Stochastic maximum principle for systems driven by local martingales with spatial parameters

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Abstract: We consider the stochastic optimal control problem for the dynamical system of the stochastic differential equation driven by a local martingale with a spatial parameter. Assuming the convexity of the control domain, we obtain the stochastic maximum principle as the necessary condition for an optimal control, and we also prove its sufficiency under proper conditions. The stochastic linear quadratic problem in this setting is also discussed.

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1. Introduction

This paper concerns the stochastic maximum principle for the dynamical system of the stochastic differential equation (SDE) driven by a local martingale with a spatial parameter.
On a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) that satisfies the usual conditions, we consider the following stochastic controlled system
\[
\begin{align*}
  dx^u(t) &= b(t, x(t), u(t)) \, dt + M(t, x(t), u(t)) \, \, dt, \\
  x^u(0) &= x^u_0,
\end{align*}
\]
where \(b : [0, T] \times \mathbb{R}^d \times U \times \Omega \to \mathbb{R}^d\) is an \(\{\mathcal{F}_t\}_{t \geq 0}\)-adapted process, and
\[
\left\{M(t, x, u), t \in [0, T]\right\}(x,u) \in \mathbb{R}^d \times U
\]
is a family of \(d\)-dimensional local martingales with the parameter \((x, u) \in \mathbb{R}^d \times U \subset \mathbb{R}^d \times \mathbb{R}^k\).

We assume that the control domain \(U\) is a convex subset of \(\mathbb{R}^k\). Let
\[
U[0, T] = \left\{u : [0, T] \times \Omega \to U : u \text{ is } \{\mathcal{F}_t\}_{t \geq 0}\text{-adapted and } \mathbb{E} \int_0^T |u(t)|^2 \, dt < \infty \right\}
\]
denote the set of all admissible controls. The cost functional \(J(u)\) is given by
\[
J(u) = \mathbb{E} \left[ \int_0^T f(t, x^u(t), u(t)) \, dt + \Phi(x^u(T)) \right], \quad u \in U[0, T],
\]
where \(f : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}\) and \(\Phi : \mathbb{R}^d \to \mathbb{R}\) are measurable functions.

For an optimal control \(\bar{u} \in U[0, T]\), i.e., a control \(\bar{u}\) satisfying \(J(\bar{u}) = \inf_{u \in U[0, T]} J(u)\), let \(\bar{x} = x^{\bar{u}}\), and we call \((\bar{x}, \bar{u})\) an optimal pair. The goal of this paper is to find the necessary condition which is the so-called stochastic maximum principle for an optimal pair \((\bar{x}, \bar{u})\) for the optimal control problem (1.1)–(1.3), and moreover, we shall prove the sufficiency of the stochastic maximum principle under proper conditions.

SDEs driven by Brownian motion have been extensively studied, in particular, by the celebrated Itô calculus. The diffusion processes described by SDEs play an important role in the study of stochastic dynamical systems. To study various problems concerning SDEs driven by random vector fields (infinite-dimensional random processes), Kunita [15] developed stochastic calculus for semimartingales with spatial parameters and studied SDEs of the following form:
\[
X_t = x_0 + \int_0^t F(ds, X_s),
\]
where \(\{F(t, x), t \in [0, T]\}_{x \in \mathbb{R}^d}\) is a family of continuous semimartingales with the spatial parameter \(x \in \mathbb{R}^d\). Note that (1.1) is a specific form of (1.4).

On the one hand, Itô’s SDE is a special case of (1.4) if we set
\[
F(t, x) = \int_0^t f_0(r, x) \, dr + \sum_{k=1}^m \int_0^t f_k(r, x) \, dB_r^k,
\]
where \((B^1, \ldots, B^m)\) is an \(m\)-dimensional Brownian motion. On the other hand, if \(F(t, x)\) is a C-Brownian motion, i.e., for any partition \(0 \leq t_0 < t_1 \cdots < t_n \leq T\) of \([0, T]\), the
increments $F(t_{i+1}, x) - F(t_i, x), i = 0, 1, \ldots, n$ are independent, Kunita [14] proved that there exist a sequence of independent Brownian motions $\{B^k\}_{k \in \mathbb{N}}$ and functions $\{f_k\}_{k \in \mathbb{N}}$ such that

$$F(t, x) = \int_0^t f_0(r, x)dr + \sum_{k=1}^\infty \int_0^t f_k(r, x)dB^k_r.$$ 

Thus, Equation (1.4) can be viewed formally as an SDE driven by infinite-dimensional Brownian motion.

Stochastic optimal control problems of dynamical systems driven by finite-dimensional Brownian motion have been studied in depth. Here, we briefly mention some literature on stochastic maximum principles, which is by no means complete. Bismut [2] obtained the local maximum principle for stochastic optimal control problems with a convex control set. Peng [22] obtained the maximum principle for the general case in which the diffusion coefficient may contain the control variable and the control domain need not be convex. More recently, stochastic maximum principles for mean-field control problems were studied in, for instance, Li [16], Buckdahn, Li, and Ma [3], Meyer-Brandis, Øksendal, and Zhou [21], and for stochastic recursive optimal control problems by employing backward stochastic differential equations (BSDEs) in Chen and Epstein [4], Ji and Zhou [13], Hu [10], etc. For stochastic maximum principles in other various situations, we also refer to, for instance, Ma and Yong [19], Hu, Ji, and Xue [11], Tang [24], Zhou [27], Wu [25], Han, Peng, and Wu [9], Yong and Zhou [26], and the references therein.

The present paper concerns the optimal control problem (1.1)–(1.3) driven by a local martingale with a spatial parameter. One obvious motivation is that, viewing (1.1) as an SDE driven by infinite-dimensional Brownian motion, it arises naturally when studying financial markets comprising numerous stocks. Indeed, optimal control problems for systems governed by infinite-dimensional stochastic evolution equations have been investigated in, for instance, [5, 7, 8, 12, 18]. Another motivation comes from the study of an illiquid financial market in which the trades of a single large investor can influence market prices. For such a market, Peter and Dietmer [1] employed a family of continuous semimartingales $\{P(t, v), t \in [0, T]\}_{v \in \mathbb{R}}$ to model the price fluctuations of the risky asset given that the large investor holds a constant stake of $v$ shares in this asset.

We would also like to point out that the existence and uniqueness of the solution to (1.4) were obtained under suitable Lipschitz conditions in Kunita [15], and this result was extended in Liang [17] to the non-Lipschitz case. Backward doubly SDEs involving martingales with spatial parameters were studied in Bally and Matoussi [20] and Song, Song, and Zhang [23], and the solutions were proved therein to be probabilistic interpretations (nonlinear Feynman-Kac formulas) for the corresponding stochastic partial differential equations.

We would like to make a few remarks on our work before ending this introduction. In our optimal control problem (1.1)–(1.3), we assume that the control domain $U \subset \mathbb{R}^k$ is a convex set, and this enables us to apply the standard variational method to derive the stochastic maximum principle. A key step of the variational method is to derive the variational equation (see eq. (3.6) in Section 3.2) for the generalized SDE (1.1), which involves calculating the derivatives of the local martingale $M$ with respect to the spatial
parameters $x$ and $u$. This is the major difference between our problem and the classical case. Further, to obtain the variational equation, we shall employ the stochastic calculus for semimartingales with parameters developed in [15]. Furthermore, the corresponding adjoint equation (see BSDE (3.18) in Section 3.3) contains an extra martingale which is orthogonal to $M$ to guarantee the existence and uniqueness of the solution. This is because the BSDE is driven by a general martingale rather than Brownian motion (see El Karoui and Huang [6]). Despite all these differences, we can show that the classical stochastic maximum principle is indeed a special case in our setting.

The rest of this paper is organized as follows. In Section 2, we provide some preliminaries on the stochastic calculus for martingales with spatial parameters. In Section 3, we formulate our optimal control problem, derive the stochastic maximum principle, and prove its sufficiency under proper conditions. Finally in Section 4, we discuss the linear quadratic optimal control problems (LQ problems) in our setting.

Throughout the article, we use $C$ to denote a generic constant which may vary in different places.

2. Preliminaries

In this section, we collect some preliminaries on regularity results and stochastic calculus for local martingales with spatial parameters. We refer to [15] for more details.

We recall some conventional notations. Denote by $\mathbb{R}^d$ the $d$-dimensional real Euclidean space. We use the notation $\partial_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right)$ for $x \in \mathbb{R}^d$. Then for $\Psi : \mathbb{R}^d \to \mathbb{R}$, $\partial_x \Psi = \left(\frac{\partial}{\partial x_j} \Psi\right)_{1 \times d}$ is a row vector, and for $\Psi : \mathbb{R}^d \to \mathbb{R}^n$, $\partial_x \Psi = \left(\frac{\partial}{\partial x_j} \Psi_i\right)_{n \times d}$ is an $n \times d$ matrix.

For two vectors $u, v \in \mathbb{R}^d$, $\langle u, v \rangle$ denotes the scalar product of $u$ and $v$, and $|v| = \sqrt{\langle v, v \rangle}$ means the Euclidean norm of $v$. We also use $\langle \cdot, \cdot \rangle$ to denote the quadratic covariation of two continuous local martingales. For $A, B \in \mathbb{R}^{d \times n}$, we denote the scalar product of $M$ and $N$ by $\langle M, N \rangle = \operatorname{tr}[M N^*]$ (resp., $\|M\| = \sqrt{\operatorname{tr}[M M^*]}$), where the superscript $*$ stands for the transpose of vectors or matrices.

2.1. Regularity of $M(t, x)$ with respect to the spatial parameter $x$

In this subsection, we shall recall some results on the differentiability of continuous local martingales with respect to the spatial parameter $x$.

Let $M := \{M(t, x), t \in [0, T]\}_{x \in \mathbb{R}^d}$ be a family of local martingales with joint quadratic variation (quadratic covariation) on the interval $[0, t]$ given by a.s.

$$\langle M(\cdot, x), M(\cdot, y) \rangle_t = \int_0^t q(s, x, y)ds,$$ \hspace{1cm} (2.1)

where $q(t, x, y)$ is a predictable process and is called the local characteristic of $M$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Let $d$ and $l$ be positive integers and $m$ be a nonnegative integer. Denote by $C^m(\mathbb{R}^d; \mathbb{R}^l)$ or simply $C^m$ the set of
\(m\)-times continuously differentiable functions \(f : \mathbb{R}^d \to \mathbb{R}^l\). We use the convention that if \(m = 0\), \(C^0(\mathbb{R}^d; \mathbb{R}^l)\) is just the set \(C(\mathbb{R}^d; \mathbb{R}^l)\) of continuous functions.

Let \(K\) be a subset of \(\mathbb{R}^d\). Denote
\[
\|f\|_{m,K} = \sup_{x \in K} \frac{|f(x)|}{1 + |x|} + \sum_{1 \leq |\alpha| \leq m}\sup_{x \in K} |D^\alpha f(x)|,
\]
where \(D^\alpha := \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_d)^{\alpha_d}}\) is the differential operator. Then \(C^m\) is a Fréchet space endowed with seminorms \(\{\| \cdot \|_{m,K} : K \subset \mathbb{R}^d \text{ is compact}\}\). When \(K = \mathbb{R}^d\), we also write \(\| \cdot \|_m := \| \cdot \|_{m,\mathbb{R}^d}\). Here \(C^m_{\mathbb{R}^d}\) denotes the set \(\{f \in C^m : \|f\|_m < \infty\}\).

For a constant \(\delta \in (0, 1]\), let \(C^{m,\delta}\) denote the set of functions \(f \in C^m\) such that the partial derivatives \(D^\alpha f\) with \(|\alpha| = m\) are \(\delta\)-Hölder continuous. Similarly, \(C^{m,\delta}\) is a Fréchet space under the seminorms,
\[
\|f\|_{m+\delta,K} := \|f\|_{m,K} + \sum_{|\alpha| = m, x,y \in K} \sup_{x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\delta},
\]
where \(K\) are compact subsets of \(\mathbb{R}^d\). Clearly \(C^{m,0} = C^m\). We also write \(\| \cdot \|_{m+\delta,\mathbb{R}^d} := \| \cdot \|_{m+\delta}\), and denote by \(C^{m,\delta}_{\mathbb{R}^d}\) the set \(\{f \in C^{m,\delta} : \|f\|_{m+\delta} < \infty\}\).

We say that a continuous function \(f(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\) belongs to the class \(C^{m,\delta}\) (or \(f(t, \cdot)\) is a \(C^{m,\delta}\)-valued function) if for each fixed \(t \in [0, T]\), \(f(t, \cdot)\) belongs to \(C^{m,\delta}\) and \(\int_0^T \|f(t, \cdot)\|_{m+\delta,K} dt < \infty\) for any compact subset \(K \subset \mathbb{R}^d\).

Similarly, the function space \(\tilde{C}^m\) consists of all \(\mathbb{R}^l\)-valued functions \(g(x, y)\) that are \(m\)-times differentiable with respect to each \(x, y \in \mathbb{R}^d\). For \(K \subset \mathbb{R}^d\), we define
\[
\|g\|_{m,K} \assign \sup_{x,y \in K} \frac{|g(x, y)|}{(1 + |x|)(1 + |y|)} + \sum_{1 \leq |\alpha| \leq m, x,y \in K} \sup_{x \neq y} |D^\alpha x D^\alpha y g(x, y)|.
\]
Then \(\tilde{C}^m\) is a Fréchet space equipped with the seminorms \(\{\| \cdot \|_{m,K} : K \subset \mathbb{R}^d\text{ is compact}\}\).

For \(\delta \in (0, 1]\), we define
\[
\|g\|_{m+\delta,K} = \|g\|_{m,K} + \sum_{|\alpha| = m} \|D^\alpha x D^\alpha y g\|_{\delta,K},
\]
where
\[
\|g\|_{\delta,K} = \sup_{x,y,x',y' \in K, x \neq x', y \neq y'} \frac{|g(x, y) - g(x', y) - g(x, y') + g(x', y')|}{|x - x'|^\delta |y - y'|^\delta}.
\]
Let \(\tilde{C}^{m,\delta}\) denote the space of functions \(g\) such that \(\|g\|_{m+\delta,K} < \infty\) for any compact subset \(K\), and thus \(\tilde{C}^{m,\delta}\) is a Fréchet space with the seminorms \(\{\| \cdot \|_{m+\delta,K} : K \subset \mathbb{R}^d\text{ is compact}\}\).

We also have \(\tilde{C}^{m,\delta} = \tilde{C}^m\).

When \(K = \mathbb{R}^d\), we write \(\| \cdot \|_{m} \assign \| \cdot \|_{m,\mathbb{R}^d}\) and \(\| \cdot \|_{m+\delta} \assign \| \cdot \|_{m+\delta,\mathbb{R}^d}\). We also define \(\tilde{C}^m_{\mathbb{R}^d} \assign \{g \in \tilde{C}^m : \|g\|_m < \infty\}\) and \(\tilde{C}^{m,\delta}_{\mathbb{R}^d} \assign \{g \in \tilde{C}^m : \|g\|_{m+\delta} < \infty\}\).
Consider a random field \( \{ F(\omega, t, x), t \in [0, T], x \in \mathbb{R}^d \} \). If \( F(\omega, t, x) \) is \( m \)-times continuously differentiable with respect to \( x \) for almost all \( \omega \in \Omega \) and for all \( t \in [0, T] \), then it is called a \( C^m \)-\textit{valued process}. Furthermore, if \( t \mapsto F(\omega, t, \cdot) \) is a continuous mapping from \([0, T]\) to \( C^m \) for almost all \( \omega \), then we call it a \textit{continuous \( C^m \)-process}. In the same way, one can define \( C^{m, \delta} \)-\textit{valued process}, \( \tilde{C}^{m, \delta} \)-\textit{valued process}, \( \bar{C}^{m, \delta} \)-\textit{valued process}, \( \widetilde{C}^{m, \delta} \)-\textit{valued process}, and \( \tilde{C}^{m, \delta} \)-\textit{process}.

The following two theorems (Theorem 2.1 and Theorem 2.2), which are adopted from [15] (Theorem 3.1.2 and Theorem 3.1.3, respectively), describe the relationship of the spatial regularity between local martingales and their joint quadratic variations.

**Theorem 2.1.** Let \( \{ M(t, x), t \in [0, T] \}_{x \in \mathbb{R}^d} \) be a family of continuous local martingales with \( M(0, x) \equiv 0 \). Assume that the joint quadratic variation \( Q(t, x, y) \) has a modification of a continuous \( \tilde{C}^{m, \delta} \)-\textit{process} for some \( m \in \mathbb{N} \) and \( \delta \in (0, 1] \). Then \( M(t, x) \) has a modification of continuous \( \bar{C}^{m, \varepsilon} \)-\textit{process} for any \( \varepsilon < \delta \). Furthermore, for any \( |\alpha| \leq m \), \( \{ D_x^\alpha M(t, x), t \in [0, T] \}_{x \in \mathbb{R}^d} \) is a family of continuous local martingales with the joint quadratic variation \( D_x^\alpha D_y^\beta Q(t, x, y) \).

**Theorem 2.2.** Let \( \{ M(t, x), t \in [0, T] \}_{x \in \mathbb{R}^d} \) and \( \{ N(t, y), t \in [0, T] \}_{y \in \mathbb{R}^d} \) be continuous local martingales with values in \( C^{m, \delta} \) for some \( m \geq 0 \) and \( \delta \in (0, 1] \). Then the joint quadratic variation has a modification of a continuous \( \tilde{C}^{m, \varepsilon} \)-\textit{process} for any \( \varepsilon < \delta \). Furthermore, the modification satisfies, for \( |\alpha|, |\beta| \leq m \),

\[
D_x^\alpha D_y^\beta (M(\cdot, x), N(\cdot, y))_t = (D_x^\alpha M(\cdot, x), D_y^\beta N(\cdot, y))_t
\]

for all \( t \in [0, T] \).

Fix some nonnegative integer \( m \) and \( \delta \in (0, 1] \), then the local characteristic \( q(t, x, y) \) of \( M \) is said to belong to \textit{the class} \( B^{m, \delta} \), if \( q(t, \cdot, \cdot) \) has a modification of a predictable \( \tilde{C}^{m, \delta} \)-\textit{valued process} with \( \int_0^T \| q(t) \|_{m, \delta} K dt < \infty \) a.s. for any compact set \( K \subset \mathbb{R}^d \). Furthermore, if \( \int_0^T \| q(t) \|_{m, \delta} dt < \infty \) a.s., we say that \( q(t, x, y) \) belongs to \textit{the class} \( B^{m, \delta}_b \), and if \( \| q(t) \|_{m, \delta} \leq c \) holds for all \( t \in [0, T] \) and \( \omega \in \Omega \), we say that \( q(t, x, y) \) belongs to \textit{the class} \( B^{m, \delta}_{ub} \).

### 2.2. Stochastic calculus with respect to local martingales with spatial parameters

Let \( \{ X_t, 0 \leq t \leq T \} \) be a \( \mathbb{R}^d \)-valued predictable process such that

\[
\int_0^T q(s, X_s, X_s) ds < \infty \quad \text{a.s.}
\]

(2.3)

Then the generalized Itô integral \( M_t(X) := \int_0^T M(ds, X_s) \) is well defined and is a local martingale. In particular, if the sample paths of \( X_t \) are continuous a.s., the integral can be approximated by Riemann sums:

\[
M_t(X) = \int_0^t M(ds, X_s) = \lim_{|\Delta| \to 0} \sum_{k=0}^{n-1} [M(t_{k+1}, X_{t_k}) - M(t_k, X_{t_k})],
\]

(2.4)
where $\Delta$ is a partition of the interval $[0, T]$ with $|\Delta|$ being the maximum length of all subintervals.

Let $Y$ be another predictable process satisfying (2.3). Then $M_t(Y)$ is also well defined, and the joint quadratic variation of $M_t(X)$ and $M_t(Y)$ is given by

$$\langle M(X), M(Y) \rangle_t = \int_0^t q(s, X_s, Y_s) ds \quad \text{a.s.} \quad (2.5)$$

**Remark 2.1.** Assume $M(t, x) = g(x)W_t$, where $W_t$ is a standard Brownian motion and $g$ is a measurable function on $\mathbb{R}^d$ such that $\int_0^T |g(X_s)|^2 ds < \infty$ a.s. The quadratic variation of $M$ is

$$\langle M(\cdot, x), M(\cdot, y) \rangle_t = g(x)g(y)t$$

with the local characteristic $q(t, x, y) = g(x)g(y)$. The stochastic integral

$$M_t(X) = \int_0^t M(ds, X_s)$$

now coincides with the classical Itô integral $\int_0^t g(X_s)dW_s$.

Let $\left\{ M(t, x) = (M^1(t, x), M^2(t, x), \ldots, M^d(t, x)), t \in [0, T] \right\}_{x \in \mathbb{R}^d}$ be a family of $d$-dimensional continuous local martingales. Here $M^i(t, x), 1 \leq i \leq q$ are one-dimensional continuous local martingales with joint quadratic variation

$$\langle M^i(\cdot, x), M^j(\cdot, y) \rangle_t = \int_0^t q_{ij}(s, x, y) ds \quad \text{a.s.} \quad (2.6)$$

Denote $q(t, x, y) = (q_{ij}(t, x, y), 1 \leq i, j \leq d)$. Then $q(t, x, y)$ is a $d \times d$-matrix-valued process such that $q_{ij}(t, x, y) = q_{ji}(t, y, x)$ a.s. for all $x, y \in \mathbb{R}^d, t \in [0, T]$ and $1 \leq i, j \leq d$. Therefore, $q(t, x, y) = q^*(t, y, x)$. Moreover, $q(t, x, x)$ is a nonnegative-definite symmetric matrix a.s. for all $(t, x) \in [0, T] \times \mathbb{R}^d$.

We introduce the following set of stochastic processes:

$$\mathcal{S}^2([0, T]; \mathbb{R}^d) := \left\{ \phi : [0, T] \times \Omega \to \mathbb{R}^d; \phi \text{ is predictable, } \mathbb{E} \left( \sup_{0 \leq t \leq T} |\phi(t)|^2 \right) < \infty \right\}.$$ 

Consider the following SDE

$$\begin{cases}
    dX_t = b(t, X_t)dt + M(dt, X_t), & t \in (0, T], \\
    X_0 = x_0,
\end{cases} \quad (2.7)$$

where $x_0 \in \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is an adapted stochastic process.

**Definition 1.** We say that $X = (X_t, t \in [0, T])$ adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ is a solution to (2.7) if $X$ satisfies the following integral equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t M(ds, X_s)$$

for $t \in [0, T]$ almost surely.
By combining Theorem 3.4.1 and Lemma 3.4.3 in [15], we obtain the following result.

**Theorem 2.3.** Assume that there exists a positive constant $K$ such that

$$
|b(t, x) - b(t, y)| \leq K |x - y|,
$$

$$
|b(t, x)| \leq K (1 + |x|),
$$

$$
\|q(t, x, x) - 2q(t, x, y) + q(t, y, y)\| \leq K |x - y|^2,
$$

$$
\|q(t, x, y)\| \leq K (1 + |x|)(1 + |y|),
$$

hold for all $x, y \in \mathbb{R}^d$ a.s. Then SDE (2.7) has a unique solution in $S^2([0, T]; \mathbb{R}^d)$.

**Remark 2.2.** If we assume $q \in B_{ab}^{0,1}$, then $q$ satisfies the conditions on $q$ in Theorem 2.3.

**Remark 2.3.** Consider the following classical SDE

$$
X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,
$$

where $b(\cdot, x)$ and $\sigma(\cdot, x)$ are adapted processes for each fixed $x \in \mathbb{R}^d$ taking values in $\mathbb{R}^d$ and $\mathbb{R}^{d \times d}$ respectively, and $W$ is a $d$-dimensional standard Brownian motion. We can write $f_0^t M(ds, X_s) = \int_0^t \sigma(s, X_s)dW_s$, where $M(t, x) = \int_0^t \sigma(s, x)dW_s$ with the joint quadratic variation

$$
q_{ij}(t, x, y) = \sum_{k=1}^d \sigma_{ik}(t, x)\sigma_{jk}(t, y).
$$

If we assume $\sigma$ is uniformly Lipschitz and linear growth as in the classical setting, then $q(t, x, y)$ satisfies the conditions in Theorem 2.3.

### 3. Stochastic maximum principle

In this section, we derive the stochastic maximum principle for the optimal control problem associated with (1.1), (1.2) and (1.3).

#### 3.1. Formulation of the stochastic optimal control problem

Recall the stochastic controlled system (1.1)

$$
\begin{align*}
\left\{ \begin{array}{l}
\quad dx^u(t) = b(t, x^u(t), u(t))dt + M(dt, x^u(t), u(t)), \\
\quad x^u(0) = x_0^u.
\end{array} \right.
\end{align*}
$$

the set of all admissible controls defined by (1.2)

$$
U[0, T] = \left\{ u : [0, T] \times \Omega \to U : u \text{ is } \mathcal{F}_t_{t \geq 0}\text{-adapted, } \mathbb{E} \int_0^T |u(t)|^2 dt < \infty \right\},
$$
and the cost functional given by (1.3)

\[ J(u) = E \left\{ \int_0^T f(t, x(t), u(t)) \right. dt + \Phi(x^n(T)) \right\}. \]

In (1.1), \( \{M(t, x, u) = (M^1(t, x, u), M^2(t, x, u), \ldots, M^d(t, x, u)), t \in [0, T]\} \) is a family of \( d \)-dimensional continuous local martingales, of which the joint quadratic variation is given by

\[ \langle M^i(\cdot, x, u), M^j(\cdot, y, v) \rangle_t = \int_0^t q_{ij}(s, x, u, y, v) ds. \tag{3.1} \]

We assume the following conditions.

(H1) The functions \( b, f \), and \( \Phi \) are continuous and continuously differentiable in \((x, u)\).

Moreover, \( b_x \) and \( b_u \) are bounded, and there exists a positive constant \( K_1 \) such that for all \( t \in [0, T] \), \((x, u) \in \mathbb{R}^{d+k}\),

\[ (|f_x| + |f_u|)(t, x, u) + |\Phi_x(x)| \leq K_1 (1 + |x| + |u|). \]

(H2) For all \((x, u), (y, v) \in \mathbb{R}^{d+k}\), \( q(t, x, u, y, v) \) belongs to \( B_{ub}^{1,\delta}(\mathbb{R}^{d+k} \times \mathbb{R}^{d+k}, \mathbb{R}^{d \times d}) \) for some \( \delta \in (0, 1) \). It follows that for \( x' = (x, u) \in \mathbb{R}^{d+k} \), \( y' = (y, v) \in \mathbb{R}^{d+k} \), the partial derivative \( \|D_{x'}D_{y'}q(t, x, u, y, v)\| \) is uniformly bounded in \((x', y')\).

In particular, Condition (H2) implies \( q \in B_{ub}^{0,1} \), i.e., there exist positive constants \( K_2 \) and \( K_3 \) such that

\[
\begin{align*}
|q(t, x, u, y, v)| &\leq K_2(1 + |x| + |u|)(1 + |y| + |v|); \\
|q(t, x, u, y, v) - q(t, x', u', y, v') - q(t, x, u, y', v') + q(t, x', u', y', v')| \\
&\leq K_3 \big(|x - x'| + |u - u'|\big)(|y - y'| + |v - v'|).
\end{align*}
\]

This (the second inequality) also yields

\[ \|q(t, x, u, v, u) - 2q(t, x, u, y, v) + q(t, y, v, y, v)\| \leq 2K_3(|x - y|^2 + |u - v|^2). \tag{3.2} \]

Therefore, assuming (H1) and (H2), we can apply Theorem 2.3 to SDE (1.1) which consequently has a unique solution \( x^n(t) \) with \( \mathbb{E}(\sup_{0 \leq t \leq T} |x^n(t)|^2) < \infty \) for each \( u \in \mathcal{U}([0, T]) \).

Recall that the goal of the optimal control problem is to minimize the cost functional \( J(u) \) over the set of admissible controls \( \mathcal{U}[0, T] \). Suppose \( \overline{u} \in \mathcal{U}[0, T] \) is an optimal control, i.e.,

\[ J(\overline{u}) = \inf_{u \in \mathcal{U}[0, T]} J(u), \]

and \( \overline{x} := x^{\overline{u}} \in \mathcal{S}^2([0, T], \mathbb{R}^d) \) is the corresponding solution of the state equation (1.1), then \((\overline{x}, \overline{u})\) is called an optimal pair. For \( u \in \mathcal{U}[0, T] \) and \( \varepsilon \in [0, 1] \), we define

\[ u^\varepsilon(t) = \overline{u}(t) + \varepsilon \left( u(t) - \overline{u}(t) \right), \quad t \in [0, T]. \]
Then, clearly $u^\varepsilon$ converges to $\pi$ in $L^2(\Omega \times [0, T])$ as $\varepsilon$ goes to zero. Recall that the control domain $U$ is convex, and hence $u^\varepsilon$ belongs to $U[0, T]$ for each $u \in U[0, T]$, and we denote by

$$x^\varepsilon(t) := x^{u^\varepsilon}(t), \quad t \in [0, T]$$

the corresponding unique solution of (1.1) in $S^2([0, T]; \mathbb{R}^d)$.

**Lemma 3.1.** Assume (H1) and (H2). Let

$$y^\varepsilon(t) = x^\varepsilon(t) - \pi(t).$$

Then, there exists a positive constant $C$ independent of $\varepsilon$ such that

$$\mathbb{E} \left[ |y^\varepsilon(t)|^2 \right] \leq C\varepsilon^2. \quad (3.3)$$

**Proof.** Clearly, $y^\varepsilon(t)$ is a semimartingale of the following form

$$y^\varepsilon(t) = \int_0^t \left[ b(s, x^\varepsilon(s), u^\varepsilon(s)) - b(s, \pi(s), \pi(s)) \right] ds$$

$$+ \int_0^t M(ds, x^\varepsilon(s), u^\varepsilon(s)) - \int_0^t M(ds, \pi(s), \pi(s)).$$

Applying Itô’s formula to $|y^\varepsilon(t)|^2$, we have

$$|y^\varepsilon(t)|^2 = 2 \int_0^t \langle y^\varepsilon(s), b(s, x^\varepsilon(s), u^\varepsilon(s)) - b(s, \pi(s), \pi(s)) \rangle ds$$

$$+ 2 \int_0^t \langle y^\varepsilon(s), M(ds, x^\varepsilon(s), u^\varepsilon(s)) \rangle - 2 \int_0^t \langle y^\varepsilon(s), M(ds, \pi(s), \pi(s)) \rangle$$

$$+ \sum_{i=1}^d \left( \int_0^t M^i(ds, x^\varepsilon(s), u^\varepsilon(s)) - \int_0^t M^i(ds, \pi(s), \pi(s)) \right) \right|_t. \quad (3.4)$$

Here, we shall prove that $\mathbb{E} \int_0^t \langle y^\varepsilon(s), M(ds, x^\varepsilon(s), u^\varepsilon(s)) \rangle$ is equal to zero. Since

$$M^i_t := \int_0^t M(ds, x^\varepsilon(s), u^\varepsilon(s))$$

is a continuous $\mathbb{R}^d$-valued local martingale and $y^\varepsilon(t)$ is square integrable, the stochastic integral $\int_0^t \langle y^\varepsilon(s), dM^i_x \rangle$ is a local martingale as well. Then, it remains to show that the local martingale $\int_0^t \langle y^\varepsilon(s), dM^i_x \rangle$ is also a martingale. The Burkholder–Davis–Gundy inequality yields

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \langle y^\varepsilon(s), M(ds, x^\varepsilon(s), u^\varepsilon(s)) \rangle \right|$$

$$\leq C \sum_{j=1}^d \mathbb{E} \left( \int_0^T |y^\varepsilon_j(t)|^2 q_{jj}(t, x^\varepsilon(t), u^\varepsilon(t), x^\varepsilon(t), u^\varepsilon(t)) dt \right)^{\frac{1}{2}}$$

for each $i = 1, 2, \ldots, d$. Therefore, the process $y^\varepsilon(t)$ is a square integrable martingale, and hence $y^\varepsilon(t)$ is a martingale as well. The Burkholder–Davis–Gundy inequality yields
\[
\leq C \sum_{j=1}^{d} E \left( \sup_{0 \leq t \leq T} |y_{j}^{\varepsilon}(t)|^2 + \int_{0}^{T} q_{jj}(t, x^{\varepsilon}(t), u^{\varepsilon}(t), x^{\varepsilon}(t), u^{\varepsilon}(t)) dt \right)
\]

\[
< \infty,
\]

where the last inequality follows from (H2) and the integrability of \(y^{\varepsilon}, x^{\varepsilon}\) and \(u^{\varepsilon}\). Hence

\[
E \int_{0}^{t} \langle y^{\varepsilon}(s), M(ds, x^{\varepsilon}(s), u^{\varepsilon}(s)) \rangle = 0.
\]

Similarly, we can also show

\[
E \int_{0}^{t} \langle y^{\varepsilon}(s), M(ds, \overline{x}(s), \overline{u}(s)) \rangle = 0.
\]

Now taking expectation for both sides of (3.4), we have

\[
E |y(t)|^2 \leq C E \left( \int_{0}^{t} |y^{\varepsilon}(s)|^2 + \varepsilon^2 |u(s) - \overline{u}(s)|^2 ds \right)
\]

\[
+ \sum_{i=1}^{d} E \int_{0}^{t} \left[ q_{ii}(s, x^{\varepsilon}(s), u^{\varepsilon}(s), x^{\varepsilon}(s), u^{\varepsilon}(s)) - 2q_{ii}(s, x^{\varepsilon}(s), u^{\varepsilon}(s), \overline{x}(s), \overline{u}(s)) + q_{ii}(s, \overline{x}(s), \overline{u}(s), \overline{x}(s), \overline{u}(s)) \right] ds
\]

\[
\leq C E \left( \int_{0}^{t} |y^{\varepsilon}(s)|^2 + \varepsilon^2 |u(s) - \overline{u}(s)|^2 ds \right),
\]

where in the first inequality we use the Lipschitz property of \(b\) and the fact that \(2|\langle x, y \rangle| \leq |x|^2 + |y|^2\), and the second inequality follows from (3.2).

Finally, the desired result (3.3) follows from applying Gronwall’s inequality to (3.5).

### 3.2. Variational equation

Assume Conditions (H1) and (H2). From Theorem 2.1, \(M(t, x, u)\) has a modification of a continuous \(C^{1,\delta'}\)-local martingale for any \(\delta' \in (0, \delta)\). In particular, the modification, denoted by \(M(t, x, u)\) again, is differentiable with respect to \(x\) and \(u\). Moreover, the partial derivatives \(\partial_{x} M(t, x, u)\) and \(\partial_{u} M(t, x, u)\) are continuous local martingales.

For notational simplicity, throughout the rest of this article, we write

\[
dM(t) = M(dt) := M(dt, \overline{x}(t), \overline{u}(t)),
\]

where \(M(t) = \int_{0}^{t} M(ds, \overline{x}(s), \overline{u}(s))\) is a continuous local martingale. We also adopt the following notations,

\[
b_{x}(t) = b_{x}(t, \overline{x}(t), \overline{u}(t)), \quad b_{u}(t) = b_{u}(t, \overline{x}(t), \overline{u}(t)),
\]
\[ \partial_x M(dt) = \partial_x M(dt, \overline{x}(t), \overline{u}(t)), \quad \partial_u M(dt) = \partial_u M(dt, \overline{x}(t), \overline{u}(t)), \]

where

\[ b_x(t) = (\partial_x b^i(t))_{d \times d} = \begin{bmatrix} b^1_{x1}(t) & \cdots & b^1_{xd}(t) \\ \vdots & \ddots & \vdots \\ b^d_{x1}(t) & \cdots & b^d_{xd}(t) \end{bmatrix}, \]

The other matrices \( b_u(t), \partial_x M(dt), \) and \( \partial_u M(dt) \) are defined similarly.

Let \( \hat{x}(t) \in \mathbb{R}^d \) be the solution to the following SDE

\[
\begin{cases}
    d\hat{x}(t) = (b_x(t)\hat{x}(t) + b_u(t)(u(t) - \overline{u}(t)))dt + \partial_x M(dt)\hat{x}(t) + \partial_u M(dt)(u(t) - \overline{u}(t)), \\
    \hat{x}(0) = 0.
\end{cases}
\]

Here the multiplication used in \( \partial_x M(dt)\hat{x}(t) \) and \( \partial_u M(dt)(u(t) - \overline{u}(t)) \) is the matrix multiplication, for instance,

\[
\partial_x M(dt)\hat{x}(t) = \begin{bmatrix} \sum_{j=1}^d \hat{x}_j(t)\partial_x M^1(dt) \\ \vdots \\ \sum_{j=1}^d \hat{x}_j(t)\partial_x M^d(dt) \end{bmatrix}.
\]

Now, we show that SDE (3.6) has a unique solution in \( \mathcal{S}^2([0, T], \mathbb{R}^d) \). If we denote

\[
\tilde{b}(t, \hat{x}(t)) = b_x(t)\hat{x}(t) + b_u(t)(u(t) - \overline{u}(t)),
\]

and

\[
\int_0^t \tilde{M}(ds, \hat{x}(s)) = \int_0^t \partial_x M(ds)\hat{x}(s) + \int_0^t \partial_u M(ds)(u(s) - \overline{u}(s)),
\]

i.e.,

\[
\tilde{M}(t, x) = \int_0^t \partial_x M(ds)x + \int_0^t \partial_u M(ds)(u(s) - \overline{u}(s)).
\]

Then, the variational equation (3.6) becomes

\[
\begin{cases}
    d\hat{x}(t) = \tilde{b}(t, \hat{x}(t))dt + \tilde{M}(dt, \hat{x}(t)), \\
    \hat{x}(0) = 0,
\end{cases}
\]

which has the same form as (1.1) with the local characteristic \( \tilde{q}(t, x, y) \) of \( \tilde{M} \) being

\[
(q(t, x, y))_{ij} = x^* \left( \frac{\partial^2 q_{ij}(t, \overline{x}(t), \overline{u}(t), \overline{u}(t), \overline{u}(t))}{\partial x \partial y} \right) y \\
+ x^* \left( \frac{\partial^2 q_{ij}(t, \overline{x}(t), \overline{u}(t), \overline{u}(t), \overline{u}(t))}{\partial x \partial v} \right) (u(t) - \overline{u}(t))
\]
\[
+ (u(t) - \overline{u}(t)) \left( \frac{\partial^2 q_{ij}(t, \overline{x}(t), \overline{u}(t), \overline{u}(t))}{\partial u \partial v} \right) y \\
+ (u(t) - \overline{u}(t)) \left( \frac{\partial^2 q_{ij}(t, \overline{x}(t), \overline{u}(t), \overline{u}(t))}{\partial u \partial v} \right) (u(t) - \overline{u}(t)).
\]

It can be easily observed that \( \tilde{b} \) and \( \tilde{q} \) are uniformly Lipschitz continuous and satisfy the following generalized linear growth condition

\[
|\tilde{b}(t, x)| \leq C(|a_t| + |x|),
\]

\[
||\tilde{q}(t, x, y)|| \leq C(1 + |a_t||x|)(1 + |a_t||y|),
\]

where \( \{a_t\}_{t \in [0, T]} \) is an adapted square integrable process. Then, using the same argument as that in Kunita’s proof in [15] for Theorem 2.3 yields that SDE (3.6) has a unique solution with

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |\hat{\eta}(t)|^2 \right] < \infty.
\]

We refer to (3.6) as the variational equation along the optimal pair \((\bar{x}, \bar{u})\). We will show in Proposition 3.1 that \( \hat{x}^{\epsilon(t)}(t) \) converges to \( \hat{x}(t) \) in \( L^2(\Omega) \) as \( \epsilon \) goes to 0. Set

\[
\eta^\epsilon(t) = \frac{x^\epsilon(t) - \overline{x}(t)}{\epsilon} - \hat{x}(t).
\]  

**Proposition 3.1.** Under assumptions (H1) and (H2), for any fixed \( T > 0 \), we have

\[
\lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E} |\eta^\epsilon(t)|^2 = 0.
\]  

**Proof.** From the state equation (1.1) and variational equation (3.6), we have

\[
\eta^\epsilon(t) = \frac{1}{\epsilon} \left\{ \int_0^t b(s, x^\epsilon(s), u^\epsilon(s)) - b(s, \overline{x}(s), \overline{u}(s)) ds + \int_0^t M(ds, x^\epsilon(s), u^\epsilon(s)) \\
- \int_0^t M(ds, \overline{x}(s), \overline{u}(s)) - \epsilon \int_0^t (b_x(s) \hat{x}(s) + b_u(s) (u(s) - \overline{u}(s))) ds \\
- \epsilon \int_0^t \partial_x M(ds, \overline{x}(s), \overline{u}(s)) \hat{x}(s) - \epsilon \int_0^t \partial_u M(ds, \overline{x}(s), \overline{u}(s)) (u(s) - \overline{u}(s)) \right\}.
\]

Denote

\[
A_\epsilon(t) = \int_0^t b_x(t, \overline{x}(t) + \lambda(x^\epsilon(t) - \overline{x}(t)), \overline{u}(t) + \lambda \epsilon (u(t) - \overline{u}(t))) d\lambda,
\]

\[
B_\epsilon(dt) = \int_0^t \partial_x M(dt, \overline{x}(t) + \lambda(x^\epsilon(t) - \overline{x}(t)), \overline{u}(t) + \lambda \epsilon (u(t) - \overline{u}(t))) d\lambda,
\]

\[
C_\epsilon(t) = \int_0^t b_u(t, \overline{x}(t) + \lambda(x^\epsilon(t) - \overline{x}(t)), \overline{u}(t) + \lambda \epsilon (u(t) - \overline{u}(t))) d\lambda,
\]

\[
D_\epsilon(dt) = \int_0^t \partial_u M(dt, \overline{x}(t) + \lambda(x^\epsilon(t) - \overline{x}(t)), \overline{u}(t) + \lambda \epsilon (u(t) - \overline{u}(t))) d\lambda,
\]

\[
\varphi_\epsilon(t) = [A_\epsilon(t) - b_x(t)] \hat{x}(t) + [C_\epsilon(t) - b_u(t)] (u(t) - \overline{u}(t)),
\]

\[
\psi_\epsilon(dt) = [B_\epsilon(dt) - \partial_x M(dt)] \hat{x}(t) + [D_\epsilon(dt) - \partial_u M(dt)] (u(t) - \overline{u}(t)).
\]
Using the fact that for a continuously differentiable function \( f(x, y) : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d \) and \( \alpha \in \mathbb{R}^d, \beta \in \mathbb{R}^k \),
\[
\int_0^1 (f_x(x + \alpha \lambda, y + \beta \lambda) \alpha + f_y(x + \alpha \lambda, y + \beta \lambda) \beta) d\lambda = f(x + \alpha, y + \beta) - f(x, y),
\]
we have
\[
\begin{aligned}
\begin{cases}
    d\eta(t) = [A_\varepsilon(t)\eta(t) + \varphi_\varepsilon(t)] dt + [B_\varepsilon(dt)\eta(t) + \psi_\varepsilon(dt)], \\
    \eta(0) = 0.
\end{cases}
\end{aligned}
\]
Therefore,
\[
\begin{aligned}
    &\mathbb{E} |\eta(t)|^2 = \sum_{i=1}^d \mathbb{E} \left| \int_0^t (A^i_\varepsilon(s)\eta^i(s) + \varphi^i_\varepsilon(s)) ds + \int_0^t (B^i_\varepsilon(ds)\eta^i(s) + \psi^i_\varepsilon(ds)) \right|^2 \\
    &\leq \sum_{i=1}^d C\mathbb{E} \left( \left| \int_0^t A^i_\varepsilon(s)\eta^i(s) ds \right|^2 + \left| \int_0^t B^i_\varepsilon(ds)\eta^i(s) \right|^2 \\
    &\quad + \left| \int_0^t \varphi^i_\varepsilon(s) ds \right|^2 + \left| \int_0^t \psi^i_\varepsilon(ds) \right|^2 \right) \\
    &\leq C \left( \mathbb{E} \int_0^T |\eta^\varepsilon(s)|^2 ds + J_\varepsilon(t) \right),
\end{aligned}
\]
where
\[
J_\varepsilon(t) = \sum_{i=1}^d \mathbb{E} \left( \left| \int_0^t \varphi^i_\varepsilon(s) ds \right|^2 + \left| \int_0^t \psi^i_\varepsilon(ds) \right|^2 \right).
\]
For simplicity of notations, we denote
\[
\begin{aligned}
x_{\lambda,\varepsilon}(t) &= \overline{x}(t) + \lambda (x^\varepsilon(t) - \overline{x}(t)), \\
u_{\lambda,\varepsilon}(t) &= \overline{u}(t) + \varepsilon (u(t) - \overline{u}(t)).
\end{aligned}
\]
Here, the last inequality holds because of the boundedness of \( b_x \) from assumption (H1) and the following estimation:
\[
\begin{aligned}
    &\sum_{i=1}^d \mathbb{E} \left| \int_0^t B^i_\varepsilon(ds)\eta^i(s) \right|^2 \\
    &= \sum_{i=1}^d \mathbb{E} \left| \sum_{j=1}^d \int_0^t \eta^i_j(s) \int_0^1 \partial_{x_j} M^i(ds, x_{\lambda,\varepsilon}(s), u_{\lambda,\varepsilon}(s)) d\lambda \right|^2 \\
    &\leq \sum_{i=1}^d \sum_{j=1}^d C\mathbb{E} \left| \int_0^t \eta^i_j(s) \int_0^1 \partial_{x_j} M^i(ds, x_{\lambda,\varepsilon}(s), u_{\lambda,\varepsilon}(s)) d\lambda \right|^2 \\
    &\leq \sum_{i=1}^d \sum_{j=1}^d C\mathbb{E} \int_0^T |\eta^i_j(s)|^2 ds \left< \int_0^1 \int_0^1 \partial_{x_j} M^i(dr, x_{\lambda,\varepsilon}(r), u_{\lambda,\varepsilon}(r)) d\lambda \right>_s
\end{aligned}
\]
\[
\sum_{i=1}^{d} \sum_{j=1}^{d} C \mathbb{E} \int_0^T |\eta_j^\varepsilon(s)|^2 \left( \int_0^1 \int_0^1 \partial^2 q_{ij}(s, x_{\lambda_1,\varepsilon}(s), u_{\lambda_1,\varepsilon}(s), x_{\lambda_2,\varepsilon}(s), u_{\lambda_2,\varepsilon}(s)) \, d\lambda_1 d\lambda_2 \right) \, ds \\
\leq C \mathbb{E} \int_0^T |\eta^\varepsilon(s)|^2 \, ds.
\]

Clearly,

\[
\sup_{0 \leq t \leq T} \mathbb{E} |\eta^\varepsilon(t)|^2 \leq C \left( \mathbb{E} \int_0^T |\eta^\varepsilon(s)|^2 \, ds + \sup_{0 \leq t \leq T} J_\varepsilon(t) \right) \\
\leq C \left( \int_0^T \sup_{0 \leq r \leq s} \mathbb{E} |\eta^\varepsilon(r)|^2 \, ds + \sup_{0 \leq t \leq T} J_\varepsilon(t) \right).
\]

From Gronwall’s lemma, we can obtain

\[
\sup_{0 \leq t \leq T} \mathbb{E} |\eta^\varepsilon(t)|^2 \leq C e^{CT} \left( \sup_{0 \leq t \leq T} J_\varepsilon(t) \right). \tag{3.12}
\]

Now, to obtain the desired result, it suffices to show that \( \sup_{0 \leq t \leq T} J_\varepsilon(t) \to 0 \) as \( \varepsilon \to 0 \). Note that

\[
\sup_{0 \leq t \leq T} J_\varepsilon(t) = \sup_{0 \leq t \leq T} \sum_{i=1}^{d} \mathbb{E} \left( \left| \int_0^t \varphi^i_\varepsilon(s) ds \right|^2 + \left| \int_0^t \psi^i_\varepsilon(s) ds \right|^2 \right) \\
\leq \sum_{i=1}^{d} \mathbb{E} \sup_{0 \leq t \leq T} \left( \left| \int_0^t \varphi^i_\varepsilon(s) ds \right|^2 + \left| \int_0^t \psi^i_\varepsilon(s) ds \right|^2 \right). \tag{3.13}
\]

For the first term on the right-hand side of (3.13), we have

\[
\sum_{i=1}^{d} \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \varphi^i_\varepsilon(s) ds \right|^2 \leq C \sum_{i=1}^{d} \mathbb{E} \int_0^T |\varphi^i_\varepsilon(s)|^2 \, ds \\
= C \sum_{i=1}^{d} \mathbb{E} \int_0^T \left| (A^i_\varepsilon(s) - b^i_\varepsilon(s)) \tilde{x}(s) + (C^i_\varepsilon(s) - b^i_\varepsilon(s)) (u(s) - \overline{u}(s)) \right|^2 \, ds \\
\leq C \mathbb{E} \int_0^T \left( \|A^i_\varepsilon(s) - b^i_\varepsilon(s)\|^2 |\tilde{x}(s)|^2 + \|C^i_\varepsilon(s) - b^i_\varepsilon(s)\|^2 |u(s) - \overline{u}(s)|^2 \right) \, ds \\
\leq C \mathbb{E} \int_0^T \int_0^1 \left( \|b^i_\varepsilon(s, x_{\lambda_\varepsilon,\varepsilon}, u_{\lambda_\varepsilon,\varepsilon}) - b^i_\varepsilon(s)\|^2 |\tilde{x}(s)|^2 \\
+ \|b^i_\varepsilon(s, x_{\lambda_\varepsilon,\varepsilon}, u_{\lambda_\varepsilon,\varepsilon}) - b^i_\varepsilon(s)\|^2 |u(s) - \overline{u}(s)|^2 \right) \, d\lambda \, ds.
\]
Thus, using the dominated convergence theorem, we can conclude that

\[
\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq T} \sum_{i=1}^{d} \mathbb{E} \left| \int_{0}^{t} \varphi_{\varepsilon}^{i}(s) ds \right|^2 = 0. \tag{3.14}
\]

For the second term on the right-hand side of (3.13),

\[
\sum_{i=1}^{d} \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \psi_{\varepsilon}^{i}(ds) \right|^2 =
\]

\[
n = \sum_{i=1}^{d} \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_{0}^{t} \left[ B_{\varepsilon}^{i}(ds) - \partial_{x} M^{i}(ds) \right] \hat{x}(s) + \left[ D_{\varepsilon}^{i}(ds) - \partial_{u} M^{i}(ds) \right] (u(s) - \bar{u}(s)) \right|^2
\]

\[
\leq \sum_{i=1}^{d} C \mathbb{E} \sup_{0 \leq t \leq T} \left( \sum_{j=1}^{d} \int_{0}^{t} \left| \tilde{x}_{j}(s) \left[ B_{\varepsilon}^{ij}(ds) - \partial_{x} M^{j}(ds) \right] \right|^2
\]

\[
+ \left| \sum_{l=1}^{k} \int_{0}^{t} \left( u_{l}(s) - \bar{u}_{l}(s) \right) \left[ D_{\varepsilon}^{il}(ds) - \partial_{u} M^{l}(ds) \right] \right|^2 \right)
\]

\[
\leq \sum_{i=1}^{d} C \mathbb{E} \sup_{0 \leq t \leq T} \left( \sum_{j=1}^{d} \left| \int_{0}^{t} \tilde{x}_{j}(s) \left[ B_{\varepsilon}^{ij}(ds) - \partial_{x} M^{j}(ds) \right] \right|^2
\]

\[
+ \left| \sum_{l=1}^{k} \int_{0}^{t} \left( u_{l}(s) - \bar{u}_{l}(s) \right) \left[ D_{\varepsilon}^{il}(ds) - \partial_{u} M^{l}(ds) \right] \right|^2 \right)
\]

\[
\leq C \sum_{i=1}^{d} \mathbb{E} \left( \sum_{j=1}^{d} \int_{0}^{T} \left| \tilde{x}_{j}(s) \right|^2 d \left( \int_{0}^{1} \int_{0}^{1} \partial_{x_{j}} M^{i}(dr, x_{\lambda, \varepsilon}(r), u_{\lambda, \varepsilon}(r)) d\lambda - \int_{0}^{1} \partial_{x_{j}} M^{i}(dr) \right)_{s}
\]

\[
+ \sum_{l=1}^{k} \int_{0}^{T} \left| u_{l}(s) - \bar{u}_{l}(s) \right|^2 d \left( \int_{0}^{1} \int_{0}^{1} \partial_{u_{l}} M^{i}(dr, x_{\lambda, \varepsilon}(r), u_{\lambda, \varepsilon}(r)) d\lambda - \int_{0}^{1} \partial_{u_{l}} M^{i}(dr) \right)_{s} \right).
\tag{3.15}
\]

Note that

\[
\left( \int_{0}^{1} \int_{0}^{1} \partial_{x_{j}} M^{i}(dr, x_{\lambda, \varepsilon}(r), u_{\lambda, \varepsilon}(r)) d\lambda - \int_{0}^{1} \partial_{x_{j}} M^{i}(dr) \right)_{s}
\]

\[
= \int_{0}^{s} \left( \int_{0}^{1} \int_{0}^{1} \frac{\partial^2 q_{ii}(s, x_{\lambda_{1}, \varepsilon}(r), u_{\lambda_{1}, \varepsilon}(r), x_{\lambda_{2}, \varepsilon}(r), u_{\lambda_{2}, \varepsilon}(r))}{\partial x_{j} \partial y_{j}} d\lambda_{1} d\lambda_{2}
\]

\[
+ \frac{\partial^2 q_{ii}(r, \bar{x}(r), \bar{u}(r), \bar{x}(r), \bar{u}(r))}{\partial x_{j} \partial y_{j}} \right) dr.
\]

Recall that in (H2), we assume $q \in B_{ub}^{1,\delta}$ which yields that the partial derivatives $\frac{\partial^2 q}{\partial x_{i} \partial y_{j}}$ of
are uniformly bounded. Thus, we have
\[
\sum_{i=1}^{d} \sum_{j=1}^{d} \mathbb{E} \int_{0}^{T} |\tilde{x}_{j}(s)|^{2} d\left(\int_{0}^{1} \int_{0}^{1} \partial_{x_{j}} M^{i}(dr, x_{\lambda, \varepsilon}(r), u_{\lambda, \varepsilon}(r)) d\lambda - \int_{0}^{1} \partial_{x_{j}} M^{i}(dr)\right)_{s} (3.16)
\]
is finite. Furthermore, (H2) implies the continuity of \(\frac{\partial^{2} q}{\partial x_{j} \partial y_{i}}\), and hence (3.16) converges to 0 as \(\varepsilon \to 0\). The same analysis can be applied to
\[
\sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{k} \mathbb{E} \int_{0}^{T} |u_{i}(s) - \overline{u}_{i}(s)|^{2} d\left(\int_{0}^{1} \int_{0}^{1} \partial_{u_{i}} M^{i}(dr, x_{\lambda, \varepsilon}(r), u_{\lambda, \varepsilon}(r)) d\lambda - \int_{0}^{1} \partial_{u_{i}} M^{i}(dr)\right)_{s}.
\]
Then, using the dominated convergence theorem, we have
\[
\lim_{\varepsilon \to 0} \sum_{i=1}^{d} \mathbb{E} \left(\sup_{0 \leq t \leq T} \left|\int_{0}^{t} \psi_{\varepsilon}(s)\right|^{2}\right) = 0.
\]
The proof is complete. \(\square\)

**Theorem 3.1.** Assume (H1) and (H2). Then we have
\[
\lim_{\varepsilon \to 0} \frac{J(u^{\varepsilon}) - J(\overline{u})}{\varepsilon} = \mathbb{E} \left\{\int_{0}^{T} \left[f_{x}(t)\tilde{x}(t) + f_{u}(t)(u(t) - \overline{u}(t))\right] dt + \Phi_{x}(\overline{x}(T))\tilde{x}(T)\right\}.
\]

**Proof.** Denote
\[
H_{\varepsilon} = \frac{1}{\varepsilon} \left(\int_{0}^{T} \left[f(t, x^{\varepsilon}(t), u^{\varepsilon}(t)) - f(t)\right] dt + \Phi(x^{\varepsilon}(T)) - \Phi(\overline{x}(T))\right) - \left(\int_{0}^{T} \left[f_{x}(t)\tilde{x}(t) + f_{u}(t)(u(t) - \overline{u}(t))\right] dt + \Phi_{x}(\overline{x}(T))\tilde{x}(T)\right).
\]
Then, to prove the desired result, it suffices to show \(\lim_{\varepsilon \to 0} \mathbb{E}[|H_{\varepsilon}|] = 0\).

From the Taylor expansion, we have, recalling the definition (3.8) of \(\eta^{\varepsilon}(t)\) and using the abbreviated notations (3.11) in the last Proposition,
\[
H_{\varepsilon} = \left(\int_{0}^{1} \Phi_{x}(x_{\lambda, \varepsilon}(T)) d\lambda\right) \eta^{\varepsilon}(T) + \left(\int_{0}^{1} \left[\Phi_{x}(x_{\lambda, \varepsilon}(T)) - \Phi_{x}(\overline{x}(T))\right] d\lambda\right) \tilde{x}(T)
+ \int_{0}^{T} \left(\int_{0}^{1} f_{x}(t, x_{\lambda, \varepsilon}(t), u_{\lambda, \varepsilon}(t)) d\lambda\right) \eta^{\varepsilon}(t) dt
+ \int_{0}^{T} \left(\int_{0}^{1} \left[f_{x}(t, x_{\lambda, \varepsilon}(t), u_{\lambda, \varepsilon}(t)) - f_{x}(t)\right] d\lambda\right) \tilde{x}(t) dt
+ \int_{0}^{T} \left(\int_{0}^{1} \left[f_{u}(t, x_{\lambda, \varepsilon}(t), u_{\lambda, \varepsilon}(t)) - f_{u}(t)\right] d\lambda\right) \left(u(t) - \overline{u}(t)\right) dt.
\]
Then, the Hölder inequality implies
\[
\mathbb{E}[|H_{\varepsilon}|] \leq \left( \mathbb{E} \left[ \int_0^1 \Phi_\varepsilon(x_{\lambda,\varepsilon}(T)) \, d\lambda \right]^2 \right)^{\frac{1}{2}} \left( \mathbb{E} [\eta^\varepsilon(T)]^2 \right)^{\frac{1}{2}}
\]
\[
+ \left( \mathbb{E} \left[ \int_0^1 [\Phi_\varepsilon(x_{\lambda,\varepsilon}(T)) - \Phi_\varepsilon(x_{\lambda,\varepsilon}(T))] \, d\lambda \right]^2 \right)^{\frac{1}{2}} \left( \mathbb{E} [\hat{\eta}^\varepsilon(T)]^2 \right)^{\frac{1}{2}}
\]
\[
+ \int_0^T \left( \mathbb{E} \left[ \int_0^1 f_\varepsilon(t, x_{\lambda,\varepsilon}(t), u_{\lambda,\varepsilon}(t)) \, d\lambda \right]^2 \right)^{\frac{1}{2}} \left( \mathbb{E} [\hat{\eta}^\varepsilon(T)]^2 \right)^{\frac{1}{2}} \, dt
\]
\[
+ \int_0^T \left( \mathbb{E} \left[ \int_0^1 [f_\varepsilon(t, x_{\lambda,\varepsilon}(t), u_{\lambda,\varepsilon}(t)) - f_\varepsilon(t)] \, d\lambda \right]^2 \right)^{\frac{1}{2}} \left( \mathbb{E} [\hat{\eta}^\varepsilon(T)]^2 \right)^{\frac{1}{2}} \, dt
\]
\[
+ \int_0^T \left( \mathbb{E} \left[ \int_0^1 [f_u(t, x_{\lambda,\varepsilon}(t), u_{\lambda,\varepsilon}(t)) - f_u(t)] \, d\lambda \right]^2 \right)^{\frac{1}{2}} \left( \mathbb{E} [\hat{\eta}^\varepsilon(T)]^2 \right)^{\frac{1}{2}} \, dt.
\]
Noting Proposition 3.1 and that the functions \( \Phi_\varepsilon, f_\varepsilon \) and \( f_u \) are continuous and satisfy the linear growth condition, we can conclude that \( \lim_{\varepsilon \to 0} \mathbb{E}[|H_{\varepsilon}|] = 0 \) by the dominated convergence theorem.

\section{3.3. Maximum principle}

Denote \( q(t, x, u, y, v) := (q_{ij}(x, u, y, v))_{d \times d} \) where \( q_{ij} \) is given by (3.1). Thus we have \( q(t, x, u, y, v) = q^*(t, x', u', x, u) \). Throughout the rest of this article, we consider both \( q := q(t, x, u, y, v) \) and \( q^* := q^*(t, x, u, y, v) \) as functions of \((t, x, u, y, v)\), and we shall use \( \frac{\partial}{\partial x}, \frac{\partial}{\partial u}, \frac{\partial}{\partial y} \) and \( \frac{\partial}{\partial v} \) to denote the partial derivatives with respect to \( x, u, y \) and \( v \), respectively. Clearly, at any point \( p_0 = (t_0, x_0, u_0, x_0, u_0) \), we have
\[
\frac{\partial}{\partial x} q^*(p_0) = \frac{\partial}{\partial y} q(p_0), \quad \frac{\partial}{\partial u} q^*(p_0) = \frac{\partial}{\partial v} q(p_0).
\]
(3.17)

Now we consider the adjoint equation, i.e., the following BSDE
\[
\begin{cases}
    dy(t) = -\left( b^*_x(t) y(t) + \left( \frac{\partial}{\partial x} \text{tr} \left[ z(t) q^*(t, x(t), u(t), x(t), u(t)) \right] \right)^* + f^*_x(t) \right) \, dt \\
    + z(t) dM(t) + dN(t),
\end{cases}
\]
(3.18)
\[y(T) = \Phi^*_x(x(T)).\]

Note that as mentioned in Section 3.1, \( dM(t) = M(dt, x(t), u(t)) \) and \((x, u) \in \mathbb{R}^{d+k}\) is an optimal pair for the control problem.

Denote
\[\mathcal{M}^2([0, T]; \mathbb{R}^d) := \left\{ \phi : [0, T] \times \Omega \to \mathbb{R}^d; \phi \text{ is predictable with } \mathbb{E} \int_0^T |\phi(t)|^2 \, dt < \infty \right\},\]
and

\[ Q^2([0, T]; \mathbb{R}^{d \times d}) := \left\{ \phi : [0, T] \times \Omega \to \mathbb{R}^{d \times d}; \phi \text{ is predictable with} \right. \]

\[ \mathbb{E} \int_0^T \text{tr} \left[ \phi(t) q(t, \bar{x}(t), \bar{u}(t), \bar{x}(t), \bar{u}(t)) \phi^*(t) \right] dt < \infty \right\}. \]

Then, according to [6], there exists a unique triple of stochastic processes

\[(y, z, N) \in M^2([0, T]; \mathbb{R}^d) \times Q^2([0, T]; \mathbb{R}^{d \times d}) \times \mathcal{L}^2\]

satisfying (3.18), where \( \mathcal{L}^2 \) is the space containing all square integrable martingales. Here, \( N \) is a \( \mathbb{R}^d \)-valued square integrable martingale orthogonal to \( M \), i.e., for \( 1 \leq i, j \leq d \),

\[ \left\langle N^i, \int_0^T M^j(ds, \bar{x}(s), \bar{u}(s)) \right\rangle_t = 0, \quad \forall t \in [0, T]. \]

**Lemma 3.2.** Let \((y, z, N)\) be the adapted solution of (3.18). Then

\[ \mathbb{E} \langle y(T), \hat{x}(T) \rangle \]

\[ = \mathbb{E} \int_0^T \left[ \left\langle b_u^*(t)y(t) + \left( \frac{\partial}{\partial u} \text{tr} \left[ z(t) q^*(t, \bar{x}(t), \bar{u}(t), \bar{x}(t), \bar{u}(t)) \right] \right)^*, u(t) - \bar{u}(t) \right\rangle - \langle f_x^*(t), \hat{x}(t) \rangle \right] dt. \]

**Proof.** Applying Itô formula to \( \langle y(t), \hat{x}(t) \rangle \), we have

\[ \mathbb{E} \langle y(T), \hat{x}(T) \rangle \]

\[ = \mathbb{E} \int_0^T \left( \langle dy(t), \hat{x}(t) \rangle + \langle y(t), d\hat{x}(t) \rangle + d\langle y, \hat{x} \rangle_t \right) dt \]

\[ = \frac{\mathbb{E}}{\mathbb{E}} \left[ - \int_0^T \left\langle b_x^*(t)y(t) + \left( \frac{\partial}{\partial x} \text{tr} \left[ z(t) q^*(t, \bar{x}(t), \bar{u}(t), \bar{x}(t), \bar{u}(t)) \right] \right)^*, u(t) - \bar{u}(t) \right\rangle dt \]

\[ + \left\langle y(t), b_x(t)\hat{x}(t) + b_u(t)(u(t) - \bar{u}(t)) \right\rangle dt \]

\[ + d \left\langle \int_0^T z(s) dM(s) + N(\cdot), \int_0^t \partial_x M(ds)\hat{x}(s) + \int_0^t \partial_u M(ds)(u(s) - \bar{u}(s)) \right\rangle_t \],

where we use the notation, for \( d \)-dimensional local martingales \( M = (M^1, \ldots, M^d) \) and \( N = (N^1, \ldots, N^d) \),

\[ \langle (M^1, \ldots, M^d), (N^1, \ldots, N^d) \rangle_t := \sum_{j=1}^d \langle M^j, N^j \rangle_t. \]

Note that from Theorem 2.2 and (3.17), it follows that

\[ d \left\langle \int_0^t z(s) M(ds), \int_0^t \partial_x M(ds)\hat{x}(s) \right\rangle_t \]
\[
\langle \left( \frac{\partial}{\partial y} \text{tr} [z(t)q(t, \overline{\pi}(t), \overline{\pi}(t), \overline{\pi}(t))] \right)^*, \widehat{x}(t) \rangle \ dt
\]

\[
= \left\langle \left( \frac{\partial}{\partial x} \text{tr} [z(t)q^*(t, \overline{\pi}(t), \overline{\pi}(t), \overline{\pi}(t))] \right)^*, \widehat{x}(t) \right\rangle \ dt
\]

and

\[
d \left\langle \int_0^T z(s)M(ds), \int_0^T \partial_u M(ds)(u(s) - \overline{u}(s)) \right\rangle_t
\]

\[
= \left\langle \left( \frac{\partial}{\partial v} \text{tr} [z(t)q(t, \overline{\pi}(t), \overline{\pi}(t), \overline{\pi}(t))] \right)^*, u(t) - \overline{u}(t) \right\rangle dt
\]

\[
= \left\langle \left( \frac{\partial}{\partial u} \text{tr} [z(t)q^*(t, \overline{\pi}(t), \overline{\pi}(t), \overline{\pi}(t))] \right)^*, u(t) - \overline{u}(t) \right\rangle dt.
\]

From the orthogonality of \( M \) and \( N \), we also have

\[
d \left\langle N, \int_0^T \partial_x M(ds)\widehat{x}(s) + \int_0^T \partial_u M(ds)(u(s) - \overline{u}(s)) \right\rangle_t = 0.
\]

By combining the above equalities, the desired result can be obtained. \( \square \)

Now, from Theorem 3.1, the adjoint equation (3.18) and Lemma 3.2, we have

\[
\lim_{\varepsilon \to 0} \frac{J(u^\varepsilon) - J(\overline{u})}{\varepsilon}
\]

\[
= \mathbb{E} \int_0^T \left\langle b^*_u(t)y(t) + \left( \frac{\partial}{\partial u} \text{tr} [z(t)q^*(t, \overline{\pi}(t), \overline{\pi}(t), \overline{\pi}(t))] \right)^* + f^*_u(t), u(t) - \overline{u}(t) \right\rangle dt.
\]

Since \( \overline{u} \) is an optimal control at which \( J(u) \) is minimized, we have for almost all \( t \in [0, T] \),

\[
\left\langle b^*_u(t)y(t) + \left( \frac{\partial}{\partial u} \text{tr} [z(t)q^*(t, \overline{\pi}(t), \overline{\pi}(t), \overline{\pi}(t))] \right)^* + f^*_u(t), u(t) - \overline{u}(t) \right\rangle \geq 0 \quad \text{a.s.} \tag{3.19}
\]

We now state our maximum principle in the following theorem, where we use the Hamiltonian defined as follows:

\[
H(t, x, u, y, z) = \langle y(t), b(t, x, u) \rangle + \text{tr}[z(t)q^*(t, \overline{\pi}, \overline{x}, x, u)] + f(t, x, u).
\tag{3.20}
\]

**Theorem 3.2.** Assume conditions (H1)-(H2). Let \( \overline{u} \) be an optimal control associated with the stochastic control problem (1.1)–(1.3) and \( (\overline{x}, \overline{u}) \) be the optimal pair. Then, there exists \( (y, z) \in \mathcal{M}^2([0, T]; \mathbb{R}^d) \times \mathcal{Q}^2([0, T]; \mathbb{R}^{d \times d}) \) satisfying the adjoint equation (3.18) such that for all \( u \in \mathcal{U}[0, T] \),

\[
H_u(t, \overline{x}(t), \overline{u}(t), y(t), z(t))(u(t) - \overline{u}(t)) \geq 0 \quad \text{a.s.} \tag{3.21}
\]

for almost all \( t \in [0, T] \), here \( H \) is given by (3.20) and \( H_u := \frac{\partial}{\partial u} H \).
Remark 3.1. We would like to remind the reader that the proof of Theorem 3.2 heavily relies on the convexity of the control domain $U$. For control problems with a general control domain which is not necessarily convex, we suspect that one may still obtain a global maximum principle as in Peng [22] by applying the second-order variational method developed therein. Noting that in our setting the noise $M$ is a local martingale with a spatial parameter rather than Brownian motion, we expect that extra difficulties would come from deriving relevant estimations and adjoint equations as well. This is a subsequent project that we plan to work on in future.

Remark 3.2. If the control domain $U$ is the whole space $\mathbb{R}^k$, let $\tilde{u}(t) = -u(t) + 2\overline{u}(t)$ for $t \in [0,T]$, and then, $\tilde{u} \in U[0,T] = M^2([0,T];\mathbb{R}^k)$. Now, Theorem 3.2 yields

$$H_u(t,\overline{x}(t),\overline{u}(t),y(t),z(t))(\tilde{u}(t) - \overline{u}(t)) \geq 0 \quad \text{a.s.},$$

i.e.,

$$H_u(t,\overline{x}(t),\overline{u}(t),y(t),z(t))(u(t) - \overline{u}(t)) \leq 0 \quad \text{a.s.}$$

This implies

$$H_u(t,\overline{x}(t),\overline{u}(t),y(t),z(t)) = 0 \quad \text{a.s.}$$

Remark 3.3. If we assume

$$M(t,x,u) = \int_0^t \sigma(s,x,u)dW_s,$$

similar to Remark 2.3, the joint quadratic variation of $M(\cdot,x,u)$ and $M(\cdot,y,v)$ is given by

$$q(t,x,u,y,v) = \sigma(t,x,u)\sigma^*(t,y,v),$$

and, the controlled system (1.1) is reduced to the classical one:

$$x^u(t) = x_0^u + \int_0^t b(s,x^u(s),u(s))ds + \int_0^t \sigma(s,x^u(s),u(s))dW_s,$$

(3.23)

The adjoint equation (3.18) becomes

$$\begin{cases}
    dy(t) = -\left( b_x^*(t)y(t) + \left( \frac{\partial}{\partial x} \text{tr} \left[ z(t)\sigma(t,\overline{x}(t),\overline{u}(t))\sigma^*(t,\overline{x}(t),\overline{u}(t)) \right] \right) + f_x^*(t) \right) dt \\
    y(T) = \Phi_x^*(\overline{x}(T)),
\end{cases}$$

(3.24)

where

$$\frac{\partial}{\partial x} \text{tr} \left[ z(t)\sigma(t,\overline{x}(t),\overline{u}(t))\sigma^*(t,\overline{x}(t),\overline{u}(t)) \right] := \frac{\partial}{\partial x} \text{tr} \left[ z(t)\sigma(t,y,v)\sigma^*(t,x,u) \right] \bigg|_{(x,y,u) = (\overline{x}(t),\overline{u}(t),\overline{x}(t))}.$$
If we assume that the filtration is generated by the Brownian motion $W$, then a mean-zero local martingale $N$ is orthogonal to $W$ if and only if $N \equiv 0$. Denoting $\tilde{z}(t) = z(t)\sigma(t, \overline{x}(t), \overline{u}(t))$, the adjoint equation (3.24) can be written as

$$
\begin{cases}
  dy(t) = - \left( b^*_x(t) y(t) + \left( \frac{\partial}{\partial x} \text{tr} \left[ \tilde{z}(t)\sigma^*(t, \overline{x}(t), \overline{u}(t)) \right] \right)^* + f^*_x(t) \right) dt + \tilde{z}(t) dW_t, \\
y(T) = \Phi^*_x(\overline{x}(T)),
\end{cases}
$$

and the variational inequality (3.19) becomes

$$
\left\langle b^*_u(t)y(t) + \left( \frac{\partial}{\partial u} \text{tr} \left[ \tilde{z}(t)\sigma^*(t, \overline{x}(t), \overline{u}(t)) \right] \right)^* + f^*_u(t), u(t) - \overline{u}(t) \right\rangle \geq 0 \quad \text{a.s.} \quad (3.25)
$$

from which the classical maximum principle can be obtained.

### 3.4. Sufficiency of the maximum principle

In this subsection, we show that the necessary condition (3.21) for an optimal control pair obtained in Theorem 3.2 is also sufficient under proper conditions.

**Theorem 3.3.** Suppose (H1)-(H2) hold. Let $\overline{u} \in U[0, T]$ satisfy that, for all $u \in U[0, T],

$$
H_u(t, \overline{x}(t), \overline{u}(t), y(t), z(t))(u(t) - \overline{u}(t)) \geq 0 \quad \text{a.s.}, \quad (3.26)
$$

for almost all $t \in [0, T]$, where $H$ is given in (3.20). We further assume that $H$ is convex with respect to $x$ and $u$ and that $\Phi$ is convex with respect to $x$. Then $\overline{u}$ is an optimal control for (1.1)-(1.3).

**Proof.** It suffices to show that $J(u) - J(\overline{u}) \geq 0$ holds for all $u \in U[0, T]$. From the convexity of $\Phi$, we have

$$
J(u) - J(\overline{u}) = \mathbb{E} \int_0^T \left[ f(t, x^u(t), u(t)) - f(t, \overline{x}(t), \overline{u}(t)) \right] dt + \mathbb{E} \left[ \Phi(x^u(T) - \overline{x}(T)) \right] \\
\geq \mathbb{E} \int_0^T \left[ f(t, x^u(t), u(t)) - f(t, \overline{x}(t), \overline{u}(t)) \right] dt + \mathbb{E} \left[ \Phi_x(\overline{x}(T))(x^u(T) - \overline{x}(T)) \right] \quad (3.27)
$$

Applying Itô’s formula to $\langle y(t), x^u(t) - \overline{x}(t) \rangle$ and then taking expectation, we obtain

$$
\begin{align*}
\mathbb{E} & \left[ \left( \Phi_x(\overline{x}(T)), x^u(T) - \overline{x}(T) \right) \right] \\
= & \mathbb{E} \int_0^T \left\langle \left( \frac{\partial}{\partial x} \text{tr} \left[ z(t)q^*(t, \overline{x}(t), \overline{u}(t), \overline{x}(t), \overline{u}(t)) \right] \right)^* - f^*_x(t), x^u(t) - \overline{x}(t) \right\rangle dt \\
+ & \mathbb{E} \int_0^T \langle y(t), b(t, x^u(t), u(t)) - b(t, \overline{x}(t), \overline{u}(t)) \rangle dt
\end{align*}
$$
Therefore, with the quadratic cost functional \( J \)
setting, where the controlled system \((x, u)\) impose the following condition on the local characteristic \( q \)

In this section, we discuss the stochastic linear quadratic problem (LQ problem) in our

where the last step follows from \((3.21)\). The proof is concluded.

\[ J(u) - J(\mathbf{u}) \geq \mathbb{E} \int_0^T \left\{ -H_x(t, \mathbf{x}(t), \mathbf{u}(t), y(t), z(t))(x^u(t) - \mathbf{x}(t)) \right. \]

It follows from the convexity of \( H \) that

\[ H(t, x^u(t), u(t), y(t), z(t)) - H(t, \mathbf{x}(t), \mathbf{u}(t), y(t), z(t)) \geq H_x(t, \mathbf{x}(t), \mathbf{u}(t), y(t), z(t))(x^u(t) - \mathbf{x}(t)) \]

\[ + H_u(t, \mathbf{x}(t), \mathbf{u}(t), y(t), z(t))(u(t) - \mathbf{u}(t)). \]

Therefore,

\[ J(u) - J(\mathbf{u}) \geq \mathbb{E} \int_0^T H_u(t, \mathbf{x}(t), \mathbf{u}(t), y(t), z(t))(u(t) - \mathbf{u}(t)) dt \geq 0. \]

where the last step follows from \((3.21)\). The proof is concluded.

\[ 4. \text{ A discussion on the stochastic LQ problem} \]

In this section, we discuss the stochastic linear quadratic problem (LQ problem) in our

setting, where the controlled system \((1.1)\) is driven by a local martingale \( M(t, x, u) \) with

\((x, u)\) as parameters. To make \((1.1)\) “linear” in terms of \((x, u)\) in the martingale part, we impose the following condition on the local characteristic \( q \) of \( M \): for any \( d \times d \) matrix \( A \) and all \((x, u), (y, v) \in \mathbb{R}^{d+k}\),

\[ \text{tr}[\mathbf{A}(q^*(t, x, u, y, v) - q^*(t, x, u, x, u))] \]

\[ = \left\langle \left( \frac{\partial}{\partial x} \text{tr}[Aq^*(t, x, u, x, u)] \right)^*, y - x \right\rangle + \left\langle \left( \frac{\partial}{\partial u} \text{tr}[Aq^*(t, x, u, x, u)] \right)^*, v - u \right\rangle. \]  

(4.1)

Now, we consider the following linear state equation,

\[ \begin{align*}
    dx^u(t) &= [A(t)x^u(t) + B(t)u(t)] dt + M(dt, x^u(t), u(t)), \\
    x^u(0) &= x^u_0,
\end{align*} \]

(4.2)

with the quadratic cost functional

\[ J(u) = \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left[ \langle Q(t)x^u(t), x^u(t) \rangle + \langle R(t)u(t), u(t) \rangle \right] dt + \langle Gx^u(T), x^u(T) \rangle \right\}. \]

(4.3)
Here, for $t \in [0, T]$, $A(t)$, and $B(t)$ are matrices with appropriate dimensions, $Q(t)$ and $G$ are symmetric nonnegative definite matrices, and $R(t)$ is a symmetric positive definite matrix. Here we use $U[0, T] = \mathcal{M}^2([0, T]; \mathbb{R}^k)$ to denote the set of admissible controls. Then, the adjoint equation (3.18) becomes

$$
\begin{align*}
\begin{cases}
    dy(t) &= -(A^*(t)y(t) + \left( \frac{\partial}{\partial x} \text{tr} \left[ z(t)q^*(t, \varpi(t), \varphi(t), \varpi(t), \varphi(t)) \right] \right)^* + Q(t)\varpi(t))dt \\
    y(T) &= G\varphi(T).
\end{cases}
\end{align*}
$$

(4.4)

Now, the Hamiltonian (3.20) is

$$
H(t, x, u, y, z) = \langle A(t)x(t) + B(t)u(t), y(t) \rangle + \text{tr} \left[ z(t)q^*(t, x(t), u(t), x(t), u(t)) \right] \\
+ \frac{1}{2} \langle Q(t)x(t), x(t) \rangle + \frac{1}{2} \langle R(t)u(t), u(t) \rangle + \frac{1}{2} \langle G(t)x(T), x(T) \rangle.
$$

Then, it follows from the stochastic maximum principle (Theorem 3.2 and Remark 3.2) that

$$
B^*(t)y(t) + \left( \frac{\partial}{\partial x} \text{tr} \left[ z(t)q^*(t, \varpi(t), \varphi(t), \varpi(t), \varphi(t)) \right] \right)^* + R(t)\varpi(t) = 0
$$

(4.5)

holds for a.e. $t \in [0, T]$ almost surely, which is a necessary condition for an optimal pair $(\varpi, \varphi)$. As in the classical situation, now we verify that $\varphi$ satisfying the necessary condition (4.5) is actually an optimal control for the generalized stochastic LQ problems.

**Theorem 4.1.** If $\varphi$ satisfies (4.5), then $\varphi$ is an optimal control for the generalized linear quadratic problem (4.2)–(4.3).

**Proof.** To prove the optimality of $\varphi$, it suffices to show $J(u) - J(\varphi) \geq 0$ for all $u \in U[0, T]$. From the nonnegative definiteness of $Q(t), R(t)$ and $G$, we have

$$
\begin{align*}
J(u) - J(\varphi) &= \frac{1}{2} \mathbb{E} \left\{ \int_0^T \left[ \langle Q(t)x^u(t), x^u(t) \rangle - \langle Q(t)\varphi(t), \varphi(t) \rangle + \langle R(t)u(t), u(t) \rangle - \langle R(t)\varphi(t), \varphi(t) \rangle \right] dt \\
& \quad + \langle Gx^u(T), x^u(T) \rangle - \langle G\varphi(T), \varphi(T) \rangle \right\} \\
& \geq \mathbb{E} \left\{ \int_0^T \left[ \langle Q(t)\varphi(t), x^u(t) - \varphi(t) \rangle + \langle R(t)\varphi(t), u(t) - \varphi(t) \rangle \right] dt + \langle G\varphi(T), x^u(T) - \varphi(T) \rangle \right\}.
\end{align*}
$$

(4.6)

Then, applying Itô’s formula to $\langle x^u(t) - \varphi(t), y(t) \rangle$, we have

$$
\mathbb{E} \langle G\varphi(T), x^u(T) - \varphi(T) \rangle
$$
where the last equality follows from (4.1). Then the desired inequality follows from (4.5), (4.6) and (4.7):

\[ J(u) - J(\overline{u}) \geq \mathbb{E} \int_0^T \left< R(t)\overline{u}(t) + B^*(x)y(t) + \left( \frac{\partial}{\partial u} \text{tr} \left[ z(t)q(t, \overline{x}(t), \overline{u}(t), \overline{x}(t), \overline{u}(t)) \right] \right)^*, u(t) - \overline{u}(t) \right> dt \]

= 0.

This concludes the proof.

\[ \square \]

**Remark 4.1.** It can be easily checked that if \( q \) is linear with respect to \( x, y, u \) and \( v \), then condition (4.1) is fulfilled, and hence the classical LQ problem is recovered. More precisely, we consider the linear form of (3.23):

\[ dx^u(t) = x_0^u + [A(t)x^u(t) + B(t)u(t)] dt + \sum_{j=1}^m \left[ C_j(t)x^u(t) + D_j(t)u(t) \right] dW^j_t, \quad (4.8) \]

where \( A, B, C_j, D_j \) are deterministic matrix-valued functions of suitable dimensions. Then, the local characteristic \( q \) of

\[ M(t, x, u) = \sum_{j=1}^m \left[ \int_0^t C_j(s)x dW^j_s + \int_0^t D_j(s)u dW^j_s \right] \]
is given by

\[ q(t, x, u, y, v) = \sum_{j=1}^{m} \left[ C^*_j(t)xy^*C^*_j(t) + C^*_j(t)xv^*D^*_j(t) + D^*_j(t)uy^*C^*_j(t) + D^*_j(t)uv^*D^*_j(t) \right], \]

which satisfies (4.1) with equality.

In this situation, condition (4.5) can be written as

\[ B^*(t)y(t) + \sum_{j=1}^{m} \text{tr} \left[ z(t)D^*_j(t)(C^*_j(t)\overline{x}(t) + D^*_j(t)\overline{u}(t)) \right] + R(t)\overline{u}(t) = 0. \]

This is consistent with the variational inequality (3.25) (which is an equality when \( U = \mathbb{R}^k \) by Remark 3.2) in the classical setting.

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