A SAT Approach to Twin-Width*

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Abstract

The graph invariant twin-width was recently introduced by Bonnet, Kim, Thomassé, and Watrigan. Problems expressible in first-order logic, which includes many prominent NP-hard problems, are tractable on graphs of bounded twin-width if a certificate for the twin-width bound is provided as an input. Computing such a certificate, however, is an intrinsic problem, for which no nontrivial algorithm is known.

In this paper, we propose the first practical approach for computing the twin-width of graphs together with the corresponding certificate. We propose efficient SAT-encodings that rely on a characterization of twin-width based on elimination sequences. This allows us to determine the twin-width of many famous graphs with previously unknown twin-width. We utilize our encodings to identify the smallest graphs for a given twin-width bound \( d \in \{1, \ldots, 4\} \).

1 Introduction

Twin-width is a new graph invariant that was recently introduced by Bonnet et al. [2, 3, 4], inspired by previous work by Guillemot and Marx [13]. Graph classes of bounded twin-width admit the fixed-parameter tractability of First-Order (FO) model checking, parameterized by the length of the FO formula, provided a witness for bounded twin-width is given. Many NP-hard problems such as “does the input graph contain an independent set of size at least \( r \)?” or “does the input graph contain a subgraph that is isomorphic to a fixed pattern graph \( H \)?” can be naturally expressed as FO model checking. Graph classes of bounded twin-width subsume and generalize several dense graph classes for which FO model checking is fixed-parameter tractable, including map graphs, bounded rank-width graphs, bounded clique-width graphs, cographs, and unit interval graphs. Thus, twin-width boundedness plays a similar role for dense graph classes as nowhere density plays for sparse graph classes [12].

Bonnet et al.’s [4] FO model checking algorithm for graphs of bounded twin-width requires a certificate that the input graph’s twin-width is bounded by a constant \( d \). The most pressing open theoretical question regarding twin-width concerns the complexity of computing such a certificate, and more generally, recognize graphs of twin-width \( \leq d \) [4]. There are no practical algorithms known to compute the twin-width of a graph exactly or approximately.

1.1 Contribution

In this paper, we take a SAT-based approach to the exact computation of twin-width. We thereby utilize the power of SAT solving (solving the propositional satisfiability problem SAT) for a combinatorial problem, continuing a compelling and successful line of research [6, 7, 15, 16, 17, 21, 24]. As a result, we can identify the exact twin-width of many graphs for which the twin-width was previously unknown.

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More specifically, we propose two SAT encodings that take a graph $G$ and an integer $d$ as input, and produce a propositional CNF formula $F(G, d)$, which is satisfiable if and only if the twin-width of $G$ is at most $d$. By running a SAT-solver on $F(G, d)$ for different values of $d$, we can determine the exact twin-width of $G$. We propose methods for computing lower and upper bounds for $d$ that allow us to reduce the interval of possible values of $d$ for running the SAT solver on. Both encodings are based on a new characterization of twin-width in terms of elimination orderings, which are somewhat related to SAT encodings used for other width measures [10, 22, 23]. However, for twin-width, the situation is more involved, because it is not sufficient to globally bound certain static values (like out-degrees in an elimination ordering for treewidth [22]).

We demonstrate the potentials and limits of our encodings by utilizing them in the following three computational experiments.

1. **Twin-width of small Random Graphs.** We determine experimentally how the twin-width of a random graph depends on its density. As one expects, the twin-width is small for dense and sparse graphs. Graphs of edge-probability 0.5 have the highest twin-width.

2. **Twin-width of Famous Named Graphs.** Over many decades of research in combinatorics, researchers have collected several special graphs, which have been used as counterexamples for conjectures or for showing the tightness of combinatorial results. We considered several of such special graphs from the literature and computed their exact twin-width. We believe that these results will be of interest to people working in combinatorics. This way, we have identified a certain class of strongly regular graphs (Paley graphs) that provide high lower bounds for twin-width.

3. **Twin-Width Numbers.** In general, it is not known how many vertices are required to form a graph of a certain twin-width. In fact, there is limited knowledge on lower-bound techniques for twin-width. We use our SAT encoding together with a graph generator to identify the smallest graphs of twin-width 1, 2, 3, and 4, respectively, and provide tight bounds for twin-width 5 and 6. This way, we can determine the first few twin-width numbers, where the $d$-th twin-width number is the smallest number of vertices of a graph with twin-width $d$. A similar computation has been conducted for clique-width [15]. Interestingly, up to isomorphism, there are unique smallest graphs of twin-width 1, 2, and 4, respectively, and there are five such graphs for twin-width 3.

### 2 Twin-width

A **trigraph** is an undirected graph $G$ with vertex set $V(G)$ whose edge set $E(G)$ is partitioned into a set $B(G)$ of black edges and a set $R(G)$ of red edges. We consider an ordinary graph as a trigraph with all its edges being black. The set $N_G(v)$ of neighbors of a vertex $v$ in a trigraph $G$ consists of all the vertices adjacent to $v$ by a black or red edge. We call $u \in N_G(v)$ a **black neighbor** of $v$ if $uv \in B(G)$ and we call it a **red neighbor** if $uv \in R(G)$. The red degree of a vertex $v \in V(G)$ of a trigraph $G$ is the number of its red neighbors. A $d$-trigraph is a trigraph where each vertex has red degree at most $d$.

#### 2.1 Twin-Width via Sequences of $d$-Contractions

We give the original definition of twin-width [2, 3, 4].

A trigraph $G'$ is obtained from a trigraph $G$ by **contraction**: two (not-necessarily adjacent) vertices $u$ and $v$ are merged into a single vertex $w$, and the edges of $G$ are updated as follows: Every vertex in the symmetric difference $N_G(u) \Delta N_G(v)$ is made a red neighbor of $w$. If a vertex $x \in N_G(u) \cap N_G(v)$ is a black neighbor of both $u$ and $v$, then $w$ is made a black neighbor of $x$; otherwise, $w$ is made a red neighbor of $x$. The other edges (not incident with $u$ or $v$) remain unchanged.

A **sequence of $d$-contractions** or **$d$-sequence** for a graph $G$ is a sequence of $d$-trigraphs $G_0, G_1, \ldots, G_n$ where $G_0 = G$, $G_{n-1}$ is the graph on a single vertex, and $G_i$ for $i \geq 1$ is obtained
from $G_{i-1}$ by contraction. We observe that $|V(G_i)| = n - i$ for $0 \leq i < n = |V(G)|$. The twin-width of a trigraph $G$, denoted $tww(G)$, is the smallest integer $d$ such that $G$ admits a $d$-sequence.

It is indeed sometimes necessary to contract non-adjacent vertices. For instance, Figure 1 shows a sequence of $2$-contractions for the Wagner graph. Without contracting non-adjacent vertices, a vertex of red degree $> 2$ would be created by the first contraction since each vertex has degree 3 and shares no neighbor with any of its neighbors.

![Figure 1: A sequence of 2-contractions for the Wagner graph. Vertices that will be contracted next are marked blue.](image)

We state here some basic properties of twin-width, observed in the original paper [4].

**Fact 2.1.** If $G'$ is an induced subgraph of a graph $G$, then $tww(G') \leq tww(G)$.

For a graph $G$, we denote by $\overline{G}$ its complement graph, which is defined by $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{ uv \mid u, v \in V(G), uv \notin E(G), u \neq v \}$.

**Fact 2.2.** For every graph $G$, we have $tww(G) = tww(\overline{G})$.

### 2.2 Twin-Width via $d$-Elimination Sequences

Next we give an alternative definition of twin-width which is better suited for formulating our SAT encodings.

Let $G$ be a graph, $T$ a tree with $V(T) = V(G)$, rooted at some vertex $r_T$, and $\prec$ a linear ordering of $V(T)$, where $u \prec v$ for two vertices $u, v \in V(T)$ such that $v$ is the parent of $u$ in $T$. We call $T$ a contraction tree, $\prec$ an elimination ordering, and the pair $(T, \prec)$ a twin-width decomposition of $G$. Thus, when we write $V(G) = \{v_1, \ldots, v_n\}$ such that $v_1 \prec \cdots \prec v_n$ and $v_n = r_T$, then $T$ and $G$ define a sequence of graphs $H_0, \ldots, H_{n-1}$ with $V(H_i) = \{v_{i+1}, \ldots, v_n\}$. We denote by $p_i$ the parent of $v_i$ in $T$. By definition, $v_i \prec p_i$.

We define the edge set $E(H_i)$ recursively as follows. For $i = 0$, we set $E(H_0) = \emptyset$, and for $1 \leq i < n$, we set

$$E(H_i) = \{ uv \in E(H_{i-1}) \mid u, v \in V(H_i) \} \cup \{ up_i \mid p_i u \notin E(G), u \in V(H_i) \} \cup \{ uv \in E(H_{i-1}) \mid u \notin V(H_i) \} \cup \{ up_i \mid v u \notin E(G), p_i u \in E(G), u \in V(H_i) \}.$$  \hspace{1cm} (1a)  

We call the sequence $H_0, \ldots, H_{n-1}$ the elimination sequence for $G$ defined by the twin-width decomposition $(T, \prec)$; if for an integer $d$, all the $H_i$ have a maximum degree $\leq d$, we call $H_0, \ldots, H_{n-1}$ a
A *d*-elimination sequence. The *width* of the twin-width decomposition \((T, \prec)\) of \(G\) is the smallest integer \(d\) such that \((T, \prec)\) defines a *d*-elimination sequence.

Figure 2 shows an example of a 2-elimination sequence, and in Figure 3 the same elimination sequence is superimposed on the graph.

**Theorem 2.1.** Let \(G\) be a graph and \(<\) an arbitrary linear ordering of \(V(G)\). \(G\) has twin-width \(\leq d\) if and only if there exists a twin-width decomposition \((T, \prec)\) of width \(\leq d\) such that

1. if \(x\) is the parent of \(y\) in \(T\), then \(x < y\);
2. the root of \(T\) is the \(<\)-maximal element of \(V(G)\).

**Proof.** Let \(G\) be a graph and assume that \(\text{tww}(G) \leq d\). By definition, there exists a \(d\)-sequence \(G_0, G_1, \ldots, G_{n-1}\), and each \(G_i\), \(i > 0\), is obtained from \(G_{i-1}\) by contracting two vertices \(u_i\) and \(v_i\), i.e., merging them into \(w_i\), a new vertex. We slightly change contraction steps. Instead of introducing a new vertex \(w_i\), we reuse one of the two vertices \(u_i, v_i\) as \(w_i\). We use the ordering \(<\) to decide which of the two vertices to reuse:

\[
 w_i = \begin{cases} 
 u_i & \text{if } u_i > v_i, \\
 v_i & \text{otherwise}. 
\end{cases}
\]  

(2)

This way, we obtain a sequence \(G'_0, G'_1, \ldots, G'_{n-1}\), with \(V(G'_i) \subseteq V(G)\), where each \(G'_i\) is isomorphic to \(G_i\). Since \(V(G) = V(G'_0) \supseteq \cdots \supseteq V(G'_{n-1})\), this gives us a linear ordering \(<\) of \(V(G)\) in a natural way. We obtain a contraction tree \(T\) by taking \(V(T) = V(G)\) and \(E(T) = \{ u_i v_i \mid 1 \leq i \leq n - 1 \}\). Because of (2), the contraction tree satisfies the two conditions claimed in the statement of the theorem. A \(d\)-elimination sequence \(H_0, \ldots, H_{n-1}\) is provided by taking \(H_i\) as the subgraph of \(G'_i\) formed by its red edges. Thus \((T, \prec)\) is a twin-width decomposition of \(G\) of width \(\leq d\).
Conversely, assume \((T, \prec)\) is a twin-width decomposition of \(G\) of width \(\leq d\). Let \(H_0, \ldots, H_{n-1}\) be the corresponding \(d\)-elimination sequence. We turn the \(d\)-elimination sequence into a \(d\)-sequence by contracting pairs of vertices as indicated by \(T\). Hence \(\text{tww}(G) \leq d\). 

\[ \square \]

3 Preprocessing

In this section, we show how to decompose a given graph \(G\) in polynomial time into a collection \(\text{prime}(G)\) of induced subgraphs of \(G\), such that \(\text{tww}(G) = \max_{H \in \text{prime}(G)} \text{tww}(H)\). This decomposition can serve as a preprocessing step for twin-width computation.

We require some definitions. A \textit{module} of a graph \(G\) is a nonempty set \(M \subseteq V(G)\) such that for any \(x, y \in M\) and \(z \in V(G) \setminus M\) we have \(xz \in E(G)\) if and only if \(yz \in E(G)\). A module \(M\) is \textit{trivial} if \(M = V(G)\) or \(|M| = 1\). \(M\) is a \textit{maximal module} if it is not strictly contained in any nontrivial module. A graph is \textit{prime} if all its maximal modules are trivial. For every graph \(G\), there exists a unique partition \(P_{\max}\) of \(V(G)\) into maximal modules \(M_1, \ldots, M_s\), and this partition can be found in linear time \([8, 19]\). This partition gives rise to the quotient graph \(G/P_{\max}\) whose vertices are the maximal modules of \(P\), and where two modules \(M_i, M_j\), \(i \neq j\), are joint by an edge if and only if all the pairs \(x_i \in M_i\), \(x_j \in M_j\) are joined by an edge in \(G\). If we select for each module \(M_i\) a representative vertex \(x_i \in M_i\), then the set \(\{x_1, \ldots, x_s\}\) of representatives induces a subgraph of \(G\) that is isomorphic to \(G/P_{\max}\). If \(G\) and its complement graph \(\overline{G}\) are connected, then \(G/P_{\max}\) is a prime graph \([2, 14]\). We recursively define the set \(\text{prime}(G)\) as follows:

1. If \(G\) is disconnected, then \(\text{prime}(G)\) is the union of the sets \(\text{prime}(C)\) for all connected components \(C\) of \(G\).
2. If \(\overline{G}\) is disconnected, then \(\text{prime}(G)\) is the union of the sets \(\text{prime}(\overline{C})\) for all connected components \(C\) of \(\overline{G}\).
3. If both \(G\) and \(\overline{G}\) are connected, then \(\text{prime}(G)\) is the union of \(\{G/P_{\max}\}\) and the sets \(\text{prime}(G[M])\) for all nontrivial \(M \in P_{\max}\).

The three cases above give rise to the \textit{modular decomposition} of the graph \(G\), represented as a rooted tree \([14]\). The root of the tree is associated with \(G\), the children of each vertex are associated with the connected components (cases 1 and 2), or the maximal modules (case 3) of the graph associated with their parent. The leaves of the tree are in a 1-to-1 correspondence with the vertices of \(G\).

**Theorem 3.1.** For every graph \(G\) we have \(\text{tww}(G) = \max_{P \in \text{prime}(G)} \text{tww}(P)\).

*Proof.* Let \(d = \max_{P \in \text{prime}(G)} \text{tww}(P)\). As observed above, \(G/P_{\max}\) is isomorphic to an induced subgraph of \(G\); by induction, this holds for all the graphs in \(\text{prime}(G)\). Because of Fact \([2,1]\) \(\text{tww}(G) \geq d\) follows.

For showing \(\text{tww}(P) \leq d\), we proceed by induction on \(|V(G)| = n\). The statement is certainly true if \(n = 1\), since then \(\text{prime}(G) = \{G\}\). Now assume \(n > 1\). We distinguish several cases.

Consider the case where \(G\) is disconnected into components \(C_1, \ldots, C_r\). For each \(1 \leq i \leq r\) we have \(\text{prime}(C_i) \subseteq \text{prime}(G)\), and so, by induction, we have \(\text{tww}(C_i) \leq \max_{P \in \text{prime}(C_i)} \text{tww}(P) \leq d\). Thus, for each \(C_i\) there is a \(d\)-sequence ending in a single-vertex graph. Using the contractions of these \(d\)-sequences we obtain a \(d\)-sequence for \(G\), which ends in an edgeless graph that consists of \(r\) isolated vertices. We can extend this \(d\)-sequence by contracting the isolated vertices pairwise in any order, obtaining eventually a single-vertex graph, without generating any red edges. Thus \(\text{tww}(G) \leq d\). The case where \(G\) is disconnected follows from the previous argument and Fact \([2,2]\).

Finally, assume that \(G\) and \(\overline{G}\) are connected. Thus \(G/P_{\max}\) is prime and is isomorphic to an induced subgraph \(G' \in \text{prime}(G)\) of \(G\). For each \(M \in P_{\max}\), \(\text{prime}(G[M]) \subseteq \text{prime}(G)\). By induction hypothesis, \(\text{tww}(G') \leq d\) and \(\text{tww}(G[M]) \leq d\). We thus obtain a \(d\)-sequence for \(G\) by putting together \(d\)-sequences for \(G[M], M \in P_{\max}\), and a \(d\)-sequence for \(G'\), which contract first each \(G[M]\) on a single vertex of \(G'\), and then contract \(G'\) on a single vertex. Hence \(\text{tww}(G) \leq d\). 

\[ \square \]
In this section, we present two SAT encodings for twin-width. Assume, we are given a graph \( G \) and determine the twin-width of all the graphs in \( \text{prime}(G) \).

4 SAT Encodings

In this section, we present two SAT encodings for twin-width. Assume, we are given a graph \( G \) with vertices \( v_1, \ldots, v_n \) and an integer \( d \). We will define a propositional formula \( F(G, d) \) in Conjunctive Normal Form (CNF) that is satisfiable if and only if \( \text{tww}(G) \leq d \). For the construction of \( F(G, d) \), we use the characterization of twin-width in terms of a twin-width decomposition \( (T, \prec) \), as established in Theorem 2.1. We use the indices \( 1 \leq i, j, k \leq n \) and subsequently omit the upper and lower bounds for readability. Furthermore, we use the mapping \( \varphi(v_i) \) to denote the position of \( v_i \) in \( \prec \). We give two different encodings for \( F(G, d) \).

### 4.1 Relative Encoding

In our first encoding, we use a relative ordering of the vertices, as used in the treewidth encoding by Samer and Veith [22]: instead of encoding \( \varphi(v_i) \) directly, we encode for vertices \( v_i, v_j \in V(G) \), whether \( \varphi(v_i) \prec \varphi(v_j) \) or not. Table 1 shows the variables utilized in the encoding. For the ordering, we use \( \binom{n}{2} \) variables \( o_{i,j} \) with \( i < j \), where \( o_{i,j} \) is true if and only if \( v_i \prec v_j \). We subsequently use the shorthand \( o_{i,j}^* \), where \( o_{i,j}^* \) is \( o_{i,j} \) if \( i < j \) and \( \neg o_{j,i} \) if \( i > j \). We encode the semantics by enforcing transitivity: for mutually distinct \( i, j, k \) we add the clauses

\[ \neg o_{i,j}^* \lor \neg o_{j,k}^* \lor o_{i,k}^* . \]

Next, we encode the contraction tree \( T \). In view of Theorem 2.1, we can assume that when \( p_i \) is the parent of \( p_j \) in \( T \), then \( i < j \) (Condition 1), and \( v_n \) is the root of \( T \) (Condition 2). Hence, we can use \( \binom{n}{2} \) variables \( p_{i,j} \) with \( i < j \), where \( p_{i,j} \) is true if and only if \( p_i = v_j \). We encode that every vertex, except the root, has exactly one parent. For that, we utilize at-least-one constraints by adding for each \( i < n \) the clause \( \bigvee_{i < j} p_{i,j} \) and at-most-one constraints by adding for mutually distinct \( i, j, k \) the clause \( \neg p_{i,j} \lor \neg p_{i,k} \). Additionally, we ensure that \( v_i \prec v_j \) holds between a vertex \( v_i \) and its parent \( v_j \), by adding for \( i < j \) the clauses

\[ \neg p_{i,j} \lor o_{i,j}^* . \]

So far we have encoded \( \prec \) and \( T \). Next, we encode the elimination sequence \( H_0, \ldots, H_n \) with two additional sets of variables. We take \( n \binom{n}{2} \) variables \( r_{i,j,k} \) with \( j < k \), where \( r_{i,j,k} \) is true if and only if after eliminating \( v_i \) it holds that \( v_j v_k \in E(H_{\varphi(v_i)}) \). We also use \( \binom{n}{2} \) auxiliary variables \( a_{i,j} \) with \( i < j \), where \( a_{i,j} \) is true if and only if there exists a \( k \) such that \( v_i v_j \in E(H_k) \). We use shorthands \( a^* \) and \( r^* \) which are defined analogously to \( o^* \).
We encode the semantics of $a$ by adding, for all mutually distinct $i, j, k, i < j$, the clause
\[-o_{i,j}^* \lor -o_{i,k}^* \lor -r_{i,j,k}^* \lor a_{j,k}^*.
\]
Furthermore, we encode the semantics of $r$ by encoding Subsets (1a)–(1d) of $E(H_i)$ according to the definition given in Section 2. Subsets (1c) and (1d) are encoded by adding for $i < j$ and $v_k \in (NC(u_i) \triangle NC(v_j)) \setminus \{u_i, v_j\}$ the clause
\[-p_{i,j} \lor -o_{i,k}^* \lor r_{i,j,k}^*.
\]
Further, Subset (1b) is encoded by adding, for mutually distinct $i, j, k, i < j$, the clause
\[-p_{i,j} \lor -o_{i,k}^* \lor -a_{i,k}^* \lor r_{i,j,k}^*.
\]
Finally, we encode Subset (1a) by adding for mutually distinct $i, j, k, m, k < m$ the clause
\[-o_{i,j}^* \lor -o_{j,k}^* \lor -o_{i,m}^* \lor -r_{i,k,m}^* \lor r_{j,k,m}^*.
\]
The $O(n^4)$ clauses required to encode the Subset (1a) dominate the size of the encoding. Unfortunately, this is unavoidable: without knowing $\varphi_o(\cdot)$, we have $O(n^2)$ possible orderings of $v_i, v_j$, and for each such ordering we have $O(n^2)$ possible edges $v_i, v_m$.

We enforce the upper bound $d$ by using cardinality constraints: sets of clauses that encode the less-than constraints with the help of auxiliary variables. For each pair $v_i, v_j$ of vertices, we limit the set $\{r_{i,j,k}^* \mid 1 \leq i, j, k \leq n\}$ to at most $d$ true values. Therefore, $v_j$ has at most $d$ neighbors in $H_{\varphi_o(v_i)}$. We achieve this by using the totalizer cardinality constraints, as they perform well with our encoding.

Since the construction of $F(G, d)$ closely follows the definitions given in Section 2, we have the following result.

\textbf{Theorem 4.1.} Given a graph $G$ with $n$ vertices and an integer $d$, we can construct in time polynomial in $n + d$ a propositional formula $F(G, d)$ which is satisfiable if and only if $tww(G) \leq d$.

### 4.2 Absolute Encoding

We can reduce the number of clauses from $O(n^4)$ to $O(n^3)$ by directly encoding the absolute position of each vertex in $\prec$. We first give the general idea behind the adapted encoding and then compare the two encodings.

We use $n(n-1)$ variables $o_{i,j}^*$, where $o_{i,j}^*$ is true if and only if $\varphi_o(v_j) = i$. We encode the semantics of these variables by assigning each vertex exactly one position that is unique among all vertices. With this modification, the indices $i, j, k, m$ refer to positions $\varphi_o(v_i), \varphi_o(v_j), \varphi_o(v_k), \varphi_o(v_m)$, respectively, rather than the indices of $v_i, v_j, v_k, v_m$. Therefore, the semantics of $r_{i,j,k}$ changes, and $r_{i,j,k}$ is true if and only if there exists an edge $uv \in E(H_i)$ such that $j = \varphi_o(u)$ and $k = \varphi_o(v)$.

The main advantage of this modification is that $E(H_i)$ can be succinctly expressed as $\neg r_{i-1,j,k} \lor r_{i,j,k}$, for $i > 1$. We also need fewer variables for $r$: since the vertex at position $i$ is eliminated before the vertex at position $j$, for $r_{i,j,k}$ it suffices to use indices in the range $i < j < k$. Finally, we only need to consider the graphs $H_1, \ldots, H_{n-d}$, as a graph with $d$ vertices cannot have a twin-width higher than $d$. This significantly reduces the number of variables and clauses.

### 4.3 Comparison

The absolute encoding’s reduced size in comparison to the relative encoding comes with the prize of making it more intricate to encode the various required properties. Most obviously, the encoding of the ordering with the variables $o_{i,j}^*$ is more complex than the encoding of the ordering with the variables $o_{i,j}$. Even more impeding is the impossibility of succinctly encoding that the parent of a vertex is lexicographically larger than the vertex itself. Without this, we are left with many symmetries in the absolute encoding,
which unnecessarily increases the search space. Encoding the edges is also considerably more intricate in
the absolute encoding: since we do not know the value of \( \varphi(v_i) \) in advance, we have to encode for each
dge \( v_i v_j \in E(G) \) that there is an edge from \( \varphi(v_i) \) to \( \varphi(v_j) \), which requires \( n(n-1) \) variables and
\( O(n^3) \) clauses.

To illustrate the encoding size, take as an example Paley-73, a graph with 73 vertices and 1314 edges
and twin-width 36. The relative encoding requires 30 million clauses and 2.5 million variables, while
the absolute encoding requires only 2.5 million clauses and 0.3 million variables.

The aforementioned disadvantages of the absolute encoding severely hinders its performance.
Paley-73’s twin-width is found by the relative encoding within three hours, while the absolute encoding
fails to find the optimal result for a 13-vertex graph within four hours.

While ill-suited for finding the optimal twin-width, the small size of the absolute encoding makes
it useful for computing upper bounds on the twin-width of larger graphs. The last unsatisfiable case
\( F(G, \text{tww}(G) - 1) \) and the first satisfiable case \( F(G, \text{tww}(G)) \) usually take an order of magnitude longer
to solve than other cases. Particularly for \( F(G, \text{tww}(G) + i), i = 1, 2, \ldots \) the solving time decreases
quickly. Thus, the absolute encoding can compute upper bounds on the twin-width for graphs that are too
large for the relative encoding.

5 Lower and Upper Bounds

In this section, we describe a simple approach for deriving lower and upper bounds for the twin-width of
graphs. We use these bounds for limiting the range for \( d \) when running the SAT solver on \( F(G, d) \).

We first discuss the lower bound. Let \( r \) be a positive integer and \( G \) a graph with at least \( r \) vertices.
We define the lower bound \( \text{lb}_r \) of order \( r \) for \( \text{tww}(G) \) as the maximum degree of the first \( r + 1 \) graphs
\( H_0, \ldots, H_{r-1} \) of any elimination sequence for \( G \). In particular, for \( r = 1 \) we have
\[
\text{lb}_1(G) = \min_{u,v \in V(G), u \neq v} |N_G(u) \triangle N_G(v)|.
\]

Clearly, \( \text{lb}_1(G) \leq \text{lb}_2(G) \leq \cdots \leq \text{lb}_r(G) = \text{tww}(G) \). If \( r \) is a constant, then \( \text{lb}_r(G) \) can be computed in
polynomial time.

For obtaining an upper bound on the twin-width of a given graph \( G \), we propose a simple greedy
algorithm. The algorithm computes an elimination ordering \( \prec \) and a contraction tree \( T \) step-by-step, by
 greedily choosing the next vertex \( v_i \) in the ordering. Assume we have already computed the first \( i \) vertices
of the elimination ordering \( v_1, \ldots, v_i-1 \) and the corresponding sequence of graphs \( H_0, \ldots, H_{i-1} \) with
\( V(H_{i-1}) = \{ v_1, \ldots, v_n \} \). We choose the next vertex \( v_i \in V(H_{i-1}) \) and the corresponding parent
\( p_i \in \{ v_{i+1}, \ldots, v_n \}, p_i < v_i \) in the lexicographic ordering of the vertices, such that the degree of \( p_i \)
in \( H_i \) is minimized; in case of a tie, we take the lexicographically minimal pair \( (v_i, p_i) \). We add the edge
\( v_ip_i \) to the contraction tree. The width of the resulting twin-width decomposition \( (T, \prec) \) gives the upper
bound \( \text{ub}_{\text{greedy}} \) on the twin-width of \( G \). Our implementation of the greedy heuristic uses caching to avoid
computing the degree of potential pairs \( (v_i, p_i) \) over and over again.

6 Experiments

We computed the twin-width of several graphs using the relative encoding\( ^1 \). We implemented and run
the encoding using Python 3.8.0 and PySAT 1.6.4\( ^3 \). As the SAT solver, we used Cadical\( ^2 \) as it worked
slightly better with the encoding than the other solvers provided by PySAT. We used a computer with an
Intel Core i5-9600KF CPU running at 3.70 GHz, 32 GB RAM and Ubuntu 20.04.

\( ^1 \)Source code can be found at https://github.com/ASchidler/twin_width
\( ^2 \)https://fmv.jku.at/cadical
\( ^3 \)https://pysatihq.github.io
\( ^\text{zenodo.5564192} \)

\( ^\text{The results can be found at https://doi.org/10.5281/} \)
6.1 Named Graphs

We computed the twin-width of several named graphs which are well-known from the literature \cite{25}. The names of the graphs either reflect their topology or their discoverer. For most of the considered graphs, the twin-width was not known. Table 2 provides an overview of our results, including lower and upper bounds as described in Section 5. Preprocessing has no effect on the named graphs, which all turned out to be prime (as one would expect, as these graphs often provide a smallest example or counterexample for a combinatorial property).

Table 2: Results for famous named graphs. For all graph not marked with *, the twin-width could be computed in at most five seconds. \( lb_1 \) gives the lower bound of order 1, \( ub_{\text{greedy}} \) gives the width of an elimination ordering computed by the greedy algorithm of Section 5.

| Graph        | \(|V|\) | \(|E|\) | \(lb_1\) | \(tww\) | \(ub_{\text{greedy}}\) | Variables | Clauses  |
|--------------|--------|--------|---------|--------|----------------|----------|---------|
| Brinkmann    | 21     | 42     | 6       | 6      | 34526          | 150770   |
| Chvátal      | 12     | 24     | 2       | 3      | 5611           | 18288    |
| Clebsch      | 16     | 40     | 6       | 6      | 15510          | 64517    |
| Desargues    | 20     | 30     | 4       | 4      | 28383          | 132636   |
| Dodecahedron | 20     | 30     | 4       | 4      | 26863          | 126244   |
| Dürer        | 12     | 18     | 2       | 3      | 5347           | 18602    |
| Errera       | 17     | 45     | 5       | 6      | 17720          | 75895    |
| FlowerSnark  | 20     | 30     | 4       | 4      | 28383          | 119176   |
| Folkman      | 20     | 40     | 2       | 3      | 10311          | 35761    |
| Franklin     | 12     | 18     | 2       | 2      | 5347           | 16354    |
| Frucht       | 12     | 18     | 2       | 3      | 5083           | 17573    |
| Goldner      | 11     | 27     | 2       | 2      | 4067           | 11813    |
| Grid 6×8*    | 48     | 82     | 2       | 3      | 396751         | 3493676  |
| Grötzsch     | 11     | 20     | 2       | 3      | 45827          | 13910    |
| Herschel     | 11     | 18     | 2       | 2      | 4067           | 13590    |
| Hoffman      | 16     | 32     | 2       | 4      | 14070          | 58051    |
| Holt         | 27     | 54     | 6       | 6      | 79513          | 405925   |
| Kittell      | 23     | 63     | 4       | 5      | 46161          | 171811   |
| McGee        | 24     | 36     | 4       | 4      | 50087          | 238494   |
| Moser        | 7      | 11     | 2       | 2      | 252            | 502      |
| Nauru        | 24     | 36     | 4       | 4      | 50087          | 239051   |
| Paley-73*    | 73     | 1314   | 36      | 36     | 2530300        | 21107035 |
| Pappus       | 18     | 27     | 4       | 4      | 20399          | 89670    |
| Peterson     | 10     | 15     | 4       | 4      | 3009           | 9388     |
| Poussin      | 15     | 39     | 3       | 4      | 11571          | 31049    |
| Robertson    | 19     | 38     | 6       | 6      | 25369          | 114592   |
| Rook 6×6*    | 36     | 180    | 10      | 10     | 216499         | 1236368  |
| Shrikhande   | 16     | 48     | 6       | 6      | 15510          | 64431    |
| Sousselier   | 16     | 27     | 4       | 4      | 14070          | 51414    |
| Tietze       | 12     | 18     | 2       | 4      | 5347           | 18628    |
| Wagner       | 8      | 12     | 2       | 2      | 1418           | 3909     |

Interestingly, the lower bound \( lb_1 \) often coincides with the exact twin-width. One possible explanation is the high level of symmetry in many of the graphs. A particularly interesting class of symmetric graphs are the strongly regular graphs: these graphs are usually parameterized by the tuple \((n, k, \lambda, \mu)\), where \(n\) is the number of vertices, \(k\) is the degree of each vertex, and every pair of vertices has either \(\lambda\) common neighbors if they are adjacent, or share \(\mu\) neighbors otherwise. For a strongly regular graph \(G\) with
Table 3: Results for Paley graphs. The twin-width agrees with the lower bound of \((|V| - 1)/2\). Time shows the number of seconds it took to solve the SAT instance.

| Name     | \(|V|\) | \(|E|\) | tww  | Variables | Clauses | Time [s] |
|----------|--------|--------|------|-----------|---------|----------|
| Paley-09 | 9      | 18     | 4    | 2080      | 6176    | <1       |
| Paley-13 | 13     | 39     | 6    | 7962      | 29205   | <1       |
| Paley-17 | 17     | 68     | 8    | 19352     | 84652   | <1       |
| Paley-25 | 25     | 150    | 12   | 73948     | 408838  | 2.8      |
| Paley-29 | 29     | 203    | 14   | 120406    | 715814  | 7.6      |
| Paley-37 | 37     | 333    | 18   | 272166    | 1916941 | 21.4     |
| Paley-41 | 41     | 410    | 20   | 384324    | 2030173 | 63.6     |
| Paley-49 | 49     | 588    | 24   | 692352    | 4513244 | 210.2    |
| Paley-53 | 53     | 689    | 26   | 893986    | 6282603 | 364.3    |
| Paley-61 | 61     | 915    | 30   | 1406886   | 11437512| 2396.8   |
| Paley-73 | 73     | 1314   | 36   | 2530300   | 21107035| 9934.3   |

parameters \((n, k, \lambda, \mu)\) we can immediately determine the lower bound of order 1

\[lb_1(G) = \min\{2(k - \mu), 2(k - \lambda - 1)\} .\]

Examples of strongly regular graphs in Table 2 are Clebsch \((16, 5, 0, 2)\), Peterson \((10, 3, 0, 1)\), Rook \(n \times n\) \((n^2, 2n-2, n-2, 2)\), and Shrikhande \((16, 6, 2, 2)\). A family of strongly regular graphs, the Paley graphs, stick out due to their high twin-width in relation to their size. For every prime power \(n\) such that \(n \equiv 1 \pmod{4}\), the Paley graph on \(n\) vertices \((\text{Paley-}n)\) is defined and is strongly regular with parameters \(k=\frac{n-1}{2}, \lambda=\frac{n-5}{4}, \mu=\frac{n-1}{4}\). Further, Paley graphs are self-complementary, i.e., Paley-\(n\) and Paley-\(n\) are isomorphic \([11]\). With our relative SAT encoding, we could verify that for Paley graphs with up to 73 vertices, the lower bound of order 1 gives the exact twin-width, see Table 3. We hope that by analyzing the twin-width decomposition provided by our encoding, one can verify that \(\text{tww}(\text{Paley-}n) = \frac{n-1}{2}\) holds in general.

Table 3 also highlights the quickly increasing size of our relative encoding. Despite the size, the solving times are comparatively short. Although the encoding can compute the twin-width for Paley-73, it often starts struggling for general graphs with more than 40 vertices. This suggests that some graphs are considerably harder for our encoding than others, independent of their size.

Two-dimensional grid graphs are interesting for twin-width. They are known to have unbounded treewidth and clique-width, but it is easy to see that their twin-width is at most 4 \([5]\). Interestingly, with our relative encoding, we found that smaller grid graphs, of size up to \(8 \times 6\), do have twin-width 3. We see it as an interesting challenge to determine the exact twin-width of all square grids. The width-3 decompositions that we found with our encodings do not suggest any obvious general pattern that could be generalized to all grid graphs, hence we still expect that at a certain size the width switches from 3 to 4.

6.2 Random Graphs

We tested the twin-width on randomly generated graphs. For this purpose, we created Erdős-Rény graphs \(G(n, p)\), where \(|V(G)| = n \in \{10, 15, 20\}\) and each edge exists with probability \(p\), where \(p\) takes values between 0 to 1 in 0.02 increments.

The results in Figure 4 show that the twin-width increases quickly with increasing graph size. Furthermore, the vertical distance between the peaks is similar. The symmetric shape is expected due to Fact 2.2.

Many of the graphs can be simplified using the preprocessing discussed in Section 3.
6.3 The Twin-Width Numbers

For every $d > 0$, let $tww_d$ be smallest integer such that there exists a graph with $tww_d$ many vertices of twin-width $d$. We call $tww_d$ the $d$-th twin-width number. In contrast to other width measures like treewidth, where similar numbers are easy to compute (the $d$-th treewidth number is $d + 1$), no uniform method is known for computing the twin-width numbers. The situation is similar for clique-width, where no uniform method is known either; Heule and Szeider [15] computed the first few clique-width numbers.

The computation of twin-width numbers provides a challenge for any exact method, as the search space grows quickly with each increment of $d$. However, with our encodings, run on prime graphs generated by Nauty[20], we were able to identify the first few twin-width numbers and give tight bounds for further ones.

Proposition 6.1. The sequence of twin-width numbers starts with $4$, $5$, $8$, $9$; the fifth twin-width number is $11$ or $12$, the sixth twin-width number is at most $13$.

For computing the twin-width numbers, we only need to consider graphs $G$ with $|E(G)| \leq \binom{n}{2}/2$, as by Fact 2.2 $|E(G)| > \binom{n}{2}/2$ implies $|E(\overline{G})| \leq \binom{n}{2}/2$. Further, according to Theorem 3.1 we only need to consider prime graphs. In particular, since every prime graph $G$ and its complement graph $\overline{G}$ are connected, we only need to consider connected graphs. The results are shown in Table 4.

The preprocessing described in Section 3 can be used for all graphs that are not prime. We can see in Table 4 that there are many connected graphs that are not prime, and thereby eligible for preprocessing.

Interestingly, for the first, second, and fourth twin-width number $tww_d$, there is a unique graph, up to isomorphism, with $tww_d$ many vertices and twin-width $d$. For the third twin-width number, there are five such graphs: $G_{8,3,i}$, $i = 1, \ldots, 5$. $G_{8,3,3}$ is self-complementary; the other four form two complementary pairs. In Figure 5 we display these graphs, together with an optimal $d$-sequence, showing only one graph from each complementary pair.

The unique graph certifying $tww_1 = 4$ is the path on 4 vertices ($P_4$). The unique graph certifying $tww_2 = 5$ is the cycle on five vertices ($C_5$). The unique graph certifying $tww_4 = 9$ is the graph Paley-9 (see Section 6.1). In fact, $C_5 = $ Paley-5, so also $tww_2$ is certified by a Paley graph. Further, if we remove any vertex from Paley-9, we obtain $G_{8,3,3}$. Similarly, we obtain $P_4$ by removing a vertex from Paley-5. Therefore, Paley graphs are related with all of the first four twin-width numbers. We could establish with our method that among all graphs with 10 vertices, there is no graph of twin-width 5, hence $tww_5 \geq 11$. We could not check all graphs with 11 vertices, as there are too many. Paley-13 shows that $tww_6 \leq 13$.

By deleting any single vertex from Paley-13, its twin-width drops to 5. This implies that $tww_5 \leq 12$, and so $11 \leq tww_5 \leq 12$ as stated in Proposition 6.1.

4http://cs.anu.edu.au/people/bdm/
Figure 5: Smallest graphs for given twin-width $d$. The integer vertex labels give a $d$-sequence, and the dashed edges give a contraction tree, as in Figure

Table 4: The number of graphs, prime graphs, and prime graphs of a specific twin-width, with a specific number of vertices.

| twin-width | $|V|$ connected | prime | 1 | 2 | 3 | 4 |
|------------|-----------------|-------|---|---|---|---|
| 4          | 3               |       | 1 | 1 | 0 | 0 |
| 5          | 11              | 4     | 3 | 1 | 0 | 0 |
| 6          | 73              | 26    | 16| 10| 0 | 0 |
| 7          | 618             | 260   | 90| 170|0 |0 |
| 8          | 8573            | 4670  | 655| 4010|5 |0 |
| 9          | 224875          | 145870| 4488| 137565| 3816|1 |
| 10         | 11716571        | 8110356| 30318| 6144756| 1935226| 56 |

7 Conclusion

We proposed the first practical approach to computing the exact twin-width of graphs, utilizing the power of state-of-the-art SAT-solvers. This allowed us to reveal the twin-width of several important graphs. Our results provide the first step for showing general twin-width bounds for infinite graph classes. For instance, our data suggests $\text{tww}(\text{Paley-}n) = (n-1)/2$. Surprisingly, up to $n = 6$, the $n \times n$ grids have twin-width 3. It would be interesting to know if and when twin-width 4 is required. Another possible application of our results is the construction of gadgets for showing the theoretical intractability of twin-width computation. Such intractability is expected [4], but no proof has yet been found.

The two proposed SAT encodings’ different performance is impressive: the relative encoding benefits from symmetry breaking and vastly outperforms the more succinct absolute encoding. Although the
relative encoding doesn’t explicitly exploit the input graph’s symmetries, it performs well on some highly symmetric graphs like Paley-73.

We hope that our results provide new insights and stimulates further theoretical investigations on twin-width. We also hope that our results provide a first step towards a practical use of twin-width. A next step would be the implementation and testing of twin-width-based dynamic programming algorithms like the algorithms for $k$-Independent Set and $k$-Dominating Set proposed by Bonnet et al. [3], which are single exponential in the twin-width.

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