The role of band-index-dependent transport relaxation times in anomalous Hall effect

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We revisit model calculations of the anomalous Hall effect (AHE) and show that, in isotropic Rashba-coupled two-dimensional electron gas (2DEG) with pointlike potential impurities, the full solution of the semiclassical Boltzmann equation (SBE) may differ from the widely-used $1/\tau^\parallel \& 1/\tau^\perp$ solution [Phys. Rev. B 68, 165311 (2003)]. Our approach to AHE is analogous to the SBE-based analysis of the anisotropic magnetoresistance leading to an integral equation for the distribution function [Phys. Rev. B 79, 045427 (2009)] but in the present case, we reduce the description to band-index-dependent transport relaxation times. When both Rashba bands are partially occupied, these are determined by solving a system of linear equations. Detailed calculations show that, for intrinsic and hybrid skew scatterings the difference between $1/\tau^\parallel \& 1/\tau^\perp$ and the full solution of SBE is notable for large Fermi energies. For coordinate-shift effects, the side-jump velocity acquired in the inter-band elastic scattering process is shown to be more important for larger Rashba coupling and may even exceed the intra-band one for the outer Rashba band. The coordinate-shift contribution to AHE in the considered case notably differs from that in the limit of smooth disorder potential analyzed before.

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I. INTRODUCTION

The extensive researches on the anomalous Hall effect (AHE) in the past twenty years have promoted some important improvements in the physical understanding and theoretical tools [1]. The importance of the inter-band coherence effects in the topologically non-trivial band structures has been highlighted theoretically [3, 4]. Some simple but topologically non-trivial band models, such as the two-dimensional (2D) massive Dirac model [3, 5, 6] and 2D spin-polarized Rashba model [4, 7–17], have been studied for analytical account of both the intrinsic and extrinsic mechanisms for AHE. While the intrinsic mechanism stems solely from the nontrivial band structure, the extrinsic mechanism relies on the existence of disorder and is the sum of two contributions known as skew scattering and side-jump. In the language of the semiclassical Boltzmann equation (SBE) framework [18], the side-jump contribution arises from the coordinate-shift effects [10, 19], whereas the skew scattering originates from the scattering asymmetry when the scattering rate is calculated beyond the lowest Born approximation. Other than the conventional skew scattering due to the non-Gaussian third-order disorder correlation, other two skew scattering contributions have been recently proposed: the intrinsic (disorder-independent) skew scattering [3, 16, 17, 20] due to the Gaussian disorder and hybrid skew scattering [16, 17] due to the fourth-order non-Gaussian disorder correlation.

The SBE approach to AHE, developed by Sinitsyn et al., in a series of excellent papers [1, 2, 10, 18, 19], is physically more transparent than quantum-mechanical Kubo-Streda [3, 5, 9, 14, 21, 23] and multiple-band Keldysh [4, 11, 12, 16, 17] approaches. For the 2D massive Dirac model, the equivalence between the Kubo-Streda and SBE approaches under the non-crossing approximation has been shown [3]. However, for the 2D spin-polarized Rashba model, although the equivalence between Kubo-Streda and Keldysh formalisms has been explicitly shown [17], existing SBE calculations [10, 13] do not produce fully same results for side-jump and skew scattering as quantum-mechanical transport theories [3, 16, 17]. Sinitsyn et al. [10] analyzed the coordinate-shift effects in the limit of smooth disorder potential, showing that the contribution of coordinate-shift has the same sign as the intrinsic one for sufficiently large Fermi energy. In the limit of smooth disorder potential, the intra-band small-angle scattering dominates the electron-impurity elastic scattering processes, leaving little room for the intra-band scattering with large momentum transfer and inter-band elastic scattering. However, the latter two elastic scattering processes may be important in the case of short-range disorder potential. Because recent convincing Kubo-Streda [3, 14] and Keldysh calculations [4, 12, 16, 17] assume pointlike potential impurities, the comparison of SBE and these calculations calls for a SBE study in the presence of pointlike potential impurities. The work of Borunda et al. [13] is an effort towards this direction, showing that when both Rashba bands are partially occupied the conventional skew scattering vanishes in both the SBE and Kubo-Streda calculations. However, the SBE calculation of coordinate-shift, intrinsic skew scattering and hybrid skew scattering were not carried out in this work. On the other hand, this work employed the $1/\tau^\parallel \& 1/\tau^\perp$ solution [24, 25] to the SBE for the distribution functions responsible for the longitudinal and skew scattering transport. The validity of $1/\tau^\parallel \& 1/\tau^\perp$ solution has been verified in the 2D massive Dirac model [3]. However, as will be shown in the present paper, this solution is not valid in a spin-polarized Rashba 2DEG with pointlike potential impurities when both Rashba bands are partially occupied.
In the present paper, we analyze the solution of SBE in isotropic 2DEG in the presence of static impurities. When considering longitudinal and skew-scattering-induced Hall transport, for multiple-Fermi-circle 2DEG we show that the solution may be different from the $1/\tau^\parallel$ & $1/\tau^\perp$ solution if a decoupling condition is unsatisfied. We prove that this decoupling condition is unsatisfied in a spin-polarized Rashba 2DEG with pointlike potential impurities when both Rashba bands are partially occupied. Focusing on this particular case, detailed calculations show that, for intrinsic and hybrid skew scatterings, the deviation of $1/\tau^\parallel$ & $1/\tau^\perp$ solution from our solution is notable for large Fermi energies. For coordinate-shift effects, the side-jump velocity acquired in the inter-band elastic scattering process is shown to be more important for larger Rashba spin-orbit coupling. Especially, the inter-band side-jump velocity may exceed the intra-band one for the outer Rashba band. The coordinate-shift contribution to the AHE in the case of pointlike impurities is significantly different from that in the limit of smooth disorder potential analyzed by Sinitsyn et al. [10].

The paper is organized as follows. To be self-contained, Sec. II briefly sketches the basic framework of the modern SBE theory. Sec. III presents general discussion on the solution of SBE in isotropic multiple-Fermi-circle 2DEG, whereas Sec. IV focuses on concrete calculations of AHE in a spin-polarized Rashba 2DEG. Sec. V concludes the paper by discussing the difference between our solution of the SBE and the $1/\tau^\parallel$ & $1/\tau^\perp$ solution. The Appendix contains the main steps of calculating scattering rates.

II. BASIC FRAMEWORK

The basic framework of SBE approach in isotropic electron gas has been formulated by Sinitsyn [15], here we sketch it very briefly. The starting point is the SBE describing the evolution of semiclassical distribution function (DF) of electron wavepackets in the presence of a weak uniform electric-field $\mathbf{E}$ and diluted static impurities. For isotropic band structures, the linearized steady-state SBE can be formulated as [3]

$$
\mathbf{F} \cdot \mathbf{v}_l^0 \frac{\partial f_0}{\partial \epsilon_l} = -\sum_{l'} \omega_{l,l'} (g_l - g_{l'}) ,
$$

(1)

$$
\mathbf{F} \cdot \mathbf{v}_l^{a_{\parallel}} \frac{\partial f_0}{\partial \epsilon_l} = \sum_{l'} \omega_{l,l'} (g_{l}^{a_{\parallel}} - g_{l'}^{a_{\parallel}}) .
$$

(2)

Here $l = (\epsilon, \mathbf{k})$ is the band index and $\mathbf{k}$ the momentum, $\mathbf{F} = e\mathbf{E}$ is the driving force, $\mathbf{v}_l^0 = \partial \epsilon_l / \hbar \partial \mathbf{k}$ is the group velocity, $\omega_{l,l'}$ the transition rate from state $l'$ to $l$ due to elastic electron-impurity scattering. The scattering rate is determined by the golden rule $\omega_{l,l'} = \frac{2\pi}{\hbar} \langle |T_{l,l'}|^2 \rangle \delta (\epsilon_1 - \epsilon')$ where $\langle \ldots \rangle$ stands for the disorder configuration average and $T_{l,l'}$ the elements of T-matrix (details in appendix). To capture the skew scattering, the T-matrix expansion beyond the 1st Born order is needed and then the principle of microscopic detailed balance breaks down $\omega_{l,l'} \neq \omega_{l',l}$. One can then define the symmetric and anti-symmetric parts of scattering rate as: $\omega_{l,l'}^{\parallel} = \frac{1}{2} (\omega_{l,l'} \pm \omega_{l',l})$. For simple isotropic bands $\sum_{l'} \omega_{l,l'}^{\parallel} = 0$ leads to the form of collision term in Eqs. (1) and (2). The semi-classical DF is decomposed into $f_1 = f_0^0 + g_l + g_l^{a_{\parallel}}$ around the equilibrium Fermi-Dirac DF $f_0^0$ with $g_l$ equilibrating the electron wavepacket acceleration resulted by the driving force between successive scattering events and the anomalous DF $g_l^{a_{\parallel}}$ describing the effect of external fields working during the coordinate-shift process. Usually the coordinate-shift effect is dealt with in the 1st Born approximation [18], thus in Eq. (2) $\omega_{l,l'}^{\parallel}$ is approximated by its first Born approximation $\omega_{l,l'}^{1_{B}}$. $v_l^{a_{\parallel}} \approx \sum_{l'} \omega_{l,l'}^{1_{B}} \hat{\mathbf{r}}_{l',l}$ is the side-jump velocity [10, 19], where $\hat{\mathbf{r}}_{l',l}$ denotes the coordinate-shift in the scattering process which scatters the electron in state $l'$ into $l$ and reads $\hat{\mathbf{r}}_{l',l} = (\langle u_l | \partial \epsilon_l \partial \mathbf{k} | u_{l'} \rangle - \langle u_{l'} | \partial \epsilon_l \partial \mathbf{k} | u_l \rangle ) | u_l \rangle - \mathbf{D} \arg (u_l | u_{l'})$ for spin-independent scalar disorder in the 1st Born approximation [19]. Here $| u_l \rangle$ is defined via $| l \rangle = | k | | u_l \rangle$, denoting the eigenstate related to the internal degrees of freedom, and $\mathbf{D} = \partial \mathbf{k} / \partial \epsilon_l$.

III. GENERAL ANALYSIS IN ISOTROPIC 2DEG

Equations (1) and (2) can be re-expressed as

$$
\mathbf{F} \cdot \mathbf{v}_l^{0_\parallel} (\epsilon, \phi) \frac{\partial f_0}{\partial \epsilon_l} = -\sum_{\eta} \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} (\epsilon_l) \times [ g_{\eta} (\epsilon, \phi) \partial (\mathbf{v}_l^{0_\parallel} (\epsilon, \phi')) - g_{\eta'} (\epsilon, \phi) \partial (\mathbf{v}_l^{0_\parallel} (\epsilon, \phi')) ]
$$

(3)

and

$$
\mathbf{F} \cdot \mathbf{v}_l^{a_{\parallel}j} (\epsilon, \phi) \frac{\partial f_0}{\partial \epsilon_l} = \sum_{\eta} \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} (\epsilon_l) \times [ g_{\eta}^{a_{\parallel}j} (\epsilon, \phi) \partial (\mathbf{v}_l^{a_{\parallel}j} (\epsilon, \phi')) - g_{\eta'}^{a_{\parallel}j} (\epsilon, \phi) \partial (\mathbf{v}_l^{a_{\parallel}j} (\epsilon, \phi')) ] ,
$$

(4)

respectively. Here the energy-integrated transition rate

$$
\omega_{\eta\phi,\eta'\phi'} (\epsilon_l) = N_{\eta\phi} (\epsilon_l) \int d\epsilon_l \omega_{l,l'}
$$

(5)

is introduced, $\epsilon = \epsilon_l$, $l = (\eta, \phi)$ is the eigenstate index, $\phi$ is the polar angle of 2D momentum, $N_{\eta\phi} (\epsilon_l)$ denotes the density of states (DOS), $\partial (\mathbf{v}_l^{0_{\parallel}j} (\epsilon, \phi))$ denotes the angle between $\mathbf{v}_l^{0_{\parallel}j} (\epsilon, \phi)$ and the external field. In isotropic bands (assuming electron-like dispersion curve) $\mathbf{v}_l^{0_{\parallel}} (\epsilon, \phi) \propto \mathbf{k}_\phi (\epsilon)$. Whereas generally one can expect $\mathbf{v}_l^{a_{\parallel}j} (\epsilon, \phi) \propto \mathbf{z} \times \mathbf{k}_\phi (\epsilon)$ due to the chirality of side-jump velocity.

In isotropic multi-band 2DEG with isotropic scatterers, the normal part $g_n$ of out-of-equilibrium DF can be
described by introducing the isotropic longitudinal and transverse transport relaxation times for electrons on iso-
energy circles with energy $\epsilon$ in the $\eta$ band as
\[
g_\eta(\epsilon, \mathbf{\partial} (\mathbf{k}_\eta(\epsilon))) = (-\partial_x f^0) \times [\mathbf{F} \cdot \mathbf{v}_\eta^0(\epsilon, \phi) \tau^L_\eta(\epsilon) + (\mathbf{z} \times \mathbf{F}) \cdot \mathbf{v}_\eta^0(\epsilon, \phi) \tau^{sk}_\eta(\epsilon)]. \tag{6}
\]
$\tau^L_\eta(\epsilon)$ describes the longitudinal transport along the driving force; whereas $\tau^{sk}_\eta(\epsilon)$ is responsible for the skewsca-
tering-induced Hall transport. Similarly, the anomalous DF is described by
\[
g^{a\text{dis}}_\eta(\epsilon, \mathbf{\partial} (\mathbf{v}^s_\eta(\epsilon, \phi))) = \partial_x f^0 \mathbf{F} \cdot \mathbf{v}^s_\eta(\epsilon, \phi) \tau^{sk}_\eta(\epsilon), \tag{7}
\]
where $\tau^{sk}_\eta$ has the dimension of time and may also be band-dependent.

Because all elastic scattering events occur in the iso-
energy circles with energy $\epsilon$, in the following derivation we suppress the variable $\epsilon$ in the arguments of DF, trans-
port time, scattering rate, group velocity and so on. Sub-
stituting Eq. (6) into (3) yields two equations:
\[
\frac{1}{\tau^L_\eta} = \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \left[ 1 - \cos (\phi' - \phi) \frac{v^0_{\eta'} \tau^L_{\eta'}}{v^0_{\eta'} \tau^L_{\eta}} - \sin (\phi' - \phi) \frac{v^0_{\eta'} \tau^{sk}_{\eta'}}{v^0_{\eta'} \tau^L_{\eta}} \right], \tag{8}
\]
\[
0 = \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \left[ \sin (\phi' - \phi) \frac{v^0_{\eta'} \tau^L_{\eta'}}{v^0_{\eta'} \tau^L_{\eta}} + \tau^{sk}_{\eta'} \frac{v^0_{\eta'} \tau^{sk}_{\eta'}}{v^0_{\eta'} \tau^L_{\eta}} - \cos (\phi' - \phi) \frac{v^0_{\eta'} \tau^{sk}_{\eta'}}{v^0_{\eta'} \tau^L_{\eta}} \right], \tag{9}
\]
where $|v^0_{\eta}(\epsilon, \phi)| \equiv v^0_{\eta}(\epsilon)$. Substituting Eq. (7) into (11) yields
\[
\frac{1}{\tau^{sk}_\eta} = \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \left[ 1 - \mathbf{F} \cdot \mathbf{v}^s_\eta(\phi') \tau^{sk}_\eta \frac{\mathbf{F} \cdot \mathbf{v}^s_\eta(\phi') \tau^{sk}_\eta}{\mathbf{F} \cdot \mathbf{v}^s_\eta(\phi') \tau^{sk}_\eta} \right]. \tag{10}
\]

The second term on the right-hand-side (rhs) is propor-
tional to $\cos (\phi' - \phi) v^{sk}_\eta/v^{sk}_\eta$, but the side-jump velocities on different Fermi circles may have preferences for differ-
et chirality (left or right).

**A. Case of single Fermi circle**

In the case of only single Fermi circle for a given Fermi
energy, $\tau^{L(sk)}_\eta$ and $\tau^{L(sk)}_\eta$ are decoupled ($\eta' \not= \eta$) in Eqs. (8) and (9), which can thus be greatly simplified into
\[
\frac{1}{\tau^L_\eta} = \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \left[ [1 - \cos (\phi' - \phi)] - \sin (\phi' - \phi) \frac{\tau^{sk}_\eta}{\tau^L_\eta} \right], \tag{11}
\]
\[
0 = \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \left[ \sin (\phi' - \phi) + [1 - \cos (\phi' - \phi)] \frac{\tau^{sk}_\eta}{\tau^L_\eta} \right]. \tag{12}
\]

One finds that two relaxation-time like quantities defined
on each isotropic Fermi circle as $F \mid \\frac{\eta}{\tau^L_\eta}$
\[
1 \equiv \frac{1}{\tau^L_\eta} = \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} [1 - \cos (\phi' - \phi)] \tag{13}
\]
and
\[
1 \equiv \frac{1}{\tau^{sk}_\eta} = \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \sin (\phi' - \phi') \tag{14}
\]
solve Eqs. (11) and (12):
\[
\tau^{L}_\eta = \frac{\tau^L_\eta}{1 + \left( \frac{\tau^{sk}_\eta}{\tau^L_\eta} \right)^2}, \quad \tau^{sk}_\eta = \frac{\tau^{sk}_\eta}{1 + \left( \frac{\tau^{sk}_\eta}{\tau^L_\eta} \right)^2}. \tag{15}
\]

This solution coincides with the “$1/\tau^L \& 1/\tau^{sk}$ approach” proposed by Schliemann and Loss $[24, 25]$. Thereby,
the validity of “$1/\tau^L \& 1/\tau^{sk}$ approach” is confirmed in
isotropic single-Fermi-circle 2DEG. The n-doped 2D mas-
sive Dirac system is just a single-Fermi-circle 2DEG, and
the application of $1/\tau^L \& 1/\tau^{sk}$ solution in this model is
successful $[3, 5]$.

Meanwhile, $\tau^{sk}_\eta$ and $\tau^{sk}_\eta$ are decoupled for $\eta' \not= \eta$ in Eq. (10), which thus reduces to
\[
1 \equiv \frac{1}{\tau^{sk}_\eta} = \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \sin (\phi' - \phi') \tag{16}
\]
We see that in the case of single Fermi circle, $\tau^{sk}_\eta$ is just the longitudinal transport time $\tau^L_\eta$ calculated in the first Born order. This relation can be easily verified in the 2D massive Dirac model $[3]$.

**B. Case of multiple Fermi circles**

In 2DEG with more than single Fermi circle for a given Fermi energy, the possible presence of inter-band elastic
scattering complicates the solution of Boltzmann equations be-
cause in this case $\tau^{L(sk)}_\eta$ and $\tau^{L(sk)}_\eta$ are coupled ($\eta' \not= \eta$) in Eqs. (8) and (9). Noticing that for isotropic bands $\int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} = 0$, Eqs. (8) and (9) reduce to the following two coupled equations
\[
1 = \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \left[ \tau^L_\eta - \cos (\phi' - \phi) \frac{v^0_{\eta'}}{v^0_{\eta'}} \tau^L_{\eta'} \right], \tag{17}
\]
\[
- \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega_{\eta\phi,\eta'\phi'} \sin (\phi' - \phi) \frac{v^0_{\eta'}}{v^0_{\eta'}} \tau^{sk}_{\eta'}.
\]
and
\[ 0 = \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega^{s}_{\eta\phi,\eta'\phi'} \left[ \tau^{sk}_\eta - \cos (\phi' - \phi) \frac{v^0_{\eta'\phi}}{v^0_\eta} \right] \]

\[ + \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega^{a}_{\eta\phi,\eta'\phi'} \sin (\phi' - \phi) \frac{v^0_{\eta'\phi}}{v^0_\eta} \tau^{L}_{\eta'\phi}. \]  

(18)

On the other hand, further analysis of Eq. (10) in general cases of multiple Fermi circles is difficult, because we do not have simpler general expression for the inter-band component of side-jump velocity.

C. Decoupling condition for multiple Fermi circles in the SBE

Although the $1/\tau^{||}$ & $1/\tau^{\perp}$ solution cannot be derived out in general cases of isotropic 2DEG with multiple Fermi circles, it may still be valid in some special multiple-Fermi-circles cases. In this subsection we examine in what cases the $1/\tau^{||}$ & $1/\tau^{\perp}$ solution solves Boltzmann equations (11) and (12) in isotropic multiple-Fermi-circle 2DEG. In the isotropic multiple-Fermi-circle case the $1/\tau^{||}$ & $1/\tau^{\perp}$ solution (15) takes the form
\[
\frac{1}{\tau^{||}_{\eta}} = \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega^{s}_{\eta\phi,\eta'\phi'} \left[ 1 - \frac{v^0_{\eta'\phi}}{v^0_\eta} \cos (\phi' - \phi) \right],
\]

\[
\frac{1}{\tau^{\perp}_{\eta}} = \sum_{\eta'} \int \frac{d\phi'}{2\pi} \omega^{a}_{\eta\phi,\eta'\phi'} \frac{v^0_{\eta'\phi}}{v^0_\eta} \sin (\phi' - \phi).
\]

(19), (20)

In this solution, $\tau^{(L)}_{\eta}$, thus $\tau^{L}_{\eta}$ on different Fermi circles are decoupled, however, they are coupled to each other in Eqs. (17) and (18). Thereby, the $1/\tau^{||}$ & $1/\tau^{\perp}$ solution is not expected to be suitable as long as the coupling between different Fermi circles exists. While, the inter-band coupling vanishes when the following decoupling condition i.e., when

\[ \int d\phi' \omega^{s}_{\eta\phi,\eta'\phi'} \cos (\phi' - \phi) = 0 \]  

(21) and

\[ \int d\phi' \omega^{a}_{\eta\phi,\eta'\phi'} \sin (\phi' - \phi) = 0 \]  

(22)

for $\eta' \neq \eta$, is satisfied, then Eqs. (17) and (18) reduce to Eqs. (12) with (19) and (20). Meanwhile, if the decoupling condition is satisfied, Eq. (10) just reduces to Eq. (15) with the 1st Born order.

In the spin-polarized Rashba 2DEG with pointlike scalar impurities when both Rashba bands are partially occupied, neither (21) nor (22) holds when both Rashba bands are partially occupied. This can be verified by noticing the form of Eqs. (48), (49), (50) and (51) in Sec. IV. It is therefore expected that the $1/\tau^{||}$ & $1/\tau^{\perp}$ solution cannot provide fully correct anomalous Hall conductivities for this system.

D. Approximate strategy and scattering rates

Usually the scattering chirality is weak \[ \tilde{2} \] $1/\tau^{||}_{\eta} \gg 1/\tau^{\perp}_{\eta}$, thus the $1/\tau^{||}$ & $1/\tau^{\perp}$ solution takes the form

\[
\tau^{L}_{\eta} = \tau^{\perp}_{\eta},
\]

\[
\tau^{sk}_{\eta} = \left( \tau^{||}_{\eta} \right)^2 / \tau^{\perp}_{\eta} + \tau^{sk}_{\eta},
\]

(23)

with $\tau^{L}_{\eta} / \tau^{sk}_{\eta} \gg 1$. Meanwhile, on the rhs of Eq. (17) the second term containing both $\omega^{s}$ and $\tau^{sk}$ may be disregarded. Eq. (17) can thus be approximated by

\[ 1 = \sum_{\eta} \int \frac{d\phi'}{2\pi} \omega^{s}_{\eta\phi,\eta'\phi'} \left[ \tau^{L}_{\eta} - \cos (\phi' - \phi) \frac{v^0_{\eta'\phi}}{v^0_\eta} \right], \]  

(24)

decoupled from Eq. (15). Once $\tau^{L}_{\eta}$ is obtained from Eq. (24), it can be substituted into Eq. (15) to get $\tau^{sk}_{\eta}$. When the decoupling condition is satisfied, Eqs. (23) and (15) just reduce to Eq. (24) with (19) and (20).

Concrete expressions of $\omega^{s}_{\eta\phi,\eta'\phi'}$ can be obtained by truncating the T-matrix expansion \[ \tilde{3} \]. Randomly distributed identical scalar $\delta$-scatterers are assumed for simplicity: $V (r) = \sum_i V_0 \delta (r - R_i)$. In calculating the skew scattering contributions, we take into account the non-Gaussian disorder correlation $\langle V_{kk'}V_{kk''}V_{kk'''}V_{kk'''} \rangle = n_{im} V_0^4$ and $\langle V_{kk'}V_{kk'}V_{kk''}V_{kk''} \rangle = n_{im} V_0^3$ other than the Gaussian correlation $\langle V_{kk'}V_{kk''} \rangle = n_{im} V_0^2$. Here $V_{kk'}$ is the spin-independent part of the disorder matrix element, $n_{im}$ is the impurity concentration, $\langle \ldots \rangle_c$ denotes the connected part of disorder correlation. The non-Gaussian correlation in $o (V^4)$ usually yields a skew scattering contribution which is a higher order quantity in terms of disorder strength relative to the conventional skew scattering arising from the $o (V^3)$ non-Gaussian correlation and thus can be neglected \[ \tilde{26} \]. However, if the conventional skew scattering contribution vanishes due to some special band structures \[ \tilde{13}, \tilde{14} \], the skew scattering induced by the $o (V^4)$ non-Gaussian correlation plays an important role in the weak scattering regime \[ \tilde{16}, \tilde{17} \].

In the golden rule when calculated to $o (V^4)$, i.e., the third Born order, the scattering rate is approximated as \[ \tilde{15} \] (details in Appendix)

\[
\omega^{s}_{l,tr} \simeq \omega^{s}_{l,tr} + \omega^{a-3nG}_{l,tr} + \omega^{a-4G}_{l,tr} + \omega^{a-4nG}_{l,tr}.
\]

(25)

In the second Born order $\omega^{a-3nG}_{l,tr}$ is related to non-Gaussian disorder correlation, whereas in the third Born order both Gaussian ($\omega^{a-4G}_{l,tr}$) and non-Gaussian ($\omega^{a-4nG}_{l,tr}$) contributions are present. The three antisymmetric scattering rates on the rhs of Eq. (25) have distinct dependence on the impurity concentration and scattering strength: $\omega^{s}_{l,tr} \sim n_{im} V_0^2$, $\omega^{a-3nG}_{l,tr} \sim n_{im} V_0^3$, $\omega^{a-4G}_{l,tr} \sim n_{im} V_0^3$. Because Eq. (15) is linear with respect to $\tau^{sk}$ and $\omega^{s}$, there exist three distinct skew scattering processes described by $\tau^{sk-3nG}_{\eta}$, $\tau^{sk-4G}_{\eta}$ and $\tau^{sk-4nG}_{\eta}$.
where \( \hat{p} \) and \( \hat{r} \) are the 2D momentum, \( \sigma \) is the Pauli matrices, \( \alpha \) the Rashba spin-orbital coupling coefficient, \( J_{\text{ex}} \) the exchange field. Inner Eigenstates of the pure system read

\[
|\eta \rangle_k = \frac{1}{\sqrt{2}} \left[ \frac{\sqrt{1 - \eta \cos \theta}}{\sqrt{1 + \eta \cos \theta}} \right] |\eta \rangle_k^0 + \frac{i}{\sqrt{2}} \left[ \frac{\sqrt{1 + \eta \cos \theta}}{\sqrt{1 - \eta \cos \theta}} \right] |\eta \rangle_k^0, \tag{29}
\]

where \( \eta = \pm \cos \theta = \frac{J_{\text{ex}}}{\Delta_k}, \Delta_k = \sqrt{\alpha^2 k^2 + J_{\text{ex}}^2}, \) \( \sin \theta = \frac{ak}{\Delta_k}, \tan \phi_k = k_y/k_x. \) In the present paper we only consider the case when \( \epsilon_F > J_{\text{ex}} \), i.e., both Rashba bands are partially occupied. For any energy \( \epsilon > J_{\text{ex}} \) there are two iso-energy circles corresponding to the two Rashba bands: \( k_y^2 (\epsilon) = \frac{2m}{\hbar^2} (\epsilon - \eta \Delta_\eta (\epsilon)) \) where we define \( \epsilon_R = m (\frac{2m}{\hbar^2})^2 \) and \( \Delta_\eta (\epsilon) = \sqrt{\epsilon^2 + J_{\text{ex}}^2 + 2\epsilon \eta - \eta \epsilon_R}. \) The DOS in the \( \eta \) band is \( \hat{N}_\eta (\epsilon) = N_0 \frac{\Delta_\eta (\epsilon)}{\Delta_\eta (\epsilon) + \eta \epsilon_R} \) with \( N_0 = \frac{m}{2\pi^2}. \) The intra-band and inter-band energy-integrated scattering rates in the lowest Born order read

\[
\omega_{\eta \phi, \eta \phi'}^{\text{sk}}(\epsilon) = \frac{N_\eta (\epsilon)}{N_0} \left| \langle \eta \rangle_{\hat{r}} \hat{u}_{\eta \phi, \eta \phi'} \right|^2, \tag{30}
\]

with \( \tau = (2\pi n_m v_0^2 N_0/h)^{-1}. \)

### IV. CALCULATIONS

#### A. Model

We consider the spin-polarized Rashba 2DEG

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\alpha}{\hbar} \hat{\sigma} \cdot (\hat{k} \times \hat{\sigma}) - J_{\text{ex}} \hat{\sigma}_z + V(\mathbf{r}), \tag{28}
\]

where \( V(\mathbf{r}) \) is the disorder potential introduced in Sec. III. D, \( m \) is the in-plane effective mass of the conduction electron, \( \hat{p} = \hbar \hat{k} \) the 2D momentum, \( \hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \) are the Pauli matrices, \( \alpha \) the Rashba spin-orbital coupling coefficient, \( J_{\text{ex}} \) the exchange field. Inner Eigenstates of the pure system read

\[
|\eta \rangle_k^{\text{sk}} = \tau^{\text{sk}}_{\eta} - 3nG \frac{2m}{\hbar^2} (\epsilon - \eta \Delta_\eta (\epsilon)) \) where we define \( \epsilon_R = m (\frac{2m}{\hbar^2})^2 \) and \( \Delta_\eta (\epsilon) = \sqrt{\epsilon^2 + J_{\text{ex}}^2 + 2\epsilon \eta - \eta \epsilon_R}. \) The DOS in the \( \eta \) band is \( \hat{N}_\eta (\epsilon) = N_0 \frac{\Delta_\eta (\epsilon)}{\Delta_\eta (\epsilon) + \eta \epsilon_R} \) with \( N_0 = \frac{m}{2\pi^2}. \) The intra-band and inter-band energy-integrated scattering rates in the lowest Born order read

\[
\omega_{\eta \phi, \eta \phi'}^{\text{sk}}(\epsilon) = \frac{N_\eta (\epsilon)}{N_0} \left| \langle \eta \rangle_{\hat{r}} \hat{u}_{\eta \phi, \eta \phi'} \right|^2, \tag{30}
\]

with \( \tau = (2\pi n_m v_0^2 N_0/h)^{-1}. \)

#### B. Coordinate-shift effects

1. **Intra-band and inter-band side-jump velocities**

   The side-jump velocity \( \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} = \sum_{\eta' \mathbf{k}'} \omega_{\eta \eta'}^{\text{sk}} \delta \mathbf{r}_{\eta \mathbf{k}} \mathbf{r}_{\eta' \mathbf{k}'} \) can be further decomposed into intra-band and inter-band components:

   \[
   \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} = \left( \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} \right)^{\text{intra}} + \left( \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} \right)^{\text{inter}}, \tag{31}
   \]

   where the intra-band and inter-band side-jump velocities are

   \[
   \left( \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} \right)^{\text{intra}} = \sum_{\mathbf{k}'} \omega_{\eta \mathbf{k}', \eta \mathbf{k}}^{\text{sk}} \delta \mathbf{r}_{\mathbf{k}', \eta \mathbf{k}} \tag{32}
   \]

   and

   \[
   \left( \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} \right)^{\text{inter}} = \sum_{\mathbf{k}'} \omega_{\eta \mathbf{k}', \eta \mathbf{k}}^{\text{sk}} \delta \mathbf{r}_{\mathbf{k}', \eta \mathbf{k}}, \tag{33}
   \]

   respectively. The intra-band (inter-band) component originates from the coordinate-shift process with initial and final states on the same Rashba band (different Rashba bands). Straightforward evaluations lead to \( \left( \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} (\epsilon, \phi) \right)^{\text{intra}} = \eta \mathbf{k} = \frac{\omega_{\eta \mathbf{k}', \eta \mathbf{k}}^{\text{sk}} \Delta_\eta (\epsilon)}{\Delta_\eta (\epsilon) + \eta \epsilon_R} \) and

   \[
   \left( \mathbf{v}_{\eta \mathbf{k}}^{\text{sj}} (\epsilon, \phi) \right)^{\text{inter}} = \frac{N_\eta (\epsilon)}{\tau N_0} \Omega_\eta (\epsilon) \mathbf{k} \times \hat{\mathbf{z}}, \tag{34}
   \]

   with \( \Omega_\eta (\epsilon) = \frac{\omega_{\eta \mathbf{k}', \eta \mathbf{k}}^{\text{sk}} \Delta_\eta (\epsilon)}{\Delta_\eta (\epsilon) + \eta \epsilon_R} \) the Berry-curvature on the isoenergy circle in momentum-space. It is known that

   •...
coordinate-shift is an inter-band coherence effect induced by the scattering potential, this is just reflected in the above formula which contains the Berry-curvature implying that the virtual vertical inter-band transition contributes to the side-jump velocity even in the intra-band scattering process. In the 2DEG with single Fermi circle such as 2D massive Dirac model \[10\], only the intra-band elastic scattering process exists. However, in the present system with two Fermi circles and short-range disorder, coordinate-shift effects also occur in the inter-band elastic scattering process. While the inter-band side-jump velocity is absent in the limit of smooth disorder potential \[10\], in the considered case it is obtained as

\[
\eta v_{\eta}^{sj}(\epsilon, \phi) = \frac{-\eta e_R}{\Delta_{-\eta}(\epsilon)} \eta v_{\eta}^{intra}(\epsilon, \phi)_{x}. \tag{35}
\]

For the inner (outer) Rashba band the inter-band side-jump velocity is anti-parallel (parallel) to the intra-band velocity. For the inner Rashba band, the magnitude of the inter-band side-jump velocity is always smaller than that of the intra-band one. However, for the outer Rashba band, the inter-band side-jump velocity exceeds the intra-band one when

\[
\epsilon < \epsilon_c = \frac{3\epsilon_R^2 - J_{ex}^2}{2eR}. \tag{36}
\]

For fixed \(J_{ex} = (J_{ex})\), \(\epsilon_c\) increases (decreases) with increasing \(J_{ex}\). One finds that the inter-band coordinate-shift contribution is more important for larger spin-orbital coupling, this can be clearly seen in figure 1, which shows the ratio \(v_{\eta}^{sj}(\epsilon)_{inter}/v_{\eta}^{sj}(\epsilon) = \epsilon_R/\sqrt{\epsilon_R^2 + J_{ex}^2 + 2eR\epsilon}\) with \(v_{\eta}^{sj}(\epsilon) \equiv |v_{\eta}^{sj}(\epsilon, \phi)|\).

The vector form of the total side-jump velocity reads

\[
v_n^{sj}(\epsilon, \phi) = \eta k_\eta(\epsilon) \times J_{ex} \eta^2 \frac{1}{2\Delta_\eta(\epsilon)(J_{ex}^2 + 2\epsilon R\epsilon)} \tau. \tag{37}
\]

The key is that \(v_n^{sj}\) is orthogonal to \(k_\eta(\epsilon)\), and the side-jump velocities of the inner and outer Rashba bands have opposite chirality.

2. Solution of the anomalous DF

Substituting Eq. (37) into (10) yields the 2 \times 2 linear system \((\eta = \pm)\) determining \(\tau_{\eta}^{sj}(\epsilon)\):

\[
1 = \sum_\eta \int \frac{d\epsilon}{2\pi} \int_{\eta_0}^{2\pi} \eta_0^2 \eta_0 \Delta_\eta(\epsilon) \cos(\phi' - \phi) \tau_{\eta}^{sj}(\epsilon). \tag{38}
\]

The factor \(\eta/\eta_0\) in the second term inside the square bracket just reflects different chirality of side-jump velocity on the two Rashba bands. It follows that

\[
\tau_{\eta}^{sj}(\epsilon) = \tau \frac{J_{ex}^2 + 2eR\epsilon}{J_{ex}^2 + eR\epsilon}, \tag{39}
\]

which does not depend on the band index \(\eta\). The anomalous part of the out-of-equilibrium DF is then obtained as

\[
g_n^{adis}(\epsilon) = \eta (\hat{t} \times \vec{F}) \cdot k_\eta(\epsilon) \frac{J_{ex}\alpha^2}{2\Delta_\eta(\epsilon)} \frac{1}{J_{ex}^2 + eR\epsilon} \frac{\partial f^0}{\partial \epsilon}, \tag{40}
\]

which is different from the one in the limit of smooth disorder potential in which only the intra-band small angle electron-impurity scattering dominates \[10\].

C. DF responsible for the skew scattering

The derivation of the needed scattering rates in this section is provided in the Appendix.

1. Longitudinal nonequilibrium DF

The correct calculation of \(\tau_{\eta}^{L}(\epsilon)\) is very crucial in order to obtain the skew scattering contributions to AHE via Eq. (27). The appropriate starting point is just the 2 \times 2 linear system \((\eta = \pm)\) Eq. (24). Substituting corresponding quantities, the longitudinal transport time is obtained as

\[
\tau_{\eta}^{L}(\epsilon) = \tau \frac{N_{-\eta}(\epsilon)}{N_0}. \tag{41}
\]

It follows that \(g_{\eta}^{L}(\epsilon, \vartheta(K_\eta(\epsilon))) = (-\partial_\epsilon f^0) \vec{F} \cdot \frac{h k_\eta(\epsilon)}{m} N_{-\eta}(\epsilon)\). This solution has the same form as that in a nonmagnetic Rashiba 2DEG \[27\].

2. Conventional skew scattering

Due to \(N_{-\eta}(\epsilon)/N_\eta(\epsilon) = \Delta_{-\eta}(\epsilon)/\Delta_\eta(\epsilon)\), the anti-symmetric part of the energy-integrated third-order
The anti-symmetric part of the energy-integrated fourth-order Gaussian scattering rate vanishes
\[ \omega^{a-3G}_{\eta\eta'}(\epsilon) \sim \sin(\phi' - \phi) \sum_{\eta''} \eta'' N_{\eta''}(\epsilon) k_{\eta''}(\epsilon) = 0, \] (42)
then \( \tau^{sk-3G}_{\eta}(\epsilon) = 0 \) and no corresponding nonequilibrium response arises when both Rashba bands are partially occupied. We emphasize that the absence of conventional skew scattering stems directly from the vanishing \( \omega^{a-3G}_{\eta\eta'}(\epsilon) \). Thus the previous SBE calculation [13] in which the multiple-Fermi-circle SBE was solved by using the \( 1/\tau^\| \) & \( 1/\tau^\perp \) solution also produced this result (via Eq. (41)).

### 3. Intrinsic skew scattering

The anti-symmetric part of the energy-integrated fourth-order Gaussian scattering rate reads
\[ \omega^{a-4G}_{\eta\eta',\eta''\eta'''}(\epsilon) = \frac{\hbar^2 N_{\eta'}(\epsilon)}{4N_0\tau^2} \eta' \frac{J_{ex}}{J_{ex} + 2\pi R\epsilon} \frac{\alpha^2 k_{\eta}(\epsilon) k_{\eta'}(\epsilon)}{\Delta_{\eta}(\epsilon) \Delta_{\eta'}(\epsilon)} \times \sin(\phi' - \phi), \] (43)
Substituting the expression of \( \tau^{L}_{\eta} \) and scattering rates into the SBE (27) for \( i = 4G \) yields the following \( 2 \times 2 \) linear system for \( \tau^{sk-4G}_{\eta} \)
\[ \begin{align*}
\frac{J_{ex}^2}{\Delta_{\eta}} + \frac{1}{4} \frac{\alpha^2 k_{\eta}^2}{\Delta_{\eta}} + \frac{N_{-\eta}}{2N_{\eta}} \left( 1 - \frac{J_{ex}^2}{\Delta_{\eta}} \right) \tau^{sk-4G}_{\eta} & = \frac{\eta J_{ex} \epsilon R}{J_{ex}^2 + 2\pi R\epsilon} \frac{\hbar}{2}. \\
\tau^{sk-4G}_{-\eta} & = \frac{\eta J_{ex} \epsilon R}{J_{ex}^2 + 2\pi R\epsilon} \frac{\hbar}{2}. \\
\end{align*} \] (44)
Then we obtain the transverse transport time induced by the 4th order Gaussian skew scattering as
\[ \tau^{sk-4G}_{\eta}(\epsilon) = \frac{\eta J_{ex} \epsilon R}{2(J_{ex}^2 + \pi R\epsilon)} \frac{1}{J_{ex}^2 + 2\pi R\epsilon}. \] (45)
This quantity does not depend on the impurity concentration and scattering strength, just the characteristic of intrinsic skew scattering. The corresponding part of the nonequilibrium DF is given by
\[ g^{sk-4G}_{\eta}(\epsilon) = -g^{dis}_{\eta}(\epsilon). \] (46)
We see that, when both Rashba bands are partially occupied, the transport contributions from the anomalous distribution related to the coordinate-shift and that from the intrinsic skew scattering cancel each other exactly.

### 4. Hybrid skew scattering

The anti-symmetric part of the energy-integrated fourth-order non-Gaussian scattering rate is obtained to be (assume \( \epsilon_R \leq J_{ex} \))
\[ \omega^{a-4G}_{\eta\eta',\eta''\eta'''}(\epsilon) = \frac{\hbar N_{\eta'}(\epsilon)}{4\pi R \tau^2} \eta' \frac{\alpha^2 k_{\eta}(\epsilon) k_{\eta'}(\epsilon)}{\Delta_{\eta}(\epsilon) \Delta_{\eta'}(\epsilon)} I(\epsilon) \sin(\phi' - \phi), \] (47)
where \( I(\epsilon) \) is given by Eq. (A17). Substituting Eq. (41) and relevant scattering rates into Eq. (27) for \( i = 4G \) yields two coupled equations (not shown) determining \( \tau^{sk-4G}_{\eta} \). Comparing these equations to Eq. (41), one can find directly that
\[ \tau^{sk-4G}_{\eta}(\epsilon) = \tau^{sk-4G}_{\eta} S(\epsilon), \] (48)
where \( S(\epsilon) = I(\epsilon) \frac{N_\alpha}{\hbar} J_{ex}^2 + 2\pi R\epsilon J_{ex} \) is band-independent.

### D. Results for AHE

The anomalous Hall conductivity in the SBE framework is expressed as \( \sigma^{yx}_{\eta} = \sigma^{in}_{yx} + \sigma^{sk}_{yx} + \sigma^{dis}_{yx} + \sigma^{adis}_{yx} \), where \( \sigma^{sk}_{yx} = e \sum_i \langle v_i \rangle \frac{g^{L}_i}{(E)_x} \) and \( \sigma^{adis}_{yx} = e \sum_i \langle v_i \rangle \frac{g^{dis}_{yx}}{(E)_x} \) represent the Hall conductivities due to the side-jump velocity and anomalous DF (skew scattering), respectively. \( \sigma^{in}_{yx} \) is the intrinsic contribution determined by the momentum-space Berry-curvature [2]. When both Rashba bands are partially occupied, \( g^{dis}_{yx} = -g^{sk-4G}_{\eta}, g^{sk-3nG}_{\eta} = 0 \) and \( \sigma^{adis}_{yx} = -\sigma^{in}_{yx} = e^2 \pi R J_{ex} \epsilon R / (2J_{ex}^2 + 2\pi R\epsilon) \). Thus \( \sigma^{in}_{yx} + \sigma^{sk}_{yx} + \sigma^{sk-4G}_{yx} + \sigma^{sk-3nG}_{yx} = 0 \). This is just the result obtained by the Kubo-Streda method in spin-sx representation [14]: \( \sigma^{yx}_{\eta} = \sigma^{yx}_{\eta} + \sigma^{yx}_{\eta} = 0, \sigma^{yx}_{\eta} = 0 \). Here \( \sigma^{yx}_{\eta} = \sigma^{in}_{yx} - \sigma^{yx}_{\eta} \) denotes the bare bubble contribution with \( \sigma^{yx}_{\eta} \) representing Streda’s Fermi sea term, and the ladder vertex correction \( \sigma^{yx}_{\eta} = \sigma^{yx}_{\eta} + \sigma^{yx}_{\eta} \). The hybrid skew scattering dominates when both Rashba bands are partially occupied [16], \( \sigma^{yx}_{\eta} = \sigma^{yx}_{\eta} \) and
\[ \sigma^{yx}_{\eta} = -\frac{e^2}{2\pi \hbar} \frac{N_\alpha \epsilon R}{\hbar \tau^{sk-4G}_{\eta}(\epsilon)} \] (49)
With Eq. (A17), this formula just reproduces the result obtained by the multi-band Keldysh formalism [16, 28].

### V. DISCUSSION AND CONCLUSION

Let us start this discussion section with a remark on the notable difference between the coordinate-shift contribution to the AHE in the presence of pointlike impurities and that in the limit of smooth disorder potential analyzed by Sinitsyn et al. [10]. In the latter case the coordinate-shift contribution \( \sigma^{sx}_{\eta} + \sigma^{adis}_{\eta} \) has the same sign as the intrinsic one for sufficiently large Fermi energy, while in the former case the coordinate-shift contribution has the opposite sign as and is twice times larger than the intrinsic one. Now we discuss the difference between \( 1/\tau^\| \) & \( 1/\tau^\perp \) solution and our solution explicitly. For \( \epsilon > J_{ex} \)
in the $1/\tau^\parallel$ & $1/\tau^\perp$ solution one obtains \cite{29} the longitudinal transport time $\tilde{\tau}_n^L = \tau_n^\parallel$ different from $\tau_n^L$ given by Eq. \cite{41}:

$$\frac{\tilde{\tau}_n^L}{\tau_n^L} = \left(\frac{N_0}{N_\eta}\right)^2 \left[\frac{J_{xx}^2 + \alpha^2 k_n^2}{\Delta_\eta^2} + \frac{\alpha^2 k_n^2}{4\Delta_n^2 + \Delta_\eta \Delta_{-\eta}}\right] + \frac{N_{-\eta}}{2N_\eta} \left(1 - \frac{J_{xx}^2}{\Delta_\eta \Delta_{-\eta}}\right)^{-1} \left(\epsilon_R^2 + J_{xx}^2 + 2\epsilon_R\epsilon\right) \left(\epsilon_R^2 + 2\epsilon_R\epsilon\right) \left(J_{xx}^2 + 2\epsilon_R\epsilon\right) \left(J_{xx}^2 + 2\epsilon_R\epsilon - \eta\epsilon_R\right) \frac{\epsilon_R^2 + J_{xx}^2 + 2\epsilon_R\epsilon - \eta\epsilon_R}{\eta\epsilon_R + \epsilon_R^2 + J_{xx}^2 + 2\epsilon_R\epsilon - \eta\epsilon_R}.$$ (50)

For the skew scattering, the $1/\tau^\parallel$ & $1/\tau^\perp$ solution

$$\tilde{\tau}_n^{s-k-i} = \left(\frac{\tilde{\tau}_n^L}{\tau_n^L}\right)^2 \int d\phi \frac{\epsilon_R}{2\pi} \omega_{n\rho,\eta'\phi} \frac{\epsilon_R^0}{\tau_n^0} \sin(\phi - \phi') \frac{\epsilon_R}{\tau_n}.$$ (51)

yields $\tilde{\tau}_n^{sk-3nG} = 0$ and

$$\frac{\tilde{\tau}_n^{sk-4G}}{\tau_n^{sk-4G}}(\epsilon) = \left(\frac{\tilde{\tau}_n^L}{\tau_n^L}\right)^2 \frac{J_{xx}^2 + 2\epsilon_R\epsilon}{\eta\epsilon_R + \epsilon_R^2 + J_{xx}^2 + 2\epsilon_R\epsilon - \eta\epsilon_R}.$$ (52)

One can verify that $\omega_{n\rho,\eta'\phi}^{sk-4G} / \omega_{n\rho,\eta'\phi}^{sk-4G} = S(\epsilon)$, thus $\tilde{\tau}_n^{sk-4nG} = \tau_n^{sk-4nG}$ and $\tilde{\tau}_n^{sk-4G} = \tau_n^{sk-4G}$.

The comparison of the $1/\tau^\parallel$ & $1/\tau^\perp$ solution and the present transport relaxation time solution are shown clearly in figure 2. Figure 2(a) and 2(b) show the ratio $\tilde{\tau}_n^L / \tau_n^L$ and $\tilde{\tau}_n^L / \tau_n^\perp$, respectively. When $\epsilon = J_{ex}$, Eq. \cite{50} tells $\tilde{\tau}_n^L / \tau_n^\perp = 1$.

In the limit of large Fermi energy, $\tilde{\tau}_n^L \approx \tau_n^\parallel \approx \tau_n^\perp$; this tendency is clear in figure 2(a) and will be clear in figure 2(b) if the curves are plotted up to $\epsilon / J_{ex} \approx 50$ (not shown). Figure 2(c) and 2(d) show the ratio $\tilde{\tau}_n^{sk} / \tau_n^\parallel$ and $\tilde{\tau}_n^{sk} / \tau_n^\perp$, respectively. Because in calculating the hybrid skew scattering we assume $\epsilon_R \leq J_{ex}$, here we follow this assumption. For the inner Rashba band, the deviation of $\tilde{\tau}_n^{sk}$ from $\tau_n^{sk}$ increases monotonically with increasing $\epsilon_R / J_{ex}$ in the regime $\epsilon_R \leq J_{ex}$. In the limit of large Fermi energy, $\tau_n^{sk} / \tau_n^{sk} \to 1/4$; this limiting behavior can be visible if the curves in figure 2(c) and 2(d) are plotted up to $\epsilon / J_{ex} \approx 50$ (not shown). Whereas when $\epsilon = J_{ex}$, Eq. \cite{62} tells $\tilde{\tau}_n^{sk} / \tau_n^{sk} = 1$.

To illustrate more transparently the difference between the $1/\tau^\parallel$ & $1/\tau^\perp$ solution and the solution of Eqs. \cite{24} and \cite{27} when the decoupling condition is not satisfied, we also consider the nonmagnetic Rashba 2DEG with pointlike potential scatterers as the simplest example \cite{27} in figure 2(e) and 2(f). In this isotropic 2DEG without AHE, the lowest Born approximation is sufficient for calculating the scattering rate, thus $1/\tau_n^L = 0$. When both Rashba bands are partially occupied, the decoupling condition \cite{24} is not satisfied and $\tilde{\tau}_n^L = \tau_n^\parallel$ given by Eq. \cite{19} differs from the solution of Eq. \cite{24} & \cite{27}. In the weak spin-orbital coupling limit or large Fermi energy limit, one finds that $\tau_n^L(\epsilon_F) / \tau_n^\perp - \tau_n^\perp \approx -\alpha \epsilon_F / 2\epsilon_F$ while $\tilde{\tau}_n^{sk}(\epsilon_F) / \tau_n^{sk} - \tau_n^{sk} \approx -\alpha \epsilon_F / 4\epsilon_F$, where $\alpha = \sqrt{2m\epsilon_F}/\hbar$. Figure 2(e) and 2(f) show that, in the nonmagnetic Rashba 2DEG, Eq. \cite{19} (some times called the “modified relaxation time approximation” \cite{30}) may be regarded as a more precise approximation than the constant relaxation time approximation $1/\tau = \sum_{\eta' \eta} \int d\phi \omega_{n\rho,\eta'\phi}^{2nG} \sin(\phi - \phi) I(\epsilon_F)$, thus Eq. \cite{50} yields $\tilde{\tau}_n^{sk-3nG} \approx 0$ and $\tilde{\tau}_n^{sk-4nG} \sim I(\epsilon_F)/n_{im}$. On the other hand, as shown by Eq. \cite{53} and figure 2(c) and 2(d), in the SBE formalism the accurate result Eq. \cite{19} does not follow from the $1/\tau^\parallel$ & $1/\tau^\perp$ solution.
Finally we mention that, it has been proposed that the analytical solution to SBE\(^1\) under the first Born approximation in general (isotropic and anisotropic) 2DEGs should be obtained following an integral equation approach instead of the \(1/\tau^\|\) & \(1/\tau^\perp\) solution or other relaxation time approximation schemes \([23]\). It can be verified that in isotropic 2DEGs our solution to SBE\(^1\) (or Eq. \(24\)) in the first Born approximation is consistent with the integral equation approach.

In summary, we have presented a systematic study on how to solve the SBE describing AHE in isotropic 2DEG with static impurities. For single-Fermi-circle 2DEG we proved that the widely-used \(1/\tau^\|\) & \(1/\tau^\perp\) solution is correct for longitudinal and skew scattering transport. While, for multiple-Fermi-circle 2DEG it was shown that the solution of SBE could differ from the \(1/\tau^\|\) & \(1/\tau^\perp\) solution if a decoupling condition is unsatisfied. The decoupling condition states in what cases the transport relaxation times (for longitudinal transport and for skew scattering) on different bands are decoupled to each other. We proved that this decoupling condition is unsatisfied in a spin-polarized Rashba 2DEG with pointlike potential impurities when both Rashba bands are partially occupied. Focusing on this specific system, when considering intrinsic and hybrid skew scatterings, it was showed that the deviation of \(1/\tau^\|\) & \(1/\tau^\perp\) solution from our solution is notable for large Fermi energies. For coordinate-shift effects, the side-jump velocity acquired in the inter-band elastic scattering process was shown to be more important for larger Rashba spin-orbit coupling. The inter-band side-jump velocity may even exceed the intra-band one for the outer Rashba band. The coordinate-shift contribution to the AHE in the presence of pointlike impurities is significantly different from that in the limit of smooth disorder potential analyzed by Sinitsyn et al. \([10]\). In the latter case the coordinate-shift contribution has the same sign as the intrinsic one for sufficiently large Fermi energy, while in the former case the coordinate-shift contribution has the opposite sign as and is two times larger in magnitude than the intrinsic one. The validity of our solution to the SBE is confirmed by the fact that the calculated result for AHE is in full agreement with the multi-band Keldysh formalism \([11, 16]\).

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### Appendix A: Calculations of the anti-symmetric part of scattering rates responsible for skew scatterings

1. **Truncating the T-Matrix expansion and Born approximations beyond the 1st order**

The T-matrix expansion according to Lippmann-Schwinger equation is

\[
T_{l,l'} = V_{l,l'} + \sum_{lm} \frac{V_{lm} V_{ml'}}{\epsilon_l - \epsilon_m + i\epsilon} + \sum_{lm} \sum_{lm'} \frac{V_{lm} V_{ml'm}}{\epsilon_l - \epsilon_{m'} + i\epsilon} \left( \epsilon_l - \epsilon_{m'} + i\epsilon \right) + \ldots
\]

For elastic scattering, the golden rule yields the scattering rate as \(\omega_{l,l'} = \omega^{(2)}_{l,l'} + \omega^{(3)}_{l,l'} + \omega^{(4)}_{l,l'} + \ldots\), where \(\omega^{(2)}_{l,l'} = \omega_{l,l'}^2 = \frac{2\pi n_G V_l^2}{\hbar} |\langle u_{l'}|u_l\rangle|^2 \delta(\epsilon_l - \epsilon_{l'})\) is symmetric, \(\omega^{(3)}_{l,l'}\) is related to non-Gaussian disorder correlation \([3, 13]\),

\[
\omega^{(3)}_{l,l'} = \frac{2\pi}{\hbar} \left( \sum_{lm} \frac{\langle V_{lm} V_{ml}' \rangle}{\epsilon_l - \epsilon_m + i\epsilon} + c.c. \right) \delta(\epsilon_{l'} - \epsilon_l),
\]

whereas \(\omega^{(4)}_{l,l'}\) contains both Gaussian (\(G\)) and non-Gaussian (\(nG\)) contributions \(\omega^{(4)}_{l,l'} = \omega^{4G}_{l,l'} + \omega^{4nG}_{l,l'}\). Here we only take into account the Gaussian part responsible for the intrinsic skew scattering in the non-crossing approximation \([3, 4, 6]\) and the non-Gaussian part responsible for the hybrid skew scattering containing only one scattering center \([10, 17]\). Then \(\omega^{4G}_{l,l'} = \omega^{4G1}_{l,l'} + \omega^{4G2}_{l,l'} + \omega^{4G3}_{l,l'}\) with

\[
\omega^{4G1}_{l,l'} = \frac{2\pi}{\hbar} \left( \sum_{lm} \sum_{lm'} \frac{\langle V_{lm} V_{ml'} \rangle}{\epsilon_l - \epsilon_m + i\epsilon} \frac{\langle V_{ml'} V_{l'm} \rangle}{\epsilon_l - \epsilon_{m'} + i\epsilon} \right) \delta(\epsilon_{l'} - \epsilon_l),
\]

\[
\omega^{4G2}_{l,l'} = \frac{2\pi}{\hbar} \left( \sum_{lm} \sum_{lm'} \frac{\langle V_{lm} V_{ml'} \rangle}{\epsilon_l - \epsilon_m - i\epsilon} \frac{\langle V_{ml'} V_{l'm} \rangle}{\epsilon_l - \epsilon_{m'} - i\epsilon} + c.c. \right) \delta(\epsilon_{l'} - \epsilon_l),
\]

...
Thus we finally get the energy-integrated 3rd order anti-symmetric scattering rate in Eq. (42)

\[
\omega_{l,tr}^{4G,3} = \frac{2\pi}{\hbar} \left( \sum_{l''} \sum_{l'''} \left\langle V_{l''} V_{l'''} \right| \left\langle V_{l'''l} V_{l''l'} \right| + c.c. \right) \delta (\epsilon_{l'} - \epsilon_l),
\]

and \(\omega_{l,tr}^{4nG} = \omega_{l,tr}^{4nG,1} + \omega_{l,tr}^{4nG,2}\) with

\[
\omega_{l,tr}^{4nG,1} = \frac{2\pi}{\hbar} \sum_{l''} \sum_{l'''} \left\langle V_{l''} V_{l'''} V_{l''} V_{l'''} \right| \delta (\epsilon_{l'} - \epsilon_l),
\]

\[
\omega_{l,tr}^{4nG,2} = \frac{2\pi}{\hbar} \left( \sum_{l''} \sum_{l'''} \left\langle V_{l''} V_{l'''} V_{l''} V_{l'''} \right| + c.c. \right) \delta (\epsilon_{l'} - \epsilon_l).
\]

The symmetric parts of \(\omega_{l,tr}^{(3)}\) and \(\omega_{l,tr}^{(4)}\) only renormalize the result of \(\omega_{l,tr}^{2}\), and thus will not be considered further, while the anti-symmetric parts lead to skew scattering. Therefore the scattering rate is approximated as Eq. (25).

2. Conventional skew scattering

The anti-symmetric part of 3rd-order scattering rate can be obtained by

\[
\omega_{l,tr}^{a-3nG} = \frac{4\pi n_{im} V_l^3}{\hbar} \delta (e_{k_l} - e_{k_k'}) \sum_{\eta'' \eta'''} \frac{1}{e_{k} - e_{k''} + i\epsilon} \text{Im} \left[ \left\langle u_{k_l}^\eta | u_{k_k'}^{\eta''} \right| \left\langle u_{k_k'}^{\eta'''} | u_{k_k}^\eta \right\rangle \right],
\]

where straightforward calculation gives \(\int \frac{dk'''}{2\pi} \text{Im} \left[ \left\langle u_{k_k}^\eta | u_{k_k'}^{\eta''} \right| \left\langle u_{k_k'}^{\eta'''} | u_{k_k}^\eta \right\rangle \right] = -\frac{1}{2} \eta'' \eta''' \sin (\phi' - \phi) \sin \theta \sin \phi \cos \phi''.\) Thus we finally get the energy-integrated 3rd order anti-symmetric scattering rate in Eq. (26).

3. Intrinsic skew scattering

Firstly, the anti-symmetric part of \(\omega_{l,tr}^{4G,1}\) is shown to be

\[
\omega_{l,tr}^{a-4G,1} = -\frac{2\pi}{\hbar} \left( n_{im} V_0^2 \right)^2 \delta (e_{k_l} - e_{k_k'}) \sum_{\eta'' \eta'''} \frac{1}{e_{k} - e_{k''} + i\epsilon} \text{Im} \left[ \left\langle u_{k_k}^{\eta''} | u_{k_k'}^{\eta'''} \right| \left\langle u_{k_k'}^{\eta'''} | u_{k_k}^\eta \right\rangle \right] \left\langle u_{k_k}^\eta | u_{k_k'}^{\eta''} \right| \left\langle u_{k_k'}^{\eta'''} | u_{k_k}^\eta \right\rangle.
\]

If \(\eta'' = \eta\), \(\text{Im} \left[ \left\langle u_{k_k}^{\eta''} | u_{k_k'}^{\eta'''} \right| \left\langle u_{k_k'}^{\eta''} | u_{k_k}^{\eta''} \right\rangle \left\langle u_{k_k'}^{\eta''} | u_{k_k}^{\eta''} \right\rangle \right] = 0\) and then only the \(\eta'' = -\eta''\) term contributes:

\[
\omega_{l,tr}^{a-4G,1} = -\frac{2\pi}{\hbar} \left( n_{im} V_0^2 \right)^2 \delta (e_{k_l} - e_{k_k'}) \sum_{\eta'' \eta'''} \frac{1}{e_{k} - e_{k''} + i\epsilon} \text{Im} \left[ \left\langle u_{k_k}^{\eta''} | u_{k_k'}^{\eta''} \right| \left\langle u_{k_k'}^{\eta''} | u_{k_k}^{\eta''} \right\rangle \right] \left\langle u_{k_k}^\eta | u_{k_k'}^{\eta''} \right| \left\langle u_{k_k'}^{\eta''} | u_{k_k}^\eta \right\rangle.
\]

Secondly, we find that it is convenient to introduce two quantities \(\omega_{l,tr}^{a-4G,2'}\) and \(\omega_{l,tr}^{a-4G,3'}\) by

\[
\omega_{l,tr}^{a-4G,2'} = \frac{\omega_{l,tr}^{4G,2} - \omega_{l,tr}^{4G,3}}{2}, \quad \omega_{l,tr}^{a-4G,3'} = \frac{\omega_{l,tr}^{4G,3} - \omega_{l,tr}^{4G,2}}{2} = -\omega_{l,tr}^{a-4G,2'},
\]

with

\[
\omega_{l,tr}^{a-4G,2'} = -\frac{4\pi}{\hbar} \left( n_{im} V_0^2 \right)^2 \delta (e_{k_l} - e_{k_k'}) \sum_{\eta'' \eta'''} \frac{1}{e_{k} - e_{k''} + i\epsilon} \text{Im} \left[ \left\langle u_{k_k}^{\eta''} | u_{k_k'}^{\eta''} \right| \left\langle u_{k_k'}^{\eta''} | u_{k_k}^{\eta''} \right\rangle \right] \left\langle u_{k_k}^\eta | u_{k_k'}^{\eta''} \right| \left\langle u_{k_k'}^{\eta''} | u_{k_k}^\eta \right\rangle.
\]
Straightforward calculations lead to

$$\text{Im} \int \frac{d \omega''}{2 \pi} \langle u^\eta_k | u^\eta_k \rangle \langle u^{\eta'}_{k'} | u^{\eta'}_{k'} \rangle \langle u^{\eta''}_{k''} | u^{\eta''}_{k''} \rangle = \frac{1}{4} \eta \sin (\phi' - \phi) \frac{\alpha k \alpha' k'}{\Delta_k \Delta_{k'} \Delta_{k''}}.$$  \hspace{1cm} (A12)

$$\text{Im} \int \frac{d \omega''}{2 \pi} \langle u^\eta_k | u^\eta_k \rangle \langle u^{\eta'}_{k'} | u^{\eta'}_{k'} \rangle \langle u^{\eta''}_{k''} | u^{\eta''}_{k''} \rangle = -\frac{1}{4} \eta \eta' \sin (\phi' - \phi) \frac{\alpha k \alpha' \eta \eta' J_{ex}}{\Delta_k \Delta_{k'} \Delta_{k''}}.$$  \hspace{1cm} (A13)

then the energy-integrated scattering rate for the intrinsic skew scattering reads

$$\omega_{\eta \phi, \eta' \phi'}^{a-4G,1}(\epsilon) = \frac{1}{4} \eta \eta' \Delta_{\eta} \Delta_{\eta'} \sin (\phi' - \phi) \frac{J_{ex}^2}{J_{ex}^2 + 2 \epsilon \Delta_{\eta} \Delta_{\eta'}}.$$  \hspace{1cm} (A14)

and \( \omega_{\eta \phi, \eta' \phi'}^{a-4G,2}(\epsilon) \sim \sin (\phi' - \phi) \sum_{\eta''} \eta'' N_{\eta''}(\epsilon) \Delta_{\eta''}(\epsilon) = 0 \) as the vanishing of \( \omega_{\eta \phi, \eta' \phi'}^{a-3G}(\epsilon) \). Thus we also have \( \omega_{\eta \phi, \eta' \phi'}^{a-4G,3}(\epsilon) = 0 \), and then the total anti-symmetric scattering rate arising from the 4th-order Gaussian disorder correlation is completely attributed to \( \omega_{\eta \phi, \eta' \phi'}^{a-4G,1}, \omega_{\eta \phi, \eta' \phi'}^{a-4G,4}(\epsilon) = \omega_{\eta \phi, \eta' \phi'}^{a-4G,1}(\epsilon) \).

4. Hybrid skew scattering

The anti-symmetric part of the non-Gaussian scattering rate Eq. (A6) reads

$$\omega_{a-4G,1}^{\eta \phi, \eta' \phi'}(\epsilon) = \frac{2 N_{\eta} N_{\eta'}^2}{\hbar} \delta (\epsilon_k - \epsilon_k') \sum_{\eta''} \sum_{\eta'''} \text{Im} \left[ \frac{1}{4} \eta \eta' \Delta_{\eta} \Delta_{\eta'} \sin (\phi' - \phi) \frac{J_{ex}^2}{J_{ex}^2 + 2 \epsilon \Delta_{\eta} \Delta_{\eta'}} \right].$$  \hspace{1cm} (A15)

Straightforward evaluation gives

$$\text{Im} \int \frac{d \omega''}{2 \pi} \int \frac{d \omega'''}{2 \pi} \langle u^\eta_k | u^{\eta''}_{k''} \rangle \langle u^{\eta''}_{k''} | u^{\eta'}_{k'} \rangle \langle u^{\eta'}_{k'} | u^\eta_k \rangle = \frac{1}{8} \eta \eta' \sin (\phi' - \phi) \sin \theta \sin \theta' (\eta'' \cos \theta'' - \eta'' \cos \theta'),$$

after some manipulations we get the energy-integrated scattering rate as

$$\omega_{a-4G,1}^{\eta \phi, \eta' \phi'}(\epsilon) = \frac{2 N_{\eta} N_{\eta'}^2}{\hbar} \delta (\epsilon_k - \epsilon_k') \sum_{\eta''} \int \text{Im} \left[ \frac{1}{4} \eta \eta' \Delta_{\eta} \Delta_{\eta'} \sin (\phi' - \phi) \frac{J_{ex}^2}{J_{ex}^2 + 2 \epsilon \Delta_{\eta} \Delta_{\eta'}} \right].$$  \hspace{1cm} (A16)

Note that the notation \( \int \omega'' \) is to emphasize the different lower integral limits of energy integral for different Rashba bands. Here we assume \( J_{ex} \geq \epsilon_R \), thus \( \int_+ = \int_{J_{ex}}^\infty, \int_- = \int_{-J_{ex}}^{-\infty} \) and then \( \sum_{\eta''} \int \omega'' \) \( J_{ex} \) \( - J_{ex} \) \( \omega'' \). Then

$$\omega_{a-4G,1}^{\eta \phi, \eta' \phi'}(\epsilon) = \frac{2 N_{\eta} N_{\eta'}^2}{\hbar} \delta (\epsilon_k - \epsilon_k') \sum_{\eta''} \left[ \int_{-J_{ex}}^{J_{ex}} \text{Im} \left( \frac{1}{4} \eta \eta' \Delta_{\eta} \Delta_{\eta'} \frac{J_{ex}^2}{J_{ex}^2 + 2 \epsilon \Delta_{\eta} \Delta_{\eta'}} \right) \right],$$

by using \( J_{ex} \geq \epsilon_R \) and \( \epsilon > J_{ex} \), we get Eq. (A17) with a minus sign where \( I(\epsilon) \) represents

$$I(\epsilon) = \int_{-J_{ex}}^{J_{ex}} \text{Im} \left( \frac{J_{ex}^2}{J_{ex}^2 + 2 \epsilon \Delta_{\eta} \Delta_{\eta'}} \right) = \frac{m}{\hbar^2 k^2 \epsilon} \ln \frac{k^2 \epsilon}{k^2 \epsilon}.$$  \hspace{1cm} (A17)

We note that it is the difference in the lower integral of the energy integral between the ± Rashba bands that contributes this \( I(\epsilon) \), which is a key factor of hybrid skew scattering. The calculation of the anti-symmetric scattering...
rate of the non-Gaussian scattering rate Eq. (A7) is similar, the factor \( I(\epsilon) \) also plays a key role. The resulting energy-integrated scattering rate is \( \omega^{(a-4nG,2)}_{\eta\phi,\eta'\phi'}(\epsilon) = -2\omega^{(a-4nG,1)}_{\eta\phi,\eta'\phi'}(\epsilon) \), then \( \omega^{(a-4nG,2)}_{\eta\phi,\eta'\phi'} = \omega^{(a-4nG,1)}_{\eta\phi,\eta'\phi'} + \omega^{(a-4nG,2)}_{\eta\phi,\eta'\phi'} = -\omega^{(a-4nG,1)}_{\eta\phi,\eta'\phi'} \).

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