How fast Dominator can win in the Maker–Breaker domination game?

Jovana Forcan *

April 29, 2020

Abstract

Consider the Maker–Breaker domination game played by Dominator and Staller on the vertex set of a graph. The aim of Dominator is to build a dominating set of the graph, and the aim of Staller is to claim a vertex and all its neighbours. Gledel, Iršič, and Klavžar introduced the Maker–Breaker domination number $\gamma_{MB}(G)$ of a graph $G$, as the minimum number of moves of Dominator to win in the game on $G$ where he is the first player. If Dominator is the second player, then the corresponding invariant is denoted by $\gamma_{MB}'(G)$. In this paper we want to find a structural characterization of the graphs $G$ with domination number $\gamma(G) = k$, where $k \geq 2$ is a fixed integer, for which $\gamma_{MB}(G) = \gamma(G) = k$ holds, answering a related question of Gledel, Iršič, and Klavžar. Also, we investigate the invariants in the game on the Cartesian product of two graphs. Especially, we are interested in finding the minimum number of moves of Dominator in the game on $P_2 \Box P_n$ for $n \geq 1$.

1 Introduction

The domination game was introduced by Brešar, Klavžar, and Rall in [2]. The game is played by two players who alternately take a turn in claiming vertices from the finite graph $G$, which were not yet chosen in the course of the game. Dominator has a goal to dominate the graph in as few moves as possible while Staller tries to prolong the game as much as possible.

Maker–Breaker games are positional games introduces by Erdős and Selfridge in [4]. For given finite set $X$ and family $\mathcal{F} \subseteq 2^X$, in Maker–Breaker game $(X, \mathcal{F})$, two players, Maker and Breaker take turns in claiming previously unclaimed elements of $X$ until all of them are claimed. The set $X$ is called the board of the game and $\mathcal{F}$, family of winning sets. Maker wins the game if, by the end of the game, claims all elements of some $F \in \mathcal{F}$.

*Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia. Email: dmi.jovana.jankovic@student.pmf.uns.ac.rs.
Otherwise, Breaker wins. For a deeper and more comprehensive analysis of Maker-Breaker games see the book of Beck [1], and the recent monograph of Hefetz, Krivelevich, Stojaković and Szabó [6].

**Maker–Breaker domination game** (or **MBD game** for short) is studied for the first time in [3]. In the MBD game on $G$, Dominator wins if he claims all vertices from some **dominating set**, that is, a set $T$ such that every vertex not in $T$ has a neighbour in $T$. If Dominator can not form a domination set, Staller wins.

In [5], Gledel et al. considered the minimum number of moves of Dominator to win the game provided that he has a winning strategy. If Dominator is the first player, then the corresponding number is denoted by $\gamma_{MB}(G)$. If Staller is the first player, then the number is denoted by $\gamma'_{MB}(G)$. We set $\gamma_{MB}(G) = \infty$ and $\gamma'_ {MB}(G) = \infty$ if Dominator does not have a winning strategy as the first and as the second player, respectively. They proved that $\gamma_{MB}(G) = \gamma(G) = 2$ if and only if $G$ has a vertex that lies in at least two $\gamma$-sets of $G$, where $\gamma(G)$ is the **domination number** of $G$, that is the order of a smallest dominating set of $G$ and $\gamma$-set is a dominating set of size $\gamma(G)$. The authors proposed finding structural characterization of the graphs $G$ for which $\gamma_{MB}(G) = \gamma(G) = k$ holds, where $k \geq 2$ is fixed. In Section 2, we provide graph $G$ with the corresponding structural characterization and prove that for every graph $G$ with $\gamma(G) = k$ holds $\gamma_{MB}(G) = k$ for each $k \geq 2$, if and only if $G \supseteq G'$.

In the same paper [5], the authors proposed finding the minimum number of moves for Dominator in the MBD game on the Cartesian product of two graphs. Motivated by a given problem, we study the Cartesian product of two graphs and prove the following theorems in Section 3.

**Theorem 1.1.** Let $G$ and $H$ be two arbitrary graphs on $n$ and $m$ vertices, respectively.

1. Suppose that Maker has a winning strategy in MBD game on at least one of these two graphs as the first and as the second player. Then

$$\gamma_{MB}(G \square H) \leq \min\{\gamma_{MB}(G) + (m-1)\gamma'_ {MB}(G), \gamma_{MB}(H) + (n-1)\gamma'_ {MB}(H)\}$$

and

$$\gamma'_{MB}(G \square H) \leq \min\{m \cdot \gamma_{MB}(G), n \cdot \gamma'_{MB}(H)\}.$$  

2. Suppose that $\gamma_{MB}(G) < \infty$, and $\gamma'_{MB}(G), \gamma'_{MB}(H) = \gamma_{MB}(H) = \infty$. Let $H$ contains at least two disjoint $K_1$s. Then, Staller wins the game even as the second player.

**Theorem 1.2.** Let $G$ be a graph on $n$ vertices. Then Dominator can win the game on $G \square K_2$ for at most $n$ moves. If Dominator has a winning strategy as the first and as the second player in the game on $G$, then $\gamma_{MB}(G \square K_2) \leq \min\{\gamma_{MB}(G) + \gamma'_ {MB}(G), n\}$ and $\gamma'_{MB}(G \square K_2) \leq \min\{2\gamma'_{MB}(G), n\}$.

Especially, we are interested in determining the minimum number of Dominator’s moves in the MBD game on $P_2 \square P_n$. So, in Section 3, we also prove the following two theorems.

**Theorem 1.3.** $\gamma'_{MB}(P_2 \square P_n) = n$ for $n \geq 1$.

**Theorem 1.4.** $\gamma_{MB}(P_2 \square P_n) = n - 2$, for $n \geq 13$.  

2
1.1 Preliminaries

For given graph $G$ by $V(G)$ and $E(G)$ we denote its vertex set and edge set, respectively. The order of graph $G$ is denoted by $v(G) = |V(G)|$, and the size of the graph by $e(G) = |E(G)|$.

Assume that the MBD game is in progress. We denote by $d_1, d_2, \ldots$ the sequence of vertices chosen by Dominator and by $s_1, s_2, \ldots$ the sequence chosen by Staller. At any given moment during this game, we denote the set of vertices claimed by Dominator by $D$ and the set of vertices claimed by Staller by $S$. As in [5], we say that the game is a $D$-game if Dominator is the first to play, i.e. one round consists of a move by Dominator followed by a move of Staller. In the $S$-game, one round consists of a move by Staller followed by a move of Dominator. We say that the vertex $v$ is isolated by Staller if $v$ and all its neighbours are claimed by Staller.

2 Relation with the domination number

Let $G$ be graph with $\gamma(G) = k$, where $k \geq 2$ is an integer. Let $U = \{a, b_2, c_2, \ldots, b_k, c_k\} \subseteq V(G)$ be a set of all vertices, which appear in $\gamma$-sets. Divide set $U$ into following subsets:

- $\{a\}$
- $\{b_i, c_i\}$, for all $i \in \{2, \ldots, k\}$.

Suppose that all vertices from $V(G) \setminus U$ can be divided into $k$ pairwise disjoint sets $A_1, A_2, \ldots, A_k$ such that all vertices from some $A_i$ are adjacent to $\{b_i, c_i\}$, for $i = 2, \ldots, k$ and $N_U(A_i) \cap N_U(A_j) = \emptyset$, for all $i \neq j$.

Vertices from $A_1$ are the leaves of the star with the center in vertex $a \in U$ and these vertices do not have other neighbours in $U$.

If $b_i c_i \in E(G)$, then at least one of the next three cases must hold

- $b_i a \in E(G)$ and $c_i a \in E(G)$
- $b_i a \in E(G)$ and there exist $j \neq i$ such that $c_i b_j, c_j c_i$, or $c_i a \in E(G)$ and there exist $j \neq i$ such that $b_j b_i, b_i c_j$,
- there exist $j, k \neq i$ such that $b_j b_i, b_i c_j, c_i b_k, c_k c_i \in E(G)$ (note that $k$ and $j$ could be equal).

This is illustrated on Figure 1.

Lemma 2.1. The number of $\gamma$-sets in graph $G$ is $2^{k-1}$. In particular, the vertex $a$ lies in every $\gamma$-set, the vertex $b_i$ lies in exactly $2^{k-2}$ $\gamma$-sets which do not contain vertex $c_i$ and vertex $c_i$ lies in other $2^{k-2}$ $\gamma$-sets which do not contain vertex $b_i$.

Proof. Denote by $\mathcal{F}$ a family of all $\gamma$-sets of graph $G$ and let $N = |\mathcal{F}|$. In every $\gamma$-set from the family $\mathcal{F}$, for each vertex define positions in the corresponding $\gamma$-set. Since every $\gamma$-set is of order $k$, denote positions in sets by $1, 2, \ldots, k$ and place vertices $a, b_2, b_3, \ldots, b_k, c_2, c_3, \ldots, c_k$,
on the corresponding positions in \( \gamma \)-sets in the following way.
Since vertices from \( A_1 \) have only one neighbour from \( U \), a vertex \( a \), it follows that each set from \( \mathcal{F} \) must contain this vertex \( a \). Its position in each \( \gamma \)-set we denote by 1.

Also, since vertices from some set \( A_i \), \( i = 2, \ldots, k \) have two common neighbours from \( U \), \( b_i \) and \( c_i \), then \( b_i \) or \( c_i \) will be placed at the position \( i \), \( i = 2, 3, \ldots, k \). More precisely, the vertex \( b_i \) will appear in \( \frac{N}{2} \gamma \)-sets and \( c_i \) will appear in other \( \frac{N}{2} \gamma \)-sets which do not contain vertex \( b_i \).

It follows that for each position \( i \) in some \( \gamma \)-set there are two possibilities, \( b_i \) or \( c_i \), \( i = 2, 3, \ldots, k \). So, we obtain that the total number of \( \gamma \)-sets is \( N = 2^{k-1} \).

**Theorem 2.2.** Let \( G \) be a graph with \( \gamma(G) = k \), \( k \geq 2 \). Then \( \gamma_{MB}(G) = \gamma(G) = k \) for all \( k \geq 2 \) if and only if \( G \supseteq \mathcal{G} \).

**Proof of Theorem 2.2.** First, suppose that \( \mathcal{G} \subseteq G \) and prove that \( \gamma_{MB}(G) = k \). It is enough to prove that \( \gamma_{MB}(\mathcal{G}) = k \).

In his first move Dominator plays \( d_1 = a \). In every other round \( 2 \leq r \leq k \), Dominator plays in the following way. If Staller in her \((r-1)^{st}\) move plays \( s_{r-1} = b_i \) (or \( s_{r-1} = c_i \)) then Dominator responses with \( d_r = c_i \) (or \( d_r = b_i \)), for each \( i = 2, 3, \ldots, k \). So, \( \gamma_{MB}(\mathcal{G}) = k \).

Suppose, now, that \( \gamma_{MB}(G) = k \) and prove that \( G \supseteq \mathcal{G} \).

After Dominator’s first move \( d_1 \), it is Staller’s turn to make a move. If she claims \( s_1 \) such that \( d_1 \) and \( s_1 \) are part of some \( \gamma \)-set, then there exists at least one more vertex \( d_2 \) such that \( d_1 \) and \( d_2 \) are part of some other \( \gamma \)-set. Otherwise, this is a contradiction with statement that Dominator wins the game. So, this gives at least two \( \gamma \)-sets: \( \{d_1, d_2, \ldots\} \) and \( \{d_1, s_1, \ldots\} \).

Since Staller plays according to her optimal strategy, the vertex she claims in each round is the best choice for her. So, for her first move she had at least two best choices \( s_1 \) and \( d_2 \). Consider separately cases when Staller claims \( s_1 \) and when she claims \( d_2 \) in the first round.

---

Figure 1: Graph \( \mathcal{G} \).
Case 1. Suppose that Staller claimed $s_1$ in her first move and Dominator claimed $d_2$ in his second move. Then Staller in her second move can claim $s_2$ such that $d_1, d_2, s_2$ is a part of some \( \gamma \)-set. Then there exists at least one more vertex $d_3$ such that $d_1, d_2, d_3$ is a part of some other \( \gamma \)-set.

Case 2. Suppose that Staller claimed $d_2$ in her first move and Dominator claimed $s_1$ in his second move. Then Staller in her second move can claim some $s_\gamma$ such that $d_1, s_1, s_\gamma$ is a part of some $\gamma$-set. Then, there exists at least one more vertex, say $d_3$ such that $d_1, s_1, d_3$ is a part of some other $\gamma$-set. Dominator claims $d_3$.

After Dominator’s third move, above analyses gives at least $4 = 2^2 \ \gamma$-sets: \{ $d_1, d_2, d_3$ \}, \{ $d_1, d_2, s_2$ \}, \{ $d_1, s_1, d_3$ \} and \{ $d_1, s_1, s_2$ \}.

Suppose that after Dominator’s $i^{th}$ move we obtain that there are $2^{i-1}$ $\gamma$-sets. Assume that after Dominator’s move in round $i$, he owns vertices: $d_1, d_2, d_3, ..., d_i$.

If in round $i$, Staller claims some $s_i$ such that $d_1, d_2, ..., d_i, s_i$ is a part of some $\gamma$-set. Then according to the statement of theorem that Dominator wins it the game, there exists a vertex $d_{i+1}$, such that $d_1, d_2, ..., d_i, d_{i+1}$ is a part of some other $\gamma$-set. So, $s_i$ or $d_{i+1}$ is the vertex on the \((i + 1)^{st}\) position of previously found $2^{i-1}$ sets. So, this gives at least $2^{i-1}$ new sets, which is in total, at least, $2 \cdot 2^{i-1} = 2^i$ $\gamma$-sets.

Since, Dominator in each round $i$, $i = 2, 3, ..., k$ can find the corresponding vertex, as response to Staller’s $(i - 1)^{st}$ move, it follows that for each position in every $\gamma$-set there are at least two possible choices. This gives at least $2^{k-1}$ $\gamma$-sets. The vertex $d_i$ (or $s_{i-1}$), for every $i = 2, 3, ..., k$, appears in at least $2^{k-2}$ $\gamma$-sets which do not contain $s_{i-1}$ (or $d_i$). The vertex $d_1$ must appear in all $\gamma$-sets. Otherwise after some number of rounds Dominator will lose the game which would be a contradiction. Also, at least one of the next four cases must hold for each $i \in \{1, ..., k - 1\}$:

\[
\begin{align*}
s_i d_{i+1} & \in E(G), \\
d_i d_{i+1}, d_i s_i & \in E(G), \\
d_i d_{i+1} & \in E(G) \text{ and there exist } j \neq i \text{ such that } s_i s_j, s_i d_{j+1} \in E(G), \text{ or } d_i s_i \in E(G) \text{ and there exist } j \neq i \text{ such that } d_{i+1} s_j, d_{i+1} d_{j+1} \in E(G), \\
& \text{there exist } j, k \neq i \text{ such that } s_i s_j, s_i d_{j+1}, d_{i+1} s_k, d_{i+1} d_{k+1} \in E(G) \text{ (where } k \text{ and } j \text{ could be equal)}.
\end{align*}
\]

So, $G \supseteq G$.

3 MBD game on $G \Box H$

First, we consider domination game on prism and prove Theorem 1.2.
Proof of Theorem 1.2. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $H$ be a copy of graph $G$ and let $V(H) = \{v'_1, v'_2, \ldots, v'_n\}$, where $v'_i = v_i$ for each $i \in \{1, 2, \ldots, n\}$. Then $V(G \square K_2) = V(G) \cup V(H) = \{v_1, v_2, \ldots, v_n, v'_1, v'_2, \ldots, v'_n\}$ and $E(G \square K_2) = E(G) \cup E(H) \cup \{v_1v'_1, v_2v'_2, \ldots, v_nv'_n\}$.

In order to win, Dominator can always use pairing strategy. That is, when Staller claims $v_i$ (or $v'_i$) for some $i \in \{1, 2, \ldots, n\}$, Dominator responds by claiming vertex $v'_i$ (or $v_i$). So, Dominator wins in at most $n$ moves. To see that bound is tight consider $G$ as the disjoint union of $K_1$.

Suppose now that Dominator can win in the game on the graph $G$ as the first and as the second player. Assume that Dominator starts the game. Note that $\gamma_{MB}(G) = \gamma_{MB}(H)$ and $\gamma'_{MB}(G) = \gamma'_{MB}(H)$.

By $S_D$ and $S'_D$ denote Dominator’s winning strategy on $G$ (and also on $H$) in $D$-game and $S$-game, respectively.

Consider the game on $G \square K_2$.

If $\gamma_{MB}(G) + \gamma'_{MB}(G) \geq n$, Dominator will just use the pairing strategy. So, suppose that $\gamma_{MB}(G) + \gamma'_{MB}(G) < n$.

Dominator for his first move chooses a vertex from $V(G)$ according to his winning strategy $S_D$. In this way he starts $D$-game on $G$.

In every other round $r \geq 2$, Dominator looks on the $(r-1)^{st}$ move of Staller. If Staller claims a vertex from $V(G)$, Dominator responds by claiming a vertex from $V(G)$ and if Staller claims a vertex from $V(H)$, Dominator also claims a vertex from $V(H)$.

If Staller claimed a vertex from $V(H)$, before Dominator, then the $S$-game is being played on $H$.

So, in the game on $G \square K_2$, Dominator can win in at most $\gamma_{MB}(G) + \gamma'_{MB}(G)$ moves.

Assume, now, that Staller starts the game on $G \square K_2$. If $2\gamma'_{MB}(G) \geq n$, Dominator will just use the pairing strategy. So, let $2\gamma'_{MB}(G) < n$. Since, in this case, Staller can make the first move on $G$ and after, also, on $H$, Dominator will need to play according to the strategy $S'_D$ on both graphs $G$ and $H$. So, to win in the game on $G \square K_2$, he needs to play at most $2\gamma'_{MB}(G)$ moves.

\[\square\]

Observation 3.1. Consider Cartesian product of two complete graphs $K_r \square K_l$ for some $r, l \geq 2$. The corresponding graph is $r \times l$ rook’s graph which domination number is equal to $\gamma = \min(r, l)$. It is easy to see that Dominator can win in $\gamma$ moves. His strategy is to claim one vertex from each row or column. Let $r \leq l$. Consider $r \times l$ rook’s graph as the graph with $r$ rows and $l$ columns. The number of vertices in each row is $l$ and the number of vertices in each column is $r$. Denote rows by $k_1, k_2, \ldots, k_r$.

Suppose that Staller starts the game. In every round $1 \leq i \leq r$ Dominator claims a vertex from a different row. If Staller in round $i$ claims a vertex $x$ from some row $k_j$, then if Dominator’s set $D$ does not contain any vertex from that row, he will claim an unclaimed vertex, say $y \neq x$ from the row $k_j$. He is able to do that since row contains $l \geq 2$ vertices.

Otherwise, if Dominator already claimed a vertex from the row $k_j$, he will select unclaimed arbitrary vertex from some other row from which he still did not claim a vertex.
Proof of Theorem 1.1. The proof for the first part of theorem is similar to the proof of Theorem 1.2.

1. Consider, first, D-game on $G \square H$. Assume that $\gamma'_M(B)(G) < \infty$. Let $\gamma'_M(B)(G) + (m - 1)\gamma'_M(B)(G) \leq \gamma'_M(B)(H) + (n - 1)\gamma'_M(B)(H)$.

By $G^{(1)}, G^{(2)}, \ldots, G^{(m)}$ denote copies of graph $G$. By $S_D$ and $S'_D$ denote Dominator's winning strategy on $G$ in D-game and S-game, respectively.

His first move Dominator will play on one copy of $G$. In every other round $i \geq 2$, he looks on the $(i-1)^{st}$ move of Staller. If Staller in his $(i-1)^{st}$ move claimed vertex from some $V(G^i)$, Dominator responds by claiming a vertex from the same set $V(G^i)$ according to the corresponding winning strategy $S_D$ or $S'_D$. Since, Staller can be the first player on at most $m-1$ copies, the statement holds.

If $\gamma'_M(B)(G) + (m - 1)\gamma'_M(B)(G) > \gamma'_M(B)(H) + (n - 1)\gamma'_M(B)(H)$, then we consider $n$ copies of graph $H$ and the proof is the same.

The proof for the S-game is similar.

2. Consider two disjoint copies of $G$ for which hold $\gamma'_M(B)(G) < \infty$ and $\gamma'_M(B)(G) = \infty$. Denote these copies by $G^0$ and $G^1$. As the first player Dominator can choose to claim a vertex from one copy, say $G^0$. Since, $\gamma'_M(B)(G^1) = \infty$, Staller can make her first move on $G^1$ and play on $G^1$ until the end of the game.

3.1 MBD game on $P_2 \square P_n$

Definition 3.2. For $1 \leq m \leq n$, let $V = \{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ and $E = \{u_iu_{i+1} : i = 1, 2, \ldots, m - 1\} \cup \{v_iv_{i+1} : i = 1, 2, \ldots, m - 1\} \cup \{u_iv_i : i = 1, 2, \ldots, m\}$. Suppose that Maker–Breaker domination game on $P_2 \square P_n$ is in progress, where $n \geq 4$.

1. By $X_m$ ($1 \leq m \leq n$) denote a subgraph of $P_2 \square P_n$, where $V(X_m) = V$ and $E(X_m) = E$, such that $u_1$ is a free vertices which is dominated by Dominator with its neighbour $v \in V(P_2 \square P_n) \setminus V(X_m)$ (Figure 3(a)).

2. By $Y_m$ ($3 \leq m \leq n$) denote a subgraph of $P_2 \square P_n$, where $V(Y_m) = V$ and $E(Y_m) = E$, such that $v_2$ is claimed by Staller and $u_1, u_m$ and $v_m$ are free vertices which are dominated by Dominator with their corresponding neighbours from the set $V(P_2 \square P_n) \setminus V(Y_m)$ (Figure 3(b)).

When consider D-game on $Y_m$, we set $s_0 = v_2$

3. By $Z_m$ ($1 \leq m \leq n$) denote a subgraph of $P_2 \square P_n$, where $V(Z_m) = V$ and $E(Z_m) = E$, such that $u_1$ and $v_1$ are free vertices which are dominated by Dominator with their corresponding neighbours from the set $V(P_2 \square P_n) \setminus V(Z_m)$ (Figure 3(c)).
4. By $W_m$ ($1 \leq m \leq n$) denote a subgraph of $P_2 \square P_n$, where $V(W_m) = V \cup \{v_0\}$ and $E(W_m) = E \cup \{v_0v_1\}$, such that $u_1$ and $v_0$ are free vertices which are dominated by Dominator with their corresponding neighbours from the set $V(P_2 \square P_n) \setminus V(W_m)$ (Figure 2(d)).

5. By $\rho_m$ ($2 \leq m \leq n$) denote a subgraph of $P_2 \square P_n$, where $V(\rho_m) = V$ and $E(\rho_m) = E$, such that $v_2$ is claimed by Staller and $u_1$ is a free vertex which is dominated by Dominator with its neighbour from $V(P_2 \square P_n) \setminus V(\rho_m)$. (Figure 2(e)). When consider D-game on $\rho_m$, we set $s_0 = v_2$.

Figure 2: Subgraph (a) $X_m$ (b) $Y_m$ (c) $Z_m$ (d) $W_m$ (e) $\rho_m$
Vertices claimed by Dominator are denoted by cycles and vertices claimed by Staller by crosses. Triangle vertices are free vertices dominated by Dominator.

Let us define two types of traps Staller can create in the MBD game.

**Trap 1 - triangle trap.** We say that Staller created triangle trap if after her move Dominator is forced to claim a vertex $v_i$ in order to dominate $v_i$, because all its neighbours
\(v_{i-1}, v_{i+1}\) and \(u_i\) are claimed by Staller and Staller can isolate \(v_i\) by claiming it in her next move. Similarly, Staller created triangle trap if Dominator is forced to claim \(u_i\) in order to dominate \(u_i\) because all its neighbours \(u_{i-1}, u_{i+1}\) and \(v_i\) are claimed by Staller.

We say that Staller creates sequence of triangle traps \(v_i - v_{j-1}\) (or \(v_i - u_j\)) if Dominator is consecutively forced to claimed vertices \(v_i, u_{i+1}, v_{i+2}, u_{i+3}, ..., v_{j-1}\) (or \(v_i, u_{i+1}, v_{i+2}, u_{i+3}, ..., u_j\)). In this sequence of triangle traps, the first triangle created by Staller is \(v_{i-1}u_iv_{i+1}\) and the last triangle created by Staller is \(v_{j-2}u_{j-1}v_j\) (or \(u_{j-1}v_ju_{j+1}\)) and \(v_j\) (or \(u_{j+1}\)) is the vertex which is claimed last by Staller in the sequence of traps. The sequence of triangle traps \(v_3 - v_7\) is illustrated on Figure 3(a).

Similarly, we say that Staller creates sequence of triangle traps from \(u_i - v_{j-1}\) (or \(u_i - u_j\)) if Dominator is consecutively forced to claimed vertices \(u_i, v_{i+1}, u_{i+2}, v_{i+3}, ..., v_{j-1}\) (or \(u_i, v_{i+1}, u_{i+2}, v_{i+3}, ..., u_j\)). In this sequence of triangle traps, the first triangle created by Staller is \(u_{i-1}v_iu_{i+1}\) and the last triangle created by Staller is \(v_{j-2}u_{j-1}v_j\) (or \(u_{j-1}v_ju_{j+1}\)) and \(v_j\) (or \(u_{j+1}\)) is the vertex which claimed last by Staller in the sequence of triangle traps.

**Trap 2 - line trap.** We say that Staller created a line trap if after her move Dominator is forced to claim a vertex \(v_i\) in order to dominate \(u_i\) because vertices \(u_{i-1}, u_i\) and \(u_{i+1}\) are claimed by Staller and Staller can isolate \(u_i\) by claiming \(v_i\) in her next move. Similarly, Staller created a line trap if Dominator is forced to claim \(u_i\) in order to dominate \(v_i\), because vertices \(v_{i-1}, v_i\) and \(v_{i+1}\) are claimed by Staller.

We say that Staller creates sequence of line traps from \(v_i - v_j\) (or \(u_i - u_j\)) if Dominator is consecutively forced to claimed vertices \(v_i, v_{i+1}, v_{i+2}, v_{i+3}, ..., v_j\) (or \(u_i, u_{i+1}, u_{i+2}, u_{i+3}, ..., u_j\)) and the last claimed vertex by Staller in this sequence is \(u_{j+1}\) (or \(v_{j+1}\)). The sequence of line traps \(u_3 - u_9\) is illustrated on Figure 3(b).

**Lemma 3.3.** Let \(m \geq 2\). Then \(\gamma_{MB}(\rho_m) = m\). Also, if Dominator skips to play his move in any round, he can not win.

**Proof.** Let \(s_0 = v_2\). For \(m = 2\), \(\rho_m\) is a cycle \(C_4\). To dominate \(v_1\) and \(u_2\), Dominator needs to play two moves. So, \(\gamma_{MB}(\rho_2) = 2\). If Dominator skips his first move on \(\rho_2\), which we denote with \(d_1 = \emptyset\), then \(s_1 = v_1\) and in her next round Staller can isolate \(v_1\) or \(v_2\).
Case 1. Consider the game on $\rho_m$. We analyse the following cases to prove that $\gamma_{MB}(\rho_3) = 3$. It is not hard to see that if Dominator skips any move on $\rho_3$, Staller can isolate some vertex.

1. $d_1 = u_3$ (or $d_1 = v_1$)
   Then $s_1$ must be equal to $v_1$ (or $s_1 = u_3$), as otherwise Dominator will need exactly one more move to win. To dominate $v_1$ and $v_2$ (or $u_2$, $u_3$ and $v_3$), Dominator needs two more moves.

2. $d_1 = v_3$ (or $d_1 = u_1$).
   Then $s_1$ must be equal to $u_1$ (or $s_1 = v_3$), as otherwise Dominator will need exactly one more move to win.
   To dominate $v_1$ and $u_2$ (or $u_3$, $v_2$ and $v_3$), Dominator needs two more moves.

3. $d_1 = u_2$.
   Dominator needs two more moves to dominate $v_1$ and $v_3$.

So, $\gamma_{MB}(\rho_3) = 3$.
Consider $D$-game on $\rho_m$ where $m \geq 4$.
First, by induction on $k$, we prove that $\gamma_{MB}(\rho_m) \leq m$.
Suppose that for $4 \leq k \leq m - 1$ statement holds, that is $\gamma_{MB}(\rho_k) \leq k$. Consider $D$-game on $\rho_{k+1}$. Dominator divides $\rho_{k+1}$ on two parts, a graph $\rho_k$ and an edge $u_{k+1}v_{k+1}$. By induction hypothesis, $\gamma_{MB}(\rho_k) \leq k$. Also, when Staller claims $u_{k+1}$ (or $v_{k+1}$), Dominator claims $v_{k+1}$ (or $u_{k+1}$). So, it follows that $\rho_{k+1} \leq k + 1$.

Next, we prove that Staller has a strategy to postpone Dominator’s winning by at least $m - 2$ moves and which ensures that Dominator can not skip any move.
Assume that $\gamma_{MB}(\rho_{k-1}) \geq k - 1$ and Dominator can not skip any move in the game on $\rho_{k-1}$ for $5 \leq k \leq m$. Consider the game on $\rho_k$ and prove that $\gamma_{MB}(\rho_k) \geq k$ and Dominator is not able to skip any move on $\rho_k$.
If $d_1 = \emptyset$, we propose the following strategy for Staller: $s_1 = v_1$ which forces $d_2 = u_1$, as otherwise Staller can isolate $v_1$ in her next move. Now, Staller starts the sequence of line traps $u_2 - u_{m-1}$. In her last move Staller claims $u_m$ and isolates $v_m$. Now, we consider all possibilities from $d_1$ and propose Staller’s strategy.

Case 1. $d_1 = u_i$, ($i \neq 1$).
Then $s_1 = v_1$ which forces $d_2 = u_1$, as otherwise Staller can isolate $v_1$ by claiming $u_1$ in her third move.
If $i = 2$, that is, if $d_1 = u_2$, then $s_2 = v_4$. Consider $D$-game on subgraph $\rho_{k-2}$ on $V(\rho_{k-2}) = \{u_3, \ldots, u_k, v_3, \ldots, v_k\}$, where $v_4 \in S$ and $u_3$ is a free vertex dominated by Dominator with $u_2$.
By induction hypothesis $\gamma_{MB}(\rho_{k-2}) \geq k - 2$ and Dominator can not skip any move.
So, Dominator needs at least $k$ moves to win on $\rho_k$ without skipping any move.

If $i > 2$, then $s_2 = v_3$ and Staller starts sequence of line traps $u_2 - u_{i-1}$.
In round $i$, Staller claims $s_i = v_{i+2}$. Consider $D$-game on subgraph $\rho_{k-i}$ on
\[ V(\rho_{k-1}) = \{u_{i+1}, \ldots, u_k, v_{i+1}, \ldots, v_k\}, \text{ where } v_{i+2} \in \mathcal{S} \text{ and } u_{i+1} \text{ is a free vertex dominated by Dominator with } u_i. \] By induction hypothesis, \( \gamma_{MB}(\rho_{k-1}) \geq k - i \) and Dominator can not skip any move. So, Dominator needs at least \( k \) moves to win on \( \rho_k \) without skipping any move.

If \( i = k \), that is, if \( d_1 = u_k \), then Dominator already played \( k \) moves, since he was forced to claim all from \( \{u_1, \ldots, u_k\} \).

If \( i = k - 1 \), that is, if \( d_1 = u_{k-1} \), then \( s_i = s_{k-1} \in \{u_k, v_k\} \). So, Dominator needs to play one more move to dominate \( v_k \). So, in total, he plays \( k \) moves.

Case 2. \( d_1 = v_i, i \geq 3 \).

**Claim 3.4.** If \( d_1 \notin \{v_3, v_4\} \), then Dominator can not win.

*Proof of Claim 3.4.* Suppose that \( d_1 \notin \{v_3, v_4\} \).

Then \( s_1 = u_2 \).

If \( d_2 = u_3 \), Staller claims \( s_2 = v_1 \) and forces \( d_3 = u_1 \) or \( d_3 = v_3 \). Since, Dominator can not dominate vertices \( v_1, u_1 \) and \( v_2 \) at the same time, in her next move Staller will isolate \( v_1 \) and \( u_1 \) by claiming \( u_1 \) or \( v_2 \) by claiming \( v_3 \).

If \( d_2 = v_3 \), Staller claims \( s_2 = u_1 \). Since Dominator can not dominate \( u_1, v_1 \) and \( u_2 \) at the same time, he will lose the game after Staller next move.

If \( d_2 = u_1 \), then \( s_2 = v_3 \) which forces \( d_3 = v_1 \). Next, \( s_3 = u_3 \). Dominator can not dominate both \( u_3, v_3 \) in his next move, since \( v_4 \notin \mathcal{S} \). In her next move, Staller isolates \( u_3 \) or \( v_3 \).

If \( d_2 = v_1 \), then \( s_2 = u_3 \) which forces \( d_3 = u_1 \). Next, \( s_3 = v_3 \). Dominator can not dominate both \( u_3 \) and \( v_3 \).

If \( d_2 \notin \{u_1, v_1, u_3, v_3\} \), then Staller claims \( s_2 = u_1 \). Since Dominator can not dominate \( u_1, v_1, u_2 \) and \( v_2 \) at the same time, he will lose the game after Staller next move. \( \square \)

So, \( d_1 \in \{v_3, v_4\} \).

Case 2.1 \( d_1 = v_3 \).

Then \( s_1 = u_1 \) which forces \( d_2 = v_1 \), and \( s_2 = u_3 \) which forces \( d_3 = u_2 \) (triangle trap). Next, \( s_3 = u_5 \). Consider \( D \)-game on subgraph \( \rho_{k-3} \). By induction hypothesis, \( \gamma_{MB}(\rho_{k-3}) \geq k - 3 \) and he can not skip any move. So, Dominator needs at least \( k \) on \( \rho_k \) moves without skipping any move.

Case 2.2 \( d_1 = v_4 \).

Then \( s_1 = u_2 \).

**Claim 3.5.** If \( d_2 \notin \{u_1, v_1\} \), then Dominator can not win.

*Proof of Claim 3.5.* Assume that \( d_2 \notin \{u_1, v_1\} \). Then, if \( d_2 = u_3 \), Staller claims \( s_2 = v_1 \). Since Dominator can not dominate \( u_1, v_1 \) and \( v_2 \) at the same time, he will lose the game after Staller next move.
If $d_2 = v_3$, Staller claims $s_2 = u_1$. Since Dominator can not dominate $u_1$, $v_1$ and $u_2$ at the same time, he will lose the game after Staller next move. If $d_2 \not\in \{u_1, v_1, u_3, v_3\}$, then Staller claims $s_2 = u_1$. Since Dominator can not dominate $u_1, v_1, u_2$ and $v_2$ at the same time, he will lose the game after Staller next move.

**Case 2.2.1** $d_2 = u_1$.

Then, $s_2 = v_3$ which forces $d_3 = v_1$ and $s_3 = u_4$ which forces $u_3$. Next, if $k \geq 6$, then $s_4 = u_6$. Consider $D$-game on subgraph $\rho_{k-4}$ on $V(\rho_{k-4}) = \{u_5, \ldots, u_k, v_5, \ldots, v_k\}$ where $v_5$ is already dominated by Dominator with $v_4$ and $u_6$ is claimed by Staller. By induction hypothesis, it holds $\gamma_{MB}(\rho_{k-4}) \geq k - 4$ and he can not skip any move, so Dominator needs at least $k$ moves without skipping any move.

If $k = 5$, then no matter what Staller claims in her fourth move, Dominator will need one more move to dominate $u_5$.

**Case 2.2.2** $d_2 = v_1$.

Then, $s_2 = u_3$ which forces $d_3 = u_1$ and $s_3 = u_4$ which forces $v_3$. Next, $s_4 = u_6$ and the rest of the proof is the same as in Case 2.2.1.

**Case 3.** $d_1 = u_1$.

Then $s_1 = v_3$.

**Case 3.1.** $d_2 = u_i$, $i > 2$.

Then, $s_2 = u_2$ which forces $d_3 = v_1$.

Next, $s_3 = v_4$ and Staller starts sequence of line traps $u_3 - u_{i-1}$. In round $i$, Staller claims $v_{i+2}$. Consider $D$-game on $\rho_{k-i}$ on $V(\rho_{k-i}) = \{u_{i+1}, \ldots, u_k, v_{i+1}, \ldots, v_k\}$ where $u_{i+1}$ is already dominated by Dominator with $u_i$ and $v_{i+2}$ is claimed by Staller. So, by induction hypothesis $\rho_{k-i} \geq k - i$ and he can not skip any move.

If $d_2 = u_k$, then Dominator already played $k$ moves.

If $d_2 = u_{k-1}$, then $s_i = s_{k-1} \in \{u_k, v_k\}$. So, Dominator needs to play one more move to dominate $v_k$.

So, in total Dominator needs at least $k$ moves on $\rho_k$ without skipping any move.

**Case 3.2.** $d_2 = v_i$, where $i > 3$.

Then, $s_2 = u_3$ which forces $d_3 = v_1$.

If $i > 4$, that is, if $d_2 = v_i \neq v_4$, then $s_3 = u_3$. Dominator can not dominate both $u_3$ and $v_3$ at the same time. In her next move, Staller isolates $u_3$ or $v_3$ and Dominator loses the game.

If $i = 4$, then, $s_3 = u_4$ which forces $d_4 = u_3$. Next, if $k \geq 6$, then $s_4 = u_6$. Consider $D$-game on $\rho_{k-4}$. By induction hypothesis $\gamma_{MB}(\rho_{k-4}) \geq k - 4$, and Dominator can not skip any move. So, Dominator needs at least $k$ moves on $\rho_k$ without skipping any move.

If $k = 5$, then no matter what Staller claims in her fourth move, Dominator will need one more move to dominate $u_5$. 

12
Case 3.3. $d_2 \in \{v_1, u_2\}$.

Then, in round 3 $\leq r \leq k - 2$, Staller claims $s_r = v_{r+2}$ and forces Dominator to claim $d_{r+1} = u_{r+1}$, as otherwise Staller can isolate $v_{r+1}$ by claiming $u_{r+1}$ in the next round, that is Staller creates sequence of line traps. In the last round $k - 1$, Staller claims $u_k$ and in this way she isolates $v_k$.

Case 4. $d_1 = v_1$.

Then, Staller claims $s_1 = u_3$.

**Claim 3.6.** If $d_2 \notin \{v_3, u_4, v_4\}$, then Dominator can not win.

*Proof of Claim 3.6.* Assume that $d_2 \notin \{v_3, u_4, v_4\}$.

Let $d_2 = u_1$ or $d_2 = u_2$.

Then, in round 2, by playing $s_2 = v_4$, Staller starts sequence of triangle traps $v_3 - v_{k-1}$ (for even $k$) or $v_3 - u_{k-1}$ (for odd $k$). In the last move, if $k$ is even Staller claims $u_k$ and isolates it. If $k$ is odd, she claims $v_k$ and isolates it.

Let $d_2 \notin \{u_1, u_2, v_3, u_4, v_4\}$.

Then, we have the following sequences of the moves: $s_2 = u_2 \Rightarrow d_3 = u_1$ and $s_3 = v_3$. Dominator can not dominate both $u_3$ and $v_3$ at the same time.

From Claim 3.6 it follows that $d_2 \in \{v_3, u_4, v_4\}$. We have $d_1 = v_1$, and $s_0 = v_2$ and $s_1 = u_3$. Next, we consider the following cases.

Case 4.1. $d_2 = v_3$.

Then, Staller claims $s_2 = u_1$ which forces Dominator to claim $d_3 = u_2$. In the next round Staller claims $u_5$. Consider $D$-game on subgraph $\rho_{k-3}$ on $\gamma_{MB}(\rho_{k-3}) \geq k - 3$ and he can not skip any move. So, Dominator needs at least $k$ moves on $\rho_k$ without skipping any move.

Case 4.2. $d_2 = u_4$.

Then, Staller claims $s_2 = u_1$ which forces Dominator to claim $d_3 = u_2$. In the next round Staller claims $s_3 = v_4$ and forces Dominator to play $d_4 = v_3$. If $k \geq 6$, then $s_4 = v_6$. Consider $D$-game on $\rho_{k-4}$ on $\gamma_{MB}(\rho_{k-4}) \geq k - 4$ and he can not skip any move. So, Dominator needs at least $k$ moves on $\rho_k$ without skipping any move.

If $k = 5$, then no matter what Staller claims in her fourth move, Dominator will need one more move to dominate $v_5$.

Case 4.3. $d_2 = v_4$.

Then, Staller claims $s_2 = u_2$ which forces Dominator to claim $d_3 = u_1$. In the
next round Staller claims \( s_3 = u_4 \) and forces Dominator to play \( d_4 = v_3 \). Next, \( s_4 = u_6 \) and the rest of the proof is similar to the proof from Case 4.2. So, Dominator needs at least \( k \) moves on \( \rho_k \) without skipping any move.

This concludes the proof of the lemma.

\[ \square \]

**Remark 3.7.** Note that \( D \)-game on graph \( \rho_m \) can be considered as \( S \)-game on \( X_m \) where \( s_1 = v_2 \). This means, that \( v_2 \) is one of the optimal choices for the first move for Staller in \( S \)-game on \( X_m \), since by playing \( v_2 \) in her first move and then following her strategy for \( \rho_m \), Staller can force Dominator to play the maximum number of moves, which is \( m \).

**Lemma 3.8.** Let \( m \geq 3 \). Then \( \gamma_{MB}(Y_m) = m - 1 \).

**Proof.** Let \( s_0 = v_2 \).

For \( m = 3 \) it is not hard to see that Dominator needs 2 moves to dominate \( v_1, v_2 \) and \( u_2 \).

By using mathematical induction on \( k \) where \( 4 \leq k \leq m \) and case analysis from Lemma 3.5 (when proving \( \gamma_{MB}(\rho_m) \geq m \)), it can be proven that \( \gamma_{MB}(Y_m) \geq m - 1 \), for \( m \geq 4 \).

The proof for the upper bound can be obtain by induction in the similar way as in Lemma 3.5.

\[ \square \]

**Lemma 3.9.** Let \( m \geq 1 \). Then \( \gamma'_{MB}(Z_m) = m - 1 \).

**Proof.** For \( m \in \{1, 2, 3\} \) it is not hard to see that Dominator needs \( m - 1 \) moves in \( S \)-game on \( Z_m \). Let \( m \geq 4 \). To prove the upper bound we use the same idea as in Lemma 3.3.

To prove the lower bound we propose the following strategy for Staller:

\( s_1 = u_m \), which forces \( d_1 \in \{u_{m-1}, v_{m-1}, v_m\} \). Otherwise in her second move Staller can choose \( v_m \) and in the third move she can isolate \( u_m \) or \( v_m \) by claiming \( u_{m-1} \) or \( v_{m-1} \), since Dominator will not be able to dominate both of \( u_m, v_m \) in his second move.

If \( d_1 = u_{m-1} \), Staller plays \( s_2 = v_{m-1} \) which forces \( d_2 = v_3 \). Then, \( s_3 = v_{m-3} \). In this way Staller creates \( Y_{m-2} \) on \( V(Y_{m-2}) = \{u_1, u_2, ..., u_{m-2}, v_1, v_2, ..., v_{m-2}\} \). From Lemma 3.8 we know that \( \gamma_{MB}(Y_{m-2}) = m - 3 \), so Dominator needs to play \( m - 1 \) moves on \( Z_m \).

If \( d_1 = v_{m-1} \), Staller plays \( s_2 = u_{m-1} \) which forces \( d_2 = v_3 \). Then, \( s_3 = u_{m-3} \). In this way Staller creates \( Y_{m-2} \). According to Lemma 3.8 \( \gamma_{MB}(Y_{m-2}) = m - 3 \), so Dominator needs to play \( m - 1 \) moves on \( Z_m \).

If \( d_1 = v_m \), Staller plays \( s_2 = u_{m-2} \) and creates \( Y_{m-1} \). According to Lemma 3.8 \( \gamma_{MB}(Y_{m-1}) = m - 2 \), so Dominator needs to play \( m - 1 \) moves on \( Z_m \).

It follows that, \( \gamma_{MB}(Z_m) = m - 1 \) for \( m \geq 4 \).

\[ \square \]

**Lemma 3.10.** Let \( m \geq 4 \). Then \( \gamma'_{MB}(W_m) = m - 1 \). In particularly, if \( m \in \{1, 2, 3\} \), then \( \gamma'_{MB}(W_m) = m \).

**Proof.** For \( m \in \{1, 2, 3\} \), it is not hard to see that Dominator needs \( m \) moves to win in \( S \)-game on \( W_m \).

Since \( W_m \) is more harder for Dominator to dominate than \( Z_m \), it follows that \( \gamma_{MB}(W_m) \geq m \).\[ \square \]
\(\gamma_{MB}(Z_m)\), so the lower bound holds.
The proof for the upper bound follows by induction. For \(n = 4\), we consider the following cases and propose Dominator’s strategy.

Case 1. \(s_1 = v_2\).
   Then, \(d_1 = u_3\).
   If \(s_2 = v_1\), then \(d_2 = v_3\) and Dominator needs one more move to dominate \(v_1\).
   Otherwise, if \(s_2 \neq v_1\), then \(d_2 = v_1\) and Dominator needs one more move to dominate \(v_4\).

Case 2. \(s_1 \neq v_2\).

Case 2.1 \(s_1 = u_4\) (or \(s_1 = v_4\)).
   Then, \(d_1 = u_3\). If \(s_2 = v_3\), then \(d_2 = v_4\) (or \(u_4\)) and \(d_3 \in \{v_1, v_2\}\). If \(s_2 = v_4\) (or \(s_1 = u_4\)), then \(d_2 = v_3\) and \(d_3 \in \{v_1, v_2\}\).

Case 2.2 \(s_1 \notin \{u_4, v_4\}\).
   Then, \(d_1 = v_2\). So, Dominator needs two more moves to dominate the remaining vertices.

Suppose that \(\gamma'_{MB}(W_{k-1}) \leq k - 2\) for \(5 \leq k \leq m - 1\). Consider \(S\)-game on \(W_k\). Dominator divides \(W_k\) on two parts \(W_{k-1}\) and an edge \(u_kv_k\). Since, \(\gamma'_{MB}(W_{k-1}) \leq k - 2\) and since he needs at most one more move to dominate \(u_k\) and \(v_k\), it follows that \(W_k \leq k - 1\).

**Lemma 3.11.** Let \(m \geq 6\). Then \(\gamma_{MB}(X_m) = m - 2\). In particularly, if \(m = 1\) then \(X_1 = 1\) and if \(m \in \{2, 3, 4, 5\}\), then \(X_m = m - 1\).

**Proof.** For \(m \in \{1, 2, 3\}\), it is not hard to see that statement holds. For \(m = 4\) and \(m = 5\) simple case analysis gives the result.

Let \(m = 6\). If in his first move Dominator plays \(d_1 = v_2\), in this way he creates subgraph \(W_4\) on \(V(W_4) = \{u_2, u_3, u_4, u_5, u_6, v_3, v_4, v_5, v_6\}\). By Lemma [3.10] we have \(\gamma'_{MB}(W_4) = 3\). So, \(\gamma_{MB}(X_6) = 4\).

Let \(m \geq 7\). To prove upper bound, suppose that \(\gamma_{MB}(X_{k-1}) \leq k - 3\) for \(8 \leq k \leq m - 1\). Consider \(D\)-game on \(X_k\). Dominator divides \(X_k\) on two parts \(X_{k-1}\) and an edge \(u_kv_k\). Since, \(\gamma_{MB}(X_{k-1}) \leq k - 3\) and since he needs at most one more move to dominate \(u_k\) and \(v_k\), it follows that \(\gamma_{MB}(X_k) \leq k - 2\).

The proof for the lower bound follows by induction on \(k\) and case analysis. Suppose that \(\gamma_{MB}(X_{k-1}) \geq k - 3\), for \(8 \leq k \leq m - 1\) and \(m \geq 7\). Consider \(D\)-game on \(X_k\) and prove that \(\gamma_{MB}(X_k) \geq k - 2\).

We analyse the following cases.

Case I \(d_1 \in \{u_1, v_1, u_2, v_2\}\).
   If \(d_1 = u_1\) (or \(d_1 = v_1\)).
   Then, consider the \(S\)-game on \(W_{k-1}\) where \(V(W_{k-1}) = \{v_1, v_2, \ldots, v_k, u_2, \ldots, u_k\}\) (or
\[ V(W_{k-1}) = \{v_2, \ldots, v_k, u_1, u_2, \ldots, u_k\} \]. By Lemma 3.10, \( \gamma'_{MB}(W_{k-1}) = k - 2 \). So, Dominator needs to play \( k - 1 \) on \( X_k \).

If \( d_2 = u_2 \) or \( v_2 \), then consider \( S \)-game on \( W_{k-2} \) where \( \{v_2, v_3, \ldots, v_k, u_3, \ldots, u_k\} \) or \( \{v_3, \ldots, v_k, u_2, u_3, \ldots, u_k\} \). By Lemma 3.10, \( \gamma'_{MB}(W_{k-2}) = k - 3 \). Also, if \( d_1 = u_2 \), Dominator needs to play one more move to dominate \( v_1 \). So, Dominator needs to play at least \( k - 2 \) moves on \( X_k \).

**Case II** \( d_1 = u_i, i \geq 3 \).

Then, \( s_1 = v_2 \).

The rest of Staller’s strategy depends on Dominator’s second move. So we analyse the following subcases.

**Case 1.** \( d_2 = u_1 \)

**Case 2.** \( d_2 = v_1 \)

**Case 3.** \( d_2 = u_j, j \geq 3 \).

**Case 4.** \( d_2 = v_j, i < j \).

**Case 5.** \( d_2 = v_j, i = j \geq 3 \).

**Case 6.** \( d_2 = u_j, i > j \geq 2 \) and \( j \) is even.

**Case 7.** \( d_2 = u_j, i > j \geq 2 \) and \( j \) is odd.

We analyse each subcase separately.

**Case 1.** If \( i = 3 \), that is \( d_1 = u_3 \), then consider \( S \)-game on \( W_{k-3} \) on \( \{u_4, u_5, v_4, v_5, \ldots, v_k\} \). By Lemma 3.10, \( \gamma'_{MB}(W_{k-3}) = k - 4 \), so Dominator needs at least \( k - 2 \) moves.

Let \( i \geq 4 \), then \( s_2 = v_3 \). Depending of Dominator’s third move, we consider the following subcases.

**Case 1.1.** \( d_3 = u_2 \) or \( d_3 = v_1 \).

Then by playing \( s_3 = v_1 \) Staller starts the sequence of line traps \( u_3 - u_{i-1} \) where \( s_{i-1} = v_i \) and \( d_i = u_{i-1} \). Then, if \( k - i \geq 2 \), Staller plays \( s_i = v_{i+2} \). Consider \( D \)-game on subgraph \( \rho_{k-i} \). According to Lemma 3.3, \( \gamma_{MB}(\rho_{k-i}) = k - i \), so Dominator needs \( k \) moves. If \( k - i = 1 \), then \( s_i = v_k \) which forces \( d_{i+1} = u_k \) and so, Dominator needs \( k \) moves. If \( k - i = 0 \), then Dominator already played \( k \) moves.

**Case 1.2.** \( d_3 = u_j, j \geq 3 \) or \( d_3 = v_j, j \geq 4 \).

**Claim 3.12.** If \( \min\{i, j\} \notin \{3, 4\} \), then Dominator can not win.

**Proof of Claim 3.12.** Assume \( \min\{i, j\} \notin \{3, 4\} \). Then \( s_3 = u_2 \) which forces \( d_4 = v_1 \). Next, \( s_4 = u_3 \). Dominator can not dominate both \( u_3 \) and \( v_3 \) at the same time. \( \square \)

**Case 1.2.1.** \( d_3 = u_j \), where \( j \geq 3 \) and \( j < i \). According to Claim 3.12, \( j \in \{3, 4\} \).
a. $j = 3$, that is $d_3 = u_3$. Then, $s_3 = u_2$ which forces $d_4 = v_1$. Consider subgraph $X_{k-3}$ on $V(X_{k-3}) = \{u_4, \ldots, u_k, v_4, \ldots, v_k\}$ where $u_4$ is already dominated with $u_3$ by Dominator. Also, $d_1 = u_i \in X_{k-3}$ and now it is Staller’s turn to make her move on $X_{k-3}$. By induction hypothesis, if $k - 3 \geq 6$, then $\gamma_{MB}(X_{k-3}) \geq k - 5$, so Dominator needs at least $k - 2$ moves. If $4 \leq k - 3 \leq 5$, then, since $\gamma_{MB}(X_{k-3}) = k - 4$, Dominator needs $k - 1$ moves.

b. $j = 4$, that is $d_3 = u_4$. Then, $s_3 = u_2$ which forces $d_4 = v_1$ and $s_4 = v_4$ which forces $d_5 = u_3$ (a line trap). Consider a subgraph $X_{k-4}$ on $V(X_{k-4}) = \{u_5, \ldots, u_k, v_5, \ldots, v_k\}$ where $u_5$ is already dominated with $u_4$ by Dominator. Also, $d_1 = u_i \in X_{k-4}$ and now it is Staller’s turn to make her move on $X_{k-4}$. By induction hypothesis, if $k - 4 \geq 6$, then $\gamma_{MB}(X_{k-4}) \geq k - 6$, so Dominator needs at least $k - 2$ moves. If $3 \leq k - 4 \leq 5$, then since $\gamma_{MB}(X_{k-4}) = k - 4$, Dominator needs $k - 1$ moves.

**Case 1.2.2.** $d_3 = u_j$, where $j \geq 3$ and $j > i$. According to Claim 3.12, $i \in \{3, 4\}$, that is $d_1 = u_3$ or $d_1 = u_4$. Staller’s strategy is the same as in Case 1.2.1.

**Case 1.2.3.** $d_3 = v_j$, $j < i$. According to Claim 3.12, and since $s_2 = v_3$, it follows that $j = 4$, that is $d_3 = v_4$.

Then, $s_3 = u_2$, which forces $d_4 = v_1$ and $s_4 = u_4$ which forces $d_5 = u_3$ (a triangle trap). Consider subgraph $X_{k-4}$. Note that $d_1 = u_i \in X_{k-4}$. The rest of the proof is the same as in Case 1.2.1.b.

**Case 1.2.4.** $d_3 = v_j$, $j = i$. According to Claim 3.12, $d_1 = u_4$ and $d_3 = v_4$.

Then, $s_3 = u_2$, which forces $d_4 = v_1$. Consider $S$-game on $Z_{k-4}$ on $V(Z_{k-4}) = \{u_5, \ldots, u_k, v_5, \ldots, v_k\}$ where $u_5$ and $v_5$ are dominated with $u_4$ and $v_4$. According to Lemma 3.9, $\gamma_{MB}(Z_{k-4}) = k - 5$, so Dominator needs $k - 1$ moves.

**Case 1.2.5.** $d_3 = v_j$, $j > i$. According to Claim 3.12, $i \in \{3, 4\}$ and the proof of this case is similar to the proof of Case 1.2.1.

**Case 2.** If, $i = 3$, that is, $d_1 = u_3$ then consider the $S$-game on subgraph $W_{k-3}$ on $V(W_{k-3}) = \{u_4, \ldots, u_k, v_3, v_4, \ldots, v_k\}$ where $v_3$ and $u_4$ are dominated with $u_3$. According to Lemma 3.10, $\gamma_{MB}(W_{k-3}) = k - 4$, so Dominator needs $k - 2$ moves.

Let $i \geq 4$. Then, $s_2 = u_3$.

Depending of Dominator’s third move, we consider the following cases.

**Case 2.1.** $d_3 = u_1$ or $d_3 = u_2$.

Let $i$ be an even number. Then, $s_4 = v_4$ and Staller starts the sequence of triangle traps $v_3 - v_{i-1}$ where $s_{i-1}$ and $d_i = v_{i-1}$. Next, if $k - i \geq 2$, $s_i = v_{i+2}$ and we have a subgraph $\rho_{k-i}$ on $V(\rho_{k-i}) = \{u_{i+1}, \ldots, u_k, v_{i+1}, \ldots, v_k\}$ where $u_{i+1}$ is dominated with $u_i$. Consider $D$-game on $\rho_{k-i}$. According to Lemma 3.3, $\gamma_{MB}(\rho_{k-i}) = k - i$, so Dominator needs $k$ moves. If $k - i = 1$, then $s_i = v_k$, which forces $d_{i+1} = u_k$, so Dominator again needs $k$ moves. If, $k - i = 0$, then Dominator already played $k$
moves.

Let $i$ be an odd number. Then, $s_4 = v_4$ and Staller starts the sequence of triangle traps $v_3 - v_{i-2}$, where $s_{i-2} = v_{i-1}$ and $d_{i-1} = v_{i-2}$. Consider graph $W_{k-i}$ on $V(W_{k-i}) = \{u_{i+1}, \ldots, u_k, v_i, v_{i+1}, \ldots, v_k\}$ where $v_i$ and $u_{i+1}$ is dominated with $u_i$. Consider $S$-game on $W_{k-i}$. This means, $s_{i-1} \in V(W_{k-i})$. According to Lemma 3.10, if $k - i \geq 4$, $\gamma_{MB}(W_{k-i}) = k - i - 1$, so Dominator needs $k - 2$ moves. If $1 \leq k - i \leq 3$, then $\gamma_{MB}(W_{k-i}) = k - i$, so Dominator needs $k - 1$ moves. If $u_i = u_k$, then Dominator already played $k - 1$ moves.

**Case 2.2.** $d_3 = u_j$, $j \geq 4$ or $d_3 = v_j$, $j \geq 3$.

It is not hard to check that Claim 3.12 can be also applied on this case. So, min\{i, j\} $\in \{3, 4\}$.

**Case 2.2.1.** $d_3 = u_j$, $j < i$. According to Claim 3.12 and since $s_2 = u_3$, it follows that $j = 4$, that is $d_3 = u_4$.

Then, $s_3 = u_1$ which forces $d_4 = u_2$ and $s_4 = v_4$ which forces $d_5 = v_3$ (triangle trap). Consider $X_{m-4}$ where $d_1 = u_i \in X_{k-4}$ and now it is Staller’s turn to make her move on $X_{k-4}$. Dominator needs at least $k - 2$ moves.

**Case 2.2.2.** $d_3 = u_j$, $j > i$. According to Claim 3.12 $i = 4$, that is $d_1 = u_4$.

The proof is the same as the proof for Case 2.2.1.

**Case 2.2.3.** $d_3 = v_j$, $j > i$. According to Claim 3.12 $i = 4$, that is $d_1 = u_4$.

The proof is the same as the proof for Case 2.2.1.

**Case 2.2.4.** $d_3 = v_j$, $j = i$. According to Claim 3.12 $i = j = 4$, that is $d_1 = u_4$ and $d_3 = v_4$.

Then, $s_3 = u_1$ which forces $d_4 = u_2$ (triangle trap). We get a subgraph $Z_{k-4}$ and the rest of the prof is the same as for Case 1.2.4. So, Dominator needs at least $k - 2$ moves.

**Case 2.2.5.** $d_3 = v_j$, $j < i$. According to Claim 3.12 $j \in \{3, 4\}$.

Let $j = 3$, that is $d_3 = v_3$.

Then, $s_3 = u_1$ which forces $d_4 = u_2$. Consider $X_{k-3}$ and the rest of the proof is the same as for Case 1.2.1a. So, Dominator needs at least $k - 2$ moves.

Let $j = 4$, that is, $d_3 = v_4$.

Then, $s_3 = u_2$, which forces $d_4 = u_1$ and $s_4 = u_4$ which forces $d_5 = v_3$ (line trap). We get a subgraph $X_{k-4}$ and the rest of the prof is the same as for Case 1.2.3.

**Case 3.** $s_2 = v_1$.

In his third move Dominator is forced to claims $u_1$, as otherwise Staller can isolate $v_1$ by claiming $u_1$ in her next move. So, $d_3 = u_1$.

Let $l = \min\{i, j\}$ and let $h = \max\{i, j\}$ . Then, $s_3 = v_3$ and in this way Staller starts her line trap $u_2 - u_{l-1}$, where $s_l = v_l$ and $d_{l+1} = u_{l-1}$. Consider subgraph $X_{k-l}$ on $V(X_{k-l}) = \{u_{i+1}, \ldots, u_k, v_{i+1}, \ldots, v_k\}$ where $u_{i+1}$ is a free vertex already dominated by
Dominator with $u_l$. Also, $u_h \in X_{k-l}$ and it is already claimed by Dominator (in his first or the second move), and now it is Staller’s turn to make a move on $X_{k-l}$. By induction hypothesis, if $k - l \geq 6$, then $\gamma_{MB}(X_{k-l}) \geq k - l - 2$, so Dominator needs at least $k - 2$ on $X_k$.

If $2 \leq k - l \leq 5$, then, since $\gamma_{MB}(X_{k-l}) \geq k - l - 1$, it follows that Dominator needs $k - 1$ moves. Also, if $k - j = 1$, Dominator needs $k$ moves.

Case 4. Then, $s_2 = v_1$.

In his third move Dominator is forced to claim $u_1$, so $d_3 = u_1$. Then, $s_3 = v_3$ and Staller starts her line trap $u_2 - u_{i-1}$ where the $s_i = v_i$ and $d_{i+1} = u_{i-1}$. Consider $X_{k-i}$ on $V(X_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$. Dominator needs at least $k - 2$ moves.

Case 5. $s_2 = v_1$.

Dominator is forced to play $d_3 = u_1$. Then, $s_3 = v_3$ and Staller starts her line trap $u_2 - u_{i-2}$ where the $s_{i-1} = v_{i-1}$ and $d_{i} = u_{i-2}$. Since, $u_i, v_i \in O$, we have graph $Z_{k-i}$ on $V(Z_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\}$. Next, $s_i \in V(Z_{k-i})$, so we consider $S$-game on $Z_{k-i}$. By Lemma 3.9, $\gamma'_{MB}(Z_{k-i}) = k - i - 1$. This means that Dominator needs to play at least $k - 1$ moves on $X_k$.

Case 6. $s_2 = u_2$.

We claim the following.

Claim 3.13. If $d_3 \notin \{u_1, v_1\}$, Dominator can not win.

Proof of Claim 3.13. Let $d_3 \notin \{u_1, v_1\}$. After Dominator’s third move at least one of the vertices $u_3, v_3$ needs to be free.

Suppose that $v_3$ is a free vertex. Then, $s_3 = v_1$, so Dominator is not able to dominate $u_1, v_1$ and $v_2$ at the same time. In her next move Staller can isolate $u_1$ and $v_1$, or $v_2$ by claiming $u_1$ or $v_3$.

If $u_3$ is free vertex, then $s_3 = u_1$ and Dominator is not able to dominate both $u_1, v_1$ and $u_2$ at the same time. In her next move Staller can isolate $u_1$ and $v_1$ or $u_2$ by claiming $v_1$ or $u_3$.

Case 6.1. $d_3 = u_1$.

Then, $s_3 = v_3$ which forces $d_4 = v_1$. By playing $s_4 = u_4$, Staller starts the sequence of triangle traps $u_3 - u_{j-1}$ where $s_j = u_j$. After Dominator’s move in round $j + 1$, $d_{j+1} = u_{j-1}$, we have graph $X_{k-j}$ on $V(X_{k-j}) = \{u_{j+1}, ..., u_k, v_{j+1}, ..., v_k\}$ where $v_{j+1}$ is dominated by Dominator with $v_j$. Also, $u_i \in X_{k-j}$ and it is already claimed by Dominator in his first move and now it is Staller’s turn to move on $X_{k-j}$. By induction hypothesis, if $k - j \geq 6$, then $\gamma_{MB}(X_{k-j}) \geq k - j - 2$, so Dominator needs at least $k - 2$ moves.

If $2 \leq k - j \leq 5$, then, since $\gamma_{MB}(X_{k-j}) \geq k - j - 1$, it follows that Dominator needs $k - 1$ moves. Also, if $k - j = 1$, Dominator needs $k$ moves.
Case 6.2. \( d_3 = v_1 \).
Then \( s_3 = u_3 \) which forces \( d_4 = u_1 \). By playing \( s_4 = u_4 \), Staller starts the sequence of line traps \( v_3 - v_{j-1} \) where \( s_j = u_j \). After Dominator’s move in round \( j+1 \), where \( d_{j+1} = v_{j-1} \), we have graph \( X_{k-j} \). The rest of the proof is the same as in Case 6.1. So, Dominator needs at least \( k-2 \) moves. Also, if \( k-j = 1 \), Dominator needs \( k \) moves.

Case 7. \( s_2 = u_1 \).
Staller’s second move \( s_2 = u_1 \) forces \( d_3 = v_1 \). By claiming \( u_3 \) Staller starts sequence of triangle traps \( u_2 - u_{j-1} \) where \( s_j = u_j \). After Dominator’s move in round \( j+1 \), we have graph \( X_{k-j} \). The vertex \( u_i \in X_{k-j} \) is already claimed by Dominator as \( d_1 \) and now it is Staller’s turn to make her move. By using induction hypothesis, we obtain that Dominator needs at least \( k-2 \) moves.

Case III. \( d_1 = v_i \), \( i \geq 3 \).
Then, \( s_1 = v_2 \). The rest of Staller’s strategy depends on Dominator’s second move. So we analyse the following subcases.

Case i. \( d_2 = u_1 \)
Case ii. \( d_2 = v_1 \)
Case iii. \( d_2 = u_j \), \( i < j \) and \( i \) is even
Case iv. \( d_2 = u_j \), \( i < j \) and \( i \) is odd
Case v. \( d_2 = u_j \), \( i = j \)
Case vi. \( d_2 = u_j \), \( i > j \)
Case vii. \( d_2 = v_j \), \( \min\{i, j\} \) is odd
Case viii. \( d_2 = v_j \), \( \min\{i, j\} \) is even

We analyse each subcase separately.

Case i. If \( i = 3 \), that is \( d_1 = v_3 \), then consider \( S \)-game on subgraph \( W_{k-3} \) on \( V(W_{k-3}) = \{u_3, u_4, \ldots, u_k, v_4, \ldots, v_k\} \) where \( u_3 \) and \( v_4 \) are dominated with \( v_3 \). Since, \( \gamma'_MB(W_{k-3}) = k - 4 \), Dominator needs to play at least \( k-2 \) moves on \( X_k \).

Let \( i \geq 4 \). Then \( s_2 = v_3 \). Depending of Dominator’s third move, we consider the following cases.

Case i.1. \( d_3 = v_1 \).
Then, \( s_3 = v_4 \) and Staller starts the sequence of line traps \( u_3 - u_{i-1} \). Consider graph \( W_{k-i} \) on \( V(W_{k-i}) = \{u_i, u_{i+1}, \ldots, u_k, v_{i+1}, \ldots, v_k\} \) where \( u_i \) and \( v_{i+1} \) is dominated with
Next, \( s_{i-1} \in V(W_{k-i}) \). According to Lemma \[3.10\] if \( k - i \geq 4 \), \( \gamma_{MB}(W_{k-i}) = k - i - 1 \), so Dominator needs \( k - 2 \) moves. If \( 1 \leq k - i \leq 3 \), then \( \gamma_{MB}(W_{k-i}) = k - i \), so Dominator needs \( k - 1 \) moves. If \( d_1 = v_i = v_k \), then Dominator already played \( k - 1 \) moves.

Case i.2. \( d_3 = v_j, j \geq 4 \) or \( d_3 = u_j, j \geq 3 \).

It is not hard to see that Claim \[3.12\] also holds in this case.

Case i.2.1. \( d_3 = v_j, j \geq 4 \). Let \( t = \min\{i, j\} \).

According to Claim \[3.12\] \( l = 4 \).

Then, \( s_3 = u_2 \) which forces \( d_4 = v_1 \) and \( s_4 = u_4 \) which forces \( d_5 = u_3 \). Consider subgraph \( X_{k-4} \). It follows that Dominator needs at least \( k - 2 \) moves.

Case i.2.2. \( d_3 = u_j, j \geq i \).

According to Claim \[3.12\] \( i = 4 \).

Then, the Staller’s strategy is the same as in Case i.2.1.

Case i.2.3. \( d_3 = u_j, i = j \).

According to Claim \[3.12\] \( i = j = 4 \).

Consider subgraph \( Z_{k-4} \). It follows that Dominator needs at least \( k - 2 \) moves.

Case i.2.4. \( d_3 = u_j, j < i \).

According to Claim \[3.12\] \( j \in \{3, 4\} \).

a. Let \( j = 3 \), that is \( d_3 = u_3 \). Then, \( s_3 = v_1 \) which forces \( d_4 = u_2 \). Consider \( X_{k-3} \). It follows that Dominator needs at least \( k - 2 \) moves.

b. Let \( j = 4 \), that is \( d_3 = u_4 \).

Then, \( s_3 = u_2 \) which forces \( d_4 = v_1 \) and \( s_4 = u_4 \) which forces \( d_5 = u_3 \). Consider subgraph \( X_{k-4} \). It follows that Dominator needs at least \( m - 2 \) moves.

Case ii. \( s_2 = u_3 \).

Depending of Dominator’s third move, we consider the following cases.

Case ii.1. \( d_3 = u_1 \) or \( d_3 = u_2 \).

a. \( i \) is even.

Then, \( s_3 = u_4 \) and Staller starts the sequence of triangle traps \( v_3 - u_{i-2} \), where \( s_{i-2} = u_{i-1} \) and \( d_{i-1} = u_{i-2} \). Consider the \( S \)-game on subgraph \( W_{k-i} \) on \( V(W_{k-i}) = \{u_i, ..., u_k, v_{i+1}, ..., v_k\} \), where \( u_i \) and \( v_{i+1} \) are dominated with \( v_i \). According to Lemma \[3.10\] if \( k - i \geq 4 \), \( \gamma_{MB}(W_{k-i}) = k - i - 1 \), so Dominator needs \( k - 2 \) moves. If \( 1 \leq k - i \leq 5 \), then since \( \gamma_{MB}(W_{k-i}) = k - i \), Dominator needs \( k - 1 \) moves.
b. \( i \) is odd.
Then, \( s_3 = v_4 \) and Staller starts the sequence of triangle traps \( v_3 - u_{i-1} \), where \( s_{i-1} = u_i \) and \( d_i = u_{i-1} \). Next, if \( k - i \geq 2 \), \( s_i = u_{i+2} \). Consider the subgraph \( \rho_{k-i} \) on \( V(\rho_{k-i}) = \{u_{i+1}, ..., u_k, v_{i+1}, ..., v_k\} \) where \( v_{i+1} \) is dominated with \( v_i \). According to Lemma 3.3 \( \gamma_{MB}(\rho_{k-i}) = k - i \), so Dominator needs \( k \) moves.
If \( k - i = 1 \), then \( s_i = u_k \) which forces \( d_{i+1} = v_k \), so Dominator again needs \( k \) moves. If, \( k - i = 0 \), then Dominator already played \( k \) moves.

Case ii.2. \( d_3 = u_j, j \geq 4 \) or \( d_3 = v_j, j \geq 3 \).
It is not hard to check that Claim 3.12 also holds for this case.

Case ii.2.1. \( d_3 = u_j, j < i \). According to Claim 3.12, \( j = 4 \), that is, \( d_3 = u_4 \).
Staller’s strategy is the same as in Case 2.2.1.

Case ii.2.2. \( d_3 = u_j, j = i \). According to Claim 3.12, \( i = j = 4 \), that is \( d_1 = v_4, d_3 = u_4 \).
Then, \( s_3 = u_2 \) which forces \( d_4 = u_1 \). Consider subgraph \( Z_{k-4} \) and the rest of the proof is the same as for Case 1.2.4.

Case ii.2.3. \( d_3 = u_j, j > i \). According to Claim 3.12, \( i \in \{3, 4\} \).

a. Let \( i = 3 \), that is \( d_1 = v_3 \). Then, \( s_3 = u_2 \) which forces \( d_4 = u_1 \). We get a subgraph \( X_{k-3} \) on \( V(X_{k-3}) = \{u_4, ..., u_k, v_4, ..., v_k\} \) where \( v_4 \) is dominated with \( v_3 \). The rest of the proof is same as in case 1.2.1.a.

b. Let \( i = 4 \), that is \( d_1 = v_4 \). Then, \( s_3 = u_2 \) which forces \( d_4 = u_1 \) and \( s_4 = u_4 \) which forces \( d_5 = v_3 \). We get a subgraph \( X_{m-4} \) on \( V(X_{k-3}) = \{u_5, ..., u_k, v_5, ..., v_k\} \) where \( v_5 \) is dominated with \( v_4 \). The rest of the proof is same as in case 1.2.3.

Case ii.2.3. \( d_3 = v_j, j < i \). According to Claim 3.12, \( j = 4 \), that is \( d_3 = v_4 \).
Staller’s strategy is the same as in Case ii.2.2.b.

Case ii.2.4. \( d_3 = v_j, j > i \). According to Claim 3.12, \( i \in \{3, 4\} \).

a. Let \( i = 3 \), that is \( d_1 = v_3 \). The proof is the same as in Case ii.2.3.a.

b. Let \( i = 4 \), that is \( d_1 = v_4 \). The proof is the same as in Case ii.2.3.b.

Case iii. \( s_2 = u_2 \).
Staller’s strategy from round 3 is the same as in Case 6.

Case iv. \( s_2 = u_1 \).
Staller’s strategy from round 3 is the same as in Case 7.

Case v. \( s_2 = v_1 \).
Staller’s strategy from round 3 is the same as in Case 5.
Case vi. $s_2 = v_1$.

Staller’s strategy from round 3 is the same as in Case 4.

Case vii. $s_2 = u_1$.

Staller’s strategy from round 3 is the same as in Case 7.

Case viii. $s_2 = u_2$.

Staller’s strategy from round 3 is the same as in Case 6.

From this case analysis it follows that $\gamma_{MB}(X_k) \geq k - 2$ for $\geq 14k \leq m$. \hfill $\square$

**Proof of Theorem 1.3** Let $V(P_2 \square P_n) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and let $E(P_2 \square P_n) = \{u_iu_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_iv_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{u_iv_i : i = 1, 2, \ldots, n\}$.

To prove that $\gamma_{MB}'(P_2 \square P_n) \leq n$ we provide Dominator with the pairing strategy. That is, when Staller claims $u_i$ (or $v_i$) for some $i \in \{1, 2, \ldots, n\}$, Dominator responds by claiming $v_i$ (or $u_i$). In this way, Dominator can win for $n$ moves in $S$-game.

Next, we prove that Staller has a strategy to postpone Dominator’s winning for at least $n$ moves.

For her first move, Staller claims vertex $v_2$, that is, $s_1 = v_2$. Since it is harder to dominate graph $P_2 \square P_n$ in $S$-game where $s_1 = v_2$ than graph $\rho_n$ in $D$-game, and since $\gamma_{MB}(\rho_n) = n$, according to Lemma 3.3 it follows that $\gamma_{MB}(P_2 \square P_n) \geq n$. \hfill $\square$

To prove Theorem 1.4 we need the following lemma which proof we give in Appendix.

**Lemma 3.14.** $\gamma_{MB}(P_2 \square P_{13}) \leq 11$.

**Proof of Theorem 1.4** Let $V(P_2 \square P_n) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and let $E(P_2 \square P_n) = \{u_iu_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_iv_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{u_iv_i : i = 1, 2, \ldots, n\}$.

First, prove that $\gamma_{MB}(P_2 \square P_n) \leq n - 2$. For $n = 13$, statement holds, according to Lemma 3.14.

Suppose that $\gamma_{MB}(P_2 \square P_{k-1}) \leq k - 3$ for $14 \leq k \leq n$ and prove that $\gamma_{MB}(P_2 \square P_k) \leq k - 2$.

Dominator’s strategy is as follows. Dominator divides graph $P_2 \square P_k$ on two parts, a graph $P_2 \square P_{k-1}$ and an edge $u_kv_k$. His first move Dominator plays on $P_2 \square P_{k-1}$ according to his winning strategy for this graph. When Staller plays on $P_2 \square P_{k-1}$, Dominator responds by claiming corresponding vertex from $P_2 \square P_{k-1}$. When Staller claims $u_k$ (or $v_k$), Dominator claims $v_k$ (or $u_k$). So, $\gamma_{MB}(P_2 \square P_k) \leq k - 2$.

To prove the lower bound, we use Lemma 3.11. Since, it is harder for Dominator to dominate $P_2 \square P_n$ than $X_n$ in $D$-game, it follows that $\gamma_{MB}(P_2 \square P_n) \geq n - 2$. \hfill $\square$

**Corollary 3.15.** Let $3 \leq m \leq n$. Then

(i) If $m$ is even, $\gamma_{MB}(P_m \square P_n) \leq \gamma_{MB}(P_2 \square P_n) + (\frac{m}{2} - 1) \gamma_{MB}'(P_2 \square P_n)$.

(ii) If $m$ and $n$ are odd, $\gamma_{MB}(P_m \square P_n) \leq \gamma_{MB}(P_n) + \left\lfloor \frac{m}{2} \right\rfloor \gamma_{MB}'(P_2 \square P_n)$. 

23
(iii) If $m$ is odd and $n$ is even, $\gamma_{MB}(P_m \Box P_n) \leq \gamma_{MB}(P_2 \Box P_m) + \left( \frac{n}{2} - 1 \right) \gamma_{MB}^*(P_2 \Box P_m)$.

Sketch of the proof. Consider $D$-game on the grid $P_m \Box P_n$.

(i) Divide graph $P_m \Box P_n$ on $\frac{m}{2}$ grids $P_2 \Box P_n$. On one grid $P_2 \Box P_n$, Dominator is the first player. On the other $\frac{m}{2} - 1$ grids $P_2 \Box P_n$, Staller can be the first player. Applying the Theorem 1.4 and 1.3, we obtain the upper bound on $\gamma_{MB}(P_m \Box P_n)$.

(ii) Divide graph $P_m \Box P_n$ on $\lfloor \frac{m}{2} \rfloor$ grids $P_2 \Box P_n$ and one path $P_n$. Dominator will start the game on the path.

The proof for the case (iii) is similar to the proof of the case (i).

References

[1] J. Beck, Combinatorial Games: Tic-Tac-Toe Theory, Encyclopedia of Mathematics and Its Applications 114, Cambridge University Press, (2008).

[2] B. Brešar, S. Klavžar and D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math., 24 (2010), pp. 979–991.

[3] E. Duchêne, V. Gledel, A. Parreau and G. Renault, Maker-Breaker domination game, arXiv:1807.09479 [cs.DM] (25 Jul 2018).

[4] P. Erdős and J. L. Selfridge, On a combinatorial game, J. Combinatorial Theory Ser. A 14 (1973) pp. 298–301.

[5] V. Gledel, V. Iršič and S. Klavžar, Maker–Breaker domination number, Bulletin of the Malaysian Mathematical Sciences Society, 42(4) (2019), pp. 1773–1789.

[6] D. Hefetz, M. Krivelevich, M. Stojaković and T. Szabó, Positional Games, Oberwolfach Seminars 44, Birkhäuser/Springer Basel, 2014.

A The proof of Lemma 3.14

First, we give two claims.

Claim A.1. Consider $S$-game on $W_4$, where $V(W_4) = \{v_0, v_1, ..., v_4, u_1, ..., u_4\}$ and $E(W_4) = \{u_iu_{i+1} : i = 1, 2, 3\} \cup \{v_iv_{i+1} : i = 1, 2, 3\} \cup \{u_iv_i : i = 1, 2, 3\} \cup \{v_0v_1\}$, and suppose that Dominator skips the first move. If $s_1 \notin \{u_3, v_3, u_4, v_4\}$, then Dominator can win in at most 4 moves.

Claim A.2. Consider $S$-game on $W_6$, where $V(W_6) = \{v_0, v_1, ..., v_6, u_1, ..., u_6\}$ and $E(W_6) = \{u_iu_{i+1} : i = 1, 2, ..., 6\} \cup \{v_iv_{i+1} : i = 1, 2, ..., 6\} \cup \{u_iv_i : i = 1, 2, ..., 6\} \cup \{v_0v_1\}$, and suppose that Dominator skips the first move. Let $s_1 = v_2$. Then Dominator can win in at most 6 moves.
The proofs for these two claims can be obtained by case analysis, so we skip it.

Suppose that game on $P_2 \Box P_{13}$ is in progress. If in some point of the game we obtain a subgraph $W_4$ with the situation described in Claim A.1 we denote this subgraph by $W_4'$. If we get a subgraph $W_6$ with the situation described in Claim A.2 we denote this subgraph by $W_6'$.

**Proof of Lemma 3.14.** Let $V(P_2 \Box P_{13}) = \{u_1, u_2, ..., u_{13}, v_1, v_2, ..., v_{13}\}$ and let $E(P_2 \Box P_{13}) = \{u_i u_{i+1} : i = 1, 2, ..., 12\} \cup \{v_i v_{i+1} : i = 1, 2, ..., 12\} \cup \{u_i v_i : i = 1, 2, ..., 13\}$. Let $L$ be a subgraph of $P_2 \Box P_{13}$ induced by the set $\{u_1, ..., u_6, v_1, ..., v_6\}$ and let $R$ be a subgraph of $P_2 \Box P_{13}$ induced by the set $\{u_8, ..., u_{13}, v_8, ..., v_{13}\}$.

For his first move Dominator plays $v_7$. Now, we consider the following cases.

Case 1. $s_1 = u_7$.

Case 2. $s_1 = u_5$.

Case 3. $s_1 \in \{u_3, v_3, u_4, v_4, u_6, v_6\}$.

Case 4. $s_1 \in \{u_2, v_2\}$.

Case 5. $s_1 \in \{u_1, v_1\}$.

The case when Staller for her first move claims a vertex from the $R$ is symmetric to the case when Staller claims a vertex from the set $L$ (cases 2-5). So, it is enough to analyse each of above cases separately.

Case 1. In his second move, Dominator plays $d_2 = u_9$.

Consider subgraph $W_4 \subset R$ where $V(W_4) = \{v_9, v_{10}, ..., v_{13}, u_{10}, ..., u_{13}\}$. Whenever Staller plays on $W_4$ (or $L$), Dominator responds on $W_4$ (or $L$). According to Lemma 3.10 $\gamma'_{MB}(W_4) = 3$. On $L$ he uses pairing strategy, where pairing sets are $\{u_i, v_i\}$, for each $i \in \{1, ..., 6\}$. So, Dominator needs at most 11 moves.

Case 2. In his second move, Dominator plays $d_2 = u_9$.

Consider $W_4 \subset R$ where $V(W_4) = \{v_9, v_{10}, ..., v_{13}, u_{10}, ..., u_{13}\}$ and consider $W_6'$, where $V(W_6') = \{u_7, u_6, ..., u_1, v_6, ..., v_1\}$ (note $u_5 \in \mathcal{G}$ and Dominator skipped to play his first move on $W_6'$).

Whenever Staller plays on $W_4$ (or $W_6'$), Dominator responds on $W_4$ (or $W_6'$). According to Lemma 3.10 $\gamma'_{MB}(W_4) = 3$. By claim A.2 Dominator needs at most 6 moves to play on $W_6'$. In total, Dominator needs at most 11 moves.

Case 3. In his second move, Dominator plays $d_2 = u_5$.

If $s_1 \in \{u_6, v_6\}$, then we have $W_4 \subset L$ on $V(W_4) = \{v_5, v_4, ..., v_1, u_4, ..., u_1\}$ and according to Lemma 3.10 $\gamma'_{MB}(W_4) = 3$. Otherwise, if $s_1 \notin \{u_6, v_6\}$, we have $W_4' \subset L$ on $V(W_4') = \{v_5, v_4, ..., v_1, u_4, ..., u_1\}$ and according to Claim A.1 $\gamma'_{MB}(W_4') \leq 4$.

Also, consider $S$-game on $W_6$ where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. By Lemma
3.10 $\gamma'_MB(W_6) = 5$. When Staller plays on $W_4$ or $W'_4$, Dominator responds on $W_4$ or $W'_4$, and when Staller plays on $W_6$, Dominator responds on $W_6$. So, Dominator needs at most 11 moves.

Case 4. In his second move, Dominator plays $d_2 = u_3$.
Consider $S$-game on $W_6$ where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. Whenever Staller plays on $W_6$ (or $L$), Dominator responds on $W_6$ (or $L$). By Lemma 3.10 $\gamma'_MB(W_6) = 5$.
On the $L$ Dominator will use pairing strategy where the pairing sets are $\{u_1, v_1\}, \{v_4, v_5\}, \{u_5, u_6\}$. Also, to dominate $v_2$ Dominator will need at most 1 more move. He will claim a free vertex from the set $\{u_2, v_2, v_3\}$. So, Dominator needs at most 11 moves.

Case 5. Dominator claims $d_2 = v_2$.
We need to consider the following subcases.

Case 5.1 $s_2 \in \{u_6, u_7, v_5, v_6, v_7\}$. Then, $d_3 = u_5$.
Whenever Staller plays on $L$ (or $R$), Dominator responds on $L$ (or $R$). To dominate all in $L$, Dominator needs at most 2 more moves (in one move he claims a vertex from $\{u_1, u_2, v_1\}$, and in other move he claims a vertex from $\{v_3, u_4\}$). On $R$ he uses pairing strategy where pairing sets are $\{u_i, v_i\}, i \in \{8, ..., 13\}$. So, Dominator needs at most 11 moves.

Case 5.2 $s_2 = u_5$. Then, $d_3 = u_4$.
Consider $W_6$ where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. Whenever Staller plays on $W_6$ (or $L$), Dominator responds on $W_6$ (or $L$).
On $L$, he needs at most 3 more moves and on $W_6$ he needs at most 5 moves. So, in total, Dominator needs at most 11 moves.

Case 5.3 $s_2 \in \{u_3, u_4, v_3, v_4\}$. Then, $d_3 = u_5$. Consider $W_6$ where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. Whenever Staller plays on $W_6$ (or $L$), Dominator responds on $W_6$ (or $L$).
On $L$, he needs at most 3 more moves and on $W_6$ he needs 5 moves. So, in total, Dominator needs at most 11 moves.

Case 5.4 $s_2 = u_2$. Then, one of the vertices $u_1$ or $v_1$ is free, and Dominator claims that free vertex as his third move. The rest of Dominator’s strategy depends on Staller’s third move.

a. $s_3 \in (V(L) \setminus \{u_3\}) \cup \{u_7\}$. Then, $d_4 = u_5$.
Consider $W_6$ where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. Whenever Staller plays on $W_6$ (or $L$), Dominator responds on $W_6$ (or $L$).
On $L$, he needs at most 2 more moves and on $W_6$ he needs 5 moves. So, in total, Dominator needs at most 11 moves.
b. $s_3 = u_5$. Then, $d_4 = u_4$. Consider $W_6$ where $V(W_6) = \{u_7, ..., u_{13}, v_8, ..., v_{13}\}$. Whenever Staller plays on $W_6$ (or $L$), Dominator responds on $W_6$ (or $L$). On $L$, he needs at most 2 more moves and on $W_6$ he needs at most 5 moves. So, in total, Dominator needs at most 11 moves.

c. $s_3 \in R \setminus \{u_9\}$. Then, $d_4 = u_9$. If $s_3 \in \{u_8, v_8\}$, consider $W_4 \subset R$ on $\{v_9, ..., v_{13}, u_{10}, u_{13}\}$ and if $s_3 \notin \{u_8, v_8\}$ consider $W_3 \subset R$ on $\{v_9, ..., v_{13}, u_{10}, u_{13}\}$. Also, on the left part of the graph $P_2 \Box P_{13}$, consider $V(W_4) = \{u_7, ..., u_3, v_6, v_3\}$. In every round Dominator responds by claiming the corresponding vertex on the subgraph on which Staller made her move. According to Claim A.1, Dominator needs at most 4 moves on the corresponding vertex on the subgraph on which Staller made her move. According to Claim A.1, Dominator needs at most 4 moves on $W_4'$ and from Lemma 3.10, Dominator needs at most 11 moves.

d. $s_3 = u_9$. Then, $d_4 = u_5$. Consider $W_6'$ on $V(W_6') = \{u_7, u_8, ..., u_{13}, v_8, v_{13}\}$. Whenever Staller plays on $W_6'$ (or $L$), Dominator responds on $W_6'$ (or $L$). On $L$, he needs to play 1 more move. On $W_6'$, he needs to play at most 6 moves, according to Claim A.2. So, Dominator needs at most 11 moves.

e. $s_3 \in \{u_{12}, v_{12}\}$. Then, $d_4 = u_{11}$. Whenever Staller plays on $R$ (or $L$), Dominator responds on $R$ (or $L$). It is not hard to see that on $R$ Dominator needs at most 4 moves, and on $L$ he needs at most 3 moves. So, Dominator needs at most 11 moves.

f. $s_3 \in \{u_{13}, v_{13}\}$. Then, $d_4 = v_{12}$. Dominator will use pairing strategy where the pairing sets are $\{u_3, u_4\}$, $\{v_4, v_5\}$, $\{u_5, u_6\}$, $\{u_8, u_9\}$, $\{v_9, v_{10}\}$, $\{u_{10}, u_{11}\}$. And to dominate $u_{13}$, Dominator will claim a free vertex from the set $\{u_{12}, u_{13}, v_{13}\}$.

Case 5.5 $s_2 \in \{u_1, v_1\}$. Then, $d_3 = u_2$. Consider $W_6$ on $V(W_6) = \{u_7, u_8, ..., u_{13}, v_8, ..., v_{13}\}$. Whenever Staller plays on $W_6$ (or $L$), Dominator responds on $W_6$ (or $L$). On $L$ Dominator needs at most 3 more moves, on $W_6$ he needs at most 5 moves. So, Dominator needs at most 11 moves.

Case 5.6 $s_2 \in \{u_8, u_{10}, u_{11}, v_8, v_{10}, v_{11}\}$. Then, $d_3 = u_9$. If $s_2 \in \{u_8, v_8\}$, consider $W_4'$ and if $s_2 \notin \{u_8, v_8\}$, consider $W_4$ on $\{v_9, v_{10}, ..., v_{13}, u_{10}, u_{13}\}$. In every round Dominator responds by claiming the corresponding vertex from the subgraph on which Staller made her move. On $L$, he needs to play at most 4 more moves. On $W_4'$ or $W_4$, he needs to play at most 4 or 3 moves, according to Claim A.1 or Lemma 3.10. So, Dominator needs at most 11 moves.
Case 5.7 $s_2 = u_9$. Then, $d_3 = u_5$.

Consider $W_6'$ on $V(W_6') = \{u_7, u_8,\ldots, u_{13}, v_8, v_{13}\}$. Whenever Staller plays on $W_6'$ (or $L$), Dominator responds on $W_6'$ (or $L$). On $L$ Dominator needs at most 2 more moves, on $W_6'$ he needs at most 6 moves, according to Claim A.2. So, Dominator needs at most 11 moves.

Case 5.8 $s_2 = \{u_{12}, v_{12}\}$. Then, $d_3 = u_{11}$.

Whenever Staller plays on $L$ (or $R$), Dominator responds on $L$ (or $R$). It is not hard to see that on $L$ Dominator needs 4 more moves and on $R$ he needs to play 4 more moves.

Case 5.9 $s_2 \in \{u_{13}, v_{13}\}$. Then, $d_3 = v_{12}$.

Dominator will use pairing strategy where the pairing sets are $\{u_3, u_4\}$, $\{v_4, v_3\}$, $\{u_5, u_6\}$, $\{u_8, u_9\}$, $\{v_9, v_{10}\}$, $\{u_{10}, u_{11}\}$. To dominate $u_{13}$, Dominator will claim a free vertex from the set $\{u_{12}, u_{13}, v_{13}\}$. Also, to dominate $u_1$, Dominator will claim a free vertex from the set $\{u_1, v_1, u_2\}$. So, Dominator needs at most 11 moves.

According to the considered cases, it follows that $\gamma_{MB}(P_2 \Box P_{13}) \leq 11$. \hfill $\square$