DISTANCE-LIKE FUNCTIONS AND SMOOTH APPROXIMATIONS

A CORRECTION TO

“LOGARITHM LAWS FOR FLOWS ON HOMOGENEOUS SPACES”

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Abstract. One of the propositions in the paper [KM], related to approximating certain sets by smooth functions, was recently found to be incorrect. Here we correct the mistake.

1. Statement of results

Let us reproduce the setting of [KM] in a slightly more general form. Let $G$ be a Lie group and $\Gamma$ a lattice in $G$. Denote by $X$ the homogeneous space $G/\Gamma$ and by $\mu$ the $G$-invariant probability measure on $X$. In what follows, $\| \cdot \|_p$ will stand for the $L^p$ norm. Fix a basis $\{Y_1, \ldots, Y_n\}$ for the Lie algebra $\mathfrak{g}$ of $G$, and, given a smooth function $h \in C^\infty(X)$ and $\ell \in \mathbb{N}$, define the “$L^2$, order $\ell$” Sobolev norm $\| h \|_{2,\ell}$ of $h$ by

$$\| h \|_{2,\ell} \overset{\text{def}}{=} \sum_{|\alpha| \leq \ell} \| D^\alpha h \|_2,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, $|\alpha| = \sum_{i=1}^n \alpha_i$, and $D^\alpha$ is a differential operator of order $|\alpha|$ which is a monomial in $Y_1, \ldots, Y_n$, namely $D^\alpha = Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$. This definition depends on the basis, however, a change of basis would only distort $\| h \|_{2,\ell}$ by a bounded factor. We also let

$$C^\infty_2(X) = \{ h \in C^\infty(X) : \| h \|_{2,\ell} < \infty \text{ for any } \ell = \mathbb{Z}_+ \}.$$

Now let $\Delta$ be a real-valued function on $X$, and for $z \in \mathbb{R}$ denote

$$\Phi_\Delta(z) \overset{\text{def}}{=} \mu(\Delta^{-1}([z, \infty))).$$

Say that $\Delta$ is DL (an abbreviation for “distance-like”) if there exists $z_0 \in \mathbb{R}$ such that $\Phi_\Delta(z_0) > 0$ and

(a) $\Delta$ is uniformly continuous on $\Delta^{-1}([z_0, \infty))$; that is, $\forall \varepsilon > 0$ there exists a neighborhood $U$ of the identity in $G$ such that for any $x \in X$ with $\Delta(x) \geq z_0$,

$$g \in U \implies |\Delta(x) - \Delta(gx)| < \varepsilon;$$

(b) the function $\Phi_\Delta$ does not decrease very fast, more precisely, if

$$\exists c, \delta > 0 \text{ such that } \Phi_\Delta(z) \geq c\Phi_\Delta(z - \delta) \quad \forall z \geq z_0.$$  

(DL)
The paper [KM] gives several examples of DL functions on homogeneous spaces of semisimple Lie groups. The main goal of that paper was to study statistics of excursions of generic trajectories of flows on $X$ into sets $\Delta^{-1}([z, \infty))$ for large enough $z$. A crucial ingredient of the argument was approximation of characteristic functions of those sets by smooth functions with uniformly bounded Sobolev norms. However, as was recently observed by Dubi Kelmer and Shucheng Yu, the argument in the main approximation statement, namely [KM, Lemma 4.2], contains a mistake. To state a corrected version below, we need to weaken the regularity assumption on the smooth functions approximating the sets $\Delta^{-1}([z, \infty))$. Namely, for $\ell \in \mathbb{Z}_+$ and $C > 0$, let us say that a nonnegative function $h \in C^\infty_2(X)$ is $(C, \ell)$-regular if

$$\|h\|_{2, \ell} \leq C\sqrt{\|h\|_1}. \quad \text{(REG)}$$

Note that the argument of [KM] used a stronger condition:

$$\|h\|_{2, \ell} \leq C\|h\|_1. \quad \text{(REG-old)}$$

Equivalently one can replace $\sqrt{\|h\|_1}$ in (REG) with $\|h\|_2$, but for technical reasons it is more convenient to use the square root of the $L^1$ norm. Here is the corrected statement of [KM, Lemma 4.2]:

**Theorem 1.1.** Let $\Delta$ be a DL function on $X$. Then for any $\ell \in \mathbb{Z}_+$ there exists $C > 0$ such that for every $z \geq z_0$ one can find two $(C, \ell)$-regular nonnegative functions $h'$ and $h''$ on $X$ such that

$$h' \leq 1_{\Delta^{-1}([z, \infty))} \leq h'' \quad \text{and} \quad c\Phi_{\Delta}(z) \leq \|h'\|_1 \leq \|h''\|_1 \leq \frac{1}{c}\Phi_{\Delta}(z), \quad (1.1)$$

with $c$ and $z_0$ as in (DL).

Fix a right-invariant Riemannian metric on $G$ and the corresponding metric ‘dist’ on $X$. For $g \in G$, let us denote by $\|g\|_s$ the distance between $g \in G$ and the identity element of $G$. (Note that $\|g\| = \|g^{-1}\|$ due to the right-invariance of the metric.) Now say that the $G$-action on $X$ is exponentially mixing if there exist $\lambda, E > 0$ and $\ell \in \mathbb{Z}_+$ such that for any $\varphi, \psi \in C^\infty_2(X)$ and for any $g \in G$ one has

$$|\langle g\varphi, \psi \rangle| \leq E e^{-\lambda \|g\|} \|\varphi\|_{2, \ell} \|\psi\|_{2, \ell}. \quad \text{(EM)}$$

Here $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(X, \mu)$.

One of the main goals of [KM] was, given a sequence $\{f_t : t \in \mathbb{N}\}$ of elements of $G$ and a sequence of non-negative functions $\{h_t : t \in \mathbb{N}\}$ on $X$ such that

$$\sum_{t=1}^\infty \|h_t\|_1 = \infty,$$

compare the growth of $\sum_{t=1}^N h_t(f_t x)$ for $\mu$-a.e. $x \in X$ with the growth of $\sum_{t=1}^N \|h_t\|_1$ as $N \to \infty$. Results like that usually go by the name ‘dynamical Borel-Cantelli lemmas’, see [CK, HNPV]. In [KM, Proposition 4.1] such a conclusion was shown to follow from the exponential mixing of the $G$-action on $X$, the exponential divergence of $\{f_t\}$, namely the condition

$$\sup_{t \in \mathbb{N}} \sum_{s=1}^\infty e^{-\lambda \|f_s f_t^{-1}\|} < \infty \quad \forall \lambda > 0, \quad \text{(ED)}$$

and the regularity assumption (REG-old) on functions $\{h_t\}$. 
In the following theorem we weaken the regularity condition \([\text{REG-old}]\) to \([\text{REG}]\) and derive the same conclusion:

**Theorem 1.2.** Suppose that the \(G\)-action on \(X\) is exponentially mixing. Let \(\{f_t : t \in \mathbb{N}\}\) be a sequence of elements of \(G\) satisfying \([\text{ED}]\), and let \(\{h_t : t \in \mathbb{N}\}\) be a sequence of non-negative \((C, \ell)\)-regular functions on \(X\) such that \(\|h_t\|_1 \leq 1\) for all \(t\), and \(\sum_{t=1}^{\infty} \|h_t\|_1 = \infty\). Then

\[
\lim_{N \to \infty} \frac{\sum_{t=1}^{N} h_t(f_t x)}{\sum_{t=1}^{N} \|h_t\|_1} = 1 \quad \text{for } \mu\text{-a.e. } x \in X.
\]

Using the above theorem in place of \([\text{KM, Proposition 4.1}]\) and Theorem \([\text{KM, Lemma 4.2}]\), one can then recover \([\text{KM, Theorem 4.3}]\), that is, prove

**Theorem 1.3.** Suppose that the \(G\)-action on \(X\) is exponentially mixing. Let \(\{f_t : t \in \mathbb{N}\}\) be a sequence of elements of \(G\) satisfying \([\text{ED}]\), let \(\Delta\) be a DL function on \(X\), and let \(\{r_t : t \in \mathbb{N}\} \subset [z_0, \infty)\) be such that

\[
\sum_{t=1}^{\infty} \Phi_{\Delta}(r_t) = \infty.
\]

Then for some positive \(c \leq 1\) and for almost all \(x \in X\) one has

\[
c \leq \liminf_{N \to \infty} \frac{\# \{1 \leq t \leq N \mid \Delta(f_t x) \geq r_t\}}{\sum_{t=1}^{N} \Phi_{\Delta}(r_t)} \leq \limsup_{N \to \infty} \frac{\# \{1 \leq t \leq N \mid \Delta(f_t x) \geq r_t\}}{\sum_{t=1}^{N} \Phi_{\Delta}(r_t)} \leq \frac{1}{c}.
\]

Consequently, for any \(\{r_t\}\) satisfying \([1.2]\) and almost all \(x \in X\) one has \(\Delta(f_t x) \geq r_t\) for infinitely many \(t \in \mathbb{N}\). That is, in the terminology of \([\text{KM}]\), the family of sets

\[
\{\Delta^{-1}([z, \infty)) : z \in \mathbb{R}\}
\]

is Borel-Cantelli for \(\{f_t\}\).

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2. Proofs

Let us state a general form of Young’s inequality, whose proof we give for the sake of self-containment of the paper. Denote by \(m\) the Haar measure on \(G\) normalized so that the quotient map \(G \to X\) locally sends \(m\) to \(\mu\). For \(\psi \in L^1(G, m)\) and \(h \in L^1(X, \mu)\), define \(\psi * h\) by

\[(\psi * h)(x) \overset{\text{def}}{=} \int_G \psi(g) h(g^{-1} x) \, dm(g).
\]

**Lemma 2.1.** Let \(\psi \in L^1(G, m)\) and \(h \in L^2(X, \mu)\). Then \(\|\psi * h\|_2 \leq \|\psi\|_1 \|h\|_2\).

**Proof.** We have

\[
|\psi * h)(x)| \leq \int_G |\psi(g)|^{1/2} |h(g^{-1} x)| \cdot |\psi(g)|^{1/2} \, dm(g)
\]

(by Cauchy-Schwarz) \leq \left( \int_G |\psi(g)| \, dm(g) \right)^{1/2} \left( \int_G |h(g^{-1} x)|^2 \psi(g) \, dm(g) \right)^{1/2}.
\]
Thus we have
\[ \|(\psi * h)(x)\|^2 \leq \|\psi\|_1 \int_G |\psi(g)| \cdot |h(g^{-1}x)|^2 \, dm(g). \]
Integrating over \( X \) and using Fubini’s Theorem gives
\[ \|\psi * h\|_2^2 \leq \|\psi\|_1 \int_X |\psi(g)| \cdot |h(g^{-1}x)|^2 \, dm(g) \, d\mu(x) \]
\[ = \|\psi\|_1 \int_G |\psi(g)| \int_X |h(g^{-1}x)|^2 \, d\mu(x) \, dm(g), \]
which, by the \( G \)-invariance of \( \mu \), is the same as
\[ \|\psi\|_1 \int_G |\psi(g)| \, dm(g) \int_X |h(x)|^2 \, d\mu(x) = \|\psi\|^2 \|h\|^2. \]
\[ \square \]

**Proof of Theorem 1.1.** We follow the proof of [KM, Lemma 4.2]. For \( z \in \mathbb{R} \), let us use the notation
\[ A(z) \overset{\text{def}}{=} \Delta^{-1}([z, \infty)). \]
Then, for \( \epsilon > 0 \), let us denote by \( A'(z, \epsilon) \) the set of all points of \( A(z) \) which are not \( \epsilon \)-close to \( \partial A(z) \), i.e.
\[ A'(z, \epsilon) \overset{\text{def}}{=} \{ x \in A(z) : \text{dist}(x, \partial A(z)) \geq \epsilon \}, \]
and by \( A''(z, \epsilon) \) the \( \epsilon \)-neighborhood of \( A(z) \), namely
\[ A''(z, \epsilon) \overset{\text{def}}{=} \{ x \in X : \text{dist}(x, A(z)) \leq \epsilon \}. \]
Choose \( z_0, \delta \) and \( c \) as in (DL). Then, using the uniform continuity of \( \Delta \) on \( \Delta^{-1}([z_0, \infty)) \), find \( \epsilon > 0 \) such that
\[ |\Delta(x) - \Delta(y)| < \delta \text{ whenever } \Delta(x) \geq z_0 \text{ and } \text{dist}(x, y) < \epsilon. \]
It follows that for all \( z \geq z_0 \),
\[ A(z + \delta) \subset A'(z, \epsilon) \subset A(z) \subset A''(z, \epsilon) \subset A(z - \delta); \]
therefore one can apply (DL) to conclude that
\[ c\mu(A(z)) \leq \mu(A'(z, \epsilon)) \leq \mu(A''(z, \epsilon)) \leq \frac{1}{c}\mu(A(z)). \tag{2.1} \]

Now take a non-negative \( \psi \in C^\infty(G) \) of \( L^1 \) norm 1 such that supp \( \psi \) belongs to the ball of radius \( \epsilon/4 \) centered in \( e \in G \). Fix \( z \geq z_0 \) and consider functions \( h' \overset{\text{def}}{=} \psi * 1_{A'(z, \epsilon/2)} \) and \( h'' \overset{\text{def}}{=} \psi * 1_{A''(z, \epsilon/2)} \). Then one clearly has
\[ 1_{A'(z, \epsilon/2)} \leq h' \leq 1_{A(z)} \leq h'' \leq 1_{A''(z, \epsilon/2)}, \]
which, together with (2.1), immediately implies (1.1). It remains to choose \( \ell \in \mathbb{Z}_+ \) and find \( C \) (independent of \( z \)) such that both \( h' \) and \( h'' \) are \((C, \ell)\)-regular. Take a multiindex \( \alpha \) with \( |\alpha| \leq \ell \), and write
\[ \|D^{\alpha} h'\|_2 = \|D^{\alpha} (\psi * 1_{A'(z, \epsilon/2)})\|_2 = \|D^{\alpha} (\psi) * 1_{A'(z, \epsilon/2)}\|_2. \]
Then, by the Young inequality,
\[ \|D^{\alpha} h'\|_2 \leq \|D^{\alpha} (\psi)\|_1 \sqrt{\mu(A'(z, \epsilon/2))} \leq \|D^{\alpha} (\psi)\|_1 \sqrt{\mu(A(z))} \leq \|D^{\alpha} (\psi)\|_1 \left( \frac{\|h'\|_1}{c} \right)^{1/2}. \]
Similarly,

\[ \|D^\alpha h''\|_2 \leq \|D^\alpha (\psi)\|_1 \sqrt{\mu(A^\alpha(z,\varepsilon/2))} \leq \|D^\alpha (\psi)\|_1 \left( \frac{\mu(A(z))}{c} \right)^{1/2} \leq \|D^\alpha (\psi)\|_1 \left( \frac{\|h''\|_1}{c} \right)^{1/2} ; \]

hence, with \( C = \frac{1}{c} \sum_{|a| \leq \ell} \|D^\alpha (\psi)\|_1 \), both \( h' \) and \( h'' \) are \((C, \ell)\)-regular, and the theorem is proven.

**Proof of Theorem 1.2.** Denote \( \int_X h_t \, d\mu = \|h_t\|_1 \) by \( a_t \). Following the argument in [KM], our goal is to show that the sequence of functions \( \{h_t \circ f_t\} \) satisfies a second-moment condition dating back to the work of Schmidt and Sprindžuk:

\[
\sup_{1 \leq M < N} \frac{\int_X \left( \sum_{t=M}^{N} h_t(f_t(x)) - \sum_{t=M}^{N} a_t \right)^2 \, d\mu}{\sum_{t=M}^{N} a_t} < \infty. \tag{SP}
\]

the conclusion of the theorem will then follow in view of [KM, Lemma 2.6], which is a special case of [SP, Chapter I, Lemma 10].

Take \( 1 \leq M < N \). As in [KM, Remark 2.7], one can rewrite the numerator as \( \sum_{s,t=M}^{N} (f_t^{-1} h_t, f_s^{-1} h_s) - a_s a_t \), and then estimate it using the exponential mixing of the \( G \)-action on \( X \):

\[
\left| \sum_{s,t=M}^{N} (f_t^{-1} h_t, f_s^{-1} h_s) - a_s a_t \right| \leq \sum_{s,t=M}^{N} \left| (f_s f_t^{-1} h_t, h_s) - a_s a_t \right| 
\]

(with \( E, \lambda, \ell \) as in (EM)) \( \leq E \sum_{s,t=M}^{N} e^{-\lambda \|f_s f_t^{-1}\|} \|h_t\|_2 \|h_s\|_2 \|a_s a_t\| \).

(by the \((C, \ell)\)-regularity of \( \{h_t\} \)) \( \leq EC^2 \sum_{s,t=M}^{N} e^{-\lambda \|f_s f_t^{-1}\|} \sqrt{a_s a_t} \).

Now, following an observation communicated to us by Shucheng Yu, split the above sum according to the comparison between \( a_s \) and \( a_t \):

\[
\sum_{a_s = a_t} e^{-\lambda \|f_s f_t^{-1}\|} \sqrt{a_s a_t} + \sum_{a_s < a_t} e^{-\lambda \|f_s f_t^{-1}\|} \sqrt{a_s a_t} + \sum_{a_s > a_t} e^{-\lambda \|f_s f_t^{-1}\|} \sqrt{a_s a_t}, \tag{2.2}
\]

where the values of \( s, t \) in the last three sums range between \( M \) and \( N \). By symmetry, the last two sums are equal. Thus (2.2) is not greater than

\[
\sum_{a_s = a_t} e^{-\lambda \|f_s f_t^{-1}\|} a_t + 2 \sum_{a_s < a_t} e^{-\lambda \|f_s f_t^{-1}\|} a_t \leq 2 \sum_{s,t=M}^{N} e^{-\lambda \|f_s f_t^{-1}\|} a_t \leq 2 \sum_{t=M}^{N} a_t \sum_{s=M}^{N} e^{-\lambda \|f_s f_t^{-1}\|} a_t \leq 2 \sum_{t=M}^{N} a_t \sup_{t \in \mathbb{N}} \sum_{s=1}^{\infty} e^{-\lambda \|f_s f_t^{-1}\|} ,
\]

and the proof of (SP) is finished in view of (ED). \qed
REFERENCES

[CK] N. Chernov and D. Kleinbock, *Dynamical Borel-Cantelli lemmas for Gibbs measures*, Israel J. Math. **122** (2001), 1–27.

[HNPV] N. Haydn, M. Nicol, T. Persson and S. Vaienti, *A note on Borel-Cantelli lemmas for non-uniformly hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **33** (2013), no. 2, 475–498.

[KM] D. Kleinbock and G.A. Margulis, *Logarithm laws for flows on homogeneous spaces*, Invent. Math. **138** (1999), no. 3, 451–494.

[Sp] V. Sprindžuk, *Metric theory of Diophantine approximations*, John Wiley & Sons, New York-Toronto-London, 1979.

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