ORDER-SHARP NORM-RESOLVENT HOMOGENISATION
ESTIMATES FOR MAXWELL EQUATIONS ON PERIODIC
SINGULAR STRUCTURES: THE CASE OF NON-ZERO CURRENT
AND THE GENERAL SYSTEM

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Abstract

For arbitrarily small values of $\varepsilon > 0$, we formulate and analyse the Maxwell system of equations
of electromagnetism on $\varepsilon$-periodic sets $S^\varepsilon \subset \mathbb{R}^3$. Assuming that a family of Borel
measures $\mu^\varepsilon$, such that supp($\mu^\varepsilon$) = $S^\varepsilon$, is obtained by $\varepsilon$-contraction of a fixed
1-periodic measure $\mu$, and for right-hand sides $f^\varepsilon \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, we prove order-sharp norm-resolvent convergence
estimates for the solutions of the system. In the resent work we address the case of non-zero current density in
the Maxwell system and complete the analysis of the general setup including non-constant permittivity
and permeability coefficients.

Keywords Homogenisation · Maxwell system · Norm-resolvent estimates · Periodic measures · Singu-
lar structures

1 Non-zero current case

1.1 Introduction

The aim of this work is to obtain norm-resolvent homogenisation estimates for the stationary Maxwell
system. We are interested in the setting of singular periodic structures described by arbitrary periodic
Borel measures. In our earlier work \cite{6} we proved such estimates for the vector problem
\begin{equation}
\text{curl } A(-/\varepsilon) \text{curl } u^\varepsilon + u^\varepsilon = f, \quad f \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \text{div } f = 0,
\end{equation}
where the $\varepsilon$-periodic measure $\mu^\varepsilon$ is given by rescaling a 1-periodic measure and $A$ is a symmetric, bounded
and uniformly positive matrix. Equation (1.1) is the resolvent form of the Maxwell system of equations in
electromagnetism in the absence of external current, where $u^\varepsilon$ represents the divergence-free magnetic field
$\mathcal{H}^\varepsilon$, the matrix $A$ is the inverse of the dielectric permittivity, and the magnetic permeability is set to unit.
Our approach is based on the study of the family of operators, parametrised by the quasimomentum $\theta$,
obtained from (1.1) by the Floquet transform. The strategy is to construct an asymptotic approximation in
powers of $\varepsilon$, analyse the homogenisation corrector as a function of $\varepsilon$ and $\theta$, and obtain an estimate uniform
with respect to $\theta$ for the reminder. The principal tool is the Poincaré-type inequality in Sobolev spaces of
quasiperiodic functions, bearing in mind that we are dealing with an arbitrary measure. The result proved
in \cite{6} allows us to estimate the magnetic field and the magnetic induction directly with the solution of the
related homogenised equation. Different estimates are proved for the electric field and electric displacement, where the approximation contains rapidly oscillating terms of zero order. Equipped with this method, the goal of the first part of this work is to obtain norm resolvent estimates for the Maxwell equation system where the external current appears, and the magnetic permeability is set to unit.

Let $\mu$ the $Q$-periodic Borel measure in $\mathbb{R}^3$, where $Q = [0,1)^3$, such that $\mu(Q) = 1$. For each $\varepsilon > 0$ the $\varepsilon$-periodic measure $\mu^\varepsilon$ is defined as $\mu^\varepsilon(B) = \varepsilon^3 \mu(\varepsilon^{-1}B)$ for every Borel set $B \subset \mathbb{R}^3$.

The aim is to analyse the asymptotic behaviour, as $\varepsilon \to 0$, of vectorial solutions for the following Maxwell equation system:

$$\begin{cases}
\text{curl}(A(\cdot/\varepsilon)D_\varepsilon) + B_\varepsilon = 0 \\
\text{curl}B_\varepsilon - D_\varepsilon = g
\end{cases} \quad (1.2)$$

where $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ represents the divergence-free current density. The magnetic induction $B_\varepsilon$ and the electric displacement $D_\varepsilon$ are divergence free. $A$, the inverse of the dielectric permittivity, is a real-valued differentiable $Q$-periodic matrix-function, symmetric, bounded and positive definite. Our goal is to obtain norm resolvent estimate for the difference between the solutions of (1.2) and a vector function linked in some sense to the solution of the homogenised system

$$\begin{cases}
\text{curl}(A^{\text{hom}}D_0) + B_0 = 0 \\
\text{curl}B_0 - D_0 = g
\end{cases} \quad (1.3)$$

where $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, div $g = 0$, and $A^{\text{hom}}$ is the constant homogenised matrix. The system of Maxwell equations has been studied by Birman and Suslina (see [1], [2]) in the whole space setting, with Lebesgue measure. The main difference with our approach is that their method is based on the analysis of the spectral properties. To rewrite the problem (1.2), we follow the idea of Birman and Suslina in [1]. Label $A^{1/2}D_\varepsilon := D_\varepsilon$ we have that (1.2) is equivalent to

$$A^{1/2} \text{curl}A^{1/2}D_\varepsilon + D_\varepsilon = -A^{1/2}g, \quad g \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \text{div} g = 0. \quad (1.4)$$

We denote with $C^\infty_0(\mathbb{R}^3)$ the set of vector functions infinitely smooth, with compact support in $\mathbb{R}^3$. The solution of (1.4) is understood as the pair $(D_\varepsilon, \text{curl}(A^{1/2}D_\varepsilon))$ in the space $H^1_{\text{curl},A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$ defined as the closure of the set of pairs

$$\{(\phi, \text{curl}A^{1/2}\phi), \forall \phi \in C^\infty_0(\mathbb{R}^3) \text{ s.t. } A^{1/2}\phi \in C^\infty_0(\mathbb{R}^3)\}$$

in the direct sum $L^2(\mathbb{R}^3, d\mu^\varepsilon) \oplus L^2(\mathbb{R}^3, d\mu^\varepsilon)$. We say that $(D_\varepsilon, \text{curl}(A^{1/2}D_\varepsilon))$ is solution of (1.4) if

$$\int_{\mathbb{R}^3} \text{curl}(A^{1/2}D_\varepsilon) \cdot \text{curl}(A^{1/2}\phi) + \int_{\mathbb{R}^3} D_\varepsilon \cdot \phi = -\int_{\mathbb{R}^3} A^{1/2}g \cdot \phi \quad \forall (\phi, \text{curl}A^{1/2}\phi) \in H^1_{\text{curl},A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon). \quad (1.5)$$

For every $\varepsilon > 0$ the left hand side of (1.5) defines an inner product in $H^1_{\text{curl},A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$. The right hand side is linear bounded functional on $H^1_{\text{curl},A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$, hence the existence and uniqueness of the solution of (1.4) is a consequence of the Riesz representation theorem.

In what follows we study the resolvent of the operator $A^\varepsilon$ with domain

$$\text{dom}(A^\varepsilon) = \{u \in L^2(\mathbb{R}^3, d\mu^\varepsilon) : \exists \text{ curl } A^{1/2}u \text{ such that } \int_{\mathbb{R}^3} \text{curl}(A^{1/2}u) \cdot \text{curl}(A^{1/2}\phi) + \int_{\mathbb{R}^3} u \cdot \phi = -\int_{\mathbb{R}^3} A^{1/2}g \cdot \phi \quad \forall \phi \in H^1_{\text{curl},A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon), \text{ for some } g \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \text{ div } g = 0\}, \quad (1.6)$$

defined by the formula $A^\varepsilon u = -A^{1/2}g - u$, where $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$, div $g = 0$, and $u \in \text{dom}(A^\varepsilon)$ are linked as in the above formula. Note that in general, for a given function $u \in L^2(Q, d\mu)$, there exists more than one element $(u, \text{curl}A^{1/2}u) \in H^1_{\text{curl},A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)$. But for each $u \in \text{dom}(A^\varepsilon)$ there exists only one $\text{curl} A^{1/2}u$.
such that (1.6) holds, since equation (1.5) has a unique solution.
\(A^\varepsilon\) is a symmetric and self-adjoint operator. Similarly to (6) we deduce that \(\text{dom}(A^\varepsilon)\) is dense in \(L^2(\mathbb{R}^3, d\mu^\varepsilon)\) \(\cap \{u|\text{div} A^{-1/2}u = 0\}\). In fact by the definition of \(\text{dom}(A^\varepsilon)\), if \(g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)\), \(\text{div} g = 0\), and \(u, v \in \text{dom}(A^\varepsilon)\) are such that \(A^\varepsilon u + u = -A^{1/2}g\) and \(A^\varepsilon v + v = -u\), we obtain
\[
\int_{\mathbb{R}^3} |u|^2 d\mu^\varepsilon = \int_{\mathbb{R}^3} A^{1/2}g \nu d\mu^\varepsilon.
\]
If \(A^{1/2}g\) is orthogonal to \(\text{dom}(A^\varepsilon)\), then \(u = 0\), therefore \(A^{1/2}g = 0\).

1.2 The Floquet transform

In order to apply the Floquet transform to (1.4) and study the related family of operators problem, we need to define the Sobolev spaces of quasi-periodic functions with respect the measure \(\mu\).

**Definition 1.1.** For each \(\kappa \in [-\pi, \pi]^3 := Q'\), the space \(H^1_{\text{curl}, A^{1/2}, \kappa}(Q, d\mu)\) is defined as the closure of the set \(\{e_\kappa \phi, \text{curl}(e_\kappa A^{1/2} \phi) : \forall \phi \in C_0^\infty(Q)\text{ s.t. } A^{1/2} \phi \in C_0^\infty(Q)\}\) in the norm \(L^2(Q, d\mu) \oplus L^2(Q, d\mu)\).

Note that there may be more than one elements in \(H^1_{\text{curl}, A^{1/2}, \kappa}(Q, d\mu)\) with the same first component. Furthermore there is a one-to-one map linking \(H^1_{\text{curl}, A^{1/2}, \kappa}(Q, d\mu)\) and \(H^1_{\text{curl}, A^{1/2}}(Q, d\mu)\). In fact for any couple \((u, v) \in H^1_{\text{curl}, A^{1/2}, \kappa}(Q, d\mu)\) the pair \((\text{curl}(e_\kappa u, \text{curl}(v - i\kappa \times A^{1/2}u)) \in H^1_{\text{curl}, A^{1/2}}(Q, d\mu)\), which is a consequence of \(\text{curl}(A^{1/2} \phi_n) = \text{curl}(e_\kappa A^{1/2} \phi_n) - i\kappa \times A^{1/2}A^{1/2} \phi_n\)

for every \(\phi_n \in C_0^\infty(Q)\) such that \(e_\kappa \phi_n \to 0\), \(\text{curl}(e_\kappa A^{1/2} \phi_n) \to 0\). On the other hand, for every \((\bar{u}, \bar{v}) \in H^1_{\text{curl}, A^{1/2}}(Q, d\mu)\) one has \(\bar{v} = \text{curl}(\bar{u} - i\kappa \times A^{1/2}u)\) for some \((u, v) \in H^1_{\text{curl}, A^{1/2}, \kappa}(Q, d\mu)\).

For every \(\kappa \in Q'\) we focus our analysis on the operator \(A_\kappa\) with domain

\(\text{dom}(A_\kappa) = \{u \in L^2(Q, d\mu) : \exists \text{curl}(e_\kappa A^{1/2} u)\text{ such that }\int_Q \text{curl}(e_\kappa A^{1/2} u) \cdot \text{curl}(e_\kappa A^{1/2} \phi) d\mu + \int_Q u \cdot \phi d\mu = -\int_Q A^{1/2} \overline{G} \cdot \bar{\phi} \forall \phi \in C_0^\infty(Q)\\text{ for some }G \in L^2(Q, d\mu), \text{curl}(e_\kappa G) = 0\}\),

defined by the formula \(A_\kappa u = -A^{1/2}G - u\), where \(G \in L^2(Q, d\mu)\) and \(u \in \text{dom}(A_\kappa)\) are linked by the above formula. \(A_\kappa\) is symmetric, self-adjoint. By an argument similar to the case \(A^\varepsilon\) we infer that his domain is dense \(L^2(Q, d\mu) \cap \{u|\text{curl}(e_\kappa A^{-1/2} u) = 0\}\).

In order to write the transformed problem of (1.4), we first recall the definition of the Floquet transformation for functions in \(L^2(\mathbb{R}^3, d\mu^\varepsilon)\). For \(\varepsilon > 0\), the \(\varepsilon\)-Floquet transform \(F_\varepsilon\) is defined for \(u \in C_0^\infty(\mathbb{R}^3)\) as:

\[(F_\varepsilon u)(\theta, z) = \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \sum_{n \in \mathbb{Z}^3} u(z + \varepsilon n) \exp(-i\varepsilon n \cdot \theta) \quad z \in \varepsilon Q, \theta \in \varepsilon^{-1}Q'\).

Note that the mapping \(F_\varepsilon\) preserves the norm and can be extended to an isometry from \(L^2(\mathbb{R}^3, d\mu^\varepsilon)\) to \(L^2(\varepsilon^{-1}Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon)\). The inverse is defined as

\[(F_\varepsilon g)^{-1}(z) = \left(\frac{\varepsilon}{2\pi}\right)^{3/2} \int_{\varepsilon^{-1}Q'} g(\theta, z) d\theta \quad g \in L^2(\varepsilon^{-1}Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon)\).

Observe that \(F_\varepsilon\) is a unitary transform. To obtain the representation for the operator \(A^\varepsilon\) (cf. (3) Section 3), we combine the \(\varepsilon\)-Floquet transform with the following unitary scaling transform:

\[(T_\varepsilon h)(\theta, y) = \varepsilon^{3/2} h(\theta, \varepsilon y) \quad \theta \in \varepsilon^{-1}Q', \: y \in Q, \: h \in L^2(\varepsilon^{-1}Q' \times \varepsilon Q, d\theta \times d\mu^\varepsilon),
\](\(T_\varepsilon h)^{-1}(\theta, z) = \varepsilon^{-3/2} h(\theta, z/\varepsilon) \quad \theta \in \varepsilon^{-1}Q', \: z \in \varepsilon Q, \: h \in L^2(\varepsilon^{-1}Q' \times Q, d\theta \times d\mu)\).
Proposition 1.2. For each \( \varepsilon > 0 \) holds the following unitary equivalence between the operator \( A^\varepsilon \) and the direct integral of the operator family \( A_\kappa \) for \( \kappa := \varepsilon \theta, \theta \in \varepsilon^{-1}Q' \):

\[
(A^\varepsilon + I)^{-1} = F_\varepsilon^{-1}T_\varepsilon^{-1} \int_{\varepsilon^{-1}Q} e_\kappa (\varepsilon^{-2}A_\kappa + I)^{-1}e_\kappa d\theta T_\varepsilon F_\varepsilon.
\]

Sketch of proof. The argument is similar to the one discussed in [6] for the Maxwell system with zero external current. Let us consider the solution \( (D_\varepsilon, \text{curl } A^{1/2}D_\varepsilon) \) \( H^1_{\text{curl } A^{1/2}}(Q, d\mu) \) of problem (1.4) with \( g \in C_0^\infty(\mathbb{R}^3) \). For such \( D_\varepsilon \) we denote the periodic amplitude of its Floquet transform as follows

\[
D_\varepsilon(y) := e_\varepsilon T_\varepsilon F_\varepsilon = \left( \frac{\varepsilon^2}{2\pi} \right)^{3/2} \sum_{n \in \mathbb{Z}^3} D_\varepsilon(\varepsilon y + \varepsilon n) \exp(-i(\varepsilon y + \varepsilon n) \cdot \theta), \quad y \in Q.
\]

For any choice of \( \text{curl } A^{1/2}D_\varepsilon \), in particular for the one in (1.4), we have that

\[
\text{curl}(e_\varepsilon D_\varepsilon^\varepsilon)(y) = \varepsilon \left( \frac{\varepsilon^2}{2\pi} \right)^{3/2} \sum_{n \in \mathbb{Z}^3} \text{curl } A^{1/2}D_\varepsilon(\varepsilon y + \varepsilon n) \exp(-i(\varepsilon y + \varepsilon n) \cdot \theta), \quad y \in Q
\]

is a curl of \( e_\varepsilon D_\varepsilon^\varepsilon \) in sense that \( (e_\varepsilon D_\varepsilon^\varepsilon, \text{curl } A^{1/2}e_\varepsilon D_\varepsilon^\varepsilon) \) is an element of \( H^1_{\text{curl } A^{1/2}, \varepsilon \theta} \). Therefore

\[
\varepsilon^{-2} \int_Q \text{curl}(e_\kappa A^{1/2}D_\varepsilon^\varepsilon) \cdot \text{curl}(e_\kappa A^{1/2}) d\mu = \int_Q D_\varepsilon^\varepsilon \cdot \overline{\varepsilon \theta} d\mu = - \int_Q A^{1/2}G \cdot \overline{\varepsilon \theta} d\mu \tag{1.7}
\]

for all \( \forall (e_\kappa \phi, \text{curl } (e_\kappa A^{1/2} \phi)) \in H^1_{\text{curl } A^{1/2}, \kappa}(Q, d\mu) \). \( G := e_\varepsilon T_\varepsilon F_\varepsilon g \) such that \( e_\kappa \text{div } e_\kappa G = 0 \) in the sense

\[
\int_Q e_\kappa G \cdot \overline{\nabla(e_\kappa \phi)} = 0 \quad \forall \phi \in C_0^\infty(Q). \tag{1.8}
\]

The density of \( f \in C_0^\infty(\mathbb{R}^3) \) in \( L^2(\mathbb{R}^3, d\mu) \) implies the claim.

In what follows we study the behaviour of the solution \( D_\varepsilon^\varepsilon \) to the problem:

\[
\varepsilon^{-2}A^{1/2}e_\varepsilon \text{curl } e_\varepsilon A^{1/2}D_\varepsilon^\varepsilon + D_\varepsilon^\varepsilon = -A^{1/2}G, \quad \varepsilon > 0, \quad \kappa \in Q'. \tag{1.9}
\]

\( G \) is a function in \( L^2(Q, d\mu) \) such that \( e_\kappa \text{div } e_\kappa G = 0 \) in the sense (1.8). The problem (1.9) is understood with the integral identity (1.7).

1.3 Helmholtz decomposition

The so-called Helmholtz decomposition for square-integrable functions, is an important tool for the asymptotic analysis of Maxwell system. In this section we provide a special version of such decomposition taking into account the quasiperiodicity of functions, the arbitrary of the measure \( \mu \) and the structure of problem (1.9). Before formulating next proposition, we recall that the notation of gradient of quasiperiodic \( L^2 \) functions with respect measures \( \mu \), and the associated Sobolev spaces \( H^1_\kappa(Q, d\mu) \) can be defined. See our earlier work [5] for the precise construction.

In what follows we assume the measure \( \mu \) such that [1]

\[
\int_Q \partial_j \phi = 0, \quad \forall \phi \in C_0^\infty(Q), \quad j = 1, 2, \ldots, d. \tag{1.10}
\]

An example of measure satisfying (1.10) is of the following type: consider a finite set \( \{\mathcal{H}_j\} \) of hyperplanes of dimension \( d \) or smaller, \( \mathcal{H}_j \) is parallel to the coordinate axis for all \( j \) and \( \mathcal{H}_j \) is not a subset of \( \mathcal{H}_k \) for all \( j, k \). Define the measure \( \mu \) on \( Q \) by the formula

\[
\mu(B) = \left( \sum_j |\mathcal{H}_j \cap Q|_j \right)^{-1} \sum_j |\mathcal{H}_j \cap B|_j \quad \text{for all Borel } B \subset Q,
\]

where \( | \cdot |_j \) represents the \( d_j \)-dimensional Lebesgue measure, \( d_j = \dim(\mathcal{H}_j) \).
We say that a vector \( v \in L^2(Q, d\mu) \) is solenoidal, or \( \mathbf{e} \cdot \text{div}(e_\kappa A^{-1/2}) \)-free, if

\[
\int_Q A^{-1/2} e_\kappa v \cdot \nabla(e_\kappa \phi) d\mu = 0 \quad \forall \phi \in C^\infty_{#,0}(Q). \tag{1.11}
\]

Furthermore, we say that a vector \( v \in L^2(Q, d\mu) \) is irrotational, or \( \mathbf{e} \cdot \text{curl}(e_\kappa A^{1/2}) \)-free, if

\[
\int_Q A^{1/2} e_\kappa v \cdot \text{curl}(e_\kappa \phi) d\mu = 0 \quad \forall \phi \in C^\infty_{#,0}(Q). \tag{1.12}
\]

Clearly, the (linear) subspaces of \( L^2(Q, d\mu) \) solenoidal and irrotational functions are orthogonal.

The measure \( \mu \) is assumed such that (see footnote 1) for any irrotational \( v \), there exist a scalar function \( \psi \in H^{1,0}(Q, d\mu) \) and \( c \in \mathbb{C}^3 \) such that

\[
v = A^{-1/2}(\mathbf{e} \cdot \nabla(e_\kappa \psi) + c). \tag{1.13}
\]

Finally, we denote by \( K \) the (closed) subspace of \( L^2(Q, d\mu) \) consisting of vectors that are both solenoidal and irrotational. It is clear from the above definitions that for any \( v \in L^2(Q, d\mu) \) there exist \( c \in \mathbb{C}^3 \) and \( \psi_c \in H^{1,0}(Q, d\mu) \) such that (1.13) holds with \( \psi = \psi_c \) and (cf. (1.11))

\[
\mathbf{e} \cdot \text{div} A^{-1}(\nabla(e_\kappa \psi_c) + e_\kappa c) = 0, \tag{1.14}
\]

in the sense that

\[
\int_Q A^{-1} \nabla(e_\kappa \psi_c) \cdot \nabla(e_\kappa \phi) = -\int_Q A^{-1} e_\kappa c \cdot \nabla(e_\kappa \phi) \quad \forall \phi \in C^\infty_{#,0}(Q). \tag{1.15}
\]

**Proposition 1.3.** For any \( c \in \mathbb{C}^3 \) there exists a unique function \( \psi_c \in H^{1,0}(Q, d\mu) \) satisfying the identity (1.15).

**Proof.** Follows from the Lax-Millgram theorem. Indeed the sesquilinear form

\[
\int_Q A^{-1} \nabla(e_\kappa u) \cdot \nabla(e_\kappa v) \quad u, v \in H^{1,0}_\#,
\]

is continuous and coercive in \( H^{1,0}_\#(Q, d\mu) \). The continuity is obtained setting \( \nabla(e_\kappa u) = e_\kappa (\nabla u + i k u) \) for all scalar functions \( u \in H^{1,0}_\#(Q, d\mu) \). The coercivity follows form the Poincaré-type inequality analysed for the scalar case in \([5\text{ Section } 5]\). The result is proved bearing in mind that the right hand side of equation (1.15) is an element in \((H^{1,0}_\#)^*\).

Writing \( \psi_c = \Psi_{\kappa} \cdot c \) for a vector function \( \Psi_{\kappa} \) we write

\[
A^{-1/2}(\mathbf{e} \cdot \nabla(e_\kappa \psi_c) + c) = A^{-1/2}(\mathbf{e} \cdot \nabla(e_\kappa \Psi_{\kappa}) + I)c, \tag{1.16}
\]

where \((\nabla \Psi_{\kappa})_{ij} := (\Psi_{\kappa})_{j,i}, i, j = 1, 2, 3\). Hence we have that \( \Psi_{\kappa} \in H^{1,0}_\#(Q, d\mu) \) is the solution of (cf. (1.14))

\[
\mathbf{e} \cdot \text{div} A^{-1}(\nabla(e_\kappa \Psi_{\kappa}) + e_\kappa I) = 0. \tag{1.17}
\]

**Proposition 1.4.** For any \( w \in L^2(Q, d\mu) \) there exists a unique solution \( \Phi_w \in H^{1,0}_\#(Q, d\mu) \) to the problem

\[
\mathbf{e} \cdot \text{div}(A^{-1} \nabla(e_\kappa \Phi_w)) = \mathbf{e} \cdot \text{div}(e_\kappa A^{-1/2} w), \tag{1.18}
\]

understood in the sense of the integral identity

\[
\int_Q A^{-1} \nabla(e_\kappa \Phi_w) \cdot \nabla(e_\kappa \phi) = \int_Q A^{-1/2} e_\kappa w \cdot \nabla(e_\kappa \phi) \quad \forall \phi \in C^\infty_{#,0}(Q). \tag{1.19}
\]
Proof. The left-hand side of equation (1.19) defines the sesquilinear form bounded and coercive in $H^1_{\#_0}(Q, d\mu)$. The coercivity follows from the Poincaré-type inequality discussed in [4, Section 5] for the scalar case. Bearing in mind that (1.19) is a bounded linear functional on $H^1_{\#_0}(Q, d\mu)$, we use the Riesz representation theorem to prove the existence and uniqueness of solution $\Phi_w$. \hfill \Box

Consider a function $w \in L^2(Q, d\mu)$. Proposition [1.4] provides a unique function $\Phi_w$ with zero mean such that $w - A^{-1/2}e_\kappa \nabla (e_\kappa \Phi_w)$ is solenoidal. We write it as the sum of its orthogonal projection (in the $L^2$ sense) on $K$ and on $K^\perp$, which yields

$$w = \bar{w} + A^{-1/2}(e_\kappa \psi_c) + A^{-1/2}e_\kappa \nabla (e_\kappa \Phi_w),$$

(1.20)

where $\bar{w}$ is solenoidal and

$$\int_Q A^{1/2} \bar{w} = 0,$$

(1.21)

$A^{-1/2}e_\kappa \nabla (e_\kappa \Phi_w)$ irrotational, and the second term is an element of $K$. It follows that all terms of (1.20) are $L^2$-orthogonal to each other.

Lemma 1.5. For any function $w \in L^2(Q, d\mu)$, the constant $c$ in the representation (1.20) is given by

$$c = \int_Q A^{1/2} w.$$

(1.22)

Proof. Multiply (1.20) for $A^{1/2}$ and integrate on $Q$, on has that

$$\int_Q A^{1/2} w = \int_Q A^{1/2} \bar{w} + \int_Q e_\kappa \nabla (e_\kappa \psi_c) + c + \int_Q e_\kappa \nabla (e_\kappa \Phi_w).$$

The claim follows from assumption (1.10) and from (1.21). \hfill \Box

1.4 Poincaré inequality

In this Section we prove a version of the Poincaré inequality for functions in the Sobolev space $H^1_{\text{curl} A^{1/2}, \kappa}(Q, d\mu)$. We restrict ourselves to the case of a scalar matrix $A$ and we assume also that

$$\max_Q \| (\text{div} A) A^{-1} \| \leq 1/2,$$

(1.23)

where at each $x \in Q$, the vector div $A$ has components $A_{ji,j}$, $i = 1, 2, 3$.

Lemma 1.6. For any $\eta \in H^1_{\#_0}(Q, d\mu)$, the zero vector is one of the curls of $A^{-1/2}e_\kappa \nabla (e_\kappa \eta)$, i.e. one has

$$(A^{-1/2}e_\kappa \nabla (e_\kappa \eta), 0) \in H^1_{\text{curl} A^{1/2}, \kappa}(Q, d\mu).$$

Proof. The statement follows from (1.12), indeed $A^{-1/2}e_\kappa \nabla (e_\kappa \eta)$ is irrotational. In fact we have that

$$\int_Q \nabla (e_\kappa \eta) \cdot \text{curl}(e_\kappa \phi) = 0 \quad \forall \phi \in C^\infty_\#_0(Q),$$

by the assumption on measure $\mu$. \hfill \Box

Theorem 1.7. Suppose that there exists $\tilde{C}_P > 0$ such that

$$\| \varphi \|_{L^2(Q, d\mu)} \leq \tilde{C}_P \| \text{curl}(e_\kappa \varphi) \|_{L^2(Q, d\mu)} \quad \forall \varphi \in C^\infty_\#_0(Q), \quad e_\kappa \text{div}(e_\kappa \varphi) = 0,$$

see [6, Proposition 5.1]. Assume $A$ a scalar matrix such that (1.23) holds.

For each $w \in L^2(Q, d\mu)$, define the constant $c = c(w)$ by the formula (1.22). There exists $C > 0$ such that for all $\kappa \in Q^\prime$ and $(e_\kappa, w, \text{curl}(e_\kappa A^{1/2} w)) \in H^1_{\text{curl} A^{1/2}, \kappa}(Q, d\mu)$ one has

$$\| w - A^{-1/2}(e_\kappa \nabla (e_\kappa \psi_c) + A^{-1/2}e_\kappa \nabla (e_\kappa \Phi_w)) \|_{L^2(Q, d\mu)} \leq C \| \text{curl}(e_\kappa A^{1/2} w) \|_{L^2(Q, d\mu)},$$

(1.24)

In [6, Section 7] we describe a class of measure that satisfy this assumption.
Proof. We apply Lemma \[1.6\] to the choices \(\eta = \psi_c\) and \(\eta = \Phi_w\) and also note that
\[
(A^{-1/2}c, e_\kappa(\kappa \times c)) \in \mathcal{H}^{1}_{\text{curl},A^{1/2},\kappa}(Q,d\mu).
\]
It follows that it suffices to show the existence of \(C > 0\) such that for all \(\kappa \in \mathcal{Q}'\), \((e_\kappa \tilde{w}, \text{curl}(e_\kappa A^{1/2} \tilde{w})) \in \mathcal{H}^{1}_{\text{curl},A^{1/2},\kappa}(Q,d\mu)\) such that \(\tau_\kappa \text{div}(e_\kappa A^{-1/2}) = 0\) (i.e. \(\tilde{w}\) is solenoidal) and
\[
\int_Q A^{1/2} \tilde{w} = 0,
\]
and \(c \in \mathbb{C}^3\), one has
\[
\|\tilde{w}\|_{L^2(Q,d\mu)} \leq C \|\text{curl}(e_\kappa A^{1/2} \tilde{w}) + e_\kappa(\kappa \times c)\|_{L^2(Q,d\mu)}.
\]  
(1.25)
In order to show the bound (1.25), notice first that \(L^2\)-orthogonality of \(\text{curl}(e_\kappa A^{1/2} \tilde{w})\) and the vector \(e_\kappa(\kappa \times c)\).

Second, we invoke [6, Proposition 5.1], which ensures the existence of \(C_P > 0\) such that for all \(\kappa \in \mathcal{Q}'\) and \((e_\kappa u, \text{curl}(e_\kappa u)) \in \mathcal{H}^{1}_{\text{curl},\kappa}(Q,d\mu)\) one has
\[
\left\| u - \int_Q u - e_\kappa \nabla(e_\kappa \Xi_u) \right\|_{L^2(Q,d\mu)} \leq C_P \|\text{curl}(e_\kappa u)\|_{L^2(Q,d\mu)},
\]  
(1.26)
where \(\Xi_u \in H^{1}_{\#\mu}(Q,d\mu)\) solves the problem
\[
\tau_\kappa \Delta (e_\kappa \Xi_u) = \tau_\kappa \text{div}(e_\kappa u),
\]  
(1.27)
understood in the sense that
\[
\int_Q \nabla(e_\kappa \Xi_u) : \nabla(e_\kappa \phi) \, d\mu = \int_Q e_\kappa u : \nabla(e_\kappa \phi) \, d\mu \quad \forall \phi \in \mathcal{C}_c^\infty(\mathcal{Q}).
\]  
(1.28)
Setting \(u = A^{1/2} \tilde{w}\) in (1.26) and using (1.21), we obtain
\[
\left\| A^{1/2} \tilde{w} - e_\kappa \nabla(e_\kappa \Xi_{A^{1/2} \tilde{w}}) \right\|_{L^2(Q,d\mu)} \leq C_P \|\text{curl}(e_\kappa A^{1/2} \tilde{w})\|_{L^2(Q,d\mu)},
\]  
(1.29)
We estimate \(\nabla(e_\kappa \Xi_{A^{1/2} \tilde{w}})\) in terms of \(A^{1/2} \tilde{w}\), by using (1.27) and observing that for a scalar matrix \(A\) the following identity holds
\[
\tau_\kappa \text{div}(e_\kappa A^{1/2} \tilde{w}) = \tau_\kappa \text{div}(e_\kappa A^{-1/2} \tilde{w}) A(x) + (\text{div } A) A^{-1} \cdot A^{1/2} \tilde{w} = (\text{div } A) A^{-1} \cdot A^{1/2} \tilde{w},
\]
where for the last equality we use the fact that \(\tilde{w}\) is solenoidal.

Using the assumption (1.23), we obtain
\[
\left\| \nabla(e_\kappa \Xi_{A^{1/2} \tilde{w}}) \right\|_{L^2(Q,d\mu)} \leq \frac{1}{2} \left\| A^{1/2} \tilde{w} \right\|_{L^2(Q,d\mu)},
\]  
(1.30)
Finally, combining (1.29) and (1.30) and invoking the ellipticity of \(A\), we obtain
\[
\|\tilde{w}\|_{L^2(Q,d\mu)} \leq 2C_P \left( \min_{Q} \|A^{1/2}\| \right)^{-1} \|\text{curl}(e_\kappa A^{1/2} \tilde{w})\|_{L^2(Q,d\mu)},
\]  
(1.31)
and hence (1.25) holds with \(C\) given by the constant on the right-hand side of (1.31).

1.5 Asymptotic approximation

In this section we present the construction of the asymptotic approximation for the solution of problem (1.9) in order to introduce the main Theorem, that is the norm resolvent estimate for the difference between \(D_0\) and the leading order term of the approximation.
1.5.1 The main result

The aim of the first part of this work is to prove the following result.

**Theorem 1.8.** The following estimate holds for $D_0^\varepsilon$ solution of (1.11) with a constant $C > 0$ independent of $\varepsilon$, $\theta$ and $G$:

$$
\|D_0^\varepsilon - A^{-1/2}(\varepsilon_\theta \nabla (e_\varepsilon \Psi) + I) d_0^\varepsilon\|_{L^2(Q, d\mu)} \leq C \varepsilon \|G\|_{L^2(Q, d\mu)}.
$$

(1.32)

The vector $d_0^\varepsilon \in \mathbb{C}^3$ is defined as

$$
d_0^\varepsilon = (\hat{A}_{e_\theta}^{\text{hom}})^{-1} (i\theta \times i\theta \times (\hat{A}_{e_\theta}^{\text{hom}})^{-1} + I)^{-1} \int Q G,
$$

(1.33)

with

$$
\hat{A}_{e_\theta}^{\text{hom}} := \int Q A^{-1}(\varepsilon_\theta \nabla (e_\varepsilon \Psi) + I).
$$

(1.34)

and $\Psi_{e_\theta}$ is the solution of (1.17).

A result analogous to Theorem 1.8 holds as well for the transformed electric field $E_0^\varepsilon := A^{1/2} D_0^\varepsilon$. In fact as a direct consequence of (1.32) we have the following theorem:

**Theorem 1.9.** The following estimate holds for the transformed electric field $E_0^\varepsilon := A^{1/2} D_0^\varepsilon$ with a constant $C > 0$ independent of $\varepsilon$, $\theta$ and $G$:

$$
\|E_0^\varepsilon - (\varepsilon_\theta \nabla (e_\varepsilon \Psi_{e_\theta}) + I) d_0^\varepsilon\|_{L^2(Q, d\mu)} \leq C \varepsilon \|G\|_{L^2(Q, d\mu)}.
$$

(1.35)

**Remark 1.10.** Define $N$ as the matrix in $H^1_\text{curl}(Q, d\mu)$ solving the cell problem

$$
curl A (\text{curl } N + I) = 0, \quad \text{div } N = 0.
$$

It can be shown that (see [8, Lemma 4.4])

$$
(\hat{A}_0^{\text{hom}})^{-1} = A^{\text{hom}} := \int Q A (\text{curl } N(y) + I).
$$

**Lemma 1.11.** There exist constants $C_1, C_2 > 0$ independent of $\kappa \in Q'$, such that the following estimates hold for $\hat{A}_\kappa^{\text{hom}}$ defined in (1.11):

$$
C_1 \leq \hat{A}_\kappa^{\text{hom}} \leq C_2.
$$

(1.36)

**Proof of Lemma 1.11.** The argument is similar to the one used in the Voigt-Reiss inequality proof (see [8, Chapter 1]). Note that $\hat{A}_\kappa^{\text{hom}}$ can be written as

$$
\hat{A}_\kappa^{\text{hom}} \cdot \lambda := \inf_{\psi_\lambda \in H^1_{\text{curl}, 0}} \int Q A^{-1}(\varepsilon_\kappa \nabla (e_\kappa \psi_\lambda) + \lambda) \cdot (\varepsilon_\kappa \nabla (e_\kappa \psi_\lambda) + \lambda), \quad \lambda \in \mathbb{C}^3.
$$

We immediately have that the upper bound holds for $\lambda \in \mathbb{C}^3$ with $C_2 := \|A^{-1}\|_{L^\infty}$, setting $\psi_\lambda = 0$.

To provide the lower bound we need to represent the inverse matrix $(\hat{A}_\kappa^{\text{hom}})^{-1}$. In order to do it we consider the space $L^2_{\text{sol}, \kappa}(Q, d\mu)$ consisting of vector functions $v \in L^2(Q, d\mu)$ such that are $e_\kappa \text{div } e_\kappa$-free, in the sense (cf. (1.11))

$$
\int_Q e_\kappa v \cdot e_\kappa \nabla \phi = 0 \quad \forall \phi \in C^\infty_0(Q).
$$

Then the matrix $(\hat{A}_\kappa^{\text{hom}})^{-1}$ can be written as

$$
(\hat{A}_\kappa^{\text{hom}})^{-1} \xi \cdot \xi = \inf_{v_\xi \in L^2_{\text{sol}, \kappa}(Q, d\mu), \langle v \rangle = 0} \int_Q A(v_\kappa + \xi) \cdot (v_\kappa + \xi), \quad \xi \in \mathbb{C}^3.
$$

(1.37)
This representation holds noting that $v_\kappa \in L^2_{sol,\kappa}$ solves uniquely the problem
\[
\int_Q e_\kappa A(v_\kappa + \xi) \cdot \text{curl}(e_\kappa \Phi) = 0 \quad \forall \Phi \in C_0^\infty(Q), \quad \int_Q v_\kappa = 0.
\]
We can express $v_\kappa$ in terms of solution of (1.14) setting $v_\kappa = A^{-1}(\tau e_\kappa \nabla (e_\kappa \psi_\lambda) + \lambda)$ where $\lambda = (\tilde{A}_\kappa^{hom})^{-1} \xi$.

In this way we obtain that
\[
(\tilde{A}_\kappa^{hom})^{-1} \xi \cdot \xi = \tilde{A}_\kappa^{hom} \lambda \cdot \lambda,
\]
hence the representation (1.37) holds.

The lower bound in (1.36) is a consequence of (1.37) for $\xi \in \mathbb{C}^3$ with $C_1 := \|A\|^{-1}_{L^\infty}$, setting $v_\kappa = 0$. □

To obtain the norm-resolvent estimate in the whole space setting for the initial problem (1.4), it remains to apply the inverse Floquet transform to the asymptotic estimate (1.32). 

**Corollary 1.12.** Let $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ and denote $g^\varepsilon_\theta := \tau e_\kappa \tau e_\kappa g(x)$ so that
\[
\int_Q g^\varepsilon_\theta d\mu = \tilde{g}(\theta), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \tilde{g}(\theta) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} g^\varepsilon_\theta d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.
\]
There exists a constant $C > 0$ such that the following estimate holds for $D^\varepsilon$ solution of (1.4)
\[
\|D^\varepsilon - (2\pi)^{-3/2} \int_{\mathbb{R}^3} A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)(\tilde{A}_\kappa^{hom})^{-1}(\Phi_\theta^{hom} + I)^{-1} \tilde{g}(\theta)e_\theta d\theta\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)} \leq C\varepsilon \|g\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)},
\]
\[
\forall \varepsilon > 0. \quad \text{Here} \quad \Phi_\theta^{hom} \quad \text{is the matrix valued quadratic form given by (1.33), and} \quad \Psi_\kappa \text{ by (1.17) for all values} \quad \theta \in \mathbb{R}^3.
\]

**Proof of Corollary 1.12** Consider $D^\varepsilon_\theta$ solution of (1.9) with $G = g^\varepsilon_\theta$ on has that
\[
D^\varepsilon_\theta - (2\pi)^{-3/2} \int_{\mathbb{R}^3} A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)d^\varepsilon_\theta d\theta =
\]
\[
[F^{-1}_\epsilon T_\epsilon^{-1}e_\kappa D^\varepsilon_\theta - F^{-1}_\epsilon T_\epsilon^{-1}e_\kappa A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)d^\varepsilon_\theta + \tilde{f}^{-1}_\epsilon T_\epsilon^{-1} e_\kappa A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)d^\varepsilon_\theta - (2\pi)^{-3/2} \int_{\varepsilon^{-1}Q'} A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)(\tilde{A}_\kappa^{hom})^{-1}(\Phi_\theta^{hom} + I)^{-1} \tilde{g}(\theta) e_\theta d\theta].
\]

To prove the Corollary we need to analyse the $L^2$ norm of the above equality. In view of the Theorem 1.8 and the unitary property of $F_\kappa$ and $T_\kappa$, we can estimate the first bracket in the right hand side as follows
\[
\|F^{-1}_\epsilon T_\epsilon^{-1}e_\kappa[D^\varepsilon_\theta - A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)d^\varepsilon_\theta]\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)} \leq C\varepsilon \|g\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)}.
\]
Noting that
\[
F^{-1}_\epsilon T_\epsilon^{-1}e_\kappa A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)d^\varepsilon_\theta =
\]
\[
(2\pi)^{-3/2} \int_{\varepsilon^{-1}Q'} A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)(\tilde{A}_\kappa^{hom})^{-1}(\Phi_\theta^{hom} + I)^{-1} \tilde{g}(\theta) e_\theta d\theta,
\]
It remains to analyse
\[
(2\pi)^{-3/2} \int_{\mathbb{R}^3 / \varepsilon^{-1}Q'} A^{-1/2}(\tau e_\kappa \nabla (e_\kappa \Psi_\epsilon) + I)(\tilde{A}_\kappa^{hom})^{-1}(\Phi_\theta^{hom} + I)^{-1} \tilde{g}(\theta) e_\theta d\theta.
\]
Note that using the estimates (1.36) we obtain
\[
\sup_{\theta \in \mathbb{R}^3 / \varepsilon^{-1}Q'} \left| (\tilde{A}_\kappa^{hom})^{-1}(\Phi_\theta^{hom} + I)^{-1} \right| \leq \frac{C_1^{-1} \varepsilon^2}{C_2^{-1} \pi^2 + \varepsilon^2}.
\]
Using Parseval identity and (1.41), we can estimate the $L^2$ norm of (1.40) as follows

$$\left\| (2\pi)^{-3/2} \int_{\mathbb{R}^3/e^\theta} A^{-1/2} (\nabla (e^\theta \Psi e^\theta) + I) \left( \hat{A}_{\theta}^\text{hom} \right)^{-1} (\hat{\Psi}_0 + I)^{-1} \hat{g}(\theta) e_\theta d\theta \right\|_{L^2(\mathbb{R}^3,d\mu^\epsilon)}$$

$$\leq \frac{C_1 e^2}{C_2 \pi^2 + \epsilon^2} \left( \left\| \nabla (e^\theta \Psi e^\theta) \right\|_{L^2(\mathbb{R}^3,d\mu^\epsilon)} + I \right) \left\| \hat{g}(\theta) \right\|_{L^2(\mathbb{R}^3,d\mu^\epsilon)} \leq \frac{C_1 e^2}{C_2 \pi^2 + \epsilon^2} \left\| \nabla (e^\theta \Psi e^\theta) \right\|_{L^2(\mathbb{R}^3,d\mu^\epsilon)}.$$ (1.42)

In the last inequality we use that from equation (1.17) we can bound uniformly $\left\| \nabla (e^\theta \Psi e^\theta) \right\|_{L^2(\mathbb{R}^3,d\mu^\epsilon)}$. Combining (1.39) and (1.42) the claim follows.

1.5.2 Formal interpretation of (1.38)

The estimate in the whole space (1.38) allows to approximate $D_\epsilon$, the solution of (1.4), with

$$(2\pi)^{-3/2} \int_{\mathbb{R}^3} A^{-1/2} (\nabla (e^\theta \Psi e^\theta) + I) \left( \hat{A}_{\theta}^\text{hom} \right)^{-1} (\hat{\Psi}_0 + I)^{-1} \hat{g}(\theta) e_\theta d\theta.$$ (1.43)

It is the correct expression of what we can naively think as the homogenised solution operator. Note that (1.43) is a pseudo-differential operator with two-scale symbol depending on $\theta$ and $\epsilon\theta$.

Here we discuss from a formal point of view the meaning of (1.43). This pseudo-differential operator can be always written as a formal series in power of $\epsilon$. The crucial point is that such series is not rigorous. In fact if we try to truncate it at some order of $\epsilon$, the series diverge. The reason of this fact resides in the structure of the leading order term of the series. In fact high-order terms have a non trivial dependence on $\theta$ for every $\epsilon$, and the presence of $\theta$ cannot be ignored.

Let us analyse the first element of this infinite order term formal series. When $\epsilon = 0$ in (1.43), we have the following standard construction:

$$A^{-1/2} (\nabla \Psi + I) \left( \hat{A}_{0}^\text{hom} \right)^{-1} (2\pi)^{-3/2} \int_{\mathbb{R}^3} (\hat{\Psi}_0 + I)^{-1} \hat{g}(\theta) e_\theta d\theta =$$

$$A^{-1/2} (\nabla \Psi + I) \left( \hat{A}_{0}^\text{hom} \right)^{-1} (\hat{\Psi}_0 + I)^{-1} \hat{g}.$$ Here $\hat{\Psi}_0 = \text{curl curl} \left( \hat{A}_{0}^\text{hom} \right)^{-1}$, and $\Psi$ is matrix function in $H^1_\#(Q,d\mu)$ solution of the following problem (cf. (1.17))

$$\text{div} A^{-1} (\nabla \Psi + I) = 0, \quad \int_Q \Psi = 0.$$ Recalling the homogenised equation for the Maxwell system in the whole space (1.3), we have that

$$\left( \text{curl curl} A^\text{hom} + I \right)^{-1} \hat{g} = \mathcal{D}_0, \quad \text{where} \quad A^\text{hom} = \left( \hat{A}_{0}^\text{hom} \right)^{-1}.$$ Hence we have that for $\epsilon = 0$ (1.43) is

$$A^{-1/2} (\nabla \Psi + I) \left( \hat{A}_{0}^\text{hom} \right)^{-1} \mathcal{D}_0.$$ (1.44)

Note that this element contains both the solution of the homogenised equation and rapidly oscillating terms. It has same structure of the limit term obtained in [6] for the setting of Maxwell system with zero external current.

High-order terms in (1.43) are solutions of some singular perturbed problems, and they are all contributive for the leading order term of the series. In fact high-order terms have a non trivial dependence on $\theta$ for every $\epsilon$, and the presence of $\theta$ cannot be ignored.

The behaviour of the solution operator in the estimate for $D_\epsilon$ highlights a dependence of $y \in Q$ and $\epsilon$. This does not appear in the case of Maxwell system with zero external current, where there is not a corrector term depending on $\epsilon$ in the estimates.

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Results about norm resolvent estimates for the system of Maxwell equation with external current and unitary magnetic permeability has been obtained by Birman and Suslina in [2] for the Lebesgue measure setting. They construct a corrector depending on $\varepsilon$ in order to obtain estimates. In our case the structure of \textbf{[1.43]} is a consequence of the representation provided for the space with the Helmholtz decomposition. With our approach it is possible to have an explicit and more compact homogenised solution operator which contains the standard construction \textbf{[1.44]} and an infinite series depending on $\varepsilon$.

\textbf{1.5.3 The approximation}

We now proceed with the proof of Theorem \textbf{1.18}. For each $\theta \in \varepsilon^{-1}Q', \varepsilon > 0$, we write the following expansion for the solution of \textbf{[1.9]}:

$$D_{\varepsilon} := U_{\varepsilon}^\gamma + z_{\varepsilon},$$ (1.45)

where

$$U_{\varepsilon}^\gamma = A^{-1/2}\left(\xi_{\varepsilon}\nabla (\xi_{\varepsilon} \Theta_{\varepsilon}) + I\right) d_{\varepsilon}^\gamma + \varepsilon^2 R_{\varepsilon}^\gamma,$$ (1.46)

$d_{\varepsilon}^\gamma \in \mathbb{C}^3$ is defined in \textbf{[1.33]}, and $\Theta_{\varepsilon}$ is the solution of \textbf{[1.17]}. The function $R_{\varepsilon}^\gamma \in H^1_{\text{curl,}A^{1/2}}(Q, d\mu)$ is defined as the solution to the problem

$$A^{1/2} \xi_{\varepsilon} \nabla \xi_{\varepsilon} A^{1/2} R_{\varepsilon}^\gamma + \varepsilon^2 A^{1/2} \left(\xi_{\varepsilon}(\nabla e_{\varepsilon} \Psi_{\varepsilon}) + c R_{\varepsilon}^\gamma\right) + \varepsilon^2 A^{-1/2} \xi_{\varepsilon} \nabla (\xi_{\varepsilon} \Psi_{\varepsilon}) (1.47)$$

understood in the sense of the integral identity

$$\int_Q \nabla (\xi_{\varepsilon} A^{1/2} R_{\varepsilon}^\gamma) \cdot \nabla (\xi_{\varepsilon} A^{1/2} \overline{\phi}) + \varepsilon^2 \int_Q A^{-1/2} \left(\xi_{\varepsilon}(\nabla e_{\varepsilon} \Psi_{\varepsilon}) + c R_{\varepsilon}^\gamma\right) \cdot \overline{\phi}$$

$$+ \varepsilon^2 \int_Q A^{-1/2} \xi_{\varepsilon} \nabla (\xi_{\varepsilon} \Psi_{\varepsilon}) \cdot \overline{\phi} = \langle H_{\varepsilon}, \overline{\phi} \rangle \forall \phi \in H^1_{\text{curl,}A^{1/2}}(Q, d\mu).$$ (1.48)

\textbf{Proposition 1.13.} There exists a unique solution $R_{\varepsilon}^\gamma \in H^1_{\text{curl,}A^{1/2}}(Q, d\mu)$ for the equation \textbf{[1.47]}.

\textbf{Proof.} Using the decomposition \textbf{[1.20]} for $\phi$ in \textbf{[1.48]}, and using the orthogonality between the element of such decomposition, we have:

$$\int_Q A^{-1/2} \left(\xi_{\varepsilon}(\nabla e_{\varepsilon} \Psi_{\varepsilon}) + c R_{\varepsilon}^\gamma\right) \cdot \overline{\phi} = \int_Q A^{-1/2} \left(\xi_{\varepsilon}(\nabla e_{\varepsilon} \Psi_{\varepsilon}) + c R_{\varepsilon}^\gamma\right) \cdot A^{-1/2} \left(\xi_{\varepsilon}(\nabla e_{\varepsilon} \overline{\psi}_{\varepsilon}) + c \overline{\phi}\right),$$

and

$$\int_Q A^{-1/2} \xi_{\varepsilon} \nabla (e_{\varepsilon} \Psi_{\varepsilon}) \cdot \overline{\phi} = \int_Q A^{-1/2} \xi_{\varepsilon} \nabla (e_{\varepsilon} \overline{\Phi}_{\varepsilon}) \cdot A^{-1/2} \xi_{\varepsilon} \nabla (e_{\varepsilon} \overline{\phi}).$$

The proof of existence and uniqueness is a consequence of Lax-Millgram theorem applied to the skew-symmetric sesquilinear form

$$b(u, v) = \int_Q \nabla (\xi_{\varepsilon} A^{1/2} u) \cdot \nabla (\xi_{\varepsilon} A^{1/2} v) + \varepsilon^2 \int_Q A^{-1/2} \left(\xi_{\varepsilon}(\nabla e_{\varepsilon} \overline{\psi}_{\varepsilon}) + c u\right) \cdot A^{-1/2} \left(\xi_{\varepsilon}(\nabla e_{\varepsilon} \overline{\psi}_{\varepsilon}) + c v\right)$$

$$+ \varepsilon^2 \int_Q A^{-1/2} \xi_{\varepsilon} \nabla (e_{\varepsilon} \overline{\Phi}_{\varepsilon}) \cdot A^{-1/2} \xi_{\varepsilon} \nabla (e_{\varepsilon} \overline{\phi}),$$

for $u, v \in H^1_{\text{curl,}A^{1/2}}(Q, d\mu)$, $\overline{\psi}_{\varepsilon}$ solving \textbf{[1.14]}, and $\overline{\Phi}_{\varepsilon}$, $\overline{\phi}$ solutions of \textbf{[1.18]}. Note that $b(u, v)$ is bounded and coercive on $H^1_{\text{curl,}A^{1/2}}(Q, d\mu)$. The coercivity follows from the Poincaré-type inequality \textbf{[1.21]}. \hfill $\Box$
The right hand side of (1.48) can be rewritten as follows:

\[
\langle H^e_\theta, \phi \rangle = -\int_Q \left( A^{1/2}G + A^{-1/2}(\overline{\epsilon_\kappa \nabla (e_\kappa \Psi_\kappa)} + I) \delta_\theta^e \right) \cdot \overline{\phi} - \varepsilon^{-1} \int_Q e_\kappa (i\theta \times \delta_\theta^e) \cdot \nabla (e_\kappa A^{1/2} \phi) \\
= -\int_Q \left( A^{1/2}G + A^{-1/2}i\theta \times (i\theta \times \delta_\theta^e) + A^{-1/2}(\overline{\epsilon_\kappa \nabla (e_\kappa \Psi_\kappa)} + I) \delta_\theta^e \right) \cdot \overline{\phi} \quad \forall \phi \in C^\infty_\#(Q).
\]

In the last equality we use that for \( \phi \in C^\infty_\#(Q) \) on has \( \overline{\epsilon_\kappa \nabla (e_\kappa A^{1/2} \phi)} = \nabla A^{1/2} \phi + i\kappa \times A^{1/2} \phi \). Furthermore \( \int_Q i\theta \times \delta_\theta^e \cdot \nabla A^{1/2} \phi = 0 \).

Thus we have that \( H^e_\theta \) is equivalent to

\[
H^e_\theta = -A^{1/2}G - A^{-1/2}i\theta \times (i\theta \times \delta_\theta^e) - A^{-1/2}(\overline{\epsilon_\kappa \nabla (e_\kappa \Psi_\kappa)} + I) \delta_\theta^e.
\]

(1.49)

1.5.4 Properties of \( H^e_\theta \)

In order to prove the main Theorem 1.18 we need estimate for \( R^e_\theta \). The tool we want to use to obtain such estimate, is the Poincaré-type inequality (1.24). To do it, we need the identity

\[
\langle H^e_\theta, R^e_\theta \rangle = \langle H^e_\theta, \tilde{R}^e_\theta \rangle
\]

where \( \tilde{R}^e_\theta \) is defined as in the decomposition (1.20).

To prove it, we are interested in two properties for \( H^e_\theta \). First of all

\[
\langle H^e_\theta, A^{-1/2} \overline{\epsilon_\kappa \nabla (e_\kappa \phi)} \rangle = 0 \quad \forall \phi \in H^1_{\#,0}(Q, d\mu).
\]

(1.50)

Starting with the definition of \( H^e_\theta \) in (1.47), we have

\[
\langle H^e_\theta, A^{-1/2} \overline{\epsilon_\kappa \nabla (e_\kappa \phi)} \rangle = \int_Q e_\kappa G \cdot \nabla (e_\kappa \phi) + \int_Q A^{-1}(\overline{\epsilon_\kappa \nabla (e_\kappa \Psi_\kappa)} + I) \delta_\theta^e \cdot \overline{\epsilon_\kappa \nabla (e_\kappa \phi)},
\]

since \( A^{1/2} \overline{\epsilon_\kappa \nabla (e_\kappa \phi)} \) is solenoidal. The first integral is zero using that \( \overline{\epsilon_\kappa \nabla (e_\kappa G)} = 0 \) (see (1.8)). The second integral is null by equation (1.11) with \( c = \delta_\theta^e \).

The second property we want to prove for \( H^e_\theta \) is

\[
\langle H^e_\theta, A^{-1/2} \overline{\epsilon_\kappa \nabla (e_\kappa \psi_\kappa)} + c \rangle = 0
\]

(1.51)

for all \( \psi_\kappa \in H^1_{\#,0}(Q, d\mu) \) and \( c \in \mathbb{C}^3 \). By linearity

\[
\langle H^e_\theta, A^{-1/2} \overline{\epsilon_\kappa \nabla (e_\kappa \psi_\kappa)} \rangle = \langle H^e_\theta, A^{-1/2} \overline{\epsilon_\kappa \nabla (e_\kappa \psi_\kappa)} \rangle + \langle H^e_\theta, A^{-1/2} c \rangle
\]

Using (1.50), it remains to analyse

\[
\langle H^e_\theta, A^{-1/2} c \rangle = -\int_Q G \cdot c - \int_Q A^{-1}(\overline{\epsilon_\kappa \nabla (e_\kappa \Psi_\kappa)} + I) \delta_\theta^e \cdot c - \int_Q i\theta \times (i\theta \times \delta_\theta^e) \cdot c,
\]

which is the equation (1.39) solved by \( \delta_\theta^e \). Hence the property (1.51) for \( H^e_\theta \) is satisfied.

1.6 Estimate for \( \varepsilon^2 R^e_\theta \)

**Theorem 1.14.** There exists \( C > 0 \) such that for all \( \varepsilon > 0 \) and \( \theta \in \varepsilon^{-1}Q' \), the solution of the equation (1.37) satisfies:

\[
\| R^e_\theta - A^{-1/2}(\overline{\epsilon_\kappa \nabla (e_\kappa \Psi_\kappa)}) + c R^e_\theta \|_{L^2(Q, d\mu)} \leq C\|G\|_{L^2(Q, d\mu)},
\]

(1.52)

\[
\| A^{1/2}(\overline{\epsilon_\kappa \nabla (e_\kappa \Psi_\kappa)}) + c R^e_\theta \|_{L^2(Q, d\mu)} \leq C\varepsilon^{-1}\|G\|_{L^2(Q, d\mu)}.
\]

(1.53)
Proof. Let $\phi_n \in C^{\infty}_0(Q)$ converging to $R_0^\epsilon$ in $L^2(Q, d\mu)$ such that $\text{curl}(e_\kappa A^{1/2} \phi_n) \to \text{curl}(e_\kappa A^{1/2} R_0^\epsilon)$ in $L^2(Q, d\mu)$. Let us write (1.48) with $\phi = \phi_n$. By (1.49) one has

$$
\int_Q \text{curl}(e_\kappa A^{1/2} R_0^\epsilon) \cdot \text{curl}(e_\kappa A^{1/2} \phi_n) + \epsilon^2 \int_Q A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \psi_{R_0^\epsilon})} + c_{R_0^\epsilon}) \cdot \overline{\phi_n} + \epsilon^2 \int_Q A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \Phi_{R_0^\epsilon})} \cdot \overline{\phi_n}
$$

$$
= - \int_Q \left( A^{1/2} G + A^{1/2} i \theta \times i \theta \times d_\theta + A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \Psi_\kappa)} + I) d_\theta \right) \cdot \overline{\phi_n} \tag{1.54}
$$

When $n \to \infty$ we can write the (1.54) as a bilinear form. In fact recalling the decomposition (1.20) and the related orthogonality conditions for $R_0^\epsilon$ we have that

$$
\int_Q A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \psi_{R_0^\epsilon})} + c_{R_0^\epsilon}) \cdot \overline{\phi_n} = \int_Q \left| A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \psi_{R_0^\epsilon})} + c_{R_0^\epsilon}) \right|^2,
$$

and

$$
\int_Q A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \Phi_{R_0^\epsilon})} \cdot \overline{R_0^\epsilon} = \int_Q \left| A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \Phi_{R_0^\epsilon})} \right|^2.
$$

Hence

$$
\int_Q \text{curl}(e_\kappa A^{1/2} R_0^\epsilon) \cdot \text{curl}(e_\kappa A^{1/2} R_0^\epsilon) + \epsilon^2 \int_Q \left| A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \psi_{R_0^\epsilon})} + c_{R_0^\epsilon}) \right|^2 + \epsilon^2 \int_Q \left| A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \Phi_{R_0^\epsilon})} \right|^2
$$

$$
= (\mathcal{H}_0^\epsilon, R_0^\epsilon). \tag{1.55}
$$

Furthermore we use the properties (1.50) and (1.51) of $\mathcal{H}_0^\epsilon$ to rewrite the left hand side of (1.55) as

$$
\int_Q \text{curl}(e_\kappa A^{1/2} R_0^\epsilon) \cdot \text{curl}(e_\kappa A^{1/2} R_0^\epsilon) + \epsilon^2 \int_Q \left| A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \psi_{R_0^\epsilon})} + c_{R_0^\epsilon}) \right|^2 + \epsilon^2 \int_Q \left| A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \Phi_{R_0^\epsilon})} \right|^2
$$

$$
= - \int_Q \left( A^{1/2} G + A^{1/2} i \theta \times i \theta \times d_\theta + A^{-1/2}(\overline{\kappa_0 \nabla (e_\kappa \Psi_\kappa)} + I) d_\theta \right) \cdot \overline{R_0^\epsilon}. \tag{1.56}
$$

Using the Poincaré-type inequality (1.24) for $R_0^\epsilon$ and the definition of $d_\theta$ (1.33) the following estimate holds

$$
\| \text{curl}(e_\kappa A^{1/2} R_0^\epsilon) \|_{L^2(Q, d\mu)} \leq C \| G \|_{L^2(Q, d\mu)}. \tag{1.57}
$$

Combining estimate (1.57) and the Poincaré-type inequality (1.24) on obtains (1.52). The same estimates and equation (1.56) imply (1.55).

**Corollary 1.15.** There exists a constant $C > 0$ such that the following estimate holds uniformly in $\epsilon$, $\theta$ and $G$:

$$
\| U_0^\epsilon - A^{-1/2} (\overline{\kappa_0 \nabla e_\kappa \Psi_\kappa}) \|_{L^2(Q, d\mu)} \leq C \| G \|_{L^2(Q, d\mu)}.
$$

**1.7 Conclusion of the convergence estimate**

**Proposition 1.16.** There exists $C > 0$ such that the function $z_0^\epsilon$ defined in (1.48) satisfies the estimate

$$
\| z_0^\epsilon \|_{L^2(Q, d\mu)} \leq C \| G \|_{L^2(Q, d\mu)} \tag{1.58}
$$

**Proof.** The function $z_0^\epsilon \in H^1_{\text{curl}, A^{1/2}}(Q, d\mu)$ solves the problem

$$
\epsilon^{-2} A^{1/2} \overline{\kappa_0} \text{curl} \text{curl}(e_\kappa A^{1/2} z_0^\epsilon) + z_0^\epsilon = -\epsilon^2 \left( R_0 - A^{-1/2} (\overline{\kappa_0 \nabla (e_\kappa \Psi_\kappa)} + c_{R_0^\epsilon}) - A^{1/2} \overline{\kappa_0 \nabla (e_\kappa \Phi_{R_0^\epsilon})} \right). \tag{1.59}
$$

Using $z_0^\epsilon$ as a test function in the integral formulation of equation (1.59), on has

$$
\epsilon^{-2} \int_Q \text{curl}(e_\kappa A^{1/2} z_0^\epsilon) \cdot \text{curl}(e_\kappa A^{1/2} z_0^\epsilon) + \int_Q \left| z_0^\epsilon \right|^2 = -\epsilon^2 \int_Q R_0^\epsilon \cdot \overline{z_0^\epsilon}.
$$

Applying Hölder inequality, Poincaré-type inequality (1.24) and (1.57) on has the (1.58).
There exists a constant $\kappa \in \mathbb{R}$.

We obtain that

$$B = \frac{1}{\kappa} \text{div}(e_\kappa A^{1/2} D_\theta) + B_\theta = 0$$

$$\kappa \text{curl} e_\kappa A^{1/2} D_\theta - D_\theta = A^{1/2} G.$$ (1.60)

Where $e_\kappa \text{div}(e_\kappa A^{1/2} D_\theta) = 0$ and $B_\theta := e_\kappa \text{curl} e_\kappa B_\theta$ is the transformed magnetic induction such that $\kappa \text{div}(e_\kappa B_\theta) = 0$. In this setting the transformed magnetic field $H_\theta = B_\theta$.

To find the approximation for $B_\theta^\varepsilon$, we use the approximation of $D_\theta^\varepsilon$ (1.49) and we plug it in system (1.60). We obtain that

$$B_\theta^\varepsilon = e_\kappa \text{curl} e_\kappa A^{1/2} (A^{1/2} (e_\kappa \Psi_\kappa + I) d_\theta^\varepsilon + \varepsilon^2 R_\theta^\varepsilon)$$

$$= i\theta \times d_\theta^\varepsilon + \varepsilon (e_\kappa \text{curl} e_\kappa A^{1/2} R_\theta^\varepsilon).$$

In the last equality we use Lemma 1.6. Here $d_\theta^\varepsilon$ solves (1.33) and $R_\theta^\varepsilon$ solves (1.46).

**Theorem 1.17.** There exists $C > 0$ independent of $\theta$, $\varepsilon$ and $G$ such that the following estimate holds for the transformed magnetic induction $B_\theta^\varepsilon$ (and consequently for the transformed magnetic field $H_\theta^\varepsilon$):

$$\|B_\theta^\varepsilon - i\theta \times d_\theta^\varepsilon\|_{L^2(Q, d\mu)} \leq C \|G\|_{L^2(Q, d\mu)}$$ (1.61)

It remains now to apply the inverse Floquet transform to (1.61) to obtain the norm-resolvent estimate in the whole space setting for $B_\varepsilon$ solution of (1.2).

**Corollary 1.18.** Let $g \in L^2(\mathbb{R}^3, d\mu^\varepsilon)$ and denote $g_\theta^\varepsilon := e_\kappa \text{curl} e_\kappa g(x)$ so that

$$\int_Q g_\theta^\varepsilon d\mu = \hat{g}(\theta), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \hat{g}(\theta) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} g e^{i\theta \cdot \varepsilon^3} d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.$$

There exists a constant $C > 0$ such that the following estimate holds for $B_\varepsilon$ solution of (1.2)

$$\|B_\varepsilon - \text{curl} \left( (2\pi)^{-3/2} \int_{\mathbb{R}^3} (\tilde{A}_\varepsilon^{\text{hom}})^{-1} (A_\varepsilon^{\text{hom}} + I)^{-1} \hat{g}(\theta) d\theta \right) \|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)} \leq C \varepsilon \|g\|_{L^2(\mathbb{R}^3, d\mu^\varepsilon)},$$ (1.62)

$\forall \varepsilon > 0$. Here $A_\varepsilon^{\text{hom}}$ is the matrix valued quadratic form given by (1.33), and $\Psi_\kappa$ by (1.17) for all values $\theta \in \mathbb{R}^3$.  

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2 The general case

2.1 Introduction

In the first part of this work, we analysed the Maxwell system equations in the case with not null external current and magnetic permeability set to unit, and the operator we analysed has the form

\[ A^{1/2} \text{curl} A^{1/2} u^\varepsilon + u^\varepsilon = -A^{1/2} g, \quad g \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \quad \text{div} \, g = 0. \] (2.1)

Such setting is an intermediate case between the case with zero external current analysed in our earlier work [6], and the general case where the magnetic permeability is arbitrary. The approach is based on the analysis of the family of operators, parametrised by the quasimomentum \( \theta \), obtained from (2.1) by Floquet transform. We produced a formal approximation in power of \( \varepsilon \) which contains rapidly oscillating terms of order zero, and we obtained an estimate uniform in \( \theta \) for the reminder. Tools developed for the non-zero current case in order to prove the estimates for the reminder, are essential for what follows. The crucial point which allows us to analyse the general case is the Helmholtz decomposition (1.20), that takes into account functions quasiperiodic which are solenoidal and irrotational in sense of definitions (1.11), (1.12).

In this second part of the work we set out to tackle the general setting of Maxwell equations system with external current and with arbitrary magnetic permeability. We label with \( A \) the inverse of the electric permittivity, with \( \tilde{A} \) the inverse of the magnetic permeability. The general Maxwell system written in terms of displacement vectors has the form

\[
\begin{aligned}
\text{curl}(A(\cdot/\varepsilon) D_\varepsilon) + B_\varepsilon &= f \\
\text{curl}(\tilde{A}(\cdot/\varepsilon) B_\varepsilon) - D_\varepsilon &= g.
\end{aligned}
\] (2.2)

\( g \) and \( f \) are divergence-free vectorial functions in \( L^2(\mathbb{R}^3, d\mu^\varepsilon) \), the magnetic induction \( B_\varepsilon \) and the electric displacement \( D_\varepsilon \) are divergence free. Our aim is to obtain norm resolvent estimate for the solutions of (2.2) in order to understand how are they linked with the solutions of the following homogenised system:

\[
\begin{aligned}
\text{curl}(A^{\text{hom}} D_0) + B_0 &= f \\
\text{curl}(\tilde{A}^{\text{hom}} B_0) - D_0 &= g.
\end{aligned}
\] (2.3)

\( A^{\text{hom}} \) and \( \tilde{A}^{\text{hom}} \) are constant matrices, representing the effective or homogenised properties of the medium. In what follows we analyse, without loss of generality, the case where \( f = 0 \).

Inspired by the work of Birman and Suslina [1], we rewrite (2.2) in a more suitable form. Define \( A^{1/2} D_\varepsilon := D_\varepsilon \) we obtain an equivalent formulation for (2.2):

\[ A^{1/2} \text{curl} \tilde{A} \text{curl} A^{1/2} D_\varepsilon + D_\varepsilon = -A^{1/2} g, \quad g \in L^2(Q, d\mu^\varepsilon), \quad \text{div} \, g = 0. \] (2.4)

Solution of (2.4) is understood as the pair \((D_\varepsilon, \text{curl}(A^{1/2} D_\varepsilon)) \in H^1_{\text{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon)\) such that

\[
\int_{\mathbb{R}^3} \tilde{A} \text{curl}(A^{1/2} D_\varepsilon) \cdot \text{curl}(A^{1/2} \phi) + \int_{\mathbb{R}^3} D_\varepsilon \cdot \phi = \int_{\mathbb{R}^3} A^{1/2} g \cdot \phi \quad \forall \phi \in H^1_{\text{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon). \] (2.5)

For the definition of \( H^1_{\text{curl} A^{1/2}} \) we refer to Section 1.1. Note that for every \( \varepsilon > 0 \) the left hand side of (2.5) is an equivalent inner product on \( H^1_{\text{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon) \) and the right hand side is linear bounded functional on \( H^1_{\text{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon) \). Hence existence and uniqueness of solution to (2.4) is a consequence of Riesz representation theorem.

In what follows we study the resolvent of the operator \( \tilde{A}^\varepsilon \) with domain

\[
\text{dom}(\tilde{A}^\varepsilon) = \{ u \in L^2(\mathbb{R}^3, d\mu^\varepsilon) : \exists \text{curl} A^{1/2} u \text{ such that } \int_{\mathbb{R}^3} \tilde{A} \text{curl}(A^{1/2} u) \cdot \text{curl}(A^{1/2} \phi) + \int_{\mathbb{R}^3} u \cdot \phi = -\int_{\mathbb{R}^3} A^{1/2} g \cdot \phi \quad \forall \phi \in H^1_{\text{curl} A^{1/2}}(\mathbb{R}^3, d\mu^\varepsilon) \text{ for some } g \in L^2(\mathbb{R}^3, d\mu^\varepsilon), \text{ div} \, g = 0 \},
\]
defined by $\tilde{A}^\varepsilon D_\varepsilon = -A^{1/2}g - D_\varepsilon$, where $g \in L^2(\mathbb{R}^3, d\mu^*)$, $\text{div} g = 0$ and $D_\varepsilon \in \text{dom}(\mathcal{A}^\varepsilon)$ are linked as in the above formula. the operator $\mathcal{A}^\varepsilon$ is symmetric, self-adjoint and $\text{dom}(\mathcal{A}^\varepsilon)$ is dense in $L^2(\mathbb{R}^3, d\mu^*) \cap \{u| \text{div} A^{-1/2}u = 0\}$ (cf with Section 1.2).

### 2.2 The Floquet transform

In this section we present the family of operator problem obtained from (2.4) throughout the Floquet transform, and we describe the unitary equivalent problem object of our analysis.

For each $\kappa \in Q'$, the operator $\tilde{\mathcal{A}}_\kappa$ with domain

$$\text{dom}(\tilde{\mathcal{A}}_\kappa) = \{u \in L^2(Q, d\mu) : \exists \text{curl}(e_\kappa A^{1/2} u)\}$$ such that

$$\int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2} u) \cdot \overline{\text{curl}(e_\kappa A^{1/2} \phi)} d\mu + \int_Q u \cdot \overline{\phi} d\mu = - \int_Q A^{1/2} G \cdot \overline{\phi} \quad \forall \phi \in C_0^\infty(Q)$$

for some $G \in L^2(Q, d\mu)$, $\text{curl}(e_\kappa G) = 0$.

is defined by the formula $\tilde{\mathcal{A}}_\kappa u = -A^{1/2} G - u$. This operator is symmetric, self-adjoint and his domain is dense if $L^2(Q, d\mu) \cap \{u| \text{div} A^{-1/2}u = 0\}$.

Bearing in mind the definitions of the $\varepsilon$-Floquet transform $\mathcal{F}_\varepsilon$ and the scaling transform $\mathcal{T}_\varepsilon$ in Section 1.2 we state the following Proposition:

**Proposition 2.1.** For each $\varepsilon > 0$ holds the following unitary equivalence between the operator $\tilde{\mathcal{A}}_\varepsilon$ and the direct integral of the operator family $\tilde{\mathcal{A}}_\kappa$ for $\kappa := \varepsilon \theta$, $\theta \in \varepsilon^{-1} Q'$:

$$(\tilde{\mathcal{A}}_\varepsilon + \mathcal{I})^{-1} = \mathcal{F}_{\varepsilon}^{-1} \mathcal{T}_{\varepsilon}^{-1} \int_{\varepsilon^{-1} Q'} \mathcal{C}_\varepsilon d\theta \mathcal{T}_{\varepsilon} \mathcal{F}_\varepsilon.$$

Therefore in what follows we study the behaviour of $D_{\tilde{\vartheta}} := \mathcal{T} \mathcal{T}_\varepsilon D_{\varepsilon}$, solution to the problem

$$\varepsilon^{-2} A^{1/2} \mathcal{C}_{\tilde{\vartheta}} \text{curl}(e_\kappa A^{1/2} D_{\tilde{\vartheta}} + D_{\tilde{\vartheta}}) = -A^{1/2} G, \quad \varepsilon > 0, \quad \kappa \in Q'.$$ (2.6)

where $G \in L^2(Q, d\mu)$ is a function such that $\overline{\text{curl}(e_\kappa G)} = 0$ in sense of (1.8).

Solution to the problem (2.6) is understood as the pair $(e_\kappa D_{\tilde{\vartheta}}, \text{curl}(e_\kappa A^{1/2} D_{\tilde{\vartheta}})) \in H^1_{\text{curl} A^{1/2}, \kappa}(Q, d\mu)$ such that

$$\varepsilon^{-2} \int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2} D_{\tilde{\vartheta}}) \cdot \overline{\text{curl}(e_\kappa A^{1/2} \phi)} d\mu + \int_Q D_{\tilde{\vartheta}} \cdot \overline{\phi} d\mu = - \int_Q A^{1/2} G \cdot \overline{\phi} d\mu$$

$\forall (e_\kappa \phi, \text{curl}(e_\kappa A^{1/2} \phi)) \in H^1_{\text{curl} A^{1/2}, \kappa}(Q, d\mu).$

### 2.3 Asymptotic approximation

In this section we present the main Theorem and the asymptotic approximation for the solution $D_{\tilde{\vartheta}}$ of equation (2.6).

We assume throughout that measure $\mu$ is such that embedding

$$H^1_{\text{curl} A^{1/2}}(Q, d\mu) \cap \{w : \text{div} A^{-1/2} w = 0\} \subset L^2(Q, d\mu)$$

is compact.

#### 2.3.1 The main result

In order to write the asymptotic approximation for $D_{\tilde{\vartheta}}$ we consider the following generalised cell problem for the matrix $\tilde{N} \in H^1_{\text{curl} A^{1/2}}(Q, d\mu)$

$$\begin{cases}
A^{1/2} \text{curl} \tilde{A} \text{curl} A^{1/2} \tilde{N} = -A^{1/2} \text{curl} \tilde{A} \\
\text{div} A^{-1/2} \tilde{N} = 0, \quad \int_Q A^{1/2} \tilde{N} = 0.
\end{cases}$$ (2.7)
Proposition 2.2. There exists a unique solution $\tilde{N} \in H^1_{\text{curl}, A^{1/2}}(Q, d\mu)$ for the equation (2.7), understood in the sense of the integral identity
\[
\int_Q \tilde{A} \text{curl}(A^{1/2} \tilde{N}) \cdot \text{curl}(A^{1/2} \phi) d\mu = - \int_Q \tilde{A} \text{curl}(A^{1/2} \phi) \quad \forall \phi \in H^1_{\text{curl}, A^{1/2}}(Q, d\mu) \text{ s.t. } \int_Q A^{1/2} \phi = 0 \quad (2.8)
\]
Proof. It follows from the compactness of the embedding $H^1_{\text{curl}, A^{1/2}}(Q, d\mu) \cap \{ w : \text{div } A^{-1/2} w = 0 \}$ into $L^2(Q, d\mu)$ that the sesquilinear form
\[
\int_Q \tilde{A} \text{curl}(A^{1/2} u) \cdot \text{curl}(A^{1/2} v) d\mu \quad u, v \in H^1_{\text{curl}, A^{1/2}}(Q, d\mu) \cap \{ u : \text{div } A^{1/2} u = 0, \int_Q A^{1/2} u = 0 \},
\]
is coercive. Note that it is also continuous. The existence and uniqueness of solution of (2.8) follows by the Riesz representation theorem. $\square$

We define the space of matrices in $L^2(Q, d\mu)$ div $A^{-1/2}$-free in the same spirit of definition (1.11) with $\kappa = 0$. Furthermore we define the space of matrices in $L^2(Q, d\mu)$ curl $A^{1/2}$-free in analogy with definition (1.12) with $\kappa = 0$.

For any matrix $V$ that is curl $A^{1/2}$-free, there exists a vector $\Psi \in H^1_{\#0}(Q, d\mu)$ and a constant matrix $a \in \mathbb{C}^{3 \times 3}$ such that (cf. (1.13) and (1.16))
\[
V = A^{-1/2}(\nabla \Psi + a), \quad (2.9)
\]
where $(\nabla \Psi)_{ij} = \Psi_{j,i}$.

We denote with $M$ the subspace of $L^2(Q, d\mu)$ consisting of matrices which are both div $A^{-1/2}$-free and curl $A^{1/2}$-free. From (2.9) follows that for any $V \in M$, there exist $a \in \mathbb{C}^{3 \times 3}$ and a vector function $\Psi \in H^1_{\#0}(Q, d\mu)$ such that
\[
\text{div } A^{-1}(\nabla \Psi + a) = 0,
\]
understood as
\[
\int_Q A^{-1} \nabla \Psi \cdot \nabla \varphi = - \int_Q A^{-1} a \cdot \nabla \varphi \quad \forall \varphi \in C^\infty_{\#0}(Q). \quad (2.10)
\]

Proposition 2.3. For any $a \in \mathbb{C}^{3 \times 3}$ there exists a unique $\Psi \in H^1_{\#0}(Q, d\mu)$ solving (2.10).

Proof. Assuming the measure $\mu$ such that the embedding $H^1_{\#0}(Q, d\mu) \subset L^2(Q, d\mu)$ is compact (see [5, Section 4]), it is clear that the sesquilinear form
\[
\int_Q A^{-1} \nabla v \cdot \nabla u \quad (v, \nabla v), (u, \nabla u) \in H^1_{\#0}(Q, d\mu)
\]
is bounded and coercive and defines an equivalent inner product on $H^1_{\#0}(Q, d\mu)$. Noting that (2.10) is a linear bounded functional on $H^1_{\#0}(Q, d\mu)$, the result follows by the Riesz representation theorem. $\square$

Finally we are ready to state the main Theorem for the general Maxwell equation system.

Theorem 2.4. There exists a constant $C > 0$ independent of $\varepsilon, \theta$ and $G$, such that for $D_\theta$ solution of (2.6), the following estimate holds
\[
\| D_\theta - A^{-1/2}(\nabla \varepsilon \Psi_\kappa + I) d_\theta \|_{L^2(Q, d\mu)} \leq C \| G \|_{L^2(Q, d\mu)}. \quad (2.11)
\]
The vector $d_\theta \in \mathbb{C}^3$ is defined as
\[
d_\theta^\varepsilon = -(\tilde{A}_\varepsilon)^{-1}(i \theta \times \tilde{A}_\varepsilon \times (\tilde{A}_\varepsilon)^{-1})^{-1} \int Q G, \quad (2.12)
\]
where
\[
\tilde{A}_\varepsilon := \int_Q \tilde{A}(\text{curl } A^{1/2} \tilde{N} + I), \quad \tilde{A}_\varepsilon := \int_Q A^{-1}(\nabla \varepsilon \Psi_\kappa + I),
\]
for $\tilde{N}$ solution of (2.7) and $\Psi_\kappa$ solution of (1.17).
An result analogous to Theorem 2.4 holds for the transformed electric field \( E'^\varepsilon := A^{1/2}D'_\varepsilon \). The following theorem is a direct consequence of (2.11).

**Theorem 2.5.** There exists \( C > 0 \) independent of \( \varepsilon, \theta \) and \( G \) such that for the transformed electric field \( E'^\varepsilon := A^{1/2}D'_\varepsilon \) holds the following estimate

\[
\|E'^\varepsilon - (\varepsilon_\kappa \nabla (e_\kappa \Psi_\kappa) + I)d'_{\theta}\|_{L^2(Q,d\mu)} \leq C\varepsilon\|G\|_{L^2(Q,d\mu)},
\]

with \( d'_\theta \) defined in (2.12).

Applying the Floquet transform back to (2.11) (analogously to (2.13)) on obtains the norm-resolvent estimate in the whole space setting.

**Corollary 2.6.** Let \( g \in L^2(\mathbb{R}^3,d\mu^\varepsilon) \) and denote \( g^\varepsilon_\theta := \varepsilon_\kappa T_\varepsilon F_\varepsilon g(x) \) so that

\[
\int_Q g^\varepsilon_\theta d\mu = \hat{g}(\theta), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \hat{g}(\theta) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} g^\varepsilon_\theta d\mu^\varepsilon, \quad \theta \in \mathbb{R}^3.
\]

There exists a constant \( C > 0 \) such that the following estimate holds for \( F_\varepsilon \) solution of (2.20)

\[
\|F_\varepsilon - (2\pi)^{-3/2} \int_{\mathbb{R}^3} A^{-1/2}(\varepsilon_\kappa \nabla (e_\kappa \Psi_\kappa) + I)(\hat{A}_{\theta}^{\text{hom}} + I)^{-1} \hat{g}(\theta)e_{\theta d\theta}\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)} \leq C\varepsilon\|\hat{g}\|_{L^2(\mathbb{R}^3,d\mu^\varepsilon)},
\]

\( \forall \varepsilon > 0 \). Here \( \hat{A}_{\theta}^{\text{hom}} \) is the matrix valued quadratic form given by (2.12), and \( \Psi_\kappa \) by (1.17) for all values \( \theta \in \mathbb{R}^3 \).

Note that the only difference with the estimate (1.38) in the setting with magnetic permeability set to unit, is in the definition of \( \hat{A}_{\theta}^{\text{hom}} \). In fact in (2.14) appears the matrix \( A^{\text{hom}} \). Such matrix is constant and does not influence the meaning of the pseudo-differential operator

\[
(2\pi)^{-3/2} \int_{\mathbb{R}^3} A^{-1/2}(\varepsilon_\kappa \nabla (e_\kappa \Psi_\kappa) + I)(\hat{A}_{\theta}^{\text{hom}} + I)^{-1} \hat{g}(\theta)e_{\theta d\theta}.
\]

### 2.3.2 The approximation

We proceed now to the proof of Theorem 2.3. For each \( \theta \in \varepsilon^{-1}Q' \), \( \varepsilon > 0 \), we consider the following approximation for the vector function \( D'_\varepsilon \) solution of (2.20):

\[
D'_\varepsilon := U'_\varepsilon + z'_\varepsilon,
\]

where

\[
U'_\varepsilon = A^{-1/2}(\varepsilon_\kappa \nabla (e_\kappa \Psi_{\kappa}) + I)d'_\theta + \varepsilon N(y)(i\theta \times d'_\theta) + \varepsilon^2 R'_\varepsilon.
\]

\( N \in H^1_{\text{curl}A^{1/2}}(Q,d\mu) \) is the matrix defined as

\[
N := \tilde{N} + A^{-1/2}(\nabla \Psi + a_\theta),
\]

where \( \tilde{N} \) is the solution of (2.7), and \( A^{-1/2}(\nabla \Psi + a_\theta) \) is an element in \( M \). Note that \( \Psi \in H^1_{\Psi_{\kappa},0}(Q,d\mu) \) is the unique solution of equation (2.10). Furthermore the constant matrix \( a_\theta \in \mathbb{C}^{3\times3} \) is chosen such that

\[
\int_Q i\theta \times \bar{A}(i\theta \times a_\theta(i\theta \times d'_\theta)) = -\int_Q i\theta \times \bar{A}i\theta \times (A^{1/2}\tilde{N} + \nabla \Psi)(i\theta \times d'_\theta),
\]

that is

\[
\int_Q i\theta \times \bar{A}i\theta \times (A^{1/2}N(i\theta \times d'_\theta)) = 0.
\]

a_\theta is such that for every \( \eta \in \Theta_{\perp} := \{ \eta \in \mathbb{C}^3 \mid \eta \cdot \theta = 0 \} \) on has \( P_{\Theta_{\perp}}(a_\theta \eta) \neq 0 \), where \( P_{\Theta_{\perp}} \) is the orthogonal projection on \( \Theta_{\perp} \).

With the following proposition we prove that there is at least a unique matrix in \( \mathbb{C}^{3\times3} \) satisfying the property (2.19).
Proposition 2.7. There exists a unique $\tilde{a}_\theta \in \mathbb{C}^{3 \times 3}$ such that
$$\tilde{a}_\theta \eta \cdot \theta = 0 \quad \tilde{a}_\theta \theta = 0,$$
and
$$\int_Q i \theta \times \bar{A}(i \theta \times \tilde{a}_\theta \eta) = - \int_Q i \theta \times \bar{A}(i \theta \times (A^{1/2} \bar{N} + \nabla \Psi) \eta) \quad \forall \eta \in \Theta^\perp$$
(2.21)

Proof. The identity (2.21) is equivalent to a linear system for the representation of the matrix $\tilde{a}_\theta$ in the
basis $\{\theta/\epsilon, e_\perp^1, e_\perp^2\}$ for any orthogonal basis $\{e_\perp^1, e_\perp^2\}$ of $\Theta^\perp$. This system is uniquely solvable subject to conditions (2.20) for any right hand side, if and only if the solution to the related homogeneous system is zero. This is verified noticing that if
$$\int_Q i \theta \times \bar{A}(i \theta \times \tilde{a}_\theta \eta) = 0 \quad \forall \eta \in \Theta^\perp,$$
then
$$\int_Q \bar{A}(i \theta \times \tilde{a}_\theta \eta) \cdot (i \theta \times \tilde{a}_\theta \eta) = 0 \quad \forall \eta \in \Theta^\perp.$$
$\tilde{A}$ is positive definite, hence from the last identity we deduce that $i \theta \times \tilde{a}_\theta \eta = 0$ and therefore $\tilde{a}_\theta \eta = 0$ by first condition in (2.20). Now, by second condition in (2.20) we obtain $\tilde{a}_\theta = 0$ as required. □

Remark 2.8. The set $\Theta^\perp$ can be characterised as $\Theta^\perp = \{\theta \times c, c \in \mathbb{C}^3\}$ (cfr [7] Lemma 6.5)).

The element $R^e_\tilde{\theta} \in H^1_{\text{curl} A^{1/2}}(Q, d\mu)$ solves the following problem
$$A^{1/2} \epsilon_i \text{curl} \epsilon_i A^{1/2} R^e_\tilde{\theta} + \epsilon^2 A^{1/2} \epsilon_i \nabla (\epsilon_i \psi R_\tilde{\theta}) + c R^e_\tilde{\theta} = 0$$
(2.22)
$$- A^{1/2} G - \epsilon^{-2} A^{1/2} \epsilon_i \text{curl} \epsilon_i A^{1/2} \epsilon_i \nabla \epsilon_i A^{1/2} N(i \theta \times d^e_\theta)$$
$$- A^{1/2} \epsilon_i \nabla (\epsilon_i \Psi_k + I) d_\theta =: H^e_\tilde{\theta} \in (H^1_{\text{curl} A^{1/2}}(Q, d\mu))^*$$
understood in the sense of integral identity
$$\int_Q \text{curl}(\epsilon_i A^{1/2} R^e_\tilde{\theta}) \cdot \text{curl}(\epsilon_i A^{1/2} \phi) + \epsilon^2 \int_Q A^{1/2} \epsilon_i \nabla (\epsilon_i \psi R_\tilde{\theta}) \cdot \phi$$
$$+ \epsilon^2 \int_Q A^{1/2} \epsilon_i \nabla (\epsilon_i \Psi_k + I) d_\theta \cdot \phi = \langle H^e_\tilde{\theta}, \phi \rangle \quad \forall \phi \in H^1_{\text{curl} A^{1/2}}(Q, d\mu).$$
(2.23)

Existence and uniqueness of solution $R^e_\tilde{\theta} \in H^1_{\text{curl} A^{1/2}}(Q, d\mu)$ for the equation (2.23) follow from the argument used in Proposition 2.7.
The right hand side of (2.23) is understood as follows
$$\langle H^e_\tilde{\theta}, \phi \rangle = - \int_Q \left( A^{1/2} G + A^{-1/2} \epsilon_i \nabla (\epsilon_i \Psi_k + I) d^e_\theta \right) \cdot \phi$$
$$- \epsilon^{-1} \int_Q \text{curl} e_i A^{1/2} \phi - \epsilon^{-1} \int_Q \text{curl} e_i A^{1/2} N(y) (i \theta \times d^e_\theta) \cdot \text{curl} e_i A^{1/2} \phi$$
for all $\phi \in H^1_{\text{curl} A^{1/2}}(Q, d\mu)$.

Setting
$$\text{curl} e_i A^{1/2} N(y) (i \theta \times d^e_\theta) = e_i (\text{curl} A^{1/2} N(y) (i \theta \times d^e_\theta) + i \theta \times A^{1/2} N(y) (i \theta \times d^e_\theta)),$$
using the equation (2.8) and that $\int_Q (i \theta \times d^e_\theta) \cdot \text{curl} A^{1/2} \phi = 0$, we can rewrite $H^e_\tilde{\theta}$ as
$$\langle H^e_\tilde{\theta}, \phi \rangle = - \int_Q \left( A^{1/2} G + i \theta \times \tilde{A} i \theta \times d^e_\theta + i \theta \times \tilde{A} \text{curl} A^{1/2} N(i \theta \times d^e_\theta) + \epsilon i \theta \times \tilde{A} (i \theta \times A^{1/2} N(i \theta \times d^e_\theta))$$
$$+ A^{-1/2} \epsilon_i \nabla (\epsilon_i \Psi_k + I) d^e_\theta \right) \cdot \phi$$
$$- \int_Q \tilde{A} i \theta \times (A^{1/2} N(i \theta \times d^e_\theta)) \cdot \text{curl} A^{1/2} \phi \quad \forall \phi \in H^1_{\text{curl} A^{1/2}}(Q, d\mu).$$
Hence we can formally write $\mathcal{H}_0$ as

$$
\mathcal{H}_0 = -A^{1/2}G - A^{1/2}i\theta \times \tilde{A}i\theta \times \tilde{d}_0 - A^{-1/2}(\nabla(e_\kappa \Psi_\kappa) + I)d_\theta - A^{1/2}\text{curl} \tilde{A}(i\theta \times A^{1/2}N(i\theta \times d_\theta))
$$

$$
- A^{1/2}i\theta \times \tilde{A}\text{curl}(A^{1/2}N(i\theta \times d_\theta)) - \varepsilon A^{1/2}i\theta \times \tilde{A}i\theta \times (A^{1/2}N(i\theta \times d_\theta)).
$$

(2.24)

### 2.3.3 Properties of $\mathcal{H}_0$

In order to prove estimates for $R_0^\varepsilon$ our main tool is the Poincaré type inequality (1.24). To take advantage of such inequality in estimates for right hand side of (2.22), we need to prove the identity

$$\langle \mathcal{H}_0, \tilde{R}_0^\varepsilon \rangle = \langle \mathcal{H}_0, \tilde{R}_0^\varepsilon \rangle$$

where $\tilde{R}_0^\varepsilon$ is defined as in the decomposition (1.21).

We need to check two properties for $\mathcal{H}_0$. The first one is

$$
\langle \mathcal{H}_0, A^{-1/2}(\overline{e_\kappa} \nabla(e_\kappa \phi)) \rangle = 0 \quad \forall \phi \in H^1_\#(Q, d\mu).
$$

(2.25)

Starting from definition (2.22) for $\mathcal{H}_0$, we have that (2.25) holds using that $\overline{e_\kappa} \text{div}(e_\kappa G) = 0$ and using the equation (1.17) with $c = d_\theta^0$ for the vector function $\Psi_\kappa$.

The second property for $\mathcal{H}_0$ is

$$
\langle \mathcal{H}_0, A^{-1/2}(\overline{e_\kappa} \nabla(e_\kappa \psi_\kappa) + c) \rangle = 0
$$

(2.26)

for all $\psi_\kappa \in H^1_\#(Q, d\mu)$ and $c \in \mathbb{C}^3$. By linearity we have

$$
\langle \mathcal{H}_0, A^{-1/2}(\overline{e_\kappa} \nabla(e_\kappa \psi_\kappa) + c) \rangle = \langle \mathcal{H}_0, A^{-1/2}\overline{e_\kappa} \nabla(e_\kappa \psi_\kappa) \rangle + \langle \mathcal{H}_0, A^{-1/2}c \rangle.
$$

Hence thanks to (2.25) and by definition (2.24), it remains to analyse

$$
\langle \mathcal{H}_0, A^{-1/2}c \rangle = -\int_Q G \cdot c - \int_Q A^{-1}(\overline{e_\kappa} \nabla(e_\kappa \Psi_\kappa) + I)d_\theta \cdot c - \int_Q i\theta \times \tilde{A}(i\theta \times d_\theta) \cdot c
$$

$$
- \int_Q i\theta \times \tilde{A}\text{curl}(A^{1/2}N(i\theta \times d_\theta)) \cdot c - \varepsilon \int_Q i\theta \times \tilde{A}i\theta \times (A^{1/2}N(i\theta \times d_\theta)) \cdot c.
$$

We have that

$$
\varepsilon \int_Q i\theta \times \tilde{A}i\theta \times (A^{1/2}N(i\theta \times d_\theta)) \cdot c = 0
$$

using (2.19).

Furthermore

$$
i\theta \times \int_Q \tilde{A}(\text{curl} A^{1/2}N + I)(i\theta \times d_\theta) \cdot c + \int_Q A^{-1}(\overline{e_\kappa} \nabla(e_\kappa \Psi_\kappa) + I)d_\theta \cdot c + \int_Q G \cdot c = 0
$$

since is the equation (2.12) solved by $d_\theta$. Therefore (2.26) holds.

### 2.4 Estimate for $\varepsilon^2 R_0^\varepsilon$

**Theorem 2.9.** There exists $C > 0$ such that for all $\varepsilon > 0$ and $\theta \in \varepsilon^{-1}Q'$, the solution of the equation (1.47) satisfies:

$$
\|R_0^\varepsilon - A^{-1/2}(\overline{e_\kappa} \nabla(e_\kappa \psi_{R_0^\varepsilon}) + c_{R_0^\varepsilon}) - A^{-1/2}(\overline{e_\kappa} \Phi_{R_0^\varepsilon})\|_{L^2(Q, d\mu)} \leq C\|G\|_{L^2(Q, d\mu)},
$$

(2.27)

$$
\|A^{-1/2}(\overline{e_\kappa} \nabla(e_\kappa \psi_{R_0^\varepsilon}) + c_{R_0^\varepsilon}) + A^{-1/2}(\overline{e_\kappa} \Phi_{R_0^\varepsilon})\|_{L^2(Q, d\mu)} \leq C\varepsilon^{-1}\|G\|_{L^2(Q, d\mu)}.
$$

(2.28)
Proof. Suppose $\phi_n \in C^\infty_0(Q)$ converging to $R_\delta^\mu$ in $L^2(Q,d\mu)$ such that $\text{curl}(A^{1/2}\phi_n) \to \text{curl}(A^{1/2}R_\delta^\mu)$ in $L^2(Q,d\mu)$. Using $\phi_n$ as test function in equation (2.23) on has:

$$\int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2}R_\delta^\mu) \cdot \text{curl}(e_\kappa A^{1/2}\phi_n) + \varepsilon^2 \int_Q A^{-1/2}(\overline{\kappa}(\nabla e_\kappa \psi R_\delta^\mu) + cR_\delta^\mu) \cdot \phi_n + \varepsilon^2 \int_Q A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Phi R_\delta^\mu) \cdot \overline{\phi_n}$$

$$= - \int_Q \left(A^{1/2}(G + i\theta \times \tilde{A}_\kappa \theta \times d_\delta^\mu + i\theta \times \tilde{A} \text{curl} A^{1/2}N(i\theta \times d_\delta^\mu)) + e^{i\theta} \times A \text{curl} A^{1/2}N(i\theta \times d_\delta^\mu)) + A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Psi + I)d_\delta^\mu \right) \cdot \overline{\phi_n}$$

Using the identity

$$\text{curl}(A^{1/2}\phi_n) = i\kappa \times A^{1/2}\phi_n + \overline{\kappa} \text{curl}(e_\kappa A^{1/2}\phi_n)$$

we can rewrite the above equation as

$$\int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2}R_\delta^\mu) \cdot \text{curl}(e_\kappa A^{1/2}\phi_n) + \varepsilon^2 \int_Q A^{-1/2}(\overline{\kappa}(\nabla e_\kappa \psi R_\delta^\mu) + cR_\delta^\mu) \cdot \phi_n + \varepsilon^2 \int_Q A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Phi R_\delta^\mu) \cdot \overline{\phi_n}$$

$$= - \int_Q \left(A^{1/2}(G + i\theta \times \tilde{A}_\kappa \theta \times d_\delta^\mu + i\theta \times \tilde{A} \text{curl} A^{1/2}N(i\theta \times d_\delta^\mu)) + A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Psi + I)d_\delta^\mu \right) \cdot \overline{\phi_n}$$

When $n \to \infty$ we use the decomposition (1.19) and the related orthogonality conditions (cf. with Theorem 1.13) to obtain a bilinear form in the right hand side of equation (2.30). Furthermore we use the properties (2.25) and (2.26) to have

$$\int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2}R_\delta^\mu) \cdot \text{curl}(e_\kappa A^{1/2}\xi_\delta^\mu) + \varepsilon^2 \int_Q A^{-1/2}(\overline{\kappa}(\nabla e_\kappa \psi R_\delta^\mu) + cR_\delta^\mu) \right)^2 + \varepsilon^2 \int_Q A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Phi R_\delta^\mu) \right)^2$$

$$= - \int_Q \left(A^{1/2}(G + i\theta \times \tilde{A}_\kappa \theta \times d_\delta^\mu + i\theta \times \tilde{A} \text{curl} A^{1/2}N(i\theta \times d_\delta^\mu)) + A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Psi + I)d_\delta^\mu \right) \cdot \overline{\phi_n}$$

In order to estimate last term in the right hand side of (2.31), we consider $\xi_\delta^\mu \in H^1_{\text{curl}}A^{1/2}(Q,d\mu)$ solution of

$$A^{1/2}\overline{\kappa} \text{curl} \tilde{A} \text{curl} e_\kappa A^{1/2}\xi_\delta^\mu + \varepsilon^2 A^{1/2}(\overline{\kappa}(\nabla e_\kappa \psi \xi_\delta^\mu) + c\xi_\delta^\mu) + \varepsilon^2 A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Phi \xi_\delta^\mu)$$

understood as the integral identity

$$\int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2}\xi_\delta^\mu) \cdot \text{curl}(e_\kappa A^{1/2}\phi) + \varepsilon^2 \int_Q A^{-1/2}(\overline{\kappa}(\nabla e_\kappa \psi \xi_\delta^\mu) + c\xi_\delta^\mu) \cdot \overline{\phi} + \varepsilon^2 \int_Q A^{-1/2}\overline{\kappa}(\nabla e_\kappa \Phi \xi_\delta^\mu) \cdot \overline{\phi}$$

$$= \int_Q e_\kappa \tilde{A}_\kappa \theta \times (A^{1/2}N(i\theta \times d_\delta^\mu)) \cdot \text{curl}(e_\kappa A^{1/2}\phi) \forall \phi \in H^1_{\text{curl}}A^{1/2}(Q,d\mu).$$

Existence and uniqueness of solution $\xi_\delta^\mu \in H^1_{\text{curl}}A^{1/2}(Q,d\mu)$ follow from the argument used in Proposition 1.13. Testing equation (2.32) with $\xi_\delta^\mu$ we obtain that

$$\| \text{curl}(e_\kappa A^{1/2}\xi_\delta^\mu) \|_{L^2(Q,d\mu)} \leq C \|G\|_{L^2(Q,d\mu)}.$$
To rewrite the right hand side of equation (2.31), we test equation (2.32) with \( \tilde{R}_\theta^\varepsilon \). We have that
\[
\int_Q e_\kappa i \theta \times (A^{1/2} \mathcal{N}(i \theta \times d_\varepsilon)) \cdot \text{curl}(e_\kappa A^{1/2} \tilde{R}_\theta^\varepsilon) = \int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2} \xi_\theta^\varepsilon) \cdot \text{curl}(e_\kappa A^{1/2} \tilde{R}_\theta^\varepsilon) + \varepsilon^2 \int_Q A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \xi_\varepsilon)} + c_{\xi_\varepsilon}) \cdot \tilde{R}_\theta^\varepsilon + \varepsilon^2 \int_Q A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \xi_\varepsilon) \cdot \tilde{R}_\theta^\varepsilon.
\]
(2.35)

Taking into account the orthogonality of elements in decomposition (1.20), we have that
\[
\int_Q A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \xi_\varepsilon)} + c_{\xi_\varepsilon}) \cdot \tilde{R}_\theta^\varepsilon = 0, \quad \int_Q A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \xi_\varepsilon) \cdot \tilde{R}_\theta^\varepsilon = 0.
\]

In order to rewrite the remaining expression in the right hand side of (2.35), we use \( \xi_\theta^\varepsilon \) as test function in equation (2.22). Note that for an arbitrary measure \( \mu \) there may be different elements in \( H^1_{\text{curl},A^{1/2}} \) with the same first component. Though for the solution \( \xi_\theta^\varepsilon \) to (2.32) there exists a natural choice of \( \text{curl}A^{1/2} \xi_\theta^\varepsilon \). In fact consider sequences \( \psi_n, \phi_n \in C_\#^\infty \) converging to \( \xi_\theta^\varepsilon \) in \( L^2(Q,d\mu) \) such that
\[
\text{curl}(e_\kappa A^{1/2} \phi_n) \rightarrow \text{curl}(e_\kappa A^{1/2} \xi_\theta^\varepsilon), \quad \text{curl}(e_\kappa A^{1/2} \psi_n) \rightarrow \text{curl}(e_\kappa A^{1/2} \xi_\theta^\varepsilon).
\]
The difference \( \text{curl}(e_\kappa A^{1/2} \phi_n) - \text{curl}(e_\kappa A^{1/2} \psi_n) \) converges to zero, and so does \( \text{curl}(A^{1/2} \phi_n) - \text{curl}(A^{1/2} \psi_n) \). Henceforth we denote with \( \text{curl}(A^{1/2} \xi_\theta^\varepsilon) \) the common \( L^2 \)-limit of \( \text{curl}(A^{1/2} \phi_n) \) for a sequence \( \phi_n \in C_\#^\infty(Q) \) with the above properties.

Since \( \text{curl}(A^{1/2} \Phi_\theta^\varepsilon) \) is chosen uniquely, we can write
\[
\int_Q \tilde{A} \text{curl}(e_\kappa A^{1/2} \tilde{R}_\theta^\varepsilon) \cdot \text{curl}(e_\kappa A^{1/2} \xi_\theta^\varepsilon) + \varepsilon^2 \int_Q A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \Phi_\theta^\varepsilon)} + c_{\xi_\theta^\varepsilon}) \cdot \tilde{R}_\theta^\varepsilon + \varepsilon^2 \int_Q A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \xi_\theta^\varepsilon) \cdot \tilde{R}_\theta^\varepsilon = \langle \mathcal{H}_\theta^\varepsilon, \xi_\theta^\varepsilon \rangle = \langle \mathcal{H}_\theta^\varepsilon, \xi_\theta^\varepsilon \rangle,
\]
where in the last equality we use the properties of \( \mathcal{H}_\theta^\varepsilon \) (2.25) and (2.26).

By the orthogonality of elements in Helmholtz decomposition (1.20), we have
\[
\int_Q A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \Phi_\theta^\varepsilon)} + c_{\Phi_\theta^\varepsilon}) \cdot \tilde{R}_\theta^\varepsilon = \int_Q A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \Phi_\theta^\varepsilon)} + c_{\Phi_\theta^\varepsilon}) \cdot A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \xi_\varepsilon)} + c_{\xi_\varepsilon}),
\]
and
\[
\int_Q A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \Phi_\theta^\varepsilon) \cdot \tilde{R}_\theta^\varepsilon = \int_Q A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \Phi \Phi_\theta^\varepsilon)}) \cdot A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \xi_\varepsilon).
\]

Hence the right hand side of (2.35) can be written as
\[
\int_Q e_\kappa i \theta \times (A^{1/2} \mathcal{N}(i \theta \times d_\varepsilon)) \cdot \text{curl}(e_\kappa A^{1/2} \tilde{R}_\theta^\varepsilon)
= \langle \mathcal{H}_\theta^\varepsilon, \xi_\theta^\varepsilon \rangle - \varepsilon^2 \int_Q A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \xi_\theta^\varepsilon) \cdot A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \xi_\varepsilon)
- \varepsilon^2 \int_Q A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \xi_\varepsilon)} + c_{\xi_\varepsilon}) \cdot A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \Phi_\varepsilon)} + c_{\Phi_\varepsilon}).
\]

The second element in the right hand side of last equation is null. In fact it is equation (2.33) tested with \( A^{-1/2} \varepsilon_\kappa \nabla (e_\kappa \Phi \Phi_\varepsilon) \). The third element in the right hand side is null as well. In fact it is equation (2.33) tested with the element \( A^{-1/2}(\overline{\varepsilon_\kappa \nabla (e_\kappa \psi \Phi_\varepsilon)} + c_{\Phi_\varepsilon}) \) that is zero taking into account 2.19.
Hence we can write equation (2.31) as
\[
\int_Q \langle \nabla \epsilon R_{\phi} \rangle \cdot \nabla e_k (A^{1/2} \epsilon R_{\phi}) + e^2 \int_Q |A^{-1/2} \epsilon R_{\phi} + \epsilon R_{\phi}|^2 + e^2 \int_Q |A^{-1/2} \epsilon R_{\phi} + \epsilon R_{\phi}|^2 \\
= - \int_Q \left( A^{1/2} (G + i \theta \times \tilde{A} \theta \times d_\theta + i \theta \times \tilde{A} \nabla A^{1/2} N(i \theta \times d_\theta)) + A^{-1/2} \epsilon R_{\phi} + \epsilon R_{\phi} \right) \cdot R_{\phi} \\
- \langle H_{\phi}, \xi \rangle.
\] (2.36)

**Lemma 2.10.** The last term in the right hand side of (2.36) is bounded uniformly in \(\epsilon, \theta\): 
\[
|\langle H_{\phi}, \xi \rangle| \leq C \|G\|_{L^2(Q, d\mu)}, \quad C > 0.
\]

**Proof.** By the definition (2.24) for \(H_{\phi}\), we have
\[
\langle H_{\phi}, \xi \rangle = - \int_Q \left( A^{1/2} (G + i \theta \times \tilde{A} \theta \times d_\theta + i \theta \times \tilde{A} \nabla A^{1/2} N(i \theta \times d_\theta)) + A^{-1/2} \epsilon R_{\phi} + \epsilon R_{\phi} \right) \cdot R_{\phi}.
\]
Using the identity (2.29) for a \(\phi_n \in C^\infty_0 (Q)\) we obtain
\[
\langle H_{\phi}, \xi \rangle = - \int_Q \left( A^{1/2} (G + i \theta \times \tilde{A} \theta \times d_\theta + i \theta \times \tilde{A} \nabla A^{1/2} N(i \theta \times d_\theta)) + A^{-1/2} \epsilon R_{\phi} + \epsilon R_{\phi} \right) \cdot \xi.
\]
Using the decomposition (1.24) for \(\xi\), we note that
\[
\int_Q \epsilon \tilde{A} \theta \times (A^{1/2} N(i \theta \times d_\theta)) \cdot \nabla e_k (A^{1/2} \xi) = \int_Q \epsilon \tilde{A} \theta \times (A^{1/2} N(i \theta \times d_\theta)) \cdot \nabla e_k (A^{1/2} \xi),
\]
since
\[
\int_Q \epsilon \tilde{A} \theta \times (A^{1/2} N(i \theta \times d_\theta)) \cdot \nabla e_k (A^{1/2} \xi) = 0.
\]
The last equality follows from (2.19), noting that \((A^{-1/2} c, e_k (\kappa \times c)) \in H^1_{\text{curl}, A^{1/2}} (Q, d\mu)\). Now, using the Hölder inequality, the Poincaré-type inequality (1.24) for \(\xi\) and taking into account the estimate (2.34), the required statement holds.

Combining Lemma 2.10 Hölder inequality and the Poincaré-type inequality (1.24) for \(R_{\phi}\) in (2.36) we obtain the following uniform bound
\[
\| \nabla \epsilon R_{\phi} \|_{L^2(Q, d\mu)} \leq \epsilon C \|G\|_{L^2(Q, d\mu)}.
\] (2.37)
The estimate (2.37) combined with (1.24) implies (2.27). Furthermore the same estimate, Lemma 2.10 and equation (2.36) imply (2.28). 

**Corollary 2.11.** There exists a constant \(C > 0\) such that the following estimate holds uniformly in \(\epsilon, \theta\) and \(G\):
\[
\| \epsilon A^{1/2} R_{\phi} \|_{L^2(Q, d\mu)} \leq \epsilon C \|G\|_{L^2(Q, d\mu)}.
\]
2.5 Conclusion of the convergence estimate

Proposition 2.12. There exists $C > 0$ such that the function $z^\varepsilon_\theta$ defined in (2.10) satisfies the estimate

$$
\|z^\varepsilon_\theta\|_{L^2(Q,d\mu)} \leq C\varepsilon\|G\|_{L^2(Q,d\mu)}.
$$

(2.38)

Proof. The vector function $z^\varepsilon_\theta \in H^1_\text{curl}A^{1/2}(Q,d\mu)$ solves

$$
\varepsilon^{-2}A^{1/2}\tau_c\varepsilon\text{curl} A\varepsilon A^{1/2}z^\varepsilon_\theta + z^\varepsilon_\theta = -\varepsilon^2\tilde{R}_\theta^\varepsilon - \varepsilon N(i\theta \times d^\varepsilon_\theta).
$$

(2.39)

Using $z^\varepsilon_\theta$ as a test function in the equation (2.39), on has

$$
\varepsilon^{-2}\int_Q \tilde{A}\varepsilon\text{curl}(\varepsilon A^{1/2}z^\varepsilon_\theta) \cdot \text{curl}(\varepsilon A^{1/2}z^\varepsilon_\theta) + \int_Q |z^\varepsilon_\theta|^2 = -\varepsilon \int_Q N(y)(i\theta \times d^\varepsilon_\theta) \cdot \bar{z}^\varepsilon_\theta - \varepsilon^2 \int_Q \tilde{R}_\theta^\varepsilon \cdot \bar{z}^\varepsilon_\theta.
$$

Using the Hölder inequality, the Poincaré-type inequality (1.24) and the definition (2.12) for $d^\varepsilon_\theta$, on has the

(2.38).

Proposition 2.12 and Corollary 2.11 imply (2.11), in fact

$$
\|D^\varepsilon_\theta - A^{-1/2}(\tau_c\varepsilon\text{curl}(\varepsilon \Psi e\theta) + I)\|_{L^2(Q,d\mu)} \leq \|U^\varepsilon_\theta - A^{-1/2}(\tau_c\varepsilon\text{curl}(\varepsilon \Psi e\theta) + I)\|_{L^2(Q,d\mu)} + \|z^\varepsilon_\theta\|_{L^2(Q,d\mu)},
$$

hence the proof of Theorem 2.4 is concluded.

2.6 Estimate for magnetic field and magnetic induction

To obtain the estimates for the magnetic field and induction, we start from equation (2.10). The transformed problem can be written as

$$
\begin{cases}
\varepsilon^{-1}\tau_c\varepsilon\text{curl} e_\kappa A^{1/2}D^\varepsilon_\theta + B^\varepsilon_\theta = 0 \\
\varepsilon^{-1}A^{1/2}\tau_c\varepsilon\text{curl} e_\kappa \tilde{A}B^\varepsilon_\theta - D^\varepsilon_\theta = A^{1/2}G,
\end{cases}
$$

(2.40)

where $\tau_c\varepsilon\text{div}(e_\kappa A^{-1/2}D^\varepsilon_\theta) = 0$ and $B^\varepsilon_\theta := \tau_c\varepsilon\mathcal{F}_*B_\varepsilon$ is the transformed magnetic induction such that $\tau_c\varepsilon\text{div}(e_\kappa B^\varepsilon_\theta) = 0$. In this setting the transformed magnetic field $H^\varepsilon_\theta = \tilde{A}B^\varepsilon_\theta$.

To find the approximation for $B^\varepsilon_\theta$ we use the approximation (2.11) we have for $D^\varepsilon_\theta$ and the first line of system (2.40). Hence we have

$$
B^\varepsilon_\theta = \varepsilon^{-1}\tau_c\varepsilon\text{curl} e_\kappa ((\tau_c\varepsilon\text{curl}(\varepsilon \Psi e\theta) + I)\varepsilon d^\varepsilon_\theta + \varepsilon N(y)(i\theta \times d^\varepsilon_\theta) + \varepsilon^2\tilde{R}_\theta^\varepsilon)
$$

$$
= (\varepsilon\text{curl} N(y) + I)(i\theta \times d^\varepsilon_\theta) + \varepsilon (i\theta \times (A^{1/2}N(y)(i\theta \times d^\varepsilon_\theta)) + \tau_c\varepsilon\text{curl} e_\kappa A^{1/2}R^\varepsilon_\theta).
$$

Here $d^\varepsilon_\theta$ is defined in (2.12) and $R^\varepsilon_\theta$ solves (2.22).

Theorem 2.13. There exists $C > 0$ independent of $\theta$, $\varepsilon$ and $G$ such that the following estimates hold for the transformed magnetic induction $B^\varepsilon_\theta$ and the transformed magnetic field $H^\varepsilon_\theta := \tilde{A}B^\varepsilon_\theta$

$$
\|B^\varepsilon_\theta - (\varepsilon\text{curl} N + I)(i\theta \times d^\varepsilon_\theta)\|_{L^2(Q,d\mu)} \leq \varepsilon C\|G\|_{L^2(Q,d\mu)}
$$

(2.41)

$$
\|H^\varepsilon_\theta - \tilde{A}(\varepsilon\text{curl} N + I)(i\theta \times d^\varepsilon_\theta)\|_{L^2(Q,d\mu)} \leq \varepsilon C\|G\|_{L^2(Q,d\mu)}
$$

(2.42)

Applying back the Floquet transform on (2.41) on obtains the following norm-resolvent estimates on the whole space setting for $B_\varepsilon$ solution of (2.22). (Analogously starting from (2.12) on obtain estimates for $H_\varepsilon := A\tilde{B}_\varepsilon$.)
Corollary 2.14. Let \( g \in L^2(R^3, d\mu^\varepsilon) \) and denote \( g^\varepsilon \theta := e^{\kappa T \varepsilon F \varepsilon} g(x) \) so that
\[
\int_Q g^\varepsilon \theta d\mu = \hat{g}(\theta), \quad \theta \in \varepsilon^{-1}Q', \quad \text{where} \quad \hat{g}(\theta) := (2\pi)^{-3/2} \int_{R^3} g e^{\theta \cdot e} d\mu^\varepsilon, \quad \theta \in R^3.
\]
There exists a constant \( C > 0 \) such that the following estimate holds for \( B_\varepsilon \) solution of (2.2)
\[
\|B_\varepsilon - (\text{curl } N + I)(2\pi)^{-3/2} \int_{R^3} (\hat{A}_h^{\text{hom}})^{-1} (\hat{A}_h^{\text{hom}} + I)^{-1} \hat{g}(\theta) e_\theta d\theta \|_{L^2(R^3, d\mu^\varepsilon)} \leq C \varepsilon \|g\|_{L^2(R^3, d\mu^\varepsilon)}, \quad (2.43)
\]
\( \forall \varepsilon > 0 \). Here \( \hat{A}_h^{\text{hom}} \) is the matrix valued quadratic form given by (2.12), and \( \Psi_\kappa \) by (1.17) for all values \( \theta \in R^3 \).

References

[1] Birman, M., Suslina, T., 2004. Second order periodic differential operators. Threshold properties and homogenization. St. Petersburg Mathematical Journal. 15(5), 639–714.

[2] Birman, M. S., Suslina, T. A. 2007. Homogenization of the stationary periodic Maxwell system in the case of constant permeability. Functional Analysis and Its Applications. 41(2), 81-98.

[3] Cherednichenko, K. D., Cooper, S., 2015. Homogenization of the system of high-contrast Maxwell equations. Mathematika, 61(2), 475-500.

[4] Cherednichenko, K. D., Cooper, S., 2016. Resolvent estimates for high-contrast elliptic problems with periodic coefficients. Archive for Rational Mechanics and Analysis 219(3), 1061–1086.

[5] Cherednichenko, K. D., D’Onofrio, S., 2018. Operator-Norm Convergence Estimates for Elliptic Homogenization Problems on Periodic Singular Structures. Journal of Mathematical Sciences 232(4), 1–15.

[6] Cherednichenko, K. D., D’Onofrio, S., 2018. Operator-norm homogenisation estimates for the system of Maxwell equations on periodic singular structures. arXiv preprint: 1811.08980.

[7] Suslina, T., 2008. Homogenization with corrector for a stationary periodic Maxwell system. St. Petersburg Mathematical Journal 19(3), 455–494.

[8] Zhikov, V. V. and Kozlov, S. M. and Oleinik, O. A., 1994. Homogenization of differential operators and integral functionals. Springer Science & Business Media

[9] Zhikov, V. V., Pastukhova S. E., 2016. Bloch principle for elliptic differential operators with periodic coefficients, Russian Journal of Mathematical Physics 23(2), 257–277.

[10] Zhikov, V. V., 2005. A note on Sobolev spaces, Journal of Mathematical Sciences 129(1), 3593–3595.