ALGEBRA IN THE STONE–ČECH COMPACTIFICATION, SELECTIONS, AND ADDITIVE COMBINATORICS

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Abstract. The algebraic structure of the Stone–Čech compactification of a semigroup, and methods from the theory of selection principles, are used to establish qualitative coloring theorems extending the Milliken–Taylor Theorem and, consequently, Hindman’s Finite Sums Theorem. The main result is the following one (definitions provided in the main text): Let $X$ be a Menger space, and $\mathcal{U}$ be a point-infinite open cover of $X$ with no finite subcover. Consider the complete graph, whose vertices are the open sets in $X$. For each finite coloring of the edges of this graph, there are disjoint finite subsets $F_1, F_2, \ldots$ of the cover $\mathcal{U}$ whose unions $V_1 := \bigcup F_1, V_2 := \bigcup F_2, \ldots$ have the following properties:

1. The family $\{V_1, V_2, \ldots\}$ is a point-infinite cover of $X$.
2. The sets $\bigcup_{n \in F} V_n$ and $\bigcup_{n \in H} V_n$ are distinct for all nonempty finite sets $F < H$.
3. All edges $\{\bigcup_{n \in F} V_n, \bigcup_{n \in H} V_n\}$, for nonempty finite sets $F < H$, have the same color.

This has a new, purely combinatorial consequence. A self-contained introduction to the necessary parts of the needed theories, modulo the definition and elementary properties of ultrafilters, is provided.

1. Background

Following is a brief, self-contained introduction to the Stone–Čech compactification of a semigroup and its necessary algebraic and combinatorial properties. All assertions made can be verified directly. More detailed introductions, with additional combinatorial applications, may be found in any of the books [6, 11]. Familiar parts may be skipped by the reader.

1.1. The Stone–Čech compactification and Hindman’s Theorem. Almost throughout, $S$ denotes an infinite semigroup. We do not assume that the semigroup $S$ is commutative; however, with the applications in mind, we use additive notation. The Stone–Čech compactification of $S$, $\beta S$, is the set of all ultrafilters on $S$. We identify each element $s \in S$ with the principal ultrafilter associated to it. Thus, we view the set $S$ as a subset of $\beta S$. A filter $\mathcal{F}$ on $S$ is free if the intersection $\bigcap \mathcal{F}$ of all elements of $\mathcal{F}$ is empty. An ultrafilter is free if and only if it is nonprincipal.

A topology on the set $\beta S$ is defined by taking the sets $[A] := \{p \in \beta S : A \in p\}$, for $A \subseteq S$, as a basis for the topology. The function $A \mapsto [A]$ respects finite unions, finite intersections,
and complements. For an element \( s \in S \) and a set \( A \subseteq S \), we have that \( s \in [A] \) if and only if \( s \in A \). In particular, the set \( S \) is dense in \( \beta S \).

The topological space \( \beta S \) is compact: If \( \beta S = \bigcup_{\alpha \in I} [A_\alpha] \) and no finite union of sets \( A_\alpha \) is \( S \), then the family \( \{ A_\alpha : \alpha \in I \} \) extends to an ultrafilter \( p \in \beta S \), so \( p \) is in some set \( [A_\alpha] \); a contradiction. Define the sum of elements \( p, q \in \beta S \) by

\[
A \in p + q \text{ if and only if } \{ b \in S : \exists C \subseteq q, b + C \subseteq A \} \in p.
\]

Then \( p + q \in \beta S \). We obtain an extension of the addition operator from \( S \) to \( \beta S \), with the following continuity properties:

1. For each element \( x \in S \), the function \( q \mapsto x + q \) is continuous.
2. For each element \( q \in \beta S \), the function \( p \mapsto p + q \) is continuous.

Fix \( x, y \in S \). Since \( (x + y) + z = x + (y + z) \) for all \( z \in S \) and the set \( S \) is dense is \( \beta S \), we have by (1) that \( (x + y) + r = x + (y + r) \) for all \( r \in \beta S \). Fixing \( r \) and unfixing \( y \), we have by (1) and (2) that \( (x + q) + r = x + (q + r) \) for all \( q \in \beta S \). Finally, fixing \( q \) and unfixing \( x \), we have by (2) that \( (p + q) + r = p + (q + r) \) for all \( p \in \beta S \). Thus, \( (\beta S, +) \) is a semigroup.

If \( e \in \beta S \) is an idempotent element, that is, if \( e + e = e \), then for each set \( A \) there are a set \( B \), and for each \( b \in B \), a set \( C \) such that \( b + C \subseteq A \). Conversely, the latter property of \( e \) implies that \( e \subseteq e + e \) and thus \( e = e + e \). In this characterization, by intersecting \( C \) with \( A \), we may assume that \( C \subseteq A \).

By the continuity of the functions \( p \mapsto p + q \), for \( q \in \beta S \), there are idempotent elements in any closed subsemigroup \( T \) of \( \beta S \). Indeed, Zorn’s Lemma provides us with a minimal closed subsemigroup \( E \) of \( T \), and it follows by minimality that \( E = \{ e \} \) for some (necessarily, idempotent) element \( e \in T \).

**Definition 1.1.** For elements \( a_1, a_2, \ldots \) in a semigroup \( S \), and a nonempty finite set \( F = \{ i_1, \ldots, i_k \} \subseteq \mathbb{N} \) with \( k \geq 1 \) and \( i_1 < \cdots < i_k \), define \( a_F := a_{i_1} + \cdots + a_{i_k} \). Let

\[
FS(a_1, a_2, \ldots) := \{ a_F : F \subseteq \mathbb{N}, \ F \text{ finite nonempty} \},
\]

the set of all finite sums, in increasing order of indices, of elements \( a_i \). Similarly, for elements \( a_1, \ldots, a_n \in S \), the set \( FS(a_1, \ldots, a_n) \) is comprised of the elements \( a_F \) for \( F \) a nonempty subset of \( \{ 1, \ldots, n \} \).

A **finite coloring** of a set \( A \) is a function \( f : A \to \{ 1, \ldots, k \} \), for \( k \in \mathbb{N} \). Given a finite coloring \( f \) of a set \( A \), a set \( B \subseteq A \) is **monochromatic** if there is a color \( i \) with \( f(b) = i \) for all \( b \in B \).

**Theorem 1.2** (Hindman [5]). For each finite coloring of \( \mathbb{N} \), there are elements \( a_1, a_2, \cdots \in \mathbb{N} \) such that the set \( FS(a_1, a_2, \ldots) \) is monochromatic.

\[1\]To see that a minimal closed subsemigroup \( E \) of \( \beta S \) must be of the form \( \{ e \} \), fix an element \( e \in E \). As \( E + e \) is a closed subsemigroup of \( E \), we have that \( E + e = E \). Thus, the stabilizer of \( e \), \( \{ t \in E : t + e = e \} \) is a (closed) subsemigroup of \( E \), and is therefore equal to \( E \). Then \( e + e = e \).
The following strikingly elegant proof of Hindman’s Theorem is due to Galvin and Glazer. Fix an idempotent element \( e \in \beta \mathbb{N} \). Let a \( k \)-coloring of \( \mathbb{N} \) be given. If \( C_i \) is the set of elements of color \( i \), then \( C_1 \cup \cdots \cup C_k \in e \), and thus there is a color \( i \) with \( A_1 := C_i \in e \). For \( n = 1, 2, \ldots \), since \( e \) is an idempotent ultrafilter, there are an element \( a_n \in A_n \) and a set \( A_{n+1} \subseteq A_n \) in \( e \) such that \( a_n + A_{n+1} \subseteq A_n \). It then follows, considering the sums from right to left, that every finite sum \( a_{i_1} + \cdots + a_{i_k} \), for \( i_1 < \cdots < i_k \), is in \( A_{i_1} \). Thus, the set \( FS(a_1, a_2, \ldots) \) is a subset of the monochromatic set \( A_1 \).

1.2. The Milliken–Taylor Theorem and proper sum sequences. For a set \( S \), let \([S]^2\) be the set of all 2-element subsets of \( S \); equivalently, the edge set of the complete graph with vertex set \( S \).

**Definition 1.3.** Let \( S \) be a semigroup. For nonempty finite sets of natural numbers \( F \) and \( H \), we write \( F < H \) if all elements of \( F \) are smaller than all elements of \( H \). A sum sequence (or sum subsystem) of a sequence \( a_1, a_2, \ldots \in S \) is a sequence of the form \( a_{F_1}, a_{F_2}, \ldots \), for nonempty finite sets of natural numbers \( F_1 < F_2 < \cdots \).

A sequence \( b_1, b_2, \ldots \in S \) is proper if \( b_F \neq b_H \) for all nonempty finite sets \( F < H \) of natural numbers. The sum graph of a proper sequence \( b_1, b_2, \ldots \in S \) is the subset of \([FS(b_1, b_2, \ldots)]^2\) consisting of the edges \( \{b_F, b_H\} \), for nonempty finite sets \( F < H \) of natural numbers.

If \( b_1, b_2, \ldots \) is a sum sequence of \( a_1, a_2, \ldots \), then \( FS(b_1, b_2, \ldots) \subseteq FS(a_1, a_2, \ldots) \). The relation of being a sum sequence is transitive.

Ramsey’s Theorem [12] asserts that, for each finite coloring of an infinite complete graph \([V]^2\) with vertex set \( V \), there is an infinite complete monochromatic subgraph, that is, an infinite set \( I \subseteq V \) such that the set \([I]^2\) is monochromatic. The Milliken–Taylor Theorem unifies Hindman’s and Ramsey’s theorems.

**Theorem 1.4 (Milliken–Taylor [10, 20]).** Let \( a_1, a_2, \ldots \) be a sequence in \( \mathbb{N} \). For each finite coloring of the set \([\mathbb{N}]^2\), there is a proper sum sequence \( b_1, b_2, \ldots \) of the sequence \( a_1, a_2, \ldots \) such that the sum graph of \( b_1, b_2, \ldots \) is monochromatic.

The Milliken–Taylor Theorem can be proved by combining the proofs of Ramsey’s and Hindman’s theorems, as can be gleaned from the proof of the forthcoming Theorem 3.6.

Our applications are in a setting where all elements of the semigroup \( S \) are idempotents. In this case, stating Hindman’s Theorem for the semigroup \( S \) instead of \( \mathbb{N} \) yields a trivial statement: for an idempotent element \( e \in S \), the set \( FS(e, e, \ldots) = \{e\} \) is obviously monochromatic. The sequence \( e, e, \ldots \) is improper, and so are all of its sum sequences. Thus, the Milliken–Taylor Theorem cannot be extended to such cases. An example of a semigroup with all elements idempotent is \( Fin(\mathbb{N}) \), the set of nonempty finite subsets of \( \mathbb{N} \), with the operation \( \cup \). For this semigroup, we have the following.
Theorem 1.5 (Milliken–Taylor). For each finite coloring of the set \([\text{Fin}(\mathbb{N})]^2\), there are elements \(F_1 < F_2 < \cdots\) in \(\text{Fin}(\mathbb{N})\) such that the sum graph of \(F_1, F_2, \ldots\) is monochromatic.

As every sequence of natural numbers has a proper sumsequence, Theorem 1.4 is a special case of the following one.

Theorem 1.6. Let \(S\) be a semigroup, and \(a_1, a_2, \ldots \in S\). If the sequence \(a_1, a_2, \ldots\) has a proper sumsequence, then for each finite coloring of the set \([S]^2\), there is a proper sumsequence of \(a_1, a_2, \ldots\) whose sum graph is monochromatic.

Theorem 1.6 is more general than Theorem 1.5, since the sequence \(\{1\}, \{2\}, \ldots\) is proper.

Theorem 1.6 follows from Theorem 1.5: By moving to a sumsequence, we may assume that the sequence \(a_1, a_2, \ldots\) is proper. Let \(\chi\) be a finite coloring of \([S]^2\). Define a coloring \(\kappa\) of \([\text{Fin}(\mathbb{N})]^2\) by \(\kappa(\{F, H\}) = \chi(\{a_F, a_H\})\) for \(F < H\), and \(\kappa(\{F, H\})\) arbitrary otherwise, and apply Theorem 1.5, using that sumsequences of proper sequences are proper.

The hypothesis of having a proper sumsequence fails only in degenerate cases.

Proposition 1.7. Let \(S\) be a semigroup, and \(a_1, a_2, \ldots \in S\). If the sequence \(a_1, a_2, \ldots\) has no proper sumsequence, then every sumsequence of \(a_1, a_2, \ldots\) has a sumsequence of the form \(e, e, \ldots\), where \(e\) is an idempotent element of \(S\); equivalently, a sumsequence whose set of finite sums is a singleton.

Proof. We use Theorem 1.5. Define a coloring of the set \([\text{Fin}(\mathbb{N})]^2\) by

\[
\{F, H\} \mapsto |\{a_F, a_H\}|.
\]

Let \(F_1 < F_2 < \cdots\) be elements of \(\text{Fin}(\mathbb{N})\) such that the sum graph of the sequence \(F_1, F_2, \ldots\) is monochromatic.

Consider the sumsequence \(b_1 := a_{F_1}, b_2 := a_{F_2}, \ldots\). Assume that the color is 2. Then the sumsequence \(b_1, b_2, \ldots\) is proper; a contradiction. Thus, the color must be 1. Then \(b_{H_1} = b_{H_2}\) for all \(H_1 < H_2\) in \(\text{Fin}(\mathbb{N})\). Let \(e := b_1\). Then \(b_n = b_1 = e\) for all \(n > 1\). For each set \(H \in \text{Fin}(\mathbb{N})\), take \(n > H\). Then \(b_H = b_n = e\). In particular, \(e + e = b_1 + b_2 = b_{\{1, 2\}} = e\). Thus, \(\text{FS}(b_1, b_2, \ldots) = \{e\}\). □

2. Idempotent filters and superfilters

Superfilters provide a convenient way to identify closed subsets of \(\beta S\).²

Definition 2.1. A family \(\mathcal{A}\) of subsets of a set \(S\) is a superfilter on \(S\) if:

1. All sets in \(\mathcal{A}\) are infinite.

²Superfilters have various names in the classic literature, including coideals, grilles, and partition-regular families, depending on the context where they are used. Some of the definitions in the literature are not equivalent to the one given here, but they are always conceptually similar. The present term is adopted from [15].
(2) For each set $A \in \mathcal{A}$, all subsets of $S$ that contain $A$ are in $\mathcal{A}$.

(3) Whenever $A_1 \cup A_2 \in \mathcal{A}$, we have that $A_1$ or $A_2$ are in $\mathcal{A}$; equivalently, for each set $A \in \mathcal{A}$ and each finite coloring of $A$, there is in $\mathcal{A}$ a monochromatic subset of $A$.

The simplest example of a superfilter on a set $S$ is the family $[S]^\infty$, consisting of all infinite subsets of $S$. Many examples of superfilters are provided by Ramsey theoretic theorems. For example, van der Waerden’s Theorem asserts that monochromatic arithmetic progressions of any prescribed finite length will be found in any long enough, finitely-colored arithmetic progression. By van der Waerden’s Theorem, the family of all sets of natural numbers containing arbitrarily long finite arithmetic progressions is a superfilter on $\mathbb{N}$.

The notions of free filter and superfilter are dual. For a family $F$ of subsets of a set $S$,

$$F^+ := \{ A \subseteq S : A^c \notin F \}.$$  

The following assertions are easy to verify.

**Lemma 2.2** (Folklore). Let $S$ be a set.

(1) For all families $F_1$ and $F_2$ of subsets of $S$, $F_1 \subseteq F_2$ implies that $F_1^+ \supseteq F_2^+$.

(2) For each family $F$ of subsets of $S$, $F^{++} = F$.

(3) For each free filter $F$ on $S$, the set $F^+$ is a superfilter containing $F$.

(4) For each superfilter $\mathcal{A}$ on $S$, the set $\mathcal{A}^+$ is a free filter contained in $\mathcal{A}$.

(5) For each filter $F$, if $A \in F^+$ and $B \in F$, then $A \cap B \in F^+$.

(6) For each ultrafilter $p$ on $S$, $p^+ = p$.

**Proof of (5).** Since $A \subseteq B^c \cup (A \cap B)$, the latter set is in $F^+$. Since $B^c \notin F^+$, we have that $A \cap B \in F^+$.

Every free ultrafilter on $S$ is a superfilter on $S$, and so is any union of free ultrafilters on $S$. Since elements of superfilters are infinite, the filter of cofinite subsets of $S$ is contained in all superfilters on $S$. By the following lemma, every superfilter $\mathcal{A}$ is a union of a closed set of free ultrafilters. Indeed, taking $F = \{N\}$ we have by the lemma that the set $C := \{ p \in \beta S : p \subseteq A \}$ is closed, and for each set $A \in \mathcal{A}$, letting $F$ be the filter generated by $A$ we see, again by the lemma, that there is an ultrafilter $p \in C$ with $A \in p$. Thus, $\bigcup C = \mathcal{A}$.

**Lemma 2.3.** Let $S$ be an infinite set. For each superfilter $\mathcal{A}$ on $S$, and each filter $F \subseteq \mathcal{A}$, the set $\{ p \in \beta S : F \subseteq p \subseteq \mathcal{A} \}$ is a nonempty closed subset of $\beta S \setminus S$.

**Proof.** It is straightforward to verify that the set is closed. We prove that it is nonempty. By Lemma 2.2(4), the set $\mathcal{A}^+$ is a filter. By Lemma 2.2(2,5) applied to the filter $\mathcal{A}^+$, we have that $A \cap B \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{A}^+$. In particular, the set $A \cap B$ is infinite for all $A \in \mathcal{F}, B \in \mathcal{A}^+$. The family $\{ A \cap B : A \in \mathcal{F}, B \in \mathcal{A}^+ \}$ is closed under finite intersections. Since its elements are infinite, it extends to a free ultrafilter $p$. Necessarily, $\mathcal{F} \subseteq p$. If there were an element $B \in p \setminus \mathcal{A}$, then $B^c \in \mathcal{A}^+ \subseteq p$; a contradiction.

**Definition 2.4.** Let $S$ be a semigroup.
(1) For a set \( A \subseteq S \) and a family \( \mathcal{F} \) of subsets of \( S \), let
\[
A^*(\mathcal{F}) := \{ b \in S : \exists C \in \mathcal{F}, b + C \subseteq A \}.
\]

(2) A filter \( \mathcal{F} \) on \( S \) is an idempotent filter if for each set \( A \in \mathcal{F} \), the set \( A^*(\mathcal{F}) \) is in \( \mathcal{F} \).

(3) A superfilter \( \mathcal{A} \) on \( S \) is an idempotent superfilter if, for each set \( A \subseteq S \) such that the set \( A^*(\mathcal{A}) \) is in \( \mathcal{A} \), we have that \( A \in \mathcal{A} \).

Thus, for ultrafilters \( p, q \) on \( S \), \( A \in p + q \) if and only if \( A^*(q) \in p \).

Let \( S \) be a semigroup. A superfilter \( \mathcal{A} \) on \( S \) is translation-invariant if \( s + A \in \mathcal{A} \) for all \( s \in S \) and \( A \in \mathcal{A} \). Every translation-invariant superfilter on a semigroup \( S \) is an idempotent superfilter.

Since ultrafilters are maximal filters, we have that, for an ultrafilter \( p \) on a semigroup \( S \), being an idempotent ultrafilter, idempotent filter, and idempotent superfilter is the same.

**Lemma 2.5.** Let \( S \) be a semigroup.

(1) For each free idempotent filter \( \mathcal{F} \) on \( S \), the superfilter \( \mathcal{F}^+ \) is idempotent.

(2) For each idempotent superfilter \( \mathcal{A} \) on \( S \), the free filter \( \mathcal{A}^+ \) is idempotent.

**Proof.** (1) Let \( A \subseteq S \), and assume that the set \( B_1 := A^*(\mathcal{F}^+) \) is in \( \mathcal{F}^+ \). Assume that \( A \notin \mathcal{F}^+ \). Then \( A^c \in \mathcal{F} \), and thus the set \( B_2 := (A^c)^*(\mathcal{F}) \) is in \( \mathcal{F} \). By Lemma 2.2(5), there is an element \( b \in B_1 \cap B_2 \). Then there are sets \( C_1 \in \mathcal{F}^+ \) and \( C_2 \in \mathcal{F} \) such that \( b + C_1 \subseteq A \) and \( b + C_2 \subseteq A^c \). Pick \( c \in C_1 \cap C_2 \). Then \( b + c \in A \cap A^c \); a contradiction.

(2) Similar. \( \square \)

**Lemma 2.6.** Let \( S \) be a semigroup, and \( \mathcal{F} \) be a free idempotent filter on \( S \). Then the set \( T := \{ p \in \beta S : \mathcal{F} \subseteq p \} \) is a closed subsemigroup of \( \beta S \) disjoint from \( S \).

**Proof.** By Lemma 2.3, with \( \mathcal{A} = [S]^\infty \), the set \( T \) is a closed subset of \( \beta S \). Since the filter \( \mathcal{F} \) is free, we have that \( T \subseteq \beta S \setminus S \). Let \( p, q \in T \), and \( A \in \mathcal{F} \). Since the filter \( \mathcal{F} \) is idempotent, \( A^*(\mathcal{F}) \in \mathcal{F} \subseteq p \). Since \( \mathcal{F} \subseteq \mathcal{F} \), \( A^*(\mathcal{F}) \subseteq A^*(\mathcal{q}) \), and therefore \( A^*(q) \in p \). By the definition of sum of ultrafilters, \( A \in p + q \). \( \square \)

**Theorem 2.7.** Let \( S \) be a semigroup, and assume that \( \mathcal{F} \) is a free idempotent filter on \( S \) contained in an idempotent superfilter \( \mathcal{A} \) on \( S \). Then there is a free idempotent ultrafilter \( e \) with \( \mathcal{F} \subseteq e \subseteq \mathcal{A} \).

**Proof.** Let \( T_1 := \{ p \in \beta S : \mathcal{F} \subseteq p \} \) and \( T_2 := \{ p \in \beta S : p \subseteq \mathcal{A} \} \). By Lemma 2.6, the set \( T_1 \) is a closed subsemigroup of \( \beta S \), and so is the set \( \{ p \in \beta S : \mathcal{A}^+ \subseteq p^+ = p \} = T_2 \).

By Lemma 2.3, the intersection \( T := T_1 \cap T_2 \) is nonempty, and is therefore a closed subsemigroup of \( \beta S \). Pick an idempotent element in \( T \). \( \square \)
3. Selection principles and an abstract partition theorem

We use the following notions from Scheepers’s seminal paper [16]. Let \( \mathcal{A} \) and \( \mathcal{B} \) be families of sets. \( S_1(\mathcal{A}, \mathcal{B}) \) is the property that, for each sequence \( A_1, A_2, \ldots \in \mathcal{A} \), one can select one element from each set, \( b_1 \in A_1, b_2 \in A_2, \ldots \), such that \( \{b_1, b_2, \ldots\} \in \mathcal{B} \). \( G_1(\mathcal{A}, \mathcal{B}) \) is a game associated to \( S_1(\mathcal{A}, \mathcal{B}) \). This game is played by two players, Alice and Bob, and has an inning per each natural number. In the \( n \)-th inning, Alice plays a set \( A_n \in \mathcal{A} \), and Bob selects an element \( b_n \in A_n \). Bob wins if \( \{b_1, b_2, \ldots\} \in \mathcal{B} \). Otherwise, Alice wins.

If Alice does not have a winning strategy in the game \( G_1(\mathcal{A}, \mathcal{B}) \), then \( S_1(\mathcal{A}, \mathcal{B}) \) holds. The converse implication holds in some important cases, including the ones in our main applications. A survey of known results of this type is provided, e.g., in Section 11 of [18].

**Example 3.1.** Let \( S \) be a set, and \( \mathcal{F} \) be a filter on \( S \) generated by countably many sets. Then Alice does not have a winning strategy in the game \( G_1(\mathcal{F}^+, \mathcal{F}^+) \); moreover, Bob has one: Fix sets \( B_1, B_2, \ldots \in \mathcal{F} \) such that every member of \( \mathcal{F} \) contains one of these sets. In each inning, by Lemma 2.2(5), Bob can pick an element \( b_n \in A_n \cap B_n \). Then \( \{b_1, b_2, \ldots\} \in \mathcal{F}^+ \).

For the filter \( \mathcal{F} \) of cofinite sets, this reproduces the simple observation that Bob has a winning strategy in the game \( G_1([S]^\infty, [S]^\infty) \).

In general, the game \( G_1(\mathcal{A}, \mathcal{B}) \) is not determined, and the property that Alice does not have a winning strategy is strictly weaker than Bob’s having one. This will be the case in our main applications [18, Section 11].

**Definition 3.2.** A free idempotent chain in a semigroup \( S \) is a descending sequence \( A_1 \supseteq A_2 \supseteq \cdots \) of infinite subsets of \( S \) such that:

1. \( \bigcap_n A_n = \emptyset \).
2. For each \( n \), the set \( A_n^*(\{A_1, A_2, \ldots\}) \) contains one of the sets \( A_m \); equivalently, there is \( m > n \) such that, for each \( a \in A_m \), there is \( k > m \) with \( a + A_k \subseteq A_m \).

For a family \( \mathcal{A} \) of subsets of \( S \), a free idempotent chain in \( \mathcal{A} \) is a free idempotent chain of elements of \( \mathcal{A} \).

**Example 3.3.** For each proper sequence \( a_1, a_2, \ldots \) in a semigroup, the sets \( FS(a_n, a_{n+1}, \ldots) \), for \( n \in \mathbb{N} \), form a free idempotent chain. Thus, if a sequence \( a_1, a_2, \ldots \) has a proper sumsequence, then there is a free idempotent chain \( A_1 \supseteq A_2 \supseteq \cdots \) with \( A_n \subseteq FS(a_n, a_{n+1}, \ldots) \) for all \( n \).

**Lemma 3.4.** Let \( S \) be a semigroup, and \( \mathcal{A} \) be a superfilter on \( S \). Every filter generated by a free idempotent chain in \( \mathcal{A} \) is a free idempotent filter contained in \( \mathcal{A} \).

**Proof.** Let \( A_1 \supseteq A_2 \supseteq \cdots \) be a free idempotent chain in \( \mathcal{A} \), and let \( \mathcal{F} \) be the filter generated by the sets \( A_1, A_2, \ldots \). Since \( \bigcap_n A_n = \emptyset \), the filter \( \mathcal{F} \) is free. Since \( A_n \in \mathcal{A} \) for each \( n \), \( \mathcal{F} \subseteq \mathcal{A} \). The filter \( \mathcal{F} \) is idempotent: For \( A \in \mathcal{F} \), let \( A_n \) be a subset of \( A \). By the definition, there is \( m \) such that \( A^*(\mathcal{F}) \supseteq A_n^*(\{A_1, A_2, \ldots\}) \supseteq A_m \). Since \( A_m \in \mathcal{F} \), we have that \( A^*(\mathcal{F}) \subseteq \mathcal{F} \).
Our theorems can be stated for any finite dimension. For clarity, we state them in the one-dimensional case, which extends Hindman’s Theorem, and in the two-dimensional case, which extends the Milliken–Taylor Theorem. The one-dimensional case always follows from the two-dimensional, for the following reason.

**Proposition 3.5.** Let $S$ be a semigroup, and $\chi$ be a finite coloring of the sets $S$ and $[S]^2$. There is a finite coloring $\eta$ of the set $[S]^2$ such that, for each proper sequence $b_1, b_2, \ldots$ with $\eta$-monochromatic sum graph, the set $\text{FS}(b_1, b_2, \ldots)$ and the sum graph of $b_1, b_2, \ldots$ are both $\chi$-monochromatic.

**Proof.** By enumerating the elements of the countable set $\text{FS}(b_1, b_2, \ldots)$, we obtain an order on this set such that every element has only finitely many smaller elements. Define a coloring $\kappa$ of $[\text{FS}(b_1, b_2, \ldots)]^2$ by
\[
\kappa(\{s, t\}) := \chi(\min\{s, t\}).
\]
Extend $\kappa$ to a coloring of $[S]^2$ in an arbitrary manner.

Assume that the set $\text{FS}(b_1, b_2, \ldots)$ is monochromatic for $\kappa$, say green. Being proper, the sequence $b_1, b_2, \ldots$ is bijective. For each nonempty finite set $F$ of natural numbers, since there are at most finitely many elements in $\text{FS}(b_1, b_2, \ldots)$ smaller than $b_F$, there is $n > F$ such that $b_F < b_n$. Then $\kappa(\{b_F, b_n\}) = \chi(b_F)$. Thus, the element $b_F$ is green.

The finite coloring $\eta$ of the set $[S]^2$, defined by
\[
\eta(\{s, t\}) := \left(\kappa(\{s, t\}), \chi(\{s, t\})\right),
\]
is as required. If $\chi$ is a $k$-coloring, we may represent the range set of $\eta$ in the form $\{1, \ldots, k^2\}$. □

The two monochromatic sets in Proposition 3.5 may be of different colors. Moreover, this can be forced by adding a coordinate to $\chi(x)$ that is 1 if $x \in S$ and 2 if $x \in [S]^2$.

**Theorem 3.6.** Let $S$ be a semigroup. Let $A$ be an idempotent superfilter on $S$, and $B$ be a family of subsets of $S$ such that Alice does not have a winning strategy in the game $G_1(A, B)$. Let $a_1, a_2, \ldots$ be a sequence in $S$, and $A_1 \supseteq A_2 \supseteq \cdots$ be a free idempotent chain in $A$ with $A_n \subseteq \text{FS}(a_n, a_{n+1}, \ldots)$ for all $n$. For each finite coloring of the sets $S$ and $[S]^2$, there are elements $b_1 \in A_1, b_2 \in A_2, \ldots$ such that:

1. The set $\{b_1, b_2, \ldots\}$ is in $B$.
2. The sequence $b_1, b_2, \ldots$ is a proper subsequence of $a_1, a_2, \ldots$.
3. The set $\text{FS}(b_1, b_2, \ldots)$ is monochromatic.
4. The sum graph of $b_1, b_2, \ldots$ is monochromatic.

**Proof.** By Proposition 3.5 it suffices to prove the two-dimensional assertion, that is, item (3) follows from item (4).
By Lemma 3.1, there is a free idempotent filter $\mathcal{F}$ such that $\{A_1, A_2, \ldots\} \subseteq \mathcal{F} \subseteq \mathcal{A}$. By Theorem 2.7, there is a free idempotent ultrafilter $e$ on $S$ such that $\mathcal{F} \subseteq e \subseteq \mathcal{A}$.

Let a finite coloring $\chi: [S]^2 \to \{1, \ldots, k\}$ be given. For each element $s \in S$, let

$$C_i(s) := \{ t \in S \setminus \{s\} : \chi(\{s, t\}) = i \}.$$ 

As $C_1(s) \cup \cdots \cup C_k(s) = S \setminus \{s\} \in e$, there is a unique $i$ with $C_i(s) \in e$. Define a finite coloring $\kappa: S \to \{1, \ldots, k\}$ by letting $\kappa(s)$ be this unique $i$ with $C_i(s) \in e$. Since $e$ is an ultrafilter, there is in $e$ a set $M \subseteq S$ that is monochromatic for the coloring $\kappa$. Assume that the color is green. Then, for each finite set $F \subseteq M$, we have that

$$G(F) := \bigcap_{s \in F} \{ t \in S \setminus \{s\} : \{s, t\} \text{ is green} \} \in e,$$

and for each element $s \in F$ and each element $t \in G(F)$, we have that $s \neq t$ and the edge $\{s, t\}$ is green.

For a set $D \in e$, define

$$D^* := \{ b \in D : \exists B \subseteq D \text{ in } e, b + B \subseteq D \} = D^*(e) \cap D.$$

Then $D^* \subseteq D$ and, since $e$ is an idempotent ultrafilter, $D^* \in e$.

We define a strategy for Alice. In this strategy, Alice makes choices from certain nonempty sets. Formally, she does that by applying prescribed choice functions to the given nonempty sets.

1. In the first inning, Alice sets $D_1 := M \cap A_1$, and plays the set $D_1^*$.
2. Assume that Bob plays an element $b_1 \in D_1^*$. Then Alice chooses a set $B \subseteq D_1$ in $e$ such that $b_1 + B \subseteq D_1$ and a set $F_1$ with $a_{F_1} = b_1$. She then chooses a natural number $m_1 > F_1$, and sets $D_2 := B \cap G(\{b_1\}) \cap A_{m_1}$. Having done that, Alice plays the set $D_2^*$.
3. Assume that Bob plays an element $b_2 \in D_2^*$. Then $b_1 + b_2 \in D_1 \subseteq M$. Alice chooses a set $B \subseteq D_2$ in $e$ such that $b_2 + B \subseteq D_2$, a set $F_2 > m_1$ with $a_{F_2} = b_2$, and a natural number $m_2 > F_2$. She sets $D_3 := B \cap G(F(b_1, b_2)) \cap A_{m_2}$, and plays $D_3^*$.
4. In the $n + 1$-st inning, Bob has picked elements $b_1 \in D_1^*, \ldots, b_n \in D_n^*$. As in the Galvin–Glazer proof of Hindman’s Theorem, by computing sums from right to left, we see that $F_n > m_{n-1}$ with $a_{F_n} = b_n$, and a natural number $m_n > F_n$. She then sets $D_{n+1} := B \cap G(F(b_1, \ldots, b_n)) \cap A_{m_n}$, and plays the set $D_{n+1}^*$.

Since Alice has no winning strategy, there is a play $(D_1^*, b_1, D_2^*, b_2, \ldots)$, according to Alice’s strategy, won by Bob. By the construction, the sequence $b_1, b_2, \ldots$ is a subsequence of $a_1, a_2, \ldots$. The set $\{b_1, b_2, \ldots\}$ is in $B$, since Bob won this play.
Let $i_1 < \cdots < i_k < j_1 < \cdots < j_l$, $F = \{i_1, \ldots, i_k\}$, and $H = \{j_1, \ldots, j_l\}$. Then
\[ b_F \in \text{FS}(b_1, \ldots, b_k), \text{ and} \]
\[ b_H = b_{j_1} + \cdots + b_{j_l}. \]
Computing the latter sum from right to left, we see that
\[ b_H \in D_{j_1} \subseteq D_{i_k+1} \subseteq \text{G} \left( \text{FS}(b_1, \ldots, b_k) \right). \]
It follows that the elements $b_F$ and $b_H$ are distinct, and the edge $\{b_F, b_H\}$ is green. \qed

We illustrate Theorem 3.6 by several examples. More substantial applications of Theorem 3.6 are provided in the next sections.

**Corollary 3.7.** Let $S$ be a semigroup. Let $a_1, a_2, \ldots$ be a sequence in $S$, and $A_1 \supseteq A_2 \supseteq \cdots$ be a free idempotent chain with $A_n \subseteq \text{FS}(a_n, a_{n+1}, \ldots)$ for all $n$. For each finite coloring of the sets $S$ and $[S]^2$, there are elements $b_1 \in A_1, b_2 \in A_2, \ldots$ such that:

1. The sequence $b_1, b_2, \ldots$ is a proper sumsequence of $a_1, a_2, \ldots$.
2. The set $\text{FS}(b_1, b_2, \ldots)$ is monochromatic.
3. The sum graph of $b_1, b_2, \ldots$ is monochromatic.

**Proof.** By Lemma 3.4 with the trivial superfilter $A = [S]^\infty$, the filter $\mathcal{F}$ on $S$ generated by the sets $A_1, A_2, \ldots$ is a free idempotent filter. By Lemma 2.5 the superfilter $\mathcal{F}^+$ is also idempotent. By Example 3.1 Bob has a winning strategy in the game $G_1(\mathcal{F}^+, \mathcal{F}^+)$. Since $\mathcal{F} \subseteq \mathcal{F}^+$, Theorem 3.6 applies with $A = B = [S]^\infty$. \qed

In most semigroups $S$ one encounters, left addition is at most finite-to-one. In this case, the superfilter $[S]^\infty$ is translation-invariant; in particular, idempotent. In this case, the proof of Corollary 3.7 reduces to one short sentence: Apply Theorem 3.6 with $A = B = [S]^\infty$.

Corollary 3.7 can also be proved in a somewhat more direct manner, using the diagonalization method of the Galvin–Glazer proof of Hindman’s theorem to construct a diagonal through the given idempotent chain, and then applying the Milliken–Taylor Theorem to the diagonal sequence to obtain a sumsequence with a monochromatic sum graph.

The Milliken–Taylor Theorem in arbitrary semigroups (Theorem 1.6) follows from Corollary 3.7, by Example 3.3.

**Corollary 3.8.** Let $\mathcal{F}_1, \mathcal{F}_2, \ldots \subseteq \text{Fin}(N)$, and $A \subseteq N$. Assume that every cofinite subset of $A$ contains a member from each family $\mathcal{F}_n$. For each finite coloring of the sets $\text{Fin}(N)$ and $[\text{Fin}(N)]^2$, there are nonempty finite subsets $F_1 < F_2 < \cdots$ of $A$ such that:

1. Each set $F_n$ contains some element of the family $\mathcal{F}_n$.
2. All nonempty finite unions $H$ of sets $F_n$ have the same color.
3. All sets $\{H_1, H_2\}$, for $H_1 < H_2$ nonempty finite unions of sets $F_n$, have the same color.
Proof. We work with the semigroup Fin(A) of all nonempty finite subsets of A. Enumerate $A = \{a_1, a_2, \ldots\}$. For each $n$, let

$$A_n := \{ F \in \text{Fin}(\{a_n, a_{n+1}, \ldots\}) : \exists H \in F_n, H \subseteq F \} \subseteq \text{FS}(\{a_n\}, \{a_{n+1}\}, \ldots).$$

Then $\bigcap_n A_n = \emptyset$. Every set $A_n$ is a subsemigroup of $S$. Thus, the sequence $A_1 \supseteq A_2 \supseteq \cdots$ is a free idempotent chain. Apply Corollary 3.7. □

Corollary 3.8 can also be proved directly: Let $m_1 := 1$. For $n = 1, 2, \ldots$, given $m_n$ we can choose a natural number $m_{n+1}$ such that $A \cap \{m_n, \ldots, m_{n+1}-1\}$ contains an element from each of the families $F_1, \ldots, F_n$. Apply Theorem 1.6 to the proper sequence $A \cap \{m_1, \ldots, m_2-1\}, A \cap \{m_2, \ldots, m_3-1\}, \ldots$ in Fin(N).

Example 3.9. Let $A \subseteq \mathbb{N}$ be a set containing arbitrarily long arithmetic progressions. For each finite coloring of the sets Fin(N) and $[\text{Fin}(\mathbb{N})]^2$, there are nonempty finite subsets $F_1 \subset F_2 \subset \cdots$ of $A$ such that:

1. The set $\bigcup_n F_n$ contains arbitrarily long arithmetic progressions.
2. All nonempty finite unions $H$ of sets $F_n$ have the same color.
3. All sets $\{H_1, H_2\}$, for $H_1 < H_2$ nonempty finite unions of sets $F_n$, have the same color.

Additional examples are provided by any notion that is captured by finite sets, e.g., entries of solutions of homogeneous systems of equations, and entries of image vectors of matrices. The upper density of a set $A \subseteq \mathbb{N}$ is the real number $\limsup_n |A \cap \{1, \ldots, n\}|/n$.

Corollary 3.10. Let $A \subseteq \mathbb{N}$ be a set of upper density $\delta$. For each finite coloring of the sets Fin(N) and $[\text{Fin}(\mathbb{N})]^2$, there are nonempty finite subsets $F_1 \subset F_2 \subset \cdots$ of $A$ such that:

1. The set $\bigcup_n F_n$ has upper density $\delta$.
2. All nonempty finite unions $H$ of sets $F_n$ have the same color.
3. All sets $\{H_1, H_2\}$, for $H_1 < H_2$ nonempty finite unions of sets $F_n$, have the same color.

Proof. The upper density of a set does not change by removing finitely many elements from that set. Take a sequence $\delta_1, \delta_2, \ldots$ increasing to $\delta$. For each $n$, let $F_n := \{ F \in \text{Fin}(A) : |F|/\max F > \delta_n \}$. Apply Corollary 3.8. □

An analogous assertion also holds for the so-called Banach density.

4. Menger spaces

A topological space $X$ is a Menger space if, for each sequence $U_1, U_2, \ldots$ of countable open covers of $X$, there are finite sets $F_1 \subseteq U_1, F_2 \subseteq U_2, \ldots$ such that the sets $\bigcup F_1, \bigcup F_2, \ldots$...
form an open cover of $X$. A property introduced by Menger in [9] was proved equivalent to this covering property by Hurewicz in [7].

Every compact space is a Menger space, and every countable union of Menger spaces is Menger. There are Menger spaces that are substantially different from countable unions of compact spaces (e.g., [22, 23]). Methods developed in the study of Menger spaces found important applications to seemingly unrelated notions in set theoretic and general topology and in real analysis. Menger spaces are central objects in the theory of Selection Principles. Several surveys of the theory are available (see [13] and references therein).

**Definition 4.1.** Let $X$ be a topological space. A countable family $\mathcal{U}$ of subsets of $X$ is an ascending cover of $X$ if it is a cover of $X$ and there is an enumeration $\mathcal{U} = \{V_1, V_2, \ldots\}$ such that $V_1 \subseteq V_2 \subseteq \cdots$. Let $\text{Asc}(X)$ be the family of open covers of $X$ that contain an ascending cover of $X$.

We consider the family $\mathcal{P}(X)$ of subsets of a set $X$ as a semigroup with the addition operator $\cup$. Thus, for a family of sets $\mathcal{U} \subseteq \mathcal{P}(X)$, the set $\text{FS}(\mathcal{U})$ is comprised of all finite unions of members of $\mathcal{U}$. Only covers with no finite subcover constitute a challenge to Menger’s property.

**Lemma 4.2.** Let $X$ be a topological space. For each countable open cover $\mathcal{U}$ with no finite subcover, we have that $\text{FS}(\mathcal{U}) \in \text{Asc}(X)$.

For a topological space $X$, let $\mathcal{O}(X)$ be the family of countable open covers of $X$. A cover of $X$ is point-infinite if every point of the space $X$ is contained in infinitely many members of the cover. Let $\Lambda(X)$ be the family of countable open point-infinite covers of $X$. The proof of [16, Corollary 6] establishes, in fact, that $S_1(\text{Asc}(X), \Lambda(X))$ holds whenever $S_1(\text{Asc}(X), \mathcal{O}(X))$ does.

**Corollary 4.3** (Folklore). A topological space $X$ is Menger if and only if $S_1(\text{Asc}(X), \Lambda(X))$ holds.

Using a game theoretic theorem of Hurewicz, Scheepers proved in [17] that a space $X$ is Menger if and only if Alice does not have a winning strategy in the game $G_{\text{fin}}(\Lambda(X), \Lambda(X))$, a variation of $G_1(\Lambda(X), \Lambda(X))$ where Bob is allowed to choose any finite number of elements in each turn. Scheepers’s theorem is used in the following proof.

**Proposition 4.4.** A topological space $X$ is Menger if and only if Alice does not have a winning strategy in the game $G_1(\text{Asc}(X), \Lambda(X))$.

---

3Our restriction to countable covers allows for more general results. For the variation without this restriction, every Menger space is a Lindelöf space (that is, one where every open cover has a countable subcover). For Lindelöf spaces, the two variations of Menger’s property coincide.
Proof. \((\Leftarrow)\) If Alice does not have a winning strategy in the game \(G_1(\text{Asc}(X), \Lambda(X))\), then \(S_1(\text{Asc}(X), \Lambda(X))\) holds. Then \(X\) is a Menger space.

\((\Rightarrow)\) Assume that Alice has a winning strategy in the game \(G_1(\text{Asc}(X), \Lambda(X))\). Using this strategy, define a strategy for Alice in the game \(G_{\text{fin}}(\text{Asc}(X), \Lambda(X))\), as follows. In the \(n\)-th inning, Alice’s strategy proposes a cover containing an ascending cover. Alice thins out this cover to make it ascending, and then removes from it the finitely many elements chosen by Bob in the earlier innings. This can only make Bob’s task harder. If Bob picks a finite subset \(F_n\) of this ascending cover, Alice takes the largest set chosen by Bob, \(B_n\), and applies her original strategy, pretending that Bob chose only this set.

Assume that Bob won a play \((U_1, F_1, U_2, F_2, \ldots)\) of the game \(G_{\text{fin}}(\text{Asc}(X), \Lambda(X))\). Then \(\bigcup_n F_n\) is a point-infinite cover of \(X\). Since the sets \(F_n\) are disjoint, the set \(\{B_1, B_2, \ldots\}\) is also a point-infinite cover of \(X\), and we obtain a play in the game \(G_1(\text{Asc}(X), \Lambda(X))\) that is won by Bob; a contradiction. Thus, Alice has a winning strategy in the game \(G_{\text{fin}}(\text{Asc}(X), \Lambda(X))\).

Since \(\text{Asc}(X) \subseteq \Lambda(X)\), Alice has a winning strategy in the game \(G_{\text{fin}}(\Lambda(X), \Lambda(X))\). By Scheepers’s Theorem, the space \(X\) is not Menger.

\[\square\]

With results proved thus far, we are ready to prove our main theorem.

**Theorem 4.5.** Let \((X, \tau)\) be a Menger space, and \(U_1 \supseteq U_2 \supseteq \cdots\) be countable point-infinite open covers of \(X\) with no finite subcover. For each finite coloring of the sets \(\tau\) and \(|\tau|^2\), there are nonempty disjoint finite sets \(F_1 \subseteq U_1, F_2 \subseteq U_2, \ldots\) such that the sets \(V_n := \bigcup F_n\), for \(n \in \mathbb{N}\), have the following properties:

1. The family \(\{V_1, V_2, \ldots\}\) is a point-infinite cover of \(X\).
2. The sets \(\bigcup_{n \in F} V_n\) and \(\bigcup_{n \in H} V_n\), for nonempty finite sets \(F \subset H\), are distinct.
3. All sets \(\bigcup_{n \in F} V_n\), for nonempty finite sets \(F \subseteq \mathbb{N}\), have the same color.
4. All sets \(\{\bigcup_{n \in F} V_n, \bigcup_{n \in H} V_n\}\), for nonempty finite sets \(F \subset H\), have the same color.

Moreover, if \(U_1 = \{U_1, U_2, \ldots\}\), we may request that the sets \(F_n := \{m : U_m \in F_n\}\) satisfy \(F_1 < F_2 < \cdots\).

**Proof.** Enumerate \(U_1 = \{U_1, U_2, \ldots\}\). Consider the semigroup \((\tau, \cup)\). We will work inside its subsemigroup \(S := \text{FS}(U_1, U_2, \ldots)\). Let

\[\mathcal{A} := \{A \subseteq S : A \in \text{Asc}(X)\}\]

The family \(\mathcal{A}\) is a superfilter: Since \(U_1\) has no finite subcover, the sequence \(U_1, U_1 \cup U_2, \ldots\) has an ascending subsequence. Thus, \(\{U_1, U_1 \cup U_2, \ldots\} \in \mathcal{A}\). If \(A \cup B \in \mathcal{A}\), then the set \(A \cup B\) contains an ascending cover \(V_1 \subset V_2 \subset \cdots\), and \(A\) or \(B\) must contain a subsequence of \(V_1, V_2, \ldots\). Thus, \(A \in \mathcal{A}\) or \(B \in \mathcal{A}\). The superfilter \(\mathcal{A}\) is translation invariant. In particular, the superfilter \(\mathcal{A}\) is idempotent.
For each \( n \), using that the cover \( U_n \) has no finite subcover, fix an element \( x_n \in X \setminus \bigcup_{i=1}^{n} U_i \).

For each \( n \), let
\[
V_n := \{ V \in \text{FS}(\{ U_m \in U_n : m \geq n \}) : x_1, \ldots, x_{n-1} \in V \} \subseteq \text{FS}(U_n, U_{n+1}, \ldots).
\]
(Note that \( V_1 = S \).) For each \( n \), let \( V_n := \{ V \in \text{FS}(U_n, U_{n+1}, \ldots) : x_1, \ldots, x_{n-1} \in V \} \subseteq \text{FS}(U_n, U_{n+1}, \ldots) \).

By Proposition 4.4, Alice does not have a winning strategy in the game \( G_1(A, \Lambda(X)) \). By Theorem 3.6, for each finite coloring of the set \( [S]^2 \), there are elements \( V_1 \in V_1, V_2 \in V_2, \ldots \) such that:

1. The set \( \{ V_1, V_2, \ldots \} \) is in \( \Lambda(X) \).
2. The sequence \( V_1, V_2, \ldots \) is a proper sum sequence of \( U_1, U_2, \ldots \) with monochromatic sum graph.

The last assertion in the theorem is clear from the proof of Theorem 3.6.

The assumption in Theorem 4.5 that the space is Menger is necessary. It is proved in [16] that being a Menger space is equivalent to the following property: For each descending sequence \( U_1 \supseteq U_2 \supseteq \cdots \) of countable point-infinite open covers of \( X \) with no finite subcover, there are nonempty finite sets \( F_1 \subseteq U_1, F_2 \subseteq U_2, \ldots \) such that the family \( \{ \bigcup F_n : n \in \mathbb{N} \} \) is a cover of \( X \).

The following example shows that the Milliken–Taylor Theorem is an instance of Theorem 4.5 where Menger’s property is obvious.

**Example 4.6.** Consider Theorem 1.5. Let \( X \) be the set of all cofinite subsets of \( \mathbb{N} \), with the discrete topology (or with the topology induced from the Cantor space \( P(\mathbb{N}) \)). Since the space \( X \) is countable, it is a Menger space.

For each \( n \), let \( O_n := \{ A \in X : n \in A \} \). The family \( \{ O_1, O_2, \ldots \} \) is a point-infinite open cover of \( X \) with no finite subcover. Let \( S := \text{FS}(O_1, O_2, \ldots) \). Then \( S \) is a semigroup, and the map \( \text{Fin}(\mathbb{N}) \to S \) defined by \( F \mapsto O_F \) is a semigroup isomorphism. Thus, a finite coloring of the set \( [\text{Fin}(\mathbb{N})]^2 \) may be viewed as a finite coloring of the set \( [S]^2 \). Let \( F_1 < F_2 < \cdots \) be nonempty finite sets such that the sets \( V_n := O_{F_n} \) satisfy assertion (4) of Theorem 4.5 and the sets \( F_n \) are as requested in Theorem 1.5.

## 5. Richer covers

Let \( X \) be a topological space, and \( A \) and \( B \) be families of covers of \( X \). Let \( \mathcal{U}_{\text{fin}}(A, B) \) be the property that, for covers \( U_1, U_2, \cdots \in A \) with no finite subcover, there are finite sets \( F_1 \subseteq U_1, F_2 \subseteq U_2, \ldots \), such that \( \{ \bigcup F_1, \bigcup F_2, \ldots \} \in B \).
Menger’s covering property is the same as $\mathcal{U}_{\text{fin}}(O(X), O(X))$. A number of important covering properties are of the form $\mathcal{U}_{\text{fin}}(O(X), \mathcal{B})$. Some examples are provided in the survey [13] and in the references therein. By Lemma 4.2, we have the following observation.

**Proposition 5.1.** Let $X$ be a topological space, and $\mathcal{B}$ be a family of covers of $X$. The assertions $\mathcal{U}_{\text{fin}}(O(X), \mathcal{B})$ and $S_1(\text{Asc}(X), \mathcal{B})$ are equivalent. □

Let $\Omega(X)$ be the family of open covers $\mathcal{U}$ of $X$ such that $X \notin \mathcal{U}$ and every finite subset of $X$ is contained in some member of the cover. This family, introduced by Gerlits and Nagy [4], is central to the study of local properties in functions spaces. The property $\mathcal{U}_{\text{fin}}(O(X), \Omega(X))$ was first considered by Scheepers [16]. By the request that $X$ does not belong to any member of $\Omega(X)$, the members of $\Omega(X)$ are infinite. Moreover, the family $\Omega(X)$ is a superfilter on the topology $\tau$ of $X$. If a cover $\mathcal{U} \in \Omega(X)$ is finer than another open cover $\mathcal{V}$ (in the sense that every member of $\mathcal{U}$ is contained in some member of $\mathcal{V}$) with $X \notin \mathcal{V}$, then $\mathcal{V} \in \Omega(X)$. To cover additional important cases, we generalize these properties.

**Definition 5.2.** Let $(X, \tau)$ be a topological space. A family $\mathcal{B}$ of open covers of $X$ is **regular** if it has the following properties:

1. Whenever $\mathcal{U} \cup \mathcal{V} \in \mathcal{B}$, we have that $\mathcal{U} \in \mathcal{B}$ or $\mathcal{V} \in \mathcal{B}$.
2. For each cover $\mathcal{U} \in \mathcal{B}$ and each finite-to-one function $f: \mathcal{U} \to \tau \setminus \{X\}$ with $U \subseteq f(U)$ for all $U \in \mathcal{U}$, the image of $f$ is in $\mathcal{B}$.

Most of the important families of rich covers are regular.

**Example 5.3.** Let $X$ be a topological space. The family $\Omega(X)$ is regular. The family $\Lambda(X)$ satisfies the second, but not the first, regularity condition. Let $\Gamma(X)$ be the family of infinite open covers of $X$ such that each point in $X$ is contained in all but finitely many members of the cover. The family $\Gamma(X)$ is regular. The property $\mathcal{U}_{\text{fin}}(O(X), \Gamma(X))$ was introduced by Hurewicz [7]. Another well-studied regular family, denoted $T^*(X)$, was introduced in [21].

In the following proof, we use the following observation. It extends, by induction, to any finite number of ascending covers.

**Lemma 5.4.** Let $\{U_1, U_2, \ldots\}$ and $\{V_1, V_2, \ldots\}$ be ascending covers of a set $X$, enumerated as such. Then the set $\{U_1 \cap V_1, U_2 \cap V_2, \ldots\}$ is an ascending cover of $X$. □

**Theorem 5.5.** Let $(X, \tau)$ be a topological space, and $\mathcal{B}$ be a regular family of open covers of $X$. The following assertions are equivalent:

1. $\mathcal{U}_{\text{fin}}(O(X), \mathcal{B})$.
2. $S_1(\text{Asc}(X), \mathcal{B})$.
3. Alice does not have a winning strategy in the game $G_1(\text{Asc}(X), \mathcal{B})$.
4. Alice does not have a winning strategy in the game associated to $\mathcal{U}_{\text{fin}}(O(X), \mathcal{B})$. 

**Lemma 5.4.** Let $\{U_1, U_2, \ldots\}$ and $\{V_1, V_2, \ldots\}$ be ascending covers of a set $X$, enumerated as such. Then the set $\{U_1 \cap V_1, U_2 \cap V_2, \ldots\}$ is an ascending cover of $X$. □
Proof. Proposition 5.1 asserts the equivalence of (1) and (2). It is immediate that (4) implies (1).

(3) ⇒ (4): Assume that Alice has a winning strategy in the game associated to \( U_{\text{fin}}(O(X), B) \). By the definition of the selection principle \( U_{\text{fin}}(A, B) \), Alice’s covers must not have finite subcovers. By taking finite unions, turn every cover in Alice’s strategy into an ascending one. This only restricts the possible moves of Bob, and turns them into moves in the game \( G_1(\text{Asc}(X), B) \). Thus, we obtain a winning strategy for Alice in the latter game.

(2) ⇒ (3): Assume that Alice has a winning strategy in the game \( G_1(\text{Asc}(X), B) \). We encode this strategy as follows. Let \( U = \{U_1, U_2, \ldots\} \) be Alice’s first move. For each choice \( U_{m_1} \) of Bob, let \( U_{m_2} = \{U_{m_1}^{m_2}, \ldots\} \) be Alice’s next move. For each choice \( U_{m_2}^{m_3} \) of Bob, let \( U_{m_2,m_3} = \{U_{m_1}^{m_2,m_3}, \ldots\} \) be Alice’s next move, etc. Thus, we have for each sequence \( m_1, m_2, \ldots, m_k \in \mathbb{N} \) a cover \( U_{m_1,m_2,\ldots,m_k} = \{U_{m_1}^{m_2,m_3,\ldots,m_k}, \ldots\} \in \text{Asc}(X) \).

Thinning out the covers Alice plays only restricts Bob’s moves. Thus, we may assume that Alice plays ascending covers, and that every cover played by Alice does not contain any of the finitely many elements played by Bob in the earlier innings. For a natural number \( n \), let \( \{1, \ldots, n\}^{\leq n} := \bigcup_{i=0}^{n} \{1, \ldots, n\}^i \), the set of all sequences of length at most \( n \) taking values in \( \{1, \ldots, n\} \), where the only sequence in \( \{1, \ldots, n\}^0 \) is the empty sequence \( \varepsilon \). We define \( U^\varepsilon_m := U_m \) for all \( m \). For each \( n \), set

\[
V_n := \left\{ \bigcap_{\sigma \in \{1, \ldots, n\}^{\leq n}} U_1^\sigma, \bigcap_{\sigma \in \{1, \ldots, n\}^{\leq n}} U_2^\sigma, \ldots \right\}.
\]

Then \( V_n \) is an ascending cover of \( X \). By the property \( S_1(\text{Asc}(X), B) \), there are elements \( V_1 \in V_1, V_2 \in V_2, \ldots \) such that \( \{V_1, V_2, \ldots\} \in B \).

The cover \( \{V_1, V_2, \ldots\} \) refines the cover \( U \). Since \( U \) has no finite subcover, the set \( \{V_1, V_2, \ldots\} \) is infinite. We construct two parallel plays,

\[
(U, U_{m_1}, U_{m_1}, U_{m_3}^{m_1}, U_{m_1,m_3}, \ldots); \quad (U, U_{m_2}, U_{m_2}, U_{m_4}^{m_2}, U_{m_2,m_4}, \ldots),
\]

according to Alice’s strategy. We use that Alice’s covers are ascending.

(1) Pick a natural number \( m_1 > 1 \) such that

\[
V_1 \subseteq U_{m_1} \in U,
\]

and \( \{V_2, \ldots, V_{m_1}\} \setminus \{V_1\} \neq \emptyset \).

(2) Each of the sets \( V_2, \ldots, V_{m_1} \) is contained in some member of the cover \( U \). Pick a natural number \( m_2 > m_1 \) such that \( U_{m_2} \neq U_{m_1} \),

\[
V_2 \cup \cdots \cup V_{m_1} \subseteq U_{m_2} \in U,
\]

and \( \{V_{m_1+1}, \ldots, V_{m_2}\} \setminus \{V_1, \ldots, V_{m_1}\} \neq \emptyset \).

(3) For \( n = 3, 4, \ldots \):
(a) If \( n \) is odd: Each of the sets \( V_{m_n-2+1}, \ldots, V_{m_n-1} \) is contained in some member of the cover \( \mathcal{U}_{m_1, m_2, \ldots, m_n} \). Pick a natural number \( m_n > m_{n-1} \) such that the set \( U := U_{m_n}^{m_1, m_2, \ldots, m_n-2} \) is distinct from all sets picked earlier,
\[
V_{m_n-2+1} \cup \cdots \cup V_{m_n-1} \subseteq U \in \mathcal{U}_{m_1, m_2, \ldots, m_n-2},
\]
and \( \{V_{m_n-1+1}, \ldots, V_{m_n}\} \setminus \{V_1, \ldots, V_{m_n-1}\} \neq \emptyset \).

(b) If \( n \) is even: Each of the sets \( V_{m_n-2+1}, \ldots, V_{m_n-1} \) is contained in some member of the cover \( \mathcal{U}_{m_2, m_4, \ldots, m_n-2} \). Pick a natural number \( m_n > m_{n-1} \) such that the set \( U := U_{m_n}^{m_2, m_4, \ldots, m_n-2} \) is distinct from all sets picked earlier,
\[
V_{m_n-2+1} \cup \cdots \cup V_{m_n-1} \subseteq U \in \mathcal{U}_{m_2, m_4, \ldots, m_n-2},
\]
and \( \{V_{m_n-1+1}, \ldots, V_{m_n}\} \setminus \{V_1, \ldots, V_{m_n-1}\} \neq \emptyset \).

Define a function
\[
f: \{V_1, V_2, \ldots\} \to \{U_{m_1}, U_{m_2}, U_{m_3}^{m_1}, U_{m_4}^{m_2}, \ldots\}
\]
as follows.

1. Map \( V_1 \) to \( U_{m_1} \).
2. Map each element of the set \( \{V_2, \ldots, V_{m_1}\} \setminus \{V_1\} \) to \( U_{m_2} \).
3. For \( n = 3, 4, \ldots \) map each element of the set \( \{V_{m_n-2+1}, \ldots, V_{m_n-1}\} \setminus \{V_1, \ldots, V_{m_n-2}\} \) to \( U_{m_n}^{m_1, m_2, \ldots, m_n-2} \) if \( n \) is odd, and to \( U_{m_n}^{m_2, m_4, \ldots, m_n-2} \) if \( n \) is even.

The function \( f \) is as needed in property (2) of regular families of covers, and it is surjective. Since the family \( \{V_1, V_2, \ldots\} \) is in \( \mathcal{B} \) and \( \mathcal{B} \) is regular, the set \( \{U_{m_1}, U_{m_2}, U_{m_3}^{m_1}, U_{m_4}^{m_2}, \ldots\} \) is in \( \mathcal{B} \). By Property (1) of regular families of covers, one of the families \( \{U_{m_1}, U_{m_3}, \ldots\} \) or \( \{U_{m_2}, U_{m_4}, \ldots\} \) is in \( \mathcal{B} \). It follows that Bob wins one of these two games against Alice’s winning strategy; a contradiction.

\( \square \)

**Theorem 5.6.** Let \( (X, \tau) \) be a topological space, and \( \mathcal{B} \) be a regular family of open covers of \( X \) (e.g., \( \Omega(X) \), \( T^*(X) \), or \( \Gamma(X) \)). Assume that \( \mathcal{U}_{\text{fin}}(O(X), \mathcal{B}) \) holds. Let \( \mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots \) be countable point-infinite open covers of \( X \) with no finite subcover. For each finite coloring of the sets \( \tau \) and \( [\tau]^2 \), there are nonempty disjoint finite sets \( \mathcal{F}_n \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots \) such that the sets \( V_n := \bigcup \mathcal{F}_n \), for \( n \in \mathbb{N} \), have the following properties:

1. The family \( \{V_1, V_2, \ldots\} \) is in \( \mathcal{B} \).
2. The sets \( \bigcup_{n \in F} V_n \) and \( \bigcup_{n \in H} V_n \), for nonempty finite sets \( F < H \), are distinct.
3. All sets \( \bigcup_{n \in F} V_n \), for nonempty finite sets \( F \subseteq \mathbb{N} \), have the same color.
4. All sets \( \bigcup_{n \in F} V_n, \bigcup_{n \in H} V_n \), for nonempty finite sets \( F < H \), have the same color.

Moreover, if \( \mathcal{U}_1 = \{U_1, U_2, \ldots\} \), we may request that the sets \( F_n := \{m : U_m \in \mathcal{F}_n\} \) satisfy \( F_1 < F_2 < \cdots \).

**Proof.** The proof is identical to that of Theorem 4.5, replacing \( \Lambda(X) \) by \( \mathcal{B} \) and using Theorem 5.3 instead of Proposition 4.4.

\( \square \)
In all of our theorems, the converse implications also hold.

**Proposition 5.7.** Let $X$ be a topological space, and $\mathcal{B}$ be a regular family of open covers of $X$. Assume that, for each descending sequence $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ of countable point-infinite open covers of $X$ with no finite subcover, there are nonempty disjoint finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots\} \in \mathcal{B}$. Then $U_{\text{fin}}(O(X), \mathcal{B})$ holds.

**Proof.** Let $\mathfrak{J}(\mathcal{B})$ be the family of open covers $\mathcal{U}$ of $X$ with no finite subcover, such that there are disjoint finite sets $\mathcal{F}_1, \mathcal{F}_2, \ldots \subseteq \mathcal{U}$ with $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots\} \in \mathcal{B}$. By the first regularity property of $\mathcal{B}$, we have that $\Lambda(X) \supseteq \mathfrak{J}(\mathcal{B})$, and by the premise of the proposition, $\Lambda(X) \subseteq \mathfrak{J}(\mathcal{B})$. By Scheepers's theorem, quoted after the proof of Theorem 4.5, the space $X$ is Menger. By Corollary 10 and Lemma 11 of [14], $U_{\text{fin}}(O(X), \mathcal{B})$ holds. \qed

6. Covers by more general sets

6.1. Borel covers. Consider the variation of Menger’s property, where covers by Borel sets are considered. Here, the restriction to countable covers is necessary to make the property nontrivial. This property has its own history and applications (see, e.g., [19] and the papers citing it). As a rule, the results known for Menger’s property extend to its Borel version [19], and thus Theorem 4.5 and its consequences also hold with “open” replaced by “Borel”. The same assertion holds for the Borel versions of the other covering properties considered above.

In addition to open or Borel, one may consider other types of sets. As long as these types are preserved by the basic operations used in the proof (mainly, finite intersections), the results obtained here apply to countable covers by sets of the considered type.

6.2. A combinatorial theorem. Order the set $\mathbb{N}^\mathbb{N}$ by coordinate-wise comparison: $f \leq g$ if $f(n) \leq g(n)$ for all $n$. Let $\diamondsuit$ be the minimal cardinality of a dominating family $D \subseteq \mathbb{N}^\mathbb{N}$, that is, such that for each function $f \in \mathbb{N}^\mathbb{N}$ there is a function $g \in D$ such that $f \leq g$. It is known that $\aleph_1 \leq \diamondsuit \leq 2^{\aleph_0}$, but it is consistent that the cardinal $\diamondsuit$ is strictly greater than $\aleph_1$ (more details are available in [2]). We may think of a cardinal number $\kappa$ as a discrete space of cardinality $\kappa$. The following assertions are equivalent:

1. $\kappa < \diamondsuit$.
2. $U_{\text{fin}}(O(\kappa), \Omega(\kappa))$ holds.
3. $U_{\text{fin}}(O(\kappa), \Omega(\kappa))$ holds.

By Theorem 5.6, we have the following purely combinatorial result.

**Theorem 6.1.** Let $\kappa$ be a cardinal number smaller than $\diamondsuit$. Let $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$ be countable point-infinite covers of $\kappa$ with no finite subcover. For each finite coloring of the sets $P(\kappa)$ and $[P(\kappa)]^2$, there are nonempty disjoint finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that the sets $A_n := \bigcup \mathcal{F}_n$, for $n \in \mathbb{N}$, have the following properties:

\footnote{Recall that we are considering only countable covers.}
(1) Every finite subset of $\kappa$ is contained in some set $A_n$.
(2) The sets $\bigcup_{n \in F} A_n$ and $\bigcup_{n \in H} A_n$, for nonempty finite sets $F < H$, are distinct.
(3) All sets $\bigcup_{n \in F} A_n$, for nonempty finite sets $F \subseteq \mathbb{N}$, have the same color.
(4) All sets $\{ \bigcup_{n \in F} A_n, \bigcup_{n \in H} A_n \}$, for nonempty finite sets $F < H$, have the same color.

Moreover, if $U_1 = \{ B_1, B_2, \ldots \}$, we may request that the sets $F_n := \{ m : B_m \in F_n \}$ satisfy $F_1 < F_2 < \cdots$. $\square$

7. Comments

7.1. Higher dimensions. Our theorems also hold in dimensions larger than 2, with minor modifications in the proofs. For a natural number $d$, let $[S]^d$ be the family of all $d$-element subsets of $S$. We state the $d$-dimensional versions of Theorem 3.6 and Theorem 4.5. For brevity, the last part of Theorem 7.2 omitted.

Theorem 7.1. Let $S$ be a semigroup, and $d$ be a natural number. Let $A$ be an idempotent superfilter on $S$, and $B$ be a family of subsets of $S$ such that Alice does not have a winning strategy in the game $G_1(A, B)$. Let $a_1, a_2, \ldots$ be a sequence in $S$, and $A_1 \supseteq A_2 \supseteq \cdots$ be a free idempotent chain in $A$ with $A_n \subseteq \text{FS}(a_n, a_{n+1}, \ldots)$ for all $n$. For each finite coloring of the set $[S]^d$, there are elements $b_1 \in A_1, b_2 \in A_2, \ldots$ such that:

(1) The set $\{b_1, b_2, \ldots\}$ is in $B$.
(2) The sequence $b_1, b_2, \ldots$ is a proper sumsequence of $a_1, a_2, \ldots$.
(3) The set $\{ \{b_{F_1}, \ldots, b_{F_d}\} : F_1, \ldots, F_d \in \text{Fin}(\mathbb{N}), F_1 < \cdots < F_d \}$ is monochromatic.

Theorem 7.2. Let $(X, \tau)$ be a Menger space, and $d$ be a natural number. For each descending sequence $U_1 \supseteq U_2 \supseteq \cdots$ of countable point-infinite open covers of $X$ with no finite subcover, and each finite coloring of the set $[\tau]^d$, there are nonempty disjoint finite sets $F_1 \subseteq U_1, F_2 \subseteq U_2, \ldots$ such that the sets $V_n := \bigcup F_n$, for $n \in \mathbb{N}$, have the following properties:

(1) The family $\{V_1, V_2, \ldots\}$ is a point-infinite cover of $X$.
(2) The sets $\bigcup_{n \in F} V_n$ and $\bigcup_{n \in H} V_n$, for nonempty finite sets $F < H$, are distinct.
(3) All sets $\{ \bigcup_{n \in F_1} V_n, \ldots, \bigcup_{n \in F_d} V_n \}$, for nonempty finite sets $F_1 < \cdots < F_d$, have the same color.

The $d$-dimensional versions of Theorems 5.6 and 6.1 are similar.

7.2. Proper sequences. We have taken the approach of proper sequences, or having proper sumsequences, to avoid pathological cases in theorems of Milliken–Taylor type. Hindman and Strauss propose an unconditional approach in [6]. Corollary 18.9 in [6] allows loops in the sum graph and considers colorings of the set $[S]^1 \cup [S]^2$. In a manner similar to the proof of Proposition 1.7, we obtain the following observation.
Proposition 7.3. Let $S$ be a semigroup, and consider the coloring $\chi$ of $[S]^1 \cup [S]^2$ defined by $\chi(\{a, b\}) := |\{a, b\}|$. If a sequence $a_1, a_2, \cdots \in S$ has no proper sumsequence, then every monochromatic sum graph of a sumsequence of $a_1, a_2, \cdots$ is a singleton.

Since we may assume that any given finite coloring of the set $[S]^1 \cup [S]^2$ is finer than the one of Proposition 7.3, there is no advantage in this approach over that of Theorem 1.6.

7.3. New covering properties. Our results suggest a number of new covering properties that were not considered thus far, and it remains unclear how exactly these relate to the classic ones. For example, the property in Theorem 4.5, in the case where $U_n = U$ for all $n$, is formally weaker than Menger’s property. Is it equivalent to it?

7.4. Additional directions. Using the selection principle $S_{\text{fin}}$ and its corresponding game, one obtains an abstract version of a theorem of Deuber and Hindman, and stronger forms of this theorem, in the spirit of the main theorem in Bergelson and Hindman. This direction will be pursued in a later project.

Acknowledgments. Marion Scheepers was the first to realize the connection between Ramsey theory and selection principles, by proving the following beautiful qualitative extension of Ramsey’s Theorem: Let $X$ be a topological space. If $S_1(\Omega(X), \Omega(X))$ holds, then for each cover $U \in \Omega(X)$ and each finite coloring of the set $[U]^2$, there is in $\Omega(X)$ a cover $V \subseteq U$ such that the graph $|V|^2$ is monochromatic. Scheepers proved a large number of results of this type, including ones jointly with Ljubiša Kočinac and others (e.g., \cite{8, 14}).

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