Mobile Robot Position Estimation using Milstein Algorithm

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Abstract. Stochastic differential equation (SDE) became a very important mathematical tool in modeling and performance analysis of many systems due to the white noise appeared in them. One of the significant problems is the time delay problem which appeared in many teleoperated systems which has a significant effect in the system performance. The target in this paper is to approximate the mobile robot future position to decrease the time delay in teleoperation. As a result of white noise, heading angle equation of the mobile robot will be converted into a SDE. The approximated solution has been improved by using a higher order of convergence numerical algorithm (Milstein algorithm) with order of convergence equal one for better estimation of future position of mobile robot. The error has been decreased in many parts of the robot path. A simulated results and a comparison between the two numerical algorithms are explained.

Keywords: Euler-Maruyama, Milstein, stochastic differential equation, teleoperation, time delay problem.

1. Background for SDEs

The phenomenon of the Brownian motion and the Langevin equation are usually considered as the origin of the SDEs. In 1827, the scientist R. Brown discovered the Brownian motion experimentally which also considered as an interesting example of random physical processes. Most physicists and mathematicians are then concerned with the mathematical modeling this phenomenon. A partial differential equation which describe the probability density of the Brownian motion displacement was derived by Einstein and Smoluchowski in between 1905 and 1906. In 1908, Paul Langevin use the Newtonian equation to drive Langevin equation which describe the erratic motion of a heavy "Brownian" particle of mass \( m \) immersed in a liquid[1].

Furthermore, this work was later resumed and extended by Ornstein and Uhlenbeck. Then, Ito and Stratonovich put the basic concepts for SDEs and also the mathematical formulations. Both of Ito and Stratonovich SDEs have been used in system modelling. It is essential to be familiar with probability theory and statistics to understand SDEs and their solutions. The solutions of SDEs are approximated by using different numerical methods because it is hard to get an explicit solution[2].

Recently SDEs have been commonly used as an essential mathematical tool in modelling and to analyze a different phenomenon in engineering, mainly in control and mechanical systems. Many engineering systems are exposed to environmental random excitations which give the stimuli for the systematic analysis of various types of SDEs to investigate the performance of these engineering systems. The importance of SDEs appears in filtering problems, stochastic approaches to fixed boundary value issues, optimal shutdown, stochastic monitoring and financing. SDE in science and engineering has a significant impact when noise affected the behavior. For example, SDEs may be used to model the fluctuating prices of the stock market[3].

2. Mobile Robots

Mobile robots are growing rapidly due to the very wide variety of their applications. The challenge in most of mobile robots applications is performing their tasks without human interface. There is actually only a small number of fully autonomous robots. Most robots differ in its control and guidance levels.

There are many levels of control such as autonomous, semi-autonomous and low level. For autonomous level, all data collected from different sensors are integrated together to perform a certain mission independently in operator or in ground control station (GCS).
The semi-autonomous level, there is a trade off in control management and guidance between the remote robot and the operator or GCS. For the low level, the operator control directly the robot with aid of different sensors mounted in the robot with direct or indirect visual feedback[4].

The time delays problem appeared in the communication channels because the operator and the teleoperated vehicle are in different sites. So, GCS will receive an inaccurate data which hampers the teleoperation. Time delay problem in teleoperating system may cause instability or weak performance of the dynamical system. In other word, there is a tradeoff between stability and high transparency in designing teleoperated systems[5].

The idea of time delay compensation is to predict vehicle in future time position. As the GCS see the vehicle in the future location after time \(\Delta t\) at which command orders arrive and applied to the vehicle to perform a certain mission.

3. Stochastic Differential Equations

As discussed in the previous section, the SDE is a differential equation in which one or more of its terms is a stochastic process (SP), resulting in a solution which is also a stochastic process (SP).

The differential formula of SDE is[6]:

\[
\begin{align*}
\frac{dM(t)}{dt} &= a(M(t))dt + \beta(M(t))dW(t) \\
M(0) &= M_0 \\
0 &\leq t \leq T
\end{align*}
\]

where \(W\) represent a Weiner process (WB) and \(a\) and \(\beta\) are functions under a certain conditions. The SDE’s solution is also a SP. The SDE can be represented as integral formula:

\[
M(t) = \int_0^t a(M(s))\,ds + \int_0^t \beta(M(s))\,dW(s)
\]

(2)

The first integral \(\int_0^t a(M(s))\,ds\) is a Riemann stochastic integral while the second integral \(\int_0^t \beta(M(s))\,dW(s)\) represents the Ito stochastic integral. From the conditions of the WBs, it is clear that any path of the WB \(W(t)\) is not differentiable, it cannot be solved by normal rules of classical calculus. So, a stochastic calculus is necessary to approximate the integral solution. Note that, If \(\beta(M(s)) \equiv 0\) and \(M\) is constant, the equation will be converted to an ordinary differential equation again[7].

\[
\frac{dM}{dt} = a(M(s)) \quad , M(0) = M_0
\]

(3)

In finance, a very important example represent the variation of financial price over time is the Black Scholes equation:

\[
\begin{align*}
\frac{dX}{dt} &= \rho X\,dt + \sigma X\,dB_t \\
X(0) &= X_0, \quad \rho \text{ and } \sigma \text{ are constants.}
\end{align*}
\]

(4)

with its solution:

\[
X(t) = X_0e^{(\rho - 0.5\sigma^2)t + \sigma B_t}
\]

(5)

This approximated solution give an estimation for the stock price over a time interval.

This work aims to apply Milstein algorithm to improve the approximated solution of mobile robot kinematics equation, future position of mobile robot, to decrease the time delay in teleoperation.

3.1. Wiener process (\(W_t\)).

Also, it is known as the Brownian motion (white noise) (BM) \(W_t \equiv B_t\). The WP is a Markov and a Gaussian process with probability distribution:
\[ p_1(y, t) = \frac{1}{\Gamma \left( \frac{1}{2} \right)} \exp \left( -\frac{y^2}{2} \right); \lim_{t \to 0^+} p_1(y, t) = \delta(y) \]

The BM over the time interval \([0, T]\) can be defined as a random variable \(W(t)\) which such that for all \(t \in [0, T]\), it must satisfies the next conditions:

1. \(W(0) = 0\) (with probability = 1).
2. For \(0 < v < t < T\) the random variable \(W(t) - W(v)\) has a Gaussian distribution with \(\mu = 0\) and \(\sigma = \sqrt{t - v}\).
3. The WB increments in any path are independent. For \(0 < t_1 < t_2 < \cdots < T\) the increments \(W_2(t) - W_1(t)\) and \(W(t) - W(t-1)\) are independent.
4. The process \(W(t)\) is continuous anywhere. All realizations (sample functions) of the WB are continuous with unbounded variation in any finite interval but it is nowhere differentiable.

The WB is a Markov process. Thus, the distribution of probability of future values of the any WB path are independent on past values but only depends on the present values. Therefore, to predict a future value only the current value is required. In spite, the WB is continuous, it is not differentiable in any path. Furthermore, the WB may have any real value regardless of how much this value is large or negative.

For computational considerations, the discretization of WB paths should be taken into account, where \(W_t\) is specific at discrete \(t\) values. So, the first step, take \(\delta t = T/M\) for some positive integer \(M\) and consider \(W_t\) denote \(W(t_i)\) with \((t_i = i\delta t)\). Condition 1 states that \(W_{t_i} = 0\) with probability 1, and conditions 2 and 3 also state that:

\[ W_{t_i} = W_{t_{i-1}} + dW_{t_i}, \quad i = 1, 2, \ldots, M \]

where each \(dW_i\) is an independent RV with formula \(\sqrt{\delta t}N(0,1)\), a simulated discretized WB path on the interval \([0,1]\) with \(M = 600\) is shown in the next Figure 1[7].

![Discretized WB path](image)

**Figure 1. Discretized WB path**

3.2. *Ito Stochastic Integral (ISI)* The stochastic integral Ito plays a unique and important role among other stochastic integrals. In interpretation and analyzing of ISI and linear SDEs, this integral is of major significance. consider an integral of the form:

\[ I(\emptyset) = \int_a^b \Phi(t, \gamma) \, dW(t, \gamma) \quad (6) \]

where \(\Phi(t, \gamma)\) is a SP which satisfying certain conditions and \(W(t, \gamma)\) is a WP. Since the WP is not differentiable with unbounded variation in any interval, the integral in equation (6) cannot be treated as the known Riemann-Stieltjes stochastic integral (RSI).
Notice that if the function $\emptyset(t, \gamma)$ is a deterministic function of $t$ (independent on $\gamma$), the integral in equation (6) may be treated as RSI, in such cases it is only essential to take into account the orthogonality of increments in the Wiener process. If $\emptyset$ is a differentiable and deterministic function, then it can be solved by classical calculus. Generally, if the two functions $\emptyset(t, \gamma)$ and $W(t, \gamma)$ are not mutually independent and the function $\emptyset(t, \gamma)$ is not continuous for $\gamma$, the integral in in equation (6) should be defined in a special matter. Ito has propose the first proper meaning of integral in equation (6). An important property of ISI construction is the dependence of $\emptyset(t, \gamma)$ on $W(t, \gamma)$ should be non-anticipative; the random function $\emptyset(t, \gamma)$, as much as possible, can depend on the present and past values of the WP $W(r, \gamma)$, $r \leq t$, but not on $W(r, \gamma)$ for $r > t$[3].

3.3.1 Properties of ISI
1. If $\emptyset_1, \emptyset_2$ belong to $H_2[c, d]$ and $\alpha_1, \alpha_2$ are random variables such that
   
   $\alpha_1 \emptyset_1 + \alpha_2 \emptyset_2 \in H_2[c, d]$, then

   \[
   \int_c^d [\alpha_1 \emptyset_1(t) + \alpha_2 \emptyset_2(t)] \, dW(t) = \alpha_1 \int_c^d \emptyset_1(t) \, dW(t) + \alpha_2 \int_c^d \emptyset_2(t) \, dW(t)
   \]

2. If $Y_{[c,d]}(t)$ is a characteristic function of the interval $[\alpha, \beta] \subset [c, d]$ then

   \[
   \int_c^d Y_{[c,d]}(t) \, dW(t) = W(\beta) - W(\alpha)
   \]

3. If $\emptyset(t) \in H_2[c, d]$ and $\int_c^d E[\emptyset^2(t)] \, dt < \infty$, then

   \[
   E\left[\int_c^d \emptyset(t) \, dW(t)\right] = 0
   \]

   \[
   E\left[\int_c^d \emptyset^2(t) \, dW(t)\right] = \int_c^d E[\emptyset^2(t)] \, dt
   \]

3.3.2 Ito chain rule For $M(t)$ possess the stochastic differential: $dM(t) = s(t) \, dt + r(t) \, dW(t)$ and let $\emptyset(t, M)$ be a continuous function in $t$ and $M$ together with its derivatives $\frac{\partial \emptyset}{\partial t}$, $\frac{\partial \emptyset}{\partial M}$ and $\frac{\partial^2 \emptyset}{\partial M^2}$. Then the process $\emptyset(t, M(t))$ has a stochastic differential (with respect to the same WP $W(t)$ given by:

   \[
   d\emptyset(t, M(t)) = \left[\frac{\partial \emptyset}{\partial t}(t, M(t)) + s(t) \frac{\partial \emptyset}{\partial M}(t, M(t)) + 0.5r^2(t) \frac{\partial^2 \emptyset}{\partial M^2}(t, M(t))\right] dt
   \]

   \[
   + \frac{\partial \emptyset}{\partial M}(t, M(t))r(t) \, dW(t)
   \]

Notice that, the formula in (7) differs from the classical calculus by Ito correction term:

\[
0.5r^2(t) \frac{\partial^2 \emptyset}{\partial M^2}(t, M(t))
\]

which often gives rise to errors in transformations of SDEs[3].

3.4 Stratonovich integral (STSI) It is also a very important part of SDEs but the difference from Ito integral is that it following the classical calculus transformation rules, and this is the main reason for its use.

\[
\int_0^T f_{\emptyset} \, dW_t
\]

for $g : \mathbb{R} \to \mathbb{R}$ which is differentiable and consider the STSI of $g(W_t)$. By the Taylor formula:

\[
g(W_{t_{i+1}}) = g(W_{t_i}) + g'(W_{t_i})[W_{t_{i+1}} - W_{t_i}] + \text{higher order terms}
\]

So the mean square limit with $\lambda = 0.5$ is
\[
\sum_{i=1}^{m} g(W_{t_i})(W_{t_{i+1}} - W_{t_i}) + 0.5 \sum_{i=1}^{m} g'(W_{t_i})(W_{t_{i+1}} - W_{t_i})^2
\]  
(10)

Higher order terms

Hence

\[
\int_{0}^{T} g(W_t) \circ dW_t = \int_{0}^{T} g(W_t) dW_t + 0.5 \int_{0}^{T} g'(W_t) dt
\]  
(11)

Now, let \( U \) be an anti-derivative of \( g \), so \( U'(x) = g(x) \) and hence \( U''(x) = g'(x) \). Applying Ito's formula to the transformation \( Y_t = U(W_t) \), then

\[
U(W_T) - U(W_0) = \int_{0}^{T} g(W_t) dW_t + 0.5 \int_{0}^{T} g'(W_t) dt
\]  
(12)

Thus the STSI:

\[
\int_{0}^{T} g(W_t) \circ dW_t = U(W_T) - U(W_0)
\]  
(13)

as in classical calculus and the classical transformation rules. The main advantage of the STSI is that normal transformation rules of calculus are considered.

The difference between ISI and STSI and SDEs resulted from a unique feature that the sample paths of a WPs are non-differentiable or even with limited variation. This is probably not that odd if these equations resulted from adding random fluctuations represented by a Gaussian white noise (GWN) to ordinary differential equations. A GWN can be considered the (nonexistent) derivative of a WP. In fact, white noises are also used for idealizing true colored noise processes, for which the autocorrelation is made arbitrarily small at different time instants[1].

Ito and Stratonovich stochastic calculus provide the mathematical valid formulations for SDEs, but the question of which interpretation should be used. Actually, it depends on how exactly the white noise processes approximate the real noise processes and on how the modelled real situation is approximated by the SDE[8].

4. Milstein Algorithm

The different numerical methods are used to approximate the SDE’s solutions because it is very hard to get it analytically or by normal rules of classical calculus. For Milstein algorithm, which has a strong order of convergence \( \gamma = 1 \). For the one-dimensional case with \( d = m = 1 \), a new term

\[
0.5 \beta (M_{j-1})(\beta (M_{j-1}))'((\Delta W)^2 - \Delta t)
\]

will be added to Euler algorithm formula:

\[
M_j = M_{j-1} + \alpha (M_{j-1})\Delta t + \beta (M_{j-1})\Delta W; J = 1,2,3,...,N
\]  
(14)

Thus, the form of Milstein algorithm is:

\[
M_j = M_{j-1} + \alpha (M_{j-1})\Delta t + \beta (M_{j-1})\Delta W + 0.5 \beta (M_{j-1})(\beta (M_{j-1}))'((\Delta W)^2 - \Delta t)
\]  
(15)

A convergence improvement is obtained by adding a term only to the Euler algorithm to produce the Milstein algorithm which improve the convergence order from \( \gamma = 0.5 \) to be \( \gamma = 1 \). With \( \gamma = 1 \), the Milstein algorithm match up with the Euler algorithm in the deterministic case but without any noise (\( \beta \equiv 0 \)). The additional term of the Milstein algorithm marked the divergence between stochastic and deterministic numerical analysis. In that context, the Milstein scheme can be considered as an appropriate generalization of Euler’s deterministic approach, given that it provides the same strong convergence as for the deterministic case[7].

As the WBs is nowhere differentiable with probability equal one, ISI will be used to get the second integral appeared in equation (2). To solve SDE numerically over the interval[0, \( T \)], a discretization of the interval is required first. For \( \Delta t = T/Q \) where Q is the discretization numbers. We let \( W(t_q) \) denote a value of the WB at \( \tau_q = q\Delta t \) with initial value \( W_0 = 0 \).[9]

So,

\[
W_{t_q} = W_{t_{q-1}} + dW_{t_q}; q = 1,2,...,Q
\]
Now, Milstein algorithm has the formula:

\[
M_j = M_{j-1} + \alpha (M_{j-1}) \Delta t + \beta (M_{j-1}) \Delta W + 0.5 \beta (M_{j-1}) (\beta (M_{j-1}))' ((\Delta W)^2 - \Delta t)
\]  (16)

where \( \Delta W = W(q + 1) - W(q) \).

To see the difference between Euler Maruyama and Milstein methods, the solution of Black Scholes diffusion equation (4) with \( \rho = 2 \) and \( \sigma = 3 \) has been approximated by both methods and compared with the analytical solution[7].

![Figure 2. Comparison between the exact, Milstein and Euler-Maruyama solution](image)

For a mobile robot, the construction of the robot’s mathematical model is very important in system design. This mathematical model convert the robot’s velocity into a generalized coordinate vector as shown[10]:

\[
x_{t+1} = x_t + V \Delta t \cos \theta_t \\
y_{t+1} = y_t + V \Delta t \sin \theta_t \\
\theta_{t+1} = \theta_t + \Delta t \omega
\]  (17)

![Figure 3. Robot model](image)

where the velocity given by the equation[11]:
The velocity of the right and the left wheel.

\[ V = r \left( \frac{w_r + w_l}{2} \right) \]
\[ w = r \left( \frac{w_r - w_l}{l} \right) \] (18)

The parameters in equation (18) are:
- \( w_r, w_l \): The velocity of the right and the left wheel.
- \( x, y \): The robot’s position relative to inertia frame.
- \( \theta \): The robot’s heading angle relative to inertia frame.
- \( V \): The robot’s linear velocity.
- \( w \): The robot’s angular velocity
- \( l \): Distance from left to right wheel
- \( r \): Wheel radius

In this work, a random heading angle is take place in kinematics equations. So, additional term \( W(t) \) WB has been added to the equation of heading angle which is converted into SDE. Thus, the kinematic model will be[12]:

\[ X_{t+1} = X_t + V \Delta t \cos \theta_t \]
\[ Y_{t+1} = Y_t + V \Delta t \sin \theta_t \]
\[ \theta_{t+1} = \theta_t + \Delta t w + W_t \] (19)

Now, a simulation for the kinematic model equations is placed to predict the future position of robot using Milstein algorithm by aid of MATLAB program to improve the solution approximated by Euler-Maruyama algorithm. The simulation parameters are[12]:

\[ w_r = 4 \text{ rad/s} , w_l = 2 \text{ rad/s} \]
\[ X_o = 1 , \quad y_o = 3 \]
\[ \theta_o = \frac{\pi}{8} \text{ rad} , \quad l = 20 \text{ cm} , \quad r = 20 \text{ cm} \] (20)

The discretized WB path is calculated over the interval \([0,1]\) with \( \delta t = 2^{-6} \) which is shown in figure (4).

![Figure 4. Milstein algorithm with \( \delta t = 2^{-6} \)]
Then we use different step size $\Delta t$ with different $\delta t = 2^{-4}$ which shown in figure (5).

![Figure 5. Milstein algorithm with $\delta t = 2^{-4}$](image)

From figures (4) and (5) the error decreased as we increase the number of points in the discretized path.

A comparison between the solution obtained by Milstein algorithm and a robot simulation by MATLAB along a predetermined path with the same parameters is shown in figure (6).

![Figure 6. Milstein and MATLAB robot simulation](image)

In figure (6), there is an error between the solution approximated by Milstein algorithm and the robot simulation by the MATLAB program in different waypoints and also in the destination point which is caused by the stochastic nature of the heading angle so it has an impact in robot approximated position in different waypoints during the time interval.
But in comparison with the solution approximated by Euler-Maruyama for the same simulation parameters, the error is decreased along the whole path and the solution accuracy is more than the solution obtained by Euler-Maruyama method as seen in figure(7)[12].

5. Conclusion
In this paper, the SDE with its main definitions and properties have been discussed in details. A solution of the Black Scholes equation is approximated by two numerical methods (Milstein and Euler-Maruyama) and compared with the exact solution. It is founded that the error due to Milstein is less than Euler-Maruyama. Then, Milstein algorithm applied to the mobile robot to estimate and predict the mobile robot future position to compensate the time delay problem in teleoperation with the aid of MATLAB program in calculations and simulation. The error between the robot simulation by MATLAB (ideal case) and the solution approximated by Milstein algorithm is a result of the stochastic nature of the robot’s heading angle. In comparison with the approximated solution by Euler-Maruyama with the same simulation parameters, the error has been decreased and improved in many parts of the robot path and also can be improved more by using other numerical methods of a higher order of convergence than one.

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