The optimal investment strategy under the disordered return and random inflation

Juan Li\textsuperscript{a} and Dengfeng Xia\textsuperscript{b}

\textsuperscript{a}School of Economics and Management, Wuhu Institute of Technology, Wuhu, Anhui, People's Republic of China; \textsuperscript{b}School of Mathematics and Physics, Anhui Polytechnic University, Wuhu, Anhui, People's Republic of China

ABSTRACT
We mainly tackle the optimal portfolio problem when investors have partial information in the financial market, and the inflation risk is taken into account, the risky asset price is suddenly changed by impacts of major events which result in the asset return disorder at a random time. Applying semimartingale and backward stochastic differential equation theory, we deduce the optimal investment strategy and value process which satisfy the exponential utility maximization of terminal real wealth. Moreover, numerical simulations are presented to illustrate the effects of the expected inflation rate and inflation volatility on the optimal investment strategy, and to compare the asset return disorder with the normal situation.

1. Introduction
It has been more than a decade since the international financial crisis broke out in 2008. The world economic environment is improving, but inflation has been lingered here and there. Since 2018, the inflation level of the world’s major economies was on the rise as a whole. To stimulate domestic consumption and investment, countries have adopted various fiscal and monetary policies. Investors have to face the depreciation of the portfolio by inflation risk in the financial market, so how to adjust the investment strategy to cope with inflation risk is very important to every investor. Therefore, some scholars use the martingale method to study the portfolio with inflation risk. Brennan and Xia (2002) studied the dynamic optimal portfolio with inflation. Siu (2011) gave the optimal allocation strategy of the long-term strategic assets with the inflation risk. Lim (2013) invested index bonds in order to hedge inflation risk, obtained the optimal consumption and portfolio, and provided the analysis of the optimal strategy which was affected by inflation risk and survival consumption. Yao, Wu, and Zeng (2014) researched the optimal investment strategy for risky assets under uncertain time-horizon and inflation. Fei, Lv, and Yu (2014) researched the decision-making of optimal consumption and portfolio under inflation with the mean-reverting process. Liang, Fei, Yao, and Rui (2015) studied the optimal investment of pension with inflation under Knight uncertain environment. Fei, Cai, and Xia (2015) investigated the impacts of the inflation and the jumps on optimal asset allocations of an investor under asset prices with a jump environment. On the basis of the framework of prospect theory, Fei, Li, and Xia (2015) examined the factors such as inflation which exerted influence on optimal investment strategies of the fund manager with incentive fees. Two optimal consumption and portfolio problems with the financial markets of Markovian switching and inflation were studied in Fei (2013) and Fei and Fei (2015). Specifically speaking, Fei (2013) solved the problem of maximizing the expected utility of consumption discounted by inflation. But Fei and Fei (2015) concerned with a class of control systems with Markovian switching, they derived an Itô formula for Markov-modulated processes, and characterized an optimal control law satisfying the generalized Hamilton–Jacobi–Bellman (HJB) equation with Markovian switching which was applied to analyse optimal consumption and portfolio under inflation and Markovian switching. Recently, Fei, Chen, and Fei (2019) investigated an optimal consumption/portfolio and retirement problem of an agent under inflation. Based on maximizing the expected discounted utility of consumption both before and after retirement, the corresponding HJB equations are established by applying the dynamic programming method. Furthermore, the strategies for the optimal
consumption/portfolio and retirement are devised. So far, research on investment portfolios with inflation has yielded fruitful results.

Significantly, with the development of economic globalization, major events have taken place in a country or a region, which impact on the economic development of other regions and countries. Objectively, it will change the pattern and situation of economic development and affect the risky assets price in the financial market. The return on investment may vary drastically at a random time. In order to characterize the dramatic impact of such a major event on the investment market, disorder problem (see p. 430 of Mania & Santacroce, 2010) is used to depict the sharp change of investors’ investment return. The random inflation risk involved, how to construct the optimal investment strategy and maximize the investment return of investors will be discussed and solved in this paper.

On the issue of the portfolio, in Merton (1971) put forward to optimum consumption and portfolio rules in a continuous time model, where the price of the stock was governed by diffusion process in the complete market. Lakner studied optimal trading strategy for an investor by utility maximization with partial information in Lakner (1995) and Lakner (1998). His work laid a foundation for the study of investment decision under partial information. Scholars continue to devote themselves to the improvement and promotion of the model. On the one hand, by introducing dividend payment and inflation, we can describe the financial market more accurately and consider its impact on the portfolio fully. For example, Karatzas and Shreve (1998) researched the income process with the dividend payment, excess return process and risk premium process of stock. Benoussan, Keppo, and Sethi (2009) introduced the stochastic inflation process and dividend payment to reflect the impact of uncertain factors in the market. The stochastic control and filtering theory were applied to work out the optimal consumption and investment strategies of investors. Fei and Wu (2000) considered a consumption and investment decision with a higher interest rate for borrowing. Fei and Wu (2002) concentrated on a stochastic control problem to maximize expected utility from terminal portfolio/consumption, where the portfolio is allowed to anticipate the future with constraints and a higher interest rate for borrowing. Hu, Fei, and Bao (2008) characterized an optimal portfolio strategy under partial information with the dividend. Fei, Fei, Rui, and Yan (2019) developed an optimal intertemporal asset allocation strategy of a multinational corporation which invested in foreign markets at exchange rate risk.

On the other hand, extending the random driving term from Brownian motion to martingale even to semimartingale, that will be applicability and generality of the model. For instance, Goll and Kallsen (2000) considered the logarithmic utility maximization problem from consumption or terminal net wealth in the semimartingale market model. Framstad, Øksendal, and Sulem (2001) considered the optimal consumption and investment when the risk asset price is semimartingale. Mania, Tevzadze, and Toronjadze (2008) considered the mean square hedging problem in which the asset price is driven by continuous semimartingale under partial information. Mania and Santacroce (2010) under partial information, assuming that the asset price was driven by semimartingale, studied the optimal investment strategy for maximizing the expected utility of terminal wealth. Li, Fei, Shi, and Li (2012, 2013) further studied optimal investment strategy with the disorder problem in the framework of semimartingale.

This paper constructs a model of the financial market in which both disorder problem and inflation risk are considered in the semimartingale framework. We apply the idea contained in literatures (see e.g. Claude & Paul-André, 1982; Karatzas & Shreve, 1998; Lakner, 1995, 1998; Li et al., 2012, 2013; Mania & Santacroce, 2010; Mania et al., 2008; Merton, 1971; Shiryayev, 1978) to evolve the optimal investment strategy and value process dealing with random inflation risk and disorder problem, when the investors only have part of information in the market. At the same time, through the numerical simulation, it is explicit how the disorder problem and inflation affect the optimal investment strategy. Compared with the existing research results, considering both stochastic inflation and disorder problem into the financial market model is an extension of the current model.

In the context of inflation risk and disorder problem, in order to seek the optimal investment strategy of the expected utility maximization of investors’ terminal real wealth, we organize the paper as follows. Section 2 constructs the financial market model with inflation and disorder problem, and considers the exponential utility maximization problem of real terminal wealth of investors. Section 3 explores theoretical derivative of optimal investment strategy and value process, and proves that the value process is the unique solution of a BSDE. Section 4 solves the concrete expression of optimal investment strategy and value process. Finally, utilizing numerical simulation, the impact of expected inflation rate and inflation volatility on the optimal investment strategy is analysed, and the differences between the disordered asset return and the normal situation are compared, and the economic explanation is given.
2. Construction of financial market model

Let us begin with the complete filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\). Here \(\mathcal{F} = \mathcal{F}_T\) \((T < \infty)\) contains all market information from 0 to terminal time \(T\). The nominal price process of risky asset is

\[
dS^*_t = S^*_t (\mu_{[t \geq t]} dt + \sigma dB^*_t),
\]

where \(B^*_t\) is a standard Brownian motion on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\), which is consistent with the driving source of the risky asset’s nominal price. We assume that \(B^*_t\) is independent of \(\tau\), and \(\tau\) satisfies \(P[\tau = 0] = p, P[\tau > t | \tau > 0] = e^{-\gamma t}, t \in [0, T], p \in [0,1], \gamma > 0\), \(\eta\) is the expected inflation rate, \(\xi > 0\) is the inflation volatility, \(\mu, \sigma, \eta, \xi\) are all constants with \(\xi \neq \sigma\).

The real price process of the risky asset is \(\tilde{S}^*_t = S^*_t / \delta_t\), an application of the Itô Lemma gives

\[
d\tilde{S}^*_t = d\left(\frac{S^*_t}{\delta_t}\right) = \frac{1}{\delta_t} dS^*_t + S^*_t d\left(\frac{1}{\delta_t}\right) + d\delta_t d\left(\frac{1}{\delta_t}\right).
\]

Substituting \(d(1/\delta_t) = -(1/\delta_t)((\eta - \xi^2) dt + \xi dB^*_t)\) into the expression above, we get

\[
d\tilde{S}^*_t = \frac{S^*_t}{\delta_t} [(\mu_{[t \geq t]} dt + \sigma dB^*_t) - ((\eta - \xi^2) dt + \xi dB^*_t) - \sigma \xi dt] = \frac{S^*_t}{\delta_t} [(\mu_{[t \geq t]} - \eta + \xi^2 - \sigma \xi) dt + (\sigma - \xi) dB^*_t].
\]

Now, the real return on risky asset is

\[
ds_t = \frac{d\tilde{S}^*_t}{S^*_t} = (\mu_{[t \geq t]} - \eta + \xi^2 - \sigma \xi) dt + (\sigma - \xi) dB^*_t.
\]

Investors can only observe the information of \(S_t\) but not the information of \(\tau\) in financial market. We denote the observable information set as \(\mathcal{G}_t \triangleq \sigma(\mathcal{S}_{0 \leq \nu \leq t}) \subset \mathcal{F}_t\) and the posterior probability process as \(p_t \triangleq P[\tau \leq t | \mathcal{G}_t]\). Then the real return process \(S_t\) relative to filtration \(\mathcal{G}_t\) is equivalent to

\[
E[S_t | \mathcal{G}_t] = E\left[\left(S_0 + \int_0^t (\mu_{[t \geq s]} - \eta + \xi^2 - \sigma \xi) ds + (\sigma - \xi) B^*_s\right) | \mathcal{G}_t\right],
\]

then

\[
S_t = S_0 + \int_0^t (\mu p - \eta + \xi^2 - \sigma \xi) ds + (\sigma - \xi) B_t,
\]

where

\[
B_t = E[B^*_t | \mathcal{G}_t] = \frac{1}{\sigma - \xi} (S_t - S_0 - \int_0^t (\mu p - \eta + \xi^2 - \sigma \xi) ds).
\]

is an innovation Brownian motion relative to information filtration \(\mathcal{G} = (\mathcal{G}_t)\).

Let \(\tilde{\pi}_t\) be the normal dollar amount invested in the risky asset at time \(t\), and \(\pi_t = \tilde{\pi}_t / \delta_t\) be the real dollar amount invested in risky asset at time \(t\). Hence, the exponential utility maximization problem from the investors’ terminal real wealth can be expressed

\[
\max_{\pi \in \pi(\mathcal{G})} E[-e^{-\alpha(T \pi_t dS_t + x - \mathcal{H})}],
\]

where \(\pi(\mathcal{G}) \triangleq \{\pi : \mathcal{G} - \text{predictable}, \pi \cdot S \in BMO(\mathcal{G})\}\) (see p. 267 of Claude & Paul-André, 1982), that is a certain class \(\pi\) which is \(\mathcal{G}\)-predictable and \(S\)-integrable, called admissible investment strategy set. \(x\) is the real initial wealth and \(\mathcal{H}\) is the real random payment at time \(T\) of investors. In order to simplify the calculation, we can let \(x = 0\). The exponential utility maximization problem (1) is

\[
\min_{\pi \in \pi(\mathcal{G})} E[e^{-\alpha(T \pi_t dS_t - \mathcal{H})}],
\]

naturally,

\[
V_t = \text{ess inf} \{E[e^{-\alpha(T \pi_t dS_t - \mathcal{H})} | \mathcal{G}_t] \}
\]

is defined as the value process at time \(t\) of problem (2), where \(\mathcal{H} = (1/\alpha) \ln E[e^{\mathcal{H}} | \mathcal{G}_T]\).

We will contribute the remainder of the paper to solving the optimal investment strategy and value process in the context of partial observable information. In order to overcome the technical difficulties, we need to put forward some hypothetical conditions and deal with the martingale transformation technically, and then we find the expression of optimal value process and optimal investment strategy which is satisfying the following conditions (a) and (b).
3. Theoretical derivation of optimal investment strategy and value process

In this section, we provide the two following assumptions.

(a) The filtration \( G_t = \sigma (S_v, 0 \leq v \leq t) \) is continuous;
(b) Terminal random payment \( H \) is a \( \mathcal{F}_T \) measurable bounded random variable. In particular, let \( H = h(\xi, S_T) \), \( h \) is a positive bounded function of two variables and \( \xi \) is a \( \mathcal{F}_T \) measurable random variable independent of \( G_T \).

When \( \xi \neq \sigma \), we define the market price of risk with random inflation to be

\[
\Theta_t = \frac{\mu_p - \eta + \xi^2 - \sigma \xi}{\sigma - \xi}.
\]

Thus, the risk neutral measure of market as \( Q \), satisfying

\[
\frac{dQ}{dP} = \mathcal{E}_t(-\Theta \cdot B)
\]

\[
= \exp \left\{ - \int_0^t \Theta_v \, dB_v - \frac{1}{2} \int_0^t \Theta^2 (v) \, dv \right\},
\]

and \( \int_0^t \Theta_v \, dt \leq C \) a.s. According to Girsanov’s theorem, \( \tilde{B}_t = B_t + \int_0^t \Theta_v \, dv \) is the Brownian motion under \( Q \), thus \( S_t = S_0 + (\sigma - \xi) \tilde{B}_t \) is the martingale under \( Q \) relative to the filtration \( G_t \).

**Theorem 3.1:** If conditions (a) and (b) are satisfied, investors can observed the information filtration \( G_t \), then the following conclusions are valid:

1. The posterior probability process \( p_t = P[\tau \leq t \mid G_t] \) is the strong Markov process which satisfies the stochastic differential equation (SDE)

\[
p_t = p_0 + \int_0^t \frac{\mu p_v}{\sigma} (1 - p_v) \, dB_v + \gamma \int_0^t (1 - p_v) \, dv.
\]

2. The value process (3) is the unique solution of the BSDE

\[
X_t = X_0 + \frac{1}{2} \int_0^t \frac{(\sigma - \xi) \varphi_v + \mu p_v - \eta + \xi^2 - \sigma \xi \, X_v}{X_v} \, dv + \int_0^t (\sigma - \xi) \varphi_v \, dB_v,
\]

\[
X_T = E[e^{aH} \mid G_T], \quad \text{and} \quad 0 < c \leq X_t \leq C.
\]

3. The optimal investment strategy for problem (2) exists in the class \( \pi (G) \), and equals to

\[
\pi_t^* = \frac{1}{a} \left( \frac{\mu p_v - \eta + \xi^2 - \sigma \xi}{(\sigma - \xi)^2} + \frac{\varphi_v}{X_t} \right).
\]

**Proof:** (1) You can see the p. 202 of Shirayev (1978) and references therein, also can refer to proof for Theorem 9.1 in Liptser and Shirayev (1977).

(2) Suppose that \( X \) is the strictly positive solution of BSDE (5), in order to prove the claim (2), first let \( Z_t = \ln X_t \), by the Itô Lemma, we deduce

\[
dZ_t = \frac{1}{X_t} \, dX_t - \frac{1}{2X_t^2} \, d[X_t, X_t]
\]

\[
= \left( \frac{(\sigma - \xi) \varphi_t + \mu p_t - \eta + \xi^2 - \sigma \xi}{X_t} \right)^2 \, dt
\]

\[
+ \left( \frac{(\sigma - \xi) \varphi_t}{X_t} \right) \, dB_t - \left( \frac{(\sigma - \xi) \varphi_t}{X_t} \right)^2 \, dt
\]

\[
= \left( \frac{(\sigma - \xi) \varphi_t + \mu p_t - \eta + \xi^2 - \sigma \xi}{X_t} \right)^2 - \left( \frac{(\sigma - \xi) \varphi_t}{X_t} \right)^2 \, dt
\]

\[
+ \frac{(\sigma - \xi) \varphi_t}{X_t} \, dB_t.
\]

Then define \( \tilde{\varphi}_t = \varphi_t / X_t \), substitute to \( dZ_t \) and simplify it,

\[
dZ_t = \left[ \left( \mu p_t - \eta + \xi^2 - \sigma \xi \right) \tilde{\varphi}_t + \frac{1}{2} \left( \frac{\mu p_t - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right)^2 \right] \, dt + (\sigma - \xi) \tilde{\varphi}_t \, dB_t,
\]

\[
Z_T = \ln E[e^{aH} \mid G_T] = a \tilde{H}, \quad \text{where}
\]

\[
\tilde{H} = \frac{1}{a} \ln E[e^{aH} \mid G_T].
\]

The existence of solutions for BSDE (6) is referred to Tevzadze (2008), so we can infer the existence of solutions for BSDE (5).

Next, we prove the solution of BSDE (5) is unique. Due to \( X_t \leq V_t \) a.s. and \( X_t \geq V_t \) a.s., we can obtain \( X_t = V_t \) a.s.

(i) Prove \( X_t \leq V_t \) a.s.

Taking now \( J_t(\pi) = -a \int_0^t \pi_j \, dS^j \) for \( \forall \pi \in \pi(G) \), the differential form is

\[
dJ_t = e^Y \, dY + \frac{1}{2} e^Y \, d[Y, Y]
\]

\[
= J_t \left( -a \pi_t \, dS_t + \frac{1}{2} a^2 \pi_t^2 (\sigma - \xi)^2 \, dt \right),
\]

using Itô’s formula, we can write

\[
d(X_t J_t) = J_t \, dX_t + X_t \, dJ_t + dX_t \, dJ_t
\]

\[
= J_t \left[ \left( \frac{(\sigma - \xi) \varphi_t + \mu p_t - \eta + \xi^2 - \sigma \xi}{X_t} \right)^2 \right] \, dt
\]

\[
+ (\sigma - \xi) \varphi_t \, dB_t.
\]
According to Theorem 8.8 in Bensoussan et al. (2009),

\[ X_t^J = X_0^J + \int_0^t \psi_t \pi_t \, dS_t + \frac{1}{2} \sigma_t^2 (\sigma - \xi)^2 \, dt \]

by Itô Lemma, and similar deduction to (7), we have

\[ \frac{d(X_t^J)}{X_t^J} = \left( \frac{\psi_t}{X_t^J} - \alpha \pi_t \right) (\sigma - \xi) \, dB_t + \frac{1}{2} (\sigma - \xi)^2 X_t^J \left[ \alpha \pi_t - \left( \frac{\mu_t - \eta + \frac{\xi^2}{2} - \sigma \xi}{(\sigma - \xi)^2} + \frac{\psi_t}{X_t^J} \right) \right]^2 \, dt. \]

Thus, we have

\[ \frac{d(X_t^J)}{X_t^J} = \left( \frac{\psi_t}{X_t^J} - \alpha \pi_t \right) (\sigma - \xi) \, dB_t + \frac{1}{2} (\sigma - \xi)^2 \left[ \alpha \pi_t - \left( \frac{\mu_t - \eta + \frac{\xi^2}{2} - \sigma \xi}{(\sigma - \xi)^2} + \frac{\psi_t}{X_t^J} \right) \right]^2 \, dt. \]

According to Theorem 8.8 in Bensoussan et al. (2009), \( X_t^J \) can be equivalent to

\[ X_t^J = X_0^J \left( \left( \frac{\psi_t}{X_t^J} - \alpha \pi_t \right) (\sigma - \xi) \cdot B \right) \times e^{(1/2)(\sigma - \xi)^2 \int_0^t [\alpha \pi_v - ((\mu_v - \eta + \xi^2 - \sigma \xi)/(\sigma - \xi)^2) + \psi_v/X_v] \, dv}. \]

(7)

We discover that (7), the uniformly integrable exponential martingale \( e_t^{((\psi/X - \alpha \pi)(\sigma - \xi) \cdot B)} \) multiplies the strictly positive incremental process, can be expressed clearly, knowing by the properties of semimartingale, \( X_t^J \) is the submartingale. Combining the boundary condition \( X_T = E[e^{aH} \mid G_T] = e^{aH} \), we have

\[ X_t^J \leq E[U_T(\pi) e^{aH} \mid G_T] \text{ a.s.}. \]

\[ X_t \leq E[e^{-a \int_0^T \pi_s \, dS_s + aH} \mid G_T] \text{ a.s.}. \]

Hence

\[ X_t \leq \inf_{\pi \in \pi(G)} E[e^{-a \int_0^T \pi_s \, dS_s + aH} \mid G_T] \text{ a.s.}. \]

Thus, we can deduce

\[ X_t \leq \inf_{\pi \in \pi(G)} E[e^{-a \int_0^T \pi_s \, dS_s + aH} \mid G_T] = V_t \text{ a.s.}. \]

(8)

(ii) Prove \( X_t \geq V_t \) a.s.

Taking \( \pi_t^* = (1/a)((\mu_t - \eta + \xi^2 - \sigma \xi)/(\sigma - \xi)^2 + \psi_t/X_t) \), by Itô Lemma, and similar deduction to (7), we obtain that \( X_t^J(\pi_t^*) \) is a strictly positive supermartingale. According to the properties of supermartingale and boundary condition \( X_T = E[e^{aH} \mid G_T] = e^{aH} \), we get

\[ X_t^J(\pi_t^*) \geq E[U_T(\pi^*) e^{aH} \mid G_T] \text{ a.s.}. \]

\[ X_t \geq E[e^{-a \int_0^T \pi^*_s \, dS_s + aH} \mid G_T] \text{ a.s.}. \]

There \((\mu_t - \eta + \xi^2 - \sigma \xi)/(\sigma - \xi) \cdot B \in BMO(G), X_t \geq c \) and \((\psi/X)(\sigma - \xi) \cdot B \in BMO(G), so that \)

\[ \frac{1}{a} \left( \frac{\mu_t - \eta + \xi^2 - \sigma \xi}{(\sigma - \xi)^2} + \frac{\psi_t}{X_t^J} \right) (\sigma - \xi) \cdot B \in BMO(G). \]

Note that \( \pi^* \in \pi(G), \]

\[ X_t \geq E[e^{-a \int_0^T \pi^*_s \, dS_s + aH} \mid G_T] \geq V_t \text{ a.s.} \]

(9)

By (8) and (9), we obtain \( X_t = V_t \) a.s. and prove that \( V_t = E[e^{-a \int_0^T \pi^*_s \, dS_s + aH}] \) is the unique solution of BSDE (5).

(3) According to claim \( X_t \geq V_t \) a.s., it is easy to obtain that

\[ \pi_t^* = \frac{1}{a} \left( \frac{\mu_t - \eta + \xi^2 - \sigma \xi}{(\sigma - \xi)^2} + \frac{\psi_t}{X_t^J} \right). \]

Hence, \( \pi_t^* \) is the investment strategy which makes the exponential utility maximization of the terminal real wealth, that is the optimal investment strategy for investors under the risk of random inflation.

\[ \square \]

4. Expression of optimal investment strategy and value process

In the light of the proof of Theorem 3.1, the value process \( V_t \) fulfils the BSDE (5). Substitute the real return process \( dS_t = (\mu_t - \eta + \xi^2 - \sigma \xi) \, dt + (\sigma - \xi) \, dB_t \) into (6), and simplify it

\[ Z_t = Z_0 + \frac{1}{2} \int_0^T \left( \frac{\mu_v - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right)^2 \, dv + \int_0^T \tilde{\varphi}_v \, dS_v, \]

(10)

\[ Z_T = \ln E[e^{aH} \mid G_T] = aH. \]

(11)

Subtract both sides of (10) from (11), we can obtain

\[ Z_T - Z_t = \frac{1}{2} \int_t^T \left( \frac{\mu_v - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right)^2 \, dv + \int_t^T \tilde{\varphi}_v \, dS_v \]

\[ = \ln E[e^{aH} \mid G_T] - Z_t, \]

or equivalently

\[ Z_t + \int_t^T \tilde{\varphi}_v \, dS_v = \ln E[e^{aH} \mid G_T] \]

\[ - \frac{1}{2} \int_t^T \left( \frac{\mu_v - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right)^2 \, dv. \]
Then take conditional expectations of two sides of equation with respect to $\mathcal{G}_t$

\[
E^\mathcal{P}\left[\left(Z_t + \int_t^T \tilde{\varphi}_t \, dS_v\right) \mid \mathcal{G}_t\right] = E^\mathcal{P}\left[\left(\ln E[e^{aH} \mid \mathcal{G}_T] - \frac{1}{2} \int_t^T \left(\frac{\mu_{hv} - \eta + \xi^2 - \sigma \xi}{\sigma - \xi}\right)^2 \, dv\right) \mid \mathcal{G}_t\right].
\]

Since

\[
\frac{dQ}{dP} = \delta_T(-\Theta \cdot B)
\]

we have two $G$-predictable $S$-integrable processes which are $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, such that

\[
\ln E[e^{aH} \mid \mathcal{G}_T] = C_1 + \int_0^T \tilde{\varphi}_1(v) \, dS_v,
\]

(14)

\[
-\frac{1}{2} \int_0^T \left(\frac{\mu_{hv} - \eta + \xi^2 - \sigma \xi}{\sigma - \xi}\right)^2 \, dv 
\]

\[
\triangleq C_2 + \int_0^T \tilde{\varphi}_2(v) \, dS_v.
\]

(15)

From (12), denote

\[
Z_1(t) \triangleq E^Q[\ln E[e^{aH} \mid \mathcal{G}_T] \mid \mathcal{G}_t],
\]

\[
Z_2(t) \triangleq -\frac{1}{2} \int_t^T \left(\frac{\mu_{hv} - \eta + \xi^2 - \sigma \xi}{\sigma - \xi}\right)^2 \, dv.
\]

(16)

First, let’s pay close attention to $Z_1(t)$. Under the condition (b), $H = h(\xi, S_t)$, $\xi$ is independent of $\mathcal{G}_t$,

\[
E[e^{aH} \mid \mathcal{G}_T] = E[e^{aH(\xi, S_T)} \mid \mathcal{G}_T] \triangleq g(S_T),
\]

\[
g(s) = E[e^{aH(\xi, S_T)} \mid S_T = s] = \int_0^\infty e^{aH(v,s)} \, dP_T(v).
\]

On account of $h > 0$, then $g(s) > 1$, and $S$ is the martingale under $(Q, \mathcal{F}^S)$, we obtain that

\[
E^Q[\ln E[e^{aH} \mid \mathcal{G}_T] \mid \mathcal{G}_t] = E^Q[\ln g(S_T) \mid S_t] = Z_1(t)
\]

is the martingale under $Q$. Moreover

\[
G(t, x) \triangleq E^Q[\ln g(S_T) \mid S_T = x] = \frac{1}{\sqrt{2\pi}} e^{-y^2/(2(\sigma - \xi)^2)} \int_0^\infty \ln g(y) e^{-(y-x)^2/(2(\sigma - \xi)^2)} \, dy,
\]

(17)

where $G \in C^{1,2}(0, T)$. Applying the Itô Lemma, we write

\[
Z_1(t) = G(t, S_t) = G(0, S_0) + \int_0^T \mathcal{G}_t(v, S_v) \, dS_v.
\]

Comparing (14) with (17), we have $C_1 = G(0, S_0), \tilde{\varphi}_1(t) = \mathcal{G}_t(v, S_v), dP \otimes dt$ a.e.

Second, let’s solve $Z_2$. To do this, we need to prove the following Lemma at first.

**Lemma 4.1:** Under the filtration $\mathcal{G}_t$, random process

\[
\frac{1}{(\sigma - \xi)} S_t
\]

is the Q-martingale.
(i) Posterior probability process \( p_t = P[t \leq t | G_t] \) of disorder time \( t \) is satisfied to SDE

\[
p_t = p_0 + \int_0^t \frac{\mu p_v}{\sigma (\sigma - \xi)} (1 - p_v) \, dS_v,
\]

\[
+ \int_0^t (1 - p_v) \left( \gamma - \frac{\mu p_v (\mu p_v - \eta + \xi^2 - \sigma \xi)}{\sigma (\sigma - \xi)} \right) \, dv.
\]

(18)

(ii) Let \( Z_2(t) = -(1/2(\sigma - \xi)^2) U(t, p_t) \), where

\[
U(t, x) \equiv E^Q \left[ \int_t^T (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv \mid p_t = x \right].
\]

Then \( U(t, x) \) satisfies partial differential equation (PDE)

\[
R_t(t, x) + \left( \frac{\mu x}{2 \sigma^2} - (1 - x)^2 \right) R_{xx}(t, x)
+ \left( \frac{\gamma - \frac{\mu x (\mu x - \eta + \xi^2 - \sigma \xi)}{\sigma (\sigma - \xi)}}{1 - x} \right) R_x(t, x)
+ (\mu x - \eta + \xi^2 - \sigma \xi)^2 = 0,
\]

where \( R_t, R_x, R_{xx} \) are partial derivatives of \( R \).

**Proof:** (i) Transformation

\[
dS_v = \left( \mu p_t - \eta + \xi^2 - \sigma \xi \right) \, dt + (\sigma - \xi) \, dB_t
\]

from measure \( P \) to \( Q \),

\[
\bar{B}_t = B_t + \int_0^t \Theta_v \, dv = B_t + \int_0^t \frac{\mu p_v - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \, dv.
\]

We have real return process under the measure \( Q \)

\[
dS_t = (\sigma - \xi) \, d\bar{B}_t.
\]

In terms of Theorem 3.1, we can obtain

\[
p_t = p_0 + \int_0^t \frac{\mu p_v}{\sigma (\sigma - \xi)} (1 - p_v)
\]

\[
\times \left( d\bar{B}_t - \frac{\mu p_v - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \, dv \right)
\]

\[
+ \gamma \int_0^t (1 - p_v) \, dv
\]

\[
= p_0 + \int_0^t \frac{\mu p_v}{\sigma (\sigma - \xi)} (1 - p_v) \, dS_v + \gamma \int_0^t (1 - p_v) \, dv
\]

\[
- \int_0^t \frac{\mu p_v}{\sigma (\sigma - \xi)} (1 - p_v)(\mu p_v - \eta + \xi^2 - \sigma \xi) \, dv,
\]

we have already proved (18).

(ii) The existence of a solution of PDE (19) follows from [Tez tadze (2008)](see p. 12 and references therein). We will prove that \( U(t, p_t) \) is satisfied to (19).

Supposing \( R(t, p_t) \) is the solution of PDE (19). By the Itô Lemma, we have

\[
dR(t, p_t) = R_t \, dt + R_x \, dp_t + \frac{1}{2} R_{xx} \, [dp_t, dp_t]
\]

\[
= R_t \, dt + R_x \left[ \frac{\mu p_v (1 - p_v)}{\sigma (\sigma - \xi)} \right] \, dS_t
+ (1 - p_v) \left( \gamma - \frac{\mu p_v (\mu p_v - \eta + \xi^2 - \sigma \xi)}{\sigma (\sigma - \xi)} \right) \, dt
\]

\[
+ \frac{1}{2} R_{xx} \left( \frac{\mu p_v (1 - p_v)}{\sigma} \right)^2 \, dt.
\]

Integration with both side

\[
R(t, p_t) - R(0, p_0) = \int_0^t R_v \, dv + \int_0^t \frac{\mu p_v (1 - p_v)}{\sigma (\sigma - \xi)} R_x \, dS_v
\]

\[
+ \int_0^t (1 - p_v) \left( \gamma - \frac{\mu p_v (\mu p_v - \eta + \xi^2 - \sigma \xi)}{\sigma (\sigma - \xi)} \right) R_x \, dv
\]

\[
+ \frac{1}{2} \int_0^t R_{xx} \left( \frac{\mu p_v (1 - p_v)}{\sigma} \right)^2 \, dv.
\]

(20)

Because of (19), the third and fourth differential of (20) can be equivalently expressed as

\[
-[R_t(t, p_t) + (\mu p_t - \eta + \xi^2 - \sigma \xi)^2).
\]

Hence, (20) becomes

\[
R(t, p_t) - R(0, p_0)
\]

\[
= \int_0^t R_v \, dv + \int_0^t \frac{\mu p_v (1 - p_v)}{\sigma (\sigma - \xi)} R_x \, dS_v
\]

\[
- \left( \left( \int_0^t R_v \, dv + \int_0^t (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv \right) \right)
\]

\[
= \int_0^t \frac{\mu p_v (1 - p_v)}{\sigma (\sigma - \xi)} R_x \, dS_v
\]

\[
- \int_0^t (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv.
\]

We may thus write

\[
R(t, p_t) + \int_0^t (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv
\]

\[
= R(0, p_0) + \int_0^t \frac{\mu p_v (1 - p_v)}{\sigma (\sigma - \xi)} R_x \, dS_v.
\]

(21)

The integral of \( S_t \) in (21) is a martingale under the measure \( Q \), it can be concluded that the left-hand side of (21) is also a martingale. Using the martingale properties and
boundary condition \( R(T, x) = 0 \), we get
\[
\left[ R(T, p_T) + \int_0^T (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv \right]
- \left[ R(t, p_t) + \int_t^T (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv \right]
= \int_0^T \frac{\mu p_v(1 - p_v)}{\sigma(\sigma - \xi)} R_x(v) \, dS_v - \int_0^t \frac{\mu p_v(1 - p_v)}{\sigma(\sigma - \xi)} R_x(v) \, dS_v
= \int_t^T \frac{\mu p_v(1 - p_v)}{\sigma(\sigma - \xi)} R_x(v) \, dS_v.
\]
We take conditional expectation with both sides relative to \( \mathcal{G}_t \),
\[
E^Q \left[ \left( \int_t^T (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv - R(t, p_t) \right) \right| \mathcal{G}_t \] = 0.
That is
\[
R(t, p_t) = E^Q \left[ \int_t^T (\mu p_v - \eta + \xi^2 - \sigma \xi)^2 \, dv \right| \mathcal{G}_t \] = U(t, p_t). (22)
So that \( U(t, p_t) = R(t, p_t) \) is the solution of PDE (19), and
\[
Z_2(t) = -\frac{1}{2(\sigma - \xi)^2} U(t, p_t).
\]
Moreover, according to proof of the Lemma 4.1, comparing (15) with (21), we then find that
\[
- \frac{1}{2} \int_0^T \left( \frac{\mu p_v - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right)^2 \, dv
= - \frac{1}{2(\sigma - \xi)^2} \left[ -R(T, p_T) + R(0, p_0) \right.
+ \int_0^T \frac{\mu p_v(1 - p_v)}{\sigma(\sigma - \xi)} R_x(v, p_v) \, dS_v \left. \right] - \frac{1}{2(\sigma - \xi)^2} R(0, p_0)
\]
\[
= C_2 + \int_0^T \varphi_2(v) \, dS_v.
\]
Of course, \( C_2 = -(1/2(\sigma - \xi)^2)R(0, p_0) \) and
\[
\varphi_2(v) = -\frac{\mu p_v(1 - p_v)}{2\sigma(\sigma - \xi)^3} R_x(v, p_v).
\]
Through the above discussion, we give the expression for process \( Z_t \) and obtain \( V_t = e^{Z_t} \). At the same time, we solve out the optimal trading strategy
\[
\pi_t^* = \frac{1}{a} \left( \frac{\mu p_t - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right) + \tilde{\varphi}_t^*.
\]
the details can be seen by the Theorem 4.1.

**Theorem 4.1:** The value process \( V_t \) of (3) which is corresponding to expected exponential utility maximization problem of terminal net wealth (1) satisfies
\[
V_t = e^{Z_t(t)} + Z_2(t) = e^{G(t, S_t) - (1/2(\sigma - \xi)^2) U(t, p_t)}. \quad (23)

**Optimal investment strategy** is
\[
\pi_t^* = \frac{1}{a} \left( \frac{\mu p_t - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right) + \tilde{\varphi}_t^* \quad \text{for} \quad t \geq 0.
\]
where
\[
\tilde{\varphi}_t = \varphi_t = G(x, S_t) - \frac{1}{2(\sigma - \xi)^2} \mu p_t(1 - p_t) R_x(t, p_t), \quad (25)
\]
\( G(x, S_t) \) is defined of (17), \( U(t, x) \) satisfies PDE (19), and \( p_t \) is given in SDE (18).

Obviously, the optimal investment strategy (24) consists of two parts, thus the risky assets return \( \int_0^t \pi_t \, dS_v \) also can be divided into two parts. One part \((1/a) \int_0^t G(t, S_t) \, dS_v\) is a hedging fund, the other part
\[
\int_0^t \left[ \frac{1}{a} \left( \frac{\mu p_t - \eta + \xi^2 - \sigma \xi}{\sigma - \xi} \right) + \varphi_t \right] \, dS_v
\]
is an investment fund.

### 5. Numerical analysis of optimal investment strategy

In Sections 3 and 4, we give the theoretical deduction and solving process of optimal investment strategy and value process. In order to more clearly and intuitively reflect the specific impact of various parameters in the financial market model on investment strategy, we will make an effort to do a numerical analysis of theoretical research results.

Before doing this, we suppose real random payment \( H = 0 \) at terminal time \( T \), then the investors’ hedging fund is 0. Moreover, \( \tau = t_0 \) is deterministic, \( \xi \neq \sigma \), the optimal investment strategy can be expressed precisely as follows
\[
\pi_t^* = \frac{\mu p_t - \eta + \xi^2 - \sigma \xi}{a(\sigma - \xi)^2}.
\]
To satisfy the above requirements, we analyse the relationship between optimal investment strategy and variables when the real terminal random payment is absent.
The impact of inflation volatility on optimal strategy.

Case 1. Let $t_0 = 0, p_t = P[\tau \leq t \mid \mathcal{G}_t] = 1$. We will study how the inflation volatility $\xi$ impacts on optimal investment strategy $\pi^*$, without considering the influence of major events. Let $\mu = 0.05, \sigma = 0.01, a = 2$, we can draw curves $\pi^*$ when $\eta_1 = 0.012, \eta_2 = 0.024, \eta_3 = 0.036$, as shown in Figure 1.

Figure 1 shows, for risk-averse investors, the increasing of inflation volatility in the financial market will result in a variety of changes for the optimal investment amount of risky assets. When the inflation volatility $\xi \in (0, 0.01)$, investors are optimistic about holding risky asset, and optimal investment amount increases with the increase of inflation volatility. At that time, the higher the expected inflation rate is, the smaller the optimal investment amount of risky asset which is at the same level of inflation volatility. Specifically, the optimal investment amount of risky assets increases relatively slowly in the range of $(0, 0.005)$, but increases rapidly in the range of $(0.005, 0.01)$. However, once the inflation volatility $\xi$ exceeds the critical point 0.01, the optimal investment of risky asset will decrease, and the investment amount will decrease with inflation volatility increasing. The optimal investment amount of risky asset decreases sharply in the range of $(0.01, 0.015)$. When inflation volatility increases gradually in the range of $(0.015, 0.025)$, the reduction of the optimal investment amount of risky asset is gentle. This can show the difference between investors’ attitudes towards the stochastic volatility risk of inflation.

Notably, when $\xi \rightarrow 0.01, \pi^* \rightarrow +\infty$ in Figure 1, we discover that $\xi = \sigma = 0.01$ is the singular point. According to dynamic representation of the nominal risky asset price and stochastic inflation in Section 2, the real risky asset price is determined by

$$dS(t) = \frac{dS_t^\pi}{S_t^\pi} = (\mu t(\xi \geq \tau) - \eta + \xi^2 - \sigma \xi) \, dt + (\sigma - \xi) \, dB^\pi_t,$$

when $\xi = \sigma$, degenerates into

$$dS(t) = \frac{dS_t^\pi}{S_t^\pi} = (\mu t(\xi \geq \tau) - \eta + \xi^2 - \sigma \xi) \, dt.$$

Obviously, when $(\xi - \sigma) \rightarrow 0$, investment risk tends to be zero, investors will get almost stable expected return. Therefore, investors will increase the amount of investment in risky asset indefinitely. This is consistent with what is shown in Figure 1, when $\xi = \sigma = 0.01$.

Case 2. Let $t_0 = 0, p_t = P[\tau \leq t \mid \mathcal{G}_t] = 1$. Leaving disorder problem out of account, we only pay close attention to the effect of inflation drift $\eta$ on optimal investment strategy $\pi^*$. Let $\mu = 0.05, \sigma = 0.01, a = 2$. We draw curves $\pi^*$ when $\xi_1 = 0.012, \xi_2 = 0.024, \xi_3 = 0.036$, see Figure 2.

In Figure 2, with the increase of expected inflation rate, the optimal investment amount of risky asset shows a downward trend as a whole. Investors will also be prudent in the face of an environment where expected inflation increases, and will reduce the optimal investment amount of risky asset. In the three cases of Figure 2, $\xi \approx 0.05$ is the critical point. When inflation volatility increases within $(0, 0.05)$, we can observe that the rate of optimal investment amount decreases will get slow. In the case of high inflation volatility, the optimal investment amount of investors is very small, even if the expected inflation rate is 0. From this, we can draw a conclusion that investors are more concerned and cautious about unknown inflation fluctuation.
Case 3. Considering $t_0 > 0$ and posterior probability of disorder time $p(1) = 0.2$, $p(2) = 0.5$, $p(3) = 0.8$, how inflation volatility $\xi$ influences on optimal investment strategy, which we will be going to focus on. Let $\mu = 0.05$, $\eta = 0.024$, $\sigma = 0.01$, $a = 2$. We draw an image of $\pi^*$ as Figure 3.

As can be seen from the (a–c) in Figure 3, when the posterior probability of disorder time $p(1) = 0.2$, with the increase of inflation volatility, the optimal investment amount of risky asset decreases first and then increases. We will let $p(2) = 0.5$ and $p(3) = 0.8$, and will discover that $\xi = 0.01$ is the demarcation point. The optimal investment amount of risky asset increases in $(0, 0.01)$ with the increase of inflation volatility, while it decreases in $(0.01, 1)$. This is consistent with the absence of disorder problem. However, with the increase of the posterior probability, it is obvious that the optimal investment level of risky asset has a great difference between disorder problem and normality.

Case 4. We want to consider that how expected inflation rate $\eta$ affects the optimal investment strategy, when $t_0 > 0$ and posterior probability of disorder time $p(1) = 0.2$, $p(2) = 0.5$, $p(3) = 0.8$. Let $\mu = 0.05$, $\xi = 0.024$, $\sigma = 0.01$, $a = 2$. Thus $\pi^*$ is depicted in the Figure 4.

In Figure 4, the influence of expected inflation rate on the optimal investment strategy is given, in the case that

---

**Figure 3.** The relationship between optimal strategy and inflation volatility at different disorder levels. (a) $0 < \xi \leq 0.02$. (b) $0.02 \leq \xi \leq 0.2$ and (c) $0.2 \leq \xi \leq 1$. 
asset price and return processes are disordered. With the increase of expected inflation rate, the optimal amount of investment in risky asset gradually decreases. And the higher the posterior probability of disorder events, the higher the optimal level of investment in risky asset.

6. Conclusions

We are committed to work out the optimal investment strategy with stochastic inflation risk and disorder problem under the partial information framework. By the application of the semimartingale and BSDE theory, we deduce the optimal investment strategy and value process. By the numerical analysis of theoretical research results, we give changes in the optimal investment strategy of risk-averse investors when dealing with stochastic inflation risk, and further analyse how inflation risk affects on investment strategies under the different probability of asset return disorder. The results of the paper will provide practical guidance for investment behaviour in the financial market, and scientific theoretical support for investment practice. Compared with the assumption that risk asset and inflation risk are driven by the same Brownian motion in this paper, if risk sources are driven by different Brownian motions, how to invest can be further studied.

Disclosure statement

No potential conflict of interest was reported by the authors.

Figure 4. The relationship between optimal strategy and expected inflation rate at different disorder levels.

Funding

This work was supported in part by the National Natural Science Foundation of China [grant number 71571001], the Key Program of University Science Research of Anhui Province in China [grant number KJ2017A550] and Plan for Excellent Young Talents in Colleges and Universities of Anhui Province in China [grant number gxyq2018186].

References

Bensoussan, A., Keppo, J., & Sethi, S. P. (2009). Optimal consumption and portfolio decisions with partially observed real prices. *Mathematical Finance*, 19(2), 215–236.

Brennan, M. J., & Xia, Y. (2002). Dynamic asset allocation under inflation. *The Journal of Finance*, 57(3), 1201–1238.

Claude, D., & Paul-André, M. (1982). *Probabilities and potential B*. Amsterdam: North-Holland.

Fei, W. Y. (2013). Optimal consumption and portfolio under inflation and Markovian switching. *Stochastics: An International Journal of Probability and Stochastic Processes*, 85(2), 272–285.

Fei, W. Y., Cai, Z. Q., & Xia, D. F. (2015). Dynamic asset allocation under jump-diffusion environment. *Journal of Management Science in China*, 18(8), 83–94. (in Chinese).

Fei, W. Y., Chen, Y. H., & Fei, C. (2019). An optimal investment and voluntary retirement choice problem with subsistence consumption constraints under inflation. *Journal of Systems Engineering*, (in Chinese, accepted).

Fei, C., & Fei, W. Y. (2015). Optimal control of Markovian switching systems with applications to portfolio decisions under inflation. *Acta Mathematica Scientia*, 35(2), 439–458.

Fei, C., Fei, W. Y., Rui, Y. Y., & Yan, L. T. (2019). International investment with exchange rate risk. *Asia-Pacific Journal of Accounting & Economics*. doi:10.1080/16081625.2019.1569539

Fei, W. Y., Li, Y. H., & Xia, D. F. (2015). Optimal investment strategies of hedge funds with incentive fees under inflationary environment. *Systems Engineering: Theory & Practice*, 35(11), 2740–2748. (in Chinese).

Fei, W. Y., Lv, H. Y., & Yu, M. X. (2014). Decision making for optimal consumption and portfolio under inflation with mean-reverting process. *Journal of Systems Engineering*, 29(6), 791–798. (in Chinese).

Fei, W. Y., & Wu, R. Q. (2000). Optimal investment consumption model with a high interest rate for borrowing. *Applied Mathematics. A Journal of Chinese Universities Series B*, 15(3), 350–358.

Fei, W. Y., & Wu, R. Q. (2002). Anticipative portfolio optimization under constraints and a higher interest rate for borrowing. *Stochastic Analysis and Applications*, 20(2), 311–345.

Framstad, N. C., Øksendal, B., & Sulem, A. (2001). Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs. *Journal of Mathematical Economics*, 35, 233–257.

Goll, T., & Kallsen, J. (2000). Optimal portfolios for logarithmic utility. *Stochastic Processes and Their Applications*, 89, 31–48.

Hu, H. M., Fei, W. Y., & Bao, P. J. (2008). Study on optimal portfolio strategy model with dividend: Under partial information. *Mathematics in Economics*, 25(4), 362–366. (in Chinese).

Karatzas, I., & Shreve, S. E. (1998). *Methods of mathematical finance*. New York, NY: Springer.

Lakner, P. (1995). Utility maximization with partial information. *Stochastic Processes and Their Applications*, 56(2), 247–273.
Lakner, P. (1998). Optimal trading strategy for an investor: The case of partial information. Stochastic Processes and Their Applications, 76(1), 77–97.

Li, J., Fei, W. Y., Shi, X. Q., & Li, Y. (2012). Optimal trading strategy under disordered asset return and partial information. Journal of Mathematics (PRC), 32(4), 693–700. (in Chinese).

Li, J., Fei, W. Y., Shi, X. Q., & Li, Y. (2013). Optimal trading strategy under disordered asset return and Knightian uncertainty. Applied Mathematics. A Journal of Chinese Universities Series A, 28(1), 13–22. (in Chinese).

Liang, Y., Fei, W. Y., Yao, Y. H., & Rui, Y. Y. (2015). Study on optimal portfolio for defined contribution pension with inflation and Knightian uncertainty. Chinese Journal of Engineering Mathematics, 32(3), 337–347. (in Chinese).

Lim, B. H. (2013). The effect of inflation risk and subsistence constraints on portfolio choice. Journal of the Korea Society for Industrial and Applied Mathematics, 17(2), 115–128.

Liptser, R. S., & Shiryaev, A. N. (1977). Statistics of random processes I general theory. New York, NY: Springer.

Mania, M., & Santacroce, M. (2010). Exponential utility maximization under partial information. Finance and Stochastics, 14, 419–448.

Mania, M., Tevzadze, R., & Toronjadze, T. (2008). Mean-variance hedging under partial information. SIAM Journal on Control and Optimization, 47(5), 2381–2409.

Merton, R. (1971). Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory, 3, 373–413.

Shiryaev, A. N. (1978). Optimal stopping rules. New York, NY: Springer.

Siu, T. K. (2011). Long-term strategic asset allocation with inflation risk and regime switching. Quantitative Finance, 11(10), 1565–1580.

Tevzadze, R. (2008). Solvability of backward stochastic differential equations with quadratic growth. Stochastic Processes and Their Applications, 118, 503–515.

Yao, H. X., Wu, H. L., & Zeng, Y. (2014). Optimal investment strategy for risky assets under uncertain time-horizon and inflation. System Engineering-Theory & Practice, 34(5), 283–290. (in Chinese).