Implementation of rigorous mathematical thinking approach to analyze the students’ ability of algebraic thinking and understanding concept and mathematical habits of mind

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Abstract. This study aims to analyze the students’ algebraic thinking skills and understanding of mathematical concept and Mathematical Habits of Mind (MHoM) students using Rigorous Mathematical Thinking (RMT) approach. To do so, we carried out three phases of a small-scale study. First, we analyzed three matrix problems, considered as tasks inviting the use of problem solving and algebraic ability and theoretically in the light of the RMT principles. Second, we tested problems to 54 students of mathematics education major of Indonesia University of Education in academic year 2016/2017. Finally we analyzed student written work, questionnaire of MHoM and compared these empirical to the theoretical results. We found that students who learn by using rigorous mathematical thinking have algebraic thinking skills and good conceptual understanding. In addition, students also have good mathematical habits of mind. We concluded that RMT approach fruitfull framework to increase students’ algebraic thinking skills and understanding of mathematical concept and MHoM students of Indonesia University of Education.

1. Introduction
Mathematics is an important basis in the development of science because mathematics acts as a logical and systematic framework. Mathematical learning is expected to be able to practice one’s reasoning and thinking skills so that they can be applied in solving everyday problems. The essential and fundamental elements needed in mathematical reasoning and systematic thinking are algebraic thinking skills and concepts deferral abilities [1].

Algebraic thinking ability is the ability of students to create, use and complete mathematical models of everyday life problems [2]. The indicators used to measure students’ algebraic thinking skills in this study are the basic concept of calculating strategies, ratios and proportions, using equivalent symbolic representation to manipulate formulas, expressions, equations, and inequalities, and generalize patterns in real world context [3, 4].

Understanding of mathematical concepts is the ability possessed by someone in order to grasp the meaning and understand correctly about the main idea or understanding and can reveal the knowledge gained about mathematics that is being studied both orally and in writing [5, 6]. The indicators of the ability to understand concepts are: (1) restating a concept, (2) classifying objects according to certain characteristic according to their nature, (3) giving examples and not example of concepts, (4) presenting concepts in various forms of mathematical representation, (5) developing the necessary re quirments or
sufficient requirements of a concept, (6) using and utilizing anf selecting certain procedures or operations, and (7) applying concepts or algorithms in problem solving [7].

Based on the results of the preliminary study, it is seen that students still experience difficulties in using a mathematical concept in solving problems related to algebraic thinking skills and understanding of concept. Difficulties in using a mathematical concept in algebraic thinking skills and concept understanding go hand in hand with habits of mind so that it strongly supports students’ performance in developing their thinking abilities [8]. Habits of mind implies that behavior requires a disciplined mind that is trained in such a way that it becomes a habit to keep trying to do more wise and intelligent actions [9].

Concept formation can develop students’ thinking habits and the tendency of mathematical theoretical thinking and metacognition by using the Rigorous Mathematical Thinking (RMT) approach [10]. The results of Zakiah study showed that Mathematical Habits of Mind (MHoM) students have not developed optimally. Pre-response results differ greatly from post-response results after students are given learning with an open ended approach. This shows that MHoM students can be developed with appropriate treatment [11]. The results of Nurcahyo also showed that the achievement of MHoM students has not shown maximum development [12].

RMT is mental activity of students who aim to formulate and solve problems, make decisions, seek understanding in a rigorous complete, accurate and valid manner that has systematic determination or certainty [13]. RMT can help improve one’s reflection on the patterns and relationships that is one of the indicators in the algebraic thinking ability, as well as habits of thinking.

Based on the results of previous studies, research examining algebraic thinking skills and understanding of concepts and capabilities with RMT approach has never been done. This has the potential to be studied further and will become a new insight into knowledge. The purpose of this study was to analyze algebraic thinking skills and understanding of mathematical concepts and MHoM students by using RMT approach.

2. Method
To analyzed matrix tasks, promoting problem solving and assessing the use of various mathematical concepts, we carried out three phases of a small-scale qualitative study involving 54 students of mathematics education major in Indonesia University of Education in academic year 2016/2017. First, we selected and adapted three matrix tasks, inviting the use of problem solving and algebraic ability and questionnaire MHoM are used as observation.

The three tasks are presented in Table 1. Tasks 1 are intended to calculate and present concept in various forms of mathematical representations. Tasks 2 are intended to uses an equivalent symbolic representation to manipulate formulas and restate a concept of inverse matrix. Tasks 2 are intended to calculate and apply concepts in solving everyday problems.

| No | Matrix tasks |
|----|--------------|
| 1 | The matrix $P = \begin{bmatrix} x & 4 \\ 5 & 2x \end{bmatrix}$ and $Q = \begin{bmatrix} 5 & 2 \\ -2 & x \end{bmatrix}$ are $2 \times 2$ order matrices as below. In order for the determinant of the matrix $P$ to be equal to twice the determinant of $Q$, the value of $x$ the meet is… |
| 2 | Known matrices $A = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 16 & 8 \\ 2 & 2 \end{bmatrix}$. Determine a $2 \times 2$ order $X$ matrix that satisfies the equation $AX = B$ |
| 3 | A mother will make 2 types of cakea. The ingredients for making cakes are prepared, that is 4 kg of flour and 3 kg of sugar. Type A cakes require 150 grams of flour and 50 grams of sugar, while type B cakes require 100 grams of flour and 100 grams of sugar. Determine how many type A cakes and type B cakes can be made with available ingredients? |

Second, we tested Tasks 1, 2 and 3 to students of mathematics education major in Indonesia University of Education in academic year 2016/2017. The time given to complete everything that is 90 minutes.
Finally, we analyzed student written work and compared these empirical to theoretical expected result. The analysis included analyzing students’ ability of identifying the use of various related concepts in the solution process in the light of RMT principles.

3. Result and Discussion

This section present result of the data analysis of the three tasks: Tasks 1, 2 and 3 assessing student problem solving skills of matrix material. We found that the three tasks are difficult for most of participated students. Of the 28 students of A class, four students solved Task 1, six students solved Task 2 and two students solved Task 3 correctly. Similarly, of the 26 students of B class, two students solved Task 1, four students solved Task 2 and one students solved Task 3 correctly. The results showed that both class of students lack of ability in dealing with matrix problems.

Figure 1 shows an example of a student’s answer to Task 1. On Task 1, this solution process requires students to apply the concept of determinant of the matrix \( |P| \) to solve this determinant to obtain \( 2x^2 - 20 = 2(5x + 4) \) and to obtain \( x^2 - 5x + 4 = 0 \), but this is not the final solution for Task 1. Students need to find the root of the quadratic equation from \( x^2 - 5x + 4 = 0 \), so get the final result of Task 1 that is \( x = 7 \) or \( x = -2 \). We found that student inabilities to produce correct solution process is caused students know well the concept used in problem solving. In this problem, in addition to the concept of matrix determinant, there are other concept that needs to be mastered by students that is the concept in the root search rules of the quadratic equations. As seen in Figure 1, students are able to complete the steps that are in accordance with the concepts used, so that students get the final results according to the question asked. Based on interviews and questionnaire, in the process of this question the students did not experience any difficulties related to the concept and calculation performed.

![Figure 1. Student’s answer to Task 1.](image)

Figure 2 shows an example of a student’s answer to Task 2. On Task 2, this solution process requires students to apply the concept of inverse matrix \( A^{-1} \) to solve this inverse to obtain \( X = B \) and to obtain \( X = \begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix} \). We found that student inabilities to produce correct solution process is caused students understand the concept that will be used and are able to develop them. Although at first the students seems hesitant about the process, they were able to develop an appropriate concept. In addition, the students also have done the overall calculation to obtain the final result that is in accordance with the desired in the problem. Based on interviews and questionnaire obtained that students initially forget that there are other concepts that can help solve the problem, but eventually recall that the concept of matrix identity is also needed in obtaining the final results of the problem.
Figure 2. Student’s answer to Task 2.

Figure 3 shows an example of a student’s answer to Task 3. On Task 3, this solution process requires students to apply the concept of a linear system of two variables. Let cake type A as x and cake type B as y, we get two linear system of two variables that is 150x + 100y = 4000 and 50x + 100y = 3000 then simplify this system to 3x + 2y = 80 and x + 2y = 60. Apply the concept of inverse matrix to solve this inverse to obtain X = B.A⁻¹ and to obtain X = \[
\begin{bmatrix}
3 & 2 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
80 \\
60
\end{bmatrix}
\]
so get the final result of Task 3 that is X = \[
\begin{bmatrix}
10 \\
25
\end{bmatrix}
\]. From Task 3 we can conclude that there are 10 type A cakes and there are 25 type B cakes can be made with available ingredients. We found that students made a mistake in using the matrix concept to solve the equation system of two variables in the formulation and problem solving procedures. Students make mistakes in implementing strategies and procedures that cause students not to get the final result asked on the question. In addition, lack of accuracy causes students to make errors in calculating. Insufficient understanding of the equation system of two variables is also the cause of errors in calculating. This is supported by the results of student interviews and questionnaires that in the process of working on questions, students forget the concept of the equation system of two variables and the concept of solving the system of equations of two variables using a matrix. Furthermore, in conducting an example for the variables that are in the problem, students make mistakes in that section, so they cannot reach the final results asked for the problem.

Figure 3. Student’s answer to Task 3.

Based on the results of the analysis of the problems above, another factor that is no less important in influencing students in solving problems related to the matrix is their understanding of the material preconditions. The weak mastery of the prerequisites material such as the concept in the rules of the search for the root of quadratic equation and the use of a matrix to solve the equation system of two variables can cause students to make prerequisites errors. In this study, students did not master the concept of using a matrix to solve the equation system of two variables, so that students made mistakes in solving the given problem.

Based on the results of the questionnaire analysis, students have familiarized themselves to ask themselves about the correct or not the answers they gave and have tried to use appropriate methods to
work on the given mathematical problems. However, students still find it difficult to get used to re-learning math topics. In reality, to get a better understanding of mathematics, we must get used to repeating mathematical topics that are considered difficult. Furthermore, students have tried to connect concepts to the newly learned material with concepts that exist in new material. In the process of working on mathematical questions, students have begun to conduct the metacognition in themselves. This can be seen from the results of this questionnaire analysis that when solving a problem, students read the question repeatedly to better understand the problem given. Then students try to solve problems in various ways, so think of other ways that can provide solution to these problems. However, students tend to avoid when asked to give reasons for each answer given. This happens because students only focused on the material provide and did not try to deepen their knowledge about the material. As a result, students difficulty in explaining their reasons for each settlement.

4. Conclusion
From the result and discussion section, we draw the following two conclusions. First, both the reality and RMT principles can be applied to analyze matrix tasks that promote student problem solving and algebraic ability. The RMT principle is applied in the analysis to explain mental activity of students who aim to formulate and solve problems, make decisions, seek understanding in a rigorous complete, accurate and valid manner that has systematic determination or certainty (see [11]). In this case matrix context, to mathematical and algebraic models; from the algebraic models to solution; and from solutions to the initial context of matrix problems. In other word, the RMT principles is able to develop students thinking habits and the tendency of mathematical theoretical thinking and metacognition (see [10]).

Second, student difficulties revealed in this study, including difficulties in mathematization and in relating various mathematical concepts, can be identified by applying RMT and mathematical concept principles. In our view, these student show that have algebraic thinking skills and good conceptual understanding.

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