}^T (\mathbb{H}_T) \lambda_T T^{-1} \|\alpha_{T,A}\|_2 + \lambda_{\min}^T (\mathbb{H}_T) \gamma T T^{-1} \|\xi_T\|_2 > \beta / 2) + \mathbb{P} (\|\theta_A - \theta_{0,A}\|_2 > (d_T/T)^{1/2} C_0) + \mathbb{P} (\lambda_{\min}^T (\mathbb{H}_T) C_0 (d_T/T)^{1/2} \|\mathbb{P}_T(\theta_0)\|_2 > \beta / 6) + \mathbb{P} (\lambda_{\min}^T (\mathbb{H}_T) \|\hat{\mathbb{G}}_T l(\theta_0, A_2\|_2 > \beta / 6)$

$$\leq \frac{2 \lambda_{\min}^T (\mathbb{H}_T)}{\beta} \{\lambda_T T^{-1} d_T^{1/2} \mathbb{E} \max_{k \in \mathcal{S}, i \in A_k} \alpha_{T,A_k,i} + \gamma T T^{-1} \mathbb{E} \max_{k \in \mathcal{S}} \xi_{T,k}\} + \frac{36 \lambda_{\min}^T (\mathbb{H}_T) K d_T}{T \beta^2} + \frac{9 \lambda_{\min}^T (\mathbb{H}_T) C_0^2 d_T^2}{4 T^2} + \frac{36 \lambda_{\min}^T (\mathbb{H}_T) C_0^2 d_T^2}{T^2} + \epsilon.$$

For $T$ and $C_0$ large enough, if $d_T^2 = o(T)$, by assumption $12$ that is if

$$\beta^{-1} T^{-1} \{\lambda_T d_T^{1/2} \mathbb{E} \max_{k \in \mathcal{S}, i \in A_k} \alpha_{T,A_k,i} + \gamma T T^{-1} \mathbb{E} \max_{k \in \mathcal{S}} \xi_{T,k}\} \longrightarrow 0,$$

then

$$\mathbb{P} (\|\hat{\theta}_A - \theta_{0,A}\|_2 > \beta) \longrightarrow 0.$$

We now turn to the second step of model selection consistency. First we prove

$$\mathbb{P} (\bigcap_{k \in \mathcal{S}^c} \{\|\hat{z}(k)\|_2 < 1\}) \longrightarrow 1 \iff \mathbb{P} (\bigcup_{k \in \mathcal{S}^c} \{\|\hat{z}(k)\|_2 \geq 1\}) \longrightarrow 0. \quad (5.9)$$

This is equivalent to proving

$$\mathbb{P} (\bigcup_{k \in \mathcal{S}^c} \{\|\hat{\mathbb{G}}_T l(\hat{\theta})(k) + \lambda_T T^{-1} \alpha_T^{(k)} \circ \hat{w}(k)\|_2 \geq \gamma T T^{-1} \xi_{T,k}\}) \longrightarrow 0.$$

We have for $k \in \mathcal{S}^c$ that $\|\hat{w}(k)\|_\infty \leq 1$, which implies by the optimality conditions of Karush-Kuhn-Tucker that

$$\mathbb{P} (\bigcup_{k \in \mathcal{S}^c} \{\|\hat{\mathbb{G}}_T l(\hat{\theta})(k) + \lambda_T T^{-1} \alpha_T^{(k)} \circ \hat{w}(k)\|_2 \geq \gamma T T^{-1} \xi_{T,k}\}) \leq \mathbb{P} (\bigcup_{k \in \mathcal{S}^c} \{|\hat{\mathbb{G}}_T l(\hat{\theta})(k)\|_2 \geq \gamma T T^{-1} \xi_{T,k} - \lambda_T T^{-1} \|\alpha_T^{(k)}\|_2\}).$$

By a Taylor expansion around $\theta_0$, let $\bar{\theta}$ such that $\|\bar{\theta} - \theta_0\| \leq \|\hat{\theta} - \theta_0\|$, we have

$$\mathbb{P} (\bigcup_{k \in \mathcal{S}^c} \{\|\hat{z}(k)\|_2 \geq 1\}) \leq \mathbb{P} (\bigcup_{k \in \mathcal{S}^c} \{\|\hat{\mathbb{G}}_T l(\theta_0)(k)\|_2 \geq \gamma T T^{-1} \xi_{T,k} - \lambda_T T^{-1} \|\alpha_T^{(k)}\|_2\}) - \|\hat{\mathbb{G}}_T l(\theta_0)(k)\|_2 \|\hat{\theta} - \theta_0\|_2 - \|\nabla (\hat{G}_T l(\hat{\theta})(k)) (k)\|_2 \|\hat{\theta} - \theta_0\|_2^2\) \leq \mathbb{P} (\bigcup_{k \in \mathcal{S}^c} \{\|\hat{\mathbb{G}}_T l(\theta_0)(k)\|_2 \geq \gamma T T^{-1} \|\hat{\theta} - \theta_0\|_2^2 - \lambda_T T^{-1} d_T^{1/2} \max_{k \in \mathcal{S}^c,i \in \mathcal{G}_k} (|\hat{\theta}(k)| - \eta) - \|\hat{\mathbb{G}}_T l(\theta_0)(k)\|_2 \|\hat{\theta} - \theta_0\|_2 - \|\nabla' (\hat{G}_T l(\hat{\theta})(k)) (k)\|_2 \|\hat{\theta} - \theta_0\|_2^2\),
where we used $\|\tilde{G}_T l(\theta_0)_{(k)}(\tilde{\theta} - \theta_0)\|_2 \leq \|\tilde{G}_T l(\tilde{\theta})_{(k)}\|_2 \|\tilde{\theta} - \theta_0\|_2$ and $
abla' \{\tilde{G}_T l(\tilde{\theta})_{(k)}\}_{(k)}2\|_2 \leq \|\tilde{G}_T l(\theta_0)_{(k)}\|_s$. Let $\epsilon > 0$, and $K_\epsilon$ strictly positive constants, we proved for $T$ large enough that

$$\mathbb{P}(\|\tilde{\theta} - \theta_0\|_2 > K_\epsilon(d_T/T)^{1/2}) < \epsilon/6.$$  

We deduce that

$$\mathbb{P}(\bigcup_{k \in S^c}\{\|\tilde{z}^{(k)}\|_2 \geq 1\}) \leq \mathbb{P}(\bigcup_{k \in S^c}\{\|\tilde{G}_T l(\theta_0)_{(k)}\|_2 \geq \gamma_T T^{-1}\|\tilde{\theta}(k)\|_2^{-\mu} - \lambda_T T^{-1}d_T^{1/2} \max_{k \in S^c,i \in G_k} (|\tilde{\theta}^{(k)}_i|^{-\eta}) - \|\tilde{G}_T l(\theta_0)_{(k)}\|_2(d_T/T)^{1/2}K_\epsilon - \|\nabla' \{\tilde{G}_T l(\tilde{\theta})_{(k)}\}_{(k)}\|_2(d_T/T)^{1/2}K_\epsilon \} + \epsilon/6.$$  

Let $M_{1,T} = (\gamma_T T)^{-1/2}$, then we obtain

$$\mathbb{P}(\bigcup_{k \in S^c}\{\|\tilde{z}^{(k)}\|_2 \geq 1\}) \leq \sum_{k \in S^c} \mathbb{P}(\|\tilde{G}_T l(\theta_0)_{(k)}\|_2 \geq \gamma_T T^{-1}\|\tilde{\theta}(k)\|_2^{-\mu} - \lambda_T T^{-1}d_T^{1/2} \max_{k \in S^c,i \in G_k} (|\tilde{\theta}^{(k)}_i|^{-\eta}) - \|\tilde{G}_T l(\theta_0)_{(k)}\|_2(d_T/T)^{1/2}K_\epsilon \} \leq M_{1,T}T) + \mathbb{P}(\|\tilde{\theta}(k)\|_2 > M_{1,T}) + \epsilon/6.$$  

Consequently, we have the relationship

$$\mathbb{P}(\bigcup_{k \in S^c}\{\|\tilde{z}^{(k)}\|_2 \geq 1\}) \leq \sum_{k \in S^c} \mathbb{P}(\|\tilde{G}_T l(\theta_0)_{(k)}\|_2 \geq \gamma_T T^{-1}M_{1,T}^{-\mu}/4) + \mathbb{P}(\lambda_T T^{-1}d_T^{1/2} \max_{k \in S^c,i \in G_k} (|\tilde{\theta}^{(k)}_i|^{-\eta}) \leq \gamma_T T^{-1}M_{1,T}^{-\mu}/4) + \mathbb{P}(\|\tilde{G}_T l(\theta_0)_{(k)}\|_2(d_T/T)^{1/2}K_\epsilon \leq \gamma_T T^{-1}M_{1,T}^{-\mu}/4) + \mathbb{P}(\|\nabla' \{\tilde{G}_T l(\tilde{\theta})_{(k)}\}_{(k)}\|_2(d_T/T)^{1/2}K_\epsilon \leq \gamma_T T^{-1}M_{1,T}^{-\mu}/4) + \mathbb{P}(\|\tilde{\theta}(k)\|_2 > M_{1,T}) + \epsilon/6 := \sum_{i=1}^5 T_i + \epsilon/6.$$  

We then focus on each $T_i$. We have by the Markov inequality

$$T_1 := \sum_{k \in S^c} \mathbb{P}(\|\tilde{G}_T l(\theta_0)_{(k)}\|_2 > \gamma_T T^{-1}M_{1,T}^{-\mu}/4) \leq \sum_{k \in S^c} 16\mathbb{E}[\|\tilde{G}_T l(\theta_0)_{(k)}\|_2^2] \{\gamma_T T^{-1}M_{1,T}^{-\mu}\}^2 \leq 16\mathbb{E}[\|\tilde{G}_T l(\theta_0)\|_2^2] \{\gamma_T T^{-1}M_{1,T}^{-\mu}\}^2 \leq \frac{16d_T}{T\{\gamma_T T^{-1}M_{1,T}^{-\mu}\}^2} = O\left(\frac{\gamma_T}{\sqrt{T}}T_{2}^{2/((1+\mu)(1-c)-1)} \right) - \frac{\epsilon^2}{\gamma_T}.$$
The quantity of interest is \( \gamma \) that (5.10) converge to zero for \( T \) sufficiently large enough. We have

\[
\gamma T \lambda_T^{-1}d_T^{-1/2}M_{1,T}^{-\mu}T^{-\kappa_\eta} \rightarrow \infty \iff \gamma T \lambda_T^{-1}d_T^{-1/2}T^{-\kappa(1+\mu)+\mu} \rightarrow \infty.
\]

As for \( T_3 \), we have by the Markov inequality

\[
T_3 := \sum_{k \in S^c} \mathbb{P}(\|H_T(k,k)\|^2 + \|R_T(k)\|^2)(d_T/T)^{1/2}K_\epsilon > \gamma T^{-1}M_{1,T}^{-\mu}/4
\]

\[
\leq \sum_{k \in S^c} \mathbb{P}(\|R_T(k)\|^2)(d_T/T)^{1/2}K_\epsilon > \gamma T^{-1}M_{1,T}^{-\mu}/4 - \|H_T(k,k)\|^2(d_T/T)^{1/2}K_\epsilon
\]

\[
\leq \sum_{k \in S^c} \{\mathbb{P}(\|R_T(k)\|^2)(d_T/T)^{1/2}K_\epsilon > \gamma T^{-1}M_{1,T}^{-\mu}/4) + \mathbb{P}(\|H_T(k,k)\|^2(d_T/T)^{1/2}K_\epsilon > \gamma T^{-1}M_{1,T}^{-\mu}/4)\}
\]

\[
\leq \sum_{k \in S^c} \left\{ \frac{64K^2d_T\mathbb{E}[\|R_T(k)\|^2]}{\gamma T^{-2}M_{1,T}^{-\mu}} + \frac{64K^2d_T\mathbb{E}[\|H_T(k,k)\|^2]}{\gamma T^{-2}M_{1,T}^{-\mu}} \right\}
\]

\[
\leq \frac{64K^2d_T\|H_T\|^2}{\gamma T^{-2}M_{1,T}^{-\mu}} + \frac{64K^2d_T^2\|R_T\|^2}{\gamma T^{-2}M_{1,T}^{-\mu}}
\]

\[
\leq \frac{64K^2d_T^2\lambda_{\max}(\|H_T\|)}{\gamma T^{-1/2}d_T^{-1/2}M_{1,T}^{-\mu}} + \frac{64K^2d_T^2}{\gamma T^{-3/2}M_{1,T}^{-\mu}}
\]

\[
= O(\frac{\gamma T^{1/2(1+\mu)(1-c)-1}}{\sqrt{T}}\frac{\gamma T^{-1/2}}{d_T^{5/2}M_{1,T}^{-\mu}}) + O(\frac{\gamma T^{1/2(2-3c)-1}}{\sqrt{T}}\frac{\gamma T^{-1}}{d_T^{5/2}M_{1,T}^{-\mu}}).
\]

We obtain for \( T_4 \) by the Markov inequality

\[
T_4 := \sum_{k \in S^c} \mathbb{P}(\|\nabla'(\tilde{G}_T(l(\bar{\theta})(k)))\|^2(d_T/T)^{2}K_\epsilon^2 > \gamma T^{-1}M_{1,T}^{-\mu}/4)
\]

\[
\leq \sum_{k \in S^c} \frac{16K^4d_T^4\mathbb{E}[\|\nabla'(\tilde{G}_T(l(\bar{\theta})(k)))\|^2]}{\gamma^2 T^{-2}M_{1,T}^{-\mu}}
\]

\[
\leq \frac{16K^4d_T^4\|\nabla'(\tilde{G}_T(l(\bar{\theta})))\|^2}{\gamma^2 T^{-2}M_{1,T}^{-\mu}}
\]

\[
\leq \frac{16K^4d_T^4\eta(K_\epsilon)}{\gamma^2 T^{-2}M_{1,T}^{-\mu}} = \frac{16K^4\eta(K_\epsilon)}{\{\gamma T^{-5/2}M_{1,T}^{-\mu}\}^2} = O(\frac{\gamma T^{1/2(1+\mu)(2-5c)-1}}{\sqrt{T}}\frac{\gamma T^{-5/2}}{d_T^{5/2}M_{1,T}^{-\mu}}).
\]
Finally, we have for $T_5$ that

$$T_5 := \sum_{k \in S^c} P(\|\tilde{\theta}^{(k)}\|_2 > M_{1,T}) \leq \sum_{k \in S^c} \frac{\mathbb{E}[\|\tilde{\theta}^{(k)}\|_2^2]}{M_{1,T}^2} \leq \frac{\mathbb{E}[\|\tilde{\theta} - \theta_0\|_2^2]}{M_{1,T}^2} = O((\frac{\gamma T}{\sqrt{T}})^{(2(1+c)-1)}).$$

Hence we obtain from these relationships and using assumption [13]

$$\frac{\gamma T}{\lambda} T^{\mu-(\frac{\gamma}{\lambda}+\kappa\eta)(1+\mu)} \longrightarrow \infty, T \to \infty,$$

$$\frac{\gamma T}{\sqrt{T}} T^{\frac{1}{2}((1+\mu)(1-c)-1)} \longrightarrow \infty, T \to \infty,$$

such that the latter implies

$$\frac{\gamma T}{\sqrt{T}} T^{\frac{1}{2}((1+\mu)(2-5c)-1)} \longrightarrow \infty, T \to \infty.$$

Consequently each $T_i$ converges to zero for $T$ large enough. Hence

$$\mathbb{P}(\bigcup_{k \in S^c} \{\|\hat{z}^{(k)}\|_2 \geq 1\}) \leq \sum_{i=1}^{5} T_i + \epsilon/6 \longrightarrow \epsilon, T \to \infty.$$

For $\epsilon \to 0$, we prove $\mathbb{P}(\bigcup_{k \in S^c} \{\|\hat{z}^{(k)}\|_2 \geq 1\}) \to 0$ for $T$ large enough.

As for the second part of the model selection procedure, we prove that

$$\mathbb{P}(\bigcap_{k \in S^c} \bigcap_{i \in A_k^c} \{\hat{w}_i^{(k)} < 1\}) \longrightarrow 1 \text{ } \Longleftrightarrow \text{ } \mathbb{P}(\bigcup_{k \in S^c} \bigcup_{i \in A_k^c} \{\hat{w}_i^{(k)} \geq 1\}) \longrightarrow 0. \quad (5.11)$$

By the optimality conditions, we have

$$\mathbb{P}(\bigcup_{k \in S} \bigcup_{i \in A_k^c} \{\hat{w}_i^{(k)} \geq 1\}) = \mathbb{P}(\bigcup_{k \in S} \bigcup_{i \in A_k^c} \{\hat{z}_T l((\hat{\theta})_{(k),i}) \geq \lambda_T T^{-1} \alpha^{(k)}_{T,i}\}).$$

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Then by a Taylor expansion around \( \theta_0 \), with \( \bar{\theta} \) between \( \hat{\theta} \) and \( \theta_0 \), we have

\[
\mathbb{P}( \bigcup_{k \in S} \bigcup_{i \in A_k^c} \{|\hat{\omega}_i^{(k)}| \geq 1\}) = \mathbb{P}( \bigcup_{k \in S} \bigcup_{i \in A_k^c} \{|\hat{G}_T(l(\theta_0))_{(k),i} + \sum_j \partial_{ij}^2 \mathbb{G}_T(l(\theta_0))(\hat{\theta}_j - \theta_{0,j})\}|_i \\
+ [\sum j,k \sum_{t=1}^T \partial_{ijk}^3 l(\epsilon_t; \bar{\theta})(\hat{\theta}_j - \theta_{0,j})^2/2_i| \geq \lambda_T T^{-1} \alpha_{T,i}^{(k)}]) \\
\leq \mathbb{P}( \bigcup_{k \in S} \bigcup_{i \in A_k^c} \{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} \alpha_{T,i}^{(k)} - [\sum j \partial_{ij}^2 \mathbb{G}_T(l(\theta_0))(\hat{\theta}_j - \theta_{0,j})]|_i \\
- [\sum j,k \sum_{t=1}^T \partial_{ijk}^3 l(\epsilon_t; \bar{\theta})(\hat{\theta}_j - \theta_{0,j})^2/2_i|])}. \]

Let \( M_{2,T} = \left( \frac{\lambda_T}{T} \right)^{\frac{1}{2+\eta}} \). Then using \( \|\hat{\theta} - \theta_0\|_2 = O_p(\frac{d_T}{T})^{\frac{1}{2}} \) and the Cauchy-Schwarz inequality, we obtain

\[
\mathbb{P}( \bigcup_{k \in S} \bigcup_{i \in A_k^c} \{|\hat{\omega}_i^{(k)}| \geq 1\}) \leq \sum_{k \in S} \sum_{i \in A_k^c} \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} \alpha_{T,i}^{(k)} - [\sum j \partial_{ij}^2 \mathbb{G}_T(l(\theta_0))(\hat{\theta}_j - \theta_{0,j})]|_i \\
- [\sum j,k \sum_{t=1}^T \partial_{ijk}^3 l(\epsilon_t; \bar{\theta})(\hat{\theta}_j - \theta_{0,j})^2/2_i|, |\tilde{\omega}_i^{(k)}| \leq M_{2,T}) + \mathbb{P}(|\tilde{\omega}_i^{(k)}| > M_{2,T})} \\
\leq \sum_{k \in S} \sum_{i \in A_k^c} \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} M_{2,T}^{-\eta} - [\sum j \partial_{ij}^2 \mathbb{G}_T(l(\theta_0))]|_i^{2}1/2 K_{\epsilon}(d_T/T)^{1/2} \\
- [\sum j,k,l,m \sum_{t,t'=1}^T \partial_{ijk}^3 l(\epsilon_t; \bar{\theta})\partial_{ilm}^3 l(\epsilon_t'; \bar{\theta})]^{1/2} K_{\epsilon}^{2}(d_T/T)) \\
+ \mathbb{P}(|\tilde{\omega}_i^{(k)}| > M_{2,T})} + \epsilon/5 \\
\leq \sum_{k \in S} \sum_{i \in A_k^c} \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} M_{2,T}^{-\eta}/3\} \\
+ \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} M_{2,T}^{-\eta}/3\}) \\
+ \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} M_{2,T}^{-\eta}/3\}) \\
+ \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} M_{2,T}^{-\eta}/3\}) \\
+ \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} M_{2,T}^{-\eta}/3\}) + \epsilon/5 := \sum_{i=1}^4 T_i + \epsilon/5.
\]

We proceed as for inactive groups. For \( T_1 \), we have by the Markov inequality

\[
T_1 := \sum_{k \in S} \sum_{i \in A_k^c} \mathbb{P}(\{|\hat{G}_T(l(\theta_0))_{(k),i} \geq \lambda_T T^{-1} M_{2,T}^{-\eta}/3\}) \leq \sum_{k \in S} \sum_{i \in A_k^c} \frac{9\mathbb{E}(\|\hat{G}_T(l(\theta_0))_{(k),i}\|^2)}{\lambda_T T^{-1} M_{2,T}^{-\eta}/3} \\
\leq \frac{9\mathbb{E}(\|\hat{G}_T(l(\theta_0))\|^2)}{\lambda_T T^{-1} M_{2,T}^{-\eta}/3} \\
= O\left(\frac{\lambda_T}{\sqrt{T}} T^{-\frac{3}{2}}(1+\eta)(1-c-1) - \frac{7}{\eta+3} \right).
\]
As for $T_2$, we have

$$T_2 := \sum_{k \in S_i \in A_k} \sum_{j} P\{(\sum_{j} (\partial_{ij}^2 G_T l(\theta_0))^2)^{1/2} K_\epsilon (d_T/T)^{1/2} > \lambda_T T^{-1/2} M_{2,T}^{-n/3}\} \leq \sum_{k \in S_i \in A_k} \sum_{j} \frac{36 d_T E[P_{T_i, (k) j}^2(\theta_0)]}{T} \frac{\lambda_T T^{-1} M_{2,T}^{-n/2}}{T} \leq \frac{36 d_T \lambda_T^2 \langle \| H \|_T \rangle^2}{T} \frac{36 d_T E[\| P_T(\theta_0) \|^2]}{T} \leq O(\frac{\lambda_T T^{-1} (1+\eta)(1-c)^{-1} - \frac{2}{1+c}}{\sqrt{\lambda_T T^{-1} (1+\eta)(1-c)^{-1} - \frac{2}{1+c}}}).$$

Furthermore, for the third order term in $T_3$, we have

$$T_3 := \sum_{k \in S_i \in A_k} \sum_{j, k, l, m} \sum_{T=1}^T \sum_{t, t'} T^{-2} \sum_{T=1}^T \partial_{ijkl}^3 l(\epsilon; T) \partial_{ilm}^3 l(\epsilon; T)^{1/2} K_\epsilon^2 (d_T/T) > \lambda_T T^{-1} M_{2,T}^{-n/3} \leq \frac{9 d_T^2 E[\| \nabla^3 \{ \tilde{G}_T l(\theta) \} \|^2]}{T^2} = O(\frac{\lambda_T T^{-1} (1+\eta)(2-5c)^{-1} - \frac{2}{1+c}}{\sqrt{\lambda_T T^{-1} (1+\eta)(2-5c)^{-1} - \frac{2}{1+c}}}).$$

Finally, we have for $T_4$ that

$$T_4 := \sum_{i \in A_k} P(\| \tilde{\theta}_i^{(k)} \| > M_{2,T}) \leq \sum_{k \in S_i \in A_k} \sum_{j} \frac{\mathbb{E}[\| \tilde{\theta}_i^{(k)} \|]}{M_{2,T}} \leq \frac{\mathbb{E}[\| \tilde{\theta} - \theta_0 \|]}{M_{2,T}} = O(\frac{\lambda_T T^{-1} (1+\eta)(1-c)^{-1} - \frac{2}{1+c}}{\sqrt{\lambda_T T^{-1} (1+\eta)(1-c)^{-1} - \frac{2}{1+c}}}).$$

We have from these relationships and by assumption\[13\] $\frac{\lambda_T}{\sqrt{T}} T^{1/2} (1+\eta)(1-c)^{-1} \to \infty$ implies

$$\frac{\lambda_T}{\sqrt{T}} T^{1/2} (1+\eta)(2-3c)^{-1} \to \infty \quad T \to \infty,$$

$$\frac{\lambda_T}{\sqrt{T}} T^{1/2} (1+\eta)(2-5c)^{-1} \to \infty \quad T \to \infty.$$

We deduce

$$P(\cup_{k \in S_i \in A_k} \{\| \tilde{w}_i^{(k)} \| \geq 1\}) \to \epsilon,$$

for $T$ sufficiently large enough. We have then concluded the model selection consistency.
We now focus on the asymptotic normality. Model selection implies that

$$\mathbb{P}(\{k \in S, i \in A_k : \hat{\theta}_i^{(k)} \neq 0\} = A) \xrightarrow{T \to \infty} 1.$$ 

As a consequence, the next relationship holds

$$\mathbb{P}(\forall k \in S, \hat{G}_T l(\hat{\theta})_{A_k} + \lambda T^{-1}\alpha_{T,A_k} \odot \text{sgn}(\hat{\theta}_{A_k}) + \gamma T^{-1}\xi_{T,k} \hat{\theta}_{A_k} \parallel \hat{\theta}_{A_k} \parallel_2 = 0) \xrightarrow{T \to \infty} 1.$$ 

By a Taylor expansion of the gradient term around $\theta_{0,A}$, we obtain

$$\mathbb{P}(\hat{G}_T l(\theta_{0,A}) + \tilde{\hat{G}}_T l(\theta_{0,A})(\hat{\theta}_A - \theta_{0,A}) + \frac{1}{2} \nabla'((\hat{\theta}_A - \theta_{0,A})'\tilde{\hat{G}}_T l(\hat{\theta})_{A_A}(\hat{\theta}_A - \theta_{0,A})) + \lambda T^{-1}\alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A) + \gamma T^{-1}\eta_{T} = 0) \xrightarrow{T \to \infty} 1,$$

where $\eta_{T} = \text{vec}(\xi_{T,k} \hat{\theta}_{A_k} \parallel \hat{\theta}_{A_k} \parallel_2, k \in S)$ and $\|\hat{\theta} - \theta_0\|_2 \leq \|\hat{\theta} - \theta_0\|_2$. As a consequence, we have

$$\mathcal{P}(\theta_0)(\hat{\theta}_A - \theta_{0,A}) + \mathbb{H}_{T,A_A}(\hat{\theta}_A - \theta_{0,A}) = -\hat{G}_T l(\theta_{0,A}) - \frac{1}{2} \nabla'((\hat{\theta}_A - \theta_{0,A})'\tilde{\hat{G}}_T l(\hat{\theta})_{A_A}(\hat{\theta}_A - \theta_{0,A})) - \lambda T^{-1}\alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A) - \gamma T^{-1}\eta_{T} + o_p(1),$$

where $\mathcal{P}(\theta_0) = \hat{G}_T l(\theta_{0,A}) - \mathbb{H}_{T,A_A}$ and $\mathbb{H}_{T,A_A} = \mathbb{E}[\nabla^2_{\theta \theta} l(\epsilon_i; \theta_0)]_{A_A}$. Then multiplying by $\sqrt{TQ_T V^{-1/2}_{T,A_A}}$, we obtain

$$\sqrt{TQ_T V^{-1/2}_{T,A_A}(\hat{\theta}_A - \theta_{0,A})} = -\sqrt{TQ_T V^{-1/2}_{T,A_A}} \mathbb{H}^{-1}_{T,A_A}(\lambda T^{-1}\alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A) + \gamma T^{-1}\eta_{T})$$

$$= \sqrt{TQ_T V^{-1/2}_{T,A_A}} \mathbb{H}^{-1}_{T,A_A} \hat{G}_T l(\theta_{0,A})$$

$$- \sqrt{T}/2 \sqrt{TQ_T V^{-1/2}_{T,A_A}} \mathbb{H}^{-1}_{T,A_A} \nabla'((\hat{\theta}_A - \theta_{0,A})'\tilde{\hat{G}}_T l(\hat{\theta})_{A_A}(\hat{\theta}_A - \theta_{0,A}))$$

$$- \sqrt{TQ_T V^{-1/2}_{T,A_A}} \mathbb{H}^{-1}_{T,A_A} \mathcal{P}(\theta_0)(\hat{\theta}_A - \theta_{0,A}) + o_p(1).$$

We focus on the $l^1$ penalty term, which can be upper bounded as

$$N_{1,T} := \|\sqrt{TQ_T V^{-1/2}_{T,A_A}} \mathbb{H}^{-1}_{T,A_A}(\lambda T^{-1}\alpha_{T,A} \odot \text{sgn}(\hat{\theta}_A))\| \leq |Q_T V^{-1/2}_{T,A_A} \| \mathbb{H}^{-1}_{T,A_A} |\lambda T^{-1/2} \max_{k \in S, i \in A_k} \alpha_{T,i,A}$$

$$\leq |Q_T V^{-1/2}_{T,A_A} |\lambda^{-1}_{\min}(\mathbb{H}_{T,A_A}) |\lambda T^{-1/2} \min_{k \in S, i \in A_k} |\hat{\theta}_i^{(k)}|^{-1/2}$$

$$\leq |Q_T V^{-1/2}_{T,A_A} |\lambda^{-1}_{\min}(\mathbb{H}_{T,A_A}) |\lambda T^{1/2} \min_{k \in S, i \in A_k} |\hat{\theta}_i^{(k)}|^{-1/2}.$$ 

If $\lambda T^{\kappa} \to 0$, then $N_{1,T} = o_p(1)$. 

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As for the $l^1/l^2$ penalty, it can be upper bounded as

$$N_{2,T} := |\sqrt{T} Q_T V^{-1/2}_{T,A,A} \frac{\gamma_T}{T} | \leq |Q_T V^{-1/2}_{T,A,A} ||H^{-1}_{T,A,A}|| \gamma_T T^{-1/2} \|\eta_T\|_2$$

$$\leq |Q_T V^{-1/2}_{T,A,A} ||H^{-1}_{T,A,A}|| \gamma_T T^{-1/2} \sqrt{\sum_{k \in S} ||\hat{\theta}(k)||^2_2}$$

$$\leq |Q_T V^{-1/2}_{T,A,A} |\lambda_{\min}^{-1}(H^{-1}_{T,A,A}) \gamma_T T^{-1/2} d_T^{1/2} \{\min_{k \in S} ||\hat{\theta}(k)||_2\}^{-\mu}$$

$$\leq |Q_T V^{-1/2}_{T,A,A} |\lambda_{\min}^{-1}(H^{-1}_{T,A,A}) \gamma_T T^{-1/2} d_T^{1/2} T^{\kappa \mu}.$$ 

Using $d_T = O(T^\epsilon)$, if $\gamma_T T^{-\epsilon} \to 0$, then $N_{2,T} = o_p(1)$. Consequently, we have $N_{1,T} + N_{2,T} = o_p(1)$.

We now turn to the hessian quantity of the Taylor expansion and prove the discrepancy $P(\theta_0)$ converges uniformly to zero in probability. For any $\epsilon > 0$, by the Markov’s inequality, we have

$$P(\|\hat{G}_T l(\theta_0),AA - H_{T,AA}\|^2 > (\epsilon/d_T)^2) \leq \frac{d_T^2}{\epsilon^2 T^2 \lambda_{\max}(H_{T,AA})} \sum_{(k,l) \in A} \{ \theta_{k,l}^2 l(\epsilon_l; \theta_0) - E[\nabla^2_{\theta_k \theta_l} l(\epsilon_l; \theta_0)] \}^2$$

As for the third order term, by the Cauchy-Schwartz inequality

$$\|\nabla'((\hat{\theta}_A - \theta_0, A),\hat{G}_T l(\theta),AA(\hat{\theta}_A - \theta_0, A))\|^2 \leq \frac{1}{T^2} \sum_{t=1}^T \left\{ \sum_{(k,l,m) \in A} \partial^3_{\theta_k \theta_l \theta_m} l_T^T(\epsilon_t; \hat{\theta}) \right\} \|\hat{\theta}_A - \theta_0, A\|^4_2$$

$$\leq \frac{1}{T^2} \sum_{t=1}^T \left\{ \sum_{(k,l,m) \in A} \psi^2_T(\epsilon_t) \right\} \|\hat{\theta}_A - \theta_0, A\|^4_2$$

$$= O_p\left( \frac{d_T}{T^2} \right) = o_p\left( \frac{1}{T} \right).$$

We now prove $X_{T,t} = \sqrt{T} Q_T V^{-1/2}_{T,A,A} H^{-1}_{T,A,A} \hat{G}_T l(\theta_0),A, t = 1, \cdots, T$, is asymptotically normal by checking the Lindeberg-Feller’s condition for applying Shiryaev’s Theorem 5.1. We remind that $\hat{G}_T l_{T,t}(\theta_0)$ is the $t$-th point of the score of the empirical criterion. Let $\beta > 0$, and to use Shiryaev’s Theorem, we need to prove that for any $\epsilon > 0$, we have

$$P\left( \sum_{t=0}^T E[\|X_{T,t}\|^2_2 |X_{T,t}\|^2 \geq \beta |\mathcal{F}_{t-1}^T] > \epsilon \right) \rightarrow 0.$$
By the Markov inequality, we obtain

$$
\Pr\left( \sum_{t=0}^{T} \mathbb{E}[||X_{t,t}||^2_2 1_{||X_{t,t}||_2 > \beta} | F_{t-1}^T] > \epsilon \right) \leq \frac{1}{\epsilon} \sum_{t=0}^{T} \mathbb{E}[||X_{t,t}||^2_2 1_{||X_{t,t}||_2 > \beta} | F_{t-1}^T]
$$

$$
\leq \frac{1}{\epsilon} \sum_{t=0}^{T} \mathbb{E}[\mathbb{E}[||X_{t,t}||^2_2 1_{||X_{t,t}||_2 > \beta} | F_{t-1}^T]]^{1/2} \Pr(\|X_{t,t}\|_2 > \beta | F_{t-1}^T)^{1/2}
$$

$$
\leq \frac{1}{\epsilon} \sum_{t=0}^{T} \frac{1}{\beta^2} \mathbb{E}[\mathbb{E}[\|\nabla l(\epsilon_t; \theta_0) \nabla l(\epsilon_t; \theta_0)\|_2 | F_{t-1}^T]]^{1/2}
$$

with $C_{st} > 0$. First, let $K_T = Q_T V_{T,A,A}^{-1} H_{T,A,A}^{-1}$, we have

$$
\mathbb{E}[\|\sqrt{T} K_T \hat{G}_T l_t(\theta_0, A) \|_2 | F_{t-1}^T] = \frac{1}{T} \mathbb{E}[\nabla l(\epsilon_t; \theta_0) K_T \nabla l(\epsilon_t; \theta_0) | F_{t-1}^T]
$$

$$
= \frac{1}{T} \mathbb{E}[\text{Trace}(\nabla l(\epsilon_t; \theta_0) K_T l(\epsilon_t; \theta_0) | F_{t-1}^T)]
$$

$$
= \frac{1}{T} \text{Trace}(\mathbb{E}[\nabla l(\epsilon_t; \theta_0) \nabla l(\epsilon_t; \theta_0) | F_{t-1}^T] K_T l(\epsilon_t; \theta_0))
$$

$$
\leq \frac{1}{T} \lambda_{\max}(H_{t-1}^T) \tilde{C}_{st}
$$

where $\tilde{C}_{st} > 0$. Furthermore, we have

$$
\mathbb{E}[\|\nabla l(\epsilon_t; \theta_0) \nabla l(\epsilon_t; \theta_0)\|_2^2 | F_{t-1}^T] = \mathbb{E}\left[ \sum_{i,j=0}^{d_T} \{ \partial_{\theta_i} l(\epsilon_t; \theta_0) \partial_{\theta_j} l(\epsilon_t; \theta_0) \}^2 | F_{t-1}^T \right]
$$

$$
\leq d_T^2 \sup_{i,j=1, \ldots, d_T} \mathbb{E}\left[ \{ \partial_{\theta_i} l(\epsilon_t; \theta_0) \partial_{\theta_j} l(\epsilon_t; \theta_0) \}^2 | F_{t-1}^T \right].
$$

By assumption 13, we have

$$
\Pr\left( \sum_{t=0}^{T} \mathbb{E}[||X_{t,t}||^2_2 1_{||X_{t,t}||_2 > \beta} | F_{t-1}^T] > \epsilon \right)
$$

$$
\leq \frac{C_{st}^2 \tilde{C}_{st}^{-1} d_T}{T^2} \sum_{t=0}^{T} \mathbb{E}\left[ \sup_{i,j=1, \ldots, d_T} \mathbb{E}\left[ \{ \partial_{\theta_i} l(\epsilon_t; \theta_0) \partial_{\theta_j} l(\epsilon_t; \theta_0) \}^2 | F_{t-1}^T \right] \lambda_{\max}(H_{t-1}^T) \right] \leq \frac{C_{st}^2 \tilde{C}_{st}^{-1} \tilde{B} T d_T}{T^2}.
$$

Consequently, we obtain

$$
\sum_{t=0}^{T} \mathbb{E}[||X_{t,t}||^2_2 1_{||X_{t,t}||_2 > \beta} | F_{t-1}^T] = o_p(1).
$$

We deduce that $X_{T,t}$ satisfies the Lindeberg-Feller condition, and by Theorem 5.1, $\sqrt{T} Q_T V_{T,A,A}^{-1} H_{T,A,A}^{-1} \hat{G}_T l(\theta_0, A)$ is asymptotically normally distributed. The asymptotic distribution of Theorem 5.4 follows.

\[\Box\]
6 Simulation Experiments

In this section, we carry out a simulation study to explore the finite sample performance of the adaptive Sparse Group Lasso. We first focus on the calibration of the adaptive weights entering the penalties. The regularization parameters must satisfy conditions to achieve the oracle property in the double asymptotic case. To do so, we suppose \( \lambda_T = T^\beta \) and \( \gamma_T = T^\alpha \), where \( \beta \) and \( \alpha \) are both strictly positive constant. Regarding assumption [13] we obtain the conditions

\[
\begin{align*}
\alpha + \frac{c}{2} + \kappa \mu - \frac{1}{2} &< 0, \\
\alpha - \frac{1}{2} + \frac{1}{2}(1 + \mu)(1 - c) - 1 &> 0, \\
\beta + \kappa \eta - \frac{1}{2} &< 0, \\
\beta - \frac{1}{2} + \frac{1}{2}(1 + \eta)(1 - c) - 1 &> 0, \\
(1 + \mu)[1 - \frac{c}{2} - \kappa \eta - \beta] + \alpha - 1 &> 0.
\end{align*}
\]

This system allows for flexibility when choosing \( \mu \) and \( \eta \) once \( \kappa, c, \alpha \) and \( \beta \) are fixed. For instance, for \( c = 1/6 \), \( \kappa = 0.05 \), \( \alpha = 1/10 \) and \( \beta = 1/10 \), then \( \mu \in [0.4, 6.3] \) and \( \eta \in [0.6, 7.9] \). If \( \alpha = \beta = 1/5 \) and for \( c = 1/6 \) and \( \kappa = 0.05 \), then \( \mu \in [0.4, 4.3] \) and \( \eta \in [0.3, 5.9] \).

We consider 6 methods in the experiment: the Lasso (L), the Adaptive Lasso (AL), the Group Lasso (GL), the Adaptive Group Lasso (AGL), the Sparse Group Lasso (SGL) and the Adaptive Sparse Group Lasso (ASGL).

There are several methods to numerically solve the non-differentiable statistical problem (5.1). Fan and Li (2001) proposed a local quadratic approximation (LQA) of the first order derivative of the penalty function and a Newton-Raphson type algorithm. To circumvent numerical instability, they suggest to shrink to zero coefficients that are close to zero, that is a coefficient \(|\theta_j| < \epsilon\), with \( \epsilon > 0 \) to be calibrated. The drawback is that once it is set to zero, it will be excluded at any step of the LQA algorithm. Hunter and Li (2005) proposed a more sophisticated version of the LQA algorithm to avoid the drawback of the stepwise selection and numerical instability. They also studied the convergence properties of the LQA method. Zou and Li (2008) proposed a local linear approximation (LLA) of the penalty function such that the estimated coefficients have naturally a sparse representation, under the condition that the penalty function enjoys the continuity condition. Zou (2006) or Zou and Zhang (2009) use the LQA algorithm for their empirical study. Other approaches are also possible such as gradient descent methods.

When one consider the OLS loss function, closed form algorithm can be applied
to our problem. Bühlmann and van de Geer (2011) compiled these methodologies for solving the Lasso and the Group Lasso using gradient descent methods for general penalized convex empirical function. We used these algorithms in our study for solving the group LASSO. As for the LASSO, we applied the shooting algorithm developed by Fu (1998), which is a particular case of the gradient descent method. Simon and al. (2013) proposed an algorithm for solving the SGL that can accommodate likelihood criteria. This is a ”two-step” method, where we first check whether the group is active, and then, if active, check if the coefficient within this group is active. In this simulation study, we used the alternative direction method of multipliers provided by Li and al. (2014).

We used a cross-validation procedure to select both parameters $\lambda_T$ and $\gamma_T$ such that both terms are defined by $\lambda_T = T^\beta$ and $\gamma_T = T^\alpha$, and $\beta = \alpha = 1/8$. The adaptive weights are computed as follows: we first compute an OLS estimator $\hat{\theta}$ such that the adaptive weights entering the penalties correspond to $\tilde{\theta} = \hat{\theta} + T^{-\kappa}$, with $\kappa = 0.2$. As for the adaptive weights, they are chosen such that the above system is satisfied: we set $\eta = 3.5$ and $\mu = 2.5$.

We report the variable selection performance through the number of zero coefficients correctly estimated, denoted as $C$ and, the number of nonzero coefficients incorrectly estimated, denoted $IC$. Besides, the mean squared error is reported as an estimation accuracy measure.

Simulated experiment. We consider a data generating process

$$y = \sum_l \beta_0^{(l)} x^{(l)} + \sigma \eta,$$

where $\eta$ is a strong white noise, normally distributed, centered with unit variance and $\sigma = 0.3$. The matrices $X^{(l)}$ follow $c_r$-dimensional multivariate normal distributions, centered and with variance covariance $\Sigma^{(l)}$ such that the entries are defined as $\Sigma^{(l)}_{ij} = \rho^{|i-j|}, 1 \leq j, i \leq c_l$. The correlation parameter $\rho$ is randomly chosen among $\{0.5, 0.8, 0.9\}$. Moreover, the dimension $d_T = [x \times T^{1/6}]$ with $T = 500, 2000, 4000$ and $x = 10, 30, 50$ respectively for the values of $T$. As $d_T = O(T^c)$ with $c = 1/6$, we can multiply by $x$ to consider more realistic settings. The number of groups is defined as $N_g = 4$ (resp. $N_g = 8$, resp. $N_g = 18$) for $n = 500$ (resp. for $n = 2000$, resp. for $n = 4000$) and the size of each of them is randomly chosen among $\{5, \cdots, 30\}$. The number of active groups is defined as $|S| = 2\alpha_T$ with $\alpha_T = [N_g/3]$. Moreover, zero coefficients are randomly chosen among the whole vector $\beta$ for active groups, such that the total number of zeros -both the zero subvectors for inactive groups
and zero components for active groups - matches the total number of inactive indices. The total number of active indices is defined as $|\mathcal{A}| = 3b_T$ with $b_T = [d_T/9]$. Finally, we generate the active indices among a uniform law $\mathcal{U}([0.1,0.99])$. Zou and Zhang (2009) experiment influenced our framework.

Table 1: Model selection and precision accuracy based on 100 replications

| T  | $d_T$ | $N_g$ | $|S|$ | $|\mathcal{A}|$ | Model   | MSE  | C   | IC |
|----|------|------|-----|-------|---------|------|-----|-----|
| 500| 28   | 4    | 2   | 9     | Truth   | 19   | 0   |     |
|    |      |      |     |       | Lasso   | 0.0178 | 13.13 | 0   |
|    |      |      |     |       | aLasso  | 0.0118 | 17.98 | 0   |
|    |      |      |     |       | GLasso  | 0.0146 | 12.77 | 0   |
|    |      |      |     |       | AGLasso | 0.0129 | 13.57 | 0   |
|    |      |      |     |       | SGL     | 0.0183 | 12.97 | 0   |
|    |      |      |     |       | ASGL    | 0.0101 | 18.83 | 0   |
| 2000| 106  | 8    | 4   | 33    | Truth   | 73   | 0   |     |
|    |      |      |     |       | Lasso   | 0.0118 | 49.65 | 0   |
|    |      |      |     |       | aLasso  | 0.0103 | 70.95 | 0   |
|    |      |      |     |       | GLasso  | 0.0150 | 57.48 | 0   |
|    |      |      |     |       | AGLasso | 0.0160 | 60.78 | 0   |
|    |      |      |     |       | SGL     | 0.0125 | 58.88 | 0   |
|    |      |      |     |       | ASGL    | 0.0095 | 72.70 | 0   |
| 4000| 199  | 18   | 12  | 66    | Truth   | 133  | 0   |     |
|    |      |      |     |       | Lasso   | 0.0105 | 87.17 | 0   |
|    |      |      |     |       | aLasso  | 0.0093 | 131.33 | 0 |
|    |      |      |     |       | GLasso  | 0.0140 | 113.42 | 0 |
|    |      |      |     |       | AGLasso | 0.0150 | 113.17 | 0 |
|    |      |      |     |       | SGL     | 0.0102 | 98.92 | 0   |
|    |      |      |     |       | ASGL    | 0.0094 | 133   | 0   |

We can highlight some interesting remarks from this simulation study. First, the adaptive versions of the Lasso, the Group Lasso or the SGL outperform their non adaptive versions. The difference is significant for the adaptive Lasso and the adaptive SGL. This is in line with the asymptotic theory. The adaptive SGL performs well as it can discard inactive groups and inactive indices among active groups and outperform other adaptive penalization methods.
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