Property of the spectrum of large-scale magnetic fields from inflation

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Abstract

The property of the spectrum of large-scale magnetic fields generated due to the breaking of the conformal invariance of the Maxwell theory through some mechanism in inflationary cosmology is studied. It is shown that the spectrum of the generated magnetic fields should not be perfectly scale-invariant but be slightly red so that the amplitude of large-scale magnetic fields can be stronger than $\sim 10^{-12}$G at the present time. This analysis is performed by assuming the absence of amplification due to the late-time action of some dynamo (or similar) mechanism.

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I. INTRODUCTION

It is observationally known that there exist magnetic fields with the field strength $\sim 10^{-6}$ G on $1-10$ kpc scale in galaxies of all types (for detailed reviews, see [1, 2, 3, 4, 5, 6, 7]) and in galaxies at cosmological distances [8]. Furthermore, magnetic fields in clusters of galaxies with the field strength $10^{-7} - 10^{-6}$ G on 10 kpc - 1 Mpc scale have been observed [9]. It is very interesting and mysterious that magnetic fields in clusters of galaxies are as strong as galactic ones and that the coherence scale may be as large as $\sim$ Mpc. The origin of these magnetic fields is not well understood yet. Although galactic dynamo mechanisms [10] have been proposed to amplify very weak seed magnetic fields up to $\sim 10^{-6}$ G, they require initial seed magnetic fields to feed on. Moreover, the effectiveness of the dynamo amplification mechanism in galaxies at high redshifts or clusters of galaxies is not well established.

Proposed generation mechanisms of seed magnetic fields fall into two broad categories. One is astrophysical processes, e.g., the Biermann battery mechanism [11] and the Weibel instability [12], which is a kind of plasma instabilities, and the other is cosmological processes in the early Universe, e.g., the first-order cosmological electroweak phase transition (EWPT) [13], quark-hadron phase transition (QCDPT) [15] (see also [16]), and the generation of the magnetic fields from primordial density perturbations before the epoch of recombination [17, 18, 19, 20, 21, 22]. However, it is difficult for these processes to generate the magnetic fields on megaparsec scales with the sufficient field strength to account for the observed magnetic fields in galaxies and clusters of galaxies without requiring any dynamo amplification.

The most natural origin of such a large-scale magnetic field is electromagnetic quantum fluctuations generated in the inflationary stage [23]. This is because inflation naturally produces effects on very large scales, larger than Hubble horizon, starting from microphysical processes operating on a causally connected volume. Since the Friedmann-Robertson-Walker (FRW) metric usually considered is conformally flat and the classical electrodynamics is conformally invariant, the conformal invariance of the Maxwell theory must have been broken in the inflationary stage in order that electromagnetic quantum fluctuations could be generated at that time [24]. Several breaking mechanisms therefore have been proposed [23, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41].

In the present paper we discuss the spectrum of large-scale magnetic fields generated due to the breaking of the conformal invariance of the Maxwell theory through some mechanism in inflationary cosmology. In particular, we perform the analysis of the spectrum of large-scale magnetic fields by assuming the absence of amplification due to the late-time action of some dynamo (or similar) mechanism. We use units in which $k_B = c = \hbar = 1$.

This paper is organized as follows. In Sec. II we consider the constraint on the amplitude of large-scale magnetic fields with a scale-invariant spectrum. In Sec. III we discuss the spectrum of large-scale magnetic fields. Finally, Sec. IV is devoted to a conclusion.

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1 In Ref. [14], it has been pointed out that causally produced stochastic magnetic fields on large scales, e.g., during EWPT or even later, are much stronger suppressed than usually assumed.

2 In Ref. [24], the breaking of conformal flatness of the FRW metric induced by the evolution of scalar metric perturbations at the end of inflation has been discussed.
II. CONSTRAINT ON THE AMPLITUDE OF LARGE-SCALE MAGNETIC FIELDS WITH A SCALE-INVARIANT SPECTRUM

We assume the spatially flat Friedmann-Robertson-Walker (FRW) space-time with the metric
\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\mathbf{x}^2, \] (1)
where \( a(t) \) is the scale factor. The background Friedmann equation is given by
\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3M_{\text{Pl}}^2} \rho_\phi \equiv H_{\text{inf}}^2, \] (2)
\[ \rho_\phi = \frac{1}{2}\dot{\phi}^2 + U[\phi], \] (3)
where a dot denotes a time derivative. Here, \( H \) is the Hubble parameter, \( \rho_\phi \) is the energy density of the inflaton field, \( U[\phi] \) is the inflaton potential, and \( M_{\text{Pl}} = G^{-1/2} = 1.2 \times 10^{19}\text{GeV} \) is the Planck mass. Moreover, \( H_{\text{inf}} \) is the Hubble constant in the inflationary stage.

It is well known that for a minimally coupled scalar field in de Sitter space, there are fluctuations in that field with energy density corresponding to that of a thermal bath at the Gibbons-Hawking temperature, \( H_{\text{inf}}/(2\pi) \) \cite{42, 43}. According to Turner and Widrow \cite{23}, it is reasonable to assume that all quantum fields in de Sitter space, in particular the electromagnetic field (if the conformal invariance of the Maxwell theory is broken through some mechanism in the inflationary stage), are excited with an energy density of order \( [H_{\text{inf}}/(2\pi)]^4 \).

Here we consider the case in which the conformal invariance of the Maxwell theory is broken through some mechanism in the inflationary stage and then magnetic fields whose origin is electromagnetic quantum fluctuations amplified during inflation are generated. Moreover, we assume that the spectrum of the generated magnetic fields is scale-invariant. The quantum degree of freedom is the photon field which will lead to the same amount of electric and magnetic energy. After inflation, however, when the conductivity of the Universe becomes high, the electric fields will be dissipated and only the magnetic fields survive. It is conjectured from the above discussion that the energy density of the magnetic fields, \( \rho_B \), is \( \rho_B \lesssim [H_{\text{inf}}/(2\pi)]^4 \). From this relation and Eq. (2), we find
\[ \frac{\rho_B}{\rho_\phi} \lesssim \frac{1}{6\pi^3} \left( \frac{H_{\text{inf}}}{M_{\text{Pl}}} \right)^2. \] (4)

The upper limit on \( H_{\text{inf}} \) is determined by the observation of the anisotropy of the cosmic microwave background (CMB) radiation. Using the Wilkinson Microwave Anisotropy Probe (WMAP) three year data on temperature fluctuation \cite{44}, we can obtain a constraint on \( H_{\text{inf}} \) from tensor perturbations \cite{45, 46},
\[ \frac{H_{\text{inf}}}{M_{\text{Pl}}} \leq 4.9 \times 10^{-5}. \] (5)

Here we consider the case in which after inflation the Universe is reheated immediately at \( t = t_R \) and then all the energy of the inflaton is reduced to radiation. Moreover, the conductivity of the Universe \( \sigma \) is negligibly small during inflation, because there are few
charged particles at that time. After reheating, however, a number of charged particles are produced, so that the conductivity immediately jumps to a large value: \( \sigma \gg H \ (t \geq t_R) \) and hence it is always much larger than the Hubble parameter at that time in the radiation-dominated stage and the subsequent matter-dominated stage. This assumption is justified by a microphysical analysis [23]. Consequently, as stated above, for a large enough conductivity at the instantaneous reheating stage, the electric fields accelerate charged particles and dissipate. Furthermore, we consider the case in which after reheating \( \rho_B \) is not supplied with any energy by some fields through the coupling between those fields and electromagnetic fields and hence \( \rho_B \) evolves in proportion to \( a^{-4}(t) \). In this case, after reheating the ratio of the energy density of the magnetic fields to that of the radiation remains constant. Consequently, it follows from Eqs. (4) and (5) that the energy density of the magnetic fields at the present time \( t_0 \) is

\[
\rho_B(t_0) \lesssim \frac{(4.9 \times 10^{-5})^2}{6\pi^3} \rho_{\gamma 0},
\]

where \( \rho_{\gamma 0} \) is the present energy density of the CMB radiation. Using Eq. (6), \( \rho_{\gamma 0} = 2.1 \times 10^{-51}(T_{0}/2.75)^4 \text{GeV}^4 \) [47], where \( T_{0} \approx 2.73 \text{K} \) is the present temperature of the CMB radiation, and \( 1[\text{GeV}^4/(8\pi)] = 1.9 \times 10^{-40} \text{GeV}^4 \), we find that the present magnetic field, \( B(t_0) \), is \( B(t_0) \lesssim 3.3 \times 10^{-12} \text{G} \).

It is known that the required strength of the cosmic magnetic fields at the structure formation, adiabatically rescaled to the present time, is \( 10^{-10} - 10^{-9} \text{G} \) in order to explain the observed magnetic fields in galaxies and clusters of galaxies without dynamo amplification mechanism. If the spectrum of the generated magnetic fields is perfectly scale-invariant, it follows from the above consideration that the present strength of the magnetic fields is at most or smaller than \( \sim 10^{-12} \text{G} \). Hence the spectrum of the magnetic fields should not be perfectly scale-invariant but be tilted in order that the amplitude of the large-scale magnetic fields can be as large as \( 10^{-10} - 10^{-9} \text{G} \) at the present time, so that the observed magnetic fields could be explained through only adiabatic compression without requiring any dynamo amplification.

### III. SPECTRUM OF LARGE-SCALE MAGNETIC FIELDS

Next, we discuss the spectrum of large-scale magnetic fields. We consider the case in which the Fourier components of the root-mean-square (rms) of the magnetic fields at the present time, \( |B(k, t_0)| \), are given by

\[
|B(k, t_0)|^2 \equiv A \left( \frac{k}{k_c} \right)^n.
\]

Here, \( k = 2\pi/L \) is comoving wave number and \( L \) is the comoving scale of the magnetic fields. Moreover, \( n \) is the spectral index of \( |B(k, t_0)|^2 \) and \( A \) is a constant. Furthermore, \( k_c = 2\pi/L_c \), where \( L_c \) is the scale on which the field strength of the magnetic fields is equal to \( 10^{-12} \text{G} \). Multiplying \( |B(k, t_0)|^2 \) by phase-space density: \( 4\pi k^3/(2\pi)^3 \), we obtain the magnetic fields in the position space at the present time

\[
|B(L, t_0)|^2 = \frac{Ak_c^3}{2\pi^2} \left( \frac{L_c}{L} \right)^{n+3}.
\]
Using Eq. (8) and taking into account the fact that \( L_c \) is the scale on which the field strength of the magnetic fields is equal to \( 10^{-12} \) G, we find

\[
|B(L_c, t_0)|^2 = \frac{A k^3}{2\pi^2} = \left( 10^{-12} \text{ G} \right)^2.
\]  

(9)

Here we consider the case in which the spectrum of the magnetic field is red, i.e., \( n < -3 \). We here note the following point: In the case of a red spectrum, a large scale cutoff of the magnetic fields is needed, otherwise the total energy of the magnetic fields is infinite. Although a mechanism which leads to this infrared cutoff (which is determined by inflationary physics) is necessary, we have not found it yet. In the following discussion, therefore, we assume that a infrared cutoff of the magnetic fields is realized by some mechanism. In this case, if the current amplitude of the magnetic fields on 1Mpc scale is larger than \( 10^{-10} \) G, \( L_c < 1 \) Mpc. It follows from Eq. (8) that in order that the current amplitude of the magnetic fields on 1Mpc scale can be stronger than \( 10^{-10} \) G, \( |B(L = 1 \text{ Mpc}, t_0)|^2 > (10^{-10} \text{ G})^2 \), the spectral index \( n \) should satisfy the following relation:

\[
n < -3 - \frac{4}{\log_{10} (1 \text{ Mpc} / L_c)}.
\]  

(10)

On the other hand, the strength of the cosmological large-scale magnetic fields is constrained by the CMB anisotropy measurements (for more detailed explanations to constraints on cosmological magnetic fields see Refs. [5, 48]). Homogeneous magnetic fields during the time of decoupling whose scales are larger than the horizon at that time cause the Universe to expand at different rates in different directions. Since anisotropic expansion of this type distorts the CMB radiation, the measurements of the CMB angular power spectrum impose limits on the strength of the cosmological magnetic fields. Barrow, Ferreira, and Silk [49] carried out a statistical analysis based on the 4-year COBE data for angular anisotropy and derived the following limit on the primordial magnetic fields that are coherent on scale larger than the present horizon:

\[
B(L = L_{\text{IR}}, t_0) < 4.8 \times 10^{-9} \text{ G},
\]  

(11)

where \( L_{\text{IR}} = \beta H_0^{-1} \) is the infrared cutoff scale of the magnetic fields. Here, \( \beta (> 1) \) is a dimensionless constant and \( H_0 = 100h \) km s\(^{-1}\) Mpc\(^{-1}\) is the Hubble constant at the present time (throughout this letter we use \( h = 0.70 \) [50]). Moreover, we have assumed the spatially flat Universe (see also [51, 52]). Hence, it follows from Eqs. (8) and (11) that the lower limit on \( n \) is given by

\[
n > -3 - \frac{2(\log_{10} 4.8 + 3)}{\log_{10} (L_{\text{IR}} / L_c)}.
\]  

(12)

If there exists the value of \( n \) which satisfies both the relations (10) and (12), the value of the right-hand side of the relation (10) is larger than that of the relation (12). In this case, using the relations (10) and (12), we obtain

\[
L_c < 4.8 \beta^{-1.2} \times 10^{-5} \text{ Mpc}.
\]  

(13)

In deriving the relation (13), we have used \( H_0^{-1} = 2997.9 h^{-1} \text{ Mpc} \).

Consequently, if the spectrum of the generated magnetic fields is slightly red and the scale \( L_c \) on which the field strength of the magnetic fields is equal to \( 10^{-12} \) G satisfies the
relation (13), the amplitude of the magnetic fields on 1Mpc scale can be larger than $10^{-10}$G at the present time without being inconsistent with the CMB anisotropy measurements. For example, if $\beta = 100$, from the relation (13) we can take $L_c = 10^{-7}$Mpc. In this case, it follows from the relations (10) and (12) that the spectral index $n$ has to satisfy the relation: $-3.58 < n < -3.57$.

Finally, for comparison, we note the case in which the spectrum of the magnetic fields is blue, i.e., $n > -3$. In the case of a blue spectrum, there exist constraints for the amplitude of the magnetic fields from the production of gravitational waves [53] and the Big Bang Nucleosynthesis (BBN) [54]. The former constraint is much more stringent than the latter. According to Ref. [53], the magnetic fields generated during an inflationary phase (reheating temperature $T \sim 10^{15}$GeV) with a spectral index $n \sim 0$, the magnetic fields have to be weaker than $10^{-39}$G. Hence the possibility of a blue spectrum is considered to be ruled out.

IV. CONCLUSION

In the present paper we have considered the spectrum of large-scale magnetic fields generated due to the breaking of the conformal invariance of the Maxwell theory through some mechanism in inflationary cosmology. As a result, we have shown that the spectrum of the generated magnetic fields should not be perfectly scale-invariant but be slightly red so that the amplitude of large-scale magnetic fields can be stronger than $\sim 10^{-12}$G at the present time.

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