HÖLDER CONTINUOUS WEAK SOLUTION OF BOUSSINESQ EQUATION WITH DIFFUSIVE TEMPERATURE

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Abstract. We show the existence of Hölder continuous periodic weak solutions of the 2d Boussinesq equation with diffusive temperature which satisfy the prescribed kinetic energy. More precisely, for any smooth $e(t) : [0,1] \to \mathbb{R}$, and $\varepsilon \in (0,\frac{1}{12})$, there exist $v \in C^{\frac{1}{12}-\varepsilon}([0,1] \times \mathbb{T}^2), \theta \in C^{\frac{1}{12}-\varepsilon}([0,1] \times \mathbb{T}^2)$ which solve (1.1) in the sense of distribution and satisfy

$e(t) = \int_{\mathbb{T}^2} |v(t,x)|^2 dx, \; \forall t \in [0,1].$

Keywords: 2d Boussinesq equation with diffusive temperature, Hölder continuous periodic weak solutions, Prescribed kinetic energy

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1. Introduction

In this paper, we consider the following 2d Boussinesq equation

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla p = \theta e_2, & \text{in } \mathbb{T}^2 \times [0,1] \\
\text{div} v = 0, \\
\partial_t \theta + v \cdot \nabla \theta - \Delta \theta = 0, & \text{in } \mathbb{T}^2 \times [0,1],
\end{cases}$$

(1.1)

where $\mathbb{T}^2$ denotes the 2-dimensional torus, and $e_2 = (0,1)$. Here in our notations, $v$ is the velocity vector, $p$ is the pressure, and $\theta$ denotes the temperature or density which is a scalar function. The Boussinesq equation was introduced to model many geophysical flows, such as atmospheric fronts and ocean circulations (see, for example, [36], [41]).

The global well-posedness of strong solution has been established by many authors for the Cauchy problem of (1.1) in 2d with regularity data (see, for example, [9], [25]). For the 3-dimensional case, the global existence of smooth solution of (1.1) remains open.

Moreover, the study of weak solutions in fluid dynamics, including those which fail to conserve energy, is quite natural in the context of turbulent flow, and has been conducted by many people in the past two decades (see [42], [43], [44], [18], [19]). The triplet $(v,p,\theta)$ on $[0,1] \times \mathbb{T}^2$ is called a weak solution of (1.1) if $\theta \in L^2_{\text{loc}}((0,1) \times \mathbb{T}^2), \ p \in L^2_{\text{loc}}((0,1) \times \mathbb{T}^2), \ \theta \in L^\infty((0,1); H^1(\mathbb{T}^2))$ and solve (1.1) in the following sense:

$$\int_0^1 \int_{\mathbb{T}^2} (\partial_t \varphi \cdot v + \nabla \varphi : v \otimes v + p \text{div} \varphi + \theta e_2 \cdot \varphi) dx dt = 0$$

for all $\varphi \in C^\infty_c((0,1) \times \mathbb{T}^2; \mathbb{R}^2)$;

$$\int_0^1 \int_{\mathbb{T}^2} (\partial_t \phi \theta + v \cdot \nabla \phi \theta - \nabla \theta \cdot \nabla \phi) dx dt = 0.$$

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for all $\phi \in C_\infty^\infty((0,1) \times T^2; R)$ and
\[ \int_0^1 \int_{T^2} v \cdot \nabla \psi \, dx \, dt = 0 \]
for all $\psi \in C_\infty^\infty((0,1) \times T^2; R)$.

The study of constructing non-unique or dissipative weak solution to fluid systems is very fashionable in recent years. The construction is based on the convex integration method pioneered by De Lellis-Székelyhidi Jr in [18, 21], where the authors tackle the Onsager conjecture for the incompressible Euler equation, showing that the incompressible Euler equation admits Hölder continuous solutions which dissipate kinetic energy. More precisely, the Onsager conjecture on incompressible Euler equation can be stated as follows:

(1) $C^{0,\alpha}$ weak solutions are energy conservative when $\alpha > \frac{1}{3}$.

(2) For any $\alpha < \frac{1}{3}$, there exist dissipative solutions with $C^{0,\alpha}$ regularity.

For this conjecture, the part (a) has been proved by P. Constantin, E, Weinan and E. Titi in [15]. Slightly weaker assumptions on the solution were subsequently shown to be sufficient for energy conservation by P. Constantin, etc. in [10, 24], also see [41]. More recently, P. Isett and Sung-jin Oh gave a proof to this part of Onsager’s conjecture for the Euler equations on manifolds by the heat flow method in [26].

Part (b) was proved by P. Isett in [29], based on a series of progress on this problem in [1, 3, 4, 5, 11, 16, 17, 22, 27], see also [6] for the construction of admissible weak solutions. Moreover, the idea and method can be used to construct dissipative weak solutions for other models, see [7, 31, 34, 45, 48, 49, 50].

Recently, Buckmaster and Vicol established the non-uniqueness of weak solution to the 3D incompressible Navier-Stokes in [8] by introducing some new ideas. Furthermore, in [2], the authors constructed wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimensions strictly less than 1. By choosing the parameter suitably, T.Luo and E.S.Titi in [32] constructed weak solutions with compact support in time for hyperviscous Navier-Stokes equation. X. Luo in [35] gave the non-uniqueness for high dimensional ($d \geq 4$) stationary Navier-Stokes equation in [37, 38]. Moreover, S.Modena and Székelyhidi established the non-uniqueness for the linear transport equation (and transport-diffusion equation) with divergence-free vector fields in some Sobolev space, see [37, 38].

Motivated by the study of Onsager’s conjecture of the incompressible Euler equation in earlier works and non-uniqueness of weak solutions to the Navier-Stokes equation, we consider the Boussinesq equation with diffusive temperature and want to know if the anomalous dissipation of kinetic energy can also happened when considering the temperature effects. The difference is that there are conversions between internal energy and mechanical energy, and we need to overcome the difficulty of interactions between velocity and temperature. To this end, we consider the existence of Hölder continuous periodic solutions of the 2d Boussinesq equations with diffusive temperature which satisfy the prescribed kinetic energy. Following the general scheme in the construction of Hölder continuous weak solution of the incompressible Euler equation and introducing some new ideas, we obtain the following result:

**Theorem 1.1.** Assume that $e(t) : [0,1] \to R$ is a given positive smooth function and $\theta^0(x)$ is any smooth function with $\int_{T^2} \theta^0(x) \, dx = 0$. Then there exist
\[ (v,p) \in C([0,1] \times T^2), \quad \theta \in C([0,1] \times T^2) \cap L^2(0,1; H^1(T^2)) \]
such that they solve the system (1.1) in the sense of distribution and
\[ e(t) = \int_{T^2} |v|^2(t,x) \, dx, \quad \forall t \in [0,1]. \]
Furthermore, for any $\varepsilon \in (0, \frac{1}{12})$, there holds

$$v \in C^{\frac{1}{12} - \varepsilon}_{t,x}, \quad p \in C^\varepsilon_{t,x}, \quad \theta \in C^{2, \frac{1}{12} - \varepsilon}_{t,x}.$$ 

Moreover, $\theta$ satisfies the following identity:

$$\frac{1}{2} \|\theta(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla \theta(s, \cdot)\|_{L^2}^2 ds = \frac{1}{2} \|\theta_0(\cdot)\|_{L^2}^2, \quad \forall t \in [0, 1].$$

Remark 1.1. We consider the following 2d Boussinesq-equation with fractional dissipation in velocity:

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla p + (-\Delta)^\alpha v = \theta e_2, \\ \text{div} v = 0, \\ \partial_t \theta + v \cdot \nabla \theta - \Delta \theta = 0,
\end{cases} \quad \text{in } T^2 \times [0, 1]. \tag{1.2}$$

For $\alpha < \frac{1}{2}$, we also can construct Hölder continuous weak solutions for (1.2) by the argument in this paper. For $\frac{1}{2} \leq \alpha < 1$, we have constructed finite energy weak solutions for (1.2) in $[33]$.

Remark 1.2. The Hölder exponent of $(v, \theta)$ is not optimal. In fact, we can obtain better regularity for $(v, p)$ by computing the material derivative as in $[3]$, using more precise bounds $(3.3)$ on $R$ and improving the decay estimate for $(2, \theta)$. However, in order to avoid making the article complicated and technical, we don’t do it in this paper.

2. Proof of main result and Plan of the paper

As in $[3]$, the proof of Theorem 1.1 will be achieved through an iteration procedure. In this and subsequent sections, $S_0^{2 \times 2}$ denotes the vector space of trace-free symmetric $2 \times 2$ matrices.

Definition 2.1. Assume $(v, p, \theta, \hat{R})$ are smooth functions on $[0, 1] \times T^2$ taking values, respectively, in $R^2, R, R, S_0^{2 \times 2}$. We say that they solve the Boussinesq-Reynold system if

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla p = \theta e_2 + \text{div} \hat{R}, \\ \text{div} v = 0, \\ \partial_t \theta + v \cdot \nabla \theta - \Delta \theta = 0,
\end{cases} \quad \text{in } T^2 \times [0, 1]. \tag{2.1}$$

We now state the main proposition of this paper, by which Theorem 1.1 is implied directly. For the statement of the main proposition, we introduce some notations.

In the following, $m = 0, 1, \cdots, \alpha \in (0, 1)$ and $\beta$ is a multi-index. We define the Hölder semi-norms as

$$[f]_{m+\alpha} = \max_{|\beta|=m} \sup_{x \neq y} \frac{\|\nabla^\beta f(t,x) - \nabla^\beta f(t,y)\|}{|x-y|^\alpha},$$

and semi-norms

$$[f]_m = \sum_{|\beta|=m} \|\nabla^\beta f\|_0,$$

where

$$\|v\|_0 := \sup_{t,x} |v(t,x)|, \quad \|v(t, \cdot)\|_0 := \sup_{x} |v(t, \cdot)|,$$

and $\nabla^\beta$ are spatial derivatives. Then, define norms

$$\|f\|_m = \sum_{k \leq m} [f]_k, \quad \|f(t, \cdot)\|_m = \sum_{k \leq m} [f(t, \cdot)]_k$$

and Hölder norms

$$\|f\|_{m+\alpha} = \|f\|_m + [f]_{m+\alpha}.$$
Moreover, we introduce two parameters:

\[ \delta_q = a^{-b^q}, \quad a^c b^q + 1 \leq \lambda_q \leq 2a^{c b^q + 1}, \]

where \( a, b, c > 1 \) are integers.

**Proposition 2.1.** Let \( e(t), \theta^0(x) \) be as in Theorem 1.1. Then we can choose two positive constants \( \eta \) and \( M \) only dependent on \( e(t) \) such that the following holds for \( c = 3 \) and \( b = 2 \), if \( a \) is sufficiently large, then there exists a sequence of functions \((v_0, p_0, \theta_0, \hat{R}_n) = (0, 0, 0, 0)\), solving the Boussinesq-Stress system (2.7) and satisfying the following estimates

\[
\begin{align*}
\|\hat{R}_q\|_0 &\leq \eta \delta_{q+1}, &\|\hat{R}_q\|_1 &\leq \eta \delta_{q+1} \lambda_q, \\
\|v_{q+1} - v_q\|_0 &\leq M \delta_{q+1}^2 \lambda_{q+1}, &\|v_{q+1} - v_q\|_1 &\leq M \delta_{q+1}^2 \lambda_{q+1}, \\
\|v_{q+1} - v_q\|_2 &\leq M \delta_{q+1}^2 \lambda_{q+1}^2, &\|v_{q+1} - v_q\|_3 &\leq M \delta_{q+1}^2 \lambda_{q+1}^3, \\
\|p_{q+1} - p_q\|_0 &\leq M^2 \delta_{q+1}, &\|p_{q+1} - p_q\|_0 &\leq M^2 \delta_{q+1} \lambda_{q+1}, \\
\frac{1}{2} \|\theta_q(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla \theta_q(s, \cdot)\|_{L^2}^2 ds &= \frac{1}{2} \|\theta^0(\cdot)\|_{L^2}^2, &\forall t \in [0, 1], \\
\|\theta_{q+1} - \theta_q(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla (\theta_{q+1} - \theta_q)(s, \cdot)\|_{L^2}^2 ds &\leq 4 M^2 \|\theta^0\|_{L^2}^2 \delta_q, &\forall t \in [0, 1], \\
\left| e(t)(1 - \delta_{q+1}) - \int_{T^2} |v_q|^2(x, t) dx \right| &\leq \frac{\delta_{q+1}}{4} e(t), &\forall t \in [0, 1],
\end{align*}
\]  

We will prove Proposition 2.1 in the subsequent sections. Here we first give a proof of Theorem 1.1 by using this proposition.

**Proof of Theorem 1.1.** From (2.2)-(2.6), we know that \((v_n, p_n, \hat{R}_n)\) are Cauchy sequences in \( C([0, 1] \times T^2) \), and \( \theta_n \) is a Cauchy sequence in \( L^\infty(0, 1; L^2(T^2)) \cap L^2(0, 1; H^1(T^2)) \), therefore there exist

\((v, p) \in C([0, 1] \times T^2), \quad \theta \in L^\infty(0, 1; L^2(T^2)) \cap L^2(0, 1; H^1(T^2))\)

such that

\[
\begin{align*}
(v_n, p_n) &\to (v, p) \quad \text{in} \quad C([0, 1] \times T^2), \\
\theta_n &\to \theta \quad \text{in} \quad L^\infty(0, 1; L^2(T^2)) \cap L^2(0, 1; H^1(T^2)), \\
\hat{R}_n &\to 0 \quad \text{in} \quad C([0, 1] \times T^2)
\end{align*}
\]

as \( n \to \infty \).

Passing into limit in (2.1), we conclude that \((v, p, \theta)\) solve (1.1) in the sense of distribution. Moreover, (2.7) implies

\[
e(t) = \int_{T^2} |v|^2(t, x) dx, \quad \forall t \in [0, 1].
\]

From (2.3), (2.4), and interpolation, we conclude

\[
\begin{align*}
\|v_{q+1} - v_q\|_0 &\leq \|v_{q+1} - v_q\|_0^{1-\alpha} \|v_{q+1} - v_q\|_0^\alpha \leq M \delta_{q+1}^2 \lambda_{q+1}^\alpha \leq 2 M a^{(-\frac{3}{2} + abc) b^q + 1}, \\
\|p_{q+1} - p_q\|_0 &\leq \|p_{q+1} - p_q\|_0^{1-\alpha} \|p_{q+1} - p_q\|_0^\alpha \leq M^2 \delta_{q+1} \lambda_{q+1}^\alpha \leq 2 M a^{(-1 + abc) b^q + 1}.
\end{align*}
\]
Thus, for every $\alpha < \frac{1}{12}$, $v_q$ is a Cauchy sequence in $C_{t,x}^\alpha$ and $p_q$ is a Cauchy sequence in $C_{t,x}^{2\alpha}$. Hence, $\theta \in C_{t,x}^\alpha$, $p \in C_{t,x}^{2\alpha}$ for every $\alpha < \frac{1}{12}$. By the Schauder estimate of linear parabolic equations, we deduce that $\theta \in C_{t,x}^{2\alpha}$.

Furthermore, by (2.5), we deduce that the temperature $\theta$ satisfies the energy equality: for every $t \in [0,1]$

$$\|\theta(t,\cdot)\|_{L^2}^2 + 2 \int_0^t \|\nabla\theta(s,\cdot)\|_{L^2}^2 = \|\theta^0(\cdot)\|_{L^2}^2.$$  

This completes the proof of Theorem 1.1. \hfill $\square$

2.1. Outline of the proof of Propositions 2.1. The rest of this paper will be dedicated to prove Proposition 2.1. We perform an inductive procedure, and construct $v_{q+1}$ from $v_q$ by adding some perturbations as follows:

$$v_{q+1} = v_q + w_0 + w_c := v_q + w,$$

where $w_0, w_c$ are smooth functions given by explicit formulas which depend on $(e(t), v_q, R_q)$. After the construction of the new velocity $v_{q+1}$, we construct the new temperature $\theta_{q+1}$ by solving the following transport-diffusion equation: there exists a $\theta_{q+1} \in C^\infty([0,1] \times T^2, \mathbb{R})$ which solves

$$\begin{cases}
\partial_t \theta_{q+1} + v_{q+1} \cdot \nabla \theta_{q+1} - \Delta \theta_{q+1} = 0, \\
\theta_{q+1}|_{t=0} = \theta^0,
\end{cases}$$

(2.8)

where $\theta^0$ is the function appeared in Proposition 2.1. After the construction of $v_{q+1}, \theta_{q+1}$, we mainly focus on finding functions $R_{q+1}, p_{q+1}$ with the desired estimates and solving the system (2.1).

The rest of the paper is organized as follows. In Section 3, we do some preliminaries. We introduce the Geometric Lemma in [21], stationary solutions of 2d Euler equation, anti-divergence operator and estimates of transport-diffusion equation with highly oscillatory forces. In Section 4, we first define the new velocity $v_{q+1}$ by constructing velocity perturbations $w_0, w_c$, and then construct the new temperature $\theta_{q+1}$ by solving the transport-diffusion equation. In Section 5 and Section 6, we establish various estimates for the perturbation. Finally, in Section 7, we give a proof of Proposition 2.1 by using the estimate which was established in Section 5 and 6.

3. Some preliminaries

We first recall the following stationary solution for the 2d Euler equation which is the building block in our iterative scheme.

3.1. Stationary flows in 2D.

**Proposition 3.1.** Let $\Lambda$ be a given finite symmetric subset of $S^1 \cap Q^2$. Then for any choice of coefficients $a_k \in \mathbb{C}$ with $\overline{a_k} = a_{-k}$, the vector field

$$W(x) = \sum_{k \in \Lambda} a_k i k^\perp e^{ik \cdot x}, \quad \Psi(x) = \sum_{k \in \Lambda} a_k e^{ik \cdot x}$$

is real-valued and satisfies

$$\text{div}(W \otimes W) = \nabla\left(\frac{|W|^2}{2} + \frac{\Psi^2}{2}\right), \quad W(x) = \nabla^\perp \Psi(x).$$

(3.1)
Here and throughout the paper, we denote $k^\perp = (-k_2, k_1)$ if $k = (k_1, k_2)$, and denote $\nabla^\perp = (-\partial_2, \partial_1)$. Furthermore,

$$
\langle W \otimes W \rangle := \int_{T^2} W \otimes W(x) dx = \sum_{k \in \Lambda} |a_k|^2 (\text{Id} - k \otimes k).
$$

The proof of this proposition can be found in [11], and for completeness, we give a direct proof here. We first prove the following lemma.

**Lemma 3.2.** Let $f_1(x) = (-\frac{b}{a}) e^{i(a,b) \cdot x}$, $f_2(x) = (-\frac{d}{c}) e^{i(c,d) \cdot x}$ with $a^2 + b^2 = c^2 + d^2$. There holds

$$
\text{div}(f_1(x) \otimes f_2(x) + f_2(x) \otimes f_1(x)) = \nabla ((ac + bd - a^2 - b^2)e^{i(a+c,b+d) \cdot x}).
$$

Here and below, we denote

$$
\begin{pmatrix}
    a \\
    b
\end{pmatrix} \otimes 
\begin{pmatrix}
    c \\
    d
\end{pmatrix} = \begin{pmatrix}
    a & c \\
    b & d
\end{pmatrix} = \begin{pmatrix}
    ac & ad \\
    bc & bd
\end{pmatrix}.
$$

**Proof.** A computation gives

$$
\text{div}(f_1(x) \otimes f_2(x) + f_2(x) \otimes f_1(x))
= \text{div} \left( \begin{pmatrix}
    -\frac{b}{a} \\
    -\frac{d}{c}
\end{pmatrix} \otimes \begin{pmatrix}
    -\frac{d}{c} \\
    -\frac{b}{a}
\end{pmatrix} e^{i(a+c,b+d) \cdot x} + \begin{pmatrix}
    -\frac{d}{c} \\
    -\frac{b}{a}
\end{pmatrix} \otimes \begin{pmatrix}
    -\frac{b}{a} \\
    -\frac{d}{c}
\end{pmatrix} e^{i(a+c,b+d) \cdot x} \right)
= \text{div} \left[ \begin{pmatrix}
    2bd & -bc - ad \\
    -ad - bc & 2ac
\end{pmatrix} e^{i(a+c,b+d) \cdot x} \right]
= i \begin{pmatrix}
    2bd & -bc - ad \\
    -ad - bc & 2ac
\end{pmatrix} \begin{pmatrix}
    a + c \\
    b + d
\end{pmatrix} e^{i(a+c,b+d) \cdot x}
= i \begin{pmatrix}
    bd(a + c) - b^2c - ad^2 \\
    ac(b + d) - ad^2 - bc^2
\end{pmatrix} e^{i(a+c,b+d) \cdot x}
= i \begin{pmatrix}
    bd(a + c) + ac(a + c) - ac^2 - ca^2 - b^2c - ad^2 \\
    ac(b + d) + bd(b + d) - bd^2 - db^2 - ad^2 - bc^2
\end{pmatrix} e^{i(a+c,b+d) \cdot x}
= i \begin{pmatrix}
    (bd + ac - a^2 - b^2)(a + c) \\
    (ac + bd - a^2 - b^2)(b + d)
\end{pmatrix} e^{i(a+c,b+d) \cdot x}
= \nabla \left( (ac + bd - a^2 - b^2)e^{i(a+c,b+d) \cdot x} \right),
$$

where we used $a^2 + b^2 = c^2 + d^2$ in the penultimate line. \qed

**Proof of Proposition 3.1**

**Proof.** We only prove the first identity in (3.1). The others are obvious. It’s direct to obtain

$$
W(x) \otimes W(x) = - \sum_{k,k' \in \Lambda} a_k a_{k'} k^\perp \otimes (k')^\perp e^{i(k+k') \cdot x}
= - \frac{1}{2} \sum_{k,k' \in \Lambda} a_k a_{k'} (k^\perp \otimes (k')^\perp + (k')^\perp \otimes k^\perp) e^{i(k+k') \cdot x},
$$

$$
\frac{|W(x)|^2}{2} = - \frac{1}{2} \sum_{k,k' \in \Lambda} a_k a_{k'} k \cdot k' e^{i(k+k') \cdot x},
$$

$$
\frac{\Psi^2(x)}{2} = \frac{1}{2} \sum_{k,k' \in \Lambda} a_k a_{k'} e^{i(k+k') \cdot x}.
$$
Thus, using the above lemma, we obtain
\[
\text{div}\left( W(x) \otimes W(x) \right) = -\frac{1}{2} \sum_{k,k' \in \Lambda} a_{k}a_{k'}(k \cdot k' - 1) \nabla e^{i(k+k') \cdot x} \\
= -\frac{1}{2} \nabla \left( \sum_{k,k' \in \Lambda} a_{k}a_{k'}(k \cdot k' - 1) e^{i(k+k') \cdot x} \right),
\]
\[
\nabla \left( \frac{|W(x)|^2}{2} + \frac{\Psi^2(x)}{2} \right) = -\frac{1}{2} \nabla \left( \sum_{k,k' \in \Lambda} a_{k}a_{k'}(k \cdot k' - 1) e^{i(k+k') \cdot x} \right),
\]
and hence
\[
\text{div}(W \otimes W) = \nabla \left( \frac{|W|^2}{2} + \frac{\Psi^2}{2} \right).
\]

3.2. Geometric Lemma. Let
\[
\Lambda_0^+ = \left\{ e_1, \frac{3}{5}e_1 + \frac{4}{5}e_2, \frac{3}{5}e_1 - \frac{4}{5}e_2 \right\}, \quad \Lambda_0^- = -\Lambda_0^+, \quad \Lambda_0 = \Lambda_0^+ \cup \Lambda_0^-,
\]
and \(\Lambda_1\) be given by the rotation of \(\Lambda_0\) counter clock-wise by \(\pi/2\):
\[
\Lambda_1^+ = \left\{ e_2, \frac{3}{5}e_2 + \frac{4}{5}e_1, \frac{3}{5}e_2 - \frac{4}{5}e_1 \right\}, \quad \Lambda_1^- = -\Lambda_1^+, \quad \Lambda_1 = \Lambda_1^+ \cup \Lambda_1^-.
\]
Clearly \(\Lambda_0, \Lambda_1 \subseteq Q^2 \cap S^1\) and we have the representation
\[
\frac{25}{32} \left( \left( \frac{3}{5}e_1 + \frac{4}{5}e_2 \right) \otimes \left( \frac{3}{5}e_1 + \frac{4}{5}e_2 \right) \right) \\
+ \left( \frac{3}{5}e_1 - \frac{4}{5}e_2 \right) \otimes \left( \frac{3}{5}e_1 - \frac{4}{5}e_2 \right) + \frac{7}{16} e_1 \otimes e_1 = \text{Id}.
\]
In fact, notice that \(S^2 \times 2(2 \times 2\text{ symmetric matrix})\) is a 3-dimensional linear space, thus by the above representation and uniqueness, we know that such representation holds for \(2 \times 2\text{ symmetric matrices near } \text{Id}\).

Lemma 3.3 (Geometric Lemma). There exists \(\varepsilon_0 > 0\), and smooth positive functions \(\gamma_k\):
\[
\gamma_k \in C^\infty(B_{\varepsilon_0}(\text{Id})), \quad k \in \Lambda_0^+,
\]
such that for every \(2 \times 2\text{ symmetric matrix } R \in B_{\varepsilon_0}(\text{Id})\), we have
\[
R = 2 \sum_{k \in \Lambda_0^+} \gamma_k^2(R) k \otimes k.
\]

Remark 3.4. By rotational symmetry, Geometric Lemma 3.3 also holds for \(k \in \Lambda_1^+\). It is convenient to introduce a small geometric constant \(c_0 \in (0,1)\) such that
\[
|k + k'| \geq 2c_0
\]
for all \(k, k' \in \Lambda_0 \cap \Lambda_1, k \neq -k'\). Moreover, for \(k \in \Lambda_l^-\) with \(l = 0,1\), we set \(\gamma_k := \gamma_{-k}\).

3.3. Anti-divergence operator. We recall the anti-divergence operator in this subsection.

Lemma 3.5. (Anti-divergence operator) There exists an operator \(\mathcal{R}\) satisfying the following property:

- For any \(v \in C^\infty(T^2; \delta^2)\), \(\mathcal{R}v(x)\) is a symmetric trace-free matrix for each \(x \in T^2\) and
\[
\text{div}\mathcal{R}v(x) = v(x) - \int_{T^2} v(x) dx.
\]
The following estimates hold: for any \( a \in C^\infty(T^2) \) and any \( m \in \mathbb{N}_+ \), \( \alpha \in (0, 1) \), there holds

\[
\| \mathcal{R}(a(x)e^{ik \cdot x}) \|_\alpha \leq C(m, \alpha) \left( \|a\|_0 + \frac{\|\nabla^m a\|_0}{\lambda^{m-\alpha}} + \frac{\|\nabla^m a\|_\alpha}{\lambda^m} \right).
\]  
(3.2)

Proof. Let \( u \in C^\infty_0(T^2) \) be a solution to

\[
\triangle u = v - \int_{T^2} v(x) dx,
\]
where \( C^\infty_0(T^2) = \{ f \in C^\infty(T^2) : \int_{T^2} f(x) dx = 0 \} \). Then set

\[
\mathcal{R}v(x) := \nabla u + (\nabla u)^T - (\text{div } u)\text{Id}.
\]

Then \( \mathcal{R} \) satisfies the above property. The detail can be found in [11], and we omit it here. \( \square \)

Moreover, \( \mathcal{R} \) satisfies the following property:

Lemma 3.6. For any \( s > 0 \), there holds

\[
\| \mathcal{R}(v) \|_0 \leq C \|v\|_{\dot{H}^s}.
\]  
(3.3)

Here and below,

\[
\|v\|_{\dot{H}^s}^2 = \sum_{k \in \mathbb{Z}^2, k \neq 0} |v_k|^2 |k|^{2s}.
\]

In particular, there holds

\[
\| \mathcal{R}(v) \|_0 \leq C \|\nabla v\|_{L^2}.
\]  
(3.4)

Proof. In fact, we can give a explicit formula for \( \mathcal{R} \) by the Fourier series expansion. Let

\[
v(x) = \sum_{k \in \mathbb{Z}^2} v_k e^{ik \cdot x}, \quad x \in T^2
\]

where \( v_k \in C^2 \) is the Fourier coefficient of the vector function \( v(x) \). Then

\[
v(x) - \int_{T^2} v(x) dx = \sum_{k \in \mathbb{Z}^2, k \neq 0} v_k e^{ik \cdot x}, \quad x \in T^2.
\]

Thus, due to \( u \in C^\infty_0(T^2) \) and \( \triangle u = v \), there holds

\[
u(x) = \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{-v_k}{|k|^2} e^{ik \cdot x}, \quad x \in T^2.
\]

Thus, we deduce that

\[
\nabla u(x) = \sum_{k \in \mathbb{Z}^2, k \neq 0} -\frac{iv_k \otimes k}{|k|^2} e^{ik \cdot x},
\]

\[
(\nabla u)^T(x) = \sum_{k \in \mathbb{Z}^2, k \neq 0} -\frac{ik \otimes v_k}{|k|^2} e^{ik \cdot x},
\]

\[
\text{div } u(x) = \sum_{k \in \mathbb{Z}^2, k \neq 0} -\frac{iv_k \cdot k}{|k|^2} e^{ik \cdot x}.
\]

Summing them together, we obtain

\[
\mathcal{R}(v)(x) = \sum_{k \in \mathbb{Z}^2, k \neq 0} \left( -\frac{iv_k \otimes k}{|k|^2} + \frac{-ik \otimes v_k}{|k|^2} + \frac{iv_k \cdot k}{|k|^2} \text{Id} \right) e^{ik \cdot x}.
\]
hence for any $s > 0$, there hold
\[
\| \mathcal{R}(v) \|_0 \leq C \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{|v_k|}{|k|} \leq C \left( \sum_{k \in \mathbb{Z}^2, k \neq 0} |v_k|^2 |k|^{2s} \right)^{\frac{1}{2}} \left( \sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{1}{|k|^{2(1+s)}} \right)^{\frac{1}{2}} \leq C \| v \|_{\dot{H}^s},
\]
where we used
\[
\sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{1}{|k|^{2(1+s)}}
\]
is convergent for any $s > 0$. In fact,
\[
\sum_{k \in \mathbb{Z}^2, k \neq 0} \frac{1}{|k|^{2(1+s)}} \leq 4 \sum_{n=1}^{\infty} \sum_{k_1+k_2=n, k_1,k_2 \geq 0} \frac{1}{(k_1^2 + k_2^2)^{1+s}} \leq 4 \sum_{n=1}^{\infty} \frac{1}{n^{2(1+s)}} + \frac{1}{(1+(n-1)^2)(1+s)} + \cdots + \frac{1}{n^{2(1+s)}} \leq C \sum_{n=1}^{\infty} \frac{2^{(1+s)(n+1)}}{n^{2(1+s)}} \leq C_s < +\infty.
\]

3.4. **Transport-diffusion equation with oscillatory force.** We consider the transport-diffusion equation with oscillatory force
\[
\begin{aligned}
\partial_t \theta + v \cdot \nabla \theta - \Delta \theta &= a(t,x) \cos(\lambda k \cdot x), \quad x \in \mathbb{T}^2, \\
\text{div} v &= 0, \\
\theta(0,x) &= 0,
\end{aligned}
\tag{3.5}
\]
where $k \in S^1 \cap Q^2$ is a vector and $\lambda k \in \mathbb{Z}^2, \lambda \neq 0$.

To simplify the formulas, we introduce the following notation: for any $n \in \mathbb{N}_+$,
\[
\|(f_1, \cdots, f_n)\|_X := \sum_{k=1}^{n} \|f_k\|_X,
\]
where $\| \cdot \|_X$ is a norm.

**Lemma 3.7.** Let $\theta(t,x)$ be a solution of \((3.7)\). Then there hold
\[
\begin{aligned}
\| \nabla \theta \|_{L^\infty L^2} &\leq C(\| v \|_0 + 1) \frac{\| (\lambda a, \partial_t a, \Delta a, \nabla a) \|_{L^\infty L^2}}{\lambda^2}, \\
\| \nabla^2 \theta \|_{L^\infty L^2} &\leq C(1 + \| v \|_0^2) \| \nabla v \|_0 \frac{\| (\lambda a, \partial_t a, \Delta a, \lambda \nabla a) \|_{L^\infty L^2}}{\lambda^2} + C \frac{\| (\nabla \partial_t a, \lambda \partial_t a, \nabla^3 a, \lambda \nabla^2 a, \lambda^2 \nabla a) \|_{L^\infty L^2}}{\lambda^2} \\
&\quad + C(\| v \|_0 + 1) \frac{\| (\nabla^2 a, \lambda \nabla a, \lambda^2 a) \|_{L^\infty L^2}}{\lambda^2},
\end{aligned}
\tag{3.6}
\]
where $C$ is an absolute constant.

**Proof.** **Step 1:** First, set
\[
\theta_1(t,x) := \frac{1 - e^{-\lambda^2 t}}{\lambda^2} a(t,x) \cos(\lambda k \cdot x).
\tag{3.7}
\]
A direct computation gives
\[
\partial_t \theta_1(t, x) = e^{-\lambda^2 t} a(t, x) \cos(\lambda k \cdot x) + \frac{1 - e^{-\lambda^2 t}}{\lambda^2} \partial_t a(t, x) \cos(\lambda k \cdot x),
\]
\[
\nabla \theta_1(t, x) = \frac{1 - e^{-\lambda^2 t}}{\lambda^2} [\nabla a(t, x) \cos(\lambda k \cdot x) - \lambda a(t, x) \sin(\lambda k \cdot x)],
\]
\[
\triangle \theta_1(t, x) = -(1 - e^{-\lambda^2 t}) a(t, x) \cos(\lambda k \cdot x) + \frac{1 - e^{-\lambda^2 t}}{\lambda^2} [\Delta a(t, x) \cos(\lambda k \cdot x) - 2 \lambda k \cdot \nabla a(t, x) \sin(\lambda k \cdot x)].
\]
Thus, we obtain that
\[
\begin{aligned}
\partial_t \theta_1 - \triangle \theta_1 &= a(t, x) \cos(\lambda k \cdot x) + f_1, \\
\theta_1(0, x) &= 0,
\end{aligned}
\]
where
\[
f_1(t, x) = \frac{1 - e^{-\lambda^2 t}}{\lambda^2} [\partial_t a(t, x) \cos(\lambda k \cdot x) - \Delta a(t, x) \cos(\lambda k \cdot x) + 2 \lambda k \cdot \nabla a(t, x) \sin(\lambda k \cdot x)].
\]
Obviously, there holds
\[
\begin{aligned}
\|f_1(t, \cdot)\|_2 &\leq \left\| (\partial_t a(t, \cdot), \triangle a(t, \cdot), \lambda \nabla a(t, \cdot)) \right\|_2, \\
\|\nabla f_1(t, \cdot)\|_2 &\leq C \left\| (\nabla \partial_t a(t, \cdot), \lambda \partial_t a(t, \cdot), \nabla^3 a(t, \cdot), \lambda \nabla^2 a(t, \cdot), \lambda^2 \nabla a(t, \cdot)) \right\|_2.
\end{aligned}
\]

**Step 2:** We consider the following transport-diffusion equation
\[
\begin{aligned}
\partial_t \theta_2 + v \cdot \nabla \theta_2 - \triangle \theta_2 &= -v \cdot \nabla \theta_1 - f_1, \\
\text{div} v &= 0, \\
\theta_2(0, x) &= 0.
\end{aligned}
\]

**$L^2$ estimate:** A direct energy estimate gives
\[
\frac{1}{2} \frac{d}{dt} \|\theta_2(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|\nabla \theta_2(t, \cdot)\|_{L^2}^2 \leq C \|v\|_{L^\infty}^2 \|\theta_1(t, \cdot)\|_{L^2}^2 + \|f_1(t, \cdot)\|_{L^2}^2 + \|\theta_2(t, \cdot)\|_{L^2}^2.
\]
By (3.7) and (3.10), we obtain
\[
\|\theta_2\|_{L^\infty L^2} \leq C \left\| v \right\|_{L^\infty}^2 \|a\|_{L^\infty L^2} + \left\| (\partial_t a, \triangle a, \lambda \nabla a) \right\|_{L^\infty L^2}.
\]

**$H^1$ estimate:** Acting $\partial_i$ on both sides of (3.11), then taking inner products with $\partial_i \theta_2$ and integrating by parts, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \|\partial_i \theta_2(t, \cdot)\|_{L^2}^2 + \|\nabla \partial_i \theta_2(t, \cdot)\|_{L^2}^2 = \int_{T^2} [-\partial_i v \cdot \nabla \theta_2 - \partial_i (v \cdot \nabla \theta_1) - \partial_i f_1] \partial_i \theta_2 dx.
\]
Integrating by parts, we deduce that the left side can be controlled by
\[
C \|v\|_{L^2} \left\| \nabla \theta_1(t, \cdot) \right\|_{L^2} + \left\| \nabla \theta_2(t, \cdot) \right\|_{L^2} + \|f_1(t, \cdot)\|_{L^2} \|\partial_i \theta_2(t, \cdot)\|_{L^2}.
\]
Summing over $i = 1, 2$, and by the Hölder inequality, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \theta_2(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|\nabla^2 \theta_2(t, \cdot)\|_{L^2}^2 \leq C \|v\|_{L^2}^2 \|\nabla \theta_2(t, \cdot)\|_{L^2}^2 + C \left( \|v\|_{L^2}^2 \|\nabla \theta_1(t, \cdot)\|_{L^2}^2 + \|f_1(t, \cdot)\|_{L^2}^2 \right).
Using the interpolation inequality $\|\nabla \theta_2(t, \cdot)\|_{L^2} \leq \|\theta_2(t, \cdot)\|_{H^2}^{1/2} \|\nabla^2 \theta_2(t, \cdot)\|_{L^2}^{1/2}$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla \theta_2(t, \cdot)\|_{L^2}^2 + \frac{1}{4} \|\nabla^2 \theta_2(t, \cdot)\|_{L^2}^2 \leq C \|v\|_0^2 \|\theta_2(t, \cdot)\|_{L^2}^2 + C(\|v\|_0^2 \|\nabla \theta_1(t, \cdot)\|_{L^2}^2 + \|f_1(t, \cdot)\|_{L^2}^2).$$

By combining (3.8) and (3.10), we arrive at

$$\|\nabla \theta_2\|_{L^\infty L^2_t} \leq C \left(\|v\|_0^2 \|\theta_2\|_{L^\infty L^2_x} + \|v\|_0 \|\nabla \theta_1\|_{L^\infty L^2_x} + \|f_1\|_{L^\infty L^2_x}\right) \leq C(1 + \|v\|_0^2) \frac{\|\lambda a, \partial_t a, \triangle a, \lambda \nabla a\|_{L^\infty L^2_x}}{\lambda^2}. \tag{3.12}
$$

**H^2 estimate:** Acting $\partial_{ij}$ on both sides of (3.11), then taking inner products with $\partial_{ij} \theta_2$ and integrating by parts, we arrive at

$$\frac{1}{2} \frac{d}{dt} \|\partial_{ij} \theta_2(t, \cdot)\|_{L^2}^2 + \|\nabla \partial_{ij} \theta_2(t, \cdot)\|_{L^2}^2 = \int_{T^2} \left[ -\partial_{ij} v \cdot \nabla \theta_2 - \partial_i v \cdot \nabla \partial_j \theta_2 - \partial_j v \cdot \nabla \partial_i \theta_2 - \partial_{ij}(v \cdot \nabla \theta_1) - \partial_{ij} f_1 \partial_{ij} \theta_2 \right] dx.$$

By integrating by parts and interpolation inequalities, we deduce that the left side can be controlled by

$$C \|\nabla v\|_0 \|\nabla \theta_2(t, \cdot)\|_{L^2} \|\nabla^3 \theta_2(t, \cdot)\|_{L^2} + C \|\nabla v\|_0 \|\nabla \theta_1(t, \cdot)\|_{L^2} \|\nabla^3 \theta_2(t, \cdot)\|_{L^2} + C \|v\|_0 \|\nabla \theta_1(t, \cdot)\|_{L^2} \|\nabla \theta_2(t, \cdot)\|_{L^2} \|f_1(t, \cdot)\|_{L^2} \|\partial_{ij} \theta_2(t, \cdot)\|_{L^2}.$$

Summing over $i, j = 1, 2$, and by the Hölder inequality, we arrive at

$$\frac{1}{2} \frac{d}{dt} \|\nabla^2 \theta_2(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|\nabla^3 \theta_2(t, \cdot)\|_{L^2}^2 \leq C \left(\|\nabla v\|_0^2 \|\nabla \theta_2\|_{L^\infty L^2_x} + \|\nabla v\|_0 \|\nabla \theta_1\|_{L^\infty L^2_x} \right) \|\nabla \theta_2\|_{L^\infty L^2_x} \|\nabla^3 \theta_2\|_{L^\infty L^2_x} + \|f_1\|_{L^\infty L^2_x} \|\partial_{ij} \theta_2(t, \cdot)\|_{L^2}.$$

Thus, by combining (3.8), (3.10) and (3.12), we arrive at

$$\|\nabla^2 \theta_2\|_{L^\infty L^2} \leq C \left(\|\nabla v\|_0^2 \|\nabla \theta_2\|_{L^\infty L^2_x} + \|\nabla v\|_0 \|\nabla \theta_1\|_{L^\infty L^2_x} \right) \|\nabla \theta_2\|_{L^\infty L^2_x} \|\nabla^3 \theta_2\|_{L^\infty L^2_x} + \|f_1\|_{L^\infty L^2_x} \|\lambda a, \partial_t a, \triangle a, \lambda \nabla a\|_{L^\infty L^2_x} \|\lambda^2 \right)^2 \tag{3.13}
$$

**Step 3:** Adding (3.9) and (3.11) together, we arrive at

$$\begin{cases} \partial_t (\theta_1 + \theta_2) + v \cdot \nabla (\theta_1 + \theta_2) - \triangle (\theta_1 + \theta_2) = a(t, x) \cos(\lambda x), \\
\text{div} v = 0, \\
(\theta_1 + \theta_2)(0, x) = 0. \end{cases}$$

Thus, by uniqueness, $(\theta_1 + \theta_2)$ is the uniqueness solution of (3.15), and by collecting (3.12) and (3.13), we arrive at (3.6). This completes the proof of this lemma. \qed
Remark 3.8. Similarly, we consider
\[
\begin{align*}
\partial_t \theta + v \cdot \nabla \theta - \Delta \theta &= a(t,x) \sin(\lambda k \cdot x), \quad x \in T^2, \\
\text{div} v &= 0, \\
\theta(0,x) &= 0,
\end{align*}
\]
(3.14)
where $k, \lambda$ are as above. Then, the following estimate still holds: Let $\theta(t,x)$ be a solution of (3.14). Then there hold
\[
\begin{align*}
\|\nabla \theta\|_{L^2} &\leq C(\|v\|_0 + 1) \frac{\|(\lambda a, \partial_t a, \Delta a, \lambda \nabla a)\|_{L^2}}{\lambda^2}, \\
\|\nabla^2 \theta\|_{L^2} &\leq C(1 + \|v\|_0^2) \|\nabla v\|_0 \frac{\|(\lambda a, \partial_t a, \Delta a, \lambda \nabla a)\|_{L^2}}{\lambda^2} \\
&\quad + C \frac{\|(\nabla \partial_t a, \lambda \partial_t a, \nabla^3 a, \lambda \nabla^2 a, \lambda^2 \nabla a)\|_{L^2}}{\lambda^2} \\
&\quad + C(\|v\|_0 + 1) \frac{\|(\nabla^2 a, \lambda \nabla a, \lambda^2 a)\|_{L^2}}{\lambda^2}.
\end{align*}
\]

Generally, if we consider the transport-diffusion equation
\[
\begin{align*}
\partial_t \theta + v \cdot \nabla \theta - \Delta \theta &= \sum_{|k|=1} a_k(t,x)e^{i \lambda k \cdot x}, \\
\text{div} v &= 0, \\
\theta(0,x) &= 0,
\end{align*}
\]
(3.15)
where $a_k(t,x)$ complex-valued functions with $\bar{a}_k(t,x) = a_{-k}(t,x)$. Then, there exists a unique solution $\theta(t,x)$ for (3.15) and it satisfies the following estimate
\[
\begin{align*}
\|\nabla \theta\|_{L^2} &\leq C(\|v\|_0 + 1) \sum_{|k|=1} \frac{\|(\lambda a_k, \partial_t a_k, \Delta a_k, \lambda \nabla a_k)\|_{L^2}}{\lambda^2}, \\
\|\nabla^2 \theta\|_{L^2} &\leq C(1 + \|v\|_0^2) \|\nabla v\|_0 \sum_{|k|=1} \frac{\|(\lambda a_k, \partial_t a_k, \Delta a_k, \lambda \nabla a_k)\|_{L^2}}{\lambda^2} \\
&\quad + C \sum_{|k|=1} \frac{\|(\nabla \partial_t a_k, \lambda \partial_t a_k, \nabla^3 a_k, \lambda \nabla^2 a_k, \lambda^2 \nabla a_k)\|_{L^2}}{\lambda^2} \\
&\quad + C(\|v\|_0 + 1) \sum_{|k|=1} \frac{\|(\nabla^2 a_k, \lambda \nabla a_k, \lambda^2 a_k)\|_{L^2}}{\lambda^2}.
\end{align*}
\]
(3.16)

4. Construction of $(v_{q+1}, p_{q+1}, \theta_{q+1}, \tilde{R}_{q+1})$

In this section, we perform the inductive procedure which allows us to construct $(v_{q+1}, p_{q+1}, \theta_{q+1}, \tilde{R}_{q+1})$ from $(v_q, p_{q}, \theta_{q}, \tilde{R}_{q})$. Recalling the choice of the sequence $\{\delta_q\}_{q \in N}$ and $\{\lambda_q\}_{q \in N}$, for sufficiently large $a$, we have, for any $q \geq 1$,
\[
\sum_{j \leq q} \delta_j \lambda_j^k \leq 2 \delta_q \lambda_q^k, \quad \sum_{j \leq q} \frac{1}{2} \lambda_j^k \leq 2 \delta_q \frac{1}{2} \lambda_q^k, \quad k = 1, 2, 3.
\]

As in [3], we write $(v, p, \theta, \tilde{R})$ instead of $(v_{q+1}, p_{q+1}, \theta_{q+1}, \tilde{R}_{q+1})$ and $(v_1, p_1, \theta_1, \tilde{R}_1)$ instead of $(v_{q+1}, p_{q+1}, \theta_{q+1}, \tilde{R}_{q+1})$. Thus, the following estimates hold:
\[
\begin{align*}
\|v\|_0 &\leq 2M, \quad \|v\|_k \leq 2M \delta_q \lambda_q^k, \quad k = 1, 2, 3, \\
\|\tilde{R}\|_0 &\leq \eta \delta_{q+1}, \quad \|\tilde{R}\|_1 \leq M \delta_{q+1} \lambda_q.
\end{align*}
\]
(4.1)
(4.2)
4.1. **Space regularization of** \(v, \hat{R}\). Let \(\psi \in C_c^\infty(\mathbb{R}^3)\) be a radial symmetry nonnegative function with \(\int_{\mathbb{R}^3} \psi(x) dx = 1\) and \(\ell\) be a small parameter. Set
\[
v_\ell = v * \psi_\ell, \quad \hat{R}_\ell = \hat{R} * \psi_\ell.
\]

Standard estimates on convolutions give
\[
\|v - v_\ell\|_0 \leq CM \delta_{q+2} \lambda_q \ell, \quad \|\hat{R} - \hat{R}_\ell\|_0 \leq CM \delta_{q+1} \lambda_q \ell, \tag{4.3}
\]
and for any \(N \geq 1\), there exists a constant \(C = C(N)\) such that
\[
\|v_\ell\|_N \leq CM \delta_{q+2} \lambda_q \ell^{1-N}, \quad \|\hat{R}_\ell\|_N \leq CM \delta_{q+1} \lambda_q \ell^{1-N}. \tag{4.4}
\]

4.2. **Partition of unity on time.** Fixing a smooth function \(\chi \in C_c^\infty((-\frac{3}{2}, \frac{3}{2}))\) such that
\[
\sum_{l \in \mathbb{Z}} \chi^2(x - l) = 1
\]
and a large parameter \(\mu \in \mathbb{N}_+\) which will be determined later.

For any \(l \in [0, \mu]\), set
\[
\rho_l := \frac{1}{2(2\pi)^2} e\left(\frac{1}{\mu}\right) (1 - \delta_{q+2}) - \int_{T^2} |v|^2 \left(x, \frac{l}{\mu}\right) dx. \tag{4.5}
\]
Due to (2.7), we deduce that there exists a universal constant \(C_0\) such that
\[
C_0^{-1} \min_{l \in [0,1]} e(t) \delta_{q+1} \leq \rho_l \leq C_0 \min_{l \in [0,1]} e(t) \delta_{q+1}. \tag{4.6}
\]

4.3. **Construction of velocity perturbation** \(w\). Firstly, for any integer \(l \in [0, \mu]\), we construct two smooth functions \(\Phi_l(t, x) : T^2 \times [0,1] \to T^2\) and \(\hat{R}_{\ell,l} : T^2 \times [0,1] \to S_0^{2 \times 2}\) by solving the following two equations:
\[
\begin{cases}
\partial_t \Phi_l + v_\ell \cdot \nabla \Phi_l = 0, \\
\Phi_l\left(\frac{l}{\mu}, x\right) = x,
\end{cases} \tag{4.7}
\]
and
\[
\begin{cases}
\partial_t \hat{R}_{\ell,l} + v_\ell \cdot \nabla \hat{R}_{\ell,l} = 0, \\
\hat{R}_{\ell,l}\left(\frac{l}{\mu}, x\right) = \hat{R}_l\left(\frac{l}{\mu}, x\right).
\end{cases} \tag{4.8}
\]

Next, for any \(k \in \Lambda_0 \cup \Lambda_1\) and any integer \(l \in [0, \mu]\), we set
\[
\chi_l(t) := \chi(\mu t - l), \\
a_{kl}(t, x) := \sqrt{\rho_l} \gamma_k \left(\frac{R_{\ell,l}(t, x)}{\rho_l}\right), \\
w_{kl}(t, x) := a_{kl}(t, x) e^{i \lambda_q \ell 1 + k^\perp \cdot \Phi_l(t, x)}. \tag{4.9}
\]
Here and throughout the paper, \(R_{\ell,l}(t, x) := \rho_l \text{Id} - \hat{R}_{\ell,l}(t, x)\).

Due to (4.6), we deduce that there exists an \(\eta = \eta(e, r_0) := C_0^{-1} r_0 \min_{l \in [0,1]} e(t)\) such that
\[
\left\| \text{Id} - \frac{R_{\ell,l}}{\rho_l} \right\|_0 \leq \frac{r_0}{2}.
\]
Hence \(a_{kl}\) in (4.9) is well-defined.

We define the principle part \(w_o\) of the velocity perturbation \(w\)
\[
w_o(t, x) := \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l(t) w_{kl}(t, x), \tag{4.10}
\]
where
\[
\Lambda(t) = \Lambda_l \mod 2.
\]
Moreover, as in [3], we set
\[
\phi_{kl}(t, x) := e^{i\lambda_{q+1}k^\perp (\Phi_l(t, x) - x)}.
\] (4.11)

Then
\[
w_o(t, x) = \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l(t) a_{kl} ik \phi_{kl} e^{i\lambda_{q+1}k^\perp x}.
\] (4.12)

Then, set the incompressibility corrector \( w_c \) as
\[
w_c(t, x) := -\sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l(t) \frac{\nabla^\perp (a_{kl}(t, x) e^{i\lambda_{q+1}k^\perp (\Phi_l(t, x) - x)})}{\lambda_{q+1}} e^{i\lambda_{q+1}k^\perp x}
\]
\[
= -\sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l(t) \left( \frac{\nabla^\perp a_{kl}(t, x) + ia_{kl} \lambda_{q+1}(\nabla^\perp \Phi_l(t, x) - \tilde{\text{Id}})k^\perp}{\lambda_{q+1}} \right) e^{i\lambda_{q+1}k^\perp \cdot \Phi_l(t, x)}. \] (4.13)

Here and throughout the paper, \( \tilde{\text{Id}} \) denotes
\[
\tilde{\text{Id}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Finally, we set
\[
w(t, x) := w_o(t, x) + w_c(t, x).
\] (4.14)

Obviously,
\[
w(t, x) = \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l(t) \frac{\nabla^\perp (a_{kl}(t, x) e^{i\lambda_{q+1}k^\perp \cdot \Phi_l(t, x)})}{\lambda_{q+1}};
\]
hence \( \text{div} w(t, x) = 0 \).

Set
\[
L_{kl}(t, x) := a_{kl}(t, x) ik - \frac{\nabla^\perp a_{kl}(t, x) + ia_{kl} \lambda_{q+1}(\nabla^\perp \Phi_l(t, x) - \tilde{\text{Id}})k^\perp}{\lambda_{q+1}};
\] (4.15)

thus the perturbation \( w \) can also be written as
\[
w = \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l(t) L_{kl} \phi_{kl} e^{i\lambda_{q+1}k^\perp x}.
\] (4.16)

After the construction of the perturbation \( w \), we choose the constant \( M \). By (4.6) and the support property of \( \chi_l(t) \), it’s direct to obtain
\[
\|w_o\|_0 \leq C_0(\varepsilon)\delta_{q+1}^2.
\]

Then, we set
\[
M = 2C_0(\varepsilon);
\]
thus, there holds
\[
\|w_o\|_0 \leq \frac{M}{2} \delta_{q+1}^2.
\] (4.17)
4.4. **Construction of new velocity** $v_1$ and **new temperature** $\theta_1$. After the construction of the velocity perturbation $w$, we define the new velocity as follows:

$$v_1(t, x) := v(t, x) + w(t, x).$$

Thus there holds $\text{div}v_1(t, x) = 0$. Then, we define the new temperature by solving the following transport-diffusion equation

$$\begin{cases}
\partial_t \theta_1 + v_1 \cdot \nabla \theta_1 - \Delta \theta_1 = 0, \\
\theta_1(0, x) = \theta^0(x).
\end{cases} \tag{4.18}$$

By the maximum principle, we deduce that

$$\|\theta_1\|_0 \leq \|\theta^0\|_0. \tag{4.19}$$

Moreover, the basic energy estimate gives

$$\frac{1}{2} \frac{d}{dt} \|\theta_1(t, \cdot)\|_{L^2}^2 + \|\nabla \theta_1(t, \cdot)\|_{L^2}^2 = 0. \tag{4.20}$$

Finally, it’s obvious that

$$\int_{T^2} \theta_1(t, x) dx = 0. \tag{4.21}$$

4.5. **The new pressure** $p_1$ and **new Reynolds stress** $\hat{R}_1$. We first compute $\text{div}(w_o \otimes w_o)$. Recalling (4.10), we deduce

$$w_o \otimes w_o = \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l^2 w_{kl} \otimes w_{-kl} + \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 w_{kl} \otimes w_{k'l}$$

$$+ \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} \chi_l \chi_{l'} \rho \frac{R_{l, l'}}{\rho_l} \otimes \rho \frac{R_{l, l'}}{\rho_{l'}},$$

$$= \sum_{l \in \mathbb{Z}} \chi_l^2 R_{l, l} + \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 w_{kl} \otimes w_{k'l}$$

$$+ \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} \chi_l \chi_{l'} \rho \frac{R_{l, l'}}{\rho_l} \otimes \rho \frac{R_{l, l'}}{\rho_{l'}}, \tag{4.22}$$

where we used the following fact

$$\sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l^2 w_{kl} \otimes w_{-kl} = \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda(l)} \chi_l^2 \rho_l \gamma_k^2 \left( \frac{R_{l, l}}{\rho_l} \right) k \otimes k$$

$$= \sum_{l \in \mathbb{Z}} \chi_l^2 \rho_l \sum_{k \in \Lambda(l)} \gamma_k^2 \left( \frac{R_{l, l}}{\rho_l} \right) k \otimes k$$

$$= \sum_{l \in \mathbb{Z}} \chi_l^2 R_{l, l},$$

Here we used Geometric Lemma 3.3 in the last step.
Furthermore,
\[
\sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 w_{kl} \otimes w_{k'l} = - \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 (w_{kl} \otimes w_{k'l} + w_{k'l} \otimes w_{kl})
\]
\[
= - \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 a_{kl} a_{k'l} \phi_{kl} \phi_{k'l} [k \otimes k' + k' \otimes k] e^{i \lambda_{q+1} (k + k') \cdot x}; \quad (4.23)
\]

thus, by Proposition 3.1 we deduce that
\[
\text{div} \left( \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 w_{kl} \otimes w_{k'l} \right)
\]
\[
= - \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 [k \otimes k' + k' \otimes k] \nabla (a_{kl} a_{k'l} \phi_{kl} \phi_{k'l}) e^{i \lambda_{q+1} (k + k') \cdot x}
\]
\[
- \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 a_{kl} a_{k'l} \phi_{kl} \phi_{k'l} [k \otimes k' + k' \otimes k] i \lambda_{q+1} (k + k') e^{i \lambda_{q+1} (k + k') \cdot x}
\]
\[
= T_{osc}^1 - \nabla \left( \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 a_{kl} a_{k'l} \phi_{kl} \phi_{k'l} (k \cdot k' + 1) e^{i \lambda_{q+1} (k + k') \cdot x} \right),
\]

where
\[
T_{osc}^1 = - \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 [k \otimes k' + k' \otimes k] \nabla (a_{kl} a_{k'l} \phi_{kl} \phi_{k'l}) e^{i \lambda_{q+1} (k + k') \cdot x}
\]
\[
+ \frac{1}{2} \sum_{l \in \mathbb{Z}} \sum_{k, k' \in \Lambda(l), k + k' \neq 0} \chi_l^2 (k \cdot k' + 1) \nabla (a_{kl} a_{k'l} \phi_{kl} \phi_{k'l}) e^{i \lambda_{q+1} (k + k') \cdot x}. \quad (4.24)
\]

Moreover, set
\[
f_{klk'l'} := \chi_l \chi_{l'} a_{kl} a_{k'l'} \phi_{kl} \phi_{k'l'}.
\]

From (4.12) and the fact \( f_{klk'l'} = f_{k'l'kl} \), we deduce that
\[
\sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} \chi_l \chi_{l'} w_{kl} \otimes w_{k'l'}
\]
\[
= - \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} f_{klk'l'} k \otimes k' e^{i \lambda_{q+1} (k + k') \cdot x}
\]
\[
= - \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} f_{klk'l'} k' \otimes k e^{i \lambda_{q+1} (k + k') \cdot x}
\]
\[
= - \frac{1}{2} \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} f_{klk'l'} (k \otimes k' + k' \otimes k) e^{i \lambda_{q+1} (k + k') \cdot x}. \quad (4.26)
\]
Thus, by Proposition 3.1 there hold

\[
\text{div} \left( \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} \chi_l \chi_{l'} w_{kl} \otimes w_{k'l'} \right)
= -\frac{1}{2} \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} (k \otimes k' + k' \otimes k) \nabla f_{klk'l'} e^{i \lambda_{\ell+1} (k+k') \cdot x - \frac{1}{2} R \cdot w_k} \chi_l \chi_{l'}
- \frac{1}{2} \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} (k \cdot k' + 1) f_{klk'l'} \nabla e^{i \lambda_{\ell+1} (k+k') \cdot x}
:= T_{osc}^2 - \frac{1}{2} \nabla \left[ \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} (k \cdot k' + 1) f_{klk'l'} e^{i \lambda_{\ell+1} (k+k') \cdot x} \right],
\]

where

\[
T_{osc}^2 = -\frac{1}{2} \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} (k \otimes k' + k' \otimes k) \nabla f_{klk'l'} e^{i \lambda_{\ell+1} (k+k') \cdot x}
+ \frac{1}{2} \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} (k \cdot k' + 1) f_{klk'l'} e^{i \lambda_{\ell+1} (k+k') \cdot x}.
\] (4.27)

Set

\[
P := \frac{1}{2} \sum_{l \in \mathbb{Z}, k,k' \in \Lambda(l), k \neq k' \neq 0} \chi_l^2 a_{kl} a_{k'l'} \phi_{kl} \phi_{k'l'} (k \cdot k' + 1) e^{i \lambda_{\ell+1} (k+k') \cdot x}
+ \frac{1}{2} \sum_{l \neq l', k \in \Lambda(l), k' \in \Lambda(l')} (k \cdot k' + 1) f_{klk'l'} e^{i \lambda_{\ell+1} (k+k') \cdot x};
\] (4.28)

thus, by (4.22)-(4.28), we deduce

\[
\text{div} \left( w_o \otimes w_o - \sum_l \chi_l^2 R_{\ell,l} + P \text{Id} \right) = T_{osc}^1 + T_{osc}^2.
\] (4.29)

After the computation of \text{div}(w_o \otimes w_o), we set

\[
\hat{R}_1 := \mathcal{R}(\partial_t w + v_{\ell} \cdot \nabla w) + \mathcal{R}(w \cdot \nabla v_{\ell}) - \mathcal{R}((\theta_1 - \theta) v_{\ell})
+ \mathcal{R} \left( \text{div} \left( w_o \otimes w_o - \sum_l \chi_l^2 R_{\ell,l} + P \text{Id} \right) \right)
+ w_o \otimes w_o + w_o \otimes w_o + w_c \otimes w_o + w_o \otimes w_c - \frac{|w_o|^2}{2} + 2 w_o \cdot w_c \text{Id}
+ w \otimes (v - v_{\ell}) + (v - v_{\ell}) \otimes w - (v - v_{\ell}) \cdot w \text{Id}
+ \hat{\tilde{R}} - \hat{R}_{\ell} + \sum_l \chi_l^2 (\hat{R}_{\ell} - \hat{R}_{\ell,l})
\] (4.30)
Obviously, there holds $\text{tr} \hat{R}_1 = 0$. Finally, put

$$p_1 = p + P - \frac{|w_c|^2 + 2w_o \cdot w_c}{2} - (v - u_t) \cdot w. \quad (4.31)$$

A direct computation gives

$$\partial_t v_1 + v_1 \cdot \nabla v_1 + \nabla p_1 = \theta_1 e_2 + \text{div} \hat{R}_1.$$ 

In fact,

$$\text{div} \hat{R}_1 = \partial_t w + v \cdot \nabla w + w \cdot \nabla v - (\theta_1 - \theta) e_2$$
$$+ \text{div} \left( w_o \otimes w_o - \sum_l \chi_l^2 R_{l,l} + P \text{Id} \right) + \text{div} \left( \hat{R} - \sum_l \chi_l^2 \hat{R}_{l,l} \right)$$
$$+ \text{div} \left( w_o \otimes w_c + w_c \otimes w_o + w_c \otimes w_c - \frac{|w_c|^2 + 2w_o \cdot w_c}{2} \text{Id} - (v - u_t) \cdot w \text{Id} \right)$$
$$= \partial_t v_1 + v_1 \cdot \nabla v_1 + \nabla p_1 - \theta_1 e_2,$$

where we used

$$-\text{div} \left( \sum_l \chi_l^2 R_{l,l} \right) = -\text{div} \left( \sum_l \chi_l^2 (\rho_l - \hat{R}_{l,l}) \right) = \text{div} \left( \sum_l \chi_l^2 \hat{R}_{l,l} \right).$$

Thus, the new functions $(v_1, p_1, \theta_1, \hat{R}_1)$ solves the Boussinesq-Reynold (2.1) system.

5. Estimate on the perturbation

5.1. Hölder space and some elementary inequalities. Recall the following elementary inequalities:

$$[fg]_\alpha \leq C([f]_\alpha ||g||_0 + [g]_\alpha ||f||_0),$$
$$||fg||_m \leq C(m) (||f||_m ||g||_0 + ||g||_m ||f||_0),$$
$$[f \circ g]_m \leq C(\|f\|_1[g]_m + \|\nabla f\|_{m-1} ||g||_m^n). \quad (5.1)$$

5.2. Condition on the parameter. To simplify the computation, as in [3], we assume the following conditions on $\mu, \ell \leq 1, \lambda_{q+1} \geq 1$:

$$\frac{1}{\delta_q^2 \lambda_q} \leq \frac{1}{\ell \lambda_{q+1}} \leq \frac{1}{\lambda_{q+1}} \leq \frac{1}{\mu}, \quad (5.2)$$

where $\beta > 0$ is a number which will be determined later. These conditions imply

$$\frac{1}{\delta_q^2 \lambda_{q+1}} \leq \frac{1}{\mu} \leq \frac{1}{\delta_q^2 \lambda_q}, \quad \frac{1}{\lambda_{q+1}} \leq \ell \leq \frac{1}{\lambda_q}. \quad (5.3)$$

5.3. Estimate on velocity perturbation. In this subsection, we collect some estimates on the velocity perturbation.

Lemma 5.1. Assume (5.2) holds. For any $l \in \mathbb{Z}$ and $t$ in the range $|\mu t - l| < 1$, we have

$$||\nabla \Phi_t||_0 \leq C, \quad ||\nabla \Phi_t - \text{Id}||_0 \leq C \delta_q^2 \lambda_q \mu^{-1},$$
$$||\nabla \Phi_t||_N \leq C(N) \delta_q^2 \lambda_q \mu^{-1} \ell^{-N}, \quad N \geq 1,$$
$$||\partial_t \Phi_t||_N \leq C(N) \delta_q^2 \lambda_q (\ell^{-N+1} + \mu^{-1} \ell^{-N}). \quad (5.4)$$
Moreover, there hold

\[ \|a_{kl}\|_0 + \|L_{kl}\|_0 \leq C\delta_{q+1}^{\frac{1}{2}}, \quad \|a_{kl}\|_N \leq C(N)\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{1-N}, \quad N \geq 1, \]
\[ \|L_{kl}\|_N \leq C(N)\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{-N}, \quad N \geq 1, \]
\[ \|\partial_t v_{\ell} \cdot \nabla L_{kl}\|_N \leq C(N)\delta_{q+1}^{\frac{1}{2}}\delta_q^{\frac{1}{2}}\lambda_{q,\ell}^{-N}, \]
\[ \|\partial_t L_{kl}\|_N \leq C(N)\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{-N-1}, \quad N \geq 0; \]
\[ \|\phi_{kl}\|_N \leq C(N)\lambda_{q+1}^{\delta_q^{\frac{1}{2}}\lambda_{q,\ell}^{-N+1}} + C\left(\delta_q^{\frac{1}{2}}\lambda_{q,\ell}^{-N+1}\right)^N \]
\[ \leq C(N)\lambda_{q+1}^{N(1-\beta)} , \quad N \geq 1, \]
\[ \|\partial_t \phi_{kl}\|_0 \leq C\delta_{q}^{\frac{1}{2}}\lambda_{q,\ell}^{-1}(\ell + \mu^{-1}), \quad \|\partial_t \phi_{kl}\|_N \leq C(N)\lambda_{q+1}^{N+1}, \quad \forall N \geq 1. \quad (5.5) \]

Consequently, there hold

\[ \|w_{\ell}\|_N \leq C(N)\delta_{q+1}^{\frac{1}{2}}\frac{\delta_q^{\frac{1}{2}}\lambda_{q,\ell}^{N}}{\mu}, \quad N \geq 0, \]
\[ \|w_{0}\|_N \leq \frac{M}{2}\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{N} + C(N)\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{N(1-\beta)}, \quad N \geq 1. \quad (5.6) \]

Proof. Estimate on \( \Phi \): The first estimate in (5.4) can be directly obtained by using (A.3), the second and third estimates in (5.4) can be directly obtained by using (A.5). By the equation (4.7), (4.1) and the elementary inequality (5.1), we deduce that

\[ \|\partial_t \Phi_{\ell}\|_N \leq \|v_{\ell} \cdot \nabla \Phi_{\ell}\|_N \leq C(N)\left(\|v_{\ell}\|_N\|\nabla \Phi_{\ell}\|_0 + \|v_{\ell}\|_0\|\nabla \Phi_{\ell}\|_N\right) \]
\[ \leq C(N)\delta_{q}^{\frac{1}{2}}\lambda_{q,\ell}(\ell^{-N+1} + \mu^{-1}\ell^{-N}). \]

This is the fourth estimate in (5.4).

Estimates on \( a_{kl}, L_{kl} \): Firstly, by (4.6), it’s easy to obtain

\[ \|a_{kl}\|_0 \leq C\delta_{q+1}^{\frac{1}{2}}. \]

Then, by (A.3), (4.2) and (4.1), we deduce that

\[ \|\hat{R}_{\ell,\ell}\|_0 \leq C\delta_{q+1}^{\frac{1}{2}}, \quad \|\hat{R}_{\ell,\ell}\|_N \leq C\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{-N+1}. \quad (5.7) \]

Recalling (4.9), (4.10), (5.7), (5.1) and (5.3), it’s easy to obtain

\[ \|a_{kl}\|_N \leq C(N)\delta_{q+1}^{\frac{1}{2}}\left(\|\hat{R}_{\ell,\ell}\|_1 \left\|\frac{\hat{R}_{\ell,\ell}}{\rho_{\ell}}\right\|_N + \|\hat{R}_{\ell,\ell}\|_1 \right) \]
\[ \leq C(N)\delta_{q+1}^{\frac{1}{2}}\left(\lambda_{q,\ell}^{-N+1} + \lambda_{q}^{N}\right) \leq C(N)\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{-N+1}, \quad N \geq 1. \]

for any \( N \geq 1 \).

Recalling (4.15) and (5.2), we deduce that

\[ \|L_{kl}\|_0 \leq C\left(\|a_{kl}\|_0 + \|a_{kl}\|_1\lambda_{q+1}^{-1} + \|a_{kl}\|_N\left(\nabla \Phi_{\ell} - \text{Id}\right)\right) \]
\[ \leq C\delta_{q+1}^{\frac{1}{2}}\left(1 + \lambda_{q,\ell}^{-1} + \delta_q^{\frac{1}{2}}\lambda_{q,\ell}^{-1}\right) \leq C\delta_{q+1}^{\frac{1}{2}}, \]
\[ \|L_{kl}\|_N \leq C(N)\left(\|a_{kl}\|_N + \|a_{kl}\|_{N+1}\lambda_{q+1}^{-1} + \|a_{kl}\|_N\left(\nabla \Phi_{\ell} - \text{Id}\right)\right) \]
\[ \leq C(N)\delta_{q+1}^{\frac{1}{2}}\lambda_{q,\ell}^{-N} \left(\lambda_{q,\ell} + \delta_q^{\frac{1}{2}}\lambda_{q,\ell}^{-1}\right), \quad \forall N \geq 1. \]
Notice that
\[(\partial_t + v_\ell \cdot \nabla) a_{kl} = 0, \quad (\partial_t + v_\ell \cdot \nabla) \nabla^\perp a_{kl} = -\sum_{j=1}^{2}(\nabla^\perp v_j^\ell) \partial_j a_{kl},\]
\[(\partial_t + v_\ell \cdot \nabla) \nabla^\perp \Phi_l = -\sum_{j=1}^{2}(\nabla^\perp v_j^\ell) \partial_j \Phi_l,\]
Thus, we obtain
\[(\partial_t + v_\ell \cdot \nabla) L_{kl} = \frac{-\sum_{j=1}^{2}(\nabla^\perp v_j^\ell) \partial_j a_{kl} - \lambda_{q+1} a_{kl} \sum_{j=1}^{2}(\nabla^\perp v_j^\ell) \partial_j \Phi_l}{\lambda_{q+1}},\]
hence by \((4.1), (5.1)\) and \((5.2)\), we deduce that
\[
\| (\partial_t + v_\ell \cdot \nabla) L_{kl} \|_N \leq C(N) \lambda_{q+1}^{-1} \| \nabla v_\ell \nabla a_{kl} \|_N + C \| a_{kl} \nabla v_\ell \nabla \Phi_l \|_N \\
\leq C(N) \lambda_{q+1}^{-1} (\| \nabla v_\ell \|_N \| \nabla a_{kl} \|_0 + \| \nabla v_\ell \|_0 \| \nabla a_{kl} \|_N ) \\
+ C(N) (\| a_{kl} \|_N \| \nabla v_\ell \|_0 \| \nabla \Phi_l \|_0 + \| a_{kl} \|_0 \| \nabla v_\ell \|_N \| \nabla \Phi_l \|_0 ) \\
\leq C(N) \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{-1} \ell^{-N} + C(N) \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} \ell^{-N} \\
\leq C(N) \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} \ell^{-N}.\]
Finally, by \((5.2)\), we deduce that
\[
\| \partial_t L_{kl} \|_N \leq \| (\partial_t + v_\ell \cdot \nabla) L_{kl} \|_N + \| v_\ell \cdot \nabla L_{kl} \|_N \\
\leq C(N) \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} \ell^{-N} + C(N) \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} \ell^{-N-1} \leq C(N) \delta_{q+1}^{\frac{1}{2}} \ell^{-N-1}.\]
**Estimate on \(\phi_{kl}\):** By \((5.1)\), we obtain
\[
\| \phi_{kl} \|_1 \leq C \lambda_{q+1} \| \nabla \Phi_l - \text{Id} \|_0 \leq C \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \mu^{-1},
\]
\[
\| \phi_{kl} \|_N \leq C(N)(\lambda_{q+1} \| \Phi_l \|_N + (\lambda_{q+1} \| \nabla \Phi_l - \text{Id} \|_0 )^N ) \\
\leq C(N) \lambda_{q+1} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \mu^{-1} \ell^{-N+1} + C(N) (\delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \mu^{-1} )^N , \quad \forall N \geq 2.
\]
Furthermore, by \((5.2)\), we arrive at
\[
\| \phi_{kl} \|_N \leq C(N) \lambda_{q+1} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \mu^{-1} \ell^{-N+1} + C(N) (\delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \mu^{-1} )^N \leq C(N) \lambda_{q+1}^{N(1-\beta)}, \quad \forall N \geq 1.
\]
A direct computation gives
\[
\partial_t \phi_{kl} = i \lambda_{q+1} k^\perp \cdot \partial_t \Phi_l \phi_{kl},
\]
thus, by \((5.1)\) and \((5.4)\), we obtain
\[
\| \partial_t \phi_{kl} \|_0 \leq C \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} (\ell + \mu^{-1}),
\]
\[
\| \partial_t \phi_{kl} \|_N \leq C(N) \lambda_{q+1} (\| \partial_t \Phi_l \|_N + \| \partial_t \Phi_l \|_0 \| \phi_{kl} \|_N ) \\
\leq C(N) \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} (\ell^{-N+1} + \mu^{-1} \ell^{-N} ) \\
+ C(N) \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} (\ell + \mu^{-1} ) (\lambda_{q+1} \delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \mu^{-1} \ell^{-N+1} + C(N) (\delta_{q+1}^{\frac{1}{2}} \lambda_{q+1} \mu^{-1} )^N ) \\
\leq C(N) \lambda_{q+1}^N, \quad \forall N \geq 1.
\]
Estimate on $w_o, w_c$: Recalling (4.12), by (5.5), we deduce that
\[
\|w_o\|_1 \leq C \sum_{l \in Z} \sum_{k \in \Lambda(l)} \chi_l(t) \|a_{kl} \phi_{kl} e^{i \lambda_{q+1} k \cdot x}\|_1
\leq C \sum_{l \in Z} \sum_{k \in \Lambda(l)} \chi_l(t) (\|a_{kl}\|_1 + \|a_{kl}\|_0 \|\phi_{kl}\|_1 + \|a_{kl}\|_0 \lambda_{q+1})
\leq \frac{M}{2} \delta_{q+1}^2 \lambda_{q+1} + C \delta_{q+1}^2 \lambda_{q+1}
\|w_o\|_N \leq C \sum_{l \in Z} \sum_{k \in \Lambda(l)} \chi_l(t) (\|a_{kl}\|_N + \|a_{kl}\|_0 \|\phi_{kl}\|_N + \|a_{kl}\|_0 \lambda_{q+1}^N)
\leq \frac{M}{2} \delta_{q+1}^2 \lambda_{q+1}^N + C (N) \delta_{q+1}^2 \lambda_{q+1}^{N(1-\beta)}, N = 2, 3.
\]

Recalling (4.13) and (5.2), we know that
\[
\|w_c\|_N \leq C(N) \sum_{l \in Z} \sum_{k \in \Lambda(l)} \chi_l(t) \lambda_{q+1}^{-1} \left(\|a_{kl}\|_{N+1} + \lambda_{q+1} \|a_{kl}(\nabla \Phi_l - \text{Id})\|_N\right)
+ \left[\|a_{kl}\|_1 + \lambda_{q+1} \|a_{kl}(\nabla \Phi_l - \text{Id})\|_0 \|e^{i \lambda_{q+1} \Phi_l \cdot k \cdot x}\|_N\right)
\leq C(N) \lambda_{q+1}^{-1} \left(\delta_{q+1}^2 \lambda_{q+1} \mu^{-1} + \lambda_{q+1} \delta_{q+1}^2 \lambda_{q+1} \mu^{-1} + \lambda_{q+1} \delta_{q+1}^2 \lambda_{q+1} \mu^{-1} \lambda_{q+1}^N\right)
\leq C(N) \delta_{q+1}^{2 \frac{1}{N}} \lambda_{q+1}^{-1}, \quad N \geq 0.
\]

5.4. Estimate on temperature perturbation. Next, we estimate the difference of temperature $	heta_1 - \theta$. A direct computation gives that
\[
\begin{cases}
\partial_t (\theta_1 - \theta) + v_1 \cdot \nabla (\theta_1 - \theta) - \Delta (\theta_1 - \theta) = -(v_1 - v) \cdot \nabla \theta,
(\theta_1 - \theta)(0, x) = 0.
\end{cases}
\]

Taking the $L^2$ inner product with $\theta_1 - \theta$ and integrating by parts, we arrive at
\[
\|(\theta_1 - \theta)(t, \cdot)\|_{L^2}^2 + 2 \int_0^t \|\nabla (\theta_1 - \theta)(s, \cdot)\|_{L^2}^2 ds
= 2 \int_0^t \int_{T^2} (v_1 - v) \cdot \nabla (\theta_1 - \theta) \theta(s, x) dx ds
\leq \int_0^t \|\nabla (\theta_1 - \theta)(s, \cdot)\|_{L^2}^2 ds + 4 \|\theta\|_{0}^2 \|v_1 - v\|_{0}^2.
\]

Thus, by (4.19), there holds
\[
\|(\theta_1 - \theta)(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla (\theta_1 - \theta)(s, \cdot)\|_{L^2}^2 ds \leq 4 \|\theta\|_{0}^2 \|v_1 - v\|_{0}^2.
\]

Moreover,
\[
-(v_1 - v) \cdot \nabla \theta = - \sum_{l} \sum_{k \in \Lambda(l)} \chi(t) \phi_{kl} L_{kl} \cdot \nabla e^{i \lambda_{q+1} k \cdot x}.
\]

By (3.16), we need estimates on $\partial^\alpha \theta$ for $|\alpha| \leq 4$. To this end, we establish the following estimates.
**Lemma 5.1.** Let $\theta$ be a smooth solution of the following transport-diffusion equation

$$
\begin{aligned}
\left\{ \begin{array}{ll}
\partial_t \theta + v \cdot \nabla \theta - \Delta \theta = 0, \\
\text{div} v = 0, \\
\theta(0, x) = \theta^0(x).
\end{array} \right.
\end{aligned}
$$

(5.10)

Then, there hold

$$
\begin{align*}
\| \nabla \theta \|_{L^\infty L^2} & \leq C \| v \|_0^2 \| \theta^0 \|_2, \\
\| \nabla^2 \theta \|_{L^\infty L^2} & \leq C (\| v \|_0^2 \| \theta^0 \|_2 \| \nabla v \|_0 + \| v \|_4 \| \theta^0 \|_2^3), \\
\| \nabla^3 \theta \|_{L^\infty L^2} & \leq C (\| v \|_0, \| \theta^0 \|_2) (\| \nabla^2 v \|_0 + \| \nabla v \|_0), \\
\| \nabla^4 \theta \|_{L^\infty L^2} & \leq C (\| v \|_0, \| \theta^0 \|_2) (\| \nabla^3 v \|_0 + \| \nabla^2 v \|_0 \| \nabla v \|_0 + \| \nabla v \|_0^2).
\end{align*}
$$

(5.11)

**Proof.** **Step 1:** Acting $\partial_j$ on both sides of (5.10), we obtain

$$
\partial_t \partial_j \theta + v \cdot \nabla (\partial_j \theta) - \Delta \partial_j \theta = -\partial_j v \cdot \nabla \theta.
$$

(5.12)

Taking the $L^2$ inner products with $\partial_j \theta$ and integrating by parts, we deduce that

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \partial_j \theta(t, \cdot) \|_{L^2}^2 + \| \nabla \partial_j \theta(t, \cdot) \|_{L^2}^2 & \leq \int_{T^2} [v \cdot \nabla \partial_j \theta \partial_j \theta + v \cdot \nabla \theta \partial_{jj} \theta] \, dx \\
& \leq \| v \|_0 \| \nabla \theta(t, \cdot) \|_{L^2} \| \partial_j \theta(t, \cdot) \|_{L^2}.
\end{align*}
$$

Summing about $j = 1, 2$, we arrive at

$$
\frac{1}{2} \frac{d}{dt} \| \theta(t, \cdot) \|_{L^2}^2 + \| \nabla \theta(t, \cdot) \|_{L^2}^2 \leq \| v \|_0 \| \nabla \theta(t, \cdot) \|_{L^2} \| \Delta \theta(t, \cdot) \|_{L^2}.
$$

Moreover, by the interpolation inequality $\| \nabla \theta(t, \cdot) \|_{L^2} \leq \| \theta(t, \cdot) \|_{L^4}^{\frac{1}{2}} \| \nabla^2 \theta(t, \cdot) \|_{L^2}^{\frac{1}{2}}$, we arrive at

$$
\frac{1}{2} \frac{d}{dt} \| \theta(t, \cdot) \|_{L^2}^2 + \| \nabla \theta(t, \cdot) \|_{L^2}^2 \leq \| v \|_0 \| \theta(t, \cdot) \|_{L^2} \| \nabla \theta(t, \cdot) \|_{L^2}^{\frac{3}{2}}.
$$

Thus, by the Hölder inequality and noticing the $\| \theta(t, \cdot) \|_{L^2} \leq \| \theta^0 \|_2$, we obtain

$$
\| \nabla \theta \|_{L^\infty L^2} \leq C \| v \|_0^2 \| \theta^0 \|_2,
$$

(5.13)

which gives the first estimate in (5.11).

**Step 2:** Acting $\partial_i$ on both sides of (5.12), we obtain

$$
\partial_i \partial_{ij} \theta + v \cdot \nabla (\partial_{ij} \theta) - \Delta \partial_{ij} \theta = -\partial_{ij} v \cdot \nabla \theta - \partial_i v \cdot \nabla \partial_j \theta - \partial_j v \cdot \nabla \partial_i \theta.
$$

(5.14)

Taking the $L^2$ inner products with $\partial_{ij} \theta$ and integrating by parts, we deduce that

$$
\int_{T^2} \partial_{ij} v \cdot \nabla \partial_{ij} \theta \, dx = \int_{T^2} [v \cdot \nabla \partial_{ij} \theta \partial_{ij} \theta + v \cdot \nabla \partial_i \theta \partial_{jj} \theta - \partial_{ij} v \cdot \nabla \theta \partial_{ijj} \theta] \, dx.
$$

We deal with the right side term by term. Firstly, integrating by parts, we obtain

$$
\int_{T^2} \partial_{ij} v \cdot \nabla \partial_i \theta \partial_{jj} \theta \, dx = \int_{T^2} v \cdot \nabla \partial_i \theta \partial_{ijj} \theta \, dx,
$$

$$
\int_{T^2} \partial_{ij} v \cdot \nabla \partial_i \theta \partial_{jj} \theta \, dx = \int_{T^2} v \cdot \nabla \partial_i \theta \partial_{ijj} \theta \, dx,
$$

$$
\int_{T^2} \partial_{ij} v \cdot \nabla \partial_i \theta \partial_{jj} \theta \, dx = - \int_{T^2} v \cdot \nabla \partial_i \theta \partial_{ijj} \theta \, dx.
$$
Summing about $i, j = 1, 2$, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^2 \theta(t, \cdot) \|_{L^2}^2 + \| \nabla^3 \theta(t, \cdot) \|_{L^2}^2 \\
\leq C \| v \|_0 \| \nabla^3 \theta(t, \cdot) \|_{L^2}^2 \leq C \| v \|_0 \| \nabla \theta(t, \cdot) \|_{L^2}^2 \| \nabla^2 \theta(t, \cdot) \|_{L^2}^2 \\
\leq C \| v \|_0 \| \nabla^3 \theta(t, \cdot) \|_{L^2}^2 + C \| v \|_0 \| \theta^0 \|_2 \| \nabla v \|_0 \| \nabla^3 \theta(t, \cdot) \|_{L^2}^2.
\]
By interpolation inequalities and (5.13), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^2 \theta(t, \cdot) \|_{L^2}^2 + \| \nabla^3 \theta(t, \cdot) \|_{L^2}^2 \\
\leq C \| v \|_0^2 \| \theta^0 \|_2^2 \| \nabla^3 \theta(t, \cdot) \|_{L^2}^2 + C \| v \|_0^2 \| \theta^0 \|_2 \| \nabla v \|_0 \| \nabla^3 \theta(t, \cdot) \|_{L^2}^2.
\]
Thus, by the Hölder inequality, we obtain
\[
\| \nabla^2 \theta \|_{L^\infty L^2} \leq C(\| v \|_0^2 \| \theta^0 \|_2 \| \nabla v \|_0 + \| v \|_0^2 \| \theta^0 \|_2^2),
\]
which is the second estimate in (5.11).

**Step 3:** Acting $\partial_k$ on both sides of (5.14), we obtain
\[
\begin{align*}
\partial_i \partial_j \partial_k \theta + v \cdot \nabla (\partial_i \partial_j \partial_k \theta) - \Delta \partial_i \partial_j \partial_k \theta \\
= -\partial_i \partial_j \partial_k v \cdot \nabla \theta - \partial_i \partial_j v \cdot \nabla \partial_k \theta - \partial_j \partial_k v \cdot \nabla \partial_i \theta \\
- \partial_i v \cdot \nabla \partial_j \partial_k \theta - \partial_j v \cdot \nabla \partial_i \partial_k \theta - \partial_k v \cdot \nabla \partial_i \partial_j \theta.
\end{align*}
\]
Taking inner products with $\partial_i \partial_j \partial_k \theta$ and integrating by parts, the left side is
\[
\frac{1}{2} \frac{d}{dt} \| \partial_i \partial_j \partial_k \theta(t, \cdot) \|_{L^2}^2 + \| \nabla \partial_i \partial_j \partial_k \theta(t, \cdot) \|_{L^2}^2.
\]
We deal with the right side term by term. Firstly,
\[
\int \partial_i \partial_j \partial_k v \cdot \nabla \partial_i \partial_j \partial_k \theta dx = - \int \partial_i \partial_j \partial_k v \cdot \nabla \partial_i \partial_j \theta + \partial_j \partial_k v \cdot \nabla \partial_i \partial_k \theta dx.
\]
By interpolation inequalities
\[
\| \nabla^2 \theta \|_{L^2} \leq \| \nabla \theta \|_{L^2}^\frac{3}{2} \| \nabla^2 \theta \|_{L^2}^{\frac{1}{2}}, \quad \| \nabla^3 \theta \|_{L^2} \leq \| \nabla \theta \|_{L^2} \| \nabla^4 \theta \|_{L^2}^{\frac{3}{2}},
\]
we deduce that
\[
\left| \int \partial_i \partial_j \partial_k v \cdot \nabla \partial_i \partial_j \partial_k \theta dx \right| \leq C \| \nabla^2 v \|_0 \| \nabla \theta \|_{L^2} \| \nabla^4 \theta \|_{L^2}.
\]
Similarly, there hold
\[
\left| \int \partial_i \partial_j \partial_k v \cdot \nabla \partial_k \theta + \partial_k v \cdot \nabla \partial_i \partial_j \theta \right| \leq C \| \nabla^2 v \|_0 \| \nabla \theta \|_{L^2} \| \nabla^4 \theta \|_{L^2}.
\]
Moreover, integrating by parts and using the divergence free condition $\text{div} v = 0$, we obtain
\[
\left| \int \partial_i v \cdot \nabla \partial_j \partial_k \theta + \partial_j v \cdot \nabla \partial_i \partial_k \theta + \partial_k v \cdot \nabla \partial_i \partial_j \theta \right| \partial_j \partial_k \theta dx \leq C \| v \|_0 \| \nabla^4 \theta \|_{L^2} \| \nabla^2 \theta \|_{L^2}^{\frac{3}{2}}.
\]
Finally, summing over $i, j, k = 1, 2$, by the Hölder inequality, we deduce that
\[
\| \nabla^3 \theta \|_{L^\infty L^2} \leq C(\| v \|_0, \| \theta^0 \|_2)(\| \nabla^2 v \|_0 + \| \nabla v \|_0).
\]

**Step 4:** Acting $\partial_i$ on equation (5.13), following the above argument, we deduce that
\[
\| \nabla^4 \theta \|_{L^\infty L^2} \leq C(\| v \|_0, \| \theta^0 \|_2)(\| \nabla^3 v \|_0 + \| \nabla^2 v \|_0 \| \nabla \theta \|_{L^\infty L^2} + \| \nabla^3 v \|_0 \| \nabla \theta \|_{L^\infty L^2}).
\]
Thus, combining (5.13)–(5.17), we arrive at
\[
\| \nabla^4 \theta \|_{L^\infty L^2} \leq C(\| v \|_0, \| \theta^0 \|_2)(\| \nabla^3 v \|_0 + \| \nabla^2 v \|_0 \| \nabla v \|_0 + \| \nabla v \|_0^2).
\]
Collecting (5.13)–(5.18) together, we complete the proof of this lemma. 

\[\Box\]
Proposition 5.2. There hold
\[
\|\nabla(\theta_1 - \theta)\|_{L^\infty L^2} \leq C\delta_{q+1}^1 (\lambda_{q+1}^{-1} \ell^{-1} + \delta_{q}^1 \lambda_0 \mu^{-1}),
\]
\[
\|\nabla^2(\theta_1 - \theta)\|_{L^\infty L^2} \leq C\delta_{q+1}^1 (\ell^{-1} + \delta_{q}^1 \lambda_0 \mu^{-1}).
\]
(5.19)

Proof. Estimate on \(\|\nabla(\theta_1 - \theta)\|_{L^\infty L^2}\): Recalling \(b_{kl} = \chi(t)\phi_{kl}L_{kl} \cdot \nabla \theta\), thus by (4.1), (5.5), (5.11), and the parameter assumption (5.22), we deduce that for any \(l \in Z\) and \(k \in \Lambda_0 \cup \Lambda_l, |\mu t - l| < 1\), there holds
\[
\|b_{kl}\|_{L^\infty L^2} \leq C\delta_{q+1}^1,
\]
\[
\|\nabla b_{kl}\|_{L^\infty L^2} \leq C \left( \|L_{kl}\|_1 \|\phi_{kl}\|_0 \|\nabla \theta\|_{L^\infty L^2} + \|L_{kl}\|_1 \|\phi_{kl}\|_1 \|\nabla \theta\|_{L^\infty L^2} \right)
\]
\[
\leq C\delta_{q+1}^1 (\ell^{-1} + \delta_{q}^1 \lambda_0 \lambda_{q+1} \mu^{-1} + \delta_{q}^1 \lambda_q),
\]
\[
\|\nabla^2 b_{kl}\|_{L^\infty L^2} \leq C \left( \|L_{kl}\|_2 \|\phi_{kl}\|_0 \|\nabla \theta\|_{L^\infty L^2} \right)
\]
\[
\leq C\delta_{q+1}^1 (\ell^{-2} + \delta_{q}^1 \lambda_0 \lambda_{q+1} \mu^{-1} \ell^{-1} + \delta_{q}^1 \lambda_0^2 \lambda_{q+1} \mu^{-2} + \delta_{q}^1 \lambda_q^2),
\]
(5.21)

Moreover, due to
\[
\partial_t b_{kl} = \chi(t)\phi_{kl}L_{kl} \cdot \nabla \theta + \chi(t)\partial_t \phi_{kl}L_{kl} \cdot \nabla \theta + \chi(t)\phi_{kl}\partial_t L_{kl} \cdot \nabla \theta + \phi_{kl}L_{kl} \cdot \nabla \partial_t \theta,
\]
thus, by (5.5), (5.11), the parameter assumption (5.22) and the equation (5.10), for any \(l \in Z\) and \(k \in \Lambda_0 \cup \Lambda_1, |\mu t - l| < 1\), there holds
\[
\|\partial_t b_{kl}\|_{L^\infty L^2} \leq C\delta_{q+1}^2 (\mu + \lambda_q + \delta_{q}^1 \lambda_0 (\ell + \mu^{-1}) + \ell^{-1} + \delta_{q}^2 \lambda_q^2),
\]
(5.22)

where we used \(|\chi(t)| \leq C \mu|\).

Collecting (5.20), (5.22) and using the parameter assumption (5.22), we deduce
\[
\lambda_{q+1} \left| \sum_l b_{kl}\right|_{L^\infty L^2} + \left| \sum_l \partial_t b_{kl}\right|_{L^\infty L^2} + \left| \sum_l \Delta b_{kl}\right|_{L^\infty L^2} + \lambda_{q+1} \left| \sum_l \nabla b_{kl}\right|_{L^\infty L^2}
\]
\[
\leq C\delta_{q+1}^1 (\lambda_{q+1}^{-1} \ell^{-1} + \delta_{q}^1 \lambda_0 \lambda_{q+1} \mu^{-1}).
\]

Thus, by (3.16), we deduce that
\[
\|\nabla(\theta_1 - \theta)\|_{L^\infty L^2} \leq C\delta_{q+1}^1 (\lambda_{q+1}^{-1} \ell^{-1} + \delta_{q}^1 \lambda_0 \mu^{-1}).
\]

Estimate on \(\|\nabla^2(\theta_1 - \theta)\|_{L^\infty L^2}\): Noticing (5.21), a direct computation gives that
\[
\|\nabla \partial_t b_{kl}\|_{L^\infty L^2} \leq C \mu \|\nabla (\phi_{kl}L_{kl} \cdot \nabla \theta)\|_{L^\infty L^2} + \|\nabla (\partial_t \phi_{kl}L_{kl} \cdot \nabla \theta)\|_{L^\infty L^2}
\]
\[
+ \|\nabla (\phi_{kl}\partial_t L_{kl} \cdot \nabla \theta)\|_{L^\infty L^2} + \|\nabla (\phi_{kl}L_{kl} \cdot \nabla \partial_t \theta)\|_{L^\infty L^2}.
\]
Moreover, by \((4.1), (5.3), (5.11)\) and the parameter assumption \((5.2)\), we deduce that

\[
\| \nabla (\phi_{kl} L_{kl} \cdot \nabla \theta) \|_{L^\infty L^2} \leq \| \phi_{kl} \|_1 \| L_{kl} \|_0 \| \nabla \theta \|_{L^\infty L^2} + \| \phi_{kl} \|_0 \| L_{kl} \|_1 \| \nabla \theta \|_{L^\infty L^2} \\
+ \| \phi_{kl} \|_0 \| L_{kl} \|_0 \| \nabla^2 \theta \|_{L^\infty L^2}
\]

\[
\leq C \delta_{q+1}^2 (\ell^{-1} + \delta_q \lambda_q \lambda_{q+1} \mu^{-1}),
\]

\[
\| \nabla (\partial_t \phi_{kl} L_{kl} \cdot \nabla \theta) \|_{L^\infty L^2} \leq \| \partial_t \phi_{kl} \|_1 \| L_{kl} \|_0 \| \nabla \theta \|_{L^\infty L^2} + \| \partial_t \phi_{kl} \|_0 \| L_{kl} \|_1 \| \nabla \theta \|_{L^\infty L^2} \\
+ \| \partial_t \phi_{kl} \|_0 \| L_{kl} \|_0 \| \nabla^2 \theta \|_{L^\infty L^2}
\]

\[
\leq C \delta_{q+1}^2 \lambda_{q+1} \delta_q \lambda_q \lambda_{q+1} \mu^{-1} + C \delta_{q+1}^2 \lambda_q \ell^{-1}
\]

\[
\leq C \delta_{q+1}^2 \lambda_{q+1} \lambda_q \lambda_{q+1} \mu^{-1} + C \delta_{q+1}^2 \lambda_q \ell^{-1},
\]

\[
\| \nabla (\phi_{kl} \partial_t L_{kl} \cdot \nabla \theta) \|_{L^\infty L^2} \leq \| \phi_{kl} \|_1 \| \partial_t L_{kl} \|_0 \| \nabla \theta \|_{L^\infty L^2} + \| \phi_{kl} \|_0 \| \partial_t L_{kl} \|_1 \| \nabla \theta \|_{L^\infty L^2} \\
+ \| \phi_{kl} \|_0 \| \partial_t L_{kl} \|_0 \| \nabla^2 \theta \|_{L^\infty L^2}
\]

\[
\leq C \delta_{q} \lambda_q \lambda_{q+1} \mu^{-1} \left( \delta_{q+1}^3 \delta_q^2 \lambda_q + \delta_{q+1}^2 \ell^{-1} \right)
\]

\[
+ C \left( \delta_{q+1}^3 \delta_q^2 \lambda_q \ell^{-1} + \delta_{q+1}^2 \ell^{-2} \right)
\]

\[
+ C \left( \delta_{q+1}^3 \delta_q^2 \lambda_q + \delta_{q+1}^2 \ell^{-1} \right) \delta_q \lambda_q
\]

\[
\leq C \delta_{q}^2 \lambda_q \lambda_{q+1} \mu^{-1} \left( \delta_{q+1}^3 \delta_q^2 \lambda_q + \delta_{q+1}^2 \ell^{-1} \right) + C \delta_{q+1} \ell^{-2},
\]

\[
\| \nabla (\phi_{kl} L_{kl} \cdot \nabla \partial_t \theta) \|_{L^\infty L^2} \leq \| \phi_{kl} \|_1 \| L_{kl} \|_0 \| \nabla \partial_t \theta \|_{L^\infty L^2} + \| \phi_{kl} \|_0 \| L_{kl} \|_1 \| \nabla \partial_t \theta \|_{L^\infty L^2} \\
+ \| \phi_{kl} \|_0 \| L_{kl} \|_0 \| \nabla^2 \partial_t \theta \|_{L^\infty L^2}
\]

\[
\leq C \delta_{q}^2 \lambda_q \lambda_{q+1} \mu^{-1} \delta_{q+1}^3 \delta_q^2 \lambda_q + C \delta_{q+1}^3 \delta_q^2 \lambda_q \ell^{-1} + C \delta_{q+1} \delta_q \lambda_q^3
\]

\[
\leq C \delta_{q+1} \delta_q \lambda_q^2 \lambda_{q+1} \mu^{-1} + \ell^{-1} \right).
\]

Finally, summing these estimate together and using the parameter assumption \((5.2)\), we obtain

\[
\| \nabla \partial_t b_{kl} \|_{L^\infty L^2} \leq C \delta_{q+1} \left( \lambda_{q+1}^2 + \delta_q \lambda_q^3 \lambda_{q+1} \mu^{-1} + \delta_q \lambda_q^2 \ell^{-1} \right).
\] (5.23)

By \((5.20), (5.22), (5.23)\) and the parameter assumption \((5.2)\), we deduce that for any \(l \in \mathcal{Z} \) and \(k \in \Lambda_0 \cup \Lambda_1, |\mu t - l| < 1\), there holds

\[
\| (\lambda_{q+1}^2 b_{kl}, \partial_t b_{kl}, \nabla b_{kl}, \lambda_{q+1} \nabla b_{kl}) \|_{L^\infty L^2}
\]

\[
\leq C \delta_{q+1} \lambda_{q+1} \left( \lambda_{q+1}^2 + \delta_q \lambda_q^3 \lambda_{q+1} \mu^{-1} + \delta_q \lambda_q^2 \ell^{-1} \right),
\]

\[
\| (\nabla \partial_t b_{kl}, \lambda_{q+1} \nabla b_{kl}, \nabla^2 b_{kl}, \lambda_{q+1} \nabla^2 b_{kl}) \|_{L^\infty L^2}
\]

\[
\leq C \delta_{q+1} \lambda_{q+1} \lambda_q \mu^{-1} \left( \delta_q \lambda_q^3 \lambda_{q+1} \mu^{-1} \right),
\]

\[
\| (\nabla^2 b_{kl}, \lambda_{q+1} \nabla b_{kl}, \lambda_{q+1}^2 b_{kl}) \|_{L^\infty L^2} \leq C \delta_{q+1} \lambda_{q+1}^2.
\]
Thus, by (3.16), we deduce that
\[
\|\nabla^2(\theta_1 - \theta)\|_{L^\infty L^2} \leq C\delta_{q+1}^{\frac{1}{4}}\lambda_{q+1}^{-\frac{1}{2}}\lambda_q(\epsilon^{-1} + \delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}\mu^{-1})
\]
\[+ C\delta_{q+1}^{\frac{1}{4}}(\epsilon^{-1} + \delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}\mu^{-1})
\]
\[\leq C\delta_{q+1}^{\frac{1}{4}}(\epsilon^{-1} + \delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}\mu^{-1}).
\]

This completes the proof of this proposition. \(\square\)

6. Estimate on the Reynold stress

In this section, we prove the following estimate for the Reynold stress.

**Proposition 6.1.** The Reynold stress \(\bar{R}_1\) defined in (4.30) satisfies the estimate
\[
\|\bar{R}_1\|_0 + \frac{\|\bar{R}_1\|_1}{\lambda_{q+1}} \leq C(\varepsilon)\delta_{q+1}^{\frac{1}{4}}(\mu\lambda_{q+1}^{-1+\varepsilon} + \lambda_{q+1}^{-1}\epsilon^{-1} + \delta_{q}^{\frac{1}{2}}\lambda_q\epsilon + \delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}\mu^{-1}).
\]

**Proof.** We deal with it term by term.

**Estimate on \(R^0\):** A direct computation gives
\[
(\partial_t + v_\ell \cdot \nabla)w = \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda_{(l)}} \left( \chi'_l(t)L_{kl} + \chi(t)(\partial_t L_{kl} + v_\ell \cdot \nabla L_{kl}) \right) \phi_{kl} e^{i\lambda_{q+1}k+x}.
\]

Set
\[
\Psi_{kl} := \chi'_l(t)L_{kl} + \chi(t)(\partial_t L_{kl} + v_\ell \cdot \nabla L_{kl});
\]

thus, by \(|\chi'_l(t)| \leq C\mu\) and (5.5), we deduce that for any \(l \in \mathbb{Z}\) and \(k \in \Lambda_0 \cup \Lambda_1\), \(|\mu t - l| < 1\), there hold
\[
\|\Psi_{kl}\|_N \leq C\mu\|L_{kl}\|_N + \|\partial_t L_{kl} + v_\ell \cdot \nabla L_{kl}\|_N
\]
\[\leq C(N)\delta_{q+1}^{\frac{1}{4}}\epsilon^{-N}(\mu + \delta_{q}^{\frac{1}{2}}\lambda_q) \leq C(N)\delta_{q+1}^{\frac{1}{4}}\epsilon^{-N}\mu, \quad \forall N \geq 0.
\]

By (6.2), we deduce that
\[
\|R^0\|_0 \leq C(N, \varepsilon)\sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda_{(l)}} \left( \frac{\|\Psi_{kl}\|_0}{\lambda_{q+1}^{1-\varepsilon}} + \frac{\|\Psi_{kl}\|_N}{\lambda_{q+1}^{N-\varepsilon}} + \frac{\|\Psi_{kl}\|_{N+\varepsilon}}{\lambda_{q+1}^N} \right)
\]
\[\leq C(N, \varepsilon)\left(\delta_{q+1}^{\frac{1}{4}}\epsilon^{-1+\varepsilon} + \delta_{q+1}^{\frac{1}{4}}\epsilon^{-N}\mu\lambda_{q+1}^{-N+\varepsilon}\right).
\]

Combining the parameter assumption (5.2) and taking \(N\) large enough such that \(N\beta > 1\), we arrive at
\[
\|R^0\|_0 \leq C(\varepsilon)\delta_{q+1}^{\frac{1}{4}}\mu\lambda_{q+1}^{-1+\varepsilon}.
\]

The same argument gives
\[
\|R^0\|_1 \leq C(\varepsilon)\delta_{q+1}^{\frac{1}{4}}\mu\lambda_{q+1}^{\varepsilon}.
\]

**Estimate on \(R^1\):** A direct computation gives
\[
w \cdot \nabla v_\ell = \sum_{l \in \mathbb{Z}} \sum_{k \in \Lambda_{(l)}} \chi(t)L_{kl} \cdot \nabla v_\ell \phi_{kl} e^{i\lambda_{q+1}k+x}.
\]

Set
\[
\Omega_{kl} = \chi(t)L_{kl} \cdot \nabla v_\ell \phi_{kl}.
\]
thus, \ref{5.12} and \ref{5.11} implies that for any \( l \in \mathbb{Z} \) and \( k \in \Lambda_0 \cup \Lambda_1 \), \(|\mu t - l| < 1\),

\[
\|\Omega_{kl}\|_0 \leq C\delta_{q+1}^{\frac{1}{2}}\delta_{q}^{\frac{1}{2}}\lambda_q,
\]

\[
\|\Omega_{kl}\|_N \leq C(N)(\|L_{kl}\|_N \|\nabla v_k\|_0 \|\phi_{kl}\|_0 + \|L_{kl}\|_N \|\nabla v_k\|_N \|\phi_{kl}\|_0
+ \|L_{kl}\|_0 \|\nabla v_k\|_0 \|\phi_{kl}\|_N)
\]

\[
\leq C\delta_{q+1}^{\frac{1}{2}}\delta_{q}^{\frac{1}{2}}\lambda_q(\ell^{-N} + \lambda_{q+1}+\delta_{q}^{\frac{1}{2}}\lambda_q\mu^{-1}\ell^{-N+1} + (\lambda_{q+1}+\delta_{q}^{\frac{1}{2}}\lambda_q\mu^{-1})^N), \quad \forall N \geq 1.
\]

Combining the parameter assumption \ref{5.2}, we obtain

\[
\|\Omega_{kl}\|_N \leq C(N)\delta_{q+1}^{\frac{1}{2}}\delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}^{N(1-\beta)}.
\]

By taking \( N \) sufficiently large, a similar argument as for \( R^0 \) gives

\[
\|R^1\|_0 \leq C(\varepsilon)\sum_{l \in \mathbb{Z}}\sum_{k \in \Lambda(0)} \left( \|\Omega_{kl}\|_0 \lambda_{q+1}^{\frac{1}{2}} + \|\Omega_{kl}\|_N \lambda_{q+1}^{\frac{1}{2}} + \frac{\|\Omega_{kl}\|_{N+\varepsilon}}{\lambda_{q+1}^{N+\varepsilon}} \right)
\]

\[
\leq C(\varepsilon)\delta_{q+1}^{\frac{1}{2}}\delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}^{N(1-\beta)},
\]

\[
\|R^1\|_1 \leq C(\varepsilon)\delta_{q+1}^{\frac{1}{2}}\delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}^{N(1-\beta)}.
\]

**Estimate on** \( R^2 \): By Lemma \ref{3.4} \ref{5.19}, we deduce that

\[
\|R^2\|_0 \leq C\|\nabla(\theta_1 - \theta)\|_{L^\infty L^2} \leq C\delta_{q+1}^{\frac{1}{2}}(\lambda_{q+1}^{\frac{1}{2}}\ell^{-1} + \delta_{q}^{\frac{1}{2}}\lambda_q\mu^{-1}),
\]

\[
\|R^2\|_1 \leq C\|\nabla^2(\theta_1 - \theta)\|_{L^\infty L^2} \leq C\delta_{q+1}^{\frac{1}{2}}(\ell^{-1} + \delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}\mu^{-1}).
\]

**Estimate on** \( R^3 \): Recalling \ref{1.29}, we know

\[
R\left(\text{div}(w_o \otimes w_o - \sum_l \chi_l^2 R_{el,l} + P\text{ld})\right) = R\left(T_{osc}^1 + T_{osc}^2\right).
\]

We first deal with \( T_{osc}^1 \). By \ref{5.12} and the parameter assumption \ref{5.3}, we deduce that for any \( l \in \mathbb{Z} \) and \( k \in \Lambda_0 \cup \Lambda_1 \), \(|\mu t - l| < 1\), there holds

\[
\|a_{kl}a_{k'l}\phi_{kl}\phi_{k'l}\|_1 \leq \|a_{kl}a_{k'l}\|_1 \|\phi_{kl}\phi_{k'l}\|_0 + \|a_{kl}a_{k'l}\|_0 \|\phi_{kl}\phi_{k'l}\|_1
\]

\[
\leq C\delta_{q+1}(\lambda_q + \delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}\mu^{-1}) \leq C\delta_{q+1}(\lambda_q + \delta_{q}^{\frac{1}{2}}\lambda_q\lambda_{q+1}\mu^{-1})
\]

\[
\leq C(N)(\|a_{kl}a_{k'l}\|_N \|\phi_{kl}\phi_{k'l}\|_0 + \|a_{kl}a_{k'l}\|_0 \|\phi_{kl}\phi_{k'l}\|_N)
\]

\[
\leq C(N)(\delta_{q+1}\lambda_q\ell^{1-N} + \delta_{q+1}\lambda_{q+1}^{N(1-\beta)}), \quad \forall N \geq 2.
\]

Recalling \ref{4.24}, by \ref{5.2}, choosing \( N \) sufficiently large, we deduce that

\[
\|R(T_{osc}^1)\|_0 \leq C(\varepsilon)\sum_{l \in \mathbb{Z}}\sum_{k,k' \in \Lambda(0), k \neq k'} \lambda_{q+1}^{\frac{1}{2}}(\|\nabla(a_{kl}a_{k'l}\phi_{kl}\phi_{k'l})\|_0
\]

\[
+ \|\nabla(a_{kl}a_{k'l}\phi_{kl}\phi_{k'l})\|_N + \|\nabla(a_{kl}a_{k'l}\phi_{kl}\phi_{k'l})\|_{N+\varepsilon})
\]

\[
\leq C(\varepsilon)\delta_{q+1}\lambda_q\lambda_{q+1}\mu^{-1}.
\]

Similarly, we can obtain

\[
\|R(T_{osc}^1)\|_1 \leq C(\varepsilon)\delta_{q+1}\lambda_q\lambda_{q+1}\mu^{-1}.
\]
Next, we deal with the second term $T^2_{osc}$. From (1.25) and (5.5), we deduce that
\[
\|f_{kk'k''}\|_N \leq \chi_2(t)\chi(t)\|a_{kk'k''}\phi_{kl}\phi_{k'l'}\|_N.
\]
Thus, the same argument as above gives that
\[
\|R(T^2_{osc})\|_0 \leq C(\varepsilon)\delta_{q+1}\delta^\frac{1}{2}\lambda_q\lambda_{q+1}^{-1},
\]
\[
\|R(T^2_{osc})\|_1 \leq C(\varepsilon)\delta_{q+1}\delta^\frac{1}{2}\lambda_q\lambda_{q+1}^{1+\varepsilon}.
\]
Summing the two parts, we obtain
\[
\|R^3\|_0 + \frac{\|R^3\|_1}{\lambda_{q+1}} \leq C(\varepsilon)\delta_{q+1}\delta^\frac{1}{2}\lambda_q\lambda_{q+1}^{-1}.
\]
(6.7)

**Estimate on $R^4$:** By (6.6), we deduce
\[
\|R^4\|_0 \leq C\|w_c\|_0\|w_c\|_0 \leq C\delta_{q+1}\delta^\frac{1}{2}\lambda_q\mu^{-1},
\]
\[
\|R^4\|_1 \leq C(\|w_c\|_1\|w_c\|_0 + \|w_c\|_1) \leq C\delta_{q+1}\delta^\frac{1}{2}\lambda_q\mu^{-1}.
\]
(6.8)

**Estimate on $R^5$:** By (6.6), (4.4), (4.3) and the parameter assumption (5.3), we deduce
\[
\|R^5\|_0 \leq C\|w_c\|_0\|v - v_c\|_0 \leq C\delta^\frac{1}{2}\delta^\frac{1}{2}\lambda_q\lambda_{q+1}^{-1},
\]
\[
\|R^5\|_1 \leq C(\|w_c\|_1\|v - v_c\|_0 + \|w_c\|_1) \leq C\delta^\frac{1}{2}\delta^\frac{1}{2}\lambda_q\lambda_{q+1}^{-1}.
\]
(6.9)

**Estimate on $R^6$:** By (4.3), we deduce
\[
\|R^6\|_0 \leq C\|\nabla R\|_0 \leq C\delta_{q+1}\lambda_q \ell,
\]
\[
\|R^6\|_1 \leq C\|\nabla R\|_1 \leq C\delta_{q+1}\lambda_q.
\]
(6.10)

Finally, summing estimates (6.2)-(6.10), we obtain (6.1).

7. PROOF OF PROPOSITION 2.1

**Proof.** Step 1: Choosing parameter $\mu, \ell$. Recalling that the sequence $\{\delta_q\}_{q \in \mathbb{N}}$ and $\{\lambda_q\}_{q \in \mathbb{N}}$ are chosen to satisfy
\[
\delta_q = a^{-bq}, \quad \lambda_q \in \left[a^{bq^{1+1}}, 2a^{bq^{1+1}}\right],
\]
where $c = 3, b = 2$ and $a > 1$ which will be determined later. Take
\[
\mu = \delta^\frac{1}{2}\lambda^\frac{1}{2} \lambda_{q+1}, \quad \ell = \delta^\frac{1}{2}\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}.
\]
(7.1)

We check that $\mu, \ell, \delta_q, \lambda_q$ satisfies (5.2). Firstly, by (7.1), it’s easy to obtain
\[
\frac{\delta^\frac{1}{2}\lambda^\frac{1}{2} \lambda_{q+1}}{\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}} = \frac{\mu}{\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}} = \frac{\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}}{\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}} = \frac{1}{\ell \lambda_{q+1}} = \frac{\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}}{\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}}.
\]
A direct computation gives that
\[
\lambda^\frac{1}{2} \delta^\frac{1}{2}\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}} \leq 2a^{-\frac{1}{2}}(\frac{1}{4} + \frac{1}{2} + \frac{\varepsilon^2}{2})b^q \leq 2a^{-bq} \leq 1,
\]
\[
\frac{1}{\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}} \leq 2a^{-\frac{1}{2}}(\frac{1}{4} + \frac{1}{2} + \frac{\varepsilon^2}{2})c^2b^q \leq 2a^{-\frac{1}{2}c^2b^q} \leq \frac{1}{\lambda_{q+1}^{-\beta}}.
\]
where we fix $\beta = \frac{1}{8}$. This justifies the parameter assumption (5.2).

Step 2: Proof of (2.2)-(2.6). Firstly, (6.1) and (7.1) imply
\[
\|\hat{R}_1\|_0 + \frac{\|\hat{R}_1\|_1}{\lambda_{q+1}} \leq C\delta^{\frac{1}{2}}\delta^\frac{1}{2}\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2} + \varepsilon} + C\delta^{\frac{1}{2}}\delta^\frac{1}{2}\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}} \leq C\delta^{\frac{1}{2}}\delta^\frac{1}{2}\lambda^\frac{1}{2} \lambda_{q+1}^{-\frac{1}{2}}.
\]
A direct computation gives
\[ \frac{1}{2} \delta_{q+1} \delta_q^{\frac{1}{2}} \lambda_q^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}} \leq 2a \left( \frac{1}{2} + \frac{1}{4} + \frac{a^2}{2} \right) \beta^{q+1} \leq 2a \left( \frac{1}{4} - \frac{1}{4} \right) \beta^{q+1} \leq 2a \frac{\beta^{q+1}}{\mu} \delta_{q+2} \leq \eta \delta_{q+2}, \]

where we take \( a \) sufficiently large. This gives (2.2).

By (5.6), we deduce
\[ \| v_1 - v \|_0 \leq \| w_o \|_0 + \| w_c \|_0 \leq \delta_{q+1}^{\frac{3}{2}} \left( \frac{M}{2} + \lambda_q^{\frac{1}{2}} \right), \]
\[ \| v_1 - v \|_1 \leq \| w_o \|_1 + \| w_c \|_1 \leq \delta_{q+1}^{\frac{1}{2}} \left( \frac{M}{2} + \lambda_q^{\frac{1}{2}} \right) \lambda_{q+1}, \]
\[ \| v_1 - v \|_2 \leq \| w_o \|_2 + \| w_c \|_2 \leq \delta_{q+1}^{\frac{1}{2}} \left( \frac{M}{2} + \lambda_q^{\frac{1}{2}} \right) \lambda_{q+1}^2, \]
\[ \| v_1 - v \|_3 \leq \| w_o \|_3 + \| w_c \|_3 \leq \delta_{q+1}^{\frac{1}{2}} \left( \frac{M}{2} + \lambda_q^{\frac{1}{2}} \right) \lambda_{q+1}^3. \]

Noticing \( \lambda_q \geq \lambda_1 \geq a \beta \), thus we obtain (2.3) by taking \( a \) sufficiently large.

Recalling (4.28), by (5.5) and the support property of \( \chi \), we obtain
\[ \| P \|_0 \leq M^2 \delta_{q+1}. \]

Thus, from (4.31), (5.6) and (7.1), we deduce
\[ \| p_1 - p \|_0 \leq C_0 \delta_{q+1} + 2 \left( \| w_o \|_0 \| w_c \|_0 + \| w \|_0 \| v - v_c \|_0 \right) \leq M^2 \delta_{q+1}; \]

this gives (2.4).

Moreover, (4.20) implies (2.5). Using the above estimate on \( \| v_1 - v \|_0 \), we easily get (2.6) from (5.9).

**Step 3: Proof of (2.7).** Finally, to complete the proof of this proposition, we only need to justify (2.7). A direct computation gives
\[
e(t) (1 - \delta_{q+2}) - \int_{T^2} |v_1(t, x)|^2 dx \\
= e(t) (1 - \delta_{q+2}) - \int_{T^2} |v(t, x)|^2 dx - \int_{T^2} |w_o|^2 dx \\
\underbrace{\quad - 2 \int_{T^2} v \cdot w dx}_{\text{Err}_2} - \int_{T^2} (2w_o \cdot w_c + |w_c|^2) dx.\
\]

We first deal with \( \text{Err}_2 \). From (5.6), it’s easy to obtain
\[ \left| \int_{T^2} (2w_o \cdot w_c + |w_c|^2) dx \right| \leq C \delta_{q+1} \lambda_q^{\frac{1}{2}} \lambda_{q+1}. \]

Moreover,
\[ v \cdot w = \sum_{l} \sum_{k \in \Lambda_{q+1}} \chi(t) L_{kl} \cdot v \phi_{kl} e^{i\lambda_{q+1} k \cdot x}. \]
Thus, by (5.3) and (15.6), we deduce that
\[
\left| \int_{T^2} v \cdot w dx \right| = \left| \sum_{l} \sum_{k \in \Lambda(l)} \chi(t) \int_{T^2} L_{kl} \cdot v \phi_{kl} e^{i\lambda_{l+1,k+1}x} dx \right|
\leq \sum_{l} \sum_{k \in \Lambda(l)} \chi(t) \left( \frac{\| \nabla (L_{kl} \cdot v \phi_{kl}) \|_0}{\lambda_{l+1}} \right)
\leq C \frac{\delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}}}{\mu} + C \frac{\delta_{q+1}^{\frac{1}{2}} \ell^{-1}}{\lambda_{l+1}}.
\]
Combining the two parts, we obtain
\[
|Err_2| \leq C \frac{\delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \lambda_{l+1}}{\mu} + C \frac{\delta_{q+1}^{\frac{1}{2}} \ell^{-1}}{\lambda_{l+1}}. \tag{7.2}
\]

For $Err_1$, we first compute $\int_{T^2} |w_0|^2 dx$. From (12.22), (12.23) and (12.26), we deduce that
\[
|w_0|^2 = 2 \sum_{l} \chi_l^2(t) \rho_l - \sum_{k \in \Lambda(l)} \sum_{k' \in \Lambda(l)} \chi_l^2 a_{kl} a_{k'l} \phi_{kl} \phi_{k'l} k \cdot k' e^{i\lambda_{l+1}(k+k') \cdot x}
- \sum_{l \neq k', l' \in \Lambda(l) \cup \Lambda(l') \cup \Lambda(l)} f_{kkl'} k \cdot k' e^{i\lambda_{l+1}(k+k') \cdot x} := 2 \sum_{l} \chi_l^2(t) \rho_l + I_1 + I_2.
\]
Recalling (4.5), we obtain
\[
2(2\pi)^2 \sum_{l} \chi_l^2(t) \rho_l = e(t) (1 - \delta_{q+2}) - \int_{T^2} |v(t, \cdot)|^2 dx
+ \sum_{l} \chi_l^2(t) \left[ \left( e\left( \frac{1}{\mu} \right) - e(t) \right) (1 - \delta_{q+2}) + \int_{T^2} \left( |v(t, x)|^2 - \left| v\left( \frac{1}{\mu}, x \right) \right|^2 \right) dx \right]. \tag{I_{31}}
\]
It’s easy to obtain that
\[
|I_{31}| \leq C(e) \mu^{-1}.
\]
As in [3], using the equation (2.1), we can deduce
\[
\int_{T^2} \left( |v(t, x)|^2 - |v(l \mu^{-1}, x)|^2 \right) dx = \int_{\frac{l}{\mu}}^{t} \int_{T^2} \partial_t |v(s, x)|^2 ds dx
= - \int_{\frac{l}{\mu}}^{t} \int_{T^2} \text{div}(v(|v|^2 + 2p)(s, x)) ds dx + 2 \int_{\frac{l}{\mu}}^{t} \int_{T^2} v \cdot \text{div} \hat{R}(s, x) ds dx
+ \int_{\frac{l}{\mu}}^{t} \int_{T^2} \theta e_2 \cdot v(s, x) ds dx
= - 2 \int_{\frac{l}{\mu}}^{t} \int_{T^2} \nabla v : \hat{R}(s, x) ds dx + \int_{\frac{l}{\mu}}^{t} \int_{T^2} \theta e_2 \cdot v(s, x) ds dx;
\]
thus, by (4.1) and (4.2), for any $l \in Z$ and $t$ in the range $|\mu t - l| < 1$, there hold
\[
|I_{32}| \leq C \frac{\delta_{q+1}^{\frac{1}{2}} \delta_{q+1}^{\frac{1}{2}} \lambda_{q}}{\mu} + C \frac{\delta_{q+1}^{\frac{1}{2}} \ell^{-1}}{\lambda_{q}}.
\]
Moreover, by (6.6) and (B.6), we deduce
\[
\left| \int_{T^2} (I_1 + I_2)dx \right| \leq C \frac{\delta_{q+1}^\frac{1}{2} \delta_q^\frac{1}{4} \lambda_q}{\mu}.
\]
Thus, collecting these estimate together, we obtain
\[
|Err_1| \leq C(e)\mu^{-1} + C\delta_{q+1}^\frac{1}{2} \delta_q^\frac{1}{4} \lambda_q \mu^{-1}.
\]
Combining (7.2) and (7.3), we deduce that
\[
|e(t)(1 - \delta_{q+2}) - \int_{T^2} v_1(t, x)^2 dx| \leq C(e)\mu^{-1} + C\delta_{q+1}^\frac{1}{2} \delta_q^\frac{1}{4} \lambda_q \mu^{-1} + C\frac{\delta_{q+1}^\frac{1}{2} \ell^{-1}}{\lambda_{q+1}}.
\]
Recalling (7.1), we obtain
\[
\left| e(t)(1 - \delta_{q+2}) - \int_{T^2} v_1(t, x)^2 dx \right| \leq C\delta_q^{-\frac{1}{2}} \lambda_q^{-\frac{1}{2}} \lambda_{q+1}^{-\frac{1}{2}} + C\delta_q^\frac{1}{2} \delta_q^\frac{1}{4} \lambda_q \lambda_{q+1}^{-\frac{1}{2}}
\]
\[
\leq C\delta_q^\frac{1}{2} \delta_{q+1}^\frac{1}{2} \lambda_q \lambda_{q+1}^{-\frac{1}{2}} \leq Ca(-\frac{1}{4} + \frac{1}{2} - \frac{1}{4}) \delta_{q+1}
\]
\[
\leq Ca^{-2\delta} \delta_{q+1},
\]
which implies (2.7) by taking \(a\) sufficiently large. Thus, we complete the proof of Proposition 2.1.

\[\square\]

\textbf{APPENDIX A. ESTIMATE FOR TRANSPORT EQUATION}

In this section we give some well known estimates for the smooth solution of transport equation:
\[
\begin{aligned}
\partial_t f + v \cdot \nabla f = g, \\
|f|_{t=0} = f_0
\end{aligned}
\]  
where \(v = v(t, x)\) is a given smooth vector field. The proof can be found in [3].

\textbf{Proposition A.1.} Assume \(|t||v||_1 \leq 1\). Then any solution of (A.1) satisfies the following estimates:
\[
\begin{aligned}
\|f(t)||_0 &\leq \|f_0||_0 + \int_0^t \|g(\cdot, \tau)||_0 d\tau, \\
\|f(t)||_\alpha &\leq 2\left(\|f_0||_\alpha + \int_0^t \|g(\cdot, \tau)||_\alpha d\tau\right)
\end{aligned}
\]
for all \(0 < \alpha \leq 1\). Generally, for any \(N \geq 1\) and \(0 \leq \alpha \leq 1\), there hold
\[
[f(\cdot, t)]_{N+\alpha} \lesssim [f_0]_{N+\alpha} + |t|[v]_{N+\alpha}[f_0] + \int_0^t (g(\cdot, \tau))_{N+\alpha} + ((t-\tau)[v])_{N+\alpha}[g(\cdot, \tau)]_{1} d\tau.
\]

Let \(\Phi(t, \cdot)\) be the inverse of the flux \(X\) of \(v\) starting at time \(t_0\) as the identity. Under the above assumption \(|t-t_0||v||_1 \leq 1\) we have
\[
\|\nabla \Phi - \text{Id}\|_0 \lesssim |t-t_0|[v]_1, \quad \|\nabla \Phi\|_N \lesssim |t-t_0|[v]_N, \quad \forall N \geq 1.
\]

\textbf{APPENDIX B. STATIONARY PHASE ESTIMATE}

In the section, we recall the following simple fact, and the proof can also be found in [3].

\textbf{Lemma B.1.} Let \(k \in \mathbb{Z}^2, k \neq 0\) and \(\lambda \geq 1\) be given. For any \(a \in C^\infty(T^2)\) and \(m \in \mathbb{N}\), there hold
\[
\left| \int_{T^2} a(x)e^{i\lambda k \cdot x} dx \right| \leq \frac{|a|_m}{\lambda^m}.
\]
(B.6)
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