PERIODIC KNOTS AND HEEGAARD FLOER CORRECTION TERMS

STANISLAV JABUKA AND SWATEE NAIK

Abstract. We derive new obstructions to periodicity of classical knots by using tools from Heegaard Floer homology. We accomplish this by a two step approach. The first studies the effect on the Heegaard Floer groups caused by modifying the underlying 3-manifold by an orientation preserving diffeomorphism. The second step applies the results of the first to the cyclic branched covers of a knot, which are naturally equipped with an additional finite order diffeomorphism when the knot is periodic. In the process we refine some of the known results regarding periodic knots. We give examples where our obstructions rule out knot periodicity even when many “classical obstructions” fail.

1. Introduction

The study of periodic knots is an extension of the usual framework of knot theory to the equivariant case. Focusing on finite cyclic group actions, we say that a knot $K$ in $S^3$ is periodic if there exists an integer $q > 1$ and an orientation preserving diffeomorphism $f : S^3 \to S^3$ such that $f(K) = K$, the order of $f$ is $q$, and the fixed point set of $f$ is a circle disjoint from $K$. Any such $q$ is called a period of $K$. The set $B = \text{Fix}(f)$ is called the axis of $f$. We shall refer to a knot of period $q$ as $q$-periodic. The positive resolution of the Smith Conjecture [28] ensures that the axis $B$ is an unknot.

Remark 1.1. It is crucial to assume that $f : S^3 \to S^3$ be smooth to ensure that $B$ is an unknot. Montgomery and Zippin [27] constructed a homeomorphism of $S^3$ of order two, with fixed point set a wild knot.
Let \( \langle f \rangle \) be the subgroup of \( \text{Diff}^+(S^3) \) generated by \( f \), then the orbit space \( S^3/\langle f \rangle \) is diffeomorphic to \( S^3 \) (see [26]), and we let \( \pi : S^3 \to S^3/\langle f \rangle \) be the associated quotient map. The knot \( \overline{K} = \pi(K) \) is called the quotient knot of \( K \) and \( \overline{B} = \pi(B) \) the quotient axis of \( K \). See Figure 1 for an illustration of 3-periodicity of the trefoil.

![Figure 1. A 3-periodic diagram of the trefoil knot \( K \) and its quotient knot \( \overline{K} \). The axis \( B \) and the quotient axis \( \overline{B} \) are also indicated.](image)

Topologists have studied how periodicity of a knot is reflected in its various invariants, including polynomial invariants such as the Alexander polynomial \([10, 29, 30]\), Jones polynomial \([31]\) and its 2-variable generalizations \([7, 8, 40, 43, 44, 46, 47]\), and the twisted Alexander polynomials \([18]\). Obstructions to periodicity have also been found in terms of hyperbolic structures on knot complements \([1]\), homology groups of branched cyclic covers \([32, 33]\), concordance invariants of Casson and Gordon \([33]\), Khovanov homology \([9]\) and Link Floer homology \([17]\).

We add to these a new knot periodicity obstruction that is based on the Heegaard Floer correction terms of a cyclic branched cover of the knot. To do this, we take advantage of the fact that if a knot \( K \) has period \( q \), any cyclic cover \( Y \) of \( S^3 \) branched along \( K \) has an order \( q \), orientation preserving self-diffeomorphism \( F : Y \to Y \) \([32]\). The latter is used in conjunction with the next theorem to obstruct periodicity.

**Theorem 1.2.** Let \( Y \) be a rational homology 3-sphere, \( s \in \text{Spin}^c(Y) \) a \( \text{spin}^c \)-structure on \( Y \), and \( F : Y \to Y \) an orientation preserving diffeomorphism. Then

\[
d(Y, F^*(s)) = d(Y, s),
\]

where \( F^*(s) \) is the pullback of \( s \) under \( F \), and where \( d(Y, s) \) is the Heegaard Floer correction term of \( Y \) associated to the \( \text{spin}^c \)-structure \( s \).
Theorem 1.2 is a consequence of a more general result describing how the Heegaard Floer groups of \((Y, s)\) and \((Y, F^*(s))\) are related; see Section 2 (and specifically Theorem 2.6) for full details.

Returning to a \(q\)-periodic knot \(K\), we let \(Y\) be a prime-power fold cyclic cover of \(S^3\) branched along \(K\). Any such \(Y\) is a rational homology 3-sphere, and features an order \(q\), orientation preserving, self-diffeomorphism \(F : Y \rightarrow Y\). As by Theorem 1.2 the Heegaard Floer correction terms \(d(Y, s)\) are invariant under the action of \(F^*\) on \(\text{Spin}^c(Y)\), understanding the fixed point set of \(F^*\) allows for a prediction of certain correction terms to appear with multiplicities divisible by \(q\), and the absence of such multiplicities obstructs \(q\)-periodicity of \(K\). For instance, if \(F^*\) is fixed-point free, then all but one value of the correction terms, appear with multiplicity \(q\).

Information about the fixed-point set of \(F^*\) can be obtained by considering the first homology of \(\overline{Y}\), the corresponding cyclic cover of \(S^3\) branched along the quotient knot \(\overline{K}\) of \(K\). If for an Abelian group \(H\) and a prime \(\ell\) we let \(H_{\ell}\) denote the \(\ell\)-primary subgroup of \(H\) (the set of elements of order a power of \(\ell\)), we have the following result from [32, Proposition 2.5] the proof of which is based on a transfer argument similar to that in Section 3.3 in the proof of Theorem 1.11.

**Remark 1.3.** Throughout this paper \(q\) and \(\ell\) will denote two distinct primes and \(n\) will be a power of some prime number.

**Theorem 1.4** (Proposition 2.5 in [32]). Let \(Y\) and \(\overline{Y}\) be the \(n\)-fold cyclic covers of \(S^3\) branched along a \(q\)-periodic knot \(K\) and its quotient knot \(\overline{K}\) respectively. Then

\[
\text{Fix} \left( F_* |_{H_1(Y;\mathbb{Z}_\ell)} \right) \cong H_1(\overline{Y};\mathbb{Z}_\ell).
\]

**Remark 1.5.** In what follows we shall rely on an affine identifications of \(\text{Spin}^c(Y)\) and \(H_1(Y;\mathbb{Z})\). In the presence of a diffeomorphism \(F : Y \rightarrow Y\), we shall further require that these identifications be “\(F\)-compatible” in the sense that if \(s \in \text{Spin}^c(Y)\) and \(s \in H_1(Y;\mathbb{Z})\) are identified with one another, then so are \(F^*(s)\) and \(F_*(s)\).

Such identification exist if and only if there is a \(\text{spin}^c\)-structure \(s_0 \in \text{Spin}^c(Y)\) with \(F^*(s_0) = s_0\), in which case the \(F\)-compatible affine identification is chosen so as to identify \(s_0\) with \(0 \in H_1(Y;\mathbb{Z})\). We force the existence of such an \(s_0\) by requiring \(H_1(Y;\mathbb{Z})_2 = 0\), the latter condition implying that \(Y\) possesses a unique spin-structure \(s_0\), and clearly \(F^*(s_0) = s_0\). The condition \(H_1(Y;\mathbb{Z})_2 = 0\) is automatic if \(n\) is a power of 2, as it will be in all of our examples.
An immediate consequence of Theorems 1.2 and 1.4 is our first obstruction to \( q \)-periodicity.

**Theorem 1.6.** Let \( Y \) and \( \overline{Y} \) be the \( n \)-fold cyclic covers of \( S^3 \) branched along a \( q \)-periodic knot \( K \) and its quotient knot \( \overline{K} \) respectively. Assume that \( H_1(\overline{Y}; \mathbb{Z}) = 0 \) and let \( F : Y \to Y \) be the diffeomorphism induced by the \( q \)-periodicity of \( K \). Then, under an \( F \)-compatible affine identification of \( \text{Spin}^c(Y) \) with \( H_1(Y; \mathbb{Z}) \) (see Remark 1.5), all but one of the Heegaard Floer correction terms \( d(Y, s) \), corresponding to Spin\(^c\)-structures \( s \in H_1(Y; \mathbb{Z})_\ell \), occur with multiplicity \( q \).

**Example 1.7.** Consider the knot \( K = 12a_{100} \) from the knot tables [6]. For its 2-fold branched cover \( Y \), we have \( H_1(Y; \mathbb{Z})_5 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \) (throughout the text we reserve the symbol \( \mathbb{Z}_m \) for the cyclic group \( \mathbb{Z}/m\mathbb{Z} \) of \( m \) elements).

If this \( K \) were 3-periodic, the Alexander polynomial of any quotient knot \( \overline{K} \) (for period \( q = 3 \)) would have to be trivial, and hence \( H_1(\overline{Y}; \mathbb{Z}) = 0 \) (see Section 3.4, Example 3.2 for a justification of this claim). According to Theorem 1.6, 3-periodicity of \( K \) would force the correction terms \( d(Y, s) \), with \( s \in H_1(Y; \mathbb{Z})_5 \) and \( s \neq 0 \), to occur with multiplicity 3. This does not happen as an explicit computation of the correction terms shows:

| \( 12a_{100} \) | \( d(Y, s) \) | \(-\frac{4}{5}\) | \(-\frac{2}{5}\) | 0 | \( \frac{2}{5}\) | \( \frac{4}{5}\) |
|---------------|----------------|-------------|-------------|----|-------------|-------------|
| Multiplicity of \( d(Y, s) \) | 2 | 6 | 6 | 4 |

\( s \in H_1(Y; \mathbb{Z})_5 - \{0\} \).

It follows that \( 12a_{100} \) cannot be 3-periodic.

When \( H_1(\overline{Y}; \mathbb{Z})_\ell \neq 0 \), that is when the fixed-point set of \( F_*|_{H_1(Y; \mathbb{Z})_\ell} \) is larger than just the zero element, it is harder to keep tally of the multiplicities of the correction terms \( d(Y, s) \) with \( s \in H_1(Y; \mathbb{Z})_\ell \), and our obstruction to \( q \)-periodicity in this case is weaker, taking the form of an inequality.

**Theorem 1.8.** Let \( K \subset S^3 \) be a \( q \)-periodic knot and let \( Y \) and \( \overline{Y} \) be the \( n \)-fold cyclic covers of \( S^3 \) branched along \( K \) and its quotient knot \( \overline{K} \) respectively. Let \( F : Y \to Y \) be the diffeomorphism induced by the \( q \)-periodicity of \( K \). Then, under an \( F \)-compatible affine identification of \( \text{Spin}^c(Y) \) with \( H_1(Y; \mathbb{Z}) \) (see Remark 1.5), there exists a subgroup \( H \) of \( H_1(Y; \mathbb{Z})_\ell \), isomorphic to \( H_1(\overline{Y}; \mathbb{Z})_\ell \), such that

(i) Each correction term \( d(Y, s) \) with \( s \in H_1(Y; \mathbb{Z})_\ell - H \), occurs with a multiplicity that is divisible by \( q \).
(ii) Let the multiplicities of correction terms \(d(Y, s)\) with \(s \in H_1(Y; \mathbb{Z})\), be \(n_1, n_2, \ldots, n_k\) and let \(m_i\) be their reductions modulo \(q\), that is \(m_i \equiv n_i \pmod{q}\), and \(0 \leq m_i < q\). Then
\[
m_1 + m_2 + \cdots + m_k \leq |H|.
\]

**Example 1.9.** Consider the knots \(7_4\) and \(9_2\) from the knot tables [6]. For each of these, the Alexander polynomial is
\[
\Delta(t) = \Delta_{7_4}(t) = \Delta_{9_2}(t) = 4t^2 - 7t + 4.
\]
Note that \(\Delta(-1) = 15\), and so the first homology of the double branched cover along any knot with this polynomial is \(\mathbb{Z}_{15} \cong \mathbb{Z}_5 \oplus \mathbb{Z}_3\).

Let \(K = 7_4 \# 7_4 \# 9_2\) and suppose that \(K\) has period 3. Then the Alexander polynomial of the quotient knot \(\overline{K}\) is forced to be \(\Delta_{\overline{K}}(t) = 4t^2 - 7t + 4\) (see Section 3.4, Example 3.3 for a justification). Let \(Y\) and \(\overline{Y}\) denote the double branched covers along \(K\) and \(\overline{K}\) respectively. Then \(H_1(Y; \mathbb{Z})_5 \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5\) and \(H_1(\overline{Y}; \mathbb{Z})_5 \cong \mathbb{Z}_5\). By Theorem 1.8, the fixed point set of the generator of the \(\mathbb{Z}_3\) action on \(H_1(Y)\) is a subgroup \(H\) isomorphic to \(\mathbb{Z}_5\) and the sum of the mod 3 multiplicities of the correction terms should be bounded above by \(|H| = 5\).

The table below shows the correction terms \(d(Y, s)\) with \(s \in H_1(Y; \mathbb{Z})\) with their corresponding multiplicities. We see 2 corrections terms with multiplicities 24 and 6, respectively, but 9 distinct correction terms which do not have multiplicities divisible by 3. By Theorem 1.8, \(K\) cannot have period 3. Specifically, adding the modulo 3 reductions of the multiplicities gives: \(2 + 2 + 2 + 1 + 2 + 1 + 1 + 1 = 10 > 5 = |H|\).

| \(d(Y, s)\) | \(-\frac{29}{10}\) | \(-\frac{5}{2}\) | \(-\frac{17}{10}\) | \(-\frac{13}{10}\) | \(-\frac{9}{10}\) | \(-\frac{1}{2}\) | \(-\frac{1}{10}\) | \frac{3}{10} | \frac{7}{10} | \frac{11}{10} | \frac{3}{2}\) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Multiplicity of \(d(Y, s)\) | 8               | 8               | 20             | 24             | 8              | 16             | 20             | 10             | 6              | 4              | 1              |

In Section 3.4 we demonstrate that Examples 1.7 and 1.9 pass several of the “classical” obstructions to knot periodicity, showing that Theorems 1.6 and 1.8 present novel periodicity obstructions.

**Remark 1.10.** The Heegaard Floer correction terms \(d(Y, s)\) of a rational homology 3-sphere, are invariant under conjugation of \(\text{Spin}^c\)-structures: \(d(Y, s) = d(Y, \bar{s})\). This “built-in” \(\mathbb{Z}_2\)-symmetry of the correction terms, unfortunately makes it difficult to use Theorems 1.6 and 1.8 to obstruct 2-periodicity of knots. However, see [17].
The use of correction terms to obstruct $q$-periodicity (with $q$ a prime) in Theorems 1.6 and 1.8 relies on the fact that the hypotheses in said theorems assure that $F_*: H_1(Y;Z) \to H_1(Y;Z)$ has fixed point set $\text{Fix}(F_*)$ smaller than $H_1(Y;Z)$. Since $q$ is prime, the cardinality of the set $\{s, F_*(s), \ldots, F_{q-1}^*(s)\}$ is $q$ for each choice $s \in H_1(Y;Z) - \text{Fix}(F_*)$, thus obtaining a $q$-fold multiple for the value of the correction terms $d(Y,s)$.

This reasoning is completely general and guarantees the existence of $q$-fold values of correction terms $d(Y,s)$, $s \in H_1(Y;Z) - \text{Fix}(F_*)$ whenever one has an order $q$ diffeomorphism $F : Y \to Y$. In the absence of a prime $\ell$ distinct from $q$ satisfying the hypotheses from Theorems 1.6 and 1.8 (as happens, for instance, when $H_1(Y;Z)$ itself is $q$-primary), ensuring that $H_1(Y;Z) - \text{Fix}(F_*)$ is nonempty becomes harder. Nevertheless, we submit the following nontriviality criterion.

**Theorem 1.11.** Let $q, \ell, n$ be as in Remark 1.3. Let $K$ be a $q$-periodic knot with quotient knot $K'$, and let $Y$ and $\overline{Y}$ be their $n$-fold cyclic branched covers. Assume that $H_1(Y;Z)_q \cong \mathbb{Z}_q^k$ for some $k \in \mathbb{N}$. If $|H_1(Y;Z)_q| < q^{k-1}$, then the fixed point set of $F_*$ intersected with the $q$-primary subgroup, i.e., $\text{Fix}(F_* : H_1(Y;Z)_q \to H_1(Y;Z)_q)$ does not equal all of $H_1(Y;Z)_q$, and each correction term $d(Y,s)$ with $s \in H_1(Y;Z)_q - \text{Fix}(F_*)$ occurs with a multiplicity that is divisible by $q$.

The proof of Theorem 1.11 is given in Section 3.3 and applications of it are supplied in Section 3.4.

We offer the following additional application of Theorems 1.6 and 1.8 and 1.11 toward obstructing odd periods of twelve crossing alternating knots. See Example 3.4 for a proof.

**Theorem 1.12.** Among the 1288 alternating twelve crossing knots, there are no examples of knots with odd period greater than 3. There are at most seven 3-periodic knots, namely

$12a_{503}, 12a_{561}, 12a_{615}, 12a_{1019}, 12a_{1022}, 12a_{1202}, \text{ and } 12a_{634}$.

Of these, the first six are 3-periodic (see Figure 4 for their 3-periodic diagrams) while the remaining $12a_{634}$, if 3-periodic, has a quotient knot with Alexander polynomial $4 - 7t + 4t^2$. 
Note that the Full Symmetry Group as listed in KnotInfo [6] for the first six knots in the above list is either $D_3$ or $D_6$, confirming 3-periodicity, whereas for $12a_{634}$, it is listed as $D_1$. However, we are not able to eliminate the possibility of period 3 for this knot using methods in this paper and we are not aware if any other methods succeed in doing so.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{knots.png}
\caption{Periodic diagrams of certain 3-periodic twelve crossing alternating knots.}
\end{figure}

**Organization.** The paper is organized as follows. Section 2 addresses the relevant Heegaard Floer homology concepts. We start by reviewing some background material after which we state the main result of the section - Theorem 2.6 - which addresses how the Heegaard Floer groups of a 3-manifold $Y$ are affected when $Y$ is replaced by a diffeomorphic copy $F(Y)$ of itself, with $F : Y \to Y$ an orientation preserving diffeomorphism.

Section 3 is devoted to knot periodicity. It describes and refines several classical obstructions to knot periodicity, and revisits Examples 1.7 and 1.9 to show that they
pass said obstructions. Proofs of Theorem 1.6, 1.8 and 1.12 are provided in this section as are several additional examples.

Section 4 briefly discusses knot concordance in the equivariant setting. We give a new obstruction, relying on the Heegaard Floer correction terms, for a knot to be equivariant slice.

2. Heegaard Floer homology

This section provides background material on the Heegaard Floer homology groups of oriented, closed 3-manifolds. The main result is Theorem 2.6 which describes how the Heegaard Floer homology groups of a Spin$^c$ 3-manifold $(Y, s)$ are affected when $Y$ is replaced by a diffeomorphic copy $F(Y)$ of itself. Theorem 1.2 is an easy consequence of Theorem 2.6. Our outline in the first half of this section closely follows the foundational papers [36, 37] by P. Ozsváth and Z. Szabó who introduced Heegaard Floer theory in 2000.

2.1. Definition of the Heegaard Floer homology groups.

2.1.1. Pointed Heegaard diagrams. Let $Y$ be a closed, oriented 3-manifold and let \( \{\Sigma_g, \alpha, \beta, z\} \) be a pointed Heegaard diagram for $Y$, that is let $\Sigma_g$ be a (closed) genus $g$ surface, let $\alpha = \{\alpha_1, \ldots, \alpha_g\}$, $\beta = \{\beta_1, \ldots, \beta_g\}$ be $2g$ embedded simple closed curves on $\Sigma_g$, and let $z \in \Sigma_g - (\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g)$ be a point. Moreover, assume that the $\alpha$-curves are mutually disjoint and linearly independent in $H_1(\Sigma_g; \mathbb{Z})$, similarly for the $\beta$-curves, and that the $\alpha$-curves intersect the $\beta$-curves transversely and hence in finitely many points. The manifold $Y$ is re-constructed from the associated Heegaard diagram $(\Sigma_g, \alpha, \beta)$ by attaching 2-dimensional 1-handles to $\Sigma_h$ along the $\alpha$- and $\beta$-curves, “thickening” the thus obtained 2-complex, and attaching a single 3-dimensional 0-handle and 3-handle. One can thus, and we shall, think of the Heegaard surface $\Sigma_g$ as embedded in $Y$.

Let $\varphi : Y \to \mathbb{R}$ be a self-indexing Morse function compatible with the pointed Heegaard diagram in that the critical points of $\varphi$ of index $i$ (with $i = 0, \ldots, 3$) lie in the level sets $\varphi^{-1}(i)$, and with $\Sigma_g = \varphi^{-1}(\frac{3}{2})$. Assume also that the $\alpha$ and $\beta$ curves on $\Sigma_g$ are the intersections of $\Sigma_g$ with the upward flowlines from the index 1 critical points and downward flow lines of the index 2 critical points, respectively. Finally, we demand that there be only one critical point of index 0 and one of index 3 (corresponding to
the unique 3-dimensional handles of indices 0 and 3). Note that there are \(g\) critical points of index 1 and 2 each.

Consider \(\text{Sym}^g(\Sigma_g)\), the \(g\)-fold symmetric product of \(\Sigma_g\) with itself, that is

\[
\text{Sym}^g(\Sigma_g) = \frac{\Sigma_g \times \cdots \times \Sigma_g}{S_g},
\]

where \(S_g\) is the symmetric group on \(g\) letters acting on \(\Sigma_g \times \cdots \times \Sigma_g\) by permutation. It is well known that \(\text{Sym}^g(\Sigma_g)\) is a \(2g\)-dimensional manifold \([24]\). Let \(T_\alpha\) and \(T_\beta\) be the \(g\)-dimensional tori \(T_\alpha = \alpha_1 \times \cdots \times \alpha_g\) and \(T_\beta = \beta_1 \times \cdots \times \beta_g\) viewed as submanifolds of \(\Sigma_g \times \cdots \times \Sigma_g\). We shall denote the images of \(T_\alpha\) and \(T_\beta\) in \(\text{Sym}^g(\Sigma_g)\) by the same symbols. An intersection point \(x \in T_\alpha \cap T_\beta\) shall thus be viewed as an unordered \(g\)-tuple \(x = \{x_1, \ldots, x_g\}\), and for convenience we shall always assume that \(x_j \in \alpha_j \cap \beta_{\sigma(j)}\) for some permutation \(\sigma \in S_g\).

2.1.2. Almost-complex structures on \(\text{Sym}^g(\Sigma_g)\) and the moduli space. The discussion in this section closely follows Section 3 in \([36]\).

Let \((\eta, j)\) be a pair consisting of a Kähler form \(\eta\) and a complex structure \(j\) on \(\Sigma_g\), of which the latter tames the former, that is \(\eta(v, j(v)) > 0\) for every nonzero tangent vector \(v\) on \(\Sigma_g\). Consider the quotient map \(\pi: \Sigma \times \Sigma \rightarrow \text{Sym}^g(\Sigma_g)\) and note that it induces a covering map away from the diagonal \(D \subset \text{Sym}^g(\Sigma_g)\), where \(D = \{\{x_1, \ldots, x_g\} \in \text{Sym}^g(\Sigma_g) : x_i \neq x_j \text{ when } i \neq j\}\).

Let \(\pi_i: \Sigma^g \rightarrow \Sigma_g\) be projection onto the \(i\)-th factor, and let \(\bar{\omega} = \pi_1^*(\eta) + \cdots + \pi_g^*(\eta)\) be the Kähler form on \(\Sigma^g\) induced by \(\eta\). Since \(\bar{\omega}\) is invariant under the action of \(S_g\) on \(\Sigma^g\), it induces a Kähler form \(\omega\) on \(\text{Sym}^g(\Sigma_g) - D\).

Choose a collection of points \(z_1, \ldots, z_m\), one from each connected component of \(\Sigma_g - (\alpha_1 \cup \cdots \cup \alpha_g \cup \beta_1 \cup \cdots \cup \beta_g)\), and find an open set \(V \subset \text{Sym}^g(\Sigma_g)\) with

\[
(\{z_1, \ldots, z_m\} \times \text{Sym}^{g-1}(\Sigma_g)) \cup D \subset V \quad \text{and} \quad V \cap (T_\alpha \cup T_\beta) = \emptyset.
\]

An almost-complex structure \(J\) on \(\text{Sym}^g(\Sigma_g)\) is called \((j, \eta, V)\)-nearly symmetric if

(i) \(J\) tames \(\omega\) over \(\text{Sym}^g(\Sigma_g) - \bar{V}\).

(ii) \(J|_V = \text{Sym}^g(j)\).

The space of all \((j, \eta, V)\)-nearly symmetric almost-complex structures shall be denoted by \(\mathcal{J}(j, \eta, V)\); it is a neighborhood of \(\text{Sym}^g(j)\) in the space of almost-complex structures which agree with \(\text{Sym}^g(j)\) over \(V\).
Let \( D = [0, 1] \times \mathbb{R} \subset \mathbb{C} \) and define the group \( \pi_2(x, y) \), associated to a pair of points \( x, y \in T_\alpha \cap T_\beta \), to be the homotopy group of Whitney disks, that is the homotopy group of maps

\[
\pi_2(x, y) = \left\{ u : D \to \text{Sym}^g(\Sigma_g) \middle| \begin{array}{l}
u(\{1\} \times \mathbb{R}) \subset T_\alpha, \\
u(\{0\} \times \mathbb{R}) \subset T_\beta, \\
\lim_{t \to -\infty} u(s + it) = x, \\
\lim_{t \to +\infty} u(s + it) = y.\end{array} \right\}.
\]

To an element \( \phi \in \pi_2(x, y) \) we associate the integer \( n_z(\phi) \) defined as the algebraic intersection number

\[
(2.2) \quad n_z(\phi) = \# \left( \text{Im}(\phi) \cap (\{z\} \times \text{Sym}^{g-1}(\Sigma_g)) \right).
\]

An element \( \phi \in \pi_2(x, x) \) with \( n_z(\phi) = 0 \) shall be called a periodic class. Indeed, it is shown in [36] that if \( g > 1 \) and for any choice of \( x \in T_\alpha \cap T_\beta \), there is an isomorphism

\[
(2.3) \quad \pi_2(x, x) \cong \mathbb{Z} \oplus H^1(Y; \mathbb{Z}),
\]

under which periodic classes map isomorphically to \( H^1(Y; \mathbb{Z}) \).

We are now in a position to define a moduli space of holomorphic “strips” connecting a pair of points \( x, y \in T_\alpha \cap T_\beta \). Namely, fix a path \( s \mapsto J_s \) of almost-complex structures over \( \text{Sym}^g(\Sigma_g) \), and define \( \mathcal{M}_{J_s}(x, y) \) and \( \mathcal{M}_{J_s}(\phi) \) (with \( \phi \in \pi_2(x, y) \)) as

\[
\mathcal{M}_{J_s}(x, y) = \left\{ u : D \to \text{Sym}^g(\Sigma_g) \middle| \frac{du}{ds} + J_s \left( \frac{du}{dt} \right) = 0, [u] \in \pi_2(x, y) \right\},
\]

\[
(2.4) \quad \mathcal{M}_{J_s}(\phi) = \left\{ u : D \to \text{Sym}^g(\Sigma_g) \middle| \frac{du}{ds} + J_s \left( \frac{du}{dt} \right) = 0, [u] = \phi \in \pi_2(x, y) \right\}.
\]

Translation in the \( i\mathbb{R} \) direction of \( D \) endows \( \mathcal{M}_{J_s}(\phi) \) with an \( \mathbb{R} \)-action, and we define the moduli space of unparametrized \( J_s \)-holomorphic curves as \( \widehat{\mathcal{M}}_{J_s}(\phi) := \mathcal{M}_{J_s}(\phi)/\mathbb{R} \).

For later use, we introduce the notation

\[
(2.5) \quad \bar{\partial}_{J_s}(u) := \frac{du}{ds} + J_s \left( \frac{du}{dt} \right),
\]

and note that the moduli spaces \((2.4)\) are cut out by this “twisted del-bar” operator.

**Theorem 2.1** (Theorem 3.4 and Theorem 3.18, [36]). Fix a Heegaard diagram \( (\Sigma_g, \alpha, \beta) \) and a triple \( (j, \eta, V) \) as above. Then, for a dense open set (in the \( C^\infty \) topology of the path space of \( J(j, \eta, V) \)) of paths \( s \mapsto J_s \) of \( (j, \eta, V) \)-nearly symmetric almost-complex structures, the moduli spaces \( \mathcal{M}_{J_s}(\phi) \) are smoothly cut out by their equations.
Additional properties of the moduli spaces $\mathcal{M}_{J_s}(\phi)$, also proved in Section 3 of [36], are listed below. Their statements involve the Maslov index $\mu(u)$ of a map $u : [0,1] \times i\mathbb{R} \to \text{Sym}^g(\Sigma_g)$ which we do not discuss here but rather refer the reader to [14, 23, 41, 42] for specifics.

(i) [Theorem 3.18, [36]] There are no non-constant $J_s$-holomorphic disks $u$ with $\mu(u) \leq 0$.

(ii) The dimension of the moduli space $\mathcal{M}_{J_s}(\phi)$ equals the Maslov index $\mu(u)$ of any map $u$ with $[u] = \phi$. As we are exclusively interested in non-empty moduli spaces for generic choices of paths $s \mapsto J_s$ (see point (i)), this dimension is given by

$$\mu(u) = \dim \mathcal{M}_{J_s}(\phi) = \dim \text{Ker}(\bar{\partial}_{J_s}), \quad [u] = \phi.$$  

(iii) For each $\phi \in \pi_2(x,y)$ with $\mu(\phi) = 1$, the quotient space $\widehat{\mathcal{M}}(\phi)$ is a compact, oriented, 0-dimensional manifold.

We defer the definition of orientations on moduli spaces to Section 2.1.4 for their explanation requires a detour into Spin$^c$-structures on $Y$, a matter we take up in the next section.

2.1.3. Spin$^c$-structures. Following Ozsváth-Szabó [36] (see also Turaev [45]), we define a Spin$^c$-structure on $Y$ to be an equivalence class $[v]$ of a nowhere vanishing vector field $v$ on $Y$, where equivalence is furnished by homotopy (through nowhere vanishing vector fields) in the complement of a ball in $Y$. We denote the set of Spin$^c$-structures on $Y$ by Spin$^c(Y)$ and note that it is an affine space associated to $H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$. We note that Spin$^c(Y)$ comes equipped with a natural conjugation map induced by sending a vector field $v$ to its negative $-v$. If $s = [v]$ we shall denote $[-v]$ by $\bar{s}$, this notation was used in Remark 1.10 in the introduction.

Given a diffeomorphism $F : Y_1 \to Y_2$ and a Spin$^c$-structure $s = [v] \in \text{Spin}^c(Y_2)$, there is an induced pull-back map $F^* : \text{Spin}^c(Y_2) \to \text{Spin}^c(Y_1)$ furnished by $F^*([v]) = [F^*(v)]$. We tacitly use here the identification $\Psi : TY_1 \xrightarrow{\cong} F^*(TY_2)$ given by

$$\Psi(y,u) = (y,F_*(u)) \quad \text{with} \quad y \in Y_1, \ u \in T_yY_1.$$ 

Thus, a vector field $v$ on $Y_2$ pulls back to $(F^*(v))(y) = F_*^{-1}(v(y)), \ y \in Y_1$.

Of the pointed Heegaard diagram $(\Sigma_g, \alpha, \beta, z)$, the function of the base point $z$ has thus far been obscured. Its relevance shall become clear shortly when we define a
function $s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \text{Spin}^c(Y)$ which depends on $z$ and on the Morse function $\varphi : Y \to \mathbb{R}$ (chosen in Section 2.1.1).

Let $x = \{x_1, \ldots, x_g\} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ be a point and consider the $g$ downward gradient flow lines of $\varphi$ originating at the $g$ index 2 critical points, terminating at the $g$ index 1 critical points, and flowing through the points $x_1, \ldots, x_g \subset \Sigma_g$. Additionally, consider also the downward gradient flowline emanating from the unique index 3 critical point, flowing to the unique index 0 critical point and passing through the base point $z$. Pick disjoint neighborhoods $B_1, \ldots, B_{g+1}$ of these $g+1$ flow lines, each homeomorphic to a 3-ball, and note that the gradient $\nabla \varphi$ is a nowhere vanishing vector field on the complement of these balls. On the boundary of each $B_i$, $\nabla \varphi$ has index 0 and can thus be extended over $B_i$ as a nowhere vanishing vector field $\tilde{\nabla} \varphi$ on all of $Y$. With this understood, we define $s_z(x)$ to be the equivalence class of the thus obtained vector field, that is $s_z(x) = [\tilde{\nabla} \varphi]$.

2.1.4. Orientations on moduli spaces. As is typical in gauge theory (see [15]), the orientability of the moduli space $\mathcal{M}_{J_s}(\phi)$ (with $\phi \in \pi_2(x, y)$ and $s \mapsto J_s$ a path in $\mathcal{J}(j, \eta, V)$) is established by showing that the determinant line bundle of the virtual index bundle of the linearization $D_u \bar{\partial}_J$ of the operator $\bar{\partial}_J$ from (2.5), is trivial in the K-theory of the path-space of $\mathcal{J}(j, \eta, V)$. A choice of orientation of $\mathcal{M}_{J_s}(\phi)$ is pinned down by a choice of a nowhere vanishing section of said trivial determinant bundle. The triviality of the determinant bundle, and thus the orientability of $\mathcal{M}_{J_s}(\phi)$, is established in Proposition 3.10 in [36], the details of which are not relevant to our discussion.

Much more relevant is the way of choosing “coherent orientations” on all the various moduli spaces $\mathcal{M}_{J_s}(\phi)$. For a choice of Spin$^c$-structure $s \in \text{Spin}^c(Y)$, this is done by first enumerating all points in $x_0, \ldots, x_m$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $s_z(x_i) = s$, for each $i = 0, \ldots, m$, and then choosing homotopy classes $\theta_i \in \pi_2(x_0, x_i)$, $i = 1, \ldots, m$, with $n_z(\theta_i) = 0$ (such a collection of of homotopy classes $\theta_i$ is referred to as a complete set of paths for $s$ in [36]).

Fix periodic classes $\phi_1, \ldots, \phi_b_1 \in \pi_2(x_0, x_0)$ which represent a basis for $H^1(Y; \mathbb{Z})$ (see (2.3)) and fix nowhere vanishing sections for the determinant line bundles of the linearized twisted del-bar operator for $\theta_1, \ldots, \theta_m, \phi_1, \ldots, \phi_b_1$. Observe that then any other homotopy class $\phi \in \pi_2(x_i, x_j)$ can uniquely be written as

$$\phi = a_1 \phi_1 + \ldots a_b_1 \phi_b_1 - \theta_i + \theta_j, \quad a_1, \ldots, a_b_1 \in \mathbb{Z}. \tag{2.7}$$
This in turn gives rise to an orientation on the moduli space associated to $\phi$ by “multiplying” the chosen sections for $\theta_1, \ldots, \theta_m, \phi_1, \ldots, \phi_{b_1}$ under the easy to establish identification

\[(2.8) \quad \det(u) \wedge \det(v) \cong \det(u \ast v), \quad u \in \pi_2(x, y), \quad v \in \pi_2(y, w), \]

where $u \ast v$ is the splicing of $u$ and $v$.

To recapitulate, compatible choices of orientations on the moduli spaces $M_{J_s}(\phi)$ for various $\phi$, are obtained by choosing a complete set of paths $\theta_1, \ldots, \theta_m$ and periodic classes $\phi_1, \ldots, \phi_{b_1}$ in an arbitrary manner, choosing orientations of $\det(\theta_1), \ldots, \det(\phi_{b_1})$ in an arbitrary manner, and perpetuating these choices to orientations on the determinant line bundles of all other homotopy classes $\phi$ by means of (2.7) and the isomorphism (2.8). We shall refer to any choice of compatible orientations on the various $M_{J_s}(\phi)$, in the above sense, as a coherent orientation system $\mathfrak{o}$, and write

\[(2.9) \quad \mathfrak{o} = \{\theta_1, \ldots, \theta_m, \phi_1, \ldots, \phi_{b_1}\}, \quad \theta_i \in \pi_2(x_0, x_i), \quad \phi_j \in \pi_2(x_0, x_0). \]

Two coherent orientation systems $\mathfrak{o}$ and $\mathfrak{o}'$ shall be called equivalent if their difference $\delta(\mathfrak{o}, \mathfrak{o}') \in \text{Hom}(H^1(Y; \mathbb{Z}), \mathbb{Z}_2)$ vanishes. The difference is defined as follows: pick an element $h \in H^1(Y; \mathbb{Z})$ and represent it by a periodic class $\phi \in \pi_2(x, x)$ as in (2.3). Then

\[
\delta(\mathfrak{o}, \mathfrak{o}')(h) = \begin{cases} 
0 & ; \quad \mathfrak{o} \text{ and } \mathfrak{o}' \text{ induce the same orientation on } \det(\phi), \\
1 & ; \quad \mathfrak{o} \text{ and } \mathfrak{o}' \text{ induce opposite orientations on } \det(\phi). 
\end{cases}
\]

While the Heegaard Floer groups of $(Y, s)$ defined in the next section depend on a choice of a coherent orientation system from among the $2^{b_1(Y)}$ possible choices, they do so only through its equivalence class. Indeed, Ozsváth and Szabó showed in [37] that there is a canonical choice of coherent orientation system for $(Y, s)$ whenever $s$ is a torsion Spin$^c$-structure (see also Section 2.1.7).

2.1.5. The chain complexes. With a Spin$^c$-structure $s \in \text{Spin}^c(Y)$ and a choice of coherent orientation system $\mathfrak{o}$ for $s$ fixed, we define the groups underlying the chain complex $CF^\infty(Y, s)$ to be the free Abelian groups generated by pairs $[x, i] \in (T_\alpha \cap T_\beta) \times \mathbb{Z}$ subject to the condition $s_+(x) = s$. These Abelian groups become $\mathbb{Z}[U]$-modules under the action defined on generators by $U \cdot [x, i] = [x, i - 1]$. 


To turn $CF^\infty(Y, s)$ into a chain complex, we endow it with the differential $\partial^\infty : CF^\infty(Y, s) \to CF^\infty(Y, s)$, defined on generators as

$$\partial^\infty([x, i]) = \sum_{y \in T_{\alpha} \cap T_{\beta}} \sum_{\phi \in \pi_2(x, y) \atop \mu(\phi) = 1} \# \widetilde{M}(\phi) \cdot [y, i - n_z(\phi)].$$

A key observation is that $n_z(\phi) \geq 0$ for any holomorphic $\phi$, being the intersection number of two complex varieties. Because of this, the subgroup $CF^{-}(Y, s)$ of $CF^\infty(Y, s)$ forms a subcomplex and we denote by $CF^+(Y, s)$ the associated quotient complex. We denote the differentials on $CF^\pm(Y, s)$ induced by $\partial^\infty$ by $\partial^\pm$. It is easy to verify that the $\mathbb{Z}[U]$-action is a chain map, inducing actions on $CF^\pm(Y, s)$. Finally, we let $\widehat{CF}(Y, s)$ denote the chain complex obtained as $\widehat{CF}(Y, s) = \text{Ker}(U : CF^+(Y, s) \to CF^+(Y, s))$ and we let $\hat{\partial}$ be its induced differential. Alternatively, $\widehat{CF}(Y, s)$ can also be defined as the quotient complex $CF^+(Y, s)/\text{Im}(U)$, a claim whose verification we leave as an exercise.

**Theorem 2.2** (Ozsáth-Szabó [36]). The homology groups $HF^\circ(Y, s)$ of the chain complexes $(CF^\circ(Y, s), \partial^\circ)$ with $\circ \in \{\infty, -, +, \widehat{\ }\}$, are topological invariants of the Spin$^c$ 3-manifold $(Y, s)$. They depend on the chosen coherent orientation system $\circ$ for $s$ only through its equivalence class.

The groups $HF^\circ(Y, s)$ are referred to as the Heegaard Floer homology groups of the pair $(Y, s)$. They come equipped with a relative cyclic $\mathbb{Z}_{\circ(s)}$-grading $gr$ which for a pair of generators $[x, i]$ and $[y, j]$ is given by

$$gr([x, i], [y, j]) = \mu(\phi) - 2n_z(\phi) + 2(i - j), \quad \text{with} \quad \phi \in \pi_2(x, y).$$

Here the integer $\mathfrak{d}(s)$ is defined as

$$\mathfrak{d}(s) = \gcd\{(c_1(s), h) \mid h \in H_2(Y; \mathbb{Z})\}.$$

In the case of $s$ torsion (and hence of $\mathfrak{d}(s) = 0$), the relative $\mathbb{Z}$-grading on $HF^\circ(Y, s)$ lifts to an absolute $\mathbb{Q}$-grading $\tilde{gr}$ in the sense that for a pair of generators, the relation

$$\tilde{gr}([x, i]) - \tilde{gr}([y, j]) = gr([x, i], [y, j])$$

holds. The precise definition or $\tilde{gr}$ won’t matter for our consideration, the interested reader is referred to Section 7 of [38].
2.1.6. Exact sequences and the correction terms. By virtue of $CF^+(Y, s)$ and \( \widehat{CF}(Y, s) \) being defined as quotient chain complexes, we obtain two short exact sequences of chain complexes, namely
\[
0 \to CF^-(Y, s) \xrightarrow{\iota} CF^\infty(Y, s) \xrightarrow{\pi} CF^+(Y, s) \to 0,
\]
\[
0 \to \widehat{CF}(Y, s) \xrightarrow{\iota} CF^+(Y, s) \xrightarrow{U} CF^+(Y, s) \to 0.
\]
Both \( \iota \)'s are inclusion maps while \( \pi \) is the quotient map. The associated long exact sequences in Heegaard Floer homology are
\[
\cdots \to HF^-(Y, s) \xrightarrow{\iota_*} HF^\infty(Y, s) \xrightarrow{\pi_*} HF^+(Y, s) \to \cdots
\]
\[
\cdots \to \widehat{HF}(Y, s) \xrightarrow{\iota_*} HF^+(Y, s) \xrightarrow{U} HF^+(Y, s) \to \cdots
\]
(2.14)

If \( s \) is torsion then the maps in the first sequence in (2.14) preserves the absolute grading \( \tilde{gr} \) except the map \( HF^+(Y, s) \to HF^-(Y, s) \) which drops the grading by 1.

**Definition 2.3.** The correction term \( d(Y, s) \) for a torsion Spin\(^c\)-structure \( s \in Spin^c(Y) \) is defined as
\[
d(Y, s) = \min \{ \tilde{gr}(\pi_*(x)) \mid x \in HF^\infty(Y, s) \},
\]
where \( \pi_* : HF^\infty(Y, s) \to HF^+(Y, s) \) is the map from the first of the two exact sequences in (2.14).

The correction terms carry a lot of information about the underlying 3-manifold \( Y \), see for instance [16, 19, 20, 25, 35] for examples of applications. We mention here that if \( Y \) is the 2-fold cyclic cover of \( S^3 \) branched along an alternating knot \( K \), then there is an algorithm for the computation of \( d(Y, s) \) for any \( s \in Spin^c(Y) \) that can easily be implemented on a computer (see [20, 39] for details). This algorithm underlies all computations of correction terms presented in this paper.

2.1.7. Twisted coefficients and canonical coherent orientation systems. To eliminate the dependence of the Heegaard Floer groups \( HF^\circ(Y, s) \) on the choice of an equivalence class of a coherent orientation system \( \sigma \) for \( s \), we present here a canonical choice for \( \sigma \). To do so, we first introduce a twisted coefficients version of Heegaard Floer homology and then appeal to a result of Ozsváth and Szabó’s from [37].
An invariant surjective additive assignment \( A \) (or an additive assignment \( A \) for short) is a collection of functions
\[
A_{x,y} : \pi_2(x,y) \to H^1(Y;\mathbb{Z}), \quad x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta,
\]
such that:

(i) For any \( \phi \in \pi_2(x,y) \) and any \( \psi \in \pi_2(y,w) \), the equality \( A_{x,y}(\phi) + A_{y,w}(\psi) = A_{x,w}(\phi \ast \psi) \) holds.

(ii) \( A_{x,x}(\phi + \psi) = A_{x,x}(\phi) \) if \( \psi \in \mathbb{Z} \subset \mathbb{Z} \oplus H^1(Y;\mathbb{Z}) \cong \pi_2(x,x) \) (see (2.3)).

(iii) \( A \) is surjective.

**Example 2.4** (Section 8.1 in [37]). Fix a Spin\(^c\)-structure \( s \in \text{Spin}^c(Y) \) and let \( x_0, \ldots, x_m \subset \mathbb{T}_\alpha \cap \mathbb{T}_\beta \) be a complete list of intersection points with \( s_z(x_i) = s \) and pick a complete set of paths \( \theta_1, \ldots, \theta_m \) for \( s \) as in Section 2.1.4. These give rise to isomorphisms \( \Theta_{i,j} : \pi_2(x_i,x_j) \to \pi_2(x_0,x_0) \) through the equality \( \theta_1 \ast \pi_2(x_i,x_j) = \pi_2(x_0,x_0) \ast \theta_j \).

We can then define \( A_{x_i,x_j} \) as
\[
A_{x_i,x_j} = \Pi \circ \Theta_{i,j},
\]
where \( \Pi : \pi_2(x_0,x_0) \cong \mathbb{Z} \oplus H^1(Y;\mathbb{Z}) \to H^1(Y;\mathbb{Z}) \) is projection onto the second summand. The thus obtained set of functions \( A = \{A_{x,y}\} \) is an example of an additive assignment.

Using the additive assignment \( A \) from Example 2.4, we are now in position to define a modified differential for the infinity version of the Heegaard Floer chain complex. Namely, letting \( CCF^\infty(Y,s) \) stand for \( CF^\infty(Y,s) \otimes_{\mathbb{Z}} \mathbb{Z}[H^1(Y;\mathbb{Z})] \), we define \( \partial^\infty : CCF^\infty(Y,s) \to CCF^\infty(Y,s) \) as
\[
\partial^\infty([x,i] \otimes \xi) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x,y)} \# \widehat{M}(\phi) \cdot [y,i - n_2(\phi)] \otimes e^{\xi + A(\phi)}.
\]
Here \( \xi \in H^1(Y;\mathbb{Z}) \) and, as customary, we write the element in the group ring \( \mathbb{Z}[H^1(Y;\mathbb{Z})] \) corresponding to \( \sigma \in H^1(Y;\mathbb{Z}) \), as \( e^{\sigma} \). The pair \( (CF^\infty(Y,s), \partial^\infty) \) is a chain complex and its homology group \( HF^\infty(Y,s) \) is the (infinity version of the) Heegaard Floer group of \( (Y,s) \) with totally twisted coefficients. It is left as an easy exercise to the reader to define totally twisted analogues \( HF^\circ(Y,s) \) for \( \circ \in \{+, -, \hat{-}\} \).

Clearly each of \( HF^\circ(Y,s) \) inherits the structure of a \( \mathbb{Z}[H^1(Y;\mathbb{Z})] \)-module (through \( n_1([x,i] \otimes \xi) = ([x,i] \otimes (\eta + \xi)) \)). In addition, the action of \( \mathbb{Z}[U] \) extends to \( HF^\circ(Y,s) \).
by setting $U \cdot ([x, i] \otimes \xi) = [x, i - 1] \otimes \xi$, making $HF^\infty(Y, s)$ into a $\mathbb{Z}[U] \otimes \mathbb{Z}[H^1(Y; \mathbb{Z})]$-module.

Our reason for this “introduction in a nutshell” to Heegaard Floer homology with totally twisted coefficients is the following result.

**Theorem 2.5** (Theorem 10.12 from [37]). There is a unique equivalence class of coherent orientation system $o$ for any torsion Spin$^c$-structure $s \in Spin^c(Y)$ for which there is an isomorphism

$$HF^\infty(Y, s) \cong \mathbb{Z}[U, U^{-1}],$$

as $\mathbb{Z}[U] \otimes \mathbb{Z}[H^1(Y; \mathbb{Z})]$-modules.

Henceforth, whenever $s \in Spin^c(Y)$ is a torsion Spin$^c$-structure, we shall agree to use the equivalence class of coherent orientation system $o$ from Theorem 2.5, without further mention.

**2.2. The maps induced by $F$.** Let $F : Y \to Y'$ be a diffeomorphism and let $(\Sigma_g, \alpha, \beta, z)$ be a pointed Heegaard diagram for $Y$. As agreed in previous section, we think of $\Sigma_g$ as embedded in $Y$. To such a diagram we associate the induced pointed Heegaard diagram for $Y'$ given by

$$\{\Sigma'_g, \alpha', \beta', z'\} \quad \text{with} \quad \Sigma'_g = F(\Sigma_g), \quad \alpha' = F(\alpha), \quad \beta' = F(\beta), \quad z' = F(z).$$

By writing $\alpha' = F(\alpha)$ we mean that $\alpha' = \{\alpha'_1, \ldots, \alpha'_g\}$ with $\alpha'_i = F(\alpha_i)$ for every $i = 1, \ldots, g$, and similarly for $\beta' = F(\beta)$.

Given a Morse function $\varphi' : Y' \to \mathbb{R}$ compatible with $(\Sigma'_g, \alpha', \beta', z')$ (see Section 2.1.1) we consider the associated compatible Morse function $\varphi = \varphi' \circ F : Y \to \mathbb{R}$ and observe that

$$s_{\varphi'}(x') = [\overline{\nabla \varphi'}] = [\overline{\nabla (F^{-1} \circ \varphi)}] = [F_*^{-1}(\overline{\nabla \varphi}) = F_*^{-1} ([\overline{\nabla \varphi}])] = F_*^{-1}(s_*(x)) = F^*(s_*(x)), \tag{2.15}$$

where by $x' \in T_{\alpha'} \cap T_{\beta'}$ we mean $F(x) = \{F(x_1), \ldots, F(x_g)\}$ with $x = \{x_1, \ldots, x_g\}$.

Let $s' \in Spin^c(Y')$ be a Spin$^c$-structure and let $[x, i] \in CF^\infty(Y, F^*(s))$ be a generator. We then define $F_* : CF^\infty(Y, F^*(s)) \to CF^\infty(Y', s)$ to be the group homomorphism which is on generators given by

$$F_*([x, i]) = [x', i].$$
Note that this is well defined according to (2.15) for if $s_z(x) = F^*(s)$ then $s_z'(x') = F_{*}^{-1}(s_z(x)) = F_{*}^{-1}(F_*(s)) = s$.

We proceed by showing that the above group homomorphism is a chain map. While this is ultimately a direct computation, we pause first to verify that $F$ induces maps between the various objects appearing in (2.10) associated to both $Y$ and $Y'$. Firstly, let $F_{*} = F_{|\Sigma_g}$ and note that $F_{*} : \Sigma_g \to \Sigma'_g$ is a surface diffeomorphism. Likewise, $\text{Sym}^g(F_{*}) : \text{Sym}^g(\Sigma_g) \to \text{Sym}^g(\Sigma'_g)$ is a diffeomorphism, one that restricts to diffeomorphisms $F_{|T_{\alpha}} : T_{\alpha} \to T'_{\alpha}$ and $F_{|T_{\beta}} : T_{\beta} \to T'_{\beta}$, and induces a bijection $F_{|T_{\alpha} \cap T_{\beta}} : T_{\alpha} \cap T_{\beta} \to T'_{\alpha} \cap T'_{\beta}$. The homomorphism $F_{*} : \pi_2(x, y) \to \pi_2(x', y')$ given by $F_*(\{u\}) = [\text{Sym}^g(F) \circ u]$ is a group isomorphism (with inverse $F_{*}^{-1} = \text{Sym}^g(F^{-1})$).

In order for this latter isomorphism to map holomorphic strips to holomorphic ones, we need to exercise care in choosing paths of almost-complex structures on $\text{Sym}^g(\Sigma_g)$ and $\text{Sym}^g(\Sigma'_g)$. Let $\eta'$ be a Kähler form on $\Sigma'_g$ and let $\eta = F^*(\eta')$. Let $j'$ be an almost-complex structure on $\Sigma'_g$ taming $\eta'$, and let $j$ be the almost-complex structure on $\Sigma_g$ (taming $\eta$) given by $j(v) = F_{*}^{-1}(j'(f_*(v)))$. As in Section 2.1.2 pick points $z_1, \ldots, z_m$, one from each connected component of $\Sigma_g - \alpha - \beta$ and let $z'_i = F(z_i), i = 1, \ldots, m$. Choose an open set $V' \subset \text{Sym}^g(\Sigma'_g)$ meeting conditions (2.1) and let $V \subset \text{Sym}^g(\Sigma_g)$ be $V = (\text{Sym}^g(F))^{-1}(V)$.

Consider the map $\mathcal{F} : \mathcal{J}(\text{Sym}^g(\Sigma'_g)) \to \mathcal{J}(\text{Sym}^g(\Sigma_g))$ given by

\[
\mathcal{F}(j') = (\text{Sym}^g(F))_{*}^{-1} \circ j' \circ \text{Sym}^g(F)_{*}.
\]

Note that if $j'_s \in \mathcal{J}(j', \eta', V')$ then $\mathcal{F}(j'_s) \in \mathcal{J}(j, \eta, V)$. The key observation to make now is that the generic condition from Theorem 2.1 on $J'_s$ is an open one. Accordingly, the open sets of generic choices $\mathcal{J}_{gen}(j, \eta, V) \subset \mathcal{J}(j, \eta, V)$ and $\mathcal{J}_{gen}(j', \eta', V') \subset \mathcal{J}(j', \eta', V')$ have the property that

\[
\mathcal{J}_{gen}(j, \eta, V) \cap F(\mathcal{J}_{gen}(j', \eta', V')) \neq \emptyset.
\]

Thus, we can and do choose a generic path $J_s \in \mathcal{J}(j, \eta, V)$ for which there exists a generic path $J'_s \in \mathcal{J}(j', \eta', V')$ with $J_S = \mathcal{F}(J'_S)$. Going forward, we shall work with such a pair of paths $J_s$ and $J'_s$.

For generic and compatible choices of $J_s$ and $J'_s$, the moduli spaces $\mathcal{M}_{J_s}(\phi)$ and $\mathcal{M}_{J'_s}(\phi')$ are cut out transversely by their defining equations (2.4), and each equal $\text{Ker}(\bar{\partial}_{J_s})$ and $\text{Ker}(\bar{\partial}_{J'_s})$ respectively (see (2.6)). As the smooth structure of the moduli
spaces can be described by a Kuranishi model \cite{12}, the map $F_* : \ker(\partial J) \to \ker(\partial J')$ becomes a diffeomorphism of smooth manifolds. In particular, relying on (2.6), we conclude

$$\mu(F_*(\phi)) = \mu(\phi).$$

We next turn to comparing the orientations of the moduli spaces $\mathcal{M}_{J}(\phi)$ and $\mathcal{M}_{J'}(\phi')$. Choose a coherent orientation system $o = \{\theta_1, \ldots, \theta_m, \phi_1, \ldots, \phi_b\}$ as in (2.9) for a Spin$^c$-structure $s \in \text{Spin}^c(Y)$. Then $o' = \{\theta'_1, \ldots, \theta'_m, \phi'_1, \ldots, \phi'_b\}$, with $\theta'_i = F_*(\theta)$ and $\phi'_j = F_*(\phi_j)$, is a coherent orientation system for $s' = (F^{-1})^*(s)$. These compatible choices of coherent orientation systems render the moduli spaces $\mathcal{M}_{J}(\phi)$ and $\mathcal{M}_{J'}(\phi')$, and hence also $\widehat{\mathcal{M}}(\phi)$ and $\widehat{\mathcal{M}}(\phi')$, diffeomorphic as oriented manifolds.

With this in place, we finally arrive at the announced computation:

$$F_*(\partial^\infty([x, i])) = F_* \left( \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \#\widehat{\mathcal{M}}(\phi) \cdot [y, i - n_z(\phi)] \right)$$

$$= \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(x, y), \mu(\phi) = 1} \#\widehat{\mathcal{M}}(\phi) \cdot F_*([y, i - n_z(\phi)])$$

$$= \sum_{y' \in \mathbb{T}_\alpha' \cap \mathbb{T}_\beta'} \sum_{\phi' \in \pi_2(x', y'), \mu(\phi') = 1} \#\widehat{\mathcal{M}}(\phi') \cdot [f(y), i - n_z(\phi')]$$

$$= F_*(\partial^\infty([x, i])).$$

Thus $F_* : CF^\infty(Y, f^*(s)) \to CF^\infty(Y', s)$ is an isomorphism of chain groups (with respect to the chosen pointed Heegaard diagrams above), one that is easily seen to commute with the action of $\mathbb{Z}[U]$. Thus, $F_*$ descends to give isomorphisms $F_* : CF^\circ(Y, F^*(s)) \to CF^\circ(Y', s)$ for each of $\circ \in \{-, +, \widehat{CF}\}$. Clearly these induce isomorphisms $F_* : HF^\circ(Y, F^*(s)) \to HF^\circ(Y', s)$ for each $\circ \in \{\infty, -, +, \widehat{CF}\}$.

With this understood, the main result of this section is captured in the next theorem.
**Theorem 2.6.** Let $Y$ be an oriented, closed 3-manifold, $s \in \text{Spin}^c(Y)$ a Spin$^c$-structure on $Y$, and $F : Y \to Y$ an orientation preserving diffeomorphism. Then there are induced isomorphisms $F^\circ : HF^\circ(Y, F^*(s)) \to HF^\circ(Y, s)$ of relatively $\mathbb{Z}_0(s)$-graded $\mathbb{Z}[U]$-modules, for any choice of $\circ \in \{\infty, \pm, \hat{\cdot}\}$. These isomorphisms fit into two commutative diagrams

$$
\cdots HF_{(d)}^-(Y, F^*(s)) \longrightarrow HF_{(d)}^+(Y, F^*(s)) \longrightarrow HF_{(d)}^+(Y, F^*(s)) \longrightarrow HF_{(d-1)}^-(Y, F^*(s)) \cdots
$$

$$
\cdots \hat{HF}_{(d)}(Y, F^*(s)) \longrightarrow \hat{HF}_{(d)}^+(Y, F^*(s)) \longrightarrow \hat{HF}_{(d)}^+(Y, F^*(s)) \longrightarrow \hat{HF}_{(d-1)}^-(Y, s) \cdots
$$

and

$$
\cdots HF_{(d)}^-(Y, s) \longrightarrow HF_{(d)}^+(Y, s) \longrightarrow HF_{(d)}^+(Y, s) \longrightarrow HF_{(d-1)}^-(Y, s) \cdots
$$

$$
\cdots \hat{HF}_{(d)}(Y, s) \longrightarrow \hat{HF}_{(d)}^+(Y, s) \longrightarrow \hat{HF}_{(d)}^+(Y, s) \longrightarrow \hat{HF}_{(d-1)}(Y, s) \cdots
$$

**Theorem 1.2** is an immediate consequence of Theorem 2.6.

It remains to prove the part of Theorem 2.6 pertaining to invariance of grading, an issue we take up next. We first note that the relative cyclic grading is preserved by $F_*$ courtesy of (2.11) and (2.16). To see that the absolute rational grading $\hat{gr}$ is also preserved in the case of a torsion Spin$^c$-structure $s \in \text{Spin}^c(Y)$, let $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ any point. Since $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a finite set and since $F_*$ is an isomorphism, it follows that we can find a sequence of points $x = x_0, x_1, \ldots, x_{m-1} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $x_{i+1} = f(x_i)$ for each $i = 0, \ldots, m-1$ (with addition of indices being cyclic modulo $m$). By preservation of relative grading, and the relation of the absolute to the relative grading (cf. (2.13)), we obtain

$$
\hat{gr}(x_2) - \hat{gr}(x_1) = gr(F(x_1), F(x_0)) = gr(x_1, x_0) = \hat{gr}(x_1) - \hat{gr}(x_0),
$$

$$
\hat{gr}(x_3) - \hat{gr}(x_2) = gr(F(x_2), F(x_1)) = gr(x_2, x_1) = \hat{gr}(x_2) - \hat{gr}(x_1),
$$

$$
\vdots
$$

$$
\hat{gr}(x_1) - \hat{gr}(x_0) = gr(F(x_0), F(x_{m-1})) = gr(x_0, x_{m-1}) = \hat{gr}(x_0) - \hat{gr}(x_{m-1}).
$$
Since \( gr(x_{i+1}, x_i) = gr(F(x_{i+1}), F(x_i)) = gr(x_i, x_{i-1}) \) for each \( i = 0, \ldots, m - 1 \), it follows that \( gr(F(x_{i+1}), F(x_i)) = gr(x_1, x_0) \). Using this, and by adding the above equations, we arrive at

\[
m \cdot (gr(x_1, x_0)) = [\tilde{g}r(x_1) - \tilde{g}r(x_0)] + [\tilde{g}r(x_2) - \tilde{g}r(x_1)] + \cdots + [\tilde{g}r(x_0) - \tilde{g}r(x_{m-1})] = 0
\]

showing that \( \tilde{g}r(x_1) = \tilde{g}r(x_0) \), as needed.

Finally, if \( s \in Spin^c(Y) \) is a torsion Spin\(^c\)-structure and \( o \) the canonical coherent orientation system for \( s \) (from Theorem 2.5), then the induced coherent orientation system \( o' \) is the canonical one \( s' = (F^{-1})^*(s) \), which is easily seen by choosing the complete sets of paths \( A \) and \( A' \) (Example 2.4) to be compatible under the correspondence \( F_* : \pi_2(x, y) \rightarrow \pi_2(x', y') \). This completes the proof of Theorem 2.6.

3. Knot periodicity

This section begins with background material on knot periodicity and reviews some of the “classical obstructions” to periodicity alluded to in the introduction. The proofs of Theorems 1.6 and 1.8 are supplied in Section 3.2.

3.1. Background and classical periodicity obstructions. Let \( K \subset S^3 \) be a knot of period \( q > 1 \) and let \( f : S^3 \rightarrow S^3 \) be the orientation preserving diffeomorphism realizing \( K \)'s \( q \)-periodicity.

For some \( n \in \mathbb{N} \), let \( \varphi : Y \rightarrow S^3 \) be an \( n \)-fold cyclic covering map of \( S^3 \) branched along \( K \). It is easy to see that \( f : S^3 \rightarrow S^3 \) lifts to an order \( q \) diffeomorphism \( F : Y \rightarrow Y \), which in turn induces a \( q \)-fold cyclic covering map \( \Pi : Y \rightarrow \bar{Y} \) (with \( \bar{Y} = Y/(F) \)) branched over \( \bar{\iota}^{-1}(\mathcal{B}) \). The Commutative Diagram 3.1 captures the descriptions from this paragraph.
With this in place, we turn to several classical knot periodicity obstructions.

3.1.1. *Edmonds’ Genus Condition.* In [13] Edmonds showed that if a knot $K$ of genus $g$ has period $q$, then

\[(3.2) \quad q \leq 2g + 1 \text{ and if } q > g, \text{ then } q = g + 1 \text{ or } 2g + 1.\]

Furthermore, if the genus of the quotient knot is $\bar{g}$, then

\[(3.3) \quad g \geq q\bar{g}.\]

This substantially limits the periods a given knot may possess.

3.1.2. *Murasugi’s Alexander Polynomial Conditions.* Note that the Alexander polynomial of a knot is a polynomial $\Delta$ with integer coefficients, such that

\[(3.4) \quad \Delta(t) = t^{\deg(\Delta)} \Delta(t^{-1}).\]

Let $K$ be a $q$-periodic knot with quotient knot $\bar{K}$ and let $\lambda = |lk(K, B)|$ be the absolute value of the linking number of $K$ with its axis $B$. Let $\Delta_K(t)$ and $\Delta_{\bar{K}}(t)$ be the Alexander polynomials of $K$ and $\bar{K}$ respectively. The next two conditions constrain periodicity of $K$ with $\bar{K}$ as a quotient knot.

\[(3.5) \quad \Delta_{\bar{K}} \mid \Delta_K \quad \text{ in } \mathbb{Z}[t, t^{-1}].\]

\[(3.6) \quad \Delta_K(t) \equiv (\Delta_{\bar{K}}(t))^q (1 + t + \cdots + t^{\lambda-1})^{q-1} (\mod q).\]

The symbol “$\equiv$” stands for congruence modulo $q$ up to multiplication by units in $\mathbb{Z}[t, t^{-1}]$. Additionally, $\gcd(\lambda, q) = 1$. The proofs of these can be found in [29, 30]. Conditions (3.5) and (3.6) are called the *Murasugi Conditions.*

The genus of any knot is necessarily greater than or equal to half of the degree of its Alexander polynomial. Therefore, using (3.3) we have

\[(3.7) \quad g(K) \geq \frac{q}{2} \deg(\Delta_{\bar{K}}).\]
3.1.3. The Homology Condition. Recall from Remark 1.3 that \( q \) and \( \ell \) denote distinct primes and \( n \) is a prime-power. Let \( g_q(\ell) \) be the smallest natural number such that

\[
g_{q}(\ell) \equiv \pm 1 \pmod{q}.
\]

Let \( Y \) and \( \overline{Y} \) be \( n \)-fold cyclic covers of \( S^3 \) branched over a \( q \)-periodic knot \( K \) and its quotient knot \( \overline{K} \) respectively. Then there exist non-negative integers \( s, b_1, \ldots b_s \) such that, after identifying \( H_1(Y; \mathbb{Z})_\ell \) with a subgroup of \( H_1(Y; \mathbb{Z})_\ell \) as allowed by Equation (1.1), we have the following isomorphism, proved in [33], and henceforth referred to as the Homology Condition.

\[
H_1(Y; \mathbb{Z})_\ell / H_1(\overline{Y}; \mathbb{Z})_\ell \cong \mathbb{Z}_{\ell}^{2b_1 g_q(\ell)} \oplus \mathbb{Z}_{\ell^2}^{2b_2 g_q(\ell)} \oplus \cdots \oplus \mathbb{Z}_{\ell^s}^{2b_s g_q(\ell)}.
\]

Prime-power fold cyclic covers \( Y \) of \( S^3 \) branched over a knot are rational homology spheres, and the degree of the cover is relatively prime to the order of the first homology of \( Y \). If the degree of the cover is odd, then \( H_1(Y; \mathbb{Z}) \) is always a double (that is \( H_1(Y; \mathbb{Z}) \cong G \oplus G \) for some \( G \)), but in general this is not the case for even-fold covers. In Corollary 3.1 below we observe that for a periodic knot we have a “double” in homology irrespective of the parity of the prime-power fold cover.

**Corollary 3.1.** Let \( q \) and \( \ell \) be two distinct primes and \( K \) a \( q \)-periodic knot with \( Y \) its \( n \)-fold cyclic branched cover (with \( n \) a prime-power of arbitrary parity). Then

\[
H_1(Y; \mathbb{Z})_\ell / H_1(\overline{Y}; \mathbb{Z})_\ell \text{ is a double.}
\]

In particular if \( Y \) is the double branched cover of \( K \), \( \Delta_K(t) \) the Alexander polynomial of the quotient knot, and \( \ell \not| \Delta_K(-1) \), then

\[
H_1(Y; \mathbb{Z})_\ell \text{ is a double.}
\]

**Proof.** The claim (3.10) is a direct consequence of (3.9) as \( \mathbb{Z}_{\ell}^{2b_q(\ell)} \cong \mathbb{Z}_{\ell^m}^{2b_q(\ell)} \oplus \mathbb{Z}_{\ell^m}^{2b_q(\ell)} \), while (3.11) is implied by (3.10) along with the observation that if \( \ell \) does not divide \( \Delta_K(-1) \), then \( H_1(\overline{Y}; \mathbb{Z})_\ell = 0 \). With regards to (3.10) and (3.11), we note that the trivial group is a double. \( \square \)

When in Subsection 3.4 we revisit Examples 1.7 and 1.9 from the introduction, we shall see that that the knots considered therein pass each of the classical obstructions (3.2), (3.5), (3.6), (3.7) and (3.9), underscoring the strength of the novel Heegaard Floer obstruction.
3.2. **Proofs of Theorems 1.6 and 1.8** For the two proofs in this section, we rely on the following notation and assumptions:

Let $K$ be a $q$-periodic knot whose periodicity is realized by an order $q$, orientation preserving diffeomorphism $f : S^3 \to S^3$. For some prime-power $n$, let $Y$ be the $n$-fold cyclic cover of $S^3$ branched over $K$ and let $F : Y \to Y$ be the lift of $f$. Note that $Y$ is a rational homology 3-sphere. Let $\overline{Y}$ be the $n$-fold cyclic cover of $S^3$ branched along the quotient knot $\overline{K}$ and let $\ell$ be a prime distinct from $q$. Recall also that we tacitly identity $Spin^c(Y)$ with $H_1(Y; \mathbb{Z})$ through an $F$-compatible affine identification (see Remark 1.5), and write $s$ or $F_*(s)$ with $s \in H_1(Y; \mathbb{Z})$ instead of $\mathfrak{s}$ or $F^*(s)$ with $\mathfrak{s} \in Spin^c(Y)$.

3.2.1. **Proof of Theorem 1.6** The hypothesis of Theorem 1.6 states that $H_1(\overline{Y}; \mathbb{Z})_\ell = 0$, showing that the fixed point set of $F_*$ restricted to $H_1(Y; \mathbb{Z})_\ell$ is $\{0\}$ (as follows from (1.1)). Thus for every non-zero element $s \in H_1(Y; \mathbb{Z})_\ell$, the set $\{F_*(s), F_2^2(s), \ldots, F_q^q(s)\}$ contains $q$ distinct spin$^c$-structures. As $d(\overline{Y}, F_*^i(s)) = d(\overline{Y}, F_*^{i+1}(s))$ for any pair $i_1, i_2 \in \{1, \ldots, q\}$ (Theorem 1.2), the claim of Theorem 1.6 follows. □

3.2.2. **Proof of Theorem 1.8** Define $H$ to be the subgroup of $H_1(Y; \mathbb{Z})_\ell$ given by $H = \text{Fix}(F_*|_{H_1(Y; \mathbb{Z})_\ell})$, and note that $H \cong H_1(Y; \mathbb{Z})_\ell$ according to (1.1). Consider the cosets $s + H$ of $H$ in $H_1(Y; \mathbb{Z})_\ell$ and for any $s + h \in s + H \neq 0 + H$, consider the associated orbit $\{s + h, F_*(s + h), F_2^2(s + h), \ldots, F_q^q(s + h)\}$ of $s + h$ under the action of $\mathbb{Z}_q = \{\text{Id}, F_*, \ldots, F_q^{-1}\}$. Each such orbit has $q$ elements since $q$ is prime (restricting the cardinality of said orbit to be either 1 or $q$) and because an orbit of cardinality 1 would lead to $s + h = F_*(s + h)$ and thus to $s + h \in H$, contrary to assumption. Accordingly, there are

$$\frac{|[H_1(Y; \mathbb{Z}) : H] - 1| \cdot |H|}{q}$$

values of correction terms (not necessarily all distinct) $d(\overline{Y}, s)$ with $s \in H_1(Y; \mathbb{Z})_\ell$, each of which occurs with multiplicity $q$. The remaining $|H|$ correction terms $d(\overline{Y}, s)$ with $s \in H$, may have arbitrary multiplicity. Theorem 1.8 now follows. □

3.3. **Proof of Theorem 1.11** In this section, assume $K$ is a $q$-periodic knot with $q$ a prime, let $\overline{K}$ be its quotient knot and let $Y$ and $\overline{Y}$ be their $n$-fold branched covers. Assume also that $H_1(Y; \mathbb{Z})_q \cong \mathbb{Z}_q^k$ for some $k \in \mathbb{N}$.
The proof of Theorem 1.11 rests on the existence of a “transfer map” \( \mu_* : H_1(\overline{Y}; \mathbb{Z})_q \to H_1(Y; \mathbb{Z})_q \) which in the case that \( F_* : H_1(Y; \mathbb{Z})_q \to H_1(Y; \mathbb{Z})_q \) is the identity, satisfies the relation (see Diagram 3.1 for a definition of \( \Pi \))

\[
(3.12) \quad \mu_* \circ \Pi_* = q \cdot \text{id}_{H_1(Y; \mathbb{Z})_q}.
\]

The existence of \( \mu_* \) and the validity of relation (3.12) follow from the results in Section III.2 of [2], see specifically relation (2.2) in said section. When \( H_1(Y; \mathbb{Z})_q \cong \mathbb{Z}_q^k \), the image of multiplication by \( q \) is a subgroup of cardinality \( q^{k-1} \), and as a consequence of (3.12), the image of \( \mu_* \) and hence \( H_1(Y; \mathbb{Z})_q \) have a cardinality greater than or equal to \( q^{k-1} \), proving Theorem 1.11.

3.4. Examples. This section illustrates the usefulness of Theorems 1.6 and 1.8 as a novel obstruction to knot periodicity. We start by re-examining Examples 1.7 and 1.9 from the introduction, through the lens of the classical periodicity obstructions from Section 3.1.

Example 3.2 (Example 1.7 revisited). As we saw in Example 1.7 from the introduction, the knot \( K = 12a_{100} \) from the knot tables, is excluded from being 3-periodic by Theorem 1.6. The irreducible factorization of its Alexander polynomial \( \Delta_K \) is given by

\[
\Delta_K(t) = 3t^6 - 21t^5 + 53t^4 - 71t^3 + 53t^2 - 21t + 3
= (t^3 - 5t^2 + 6t - 3)(3t^3 - 6t^2 + 5t - 1).
\]

(3.13)

The Murasugi Condition (3.6) for \( \Delta_K(t) \) and with \( q = 3 \), reads

\[
(3.14) \quad (1 + t)^2 \equiv (1 + t + \cdots + t^{\lambda-1})^2 \cdot (\Delta_K(t))^3 \pmod{3}.
\]

As by (3.4) and (3.5) \( \Delta_K \) is a symmetric factor of \( \Delta_K \), the only possibilities for \( \Delta_K(t) \) are 1 or \( \Delta_K(t) \). Of these, the only one that fits condition (3.14) is \( \Delta_K(t) = 1 \) (verifying the claim made in Example 1.7). Thus \( K = 12a_{100} \) passes the Murasugi condition with \( q = 3 \), \( \Delta_K(t) = 1 \) and \( \lambda = 2 \) (note that, as required, \( \gcd(\lambda, q) = 1 \)).

The first homology of the 2-fold cyclic cover \( Y \) of \( S^3 \) branched over \( K \) is

\[
H_1(Y; \mathbb{Z}) \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_9.
\]
Given this and given \( q = 3 \), the only meaningful choice of \( \ell \) is 5. Since \( g_3(5) = 1 \) (see (3.8)), condition (3.9) gives us (keeping in mind that \( \Delta_K (t) = 1 \) implies \( H_1 (Y ; Z) = 0 \))

\[
H_1 (Y ; Z)_5 \cong \mathbb{Z}^{2a_1} \oplus \mathbb{Z}^{2b_1} \oplus \cdots \oplus \mathbb{Z}^{2b_t},
\]

which is clearly satisfied with \( t = 1, a_1 = 1 \) and \( a_i = 0 \) for \( i \geq 2 \). Accordingly, \( K = 12a_{100} \) passes the Homology Condition.

**Example 3.3** (Example 1.9 revisited). Let \( K = 7_4 \# 7_4 \# 9_2 \) as in Example 1.9. As already noted, the knots \( K_1 = 7_4 \) and \( K_2 = 9_2 \) have the same Alexander polynomial \( \Delta_{K_i} (t) = 4t^2 - 7t + 4 \), so that the Alexander polynomial of \( K \) is \( \Delta_K (t) = (4t^2 - 7t + 4)^3 \). As \( 4t^2 - 7t + 4 \) is irreducible, the only possibilities for the Alexander polynomial of a quotient knot \( \overline{K} \) are 1 or powers of \( 4t^2 - 7t + 4 \).

The Murasugi condition (3.6) with \( q = 3 \) becomes

\[
(1 + t)^6 \equiv (1 + t + \cdots + t^{\lambda - 1})^2 \cdot (\Delta_{\overline{K}} (t))^3 \pmod{3},
\]

which forces \( \Delta_{\overline{K}} (t) = 4t^2 - 7t + 4 \) and \( \lambda = 1 \) (verifying a claim made in Example 1.9). With these choices, \( K \) satisfies the Murasugi Conditions (3.5) and (3.6).

It is easily seen that \( K \) also meets the Homology Condition (3.9) since

\[
H_1 (Y ; Z) \cong \mathbb{Z} \oplus \mathbb{Z}_5,
\]

(\( Y_i \) is the 2-fold cyclic cover of \( S^3 \) branched along \( K_i, i = 1, 2 \)), rendering \( H_1 (Y ; Z) \) (with \( Y \) the 2-fold cyclic cover of \( S^3 \) branched along \( K \)) isomorphic to \( H_1 \) of the 2-fold cyclic branched cover of \( S^3 \) branched along the 3-periodic knot \( K' = 7_4 \# 7_4 \# 7_4 \).

**Example 3.4** (Twelve crossing alternating knots). The largest genus of any knot with 12 crossings is \( g = 5 \). This is observed in [6], and is proved by the following simple calculation.

Seifert’s algorithm to find a Seifert surface for a 12-crossing knot will generate at least three Seifert circuits. (Only the trivial knot has exactly one Seifert circuit and only the torus knots of type \((n, 2)\) have exactly two Seifert circuits and \( n \) in that case has to be odd.) So the Euler characteristic of a Seifert surface bounded by a 12-crossing knot is \( \chi \geq -12 + 3 \). As \( \chi = 1 - 2g \), it follows that \( g \leq 5 \).

According to Edmonds’ Condition (3.2), the largest possible period of any 12 crossing knot is then \( q = 11 \). Furthermore, for a genus 5 knot, the only possible periods larger
than 5 are 6 and 11 and a period 6 knot is necessarily period 3. So the only prime
periods we need to investigate are 2, 3, 5 and 11.

Of the 1288 alternating twelve crossing knots from the knot tables [6], there are no
knots that pass the Murasugi Conditions (3.5) and (3.6) and the Homology Condition
(3.9) with $q = 5$ or 11. Thus excluding odd $q$-periodicity with $q > 3$ does not require
the use of correction terms at all. We now focus on period 3.

First note that if $\Delta_K(t) \equiv 1 \mod 3$, the knot $K$ would satisfy the Murasugi Condi-
tions (3.5) and (3.6) with $\Delta_K$ equal to any of the symmetric factors of $\Delta_K$. There are
twenty nine 12-crossing alternating knots that have such an Alexander polynomial.
These are: 12a37, 12a52, 12a145, 12a170, 12a183, 12a255, 12a314, 12a320, 12a330, 12a339, 12a379, 12a400, 12a407, 12a543, 12a555, 12a613, 12a619, 12a753, 12a792, 12a797, 12a877, 12a978, 12a1063, 12a1106, 12a1113, 12a1148, 12a1202, 12a1287.

One of these, 12a1202, is shown to be period 3 in Figure 2. The remaining 28 knots
have irreducible Alexander polynomials, and therefore choices for $\Delta_K$ are 1 or $\Delta_K$
itself.

The knot 12a1287 has a quadratic Alexander polynomial $\Delta_K(t) = 9t^2 - 19t + 9$. It is
easy to see that this knot has genus one. Now, (3.7) tells us that $\Delta_K(t) = 1$, and the
Homology Condition in (3.11), with $\ell = \Delta_K(-1) = 37$, rules out period 3.

The other 27 knots have Alexander polynomials with degree 4 or higher. As genus
of a 12-crossing knot is at most 5, from (3.7) we know that the degree of $\Delta_K$ has to
be at most 2. Therefore, in each of these cases, we have $\Delta_K = 1$. Using the Homology
Condition (3.9) we are able to eliminate period 3 for all 27.

There are 17 remaining knots that pass the Murasugi and Homology Conditions
with $q = 3$. These include the remaining five period 3 knots from Figure 2 namely,
12a503, 12a561, 12a615, 12a1019, 12a1022, and the 12 knots listed below:

\begin{align}
12a735, \\
12a100, 12a348, 12a376, 12a1206, \\
12a390, 12a425, 12a459, 12a596 12a672, \\
12a634, 12a780.
\end{align}

(3.15)

We first deal with 12a735, as this knot can be handled using our refinements of the
classical obstructions. It has Alexander polynomial $\Delta_K(t) = 6 - 25t + 37t^2 - 25t^3 + 6t^4$, which satisfies the Murasugi conditions with $\Delta_K(t)$ equal to 1 or $3t^2 - 5t + 3$. Homology
Conditions are satisfied if it is the latter. However, this knot has genus 2, and therefore (3.7) forces $\Delta_K(t)$ to be 1. As we have $\Delta_K(-1) = 9 \times 11$, with $\ell = 11$, (3.11) rules out period 3.

Of the knots in the second row of (3.15), $12a_{100}$ was already shown to not have period 3 in Example 1.7 using correction terms. Using the same method, we shall demonstrate next that the other three knots in this row also cannot have period 3.

The Alexander polynomial of knot $K = 12a_{348}$ is

$$\Delta_K(t) = 2 - 17t + 54t^2 - 79t^3 + 54t^4 - 17t^5 + 2t^6,$$

(3.16)

$$= (t - 2)(2t - 1)(t^2 - 3t + 1)^2,$$

and its mod 3 reduction is

$$\Delta_K(t) \equiv (1 + t + \cdots + t^3)^2 \pmod{3}.$$  

(3.17)

This with (3.6) forces $\lambda = 4$ and $\Delta_R(t) \equiv 1$. Examining the factors of $\Delta_K(t)$ in (3.16), we find that $\Delta_R(t) = 1$. Accordingly, the 2-fold cyclic cover $Y$ of $S^3$ branched along $K$, has trivial first homology. Choosing $\ell = 5$, Theorem 1.6 shows that all correction terms $d(Y, s)$ with $s \in H_1(Y; \mathbb{Z}) - \{0\}$ (and with $Y$ the 2-fold cyclic cover of $S^3$ branched along $K$) must come with multiplicities divisible by 3. However, the correction terms and their multiplicities in the next table show that this is not the case.

| $d(Y, s)$ | $-\frac{94}{45}$ | $\frac{58}{45}$ | $-\frac{8}{9}$ | $-\frac{28}{45}$ | $\frac{16}{45}$ | $-\frac{2}{9}$ | $\frac{14}{45}$ | $\frac{4}{9}$ |
|-----------|------------------|------------------|--------------|---------------|------------|----------|------------|----------|
| Multiplicity of $d(Y, s)$ | 1 | 2 | 4 | 2 | 2 | 2 | 6 | 4 |

Turning to $K = 12a_{376}$, we note that its two-fold cover $Y$ has $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}_{27} \oplus \mathbb{Z}_5$ while its quotient knot $\overline{K}$ has two-fold cover $\overline{Y}$ with $H_1(\overline{Y}; \mathbb{Z}) \cong \mathbb{Z}_{15} \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5$.

Equation (3.12) prevents $F_* : H_1(Y; \mathbb{Z}) \to H_1(Y; \mathbb{Z})$ from being the identity, since the cardinality of the image of $3 \cdot \text{id}_{H_1(Y; \mathbb{Z})}$ is 45 while $|H_1(\overline{Y}; \mathbb{Z})| = 15 < 45$. This forces $F_* = 46 \cdot \text{id}$ or $F_* = 91 \cdot \text{id}$, each of which has a fixed point set of cardinality 45. This in turn forces the $135 - 45 = 90$ remaining correction terms of $Y$ to have values that come with multiplicities that are multiples of 3. An explicit computation shows that this is not the case:

| $d(Y, s)$ | $1/2$ | $-\frac{11}{10}$ | $-\frac{1}{6}$ | $\frac{1}{10}$ | $-\frac{17}{30}$ | $\frac{7}{30}$ | All others |
|-----------|--------|------------------|--------------|---------------|------------|----------|----------|
| Multiplicity of $d(Y, s)$ | 3 | 6 | 6 | 12 | 12 | 2 or 4 |
Lastly, the knot $K = 12a_{1206}$ from (3.15) has two-fold cover $Y$ with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}_7 \oplus \mathbb{Z}_{35}$ and has quotient knot $\overline{K}$ with two-fold cover $\overline{Y}$ with $H_1(\overline{Y}; \mathbb{Z}) \cong \mathbb{Z}_5$. This allows for an application of Theorem 1.8 with the choice of $\ell = 7$, forcing the 48 correction terms $d(Y, s)$ of $Y$ corresponding to $s \in H_1(Y; \mathbb{Z})_7 - \{0\}$ to come with values with multiplicities a multiple of 3. An explicit computation shows this does not happen, precluding $12a_{1206}$ from being 3-periodic:

| $d(Y, s)$ | $-\frac{9}{7}$ | $-1$ | $-\frac{5}{7}$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{5}{7}$ | $1$ | $\frac{9}{7}$ |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Multiplicity of $d(Y, s)$ | 2 | 4 | 6 | 6 | 6 | 6 | 8 | 2 |

Somewhat similar to the case of $12a_{476}$ above, the knots in row three of (3.15) have two-fold covers $Y$ with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}_k^2$ with $k = 4, 5$. However, their corresponding quotient knots $\overline{K}$ have two-fold covers $\overline{Y}$ with $|H_1(\overline{Y}; \mathbb{Z})| = 3, 9$, and thus by Theorem 1.11 the action of $F_*$ on $H_1(Y; \mathbb{Z})$ cannot be the identity. It thus must equal $F_* = 28 \cdot \text{id}$ or $F_* = 55 \cdot \text{id}$ (if $k = 4$) and $F_* = 82 \cdot \text{id}$ or $F_* = 162 \cdot \text{id}$ (if $k = 5$). The fixed point set of $F_*$ has cardinality 27 (if $k = 4$) or 81 (if $k = 5$), showing that $81 - 27 = 54$ (if $k = 4$) or $243 - 81 = 162$ (if $k = 5$) of the correction terms of $Y$ must come with values that have multiplicity a multiple of 3. An explicit computation shows that this is not the case for any of these knots. For example, for $12a_{425}$, we have $H_1(Y) = \mathbb{Z}_{34}$, $H_1(\overline{Y}) = 0$, and the correction terms and their multiplicities are given by

| $d(Y, s)$ | $-\frac{8}{9}$ | $\frac{4}{9}$ | $-\frac{2}{9}$ | 0 | All others |
| --- | --- | --- | --- | --- | --- |
| Multiplicity of $d(Y, s)$ | 4 | 4 | 6 | 9 | 2 |

Finally, the two knots in the last row, if 3-periodic, are forced by the Murasugi Conditions to have quotient knots with Alexander polynomials $4 - 7t + 4t^2$ (in the case of $12a_{634}$) and $1 - t + t^2$ (in the case of $12a_{750}$). The knot $K = 12a_{634}$ has 2-fold cyclic branched cover $Y$ with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_5$ and quotient knot $\overline{K}$ with 2-fold cyclic branched cover $\overline{Y}$ with $H_1(\overline{Y}; \mathbb{Z}) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_5$. Thus the only sensible choice of $\ell$ in Theorem 1.8 is that of $\ell = 5$, rendering the said theorem ineffective as $H_1(Y; \mathbb{Z})_5 \cong H_1(\overline{Y}; \mathbb{Z})_5$. On the other hand, condition (3.12) in Theorem 1.11 can be satisfied for certain choices of maps $\Pi_*$ and $\mu_*$, not allowing us to preclude the equality $F_* = \text{id}$. Accordingly, the correction term methods cannot be brought to bear on this knot.

The knot $K = 12a_{750}$ has two-fold cover $Y$ with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_9$ and quotient knot $\overline{K}$ with two-fold cover $\overline{Y}$ with $H_1(\overline{Y}; \mathbb{Z}) \cong \mathbb{Z}_3$. Thus Theorem 1.8 with
ℓ = 5 applies here, but alas all the correction terms corresponding to \( \mathbb{Z}_5 \oplus \mathbb{Z}_5 - \{0\} = H_1(Y; \mathbb{Z})_5 - \{0\} \) do come with values that all have multiplicity 3. Theorem 1.11 does not apply to the 3-torsion here. This makes \( K = 12a_{780} \) the only knot among twelve crossing alternating knots that passes the Murasugi Conditions, the Homology Condition and the correction terms condition, but the stronger factorization conditions over cyclotomic integers obtained by Davis-Livingston in [10] show that this knot is not 3-periodic. (The knot 12a_{634} satisfies Davis-Livingston conditions.)

We finish this section by mentioning an application of Theorem 1.2 in a more general setting of branched covers of \( S^3 \) over knots that are not necessarily periodic.

**Theorem 3.5.** Let \( q \) be a prime and let \( Y \) be the \( q \)-fold cyclic cover of \( S^3 \) branched over a knot \( K \) (not necessarily periodic) and assume that \( H_1(Y; \mathbb{Z})_2 = 0 \). Then each Heegaard Floer group \( HF^c(Y, s) \), \( s \in \text{Spin}^c(Y) - \text{Spin}(Y), \sigma \in \{\infty, - , +, \hat{\sigma}\} \) occurs (up to isomorphism of absolutely \( \mathbb{Q} \)-graded \( \mathbb{Z}[U] \)-modules) with multiplicity that is divisible by \( q \). In particular, the correction terms \( d(Y, s) \), \( s \in \text{Spin}^c(Y) - \text{Spin}(Y) \) occur with multiplicity divisible by \( q \).

**Proof.** We refer to Equation (1.1) for the notation. Pick an \( F \)-compatible affine identification of \( \text{Spin}^c(Y) \) with \( H_1(Y; \mathbb{Z}) \) which maps the unique spin-structure of \( Y \) to \( 0 \in H_1(Y; \mathbb{Z}) \). Here \( F \) is a generator of \( \mathbb{Z}_q \) – the group of deck transformations of \( Y \).

As \( H_1(Y; \mathbb{Z}) \) does not contain \( q \)-primary torsion, any prime \( \ell \) for which the \( \ell \)-primary part of \( H_1(Y; \mathbb{Z}) \) is nontrivial, is distinct from \( q \). As the quotient space \( \overline{Y} \) of \( Y \) under deck transformations is \( S^3 \), the action of \( F_* \) on \( H_1(Y; \mathbb{Z}) \) is free. The claim then follows from Theorems 2.6 and 1.2. \( \square \)

Unfortunately, at present for \( q > 2 \), there are no effective methods to compute the Heegaard-Floer groups or the associated correction terms for a \( q \)-fold cover of \( S^3 \) branched over a knot \( K \).

4. Equivariant Slice Knots

A knot \( K \subset S^3 \) is *slice* if it bounds a properly embedded smooth 2-disk \( D \) in the 4-ball \( B^4 \). In such a case, a prime-power fold cyclic cover \( Y \) of \( S^3 \), branched over the slice knot, is the boundary of the corresponding cover of the 4-ball branched over the slice disk. Correction terms of \( Y \) provide obstructions to sliceness via the theorem below, proved in [36]. (Also see [20, Theorem 2.3].)
Theorem 4.1. Let $Y$ be a rational homology 3-sphere which bounds a rational homology 4-ball $X$. Then $|H_1(Y; \mathbb{Z})| = m^2$ for some $n$ and there is a subgroup $P$ of $H_1(Y; \mathbb{Z})$ of order $m$ such that

$$d(Y, s) = 0 \quad \forall s \in P$$

under a suitable identification $\text{Spin}^c(Y) \cong H_1(Y; \mathbb{Z})$.

The subgroup $P$ is the kernel of the map from $H_1(Y; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ induced by inclusion of $Y$ into $X$. As a consequence, the non-singular linking form on $H_1(Y)$ is identically 0 when restricted to $P$ and the Casson-Gordon invariants vanish for characters that are obtained by linking with elements of $P$. See [3, 4].

A knot is equivariant slice or $q$-equivariant slice, if it is slice as well as $q$-periodic, and the periodic map of $S^3$ extends to a period $q$ self-diffeomorphism of $B^4$ that leaves the slice disk invariant. It follows that the set of fixed points of the $\mathbb{Z}_q$ action on $B^4$ is a 2-disk that intersects the slice disk in one point. Therefore, if $K$ is equivariant slice with axis $B$, then $\text{lk}(K, B) = 1$ and $K \cup B$ is concordant to the Hopf link. See [34] for more details. It was shown in [5] that there are knots which are slice and periodic but not equivariant slice.

As we have successfully used correction terms of double branched covers to obstruct periodicity of a knot, it is natural to ask whether these obstructions can be further refined to distinguish a slice, periodic knot from an equivariant slice knot.

Theorem 4.2. Let $K$ be a $q$-equivariant slice knot with a prime-power fold cyclic branched cover $Y$. Suppose that $q$ and $\ell$ are distinct primes. Let $f_q(\ell)$ be the smallest natural number such that $\ell f_q(\ell) \equiv 1 \pmod{q}$. Then, if $H_1(Y)_{\ell}$ is trivial, there exist non-negative integers $t, a_1, \ldots, a_t$ and a subgroup $P_{\ell} \leq H_1(Y)_{\ell}$ such that,

$$P_{\ell} \cong \mathbb{Z}^{a_1 f_q(\ell)}_{\ell} \oplus \mathbb{Z}^{a_2 f_q(\ell)}_{\ell} \oplus \cdots \oplus \mathbb{Z}^{a_t f_q(\ell)}_{\ell},$$

and $d(Y, s) = 0$ for $s \in P_{\ell}$, under a suitable identification of $\text{Spin}^c(Y) \cong H_1(Y; \mathbb{Z})$.

Proof. The subgroup denoted $P$ in Theorem 4.1 is $\text{Ker}(H_1(Y) \to H_1(X))$, where $Y$ (resp. $X$) is a cyclic cover of $S^3$ (resp. $B^4$) branched over $K$ (resp. a slice disk for $K$). It is easily seen that if the knot $K$ is equivariant slice, then this kernel is equivariant. This, along with Equation (4.1) was proved in [34]. The statement about correction terms follows from Theorem 4.1. □
Example 4.3. The knot 9_{41} is 3-equivariant slice (in fact, equivariant ribbon) as shown in [34]. The homology of the double branched cover is $\mathbb{Z}_7 \times \mathbb{Z}_7$, and the quotient knot is trivial. We see below the correction terms for this knot along with their multiplicity exhibiting the 3-fold symmetry and the value 0 on a subgroup isomorphic to $\mathbb{Z}_7$.

| $d(Y, s)$ | $-\frac{8}{7}$ | $-\frac{6}{7}$ | $-\frac{4}{7}$ | $-\frac{2}{7}$ | 0 | $\frac{2}{17}$ | $\frac{4}{7}$ |
|-----------|--------------|--------------|--------------|--------------|---|-------------|-------------|
| Multiplicity of $d(Y, s)$ | 6 | 6 | 6 | 6 | 13 | 6 | 6 |

Example 4.4. Consider the polynomial

$$\Delta(t) = 6t^2 - 13t + 6 = (2t - 3)(3t - 2) \equiv -t \mod 3.$$ 

Let $K$ be a knot with $\Delta_K = \Delta$. This polynomial satisfies the Murasugi conditions with $\Delta_K$ equal to either 1 or $\Delta$ itself.

Suppose that $K$ is 3-equivariant slice with $\Delta_K = 1$. In this case the homology of the double branched cover over the quotient knot is trivial. As $\Delta(-1) = 25$, the order of the first homology group of the double branched cover over $K$ is 25, and therefore the order of the metabolizer $\mathcal{P}_5$ is 5. However, $f_3(5) = 2$, so by Equation (4.1), $|\mathcal{P}_5|$ should be divisible by 25. It follows that $\Delta_K \neq 1$.

In case $\Delta_K = \Delta$, none of our results give any information, and in fact by Theorem 1.10 of [11] there is a 3-equivariant slice knot with this polynomial. As this polynomial is quadratic, the genus of the quotient knot would be at least one, and by the Riemann-Hurwitz formula relating Euler characteristics of a branched cover to its quotient, such an equivariant slice knot would necessarily have genus at least 3.

References

[1] C. Adams, M. Hildebrand and J. Weeks, *Hyperbolic invariants of knots and links*, Trans. AMS 326 (1991) 1-56.
[2] G. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972. xiii+459 pp.
[3] A. Casson, C. McA. Gordon, *Cobordism of Classical Knots*, Preprint, Orsay (1975). (Reprinted in: A la Recherche de la Topologie Perdue, eds. A. Martin, L. Guillou, Progress in Mathematics 62, Birkhauser, 1986.)
A. Casson, C. McA. Gordon, On Slice Knots in Dimension 3, Proc. Sympos. Pure Math. 32, Part 2, A. M. S., Providence, R.I., 1978, pp. 39–53.

J. C. Cha and K. H. Ko, On equivariant slice knots, Proc. Amer. Math. Soc. 127 (1999) 2175–2182.

Cha, J. C., and Livingston, C., Knotinfo: Table of knot invariants, http://www.indiana.edu/~knotinfo.

Chbili, Nafaa, Strong periodicity of links and the coefficients of the Conway polynomial, Proc. Amer. Math. Soc. 136 (2008), no. 6, 2217–2224.

Chbili, Nafaa, Le polynome de Homfly des nuds librement periodiques, C. R. Acad. Sci. Paris Sr. I Math. 325 (1997), no. 4, 411–414.

Chbili, Nafaa, Equivalent Khovanov homology associated with symmetric links, Kobe J. Math. 27 (2010), no. 1-2, 73–89.

J. F. Davis and C. Livingston, Alexander polynomials of periodic knots, Topology 30 (1991) 551–564.

J. F. Davis and S. Naik, Alexander polynomials of equivariant slice knots, Trans. Amer. Math. Soc. 358 (2006) 2949-2964.

S. Donaldson and P. Kronheimer, The geometry of four-manifolds, Oxford Mathematical Monographs. Oxford Science Publications. The Claredon Press, Oxford University Press, New York, 1990. x+440 pp. ISBN: 0-19-853553-8.

A. Edmonds, Least area Seifert surfaces and periodic knots, Topology and its Appl. 18 (1984) 109-113.

A. Floer, A relative Mrose index for the symplectic action, Comm. Pure Appl. Math 41 (1988), no. 4, 393-407.

A. Floer and H. Hofer, Coherent orientations for periodic orbit problems in symplectic geometry, Math. Z. 212 (1993), no. 1, 13–38.

J. Greene and S. Jabuka, The slice-ribbon conjecture for 3-stranded pretzel knots, Amer. J. Math. 133 (2011), no. 3, 555–580.

Hendricks, Kristen, A note on the link Floer homology of doubly-periodic knots, Preprint, arXiv:1206.5989v1 [math.GT].

J. Hillman and C. Livingston and S. Naik The Twisted Alexander Polynomial and Periodicity of Knots, Algebraic and Geometric Topology 6 (2006) 145-169.

S. Jabuka, Concordance invariants from higher order covers, Topology Appl. 159 (2012), no. 10–1, 2694–2710.

S. Jabuka and S. Naik, Order in the concordance group and Heegaard Floer homology, Geometry & Topology 11 (2007) 979–994.

I. Kaplansky, Modules over Dedekind rings and valuation rings, Trans. Amer. Math. Soc., 72 (1952) 327–340.
[22] K. H. Ko, D. H. Choi, and W. T. Song, *Seifert matrices and equivariant concordances of periodic Knots*, (1997) preprint.

[23] R. Lipshitz, *A cylindrical reformulation of Heegaard Floer homology*, Geom. Topol. **10** (2006), 955–1097.

[24] I. G. Macdonald, *Symmetric products of an algebraic curve*, Topology **1** (1962) 319–343.

[25] C. Manolescu and B. Owens, *A concordance invariant from the Floer homology of double branched covers*, Int. Math. Res. Not. (2007), no. 20.

[26] E. Moise, *Periodic Homeomorphisms of the 3-Sphere*, Illinois J. Math. **6** (1962) 206–225.

[27] D. Montgomery and L. Zippin, *Examples of transformation groups*, Proc. Amer. Math. Soc. **5**, (1954), 460–465.

[28] J. Morgan, *The Smith Conjecture*, (ed. J. Morgan and H. Bass) Pure and Applied Math. 112, Academic Press, Orlando, 1984, pp. 3–18.

[29] K. Murasugi, *On Periodic Knots*, Comment. Math. Helv. **46** (1971) 162–174.

[30] K. Murasugi, *On Symmetries of Knots*, Tsukuba J. Math. **4** (1980) 331–347.

[31] K. Murasugi, *Jones polynomials of periodic links*, Pacific J. Math. **131** (1988) 319–329.

[32] S. Naik, *Periodicity, Genera, and Alexander polynomials of Knots*, Pacific J. Math. **166** (1994) 357–371.

[33] S. Naik, *New Invariants of Periodic Knots*, Math. Proc. Camb. Phil. Soc. **122** (1997) 281-290.

[34] S. Naik, *Equivariant slice knots in $S^3$*, *Knots ’96 (Tokyo)*, 81–89, Proceedings of the Fifth International Research Institute of Mathematical Society of Japan, held at Waseda University, Tokyo, July 22-26, 1996, ed. S. Suzuki, World Sci. Publishing, 1997.

[35] Y. Ni and Z. Wu, *Cosmetic surgeries on knots in $S^3$*, arXiv:1009.4720.

[36] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. **2** (2004), no. 3, 1027–1158.

[37] P. Ozsváth and Z. Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. **2** (2004), no. 3, 1159–1245.

[38] P. Ozsváth and Z. Szabó, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math., **202** (2006), no. 2, 326–400.

[39] P. Ozsváth and Z. Szabó, *On the Floer homology of plumbed three-manifolds*, Geom. Topol. **7** (2003), 185–224.

[40] J. Przytycki, *On Murasugi’s and Traczyk’s criteria for periodic links*, Math. Ann. **283** (1989) 465–478.

[41] J. Robbin and D. Salamon, *The spectral flow and the Maslov index*, Bull. London Math. Soc. **27** (1995), no. 1, 1–33.

[42] D. Salamon and E. Zehnder, *Morse theory for periodic solutions of Hamiltonian systems and the Maslov index*, Comm. Pure Appl. Math **45** (1992), no. 10, 1303–1360.

[43] P. Traczyk, $10_{101}$ has no period 7: A Criterion for Periodicity of Links, Proc. AMS. **108** (1990) 845–846.
[44] P. Traczyk, *Periodic knots and the skein polynomial*, Inv. Math. **106** (1991) 73–85.

[45] V. Turaev, *Torsion invariants of Spin<sup>c</sup>-structures on 3-manifolds*, Math. Res. Lett. **4** (1997), no. 5, 679–695.

[46] Y. Yokota, *The Jones polynomial of periodic knots*, Proc. AMS **113** (1991) 889–894.

[47] Y. Yokota, *Kauffman Polynomial of Periodic Links*, Topology **32** (1993) 309–324.

Department of Mathematics and Statistics, University of Nevada, Reno, NV 89557

E-mail address: jabuka@unr.edu

University of Nevada, Reno

E-mail address: naik@unr.edu