ENERGY OF A KNOT: VARIATIONAL PRINCIPLES; MM-ENERGY.

OLEG KARPENKOV

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1. INTRODUCTION

Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and $\tau : S^1 \to \mathbb{R}^3$ be a smooth knot. We will assume that $\tau(t)$ is the arc length parametrization. Denote by $D(t_1, t_2)$ the length of the minimal subarc between $t_1$ and $t_2$ on the circle. Let $|\ast|$ denote the absolute value of vectors in $\mathbb{R}^3$.

Following [1], we denote by

$$E(\tau) = E_f(\tau) = \iint_{S^1 \times S^1} f(|\tau(t_1) - \tau(t_2)|, D(t_1, t_2)) dt_1 dt_2$$

the energy of the knot $\tau$, where $f(\rho, \alpha)$ satisfies the following conditions:

1) $f(\rho, \alpha) \in C^{1,1}(U)$, where $U = \{ (\rho, \alpha) | 0 < \rho \leq \alpha, \alpha \leq \pi \}$;
2) there exist the following limits:

$$\lim_{\rho \to 0, \rho/\alpha \to 1} f(\rho, \alpha), \quad \lim_{\rho \to 0, \rho/\alpha \to 1} \frac{\partial f(\rho, \alpha)}{\partial \rho}, \quad \lim_{\rho \to 0, \rho/\alpha \to 1} \frac{\partial f(\rho, \alpha)}{\partial \rho}.$$ 

Almost all energies are not homothety invariant, so we will consider only knots of length $2\pi$.

The energy of a knot is not an invariant of the topological class of this knot. If we make a smooth perturbation of a knot, its energy smoothly changes. We will consider energies with the following important properties. The energy is always positive. When a knot crossing tends to a double point, the energy tends to infinity. So every topological class of knots has a representative with the minimal value of energy. This knot is called a
normal form of the class. It is unknown whether each class has a unique normal form or not, i.e., whether the normal form for some energy is an invariant of the topological class or not. The normal forms satisfy the variational equations considered below.

Some energies have a physical meaning. For example $f = 1/(|\tau(t_1) - \tau(t_2)|)$ is the energy of a charged knot. Unfortunately, this energy is always infinite. As long as the charged knot does not break there must be some other forces which save the knot. Let us consider a model of such a restriction:

$$f = \frac{(D^2(t_1, t_2))}{(\tau(t_1) - |\tau(t_2)|)}.$$  

For this energy we will develop our variational principles.

The study of knot energies began with the work of Moffatt (1969) [7], and was developed by him in [8] following Arnold’s work [2]. The first steps in studying properties of the energies of knots were made by O’Hara [9, 10, 11] and the first variational principles for polygons in space were studied by Fukuhara [4].

The aim of this article is to prove that any extremal knot $\tau$ satisfies certain variational equations. The paper is organized as follows. We start in Section 2 with the definitions and formulations of the main theorem. In Section 3 we prove this theorem. In Section 4 we prove that the circle unknot always satisfies our extremal conditions. Unfortunately the integrals in the equations do not converge for all possible energies. For example, they do not converge in the case of the most famous energy: Möbius energy. We discuss this also in Section 4. Section 5 seems to be independent from the previous sections. In Section 5 we represent Mn-energy. The definition of this energy differs with one regarded above. Nevertheless besides its own properties Mn-energy has some similar with Möbius energy properties.

This work is partially published (see [5] and [6]).

The author is grateful to professor A. B. Sossinsky for constant attention to this work.

2. Notation and definitions

Mostly we will work with knots of fixed length $2\pi$. So let $S^1 = \mathbb{R}/(2\pi \mathbb{Z})$ be the circle and let $\tau : S^1 \rightarrow \mathbb{R}^3$ denote some smooth knot of length $2\pi$. Let $\tau(t)$ be the arc length parametrization.

By $\kappa(t)$ we denote the curvature at $t$ and $R(t) = 1/\kappa(t)$, the radius of curvature at $t$.

**Definition 2.1.** Given a smooth knot $\tau : S^1 \rightarrow \mathbb{R}^3$ and a point $t_0 \in S^1$, a **locally perturbed knot** is a knot (denoted by $\tau_{t_0, \epsilon}$) such that

a) $|\tau(t) - \tau_{t_0, \epsilon}(t)| < \epsilon^2$ if $D(t_0, t) \leq \epsilon$ and $\tau(t) = \tau_{t_0, \epsilon}(t)$ if $D(t_0, t) > \epsilon$;

b) $|\kappa(t) - \kappa_{t_0, \epsilon}(t)| < \epsilon$ for $D(t_0, t) < \epsilon$;

c) $\tau_{t_0, \epsilon}(t_0 + \lambda) = \tau_{t_0, \epsilon}(t_0) + \lambda \tau_{t_0, \epsilon}'(t_0) + (\lambda^2/2) \tau_{t_0, \epsilon}''(t_0) + o(\epsilon^2)$ if $D(t_0, t_0 + \lambda) \leq \epsilon$.

Note that at the points $t_0 - \epsilon$ and $t_0 + \epsilon$ the curvature is not restricted.

The length of the knot $\tau_{t_0, \epsilon}$ can change, but we regard knots of length $2\pi$ only. One of the ways to solve this problem is to consider the restriction of the set of locally perturbed
knots to the set of knots of constant length $2\pi$, but this definition is unsatisfactory. Indeed,
let a knot $\tau$ in some neighborhood of the point $t_0$ be a piece of a straight line. Then the
set of locally perturbed knots at the point $t_0$ of length $2\pi$ consists of the knot $\tau$ only.

We will extend this set in the following way.

**Definition 2.2.** Let the length of $\tau_{t_0,\varepsilon}$ be $(1 + \delta)2\pi$. The *locally perturbed length $2\pi$ knot $\tilde{\tau}_{t_0,\varepsilon}$* is the knot obtained from $\tau_{t_0,\varepsilon}$ by homothety with coefficient $1/(1 + \delta)$ and center at
the origin. We also say that the knot $\tilde{\tau}$ is *associated* with the knot $\tau$.

Consider any $\tau_{t_0,\varepsilon}$. We will show later that $\delta = c_1\varepsilon^3 + o(\varepsilon^3)$. Thus by Definition 2.1 we have

$$|\tau_{t_0,\varepsilon}(t_1) - \tau_{t_0,\varepsilon}(t_2)| = |\tau(t_1) - \tau(t_2)| + c_2(t_1, t_2)\varepsilon^2 + o(\varepsilon^2)$$

if $D(t_0, t_1) < \varepsilon$ or $D(t_0, t_2) < \varepsilon$. Then we may conclude that

$$E(\tau_{t_0,\varepsilon}) = E(\tau) + c_3\varepsilon^3 + o(\varepsilon^3) \quad \text{and} \quad E(\tilde{\tau}_{t_0,\varepsilon}) = E(\tau) + c_4\varepsilon^3 + o(\varepsilon^3).$$

The coefficients $c_3$ and $c_4$ of the term $\varepsilon^3$ will be called the *variation* and denoted by $Var(\tau_{t_0,\varepsilon})$ and $Var(\tilde{\tau}_{t_0,\varepsilon})$ respectively.

Now all is prepared for the definition of a locally extremal point of a knot.

**Definition 2.3.** Any $t_0 \in S^1$ is called *locally extremal point of $\tau$* if $Var(\tilde{\tau}_{t_0,\varepsilon}) = 0$ for each
locally perturbed knot $\tilde{\tau}_{t_0,\varepsilon}$ of length $2\pi$.

**Definition 2.4.** The knot $\tau$ is said to be *locally extremal* if all its points are locally extremal.

Let us find necessary and sufficient conditions for the point $t_0$ to be locally extremal. We
denote the vector product of two vectors $a$ and $b$ by $[a, b]$. By $(a, b, c)$ we denote the mixed
product (oriented volume) of the vectors $a$, $b$ and $c$. Let $\dot{\tau}(t)$ be the velocity vector and $\ddot{\tau}(t)$ be the acceleration vector. Now we define the functions $\Psi(t_0, t)$ and $\Phi(t_0, t)$.

$$\Psi(t_0, t) = \begin{cases} 
\left( \frac{\dot{\tau}(t_0)}{|\dot{\tau}(t_0)|}, \frac{\dot{\tau}(t_0)}{|\dot{\tau}(t_0)|}, \frac{\tau(t) - \tau(t_0)}{|\tau(t) - \tau(t_0)|} \right), & \text{if } \ddot{\tau}(t_0) \neq 0; \\
\left( \frac{\tau(t) - \tau(t_0)}{|\tau(t) - \tau(t_0)|}, \frac{\ddot{\tau}(t_0)}{|\ddot{\tau}(t_0)|} \right), & \text{if } \ddot{\tau}(t_0) = 0.
\end{cases}$$

$$\Phi(t_0, t) = \begin{cases} 
\left( \frac{\ddot{\tau}(t_0)}{|\ddot{\tau}(t_0)|}, \frac{\dot{\tau}(t) - \tau(t_0)}{|\dot{\tau}(t) - \tau(t_0)|}, \frac{\dot{\tau}(t_0)}{|\dot{\tau}(t_0)|}, \frac{\ddot{\tau}(t_0)}{|\ddot{\tau}(t_0)|} \right), & \text{if } \dot{\tau}(t_0) \neq 0; \\
0, & \text{if } \dot{\tau}(t_0) = 0.
\end{cases}$$

Note that $|\ddot{\tau}(t_0)| = 1$ and $|\tau(t) - \tau(t_0)| \neq 0$ if $t \neq t_0$. Thus $\Psi$ and $\Phi$ are well defined.

We also remark that $\Psi(t_0, t) = \sin \psi(t_0, t)$, where $\psi(t_0, t)$ is the angle between the vector
$\tau(t) - \tau(t_0)$ and the oriented plane spanning of $\dot{\tau}(t_0)$ and $\ddot{\tau}(t_0)$. The function $\Phi$ has a
similar representation: $\Phi(t_0, t) = \sin \phi(t_0, t)$, where $\phi(t_0, t)$ is the angle between the vector
$\tau(t) - \tau(t_0)$ and the oriented plane spanning of $\dot{\tau}(t_0)$ and $[\dot{\tau}(t_0), \ddot{\tau}(t_0)]$. (See Fig. 1). These
angles can be either positive or negative.
Theorem 2.1. Let $\tau$ be a smooth knot. The point $t_0$ is a locally extremal point of $\tau$ if and only if the following conditions hold:

$$V_1(t_0) := \frac{2}{3R(t_0)} \left( \int_{S^1} (f + R(t_0)\Phi(t_0, t)\frac{\partial f}{\partial \rho}) dt - \frac{1}{\pi} \int_{S^1 \times S^1} \left( 2f + D(t_1, t_2)\frac{\partial f}{\partial \rho} + \middle| \tau(t_1) - \tau(t_2) \middle| \frac{\partial f}{\partial \alpha} \right) dt_1 dt_2 \right) = 0;$$

$$V_2(t_0) := \frac{4}{3R(t_0)} \int_{S^1} \frac{\partial f}{\partial \rho} \Psi(t_0, t) dt = 0.$$ 

Here $A \subset S^1 \times S^1$ is the set of points $(t_1, t_2)$ such that $D(t_1, t_2) = D(t_1, t_0) + D(t_0, t_2)$.

Corollary 2.1. A knot $\tau$ is locally extremal if and only if almost all of its points are locally extremal, i.e.,

$$\int_{S^1} \left( V_1^2(t) + V_2^2(t) \right) dt = 0.$$ 

3. Proofs

Let $t_0$ be any point of $S^1$. We choose orthonormal coordinates in $\mathbb{R}^3$ such that $\tau(t_0)$ is on the $(X, Y)$-plane, $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ lie symmetrically on the $X$-axis. If $\tau(t_0 - \varepsilon)$, $\tau(t_0)$ and $\tau(t_0 + \varepsilon)$ are on the same line, then we make any possible choice of the $Y$-axis. Finally, we choose the $Z$-axis such that the orientation of the $(X, Y, Z)$-space is positive (see Fig. 2a)).
Let $P_\varepsilon$ be the class of parabolic arcs and one segment such that all the parabolas have their vertex in the $(Y, Z)$-plane, $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ are the endpoints of the arcs, and the endpoints of the segment are $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$. Each parabola can be specified by two parameters $(\lambda, \gamma)$, where $2\lambda$ is the “acceleration” and $\gamma$ is the angle between the $(X, Y)$-plane and the plane containing the parabola (see Fig. 2b)). Notice also that $(0, \gamma)$ is some segment.

Denote by $M_{P, t_0, \varepsilon}$ the 2-dimensional set of knots $\tau(t_0, \varepsilon, \lambda, \gamma)$, where the curve connecting $\tau(t_0 - \varepsilon)$ and $\tau(t_0 + \varepsilon)$ belongs to the class $P_\varepsilon$ with the following property: the knot $(\tau_{t_0, \varepsilon, \lambda, \gamma} + \tau)/2$ is a locally perturbed knot. Denote by $\tilde{M}_{P, t_0, \varepsilon}$ the set of knots associated with the knots in the class $P_\varepsilon$.

**Theorem 3.1.** Let $\tau$ be a smooth knot. The point $t_0$ is a locally extremal point if and only if $\text{Var}(\tilde{\tau}_{t_0, \varepsilon}) = 0$ for each locally perturbed (at $t_0$) knot $\tilde{\tau}_{t_0, \varepsilon} \in \tilde{M}_{P, t_0, \varepsilon}$.

**Proof of Theorem 3.1.**

We begin the proof with the following lemma.

**Lemma 3.1.** Let $C = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{y^2 + z^2} < r, |x| < s\}$ be a cylinder. Suppose a point moves inside $C$ with velocity of constant modulus 1 and so that the absolute value of its acceleration is bounded by $K$ (see Fig. 3). Let $x(0) = -s$, $x(T) = s$, $s \gg r$ and $K < 1/(4r)$. Then the length of the trajectory of a point (i.e. $T$) is bounded:

$$T < \frac{2s}{\sqrt{1 - 4Kr}}.$$
First let us prove that $\dot{y}^2(t_0) < 2Kr$. We first consider the case for which $x(t_0) < 0$, $\dot{x}(t_0) > 0$ and $y(t_0) > 0$; then

$$y(t) = y(t_0) + \int_{t_0}^{t} \dot{y}(\xi)d\xi < r.$$ 

By the assumption, we have

$$\dot{y}(\xi) = \dot{y}(t_0) + \int_{t_0}^{\xi} \ddot{y}(\zeta)d\zeta > \dot{y}(t_0) - \int_{t_0}^{\xi} Kd\zeta = \dot{y}(t_0) - (\xi - t_0)K.$$ 

It follows that

$$y(t) > y(t_0) + \int_{t_0}^{t} \dot{y}(x_0) - (\xi - t_0)Kd\xi = y(t_0) + (t - t_0)\dot{y}(t_0) - \frac{(t - t_0)^2}{2}K.$$ 

But $y(t_0) > -r$ and $y(t) < r$, so

$$(t - t_0)\dot{y}(t_0) - \frac{(t - t_0)^2}{2}K - 2r < 0.$$ 

By assumption $x < 0$ and $s \gg r$, so the vertex of the parabola is at the point $t - t_0 = \dot{y}(t_0)/K < s$. This yields the inequality $\dot{y}^2(t_0) < 2Kr$.

The proof for the cases in which $\dot{x}(t_0) > 0$ and $y(t_0) < 0$; $\dot{x}(t_0) < 0$ and $y(t_0) > 0$; $\dot{x}(t_0) < 0$ and $y(t_0) < 0$ is similar.

Secondly, we claim that $\dot{z}^2(t_0) < 2Kr$. The proof is similar to the inequality for $\dot{y}^2(t_0)$.

By the previous statements, it follows that

$$\dot{x}^2(t_0) = 1 - \dot{y}^2(t_0) - \dot{z}^2(t_0) > 1 - 4Kr > 0$$ 

for every $t_0 \in [0, T]$. So we have $T < 2s/(1 - 4Kr)$.

This completes the proof of Lemma 3.1.
Figure 4. The trajectory of the point $p(t)$ inside the cylinder $C_f$.

We continue the proof with a generalization of the previous lemma.

**Lemma 3.2.** Let $f : [-s, s] \mapsto \mathbb{R}^3$ be a unit-length smooth map, let the curvature of $f$ be bounded ($|\dot{f}(t)| < K_1$) and $sK_1 < 1$. Let $D^2(t) \in \mathbb{R}^3$, where $t \in [-s, s]$ is the disk of radius $r$ centered at $f(t)$ with the plane of the disc orthogonal to $\dot{f}$. Let also $rK_1 < 1$. Denote by $C_f = \bigcup_{-s, s} D^2(t)$ the tubular neighborhood of the curve $f$. Suppose a point $p(t) = (x(t), y(t), z(t))$ moves inside $C_f$ (see Fig. 4) with velocity of constant absolute value 1 and let the absolute value of its acceleration be bounded by $K_2$. Let $p(0) \in D^2(-s)$, $p(T) \in D^2(s)$. Let $s \gg r$ and

$$K_2 + \frac{1}{1-rK_1} K_1 < \frac{1}{4r}.$$  

Then the length of the trajectory of the point (i.e., $T$) is bounded and

$$2s(1-rK_1) < T < \frac{2s(1+rK_1)}{\sqrt{1-4K_2rr}}.$$

Let us define $\tilde{x} = t$.

Now we describe some map $\pi$ from $C_f$ to the standard cylinder $C$ (see Fig. 3). Let

$$\pi(D^2(\tilde{x})) = \{(\tilde{x}, y, z) \in \mathbb{R}| \sqrt{y^2 + z^2}\}$$

be isometric images of the disk $D^2(\tilde{x})$ for each $\tilde{x} \in [-s, s]$. If we fix a preimage $\tilde{y}$-axis of the $y$-axis and a preimage $\tilde{z}$-axis of the $z$-axis in the disc $D^2(\tilde{x})$ for each $\tilde{x} \in [-s, s]$, then the map will be completely described. As long as $sK_1 < 1$ and $rK_1 < 1$, this map is well defined and the manifold $N_f = \bigcup_{-s, s} \partial D^2(t)$ with boundary $\partial D^2(-s) \cup \partial D^2(s)$ is smooth.

Let $\pi(\tilde{y}_{-s}) = (-s, r, 0)$ for some $\tilde{y}_{-s} \in \partial D^2(-s)$. Consider the vector field on $N_f$ with the following property: if the point $q$ lies on the circle $\partial D^2(\tilde{x})$, then the vector $v_q$ equals $\dot{f}(\tilde{x})$; this means that $v_q$ is the unit-length vector orthogonal to the disc $D^2(\tilde{x})$ with the corresponding direction. Denote the integral trajectory of this field passing through the point $\tilde{y}_{-s}$ by $\tilde{y} = \{\tilde{y}(\tilde{x})| \tilde{x} \in [-s, s]\}$. This trajectory defines the $\tilde{y}$ coordinate in each disc.
$D^2(\tilde{x})$. Finally we define the unit-length $\check{z}$-vector as the vector product of the unit-length $\check{x}$-vector and unit-length $\check{y}$-vector (in each $D^2(\tilde{x})$).

The image $\pi(p)$ of the point $p$ moves inside $C$. We denote $\pi(p)$ by $\hat{p}$. Notice that
\[
\frac{|\hat{p}(t)|}{|\hat{p}(t)|} = \frac{1}{|\hat{p}(t)|} \in [1 - rK_1, 1 + rK_1].
\]
Note also that if the curvature of the trajectory is $K$ at some point $p(t)$, then the curvature of the image of this trajectory will be
\[
\hat{K} < K + \frac{1}{K_1 - r}
\]
at the point $\hat{p}(t)$, as can be easily shown.

Now Lemma 3.2 follows from Lemma 3.1.

We continue the proof of Theorem 3.1. Let $\tau_{t_0, \varepsilon}$ be any locally perturbed knot at the point $t_0$ and let $t \in S^1$ such that $D(t_0, t) < \varepsilon$. Consider
\[
P_{\tau_{t_0, \varepsilon}}(t) = \tau_{t_0, \varepsilon}(t_0) + (t - t_0)(\hat{\tau}_{t_0, \varepsilon}(t_0) + c_1) + \frac{(t - t_0)^2}{2}(\hat{\tau}(t_0) + c_2)
\]
We choose the constants $c_1 = o(\varepsilon^2)$ and $c_2 = o(\varepsilon^2)$ so that
\[
P_{\tau_{t_0, \varepsilon}}(t + \varepsilon) = \tau_{t_0, \varepsilon}(t + \varepsilon), \quad P_{\tau_{t_0, \varepsilon}}(t - \varepsilon) = \tau_{t_0, \varepsilon}(t - \varepsilon).
\]
Here we take the unit-length parametrization and denote the length of curves by $l(*)$. Then $P_{\tau_{t_0, \varepsilon}}(t)$ is a parabolic arc in the $\varepsilon$-neighborhood of the point $t_0$. From Lemma 3.2 it follows that $|P_{\tau_{t_0, \varepsilon}}(t) - \tau_{t_0, \varepsilon}(t)| = o(\varepsilon^2)$ and also $l(P_{\tau_{t_0, \varepsilon}}(t)) = l(\tau_{t_0, \varepsilon}) + o(\varepsilon^3)$. So $E_{\tau_{t_0, \varepsilon}} - E_{P_{\tau_{t_0, \varepsilon}}(t)} = o(\varepsilon^3)$.

Consider the perturbed curve $\tau_{t_0, \varepsilon, \lambda, \gamma}$ passing through the point $\tau_{t_0, \varepsilon}(t_0)$. We have
\[
|\tau_{t_0, \varepsilon, \lambda, \gamma}(t) - \tau_{t_0, \varepsilon}(t)| < \varepsilon.
\]
We also have $E_{\tau_{t_0, \varepsilon, \lambda, \gamma}} - E_{\tau_{t_0, \varepsilon}} = o(\varepsilon^3)$.

Finally we conclude that $E_{\tau_{t_0, \varepsilon, \lambda, \gamma}} - E_{\tau_{t_0, \varepsilon}} = o(\varepsilon^3)$.

One can see that the knot $\tau_{t_0, \varepsilon, \lambda, \gamma}$ belongs $M_{P, t_0, \varepsilon}$. We note again that $l(P_{\tau_{t_0, \varepsilon}}(t)) = l(\tau_{t_0, \varepsilon}) + o(\varepsilon^3)$. Hence
\[
E_{\tau_{t_0, \varepsilon, \lambda, \gamma}} - E_{\tau_{t_0, \varepsilon}} = o(\varepsilon^3).
\]
By definition, the knot $\tau_{t_0, \varepsilon, \lambda, \gamma}$ belongs $\tilde{M}_{P, t_0, \varepsilon}$. This completes the proof of Theorem 3.1.

**Proof of Theorem 2.1** Without loss of generality, we put
\[
t_0 = 0, \quad \gamma = o(1), \quad \text{and} \quad \lambda = 1/(2R(0)) + o(1),
\]
where $R(0)$ is the radius of curvature at the point 0. According to Theorem 3.1, we can consider only the class $M_P$ of knots. Let $\tau_{0, \varepsilon, \lambda, \gamma}$ be a knot in $\tilde{K}_P$. Denote
\[
\Delta := \left[ \frac{\varepsilon}{1 + \delta}, \frac{\varepsilon}{1 + \delta} \right] \subset S^1.
\]
Now note that for any $\tau$ we have
\[
E(\tau) = \int\int_{S^1 \times S^1} f \, dx \, dy = 2 \int\int_{\Delta \times S^1} f \, dx \, dy - \int\int_{A \times \Delta} f \, dx \, dy + \int\int_{S^1 \times S^1 \setminus A} f \, dx \, dy =: 2E_1(\tau) - E_2(\tau) + E_3(\tau) + E_4(\tau).
\]

Here \( f = f(\rho(\tau(x), \tau(y)), \alpha(\tau(x), \tau(y))) \). Further note that

\[
\text{Var}(\tau) = 2\text{Var}_1(\tau) - \text{Var}_2(\tau) + \text{Var}_3(\tau) + \text{Var}_4(\tau),
\]

where \( \text{Var}_i \) is the variation of \( E_i \).

First we calculate \( \text{Var}_1 \). We recall that \( \sin \phi = \Phi \) and \( \sin \psi = \Psi \).

Lemma 3.3.

\[
\text{Var}_1(\tau_0, \epsilon, \lambda, \gamma) = 4 \left( \frac{1}{3} \int_{S^1} f \frac{\partial f}{\partial \rho} \, dy \right) \left( \lambda - \frac{1}{2} R(0) \right) + \frac{2}{3} \int_{S^1} \sin \psi \frac{\partial f}{\partial \rho} \, dy \gamma.
\]

The length of the arc of the parabola is \( 2\epsilon + \frac{2}{3} \lambda^2 \epsilon^3 + o(\epsilon^3) \). So \( \delta = \frac{2}{3} \lambda^2 \epsilon^3 \). Note also that the coefficient of homothety is \( o(\epsilon^2) \) and thus \( \text{Var}_1(\tau_0, \epsilon, \lambda, \gamma) = \text{Var}_1(\tau_0, \epsilon, \lambda, \gamma) \). Let

\[
(a, b, c) = (a(t), b(t), c(t)) = \tau_0, \lambda, \gamma(t), \quad \ell = \ell(t) = \sqrt{a(t)^2 + b(t)^2 + c(t)^2}, \quad f = f(\rho, \alpha).
\]

Thus we have

\[
E_1(\tau_0, \epsilon, \lambda) = \int_{S^1} \int_{-\epsilon}^{\epsilon} \left( 1 + (2\lambda t_1)^2 \right)^{1/2} f \left( \left( (t_1 - a(t_2))^2 + (\lambda t_1^2 - \lambda \epsilon^2) \cos \gamma - b(t_1) \right)^2 + (\lambda t_1^2 - \lambda \epsilon^2) \sin \gamma - c(t_1) \right)^{1/2},
\]

\[
dt_1 dt_2 + o(\epsilon^3) = \int_{S^1} \int_{-\epsilon}^{\epsilon} \left( 1 + 2\lambda^2 t_1^2 + o(t_1^2) \right) \left( f + \frac{t_1^2 - at_1 + \lambda(\epsilon^2 - t_1^2)(b \cos \gamma + c \sin \gamma)}{\ell} - \frac{a^2 t_1^2}{2\ell^2} \right) \times \frac{\partial f}{\partial \rho} + \frac{a^2 t_1^2}{2\ell} \frac{\partial f}{\partial \rho} + D(0, t_1) \frac{\partial f}{\partial \alpha} \right) dt_1 dt_2 + o(\epsilon^3) =
\]
\[
\begin{align*}
= & \int \int_{S^1} \left( 2\lambda^2 t_1^2 f + \frac{t_1^2 - at_1 + \lambda(\varepsilon^2 - t_1^2)(b \cos \gamma + c \sin \gamma)}{\ell} \right. \\
& \left. + \frac{a^2 t_1^2}{2\ell^2} \frac{\partial f}{\partial \rho} + \frac{a^2 t_1^2 \partial^2 f}{2\ell^2} + D(t_1,0) \frac{\partial f}{\partial \alpha} \right) dt_1 dt_2 + o(\varepsilon^3) = \\
= & \int_{S^1} \left( \frac{2\lambda^2 \varepsilon^3}{3} f + \varepsilon f + \left( \frac{2\varepsilon^3 + 4\lambda \varepsilon^3 (b \cos \gamma + c \sin \gamma)}{3\ell} \right) - \frac{a^2 \varepsilon^3}{2\ell^2} \frac{\partial f}{\partial \rho} + \frac{a^2 \varepsilon^3 \partial^2 f}{2\ell^2} \right) dt_2 + \\
& \int \int_{S^1} D(t_1,0) \frac{\partial f}{\partial \alpha} dt_1 dt_2 + o(\varepsilon^3) \\
\end{align*}
\]

This yields
\[
\begin{align*}
\frac{dE_1(\lambda,\gamma)}{d\lambda} = & d \left( \int \left( \frac{2\lambda^2 \varepsilon^3}{3} f + \left( \frac{4\lambda \varepsilon^3 (b \cos \gamma + c \sin \gamma)}{3\ell} \right) \frac{\partial f}{\partial \rho} \right) dt_2 + o(\varepsilon^3) \right) = \\
& \left( \int \left( \frac{4\lambda \varepsilon^3}{3} f + \left( \frac{4\varepsilon^3 (b \cos \gamma + c \sin \gamma)}{3\ell} \right) \frac{\partial f}{\partial \rho} \right) dt_2 + o(\varepsilon^3) \right) d\lambda + \\
& \left( \int \frac{4\varepsilon^3 \lambda(-b \sin \gamma + c \cos \gamma)}{3\ell} \frac{\partial f}{\partial \rho} dt_2 + o(\varepsilon^3) \right) d\gamma.
\end{align*}
\]

Finally we substitute
\[
\frac{b}{\ell} = \sin \phi, \quad \frac{c}{\ell} = \sin \psi, \quad \gamma = o(1), \quad \lambda = \frac{1}{2R(0)} + o(1),
\]
where \(R(0)\) is the radius of curvature at the point 0, obtaining
\[
\begin{align*}
Var_1(\tilde{\tau}_{0,\varepsilon},\lambda,\gamma) = & \frac{4}{3} \left( \int_{S^1} \frac{f}{R(0)} + \sin \phi \frac{\partial f}{\partial \rho} dt_2 \right) \left( \lambda - \frac{1}{2R(0)} \right) + \frac{2}{3R(0)} \left( \int_{S^1} \sin \psi \frac{\partial f}{\partial \rho} dt_2 \right) \gamma.
\end{align*}
\]

The proof of Lemma 3.3 is complete.

**Lemma 3.4.** \(Var_2 = 0.\)

Since \(E_2(\tau) - E_2(\tilde{\tau}_{0,\varepsilon}) = o(\varepsilon^3),\) we immediately have \(Var_2 = 0.\)

**Lemma 3.5.**
\[
\begin{align*}
Var_3 = & \left( \frac{2}{3\pi R(0)} \right) \int \int_{A \setminus (\Delta \times S^1 \cup S^1 \times \Delta)} -2f - \ell \frac{\partial f}{\partial \rho} + (2\pi - D(t_1,t_2)) f \chi dt_1 dt_2 \left( \lambda - \frac{1}{2R(0)} \right).
\end{align*}
\]

The following calculations prove this lemma.
\[
E_3(\tau_0, \varepsilon, \lambda) = \iint_{A(\Delta \times S^1 \cup S^1 \times \Delta)} \left( \ell \left( 1 - \frac{2\lambda^2 \varepsilon^3}{3 \pi} \right) D \left( \left| t_2 - t_1 \right| + \frac{4\lambda^2 \varepsilon^3}{3} \left( 1 - \frac{2\lambda^2 \varepsilon^3}{3 \pi} \right), 0 \right) \right) \quad d\left( 1 - \frac{2\lambda^2 \varepsilon^3}{3 \pi} \right)t_1 d\left( 1 - \frac{2\lambda^2 \varepsilon^3}{3 \pi} \right)t_2 + o(\varepsilon^3) = \\
\iint_{A(\Delta \times S^1 \cup S^1 \times \Delta)} \left( f + \left( \frac{2 \ell}{\pi} - \frac{\ell \partial f}{\partial \rho} + \frac{2}{3} \frac{D(t_1, t_2)}{\pi} f \right) \right) \frac{2\lambda^2 \varepsilon^3}{3} dt_1 dt_2 + o(\varepsilon^3).
\]

Let us remark that
\[
\iint_{\Delta \times S^1 \cup S^1 \times \Delta} \left( -\frac{2 f}{\pi} - \frac{\ell \partial f}{\partial \rho} + \frac{2}{3} \frac{D(t_1, t_2)}{\pi} f \right) \frac{2\lambda^2 \varepsilon^3}{3} dt_1 dt_2 = o(\varepsilon^3).
\]

Therefore
\[
\text{Var}_3 = \left( \frac{2}{3\pi R(0)} \right) \iint_{A(\Delta \times S^1 \cup S^1 \times \Delta)} -2f - \ell \frac{\partial f}{\partial \rho} + \left( 2\pi - D(t_1, t_2) \right) f \partial \rho dt_1 dt_2 \left( \lambda - \frac{1}{2R(0)} \right).
\]

Lemma 3.6.
\[
\text{Var}_4 = \left( \frac{2}{3\pi R(0)} \right) \iint_{S^1 \times S^1 \setminus A} -2f - \ell \frac{\partial f}{\partial \rho} - D(t_1, t_2) \partial \alpha \partial \alpha dt_1 dt_2 \left( \lambda - \frac{1}{2R(0)} \right).
\]

The proof of this lemma is similar to the previous one.

Lemmas 3.3-3.6 complete the proof of Theorem 2.1.

4. Corollaries

In [1] it is shown that the circle is not always the global maximum, or the global minimum for the energy considered. Let us show that circle is a locally extremal knot for any energy \( E \) satisfying the conditions 1), 2) of the Introduction.

Corollary 4.1. The circle is always a locally extremal knot.

If \( \tau \) is a circle, then
\[
\ell(t_1, t_2) = 2 \sin \frac{t_2 - t_1}{2}, \quad R(t_1) = 1, \quad \psi(t_1, t_2) = 0, \quad \phi = \frac{t_2 - t_1}{2}.
\]

So \( V_2(t_1) = 0 \) for any \( t_1 \in S^1 \). Further
\[ V_1(t_1) = \frac{1}{3} \left( 8 \int_{S^1} f + \sin \left( \frac{|t_1|}{2} \right) f_\rho \, dt_1 - \frac{2}{\pi} \int \int_{S^1 \times S^1} 2f + 2 \sin \left( \frac{|t_2 - t_1|}{2} \right) \frac{\partial f}{\partial \rho} \right. \\
+ \left. D(t_1, t_2) \frac{\partial f}{\partial \alpha} \, dt_1 \, dt_2 + \frac{2}{\pi} \int \int_{S^1 \times S^1} D(0, t_1) \frac{\partial f}{\partial \alpha} \, dt_1 \right) = \\
\frac{1}{3} \left( 8 \int_{S^1} f + \sin \left( \frac{|t_1|}{2} \right) \frac{\partial f}{\partial \rho} \, dt_1 - \frac{2}{\pi} \int \int_{S^1 \times S^1} 2f + 2 \sin \left( \frac{|t_1|}{2} \right) \frac{\partial f}{\partial \rho} + D(0, t_1) \frac{\partial f}{\partial \alpha} \, dt_1 + \\
4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2D(t_1, t_2) \frac{\partial f}{\partial \alpha} \, dt_1 \, dt_2 \right) f(\rho, \alpha) = f(\rho, 2\pi - \alpha) = -4 \int_{S^1} D(0, t_1) \frac{\partial f}{\partial \alpha} \, dt_1 + 4 \int_{S^1} D(0, t_1) \frac{\partial f}{\partial \alpha} \, dt_1 = 0. \]

Therefore any point of the circle is a locally extremal point. Hence the circle is locally extremal. The corollary is proved.

Now let us say a few words about Möbius energy which is (in the version from [3])

\[ f_M = \frac{1}{|\tau(t_1) - \tau(t_2)|^2} - \frac{1}{D^2(t_1, t_2)}. \]

It has many remarkable properties (see [9] and [3]). Möbius energies of homothetic knots are equal. This energy is invariant for Möbius transformations (see also Section 5). The variational equations and the gradient flow equation of Möbius energy was studied in [3].

Unfortunately, for Möbius energy, the variation \( \text{Var} \) is always infinite, and this means that we cannot perturb the knot in the way considered above.

The main property of Möbius energy is as follows. When a knot crossing tends to a double point, the energy tends to infinity. The energy is always positive. So every topological type of knot has a representative with minimal value of energy, some normal form.

Notice that the main part of Möbius energy is \( 1/|\tau(t_1) - \tau(t_2)|^2 \). The other part \( 1/D^2(t_1, t_2) \) is only a normalization that makes the integral convergent. So let us make another normalization of the “main part” of Möbius energy. In this case we often lose the invariance for Möbius transformations. Let us consider the following energy:

\[ \tilde{f} = \frac{D^3(x, y)}{|\tau(x), \tau(y)|^2}. \]

It is easily seen that this energy on one hand has the above property and on the other we can use our variational principles. Note also that such an energy is the same for homothetic knots.
Corollary 4.2. We present $V_1$ and $V_2$ for this energy:

$$V_1(t_0) = \frac{2}{3R(t_0)} \left( 4 \int_{S^1} \left( \frac{|\tau(t) - \tau(t_0)|^3}{D(t,t_0)^2} \left( 1 - 2 \frac{R(t_0)}{D(t,t_0)} \Phi(t_0,t) \right) \right) dt - \frac{3}{\pi} \iint_{S^1 \times S^1} \frac{|\tau(t_2) - \tau(t_1)|^3}{D(t_2,t_1)^2} dt_1 dt_2 + 6 \iint_A \frac{|\tau(t_2) - \tau(t_1)|^2}{D(t_2,t_1)^2} dt_1 dt_2 \right);$$

$$V_2(t_0) = -\frac{8}{3R(t_0)} \int_{S^1} \frac{|\tau(t_1) - \tau(t_2)|^3}{D(t_0,t)^3} \Psi(t_0,t) dt.$$

5. Definition and some basic properties of Mm-energy

In this section we define the Mm-energy of a knot. The nature of this energy differs from the energies considered in the previous sections.

Let us fix some point $t_0$ on the circle and define the real number $f_{Mm}(t_0)$. Consider the map $\rho_{t_0} : S^1 \rightarrow \mathbb{R}$ such that $\rho_{t_0}(t) = |\tau(t) - \tau(t_0)|$. Let us note that the map $\tau$ is smooth. Hence $\rho_{t_0}$ is also smooth except for one point $t_0$. If the number of maximums and minimums is finite, then we define the function $f_{Mm}$ as follows:

$$f_{Mm}(t_0) = \frac{1}{\rho_{t_0}(t_M)} + \sum_{t_{m_i} \in U_1} \frac{1}{\rho_{t_0}(t_{m_i})} - \sum_{t_{M_j} \in U_2} \frac{1}{\rho_{t_0}(t_{M_j})},$$

where $t_M$ is one of the points where the function $\rho_{t_0}$ achieves its global maximum; $U_1$ is the set of all points of the circle, except the point $t_0$, where the function $\rho_{t_0}$ has local minimums; $U_2$ is the set of all points of the circle, except the point $t_M$, where the function $\rho_{t_0}$ has local maximums (see Fig. 5). Here we suppose $t_0 < t_* < t_0 + 2\pi$. In the case of an infinite number of maximums and minimums we make a small smooth perturbation $\tilde{\rho}_{t_0}$ so that the number of minimums and maximums becomes finite. Now we can calculate the value of $\tilde{f}_{Mm}(t_0)$ for the function $\tilde{\rho}_{t_0}$ as it was made before. Finally we define the $f_{Mm}(t_0)$ as the limit of $\tilde{f}_{Mm}(t_0)$ in the $C^\infty$-topology.

Now we define the Mm-energy.

**Definition 5.1.** We call Mm-energy of the given knot the following number:

$$E_{Mm}(\tau) = \int_{S^1} f_{Mm}(t) dt,$$

if the integral converges.

**Remark 5.1.** Consider some small smooth perturbation of a knot. Then for any point $t_0$ of the circle the function $\rho_{t_0}$ is also perturbed in a smooth way. At a generic point four possible modifications in the sums of $f_{Mm}$ can occur: small changes of the values of the
maximums and minimums; the death of one maximum and of the neighboring minimum; conversely, the birth of one maximum and minimum at some point; a local maximum close to the global maximum can become the global maximum. In all these cases the variation of the resulting $f_{Mm}$ is small. This is the reason why the Mm-energy depends on small perturbations of knots continuously.

Further we formulate the basic properties of Mm-energy.

**Proposition 5.1.** The Mm-energy is greater than or equal to 2.

Consider the sum

$$f_{Mm}(t_0) = \frac{1}{\rho_{t_0}(t_M)} + \sum_{t_{m_i} \in U_1} \frac{1}{\rho_{t_0}(t_{m_i})} - \sum_{t_{M_j} \in U_2} \frac{1}{\rho_{t_0}(t_{M_j})}$$

We can fix the ordering of the minimums and the maximums in the standard way:

$$t_0 < t_{M_1} < t_{m_1} < \ldots < t_{M_k} < t_{m_k} < t_M < t_{m_{k+1}} < t_{M_{k+1}} < \ldots < t_{m_n} < t_{M_n} < t_0 + 2\pi.$$ 

Then we have

$$f_{Mm}(t_0) = \sum_{i=0}^{k} \left( \frac{1}{\rho_{t_0}(t_{m_i})} - \frac{1}{\rho_{t_0}(t_{M_i})} \right) + \frac{1}{\rho_{t_0}(t_M)} + \sum_{i=k+1}^{n} \left( \frac{1}{\rho_{t_0}(t_{m_i})} - \frac{1}{\rho_{t_0}(t_{M_i})} \right) \geq 0 + \frac{1}{\rho_{t_0}(t_M)} + 0 = \frac{1}{\rho_{t_0}(t_M)}.$$ 

Finally, note that the length of the knot is $2\pi$, hence the function $\rho_{t_0}(t_M)$ is smaller than or equal to $\pi$. Therefore

$$E_{Mm}(\tau) = \int_{S^1} f_{Mm}(t) dt \leq \int_{S^1} \frac{1}{\rho_t(t_{M_i})} dt \leq \int_{S^1} \frac{1}{\pi} dt = \frac{2\pi}{\pi} = 2.$$ 

This completes the proof of Proposition 5.1.

**Proposition 5.2.** The Mm-energy is an invariant of homothety.
Suppose \( \tau \) is a knot of length \( 2\pi \) and \( \tilde{\tau} \) is a homothetic knot of length \( 2l\pi \), where \( l \) is the coefficient of homothety. Then \( d\tilde{t} = ldt \) and \( \tilde{\rho}(\tilde{t}) = l\rho(t) \) for any \( t \), and so \( \tilde{f}_{Mm}(\tilde{t}) = f_{Mm}(t)/l \). Thus we obtain

\[
E_{Mm}(\tilde{\tau}) = \int_{S^1} \tilde{f}_{Mm}(\tilde{t})d\tilde{t} = \int_{S^1} \frac{f_{Mm}(t)}{l}ldt = \int_{S^1} f_{Mm}(t)dt = E_{Mm}(\tau).
\]

Proposition 5.2 is proven.

So we can consider knots without any restriction on their lengths.

**Proposition 5.3.** When two branches of the knot tends to a double crossing, the Mm-energy tends to infinity.

Consider a smooth family \( \{\tau_\lambda|\lambda \in [0,1]\} \) such that \( \tau_0 \) is a smooth knot with double crossing and \( \tau_\lambda, \lambda \neq 0 \) is a smooth knot without any double crossing. For every \( \varepsilon \) we can choose a sufficiently small \( \lambda \) satisfying the following conditions: there exist two points \( t_1 \) and \( t_2 \) with \( |t_1 - t_2| < \varepsilon^2 \) such that the functions \( \rho_{t_1} \) and \( \rho_{t_2} \) have global minima at the points \( t_2 \) and \( t_1 \) correspondingly; and the ball \( B_{\varepsilon,p} \) of radius \( \varepsilon \) with center at the midpoint \( p \) of the segment \([\tau_\lambda(t_1), \tau_\lambda(t_2)]\) has only two connected components of a knot \( \tau_\lambda \) inside.

The family is smooth, hence the curvature of all knots is bounded by some \( N \). If \( \varepsilon < 1/N \), then every point \( t \) of the knot \( \tau_\lambda \) inside the ball \( B_{\varepsilon/2,p} \) has one extremum (i.e., the global minimum) of the function \( \rho_t \) inside the ball \( B_{\varepsilon,p} \), and every point \( t \) of this knot inside the ball \( B_{\varepsilon,p} \) has no more than one extremum (i.e., the global minimum) of \( \rho_t \) inside the ball \( B_{\varepsilon,p} \). Let us estimate the energy inside the ball \( B_{\varepsilon,p} \).

\[
E_{Mm}(\tau_\lambda \cap B_{\varepsilon,p}) > 4 \int_{\varepsilon^2}^{\varepsilon^2} \frac{1}{t + \frac{\varepsilon^2}{2}} dt = 4 \ln(t + \frac{\varepsilon^2}{2})\bigg|_{\varepsilon^2}^{\varepsilon^2} = 4 \ln \frac{\varepsilon^2 + \varepsilon^2}{\varepsilon^2} > 4 \ln \frac{2}{\varepsilon}.
\]

The other terms (we ignore the global minimum of \( \rho_t \)) of the function \( f_{Mm} \) changes in a smooth way, hence the Mm-energy grows to infinity.

Therefore Mm-energy separates knots from different topological classes.

The following property is an essential property of Mm-energy.

**Proposition 5.4.** The Mm-energy is well defined for piecewise smooth knots with obtuse angles.

If some point \( t \) is “near” the angle then the function \( \rho_t \) is monotone function in some neighborhood of the vertex of an angle and hence there are no minima or maxima of \( \rho_t \) in this neighborhood.

In particular, the Mm-energy is well defined for piecewise linear knots with obtuse angles. So we can consider piecewise linear approximations of smooth knots and take the restriction to the set of piecewise linear knots. This property allows us to develop computer experiments in calculating normal forms for Mm-energies of topological classes of knots and the values of Mm-energies for this normal forms.
Now we calculate Mm-energy for some knots. First we find the Mm-energy of the circle $\tau_0$

$$E_{Mm}(\tau_0) = \int_{S^1} \frac{1}{2} dt = \pi.$$ 

Unfortunately the circle is not the normal form for the class of trivial knots. An example of the trivial knot with Mm-energy less than $\pi$ is shown on Figure 6. This knot is a union of two arcs of the circle. Direct calculations shows that the Mm-energy of this knot is $2 \ln \left( \frac{7 + 4\sqrt{3}}{3} \right) \approx 3.070607 < \pi$.

Computer experiments provide upper bounds for the Mm-energies of the normal forms for some topological classes (see the table behind).

| CLASSES OF KNOTS | THE UPPER BOUNDS FOR THE ENERGIES OF NORMAL FORMS |
|------------------|-----------------------------------------------|
| the class of the circle | 3.044012 |
| the class of the trefoil | 13.152759 |
| the class of the figure-eight | 19.450447 |
| the class of $5_1$ | 26.498108 |
| the class of $5_2$ | 27.168222 |
| the class of $6_1$ | 34.469191 |
| the class of $6_2$ | 35.466138 |
| the class of $6_3$ | 37.683129 |
| the class of the connected sum of right and left trefoils | 25.734616 |
| the class of the connected sum of two right trefoils | 26.748901 |
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E-mail address, Oleg Karpenkov: karpenk@mccme.ru