A NEW FAMILY OF GAUGES IN
LINEARIZED GENERAL RELATIVITY

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Abstract. For vacuum Maxwell theory in four dimensions, a supplementary condition
exists (due to Eastwood and Singer) which is invariant under conformal rescalings of the
metric, in agreement with the conformal symmetry of the Maxwell equations. Thus, start-
ing from the de Donder gauge, which is not conformally invariant but is the gravitational
counterpart of the Lorenz gauge, one can consider, led by formal analogy, a new family of
gauges in general relativity, which involve fifth-order covariant derivatives of metric per-
turbations. The admissibility of such gauges in the classical theory is first proven in the
cases of linearized theory about flat Euclidean space or flat Minkowski space-time. In the
former, the general solution of the equation for the fulfillment of the gauge condition after
infinitesimal diffeomorphisms involves a 3-harmonic 1-form and an inverse Fourier trans-
form. In the latter, one needs instead the kernel of powers of the wave operator, and a
contour integral. The analysis is also used to put restrictions on the dimensionless param-
eter occurring in the DeWitt supermetric, while the proof of admissibility is generalized to
a suitable class of curved Riemannian backgrounds. Eventually, a non-local construction
is obtained of the tensor field which makes it possible to achieve conformal invariance of
the above gauges.
1. Introduction

The transformation properties of classical and quantum field theories under conformal rescalings of the metric have led, over the years, to many deep developments in mathematics and theoretical physics, e.g. conformal-infinity techniques in general relativity,\(^1\) twistor methods for gravitation and Yang–Mills theory,\(^2,3\) the conformal-variation method in heat-kernel asymptotics,\(^4\) the discovery of conformal anomalies in quantum field theory.\(^5\) All these topics are quite relevant for the analysis of theories which possess a gauge freedom. As a first example, one may consider the simplest gauge theory, i.e. vacuum Maxwell theory in four dimensions in the absence of sources. At the classical level, the operator acting on the potential \(A^b\) is found to be

\[
P^b_a = -\delta^b_a \Box + R^b_a + \nabla_a \nabla^b, \tag{1.1}
\]

where \(\nabla\) is the Levi–Civita connection on space-time, \(\Box \equiv g^{ab} \nabla_a \nabla_b\), and \(R^{ab}\) is the Ricci tensor. Thus, the supplementary (or gauge) condition of the Lorenz type, i.e.

\[
\nabla^b A_b = 0, \tag{1.2a}
\]

is of crucial importance to obtain a wave equation for \(A^b\). The drawback of Eq. (1.2a), however, is that it is not preserved under conformal rescalings of the metric:

\[
\hat{g}_{ab} = \Omega^2 g_{ab}, \quad \hat{g}^{ab} = \Omega^{-2} g^{ab}, \tag{1.3}
\]

whereas the Maxwell equations

\[
\nabla^b F_{ab} = 0 \tag{1.4}
\]

are invariant under the rescalings (1.3). This remark was the starting point of the investigation by Eastwood and Singer,\(^6\) who found that a conformally invariant supplementary condition may be imposed, i.e.

\[
\nabla_b \left[ \left( \nabla^b \nabla^a - 2R^{ab} + \frac{2}{3} Rg^{ab} \right) A_a \right] = 0. \tag{1.5a}
\]
As is clear from Eq. (1.5a), conformal invariance is achieved at the price of introducing
third-order derivatives of the potential. In flat backgrounds, such a condition reduces to
\[ \Box \nabla_b A_b = 0. \] (1.6)

Of course, all solutions of the Lorenz gauge are also solutions of Eq. (1.6), whereas the
converse does not hold.

Leaving aside the severe technical problems resulting from the attempt to quantize in
the Eastwood–Singer gauge, we are now interested in understanding the key features of
the counterpart for Einstein’s theory of general relativity. In other words, although the
vacuum Einstein equations
\[ R_{ab} - \frac{1}{2} g_{ab} R = 0 \] (1.7)
are not invariant under the conformal rescalings (1.3), we would like to see whether the
geometric structures leading to Eq. (1.5a) admit a non-trivial generalization to Einstein’s
theory, so that a conformally invariant supplementary condition with a higher order oper-
ator may be found as well. For this purpose, we re-express Eqs. (1.2a) and (1.5a) in the
form
\[ g^{ab} \nabla_a A_b = 0, \] (1.2b)
\[ g^{ab} \nabla_a \nabla_b \nabla_c A_c + \left[ \nabla_b \left( -2 R^{ba} + \frac{2}{3} R g^{ba} \right) \right] A_a \]
\[ + \left( -2 R^{ba} + \frac{2}{3} R g^{ba} \right) \nabla_b A_a = 0. \] (1.5b)

Eq. (1.2b) involves the space-time metric in its contravariant form, which is also the metric
on the bundle of 1-forms. In Einstein’s theory, one deals instead with the vector bundle of
symmetric rank-two tensors on space-time with DeWitt supermetric
\[ E^{abcd} \equiv \frac{1}{2} \left( g^{ac} g^{bd} + g^{ad} g^{bc} + \alpha g^{ab} g^{cd} \right), \] (1.8)
\[ \alpha \text{ being a real parameter different from } -\frac{2}{m}, \text{ where } m \text{ is the dimension of space-time (this restriction on } \alpha \text{ is necessary to make sure that the metric } E^{abcd} \text{ has an inverse). One is thus led to replace Eq. (1.2b) with the de Donder gauge} \]

\[ W^a \equiv E^{abcd} \nabla_b h_{cd} = 0. \quad (1.9) \]

Hereafter, \( h_{ab} \) denotes metric perturbations, since we are interested in linearized general relativity. The supplementary condition (1.9) is not invariant under conformal rescalings, but the expression of the Eastwood–Singer gauge in the form (1.5b) suggests considering as a “candidate” for a conformally invariant gauge involving a higher-order operator the equation

\[ E^{abcd} \nabla_a \nabla_b \nabla_c \nabla_d W^e + \left[ \left( \nabla_p T^{pebc} \right) + T^{pebc} \nabla_p \right] h_{bc} = 0. \quad (1.10) \]

More precisely, Eq. (1.10) is obtained from Eq. (1.5b) by applying the replacement prescriptions

\[ g^{ab} \to E^{abcd}, \ A_b \to h_{ab}, \ \nabla^b A_b \to W^e, \]

with \( T^{pebc} \) a rank-four tensor field obtained from the Riemann tensor, the Ricci tensor, the trace of Ricci and the metric. In other words, \( T^{pebc} \) is expected to include all possible contributions of the kind \( R^{pebc}, R^{pe} g^{bc}, R g^{pe} g^{bc} \). We will however see in Sec. 5 that \( T^{pebc} \) is even more involved.

When a supplementary (or gauge) condition is imposed in a theory with gauge freedom, one of the first problems is to make sure that such a condition is preserved under the action of the gauge symmetry. More precisely, either the gauge is originally satisfied, and hence also the gauge-equivalent field configuration should fulfill the condition, or the gauge is not originally satisfied, but one wants to prove that, after performing a gauge transformation, it is always possible to fulfill the supplementary condition, eventually. The latter problem is the most general, and has a well known counterpart already for Maxwell theory (see Sec. 6.5 of Ref. 8). For linearized classical general relativity in the family of gauges described by Eq. (1.10), the gauge symmetry remains the request of invariance under infinitesimal diffeomorphisms. Their effect on metric perturbations is given by

\[ \varphi h_{ab} \equiv h_{ab} + (L_\varphi h)_{ab} = h_{ab} + \nabla_{(a} \varphi_{b)}. \quad (1.11) \]
For some smooth metric perturbation one might indeed have (cf. Eq. (1.10))

$$E^{abcd}\nabla_a \nabla_b \nabla_c \nabla_d W^e(h) + \left[ (\nabla_p T^{pebc}) + T^{pebc} \nabla_p \right] h_{bc} \neq 0. \quad (1.12)$$

We would like to prove that one can, nevertheless, achieve the condition

$$E^{abcd}\nabla_a \nabla_b \nabla_c \nabla_d W^e(\varphi h) + \left[ (\nabla_p T^{pebc}) + T^{pebc} \nabla_p \right] \varphi h_{bc} = 0. \quad (1.13)$$

Equation (1.13) is conveniently re-expressed in a form where the left-hand side involves a differential operator acting on the 1-form $\varphi q$, and the right-hand side depends only on metric perturbations, their covariant derivatives and the Riemann curvature. Explicitly, one finds

$$P^q\varphi_q = -F_e, \quad (1.14)$$

where (hereafter $h$ is the trace $g^{ab}h_{ab}$)

$$P^q = (\nabla^{c} \nabla^{d})\nabla_{c} \nabla_{d} + \frac{\alpha}{2} \Box^2 \left( \delta^q \Box + \nabla^q \nabla_e + \alpha \nabla_e \nabla^q \right)$$

$$+ 2T^{pq}_{\ b} \nabla_b + 2T^{pq}_{\ c} \nabla_p \nabla_b, \quad (1.15)$$

$$F_e \equiv 2 \left( \nabla^{c} \nabla^{d} \nabla_{c} \nabla_{d} + \frac{\alpha}{2} \Box^2 \right) \left( \nabla^q h_{qe} + \frac{\alpha}{2} \nabla_e h \right)$$

$$+ 2T^{pq}_{\ bc} h_{bc} + 2T^{pq}_{\ b} \nabla_p h_{bc}. \quad (1.16)$$

Section 2 solves Eq. (1.14) via Fourier transform when the Riemann curvature of the background vanishes and the metric $g$ is positive-definite. Linearized theory about $m$-dimensional Minkowski space-time is studied in Sec. 3. Section 4 shows how to solve Eq. (1.14) in curved Riemannian backgrounds without boundary, and the construction of conformally invariant gauges is obtained in Sec. 5 in non-local form. Concluding remarks are presented in Sec. 6.

2. Linearized Theory about Flat Euclidean Space

It may be helpful to begin the analysis of Eq. (1.14) in the limiting case when the Riemann curvature of the background geometry $(M, g)$ vanishes. This means that one is considering
linearized theory about flat space-time, or flat space if $g$ is taken to be positive-definite. It remains useful, however, to use a notation in terms of covariant derivatives in $P^e_q$ and $F^e$, not only to achieve covariance, but also because the flat background might have a curved boundary $\partial M$, and hence $\nabla_c$ might be re-expressed in terms of covariant derivatives with respect to the induced connection on $\partial M$, after taking into account the extrinsic curvature of $\partial M$.

Under the above assumptions, Eq. (1.14) becomes a partial differential equation involving a sixth-order differential operator with constant coefficients. It is therefore convenient to take the Fourier transform (denoted by a tilde) of both sides of Eq. (1.14), because it is well known that the Fourier transform turns a constant coefficient differential operator into a multiplication operator. Bearing in mind the definitions (hereafter $E^m$ is flat $m$-dimensional Euclidean space)

\[
\tilde{H}_q(\xi) \equiv (2\pi)^{-\frac{m}{2}} \int_{E^m} H_q(x)e^{-i\xi_a x^a} dx^1...dx^m,
\]

(2.1)

\[
H_q(x) \equiv (2\pi)^{-\frac{m}{2}} \int_{E^m} \tilde{H}_q(\xi)e^{i\xi_a x^a} d\xi_1...d\xi_m,
\]

(2.2)

where $H_q$ may be $\varphi_q$ or $F_q$, and $\xi \in T^*(M)$, one then finds the equation

\[
\sigma[P^e_q](\xi)\tilde{\varphi}_q(\xi) = -\tilde{F}_e(\xi),
\]

(2.3)

where $\sigma[P^e_q](\xi)$ is the symbol of the operator $P^e_q$, obtained by replacing $\nabla_a$ with $i\xi_a$. In other words, we insert Eq. (2.2) and the first lines of Eqs. (1.15) and (1.16) into Eq. (1.14). This leads to Eq. (2.3), where

\[
\sigma[P^e_q](\xi) = -\left(1 + \frac{\alpha}{2}\right)(\xi_a\xi^a)^3 \left[\delta^q_e + (1 + \alpha)\frac{\xi^e\xi^q}{\xi_a\xi^a}\right].
\]

(2.4)

Our aim is now to solve Eq. (2.3) for $\tilde{\varphi}_q(\xi)$, and eventually anti-transform to get $\varphi_q(x)$. For this purpose, we first study a background metric $g$ which is Riemannian. This implies that $\xi_a\xi^a = g(\xi, \xi) \neq 0$ for all $\xi \neq 0$, and hence one has to rule out the value $\alpha = -2$ of
the parameter $\alpha$ in the supermetric (1.8) to obtain a well defined inverse of $\sigma[P_e^q]$. Under these assumptions one finds, after setting $\rho q(e)(\xi) \equiv \sigma[P_q^e](\xi)$, (2.5)

that the inverse of the symbol is

$$
\rho^{-1} q(e) = -\left(1 + \frac{\alpha}{2}\right)^{-1} (\xi_a \xi^a)^{-3} \left[ \delta q^s - \frac{(1 + \alpha) \xi q \xi^s}{(2 + \alpha) \xi_a \xi^a} \right].
$$

(2.6)

Thus, equation (2.3) can be solved for the Fourier transform of $\varphi_q$ as

$$
\tilde{\varphi}_q(\xi) = \left(1 + \frac{\alpha}{2}\right)^{-1} (\xi_a \xi^a)^{-3} \left[ \delta q^s - \frac{(1 + \alpha) \xi q \xi^s}{(2 + \alpha) \xi_a \xi^a} \right] \tilde{F}_s(\xi),
$$

(2.7)

for all $\xi \neq 0$. This leads, by inverse Fourier transform, to the desired formula for the 1-form $\varphi_q(x)$, i.e. (see (1.16) and (2.2))

$$
\varphi_q(x) = \frac{(2\pi)^{-\frac{m}{2}}}{(1 + \frac{\alpha}{2})} \int_{E^m} |\xi|^{-6} \left[ \delta q^s - \frac{(1 + \alpha) \xi q \xi^s}{(2 + \alpha) |\xi|^2} \right] \tilde{F}_s(\xi)e^{i\xi \cdot x} d\xi,
$$

(2.8)

where we have defined $|\xi|^2 \equiv \xi_a \xi^a$, $\xi \cdot x \equiv \xi_a x^a$ and $d\xi \equiv d\xi_1...d\xi_m$. The only poles of the integrand occur when

$$
\xi_0 = \pm i \sqrt{\sum_{k=1}^{m-1} \xi_k \xi^k},
$$

i.e. on the imaginary $\xi_0$ axis. Thus, integration on the real line for $\xi_0$, and subsequent integration with respect to $\xi_1, ..., \xi_{m-1}$, yields a well defined integral representation of $\varphi_q$. This is entirely analogous to the integration performed in quantum field theory to define the Euclidean form of the Feynman Green function (see section 2.7 of Ref. 9, first item).

In particular, when $\alpha = -1$, the operator on $\varphi_q$ reduces to the cubic Laplacian, i.e. the Laplacian composed twice with itself:
Note, however, that Eq. (2.8) does not yield the general solution of Eq. (1.14). For this purpose, one has to add to (2.8) the general solution of the homogeneous equation

\[ P^b_a \varphi_b(x) = 0. \quad (2.9) \]

In particular, if \( \alpha \) is set equal to \(-1\), Eqs. (1.15) and (2.9) lead to

\[ \Box^3 \varphi_a(x) = 0 \quad \forall \ a = 1, \ldots, m. \quad (2.10) \]

Since we are considering flat space in Cartesian coordinates, as is clear already from the definitions (2.1) and (2.2), Eq. (2.10) reads, explicitly,

\[ g^{bc} g^{de} g^{fh} \frac{\partial^6 \varphi_a(x)}{\partial x^b \partial x^c \partial x^d \partial x^e \partial x^f \partial x^h} = 0 \quad \forall \ a = 1, \ldots, m. \quad (2.11) \]

Thus, all components of \( \varphi_a \) should be represented by 3-harmonic functions, which, by definition, satisfy the equation

\[ \Box^3 f(x) = 0. \quad (2.12) \]

What we need is the following structural property:

**Theorem 2.1.** Every 3-harmonic function \( f \) in \( \mathbf{E}^m \) is completely determined by three harmonic functions and by the Green kernel of the Laplacian.

**Proof.** Let us define the functions \( u \) and \( v \) by the equations

\[ u(x) \equiv \Box f(x), \quad (2.13) \]
\[ v(x) \equiv \Box u(x). \quad (2.14) \]

We then find, by virtue of Eq. (2.12), that \( v \) is harmonic:

\[ 0 = \Box^3 f(x) = \Box^2 u(x) = \Box v(x). \quad (2.15) \]

We now use the Green kernel \( G(x, y) \) of the Laplacian, for which \( (\delta(x, y) \) being the Dirac distribution)

\[ \Box_x G(x, y) = \delta(x, y), \quad (2.16) \]
where the subscript for the Laplacian is used to denote its action as a differential operator on the coordinates of the first argument of the kernel. The function $u$ can be then expressed as ($dy = dy_1...dy_m$ being the integration measure on $E^m$)

$$u(x) = \Box^{-1}v(x) = \int_{E^m} G(x, y) v(y) dy,$$

and hence, from (2.13),

$$f(x) = \Box^{-1}u(x) = \int_{E^m} G(x, y) u(y) dy = \int_{E^m} \int_{E^m} G(x, y)G(y, z)v(z) dy dz.$$  

(2.18)

This is not, however, the most general solution of Eq. (2.12). One can in fact add to the integral (2.18) a harmonic function $f_1$ and a bi-harmonic function $f_2$, because, if $h$ is a $n$-harmonic function, for which ($n$ being $\geq 1$)

$$\Box^n h(x) = 0,$$

then $h$ is also $(n + m)$-harmonic, with $m \geq 1$, whereas the converse does not necessarily hold. In our case, since $n = 3$, this implies that the general solution of Eq. (2.12) can be written as

$$f(x) = f_1(x) + f_2(x) + \int_{E^m} \int_{E^m} G(x, y)G(y, z)v(z) dy dz,$$

(2.20)

where $f_1$ is harmonic and $f_2$ is bi-harmonic:

$$\Box f_1(x) = 0,$$

(2.21)

$$\Box^2 f_2(x) = 0.$$  

(2.22)

We now apply the same procedure to $f_2$, to write it as

$$f_2(x) = g_1(x) + \int_{E^m} G(x, y)w(y) dy,$$

(2.23)
where $g_1$ and $w$ are harmonic. By virtue of (2.20) and (2.23) we can write

$$f(x) = \Omega(x) + \int_{E^m} G(x, y)w(y)dy + \int_{E^m} \int_{E^m} G(x, y)G(y, z)v(z)dy dz,$$

(2.24)

where $\Omega$ is the harmonic function equal to $f_1 + g_1$. This completes the proof of the result we needed (cf. Ref. 10).

The general solution of Eq. (1.14) reads therefore, when $\alpha = -1$ (see (2.8)),

$$\varphi_a(x) = \Omega_a(x) + \int_{E^m} G^b_a(x, y)w_b(y)dy + \int_{E^m} \int_{E^m} G^b_a(x, y)G^c_b(y, z)v_c(z)dy dz + 2(2\pi)^{-\frac{m}{2}} \int_{E^m} |\xi|^{-6} \tilde{F}_a(\xi)e^{i\xi \cdot x} d\xi,$$

(2.25)

where $\Omega_a, w_a$ and $v_a$ are harmonic 1-forms in $E^m$.

In the applications, it may be useful to consider the Euclidean 4-ball, i.e. a portion of flat Euclidean 4-space bounded by a 3-sphere of radius $a$. The cubic Laplacian on normal and tangential components of $\varphi_a$ leads to sixth-order operators with a regular singular point at $\tau = 0$, where $\tau$ is the radial coordinate lying in the closed interval $[0, a]$. After defining the new independent variable $T \equiv \log(\tau)$, the differential equations involving such operators are mapped into equations with constant coefficient operators (see Ref. 11 for the $\Box$ and $\Box^2$ operators), and hence one can use again Fourier transform techniques to find a particular solution of Eq. (1.14). The general structure of $\varphi_a(x)$ is therefore elucidated quite well by the result (2.25) in Cartesian coordinates. Strictly speaking, we should of course replace $\varphi_a$ by $\varphi_a dx^a$, since only the latter represents a 1-form. The same holds for all the other 1-forms considered hereafter.
3. Linearized Theory About Flat Space-Time

If the background metric is Minkowskian, one can first define the partial Fourier transform with respect to the time variable, according to the rule

\[
\tilde{\varphi}_a(\vec{x}, \omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_a(\vec{x}, t)e^{i\omega t} dt,
\]

(3.1)

where \(\vec{x} \equiv (x^1, ..., x^{m-1})\), with the corresponding anti-transform

\[
\varphi_a(\vec{x}, t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\varphi}_a(\vec{x}, \omega)e^{-i\omega t} d\omega.
\]

(3.2)

The operator \(P^b_a\) is studied in \(m\)-dimensional Minkowski space-time and, for simplicity, we set \(\alpha = -1\) (it remains necessary to rule out \(\alpha = -2\) to obtain a meaningful solution). Equation (1.14) becomes, therefore,

\[
\Box^3 \varphi_a(\vec{x}, t) = -2F_a(\vec{x}, t),
\]

(3.3)

where \(\Box\) is now the wave operator, i.e. (here \(x^0 = ct\))

\[
\Box \equiv -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \sum_{i,j=1}^{m-1} g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \equiv -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \Delta.
\]

(3.4)

By virtue of (3.2) and (3.4), Eq. (3.3) leads to the following equation for the partial Fourier transform \(\tilde{\varphi}_a(\vec{x}, \omega)\):

\[
\left[ \triangle^3 + \frac{3 \omega^2}{c^2} \triangle^2 + \frac{3 \omega^4}{c^4} \triangle + \frac{\omega^6}{c^6} \right] \tilde{\varphi}_a(\vec{x}, \omega) = -2\tilde{F}_a(\vec{x}, \omega).
\]

(3.5)

One can now use a Green-function approach, and look for the Green function of the operator on the left-hand side of (3.5). On setting \(R \equiv |\vec{x} - \vec{x}'|\), \(k \equiv \frac{\omega}{c}\), this is a solution, for all \(R \neq 0\), of the equation

\[
\left[ \triangle^3 + 3k^2 \triangle^2 + 3k^4 \triangle + k^6 \right] G_k(R) = 0,
\]

(3.6)
where

\[ \triangle = \frac{d^2}{dR^2} + \frac{(m - 1)}{R} \frac{d}{dR}. \]  

(3.7)

The desired Green function can be factorized in the form

\[ G_k(R) = R^\delta F(R), \]  

(3.8)

and the parameter \( \delta \) is eventually found to depend on \( m \), to obtain a simplified form of the sixth-order differential operator under investigation. For example, for the Helmholtz equation

\[ (\triangle + k^2)G_k(R) = 0 \quad \forall R \neq 0, \]

associated to the wave equation, this method yields

\[ G_k(R) = r^{-\frac{(m-1)}{2}} e^{\pm ikR}. \]

In our case, however, Eqs. (3.6) and (3.7) lead to a lengthy calculation, whereas the partial Fourier transform of Eq. (3.5) with respect to the spatial variables leads more quickly to the Lorentzian counterpart of Eq. (2.8). In other words, it is convenient to define (here \( \vec{\xi} \equiv (\xi_1, ..., \xi_{m-1}) \))

\[ \hat{\varphi}_a(\vec{\xi}, \omega) \equiv (2\pi)^{-\frac{(m-1)}{2}} \int_{\mathbb{R}^{m-1}} \tilde{\varphi}_a(\vec{x}, \omega)e^{-i\xi_k x^k} d\vec{x}^1 ... d\vec{x}^{m-1}, \]  

(3.9)

and the associated Fourier anti-transform

\[ \tilde{\varphi}_a(\vec{x}, \omega) \equiv (2\pi)^{-\frac{(m-1)}{2}} \int_{\mathbb{R}^{m-1}} \hat{\varphi}_a(\vec{\xi}, \omega)e^{i\xi_k x^k} d\xi_1 ... d\xi_{m-1}. \]  

(3.10)

Equations (3.5) and (3.10) lead to

\[ \left[ -|\vec{\xi}|^6 + 3\xi_0^2|\vec{\xi}|^4 - 3\xi_0^4|\vec{\xi}|^2 + \xi_0^6 \right] \hat{\varphi}_a(\vec{\xi}, \omega) = -2\hat{F}_a(\vec{\xi}, \omega), \]  

(3.11)
where $|\vec{\xi}|^2 \equiv \sum_{i,j=1}^{m-1} g^{ij} \xi_i \xi_j$, $\xi_0 = \frac{\omega}{c}$. Some care is necessary to invert this equation, because the polynomial on the left-hand side is equal to $-\left[|\vec{\xi}|^2 - \xi_0^2 \right]^3$, which vanishes if

$$\xi_0 = -\sqrt{|\vec{\xi}|^2} = \xi_{0,1}, \quad (3.12)$$

or

$$\xi_0 = +\sqrt{|\vec{\xi}|^2} = \xi_{0,2}. \quad (3.13)$$

This means that the $\xi_0$ integration is singular and there are two poles on the real $\xi_0$ axis at the points given by (3.12) and (3.13). In other words, the integration with respect to $\xi_0$ is a contour integral which may be performed by deforming the contour around the poles. The way in which this deformation is performed determines the particular solution of Eq. (3.3). To deal with the poles, we choose a Feynman-type contour that comes from $-\infty - i\varepsilon$, encircles the points $\xi_{0,1}$ and $\xi_{0,2}$ from below and from the above, respectively, and goes off to $+\infty + i\varepsilon$. This operation is denoted with the standard $+i\varepsilon$ symbol, with a contour that we call $\gamma_m$. Thus, by virtue of (3.2), (3.10) and (3.11) we find, if $\alpha = -1$, a particular solution of Eq. (1.14) in the form (cf. the third line of (2.25))

$$\varphi_a(\vec{x}, t) = 2(2\pi)^{-\frac{m}{2}} \int_{\gamma_m} \frac{1}{[(|\vec{\xi}|^2 - \xi_0^2)^3 + i\varepsilon]} \hat{F}_a(\vec{\xi}, \omega) e^{ig^{cd}\xi_c x_d} d\xi. \quad (3.14)$$

With this understanding, the complete solution of Eq. (1.14) when $\alpha = -1$ is formally similar to Eq. (2.25), but bearing in mind that $G_a^b(\vec{x}, t; \vec{y}, t')$ is now the Green kernel of the wave operator, and that $\Omega_a, w_a$ and $v_a$ are three solutions of the homogeneous wave equation $\Box A_a = 0$. Unlike the case of Euclidean background, various boundary conditions, and hence various choices of contour, lead to different solutions.

4. Generalization to Curved Backgrounds

If one studies linearized gravity about curved backgrounds $M$, it is no longer possible to express the Fourier transform of Eq. (1.14) in the form (2.3), because the second line of Eq. (1.15) contributes terms which cannot be factorized under Fourier transform. However,
it remains true that the operator $P_e^q$ is the sum of a sixth-order differential operator $A_e^q$ with constant coefficients and other terms

$$
B_e^q \equiv 2T_e^{(bq)} \nabla_b + 2T_e^{(bq)} \nabla_p \nabla_b, \tag{4.1}
$$

involving curvature and lower-order covariant derivatives. We are therefore studying the partial differential equation

$$
\left( A_e^q + B_e^q \right) \varphi_q(x) = -F_e(x). \tag{4.2}
$$

Hereafter we focus on Riemannian backgrounds (i.e. with positive-definite metric) so as to be able to use some well established results which rely on the theory of elliptic operators on compact Riemannian manifolds without boundary. By construction, the symbol of $A_e^q$ coincides with the expression (2.4), and hence the operator $A_e^q$ is invertible if and only if $\alpha \neq -2$. Let $G_e^q(x,y)$ denote the corresponding Green kernel. The application of the inverse operator $A^{-1}$ to both sides of Eq. (4.2) yields therefore the integral equation

$$
\varphi_e(x) + \int_M G_e^p(x,y) \left( B_p^r \varphi_r \right)(y) \sqrt{g(y)} dy + \int_M G_e^p(x,y) F_p(y) \sqrt{g(y)} dy = 0, \tag{4.3}
$$

where $g(y)$ denotes the determinant of the background metric. To solve for $\varphi_e(x)$ we have to study the action of $B_p^r$ on $\varphi_r$. For this purpose, we assume that the curvature of the background is such that $B_p^r$ is a symmetric elliptic operator, so that a discrete spectral resolution with eigenvectors of class $C^\infty$ exists (see Sec. 1.6 of Ref. 4). The eigenvectors $\varphi_r^{(n)}(x)$, for which ($\lambda(n)$ denoting the eigenvalues)

$$
B_p^r \varphi_r^{(n)}(x) = \lambda(n) \varphi_r^{(n)}(x), \tag{4.4}
$$

make it then possible to expand $\varphi_r(x)$ in the form

$$
\varphi_r(x) = \sum_{n=1}^{\infty} a_{(n)} \varphi_r^{(n)}(x) + \overline{\varphi_r}(x), \tag{4.5}
$$
where \( \varphi_r(x) \in \text{Ker} B^r_p \), i.e. \( B^r_p \varphi_r(x) = 0 \). Note that the ellipticity condition means that the leading symbol of \( B^r_p \) is non-vanishing for all \( \xi_a \neq 0 \), i.e.

\[
-2 T^{(bq)}_e \xi_p \xi_b \neq 0 \text{ for } \xi \neq 0. \tag{4.6}
\]

This condition receives contributions from the parts of \( T \) involving the Ricci tensor and the scalar curvature, but not from the Riemann tensor, which is antisymmetric in \( b \) and \( q \). For a given choice of background with associated curvature and tensor \( T \), the above condition provides a useful operational criterion to check the admissibility of our supplementary condition (1.10). If the leading symbol of \( B^r_p \) is further taken to be positive-definite for \( \xi \neq 0 \), so as to ensure that the spectrum of \( B^r_p \) is bounded from below,\(^4\) one gets another useful operational criterion expressed by a majorization, i.e (cf. (4.6))

\[
- T^{(bq)}_e \xi_p \xi_b > 0 \text{ for } \xi \neq 0. \tag{4.7}
\]

By virtue of the above assumptions, on defining

\[
f_r(x) \equiv \int_M G_r(x, y) F_q(y) \sqrt{g(y)} dy, \tag{4.8}
\]

its expansion involves a set of coefficients \( f_{(n)} \) (replacing \( a_{(n)} \)), and possibly a remainder term \( \tilde{f}_r(x) \) (replacing \( \varphi_r(x) \)) in the kernel of \( B^r_p \):

\[
f_r(x) = \sum_{n=1}^{\infty} f_{(n)} \varphi_r^{(n)}(x) + \tilde{f}_r(x). \tag{4.9}
\]

Moreover, the insertion of (4.5) into Eq. (4.3) leads to the analysis of 1-form-valued “coefficients” \( \gamma_r^{(m)}(x) \) defined by

\[
\gamma_r^{(m)}(x) \equiv \int_M G_r(x, y) \varphi_p^{(m)}(y) \sqrt{g(y)} dy, \tag{4.10}
\]

which can be expanded in the form (cf. (4.5))

\[
\gamma_r^{(m)}(x) = \sum_{n=1}^{\infty} \gamma_r^{(m)(n)} \varphi_r^{(n)}(x) + \tilde{\gamma}_r^{(m)}(x), \tag{4.11}
\]
where $\gamma_r^{(m)}(x)$ belongs to the kernel of $B_p^r$. By virtue of (4.4)–(4.11), Eq. (4.3) may be re-expressed in the form

$$\sum_{n=1}^{\infty} \left( a_{(n)} + \sum_{m=1}^{\infty} \lambda_{(m)} a_{(m)} \gamma^{(m)(n)}(x) + f_{(n)} \right) \varphi_r^{(n)}(x)$$

$$+ \varphi_r(x) + \overline{f}_r(x) + \sum_{m=1}^{\infty} \lambda_{(m)} a_{(m)} \gamma_r^{(m)}(x) = 0.$$  \hspace{1cm} (4.12)

The first and second line of Eq. (4.12) should vanish separately, and hence one finds an infinite set of equations for the coefficients $a_{(n)}$:

$$\sum_{m=1}^{\infty} \left( \delta_{mn} + \lambda_{(m)} \gamma^{(m)(n)} \right) a_{(m)} = -f_{(n)}, \hspace{1cm} (4.13)$$

jointly with the equation

$$\varphi_r(x) = -\overline{f}_r(x) - \sum_{m=1}^{\infty} \lambda_{(m)} a_{(m)} \gamma_r^{(m)}(x). \hspace{1cm} (4.14)$$

Although the technical details remain elaborated, we have obtained a complete prescription for finding $\varphi_e(x)$ and hence proving the admissibility of our gauges in curved backgrounds.

First, we solve the inhomogeneous system (4.13) once the eigenvalues $\lambda_{(m)}$ of the operator $B_p^r$ are determined in the given background (this is already a hard task). The remaining part (if any) of $\varphi_e(x)$, i.e. the one in the kernel of $B_p^r$, is then evaluated from Eq. (4.14), and eventually the full $\varphi_e(x)$ is obtained from the expansion (4.5).

5. Construction of a Conformally Invariant Gauge

We now consider a metric $\gamma = \gamma_{ab} dx^a \otimes dx^b$ solving the full Einstein equations in vacuum ($\gamma$ is therefore the “physical metric”):

$$R_{ab}(\gamma) - \frac{1}{2} \gamma_{ab} R(\gamma) = 0. \hspace{1cm} (5.1)$$
On denoting again by $g_{ab}$ a background metric,\textsuperscript{12} we study a gauge condition linear in $\gamma$ having the form (cf. (1.10))

$$S^e(\gamma) \equiv E^{abcd}(g) \nabla_a \nabla_b \nabla_c \nabla_d W^e(\gamma) + \nabla_p \tilde{T}^{pe}(\gamma) = 0,$$

\hspace{2cm} (5.2)

where

$$W^e(\gamma) \equiv E^{epqr}(g) \nabla_p \gamma_{qr}.$$  \hspace{2cm} (5.3)

The tensor $\tilde{T}^{pe}(\gamma)$ is linear in $\gamma$ but can be more general than the combination $T^{pebc} \gamma_{bc}$, where $T^{pebc}$ is obtained by all possible permutations with different coefficients of $R^{a}{}_{bcd}(g)$, $R^{a}{}_{pe}(g) g^{bc}$, $R(g) g^{pe} g^{bc}$ (cf. comments following Eq. (1.10)). In other words, $\tilde{T}^{pe}(\gamma)$ may contain terms like

$$R^{pebc}(g), R^{pe}(g) g^{bc}, R(g) g^{pe} g^{bc}$$

plus many other contributions where the Riemann tensor $R^{a}{}_{bcd}$ built from the background metric is replaced by any tensor field $F^{a}{}_{bcd}$ of type $(1,3)$ and independent of the physical metric. Upon considering the conformal rescalings

$$\gamma_{ab} \rightarrow \Omega^2 \gamma_{ab}, \quad \gamma^{ab} \rightarrow \Omega^{-2} \gamma^{ab},$$

the terms we have written down explicitly have conformal weights $-2, 2, 2, 2$, respectively. Note that the connection $\nabla$ leads to covariant derivatives $\nabla_a$ with respect to the background metric $g$, subject to the condition

$$\nabla g = 0 \implies g_{bc;a} = 0,$$

\hspace{2cm} (5.4)

whereas the covariant derivatives of the physical metric $\gamma$ with respect to the background metric $g$ do not vanish: $\nabla \gamma \neq 0$.\textsuperscript{12} It will be shown that $\tilde{T}^{pe}(\gamma)$ depends also on $\nabla \gamma$ and is obtained from a non-local construction (see below).

We now study the behaviour of $S^e(\gamma)$ under conformal rescalings of the physical metric which solves Eq. (5.1). For this purpose, it is convenient to define

$$Q^e(\gamma) \equiv E^{abcd}(g) \nabla_a \nabla_b \nabla_c \nabla_d W^e(\gamma).$$

\hspace{2cm} (5.5)
We therefore find that

\[ Q^e(\Omega^2\gamma) = \Omega^2 Q^e(\gamma) + U^e(\gamma, \nabla^{(k)}\gamma, \Omega, \nabla^{(k)}\Omega), \quad (5.6) \]

where \( \nabla^{(k)} \) denotes covariant derivative of \( k \)-th order, with \( k = 1, 2, 3, 4 \), and \( U^e \) is a vector given by

\[ U^e \equiv E^{abcd}(g) E^{epqr}(g) Z_{abcdpqr}, \quad (5.7) \]

having defined

\[ Z_{abcdpqr} \equiv \sum_{k=1}^{5} Z^{(k)}_{abcdpqr}, \quad (5.8) \]

with

\[ Z^{(1)}_{abcdpqr} \equiv 2\Omega \left( \Omega; a \gamma_{qr; pdc} + \Omega; b \gamma_{qr; pda} + \Omega; c \gamma_{qr; pdb} + \Omega; d \gamma_{qr; pca} + \Omega; e \gamma_{qr; pdb} \right), \quad (5.9) \]

\[ Z^{(2)}_{abcdpqr} \equiv 2 \left[ \left( \Omega; d \Omega; p + \Omega; p \Omega; d \right) \gamma_{qr; cba} + \left( \Omega; c \Omega; p + \Omega; p \Omega; c \right) \gamma_{qr; dba} \right. \]
\[ + \left. \left( \Omega; b \Omega; p + \Omega; p \Omega; b \right) \gamma_{qr; dca} + \left( \Omega; a \Omega; p + \Omega; p \Omega; a \right) \gamma_{qr; dcb} \right. \]
\[ + \left. \left( \Omega; c \Omega; d + \Omega; d \Omega; c \right) \gamma_{qr; pba} + \left( \Omega; a \Omega; d + \Omega; d \Omega; a \right) \gamma_{qr; pca} \right. \]
\[ + \left. \left( \Omega; a \Omega; d + \Omega; d \Omega; a \right) \gamma_{qr; pcb} + \left( \Omega; b \Omega; c + \Omega; c \Omega; b \right) \gamma_{qr; pda} \right. \]
\[ + \left. \left( \Omega; a \Omega; c + \Omega; c \Omega; a \right) \gamma_{qr; pdb} + \left( \Omega; a \Omega; b + \Omega; b \Omega; a \right) \gamma_{qr; pdc} \right], \quad (5.10) \]

\[ Z^{(3)}_{abcdpqr} \equiv 2 \left[ \left( \Omega; d \Omega; p + \Omega; p \Omega; d + \Omega; e \Omega; p + \Omega; p \Omega; e \right) \gamma_{qr; ba} \right. \]
\[ + \left. \left( \Omega; d \Omega; p + \Omega; p \Omega; d + \Omega; b \Omega; p + \Omega; p \Omega; b \right) \gamma_{qr; ca} \right. \]
\[ + \left. \left( \Omega; d \Omega; p + \Omega; p \Omega; d + \Omega; a \Omega; p + \Omega; p \Omega; a \right) \gamma_{qr; cb} \right. \]
\[ + \left. \left( \Omega; c \Omega; d + \Omega; d \Omega; c \right) \gamma_{qr; da} \right. \]
\[Z_{abcdpqr}^{(4)} \equiv 2 \left[ (\Omega_{dcb}\Omega_{:p} + \Omega_{:d}\Omega_{:pb} + \Omega_{:db}\Omega_{:pc} + \Omega_{:d}\Omega_{:pcb}) \gamma_{qr;db} + \Omega_{:cb}\Omega_{:pd} + \Omega_{:c}\Omega_{:pdb} + \Omega_{:b}\Omega_{:pdc} + \Omega_{:d}\Omega_{:pdc}) \gamma_{qr;a} + (\Omega_{dca}\Omega_{:p} + \Omega_{:d}\Omega_{:pa} + \Omega_{:da}\Omega_{:pc} + \Omega_{:d}\Omega_{:pca}) \gamma_{qr;b} + \Omega_{:b}\Omega_{:pd} + \Omega_{:b}\Omega_{:pdb} + \Omega_{:d}\Omega_{:pdc} + \Omega_{:d}\Omega_{:pdb}) \gamma_{qr;c} + \Omega_{:cba}\Omega_{:p} + \Omega_{:cb}\Omega_{:pa} + \Omega_{:ca}\Omega_{:pb} + \Omega_{:c}\Omega_{:pba} + \Omega_{:ba}\Omega_{:pc} + \Omega_{:b}\Omega_{:pca} + \Omega_{:a}\Omega_{:pcb} + \Omega_{:c}\Omega_{:pca} \right \gamma_{qr:d} + \Omega_{:cb}\Omega_{:id} + \Omega_{:cb}\Omega_{:da} + \Omega_{:ca}\Omega_{:db} + \Omega_{:c}\Omega_{:dba} + \Omega_{:ba}\Omega_{:dc} + \Omega_{:b}\Omega_{:dca} + \Omega_{:a}\Omega_{:dcb} + \Omega_{:c}\Omega_{:dcba}) \gamma_{qr;p} \right], \quad (5.11)\]

\[Z_{abcdpqr}^{(5)} \equiv 2 \left[ (\Omega_{dcb}\Omega_{:p} + \Omega_{:c}\Omega_{:pa} + \Omega_{:d}\Omega_{:pb} + \Omega_{:d}\Omega_{:pca}) \gamma_{qr;db} + \Omega_{:cb}\Omega_{:pd} + \Omega_{:c}\Omega_{:pdb} + \Omega_{:b}\Omega_{:pdc} + \Omega_{:d}\Omega_{:pdc}) \gamma_{qr;a} \right], \quad (5.12)\]
\[ + \Omega;_{cb}\Omega;_{pda} + \Omega;_{da}\Omega;_{pcb} + \Omega;_{db}\Omega;_{pca} + \Omega;_{dc}\Omega;_{pba} + \Omega;_{a}\Omega;_{pdcb} + \Omega;_{b}\Omega;_{pdca} + \Omega;_{c}\Omega;_{pdba} + \Omega;_{d}\Omega;_{pcba} + \Omega;\Omega;_{pdcb} \right] \gamma_{qr}. \tag{5.13} \]

The previous formulae imply that
\[
S^e(\Omega^2 \gamma) \equiv Q^e(\Omega^2 \gamma) + \nabla_p \tilde{T}^{pe}(\Omega^2 \gamma) \\
= \Omega^2 Q^e(\gamma) + U^e + \nabla_p \tilde{T}^{pe}(\Omega^2 \gamma). \tag{5.14} \]

On the other hand, our gauge is invariant under conformal rescalings of \( \gamma \) if and only if
\[
S^e(\Omega^2 \gamma) = \Omega^2 \left( Q^e(\gamma) + \nabla_p \tilde{T}^{pe}(\gamma) \right). \tag{5.15} \]

By virtue of Eqs. (5.14) and (5.15), the desired tensor \( \tilde{T}^{pe}(\gamma) \) should obey the equation
\[
\nabla_p \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \nabla_p \tilde{T}^{pe}(\gamma) = -U^e(\gamma, \nabla^{(k)} \gamma, \Omega, \nabla^{(k)} \Omega). \tag{5.16} \]

We now use the identity
\[
\nabla_p(\Omega^2 \tilde{T}^{pe}(\gamma)) = 2\Omega \Omega_p \tilde{T}^{pe}(\gamma) + \Omega^2 \nabla_p \tilde{T}^{pe}(\gamma) \tag{5.17} \]

to re-express Eq. (5.16) in the form
\[
\nabla_p \left( \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right) + 2\Omega \Omega_p \tilde{T}^{pe}(\gamma) = -U^e. \tag{5.18} \]

At this stage we consider a vector \( f^e \) on \((M, g)\), with associated covector \( f_e \equiv g_{ep}f^p \) having non-vanishing contraction with \( U^e \). The Leibniz rule yields therefore
\[
\nabla_p \left( f_e \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 f_e \tilde{T}^{pe}(\gamma) \right) - \left( \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right) f_{e;p} \\
+ 2\Omega \Omega_p f_e \tilde{T}^{pe}(\gamma) = -U^e f_e. \tag{5.19} \]

Thus, on defining
\[
C^p \equiv f_e \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 f_e \tilde{T}^{pe}(\gamma), \tag{5.20} \]

\[ + \Omega;_{cb}\Omega;_{pda} + \Omega;_{da}\Omega;_{pcb} + \Omega;_{db}\Omega;_{pca} + \Omega;_{dc}\Omega;_{pba} + \Omega;_{a}\Omega;_{pdcb} + \Omega;_{b}\Omega;_{pdca} + \Omega;_{c}\Omega;_{pdba} + \Omega;_{d}\Omega;_{pcba} + \Omega;\Omega;_{pdcb} \right] \gamma_{qr}. \tag{5.13} \]

The previous formulae imply that
\[
S^e(\Omega^2 \gamma) \equiv Q^e(\Omega^2 \gamma) + \nabla_p \tilde{T}^{pe}(\Omega^2 \gamma) \\
= \Omega^2 Q^e(\gamma) + U^e + \nabla_p \tilde{T}^{pe}(\Omega^2 \gamma). \tag{5.14} \]

On the other hand, our gauge is invariant under conformal rescalings of \( \gamma \) if and only if
\[
S^e(\Omega^2 \gamma) = \Omega^2 \left( Q^e(\gamma) + \nabla_p \tilde{T}^{pe}(\gamma) \right). \tag{5.15} \]

By virtue of Eqs. (5.14) and (5.15), the desired tensor \( \tilde{T}^{pe}(\gamma) \) should obey the equation
\[
\nabla_p \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \nabla_p \tilde{T}^{pe}(\gamma) = -U^e(\gamma, \nabla^{(k)} \gamma, \Omega, \nabla^{(k)} \Omega). \tag{5.16} \]

We now use the identity
\[
\nabla_p(\Omega^2 \tilde{T}^{pe}(\gamma)) = 2\Omega \Omega_p \tilde{T}^{pe}(\gamma) + \Omega^2 \nabla_p \tilde{T}^{pe}(\gamma) \tag{5.17} \]

to re-express Eq. (5.16) in the form
\[
\nabla_p \left( \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right) + 2\Omega \Omega_p \tilde{T}^{pe}(\gamma) = -U^e. \tag{5.18} \]

At this stage we consider a vector \( f^e \) on \((M, g)\), with associated covector \( f_e \equiv g_{ep}f^p \) having non-vanishing contraction with \( U^e \). The Leibniz rule yields therefore
\[
\nabla_p \left( f_e \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 f_e \tilde{T}^{pe}(\gamma) \right) - \left( \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right) f_{e;p} \\
+ 2\Omega \Omega_p f_e \tilde{T}^{pe}(\gamma) = -U^e f_e. \tag{5.19} \]

Thus, on defining
\[
C^p \equiv f_e \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 f_e \tilde{T}^{pe}(\gamma), \tag{5.20} \]
integration over \((M, g)\) of both sides of Eq. (5.19) and application of the divergence theorem yield \((N_b \text{ being the normal to } \partial M)\)

\[
\int_{\partial M} \left[ f_e \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 f_e \tilde{T}^{pe}(\gamma) \right] N_p \, d\sigma = - \int_M U^e f_e \, dV
\]

\[=- 2 \int_M \Omega \Omega_p f_e \tilde{T}^{pe}(\gamma) \, dV\]

\[+ \int_M f_{e;p} \left[ \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right] \, dV,
\]

(5.21)

where the left-hand side results from

\[
\int_M C^{p;p} \, dV.
\]

The tensor \(\tilde{T}^{pe}(\gamma)\) which solves the problem of finding a “compensating term” \(\nabla_p \tilde{T}^{pe}(\gamma)\) such that the conformal invariance condition (5.15) is fulfilled is therefore obtained implicitly in non-local form, by solving the integral equation (5.21). The integration measure \(dV\) in Eq. (5.21) involves the background metric \(g\) because we started from Eq. (5.18) which contains covariant derivatives that annihilate \(g\). Note that, if \(f^e\) were a Killing vector for \((M, g)\), only the antisymmetric part of \(\tilde{T}^{pe}(\gamma)\) would contribute to the last integral in Eq. (5.21).

At this stage, the reader might still be wondering whether the proof of conformal invariance has been actually obtained, because Eq. (5.21) is very complicated, and both members involve integrals. We can now take advantage of the arbitrariness in the choice of \(\tilde{T}^{pe}(\gamma)\), since it is sufficient to show that a class of \(\tilde{T}^{pe}(\gamma)\) tensors exist for which Eq. (5.18) (and hence Eq. (5.15)) is satisfied. For this purpose, we assume that

\[
\nabla_q \left[ \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right] + 2\Omega \Omega_q \tilde{T}^{pe}(\gamma) = - \frac{1}{m} \delta^p_q U^e.
\]

(5.22)

Of course, Eq. (5.22) is not implied by Eq. (5.18), but on setting \(q = p\) and summing over \(p\) one recovers Eq. (5.18). We can now multiply both sides of Eq. (5.22) by \(f^q\) and contract over the index \(q\). Upon setting

\[
\mathcal{D} \equiv f^q \nabla_q
\]

(5.23)
the resulting equation reads

$$D \left[ \tilde{T}^{pe}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{pe}(\gamma) \right] + 2\Omega(D\Omega)\tilde{T}^{pe}(\gamma) = -\frac{1}{m} f^p U^e. \quad (5.24)$$

This suggests applying the inverse operator $D^{-1}$ to both sides of Eq. (5.24) to solve for $\tilde{T}^{pe}(\gamma)$ with the help of an integral equation. However, we have to turn the resulting equation into an equation for differential forms, since otherwise the integration over $M$ is ill-defined. For this purpose, we define (the symbol $\otimes_s$ denotes symmetrized tensor product)

$$\tilde{T}^{(\_)} \equiv \tilde{T}_{pe} dx^p \otimes_s dx^e = \tilde{T}_{(pe)} dx^p \otimes_s dx^e, \quad (5.25)$$

$$\tilde{T}^{\wedge} \equiv \tilde{T}_{pe} dx^p \wedge dx^e = \tilde{T}_{[pe]} dx^p \wedge dx^e, \quad (5.26)$$

$$(fU)^{\_} \equiv f_p U_e dx^p \otimes_s dx^e = f_{[pe]} U_e dx^p \otimes_s dx^e, \quad (5.27)$$

$$(fU)^{\wedge} \equiv f_p U_e dx^p \wedge dx^e = f_{[pe]} U_e dx^p \wedge dx^e. \quad (5.28)$$

Thus, since the operator $D^{-1}$ is an integral operator with kernel equal to the Green function $G_D(x, y)$ of the operator $D$, we find

$$\left[ \tilde{T}^{(\_)}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{(\_)}(\gamma) \right](x) + 2 \int_M G_D(x, y)[\Omega D\Omega](y)[\tilde{T}^{(\_)}(\gamma)](y)dV(y)$$

$$= -\frac{1}{m} \int_M G_D(x, y)(fU)^{\_}(y)dV(y), \quad (5.29)$$

$$\left[ \tilde{T}^{\wedge}(\Omega^2 \gamma) - \Omega^2 \tilde{T}^{\wedge}(\gamma) \right](x) + 2 \int_M G_D(x, y)[\Omega D\Omega](y)[\tilde{T}^{\wedge}(\gamma)](y)dV(y)$$

$$= -\frac{1}{m} \int_M G_D(x, y)(fU)^{\wedge}(y)dV(y), \quad (5.30)$$

where $dV(y) \equiv \sqrt{\det g(y)} \ dy_1...dy_m$ if $g$ is positive-definite. This form of the integral equations for the symmetric and anti-symmetric parts of $\tilde{T}_{pe}$ suggests defining the kernel

$$K_{\Omega}(x, y) \equiv 2G_D(x, y)[\Omega D\Omega](y) - \delta(x, y)\Omega^2(y), \quad (5.31)$$
where \( \delta(x, y) \) is the Dirac \( \delta \)-distribution. One can therefore re-express Eqs. (5.29) and (5.30) in the form

\[
\begin{align*}
\left[ \tilde{T}_\circ(\Omega^2 \gamma) \right](x) &+ \int_M K_\Omega(x, y) \left[ \tilde{T}_\circ(\gamma) \right](y) dV(y) \\
&= -\frac{1}{m} \int_M G_D(x, y)(fU)_\circ(y) dV(y),
\end{align*}
\]

where the symbol \( \circ \) is a concise notation for the subscript \( ( ) \) used in Eq. (5.29) or the subscript \( \wedge \) used in Eq. (5.30). The right-hand side of Eq. (5.32) is completely known for a given choice of the vector \( f^p \) and of the dimension \( m \) of \( M \) (see (5.7)–(5.13)).

We can now use a method similar to the one applied in the end of Sec. 4. For this purpose, we assume that the metric \( \gamma \) is positive-definite. If the operator \( D \) defined in (5.23) is symmetric and elliptic on a compact Riemannian manifold without boundary, it admits a discrete spectral resolution with \( C^\infty \) eigenvectors \( \beta^{(n)}(x) \) satisfying the eigenvalue equation

\[
D \beta^{(n)}(x) = \lambda^{(n)} \beta^{(n)}(x).
\]

It is then possible to consider the expansions

\[
\begin{align*}
\left[ \tilde{T}_\circ(\gamma) \right](x) &\equiv \sum_{n=1}^\infty \tilde{T}_{(n, \circ)}(\gamma) \beta^{(n)}(x), \\
\left[ \tilde{T}_\circ(\Omega^2 \gamma) \right](x) &\equiv \sum_{n=1}^\infty \tilde{T}_{(n, \circ)}(\Omega^2 \gamma) \beta^{(n)}(x),
\end{align*}
\]

where we assume that the right-hand sides have no part belonging to the kernel of \( D \) (since otherwise the resulting algorithm would not lead to algebraic equations, because \( K_\Omega(x, y) \) does not annihilate any function \( u \) such that \( Du = 0 \)). This is equivalent to choosing the vector \( f^e \) so that the resulting operator \( D \) (see (5.23)) has no zero-modes. Moreover, on defining

\[
f(x) \equiv \frac{1}{m} \int_M G_D(x, y)(fU)_\circ(y) dV(y),
\]

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we can further expand the coefficient \( \rho^{(n)}(x) \) in terms of the eigenvectors of \( \mathcal{D} \) (cf. (4.11)), i.e.

\[
\rho^{(n)}(x) = \sum_{q=1}^{\infty} \rho^{(q)}(x) \beta^{(q)}(x),
\]

while for \( f(x) \) we write

\[
f(x) = \sum_{n=1}^{\infty} f^{(n)}(x).
\]

By virtue of (5.34)–(5.39), Eq. (5.32) is equivalent to the infinite system of algebraic equations (cf. (4.13))

\[
\left[ \bar{T}_{(n,\circ)}(\Omega^2 \gamma) \right] + \sum_{q=1}^{\infty} \bar{T}_{(q,\circ)}(\gamma) \rho^{(q)(n)} = -f^{(n)}.
\]

In Eq. (4.3), the first integrand involves \( B^r \varphi^r \). This is why the inclusion of \( \varphi^r(x) \in \text{Ker} \, B^r \) in the expansion (4.5) does not affect the evaluation of the corresponding integral. However, in Eq. (5.32) the first integrand does not involve any differential operator acting on \( \bar{T}_o(\gamma)(y) \). This is why only the form (5.34) of the expansion leads to an useful algorithm. One can then check that in the expansion (5.39) any \( \bar{f}(x) \in \text{Ker} \, \mathcal{D} \) has to vanish, upon imposing Eq. (5.32) and the remaining expansions. In Eq. (5.40) both \( f^{(n)} \) and \( \rho^{(q)(n)} \) are known. Hence one gets an elegant but complicated solution, where the coefficients \( \bar{T}_{(n,\circ)}(\gamma) \) and \( \bar{T}_{(n,\circ)}(\Omega^2 \gamma) \) obey an infinite system of equations.

6. Concluding Remarks

The original contributions of our paper are as follows.

(i) From the analysis of general mathematical structures (i.e. vector bundles over Riemannian manifolds and conformal symmetries), we have suggested that, if the Lorenz gauge
can be replaced by the Eastwood–Singer gauge in Maxwell theory, the de Donder gauge can be replaced by a broader family of supplementary conditions in general relativity.

(ii) A restriction on the parameter in the DeWitt supermetric (1.8) has been obtained in a simple and elegant way.

(iii) When the background is flat and Riemannian, i.e. $m$-dimensional Euclidean space $\mathbb{E}^m$, the admissibility of such gauges in linearized theory has been proven by using the Green kernel of the Laplacian, the theory of polyharmonic functions and polyharmonic forms, and Fourier transform techniques. The result (2.25) is non-local in that it involves integrals over $\mathbb{E}^m$. The solution $\varphi_a(x)$ is unique since the operator $\Box^3$ is elliptic.

(iv) When the background is flat and Lorentzian, i.e. $m$-dimensional Minkowski space-time, the admissibility of such gauges in linearized theory has been proven by using the Green kernel of the wave operator and partial Fourier transforms with respect to time and space. The solution $\varphi_a(\vec{x}, t)$ depends on the contour of integration, i.e. on suitable conditions imposed on $\varphi_a(\vec{x}, t)$. We have considered, in particular, a contour integral of the Feynman type, but it remains to be seen whether other choices of contour integral give rise to solutions which are more relevant for physics.

(v) When the background is curved with positive-definite metric, the gauges (1.10) remain admissible provided that $\alpha \neq -2$ (see (ii)) and if the operator $B_{e^q}$ defined in (4.1) admits a discrete spectral resolution. Useful operational criteria to check the admissibility of our gauges are then provided by (4.6) or (4.7) for a given choice of Riemann curvature and of the tensor field $T^{pebc}$.

(vi) The condition (5.15) for conformal invariance of the gauge conditions is fulfilled provided that $\tilde{T}^{pe}(\gamma)$ is chosen to satisfy the integral equation (5.21). The counterpart of the Eastwood–Singer gauge for gravity is therefore far more elaborated. In particular, if the assumption expressed by Eq. (5.22) holds, we have obtained the integral equation (5.32), and the solution is found (in principle) by solving the infinite system of algebraic equations (5.40).
In our proof of conformal invariance of gauge conditions, it is crucial to consider conformal rescalings of the physical metric $\gamma_{ab}$, while the background metric $g_{ab}$ is kept fixed. We have done so because it is $\gamma_{ab}$ which solves the Einstein equations, which are not conformally invariant. The consideration of general mathematical structures seems to suggest that a key ingredient is the addition of a “compensating term” $\nabla_p \tilde{T}^{pe}(\gamma)$ to the higher-order covariant derivatives of the original gauge condition (see (1.5b) and (5.2)). Unlike the case of Maxwell theory in curved backgrounds, where conformal rescalings of the background metric are considered, we have therefore studied conformal rescalings of the physical metric only in general relativity. Still, it remains of interest for further research to consider conformal rescalings of both background and physical metric.

Indeed, the importance of integral equations in general relativity was already investigated, although from a completely different perspective, in Ref. 13 (where the metric tensor was taken to be linearly related to the energy-momentum tensor through an integral involving a kernel), whereas equations similar to our Eq. (5.18) occur when the formulation of conservation laws is considered in Einstein’s theory of gravity.14 It now remains to be seen how to extend the results of Secs. 4 and 5, proved for positive-definite metrics, to the case of Lorentzian metrics, which are of course the object of interest in general relativity. We can however point out that the derivation of the integral equations (5.21) and (5.32) does not depend on the signature of the metric, and hence the construction of $\tilde{T}^{pe}(\gamma)$ remains non-local also in the Lorentzian case. The Green functions that one may want to use will be distinguished by various boundary conditions (cf. Sec. 3), and hence the Lorentzian framework will be actually richer in this respect.

The above results seem to suggest that new perspectives are in sight in the investigation of supplementary conditions in general relativity. They might have applications both in classical theory (linearized equations in gravitational wave theory, symmetry principles and their impact on gauge conditions), and in the attempts to quantize the gravitational field (at least as far as its semiclassical properties are concerned). Hopefully, in the years to come it will become clear whether the necessary mathematics can be used to rule out (or verify) the existence of some properties of the universe (see, in particular, the restrictions
on the DeWitt supermetric found in Ref. 15, where big-bang nucleosynthesis has been used as a probe of the geometry of superspace).

Acknowledgments

This work has been partially supported by PRIN97 ‘Sintesi’, and the authors are indebted to Ivan Avramidi for enlightening correspondence.

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