Explosive percolation in graphs

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Abstract. Percolation is perhaps the simplest example of a process exhibiting a phase transition and one of the most studied phenomena in statistical physics. The percolation transition is continuous if sites/bonds are occupied independently with the same probability. However, alternative rules for the occupation of sites/bonds might affect the order of the transition. A recent set of rules proposed by Achlioptas et al. [Science 323, 1453 (2009)], characterized by competitive link addition, was claimed to lead to a discontinuous connectedness transition, named “explosive percolation”. In this work we survey a numerical study of the explosive percolation transition on various types of graphs, from lattices to scale-free networks, and show the consistency of these results with recent analytical work showing that the transition is actually continuous.

1. Introduction
Phase transitions lie at the heart of the modern development of statistical physics [1, 2]. They are changes in the state of order of a system and can be classified based on their properties in the immediate proximity of the critical point. Phase transitions are continuous (or second-order), if the order parameter changes continuously across the two phases, with an infinite correlation length and consequent power law decay of correlations and divergence of higher moments of the order parameter at the critical point. Otherwise one speaks of discontinuous (first-order) phase transitions, which are typically characterized by a discontinuous jump of the order parameter at the critical point.

The purely geometric process known as random percolation [3] offers a paradigmatic example of a continuous phase transition. The starting point is a graph, e.g. a lattice. The sites or the links of the lattice are occupied independently with some probability $p$. Nearest-neighboring occupied sites/links form structures called clusters. For low values of the occupation probability $p$ just a few small clusters are formed, but if $p$ increases the number and size of the clusters will increase as well. When the occupation probability exceeds a critical value $p_c$, a macroscopic cluster, occupying a finite fraction of the total number of sites/links, emerges. This macroscopic structure is called percolation cluster and its relative size $P$, the percolation strength, is the order parameter of the transition: $P = 0$ indicates the phase with only microscopic clusters, whereas $P > 0$ indicates the phase with (at least) one macroscopic cluster. Random percolation has
been studied on lattices, random graphs [4] and scale-free networks [5, 6, 7, 8, 9]. Analytical and numerical studies have proved that the percolation transition is continuous, without exceptions. This however holds for random percolation. It cannot be excluded a priori that alternative processes of occupation of sites/links might lead to different types of geometric transitions.

In a recent paper [10], Achlioptas et al. have introduced a special set of rules, in which links are occupied as a result of a competitive process between pairs of links (Achlioptas processes). The idea is to slow down the process of cluster growth, by inserting links leading to the merge of small clusters. This can be done in several ways. Achlioptas et al. focused on the so-called product rule (PR): given a pair of links, randomly selected among those which are not yet occupied, one occupies the link merging the two clusters with smaller product size (Fig. 1).

One could consider variants of this rule, like taking the sum instead of the product, or just the minimum size of the clusters of each pair. Also, the competition can be extended to more than two links. In any case, the result of such processes is a slow growth of the cluster sizes, which causes a delay in the onset of the percolation transition. On the other hand, since the density of links and, consequently, of clusters at the onset is higher than for random percolation, it is natural to expect that the percolation cluster has a very rapid growth. This is indeed confirmed by numerical studies; in fact, the growth of the percolation cluster is so quick that the percolation strength \( P \) appears to vary discontinuously at the onset. The sudden jump in the order parameter has motivated the name “explosive percolation”. In the last two years, Achlioptas processes have been extensively studied and meanwhile a lot is known about the explosive percolation transition [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

The main issue concerned the order of the transition: is it continuous or discontinuous? Achlioptas et al. claimed that it is discontinuous, and this is what has mostly attracted the attention of scholars. As a matter of fact, it was soon shown that, despite the alleged jump of the order parameter at the threshold \( p_c \), the explosive percolation transition has peculiar features of continuous phase transitions, like power law distributions of cluster sizes [15, 17, 19] and power law scaling of the mean cluster size at \( p_c \) [12, 13, 15]. Indeed, in recent works by Nagler et al. [21] and da Costa et al. [22] it was proven that the transition is actually continuous, and that the jump of the order parameter is only apparent, and due to the very small critical exponent of the order parameter \( \beta \).

In this paper we summarize the finite size scaling analysis of Ref. [15], and verify that the results are indeed consistent with the analytical findings of da Costa et al.. In Section 2 we introduce finite size scaling, in Section 3 we discuss the results for different types of graphs. A summary is reported in Section 4.

2. Finite size scaling

Finite size scaling [23] is a well-known technique adopted in numerical studies of phase transitions. For continuous phase transitions, the correlation length is infinite at the critical threshold \( p_c \), so every variable \( X \) is scale-independent in the proximity of that point and has a power law form,

\[ X \sim |p - p_c|^{\omega}, \]

where \( \omega \) is a critical exponent. If the system has a finite size \( N \), the variable \( X \) near the threshold has the following scaling form

\[ X = N^{-\omega/\nu} F \left[ (p - p_c) N^{1/\nu} \right]. \]

In Eq. 2, \( \nu \) is a critical exponent and \( F \) a universal function. Exactly at the critical point \( (p = p_c) \) the variable displays the simple scaling \( X \sim N^{-\omega/\nu} \), which can be used to extract the exponents’ ratio \( \omega/\nu \), by using systems of different sizes. Moreover, if \( p_c, \nu \) and \( \omega \) are known,
Figure 1. Scheme of an Achlioptas process with product rule. Two links (dashed lines) are selected at random among all possible pairs of non-adjacent nodes. The link joining the pair of clusters with the smaller product size is eventually added to the system. Here the winning link is that between clusters $c_1$ and $c_2$ (product size $7 \cdot 2 = 14 < 4 \cdot 4 = 16$).

The expression $X N^{\omega/\nu}$ as a function of $(p - p_c) N^{1/\nu}$ is just the universal function $F$, which is independent of $N$, so curves referring to different system sizes collapse.

We investigated the two main variables of percolation [3], i.e. the percolation strength $P$ and the average cluster size $S$. The percolation strength $P$, as we have said above, is the order parameter of the transition, and equals the relative size of the percolating cluster(s) with respect to the total system size $N$. While on lattices there are operative criteria to define a percolating cluster (e.g. if it runs from one edge of the lattice to the opposite one, say), on generic graphs this is not the case, so $P$ is defined as the relative size of the largest connected cluster. The scaling ansatz of the percolation strength is

$$P = N^{-\beta/\nu} F^{(1)} \left[ (p - p_c) N^{1/\nu} \right]. \quad (3)$$

The average cluster size $S$ is defined as

$$S = \frac{\sum_s n_s s^2}{\sum_s n_s s}, \quad (4)$$

where $n_s$ is the number of clusters of size $s$ per node. The sums run over all possible values of $s$ except for the one of the largest cluster. The scaling ansatz of $S$ is

$$S = N^{\gamma/\nu} F^{(2)} \left[ (p - p_c) N^{1/\nu} \right]. \quad (5)$$

The universal functions $F^{(1)}$ and $F^{(2)}$ of Eqs. (3) and (5) are not the same, but they are related.

In random percolation, the probability distribution $P(s)$ of cluster sizes (except the largest), decays at $p_c$ as the power law $P(s) \sim s^{-\tau}$ with the cluster size $s$. We have computed the cluster size distribution $P(s)$ at $p_c$ and measured the Fisher exponent $\tau$. For a given system, $P(s)$ is
related to \( n_s \) by the relation \( P(s) = N n_s / n_c \), where \( n_c \) is the total number of “finite” clusters. We shall use the symbol \( n_s \) to indicate \( P(s) \) as well, but in the plots \( n_s \) is normalized as \( P(s) \), for consistency.

The percolation threshold \( p_c \) is localized in two independent ways. The first method exploits the scaling of the pseudo-critical points \( p_c(N) \)

\[
p_c = p_c(N) + b N^{-1/\nu}.
\]

By using several system sizes, one can perform a fit with the three parameters \( b, \nu \) and \( p_c \). The pseudocritical point for a system with finite size \( N \) can be defined in several ways, for us it indicates the value of \( p \) at which the average cluster size \( S \) peaks.

An alternative procedure relies on Eq. (3). The percolation strength \( P \) is plotted as a function of the system size \( N \) for a given value of \( p \). Since, for \( p = p_c \) the scaling follows a power law, the correct value of the percolation threshold can be determined by finding the value of \( p \) which yields the best power law fit.

### 3. Results

#### 3.1. Implementing Achlioptas processes with product rule

The starting point is just a graph with \( N \) nodes and no links. Links are added one by one, according to the competitive rule previously described, i.e. by selecting each time a pair of links at random and picking the one yielding the smaller product for the sizes of the clusters it merges. For scale-free networks the situation is a bit more involved, and we describe it in Section 3.4. The procedure goes on until the desired density of links \( p \) is reached. We defined \( p \) as the number of links of the graph divided by the total number of links present in the graph when it has been “completed”, i.e., after the addition of the last link. All graphs considered here are “sparse”, i.e., the average degree \( \langle k \rangle \), expressing the ratio between (twice) the number of links and the number of nodes \( N \), does not depend on \( N \). At time \( t \) of the growth process there are exactly \( t \) links in the system: their density \( p \), according to our definition, is then \( t/(N\langle k \rangle) \).

#### 3.2. Lattices

Achlioptas processes on the square lattice were first studied by Ziff [11], who found similar properties for the explosive transition as Achlioptas et al. had found for Erdős-Rényi random graphs. The criterion to assess the nature of the transition was the same one proposed by Achlioptas et al., namely the scaling with the system size \( N \) of the transition window \( \Delta p = p_2 - p_1 \), where \( p_2 \) is the lowest value of \( p \) for which \( P > 0.5 \) and \( p_1 \) the lowest value of \( p \) for which \( P > 1/\sqrt{N} \). In a recent paper Ziff has performed a finite size scaling analysis as well [18]. The results of our analysis are shown in Fig. 2. The order parameter \( P \) yields a very small exponents’ ratio \( \beta/\nu \), compatible with zero \( [0.07(3)] \) (Fig. 2, left), which is what one would expect to find for a discontinuous transition. The average cluster size \( S \), instead, has a non-trivial power law scaling at \( p_c \), with exponent \( \gamma/\nu = 1.7(1) \) (Fig. 2, center). This had first been observed by Cho et al. in scale-free networks [12]. Fig. 2 (right), showing the distribution of sizes \( n_s \) for all clusters except the largest one, provides an explanation of the scaling of \( S \). The distribution is a clear power law [exponent \( 1.9(1) \)], which is incompatible with a classic discontinuous transition, as it usually occurs for continuous transitions. Since \( n_s \) is a power law, all variables derived from \( n_s \), including the average cluster size \( S \), have power law scaling.

In 3d-lattices, the general picture is consistent with that in two dimensions (Fig. 3). Again, the scaling at \( p_c \) of the order parameter \( P \) yields a very small exponent \( \beta/\nu = 0.02(2) \), compatible with zero (Fig. 3, left). Still, \( S \) scales with an exponent \( \gamma/\nu = 2.1(1) \) (Fig. 3, center), again due to the power law shape of the distribution of cluster sizes (Fig. 3, right). The Fisher exponent \( \tau = 1.99(4) \) is compatible with the value we measured in two dimensions \([1.9(1)]\).
### Figure 2.
Explosive percolation on 2d-lattices. (Left) Percolation strength $P$ as a function of the lattice side $L$ for three different values of the occupation probability: $p = 0.5256$ (violet diamonds), $p = 0.5266$ (orange circles) and $p = 0.5276$ (grey squares). The dashed line indicates the best fit obtained at the critical point $p = p_c = 0.5266(2)$, from which we get $\beta/\nu = 0.07(3)$. (Center) The average cluster size $S$ as a function of the lattice side $L$ for the same values of $p$ used for $P$. The dashed line has slope $\gamma/\nu = 1.7(1)$. (Right) Cluster size distributions measured at $p_c$ for PR (grey squares) and random percolation (orange circles). The exponents are $\tau = 1.9(1)$ for PR (red dotted line), while for random percolation $\tau = 2.05(1)$ (black dashed line). Simulations have been performed on lattices with side $L = 4096$.

### Figure 3.
Explosive percolation on 3d-lattices. (Left) Percolation strength $P$ as a function of the lattice side $L$ for three different values of the occupation probability: $p = 0.3866$ (violet diamonds), $p = 0.3876$ (orange circles) and $p = 0.3886$ (grey squares). The dashed line indicates the best fit obtained at the critical point $p = p_c = 0.3876(2)$, from which we get $\beta/\nu = 0.02(2)$. (Center) The average cluster size $S$ as a function of the lattice side $L$ for the same values of $p$ used for $P$. The dashed line has slope $\gamma/\nu = 2.1(1)$. (Right) Cluster size distributions at the critical point for PR (grey squares) and random percolation (orange circles). The exponents of the power law fits are $\tau = 1.99(4)$ for PR (red dotted line) and $\tau = 2.20(1)$ for random percolation (black dashed line). Simulations have been performed on lattices with side $L = 256$.

#### 3.3. Erdős-Rényi graphs

Fig. 4 presents the results of our analysis. For PR we see again a flat profile of the order parameter $P$ with $N$ ($\beta/\nu = 0.02(1)$), consistent with a discontinuous transition, along with a power law scaling of the average cluster size $S$, with exponent $\gamma/\nu = 0.48(4)$. The exponent $\tau = 2.08(5)$ (Fig. 4, right) is still compatible with the values found for PR on lattices, both in two and three dimensions (see Section 3.2 and Table 1).

#### 3.4. Scale-free networks

Many natural, social and man-made systems, if represented as graphs, display common features. The most striking is a broad distribution of the degree of the nodes, with a large majority of
Figure 4. Explosive percolation on Erdős-Rényi random graphs. (Left) Percolation strength $P$ as a function of the network size $N$ for three different values of the occupation probability: $p = 0.8872$ (violet diamonds), $p = 0.8882$ (orange circles) and $p = 0.8892$ (grey squares). The dashed line indicates the best fit obtained at the critical point $p = p_c = 0.8882(2)$, from which we get $\beta/\nu = 0.02(1)$. (Center) The average cluster size $S$ as a function of the network size $N$ for the same values of $p$ used for $P$. The dashed line has slope $\gamma/\nu = 0.48(4)$. (Right) Cluster size distributions $n_s$ measured at the critical point for PR (grey squares) and random percolation (orange circles). The exponents are $\tau = 2.08(5)$ (red dotted line) for PR and $\tau = 2.51(2)$ (black dashed line) for random percolation. Simulations have been performed on graphs with $N = 8\,192\,000$.

Figure 5. Explosive percolation on random SF networks. Percolation threshold $p_c(N)$ as a function of the degree exponent $\lambda$ for different network sizes $N$. The black line is the infinite size limit extrapolation of the critical threshold, performed by applying Eq. (6). The percolation threshold vanishes for $\lambda < \lambda_c \sim 2.3$, while it is non-zero for $\lambda > \lambda_c$. 
Figure 6. Explosive percolation on SF networks for degree exponents $\lambda = 2.5$ (top diagrams) and $\lambda = 2.8$ (bottom diagrams). (Left) Percolation strength $P$ as a function of the system size $N$ for three different values of the occupation probability. Top: $p = 0.0529$ (violet diamonds), $p = 0.0629$ (orange circles) and $p = 0.0729$ (grey squares). Bottom: $p = 0.1229$ (violet diamonds), $p = 0.1329$ (orange circles) and $p = 0.1349$ (grey squares). The dashed lines indicate the best fits obtained at the critical points $p = p_c = 0.0629(1)$ (top) and $p = p_c = 0.1329(1)$ (bottom), from which we get $\beta/\nu = 0.59(1)$ (top) and $\beta/\nu = 0.50(1)$ (bottom). (Right) The average cluster size $S$ as a function of the network size $N$ for the same values of $p$ used for $P$. The dashed lines have slopes $\gamma/\nu = 0.24(1)$ (top) and $\gamma/\nu = 0.42(1)$ (bottom). Simulations have been performed on graphs with $N = 8192000$.

nodes having low degree and a small subset of nodes having high degree. The tails of the degree distributions are often well fitted by power laws, which lack a characteristic scale, justifying the name “scale-free networks” given to such systems [24, 25, 26, 27, 28, 29, 30, 31]. We indicate the exponent of the power law degree distribution with $\lambda$. Nodes with large degree, called “hubs”, have a fundamental role for the structure and dynamics of networks. For instance, they “keep” a large portion of the system in the same connected component, guaranteeing stability and compactness to the structure. In particular, in random scale-free (SF) networks with degree exponent $\lambda < 3$, there are many hubs and a very small fraction of links (vanishing in the limit of systems of infinite size) manages to keep a macroscopic fraction of nodes of the graph together in the same connected component. This amounts to saying that the percolation threshold is zero [5, 6, 7, 8, 9].

The Achlioptas process with product rule for SF networks is implemented as follows. We
Explosive percolation on SF networks for degree exponent $\lambda = 3.5$. (Left) Percolation strength $P$ as a function of the system size $N$ for three different values of the occupation probability: $p = 0.2214$ (violet diamonds), $p = 0.2224$ (orange circles) and $p = 0.2234$ (grey squares). The dashed line indicates the best fit obtained at the critical point $p = p_c = 0.2224(2)$, from which we get $\beta/\nu = -0.06(3)$. (Center) The average cluster size $S$ as a function of the network size $N$ for the same values of $p$ used for $P$. The dashed line has slope $\gamma/\nu = 0.40(9)$. (Right) Cluster size distributions $n_s$ measured at the critical point for PR (grey squares) and random percolation (orange circles). The exponents are $\tau = 2.2(1)$ (red dotted line) for PR and $\tau = 2.94(1)$ (black dashed line) for random percolation. Simulations have been performed on graphs with $N = 8,192,000$.

start with a set of $N$ nodes and a given degree sequence $\{k_1, k_2, \ldots, k_N\}$. The degrees of the sequence are extracted from a power law distribution with exponent $\lambda$. The average degree $\langle k \rangle$ is set equal to 5. Every node $i$ holds $k_i$ stubs (half-links), where $k_i$ is the degree of $i$. By attaching such stubs in pairs one builds a network with the desired power law degree distribution with exponent $\lambda$. If stubs are randomly connected we would have the simple configuration model [32], which yields a random SF network. For the Achlioptas process at each iteration two pairs of stubs are selected and the PR determines which pair of stubs has to be eventually joined in a link (the PR applies as shown in Fig. 1).

The first remarkable result concerns the threshold. For random percolation it is zero for $\lambda < 3$ and non-zero for $\lambda > 3$ [5]. However, for explosive percolation, the threshold is non-zero already for $\lambda > \lambda_c \sim 2.3$ (Fig. 5) [12, 13].

In Fig. 6 we show the scalings at $p_c$ of $P$ and $S$ for $\lambda = 2.5$ and 2.8. Here we see an important difference with the cases of lattices and Erdős-Rényi random graphs, namely that the scaling of $P$ at $p_c$ is non-trivial, as $P$ decreases with $N$ as a power law in both cases. This is very different from what one would find for a discontinuous transition, in which $P$ would be approximately constant with $N$. For $S$ we also find power law scaling (Fig. 6, right panels), as we had previously seen for lattices and Erdős-Rényi random graphs. Overall, the transition looks like a standard continuous transition.

For $\lambda > 3$, however, the scenario changes. In Fig. 7 we show the results of our finite size scaling analysis for SF networks with exponent $\lambda = 3.5$. For random percolation on SF networks it is well known [33] that $\beta/\nu = 1/(\lambda - 1)$ and $\gamma/\nu = (\lambda - 3)/(\lambda - 1)$ for $3 \leq \lambda \leq 4$. For $\lambda > 4$, we are in the mean field limit and the exponents are independent of the degree exponent $\lambda$: $\beta/\nu = \gamma/\nu = 1/3$. These values coincide with the exponents for the percolation transition of Erdős-Rényi random networks. SF networks are very similar to Erdős-Rényi random networks in the limit $\lambda \to \infty$. The explosive percolation transition looks as on lattices and ER random networks. The scaling of $P$ at $p_c$ is trivial (Fig. 7, left), with $\beta/\nu = -0.06(3)$, which is essentially zero. The power law scaling of $S$ at $p_c$ is non-trivial (Fig. 7, center), with $\gamma/\nu = 0.40(9)$. The Fisher exponent $\tau = 2.2(1)$ (Fig. 7, right).
We have found two different classes of phase transitions for explosive percolation, that we shall call class A and B. Class A includes ER random networks (the original system studied by Achlioptas et al. [10]), lattices and SF networks with degree exponent $\lambda > 3$. Here we have observed an apparent saturation of the order parameter $P$ at $p_c$ with the size of the system $N$, due to an exponent $\beta$ that is very small, compatible with zero. Class B includes SF networks with $\lambda < 3$, where there seems to be no room for a discontinuous transition, as the results of the finite size scaling analysis are fully compatible with a standard continuous transition. Still, apart from the anomalous scaling behavior of the order parameter $P$ at $p_c$ observed in class A, we found that the other percolation variables display power law scaling in all cases, without exceptions, just as in second-order phase transitions. The most striking feature is the fact that the size distribution $n_s$ of the “finite” clusters at $p_c$ is a power law, not exponential or Gaussian as one expects from first-order phase transitions.

Da Costa et al. [22] have recently proved that an aggregation process in the same spirit as Achlioptas processes leads to a continuous phase transition, characterized by an exponent $\beta$ for the order parameter $P$ that is very small, $\beta \sim 0.0555$. They also demonstrated that the fact that the transition is continuous for the special process they considered enforces the same type of transition for Achlioptas processes too, with generally different but still small values for $\beta$. This is fully consistent with what we have found for the systems of class A. Da Costa et al. have derived important relations for the critical exponents. For instance, $\beta/\nu = \beta/(4\beta + 1)$ and the Fisher exponent $\tau = 2 + \beta/(3\beta + 1)$. For $\beta \sim 0.0555$ the exponents’ ratio $\beta/\nu \sim 0.0455$. This value is compatible, within errors, with the values of $\beta/\nu$ computed for the systems of class A (see Table 1). So, the apparent saturation of the order parameter $P$ with the system size $N$ could be indeed due to the smallness of the exponent $\beta$. Moreover, the Fisher exponent $\tau$ for every transition we have investigated is very close to 2. From the relation $\tau = 2 + \beta/(3\beta + 1) < 2 + \beta$, and the fact that $\beta$ is small, we see that $\tau$ will always be very close to 2, just as we found. The difference between $\tau$ and 2 is hard to determine numerically. Likewise, the values of $\tau$ derived from different Achlioptas processes can be hardly distinguished from each other. We remark that this holds for lattices, Erdős-Rényi graphs and SF networks with degree exponent $\lambda > 3$, while the explosive percolation transition on SF networks with $\lambda < 3$ appears as something very different ($\beta$ is not small here), and deserves further investigations in the future.

Table 1. Percolation thresholds and critical exponents for random percolation (RP) and Achlioptas process with product rule (PR).

| System        | Model | $p_c$ | $\beta/\nu$ | $\gamma/\nu$ | $\tau$ |
|---------------|-------|-------|-------------|-------------|--------|
| 2d-lattice    | RP    | 0.5   | 0.11(1)     | 1.76(1)     | 2.05(1) |
|               | PR    | 0.5266(2) | 0.07(3) | 1.7(1)     | 1.9(1)  |
| 3d-lattice    | RP    | 0.2488(3) | 0.48(1) | 2.0(1)     | 2.20(1) |
|               | PR    | 0.3876(2) | 0.02(2) | 2.1(1)     | 1.99(4) |
| ER network    | RP    | 0.5   | 0.33(1)     | 0.34(1)     | 2.51(2) |
|               | PR    | 0.8882(2) | 0.02(1) | 0.48(4)    | 2.08(5) |
| SF network $\lambda = 2.5$ | RP    | 0     | –          | –          | –      |
|               | PR    | 0.0629(1) | 0.59(1) | 0.24(1)    | 2.15(2) |
| SF network $\lambda = 2.8$ | RP    | 0    | –          | –          | –      |
|               | PR    | 0.1329(1) | 0.50(1) | 0.42(1)    | 2.13(6) |
| SF network $\lambda = 3.5$ | RP    | 0.078(1) | 0.38(1) | 0.15(2)    | 2.94(1) |
|               | PR    | 0.2224(2) | –0.06(3) | 0.40(9)    | 2.2(1)  |
We have also found that there is a non-zero explosive percolation threshold for SF networks for $\lambda > \lambda_c \sim 2.3$ (Fig. 5), while the threshold for random percolation is zero as long as $\lambda \leq 3$. Cho et al. [12] suggested that this is due to the non-random addition of links during the Achlioptas process, because of which the degree distribution of the system during the growth deviates from that imposed by construction, which will be eventually reached at the end of the process. This has been indeed verified numerically [12, 15].

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