FIRST INTEGRALS OF A CLASS OF $n$-DIMENSIONAL LOTKA-VOLTERRA DIFFERENTIAL SYSTEMS

JAUME LLIBRE, ADRIAN C. MURZA AND ANTONIO E. TERUEL

Abstract. Lotka-Volterra model is one of the most popular in biochemistry. It is used to analyze cooperativity, autocatalysis, synchronization at large scale and especially oscillatory behavior in biomolecular interactions. These phenomena are in close relationship with the existence of first integrals in this model. In this paper we determine the independent first integrals of a family of $n$-dimensional Lotka-Volterra systems. We prove that when $n = 3$ and $n = 4$ the system is completely integrable. When $n \geq 6$ is even, there are three independent first integrals, while when $n \geq 5$ is odd there exist only two independent first integrals. In each of these mentioned cases we identify in the parameter space the conditions for the existence of Darboux first integrals. We also provide the explicit expressions of these first integrals.

1. Introduction and formulation of the problem

The real nonlinear ordinary differential systems are widely used to model processes or reactions in a variety of fields of science, from biology and chemistry to economy, physics and engineering. The qualitative theory of dynamical systems is employed to analyze the behavior of these dynamical systems. Within this analysis one of the important features is the existence of first integrals of the differential systems defined in $\mathbb{R}^n$. This is mainly due to the fact that the existence of a first integral allows to reduce the dimension of the system by one. So in the qualitative theory of the differential systems are important the methods allowing to detect the presence of first integrals.

In this paper we shall apply the Darboux theory of integrability to real polynomial Lotka-Volterra differential systems. This theory provides a method of constructing first integrals of polynomial differential systems, based on the number of invariant algebraic hypersurfaces that they have. Since its publication in 1878, this theory originally

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developed by Darboux, has been extended and/or refined by many authors first for polynomial differential systems in $\mathbb{R}^2$ see for instance [2, 3, 4, 5, 7, 9, 13, 19], and later on for polynomial differential systems in $\mathbb{R}^n$ see [11, 14, 15, 16, 17, 18, 20, 21].

The Lotka-Volterra differential systems (see [22, 25]) also called Kolmogorov differential systems (see [12]) are used to model a wide range of experimental processes [1, 6, 8, 26]. In biochemistry, for example the pioneering work of Wyman [26] models the autocatalytic chemical reactions, called by Di Cera et al. [8] a “turning wheel” of one-step transitions of the macromolecule. Turning wheels have multiple applications in biochemistry. For instance, enzyme kinetics [26], circadian clocks [23] and genetic networks [1, 6] are just a few of them.

In [8] it has been proved that when the law of mass conservation applies, the autocatalytic chemical reactions between $x_i$, $i = 1, \ldots, n$ are governed by the following $n$-parameter family of nonlinear differential equations

\begin{align}
\dot{x}_1 &= x_1(k_1x_2 - k_nx_n) = P_1(x_1, \ldots, x_n) = x_1K_1(x_1, \ldots, x_n), \\
\dot{x}_2 &= x_2(k_2x_3 - k_1x_1) = P_2(x_1, \ldots, x_n) = x_2K_2(x_1, \ldots, x_n), \\
&\vdots \\
\dot{x}_n &= x_n(k_nx_1 - k_{n-1}x_{n-1}) = x_nK_n(x_1, \ldots, x_n) = P_n(x_1, \ldots, x_n),
\end{align}

where the parameters satisfy $k_i \in \mathbb{R}\setminus\{0\}$. This is only one of the multiple examples of $n$-dimensional Lotka-Volterra differential systems.

In the rest of the paper we will study the first integrals of the differential system (1) using the Darboux theory of integrability.

Let

\begin{equation}
X = P_1 \frac{\partial}{\partial x_1} + P_2 \frac{\partial}{\partial x_2} + \ldots + P_n \frac{\partial}{\partial x_n}
\end{equation}

be the vector field associated to system (1). Let $U$ be an open and dense subset of $\mathbb{R}^n$. A first integral of system (1) is a non-constant function $H : U \to \mathbb{R}$ such that it is constant on the solutions of system (1), i.e. $XH = 0$ in the points of $U$. Two first integrals $H_i : U \to \mathbb{R}$ for $i = 1, 2$ of system (1) are independent if their gradients $\nabla H_1$ and $\nabla H_2$ are independent in all the points of $U$ except perhaps in a zero Lebesgue measure set of $U$. System (1) is completely integrable if there exist $n - 1$ independent first integrals.

Our main result is the following one.
Theorem 1. For the Lotka-Volterra differential system \( \text{(1)} \) with \( k_i \neq 0 \) for \( i = 1, 2, \ldots, n \) the following statements hold.

(a) \( H = \sum_{i=1}^{n} x_i \) is a first integral. So in particular for \( n = 2 \) the system is completely integrable.

(b) For \( n \geq 3 \) odd
\[
H_1 = \sum_{i=1}^{n} x_i, \quad H_2 = x_1 \prod_{i=2}^{n} x_i^{\mu_i},
\]
where
\[
\mu_j = \begin{cases} k_1k_3 \ldots k_{j-2} & \text{if } j \geq 3 \text{ odd,} \\ \frac{k_1k_3 \ldots k_j}{k_2k_4 \ldots k_{j-1}} & \text{if } j \geq 2 \text{ even,} \end{cases}
\]
are two independent first integrals.

(c) For \( n = 3 \) the system is completely integrable with the two independent first integrals
\[
H_1 = x_1 + x_2 + x_3, \quad H_2 = x_1x_2x_3^{k_4/k_2}x_2^{k_4/k_2}.
\]

(d) If \( n \geq 4 \) is even and \( k_1k_3 \ldots k_{n-1} = k_2k_4 \ldots k_n \), then
\[
H_1 = \sum_{i=1}^{n} x_i, \quad H_2 = x_1x_3^{\mu_3} \ldots x_{n-1}^{\mu_{n-1}}, \quad H_3 = x_2x_4^{\mu_4} \ldots x_n^{\mu_n},
\]
where
\[
\mu_j = \begin{cases} k_{j+1}k_{j+3} \ldots k_n & \text{if } j \geq 3 \text{ odd,} \\ \frac{k_2k_4 \ldots k_{j-2}}{k_3k_5 \ldots k_{j-1}} & \text{if } j \geq 4 \text{ even,} \end{cases}
\]
are three independent first integrals.

(e) For \( n = 4 \) the system is completely integrable if \( k_1k_3 = k_2k_4 \), with the three independent first integrals
\[
H_1 = x_1 + x_2 + x_3 + x_4, \quad H_2 = x_1x_3^{k_4/k_3}, \quad H_3 = x_2x_4^{k_4/k_3}.
\]

Theorem 1 is proved in the next section.

Let \( U \) be an open and dense subset of \( \mathbb{R}^n \). The function \( M: U \to \mathbb{R} \) is a Jacobi multiplier for the Lotka-Volterra differential system \( \text{(1)} \) if
\[
\sum_{i=1}^{n} \frac{\partial(MP_i)}{\partial x_i} = 0.
\]
The so-called Jacobi Theorem, see Theorem 2.7 of [10], applied to our system (1) says that if system (1) admits a Jacobi multiplier $M$ and $n - 2$ independent first integrals, then the system admits an extra first integral. An easy computation shows that the function

$$M(x_1, \ldots, x_n) = \frac{1}{\prod_{i=1}^{n} x_i}$$

is a Jacobi multiplier of system (1). However we cannot use it to improve the number of independent first integrals, because when the dimension is $n = 3$ and $n = 4$ the system is completely integrable, while for $n \geq 5$ we do not know $n - 2$ independent first integrals [24].

2. Proof of Theorem 1

Let $f \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ be a polynomial. The algebraic hypersurface $f = 0$ of $\mathbb{R}^n$ is an invariant algebraic hypersurface of the system (1) if there exists a polynomial $K \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ such that $Xf = Kf$. The polynomial $K$ is called the cofactor of $f$. We note that an invariant hypersurface $f = 0$ has the property that if an orbit of system (1) has a point in $f = 0$, then the whole orbit is contained in $f = 0$, for more details see for instance Chapter 8 of [9].

From the definition of invariant algebraic hypersurface it follows immediately that for $i = 1, \ldots, n$ the hyperplanes $x_i = 0$ are invariant hyperplanes of system (1), and their corresponding cofactors are $K_i(x_1, x_2, \ldots, x_n)$.

The following result is due to Darboux, see [7], or Chapter 8 of [9].

**Theorem 2.** Suppose that the polynomial vector field (1) admits $n$ invariant algebraic surfaces $f_i = 0$ with cofactors $K_i$ for $i = 1, 2, \ldots, n$. If there exist $\lambda_i \in \mathbb{R}$ not all zero such that $\sum_{i=1}^{n} \lambda_i K_i = 0$, then the function $f_1^{\lambda_1} f_2^{\lambda_2} \ldots f_n^{\lambda_n}$ is a first integral of the vector field (1).

**Proof of statement (a) of Theorem 1** Let $H = \sum_{i=1}^{n} x_i$. Then an easy calculation shows that $XH = 0$, where $X$ has been defined in (2). □

**Proof of statement (b) of Theorem 1** Assume that $n \geq 3$ is odd. From statement (a) of Theorem 1 the function $H_1$ is a first integral. Now we calculate the other first integral.
For $i = 1, 2, \ldots, n$ we know that the hyperplane $x_i = 0$ is invariant for system (1), and that its cofactor is $K_i(x_1, x_2, \ldots, x_n) = k_ix_{i+1} - k_{i-1}x_{i-1}$. From Theorem 2 if there exist $\lambda_i$ not all zero and such that 
\[ \sum_{i=1}^{n} \lambda_i K_i = 0, \]
then $H = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$ is a first integral of system (1). Then we have 
\[ \sum_{i=1}^{n} \lambda_i K_i = x_1(k_n \lambda_n - k_1 \lambda_2) + x_2(k_1 \lambda_1 - k_2 \lambda_3) + x_3(k_2 \lambda_2 - k_3 \lambda_4) + x_4(k_3 \lambda_3 - k_4 \lambda_5) + \ldots + x_{n-1}(k_{n-2} \lambda_{n-2} - k_{n-1} \lambda_n) + x_n(k_{n-1} \lambda_{n-1} - k_n \lambda_1) = 0, \]
or equivalently 
\[ k_n \lambda_n - k_1 \lambda_2 = k_1 \lambda_1 - k_2 \lambda_3 = k_2 \lambda_2 - k_3 \lambda_4 = k_3 \lambda_3 - k_4 \lambda_5 = \ldots \]
\[ k_{n-2} \lambda_{n-2} - k_{n-1} \lambda_n = k_{n-1} \lambda_{n-1} - k_n \lambda_1 = 0. \]

Then it is easy to check that the solutions $\lambda_j$’s of system (3) are 
\[ \lambda_j = \frac{k_1 k_3 \ldots k_{j-2}}{k_2 k_4 \ldots k_{j-1}} \lambda_1 \text{ if } j \geq 3 \text{ odd}, \quad \lambda_j = \frac{k_{j+1} k_{j+3} \ldots k_n}{k_j k_{j+2} \ldots k_{n-1}} \lambda_1 \text{ if } j \geq 2 \text{ even}. \]

Since the unique free lambda is $\lambda_1$, by Theorem 2 we can choose $\lambda_1 = 1$ for obtaining the first integral $H_2$ of system (1) given in statement (b).

Clearly that the integrals $H_1$ and $H_2$ are independent because the gradient $\nabla H_1 = (1, 1, \ldots, 1)$ is independent of the gradient $\nabla H_2$. □

Proof of statement (c) of Theorem 7. Statement (c) follows immediately from statement (b). □

Proof of statement (d) of Theorem 7. Assume that $n \geq 4$ even. Now we calculate the two additional first integrals to the integral $H_1$.

Taking into account that $k_1 k_3 \ldots k_{n-1} = k_2 k_4 \ldots k_n$ it is easy to check that the solutions $\lambda_j$’s of system (3) can be written as 
\[ \lambda_j = \frac{k_{j+1} k_{j+3} \ldots k_n}{k_j k_{j+2} \ldots k_{n-1}} \lambda_1 \text{ if } j \geq 3 \text{ odd}, \quad \lambda_j = \frac{k_2 k_4 \ldots k_{j-2}}{k_3 k_5 \ldots k_{j-1}} \lambda_2 \text{ if } j \geq 4 \text{ even}. \]
Since the unique free lambdas are $\lambda_1$ and $\lambda_2$, we can choose the following two choices: $(\lambda_1, \lambda_2) = (1, 0)$ and $(\lambda_1, \lambda_2) = (0, 1)$, and by applying Theorem 2 we obtain the two independent first integrals $H_2$ and $H_3$ given in the statement (d).

Clearly that these third integrals are independent since $H_2$ has only even coordinates, $H_3$ only odd coordinates, and the combination of the gradient vectors of $H_2$ and $H_3$ cannot provide the gradient of $H_1$. □

Proof of statement (e) of Theorem 1

Statement (e) follows immediately from statement (d). □

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FIRST INTEGRALS FOR \(n\)-DIMENSIONAL LOTKA-VOLterra SYSTEMS

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Jaume Llibre, Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat

Adrian C. Murza, Institute of Mathematics “Simion Stoilow” of the Romanian Academy, Calea Griviței 21, 010702 Bucharest, Romania

E-mail address: adrian_murza@hotmail.com
Antonio E. Teruel, Departament de Matemàtiques i Informàtica, Universitat de les Illes Balears, Crta. de Valldemoosa km. 7.5, 07122 Palma de Mallorca, Spain

E-mail address: antonio.teruel@uib.es