Abstract

In three-dimensional Euclidean geometry, the scalar product produces a number associated to two vectors, while the vector product computes a vector perpendicular to them. These are key tools of physics, chemistry and engineering and supported by a rich vector calculus of 18th and 19th century results. This paper extends this calculus to arbitrary metrical geometries on three-dimensional space, generalising key results of Lagrange, Jacobi, Binet and Cauchy in a purely algebraic setting which applies also to general fields, including finite fields. We will then apply these vector theorems to set up the basic framework of rational trigonometry in the three-dimensional affine space and the related two-dimensional projective plane, and show an example of its applications to relativistic geometry.

Keywords: scalar product; vector product; symmetric bilinear form; rational trigonometry; affine geometry; projective geometry; triangle

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1 Introduction

In three-dimensional Euclidean vector geometry, the **scalar product** of two vectors

\[ v_1 \equiv (x_1, y_1, z_1) \quad \text{and} \quad v_2 \equiv (x_2, y_2, z_2) \]

is the number

\[ v_1 \cdot v_2 \equiv x_1x_2 + y_1y_2 + z_1z_2 \]

while the **vector product** is the vector

\[ v_1 \times v_2 = (x_1, y_1, z_1) \times (x_2, y_2, z_2) \equiv (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2, x_1y_2 - x_2y_1). \]

These definitions date back to the late eighteenth century, with Lagrange’s [12] study of the tetrahedron in three-dimensional space. Hamilton’s quaternions in four-dimensional space combine both operations, but Gibbs [7] and Heaviside [9] choose to separate the two quantities and introduce the current notations, where they were particularly effective in simplifying Maxwell’s equations.
Implicitly the framework of Euclidean geometry from Euclid’s Elements \[8\] underlies both concepts, and gradually in the 20th century it became apparent that the scalar, or dot, product in particular could be viewed as the starting point for Euclidean metrical geometry, at least in a vector space. However the vector, or cross, product is a special construction more closely aligned with three-dimensional space, although the framework of Geometric Algebra (see \[10\] and \[6\]) does provide a more sophisticated framework for generalizing it.

In this paper, we initiate the study of vector products in general three-dimensional inner product spaces over arbitrary fields, not of characteristic 2, and then apply this to develop the main laws of rational trigonometry for triangles both in the affine and projective settings, now with respect to a general symmetric bilinear form, and over an arbitrary field.

We will start with the associated three-dimensional vector space \(V^3\), regarded as row vectors, of a three-dimensional affine space \(A^3\) over an arbitrary field \(F\), not of characteristic 2. A general metrical framework is introduced via a \(3 \times 3\) invertible symmetric matrix \(B\), which defines a non-degenerate symmetric bilinear form on the vector space \(V^3\) by

\[v \cdot_B w \equiv v B w^T.\]

We will call this number, which is necessarily an element of the field \(F\), the \(B\)-scalar product of \(v\) and \(w\).

If \(B\) is explicitly the symmetric matrix

\[B \equiv \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix},\]

then the adjugate of \(B\) is defined by

\[\text{adj} B = \begin{pmatrix} a_2a_3 - b_1^2 & b_1b_2 - a_3b_3 & b_1b_3 - a_2b_2 \\ b_1b_2 - a_3b_3 & a_1a_3 - b_2^2 & b_2b_3 - a_1b_1 \\ b_1b_3 - a_2b_2 & b_2b_3 - a_1b_1 & a_1a_2 - b_3^2 \end{pmatrix},\]

and in case \(B\) is invertible,

\[\text{adj} B = (\det B) B^{-1}.\]

Then we define the \(B\)-vector product between the two vectors \(v_1\) and \(v_2\) to be the vector

\[v_1 \times_B v_2 \equiv (v_1 \times v_2) \text{adj} B.\]

We will extend well-known and celebrated formulas from Binet \[1\], Cauchy \[3\], Jacobi \[11\] and Lagrange \[12\] involving also the \(B\)-scalar triple product

\[[v_1, v_2, v_3]_B \equiv v_1 \cdot_B (v_2 \times_B v_3)\]
the $B$-vector triple product
\[
\langle v_1, v_2, v_3 \rangle_B \equiv v_1 \times_B (v_2 \times_B v_3)
\]

the $B$-scalar quadruple product
\[
[v_1, v_2; v_3, v_4]_B \equiv (v_1 \times_B v_2) \cdot_B (v_3 \times_B v_4)
\]

and the $B$-vector quadruple product
\[
\langle v_1, v_2; v_3, v_4 \rangle_B \equiv (v_1 \times_B v_2) \times_B (v_3 \times_B v_4).
\]

These tools will then be the basis for a rigorous framework of vector trigonometry over the three-dimensional vector space $\mathbb{V}^3$, following closely the framework of affine rational trigonometry in two-dimensional Euclidean space formulated in [17], but working more generally with vector triangles in three-dimensional vector space, by which we mean an unordered collection of three vectors whose sum is the zero vector. In this framework, the terms "quadrance" and "spread" are used rather than "distance" and "angle" respectively; though in Euclidean geometry quadrances and spreads are equivalent to squared distances and squared sines of angles respectively, when we start to generalise to arbitrary symmetric bilinear forms on vector spaces over general fields the definitions of quadrance and spread easily generalise using only linear algebraic methods. To highlight this in the case of Euclidean geometry over a general field, the quadrance of a vector $v$ is the number
\[
Q(v) \equiv v \cdot v
\]
and the spread between two vectors $v$ and $w$ is the number
\[
s(v, w) \equiv 1 - \frac{(v \cdot w)^2}{(v \cdot v)(w \cdot w)}.
\]

By using these quantities, we avoid the issue of taking square roots of numbers which are not square numbers in such a general field, which becomes problematic when working over finite fields.

We also develop a corresponding framework of projective triangles in the two-dimensional projective space associated to the three-dimensional vector space, where geometrically a projective triangle can be viewed as a triple of one-dimensional subspaces of $\mathbb{V}^3$. Mirroring the setup in [16] and [18], the tools presented above will be sufficient to develop the metrical geometry of projective triangles and extend known results from [16] and [18] to arbitrary non-degenerate $B$-scalar products. This then gives us a basis for both general elliptic and hyperbolic rational geometries, again working over a general field.

Applying the tools of $B$-scalar and $B$-vector products together gives a powerful method in which to study two-dimensional rational trigonometry in three-dimensional space over a general metrical framework and ultimately the tools presented in the paper could be used to study three-dimensional rational trigonometry in three-dimensional space over said general metrical framework, with emphasis on the trigonometry of a general tetrahedron. This is presented in more detail in the follow-up paper *Generalised vector products applied to affine rational trigonometry of a general tetrahedron.*
We will illustrate this new technology with several explicit examples where all the calculations are completely visible and accurate, and can be done by hand. First we look at the geometry of a methane molecule $\text{CH}_4$ which is a regular tetrahedron, and derive new (rational!) expressions for the separation of the faces, and the solid spreads. We also analyse both a vector triangle and a projective triangle in a vector space with a Minkowski bilinear form; this highlights how the general metrical framework of affine and projective rational trigonometry can be used to study relativistic geometry.

2 Vector algebra over a general metrical framework

We start by considering the three-dimensional vector space $\mathbb{V}^3$ over a general field $\mathbb{F}$ not of characteristic 2, consisting of row vectors

$$v = (x, y, z).$$

2.1 The $B$-scalar product

A $3 \times 3$ symmetric matrix

$$B \equiv \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}$$

determines a symmetric bilinear form on $\mathbb{V}^3$ defined by

$$v \cdot_B w \equiv vBw^T.$$ We will call this the $B$-scalar product. The $B$-scalar product is non-degenerate precisely when the condition that $v \cdot_B w = 0$ for any vector $v$ in $\mathbb{V}^3$ implies that $w = 0$; this will occur precisely when $B$ is invertible, that is when

$$\det B \neq 0.$$ We will assume that the $B$-scalar product is non-degenerate throughout this paper.

The associated $B$-quadratic form on $\mathbb{V}^3$ is defined by

$$Q_B (v) \equiv v \cdot_B v$$

and the number $Q_B (v)$ is the $B$-quadrance of $v$. A vector $v$ is $B$-null precisely when

$$Q_B (v) = 0.$$ The $B$-quadrance satisfies the obvious properties that for vectors $v$ and $w$ in $\mathbb{V}^3$, and a number $\lambda$ in $\mathbb{F}$

$$Q_B (\lambda v) = \lambda^2 Q_B (v)$$

as well as

$$Q_B (v + w) = Q_B (v) + Q_B (w) + 2 (v \cdot_B w)$$
\[ Q_B (v - w) = Q_B (v) + Q_B (w) - 2 (v \cdot_B w) . \]

Hence the \( B \)-scalar product can be expressed in terms of the \( B \)-quadratic form by either of the two polarisation formulas
\[
v \cdot_B w = \frac{Q_B (v + w) - Q_B (v) - Q_B (w)}{2} = \frac{Q_B (v) + Q_B (w) - Q_B (v - w)}{2}.
\]

Two vectors \( v \) and \( w \) in \( \mathbb{V}^3 \) are \( B \)-perpendicular precisely when \( v \cdot_B w = 0 \), in which case we write \( v \perp_B w \).

When \( B \) is the \( 3 \times 3 \) identity matrix, we have the familiar Euclidean scalar product, which we will write simply as \( v \cdot w \), and the corresponding Euclidean quadratic form will be written just as \( Q(v) = v \cdot v \), which will be just the quadrance of \( v \). Our aim is to extend the familiar theory of scalar and quadratic forms from the Euclidean to the general case, and then apply these in a novel way to establish a purely algebraic, or rational, trigonometry in three dimensions.

2.2 The \( B \)-vector product

Given two vectors \( v_1 \equiv (x_1, y_1, z_1) \) and \( v_2 \equiv (x_2, y_2, z_2) \) in \( \mathbb{V}^3 \), the Euclidean vector product of \( v_1 \) and \( v_2 \) is the vector
\[
v_1 \times v_2 = (x_2, y_2, z_2) \times (x_2, y_2, z_2) \equiv (y_1 z_2 - y_2 z_1, x_2 z_1 - x_1 z_2, x_1 y_2 - x_2 y_1).
\]

We now extend this notion to the case of a general symmetric bilinear form. Let \( v_1, v_2 \) and \( v_3 \) be vectors in \( \mathbb{V}^3 \), and let
\[
M \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}
\]
be the matrix with these vectors as rows. We define the adjugate of \( M \) to be the matrix
\[
\text{adj} M \equiv \begin{pmatrix} v_2 \times v_3 \\ v_3 \times v_1 \\ v_1 \times v_2 \end{pmatrix}^T.
\]

If the \( 3 \times 3 \) matrix \( M \) is invertible, then the adjugate is characterized by the equation
\[
\frac{1}{(\det M)} \text{adj} M \equiv M^{-1}.
\]

In this case the properties
\[
\text{adj} (MN) = (\text{adj} N) (\text{adj} M)
\]
and
\[
M (\text{adj} M) = (\text{adj} M) M = (\det M) I
\]

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are immediate, where \( I \) the \( 3 \times 3 \) identity matrix, and in fact they hold more generally for arbitrary \( 3 \times 3 \) matrices \( M \) and \( N \). In the invertible case we have in addition

\[
\text{adj} (\text{adj} M) = \det (\text{adj} M) (\text{adj} M)^{-1} = \det ((\det M)^{-1}) ((\det M)^{-1})^{-1} = (\det M)^3 (\det M^{-1}) (\det M)^{-1} M = (\det M) M.
\]

For the fixed symmetric matrix \( B \) from \( \bullet \), we write

\[
\text{adj} B = \begin{pmatrix}
a_2 a_3 - b_1^2 & b_1 b_2 - a_3 b_3 & b_1 b_3 - a_2 b_2 \\
b_1 b_2 - a_3 b_3 & a_1 a_3 - b_2^2 & b_2 b_3 - a_1 b_1 \\
b_1 b_3 - a_2 b_2 & b_2 b_3 - a_1 b_1 & a_1 a_2 - b_3^2
\end{pmatrix} = \begin{pmatrix}
\alpha_1 & \beta_3 & \beta_2 \\
\beta_3 & \alpha_2 & \beta_1 \\
\beta_2 & \beta_1 & \alpha_3
\end{pmatrix}.
\]

Now define the \( B \)-vector product of vectors \( v_1 \) and \( v_2 \) to be the vector

\[
v_1 \times_B v_2 \equiv (v_1 \times v_2) \text{adj} B.
\]

The motivation for this definition is given by the following theorem. A similar result has been obtained in \( \bullet \).

**Theorem 1 (Adjugate vector product theorem)** Let \( v_1 \), \( v_2 \) and \( v_3 \) be vectors in \( \mathbb{R}^3 \), and let \( M \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \) be the matrix with these vectors as rows. Then for any \( 3 \times 3 \) invertible symmetric matrix \( B \),

\[
\text{adj} (MB) \equiv \begin{pmatrix} v_2 \times_B v_3 \\ v_3 \times_B v_1 \\ v_1 \times_B v_2 \end{pmatrix}^T.
\]

**Proof.** By the definition of adjugate matrix, \( \text{adj} M \) is

\[
\text{adj} M = \begin{pmatrix}
(v_2 \times v_3)^T \\
(v_3 \times v_1) \\
(v_1 \times v_2)
\end{pmatrix}.
\]

Since \( \text{adj} (MB) = \text{adj} B \text{adj} M \) and \( B \) is symmetric,

\[
(\text{adj} (MB))^T = (\text{adj} M)^T \text{adj} B = \begin{pmatrix}
(v_2 \times v_3) \\
(v_3 \times v_1) \\
(v_1 \times v_2)
\end{pmatrix} \text{adj} B
\]

\[
= \begin{pmatrix}
(v_2 \times v_3) \text{adj} B \\
(v_3 \times v_1) \text{adj} B \\
(v_1 \times v_2) \text{adj} B
\end{pmatrix} = \begin{pmatrix}
(v_2 \times_B v_3) \\
(v_3 \times_B v_1) \\
(v_1 \times_B v_2)
\end{pmatrix}.
\]
Now take the matrix transpose on both sides. ■

The usual linearity and anti-symmetric properties of the Euclidean vector product hold for $B$-vector products.

2.3 The $B$-scalar triple product

The Euclidean scalar triple product of three vectors $v_1$, $v_2$ and $v_3$ in $\mathbb{R}^3$ (see [7, pp. 68-71]) is

$$[v_1, v_2, v_3] \equiv v_1 \cdot (v_2 \times v_3) = \det \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$  

We can generalise this definition for an arbitrary symmetric bilinear form with matrix representation $B$; so we define the $B$-scalar triple product of $v_1$, $v_2$ and $v_3$ to be

$$[v_1, v_2, v_3]_B \equiv v_1 \cdot_B (v_2 \times_B v_3).$$

The following result allows for the evaluation of the $B$-scalar triple product in terms of determinants.

**Theorem 2 (Scalar triple product theorem)** Let $M \equiv \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ for vectors $v_1$, $v_2$ and $v_3$ in $\mathbb{R}^3$. Then

$$[v_1, v_2, v_3]_B = (\det B) (\det M).$$

**Proof.** From the definitions of the $B$-scalar product, $B$-vector product and the $B$-scalar triple product,

$$[v_1, v_2, v_3]_B = v_1 B ((v_2 \times v_3) \ adj B)^T = v_1 (B \ adj B) (v_2 \times v_3)^T.$$  

As $\ adj B = (\det B) B^{-1}$ and $v_1 \cdot (v_2 \times v_3) = \det M$,

$$[v_1, v_2, v_3]_B = (\det B) v_1 (v_2 \times v_3)^T = (\det B) (v_1 \cdot (v_2 \times v_3)) = (\det B) (\det M)$$  

as required. ■

We can now relate $B$-vector products to $B$-perpendicularity.

**Corollary 3** The vectors $v$ and $w$ in $\mathbb{R}^3$ are both $B$-perpendicular to $v \times_B w$, i.e.

$$v \perp_B (v \times_B w) \quad \text{and} \quad w \perp_B (v \times_B w).$$
Proof. By the Scalar triple product theorem,

\[ v \cdot_B (v \times_B w) = [v, v, w]_B = (\det B) \det \begin{pmatrix} v \\ w \end{pmatrix} = 0. \]

Similarly, \([w, v, w]_B = 0\) and so both \(v \perp_B (v \times_B w)\) and \(w \perp_B (v \times_B w)\). ■

We could also rearrange the ordering of \(B\)-scalar triple products as follows.

**Corollary 4** For vectors \(v_1, v_2, \text{ and } v_3\) in \(\mathbb{V}^3\),

\[ [v_1, v_2, v_3]_B = [v_2, v_3, v_1]_B = [v_3, v_1, v_2]_B = - [v_1, v_3, v_2]_B = - [v_2, v_1, v_3]_B = - [v_3, v_2, v_1]_B. \]

**Proof.** This follows from the corresponding relations for \([v_1, v_2, v_3]\), or equivalently the transformation properties of the determinant upon permutation of rows. ■

### 2.4 The \(B\)-vector triple product

Recall that the **Euclidean vector triple product** of vectors \(v_1, v_2, \text{ and } v_3\) in \(\mathbb{V}^3\) (see [7, pp. 71-75]) is

\[ \langle v_1, v_2, v_3 \rangle \equiv v_1 \times (v_2 \times v_3). \]

The **\(B\)-vector triple product** of the vectors is similarly defined by

\[ \langle v_1, v_2, v_3 \rangle_B \equiv v_1 \times_B (v_2 \times_B v_3). \]

We can evaluate this by generalising a classical result of Lagrange [12] from the Euclidean vector triple product to \(B\)-vector triple products, following the general lines of argument of [4] and [15, pp. 28-29] in the Euclidean case; the proof is surprisingly complicated.

**Theorem 5 (Lagrange’s formula)** For vectors \(v_1, v_2, \text{ and } v_3\) in \(\mathbb{V}^3\),

\[ \langle v_1, v_2, v_3 \rangle_B = (\det B) \left[ (v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3 \right]. \]

**Proof.** Let \(w \equiv \langle v_1, v_2, v_3 \rangle_B\). If \(v_2 \text{ and } v_3\) are linearly dependent, then \(v_2 \times_B v_3 = 0\) and thus \(\langle v_1, v_2, v_3 \rangle_B = 0\). Furthermore, we are able to write one of them as a scalar multiple of the other, which implies that

\[ (v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3 = 0 \]

and thus the required result holds. So we may suppose that \(v_2 \text{ and } v_3\) are linearly independent. From Corollary 4, \((v_2 \times_B v_3) \perp_B w\) and thus

\[ v_2 \perp_B (v_2 \times_B v_3) \quad \text{and} \quad v_3 \perp_B (v_2 \times_B v_3). \]

As \(w\) is parallel to \(v_2 \text{ and } v_3\), we can deduce that \(w\) is equal to some linear combination of \(v_2 \text{ and } v_3\).
So, for some scalars $\alpha$ and $\beta$ in $\mathbb{F}$, we have

$$w = \alpha v_2 + \beta v_3.$$  

Furthermore, since $v_1 \perp_B w$, the definition of $B$-perpendiculality implies that

$$w \cdot_B v_1 = \alpha (v_1 \cdot_B v_2) + \beta (v_1 \cdot_B v_3) = 0.$$  

This equality is true precisely when $\alpha = \lambda (v_1 \cdot_B v_3)$ and $\beta = -\lambda (v_1 \cdot_B v_2)$, for some non-zero scalar $\lambda$ in $\mathbb{F}$. Hence,

$$w = \lambda [(v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3].$$  

To proceed, we first want to prove that $\lambda$ is independent of the choices $v_1$, $v_2$ and $v_3$, so that we can compute $w$ for arbitrary $v_1$, $v_2$ and $v_3$. First, suppose that $\lambda$ is dependent on $v_1$, $v_2$ and $v_3$, so that we may define $\lambda \equiv \lambda (v_1, v_2, v_3)$. Given another vector $d$ in $\mathbb{V}^3$, we have

$$w \cdot_B d = \lambda (v_1, v_2, v_3) [(v_1 \cdot_B v_3) (v_2 \cdot_B d) - (v_1 \cdot_B v_2) (v_3 \cdot_B d)].$$

Directly substituting the definition of $w$, we use the Scalar triple product theorem to obtain

$$w \cdot_B d = (v_1 \times_B (v_2 \times_B v_3)) \cdot_B d = v_1 \cdot_B ((v_2 \times_B v_3) \times_B d) = -v_1 \cdot_B (d, v_2, v_3)_B.$$  

Based on our calculations of $w$, we then deduce that

$$-v_1 \cdot_B (d, v_2, v_3)_B = -v_1 \cdot_B \left( \lambda (d, v_2, v_3) [(d \cdot_B v_3) v_2 - (d \cdot_B v_2) v_3] \right) = \lambda \left( d, v_2, v_3 \right) \left( (v_1 \cdot_B v_3) (v_2 \cdot_B d) - (v_1 \cdot_B v_2) (v_3 \cdot_B d) \right).$$

Because this expression is equal to $w \cdot_B d$, we deduce that $\lambda (v_1, v_2, v_3) = \lambda (d, v_2, v_3)$ and hence $\lambda$ must be independent of the choice of $v_1$. With this, now suppose instead that $\lambda \equiv \lambda (v_2, v_3)$, so that

$$w \cdot_B d = \lambda (v_2, v_3) [(v_1 \cdot_B v_3) (v_2 \cdot_B d) - (v_1 \cdot_B v_2) (v_3 \cdot_B d)]$$

for a vector $d$ in $\mathbb{V}^3$. By direct substitution of $w$, we use the Scalar triple product theorem to obtain

$$w \cdot_B d = (v_1 \times_B (v_2 \times_B v_3)) \cdot_B d = (v_2 \times_B v_3) \cdot_B (d \times_B v_1) = v_2 \cdot_B (v_3, d, v_1)_B.$$  

Similarly, based on the calculations of $w$ previously, we have

$$v_2 \cdot_B (v_3, d, v_1)_B = v_2 \cdot_B \lambda (d_2, v_3) [(v_1 \cdot_B v_3) d - (v_3 \cdot_B d) v_1] = \lambda (d, v_1) \left( (v_1 \cdot_B v_3) (v_2 \cdot_B d) - (v_1 \cdot_B v_2) (v_3 \cdot_B d) \right).$$

Because this expression is also equal to $w \cdot_B d$, we deduce that $\lambda (v_2, v_3) = \lambda (d, v_1)$ and conclude that $\lambda$ is indeed independent of $v_2$ and $v_3$, in addition to $v_1$. So, we substitute any choice of vectors $v_1$, $v_2$
and $v_3$ in order to find $\lambda$. With this, suppose that $v_2 \equiv (1, 0, 0)$ and $v_1 = v_3 \equiv (0, 1, 0)$. Then,

$$v_2 \times_B v_3 = [(1, 0, 0) \times (0, 1, 0)] \text{adj} B$$

$$= (0, 0, 1) \begin{pmatrix} \alpha_1 & \beta_2 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix}$$

$$= (\beta_2, \beta_1, \alpha_3)$$

and hence

$$\langle v_1, v_2, v_3 \rangle_B = [(0, 1, 0) \times (\beta_2, \beta_1, \alpha_3)] \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix}$$

$$= (\alpha_3, 0, -\beta_2) \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix}$$

$$= (\alpha_1 \alpha_3 - \beta_2^2, \alpha_3 \beta_3 - \beta_1 \beta_2, 0).$$

Now use the fact that \text{adj} (\text{adj} B) = (\det B) B to obtain

$$\text{adj} \begin{pmatrix} \alpha_1 & \beta_3 & \beta_2 \\ \beta_3 & \alpha_2 & \beta_1 \\ \beta_2 & \beta_1 & \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_2 \alpha_3 - \beta_2^2 & \beta_1 \beta_2 - \alpha_3 \beta_3 & \beta_1 \beta_3 - \alpha_2 \beta_2 \\ \beta_1 \beta_2 - \alpha_3 \beta_3 & \alpha_1 \alpha_3 - \beta_2^2 & \beta_2 \beta_3 - \alpha_1 \beta_1 \\ \beta_1 \beta_3 - \alpha_2 \beta_3 & \beta_2 \beta_3 - \alpha_1 \beta_1 & \alpha_1 \alpha_2 - \beta_2^2 \end{pmatrix}$$

$$= (\det B) \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix}$$

so that

$$\langle v_1, v_2, v_3 \rangle_B = (\det B) (a_2, -b_3, 0).$$

Since $v_1 \cdot_B v_2 = e_1 B e_2^T = b_3$ and $v_1 \cdot_B v_3 = e_2 B e_2^T = a_2$, it follows that

$$(\det B) (a_2, -b_3, 0) = (\det B) [(v_1 \cdot_B v_3) e_1 - (v_1 \cdot_B v_2) e_2]$$

$$= (\det B) [(v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3].$$

From this, we deduce that $\lambda = \det B$ and hence

$$\langle v_1, v_2, v_3 \rangle_B = (\det B) [(v_1 \cdot_B v_3) v_2 - (v_1 \cdot_B v_2) v_3]$$

as required. ■

The $B$-vector product is not an associative operation, but by the anti-symmetric property of $B$-vector products, we see that

$$\langle v_1, v_2, v_3 \rangle_B = -\langle v_1, v_3, v_2 \rangle_B.$$
The following result, attributed in the Euclidean case to Jacobi [11], connects the theory of $B$-vector products to the theory of Lie algebras and links the three $B$-vector triple products which differ by an even permutation of the indices.

**Theorem 6 (Jacobi identity)** For vectors $v_1$, $v_2$ and $v_3$ in $\mathbb{V}^3$,

$$\langle v_1, v_2, v_3 \rangle_B + \langle v_2, v_3, v_1 \rangle_B + \langle v_3, v_1, v_2 \rangle_B = 0.$$  

**Proof.** Apply Lagrange’s formula to each of the three summands to get

$$\langle v_1, v_2, v_3 \rangle_B = (\det B) [(v_1 \cdot B v_3) v_2 - (v_1 \cdot B v_2) v_3]$$

as well as

$$\langle v_2, v_3, v_1 \rangle_B = (\det B) [(v_1 \cdot B v_2) v_3 - (v_2 \cdot B v_3) v_1]$$

and

$$\langle v_3, v_1, v_2 \rangle_B = (\det B) [(v_2 \cdot B v_3) v_1 - (v_1 \cdot B v_3) v_2].$$

So,

$$\langle v_1, v_2, v_3 \rangle_B + \langle v_2, v_3, v_1 \rangle_B + \langle v_3, v_1, v_2 \rangle_B = (\det B) [(v_1 \cdot B v_3) v_2 - (v_1 \cdot B v_2) v_3] + (\det B) [(v_1 \cdot B v_2) v_3 - (v_2 \cdot B v_3) v_1] + (\det B) [(v_2 \cdot B v_3) v_1 - (v_1 \cdot B v_3) v_2] = 0$$

as required. ■

2.5 The $B$-scalar quadruple product

Recall that the **Euclidean scalar quadruple product** of vectors $v_1$, $v_2$, $v_3$ and $v_4$ in $\mathbb{V}^3$ (see [7 pp. 75-76]) is the scalar

$$[v_1, v_2; v_3, v_4] \equiv (v_1 \times v_2) \cdot (v_3 \times v_4).$$

We define similarly the **$B$-scalar quadruple product** to be the quantity

$$[v_1, v_2; v_3, v_4]_B \equiv (v_1 \times_B v_2) \cdot_B (v_3 \times_B v_4).$$

The following result, which originated from separate works of Binet [1] and Cauchy [3] in the Euclidean setting (also see [2] and [15, p. 29]), allows us to compute $B$-scalar quadruple products purely in terms of $B$-scalar products.

**Theorem 7 (Binet-Cauchy identity)** For vectors $v_1$, $v_2$, $v_3$ and $v_4$ in $\mathbb{V}^3$,

$$[v_1, v_2; v_3, v_4]_B = (\det B) [(v_1 \cdot_B v_3) (v_2 \cdot_B v_4) - (v_1 \cdot_B v_4) (v_2 \cdot_B v_3)].$$

**Proof.** Let $w \equiv v_1 \times_B v_2$, so that by the Scalar triple product theorem and Corollary 3,

$$[v_1, v_2; v_3, v_4]_B = [w, v_3, v_4]_B = [v_4, w, v_3]_B.$$
By Lagrange’s formula,

\[ w \times_B v_3 = - \langle v_3, v_1, v_2 \rangle_B = (\det B) \left[ (v_1 \cdot_B v_3) v_2 - (v_2 \cdot_B v_3) v_1 \right] \]

and hence

\[
[v_1, v_2; v_3, v_4]_B = ((\det B) \left[ (v_1 \cdot_B v_3) v_2 - (v_2 \cdot_B v_3) v_1 \right]) \cdot v_4 \\
= (\det B) \left[ (v_1 \cdot_B v_3) (v_2 \cdot_B v_4) - (v_1 \cdot_B v_4) (v_2 \cdot_B v_3) \right]
\]

as required. ■

An important special case of the classical Binet-Cauchy identity is a result of Lagrange [12], which we now generalise. We distinguish this from Lagrange’s formula, which computes the \(B\)-vector triple product of three vectors, by calling it Lagrange’s identity.

**Theorem 8 (Lagrange’s identity)** Given vectors \(v_1\) and \(v_2\) in \(\mathbb{V}^3\),

\[ Q_B (v_1 \times_B v_2) = (\det B) \left[ Q_B (v_1) Q_B (v_2) - (v_1 \cdot_B v_2)^2 \right]. \]

**Proof.** This immediately follows from the Binet-Cauchy identity by setting \(v_1 = v_3\) and \(v_2 = v_4\). ■

Here is another consequence of the Binet-Cauchy identity, which is somewhat similar to the Jacobi identity for \(B\)-vector triple products.

**Corollary 9** For vectors \(v_1\), \(v_2\), \(v_3\) and \(v_4\) in \(\mathbb{V}^3\), we have

\[
[v_1, v_2; v_3, v_4]_B + [v_2, v_3; v_1, v_4]_B + [v_3, v_1; v_2, v_4]_B = 0.
\]

**Proof.** From the Binet-Cauchy identity, the summands evaluate to

\[
[v_1, v_2; v_3, v_4]_B = (\det B) \left[ (v_1 \cdot_B v_3) (v_2 \cdot_B v_4) - (v_1 \cdot_B v_4) (v_2 \cdot_B v_3) \right]
\]

as well as

\[
[v_2, v_3; v_1, v_4]_B = (\det B) \left[ (v_2 \cdot_B v_1) (v_3 \cdot_B v_4) - (v_2 \cdot_B v_4) (v_3 \cdot_B v_1) \right]
\]

and

\[
[v_3, v_1; v_2, v_4]_B = (\det B) \left[ (v_3 \cdot_B v_2) (v_1 \cdot_B v_4) - (v_3 \cdot_B v_4) (v_1 \cdot_B v_2) \right].
\]

When we add these three quantities, we get 0 as required. ■

**2.6 The \(B\)-vector quadruple product**

Recall that the **Euclidean vector quadruple product** of vectors \(v_1, v_2, v_3\) and \(v_4\) in \(\mathbb{V}^3\) (see [14 pp. 76-77]) is the vector

\[
\langle v_1, v_2; v_3, v_4 \rangle \equiv (v_1 \times v_2) \times (v_3 \times v_4).
\]

Define similarly the **\(B\)-vector quadruple product** to be

\[
\langle v_1, v_2; v_3, v_4 \rangle_B \equiv (v_1 \times_B v_2) \times_B (v_3 \times_B v_4).
\]
The key result here is the following.

**Theorem 10 (B-vector quadruple product theorem)** For vectors \(v_1, v_2, v_3\) and \(v_4\) in \(\mathbb{V}^3\),

\[
\langle v_1, v_2; v_3, v_4 \rangle_B = (\det B) \left( [v_1, v_2, v_4]_B v_3 - [v_1, v_2, v_3]_B v_4 \right)
\]

\[
= (\det B) \left( [v_1, v_3, v_4]_B v_2 - [v_2, v_3, v_4]_B v_1 \right).
\]

**Proof.** If \(u \equiv v_1 \times_B v_2\), then use Lagrange’s formula to get

\[
\langle v_1, v_2; v_3, v_4 \rangle_B = \langle u, v_3, v_4 \rangle_B = (\det B) [u \cdot_B v_4] v_3 - (u \cdot_B v_3) v_4.
\]

From the Scalar triple product theorem,

\[
u \cdot_B v_3 = [v_1, v_2, v_3]_B\quad\text{and}\quad u \cdot_B v_4 = [v_1, v_2, v_4]_B.
\]

Therefore,

\[
\langle v_1, v_2; v_3, v_4 \rangle_B = (\det B) \left( [v_1, v_2, v_4]_B v_3 - [v_1, v_2, v_3]_B v_4 \right).
\]

Since

\[
\langle v_1, v_2; v_3, v_4 \rangle_B = -\langle v_3, v_4; v_1, v_2 \rangle_B = (v_3 \times_B v_4) \times_B (v_2 \times_B v_1)
\]

Corollary 3 gives us

\[
\langle v_1, v_2; v_3, v_4 \rangle_B = (\det B) \left( [v_3, v_4, v_1]_B v_2 - [v_3, v_4, v_2]_B v_1 \right)
\]

\[
= (\det B) \left( [v_1, v_3, v_4]_B v_2 - [v_2, v_3, v_4]_B v_1 \right).
\]

\[\square\]

As a corollary, we find a relation satisfied by any four vectors in three-dimensional vector space, extending the result in [7, p. 76] to a general metrical framework.

**Corollary 11 (Four vector relation)** Suppose that \(v_1, v_2, v_3\) and \(v_4\) are vectors in \(\mathbb{V}^3\). Then

\[
[v_2, v_3, v_4]_B v_1 - [v_1, v_3, v_4]_B v_2 + [v_1, v_2, v_4]_B v_3 - [v_1, v_2, v_3]_B v_4 = 0.
\]

**Proof.** This is an immediate consequence of equating the two equations from the previous result, after cancelling the non-zero factor \(\det B\). \[\square\]

As another consequence, we get an expression for the meet of two distinct two-dimensional subspaces.

**Corollary 12** If \(U \equiv \text{span} (v_1, v_2)\) and \(V \equiv \text{span} (v_3, v_4)\) are distinct two-dimensional subspaces then \(v \equiv \langle v_1, v_2; v_3, v_4 \rangle_B\) spans \(U \cap V\).

**Proof.** Clearly \(v\) is both in \(U\) and in \(V\) from the \(B\)-vector quadruple product theorem. We need only show that it is non-zero, but this follows from

\[
\langle v_1, v_2; v_3, v_4 \rangle_B = (\det B) \left( [v_1, v_2, v_4]_B v_3 - [v_1, v_2, v_3]_B v_4 \right)
\]

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since by assumption $v_3$ and $v_4$ are linearly independent, and at least one of $[v_1,v_2,v_4]_B$ and $[v_1,v_2,v_3]_B$ must be non-zero since otherwise both $v_4$ and $v_3$ lie in $U$, which contradicts the assumption that the $U$ and $V$ are distinct. \[\square\]

A special case occurs when each of the factors of the $B$-quadruple vector product contains a common vector. This extends the result in [7, p. 80] to a general metrical framework.

**Corollary 13** If $v_1$, $v_2$ and $v_3$ are vectors in $\mathbb{V}^3$ then

$$\langle v_1, v_2; v_1, v_3 \rangle_B = (\det B) [v_1, v_2, v_3]_B v_1.$$  

**Proof.** This follows from

$$\langle v_1, v_2; v_1, v_3 \rangle_B = (\det B) ([v_1, v_2, v_3]_B v_1 - [v_1, v_2, v_1]_B v_3)$$  

together with the fact that $[v_1, v_2, v_1]_B = 0$. \[\square\]

Yet another consequence is given below, which was alluded to [7, p. 86] in for the Euclidean case.

**Theorem 14** (Triple scalar product of products) For vectors $v_1,v_2$ and $v_3$ in $\mathbb{V}^3$,

$$[v_2 \times_B v_3, v_3 \times_B v_1, v_1 \times_B v_2]_B = (\det B) ([v_1, v_2, v_3]_B)^2.$$  

**Proof.** From the previous corollary,

$$[v_2 \times_B v_3, v_3 \times_B v_1, v_1 \times_B v_2]_B = -(v_2 \times_B v_3) \cdot_B ((v_1 \times_B v_3) \times_B (v_1 \times_B v_2))$$

$$= -(v_2 \times_B v_3) \cdot_B (\det B) [v_1, v_3, v_2]_B v_1$$

$$= (\det B) ([v_1, v_2, v_3]_B)^2.$$  

\[\square\]

It follows that if $v_1$, $v_2$ and $v_3$ are linearly independent, then so are $v_1 \times_B v_2$, $v_2 \times_B v_3$ and $v_3 \times_B v_1$. This also suggests there is a kind of duality here, which we can clarify by the following result, which is a generalization of Exercise 8 of [14, p. 116].

**Theorem 15** Suppose that $v_1,v_2$ and $v_3$ are linearly independent vectors in $\mathbb{V}^3$, so that $[v_1,v_2,v_3]_B$ is non-zero. Define

$$w_1 = \frac{v_2 \times_B v_3}{[v_1,v_2,v_3]_B}, \quad w_2 = \frac{v_3 \times_B v_1}{[v_1,v_2,v_3]_B} \quad \text{and} \quad w_3 = \frac{v_1 \times_B v_2}{[v_1,v_2,v_3]_B}.$$  

Then

$$v_1 \times_B w_1 + v_2 \times_B w_2 + v_3 \times_B w_3 = 0$$  

and

$$v_1 \cdot_B w_1 + v_2 \cdot_B w_2 + v_3 \cdot_B w_3 = 3$$  

and

$$[v_1,v_2,v_3]_B [w_1,w_2,w_3]_B = \det B.$$
and 
\[ v_1 = \frac{w_2 \times w_3}{[w_1, w_2, w_3]_B}, \quad v_2 = \frac{w_3 \times w_1}{[w_1, w_2, w_3]_B} \quad \text{and} \quad v_3 = \frac{w_1 \times w_2}{[w_1, w_2, w_3]_B}. \]

**Proof.** By the Jacobi identity,

\[
v_1 \times_B w_1 + v_2 \times_B w_2 + v_3 \times_B w_3 = \frac{1}{[v_1, v_2, v_3]_B} \left( v_1 \times_B (w_2 \times_B v_3) + v_2 \times_B (w_3 \times_B v_1) + v_3 \times_B (v_1 \times_B v_2) \right)
\]

\[
= \frac{1}{[v_1, v_2, v_3]_B} \left( \langle v_1, v_2, v_3 \rangle_B + \langle v_2, v_3, v_1 \rangle_B + \langle v_3, v_1, v_2 \rangle_B \right)
\]

\[
= 0.
\]

Moreover, we use the Scalar triple product to obtain

\[
v_1 \cdot_B w_1 + v_2 \cdot_B w_2 + v_3 \cdot_B w_3 = \frac{1}{[v_1, v_2, v_3]_B} \left( [v_1, v_2, v_3]_B + [v_2, v_3, v_1]_B + [v_3, v_1, v_2]_B \right)
\]

\[
= \frac{1}{[v_1, v_2, v_3]_B} \left( [v_1, v_2, v_3]_B + [v_2, v_3, v_1]_B + [v_3, v_1, v_2]_B \right)
\]

\[
= 3.
\]

By the Triple scalar product of products theorem,

\[
[w_1, w_2, w_3]_B = \left[ \frac{v_2 \times_B v_3}{[v_1, v_2, v_3]_B}, \frac{v_3 \times_B v_1}{[v_1, v_2, v_3]_B}, \frac{v_1 \times_B v_2}{[v_1, v_2, v_3]_B} \right]_B
\]

\[
= \frac{1}{[v_1, v_2, v_3]_B^3} [v_2 \times_B v_3, v_3 \times_B v_1, v_1 \times_B v_2]_B
\]

\[
= \frac{1}{[v_1, v_2, v_3]_B} (\det B) \left( [v_1, v_2, v_3]_B \right)^2
\]

\[
= \frac{\det B}{[v_1, v_2, v_3]_B}.
\]

Hence,

\[
[v_1, v_2, v_3]_B \cdot [w_1, w_2, w_3]_B = \det B.
\]
Given this result, we have

\[
w_2 \times_B w_3 = \frac{v_3 \times_B v_1}{[v_1, v_2, v_3]_B} \times_B \frac{v_1 \times_B v_2}{[v_1, v_2, v_3]_B} = \frac{1}{[v_1, v_2, v_3]_B^2} [(v_3 \times_B v_1) \times_B (v_1 \times_B v_2)]
\]

\[
= - \frac{1}{[v_1, v_2, v_3]_B^2} [(v_1 \times_B v_3) \times_B (v_1 \times_B v_2)]
\]

\[
= - \frac{1}{[v_1, v_2, v_3]_B^2} (\det B) [v_1, v_3, v_2]_B v_1 = \frac{(\det B) v_1}{[v_1, v_2, v_3]_B} = [w_1, w_2, w_3]_B v_1
\]

where the fourth line results from Corollary 13 and the fifth line results from the Scalar triple product theorem. Thus

\[
v_1 = \frac{w_2 \times w_3}{[w_1, w_2, w_3]_B}
\]

and the results for \(v_2\) and \(v_3\) are similar.

The first and last of the results of [14, p. 116] was also proven in [7, p. 86] for the Euclidean situation.

3 Rational trigonometry for a vector triangle

We would like to extend the framework of [17] and configure three-dimensional rational trigonometry so it works over a general field and general bilinear form, centrally framing our discussion on the tools we have developed above.

We assume as before a \(B\)-scalar product on the three-dimensional vector space \(V^3\). A vector triangle \(v_1v_2v_3\) is an unordered collection of three vectors \(v_1, v_2\) and \(v_3\) satisfying

\[
v_1 + v_2 + v_3 = 0.
\]

The \(B\)-quadrances of such a triangle are the numbers

\[
Q_1 \equiv Q_B(v_1), \quad Q_2 \equiv Q_B(v_2) \quad \text{and} \quad Q_3 \equiv Q_B(v_3).
\]

Define Archimedes’ function [17, p. 64] as

\[
A(a, b, c) \equiv (a + b + c)^2 - 2(a^2 + b^2 + c^2).
\]

The \(B\)-quadra of \(v_1v_2v_3\) is then

\[
A_B(v_1v_2v_3) = A(Q_1, Q_2, Q_3).
\]
Theorem 16 (Quadrea theorem) Given a vector triangle \( \overline{v_1v_2v_3} \)

\[
Q_B (v_1 \times_B v_2) = Q_B (v_2 \times_B v_3) = Q_B (v_3 \times_B v_1) = \frac{(\det B)}{4} A_B (v_1v_2v_3).
\]

**Proof.** By bilinearity, we have

\[
Q_B (v_2 \times_B v_3) = Q_B (v_2 \times_B (-v_1 - v_2)) = Q_B (-v_2 \times_B v_1) = Q_B (v_1 \times v_2)
\]

and similarly

\[
Q_B (v_2 \times_B v_3) = Q_B (v_3 \times_B v_1).
\]

From Lagrange’s identity

\[
Q_B (v_1 \times_B v_2) = (\det B) \left[ Q_B (v_1) Q_B (v_2) - (v_1 \cdot_B v_2)^2 \right]
\]

we use the polarisation formula to obtain

\[
Q_B (v_1 \times_B v_2) = (\det B) \left[ Q_1 Q_2 - \left( \frac{Q_1 + Q_2 - Q_3}{2} \right)^2 \right] = \frac{\det B}{4} \left[ 4Q_1 Q_2 - (Q_1 + Q_2 - Q_3)^2 \right] = \frac{\det B}{4} A (Q_1, Q_2, Q_3)
\]

and then a simple rearrangement gives one of our desired results; the others follow by symmetry. ■

Over a general field the angle between vectors is not well-defined. So in rational trigonometry we replace angle with the algebraic quantity called a spread which is more directly linked to the underlying scalar products. The **B-spread** between vectors \( v_1 \) and \( v_2 \) is the number

\[
s_B (v_1, v_2) \equiv 1 - \frac{(v_1 \cdot_B v_2)^2}{Q_B (v_1) Q_B (v_2)}.
\]

By Lagrange’s identity this can be rewritten as

\[
s_B (v_1, v_2) = \frac{Q_B (v_1 \times_B v_2)}{(\det B) Q_B (v_1) Q_B (v_2)}.
\]

The **B-spread** is invariant under scalar multiplication of either \( v_1 \) or \( v_2 \) (or both). If one or both of \( v_1, v_2 \) are null vectors then the spread is undefined.

In what follows, we will consider a vector triangle \( \overline{v_1v_2v_3} \) with **B-quadrances**

\[
Q_1 \equiv Q_B (v_1), \quad Q_2 \equiv Q_B (v_2) \quad \text{and} \quad Q_3 \equiv Q_B (v_3)
\]

as well as **B-spreads**

\[
s_1 \equiv s_B (v_2, v_3), \quad s_2 \equiv s_B (v_1, v_3) \quad \text{and} \quad s_3 \equiv s_B (v_1, v_2)
\]
and $B$-quadrea

$$A \equiv A_B (v_1 v_2 v_3).$$

We now present some results of planar rational trigonometry which connect these fundamental quantities in three dimensions, with proof. These are generalisations of the results in [17, pp. 89-90] to an arbitrary symmetric bilinear form and to three dimensions.

**Theorem 17 (Cross law)** For a vector triangle $v_1v_2v_3$ with $B$-quadrances $Q_1$, $Q_2$ and $Q_3$, and corresponding $B$-spreads $s_1$, $s_2$ and $s_3$, we have

$$(Q_2 + Q_3 - Q_1)^2 = 4Q_2Q_3 (1 - s_1)$$

as well as

$$(Q_1 + Q_3 - Q_2)^2 = 4Q_1Q_3 (1 - s_2)$$

and

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2 (1 - s_3).$$

**Proof.** We just prove the last formula, the others follow by symmetry. Rearrange the polarisation formula to get

$$Q_1 + Q_2 - Q_3 = 2 (v_1 \cdot_B v_2)$$

and then square both sides to obtain

$$(Q_1 + Q_2 - Q_3)^2 = 4 (v_1 \cdot_B v_2)^2.$$

Rearrange the definition of $s_3$ to obtain

$$(v_1 \cdot_B v_2)^2 = Q_B (v_1) Q_B (v_2) (1 - s_3) = Q_1Q_2 (1 - s_3).$$

Putting these together we get

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2 (1 - s_3).$$

We use the Cross law as a fundamental building block for a number of other results. For instance, we can express the $B$-quadrea of a triangle in terms of its $B$-quadrances and $B$-spreads.

**Theorem 18 (Quadrea spread theorem)** For a vector triangle $v_1v_2v_3$ with $B$-quadrances $Q_1$, $Q_2$ and $Q_3$, corresponding $B$-spreads $s_1$, $s_2$ and $s_3$, and $B$-quadrea $A$, we have

$$A = 4Q_1Q_2s_3 = 4Q_1Q_3s_2 = 4Q_2Q_3s_1.$$

**Proof.** Rearrange the equation

$$(Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2 (1 - s_3)$$
from the Cross law as

\[ 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2s_3. \]

We recognize on the left hand side an asymmetric form of Archimedes’ function, so that

\[ (Q_1 + Q_2 + Q_3)^2 - 2 (Q_1^2 + Q_2^2 + Q_3^2) = 4Q_1Q_2s_3 \]

which is

\[ A(Q_1, Q_2, Q_3) = A = 4Q_1Q_2s_3. \]

We can use the Quadrea spread theorem to determine whether a vector triangle \( \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \) is degenerate, i.e. when the vectors \( \mathbf{v}_1, \mathbf{v}_2 \) and \( \mathbf{v}_3 \) are collinear.

**Theorem 19 (Triple quad formula)** Let \( \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \) be a vector triangle with \( B \)-quadrances \( Q_1, Q_2 \) and \( Q_3 \). If \( \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \) is degenerate then

\[ (Q_1 + Q_2 + Q_3)^2 = 2 (Q_1^2 + Q_2^2 + Q_3^2). \]

**Proof.** If \( \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \) is degenerate, then we may suppose that \( \mathbf{v}_2 \equiv \lambda \mathbf{v}_1 \), for some \( \lambda \) in \( F \), so that \( \mathbf{v}_3 = -(1 + \lambda) \mathbf{v}_1 \). Thus, by the properties of the \( B \)-quadratic form,

\[ Q_2 = \lambda^2 Q_1 \quad \text{and} \quad Q_3 = (1 + \lambda)^2 Q_1. \]

So,

\[
(Q_1 + Q_2 + Q_3)^2 - 2 (Q_1^2 + Q_2^2 + Q_3^2) \\
= \left[ (1 + \lambda^2 + (1 + \lambda)^2)^2 - 2 (1 + \lambda^4 + (1 + \lambda)^4) \right] Q_1^2 \\
= \left[ 4 (\lambda^2 + \lambda + 1)^2 - 4 (\lambda^2 + \lambda + 1)^2 \right] Q_1^2 \\
= 0.
\]

The result immediately follows. ■

The Cross law also gives the most important result in geometry and trigonometry: Pythagoras’ theorem.

**Theorem 20 (Pythagoras' theorem)** For a vector triangle \( \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \) with \( B \)-quadrances \( Q_1, Q_2 \) and \( Q_3 \), and corresponding \( B \)-spreads \( s_1, s_2 \) and \( s_3 \), we have \( s_3 = 1 \) precisely when

\[ Q_1 + Q_2 = Q_3. \]

**Proof.** The Cross law relation

\[ (Q_1 + Q_2 - Q_3)^2 = 4Q_1Q_2 (1 - s_3) \]
together with the assumption that $Q_1$ and $Q_2$ are non-zero implies that $s_3 = 1$ precisely when

$$Q_1 + Q_2 = Q_3.$$ 

One other use of the Quadrea spread theorem is in determining ratios between $B$-spreads and $B$-quadrances, which is a rational analog of the sine law in classical trigonometry.

**Theorem 21 (Spread law)** For a vector triangle $v_1v_2v_3$ with $B$-quadrances $Q_1$, $Q_2$ and $Q_3$, and corresponding $B$-spreads $s_1$, $s_2$ and $s_3$, and $B$-quadrea $A$, we have

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{A}{4Q_1Q_2Q_3}.$$ 

**Proof.** Rearrange the Quadrea spread theorem to get

$$s_1 = \frac{A}{4Q_2Q_3}, \quad s_2 = \frac{A}{4Q_1Q_3} \quad \text{and} \quad s_3 = \frac{A}{4Q_1Q_2}.$$ 

If the three $B$-spreads are defined, then necessarily all three $B$-quadrances are non-zero. So divide $s_1$, $s_2$ and $s_3$ by $Q_1$, $Q_2$ and $Q_3$ respectively to get

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{A}{4Q_1Q_2Q_3}$$

as required. 

Finally we present a result that gives a relationship between the three $B$-spreads of a triangle, following the proof in [17, pp. 89-90].

**Theorem 22 (Triple spread formula)** For a vector triangle $v_1v_2v_3$ with $B$-spreads $s_1$, $s_2$ and $s_3$, we have

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$ 

**Proof.** If one of the $B$-spreads is 0, then from the Spread law they will all be zero, and likewise with $A$; thus the formula is immediate in this case. Otherwise, with the $B$-quadrances $Q_1$, $Q_2$ and $Q_3$ of $v_1v_2v_3$ as previously defined, the Spread law allows us to define the non-zero quantity

$$D = \frac{4Q_1Q_2Q_3}{A},$$

so that

$$Q_1 = Ds_1, \quad Q_2 = Ds_2 \quad \text{and} \quad Q_3 = Ds_3.$$ 

We substitute these into the Cross law and pull out common factors to get

$$D^2(s_1 + s_2 - s_3)^2 = 4D^2s_1s_2(1 - s_3).$$ 

Now divide by $D^2$ and rearrange to obtain

$$4s_1s_2 - (s_1 + s_2 - s_3)^2 = 4s_1s_2s_3.$$ 

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We use the identity discussed earlier in the context of Archimedes’ function to rearrange this to get
\[(s_1 + s_2 + s_3)^2 = 2 \left( s_1^2 + s_2^2 + s_3^2 \right) + 4s_1s_2s_3. \]

4 Rational trigonometry for a projective triangle

Rational trigonometry has an affine and projective version. The projective version is typically more algebraically involved. The distinction was first laid out in [18] by framing hyperbolic geometry in a projective setting, and is generally summarised in [16]. So, the projective results are the essential formulas for the rational trigonometric approach to both hyperbolic and spherical, or elliptic, trigonometry. In this paper, the spherical or elliptic interpretation is primary, as it is most easily accessible from a Euclidean orientation.

A projective vector \( p = [v] \) is an expression involving a non-zero vector \( v \) with the convention that \( [v] = [\lambda v] \) for any non-zero number \( \lambda \). If \( v = (a, b, c) \) then we will write \([v] = [a : b : c]\) since it is only the proportion between these three numbers that is important. Clearly a projective vector can be identified with a one-dimensional subspace of \( V^3 \), but it will not be necessary to do so.

A projective triangle, or tripod, is a set of three distinct projective vectors, namely \( \{p_1, p_2, p_3\} = \{[v_1], [v_2], [v_3]\} \) which we will write as \( p_1p_2p_3 \). Such a projective triangle is degenerate precisely when \( v_1, v_2 \) and \( v_3 \) are linearly dependent.

If \( p_1 = [v_1] \) and \( p_2 = [v_2] \) are projective points, then we define the \( B \)-normal of \( p_1 \) and \( p_2 \) to be the projective point
\[ p_1 \times_B p_2 \equiv [v_1 \times_B v_2] \]
and this is clearly well-defined. We now define the \( B \)-dual of the non-degenerate tripod \( \overline{p_1p_2p_3} \) to be the tripod \( \overline{r_1r_2r_3} \), where
\[ r_1 \equiv p_2 \times_B p_3, \quad r_2 \equiv p_1 \times_B p_3 \quad \text{and} \quad r_3 \equiv p_1 \times_B p_2. \]

Such a tripod will also be called the \( B \)-dual projective triangle \([18]\) of the projective triangle \( p_1p_2p_3 \). The following result highlights the two-fold symmetry of such a concept.

**Theorem 23** If the \( B \)-dual of the non-degenerate tripod \( \overline{p_1p_2p_3} \) is \( \overline{r_1r_2r_3} \), then the \( B \)-dual of the tripod \( \overline{r_1r_2r_3} \) is \( \overline{p_1p_2p_3} \).

**Proof.** Let \( p_1 \equiv [v_1], p_2 \equiv [v_2] \) and \( p_3 \equiv [v_3] \), so that
\[ r_1 = p_2 \times_B p_3 = [v_2 \times_B v_3], \quad r_2 = p_1 \times_B p_3 = [v_1 \times_B v_3] \]
and
\[ r_3 = p_1 \times_B p_2 = [v_1 \times_B v_2]. \]
Suppose that the $B$-dual of $\overrightarrow{r_1r_2r_3}$ is given by $t_1t_2t_3$, where

$$t_1 \equiv r_2 \times_B r_3, \quad t_2 \equiv r_1 \times_B r_3 \quad \text{and} \quad t_3 \equiv r_1 \times_B r_2.$$ 

By the definition of the $B$-normal, we use Corollary 13 to get

$$t_1 = [(v_1 \times_B v_3) \times_B (v_1 \times_B v_2)] = [(\det B)[v_1, v_3, v_2]_B v_1].$$

Since $B$ is non-degenerate and the vectors $v_1, v_2$ and $v_3$ are linearly independent, $(\det B)[v_1, v_3, v_2]_B \neq 0$ and by the definition of a projective point

$$t_1 = [v_1] = p_1.$$ 

By symmetry, $t_2 = p_2$ and $t_3 = p_3$, and hence $t_1t_2t_3 = p_1p_2p_3$. Thus, the $B$-dual of $\overrightarrow{r_1r_2r_3}$ is $p_1p_2p_3$.

The $B$-projective quadrance between two projective vectors $p_1 = [v_1]$ and $p = [v_2]$ is

$$q_B(p_1, p_2) \equiv s_B(v_1, v_2).$$

Clearly a $B$-projective quadrance is just the $B$-spread between the corresponding vectors. So, the following result should not be a surprise.

**Theorem 24 (Projective triple quad formula)** If $\overrightarrow{p_1p_2p_3}$ is a degenerate tripod with projective quadrances

$$q_1 \equiv q_B(p_2, p_3), \quad q_2 \equiv q_B(p_1, p_3) \quad \text{and} \quad q_3 \equiv q_B(p_1, p_2),$$

then

$$(q_1 + q_2 + q_3)^2 = 2(q_1^2 + q_2^2 + q_3^2) + 4q_1q_2q_3.$$ 

The Projective triple quad formula is analogous and parallel to the Triple spread formula in affine rational trigonometry, due to this fact.

Given a projective triangle $\overrightarrow{p_1p_2p_3}$ it will have three projective quadrances $q_1, q_2$ and $q_3$. We now introduce the projective spreads $S_1, S_2$ and $S_3$ of the projective triangle $\overrightarrow{p_1p_2p_3}$ to be the projective quadrances of its $B$-dual $\overrightarrow{r_1r_2r_3}$, that is

$$S_1 \equiv q_B(r_2, r_3), \quad S_2 \equiv q_B(r_1, r_3) \quad \text{and} \quad S_3 \equiv q_B(r_1, r_2).$$

We now proceed to present results in projective rational trigonometry, which draw on the results from [10], but will be framed in the three-dimensional framework using $B$-vector products and a general symmetric bilinear form.

**Theorem 25 (Projective spread law)** Given a tripod $\overrightarrow{p_1p_2p_3}$ with $B$-projective quadrances $q_1$, $q_2$ and $q_3$, and $B$-projective spreads $S_1, S_2$ and $S_3$, we have

$$\frac{S_1}{q_1} = \frac{S_2}{q_2} = \frac{S_3}{q_3}.$$
Theorem 26 (Projective cross law)

Let \( p_1 \equiv [v_1] \), \( p_2 \equiv [v_2] \) and \( p_3 \equiv [v_3] \) be the points of \( p_1 p_2 p_3 \). Also consider its \( B \)-dual \( \overline{1123} \), where
\[
    r_1 \equiv [v_2 \times_B v_3], \quad r_2 \equiv [v_1 \times_B v_3] \quad \text{and} \quad r_3 \equiv [v_1 \times_B v_2].
\]

By the definition of the \( B \)-projective quadrance,
\[
    q_1 = \frac{Q_B(v_2 \times_B v_3)}{(\det B) Q_B(v_2) Q_B(v_3)}
\]
and similarly
\[
    q_2 = \frac{Q_B(v_1 \times_B v_3)}{(\det B) Q_B(v_1) Q_B(v_3)} \quad \text{and} \quad q_3 = \frac{Q_B(v_1 \times_B v_2)}{(\det B) Q_B(v_1) Q_B(v_2)}.
\]

By the definition of the \( B \)-projective spread and Corollary 13,
\[
    S_1 = \frac{Q_B([v_1, v_2, v_3]_B)}{(\det B) Q_B(v_1 \times_B v_2) Q_B(v_1 \times_B v_3)} = \frac{(\det B) [v_1, v_2, v_3]_B^2 Q_B(v_1)}{Q_B(v_1 \times_B v_2) Q_B(v_1 \times_B v_3)}
\]
and similarly
\[
    S_2 = \frac{(\det B) [v_1, v_2, v_3]_B^2 Q_B(v_2)}{Q_B(v_1 \times_B v_2) Q_B(v_1 \times_B v_3)} \quad \text{and} \quad S_3 = \frac{(\det B) [v_1, v_2, v_3]_B^2 Q_B(v_3)}{Q_B(v_1 \times_B v_2) Q_B(v_1 \times_B v_3)}.
\]

So,
\[
    \frac{S_1}{q_1} = \frac{(\det B)^2 [v_1, v_2, v_3]_B^2 Q_B(v_1) Q_B(v_2) Q_B(v_3)}{Q_B(v_1 \times_B v_2) Q_B(v_1 \times_B v_3) Q_B(v_1 \times_B v_3)} = \frac{S_2}{q_2} = \frac{S_3}{q_3}
\]
as required. \( \blacksquare \)

If we balance each side of the result of the Projective spread law to its lowest common denominator, multiplying through by the denominator motivates us to define the quantity
\[
    S_1 q_2 q_3 = S_2 q_1 q_3 = S_3 q_1 q_2 \equiv a_B (p_1 p_2 p_3) \equiv a_B,
\]
which will be called the \( B \)-projective quadrea of the tripod \( p_1 p_2 p_3 \).

There is a relationship between the \( B \)-projective quadrea and the projective quadrae discovered in [16], which is central to our study of projective rational trigonometry. We extend this result to an arbitrary symmetric bilinear form, using a quite different argument.

Theorem 26 (Projective cross law) Given a tripod \( p_1 p_2 p_3 \) with \( B \)-projective quadrae \( q_1, q_2 \) and \( q_3 \), \( B \)-projective spreads \( S_1, S_2 \) and \( S_3 \), and \( B \)-projective quadrea \( a_B \),
\[
    (a_B - q_1 - q_2 - q_3 + 2)^2 = 4 (1 - q_1) (1 - q_2) (1 - q_3).
\]

Proof. Let \( p_1 \equiv [v_1] \), \( p_2 \equiv [v_2] \) and \( p_3 \equiv [v_3] \) be the points of \( p_1 p_2 p_3 \) and \( \overline{1123} \) be the \( B \)-dual of \( p_1 p_2 p_3 \), so that from the proof of the Projective spread law the \( B \)-projective quadrae and spreads are
\[
    q_1 = \frac{Q_B(v_2 \times_B v_3)}{(\det B) Q_B(v_2) Q_B(v_3)}, \quad q_2 = \frac{Q_B(v_1 \times_B v_3)}{(\det B) Q_B(v_1) Q_B(v_3)}, \quad q_3 = \frac{Q_B(v_1 \times_B v_2)}{(\det B) Q_B(v_1) Q_B(v_2)},
\]
with
Given that
\[ S_1 = \frac{(\det B)[v_1, v_2, v_3]^2}{Q_B(v_1 \times B v_2) Q_B(v_1 \times B v_3)} \]
\[ S_2 = \frac{(\det B)[v_1, v_2, v_3]^2}{Q_B(v_1 \times B v_2) Q_B(v_2 \times B v_3)} \]
and
\[ S_3 = \frac{(\det B)[v_1, v_2, v_3]^2}{Q_B(v_1 \times B v_3) Q_B(v_2 \times B v_3)}. \]

Furthermore,
\[ a_B = \frac{S_1 q_2 q_3 = S_2 q_1 q_3 = S_3 q_1 q_2}{(\det B) Q_B(v_1) Q_B(v_2) Q_B(v_3)}. \]

Noting that
\[ 1 - q_1 = \frac{(v_2 \cdot B v_3)^2}{Q_B(v_2) Q_B(v_3)}, \quad 1 - q_2 = \frac{(v_1 \cdot B v_3)^2}{Q_B(v_1) Q_B(v_3)} \quad \text{and} \quad 1 - q_3 = \frac{(v_1 \cdot B v_2)^2}{Q_B(v_1) Q_B(v_2)} \]
we see from the Scalar triple product theorem that
\[ (a_B - q_1 - q_2 - q_3 + 2)^2 = \left( \frac{(\det M)^2 \det B}{Q_B(v_1) Q_B(v_2) Q_B(v_3)} + (1 - q_1) + (1 - q_2) + (1 - q_3) - 1 \right)^2 \]
\[ = \left( \frac{(\det M)^2 \det B}{Q_B(v_1) Q_B(v_2) Q_B(v_3)} + \frac{(v_2 \cdot B v_3)^2}{Q_B(v_2) Q_B(v_3)} + \frac{(v_1 \cdot B v_3)^2}{Q_B(v_1) Q_B(v_3)} + \frac{(v_1 \cdot B v_2)^2}{Q_B(v_1) Q_B(v_2)} - 1 \right)^2 \]
where
\[ M \equiv \begin{pmatrix} - & v_1 & - \\ - & v_2 & - \\ - & v_3 & - \end{pmatrix}. \]

Given that
\[ (\det M)^2 \det B = \det (MBM^T) \]
\[ = \det \begin{pmatrix} Q_B(v_1) & v_1 \cdot B v_2 & v_1 \cdot B v_3 \\ v_1 \cdot B v_2 & Q_B(v_2) & v_2 \cdot B v_3 \\ v_1 \cdot B v_3 & v_2 \cdot B v_3 & Q_B(v_3) \end{pmatrix} \]
\[ = Q_B(v_1) Q_B(v_2) Q_B(v_3) + 2(v_1 \cdot B v_2)(v_1 \cdot B v_3)(v_2 \cdot B v_3) \]
\[ - (v_1 \cdot B v_2)^2 Q_B(v_3) - (v_1 \cdot B v_3)^2 Q_B(v_2) - (v_2 \cdot B v_3)^2 Q_B(v_1) \]
we obtain
\[ (a_B - q_1 - q_2 - q_3 + 2)^2 = \left( \frac{2(v_1 \cdot B v_2)(v_1 \cdot B v_3)(v_2 \cdot B v_3)}{Q_B(v_1) Q_B(v_2) Q_B(v_3)} \right)^2 \]
\[ = 4 \left( \frac{(v_1 \cdot B v_2)^2}{Q_B(v_1) Q_B(v_2)} \right) \left( \frac{(v_1 \cdot B v_3)^2}{Q_B(v_1) Q_B(v_3)} \right) \left( \frac{(v_2 \cdot B v_3)^2}{Q_B(v_2) Q_B(v_3)} \right) \]
\[ = 4(1 - q_1)(1 - q_2)(1 - q_3) \]
as required. ■

The Projective cross law can also be expressed in various asymmetric forms.

**Corollary 27** Given a tripod \( \overline{p_1 p_2 p_3} \) with \( B \)-projective quadrances \( q_1, q_2 \) and \( q_3 \), and \( B \)-projective spreads \( S_1, S_2 \) and \( S_3 \), the Projective cross law can be rewritten as either

\[
(S_1 q_2 q_3 + q_1 - q_2 - q_3)^2 = 4q_2 q_3 (1 - q_1) (1 - S_1),
\]
\[
(S_2 q_1 q_3 - q_1 + q_2 - q_3)^2 = 4q_1 q_3 (1 - q_2) (1 - S_2)
\]

or

\[
(S_3 q_1 q_2 - q_1 - q_2 + q_3)^2 = 4q_1 q_2 (1 - q_3) (1 - S_3).
\]

**Proof.** Let \( C_1 \equiv 1 - S_1 \). Substitute \( a_B = S_1 q_2 q_3 = (1 - C_1) q_2 q_3 \) into the Projective cross law to get

\[
((1 - C_1) q_2 q_3 - q_1 - q_2 - q_3 + 2)^2 - 4 (1 - q_1) (1 - q_2) (1 - q_3) = 0.
\]

Expand the left-hand side and simplify the result as a polynomial in \( C_1 \) to obtain

\[
(q_2^2 q_3^2) C_1^2 + 2 q_2 q_3 (q_2 + q_3 - q_1 - q_2 q_3 - 2) C_1 + (q_2 q_3 + q_1 - q_2 - q_3)^2 = 0.
\]

Now insert the term \(-4q_1 q_2 q_3 C_1 \) into the last equation and balance the equation as required. Further rearrange to get

\[
(q_2^2 q_3^2) C_1^2 + 2 q_2 q_3 (q_2 + q_3 - q_1 - q_2 q_3) C_1 + (q_2 q_3 + q_1 - q_2 - q_3)^2
\]
\[
= 4 q_2 q_3 C_1 - 4 q_1 q_2 q_3 C_1 = 4 q_2 q_3 (1 - q_1) C_1.
\]

As the left-hand side is a perfect square, factorise this to get

\[
(q_2 + q_3 - q_1 - q_2 q_3 + q_2 q_3 C_1)^2 = 4 q_2 q_3 (1 - q_1) C_1.
\]

Replace \( C_1 \) with \( 1 - S_1 \) and simplify to obtain

\[
(q_2 + q_3 - q_1 - q_2 q_3 + (1 - S_1) q_2 q_3)^2 = 4 q_2 q_3 (1 - q_1) (1 - S_1).
\]

The other results follow by symmetry. ■

Note that the \( B \)-projective quadrea \( a_B \) also features in the reformulation, and can replace the quantity \( S_1 q_2 q_3 \) (as well as its symmetrical reformulations). Also, the Projective triple quad formula from earlier follows directly from the Projective cross law, so it can be proven in this way; this will be omitted from the paper.

In addition to the \( B \)-projective quadrea, we can also discuss the dual analog of it. This quantity is called the \( B \)-quadreal \([18]\) and is defined by

\[
l_B \equiv l_B (\overline{p_1 p_2 p_3}) \equiv q_1 S_2 S_3 = q_2 S_1 S_3 = q_3 S_1 S_2.
\]
We can also say from Theorem 23 that \( l_B \) is the \( B \)-projective quadrea of the \( B \)-dual tripod \( r_1 r_2 r_3 \) of \( p_1 p_2 p_3 \) and \( a_B \) is the \( B \)-quadreal of \( r_1 r_2 r_3 \). The following extends the result in [18] for \( B \)-quadratic forms.

**Corollary 28** For a tripod \( p_1 p_2 p_3 \) with \( B \)-projective quadrances \( q_1, q_2 \) and \( q_3 \), \( B \)-projective spreads \( S_1, S_2 \) and \( S_3 \), \( B \)-projective quadrea \( a_B \) and \( B \)-quadreal \( l_B \),

\[
a_B l_B = q_1 q_2 q_3 S_1 S_2 S_3.
\]

**Proof.** Given

\[
a_B = S_1 q_2 q_3 = S_2 q_1 q_3 = S_3 q_1 q_2
\]

and

\[
l_B = q_1 S_2 S_3 = q_2 S_1 S_3 = q_3 S_1 S_2,
\]

we get

\[
a_B l_B = (S_1 q_2 q_3) (q_1 S_2 S_3) = (S_2 q_1 q_3) (q_2 S_1 S_3)
\]

\[
= (S_3 q_1 q_2) (q_3 S_1 S_2) = q_1 q_2 q_3 S_1 S_2 S_3,
\]

as required.

We now present a projective version of Pythagoras’ theorem. This is an extension of the result in [16] and [18] to arbitrary symmetric bilinear forms.

**Theorem 29 (Projective Pythagoras’ theorem)** Take a tripod \( p_1 p_2 p_3 \) with \( B \)-projective quadrances \( q_1, q_2 \) and \( q_3 \), \( B \)-projective spreads \( S_1, S_2 \) and \( S_3 \). If \( S_1 = 1 \), then

\[
q_1 = q_2 + q_3 - q_2 q_3.
\]

**Proof.** Substitute \( S_1 = 1 \) into the Projective cross law

\[
(S_1 q_2 q_3 - q_1 - q_2 - q_3 + 2)^2 = 4 (1 - q_1) (1 - q_2) (1 - q_3)
\]

and rearrange the result to get

\[
(q_2 q_3 - q_1 - q_2 - q_3 + 2)^2 - 4 (1 - q_1) (1 - q_2) (1 - q_3) = 0.
\]

The left-hand side is factored into

\[
(q_2 + q_3 - q_1 - q_2 q_3)^2 = 0,
\]

so that solving for \( q_1 \) gives

\[
q_1 = q_2 + q_3 - q_2 q_3.
\]

Note the term \(-q_2 q_3\) involved in the Projective Pythagoras’ theorem; this is not present in Pythagoras’ theorem in affine rational trigonometry. The Projective Pythagoras’ theorem can be restated [18]...
as

\begin{align*}
1 - q_1 &= 1 - q_2 - q_3 + q_2q_3 \\
&= (1 - q_2)(1 - q_3).
\end{align*}

As for the converse of the Projective Pythagoras’ theorem, start with the asymmetric form of the Projective cross law

\[(S_1q_2q_3 + q_1 - q_2 - q_3)^2 = 4q_2q_3(1 - q_1)(1 - S_1).\]

If \( q_1 = q_2 + q_3 - q_2q_3 \) then

\[(q_2q_3(1 - S_1))^2 = 4q_2q_3(1 - q_2)(1 - q_3)(1 - S_1),\]

which can also be rearranged and factorised as

\[q_2q_3(1 - S_1)(4q_2 + 4q_3 - 3q_2q_3 - S_1q_2q_3 - 4) = 0.\]

Here, we see that \( S_1 = 1 \) is not the only solution; we can also have the solution

\[S_1 = \frac{4(q_2 + q_3 - 1)}{q_2q_3} - 3.\]

So, the converse of the Projective Pythagoras’ theorem may not necessarily hold. It is of independent interest to deduce the meaning of the latter solution for \( S_1 \).

5 The methane molecule for chemists

To illustrate the practical aspect and attractive values that this technology gives, we apply the results of this paper to the methane molecule \( \text{CH}_4 \) consisting of four hydrogen atoms arranged in the form of a regular tetrahedron, and a central carbon atom. The geometry of this configuration is well-known to chemists, at least using the classical measurements; however with rational trigonometry a new picture emerges which illustrates the advantages of thinking algebraically.

Note that we do not assume a particular field here; to make things precise mathematically we would need an appropriate quadratic extension of the rationals to fix the vectors in an appropriate vector space, but we do work over the familiar Euclidean geometry, so we remove the \( B \) prefix from the subsequent discussion. Once we have built the regular tetrahedron, all the measurements are rational expressions in the common quadrance of the six sides, which we will denote by \( Q \). Then the faces are equilateral triangles with quadreas

\[A = A(Q, Q, Q) = (3Q)^2 - 2(3Q^2) = 3Q^2\]

which is 16 times the square of the classical area. The spreads \( s \) in any such equilateral triangle satisfy the Cross law

\[(Q + Q - Q)^2 = 4Q \times Q \times (1 - s)\]
so that
\[ s = \frac{3}{4} \]
which is the (rational) analog of approximately 1.047 20 in the radian system, or exactly 60° in the much more ancient Babylonian system.

This will also be the projective quadrance \( q \) of any two sides meeting at a common vertex, so \( q = 3/4 \). Three such concurrent sides gives an equilateral projective triangle, and the projective formulas we have developed apply also to the (equal) projective spreads \( S \) of this projective triangle. In particular the Projective cross law in terms of the projective quadrea \( a \) gives

\[
\left( a - 3 \times \frac{3}{4} + 2 \right)^2 = 4 \left( 1 - \frac{3}{4} \right)^3
\]

or

\[
\left( a - \frac{1}{4} \right)^2 = \frac{1}{16}
\]

and since \( a \neq 0 \) we get

\[ a = \frac{1}{2}. \]

This quantity can be considered as the solid spread formed by the three lines meeting at a vertex, which is a rational analog of the solid angle of spherical trigonometry. But from the definition of the projective quadrea, and the symmetry of the Projective spread law, we deduce that \( a = Sq^2 \) so that

\[ S = \frac{8}{9}. \]

Geometrically this is the spread between two faces of the tetrahedron, which is the (rational) analog of approximately \( \arcsin \left( \sqrt{8/9} \right) \approx 1.23096 \) in the radian system, or approximately 70.528 8° in the Babylonian system. The supplement of this angle, which is approximately 109.471°, corresponds to the same spread of 8/9, which geometrically is formed by the central lines from the carbon atom to any two hydrogen atom. So we see that the rational trigonometry of this paper allows more natural, rational and exact expressions for an important measurement in chemistry, with the calculations in the realm of high school algebra, without use of a calculator. This is a powerful indicator that for more complicated calculations, we can expect a significant speed up of processing by adopting the language and concepts of rational trigonometry.

In our follow up paper we will be investigating the rich trigonometry of a general tetrahedron, for which this is just a very simple example.

6 Affine and projective relativistic trigonometry in relativistic geometries

To illustrate both affine and projective formulas in a less symmetrical situation, we shift to a relativistic-three dimensional geometry, and consider two examples of triangles: one of a vector triangle in \( \mathbb{V}^3 \) and the other of a projective triangle in \( \mathbb{P}^2 \), both over the rational number field. Here the metric structure
is given by the Minkowski scalar product \[13\] on \( \mathbb{V}^3 \) defined by
\[
(x_1, y_1, z_1) \cdot_B (x_2, y_2, z_2) \equiv x_1x_2 + y_1y_2 - z_1z_2.
\]
The matrix
\[
B \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]
which represents this symmetric bilinear form, is often called the relativistic bilinear form.

**Affine relativistic example**

In the first example, consider the vector triangle \( \overline{v_1v_2v_3} \) with
\[
v_1 \equiv (-1, 3, -2), \quad v_2 \equiv (2, -5, 4) \quad \text{and} \quad v_3 \equiv -v_1 - v_2 = (-1, 2, -2).
\]
The \( B \)-quadrances of \( \overline{v_1v_2v_3} \) are
\[
Q_1 = ((-1, 3, -2) \cdot_B (1, 0, 0)) = 6,
\]
and similarly
\[
Q_2 = 13 \quad \text{and} \quad Q_3 = 1.
\]
The \( B \)-quadrea of \( \overline{A_0A_1A_2} \) is then
\[
A = (6 + 13 + 1)^2 - 2 (6^2 + 13^2 + 1^2) = -12,
\]
and hence, by the Quadrea spread theorem, the \( B \)-spreads are
\[
s_1 = \frac{-12}{4 \times 13 \times 1} = -3, \quad s_2 = \frac{-12}{4 \times 6 \times 1} = -\frac{1}{2} \quad \text{and} \quad s_3 = \frac{-12}{4 \times 6 \times 13} = -\frac{1}{26}.
\]
To verify our calculations, we observe that
\[
\frac{s_1}{Q_1} = -\frac{3}{13 \times 6} = -\frac{1}{26}, \quad \frac{s_2}{Q_2} = -\frac{1}{13 \times 2} = -\frac{1}{26} \quad \text{and} \quad \frac{s_3}{Q_3} = -\frac{1}{26 \times 1} = -\frac{1}{26}.
\]
As each ratio is equal, the Spread law holds for \( \overline{A_0A_1A_2} \). Furthermore,
\[
(s_0 + s_1 + s_2)^2 - 2 (s_0^2 + s_1^2 + s_2^2) = \left( -\frac{3}{13} - \frac{1}{2} - \frac{1}{26} \right)^2 - 2 \left( \left( -\frac{3}{13} \right)^2 + \left( -\frac{1}{2} \right)^2 + \left( -\frac{1}{26} \right)^2 \right)
\]
\[
= -\frac{3}{169}
\]
and
\[4s_0s_1s_2 = 4 \left(-\frac{3}{13}\right) \left(-\frac{1}{2}\right) \left(-\frac{1}{26}\right) = -\frac{3}{169}.\]

Because of the equality of these two identities, the Triple spread formula thus holds.

**Projective relativistic example**

Now for a second projective example, let
\[v_1 \equiv (2, -1, 3), \quad v_2 \equiv (-2, 5, 0) \quad \text{and} \quad v_3 \equiv (3, 0, 4)\]
be three vectors in \(V^3\), so that we may define \(\overline{p_1p_2p_3}\) to be a projective triangle in \(P^2\) with projective points
\[p_1 \equiv [v_1], \quad p_2 \equiv [v_2] \quad \text{and} \quad p_3 \equiv [v_3].\]

The \(B\)-projective quadrances of \(\overline{p_1p_2p_3}\) are
\[q_1 = 1 - \left(-\frac{6}{29}\right)^2 = \frac{239}{203}\]
and similarly
\[q_2 = -\frac{2}{7} \quad \text{and} \quad q_3 = \frac{197}{116}.

Let \(r_1r_2r_3\) be the \(B\)-dual of \(\overline{p_1p_2p_3}\), so that
\[r_1 = [(20, 8, 15)], \quad r_2 = [(4, -1, 3)] \quad \text{and} \quad r_3 = [(15, 6, 8)].\]

Then, the \(B\)-projective spreads of \(\overline{p_1p_2p_3}\) are
\[S_1 = 1 - \frac{30^2}{197 \times 8} = \frac{169}{394},\]
and similarly
\[S_2 = -\frac{4901}{47083} \quad \text{and} \quad S_3 = \frac{1183}{1912}.

To verify our calculations, we observe that
\[\frac{S_1}{q_1} = \frac{169}{394} \div \frac{239}{203} = \frac{34307}{94166},\]
and similarly
\[\frac{S_2}{q_2} = \left(-\frac{4901}{47083}\right) \div \left(-\frac{2}{7}\right) = \frac{34307}{94166} \quad \text{and} \quad \frac{S_3}{q_3} = \frac{1183}{1912} \div \frac{197}{116} = \frac{34307}{94166}.

Since they are all equal, the Projective spread law holds. Furthermore, with the \(B\)-projective quadra of \(\overline{p_1p_2p_3}\) given as
\[a_B = \frac{197}{116} \times \left(-\frac{2}{7}\right) \times \frac{169}{394} = -\frac{169}{812},\]

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we observe that
\[
\begin{align*}
&(a_B - q_{12} - q_{13} - q_{23} + 2)^2 \\
&= \left( -\frac{169}{812} - \frac{197}{116} + \frac{2}{7} - \frac{239}{203} + 2 \right)^2 \\
&= \frac{26244}{41209}
\end{align*}
\]
and
\[
\begin{align*}
4(1 - q_{12})(1 - q_{13})(1 - q_{23}) \\
&= 4 \left( 1 - \frac{197}{116} \right) \left( 1 + \frac{2}{7} \right) \left( 1 - \frac{239}{203} \right) \\
&= \frac{26244}{41209}.
\end{align*}
\]
As we have equality, the Projective cross law is verified. Note that for the $B$-dual tripod $r_{123}$ of $p_{123}$, its $B$-projective quadrances are the $B$-projective spreads of $p_{123}$ and its $B$-projective spreads are the $B$-projective quadrances of $p_{123}$. Furthermore, the $B$-quadreal of $p_{123}$, which is
\[
l_B = \frac{169}{394} \times \left( -\frac{4901}{47083} \right) \times \frac{197}{116} = \frac{-28561}{376664}
\]
is the $B$-projective quadrea of $r_{123}$, and the $B$-quadreal of $r_{123}$ is the $B$-projective quadrea of $p_{123}$.

7 Using general metrics for linear algebra problems

We make just a few simple observations that give more motivation for the utility of having a general trigonometry valid for an arbitrary quadratic form. Suppose we are working in three-dimensional Euclidean space $E^3$, and we have an application that crucially involves a linear change of coordinates, given by a linear transformation $E : V^3 \to V^3$ represented by a matrix of the same name.

In classical geometry, we may find it awkward to make the transition to this new change of coordinates if we are primarily interested in metrical properties, for the simple reason that
\[
(v_1 L) \cdot (v_2 L) = (v_1 L)(v_2 L)^T = v_1 B v_2^T = v_1 \cdot_B v_2
\]

involves a new scalar product associated to the matrix
\[
B \equiv LL^T.
\]

But with the set up of this paper, we are completely in control of such a general metrical situation, so we may apply the desired linear transformation without weakening our ability to apply the powerful geometrical tools provided by the $B$-vector calculus.

In effect we are introducing the metric as a variable quantity into our geometrical three-dimensional theories. This is a familiar approach in modern differential geometry, but it has seen little development
in classical geometry.

As a simple example, suppose we are interested in the geometry of a lattice in three dimensions and associated theta functions. It is traditional to consider different lattices, but always with respect to a Euclidean quadratic form. With this more general technology, we may perform a linear transformation so that the lattice itself is just the standard lattice, and all the complexity and variability is then inherent in the now variable quadratic form.

8 Further applications to the trigonometry of a tetrahedron

The affine and projective triangle geometries are both necessary ingredients for the systematic rational investigation of a tetrahedron in three-dimensional affine/vector space. As an elementary example, we may use the $B$-quadrance to develop a new metrical quantity associated to a general tetrahedron itself. We can also use the projective tools developed in this paper to develop two new affine metrical quantities: a $B$-dihedral spread and a $B$-solid spread. These two quantities extend the projective notions of $B$-projective spread and $B$-projective quadrea respectively into the three-dimensional affine space. We will deal with these quantities and their results in a more formal way in the subsequent paper Generalised vector products applied to affine rational trigonometry of a general tetrahedron.

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