Null-controllability for parabolic equations in Banach spaces

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Abstract

We study null-controllability of parabolic equations in Banach spaces. We show that a generalized uncertainty principle and a dissipation estimate imply (approximate) null-controllability. Our result unifies and generalizes earlier results obtained in the context of Hilbert and Banach spaces. In particular, we do not assume reflexivity of the underlying Banach space, thus allowing to apply our result, e.g., to $L_1$-spaces. As an application we consider parabolic equations of the form
\[
\dot{x}(t) = -Ax(t) + 1_E u(t), \quad x(0) = x_0 \in L^p(\mathbb{R}^d),
\]
with interior control on a so-called thick set $E \subset \mathbb{R}^d$, where $p \in [1, \infty)$, and where $A$ is an elliptic operator of order $m \in \mathbb{N}$ in $L^p(\mathbb{R}^d)$. In particular, our result applies to the case $p = 1$.

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1 Introduction

We consider inhomogeneous abstract Cauchy problems of the form
\[
\dot{x}(t) = -Ax(t) + Bu(t), \quad t \in (0, T], \quad x(0) = x_0 \in X,
\] (1)
where $X$ and $U$ are Banach spaces, $-A$ is the generator of a $C_0$-semigroup $(S_t)_{t \geq 0}$ on $X$, $B: U \to X$ is a bounded linear operator, $T > 0$, and where $u \in L^r((0, T); U)$ with some $r \in [1, \infty]$. Hence, the influence of the control function $u$ is restricted to the range of the operator $B$. The focus of this paper is laid on null-controllability, that is, for any initial condition $x_0 \in X$ there is a control function $u$ such that the solution of (1) at time $T$ equals zero. We will also be concerned with the notion of approximate null-controllability, which means that for any $\varepsilon > 0$ and any $x_0 \in X$ we can find a control function $u$ such that the solution of (1) at time $T$ has norm smaller than $\varepsilon$. By linearity, (approximate) null-controllability implies that any target state in the range of the operator $S_T$ can be reached (up to an error $\varepsilon$) within time $T$. An equivalent formulation of (approximate) null-controllability is final-state observability of
the adjoint system to (1). This means that there is a constant $C_{\text{obs}} \geq 0$ such that for all $\varphi_0 \in X'$ we have

$$\| S_T^{r'} \varphi_0 \|_{X'} \leq \begin{cases} C_{\text{obs}} \left( \int_0^T \| B' S_t^{r'} \varphi_0 \|_{U'}^r \, dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\ C_{\text{obs}} \sup_{t \in [0,T]} \| B' S_t^{r'} \varphi_0 \|_{U'} & \text{if } r' = \infty, \end{cases}$$

where $r' \in [1, \infty]$ is such that $1/r + 1/r' = 1$. This equivalence follows from Douglas’ lemma. It has first been formulated in Hilbert spaces [Dou66], and subsequently studied in Banach spaces, e.g. in [Emb73, DR77, Har78, CP78, Car85, Car88, For14].

One motivating example for the described setting is the classical heat equation, i.e. $A = -\Delta$ in $L_p(\mathbb{R}^d)$ with $p \in [1, \infty)$, and with interior control, i.e. the inhomogeneity is given by the embedding $B = \mathbf{1}_E: L_p(E) \to L_p(\mathbb{R}^d)$ with some measurable set $E \subset \mathbb{R}^d$. In this case $(S_t)_{t \geq 0}$ is the Gauß-Weierstraß semigroup. Note that from the physical point of view the case $p = 1$ is probably most interesting, since we can then interpret the states as heat densities (and its norms will be the total heat content). The paper is split into two parts.

In Section 2 we provide sufficient criteria for final-state observability, and thus (approximate) null-controllability, in the abstract framework of Banach spaces. For observability these conditions will be given by a generalized uncertainty principle (or unique continuation estimate) of the form

$$\forall \varphi \in X' \ \forall \lambda > \lambda^*: \quad \| P_{\lambda'} \varphi \|_{X'} \leq d_0 e^{d_1 \lambda^1} \| B' P_{\lambda} \varphi \|_{U'}, \quad (2)$$

and a dissipation estimate of the form

$$\forall \varphi \in X' \ \forall \lambda > \lambda^* \ \forall t \in (0, T/2]: \quad \| (\text{Id} - P_{\lambda}) S_t^{r'} \varphi \|_{X'} \leq d_2 e^{-d_3 \lambda^{2} t^3} \| \varphi \|_{X'}, \quad (3)$$

cf. Theorem 2.8. By Douglas’ lemma, we obtain the corresponding criteria for (approximate) null-controllability, cf. Theorem 2.3. In the case where $X$ and $U$ are Hilbert spaces and $r = 2$, such a strategy for proving observability has first been described in [LR95, LZ98, JL99], and further studied, e.g., in [Mil10, TT11, WZ17, BPS18, NTTV]. However, far less is known on its generalization to Banach spaces. To the best of our knowledge, the first paper which turns (2) and (3) into an observability estimate in Banach spaces is [GST]. A drawback of this result is that it assumes strong continuity of the dual semigroup $(S_t')_{t \geq 0}$. Since we are interested in applications to differential operators on $L_1$ (as noted above), the dual system will act on $L_\infty$, where strong continuity of semigroups is rather rare [Lot85]. Our theorems in Section 2 generalize the main result from [GST] to the case of semigroups which are not necessarily strongly continuous.

In Sections 3 and 4 we study applications of our abstract result from Section 2. In particular, we consider the parabolic system (1) on $X = L_p(\mathbb{R}^d)$, where $p \in [1, \infty)$, $r \in [1, \infty]$, $-A$ is a strongly elliptic differential operator of order $m \in \mathbb{N}$ with constant coefficients, and where $B = \mathbf{1}_E: L_p(E) \to L_p(\mathbb{R}^d)$ is the embedding from a measurable set $E \subset \mathbb{R}^d$ to $\mathbb{R}^d$. Note that we allow for lower order terms in the strongly elliptic differential operator. We will show that if $E$ is a so-called thick set, then the system...
is (approximately) null-controllable. Moreover, we provide explicit upper bounds on
the norm of the control function \( u \) which steers the system
(approximately) to zero at time \( T \). Our bounds are explicit in terms of geometric
properties of the thick set \( E \) and of the final time \( T \). Controllability with respect
to \( L_p((0,T);L_p(\Omega)) \) spaces, where \( \Omega \) is a bounded domain and
\( p \in [1,\infty) \) has been studied earlier in the literature, for instance in [FPZ95]
in the context of semilinear heat equations. The results obtained in [FPZ95]
include as a special case approximate null-controllability for the linear heat equation in
\( L_p((0,T);L_p(\Omega)) \) with \( p \in [1,\infty) \). For further results in this
direction, we refer to [FCZ00] and the survey article [Zua06]. In comparison, we obtain
approximate null-controllability of linear differential operators of
higher orders on \( L_r((0,T);L_p(\mathbb{R}^d)) \) with \( r \in [1,\infty] \) and \( p = 1 \). In the case \( p \in (1,\infty) \) we
show exact null-controllability on \( L_r((0,T);L_p(\mathbb{R}^d)) \). Moreover, we provide sharp upper
bounds on the control cost with respect to the final time \( T \) and the geometric properties
of the control set \( E \). In order to obtain this result, we verify the assumptions of our
abstract theorem from Section 2. Assumption (2) follows directly from a Logvinenko–
Sereda theorem [LS74, Kov01]. For the proof of Assumption (3) we employ Gaussian
heat kernel estimates for the semigroup. This allows us to also deal with systems on
\( L_1(\mathbb{R}^d) \).

2 Abstract framework and sufficient criteria for observability and null-controllability

For normed spaces \( V \) and \( W \) we denote by \( \mathcal{L}(V,W) \) the space of bounded linear operators
from \( V \) to \( W \), and by \( B_V(r) = \{ x \in V : \|x\| \leq r \} \) the closed ball in \( V \) of center 0 and
radius \( r > 0 \).

Let \( X \) and \( U \) be Banach spaces, \((S_t)_{t \geq 0}\) be a \( C_0 \)-semigroup on \( X \), \( -A \) the corresponding
infinitesimal generator on \( X \), and \( B \in \mathcal{L}(U,X) \). For \( T > 0 \) we consider the linear control system

\[
\dot{x}(t) = -Ax(t) + Bu(t), \quad t \in (0,T], \quad x(0) = x_0 \in X,
\]

where \( u \in L_r((0,T);U) \) with \( r \in [1,\infty] \). The function \( x \) is called state function and \( u \) is
called control function. The unique mild solution of (4) is given by Duhamel’s formula

\[
x(t) = S_t x_0 + \int_0^t S_{t-\tau} B u(\tau) d\tau, \quad t \in [0,T].
\]

The controllability map is given by \( \mathcal{B}^T : L_r((0,T);U) \to X \),

\[
\mathcal{B}^T u = \int_0^T S_{T-\tau} B u(\tau) d\tau.
\]

We recall several manifestations of null-controllability from the literature and discuss
their relation.

**Definition 2.1.** Let \( T > 0 \), \( r \in [1,\infty] \), and \( \rho > 0 \). We say that the system (4) is
null-controllable on \([0, T]\) with respect to \(L_r((0, T); U)\), if for all initial states \(x_0 \in X\) there exists a control function \(u \in L_r((0, T); U)\) such that \(x(T) = S_T x_0 + B^T u = 0\), or equivalently, if \(\text{Ran } S_T \subset \text{Ran } B^T\). In that case we define the control cost in time \(T\) by
\[
C = \sup_{x_0 \in X} \inf_{\|x_0\|_X = 1} \{\|u\|_{L_r((0,T);U)} : S_T x_0 + B^T u = 0\}.
\]

null-controllable on \([0, T]\) in the constraint set \(B_{L_r((0,T);U)}(1)\) with radius \(\rho\), if for all initial states \(x_0 \in B_X(\rho)\) there is a control function \(u \in B_{L_r((0,T);U)}(1)\) such that \(x(T) = S_T x_0 + B^T u = 0\), or equivalently, if \(S_T(B_X(\rho)) \subset B^T(B_{L_r((0,T);U)}(1))\).

approximately null-controllable on \([0, T]\) with respect to \(L_r((0, T); U)\), if for each \(\varepsilon > 0\) and each \(x_0 \in X\), there exists an \(u \in L_r((0, T); U)\) such that \(\|x(T)\|_X = \|S_T x_0 + B^T u\|_X < \varepsilon\), or equivalently, if \(\text{Ran } S_T \subset \text{Ran } B^T\). In that case we define the control cost in time \(T\) by
\[
C = \sup_{\varepsilon > 0} \sup_{x_0 \in X} \inf_{\|x_0\|_X = 1} \{\|u\|_{L_r((0,T);U)} : \|S_T x_0 + B^T u\|_X < \varepsilon\}.
\]

approximately null-controllable on \([0, T]\) in the constraint set \(B_{L_r((0,T);U)}(1)\) with radius \(\rho\), if for all initial states \(x_0 \in B_X(\rho)\) and all \(\varepsilon > 0\) there is a control function \(u \in B_{L_r((0,T);U)}(1)\) such that \(\|x(T)\|_X = \|S_T x_0 + B^T u\|_X < \varepsilon\), or equivalently, if \(S_T(B_X(\rho)) \subset B^T(B_{L_r((0,T);U)}(1))\).

That null-controllability implies approximate null-controllability is obvious. The positive parameter \(\rho\) in the definition of (approximate) null-controllability has the geometric interpretation that any initial state with norm at most \(\rho\) can be (approximately) controlled to zero by a control function with norm at most one. By linearity, this is equivalent to the fact that any initial state can be controlled (approximately) to zero, with control costs satisfying \(C \leq \rho^{-1}\). This is formulated in the following lemma.

**Lemma 2.2.** Let \(T > 0\), \(r \in [1, \infty]\), and \(\rho > 0\). Then:

(a) The system (4) is null-controllable on \([0, T]\) with respect to \(L_r((0, T); U)\) and the control cost in time \(T\) satisfies \(C \leq \rho^{-1}\) if and only if the system (4) is null-controllable on \([0, T]\) in the constraint set \(B_{L_r((0,T);U)}(1)\) with radius \(\rho\).

(b) The system (4) is approximately null-controllable on \([0, T]\) with respect to \(L_r((0, T); U)\) and the control cost in time \(T\) satisfies \(C \leq \rho^{-1}\) if and only if the system (4) is approximately null-controllable on \([0, T]\) in the constraint set \(B_{L_r((0,T);U)}(1)\) with radius \(\rho\).

In certain situations null-controllability is equivalent to approximate null-controllability, see Remark 2.7.
Proof of Lemma 2.2. Suppose that the system (4) is null-controllable on \([0,T]\) with respect to \(L_r((0,T);U)\) and the control cost in time \(T\) satisfies \(\mathcal{C} \leq \rho^{-1}\). This implies that for each \(x_0 \in X\) there exists \(u_{x_0} \in L_r((0,T);U)\) such that \(S_T x_0 + B_T u_{x_0} = 0\) and \(\|u_{x_0}\|_{L_r((0,T);U)} \leq \rho^{-1}\|x_0\|_{X}\). If \(\|x_0\|_{X} \leq \rho\), then \(\|u_{x_0}\|_{L_r((0,T);U)} \leq 1\), which proves the implication “\(\Rightarrow\)” of (a). Suppose now that the system (4) is null-controllable on \([0,T]\) in the constraint set \(B_{L_r((0,T);U)}(1)\) with radius \(\rho\). Let \(x_0 \in X\). If \(x_0 = 0\) we choose \(u = 0\). For \(x_0 \neq 0\) we define \(\tilde{x}_0 = (\rho/\|x_0\|_{X})x_0\). Then by assumption there exists \(\tilde{u} \in B_{L_r((0,T);U)}(1)\) such that \(S_T \tilde{x}_0 + B_T \tilde{u} = 0\). Setting \(u = (\|x_0\|_{X}/\rho)\tilde{u}\) we find \(S_T x_0 + B_T u = 0\) and \(\|u\|_{L_r((0,T);U)} \leq \rho^{-1}\|x_0\|_{X}\). This proves the implication “\(\Leftarrow\)” of part (a). The proof of (b) is similar.

Our main result in this section is the following theorem. It provides sufficient criteria such that the system (4) is approximately null-controllable on \([0,T]\) with respect to \(L_r((0,T);U)\).

Theorem 2.3. Let \(X, U\) be Banach spaces, \(B \in \mathcal{L}(U,X)\), \((S_t)_{t \geq 0}\) a \(C_0\)-semigroup on \(X\) and \(M \geq 1\), \(\omega \in \mathbb{R}\) such that \(\|S_t\| \leq Me^{\omega t}\) for all \(t \geq 0\). Let further \(\lambda^* \geq 0\) and \((P_\lambda)_{\lambda \geq \lambda^*}\) be a family of bounded linear operators in \(X\), \(d_0, d_1, d_3, \gamma_1, \gamma_2, \gamma_3, T > 0\) with \(\gamma_1 < \gamma_2, \gamma_3, d_2 \geq 1\), and assume that

\[
\forall \lambda > \lambda^*: \quad P_\lambda(B_X(1)) \subset P_\lambda(B_U(d_0 e^{d_1 \lambda^{\gamma_1}})) \quad (5)
\]

and

\[
\forall x \in X \forall \lambda > \lambda^* \forall t \in (0, T/2]: \quad \|S_t(\text{Id} - P_\lambda)x\|_X \leq d_2 e^{-d_3 \lambda^{\gamma_2} t^{\gamma_3}}\|x\|_X. \quad (6)
\]

Then for all \(r \in [1, \infty]\) we have the range inclusion \(S_T(B_X(\rho)) \subset B_T(B_{L_r((0,T);U)}(1))\), where

\[
\rho = \frac{T^{1/r}}{C_1} \exp \left( - \frac{C_2}{T^{\frac{\gamma_2}{\gamma_3} - \gamma_1}} - C_3 T \right),
\]

\(T^{1/r} = 1\) if \(r = \infty\), and where

\[
C_1 = (4Md_0) \max \left\{ \left( (4d_2 M^2)(d_0 \|B\|_{\mathcal{L}(U,X)} + 1) \right)^{8/(\omega \ln 2)}, e^{4d_1 (2\lambda^*)^{\gamma_1}} \right\},
\]

\[
C_2 = 4(2^{\gamma_1}(2 \cdot 4^{\gamma_3})(\frac{\gamma_2}{\gamma_3} - \gamma_1) d_1^2 / d_3^{\gamma_3})^{\frac{1}{\gamma_2 - \gamma_1}},
\]

\[
C_3 = \max \{\omega, 0\}(1 + 10/(\omega \ln 2)).
\]

In particular, the system (4) is approximately null-controllable on \([0,T]\) with respect to \(L_r((0,T);U)\) and the control cost in time \(T\) satisfies \(\mathcal{C} \leq \rho^{-1}\).

A discussion on the assumptions of Theorem 2.3 and its novel aspects compared to earlier results in the literature are postponed to Remark 2.10. The proof of Theorem 2.3 is given at the end of this section. For these purposes, we recall another well known concept called final-state observability, and discuss its relation to null-controllability using a variant of Douglas’ lemma.

We denote by \(X'\) and \(U'\) the dual spaces of \(X\) and \(U\), respectively, and by \(A'\) in \(X'\) and \(B' \in \mathcal{L}(X', U')\) the corresponding dual operators of \(A\) and \(B\). On \(X'\) we define
the semigroup \((S'_t)_{t \geq 0}\), where \(S'_t\): \(X' \to X'\) is the dual operator of \(S_t\). Note that the semigroup \((S'_t)_{t \geq 0}\) is in general not strongly continuous, but merely weak*-continuous. The weak* generator of \((S'_t)_{t \geq 0}\) is given by \(-A'\). If \(X\) is reflexive then \((S'_t)_{t \geq 0}\) is strongly continuous and \(-A'\) is the infinitesimal generator of \((S'_t)_{t \geq 0}\). We consider the adjoint or dual system to (4); that is,

\[
\begin{align*}
\dot{\psi}(t) &= -A'\psi(t) & t \in (0, T], \\
\psi(0) &= \varphi_0 \in X', \\
\psi(t) &= B'\varphi(t) & t \in [0, T].
\end{align*}
\]

(7)

Recall that the mild solution of (7) is given by \(\varphi(t) = S'_t\varphi_0, \psi(t) = B'S'_t\varphi_0\) for \(t \in [0, T]\).

**Definition 2.4.** Let \(r' \in [1, \infty]\) and \(C_{\text{obs}} \geq 0\). We say that the system (7) satisfies a final-state observability estimate in \(L_{r'}((0, T); U')\) with observability constant \(C_{\text{obs}}\), if for all \(\varphi_0 \in X'\) we have

\[
\|\varphi(T)\|_{X'} \leq \begin{cases} 
C_{\text{obs}} \left( \int_0^T \|\psi(t)\|_{U'}^{r'} dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\
C_{\text{obs}} \sup_{t \in [0, T]} \|\psi(t)\|_{U'} & \text{if } r' = \infty;
\end{cases}
\]

that is,

\[
\|S'_T\varphi_0\|_{X'} \leq \begin{cases} 
C_{\text{obs}} \left( \int_0^T \|B'S'_t\varphi_0\|_{U'}^{r'} dt \right)^{1/r'} & \text{if } r' \in [1, \infty), \\
C_{\text{obs}} \sup_{t \in [0, T]} \|B'S'_t\varphi_0\|_{U'} & \text{if } r' = \infty.
\end{cases}
\]

**Remark 2.5.** Note that \(\|\psi(\cdot)\|_{U'}\) is measurable since by duality and a consequence of the Hahn–Banach theorem this function is actually lower semicontinuous. Moreover, the right-hand side of the final-state observability estimate describes \(C_{\text{obs}}\) times the norm of \(\psi\) in \(L_{r'}((0, T); U')\), provided \(\psi\) is Bochner-measurable and \(r' < \infty\).

It is a classical fact that the two concepts null-controllability (in its various manifestations) and observability are closely related. In the setting of Hilbert spaces, Douglas’ lemma [Dou66] on majorization, factorization and range inclusion implies that null-controllability is indeed equivalent to observability. Unfortunately, the original version of Douglas’ lemma in [Dou66] does not hold verbatim in the general framework of Banach spaces, see e.g. [Bou78, AP91]. However, alternative formulations of this fundamental relation in Banach spaces have been studied, e.g. in [Emb73, DR77, Har78, CP78, Car85, Car88, For14]. We formulate a variant which follows from a combination of the results in [Har78] and [Car85], see also [Car88] for a detailed discussion.

**Theorem 2.6** ([Har78, Car85]). Let \(X, Y, \) and \(Z\) be Banach spaces, \(C \in \mathcal{L}(X, Y),\) and \(F \in \mathcal{L}(Z, Y)\). Consider the following statements:

(a) \(\text{Ran}(F) \subset \text{Ran}(C),\)

(b) there exists \(\rho_1 > 0\) such that \(FB_Z(\rho_1) \subset CB_X(1),\)

6
(c) Ran(F) ⊂ \overline{\text{Ran}(C)}.

(d) there exists \( \rho_2 > 0 \) such that \( FB_Z(\rho_2) \subset C B_X(1) \),

(e) there exists \( \rho_3 > 0 \) such that for all \( y' \in \mathcal{Y}' \) we have \( \rho_3 \| F'y' \|_{\mathcal{Z}'} \leq \| C'y' \|_{\mathcal{X}'} \).

Then (a)\( \Leftrightarrow \) (b), (b)\( \Rightarrow \) (d) with \( \rho_2 = \rho_1 \), and (c)\( \Leftrightarrow \) (d)\( \Leftrightarrow \) (e) with \( \rho_2 = \rho_3 \).

Remark 2.7. Theorem 2.6 provides strong relations between our various manifestations of null-controllability and a final-state observability estimate. In particular, let us apply Theorem 2.6 with \( X = L_r((0, T); U) \) with \( r \in [1, \infty] \), \( \mathcal{Y} = \mathcal{Z} = X \), \( F = S_T \), and \( C = B_T \), and use the fact that for all \( x' \in X' \) we have

\[ \|(B^T)'x'\|_{\mathcal{X}'} = \sup_{\tau \in [0, T]} \|B'S'_{T-\tau}x'\|_{U'} = \sup_{t \in [0, T]} \|B'S'_{t}x'\|_{U'} \]

if \( r = 1 \), and

\[ \|(B^T)'x'\|_{\mathcal{X}'} = \left( \int_0^T \|B'S'_{t}x'\|_{U'}^r dt \right)^{1/r'} = \left( \int_0^T \|B'S'_{t}x'\|_{U'}^r dt \right)^{1/r'} \]

if \( r \in (1, \infty) \), where \( r' \in [1, \infty) \) is such that \( 1/r + 1/r' = 1 \), see [Vie05, Theorem 2.1]. In this situation we have the following consequences of Theorem 2.6:

(a) The statements (a)–(d) of Theorem 2.6 are equivalent to the fact that the system (4) is (a) null-controllable on \([0, T]\) with respect to \( L_r((0, T); U) \), (b) null-controllable on \([0, T]\) in the constraint set \( B_{L_r((0, T); U)}(1) \) with radius \( \rho_1 \), (c) approximately null-controllable on \([0, T]\) with respect to \( L_r((0, T); U) \), and (d) approximately null-controllable on \([0, T]\) in the constraint set \( B_{L_r((0, T); U)}(1) \) with radius \( \rho_2 \), respectively.

The equivalences “(a)\( \Leftrightarrow \) (b)” and “(c)\( \Leftrightarrow \) (d)”, as well as the implication “(b)\( \Rightarrow \) (d)” of Theorem 2.6 have already been discussed in Lemma 2.2 and its preceding paragraph. Moreover, the equivalence “(d)\( \Leftrightarrow \) (e)” in Theorem 2.6 yields that the system (4) is approximately null-controllable on \([0, T]\) in the constraint set \( B_{L_r((0, T); U)}(1) \) with radius \( \rho_2 \) if and only if the system (7) satisfies a final-state observability estimate in \( L_{r'}((0, T); U') \) with observability constant \( \rho_2^{-1} \). In view of Lemma 2.2 the latter is equivalent to the fact that the system (4) is approximately null-controllable on \([0, T]\) with respect to \( L_r((0, T); U) \) and the control cost in time \( T \) satisfies \( C \leq \rho_2^{-1} \).

(b) If either

- \( \mathcal{X} \) is reflexive,
- \( r \in (1, \infty] \) and \( U \) is reflexive,
- \( S_T \) is surjective, or
- for all \( r > 0 \) the set \( B^T(B_{L_r((0, T); U)}(r)) \) is closed,
then the system (4) is null-controllable on $[0,T]$ with respect to $L_r((0,T);U)$ if and only if there is $\rho_3 > 0$ such that the system (7) satisfies a final-state observability estimate in $L_r((0,T);U')$ with observability constant $\rho_3^{-1}$. This follows from the fact that then (b) and (d) in Theorem 2.6 are under one of the above assumptions equivalent, see [Car88, Remark 2.1], [Vie05, Remark 1.2 and Remark 1.3], and [YLC06, Theorem 2.1]. As above, we infer from Lemma 2.2 that $\rho_3^{-1}$ is an upper bound for the control cost in time $T$.

In view of Lemma 2.2, Theorem 2.6 and Remark 2.7, Theorem 2.3 will be a consequence of the following theorem. Note that this is an improvement of [GST, Theorem 2.1] since we do not require the semigroup to be strongly continuous.

**Theorem 2.8.** Let $X$ and $Y$ be Banach spaces, $C \in \mathcal{L}(X,Y)$, $(S_t)_{t \geq 0}$ a semigroup on $X$, $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_t\| \leq Me^{\omega t}$ for all $t \geq 0$, and assume that for all $x \in X$ the mapping $t \mapsto \|CS_t x\|_Y$ is measurable. Further, let $\lambda^* \geq 0$, $(P_\lambda)_{\lambda > \lambda^*}$ a family of bounded linear operators in $X$, $r \in [1, \infty]$, $d_0, d_1, d_2, \gamma_1, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$, and $d_2 \geq 1$, and assume that

$$\forall x \in X \forall \lambda > \lambda^*: \quad \|P_\lambda x\|_X \leq d_0 e^{d_1 \lambda \gamma_1} \|CP_\lambda x\|_Y,$$

and

$$\forall x \in X \forall \lambda > \lambda^* \forall t \in (0,T/2]: \quad \|(I - P_\lambda)S_t x\|_X \leq d_2 e^{-d_3 \lambda \gamma_1 T} \|x\|_X.$$

Then we have for all $x \in X$

$$\|S_T x\|_X \leq \begin{cases} C_{\text{obs}} \left( \frac{T}{r} \right)^{1/r} \exp \left( \frac{C_2}{T^{\gamma_2 - \gamma_1}} + C_3 T \right), & \text{if } r \in [1, \infty), \\ C_{\text{obs}} \text{ ess sup}_{t \in [0,T]} \|CS_t x\|_Y & \text{if } r = \infty, \end{cases}$$

with

$$C_{\text{obs}} = \frac{C_1}{T^{\gamma_1/r}} \exp \left( \frac{C_2}{T^{\gamma_2 - \gamma_1}} + C_3 T \right),$$

where $T^{1/r} = 1$ if $r = \infty$, and

$$C_1 = (4Md_0) \max \left\{ \left( \frac{4d_2 M^2 (d_0 \|C\|_{\mathcal{L}(X,Y)} + 1)}{4/\ln 2} \right)^{8/(\ln 2)} e^{4d_1 (2\lambda^*)^{\gamma_1}}, \right\},$$

$$C_2 = 4\left( \frac{2^{\gamma_1} (2 \cdot 4^{\gamma_1})^{\frac{\gamma_2 - \gamma_1}{2}} d_1^2 / d_3^{\gamma_1}}{\gamma_2 - \gamma_1} \right)^{\frac{1}{\gamma_2 - \gamma_1}},$$

$$C_3 = \max \{\omega, 0\} \left( 1 + 10/(\ln 2) \right).$$

**Proof.** Since we do not assume the semigroup $(S_t)_{t \geq 0}$ to be strongly continuous, we cannot apply Theorem 2.1 in [GST] directly. The strong continuity of $(S_t)_{t \geq 0}$ was assumed in [GST] in order to ensure that for all $x \in X$ and $\lambda > \lambda^*$ the functions

$$F(t) = \|S_t x\|_X, \quad F_\lambda(t) = \|P_\lambda S_t x\|_X, \quad F_\lambda^-(t) = \|(I - P_\lambda)S_t x\|_X,$$
\[ G(t) = \|CS_t x\|_Y, \quad G_\lambda(t) = \|CP_\lambda S_t x\|_Y, \quad G_\lambda^+(t) = \|C(\text{Id} - P_\lambda) S_t x\|_Y, \]

are measurable. The measurability of these six functions was used to obtain the estimate

\[ F(t) \leq \frac{2Me^{\omega t}d_0e^{d_1\lambda t}}{t} \int_{t/2}^t G(\tau) d\tau + \frac{d_2 M^2e^{5\omega t/4}e^{d_1\lambda t}}{e^{d_2\lambda t/2}(t/4)^{\gamma_2}} \left( d_0\|C\|_{L(X,Y)} + 1 \right) F(t/4), \]

where \( \omega_+ = \max\{0, \omega\} \). Such an inequality implies the statement of the theorem by iteration, see \[GST\]. Thus it suffices to show the last displayed inequality by assuming merely measurability of the mapping \( t \mapsto G(t) \). Let \( t > 0, \tau \in [t/2, t] \) and \( x \in X \). Since \( F(\tau) \leq F_\lambda(\tau) + F_\lambda^+(\tau) \), by our assumptions and by the semigroup property we obtain

\[ F(\tau) \leq d_0 e^{d_1\lambda t}G_\lambda(\tau) + d_2e^{-d_3\lambda t/2} F(\tau/2). \]

Using \( G_\lambda(\tau) \leq G(\tau) + G_\lambda^+(\tau) \leq G(\tau) + \|C\|_{L(X,Y)} F_\lambda^+(\tau) \), our assumption, \( e^{d_1\lambda t} \geq 1 \), and \( F(\tau/2) \leq Me^{\omega t/4} F(t/4) \) we obtain

\[ F(\tau) \leq d_0 e^{d_1\lambda t}G(\tau) + (d_0\|C\|_{L(X,Y)} + 1)d_2e^{-d_3\lambda t/2} e^{d_1\lambda t} M e^{\omega t/4} F(t/4). \]

Since \( F(t) \leq Me^{\omega t} F(\tau) \), we obtain

\[ F(t) \leq Me^{\omega t}d_0e^{d_1\lambda t}G(\tau) + (d_0\|C\|_{L(X,Y)} + 1)d_2e^{-d_3\lambda t/2} e^{d_1\lambda t} M^2 e^{\omega t/4} F(t/4). \]

Since the mapping \( \tau \mapsto G(\tau) \) is measurable by assumption, we can integrate this inequality with respect to \( \tau \), and obtain the desired estimate. \( \square \)

**Remark 2.9.** In the situation of Theorem 2.8, if we assume that \( t \mapsto CS_t x \) is Bochner measurable, we can rewrite the statement of the theorem as

\[ \|S_T x\|_X \leq C_{\text{obs}}\|CS_1 x\|_{L_r((0,T); Y)}. \]

Note, however, that we refrain to write the norm in \( L_r((0,T); Y) \) on the right-hand side in (10) due to the possible lack of Bochner measurability.

**Remark 2.10.** Assumption (8) can be called generalized uncertainty principle or generalized unique continuation, as it implies that if \( P_\lambda x \) is in the kernel of \( C \) then \( P_\lambda x \) has to be zero. Assumption (9) is a dissipation estimate as it requires exponential decay of \( (\text{Id} - P_\lambda) S_t x \) with respect to \( \lambda \) and \( t \).

Thus, an alternative formulation of Theorem 2.8 is that the generalized uncertainty principle (8) together with the dissipation estimate (9) implies for all \( r \in [1, \infty] \) an observability estimate in \( L_r((0,T); Y) \). Such statements have been studied earlier in the case where \( X \) and \( Y \) are Hilbert spaces, starting with the seminal papers \[LR95, LZ98, JL99\], and further studied, e.g., in \[Mil10, TT11, WZ17, BPS18, NTTV\]. Note that if \( (S_t)_{t \geq 0} \) is strongly continuous, the generator is self-adjoint and bounded from below and the \( P_\lambda \)'s are spectral projection of the generator, the dissipation estimate (9) is automatically satisfied. In the case where \( X \) and \( Y \) are Banach spaces, and under the additional assumption that the semigroup \( (S_t)_{t \geq 0} \) is strongly continuous, the statement
of Theorem 2.3 has already been proven in [GST, Theorem 2.1]. The main advantage of our Theorem 2.8 is that we do not require that \((S_t)_{t \geq 0}\) is strongly continuous. This extends the applicability to the case of dual semigroups on \(L_\infty\), or put differently, we can investigate null-controllability for systems in \(L_1\) by means of Theorem 2.3.

**Proof of Theorem 2.3.** We apply Theorem 2.8 to the dual semigroup \((S'_t)_{t \geq 0}\) on \(X'\), \(Y := U'\), \(C := B'\), and \(P_\lambda\) replaced by its dual operator \(P'_\lambda\). Note that \((S'_t)_{t \geq 0}\) is exponentially bounded since \((S_t)_{t \geq 0}\) is a \(C_0\)-semigroup (and hence exponentially bounded). Further, in view of Theorem 2.6, (5) yields (8), and (6) implies (9) by duality. Moreover, the measurability of the functions \(t \mapsto \|B'S'_tx'\|_{U'}\) for all \(x' \in X'\) follows from duality and the description of dual norms via the Hahn–Banach theorem. Thus, by Theorem 2.8 we obtain

\[
\|S'_T x'\|_{X'} \leq \begin{cases} 
C_{\text{obs}} \left( t^\frac{q'}{q} \|B'S'_tx'\|_{U'}^r \right)^{1/r'} & \text{if } r' \in [1, \infty), \\
C_{\text{obs}} \|C'S'_tx'\|_{U'} & \text{if } r' = \infty
\end{cases}
\]

for all \(x' \in X'\), and Remark 2.7 implies the assertion. \(\square\)

### 3 Observability and null-controllability in \(L_p\)-Spaces

In order to formulate our main theorems we review some basic facts from Fourier analysis. For details we refer, e.g., to the textbook [Gra14]. We denote by \(S(\mathbb{R}^d)\) the Schwartz space of rapidly decreasing functions, which is dense in \(L_p(\mathbb{R}^d)\) for all \(p \in [1, \infty)\). The topological dual space of \(S(\mathbb{R}^d)\) is denoted by \(S'(\mathbb{R}^d)\). Elements of this space are called tempered distributions. For \(f \in S(\mathbb{R}^d)\) let \(\mathcal{F} f : \mathbb{R}^d \to \mathbb{C}\) be the Fourier transform of \(f\) defined by

\[
\mathcal{F} f(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, dx.
\]

Then \(\mathcal{F} : S(\mathbb{R}^d) \to S(\mathbb{R}^d)\) is bijective, continuous and has a continuous inverse, given by

\[
\mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(\xi) e^{ix \cdot \xi} \, d\xi
\]

for all \(f \in S(\mathbb{R}^d)\). For \(f \in S'(\mathbb{R}^d)\) the Fourier transform is again denoted by \(\mathcal{F}\) and is given by \((\mathcal{F} \phi)(f) = f(\mathcal{F} \phi)\) for \(\phi \in S(\mathbb{R}^d)\). By duality, the Fourier transform is bijective on \(S'(\mathbb{R}^d)\) as well.

For \(p \in [1, \infty]\), \(g \in L_1(\mathbb{R}^d)\), and \(f \in L_p(\mathbb{R}^d)\) we define the convolution \(g * f\) by

\[
g * f = \int_{\mathbb{R}^d} g(x - y) f(y) \, dy. \tag{11}
\]

Then Young’s inequality states for all \(p \in [1, \infty]\), \(g \in L_1(\mathbb{R}^d)\), and \(f \in L_p(\mathbb{R}^d)\) that

\[
\|g * f\|_{L_p(\mathbb{R}^d)} \leq \|g\|_{L_1(\mathbb{R}^d)} \|f\|_{L_p(\mathbb{R}^d)}.
\]

It follows from Young’s inequality that for every \(g \in L_1(\mathbb{R}^d)\) and \(p \in [1, \infty]\) the operator \(T_g : L_p(\mathbb{R}^d) \to L_p(\mathbb{R}^d)\) given for \(f \in L_p(\mathbb{R}^d)\)
by $T_g f = g * f$ is bounded with $\|T_g\| = \|g\|_{L^1(\mathbb{R}^d)}$. In fact, one can show that $\|T_g\| = \|g\|_{L^1(\mathbb{R}^d)}$ by employing a suitable approximation to the identity.

It is not possible to extend (11) to general $g, f \in S'(\mathbb{R}^d)$. However, for $g \in S'(\mathbb{R}^d)$ and $f, \phi \in S(\mathbb{R}^d)$, we may define $(g * f)(\phi) = g(f(\cdot) \ast \phi)$ which is consistent with (11) if $g \in L^p(\mathbb{R}^d)$ for some $p \in [1, \infty]$.

For $\mu \in \mathbb{R}$ we say that a smooth function $\lambda: \mathbb{R}^d \to \mathbb{C}$ is in the symbol class $\mathcal{S}_\mu$ if for each multi-index $\alpha \in \mathbb{N}^d_0$ there exists a constant $C_\alpha > 0$ such that for all $\xi \in \mathbb{R}^d$ we have

$$|\partial^\alpha \lambda(\xi)| \leq C_\alpha (1 + |\xi|)^{\mu - |\alpha|}.$$ 

Given $f \in S'(\mathbb{R}^d)$ and $\lambda \in \mathcal{S}_\mu$ for some $\mu \in \mathbb{R}$, we define $\lambda f \in S'(\mathbb{R}^d)$ on test functions $\phi \in S(\mathbb{R}^d)$ by setting $\lambda f(\phi) = f(\lambda \phi)$. Let $D = -i \nabla$. For $\lambda \in \mathcal{S}_\mu$ with $\mu \in \mathbb{R}$ we define the associated Fourier multiplier

$$\lambda(D): S'(\mathbb{R}^d) \to S'(\mathbb{R}^d), \ f \mapsto \mathcal{F}^{-1}(\lambda \mathcal{F} f).$$

For $\lambda \in \mathcal{S}(\mathbb{R}^d)$, it is a standard exercise to check that

$$\lambda(D) f = (\mathcal{F}^{-1} \lambda) * f$$

holds for every $f \in S'(\mathbb{R}^d)$. Thus, we may view $\lambda(D)$ as a convolution operator with kernel $\mathcal{F}^{-1} \lambda$.

Let $m \in \mathbb{N}$ and

$$a(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha, \ \xi \in \mathbb{R}^d,$$

be a polynomial of degree $m$ with coefficients $a_\alpha \in \mathbb{C}$. Then

$$a_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha, \ \xi \in \mathbb{R}^d,$$

is called the principal symbol of $a$. We say that the polynomial $a$ is strongly elliptic if there is a constant $c > 0$ such that the principal symbol satisfies for all $\xi \in \mathbb{R}^d$ the lower bound

$$\text{Re } a_m(\xi) \geq c |\xi|^m.$$ 

(12)

Note that strong ellipticity implies that $m$ is even.

It is clear that $a \in \mathcal{S}^m$, thus the Fourier multiplier $a(D)$ is well-defined. It is a differential operator with constant coefficients of the form $a(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$. We say that the differential operator $a(D)$ is strongly elliptic if the polynomial $a$ is strongly elliptic.

Let $T > 0$, $m \in \mathbb{N}$, and $a$ be a strongly elliptic polynomial of order $m$. On the time interval $[0, T]$ we consider the generalized heat equation

$$\dot{\phi}(t) = -a(D) \phi(t), \ t \in (0, T], \ \phi(0) = \phi_0 \in S'(\mathbb{R}^d)$$ 

(13)
for a function $\phi \in C^1([0,T];S'(\mathbb{R}^d))$. It is easy to check that $e^{-ta} \in \mathcal{S}(\mathbb{R}^d)$ for every $t > 0$. Therefore, we define the semigroup $(S_t)_{t \geq 0}$ on $S'(\mathbb{R}^d)$ by

$$S_t = e^{-ta(D)}.$$  

(14)

Then $S_t$ is given by the convolution with the integral kernel $k_t = \mathcal{F}^{-1}e^{-ta}$. Moreover, $\phi(t) = S_t\phi_0$ solves (13) in the sense that $\dot{\phi}(t) = -a(D)\phi(t)$ for $t \in (0,T]$, and $\phi(t) \to \phi_0$ in the sense of distributions as $t \to 0$. If $a$ is strongly elliptic, the binomial theorem and the fact that one can bound real polynomials by its term of maximal degree plus a constant implies that there exists $\omega \in \mathbb{R}$ and $\sigma \geq 0$ such that

$$\text{Re} \, a(\xi + i\eta) \geq (3/4)c|\xi|^m - \sigma|\eta|^m - \omega$$

(15)

for all $\xi, \eta \in \mathbb{R}^d$. (Note that since $a$ is a polynomial, we can extend $a$ to $\mathbb{C}^d$ even if we only defined it on $\mathbb{R}^d$.) As a consequence, we obtain the following heat kernel bound for $k_t$, where the proof can be found, e.g., in [TR96].

**Proposition 3.1.** Let $a$ be a strongly elliptic polynomial satisfying (15). Then there exist $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}^d$ and $t > 0$ we have

$$|k_t(x)| \leq c_1e^{\omega t}t^{-d/m}e^{-c_2\left(\frac{|x|^m}{t}\right)^{\frac{1}{m-1}}}.$$  

It follows from this generalized heat kernel bound and Young’s inequality that $S_t$ defines for all $q \in [1, \infty]$ a bounded operator from $L_q(\mathbb{R}^d)$ to $L_q(\mathbb{R}^d)$ and that there exists $M \geq 1$ such that $\|S_t\| \leq Me^{\omega t}$ for all $t \geq 0$. Indeed, setting

$$M = \int_{\mathbb{R}^d} c_1e^{-c_2|y|^{m/(m-1)}} \, dy,$$

(16)

we observe

$$\|S_t\| = \|k_t\|_1 \leq \int_{\mathbb{R}^d} c_1e^{\omega t}t^{-d/m}e^{-c_2\left(\frac{|x|^m}{t}\right)^{\frac{1}{m-1}}} \, dx = e^{\omega t} \int_{\mathbb{R}^d} c_1e^{-c_2|y|^{m/(m-1)}} \, dy = Me^{\omega t}.$$  

Note that $M$ is independent of $q$. Moreover, $(S_t)_{t \geq 0}$ is a semigroup on $L_q(\mathbb{R}^d)$ for all $q \in [1, \infty]$, and strongly continuous for $q < \infty$.

We are now ready to formulate our main result in this section. For this purpose, we introduce the notion of a thick subset $E$ of $\mathbb{R}^d$.

**Definition 3.2.** Let $\rho \in (0,1]$ and $L \in (0,\infty)^d$. A set $E \subset \mathbb{R}^d$ is called $(\rho, L)$-thick if $E$ is measurable and for all $x \in \mathbb{R}^d$ we have

$$\left| E \cap \left( \bigtimes_{i=1}^d (0, L_i) + x \right) \right| \geq \rho \prod_{i=1}^d L_i.$$  

Here, $|\cdot|$ denotes Lebesgue measure in $\mathbb{R}^d$. Moreover, $E \subset \mathbb{R}^d$ is called thick if there are $\rho \in (0,1]$ and $L \in (0,\infty)^d$ such that $E$ is $(\rho, L)$-thick.
Theorem 3.3. Let $m \in \mathbb{N}$, $a: \mathbb{R}^d \to \mathbb{C}$ a strongly elliptic polynomial $a$ of order $m$, $c > 0$ as in (12), $\omega \in \mathbb{R}$ as in (15), and $(S_t)_{t \geq 0}$ as in (14). Let $\rho \in (0, 1]$, $L \in (0, \infty)^d$, $E \subset \mathbb{R}^d$ a $(\rho, L)$-thick set, $q, r \in [1, \infty]$, and $T > 0$. Then we have for all $u \in L_q(\mathbb{R}^d)$

$$
\|S_T u\|_{L_q(\mathbb{R}^d)} \leq \begin{cases}
C_{\text{obs}} \left( \int_0^T \| (S_t u) \|_{L_q(E)} \right)^{1/r} dt, & \text{if } r \in [1, \infty), \\
C_{\text{obs}} \text{ess sup} \| (S_t u) \|_{L_q(E)} & \text{if } r = \infty,
\end{cases}
$$

where

$$
C_{\text{obs}} = \frac{K_a}{T^{1/r}} \left( \frac{K_d}{\rho} \right)^{K_d(1+|L|/\lambda^*)} \exp \left( \frac{K_m(\|L\| \ln(K_d/\rho))^{m/(m-1)}}{(cT)^{1/(m-1)}} + K \max\{\omega, 0\} T \right).
$$

Here, $\lambda^* = (2^{m+4} \max\{\omega, 0\}/c)^{1/m}$, $K > 0$ is an absolute constant, and $K_a, K_d, K_m > 0$ are constants depending only on the polynomial $a$, on $d$, or on $m$, respectively.

Remark 3.4. The statement of Theorem 3.3 can be interpreted that for all $q \in [1, \infty]$ the system

$$
\dot{\phi}(t) = a(D)\phi(t), \quad t \in (0, T], \quad \psi(t) = \phi(t)|_{E}, \quad t \in [0, T], \quad \phi(0) = \phi_0 \in L_q(\mathbb{R}^d)
$$

satisfies a final-state observability estimate in $L_r((0, T); L_q(E))$ with observability constant $C_{\text{obs}}$.

As a consequence of Theorem 3.3 we obtain the following null-controllability result in Theorem 3.6. Suppose that $p \in [1, \infty)$, $m \in \mathbb{N}$, and $a(D)$ is a strongly elliptic differential operator with symbol $a = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ of order $m$. Then the semigroup $(e^{-ta(D)})_{t \geq 0}$ is strongly continuous on $L_p(\mathbb{R}^d)$. We denote the associated generator by $-A_p$.

Remark 3.5. Note that $-A_p = -a(D)|_{L_p(\mathbb{R}^d)}$ is the part of $-a(D)$ (as an operator on $S'(\mathbb{R}^d)$) in $L_p(\mathbb{R}^d)$. Moreover, it is easy to see that $A'_p = -\tilde{a}(D)|_{L_{p'}(\mathbb{R}^d)}$, where

$$
\tilde{a}(\xi) = a(-\xi) = \sum_{|\alpha| \leq m} a_\alpha (-1)^{|\alpha|} \xi^\alpha.
$$

Indeed, for $f \in L_p(\mathbb{R}^d)$, $g \in L_{p'}(\mathbb{R}^d)$ we compute

$$
\int_{\mathbb{R}^d} e^{-ta(D)} f(x)g(x)dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F^{-1} e^{-ta})(x-y)f(y)g(x)dydx
$$

$$
= \int_{\mathbb{R}^d} f(x)e^{-ta(-D)}g(x)dx,
$$

which yields the claim. Since $m$ is even, strong ellipticity of $a$ implies strong ellipticity of $\tilde{a}$.
We consider the system
\[ \dot{x}(t) = -A_p x(t) + 1_E u(t), \quad t \in (0, T), \quad x(0) = x_0 \in L_p(\mathbb{R}^d), \] (17)
where \( u \in L_r((0, T); L_p(E)) \) with \( r \in [1, \infty] \), and where \( E \subset \mathbb{R}^d \) is measurable and \( 1_E : L_p(E) \to L_p(\mathbb{R}^d) \) is the operator which extends a function on \( E \) to \( \mathbb{R}^d \) by zero.

**Theorem 3.6.** Let \( p \in [1, \infty), \rho \in (0, 1], L \in (0, \infty)^d, r \in [1, \infty], T > 0, \) and suppose that \( E \subset \mathbb{R}^d \) is \((\rho, L)\)-thick.

(a) If \( p \in (1, \infty) \), then the system (17) is null-controllable on \([0, T]\) with respect to \( L_r((0, T); L_p(E)) \).

(b) If \( p = 1 \), then the system (17) is approximately null-controllable on \([0, T]\) with respect to \( L_r((0, T); L_p(E)) \).

Moreover, the control cost in time \( T \) satisfies \( \mathcal{C} \leq C_{\text{obs}} \), where \( C_{\text{obs}} \) is as in Theorem 3.3 with \( r \) replaced by \( r' \), where \( r' \in [1, \infty] \) is such that \( 1/r + 1/r' = 1 \).

**Proof.** First we consider the case \( p = 1 \). Let \((V_t)_{t \geq 0}\) be the semigroup generated by \(-A_p\) and
\[ B^T : L_r((0, T); L_1(E)) \to L_1(\mathbb{R}^d), \quad u \mapsto \int_0^T V_{T-t} 1_E u(t) dt. \]

By Lemma 2.2(b), the system (17) is approximately null-controllable on \([0, T]\) with respect to \( L_r((0, T); L_1(E)) \) and the control cost in time \( T \) satisfies \( \mathcal{C} \leq C_{\text{obs}} \) if and only if
\[ V_T(B_{L_1(\mathbb{R}^d)}(C_{\text{obs}}^{-1})) \subset B^T(B_{L_r((0, T); L_1(E))}(1)). \] (18)

Let \( a \) be the symbol of \( A_p \). By Remark 3.5, we have that \( V'_t : L_{r'}(\mathbb{R}^d) \to L_{r'}(\mathbb{R}^d) \) is given by \( e^{-ta(D)} = e^{-\tilde{a}(D)} \). An application of Theorem 2.6 (and Remark 2.7 in the case \( r = \infty \)) shows that the range inclusion (18) is equivalent to the assertion of Theorem 3.3 with \( S_t = V_t' = e^{-\tilde{a}(D)}, q = p' = \infty \), and \( r \) replaced by \( r' \).

If \( p \in (1, \infty) \), the corresponding \( L_p \)-spaces are reflexive. Thus we conclude from Theorem 2.6 and Remark 2.7 that the assertion of Theorem 3.3 is equivalent to the fact that the system (17) is null-controllable on \([0, T]\) with respect to \( L_r((0, T); L_p(E)) \), and the control cost satisfies \( \mathcal{C} \leq C_{\text{obs}} \) with \( r \) replaced by \( r' \). \( \square \)

It remains to prove Theorem 3.3. For this purpose, we apply Theorem 2.8 with \( C : L_q(\mathbb{R}^d) \to L_q(E) \), \( Cf := f|_E \) being the restriction to \( E \), \( (S_t)_{t \geq 0} = (e^{-ta(D)})_{t \geq 0} \), and \( P_\lambda = \chi_\lambda(D) \) where \( \chi_\lambda \in C^\infty(\mathbb{R}^d) \) is some suitable cutoff function supported in the ball \( B_{\mathbb{R}^d}(\lambda) \subset \mathbb{R}^d \). This is postponed to Section 4.
4 Dissipation, uncertainty principle and the proof of Theorem 3.3

Let $\eta \in C_c^\infty((0, \infty))$ with $0 \leq \eta \leq 1$ such that $\eta(r) = 1$ for $r \in [0, 1/2]$ and $\eta(r) = 0$ for $r \geq 1$. For $\lambda > 0$ we define $\chi_\lambda : \mathbb{R}^d \to \mathbb{R}$ by $\chi_\lambda(\xi) = \eta(|\xi|/\lambda)$. Since $\chi_\lambda \in \mathcal{S}(\mathbb{R}^d)$, we have $\mathcal{F}^{-1}\chi_\lambda \in \mathcal{S}(\mathbb{R}^d) \subset L_1(\mathbb{R}^d)$ and for all $q \in [1, \infty]$ we define $P_\lambda : L_q(\mathbb{R}^d) \to L_q(\mathbb{R}^d)$ by $P_\lambda f = (\mathcal{F}^{-1}\chi_\lambda) * f$. By taking Fourier transforms it is clear that $P_\lambda f = \chi_\lambda(D)f$ for all $f \in L_q(\mathbb{R}^d)$. By Young’s inequality we have for all $f \in L_q(\mathbb{R}^d)$

$$\|P_\lambda f\|_{L_q(\mathbb{R}^d)} = \|\mathcal{F}^{-1}(\chi_\lambda) * f\|_{L_q(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} \|f\|_{L_q(\mathbb{R}^d)}.$$  

Moreover, the norm $\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)}$ is independent of $\lambda > 0$. Indeed, by the scaling property of the Fourier transform and by change of variables we have for all $\lambda > 0$

$$\|\mathcal{F}^{-1}\chi_\lambda\|_{L_1(\mathbb{R}^d)} = |\lambda|^d \|\mathcal{F}^{-1}\chi_1\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\chi_1\|_{L_1(\mathbb{R}^d)}.$$  

Hence, for all $\lambda > 0$ the operator $P_\lambda$ is a bounded linear operator and the family $(P_\lambda)_{\lambda > 0}$ is uniformly bounded by $\|\mathcal{F}^{-1}\chi_1\|_{L_1(\mathbb{R}^d)}$.

Our main result in this section is the following dissipation estimate:

**Proposition 4.1.** Let $m \in \mathbb{N}$, $a : \mathbb{R}^d \to \mathbb{C}$ a strongly elliptic polynomial of order $m$, $c > 0$ as in (12), $\omega \in \mathbb{R}$ as in (15), $(S_t)_{t \geq 0}$ as in (14), and $(P_\lambda)_{\lambda > 0}$ as above. Then for all $q \in [1, \infty]$, $f \in L_q(\mathbb{R}^d)$, $\lambda > (2^{m+4}\max\{\omega, 0\}/c)^{1/m}$, and $t \geq 0$ we have

$$\|(\text{Id} - P_\lambda)S_tf\|_{L_q(\mathbb{R}^d)} \leq K_a e^{-2m-4ct\lambda^m} \|f\|_{L_q(\mathbb{R}^d)},$$

where $K_a \geq 1$ is a constant depending only on the polynomial $a$ (and therefore also on $m$ and $d$).

We first prove the dissipation estimate for the semigroups that are associated to powers of the Laplacian on $L_q(\mathbb{R}^d)$.

**Proposition 4.2.** Let $m \geq 2$ be even, $(G_t)_{t \geq 0} = (e^{-tD}f)_{t \geq 0}$, and $P_\lambda$ as above. Then for all $q \in [1, \infty]$, $f \in L_q(\mathbb{R}^d)$, $\lambda > 0$, and $t \geq 0$ we have

$$\|(\text{Id} - P_\lambda)G_tf\|_{L_q(\mathbb{R}^d)} \leq K_{m,d} e^{-2m-4t\lambda^m} \|f\|_{L_q(\mathbb{R}^d)},$$

where $K_{m,d} > 0$ is a constant depending only on $m$ and $d$.

**Proof.** Let us set $a = |\cdot|^m$. The heat semigroup $(G_t)_{t \geq 0}$ is then given by $G_tf = e^{-t(D)^m}f$ for $t \geq 0$ and $f \in L_q(\mathbb{R}^d)$. Hence we have for all $f \in L_q(\mathbb{R}^d)$ that

$$(\text{Id} - P_\lambda)G_tf = \mathcal{F}^{-1}((1 - \chi_\lambda)e^{-t\lambda}) * f,$$

and by Young’s inequality we obtain for all $\lambda, t > 0$ and all $f \in L_q(\mathbb{R}^d)$

$$\|(\text{Id} - P_\lambda)G_tf\|_{L_q(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}((1 - \chi_\lambda)e^{-t\lambda})\|_{L_1(\mathbb{R}^d)} \|f\|_{L_q(\mathbb{R}^d)}.$$
For $\mu > 0$ we define $k_\mu : \mathbb{R}^d \to \mathbb{R}$ by $k_\mu = \mathcal{F}^{-1}((1 - \chi_\mu)e^{-a})$. By substitution first in Fourier space, and then in direct space we obtain, using $|t|^{1/m}|\xi|^m = |t| |\xi|^m$, for all $\lambda, t > 0$

$$\|\mathcal{F}^{-1}((1 - \chi_\lambda)e^{-a})\|_{L_1(\mathbb{R}^d)}$$

$$= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \frac{1}{t^{d/m}} \left| \int_{\mathbb{R}^d} e^{ix \cdot (t^{-1/m}\xi)}(1 - \chi_{t^{1/m}\lambda}(\xi))e^{-|\xi|^m} d\xi \right| dx$$

$$= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} e^{iy \cdot \xi}(1 - \chi_{t^{1/m}\lambda}(\xi))e^{-|\xi|^m} d\xi \right| dy = \|k_{t^{1/m}\lambda}\|_{L_1(\mathbb{R}^d)}.$$  

We denote by $K_{m,d} > 0$ constants which depend only on $m$ and the dimension $d$. We allow these constants to change with each occurrence. By Young’s inequality and (19), we have

$$\|\mathcal{F}^{-1}(\chi_\mu e^{-a})\|_{L_1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}(\chi_\mu * \mathcal{F}^{-1}(e^{-a}))\|_{L_1(\mathbb{R}^d)} \leq K_{m,d}. $$

Hence we find for all $\mu > 0$ the uniform bound

$$\|k_\mu\|_{L_1(\mathbb{R}^d)} \leq \|\mathcal{F}^{-1}e^{-a}\|_{L_1(\mathbb{R}^d)} + \|\mathcal{F}^{-1}(\chi_\mu e^{-a})\|_{L_1(\mathbb{R}^d)} \leq K_{m,d}.$$  

Next we show that the $L_1$-norm of $k_\mu$ decays even exponentially as $\mu$ tends to infinity. For this purpose, let now $\mu \geq 1$, $\alpha \in \mathbb{N}_0^d$ with $|\alpha|_1 \leq d + 1$, and denote by $M_\alpha$ the multiplication with $x^\alpha$. By differentiation properties of the Fourier transform we have

$$M_\alpha k_\mu = M_\alpha \mathcal{F}^{-1}[(1 - \chi_\mu)e^{-a}] = \mathcal{F}^{-1}D_{\xi}^\alpha[(1 - \chi_\mu)e^{-a}]$$

and hence for all $x \in \mathbb{R}^d$

$$|x^\alpha k_\mu(x)| = \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} D_{\xi}^\alpha[(1 - \chi_\mu(\xi))e^{-|\xi|^m}] d\xi \right|$$

$$\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |D_{\xi}^\alpha[(1 - \chi_\mu(\xi))e^{-|\xi|^m}]| d\xi.$$  

On the integrand of the right-hand side in (21) we apply the product rule and the triangle inequality to obtain

$$|D_{\xi}^\beta[(1 - \chi_\mu(\xi))e^{-|\xi|^m}]| \leq \sum_{\beta \in \mathbb{N}_0^d, \beta \leq \alpha} \binom{\alpha}{\beta} |D_{\xi}^{\alpha - \beta}(1 - \chi_\mu(\xi))||D_{\xi}^\beta e^{-|\xi|^m}|.$$  

For all $\beta \in \mathbb{N}_0^d$ and $\beta \leq \alpha$ we have

$$|D_{\xi}^\beta e^{-|\xi|^m}| \leq K_{m,d}(1 + |\xi|)^{|\beta|_1(m-1)}e^{-|\xi|^m} \leq K_{m,d}e^{-|\xi|^m/2},$$

where for the last inequality we used that $\xi \mapsto (1 + |\xi|)^{|\beta|_1(m-1)}e^{-|\xi|^m/2}$ is bounded on $\mathbb{R}^d$. Since $\mu \geq 1$, for all $\beta \in \mathbb{N}_0^d, \beta \leq \alpha$ we have

$$|D_{\xi}^{\alpha - \beta}(1 - \chi_\mu(\xi))| \leq \mu^{1 - |\alpha|_1} \sup_{\gamma \leq \alpha} \sup_{\xi \in \mathbb{R}^d} |D_{\xi}^\gamma \chi_1(\xi/\mu)| \leq \sup_{\gamma \leq \alpha} \sup_{\xi \in \mathbb{R}^d} |D_{\xi}^\gamma \chi_1(\xi/\mu)| 1_{\mathbb{R}^d \setminus B_{Rd}(\mu/2)}(\xi).$$
and hence
\[ |D^\alpha_{\xi}(1 - \chi_\mu(\xi))||D^\beta_{\xi}e^{-|\xi|^m}| \leq K_{m,d}e^{-|\xi|^m/2}\mathbf{1}_{\mathbb{R}^d\setminus B_{2\xi}(\mu/2)}(\xi) \leq K_{m,d}e^{-|\xi|^m/4}e^{-\mu^m/2^{m+2}}. \]

Thus, (22) and |\alpha|_1 \leq d + 1 imply for all \( \xi \in \mathbb{R}^d \) that
\[ |D^\alpha_{\xi}(1 - \chi_\mu(\xi))e^{-|\xi|^m}| \leq K_{m,d}e^{-|\xi|^m/4}e^{-\mu^m/2^{m+2}} \sum_{\beta \in \mathbb{N}_0^d, \beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) \leq K_{m,d}e^{-|\xi|^m/4}e^{-\mu^m/2^{m+2}}. \]

Hence, from (21), for all \( x \in \mathbb{R}^d \) we obtain
\[ |x^\alpha k_\mu(x)| \leq K_{m,d}e^{-\mu^m/2^{m+2}} \int_{\mathbb{R}^d} e^{-|\xi|^m/4}d\xi = K_{m,d}e^{-\mu^m/2^{m+2}}. \quad (23) \]

In particular, for \( j \in \{1, 2, \ldots, d\} \) and \( \alpha_j = (d+1)e_j \), where \( e_j \) denotes the \( j \)-th canonical unit vector in \( \mathbb{R}^d \), we obtain \( |x_j|^{d+1}|k_\mu(x)| \leq K_{m,d}e^{-\mu^m/2^{m+2}} \), hence \( \|x\|_{\infty}^{d+1}|k_\mu(x)| \leq K_{m,d}e^{-\mu^m/2^{m+2}} \), and consequently for all \( x \in \mathbb{R}^d \) and all \( \mu \geq 1 \) we find
\[ |x|^{d+1}|k_\mu(x)| \leq K_{m,d}e^{-\mu^m/2^{m+2}}. \quad (24) \]

From (23) with \( \alpha = 0 \) and (24) we obtain for all \( \mu \geq 1 \) that
\[ \|k_\mu\|_{L_1(\mathbb{R}^d)} \leq K_{m,d}e^{-\mu^m/2^{m+2}} \int_{\mathbb{R}^d} dx + K_{m,d}e^{-\mu^m/2^{m+2}} \int_{\mathbb{R}^d\setminus B_1(1)} |x|^{d-1}dx \leq K_{m,d}e^{-\mu^m/2^{m+2}}. \]

From this inequality and (20) we obtain for all \( \mu > 0 \) that
\[ \|k_\mu\|_{L_1(\mathbb{R}^d)} \leq K_{m,d}e^{-\mu^m/2^{m+2}}. \]

**Proof of Proposition 4.1.** We introduce \( \tilde{a}: \mathbb{R}^d \to \mathbb{C}, \tilde{a}(\xi) = (c/2)|\xi|^m \). Then \( \tilde{a} \) and \( (a - \tilde{a}) \) are strongly elliptic polynomials of order \( m \in \mathbb{N} \). Note that \( e^{-t\tilde{a}(D)} = G(c/2)t \) for all \( t \geq 0 \), where \( (G(t))_{t \geq 0} \) is as in Proposition 4.2. Moreover, let \( T_t = e^{-t(a - \tilde{a})}G(c/2)t = e^{-t(a(D)-\tilde{a}(D))}G(c/2)t \) for \( t \geq 0 \). Since Fourier multipliers commute, we have
\[ S_t = e^{-t\tilde{a}(D)} = e^{-t(a(D)-\tilde{a}(D))}e^{-t\tilde{a}(D)} = T_tG(c/2)t. \]

As in Proposition 3.1 we obtain a corresponding heat kernel bound for the kernel of the semigroup \( (T_t)_{t \geq 0} \) with the same growth rate \( \omega \) as for \( (S_t)_{t \geq 0} \). Indeed, by (15) there exists \( \sigma \geq 0 \) such that
\[ \text{Re}(a - \tilde{a})(\xi + \eta) = \text{Re}a(\xi + \eta) - (c/2)|\xi + \eta|^m \]
\[ \geq (3/4)c|\xi|^m - \sigma|\eta|^m - \omega - (c/2)|\xi + \eta|^m \]
\[ \geq (3/4)c|\xi|^m - \sigma|\eta|^m - \omega - (c/2)|\xi|^m - (c/2)|\eta|^m \]
\[ \geq (1/4)c|\xi|^m - (\sigma + c/2)|\eta|^m - \omega, \]

17
with yields (15) with $a$ replaced by $a - \tilde{a}$. Thus, as a consequence of Proposition 3.1 there exists $M \geq 1$ such that $\|T_t\| \leq M e^{\omega t}$ for all $t \geq 0$. By Proposition 4.2 and since Fourier multipliers commute, we obtain for all $f \in L_q(\mathbb{R}^d)$

$$\|(\text{Id} - P_\lambda)S_t f\|_{L_q(\mathbb{R}^d)} = \|S_t (\text{Id} - P_\lambda) f\|_{L_q(\mathbb{R}^d)} \leq \|T_t\|_{L_q(\mathbb{R}^d)} \|G_{(c/2)t}(\text{Id} - P_\lambda)f\|_{L_q(\mathbb{R}^d)} \leq M K_{m,d} e^{-(2^{-m-2}(c/2)\lambda^m - \omega)},$$

where $K_{m,d} > 0$ is a constant depending only on $m$ and $d$. Since $\lambda > (2^{m+4} \max\{\omega, 0\}/c)^{1/m}$, we have $2^{-m-2}(c/2)\lambda^m - \omega > 2^{-m-2}c\lambda^m/4 = 2^{-m-4}c\lambda^m$. \hfill \square

The uncertainty principle that is appropriate for our purposes is obtained as a consequence of the following theorem. It has originally been proven by Logvinenko and Sereda in [LS74], and significantly improved by Kovrijkine in [Kov00, Kov01]. Recently, it has been adapted to functions on the torus instead of $\mathbb{R}^d$, see [EV]. We quote a special case from [Kov01].

**Theorem 4.3** (Logvinenko–Sereda theorem). There exists $K \geq 1$ such that for all $q \in [1, \infty]$, all $\lambda > 0$, all $\rho \in (0, 1]$, all $L \in (0, \infty)^d$, all $(\rho, L)$-thick sets $E \subset \mathbb{R}^d$, and all $f \in L_q(\mathbb{R}^d)$ satisfying $\text{supp}\mathcal{F}f \subset [-\lambda, \lambda]^d$ we have

$$\|f\|_{L_q(\mathbb{R}^d)} \leq d_0 e^{d_1\lambda} \|f\|_{L_q(E)},$$

where

$$d_0 = e^{Kd \ln(K^d/\rho)} \quad \text{and} \quad d_1 = 2|L|_1 \ln(K^d/\rho). \quad (25)$$

We are now in a position to prove Theorem 3.3.

**Proof of Theorem 3.3.** Let $(P_\lambda)_{\lambda > 0}$ be the family of operators defined at the beginning of Section 4. Then we have $\text{supp}\mathcal{F}(P_\lambda f) \subset [-\lambda, \lambda]^d$ for all $\lambda > 0$ and all $f \in L_q(\mathbb{R}^d)$. Thus, Theorem 4.3 implies that for all $f \in L_q(\mathbb{R}^d)$ and all $\lambda > 0$ we have

$$\|P_\lambda f\|_{L_q(\mathbb{R}^d)} \leq d_0 e^{d_1\lambda} \|P_\lambda f\|_{L_q(E)},$$

where $d_0$ and $d_1$ are as in (25). Moreover, according to Proposition 4.1, for all $\lambda > \lambda^*$ and all $f \in L_q(\mathbb{R}^d)$ we have

$$\|(I - P_\lambda)S_t f\|_{L_q(\mathbb{R}^d)} \leq d_2 e^{-d_3\lambda^m t} \|f\|_{L_q(E)},$$

where $\lambda^* = (2^{m+4} \max\{\omega, 0\}/c)^{1/m}$, $d_2 \geq 1$ depends only on the polynomial $a$, and $d_3 = 2^{-m-4}c$. Moreover, the function $t \mapsto \|(S_t f)|_E\|_{L_q(E)}$ is Borel-measurable for all $f \in L_q(\mathbb{R}^d)$. Indeed, if $q \in [1, \infty)$ the semigroup $(S_t)_{t \geq 0}$ is strongly continuous and the measurability follows. If $q = \infty$, measurability is a consequence of duality and the representation of the norm in $L_\infty(E)$ by means of the Hahn–Banach theorem.
Hence we can apply Theorem 2.8 with \( X = L_q(\mathbb{R}^d), Y = L_q(E), C : X \to Y \) given by the restriction map on \( E \), and obtain that the statement of the theorem holds with \( C_{\text{obs}} \) replaced by

\[
\hat{C}_{\text{obs}} = \frac{C_1}{T^{1/r}} \exp\left( \frac{C_2}{T^{1/(m-1)}} + C_3 T \right),
\]

where \( T^{1/r} = 1 \) if \( r = \infty \), and

\[
\begin{align*}
C_1 &= (4Md_0) \max\left\{ \left( 4d_2M^2(d_0 + 1) \right)^{8/(e \ln 2)}, e^{4d_1\lambda^*} \right\}, \\
C_2 &= 4 \left( 2 \cdot 8^{m-1} d_m^m / d_3 \right)^{m-1}, \\
C_3 &= \max\{\omega, 0\} \left( 1 + 10/(e \ln 2) \right),
\end{align*}
\]

with \( M \) as in (16). We denote by \( K_d, K_m, \) and \( K_a \) positive constants which depend only on the dimension \( d \), on \( m \), or on the polynomial \( a \), respectively. A straightforward calculation shows that

\[
C_1 \leq K_a \left( \frac{K_d}{\rho} \right)^{K_d(1+|L|\lambda^*)} \quad \text{and} \quad C_2 \leq \frac{K_m(|L|_1 \ln(K_d/\rho))^{m/(m-1)}}{c^{1/(m-1)}}.
\]

Thus we obtain

\[
\hat{C}_{\text{obs}} \leq \frac{K_a}{T^{1/r}} \left( \frac{K_d}{\rho} \right)^{K_d(1+|L|\lambda^*)} \exp\left( \frac{K_m(|L|_1 \ln(K_d/\rho))^{m/(m-1)}}{(cT)^{1/(m-1)}} + C_3 T \right) =: C_{\text{obs}}. \quad \Box
\]

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