Sensitivity Analysis with respect to a Stock Price Model with Rough Volatility via a Bismut-Elworthy-Li Formula for Singular SDEs

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Abstract

In this paper, we show the existence of unique Malliavin differentiable solutions to SDE’s driven by a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$ and singular, unbounded drift vector fields, for which we also prove a stability result. Further, using the latter results, we propose a stock price model with rough and correlated volatility, which also allows for capturing regime switching effects. Finally, we also derive a Bismut-Elworthy-Li formula with respect to our stock price model for certain classes of vector fields.

keywords: Bismut-Elworthy-Li formula, singular SDEs, fractional Brownian motion, Malliavin calculus, stochastic flows, stochastic volatility

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1 Introduction

Consider the stochastic differential equation (SDE)

$$dX^x_t = b(t, X^x_t)dt + dB^H_t, 0 \leq t \leq T, X^x_0 = x \in \mathbb{R}^d,$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Borel-measurable function and $B^H_t, 0 \leq t \leq T$ is a fractional Brownian motion with Hurst parameter $H < \frac{1}{2}$.

It was shown in [7] by using techniques from Malliavin calculus (see e.g. [27]) that the SDE (1) admits the existence of a unique (global) strong solution $X^x$, when

$$b \in L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)).$$

(2)

Here, a solution to (1) is called strong, if it can be represented as a (progressively) measurable functional of the driving noise $B^H$.

Related results in the direction of [7], whose proofs, however, are based on different methods, can be e.g. found in [28], [11] and [24].

By employing the Malliavin calculus approach developed in the fundamental papers of [18] and [19] in the Wiener process case, the authors in [2] derive a so-called Bismut-Elworthy-Li formula (BEL-formula) for (strong) solutions to (1), when the vector field $b$ is singular in the sense of (2). Roughly speaking, such a formula, gives a representation of expressions of the form

$$\frac{d}{dx} E[\Phi(X^x_T)].$$

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for functions Φ : \( \mathbb{R}^d \rightarrow \mathbb{R} \), which doesn't involve the derivative of Φ. In this context, we also mention the interesting work \[17\], where a BEL-formula is established for (differentiable) functional drift coefficients \( b \) and used to study Harnack type of inequalities. See also the article \[26\] in the case of a Wiener process.

In this paper, we aim at extending the result in \[2\] based on Malliavin calculus to the case, when \( b \in L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)) \) (see Theorem \[3\]). Further, we also prove as one of our main results the existence of a Malliavin differentiable unique solution to a SDE of the type \( (1) \) in the case of vector fields \( b \) given by a sum of a merely bounded Borel-measurable function and a Lipschitz continuous function \( b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) (Theorem \[13\]). We remark here that the latter result, actually also provides an alternative construction method of strong solutions in \[28\], where the authors use a comparison theorem for SDE’s. We also refer to \[6\] in the Wiener process case.

Further, we propose a stock price model of Black-Scholes type with a rough stochastic volatility, whose dynamics is subject to a SDE of the type \( (1) \) and is correlated with the driving noise of the stock price SDE. Here, we allow for singular drift coefficients, which are either in \( L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)) \) or a sum of a bounded and a Lipschitz continuous function. In fact, the selection of a singular or discontinuous coefficient \( b \) in our model is used to capture "regime switching effects" with respect to the volatility dynamics, which may be due to financial crashes, market regulations or even natural disasters. Moreover, we derive with respect to our model, which generalizes the stock price model in \[2\] to the case of correlated volatility and stock price noise and which allows (compared to \[2\]) for an Ornstein-Uhlenbeck type of volatility dynamics with regime switching feature, a BEL-formula for both cases of vector fields \( b \) (Theorem \[7\], Theorem \[10\] and Theorem \[12\]). We comment on here that such representations are known in finance as greeks which are sensitivity parameters for measuring e.g. the changes of fair values of options with respect to the initial underlying stock price (i.e. delta) or with respect to the initial volatility (i.e. vega). See \[18\], \[19\] or \[15\] for more information.

Finally, we prove a stability estimate for solutions to \( (1) \) with unbounded coefficients (see Proposition \[21\]), which in fact plays a crucial role in \[30\] and \[31\] for showing path-by-path uniqueness of solutions to \( (1) \) in the case of a Wiener process and bounded vector fields and which may be also applied to the situation in our paper. Path-by-path uniqueness, which is a much stronger concept than pathwise uniqueness and which goes back to \[13\], actually entails the existence of a measurable set \( \Omega^* \) with probability mass 1 such that for all \( \omega \in \Omega^* \) there exists a unique deterministic solution \( X^\omega(\omega) \) to \( (1) \) in the space of continuous functions (uniformly in the initial condition).

Our paper is organized as follows: In Section 2 we prove a BEL-formula (Theorem \[7\]) for vector fields \( b \in L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)) \) based on an existence and a uniqueness result for strong solutions (Theorem \[3\]). In Section 3 we introduce our stock price model, with respect to which we establish BEL-formulas (Theorem \[10\], Theorem \[12\]). We also prove the Malliavin differentiability solu-
tion of solutions in the case of vector fields which are a sum of a bounded and Lipschitz continuous function (Theorem 13) and the above mentioned stability result in this case.

2 A Bismut-Elworthy-Li formula for integrable vector fields

In this Section we first generalize a result in [7] on the existence of a unique global strong solution of (1) to the case of integrable vector fields \( b \). Further, we show that such a solution is Malliavin differentiable and Sobolev differentiable with respect to the initial condition, if the Hurst parameter \( H \) is small enough. Finally, based on the latter result, we establish a Bismut-Elworthy-Li formula (BEL-formula) for solutions to (1).

Let \( B_H^t, t \geq 0 \) be a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( H \in (0, 1) \), that is a \( d \)-dimensional stochastic process with components given by independent one-dimensional fractional Brownian motions with Hurst parameter \( H \in (0, 1) \) on some complete probability space \( (\Omega, \mathcal{F}, \mu) \), which are centered Gaussian processes with a covariance structure \( R_H(t, s) \) of the form

\[
R_H(t, s) = E[B_H^t B_H^s] = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H})
\]

for all \( t, s \geq 0 \). If \( H = \frac{1}{2} \) the fractional Brownian motion is a Wiener process. See the Appendix for more details. We also recall \( B_H^t \) the representation

\[
B_H^t = \int_0^t K_H(t, s) I_d \times d dB_s
\]

for a \( d \)-dimensional Brownian motion \( B_\cdot \), where \( I_d \times d \in \mathbb{R}^{d \times d} \) is the unit matrix and \( K_H \) the kernel as given in (48) in the Appendix.

Consider the SDE

\[
dX^x_t = b(t, X^x_t)dt + dB_H^t, X^x_0 = x, 0 \leq t \leq T.
\]

for \( H < \frac{1}{2} \).

In what follows, we will also make use of the following function spaces:

\[
\begin{align*}
L^1_{\infty} & : = L^1(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)), \\
L^\infty & : = L^\infty(\mathbb{R}^d; L^\infty([0, T]; \mathbb{R}^d)), \\
L^{1, \infty} & : = L^1_{\infty} \cap L^\infty.
\end{align*}
\]

In order to prove our first result on a regular unique strong solution to (1), we need the following auxiliary result:
Lemma 1 Let $H < \frac{1}{d+1}$ and $b \in L^\infty([0,T];L^1(\mathbb{R}^d;\mathbb{R}^d)) \supset L^1_{\infty}$. Then there exists for all $k \geq 0$ a continuous function $L : [0, \infty) \to [0, \infty)$ depending on $H, T, d$ and $k$ such that

$$
E \left[ \exp \left( k \int_0^T \left\| K_H^{-1} \left( \int_0^u b(s, x + B^H_s) ds \right) \right\|^2 du \right) \right] 
\leq L(||b||_{L^\infty([0,T];L^1(\mathbb{R}^d)))},
$$

where the operator $K_H^{-1}$ is defined as in (51) in the Appendix.

Proof. Assume without loss of generality that $b \in L^\infty([0,T];L^1(\mathbb{R}^d;\mathbb{R}^d))$. Using the definition of $K_H^{-1}$, we find that

$$
K_H^{-1} \left( \int_0^u b(s, x + B^H_s) ds \right) (u) 
= u^{H-\frac{1}{2}} H_{0+} \left( H - \frac{1}{2} \right) \frac{1}{\Gamma (\frac{H}{2} - H)} \int_0^u s^{\frac{1}{2}-H} (u-s)^{-\frac{1}{2}-H} b(s, x + B^H_s) ds
$$

\[= \frac{1}{\Gamma (\frac{H}{2} - H)} u^{-\frac{1}{2}-H} \int_0^u \left( \frac{s}{u} \right)^{\frac{1}{2}-H} (1 - \frac{s}{u})^{-\frac{1}{2}-H} b(s, x + u^H B^H_s) ds \]

\[\overset{law}{=} \frac{1}{\Gamma (\frac{H}{2} - H)} u^{\frac{1}{2}-H} \int_0^1 \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, s) b(su, x + u^H B^H_s) ds, \]

where

$$
\gamma_{\alpha, \beta}(u, s) := s^\alpha (u - s)^\beta, u > s.
$$

So

$$(K_H^{-1} \left( \int_0^u b(s, x + B^H_s) ds \right) (u) )^{2m}$$

\[\overset{law}{=} \frac{1}{\Gamma (\frac{H}{2} - H)^{2m}} u^{2m(\frac{1}{2}-H)} \left( \int_0^1 \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, s) b(su, x + u^H B^H_s) ds \right)^{2m} \]

\[= \frac{1}{\Gamma (\frac{H}{2} - H)^{2m}} u^{2m(\frac{1}{2}-H)} (2m)! \times \]

$$
\times \int_{\Delta_{n;1}}^m \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, s_1) b(s_1 u, x + u^H B^H_{s_1}) \cdots \gamma_{-\frac{1}{2}-H, \frac{1}{2}-H}(1, s_{2m}) b(s_{2m} u, x + u^H B^H_{s_{2m}}) ds_1 \cdots ds_{2m}.
$$
Hence,

\[
E \left[ \left( \int_0^T \left( K_{\frac{1}{H}}^{-1} \left( \int_0^T b(s, x + B^H_s) ds \right)(u) \right)^2 du \right)^m \right] \\
\leq T^{m-1} \int_0^T E \left[ K_{\frac{1}{H}}^{-1} \left( \int_0^T b(s, x + B^H_s) ds \right)^{2m} \right] du \\
= T^{m-1} \int_0^T \frac{1}{\Gamma(\frac{1}{H} - 2m)^2m} u^{2m(\frac{1}{H} - \frac{1}{2})} (2m)! \times \\
\times \int_{\Delta_0^T} \prod_{j=1}^{2m} \gamma_{\frac{1}{H}, \frac{1}{H}}(1, s_j) E \left[ \prod_{j=1}^{2m} b(s_j u, x + u^H B^H_{s_j}) \right] ds_1 \ldots ds_{2m} du.
\]

On the other hand,

\[
E \left[ \prod_{j=1}^{2m} b(s_j u, x + u^H B^H_{s_j}) \right] \\
= \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^{2m} b(s_j u, x + u^H y_j) \prod_{l=1}^d \frac{1}{(2\pi)^m \det(Q(s))^m} \exp \left( -\frac{1}{2} \left( y^{(l)} \right)^T Q^{-1}(s) y^{(l)} \right) dy^{(1)} \ldots dy^{(d)},
\]

where

\[ y_j := (y_j^{(1)}, \ldots, y_j^{(d)}), j = 1, \ldots, 2m, y^{(l)} := (y_1^{(l)}, \ldots, y_{2m}^{(l)}), l = 1, \ldots, d \]

and

\[ Q(s) := \text{Cov} \left[ B^{H,1}_{s_1}, \ldots, B^{H,1}_{s_{2m}} \right], \]

where \( B^{H}_u = (B^{H,1}_u, \ldots, B^{H,d}_u)^* \). It follows from the strong local non-determinism of the fractional Brownian motion (see e.g. Lemma 4.1 and 4.2 in \[8\]) that

\[ \det(Q(s)) \geq K(H)s_1^{2H}(s_2 - s_1)^{2H} \ldots (s_{2m} - s_{2m-1})^{2H} \]

for a constant \( K(H) \) not depending on \( m \). Thus

\[
\left| E \left[ \prod_{j=1}^{2m} b(s_j u, x + u^H B^H_{s_j}) \right] \right| \\
\leq \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^{2m} \left| b(s_j u, x + u^H y_j) \right| \prod_{l=1}^d \frac{1}{(2\pi)^m \det(Q(s))^m} \exp \left( -\frac{1}{2} \left( y^{(l)} \right)^T Q^{-1}(s) y^{(l)} \right) dy^{(1)} \ldots dy^{(d)} \\
\leq \frac{1}{(2\pi)^{md} \det(Q(s))^d} \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^{2m} \left| b(s_j u, y_j) \right| dy_j \ldots dy_{2m} \\
= \frac{1}{(2\pi)^{md} \det(Q(s))^d} u^{-2mHd} \int_{(\mathbb{R}^d)^{2m}} \prod_{j=1}^{2m} \left| b(s_j u, y_j) \right| dy_1 \ldots dy_{2m} \\
\leq \frac{1}{(2\pi)^{md} \det(Q(s))^d} u^{-2mHd} \left( \sup_{0 \leq t \leq T} \int_{\mathbb{R}^d} |b(t, y)| dy \right)^{2m}.
\]
So we obtain that
\[
E \left[ \left( \int_0^T (K_H^{-1} \int_0^T b(s, x + B^H_s)ds)(u) \right)^2 du \right]^m
\]
\[
\leq T^{m-1} \int_0^T \frac{1}{\Gamma(\frac{1}{2} - H)^{2m}} u^{2m(\frac{1}{2} - H)}(2m)! \times
\]
\[
\times \int \Delta_{\delta_1,1}^{2m} \prod_{j=1}^{2m} \gamma_{\frac{1}{2} - H, \frac{1}{2} - H}(1, s_j) \frac{1}{(2\pi)^{md} \det(Q(s))^{\frac{1}{2}}} u^{-2mH} \left( \sup_{0 \leq t \leq T} \int_{R^d} |b(t, y)| dy \right)^{2m} ds_1 ... ds_{2m} du
\]
\[
= T^{m-1} \left( \sup_{0 \leq t \leq T} \int_{R^d} |b(t, y)| dy \right)^{2m} \int_0^T \frac{1}{\Gamma(\frac{1}{2} - H)^{2m}} u^{2m(\frac{1}{2} - H(d+1))}(2m)! \times
\]
\[
\times \int \Delta_{\delta_1,1}^{2m} \prod_{j=1}^{2m} \gamma_{\frac{1}{2} - H, \frac{1}{2} - H}(1, s_j) \frac{1}{(2\pi)^{md} \det(Q(s))^{\frac{1}{2}}} ds_1 ... ds_{2m} du.
\]
Further, we know from Lemma A.5 in \cite{8} that
\[
(2m)! \int \Delta_{\delta_1,1}^{2m} \prod_{j=1}^{2m} \gamma_{\frac{1}{2} - H, \frac{1}{2} - H}(1, s_j) \frac{1}{(2\pi)^{md} \det(Q(s))^{\frac{1}{2}}} ds_1 ... ds_{2m}
\]
\[
\leq C_{H,d}^m(m!)^{2H(1+d)}.
\]
Therefore, we get that
\[
E \left[ \left( \int_0^T (K_H^{-1} \int_0^T b(s, x + B^H_s)ds)(u) \right)^2 du \right]^m
\]
\[
\leq T^{m-1} \frac{1}{\Gamma(\frac{1}{2} - H)^{2m}} \left( \sup_{0 \leq t \leq T} \int_{R^d} |b(t, y)| dy \right)^{2m} TT^{2m(\frac{1}{2} - H(d+1))} C_{H,d}^m(m!)^{2H(1+d)}.
\]
Hence,
\[
E \left[ \exp(k \int_0^T (K_H^{-1} \int_0^T b(s, x + B^H_s)ds)(u) \right)^2 du \]
\[
\leq \sum_{m \geq 0} \frac{k^m C_{H,d}^m(m!)^{2H(1+d)} T^{m-1} \frac{1}{\Gamma(\frac{1}{2} - H)^{2m}} \left( \sup_{0 \leq t \leq T} \int_{R^d} |b(t, y)| dy \right)^{2m} TT^{2m(\frac{1}{2} - H(d+1))} }
\]
\[
= L \|b\|_{L^\infty([0,T];L^1(R^d))}
\]
for a continuous function \( L : [0, \infty) \rightarrow [0, \infty) \) depending on \( H, T, d \) and \( k \).

**Corollary 2** Assume that \( H < \frac{1}{2(d+1)} \) and \( b \in L^\infty([0,T];L^1(R^d,R^d)) \). Then, there exists a weak solution to \( \mathcal{H} \). Further, suppose that
\[
(X_i^{(i)}, B_i^{(i), H}), (\Omega^{(i)}, \mathcal{F}^{(i)}, \mu^{(i)}), \{ \mathcal{F}_t^{(i)} \}_{0 \leq t \leq T}, i = 1, 2
\]

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are two weak solutions to (4) \((B^{i,H} \cdot \cdot \cdot \text{denotes a fractional Brownian motion with Hurst parameter } H \text{ with respect to } X^{(i)})\). Require that

\[
\int_0^T \left\| \mathcal{K}_H^{-1} \left( \int_0^s b(s, X^{(i)}_s) ds \right) \right\|^2 du < \infty \mu^{(i)} - \text{a.e., } i = 1, 2. \tag{5}
\]

Then both solutions are equal in law.

**Proof.** The proof of the existence of a weak solution to (4) is a direct consequence of Lemma 1 in connection with Girsanov’s theorem (Theorem 2.4) and the Novikov condition. The uniqueness of weak solutions to (4) follows from a very similar proof of Proposition 5.3.10 in [22].

We are now coming to an existence and uniqueness result of strong solutions to the SDE (4) which extends the result in [7] for \(b \in L_{\infty, \infty}^1\) and \(d > 1\):

**Theorem 3** Suppose that \(b \in L_{\infty, \infty}^1\) and \(H < \frac{1}{2(d+2)}\). Then there exists a unique (global) strong solution \(X_x\) of the SDE (4) in the class of stochastic processes satisfying condition (5). Furthermore the solution \(X_x\) is Malliavin differentiable in the direction of the Brownian motion \(B\) at each time point \(t \in [0, T]\) and for all initial conditions \(x \in \mathbb{R}^d\). Moreover, \(X_t\) is locally Sobolev differentiable \(\mu - \text{a.e.},\) that is more precisely

\[
X_t \in \bigcap_{p \geq 2} L^2(\Omega; W^{1,p}(U))
\]

for bounded and open sets \(U \subset \mathbb{R}^d\).

**Remark 4** Compare also the results in [11] and [28], which cannot be applied to the case, when \(b \in L_{\infty, \infty}^1\) for \(d > 1\).

**Remark 5** We also mention the interesting result in [24], which contains the existence of a unique strong solution in Theorem 3 as a special case. However, the method for proving this result, which is based on a stochastic sewing lemma and the Yamada-Watanabe approach, doesn’t yield, as in our case, regularity of solutions in the sense of Malliavin and Sobolev differentiability.

**Remark 6** Using the same line of reasoning as in the proof of Theorem 3 (see below), we remark that the above result also holds, if the driving noise \(B^H\) is replaced by \(\rho_1 B^H + \rho_2 W\), where \(W\) is a Wiener process independent of \(B^H\) and \(\rho_1, \rho_2 \in \mathbb{R} \setminus \{0\}\).

**Proof of Theorem 3** The proof of this result is very similar to that of [7] (see also [8]). Therefore, we give here a sketch of the proof, where we indicate which modifications in the proof are needed.
The idea for the proof of the existence of a unique Malliavin differentiable solution \( X^x \) of the SDE (3) relies on a compactness criterion for square integrable functionals of Wiener processes in (12). The proof consists of the following steps:

1. We consider a sequence of compactly supported smooth functions \( b_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, n \geq 1 \) such that

\[
\lim_{n \to \infty} b_n \to b \text{ in } L^1_b.
\]

The objective here is to show that for each \( 0 \leq t \leq T \) the sequence \( X_{t}^{x, n} \) associated with the SDE

\[
X_{u}^{x, n} = x + \int_0^u b_n(s, X_{s}^{x, n}) \, ds + B_u, \quad 0 \leq u \leq T
\]

for \( H < \frac{1}{d(d+2)} \) is relatively compact in \( L^2(\mu; \mathbb{R}^d) \) by employing the compactness criterion for \( L^2(\mu; \mathbb{R}^d) \) (\( \mu \) Wiener measure) in (12). In fact, the following estimates in (7) give a sufficient criterion for the relative compactness of the sequence \( X_{t}^{x, n} \), \( n \geq 1 \):

\[
\| D^B X_{t}^{x, n} \|_{L^2([0, t] \times \Omega)}^2 \leq C_1(\| b_n \|_{L^1_b}) \tag{7}
\]

and

\[
\int_0^t \int_0^u \| D^B X_{s}^{x, n} - D^B X_{s'}^{x, n} \|_{L^2(\mu)}^2 \, ds'ds \leq C_2(\| b_n \|_{L^1_b}), \tag{8}
\]

where \( D^B \) denotes the Malliavin derivative in the direction of the Brownian motion \( B \) and where \( C_1, C_2 : [0, \infty) \to [0, \infty) \) are continuous functions depending on \( H, T \) and \( d \). By using Girsanov’s theorem in connection with Lemma (1) (see for more detailed explanations the next steps) we obtain under the assumptions of Theorem (3) the same estimates as in (7), (8) with the only difference that \( \| b_n \|_{L^1_b} \) in the functions \( C_1, C_2 \) is replaced by \( \| b_n \|_{L^1_b} \).

2. By applying the Malliavin derivative \( D^B \) to both sides of (6) and the chain rule for the Malliavin derivative (see (27) or (15)), we find that

\[
D^B X_{u}^{x, n} = x + \int_0^u b_n'(s, X_{s}^{x, n}) D^B X_{s}^{x, n} \, ds + \int_0^u K_H(s, \theta) \, Id, \quad 0 \leq \theta < u \leq T, \quad n \geq 1, \text{ a.e.,}
\]

where \( b_n' \) is the spatial Fréchet derivative of \( b_n \). Then, Picard iteration gives

\[
D^B_0 X_{u}^{x, n} = K_H(u, \theta) Id + 
\]

\[
\sum_{m \geq 1} \int_{\theta < s_1 < \ldots < s_m < u} b_n'(s_m, X_{s_m}^{x, n}) \ldots b_n'(s_1, X_{s_1}^{x, n}) K_H(s_1, \theta) ds_1 \ldots ds_m
\]

where the convergence is in \( L^p \)-sense. Then, in order to "eliminate" the derivatives \( b_n' \) in (10) we can apply Girsanov’s change of measure (see Theorem (24) in the Appendix) combined with Lemma (1) and the following "local time variational calculus argument":

\[
\int_{\Delta_{\theta, t}} D^\alpha f(s, B_s^H) \, ds = \int_{(\mathbb{R}^d)^n} \Lambda^f_{\alpha}(\theta, t, z) \, dz \tag{11}
\]
for a random field $\Lambda^f_{\alpha}$ which (in the case of time-homogeneous vector fields) can be interpreted as a (scaled) local time on the $m$-dimensional simplex

$$\Delta^m_{\theta,t} := \{(s_m, \ldots, s_1) \in [0,T]^m : \theta < s_m < \ldots < s_1 < t\}.\quad (12)$$

See [52] in the Appendix for the precise definitions of the notation involved. Here we apply (11) to the case, when

$$D^\alpha f(s, z) = \prod_{j=1}^m D^\alpha_j f_j(s_j, z_j) = \frac{b_n}{\text{barshort}}(s_m, x + z_m)\ldots\frac{b_n}{\text{barshort}}(s_1, x + z_1)K_H(s_1, \theta).$$

Finally, certain estimates with respect to $\Lambda^f_{\alpha}$ (see Theorem 26 in the Appendix) yield the bounds (7), (8) (with $\|b_n\|_{L^1,\infty,\infty}$ in the functions $C_1, C_2$ is replaced by $\|b_n\|_{L^1,\infty}$).

3. step: The bounds of the type (7), (8) in step 2 enable us to apply the compactness criterion in [12] and we obtain that

$$X_t^{x,n} \xrightarrow{n \to \infty} Y_t \text{ in } L^2(\mu)$$

for a subsequence $n_l(t), l \geq 1$. However, by using a very similar proof of Lemma 5.5 in [8] in connection with Lemma 1 (or see [7]) it turns out that

$$Y_t = E[X_t^x | F_t],$$

where $X_t^x$ is the weak solution to (4) of Corollary 2 and where $F_t, 0 \leq t \leq T$ is the (augmented) filtration generated by $B^H$. Thus

$$X_t^{x,n} \xrightarrow{n \to \infty} Y_t \text{ in } L^2(\mu)\quad (13)$$

Using the latter, one shows for all bounded and continuous functions $\varphi$ that

$$\varphi(E[X_t^x | F_t]) = E[\varphi(X_t^x) | F_t] \text{ a.e.,}$$

which implies the $F_t$-adaptedness of $X_t^x$. Hence, the weak solution in Corollary 2 is a strong solution. Strong uniqueness of solutions of (4) is also a consequence of Corollary 2. Further, Malliavin differentiability of the solution directly follows from the compactness criterion in [12].

As for the assertion of the Sobolev regularity of the solution with respect to the initial condition, one can invoke the "local time variational argument" combined with Girsanov's theorem (Theorem 24) and Lemma 1 in step 2 in the same way and derive the following estimate: For $p \geq 2$ and $H < \frac{1}{2(d+2)}$, we have

$$\sup_{x \in \mathbb{R}^d} E\left[\left\|\frac{\partial}{\partial x} X_t^{x,n} \right\|^p\right] \leq C_{p,H,d,T}(\|b_n\|_{L^1_\infty}, \|b_n\|_{L^1_\infty}) < \infty, n \geq 1$$

(14)

for some continuous function $C_{p,H,d,T} : [0, \infty)^2 \to [0, \infty)$. However, because of Lemma 1 we obtain under assumptions of Theorem 3 the bound

$$\sup_{x \in \mathbb{R}^d} E\left[\left\|\frac{\partial}{\partial x} X_t^{x,n} \right\|^p\right] \leq L_{p,H,d,T}(\|b_n\|_{L^1_\infty}) < \infty, n \geq 1$$

(15)
for some continuous function $L_{p,H,d,T} : [0, \infty) \rightarrow [0, \infty)$, which yields the spatial regularity of the solution.

The next result is a generalization of that in [2] to the case of unbounded vector fields in $L^1_{\infty} = L^1([0,\infty);\mathbb{R}^d)$:

**Theorem 7 (Bismut-Elworthy-Li formula for $b \in L^1_{\infty}$)** Assume that $b \in L^1_{\infty}$ and $H < \frac{1}{2(d+2)}$. Let $X_t^x$ be the unique strong solution to the SDE

$$dX_t^x = b(t, X_t^x)dt + dB^H_t, \quad X_0^x = x,$$

Suppose that $U$ is a bounded and open subset of $\mathbb{R}^d$ and $\Phi : \mathbb{R}^d \rightarrow \mathbb{R}$ a Borel measurable function with

$$\Phi(X_T^x) \in L^2(\Omega \times U, \mu \times dx).$$

Further, let $a : [0, T] \rightarrow \mathbb{R}$ be a bounded Borel measurable function such that

$$\int_0^T a(s)ds = 1.$$

Then, we have the following representation

$$\frac{\partial}{\partial x} E[\Phi(X_T^x)] = C_H E[\Phi(X_T^x)] \int_0^T u^{-H/2} \int_0^T a(s-u)(s-u)^{1/2-H} s^{H/2} (\frac{\partial}{\partial x} X_{s-u}^x) \ast dB^H_s\mu(d\omega),$$

for all $x \in U$ a.e., $0 < t \leq T$, where $\ast$ is the transpose of a matrix and where $C_H = 1/(c_H\Gamma(\frac{1}{2} + H)\Gamma(\frac{1}{2} - H))$ for

$$c_H = \frac{2H}{(1-2H)B(1-2H, H+1/2)^{1/2}}.$$

Here $\Gamma$ and $B$ are the Gamma and Beta function, respectively.

**Proof.** The proof is very similar to that in [2]. However, since we will need parts of the proof in Section 3, it can be found in the Appendix. ■

**Remark 8** We mention that $\frac{\partial}{\partial x} X_t^x, 0 \leq t \leq T$ in the BEL-formula is a process $Y : [0, T] \times \Omega \times U \rightarrow \mathbb{R}^{d \times d}$ in $L^2([0, T] \times \Omega \times U, \mathcal{P} \otimes \mathcal{B}(U); \mathbb{R}^{d \times d})$ such that $Y_t(\omega)$ coincides with the Sobolev derivative of $X_t(\omega)$ (t, $\omega$)-a.e. Here, $\mathcal{P}$ denotes the predictable $\sigma$-algebra with respect to the $\mu$-augmented filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ generated by $B^H$.

**Remark 9** From a financial mathematics point of view the expression on the right hand side of (16) has the interpretation of the greek delta, that is a sensitivity measure, which measures changes of the fair value of a financial claim with
payoff function $\Phi$ and underlying stock price processes $X^x$ with respect to the initial prices $x \in \mathbb{R}^d$ of the stocks. However, since $\frac{\partial}{\partial x} X^x_t$ is a Sobolev derivative, this sensitivity measure is only defined for $x \in U$ a.e., which makes it rather unusable in financial applications. In order to overcome this problem, one may choose as in [2] a continuous version of $\frac{\partial}{\partial x} X^x_t$ by using the following estimate, which can derived (by means of Girsanov’s theorem and Lemma 7) in the same way as in Proposition 10, [2] in the case of $b \in L^1((0, T) \times \mathbb{R}^d)$ and $p > 2$. Then, if $H < \frac{1}{2(d+3)}$, we have

$$\sup_{x \in \mathbb{R}^d} E\left[\left\| \frac{\partial^2}{\partial x^2} X^{x,x}_t \right\|^p \right] \leq C_{p,H,d,T}(\|b\|_{L^\infty}) < \infty \quad (17)$$

for some continuous function $C_{p,H,d,T} : [0, \infty) \rightarrow [0, \infty)$.

3 A regularity result for SDE’s with non-integrable, unbounded vector fields and a stock price model with regime switching correlated rough volatility

As one of the main results (Theorem 13) in this Section we prove the Malliavin differentiability to (4) for $d = 1$, when the vector field $b$ is given by a sum of a bounded and Lipschitz continuous function. This result, whose proof is based on the compactness in [12] and a transfer principle between a Wiener process and a fractional Brownian motion, provides a generalization of Theorem 3.1 in [6] in the case of fractional noise for $H < \frac{1}{2}$. Further, we introduce an extension of a stock price model with regime switching rough volatility in [2], which allows for correlation between the driving noise of the stock price SDE and that of the stochastic volatility. Then, we derive (as in [2]) a BEL-formula with respect to this model. Here, we first study the case, when the drift $b$ of the SDE for the volatility process belongs to $L^\infty$. Then, we examine the case, when $b$ can be decomposed as a sum of a bounded and Lipschitz continuous function. Finally, we conclude the paper with the proof of a stability result for solutions to (4) in the setting of Theorem 13.

Let us now consider the following model for stock prices $S^{x_1,x_2}_t, 0 \leq t \leq T$ with stochastic volatility $\sigma^{x_2}_t, 0 \leq t \leq T$ given by the solution to the SDE

$$S^{x_1,x_2}_t = x_1 + \int_0^t \mu S^{x_1,x_2}_u du + \int_0^t g(\sigma^u) S^{x_1,x_2}_u dW_u$$

$$\sigma^x_t = x_2 + \int_0^t b(u, \sigma^u) du + \sqrt{1 - \rho^2} B^H_t + \rho W_t, x_1, x_2 \in \mathbb{R}, 0 \leq t \leq (18)$$

where $W_t$ is a Wiener process, which is independent of a fractional Brownian motion $B^H_t$ with Hurst parameter $H < \frac{1}{2(d+3)} = \frac{1}{6}$ for $d = 1$. Here $\mu \in \mathbb{R},$
\( b \in L^1_\infty \) and \( g : \mathbb{R} \rightarrow (\alpha, \infty) \) is a function from \( C^2_\rho(\mathbb{R}) \) for some \( \alpha > 0 \). Further, \( \rho \in (-1, 1) \) describes the correlation between \( B^H \) and \( W \).

We mention that the stock price model \([18]\) covers that in \([2]\) as a special case, when \( \rho = 0 \).

In order to establish a BEL-formula for the stock price model \([18]\) in the case of vector fields \( b \in L^1_\infty \), we shall follow here closely the arguments and exposition in \([2]\).

In the sequel, let \( \Omega = \Omega_1 \times \Omega_2 \) for sample spaces \( \Omega_1, \Omega_2 \), which accommodate \( W \) and \( B^H \).

For the time being, require that \( b \in C^\infty_c((0, T) \times \mathbb{R}^d) \). Then, one can show (see e.g. \([27]\)) that \( X_t^x := (S_t^{x_1,x_2}, \sigma_t^{x_2}) \), \( x = (x_1, x_2) \) is Malliavin differentiable with respect to \( Z = (Z(1), Z(2)) = (W, B^H)^* \) with Malliavin derivative \( D = (D^W, D^H)^* \) and we obtain that

\[
D_s X_t^x = \int_s^t \begin{pmatrix} \mu & 0 \\ 0 & b'(u, \sigma_u^{x_2}) \end{pmatrix} D_s X_u^x du \\
+ \left( \sum_{j=1}^2 \int_s^t \sum_{l=1}^2 \frac{\partial}{\partial x_l} a_{ij}(S_u^{x_1,x_2}, \sigma_u^{x_2})(D_s X_u^x)_{1l} dZ_j(u) \right)_{1 \leq i,j \leq 2} \\
+ \chi_{(0,1)}(s) \left( a_{ij}(S_s^{x_1,x_2}, \sigma_s^{x_2}) \right)_{1 \leq i,j \leq 2}
\]

where

\[
(a_{ij}(x_1, x_2))_{1 \leq i,j \leq 2} = \begin{pmatrix} g(x_2) x_1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}.
\]

Using very similar arguments as e.g. in \([23]\), we find that \( X_t^{x,y} \) is twice continuously differentiable with respect to \( (x, y) \). Then, by following a similar line of reasoning as in the proof of Theorem \([4]\) in combination with a substitution formula for Wiener integrals \([27]\), Theorem 3.2.9], we get that

\[
D_s X_t^x = \frac{\partial}{\partial x} X_t^{x,y} \chi_{(0,1)}(s) \left( a_{ij}(S_s^{x_1,x_2}, \sigma_s^{x_2}) \right)_{1 \leq i,j \leq 2}.
\]

In a similar manner, we observe that

\[
\frac{\partial}{\partial x} E[\Phi(X_T^{x,y})] = E[\Phi'(X_T^x) \frac{\partial}{\partial x} X_T^{x,y} \frac{\partial}{\partial x} X_T^x]
\]

for payoff functions \( \Phi \in C^\infty_c(\mathbb{R}^2) \). Hence,

\[
\frac{\partial}{\partial x} E[\Phi(X_T^x)] = E[\Phi'(X_T^x) D_x X_T^x \left( a_{ij}(S_s^{x_1,x_2}, \sigma_s^{x_2}) \right)^{-1}_{1 \leq i,j \leq 2} \frac{\partial}{\partial x} X_T^{x}].
\]
So, for $a \in L^\infty([0, T])$ with $\int_0^T a(s)ds = 1$ we can employ the chain rule with respect to $D_s$ and get that

$$\frac{\partial}{\partial x} E[\Phi(X_T^x)]$$

$$= E\left[ \int_0^T \{a(s)\Phi(X_T^x)D_sX_T^x (a_{ij}(S_s^{x_1,x_2}, \sigma_s^{x_2}))^{-1}_{1\leq i,j \leq 2} \frac{\partial}{\partial x} X_s^x \} ds \right]$$

We also see that

$$(a_{ij}(S_s^{x_1,x_2}, \sigma_s^{x_2}))^{-1}_{1\leq i,j \leq 2} \frac{\partial}{\partial x} X_s^x$$

$$= (S_s^{x_1,x_2} g(\sigma_s^{x_2})) \sqrt{1 - \rho^2}^{-1}$$

$$\cdot \begin{pmatrix} \sqrt{1 - \rho^2} \frac{\partial}{\partial x_1} S_s^{x_1,x_2} & \sqrt{1 - \rho^2} \frac{\partial}{\partial x_2} S_s^{x_1,x_2} \\ -\rho \frac{\partial}{\partial x_1} S_s^{x_1,x_2} + \frac{\partial}{\partial x_2} \sigma_s^{x_2} & -\rho \frac{\partial}{\partial x_2} S_s^{x_1,x_2} + \frac{\partial}{\partial x_1} \sigma_s^{x_2} \end{pmatrix}.$$ 

So it follows that

$$D_s \Phi(X_T^x) (a_{ij}(S_s^{x_1,x_2}, \sigma_s^{x_2}))^{-1}_{1\leq i,j \leq 2} \frac{\partial}{\partial x} X_s^x$$

$$= (D_s^W \Phi(X_T^x))(S_s^{x_1,x_2} g(\sigma_s^{x_2}))^{-1}_{1\leq i,j \leq 2} \frac{\partial}{\partial x_1} S_s^{x_1,x_2}$$

$$- D_s^H \Phi(X_T^x) \sqrt{1 - \rho^2}^{-1} (S_s^{x_1,x_2} g(\sigma_s^{x_2}))^{-1}_{1\leq i,j \leq 2} \frac{\partial}{\partial x_1} S_s^{x_1,x_2},$$

$$D_s^W \Phi(X_T^x)(S_s^{x_1,x_2} g(\sigma_s^{x_2}))^{-1}_{1\leq i,j \leq 2} \frac{\partial}{\partial x_2} S_s^{x_1,x_2}$$

$$+ D_s^H \Phi(X_T^x)(-\frac{\rho}{\sqrt{1 - \rho^2}} \frac{\partial}{\partial x_2} S_s^{x_1,x_2} (S_s^{x_1,x_2} g(\sigma_s^{x_2}))^{-1}$$

$$+ \frac{1}{\sqrt{1 - \rho^2}} \frac{\partial}{\partial x_2} \sigma_s^{x_2} )^*.$$
Therefore, we obtain that

\[
\frac{\partial}{\partial x} E[\Phi(X_T^x)] = (E[\int_0^T a(s)D_s^W \Phi(X_T^x)(S_{s_1}^{x_1}(\sigma_{s_1}^{x_2}))^{-1} \frac{\partial}{\partial x_1} S_{s_1}^{x_1} ds] \]

\[
-\frac{\rho}{\sqrt{1-\rho^2}} \frac{\partial}{\partial x_1} S_{s_1}^{x_1} ds, \]

\[
E[\int_0^T a(s)(D_s^H \Phi(X_T^x)(S_{s_1}^{x_1}(\sigma_{s_1}^{x_2}))^{-1} \frac{\partial}{\partial x_2} S_{s_1}^{x_1} ds] \]

\[
+ E[\int_0^T a(s)D_s^H \Phi(X_T^x)(\frac{-\rho}{\sqrt{1-\rho^2}} \frac{\partial}{\partial x_2} S_{s_1}^{x_1} (S_{s_1}^{x_1}(\sigma_{s_1}^{x_2}))^{-1} S_{s_1}^{x_1} ds] \]

\[
+ \frac{1}{\sqrt{1-\rho^2}} \frac{\partial}{\partial x_2} \sigma_{s_1}^{x_2} ds] \]

\[
The \ by \ exploiting \ the \ independence \ of \ W. \ and \ B^H, \ we \ can \ use \ a \ similar \ reasoning \ as \ in \ the \ proof \ of \ Theorem 7 \ and \ find \ that \]

\[
E[\int_0^T a(s)D_s^H \Phi(X_T^x)(\frac{-\rho}{\sqrt{1-\rho^2}} \frac{\partial}{\partial x_2} S_{s_1}^{x_1} (S_{s_1}^{x_1}(\sigma_{s_1}^{x_2}))^{-1}) \]

\[
\cdot \left( \frac{-\rho}{\sqrt{1-\rho^2}} \frac{\partial}{\partial x_2} S_{s_1}^{x_1} (S_{s_1}^{x_1}(\sigma_{s_1}^{x_2}))^{-1} \right) \]

\[
+ \frac{1}{\sqrt{1-\rho^2}} \frac{\partial}{\partial x_2} \sigma_{s_1}^{x_2} dB_s du], \]

where \( B_s \) is a one-dimensional Brownian motion in the stochastic integral representation \( (3) \).

In the last step, we can use the duality formula with respect to \( W \) and similar arguments as in the proof of Theorem 7 (see also Remark 6) with respect to regular functions \( g, b, \Phi \) and derive the following BEL-formula for the stock price model \( (18) \):

**Theorem 10** Assume that \( U \subset \mathbb{R}^2 \) is a bounded, open set and \( b \in L^1_{\infty} \) in the stock price model \( (18) \). In addition, require that \( g : \mathbb{R} \rightarrow (\alpha, \infty) \) is in \( C^2_b(\mathbb{R}) \) for some \( \alpha > 0 \) and that \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a Borel-measurable function with

\[
\Phi(S_T^x, \sigma_T^x) \in L^2(\Omega \times U, \mu \times dx). \]

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Let \( a \in L^\infty([0,T]) \) with \( \int_0^T a(s)\,ds = 1 \). Then

\[
\frac{\partial}{\partial x} E[\Phi(S_{x_1}^{x_2}, \sigma_t^{x_2})] = (E[\Phi(X_T^{x_2}) \int_0^T a(s)(S_s^{x_1}g(\sigma_s^{x_2}))^{-1} \frac{\partial}{\partial x_1} S_s^{x_1} \,dW_s] + C_H E[\Phi(X_T^{x_2}) \int_0^T a(s)(S_s^{x_1}g(\sigma_s^{x_2}))^{-1} \frac{\partial}{\partial x_2} S_s^{x_1} \,dW_s])
\]

for almost all \( x = (x_1, x_2) \in U \), where \( C_H \) is a constant defined as in Theorem 7.

Remark 11 In fact, one can prove as in [2] by using the estimate 17 that the right hand side of (19) has a continuous modification, if \( H < \frac{1}{2(d+3)} = \frac{1}{8} \) (for \( d = 1 \)).

We mention that our stock price model coincides with the model of Gatheral, Jaisson, Rosenbaum [21] in the case of independent \( \sigma_t^{x_2} \), \( W_t, 0 \leq t \leq T \), when (formally)

\[ g(x) = \exp(x/\nu), b(t, x) = -a\nu(x - b) \] and \( \rho = 0 \)

or more explicitly, when

\[
S_t^{x_1, x_2} = x_1 + \int_0^t \mu S_u^{x_1, x_2} \,du + \int_0^t \exp(\sigma_x^{x_2}/\nu) S_u^{x_1, x_2} \,dW_u \\
\sigma_t^{x_2} = x_2/\nu - \int_0^t a\nu(\sigma_u^{x_2} - b) \,du + B_t^H, x_1, x_2 \in \mathbb{R}, 0 \leq t \leq T, (20)
\]

for \( a, \nu > 0, b \in \mathbb{R} \).

The process \( \sigma_t^{x_2}, 0 \leq t \leq T \) in (20) is a stationary and mean reverting process, which can be regarded as a generalization of the Vasicek model for short rates in the case of stochastic log-volatility with (log-volatility) mean reversion \( a\nu \) and long-run average level \( b \).
It turns out that $\sigma_t^{x_2}, 0 \leq t \leq T$ has the explicit representation

$$\sigma_t^{x_2} = (x_2/\nu) + b(1 - e^{-\alpha t}) + \int_0^t e^{-\alpha(t-s)} dB_s^H,$$

where the last term on the right hand side is defined as a Young integral with respect to the integrator $B_s^H$ for $H < 1/2$. See [32].

However, due to economical crises, financial disasters or market regulations one would expect to observe a change with respect to the behaviour of the dynamics $\sigma_t^{x_2}$, that is e.g. a "regime change" from a log-volatility long-run average level $b_1$ to $b_2$ or from a mean reversion $a_1$ to $a_2$, provided $\sigma_t^{x_2}$ exceeds a certain threshold $R$. In order to capture such "regime switching" effects and the roughness of paths of $\sigma_t^{x_2}$, whose empirical evidence was found in [21] and which is modelled by means of $B_t^H$ for small Hurst parameters $H < 1/2$, it is natural to take the stochastic volatility model (20) as a starting point and to modify it as follows:

$$\sigma_t^{x_2} = x_2/\nu - \int_0^t b(u, \sigma_u^{x_2}) du + B_t^H, 0 \leq t \leq T,$$

where $b$ is a discontinuous vector field of linear growth given by

$$b(t, y) = a_1(y - b_1)1_{(-\infty, R]}(y) + a_2(y - b_2)1_{[R, \infty)}(y)$$

for some $a_1, a_2 > 0, R, b_1, b_2 \in \mathbb{R}$.

If $g(x) = \exp(x)$ and if $\sigma_t^{x_2}$ follows the dynamics in (21), one observes that the stock price process $S_{x_1,x_2}^t$ in (18) is not square integrable in general and hence not Malliavin differentiable. Therefore we cannot directly use Malliavin techniques here to derive a BEL-formula as in Theorem 10 for this situation. In order to overcome this deficiency, one may in view of applications instead replace the exponential function by a function $g : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(x) = \exp(f(x)),$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth compactly supported function with $f(x) = x$ on $[-l, l]$ for some large $l > 0$.

Hence, in summary a reasonable applicable stochastic volatility model in our setting, which takes into account both volatility roughness and regime switching effects, could be the following:

$$S_{x_1,x_2}^t = x_1 + \int_0^t \mu S_{x_1,x_2}^u du + \int_0^t g(\sigma_u^{x_2}/\nu) S_{x_1,x_2}^u dW_u$$

$$\sigma_t^{x_2} = x_2/\nu - \int_0^t b(u, \sigma_u^{x_2}) du + B_t^H, x_1, x_2 \in \mathbb{R}, 0 \leq t \leq T,$$

where $g$ is a smooth function of the form (23) and where the vector field $b$ is given by (22).

Since the coefficient $b$ in (24) can be decomposed as a sum of measurable bounded function and a Lipschitz function (of linear growth), one can in fact prove the following BEL-representation:
Theorem 12 Let $U \subset \mathbb{R}^2$ be a bounded, open. Suppose that $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ in the stock price model (18) has the decomposition
\[ b = \tilde{b} + \hat{b}, \tag{25} \]
where $\tilde{b} \in L^\infty([0, T] \times \mathbb{R})$ and where $\hat{b} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies a linear growth and Lipschitz condition uniformly in time. Further, require that $g : \mathbb{R} \rightarrow (0, \infty)$ is given as in (23) and that $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a Borel-measurable function such that
\[ \Phi(S_t^x, \sigma_t^x) \in L^2(\Omega \times U, \mu \times dx). \]
Let $a$ be a bounded and measurable function on $[0, T]$, which sums up to 1. Then for $H < \frac{1}{2}$ and $\rho \in (-1, 0) \cup (0, 1)$ the function $u : U \rightarrow \mathbb{R}$ defined by
\[ u(x) = E[\Phi(S_t^x, \sigma_t^x)], x \in U \]
belongs to $C^1(U)$ and $\frac{\partial}{\partial x} u(x)$ has the representation (19).

The proof of Theorem 12 requires some notions and auxiliary results.

Using the compactness criterion for square integrable functionals of Wiener processes from Malliavin calculus [12], the proof of the next result gives an alternative method for the construction of unique strong solutions of (26) to the work of [28] in the case of vector fields given by (25).

Theorem 13 Consider the SDE
\[ X_t^x = x + \int_0^t b(u, X_s^x)du + \rho_1 B_t^H + \rho_2 W_t, x \in \mathbb{R}, 0 \leq t \leq T, \tag{26} \]
where $B^H$ is a fractional Brownian motion with $H < \frac{1}{2}$ being independent of a Wiener process $W$, where $\rho_1, \rho_2 \in \mathbb{R} \setminus \{0\}$. Assume for $b$ the decomposition (25) with respect to coefficients $\tilde{b}, \hat{b}$ satisfying the conditions of Theorem 12. Then there exists a unique strong solution $X_t^x$ to the SDE (26). Moreover, $X_t^x$ is Malliavin differentiable in the direction of $(B, W)^*$ and $(B^H, W)^*$ for all $t$, where $B$ is the Wiener process in the stochastic integral representation of $B^H$.

Let us now recall the concept of the local time-space integral, which goes back to [16] and which in the following form was given in [6]:

Definition 14 Let $(\mathcal{H}^\infty, \|\cdot\|)$ be the Banach space of Borel measurable functions $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ endowed with the norm $\|f\|_x$ given by
\[ \|f\|_x = 2 \left( \int_0^T \int_\mathbb{R} f^2(s, y) \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{|y - x|^2}{2s}\right) dy ds \right)^{1/2} + \int_0^T \int_\mathbb{R} |y - x| |f(s, y)| \frac{1}{s^{1/2} \sqrt{2\pi s}} \exp\left(-\frac{|y - x|^2}{2s}\right) dy ds. \]
Denote by \( f_{\Delta} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) a simple function of the form

\[
f_{\Delta}(s, x) = \sum_{1 \leq i \leq n-1, 1 \leq j \leq m-1} f_{ij}^{1(y_i, y_{i+1})}(y) 1_{(s_j, s_{j+1})}(s),
\]

where \((s_j)_{1 \leq j \leq m}\) is a partition of \([0, T]\) and where \((y_i)_{1 \leq i \leq n}\) and \((f_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}\) are finite sequences of real numbers. Let \( L^{X^\alpha}(t, y) \) be the local time of the solution \( X^\alpha \) to (26) for \( \rho_1 = 0 \) and \( \rho_2 = 1 \) and vector fields \( b \) as in (25). Then the local time-space integral of a simple function \( f_{\Delta} \) with respect to the integrator \( L^{X^\alpha}(dt, dy) \) is defined by

\[
\int_0^T \int_{\mathbb{R}} f_{\Delta}(s, y) L^{X^\alpha}(ds, dy) = \sum_{1 \leq i \leq n-1, 1 \leq j \leq m-1} f_{ij}(L^{X^\alpha}(s_{j+1}, y_{i+1}) - L^{X^\alpha}(s_j, y_{i+1}) - L^{X^\alpha}(s_{j+1}, y_i) + L^{X^\alpha}(s_j, y_i)).
\]

The class of simple functions is dense in \( (\mathcal{H}^\alpha, \|\cdot\|) \). For \( f \in \mathcal{H}^\alpha \) let \( f_n, n \geq 1 \) be a sequence of simple functions converging to \( f \) in \( \mathcal{H}^\alpha \). Then the local time-space integral of \( f \) can be defined as the following (existing) limit in probability:

\[
\int_0^T \int_{\mathbb{R}} f(s, y) L^{X^\alpha}(ds, dy) := \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}} f_n(s, y) L^{X^\alpha}(ds, dy).
\]

See Lemma 2.7 in [6].

In the sequel, we define

\[
\int_0^t \int_{\mathbb{R}} f(s, y) L^{X^\alpha}(ds, dy) := \int_0^T \int_{\mathbb{R}} 1_{[0, t]}(s) f(s, y) L^{X^\alpha}(ds, dy)
\]

for \( 0 \leq t \leq T \), if \( f \in \mathcal{H}^\alpha \).

We also need the following representation of local time-space integrals (27) in the case of \( X^\alpha_t = W^\alpha_t := x + W_t \), which is due to [10]:

**Lemma 15** If \( f \in \mathcal{H}^0 \), then

\[
\int_0^t \int_{\mathbb{R}} f(s, y) L^{W^\alpha}(ds, dy) = \int_0^t f(s, W^\alpha_s) dW_s + \int_{T-t}^T f(T - s, \widehat{W}^\alpha_s) dW^\alpha_s - \int_{T-t}^T f(T - s, \widehat{W}^\alpha_s) \frac{\widehat{W}^\alpha_s}{T - s} ds,
\]

where \( \widehat{W}_t := W_{T-t}, 0 \leq t \leq T \) is the time-reversed Wiener process and

\[
W^*_t := \widehat{W}_t - W_T + \int_0^t \frac{\widehat{W}_s}{T - s} ds, 0 \leq t \leq T
\]

is a Wiener process with respect to the filtration of \( \widehat{W} \).
Later on we will also make use of the following integration by parts relation with respect to local time-space integrals (see [16], [6]):

**Lemma 16** Suppose \( f \in \mathcal{H}^x \) is Lipschitz continuous with respect to the spatial variable and denote by \( f' \) its spatial weak derivative. Then all \( 0 \leq t \leq T \) \( X^x_t \) is Malliavin differentiable and

\[
- \int_0^t \int_{\mathbb{R}} f(s,y) L^{X^x} (ds, dy) = \int_0^t f(s, X^x_s) ds \quad \text{a.e.}
\]

Using mollification let us now in view of the next auxiliary result consider smooth functions \( \hat{b}_n, n \geq 1 \) such that

\[
\left| \hat{b}_n(t,x) - \hat{b}_n(t,y) \right| \leq K |x-y|, \ x, y \in \mathbb{R}, 0 \leq t \leq T, n \geq 1
\]

where \( C > 0 \) and \( K \) is the Lipschitz constant of \( \hat{b} \). So

\[
\left| \hat{b}_n(t,x) \right| \leq K
\]

for all \( x \in \mathbb{R}, 0 \leq t \leq T, n \geq 1 \). Further, let \( \tilde{b}_n, n \geq 1 \) be a sequence of smooth and compactly supported functions such that

\[
\tilde{b}_n(t,x) \xrightarrow{n \to \infty} \bar{b}(t,x) \quad \text{a.e.}
\]

and

\[
\left| \tilde{b}_n(t,x) \right| \leq L
\]

for all \( t, x \) for some constant \( L < \infty \).

**Lemma 17** Let \( \hat{b}_n, \tilde{b}_n, n \geq 1 \) be as in (29), (30a) and \( b_n = \tilde{b}_n + \hat{b}_n, n \geq 1 \). Assume that \( X^{x,n} \) is the strong solution associated with the vector field \( b_n, n \geq 1 \). Further, let \( X^x \) be the weak solution to (26) and \( \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) the (\( \mu \)-completed) filtration generated by \( B^H_t, W_t, 0 \leq t \leq T \). Then, for all \( 0 \leq t \leq T \)

\[
X^{x,n}_t \xrightarrow{n \to \infty} E \left[ X^x_t \mid \mathcal{F}_t \right]
\]

as well as

\[
(X^{x,n}_t)^2 \xrightarrow{n \to \infty} E \left[ (X^x_t)^2 \mid \mathcal{F}_t \right]
\]

weakly in \( L^2(\Omega, \mathcal{F}_t) \).
Proof. The proof is the same as that of Lemma A.3 in [6].

Proof of Theorem 13. Assume for notational convenience that $\rho_1 = \rho_2 = 1$.

Let $\tilde{b}_n, \tilde{b}_n, n \geq 1$ the smooth approximating sequences of functions in (29) and (30a).

In proving this result, we aim at applying as in Section 2 the compactness criterion in [12] to the sequence $X^{x,n}, n \geq 1$ for each fixed $t$. Just as in [6] we can show for $b_n := \tilde{b}_n + \tilde{b}_n$ that

$$D_s^W X^{x,n} = \exp\left\{ \int_s^t \tilde{b}_n(u, X_u^{x,n}) + \tilde{b}_n(u, X_u^{x,n}) du \right\} \mu - a.e., s \leq t \ ds - a.e.,$$

where $\tilde{b}$ is the weak spatial derivative. Further, by using Girsanov’s theorem with respect $W$, the mean value theorem and Hölder’s inequality we obtain similarly to the proof of Theorem A.4 in [6] that for $0 \leq s \leq s' \leq t$ and $x \in K \subset \mathbb{R} (K$ compact)

$$\|D_s^W X^{x,n}_t - D_{s'}^W X^{x,n}_t\|_{L^2(\mu)}^2$$

$$\leq E[\exp\left\{ \frac{1+\varepsilon}{\varepsilon} \int_s^t b'(u, x + B^H_u + W_u) du \right\} \sup_{0 \leq \theta \leq 1} \exp\left\{ \frac{1+\varepsilon}{\varepsilon} \theta \int_s^{s'} b'(u, x + B^H_u + W_u) du \right\} \left( \int_s^{s'} b''(u, x + B^H_u + W_u) du \right)^{\frac{1+\varepsilon}{2}} E[\mathcal{E}(b_n)^{1+\varepsilon}]^{\frac{1}{1+\varepsilon}},$$

where

$$\mathcal{E}(b)_T := \exp\left\{ \int_0^T b(u, x + B^H_u + W_u) dW_u - \frac{1}{2} \int_0^T (b(u, x + B^H_u + W_u))^2 \ du \right\}$$

and where $\varepsilon$ is chosen such that

$$\sup_{x \in K} \sup_{n \geq 1} E[\mathcal{E}(b_n)^{1+\varepsilon}] < \infty.$$

Using the latter estimate combined with the same arguments as in Theorem A.4 in [6], we find that

$$\|D_s^W X^{x,n}_t - D_{s'}^W X^{x,n}_t\|_{L^2(\mu)}^2$$

$$\leq C\left( \|\tilde{b}_n\|_\infty^2 T |s - s'| + \|\tilde{b}_n\|_\infty^2 |s - s'| + \tilde{C} \|\tilde{b}_n\|_\infty^2 |s - s'| \right)$$

$$\leq C\left( \|\tilde{b}_n\|_\infty^2 T |s - s'| + \|\tilde{b}_n\|_\infty^2 |s - s'| + \tilde{C} \|\tilde{b}_n\|_\infty^2 |s - s'| \right).$$
So
\[ \|D_s^W X^x,n_t - D_{s'}^W X^x,n_t\|_{L^2(\mu)}^2 \leq C |s - s'| \] (32)
for all \(0 \leq s \leq s' \leq t\), where \(C\) is a constant depending on \(H,T,\varepsilon\) and \(b\). Therefore, there exists a \(\beta \in (0,\frac{1}{2})\) and a constant \(C < \infty\) depending on \(H,T,\varepsilon\) and \(b\) such that for all \(0 \leq s \leq s' \leq t\):
\[
\sup_{x \in K} \sup_{n \geq 1} \int_0^T \int_0^T \frac{\|D_s^W X^x,n_t - D_{s'}^W X^x,n_t\|_{L^2(\mu)}^2}{|s' - s|^{1+\alpha}} ds dt' \leq C. \tag{33}
\]
In the same way, we can also verify that
\[
\sup_{x \in K} \sup_{n \geq 1} \|D_s^W X^x,n_t\|_{L^2([0,t]\times\Omega)}^2 \leq C \tag{34}
\]
for a constant \(C < \infty\).

In order to employ the compactness criterion in [12], we next also have to prove the estimates (34), (33) for the Malliavin derivative \(D_B^s\) in the direction of the Wiener process \(B\) in the stochastic integral representation of \(B^H\). To this end, we can use the transfer principle for Malliavin derivatives of Proposition 5.2.1 in [27] and get the following representation
\[
D_s^B X^x,n_t = K_H(t,s) D_s^H X^x,n_t + c_H(H - \frac{1}{2}) \int_s^t (D_u^H X^x,n_t - D_s^H X^x,n_t) \left( \frac{u}{s} \right)^{H - \frac{1}{2}} \frac{1}{(u - s)^{H - \frac{3}{2}} - H} du
\]
\[
\mu - a.e., s \in [0,t] \ a.e. \quad \text{Note here that}
\]
\[
\frac{\partial}{\partial t} K(t,s) = c_H(H - \frac{1}{2}) \left( \frac{t}{s} \right)^{H - \frac{1}{2}} \frac{1}{(t - s)^{H - \frac{3}{2}} - H}, t > s. \tag{35}
\]
So
\[
\|D_s^B X^x,n_t\|_{L^2(\mu)} \leq K_H(t,s) \|D_s^H X^x,n_t\|_{L^2(\mu)} + c_H \left( \frac{1}{2} - H \right) \int_s^t \|D_u^H X^x,n_t - D_s^H X^x,n_t\|_{L^2(\mu)} \left( \frac{u}{s} \right)^{H - \frac{1}{2}} \frac{1}{(u - s)^{H - \frac{3}{2}} - H} du.
\]
On the other hand, by using the chain rule for the Malliavin derivative we obtain just as in (31) the representation
\[
D_s^H X^x,n_t = \exp \{ \int_s^t \hat{b}'_n(u, X^x,u) du \} \mu - a.e., s \leq t \ ds - a.e.
\]

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So as in the case of $D^H$ we can get the estimate

$$\|D^H_s X_{t,n} - D^H_{s'} X_{t,n}\|_{L^2(\mu)}^2 \leq C |s - s'|$$

(36)

for all $0 \leq s \leq s' \leq t$, where $C$ is a constant depending on $H, T, \varepsilon$ and $b$.

Thus for all compact sets $K \subset \mathbb{R}$ there is a constant $C$ depending on $T, K, \varepsilon$ and $b$ such that

$$\sup_{x \in K} \sup_{0 \leq u \leq t} \sup_{n \geq 1} \|D^H_u X_{t,n}\|_{L^2(\mu)} \leq C$$

(37)

as well as

$$\sup_{x \in K} \sup_{n \geq 1} \|D^H_u X_{t,n} - D^H_s X_{t,n}\|_{L^2(\mu)} \leq C |u - s|^\frac{1}{2}$$

(38)

for all $0 \leq s \leq u \leq t$. Hence,

$$\|D^B_s X_{t,n}\|_{L^2(\mu)} \leq C(K_H(t,s) + c_H(\frac{1}{2} - H) \int_s^T \left(\frac{u}{s}\right)^{H-\frac{1}{2}} \frac{1}{(u-s)^{1-H}} du)$$

$$\leq C(K_H(t,s) + c_H(\frac{1}{2} - H) \int_s^T \frac{s^{H-\frac{1}{2}}}{(u-s)^{1-H}} du)$$

$$= C(K_H(t,s) + c_H(\frac{1}{2} - H) \frac{1}{H}(T-s)^H).$$

The latter entails that

$$\sup_{x \in K} \sup_{n \geq 1} \|D^B X_{t,n}\|_{L^2(\mu \times [0,T])} \leq C^*$$

for a constant $C^* = C^*(H, T, \varepsilon, b) > 0$. Further, we get for $s_2 \geq s_1$ that
\[
D_{s_2}^{H}x_{t}^{\tau,n} - D_{s_1}^{H}x_{t}^{\tau,n} = K_{H}(t,s_2)D_{s_2}^{H}x_{t}^{\tau,n} - K_{H}(t,s_1)D_{s_1}^{H}x_{t}^{\tau,n}
\]
\[
+ c_{H}(H - \frac{1}{2}) \int_{s_2}^{T} (D_{u}^{H}x_{t}^{\tau,n} - D_{s_2}^{H}x_{t}^{\tau,n}) \frac{u}{s_2} H^{-\frac{1}{2}} \frac{1}{(u-s_2)^{\frac{3}{2} - H}} du
\]
\[
- c_{H}(H - \frac{1}{2}) \int_{s_1}^{T} (D_{u}^{H}x_{t}^{\tau,n} - D_{s_1}^{H}x_{t}^{\tau,n}) \frac{u}{s_1} H^{-\frac{1}{2}} \frac{1}{(u-s_1)^{\frac{3}{2} - H}} du
\]
\[
= K_{H}(t,s_2)(D_{s_2}^{H}x_{t}^{\tau,n} - D_{s_1}^{H}x_{t}^{\tau,n}) + (K_{H}(t,s_2) - K_{H}(t,s_1))D_{s_1}^{H}x_{t}^{\tau,n}
\]
\[
+ \int_{s_2}^{T} (D_{u}^{H}x_{t}^{\tau,n} - D_{s_2}^{H}x_{t}^{\tau,n})(\frac{\partial}{\partial u} K(u,s_2) - \frac{\partial}{\partial u} K(u,s_1)) du
\]
\[
- c_{H}(H - \frac{1}{2}) \int_{s_2}^{T} (D_{u}^{H}x_{t}^{\tau,n} - D_{s_1}^{H}x_{t}^{\tau,n}) \frac{u}{s_1} H^{-\frac{1}{2}} \frac{1}{(u-s_1)^{\frac{3}{2} - H}} du
\]
\[
- c_{H}(H - \frac{1}{2}) \int_{s_1}^{T} (D_{u}^{H}x_{t}^{\tau,n} - D_{s_1}^{H}x_{t}^{\tau,n}) \frac{u}{s_1} H^{-\frac{1}{2}} \frac{1}{(u-s_1)^{\frac{3}{2} - H}} du
\]
\[
= \sum_{j=1}^{5} I_{j}(s_1,s_2).
\]

Let us first have a look at the most difficult term, that is
\[
I_3(s_1,s_2) = \int_{s_2}^{T} (D_{u}^{H}x_{t}^{\tau,n} - D_{s_2}^{H}x_{t}^{\tau,n}) \frac{\partial}{\partial u} (K_{H}(u,s_2) - K_{H}(u,s_1)) du
\]

In what follows, we aim at using the following inequality (see the proof of Lemma A.4 in [4]):

For all \( \gamma \in (0,H) \), \( 0 < \theta_1 < \theta_2 < T \) we have that
\[
|K_{H}(t,\theta_1) - K_{H}(t,\theta_2)| \leq C_{H,T} \frac{(\theta_2 - \theta_1)^{\gamma}}{(\theta_2 \theta_1)^{\gamma}} \theta_2^{H - \frac{1}{2} - \gamma} (t - \theta_2)^{H - \frac{1}{2} - \gamma}. \tag{39}
\]

It is shown in [4] (Proof of Lemma A.4)) that there exists a \( \beta > 0 \) depending on \( H \) and \( \gamma \) such that
\[
\int_{0}^{t} \int_{0}^{t} \left( \frac{|\theta_2 - \theta_1|^{2\gamma}}{(\theta_2 \theta_1)^{2\gamma}} \theta_2^{H - 1 - 2\gamma} (t - \theta_2)^{2H - 1 - 2\gamma} \right) |\theta_2 - \theta_1|^{-1 + \beta} d\theta_1 d\theta_2 < \infty
\]
\[
(40)
\]

Since
\[
|K(t,s)| \leq C(t-s)^{H - \frac{3}{2}}, t > s
\]
for a constant \( C \) and since \( D_{u}^{H}x_{t}^{\tau,n} \) is Lipschitz continuous in \( u \), we obtain from
the definition of $K_H$, integration by parts and inequality (39) that

$$\|I_3(s_1, s_2)\|_{L^2(\mu)}$$

$$\leq c_H(1 - H) \int_{s_2}^T \|D_u X_{i,n} - D_{s_2} X_{i,n}\|_{L^2(\mu)} \left(\left(\frac{s_2}{u}\right)^{1-H} \frac{1}{(u-s_2)^{1/2-H}} - \left(\frac{s_1}{u}\right)^{1-H} \frac{1}{(u-s_1)^{1/2-H}}\right) du$$

$$\leq c_H(1 - H) \int_{s_2}^T C(u-s_2)^{1/2-H} \left(\left(\frac{s_2}{u}\right)^{1-H} \frac{1}{(u-s_2)^{1/2-H}} - \left(\frac{s_1}{u}\right)^{1-H} \frac{1}{(u-s_1)^{1/2-H}}\right) du$$

$$= \lim_{s \searrow s_2} C(u-s_2)^{1/2} \frac{1}{(u-s_2)^{1/2-H}} \left(K(u,s_2) - K(u,s_1)\right)\bigg|_{u=s} - \int_{s_2}^T \frac{C}{2} (u-s_2)^{-1/2} (K(u,s_2) - K(u,s_1)) du$$

$$= C(T-s_2)^{1/2} \frac{1}{s_2} C_{H,T} \frac{(s_2-s_1)^{1/2-H-\gamma}}{(s_2s_1)^{1/2-H}} (T-s_2)^{H-1/2}$$

$$+ \int_{s_2}^T \frac{C}{2} (u-s_2)^{-1/2} C_{H,T} \frac{(s_2-s_1)^{1/2-H-\gamma}}{(s_2s_1)^{1/2-H}} (u-s_2)^{H-1/2} du$$

$$= C(T-s_2)^{1/2} \frac{1}{s_2} C_{H,T} \frac{(s_2-s_1)^{1/2-H-\gamma}}{(s_2s_1)^{1/2-H}} (T-s_2)^{H-1/2}$$

$$+ \int_{s_2}^T \frac{C}{2} C_{H,T} \frac{(s_2-s_1)^{1/2-H-\gamma}}{(s_2s_1)^{1/2-H}} (u-s_2)^{H-1/2} du$$

$$= C C_{H,T} \frac{(s_2-s_1)^{1/2-H-\gamma}}{(s_2s_1)^{1/2-H}} (T-s_2)^{H-1/2}$$

$$+ \frac{C}{2} C_{H,T} \frac{(s_2-s_1)^{1/2-H-\gamma}}{(s_2s_1)^{1/2-H}} \frac{1}{H-1/2} (T-s_2)^{H-1/2}$$

Hence, we conclude from (40) that there exists a $\beta > 0$ depending on $H$ and $\gamma$ such that

$$\int_0^T \int_0^T \frac{\|I_3(s_1, s_2)\|^2_{L^2(\mu)}}{|s_2-s_1|^{1+2\beta}} ds_1 ds_2 < \infty.$$
Further, we see that

\[ \| I_4(s_1, s_2) \|_{L^2(\mu)} \leq c_H \left( \frac{1}{2} - H \right) \int_{s_2}^T \left\| D_{s_2}^H X_t^{x,n} - D_{s_1}^H X_t^{x,n} \right\|_{L^2(\mu)} \left( \frac{u}{s_1} \right)^{H - \frac{1}{2}} (u - s_1)^{- \frac{1}{2} - H} du \]

\[ \leq c_H \left( \frac{1}{2} - H \right) C (s_2 - s_1)^{\frac{1}{2}} \int_{s_2}^T \frac{1}{(u - s_1)^{\frac{1}{2} - H}} du \]

\[ = c_H \left( \frac{1}{2} - H \right) C (s_2 - s_1)^{\frac{1}{2}} \left( (s_2 - s_1)^{\frac{1}{2}} (T - s_1)^{\frac{1}{2} - H} + (s_2 - s_1)^H \right) \]

\[ \leq 2c_H \left( \frac{1}{2} - H \right) C \frac{1}{H - 3/2} (s_2 - s_1)^H. \]

So

\[ \int_0^T \int_0^T \frac{\| I_4(s_1, s_2) \|^2_{L^2(\mu)}}{|s_2 - s_1|^{1+2\beta}} ds_1 ds_2 < \infty \]

for a \( \beta > 0 \) depending on \( H \).

In addition, it also follows that

\[ \| I_5(s_1, s_2) \|_{L^2(\mu)} \leq c_H \left( \frac{1}{2} - H \right) \int_{s_1}^{s_2} \left\| D_{u}^H X_t^{x,n} - D_{s_1}^H X_t^{x,n} \right\|_{L^2(\mu)} \left( \frac{u}{s_1} \right)^{H - \frac{1}{2}} (u - s_1)^{- \frac{1}{2} - H} du \]

\[ \leq c_H \left( \frac{1}{2} - H \right) C \int_{s_1}^{s_2} (u - s_1)^{\frac{1}{2}} (u - s_1)^{- \frac{1}{2} - H} du \]

\[ = c_H \left( \frac{1}{2} - H \right) C \frac{1}{1 - H} (s_2 - s_1)^H. \]

Hence,

\[ \int_0^T \int_0^T \frac{\| I_5(s_1, s_2) \|^2_{L^2(\mu)}}{|s_2 - s_1|^{1+2\beta}} ds_1 ds_2 < \infty \]

for a \( \beta > 0 \) depending on \( H \).

By using (37), (38), (39) and (40) we can treat the terms \( I_1(s_1, s_2) \) and \( I_2(s_1, s_2) \) similarly and we get altogether for compact sets \( K \subset \mathbb{R} \) that

\[ \sup_{x \in K} \sup_{n \geq 1} \int_0^T \int_0^T \left\| D_s^B X_t^{x,n} - D_{s}^B X_t^{x,n} \right\|^2_{L^2(\mu)} ds ds' \leq C < \infty. \]

Using the above estimates with respect to \( D^B \) and \( D^W \) we can now apply the compactness criterion in (12) and obtain that for all \( 0 \leq t \leq T \) there exists a subsequence \( n_k, k \geq 1 \) depending on \( t \) and \( Y_t^{x} \in L^2(\Omega, \mathcal{F}_t) \)

\[ X_t^{x,n_k} \xrightarrow{k \to \infty} Y_t^{x} \]
in $L^2(\Omega, \mathcal{F}_t)$. However, it follows from Lemma 17 that $Y^x_t = E[X^x_t | \mathcal{F}_t]$ a.e. Hence,

$$X^{x,n}_t \xrightarrow{n \to \infty} E[X^x_t | \mathcal{F}_t]$$

in $L^2(\Omega, \mathcal{F}_t)$. The latter implies in connection with Lemma 17 that

$$(X^x_t)^2 = E[(X^x_t)^2 | \mathcal{F}_t] \ a.e.$$ 

So $X^x_t = E[X^x_t | \mathcal{F}_t]$ a.e., which shows that the weak solution must be adapted to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ and therefore a strong solution. Strong uniqueness is a consequence of Girsanov’s theorem. Further, the Malliavin differentiability of $X^x_t$ in the direction of $(B \cdot, W \cdot)^*$, but also $(B^H \cdot, W \cdot)^*$ follows from Lemma 1.2.3 in [27] in connection with (41) and the uniform estimates with respect to the Malliavin derivatives of $X^{x,n}_t$.

**Lemma 18** Recall that $\Omega = \Omega_1 \times \Omega_2$, where $B^H$ is defined on $\Omega_2$ and $W$ on $\Omega_2$. Let $X^x$ be the Malliavin differentiable solution of Theorem 13. Then for all $0 \leq t \leq T$:

$$D^W_s X_t = \rho_2 \exp\left\{-\int_s^t \int_{\mathbb{R}} f(r, y)L^{\rho_2^{-1}(X^x-x-\rho_1 B^H(\omega_2))}(\omega_1, dr, dy)\right\} \omega_2 - a.e, \omega_1 - a.e, s \leq t ds - a.e$$

as well as

$$D^H_s X_t = \rho_1 K_H(t, s) \exp\left\{-\int_s^t \int_{\mathbb{R}} f(r, y)L^{\rho_2^{-1}(X^x-x-\rho_1 B^H(\omega_2))}(\omega_1, dr, dy)\right\}$$

$$+ \rho_1 c_H(\frac{1}{2} - H) \int_s^T \exp\left\{-\int_u^t \int_{\mathbb{R}} f(r, y)L^{\rho_2^{-1}(X^x-x-\rho_1 B^H(\omega_2))}(\omega_1, dr, dy)\right\}$$

$$\exp\left\{-\int_s^t \int_{\mathbb{R}} f(r, y)L^{\rho_2^{-1}(X^x-x-\rho_1 B^H(\omega_2))}(\omega_1, dr, dy)\right\} \left(\frac{u}{s}\right)^{H-\frac{1}{2}} \frac{1}{(u-s)^{\frac{1}{2}-H}} du,$$

$\omega_2 - a.e, \omega_1 - a.e, s \leq t ds - a.e$, where

$$f(s, y) = \rho_2^{-1} b(s, x + \rho_1 B^H_s(\omega_1) + \rho_2 y).$$

**Proof.** The first representation is a consequence of relation (51) in connection with Lemma 18 and Theorem 13 if we use the sample space splitting $\Omega = \Omega_1 \times \Omega_2$ for sample spaces $\Omega_1, \Omega_2$ on which $W$ and $B^H$ are defined, respectively. The second assertion follows from the above mentioned transfer principle for Malliavin derivatives. ■
Lemma 19 Retain the conditions and notation of Theorem 13. Then for all $0 \leq t \leq T$

$$X^t \in L^2(\Omega; W^{1,2}_{loc}(\mathbb{R}))$$

and

$$\frac{\partial}{\partial x} X^t = \exp \left\{ \int_s^t \int_\mathbb{R} f(s, y) L^{\rho_2^{-1}}(X^t - x - \rho_1 B^H(\omega_2))(\omega_1, d\tau, dy) \right\} \omega_2 - a.e, \omega_1 - a.e, x - a.e,$$

where

$$f(s, y) = \rho_2^{-1} b(s, x + \rho_1 B^H(\omega_1) + \rho_2 y).$$

**Proof.** The proof is a direct consequence of the proof of Proposition 3.5 in [6], if one uses the sample space splitting $\Omega = \Omega_1 \times \Omega_2$. ■

Lemma 20 Adopt the conditions and notation of Theorem 13. Let $\Phi \in C_0^\infty(\mathbb{R})$ and define the functions $u_n, u$ for $T > 0$ given by

$$u_n(x) := E[\Phi(X^t_{x,n})] \text{ and } u(x) := E[\Phi(X^t_x)],$$

where $X^z_{x,n}, n \geq 1$ is the approximating sequence of solutions associated with the vector fields $b_n, n \geq 1$ given by (29) and (30a). Further, let $\overline{\pi}$ be the function defined as

$$\overline{\pi}(x) := E[\Phi'(X^t_x) \frac{\partial}{\partial x} X^t_x].$$

Then

$$u_n(x) \xrightarrow{n \to \infty} u(x) \text{ for all } x$$

as well as

$$u'_n(x) \xrightarrow{n \to \infty} \overline{\pi}(x)$$

uniformly on compact subsets $K \subset \mathbb{R}$. So $u \in C^1(\mathbb{R})$ with $u' = \overline{\pi}(x)$.

**Proof.** The proof is the same as that of Lemma 4.1 in [6] applied to the splitting $\Omega = \Omega_1 \times \Omega_2$. ■

We are coming now to the proof of Theorem 12

**Proof of Theorem 12.** The proof is a consequence of that of Theorem 10 in combination with Theorem 13 and Lemma 20. ■
Proposition 21 Let $H < \frac{1}{6}$ and let $X^{s,x}$ be the unique strong solution to the SDE

$$dX^{s,x}_t = b(t, X^{s,x}_t)dt + \rho_1 dB^H_t + \rho_2 dW_t, X^s_x = x, 0 \leq t \leq T,$$  \hspace{1cm} (42)

where $b$ is of the form (27) and $\rho_1 \in \mathbb{R} \setminus \{0\}, \rho_2 \in \mathbb{R}$. Let $K$ be a compact cube in $\mathbb{R}$ and $r \in \mathbb{N}$. Then for all $s \in [0, T)$ and $x, y \in K$:

$$E \left[ \sup_{t \in [s,T]} |X^{s,x}_t - X^{s,y}_t|^2 r \right] \leq C_{r,H,T}(K) |x - y|^{2^r}. \hspace{1cm} (43)$$

Proof. Let us assume without loss of generality that $s = 0, T = 1, \rho_1, \rho_2 = 1$. In proving this result we aim at employing the inequality of Garsia-Rodemich-Rumsey (see Lemma 30 in the Appendix) in the the case, when $d(t,s) = |t - s|^{\frac{1}{1+\varepsilon}}, 0 < \varepsilon < 1, \Psi(x) = x^{\frac{4(1+\varepsilon)}{\varepsilon}}, x \geq 0, \Lambda = [0, 1], f(t) = |X^r_t - X^r_x|, x, y \in K$, where $K$ is a compact cube in $\mathbb{R}$. Then, $\sigma(r) \geq r^{\frac{1}{1+\varepsilon}}$ and we obtain that

$$|f(t) - f(s)| \leq 18 \int_0^{d(t,s)/2} \Psi^{-1} \left( \frac{U}{(\sigma(r))^2} \right) dr,$$

where

$$U = \int_0^1 \int_0^1 \Psi \left( \frac{|f(t_2) - f(t_1)|}{d(t_2, t_1)} \right) dt_2 dt_1.$$  \hspace{1cm} (43)

For $s = 0$ we get that

$$|f(t)| \leq 18 \int_0^{d(t,0)/2} \Psi^{-1} \left( \frac{U}{(\sigma(r))^2} \right) dr + |f(0)| = 18 \int_0^{d(t,0)/2} \Psi^{-1} \left( \frac{U}{(\sigma(r))^2} \right) dr + |x - y|.$$  \hspace{1cm} (42)

Let $p = 2^r > 1$ with $r \in \mathbb{N}$ such that $p^{\frac{1}{2(1+\varepsilon)}} > 1$. Then

$$|f(t)|^p \leq C_p(\int_0^{d(t,0)/2} \Psi^{-1} \left( \frac{U}{(\sigma(r))^2} \right) dr)^p + |x - y|^p.$$  \hspace{1cm} (44)

Thus,

$$\sup_{0 \leq t \leq 1} |f(t)|^p \leq C_p(\int_0^1 \Psi^{-1} \left( \frac{U}{(\sigma(r))^2} \right) dr)^p + |x - y|^p \leq C_p(\int_0^1 \left( \frac{1}{r} \right)^{\frac{4(1+\varepsilon)}{1+\varepsilon}} dr)^p U^p \frac{1}{(1+\varepsilon)^p} + |x - y|^p \leq C_p(U^p \frac{1}{(1+\varepsilon)^p} + |x - y|^p).$$  \hspace{1cm} (44)
Further,

\[
U^{p,\varepsilon(1+\varepsilon)} \leq \int_0^1 \int_0^1 \left( \Psi \left( \frac{|f(t_2) - f(t_1)|}{d(t_2, t_1)} \right) \right)^{p(1+\varepsilon)} dt_2 dt_1
\]

\[
= \int_0^1 \int_0^1 \left( \frac{|f(t_2) - f(t_1)|}{d(t_2, t_1)} \right)^{\frac{p(1+\varepsilon)}{2}} dt_2 dt_1
\]

\[
= \int_0^1 \int_0^1 \left( |f(t_2) - f(t_1)| \right)^{p} dt_2 dt_1.
\]

Hence, we find that

\[
E \left[ \sup_{0 \leq t \leq 1} |f(t)|^p \right] \leq C_p \left[ U^{p,\varepsilon(1+\varepsilon)} \right] + |x - y|^p
\]

\[
\leq C_p \int_0^1 \int_0^1 E \left[ \left( \frac{|f(t_2) - f(t_1)|}{d(t_2, t_1)} \right)^p \right] dt_2 dt_1 + |x - y|^p
\]

\[
\leq C_p \int_0^1 \int_0^1 E \left[ \left( \frac{|X_{x,n}^z - X_{x,n}^y - (X_{x,n}^z - X_{y,n}^z)|}{d(t_2, t_1)} \right)^p \right] dt_2 dt_1 + |x - y|^p.
\]

Further, by applying (41) in the proof of Theorem (13) the uniform integrability of \(|X_{t,n}^x|^p, n \geq 1, 0 \leq t \leq 1\) and Fatou's Lemma, we see that

\[
E \left[ \sup_{0 \leq t \leq 1} |f(t)|^p \right]
\]

\[
\leq C_p \left( \lim_{n \to \infty} \int_0^1 \int_0^1 E \left[ \left( \frac{|X_{x,n}^z - X_{y,n}^z - (X_{x,n}^z - X_{y,n}^z)|}{d(t_2, t_1)} \right)^p \right] dt_2 dt_1 + |x - y|^p \right),
\]

where \(X_{t,n}^x, 0 \leq t \leq 1\) is the strong solution to (42) associated with the approximating sequence of smooth vector fields \(b_n, n \geq 1\) given by (29) and (30a).

Assume without loss of generality that \(x > y\). Then, by using the fundamental theorem of calculus, we see that

\[
X_{t_2}^x - X_{t_2}^y - (X_{t_1}^x - X_{t_1}^y) = \int_y^x \left( \frac{\partial}{\partial z} X_{t_2}^z - \frac{\partial}{\partial z} X_{t_1}^z \right) dz.
\]

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Hence,

\[
E \left[ \left( \frac{X_{t_2}^{x,n} - X_{t_2}^{y,n} - (X_{t_1}^{x,n} - X_{t_1}^{y,n})}{d(t_2, t_1)} \right)^p \right] \\
\leq E \left[ \left( \int_y^x \frac{\partial X_{t_2}^{x,n} - \partial X_{t_1}^{x,n}}{d(t_2, t_1)} dz \right)^p \right] \\
\leq |x - y|^{p-1} E \left[ \int_y^x \left( \frac{\partial X_{t_2}^{x,n} - \partial X_{t_1}^{x,n}}{d(t_2, t_1)} \right)^p dz \right] \\
= |x - y|^{p-1} \int_y^x E \left[ \left( \frac{\partial X_{t_2}^{x,n} - \partial X_{t_1}^{x,n}}{d(t_2, t_1)} \right)^p \right] dz \\
\leq |x - y|^p \sup_{z \in K} E \left[ \left( \frac{\partial X_{t_2}^{x,n} - \partial X_{t_1}^{x,n}}{d(t_2, t_1)} \right)^p \right]
\]

Thus

\[
E \left[ \sup_{0 \leq t \leq 1} |f(t)|^p \right] \\
\leq C_{p,d} \lim_{n \to \infty} \int_0^1 \int_0^1 \sup_{z \in K} E \left[ \left( \frac{\partial X_{t_2}^{x,n} - \partial X_{t_1}^{x,n}}{d(t_2, t_1)} \right)^p \right] dt_2 dt_1 |x - y|^p + |x - y|^p.
\] (44)

By Lemma 2.6 in [6] that there exists a \( \beta > 0 \) depending on \( K \) such that

\[
\sup_{x \in K} E \left[ E \left( \int_0^T b_n(u, x + B_u^H + W_u)dW_u \right)^{1+\beta} \right] < \infty
\]

where \( \mathcal{E}(M_T) = \mathcal{E}_T(M) \) denotes the Doleans-Dade exponential of a martingale \( M \). Moreover, the proof of Lemma 2.6 in [6] in connection with the properties (29), (30) show that

\[
E \left[ E \left( \int_0^T b_n(u, x + B_u^H + W_u)dW_u \right)^{1+\beta} \right] \\
\leq e^{\tilde{C}_{\beta,T}(1+|x|)^2} \times \\
E \left[ \exp \left\{ 2\tilde{C}_{\beta,T}(1+|x|) \int_0^T |B_u| \, du + \tilde{C}_{\beta,T} \right\} \int_0^T |B_u|^2 \, du \right],
\] (45)

where \( \tilde{C}_{\beta,T} \) is a constant with \( \lim_{\beta \searrow 0} \tilde{C}_{\beta,T} = 0 \).

We know that

\[
\frac{\partial}{\partial x} X_t^{x,n} = \exp(\int_0^t b_n(u, X_u^{x,n})du), 0 \leq t \leq 1.
\]

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Then, using Girsanov’s theorem with respect to the Brownian motion $W$, and Hölder’s inequality, we find that

$$
E \left[ \left( \frac{\partial}{\partial z} X_{t_2}^{x,n} - \frac{\partial}{\partial z} X_{t_1}^{x,n} \right)^p \right] \\
\leq E \left[ \left( \frac{\partial}{\partial z} X_{t_2}^{x,n} - \frac{\partial}{\partial z} X_{t_1}^{x,n} \right)^{p + \beta} \right]^{\frac{1}{1 + \beta}} E \left[ \mathcal{E} \left( \int_0^T b_n(u, x + B^H_u + W_u) \, dW_u \right)^{1 + \beta} \right]^{\frac{1}{1 + \beta}} \\
= E \left[ \left( \exp \left( \int_0^{t_2} b_n(u, x + B^H_u + W_u) \, du \right) - \exp \left( \int_0^{t_1} b_n(u, x + B^H_u + W_u) \, du \right) \right)^{\frac{p + \beta}{1 + \beta}} \right]^{\frac{1}{1 + \beta}}.
$$

Further, by applying the inequality $|e^{z_1} - e^{z_2}| \leq (e^{z_1} \vee e^{z_2}) |z_1 - z_2|$ and Hölder’s inequality, we have for $t_2 > t_1$ that

$$
E \left[ \left( \frac{\exp \left( \int_0^{t_2} b_n(u, x + B^H_u + W_u) \, du \right) - \exp \left( \int_0^{t_1} b_n(u, x + B^H_u + W_u) \, du \right)}{d(t_2, t_1)} \right)^{\frac{p + \beta}{1 + \beta}} \right]^{\frac{1}{1 + \beta}} \\
\leq E \left[ \left( \frac{\int_{t_1}^{t_2} b_n(u, x + B^H_u + W_u) \, du}{d(t_2, t_1)} \right)^{2p + \beta} \right]^{\frac{1}{2(1 + \beta)}} \times \\
E \left[ \left( \exp \left( \int_0^{t_2} b_n(u, x + B^H_u + W_u) \, du \right) \vee \exp \left( \int_0^{t_1} b_n(u, x + B^H_u + W_u) \, du \right) \right)^{2p + \beta} \right]^{\frac{1}{2(1 + \beta)}} \\
= I_1(x) \cdot I_2(x).
$$
Suppose without loss of generality that $p^{1+\frac{H}{p}} = m$ for $m \in \mathbb{N}$. We observe that

$$E \left[ \left( \frac{\int_{t_1}^{t_2} b_n(u, x + B^H_u + W_u)du}{d(t_2, t_1)} \right)^{2p^{1+\frac{H}{p}}} \right]$$

$$\leq C_m E \left[ \left( \frac{\int_{t_1}^{t_2} \tilde{b}_n(u, x + B^H_u + W_u)du}{d(t_2, t_1)} \right)^{2p^{1+\frac{H}{p}}} \right] + K^{2p^{1+\frac{H}{p}}} \left( \frac{\left| t_2 - t_1 \right|}{d(t_2, t_1)} \right)^{2p^{1+\frac{H}{p}}}.$$ 

We now choose $\varepsilon > 0$ such that $\frac{\varepsilon}{1+\varepsilon} = 1 - 3H$. So

$$d(t, s) = |t - s|^{1+\varepsilon} = |t - s|^{1-3H}.$$ 

On the other hand, it follows from inequality (34) in connection with Lemma 25 in the Appendix and (39a) that

$$E \left[ \left( \frac{\int_{t_1}^{t_2} b_n(u, x + B^H_u + W_u)du}{d(t_2, t_1)} \right)^{2p^{1+\frac{H}{p}}} \right]$$

$$\leq L_m E \left[ \exp(\alpha C(H, d, T) \| b_n \|_{L^2}^2) \left( 1 + \sup_{0 \leq t \leq T} |B_t| \right)^2 \right]$$

$$\leq L_m E \left[ \exp(\alpha C(H, d, T) L^2 (1 + \sup_{0 \leq t \leq T} |B_t|^2)) \right] < \infty$$

for constants $L_m, C(H, d, T), L, \alpha$, where $\alpha = \alpha(H, d, T) > 0$ is sufficiently small and $d = 1$. The latter implies that

$$\sup_{x \in K} I_1(x) \leq (L_m E \left[ \exp(\alpha C(H, d, T) L^2 (1 + \sup_{0 \leq t \leq T} |B_t|^2)) \right])^{\frac{1}{1+\varepsilon}} < \infty.$$ 

As for the factor $I_2(x)$ we obtain that

$$(I_2(x))^{2+\frac{H}{p}}$$

$$\leq E \left[ \left( \exp(\int_0^{t_2} b_n(u, x + B^H_u + W_u)du) \vee \exp(\int_0^{t_1} b_n(u, x + B^H_u + W_u)du) \right)^{2p^{1+\frac{H}{p}}} \right]$$

$$\leq E \left[ \exp(2m \int_0^{t_2} b_n(u, x + B^H_u + W_u)du) \right] + E \left[ \exp(2m \int_0^{t_1} b_n(u, x + B^H_u + W_u)du) \right].$$
We also see that
\[
E \left[ \exp(2m \int_0^t \tilde{b}_n(u, x + B^H_u + W_u) \, du) \right] \\
\leq C_{m, K^*} E \left[ \exp(2m \left| \int_0^t \tilde{b}_n(u, x + B^H_u + W_u) \, du \right|) \right],
\]
where $K^*$ is the uniform Lipschitz constant of $\tilde{b}_n$ in (29). Without loss of generality consider the case, when $t = 1$. Then inequality (54) in the Appendix combined with (29), (30a) entail that
\[
E \left[ \exp(2m \left| \int_0^1 \tilde{b}_n(u, x + B^H_u + W_u) \, du \right|) \right] \\
\leq E \left[ \exp \left( \frac{2m}{\sqrt{\alpha}} \left| \int_0^1 \tilde{b}_n(u, x + B^H_u + W_u) \, du \right|^2 \right) \right] \\
\leq C_{\alpha, m} E \left[ \exp(\alpha C(H, d, T) L^2(1 + \sup_{0 \leq t \leq T} |B_t|)^2) \right],
\]
for constants $C_{\alpha, m}, C(H, d, T), L, \alpha$, where $\alpha = \alpha(H, d, T) > 0$ is sufficiently small and $d = 1$. Hence, we find that
\[
\sup_{x \in K} I_2(x) < \infty.
\]
Finally, the proof follows from (44), (46) and (45).

\[\text{Remark 22} \quad \text{An estimate of the form (43) can be e.g. found in [30], ?? in the case of a Wiener process and } b \in L^\infty([0, T] \times \mathbb{R}^d). \text{ See also [32] in the case of a fractional Brownian motion with Hurst parameter } H < \frac{1}{2(d+2)} \text{ and } b \in L^{1, \infty}_H = L^\infty_H \cap L^\infty_\alpha. \text{ It turns out that such an estimate-as already mentioned in the Introduction-plays a central role for proving path-by-path uniqueness of solutions (see [13]) to SDE’s with additive Wiener or fractional Brownian noise for bounded vector fields } b, \text{ which is a much stronger property than pathwise uniqueness. See [30], [31] and also [3] for more details.}\]

\[\text{4 Appendix}\]

For some of the proofs in this article we need a version of Girsanov’s theorem for the fractional Brownian motion. In order to state this result, let us recall some basic concepts from fractional calculus (see [29] and [25]).

Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in L^p([a, b])$ with $p \geq 1$ and $\alpha > 0$. Define the left- and right-sided Riemann-Liouville fractional integrals as
\[
I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha-1} f(y) \, dy.
\]
and
\[ I^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy \]
for almost all \( x \in [a, b] \). Here \( \Gamma \) denotes the Gamma function.

Let \( p \geq 1 \) and let \( I^\alpha_a (L^p) \) (resp. \( I^\alpha_b (L^p) \)) be the image of \( L^p([a, b]) \) of the operator \( I^\alpha_a \) (resp. \( I^\alpha_b \)). If \( f \in I^\alpha_a (L^p) \) (resp. \( f \in I^\alpha_b (L^p) \)) and \( 0 < \alpha < 1 \) then we can introduce the left- and right-sided Riemann-Liouville fractional derivatives by
\[ D^\alpha_a f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^\alpha} dy \]
and
\[ D^\alpha_b f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(y)}{(y-x)^\alpha} dy. \]

The left- and right-sided derivatives of \( f \) have the representations
\[ D^\alpha_a f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^\alpha+1} dy \right) \]
and
\[ D^\alpha_b f(x) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{f(x)}{(b-x)^\alpha} + \alpha \int_x^b \frac{f(x) - f(y)}{(y-x)^\alpha+1} dy \right). \]

The above definitions entail that
\[ I^\alpha_a (D^\alpha_a f) = f \]
for all \( f \in I^\alpha_a (L^p) \) and
\[ D^\alpha_a (I^\alpha_a f) = f \]
for all \( f \in L^p([a, b]) \) and similarly for \( I^\alpha_b \) and \( D^\alpha_b \).

Denote by \( B^H = \{ B^H_t, t \in [0, T] \} \) a \( d \)-dimensional fractional Brownian motion with Hurst parameter \( H \in (0, 1/2) \). Here this means that \( B^H \) is a centered Gaussian process with a covariance function given by
\[ (R_H(t, s))_{i,j} := E[B^H_t, B^H_s] = \delta_{i,j} \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}), \quad i, j = 1, \ldots, d, \]
where \( \delta_{i,j} \) is one, if \( i = j \), or zero else.

In what follows we briefly pass in review the construction of the fractional Brownian motion, which can be found in [27]. For convenience, we confine ourselves to the case \( d = 1 \).

Let \( \mathcal{E} \) be the class of step functions on \([0, T] \) and denote by \( \mathcal{H} \) the Hilbert space which obtained through the completion of \( \mathcal{E} \) with respect to the inner product
\[ \langle 1_{[0,t]}, 1_{[0,s]} \rangle_\mathcal{H} = R_H(t, s). \]
The latter gives an extension of the mapping $1_{[0,t]} \mapsto B_t$ to an isometry between $\mathcal{H}$ and a Gaussian subspace of $L^2(\Omega)$ with respect to $B^H$. Let $\varphi \mapsto B^H(\varphi)$ be this isometry.

If $H < 1/2$, one verifies that the covariance function $R_H(t,s)$ can be represented as

$$R_H(t,s) = \int_0^{t \wedge s} K_H(t,u)K_H(s,u)du,$$  \hspace{1cm} (47)

where

$$K_H(t,s) = c_H \left[ \left( \frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{-\frac{1}{2}} + \left( \frac{1}{2} - H \right)s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{1}{2}}(u-s)^{-\frac{1}{2}}du \right].$$  \hspace{1cm} (48)

Here $c_H = \sqrt{\frac{2H}{(1-2H)b(1-2H,H+1/2)}}$ and $b$ is the Beta function. See \cite[Proposition 5.1.3]{27}.

Using the kernel $K_H$, one can define via (47) an isometry $K_H^*$ between $E$ and $L^2([0,T])$ such that $(K_H^* \varphi)(s) = K_H(t,s)1_{[0,t]}(s)$. This isometry admits for an extension to the Hilbert space $\mathcal{H}$, which has the following representations in terms of fractional derivatives

$$(K_H^* \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) s^{\frac{1}{2}-H} \left( D_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} \varphi(u) \right)(s)$$

and

$$(K_H^* \varphi)(s) = c_H \Gamma \left( H + \frac{1}{2} \right) \left( D_{T-}^{\frac{1}{2}-H} \varphi(s) \right)(s)
\hspace{0.5cm} + c_H \left( \frac{1}{2} - H \right) \int_s^T \varphi(t)(t-s)^{\frac{1}{2}-H} \left( 1 - \left( \frac{t}{s} \right)^{H-\frac{1}{2}} \right) dt.$$

for $\varphi \in \mathcal{H}$. One can also show that $\mathcal{H} = L^2_{T-}(H)$. See \cite[Proposition 6]{14}.

We know that $K_H^*$ is an isometry from $\mathcal{H}$ into $L^2([0,T])$. Hence, the $d$-dimensional process $W = \{W_t, t \in [0,T]\}$ defined by

$$W_t := B^H((K_H^*)^{-1}(1_{[0,t]}))$$  \hspace{1cm} (49)

is a Wiener process and the process $B^H$ has the representation

$$B^H_t = \int_0^t K_H(t,s)dW_s.$$  \hspace{1cm} (50)

See \cite{1}.

In the sequel, we also need the Definition of a fractional Brownian motion with respect to a filtration.
**Definition 23** Let $\mathcal{G} = \{\mathcal{G}_t\}_{t \in [0,T]}$ be a filtration on $(\Omega, \mathcal{F}, P)$ satisfying the usual conditions. A fractional Brownian motion $B^H$ is called a $\mathcal{G}$-fractional Brownian motion if the process $W$ defined by (40) is a $\mathcal{G}$-Brownian motion.

Let $W$ be a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ is the natural filtration generated by $W$ and augmented by all $P$-null sets. Denote by $B := B^H$ the fractional Brownian motion with Hurst parameter $H \in (0,1/2)$ as in (30).

We shall also apply a version of Girsanov’s theorem for fractional Brownian motion which can be found in [14, Theorem 4.9]. See also [28, Theorem 2]. This version relies on the following definition of an isomorphism $K_H$ from $L^2([0,T])$ onto $I^{H+\frac{1}{2}}_0(L^2)$ with respect to the kernel $K_H(t,s)$ based on fractional integrals (see [14, Theorem 2.1]):

$$(K_H \varphi)(s) = I^{2H}_0 s^{\frac{1}{2} - H} I^{\frac{1}{2} - H}_{0+} s^{H - \frac{1}{2}} \varphi, \quad \varphi \in L^2([0,T]).$$

The latter combined with the properties of the Riemann-Liouville fractional integrals and derivatives can be used to prove the following representation of inverse of $K_H$:

$$(K_H^{-1} \varphi)(s) = s^{\frac{1}{2} - H} D_{0+}^{\frac{1}{2} - H} s^{H - \frac{1}{2}} D_{0+}^{2H} \varphi(s), \quad \varphi \in I^{H+\frac{1}{2}}_0(L^2). \quad (51)$$

Using this we find for absolutely continuous functions $\varphi$ (see [28]) that

$$(K_H^{-1} \varphi)(s) = s^{H - \frac{1}{2}} I^{\frac{1}{2} - H}_{0+} s^{\frac{1}{2} - H} \varphi'(s).$$

**Theorem 24 (Girsanov’s theorem for fBm)** Let $u = \{u_t, t \in [0,T]\}$ be an $\mathcal{F}$-adapted process with integrable trajectories and set $B^H_t = B^H_t + \int_0^t u_s ds, \quad t \in [0,T]$. Suppose that

(i) $\int_0^T u_s ds \in I^{H+\frac{1}{2}}_0(L^2([0,T]))$, $P$-a.s.

(ii) $E[\xi_T] = 1$ where

$$\xi_T := \exp \left\{ - \int_0^T K_H^{-1} \left( \int_0^t u_r dr \right)(s) dW_s - \frac{1}{2} \int_0^T K_H^{-1} \left( \int_0^t u_r dr \right)^2(s) ds \right\}.$$

Then the shifted process $\tilde{B}^H$ is an $\mathcal{F}$-fractional Brownian motion with Hurst parameter $H$ under the new probability $\tilde{P}$ defined by $\frac{d\tilde{P}}{dP} = \xi_T$.

**Remark 25** In the the multi-dimensional case, we define

$$(K_H \varphi)(s) := ((K_H \varphi^{(1)})(s), \ldots, (K_H \varphi^{(d)})(s))^*, \quad \varphi \in L^2([0,T]; \mathbb{R}^d),$$

where $*$ denotes transposition. Similarly for $K_H^{-1}$ and $K_H$. 

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In this Appendix we also recapitulate an integration by parts formula from [7], which is based on a sort of local time of the Gaussian process $B^H$.

Let $m \in \mathbb{N}$ and let $f : [0, T]^m \times (\mathbb{R}^d)^m \to \mathbb{R}$ be a function defined by

$$f(s, z) = \prod_{j=1}^{m} f_j(s_j, z_j), \quad s = (s_1, \ldots, s_m) \in [0, T]^m, \quad z = (z_1, \ldots, z_m) \in (\mathbb{R}^d)^m,$$

where $f_j : [0, T] \times \mathbb{R} \to \mathbb{R}$, $j = 1, \ldots, m$ are (spatially) smooth functions with compact support. Further, let $\kappa : [0, T]^m \to \mathbb{R}$ be a function given by

$$\kappa(s) = \prod_{j=1}^{m} \kappa_j(s_j), \quad s \in [0, T]^m,$$

where $\kappa_j : [0, T] \to \mathbb{R}$, $j = 1, \ldots, m$ are integrable.

Further, denote by $\alpha_j$ a multiindex and $D^{\alpha_j}$ its corresponding differential operator. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^d \times m$. Then define $|\alpha| = \sum_{j=1}^{m} \sum_{l=1}^{d} \alpha_j^{(l)}$ and

$$D^{\alpha}f(s, z) = \prod_{j=1}^{m} D^{\alpha_j}f_j(s_j, z_j).$$

In [7] the following integration by parts formula was shown

$$\int_{\Delta^m_{\theta, t}} D^{\alpha}f(s, B^H_s)ds = \int_{(\mathbb{R}^d)^m} \Lambda^f_{\alpha}(\theta, t, z)dz,$$

where $\Lambda^f_{\alpha}$ is a suitable random field, $\Delta^m_{\theta, t}$ is the $m$-dimensional simplex (see [12] and $B^H_s := (B^H_{s_1}, \ldots, B^H_{s_m})$ is a fractional Brownian on that simplex. More specifically, we have that

$$\Lambda^f_{\alpha}(\theta, t, z) = (2\pi)^{-dm} \int_{(\mathbb{R}^d)^m} \int_{\Delta^m_{\theta, t}} \prod_{j=1}^{m} f_j(s_j, z_j)(-iu_j)^{\alpha_j} \exp\{-i(u_j, B^H_{s_j} - z_j)\}dsdu. \quad (53)$$

It turns out that the random field $\Lambda^f_{\alpha}(\theta, t, z)$ is a well-defined element of $L^2(\Omega)$.

In this paper, we also need the notion of shuffle permutations: Let $m$ and $n$ be integers. Denote by $S(m+n)$ the set of shuffle permutations, that is the set of permutations $\sigma : \{1, \ldots, m+n\} \to \{1, \ldots, m+n\}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$.

We introduce the following notation: Given $(s, z) = (s_1, \ldots, s_m, z_1, \ldots, z_m) \in [0, T]^m \times (\mathbb{R}^d)^m$ and a shuffle $\sigma \in S(m, m)$ we write

$$f_\sigma(s, z) := \prod_{j=1}^{2m} f_{[\sigma(j)]}(s_j, z_{[\sigma(j)]})$$

and

$$\kappa_\sigma(s) := \prod_{j=1}^{2m} \kappa_{[\sigma(j)]}(s_j),$$

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where \( j \) is equal to \( j \) if \( 1 \leq j \leq m \) and \( j - m \) if \( m + 1 \leq j \leq 2m \).

Define the expressions

\[
\Psi_{f_k}^{f}(\theta, t, z) = \prod_{l=1}^{d} \sqrt{2^{|\alpha(l)|}}! \sum_{\sigma \in S(m, m)} \int_{\Delta^2_{0,1}} |f_\sigma(s, z)| \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{H(d + 2 \sum_{l=1}^{d} \alpha(l))}} ds_1 ... ds_{2m},
\]

\[
\Psi_{\kappa}^{\kappa}(\theta, t) = \prod_{l=1}^{d} \sqrt{2^{|\alpha(l)|}}! \sum_{\sigma \in S(m, m)} \int_{\Delta^2_{0,1}} |\kappa_\sigma(s)| \prod_{j=1}^{2m} \frac{1}{|s_j - s_{j-1}|^{H(d + 2 \sum_{l=1}^{d} \alpha(l))}} ds_1 ... ds_{2m}.
\]

**Theorem 26** Suppose that \( \Psi_{f_k}^{f}(\theta, t, z), \Psi_{\kappa}^{\kappa}(\theta, t) < \infty \). Then, \( \Lambda_{\alpha}^{f}(\theta, t, z) \) given by (53) is a random variable in \( L^2(\Omega) \) and there exists a universal constant \( C = C(T; H, d) > 0 \) such that

\[
E \left[ |\Lambda_{\alpha}^{f}(\theta, t, z)|^2 \right] \leq C^{m+|\alpha|} \Psi_{f_k}^{f}(\theta, t, z).
\]

Moreover, we have

\[
\left| E \left[ \int_{(\mathbb{R}^d)^m} \Lambda_{\alpha}^{f}(\theta, t, z) dz \right] \right| \leq C^{m/2+|\alpha|/2} \prod_{j=1}^{m} \|f_j\|_{L^1(\mathbb{R}^d; L^\infty([0, T]))} (\Psi_{\kappa}^{\kappa}(\theta, t))^{1/2}.
\]

We also need the following auxiliary result in connection with proof of Theorem 7.

**Lemma 27** Let \( U \subset \mathbb{R}^d \) be an open and bounded subset. Consider the sequence \( X^{x, n}, n \geq 1 \) in Proposition 13. Then

\[
\frac{\partial}{\partial x} X^{x, n} \rightharpoonup \frac{\partial}{\partial x} X
\]

in \( L^2([0, T] \times \Omega \times U) \) weakly.

**Proof.** This result is a consequence of relation (13) and the estimate (14) in step 3 of the proof sketch of Theorem 7. ■

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Proof of Theorem 7 Suppose that $\Phi \in C^\infty_c(\mathbb{R}^d)$ and let $b_n \in C^\infty_c((0, T) \times \mathbb{R}^d)$ be a sequence of functions such that

$$b_n \xrightarrow{n \to \infty} b \text{ in } L^1_{\infty}.$$ 

Let $X^{s,x,n}_t$ be the unique strong solution to

$$dX^{s,x,n}_t = b_n(t, X^{s,x,n}_t)dt + dB^H_t, X^{s,x,n}_s = x, s \leq t \leq T$$

for all $n$. Since $b_n$ is smooth and compactly supported, we know (see e.g. [23]) that there exists a $\Omega^*$ with $\mu(\Omega^*) = 1$ such that for all $\omega \in \Omega^*$,

$$(x \mapsto X^{s,x,n}_t(\omega)) \in C^\infty(\mathbb{R}^d).$$

Hence dominated convergence implies that

$$\frac{\partial}{\partial x} E[\Phi(X^{s,n}_T)] = E[\Phi'(X^{s,n}_T)\frac{\partial}{\partial x} X^{s,n}_T].$$

Here $\Phi'$ denotes the derivative of $\Phi$ and $X^{s,n}_t = X^{0,x,n}_t$. Further, we obtain for all $0 \leq s \leq t \leq T, x \in U$ that

$$X^{s,n}_t = X^{s,X^{s,n}_s,n}_t \text{ a.e.}$$

Thus

$$\frac{\partial}{\partial x} E[\Phi(X^{s,n}_T)] = E[\Phi'(X^{s,n}_T)\frac{\partial}{\partial x} X^{s,X^{s,n},n}_T, \frac{\partial}{\partial x} X^{s,n}_T].$$

It is known that the Malliavin derivative $D^H u X^{s,n}_t$ of $X^{s,n}_t$ in the direction of $B^H$ exists (see e.g. [24]) and that

$$D^H u X^{s,n}_t = \int_u^t b_n(t, X^{s,n}_r)D^H u X^{s,n}_r dr + \chi_{(s,t)}(u)I_d \times d,$$

where $I_d \times d$ is the identity matrix. We also observe that $\frac{\partial}{\partial x} X^{u,n,X^{s,n},n}_t$ is a solution of the latter equation for $s = 0$. So we conclude that

$$D^H u X^{s,n}_t = \frac{\partial}{\partial x} X^{u,n,X^{s,n},n}_t$$

a.e., which yields

$$\frac{\partial}{\partial x} E[\Phi(X^{s,n}_T)] = E[\Phi'(X^{s,n}_T)D^H s X^{s,n}_T, \frac{\partial}{\partial x} X^{s,n}_T].$$

Choose $\varphi \in C^\infty_c(U)$. Then, we see that

$$-\int_U E[\Phi(X^{s,n}_T)]\frac{\partial}{\partial x} \varphi(x)dx = \int_U \varphi(x)E[\Phi(X^{s,n}_T)D^H s X^{s,n}_T, \frac{\partial}{\partial x} X^{s,n}_T]dx.$$
Since the function \(a\) sums up to one, we can then use the chain rule for \(D^H\) (see [27]) and get that

\[
- \int_U E[\Phi(X_{T}^{x,n})] \frac{\partial}{\partial x} \varphi(x) dx
= \int_U \varphi(x) E[\int_0^T \{a(s)\Phi'(X_{T}^{x,n}) D^H_s X_{T}^{x,n} \frac{\partial}{\partial x} X_{T}^{x,n}\} ds] dx
= \int_U \varphi(x) E[\int_0^T \{a(s) D^H_s \Phi(X_{T}^{x,n}) \frac{\partial}{\partial x} X_{T}^{x,n}\} ds] dx
\]

Further, we know from Proposition 5.2.1 and p. 285 in [27] that

\[
D^H_s \Phi(X_{T}^{x,n}) = C s^{\frac{1}{2} - H} (\int_s^T (u - s)^{-H - \frac{1}{2}} u^{H - \frac{1}{2}} D_u \Phi(X_{T}^{x,n}) du) X_{T}^{x,n} ds
\]

for a constant \(C\) depending on \(H\). Therefore, using substitution (first for \(u\) substituted by \(u + s\) in the above relation and then for \(s\) by \(s - u\) in the next step), Fubini's theorem and the duality formula with respect to the Malliavin derivative \(D\), we obtain that

\[
- \int_U E[\Phi(X_{T}^{x,n})] \frac{\partial}{\partial x} \varphi(x) dx
= C \int_U \varphi(x) E[\int_0^T \{a(s) C s^{\frac{1}{2} - H} (\int_s^T (u - s)^{-H - \frac{1}{2}} u^{H - \frac{1}{2}} D_u \Phi(X_{T}^{x,n}) du) X_{T}^{x,n} ds\} ds] dx
= C \int_U \varphi(x) E[\Phi(X_{T}^{x,n})]
\]

\[
\times \int_0^T u^{-H - \frac{1}{2}} \int_u^T a(s - u)(s - u)^{\frac{1}{2} - H} s^{H - \frac{1}{2}} D_s \Phi(X_{T}^{x,n}) \frac{\partial}{\partial x} X_{T}^{x,n} ds du] dx
= C \int_U \varphi(x) E[\Phi(X_{T}^{x,n})]
\]

\[
\times \int_0^T u^{-H - \frac{1}{2}} \int_u^T a(s - u)(s - u)^{\frac{1}{2} - H} s^{H - \frac{1}{2}} \left( \frac{\partial}{\partial x} X_{T}^{x,n} \right)^* dB_s du)^* dx
= I_1(n) + I_2(n),
\]

where

\[
I_1(n) : = C \int_U \varphi(x) E[(\Phi(X_{T}^{x,n}) - \Phi(X_{T}^{x,n}))]
\]

\[
\times \int_0^T u^{-H - \frac{1}{2}} \int_u^T a(s - u)(s - u)^{\frac{1}{2} - H} s^{H - \frac{1}{2}} \left( \frac{\partial}{\partial x} X_{T}^{x,n} \right)^* dB_s du)^* dx
\]
and

\[ I_2(n) = C \int_U \varphi(x) E[\Phi(X_T^x)] \int_0^T u^{-H - \frac{1}{2}} \times \int_u^T a(s - u)(s - u)^{\frac{1}{2} - H - \frac{1}{2}} s^{H - \frac{1}{2}} \left( \frac{\partial}{\partial x} X_{s-u}^{x,n} \right)^* dB_s du ]^* dx \]

\[ = C \int_U \varphi(x) E[\Phi(X_T^x)] \int_0^T u^{-H - \frac{1}{2}} \times \int_u^T a(s - u)(s - u)^{\frac{1}{2} - H - \frac{1}{2}} s^{H - \frac{1}{2}} \left( \frac{\partial}{\partial x} X_{s-u}^{x,n} \right)^* dB_s du ]^* dx \]

\[ + I_3(n), \]

where

\[ I_3(n) = C \int_U \varphi(x) E[\Phi(X_T^x)] \int_0^T u^{-H - \frac{1}{2}} \int_u^T a(s - u)(s - u)^{\frac{1}{2} - H - \frac{1}{2}} \times \left( \frac{\partial}{\partial x} X_{s-u}^{x,n} \right)^* - \left( \frac{\partial}{\partial x} X_{s-u}^{x} \right)^* \] dB_s du]^* dx.

By applying Fubini’s theorem, Hölder’s inequality, the Itô isometry, the estimate (15), the relation (13) in step 3 of the proof sketch of Theorem 3 and dominated
convergence that
\[
\|I_1(n)\| 
\leq \|\varphi\|_\infty \int_U (E[|\Phi(X_T^{x,n}) - \Phi(X_T^x)|^2])^{1/2} 
\times \left( \int_0^T s^{2H-1} E\left[ \int_0^s u^{-H-\frac{1}{2}} |a(s-u)| (s-u)^{\frac{1}{2}-H} \left\| \frac{\partial}{\partial x} X_{s-u} \right\| du \right] ds \right)^{1/2} dx 
\]
\[
\leq \|\varphi\|_\infty \int_U (E[|\Phi(X_T^{x,n}) - \Phi(X_T^x)|^2])^{1/2} \left( \int_0^T s^{2H-1} \right) 
\times \left( \int_0^s u^{-H-\frac{1}{2}} |a(s-u_1)| (s-u_1)^{\frac{1}{2}-H} u_2^{-H-\frac{1}{2}} |a(s-u_2)| (s-u_2)^{\frac{1}{2}-H} 
\times E\left[ \left\| \frac{\partial}{\partial x} X_{s-u_1} \right\|^2 \right]^{1/2} E\left[ \left\| \frac{\partial}{\partial x} X_{s-u_2} \right\|^2 \right]^{1/2} du_1 du_2 \right)^{1/2} dx 
\]
\[
= \|\varphi\|_\infty \int_U (E[|\Phi(X_T^{x,n}) - \Phi(X_T^x)|^2])^{1/2} \left( \int_0^T s^{2H-1} \right) 
\times \left( \int_0^s u^{-H-\frac{1}{2}} |a(s-u)| (s-u)^{\frac{1}{2}-H} E\left[ \left\| \frac{\partial}{\partial x} X_{s-u} \right\|^2 \right]^{1/2} du \right)^{1/2} dx 
\]
\[
\leq \|\varphi\|_\infty \int_U (E[|\Phi(X_T^{x,n}) - \Phi(X_T^x)|^2])^{1/2} dx \left( \int_0^T s^{2H-1} \right) 
\times \sup_{n \geq 1} L_{2,H,d,T}(\|b_n\|_{L_1})^{1/4} \left( \int_0^s u^{-H-\frac{1}{2}} |a(s-u)| (s-u)^{\frac{1}{2}-H} du \right)^{1/2} 
\]
\[
\leq C \|\varphi\|_\infty \int_U (E[|\Phi(X_T^{x,n}) - \Phi(X_T^x)|^2])^{1/2} dx \left( \int_0^T s^{H-\frac{1}{2}} ds \right)^{1/2} 
\rightarrow 0, \quad n \rightarrow \infty 
\]
where used the boundedness of the function \(a\) in the last estimate.

Using the Clark-Ocone formula (see e.g. [27]) combined with the Itô isometry
and the chain rule for Malliavin derivatives, we find that

\[
I_3(n) = C \int_U \varphi(x) E[E(\Phi(X_T^n))] \int_0^T u^{-H-\frac{1}{2}} \int_u^T a(s-u)(s-u)^{\frac{1}{2}-H} s^{H-\frac{1}{2}} \\
\times \left( \frac{\partial}{\partial x} X_{s-u}^{z,n} \right)^* \left( \frac{\partial}{\partial x} X_{s-u}^{z,n} \right)^* dB_s du \right| dx \\
+ C \int_U \varphi(x) E\left[ \int_0^T u^{-H-\frac{1}{2}} \int_u^T a(s-u)(s-u)^{\frac{1}{2}-H} s^{H-\frac{1}{2}} D_s \Phi(X_T^n) \\
\times \left( \frac{\partial}{\partial x} X_{s-u}^{z,n} \right)^* ds du \right| dx \\
= C \int_U \varphi(x) E\left[ \int_0^T u^{-H-\frac{1}{2}} \int_u^T a(s-u)(s-u)^{\frac{1}{2}-H} s^{H-\frac{1}{2}} \Phi(X_T^n) D_s X_T^n \\
\times \left( \frac{\partial}{\partial x} X_{s-u}^{z,n} \right)^* ds du \right| dx.
\]

Then Lemma 27 and dominated convergence combined with the estimate (14) give

\[
\|I_3(n)\| \xrightarrow{n \to \infty} 0.
\]

We also have that

\[
- \int_U E[E(\Phi(X_T^n))] \frac{\partial}{\partial x} \varphi(x) dx \xrightarrow{n \to \infty} \int_U E[E(\Phi(X_T^n))] \frac{\partial}{\partial x} \varphi(x) dx.
\]

Hence,

\[
- \int_U E[E(\Phi(X_T^n))] \frac{\partial}{\partial x} \varphi(x) dx \\
= C \int_U \varphi(x) E[E(\Phi(X_T^n))] \\
\times \int_0^T u^{-H-\frac{1}{2}} \int_u^T a(s-u)(s-u)^{\frac{1}{2}-H} s^{H-\frac{1}{2}} \left( \frac{\partial}{\partial x} X_{s-u}^{z,n} \right)^* dB_s ds \right| dx
\]

Finally, using the monotone class theorem combined with dominated convergence and the Cauchy-Schwarz inequality, we can show the above relation for Borel measurable functions \( \Phi : \mathbb{R}^d \to \mathbb{R} \) such that

\( \Phi(X_T^n) \in L^2(\Omega \times U, \mu \times dx) \).

So the proof follows.

**Lemma 28** Let \( b \in C_b^1([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \) and \( \|b\|_\infty \leq 1 \). Suppose that \( H < \frac{1}{6} \). Then there exists a \( C < \infty \) and a sufficiently small \( \alpha > 0 \) (which depend on \( H, d, T, \) but not on \( b \)) such that for \( 0 \leq s < t \leq T \):

\[
E \left[ \exp\left(\frac{\alpha}{|t-s|^{2(1-3H)}} \left\| \int_s^t D_x b(u, B_u^H) du \right\|^2 \right) \right] < C,
\]
where $D_x$ is the Fréchet derivative of $b$ with respect to the spatial variable $x$.

**Remark 29** In fact, the proof of Lemma 2.12 in (Amine, Mansouri, Proske) gives the following bound

$$
E \left[ \exp(\frac{\alpha}{|t-s|^{2(1-3H)}} \left\| \int_s^t D_x b(u, B^H_u) \, du \right\|^2) \right] 
\leq E \left[ \exp(\alpha C(H, d, T) \|b\|_\infty^2 (1 + \sup_{0 \leq t \leq T} |B_t|)^2) \right] < \infty.
$$

(54)

for a constant $C(H, d, T) < \infty$.

We also want to apply the following special version of a lemma, which is due to Garcia, Rodemich and Rumsey (see e.g. [20])

**Lemma 30** Let $\Lambda$ be a compact interval endowed with a metric $d$. Define $\sigma(r) = \inf_{x \in \Lambda} \lambda(B(x, r))$, where $B(x, r) := \{y \in \Lambda : d(x, y) \leq r\}$ denotes the ball of radius $r$ centered in $x \in \Lambda$ and where $\lambda$ is the Lebesgue measure. Assume that $\Psi : [0, \infty) \rightarrow [0, \infty)$ is positive, increasing and convex with $\Psi(0) = 0$ and denote by $\Psi^{-1}$ its inverse, which is a positive, increasing and concave function. Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is continuous on $(\Lambda, d)$ and let

$$
U = \int_{\Lambda \times \Lambda} \Psi \left( \frac{|f(t) - f(s)|}{d(t, s)} \right) \, dtds.
$$

Then

$$
|f(t) - f(s)| \leq 18 \int_0^{d(t, s)/2} \Psi^{-1} \left( \frac{U}{(\sigma(r))^2} \right) \, dr.
$$

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