Squeezed states of the harmonic oscillator are a common resource in applications of quantum technology. If the noise is suppressed in a nonlinear combination of quadrature operators below threshold for all possible up-to-quadratic Hamiltonians, the quantum states are non-Gaussian and we refer to the noise reduction as nonlinear squeezing. Non-Gaussian aspects of quantum states are often more vulnerable to decoherence due to imperfections appearing in realistic experimental implementations. Therefore, a stability of nonlinear squeezing is essential. We analyze the behavior of quantum states with cubic nonlinear squeezing under loss and dephasing. The properties of nonlinear squeezed states depend on their initial parameters which can be optimized and adjusted to achieve the maximal robustness for the potential applications.

1 Introduction

Quantum optical systems are well suited for testing fundamental aspects of quantum physics as well as for developing practical applications, such as quantum communication [7, 39, 63] that is quickly approaching practical applications [14, 77], quantum metrology [46, 60] that found its place in gravitational wave detection [1], and even quantum computation [31, 51], in which the boson sampling protocol is reaching quantum supremacy [84]. Furthermore, compared to other physical systems, light can be manipulated with high speeds [52]. There are two main directions for exploiting the quantum nature of light for fundamental tests and applications. The first approach utilizes a microscopic control of discrete photons by single-photon detectors and encodes information into their polarization, spatial, or temporal properties [32]. The second approach employs the continuous variables (CV) of light represented by the quadratures of the field. Quantum features of such light manifest as squeezed fluctuations in individual modes of multi-photon light that are collectively controlled and measured by homodyne detection [5, 10, 74].

Such squeezed light can be obtained by nonlinear processes in optical crystals, fibers, and atomic ensembles and it is a prominent resource for CV quantum optics [3, 4]. Squeezed light is broadly used as a way to generate quantum entanglement [78], gain advantage in metrology applications [1], or control quantum systems [30, 47]. Together with the tools of linear optics, homodyne detection, and linear feedforward electro-optical control, squeezing is a sufficient resource for the implementation of any Gaussian operation [19, 74]. However, Gaussian processing alone is not capable of realizing the ultimate goals of quantum technology, such as quantum computation [43]. For that, Gaussian processing with traveling light needs to be supplemented by experimentally challenging non-Gaussian elements [8, 25, 38].

One such possible non-Gaussian element is the deterministic quadrature phase gate [25] that can unconditionally implement non-Gaussian non-classical quantum operation, thus opening the first door towards deterministic quantum processing with traveling CV states of light [38]. This sets it apart from the usual probabilistic methods for injecting non-Gaussianity into a quantum system that rely on approximating addition or subtraction of individual photons by using single-photon detectors [6, 15, 42, 53, 69, 76, 81, 82]. Direct implementation of the quadrature phase gates by nonlinear crystals, optical fibers, and atomic ensembles is currently unavailable, but they can be deterministically realized in a measurement induced fashion [40, 49, 64]. Using this approach, the required cubic nonlinearity is imprinted on the target system by coupling it to a different system in a specific quantum state, measuring this auxiliary system, and completing the process by deterministic nonlinear feed-forward. The approach requires the auxiliary quantum systems prepared in highly non-classical cubic phase states. Preparation of such quantum states is currently discussed both in the theoretical [29, 35, 50, 57, 58, 71, 83] and the experimental [33, 80] context.

The key property of realistically prepared cubic phase states is their nonlinear squeezing [33, 71] which determines how close they are to the ideal infinite energy eigenstates [25]. The nonlinear squeezing is a new type of operationally defined quantum non-Gaussian resource [12] that enables deterministic implementation of the cubic phase gate. To fully under-
stand this resource and to enable its successful utilization it is necessary to know its behavior under decoherence, which is already the limiting factor in the experimental tests [33, 80]. Decoherence is generally unavoidable but its effects can be marginalized if they are properly understood, as was recently demonstrated in [27, 30, 45].

In this paper, we expand the concept of nonlinear squeezing introduced in [33, 71] and analyze its behavior under the losses and phase noise - the two most prevalent sources of imperfections in optical experiments. For several classes of approximate cubic phase states, we show how their nonlinear squeezing transforms under decoherence relative to their initial nonlinear squeezing and how their robustness depends on the cubicity [29]. Based on this analysis we show that for any specific decoherence channel, there is a set of parameters that minimizes the effects of decoherence and maximizes the final nonlinear squeezing. We also show how local Gaussian operations can be used to convert the cubicity [29] of the quantum states with nonlinear squeezing into the robust form and thus safeguard it against decoherence.

## 2 Linear and nonlinear squeezing

In quantum optics, squeezed states are those with variance of linear quadrature operator \( \hat{x}(\theta) = \hat{x} \cos \theta + \hat{p} \sin \theta \), where \([\hat{x}, \hat{p}] = i\), reduced below the value given by the vacuum state. Such states are most often generated by means of second-order nonlinear processes [70, 75], which can be even used to generate multimode squeezed states [17, 20], but it is also possible to arrive at squeezed states by employing four wave mixing [44, 65], or Kerr nonlinearity [28, 61]. Historically, nonlinear coherent and squeezed states were introduced to describe protocols in a nonlinear regime of trapped ion oscillators [13, 16, 36, 62, 73]. In many applications in the quantum technologies [19, 47, 48, 59] the only relevant characteristics of squeezed states is their squeezed variance. This is because such scenarios ideally require eigenstates of the respective quadrature operator \( \hat{x}(\theta) \) and squeezed states play the role of approximation of these states.

In a similar vein, there are quantum protocols that require eigenstates of nonlinear operators. The hierarchy of nonlinear phase gates implementing unitary operations with Hamiltonians \( \hat{H} = \chi \hat{x}^n \) [40, 41] requires ancillary modes prepared in eigenstates of operators \( \hat{p} + n \chi \hat{x}^{n-1} \). In practice, such eigenstates are, due to their infinite energy, impossible to prepare and any practical realization needs to accept states which are only approximations. There are many ways to determine overlap or distance between different quantum states [66, 68]. However, in this particular scenario of measurement-induced protocols for traveling beams of light, in which the ultimate goal is the unnormalizable eigenstate of some operator, it is best to utilize the concept of squeezing. Advantageously, it is not required that the overall state is pure, or that it saturates any minimal uncertainty principle. Eigenstates \(|\phi\rangle\) of an arbitrary operator \( \hat{O} \) always have

\[
\var_{\langle\phi\rangle} \hat{O} = \text{Tr} \left[ |\phi\rangle \langle\phi| \hat{O}^2 \right] - \text{Tr} \left[ |\phi\rangle \langle\phi| \hat{O} \right]^2 = \langle \langle \hat{O} - \langle\hat{O}\rangle \rangle^2 \rangle,
\]

which is vanishing, so when quantum states are squeezed in this operator they are also approaching the required eigenstate.

The quantum states, required for the hierarchy of nonlinear phase gates [41, 49], are the eigenstates of operators

\[
\hat{O}_n(z) = \hat{p} + z \hat{x}^{n-1},
\]

where \( n \) represents the order of the nonlinearity and \( z \) is a real parameter that determines the phase space shape of the quantum state. In the case of \( n = 3 \), \( z \) is sometimes called cubicity [29]. Note that for \( n = 2 \), the operator \( \hat{O}_2 \) is just a re-scaled rotated quadrature operator, \( \hat{O}_2 = \hat{x}(\theta) = \hat{x} \cos \theta + \hat{p} \sin \theta \) with \( z = \tan \theta \) representing only rotation in phase space. For \( n > 2 \), the eigenstates of operators (3) start as eigenstates of \( \hat{p} \) for \( z = 0 \) and then get progressively more deformed in the phase space as \( |z| \) increases. For example, Wigner function of the ideal cubic state \( \hat{C}(z) [p = 0] \) is proportional to the Airy function with argument \( zz^2 - p \) [24]. The dominant feature of the function is a parabolically shaped ridge along the phase space curve given by \( p = zz^2 \). Moreover, for \( n > 2 \) the eigenstates of (3) are quantum non-Gaussian, they cannot be represented by a Gaussian Wigner function or their mixtures [26]. As a consequence, any sufficiently good approximation needs to be quantum non-Gaussian as well. This is an important distinction as quantum non-Gaussianity is necessary for quantum behavior which cannot be simulated classically [43] nor semiclassically.

In the following we are going to focus on the cubic nonlinearity with \( n = 3 \) and define the cubic nonlinear squeezing of a quantum state \( \hat{p} \) by a ratio of variances

\[
\xi_3(z) = \frac{\var_{\hat{p}} \hat{O}_3(z)}{\min_{\hat{p}} \var_{\hat{p}} \hat{O}_3(z)},
\]

where the minimum in the denominator is taken over the set of all Gaussian states and their mixtures and it is equal to \( \min_{\hat{p}} \var_{\hat{p}} \hat{O}_3(z) = 3/4(z^2)^{1/3} \) (see Appendix A for the derivation of the optimal states). We say that quantum state has nonlinear squeezing if \( \xi(z) < 1 \) for some \( z \). The Gaussian renormalization that sets the definition apart from metrological nonlinear squeezing of [23] ensures that states with nonlinear squeezing are genuine non-Gaussian quantum states [11, 18, 37], which is a necessary condition for quantum computation [43]. The quantity (4) also directly determines the amount of noise added in
the process of deterministic implementation of nonlinear quadrature gates [40, 41, 49], normalized with respect to the optimal Gaussian ancilla. The nonlinear squeezing is therefore an operationally defined formula that determines whether the nonlinear quadrature phase gates can overcome the Gaussian variants. The variance of the $\hat{O}_3(z)$ operator can be decomposed into three terms:

$$\text{var}_\rho \hat{O}_3(z) = ((\hat{p}^2) - \langle \hat{p} \rangle) + z^2((\hat{x}^4) - \langle \hat{x}^2 \rangle^2) + z((\hat{p}\hat{x}^2 + \hat{x}^2\hat{p}) - \langle \hat{p} \rangle \langle \hat{x}^2 \rangle),$$

(5)

where the first two terms stand for the variances of $\hat{p}$ and $\hat{x}^2$, and the third term represents correlation between these operators that is zero for all Gaussian states. Due to uncertainty relations, requiring that $\text{var}_\rho \hat{O}_3(z) \to 0$ necessitates that all the three terms diverge (see Appendix A for details). Interestingly, the cubicity parameter $z$ determines the shape of the quantum state but not the amount of quantum non-Gaussianity useful for the measurement induced gates [40, 41, 49]. This rather counterintuitive behavior is the consequence of a simple feature - parameter $z$ can be altered by a squeezing operation that transforms the quadratures as $\hat{x} \to \hat{x}/|\lambda|$ and $\hat{p} \to \lambda \hat{p}$ than $\xi_n(z) \to \xi_n(|\lambda|^n)$. As a consequence, a quantum state can have non-Gaussian features sufficient for applications even if $z$ is very small, as long as it is not zero. However, when $|z| \to 0$ or $|z| \to \infty$ the optimal Gaussian states become highly squeezed states and surpassing their variance might be experimentally difficult. On the other hand, for $z = \pm \frac{1}{\sqrt{\eta}}$ the optimal Gaussian state is the vacuum state which makes direct experimental observation of the nonlinear squeezing more feasible as will be seen in the following analysis.

For any quantum state, the variance of the cubic nonlinear operator (3) can be obtained in several ways. One is, as usual, quantum tomography and subsequent numerical evaluation. However, it is also possible to estimate the quantity much more efficiently by measuring statistics of generalized quadrature operator $\hat{x}(\theta) = \hat{x} \cos \theta \hat{p} \sin \theta$ for $\theta = (0, \pi/2, \pi/4, -\pi/4)$. With these data, the nonlinear variance can be directly constructed as [50]

$$\text{var}_\rho \hat{O}_3(z) = \langle \hat{x}(\pi/2)^2 \rangle + z^2 \langle \hat{x}(0)^4 \rangle + 2\sqrt{\eta} \frac{\sqrt{2z}}{3} \left[[\hat{x}(\pi/4)^3] - \langle \hat{x}(-\pi/4)^3 \rangle \right] - \frac{2z}{3} \langle \hat{x}(\pi/2)^3 \rangle - \langle \hat{x}(\pi/2) \rangle^2 + z \langle \hat{x}(0)^2 \rangle^2.$$  

(6)

Finally, the nonlinear variance can be directly measured by a nonlinear measurement of the nonlinear quadrature operator $\hat{O}_3(z)$. Such measurement employs direct joint detection of quadrature operators $\hat{x}$ and $\hat{p}$, similarly to heterodyne detection, with the caveat that feed-forward phase shift depending on the value of the first measurement is placed in front of the second one [48]. The output of the second detector then directly returns the value of the nonlinear quadrature operator, burdened by a portion of additive noise, that can be used for estimation of the nonlinear squeezing as well as for projective implementation of quantum operations [48]. The additive noise added during the measurement is given by the nonlinear squeezing of the ancilla used in the measurement. When the measurement is used for projective implementation of quantum operations the ancilla needs to be nonlinearly squeezed, but for the observation of the nonlinear squeezing itself, this is not necessary.

### 3 Decoherence of nonlinear squeezing

Decoherence is an unavoidable part of realistic physical systems. In quantum optics, the most common sources of imperfection are losses and dephasing, see Fig. 1. Loss emerges when some part of the optical mode is lost to the environment, which can happen due to reflection on optical components, imperfect mode matching, or many other factors. Dephasing, also known as phase noise [21, 22], appears when the phase reference between different parts of the system becomes imperfect and the system needs to be described as a mixture of states randomly rotated in phase space.

To characterize the effects of decoherence for the given quantum state $\hat{\rho}$, which decoheres into state $\hat{\rho}'$, we need to calculate the nonlinear variance

$$\text{var}_\rho \hat{O} = \langle \hat{p}^2 \rangle + z \langle \hat{p}\hat{x}^2 + \hat{x}^2\hat{p} \rangle + z^2 \langle \hat{x}^4 \rangle - \langle \hat{p} \rangle^2 - 2z \langle \hat{p} \rangle \langle \hat{x}^2 \rangle + z^2 \langle \hat{x}^2 \rangle^2,$$

(7)

where the individual moments, the mean values of symmetrically ordered polynomials of quadrature operators, $M(\hat{x}, \hat{p})$, are evaluated with respect to the decohered state $\hat{\rho}'$. The loss can be modeled by a virtual beam splitter with vacuum in the idle input port, and tracing over the idle output port. The respective moments can then be calculated as

$$\langle M(\hat{x}, \hat{p}) \rangle = \text{Tr} \left[M(\sqrt{\eta}\hat{x} + \sqrt{1-\eta}\hat{x}_0, \sqrt{\eta}\hat{p} + \sqrt{1-\eta}\hat{p}_0) \hat{\rho} \otimes |0\rangle\langle0| \right],$$

(8)

where the quadrature operators $\hat{x}_0$ and $\hat{p}_0$ belong to the auxiliary environment mode in the vacuum state and $\eta$ is an intensity transmission coefficient representing the loss. Similarly any moment of a state affected by dephasing, which is modeled by subjecting the state to a random phase shift, can be calculated as

$$\langle M(\hat{x}, \hat{p}) \rangle = \int \text{Tr} \left[M(\hat{x} \cos \phi + \hat{p} \sin \phi, \hat{p} \cos \phi - \hat{x} \sin \phi) \hat{\rho} P(\phi) d\phi \right].$$

(9)

The formulae represents mix of phase shifts governed by probability distribution $P(\phi)$. We can consider this probability to be Gaussian, $P(\phi) =$
\[ e^{-\phi^2/2\Delta^2}/\sqrt{2\pi\Delta^2} \], and fully defined by the standard deviation of the fluctuations \( \Delta \).

The two decoherence mechanisms, loss and dephasing, are the main factors limiting the linear squeezing achievable in contemporary experiments [75] and the main reason why the generated states are never pure. Pure squeezed vacuum state, \( \hat{S}(r)|0\rangle \) with \( \hat{S}(r) = e^{-i\phi(r\hat{x} + \hat{p})/2} \) has zero mean values of quadrature operators and is therefore fully defined by their second moments, \( \langle \hat{x}^2 \rangle = e^{2r}/2 \) and \( \langle \hat{p}^2 \rangle = e^{-2r}/2 \). The effects of both kinds of decoherence can be quantified by a pair of quadrature variances \( V_{L,\Delta} = \text{Tr} \left[ \hat{x}^2 \rho_{L,\Delta} \right] \), where the subscripts differentiate between loss and dephasing, respectively, which can be straightforwardly evaluated with help of relations (8) and (9):

\[ V_L = \eta \frac{e^{-2r}}{2} + \frac{1 - \eta}{2}, \quad V_\Delta = \frac{\cosh 2r + e^{-2\Delta} \sinh 2r}{2}. \]

We can immediately see that the two kinds of decoherence affect the squeezed state differently. In the case of losses, the resulting variance depends only on the squeezed variance of the initial state. As a consequence, no matter what the loss is, it is always to one’s advantage to have the squeezing as large as is possible. Another interesting feature of the loss scenario is that the resulting state is always squeezed, as in having fluctuations below the vacuum level, as long as \( \eta > 0 \). Neither of these properties holds for random dephasing, which mixes together the squeezed and the anti-squeezed quadratures. For any level of dephasing, there is an optimal amount of squeezing, represented by the parameter \( r \), which results in optimal squeezed variance \( V_\Delta \). Both the larger and the smaller initial squeezing lead to increased variance. Also, for any level of dephasing, there is a maximal squeezing of the input state which the phase randomized state no longer shows any nonclassical squeezing, keeping in mind, dephasing does not completely erase the non-classical properties of the photon number distribution of the state, only the non-classicality manifesting in the squeezing of the noise of quadrature operators.

Let us now compare the effects of decoherence for states exhibiting nonlinear squeezing. We shall start with states prepared by applying ideal cubic nonlinearity to pure squeezed states. Such states are Heisenberg limited and can be therefore considered optimal approximations for the given level of nonlinear squeezing. They can be expressed as

\[ |\chi_3\rangle = \hat{C}(\chi)|\hat{S}(r)|0\rangle, \]

where \( \hat{C}(\chi) = \exp \left[ -\frac{ir\chi \hat{x}^3}{3} \right] \) is the operator for cubic nonlinearity. When this state is affected by loss we can, with help of (8), find the cubic variance of the decohered state as

\[ \text{var}_{\chi}(\hat{O}_3(z)) = \frac{1}{2}(\eta g^2 + 1 - \eta) + \frac{1}{2} g^2 \eta(\chi - z\sqrt{\eta})^2 + \frac{1}{2} \eta^2 (1 - \eta)^2, \]

where \( g = e^{-r} \) is a shorthand notation for the squeezing parameter of the state. There are two parameters, \( z \) and \( \chi \). Final cubicity \( z \) relates to decohered state which is either measured or utilized for further processing [40, 64]. In turn, the initial cubicity \( \chi \) relates to the initially prepared cubic nonlinear state. This allows us to optimize the values of the initial parameters \( \chi \) and \( g \) in order to obtain a quantum state that will have the lowest possible value of \( \xi_3(z) \) for the desired value of \( z \) and the given loss with transmission coefficient \( \eta \). We can also see that the final nonlinear variance has contributions of both \( g \) and \( g^{-1} \). This is the consequence of the commutation relation \([\hat{p} + z\hat{x}^2, \hat{x}] = i\) and the corresponding bound on their variances. It is also a significant contribution to the deterioration of the nonlinear variance.

For any particular value of \( z \) and \( \eta \) the variance (12) is minimized for \( g = \sqrt{2\eta(1 - \eta)} \) and \( \chi = z\sqrt{\eta} \). This means that in order to prepare the optimal nonlinear squeezed state with the final cubicity \( z \), a state with a lower value of the initial cubicity \( \chi \) needs to be prepared. This might seem counterintuitive, but we need to keep in mind that the parameter \( z \) determines only the shape of the quantum state. The non-Gaussian nonclassical properties of the state are determined by the relative variance (4), which always increases under losses. Fig. 2 shows the maximal obtainable nonlinear squeezing for any given combination of \( \eta \) and \( z \). The optimal value of \( \xi_3(z) \) was minimized with respect to \( \chi \) and \( g \) of the initial state. The color lines in Fig. 2 highlight specific levels of the output cubic squeezing within the measurable range [34].

We can see that robustness of the nonlinear squeezing significantly depends on the desired value of final
...cubicity $z$. In fact, for a specific value of $z = \frac{1}{\sqrt{2}}$, the nonlinear squeezing behaves similarly to the linear one; it is preserved for any amount of loss. This is because the Gaussian state with the optimal nonlinear variance for $z = \frac{1}{\sqrt{2}}$ is the vacuum state, which is also the pointer state for loss channels [9]. During the decoherence, the cubic state approaches the vacuum state, but, as long as $\eta > 0$ it does not reach it and the final state shows better nonlinear squeezing.

It should be also noted that the cubic squeezing of cubic states is not inherently less robust than that of Gaussian states. This can be verified by using ratio

$$\xi_n'(z, \eta) = \frac{\text{var}_{\rho_n} \hat{O}_n(z)}{\text{min}_{\rho_n, \Delta} \text{var}_{\rho_n, \Delta} \hat{O}_n(z)},$$

where the minimization is over all Gaussian states which can be produced from pure Gaussian states by lossy channels with the fixed transmission $\eta$, rather than pure Gaussian states as in (4). Such states need to be found numerically, but our analysis revealed that, for state (11) optimized with respect to $\chi$ and $g$, $\xi_n'(z, \eta) < 1$ for all $z$ and $\eta$.

Similar analysis can be performed for the case of dephasing channel (9) affecting the initial cubic squeezed state (11). Analytical formula for the nonlinear variance can be found in Appendix B and Fig. 3 shows the nonlinear squeezing variance $\xi_n(z)$, minimized with respect to $\chi$ and $g$ of the initial state, for given final cubicity $z$ and dephasing coefficient $\Delta$. Again we can see that the nonlinear squeezing of the optimized state survives largest dephasing for $z = \frac{1}{\sqrt{2}}$. Unlike the loss channel, however, as $\Delta$ increases, dephasing will eventually completely destroy the advantage the nonlinear squeezed states have over the optimal Gaussian states. Interestingly, the nonlinear squeezing of cubic states is again no less robust than that of the Gaussian states when both pass through the same dephasing channel. This can be observed by evaluating

$$\xi_n'(z, \Delta) = \frac{\text{var}_{\rho_n} \hat{O}_n(z)}{\text{min}_{\rho_n, \Delta} \text{var}_{\rho_n, \Delta} \hat{O}_n(z)},$$

where the minimization in the denominator is now taken over Gaussian states that were subjected to dephasing channel with coefficient $\Delta$. Similarly to the loss scenario, $\xi_n'(z, \Delta) \leq 1$ for all values of $z$ and $\Delta$.

4 Cubic squeezing of various approximative quantum states and its decoherence

Until now we have analyzed only decoherence of the ideal cubic state. Let us now relax this assumption and investigate quantum states that are closer to experimental reality. Let us start by quantum states that are produced by perfect nonlinearity but are no longer initially pure minimal uncertainty states. Such states can be described by density operator

$$\tilde{\rho}_D = C(\chi) S(r) \tilde{\rho}_h S^\dagger(r) C^\dagger(\chi),$$

where $\tilde{\rho}_h$ is a thermal state with $\text{var}_{\rho_h} \hat{x} = \text{var}_{\rho_h} \hat{p} = \frac{\hbar}{2}$. In the Heisenberg representation we can now arrive at the variance of the nonlinear operator in the...
form
\[
\text{var}_{\psi_c}(\hat{O}_3(z)) = \frac{g^2 \eta (D + 1)}{2} + \frac{\eta (D + 1)^2}{2g^4} \left( \chi - z \sqrt{\eta} \right)^2 + \frac{z^2 \eta (1 - \eta)}{g^2} (D + 1) + \frac{(1 - \eta) + z^2 (1 - \eta)^2}{2}.
\]

(16)

Similarly to the ideal scenario, the variance is minimized for \( \chi = \sqrt{\eta} z \). However, even under the optimization over \( g \) and \( \chi \), the added noise \( D \) reduces the amount of losses that can be tolerated before the imperfect cubic state loses advantage over the optimal Gaussian state. This is demonstrated in Fig. 4a), which shows the highest amount of losses for which the quantum state (15) with optimized parameters \( \chi \) and \( g \) produces \( \xi_3(z) < 1 \). We can also see that cubicity \( z \) for which the state shows the highest robustness against losses, drifts towards lower values as \( D \) increases. We can perform similar analysis for mixed states affected by dephasing. The calculation is more involved as it requires polynomials of both \( \hat{x} \) and \( \hat{p} \) up to eighth order and it can be found in Appendix B. Fig. 4b) now shows the maximal amount of phase fluctuations, \( \Delta \), that still allow the initial state (15), optimized over \( g \) and \( \chi \) and affected by dephasing channel (9), to show \( \xi_3(z) < 1 \). Similarly to the case of losses, additional noise \( D \) lowers the tolerance of the mixed state to dephasing and causes drift of the most robust cubicity \( z \) towards lower values.

Let us now turn to analysis of quantum states that can be realized experimentally in quantum optics. The minimal experimental implementation [34] takes form of a superposition of zero- and one-photon states:
\[
|\psi\rangle = u|0\rangle + i \sqrt{1 - u^2} |1\rangle,
\]

(17)

where \( 0 \leq u \leq 1 \). The pure imaginary coefficient of the Fock state \( |1\rangle \) leads to Wigner function invariant under transformation \( \hat{x} \rightarrow -\hat{x} \), which is what we expect of approximative eigenstate of operator \( \hat{p} + z \hat{x}^2 \).

In practical scenarios, the performance of the approximate superposition can be enhanced by squeezing operation, either by actively squeezing the prepared state, or by directly preparing it by displaced subtraction from a squeezed state [67]. In this case, the approximate state can be expressed as
\[
|\psi\rangle = \hat{S}(r)(u|0\rangle + i \sqrt{1 - u^2} |1\rangle)
\]

(18)

and the state can be optimized over both parameters, \(-\infty < r < \infty \) and \( 0 \leq u \leq 1 \), to minimize the relative nonlinear squeezing variance \( \xi(z) \).

Fig. 5 illustrates the robustness of approximative states (17) and (18) in comparison to the ideal cubic state. Fig. 5a) shows the combination of loss \( \eta \) and final cubicity \( z \) for which the compared quantum states can achieve nonlinear squeezing, \( \xi(z) < 1 \). Similarly, Fig. 5b) shows the same for the case of phase noise with parameter \( \Delta \). The results were obtained by fixing the displayed parameters and optimizing the free parameters of the states. For the case of loss, Fig. 5a), we can see that all three types of states are most robust for final cubicity \( z = \frac{1}{\sqrt{2}} \). For this parameter, the ideal cubic state as well as the squeezed superposition state theoretically show nonlinear squeezing for any losses, even though the violation is negligible for \( \eta \rightarrow 0 \). Supersposed state without squeezing (17) is not as robust, for \( z = \frac{1}{\sqrt{2}} \) its nonlinear squeezing vanishes roughly for \( \sqrt{\eta} \approx 0.2 \) and even for higher transmission coefficient the area in which the squeezing exists at all is significantly narrower. Comparison between (17) and (18) therefore shows that the additional squeezing can significantly expand the robustness [30] and thus the applicability of the approximative cubic phase states. The optimal Gaussian squeezing required depends on \( z \) as well as the losses and it can be found in Appendix C. In the most robust regime of \( z \approx \frac{1}{\sqrt{2}} \) the required linear squeezing is around \(-1.3 \text{ dB} \), which can be considered experimentally feasible.

The difference between the ideal states and the superposition states is more pronounced for the case of phase noise shown in Fig. 5b). The most prominent difference is that the superposition states (17) and
(18) exhibit the greatest robustness at nonlinearity parameter $z \approx 0.55$, rather than for $z = \sqrt{2}$, which is the most robust point for the ideal cubic state. Furthermore, this even allows the approximate superpositions to reach better nonlinear squeezing than the ideal cubic state. The reason for this behavior is given by the photon number distribution of the states. At the point of the highest robustness, nonlinear squeezing of both superpositions (17) and (18) is fully determined by the off-diagonal terms $\langle 0 | \hat{\rho} | 1 \rangle$. In the density matrices. For the ideal state, however, the nonlinear squeezing also depends on all the other off-diagonal terms, $\langle k | \hat{\rho} | l \rangle$ with $k > l + 1$, which vanish more quickly under the phase noise. For some parameters, the higher nonlinear squeezing of the ideal state therefore vanishes before that of the approximate state. Finally, we can see that the additional squeezing operation in (18) significantly increases the range of parameters $z$ for which the nonlinear quantum state survives the decoherence. The optimal Gaussian squeezing required for achieving this robustness can again be found in Appendix C. In the relevant domain $0.5 \lesssim z \lesssim 1$ it is at most -3 dB which can be considered feasible.

It is also possible to consider superpositions of Fock states that go up to higher $n$. Such states can be prepared either through repeated photon subtraction or addition [6, 56, 67] or from an entangled state by projective single photon measurements [79, 80]. For any specific dimension it is possible to find the optimal state that minimizes the nonlinear variance [49]. In Fig. 6a) we can see the effects of loss on these states. The higher dimension of the Hilbert space comes with initially higher nonlinear squeezing, but also with faster deterioration under losses. Interestingly, for all the states the nonlinear squeezing gets lost roughly for $\eta \approx 0.8$. Note that this is because the states were optimized only for the lossless regime. It is possible to optimize the states for any given level of loss and in this case the higher dimension of the states’ Hilbert space is always an advantage, but the complexity of optimization rapidly increases with the dimension. Similarly, Fig. 6b) shows the effects of phase noise and we can see a very similar behavior in which the quantum states with higher initial nonlinear squeezing are more prone to the effects of decoherence. Proper numerical optimization will remove this issue in practical considerations.

Finally, a high fidelity approximation of cubic state can be also created by applying suitable Gaussian operations to the tri-squeezed state $\hat{U}_3(t) = e^{i(t^2 \hat{a}^3 + t \hat{a} \hat{a}^3)} | 0 \rangle$ [83]. Within the context of our work, high fidelity is not relevant, as for example a fidelity between vacuum state and vacuum squeezed by 1 dB only slightly decreases to 0.9967, and thus is not suitable for squeezing quantification. However, the proposed approach can also be optimized with respect to the nonlinear squeezing of the prepared states. The minimal attainable nonlinear squeezing parameter $\xi(z)$ in the presence of loss and phase noise is shown in Fig. 7a) and Fig. 7b), respectively. It was obtained by considering quantum state $\hat{S}(r) \hat{U}_3(t) | 0 \rangle$ subjected to the decoherence effects and optimizing over parameters $r$ and $t$. The behavior is qualitatively similar to ideal cubic phase states in Fig. 2 and Fig. 3, however the actual values of achievable squeezing are lower. For example, for $z = 1/\sqrt{2}$ and $\eta = 0.9$, the tri-squeezed state can achieve roughly 0.5 dB of nonlinear squeezing, while for the ideal state it is 2 dB. We can therefore see that, despite the high fidelity, the nonlinear squeezing properties of the states differ.

5 Protection against decoherence

In the previous section, we have seen that the ability of nonlinear squeezing to persevere under physical imperfections, both loss and phase noise, is to a large extent determined by the value of the final cubicity $z$. For losses, for example, preparing a nonlinearly
squeezed state with the correct initial cubicity $\chi$ can mean the difference between the non-classical behavior being lost for arbitrarily small imperfections and it staying preserved no matter how large the losses are. This is quite unlike the situation with the other commonly employed non-classical quantum states. For example, squeezed states will never lose all squeezing under losses, but they are consistent in this. The other hand, superposition states, such as cat states or GKP qubits [25, 55, 72], are very fragile, losing a significant amount of non-classicality under even a small amount of losses. This behavior can be slowed down by suitable squeezing operation, [30], but can never be completely prevented.

A suitable squeezing operation can also help to preserve the nonlinear squeezing of quantum states (4). Applying squeezing operation $\hat{S}(r)$ to a quantum state exhibiting nonlinear squeezing changes the value of the $n$-th order nonlinear parameter as $z \rightarrow \frac{\chi}{\sqrt{2}}$, so that the parameter can be freely adjusted by the Gaussian operation [2]. Protection against decoherence then consists of transforming the quantum state into the form most resilient against the given form of imperfection. For example, when given an ideal cubic state with nonlinear squeezing in variable $\bar{p} - \chi x^2$ that is expected to interact with a lossy channel with parameter $\eta$, the protection routine consists of applying squeezing operation with $g = (\chi\sqrt{2\eta})^{1/3}$ that transforms the state into its most robust form. After the channel, another squeezing operation can be used to transform the new cubicity, $z = \frac{1}{\sqrt{2}}$, into whatever is needed for the particular application. The particular choice of the squeezing depends on the nature of the state. It will be the same for the approximate states (17) and (18) and it will differ for mixed cubic state (15), as per the behavior shown in Fig. 5a) and Fig. 4a). A similar approach can be employed for protection against the phase noise. However, even though adjusting the cubicity by squeezing operation is again a viable approach, in certain cases it might be useful to employ either different methods of preparation or quantum scissors [54] to adjust the dimension of the quantum state’s Hilbert space.

6 Conclusion

Nonlinear squeezing is an operationally defined quantifier of quantum non-Gaussianity for the measurement-induced nonlinear quantum processing [40, 41, 49]. It is defined as the noise-reduction non-Gaussian states can achieve over Gaussian states when used for implementation of deterministic non-

Figure 6: Nonlinear squeezing of quantum states generated as optimal superpositions of photon number states $|0\rangle, \cdots, |n_{\max}\rangle$, [49], when subjected to a) losses, and b) phase noise. The target cubicity was set to $z = \frac{1}{\sqrt{2}}$.

Figure 7: Nonlinear squeezing of a trisqueezed state that is transformed by a Gaussian squeezing into the approximation of the cubic state $S(r)U_3(t)|0\rangle$ and then subjected to a) loss, and b) phase noise. Parameters $t$ and $r$ are optimized over to obtain the lowest possible nonlinear squeezing for the given level decoherence and cubicity $z$. 
linear phase gate, and it is therefore also a resource for quantum technologies [12]. Nonlinear squeezing of operator $\hat{p} + z \hat{x}^2$, defined by the normalized quantifier (4) is a non-Gaussian effect. It does not depend on the value of $z$ that can be tailored by Gaussian operations as long as $z \neq 0$. However, the role of $z$ importantly rises when the state with nonlinear squeezing is subjected to decoherence, such as loss or phase noise, typical in quantum optics.

The most important observation that clearly separates the nonlinear from the linear squeezing is that the effects of decoherence on the nonlinear squeezing are also nonlinear. For any given value of the decoherence parameters, for both losses and phase noise, there exists the optimal amount of initial nonlinear squeezing. As a consequence, in the presence of strong decoherence, preparing a quantum state with very high nonlinear squeezing is not only redundant, it is counterproductive. This is the consequence of the structure of the nonlinear operator. Low variance of $\hat{p} + z \hat{x}^2$ needs to be compensated by high variance of $\hat{x}$ due to uncertainty relations. High nonlinear squeezing is a manifestation of strong correlations between $\hat{p}$ and $\hat{x}^2$. When these correlations diminish under decoherence, the large initial variances of $\hat{p}$ and $\hat{x}^2$ result in larger variance of the joint nonlinear operator. The second realization is that the robustness of the nonlinear squeezing depends on the value of the final cubicity $z$. For many quantum states, the most robust regime was found in the neighborhood of $z = \frac{1}{\sqrt{2}}$, for which the optimal Gaussian state is the vacuum state. For the case of loss, it is the consequence of the vacuum state being the pointer state for the process. Loss gradually transforms the states with nonlinear squeezing into the vacuum state, but as long as the transformation is not complete, some of the initial nonlinear squeezing can be preserved, similarly to the linear variance than its constituents. We define $\rho_3$ as:

$$\hat{p}_3 = w_1\hat{p}_1 + w_2\hat{p}_2.$$  

(19)

The required property is:

$$\text{var}_{\rho_3}\hat{O} \geq \min(\text{var}_{\hat{p}_1}\hat{O}, \text{var}_{\hat{p}_2}\hat{O}),$$  

(20)

where $\hat{O} = \hat{p} + z\hat{x}^2$. Without loss of generality we can assume that:

$$\text{var}_{\rho_3}\hat{O} \leq \text{var}_{\rho_2}\hat{O}.  

(21)

Variance $\text{var}_{\rho_2}\hat{O}$ can be rewritten with use of (19) as:

$$\text{var}_{\rho_2}\hat{O} = \text{var}_{\rho_1}\hat{O} + w_2(\text{Tr}[O^2\hat{p}_2] - \text{Tr}[O^2\hat{p}_1]) + 2w_2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_2]^2 - 2(w_2 - w_2^2)\text{Tr}[O\hat{p}_1]\text{Tr}[O\hat{p}_2],$$  

(22)

which can be also written as $\text{var}_{\rho_1}\hat{O} + \Omega$. With help of auxiliary inequality $a^2 - b^2 > 2ab$ we can now find a lower bound of $\Omega$:

$$\Omega > w_2(\text{Tr}[O^2\hat{p}_2] - \text{Tr}[O^2\hat{p}_1]) + 2w_2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_2]^2 - (w_2 - w_2^2)(\text{Tr}[O\hat{p}_1]^2 + \text{Tr}[O\hat{p}_2]^2) = w_2(\text{var}_{\rho_1}\hat{O} - \text{var}_{\rho_2}\hat{O}) > 0$$  

(23)

The inequality in the last line of (23) is valid due to the assumption (21). The optimal Gaussian state is therefore pure and it can be possibly generated from

Acknowledgements

P.M. and V. K. acknowledge Grant No. 22-08772S of the Czech Science Foundation and also support by national funding from MEYS and the European Union’s Horizon 2020 (2014-2020) research and innovation framework programme under grant agreement program under Grant No. 731473 (project 8C20002 ShoQC). Project ShoQC has received funding from the QuantERA ERA-NET Cofund in Quantum Technologies implemented within the European Union’s Horizon 2020 program. R.F. acknowledges the project 21-13265X of the Czech Science Foundation. V. K. acknowledges project IGA-PrF-2022-005 of Palacky University Olomouc.

A Optimal Gaussian state and properties of the nonlinear operator

While searching for the optimal Gaussian state as the benchmark for non-linear states, we can in general assume pure states or mixed states which are arbitrarily displaced, phase shifted and squeezed. However, the optimal state is a pure squeezed state.

To prove optimality of pure states it is sufficient to show, that any mixture $\rho_1$ of arbitrary states described by density matrices $\rho_1$ and $\rho_2$ has higher nonlinear variance than its constituents. We define $\rho_3$ as:

$$\rho_3 = w_1\rho_1 + w_2\rho_2.$$  

(19)

The required property is:

$$\text{var}_{\rho_3}\hat{O} \geq \min(\text{var}_{\rho_1}\hat{O}, \text{var}_{\rho_2}\hat{O}),$$  

(20)

where $\hat{O} = \hat{p} + z\hat{x}^2$. Without loss of generality we can assume that:

$$\text{var}_{\rho_3}\hat{O} \leq \text{var}_{\rho_2}\hat{O}.  

(21)

Variance $\text{var}_{\rho_2}\hat{O}$ can be rewritten with use of (19) as:

$$\text{var}_{\rho_2}\hat{O} = \text{var}_{\rho_1}\hat{O} + w_2(\text{Tr}[O^2\hat{p}_2] - \text{Tr}[O^2\hat{p}_1]) + 2w_2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_2]^2 - 2(w_2 - w_2^2)\text{Tr}[O\hat{p}_1]\text{Tr}[O\hat{p}_2],$$  

(22)

which can be also written as $\text{var}_{\rho_1}\hat{O} + \Omega$. With help of auxiliary inequality $a^2 - b^2 > 2ab$ we can now find a lower bound of $\Omega$:

$$\Omega > w_2(\text{Tr}[O^2\hat{p}_2] - \text{Tr}[O^2\hat{p}_1]) + 2w_2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_1]^2 - w_2^2\text{Tr}[O\hat{p}_2]^2 - (w_2 - w_2^2)(\text{Tr}[O\hat{p}_1]^2 + \text{Tr}[O\hat{p}_2]^2) = w_2(\text{var}_{\rho_1}\hat{O} - \text{var}_{\rho_2}\hat{O}) > 0$$  

(23)

The inequality in the last line of (23) is valid due to the assumption (21). The optimal Gaussian state is therefore pure and it can be possibly generated from
vacuum state by squeezing, displacement and phase shift.

If we displace state in the \( \hat{p} \) quadrature the variance \( \text{var}_\rho \hat{O} \) wouldn’t change. If we displace state from origin of the phase space the variance will transform as:

\[
\text{var}_\rho \hat{O} \rightarrow \text{var}_\rho \hat{O} + 4z^2d_2^2 \text{var}(\hat{x}) + 4zd_2 \text{cov}(\hat{x}, \hat{p}).
\]  

(24)

Thus the additional terms could be negative depending on the covariance i.e., the rotation and squeezing of the original state. Let’s use the formalism of covariance matrices, we can represent state in the form:

\[
V_\rho = \begin{pmatrix} A & C \\ C & B \end{pmatrix}
\]  

(25)

which will cover arbitrary squeezing and rotation. The vector of mean values:

\[
\hat{\xi} = \left( \begin{array}{c} a \\ 0 \end{array} \right)
\]  

(26)
describes, yet unknown, displacement in the \( \hat{x} \) quadrature and \( \hat{X} \) is vector of quadrature operators.

We need moments \( \langle \hat{x}^4 \rangle, \langle \hat{x}^2 \hat{p} \rangle, \langle \hat{x}^2 \rangle \) and \( \langle \hat{p}^2 \rangle \) for obtaining the variance \( \text{var}_\rho \hat{O} \), other terms will be zero if we begin with vacuum state with \( \langle \hat{p} \rangle = 0 \).

These moments can be calculated with help of Wigner function, which is related to the covariance matrix:

\[
W(x,p) = \frac{\exp\left(-\frac{1}{2}(\hat{X} - \hat{\xi})^T V^{-1}(\hat{X} - \hat{\xi}) \right)}{\pi \sqrt{\det(V)}}.
\]  

(27)

A moment of symmetric operator \( f(\hat{x}, \hat{p}) \) can be calculated from

\[
\langle f(\hat{x}, \hat{p}) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,p) W(x,p) dp dx.
\]  

(28)

Using (28), required moments are expressed in the parameters of the covariance matrix as follows:

\[
\langle \hat{x}^4 \rangle = a^4 + 6a^2A + 3A^2
\]  

(29)

\[
\frac{1}{2} \langle \hat{x}^2 \hat{p} + \hat{p} \hat{x}^2 \rangle = 2aC
\]  

(30)

\[
\langle \hat{x}^2 \rangle = a^2 + A
\]  

(31)

\[
\langle \hat{p}^2 \rangle = B.
\]  

(32)

Using these expressions and relation of nonlinear variance to lower moments:

\[
\text{var}_\rho \hat{O} = \langle \hat{p}^2 \rangle - z \langle \hat{p} \hat{x}^2 + \hat{x}^2 \hat{p} \rangle + z^2 \langle \hat{x}^4 \rangle - \langle \hat{p} \rangle^2 - 2z \langle \hat{p} \rangle \langle \hat{x}^2 \rangle + z^2 \langle \hat{x}^2 \rangle^2,
\]  

(33)

nonlinear variance yields:

\[
\text{var}_\rho \hat{O} = B - 4zaC + 4z^2a^2A + 2z^2A^2.
\]  

(34)

Optimisation of the displacement \( a \) leads to \( a = \frac{C}{2zA} \) and simpler form of the nonlinear variance:

\[
\text{var}_\rho \hat{O} = B + 2z^2A^2 - \frac{C^2}{A}.
\]  

(35)

Now we can use a relation which is consequence of the uncertainty principle and valid for pure states:

\[
AB - C^2 = \frac{1}{4},
\]  

(36)

and discuss two possible cases. At first we can consider \( C = 0 \). Then \( AB = \frac{1}{4} \) and whole variance reads:

\[
\text{var}_\rho \hat{O} = \frac{1}{4A} + 2z^2A^2.
\]  

(37)

While considering non-zero \( C \), we can express it from the equation (36) and substitute result into (35). Final expression is of the form:

\[
\text{var}_\rho \hat{O} = \frac{1}{4A} + 2z^2A^2.
\]  

(38)

Therefore the variance is governed by the same equation independently on the value of \( C \). However the same value of nonlinear variance is obtained with different values of \( C \). As a consequence displacement or phase shift can’t contribute to higher level of nonlinear squeezing. However it is possible that it could reduce the requirements on linear squeezing in state preparation.

Let us now comment on the properties of the quantum states that have diminishing variance

\[
\text{var}_\rho \hat{O}_3(z) = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle) + z^2(\langle \hat{x}^4 \rangle - \langle \hat{x}^2 \rangle^2) + z(\langle \hat{p} \hat{x}^2 + \hat{x}^2 \hat{p} \rangle - \langle \hat{p} \rangle \langle \hat{x}^2 \rangle).
\]  

(39)

Due to commutation relations \([\hat{O}_3(z), \hat{x}]\), the variance of \( \hat{x} \) has to diverge. However, this also means that variance of \( \hat{x}^2 \) has to diverge as well. To see this, consider application of unitary operation \( \hat{U} = \exp\left\{ i \frac{z^2 \hat{x}^2}{4} \right\} \) that transforms \( \hat{O}_3(z) \) into \( \hat{p}' \) and leaves \( \hat{x}' = \hat{x} \) unchanged. Zero variance of \( \hat{O}_3(z) \) thus translates into zero variance of \( \hat{p}' \) that necessitates the state is a quadrature eigenstate that has diverging variances of both \( \hat{x} \) and \( \hat{x}^2 \). The third correlation term in (39) now has to diverge as well to compensate the first two terms.
B Analytical formulas for calculation of decohered nonlinear squeezing.

For the ideal cubic state \( \hat{\mathcal{C}}(\chi)\hat{S}(\tau)|0\rangle \) affected by dephasing, the variance can be analytically found to be:

\[
\operatorname{var}_|\chi\rangle(\hat{\mathcal{O}}_3(z)) = \frac{5}{4}\chi^2 - \frac{e^{-2\Delta^2}}{2} + \frac{e^{-8\Delta^2}}{8} + \frac{3}{8} + (1 - e^{-8\Delta^2}) \frac{3}{16} z^2 - \left(e^{-\frac{\Delta^2}{2}} - e^{-\frac{2\Delta^2}{3}}\right) \chi z + \left(\frac{e^{-2\Delta^2}}{2} + e^{-8\Delta^2} \frac{8}{8} + \frac{3}{16} \chi^2 z^2 + e^{-2\Delta^2} \frac{27}{16} \chi^2 g^2\right) + (1 - e^{-2\Delta^2}) \frac{25}{32} \chi^2 g^2 - \left(e^{-\frac{2\Delta^2}{3}} - e^{-\frac{2\Delta^2}{2}}\right) \frac{15}{16} \chi^3 z + (\chi^2 - e^{-8\Delta^2}) \frac{3}{4} \chi z + (1 + e^{-2\Delta^2}) \frac{g^2}{2} (1 - e^{-2\Delta^2}) + \frac{1}{2} \left(\frac{2g^2 (1 - e^{-2\Delta^2}) + g^2}{2} + g^2 (1 - e^{-2\Delta^2})\right) + \frac{3}{4} \left(-\frac{e^{-2\Delta^2}}{2} + e^{-8\Delta^2} \frac{8}{8} +\right) + \frac{3}{8} \chi^2 \left(1 - 2\Delta^2\right) + g^2 \frac{3}{4} \chi \frac{(1 - e^{-2\Delta^2})}{4g^2}\]

For quantum state approximatively prepared as superposition \( u|0\rangle + iv|1 - u^2|1\rangle \), the variance after the decoherence can be found to be

\[
\operatorname{var}_|\chi\rangle(\hat{\mathcal{O}}_3(z)) = -\chi^2 \left(u^2 - 1\right)^2 z^2 + \eta \left(u^2 - 1\right) \left(2u^2 - 2z^2 - 1\right) + 2\eta \chi^2 \left|u^2 - 1\right| \sqrt{2 - 2u^2 z} + \frac{1}{2} \left(z^2 + 1\right). \tag{41}
\]

in the case of loss, and

\[
\operatorname{var}_|\chi\rangle(\hat{\mathcal{O}}_3(z)) = 2e^{-\Delta^2} u^4 - 2e^{-\frac{\Delta^2}{2}} \sqrt{2 - 2u^2 z} - 2e^{-\Delta^2} u^2 + 2e^{-\Delta^2} \sqrt{2 - 2u^2 z} - u^4 z^2 - u^2 + \frac{3}{2} z^2 + \frac{3}{2} \tag{42}
\]

in the case of phase noise.

C Optimal parameters for nonlinear squeezing minimisation.

Fig. 8 and Fig. 9 show optimal initial cubicity and squeezing of the cubic state (11) when undergoing losses or dephasing, respectively. Fig. 10 shows optimal Gaussian squeezing of the decohered squeezed superposition (18).

![Figure 8](image1.png)  
**Figure 8:** Optimal initial cubicity \( \chi = z\sqrt{\eta} \) a) and squeezing g = \( \sqrt{2z^2(1 - \eta)} \) b) of the ideal cubic state (11), when considering losses.

![Figure 9](image2.png)  
**Figure 9:** Optimal initial cubicity a) and squeezing b) of the ideal cubic state (11), when considering dephasing.
Figure 10: Optimal squeezing $g$ in dB for the squeezed superposition (18) in the case of losses a) and dephasing b).

References

[1] J. Aasi, J. Abadie, B. Abbott, et al. Enhanced sensitivity of the ligo gravitational wave detector by using squeezed states of light. *Nat. Phot.*, (7): 613–619, 2013. DOI: 10.1038/nphoton.2013.177.

[2] Francesco Albarelli, Marco G. Genoni, Matteo G. A. Paris, and Alessandro Ferraro. Resource theory of quantum non-gaussianity and wigner negativity. *Phys. Rev. A*, 98: 052350, Nov 2018. DOI: 10.1103/PhysRevA.98.052350.

[3] Ulrik L Andersen, Tobias Gehring, Christoph Marquardt, and Gerd Leuchs. 30 years of squeezed light generation. *Physica Scripta*, 91(5):053001, apr 2016. DOI: 10.1088/0031-8949/91/5/053001. URL: https://doi.org/10.1088/0031-8949/91/5/053001.

[4] Ulrik L Andersen, Tobias Gehring, Christoph Marquardt, and Gerd Leuchs. 30 years of squeezed light generation. *Physica Scripta*, 91(5):053001, apr 2016. DOI: 10.1088/0031-8949/91/5/053001. URL: https://doi.org/10.1088/0031-8949/91/5/053001.

[5] Ulrik Lund Andersen, Jonas Schou Neergaard-Nielsen, Peter van Loock, and Akira Furusawa. Hybrid discrete- and continuous-variable quantum information. *Nature Physics*, 11(9):713–719, 2015. ISSN 1745-2473. DOI: 10.1038/nphys3410.

[6] Francesco Arzani, Nicolas Treps, and Giulia Ferrini. Polynomial approximation of non-gaussian unitaries by counting one photon at a time. *Phys. Rev. A*, 95:052352, May 2017. DOI: 10.1103/PhysRevA.95.052352. URL: https://link.aps.org/doi/10.1103/PhysRevA.95.052352.

[7] Koji Azuma, Kiyoshi Tamaki, and William J. Munro. All-photonic intercity quantum key distribution. *Nat. Comm.*, (6):10171, 2015. DOI: 10.1038/ncomms10171.

[8] Ben Q. Baragiola, Giacomo Pantaleoni, Rafael N. Alexander, Angela Karanjai, and Nicolas C. Menicucci. All-gaussian universality and fault tolerance with the gottesman-kitaev-preskill code. *Phys. Rev. Lett.*, 123:200502, Nov 2019. DOI: 10.1103/PhysRevLett.123.200502. URL: https://link.aps.org/doi/10.1103/PhysRevLett.123.200502.

[9] Carlos Alexandre Brasil and Leonardo Andretta de Castro. Understanding the pointer states. *European Journal of Physics*, 36 (6):065024, sep 2015. DOI: 10.1088/0143-0807/36/6/065024. URL: https://doi.org/10.1088/0143-0807/36/6/065024.

[10] Samuel L. Braunstein and Peter van Loock. Quantum information with continuous variables. *Rev. Mod. Phys.*, 77:513–577, Jun 2005. DOI: 10.1103/RevModPhys.77.513. URL: https://link.aps.org/doi/10.1103/RevModPhys.77.513.

[11] Ulysse Chabaud, Damian Markham, and Frédéric Grosshans. Stellar representation of non-gaussian quantum states. *Physical Review Letters*, 124(6), feb 2020. DOI: 10.1103/physrevlett.124.063605. URL: https://doi.org/10.1103/physrevlett.124.063605.

[12] Eric Chitambar and Gilad Gour. Quantum resource theories. *Rev. Mod. Phys.*, 91: 025001, Apr 2019. DOI: 10.1103/RevModPhys.91.025001. URL: https://link.aps.org/doi/10.1103/RevModPhys.91.025001.

[13] Jeremie J Choquette, John G. Cordes, and David Kiang. Nonlinear coherent states: nonclassical properties. *Journal of Optics B: Quantum and Semiclassical Optics*, 5(1):56–59, jan 2003. DOI: 10.1088/1464-4266/5/1/308. URL: https://doi.org/10.1088/1464-4266/5/1/308.

[14] Hui Dai, Qi Shen, Chao-Ze Wang, Shuang-Lin Li, Wei-Yue Liu, Wen-Qi Cai, Sheng-Kai Liao, Ji-Gang Ren, Juan Yin, Yu-Ao Chen, Qiang Zhang, Feihu Xu, Cheng-Zhi Peng, and Jian-Wei Pan. Towards satellite-based quantum-secure time transfer. *Nat. Phys.*, (16):848–52, 2020. DOI: 10.1038/s41567-020-0892-y.

[15] Mohammed Dakna, Tiemo Anhut, Tomáš Opatrný, Ludwig Knöll, and Dirk-Gunnar Welsch. Generating schrödinger-cat-like states...
by means of conditional measurements on a beam splitter. *Phys. Rev. A*, 55:3184–3194, Apr 1997. DOI: 10.1103/PhysRevA.55.3184. URL https://link.aps.org/doi/10.1103/PhysRevA.55.3184.

[16] Ruynet Lima de Matos Filho and Werner Vogel. Nonlinear coherent states. *Phys. Rev. A*, 54:4560–4563, Nov 1996. DOI: 10.1103/PhysRevA.54.4560. URL https://link.aps.org/doi/10.1103/PhysRevA.54.4560.

[17] Tobias Eberle, Vitus Händchen, and Roman Schnabel. Stable control of 10 db two-mode squeezed vacuum states of light. *Opt. Express*, 21(9):11546–11553, May 2013. DOI: 10.1364/OE.21.011546. URL http://www.opticsexpress.org/abstract.cfm?URI=oe-21-9-11546.

[18] Radim Filip and Ladislav Mišta. Detecting quantum states with a positive wigner function beyond mixtures of gaussian states. *Phys. Rev. Lett.*, 106:290401, May 2011. DOI: 10.1103/PhysRevLett.106.290401. URL https://link.aps.org/doi/10.1103/PhysRevLett.106.290401.

[19] Radim Filip, Petr Marek, and Ulrik L. Andersen. Measurement-induced continuous-variable quantum interactions. *Phys. Rev. A*, 71:042308, Apr 2005. DOI: 10.1103/PhysRevA.71.042308. URL https://link.aps.org/doi/10.1103/PhysRevA.71.042308.

[20] Akira Furusawa, Jens Lykke Sorensen, Samuel L Braunstein, Christopher A Fuchs, H Jeff Kimble, and Eugene S Polzik. Unconditional quantum teleportation. *Science*, 282(5389):706–709, 1998. ISSN 0036-8075. DOI: 10.1126/science.282.5389.706. URL https://science.sciencemag.org/content/282/5389/706.

[21] Marco G. Genoni, Stefano Olivares, and Matteo G. A. Paris. Optical phase estimation in the presence of phase diffusion. *Phys. Rev. Lett.*, 106:153603, Apr 2011. DOI: 10.1103/PhysRevLett.106.153603. URL https://link.aps.org/doi/10.1103/PhysRevLett.106.153603.

[22] Marco G. Genoni, Stefano Olivares, Davide Brivio, Simone Cialdi, Daniele Cipriani, Alberto Santamato, Stefano Vezzoli, and Matteo G. A. Paris. Optical interferometry in the presence of large phase diffusion. *Phys. Rev. A*, 85:043817, Apr 2012. DOI: 10.1103/PhysRevA.85.043817. URL https://link.aps.org/doi/10.1103/PhysRevA.85.043817.

[23] Manuel Gessner, Augusto Smerzi, and Luca Pezzè. Metrological nonlinear squeezing parameter. *Phys. Rev. Lett.*, 122:090503, Mar 2019. DOI: 10.1103/PhysRevLett.122.090503. URL https://link.aps.org/doi/10.1103/PhysRevLett.122.090503.

[24] Shohini Ghose and Barry C. Sanders. Non-gaussian ancilla states for continuous variable quantum computation via gaussian maps. *Journal of Modern Optics*, 54(6):855–869, 2007. DOI: 10.1080/09500340601101575. URL https://doi.org/10.1080/09500340601101575.

[25] Daniel Gottesman, Alexei Kitaev, and John Preskill. Encoding a qubit in an oscillator. *Phys. Rev. A*, 64:012310, Jun 2001. DOI: 10.1103/PhysRevA.64.012310. URL https://link.aps.org/doi/10.1103/PhysRevA.64.012310.

[26] Lucas Happ, Maxim A Efremov, Hyunchul Nha, and Wolfgang P Schleich. Sufficient condition for a quantum state to be genuinely quantum non-gaussian. *New Journal of Physics*, 20(2):023046, Feb 2018. DOI: 10.1088/1367-2630/aaac25. URL https://doi.org/10.1088/1367-2630/aaac25.

[27] Sajede Harraz and Shuang Cong. $n$-qubit state protection against amplitude damping by quantum feed-forward control and its reversal. *IEEE Journal of Selected Topics in Quantum Electronics*, 26(3):1–8, 2020. DOI: 10.1109/JSTQE.2020.2969574.

[28] Joel Heersink, Vincent Josse, Gerd Leuchs, and Ulrik L. Andersen. Efficient polarization squeezing in optical fibers. *Opt. Lett.*, 30(10):1192–1194, May 2005. DOI: 10.1364/OL.30.001192. URL http://ol.osa.org/abstract.cfm?URI=ol-30-10-1192.

[29] Timo Hillmann, Fernando Quijandria, Gørn Johansson, Alessandro Ferraro, Simone Gasparinetti, and Giulia Ferrini. Universal gate set for continuous-variable quantum computation with microwave circuits. *Phys. Rev. Lett.*, 125:160501, Oct 2020. DOI: 10.1103/PhysRevLett.125.160501. URL https://link.aps.org/doi/10.1103/PhysRevLett.125.160501.

[30] Hanna Le Jeannic, Adrien Cavailles, Kun Huang, Radim Filip, and Julien Laurat. Slowing quantum decoherence by squeezing in phase space. *Phys. Rev. Lett.*, 120:073603, Feb 2018. DOI: 10.1103/PhysRevLett.120.073603. URL https://link.aps.org/doi/10.1103/PhysRevLett.120.073603.

[31] Emmanuel Knill, Raymond Laflamme, and Gerard Milburn. A scheme for efficient quantum computation with linear optics. *Nature*, (409):46–52, 2001. DOI: 10.1038/35051009.

[32] Pieter Kok, William J. Munro, Kae Nemoto, Timoth C. Ralph, Jonathan P. Dowling, and Gerard Milburn. Linear optical quantum computing with photonic qubits. *Rev. Mod. Phys.*, 79:135–174, Jan 2007. DOI: 10.1103/RevModPhys.79.135. URL https://link.aps.org/doi/10.1103/RevModPhys.79.135.

[33] Shunya Konno, Atsushi Sakaguchi, Warit Asavanant, Hisashi Ogawa, Masaya Kobayashi,
Petr Marek, Radim Filip, Jun-ichi Yoshikawa, and Akira Furusawa. Nonlinear squeezing for measurement-based non-gaussian operations in time domain. Phys. Rev. Applied, 15:024024, Feb 2021. DOI: 10.1103/PhysRevApplied.15.024024. URL https://link.aps.org/doi/10.1103/PhysRevApplied.15.024024.

[34] Shunya Konno, Atsushi Sagakuchi, Warit Asavanant, Hisashi Ogawa, Masaya Kobayashi, Petr Marek, Radim Filip, Jun-ichi Yoshikawa, and Akira Furusawa. Nonlinear squeezing for measurement-based non-gaussian operations in time domain. Phys. Rev. Applied, 15:024024, Feb 2021. DOI: 10.1103/PhysRevApplied.15.024024. URL https://link.aps.org/doi/10.1103/PhysRevApplied.15.024024.

[35] Marina Kudra, Mikael Kervinen, Ingrid Strandberg, Shahnawaz Ahmed, Marco Scigliuzzo, Amr Osman, Daniel Pérez Lozano, Giulia Ferrini, Jonas Bylander, Anton Frisk Kockum, Fernando Quijandría, Per Delsing, and Simone Gasparinetti. Robust preparation of wigner-negative states with optimized snapshot sequences. arXiv:2111.07965. DOI: https://arxiv.org/abs/2111.07965.

[36] Leong Chuan Kwek and David Kiang. Nonlinear squeezed states. Journal of Optics B: Quantum and Semiclassical Optics, 5(5):383–386, aug 2003. DOI: 10.1088/1464-4266/5/5/301. URL https://doi.org/10.1088/1464-4266/5/5/301.

[37] Lukáš Lachman, Ivo Straka, Josef Hlášek, Petr Marek, Radim Filip, Jan Provaník, and Radim Filip. Loop-based subtraction of a single photon from a traveling beam of light. Optics Express, (23):29837–29847, 2018. DOI: 10.1364/OE.26.029837.

[38] Andrea Mari and Jens Eisert. Positive wigner functions render classical simulation of quantum computation efficient. Phys. Rev. Lett., 109:230503, Dec 2012. DOI: 10.1103/PhysRevLett.109.230503. URL https://link.aps.org/doi/10.1103/PhysRevLett.109.230503.

[39] Colin McCormick, Vincent Boyer, Emilio Amurdo, and Paul Lett. Strong relative intensity squeezing by four-wave mixing in rubidium vapor. Optics letters, 32:178–80, 02 2007. DOI: 10.1364/OL.32.000178.

[40] M. Mičuda, L. Straka, M. Miková, M. Dušek, N. J. Cerf, J. Fiurášek, and M. Ježek. Noiseless loss suppression in quantum optical communication. Phys. Rev. Lett., 109:180503, Nov 2012. DOI: 10.1103/PhysRevLett.109.180503. URL https://link.aps.org/doi/10.1103/PhysRevLett.109.180503.

[41] Morgan Mitchell, Jeff Lundeen, and Ephraim Steinberg. Super-resolving phase measurements with a multiphoton entangled state. Nature, (429):161–164, 2004. DOI: 10.1038/nature02493.

[42] Yoshichika Miwa, Jun-ichi Yoshikawa, Noriaki Iwata, Mamoru Endo, Petr Marek, Radim Filip, Peter van Loock, and Akira Furusawa. Exploring a new regime for processing optical qubits: Squeezing and unsqueezing single photons. Phys. Rev. Lett., 113:013601, Jul 2014. DOI: 10.1103/PhysRevLett.113.013601. URL https://link.aps.org/doi/10.1103/PhysRevLett.113.013601.

[43] Kazunori Miyata, Atsushi Sakaguchi, Jin-yu Konno, Atsushi Sakaguchi, Jun-ichi Yoshikawa, and Akira Furusawa. General implementation of arbitrary nonlinear quadrature phase gates. Phys. Rev. A, 97:022329, Feb 2018. DOI: 10.1103/PhysRevA.97.022329. URL https://link.aps.org/doi/10.1103/PhysRevA.97.022329.

[44] Petr Marek, Jan Provaník, and Radim Filip. Deterministic implementation of weak quantum cubic nonlinearity. Phys. Rev. A, 84:053802, Nov 2011. DOI: 10.1103/PhysRevA.84.053802. URL https://link.aps.org/doi/10.1103/PhysRevA.84.053802.

[45] Petr Marek, Radim Filip, Hisashi Ogawa, Atsushi Sakaguchi, Shuntaro Takeda, Jun-ichi Yoshikawa, and Akira Furusawa. General implementation of arbitrary nonlinear quadrature phase gates. Phys. Rev. A, 97:022329, Feb 2018. DOI: 10.1103/PhysRevA.97.022329. URL https://link.aps.org/doi/10.1103/PhysRevA.97.022329.

[46] Colin McCormick, Vincent Boyer, Emilio Amurdo, and Paul Lett. Strong relative intensity squeezing by four-wave mixing in rubidium vapor. Optics letters, 32:178–80, 02 2007. DOI: 10.1364/OL.32.000178.

[47] Yoshichika Miwa, Jun-ichi Yoshikawa, Noriaki Iwata, Mamoru Endo, Petr Marek, Radim Filip, Peter van Loock, and Akira Furusawa. Exploring a new regime for processing optical qubits: Squeezing and unsqueezing single photons. Phys. Rev. Lett., 113:013601, Jul 2014. DOI: 10.1103/PhysRevLett.113.013601. URL https://link.aps.org/doi/10.1103/PhysRevLett.113.013601.

[48] Kazunori Miyata, Hisashi Ogawa, Petr Marek, Radim Filip, Hidehiro Yonezawa, Jun-ichi Yoshikawa, and Akira Furusawa. Experimental realization of a dynamic squeezing gate. Phys. Rev. A, 90:060302, Dec 2014. DOI: 10.1103/PhysRevA.90.060302. URL https://link.aps.org/doi/10.1103/PhysRevA.90.060302.

[49] Kazunori Miyata, Hisashi Ogawa, Petr Marek, Radim Filip, Hidehiro Yonezawa, Jun-ichi Yoshikawa, and Akira Furusawa. Implementation of a quantum cubic gate by an adaptive non-gaussian measurement. Phys. Rev. A, 93:022301, Feb 2016. DOI: 10.1103/PhysRevA.93.022301. URL https://link.aps.org/doi/10.1103/PhysRevA.93.022301.

[50] Darren W Moore, Andrey A Rakhubovsky, and Radim Filip. Estimation of squeezing in a nonlinear quadrature of a mechanical oscillator. New
https://science.sciencemag.org/content/370/6523/1460.