In this paper, we discuss the internal and external metric of the semi-realistic stars in relativistic MOND theory. We show the Oppenheimer-Volkoff equation in relativistic MOND theory and get the metric and pressure inside the stars to order of post-Newtonian corrections. We study the features of motion around the static, spherically symmetric stars by Hamilton-Jacobi method, and find there are only some small corrections in relativistic MOND theory.

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1. Introduction

The "missing mass" problem comes of the discrepancy between two methods which relate to the measurements of mass to luminosity ratio. One is to measure the cluster's mass to luminosity by dynamic method; and the other is to directly measure the mass to luminosity through the rotation curve of spiral galaxies. To solve the problem, a prevalent way is to assume that there is much more invisible non-baryonic dark matter than visible baryonic matter in the universe. However, as an alternative to dark matter, Milgrom[1] proposed a Modified Newtonian Dynamics(MOND) which implies that Poisson equation should be rewritten as $\nabla (\nabla \Phi |\nabla \Phi|/a_0) = 4\pi G \rho$ when $|\nabla \Phi| \ll a_0$, where $a_0 \approx 1 \times 10^{-8} \text{cm}^2/\text{s}^2$ from empirical data. One found that MOND can explain not only the rotation curve[2] of spiral galaxies but also the Tully-Fisher law[3]: $L_K \propto v_a^4$, where $L_K$ is the infrared luminosity of a disk galaxy and $v_a$ is the asymptotic rotational velocity.

Though MOND can explain the experiment successfully, it is not a theory and there exists some theoretical problems, such as the conservation of momentum and angular momentum. Recently, Bekenstein[4] has developed MOND to be a relativistic theory of gravity, abbreviated TeVeS, which depends on a tensor field $g_{\alpha\beta}$, a vector field $U_\alpha$, a dynamical scalar field $\phi$ and non-dynamical scalar field $\sigma$. The theory also involves a free function $F$, a length scale $\ell$ and two positive dimensionless parameters, $k$ and $K$. Bekenstein has shown that TeVeS has general relativity(GR) as its limit when $k \to 0$ with $\ell \propto k^{-3/2}$ and $K \propto k$ in Friedmann-Robertson-Walker(FRW) cosmology, and TeVeS has a MOND and Newtonian limit under the proper circumstance. TeVeS also passes the elementary solar system tests of gravity theory[5]. Even though TeVeS is not a perfect theory, such as the violation of the local Lorentz invariance and the free choosing of the function $F$, TeVeS is still strong interesting recently. A recent paper[6] discussed the large scale structure in TeVeS theory which show that it may be possible to reproduce observations of the cosmic microwave background and galaxy distributions.

The external spacetime of the spherically symmetric objects in TeVeS has been explored by Bekenstein[4] and Giannios[7]. Bekenstein has studied the post-Newtonian corrections of the static, spherically symmetric metric outside the spherically symmetric objects, and Giannios has derived the analytic expression for the physical metric. Up to now, there is no research on the stellar structure in TeVeS. In this paper we study analytically a semi-realistic star to get the primary characteristic of stellar structure in TeVeS. Firstly, we discuss the asymptotic vacuum spacetimes in the exterior of the stars by use of a static, spherically symmetric metric which can be transformed into the metric Bekenstein adopted. Secondly, we consider the Oppenheimer-Volkoff equation in TeVeS, and use the series expansion to compute the metric and pressure inside the stars. Finally, to get the observable correct effect of TeVeS, we explore the features of motion around the spherically symmetric stars by Hamilton-Jacobi method.

2. The Brief Review of TeVeS

The basic equations of TeVeS can be derived from the action[4]:

$$S = S_g + S_v + S_\phi + S_m,$$

(1)
where the actions of the tensor field \( S_g \), the vector field \( S_v \), the scalar field \( S_s \) and the matter \( S_m \) are:

\[
S_g = (16\pi G)^{-1} \int \sigma^{\alpha\beta} R_{\alpha\beta} (-g)^{1/2} dx^4,
\]

\[
S_s = -\frac{1}{2} \int \left[ \sigma^2 h^{\alpha\beta} \phi_\alpha \dot{\phi}_\beta + \frac{1}{2} G\sigma^2 F(kG\sigma^2) \right] (-g)^{1/2} dx^4,
\]

\[
S_v = -\frac{K}{32\pi G} \int [g^{\alpha\beta} g^{\mu\nu} \mathcal{U}_{[\alpha,\mu]} \mathcal{U}_{[\beta,\nu]} - 2(\lambda/K)(g_{\mu\nu} \mathcal{U}_\nu + 1)] (-g)^{1/2} dx^4,
\]

\[
S_m = \int \mathcal{L}(\tilde{g}_{\mu\nu}, f^\alpha, f^\alpha_{[\mu\nu]}, \cdots) (-\tilde{g})^{1/2} dx^4.
\]

In the equations above, \( g \) is the determinant of Einstein metric \( g_{\alpha\beta} \), \( R_{\alpha\beta} \) is the Ricci tensor of \( g_{\alpha\beta} \) just as in GR, \( h^{\alpha\beta} = g^{\alpha\beta} - \mathcal{U}^\alpha \mathcal{U}^\beta \), and \( \tilde{g}_{\alpha\beta} \) is the physical metric which is related to the Einstein metric by \( \tilde{g}_{\alpha\beta} = e^{-2\phi}(g_{\alpha\beta} + \mathcal{U}_{\alpha} \mathcal{U}_{\beta}) - e^{2\phi} \mathcal{U}_{\alpha} \mathcal{U}_{\beta} \). \( F(\mu) \) is a free dimensionless function, with \( \mu = kG\sigma^2 \). \( \lambda \) is a spacetime dependent Lagrange multiplier and \( f^\alpha \) symbolically represents the matter fields. The symbol \( | \) denotes the covariant derivatives with respect to \( \tilde{g}_{\mu\nu} \) and a pair of indices surrounded by brackets stands for antisymmetrization, i.e. \( A_{[\mu}B_{\nu]} = A_{\mu}B_{\nu} - A_{\nu}B_{\mu} \).

Varying the action \( S \) with respect to \( g^{\alpha\beta}, \mathcal{U}_\alpha \) and \( \phi \), one will get the metric equation

\[
G_{\alpha\beta} = 8\pi G \left[ \mathcal{T}_{\alpha\beta} + (1 - e^{-4\phi}) \mathcal{M}_{\mu} \mathcal{T}_{\mu\alpha\beta} + \tau_{\alpha\beta} \right] + \Theta_{\alpha\beta},
\]

the vector equation

\[
K\mathcal{U}^{[\alpha_{[\beta}} \mathcal{U}^{\gamma_{\beta]}} + \tilde{\lambda}\mathcal{U}^\alpha + 8\pi G\sigma^2 \mathcal{U}^\beta \phi_\alpha \phi_\gamma = 8\pi G(1 - e^{-4\phi}) g^{\alpha\mu} \mathcal{U}^\beta \mathcal{T}_{\mu\beta},
\]

and the scalar equation

\[
[\mu h^{\alpha\beta} \phi_\alpha]_{[\beta} + \frac{1}{2} G\sigma^2 (1 + e^{-4\phi}) \mathcal{U}_{[\alpha\beta]} \mathcal{T}_{\beta],
\]

where

\[
\tau_{\alpha\beta} \equiv \sigma^2 \left[ \phi_\alpha \phi_\beta - \frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu g_{\alpha\beta} - \mathcal{U}^\mu \phi_{\mu} \left( \mathcal{U}_{[\alpha} \phi_{\beta]} - \frac{1}{2} \mathcal{U}^\tau \phi_{\tau} g_{\alpha\beta} \right) \right] - \frac{1}{4} G\sigma^2 F(\mu) g_{\alpha\beta},
\]

\[
\Theta_{\alpha\beta} \equiv K \left( g^{\mu\nu} \mathcal{U}_{[\mu,\alpha]} \mathcal{U}_{[\nu,\beta]} \right) - \frac{1}{2} g^{\tau\gamma} g^{\mu\nu} \mathcal{U}_{[\sigma,\mu]} \mathcal{U}_{[\tau,\nu]} g_{\alpha\beta} \right) - \tilde{\lambda} \mathcal{U}_{\alpha\beta},
\]

and a pair of indices surrounded by parenthesis stands for symmetrization, i.e. \( A_{[\mu}B_{\nu]} = A_{\mu}B_{\nu} + A_{\nu}B_{\mu} \). If one models the matter as a perfect fluid, then the energy-momentum tensor has the form

\[
\mathcal{T}_{\alpha\beta} = \tilde{\rho} \mathcal{U}_{\alpha\beta} + \tilde{p} (\tilde{g}_{\alpha\beta} + \mathcal{U}_{\alpha} \mathcal{U}_{\beta}),
\]

where \( \tilde{\rho} \) is the proper energy density, \( \tilde{p} \), the pressure and \( \mathcal{U}_{\alpha} \), the 4-velocity, with \( \mathcal{U}_{\alpha} = e^{\phi} \mathcal{U}_{\alpha} \), all three expressed in the physical metric. If varying the action \( S_v \) with respect to \( \sigma \), one can arrive at the equation, \(-\mu F(\mu) - \frac{1}{2} \mu^2 F'(\mu) = k\mathcal{E} h^{\alpha\beta} \phi_\alpha \phi_\beta \). There is some discussion about the free function \( F(\mu) \) [4]. Because in this paper, we are interested in the post-Newtonian corrections not the purely MOND case, we shall take \( \mu = 1 \), i.e. \( \sigma^2 = 1/(kG) \).

### 3. External Spacetime of a Static, Spherically Symmetric Star in TeVeS

The metric of the static and spherically symmetric stars can be written as

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

where \( \nu \) and \( \lambda \) are the function of \( r \) only. It is worth noting that metric [12] has the different form with that in Ref. [4], but both metric can be transformed between each other. As we are looking for the static solutions, we will take the vector field to be pointing in the timelike direction. Normalized to \( \mathcal{U}_{\alpha} \mathcal{U}^\alpha = -1 \), it has

\[
\mathcal{U}^\alpha = \{ e^{-\nu/2}, 0, 0, 0 \}.
\]
Using Eqs. (6) and (13), we get easily the components of gravitational equation. The \(tt\) component is:

\[
\frac{e^{-\lambda}}{r^2} \left( -1 + e^\lambda + r' \right) = 8\pi G [\tilde{p} e^{2\phi} + \frac{1}{kG} \frac{e^{-\lambda}}{2} \left( \phi' \right)^2] + Ke^{-\lambda} [\frac{(\nu')^2}{8} + \frac{\nu''}{2} - \frac{\nu' \chi'}{4} + \frac{\nu'}{r}],
\]

the \(rr\) component is:

\[
\frac{e^{-\lambda}}{r^2} (1 - e^\lambda + r') = 8\pi G [\tilde{p} e^{-2\phi} + \frac{1}{kG} \frac{e^{-\lambda}}{2} \left( \phi' \right)^2] + Ke^{-\lambda} [-\frac{(\nu')^2}{8}],
\]

and the \(\theta\theta\) component is:

\[
\frac{e^{-\lambda}}{2} \left[ -\frac{(\nu')^2}{2} + \frac{\nu''}{2} - \frac{\nu' - \chi'}{r} \right] = 8\pi G [\tilde{p} e^{-2\phi} - \frac{1}{kG} \frac{e^{-\lambda}}{2} \left( \phi' \right)^2] + Ke^{-\lambda} [-\frac{(\nu')^2}{8}].
\]

The \(\varphi\varphi\) component is proportional to the \(\theta\theta\) component, so there is no need to consider it separately. In principle, the first two components of gravitational equations are enough to solve the metric. Furthermore the scalar equation becomes

\[
\left[ e^{(\lambda - \nu)/2} \right]' = kG (\tilde{p} + 3\tilde{\rho}) e^{(\nu + \lambda)/2} e^{-2\phi} r^2.
\]

Similar to Bekenstein’s definition [4], we define the "scalar mass" \(m_s\) as

\[
m_s = m_s(R) = 4\pi \int_0^R (\tilde{\rho} + 3\tilde{\rho}) e^{(\nu + \lambda)/2} e^{-2\phi} r^2 dr,
\]

where \(R\) is the radius of a spherical symmetric star.

We consider the case that the exterior of the star is vacuum, and have \(\tilde{\rho} = 0\) and \(\tilde{\rho} = 0\). For \(r > R\), Eq. (17) becomes

\[
\phi' = k \frac{G m_s e^{(\lambda - \nu)/2}}{4\pi r^2}.
\]

Substitute \(\tilde{\rho}, \tilde{\rho}\) and \(\phi'\) into Eqs. (14) and (15), we have

\[
\frac{e^{-\lambda}}{r^2} \left( -1 + e^\lambda + r' \right) = k \frac{(G m_s)^2 e^{-\nu}}{4\pi r^4} + Ke^{-\lambda} [-\frac{(\nu')^2}{8} + \frac{\nu''}{2} - \frac{\nu' \chi'}{4} + \frac{\nu'}{r}],
\]

and

\[
\frac{e^{-\lambda}}{r^2} (1 - e^\lambda + r') = k \frac{(G m_s)^2 e^{-\nu}}{4\pi r^4} + Ke^{-\lambda} [-\frac{(\nu')^2}{8}].
\]

For getting the post-Newtonian corrections, one can expand \(e''\) and \(e^\lambda\) as

\[
e'' = \alpha_0 [1 - \frac{r_e}{r}] + \alpha_2 \left( \frac{r_e}{r} \right)^2 + \alpha_3 \left( \frac{r_e}{r} \right)^3 + \ldots,
\]

\[
e^\lambda = \beta_0 [1 + \beta_1 \left( \frac{r_e}{r} \right) + \beta_2 \left( \frac{r_e}{r} \right)^2 + \beta_3 \left( \frac{r_e}{r} \right)^3 + \ldots,
\]

where \(r_e\) is a length scale to be determined in the next section, and the size of the coefficient of the \(r_e/r\) term in Eq. (22) has been absorbed into \(r_e\). Substituting them into Eqs. (20) and (21), matching the coefficient of like powers of \(1/r\), and solving the recurrence relation, we obtain

\[
\alpha_2 = 0,
\]

\[
\alpha_3 = -\frac{K}{48} + \frac{k}{a_0} \frac{(G m_s)^2}{4\pi} \frac{1}{6r_e^2}, \ldots
\]

\[
\beta_0 = 1,
\]

\[
\beta_1 = 1,
\]

\[
\beta_2 = 1 + \frac{K}{8} - \frac{k}{a_0} \frac{(G m_s)^2}{4\pi} \frac{1}{r_e^2},
\]

\[
\beta_3 = 1 + \frac{5K}{16} - \frac{k}{a_0} \frac{(G m_s)^2}{4\pi} \frac{5}{2r_e^2}, \ldots
\]
Comparing the expansion with the Schwarzschild metric in GR, we get

\[ \phi = \phi_c - \frac{kGm_r}{4\pi r} - \frac{kGmr_e}{8\pi r^2} + \mathcal{O}(r^{-3}), \]

(26)

where the constant \( \phi_c \) is a cosmological value and can be absorbed in the rescaling \( t \) and \( r \) coordinates.

Because of the relation \( \tilde{g}_{tt} = -e^{2\phi/\nu} \), \( \tilde{g}_{rr} = e^{-2\phi/\lambda} \) and \( \tilde{g}_{\theta\theta} = \tilde{g}_{\phi\phi}/\sin^2 \theta = r^2 e^{-2\phi} \), the physical metric becomes

\[ \tilde{g}_{tt} = -1 + \frac{2G_N m}{r} - \frac{2G_N m(G_N m - \frac{r_e}{2})}{r^2} + \mathcal{O}(r^{-3}), \]

(27)

\[ \tilde{g}_{rr} = 1 + \frac{2G_N m}{r} + \frac{2G_N m(G_N m + \frac{r_e}{2})}{r^2} - \frac{k(Gm_r)^2}{4\pi r^2} + \frac{kGm_N}{8\pi r^2} + \mathcal{O}(r^{-3}), \]

\[ \tilde{g}_{\theta\theta} = r^2 + (2G_N m - r_e) r + 2G_N m(G_N m - \frac{r_e}{2}) + \mathcal{O}(r^{-1}), \]

\[ \tilde{g}_{\phi\phi} = \sin^2 \theta \tilde{g}_{\theta\theta}, \]

where \( 2G_N m = r_e + 2kGm_r/4\pi \).

4. Oppenheimer-Volkoff equation in TeVeS

To get some characteristic of stellar structure in TeVeS, we will research a semi-realistic stars in this section. In other words we can get the kernel information about the stellar structure in TeVeS through the simplified model, and to find how difference the stellar structure is between in TeVeS and in GR. The simplified model of star comes from assuming that the fluid is incompressible: the density is a constant \( \tilde{\rho}_0 \) at \( r < R \) and it vanishes at \( r > R \). In the internal of a star with ideal fluid \( \tilde{T}_{\alpha\beta} \), the \( r \) component of energy-momentum conservation law \( \tilde{T}_{\alpha r} = 0 \) gives

\[ \frac{d (\tilde{\rho} e^{-2\phi})}{dr} = -\frac{(\tilde{\rho} e^{-2\phi} + \rho e^{2\phi})}{2} \frac{d\nu}{dr}. \]

(28)

Rewriting Eq. (24) and Eq. (25), we have

\[ \frac{e^{-\lambda}}{r^2} (-1 + e^\lambda + r\nu') = 8\pi G\tilde{\rho}, \]

(29)

and

\[ \frac{e^{-\lambda}}{r^2} (1 - e^\lambda + r\nu') = 8\pi G\tilde{\rho}. \]

(30)

where \( \tilde{\rho} = \tilde{\rho} e^{2\phi} + \frac{K}{8\pi G} e^{-\lambda} \left[ \frac{(\nu')^2}{8} + 2\nu' - \frac{\nu'^2}{4} + \nu'' \right] \) and \( \tilde{\rho} = \left[ \rho e^{-2\phi} + \frac{K}{8\pi G} e^{-\lambda} \left[ -\frac{(\nu')^2}{8} \right] \right] \). According to Eq. (29), it is convient to replace \( \lambda(r) \) with a new function \( m(r) \), given by

\[ m(r) = \frac{1}{2G} \left( r - re^{-\lambda} \right), \]

(31)

or equivalently

\[ e^{-\lambda} = 1 - \frac{2Gm(r)}{r}. \]

(32)

Then Eq. (29) becomes

\[ \frac{dm(r)}{dr} = 4\pi r^2 \tilde{\rho}, \]

(33)

which can be integrated to obtain

\[ m(r) = 4\pi \int_0^r \tilde{\rho}(r')r'^2dr'. \]

(34)
For $r \geq R$,
\[
e^{-\lambda} = 1 - \left(1 - \frac{2G \left[4\pi \int_{0}^{R} \tilde{\rho}(r')r'^2 dr' + 4\pi \int_{R}^{\infty} \tilde{\rho}(r')r'^2 dr'\right]}{r} \right) = 1 - \frac{2Gm_g(R)}{r} + \frac{8\pi G \int_{R}^{\infty} \tilde{\rho}(r')r'^2 dr'}{r},
\]
(35)

where $m_g$ is the "gravitational mass", with $m_g = m_g(R) = 4\pi \int_{0}^{R} \tilde{\rho}(r')r'^2 dr'$, and in the integration the expression of $\tilde{\rho}(r')$ depends on the internal metric which has a complex form. At $r = R$, using Eqs. (15), (25) and (35), we have the relation between $m_s, m_g$ and $r_e$

\[
r_e = \frac{K r_e^2}{8R} + \frac{kG^2 m_s^2}{4\pi R^2} + O\left(\frac{r_e^3}{R^2}\right) = 2Gm_g.
\]
(36)

In terms of $m(r)$, Eq. (36) can be rewritten

\[
d\nu = \frac{2Gm(r) + 8\pi G\tilde{\rho}^3}{r[r - 2Gm(r)]}.
\]
(37)

Combining Eq. (37) with Eq. (29) allows us to obtain the Oppenheimer-Volkoff equation in TeVeS

\[
d(\tilde{\rho}e^{-2\phi}) = -(\tilde{\rho}e^{-2\phi} + \tilde{\rho}e^{2\phi})\frac{Gm(r) + 4\pi G\tilde{\rho}^3}{r[r - 2Gm(r)]}.
\]
(38)

To post-Newtonian corrections, in the interiors of static, spherical symmetry stars, we expand $e^\nu, e^\lambda, \phi$ and $\tilde{\rho}$ as

\[
e^\nu = a_0 [1 + \left(\frac{r}{r_i}\right)^2 + a_2 \left(\frac{r}{r_i}\right)^4 + a_4 \left(\frac{r}{r_i}\right)^6 + \ldots],
\]
(39)

\[
e^\lambda = b_0 [1 + b_1 \left(\frac{r}{r_i}\right)^2 + b_2 \left(\frac{r}{r_i}\right)^4 + b_4 \left(\frac{r}{r_i}\right)^6 + \ldots],
\]
(40)

\[\tilde{\rho} = c_0 [1 + c_1 \left(\frac{r}{r_i}\right)^2 + c_2 \left(\frac{r}{r_i}\right)^4 + c_4 \left(\frac{r}{r_i}\right)^6 + \ldots],
\]
(41)

\[
\phi = d_1 \left(\frac{r}{r_i}\right)^2 + d_2 \left(\frac{r}{r_i}\right)^4 + d_4 \left(\frac{r}{r_i}\right)^6 + \ldots,
\]
(42)

where $r_i$ is a length scale to be determined later and the size of the coefficient of the $r/r_i$ term in Eq. (39) has been absorbed into $r_i$. In Eq. (12), we need not write down the constant term because the constant is a cosmological value and can be absorbed in the rescaling $t$ and $r$ coordinates. Substituting them into Eq. (14), Eq. (15), Eq. (17) and Eq. (18), matching the coefficient of like powers of $r^2$, and solving the recurrence relation, we have

\[
a_2 = \frac{27k(2 - K)}{80\pi} + \frac{8\pi G\tilde{\rho}_0(5 - 4K)r_i^2}{30(2 - K)} + \frac{5 - 4K^2 + 6kG\tilde{\rho}_0 r_i^2 + K(8 - 3kG\tilde{\rho}_0 r_i^2)}{10(2 - K)}, \ldots
\]
(43)

\[
b_0 = 1,
\]
(44)

\[
b_1 = K + \frac{8\pi G\tilde{\rho}_0}{3} r_i^2,
\]
(45)

\[
b_2 = \frac{K^3(16\pi - 9k)}{16\pi(-2 + K)} + \frac{K^2(-348\pi + 243k + 640\pi^2 Gr_i^2 \tilde{\rho}_0)}{120\pi(-2 + K)} - \frac{K[-243k - 1632\pi^2 Gr_i^2 \tilde{\rho}_0 + 1280\pi^3 Gr_i^4 \tilde{\rho}_0^2 + 72\pi(2 + 3kGr_i^2 \tilde{\rho}_0)]}{180\pi(-2 + K)} - \frac{1280\pi^3 Gr_i^2 \tilde{\rho}_0^2 + 27k(3 + 8\pi Gr_i^2 \tilde{\rho}_0)}{90\pi(-2 + K)}, \ldots
\]

\[
c_0 = \frac{\tilde{\rho}_0}{3} + \frac{2 - K}{8\pi Gr_i^2}.
\]
\[ c_1 = \frac{1}{3} \left( \frac{kG(2-K)}{16\pi G} + \frac{(10-5K)[8\pi G - 3kG(2-K)]}{48\pi G(-2 + K + 8\pi \rho_0 r_i^2)} \right), \]

\[ c_2 = \frac{27k^2(-2 + K)^3}{128\pi^2(-6 + 3K + 8\pi \rho_0 r_i^2)} + \frac{32(-5 + 4K)\pi^2 G^2 r_i^4 \rho_0^2}{15(-2 + K)(-6 + 3K + 8\pi \rho_0 r_i^2)} - \frac{4\pi \rho_0^2 K(-15 + 48\pi \rho_0^2 K + 2K(-3 + 5k\rho_0^2 K))}{15(-2 + K)(-6 + 3K + 8\pi \rho_0 r_i^2)} - \frac{3k(-2 + K)^2(5K - 2(9 + 5k\rho_0^2 K))}{40(-6 + 3K + 8\pi \rho_0 r_i^2)} - \frac{60 - 492k\rho_0^2 K + 40k^2 G^2 r_i^4 \rho_0^2 + 8K^2(3 + 10k\rho_0^2 K) + K(-93 + 86k\rho_0^2 K - 20k^2 G^2 r_i^4 \rho_0^2)}{40(-6 + 3K + 8\pi \rho_0 r_i^2)}, \ldots \]

\[ d_1 = \frac{3k(-2 - K)}{8\pi}, \] \hspace{1cm} \[ d_2 = \frac{k}{192\pi} \left[ -12K^2 + 6(-3 + 2(4\pi - K)G^2 r_i^2 \rho_0) + K(33 + 2(-16\pi + 3k)G^2 r_i^2 \rho_0) \right], \ldots \]

To get the value of the coefficient of \( a_0 \), we also use the fact that the limit of TeVeS is GR when \( k \to 0 \) and \( K \to 0 \). Comparing the expansion with the internal metric of a static, spherically symmetry star in GR we have

\[ a_0 = \frac{5}{2} - \frac{2G\rho}{2R} - 3 \sqrt{1 - \frac{2G\rho}{R}}. \] At \( r = R \), using Eq.(15), Eq.(44) and Eq.(35), we have the relation between \( m_g \) and \( r_i \)

\[ \left( K + \frac{8}{3\pi G\rho_0 r_i^2} \right) \frac{R^3}{r_i^3} = 2Gm_g, \]

\[ \frac{45K^3k}{80\pi(-2 + K)} + \frac{18K^2(4\pi - 9k)}{80\pi(-2 + K)} - \frac{4K[-27k + 32\pi^2 G^2 \rho_0 r_i^2 + 8\pi(2 + 3k\rho_0^2 K)]}{80\pi(-2 + K)} + \frac{24k(3 + 8\pi G^2 \rho_0 r_i^2)}{80\pi(-2 + K)} \]

\[ + O \left( \frac{R^3}{r_i^3} \right) = 2Gm_g. \]

Using the relation \( \tilde{g}_{tt} = e^{2\phi + \nu}, \tilde{g}_{rr} = e^{-2\phi + \lambda} \) and \( \tilde{g}_{\theta\theta} = \tilde{g}_{\phi\phi} + \sin^2 \theta = r^2 e^{-2\phi} \), we have

\[ \tilde{g}_{tt} = -\left( \frac{5}{2} - \frac{9G\rho}{2R} - \frac{3}{2} \sqrt{1 - \frac{2G\rho}{R}} \right) - \left( \frac{5}{2} - \frac{9G\rho}{2R} - \frac{3}{2} \sqrt{1 - \frac{2G\rho}{R}} \right) \left[ 1 - \frac{3k(-2 + K)}{8\pi} \right] \frac{r}{r_i} \] \hspace{1cm} \[ + O \left[ \frac{r}{r_i} \right]^4, \]

\[ \tilde{g}_{rr} = 1 + \left[ K + \frac{3k(-2 + K)}{8\pi} + \frac{8}{3\pi G\rho_0 r_i^2} \right] \frac{r}{r_i} \] \hspace{1cm} \[ + O \left[ \frac{r}{r_i} \right]^4, \]

\[ \tilde{g}_{\theta\theta} = r^2 \left[ 1 + \frac{3k(-2 + K)}{8\pi} \right] \frac{r}{r_i} \] \hspace{1cm} \[ + O \left[ \frac{r}{r_i} \right]^4, \]

\[ \tilde{g}_{\phi\phi} = \sin^2 \theta \tilde{g}_{\theta\theta}. \]

It is clearly that the internal physical metric of spherical star in TeVeS will get back that in GR when \( k \to 0 \) and \( K \to 0 \).

Though the stellar model discussed in this section is not a perfect model, it is still a first-order approximation of small star of which the pressure is not large. The procedure above shows us that theoretically we can deal with a real star in TeVeS in the similar way, but the numerical calculation will be more complex.

5. Features of Motion Around a Spherically Symmetric Object in TeVeS

Based on the analyses of the sections above, the features of motion around any spherically symmetric compacted objects not only the black holes can be discussed now. An analytic solution has been gotten in spherically symmetric.
spacetime in TeVeS for arbitrary $k$ and $K$, where the metric in TeVeS is consistent with these in general relativity, if $k = 0$ and $K = 0$. The solution is based on the line element

$$ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -e^{\nu(r)} dt^2 + e^{\xi(r)} (d\rho^2 + \delta^2 (d\theta^2 + \sin^2 \theta d\phi^2)),$$

which is equivalent to the line element in Eq.(12). Between Eq.(12) and Eq.(49), it is easy to find the relation:

$$ e^{\nu(r)} = e^{\nu(r)}, $$

$$ e^{\xi(r)} d\rho^2 = e^{\lambda(r)} d\rho^2, $$

$$ e^{\xi(r)} \rho^2 = r^2. $$

And the static, spherically symmetric analytic solutions are:

$$ \phi(\rho) = \phi_c + \frac{k G m_s}{8 \pi \rho_c} \ln \left( \frac{\rho - \rho_c}{\rho + \rho_c} \right), $$

$$ e^\nu = \left( \frac{\rho - \rho_c}{\rho + \rho_c} \right)^{\rho_y/2\rho_c}, $$

$$ e^\xi = \left( \frac{\rho^2 - \rho_c^2}{\rho^2} \right)^{\pi \rho_y/2\rho_c}, $$

where $\rho_c = \frac{R}{4} \sqrt{1 + \frac{k}{\pi} \left( \frac{G m_s}{\rho_g} \right)^2} - \frac{k}{2} \rho_g$ is a length scale to be determined. And $m_s$ is the "scalar" mass which is defined by $m_s \equiv 4 \pi \int_0^R (\rho + 3\rho_e) e^{\nu/2 + 3\lambda/2} \rho^2 d\rho$, where $R$ is radius of the matter’s boundary. Using the results of post-Newtonian corrections, one has the relation between $m_s, m_g$ and $\rho_g$

$$ \rho_g - \frac{3k \rho_g^2}{8 R} - \frac{k G m_s^2}{4 \pi R} + O(\rho_g^3/R^2) = 2 G m_g, $$

where $m_g$ is the "gravitational mass". For the sun $\rho_g/R \sim G m_s/R \sim 10^{-5}$, one will find that $\rho_g \approx 2 G m_g$ with fractional accuracy much better than $10^{-5}$. The physical metric may reduce to $\tilde{g}_{tt} = g_{tt} e^{2\phi}$ and $\tilde{g}_{ii} = g_{ii} e^{-2\phi}$. And one will arrive at

$$ \tilde{g}_{tt} = -\left( \frac{\rho - \rho_c}{\rho + \rho_c} \right)^a $$

$$ \tilde{g}_{\rho\rho} = \tilde{g}_{\theta\theta} = \tilde{g}_{\phi\phi} = \left( \frac{\rho^2 - \rho_c^2}{\rho^2} \right)^2 \left( \frac{\rho - \rho_c}{\rho + \rho_c} \right)^{-a} $$

where $a \equiv \frac{\rho_g}{2\rho_c} + \frac{k G m_s}{4 \pi \rho_g \rho_c}$. When $k = 0$ and $K = 0$, the metric will reduce to the metric in GR.

The relation between energy and momentum for a test particle of rest mass $m$ in curved space is

$$ \tilde{g}^{\alpha\beta} p_\alpha p_\beta + m^2 = 0, $$

where $p_\alpha = \partial S/\partial x^\alpha$ and $S$ is the Hamilton-Jacobi function. Thus Hamilton-Jacobi equation for propagation of wave crests in TeVeS spherically symmetric geometry becomes

$$ \tilde{g}^{tt} \left( \frac{\partial S}{\partial t} \right)^2 + \tilde{g}^{\rho\rho} \left( \frac{\partial S}{\partial \rho} \right)^2 + \tilde{g}^{\theta\theta} \rho^{-2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \tilde{g}^{\phi\phi} \rho^{-2} \sin^{-2} \theta \left( \frac{\partial S}{\partial \phi} \right)^2 - m^2 = 0, $$

where $\dot{\rho} = \sqrt{\tilde{g}_{tt}}$. Set

$$ S = -\tilde{E} t + S_1(\rho) + S_2(\theta) + S_3(\phi). $$

Substitute Eq.(59) into Eq.(68), we have

$$ \tilde{g}^{tt} \tilde{E}^2 + \tilde{g}^{\rho\rho} \left( \frac{\partial S_1}{\partial \rho} \right)^2 + \tilde{g}^{\theta\theta} \rho^{-2} \left( \frac{\partial S_2}{\partial \theta} \right)^2 + \tilde{g}^{\phi\phi} \rho^{-2} \sin^{-2} \theta \left( \frac{\partial S_3}{\partial \phi} \right)^2 + m^2 = 0. $$
Solving the equation we find the solution of Hamilton-Jacobi function

\[
S = -\tilde{E}t + \int^\rho \sqrt{-\frac{\tilde{g}^{tt}}{\tilde{g}^{pp}} \tilde{E}^2 - \frac{m^2}{\rho^2}} - \frac{\tilde{L}^2}{\rho^2} \, d\tilde{\rho} + \int^\rho \sqrt{\frac{\tilde{L}^2}{\rho^2} - \frac{p_c}{\sin^2 \theta}} \, d\theta + p_c \varphi, \tag{61}
\]

where \(\tilde{L}\) is the angular momentum and \(\tilde{E}\) is the energy of the system.

Find the relation between \(t\) and \(\tilde{\rho}\) by considering "interference of wave crests" belonging to slight different \(\tilde{E}\) values

\[
\frac{\partial S}{\partial \tilde{E}} = -t + \int^\rho \frac{-\tilde{g}^{tt} \tilde{E}}{\sqrt{-\tilde{g}^{tt} \tilde{E}^2 - \frac{m^2}{\rho^2} - \frac{\tilde{L}^2}{\rho^2}}} \, d\tilde{\rho} = 0. \tag{62}
\]

Then one has

\[
d\tilde{\rho} = \sqrt{\frac{\tilde{g}^{tt} \tilde{E}^2 - \frac{m^2}{\rho^2} - \frac{\tilde{L}^2}{\rho^2}}{\frac{\tilde{g}^{tt}}{\rho^2} \tilde{E}}}. \tag{63}
\]

When \(d\tilde{\rho}/dt = 0\), Eq. (63) infers the effective potential energy of the system:

\[
\tilde{U} = \left[ \frac{\tilde{g}^{\rho \rho}}{\tilde{g}^{tt}} \left( \frac{m^2}{\tilde{g}^{tt}} + \frac{\tilde{L}^2}{\rho^2} \right) \right]^{1/2}. \tag{64}
\]

Substituting the physical metric \(\tilde{g}^{\mu \nu}\) and \(\tilde{g}^{\rho \rho}\) into Eq. (64), we have the effective potential for unit mass

\[
U = \left( \frac{\rho - \rho_c}{\rho + \rho_c} \right)^{a/2} \left[ 1 + \frac{\tilde{L}^2}{m^2 (\rho^2 - \rho_c^2)} \left( \frac{\rho - \rho_c}{\rho + \rho_c} \right)^a \right]^{1/2}, \tag{65}
\]

where \(U = \tilde{U}/m\) is the effective potential for unit mass of the test particle. Defining three dimensionless quantities \(\tau \equiv \rho/\rho_0, \tau_c \equiv \rho_c/\rho_0\) and \(L = \tilde{L}^2/(m^2 \rho_0^2)\), we get

\[
U = \left( \frac{\tau - \tau_c}{\tau + \tau_c} \right)^{a/2} \left[ 1 + \frac{\tilde{L}^2}{\tau^2 (\tau^2 - \tau_c^2)} \left( \frac{\tau - \tau_c}{\tau + \tau_c} \right)^a \right]^{1/2}, \tag{66}
\]

where \(a = \frac{1}{2} \left( 1 + \frac{k}{2\pi} \frac{G m_s}{\rho_0} \right)\) and \(\tau_c = \frac{1}{\tau} \sqrt{1 + \frac{k}{\pi} \left( \frac{G m_s}{\rho_0} \right)^2 - \frac{k}{2}}\).

Energy, in units of the rest mass \(m\) of the particle, is denoted \(E = \tilde{E}/m\). It is worth noticing that one can define a turning point by the condition \(U^2 = E^2\). Obviously, the minimum and maximum of the potential depend on the \(L\). The roots of \(\partial U/\partial \rho = 0\) are given in terms of the \(L\) by numerical calculation, so that we can define the lowest reduced angular momentum \(L_{\text{crit}}\) by the condition \(U_{\text{min}}(L_{\text{lowest}}) = U_{\text{max}}(L_{\text{lowest}})\). In other words, there is no periastron for \(L \leq L_{\text{lowest}}\), and any incoming particle is necessarily pulled into \(\rho = \rho_c\). The radius of stable circular orbits represents the point sitting at minimum of effective potential. The last stable circular orbit corresponds to \(L_{\text{lowest}}\). It is easily found that \(L_{\text{lowest}}\) and the radius of stable circular orbits depend on \(k\) and \(K\). We also define \(L_{\text{crit}}\) by the condition \(U_{\text{max}}(L_{\text{crit}}) = 1\), then there are bound orbits for \(L > L_{\text{crit}}\) and test particles coming in from \(\rho = \infty\) with \(E^2 < U_{\text{max}}^2\) reach periastrons and then return to \(\rho = \infty\), but particles from \(\tau = \infty\) with \(E^2 > U_{\text{max}}^2\) get pulled into \(\rho = \rho_c\). There are unstable circular orbits at the maximum of the effective potential, in which maximum moves outward form the radius of stable circular orbits for \(L = \infty\) to the radius of stable circular orbits for \(L_{\text{lowest}}\).

Giannios \[7\] had proved that the solution for the metric of static, spherically symmetric black holes is identical to the Schwarzschild solution in GR. To show the difference of features of motion around a spherically symmetric object between in TeVeS and in GR, we can discuss the corrections of stars, such as sun, but not black holes in TeVeS. For sun \(G m_s/\rho_0 \sim 10^{-10}\), so it is reasonable to neglect the terms with \(G m_s/\rho_0\) in the expression of \(a\) and \(\tau_c\), i.e. \(k\) is insensitive to \(L_{\text{lowest}}\) and \(L_{\text{crit}}\). We define the relative errors of \(L_{\text{lowest}}\) and \(L_{\text{crit}}\)

\[
\Delta L_{\text{lowest}} = \frac{L_{\text{lowest}}(K) - L_{\text{lowest}}(K = 0)}{L_{\text{lowest}}(K = 0)}, \tag{67}
\]

\[
\Delta L_{\text{crit}} = \frac{L_{\text{crit}}(K) - L_{\text{crit}}(K = 0)}{L_{\text{crit}}(K = 0)}. \tag{68}
\]
TABLE I: Values of $L_{\text{lowest}}$, $\Delta L_{\text{lowest}}$ and $L_{\text{crit}}$, $\Delta L_{\text{crit}}$ for different $K$

| $K$ | $L_{\text{lowest}}$ | $\Delta L_{\text{lowest}}$ | $L_{\text{crit}}$ | $\Delta L_{\text{crit}}$ |
|-----|----------------------|-----------------------------|-------------------|-----------------------------|
| 0   | 1.73205              | 0                           | 2                 | 0                           |
| 0.01| 1.73221              | 0.009013%                   | 2.0002            | 0.00992684%                 |
| 0.02| 1.73236              | 0.018022%                   | 2.0004            | 0.019847%                   |
| 0.03| 1.73252              | 0.027025%                   | 2.0006            | 0.0297605%                  |
| 0.04| 1.73267              | 0.036023%                   | 2.00079           | 0.0396673%                  |
| 0.05| 1.73283              | 0.045016%                   | 2.00099           | 0.0495674%                  |
| 0.06| 1.73299              | 0.054003%                   | 2.00119           | 0.059461%                   |
| 0.07| 1.73314              | 0.062985%                   | 2.00139           | 0.0693478%                  |
| 0.08| 1.73330              | 0.071961%                   | 2.00158           | 0.0792281%                  |
| 0.09| 1.73345              | 0.080932%                   | 2.00178           | 0.0891018%                  |
| 0.1 | 1.73361              | 0.089899%                   | 2.00198           | 0.0989688%                  |
| 0.2 | 1.73516              | 0.179271%                   | 2.00395           | 0.19728%                    |
| 0.3 | 1.73699              | 0.268126%                   | 2.00559           | 0.292944%                   |
| 0.4 | 1.73822              | 0.356470%                   | 2.00784           | 0.391974%                   |
| 0.5 | 1.73975              | 0.444313%                   | 2.00977           | 0.488381%                   |
| 0.6 | 1.74126              | 0.531661%                   | 2.01168           | 0.584175%                   |
| 0.7 | 1.74276              | 0.618522%                   | 2.01359           | 0.679367%                   |
| 0.8 | 1.74426              | 0.704904%                   | 2.01548           | 0.773968%                   |
| 0.9 | 1.74575              | 0.790813%                   | 2.01736           | 0.867986%                   |
| 1   | 1.74723              | 0.876256%                   | 2.01923           | 0.961433%                   |
| 1.1 | 1.74870              | 0.961241%                   | 2.02109           | 1.05432%                    |
| 1.2 | 1.75016              | 1.045774%                   | 2.02293           | 1.14665%                    |
| 1.3 | 1.75162              | 1.129861%                   | 2.02477           | 1.23843%                    |
| 1.4 | 1.75307              | 1.213508%                   | 2.02659           | 1.32968%                    |
| 1.5 | 1.75451              | 1.296722%                   | 2.02841           | 1.4204%                     |
| 1.6 | 1.75594              | 1.379509%                   | 2.03021           | 1.5106%                     |
| 1.7 | 1.75737              | 1.461874%                   | 2.03201           | 1.6003%                     |
| 1.8 | 1.75879              | 1.543824%                   | 2.03379           | 1.68948%                    |
| 1.9 | 1.76020              | 1.625363%                   | 2.03556           | 1.77817%                    |

With the numerical calculation, the values of $L_{\text{lowest}}$, $\Delta L_{\text{lowest}}$ and $L_{\text{crit}}$, $\Delta L_{\text{crit}}$ for different $K$ are listed in Table I. Note that $K = 0$ corresponds to GR. We find $L_{\text{lowest}}$ and $L_{\text{crit}}$ have a small difference between GR and TeVeS.

6. Conclusion

Based on relativistic MOND theory, we have studied the spacetime in the external and internal of a semi-realistic star with a constant density inside the star in this paper. We show the Oppenheimer-Volkoff equation in TeVeS, and calculate the metric coefficient and pressure inside the star to post-Newtonian order. Therefore, one can determine the relation between $m_s$, $m_g$ and $R$ by integrating the metric and scalar equation inside the star. The more complex equation of state to approach the stellar structure is left for future work. Furthermore, we study the features of a test particle around the star in TeVeS and get the effective potential which can determine the features of particle’s motion. We define two relative errors to discuss the corrections of TeVeS and show that there is a small correction of TeVeS which can be tested by the future experiments.
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[1] Milgrom M 1983 Astrophys. J. 270 365; ibit. 1983 270 371; ibit. 1983 270 384
[2] Sarders R H and McGaugh S S 2002 Ann. Rev. Astron. Ap. 40 263
[3] Tully R P and Fisher J R 1977 Astro. Astrophys. 54 661
[4] Bekenstein J D 2004 Phys. Rev. D70 083509
[5] Bekenstein J D arXiv: astro-ph/0412652
[6] Skordis C, Mota D F, Ferreira P G and Boehm C 2006 Phys. Rev. Lett. 96 011301
[7] Giannios D 2005 Phys. Rev. D71 103511
[8] Li X Z and Hao J G 2002 Phys. Rev. D66 107701; Hao J G and Li X Z 2003 Class. Quantum Grav. 20 1703; Shi X and Li X Z 1991 Class. Quantum Grav. 8 761
[9] Oppenheimer J R and Volkoff G M 1939 Phys. Rev. 55 374