Theorem 1. If $M$ is a non-orientable triangulable $(n-1)$-manifold (possibly with boundary), then $\mathbb{R}^n$ does not contain uncountably many pairwise disjoint copies of $M$.

The case $n = 3$ is due to Frolkina [2]; her proof involves replacing the given copies of $M$ by tame ones. She also proves that $\mathbb{R}^n$ does not contain uncountably many pairwise disjoint tame copies of $M$. The proof of Theorem 1 is an exercise in elementary algebraic topology, which I found myself doing as I was asked to referee Frolkina’s paper.

Remark 2. It is well-known that if $X$ is a compactum such that $\mathbb{R}^n$ contains uncountably many pairwise disjoint copies of $X$, then $X \times \mathbb{N}^+$ embeds in $\mathbb{R}^n$, where $\mathbb{N}^+$ is the one-point compactification of the countable discrete space $\mathbb{N}$. Indeed, the space of maps $C^0(X, \mathbb{R}^n)$ is separable, so any its uncountable subset $S$ contains a convergent sequence $(f_i)$ whose limit $f_\infty$ also belongs to $S$.

Lemma 3. If $X$ is a compact polyhedron such that $X \times \mathbb{N}^+$ embeds in $\mathbb{R}^n$, then the deleted product $\widetilde{X} \times I$ admits a $\mathbb{Z}/2$-map to $S^{n-1}$.

Here $\mathbb{Z}/2$ acts on $\tilde{Y} := Y \times Y \setminus \Delta$ by exchanging the factors and on $S^{n-1}$ by the antipodal involution $x \mapsto -x$.

Proof. Let $g = (g_1, g_2, \ldots, g_\infty): X \times \mathbb{N}^+ \hookrightarrow \mathbb{R}^n$ be an embedding. Let us endow $X$ with the metric induced by $g_\infty$. Let $R$ be a $\mathbb{Z}/2$-invariant regular neighborhood of the diagonal in $X \times X$, so that $X \times X \setminus R$ is $\mathbb{Z}/2$-homeomorphic to $\tilde{X}$. Then there exists an $\varepsilon > 0$ such that $R$ contains the $2\varepsilon$-neighborhood of $\Delta$ in the $l_\infty$ product metric on $X \times X$. Then there exists an $i \in \mathbb{N}$ such that $d(g_i, g_\infty) < \varepsilon$. Let $\tilde{Y} = X \times [0, 1]$ and let us define $g: \tilde{Y} \to \mathbb{R}^n$ by $g(x, t) = t g_i(x) + (1 - t) g_\infty(x)$. Let $\tilde{Y}_R = \{(x, s), (y, t)\} \in Y \times Y \setminus (x, y) \notin R$ or $(t, s) \in \{0, 1\}$}. Since $(x, y) \notin R$ implies $d(x, y) \geq 2\varepsilon$, we have $g(p) \neq g(q)$ for any $(p, q) \in \tilde{Y}_R$. Hence we may define an equivariant map $\tilde{g}: \tilde{Y}_R \to S^{n-1}$ by $\tilde{g}(p, q) = \frac{|g(q) - g(p)|}{|g(q) - g(p)|}$. Clearly, $\tilde{Y}_R$ is a $\mathbb{Z}/2$-deformation retract of $\tilde{Y}$. Thus we obtain an equivariant map $\tilde{Y} \to S^{n-1}$. □

Proof of Theorem 1. Let us note that $M$ is a pseudo-manifold (possibly with boundary). By considering a regular neighborhood in $M$ of an embedded orientation-reversing loop in the dual 1-skeleton of $M$ we may assume that $M$ is homeomorphic to the total space of the nonorientable $(n-2)$-disc bundle over $S^1$. By Lemma 3 it suffices show that if $N$ is a non-orientable smooth $n$-manifold (namely, $N = M \times I$), then $\tilde{N}$ admits no equivariant map to $S^{n-1}$. By considering the interior of $N$ we may assume that it has no
boundary. Let $SN$ be the total space of the spherical tangent bundle of $N$ and let $t$ be the involution on $SN$ that is antipodal on each fiber $S^{n-1}$. Let $c \in H^1(PN; \mathbb{Z}/2)$ be the fundamental class of $t$, i.e. the first Stiefel–Whitney class of the line bundle associated to the double covering $SN \to PN := SN/t$. Then according to one of the definitions of the Stiefel–Whitney classes $w_i = w_i(N)$ [1; (6.2)], $c^n = q^*(w_n) + q^*(w_{n-1})c + \cdots + q^*(w_1)c^{n-1}$, where $q: PN \to N$ is the projectivized tangent bundle of $N$. Since $N$ is non-orientable, we have $w_1 \neq 0$. Then, since the $w_i$ are uniquely determined by the previous formula (see [1] and note that the authors are implicitly using the Leray–Hirsch theorem), we must have $c^n \neq 0$. But if there exists an equivariant map $\varphi: SN \to S^{n-1}$, then $c = \varphi^*(d)$, where $d \in H^1(\mathbb{RP}^{n-1}; \mathbb{Z}/2)$ is the fundamental class of the antipodal involution on $S^{n-1}$, and hence $c^n = \varphi^*(d^n) = 0$.

In conclusion, let us note another application of Lemma 3 (which is used in [2]).

**Lemma 4.** [4] If $X$ is a contractible compact polyhedron such that $X \times \mathbb{N}^+$ embeds in $\mathbb{R}^n$, then the suspension $\Sigma X$ admits a $\mathbb{Z}/2$-map to $S^{n-1}$.

**Proof.** Since $X$ is contractible, $\Sigma X$ admits a $\mathbb{Z}/2$-map to the double mapping cylinder of the inclusions $X \supset X \subset X$. The latter $\mathbb{Z}/2$-embeds in $\tilde{X} \times I$. \hfill $\Box$

**Theorem 5.** [5] $\mathbb{R}^n$ does not contain uncountably many pairwise disjoint copies of the $(n-1)$-dimensional umbrella $U_{n-1}$, that is, the cone over $S^{n-2} \sqcup pt$.

A different proof of Theorem 5, also based on Lemma 4, is given in [4].

**Proof.** Let us triangulate $U := U^{n-1}$ as the cone over $\partial \Delta^{n-1} \sqcup pt$. Then it is self-dual as a subcomplex of $\Delta^{n+1}$, i.e. contains precisely one face out of each pair $\Delta^k$, $\Delta^{n-k}$ of complementary faces of $\Delta^{n+1}$. Consequently its simplicial deleted join $U \ast U$ is $\mathbb{Z}/2$-homeomorphic to $S^n$ [3; Corollary 3.16]. Also there exists a $\mathbb{Z}/2$-map from $U \ast U$ to the suspension over the simplicial deleted product $U \otimes U$ (see [3; Lemma 3.25]). Since $U \otimes U \subset \tilde{U}$, by the Borsuk–Ulam theorem there exists no $\mathbb{Z}/2$-map $\Sigma \tilde{U} \to S^{n-1}$.

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