pp-wave initial data

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Abstract
An initial data characterization for vacuum pp-wave spacetimes in dimension four is constructed. This is a vacuum initial data set plus some extra conditions guaranteeing that the data development is a subset of a vacuum pp-wave. Some of the extra conditions only depend on the same quantities used to construct the vacuum initial data, namely the first and the second fundamental forms while others are related to a conformal Killing initial data characterization (CKID).

Keywords Initial value problem · Gravitational wave · Conformal Killing vector

1 Introduction
An initial data characterization is a set of conditions that guarantee that an initial data set of the Einstein equations corresponds to a given exact solution of the Einstein’s field equations. The simplest example are the standard vacuum constraints that guarantee that the data development is a subset of a vacuum solution.

Other more specific examples are the construction of initial data for the Schwarzschild solution [9], the construction of initial data for the Kerr solution [10], construction of initial data for a general vacuum type D solution [11] the construction of vacuum Killing initial initial data [1, 4, 14], its recent generalization for vacuum conformal Killing initial data [7] and the initial data for the closed conformal Killing-Yano equation obtained in [8]. Some of these initial data characterizations (Schwarzschild, Kerr, type D) share the common property of being ideal in the sense that they only involve the natural variables used in the set of vacuum constraints, namely, the first and second fundamental forms. These natural variables are used in an algorithmic way, i.e., there is an algorithm to check whether a vacuum initial data set belongs to one of the initial data characterizations mentioned before.
The purpose of this letter is to extend the previous results by providing a new vacuum initial data characterization for the case of vacuum plane fronted waves with parallel rays or in short vacuum pp-waves solutions. This initial data characterization is derived from the conformal Killing initial data characterization found in \cite{7}. This is because we shall take advantage of the existence of proper conformal Killing vectors in a vacuum pp-wave spacetime to find our main result (Theorem 4).

If a vacuum pp-wave admits a parallel spinor \cite{15} then one can take advantage of the initial data characterization of parallel spinors developed in \cite{16} to obtain an alternative initial data characterization of a pp-wave in that particular case. Necessary and sufficient conditions for a pp-wave to admit a parallel spinor can be found in Proposition 2.5 of \cite{16}.

As is well-known since Penrose’s seminal work \cite{17}, a pp-wave is not always globally hyperbolic. A detailed study of the global properties of a general pp-wave spacetime can be found in \cite{6}. In particular this reference addresses under which conditions a pp-wave is globally hyperbolic. Note that in this work we deal with local solutions of the vacuum Einstein equations which in principle might be globally hyperbolic, even though its maximal analytic extension is not.

The plan of this work is as follows: in Sect. 2 we introduce our conventions, in Sect. 3 we review the conformal Killing initial data characterization presented in \cite{7} that is the starting point to obtain our main result. This is in Theorem 4, presented in Sect. 4.

2 Geometric framework

We shall work with two manifolds: a 4-dimensional Lorentzian manifold \((M, g_{ab})\) and a 3-dimensional Riemannian manifold \((\Sigma, h_{AB})\). All their structures are assumed, unless otherwise stated, to be smooth. In this work we will use abstract indices to denote tensor fields. The signature convention for the Lorentzian manifold is \((-\),\(+\),\(+\),\(+\)). Small Latin indices \(a, b, c, \ldots\) will be used for tensors on \(M\) and capital Latin indices \(A, B, C, \ldots\) for tensors defined in \(\Sigma\).

Definition 1 Let \((M, g_{ab})\) be a 4-dimensional smooth connected Lorentzian manifold whose Levi-Civita connection is given by \(\nabla_a\) and \((\Sigma, h_{AB})\) a 3-dimensional Riemannian manifold whose Levi-Civita connection is given by \(D_A\). An isometric embedding is an embedding map \(\phi : \Sigma \to M\) such that \(\phi^* g = h\).

One can use the embedding map \(\phi\) to define the pull-back bundles \(\phi^*(T(M))\), \(\phi^*(T^*(M))\), each with base manifold \(\Sigma\), and take them as the starting point to construct a tensor bundle in the usual fashion. We will also use small Latin indices as the abstract indices to denote tensors from this tensor bundle. For example, the unit vector field \(N^a\) to \(\phi(\Sigma)\) can be regarded as a section of \(\phi^*(T(M))\). However, it is possible to extend \(N^a\) smoothly off the hypersurface \(\phi(\Sigma)\) and get a smooth section of \(T(M)\) if we set up a foliation of \(M\) containing \(\phi(\Sigma)\) as one of its leaves and define the normal vector field to each leaf. We shall still use the same symbol \(N^a\) for such extended vector field, leaving to the context to decide whether we are dealing with the extended vector field or its restriction.
The differential of the map $\phi$ induces solders $\phi_a^A, \phi_A^a$ between the vector bundles $\phi^*(T(M))$ and $T(\Sigma)$ that allow us to relate tensor fields arising from the respective tensor bundles. Since $\phi$ is an embedding, the solders have the properties

$$\phi_a^A \phi_A^b \equiv \delta_a^b, \quad \phi_A^b \phi_b^B = \delta_A^B,$$

(1)

(the symbol $\equiv$ means equality among tensor fields belonging to a tensor bundle arising from $\phi^*(T(M))$). For example the relation $\phi^* g = h$ can be written as

$$h_{AB} = \phi_A^a \phi_B^b g_{ab},$$

(2)

from which we deduce using (1)

$$g_{ab} \equiv \phi_a^A \phi_b^B h_{AB}.$$  

(3)

**Definition 2** If the image $\phi(\Sigma)$ of $\Sigma$ under the isometric embedding $\phi$ is a Cauchy hypersurface of a (globally hyperbolic subset) of $M$, then the triple $\{(M, g_{ab}), (\Sigma, h_{AB}, \pi_{AB}), \phi\}$ is an initial data characterization of $(M, g_{ab})$. In that case we say that the Riemannian manifold $(\Sigma, h_{AB})$ is an initial data set for the Lorentzian manifold $(M, g_{ab})$.

### 2.1 The Cauchy problem in general relativity

Vacuum Einstein equations $R_{ab} = 0$ can be written as a hyperbolic system of equations whose initial data must fulfill a set of constraints. According to Definition 2 this result can be regarded as a vacuum initial data characterization

**Theorem 1** (Vacuum initial data characterisation) Let $(\Sigma, h_{AB})$ be a 3-dimensional Riemannian manifold and suppose that there exists a symmetric tensor field $\pi_{AB}$ on it which satisfies the conditions (vacuum constraints)

$$r + \pi^2 - \pi^{AB} \pi_{AB} = 0,$$

(4)

$$D^B \pi_{AB} - D_A \pi = 0,$$

(5)

where $\pi \equiv \pi_A^A$. Provided that $h_{AB}$ and $\pi_{AB}$ are smooth there exists an isometric embedding $\phi$ of $\Sigma$ into a globally hyperbolic, vacuum solution $(M, g_{ab})$ of the Einstein field equations. The set $(\Sigma, h_{AB}, \pi_{AB})$ is then called a vacuum initial data set and the spacetime $(M, g_{ab})$ is the data development. Furthermore the spacelike hypersurface $\phi(\Sigma)$ is a Cauchy hypersurface in $M$ and $D(\phi(\Sigma)) = M$.

See Theorem 8.9 of [3] for a formulation of this result with more general differentiability assumptions.
3 CKID initial data

The existence of an isometric embedding of a $n-1$-dimensional Riemannian manifold $(\Sigma, h_{AB})$ into a $n$-dimensional Lorentzian manifold $(M, g_{ab})$ admitting a conformal Killing vector is addressed by the following result proven in [7] (we adopt the same notation and conventions as when $n = 4$).

**Theorem 2** (AGP, I. Khavkine, 2019, conformal Killing initial data (CKID)) Consider a globally hyperbolic Einstein vacuum Lorentzian manifold, $(M, g_{ab})$ of dimension $n > 2$ with $R_{ab} = 0$, and a Cauchy hypersurface given by the $n-1$ dimensional Riemannian manifold $\Sigma \subset M$ with Riemannian metric $h_{AB}$. Let $v_0$ and $v^A$ be respectively a scalar and a vector field on $\Sigma$ and define in terms of them the following quantities on $\Sigma$

$$u \equiv (D_C v^C - \pi v_0),$$
$$\nabla_0 u \equiv \frac{1}{n-1} \pi u + \left( -D^A D_A v_0 + (\pi_{AB} \pi^{AB}) v_0 + (D^A \pi) v_A \right).$$

Using the above, the necessary and sufficient conditions yielding a set of conformal Killing initial data (CKID) for $v_a$ on $\Sigma$ are given by the following differential conditions:

$$D_A v_B + D_B v_A - 2\pi_{AB} v_0 - \frac{2}{n-1} g_{AB} u = 0,$$  

$$\left( D_B D_A v_0 + (2\pi_{AC} \pi^C_B - \pi^C_C \pi_{AB} - r_{AB}) v_0 - 2\pi_{(A} D_{B)} v^C \right) - v^C (D_C \pi_{AB}) + \frac{1}{n-1} \left( u \pi_{AB} + g_{AB} \nabla_0 u \right) = 0,$$

$$D_A D_B u - \pi_{AB} (\nabla_0 u) = 0,$$

$$\left( r_{AB} + \pi \pi_{AB} - \pi_{AC} \pi^C_B \right) (\nabla_0 u) - \left( D_{(A} \pi_{B)C} - D_C \pi_{AB} \right) D^C u = 0.$$

**Remark 1** As it was shown in [7] one can find initial data characterizations for the specializations of a conformal Killing vector: if the scalar $u$ defined by (6a) is constant in $\Sigma$ then the corresponding conformal Killing vector is the homothetic specialization (and if the constant is zero then we have the Killing specialization).

4 Main result

Vacuum Einstein equations are not conformally invariant. Therefore if a vacuum solution admits a proper conformal Killing vector then it must belong to a very specific class as the following theorem proven in [5] shows.

**Theorem 3** (Eardley, Isenberg, Mardsen and Moncrief (1986)) Let $(M, g)$ be a four-dimensional spacetime which satisfies the vacuum Einstein equations and admits a conformal killing vector field $v$. Then either
1. \((M, g)\) is everywhere locally flat.

2. \((M, g)\) is a plane-fronted wave.

3. \(v\) is a homothetic Killing vector field.

A plane-fronted wave is a vacuum solution \((M, g_H)\) characterized by the existence of a null vector field \(k^a\) that is covariantly constant \(\nabla_a k^b = 0\). Under these conditions, a local coordinate system \((u, r, x^1, x^2)\) can be introduced such that the metric \(g_H\) adopts the form

\[
g_H = -2H(u, x)du \otimes du - du \otimes dr - dr \otimes du + \delta, \tag{8}
\]

where \(H(u, x) = H(u, x^1, x^2)\) is a smooth function and \(\delta(x) = \delta(x^1, x^2)\) is the Euclidean metric in dimension 2. In these coordinates the null vector field \(k^a\) is given by \(k^a = \frac{\partial}{\partial r}\) and the vacuum condition reads

\[
\Box_\delta H(u, x) = 0, \tag{9}
\]

where \(\Box_\delta\) is the Hodge operator computed with respect to the metric \(\delta\). Indeed, the solutions of the conformal Killing equation on a \(pp\)-wave background are explicitly computed in [12] generalizing previous results of [13] and [5].

Combining Theorem 3 with Theorem 2, we can give an initial data characterization of a plane-fronted wave in dimension four. To that end, we need to make sure that our vacuum solution is non-flat and has a conformal Killing vector that it is not homothetic. The necessary and sufficient conditions for that to happen are found next. First of all, we need an initial data characterization for the flat Minkowski space-time in dimension 4.

**Proposition 1** If \(n = 4\), then a vacuum initial data set \((\Sigma, h_{AB}, \pi_{AB})\) is an initial data set of the flat Minkowski spacetime, iff the following tensors defined on \(\Sigma\) vanish identically

\[
E_{AB} \equiv r_{AB} + \pi \pi_{AB} - \pi_{AC} \pi_{CB}, \tag{10}
\]

\[
B_{AB} \equiv \varepsilon^{KL} (A D_K \pi_{L})_{B}, \tag{11}
\]

where \(\varepsilon_{ABC} = \varepsilon_{[ABC]}\) is the volume form of \((\Sigma, h_{AB})\).

**Proof** The tensors \(E_{AB}, B_{AB}\) are related to the space-time electric and magnetic parts \(E_{ac} \equiv W_{abcd} N^b N^d, B_{ac} \equiv (\ast W)_{abcd} N^b N^d\) of the Weyl tensor \(W_{abcd}\) by (see e.g. eqns. (21a)–(21b) of [9])

\[
E_{ab} \equiv \phi_a^A \phi_b^B E_{AB}, \quad B_{ab} \equiv \phi_a^A \phi_b^B B_{AB}. \tag{12}
\]

If \(E_{ab}, B_{ab}\) vanish on \(\phi(\Sigma)\) and \((M, g_{ab})\) is a vacuum solution then it is known [2] that the Weyl tensor \(W_{abcd}\) is zero on \(D(\phi(\Sigma))\) and hence we deduce that the Riemann tensor \(R_{abcd}\) vanish on \(D(\phi(\Sigma))\). Therefore \((D(\phi(\Sigma)), g_{ab})\) is a subset of the Minkowski spacetime. \(\square\)
Theorem 4 (Vacuum plane-fronted wave initial data characterization) Under the conditions of Theorem 2 for \( n = 4 \), if the quantity \( u \) is non-constant, (7) has a non-trivial solution and the tensors \( E_{AB} \), \( B_{AB} \) do not vanish identically on \( \Sigma \), then the data development \( D(\phi(\Sigma)) \) must be a subset of a plane-fronted wave \((M, g_H)\) for some function \( H \) fulfilling the condition (9).

**Proof** Given that by assumption the conditions of Theorem 2 are fulfilled by a non-trivial \( v, v_A \), the vacuum solution \((D(\phi(\Sigma)), g)\) has a conformal Killing vector that cannot be a homothetic or Killing specialization, since \( u \) is non-constant (see Remark 1). Therefore \((D(\phi(\Sigma)), g)\) has a proper conformal Killing vector and since \((D(\phi(\Sigma)), g)\) is a four-dimensional vacuum solution, then the result follows from Theorem 3.

\[ \square \]

The previous result is a characterization of an isometric embedding of a Riemannian manifold into a four-dimensional vacuum plane-fronted wave because \((D(\phi(\Sigma)), g) \subseteq (M, g_H)\) for some \( H \) solving (9). The equality can only hold in those cases in which \((M, g_H)\) is globally hyperbolic.

5 Conclusions

An initial data characterization of a vacuum \( pp \)-wave spacetime has been found in Theorem 4. This initial data characterization involves the first and the second fundamental forms \( h_{AB}, K_{AB} \), fulfilling the vacuum constraints (4)–(5), the quantities \( v, v_A \) fulfilling the CKID conditions (6)–(7) and a check that (10)–(11) are not identically zero.

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