Texture matrices are a way of mitigating the redundancy inherent in the description of flavor physics via Yukawa couplings by eliminating some entries in order to identify relevant parameters. A four-zero texture scheme has been used in the literature to successfully describe fermion masses and mixing. However, as we show in this work, improving experimental constraints require an update to this analysis. In this paper the implications of a 2-zero texture mass matrix is studied for quarks and leptons. We show that the introduction of a new parameter in each mass matrix allow us to reach good results with relative low cost in predictability. We report a numerical study using a hybridized nature-inspired/cellular automata search algorithm. We find that leptons and quarks can be described by the same 1-zero structure. We describe some scenarios where a simplified description can be achieved, including a narrow region in parameter space where the same values describe charged leptons and neutrinos, which is a stronger version of a previously proposed Universal Texture Constraint.
I. INTRODUCTION

The Standard Model (SM) provides a remarkable description of elementary particle physics up to the sub-TeV range. One of its shortcomings, however, is the lack of a proper understanding of flavor; since the symmetries of the SM do not restrict the mass matrices elements in the flavor basis, the relation between masses and mixing angles of fermions is not understood at all. As has been discussed by many authors \cite{1}, a satisfactory explanation of flavor can only be possible in the context of a model with additional flavor dynamics; insofar as the mixings and masses are described, but not explained by the SM, flavor physics is New Physics (NP).

The history of the attempts to face the flavor problem through texture matrices is very long. Because the entries of the mass matrices are not directly measured, the description of flavor through the Yukawa couplings is characterized by redundancy and a degree of arbitrariness; an early effort to reduce this freedom involved setting to zero some elements of the mass matrix, called texture zeroes. Since the pioneering paper of H. Fritzsch \cite{2}, different matrix-shapes (i.e., different sets of texture zeroes, or in short textures) have been studied in order to understand from the bottom--up the hierarchy between fermions masses and its relation with discrete symmetries.

For the quark mass matrices, the strong hierarchy of quark masses and the supressed off-diagonal elements of the CKM matrix have historically suggested small mixing angles coming from radiative corrections \cite{3}, higher order terms \cite{4} or even the overlap of wavefunctions in extra dimensional models \cite{5}. On the other hand, the discovery of neutrino oscillations, which required the addition to the SM of scalar couplings to generate neutrino mass terms (and potentially Majorana masses) prompted the exploration of Yukawa textures in the leptonic sector. Since the neutrino mixings are substantial, an approximately tribimaximal mixing pattern was a promising avenue of research \cite{6,7}. Since only three angles and two mass differences are presently known, fixing a texture gives a prediction for the neutrino mass scale \cite{8}.

If some fermion family structure survives at very high energies and scalar bosons remain elementary, then Yukawa couplings are fundamental constants of Nature \cite{9}. On the other hand, if the UV completion of the Standard Model is some GUT where quarks and leptons (and probably some other particles) transform as a multiplet, Yukawa couplings need also be unified, and the low-energy hierarchy could carry an imprint of the high-energy gauge symmetry breaking pattern.

In general, the number of free parameters in a fermion mass matrix is 18, before imposing any physical constraints. Taking into account that it is possible to make a unitary weak-basis transformation of fermions, and that right-handed fields in the Lagrangian are singlets under \(SU(2)\), we can choose a basis where mass matrices are Hermitian \cite{10}. For a pair of mass matrices, using a weak basis transformation to make them Hermitian, and removing non-physical phases, one ends up with ten free parameters that relate masses and mixing angles for every sector. Further restrictions have a non-redundant effect in the model.

The starting point for an analysis of texture zeroes is then a 1-zero mass matrix. In this paper we study the general properties of this texture mass matrix for both the quark and leptonic sectors. The main goal of this work is to establish the parameter space constrained by the updated experimental measurements in both sectors. Also, we aim to explore the idea of common flavor structures for charged and neutral fermions; for this reason, we pay special attention to scenarios where flavor texture universality can be realized, like in the positive--parameters case we present in section \textit{IV}.

As is widely known, the neutrino sector also allows for Majorana terms, which stem from the gauge-neutrality of sterile right-handed neutrinos. Through the Seesaw mechanism \cite{11, 12, 13}, large Majorana masses could explain the mass hierarchy between charged lepton and neutrinos. Nevertheless, in order to simplify our analysis, in this work we only consider neutrino masses of Dirac type. This also has the effect of preserving the similarity (at least in the electroweak sector) of the interaction structure of leptons and quarks.

Other choices are needed. Updated experimental results cannot determine the sign of the \(\Delta m_{13}^2\) difference, which leads to two possibilities for the neutrino mass ordering; they are called the normal and inverse order. In this work, we will assume normal ordering, again with the goal of exploring a common description for the quark and lepton sectors of the SM. Recent experimental data, including the Super-Kamiokande results favors Normal Order over the inverse order \cite{15, 16}.

In order to quantify the possible common origin of the flavor interaction in all sectors, we analyze the scaled parameters space in the lepton sector to look for universal values that generate mass matrices both in the charged lepton and the neutrino sector, the so--called Universal Texture Constraint (UTC) \cite{17}.

In section \textit{II} the 1-zero texture matrix is defined and its general properties are studied in the context of the SM symmetries. In section \textit{III} the Yukawa sector and the number of parameters involved in the relation between masses and mixing angles is discussed, and exact expressions for the CKM and PMNS in terms of the 1-zero parameters are given. In section \textit{IV} a complete numerical analysis for the free parameters in the quark sector is presented, using updated experimental measurements and novel bio-inspired methods to minimize the \(\chi^2\) function build with non correlated VCKM matrix elements. In section \textit{V} it is shown a numerical analysis for the neutrino sector using a global
fit of experimental measurement from [13] including the theoretical restrictions and for every possible parametrization found. Section [VI] shows the region of parameters allowed by the Universal Texture Constraint [17] in the leptonic sector for real positive parameters. In section [VII] it is given the conclusions.

II. THE 1-ZERO TEXTURE MATRIX

Taking into account the invariance of the Lagrangian under a common unitary transformation \( M \rightarrow UMU^\dagger \), the mass matrix in the flavor basis takes the general form

\[
M_F = \begin{pmatrix} E_F & D_F & 0 \\ D_F^\ast & C_F & B_F \\ 0 & B_F^\ast & A_F \end{pmatrix} ; \quad F = u, d, \ell.
\]

(1)

This is the 1-zero texture mass matrix; the zeroes at (1, 3) and (3, 1) are the only reduction of parameters that can be made by a symmetry already present in the SM Lagrangian. The matrix (1) has, in general, 7 parameters including phases. We write \( B_F (= |B_F| e^{i\phi_{DF}}) \) and \( D_F (= |D_F| e^{i\phi_{DF}}) \) in polar form, and we separate the \( M_F \) matrix in the product of an orthogonal matrix and a unitary diagonal phase matrix:

\[
M_F = P_F^\dagger \bar{M}_F P_F,
\]

(2)

where

\[
P_F = \text{diag} \left( 1, e^{i\phi_{DF}}, e^{i(\phi_{DF} + \phi_{BF})} \right).
\]

(3)

The orthogonal matrix \( \bar{M}_F \) is then written as

\[
\bar{M}_F = \begin{pmatrix} E_F & |D_F| & 0 \\ |D_F| & C_F & |B_F| \\ 0 & |B_F| & A_F \end{pmatrix}.
\]

(4)

The eigenvalues \( \lambda_i^F \) of \( \bar{M}_F \) are related to the fermion masses through \( \lambda_i^F = \pm m_i^F \); the sign depends on the chosen parameterization. We diagonalize the Hermitian matrix \( H_F \equiv M_F \bar{M}_F^\dagger \). By construction, its eigenvalues are the squared fermion masses \( \{ (m_1^F)^2, (m_2^F)^2, (m_3^F)^2 \} \). It is possible to isolate the phases through \( H_F = P_F H_F P_F^\dagger \), with \( H_F = M_F \bar{M}_F^\dagger \) a symmetrical matrix. The invariants under similarity transformations are

\[
\text{Tr}(\bar{H}_F) = (m_1^F)^2 + (m_2^F)^2 + (m_3^F)^2,
\]

(5)

\[
\text{Det}(\bar{H}_F) = (m_1^F)^2 (m_2^F)^2 (m_3^F)^2,
\]

(6)

\[
\frac{1}{2} [\text{Tr}^2(\bar{H}_F) - \text{Tr}(\bar{H}_F^2)] = (m_1^F)^2 (m_2^F)^2 + (m_1^F)^2 (m_3^F)^2 + (m_2^F)^2 (m_3^F)^2. \quad (7)
\]

In general, it is possible to solve this system of equations for \( |B_F|, |D_F| \) and \( C_F \) in terms of the parameters \( E_F \) and \( A_F \); for comparison, we can recover the 4-zero texture model by setting \( E_F = 0 \). The system (4-7) has \( 2^3 \) solutions taking into account the restrictions \( |D_F| > 0 \) and \( |B_F| > 0 \). Thus, there are \( 2^6 \) ways to parameterize the mixing matrix of fermions for 2-zero texture problem to determine the mixing of fermions on every sector. The solutions of (5-7) are:

\[
|B_F| = \sqrt{\frac{(A_F - \eta_1 m_1^F)(A_F - \eta_2 m_2^F)(A_F - \eta_3 m_3^F)}{E_F - A_F}}, \quad (8)
\]

\[
|D_F| = \sqrt{\frac{(E_F - \eta_1 m_1^F)(E_F - \eta_2 m_2^F)(E_F - \eta_3 m_3^F)}{A_F - E_F}}, \quad (9)
\]

\[
C_F = - (A_F + E_F - \eta_1 m_1^F - \eta_2 m_2^F - \eta_3 m_3^F), \quad (10)
\]
where \( \eta_i = \pm 1 \) for \( i = 1, 2, 3 \). These solutions are left unchanged by

\[
A_F \leftrightarrow E_F \implies |D_F| \leftrightarrow |B_F|,
\]

(11)

leaving invariant the element \( C_F \).

The simplest discrete symmetry behind this exchange is \( S_3 \), which we can represent as:

\[
[R_0] = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
[R_1] = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
[R_2] = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]

\[
[r_1] = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
[r_2] = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix},
[r_3] = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

These satisfy the following properties:

\( [r_i] = [r_i]^{-1} = [r_i]^T \).

\( [R_j] = [R_i]^{-1} = [R_j]^T, \ (i, j) \in (1, 2). \)

The action over 1-zero texture matrices is:

\[
[X_i]M_F[X_i]; \quad X = R, r.
\]

When \( [X_i] = [r_2] \) we have:

\[
[r_2]M_F[r_2] = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} \begin{pmatrix}
E_F & D_F & 0 \\
D_F^T & C_F & B_F \\
0 & B_F^T & A_F
\end{pmatrix} \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
A_F & B_F & 0 \\
B_F^T & C_F & D_F \\
0 & D_F^T & E_F
\end{pmatrix},
\]

reflecting the fact that this combination defines a family of matrices related by an equivalence class. If we choose either Normal Ordering of masses, i.e. \( 0 < m_1^F < m_2^F < m_3^F \), or inverse ordering (IO) , i.e. \( 0 < m_3^F < m_1^F < m_2^F \), the viable intervals for \( E_F \) and \( A_F \) are already determined for every solution. In table (I), we show the different cases allowed by the condition \( A_F \geq E_F \) (NO case).

| Case | \( \eta_1 \) | \( \eta_2 \) | \( \eta_3 \) | \( A_F \) and \( E_F \) interval |
|------|-------------|-------------|-------------|--------------------------------|
| 1    | +1          | +1          | +1          | \( m_1^F < E_F < m_2^F < A_F < m_3^F \) |
| 2    | +1          | -1          | +1          | \( -m_2^F < E_F < m_1^F < A_F < m_3^F \) |
| 3    | -1          | +1          | +1          | \( -m_1^F < E_F < m_2^F < A_F < m_3^F \) |
| 4    | -1          | -1          | +1          | \( -m_2^F < E_F < -m_1^F < A_F < m_3^F \) |
| 5    | +1          | +1          | -1          | \( -m_3^F < E_F < m_1^F < A_F < m_2^F \) |
| 6    | +1          | -1          | -1          | \( -m_3^F < E_F < -m_1^F < A_F < m_2^F \) |
| 7    | -1          | +1          | -1          | \( -m_3^F < E_F < -m_1^F < A_F < m_2^F \) |
| 8    | -1          | -1          | -1          | \( -m_3^F < E_F < -m_2^F < A_F < -m_1^F \) |

As expected, to reproduce the canonical 4-zero Fritzsch texture case, it is necessary to analyze the allowed values
for $E_F$ and take the limit $E_F \to 0$ in equations \([8,10]\). Setting $\eta_3 = 1$, we obtain:

\[
\begin{align*}
\lim_{E_F \to 0} |B_F|^2 &= \frac{(A_F - \eta_1 m_{1}^F)(A_F - \eta_2 m_{2}^F)(m_{3}^F - A_F)}{A_F}, \\
\lim_{E_F \to 0} |D_F|^2 &= -\eta_1 \eta_2 m_{1}^F m_{2}^F m_{3}^F, \\
\lim_{E_F \to 0} C_F &= -A_F + \eta_1 m_{1}^F + \eta_2 m_{2}^F + m_{3}^F.
\end{align*}
\]

For the quark sector, our parameterization 2 is reduced to the 4-zero Fritzsch texture when $\eta = -1$ in \([18]\) and our parameterization 3 is reduced to the 4-zero Fritzsch texture when $\eta = 1$ in \([18]\). When neutrinos are introduced, our parameterization 5 is reduced to parameterization 1 (see eq.11) in \([19]\), our parameterization 2 is reduced to parameterization 1 (see eq.13) in \([19]\), our parameterization 2 is reduced to parameterization 1 (see eq.12) in \([19]\) and our parameterization 5 is reduced to parameterization 1 (see eq.13) in \([19]\).

The analysis in \([20]\), where the 1-zero texture was previously diagonalized, corresponds to our parameterization 2. In the following, we proceed to analyze every possible parameterization for the mixing matrices.

Concerning leptons, we make the following assumptions:

1. A Normal Ordering of the lepton masses, so that $|\lambda_3| > |\lambda_2| > |\lambda_1| > 0$.
2. Following the idea that the elements of the mass matrix arise through some perturbative process, we impose a hierarchy in the parameters such that the $E_F < A_F$.
3. Implicitly, we are assuming a Hermitian mass matrix, so that the diagonalization is given by a unitary transformation.
4. The eigenvalues of the mass matrix are related to the masses via

\[
\begin{align*}
\lambda_1 &= \eta_1 m_{1}, \\
\lambda_2 &= \eta_2 m_{2}, \\
\lambda_3 &= m_{3}.
\end{align*}
\]

III. THE MIXING MATRICES FOR QUARKS AND LEPTONS

The Yukawa sector of the SM in the flavor basis has the form

\[
\mathcal{L}_Y = -\bar{Q}'_L \left( Y^d \phi d_R + Y^u \phi u_R \right) - L'_L Y^d \phi \ell'_R + \text{h.c.}
\]

where $Q'_L = (u'_L, d'_L)^T$ and $L'_L = (\nu'_L, \ell'_L)^T$, $\phi = (\varphi^\pm, \varphi^0)^T$ and $|\varphi^0\rangle = v$. Here $\bar{\phi}$ is the charge conjugated of $\phi$, with its neutral component written as $\varphi^0 = v + h^0$, and the fields $d_R, u_R$ are $SU(2)$ singlets.

The existence of neutrino masses requires to go beyond the Standard Model, adding right-handed neutrinos with Yukawa terms plus potentially Majorana masses:

\[
\mathcal{L}_{Y\nu} = -L'_L Y'\nu\phi'_{R} + M_R \bar{\nu}_R' \nu_{R}.
\]

In the following, we will only consider Dirac masses for the neutrinos.

After Spontaneous Symmetry Breaking (SSB), this Lagrangian can be separated in

\[
\mathcal{L}_Y = \mathcal{L}_\text{mass} + \mathcal{L}_\text{CC} + \mathcal{L}_\text{NC}.
\]

Changing to the mass basis through $u_{R,L} = U_{R,L} u_{R,L}, d_{R,L} = U_{R,L} d_{R,L}, \ell_{R,L} = U_{R,L} \ell_{R,L},$ and $\nu_{R,L} = U^{\nu} R, L \nu_{R,L}$, then omitting the contribution from Goldstone modes, this Lagrangian becomes:

\[
\begin{align*}
\mathcal{L}_\text{mass} &= -\bar{d}_L M_d d_R - \bar{u}_L M_u u_R - \bar{\ell}_L M_{\ell} \ell_R - \bar{\nu}_L M_{\nu} \nu_R + \text{h.c} \quad (22) \\
\mathcal{L}_\text{NC} &= -\frac{1}{v} \left( \bar{d}_L M_d d_R + \bar{u}_L M_u u_R + \bar{\ell}_L M_{\ell} \ell_R + \bar{\nu}_L M_{\nu} \nu_R \right) h^0 + \text{h.c} \quad (23)
\end{align*}
\]

\footnote{The general case is when $\text{Diag}(\lambda_1, \lambda_2, \lambda_3) = U^\dagger_L M U_R$, with $U_L \neq U_R$, is a bilinear transformation. We omit this general case.}
where $\tilde{M} = U^F U^F Y U^F_R = \text{diag}(m_1^F, m_2^F, m_3^F)$.

The mass matrix is diagonalized through a bilinear transformation $U_{L}^{f\dagger} M^f U_R^f = \text{Diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_i$ are the mass matrix eigenvalues and $f = u, d, \ell, \nu$. In reverse, to find the matrices $U^f_R$ and $U^f_L$ we consider the diagonalization of the squared matrices $M^f M^{f\dagger}$ and $M^{f\dagger} M^f$. These are Hermitian matrices by definition, diagonalized by $U^f_L$ and $U^f_R$ respectively. The mixing matrices are built by the left-hand matrix $U^f_L$ through the relations

$$V_{\text{CKM}} = U^u_L U^d_L,$$
$$V_{\text{PMNS}} = U^\nu_L U^\ell_L,$$  

for quarks and leptons respectively. Since the measured observables are just the masses and mixing angles, the matrix $U^f_R$ is non-observable. When the mass matrix is Hermitian then $U^f_L = U^f_R$. Mixing matrices (24) and (25) assume Dirac masses for the fermions; additionally, the neutrino sector also allows for Majorana mass terms.

It is possible to chose a special flavor basis where the flavor symmetries are explicit. In particular, the reduction of free parameters needed to describe the relation between fermions masses and mixing angles introduce significant zeroes in the mass matrices; this is what the literature refers to as choosing a texture mass matrix. Once the Hermitian nature of the mass matrix is established, the position of the zeroes in this matrix determines the number of free parameters. A zero outside the diagonal represent a reduction by 2 parameters, i.e. the norm and the phase; a zero in the diagonal only removes one real parameter.

In order to make a predictive model from a texture mass matrix two objectives are pursued:

- To reproduce the measured mixing angles through suitable functions depending on the measured fermion masses. It is important to include only the non-correlated measurements, as the mixing angles and the Jarlskog index or the independent mixing matrix elements in [21] and [22]. In this work, we have chosen a parametrization proposed by M. Kobayashi and T. Maskawa as is described in [21, 22] for the case of quarks. For the leptonic sector the PMNS matrix was parameterized as in [20], where the mixing matrix is separated as the product of three rotation matrices and a CP violating phase matrix. In the quark sector, for convenience, we have taken the matrix elements of the first row of the CKM matrix and the Jarlskog index to determine allowed regions of values for the free parameters of the model. For the leptonic sector, we have chosen also the first-row PMNS matrix elements and the element (2, 3) that contains the CP phase in this sector. Both choices are equivalent because the introduction of a texture matrix relates these parameters to mixing angles and the CP phase.

- To introduce the smallest number of free parameters. In general, a $3 \times 3$ Hermitian matrix has 9 free parameters. If all eigenvalues are known, then this is reduced by 3; an additional restriction can be obtained from the CP violating phase through the Jarlskog index. Therefore, a mass matrix has 5 free parameters before the introduction of a texture. In this work we have considered non-degenerated masses in the leptonic sector as they are in the quark sector. Below, we present a detailed description of the remaining free parameters.

To build a mixing matrix two matrices are needed, $U^u_L$ and $U^d_L$, where $f_d$ and $f_u$ represent the down-type and up-type fermions: $f_u = u, \nu$ and $f_d = d, \ell$. For every type of fermions a mass matrix texture is introduced. Every Hermitian matrix has 9 parameters, so mixing matrices are expressed in terms of 18 parameters. Once we remove some parameters with 3 invariants (3 parameters), one CP violating phase (1 parameter), the mixing angles (3 parameters) and by using the rephasing freedom, 10 parameters remain to be fixed.

Imposing the same texture matrix for the up-type and down-type fermions produces what is called a parallel texture; this duplicates the number of restrictions on the free parameters. For instance, if a non-diagonal zero is put on a texture, the number of free parameters is reduced by 4, leaving 6 parameters to describe the mixing matrix. This case is studied in this work.

There are additional possibilities to further reduce the number of free parameters; two interesting examples will be commented on.

- The first is the Universal Texture Constraint, where a relation between different types of fermions is considered such that the same values used for the parameters in the up-type texture mass matrix are used in the down-type without taking into account the phases; in this way, the number of parameters is reduced to 3 parameters and one phase.

The quark sector is one of the most studied, and the mixing angles and CP phase have been measured with remarkable precision. This measurements strongly restrict the viable textures for the mass matrix. In particular, the 4-zero texture, where 2 zeroes come from the up-type quark sector and the rest from the down-type quarks has been extensively studied.
On the leptonic sector, on the other hand, many open questions remain. One is the scale of the masses of the neutrinos, and the Dirac or Majorana nature of such masses. Unlike the quark sector, where all masses are known, in the leptonic sector only the mixing angles and the differences of the squared masses are known, so there is one extra free parameter. It is possible to choose this free parameter as the heaviest neutrino mass ($m_{\nu_3}$), and we have done so.

Because there is some freedom in the election of the flavor basis, in this work two cases are analyzed. First, it is assumed that the flavor basis where the mass matrix for neutrinos have some texture zeroes, and the one for the charged lepton is diagonal. In other words, there are not free parameters coming from the charged lepton sector. Nevertheless, as it was shown below, in light of the updated fitting from the NuFit2019 collaboration, it is no possible to fulfill the experimental condition on the mixing angles and the Jarlskog index with this small set of free parameters.

- The second analyzed possibility is the mass matrix with texture zeroes for the charged lepton sector. Although there is a large number of possibilities to chose a texture zero with a smaller number of new parameters, in this work the Hermitian mass matrix with a non-diagonal zero in the charged lepton sector is studied, because it is the most general matrix that preserves all symmetries of the SM. Although some authors claim that this is not a very predictive model because it introduce many new parameters, it is interesting to explore the common origin of flavor analyzing the case of a parallel texture between charged leptons and neutrino sector.

In order to analyze how is the free parameters space of the 1-zero texture mass matrix, we have made the assumption that there is a parallel structure between charged leptons and neutrinos sectors, this is, the same texture for the neutrino mass matrix of the charged leptons is used in the neutrino sector. This duplicates the number of free parameters introduced in the model of the PMNS matrix respect the one where the charged leptons basis is the mass basis. Nevertheless, to reproduce the updated experimental results, more parameters are needed, because we can demonstrate that if we work in the mass basis for the charged lepton and the flavor basis in the neutrino sector, this set is not enough. In this analysis we have adopt an scheme without approximations.

Invariants of the mass matrix under a similarity transformation produce $n$ relations between free parameters. The number of free parameters for a Hermitian texture is given by

$$
N(n, n_d, n_{nd}) = n^2 - n - n_d - n_{nd},
$$

where $n$ is the number of generations (number of invariants under unitary transformations), $n_d$ the number of zeroes in the diagonal and $n_{nd}$ the number of parameters reduced by the off-diagonal zeroes (for one off-diagonal zero $n_{nd} = 2$, see ref. [23]). In the quark and the charged lepton sector, the masses of fermions are known, but in the neutrino sector only the mass differences are known and the heaviest neutrino mass is yet to be determined. It is however possible to deduce through cosmological observations an upper bound on this heaviest neutrino mass.

As previously discussed, without loss of generality we can reduce the shape of the mass matrix to the texture with one off-diagonal zero. For three generations ($n = 3$) and with fermion masses given as known, as in the case of quarks and charged leptons, using the three restrictions coming from the matrix invariants, and exploiting the rephasing of the fermion fields, the number of free parameters $P$ of a Hermitian texture is given by

$$
P = N(3, 0, 2) - n_\alpha = 3,
$$

where $n_\alpha = 1$ is the number of phases removed by rephasing fermion fields. The mixing matrix for every sector thus has $2P$ free parameters that have to be restricted by additional measurements. Another source of information on the parameters of the texture is the mixing angles of fermions and the CP violating phase. Thus in the quark sector when an off-diagonal 1-zero texture is used to model the mass matrix we have 2 free parameters to explain the relation between the quarks masses and the mixing angles, or equivalently the non–correlated $V_{\text{CKM}}$ matrix elements.

In the lepton sector, the known observables are $\Delta m^2_{21}, \Delta m^2_{32}$, three mixing angles and a CP violating phase, so there is one less observable with respect to the quark sector. In the next section we describe the issues introduced by the simplest scheme where the mass matrix of the charged leptons is diagonal, and discuss the phenomenological viability of the parallel texture assumption when the updated experimental measurements on the neutrino sector are taken into account.
Finally, the theoretical flavor mixing matrices $V_{CKM}$ and $U_{PMNS}$ arising from 2-zero textures are:

\[
V_{CKM}^{th} = U_L^{th} U_R^{th} = O_T^T (P_u P_d^T) O_d,
\]

\[
(V_{CKM}^{th})_{1\alpha} = (O_u)_{1i} (O_d)_{\alpha i} + (O_u)_{2i} (O_d)_{\alpha 2} e^{i\phi_1^q} + (O_u)_{3i} (O_d)_{\alpha 3} e^{i(\phi_1^q + \phi_2^q)},
\]

\[
V_{PMNS}^{th} = U_L^{th} U_R^{th} = O_T^T (P_e P_l^T) O_l,
\]

\[
(V_{PMNS}^{th})_{j \beta} = (O_e)_{1j} (O_l)_{1\beta} + (O_e)_{2j} (O_l)_{2\beta} e^{i\phi_1^q} + (O_e)_{3j} (O_l)_{3\beta} e^{i(\phi_1^q + \phi_2^q)},
\]

where the phases are defined as $\phi_1^q = \phi_{Du} - \phi_{Dd}$, $\phi_2^q = \phi_{Bu} - \phi_{Bd}$, $\phi_1^l = \phi_{Dv} - \phi_{Dl}$, $\phi_2^l = \phi_{Bv} - \phi_{Bl}$ and the $O$ matrix elements are:

\[
(O_{F_{11}}) = 1 + \frac{(E_F - \eta_1 m_1^F)(A_F - E_F)}{(\eta_2 m_2^F - E_F)(\eta_3 m_3^F - E_F)} + \frac{(E_F - \eta_1 m_1^F)(A_F - \eta_2 m_2^F)(\eta_3 m_3^F - A_F)}{(\eta_2 m_2^F - E_F)(\eta_3 m_3^F - E_F)(\eta_1 m_1^F - E_F)}^{-1/2}
\]

\[
(O_{F_{22}}) = 1 + \frac{(E_F - \eta_1 m_1^F)(\eta_3 m_3^F - E_F)}{(\eta_2 m_2^F - E_F)(A_F - E_F)} + \frac{(E_F - \eta_1 m_1^F)(\eta_3 m_3^F - E_F)(A_F - \eta_2 m_2^F)}{(\eta_2 m_2^F - E_F)(A_F - \eta_2 m_2^F)(\eta_1 m_1^F - E_F)}^{-1/2}
\]

\[
(O_{F_{33}}) = 1 + \frac{(A_F - \eta_3 m_3^F - E_F)}{(\eta_2 m_2^F - E_F)(A_F - \eta_2 m_2^F)} + \frac{(E_F - \eta_1 m_1^F)(\eta_3 m_3^F - E_F)(\eta_3 m_3^F - A_F)}{(\eta_2 m_2^F - E_F)(\eta_3 m_3^F - E_F)(\eta_1 m_1^F - E_F)}^{-1/2}
\]

\[
(O_{F_{21}}) = \frac{\eta_1 m_1^F - E_F}{|D_F|} (O_{F_{11}}) \quad i = 1, 2, 3.
\]

\[
(O_{F_{31}}) = \frac{B_F}{\eta_1 m_1^F - E_F} (O_{F_{21}}) \quad i = 1, 2, 3.
\]

**IV. NUMERICAL ANALYSIS OF THE 2 ZERO TEXTURE IN THE QUARK SECTOR**

Equation (28), gives the theoretical quark flavour mixing matrix $V_{CKM}^{th}$ as an explicit function of texture quark model parameters $A_u$, $A_d$, $E_u$, $E_d$, $\phi_1^q$ and $\phi_2^q$. In order to find a numerical range for these parameters according with the experimental $V_{CKM}$ values a $\chi^2$ minimization procedure is performed. Because they are related by the unitary conditions ($V_{CKM} V_{CKM}^\dagger = I$), not all $V_{CKM}$ elements are independent. We can choose the four independent parameters as follows: three mixing angles and one CP-violating complex phase. Following [24, 25], we take $|V_{ud}|$, $|V_{us}|$, $|V_{ub}|$ and the Jarlskog invariant as experimental parameters for the $\chi^2$ analysis. We define:

\[
\chi^2(A_u, A_d, E_u, E_d, \phi_1^q, \phi_2^q) = \frac{|V_{ud}^{th} - |V_{ud}||^2}{\sigma_{V_{ud}}^2} + \frac{|V_{us}^{th} - |V_{us}||^2}{\sigma_{V_{us}}^2} + \frac{|V_{ub}^{th} - |V_{ub}||^2}{\sigma_{V_{ub}}^2} + \frac{(J_q^{th} - J_q)^2}{\sigma_{J_q}^2},
\]

where the terms with super-index “th” are given in (28) and the quantities without super-index are the experimental data with uncertainty $\sigma_{V_{ij}}$ taken from (22).

Since we have 4 experimental observables, as given in the above equation (35), $(N_{obs} = 4)$, a good criteria is to find the numerical range of parameters $A_u, A_d, E_u, E_d, \phi_1^q$ and $\phi_2^q$, that gives a value of $\chi^2_{N_{obs}}$ less or equal to 1.

\[
\chi^2(A_u, A_d, E_u, E_d, \phi_1^q, \phi_2^q) \leq 1
\]

In this analysis, we will consider that u-quark mass matrix and d-quarks mass matrix have the same parameterization (see table I). This automatically gives us eight working scenarios (see table II), which we will analyze numerically. To search the set of parameters that satisfy the cost function criteria (36), it is necessary to implement a specific search technique and, in this area, it is common to use exhaustive search techniques which work very well for optimization of small number of parameters (6-textures, 5-textures, 4-textures) and assuming certain simplifying conditions [26, 27]. On the other hand, if we implement traditional optimization techniques like the gradient method, Davidon-Fletcher-Powell method or Newton method, great difficulties ensue because the nonlinear dependence among the quantities (O matrix elements) in general leads to convergence problems.

In the last two decades, meta-heuristic algorithms have been applied to many applications due to their efficiency, reliability and relatively low computation time [28]. There is a great diversity of heuristic algorithms, which imitate strategies developed by animals and plants for their survival. One of the most important algorithms is the Particle Swarm Optimization (PSO) [29], due to its relatively ease of implementation and good convergence speed, although the
particles are easily trapped by local optima. There are many variants of the PSO that improve its performance [30]. The Differential Evolution algorithm (DE) [31] is another widely used heuristic due to its robustness [32], although the speed of convergence is quite low. To improve the two main disadvantages of PSO and DE, various hybrid methods have been presented, such as CPSO-DE [33], that combines the nature inspired PSO with a cellular automata [34], and an evolution rule based on DE. In this paper, we use a variant of CPSO-DE that uses a mutation probability to create a mutation/donor vector using the current best individual found so far (DE/best) and a random vector (DE/rand/1). A methodology based on the optimization of constraints is also used.

In our numerical algorithm, we use the following quark masses values [22]:

\[
\begin{align*}
m_u &= 2.16^{+0.49}_{-0.26} \text{ MeV} \\
m_c &= 1.27^{+0.02}_{-0.02} \text{ GeV} \\
m_t &= 172.76^{+0.39}_{-0.30} \text{ GeV} \\
m_d &= 4.67^{+0.48}_{-0.12} \text{ MeV} \\
m_s &= 93^{+11}_{-5} \text{ MeV} \\
m_b &= 4.18^{+0.03}_{-0.02} \text{ GeV}
\end{align*}
\]

And the $V_{CKM}$ current values [22]:

\[
V_{CKM} = 
\begin{pmatrix}
0.97401 \pm 0.00011 & 0.22650 \pm 0.00048 & 0.00361 \pm 0.00009 \\
0.22636 \pm 0.00048 & 0.97320 \pm 0.00011 & 0.04053 \pm 0.00083 \\
0.00854 \pm 0.00023 & -0.00016 & 0.03978 \pm 0.00082 \\
0.00094 & -0.00006 & 0.999172 \pm 0.00021 \\
-0.00036 & -0.00037 & 0.99990353
\end{pmatrix}
\]

and the Jarlskog invariant value $J_{\phi} = (3.00^{+0.15}_{-0.09}) \times 10^{-5}$.

Among these scenarios (see table II) for the quark mixing matrix in quarks, we have selected the one with real positive free parameters for in-depth study; this restriction leads to the scenario 1. In any numerical analysis an associated numerical error will appear. We analyze the dependence of $\chi^2$ value with the significant digits in the free parameters. Our result is shown in Figure 1 where we can note that, after six significant digits the $\chi^2$ is almost constant. In Figure 2 we show plots of the parameter space color-coded to show the value of the cost function, and projected on the $A_u/m_t$ vs. $A_d/m_b$, $E_d/m_b$ vs. $E_u/m_t$, $E_u/m_t$ vs. $A_u/m_t$ and $E_u/m_t$ vs. $A_d/m_b$ planes.

A look at subfigure (a), is suggestive of a linear relation between the scaled parameters $A_d/m_b$ and $A_u/m_t$. We compute the Pearson Coefficient ($\rho$), which has the value $\rho = 0.998$; the linear fit parameters produce a slope $m = 0.979$ and a $y$-intercept $b = 0.001$, essentially a straight line at 45°. The correlation between the Yukawa elements $A_u$ and $A_d$ is the form:

\[
\frac{A_d}{m_b} = m \frac{A_u}{m_u} + b.
\]

In subfigure (b) and (d) of Figure 2, the boundaries of the parameter space are clear and the lower bound of $E_u/m_t$ is compatible with zero. This result implies that the texture with a zero in the (1,1) place is a possible scenario for the up-type quark sector. Nevertheless, as it is shown in these plots, the lower bound of the $E_d/m_b$ parameter never reaches the zero value for this scenario. This is interesting because it seemingly excludes the 2-zero texture mass matrix for the down type sector.

A linear correlation between parameters is also present in other scenarios. We found a linear behavior between $A_d/m_b$ and $A_u/m_t$ for the scenario 3, and for $E_d/m_b$ and $E_u/m_t$ in scenarios 5 and 7, with Pearson coefficients 0.990, 0.986 and 0.984 respectively.
In figure 3 we have plotted the phases $\phi_u$ versus $\phi_d$. The bulk of the points lies on the straight lines $\phi_d = 0$, $\phi_d = 2\pi$ and $\phi_d = 4\pi$. This implies a correlation between the Yukawa phases $\phi_{B_u}$ and $\phi_{B_d}$ in the form:

$$\phi_2 = \phi_{B_u} - \phi_{B_d} = 2n \pi, \quad \rightarrow \quad \phi_{B_u} = 2n \pi + \phi_{B_d},$$

where $n$ is an integer number. These two phases are equal in the case $n = 0$. 

Table III. $\chi^2$ minimum and their corresponding numerical values of the parameters.

| Scenario | $\chi^2$ Minimum | Quarks Parameters Value |
|----------|-------------------|-------------------------|
| 1        | 0.00125           | $A_u = 5.382829 \times 10^4 \text{MeV}, A_d = 1.225015 \times 10^3 \text{MeV}, E_u = 6.995365 \times 10^4 \text{MeV}, E_d = 2.177576 \times 10^5 \text{MeV}, \phi_u^2 = 6.045549, \phi_d^2 = 6.236893.$ |
| 2        | 0.0025            | $A_u = 9.416653 \times 10^4 \text{MeV}, A_d = 7.763994 \times 10^1 \text{MeV}, E_u = -1.648069 \times 10^2 \text{MeV}, E_d = 2.666386 \text{MeV}, \phi_u^2 = 5.127067, \phi_d^2 = 5.356038.$ |
| 3        | 0.00047           | $A_u = 1.652226 \times 10^4 \text{MeV}, A_d = 4.194742 \times 10^2 \text{MeV}, E_u = 7.206153 \times 10^4 \text{MeV}, E_d = 1.417096 \text{MeV}, \phi_u^2 = 5.217658, \phi_d^2 = 6.168979.$ |
| 4        | 0.00389           | $A_u = 1.726981 \times 10^5 \text{MeV}, A_d = 6.790165 \times 10^1 \text{MeV}, E_u = -4.156814 \text{MeV}, E_d = -9.485004 \text{MeV}, \phi_u^2 = 3.166315, \phi_d^2 = 5.648340.$ |
| 5        | 0.0025            | $A_u = 5.646824 \times 10^1 \text{MeV}, A_d = 1.824668 \times 10^1 \text{MeV}, E_u = -1.05353 \times 10^3 \text{MeV}, E_d = -2.680576 \times 10^5 \text{MeV}, \phi_u^2 = 3.114467, \phi_d^2 = 3.572330.$ |
| 6        | 1.38 $\times 10^5$ | $A_u = 5.519055 \times 10^1 \text{MeV}, A_d = 1.214369 \times 10^1 \text{MeV}, E_u = -4.582454 \times 10^3 \text{MeV}, E_d = -1.107061 \times 10^5 \text{MeV}, \phi_u^2 = 3.086220, \phi_d^2 = 4.077284.$ |
| 7        | 0.0025            | $A_u = 5.519055 \times 10^1 \text{MeV}, A_d = 1.214369 \times 10^1 \text{MeV}, E_u = -4.582454 \times 10^3 \text{MeV}, E_d = -1.107061 \times 10^5 \text{MeV}, \phi_u^2 = 3.086220, \phi_d^2 = 4.077284.$ |
| 8        | 1.38 $\times 10^5$ | $A_u = 5.519055 \times 10^1 \text{MeV}, A_d = 1.214369 \times 10^1 \text{MeV}, E_u = -4.582454 \times 10^3 \text{MeV}, E_d = -1.107061 \times 10^5 \text{MeV}, \phi_u^2 = 3.086220, \phi_d^2 = 4.077284.$ |
Figure 2. In this plot, the parameter space for the scenario 1 of the table III is shown, where all parameters are positive. Sub figure (a) suggest a linear relation between the scaled parameters $A_d/m_b$ and $A_u/m_t$ (See further discussion in the text). The color bar indicate the values of the function in (35).

Figure 3. Plot showing the allowed range of $\phi_1^q$ versus $\phi_2^q$, in $\pi$ units. While $\phi_2 \sim 0$ or $\phi_2 \sim 2\pi$, the values for $\phi_1$ are not restricted by this analysis.
V. NUMERICAL ANALYSIS OF THE 2 ZERO TEXTURE IN THE LEPTONIC SECTOR

A. Comparison between diagonal mass matrix and 1-zero texture mass matrix in the charged lepton sector.

To reduce the number of free parameters, a popular choice of constraints is a 1-zero neutrino mass matrix in the flavor basis, and a diagonal mass matrix for the charged leptons [35]. This choice intends to describe all phenomenology of lepton sector with a reduced number of free parameters. Nevertheless with the advance on the precision of experimental results, such number of free parameters and the restriction imposed by the texture in the neutrino mixing matrix, it is no longer possible to accommodate the updated experimental fittings reported by the collaboration NuFit [15].

In the parametrization described above, the mass matrix neutrinos can be written as

\[ M^\nu = U_{\text{PMNS}} \text{Diag}(m_1^\nu, m_2^\nu, m_3^\nu) U_{\text{PMNS}}^\dagger, \]

where the matrix \( M^\nu \) is a texture matrix as in \([1] \). In order to check the viability of these results, the fitting in \([15]\) was taken for the elements of the PMNS matrix. These values are shown in figure 4.

![Figure 4](image-url)

Figure 4. Matrix elements of the PMNS mixing as result of the analysis of the group NuFit. Two cases are presented, with and without Super Kamiokande experimental results. As explained in \([15]\), Super Kamiokande is considered apart from the global fit since there is not enough public information available to reproduce its analysis. Nevertheless, as can be seen in the figure, this mainly produces a small difference in the (2,3) entry of the mixing matrix. This difference does not drastically change the results.

In order to obtain the experimentally allowed parameter regions, the following function is defined as

\[ \chi^2_{\eta_1, \eta_2} = \sum_{i=1}^{3} \frac{[U_{ei}^{\text{th}} - U_{ei}^{\text{exp}}]^2}{(\Delta U_{ei}^{\text{exp}})^2} + \frac{[U_{\mu 3}^{\text{th}} - U_{\mu 3}^{\text{exp}}]^2}{(\Delta U_{\mu 3}^{\text{exp}})^2}, \]

where the theoretical mixing matrix is \( U_{ei}^{\text{th}} = U_{ei}^{\text{th}}(A_{\nu}, E_\nu, m_\nu, \phi^1, \phi^2, \eta_1, \eta_2) \) (for \( \alpha = e, \mu \)). In the parametrization in use here, the theoretical PMNS matrix elements in \([38]\) are functions of the mixing angles and the CP phase; this expressions therefore omits redundant experimental information.

The \( U_{\alpha i}^{\text{exp}} \) and \( \Delta U_{\alpha i}^{\text{exp}} \) are the fit from experimental measurement of \( U_{\text{PMNS}} \) matrix element and the error associated respectively as reported in \([16]\). The dependence on the chiral parameters \( \eta_1 \) and \( \eta_2 \) has been taken into account through the eigenvalues

\[ \lambda_1 = \eta_1 \sqrt{m_{\mu 3}^2 - \Delta m_{23}^2} - \Delta m_{12}^2 \]
\[ \lambda_2 = \eta_2 \sqrt{m_{\mu 3}^2 - \Delta m_{23}^2} \]

with \( \Delta m_{23}^2 = 2.525 \times 10^{-3}\text{eV}^2 \) and \( \Delta m_{12}^2 = 7.39 \times 10^{-3}\text{eV}^2 \). There is a lower bound from Hermiticity: \( m_{\nu 3} \geq \sqrt{\Delta m_{23}^2 + \Delta m_{12}^2} \). As can be seen in the expression \([38]\), the free parameters are restricted using 4 observables, namely \( U_{e1}, U_{e2}, U_{e3} \) and \( U_{\mu 3} \). The experimentally allowed region must fulfill \( \chi^2_{\eta_1, \eta_2} / N_{\text{obs}} \leq 1 \), where \( N_{\text{obs}} = 4 \).

The upper bound for \( m_{\nu 3} \) was taken from the model-independent analysis in \([36]\) where it is shown that for Normal Ordering, the sum of neutrino masses fulfills the bound

\[ \sum_i m_{\nu i} \lesssim 0.26 \text{ eV}. \]

The definitions of \( \Delta m_{23}^2 \) and \( \Delta m_{12}^2 \) allow to determined the approximated upper bound \( m_{\nu 3} \lesssim 0.0959 \text{ eV} \). In the following, our analysis will be made using this range of values for the mass of the heaviest neutrino.
It was proceed to compare the behavior of $\chi^2$ using two cases; the diagonal case represented by the expression \cite{37}, and the parallel case where the mixing matrix is written with one zero texture for both the charged lepton sector and neutrino sector.

The diagonal case only consider the free parameters coming from the texture of the neutrino sector and the parallel case increased the number of free parameters. Although this is not the minimal choice, it is consistent with the search of similar flavor structures between the studied sectors.

The analysis was made as follows: first, a random set of parameters was generated, fixing the value for $m_{\nu_3}$ to calculate the function \cite{38} between the limits imposed by the properties of the mixing matrix and the eigenvalues \cite{39,40}. This process was repeated until obtained a minimal value for the $\chi^2$. This method was performed for several values of the heaviest neutrino mass.

In figure 5, the plot shows the minimal values for $\chi^2$ with different $m_{\nu_3}$ values for the case with $\eta_1 = 1$ and $\eta_2 = 1$. In both cases a 1-zero texture mass matrix was used for the neutrino sector.

With a parallel 1-zero texture, the scaled $\tilde{\chi}^2 = \chi^2_{\eta_1,\eta_2}/4$ function depends on 7 continuous parameters. The mixing matrix that comes from a diagonal mass matrix in the charged lepton sector depends only on the parameters of the neutrino sector. In order to find a scenario allowed by the experimental bounds, we searched for a combinations of free parameters where $\tilde{\chi}^2_{\eta_1,\eta_2} < 1$. Evaluating both cases, it is observed that in the case of a diagonal mass matrix in the charged lepton sector, it is not possible to obtain a low enough $\tilde{\chi}^2$ function. In contrast, with the introduction of the new parameters from the 1-zero mass matrix in the same sector some combination of parameters lying appears in the experimental allowed region. The cases $(\eta_1, \eta_2) = ( -, + ), (+, -), (-, -)$ were analyzed, obtaining an allowed region only for the case $(-, +)$ with a similar result as in figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Behavior of $\chi^2/N_{\text{obs}}$ with the mass of the heaviest neutrino $m_{\nu_3}$ for the case $\eta_1 = 1$ and $\eta_2 = 1$ using a 1-zero parallel textures and the diagonal. The shaded zone shows the allowed experimental region for the $\tilde{\chi}^2$ function. Every point is the smallest value of the $\tilde{\chi}^2$ function obtained from a random sample of free parameters. As can be seen, as a consequence of having bigger number of free parameters, the one-zero texture parallel case, have some scenarios allowed compatible with the fitting in \cite{37}. A diagonal mass matrix in the charged leptonic sector cannot reproduce the experimental fitting of NuFIT 2019.}
\end{figure}

Once we have numerically demonstrated the viability of the 1-zero texture mass matrix in the charged lepton and neutrino sectors, in contrast with the diagonal mass matrix, we test how robust is this parallel texture independently of the chosen combination of parameters.

It was selected the combination of parameters that generates the lowest value for the scaled $\chi^2$ function in both parametrizations, diagonal and parallel textures. In order to test the how sensitive is the dependence of the $\chi^2/N_{\text{obs}}$ on the heaviest neutrinos mass $m_{\nu_3}$ it was varied respect this parameter for both cases.

The scenarios are characterized by the six parameters $S_i = (A_{\ell}, \theta, \Delta m^{2}_{\ell}, \phi^{1}_{\ell}, \phi^{2}_{\ell})$.

To illustrate the behavior, we have chosen the allowed scenarios in Table IV representing local minima of the cost function \cite{38} calculated numerically.

In figure 6 we plot the $\chi^2/N_{\text{obs}}$ function varying $m_{\nu_3}$ around the best fit values for the scenarios described above. As can be seen, the 1-zero texture mass matrix for both sectors, charged leptons and neutrinos, allows a suitable parameterization for the scale of the heaviest neutrino mass in the Normal Order case. In the next section, we analyse...
Table IV. Scenarios taken to illustrate the behavior of $\chi^2/N_{\text{obs}}$ in terms of $m_{\nu_3}$. These are plotted in Figure 6. The prediction for the heaviest neutrino mass depends strongly of the combination of free parameters. In both scenarios are taken to reach its minimum at $m_{\nu_3} = 0.0514286$ eV.

all possible parametrizations coming from the parallel texture.

Figure 6. Scenarios in table IV that represent the local minimum of the $\chi^2$ function in (42). The mass of the heaviest neutrino was varied for every scenario. As can be seen in the plot, the minimum of the cost function determines different allowed region for the $m_{\nu_3}$ values.

Parallel textures are up to the task of describing the experimental data; this analysis shows also that present data disfavors a diagonal shape for the charged lepton mass matrix. Although this result is expected due to the introduction of additional parameters, this scenario maintains the idea of the common origin of the mass matrices of the charged lepton and neutrino sector. Even with the introduction of more parameters in the parallel case, not all possible theoretical parametrization can satisfy the updated experimental constraints.

**B. Numerical analysis of parallel texture between neutrino and charged lepton sector.**

In this section, we analyze the case where the mass matrix for the neutrino sector and charged lepton sector have a parallel 1-zero texture form. As was mentioned above, the combination of this textures leads to a mixing matrix with the 2-zero texture. This is the same as the quark sector case analysed before.

A complete analysis consist of finding the $8^2$ possibilities to parameterize the mixing matrix corresponding to the 8 possible solutions of the system of equations for the invariants. Nevertheless, in the parallel case where the same mass matrix texture is used in both sector, charged leptons and neutrinos, only 8 choices are possible. This is the case that we will study.

A general treatment of the problem starts by finding the PMNS mixing matrix $U_{\alpha i}^{\text{th}}$, diagonalizing the matrices $M_\ell M_\nu^\dagger$ and $M_\nu M_\ell^\dagger$ where $M_{(\ell, \nu)}$ have a 1-zero texture shape. In total, the mixing matrix elements are functions on 7 parameters, i.e. $A_{\ell, \nu}, E_{\ell, \nu}, m_{\nu_3}^2$ and the phases $\phi'_1, \phi'_2$. In order to describe the allowed regions in the parameter space, a $\chi^2$ function was defined as

$$\chi^2(\eta_1, \eta_2, \eta_3) = \sum_{i=1}^{3} \frac{[U_{\alpha i}^{\text{th}}(\eta_1, \eta_2, \eta_3) - U_{\alpha i}^{\text{exp}}]^2}{(\Delta U_{\alpha i}^{\text{exp}})^2} + \frac{[U_{\mu i}^{\text{th}}(\eta_1, \eta_2, \eta_3) - U_{\mu i}^{\text{exp}}]^2}{(\Delta U_{\mu i}^{\text{exp}})^2}. \quad (42)$$
Because the main goal of this work is to explore the common origin of the mass matrices of all sectors of the SM, we have restricted the analysis to the case of Normal Ordering of the fermion masses where $m_1^F < m_2^F < m_3^F$. At present, experimental results in the neutrino sector also allows us to consider Inverse Ordering of neutrino masses, with $m_1^\nu < m_2^\nu < m_3^\nu$. The IO of masses not only changes the sign of the $\Delta m_{21}^\nu$ observable but also the theoretical prediction for the $V_{PMNS}$ matrix elements. Although we aim to address this question in a later work, it can be mentioned that there is a relation between the NO and IO cases in the neutrino sector through $S_3$ transformations on the neutrino mass matrix.

The $2^{43}$ scenarios are characterized by the allowed range of values for the free parameters and in this section we have established the notation $E_{\nu,\ell} = e_{\nu,\ell}$ and $A_{\nu,\ell} = a_{\nu,\ell}$ to distinguish these parameters from those analysed in the quark case. In table V we show those intervals for the scenarios defined in the leptonic sector. Some intervals are contained into others, nevertheless it should be pointed out that in every scenario, the expressions for the theoretical mixing matrix elements differs by the combination of the $\eta_i$ parameters defined through equations (8-10). The masses of the neutrinos are defined by $m_1^\nu = |\lambda_1|$ and $m_2^\nu = |\lambda_2|$ using the equations (39) and (40), leaving the heaviest neutrino mass $m_{\nu_3}$ as a free parameter. This characteristic makes the parameter space of the leptonic sector quite different to the quark sector, where the range of values for the parameters are fixed by the known quarks masses.

| Scenario | Intervals |
|----------|-----------|
| 1:       | $-m_1^F < e_F < -m_2^F < a_F < -m_3^F$ |
| 2:       | $-m_3^F < e_F < -m_2^F < a_F < m_1^F$ |
| 3:       | $-m_3^F < e_F < -m_1^F < a_F < m_2^F$ |
| 4:       | $-m_2^F < e_F < m_1^F < m_2^F < a_F < m_3^F$ |
| 5:       | $-m_2^F < e_F < -m_2^F < a_F < m_3^F$ |
| 6:       | $-m_2^F < e_F < m_1^F < a_F < m_3^F$ |
| 7:       | $m_1^F < e_F < m_3^F < a_F < m_3^F$ |
| 8:       | $m_1^F < e_F < m_2^F < m_3^F < a_F < m_3^F$ |

Table V. Interval of values for $a_{\nu}$, $a_{\mu}$, $e_\nu$ and $e_\mu$ in different scenarios for the charged lepton and neutrino sectors. Has been taken Normal Order for the neutrino masses as the charged lepton masses, this is $m_1^F < m_2^F < m_3^F$ with $F = \ell, \nu$.

To select a region of allowed parameters, the $\chi^2$ function was compared to the number of non-correlated observables. In this analysis those were 4, i.e. $U_{e1}$, $U_{e2}$, $U_{\mu 3}$ and $U_{\mu 1}$. The interval of possible values for $a_{\nu}$ and $e_\nu$ depends on the observables $\Delta m_{32}^2$, $\Delta m_{21}^2$ and $m_{\nu_3}^\nu$.

As can be seen in the figures (12), some region have clear boundaries. The shape and extension of these regions are determined by the specific form of the function (42) and the interval for the parameters in the table V. Samples to generate points in all scenarios were calculated with the same statistical conditions, i.e., the number of points in the plot are proportional to the probability of having a combination of values compatible with the chosen observables. Thus the boundaries of the region represent the combination of restrictions for the intervals in table V with the experimental observables.

The main result of this analysis is that scenarios 1 and 6 are discarded because the restriction on free parameters can not generate $\chi^2$ values smaller than one. In figures 7 to 12, we present scatter plots of the parameter space for the experimentally allowed scenarios. The fitting used to find the data were taken from figure 4.

Figure 7 shows points that represent experimentally allowed parameter values for the scenario 2. It shows scaled parameters with respect to the heaviest fermion mass for every sector, and leads to similar values in a narrow region determined mainly by the restriction on $a_{\nu}/m_{\nu_3}$. As a consequence we can explore the possibility of a Universal Texture Constraint (UTC) as described in [17] for the Yukawa sector in extended scalar models, but in this case for the SM. The UTC was introduced in [27] to reduce drastically the number of free parameters in models with parallel textures.

As in scenario 2, it is also possible to find a matching region in the scenario 3 where the values of the scaled parameters can be equal. For this scenario 3, the matching region lies in the right hand side of the vertical line at $a_{\nu}/m_{\nu_3} = 0$ (see the plot in Figure 8). The matching region is thus $-0.5 \leq a_{\nu}/m_{\nu_3} \leq 0$ and positive values for $a_{\mu}/m_{\nu_3}$.

Similar observations can be made for the scenarios 4, 5, 7 and 8 shown in figures 9, 10, 11 and 12 respectively. Interestingly, in the scenario 8 (as seen in Figure 12), the allowed region for the neutrino sector is completely contained in the allowed region of the charged lepton sector. In the following section, we analyze this matching to support the possible presence of an UTC.

A possible common origin of the mass matrices in leptonic and quark sectors is not evident in the mixing matrix because the mixing angles are very different, and the masses span a wide range of order of magnitude.
Figure 7. In this plot it is shown the scaled scatter plot for $e_{\nu}/m_{\nu_{3}}$ vs $a_{\nu}/m_{\nu_{3}}$ Neutrino Sector (NS) in the scenario 2, with random values for the other parameters. The limits of the plot are determined by the values of the table $\nabla$ calculated for a random value of the heaviest neutrino mass fulfilling the experimental bounds. Also it is shown the projection on the space $e_{\ell}/m_{\tau}$ vs $a_{\ell}/m_{\tau}$ Leptonic Sector (LS). The interval of the parameters in the charged lepton sector are fixed and scaled with $m_{\tau}$. As can be seen in the plot, there is a narrow common region where the values for the LS are similar to the NS.

Figure 8. In this plot we show the behavior of the experimentally allowed values in the scenario 3 for the projections $e_{\nu}/m_{\nu_{3}}$ vs $a_{\nu}/m_{\nu_{3}}$ (NS) and $e_{\ell}/m_{\tau}$ vs $a_{\ell}/m_{\tau}$ (LS), using random values for the other parameters. The region corresponding to the parameters for the LS have a common region with the obtained for the NS. This region in on the positive values for the $a$’s parameters in both sector and the $-0.5 \lesssim \frac{\sqrt{\nu_{e}(0)}}{m_{\nu_{3}}m_{\tau}} \lesssim 0$ region.
Figure 9. Allowed points corresponding to the scenario 4 for $e_\nu$ vs $a_\nu$ using random values for the other parameters. The matching region between scaled parameters in the charged lepton and neutrino sector is below the horizontal line $e_\nu/m_\nu^3 = e_\ell/m_\tau = 0$.

Figure 10. Allowed points corresponding to the scenario 5 for $e_\nu$ vs $a_\nu$ using random values for the other parameters. The matching region between scaled parameters in the charged lepton and neutrino sector is in the right-hand side of the vertical line $a_\nu/m_\nu^3 = 0$ in the figure of the left.

If there is a relation between these sectors it could be better manifested in scaled parameters over the heaviest fermion masses, which we endeavor to study next.
VI. THE SCENARIO WITH POSITIVE PARAMETERS IN THE LEPTONIC SECTOR

In this section, we study the scenario where \( a_\nu \), \( a_\ell \), \( e_\nu \) and \( e_\ell \) are all positive to find the region of free parameters that can reproduce both the neutrino and the charged lepton texture compatible with the mixing angles and the CP phase. To use the same scheme as in the quark sector, we have only considered NO of the neutrino masses.

As can be seen in the Figure 2a, the results suggest a linear relationship between the scaled parameters \( A_d/m_b \) and \( A_u/m_t \). This is in the scenario when all parameters are positive, and for this reason we have chosen the same case but in the leptonic sector in order to look for similar behaviors in the parameters.

In order to determine the behavior of the regions of the \( a_\nu/m_{\nu_3} \) vs \( e_\nu/m_{\nu_3} \) parameter space in the leptonic sector
with respect to the heaviest neutrino mass, we selected the minimum point for the $\chi^2$ function. Then, analytical regions in the $a_\ell/m_\tau$ vs $e_\ell/m_\tau$ for several values of $m_{\nu_3}$ were calculated while using the values previously determined for other parameters. These results are summarized in figure 13 where it is noticed how with the increase of $m_{\nu_3}$, the interval of possible values for $a_\nu/m_{\nu_3}$ narrows down. On the other hand, the range for $e_\nu/m_{\nu_3}$ remains approximately constant. Increasing the values for $m_{\nu_3}$ out of the range shown in the plot reduces the allowed region until it disappears. An experimental measurement of the mass $m_{\nu_3}$ out of the range generating the allowed regions would clearly discard this scenario.

![Figure 13. Dependence on the heaviest mass neutrino $m_{\nu_3}$ of the allowed regions in the $a_\nu/m_{\nu_3}$ vs $e_\nu/m_{\nu_3}$ parameter space. This plot was generated using the values $a_\ell/m_\tau = 0.597$, $e_\ell/m_\tau = 0.05094$, $\phi_1 \simeq 1.86\pi$ and $\phi_2 \simeq 1.65\pi$](image)

Figure 13. Dependence on the heaviest mass neutrino $m_{\nu_3}$ of the allowed regions in the $a_\nu/m_{\nu_3}$ vs $e_\nu/m_{\nu_3}$ parameter space. This plot was generated using the values $a_\ell/m_\tau = 0.597$, $e_\ell/m_\tau = 0.05094$, $\phi_1 \simeq 1.86\pi$ and $\phi_2 \simeq 1.65\pi$

Using the same methodology, we analysed the behavior of the allowed regions in the phases $\phi_1$ vs $\phi_2$. The allowed region is very sensitive to the mass $m_{\nu_3}$ as presented in Figure 14.

![Figure 14. Allowed regions in the phase space obtained varying the mass $m_{\nu_3}$. To obtain this regions the parameters $a_\ell/m_\tau$, $e_\ell/m_\tau$, $a_\nu/m_{\nu_3}$ and $e_\nu/m_{\nu_3}$ were fixed by the $\chi^2$ minimum.](image)

Figure 14. Allowed regions in the phase space obtained varying the mass $m_{\nu_3}$. To obtain this regions the parameters $a_\ell/m_\tau$, $e_\ell/m_\tau$, $a_\nu/m_{\nu_3}$ and $e_\nu/m_{\nu_3}$ were fixed by the $\chi^2$ minimum.

Using the same methodology, we analysed the behavior of the allowed regions in the phases $\phi_1$ vs $\phi_2$. The allowed region is very sensitive to the mass $m_{\nu_3}$ as presented in Figure 14.

In order to evaluate the validity of the UTC in the parallel texture, in Figure 15 we present together the allowed regions using the condition $a_\ell/m_\tau = a_\nu/m_{\nu_3}$ and $e_\ell/m_\tau = e_\nu/m_{\nu_3}$ for different values of the heaviest neutrino mass.
The allowed region gets narrow and small when the heaviest neutrino mass approaches the experimental bound mentioned in section V A. This result suggests that the 1-zero texture with UTC makes the parallel texture parametrization a more predictive model for the leptonic sector. It is worth to mention that a radical application of the UTC can relate the lepton and quark sectors of the SM shedding light on an underlying flavor physics beyond SM.

**VII. CONCLUSIONS**

We have obtained exact expressions for the VCKM and PMNS matrix elements in terms of the free parameters in the scenario of a 1-zero parallel texture for all leptons and quarks.

Numerical work was done using a hybridized nature-inspired/cellular automata search algorithm CPSO-DE, supplemented with constraints. We studied the limit where some parameters go to zero, to compare with previous texture matrices used in the literature. This analysis concludes that narrowing experimental constraints are in tension with the diagonal plus 1-zero texture case studied in [38] and [27]. It also shows that the zero-parameter limit for type-down quarks, when the 1-zero texture reduces to the two-zero texture, is excluded.

Similar to the UTC discussed in [17], we find that leptons and quarks can be described by the same structure. Eight scenarios, corresponding to the eight possibilities in (42), were considered. In six of those, as detailed above, parameter values consistent with all experimental constraints were found, and the minima for the cost function reported; for two scenarios, the local minima found are excluded by this criteria.

Even though this model requires an enlarged parameter space, it provides in exchange a common framework for the description of leptons and quarks. Furthermore, the explored regions for the parameters describing the quark mass matrices suggest that, once scaled, some parameters for up-type and down-type quarks are approximately equal. A reduction on the number of parameters can be achieved by imposing some sort of universality constraint. For the positive–parameters case studied, there is a narrow region where the same values of the charged leptons and neutrinos parameters (with Dirac masses and Normal Ordering) provide reasonable agreement with experimental data. This is a more extreme version of a UTC, with a significant parametric reduction.
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