Investigations on hoop conjecture for horizonless spherical charged stars

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Abstract

For horizonless spherical stars with uniform charge density, the hoop conjecture was tested based on the interior solution. In this work, we are interested in more general horizonless spherical charged stars. We test hoop conjecture using the exterior solution since all types of interior solutions correspond to the same exterior Reissner-Nordström solution. Our analysis shows that the hoop conjecture is violated for very compact stars if we express the conjecture with the total ADM mass. And the hoop conjecture holds if we express the conjecture using the mass in the sphere.

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I. INTRODUCTION

The famous hoop conjecture introduced almost five decades ago asserts that the existence of black hole horizons is characterized by the mass and circumference relation $\frac{C}{4\pi M} \leq 1$ [1, 2]. Here $C$ is the circumference of the smallest ring that can engulf the black hole in all azimuthal directions and $M$ is usually interpreted as the asymptotically measured total ADM mass [3]-[31].

For horizonless curved spacetimes, the hoop conjecture should be characterized by the opposite inequality $\frac{C}{4\pi M} > 1$ [1, 2]. Then it is a question whether the mass in the ratio can still be interpreted as the ADM mass. For a specific interior solution of horizonless charged star with uniform charge density, however, if $M$ is interpreted as the ADM mass, the relation $\frac{C}{4\pi M} > 1$ can be violated for certain set of parameters and this relation holds if $M$ is the mass contained in the engulfing sphere [32–35]. In contrast, black holes can violate the hoop relation if the mass term is interpreted as the mass in the sphere [36]. Considering the different appearances of hoop conjecture in black holes and horizonless stars with uniform charge density, it is interesting to test hoop conjecture in the background of more general horizonless compact stars. In particular, it is meaningful to examine the case of horizonless stars compact nearly to form horizons.

We extend the discussion in [32–35] by considering the exterior solution since all types of interior solutions correspond to the same exterior Reissner-Nordström solution. For general horizonless spherical charged stars, our analysis shows that the hoop relation is violated if the mass term is interpreted as the ADM mass and the hoop relation holds if we use the gravitating mass within the sphere.

II. STUDIES OF THE MASS IN HOOP CONJECTURE

We are interested in general horizonless spherical charged stars. And the spacetime reads [32–35]

$$ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The metric functions $\nu$ and $\lambda$ only depend on the radial coordinate $r$. The sphere surface radius is located at $r_0$. In the exterior region $r \geq r_0$, the background is the Reissner-Nordström solution

$$e^\nu = e^{-\lambda} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

where $M$ is the ADM mass of the spacetime and $Q$ is the star charge. In this work, we pay attention to the case of $M \geq Q$. The would be horizon position is at $r_h = M + \sqrt{M^2 - Q^2}$. Since we are interested in
horizonless stars, there is the relation $r_0 > r_h$. We can simply set the surface radius as $r_0 = (1 + \varepsilon)r_h$, where $\varepsilon$ is a small positive parameter.

The circumference $C$ is given by

$$C = 2\pi r_0 = 2\pi(1 + \varepsilon)(M + \sqrt{M^2 - Q^2}).$$

(3)

For horizonless curved spacetimes, the hoop conjecture should be characterized by $[1, 2]$

$$\frac{C}{4\pi \mathcal{M}} > 1.$$

(4)

If we interpret the mass $\mathcal{M}$ as the ADM mass $M$. The mass to circumference ratio is

$$\frac{C}{4\pi \mathcal{M}} = \frac{2\pi(1 + \varepsilon)(M + \sqrt{M^2 - Q^2})}{4\pi M} = \frac{(1 + \varepsilon)(M + \sqrt{M^2 - Q^2})}{2M}.$$

(5)

For parameters satisfying $\varepsilon \leq \frac{M - \sqrt{M^2 - Q^2}}{M + \sqrt{M^2 - Q^2}}$, the relation (5) yields that

$$\frac{C}{4\pi \mathcal{M}} \leq 1.$$

(6)

So for very compact stars with $\varepsilon \leq \frac{M - \sqrt{M^2 - Q^2}}{M + \sqrt{M^2 - Q^2}}$, the hoop conjecture is violated if $\mathcal{M}$ is the ADM mass.

If we interpret $\mathcal{M}$ as the mass in the engulfing sphere, the mass is

$$\mathcal{M} = M - \frac{Q^2}{2r_0}.$$  

(7)

The hoop conjecture is expressed by the mass to circumference ratio

$$\frac{C}{4\pi \mathcal{M}} = \frac{2\pi r_0}{4\pi(M - \frac{Q^2}{2r_0})} \left( \frac{1 + \varepsilon}{2M(1 + \varepsilon)r_h - Q^2} \right) \left( \frac{r_h^2 + 2\varepsilon r_h^2 + \varepsilon^2 r_h^2}{2Mr_h - Q^2 + 2M\varepsilon r_h} \right) > \frac{r_h^2 + 2\varepsilon r_h^2}{2Mr_h - Q^2 + 2M\varepsilon r_h}.$$  

(8)

Since $r_h$ is the horizon satisfying $1 - \frac{2M}{r_h} + \frac{Q^2}{r_h^2} = 0$, there is the relation

$$r_h^2 = 2Mr_h - Q^2.$$  

(9)

Considering $r_h = M + \sqrt{M^2 - Q^2} \geq M$, we get

$$2\varepsilon r_h^2 \geq 2\varepsilon Mr_h.$$  

(10)

The relations (8), (9) and (10) imply that

$$\frac{C}{4\pi \mathcal{M}} > 1.$$  

(11)
It means the hoop conjecture holds for horizonless stars if we use the mass within the engulfing sphere. Since we consider the exterior solution, our conclusion holds in the exterior region of various horizonless stars. Here we analytically show that for exterior solutions of compact stars, Thorne hoop conjecture may generally hold if the mass term is interpreted as mass contained within the engulfing sphere.

III. CONCLUSIONS

The famous hoop conjecture is expressed by the mass to circumference ratio. For horizonless spherical stars with uniform charge density, the hoop conjecture was tested based on the interior solution in [32–35]. We investigated hoop conjecture in the exterior region of more general spherical horizonless compact stars. We tested hoop conjecture using the exterior solution since all types of interior solutions correspond to the same exterior Reissner-Nordsrøm solution. Our analysis showed that the hoop conjecture cannot hold for very compact stars if the mass is interpreted as the ADM mass in the total spacetime. And the hoop conjecture holds if we interpret the mass as the gravitating mass contained within the engulfing sphere.

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