Limit of Relativistic Quantum Brayton Engine of Massless Boson Trapped 1 Dimensional Potential Well

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Abstract. A quantum Brayton engine using a massless Boson trapped in a one-dimensional potential well as a working substance has been constructed to be studied. We used a modified analogical model by the first law of thermodynamics for a quantum system implementation. The system was built with Klein Gordon equation, relativistic quantum mechanics Hamiltonian, without the mass term of the Hamiltonian and it was trapped in a one-dimensional potential well. Then, it underwent the quantum ideal Brayton cycle process. The quantum Brayton process consisted of adiabatic compression, isobaric expansion, adiabatic expansion, and isobaric compression processes. This relativistic quantum mechanics Brayton engine has found that the efficiency formulation was similar to the classical one. This study obtained the conformity results between the quantum relativistic and classical Brayton engines. By this cycle process, we also invented that the ratio of heat capacity under constant pressure and volume of the system was 2. It could indicate that the efficiency of quantum relativistic Brayton engine is higher than the classical one.

Keywords: Brayton, Klein-Gordon equation, relativistic, quantum.

1. Introduction

A heat engine is a device to convert a heat energy from a high-temperature reservoir to other work [1–3], but the heat energy is not all converted into a mechanical work because some energy is rejected into a low-temperature reservoir, so the efficiency of device is always lower than 100%, same as the thermodynamic second law [4]. Generally, the efficiency of a heat engine has more than 50% [5]. The presence of quantum theory is expected to increase of heat engine efficiency. The performance of the heat engine can be reviewed by the principle of quantum mechanics, which is known as the quantum heat engine[2,3,6–10].

In this cases, a quantum heat engine system is the quantum particle (single mass-less Boson) trapped in a one-dimensional potential well, and one of the dividing wall on the potential-well can be friction-less move, so this system is similar to the piston in the cylinder. The expectation value of the Hamiltonian system is the internal energy of the system. The quantity volume of the cylinder in the classical thermodynamics is replaced by the length of the distance between the dividing walls on the potential well in quantum mechanics system. The Pressure of the piston in the classical thermodynamic system is replaced by the expectancy value of force in dividing the wall potential well [1,6,11]. The quantum heat engine is different from the classical heat engine, the working substance of quantum heat engine is quantum particles so it implies that the energy state of the system is discrete[11].
In previous studies [1,2] it has a disadvantage that does not apply of first law thermodynamics for the quantum system, therefore the expectation value of force and energy are not well defined. Carnot quantum engine for two states of the system has an efficiency value larger than 2 more states of the system, this shows that the more the number of states involved, the lower the efficiency [2]. The efficiency value of a quantum heat engine depends on the ratio of adiabatic expansion process [1,2,6,12] and the other studies for other engines such as Otto, Brayton, and Diesel [3,8,13] has a similar efficiency as the classical system. The study of relativistic Dirac particles as a working substance of quantum Carnot engine has found that the efficiency value depends on the root square of Compton wavelength, the ratio of adiabatic expansion and the number of state system [9,12].

The study of massless boson as a working substance in a quantum heat engine is rarely conducted, so further research is needed. Therefore, a research using relativistic particle such as massless boson as working substance of quantum heat engine is required, especially using Brayton cycle. Thus, we studied about the limit of relativistic quantum Brayton engine of massless boson trapped 1 dimensional well. This study is expected to develop results from previous theories and can be a consideration for a further research development.

2. Experimental Methods

2.1. Single mass-less boson in a one-dimensional potential well
This study considered the case of a classical thermodynamic system which has a proximity analogy with a quantum system. The analogy model implemented the first law of thermodynamics modified for such quantum system, is intended to describe a change of a physical quantity throughout in the process [3]. The potential well or one-dimensional box system in a quantum mechanics is the closest analogy with a piston-cylinder in a classical thermodynamic system. The potential well of the system, in this case, is a particle trapped in a 1-dimensional box system. The case taken was the potential system which was infinite ($V = \infty$) at $x \leq 0$ dan $x \geq 0$, and the zero ($V = 0$) at $0 < x < L$ in 1-dimensional potential well (1-Dimensional box system). The illustration of an analogical mechanical quantum system as a piston-cylinder system is Figure 1 showed.

![Figure 1](image_url)  
**Figure 1.** The piston system between the classical thermodynamics and mechanical quantum system; (a) piston inside the cylinder in a classical system (b) the analogy model of a piston in a mechanical quantum.

The generality of a free-state motion equation for a mass-less Boson at Minkowsky space (space-time dimension) is described as Klein- Gordon equation [14]. The Klein-Gordon equation is

$$\hat{p}^\mu \hat{p}_\mu \Psi = \left( \frac{E^2}{c^2} - \vec{p} \cdot \vec{p} \right) \Psi = m_0^2 c^2 \Psi.$$

In this equation (1), $m_0$ is a motionless mass of the particle, each momentum operator in Minkowski space is
\[\hat{p}^\mu = i\hbar \frac{\partial}{\partial x_\mu} = i\hbar \left( \frac{\partial}{\partial (ct)}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = i\hbar \left( \frac{\partial}{\partial (ct)}, -\vec{\nabla} \right) = i\hbar (p_0, \vec{p}) \quad (2)\]

\[\hat{p}^\mu \hat{p}_\mu = -\hbar^2 \frac{\partial^2}{\partial x_\mu \partial x^\mu} = -\hbar^2 \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right) \]

\[\simeq -\hbar^2 \Delta = -\hbar^2 \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \quad (3)\]

So with considering the equation (3) and (2), we can rewrite the Klein-Gordon equation as follow

\[\left( \Delta + \frac{m_0^2 c^2}{\hbar^2} \right) \Psi = \left( \frac{\partial^2}{c^2 \partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} + \frac{m_0^2 c^2}{\hbar^2} \right) \Psi = 0. \quad (4)\]

Consider the case of massless boson trapped 1-dimensional potential well, we obtain the equation motion is

\[\hat{H}^2 \psi(x) = -\hbar^2 c^2 \frac{\partial^2 \psi(x)}{\partial x^2}. \quad (5)\]

Equation (5) is a second order linear differential equation and the general solution of the equation is

\[\psi(x) = A \sin kx + B \cos kx, \quad k = \frac{E}{\hbar c}. \quad (6)\]

Consider of applying boundary and normalized condition that probability of a state system is zero at \(x = 0\) and \(x = L\), and 1 for \(0 < x < L\), we obtained the solution which is a wave-function as Eigen-vector is

\[\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}. \quad (7)\]

eigen-vector (Eigen-Function) functions as representations of each Eigen-state level. Eigen-function at equation (7) associate with Eigen-energy system, is

\[E_n(L) = \frac{n\pi \hbar c}{L} = nE_i(L). \quad (8)\]

If the system has the state which is unsure or probability of state is not 100% at one of the state systems, then the state of the system is superposition by each other of Eigen-state. The mathematical form of superposition by Eigen-function at stationary condition is

\[\Psi(x,t) = \sum_{n=1}^{\infty} a_n \psi_n(t) \theta(t) = (a_1 \psi_1(x) + a_2 \psi_2(x) + \ldots) \exp \left( -i \frac{E_1}{\hbar} \right). \quad (9)\]

The expectation value of energy possessed by the particle massless Boson trapped in a one-dimensional potential well is

\[\langle H \rangle = \int_{-\infty}^{\infty} \psi^* (x,t) \hat{H} \psi(x,t) dx = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_n^* a_m E_m \delta_{mn} = \left( \sum_{n=1}^{\infty} |a_n(L)|^2 \right) \frac{n\pi \hbar c}{L}. \quad (10)\]

where \(\delta_{mn}\) describes the orthogonal state property of the system.
2.2. **The first law thermodynamics in quantum Brayton engine**

The description of the physical quantity between the classical thermodynamics and quantum mechanics system is different. The classical thermodynamics considers macroscopic quantities (such as pressure, volume, and temperature), while the quantum mechanics and mechanical statistic consider the microscopic quantities (such as internal energy, distribution function, and the probability of state system). However, from the difference between the classical thermodynamics and quantum mechanics system, there is a relation that can attribute to each other, such as the quantity of temperature. The quantity of temperature is associated with the internal energy system, with considering the microscopic for the quantum system shown the energy internal system is manifested of the expectation value of the Hamiltonian for particle trapped in a one-dimensional potential well system. The quantity of Pressure in the classical thermodynamics system is replaced by expectation-value of Force in dividing wall potential well. The relation of the physical quantity of classical thermodynamics and quantum mechanics is shown by table 1. The general form for the First law of thermodynamics is

\[
dU = \delta Q - \delta W.
\]

Previously research has not defined the expectation value of force well since it has not developed and implemented the modified first law of thermodynamics for the quantum mechanics system [1,2,6]. The modified first law of thermodynamics for a quantum mechanics system is

\[
d\langle H \rangle = \sum_{n=1}^{N} E_n(L) dP_n(L) + \sum_{n=1}^{N} P_n(L) dE_n(L).
\]

### Table 1. The association of quantities for classical thermodynamics and quantum mechanics system

| No | Quantities of classical thermodynamics | Quantities of quantum mechanics |
|----|--------------------------------------|---------------------------------|
| 1  | Pressure (P)                         | Force (F)                       |
| 2  | Volume (V)                           | Length of the system (L)        |
| 3  | Internal Energy (U)                  | Expectation value of Hamiltonian|
| 4  | Heat energy (Q)                      | The change of probability in the expectation value of Hamiltonian |
| 5  | Work (W)                             | The change of Eigen-energy system |

The heat energy absorbed to the system or released to the surrounding of the system will imply the change of probability in the expectation value of Hamiltonian (internal energy system), therefore, the quantities of heat energy for quantum mechanics system is

\[
\delta Q = \sum_{n=1}^{N} E_n(L) dP_n(L).
\]

The Quantities work of system expanding the dividing wall of the potential well or imposing on the system can be writen as follow

\[
\delta W(L) = - \sum_{n=1}^{N} P_n(L) dE_n(L)
\]

### 2.2.1. Adiabatic Process

An adiabatic process for the classical thermodynamic system is characterized by the absence flow of the heat energy between system and environment (reservoir), so the work of system to expand comes from the change of internal energy system. In addition, the adiabatic compression will increase the internal energy system. The first law of thermodynamics for the adiabatic process is

\[
dU = -\delta W = -\ddot{F}(L) \bullet d\dot{L}.
\]
The quantum mechanics system for the adiabatic process characterized by the probability of a level state is constant [3]. Thus, the modified first law of thermodynamics for quantum mechanics system is

\[ d\langle H \rangle = \sum_{n=1}^{N} P_n(L) dE_n(L). \]  

(16)

The expectation value of force to moving the dividing wall of potential well for massless Boson as working substance in a quantum heat engine is

\[ F(L) = \left\{ \sum_{n=1}^{N} P_n(L)n \right\} \frac{\pi \hbar c}{L^2}. \]  

(17)

The Quantities of work throughout the adiabatic process is obtained by integrating equation (17) to infinitesimal partition of length \((dL)\) from the initial length \((L_i)\) to the final length \((L_f)\), i.e.

\[ W_{i\rightarrow f} = \int_{L_i}^{L_f} F(L) dL = \left\{ \sum_{n=1}^{N} P_n(L_i)n \right\} \frac{\pi \hbar c}{L^2} \left( 1 - \frac{L_i}{L_f} \right). \]  

(18)

2.2.2. Isobaric Process

The isobaric process for the classical thermodynamic system is characterized by absence the change of pressure (the pressure throughout is constant), the system is expanded or compressed in conjunction by the flow of the heat energy between the system and the environment (reservoir). The first law of thermodynamics for the Isobaric process is

\[ dU = \delta Q - \delta W. \]  

(19)

The quantum mechanics system for the isobaric process is characterized by absence the change of expectation-value of force (expectation-value of force throughout is constant). The modified First law of thermodynamics for the isobaric process is

\[ d\langle H \rangle = \delta Q - \int_{L_i}^{L_f} FdL. \]  

(20)

The quantities of work throughout the Isobaric process is characterized by expectation-value of force for initial-state and final-state is constant. Thus, for massless Boson as a working substance of a quantum heat engine in the Isobaric process occurs as follow:

\[ F(L_i) = F(L) \]

\[ \left( \sum_{n=1}^{N} |a_n(L)|^2 n \right) \frac{\pi \hbar c}{L_i^2} = \left( \sum_{n=1}^{N} |a_n(L)|^2 n \right) \frac{\pi \hbar c}{L^2} \]  

(21)

\[ \left( \frac{L}{L_i} \right)^2 \sum_{n=1}^{N} P_n(L_i)n = \sum_{n=1}^{N} P_n(L)n. \]

To find the Quantities of work throughout the Isobaric process is obtained by integrating the expectation-value of force to the infinitesimal partition of length \((dL)\) from the initial to final length. Consider the equation (21) to integrating operation obtain the Quantities of work, i.e.

\[ W_{i\rightarrow f} = \int_{L_i}^{L_f} F(L) dL = \int_{L_i}^{L_f} \left\{ \sum_{n=1}^{N} P_n(L)n \right\} \frac{\pi \hbar c}{L^2} dL \]

\[ = \int_{L_i}^{L_f} \left\{ \sum_{n=1}^{N} P_n(L_i)n \right\} \frac{L}{L_i} \frac{\pi \hbar c}{L_i^2} dL \]  

(22)

\[ = \left( \sum_{n=1}^{N} P_n(L_i)n \right) \frac{\pi \hbar c}{L_i} \left( \frac{L_f}{L_i} - 1 \right). \]
The amount of quantities of heat energy flowing between the system and environment (reservoir) along the isobaric process from initial length to final length condition is obtained by implying the modified of the first law thermodynamics, i.e.

\[ Q = \langle H \rangle_f - \langle H \rangle_i + W_{f \rightarrow f}, \]

\[ = \left( \sum_{n=1}^{\infty} |P_n(L_f)|^2 \right) \frac{\pi c}{L_f} - \left( \sum_{n=1}^{\infty} |P_n(L_i)|^2 \right) \frac{\pi c}{L_i} + W_{f \rightarrow f}. \]  

(23)

3. Results and Discussion

3.1. The Cycle of Quantum Brayton Engine

The Brayton engine is an idealization of a turbine engine. However, this cycle is complicated to realize because the turbine machine has not closed process in that cycle, so it is possible for the change of working substance. George Brayton (1870) is a first-engineer to present of the Brayton cycle. The Brayton cycle compared to the Otto and Diesel cycle, the Brayton cycle acts on a larger volume condition but a smaller pressure[15].

The Brayton cycle process consisted of adiabatic compression, isobaric Expansion, adiabatic expansion and isobaric compression. The Diagram F-L and scenario of quantum Brayton engine has shown by Figure 2.

The Initial condition of the Brayton cycle starting by adiabatic compression process from D→A condition, and then the heat energy from surrounding (high-temperature reservoir) will be absorbed by the system through the process of isobaric expansion process (at A→B condition). Afterward, the system undergoes an adiabatic expansion process (at B→C condition). The last cycle process is an isobaric compression where the system will return to the initial condition (at C→D condition).

Figure 2. The cycle of quantum brayton engine: (a) diagram F-L of the brayton cycle (b) scenario state of the brayton cycle

3.2. Initial State of System

The initial state for this Brayton cycle is started by D-condition, where the state-probability of the system is 100% at the ground-state (n=1). The wave function to describe the system for the initial condition is

\[ \psi_{\text{initial}} = \left| \Psi_1 \right\rangle \]

where \( \left| \Psi_1 \right\rangle \) is the ground-state wave function.
\begin{equation}
\psi_D = \sqrt{\frac{2}{L_D}} \sin \left( \frac{\pi x}{L_D} \right) .
\end{equation}

The expectation value of Hamiltonian in this system for initial condition \((L=L_D)\) is
\begin{equation}
\langle H \rangle_D = \left \{ \sum_{n=1}^{N} |a_n(L_D)|^2 \right \} \frac{\pi \hbar c}{L_D} = \frac{\pi \hbar c}{L_D} .
\end{equation}

3.3. Adiabatic Compression \(D\rightarrow A\)

The \(D\rightarrow A\) process is adiabatic compression. In this process is characterized by the state-probability of the system is constant, so the state-probability of the system is fixed (100% at the ground state). Nevertheless, the internal energy of the system increase because the length of the system decreased (compressed). The quantities of work throughout the adiabatic compression process (from \(D\rightarrow A\) condition) is
\begin{equation}
W_{D\rightarrow A} = \int_{L_B}^{L_A} \left \{ \sum_{n=1}^{N} |a_n(L_D)|^2 \right \} \frac{\pi \hbar c}{L} dL = \frac{\pi \hbar c}{L_A} \left( \frac{L_A}{L_D} - 1 \right) = \frac{\pi \hbar c}{L_A} \left( \frac{1}{\beta_1} - 1 \right) .
\end{equation}

The \(\beta_i\) coefficient is the ratio of \(L_D\) and \(L_A\). The value of work throughout adiabatic compression is minus (\(-\)) that shown by \(L_D>\)\(L_A\) or \(\beta_i>1\). This proves that the system is subject of work throughout the adiabatic compression process.

3.4. Isobaric expansion \(A\rightarrow B\)

The \(A\rightarrow B\) process is an isobaric process. In this process is characterized by the expectation values constant throughout \(A\rightarrow B\) isobaric expansion. The heat energy from the boundary (high-temperature reservoir) will be absorbed by the system in such a way the state of the system increases, this study assumes that the state probability of the highest state \((n=N)\) system increased until 100%. The initial process of isobaric expansion started at \(A\)-condition, the state of state-probability of the system is still 100% at the ground-state. The wave function to describe the system at \(A\) condition is
\begin{equation}
\psi_A(x) = \sqrt{\frac{2}{L_A}} \sin \left( \frac{\pi x}{L_A} \right) .
\end{equation}

The expectation value of the Hamiltonian system at \(A\)-condition is
\begin{equation}
\langle H \rangle_A = \left \{ \sum_{n=1}^{N} |a_n(L_A)|^2 \right \} \frac{\pi \hbar c}{L_A} = \frac{\pi \hbar c}{L_A} .
\end{equation}

Along the isobaric expansion process, state-probability at the ground state will decrease, however, the state-probability of a higher state increases. An increase of state-probability in the highest state \((n=N)\) reaches 100% when the system at \(B\)-condition. The wave function to describe the system at \(B\) condition is
\begin{equation}
\psi_B(x) = \sqrt{\frac{2}{L_B}} \sin \left( \frac{N \pi x}{L_B} \right) .
\end{equation}

The expectation value of the Hamiltonian system at \(B\)-condition is
\begin{equation}
\langle H \rangle_B = \left \{ \sum_{n=1}^{N} |a_n(L_B)|^2 \right \} \frac{\pi \hbar c}{L_B} = N \frac{\pi \hbar c}{L_B} .
\end{equation}

The quantity of work and heat energy throughout the isobaric expansion process from \(A\rightarrow B\) condition is
\begin{equation}
W_{A\rightarrow B} = \int_{L_A}^{L_B} \left \{ \sum_{n=1}^{N} P_n(L) n \right \} \frac{\pi \hbar c}{L^2} dL = \frac{\pi \hbar c}{L_A} \left( \frac{L_B}{L_A} - 1 \right) = \frac{\pi \hbar c}{L_A} \left( \sqrt{N} - 1 \right) .
\end{equation}
\[ Q_H = \langle H \rangle_B - \langle H \rangle_A + W_{A \rightarrow B} = 2 \frac{\pi \hbar c}{L_A} \left( \sqrt{N} - 1 \right). \]

(32)

The value of work throughout A→B isobaric expansion process is positive (+) showed by L_B>L_A, this proves that the system does the work to expansion. The value of heat energy throughout A→B process is positive (+) shown by √N > 1, this proves that the heat energy from surrounding (high-temperature reservoir) is absorbed by the system to increase the state-system. √N is the ratio of L_B and L_A, this relation has obtained by considering the equation (21) that expectation-value of force is constant.

3.5. Adiabatic Expansion B→C

The B→C process is an adiabatic expansion that absent of heat energy flow between the system and environment, so the state-probability of the system is fixed (100% at the N-state). The quantity of work throughout the adiabatic expansion (B→C) is

\[ W_{B \rightarrow C} = N \frac{\pi \hbar c}{L_B} \left( 1 - \frac{L_B}{L_C} \right) = \frac{\pi \hbar c}{L_A} \left( \sqrt{N} - \frac{\sqrt{N}}{\beta_2} \right). \]

(33)

The β_2 coefficient is the ratio of L_C and L_A. The value of work is positive (+) showed by L_C>L_B or β_2>1, this proves that the system does the work for expansion.

3.6. Isobaric Compression C→D

The C→D process is an isobaric compression. In this process is characterized by the expectation value of force is constant throughout C→D isobaric compression. Heat energy from the system is released to the environment/boundary (low-temperature reservoir), so the state of the system will decrease until the state-probability of the system is 100% at the ground-state when the system reaches to D-condition. The state system at C-condition is still in the highest-state (state-probability of the system is 100% at n=N-state). The wave-function to describe the system at C-condition is

\[ \psi_C(x) = \sqrt{\frac{2}{L_C}} \sin \frac{N\pi x}{L_C}. \]

(34)

The expectation value of the Hamiltonian system at C-condition is

\[ \langle H \rangle_C = \left\{ \sum_{n=1}^{N} a_n (L_C)^2 \right\} \frac{\pi \hbar c}{L_C} = N \frac{\pi \hbar c}{L_C}. \]

(35)

Along the isobaric compression process, state-probability at N-state will decrease, however, the state-probability at the ground state (n=1) increases up to 100% when the cycle back to D-condition. The wave function to describe the system at D-condition shown by equation (24), and the expectation value of the Hamiltonian system showed by equation (25). The quantity of work and the flow of heat energy between the system and environment throughout the isobaric compression process i.e.

\[ W_{C \rightarrow D} = N \frac{\pi \hbar c}{L_C} \left( \frac{L_C}{L_D} - 1 \right) = \frac{\pi \hbar c}{L_A} \left( \frac{1}{\beta_2} - \frac{\sqrt{N}}{\beta_2} \right). \]

(36)

\[ Q_C = \langle H \rangle_D - \langle H \rangle_C + W_{C \rightarrow D} = 2 \frac{\pi \hbar c}{L_A \beta_2^2} \left( 1 - \sqrt{N} \right). \]

(37)

The value of work throughout C→D isobaric compression process is negative (-) showed by L_C>L_D, this proves that a system is a subject of work. The value of heat energy throughout C→D isobaric compression process is negative (-) showed √N > 1, this proves that the heat energy system is released to environment/surrounding (low-temperature reservoir) and then the internal energy system will decrease until the ground-state.
3.7. The Efficiency of Quantum Brayton Engine

The efficiency is ratio the quantity total value of the working system and absorbed heat energy system by the environment (high-temperature reservoir). The expansion ratio \( \beta_1 \) and \( \beta_2 \) is same-value because \( L_B/L_A = L_C/L_D = \sqrt{N} \), so the total value of work and efficiency i.e.

\[
W_{total} = \int F(L) dL = \frac{\pi hc}{L_A} \left[ 2\sqrt{N} - 2 + \left( \frac{1}{\beta_1} + \frac{1}{\beta_2} \right) - 2\sqrt{N} \right] = 2\frac{\pi hc}{L_A} \left( \sqrt{N} - 1 \right) \left( 1 - \frac{1}{\beta} \right)
\]

\( \eta_{Brayton} = \frac{W_{total}}{Q_H} = 1 - \frac{L_B}{L_C} \vee 1 - \frac{L_A}{L_D}. \) (38)

3.8. Heat Capacity of Quantum Brayton Engine

The heat capacity is the ratio of heat energy to increase the temperature system. A heat capacity depends on the amount and the characteristic of the working substance system. For a specific property of working substance heat capacity is known as specific heat (C). Specific heat property based on the process in the thermodynamic system divided by two properties, that specific heat at constant volume condition (C\(_V\)) and specific heat at constant pressure condition (C\(_P\)). Specific heat at constant volume is obtained from the partial differential of internal energy by temperature. Specific heat at constant pressure is obtained from the partial differential of Helmholtz free-energy by temperature \([16]\), the mathematics form of specific heat is

\[
C_v = \frac{\partial U}{\partial T}
\]

(40)

\[
C_p = \frac{\partial H}{\partial T} = \frac{\partial (U + PV)}{\partial T}.
\]

(41)

Specific heat is related to a temperature, therefore it needs to introduce the temperature function approach to statistical mechanics. The working substance particle in this study is only a single mass-less Boson, so the properties of the particle following the Bose-Einstein distribution. Bose-Einstein distribution can be approached by Maxwell-Boltzmann distribution because the working substance is a single particle. The mathematics form of distribution-function is

\[
f(E, T) = \left( \exp \left( \frac{E_n(L)}{k_bT} \right) - 1 \right)^{-1} \approx \exp \left( - \frac{E_n(L)}{k_bT} \right).
\]

(42)

Statistical mechanics system present of partition-function. The partition function is the amount of state-partition in the system, the mathematical form of a partition-function is

\[
Z(\beta, L) = \sum_{n=1}^{\infty} \exp (-\beta E_n(L)), \beta = \frac{1}{k_bT}
\]

(43)

The internal energy of the system (U) is the expectation value of the Hamiltonian system. The internal energy of the system is obtained by considering the Maxwell-Boltzmann distribution and partition-function, \((Z(L,\beta))\), i.e.

\[
U(L, T) = \langle H \rangle = \frac{\sum_{n=1}^{\infty} E_n(L) \exp (-\beta E_n(L))}{\sum_{n=1}^{\infty} \exp (-\beta E_n(L))} = -\left( \frac{\partial \ln Z(\beta, L)}{\partial \beta} \right).
\]

(44)
The determination of a specific heat at constant volume condition is complicated if we following the equation (44) because the partition-function have sigma notation form. The partition-function is following the amount of geometric series, so mathematic form of partition-function can be re-write as

\[ Z(\beta, L) = e^{-\beta E_1} + e^{-2\beta E_1} + e^{-3\beta E_1} + \ldots = \frac{\exp(-\beta E_1(L))}{1-\exp(-\beta E_1(L))}. \] (45)

The internal Energy of the system is

\[ U(\beta, L) = -\frac{\partial \ln Z(\beta, L)}{\partial \beta} = \frac{E_1(L)}{1-e^{-\beta E_1(L)}}, \] (46)

Specific heat at constant volume condition in Quantum Brayton Engine can be analogized by specific heat at constant length condition because the quantity of volume in the classical system is replaced by the length of the dividing wall potential-well. Specific heat at constant length condition is

\[ C_L = \frac{\partial}{\partial T} \left( \frac{E_1(L)}{1-\exp(-\beta(T)E_1(L))} \right) = \frac{E_1^2(L)e^{-\beta(T)E_1(L)}}{k_B T^2 \left( 1-e^{-\beta(T)E_1(L)} \right)^2}. \] (47)

Specific heat at constant pressure condition in Quantum Brayton Engine can be analogized by a specific heat at the expectation value of force is a constant condition because the quantity of pressure in the classical system is replaced by expectation value of force. Expectation value of force is obtained by considering partition-function, i.e.

\[ F(\beta, L) = \sum_{n=1}^{\infty} \frac{1}{\beta} \frac{\partial}{\partial L} \exp(-\beta E_n(L)) = \frac{1}{\beta} \frac{\partial \ln Z(\beta, L)}{\partial L} \] (48)

Substitution of equation (45) to equation (48), obtained the expectation-value of force

\[ F(\beta, L) = \frac{1}{\beta} \frac{\partial}{\partial L} \ln \left( \frac{e^{-\beta E_1(L)}}{1-e^{-\beta E_1(L)}} \right) = -\left( \frac{1}{1-e^{-\beta E_1(L)}} \right) \frac{\partial E_1(L)}{\partial L}. \] (49)

Helmholtz free energy is related by specific heat at constant force condition. Helmholtz free energy is an addition of internal energy (expectation value of energy) and the multiplication expectation value of force by the length of the system. The Helmholtz free energy is

\[ H = U(\beta, L) + F(\beta, L)L = \left( \frac{1}{1-e^{-\beta E_1(L)}} \right) \left( E_1(L) - L \frac{\partial E_1(L)}{\partial L} \right). \] (50)

Specific heat at the expectation value of force is constant condition is

\[ C_f = \frac{\partial H}{\partial T} = \frac{E_1^2(L)}{k_B T^2} \left( 1-e^{-\beta(T)E_1(L)} \right) \left( \frac{e^{-\beta(T)E_1(L)}}{\left( 1-e^{-\beta(T)E_1(L)} \right)^2} \right) \] (51)

The ratio of specific heat at expectation value of force is constant and specific heat at the length of the system is at a constant condition as follow

\[ \frac{C_f}{C_L} = \left( 1- \frac{L}{E_1(L)} \frac{\partial E_1(L)}{\partial L} \right), \] (52)
The results of the AP.

The general form efficiency for Brayton engine is

\[ \eta_{Brayton} = 1 - \left( \frac{V_A}{V_D} \right)^{\gamma - 1}, \quad \gamma = \frac{C_P}{C_V} \]  

The general form efficiency of classical thermodynamics [5,15], non-relativistic quantum mechanics [1–3,6,8] and ultra-relativistic quantum mechanics [9,12][13] showed Table 2.

**Table 2.** The Comparison efficiency of heat engine among classical thermodynamics, non-relativistic quantum mechanics and ultra-relativistic quantum.

| No | Based on the system                  | Efficiency of the Brayton Engine |
|----|-------------------------------------|----------------------------------|
| 1  | Classical Thermodynamic             | \( 1 - \left( \frac{V_A}{V_D} \right)^{\gamma - 1} \) |
| 2  | Non-Relativistic quantum heat engine| \( 1 - \left( \frac{L_A}{L_D} \right)^2 \) |
| 3  | Ultra-relativistic quantum heat engine | \( 1 - \left( \frac{L_A}{L_D} \right) \) |

The general form of the Brayton efficiency at the classical thermodynamics have a relation with the efficiency of Ultra-relativistic Quantum Brayton Engine in equation (39), it can be taken correlation that the existence of correspondence between the quantum system and the classical system. Based on Tabel 2, shown that the existence of correspondence between the quantum system and the classical system, the difference is that there is a system which is Laplace constant (\( \gamma \)) or the ratio of specific heat at constant volume and specific heat at constant. The Laplace constant for classical Brayton engine depended by working substance (diatomic or monoatomic). The Laplace constant of non-relativistic quantum Brayton engine from the results of the previous study [3] show the Laplace constant (\( \gamma \)) or the ratio of specific heat at constant pressure and specific heat at constant volume is 3, it showed that the powered of \( L_A/L_D \) ratio is 2. The Laplace constant of Ultra-relativistic quantum Brayton engine forms a mass-less Boson as working substance is 2.

4. Conclusion
In this paper, we have explored the limit of relativistic quantum Brayton engine based on the massless boson trapped in a one-dimensional box system. We used the Klein Gordon equation to describe the boson motion. The boundary condition is applied for the system to obtain the Eigenvalue and associated by Eigenstate of the system. The motion walls of a one-dimensional box satisfy the Brayton cycle. The Brayton cycle occurred some of the adiabatic and isobaric processes. The results of the study showed the efficiency of quantum Brayton engine based on massless boson as working substance is larger than the classical Brayton engine but lower than non-relativistic quantum Brayton engine. The efficiency of ultra-relativistic Brayton engine is larger than the classical Brayton engine but lower than non-relativistic Brayton engine can be shown by a ratio of the specific heat at a constant pressure by a specific heat at a constant volume.
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