3-form Yang-Mills based on 2-crossed modules

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Abstract

In this paper, we study the higher Yang-Mills theory in the framework of higher gauge theory. It was shown that the 2-form electromagnetism can be generalized to the 2-form Yang-Mills theory with the group $U(1)$ replaced by a crossed module of Lie groups. To extend this theory to even higher structure, we develop a 3-form Yang-Mills theory with a 2-crossed module of Lie groups. First, we give an explicit construction of non-degenerate symmetric $G$-invariant forms on the 2-crossed module of Lie algebras. Then, we derive the 3-Bianchi-Identities for 3-curvatures. Finally, we create a 3-form Yang-Mills action and obtain the corresponding field equations.

Keywords: crossed module, 2-crossed module, 3-connection, 3-Bianchi-Identities, 3-form Yang-Mills

In modern mathematical physics a certain idea is very productive: the higher-dimensional extended objects are thought to be the basic constituents of matter and mediators of fundamental interactions. At present, higher gauge theory [1] seems to be the geometrically most promising technique to describe the dynamics of the higher-dimensional extended objects where gauge fields and field strengths are higher degree forms. Among them, the 2-gauge theory [2, 3, 4] and the 3-gauge theory [5, 6, 7, 8] have been studied deeply, whose higher algebraic structures and higher geometrical structures can be found in, e.g. [9, 4, 10, 11, 12].

In this article, we are concerned about the non-abelian higher gauge fields arising in a great deal of physical contexts, such as six dimensional superconformal field theory [6], quantum gravity [9], string theory [13] and M-theory [14, 15], and so on.

The electromagnetic theory is the simplest abelian gauge theory where the gauge field is described by the connection 1-form $A$ on a $U(1)$ bundle. When the charged particles are extended to 1-dimensional charged strings, one can get the 2-form electrodynamics [16] where the 2-form gauge field can be described in terms of the 2-form $B$ known as the Kalb-Ramond field [17]. The 2-form gauge field $B$ is described by a connection on a $U(1)$ gerbe as the categorified version of a $U(1)$ bundle [18]. Furthermore, there exists a generalization known as $p$-form electrodynamics, where the $p$-form gauge fields play the role of gauge fields [19, 20, 21]. When the group $U(1)$ is replaced by a nonabelian counterpart, one can obtain the Yang-Mills theory [22, 23]. Therefore, it raises a natural question of whether one can generalize Yang-Mills theory to a kind of higher-form Yang-Mills theory. The main contribution of this work is to develop a general method for 2-form, 3-form Yang-Mills theories.

In the related work [24], Baez generalized the Yang-Mills theory to the 2-form Yang-Mills
theory based on the 2-form electrodynamics. The gauge group is replaced by a crossed module of Lie groups \((H, G; \alpha, \triangleright)\) \([25, 26, 27]\) satisfying the following two conditions:

\[
\begin{align*}
\alpha(g \triangleright h) &= g\alpha(h)g^{-1}, \quad \forall g \in G, h \in H, \\
\alpha(h_1) \triangleright h_2 &= h_1 h_2 h_1^{-1}, \quad \forall h_1, h_2 \in H,
\end{align*}
\]

where \(\alpha : H \to G\) is a Lie group homomorphism and \(\triangleright\) is a smooth action of the Lie group \(G\) on the Lie group \(H\) by automorphisms. A simple example of a Lie crossed module is \(G = H = U(N)\) with \(\alpha\) the identity map and the \(\triangleright\) the adjoint action. An associated differential crossed module \((\mathfrak{h}, \mathfrak{g}; \tilde{\alpha}, \tilde{\triangleright})\) can be constructed, where \(\tilde{\alpha} : \mathfrak{h} \to \mathfrak{g}\) is a Lie algebra morphism and \(\tilde{\triangleright}\) is a left action of the Lie algebra \(\mathfrak{g}\) of \(G\) on the Lie algebra \(\mathfrak{h}\) of \(H\) by derivations. Obviously, the differential version of the above example is \(\mathfrak{g} = \mathfrak{h} = u(N)\) with \(\tilde{\triangleright}\) being the adjoint action and \(\tilde{\alpha}\) being the identity map. We represent the crossed module of Lie groups using squares of the form

\[
\begin{array}{ccc}
g_1 & h & g_2 \\
g_3 & h_1 & g_4
\end{array}
\]

where \(g_1, g_2 \in G\), \(h \in H\) and \(\alpha(h) = g_2 g_1^{-1}\). Thus, we can calculate the crossed module with squares given by horizontally

\[
\begin{array}{ccc}
g_1 & h_1 & g_2 \\
g_3 & h_2 & g_4
\end{array}
\]

\[
= \begin{array}{ccc}
g_1 & h_1 \circ h_2 & g_3 \\
g_1 & h_1 \circ h_2 := h_2 h_1
\end{array}
\]

and vertically

\[
\begin{array}{ccc}
g_1 & h_1 & g_2 \\
g_3 & h_2 & g_4
\end{array}
\]

\[
= \begin{array}{ccc}
g_1 & h_1 \star h_2 & g_2 g_4 \\
g_3 & h_1 \star h_2 := h_1 (g_1 \triangleright h_2).
\end{array}
\]

The two compositions are defined properly, since the squares resulting from the compositions satisfy

\[
\begin{align*}
\alpha(h_1 \circ h_2) &= g_3 g_1^{-1}, \\
\alpha(h_1 \star h_2) &= g_2 g_4 g_3^{-1} g_1^{-1},
\end{align*}
\]

by using the homomorphism \(\alpha\) and the identity \((1)\). Here, the horizontal composition of squares is denoted by a circle \(\circ\), and the vertical composition is denoted by a star \(\star\). Squares admit horizontal and vertical inverses, defined by
\( g_2 h^{-1} \quad g_1 \quad g_1^{-1} h^{-1} \quad g_2^{-1} \)

where \( h^{-1} = h^{-1} \) and \( h^{-1} = g_1^{-1} \triangleright h^{-1} \). The above definitions of inverses are reasonable since they satisfy

\[
\alpha(h^{-1}) = g_1 g_2^{-1}, \quad \alpha(h^{-1}) = g_2^{-1} g_1.
\]

In addition, the vertical multiplication is the group multiplication of the group \( G \rtimes H \), but the horizontal one is not. There are some slightly different descriptions, see [1, 16, 28] for more details. Based on this algebraic structure, Baez constructed the 2-form Yang-Mills action and obtained the corresponding field equations. More information about the 2-form Yang-Mills theory can be found in [24]. In the special case when \( H \) is trivial, the 2-form Yang-Mills equations reduce to those of \( G \) [23].

When \( G \) is trivial and \( H = U(1) \), the 2-form Yang-Mills equations reduce to those of 2-form electromagnetism [16].

In another recent work [29], Gastel gave two definitions of 2-Yang-Mills and 3-Yang-Mills theory which are linear if the good gauges are chosen. However, what we focus on here is different from his ideas. In this paper, we extend the 2-form Yang-Mills theory of Baez’s work to a 3-form Yang-Mills theory. Consequently, we need to introduce a new algebraic structure, known as 2-crossed module of Lie groups, which is described by three groups [9, 5, 7, 30, 31, 32, 33, 34, 35]. We denote a 2-crossed module of Lie groups as \((L, H, G; \beta, \alpha, \triangleright, \{,\})\) consisting of three Lie groups \( L, H \) and \( G \), and two group homomorphisms \( \beta \) and \( \alpha \)

\[
L \xrightarrow{\beta} H \xrightarrow{\alpha} G,
\]

where \( \alpha \beta = 1 \), and the smooth left action \( \triangleright \) by automorphisms of \( G \) on \( L \) and \( H \), and on itself by conjugation, i.e.

\[
g \triangleright (e_1 e_2) = (g \triangleright e_1)(g \triangleright e_2), \quad (g_1 g_2) \triangleright e = g_1 \triangleright (g_2 \triangleright e),
\]

for any \( g, g_1, g_2 \in G, e, e_1, e_2 \in H \) or \( L \), and a \( G \)-equivariant smooth function \( \{,\} : H \times H \rightarrow L \) called the Peiffer lifting, such that

\[
g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\},
\]

for any \( g \in G \) and \( h_1, h_2 \in H \) and they satisfy certain relations. There is a left action of \( H \) on \( L \) by automorphisms \( \triangleright' \) which is defined by

\[
h \triangleright' l = l \{\beta(l)^{-1}, h\}, \quad \forall l \in L, h \in H.
\]

This together with the homomorphism \( \beta : L \rightarrow H \) determines a crossed module \((L, H; \beta, \triangleright')\).

There is a differential 2-crossed module, also called a 2-crossed module of Lie algebras, corresponding to the 2-crossed module of Lie groups \((L, H, G; \beta, \alpha, \triangleright, \{,\})\). This definition appeared in [36]. We mark it with \((\ell, \hbar, g; \tilde{\beta}, \tilde{\alpha}, \tilde{\triangleright}, \{,\})\) consisting of three Lie algebras \( \ell, \hbar \) and \( g \) respectively corresponding to the three Lie groups \( L, H \) and \( G \), and two Lie algebra maps \( \tilde{\beta} \) and \( \tilde{\alpha} \)

\[
\ell \xrightarrow{\tilde{\beta}} \hbar \xrightarrow{\tilde{\alpha}} g,
\]
where \( \tilde{\alpha} \tilde{\beta} = 0 \), and left action \( \tilde{\cdot} \) of \( g \) on \( \ell, h \) and \( g \) by automorphisms, and a \( g \)-equivariant bilinear map \( \{, \} : h \times h \rightarrow \ell \) called the Peiffer lifting, such that

\[
X \tilde{\cdot} \{Y_1, Y_2\} = \{X \tilde{\cdot} Y_1, Y_2\} + \{Y_1, X \tilde{\cdot} Y_2\}, \quad \forall X \in g, Y_1, Y_2 \in h, \tag{9}
\]

and they satisfy appropriate conditions \([5]\). Analogously to the 2-crossed module of Lie groups case, there is a left action \( \tilde{\cdot}' \) of \( h \) on \( \ell \) which is defined by

\[
Y \tilde{\cdot}' Z = -\left\{\tilde{\beta}(Z), Y\right\}, \quad \forall Y \in h, Z \in \ell. \tag{10}
\]

This together with the homomorphism \( \tilde{\beta} : \ell \rightarrow h \) defines a differential crossed module \((\ell, h; \tilde{\beta}, \tilde{\cdot}')\).

In consideration of the crossed module of Lie groups \((L, H; \beta, \cdot')\) satisfying

\[
\beta(h \cdot' l) = h\beta(l)h^{-1}, \quad \forall h \in H, l \in L, \tag{11}
\]

\[
\beta(l) \cdot' l' = ll'\beta(l')^{-1}, \quad \forall l, l' \in L, \tag{12}
\]

and its calculations with squares as shown above, we represent the 2-crossed module of groups by using cubes as Figure 1, where \( g_1, g_2, g_3, g_4 \in G, h_1, h_2 \in H, l \in L \) and \( \beta(l) = h_2h_1^{-1} \).

![Figure 1: Cube of 2-crossed module](Image)

There are two types of composition between these cubes. The first one is the horizontal composition given by Figure 2 which satisfies

\[
\beta(l \circ l') = \beta(l')\beta(l) = h_3h_1^{-1} \tag{13}
\]

by using \( \beta(l) = h_2h_1^{-1} \) and \( \beta(l') = h_3h_2^{-1} \). And the second one is the vertical composition given by Figure 3 which satisfies

\[
\beta(l \star l') = \beta(l(h_1 \cdot' l')) = h_2h_4h_3^{-1}h_1^{-1} \tag{14}
\]

by using \( \beta(l) = h_2h_1^{-1}, \beta(l') = h_4h_3^{-1} \) and \( \beta(h_1 \cdot' l') = h_1\beta(l')h_1^{-1} \). Correspondingly, there are two different inverses under the above two compositions. The horizontal inverse is defined by Figure 4 satisfying

\[
\beta(l^{-h}) = h_1h_2^{-1}, \tag{15}
\]

and the vertical one is given by Figure 5 satisfying

\[
\beta(l^{-v}) = h_2^{-1}h_1. \tag{16}
\]

Here, we restrict ourselves to the particular 2-crossed module of groups in which \((H, G; \alpha, \cdot')\) is also a crossed module.
In order to construct the 3-form Yang-Mills action, we introduce some kind of $G$-invariant bilinear forms on the 2-crossed module of Lie algebras. More precisely speaking, we generalize the $G$-invariant bilinear forms on the crossed module of Lie algebras [37, 38]. Firstly, we generalize the mixed relations of $(H, G; \alpha, \triangleright)$ and $(\bar{h}, g; \tilde{\alpha}, \triangleright)$ to the mixed relations of $(L, H, G; \beta, \alpha, \triangleright, \{, \})$ and $(l, h, g; \bar{\beta}, \tilde{\alpha}, \triangleright, \{, \})$. When $(H, G; \alpha, \triangleright)$ is a crossed module of Lie groups, there are two mixed relations [37] induced by the action via $\triangleright$ on $H$ and also denoted by $\triangleright$

\[ \tilde{\alpha}(g \triangleright Y) = g\tilde{\alpha}(Y)g^{-1}, \quad \forall g \in G, Y \in \bar{h}, \]  

\[ (17) \]
\[ \alpha(h) \triangleright Y = hYh^{-1}, \quad \forall h \in H, Y \in \mathfrak{h}. \quad (18) \]

Besides, \((L, H; \beta, \triangleright')\) is also a crossed module of Lie groups. Thus there are also the mixed relations, corresponding to the crossed module of Lie algebras \((\ell, \mathfrak{h}; \tilde{\beta}, \triangleright'')\), induced by the action via \(\triangleright''\) on \(L\) and also denoted by \(\triangleright'\)

\[ \tilde{\beta}(h \triangleright' Z) = h\tilde{\beta}(Z)h^{-1}, \quad \forall h \in H, Z \in \ell, \quad (19) \]

\[ \beta(l) \triangleright' Z = lZl^{-1}, \quad \forall l \in L, Z \in \ell. \quad (20) \]

A symmetric non-degenerate \(G\)-invariant form in \((\ell, \mathfrak{h}, g; \tilde{\beta}, \tilde{\alpha}, \tilde{\triangleright'}, \{,\})\) is given by a triple of non-degenerate symmetric bilinear forms \(\langle,\rangle_g\) in \(g\), \(\langle,\rangle_{\mathfrak{h}}\) in \(\mathfrak{h}\) and \(\langle,\rangle_\ell\) in \(\ell\) such that

1. \(\langle,\rangle_g\) is \(G\)-invariant, i.e.
   \[ \langle gxg^{-1}, gx'g^{-1} \rangle_g = \langle X, X' \rangle_g, \quad \forall g \in G, X, X' \in g; \]

2. \(\langle,\rangle_{\mathfrak{h}}\) is \(G\)-invariant, i.e.
   \[ \langle g \triangleright Y, g \triangleright Y' \rangle_{\mathfrak{h}} = \langle Y, Y' \rangle_{\mathfrak{h}}, \quad \forall g \in G, Y, Y' \in \mathfrak{h}; \]

3. \(\langle,\rangle_\ell\) is \(G\)-invariant, i.e.
   \[ \langle g \triangleright Z, g \triangleright Z' \rangle_\ell = \langle Z, Z' \rangle_\ell, \quad \forall g \in G, Z, Z' \in \ell. \]

Note that \(\langle,\rangle_{\mathfrak{h}}\) is necessarily \(H\)-invariant. Since

\[ \langle hyh^{-1}, hy'h^{-1} \rangle_{\mathfrak{h}} = \langle \alpha(h) \triangleright Y, \alpha(h) \triangleright Y' \rangle_{\mathfrak{h}} = \langle Y, Y' \rangle_{\mathfrak{h}}, \quad \forall h \in H, Y, Y' \in \mathfrak{h}, \]

where we have used the mixed relation \((18)\). In the case when the Peiffer lifting or the map \(\beta\) is trivial consequently, \(\langle,\rangle_\ell\) is \(H\)-invariant, i.e.

\[ \langle h \triangleright' Z, h \triangleright' Z' \rangle_\ell = \langle Z, Z' \rangle_\ell, \quad \forall h \in H, Z, Z' \in \ell. \]

Besides, \(\langle,\rangle_\ell\) is necessarily \(L\)-invariant based on the \(H\)-invariance of \(\langle,\rangle_\ell\), i.e.

\[ \langle lZl^{-1}, lZ'l^{-1} \rangle_\ell = \langle \beta(l) \triangleright' Z, \beta(l) \triangleright' Z' \rangle_\ell = \langle Z, Z' \rangle_\ell, \quad \forall l \in L, Z, Z' \in \ell, \]

where the mixed relation \((20)\) is used. See \(38\) for details.

There are no compatibility conditions among the bilinear forms \(\langle,\rangle_\ell\), \(\langle,\rangle_{\mathfrak{h}}\) and \(\langle,\rangle_g\). From the well-known fact that any representation of \(G\) can be made unitary if \(G\) is a compact group, one can get the following Lemma.

**Lemma 1.** Let \((L, H, G; \beta, \alpha, \triangleright, \{,\})\) be a 2-crossed module of Lie groups with the group \(G\) being compact in the real case, or having a compact real form in the complex case. Then one can construct \(G\)-invariant symmetric non-degenerate bilinear forms \(\langle,\rangle_g\), \(\langle,\rangle_{\mathfrak{h}}\) and \(\langle,\rangle_\ell\) in the associated differential 2-crossed module \((\ell, \mathfrak{h}, g; \tilde{\beta}, \tilde{\alpha}, \tilde{\triangleright'}, \{,\})\). Furthermore these forms can be chosen to be positive definite.
These invariance conditions imply that:

$$\langle [X, X'], X'' \rangle_g = -\langle X', [X, X''] \rangle_g, \quad (21)$$

$$\langle [Y, Y'], Y'' \rangle_h = -\langle Y', [Y, Y''] \rangle_h, \quad (22)$$

$$\langle [Z, Z'], Z'' \rangle_\ell = -\langle Z', [Z, Z''] \rangle_\ell. \quad (23)$$

One can define two bilinear antisymmetric maps $\sigma : \mathfrak{h} \times \mathfrak{h} \rightarrow g$ by the rule:

$$\langle \sigma(Y, Y') , X \rangle_g = -\langle Y, X \tilde{\sigma} Y' \rangle_h, \quad \forall X \in g, Y, Y' \in \mathfrak{h}, \quad (24)$$

and $\kappa : \ell \times \ell \rightarrow g$ by the rule:

$$\langle \kappa(Z, Z') , X \rangle_g = -\langle Z, X \tilde{\sigma} Z' \rangle_\ell, \quad \forall X \in g, Z, Z' \in \ell. \quad (25)$$

Especially, when $\mathfrak{h} = \ell = g$ and the bilinear form is the Killing form i.e. $\langle -, - \rangle_g = K(-,-)$, the above definitions will become very natural as follows:

$$\sigma(Y, Y') = [Y, Y'], \quad X \tilde{\sigma} Y' = [X, Y'], \quad \kappa(Z, Z') = [Z, Z'], \quad X \tilde{\sigma} Z' = [X, Z'],$$

and (24) and (25) will become Killing identities

$$K([Y, Y'], X) + K(Y, [X, Y']) = 0,$$

$$K([Z, Z'], X) + K(Z, [X, Z']) = 0.$$

Due to $\sigma(Y', Y) = -\sigma(Y, Y')$ and $\kappa(Z', Z) = -\kappa(Z, Z')$, then one has

$$\langle Y, X \tilde{\sigma} Y' \rangle_h = -\langle Y', X \tilde{\sigma} Y \rangle_h = -\langle X \tilde{\sigma} Y, Y' \rangle_h, \quad (26)$$

$$\langle Z, X \tilde{\sigma} Z' \rangle_\ell = -\langle Z', X \tilde{\sigma} Z \rangle_\ell = -\langle X \tilde{\sigma} Z, Z' \rangle_\ell. \quad (27)$$

Further, one needs to define two bilinear maps $\eta_1 : \ell \times \mathfrak{h} \rightarrow \mathfrak{h}$ and $\eta_2 : \ell \times \mathfrak{h} \rightarrow \mathfrak{h}$ by the rule:

$$\langle \{Y, Y'\} , Z \rangle_\ell = -\langle Y', \eta_1(Z, Y) \rangle_h = -\langle Y, \eta_2(Z, Y') \rangle_h, \quad (28)$$

for each $Y, Y' \in \mathfrak{h}$, and $Z \in \ell$. See [38] for more information about these maps. To obtain the 2-form Yang-Mills equations, one defines a map $\alpha^* : g \rightarrow \mathfrak{h}$ in [24] by the rule:

$$\langle Y, \alpha^*(X) \rangle_h = \langle \tilde{\alpha}(Y), X \rangle_g, \quad \forall X \in g, Y \in \mathfrak{h}. \quad (29)$$

Here, to obtain the 3-form Yang-Mills equations, we have to define a map $\beta^* : \mathfrak{h} \rightarrow \ell$ by the rule:

$$\langle Z, \beta^*(Y) \rangle_\ell = \langle \tilde{\beta}(Z), Y \rangle_h, \quad \forall Y \in \mathfrak{h}, Z \in \ell. \quad (30)$$

The maps $\sigma$, $\kappa$, $\eta_1$, $\eta_2$, $\alpha^*$ and $\beta^*$ are very closely linked with our approach to the construction of the 3-form Yang-Mills.

In order to calculate more efficiently, we introduce the component notation. Given a Lie algebra $g$, there is a vector space $\Lambda^k(M, g)$ of $g$-valued differential $k$-forms on the manifold
M. For \( A = \sum_a A^a X_a \in \Lambda^k(M, g) \), \( A' = \sum_b A^{b} X_b \in \Lambda^k(M, g) \) for some scalar differential \( k_1 \)-form \( A^a \), and \( k_2 \)-form \( A^b \), and elements \( X_a, X_b \in g \), define

\[
A \wedge A' := \sum_{a,b} A^a \wedge A^b X_a X_b, \quad A \wedge [\ ] A' := \sum_{a,b} A^a \wedge A^b [X_a, X_b], \quad dA = \sum_a dA^a X_a,
\]

then there is an identity

\[
A \wedge [\ ] A' = A \wedge A' - (-1)^{k_1 k_2} A' \wedge A.
\]

In the above, we can also choose \( g \) to be \( h \) or \( f \). For \( B = \sum_b B^b Y_b \in \Lambda^{k_1}(M, h) \), \( B' = \sum_b B^b Y_b \in \Lambda^{k_2}(M, h) \), \( \forall Y_a, Y_b \in h \), define

\[
A \wedge h \ B := \sum_{a,b} A^a \wedge B^b X_a \ddot{\triangledown} Y_b, \quad B \wedge [\ ] h \ B := \sum_{a,b} B^a \wedge B^b \{Y_a, Y_b\}, \quad \tilde{\alpha}(B) := \sum_a B^a \tilde{\alpha}(Y_a).
\]

For \( C = \sum_a C^a Z_a \in \Lambda^{k_1}(M, f) \), define

\[
A \wedge h \ C := \sum_{a,b} A^a \wedge C^b X_a \ddot{\triangledown} Z_b, \quad B \wedge h \ C := \sum_{a,b} B^a \wedge C^b Y_a \ddot{\triangledown} Z_b, \quad \tilde{\beta}(C) := \sum_a C^a \tilde{\beta}(Z_a), \quad \text{where } Y_a \ddot{\triangledown} Z_b = -\left\{ \tilde{\beta}(Z_b), Y_a \right\} \text{ by using \((10)\).}
\]

Furthermore, we have non-degenerate \( G \)-invariant forms in \( \Lambda^k(M, g) \), \( \Lambda^k(M, h) \) and \( \Lambda^k(M, f) \) induced by \( \langle \cdot, \cdot \rangle_g \), \( \langle \cdot, \cdot \rangle_h \) and \( \langle \cdot, \cdot \rangle_f \) and we denote them by \( \langle \cdot, \cdot \rangle \). Then we have

\[
\langle A, A' \rangle := \sum_{a,b} A^a \wedge A^b \langle X_a, X_b \rangle, \quad \langle B, B' \rangle := \sum_{a,b} B^a \wedge B^b \langle Y_a, Y_b \rangle, \quad \langle C, C' \rangle := \sum_{a,b} C^a \wedge C^b \langle Z_a, Z_b \rangle,
\]

where \( C' = \sum_b C^b Z_b \in \Lambda^{k_2}(M, f) \), and \( A, A', B, B' \) are as above. There are identities

\[
\langle A, A' \rangle = (-1)^{k_1 k_2} \langle A', A \rangle, \quad \langle B, B' \rangle = (-1)^{k_1 k_2} \langle B', B \rangle, \quad \langle C, C' \rangle = (-1)^{k_1 k_2} \langle C', C \rangle,
\]

using the symmetry of \( \langle \cdot, \cdot \rangle_g \), \( \langle \cdot, \cdot \rangle_h \) and \( \langle \cdot, \cdot \rangle_f \).

There is an important proposition in \([6]\), and we use it to calculate the 3-form Yang-Mills equations.

**Proposition 1.**

1. For \( A \in \Lambda^k(M, g) \), \( A' \in \Lambda^k(M, g) \) and \( C \in \Lambda^*(M, f) \),

\[
\tilde{\beta}(A \wedge h \ C) = A \wedge h \tilde{\beta}(C), \quad \text{(31)}
\]

\[
A \wedge h \ A' = A \wedge A' + (-1)^{k_1 k_2 + 1} A' \wedge A. \quad \text{(32)}
\]

2. For \( A \in \Lambda^k(M, g) \), \( B_1 \in \Lambda^l(M, h) \), \( B_2 \in \Lambda^l(M, h) \) and \( W \in \Lambda^*(M, w) \) \((w = g, h, f)\),

\[
d(A \wedge h \ W) = dA \wedge h \ W + (-1)^k A \wedge h dW, \quad \text{(33)}
\]

\[
d(B_1 \wedge [\ ] h \ B_2) = dB_1 \wedge [\ ] h \ B_2 + (-1)^{l_1} B_1 \wedge [\ ] h dB_2, \quad \text{(34)}
\]

\[
A \wedge h (B_1 \wedge [\ ] h \ B_2) = (A \wedge h B_1) \wedge [\ ] h B_2 + (-1)^{k l_1} B_1 \wedge [\ ] h (A \wedge h B_2), \quad \text{(35)}
\]

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The following propositions provide identities of non-degenerate symmetric $G$-invariant forms in a differential 2-crossed module.

**Proposition 2.** For $A_1 \in \Lambda^{k_1}(M, g)$, $A_2 \in \Lambda^{k_2}(M, g)$, $A_3 \in \Lambda^{k_3}(M, g)$, $B_1 \in \Lambda^{t_1}(M, h)$, $B_2 \in \Lambda^{t_2}(M, h)$, $B_3 \in \Lambda^{t_3}(M, h)$, $C_1 \in \Lambda^{q_1}(M, \ell)$, $C_2 \in \Lambda^{q_2}(M, \ell)$, and $C_3 \in \Lambda^{q_3}(M, \ell)$ we have

\[
\langle A_1 \wedge [] A_2, A_3 \rangle = (-1)^{k_1k_2+1}\langle A_2, A_1 \wedge [] A_3 \rangle, 
\]

\[
\langle B_1 \wedge [] B_2, B_3 \rangle = (-1)^{t_1t_2+1}\langle B_2, B_1 \wedge [] B_3 \rangle, 
\]

\[
\langle C_1 \wedge [] C_2, C_3 \rangle = (-1)^{q_1q_2+1}\langle C_2, C_1 \wedge [] C_3 \rangle. 
\]

**Proof:** We can get these identities easily by using (21), (22) and (23). \[\square\]

**Proposition 3.** For $A \in \Lambda^{k}(M, g)$, $B_1 \in \Lambda^{t_1}(M, h)$, $B_2 \in \Lambda^{t_2}(M, h)$, $B_3 \in \Lambda^{q_1}(M, \ell)$, $C_1 \in \Lambda^{q_2}(M, \ell)$, and $C_2 \in \Lambda^{q_3}(M, \ell)$, we have

\[
\langle B_1, A \wedge \tilde{\wedge} B_2 \rangle = (-1)^{t_1(k+t_1)+m_1+1}\langle B_2, A \wedge \tilde{\wedge} B_1 \rangle = (-1)^{m_1+1}\langle A \wedge \tilde{\wedge} B_1, B_2 \rangle, 
\]

\[
\langle C_1, A \wedge \tilde{\wedge} C_2 \rangle = (-1)^{q_1(k+q_1)+m_1+1}\langle C_2, A \wedge \tilde{\wedge} C_1 \rangle = (-1)^{m_1+1}\langle A \wedge \tilde{\wedge} C_1, C_2 \rangle. 
\]

**Proof:** We can get those identities easily by using (26) and (27). \[\square\]

We can define a bilinear map $\tilde{\sigma}: \Lambda^{t_1}(M, h) \times \Lambda^{t_2}(M, h) \rightarrow \Lambda^{t_1+t_2}(M, g)$ by

\[
\tilde{\sigma}(B_1, B_2) := \sum_{a,b} B_1^a \wedge B_2^b \sigma(Y_a, Y_b),
\]

for $B_1 = \sum_a B_1^a Y_a \in \Lambda^{t_1}(M, h)$, $B_2 = \sum_b B_2^b Y_b \in \Lambda^{t_2}(M, h)$. Since $\sigma$ is antisymmetric, we have

\[
\tilde{\sigma}(B_1, B_2) = (-1)^{t_1t_2+1}\tilde{\sigma}(B_2, B_1).
\]

Let $A = \sum_c A_cX_c \in \Lambda^{k}(M, g)$, then

\[
\langle \tilde{\sigma}(B_1, B_2), A \rangle = \sum_{a,b,c} B_1^a \wedge B_2^b \wedge A_c \langle \sigma(Y_a, Y_b), X_c \rangle_g = -\sum_{a,b,c} B_1^a \wedge B_2^b \wedge A_c \langle Y_a, X_c \tilde{\wedge} Y_b \rangle_h
\]

\[
= (-1)^{k_1t_2+1} \sum_{a,b,c} B_1^a \wedge A_c \wedge B_2^b \langle Y_a, X_c \tilde{\wedge} Y_b \rangle_h
\]

\[
= (-1)^{k_1t_2+1}\langle B_1, A \wedge \tilde{\wedge} B_2 \rangle,
\]

by using (24), i.e.

\[
\langle \tilde{\sigma}(B_1, B_2), A \rangle = (-1)^{k_1t_2+1}\langle B_1, A \wedge \tilde{\wedge} B_2 \rangle,
\]

(41)
and the following identity holds

$$\langle A, \overline{\sigma}(B_1, B_2) \rangle = (-1)^{t_1 t_2+1} \langle A \land \overline{\land} B_2, B_1 \rangle. \quad (42)$$

Similarly, we can define a bilinear map $\overline{\sigma} : \Lambda^{q_1} (M, \ell) \times \Lambda^{q_2} (M, \ell) \longrightarrow \Lambda^{q_1+q_2} (M, g)$ by

$$\overline{\sigma}(C_1, C_2) := \sum_{a, b} C_1^a \land C_2^b \kappa(Z_a, Z_b),$$

for $C_1 = \sum_a C_1^a Z_a \in \Lambda^{q_1} (M, \ell)$ and $C_2 = \sum_b C_2^b Z_b \in \Lambda^{q_2} (M, \ell)$. Since $\kappa$ is antisymmetric, we have

$$\overline{\sigma}(C_1, C_2) = (-1)^{q_1 q_2 + 1} \overline{\sigma}(C_2, C_1).$$

Let

$$A = \sum_c A^c X_c \in \Lambda^k (M, g),$$

then

$$\langle \overline{\sigma}(C_1, C_2), A \rangle = \sum_{a, b, c} C_1^a \land C_2^b \land A^c (\kappa(Z_a, Z_b), X_c)_g = - \sum_{a, b, c} C_1^a \land C_2^b \land A^c (Z_a, X_c \overline{\land} Z_b)_\ell$$

$$= (-1)^{k q_2 + 1} \sum_{a, b, c} C_1^a \land A^c \land C_2^b (Z_a, X_c \overline{\land} Z_b)_\ell$$

$$= (-1)^{k q_2 + 1} \langle C_1, A \land \overline{\land} C_2 \rangle,$$

by using (25), i.e.

$$\langle \overline{\sigma}(C_1, C_2), A \rangle = (-1)^{k q_2 + 1} \langle C_1, A \land \overline{\land} C_2 \rangle, \quad (43)$$

and there is an identity

$$\langle A, \overline{\sigma}(C_1, C_2) \rangle = (-1)^{q_1 q_2 + 1} \langle A \land \overline{\land} C_2, C_1 \rangle. \quad (44)$$

Finally, we define bilinear maps $\overline{\eta} : \Lambda^q (M, \ell) \times \Lambda^l (M, \tilde{h}) \longrightarrow \Lambda^{q+l} (M, \tilde{h})$ by

$$\overline{\eta}(C, B) := \sum_{a, b} C^b \land B^a \eta_i(Z_b, Y_a), \quad i = 1, 2$$

for $B_i = \sum_a B_i^a Y_a \in \Lambda^i (M, \tilde{h})$, $C = \sum_b C^b Z_b \in \Lambda^q (M, \ell)$. There are two identities

$$\langle B_1 \land \{1\}, B_2, C \rangle = (-1)^{t_1 (t_2 + q) + 1} \langle B_2, \overline{\eta}(C, B_1) \rangle = (-1)^{t_2 q + 1} \langle B_1, \overline{\eta}(C, B_2) \rangle, \quad \quad (45)$$

by using (28).

For $A \in \Lambda^k (M, g)$, $B \in \Lambda^l (M, \tilde{h})$ and $C \in \Lambda^q (M, \ell)$, we have

$$\langle B, \alpha^* (A) \rangle = \langle \overline{\alpha}(B), A \rangle, \quad (46)$$

$$\langle C, \beta^* (B) \rangle = \langle \overline{\beta}(C), B \rangle, \quad (47)$$

being induced by (29) and (30).
Up to present, we have established the algebraic structures of 3-form Yang-Mills theory. Next, we derive the 3-Bianchi-Identities on account of the above properties for Lie algebra valued differential forms. The 3-connection, on \(d\)-dimensional spacetime manifold \(M\) (\(d \geq 4\)), is described by a 1-form \(A\) valued in the Lie algebra \(g\), a 2-form \(B\) valued in the Lie algebra \(\hat{h}\), and a 3-form \(C\) valued in the Lie algebra \(l\). We refer the interested readers to [7] for more details. The corresponding 3-curvature in this case is given by

\[
\Omega_1 := dA + A \wedge A, \quad \Omega_2 := dB + A \wedge \tilde{B}, \quad \Omega_3 := dC + A \wedge \tilde{C} + B \wedge \{^1\} B.
\]

There are a fake 1-curvature and a fake 2-curvature defined as

\[
F_1 := dA + A \wedge A - \tilde{\alpha}(B), \quad F_2 := dB + A \wedge \tilde{B} - \tilde{\beta}(C).
\]

We call the 3-connection \((A, B, C)\) fake 1-flat, if the fake 1-curvature vanishes, i.e. \(F_1 = 0\). Similarly, we call the 3-connection \((A, B, C)\) fake 2-flat, if the fake 2-curvature vanishes, i.e. \(F_2 = 0\). If the 3-curvature 4-form vanishes, i.e. \(\Omega_3 = 0\), the 3-connection \((A, B, C)\) will be called 3-flat. When \((A, B, C)\) is both fake 1-flat and fake 2-flat, we say that the 3-connection \((A, B, C)\) is fake-flat. Otherwise, the 3-connection \((A, B, C)\) is not fake-flat.

**Theorem 1.** If \((A, B, C)\) is any 3-connection on \(M\), then its fake 3-curvature \((\Omega_3, F_1, F_2)\) satisfies the 3-Bianchi-Identities:

\[
dF_1 + A \wedge[^{[1]}] F_1 = -\tilde{\alpha}(F_2),
\]

\[
dF_2 + A \wedge \tilde{B} F_2 = (F_1 + \tilde{\alpha}(B)) \wedge \tilde{B} - \tilde{\beta} (\Omega_3 - B \wedge[^{1]} B),
\]

\[
d\Omega_3 + A \wedge \tilde{C} \Omega_3 = (F_1 + \tilde{\alpha}(B)) \wedge \tilde{C} + (F_2 + \tilde{\beta}(C)) \wedge[^{1]} B + B \wedge[^{1]} (F_2 + \tilde{\beta}(C)).
\]

**Proof:**

\[
dF_1 = d(dA + A \wedge A - \tilde{\alpha}(B)) = dA \wedge A - A \wedge dA - d(\tilde{\alpha}(B))
= -A \wedge[^{[1]}] F_1 - \tilde{\alpha}(F_2 + \tilde{\beta}(C)),
\]

using \(\alpha(A \wedge \tilde{B}) = A \wedge[^{[1]}] \tilde{\alpha}(B)\), and

\[
dF_2 = d(dB + A \wedge \tilde{B} - \tilde{\beta}(C)) = dA \wedge \tilde{B} - A \wedge \tilde{B} dB - d(\tilde{\beta}(C))
= (F_1 + \tilde{\alpha}(B)) \wedge \tilde{B} - \tilde{\beta}(\Omega_3 - B \wedge[^{1]} B) - A \wedge \tilde{B} F_2,
\]

using \((33)\) and \(\tilde{\alpha}\) is homomorphism, and

\[
d\Omega_3 = d(dC + A \wedge \tilde{C} + B \wedge[^{1]} B) = dA \wedge \tilde{C} - A \wedge \tilde{C} dC + d(B \wedge[^{1]} B)
= (F_1 + \tilde{\alpha}(B)) \wedge \tilde{C} - A \wedge \tilde{C} \Omega_3 + (F_2 + \tilde{\beta}(C)) \wedge[^{1]} B + B \wedge[^{1]} (F_2 + \tilde{\beta}(C)),
\]

using \((33), (34), (35)\) and \(\tilde{\alpha}\) is homomorphism.

□
When the maps $\tilde{\alpha}$ and $\tilde{\beta}$ are trivial, i.e. $\tilde{\alpha}(B) = 0$ and $\tilde{\beta}(C) = 0$, we consider the ordinary 3-curvature $(\Omega_1, \Omega_2, \Omega_3)$ satisfying the $3$-Bianchi-Identities which can be derived from Theorem 1

\[ d\Omega_1 + A \wedge [\Omega_1] = 0, \]
\[ d\Omega_2 + A \wedge [\Omega_2] = \Omega_1 \wedge _B \]
\[ d\Omega_3 + A \wedge [\Omega_3] = \Omega_1 \wedge _B C + \Omega_2 \wedge [\Omega_2] B + B \wedge [\Omega_2] \Omega_2. \]

Generalizing the Yang-Mills action and the 2-form Yang-Mils action in [24], we write down the following 3-form Yang-Mills action as a function of the 3-connection $(A, B, C)$ in 3-form Yang-Mills gauge theory:

\[ S = \int_M \langle F_1, *F_1 \rangle + \langle F_2, *F_2 \rangle + \langle \Omega_3, *\Omega_3 \rangle, \]
by setting the variation of the action to zero:

\[ \delta S = 2 \int_M \langle \delta F_1, *F_1 \rangle + \langle \delta F_2, *F_2 \rangle + \langle \delta \Omega_3, *\Omega_3 \rangle = 0. \]

The first section is as follows:

\[ \langle \delta F_1, *F_1 \rangle = \delta (dA + A \wedge A - \tilde{\alpha}(B), *F_1) \]
\[ = \delta A, d * F_1 + A \wedge [\delta F_1] - \langle \delta B, \alpha^*(F_1) \rangle, \]
using $\delta (A \wedge A) = A \wedge [\delta A]$, (36) and (46). The second section is as follows:

\[ \langle \delta F_2, *F_2 \rangle = \delta (dB + A \wedge _B B - \tilde{\beta}(C), *F_2) \]
\[ = -\langle \delta A, \overline{\sigma}(F_2, B) \rangle - \langle \delta B, d * F_2 + A \wedge _B * F_2 \rangle - \langle \delta C, \beta^*(F_2) \rangle, \]
using $\delta (A \wedge _B B) = \delta A \wedge _B B + A \wedge _B \delta B$, [42], [39] and [47]. The third section is as follows:

\[ \langle \delta \Omega_3, *\Omega_3 \rangle = \langle \delta (dC + A \wedge _B C + B \wedge [\Omega_2]), *\Omega_3 \rangle \]
\[ = (-1)^{d-1} \langle \delta A, \overline{\rho}(\Omega_3, C) \rangle - \langle \delta B, \overline{\eta}_{[2]}(\Omega_3, B) + \overline{\eta}_{[1]}(\Omega_3, B) \rangle + \langle \delta C, d * \Omega_3 + A \wedge _B * \Omega_3 \rangle, \]
using (41), (40) and (45). Thus we have

\[ \delta S = 2 \int_M \langle \delta A, d * F_1 + A \wedge [\Omega_3] *, F_1 \rangle + (-1)^{d-1} \langle \delta A, \overline{\rho}(\Omega_3, C) \rangle - \overline{\sigma}(F_2, B) \]
\[ - \langle \delta B, d * F_2 + A \wedge _B * F_2 + \overline{\eta}_{[2]}(\Omega_3, B) + \overline{\eta}_{[1]}(\Omega_3, B) + \alpha^*(F_1) \rangle \]
\[ + \langle \delta C, d * \Omega_3 + A \wedge _B * \Omega_3 - \beta^*(F_2) \rangle. \]
We see that the variation of the action vanishes for $\delta A$, $\delta B$ and $\delta C$ if and only if the following field equations hold:

$$d \ast F_1 + A \wedge [\cdot] \ast F_1 = \sigma(*F_2, B) + (-1)^d \rho(*\Omega_3, C), \quad (54)$$

$$d \ast F_2 + A \wedge \tilde{\wedge} \ast F_2 = -\mu_2(*\Omega_3, B) - \eta_1(*\Omega_3, B) - \alpha^*(\ast F_1), \quad (55)$$

$$d \ast \Omega_3 + A \wedge \tilde{\wedge} \ast \Omega_3 = \beta^*(\ast F_2). \quad (56)$$

And when the 3-connection $(A, B, C)$ is fake-flat, the field equations become

$$d \ast \Omega_1 + A \wedge [\cdot] \ast \Omega_1 = \sigma(*\Omega_2, B) + (-1)^d \rho(*\Omega_3, C), \quad (57)$$

$$d \ast \Omega_2 + A \wedge \tilde{\wedge} \ast \Omega_2 = -\mu_2(*\Omega_3, B) - \eta_1(*\Omega_3, B), \quad (58)$$

$$d \ast \Omega_3 + A \wedge \tilde{\wedge} \ast \Omega_3 = 0. \quad (59)$$

Though one may wonder about 4-gauge theory, to the best of our knowledge, it has not been defined yet. The notion of a 3-crossed module, which should be the foundation of 4-gauge theory, has been developed in [39, 40]. Ideally, in higher gauge theory, the Yang-Mills theory may be generalized to a kind of “n-form Yang-Mills theory” in accordance with the chosen $n$-group structure ($n > 3$) [41]. There is no doubt that one may encounter a lot of difficulties.

To summarize, the relationships between the related models are concluded in Figure 6.

![Figure 6: The relationships](image-url)

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3-form Yang-Mills based on 2-crossed modules

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In this paper, we study the higher Yang-Mills theory in the framework of higher gauge theory. It was shown that the 2-form electromagnetism can be generalized to the 2-form Yang-Mills theory with the group $U(1)$ replaced by a crossed module of Lie groups. To extend this theory to even higher structure, we develop a 3-form Yang-Mills theory with a 2-crossed module of Lie groups. First, we give an explicit construction of non-degenerate symmetric $G$-invariant forms on the 2-crossed module of Lie algebras. Then, we derive the 3-Bianchi-Identities for 3-curvatures. Finally, we create a 3-form Yang-Mills action and obtain the corresponding field equations.

Keywords: crossed module of groups, 2-crossed module of groups, 3-connection, 3-Bianchi-Identities, 3-form Yang-Mills

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In modern formal theoretical physics a certain idea is very productive: the higher-dimensional extended objects are thought to be the basic constituents of matter and mediators of fundamental interactions. At present, higher gauge theory \cite{1} seems to be the geometrically most promising technique to describe the dynamics of the higher-dimensional extended objects where gauge fields and field strengths are higher degree forms. Among them, the 2-gauge theory \cite{2,4} and the 3-gauge theory \cite{5,7} have been studied deeply, whose higher algebraic structures and higher geometrical structures can be found in, e.g. \cite{4,9,11}. In this article, we are concerned about the non-abelian higher gauge fields arising in a great deal of physical contexts, such as six dimensional superconformal field theory \cite{6}, quantum gravity \cite{8}, string theory \cite{13} and M-theory \cite{14,15}, and so on.

The electromagnetic theory is the simplest abelian gauge theory where the gauge field is described by the connection 1-form \( A \) on a \( U(1) \) bundle. When the charged particles are extended to 1-dimensional charged strings, one can get the 2-form electrodynamics \cite{16} where the 2-form gauge field can be described in terms of the 2-form \( B \) known as the Kalb-Ramond field \cite{17}. The 2-form gauge field \( B \) is described by a connection on a \( U(1) \) gerbe as the categorized version of a \( U(1) \) bundle \cite{18}. Furthermore, there exists a generalization known as \( p \)-form electrodynamics, where the \( p \)-form gauge fields play the role of gauge fields \cite{19,21}. When the group \( U(1) \) is replaced by a nonabelian counterpart, one can obtain the Yang-Mills theory \cite{22,23}. Therefore, it raises a natural question of whether one can generalize Yang-Mills theory to a kind of higher-form Yang-Mills theory. The main contribution of this work is to develop a general method for 2-form, 3-form Yang-Mills theories.

In the related work \cite{24}, Baez generalized the Yang-Mills theory to the 2-form Yang-Mills theory based on the 2-form electrodynamics. The gauge group is replaced by a crossed module of groups \((H,G;\alpha,\triangleright)\) \cite{25,27}, where \( \alpha : H \rightarrow G \) is a Lie group homomorphism and \( \triangleright \) is a smooth action of the Lie group \( G \) on the Lie group \( H \) by automorphisms. A simple example of a Lie crossed module is \( G = H = U(N) \) with \( \alpha \) the identity map and the \( \triangleright \) the adjoint action. An associated differential crossed module \((\tilde{h},\tilde{g};\tilde{\alpha},\tilde{\triangleright})\) can be constructed, where \( \tilde{\alpha} : \tilde{h} \rightarrow \tilde{g} \) is a Lie algebra map and \( \tilde{\triangleright} \) is a left action of the Lie algebra \( \mathfrak{g} \) of \( G \) on the Lie algebra \( \mathfrak{h} \) of \( H \) by derivations. Obviously, the differential version of the above example is \( g = \tilde{h} = \mathfrak{u}(N) \) with \( \tilde{\triangleright} \) being the adjoint action and \( \tilde{\alpha} \) being the identity map. We perform the crossed module of groups using squares of the form

\[
\begin{array}{ccc}
g_1 & h & g_2 \\
g_3 & h_1 & g_2 \\
g_1 & h_2 & g_3 \\
g_1 & h_3 & g_2
\end{array}
\]

where \( g_1, g_2 \in G, h \in H \) and \( \alpha(h) = g_2g_1^{-1} \). Thus, we can calculate the crossed module with squares given by horizontally

\[
\begin{array}{ccc}
g_1 & h_1 & g_2 \\
g_1 & h_2 & g_3 \\
g_1 & h_1 \circ h_2 & g_3 \\
g_1 \circ h_2 & g_1 \circ h_2 & h_1 \circ h_2 := h_2h_1
\end{array}
\]

and vertically

\[
\begin{array}{ccc}
g_1 & h_1 & g_2 \\
g_3 & h_1 & g_2 \\
g_1 & h_2 & g_4 \\
g_1 & h_3 & g_2
\end{array}
\]

= \[
\begin{array}{ccc}
g_1 & h_1 \ast h_2 & g_2 \\
g_1 & h_2 \ast h_4 & g_4 \\
g_1 & h_2 \ast h_3 & h_1 \ast h_2 := h_1(g_1 \triangleright h_2)
\end{array}
\]

Here, horizontal composition of squares is denoted by a circle “\( \circ \)”, and vertical composition is denoted by a star “\( \ast \)”. Squares admit horizontal and vertical inverses, defined by
where $h^{-h} = h^{-1}$ and $h^{-v} = g^{-1} \triangleright h^{-1}$. In addition, the vertical multiplication is the group multiplication of the group $H$, but the horizontal one is not. There are some slightly different descriptions, see [11, 16, 28] for more details. Based on this algebraic structure, Baez constructed the 2-form Yang-Mills action and obtained the corresponding field equations. More information about the 2-form Yang-Mills theory can be found in [24]. In the special case when $H$ is trivial, the 2-form Yang-Mills equations reduce to the ordinary Yang-Mills equations with gauge group $G$, [23]. When $G$ is trivial and $H = U(1)$, the 2-form Yang-Mills equations reduce to those of 2-form electromagnetism [10].

In another recent work [29], Gastel gave two definitions of 2-Yang-Mills and 3-Yang-Mills theory which are linear if the good gauges are chosen. However, what we focus on here is different from his ideas. In this paper, we extend the 2-form Yang-Mills theory of Baez’s work to a 3-form Yang-Mills theory. Consequently, we need to introduce a new algebraic structure, known as 2-crossed module of groups, which is described by three groups [5, 7, 9, 30–35]. We denote a 2-crossed module of Lie groups as $(L, H, G; \beta, \alpha, \triangleright, \{, \})$ consisting of three Lie groups $L$, $H$ and $G$, and two group homomorphisms $\beta$ and $\alpha$.

$$L \xrightarrow{\beta} H \xrightarrow{\alpha} G,$$

and the smooth left action $\triangleright$ by automorphisms of $G$ on $L$ and $H$, and on itself by conjugation, i.e.

$$g \triangleright (e_1 e_2) = (g \triangleright e_1)(g \triangleright e_2), \quad (g_1 g_2) \triangleright e = g_1 \triangleright (g_2 \triangleright e), \quad (1)$$

for any $g, g_1, g_2 \in G, e, e_1, e_2 \in H$ or $L$, and a $G$-equivariant smooth function $\{, \} : H \times H \rightarrow L$ called the Peiffer lifting, such that

$$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\}, \quad (2)$$

for any $g \in G$ and $h_1, h_2 \in H$ and they satisfy certain relations. There is a left action of $H$ on $L$ by automorphisms $\triangleright'$ which is defined by

$$h \triangleright' l = l \{\beta(l)^{-1}, h\}, \quad \forall l \in L, h \in H. \quad (3)$$

This together with the homomorphism $\beta : L \rightarrow H$ determines a crossed module $(L, H; \beta, \triangleright')$.

There is a differential 2-crossed module, also called a 2-crossed module of Lie algebras, corresponding to the 2-crossed module of Lie groups $(L, H, G; \beta, \alpha, \triangleright, \{, \})$. We mark it with $(\ell, \mathfrak{h}, g; \mathfrak{b}, \mathfrak{a}, \triangleright, \{, \})$ consisting of three Lie algebras $\ell, \mathfrak{h}$ and $g$ respectively corresponding to the three Lie groups $L$, $H$ and $G$, and two Lie algebra maps $\mathfrak{b}$ and $\mathfrak{a}$.

$$\ell \xrightarrow{\mathfrak{b}} \mathfrak{h} \xrightarrow{\mathfrak{a}} g,$$

and left action $\triangleright$ of $g$ on $\ell, \mathfrak{h}$ and $g$ by automorphisms, and a $g$-equivariant bilinear map $\{, \} : \mathfrak{h} \times \mathfrak{h} \rightarrow \ell$ called the Peiffer lifting, such that

$$X \triangleright \{Y_1, Y_2\} = \{X \triangleright Y_1, Y_2\} + \{Y_1, X \triangleright Y_2\}, \quad \forall X \in g, Y_1, Y_2 \in \mathfrak{h}, \quad (4)$$

and they satisfy appropriate conditions [3]. Analogously to the 2-crossed module of Lie groups case, there is a left action $\triangleright'$ of $\mathfrak{h}$ on $\ell$ which is defined by

$$Y \triangleright' Z = -\{\mathfrak{b}(Z), Y\}, \quad \forall Y \in \mathfrak{h}, Z \in \ell. \quad (5)$$

This together with the homomorphism $\mathfrak{b} : \ell \rightarrow \mathfrak{h}$ defines a differential crossed module $(\ell, \mathfrak{h}; \mathfrak{b}, \triangleright')$. In consideration of the crossed module of groups $(L, H; \beta, \triangleright')$ and its calculations with squares as shown above, we perform the 2-crossed module of groups using cubes of the form.
where \( g_1, g_2, g_3, g_4 \in G, h_1, h_2 \in H, l \in L \) and \( \beta(l) = h_2 h_1^{-1} \). There are three types of compositions between cubes given by FIG. 1 where \( h_1 h_3 = h_3 h_1, h_2 h_4 = h_4 h_2, l \circ l' = l(h_1 \triangleright l') \), and FIG. 2 where \( l \circ l' = l'l \), and FIG. 3 where \( h_1 h_3 = h_1, h_2 h_4 = h_2(g_1 \triangleright h_4), l \circ l' = l(h_1 \triangleright l') \). Cubes admit horizontal and vertical inverses, defined by FIG. 4 where \( h_1^{-1} h_1 = h_1^{-1}, h_2^{-1} h_2 = h_2^{-1}, l^{-v} = h_1^{-1} \triangleright l^{-1} \), and FIG. 5 where \( l^{-h} = l^{-1} \), and FIG. 6 where \( h_1^{-v} = g_1^{-1} \triangleright h_1^{-1}, h_2^{-v} = g_2^{-1} \triangleright h_2^{-1}, l^{-v} = h_1^{-1} \triangleright l^{-1} \).

Here, we restrict ourselves to the particular 2-crossed module of groups in which \( (H, G; \alpha, \triangleright) \) is also a crossed module.

In order to construct the 3-form Yang-Mills action, we introduce some kind of \( G \)-invariant bilinear forms on the 2-crossed module of Lie algebras. More precisely speaking, we generalize the \( G \)-invariant bilinear forms on the a crossed module of Lie algebras \([38, 39]\). Firstly, we generalize the mixed relations of \( (H, G; \alpha, \triangleright) \) and \( (h, g; \tilde{\alpha}, \triangleright) \) to the mixed relations of \( (L, H, G; \beta, \alpha, \triangleright, \{, \}) \) and \( (\ell, \tilde{h}, \gamma; \tilde{\beta}, \tilde{\alpha}, \tilde{\triangleright}, \{, \}) \). When \( (H, G; \alpha, \triangleright) \) is a crossed module of groups, there are two mixed relations \([38]\) induced by the action via \( \triangleright \) on \( H \) and also denoted by \( \triangleright \)

\[
\tilde{\alpha}(g \triangleright Y) = g \tilde{\alpha}(Y) g^{-1}, \quad \forall g \in G, Y \in \tilde{h},
\]

(6)
Thus there are also the mixed relations, corresponding to the crossed module of Lie algebras \((\alpha, \beta, \triangleright')\) induced by the action via \(\triangleright\) where we have used the mixed relation (7). In the case when the Peiffer lifting or the map \(L, H\) besides, \((\beta(h \triangleright' l) = h\beta(l)h^{-1}, \quad \forall h \in H, l \in L, (8)\)

\[\beta(l) \triangleright' l' = ll'l^{-1}, \quad \forall l, l' \in L. (9)\]

Thus there are also the mixed relations, corresponding to the crossed module of Lie algebras \((\ell, \bar{\beta}, \triangleright')\), induced by the action via \(\triangleright'\) on \(L\) and also denoted by \(\triangleright'\)

\[\bar{\beta}(h \triangleright' Z) = h\bar{\beta}(Z)h^{-1}, \quad \forall h \in H, Z \in \ell, (10)\]

\[\beta(l) \triangleright' Z = lZl^{-1}, \quad \forall l \in L, Z \in \ell. (11)\]

A symmetric non-degenerate \(G\)-invariant form in \((\ell, \bar{\beta}, g; \tilde{\alpha}, \tilde{\beta}, \alpha, \tilde{\beta}, \{\}, \\{\}\) is given by a triple of non-degenerate symmetric bilinear forms \(\langle, \rangle_g\) in \(g\), \(\langle, \rangle_h\) in \(h\) and \(\langle, \rangle_{\ell}\) in \(\ell\) such that

1. \(\langle, \rangle_g\) is \(G\)-invariant, i.e.
   \[\langle gxg^{-1}, gx'g^{-1} \rangle_g = \langle x, x' \rangle_g, \quad \forall g \in G, X, X' \in g;\]

2. \(\langle, \rangle_h\) is \(H\)-invariant, i.e.
   \[\langle g \triangleright Y, g \triangleright Y' \rangle_h = \langle Y, Y' \rangle_h, \quad \forall g \in G, Y, Y' \in h;\]

3. \(\langle, \rangle_{\ell}\) is \(\ell\)-invariant, i.e.
   \[\langle g \triangleright Z, g \triangleright Z' \rangle_{\ell} = \langle Z, Z' \rangle_{\ell}, \quad \forall g \in G, Z, Z' \in \ell.\]

Note that \(\langle, \rangle_h\) is necessarily \(H\)-invariant. Since

\[\langle hYh^{-1}, hY'h^{-1} \rangle_h = \langle \alpha(h) \triangleright Y, \alpha(h) \triangleright Y' \rangle_h = \langle Y, Y' \rangle_h, \quad \forall h \in H, Y, Y' \in h,\]

where we have used the mixed relation (7). In the case when the Peiffer lifting or the map \(\beta\) is trivial consequently, \(\langle, \rangle_{\ell}\) is \(H\)-invariant, i.e.

\[\langle h \triangleright' Z, h \triangleright' Z' \rangle_{\ell} = \langle Z, Z' \rangle_{\ell}, \quad \forall h \in H, Z, Z' \in \ell.\]

Besides, \(\langle, \rangle_{\ell}\) is necessarily \(L\)-invariant based on the \(H\)-invariance of \(\langle, \rangle_{\ell}\), i.e.

\[\langle lZl^{-1}, lZ'l^{-1} \rangle_{\ell} = \langle \beta(l) \triangleright Z, \beta(l) \triangleright Z' \rangle_{\ell} = \langle Z, Z' \rangle_{\ell}, \quad \forall l \in L, Z, Z' \in \ell,\]

where the mixed relation (11) is used. See 39 for details.

There are no compatibility conditions among the bilinear forms \(\langle, \rangle_{\ell}, \langle, \rangle_h\) and \(\langle, \rangle_g\). From the well-known fact that any representation of \(G\) can be made unitary if \(G\) is a compact group, one can get the following Lemma.
Lemma 1 Let \((L, H, G; \beta, \alpha, \triangleright, \{, \})\) be a 2-crossed module of Lie groups with the group \(G\) being compact in the real case, or having a compact real form in the complex case. Then one can construct \(G\)-invariant symmetric non-degenerate bilinear forms \(\langle , \rangle_g, \langle , \rangle_h\) and \(\langle , \rangle_\ell\) in the associated differential 2-crossed module \((\ell, \hbar, g; \beta, \alpha, \triangleright, \{, \})\). Furthermore these forms can be chosen to be positive definite.

These invariance conditions imply that:
\[
\langle [X, X'], [X''] \rangle_g = -\langle X', [X, X''] \rangle_g, \tag{12}
\]
\[
\langle [Y, Y'], [Y''] \rangle_h = -\langle Y', [Y, Y''] \rangle_h, \tag{13}
\]
\[
\langle [Z, Z'], [Z''] \rangle_\ell = -\langle Z', [Z, Z''] \rangle_\ell. \tag{14}
\]

One can define two bilinear antisymmetric maps \(\sigma : \hbar \times \hbar \rightarrow g\) by the rule:
\[
\langle \sigma(Y, Y') , X \rangle_g = -\langle Y, X \triangleright Y' \rangle_h, \quad \forall X \in g, Y, Y' \in \hbar, \tag{15}
\]
and \(\kappa : \ell \times \ell \rightarrow g\) by the rule:
\[
\langle \kappa(Z, Z') , X \rangle_g = -\langle Z, X \triangleright Z' \rangle_\ell, \quad \forall X \in g, Z, Z' \in \ell. \tag{16}
\]

Due to \(\sigma(Y', Y) = -\sigma(Y, Y')\) and \(\kappa(Z', Z) = -\kappa(Z, Z')\), then one has
\[
\langle Y, X \triangleright Y' \rangle_h = -\langle Y', X \triangleright Y \rangle_h = -\langle X \triangleright Y, Y' \rangle_h, \tag{17}
\]
\[
\langle Z, X \triangleright Z' \rangle_\ell = -\langle Z', X \triangleright Z \rangle_\ell = -\langle X \triangleright Z, Z' \rangle_\ell. \tag{18}
\]

Further, one needs to define two bilinear maps \(\eta_1 : \ell \times \hbar \rightarrow \hbar\) and \(\eta_2 : \ell \times \hbar \rightarrow \hbar\) by the rule:
\[
\langle \{Y, Y'\}, Z \rangle_\ell = -\langle Y, \eta_1(Z, Y') \rangle_h = -\langle Y', \eta_2(Z, Y) \rangle_h, \tag{19}
\]
for each \(Y, Y' \in \hbar\), and \(Z \in \ell\). See \[39\] for more information about these maps. To obtain the 2-form Yang-Mills equations, one defines a map \(\alpha^* : g \rightarrow \hbar\) in \[24\] by the rule:
\[
\langle Y, \alpha^*(X) \rangle_h = \langle \tilde{\alpha}(Y), X \rangle_g, \quad \forall X \in g, Y \in \hbar. \tag{20}
\]

Here, to obtain the 3-form Yang-Mills equations, we have to first define a map \(\beta^* : \hbar \rightarrow \ell\) by the rule:
\[
\langle Z, \beta^*(Y) \rangle_\ell = \langle \tilde{\beta}(Z), Y \rangle_h, \quad \forall Y \in \hbar, Z \in \ell. \tag{21}
\]

The maps \(\sigma, \kappa, \eta_1, \eta_2, \alpha^*\) and \(\beta^*\) are very closely linked with our approach to the construction of the 3-form Yang-Mills.

In order to calculate more efficiently, we introduce the component notation. Given a Lie algebra \(g\), there is a vector space \(\Lambda^k(M, g)\) of \(g\)-valued differential \(k\)-forms on the manifold \(M\). For \(A = \sum_a A^a X_a \in \Lambda^k(M, g)\), \(A' = \sum_a A'^a X_a \in \Lambda^k(M, g)\) for some scalar differential \(k\)-forms \(A^a, A'^a\) and elements \(X_a, X_b \in g\), define
\[
A \wedge A' := \sum_{a,b} A^a \wedge A'^b X_a X_b, \quad A \wedge^{[1]} A' := \sum_{a,b} A^a \wedge A'^b [X_a, X_b], \quad dA = \sum_a dA^a X_a,
\]
then there is an identity
\[
A \wedge^{[1]} A' = A \wedge A' - (-1)^{k_1 k_2} A' \wedge A.
\]

In the above, we can also choose \(g\) to be \(\hbar\) or \(\ell\). For \(B = \sum_b B^b Y_b \in \Lambda^k(M, \hbar)\), \(B' = \sum_b B'^b Y_b \in \Lambda^k(M, \hbar)\), \(\forall Y_a, Y_b \in \hbar\), define
\[
A \wedge^{\tilde{\hbar}} B := \sum_{a,b} A^a \wedge B^b X_a \triangleright Y_b, \quad B \wedge^{\{1\}} B' := \sum_{a,b} B^a \wedge B'^b \{Y_a, Y_b\}, \quad \tilde{\alpha}(B) := \sum_a B^a \tilde{\alpha}(Y_a).
\]
For $C = \sum_a C_a \in \Lambda^k(U, \mathcal{L})$, $C' = \sum_b C_b \in \Lambda^k(U, \mathcal{L})$, define

$$A \wedge^\nabla C := \sum_{a,b} A^a \wedge C^b X_a \bar{\delta} Z_b , \quad B \wedge^{\nabla'} C := \sum_{a,b} B^a \wedge C^b Y_a \bar{\delta}' Z_b , \quad \tilde{\beta}(C) := \sum_a C_a \tilde{\beta}(Z_a),$$

where $Y_a \bar{\delta}' Z_b = - \left\{ \tilde{\beta}(Z_b), Y_a \right\}$ by using \( \square \).

Furthermore, we have non-degenerate $G$-invariant forms in $\Lambda^k(M, g)$, $\Lambda^k(M, \mathfrak{h})$ and $\Lambda^k(M, \mathcal{L})$ induced by $\langle \cdot, \cdot \rangle_g$, $\langle \cdot, \cdot \rangle_h$ and $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ and we denote them by $\langle \cdot, \cdot \rangle$. Then we have

$$\langle A, A' \rangle := \sum_{a,b} A^a \wedge A'^b \langle X_a, X_b \rangle_g, \quad \langle B, B' \rangle := \sum_{a,b} B^a \wedge B'^b \langle Y_a, Y_b \rangle_h, \quad \langle C, C' \rangle := \sum_{a,b} C^a \wedge C'^b \langle Z_a, Z_b \rangle_{\mathcal{L}}.$$

There are identities

$$\langle A, A' \rangle = (-1)^{k_1 k_2} \langle A', A \rangle, \quad \langle B, B' \rangle = (-1)^{k_1 k_2} \langle B', B \rangle, \quad \langle C, C' \rangle = (-1)^{k_1 k_2} \langle C', C \rangle,$$

using the symmetry of $\langle \cdot, \cdot \rangle_g$, $\langle \cdot, \cdot \rangle_h$ and $\langle \cdot, \cdot \rangle_{\mathcal{L}}$.

There is an important proposition in \[17\], and we use it to calculate the 3-form Yang-Mills equations.

**Proposition 1**  
1. For $A \in \Lambda^k(M, g)$, $A' \in \Lambda^k(M, g)$ and $C \in \Lambda^3(M, \mathcal{L})$,

$$\tilde{\beta}(A \wedge^\nabla C) = A \wedge^\nabla \tilde{\beta}(C),$$

$$A \wedge^\nabla A' = A \wedge A' + (-1)^{kk+1} A' \wedge A.$$

2. For $A \in \Lambda^k(M, g)$, $B_1 \in \Lambda^{k_1}(M, \mathfrak{h})$, $B_2 \in \Lambda^{k_2}(M, \mathfrak{h})$ and $W \in \Lambda^w(M, w)$ ($w = g, h, \mathcal{L}$),

$$d(A \wedge^\nabla W) = dA \wedge^\nabla W + (-1)^k A \wedge^\nabla dW,$$

$$d(B_1 \wedge^{\{1\}} B_2) = dB_1 \wedge^{\{1\}} B_2 + (-1)^{t_1} B_1 \wedge^{\{1\}} dB_2,$$

$$A \wedge^\nabla (B_1 \wedge^{\{1\}} B_2) = (A \wedge^\nabla B_1) \wedge^{\{1\}} B_2 + (-1)^{k_1 t_1} B_1 \wedge^{\{1\}} (A \wedge^\nabla B_2).$$

The following propositions provide identities of non-degenerate symmetric $G$-invariant forms in a Lie 2-crossed module.

**Proposition 2** For $A_1 \in \Lambda^{k_1}(M, g)$, $A_2 \in \Lambda^{k_2}(M, g)$, $A_3 \in \Lambda^{k_3}(M, g)$, $B_1 \in \Lambda^{k_1}(M, \mathfrak{h})$, $B_2 \in \Lambda^{k_2}(M, \mathfrak{h})$, $B_3 \in \Lambda^{k_3}(M, \mathfrak{h})$, $C_1 \in \Lambda^{k_1}(M, \mathcal{L})$, $C_2 \in \Lambda^{k_2}(M, \mathcal{L})$, and $C_3 \in \Lambda^{k_3}(M, \mathcal{L})$ we have

$$\langle A_1 \wedge^{\{1\}} A_2, A_3 \rangle = (-1)^{k_1 k_2 + 1} \langle A_2, A_1 \wedge^{\{1\}} A_3 \rangle,$$

$$\langle B_1 \wedge^{\{1\}} B_2, B_3 \rangle = (-1)^{t_1 t_2 + 1} \langle B_2, B_1 \wedge^{\{1\}} B_3 \rangle,$$

$$\langle C_1 \wedge^{\{1\}} C_2, C_3 \rangle = (-1)^{q_1 q_2 + 1} \langle C_2, C_1 \wedge^{\{1\}} C_3 \rangle.$$

**Proof:** We can get these identities easily by using \[12\], \[13\] and \[14\].

\[\square\]
Proposition 3 For $A \in \Lambda^k(M, g)$, $B_1 \in \Lambda^l(M, \kappa)$, $B_2 \in \Lambda^q(M, \ell)$, $C_1 \in \Lambda^q(M, \ell)$ and $C_2 \in \Lambda^q(M, \ell)$, we have
\[
\langle B_1, A \wedge \tilde{\wedge} B_2 \rangle = (-1)^{(l+q_2)+k_1+1} \langle B_2, A \wedge \tilde{\wedge} B_1 \rangle = (-1)^{k_2+1} \langle A \wedge \tilde{\wedge} B_1, B_2 \rangle,
\]
(30)
\[
\langle C_1, A \wedge \tilde{\wedge} C_2 \rangle = (-1)^{q_2(q_2+q_2+1)} \langle C_2, A \wedge \tilde{\wedge} C_1 \rangle = (-1)^{k_2+1} \langle A \wedge \tilde{\wedge} C_1, C_2 \rangle.
\]
(31)

Proof: We can get those identities easily by using (17) and (18).

We can define a bilinear map $\pi: \Lambda^l(M, \kappa) \times \Lambda^q(M, \ell) \rightarrow \Lambda^{1+(l+q_2)}(M, \ell)$ by
\[
\pi(B_1, B_2) := \sum_{a,b} B_1^a \wedge B_2^b \sigma(Y_a, Y_b),
\]
for $B_1 = \sum_a B_1^a Y_a \in \Lambda^l(M, \kappa)$, $B_2 = \sum_b B_2^b Y_b \in \Lambda^q(M, \ell)$. Since $\sigma$ is antisymmetric, we have
\[
\pi(B_1, B_2) = (-1)^{l+q_2+1} \pi(B_2, B_1).
\]
Let $A = \sum_c A^c X_c \in \Lambda^k(M, g)$, then
\[
\langle \pi(B_1, B_2), A \rangle = \sum_{a,b,c} B_1^a \wedge B_2^b \wedge A^c \langle \sigma(Y_a, Y_b), Y_c \rangle_g = - \sum_{a,b,c} B_1^a \wedge B_2^b \wedge A^c \langle \sigma(Y_a, X_c \tilde{\wedge} Z_b) \rangle_g
\]
\[
= (-1)^{k_2+1} \sum_{a,b,c} B_1^a \wedge A^c \wedge B_2^b \langle Y_a, X_c \tilde{\wedge} Z_b \rangle_g
\]
\[
= (-1)^{k_2+1} \langle B_1, A \wedge \tilde{\wedge} B_2 \rangle,
\]
by using (15), i.e.
\[
\langle \pi(B_1, B_2), A \rangle = (-1)^{k_2+1} \langle B_1, A \wedge \tilde{\wedge} B_2 \rangle,
\]
(32)
and the following identity holds
\[
\langle A, \pi(B_1, B_2) \rangle = (-1)^{l+q_2+1} \langle A \wedge \tilde{\wedge} B_2, B_1 \rangle.
\]
(33)
Similarly, we can define a bilinear map $\pi: \Lambda^q(M, \ell) \times \Lambda^q(M, \ell) \rightarrow \Lambda^{1+q_2+q_2}(M, \ell)$ by
\[
\pi(C_1, C_2) := \sum_{a,b} C_1^a \wedge C_2^b \kappa(Z_a, Z_b),
\]
for $C_1 = \sum_a C_1^a Z_a \in \Lambda^q(M, \ell)$ and $C_2 = \sum_b C_2^b Z_b \in \Lambda^q(M, \ell)$. Since $\kappa$ is antisymmetric, we have
\[
\pi(C_1, C_2) = (-1)^{q_2+q_2+1} \pi(C_2, C_1).
\]
Let
\[
A = \sum_c A^c X_c \in \Lambda^k(M, g),
\]
then
\[
\langle \pi(C_1, C_2), A \rangle = \sum_{a,b,c} C_1^a \wedge C_2^b \wedge A^c \langle \pi(Z_a, Z_b), Y_c \rangle_g = - \sum_{a,b,c} C_1^a \wedge C_2^b \wedge A^c \langle Z_a, X_c \tilde{\wedge} Z_b \rangle_g
\]
\[
= (-1)^{k_2+1} \sum_{a,b,c} C_1^a \wedge A^c \wedge C_2^b \langle Z_a, X_c \tilde{\wedge} Z_b \rangle_g
\]
\[
= (-1)^{k_2+1} \langle C_1, A \wedge \tilde{\wedge} C_2 \rangle.
\]
by using (16), i.e.
\[ \langle \pi(C_1, C_2), A \rangle = (-1)^{h_1 q_2 + 1} (C_1, A \wedge^\pi C_2), \]
and there is an identity
\[ \langle A, \pi(C_1, C_2) \rangle = (-1)^{h_1 q_2 + 1} (A \wedge^\pi C_2, C_1). \] (35)

Finally, we define bilinear maps \( \pi_i : \Lambda^q(M, \ell) \times \Lambda^i(M, \mathfrak{g}) \to \Lambda^{q+i}(M, \mathfrak{g}) \) by
\[ \pi_i(C, B) := \sum_{a,b} C^b \wedge B^a \eta_i(Z_a, Y_b), \quad i = 1, 2 \]
for \( B = \sum_a B^a Y_a \in \Lambda^i(M, \mathfrak{g}), \ C = \sum_b C^b Z_b \in \Lambda^q(M, \ell) \). There are two identities
\[ \langle B_1 \wedge (\cdot) B_2, C \rangle = (-1)^{t_2 q + 1} \langle B_1, \pi_i(C, B_2) \rangle = (-1)^{t_1 (t_2 + q) + 1} \langle B_2, \pi_i(C, B_1) \rangle, \]
by using (19).

For \( A \in \Lambda^k(M, g), \ B \in \Lambda^l(M, \mathfrak{h}) \) and \( C \in \Lambda^q(M, \ell) \), we have
\[ \langle B, \alpha^*(A) \rangle = \langle \alpha(B), A \rangle, \]
(37)
\[ \langle C, \beta^*(B) \rangle = \langle \beta(C), B \rangle, \] (38)
being induced by (20) and (21).

Up to present, we have established the algebraic structures of 3-form Yang-Mills theory. Next, we derive the 3-Bianchi-Identities on account of the above properties for Lie algebra valued differential forms. The 3-connection, on \( d \)-dimensional spacetime manifold \( M \) (\( d \geq 4 \)), is described by a 1-form \( A \) valued in the Lie algebra \( g \), a 2-form \( B \) valued in the Lie algebra \( \mathfrak{h} \), and a 3-form \( C \) valued in the Lie algebra \( \ell \). We refer the interested readers to [7] for more details. The corresponding 3-curvature in this case is given by
\[ \Omega_1 := dA + A \wedge A, \quad \Omega_2 := dB + A \wedge^\pi B, \quad \Omega_3 := dC + A \wedge^\pi C + B \wedge (\cdot) B. \]

There are a fake 1-curvature and a fake 2-curvature defined as
\[ F_1 := dA + A \wedge A - \tilde{\alpha}(B), \quad F_2 := dB + A \wedge^\pi B - \tilde{\beta}(C). \]

We call the 3-connection \((A, B, C)\) fake 1-flat, if the fake 1-curvature vanishes, i.e. \( dA + A \wedge A = \tilde{\alpha}(B) \). Similarly, we call the 3-connection \((A, B, C)\) fake 2-flat, if the fake 2-curvature vanishes, i.e. \( dB + A \wedge^\pi B = \tilde{\beta}(C) \). If the 3-curvature 4-form vanishes, i.e. \( \Omega_3 = 0 \), the 3-connection \((A, B, C)\) will be called 3-flat. When \((A, B, C)\) is both fake 1-flat and fake 2-flat, we say that the 3-connection \((A, B, C)\) is fake-flat. Otherwise, the 3-connection \((A, B, C)\) is not fake-flat.

**Theorem 1** If \((A, B, C)\) is any 3-connection on \( M \), then its fake 3-curvature \((\Omega_1, F_1, F_2)\) satisfies the 3-Bianchi-Identities:
\[ dF_1 + A \wedge (\cdot) F_1 = -\tilde{\alpha}(F_2 + \tilde{\beta}(C)), \] (39)
\[ dF_2 + A \wedge^\pi F_2 = (F_1 + \tilde{\alpha}(B)) \wedge^\pi B - \tilde{\beta}(\Omega_3 - B \wedge (\cdot) B), \]
(40)
\[ d\Omega_3 + A \wedge^\pi \Omega_3 = (F_1 + \tilde{\alpha}(B)) \wedge^\pi C + (F_2 + \tilde{\beta}(C)) \wedge (\cdot) B + B \wedge (\cdot) (F_2 + \tilde{\beta}(C)). \] (41)
Proof:

\[
dF_1 = d(da + A \wedge A - \tilde{\alpha}(B)) = da \wedge A - A \wedge da - d(\tilde{\alpha}(B)) \\
= (F_1 - A \wedge A + \tilde{\alpha}(B)) \wedge A - A \wedge (F_1 - A \wedge A + \tilde{\alpha}(B)) - \tilde{\alpha}(dB) \\
= F_1 \wedge A + \tilde{\alpha}(B) \wedge A - A \wedge F_1 - A \wedge \tilde{\alpha}(B) - \tilde{\alpha}(dB) \\
= -A \wedge [i] F_1 - A \wedge [i] \tilde{\alpha}(B) - \tilde{\alpha}(dB) \\
= -A \wedge [i] F_1 - \tilde{\alpha}(A \wedge \tilde{\wedge} B + dB) \\
= -A \wedge [i] F_1 - \tilde{\alpha}(F_2 + \tilde{\beta}(C)),
\]

by using \(a(A \wedge \tilde{\wedge} B) = A \wedge [i] \tilde{\alpha}(B)\), and

\[
dF_2 = d(db + A \wedge \tilde{\wedge} B - \tilde{\beta}(C)) = da \wedge \tilde{\wedge} B - A \wedge \tilde{\wedge} dB - d(\tilde{\beta}(C)) \\
= (F_1 - A \wedge A + \tilde{\alpha}(B)) \wedge \tilde{\wedge} B - A \wedge \tilde{\wedge} F_2 + \tilde{\alpha}(B) \wedge \tilde{\wedge} B - A \wedge \tilde{\wedge} F_2 + \tilde{\alpha}(B) \wedge \tilde{\wedge} B - A \wedge \tilde{\wedge} \tilde{\beta}(C) - \tilde{\beta}(dB) \\
= (F_1 + \tilde{\alpha}(B)) \wedge \tilde{\wedge} B - \tilde{\beta}(A \wedge \tilde{\wedge} C + dB) - A \wedge \tilde{\wedge} F_2 + \tilde{\alpha}(B) \wedge \tilde{\wedge} B - \tilde{\beta}(A \wedge \tilde{\wedge} C + dB) - A \wedge \tilde{\wedge} F_2,
\]

by using \(22\) and \(\tilde{\wedge}\) is homomorphism, and

\[
d\Omega_3 = d(dc + A \wedge \tilde{\wedge} C + B \wedge [i] B) = da \wedge \tilde{\wedge} C - A \wedge \tilde{\wedge} dc + d(b \wedge [i] B) \\
= da \wedge \tilde{\wedge} C - A \wedge \tilde{\wedge} (\Omega_3 - A \wedge \tilde{\wedge} C - B \wedge [i] B) + dB \wedge [i] B + B \wedge [i] dB \\
= da \wedge \tilde{\wedge} C - A \wedge \tilde{\wedge} \Omega_3 + (A \wedge \Omega_3 + A \wedge \tilde{\wedge} C + A \wedge \tilde{\wedge} (B \wedge [i] B) + dB \wedge [i] B + B \wedge [i] dB \\
= (F_1 + \tilde{\alpha}(B)) \wedge \tilde{\wedge} C - A \wedge \tilde{\wedge} \Omega_3 + (\Omega_3 + (F_2 + \tilde{\beta}(C)) \wedge [i] B + B \wedge [i] (A \wedge \tilde{\wedge} B) + dB \wedge [i] B + B \wedge [i] dB \\
= (F_1 + \tilde{\alpha}(B)) \wedge \tilde{\wedge} C - A \wedge \tilde{\wedge} \Omega_3 + (F_2 + \tilde{\beta}(C)) \wedge [i] B + B \wedge [i] (F_2 + \tilde{\beta}(C)),
\]

by using \(22\), \(25\), \(26\) and \(\tilde{\wedge}\) is homomorphism.

When the 3-connection \((A, B, C)\) is fake-flat, the **3-Bianchi-Identities** become

\[
d\Omega_1 + A \wedge [i] \Omega_1 = 0,
\]

\[
d\Omega_2 + A \wedge \tilde{\wedge} B = \Omega_1 \wedge \tilde{\wedge} B,
\]

\[
d\Omega_3 + A \wedge \tilde{\wedge} \Omega_3 = \Omega_1 \wedge \tilde{\wedge} \Omega_3 + \Omega_2 \wedge [i] B + B \wedge [i] \Omega_2.
\]

Generalizing the Yang-Mills action and the 2-form Yang-Mills action in [24], we write down the following 3-form Yang-Mills action as a function of the 3-connection \((A, B, C)\) in 3-form Yang-Mills gauge theory:

\[
S = \int_M \langle F_1, *F_1 \rangle + \langle F_2, *F_2 \rangle + \langle \Omega_3, *\Omega_3 \rangle.
\]

One can obtain the field equations by setting the variation of the action to zero:

\[
\delta S = 2 \int_M \langle \delta F_1, *F_1 \rangle + \langle \delta F_2, *F_2 \rangle + \langle \delta \Omega_3, *\Omega_3 \rangle = 0.
\]
The first section is as follows:

\[
(\delta F_1, *F_1) = \langle \delta (dA + A \wedge A - \alpha B), *F_1 \rangle \\
= \langle \delta \alpha A, *F_1 \rangle + \langle A \wedge [\delta A, *F_1] \rangle - \langle \alpha \delta B, *F_1 \rangle \\
= \langle \delta A, d * F_1 \rangle + \langle \delta A, A \wedge [\delta F_1] \rangle - \langle \delta B, \alpha^*(*F_1) \rangle \\
= \langle \delta A, d * F_1 + A \wedge [\delta F_1] \rangle - \langle \delta B, \alpha^*(*F_1) \rangle,
\]

by using \(\delta (A \wedge A) = A \wedge [\delta A] \) and \(\alpha \). The second section is as follows:

\[
(\delta F_2, *F_2) = \langle \delta (dB + A \wedge B - \beta C), *F_2 \rangle \\
= \langle \delta B, *F_2 \rangle + \langle \delta A \wedge B, *F_2 \rangle + \langle A \wedge \delta B, *F_2 \rangle - \langle \beta \delta C, *F_2 \rangle \\
= -\langle \delta B, d * F_2 \rangle - \langle \delta A, \pi(*F_2, B) \rangle - \langle \delta B, A \wedge \delta *F_2 \rangle - \langle \delta C, \beta^*(*F_2) \rangle \\
= -\langle \delta A, \pi(*F_2, B) \rangle - \langle \delta B, d * F_2 + A \wedge \delta *F_2 \rangle - \langle \delta C, \beta^*(*F_2) \rangle,
\]

by using \(\delta (A \wedge B) = \delta A \wedge \delta B + A \wedge \delta B \), \(\beta \), \(\alpha \) and \(\beta \). The third section is as follows:

\[
(\delta \Omega_3, *\Omega_3) = \langle \delta (dC + A \wedge C + B \wedge (\delta B), *\Omega_3 \rangle \\
= \langle \delta \alpha C, *\Omega_3 \rangle + \langle \delta A \wedge \beta C, *\Omega_3 \rangle \\
+ \langle A \wedge \delta \Omega_3, *\Omega_3 \rangle + \langle \delta B \wedge (\delta B), *\Omega_3 \rangle + \langle B \wedge (\delta B), *\Omega_3 \rangle \\
= \langle \delta C, d * \Omega_3 \rangle + (-1)^d \langle \delta A, \pi(*\Omega_3, C) \rangle \\
+ \langle \delta C, A \wedge \beta \Omega_3 \rangle - \langle \delta B, \pi_1(*\Omega_3, B) \rangle - \langle \delta B, \pi_2(*\Omega_3, B) \rangle \\
= (-1)^d \langle \delta A, \pi(*\Omega_3, C) \rangle - \langle \delta B, \pi_1(*\Omega_3, B) + \pi_2(*\Omega_3, B) \rangle \\
+ \langle \delta C, d * \Omega_3 + A \wedge \beta \Omega_3 \rangle,
\]

by using \(\pi_1 \), \(\pi_2 \) and \(\beta \). Thus we have

\[
\delta S = 2 \int_M \delta A, d * F_1 + A \wedge [\delta F_1] + (-1)^d \langle \pi(*\Omega_3, C) - \pi(*F_2, B) \rangle \\
- \langle \delta B, d * F_2 + A \wedge \beta F_2 + \pi(*\Omega_3, B) + \pi_2(*\Omega_3, B) + \alpha^*(*F_1) \rangle \\
+ \langle \delta C, d * \Omega_3 + A \wedge \beta \Omega_3 - \beta^*(*F_2) \rangle.
\]

We see that the variation of the action vanishes for \(\delta A, \delta B \) and \(\delta C \) if only if the following field equations hold:

\[
d * F_1 + A \wedge [\delta F_1] = \pi(*F_2, B) + (-1)^d \pi(*\Omega_3, C), \tag{45}
\]
\[
d * F_2 + A \wedge \beta F_2 = -\pi_1(*\Omega_3, B) - \pi_2(*\Omega_3, B) - \alpha^*(*F_1), \tag{46}
\]
\[
d * \Omega_3 + A \wedge \beta \Omega_3 = \beta^*(*F_2). \tag{47}
\]

And when the 3-connection \((A, B, C)\) is fake-flat, the field equations become

\[
d * \Omega_1 + A \wedge [\delta \Omega_1] = \pi(*\Omega_2, B) + (-1)^d \pi(*\Omega_3, C), \tag{48}
\]
\[
d * \Omega_2 + A \wedge \beta \Omega_2 = -\pi_1(*\Omega_3, B) - \pi_2(*\Omega_3, B), \tag{49}
\]
\[
d * \Omega_3 + A \wedge \beta \Omega_3 = 0. \tag{50}
\]
Though one may wonder about 4-gauge theory, to the best of our knowledge, it has not been defined yet. The notion of a 3-crossed module, which should be the foundation of 4-gauge theory, has been developed in [40, 41]. Ideally, in higher gauge theory, the Yang-Mills theory may be generalized to a kind of "n-form Yang-Mills theory" in accordance with the chosen n-group structure \((n > 3)\) [42]. There is no doubt that one may encounter a lot of difficulties.

To summarize, the relationships between the related models are concluded in FIG. 7.

\[
\begin{align*}
L, H : & \text{ trivial} \\
G : & \text{ trivial} \\
3-YM : & 3\text{-form Yang-Mills} \\
2-YM : & 2\text{-form Yang-Mills} \\
(1-)YM : & (1\text{-form})\text{Yang-Mills} \\
3-Elec : & 3\text{-form Electromagnetism} \\
2-Elec : & 2\text{-form Electromagnetism} \\
(1-)Elec : & (1\text{-form})\text{ Electromagnetism}
\end{align*}
\]

FIG. 7. The relationships

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