A HAMILTONIAN FORMULATION OF TOPOLOGICAL GRAVITY*

H. WAELBROECK† and J. A. ZAPATA‡

Institute for Theoretical Physics,
University of California at Santa Barbara,
Santa Barbara, CA.

March 24, 2022

Abstract

Topological gravity is the reduction of Einstein’s theory to space-times with vanishing curvature, but with global degrees of freedom related to the topology of the universe. We present an exact Hamiltonian lattice theory for topological gravity, which admits translations of the lattice sites as a gauge symmetry. There are additional symmetries, not present in Einstein’s theory, which kill the local degrees of freedom. We show that these symmetries can be fixed by choosing a gauge where the torsion is equal to zero. In this gauge, the theory describes flat space-times. We propose two methods to advance towards the holy grail of lattice gravity: A Hamiltonian lattice theory for curved space-times, with first-class translation constraints.

*This research was supported in part by the National Science Foundation Grant No. PHY89-04035, by CONACyT (Mexico) and by the A.G.C.D. (Belgium).

†Permanent address: Inst. C. Nucleares, UNAM, Circ Ext, C.U., Mexico D.F. 04510

‡P. a.: Dept. of Physics, Penn. St. University, 104 Davey Lab, Univ. Park, PA 16802
1 Introduction

The nonlinearity of Einstein’s equations imply that, in many situations of physical interest, one must turn to nonperturbative methods, typically numerical computations based on a lattice theory [1]. For similar reasons, perturbative quantum gravity is plagued with nonrenormalizable infinities [2], a fact which has convinced many that a nonperturbative approach [3] must be developed before one can even discuss the existence of the theory. In view of the difficulties associated with nonperturbative quantum field theory, physicists have turned increasingly to alternative discrete theories [4], among them Hamiltonian lattice theories [5].

In both, numerical relativity, and lattice theories of quantum gravity, a key problem has been that of finding constraints, analogous to the diffeomorphism constraints $\mathcal{H}_\mu(x) \approx 0$, that are consistent with time evolution even for a finite lattice. In the constrained Hamiltonian formalism, this requires that the constraints form a closed, or “first-class”, algebra [6]. Indeed,

$$[H, \mathcal{H}_\nu] = \left[ \sum_x N^\mu(x) \mathcal{H}_\mu(x), \mathcal{H}_\nu \right]$$

which vanishes for all $\{x, N^\mu(x)\}$ if and only if $[\mathcal{H}_\mu, \mathcal{H}_\nu] \approx 0$.

One can achieve consistency with time evolution for fixed $N^\mu(x)$ by solving (I) for these lapse-shift functions. However, this would increase the number of degrees of freedom per point, since the number of symmetries would be reduced. Also when the lattice brackets analogous to $[\mathcal{H}_\mu, \mathcal{H}_\nu]$ are non-zero, they are usually very small, so that $\mathcal{H}_\mu$ generates a flow which is almost a symmetry of the action, which will be nearly flat along this flow. This causes a problem both for the convergence of numerical relativity schemes, and for the calculation of the quantum gravity path integral. In light of these facts, the task of finding a Hamiltonian lattice theory with first-class constraints has drawn a great deal of attention in the past decades [5]. Believing that the problem is too difficult to tackle head on, we have turned our attention first to the model of 2+1-dimensional gravity [7], [8], since in this model the space-time curvature vanishes [9], and the quantum theory can be formulated elegantly [10]. We then showed how curvature could be introduced through
the reality conditions in the complexified theory \[11\]. We are now aiming to generalize this work to 3+1-dimensional space-times.

The purpose of this article is to present a lattice theory for topological gravity, with first-class constraints. To achieve this, we generalize previous work in 2+1 dimensions by exploiting Horowitz’s results, which show that a generalization of Witten’s work from 2+1 to 3+1 dimensions \[12\], leads to a topological theory for flat spacetimes coupled to closed two-forms.

Horowitz proposed going from the Palatini action for Einstein gravity \[13\],

\[
S = \int_M e^a \wedge e^b \wedge R^{cd} \epsilon_{abcd},
\]

(2)

to the topological action, replacing \((e \wedge e)^{*}\) by a two form \(B\).

\[
S_T = \int_M B_{cd} \wedge R^{cd}.
\]

(3)

One can consider (3) as an action functional over the fields \(B\) and an \(SO(3,1)\) connection \(\omega\) \((R = d\omega + [\omega, \omega])\), and drop the constraint

\[
B^* = e \wedge e.
\]

(4)

The action (3) should give a larger set of solutions than (2), since there are fewer constraints. Yet at first sight (3) has only global degrees of freedom. Indeed, the variational equations from (3) state that \(\omega\) is a flat \(SO(3,1)\) connection, and \(B\) a closed two-form with values in \(SO(3,1)\). Furthermore, (3) is invariant under the usual Maxwell gauge transformations and under the translations of \(B\) by an exact form. Thus, the space of solutions is the set of inequivalent flat \(SO(3,1)\) connections on \(M\), together with the inequivalent closed two-forms. These are related to the first and second cohomology vector spaces \(H_1(M)\) and \(H_2(M)\). For \(M = \Sigma \times (0,1)\) the existence of a canonical structure rests on the Poincaré duality between the vector spaces \(H_1(\Sigma)\) and \(H_2(\Sigma)\). The local degrees of freedom of (2), that do not appear in the topological action (3), are hidden in the torsion, as Horowitz pointed out. From this point of view, topological gravity is seen to be a theory of teleparallelism \[14\], where one chooses a connection with torsion but no curvature, rather than the contrary\[\]
The lattice theory, analogous to Horowitz’s topological gravity, is based on the following lattice analogues of $\omega$ and $B$: A set of $SO(3,1)$ matrices, which define parallel-transport through each lattice face, and the area bivector of each face. The first-class constraints demand the closure of each lattice cell (the sum of area bivectors on the boundary of a cell must be zero) and the vanishing of the curvature at each bone of the lattice. The former generate $SO(3,1)$ transformations of the frame at each cell, while the latter split into two groups: The generators of spacetime translations at each lattice site, which reduce to $\mathcal{H}_\mu(x)$ in the continuum limit, and generators of additional symmetries which are responsible for eliminating the local degrees of freedom of Einstein’s theory. We will fix these last symmetries by demanding that the lattice analogue of torsion vanish, thereby reducing the theory to flat spacetimes.

The article is organized as follows. The lattice theory is introduced in section two, by constructing a lattice analogue of the action (3). In section three, we will impose “geometricity conditions,” which imply that the variables represent a flat spacetime (with no torsion). The possibility of extending this work to space times with curvature (or torsion) is discussed in the concluding section.

2 Topological Lattice Theory

For topological theories, which have only a finite number of degrees of freedom, it is logical to put the settings of the theory in a discrete context. We follow this approach, and formulate a theory for topological gravity that in a certain gauge has the interpretation of a geometrical lattice.

One can think of the variables of the discrete $3+1$-dimensional gravity as the result of integrating the continuum ones in a space-like direction. That is, instead of the connection $\omega$ we take Lorentz matrices $M$ which arise from integrating the connection in a space-like direction. Such matrices realize parallel transport along noninfinitesimal distances. In the place of the bivector valued two-form $B$ we consider its integral over noninfinitesimal areas, and denote these by $E$. The continuous variable $\omega_0$ has no discrete counterpart because

---

1 Given any pseudoriemannian metric one can construct a flat connection compatible with the metric; but it is usually a connection with torsion. This implies that every space time is a solution of (3); the converse is not necessarily true.
it describes a time-like aspect of the continuous theory and we are seeking a
discrete theory with continuous time. For the same reason the variable $B_{k0}$
has a discrete-continuous counterpart $B(I)^{[\mu\nu]} = (\int_{l(I)} e_i^{[\mu} dx^i) e_{0]}^{\nu] = A(I)^{[\mu} e_{0]}^{\nu]}$, were $A(I)$ is a spacelike noninfinitesimal segment. There are two important
continuous quantities which have discrete counterparts that may also be de-
duced from the discrete variables. The discrete counterpart of the curvature
two-form is its integral over non-infinitesimal surfaces dual to the segment
$I: \, P(I)$. In the place of the exterior derivative of the two-form $B$ we take
its integral over the boundary of a boundary of a non-infinitesimal volume $i: \, J(i)$.

If one separates the time and spatial directions and integrates by parts
one of the resulting terms, the continuum action for topological gravi
ty (5) can be rewritten in a form which makes it easier to change to the Hamiltonian
formalism and identify the discrete counterparts of each term.

$$S_T = 3 \int dt \int \Sigma (\omega^A_{[ij}] B_{jk]} A - R^A_{[ij} B_{k]0A} + \omega_0^A D_{[i} B_{jk]A}) dx^i dx^j dx^k \tag{5}$$

The Lagrangian has a natural discrete analog which will be our starting
point

$$L = \sum_{ij} E_{ij} \cdot \dot{\omega}_{ij} - \sum_I P(I) \cdot B(I) + \sum_i \omega(i)_0 \cdot J(i) \tag{6}$$

With the help of a little symbolic manipulation it is possible to rearrange the
first term:\

$$\omega = \ln M \tag{7}$$
$$\dot{\omega} = M^{-1} \dot{M} \tag{8}$$
$$L = \sum_{ij} C^B_{CA} E_{(ij)}^A M_{(ij)}^D B, \dot{M}_{(ij)}^C D + \ldots \tag{9}$$

where $C^B_{CA}$ are the structure constants of the Lie algebra of $SO(3,1)$; they
are also the generators of the Lorentz transformations for bivectors, i.e.

\footnote{Equation (8) is strictly correct only for an Abelian group, since it neglects the ordering ambiguity of the two matrices. However, it is easy to see that there are only two ways to write the dynamical term (4) considering that indices can be contracted only if they live in the same frame, the only other possibility being equivalent to this after integration by parts.}
\[ M_{ij}^A B = \exp(\omega_{ij} C^{CA} B). \] The latin indices \((i, j, \ldots)\) denote lattice cells, and a pair of indices \((ij)\) denotes the lattice face which separates cells \((i)\) and \((j)\).

At this stage we consider the \(12N_2^2\) numbers \(E_{(ij)}^{[ab]}, E_{(ji)}^{[ab]}\), in addition to the \(72N_2^2\) numbers \(M_{(ij)}^{[ab]}, M_{(ji)}^{[cd]}, M_{(ij)}^{[cd]}\), as independent dynamical variables. The first term of the Lagrangian (15) leads to constraints linear in the momenta, which can be solved in conjunction with the following constraints, that restrict the matrices \(M_{ij}\) to be orthogonal and all the variables to be antisymmetric in \((ij)\),

\[
E_{(ij)}^A = -M_{(ij)}^A B E_{(ji)}^B , \tag{10}
\]
\[
M_{(ij)}^A C M_{(ji)}^B = \delta^A_B , \tag{11}
\]
\[
M_{(ij)}^A C M_{(ij)}^{BC} = \eta^{AB} . \tag{12}
\]

The Dirac procedure gives rise to the brackets [7]

\[
[E_{(ij)}^A, E_{(ij)}^B] = C^{AB} D E_{(ij)}^D , \tag{13}
\]
\[
[E_{(ij)}^A, M_{(ji)}^B C] = C^{AB} D M_{(ij)}^D , \tag{14}
\]
\[
[E_{(ij)}^A, M_{(ji)}^{BC}] = -C^{AB} C M_{(ji)}^B D , \tag{15}
\]

where \(C^{[ab]} [cd] [ef] = \varepsilon^{[ab]} [rs] \varepsilon^{[cd]} [st] \varepsilon^{[ef]} [rt]\) are the structure constants of the Lorentz group.

The second and third terms of the Lagrangian (13) are responsible of the constraints

\[
J(i)^A = E_{(ij)}^A + E_{(ik)}^A + \ldots \approx 0 , \tag{16}
\]
\[
P(I)^A = \frac{1}{4} C_B^C A W(I)_C^B = \frac{1}{4} C_B^C A (M_{ij} M_{jk} \ldots M_{ni})_C^B \approx 0 . \tag{17}
\]

It is useful to notice that for matrices \(W(I)\) near the the identity we have \(W(I)^A_B = \exp(C^A_B C^{(i)}).\) As we will see, within the geometrical gauge, the constraints (16,17) can be interpreted as the requirement that the faces close and that the curvature of the connection vanishes. For these brackets the
restrictions (10)-(12) are identities, and one can show that the constraints (16) and (17) are first-class. The first generates Lorentz transformations, and the second, translations of the bivectors \( E \) by the analogous of an exact form, which in the lattice context means a translation of the bivectors in a way that does not break the closure conditions. In the next section, we will show how some of these transformations are related to translations of lattice sites.

\[
\begin{align*}
[J^A_{(i)}, E_{(ij)}^B] &= C^{AB}_D E_{(ij)}^D, \\
[J^A_{(i)}, M_{(ij)}^B_C] &= C^{AB}_D M_{(ij)}^D_C, \\
[\xi^A P_{(I)} A, E_{(ij)}^B] &\approx \xi^B.
\end{align*}
\]

Not all the translations generated by \( P(I) \) are independent: The Bianchi identities imply that the sum of the flatness conditions (17) associated to all the links that flow into a given vertex, is redundant. We now know the number of variables, the number of constraints and the number of symmetries they generate. Hence, we can now compute the total number of configuration space degrees freedom of the theory:

\[
d = 6N_2 - 6N_3 - 6(N_1 - N_0) = 6\chi.
\]

This is just six times the Euler number \( \chi \), which is always equal to zero in three dimensions because of the Poincaré duality for the Betti numbers, \( b_i = b_{n-i} \).

\[
\chi = b_0 - b_1 + b_2 - b_3 = 0
\]

Thus, all of the degrees of freedom of the lattice can be gauged away. The counting given above fails at the global level when, for some topologies, some of the constraints become redundant. This implies that, for certain topologies, \( 3 + 1 \) dimensional flat spacetimes can have degrees of freedom. We discuss this further in a separate article [15], which describes a reduced version of this theory with a minimal lattice, which has only one vertex and one cell \( (N_0 = N_1 = 1) \). The reduction of the lattice skeleton to minimal form follows the same procedure as in \( 2 + 1 \) dimensional gravity [16].
3 Geometrical Gauge and Translation Symmetry

By the geometrical gauge we mean a gauge in which the configuration variables of the theory, \( E_{ij} \) are area bivectors for the faces of a simplicial lattice, with vertices that are linked by straight lines, which can be described by four-vectors \( A(I) \). The bivectors \( E^*_{ij} \) are then expressed as a wedge product of the vectors \( A(J) \), \( A(K) \) and \( A(L) \) that correspond to the frontier of the face \((ij)\)

\[
E^*_{ij} = A(J) \wedge A(K) = A(K) \wedge A(L) = A(L) \wedge A(J) .
\]  

(23)

Since the linking vectors \( A \) form the boundary of the face \((ij)\), they satisfy the closure conditions

\[
A(J) + A(K) + A(L) = 0
\]  

(24)

Equation (23) leads to a first set of geometricity conditions, that guarantee the geometricity of each separate cell: The requirements that the bivectors \( E^* \) represent a plane face between neighboring cells, and that the faces of a cell intersect in pairs, so that each cell be totally contained in a three-dimensional subspace of Minkowski space-time. These geometricity conditions, for cell \((i)\) are

\[
F(ij) \equiv \varepsilon_{abcd} E_{(ij)}^{ab} E_{(ij)}^{cd} = 0 ,
\]  

(25)

\[
C(ijk) \equiv \varepsilon_{abcd} E_{(ij)}^{ab} E_{(ik)}^{cd} = 0 .
\]  

(26)

We also want to have a covariant description in which parallel transport between neighboring faces is described by Lorentz matrices \( M^{a}_{(ij) b} \). We refer to the variables \( A(I) \) and \( M_{ij} \) as the geometrical variables. The geometricity conditions for the variables of the theory must imply that the geometrical variables satisfy the compatibility conditions

\[
A(I_i) = -M_{ij} A(I_j)
\]  

(27)

\(^3\)We will use the same notation \( (M_{(ij)^{a}_{b}}) \) for these matrices, which act on vectors, as for the previously defined \( (M_{(ij)^{A}_{B}}) \) in the bivector representation; it is the same element of \( SO(3,1) \), but in a different representation.
where $A(I_i)$ is the vector associated to the link $(I)$ in the frame $(i)$, and $A(I_j)$ is the same vector, parallel transported to the frame $(j)$.

This geometrical requirement will be turned into constraints for the variables $E, M$ of the theory. Some of these requirements are contained in the identity $E_{ij} = -M_{ij}E_{ji}$, but there are other conditions. These new restrictions reflect the different ways that $E^*$ can be written as $A \wedge A$; in particular they reflect the degree of freedom corresponding to the rotations within the plane defined by $E^*$. The constraint on the connection matrices which freezes this degree of freedom is just the requirement that the lattice torsion vanishes. This requires that links on the boundary between two cells be the same when seen from either cell, i.e. that if $A(I_i)$ is the intersection of the faces $E^*_{ij}$ and $E^*_{ik}$ of cell $(i)$, and $A(I_j)$ is the intersection of two faces of cell $(j)$, $E^*_{ji}$ and $E^*_{jl}$, then equation (27) holds: $A(I_i) = -M_{ij}A(I_j)$. Since one has $E^*_{ij} = -M_{ij}E^*_{ji}$, it is sufficient that the planes defined by $E^*_{ij}, E^*_{ik}$ and $M_{ij}E^*_{jl}$ all intersect along the same line:

$$C_{ABCE(ij)^A}E_{(ik)^B}M_{(ij)^C}E_{(jl)^D} = 0 \quad .$$

(28)

These conditions, together with (25, 26), guarantee the geometricity of the lattice. They form a redundant set of constraints, which fix all but $4N_0$ of the flatness constraints $P(I) \approx 0$. The remaining first-class constraints generate translations of each lattice vertex $(v)$ in the embedding Minkowski space, and are given by

$$H_a(v) = \sum_{I \rightarrow v} \varepsilon_{abcd}A(I)^bP(I)^{cd}$$

(29)

where $A(I)$ is the vector associated to the link $(I)$ and pointing to the vertex $(v)$. Within the geometrical gauge one can obtain the vectors $A(I) = A(ijk)$ as functions of the bivectors $E^*_{ij}$ and $E^*_{ik}$, which define the direction of $A(I)$ as their intersection place, and as function of a 3-volume element for cell $(i)$ which is a function of the four bivectors $E^*$ that conform cell $(i)$.

$$A(I)^a = -\varepsilon(i)^{abc}\varepsilon(i)^{def}E^*_{(ij)^e}E^*_{(ik)^f}E^*_{(ij)^g}E^*_{(ik)^h} ,$$

(30)

$^4$The identities $E_{ij} = -M_{ij}E_{ji}$ and $E^*_{ij} = -M_{ij}E^*_{ji}$ are equivalent.
were

\[ \varepsilon(i)_{abc} = \frac{\phi(i)_{abc}}{\sqrt{\phi(i)_{efg} \phi(i)^{efg}}} \]  

where \( \mathbf{x}(i) \) is a four-vector that is transverse to the tetrahedron \((i)\). One can show that the 3-volume element \( \varepsilon(i) \) doesn’t depend on the choice of \( \mathbf{x}(i) \).

Now it is possible to get an expression for the translation generator \( s \) in terms of the bivectors \( \mathbf{E} \). The action of the constraints \( H(v)_a \) on the bivector \( \mathbf{E}_{ij}^* = \mathbf{A}(J) \wedge \mathbf{A}(K) = \mathbf{A}(K) \wedge \mathbf{A}(L) = \mathbf{A}(L) \wedge \mathbf{A}(J) \), which contains the vertex \((v)\) as the place of intersection of \( \mathbf{A}(J) \) and \( \mathbf{A}(K) \), is given by (\( N^a \) is the “lapse-shift” vector for vertex \((v)\))

\[ [N^a H_a(v), E_{(ij)}^*] \approx \varepsilon_{abcd} \varepsilon_{gh} N^a A(J)^b [P(J)^{cd}, E_{(ij)}^{gh}] \]

\[ + \varepsilon_{abcd} \varepsilon_{gh} N^a A(K)^b [P(K)^{cd}, E_{(ij)}^{gh}] \]

\[ = N^a \left( \frac{1}{2} A(J)^b \delta_{ab}^{ef} - \frac{1}{2} A(K)^b \delta_{ab}^{ef} \right) \]

\[ = \frac{1}{2} N^a \delta_{ab}^{ef} (A(J)^b - A(K)^b) = \frac{1}{2} \delta_{ab}^{ef} N^a A(L)^b \]  

In the geometrical picture, a translation of the vertex \((v)\) by \( N^a \) would have the same effect. This means that \( H_a(v) \) is the generator of translations of the vertex \((v)\).

The claim that the translation constraints \( H_a(v) \) are the lattice counterpart of the diffeomorphism constraints is not only based in the fact that the lattice counterpart of diffeomorphisms are translations of the lattice vertices, it is supported by an analogy between the dynamical components of the Einstein tensor \( G_{ab} \) and the translation constraints of the lattice.

\[ R_{b\,cd}^a \rightarrow P(I)_b^a P(I)_{cd}/|P(I)| \]  

\[ G_{ab} \rightarrow \varepsilon_{[a\,rs]}^c P(I)^{[rs]} \varepsilon_{bl[tu]} P(I)^{[tu]}/|P(I)| \]  

In the right-hand sides of these expressions, we have used the fact that for a geometrical lattice, a small spacelike area orthogonal to the link \( \mathbf{A}(I) \) can
be given by $P(I)^{ab} = \varepsilon^{ab}_{cd} A(I)^c X(I)^d$, where $X(I)$ is some timelike vector. Using this expression of $P(I)$, and a four-vector $t$ orthogonal to $A(I)$, we get

$$G_{ab}^b ightarrow -\frac{1}{2} \varepsilon^{c}_{ars} P(I)^r^s A(I)_c t^b X(I)_b / |P(I)|$$

(36)

The analogy can be used to see that the continuum limit of the translation constraints are the diffeomorphism constraints. If we use the correct vector $X(I)$ and pass to the continuum limit, where we can choose a time four-vector $t(v)$ orthogonal to all the lattice links flowing into the lattice vertex $(v)$, we can evaluate the counterpart of the Einstein tensor in the lattice vertex $(v)$ to get

$$G_{a0} = G_{ab}^b \rightarrow k(v) \sum_{I \rightarrow v} \varepsilon^{acrs} A(I)^c P(I)^r^s$$

(37)

We have arrived at a geometrical theory that can be described using the bivectors $E$, but which has as natural variables the $N_1$ vectors $A(I)$ that describe the lattice links. These vectors satisfy a closure condition for each face, minus a redundancy for each cell. This, in addition to the $4N_0$ symmetries of the theory, means that the total number of degrees of freedom of the theory is

$$4N_1 - 4(N_2 - N_3) - 4N_0 = 0$$

(38)

In the original, non geometrical theory, there were other “translational” gauge freedoms, which did not commute with the geometricity conditions. These were fixed by imposing the torsion $= 0$ constraints. Since the $4N_0$ translation constraints commute with the torsion $= 0$ conditions, one simply chose an initial surface which is geometrical, and evolve it in time with the hamiltonian constraints $H_a(v)$, using the original Poisson Brackets $[[3], [3]]$.

5 The lattice theory in its present form describes spacetimes with vanishing curvature. The requirement of an appropriate choice of the vector $X(I)$ for the continuum limit, hints that this vector will probably be rigidly defined in a future extension of this lattice theory to spacetimes with curvature. However, it is important to notice that in the present theory we can choose $X(I)$ to be an arbitrary timelike vector.

6 This counting of the degrees of freedom does not include an analysis of the redundancies of the constraints and symmetries. This is the reason why the degrees of freedom of some spacetimes with specific topologies do not appear in the counting. For ”generic” topologies (specifically, those that do not admit a Seifert bundle structure $[[7]]$, the counting $[[3]]$ is correct and is related to the “rigidity theorem” of Mostow $[[20]]$. 

10
4 Conclusion

The principal aim of this article has been to show how the exact translation symmetry of 2+1 dimensional lattice gravity carries over to 3+1 dimensions. The latticized Chern-Simons theory in 2+1 dimensions indeed generalizes nicely to a lattice theory for topological gravity. The latter is exactly solvable as a classical theory, and a Hilbert space of quantum states can be identified more or less explicitly [17], [18].

One way to understand the reduction of gravity to topological gravity, is by taking note of the increase in the number of constraints, from four translation constraints per point (i.e., $4N_0$) to $6(N_1 - N_0)$ flatness conditions, where $N_1$ is the number of lattice links and $N_0$ is the number of lattice sites. As we saw, the translation constraints, analogous to $H_\mu \approx 0$, can be reconstructed by forming $4N_0$ projections from the larger set of $6(N_1 - N_0)$ constraints. The remaining $6N_1 - 10N_0$ constraints of topological lattice gravity, are the responsible for killing the local degrees of freedom of the theory.

This insight suggests a novel approach to the longstanding problem of finding a lattice theory for ordinary gravity, with $4N_0$ first-class constraints which would reduce to $H_\mu(x)$ in the continuum limit. The proposal is summarized in the diagram below [Figure 4.1]: One first reduces to topological gravity by increasing the number of constraints. This reduced theory can be placed on a lattice, without breaking the translation symmetry, as we have shown in this article. Given the lattice theory, and $4N_0$ translation constraints, one then drops the extra $6N_1 - 10N_0$ constraints, which were responsible for reducing the theory to flat spacetimes.

Unfortunately, one easily shows that that the $4N_0$ translation constraints form a closed (first-class) algebra only when the other $6N_1 - 10N_0$ constraints are satisfied (i.e., when the spacetime is flat). If the flatness conditions are to be dropped, one would have to modify the $4N_0$ constraints by adding terms proportional to the curvature, so that the algebra of the constraints closes even for non-vanishing curvature. Note that such extended constraints would automatically reduce to the $4N_0$ constraints given above when the curvature does vanish.

Finding such extended constraints is a difficult task: All we have done is to translate the difficulty of finding first-class lattice constraints, into this new context. However, there are general methods designed for the problem
of finding first-class extensions of constraints which close only “on-shell” (i.e., when other constraints, the “shell-conditions,” hold), such as that of adding terms linear in BRST “ghost” variables, and the existence of a solution to this general problem has been proved [19]. This existence proof is in itself an important point, since it implies that our failure to find first-class constraints for lattice gravity is really our own failure, rather than a signal that lattice gravity could not possibly admit such symmetry. This contradicts previous arguments by various authors, including one of us (HW), that lattice gravity should not have translation symmetry because, in a piecewise-flat lattice, displacing lattice sites clearly leads to a change in the metric properties of the manifold. It is not a priori impossible that such a deformation, coupled to an appropriate change in the parallel transport matrices, could leave a lattice-gravity action invariant. To see just what combinations of lattice deformations and parallel-transport changes are “symmetries,” is precisely what one would accomplish by finding first-class constraints. We stress, however, that the method of finding first-class extensions of second-class constraints has not been previously applied to lattice theories, as far as we are aware; there is no guarantee that the method is practically applicable in this context.

Another approach to the problem of first-class lattice constraints is suggested by the theory of teleparallelism. One can view the lattice theory without geometricity conditions, as a framework for a lattice theory of curved spacetimes, but described with a “connection” that has torsion and no curvature. From this point of view, the lack of local degrees of freedom of topological gravity reflects only our failure to count the torsion degrees of freedom, where the dynamics lies. One could attempt to include lattice “torsion” variables and extend the constraints and bracket algebra to act non-trivially on the new variables. The task is, again, to find first-class extensions of the translation constraints.

Thus, the dream of finding a consistent Hamiltonian lattice gravity theory remains very much alive; we hope that this article will turn out to be a constructive step in that direction.

References
[1] Regge T 1961 Nuovo Cimento 19 558
Collins P A y Williams R M 1974 Phys. Rev. D 10 3537
Hartle J B y Sorkin R 1981 Gen. Rel. Grav. 13 541
Ellis G F R y Williams R M 1984 Gen. Rel. Grav. 13 541
Hamber H y Williams R M 1984 Nucl. Phys. B 248 392
Cheeger J et al 1984 Commun. Math. Phys. 92 405
Berg B 1985 Phys. Rev. Lett. 55 904
Romer H y Zahringer M 1986 Class. Quantum Grav. 3 897
Letho M, Nielsen H B y Nimomiya M 1986 Nucl. Phys. B 272 213
Miller W A 1986 Found. Phys. 16 97
Piran T y Strominger A 1986 Class. Quantum Grav. 3 97
Barrett J W 1988 Class. Quantum Grav. 5 1187
Dubal M R 1989 Class. Quantum Grav. 6 1925
Dubal M R 1990 Class. Quantum Grav. 7 371
Kheyferts A, La Fave N J y Miller W A 1988 Int. J. Phys. 27 133

Kheyferts A, Miller W y Weeler J A 1988 Phys. Rev. Lett. 2042

[2] Collins J "Renormalization: an introduction to renormalization, the renormalization group, and the operator product expansion." (Cambridge University Press, 1984)

[3] Ashtekar A, Rovelli C and Smolin L 1992 “Gravitons and loops” Phys. Rev.D(to appear)
Jacobson T and Smolin L 1988 Nucl. Phys. B299 295

[4] Penrose R "On the Nature of Quantum Geometry", in Magic without magic: J. A. Wheeler. Ed. Klauder (1973)
Sorkin R, Proceedings of the Silarg VII symposiumm (World Scientific, 1990)

[5] Friedman J and Jack I 1986 J. Math. Phys. 27 2973
Piran T and Williams R M 1986 Phys. Rev. D 33 1622

[6] Dirac P A M 1964 Lectures on Quantum Mechanics (Graduate Shool of Science Monograph Series) (New York: Belfer)
[7] Waelbroeck H 1990 *Class. Quantum Grav.* **7** 751

[8] Waelbroeck H 1991 *Nucl. Phys.* **B 364** 475

[9] Brown J D “Lower Dimensional Gravity” (World Scientific, 1988)

[10] Witten E 1988 *Nucl. Phys.* **B 311** 46
    Carlip S 1989 *Nucl. Phys.* **B 324** 106

[11] Waelbroeck H, Urrutia L and Zertuche F “A Lattice Theory With Curvature and Translation Symmetry, Proc.Sixth Marcel Grossman Meeting, World Scientific 1992

[12] Horowitz G 1989 *Commun. Math. Phys.* **125** 417

[13] Misner C, Thorne K and Wheeler J A *Gravitation* (W H Freeman, 1973)

[14] Hayashi N and Shirafuji T 1979 *Phys. Rev.* **D 19** 3524

[15] Waelbroeck H “BF theory and Flat Spacetimes”, Mexico Preprint ICN-UNAM-93-12 [gr-qc/9311033].

[16] Waelbroeck H 1990 *Phys. Rev. Lett.* **64** 2222

[17] Nelson J E and Regge T 1991 *Commun. Math. Phys.* **141** 211
    Urrutia L and Zertuche F 1992 *Class. Quantum Grav.* **8** 641

[18] Carlip S 1990 *Phys. Rev.* **D 42** 2647
    Carlip S 1991 *Class. Quantum Grav.* **8** 5

[19] Batalin I A and Fradkin E S 1987 *Nucl. Phys.* **B279** 514

[20] Mostow G D “Strong rigidity of locally symmetric spaces”, *Ann. Math. Studies* 78, Princeton Univ. Press 1973
Figure captions

Figure 4.1 The detour through topological gravity is suggested as a way to find a consistent Hamiltonian lattice theory of gravity. The first step, to define the topological gravity, was carried out by Horowitz [12]. The second, to topological lattice gravity, was completed in this article. The remaining task (dotted arrow) is to seek first-class extensions of the $4N_0$ translation constraints, possibly by adding terms linear in the curvature and the BRST ghosts.
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9311035v1