An Improved Bound for Weak Epsilon-Nets in the Plane*

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Abstract

We show that for any finite point set $P$ in the plane and $\epsilon > 0$ there exist $O\left(\frac{1}{\epsilon^{3/2+\gamma}}\right)$ points in $\mathbb{R}^2$, for arbitrary small $\gamma > 0$, that pierce every convex set $K$ with $|K \cap P| \geq \epsilon |P|$. This is the first improvement of the bound of $O\left(\frac{1}{\epsilon^2}\right)$ that was obtained in 1992 by Alon, Bárány, Füredi and Kleitman for general point sets in the plane.

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1 Introduction

**Transversals and \( \epsilon \)-nets.** Given a family \( \mathcal{K} \) of geometric ranges in \( \mathbb{R}^d \) (e.g., lines, triangles, or convex sets), we say that \( Q \subset \mathbb{R}^d \) is a transversal to \( \mathcal{K} \) (or \( Q \) pierces \( \mathcal{K} \)) if each \( K \in \mathcal{K} \) is pierced by at least one point of \( Q \). Given an underlying set \( P \) of \( n \) points, we say that a range \( K \in \mathcal{K} \) is \( \epsilon \)-heavy if \( |P \cap K| \geq \epsilon n \). We say that \( Q \) is an \( \epsilon \)-net for \( \mathcal{K} \) if it pierces every \( \epsilon \)-heavy range in \( \mathcal{K} \). We say that an \( \epsilon \)-net for \( \mathcal{K} \) is a strong \( \epsilon \)-net if \( Q \subset P \), that is, the points of the net are drawn from the underlying point set \( P \). Otherwise (i.e., if \( Q \) includes additional points outside \( P \)), we say that \( Q \) is a weak \( \epsilon \)-net.

The study of \( \epsilon \)-nets was initiated by Vapnik and Chervonenkis [41], in the context of Statistical Learning Theory. Following a seminal paper of Haussler and Welzl [25], \( \epsilon \)-nets play a central role in Discrete and Computational Geometry [29]. For example, bounds on \( \epsilon \)-nets determine the performance of the best-known algorithms for Minimum Hitting Set/Set Cover Problem in geometric hypergraphs [7, 10, 20, 21], and the transversal numbers of families of convex sets [3, 4, 6, 27].

Informally, the cardinality of the smallest possible \( \epsilon \)-net for the range set \( \mathcal{K} \) determines the integrality gap of the corresponding transversal problem – the ratio between (1) the size of the smallest possible transversal \( Q \) to \( \mathcal{K} \) and (2) the weight of the “lightest” possible fractional transversal to \( \mathcal{K} \) [6, 3, 21].

Haussler and Welzl [25] proved the existence of strong \( \epsilon \)-nets of cardinality \( O \left( \frac{1}{\epsilon} \log \frac{1}{\epsilon} \right) \) for families of simply-shaped, or semi-algebraic geometric ranges in \( \mathbb{R}^d \)-space, for a fixed \( d > 0 \) (e.g., boxes, spheres, halfspaces, simplices), by observing that their induced hypergraphs have a bounded Vapnik-Chervonenkis dimension (so called VC-dimension). While the bound is generally tight for a fixed VC-dimension [28], better constructions are known for several special families of ranges, including tight bounds for discs in \( \mathbb{R}^2 \), halfplanes in \( \mathbb{R}^2 \) and halfspaces in \( \mathbb{R}^3 \) [20, 28, 33], and rectangles in \( \mathbb{R}^2 \) and boxes in \( \mathbb{R}^3 \) [7, 37]. We refer the reader to a recent state-of-the-art survey [36] for the best known bounds.

**Weak \( \epsilon \)-nets for convex sets.** In sharp contrast to the case of simply-shaped ranges, no constructions of small-size strong \( \epsilon \)-nets exist for general families of convex sets in \( \mathbb{R}^d \), for \( d \geq 2 \). For example, given an underlying set of \( n \) points in convex position in \( \mathbb{R}^2 \), any strong \( \epsilon \)-net with respect to convex ranges must include at least \( n - \epsilon n \) of the points. This phenomenon can be attributed to the fact that such families of ranges determine hypergraphs of unbounded VC-dimension. Nevertheless, Bárány, Füredi and Lovász [9] observed in 1990 that families of convex sets in \( \mathbb{R}^2 \) still admit weak \( \epsilon \)-nets of cardinality \( O(\epsilon^{-1026}) \). Alon et al. [1] were the first to show in 1992 that families of convex sets in any dimension \( d \geq 1 \) admit weak \( \epsilon \)-nets whose cardinality is bounded in terms of \( 1/\epsilon \) and \( d \). The subsequent study and application of weak \( \epsilon \)-nets bear strong relations to convex geometry, including Helly-type, Centerpoint and Selection Theorems; see [30] Sections 8 – 10 for a comprehensive introduction. For example, Alon and Kleitman [6] used the boundedness of weak \( \epsilon \)-nets to prove Hadwiger-Debrunner \((p,q)\)-conjecture, which concerns transversal numbers of convex sets in \( \mathbb{R}^d \).

**Bounds on weak \( \epsilon \)-nets.** For any \( \epsilon > 0 \) and \( d \geq 0 \), let \( f_d(\epsilon) \) be the smallest number \( f > 0 \) so that, for any underlying finite point set \( P \), one can pierce all the \( \epsilon \)-heavy convex sets using only \( f \) points in \( \mathbb{R}^d \). It is an outstanding open problem in Discrete and Computational geometry to
determine the true asymptotic behaviour of \( f_d(\epsilon) \) in dimensions \( d \geq 2 \). Alon et al. [1] (see also [6]) used Tverberg-type results to show that \( f_d(\epsilon) = O(1/\epsilon^{d+1-1/s_d}) \) (where \( 0 < s_d < 1 \) is a selection ratio which is fixed for every \( d \)), and \( f_2(\epsilon) = O(1/\epsilon^2) \). The bound in higher dimensions \( d \geq 3 \) has been subsequently improved in 1993 by Chazelle et al. [16] to roughly \( O^* \left( \frac{1}{\epsilon} \right) \) (where \( O^*(\cdot) \)-notation hides multiplicative factors that are polylogarithmic in \( \log 1/\epsilon \)). Though the latter construction was somewhat simplified in 2004 by Matoušek and Wagner [32] using simplicial partitions with low hyperplane-crossing number [31], no improvements in the upper bound for general families of convex sets and arbitrary finite point sets occurred for the last 25 years, in any dimension \( d \geq 2 \).

In view of the best known lower bound of \( \Omega \left( \frac{1}{\epsilon} \log d - 1 \left( \frac{1}{\epsilon} \right) \right) \) for \( f_d(\epsilon) \) due to Bukh, Matoušek and Nivasch [11], it still remains to settle whether the asymptotic behaviour of this quantity substantially deviates from the long-known “almost-\((1/\epsilon)\)” bounds on strong \( \epsilon \)-nets (e.g., for triangles in \( \mathbb{R}^2 \) or simplices in \( \mathbb{R}^d \))?

The only interesting instances in which the gap has been essentially closed, involve special point sets [16, 12, 5]. For example, Alon et al. [5] showed in 2008 that any finite point set in a convex position in \( \mathbb{R}^2 \) allows for a weak \( \epsilon \)-net of cardinality \( O \left( \frac{\alpha(\epsilon)}{\epsilon} \right) \) with respect to convex sets.

Our result and organization. We provide the first improvement of the general bound in \( \mathbb{R}^2 \).

**Theorem 1.1.** We have \( f_2(\epsilon) = O \left( \frac{1}{\epsilon^{3/2+\gamma}} \right) \), for any \( \gamma > 0 \).

That is, for any underlying set of \( n \) points in \( \mathbb{R}^2 \), and any \( \epsilon > 0 \), one can construct a weak \( \epsilon \)-net with respect to convex sets whose cardinality is \( O \left( \frac{1}{\epsilon^{3/2+\gamma}} \right) \); here \( \gamma > 0 \) is an arbitrary small constant which does not depend on \( \epsilon \).

The rest of the paper is organized as follows:

In Section 2 we provide a comprehensive overview of our approach, lay down the recursive framework, and establish several basic properties that are used throughout the proof of Theorem 1.1.

In Section 3 we use the recursive framework of Section 2 to give a constructive proof of Theorem 1.1. The eventual net combines the following elementary ingredients: (1) vertices of certain trapezoidal decompositions of \( \mathbb{R}^2 \), (2) 1-dimensional \( \hat{\epsilon} \)-nets, for \( \hat{\epsilon} = \omega(\epsilon^2) \), which are constructed within few vertical lines with respect to carefully chosen point sets, and (3) strong \( \hat{\epsilon} \)-nets with respect to triangles in \( \mathbb{R}^2 \).

In Section 4 we briefly summarize the properties of our construction and survey the future lines of work.

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1. As Alon, Kalai, Matoušek, and Meshulam noted in 2001: “Finding the correct estimates for weak \( \epsilon \)-nets is, in our opinion, one of the truly important open problems in combinatorial geometry” [4].
2. The constant of proportionality within \( O(\cdot) \) may heavily depend on \( \gamma \).
2 Preliminaries

2.1 Proof outline

We briefly outline the main ideas behind our proof of Theorem \[1.1\]. We begin by sketching the \(O(1/\epsilon^2)\) planar construction of Alon et al. \[1\] (or, rather, its more comprehensive presentation by Chazelle \[15\]).

The quadratic construction. Refer to Figure 1(left). We split the underlying point set \(P\) by a vertical median line into subsets \(P^-\) and \(P^+\) (of cardinality \(n/2\) each), and recursively construct a weak \((4\epsilon/3)\)-net with respect to each of these sets. Let \(K\) be an \(\epsilon\)-heavy convex set. If at least \(3\epsilon n/4\) points of \(P\) lie to the same side of \(P\), we pierce \(K\) by one of the auxiliary \((4\epsilon/3)\)-nets. Otherwise, the points of \(P_K := P \cap K\) span at least \(\epsilon^2 n^2/16\) edges that cross \(K \cap L\), so we can pierce \(P\) by adding to our net each \((\epsilon^2 n^2/16)\)-th crossing point of \(L\) with the edges of \(\binom{P}{2}\).

The above argument yields a recurrence of the form \(f_2(\epsilon) \leq 2f_2(4\epsilon/3) + 16/\epsilon^2\) which bottoms out when \(\epsilon\) surpasses 1 (in which case we use the trivial bound \(f(\epsilon) \leq 1\) for all \(\epsilon \geq 1\)).

Notice that the above approach immediately yields a net of size \(o(1/\epsilon^2)\) for sets \(K\) that fall into one of the following favourable categories:

1. The interval \(K \cap L\) is crossed by more than \(\Theta(\epsilon^2 n^2)\) edges of \(\binom{P}{2}\), with either one or both of their endpoints lying outside \(K\).

   For example, we need only \(1/\delta = o\left(1/\epsilon^2\right)\) points to pierce such sets \(K\) whose cross-sections \(K \cap L\) contain at least \(\delta n^2 = \omega\left(\epsilon^3 n^2\right)\) intersection points of \(L\) with the edges of \(\binom{P}{2}\).\(^4\)

2. At least a fixed fraction of the \(\Omega(\epsilon^2 n^2)\) edges spanned by \(P_K\) belong to a relatively sparse subset \(\Pi \subset \binom{P}{2}\) of cardinality \(m = o(n^2)\). This subset \(\Pi\) is carefully constructed in advance and does not depend on the choice of \(K\).

   This too leads to a net of size \(O\left(m/(\epsilon^2 n^2)\right) = o\left(1/\epsilon^2\right)\) provided that a large fraction of these edges of \(\Pi\) end up crossing \(L\). (In other words, the endpoints of these edges must be sufficiently spread between the halfplanes of \(\mathbb{R}^2 \setminus L\).)

\(^4\)In the sequel we use \(\binom{A}{2}\) to denote the complete set of edges spanned by a (finite) point set \(A \subset \mathbb{R}^2\).
\textbf{Figure 2:} Left: We partition the plane into $O(1/\epsilon)$ sectors $W_j(p)$, each containing roughly $\epsilon n$ outgoing edges $pq$, and an average amount of $O(\epsilon n/r^2)$ outgoing short edges. Right: The point $p$ with the outgoing short edges that are “parallel” to $ab$, and whose supporting lines are roughly tangent to $K$. In Case 1, the $\Omega(\epsilon n/r)$ outgoing short edges of $p$ within $\Delta \cap K$ occupy multiple sectors $W_j(p)$ which are almost tangent to $K$. This yields $\omega(\epsilon^2 n^2)$ segments that cross the intercept $K \cap L$.

\textbf{Decomposing $\mathbb{R}^2$.} To force at least one of the above favourable scenarios, we devise a randomized decomposition of $\mathbb{R}^2$ and $\mathcal{P}$. Rather than using a single line to split $\mathbb{R}^2$ into halfplanes, we use a subset $\mathcal{R}$ of $r = o(1/\epsilon)$ lines that are chosen at random from among the lines that support the edges of $\binom{\mathcal{P}}{2}$, and consider their entire arrangement $\mathcal{A}(\mathcal{R})$ – the decomposition of $\mathbb{R}^2 \setminus \bigcup \mathcal{R}$ into open $2$-dimensional faces. (See Section 2.3 for the precise definition of an arrangement, and its essential properties.) We use the $\binom{\mathcal{P}}{2} = o(1/\epsilon^2)$ vertices of $\mathcal{A}(\mathcal{R})$ to construct a small-size point set $Q$ with the following property: Every convex set $K$ that is not pierced by $Q$ must demonstrate a “line-like” behaviour with respect to $\mathcal{A}(\mathcal{R})$: its zone (namely, the $2$-faces intersected by $K$) is contained, to a large extent, in the zone of a single edge $ab \in \binom{\mathcal{P}}{2}$; furthermore, there exist $\Omega(\epsilon^2 n^2)$ such “proxy” edges $ab$ in $\binom{\mathcal{P}}{2}$. See Figure 1 (right). In what follows, we refer to such sets as narrow.

\textbf{Representing narrow convex sets by edges.} The fundamental difficulty of representing and manipulating convex sets (as opposed to lines, segments, simplices, and other simply-shaped geometric objects) is that they can cut the underlying point set $\mathcal{P}$ out into exponentially many subsets $\mathcal{P}_K$, so the standard divide-and-conquer schemes [19] hardly apply in this setting. Fortunately, every narrow convex set $K$ can be largely described by its “proxy” edges $ab \in \binom{\mathcal{P}_K}{2}$. (For example, $K$ cannot include points outside the respective zones of these edges.)

\textbf{From narrowness to expansion.} The main geometric phenomenon behind our choice of the sparse (i.e., non-dense) subset $\Pi \subset \binom{\mathcal{P}}{2}$ is that the “expected” rate of expansion of $\mathcal{P}_K$ within the arrangement $\mathcal{A}(\mathcal{R})$ from a point $p \in \mathcal{P}_K$, for a narrow convex set $K$, is generally lower than that of the entire set $\mathcal{P}$ from that same point $\text{[4]}$.

\text{To illustrate this behaviour, assume first that the points of $\mathcal{P}$ are evenly distributed among the cells of $\mathcal{A}(\mathcal{R})$, so each cell contains roughly $n/r^2$ points. We say that an edge $pq \in \binom{\mathcal{P}}{2}$ is short if both of its endpoints lie in the same cell of $\mathcal{A}(\mathcal{R})$.}
For each point $p$ of $P$ we partition the surrounding plane into $z = \Theta\left(\frac{1}{\epsilon}\right)$ sectors $W_1(p), W_2(p), \ldots, W_z(p)$ so that each sector encompasses $\Theta(\epsilon n)$ outgoing edges $pq \in \left(\frac{r}{2}\right)$; see Figure 2 (left).

To pierce a narrow convex set $K$ whose zone in $\mathcal{A}(\mathcal{R})$ is traced by an edge $ab \in \left(\frac{P_K}{2}\right)$, we combine the following key observations:

- For an average edge $pq \in \left(\frac{r}{2}\right)$, the respective sector $W_j(p)$ contains only $O(\epsilon n/r^2)$ short edges.

- For an average point $p$ in $P_K$, its cell $\Delta$ contains at least $\epsilon n/r$ points of $P_K$, which are connected to $p$ by short edges (because $K$ crosses at most $r + 1$ cells of $\mathcal{A}(\mathcal{R})$).

We further guarantee that the points of $P_K$ are in a sufficiently convex position, and are substantially distributed in the zone of $K$: The former property is enforced by using a strong $\tilde{\epsilon}$-net \cite{25}, with $\tilde{\epsilon} = \Theta(\epsilon/r)$, to eliminate the forbidden convex sets $K$, whereas the latter condition is enforced using a suitably amplified version of the prior line-splitting argument. Thus, for $\Omega(\epsilon n)$ choices of $p \in P_K$, we can assume that both endpoints of the “proxy” edge $ab \in \left(\frac{r}{2}\right)$ of $K$ lie outside the cell $\Delta$ of $p$, and at least half of the $\Omega(\epsilon n/r)$ points $q \in P_K \setminus \{p\}$ within $\Delta$ lie to the same side of $ab$ as $p$. By the near convex position of $P_K$, most lines spanned by such short edges $pq$ within $\Delta$ are roughly tangent to the convex hull of $P_K$; see Figure 2 (right). (In particular, the four points $a, p, q, b$ form a convex quadrilateral.)

Assume with no loss of generality that at least half of the above short edges $pq$ are parallel to $ab$, in the sense that the four points $a, p, q, b$ appear in this order along their convex hull. Since an average sector $W_j(p)$ contains only $O(\epsilon n/r)$ such edges, we interpolate between the following scenarios.

Case 1. The wedge spanned by the above $\Omega(\epsilon n/r)$ short edges $pq \in \left(\frac{P_K}{2}\right)$ (along with $pb$) occupies $r$ “average” sectors $W_j(p), W_{j+1}(p), \ldots, W_{j+r}(p)$, which are almost tangent to $K$. We show that the points $P$ within $W_j(p) \cup W_{j+1}(p) \cup \ldots \cup W_{j+r}(p)$ yield $r\epsilon^2 n^2$ edges that cross the intercept $K \cap L$ of $K$ with the “middle” vertical line $L$ that we use to split the points of $P$. (Again, see Figure 2 (right);) Hence, the intersection of $K \cap L$ is relatively “thick”, so we can pierce such sets using $O\left(1/(\epsilon r^2)\right)$ points.

Case 2. The previous scenario does not occur. Using the near-convexity of $P_K$, we find $\Omega(\epsilon n)$ outgoing edges of $p$ within $\left(\frac{P_K}{2}\right)$ that are parallel to $ab$ in the above sense and occupy a constant number of rich sectors $W_j(p)$ with at least $\Omega(\epsilon n/r)$ short edges.

The first property implies that exist $O\left(1/(\epsilon r)\right)$ rich sectors $W_j(p)$, which encompass a total of $O(n/r)$ edges that emanate from $p$. To pierce such convex sets $K$ that fall into Case 2, we define our sparse set $\Pi \subset \left(\frac{r}{2}\right)$ as the set of edges $pq$ which lie in rich sectors $W_j(p), W_j(q)$ (for at least one of the respective endpoints $p$ or $q$). It is easy to check that $P_K$ spans at least $\Omega(\epsilon^2 n^2)$ such edges within $\Pi$, and sufficiently many of these edges must cross $L$. Hence, $K$ falls into the second favourable case.

The vertical decomposition. Since the actual distribution of $P$ in $\mathcal{A}(\mathcal{R})$ is not necessarily uniform, we subdivide the cells of $\mathcal{A}(\mathcal{R})$ into a total of $O(r^2)$ more homogeneous trapezoidal cells, so that each cell contains at most $n/r^2$ points of $P$.\footnote{A similar decomposition was used, e.g., by Clarkson et al. \cite{19} to tackle the closely related problem of bounding generalized point-line incidences (e.g., incidences between points and unit circles, or incidences between lines and certain cells of their arrangement); the relation between the two problems is briefly discussed in Section 4.} To adapt the preceding expansion argument to
the faces of the resulting decomposition $\Sigma$, we extend the notion of narrowness to $\Sigma$ and guarantee that every narrow convex set $K$ crosses only a small fraction of the faces in $\Sigma$. More specifically, the (strong) Epsilon Net Theorem implies that any trapezoidal cell $\tau$ is crossed by $O(n^2 \log r/r)$ of the lines that support the edges of $(P_2' \bigotimes)$ so an average edge of $(P_2')$ crosses only $O(r \log r)$ trapezoidal cells of $\Sigma$. As a result, a “typical” narrow convex set $K$ (whose zone in $\Sigma$ can “read off” from any of its $\Omega((\epsilon^2 n^2)$ proxy edges within $(P_2'))$ crosses relatively few faces of $\Sigma$; in other words, $K$ has a low crossing number with respect to $\Sigma$.

The “exceptional” convex sets $K$, which cross too many faces of $\Sigma$, are dispatched separately using that, for $\Omega(\epsilon^2 n^2)$ of their edges, their supporting lines cross too many cells $\Sigma$ and, thereby, belong to another sparse subset of $(P_1')$.

Discussion. Our trapezoidal decomposition $\Sigma$ of $\mathbb{R}^2$ overly resembles the first step of the proof of the Simplicial Partition Theorem of Matoušek [31] in dimension $d = 2$, which provides $s = O(r^2)$ triangles $\Delta_1, \ldots, \Delta_s$ so that each triangle $\Delta_i$ contains $\Theta(n/s)$ points, and any line in $\mathbb{R}^2$ crosses $O(\sqrt{s}) = O(r)$ of these triangles.

It is instructive to compare our approach to the partition-based technique of Matoušek and Wagner [32], which directly uses the above theorem in $\mathbb{R}^d$ to re-establish the near-1/$\epsilon^d$ bounds of Alon et al. [1] (in $\mathbb{R}^2$) and Chazelle et al. [16] (in any dimension $d \geq 2$), via a simple recursion on the point set and the parameter $\epsilon$.

Notice that the triangles $\Delta_i$ in the Simplicial Partition Theorem, for $1 \leq i \leq s$, cannot be related to particular cells of any single arrangement of lines. To enforce a low crossing number among the convex sets $K$ with respect to the partition $\{\Delta_1, \ldots, \Delta_s\}$, Matoušek and Wagner pierce the “exceptional” sets by an auxiliary net of $O\left(s^{O(d^2)}\right)$ points which are obtained in a rather elaborate manner via Rado’s Centerpoint Theorem [30]. Hence, the cardinality of their net heavily depends on the size $s$ of the partition. Unfortunately, their simplicial partition does not quite suit our analysis, which needs a relatively large number of cells to achieve a substantial improvement over the $O\left(1/\epsilon^2\right)$ bound.

2.2 The recursive framework.

We refine the notation of Section [1] and lay down the formal framework in which our analysis is cast.

Definition. For a finite point set $P$ in $\mathbb{R}^2$ and $\epsilon > 0$, let $K(P, \epsilon)$ denote the family of all the $\epsilon$-heavy convex sets with respect to $P$. We then say that $Q \subset \mathbb{R}^2$ is a weak $\epsilon$-net for a family of convex sets $\mathcal{G}$ in $\mathbb{R}^2$ if it pierces every set in $\mathcal{G} \cap K(P, \epsilon)$.

If the parameter $\epsilon$ is fixed, we can assume that each set in $K$ is $\epsilon$-heavy, so $Q$ is simply a point transversal to $K$. Note also that every weak $\epsilon$-net with respect to $P$ is, in particular, a weak $\epsilon$-net with respect to any subfamily $\mathcal{K}$ of convex sets in $\mathbb{R}^2$.

Notice that the previous constructions [5, 16, 32] employed recurrence schemes in which every problem instance $(P, \epsilon)$ was defined over a finite point set $P$, and sought to pierce each $\epsilon$-heavy convex set $K \in K(P, \epsilon)$ using the smallest possible number of points. This goal was achieved in a divide-and-conquer fashion, by tackling a number of simpler sub-instances $(P', \epsilon')$ with a smaller point set $P' \subset P$ and a larger parameter $\epsilon' > \epsilon$.

To amplify our sub-quadratic bound on $f_2(\epsilon)$, we employ a somewhat more refined framework: each recursive instance is now endowed not only with the underlying point set $P$, but also with a

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6In other words, $\Sigma$ is a $\Theta(r/\log r)$-cutting [17] of $\mathbb{R}^2$ with respect to these lines.
certain subset of edges $\Pi \subset \binom{P}{2}$ which contains a large fraction of the edges spanned by the points of $P \cap K$. Thus, our recurrence can advance not only by increasing the parameter $\epsilon$, but also by restricting the convex sets to “include” many edges of the (typically, sparse) subset $\Pi$.

**Definition.** Let $\Pi \subset \binom{P}{2}$ be a subset of edges spanned by the underlying $n$-point set $P$. Let $\sigma > 0$. We say that a convex set $K$ is $(\epsilon, \sigma)$-restricted to the graph $(P, \Pi)$ if $P \cap K$ contains a subset $P_K$ of $\lceil\epsilon n\rceil$ points so that the induced subgraph $\Pi_K = \binom{P_K}{2} \cap \Pi$ contains at least $\sigma(\epsilon n)$ edges. (In particular, each $(\epsilon, \sigma)$-restricted set $K$ must be $\epsilon$-heavy with respect to $P$.)

Notice that the choice of the set $P_K$ may not be unique, and that $K$ may enclose additional points of $P$. To simplify the presentation, in the sequel we select a unique witness set $P_K$ for every convex set $K$ that is $(\epsilon, \sigma)$-restricted to $(P, \Pi)$.

At each recursive step we construct a weak $\epsilon$-net $Q$ with respect to a point set $P$ and a certain family $K = K(P, \Pi, \epsilon, \sigma)$ of convex sets which is determined by $\epsilon > 0$, a finite point set $P \subset \mathbb{R}^2$, a set of edges $\Pi \subset \binom{P}{2}$, and a threshold $0 < \sigma$. This family $K$ consists of all the convex sets $K$ that are $(\epsilon, \sigma)$-restricted to $(P, \Pi)$.

In what follows, we refer to $(P, \Pi)$ (or simply to $\Pi$) as the restriction graph, and to $\sigma$ as the restriction threshold of the recursive instance.

The topmost instance of the recurrence involves $\Pi = \binom{P}{2}$ and $\sigma = 1$. Each sub-sequent sub-instance $K' = K(P', \Pi', \epsilon', \sigma')$ involves a larger $\epsilon'$ and/or a much sparser restriction graph $(P', \Pi')$. Each such increase in $\epsilon$ or decrease in the density $\lambda := |\Pi|/\binom{n}{2}$ is accompanied only by a comparatively mild decrease in the restriction threshold $\sigma$, which is bounded from below by a certain positive constant throughout the recurrence. $\Box$

The above recurrence bottoms out when either $\epsilon$ surpasses a certain (suitably small) constant $0 < \tilde{\epsilon} < 1$, or the density $\lambda$ of the restriction graph falls below $\epsilon$. In the former case we can use the $O\left((1/\tilde{\epsilon})^2\right) = O(1)$ bound of Alon et al. [1], and in the latter we resort to a much simpler sub-recurrence which is effectively near-linear in $1/\epsilon$.

In the course of our analysis we stick with the following notation. We use $f(\epsilon, \lambda, \sigma)$ to denote the smallest number $f$ so that for any finite point set $P$ in $\mathbb{R}^2$, and any subset $\Pi \subset \binom{P}{2}$ of density $\lambda \leq |\Pi|/\binom{n}{2}$, there is a point transversal of size $f$ to $K(P, \Pi, \epsilon, \sigma)$. We set $f(\epsilon, \lambda, \sigma) = 1$ whenever $\epsilon \geq 1$. Since the underlying dimension $d = 2$ is fixed, for the sake of brevity we use $f(\epsilon)$ to denote the quantity $f_2(\epsilon) = f(\epsilon, 1, 1)$, and note that the trivial bound $f(\epsilon, \lambda, \sigma) \leq f(\epsilon)$ always holds.

### 2.3 Geometric essentials: Arrangements and strong $\epsilon$-nets

**Strong $\epsilon$-nets.** Let $X$ be a (finite) set of elements and $\mathcal{F} \subset 2^X$ be a set of hyperedges spanned by $X$. A strong $\epsilon$-net for the hypergraph $(X, \mathcal{F})$ is a subset $Y \subset X$ of elements so that $F \cap Y \neq \emptyset$ is satisfied for all hyperedges $F \in \mathcal{F}$ with $|F| \geq \epsilon n$.

**Definition.** Let $X$ be a set of $n$ elements, and $r > 0$ be an integer. An $r$-sample of $X$ is a subset $Y \subset X$ of $r$ elements chosen at random from $X$, so that each such subset $Y \in \binom{X}{r}$ is selected with uniform probability $1/\binom{n}{r}$.

The Epsilon-Net Theorem of Haussler and Welzl [25] states that any such hypergraph $(X, \mathcal{F})$, that is drawn from a so called range space of a bounded VC-dimension $D > 0$, admits a strong

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[7] The preceding discussion in Section 2.1, which we formalize in Lemma 2.5, implies then that $f(\epsilon, \sigma, \delta) = o\left(1/\epsilon^2\right)$ once the density $\lambda$ falls substantially below 1 (given that the restriction threshold $\sigma$ remains close to 1). Hence, our further recurrence over $\Pi$ is used to merely amplify this gain.
Figure 3: Left: Lemma 2.2 – the set $K$ meets the three boundary edges $e_1, e_2, e_3$ of the cell $\Delta = L_1^- \cap L_2^- \cap L_3^-$. The point $p \in K$ lies in $L_1^+ \cap L_2^- \cap L_3^-$. The segment between $p$ and $L_2 \cap L_3$ crosses $K \cap L$. Right: The trapezoidal decomposition $\Sigma(L)$.

$\epsilon$-net $Y$ of cardinality $r = O\left(\frac{D}{\epsilon} \log \frac{D}{\epsilon}\right)$. Moreover, such a net $Y$ can obtained, with probability at least $1/2$, by choosing an $r$-sample of $X$.

In particular, this implies the following result.

**Theorem 2.1.** Let $P$ be a finite set of points in $\mathbb{R}^2$, then one can pierce all the $\epsilon$-heavy triangles with respect to $P$ using only $O\left(\frac{1}{\epsilon} \log \frac{1}{\epsilon}\right)$ points of $P$.

**Arrangements of lines in $\mathbb{R}^2$.** Our divide-and-conquer approach uses cells in the arrangement of lines that are sampled at random from among the lines spanned by the edges of our restriction graph $(P, \Pi)$.

To simplify the exposition, we can assume that the points of $P$ are in a general position. In particular, no three of them are collinear, and no two of them span a vertical line.

**Definition.** Any finite family $\mathcal{L}$ of $m$ lines in $\mathbb{R}^2$ induces the arrangement $\mathcal{A}(\mathcal{L})$ – the partition of $\mathbb{R}^2$ into 2-dimensional cells, or 2-faces – maximal connected regions of $\mathbb{R}^2 \setminus (\bigcup \mathcal{L})$. Each of these cells is a convex polygon whose boundary is composed of edges – portions of the lines of $\mathcal{L}$, which connect vertices – crossings among the lines of $\mathcal{L}$. The complexity of a cell is the total number of edges and vertices that lie on its boundary.

**Lemma 2.2.** Let $L_1, L_2$ and $L_3$ be three lines in $\mathbb{R}^2$, and $\Delta \subset \mathbb{R}^2 \setminus (L_1 \cup L_2 \cup L_3)$ be a cell in their arrangement. For each $1 \leq i \leq 3$, let $L_i^-$ and $L_i^+$ be the two halfplanes of $\mathbb{R}^2 \setminus L_i$ so that $\Delta \subset L_i^-$ (see Figure 3 (left)). Suppose that each line $L_i$ contains a boundary edge $e_i$ of $\Delta$ so that the three edges $e_1, e_2$, and $e_3$, appear in this clockwise order along the boundary of $\Delta$. Then for any convex set $K$ that meets all the three sides $e_1, e_2, e_3$ of $\Delta$, and any point $p \in K \cap L_1^+ \cap L_2^- \cap L_3^-$, the segment between $L_2 \cap L_3$ and $p$ must cross $L_1$ within the interval $K \cap L_1$.

**The trapezoidal decomposition.** We further subdivide each cell $\Delta$ of the above arrangement $\mathcal{A}(\mathcal{L})$ by raising a vertical wall from every boundary vertex of $\Delta$ that is not $x$-extremal (i.e., if the vertical line through the vertex enters the interior of $\Delta$); see Figure 3 (right). As is easy to check, the resulting decomposition $\Sigma(\mathcal{L})$ is composed of $O(m^2)$ open trapezoidal cells. The boundary of

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8To construct a weak $\epsilon$-net for a degenerate point set $P$, we perform a routine symbolic perturbation of $P$ into a general position. A weak $\epsilon$-net with respect to the perturbed set would immediately yield such a net with respect to the original set.

9In the sequel, we apply the lemma only in the special case where $L_1$ and $L_3$ are vertical lines.
each cell $\mu$ in $\Sigma(L)$ consists of at most 4 edges\footnote{Some of the trapezoidal cells can be triangles, or unbounded.}, including at most 2 vertical edges, and the at most 2 other edges that are contained in non-vertical lines of $L$.

**Theorem 2.3.** Let $L$ be a family of $m$ lines in $\mathbb{R}^2$, and $0 < r \leq m$ integer. Then, with probability at least $1/2$, an $r$-sample $R \in \binom{L}{r}$ of $L$ crosses every segment in $\mathbb{R}^2$ that is intersected by at least $C(m/r) \log r$ lines of $L$. Here $C > 0$ is a sufficiently large constant that does not depend on $m$ or $r$.

The proof of Theorem 2.3 can be found, e.g., in [15]. It is established by applying the Epsilon Net Theorem to the range space in which every vertex set is a finite family $L$ of lines in $\mathbb{R}^2$, and each hyperedge consists of all the lines spanned by the edges of $\Pi$. If the underlying restriction graph $(P, \Pi)$ is $r$-crowded in $\Pi$, then $\Pi$ cuts out a subset $\Sigma(R)$ of at least $2 C \log r$ lines of $L$; in other words, it serves as an $\left( \frac{4C \log r}{r} \right)$-cutting of $L$.

**The zone.** Let $\Sigma$ be a family of open cells in $\mathbb{R}^2$ (e.g., the above arrangement $A(L)$ or its refinement $\Sigma(L)$). The zone of a convex set $K \subset \mathbb{R}^2$ in $\Sigma$ is the subset of all the cells in $\Sigma$ that intersect $K$.

The crossing number of a convex set $K$ with respect to $\Sigma$ is the cardinality of its zone within $\Sigma$, that is, the number of the cells in $\Sigma$ that are intersected by $K$.

**Definition.** For every pair $p, q \in \mathbb{R}^2$ let $L_{p,q}$ denote the line through $p$ and $q$. Given an edge set $\Pi \subset \binom{P}{2}$, let

$$L(\Pi) := \{ L_{p,q} \mid \{p, q\} \in \Pi \}$$

be the set of all the lines spanned by the edges of $\Pi$. If the underlying restriction graph $(P, \Pi)$ is clear from the context, we resort to a simpler notation $L := L(\Pi)$.

**Decomposing $\mathbb{R}^2$ into vertical slabs.** For each integer $r > 0$ we construct a collection $Y(r)$ of $r$ vertical (i.e., $y$-parallel) lines so that every vertical slab of the arrangement $A(Y(r))$ contains between $[n/(r + 1)]$ to $[n/(r + 1)]$ points of the underlying $n$-point set $P$, and no line of $Y(r)$ passes through a point of $P$. (The two extremal slabs of $A(Y)$ are halfplanes, and each of them is delimited by a single line of $Y(r)$.)

In what follows, we use $\Lambda(r)$ to denote the above slab decomposition $A(Y(r))$.

We say that a convex set $K$ is $\epsilon'$-crowded in $\Lambda(r)$ if there a slab in $\Lambda(r)$ that contains at least $\epsilon' n$ points of $P \cap K$; otherwise, we say that $K$ is $\epsilon'$-spread in $\Lambda(r)$.

The following main property of the decompositions $\Lambda(r)$ is used throughout our proof of Theorem 1.1.

**Lemma 2.4.** Let $P$ be an underlying set of $n$ points in $\mathbb{R}^2$ and $r > 0$ be an integer. For each $\epsilon' \geq 0$ there is a set of $O(r \cdot f(\epsilon' \cdot r))$ points that pierce every convex set $K$ that is $\epsilon'$-crowded in $\Lambda(r)$.

Notice that the recursive term in Lemma 2.4 is essentially linear in $\epsilon$ for $\epsilon'$ close enough to $\epsilon$; see, e.g., [16] Section 3.

**Proof of Lemma 2.4.** Assume with no loss of generality that $r > 2n$, for otherwise our net consists of $P$. Recall that each slab $\tau \in \Lambda(r)$ cuts out a subset $P_\tau := P \cap \tau$ of cardinality $n_\tau := |P_\tau| \leq \left\lceil n/r \right\rceil = \Theta(n/r)$. \hfill $\square$
The crucial observation is that each \( \epsilon' \)-crowded convex set \( K \) must belong to the family \( \mathcal{K}(P, \epsilon, \lambda, \sigma) \) for some slab \( \tau \) in \( \Lambda(r) \). (In particular, we can further assume that \( \epsilon' = O(1/r) \).) For each slab \( \tau \in \mathcal{A}(R) \) we recursively construct the net \( Q_{\tau} \) for the above instance \( \mathcal{K}(P, \epsilon, \lambda, \sigma) \). Using the definition of the function \( f(\cdot) \), and that \( n_r = \Theta(n/r) \), it is easy to check that the total cardinality of these nets \( Q_{\tau} \) is indeed \( O(r \cdot f(r \cdot \epsilon')) \). \( \square \)

The following lemma implies that the recursive instance \( \mathcal{K} = (P, \Pi, \epsilon, \sigma) \) admits a net of size \( \alpha(1/\epsilon^2) \) given that the underlying restriction graph \((P, \Pi)\) is not dense (and that the restriction threshold \( \sigma \) is sufficiently close 1).

**Lemma 2.5.** Let \( r \geq 1 \) be an integer. Then any family \( \mathcal{K} \subset \mathcal{K}(P, \Pi, \epsilon, \sigma) \) admits a point transversal of size

\[
O \left( r \cdot f(\epsilon \cdot \sigma \cdot r) + \frac{r^2 |\Pi|}{\sigma \epsilon^2 n^2} \right).
\]

**Proof.** Assume with no loss of generality that \(|P| \geq 2r\), for otherwise the claim follows trivially. We consider the slab decomposition \( \Lambda(r) \) and apply Lemma 2.4 with \( \epsilon' = \sigma/4 \) to obtain a net \( Q' \) of size \( O(r \cdot f(\epsilon \cdot \sigma \cdot r)) \) that pierces every set \( K \in \mathcal{K} \) that is \( \epsilon' \)-crowded in \( \Lambda(r) \).

In addition, for each vertical line \( L \in \mathcal{Y}(r) \) we construct an auxiliary net \( Q_L \) by choosing every \( |\sigma(\epsilon n)/(2r)| \)-th crossing point of \( L \) with the edges of \( \Pi \). Notice that

\[
\sum_{L \in \mathcal{Y}(r)} |Q_L| = O \left( \frac{r^2 |\Pi|}{\sigma \epsilon^2 n^2} \right)
\]

It suffices to check that every convex set \( K \in \mathcal{K} \) is stabbed by one of the above nets. To this end, we distinguish between two cases.

1. If at least half of the segments of \( \Pi_K = \left( \frac{r_K}{2} \right) \cap \Pi \) do not cross any line of \( \mathcal{Y}(r) \), we find a point \( p \in P_K \) so that at least \( 2\sigma(\epsilon n)/(en) \geq \sigma n/4 \) of its neighbors in the graph \((P_K, \Pi_K)\) lie in the same slab \( \tau \in \Lambda(r) \) that contains \( p \). Hence, \( K \) is \( \epsilon' \)-crowded in \( \Lambda(r) \) and, therefore, pierced by a point of \( Q' \). See Figure 4 (left).

2. At least half of the segments of \( \Pi_K \) cross a line of \( \mathcal{Y}(r) \). Since there are at least \( (\sigma/2)(\epsilon n)/(2r) \) intersection points between the edges of \( \Pi_K \) and the lines of \( \mathcal{Y}(r) \), there must be a line \( L \in \mathcal{Y}(r) \) which contains at least \( \sigma(\epsilon n)/(2r) \) of these intersections. Hence, \( K \) is hit by the corresponding net \( Q_L \). See Figure 4 (right).

\( \square \)

In what follows, \( \sigma \) remains bounded from below by a certain positive constant, and we apply Lemma 2.5 with \( r \) that is a very small (albeit, fixed) constant power \( 1/\epsilon \). (In particular, \( r \) is much larger than \( 1/\sigma \).) Notice that this yields the following bound

\[
f(\epsilon, \lambda, \sigma) = O \left( r \cdot f(\epsilon \cdot \sigma \cdot r) + \frac{r^2 \lambda}{\sigma \epsilon^2} \right)
\]

in which the recursive term on the right side is essentially linear in \( 1/\epsilon \), and the constants of proportionality that are hidden by the \( O(\cdot) \)-notation do not depend on \( \epsilon, \sigma, \) and \( \lambda \). Moreover, the

\( ^{11} \)To simplify the presentation, we routinely omit the constant factors within the recursive terms of the form \( f(\epsilon \cdot r) \) as long as these constants are much larger than \( 1/h \).

\( ^{12} \)If \( \lfloor \sigma(\epsilon n)/(2r) \rfloor = 0 \), then no point is chosen from \( L \).
A non-recursive term is \( o(1/\epsilon^2) \) provided that the density \( \lambda \) is substantially smaller than 1, and it is close to \( 1/\epsilon \) if \( \lambda = O(\epsilon) \). A standard inductive approach to solving recurrences of this kind is presented, e.g., in [23] and [39, Section 7.3.2].

3 Proof of Theorem 1.1

To establish Theorem 1.1, we derive a recursive bound for the quantity \( f(\epsilon, \lambda, \sigma) \) (defined in Section 2.2) which implies that \( f(\epsilon) = f(\epsilon, 1, 1) = O(1/\epsilon^{3/2+\gamma}) \), for any \( \gamma > 0 \). To this end, we fix the family \( K := K(P, \Pi, \epsilon, \sigma) \) for arbitrary \( P \subset \mathbb{R}^2 \), \( 0 \leq \epsilon, \sigma \leq 1 \), and \( \Pi \subset A(\mathbb{R}_1) \). We then bound the piercing number of \( K \) in terms of the simpler quantities \( f(\epsilon, \lambda', \sigma') \) and \( f(\epsilon) \), for \( \lambda' < \lambda \) and \( \epsilon' > \epsilon \).

Throughout our analysis, the restriction threshold \( \sigma \) is bounded from below by an absolute positive constant which does not depend on \( \epsilon \) and \( \lambda \). We can also assume that \( \epsilon \) is bounded from above by a sufficiently small absolute constant \( \tilde{\epsilon} > 0 \); otherwise, we can use the previous \( O(1/\epsilon^2) = O(1) \) bound [1]. In addition, we can assume that \( |P| \geq 1/\epsilon \); otherwise our transversal consists of \( P \). For most of this section we also assume that \( |\Pi| \geq \epsilon'(\delta^2) = \Omega(1/\epsilon) \) (or, else, Lemma 2.5 would provide a much simpler recurrence (1), which is essentially linear in \( 1/\epsilon \)).

To bound the piercing number of \( K \), we gradually construct a net \( Q \) which pierces every \( \epsilon \)-heavy set \( K \in K \). Our construction begins with an empty net \( Q = \emptyset \) and proceeds through several stages. At each stage we add a small number of points to the net \( Q \) and immediately eliminate the already pierced convex sets from the family \( K \). The surviving sets \( K \in K \), which have yet not been pierced by \( Q \), satisfy additional restrictions which facilitate their treatment at the subsequent stages.

Our main decomposition \( \Sigma = \Sigma(r_1) \) of \( \mathbb{R}^2 \) in Section 3.2 is based on cells in the arrangement of an \( r_1 \)-sample \( R_1 \) of \( \mathcal{L} = \mathcal{L}(\Pi) \), for a fairly large value \( r_1 = \Theta \left( \sqrt{1/\epsilon} \right) \). Informally, the lines of \( R_1 \) are sampled from \( \mathcal{L} \) so as to control the crossing number (i.e., size of the respective zone in \( \Sigma(r_1) \)) of an average edge \( pq \) of \( \Pi \). This bound readily extends to the narrow convex sets \( K \) whose zones are traced by such edges \( pq \). Recall that our main argument (which was sketched in Section 2.1) requires that the points \( P_K \) of each set \( K \in K \) are in a “sufficiently convex” position, and are substantially spread within the zone of \( K \) in \( \mathcal{A}(R_1) \). To this end, we use the auxiliary slab decomposition \( \Lambda(r_0) \) of Lemma 2.4, with a suitable \( r_0 \gg 1/\sigma \), in combination with The Epsilon Net Theorem 2.1.13

13For \( x, y \geq 1 \), the notation \( x \ll y \) means that \( x = O(y^{\eta}) \). Here \( \eta > 0 \) is an arbitrary small but constant parameter to be fixed in the sequel, and the constants hidden by the \( O(\cdot) \)-notation do not depend on \( x \) and \( y \). (For \( 0 < x, y \leq 1 \), the notation \( x \ll y \) means that \( 1/y \ll 1/x \).)
The roadmap. The rest of this section is organized as follows.

In Section 3.1 we construct an auxiliary slab decomposition \( \Lambda(r_0) \), where \( r_0 \) is bounded by an arbitrary small (albeit, fixed) positive power of \( 1/\epsilon \), and use Lemma 2.4 to guarantee that the points of our convex sets \( K \) are sufficiently spread among the slabs of \( \Lambda(r_0) \). This is achieved at the expense of adding to \( Q \) a small-size auxiliary net \( Q_0 \).

In Section 3.2 we use the larger sample \( R_1 \) of \( r_1 \) lines from \( L = L(\Pi) \) to define the finer main decomposition \( \Sigma(r_1) \) of \( \mathbb{R}^2 \). As mentioned in Section 2.1, \( \Sigma(r_1) \) is obtained by vertically subdividing the cells of \( \Lambda(R_1) \) into trapezoidal sub-cells. By the properties of \( \Sigma(r_1) \) as a \( \Theta(\log r_1/r_1) \)-cutting for \( L \) [17], an average line of \( L \) crosses only \( O(r_1 \log r_1) \) cells of \( \Sigma(r_1) \). We further “normalize” \( \Pi \) by omitting a relatively small fraction of its edges whose supporting lines in \( L \) cross too many of the cells of \( \Sigma(r_1) \). We then remove from \( K \) every convex set that is not \((\epsilon, \sigma/2)\)-restricted to the surviving graph \((P, \Pi)\). To that end, we add to \( Q \) another auxiliary net \( Q_1 \) which is obtained by solving a simpler recursive instance \( K(P, \Pi', \epsilon, \sigma/2) \), with a much sparser restriction graph \( \Pi' \).

In Section 3.3 make sure that every remaining set \( K \in K \) is narrow in the decomposition \( \Sigma(r_1) \) (in the sense described in Section 2.1) and, therefore, it crosses roughly \( O(r_1 \log r_1) \) of the decomposition cells\(^{14}\). The leftover convex sets, that are not sufficiently narrow in \( \Sigma(r_1) \), are pierced by an auxiliary net \( Q_2 \) whose size is close to \( r_1/\epsilon \).

In Section 3.4 we use the properties of \( \Sigma(r_1) \) to construct the final net \( Q_3 \) which pierces all the remaining sets \( K \in K \) (missed by the auxiliary nets \( Q_i \) of the previous stages \( 0 \leq i \leq 2 \)). This is achieved through a skillful combination of the two paradigms sketched in Section 2.1. Thus, the eventual net \( Q \) for our family \( K \) is given by the union \( \bigcup_{i=0}^3 Q_i \).

In Section 3.5 combine the bounds of the preceding Sections 3.1-3.4 to bound the overall cardinality \( Q \), and then derive the final recurrence for the quantities \( f(\epsilon, \lambda, \sigma) \) and \( f(\epsilon) \).

3.1 Stage 0: The strip decomposition \( \Lambda(r_0) \)

At this stage we construct an auxiliary, almost constant-size slab decomposition \( \Lambda(r_0) \) and use Lemma 2.4 to guarantee for each convex set \( K \in K \) that the points of \( P_K \) are sufficiently spread among the slabs of \( \Lambda(r_0) \). This is achieved at the expense of adding to \( Q \) a certain auxiliary net \( Q_0 \), which is provided by Lemma 2.4 and immediately removing from \( K \) the sets already pierced by \( Q_0 \).

To this end, we select a set \( \mathcal{Y}(r_0) \) of vertical lines, as detailed in Section 2.3 each slab \( \tau \) of the resulting arrangement \( \Lambda(r_0) \) contains between \( [n/(r_0 + 1)] \) and \( [n/(r_0 + 1)] \) points of \( P \).

Let \( 0 < C_0 < 1/4 \) be a sufficiently small absolute constant which does not depend on \( \sigma \). By Lemma 2.4 we can pierce (and subsequently remove from \( K \)) every \( \epsilon' \)-crowded convex set \( K \), for \( \epsilon' = C_0 \sigma \epsilon \), using an auxiliary net \( Q_0 \) of cardinality\(^{15}\)

\[
O \left( r_0 \cdot f \left( \epsilon \cdot \sigma \cdot r_0 \right) \right).
\]

Definition. Let \( \tau \) be a cell in an arrangement of lines. The edge \( pq \) crosses \( \tau \) transversally if \( pq \) intersects the interior of \( \tau \), and none of \( p, q \) lies in \( \tau \); see Figure 5.

Denote

\[
\epsilon_0 := \frac{\sigma \epsilon}{100 r_0}.
\]

\(^{14}\)More precisely, we clip every set \( K \) to a carefully chosen slab \( \tau \in \Lambda(r_0) \) and apply a similar restriction to \( \Sigma(r_1) \).

\(^{15}\)Throughout our recurrence, \( \sigma \) remains bounded from below by an absolute positive constant. In the sequel, we choose \( r_0 \gg 1/\sigma \) to be an arbitrary small constant positive power of \( 1/\epsilon \). The constants of proportionality hidden by the \( O(\cdot) \)-notation do not depend on \( \sigma \).
Let $K \in \mathcal{K}$ be a convex set. We say that a slab $\tau \in \Lambda(r_0)$ is a *middle slab* with respect to $K$ if it satisfies the following conditions:

(M1) $\epsilon_0 n \leq |P_K \cap \tau| \leq C_0 \sigma \epsilon n \leq \epsilon n/4$, and

(M2) $\Omega\left(\sigma \epsilon^2 n^2/r_0\right)$ of the edges of $\Pi_K$ cross $\tau$ transversally. (As before, $\Pi_K$ denotes the induced sub-graph $\Pi \cap \left(\mathcal{P}_K^2\right)$.)

![Figure 5: The slab $\tau \in \Lambda(r_0)$ is a middle slab for $K$. The depicted edge $pq \in \Pi_K$ crosses $\tau$ transversally.](image)

Notice that the second property (M2) depends on the underlying restriction graph $(P, \Pi)$. In Section 3.3, we further restrict the edge set $\Pi$; as a result, some convex sets in $K$ may cease to be $(\epsilon, \sigma)$-restricted to the refined graph $(P, \Pi)$. Thus, the following property is stated in a greater generality.

**Proposition 3.1.** With the previous choice of $0 < \epsilon < 1$, $0 < \sigma \leq 1$, and $P$, and an arbitrary edge set $\Pi \subseteq \left(\mathcal{P}_2\right)$, the following property holds: For every convex set $K \in \mathcal{K}(P, \Pi, \epsilon, \sigma/2)$ that is missed by the previously defined net $Q_0$, there is at least one middle slab in $\Lambda(r_0)$.

**Proof.** By definition, any convex set $K \in \mathcal{K}$ with at least $C_0 \sigma \epsilon n$ points in a single slab $\tau \in \Lambda(r_0)$ is $\epsilon'$-crowded and, therefore, already pierced by $Q_0$. Hence, the second inequality in (M1) holds for any slab $\tau \in \Lambda(r_0)$.

Let $\Lambda_K$ be the set of all the slabs $\tau$ in $\Lambda(r_0)$ that are crossed by $K$, and let $\Lambda'_K \subseteq \Sigma_K$ denote the subset of these slabs that satisfy $|P_K \cap \tau| \geq \epsilon_0 n = \sigma \epsilon n/(100r_0)$. Notice that every slab of $\Lambda'_K$ satisfies condition (M1), and the points in the slabs of $\Lambda_K \setminus \Lambda'_K$ are involved in a total of at most

$$\frac{\sigma \epsilon n}{100r_0} \cdot (r_0 + 1) \cdot \left\lceil \frac{\epsilon n}{4} \right\rceil$$

adjacencies with the edges of $\Pi_K$. Using that $|\Pi_K| = |\left(\mathcal{P}_2\right) \cap \Pi| \geq \frac{\sigma(\epsilon n)}{2}$, we obtain a subset $\Pi'_K \subseteq \Pi_K$ of at least $\frac{\sigma(\epsilon n)}{4}$ edges so that both of their endpoints lie in the slabs of $\Lambda'_K$.

If no cell in $\Lambda'_K$ satisfies condition (M2), we obtain at least $|\Pi'_K|/2 = \Omega(\sigma \epsilon^2 n^2)$ edges of $\Pi_K$ so that no one of them has a transversal crossing with a slab of $\Lambda'_K$. Thus, by the pigeonhole principle, there must be a slab $\tau \in \Lambda'_K$ and a point $p \in P_K \cap \tau$ so that $\Omega(\sigma \epsilon n)$ of its neighbors in the graph $\Pi'_K$ lie either in $\tau$ or in one of its (at most) two neighboring slabs within $\Lambda'_K$. (Notice that these slabs need not be consecutive in $\Lambda(r_0)$ or $\Lambda_K$.) Since one of these three slabs of $\Lambda'_K$ must then contain $\Omega(\sigma \epsilon n)$ neighbors of $p$ in $\Pi'_K$, $K$ must be pierced by $Q_0$ (given a small enough constant $C_0$). This contradiction establishes the claim. $\square$
3.2 Stage 1: The main decomposition of $\mathbb{R}^2$

At this stage we construct the main decomposition $\Sigma(r_1)$ of $\mathbb{R}^2$ into $O(r_1^2)$ cells, for $r_1 \gg r_0$. Since $\Sigma(r_1)$ is a refinement of the auxiliary slab decomposition $\Lambda(r_0)$, we can use the properties of $\Lambda(r_0)$ to show that the points of $P_K$ are sufficiently spread in the finer decomposition $\Sigma(r_1)$.

The decomposition $\Sigma(r_1)$. We select the parameter $r_1$ so that $1/\sigma \ll r_0 \ll r_1 = \Theta \left( \sqrt{1/\epsilon} \right)$ and sample a subset $\mathcal{R}_1$ of $r_1$ lines from $\mathcal{L} = \mathcal{L}(\Pi)$. We can assume with no loss of generality that no line of $\mathcal{Y}(r_0)$ passes through a vertex of $A(\mathcal{R}_1)$.

We then construct a trapezoidal decomposition $\Sigma(\mathcal{R}_1)$ of $A(\mathcal{R}_1)$ which was described in Section 2.3; see Figure 6 (left). We further subdivide each cell $\hat{\mu} \in \Sigma(\mathcal{R}_1)$ (where necessary) into subtrapezoids $\mu$ so that $|P \cap \mu| \leq n/r_1^2$; this can be achieved using at most $\left\lceil r_1^2 |P \cap \hat{\mu}| / n \right\rceil$ additional vertical walls. Furthermore, we can assume that none of these walls coincides with a point of $P$.

A standard calculation (see, e.g., [13]) shows that the resulting finer partition $\Sigma(r_1)$ encompasses a total of $O(r_1^2)$ trapezoids. Since $\Sigma(r_1)$ is a refinement of $\Sigma(\mathcal{R}_1)$, each of its cells is still crossed by $O((m \log r_1)/r_1)$ lines of $\mathcal{L}$, where $m$ denotes the cardinality of $\Pi$ and $\mathcal{L} = \mathcal{L}(\Pi)$.

Refining the restriction graph $\Pi$. Since every trapezoidal cell of $\Sigma(r_1)$ is crossed by $O((m \log r_1)/r_1)$ lines of $\mathcal{L}$, the zone of an “average” line in $\mathcal{L} = \mathcal{L}(\Pi)$ consists of $O(r_1 \log r_1)$ cells of $\Sigma(r_1)$. More precisely, we have the following property.

For $t \geq 1$ let $\mathcal{L}(t)$ be the subset of all the lines in $\mathcal{L}$ that cross more than $tr_1 \log r_1$ cells of $\Sigma(r_1)$.$^{16}$

**Proposition 3.2.** We have

$$|\mathcal{L}(t)| = O \left( \frac{m}{t} \right).$$

$^{16}$If $m < r_1$ then we obtain the desired decomposition by choosing $\mathcal{R}_1 = \mathcal{L}$. Note that the lines of $\mathcal{R}_1$ are not necessarily in a general position: many of them can pass through the same point of $P$. Nevertheless, there exist at most $2r_1$ such points in $P$ that lie on one or more lines of $\mathcal{R}_1$.

$^{17}$In the sequel $\log x$ denotes the binary logarithm $\log_2 x$. 

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For the sake of completeness, we spell out the fairly standard proof of Proposition 3.2.

**Proof.** Since any trapezoidal cell $\mu$ in $\Sigma(r_1)$ is crossed by $O((m \log r_1)/r_1)$ lines of $L$, the bipartite graph of pairwise intersections between the lines of $L$ and the cells of $\Sigma(r_1)$ contains

$$O\left(r_1^2 \cdot \frac{m \log r_1}{r_1}\right) = O(r_1 m \log r_1)$$

edges. Since every line of $L(t)$ contributes at least $tr_1 \log r_1$ intersections, the claim now follows by the pigeonhole principle (or Markov’s inequality).

Let $\Pi_t$ be the set of edges that span the lines of $L(t)$. We fix a sufficiently large parameter $t$, so that $r_1 \gg t \gg r_0$. Consider the recursive instance

$$K(t) := K(P, \Pi_t, \epsilon, \sigma/2).$$

Using the bound of Proposition 3.2 on $|\Pi_t| = |L(t)|$, we can pierce the sets of $K(t)$ by an auxiliary net $Q_2$ of size\(^1\)

$$|Q_1| = f\left(\epsilon, \frac{|\Pi_t|}{n^2}, \frac{\sigma}{2}\right) \leq f\left(\epsilon, \frac{m}{t}, \frac{\sigma}{2}\right) = f\left(\epsilon, \frac{\lambda}{t}, \frac{\sigma}{2}\right). \quad (4)$$

We immediately add the points of $Q_1$ to our net $Q$, and remove the sets of $K(t)$ from our family $K$. Note that choosing $t$ to be a very small (albeit, constant) positive power of $1/\epsilon$ guarantees that the recurrence (4) on the density $\lambda = m/n^2$ is invoked only a fixed number of times before $\lambda$ falls below $\epsilon$; thus, $\sigma$ remains bounded from below by a sufficiently small constant.

Notice that every remaining set $K \in K$ belongs to the family $K(P, \Pi(t), \epsilon, \sigma/2)$. We thus remove the edges of $\Pi(t)$ from $\Pi$. In doing so, we stick with the same remaining family $K$ even if some of its sets $K \in K$ are only $(\epsilon, \sigma/2)$-restricted with respect to the refined graph $(P, \Pi)$.

For every remaining set $K \in K$ that is missed by the auxiliary net $Q_1$, the induced edge set $\Pi_K = \Pi \cap \binom{\Pi}{2}$ still contains at least $(\sigma/2)(n^2)$ edges, each of them crossing at most $tr_1 \log r_1$ cells of the decomposition $\Sigma(r_1)$ (again, see Figure 6 (left)). In the following Section 3.3 we use this property to guarantee that every set $K \in K$ intersects at most $tr_1 \log r_1$ cells of $\Sigma(r_1)$ within some middle slab $\tau$ of $K$. As before, this is achieved at expense of adding an additional small-size auxiliary net to $Q$.

### 3.3 Stage 2: Controlling the crossing number in $\Sigma(r_1)$

For each slab $\tau \in \Lambda(r_0)$ we consider the subfamily $K_{\tau}$ of all the convex sets $K \in K$ so that $\tau$ is their middle slab. By Proposition 3.1 (and since every remaining set $K \in K$ is missed by the net $Q_0$ of Stage 0), we have $K = \bigcup_{\tau \in \Lambda(r_0)} K_{\tau}$. Notice that a single convex set $K$ can belong to several such sub-families $K_{\tau}$.

In Section 3.4 we use the decomposition $\Sigma(r_1)$ to construct a small-size net $Q_{\tau}$ for each sub-family $K_{\tau}$. To this end, for every slab $\tau \in \Lambda(r_0)$ we consider the restriction

$$\Sigma_{\tau} := \{\mu \in \Sigma(r_1) \mid \mu \subset \tau\}.$$
Definition. We say that a convex set \( K \in \mathcal{K}_\tau \) is narrow in \( \Sigma_\tau \) if for every segment \( pq \subset K \) that crosses \( \tau \) transversally, the restriction \( K \cap \tau \) is contained in the zone of \( pq \) within \( \Sigma_\tau \). (Notice that the cells of this zone lie in the zone of \( pq \) within the arrangement \( \mathcal{A}(R_1) \).) See Figure 3 (right).

Informally, the \( \Sigma_\tau \)-narrowness of \( K \) means that its behaviour is “line-like” in \( \Sigma_\tau \), so the zone of \( K \) in \( \Sigma_\tau \) can be completely “read off” from any edge of \( \Pi_K \) that crosses \( \tau \) transversally.

Proposition 3.3. Let \( \tau \) be a slab of \( \Lambda(r_0) \) and \( K \) be a set of \( \mathcal{K}_\tau \) that is narrow in \( \Sigma_\tau \). Then \( K \) intersects at most \( tr_1 \log r_1 \) cells of \( \Sigma_\tau \).

Proof. As \( \tau \) is a middle slab for \( K \), there exist \( \Omega \left( \sigma^2 n^2 / r_0 \right) \) edges \( pq \in \Pi_K \) that belong to \( \Pi \) and cross \( \tau \) transversally. Each of these edges crosses the same subset of at most \( tr_1 \log r_1 \) cells of \( \Sigma_\tau \) (since none of them belongs to the set \( \Pi_t \) that we removed at Stage 1). By the \( \Sigma_\tau \)-narrowness of \( K \), these cells form the zone of \( K \cap \tau \) within \( \Sigma_\tau \).

We now get rid of the sets \( K \in \mathcal{K}_\tau \) that are not narrow in \( \Sigma_\tau \).

Proposition 3.4. With the previous definitions, there is a set \( Q_2 \) of cardinality \( O \left( r_0^2 r_1 / \epsilon \right) \) points that, for each slab \( \tau \in \Lambda(r_0) \), pierce every convex set \( K \in \mathcal{K}_\tau \) that is not narrow in \( \Sigma_\tau \).

Proof. We first add to \( Q_2 \) all the \( O \left( r_0^2 \right) \) vertices of the trapezoids of \( \Sigma(r_1) \). We then add to \( Q_2 \) the set \( X \) of the \( r_0 r_1 \) intersection points of the lines of \( \mathcal{Y}(r_0) \) with the lines of \( R_1 \), and construct an even larger family \( Y \subset \bigcup \mathcal{Y}(r_0) \) by intersecting each line of \( \mathcal{Y}(r_0) \) with the edges of \( P \times X \). Notice that the resulting set has cardinality at most \( O \left( r_0^2 r_1 n \right) \), as each line of \( \mathcal{Y}(r_0) \) contains at most \( r_0 r_1 n \) crossing points. Let \( C_1 > 0 \) be a sufficiently small constant (that does not depend on \( \sigma \)). For each line \( L \in \mathcal{Y}(r_0) \) we add to \( Q_2 \) every \( \lfloor C_1 \epsilon n \rfloor \)-th point of \( L \cap Y \), for a total of \( O \left( r_0^2 r_1 / \epsilon \right) \) such points.

Since \( r_1 = o(1/\epsilon) \), the overall cardinality of our auxiliary net \( Q_2 \) is bounded by \( O \left( r_1^2 + r_0^2 r_1 / \epsilon \right) = O \left( r_0^2 r_1 / \epsilon \right) \). It, therefore, suffices to check that \( Q_2 \) satisfies the asserted properties. To this end, we fix a slab \( \tau \in \Lambda(r_0) \). Recall that \( \tau \) is a middle slab for each set \( K \in \mathcal{K}_\tau \). Let \( K \in \mathcal{K}_\tau \) be a set missed by \( Q_2 \). We are to show that \( K \) is narrow in \( \Sigma_\tau \). Pick any segment \( pq \in K \) that crosses \( \tau \) transversally. Assume with no loss of generality that \( q \) lies to the right of \( p \).

![Figure 7: Proof of Proposition 3.4](image_url)

Assume for a contradiction that there is a point \( u \in K \cap \tau \) that lies in a cell \( \mu \in \Sigma_\tau \) outside the zone of \( pq \). Let \( \rho \) be the parent cell of \( \mu \) in the arrangement \( \mathcal{A}(R_1) \). Let \( L_0 \) (resp., \( L_1 \)) be
the vertical line adjacent to $\tau$ from the left (resp., right). Let $L_0^-$ (resp., $L_1^+$) denote the halfplane of $\mathbb{R}^2 \setminus L_0$ (resp., $\mathbb{R}^2 \setminus L_1$) containing $p$ (resp., $q$). We distinguish between three possible cases as illustrated in Figure 7.

1. Both cells $\mu$ and $\rho$ are separated from $pq \cap \tau$ by a line $L_\mu \in \mathcal{R}_1$ that misses $pq \cap \tau$.

   Since $K$ is not pierced by $X$, $K \cap L_0^-$ must lie to the same side of $L_\mu$ as $p$ (or, else, $K$ would contain the point $L_0 \cap L_\mu \in Q_2$), and a symmetric property must hold for $K \cap L_1^+$. Since $|P_\tau \cap K| \leq \epsilon n/4$ (by the property (M1) of a middle slab $\tau$ with respect to $K$), at least one of the subsets $P_K \cap L_0^-, P_K \cap L_1^+$, let it be the former set, must contain $\Omega(\epsilon n)$ points of $P_K$. Applying Lemma 2.2 to the cell $\Delta \subset \mathbb{R}^2 \setminus (L_0 \cup L_1 \cup L_\mu)$ that contains $pq \cap \tau$ readily implies that $L_0 \cap K$ must be crossed by all the edges connecting the vertex $L_1 \cap L_\mu \in X$ and the $\Omega(\epsilon n)$ points of $P_K \cap L_0^-$. Given a sufficiently small choice of $C_1$, the intercept $K \cap L$ must contain a point of $Q_2$.

2. There exist lines $L_\mu, L'_\mu$, each crossing $pq \cap \tau$, so that $\rho$ and $\mu \subset \rho$ lie in the only wedge of $\mathbb{R}^2 \setminus (L_\mu \cup L'_\mu)$ that does not meet $pq$. In this case, $K$ must be pierced by the vertex $L_\mu \cap L'_\mu \in X$.

3. The edge $pq$ crosses $\rho$ but the cell $\mu$ is separated from $pq \cap \rho$ by a vertical line $L_\mu$ which supports a vertical wall $\sigma$ on the boundary of $\mu$. Since $pq$ crosses $\tau$ transversally, it must cross the line $L_\mu$ which is “sandwiched” within $\tau$, and this crossing must happen outside $g$.

Therefore, and due to its convexity, $K$ must contain at least one of the endpoints of $g$.

We conclude that, in either of the above three cases, $K$ must contain a point of $Q_2$. This contradiction confirms that $K$ is indeed narrow in $\Sigma_\tau$. \hfill \Box

We immediately add the points of $Q_2$ to our net $Q$, and remove from $\mathcal{K}$ (and, thus, from each subset $\mathcal{K}_\tau$) every set that is pierced by $Q_2$. As a result, for every $\tau \in \Lambda(r_0)$, every remaining set of $\mathcal{K}_\tau$ is narrow in $\Sigma_\tau$.

Combing the bound $|Q_2| = O\left(\frac{r_0^2 r_1}{\epsilon}\right)$ of Proposition 3.4 with the bounds $[4]$ and $[4]$ on the auxiliary nets $Q_0$ and $Q_1$ that were constructed at the previous Stages 0 and 1, so far we have added a total of

$$f\left(\epsilon, \frac{\lambda}{t}, \frac{\sigma}{2}\right) + O\left(r_0 \cdot f(\epsilon, \sigma \cdot r_0) + \frac{r_0^2 r_1}{\epsilon}\right)$$

(5)

points to the net $Q$. As previously mentioned, choosing $t$ to be a very small (albeit, constant) positive power of $1/\epsilon$ guarantees that our recurrence $[4]$ in $\lambda$ has only constant depth; thus, $\sigma$ remains bounded from below by a certain positive constant. Hence, the second recursive term is essentially linear in $1/\epsilon$. Therefore, the contribution of (5) to the cardinality of $Q$ is effectively dominated by the non-recursive term, which is roughly bounded by $1/\epsilon^{3/2}$ for $r_0 \ll r_1 = \Theta(\sqrt{1/\epsilon})$.

**Discussion.** To optimize the overall bound on the cardinality of the $\epsilon$-net, in Section 3.6 we set $r_1 = \Theta\left(\sqrt{1/\epsilon}\right)$. Since $r_0$ is a an arbitrary small positive power of $1/\epsilon$, the cardinality of the auxiliary net $Q_2$ in Proposition 3.4 becomes close to $1/\epsilon^{3/2}$. Note that a more economical construction of the sets $Y_L$, for $L \in \mathcal{Y}(r_0)$, would have resulted in an auxiliary net of size $O(r_0 r_1/\epsilon)$, and with exactly same properties as argued in Proposition 3.4. However, the actual polynomial dependence on $r_0$ is immaterial for the eventual recurrence that we derive in Section 3.6.
3.4 Stage 3: The set $P_K$ – from the low crossing number to expansion in $\Sigma(r_1)$

At this stage we complete the construction of the net $Q$ for $K(P, \Pi, \epsilon, \sigma)$ by building a “local” net $Q_\tau$ for each family $K_\tau$, which remains fixed for most of this section. To this end, we implement the paradigm of Section 2.1 within each slab $\tau$ of $\Lambda(r_0)$.

**The setup.** By definition, the slab $\tau \in \Lambda(r_0)$ is a middle slab for each convex set $K \in K_\tau$. Namely, we have $|P_K \cap \tau| \geq \epsilon_0 n$ and the graph $\Pi_K$ contains $\Omega\left(\frac{\sigma}{\epsilon_0} \left(\frac{r_1}{2}\right)\right)$ edges $pq$ that cross $\tau$ transversally. By the definition of $K_\tau$, the slab $\tau$ contains at least $\epsilon_0 n$ points of $P_K$. In addition, we are given a sub-partition $\Sigma_\tau$ of $\tau$ into trapezoidal cells so that each of these cells contains at most $n/r_1^2$ points of $P$. We also assume that each $K \in K_\tau$ is narrow in $\Sigma_\tau$. By Proposition 3.3, the zone of $K$ in $\Sigma_\tau$ is composed of at most $tr_1 \log r_1$ cells; all of these cells are intersected by each of the above “witness” edges $pq \in \Pi_K$ that cross $\tau$ transversally.

We pick such an edge $pq \in \Pi_K$ that crosses $\tau$ transversally and assume, with no loss of generality, that $p$ lies to the left of $q$, and that at least $\epsilon_0 n/2 - 2$ of the points of $P_K \cap \tau$ lie above the line $L_{p,q}$ from $p$ to $q$. (For each $K \in K_\tau$ we choose exactly one such “witness” edge, which remains fixed throughout the analysis.)

Let $P_\tau^+$ be this portion of $P_{\tau} = P \cap \tau$ above the line $L_{p,q}$ and put $P_K^+ = P_K \cap P_\tau^+$. Denote $K^+ := \text{conv}(P_K^+ \cup \{p, q\})$. (Notice that $K^+$ is supported by the line $L_{p,q}$ at its boundary edge $pq$; see Figure 8. Note also that $K^+$ too is narrow in $\Sigma_\tau$.)

**Definition.** We set

$$\epsilon_1 := \frac{\epsilon_0}{80 \log 1/\epsilon} \quad \text{and} \quad \hat{\epsilon} := \frac{\epsilon_0}{8tr_1 \log r_1}$$

For each cell $\mu \in \Sigma_\tau$ we denote $P_K(\mu) := P_K^+ \cap \mu$ and $g_{\mu} := |P_K(\mu)|$.

We say that a cell $\mu \in \Sigma_\tau$ is *full* (with respect to $K^+$) if $g_{\mu} \geq \hat{\epsilon} n$. By the Pigeonhole Principle, at least $\epsilon_0 n/5 \geq \epsilon_0 n/4 - 2r_1$ points of $P_K^+$ lie in (the respective interiors of) the full cells of $\Sigma_\mu$, whose set we denote by $\Sigma_{K}$.

To implement the paradigm of Section 2.1 for the set $P_{\tau} = P \cap \tau$, the decomposition $\Sigma_\tau$ of $\tau$, and the convex set $K^+$, we first guarantee that the points of $P_K^+$ are in a sufficiently convex position,
and that they are sufficiently spread within $\Sigma\tau$. (The latter property is essential for guessing the splitting line $L$, whose intercept $K^+ \cap L$ is crossed by many edges of $Pr_2$.) To this end, we construct two auxiliary nets.

1. We construct a finer slab decomposition $\Lambda(s_0)$, where $s_0 \gg r_0$ is again an arbitrary small (albeit, fixed) constant power of $1/\sigma$. We can assume with no loss of generality that $\Lambda(s_0)$ is a refinement of $\Lambda(r_0)$, that is, we have $\mathcal{Y}(s_0) \supset \mathcal{Y}(r_0)$. Furthermore, since $s_0 \ll r_1 = \Theta\left(\sqrt{1/\sigma}\right)$, we can add the lines of $\mathcal{Y}(s_0)$ to the sample $R_1$ with no affect on the asymptotic properties of $A(R_1)$ and its vertical decomposition $\Sigma(r_1)$.

We then apply Lemma 3.4 to construct an auxiliary net $Q(s_0)$ that pierces every convex set that is $(C\epsilon_1)$-crowded in $\Lambda(s_0)$. Here $C > 0$ is a sufficiently small constant to be determined in the sequel. Notice that

$$|Q(s_0)| = O\left(s_0 \cdot f(\epsilon_1 \cdot s_0)\right) = O\left(s_0 \cdot f\left(\epsilon \cdot \frac{s_0 \cdot \sigma}{r_0 \log 1/\epsilon}\right)\right),$$

where the last inequality uses the definition (3) of $\epsilon_0$ in Section 3.1.

Upon adding $Q(s_0)$ to $Q$, we can assume that each convex set $K$ (and its restriction $K^+$) is $(C\epsilon_1)$-spread in $\Lambda(s_0)$.

2. We invoke Theorem 2.11 to construct a strong $(\hat{C}\epsilon)$-net $Q^\Delta(\hat{\epsilon})$ over the set $P$ with respect to triangles, and add its points to the nets $Q$ and $Q_{\tau}$.

Notice that this step increases the cardinality of $Q$ by

$$|Q^\Delta(\hat{\epsilon})| = O\left(\frac{1}{\hat{\epsilon}} \log \frac{1}{\epsilon}\right) = O\left(\frac{t\tau_0 r_1}{\epsilon \sigma} \log^2 \frac{1}{\epsilon}\right).$$

We can remove from $K$ and $K_{\tau}$ every convex set that contains a triangle whose interior encloses at least $\hat{C}\epsilon n$ points of $P$.

We now summarize the key properties of the remaining sets $K \in K_{\tau}$, which are not pierced by the auxiliary nets $Q(s_0)$ and $Q^\Delta(\hat{\epsilon})$.

**Definition.** We say that an edge $uv \in P_{r_2}$ is short if its endpoints lie in the same cell $\mu \in \Sigma_{\tau}$.

Notice that the set $K^+$ contains $\left(g_{\frac{3}{2}}\right) = \Omega(\epsilon^2 n^2)$ short edges within every full cell $\mu \in \Sigma_K$. Let $uv$ be such a short edge whose endpoints belong to $P_K(\mu)$, for some cell $\mu$ of $\Sigma_K$. We say that $uv$ is good for $K^+$ if all the points of $P_K^+ \cup \{p,q\}$ outside $\mu$ lie to the same side of the line $L_{u,v}$, and otherwise we say that $uv$ is bad for $K^+$; see Figure 9.

Informally, the good edges span lines that are nearly tangent to $K$. In particular, for every good edge $uv$ the corresponding line $L_{u,v}$ must miss $pq$. Since $uv$ lies above $pq$, the edges $uv$ and $pq$ are boundary edges of a convex quadrilateral.

**Proposition 3.5.** (i) Let $u_1v_1, u_2v_2, \ldots, u_kv_k$ be good edges with respect to $K$ so that no two of these edges lie in the same cell of $\Sigma_K$. Then the $k+1$ edges of $\{u_jv_j \ | \ 1 \leq j \leq k\} \cup \{pq\}$ lie on the boundary of the same convex $(2k+2)$-gon; see Figure 11.

(ii) Let $\mu \in \Sigma_K$ be a full cell. Then the points of $P_K(\mu)$ determine at least $\frac{3}{4} \left(\frac{g_{\mu}}{2}\right)$ good edges.
Figure 9: Left: The short edge $uv$ is good for $K$ because all the points of $P_K^+ \cup \{p, q\}$ that lie outside $\mu$ are to the same side of $L_{u,v}$. Right: The short edge $uv$ is bad for $K$.

Figure 10: Proposition 3.5 (i) – The good edges $u_jv_j$ with supporting lines $L_{u_j,v_j}$ are depicted. Since these edges lie in distinct cells, they bound a convex polygon (together with $pq$).

Proof. The first part readily follows by the definition of a good edge. To see the second part, we consider the set $E_{bad}$ of all the bad edges that are spanned by the points of $P_K(\mu)$. To bound its cardinality, we represent $E_{bad}$ as the union of the following subsets:

- $E_1$ consists of all the bad edges $uv$ whose respective lines $L_{u,v}$ cross $pq$; see Figure 11 (left).
- $E_2$ (resp., $E_3$) consists of all the bad edges $uv \in E_{bad} \setminus E_1$ for which there is a point $w \in P_K^+$ that lies in a cell $\mu' \in \Sigma_\tau \setminus \{\mu\}$ crossed by $pq$ to the right (resp., left) of $\mu \cap pq$, so that $w$ is separated by $L_{u,v}$ from $pq$. See Figure 11 (center and right).

Provided that $\hat{C} < 1/40$, it suffices to show that each of these sets $E_1, E_2, E_3$ has cardinality at most $10\hat{C}(\frac{n}{2})$. (Notice that $E_2$ and $E_3$ may overlap, and for every edge $uv \in E_2 \cup E_3$ the respective line $L_{u,v}$ misses $pq$.)

Bounding $|E_1|$. Assume for a contradiction that $|E_1| \geq 10\hat{C}(\frac{n}{2})$. We direct every edge $uv \in E_1$ from $u$ to $v$ if the intersection of $L_{u,v}$ with $pq$ is closer to $v$ than to $u$ (and otherwise we direct the edge from $v$ to $u$). By the pigeonhole principle, there is a vertex $u \in P_K(\mu)$ whose out-degree is at least $\hat{C}en$. Hence, the triangle $T = \triangle pqu \subset K^+$ contains at least $\hat{C}en$ points of $P$, so $K$ must have been previously pierced by the auxiliary net $Q^{\wedge}(\hat{e})$ and removed from $K_\tau$. See Figure 12 (left).

\footnote{We emphasize that the definition of a short edge is independent of $K$ whereas the notion of a good edge assumes both the underlying convex set $K$, and the witness edge $pq \in \Pi_K$ which crosses $\tau$ transversally.}
Figure 11: Proof of Proposition 3.5 (ii). The bad edges of $E_1$, $E_2$, and $E_3$ are depicted (resp., left, center, and right). Notice that we direct every edge $uv \in E_1$ towards the intersection $L_{u,v} \cap pq$, whereas the edges of $E_2 \cup E_3$ are directed rightwards.

Figure 12: Proof of Proposition 10. The point $u$ and its outgoing bad edges of $E_1$ and $E_2$ (resp., left and right). In each case, the other endpoints of the outgoing edges lie inside a triangle $T \subset K$ with apex $u$. 
Bounding $|E_2|$ and $|E_3|$. Since the definitions of $E_2$ and $E_3$ are symmetrical, we bound only the cardinality of the former set. We direct every edge $uv \in E_2$ rightwards; see Figure 11 (center).

Once again, we assume for a contradiction that $|E_2| \geq 10C\left(\frac{g_\mu}{2}\right)$, so there is a vertex $u$ whose out-degree $d(u)$ is at least $\hat{C}\hat{\epsilon}n$. Let $uv_1, uv_2, \ldots, uv_{d(u)} = uv^*$ be the counterclockwise sequence of all the outgoing edges of $u$ in $E_2$ (so that the occupied sector of $\mathbb{R}^2$ does not contain any of the points $p, q$). As in the previous case, we find a triangle $T \subset K$ which contains all the $d(u) \geq \hat{C}\hat{\epsilon}n$ endpoints of the edges of $E_2$ that emanate from $u$; see Figure 12 (right).

By the definition of $E_2$, we can choose a point $w$ that lies in a cell $\mu^* \in \Sigma_K$ that is crossed by $pq$ to the right of $\mu \cap pq$ and is separated by $L_{u,v^*}$ from the edge $pq$. Our analysis is assisted by the following property.

**Claim 3.6.** There is a line $L^*$ that crosses both edges $uw$ and $uq$ and so that the entire segment $\triangle uwq \cap L^*$ lies outside the interior of $\mu$.

**Proof of Claim 3.6.** If $\mu$ and $\mu^*$ are separated by a line $L^* \in \mathcal{R}_1$, then this line must also cross $uq$; see Figure 13. (This is because $L^*$ meets $pq$ in-between the intersections $pq \cap \mu$ and $pq \cap \mu^*$; hence, $q$ must lie to the same side of $L^*$ as $\mu^*$.)

![Figure 13: Proof of Claim 3.6](image)

By the definition of $\mathcal{R}_1$, we assume for a contradiction that $|E_2| \geq 10C\left(\frac{g_\mu}{2}\right)$, so there is a vertex $u$ whose out-degree $d(u)$ is at least $\hat{C}\hat{\epsilon}n$. Let $uv_1, uv_2, \ldots, uv_{d(u)} = uv^*$ be the counterclockwise sequence of all the outgoing edges of $u$ in $E_2$ (so that the occupied sector of $\mathbb{R}^2$ does not contain any of the points $p, q$). As in the previous case, we find a triangle $T \subset K$ which contains all the $d(u) \geq \hat{C}\hat{\epsilon}n$ endpoints of the edges of $E_2$ that emanate from $u$; see Figure 12 (right).

By the definition of $E_2$, we can choose a point $w$ that lies in a cell $\mu^* \in \Sigma_K$ that is crossed by $pq$ to the right of $\mu \cap pq$ and is separated by $L_{u,v^*}$ from the edge $pq$. Our analysis is assisted by the following property.

**Claim 3.6.** There is a line $L^*$ that crosses both edges $uw$ and $uq$ and so that the entire segment $\triangle uwq \cap L^*$ lies outside the interior of $\mu$.

**Proof of Claim 3.6.** If $\mu$ and $\mu^*$ are separated by a line $L^* \in \mathcal{R}_1$, then this line must also cross $uq$; see Figure 13. (This is because $L^*$ meets $pq$ in-between the intersections $pq \cap \mu$ and $pq \cap \mu^*$; hence, $q$ must lie to the same side of $L^*$ as $\mu^*$.)

On the other hand, if $u$ and $w$ lie within the same cell of $\mathcal{A}(\mathcal{R}_1)$, then they must be separated by a vertical wall. Since $q$ lies to the right of $\tau$, both $uw$ and $uq$ must cross the vertical line $L^*$ which supports that wall. See Figure 12 (right).

In either case, the intersection $\triangle uwq \cap L^*$ lies outside the interior of $\mu$ by the definition of $\mathcal{A}(\mathcal{R}_1)$ and $\Sigma(\mathcal{R}_1)$.

Let $a$ and $b$ be the respective $L^*$-intercepts of $uw$ and $uq$ as depicted in Figure 13. Claim 3.6 (along with the convexity of $\mu$) implies that the triangle $T = \triangle uab$ indeed contains the $d(u) \geq \hat{C}\hat{\epsilon}n$ points $v_1, \ldots, v_{d(u)}$ within $\mu$. As before, this is contrary to the assumption that $K$ is missed by the strong $(\hat{C}\hat{\epsilon})$-net $Q^\hat{\epsilon}$. This contradiction completes the proof of Proposition 3.5.

**Definition.** Let $\mu$ be a cell of $\Sigma_K$. We orient every good edge within $\mu$ to the right. We say that a point $u \in P_K(\mu)$ is **good** if it is adjacent to at least $g_\mu/10$ outgoing good edges.

The second part of Proposition 3.5 implies the following property:

**Proposition 3.7.** Every full cell $\mu \in \Sigma_K$ contains at least $g_\mu/4$ good points of $P_K^+$, for a total of at least $\epsilon_0 n/20$ such points.

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20Specifically, if this edge is directed from $u$ to $v$ then $pu$ and $vq$ are edges of the convex quadrilateral $\text{conv}(p, q, u, v)$. 

22
Definition. For good point \( u \in P_K^+ \), let \( W_u \) denote the smallest planar wedge with apex \( u \) that contains \( uq \) and all the outgoing good edges \( uv \) of \( u \) (within \( \tau \)) but does not contain \( up \); see Figure 14 (left). Note that \( W_u \) lies entirely in the halfplane to the right of \( u \). Let \( D_u \) denote the cardinality of \( (P_\tau \cap W_u) \setminus \{u\} \), that is, the number of the edges in \( uw \in (P_\tau) \) that are adjacent to \( u \) and lie within \( W_u \cap \tau \).

Since \( W_u \) encompasses all the outgoing good edges of \( u \in P_K(\mu) \), we trivially have \( D_u \geq g_\mu/4 \geq \epsilon n/10 \), and \( D_u \) can be much larger than \( \epsilon n \) due to the additional points of \( P_\tau \setminus P_K \) that potentially lie within \( W_u \cap \tau \).

To interpolate between the two favourable scenarios sketched in Section 2.1, we subdivide the good points \( u \in P_K^+ \) into \( O(\log 1/\epsilon) \) classes according to their degrees \( D_u \).

Definition. For each \( i \) in the interval \( I := \lceil \log (2\epsilon/5\epsilon_1) \rceil, \log 4/\epsilon_1 \rceil \) denote \( \delta_i = 2^i \epsilon_1/4 \).

We say that a good point \( u \in P_K^+ \) is of type \( i \) if \( \delta_i n \leq D_u < \delta_{i+1} n \). For each \( i \in I \) we use \( P_K(i) \) to denote the subset of all the \( i \)-type good points in \( P_K^+ \). Since \( D_u \geq \epsilon n/10 \) holds for any good point, the union \( \bigcup_{i \in I} P_K(i) \) covers all the good points in \( P_K^+ \). (In the sequel, we show that it is enough to consider only the non-negative types \( i \geq 0 \).)

Since \( \epsilon = \omega(\epsilon^2) \), we have \( |I| \leq 4 \log 1/\epsilon \). Hence, the pigeonhole principle guarantees that there is \( i \in I \) so that \( |P_K(i)| \geq \epsilon_0 n/(80 \log 1/\epsilon) = \epsilon_1 n \), in which case we say that the set \( K \in K_\tau \) is of type \( i \). We keep the type \( i \) of our convex sets \( K \) (or, rather, their restrictions \( K^+ \)) fixed during the rest of the analysis, and note that a set may belong to \( O(\log 1/\epsilon) \) distinct types.

Since \( K \) is \( \hat{C} \epsilon_1 \)-spread in \( \Lambda(s_0) \) (and \( \hat{C} \) is a sufficiently small constant), there must be a line \( L \in \mathcal{Y}(s_0) \) so that at least \( \epsilon_1 n/4 \) good points in \( P_K(i) \) lie to each side of \( L \). Let \( A_K(i) \) (resp., \( B_K(i) \)) denote the subset of the good points in \( P_K(i) \) that lie to the left (resp., right) of \( L \); see Figure 14 (right).

Proposition 3.8. For every point \( u \in A_K(i) \) the respective wedge \( W_u \) contains at least \( g_\mu/10 \geq \epsilon n/10 \) outgoing good edges \( uv \), and all the points of \( B_K(i) \).

Since \( |B_K(i)| \geq \epsilon_1 n/4 \), the proposition implies that \( \delta_i \geq \epsilon_1/4 \), or \( i \geq 0 \).

\(^{21}\)Recall that \( P_\tau \) denotes the set \( P \cap \tau \). Notice that many of these points \( w \in (P_\tau \cap W_u) \setminus \{u\} \), which contribute to the count \( D_u \), may not belong to \( P_K \) or even to \( P \cap K \).
Proof of Proposition 3.8. The desired number of the good edges in $W_u$ follows by the construction of $W_u$ (and because all the points in $A_K(i)$ are good).

To show that $W_u$ contains all the points of $B_K(i)$, let $\mu \in \Sigma_K$ be the full cell that contains $u$. Since every point $w \in B_K(i)$ lies in a cell $\mu' \in \Sigma_K$ to the right of $\mu$ and $L$, the desired property follows by the first part of Proposition 3.5 (as each of the points $u$ and $w$ is adjacent to a good edge within the respective cell).

To pierce the remaining sets $K \in \mathcal{K}_\tau$ (of type $i$), we combine the following two lemmas whose somewhat technical proofs are relegated to Section 3.5.

**Lemma 3.9.** For each $u \in A_K(i)$, its respective wedge $W_u$ contains $\Omega(\delta_i n)$ edges that are adjacent to $u$ and cross $L$ within $K \cap L$ (for a total of $\Omega(\epsilon_1 \delta_i n^2)$ such edges that cross $K \cap L$). See Figure 15.

![Figure 15: Lemma 3.9 – a schematic illustration. For each $u \in A_K(i)$, its respective wedge $W_u$ contains $\Omega(\delta_i n)$ edges that are adjacent to $u$ and cross $K \cap L$.](image)

**Lemma 3.10.** There is a subset $\Pi(i) \subset \binom{P_2}{2}$ with the following properties:

i. $\Pi(i)$ does not depend on the set $K \in \mathcal{K}_\tau$, and has cardinality

$$|\Pi(i)| = O\left(\frac{\delta_i n^2}{r_1^2 \epsilon^2}\right).$$

ii. For each point $u \in A_K(i)$, the set $\Pi(i)$ contains all the edges $uw \in \binom{P_2}{2}$ that are adjacent to $u$ and lie within the respective wedge $W_u$.

Notice that the density of the graph $\Pi(i)$ is proportional to $\delta_i$, giving rise to the following tradeoff:

1. If $\delta_i$ exceeds $r_1 \epsilon_1$ then we are in the first favourable scenario of Section 2.1 – combining Lemma 3.9 and Lemma 3.10 (ii) for each $u \in A_K(i)$ yields that the intercept $K \cap L$ is crossed by roughly $(r_1 \epsilon_1 n) \cdot (\epsilon_1 n) \simeq r_1 \epsilon_2^2 n$ edges.

2. On the other hand, as $\delta_i$ approaches $\epsilon$, the set $\Pi(i)$ contains roughly $n^2/r_1$ edges, which gives rise to the second favourable scenario of Section 2.1 (e.g., via Lemma 2.5, or through a direct application of Lemma 3.9).

Our net $Q_\tau(i)$ for the convex sets $K \in \mathcal{K}_\tau$ of type $i$ interpolates between the above extreme cases.
The net. For every $i \in I$, and every line $L \in \mathcal{Y}(s_0)$ within $\tau$, we select every intersection of $L$ with an edge of $\Pi(i)$ into the set $X_L(i)$. We then select every $[C' \epsilon_1 \delta_n^2]$-th point of $X_L(i)$ into our net $Q_L(i)$, for a sufficiently small constant $C' > 0$.

We then define

$$Q_\tau(i) := \bigcup \{Q_L(i) \mid L \in \Lambda(s_0), L \subset \tau \}.$$ 

and $Q_\tau := \bigcup_{i \in I} Q_\tau(i)$.

Hence, our last net $Q_3$ at Stage 3 is given by

$$Q_3 := Q(s_0) \cup Q^\Delta(\hat{t}) \cup \bigcup_{\tau \in \Lambda(r_0)} Q_\tau.$$ 

The analysis. We show that $Q_3$ pierces the remaining sets of $\mathcal{K}$, which are missed by $Q_0 \cup Q_1 \cup Q_2$. Using the definition of $Q_3$, we argue for each slab $\tau \in \Lambda(r_0)$ that the respective net $Q_\tau$ pierces all the sets of $\mathcal{K}_\tau$ which were missed by the previous nets $Q_0, Q_1, Q_2, Q(s_0)$ and $Q^\Delta(\hat{t})$. It suffices to check, for all $\tau \in \Lambda(r_0)$ and $i \in I$, that every type-$i$ set $K \in \mathcal{K}_\tau$ is pierced by some net $Q_L(\tau)$ whose line $L \in \Lambda(s_0)$ lies within $\tau$.

Indeed, according to Lemma [3.9] every point $u \in A_K(i)$ gives rise to $\Omega(\delta_n)$ outgoing edges that cross the intercept $K \cap L$ with some line $L \in \mathcal{Y}(s_0)$ which separates $A_K(i)$ and $B_K(i)$, for a total of $\Omega(\epsilon_1 \delta_n^2)$ such edges. Hence, choosing a small enough constant $C'$ (which may depend on $C'$) guarantees that $K$ is pierced by $Q_L(i)$.

Stage 3: Wrap up. For every type $i \in I$, and every line $L \in \Lambda(s_0)$ within $\tau$, the cardinality of $Q_L(i)$ is bounded by

$$O\left(\frac{|X_L(i)|}{\delta_i \epsilon_1 n^2}\right) = O\left(\frac{|\Pi(i)|}{\delta_i \epsilon_1 n^2}\right) = O\left(\frac{\delta_i n^2}{r_1^2 \epsilon_0^2 \epsilon_1 n^2}\right) = O\left(\frac{t \log r_1 \log 1/\epsilon}{r_1 \epsilon_0^2}\right).$$ 

where the second inequality uses the bound of Lemma [3.10] (ii), and the third one uses the definitions of $\epsilon_1$ and $\hat{t}$.

Recall that $\Lambda(s_0)$ is a refinement of $\Lambda(r_0)$, every slab $\tau \in \Lambda(r_0)$ contains $O(s_0/r_0)$ lines of $\mathcal{Y}(s_0)$. Using this and the definition [3] we can bound the cardinality of $Q_\tau$ by

$$O\left(\frac{s_0 |I|}{r_0} \cdot \frac{t \log r_1 \log 1/\epsilon}{r_1 \epsilon_0^2}\right) = O\left(\frac{s_0 r_0 t \log 3/\epsilon}{\sigma^2 r_1 \epsilon^2}\right).$$ 

Repeating this bound for each slab $\tau \in \Lambda(r_0)$ and combining it with the prior bounds [6] and [7] on the respective cardinalities of the nets $Q(s_0)$ and $Q^\Delta(\hat{t})$, we conclude that the overall increase in the size of $Q$ at Stage 3 is bounded by

$$O\left(s_0 \cdot f\left(\epsilon \cdot \frac{s_0 \cdot \sigma}{r_0 \log 1/\epsilon}\right) + \frac{t r_0 r_1}{\epsilon \sigma} \log^2 1/\epsilon + \frac{s_0 r_0^2 t \log 3/\epsilon}{\sigma^2 r_1 \epsilon^2}\right).$$ 

(9)

Given that $s_0$ is very small (albeit, fixed) positive power of $1/\epsilon$ that satisfies $s_0 \gg r_0 \gg 1/\sigma$, the recursive term is again near-linear in $1/\epsilon$. Furthermore, the two non-recursive terms sum up to roughly $r_1/\epsilon + 1/(r_1 \epsilon^2)$.

In Section 3.6 we combine (9) with the bounds on the sizes of the auxiliary nets $Q_0, Q_1$, and $Q_2$ of the previous Stages $0 \to 2$ to derive a recurrence for $f(\epsilon)$ whose solution is close to $f(\epsilon) = O(1/\epsilon^{3/2})$. 25
3.5 Proofs of Lemmas 3.9 and 3.10

Proof of Lemma 3.9. Refer to Figure 16. Fix a point \( u \in A_K(i) \), and let \( uv \) be the good edge that delimits its respective wedge \( \mathcal{W}_u \). (In other words, \( uv \) attains the largest slope among the good edges that emanate from \( u \) to the right.)

By Proposition 3.8, the wedge \( \mathcal{W}_u \) contains all the points of \( B_K(i) \). We fix any of these points \( u' \in B_K(i) \) together with the edge \( u'u' \) which delimits the respective wedge \( \mathcal{W}_{u'} \). Since the point \( u' \) too is of type \( i \), the respective wedge \( \mathcal{W}_{u'} \) contains at least \( \delta_in \) points \( w \in P_\tau \). It, therefore, suffices to show that all the resulting edges \( uw \) cross the intercept \( G \cap L \subset K \cap L \).

Since the edges \( uv \) and \( u'u' \) are good, Proposition 3.5 implies that the three edges \( uv, uu' \) and \( pq \) form a convex 6-gon \( G \). Let \( L_0 \in \mathcal{Y}(r_0) \) (resp., \( L_1 \in \mathcal{Y}(r_0) \)) be the line that supports \( \tau \) from the left (resp., right). Notice that \( L_0 \) is crossed by the edges \( pq \) and \( pu \), and \( L_1 \) is crossed by the edges \( pq \) and \( v'q \), and none of the remaining edges \( uw, vu', uu' \), of \( G \) crosses \( L_0 \) or \( L_1 \). Thus, the intersection \( G_\tau := G \cap \tau \) is a convex 8-gon. The claim now follows since (1) \( \mathcal{W}_{u'} \) is separated from \( u \) by \( L_1 \), and (2) every point \( w \in \mathcal{W}_u \cap \tau \) lies either inside \( G_\tau \subset K \), or in the triangular “ear” that is adjacent to the edge \( v'q \cap \tau \) of \( G_\tau \) and delimited by \( L_1 \) and \( L_{u'u'} \). □

![Figure 16: Proof of Lemma 3.9](image)

The convex 6-gon \( G = \text{conv}(p, q, u, v, u', v') \) is depicted. Every point \( w \in \mathcal{W}_{u'} \) is separated from \( u \) by \( L \). It lies either inside \( K \), or in the triangular “ear” that is adjacent to the edge \( v'q \cap \tau \) of \( G_\tau = G \cap \tau \) and delimited by \( L_1 \) and \( L_{u'u'} \).

Proof of Lemma 3.10. We first describe the sparse subgraph \( \Pi(i) \subset \left( P_\tau \right) \) which does not depend on the choice of the convex set \( K \).

The graph \( \Pi(i) \). Denote \( P_\tau := P \cap \tau \) and \( n_\tau := |P \cap \tau | \). For each \( p \in P_\tau \) which lies in some cell \( \mu \in \Sigma_\tau \) we partition the \( n_\tau - 1 \) adjacent edges \( pp_1, \ldots, pp_{n_\tau - 1} \in \left( P_\tau \right) \) (which appear in this clockwise order around \( p \)) into \( z = O(1/\delta_i) \) blocks \( \mathcal{E}_j \), for \( 0 \leq j \leq z - 1 \), so that every block but the last one contains \( [2\delta_in] \) edges, and the last block contains at most \( [2\delta_in] \) edges. If \( z \geq 3 \), we partition \( \mathbb{R}^2 \) around \( p \) into \( z \) canonical sectors, where each sector \( \mathcal{W}_j(p) \) encompasses three consecutive blocks \( \mathcal{E}_j, \mathcal{E}_{j+1}, \mathcal{E}_{j+3} \) of edges, and the indexing is modulo \( z \). See Figure 17. Otherwise (i.e., if \( [2\delta_in] > (n - 1)/2 \)), we define only one sector \( \mathcal{W}_0(p) = \mathbb{R}^2 \).

Notice that, given that \( z \geq 3 \), the neighboring sectors overlap, each sector \( \mathcal{W}_j(p) \) satisfies \( 2[2\delta_in] \leq |(\mathcal{W}_j(p) \cap P_\mu) \setminus \{p\}| \leq 3[2\delta_in] \), and an edge \( pq \) lies in exactly three of the sectors of \( p \).

We say that the sector \( \mathcal{W}_j(p) \) is rich if \( |(\mathcal{W}_j(p) \cap P_\mu) \setminus \{p\}| \geq \hat{\epsilon}n/10 \). In other words, the sector \( \mathcal{W}_j(p) \) must contain at least \( \hat{\epsilon}n/10 \) short edges \( pq \).

We add to \( \Pi(i) \) every edge \( pq \) that lies a rich sector of at least one of its endpoints \( p \) or \( q \).
Figure 17: Proof of Lemma 3.10—defining the sparse graph \( \Pi(i) \subset (P_2)^{(P_2)} \). We partition \( \mathbb{R}^2 \) into \( z = O(1/\delta_i) \) sectors \( W_j(p) \). In each sector, the number of the edges of \( (P_2)^{(P_2)} \) that are adjacent to \( p \) ranges between \( 2\lceil 2\delta_i n \rceil \) and \( 3\lceil 2\delta_i n \rceil \). We add the edges of \( W_j(p) \) to \( \Pi(i) \) only if this sector is rich and encompasses at least \( \hat{e}n/10 \) short edges.

Analysis. To see the first property of \( \Pi(i) \), it is sufficient to show that any point \( p \) that lies in \( \tau \) contributes \( O\left( \frac{\delta_i n}{r_1^2} \right) \) edges to the set \( \Pi(i) \).

Indeed, recall that for each cell \( \mu \in \Sigma_{\tau} \) we have that \( n_\mu = |P_\mu| \leq n/r_1^2 \). Therefore, the pigeonhole principle implies for each \( p \in P_\mu \) there can be only \( O\left( \frac{n}{r_1^2\hat{e}n} \right) = O\left( \frac{1}{r_1^2\hat{e}} \right) \) rich sectors \( W_j(p) \), which satisfy \( |W_j(p) \cap P_\mu \setminus \{p\}| \geq \hat{e}n/10 \), and any such sector contributes \( O(\delta_i n) \) edges to \( \Pi(i) \).

For the second property, we recall that, for every good point \( u \in A_{K}(i) \) that lies in some full cell \( \mu \in \Sigma_{K} \), the respective wedge \( W_u \) contains at most \( 2\delta_i n \) outgoing edges \( uw \) within \( \tau \) and, therefore, is contained in (at least) one of the sectors \( W_j(u) \). Proposition 3.7 now implies that this sector \( W_j(u) \) is rich, for it contains at least \( g_\mu/10 \geq \hat{e}n/10 \) outgoing short edges \( uv \).

3.6 The final recurrence

In this section we develop the complete recurrence for the quantity \( f_2 = f(\epsilon) \), which solves to \( f(\epsilon) = O\left( \frac{1}{\epsilon^{3/2+\gamma}} \right) \), for arbitrary small \( \gamma > 0 \).

To simplify our exposition, we stick with the previous convention: For \( x, y \geq 1 \), we say that \( x \ll y \) whenever \( x = O(y^n) \) for some arbitrary small (albeit, constant) positive parameter \( \eta \). For \( 0 < x, y \leq 1 \), we say that \( x \ll y \) if \( 1/y \ll 1/x \).

As mentioned in Section 2 we fix a suitably small constant \( 0 < \bar{\epsilon} < 1 \) and use the old bound \( f(\epsilon) = O\left( \frac{1}{\epsilon^2} \right) = O(1) \) of Alon et al. \[1\] whenever \( \epsilon > \bar{\epsilon} \). (The choice of \( \bar{\epsilon} \) will affect the multiplicative constant in the eventual asymptotic bound on \( f(\epsilon) \).) Assume then that \( \epsilon < \bar{\epsilon} \).

Bounding \( f(\epsilon, \lambda, \sigma) \). To obtain the desired recursion for \( f(\epsilon) = f(\epsilon, 1, 1) \), we first express \( f(\epsilon, \lambda, \sigma) \) in terms of the simpler quantities \( f(\epsilon') \), for \( \epsilon' > \epsilon \). To this end, we fix a family
\( K = K(P, \Pi, \epsilon, \sigma) \) that satisfies \(|\Pi|/(P^2) \leq \lambda \), and bound the overall cardinality of the point transversal \( Q \) for \( K \) that was constructed in Sections 3.1 through 3.4. As explained in the beginning of Section 3, we can assume with no loss of generality that \(|P| \geq 1/\epsilon \).

Assume first that \( \lambda > \epsilon \). Combining the bounds in (5) and (9), we obtain the following bound on the overall cardinality of our net \( Q \):

\[
\begin{align*}
&f \left( \epsilon, \frac{\lambda}{t}, \frac{\sigma}{2} \right) + \\
&+ O \left( r_0 \cdot f(\epsilon \cdot \sigma \cdot r_0) + \frac{r_1^2 r_0}{\epsilon} + s_0 \cdot f \left( \epsilon \cdot \frac{s_0 \cdot \sigma}{r_0 \log 1/\epsilon} \right) + \frac{tr_0 r_1}{\epsilon \sigma} \log^2 \frac{1}{\epsilon} + \frac{s_0 r_0^2 t \log^3 1/\epsilon}{\sigma^2 r_1 \epsilon^2} \right) \quad (10)
\end{align*}
\]

By substituting \( r_1 = \Theta \left( \sqrt{1/\epsilon} \right) \) and rearranging the terms, we obtain for all \( \epsilon \leq \tilde{\epsilon} \) and \( \lambda > \epsilon \) that

\[
f(\epsilon, \lambda, \sigma) \leq f \left( \epsilon, \frac{\lambda}{t}, \frac{\sigma}{2} \right) + \Psi(\epsilon, \sigma). \tag{11}
\]

We begin with \( f(\epsilon) = f(\epsilon, 1, 1) \) and recursively apply the inequality (11) to the “leading” term, which involves the density \( \lambda \). Notice that \( \sigma \) is initially equal to 1, and it will be bounded from below throughout this recurrence in \( \lambda \) by a fixed positive constant. Hence, all the parameters \( r_0, t \) and \( s_0 \) can be chosen to be arbitrary small (albeit, constant) positive powers of \( 1/\epsilon \) that satisfy

\[
1/\sigma \ll r_0 \ll s_0 \ll r_1 = \Theta \left( \sqrt{1/\epsilon} \right). \tag{12}
\]

This relation between \( s_0, r_0, \) and \( t \) will guarantee for a suitable constant \( \eta' = \Theta(\eta) \) that

\[
\Psi(\epsilon, \sigma) = O \left( s_0 \cdot f \left( \epsilon \cdot s_0^{1-\eta'} \right) + r_0 \cdot f \left( \epsilon \cdot r_0^{1-\eta'} \right) + \frac{1}{\epsilon^{3/2 + \eta'}} \right). \tag{13}
\]

This recurrence in \( \lambda \) bottoms out when the value of \( \lambda \) falls below \( \epsilon \). Since \( t \) is a fixed (though arbitrary small) positive power of \( 1/\epsilon \), the inequality (11) is applied \( k = O \left( \log_\epsilon 1/\epsilon \right) = O(1) \) times. Hence, the value of the restriction threshold \( \sigma \) in the \( i \)-th application of (11) is bounded from below by \( 1/2^{i-1} = \Theta(1) \), and the crucial relations (12) and (13) can be preserved in each iteration. Using the trivial property that \( f(\epsilon, \lambda', \sigma') \leq f(\epsilon, \lambda, \sigma) \) for all \( 0 \leq \lambda' \leq \lambda \) and \( 0 < \sigma' \leq \sigma \), and that \( k = O(1) \), we conclude that

\[\text{As previously mentioned, we routinely omit the constant factors within the recursive terms of the form } f(\epsilon \cdot h) \text{ as long as these constants are much larger than } 1/h. \text{ A suitably small choice of the constant } \tilde{\epsilon} \text{ (and thereby } \epsilon \leq \tilde{\epsilon} \text{) guarantees that } \epsilon \text{ indeed increases with each invocation of the recurrence.}
\]

\[\text{Recall that } \eta > 0 \text{ is an arbitrary small parameter that is hidden by our } \ll \text{-notation. The constant factors within } \Theta(\eta) \text{ do not depend on the choice of } r_0, s_0, t \text{ and } \eta.\]
\[ f(\epsilon) = f(\epsilon, 1, 1) \leq f(\epsilon, \epsilon, 2^{-k}) + \sum_{i=1}^{k} \Psi(\epsilon, 2^{-i+1}) = O\left(f(\epsilon, \epsilon, 2^{-k}) + \Psi(\epsilon, 2^{-k})\right). \quad (14) \]

To bound \( f(\epsilon, \epsilon, 2^{-k}) \), we invoke Lemma \ref{lem:bound} with an arbitrary small (albeit, fixed) positive power \( r \) of \( \frac{1}{\epsilon} \) that satisfies \( s_0 \ll r \ll r_1 = \Theta\left(\sqrt{1/\epsilon}\right) \). A suitable choice of \( \eta' = \Theta(\eta) \) again yields
\[ f(\epsilon, \epsilon, 2^{-k}) = O\left(\frac{r^2}{\sigma \epsilon} + r \cdot f(\epsilon \cdot \sigma \cdot r)\right) = O\left(r \cdot f(\epsilon \cdot r^{1-\eta'}) + \frac{1}{e^{1+\eta'}}\right). \quad (15) \]

Substituting (15) and (13) into (14) readily gives
\[ f(\epsilon) = O\left(r \cdot f(\epsilon \cdot r^{1-\eta'}) + s_0 \cdot f(\epsilon \cdot s_0^{1-\eta'}) + r_0 \cdot f(\epsilon \cdot r_0^{1-\eta'}) + O\left(\frac{1}{e^{3/2 + \eta'}}\right)\right). \quad (16) \]

Bounding \( f(\epsilon) \). We emphasize that the final recurrence (16) does not involve \( \lambda \) and \( \sigma \). Furthermore, each recursive term on its right hand side is of the form \( h^{1+O(\eta')} \cdot f(\epsilon \cdot h) \), where \( h \ll 1/\epsilon \) is some arbitrary small (albeit, fixed) degree of \( 1/\epsilon \). This last recurrence terminates when \( \epsilon \geq \bar{\epsilon} \), in which case we have \( f(\epsilon) = O(1) \). By following the standard inductive approach which applies to recurrences of this type (see, e.g., \cite{32}, and also \cite{23, 38} and \cite{39, Section 7.3.2}), and fixing suitably small constants \( \eta \) and \( \bar{\epsilon} \) (and, thereby, also \( \eta' = \Theta(\eta) \)), the recurrence solves to
\[ f(\epsilon) = O\left(\frac{1}{e^{3/2 + \gamma}}\right), \quad (17) \]

where \( \gamma > 0 \) is an arbitrary small constant, and the constant of proportionality depends on \( \gamma \). (Informally, all the recursive terms in (16) are near-linear in \( 1/\epsilon \), so the recurrence is dominated by its non-recursive term. To establish the bound for a particular value of \( \gamma > 0 \), the parameters \( \eta > 0 \) and \( \eta' = \Theta(\eta) \) and, therefore, the positive powers of \( 1/\epsilon \) in \( r_0, s_0, t \) and \( r \), are all set to be much smaller than \( \gamma \). Furthermore, the value of \( \bar{\epsilon} \) is chosen so as to guarantee that \( 1/\bar{\epsilon} \) is much larger than these parameters. Note that the resulting constant of proportionality in \( O(\cdot) \) may be exponential in \( 1/\gamma \).)

This concludes the proof of Theorem 1.1. \( \square \)

4 Concluding remarks

- Our analysis is largely inspired by the partition-based proof \cite{19} of the Szemerédi-Trotter Theorem \cite{40} on the number of point-line incidences in the plane. In the case at hand, narrow convex sets are viewed as abstract lines, so a non-trivial incidence bound implies that a typical point of \( P \) is involved in \( o(1/\epsilon) \) such canonical sets (which are naturally associated with the surrounding sectors \( W_j(p) \)). This gives rise to a sparse restriction graph \( \Pi \). Therefore, it is no surprise that our main decomposition \( \Sigma(r_1) \) overly repeats the one that was used by Clarkson et al. \cite{19} in order to extend the Szemerédi-Trotter bound to more general settings (e.g., incidences between points and unit circles, and incidences between lines and certain cells in their arrangement).

- Our proof of Theorem 1.1 is fully constructive, and the resulting net includes points of the following types:
1. The vertices of the decompositions $\Sigma(r_1)$ which arise in the various recursive instances.
2. 1-dimensional $\hat{\epsilon}$-nets within lines $L \in \mathcal{Y}(r_0)$, for $\hat{\epsilon} = \tilde{\Omega}(\epsilon^{3/2})$. In each net of this kind, the underlying point set is composed of the $L$-intercepts of the edges of $\binom{P}{2}$. These edges typically belong to one of the sparser graphs $\Pi_i$ (in Section 3.3) or $\Pi(i)$ (in Section 3.4).
3. 1-dimensional $\hat{\epsilon}$-nets within lines $L \in \mathcal{Y}(r_0)$, for $\hat{\epsilon} = \tilde{\Omega}(\epsilon^{3/2})$, where the underlying point sets are composed of the $L$-intercepts of the “mixed” edges, which connect the vertices of $\Sigma(r_1)$ to the points of $P$.
4. 2-dimensional $\hat{\epsilon}$-nets of Theorem 2.1 with respect to triangles in $\mathbb{R}^2$.

- Our construction and its analysis combine classical elements of the 30-year old theory of linear arrangements in computational geometry (which generalize to any dimension) with a few ad-hoc arguments in $\mathbb{R}^2$ (which do not immediately extend to higher dimensions). Nevertheless, we anticipate that it is only a matter of time until the current analysis is cast in a more abstract framework and extended to any fixed dimension.

The author conjectures that the actual asymptotic behaviour of the functions $f_d(\epsilon)$ in any dimension $d \geq 1$ is close to $1/\epsilon$, as is indeed the case for their “strong” counterparts with respect to simply shaped objects in $\mathbb{R}^d$ [25]. It’s worth mentioning that our main argument in Section 3.4 exploits the delicate interplay between the two notions of $\epsilon$-nets which was previously explored by Mustafa and Ray [35] and, more recently, by Har-Peled and Jones [24].

- As the primary focus of this study is on the combinatorial aspects of weak $\epsilon$-nets, we did not seek to optimize the construction cost of our net $Q$.

A straightforward implementation of the recursive construction of $Q$ runs in time $\tilde{O}\left(\frac{n^2}{\sqrt{\epsilon}}\right)$. The construction of $\Sigma(R_1)$ from the sample $R_1 \subset \mathcal{L}(\Pi)$, the assignment of the points of $P$ to the trapezoidal cells, and the zones of the lines of $\mathcal{L}(\Pi)$, can all be performed using the standard textbook algorithms [34, 39]. Most of the running time is spent on explicitly maintaining the restriction graphs $\Pi$ along with the sparse graphs of Section 3.4 and tracing the zones of the lines of $\mathcal{L}(\Pi)$ in $\Sigma(r_1)$.

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