Late time acceleration in 3-brane Brans-Dicke cosmology

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Abstract

In order to investigate more features of the Brans-Dicke cosmology in
the five-dimensional space-time, we explore the solutions of its dynamical
systems. A behavior of the universe in its early and late time by
means of the scale factor is considered. As a results, we show that it is
possible to avoid the big rip singularity and to cross the phantom divide
line. Furthermore, we review the dark energy component of the universe
and its agreement with the observation data for this 3-brane Brans Dicke
cosmology by means of the cosmological parameters.

1 Introduction

The Brans-Dicke (BD) theory [1], the simplest case of the scalar tensor theory
[2], is defined by a constant coupling parameter ω and a scalar field φ. General
relativity is recovered when ω goes to infinity [3] and from timing experiments
using the Viking space probe [4], ω must exceed 500. This constraint ruled out
many of extended inflation [5, 6] and provides a succession of improved models
of extended inflation [7, 8, 9, 10]. Furthermore, all important features of the
evolution of the Universe such as: inflation [11], early and late time behavior of
universe [12], cosmic acceleration and structure formation [13], quintessence and
coincidence problem [14], self-interacting potential and cosmic acceleration [15],
High energy description of dark energy in an approximate 3-brane [16] could
be explained successfully in the BD formalism. For a large value of the ω-
parameter, BD theory gives the correct amount of inflation and early and late
time behaviors, while small and negative values explain cosmic acceleration,
structure formation and coincidence problem.

The dark energy, qualified as responsible for the cosmic acceleration deter-
mines the feature of a future evolution of the universe. The nature of this kind
of energy may lead to the improvement of our pictures on particle physics and
gravitation. The investigations on the nature of the dark energy lead to various

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candidates. Among them, the most popular one, the cosmological constant [17], the dynamical scalar fields, described by the equation of state \( p = (\gamma - 1)\rho \), like quintessence [15, 18, 19, 20] (\( \gamma > 0 \)) or like phantom [21, 22, 23] (\( \gamma < 0 \)). However, many problems associated with the phantom candidate have to be explained. The first one is the big rip [24], the appearance of a future singularity in a finite time, at which all cosmological parameters blow up. To overcome this problem, many models, whose solutions do not suffer from big rip, have been proposed such as the model which stated that the final state of phantom cosmology may be inflation [25], the one inspired from the string theory [26], and the model in which the de-Sitter solution is attractor of phantom cosmology [27]. The second one is how to cross the phantom divide line \( \gamma = 0 \). However, the dynamical transition from \( \gamma > 0 \) to \( \gamma < 0 \) or vice versa is possible [28, 29, 30, 31]. We are thus motivated to explore the cosmological behavior of the phantom field, in such a way that inflation is the generic feature of phantom cosmology rather than the big rip as in Ref. [25] and also to explain the possible crossing of phantom divide line.

More recently, a great deal of interest is devoted to high dimension space-time. Superstring, which predicts a new type of nonlinear structure, called a brane [32, 33, 34], suggests that our universe might be of higher dimensions which are compactified. The matter field content of our universe is confined to a four dimensional space-time called a 3-brane in the case of (5D) space-time. In addition, several works have studied higher dimensional BD theory in order to use the advantages of the combination of the high dimensional cosmology and the BD theory.

In a recent paper [16], we have generalized the BD cosmology to a 3-brane with a nonzero cosmological constant, \( \Lambda_4 \), derived from 5-dimensional (5D) bulk space-time. In this paper, we discuss the evolution of the scale factor at the very early stage of the expansion of the universe and at the late stage, where a new kind of matter (dark energy) is suspected to be responsible for the cosmic acceleration. We show that a possible accelerated era could appear if the universe undergoes a bounce state in the past, and avoids the big bang singularity or a turnaround state avoiding consequently the big rip singularity. Otherwise, the accelerating era appears with this two singularities in the past or/and in the future. Furthermore, in order to generalize our previous work [16] and to complete this study to late time accelerating universe, we linearize the dynamic system, by means of the cosmological parameters, in the intermediate energy levels.

This paper is organized as follows. In section 2 we present the equations of the fields in the 3-brane Brans-Dicke cosmology. These expressions will be given in the shape of a dynamic system by introducing some variables. In section 3, we discuss the evolution of the scale factor and the possibility to overcome both the big rip singularity and the crossing divide line in phantom cosmology, by assuming that the universe started out small. In section 4 we examine the late time accelerating universe by giving the solution in the intermediate energy levels around the stable equilibrium solution. Section 5 is devoted to the conclusions.
2 Field Equations in 3-brane Brans-Dicke cosmology

The detail of the 3-brane Brans-Dicke cosmology derived from 5-dimensional bulk space-time is given in [16]. In this section we shall review the main results. First, we consider that the behavior of BD field is sensitive only to a physical 3-brane. Thus, it is described by the same action as in 4-dimension (4D)

\[ S = \int d^4x \sqrt{-g} \left( \phi [R - 2\Lambda_4] - \frac{\omega}{\phi} \partial_{\mu}\phi \partial^{\mu}\phi - 16\pi E_m \right). \] (1)

Furthermore, to recover the BD cosmology at low energy, we add simply a BD stress energy tensor to the Einstein equation, 4\( G_{\mu\nu} \) [32], where all quadratic and mixed terms of this stress-energy tensor are canceled. Therefore, the modified Einstein equation takes the form:

\[ 4G_{\mu\nu} = -\Lambda_4 \gamma_{\mu\nu} + 8\pi G_N \gamma_{\mu\nu} + k_5^4 \Pi_{\mu\nu} + k_5^2 \phi^2 \partial_{\mu\nu}\phi + \frac{1}{4\phi} \partial_{\mu\nu}\phi \partial^{\mu\nu}\phi \] (2)

where \( \Lambda_4 \) is the 4D cosmological constant, 8\( \pi G_N = k_4^2 \) is the 4D gravitational constant (\( G_N \) is the Newton’s constant of gravity), \( k_5^2 \) is the 5D gravitational constant and \( \gamma_{\mu\nu}, \Pi_{\mu\nu} \) are respectively the energy momentum, the quadratic tensors on the brane.

We have shown that the equation of motion of the BD field and the equation of state are modified. Indeed, by varying the action (1) versus the metric tensor and BD field \( \phi \), in the homogeneous and isotropic Friedman-Robertson-Walker metric with scale factor \( a(t) \) and spatial curvature index \( k \) (with \( G_N = \frac{1}{\phi} \)), one obtain the following BD field equations:

\[ \frac{\dot{a}}{a} = \frac{\Lambda_4}{3} - \frac{4\pi}{3\phi}(3\gamma - 2)\rho - \frac{k_5^4}{36}(3\gamma - 1)\rho^2 - \frac{\omega}{3} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{1}{2} \frac{\dot{\phi}}{a} - \frac{1}{2} \frac{\ddot{\phi}}{\phi} \] (3)

\[ \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{\Lambda_4}{3} + 8\pi \rho + \frac{k_5^4}{36}\rho^2 + \frac{\omega}{6} \left( \frac{\dot{\phi}}{\phi} \right)^2 - \frac{\dot{\phi}}{a} - \frac{\ddot{\phi}}{a} \] (4)

\[ -\frac{1}{a^3} \frac{d(\dot{\phi}^3)}{dt} = \frac{8\pi}{3 + 2\omega}((3\gamma - 4)\rho + \frac{k_5^4}{48\pi}(3\gamma - 2)\rho^2 - \frac{\Lambda_4}{4\pi}) \] (5)

\[ \dot{\rho} = -3 \frac{\dot{a}}{a}(p + \rho) \] (6)

where \( \rho \) and \( p \) are respectively the energy density and the pressure of the cosmic fluid with the equation of state \( p = (\gamma - 1)\rho \).

Equation (5) is the modified BD field equation, while in the standard BD cosmology, it can be written as \( (\omega \neq -\frac{3}{2}) \),

\[ -\frac{1}{a^3} \frac{d(\dot{\phi}^3)}{dt} = \frac{8\pi}{3 + 2\omega}((3\gamma - 4)\rho - \frac{\Lambda_4}{4\pi}) \] (7)
Comparing the BD field equations (3)-(5), we conclude that at high energy limit \( \rho^2 \gg \rho \), the 3-brane BD cosmology is described by the same manner as in the 4D BD cosmology with the equation of state of a perfect fluid is [16] \( p = (2\gamma - 1)\rho \).

By introducing the variables, \( H = \dot{a} / a \), \( F = \dot{\phi} / \phi \) and \( Z = \frac{8\pi\rho}{\phi} \), the field equations (4), (5) and (6) become:

\[
\frac{dH}{dt} = -H^2 - \frac{\omega}{3} F^2 + HF + \frac{2\omega}{3(3 + 2\omega)} \Lambda_4 - \frac{(3\gamma - 2\omega + 3)}{(3 + 2\omega)} Z - \frac{(6\gamma - 2\omega + 3)}{(3 + 2\omega)} k^4 \rho^2 \quad (8)
\]

\[
\frac{dF}{dt} = -F^2 - 3HF + \frac{4}{(3 + 2\omega)} \Lambda_4 - \frac{(3\gamma - 4)}{(3 + 2\omega)} Z - \frac{(3\gamma - 2)}{(3 + 2\omega)} k^4 \rho^2 \quad (9)
\]

\[
0 = H^2 - \frac{\omega}{6} F^2 + HF + \frac{k^4}{a^2} - \frac{\Lambda_4}{3} Z - \frac{k^4}{36} \rho^2 \quad (10)
\]

3 Scale factor evolution in 3-brane Brans-Dike theory

In this section we use the preceding dynamic equations to discuss the evolution of the scale factor in three cases: low, high and intermediate energy levels. We suppose that near the big-bang time, the universe is located in the equilibrium state, as in Ref. [35], namely \( \dot{a} = 0 \).

3.1 Low energy case:

When the energy density \( \rho \) is sufficiently diluted in the universe, we are practically in low energy limits, i.e. \( \rho \gg \rho^2 \). The preceding field equations (8), (9) and (10) become by eliminating the \( Z \)-variable:

\[
(2\omega + 3) \frac{dH}{dt} = -3H^2(\gamma \omega + 2) + \frac{1}{2} F^2 \omega (\gamma \omega - 2\omega - 1) + FH\omega (4 - 3\gamma) + \Lambda_4 (\gamma \omega + 1) - (3\gamma \omega - 2\omega + 3) \frac{k}{a^2} \quad (11)
\]

\[
(2\omega + 3) \frac{dF}{dt} = -3(3\gamma - 4) H^2 + \frac{1}{2} F^2 (3\gamma \omega - 8\omega - 6) - 3FH (3\gamma + 2\omega - 1) + \Lambda_4 (3\gamma - 2) - 3(3\gamma - 4) \frac{k}{a^2} \quad (12)
\]

which are exactly the field equations in the 4-dimensional BD cosmology.

For a static universe (\( \dot{a} = 0 \)), the field equations (11) and (12) have a solution
and the BD field behavior depends on the 4D cosmological constant
\[ \phi = \phi_0 e^{\alpha t} \] (14)

and, for \( \omega \neq -1 \)
\[ a = a_* = \sqrt{\frac{1}{\Lambda_4} \frac{(2\omega + 3)}{(\omega + 1)}} = \frac{1}{\alpha} \sqrt{\frac{2}{(\omega + 1)}} \quad \text{with} \quad k = 1 \] (15)

Which means that in a static universe, the BD field evolves exponentially as \( e^{\alpha t} \).

Let us notice that the two solutions \( \phi(t) \) and \( a(t) \) are expected to be stable and we refer to [35] where the stability of such solutions is well established.

Now, we investigate how the matter content of the universe, with an equation of state \( p = (\gamma - 1)\rho \), affects its behavior compared to this stable solution.

In this aim, we keep the BD field varying as \( e^{\alpha t} \) and the previous equations become:

\[
(2\omega + 3) \frac{dH}{dt} = -3H^2(\gamma\omega + 2) + \frac{1}{2} \alpha^2 \omega (\gamma\omega - 2\omega - 1) + \alpha H\omega(4 - 3\gamma) + \Lambda_4 (\gamma\omega + 1) - (3\gamma\omega - 2\omega + 3) \frac{k}{a^2} \]

\[0 = -3(3\gamma - 4)H^2 + \frac{1}{2} \alpha^2 (3\gamma\omega - 8\omega - 6) - 3\alpha H (3\gamma + 2\omega - 1) + \Lambda_4 (3\gamma - 2) - 3(3\gamma - 4) \frac{k}{a^2}.\]

Or, by eliminating the term \( \frac{k}{a^2} \):

\[3(3\gamma - 4) \frac{dH}{dt} = -3H^2 (3\gamma - 4) + 3\alpha H (3\gamma - 2\omega + 3\gamma\omega - 1),\] (18)

we obtain the following equation
\[
(3\gamma - 4) \frac{\ddot{a}}{a} = \alpha \frac{\dot{a}}{a} (3\gamma - 2\omega + 3\gamma\omega - 1).\] (19)

If the universe is dominated by the radiation (i.e., \( \gamma = \frac{4}{3} \)), we recover the static universe and it remains eternally. Otherwise, we rewrite this equation, by putting \( \theta = \dot{a} \), in the form
\[
\frac{\dot{\theta}}{\theta} = \alpha \frac{(3\gamma - 2\omega + 3\gamma\omega - 1)}{(3\gamma - 4)} \]

\[1\text{the case } \omega = -1, \text{ which gives an eternally flat and static universe, is excluded in this study.} \]
which gives easily

$$\theta = \theta_* e^{\frac{\beta}{\beta} (3\gamma - 2\omega + 3\gamma\omega - 1) (t-t_*)},$$

(21)

where $\theta_*$ is a constant of integration. Hence the scale factor varies as:

$$a(t) = \frac{\theta_*}{\beta} e^{\beta (t-t_*)} + c$$

(22)

where $\beta = \sqrt{\frac{2\Lambda}{2\omega+3}} \frac{(3\gamma - 2\omega + 3\gamma\omega - 1)}{(3\gamma - 4)}$ and $c$ is the integration constant.

The behavior of the universe depends on the product $\beta \theta_*$. Indeed, if the universe undergoes an era in which $\theta_*$ is positive, it begins to accelerate for $\beta > 0$. The condition, in which $\theta_*$ is positive, is a feature of an era following the one which we suppose to be a bounce state characterized by $\dot{a}(t) \rightarrow 0$ and $\ddot{a}(t) > 0$ [36]. Otherwise, the accelerating era is for $\beta < 0$, which can be a feature of the era preceding a turnaround state characterized by $\dot{a}(t) \rightarrow 0$ and $\ddot{a}(t) < 0$ [37].

The time in which $\theta_*>0$ could happen in the past, as in the high energy limit (see the next section) or in the future as in the low energy limit. In the former, the existence of the bounce state means that the big bang singularity could be avoided. In the future, however, it is the turnaround state which is important since it may avoid the big rip singularity if never it exists.

To illustrate this situation, we consider the case $\omega \gg 1$ i.e. $\beta = \sqrt{\Lambda_4} \frac{\gamma - 2}{\gamma - 2} \omega^{1/2}$. For $\beta > 0$, the parameter $\gamma$ varies in the range $\gamma \in \left[-\infty, \frac{2}{3}\right] \cup \left[\frac{4}{3}, +\infty\right]$. This range excludes the ordinary forms of matter/energy (like dust or radiation) and the one like a cosmic string, $\gamma = \frac{2}{3}$, in which the universe remains static and eternally. In other word, the accelerating universe caused by an exotic form of matter/energy is expected in the interval range where $\gamma < \frac{2}{3}$ ($p < -\frac{1}{3} \rho$) or $\gamma > \frac{4}{3}$. In the case $\gamma < \frac{2}{3}$, a possible nature of the dark energy will be a quintessence, ($\gamma > 0$), a domain wall, $\gamma = \frac{1}{3}$, a cosmological constant, $\gamma = 0$, or a phantom, ($\gamma < 0$). While for $\gamma > \frac{4}{3}$ the only possible candidate is the quintessence. In the case where $\beta < 0$, the accelerating universe is caused by the dark energy component of the energy density of the universe, for $\gamma \in \left[\frac{2}{3}, \frac{4}{3}\right]$, and the quintessence is the only possible candidate for the nature of this kind of matter/energy.

### 3.2 High energy case

in the Ref. [16], We have shown that, at high energy limit, the 3-brane BD cosmology could be described by the 4D BD cosmology with the following equation of state $p = (2\gamma - 1)\rho$, i.e. the $\gamma$-parameter in the standard cosmology is equal to twice the one in 5D bulk space-time.

The field equations in this limit are the same as (11) and (12) where the $\gamma$-parameter should be replaced by $2\gamma$.

The solutions, in this case, are the same as the one in the low limit with the equation of state $p = (2\gamma - 1)\rho$ and $\beta = \sqrt{\frac{2\Lambda}{2\omega+3}} \frac{(6\gamma - 2\omega + 6\gamma\omega - 1)}{2(3\gamma - 2)}$

We can draw the same conclusion by replacing the $\gamma$-parameter by $2\gamma$. Indeed, for $\beta > 0$, the parameter $\gamma$ varies in the range $\gamma \in \left[-\infty, \frac{1}{3}\right] \cup \left[\frac{2}{3}, +\infty\right]$. 




for $\omega \gg 1$. This range excludes the forms of matter/energy like a domain wall, $\gamma = \frac{1}{3}$, a cosmic string; $\gamma = \frac{2}{3}$, or a quintessence, for which $\gamma \in \left[\frac{1}{3}, \frac{2}{3}\right]$. In other word, the accelerating universe is caused by an exotic form of matter/energy, like a phantom or a cosmological constant, for $\gamma < \frac{1}{3}$ in agreement with [38] or like a quintessence for $\gamma > \frac{2}{3}$. Otherwise, the cosmic acceleration is caused only by a dark energy, for which $\gamma \in \left[\frac{1}{3}, \frac{2}{3}\right]$, and the only possible candidate for the nature of this kind of matter/energy is, once more, the quintessece.

### 3.3 The intermediate energy level

In the intermediate energy level case, where both $\rho$ and $\rho^2$ contribute to the evolution of the universe, the field equations (8), (9) and (10) become:

\begin{align*}
(3 + 2\omega) \frac{dH}{dt} &= -6 (\gamma \omega + 1) H^2 - \omega \left(\frac{\omega (2 - 2\gamma) + 1}{2}\right) F^2 - \omega (6\gamma - 4) H F \\
&\quad - \frac{k}{a^2} (\omega (6\gamma - 2) + 3) + \Lambda_4 (2\gamma \omega + 1) + \gamma \omega Z \tag{23}
\end{align*}

\begin{align*}
(3 + 2\omega) \frac{dF}{dt} &= -3 (6\gamma - 4) H^2 - (\omega (4 - 3\gamma) + 3) F^2 - 3(6\gamma - 1 + 2\omega) H F \\
&\quad - \frac{3k}{a^2} (6\gamma - 4) + \Lambda_4 (6\gamma - 2) + 3\gamma Z. \tag{24}
\end{align*}

Since, in the low and high energy limits, the BD field increases as $e^{\alpha t}$, we suggest that, in the intermediate energy limit, this BD field behaves in the same manner i.e. as $e^{\alpha t}$ and letting, only, the scale factor to be variable. With this assumption the equations (23) and (24) become:

\begin{align*}
(3 + 2\omega) \frac{dH}{dt} &= -6 (\gamma \omega + 1) H^2 - \omega \left(\frac{\omega (2 - 2\gamma) + 1}{2}\right) \alpha^2 - \omega (6\gamma - 4) H \alpha \\
&\quad - \frac{k}{a^2} (\omega (6\gamma - 2) + 3) + \Lambda_4 (2\gamma \omega + 1) + \gamma \omega Z \tag{25}
\end{align*}

\begin{align*}
0 &= -3 (6\gamma - 4) H^2 - (\omega (4 - 3\gamma) + 3) \alpha^2 - 3(6\gamma - 1 + 2\omega) H \alpha \\
&\quad - \frac{3k}{a^2} (6\gamma - 4) + \Lambda_4 (6\gamma - 2) + 3\gamma Z \tag{26}
\end{align*}

or by eliminating the $Z$-term between the two latter equations, one finds:

\[ 3 \left( \ddot{a}a + \dot{a}^2 \right) = 3\omega \dot{a}a - 3k + \left( \frac{1}{2} \omega \alpha^2 + \Lambda_4 \right) \alpha^2 \tag{27} \]

which becomes in term of a new variable $\theta = a^2$:

\[ \ddot{\theta} - \omega \alpha \dot{\theta} - 2\Lambda_4 \frac{\omega + 1}{2\omega + 3} \theta = -2 \tag{28} \]

We notice that this dynamical equation, and therefore its solution, is free of $\gamma$-parameter and all kinds of matter/energy are welcomed. Its solution is:
\[ \theta = a^2 = a^2_* + c_1 e^{-\alpha t} + c_2 e^{\alpha (\omega + 1) t} \]  

(29)

where \( a_* = \frac{1}{\alpha} \sqrt{\frac{2}{(\omega + 1)}} \) for \( \omega \neq -1 \).

One can notice that at \( t \to 0 \) the scale factor varies as \( \sim \sqrt{t} \) and at \( t > 0 \) but not too much, the evolution of the scale factor, described by \( a(t) \sim e^{\frac{\omega}{2} (\omega + 1) t} \), is consistent with the primordial rapid inflation.

We conclude that, the 3-brane BD cosmology at the intermediate limit where \( \rho \) and \( \rho^2 \) are both considered, we recover the standard like expansion of the universe for all kind of matter/energy, at \( t \to 0 \), as in the standard cosmology but for radiation era only. For a late time \( t > 0 \), we recover an exponential expansion for all kind of matter/energy, however it is only for vacuum energy in the standard cosmology.

We stress that the last two approximations, denoted by low and high energy limit, should be a good and simple illustration to overcome the big rip problem and may explain how to cross the divide line \( \gamma = 0 \) in phantom cosmology. Indeed, for the big rip problem, if we replace the equation of state parameter \( \gamma \) by \( \gamma - \frac{1}{3} \) in the low energy limit, i.e. we exclude the radiation from the matter/energy content of the present universe which is well justified by observation [39], we conclude the following: The low energy limit corresponds to the late time evolution of the universe and the high energy limit to the early time. For the late time, we assume that the acceleration occurs in the case when \( \beta < 0 \) while for the early time we assume that it happens in the case \( \beta > 0 \). Hence the early time acceleration is caused by the matter/energy content with \( \gamma \in \left[ -\infty, \frac{5}{3} \right] \cup ]2, +\infty[ \) while for the late time the acceleration is due to the matter/energy content with \( \gamma \in ]-\frac{2}{3}, 0[ \). We notice the possibility of a transition from a state of the universe, dominated by either a dust \( (\gamma = 1) \), a cosmological constant \( (\gamma = 0) \), or a quintessence \( (\gamma \in ]0, \frac{5}{3} [ \cup ]2, +\infty[ ) \) to a state of the universe, dominated by a phantom \( (\gamma < 0) \), but undergoing a turnaround state avoiding therefore the big rip singularity. Hence, the final state of phantom cosmology may be inflation rather than big rip since the turnaround state stops the acceleration.

Furthermore, it is also possible to cross the phantom divide line \( (\gamma = 0) \). Indeed, if the universe, at early time, is described by a matter/energy content as dust, cosmological constant, or quintessence \( (\gamma \in ]0, \frac{5}{3} [ \cup ]2, +\infty[ ) \), the acceleration of the universe at late time is caused necessarily by a phantom like field with \( \gamma \in ]-\frac{2}{3}, 0[ \) i.e. a transition from \( \gamma > 0 \) to \( \gamma < 0 \).

4 Late time accelerating universe

In this section, we consider the intermediate case in which \( \rho \) and \( \rho^2 \) are both considered. The limit case, low and high energy, were considered in our previous work [16]. We have shown that our results are in agreement with the observation data, more precisely with the dark energy via the cosmological parameters.

We can analyze how much today’s the universe is far from late-time inflation
by linearizing the dynamical system about the stable cosmological solution with flat space and show how the Hubble parameter varies with the scale factor $a(t)$.

### 4.1 Stability

Since the combined results of the cosmic microwave background and type Ia SNe [40, 41, 42] conclude that the universe undergoes a flat period today, we neglect the curvature parameter $k/a^2$ as $a(t)$ increases with the expansion of the universe. Under these considerations, and in analogy with the previous section, the stable solution for (23) and (24) is:

$$
(H_2, F_2, Z_2) = \sqrt{\frac{2\Lambda_4}{(2\omega + 3)(3\omega + 4)}} (\omega + 1, 1, 0).
$$

Indeed, the stable solutions are obtained from the equilibriums points. To this end, we add to (23) and (24), the equation$^2$

$$
\frac{dZ}{dt} = -(3\gamma H + F)Z.
$$

Neglecting the $k$-term, the equations (23) and (24) at the equilibrium points become

\begin{align*}
0 & = -6(\gamma \omega + 1) H^2 - \omega \left(\frac{(2 - 2\gamma) + 1}{2}\right) F^2 - \omega (6\gamma - 4) HF \\
& \quad + \Lambda_4 (2\gamma \omega + 1) + \gamma \omega Z \\
0 & = -3(6\gamma - 4) H^2 - (\omega (4 - 3\gamma) + 3) F^2 - 3(6\gamma - 1 + 2\omega) HF \\
& \quad + \Lambda_4 (6\gamma - 2) + 3\gamma Z \\
0 & = -(3\gamma H + F)Z.
\end{align*}

In this analysis we discuss two cases, corresponding to $Z \neq 0$ and to $Z = 0$.

- In the case where $Z \neq 0$, the first equilibrium point is given by

$$
F_1 = -3\gamma H_1
$$

$$
H_1^2 = \frac{2\Lambda_4}{18\gamma \omega - 9\gamma^2 \omega + 12}
$$

and

\footnote{We have $Z = \frac{8\pi \rho}{\phi}$ and $\dot{\rho} = -3\gamma \rho \frac{\dot{a}}{a} = -3\gamma H\rho$. Hence $\frac{dZ}{dt} = \frac{8\pi \rho}{\phi} \dot{\rho} - \frac{8\pi \rho}{\phi^2} \rho \phi = -\frac{8\pi \rho}{\phi} (3\gamma H + F)$}
\[ Z_1 = -\frac{(2\Lambda_4 + 6\gamma\Lambda_4 + 6\gamma\omega\Lambda_4)}{6\gamma\omega - 3\gamma^2\omega + 4} \]  

in compact form

\[
\begin{pmatrix}
H_1 \\
F_1 \\
Z_1
\end{pmatrix} =
\begin{pmatrix}
\sqrt{\frac{2\Lambda_4}{18\gamma\omega - 9\gamma^2\omega + 12}} \\
-3\gamma \sqrt{\frac{2\Lambda_4}{18\gamma\omega - 9\gamma^2\omega + 12}} \\
-\frac{(2\Lambda_4 + 6\gamma\Lambda_4 + 6\gamma\omega\Lambda_4)}{6\gamma\omega - 3\gamma^2\omega + 4}
\end{pmatrix}
\]

- In the case where \( Z = 0 (\rho = 0) \),

\[
0 = -6(\gamma\omega + 1)H^2 - \omega \left( \frac{\omega(2 - 2\gamma) + 1}{2} \right) F^2 - \omega(6\gamma - 4)HF + \Lambda_4(2\gamma\omega + 1) \\
0 = -3(6\gamma - 4)H^2 - (\omega(4 - 3\gamma) + 3)F^2 - 3(6\gamma - 1 + 2\omega)HF + \Lambda_4(6\gamma - 2)
\]

and for \( \omega > -4/3 \) or \( \omega < -3/2 \), the equilibrium point is

\[
(H_2, F_2, Z_2) = \sqrt{\frac{2\Lambda_4}{(2\omega + 3)(3\omega + 4)}}(\omega + 1, 1, 0).
\]

Note that for the second equilibrium point, the values of \( H_2 \) and \( F_2 \) are the same as the one in the 4-dimension case [16]. Therefore, the \( Z \)-term behaves like a corrective term for the 4-dimension case. In this sense we check the stability of this point\(^3\) by writing

\[
\begin{align*}
&h = H - H_2 \\
f = F - F_2 \\
z = Z - Z_2.
\end{align*}
\]

The equations (23) and (24) become

\[
\begin{pmatrix}
\frac{dh}{dt} \\
\frac{df}{dt} \\
\frac{dz}{dt}
\end{pmatrix} =
\begin{pmatrix}
-2(3\gamma\omega + 2) & -\omega (2\gamma - 1) & \omega(\omega + 1)\gamma \\
-9(2\gamma - 1) & -(6\gamma + 3\omega + 1) & \frac{\omega(\omega + 1)\gamma}{(2\omega + 3)H_2} \\
0 & 0 & -(3\gamma + 3\gamma\omega + 1)
\end{pmatrix}
\begin{pmatrix}
h \\
f \\
z
\end{pmatrix} + \ldots.
\]

In the case where all the eigenvalues of the Jacobian have a non vanishing real part, the fixed point is called hyperbolic and the signs of this real parts

\(^3\)Since the eigenvalues of the Jacobian of the first equilibrium point \((H_1, F_1, Z_1)\) have a complicated expression, its stability is not considered.
determine its stability. Indeed, if the real part of each eigenvalue has a negative sign then the equilibrium point is stable. While if the sign of the real part of each eigenvalue is positive, or if the sign of one of them is positive and negative for other, then the equilibrium point is unstable. Finally, if the real part of any of the eigenvalues is zero, then the equilibrium point is called nonhyperbolic and its stability in the neighborhood of that point cannot be determined by this method.

The eigenvalues of the Jacobian at the equilibrium point (39) are:

\[ \lambda_1 = -(6\gamma + 6\gamma\omega + 1)\sqrt{\frac{2A_4}{(2\omega + 3)(3\omega + 4)}}, \]

\[ \lambda_2 = -(3\omega + 4)\sqrt{\frac{2A_4}{(2\omega + 3)(3\omega + 4)}}, \]

\[ \lambda_3 = -(3\gamma + 3\gamma\omega + 1)\sqrt{\frac{2A_4}{(2\omega + 3)(3\omega + 4)}}. \]

We notice that, for \( \gamma \geq 0 \) and \( \omega > -4/3 \), the sign of this eigenvalues is negative and hence the equilibrium point \((H_2, F_2, Z_2)\) is stable.

In this case, we should have \( \omega > -4/3 \) or \( \omega < -3/2 \), and in the limit \( \omega \to +\infty \) we obtain:

\[ H_2 \approx \sqrt{\frac{\Lambda_4}{3}} \approx \omega F_2 \]  

(40)

4.2 Linearized dynamical system

To solve the dynamical system (8), (9) and (10) we linearize the solution as in [43]:

\[ H = H_2 + h(a) \]  

(41)

\[ F = F_2 + f(a) \]  

(42)

\[ Z = z(a) \]  

(43)

where \( h(a) \), \( f(a) \) and \( z(a) \) are linearized perturbation functions to be determined later.

Putting (41), (42) and (43) into the field equations (8), (9), (10) and neglecting higher terms in \( h(a), f(a) \) and the product \( h(a)f(a) \) one obtains the following system:

\[
\begin{pmatrix}
\frac{dh}{da} \\
\frac{df}{da} \\
\frac{dz}{da}
\end{pmatrix} = \frac{1}{a(\omega + 1)} \begin{pmatrix}
-2(3\gamma + 2) & -\omega(2\gamma - 1) & \frac{\omega(\omega + 1)\gamma}{(2\omega + 3)H_2} \\
-9(2\gamma - 1) & -(6\gamma + 3\omega + 1) & \frac{3(\omega + 1)\gamma}{(2\omega + 3)H_2} \\
0 & 0 & -(3\gamma + 3\gamma\omega + 1)
\end{pmatrix} \begin{pmatrix}
h \\
f \\
Z
\end{pmatrix}
\]
This system becomes

\[
\begin{pmatrix}
\frac{dx}{da} \\
\frac{dy}{da} \\
\frac{dz}{da}
\end{pmatrix} = \frac{1}{a(\omega + 1)} \begin{pmatrix}
-3\omega - 4 & 0 & 0 \\
0 & -6\gamma - 6\gamma\omega - 1 & 0 \\
0 & 0 & -3\gamma - 3\gamma\omega - 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} + \frac{3}{a^3 \omega^2} \begin{pmatrix}
-6\omega - 2\omega + 6\gamma\omega - 1
\end{pmatrix}
\]

with

\[
\begin{pmatrix}
h \\
f \\
z
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{3}x + \frac{1}{3}y\omega + \frac{z}{a\omega + 6\omega^2} \\
x + y + \frac{z}{a\omega + 6\omega^2}
\end{pmatrix}
\]

The solutions of (45) are:

\[
\begin{cases}
x = C_1 + BC_2 + \frac{B}{(A'-2)^{a^2}} \\
y = C_1' + B'C_2' + \frac{B'}{(A'-2)^{a^2}} \\
Z = C_3 \left( \frac{1}{a} \right)
\end{cases}
\]

with

\[
A = \frac{3\omega + 4}{\omega + 1}; \quad B = \frac{3k}{H_2 (\omega + 1)};
\]

\[
A' = \frac{(6\gamma + 6\gamma\omega + 1)}{\omega}; \quad B' = \frac{3k (6\gamma - 2\omega + 6\gamma\omega - 1)}{H_2 (\omega + 1) (2\omega + 3)};
\]

and \(C_1, C_2, C_3, C_1'\) and \(C_2'\) are integration constants. The linearized solutions (41), (42) and (43) become then:

\[
H = H_2 - \frac{k}{a^3 H_2} \left( \frac{(\omega + 1)(\omega + 3)}{(\omega + 2)(2\omega + 3)} \left( \frac{a_0}{a} \right)^2 + H_0 K_1 \left( \frac{a_0}{a} \right) + \frac{6\gamma + 6\gamma\omega + 1}{\omega + 1} \right) + H_0 K_2 \left( \frac{a_0}{a} \right) + H_0 K_3 \left( \frac{a_0}{a} \right)
\]

\[
F = F_2 + \frac{3k}{a^2 H_2} \left( \frac{\omega + 1}{(\omega + 2)(2\omega + 3)} \left( \frac{a_0}{a} \right)^2 + 3H_0 K_1 \left( \frac{a_0}{a} \right) + \frac{6\gamma + 6\gamma\omega + 1}{\omega + 1} \right) + H_0 K_2 \left( \frac{a_0}{a} \right) + H_0 K_3 \left( \frac{a_0}{a} \right)
\]
where the subscript '0' indicates the present value. $K_1$, $K_2$ and $K_3$ are dimensionless integration constants.

Letting $\omega \to \infty$, the linearized solutions (48), (49) and (50) are written in the form:

\[
H = H_2 - \frac{k}{2a_0^2H_2} \left( \frac{a_0}{a} \right)^2 + H_0K_1 \left( \frac{a_0}{a} \right)^{3+\frac{1}{\omega}} + H_0K_2 \left( \frac{a_0}{a} \right)^{3+\frac{1}{\omega}} + H_0K_3 \left( \frac{a_0}{a} \right)^{6+\frac{1}{\omega}} \tag{51}
\]

\[
F = F_2 + 3H_0K_1 \left( \frac{a_0}{a} \right)^{3+\frac{1}{\omega}} + 3H_0K_2 \left( \frac{a_0}{a} \right)^{3+\frac{1}{\omega}} + 3H_0K_3 \left( \frac{a_0}{a} \right)^{6+\frac{1}{\omega}} \tag{52}
\]

\[
Z = C_3 \left( \frac{1}{a} \right)^{\frac{3\gamma+3\omega+1}{(\omega+1)}} \tag{53}
\]

### 4.3 Cosmological parameters and dark energy

In what follows, we define the individual ratios in terms of the density parameter $\rho$ ($\Omega_i \equiv \rho_i/\rho_c$) where $\rho_i$ run for matter, radiation, cosmological constant and even curvature; $\rho_c = \frac{3H_0^2}{8\pi G}$ is the critical density and $H_0$ is the Hubble parameter today

\[
\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \quad \Omega_k = -\frac{k}{a_0^2H_0^2}, \quad \Omega_M = \frac{8\pi G\rho_M}{3H_0^2}, \quad \Omega_R = \frac{8\pi G\rho_R}{3H_0^2}. \tag{54}
\]

And from the standard Friedmann equations we have [44, 39]:

\[
\left( \frac{H}{H_0} \right)^2 = \Omega_\Lambda + \Omega_R \left( \frac{a_0}{a} \right)^4 + \Omega_M \left( \frac{a_0}{a} \right)^3 + \Omega_k \left( \frac{a_0}{a} \right)^2 \tag{55}
\]

Substituting the solution (51) in equation (55) and, in order to recover the different exponents of the equation (55), we neglect terms for which the power is higher than 4. Hence one gets, for each $\gamma$, in 3-brane space-time the expressions of the constants $K_1$, $K_2$ and $K_3$ by comparing respectively the expressions of $\Omega_i$ in (55) and those of B.D cosmology in (51) for $\omega \to \infty$.

First, let us mention that all forms of matter/energy are possible and we restrict ourselves to the $\gamma$-parameter of the equation of state for which the different exponents of the equation (55) are recovered. From (40) we have:

\[
\left( \frac{H_2}{H_0} \right)^2 = \Omega_\Lambda = \frac{\Lambda}{3H_0^2}. \tag{56}
\]
The following table summarizes the main results:

| $\gamma$ | $-1/3$ | 0 | $1/3$ | $1$ | $2/3$ | $1$ | $4/3$ | $2$ |
|----------|-------|---|------|-----|------|-----|------|-----|
| $K_1$    | $\frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$ | $K_1$ | $\frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$ | $K_1$ | $\frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$ | $\frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$ | $\frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$ | $\frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$ |
| $K_2$    | $\forall$ | $K_2$ | 0 | 0 | 0 | $\forall$ | $\forall$ | $\forall$ |
| $K_3$    | $\forall$ | $\forall$ | $\forall$ | $K_3$ | 0 | $K_3$ | 0 | $\forall$ |

The character $\forall$ means that all values of $K_i$ are possible.

In the case $\gamma = 0$ we have $K_1 + K_2 = 0$, and if we take $K_1 = \frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$, then $K_2$ should have the value $-\frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$. And for $\gamma = \frac{1}{2}$ and 1, we have $K_1 + K_3 = \frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$ and if we take $K_1 = \frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$, then $K_3$ should have the value 0.

According to the present CMB observations and type Ia SNe [40, 41, 42], our universe seems to be spatially flat and possess a non vanishing cosmological constant [45]. For a flat matter dominated universe, cosmological measurements imply that the fraction $\Omega_\Lambda$ of the contribution of the cosmological constant $\Lambda$ to present energy density of the universe is $\Omega_\Lambda \simeq 0.75$ and $\Omega_M \simeq 0.25$.

On the other hand and from the density of the microwave background photons, $\rho_R = 4.5 \times 10^{-34} \text{g/cm}^3$ which gives $\Omega_R = 2.4h^{-2} 10^{-5}$ where $0.4 < h < 1$ [39]. Therefore, we can safely neglect the contribution of relativistic particles to the total density of the universe today, which is dominated by either a non-relativistic particles (baryons, dark matter or massive neutrinos), a cosmological constant or an exotic form of matter/energy.

An interesting consequence of these considerations is that one can write the Friedmann equation (55) today as:

$$1 = \Omega_\Lambda + \Omega_M + \Omega_k$$

In what follows, we discuss all possible form of matter/energy, so that we recover the different exponents of the equation (55).

### 4.3.1 Flat universe

According to equation (58), the line $\Omega_\Lambda = 1 - \Omega_M$ corresponds to a flat universe ($\Omega_k = 0$), and separates the open universe from the closed one. Except the cases where $\gamma = 0$, $\frac{1}{2}$ and 1, the table (57) shows that $K_1 = \frac{\Omega_M}{2\sqrt{\Omega_\Lambda}}$.

If $K_1 = \frac{1-\Omega_\Lambda}{2\sqrt{\Omega_\Lambda}}$, the universe becomes flat. If $K_1 < \frac{1-\Omega_\Lambda}{2\sqrt{\Omega_\Lambda}}$, we obtain an open universe, otherwise the universe is close.

With the numerical value, $K_1 \simeq 0.14434$, we conclude that our universe is flat and the theory is in agreement with the observation data.

### 4.3.2 Accelerating universe

Consider now the deceleration parameter [3, 44, 39]

$$q_0 = \Omega_k + \frac{1}{2}\Omega_M - \Omega_\Lambda.$$  

Using the present results obtained on density parameters we neglect $\Omega_k$ and one can parameterize the matter/energy content of the universe with just two
components: the matter, characterized by $\Omega_M$, and the vacuum energy characterized by $\Omega_\Lambda$, i.e., $q_0 = \frac{1}{2} \Omega_M - \Omega_\Lambda$.

A uniform expansion ($q_0 = 0$) corresponds to the line $\Omega_\Lambda = \frac{\Omega_M}{2}$ separating the accelerating from the decelerating universe and $K_1$ verify:

$$K_1 = \frac{\Omega_M}{2\sqrt{\Omega_\Lambda}} = \sqrt{\Omega_\Lambda}. \quad (60)$$

If $K_1 < \sqrt{\Omega_\Lambda}$, the universe is in an accelerating phase while $K_1 > \sqrt{\Omega_\Lambda}$ corresponds to a decelerating phase of the universe.

Consequently, the $(\Omega_M, \Omega_\Lambda)$ plane shows that we live in an accelerating flat universe, since numerical calculations show that $K_1 < \sqrt{\Omega_\Lambda}$, which is in accordance with the experimental data of Ia SNe [40].

5 Conclusions

In this work we have examined the behavior of 3-brane of Brans-Dicke cosmology which differs from other Brans-Dicke cosmology by the fact that the 5D approach affects the ordinary matter (by the square of energy density) but not the Brans-Dicke field. This approach clearly shows how to describe the early and late time behavior of the universe first by means of the scale factor (section 3) and second by means of the cosmological parameters (section 4) for large value of the $\omega$-parameter. Let us notice also that the present work is a generalization of our previous work [16] in which we did not consider the intermediate case.

Furthermore, this approach gives two possibilities to describe the universe. The first one, consider that the universe underwent a bounce state and hence avoided the big bang singularity. The second one in which the universe will undergo a turnaround state and therefore avoiding, probably, the big rip singularity. In the other cases where no bounce nor turnaround state are present, the universe began to expand from the big bang singularity in the past, or/and will meet the big rip singularity and its dramatic consequences. The two possibilities are consistent with the fact that, today, the universe undergoes an accelerating period. However, we can opt for the case in which the universe underwent a bounce state in the past and will undergo a turnaround state in the future in order to avoid the dramatic consequences of the big rip and to have the possibility of crossing the phantom divide line.

Finally, we conclude that the assumption of 3-brane behavior of Brans-Dicke cosmology gives an interesting results and enables us to explore this approach in more detail in future investigations.

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