Abstract
In this paper we define a new product-like binary operation on directed graphs, and we discuss some of its properties. We also briefly discuss its application in constructing the subtyping relation in generic nominally-typed object-oriented programming languages.

Keywords Graph Product, Partial Graph Product, Object-Oriented Programming (OOP), Nominal Typing, OO Subtyping, OO Generics, Variance Annotations, Java

1. Introduction
Computer science is one of the many fields in which graph products are becoming commonplace [16], where graph products are often viewed as a convenient language with which to describe structures. The notion of a product in any mathematical science enables the combination or decomposition of its elemental structures. In graph theory there are four main products: the Cartesian product, the direct/tensor/categorical product, the strong product and the lexicographic product, each with its own set of applications and theoretical interpretations.

The applications of graph theory and graph products in researching programming languages, in particular, are numerous. In this paper we present a notion of a partial Cartesian graph product and discuss some of its properties.

We conjecture partial Cartesian graph products may have a number of applications and uses in computer science, mathematics, and elsewhere. In particular, we briefly demonstrate how the notion of a partial Cartesian graph product we present in this paper can be applied to accurately construct the subtyping relation in generic nominally-typed object-oriented (OO) programming languages such as Java [13], C# [2], C++ [11], Scala [18] and Kotlin [3].

As such, this paper is structured as follows. In Section 2 we present the definition of the partial Cartesian graph product of two graphs and the intuition behind it (we present two equivalent views of the partial product), then in Section 3 we present examples for partial Cartesian graph products that illustrate our definition (in Appendix A we present SageMath code implementations of our definition/intuitions). In Section 4 we then discuss some of the basic properties of partial Cartesian graph products.

The, in Section 5 we discuss some earlier work similar to ours, and discuss the similarities and differences between their properties. In Section 6 we then discuss, in brief, how the partial Cartesian graph product operation can be used to construct the subtyping relation in Java. We conclude in Section 7 with some final remarks and a brief discussion of some research that can possibly extend the theoretical and practical reach of the research presented in this paper.

2. Partial Cartesian Graph Product

Definition 1. (Partial Cartesian Graph Product, ×). For two directed graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) where

- \( V_1 = V_p + V_n \) such that \( V_p \) and \( V_n \) partition \( V_1 \) (i.e., \( V_p \subseteq V_1 \) and \( V_n = V_1 \setminus V_p \)),
- \( E_1 = E_{pp} + E_{pn} + E_{ap} + E_{an} \) such that \( E_{pp}, E_{pn}, E_{ap}, \) and \( E_{an} \) partition \( E_1 \),
- \( G_p = (V_p, E_{pp}) \) and \( G_n = (V_n, E_{nn}) \) are disjoint subgraphs of \( G_1 \) (the ones induced by \( V_p \) and \( V_n \), respectively, which guarantees that edges of \( E_{pp} \) connect only vertices of \( V_p \) and edges of \( E_{nn} \) connect only vertices of \( V_n \)), and \( E_{pn} \) and \( E_{ap} \) connect vertices from \( V_p \) to \( V_n \) and vice versa, respectively, and
- \( G_1 \) is any directed graph (i.e., \( G_2 \), unlike \( G_1 \), need not have some partitioning of its vertices and edges),

we define the partial Cartesian graph product of \( G_1 \) and \( G_2 \) relative to the set of vertices \( V_p \subseteq V_1 \) as

\[
G = G_1 \times_{V_p} G_2 = (V, E) = G_p \sqcup G_2 + G_n
\]

where
- \( V = V_p \times V_2 + V_n \) (× and + are the standard Cartesian set product and disjoint union operations),
- \( G_p \sqcup G_2 = (V_{p2}, E_{p2}) \) is the standard Cartesian graph product [19] of \( G_p \) and \( G_2 \), and,
- for defining \( E \), the operator + is defined (implicitly relative to \( G_1 \)) such that we have [4]

\[
\begin{align*}
(u_1, v_1) &\sim (u_2, v_2) \in E & \text{if} & (u_1, v_1) &\sim (u_2, v_2) \in E_{p2} \\
(u_1, v) &\sim u_2 \in E & \text{if} & u_1 &\sim u_2 \in E_{pn}, v \in V_2 \\
u_1 &\sim (u_2, v) \in E & \text{if} & u_1 &\sim u_2 \in E_{ap}, v \in V_2 \\
u_1 &\sim u_2 \in E & \text{if} & u_1 &\sim u_2 \in E_{nn}
\end{align*}
\]

Notes:

1. As revealed, for example, by doing an online search on ‘graph theory and programming languages research’.
2. We discuss this application in much more detail in [20].
3. We may call + a “Cartesian disjoint union” (hence the addition-like symbol +), since + effects adding or attaching subgraph \( G_n \) to the Cartesian product \( G_p \sqcup G_2 \), using edges between \( G_n \) and \( G_p \) (in \( G_1 \)) in the same way as these edges are used to define edges in the Cartesian product \( G_1 \sqcup G_2 \).
As expressed by the definition of the partial Cartesian graph product, each edge \( e \in E \) in \( G_1 \times_{V_p} G_2 \) falls under exactly one of four cases: either \( e \) comes from \( G_p \times G_2 \), or \( e \) connects \( G_1 \bowtie G_2 \) to \( G_n \), or \( e \) connects \( G_n \) to \( G_p \times G_2 \), or \( e \) comes from \( G_n \).

The vertices in set \( V_p \) are called the product vertices (of \( G_1 \)), i.e., vertices that participate in the product \( G_p \times G_2 \), while vertices in its complement, \( V_n \) (which we sometimes also write as \( V'_p \)), are called the non-product vertices (of \( G_1 \)) since these vertices are not paired with vertices of \( G_2 \) in the construction of \( G_1 \times_{V_p} G_2 \).

We call \( G_1 \times_{V_p} G_2 \) a partial graph product since, in comparison with the standard (full/total) Cartesian graph product \( G_1 \times G_2 \), the main component of \( G_1 \times_{V_p} G_2 \) (namely, the component \( G_p \times G_2 \)) is typically the Cartesian product of a proper subgraph (namely, \( G_p \)) of \( G_1 \) with \( G_2 \).

Sometimes we omit the subscript \( V_p \) and write \( G_1 \times G_2 \), assuming \( V_p \) is constant and implicit in the definition of \( G_1 \) (as is the case, for example, when using \( \times \) to model generic OO subtyping).

In the partial graph product \( G_1 \times_{V_p} G_2 \), if we have \( V_p = V_1 \) then we will have \( G_p = G_1 \) and \( G_n \) will be the empty graph, and in this case we have \( G_1 \times_{V_p} G_2 = G_1 \bowtie G_2 \). If, on the other hand, we have \( V_p = \emptyset \) then \( G_p \) will be the empty graph and we will have \( G_n = G_1 \), and in this case we have \( G_1 \times_{V_p} G_2 = G_1 \).

In other words, in case all vertices of \( G_1 \) are product vertices, then, as might be expected, \( G_1 \times_{V_p} G_2 \) will be the standard Cartesian product of \( G_1 \) and \( G_2 \), while in case all vertices of \( G_1 \) are non-product vertices then \( G_1 \times_{V_p} G_2 \) will be just \( G_1 \) (i.e., graph \( G_2 \) is disregarded).

**Intuition**  The intuition behind the definition of \( \times \) is simple. The partial product \( G_1 \times G_2 \) of two graphs \( G_1 \) and \( G_2 \) can be equivalently viewed as either:

- A graph that is based on the Cartesian product of the subgraph \( G_p \) (of \( G_1 \)) with \( G_2 \) that further includes \( G_n \), while appropriately respecting how \( G_n \) is connected to \( G_1 \) (which is the view reflected in our definition of \( \times \) above\(^4\)), or as
- Some sort of a special “subgraph” of the graph \( G_1 \bowtie G_2 \), the standard Cartesian product of \( G_1 \) and \( G_2 \), where some specified set of vertices of \( G_1 \bowtie G_2 \) (namely those of \( V_1 \times V_2 \)) gets “coalesced” into a smaller set (one isomorphic to \( V_1 \)), i.e., where some vertices of \( G_1 \) (namely, vertices of \( G_n \), i.e., members of \( V_n \)) do not fully participate in the product graph (participate only with their edges\(^5\))
- The equivalence of these two informal views of \( \times \) can be proven by showing that the product graphs resulting from the two views are always isomorphic.

### 3. Partial Graph Product Examples

We illustrate the definition of \( \times \) by presenting the partial Cartesian product of some sample graphs.

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\(^4\) It is also the view reflected in our standard SageMath implementation of \( \times \). (See Appendix [A]).

\(^5\) This was the view reflected in our initial SageMath implementation of \( \times \). (See Appendix [A]).

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Consider the graphs \( G_1 \) and \( G_2 \) depicted in Figure\(^2\) and\(^3\) and present the graphs of some partial products of \( G_1 \) and \( G_2 \). The reader should ensure he or she sees the product graphs in Figures\(^2\) and\(^3\) as intuitively clear.

Appendix [A] presents the SageMath code we used to help generate the diagrams in Figures\(^2\) and\(^3\).

### 4. Basic Properties of \( \times \)

In this section we discuss some of the fundamental properties of partial Cartesian graph products, particularly the size and order of constructed graphs.

To calculate the number of vertices and number of edges in partial product graphs, let \( |S| \) denote the size (i.e., cardinality) of a set \( S \), and for a graph \( G = (V, E) \) let \( |G| = |V| \) denote the number of vertices in \( G \) (usually also called the size of \( G \)) and let \( |G| = |E| \) denote the number of edges (usually called the order of \( G \)).

Then, for a graph \( G_1 = (V_1, E_1) \) with size \( v_1 \) and order \( e_1 \), a graph \( G_2 = (V_2, E_2) \) with size \( v_2 \) and order \( e_2 \), and for a set \( V_p \subseteq V_1 \) with size \( v_p \leq v_1 \) and a complement \( V_n = V_1 \setminus V_p \) with size \( v'_p = v_1 - v_p \) that induces a partitioning of \( E_1 = E_{pp} + E_{pn} + E_{pn} + E_{nn} \) such that \( G_p = (V_p, E_{pp}) \), \( e_p = |E_{pp}| + |E_{pn}| + |E_{np}| \) (as in Definition\(^1\)) and \( e'_p = e_1 - e_p = |E_{nn}| \) (i.e., \( e_1 = e_p + e'_p \)), the number of vertices of the partial Cartesian product graph is expressed by the equation

\[
|G_1 \times_{V_p} G_2| = |V_p| \cdot |G_2| + |V'_p| = v_p \cdot v_2 + v'_p
\]

while the number of edges is expressed by the equation

\[
\langle G_1 \times_{V_p} G_2 \rangle = |V_p| \cdot \langle G_2 \rangle + (\langle G_1 \rangle - \langle G_n \rangle) \cdot \langle G_2 \rangle + \langle G_n \rangle
\]

\[
= (v_p \cdot e_2 + e_p \cdot v_2) + e'_p
\]

Note that we also have

\[
\langle G_1 \times_{V_p} G_2 \rangle = (v_p \cdot e_2 + e_1 \cdot v_2) - e'_p \cdot (v_2 - 1)
\]

\[
= |V_p| \cdot \langle G_2 \rangle + \langle G_1 \rangle \cdot \langle G_2 \rangle - \langle G_n \rangle \cdot \langle G_2 \rangle + \langle G_n \rangle
\]

which could be a more intuitive equation for \( G_1 \times_{V_p} G_2 \) given that it indicates that edges of the partial product connecting vertices of the product corresponding to \( G_n \) get “coalesced” into one edge (i.e., multiedges are disallowed).

For the sake of comparison, for the standard Cartesian product \( G_1 \bowtie G_2 \) (which is a commutative operation, up to graph isomorphism) we have

\[
|G_1 \bowtie G_2| = |G_1| \times |G_2| = v_1 \cdot v_2
\]

\[
\langle G_1 \bowtie G_2 \rangle = \langle G_1 \rangle \times \langle G_2 \rangle = v_1 \cdot e_2 + e_1 \cdot v_2.
\]

As we briefly illustrate in Section\(^6\) the fact that the size of \( G_1 \bowtie G_2 \) can be smaller than the multiplication of the sizes of \( G_1 \) and \( G_2 \) (as in the standard Cartesian graph product) makes \( \times \) perfectly suited for modeling generic OO subtyping.

Note that, in the equations above, we intentionally depart from the more common notation for graph sizes where \( n \) is used to denote the size of a graph and \( m \) is used to denote its order, so as to make the equations for sizes and orders of product graphs, particularly Equations\(^2\) and\(^3\), readily memorizable and reminiscent of the graph equations defining the product graphs themselves (e.g., Equation\(^1\) on the preceding page).

\(^6\) Even though some better layout of the graphs could make their task even easier.
5. Related Work

The closest work to our work in this paper seems to be that of [26]. In [26] a definition of another partial Cartesian graph product operation, denoted □_S, is presented. Driven by our use of the partial Cartesian graph product ⊸ in constructing the generic OO subtyping relation, our definition of ⊸ differs from that of □_S presented in [26], as we present below.

7 We had not known this work existed until after we defined ⊸ and named it.
5.1 A Comparison of $\times$ and $\Box_S$

First, it should be noted that the order of the factors $G_1$ and $G_2$ in the partial products $G_1 \times_S G_2$ and $G_2 \Box_S G_1$ is reversed (due to the set $S$ being a subset of the vertices of graph $G_1$, compared to graph $G_2$ graph $G_1$ has a special status in the products, and thus both partial products are non-commutative operations. For both operations, the order of the factors of the products matters).

More significantly, as we explain using equations in the sequel, while $G_1 \times_S G_2$ and $G_2 \Box_S G_1$ can have the same number of edges, $G_1 \times_S G_2$ typically has less vertices than $G_2 \Box_S G_1$.

Using the same notation as that of Section 4, the number of vertices of a partial product graph $G_2 \Box_{V_p} G_1$ is expressed by the equation

$$|G_2 \Box_{V_p} G_1| = |G_2| \cdot |G_1| = v_2 \cdot v_1$$

while the number of its edges is expressed by the equation

$$\langle G_2 \Box_{V_p} G_1 \rangle = |G_2| \cdot (|G_1| + |G_2|) \cdot |V_p|$$

$$= v_2 \cdot e_1 + e_2 \cdot v_p$$

$$= v_p \cdot e_2 + e_1 \cdot v_2.$$  

Note also that if multiedges were allowed for $\times$ we would have

$$\langle G_1 \times_{V_p} G_2 \rangle = v_p \cdot e_2 + e_p \cdot v_2$$

and the order of $G_1 \times_{V_p} G_2$ will then be the same as that of $G_2 \Box_{V_p} G_1$ (which, when multiedges are disallowed, happens only if $G_n$ has no edges, i.e., when none of the vertices of $G_n$ is connected to another vertex of $G_n$, sometimes called a discrete graph).

Also it should be noted that the full (i.e., standard) Cartesian graph product can be obtained using either of the two partial Cartesian graph products by setting $V_p = V_1$. This illustrates that, compared to the standard Cartesian graph product, if $V_p \neq V_1$ then the partial product operation $\times$ decreases both the vertices and the edges of the product while the partial product operation $\Box_{V_p}$ decreases only the edges of the product.

To visually illustrate the difference between $\times$ and $\Box_S$ we adapt the example presented in [29] for illustrating $\Box_S$. The graph diagrams presented in Figure 4 help illustrate the differences between the two operations we discussed above.

Further adding to the differences between $\times$ and $\Box_S$, the main motivation for defining $\times$ is to apply it in modeling generic OO subtyping, while the motivation behind defining $\Box_S$—as presented in [28]—seems to be a purely theoretical motivation, namely, studying Vizing’s conjecture (a famous conjecture in graph theory, relating the domination number of a product graph to the domination number of its factors).

Finally, our choice of the symbol $\times$ for denoting the partial product operation allows for making $S$ implicit while indicating that the product operation is partial. For the notation $\Box_S$ doing this is not possible, given that the symbol $\Box$—which will result if $S$ is dropped from the notation—is the symbol for the standard Cartesian graph product, i.e., for a different operation.

6. An Application of $\times$: Modeling Generic OO Subtyping

Generic types [1] [2] [3] [4] [5] add to the expressiveness and type safety of industrial-strength object-oriented programming (OOP) languages such as Java, C#, Scala, Kotlin and other nominally-typed OO programming languages [6].

Figure 4: Comparing $\times_S$ to $\Box_S$ (layout by SageMath)
As we detail in [4, 7], many models for generics have been proposed, particularly for modeling features such as wildcard types [11, 13, 15, 21, 23]. However, as expressed by their authors, none of these models seem to be a fully satisfactory model.

This situation, in our opinion, is due to these models and the mathematical foundations they build upon distancing themselves (unnecessarily) from the nominal-typing of generic OOP languages and, accordingly, them being unaware of the far-reaching implications nominal-typing has on the type systems of these languages and on analyzing and understanding them, which—as again, in our opinion—includes analyzing and understanding generics and generic variance annotations (of which wildcard types are instances).

To demonstrate the direct effect of nominal-typing on the Java type system and on generics in particular, we illustrate how the generic *subtyping* relation in Java can be constructed, using \( \kappa \) and the *subclassing* relation (which is an inherently nominal relation, in Java and in all OO languages) based on the nominality of the subtyping relation in Java (i.e., due to the nominal typing and nominal subtyping of Java, the subclassing relation is the basis for defining the subtyping relation).

In brief, with some simplifying assumptions that we describe in [9, 10], the generic subtyping relation in Java can be constructed iteratively using the nominal subclassing relation and the partial Cartesian graph product \( \kappa \), as follows.

Let \( C \) be the graph of the subclassing relation in some Java program. Let \( C_\delta \) be the generic classes subset of classes in \( C \). Then the graph \( S \) of the subtyping relation in the Java program (typically \( S \) is infinite, if there is at least one generic class in \( C \)) can be constructed as the limit of the sequence of graphs \( S_i \) of subtyping relations constructed iteratively using the equation

\[
S_{i+1} = C \kappa C_\delta S_i^{\triangle}
\]

(4)

where \( S_i^{\triangle} \) is the graph of the containment relation between wildcard type arguments derived from \( S_i \) (as explained in [12]) and \( S_i^{\triangle} = \text{Graph('?')} \) is the one-vertex graph having the default wildcard type argument, ‘?’ as its only vertex and no containment relation edges (again as explained in [9]).

It should be noted that Equation (4) tells us that in the construction of the graph of the subtyping relation \( S \) the generic classes in \( C \) (i.e., \( C_\delta \)) correspond to product vertices, while the non-generic classes in \( C \) correspond to non-product vertices in the partial product graph of each approximation \( S_{i+1} \) of \( S \). This property of \( \kappa \) preserves non-generic types (and the subtyping relations between them) during the construction of \( S \), meaning that non-generic types in \( S \) remain as non-generic types in \( S_{i+1} \), and thus, ultimately, are non-generic types in \( S \) as well.

### 6.1 Java Subtyping Example

Figure 5 illustrates the use of \( \kappa \) to construct the Java subtyping relation. To decrease clutter, given that OO subtyping is a transitive relation, we present the transitive reduction of the subtyping graphs in Figure 5.

The three graphs in Figure 5 illustrate the construction of the subtyping relation \( S \) of a Java program that only has the generic class definition

```java
class C<T> {}
```

As defined by Equation (4), the graph of \( S_2 = C \kappa (C_\delta S_1^{\triangle}) \) in Figure 5 is constructed as the partial product of the graph of the sublassing/inheritance relation \( C \) and the graph of \( S_1^{\triangle} \) (of wildcard types over \( S_1 \), ordered by containment) relative to the set \{ \( C \) \} of generic classes in \( C \).

More details and examples on the use of \( \kappa \) to construct the generic OO subtyping relation can be found in [9, 10].

### 7. Concluding Remarks and Future Work

In this paper we defined a new binary operation \( \kappa \) on graphs that constructs a partial product of its two input graphs, we presented few examples that illustrate the definition of \( \kappa \), and we discussed some of the basic properties of the operation. We also compared the \( \kappa \) operation to the closest similar work. Finally, we also discussed how the partial graph product operation \( \kappa \) may be used in understanding the subtyping relation in generic nominally-typed OO programming languages.

As of the time of this writing, we do not know of any other application of the new graph operation we present. Nevertheless, in this paper we presented the partial product operation over graphs in abstract mathematical terms, in the hope that the operation may prove to be useful in other mathematical contexts and domains.

Although we have not done so here, we believe the notion of partial Cartesian graph products, as presented here, can be easily adapted to apply to other mathematical notions such as sets, partial orders, groups (or even categories, more generally). To model infinite self-similar graphs (or groups or categories) we also believe partial products, over graphs, groups, or categories, can in some way be modeled by operads, which are category-theoretic tools that have proved to be useful in modeling self-similar phenomena [8, 10].

Finally, studying in more depth properties of partial Cartesian graph products such as the size, order (as we hinted at in Section 5) and rank of elements of the products and of infinite applications of them, is work that can build on work we presented in this paper, and which can be of both theoretical and practical significance, particularly in computer science graph theoretic applications. Also, we believe a notion of ‘degree of partialness’ of a partial product...
can be a useful notion, even though we do not immediately see an application of this notion.

References

[1] ISO/IEC 14882:2011: Programming Languages: C++. 2011.
[2] C# language specification, version 5.0. http://msdn.microsoft.com/vsharpx 2015.
[3] Kotlin language documentation, v. 1.2. http://www.kotlinlang.org 2018.
[4] Moez A. AbdelGawad. Towards an accurate mathematical model of generic nominally-typed OOP (extended abstract). arXiv:1610.05114 [cs.PL], 2016.
[5] Moez A. AbdelGawad. Towards understanding generics. Technical report, arXiv:1605.01480 [cs.PL], 2016.
[6] Moez A. AbdelGawad. Why nominal-typing matters in OOP. Preprint available at http://arxiv.org/abs/1606.03809 2016.
[7] Moez A. AbdelGawad. Novel uses of category theory in modeling OOP (extended abstract). Accepted at The Nordic Workshop on Programming Theory (NWPT’17), Turku, Finland, November 1-3, 2017. (Full version available at arXiv.org: 1709.08056 [cs.PL]), 2017.
[8] Moez A. AbdelGawad. Towards a Java subtyping operad. Proceedings of FTfJP’17, Barcelona, Spain, June 18-23, 2017.
[9] Moez A. AbdelGawad. Java subtyping as an infinite self-similar partial graph product. Available as arXiv preprint at http://arxiv.org/abs/1805.06893 2018.
[10] Moez A. AbdelGawad. Towards taming Java wildcards and extending Java with interval types. Available as arXiv preprint at http://arxiv.org/abs/1805.10931 2018.
[11] Nicholas Cameron, Sophia Drossopoulou, and Erik Ernst. A model for Java with wildcards. In ECOOP’08, 2008.
[12] Nicholas Cameron, Erik Ernst, and Sophia Drossopoulou. Towards an existential types model for Java wildcards. 9th Workshop on Formal Techniques for Java-like Programs, 2007.
[13] James Gosling, Bill Joy, Guy Steele, and Gilad Bracha. The Java Language Specification. Addison-Wesley, 2005.
[14] James Gosling, Bill Joy, Guy Steele, Gilad Bracha, and Alex Buckley. The Java Language Specification. Addison-Wesley, 2014.
[15] Ben Greenman, Fabian Muehlboeck, and Ross Tate. Getting f-bounded polymorphism into shape. In PLDI ’14: Proceedings of the 2014 ACM SIGPLAN conference on Programming Language Design and Implementation, 2014.
[16] Richard Hammack, Willfried Imrich, and Sandi Klavzar. Handbook of Product Graphs. CRC Press, second edition edition, 2011.
[17] Angelika Langer. The Java Generics FAQ. www.angelikalanger.com/GenericsFAQ/JavaGenericsFAQ.html, 2015.
[18] Martin Odersky. The Scala language specification, v. 2.9. http://www.scala-lang.org 2014.
[19] David Spivak. Category theory for the sciences. MIT Press, 2014.
[20] William Stein. Sagemath 8.1. http://www.sagemath.org 2017.
[21] Alexander J. Summers, Nicholas Cameron, Mariangiola Dezani-Ciancaglini, and Sophia Drossopoulou. Towards a semantic model for Java wildcards. 10th Workshop on Formal Techniques for Java-like Programs, 2010.
[22] Ross Tate. Mixed-site variance. In FOOL ’13: Informal Proceedings of the 20th International Workshop on Foundations of Object-Oriented Languages, 2013.
[23] Ross Tate, Alan Leung, and Sorin Lerner. Taming wildcards in Java’s type system. PLDI’11, June 4–8, 2011, San Jose, California, USA., 2011.
[24] Mads Torgersen, Erik Ernst, and Christian Plesner Hansen. Wild FJ. In Foundations of Object-Oriented Languages, 2005.
[25] Mads Torgersen, Christian Plesner Hansen, Erik Ernst, Peter von der Ahé, Gilad Bracha, and Neal Gafta. Adding wildcards to the Java programming language. In SAC, 2004.
[26] Ismael Gonzalez Yero. Partial product of graphs and Vizing’s conjecture. Ars Mathematica Contemporanea, 2015.

A. SageMath Code

To generate the graph examples presented in this paper we implemented the definition of $\times$ (as presented in Section 2) in SageMath 8.1 [21]. For those interested, we present in this appendix our SageMath implementation code. The code presented here is not optimized for speed of execution but rather for clarity and simplicity of implementation.

```python
# PGP

def comp(pv, g):
    """ Computes the complement of pv relative to vertices of g """
    return filter(lambda v: v not in pv, g.vertices())

def PGP(g1, pv, g2):
    """ Computes the partial cartesian product of graphs g1 and g2. """

    INPUT:
    - `{pv}` (list) -- is the list of product vertices in g1.

    """

    # 1st step
    gp = g1.subgraph(pv)
    g = gp.cartesian_product(g2)

    # 2nd step
    npv = comp(pv, g1)
    gn = g1.subgraph(npv)
    g = g.union(gn)

    # 3rd step
    gpn = g1.subgraph(edge_property=(lambda e: e[0] in pv and e[1] in npv))
    for u1, u2 in gpn.edge_iterator(labels=None):
        for v in g2:
            g.add_edge((u1, v), u2)

    # 4th step
    gnp = g1.subgraph(edge_property=(lambda e: e[1] in pv and e[0] in npv))
    for u1, u2 in gnp.edge_iterator(labels=None):
        for v in g2:
            g.add_edge((u1, u2, v))
```

For example, the degree of partialness can be a value (a real number) between 0 and 1, defined possibly as the size of the set of product vertices, $|V_p|$, divided by the size of the set of all vertices of the first factor graph, $|V_1|$ (i.e., the degree of partialness of a partial product will be $|V_p|/|V_1|$. In this case a degree of partialness with value 1 means the standard Cartesian product, while a value of 0 means no product.)
For convenience, our initial shorter (but equivalent) implementation code for \( \times \) (where GSP stands for 'Generic Subtyping Product') is as follows. The code corresponds to the second informal view of the partial Cartesian graph product we presented in Section 2.

```python
# GSP
def GSP(g1, pv, g2):
g = DiGraph.cartesian_product(g1, g2)  # main step

lnpvc = map(lambda npv: filter(lambda (v, _): v==npv, g.vertices()), comp(pv, g1))
# lnpvc is list of non-product vertex clusters

# merge the clusters
map(lambda vc: g.merge_vertices(vc), lnpvc)

return g
```

For any two graphs \( g_1 \), \( g_2 \) and any list \( pv \) (listing the product vertices subset of the vertices of \( g_1 \)) we have

\[
GSP(g1,pv,g2).is_isomorphic(PCGP(g1,pv,g2))
\]