Boundedness and large time behavior in a higher-dimensional Keller–Segel system with singular sensitivity and logistic source

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Abstract

This paper focuses on the following Keller-Segel system with singular sensitivity and logistic source

\[
\begin{aligned}
    u_t &= \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) + au - \mu u^2, \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, t > 0
\end{aligned}
\]

in a smoothly bounded domain \( \Omega \subset \mathbb{R}^N \) \((N \geq 1)\), with zero-flux boundary conditions, where \( a > 0, \mu > 0 \) and \( \chi > 0 \) are given constants. If \( \chi \) is small enough, then, for all reasonable regular initial data, a corresponding initial-boundary value problem for (*) possesses a global classical solution \((u, v)\) which is bounded in \( \Omega \times (0, +\infty) \). Moreover, if \( \mu \) is large enough, the solution \((u, v)\) exponentially converges to the constant stationary solution \((\frac{a}{\mu}, \frac{a}{\mu})\) in the norm of \(L^\infty(\Omega)\) as \( t \to \infty \). To the best of our knowledge, this new result is the first analytical work for the boundedness and asymptotic behavior of Keller–Segel system with singular sensitivity and logistic source in higher dimension case \((N \geq 3)\).

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1 Introduction

Chemotaxis systems in the form of the classical Keller–Segel system (see Winkler et al. [3] and Keller and Segel [21, 22]) model aggregation phenomena in situations where cells are attracted by a signal they themselves emit. Following experimental works of Adler (see Adler et al. [1, 2]), in 1971, Keller and Segel ([22]) introduced a phenomenological model to capture this kind of behaviour, a prototypical version of which is given by

\[
\begin{cases}
  u_t = \Delta u - \nabla \cdot (u\chi_0(v)\nabla v), & x \in \Omega, t > 0, \\
  \tau v_t = \Delta v + u - v, & x \in \Omega, t > 0,
\end{cases}
\]

(1.1)

where \(\tau \in \{0, 1\}\), the function \(\chi_0\) measures the chemotactic sensitivity, \(u\) denotes the cell density and \(v\) describes the concentration of the chemical signal which is directly produced by cells themselves. Systems of type (1.1) and their variants are used in mathematical biology as macroscopic models for cell populations, within which individual cells partially orient their movement toward increasing concentrations of a signal substance. In the last 40 years, a variety of chemotaxis models have been extensively studied with various mechanisms from the cells diffusivity, the chemotactic sensitivity, and the cells growth-death (see [3, 17]). We refer to the review papers [3, 17, 18] for detailed descriptions of the models and their developments. The most peculiar features of (1.1) are the global existence, blowup and asymptotic behavior to the solutions under some suitable initial data (see e.g., [18, 29, 50] for \(\chi_0 := \chi > 0\) and [4, 14, 24, 48, 37, 28] for \(\chi_0 := \frac{\chi}{v}\)). In fact, if \(\chi_0(v) := \chi > 0\), it is known that for all reasonably regular initial data the solutions of the corresponding Neumann initial boundary value problem for (1.1) are global and remain bounded when either \(N = 1\), or \(N = 2\) and \(\int_{\Omega} u_0 < 4\pi\), or \(N \geq 3\) and the initial is sufficiently small ([32, 46, 20]). However, the sensitivity function \(\chi_0(v)\) can not always be a constant, for example, in accordance with the Weber-Fechner’s law of stimulus perception in the process of chemotactic response (see Fujie and Senba [14]), the sensitivity function \(\chi_0(v)\) will be chosen by \(\chi_0(v) = \frac{\chi}{v}\). For system (1.1) with \(\chi_0(v) = \frac{\chi}{v}\), it is known that all radial classical solutions are global-in-time if either \(N \geq 3\) with \(\chi < \frac{2}{N-2}\), or \(N = 2\) with \(\chi > 0\) arbitrary (see Nagai and Senba [27]). When \(N \geq 2\), there exist globally bounded classical solutions...
if $\chi < \sqrt{\frac{2}{N}}$ (see Fujie [11]). The proof of boundedness of solutions for $\chi < \sqrt{\frac{2}{N}}$ in [11] even relying on the second equation actually ensures a positive pointwise lower bound for $v$. The lower bound for $v$ can be obtained by the lower bound for $\int_{\Omega} u$, which is a clear result for (1.1). For more results with various sensitivity functions, we refer to [13, 24, 26].

Apart from the aforementioned system, a source of logistic type is included in (1.1) to describe the spontaneous growth of cells (see Winkler [47] and see also [41] and Zheng [60]). In this paper, we deal with the fully parabolic Keller–Segel system with singular sensitivity and logistic source

$$
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u\chi_0(v)\nabla v) + au - \mu u^2, \quad x \in \Omega, t > 0, \\
v_t &= \Delta v + u - v, \quad x \in \Omega, t > 0, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega
\end{align*}
$$

(1.2)

in a bounded domain $\Omega \subset R^N (N \geq 1)$ with smooth boundary $\partial \Omega$. Our primary interest is in the case in which $a > 0$ as well as $\mu > 0$ and

$$\chi_0(v) = \frac{\chi}{v}, \quad \text{for} \quad v > 0$$

(1.3)

with a constant $\chi > 0$.

For (1.2) with $\chi_0(v) := \chi > 0$, a large quantities of literatures is devoted to investigating boundedness and blow-up of the solutions (see e.g., Cieślak et al. [8, 9, 10], Burger et al. [6], Calvez and Carrillo [7], Keller and Segel [21, 22], Horstmann et al. [18, 19, 20], Osaki [32], Painter and Hillen [33], Perthame [34], Rascle and Ziti [36], Wang et al. [42], Winkler [44, 45, 47, 49, 50, 52], Xiang [53] Zheng [56]). In fact, for any $\mu > 0$, it is also shown that the logistic source can prevent blow up whenever $N \leq 2$, or $\mu$ is sufficiently large (see Osaki and Yagi [32], Osaki et al. [31], Winkler [47], Zheng [60]).

The mathematical challenge to (1.2) with $\chi_0(v) = \frac{\chi}{v}$ is that we must avoid the singular value $v = 0$. Therefore, in order to show the global existence and boundedness to problem (1.2), we should gain a positive pointwise lower bound for $v$, which is a well-known fact for problem (1.1), due to the variation-of-constants formula for $v$ and the fact that

$$\int_{\Omega} u(x, t) = \int_{\Omega} u_0(x) > 0 \text{ for all } t > 0.$$
As for logistic sources contains in (1.2) with quadratic absorption, however, nothing seems to be known in this direction so far (see Zhao and Zheng [54] and Winkler et al. [15] for $N = 2$). Up to now, however, global existence results seem to be available only for certain simplified variants such as e.g. the two-dimensional analogue of (1.2) (see Zhao and Zheng [54] and Winkler et al. [15]). Therefore, very few results appear to be available on system (1.2) with such singular sensitivities and logistic source (see, e.g., Zhao and Zheng [54] and Winkler et al. [15]). In fact, in the spatially two-dimensional case, the knowledge about systems of type (1.3) is expectedly much further developed. The parabolic-elliptic system (1.2) (the second equation of (1.2) is replaced by $\Delta v = v - u$) was considered in [15], where it was obtained that there exists a unique globally bounded classical solution whenever

$$a > \begin{cases} \frac{\chi^2}{4} & \text{if } 0 < \chi \leq 2, \\ \chi - 1 & \text{if } \chi > 2. \end{cases}$$

(1.4)

Recently, if $a$ satisfies (1.4), Zhao and Zheng ([54]) obtained the global bounded classical solution for the fully parabolic system (1.2) in the 2-dimensional setting. As far as we can tell, however, despite a result on global existence established in [54] and [15], the question of boundedness of solutions is completely open in higher dimensions ($N \geq 3$). With some carefully analysis, the purpose of the present work is to investigate the convergence of all solution components in (1.3) under some conditions, possibly involving the initial data or the interaction between chemotactic cross-diffusion and the limitation of cell growth. Without any restriction on the space dimension, the first object of the present paper is to address the global boundedness of solutions to (1.3). Our main result in this respect is the following.

**Theorem 1.1.** Assume that the initial data $(u_0, v_0)$ fulfills

$$\begin{cases} u_0 \in C^0(\Omega) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \neq 0, \ x \in \bar{\Omega}, \\ v_0 \in C^{2,\infty}(\Omega) \text{ with } v_0 > 0 \text{ in } \bar{\Omega}, \text{ and } \frac{\partial v_0}{\partial \nu} = 0, \ x \in \partial \Omega. \end{cases}$$

(1.5)

Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain. If $a$ and $\chi$ satisfies (1.4) and

$$\begin{cases} \chi > 0 & \text{if } N = 1, \\ 0 < \chi < \sqrt{\frac{2}{N}} & \text{if } N \geq 2, \end{cases}$$

(1.6)
respectively, then there exists a unique pair \((u, v)\) of non-negative functions:

\[
\begin{cases}
    u \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)), \\
v \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)),
\end{cases}
\]

which solves (1.2) classically. Moreover, the solution of (1.2) is bounded in \(\Omega \times (0, \infty)\).

**Remark 1.1.** (i) We should point that in view of singular sensitivities, a variation of Maximal Sobolev Regularity cannot be used to solve problem (1.2) (see [5, 60]), since, it is hard to estimate \(\int_{\Omega} |\nabla v|^2 q\) by using the boundedness of \(\int_{\Omega} |\Delta v|^q\) for \(N \geq 3\).

(ii) We should pointed that the idea of this paper can be also solved with other types of models, e.g. an chemotaxis-growth model with indirect attractant production and singular sensitivity (see [57]) and singular sensitivity in a Keller-Segel-fluid system with logistic source.

Going beyond these boundedness statements, a number of results are available which show that the cell kinetics of logistic-type may lead to quite colorful dynamics (see e.g. Winkler et al. [40, 52, 51], Galakhov et al. [16]). For instance, if \(\chi_0(v) = \chi > 0\), Winkler ([52]) found that the solutions of one dimensional parabolic-elliptic models (1.2) may become large at intermediate time scales provided that \(a = \mu \geq 1\). On the other hand, in [51] it was found that all the solutions of the Keller-Segel system (1.2) with \(a = 1\) and \(\chi_0(v) = \chi > 0\) converge to \((\frac{1}{\mu}, \frac{1}{\mu})\) exponentially for a suitable small value of \(\frac{\chi}{\mu}\) and convex domain \(\Omega\). Recently, by applying a variation of Maximal Sobolev Regularity, [53] and [59] (see also [5]) improve the results of [51] to a bounded non-convex domain. As compared to this, the large time behavior to Keller-Segel system (1.2) with singular sensitivity seems to be much less understood. To the best of our knowledge, not even one dimensional result for large time behavior seems available, due to the challenges lies in this problem.

Motivated by the above works, it seems natural and inevitable that our second result, addressing asymptotic homogenization of all solution components, requires \(\mu\) to be appropriately large. Our result in this direction can be stated as follows:

**Theorem 1.2.** Assume the hypothesis of Theorem 1.1 holds. Then there exists \(\mu_0 > 0\) with
the property that if
\[ \mu > \mu_0, \]  
(1.7)
one can find \( \gamma > 0 \) as well as \( t_0 \) and \( C > 0 \) such that the global classical solution \((u, v)\) of (1.2) satisfies
\[ \|u(\cdot, t) - \frac{a}{\mu} \|_{L^\infty(\Omega)} \leq Ce^{-\gamma t}, \text{ for all } t > t_0 \]  
(1.8)
and
\[ \|v(\cdot, t) - \frac{a}{\mu} \|_{L^\infty(\Omega)} \leq Ce^{-\gamma t}, \text{ for all } t > t_0. \]  
(1.9)

Remark 1.2. (i) Theorem 1.2 extends the results of Theorem 1.1 [51], where the convexity of \( \Omega \) required in [51].

(ii) We should also pointed that the idea of this paper can be also solved with other types of models, e.g. an chemotaxis-growth model with indirect attractant production and singular sensitivity (see [57]).

It is worth to remark the main idea underlying the proof of our results. The key step to the proof of Theorem 1.1 is to establish a positive uniform-in-time lower bound for \( v \), which is equivalent to obtain \( \inf_{0 \leq t < \infty} \|u(\cdot, t)\|_{L^1(\Omega)} > 0 \) (see Lemma 3.4), and can be transformed to build the global boundedness for a weighted integral of the form \( \int_\Omega u^{-p}v^{-q}dx \) introduced for system (1.2) with suitable \( p, q > 0 \) to be determined (see Lemmas 3.2–3.3). The technical advantage of small values of \( \chi \) (see (1.6)) is that these will allow us to pick some \( \kappa > \frac{N}{2} \), \( q_0 \in (0, \frac{N}{2}) \) and \( C_0 > 0 \) such that
\[ \int_\Omega u^\kappa v^{-q_0} \leq \frac{C_0}{\mu} \text{ for all } t \in (0, T_{max}), \]
so that, implies the boundedness of \( L^{\frac{N}{2}+\varepsilon}(\Omega) \) by using the variation-of-constants formula. Then we use the standard estimate for Neumann semigroup and the standard Alikakos–Moser iteration (see e.g. Lemma A.1 of [38]) to show Theorem 1.1.

In order to prove Theorem 1.2 we will find a nonnegative function \( F \) satisfying
\[ F'(t) := \frac{d}{dt} \left( \int_\Omega (U - 1 - \ln U) + \frac{L}{2} \int_\Omega V^2 \right) \leq -G_0(\int_\Omega (U - 1)^2 + \frac{L}{2} \int_\Omega V^2) \]
with some suitable positive numbers \( L \) and \( G_0 \) (see Lemma 4.1) depending on the positive pointwise lower bound of \( v \). Then, by means of an analysis of the above inequality and the
uniform Hölder estimates (see Lemma 4.2), one can establish \( \lim_{t \to +\infty} \left( \|u(\cdot, t) - a\|_{L^\infty(\Omega)} + \|v(\cdot, t) - a\|_{L^\infty(\Omega)} \right) = 0 \) (see Lemmas 4.2 and 4.3). By interpolation, we can thus assert the claimed uniform exponential stabilization property. We can thereupon make use of the interpolation and the spatial regularity the solution \((u, v)\) (Lemma 4.2) to show that the above convergence actually takes place at an exponential rate (Lemma ??).

The plan of this paper is as follows. In Section 2, we give some preliminaries, prove estimates for the Neumann heat semigroup in our setting, and state our local existence results of solution to (1.2). In Section 3, with the aid of the weighted integral of the form \( \int_\Omega u^{-p} v^{-q} \, dx \) and \( \int_\Omega u^p v^{-q} \, dx \) for some positive constants \( p, q \), we consider the boundedness of solutions to (1.2) by the variation-of-constants formula. In Section 4, we show the uniform convergence of solution to (1.2) with a suitable energy functional.

## 2 Preliminaries

In this section, we first state several elementary lemmas which will be needed later.

**Lemma 2.1.** *(Page 126 of [30])* Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \), \( p, q, r, s \geq 1 \), \( j, m \in \mathbb{N}_0 \) and \( \alpha \in \left[ \frac{j}{m}, 1 \right] \) satisfying \( \frac{1}{p} = \frac{j}{m} + \left( \frac{1}{r} - \frac{m}{N} \right) \alpha + \frac{1-\alpha}{q} \). Then there are positive constants \( C_1 \) and \( C_2 \) such that for all functions \( w \in L^q(\Omega) \) with \( \nabla w \in L^r(\Omega) \), \( w \in L^s(\Omega) \),

\[
\|D^j w\|_{L^p(\Omega)} \leq C_1 \|D^m w\|^\alpha_{L^r(\Omega)} \|w\|^{1-\alpha}_{L^q(\Omega)} + C_2 \|w\|_{L^s(\Omega)}.
\]

**Lemma 2.2.** *(\[12, 51, 46, 58\])* Let \((e^{\tau \Delta})_{\tau \geq 0}\) be the Neumann heat semigroup in \( \Omega \), and \( \lambda_1 > 0 \) is the first nonzero eigenvalue of \(-\Delta\) in \( \Omega \subset \mathbb{R}^N \) under the Neumann boundary condition. Then there exist \( c_i = c_i(\Omega) \) depending on \( \Omega \) such that the following estimates hold. (i) If \( 1 \leq q \leq p \leq \infty \), then

\[
\|e^{\tau \Delta} \varphi\|_{L^p(\Omega)} \leq c_1 (1 + \tau^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 \tau} \|\varphi\|_{L^q(\Omega)} \quad \text{for all } \tau > 0 \quad \text{and any } \varphi \in L^q(\Omega) \quad \text{and} \quad \int_\Omega \varphi = 0.
\]

(ii) If \( 1 \leq q \leq p \leq \infty \), then

\[
\|\nabla e^{\tau \Delta} \varphi\|_{L^p(\Omega)} \leq c_2 (1 + \tau^{-\frac{1}{2} - \frac{N}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 \tau} \|\varphi\|_{L^q(\Omega)} \quad \text{for all } \tau > 0
\]

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holds and any $\varphi \in L^q(\Omega)$.

(iii) If $2 \leq q \leq p < \infty$, then
\[
\|\nabla e^{\tau \Delta} \varphi\|_{L^p(\Omega)} \leq C_3 (1 + \tau^{-\frac{N}{2} \left(1 - \frac{1}{p}\right)}) e^{-\lambda_1 \tau} \|\nabla \varphi\|_{L^q(\Omega)} \quad \text{for all } \tau > 0
\]
holds and any $\varphi \in W^{1,p}(\Omega)$.

If $1 < q \leq p \leq \infty$, then
\[
\|e^{\tau \Delta} \nabla \cdot \varphi\|_{L^p(\Omega)} \leq C_4 (1 + \tau^{-\frac{\lambda_1 N}{2} \left(1 - \frac{1}{p}\right)}) \|\varphi\|_{L^q(\Omega)} \quad \text{for all } \tau > 0
\]
holds for all $\varphi \in (L^q(\Omega))^N$.

The following local existence result is rather standard, since a similar reasoning in [8, 12, 43, 55, 54]. Therefore, we omit it here.

**Lemma 2.3.** Assume that the nonnegative functions $u_0$ and $v_0$ satisfies (1.5). Then for any $a \in \mathbb{R}$ and $\mu > 0$, there exists a maximal existence time $T_{\text{max}} \in (0, \infty]$ and a pair of nonnegative functions
\[
\begin{cases}
    u \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})), \\
v \in C^0(\bar{\Omega} \times [0, T_{\text{max}})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\text{max}})),
\end{cases}
\]
which solves (1.2) classically and satisfies $u, v > 0$ in $\Omega \times (0, T_{\text{max}})$. Moreover, if $T_{\text{max}} < +\infty$, then
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad \text{as } \quad t \nearrow T_{\text{max}}.
\]

**Lemma 2.4.** ([25]) Let $T \in (0, \infty]$, let $y \in C^1((0, T)) \cap C^0([0, T))$, $h \in C^0([0, T))$, $B > 0, A > 0$ satisfy
\[
y'(t) + Ay(t) \leq h(t) \quad \text{and} \quad \int_{(t-1)^+}^t h(s)ds \leq B \quad \text{for a.e. } t \in (0, T). \quad (2.2)
\]
Then
\[
y(t) \leq y_0 + \frac{B}{1 - e^{-A}} \quad \text{for all } t \in (0, T).
\]
3 The boundedness and classical solution of (1.2)

3.1 Some well-known result about (1.2)

In order to discuss the boundedness and classical solution of (1.2), firstly, we will recall some well-known result about the solutions to (1.2).

Lemma 3.1. Under the assumptions in Lemma 2.3, we derive that there exists a positive constant \( \lambda \) independent of \( a \) and \( \mu \) such that the solution of (1.2) satisfies

\[
\int_{\Omega} u + \int_{\Omega} v^2 + \int_{\Omega} |\nabla v|^2 \leq \lambda \quad \text{for all } t \in (0, T_{\text{max}})
\]

and

\[
\int_{(t-1)^+}^{t} \int_{\Omega} [||\nabla v||^2 + u^2 + |\Delta v|^2] \leq \lambda \quad \text{for all } t \in (0, T_{\text{max}}).
\]

Proof. From integration of the first equation in (1.2) we obtain

\[
\frac{d}{dt} \int_{\Omega} u = \int_{\Omega} (au - \mu u^2) \quad \text{for all } t \in (0, T_{\text{max}}),
\]

which implies that

\[
\frac{d}{dt} \int_{\Omega} u \leq a \int_{\Omega} u - \frac{\mu}{|\Omega|} \left( \int_{\Omega} u \right)^2 \quad \text{for all } t \in (0, T_{\text{max}})
\]

by using the Cauchy-Schwarz inequality. Hence, employing the Young inequality to (3.4) and integrating the resulted inequality in time, we derive that there exists a positive constant \( C_1 \) such that

\[
\int_{\Omega} u \leq C_1 \quad \text{for all } t \in (0, T_{\text{max}}).
\]

For each \( t \in (0, T_{\text{max}}) \), integration with respect to time results in

\[
\int_{(t-1)^+}^{t} \int_{\Omega} u^2 \leq C_2
\]

by (3.5). Now, multiplying the second equation of (1.2) by \(-\Delta v\), integrating over \( \Omega \) and using the Young inequality, we get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla v\|^2_{L^2(\Omega)} + \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2 = - \int_{\Omega} u \Delta v \leq \frac{1}{2} \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} |\Delta v|^2 \quad \text{for all } t \in (0, T_{\text{max}}),
\]
from Lemma 2.4 we infer that
\[ \int_{\Omega} |\nabla v|^2 \leq C_3 \text{ for all } t \in (0, T_{\text{max}}) \quad (3.7) \]
and
\[ \int_{(t-1)_+}^t \left[ |\nabla v|^2 + |\Delta v|^2 \right] \leq C_4 \quad (3.8) \]
by (3.6). Next, testing the second equation of (1.2) by \( v \), we conclude that
\[ \int_{\Omega} v^2 \leq C_5 \text{ for all } t \in (0, T_{\text{max}}). \quad (3.9) \]
by applying (3.6). Now, collecting (3.5)–(3.9) yields to (3.1) and (3.2).

3.2 A lower bounded estimate of \( v \)

In order to deal with the singular sensitivity, in this subsection, we will derive a lower bounded estimate of \( v \). To achieve this, we transform this into time-independent lower bound for \( \int_{\Omega} u^{-\alpha} \) for some \( \alpha > 0 \). Indeed, we firstly conclude a bound on \( \int_{\Omega} u^p v^q \) with some negative exponents \( p \) and \( q \).

Lemma 3.2. \( \text{Let } \Omega \subset \mathbb{R}^N (N \geq 1) \text{ be a smooth bounded domain. Let } (u, v) \text{ be a solution to (1.2) on } (0, T_{\text{max}}) \text{. Then for all } \tilde{p}, \tilde{q} \in \mathbb{R}, \text{ on } (0, T_{\text{max}}) \text{ we have} \)
\[
\frac{d}{dt} \int_{\Omega} u^{\tilde{p}} v^{\tilde{q}} = -\tilde{p}(\tilde{p} - 1) \int_{\Omega} u^{\tilde{p}-2} |\nabla u|^2 + [\tilde{p}(\tilde{p} - 1) \chi - 2\tilde{p}\tilde{q}] \int_{\Omega} u^{\tilde{p}-1} v^{\tilde{q}-1} \nabla u \cdot \nabla v \\
+ [-\tilde{q}(\tilde{q} - 1) + \tilde{p}\tilde{q}\chi] \int_{\Omega} u^{\tilde{p}} v^{\tilde{q}-2} |\nabla v|^2 + [a\tilde{p} - \tilde{q}] \int_{\Omega} u^{\tilde{p}} v^{\tilde{q}} \\
- \mu \tilde{p} \int_{\Omega} u^{\tilde{p}+1} v^{\tilde{q}} + \tilde{q} \int_{\Omega} u^{\tilde{p}+1} v^{\tilde{q}-1} \text{ for all } t \in (0, T_{\text{max}}). \quad (3.10) \]
\[ \]
Proof. Proceeding analogously to Lemma 2.3 of [48], we can prove the desired identity. \( \square \)

With the help of Lemma 3.2 we can estimate \( \int_{\Omega} u^{-p} v^{-q} \) (for some negative exponents \( p \) and \( q \)) in the following format:

Lemma 3.3. \( \text{Let } \Omega \subset \mathbb{R}^N (N \geq 1) \text{ be a smooth bounded domain and } (u, v) \text{ be a solution to (1.2) on } (0, T_{\text{max}}) \text{. Then for } a \in \mathbb{R}, \text{ there exist } p \in (0, 1), C \text{ and } q > q_{1,+} := \frac{p+1}{2} (\sqrt{1 + px^2} - 1) \text{ such that} \)
\[
\frac{d}{dt} \int_{\Omega} u^{-p} v^{-q} \leq (q - ap) \int_{\Omega} u^{-p} v^{-q} + C \text{ for all } t \in (0, T_{\text{max}}). \quad (3.11) \]

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Proof. Firstly, choosing $\bar{p} := -p > 0$ and $\bar{q} := -q > 0$ and in Lemma 3.2, we obtain
\[
\frac{d}{dt} \int_{\Omega} u^{-p} v^{-q} \\
= -p(p + 1) \int_{\Omega} u^{-p-2} v^{-q} |\nabla u|^2 + [p(p + 1) \chi - q] \int_{\Omega} u^{-p-1} v^{-q-1} \nabla u \cdot \nabla v \\
+ (pq \chi - q(q + 1)) \int_{\Omega} u^{-p-2} v^{-q-2} |\nabla v|^2 + (q - ap) \int_{\Omega} u^{-p} v^{-q} \\
+ \mu p \int_{\Omega} u^{-p+1} v^{-q-1} - q \int_{\Omega} u^{-p+1} v^{-q-1} \text{ for all } t \in (0, T_{\text{max}}).
\] (3.12)

Next, by the Young inequality, the second term of (3.12) can be estimated by
\[
[p(p + 1) \chi - 2pq] \int_{\Omega} u^{-p-1} v^{-q-1} \nabla u \cdot \nabla v \\
\leq p(p + 1) \int_{\Omega} u^{-p-2} v^{-q} |\nabla u|^2 + \frac{p[(p + 1) \chi - 2q]^2}{4(p + 1)} \int_{\Omega} u^{-p-2} v^{-q-2} |\nabla v|^2 \\
\text{for all } t \in (0, T_{\text{max}}). \tag{3.13}
\]

Inserting (3.13) into (3.12) implies that
\[
\frac{d}{dt} \int_{\Omega} u^{-p} v^{-q} \\
\leq \left\{ \frac{p[(p + 1) \chi - 2q]^2}{4(p + 1)} + pq \chi - q(q + 1) \right\} \int_{\Omega} u^{-p-2} v^{-q-2} |\nabla v|^2 \\
+ [q - ap] \int_{\Omega} u^{-p} v^{-q} + \mu p \int_{\Omega} u^{-p+1} v^{-q-1} \text{ for all } t \in (0, T_{\text{max}}). \tag{3.14}
\]

Now, denote
\[
4(p + 1)f(p; q, \chi) := \tilde{f}(q) = -4q^2 - 4(p + 1)q + p(p + 1)^2 \chi^2,
\]
where
\[
f(p; q, \chi) := \frac{p[(p + 1) \chi - 2q]^2}{4(p + 1)} + pq \chi - q(q + 1).
\]

Therefore,
\[
f(p; q, \chi) < 0
\]
by the Viète formula and $q > q_{1, +} = \frac{p + 1}{2}(\sqrt{1 + p \chi^2} - 1)$. Combine with (3.14) to get
\[
\frac{d}{dt} \int_{\Omega} u^{-p} v^{-q} \leq (q - ap) \int_{\Omega} u^{-p} v^{-q} + \mu p \int_{\Omega} u^{-p+1} v^{-q-1} \text{ for all } t \in (0, T_{\text{max}}). \tag{3.15}
\]

Take $p \in (0, 1)$, in view of the Young inequality and using (3.14), we conclude that there exists some $C_1 > 0$, such that
\[
\mu p \int_{\Omega} u^{-p+1} v^{-q} \leq q \int_{\Omega} u^{-p+1} v^{-q-1} + (\frac{\mu p}{q + 1})^{q+1} \int_{\Omega} u^{(1-p)} \\
\leq q \int_{\Omega} u^{-p+1} v^{-q-1} + C_1, \tag{3.16}
\]

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from which (3.11) immediately follows by (3.32) and some basic calculation.

Thanks to the Hölder inequality and $L^p$-$L^q$ for the Neumann heat semigroup, Lemma 2.3 directly entails a uniform lower bound for $v$ in $\Omega$ with $a$ satisfying (1.2).

**Lemma 3.4.** Let $\Omega \subset \mathbb{R}^N (N \geq 1)$ be a smooth bounded domain and $(u, v)$ be a solution to (1.2) on $(0, T_{\text{max}})$. If $a$ satisfies (1.4), then there exists a positive constant $\eta_0$ independent of $\mu$ such that

$$v(x, t) \geq \eta_0 \quad \text{for all} \quad (x, t) \in \Omega \times (0, T_{\text{max}}).$$  \hspace{1cm} (3.17)

**Proof.** Let $\delta_1 = \frac{1}{2} \inf_{x \in \Omega} v_0(x)$. In view of Lemma 2.3, there exists $t_0 \in (0, T_{\text{max}})$, such that

$$v(x, t) > \delta_1 \quad \text{for all} \quad (x, t) \in \Omega \times (0, t_0].$$  \hspace{1cm} (3.18)

So we only need to prove (3.17) for $t \in (t_0, T_{\text{max}})$. In fact, let

$$\tilde{g}(p) := \frac{p + 1}{2} (\sqrt{1 + p\chi^2} - 1) - ap, \quad p > 0.$$

Due to $a$ satisfy (1.4), we derive that

$$(0, 1) \cap (p_{\tilde{g}_-}, p_{\tilde{g}_+}) \neq \emptyset$$

by using the Viète formula again, where

$$p_{\tilde{g}, \pm} = \frac{2a^2 + 2a - \chi^2 \pm 2a \sqrt{(1 + a)^2 - \chi^2}}{\chi^2}.$$

Taking $\alpha \in (0, \min\{\frac{pN}{qN - 2q + N}, p\})$, then

$$- \frac{N}{2} (1 - \frac{p - \alpha}{q\alpha}) > -1$$  \hspace{1cm} (3.19)

and

$$\int_{\Omega} u^{-\alpha} \leq \left( \int_{\Omega} u^{-p} v^{-q} \right)^{\frac{\alpha}{p}} \left( \int_{\Omega} u^{\frac{\alpha}{p\alpha}} v^{\frac{p - \alpha}{p}} \right)^{\frac{p - \alpha}{p}}$$  \hspace{1cm} (3.20)

by the Hölder inequality. Integrating (3.11) from $t_0$ to $t$ yields

$$\int_{\Omega} u^{-p} v^{-q} \leq e^{(q - ap)(t - t_0)} \int_{\Omega} u^{-p}(x, t_0) v^{-q}(x, t_0) + C_1$$  \hspace{1cm} (3.21)

with some $C_1 > 0$. 

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On the other hand, by Lemma 2.2 with (3.1) and notation \( \bar{u} = \bar{u}(t) := \frac{1}{|\Omega|} \int_{\Omega} u \), we have

\[
\begin{align*}
\|v(\cdot, t)\|_{L^{p(\infty)}(\Omega)} &\leq \|e^{t(\Delta-1)}v_0\|_{L^{p(\infty)}(\Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}(u(\cdot, s) - \bar{u})\|_{L^{p(\infty)}(\Omega)} \, ds + \int_0^t \|e^{(t-s)(\Delta-1)}\bar{u}\|_{L^{p(\infty)}(\Omega)} \, ds \\
&\leq c_1\|v_0\|_{L^\infty(\Omega)} + c_1 \int_0^t (1 + (t-s)^{-N(1-\frac{p-\alpha}{q\alpha})}) e^{-t(\lambda_1+1)(t-s)}\|u(\cdot, s) - \bar{u}\|_{L^1(\Omega)} \, ds \\
&\quad + c_1 \lambda \int_0^t e^{-t(\lambda_1+1)(t-s)} \, ds \\
&\leq C_2 \quad \text{for all } t \in (t_0, T_{\max}).
\end{align*}
\]

(3.22)

with some \( C_2 > 0 \), where \( c_1 \) is the same as Lemma 2.2. Here we have used the fact that

\[
\int_0^t (1 + (t-s)^{-N(1-\frac{p-\alpha}{q\alpha})}) e^{-t(\lambda_1+1)(t-s)} \, ds \leq \int_0^\infty (1 + (t-s)^{-N(1-\frac{p-\alpha}{q\alpha})}) e^{-t(\lambda_1+1)(t-s)} \, ds < +\infty,
\]
due to (3.19).

Combine (3.20)–(3.22) to know there exists \( C_3 > 0 \) such that

\[
\int_\Omega u^{-\alpha}(x, t) \leq C_3 \quad \text{for all } t \in (t_0, T_{\max})
\]

(3.23)

and hence

\[
\begin{align*}
\int_\Omega u(x, t) &\geq |\Omega|^{\frac{1}{\alpha}} \left( \int_\Omega u^{-\alpha} \right)^{-\frac{1}{\alpha}} \\
&\geq |\Omega|^{\frac{1}{\alpha}} C_3^{-\frac{1}{\alpha}} \\
&:= \delta_2 \quad \text{for all } t \in (t_0, T_{\max})
\end{align*}
\]

(3.24)

by the H"older inequality. The representation of \( v \) as

\[
v(\cdot, t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot, s) \, ds \quad \text{for all } t \in (t_0, T_{\max})
\]

(3.25)

makes it possible to apply well-known estimates for the Neumann heat-semigroup \( \{e^{t\Delta}\}_{t \geq 0} \), which provides a positive constant \( \delta_3 \) such that

\[
\begin{align*}
v(\cdot, t) &= e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot, s) \, ds \\
&\geq \int_0^t \left( \frac{4\pi(t-s)}{\text{diam} \Omega} \right)^{\frac{N}{2}} e^{-t(\lambda_1+1)(t-s)} \int_\Omega u(x, s) \, ds \\
&\geq \delta_2 \int_0^t \left( \frac{4\pi\sigma}{\text{diam} \Omega} \right)^{\frac{N}{2}} e^{-t(\lambda_1+1)(t-s)} \, ds \\
&:= \delta_3 \quad \text{for all } t \in (t_0, T_{\max}) \quad \text{and } x \in \Omega
\end{align*}
\]

(3.26)

by using (3.23) and (3.24). Let \( \eta_0 = \min\{\delta_1, \delta_3\} \) to complete the proof. \( \square \)
In order to obtain a bound for \(u\) with respect to the norm in \(L^\infty(\Omega)\), we need to obtain an \(L^p(\Omega)\)-estimate for \(u\) for some \(p > \frac{N}{2}\). To this end, we transform this into time-independent upper bound for \(\int_{\Omega} u^\beta\) for some \(\beta > 0\). In fact, we firstly conclude a bound on \(\int_{\Omega} u^p v^q\) with some positive exponents \(p\) and \(q\).

**Lemma 3.5.** Let \(\Omega \subset \mathbb{R}^N (N \geq 1)\) be a smooth bounded domain. Assume that \(\chi < 1\). Let \(a\) satisfy (1.4) and \((u, v)\) be a solution to (1.2) on \((0, T_{\text{max}})\). Then for all \(p \in (1, \frac{1}{\chi})\), for each \(q \in (q_2_-(p), q_2_+(p))\), one can find \(C > 0\) independent of \(\mu\) such that

\[
\int_{\Omega} u^p v^{-q} \leq \frac{C}{\mu} \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]

where

\[
q_2_\pm(p) := q_2_\pm = \frac{p - 1}{2}(1 \pm \sqrt{1 - p\chi^2}).
\]

**Proof.** Firstly, choosing \(\tilde{p} := p > 1\) and \(\tilde{q} := -q > 0\) and in Lemma (3.2), we obtain

\[
\frac{d}{dt} \int_{\Omega} u^p v^{-q} = -p(p - 1) \int_{\Omega} u^{p-2} v^{-q} |\nabla u|^2 + [2pq + p(p-1)\chi] \int_{\Omega} u^{p-1} v^{-q-1} \nabla u \cdot \nabla v
\]

\[
- (q(q+1) + pq\chi) \int_{\Omega} u^{p-2} v^{-q} |\nabla v|^2 + (q + ap) \int_{\Omega} u^p v^{-q}
\]

\[
- \mu p \int_{\Omega} u^{p+1} v^{-q} - q \int_{\Omega} u^{p+1} v^{-q-1} \quad \text{for all} \quad t \in (0, T_{\text{max}}),
\]

where by the Young inequality,

\[
[p(p-1)\chi + 2pq] \int_{\Omega} u^{p-1} v^{-q-1} \nabla u \cdot \nabla v
\]

\[
\leq p(p-1) \int_{\Omega} u^{p-2} v^{-q} |\nabla u|^2 + \frac{p[(p-1)\chi + 2q]^2}{4(p-1)} \int_{\Omega} u^{p-2} v^{-2} |\nabla v|^2
\]

for all \(t \in (0, T_{\text{max}})\). Inserting (3.30) into (3.29) implies that

\[
\frac{d}{dt} \int_{\Omega} u^p v^{-q} \leq \{ \frac{p[(p-1)\chi + 2q]^2}{4(p-1)} - q(q + 1) - pq\chi \} \int_{\Omega} u^{p-2} v^{-2} |\nabla v|^2
\]

\[
+ [q + ap] \int_{\Omega} u^p v^{-q} - \mu p \int_{\Omega} u^{p+1} v^{-q} - q \int_{\Omega} u^{p+1} v^{-q-1} \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\]

Denote

\[
g(p; q, \chi) := \frac{p[(p-1)\chi + 2q]^2}{4(p-1)} - pq\chi - q(q + 1),
\]

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and rewrite it as the quadric expression in $q$ that
\[ 4(p - 1)g(p; q, \chi) := \bar{g}(q) = -4q^2 + 4(p - 1)q - p(p - 1)^2\chi^2. \]

According to $\Delta_{\bar{g}, q} := 16(p - 1)^2(1 - p\chi^2) > 0$, our assumption $q \in (q_{2,-}, q_{2,+})$ ensures that
\[ g(p; q, \chi) < 0, \]
where $q_{2,\pm}$ is given by (3.28). Combine with (3.31) to get
\[ \frac{d}{dt} \int_{\Omega} u^p v^{-q} \leq (q + ap) \int_{\Omega} u^p v^{-q} - \mu p \int_{\Omega} u^{p+1} v^{-q} - q \int_{\Omega} u^{p+1} v^{-q-1} \quad \text{for all} \quad t \in (0, T_{\max}). \quad (3.32) \]

We now invoke the Young inequality and use (3.17) in estimating
\[ (q + ap + 1) \int_{\Omega} u^p v^{-q} \leq \mu p \int_{\Omega} u^{p+1} v^{-q} + \frac{1}{p+1}(\mu(q + 1))^{-p}(q + ap + 1)^{p+1} \int_{\Omega} v^{-q} \]
\[ \leq \mu p \int_{\Omega} u^{p+1} v^{-q} + \frac{1}{p+1}(\mu(q + 1))^{-p}(q + ap + 1)^{p+1} \eta_0^{-q}|\Omega|, \]
which together with (3.32) implies
\[ \frac{d}{dt} \int_{\Omega} u^p v^{-q} + \int_{\Omega} u^p v^{-q} \leq \frac{1}{p+1}(\mu(q + 1))^{-p}(q + ap + 1)^{p+1} \eta_0^{-q}|\Omega| \quad \text{for all} \quad t \in (0, T_{\max}). \quad (3.33) \]

For all $t \in (0, T_{\max})$, integrating this between 0 and $t$, taking into account Lemma 2.4, we obtain
\[ \int_{\Omega} u^p(\cdot, t) v^{-q}(\cdot, t) \leq e^{-t} \int_{\Omega} u_0^p v_0^{-q} + \frac{1}{p+1}(\mu(q + 1))^{-p}(q + ap + 1)^{p+1} \eta_0^{-q}|\Omega|(1 - e^{-t}). \quad (3.35) \]

Therefore, (3.27) holds due to $p > 1$. \hfill \Box

**Corollary 3.1.** Assume that $0 < \chi < \sqrt{\frac{2}{N}}$ with $N \geq 2$. Then there exist $\kappa > \frac{N}{2}$, $q_0 \in (0, \frac{N}{2})$ and $C_0 > 0$ independent of $\mu$ such that
\[ \int_{\Omega} u^\kappa v^{-q_0} \leq \frac{C_0}{\mu} \quad \text{for all} \quad t \in (0, T_{\max}). \quad (3.36) \]

**Proof.** Firstly, we derive that $\frac{1}{\chi} > \frac{N}{2}$ by $N \geq 2$ and $0 < \chi < \sqrt{\frac{2}{N}}$. Therefore, we may choose $p := \kappa > \frac{N}{2}$ such that $p \in (1, \frac{1}{\chi^2})$ and $q_0 \in (\frac{p-1}{2}(1 - \sqrt{1 - p\chi^2}), \frac{p-1}{2}(1 + \sqrt{1 - p\chi^2})) \subset (0, \frac{N}{2})$. The claimed inequality (3.36) thus results from Lemma 3.5. \hfill \Box
3.3 The proof of Theorem 1.1

The goal of this subsection is to establish a bound for \( u \) with respect to the norm in \( L^\infty(\Omega) \) in quantitative dependence on a supposedly known pointwise lower bound for \( v \). Indeed, using that boundedness properties of \( u \) (see Corollary 3.1) and a pointwise lower bound for \( v \) (see Lemma 3.4) imply boundedness properties of \( \|u(\cdot, t)\|_{L^\infty(\Omega)} \) and \( \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \) by the variation-of-constants formula.

**Lemma 3.6.** Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be a smooth bounded domain. Let \( \lambda \) satisfy (1.4) and \((u, v)\) be a solution to (1.2) on \((0, T_{max})\). If \( \chi \) satisfies (1.6), then

\[
\sup_{t \in (0,T_{max})} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot, t)\|_{L^\infty(\Omega)}) < +\infty. \tag{3.37}
\]

**Proof.** Firstly, according to Corollary 3.1 we pick \( \kappa > \frac{N}{2} \) and \( q_0 \in (0, \frac{N}{2}) \) such that

\[
\int_\Omega u^\kappa v^{-q_0} \leq C_1 \quad \text{for all } t \in (0, T_{max}). \tag{3.38}
\]

holds with some \( C_1 > 0 \). Since \( q_0 < \frac{N}{2} \) and \( \kappa > \frac{N}{2} \), it is possible to fix \( l_0 \in (\frac{N}{2}, \kappa) \) such that \( l_0 < \frac{N(\kappa-q_0)}{N-2q_0} \). Using (3.38), we find that

\[
\left( \int_\Omega u^\kappa v^{-q_0} \right)^{\frac{1}{l_0}} \leq \left( \int_\Omega u^\kappa v^{-q_0} \right)^{\frac{1}{\kappa}} \left( \int_\Omega v^{\kappa-\frac{l_0}{l_0-q_0}} \right)^{\frac{\kappa-l_0}{q_0-k}} \leq C_1 \left( \int_\Omega v^{\frac{l_0}{l_0-q_0}} \right)^{\frac{1}{\kappa}} \tag{3.39}
\]

\[
= C_1 \|v(\cdot, t)\|_{L^{\frac{l_0}{l_0-q_0}}(\Omega)} \quad \text{for all } t \in (0, T_{max}),
\]

here the Hölder inequality has been used. Since, \( l_0 < \frac{N(\kappa-q_0)}{N-2q_0} \) implies that

\[
\frac{N}{2} \left[ \frac{1}{l_0} - \frac{\kappa-l_0}{l_0q_0} \right] < 1,
\]
so that, by Lemma 2.2 with (3.1), we have

$$\|v(\cdot, t)\|_{L^{\infty}(0, \Omega)} \leq \|e^{t(\Delta-1)}v_0\|_{L^{\infty}(0, \Omega)} + \int_0^t \|e^{(t-s)(\Delta-1)}(u(\cdot, s) - \bar{u})\|_{L^{\infty}(0, \Omega)} ds + \int_0^t \|e^{(t-s)(\Delta-1)}\bar{u}\|_{L^{\infty}(0, \Omega)} ds$$

$$\leq c_1 \|v_0\|_{L^{\infty}(\Omega)} + c_1 \int_0^t (1 + (t - s))^{-\frac{N}{2}}(\frac{1 - q_0}{0^{q_0}})e^{-\lambda_1(t-s)}\|u(\cdot, s) - \bar{u}\|_{L^0(\Omega)} ds$$

$$+ c_1 \lambda \int_0^t e^{-\lambda_1(t-s)} ds$$

$$\leq c_1 \|v_0\|_{L^{\infty}(\Omega)} + c_1 \sup_{s \in (0, T_{\max})} \|u(\cdot, s)\|_{L^0(\Omega)} \int_0^t (1 + (t - s))^{-\frac{N}{2}}(\frac{1 - q_0}{0^{q_0}})e^{-\lambda_1(t-s)} ds$$

$$+ c_1 \lambda \int_0^t e^{-\lambda_1(t-s)} ds$$

$$\leq C_2(1 + \sup_{s \in (0, T_{\max})} \|u(\cdot, s)\|_{L^0(\Omega)})$$

for all $t \in (0, T_{\max})$, \(3.40\)

where $\bar{u} = \bar{u}(t) := \frac{1}{|\Omega|} \int_\Omega u$ and

$$C_2 = c_1 \max \left\{ \|v_0\|_{L^{\infty}(\Omega)} + \frac{\lambda}{\lambda_1 + 1} \int_0^\infty (1 + (t - s))^{-\frac{N}{2}}(\frac{1 - q_0}{0^{q_0}})e^{-\lambda_1(t-s)} ds \right\}.$$ 

Here $c_1$ is the same as Lemma 2.2. Therefore,

$$\sup_{t \in (0, T_{\max})} \|v(\cdot, t)\|_{L^{\infty}(0, \Omega)} \leq C_2(1 + \sup_{s \in (0, T_{\max})} \|u(\cdot, s)\|_{L^0(\Omega)}). \quad 3.41$$

Therefore, there is $C_3 > 0$ fulfilling

$$\sup_{t \in (0, T_{\max})} \|u(\cdot, t)\|_{L^0(\Omega)} \leq C_3 \left( 1 + \left( \sup_{s \in (0, T_{\max})} \|u(\cdot, s)\|_{L^0(\Omega)} \right)^{\frac{q_0}{\kappa}} \right) \quad 3.42$$

by using (3.39). Upon the observation that $\frac{q_0}{\kappa} < 1$ due to $\kappa > \frac{N}{2} > q_0$, we can conclude

$$\sup_{t \in (0, T_{\max})} \|u(\cdot, t)\|_{L^0(\Omega)} \leq \tilde{\lambda}. \quad 3.43$$

Now, collecting (3.1) and (3.43), we derive that for some $r_0 \geq 1$ satisfying $r_0 > \frac{N}{2}$,

$$\sup_{t \in (0, T_{\max})} \|u(\cdot, t)\|_{L^{r_0}(\Omega)} \leq \tilde{\lambda}_1. \quad 3.44$$

Now, involving the variation-of-constants formula for $v$ and $L^p$-$L^q$ estimates for the heat semigroup again, we derive that for $\theta \in \left[ 1, \frac{Nr_0}{N-r_0} \right]$, there exists a positive constant $C_4$ such
that
\[
\|\nabla v(t, \cdot)\|_{L^p(\Omega)} \\
\leq \|\nabla e^{(t-1,1)\Delta} v(0, \cdot)\|_{L^p(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta} u(\cdot, s)\|_{L^p(\Omega)} ds \\
\leq c_2 \|v_0\|_{L^\infty(\Omega)} + c_2 \int_0^t (1 + (t-s)^{-\frac{1}{2} + \frac{N}{2}(\frac{1}{r_0} - \frac{1}{p})}) e^{-\lambda_1(t-s)} ds \leq c_2 \|v_0\|_{L^\infty(\Omega)} + c_2 \lambda_1 \int_0^t (1 + (t-s)^{-\frac{1}{2} + \frac{N}{2}(\frac{1}{r_0} - \frac{1}{p})}) e^{-\lambda_1(t-s)} ds \\
\leq c_4 \quad \text{for all} \quad t \in (0, T_{\max})
\]

by combining (3.44) with (3.44), where \(c_2\) is the same as Lemma 2.2. Here we have used the fact that
\[
\int_0^t (1 + (t-s)^{-\frac{1}{2} + \frac{N}{2}(\frac{1}{r_0} - \frac{1}{p})}) e^{-\lambda_1(t-s)} ds \leq \int_0^\infty (1 + s^{-\frac{1}{2} + \frac{N}{2}(\frac{1}{r_0} - \frac{1}{p})}) e^{-\lambda_1 s} ds < +\infty.
\]

Therefore, there is \(C_5 > 0\) satisfies
\[
\int_\Omega |\nabla v|^\theta \leq C_5 \quad \text{for all} \quad t \in (0, T_{\max}) \quad \text{and} \quad \theta \in \left[1, \frac{Nr_0}{N - r_0}\right] \quad (3.46)
\]

by (3.45). Next, fix \(T \in (0, T_{\max})\), let \(M(T) := \sup_{t \in (0, T)} \|u(\cdot, t)\|_{L^\infty(\Omega)}\) and \(h := \nabla v\). Then by (3.46), there exists \(C_6 > 0\) such that
\[
\|\nabla v\|_{L^p(\Omega)} \leq C_6 \quad \text{for all} \quad t \in (0, T_{\max}) \quad \text{and some} \quad N < \theta_0 < \frac{Nr_0}{N - r_0}. \quad (3.47)
\]

Next, by means of an associate variation-of-constants formula once again, one can derive that for any \(t \in (t_0, T),\)
\[
u(t) = e^{(t-t_0)\Delta} u(\cdot, t_0) - \chi \int_{t_0}^t e^{(t-s)\Delta} \nabla \cdot \left(\frac{u(\cdot, s)}{v(\cdot, s)} h(\cdot, s)\right) ds + \int_{t_0}^t e^{(t-s)\Delta} (au(\cdot, s) - \mu u^2(\cdot, s)) ds,
\]

where \(t_0 := (t - 1)_+\). If \(t \in (0, 1],\) by virtue of the maximum principle, we derive that
\[
\|e^{(t-t_0)\Delta} u(\cdot, t_0)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \quad (3.49)
\]

while if \(t > 1,\) we estimate the first integral on the right of (3.48) by means of the Neumann heat semigroup and Lemma 2.2 according to
\[
\|e^{(t-t_0)\Delta} u(\cdot, t_0)\|_{L^\infty(\Omega)} \leq C_7 (t - t_0)^{-\frac{N}{2}} \|u(\cdot, t_0)\|_{L^1(\Omega)} \leq C_8.
\]

(3.50)
Now, in view of (3.1) and (3.17), we fix an arbitrary $p \in (N, \theta)$ and then once more invoke known smoothing properties of the Stokes semigroup and the Hölder inequality to find $C_9 > 0$ such that

$$\chi \int_t^t \| e^{(t-s)\Delta} \nabla \cdot \frac{u(\cdot, s)}{v(\cdot, s)} \tilde{h}(\cdot, s) \|_{L^\infty(\Omega)} ds$$

$$\leq C_9 \int_t^t \left( 1 + (t-s)^{-\frac{1}{2} - \frac{N}{2p}} \right) e^{-\lambda_1(t-s)} \| u(\cdot, s) \tilde{h}(\cdot, s) \|_{L^p(\Omega)} ds$$

$$\leq C_9 \int_t^t \left( 1 + (t-s)^{-\frac{1}{2} - \frac{N}{2p}} \right) e^{-\lambda_1(t-s)} \| u(\cdot, s) \|_{L^p(\Omega)} \| \tilde{h}(\cdot, s) \|_{L^\theta(\Omega)} ds$$

$$\leq C_9 \int_t^t \left( 1 + (t-s)^{-\frac{1}{2} - \frac{N}{2p}} \right) e^{-\lambda_1(t-s)} \| u(\cdot, s) \|_{L^p(\Omega)} \| \tilde{h}(\cdot, s) \|_{L^\theta(\Omega)} ds$$

$$\leq C_{10} M^b(T) \quad \text{for all} \quad t \in (0, T),$$

where $b := \frac{p \theta - \theta + p}{p\theta} \in (0, 1)$ and

$$C_{10} := C_9 \lambda_1^{1-b} C_6 \int_0^1 \left( 1 + \sigma^{-\frac{1}{2} - \frac{N}{2p}} \right) e^{-\lambda_1 \sigma} d\sigma.$$

Since $p > N$, we conclude that $-\frac{1}{2} - \frac{N}{2p} > -1$. Similarly, due to Lemma 2.2, we can estimate the third integral on the right of (3.48) as follows:

$$\int_t^t \| e^{(t-s)\Delta} (au(\cdot, s) - \mu u^2(\cdot, s)) \|_{L^\infty(\Omega)} ds \leq \int_t^t \sup_{u \geq 0} (au - \mu u^2)_+ ds$$

$$\leq \int_t^t a^2 \mu^{-1} 2^{-2}$$

so that, in view of the definition of $M(T)$, there exists a positive $C_{11}$ such that

$$M(T) \leq C_{11} + C_{11} M^b(T) \quad \text{for all} \quad T \in (0, T_{\text{max}})$$

by using (3.48)–(3.51). By comparison, this implies that

$$\| u(\cdot, t) \|_{L^\infty(\Omega)} \leq C_{12} \quad \text{for all} \quad t \in (0, T_{\text{max}}),$$

due to $b < 1$ and $T \in (0, T_{\text{max}})$ was arbitrary. Finally, with the regularity properties from (3.51) at hand, one can readily derive

$$\| v(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq C_{13} \quad \text{for all} \quad t \in (0, T_{\text{max}})$$

by means of standard parabolic regularity arguments applied to the second equation in (1.2).

The proof Lemma 3.6 is completed. \qed
We are now in a position to prove Theorem 1.1.

The proof of Theorem 1.1. In view of (3.37), we apply Lemma 2.3 to reach a contradiction. Hence the classical solution \((u, v)\) of (1.2) is global in time and bounded. Finally, employing the same arguments as in the proof of Lemma 1.1 in [47], and taking advantage of Lemma 3.6 we conclude the uniqueness of solution to (1.2).

4 Asymptotic behavior

In this section we study the long-time behavior for (1.2) in the case \(\mu\) is large enough. The goal of this section will be to establish the convergence properties stated in Theorem 1.1. The key idea of our approach is to use the variation-of-constants formula, the form of which is inspired by [51] (see also [39, 59]).

To show the global asymptotic stability of \((\frac{a}{\mu}, \frac{a}{\mu})\), it will be convenient to introduce the following notation:

\[
U(x, t) = \frac{\mu}{a} u(x, t) \quad \text{and} \quad V(x, t) = v(x, t) - \frac{a}{\mu}.
\]

Accordingly, we see \((U, V)\) have the following properties:

\[
\begin{align*}
U_t &= \Delta U - \chi \nabla \cdot \left( \frac{U}{\eta} \nabla V \right) + aU(1 - U), \quad x \in \Omega, t > 0, \\
V_t &= \Delta V - V + a\mu(U - 1), \quad x \in \Omega, t > 0, \\
\frac{\partial U}{\partial \nu} &= \frac{\partial V}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
U(x, 0) &= U_0(x) = \frac{\mu}{a} u_0(x), \quad V(x, 0) = V_0(x) = v(x, t) - \frac{a}{\mu}, \quad x \in \Omega
\end{align*}
\]  

by (1.2) and a straightforward computation.

From the proof of Lemma 3.4 we derive that: there exists a positive constant \(k_0\) independent of \(\mu\) such that

\[
\frac{1}{v^2} \leq k_0 \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t > 0,
\]

where \(k_0 = \frac{1}{\eta^2}\).

Lemma 4.1. Let \((u, v)\) be a global classical solution of (1.2). Then if

\[
\mu > \max \left\{ 1, a\chi k_0 \frac{\sqrt{2}}{4} \right\},
\]

\[
\quad \text{(4.4)}
\]
then for all $t > 0$ the function

$$F(t) := \int_{\Omega} (U - 1 - \ln U) + \frac{L}{2} \int_{\Omega} V^2$$

(4.5)

satisfies

$$F'(t) \leq -G(t),$$

(4.6)

with

$$G(t) = G_0\left(\int_{\Omega} (U - 1)^2 + \frac{L}{2} \int_{\Omega} V^2\right),$$

(4.7)

and

$$G_0 = \min\{a - \frac{L}{2} \frac{a^2}{\mu^2}, L - \frac{\chi^2 k_0}{4}\} > 0,$$

where $L$ is a positive constant which satisfies that

$$\frac{\chi^2 k_0}{4} < L < \frac{2\mu^2}{a^2}$$

(4.8)

and $k_0$ is the same as (4.3).

**Proof.** Firstly, multiplying the second equation in (4.2) by $V$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} V^2 + \int_{\Omega} |\nabla V|^2 + \int_{\Omega} V^2 = \frac{a}{\mu} \int_{\Omega} V(U - 1)$$

$$\leq \frac{1}{2} \int_{\Omega} V^2 + \frac{1}{2} \left(\frac{a}{\mu}\right)^2 \int_{\Omega} (U - 1)^2$$

for all $t > 0$.

(4.9)

by the Young inequality. We have from (4.9) that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} V^2 + \int_{\Omega} |\nabla V|^2 + \frac{1}{2} \int_{\Omega} V^2 \leq \frac{1}{2} \left(\frac{a}{\mu}\right)^2 \int_{\Omega} (U - 1)^2$$

for all $t > 0$.

(4.10)

The strong maximum principle along with the assumption $U_0 \not\equiv 0$ yields $U > 0$ in $\bar{\Omega} \times (0, +\infty)$. Relying on this, we multiply the first equation in (4.2) by $1 - \frac{1}{U}$ and integrate by parts, then by (4.3),

$$\frac{d}{dt} \int_{\Omega} (U - 1 - \ln U)$$

$$= -\int_{\Omega} \frac{|\nabla U|^2}{U^2} + \chi \int_{\Omega} \frac{1}{Uv} \nabla U \cdot \nabla v - a \int_{\Omega} (U - 1)^2$$

$$\leq -\int_{\Omega} \frac{|\nabla U|^2}{U^2} + \int_{\Omega} \frac{|\nabla U|^2}{U^2} + \frac{\chi^2}{4} \int_{\Omega} \frac{|\nabla V|^2}{v^2} - a \int_{\Omega} (U - 1)^2$$

$$\leq \frac{\chi^2 k_0}{4} \int_{\Omega} |\nabla V|^2 - a \int_{\Omega} (U - 1)^2$$

for all $t > 0$.

(4.11)
by the Young inequality, where $k_0$ is the same as (4.3). Observe that (4.8), let $\frac{L}{2} \times L + (4.10)$, then we have

$$
\begin{align*}
\frac{d}{dt} \int_{\Omega} (U - 1 - \ln U) + (a - \frac{L}{2} \frac{\mu}{\rho}) \int_{\Omega} (U - 1)^2 \\
+ \frac{L}{2} \frac{d}{dt} \int_{\Omega} V^2 + (L - \frac{\chi^2_k}{4}) \int_{\Omega} |\nabla V|^2 + \frac{L}{2} \int_{\Omega} V^2 \\
\leq 0 \quad \text{for all} \quad t > 0,
\end{align*}
$$

(4.12)

which together with the definition of $F$ and $G$ implies that (4.6) holds.

Lemma 4.2. Assume that the conditions in Theorem 1.1 are satisfied. Let $(u, v)$ be a global classical solution of (1.2). There is $\alpha > 0$ such that $u, v \in C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times (1, +\infty))$. Moreover, there exists a positive constant $C$ such that for every $t > 1$,

$$
\|u(\cdot, t)\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times (1, +\infty))} + \|v(\cdot, t)\|_{C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times (1, +\infty))} \leq C
$$

(4.13)

and

$$
\|u(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C.
$$

(4.14)

Proof. Firstly, based on the regularity of $u$ and $v$, one can readily get a constant $C_1 > 0$ such that

$$
\|u(\cdot, t)\|_{L^{\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C_1 \quad \text{for all} \quad t > 0.
$$

(4.15)

Next, we can rewrite the first equation of (1.2) as

$$
\frac{\partial}{\partial t} u = \nabla a(x, t, u, \nabla u) + b(x, t, u)
$$

(4.16)

with boundary data $a(x, t, u, \nabla u) \cdot \nu = 0$ on $\partial \Omega \times (0, \infty)$, where $a(x, t, u, \nabla u) := \nabla u - \frac{\mu}{\rho} \nabla v$, $b(x, t, u) = au - \mu u^2$, $(x, t) \in \Omega \times (0, \infty)$. Therefore, in view of (4.3) and (4.15), applying Lemma 1.3 of [35] to (4.16), we drive that

$$
u \in C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega} \times [1, +\infty)),
$$

(4.17)

so that, by the second equation of (1.2), we can get that $v \in C^{\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times [1, +\infty))$. Finally, with the aforementioned regularity properties of $u$ and $v$ at hand, we can obtain form Theorem IV.5.3 of [23] that (4.14) holds.
Lemma 4.3. Assume the hypothesis of Theorem 1.2 holds. Then if \((u, v)\) is a nonnegative global classical solution of (1.2), we have

\[
\lim_{t \to +\infty} \|U(\cdot, t) - 1\|_{L^\infty(\Omega)} = 0
\]  

(4.18)

as well as

\[
\lim_{t \to +\infty} \|V(\cdot, t)\|_{L^\infty(\Omega)} = 0.
\]  

(4.19)

Proof. Starting from the functional inequality (4.6) and Lemma 4.2, Lemma 4.3 can be proved in the same way as in Ref. [39]. Therefore, we omit it here.

With the above preparation, we can now integrate the energy inequality (see Lemma 4.1) and make use of the Gagliardo-Nirenberg inequality as well as Lemma 4.3 to achieve that the solution \((u, v)\) exponentially converges to the constant stationary solution \((\frac{a}{\mu}, \frac{a}{\mu})\) in the norm of \(L^\infty(\Omega)\) as \(t \to \infty\).

The proof of Theorem 1.2

Proof. Denote \(h(s) := s - 1 - \ln s\). Noticing that \(h'(s) = 1 - \frac{1}{s}\) and \(h''(s) = 1 + s^{-2} > 0\) for all \(s > 0\), we obtain that \(h(s) \geq h(1) = 0\) and \(F(t)\) is nonnegative. From Lemma 4.1, we have

\[
\int_{\tau_0 + 1}^{t} G(s) \leq F(\tau_0 + 1) - F(t) \leq F(\tau_0 + 1) \quad \text{for all } t > \tau_0 + 1,
\]  

(4.20)

from the definition of \(G\) and \(F\), we also have

\[
\int_{\tau_0 + 1}^{t} \left\{ \int_{\Omega} (U - 1)^2 + \frac{L}{2} \int_{\Omega} V^2 \right\} < +\infty.
\]  

(4.21)

Observe that

\[
\lim_{s \to 1} \frac{s - 1 - \ln s}{(s - 1)^2} = \frac{1}{2},
\]

so that, for \(\varepsilon = \frac{1}{6}\), there exists a positive constant \(\delta < \frac{1}{4}\) such that for any \(|s - 1| < \delta\),

\[
-\frac{1}{6} < \frac{s - 1 - \ln s}{(s - 1)^2} - \frac{1}{2} < \frac{1}{6},
\]

thus,

\[
\frac{1}{3}(s - 1)^2 < s - 1 - \ln s < \frac{2}{3}(s - 1)^2 \quad \text{for any } |s - 1| < \delta.
\]
For the above $\delta > 0$, then there exists $t_0 > 0$ such that for all $t > t_0$,
\[
\|U(\cdot, t) - 1\|_{L^\infty(\Omega)} < \delta \tag{4.22}
\]
by (4.18). Therefore, (4.22) implies that for all $x \in \Omega$ and $t > t_0$,
\[
\frac{1}{3}(U(x, t) - 1)^2 < U(x, t) - 1 - \ln U(x, t) < \frac{2}{3}(U(x, t) - 1)^2 \leq (U(x, t) - 1)^2, \tag{4.23}
\]
which in view of the definition of $F$ and $G$ yields to
\[
\frac{1}{3} \int_\Omega (U - 1)^2 + \frac{L}{2} \int_\Omega V^2 \leq F(t) \leq \frac{1}{G_0} G(t). \tag{4.24}
\]
Hence
\[
F'(t) \leq -G(t) \leq G_0 F(t), \tag{4.25}
\]
from which we obtain
\[
F(t) \leq F(t_0) e^{-G_0(t-t_0)}, \tag{4.26}
\]
Substituting (4.26) into (4.24), we obtain
\[
\frac{1}{3} \int_\Omega (U - 1)^2 + \frac{L}{2} \int_\Omega V^2 \leq F(t_0) e^{-G_0(t-t_0)}, \tag{4.27}
\]
which implies that there is $C_1 > 0$ fulfilling such that
\[
\|U(\cdot, t) - 1\|_{L^2(\Omega)} \leq C_1 e^{-\frac{G_0}{2} t} \text{ for all } t > t_0 \tag{4.28}
\]
as well as
\[
\|v(\cdot, t) - \frac{a}{\mu}\|_{L^2(\Omega)} \leq C_1 e^{-\frac{G_0}{2} t} \text{ for all } t > t_0. \tag{4.29}
\]
Furthermore, we also derive that there exist constants $C_2 > 0$ and $t_0 > 1$ such that
\[
\|U(\cdot, t) - 1\|_{W^{1,\infty}(\Omega)} + \|V(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C_2 \text{ for all } t > t_0 \tag{4.30}
\]
by (4.14) and (1.1). We also recall from the Gagliardo-Nirenberg inequality that there exist positive constants $C_4$ and $C_5$ such that
\[
\|U(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq C_3(\|U(\cdot, t) - 1\|_{W^{1,\infty}(\Omega)}^{\frac{N}{2}} \|U(\cdot, t) - 1\|_{L^2(\Omega)} + \|U(\cdot, t) - 1\|_{L^2(\Omega)}^{\frac{N}{2}}) \leq C_4\|U(\cdot, t) - 1\|_{L^2(\Omega)} + C_5 e^{-\frac{G_0}{N+2} t} \text{ for all } t > t_0. \tag{4.31}
\]
Similarly, we can obtain
\[\|V(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C_6 e^{-\frac{G_0}{N}t} \quad \text{for all } t > t_0.\] \hfill (4.32)

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