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THE MULTIPLICITIES OF THE EQUIVARIANT INDEX OF TWISTED DIRAC OPERATORS

PAUL-EMILE PARADAN, MICHELE VERGNE

RÉSUMÉ. In this note, we give a geometric expression for the multiplicities of the equivariant index of a Dirac operator twisted by a line bundle.

1. Introduction

This note is an announcement of work whose details will appear later.

Let $M$ be a compact connected manifold. We assume that $M$ is even dimensional and oriented. We consider a spin$^c$ structure on $M$, and denote by $\mathcal{S}$ the corresponding irreducible Clifford module. Let $K$ be a compact connected Lie group acting on $M$, and preserving the spin$^c$ structure. We denote by $D : \Gamma(M, \mathcal{S}^+) \to \Gamma(M, \mathcal{S}^-)$ the corresponding twisted Dirac operator. The equivariant index of $D$, denoted $Q^{\text{pin}}_K(M)$, belongs to the Grothendieck group of representations of $K$,

$$Q^{\text{pin}}_K(M) = \sum_{\pi \in \hat{K}} m(\pi) \pi.$$ 

An important example is when $M$ is a compact complex manifold, $K$ a compact group of holomorphic transformations of $M$, and $\mathcal{L}$ any holomorphic $K$-equivariant line bundle on $M$ (not necessarily ample). Then the Dolbeault operator twisted by $\mathcal{L}$ can be realized as a twisted Dirac operator $D$. In this case $Q^{\text{pin}}_K(M) = \sum_{\pi}(\pi) \rho H^{0,n}(M, \mathcal{L}).$

The aim of this note is to give a geometric description of the multiplicity $m(\pi)$ in the spirit of the Guillemin-Sternberg phenomenon $[Q, R] = 0$ [3, 7, 8, 11, 9].

Consider the determinant line bundle $L = \det(\mathcal{S})$ of the spin$^c$ structure. This is a $K$-equivariant complex line bundle on $M$. The choice of a $K$-invariant hermitian metric and of a $K$-invariant hermitian connection $\nabla$ on $L$ determines an abstract moment map

$$\Phi_\nabla : M \to \mathfrak{k}^*$$

by the relation $\mathcal{L}(X) - \nabla_{X_M} = \frac{i}{2}(\Phi_\nabla, X)$, for all $X \in \mathfrak{k}$. We compute $m(\pi)$ in term of the reduced "manifolds" $\Phi^{-1}_\nabla(f)/K_f$. This formula extends the result of [10].
However, in this note, we do not assume any hypothesis on the line bundle $L$, in particular we do not assume that the curvature of the connection $\nabla$ is a symplectic form. In this pre-symplectic setting, a (partial) answer to this question has been obtained by [6, 4, 5, 1] when $K$ is a torus. Our method is based on localization techniques as in [9], [10].

2. Admissible coadjoints orbits

We consider a compact connected Lie group $K$ with Lie algebra $\mathfrak{k}$. Consider an admissible coadjoint orbit $O$ (as in [2]), oriented by its symplectic structure. Then $O$ carries a $K$-equivariant bundle of spinors $S_O$, such that the associated moment map is the injection $O \hookrightarrow \mathfrak{k}^*$. We denote by $Q^{\text{spin}}_K(O)$ the corresponding equivariant index.

Let us describe the admissible coadjoint orbits with their spin$^c$ index.

Let $T$ be a Cartan subgroup of $K$ with Lie algebra $\mathfrak{t}$. Let $\Lambda = \mathfrak{t}^*$ be the lattice of weights of $T$ (thus $e^{i\lambda}$ is a character of $T$). Choose a positive system $\Delta^+ \subset \mathfrak{t}^*$, and let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Let $t^*_\geq 0$ be the closed Weyl chamber and we denote by $F$ the set of the relative interiors of the faces of $t^*_\geq 0$. Thus $t^*_\geq 0 = \bigcup_{\sigma \in F} \sigma$, and we denote $t^*_0 = F$ the interior of $t^*_\geq 0$.

We index the set $\hat{K}$ of classes of finite dimensional irreducible representations of $K$ by the set $(\Lambda + \rho) \cap t^*_0$. The irreducible representation $\pi_\lambda$ corresponding to $\lambda \in (\Lambda + \rho) \cap t^*_0$ is the irreducible representation with infinitesimal character $\lambda$. Its highest weight is $\lambda - \rho$.

Let $\sigma \in F$. The stabilizer $K_\xi$ of a point $\xi \in \sigma$ depends only of $\sigma$. We denote it by $K_\sigma$, and by $\mathfrak{k}_\sigma$ its Lie algebra. We choose on $\mathfrak{k}_\sigma$ the system of positive roots contained in $\Delta^+$, and let $\rho_\sigma$ be the corresponding $\rho$.

When $\mu \in \sigma$, the coadjoint orbit $K \cdot \mu$ is admissible if and only if $\mu - \rho + \rho_\sigma \in \Lambda$. The spin$^c$ equivariant index of the admissible orbits is described in the following lemma.

Lemma 2.1. Let $K \cdot \mu$ be an admissible orbit : $\mu \in \sigma$ and $\mu - \rho + \rho_\sigma \in \Lambda$. If $\mu + \rho_\sigma$ is regular, then $\mu + \rho_\sigma \in \rho + \overline{\sigma}$. Thus we have

$$Q^{\text{spin}}_K(K \cdot \mu) = \begin{cases} 0 & \text{if } \mu + \rho_\sigma \text{ is singular}, \\ \pi_{\mu + \rho_\sigma} & \text{if } \mu + \rho_\sigma \text{ is regular}. \end{cases}$$

In particular, if $\lambda \in (\Lambda + \rho) \cap t^*_0$, then $K \cdot \lambda$ is admissible and $Q^{\text{spin}}_K(K \cdot \lambda) = \pi_\lambda$. 
Let \( H_k \) be the set of conjugacy classes of the reductive algebras \( f \in \mathfrak{t}^* \). We denote by \( S_k \) the set of conjugacy classes of the semi-simple parts \( [h, h] \) of the elements \( (h) \in H_k \). The map \( (h) \mapsto ([h, h]) \) induces a bijection between \( H_k \) and \( S_k \).

The map \( F \rightarrow H_k, \sigma \mapsto (f_{\sigma}) \), is surjective and for \( (h) \in H_k \) we denote by \( \mathfrak{t}_{h}^* \) the set of elements \( f \in \mathfrak{t}^* \) with infinitesimal stabilizer \( f \) belonging to the conjugacy class \( (h) \).

We have \( \mathfrak{t}_{h}^* = K (\cup_{\sigma \in F(h)} \mathfrak{t}) \). In particular all coadjoint orbits contained in \( \mathfrak{t}_{h}^* \) have the same dimension. We say that such a coadjoint orbit is of type \( (h) \). If \( (h) = (0) \), then \( \mathfrak{t}_{h}^* \) is the open subset of regular elements.

We denote by \( A(h) \) the set of admissible coadjoint orbits of type \( (h) \). This is a discrete subset of orbits in \( \mathfrak{t}_{h}^* \).

**Example 1:** Consider the group \( K = SU(3) \) and let \( (h) \) be the conjugacy class such that \( \mathfrak{t}_{h}^* \) is equal to the set of subregular elements \( f \in \mathfrak{t}^* \) (the orbit of \( f \) is of dimension \( \text{dim}(K/T) - 2 \)). Let \( \omega_1, \omega_2 \) be the two fundamental weights. Let \( \sigma_1, \sigma_2 \) be the half lines \( \mathbb{R}_{>0} \omega_1, \mathbb{R}_{>0} \omega_2 \). Then \( \mathfrak{t}_{h}^* \cap t_{\geq 0}^* = \sigma_1 \cup \sigma_2 \). The set \( A(h) \) is equal to the collection of orbits \( K \cdot (\frac{1+2n}{2} \omega_i) \), \( n \in \mathbb{Z}_{\geq 0}, i = 1, 2 \). The representation \( Q_{spin}^K(M) \) is 0 if \( n = 0 \), otherwise it is the irreducible representation \( \pi_{\rho + (n-1)\omega_i} \). In particular, both representations associated to the admissible orbits \( \frac{3}{2} \omega_1 \) and \( \frac{3}{2} \omega_2 \) are the trivial representation \( \pi_{\rho} \).

3. **The theorem**

Consider the action of \( K \) in \( M \). Let \( (\mathfrak{t}_M) \) be the conjugacy class of the generic infinitesimal stabilizer. On a \( K \)-invariant open and dense subset of \( M \), the conjugacy class of \( \mathfrak{t}_m \) is equal to \( (\mathfrak{t}_M) \). Consider the (conjugacy class) \( ([\mathfrak{t}_A, \mathfrak{t}_M]) \).

We start by stating two vanishing lemmas.

**Lemma 3.1.** If \( ([\mathfrak{t}_A, \mathfrak{t}_M]) \) does not belong to the set \( S_k \), then \( Q_{spin}^K(M) = 0 \) for any \( K \)-invariant spin\(^c\) structure on \( M \).

If \( ([\mathfrak{t}_A, \mathfrak{t}_M]) = ([h, h]) \) for some \( h \in H_k \), any \( K \)-invariant map \( \Phi : M \rightarrow \mathfrak{t}^* \) is such that \( \Phi(M) \) is included in the closure of \( \mathfrak{t}_h^* \).

**Lemma 3.2.** Assume that \( ([\mathfrak{t}_A, \mathfrak{t}_M]) = ([h, h]) \) with \( h \in H_k \). Let us consider a spin\(^c\) structure on \( M \) with determinant bundle \( L \). If there exists a \( K \)-invariant hermitian connection \( \nabla \) on \( L \) such that \( \Phi_{\nabla}(M) \cap \mathfrak{t}_h^* = \emptyset \), then \( Q_{spin}^K(M) = 0 \).
Thus from now on, we assume that the action of $K$ on $M$ is such that $([\mathfrak{t}_M, \mathfrak{t}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ for some $\mathfrak{h} \in \mathfrak{h}_t$. Let us consider a spin$^c$ structure on $M$ with determinant bundle $L$ and a $K$-invariant hermitian connection with moment map $\Phi_\nabla : M \to \mathfrak{t}^*$.

We extend the definition of the index to disconnected even dimensional oriented manifolds by defining $Q_{\text{spin}}^K(M)$ to be the sum over the connected components of $M$. If $K$ is the trivial group, $Q_{\text{spin}}^K(M) \in \mathbb{Z}$ and is denoted simply by $Q_{\text{spin}}(M)$.

Consider a coadjoint orbit $O = K \cdot f$. The reduced space $M_O$ is defined to be the topological space $\Phi_\nabla^{-1}(O)/K = \Phi_\nabla^{-1}(f)/K_f$. We also denote it by $M_f$. This space might not be connected.

In the next section, we define a $\mathbb{Z}$-valued function $O \mapsto Q_{\text{spin}}^K(M_O)$ on the set $A(h)$ of admissible orbits of type $h$. We call it the reduced index:

- if $M_O = \emptyset$, then $Q_{\text{spin}}^K(M_O) = 0$,
- when $M_O$ is an orbifold, the reduced index $Q_{\text{spin}}^K(M_O)$ is defined as an index of a Dirac operator associated to a natural “reduced” spin$^c$ structure on $M_O$.

Otherwise, it is defined via a limit procedure. Postponing this definition, we have the following theorem.

**Theorem 3.3.** Assume that $([\mathfrak{t}_M, \mathfrak{t}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $\mathfrak{h} \in \mathfrak{h}_t$. Then

$$Q_{\text{spin}}^K(M) = \sum_{O \in A(h)} Q_{\text{spin}}^K(M_O) Q_{\text{spin}}^K(O).$$

In the expression above, when $\mathfrak{h}$ is not abelian, $Q_{\text{spin}}^K(O)$ can be 0, and several orbits $O \in A(h)$ can give the same representation.

Theorem 3.3 is in the spirit of the $[Q, R] = 0$ theorem. However it has some radically new features. First, as $\Phi_\nabla$ is not the moment map of a Hamiltonian structure, the definition of the reduced space requires more care. For example, the fibers of $\Phi_\nabla$ might not be connected, and the Kirwan set $\Phi_\nabla^t(M) \cap \mathfrak{t}_t^* \geq 0$ is not a convex polytope. Furthermore, this Kirwan set depends of the choice of connection $\nabla$. Second, the map $O \in A(h) \to Q_{\text{spin}}^K(O)$ is not injective, when $\mathfrak{h}$ is not abelian. Thus the multiplicities $m_\lambda$ of the representation $\pi_\lambda$ in $Q_{\text{spin}}^K(M)$ will be eventually obtained as a sum of reduced indices involving several reduced spaces.

We explicit this last point.

**Theorem 3.4.** Assume that $([\mathfrak{t}_M, \mathfrak{t}_M]) = ([\mathfrak{h}, \mathfrak{h}])$ with $\mathfrak{h} \in \mathfrak{h}_t$. Let $m_\lambda \in \mathbb{Z}$ be the multiplicity of the representation $\pi_\lambda$ in $Q_{\text{spin}}^K(M)$. We have

$$m_\lambda = \sum_{\sigma \in \pi(h) \lambda - \rho_\sigma \in \sigma} Q_{\text{spin}}^K(M_{\lambda - \rho_\sigma}).$$
More explicitly, the sum is taken over the (relative interiors of) faces $\sigma$ of the Weyl chamber such that

\[(2) \quad ([k_M],[k_M]) = ([k_\sigma],[k_\sigma]), \quad \Phi_\nabla(M) \cap \sigma \neq \emptyset, \quad \lambda \in \{\sigma + \rho_\sigma\}.\]

If $k_M$ is abelian, we have simply $m_\lambda = Q^{\text{spin}}(\Phi_\nabla^{-1}(\lambda)/T)$. In particular, if the group $K$ is the circle group, and $\lambda$ is a regular value of the moment map $\Phi_\nabla$, this result was obtained in [1].

If $k_M$ is not abelian, and the curvature of the connection $\nabla$ is symplectic, Kirwan convexity theorem implies that the image $\Phi_\nabla(M) \cap t_{\geq 0}$ is contained in the closure of one single $\sigma$. Thus there is a unique $\sigma$ satisfying Conditions (2). In this setting Theorem 3.4 is obtained in [10].

Let us give an example where several $\sigma$ contribute to the multiplicity of a representation $\pi_\lambda$.

We take the notations of Example 1. We label $\omega_1, \omega_2$ so that $\ell_{\omega_1}$ is the group $S(U(2) \times U(1))$ stabilizing the line $\mathbb{C}e_3$ in the fundamental representation of $SU(3)$ in $\mathbb{C}^4 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$.

Let $P = \{0 \subset L_2 \subset L_3 \subset \mathbb{C}^4\}$ be the partial flag manifold with $L_2$ a subspace of $\mathbb{C}^4$ of dimension 2 and $L_3$ a subspace of $\mathbb{C}^4$ of dimension 3. Denote by $L_1, L_2$ the equivariant line bundles on $P$ with fiber at $(L_2, L_3)$ the one-dimensional spaces $\wedge^2 L_2$ and $L_3/L_2$ respectively. Let $M$ be the subset of $P$ where $L_2$ is assumed to be a subspace of $\mathbb{C}^3$. Thus $M$ is fibered over $P_2(\mathbb{C})$ with fiber $P_1(\mathbb{C})$. The group $SU(3)$ acts naturally on $M$, and the generic stabilizer of the action is $SU(2)$. We denote by $L_{a,b}$ the bundle $L_1^a \otimes L_2^b$ restricted to $M$. This line bundle is equipped with a natural holomorphic and hermitian connection $\nabla$. Consider the spin$^c$ structure with determinant bundle $L = L_{2a+1,2b+1}$, where $a, b$ are positive integers. If $a \geq b$, the curvature of the line bundle $L$ is non degenerate, and we are in the symplectic case. Let us consider $b > a$. It is easy to see that, in this case, the Kirwan set $\Phi_\nabla(M) \cap t_{\geq 0}$ is the non convex set $[0, b - a] \omega_1 \cup [0, a + 1] \omega_2$. We compute the character of the representation $Q^{\text{spin}}_K(M)$ by the Atiyah-Bott fixed point formula, and find

\[Q^{\text{spin}}_K(M) = \sum_{j=0}^{b-a-2} \pi_{\rho + j\omega_1} \oplus \sum_{j=0}^{a-1} \pi_{\rho + j\omega_2}.\]
In particular the multiplicity of $\pi_\rho$ (the trivial representation) is equal to 2. We use now Theorem 3.3 and the discussion of Example 1, and obtain (reduced multiplicities are equal to 1)

$$Q^\text{spin}_K(M) = \sum_{j=0}^{b-a-1} Q^\text{spin}_K(K \cdot (1 + 2j \omega_1)) \oplus \sum_{j=0}^{a} Q^\text{spin}_K(K \cdot (1 + 2j \omega_2)).$$

Using the formulae for $Q^\text{spin}_K(K \cdot (1 + 2n \omega_i))$ given in Example 1, these two formulae (fortunately) coincide. Furthermore we see that both faces $\sigma_1, \sigma_2$ give a non zero contribution to the multiplicity of the trivial representation.

4. Definition of the reduced index

We start by defining the reduced index for the action of an abelian torus $H$ on a connected manifold $Y$. Denote by $\Lambda$ the lattice of weights of $H$. We do not assume $Y$ compact, but we assume that the set of stabilizers $H_m$ of points in $Y$ is finite. Let $\mathfrak{h}_Y$ be the generic infinitesimal stabilizer of the action $H$ on $Y$, and $H_Y$ be the connected subgroup of $H$ with Lie algebra $\mathfrak{h}_Y$. Thus $H_Y$ acts trivially on $Y$. Let us consider a spin$^c$ structure on $Y$ with determinant bundle $L$, and a $H$ invariant connection $\nabla$ on $L$. The image $\Phi_{\Delta}(Y)$ spans an affine space $I_Y$ parallel to $\mathfrak{h}_Y^\perp$. We assume that the fibers of the map $\Phi_{\Delta}$ are compact. We can easily prove that there exists a finite collection of hyperplanes $W_1, \ldots, W_p$ in $I_Y$ such that the group $H/H_Y$ acts locally freely on $\Phi_{\Delta}^{-1}(f)$, when $f$ is in $\Phi_{\nabla}(Y)$, but not on any of the hyperplanes $W_i$.

Proposition 4.1. • When $\mu \in I_Y \cap \Lambda$ is a regular value of $\Phi_{\nabla} : Y \to I_Y$, the reduced space $Y_{\mu}$ is an oriented orbifold equipped with an induced spin$^c$ structure : we denote $Q_{\mu}^{\text{spin}}(Y_{\mu})$ the corresponding spin$^c$ index.

• For any connected component $C$ of $I_Y \setminus \cup_{k=1}^p W^k$, we can associate a periodic polynomial function $q_c : \Lambda \cap I_Y \to \mathbb{Z}$ such that

$$q_c(\mu) = Q_{\mu}^{\text{spin}}(Y_{\mu})$$

for any element $\mu \in \Lambda \cap C$ which is a regular value of $\Phi : Y \to I_Y$.

• If $\mu \in \Lambda$ belongs to the closure of two connected components $C_1$ and $C_2$ of $I_Y \setminus \cup_{k=1}^p W^k$, we have

$$q_{C_1}(\mu) = q_{C_2}(\mu).$$

We can now state the definition of the “reduced” index on $\Lambda$ :

• $Q_{\mu}^{\text{spin}}(Y_{\mu}) = 0$ if $\mu \notin \Lambda \cap I_Y$, 

• For any connected component $C$ of $I_Y \setminus \cup_{k=1}^p W^k$, we can associate a periodic polynomial function $q_c : \Lambda \cap I_Y \to \mathbb{Z}$ such that

$$q_c(\mu) = Q_{\mu}^{\text{spin}}(Y_{\mu})$$

for any element $\mu \in \Lambda \cap C$ which is a regular value of $\Phi : Y \to I_Y$.

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• For any connected component $C$ of $I_Y \setminus \cup_{k=1}^p W^k$, we can associate a periodic polynomial function $q_c : \Lambda \cap I_Y \to \mathbb{Z}$ such that

$$q_c(\mu) = Q_{\mu}^{\text{spin}}(Y_{\mu})$$

for any element $\mu \in \Lambda \cap C$ which is a regular value of $\Phi : Y \to I_Y$.
• for any $\mu \in \Lambda \cap I_Y$, we define $Q^{\text{pin}}(Y_\mu)$ as being equal to $q^C(\mu)$ where $C$ is any connected component containing $\mu$ in its closure. In fact $Q^{\text{pin}}(Y_\mu)$ is computed as an index of a particular spin$^c$ structure on the orbifold $\Phi_C^{-1}(\mu + \epsilon)/H$ for any $\epsilon$ small and such that $\mu + \epsilon$ is a regular value of $\Phi_C$.

If $Y$ is not connected, we define the reduced index at a point $\mu \in \Lambda$ as the sum of reduced indices over all connected components of $Y$.

More generally, let $H$ be a compact connected group acting on $Y$ and such that $[H,H]$ acts trivially on $Y$. Let $\mathcal{S}_Y$ be an equivariant spin$^c$ structure on $Y$ with determinant bundle $L$. For any $\mu \in \mathfrak{h}^*$ such that $\mu([\mathfrak{h},\mathfrak{h}]) = 0$, and admissible for $H$, it is then possible to define $Q^{\text{pin}}(Y_\mu)$. Indeed eventually passing to a double cover of the torus $H/[H,H]$ and translating by the square root of the action of $H/[H,H]$ on the fiber of $L$, we are reduced to the preceding case of the action of the torus $H/[H,H]$, and a $H/[H,H]$-equivariant spin$^c$ structure on $Y$.

Consider now the action of a connected compact group $K$ on $M$. Let $\sigma$ be a (relative interior) of a face of $t_{>0}$ which satisfies the following conditions

\begin{equation}
(\{t_M, t_M\}) = (\{t_\sigma, t_\sigma\}), \quad \Phi_C^{-1}(\sigma) \neq \emptyset.
\end{equation}

Let us explain how to compute the “reduced” index map $\mu \to Q^{\text{pin}}(M_\mu)$ on the set $\sigma \cap \{\Lambda + \rho - \rho_\sigma\}$ that parameterizes the admissible orbits intersecting $\sigma$. We work with the “slice” $Y$ defined by $\sigma$. The set $U_\sigma := K_\sigma(\cup_{\tau \in \mathcal{S}} \tau)$ is an open neighborhood of $\sigma$ in $t_\sigma^*$ such that the open subset $KU_\sigma \subset t^*$ is isomorphic to $K \times_{K_\sigma} U_\sigma$. We consider the $K_\sigma$-invariant subset $Y = \Phi_C^{-1}(U_\sigma)$. The following lemma allows us to reduce the problem to the abelian case.

**Lemma 4.2.** • $Y$ is a non-empty submanifold of $M$ such that $KY$ is an open subset of $M$ isomorphic to $K \times_{K_\sigma} Y$.

• The Clifford module $\mathcal{S}_M$ on $M$ determines a Clifford module $\mathcal{S}_Y$ on $Y$ with determinant line bundle $L_Y = L_M|Y \otimes \mathbb{C}_{-2(\rho - \rho_\sigma)}$. The corresponding moment map is $\Phi_C|Y = \rho + \rho_\sigma$.

• The group $[K_\sigma, K_\sigma]$ acts trivially on $Y$ and on the bundle of spinors $\mathcal{S}_Y$.

We thus consider $Y$ with action of $K_\sigma$, and Clifford bundle $\mathcal{S}_Y$. If $\mu \in \sigma$ is admissible for $K$, then $\mu - \rho + \rho_\sigma \in \Lambda$ is admissible for $K_\sigma$. The reduced space $M_\mu = \Phi_C^{-1}(\mu)/K_\sigma$ is equal to the reduced space $Y_{\mu - \rho + \rho_\sigma}$. As $[K_\sigma, K_\sigma]$ acts trivially on $(Y, \mathcal{S}_Y)$, we are in the abelian case, and we define $Q^{\text{pin}}(M_\mu) := Q^{\text{pin}}(Y_{\mu - \rho + \rho_\sigma})$. 
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