On the Complexity of Realizing Facial Cycles

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Abstract. We study the following combinatorial problem. Given a planar graph \( G = (V, E) \) and a set of simple cycles \( \mathcal{C} \) in \( G \), find a planar embedding \( \mathcal{E} \) of \( G \) such that the number of cycles in \( \mathcal{C} \) that bound a face in \( \mathcal{E} \) is maximized. We establish a tight border of tractability for this problem in biconnected planar graphs by giving conditions under which the problem is NP-hard and showing that relaxing any of these conditions makes the problem polynomial-time solvable. Moreover, we give a 2-approximation algorithm for series-parallel graphs and a \((4 + \varepsilon)\)-approximation for biconnected planar graphs.

1 Introduction

A planar graph is a graph that can be embedded into the plane without crossings. While there exist infinitely many such embeddings, the embeddings for connected graphs can be grouped into finitely many equivalence-classes of \textit{combinatorial embeddings}, where two embeddings are \textit{equivalent} if the clockwise cyclic order of the edges around each vertex is the same. Since a graph may admit exponentially many different such embeddings, many drawing algorithms for planar graphs simply assume that one embedding has been fixed beforehand and draw the graph with this fixed embedding. Often, however, the quality of the resulting drawing depends strongly on this embedding; examples are the number of bends in orthogonal drawings, or the area requirement of planar straight-line drawings.

Consequently, there is a long line of research that seeks to optimize quality measures over all combinatorial embeddings. Not surprisingly, except for a few notable cases such as minimizing the radius of the dual graph \cite{13,13}, many of these problems have turned out to be NP-complete. For example it is NP-complete to decide whether there exists a planar embedding that allows for a planar orthogonal drawing without bends or for an upward planar drawing \cite{11}. While there has been quite a bit of work on solving these problems for special cases, e.g., for the orthogonal bend minimization problem \cite{15}, to the best of our knowledge, approximation algorithms have rarely been considered.

Another way of describing a combinatorial embedding of a connected graph \( G \) is by describing its \textit{facial walks}, i.e., by listing the walks of \( G \) that bound a face. In the case of biconnected planar graphs, the facial walks are simple,
and we refer to them as facial cycles. In this paper we consider the problem of optimizing the set of facial cycles, i.e., given a list $C$ of cycles in a biconnected graph $G$, we seek an embedding $E$ of $G$ such that as many cycles of $C$ as possible are facial cycles of $E$. The research on this problem was initiated by Mutzel and Weiskircher [14], who gave an integer linear program (ILP) for a weighted version of the problem. Woeginger [16] showed that the problem is NP-complete by showing that it is NP-complete to maximize the number of facial cycles that have size at most 4. Da Lozzo et al. [6] consider the problem of deciding whether there exists an embedding such that the maximum face size is $k$. They give polynomial-time algorithms for $k \leq 4$ and show NP-hardness for $k \geq 5$ and give a factor-6 approximation for minimizing the size of the largest face. Finally, Dornheim [9] studies a decision problem subject to so-called topological constraints, which specify for certain cycles of a planar graph two subsets of edges of the graph that have to be embedded inside and outside the respective cycle; note that a cycle is a facial cycle if its interior is empty. He proved NP-completeness and reduced the connected case to the biconnected case.

We note that, given a biconnected planar graph $G$ and a set $C$ of cycles of $G$, it can be efficiently decided whether there exists a planar embedding of $G$ in which all cycles of $C$ are facial cycles; for each cycle $C \in C$, we subdivide each edge of $C$ once and connect the subdivision vertex to a new vertex $v_C$. If the resulting graph is planar, the desired embedding of $G$ can be obtained by removing all vertices $v_C$ and their incident edges.

**Contribution and Outline.** In this paper, we thoroughly study the problem $\text{Max Facial } C\text{-Cycles}$ of maximizing the number of cycles from a given set $C$ that bound a face of a biconnected planar graph. We start with preliminaries concerning graphs and their combinatorial embeddings in Section 2. In Section 3, we show that $\text{Max Facial } C\text{-Cycles}$ is NP-complete even if each cycle in $C$ intersects any other cycle in $C$ in at most two vertices and intersects at most three other cycles of $C$. In Section 4, we complement these results with efficient algorithms for series-parallel and general planar graphs when the cycles intersect only few other cycles in more than one vertex. Finally, in Section 5, we develop an efficient approximation algorithm for the problem. For series-parallel graphs we give a 2-approximation, and for biconnected planar graphs we achieve a $(4 + \varepsilon)$-approximation for $\varepsilon > 0$.

**2 Preliminaries**

A planar drawing $\Gamma$ of a graph maps vertices to points in the plane and edges to internally disjoint curves. Drawing $\Gamma$ partitions the plane into topologically connected regions, called faces. The bounded faces are internal and the unbounded face is the outer face. A planar drawing determines a circular ordering of the edges incident to each vertex. Two planar drawings of a connected planar graph are equivalent if they determine the same orderings and have the same outer face. A combinatorial embedding is an equivalence class of planar drawings.
3 Complexity

In this section we study the computational complexity of the underlying decision problem FACIAL C-CYCLES of MAX FACIAL C-CYCLES, which given a biconnected planar graph $G$, a set $C$ of simple cycles of $G$, and a positive integer $k \leq |C|$ asks whether there exists a planar embedding $E$ of $G$ such that at least $k$ cycles in $C$ are facial cycles of $E$. FACIAL C-CYCLES is in NP, as we can guess a set $C' \subseteq C$ of $k$ cycles and then check whether an embedding of $G$ exists in which all cycles in $C'$ are facial cycles in polynomial time. We show NP-hardness for general graphs and for series-parallel graphs.

**Theorem 1.** FACIAL C-CYCLES is NP-complete, even if each cycle $C \in C$

(i) intersects any other cycle in $C$ in at most two vertices, and

(ii) intersects at most three other cycles of $C$ in more than one vertex.

**Proof (sketch).** We give a reduction from MAXIMUM INDEPENDENT SET in 3-connected cubic planar graphs, which we recently showed to be NP-complete [7]. Let $H$ be a 3-connected cubic planar graph. Observe that $H$ has a unique combinatorial embedding up to a flip. We construct an instance $\langle G, C, k \rangle$ of FACIAL C-CYCLES as follows; see Fig. 1. Take the planar dual $H^*$ of $H$ and take $C$ as the set of facial cycles of $H^*$. Observe that $H^*$ is a planar triangulation, since $H$ is cubic and 3-connected. The graph $G$ is obtained from $H^*$ by adding for each edge $e = uv \in E(H^*)$ an edge vertex $v_e$ with neighbors $u$ and $v$. It is not hard to see that $H$ admits an independent set of size $k$ if and only if $G$ admits a combinatorial embedding where $k$ cycles in $C$ are facial (see appendix). By construction $C$ satisfies the restrictions in the statement of the theorem. □
Theorem 2. FACIAL $C$-Cycles is NP-complete for series-parallel graphs, even if any two cycles in $C$ share at most three vertices.

Proof (sketch). We reduce from HAMILTONIAN CIRCUIT, which is known to be NP-complete even for cubic graphs \cite{10}. Let $H$ be any such a graph.

Each vertex $a \in V(H)$ is represented by the following gadget $G_a$. It consists of the graph $K_{2,3}$, where the vertices in the partition of size 2 are denoted $s^a$ and $v^a$ and the other vertices are denoted $u^a_1, u^a_2, u^a_3$, and of an additional vertex $t^a$ adjacent to $v^a$; see Fig. 2b. The graph $G$ is obtained by merging the vertices $s^a$ into a single vertex $s$ and the vertices $t^a$ into a single vertex $t$.

To define $C$, we number the incident edges of each vertex of $H$ from 1 to 3. If $ab$ is the $i$-th edge for $a$ and the $j$-th edge for $b$, we define $C_{ab} \in C$ as the cycle $(s, u^a_i, v^a, t, u^b_j, s)$; see Fig. 2a and 2c. We claim that $G$ admits a combinatorial embedding with $|V(H)|$ facial cycles in $C$ if and only if $H$ is Hamiltonian.

If $Q$ is a Hamiltonian circuit of $H$, we embed $G$ such that the order of the gadgets $G_a$ is the same as the order of the vertices in $Q$. We then choose embedding of the gadgets such that for each edge $ab$ of $Q$ the cycle $C_{ab}$ bounds the face between $G_a$ and $G_b$; this yields the claimed number of facial cycles in $C$. Conversely, observe that if $C_{ab}$ is a facial cycle of an embedding of $G$, then $G_a$ and $G_b$, where $ab$ is an edge of $H$, must be consecutive in the circular order around $s$. If $G$ has $|V(H)|$ facial cycles in $C$ it follows that the vertices corresponding to the gadgets form a Hamiltonian circuit in this order. \qed

4 Polynomial-time Solvable Cases

In this section we discuss special cases of MAX FACIAL $C$-CYCLES that admit a polynomial-time solution. In particular, we show that strengthening any of the conditions in Theorem 1 or Theorem 2 makes the problem tractable.
4.1 General Planar Graphs

In this section we study Max Facial $C$-Cycles when each cycle in $C$ intersects at most two other cycles in $C$ in more than one vertex. In this setting we give in Theorem 5 a linear-time algorithm for biconnected planar graphs. Further, for the class of series-parallel graphs we present in Theorem 4 an FPT-algorithm with respect to the maximum number of cycles in $C$ sharing two or more vertices with any cycle in $C$. We remark that our algorithms imply that strengthening any of the two conditions of Theorem 1 results in a polynomial-time solvable problem. In particular, Max Facial $C$-Cycles is polynomial-time if any two cycles in $C$ share at most one vertex.

We compute the optimal solution in these cases by a dynamic program that works bottom-up in the SPQR-tree $T$ of $G$. Let $\mu$ be a node of $T$. We call a cycle $C \in C$ relevant for $\mu$ (or for skel($\mu$)) if it projects to a cycle in skel($\mu$), that is, the vertices of $C$ in skel($\mu$) and the edges of skel($\mu$) that contain vertices or edges in $C$ form a cycle $C'$ in skel($\mu$) with at least two edges. The cycle $C'$ is the projection of the cycle $C$ in skel($\mu$). Similarly, we also define the projection of a cycle $C \in C$ to pert($\mu$). We denote the set of relevant cycles and of interface cycles of a node $\mu$ by $R(\mu)$ and by $I(\mu)$, respectively. Clearly, $I(\mu) \subseteq R(\mu)$.

Let $\mu$ be a node of $T$. We have the following two important observations.

Observation 1 If each cycle in $C$ intersects at most two other cycles in $C$ in more than one vertex, then $|I(\mu)| \leq 3$.

Observation 2 In any combinatorial embedding $E$ of $G$ at most two interface cycles of $\mu$ can simultaneously bound a face in $E$.

Observation 1 holds since all interface cycles of a node $\mu$ share at least the poles of $\mu$. Observation 2 holds since each interface cycle can only bound one of the two faces incident to the virtual edge representing the parent of $\mu$ in skel($\mu$).

Thus to the rest of $G$ the only relevant information about a combinatorial embedding of pert($\mu$) is (a) the number of facial cycles in $C$ and (b) the set of cycles in $C$ projecting to the facial cycles incident to the parent edge.

If $E$ is a combinatorial embedding of pert($\mu$) and the elements of $I \in I(\mu)$ have project to distinct faces incident to the parent edge in pert($\mu$), we say that $E$ realizes $I$; see Fig. 3.

For any node $\mu$ and any set $I \in I(\mu)$, we denote by $T[\mu, I]$ the maximum number $k$ such that there exists a combinatorial embedding $E$ of pert($\mu$) that realizes $I$ and such that $k$ cycles

Fig. 3: Graph and a P-node skeleton (shaded) with three virtual edges corresponding to children from left to right realizing none, the green and the red and only the red cycle, respectively. The red cycle bounds a face since, in addition, the second and third child are adjacent in the embedding of the skeleton.
in $C$ bound a face of $\mathcal{E}$ that is not incident to the parent edge of $\text{pert}(\mu)$. If no such embedding exists, we set $T[\mu, I] = -\infty$. Due to Observation 2, for convenience we extended the definition of $T$ to the case in which the size of $I$ is larger than 2; in this case, we define $T[\mu, I] = -\infty$.

We show how to compute the entries of $T$ in a bottom-up fashion in the SPQR-tree $T$ of $G$. It is not hard to modify the dynamic program to additionally output a corresponding combinatorial embedding of $G$. We root $T$ at an arbitrary Q-node $\rho$. Let $\phi$ be the unique child of $\rho$. Note that the maximum number of facial cycles in $C$ for any combinatorial embedding of $G$ is $\max_{f \in E(\phi)} |I| + T[\phi, I]$.

For any leaf Q-node $\mu$, we have that $T[\mu, I] = 0$ for each $I \in I(\mu)$. The following lemmata deal with the different types of inner nodes in an SPQR-tree.

**Lemma 1.** Let $\mu$ be an S-node with children $\mu_i$, $i = 1, \ldots, k$. Then $T[\mu, I] = \sum_{i=1}^{k} T[\mu_i, I]$, for $I \in I(\mu)$. Each entry $T[\mu, I]$ can be computed in $O(k)$ time.

**Proof.** The lemma follows easily from the observation that a combinatorial embedding of $\text{pert}(\mu)$ realizes $I$ if and only if each of its children realizes $I$. \qed

**Lemma 2.** Let $\mu$ be a P-node with children $\mu_1, \ldots, \mu_k$. Then

$$T[\mu, I] = \max_{I \subseteq C \subseteq \mathcal{R}(\mu)} \left( \sum_{i=1}^{k} T[\mu_i, C_{\mu_i}] + f(C) \right),$$

where (i) $C_{\mu_i} = C \cap I(\mu_i)$ and (ii) $f(C) = |C \setminus I|$ if $\text{skel}(\mu)$ admits a planar embedding $\mathcal{E}$ where (a) each two virtual edges $e_i$ and $e_j$ corresponding to children $\mu_i$ and $\mu_j$ of $\mu$, respectively, such that $|C_{\mu_i} \cap C_{\mu_j}| = 1$ are adjacent in $\mathcal{E}$, and where (b) the virtual edges $c'$ and $c''$ corresponding to the children $\mu'$ and $\mu''$ of $\mu$ such that $C_{\mu'} \cap I \neq \emptyset$ and $C_{\mu''} \cap I \neq \emptyset$, respectively, are incident to the outer face of $\mathcal{E}$, and $f(C) = -\infty$ otherwise.

**Proof.** Consider an embedding of $\text{pert}(\mu)$ that embeds $T[\mu, I]$ cycles of $C$ as facial cycles and the corresponding embedding $\mathcal{E}$ of $\text{skel}(\mu)$. Let $C \subseteq \mathcal{R}(\mu)$ denote the set of cycles in $C$ that are facial cycles in $\mathcal{E}$ or that are in $I$. Obviously, to make a cycle $c \in C \setminus I$ a facial cycle, each of the two children of $\mu$ that contain $c$ in their interface (i) must be adjacent in $\mathcal{E}$ and (ii) must both realize cycle $c$. Also, in order for the cycles in $I$ to bound the outer-face of the embedding of $\text{pert}(\mu)$, the two children of $\mu$ containing such interface cycles (i) must be incident to the outer-face of $\mathcal{E}$ and (ii) must each realize one of these cycles in their interface. Hence $T[\mu, C]$ is a lower bound on the number of facial cycles in $C$ in the embedding of $\text{pert}(\mu)$. On the other hand, it is not hard to see that by picking the maximum over all subsets $C \subseteq \mathcal{R}(\mu)$ this bound is attained for the correct set of cycles $C$. \qed

We note that the existence of a corresponding embedding for a P-node $\mu$ with $k$ children can be tested in $O(k)$ time for any set $C \subseteq \mathcal{R}(\mu)$, thus allowing us to evaluate $f(C)$ efficiently as follows. Consider the auxiliary multigraph $O$ that contains a vertex for each virtual edge of $\text{skel}(\mu)$, except for the edge representing
the parent of \( \mu \), and two such edges are adjacent if and only if there is a cycle in \( C \setminus I \) that contains edges from both expansion graphs. Also, if there exist two virtual edges in \( \text{ske}l(\mu) \) containing edges from cycles in \( I \), multigraph \( O \) contains an edge connecting them. A corresponding embedding exists if and only if \( O \) is either a simple cycle or it is a collection of paths. In latter case, \( O \) can be augmented to a simple cycle and the order of the virtual edges along this cycle defines a suitable embedding of \( \text{ske}l(\mu) \).

Generally, the number of cycles in \( \mathcal{R}(\mu) \) can be large. However, if every cycle \( C \in \mathcal{C} \) shares two or more vertices with at most \( r \) other cycles in \( \mathcal{C} \), the running time can be bounded as follows.

**Lemma 3.** Let \( \mu \) be a \( P \)-node with children \( \mu_1, \ldots, \mu_k \) such that any cycle of \( \mathcal{R}(\mu) \) shares two or more vertices with at most \( r \) other cycles in \( \mathcal{R}(\mu) \). For each set \( I \in \mathcal{I}(\mu) \), table \( T[\mu, I] \) can be computed in \( O(r^2 2^r \cdot k) \) time from \( T[\mu_i, \cdot] \) with \( i = 1, \ldots, k \).

**Proof.** We employ Lemma 2. It is \( |\mathcal{R}(\mu)| \leq r + 1 \), and \( |\mathcal{I}(\mu)| = O(r^2) \). For each \( I \in \mathcal{I}(\mu) \) we need to consider all the sets \( C \subseteq \mathcal{R}(\mu) \) such that \( I \subseteq C \). There are \( O(2^r) \) such sets \( C \) and for each of them we evaluate \( f(C) \) in \( O(k) \) time. \( \square \)

We now deal with \( R \)-nodes. Let \( \mu \) be an \( R \)-node with \( k \) children \( \mu_1, \ldots, \mu_k \), let \( I \in \mathcal{I}(\mu) \) and let \( C \subseteq \mathcal{R}(\mu) \) with \( C \supseteq I \) be a set of cycles that project to distinct facial cycles of \( \text{ske}l(\mu) \). Note that relevant cycles of \( \mu \) that do not project to a facial cycle of \( \text{ske}l(\mu) \) can never bound a face, and we can hence assume that such cycles have been removed from \( \mathcal{C} \) in a preprocessing step, i.e., every relevant cycle of \( \mu \) projects to a facial cycle of \( \text{ske}l(\mu) \). We define

\[
\text{gain}(C, I) = \sum_{i=1}^{k} \left( T[\mu_i, C \cap \mathcal{I}(\mu_i)] - T[\mu_i, I \cap \mathcal{I}(\mu_i)] \right) + |C \setminus I(\mu)|.
\]

It is not hard to see that, for two such sets of cycles \( C_1, C_2 \subseteq \mathcal{R}(\mu) \) with \( I \subseteq (C_1 \cup C_2) \) and such that no two cycles \( C' \in C_1 \) and \( C'' \in C_2 \) share a virtual edge of \( \text{ske}l(\mu) \), we have \( \text{gain}(C_1 \cup C_2, I) = \text{gain}(C_1, I) + \text{gain}(C_2, I) \).

Let \( H \) be the subgraph of the dual of \( \text{ske}l(\mu) \) induced by the faces that are projections of cycles in \( \mathcal{R}(\mu) \). Since we assume that any two cycles in \( \mathcal{C} \) share two or more vertices with at most two other cycles in \( \mathcal{C} \), the maximum degree of \( H \) is at most 2. Once we have chosen an interface \( I \in \mathcal{I}(\mu) \) and for each of the connected components \( H_1, \ldots, H_c \) of \( H \) a set \( C_i \) of cycles that we want to realize as faces for \( H_i \), the overall number of faces can be expressed as \( \sum_{i=1}^{k} T[\mu_i, I \cap \mathcal{I}(\mu_i)] + \sum_{i=1}^{c} \text{gain}(C_i) \). In particular,

\[
T[\mu, I] = \sum_{i=1}^{k} T[\mu_i, I \cap \mathcal{I}(\mu_i)] + \max_{I \subseteq \bigcup_{i=1}^{c} C_i \subseteq \mathcal{R}(\mu)} \sum_{i=1}^{c} \text{gain}(C_i, I),
\]

where the maximization considers only those sets \( C_i \) whose cycles project to distinct faces of \( \text{ske}l(\mu) \) and whose dual vertices are in \( H_i \).
It thus remains to choose for each connected component $H_i$ of $H$ a set of cycles in $C$ that project to faces that are vertices of $H_i$ and that maximize the gain. We exploit the fact that these graphs have maximum degree 2 to give an efficient algorithm via dynamic programming.

Let $H'$ be such a connected component, which is either a path or a cycle. We observe that, if $H$ contains vertices corresponding to the faces incident to the parent edge, then they are contained in the same connected component of $H$. In the following, we assume that $H'$ does not contain these vertices. The other case can be treated similarly, but requires also to take into account that a set $I \in I(\mu)$ of cycles has to be realized.

Assume that $H'$ is a path $v_1, \ldots, v_h$. Each vertex $v_i$ is associated with a set $C_i \subseteq C$ of potential cycles that can realize the face that corresponds to $v_i$. Observe that, since each such vertex corresponds to a facial cycle that is the projection of some cycle in $C$, it follows that $|C_i| = 1$ for $i = 2, \ldots, h - 1$ and that $|C_1|, |C_h| \leq 2$ for $h \geq 2$ and $|C_1| \leq 3$ if $h = 1$. Otherwise one such cycle would intersect too many other cycles in two or more vertices.

We now compute the optimal solution by dynamic programming along the path. More precisely, we define $P[i, C']$ with $i \in \{1, \ldots, h\}$ and $C' \subseteq C_i$ with $|C'| \leq 1$ as the maximum gain obtainable by any set of cycles in $C_1 \cup \cdots \cup C_{i-1} \cup C'$. Clearly $P[1, \emptyset] = 0$ and $P[1, C] = \text{gain}(\{C\})$. For $i > 1$, observe that $P[i, \emptyset] = \max_{C' \subseteq C_1} P[i-1, C']$ and for $C \in C_i$, it is $P[i, \{C\}] = \max_{C' \subseteq C_1} \max_{|C'| \leq 1} P[i-1, C'] + \text{gain}(C' \cup \{C\}, C')$. Note that $\text{gain}(C' \cup \{C\}, C')$ describes the gain of realizing $C$ in addition to $C'$.

This recurrence allows us to compute the optimal gain value in $O(k)$ time if $H'$ is a path of length $k$. Now assume that $H'$ is a cycle of length $k$. Observe that, in this case, each facial cycle has at most one candidate cycle in $C$. We exploit that either all these cycles are chosen, or at least one of them is not chosen. It is not hard to compute the gain of the solution that chooses all facial cycles. Further, we try each facial cycle as the one that is not chosen, leaving us with an instance that forms a path, to which we apply the previous algorithm. Altogether, in this way we can compute the optimal gain value when $H'$ is a cycle of length $k$ in $O(k^2)$ time. It is not hard to adapt the dynamic program to realize a given set of cycles in $I(\mu)$. We thus have the following lemma.

**Lemma 4.** Let $\mu$ be an $R$-node with children $\mu_1, \ldots, \mu_k$. There is an $O(k^2)$-time algorithm for computing $T[\mu, \cdot]$ from $T[\mu_i, \cdot]$ for $i = 1, \ldots, k$, provided that cycles in $C$ shares two or more vertices with at most two other cycles from $C$.

Altogether, Lemmas 1, 3, and 4 imply the following theorem.

**Theorem 3.** Max Facial $C$-Cycles can be solved in $O(n^2)$ time if every cycle in $C$ intersects at most two other cycles in more than one vertex.

### 4.2 Series-Parallel Graphs

In this section, we consider Max Facial $C$-Cycles on series-parallel graphs. Note Combining the results from Lemma 1 and Lemma 3 yields the following.
Theorem 4. Max Facial $C$-Cycles is solvable in $O(r^22^r \cdot n)$ time for series-parallel graphs if any cycle in $C$ intersects at most $r$ other cycles.

Corollary 1. Max Facial $C$-Cycles is solvable in $O(n)$ time for series-parallel graphs if any cycle in $C$ intersects at most two other cycles.

In the following we show that Max Facial $C$-cycles can be solved in polynomial time for series-parallel graphs if any two cycles in $C$ share at most two vertices. The next lemma shows the special structure of relevant cycles in P-nodes of the SPQR-tree in this case.

Lemma 5. Let $G$ be a series-parallel graph and $C$ be a set of cycles in $G$ such that any two cycles share at most two vertices. For each P-node $\mu$ any two relevant are either edge-disjoint in $\text{skel}(\mu)$ or they share the unique virtual edge of $\text{skel}(\mu)$ that corresponds to a Q-node child of $\mu$, if any.

We again use a bottom-up traversal traversal of the SPQR-tree of a series-parallel graph to obtain the following theorem. The S-nodes are handled using Lemma 1 and the structural properties guaranteed by Lemma 5 allow for a simple handling of the P-nodes.

Theorem 5. Max Facial $C$-Cycles is solvable in $O(n)$ for series-parallel graphs if any two cycles in $C$ share at most two vertices.

5 Approximation Algorithms

In this section we derive constant-factor approximations for Max Facial $C$-Cycles in series-parallel graphs and in biconnected planar graphs. Again, we use dynamic programming on the SPQR-tree. This time, however, instead of computing $T[\mu, I]$, we compute an approximate version $\tilde{T}[\mu, I]$ of it. A table $\tilde{T}[\mu, I]$ is a $c$-approximation of $T[\mu, I]$ if $1/c \cdot T[\mu, I] \leq \tilde{T}[\mu, I] \leq T[\mu, I]$, for all $I \in I(\mu)$.

For P-nodes, we give an algorithm that approximates each entry within a factor of 2, for R-nodes, we achieve an approximation ratio of $(4 + \varepsilon)$ for any $\varepsilon > 0$. In the following lemmas we deal separately with S-, P-, and R-nodes.

Lemma 6. Let $\mu$ be an S-node with children $\mu_1, \ldots, \mu_k$. Assume that $\tilde{T}[\mu_i, I]$ is a $c$-approximation of $T[\mu_i, I]$ for $i = 1, \ldots, k$. Then setting $\tilde{T}[\mu, I] = \sum_{i=1}^k \tilde{T}[\mu_i, I]$ yields a $c$-approximation of $T[\mu, I]$.

Proof. To see this, observe that by Lemma 4 it is $1/c \cdot T[\mu, I] = 1/c \cdot \sum_{i=1}^k T[\mu_i, I] \leq \sum_{i=1}^k \tilde{T}[\mu_i, I]$ and $\sum_{i=1}^k \tilde{T}[\mu_i, I] \leq \sum_{i=1}^k T[\mu_i, I] = T[\mu, I]$. \qed

Next we deal with a P-node $\mu$ with children $\mu_1, \ldots, \mu_k$. The algorithm works as follows. Fix an set $I \in I(\mu)$. We construct an auxiliary graph $H$ as follows. The vertices of $H$ are the children $\mu_1, \ldots, \mu_k$ of $\mu$. Two vertices $\mu_i$ and $\mu_j$ are adjacent in $H$ if and only if there exists a cycle $C \in C$ that intersects $\mu_i$ and $\mu_j$ such that $\tilde{T}[\mu_x, I \cap I(\mu_x) \cup \{C\}] = \tilde{T}[\mu_x, I \cap I(\mu_x)]$ for $x \in \{i, j\}$, i.e.,
intersected by a cycle in $M$ since at least half of the cycles of the optimum. However, this argument is not valid, and hence the intuition about comparing the matching sizes indeed applies.

We claim that this gives a max$\{2, c\}$-approximation of $T[\mu, \cdot]$ if the input is a $c$-approximation for $T[\mu_i, \cdot]$ for $i = 1, \ldots, k$.

**Lemma 7.** Let $\mu$ be a P-node an let $\hat{T}[\mu, \cdot]$ denote the table computed in the above fashion. Then $\hat{T}[\mu, \cdot]$ is a max$\{2, c\}$-approximation of $T[\mu, \cdot]$ if $\hat{T}[\mu_i, \cdot]$ is a $c$-approximation of $T[\mu_i, \cdot]$.

**Proof.** We first show that $\hat{T}[\mu, \cdot] \leq T[\mu, \cdot]$. To this end, it suffices to show that, for any $I \in I(\mu)$, there exists an embedding of $\text{pert}(\mu)$ that realizes $I$ and has $\hat{T}[\mu, I]$ realized cycles from $C$. Consider the multigraph with vertex set $\{\mu_0, \mu_1, \ldots, \mu_k\}$ and edge set $C_M \cup I$. This graph has maximum degree 2 and, due to our special treatment, unless $k = 2$, none of its connected components is a cycle. We can thus always complete this graph into a cycle, which defines a circular order of $\mu_0, \ldots, \mu_k$, and hence an embedding of $\text{skel}(\mu)$. In this embedding, all the cycles in $C_M \cup I$ project to facial cycles. Realizing all these cycles yields $\hat{T}[\mu, I] = \sum_{i=1}^{k} \hat{T}[\mu_i, (I \cup C_M) \cap I(\mu_i)] + |M| \leq \sum_{i=1}^{k} T[\mu_i, (I \cup C_M) \cap I(\mu_i)] + |M|$ realized cycles. By the definition of the $T[\mu_i, \cdot]$ we get embeddings for the $\text{pert}(\mu_i)$ with a corresponding number of cycles in $C$ and by combining them according to the embedding of $\text{skel}(\mu)$ chosen above we obtain an embedding of $\text{pert}(\mu)$ that realizes $I$ and has at least $\hat{T}[\mu, I]$ facial cycles in $C$. Hence $\hat{T}[\mu, I] \leq T[\mu, I]$.

Conversely, consider $T[\mu, I]$ and a corresponding embedding of $\text{skel}(\mu)$. Denote by $C_{\text{opt}}$ the set of cycles realized by an optimal solution that project to facial cycles of $\text{skel}(\mu)$. We consider two cycles in $C_{\text{opt}}$ as adjacent if they intersect the same child of $\mu$. Clearly, each child $\mu_i$ is intersected by at most two cycles in $C_{\text{opt}}$ and, moreover, the two faces of $\text{skel}(\mu)$ incident to the parent edge are not realized. Hence the corresponding graph is a collection of paths. It is hence possible to edge-color it with two colors. Let $C_{\text{opt}}'$ be the cycles in the larger color class. We have $|C_{\text{opt}}'| \geq |C_{\text{opt}}|/2$ and no two distinct cycles in $C_{\text{opt}}'$ intersect the same child $\mu_i$ of $\mu$, i.e., interpreting the cycles in $C_{\text{opt}}'$ as edges on the vertex set $\{\mu_1, \ldots, \mu_k\}$ yields a matching $M'$. We would like to argue that our matching $M$ in the auxiliary graph $H$ is larger than $M'$, and hence we realize at least half of the cycles of the optimum. However, this argument is not valid, since $M'$ may contain edges that are not present in $H$ due to approximation errors in the $\hat{T}[\mu_i, \cdot]$. We will show that the contribution of these edges is irrelevant and hence the intuition about comparing the matching sizes indeed applies.

Let $M_1' = M' \setminus E(H)$ and $M_2' = M' \cap E(H)$. Let $J = \{1, \ldots, k\}$ and let $J_1 = \{i \in J \mid \exists C \in M_1' \text{ that intersects } \mu_i\}$ be the indices of children that are intersected by a cycle in $M_1'$. The set $J_2 = J \setminus J_1$ contains the remaining indices.
Clearly, we have \( T[\mu, I] = \sum_{i=1}^{k} T[\mu_i, (I \cup C_{opt}) \cap \mathcal{I}(\mu_i)] + |C_{opt}| \) according to Lemma 2. Realizing instead of \( C_{opt} \) just the set of cycles \( C'_{M'} = C'_{opt} \) corresponding to \( M' \) drops at most \( |C_{opt}|/2 \) facial cycles in \( C \), while imposing weaker interface constraints on the children. We therefore have

\[
T[\mu, I] = \sum_{i=1}^{k} T[\mu_i, (I \cup C_{opt}) \cap \mathcal{I}(\mu_i)] + |C_{opt}| \leq \sum_{i=1}^{k} T[\mu_i, (I \cup C_{M'}) \cap \mathcal{I}(\mu_i)] + 2|M'|
\]

We now use the fact that the \( \tilde{T}[\mu_i, \cdot] \) are a \( c' \)-approximation of the \( T[\mu_i, \cdot] \), and hence also a \( c' \)-approximation for \( c' = \max\{c, 2\} \), and we also separate the sum by the index set \( J_1 \) and \( J_2 \) and consider the two matchings \( M'_1 \) and \( M'_2 \) separately.

\[
\sum_{i=1}^{k} T[\mu_i, (I \cup C_{M'}) \cap \mathcal{I}(\mu_i)] + 2|M'| \leq c' \sum_{i \in J_1} \tilde{T}[\mu_i, (C_{M'_1} \cup I) \cap \mathcal{I}(\mu_i)] + 2|M'_1| + c' \sum_{i \in J_2} \tilde{T}[\mu_i, (C_{M'_2} \cup I) \cap \mathcal{I}(\mu_i)] + 2|M'_2|. \tag{1}
\]

Observe that the indices of the children intersected by cycles that form a matching \( M_2 \) in \( H \) are all contained in \( J_2 \). By the definition of \( H \), we thus have \( \tilde{T}[\mu_i, (C_{M'_2} \cup I) \cap \mathcal{I}(\mu_i)] = T[\mu_i, I \cap \mathcal{I}(\mu_i)], \) for \( i \in J_2 \).

For the first term, observe that, for each edge \( \mu_i \mu_j \in M'_1 \), we have \( \tilde{T}[\mu_i, (M'_1 \cup I) \cap \mathcal{I}(\mu_x)] \leq \tilde{T}[\mu_x, I \cap \mathcal{I}(\mu_x)] - 1 \) for at least one \( x \in \{i, j\} \). Otherwise the edge would be in \( H \), and hence in \( M'_2 \). Let \( J'_1 \subseteq J_1 \) denote the set of indices where this happens and let \( J'_2 = J_1 \setminus J'_1 \). Observe that \( |J'_2| \geq |M'_1| \). We thus have

\[
c' \cdot \sum_{i \in J_1} \tilde{T}[\mu_i, (C_{M'_1} \cup I) \cap \mathcal{I}(\mu_i)] + 2|M'_1| = c' \sum_{i \in J'_1} \tilde{T}[\mu_i, (C_{M'_1} \cup I) \cap \mathcal{I}(\mu_i)] + c' \sum_{i \in J'_2} \tilde{T}[\mu_i, (C_{M'_1} \cup I) \cap \mathcal{I}(\mu_i)] + 2|M'_1| \leq c' \sum_{i \in J'_1} (\tilde{T}[\mu_i, I \cap \mathcal{I}(\mu_i)] - 1) + c' \sum_{i \in J'_2} \tilde{T}[\mu_i, I \cap \mathcal{I}(\mu_i)] + 2|M'_1| \leq c' \left( \sum_{i \in J_1} \tilde{T}[\mu_i, I \cap \mathcal{I}(\mu_i)] - |J'_1| \right) + 2|M'_1| \leq c' \sum_{i \in J_1} \tilde{T}[\mu_i, I \cap \mathcal{I}(\mu_i)],
\]

where the last step uses the fact that \( c' \geq 2 \). Plugging this information into Eq. (1) yields the following.

\[
c' \cdot \sum_{i \in J_1} \tilde{T}[\mu_i, (C_{M'_1} \cup I) \cap \mathcal{I}(\mu_i)] + 2|M'_1| + c' \cdot \sum_{i \in J_2} \tilde{T}[\mu_i, (C_{M'_2} \cup I) \cap \mathcal{I}(\mu_i)] + 2|M'_2| \leq c' \sum_{i=1}^{k} \tilde{T}[\mu_i, I \cap \mathcal{I}(\mu_i)] + 2|M'_1| \leq c' \sum_{i=1}^{k} \tilde{T}[\mu_i, I \cap \mathcal{I}(\mu_i)] + 2|M| \leq c' \left( \sum_{i=1}^{k} \tilde{T}[\mu_i, I \cap \mathcal{I}(\mu_i)] + |M| \right).
\]
Here the last two steps use the fact that $M \subseteq E(H)$ is a maximum matching, and hence larger than $M'_2$, and that $c' \geq 2$, respectively.

We note that the bottleneck for computing $T[\mu, I]$ is finding a maximum matching in a graph with $O(|\text{ske}l(\mu)|)$ vertices and $O(|C|)$ edges. Hence the running time for one step is $O(|\text{ske}l(\mu)| + \sqrt{|\text{ske}l(\mu)|} \cdot |C|)$. Since $|I(\mu)| \leq |C|^2$, the running time for processing a single P-node $\mu$ is $O(|\text{ske}l(\mu)|^2 + \sqrt{|\text{ske}l(\mu)|} \cdot |C|^3)$. The total time for processing all P-nodes then is $O(n|C|^2 + \sqrt{n}|C|^3)$.

**Theorem 6.** There is a 2-approximation algorithm with running time $O(n|C|^2 + \sqrt{n}|C|^3)$ for MAX FACIAL $C$-Cycles in series-parallel graphs.

Next we deal with R-nodes. Let $\mu$ be an R-node with children $\mu_1, \ldots, \mu_k$ and let $J = \{1, \ldots, k\}$. For each face $f$ of $\text{ske}l(\mu)$ let $J_f$ denote the indices of the children $\mu_i$ whose corresponding virtual edge in $\text{ske}l(\mu)$ is incident to $f$.

Fix $I \in I(\mu)$. We propose the following algorithm for computing $\hat{T}[\mu, I]$. Consider the subgraph $H$ of the dual of $\text{ske}l(\mu)$ induced by those vertices $v$ corresponding to a face $f$ not incident to the parent edge of $\text{ske}l(\mu)$ and such that there exists a cycle $C_v \in C$ that projects to the boundary of $f$ and such that $\hat{T}[^{\mu_i}, (\{C_v\} \cup I) \cap I(\mu)] = \hat{T}[^{\mu_i}, I \cap I(\mu)]$, i.e., requiring that $C_v$ is realized in $\mu_i$ does not change the approximate number of faces realized by $\text{pert}(\mu_i)$.

Now we compute a $(1 + \varepsilon/4)$-approximation of a maximum independent set of $H$, which can be done in time polynomial in $|\text{ske}l(\mu)|$ (and exponential in $(1/\varepsilon)$) \[2\]. Let $X$ denote this independent set, and let $C_X = \{C_v \mid v \in X\}$ be a set of corresponding cycles in $C$. We set $\hat{T}[\mu, I] = \sum_{i=1}^{k} \hat{T}[^{\mu_i}, (I \cup X) \cap I(\mu_i)] + |X|$, and claim that in this fashion $\hat{T}[\mu, \cdot]$ is a $\max\{c, (4 + \varepsilon)\}$-approximation provided that $\hat{T}[^{\mu_i}, \cdot]$ is a $c$-approximation of $T[^{\mu_i}, \cdot]$. The proof 4-colors the facial cycles $C_{opt}$ that are realized by an optimal solution and considers the largest color class, which is an independent set of size at least $|C_{opt}|/4$. The proof is similar to that of Lemma 7.

**Lemma 8.** Let $\hat{T}[\mu, \cdot]$ denote the table computed in the above fashion. Then $\hat{T}[\mu, \cdot]$ is a $\max\{c, (4 + \varepsilon)\}$-approximation of $T[^{\mu_i}, \cdot]$ provided that $\hat{T}[^{\mu_i}, \cdot]$ is a $c$-approximation of $T[^{\mu_i}, \cdot]$.

Overall, we obtain the following theorem.

**Theorem 7.** MAX FACIAL $C$-Cycles for biconnected planar graphs admits an efficient $(4 + \varepsilon)$-approximation algorithm for any $\varepsilon > 0$.

### 6 Conclusion

In this paper we showed NP-hardness of MAX FACIAL $C$-Cycles under restrictive conditions, showed that even stronger conditions make the problem tractable and gave constant-factor approximations for series-parallel and biconnected planar graphs with approximation guarantees of 2 and $4 + \varepsilon$, respectively.
We remark that it is possible to adapt all our algorithmic results to the weighted case where each facial cycle has a positive weight and one seeks a planar embedding that maximizes the total weight of the facial cycles in $\mathcal{C}$. We leave open the question whether similar algorithmic results can be obtained for arbitrary, not necessarily biconnected, planar graphs.
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Connectivity and SPQR-trees

A graph $G$ is connected if there is a path between any two vertices. A cutvertex is a vertex whose removal disconnects the graph. A separating pair is a pair of vertices $\{u,v\}$ whose removal disconnects the graph. A connected graph is 2-connected if it does not have a cutvertex and a 2-connected graph is 3-connected if it does not have a separating pair. A 2-connected plane graph $G$ is internally 3-connected if $G$ can be extended to a 3-connected planar graph by adding a vertex in the outer face and joining it to all the vertices incident to the outer face.

We consider $uv$-graphs with two special pole vertices $u$ and $v$, which can be constructed in a fashion very similar to series-parallel graphs. Namely, an edge $(u,v)$ is an $uv$-graph with poles $u$ and $v$. Now let $G_i$ be an $uv$-graph with poles $u_i,v_i$ for $i = 1, \ldots, k$ and let $H$ be a planar graph with two designated vertices $u$ and $v$ and $k + 1$ edges $uv, e_1, \ldots, e_k$. We call $H$ the skeleton of the composition and its edges are called virtual edges; the edge $uv$ is the parent edge and $u$ and $v$ are the poles of the skeleton $H$. To compose the $G_i$ into an $uv$-graph with poles $u$ and $v$, we remove the edge $uv$ and replace each $e_i$ by $G_i$ for $i = 1, \ldots, k$ by removing $e_i$ and identifying the poles of $G_i$ with the endpoints of $e_i$. In fact, we only allow three types of compositions: in a series composition the skeleton $H$ is a cycle of length $k + 1$, in a parallel composition $H$ consists of two vertices connected by $k + 1$ parallel edge, and in a rigid composition $H$ is 3-connected.

It is known that for every 2-connected graph $G$ with an edge $uv$ the graph $G - st$ is an $uv$-graph with poles $u$ and $v$. Much in the same way as series-parallel graphs, the $uv$-graph $G \setminus uv$ gives rise to a (de-)composition tree $T$ describing how it can be obtained from single edges. The nodes of $T$ corresponding to edges, series, parallel, and rigid compositions of the graph are $Q$-, $S$-, $P$-, and $R$-nodes, respectively. To obtain a composition tree for $G$, we add an additional root $Q$-node representing the edge $uv$. To fully describe the composition, we associate with each node $\mu$ its skeleton denoted by $\text{skel}(\mu)$. For a node $\mu$ of $T$, the pertinent graph $\text{pert}(\mu)$ is the subgraph represented by the subtree with root $\mu$. Similarly, for a virtual edge $\epsilon$ of a skeleton $\text{skel}(\mu)$, the expansion graph of $\epsilon$, denoted by $\text{exp}(\epsilon)$ is the pertinent graph $\text{pert}(\mu')$ of the neighbour $\mu'$ of $\mu$ corresponding to $\epsilon$ when considering $T$ rooted at $\mu$.

The SPQR-tree of $G$ with respect to the edge $uv$, originally introduced by Di Battista and Tamassia [8], is the (unique) smallest decomposition tree $T$ for $G$. Using a different edge $u'v'$ of $G$ and a composition of $G - u'v'$ corresponds to rerooting $T$ at the node representing $u'v'$. It thus makes sense to say that $T$ is the SPQR-tree of $G$. The SPQR-tree of $G$ has size linear in $G$ and can be computed in linear time [12]. Planar embeddings of $G$ correspond bijectively to planar embeddings of all skeletons of $T$; the choices are the orderings of the parallel edges in $P$-nodes and the embeddings of the $R$-node skeletons, which are unique up to a flip. When considering rooted SPQR-trees, we assume that the embedding of $G$ is such that the root edge is incident to the outer face, which is equivalent to the parent edge being incident to the outer face in each skeleton. We remark that in a planar embedding of $G$, the poles of any node $\mu$ of $T$ are
incident to the outer face of $\text{pert}(\mu)$. Hence, in the following we only consider embeddings of the pertinent graphs with their poles lying on the same face and refer to such embeddings as regular.

Let $\mu$ be a node of $T$, we denote the poles of $\mu$ by $u(\mu)$ and $v(\mu)$, respectively. In the remainder of the paper, we will assume edge $(u(\mu), v(\mu))$ to be part of $\text{skel}(\mu)$ and $\text{pert}(\mu)$. The outer face of a (regular) embedding of $\text{pert}(\mu)$ is the one obtained from such an embedding after removing the $(u(\mu), v(\mu))$ connecting its poles. Also, the two paths incident to the outer face of $\text{pert}(\mu)$ between $u(\mu)$ and $v(\mu)$ are called boundary paths of $\text{pert}(\mu)$.

B Omitted Proofs from Section 3

Lemma 9. Facial $\mathcal{C}$-Cycles is in NP.

Proof. Let $(G, \mathcal{C}, k)$ be an instance of Facial $\mathcal{C}$-Cycles. A non-deterministic Turing machine can guess in polynomial-time a combinatorial embedding $\mathcal{E}$ of $G$ and test whether at least $k$ cycles in $\mathcal{C}$ are facial cycles in $\mathcal{E}$. □

Theorem 1. Facial $\mathcal{C}$-Cycles is NP-complete, even if

(i) each cycle in $\mathcal{C}$ intersects any other cycle in $\mathcal{C}$ in at most two vertices, and

(ii) each cycle in $\mathcal{C}$ intersects at most three other cycles of $\mathcal{C}$ in more than one vertex.

Proof. We give a reduction from Maximum Independent Set in 3-connected cubic planar graphs, which is NP-complete [7]. Let $H$ be a 3-connected cubic planar graph. Observe that $H$ has a unique combinatorial embedding up to a flip. We construct an instance $(G, \mathcal{C}, k)$ of Facial $\mathcal{C}$-Cycles as follows. We take the planar dual $H^*$ of $H$ and take $\mathcal{C}$ as the set of facial cycles of $H^*$. Observe that $H^*$ is a planar triangulation, since $H$ is cubic and 3-connected. The graph $G$ is obtained from $H^*$ by adding for each edge $e = uv \in E(H^*)$ an edge vertex $v_e$ that is adjacent to both $u$ and $v$; see Fig. We claim that $H$ admits an independent set of size $k$ if and only if $G$ admits a combinatorial embedding where $k$ cycles in $\mathcal{C}$ are facial.

Note that the embedding of $H^* \subseteq G$ is unique up to a flip. The only embedding choices for $G$ are to decide, for each edge $e \in E(H^*)$, in which of the two faces incident to $e$ in $H^*$ the vertex $v_e$ is embedded. A cycle in $\mathcal{C}$ bounding a face of $H^*$ forms a facial cycle in the embedding of $G$ if and only if no edge vertex is embedded inside it. Note that no two cycles in $\mathcal{C}$ sharing an edge $e \in E(H^*)$ can both bound a face of $G$ since the shared edge vertex $v_e$ must be embedded in the interior of one of the two faces of $H^*$ incident to $e$. It follows that an embedding with $k$ facial cycles in $\mathcal{C}$ induces a set of independent faces in $H^*$, and thus an independent set of size $k$ in $H$. Conversely, by embedding the edge vertices outside the faces of $H^*$ corresponding to an independent set of size $k$ in $H$, we obtain an embedding of $G$ with $k$ facial cycles in $\mathcal{C}$.

Observe that since $H^*$ is a 3-connected, no two faces of $H^*$ (resp. no two cycles in $\mathcal{C}$ sharing an edge) share more than two vertices and, moreover, since
$H^*$ is a planar triangulation, no cycle of $C$ shares two vertices with more than three other cycles. By construction $C$ satisfies the restrictions in the statement of the theorem.

**Theorem 2**  **Facial $C$-Cycles is NP-complete for series-parallel graphs, even if any two cycles in $C$ share at most three vertices.**

**Proof.** We give a reduction from Hamiltonian Circuit, which is known to be NP-complete even for cubic graphs [10]. Let $H$ be any such a graph.

Each vertex $a \in V(H)$ is represented by the following gadget $G_a$. It consists of the graph $K_{2,3}$, where the vertices in the partition of size 2 are denoted $s^a$ and $v^a$ and the other vertices are denoted $u^a_1, u^a_2, u^a_3$, and of an additional vertex $t^a$ adjacent to $v^a$; see Fig. 2a. The graph $G$ is obtained by merging the vertices $s_a$ into a single vertex $s$ and the vertices $t_a$ into a single vertex $t$.

To define $C$, we number the incident edges of each vertex of $H$ from 1 to 3. If $ab$ is the $i$-th edge for $a$ and the $j$-th edge for $b$, we define $C_{ab} \in C$ as the cycle $(s, u^a_1, v^a, t, v^b, u^b_1, s)$; see Fig. 2a and 2c. We claim that $G$ admits a combinatorial embedding with $|V(H)|$ facial cycles in $C$ if and only if $H$ is Hamiltonian.

Assume that $Q$ is a Hamiltonian circuit of $H$. We embed the graph $G$ such that the order of the gadgets $G_a$ around $s$ is the same as the order of the vertices along $Q$. Now, for each edge $ab \in Q$ the gadgets $G_a$ and $G_b$ are adjacent in this order, say with $G_a$ before $G_b$. Assume that $ab$ is the $i$-th edge for $a$ and the $j$-th edge for $b$. We choose the order of the vertices $u^a_i$ in $G_a$ and the vertices $u^b_j$ in $G_b$ such that $u^a_i$ and $u^b_j$ are incident to the face shared by $G_a$ and $G_b$. Thus $C_{ab}$ bounds a face. The resulting embedding clearly has $|V(H)|$ facial cycles in $C$.

Conversely, assume that $G$ has a combinatorial embedding with at least $k = |V(H)|$ facial cycles in $C$. Consider any two adjacent gadgets $G_a$ and $G_b$. Since there are $|V(H)|$ facial cycles in $C$, it follows that the face between $G_a$ and $G_b$ must be bounded by the cycle $C_{ab}$ in $C$. But this implies that $a$ and $b$ are adjacent in $H$. Hence the circular order of the gadgets around $s$ determines a Hamiltonian circuit of $H$. Observe that any two cycles of $C$ share at most three vertices. □

**C  Omitted proofs from Section 4**

**Lemma 5**  **Let $G$ be a series-parallel graph and $C$ be a set of cycles in $G$ such that any two cycles share at most two vertices. For each $P$-node $\mu$ any two relevant are either edge-disjoint in skel($\mu$) or they share the unique virtual edge of skel($\mu$) that corresponds to a $Q$-node child of $\mu$, if any.**

**Proof.** Let $C$ and $C'$ be two relevant cycles for some $P$-node $\mu$ with poles $u$ and $v$. Clearly $C$ and $C'$ share the two poles $u$ and $v$. Now assume that $C$ and $C'$ additionally share a virtual edge $e$ of skel($\mu$). Consider the expansion graph $G_e$ of $e$ and observe that $\{u, v\}$ cannot be a separation pair of $G_e$, since $\mu$ is a $P$-node. Thus the corresponding child $\nu$ of $\mu$ must be either a $Q$- or an $S$-node. If it is an $S$-node, however, then $G_e$ contains a cutvertex $c$, which is contained in both $C$ and $C'$, a contradiction. Further observe that a $P$-node may have at most one child that is a $Q$-node. This concludes the proof. □
Theorem 5 Max Facial $C$-Cycles is solvable in $O(n)$ for series-parallel graphs if any two cycles in $C$ share at most two vertices.

Proof. We use again a bottom-up approach as in the previous section. Q-nodes can be handled trivially as before, and S-nodes can be handled by Lemma 4. It remains to deal with the P-nodes. Here we use the special structure guaranteed by Lemma 5.

Let $\mu$ be a P-node with $k$ children and let $I \in I(\mu)$. First consider all cycles in $R(\mu)$ whose projections do not traverse a Q-node child of $\mu$. By Lemma 5, they are pairwise disjoint, and we realize each such cycle $C$ if it has positive gain, i.e., if and only if $\text{gain}(\{C\}, I) > 0$. For the remaining cycles, which all share the same virtual edge $e$ that corresponds to a Q-node child $\nu$ of $\mu$, we observe that at most two of them can be realized, and their gains are again independent since they share only $e$ and they are disjoint from the other realized cycles. Altogether, this allows to fill the table $T[\mu, I]$ in $O(k)$ time for each $I \in I(\mu)$. Note that $|I(\mu)| \leq 1$ unless the parent $\mu_0$ of $\mu$ is a Q-node, in which case the algorithm has reached the root of the SPQR-tree, and we simply choose $I$ greedily so that it contains up to two cycles whose projections contain the parent edge and that have positive gain.

D Omitted Proofs from Section 5

Lemma 8 Let $\hat{T}[\mu, \cdot]$ denote the table computed in the above fashion. Then $\hat{T}[\mu, \cdot]$ is a $\max \{e, (4 + \varepsilon)\}$-approximation of $T[\mu, \cdot]$ provided that $\hat{T}[\mu_i, \cdot]$ is a $c$-approximation of $T[\mu_i, \cdot]$.

Proof. We first show $\hat{T}[\mu, \cdot] \leq T[\mu, \cdot]$ by constructing for each $I \in I(\mu)$ an embedding of $\text{pert}(\mu)$ that realizes $I$ and that has $\hat{T}[\mu, I]$ realized cycles from $C$. Fix $I \in I(\mu)$ and let $C_X$ denote the set of cycles in $C$ determined as above. Then, it is $\hat{T}[\mu, I] = \sum_{i=1}^{k} \hat{T}[\mu_i, (I \cup C_X) \cap I(\mu)] + |X| \leq \sum_{i=1}^{k} T[\mu_i, (I \cup C_X) \cap I(\mu)] + |X|$ since the $\hat{T}[\mu_i, \cdot]$ are $c$-approximations of $T[\mu_i, \cdot]$. Thus, there exist embeddings of the $\text{pert}(\mu_i)$ with the corresponding number of facial cycles in $C$. By combining them according to the embedding of $\text{skel}(\mu)$, we obtain a planar embedding of $\text{pert}(\mu)$, which in addition to the facial cycles of the $\text{pert}(\mu_i)$ in $C$ has $|X|$ facial cycles in $C$ that project to faces of $\text{skel}(\mu)$. This proves the claim.

Conversely, consider $T[\mu, I]$ and a corresponding embedding of $\text{pert}(\mu)$. Let $C_{\text{opt}}$ denote the faces of $C \setminus I$ that bound faces of $\text{pert}(\mu)$. By the 4-color theorem 15, it is possible to 4-color the faces in $C_{\text{opt}}$ such that two faces have the same color only if they are disjoint. Let $C'$ denote the largest color class and observe that $|C'| \geq |C_{\text{opt}}|/4$.

Again, as in Lemma 7 we would like to compare the sizes of $C'$ and $X$ and argue that $X$ cannot be much smaller than $C'$ since it is an approximation of a maximum independent set in $H$ and $C'$ corresponds to an independent set of faces in $\text{skel}(\mu)$. Again, the problem is that $C'$ may contain cycles that project to faces of $\text{skel}(\mu)$ for which no vertex is contained in $H$ due to approximation
errors in the tables $\hat{T}[^{\mu_i}]$. As before, we argue that the contribution of these cycles is irrelevant in the approximation, and hence this consideration applies.

Let $C'_1 = C' \setminus V(H)$ and let $C'_2 = C' \setminus C'_1$. Recall that $J = \{1, \ldots, k\}$ and let $J_1 = \{i \in J \mid \exists C \in C'_i \text{ that intersects } \mu_i \}$ be the indices of children that are intersected by a cycle in $C'_i$. The set $J_2 = J \setminus J_1$ contains the remaining indices.

Clearly, we have $T[^{\mu_i}, I] = \sum_{i=1}^k T[^{\mu_i}, (I \cup C_{\text{opt}}) \cap I(\mu_i)] + |C_{\text{opt}}|$ according to Lemma 4. Realizing instead of $C_{\text{opt}}$ just the set of cycles $C'_2$ drops at most $|C_{\text{opt}}|/4$ facial cycles in $C$, while imposing weaker interface constraints on the children. We therefore have

$$T[^{\mu_i}, I] = \sum_{i=1}^k T[^{\mu_i}, (I \cup C_{\text{opt}}) \cap I(\mu_i)] + |C_{\text{opt}}| \leq \sum_{i=1}^k T[^{\mu_i}, (I \cup C') \cap I(\mu_i)] + 4|C'|$$

(2)

We now use the fact that the $\hat{T}[^{\mu_i}, \cdot]$ are a $c$-approximation of the $T[^{\mu_i}, \cdot]$, and hence also a $c'$-approximation for $c' = \max\{c, 4\}$, and we also separate the sum by the index set $J_1$ and $J_2$ and consider the two sets of cycles $C'_1$ and $C'_2$ separately.

$$\sum_{i=1}^k T[^{\mu_i}, (I \cup C') \cap I(\mu_i)] + 4|C'| \leq c' \cdot \sum_{i \in J_1} T[^{\mu_i}, (C'_1 \cup I) \cap I(\mu_i)] + 4|C'_1|$$

$$+ c' \cdot \sum_{i \in J_2} T[^{\mu_i}, (C'_2 \cup I) \cap I(\mu_i)] + 4|C'_2|.$$  

(3)

Observe that the indices of the children intersected by cycles that correspond to an vertex in $H$ are all contained in $J_2$. By the definition of $H$, we thus have $\hat{T}[^{\mu_i}, (C'_2 \cup I) \cap I(\mu_i)] = \hat{T}[^{\mu_i}, I \cap I(\mu_i)]$, for $i \in J_2$.

For the first term, observe that, for each cycle $C \in C'_1$, we have $\hat{T}[^{\mu_x}, ((C'_1) \cup I) \cap I(\mu_x)] \leq \hat{T}[^{\mu_x}, I \cap I(\mu_x)] - 1$ for at least one $x \in J_I$, where $f$ is the facial cycle of $\text{skel}(\mu)$ to which $C$ projects. Otherwise the vertex would be in $H$, and hence in $C'_1$. Let $J'_I \subseteq J_1$ denote the set of indices where this happens and let $J''_I = J_1 \setminus J'_I$. Observe that $|J'_I| \geq |C'_1|$. We thus have the following.

$$c' \cdot \sum_{i \in J_1} \hat{T}[^{\mu_i}, (C'_1 \cup I) \cap I(\mu_i)] + 4|C'_1|$$

$$= c' \cdot \sum_{i \in J'_I} \hat{T}[^{\mu_i}, (C'_1 \cup I) \cap I(\mu_i)] + c' \cdot \sum_{i \in J''_I} \hat{T}[^{\mu_i}, (C'_1 \cup I) \cap I(\mu_i)] + 4|C'_1|$$

$$\leq c' \cdot \sum_{i \in J'_I} (\hat{T}[^{\mu_i}, I \cap I(\mu_i)] - 1) + c' \cdot \sum_{i \in J''_I} \hat{T}[^{\mu_i}, I \cap I(\mu_i)] + 4|C'_1|$$

$$\leq c' \cdot \left( \sum_{i \in J'_I} \hat{T}[^{\mu_i}, I \cap I(\mu_i)] - |J'_I| \right) + 2|M'_I| \leq c' \cdot \sum_{i \in J_I} \hat{T}[^{\mu_i}, I \cap I(\mu_i)]$$

Where the last step uses the fact that $c' \geq 4$. Plugging this information into Eq. 3 yields the following.
\[ c' \cdot \sum_{i \in J_1} \tilde{T}[\mu_i, (C'_1 \cup I) \cap I(\mu_i)] + 4|C'_1| + c' \cdot \sum_{i \in J_2} \tilde{T}[\mu_i, (C'_2 \cup I) \cap I(\mu_i)] + 4|C'_2| \]

\[ \leq c' \cdot \sum_{i=1}^{k} \tilde{T}[\mu_i, I \cap I(\mu_i)] + 4|C'_2| \leq c' \cdot \sum_{i=1}^{k} \tilde{T}[\mu_i, I \cap I(\mu_i)] + 4(1 + \varepsilon/4)|X| \]

\[ \leq \max\{c', 4 + \varepsilon\} \cdot \left( \sum_{i=1}^{k} \tilde{T}[\mu_i, I \cap I(\mu_i)] + |X| \right) = \max\{c, 4 + \varepsilon\} \tilde{T}[\mu_i, I] \]

Where the last third and second to last steps use the fact that \( X \subseteq V(H) \) is a \((1 + \varepsilon/4)\) approximation of a maximum independent set in \( H \) maximum matching, and hence \((1 + \varepsilon/4)|X|\) is at least as large as \( |C'_2| \), and that \( c' \geq 4 \), respectively.