FINITE SPEED OF PROPAGATION AND OFF-DIAGONAL BOUNDS FOR ORNSTEIN-UHLENBECK OPERATORS IN INFINITE DIMENSIONS

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Abstract. We study the Hodge-Dirac operators $D$ associated with a class of non-symmetric Ornstein-Uhlenbeck operators $L$ in infinite dimensions. For $p \in (1, \infty)$ we prove that $iD$ generates a $C_0$-group in $L^p$ with respect to the invariant measure if and only if $p = 2$ and $L$ is self-adjoint. An explicit representation of this $C_0$-group in $L^2$ is given and we prove that it has finite speed of propagation. Furthermore we prove $L^2$ off-diagonal estimates for various operators associated with $L$, both in the self-adjoint and the non-self-adjoint case.

1. Introduction

In this paper we establish analogues of several well-known $L^p$-results for the wave group $(e^{it\sqrt{-\Delta}})_{t \geq 0}$, the Schrödinger group $(e^{it\Delta})_{t \geq 0}$, and the heat semigroup $(e^{t\Delta})_{t \geq 0}$ by replacing the Laplace operator $\Delta$ by a (possibly infinite-dimensional and non-symmetric) Ornstein-Uhlenbeck operator. Our principal tool is the first-order approach introduced by Axelsson, Keith, and McIntosh [9] and developed in many recent papers [3, 4, 5, 6, 7, 8, 25, 35, 36, 49], which looks at these objects through the functional calculus of Hodge-Dirac operators such as

$D := \begin{bmatrix} 0 & -\text{div} \\ \nabla & 0 \end{bmatrix}$

acting on the direct sum $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d; \mathbb{C}^d)$. This approach has already been used in the Ornstein-Uhlenbeck context in [42, 43] to obtain necessary and sufficient conditions for the $L^p$-boundedness of Riesz transforms. The relevant Hodge-Dirac operator is given by

$\mathcal{D} := \begin{bmatrix} 0 & \nabla_H B \\ \nabla_H & 0 \end{bmatrix}$,

acting on $L^2(E, \mu) \oplus L^2(E, \mu; H)$, where $E$ is a Banach space, $\mu$ is an invariant measure on $E$, $H$ is a Hilbert subspace of $E$, $\nabla_H$ is the gradient in the direction of $H$, and $B$ is a bounded linear operator acting on $H$ (see Section 2 for precise...
The corresponding Ornstein-Uhlenbeck operator is then given by
\[ \mathcal{L} = -\frac{1}{2} \nabla_H B \nabla_H. \]

The first result we prove is a version for Ornstein-Uhlenbeck operators of the following theorem on \( L^p \)-extendability of the wave group. It can be viewed as an analogue of the classical result of Hörmander [34] (see also [2, Theorem 3.9.4]) stating that the Schrödinger group \( (e^{it\Delta})_{t \in \mathbb{R}} \) extends to \( L^p(\mathbb{R}^d) \) if and only if \( p = 2 \).

**Theorem 1.1.** Let \( 1 < p < \infty \) and \( d \geq 1 \). The following assertions are equivalent:
(i) the operator \( i\sqrt{-\Delta} \) generates a \( C_0 \)-group on \( L^p(\mathbb{R}^d) \);
(ii) \( p = 2 \) or \( d = 1 \).

This equivalence is due to Littman [40]; a proof by Fourier multiplier methods can be found in [2, Theorem 8.3.13].

Theorem 1.1 shows that, even in the setting of \( \mathbb{R}^n \) and the Euclidean Laplacian, simple oscillatory Fourier multipliers can fail to be bounded in \( L^p \) for \( p \neq 2 \). The study of such operators that are beyond the reach of classical results on Fourier multipliers such as the Mihlin-Hörmander theorem, is an important objective of Fourier integral operator theory. One of the first results in this direction is the following theorem of Miyachi [51, Corollary 1] and Peral [55], that shows that a suitably regularised version of the wave group is \( L^p \)-bounded.

**Theorem 1.2.** Let \( 1 < p < \infty \), and fix \( \lambda > 0 \). The regularised operators
\[ (\lambda - \Delta)^{-\alpha/2} \cos(t\sqrt{-\Delta}), \quad t \in \mathbb{R}, \]
are bounded on \( L^p(\mathbb{R}^d) \) if and only if \( \alpha \geq (d - 1)|\frac{1}{p} - \frac{1}{2}| \).

This result has been extended in many directions, and included in a general theory of Fourier integral operators (see, in particular, the celebrated paper by Seeger, Sogge, and Stein [57], and Section IX.5 of Stein’s book [59]).

Our paper is part of a long term programme (see also the Hardy space theory developed in [46, 47, 48] and [44, 45, 56]) to expand harmonic analysis of Ornstein-Uhlenbeck operators beyond Fourier multipliers and towards Fourier integral operators. We first remark that no analogue of Miyachi-Peral’s result can be found in [2, Theorem 8.3.13] and [Peral [55]] (except if \( p = 2 \)). This is related to the fact that there are no Sobolev embeddings in the Ornstein-Uhlenbeck context, and, in a sense, no non-holomorphic functional calculus in \( L^p \) for \( p \neq 2 \) (see [33]).

Perhaps surprisingly (given that our space of variables is not geometrically doubling), we can nonetheless establish the fundamental estimates that underpin spectral multiplier theory (see e.g. [13] and the references therein), namely the finite speed of propagation of \( (e^{it\mathcal{L}})_{t \in \mathbb{R}} \), and the \( L^2 - L^2 \) off-diagonal bounds of Davies-Gaffney type for \( (e^{it\mathcal{L}})_{t \geq 0} \). The former generalises to the Ornstein-Uhlenbeck context the following classical result for the wave group. Let \( D \) be the Dirac operator on \( L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d, \mathbb{C}^d) = L^2(\mathbb{R}^d; \mathbb{C}^{d+1}) \) defined by (1.1).

**Theorem 1.3.** The \( C_0 \)-group \( (e^{itD})_{t \in \mathbb{R}} \) on \( L^2(\mathbb{R}^d; \mathbb{C}^{d+1}) \) has unit speed of propagation, meaning that, if \( f \in L^2(\mathbb{R}^d; \mathbb{C}^{d+1}) \) is supported in a set \( K \), then \( e^{itD}f \) is supported in \( \{ x \in \mathbb{R}^d : \text{dist}(x, K) \leq |t| \} \).
The $L^2$-$L^2$ off-diagonal estimates (which can be deduced from Theorem 1.3) are integrated heat kernel bounds such as

$$
\|1_G e^{t\Delta}(1_F u)\|_2 \lesssim \exp \left(-\frac{d(F,G)^2}{t}\right)\|u\|_2,
$$

for $F, G \subseteq \mathbb{R}^d$, $u \in L^2(\mathbb{R}^d)$, and $t > 0$. These bounds play a key role in spectral multiplier theory, but hold far more generally than standard pointwise heat kernel bounds (which do not hold, in particular, for Ornstein-Uhlenbeck operators, even in finite dimension).

In a future project, we plan to use the off-diagonal estimates, together with the aforementioned Hardy space theory, to study perturbations of Ornstein-Uhlenbeck operators arising from non-linear stochastic PDE.

Let us now turn to a summary of the results of this paper. After a brief introduction to Ornstein-Uhlenbeck operators $L$ in an infinite-dimensional setting in Section 2, we begin in Section 3 by proving analogues of Theorems 1.1 and 1.2 for the operators $iL$. This is somewhat easier than proving analogues for $i\sqrt{-L}$, which is done in Section 4. Roughly speaking, we find that both $iL$ and $i\sqrt{-L}$ generate groups in $L^p$ with respect to the invariant measure if and only if $p = 2$ and $L$ is self-adjoint. Moreover, in contrast with the Euclidean case, we show that no amount of resolvent regularisation will push the groups into $L^p$.

We turn to the analogue of Theorem 1.3 in Section 5 and prove that the group generated $iD$ has finite speed of propagation, whereas the group generated by $iL$ does not. To the best of our knowledge, the former is the first result of this kind in an infinite-dimensional setting.

In Section 6, we prove $L^2$-$L^2$ off-diagonal bounds for various operators associated with $\mathcal{L}$, such as $e^{t\mathcal{L}}$ and $\nabla_He^{t\mathcal{L}}$, where $\nabla_H$ is a suitable directional gradient introduced in Section 2. In the symmetric case, this is done as an application of finite speed of propagation, and the off-diagonal bounds are of Gaffney-Davies type. In the non-symmetric case, we obtain off-diagonal bounds for the resolvent operators $(I - t^2\mathcal{L})^{-1}$ by a direct method.

2. Non-symmetric Ornstein-Uhlenbeck operators

We begin by describing the setting that we will be using throughout the paper. We fix a real Banach space $E$ and a real Hilbert space $H$, which is continuously embedded in $E$ by means on an inclusion operator

$$
i_H : H \hookrightarrow E.
$$

Identifying $H$ with its dual via the Riesz representation theorem, we define $Q_H := i_H \circ i_H^*$. Let $S = (S(t))_{t \geq 0}$ be a $C_0$-semigroup on $E$ with generator $A$.

Assumption 2.1. There exists a centred Gaussian Radon measure $\mu$ on $E$ whose covariance operator $Q_\mu \in \mathcal{L}(E^*, E)$ is given by

$$
\langle Q_\mu x^*, y^* \rangle = \int_0^\infty \langle Q_H S(s)x^*, S(s)y^* \rangle \, ds, \quad x^*, y^* \in E^*,
$$

the convergence of the integrals on the right-hand side being part of the assumption.

The relevance of Assumption 2.1 is best explained in terms of its meaning in the context of stochastic evolution equations. For this we need some terminology. Let $W_H$ be an $H$-cylindrical Brownian motion on an underlying probability space $(\Omega, \mathcal{F})$. 
By definition, this means that $W_H$ is a bounded linear operator from $L^2(\mathbb{R}_+; H)$ to $L^2(\Omega)$ such that for all $f, g \in L^2(\mathbb{R}_+; H)$ the random variables $W_H(f)$ and $W_H(g)$ are centred Gaussian variables and satisfy
\[ \mathbb{E}(W_H(f)W_H(g)) = \langle f, g \rangle, \]
where $\langle f, g \rangle$ denotes the inner product of $f$ and $g$ in $L^2(\mathbb{R}_+; H)$. The operators $W_H(t) : H \to L^2(\Omega)$ defined by $W_H(t)h := W_H(1_{[0,t]} \otimes h)$ are then well defined, and for each $h \in H$ the family $(W(t)h)_{t \geq 0}$ is a Brownian motion; it is a standard Brownian motion if the vector $h$ has norm one. Moreover, for orthogonal unit vectors $h_n$, the Brownian motions $(W(t)h_n)_{t \geq 0}$ are independent. For more information the reader is referred to [52].

It is well known that Assumption 2.1 holds if and only if the linear stochastic evolution equation
\begin{equation}
(\text{SCP}) \quad dU(t) = AU(t) + i_H \, dW_H(t), \quad t \geq 0,
\end{equation}
is well-posed and admits an invariant measure. More precisely, under Assumption 2.1 the problem (SCP) is well-posed and the measure $\mu$ is invariant, and conversely if (SCP) is well-posed and admits an invariant measure, then Assumption 2.1 holds and the measure $\mu$ is invariant for (SCP). In particular, if (SCP) has a unique invariant measure, it must be the measure $\mu$ whose existence is guaranteed by Assumption 2.1. Details may be found in [19, 54], where also the rigorous definitions are provided for the notions of solution and invariant measure for (SCP).

Remark 2.2. More generally one may consider Ornstein-Uhlenbeck operators associated with the problem
\begin{equation}
(\text{SCP}) \quad dU(t) = AU(t) + \sigma \, dW_H(t), \quad t \geq 0,
\end{equation}
where $\sigma : H \to E$ is a given bounded operator. This does not add any generality, however, as can be seen from the following reasoning. First, by the properties of the Itô stochastic integral, replacing $H$ by $H \ominus \text{N}(\sigma)$ (the orthogonal complement of the kernel of $\sigma$) affects neither the solution process $(U(t, x))_{t \geq 0}$ nor the invariant measure $\mu$, and therefore this replacement leads to the same operator $\mathcal{L}$. Thus we may assume $\sigma$ to be injective. But once we have done that, we may identify $H$ with its image $\sigma(H)$ in $E$, which amounts to replacing $\sigma$ by the inclusion mapping $i_{\sigma(H)}$ of $\sigma(H)$ into $E$.

In what follows, Assumption 2.1 will always be in force even if it is not explicitly mentioned. Let $(U(t, x))_{t \geq 0}$ denote the solution of (SCP) with initial value $x \in E$. The formula
\[ P(t)f(x) := \mathbb{E}(f(U(t, x))), \quad t \geq 0, \quad x \in E, \]
defines a semigroup of linear contractions $P = (e^{t\mathcal{L}})_{t \geq 0}$ on the space $B_b(E)$ of bounded scalar-valued Borel functions on $E$, the so-called Ornstein-Uhlenbeck semigroup associated with the data $(A, H)$. By Jensen’s inequality, this semigroup extends to a $C_0$-semigroup of contractions on $L^p(E, \mu)$. Its generator will be denoted by $\mathcal{L}$, and henceforth we shall write $P(t) = e^{t\mathcal{L}}$ for all $t \geq 0$.

In most of our results we will make the following assumption.

Assumption 2.3. For some (equivalently, for all) $1 < p < \infty$ the semigroup $(e^{t\mathcal{L}})_{t \geq 0}$ extends to an analytic $C_0$-semigroup on $L^p(E, \mu)$. 
Here we should point out that, although the underlying spaces $E$ and $H$ are real, function spaces over $E$ will always be taken to be complex. The independence of $p \in (1, \infty)$ is a consequence of the Stein interpolation theorem.

The problem of analyticity of $\left( e^{t\mathcal{L}} \right)_{t \geq 0}$ has been studied by various authors in [26, 28, 30, 41]. In these papers, various necessary and sufficient conditions for analyticity were obtained. Analyticity always fails for $p = 1$; this observation goes back to [20] where it was phrased for the harmonic oscillator; the general case follows from [14, 41].

Under Assumption 2.3 it is possible to represent $\mathcal{L}$ in divergence form. For the precise statement of this result we need to introduce the following terminology. A $C^1_b$-cylindrical function is a function $f : E \to \mathbb{R}$ of the form

$$f(x) = \phi((x, x^*_1), \ldots, (x, x^*_n))$$

for some $n \geq 1$, with $x^*_j \in E^*$ for all $j = 1, \ldots, n$ and $\phi \in C^1_b(\mathbb{R}^n)$. The gradient in the direction of $H$ of such a function is defined by

$$\nabla_H f(x) := \sum_{j=1}^n \frac{\partial \phi}{\partial x_j}(((x, x^*_1), \ldots, (x, x^*_n)) i_H^* x_j, x \in E.$$

If $\left( e^{t\mathcal{L}} \right)_{t \geq 0}$ is analytic on $L^p(E, \mu)$ for some/all $1 < p < \infty$, then $\nabla_H$ is closable as a densely defined operator from $L^p(E, \mu)$ to $L^p(E, \mu; H)$ [30, Proposition 8.7]. In what follows, $\nabla_H$ will always denote this closure and $D_p(\nabla_H)$ and $R_p(\nabla_H)$ denote its domain and range. For $p = 2$ we usually omit the subscripts and write $D(\nabla_H) = D_2(\nabla_H)$ and $R(\nabla_H) = R_2(\nabla_H)$.

It was shown in [41] that if $\left( e^{t\mathcal{L}} \right)_{t \geq 0}$ is analytic on $L^2(E, \mu)$, then $-\mathcal{L}$ admits the ‘gradient form’ representation

$$-\mathcal{L} = \frac{1}{2} \nabla_H^* B \nabla_H$$

for a unique bounded operator $B \in \mathcal{L}(H)$ which satisfies

$$B + B^* = 2I.$$ 

Note that this identity implies the coercivity estimate $\langle Bh, h \rangle_H \geq \|h\|^2_H$ for all $h \in H$.

The rigorous interpretation of (2.1) is that for $p = 2$ the operator $-\mathcal{L}$ is the sectorial operator associated with the sesquilinear form

$$(f, g) \mapsto \frac{1}{2} \langle B \nabla_H f, \nabla_H g \rangle.$$ 

Therefore $\mathcal{L}$ generates an analytic $C_0$-semigroup of contractions on $L^2(E, \mu)$.

It is not hard to show (see [30]) that

$$\mathcal{L}$$

is self-adjoint on $L^2(E, \mu)$ if and only if $B = I_H$ where $I_H$ is the identity operator on $H$. In that case we have $D(\sqrt{-\mathcal{L}}) = D(\nabla_H)$ and

$$\|\sqrt{-\mathcal{L}} f\|_2^2 = \frac{1}{2} \|\nabla_H f\|_2^2.$$

Remark 2.4. Necessary and sufficient conditions for equivalence of homogeneous norms $\|\sqrt{-\mathcal{L}} f\|_p \approx \|\nabla_H f\|_p$ in the non-symmetric case have been obtained in
[42], thereby unifying earlier results for the symmetric case in infinite dimensions [16, 58] and the non-symmetric case in finite dimensions [50].

3. The $C_0$-group generated by $i\mathcal{L}$

We start with an analogue of Hörmander’s theorem:

**Theorem 3.1.** Let Assumptions 2.1 and 2.3 hold and let $1 \leq p < \infty$. The operator $i\mathcal{L}$ generates a $C_0$-group on $L^p(E, \mu)$ if and only if $p = 2$ and $\mathcal{L}$ is self-adjoint on $L^2(E, \mu)$.

**Proof.** Let $1 < p < \infty$ be fixed and suppose that $i\mathcal{L}$ generates a $C_0$-group on $L^p(E, \mu)$. Then, by [2, Corollary 3.9.10], the semigroup $(e^{t\mathcal{L}})_{t \geq 0}$ on $L^p(E, \mu)$ generated by $\mathcal{L}$ is analytic of angle $\pi/2$ and the group generated by $i\mathcal{L}$ is its boundary group, i.e.,

$$e^{it\mathcal{L}} f = \lim_{\varepsilon \to 0} e^{i(s+\varepsilon t)\mathcal{L}} f$$

for all $f \in L^p(E, \mu)$. But it is well known [14, 41] that $(e^{t\mathcal{L}})_{t \geq 0}$ fails to be analytic on $L^1(E, \mu)$ and that for $1 < p < \infty$ the optimal angle of analyticity $\theta_p$ of $(e^{t\mathcal{L}})_{t \geq 0}$ in $L^p(E, \mu)$ is given by

$$\cot \theta_p := \frac{\sqrt{(p-2)^2 + p^2 \| B - B^* \|^2}}{2\sqrt{p-1}},$$

with $B \in \mathcal{L}'(H)$ the operator appearing in (2.1). If either $p \neq 2$ or $B \neq B^*$, this angle is strictly less than $\pi/2$. □

**Remark 3.2.** An alternative proof of self-adjointness can be given that does not rely on the formula (3.1) for the optimal angle. It relies on the following result on numerical ranges. If $G$ is the generator of a $C_0$-semigroup on a complex Hilbert space $\mathcal{H}$ such that $(Gx, x) \in \mathbb{R}$ for all $x \in \mathcal{D}(G)$, then $G$ is self-adjoint. Indeed, for any $\lambda \in \mathbb{R}$ the operator $\lambda - G$ has real numerical range. Therefore, for any real $\lambda \in \rho(G)$ the resolvent operator $R(\lambda, G)$ has real numerical range. Hence, by [31, Theorem 1.2-2], $R(\lambda, G)$ is self-adjoint, and then the same is true for $G$.

Now let us revisit the proof of self-adjointness in the theorem for $p = 2$. By second quantisation [15, 30], the analytic semigroup generated by $\mathcal{L}$ on $L^2(E, \mu)$ is contractive in the right half-plane $\{ z \in \mathbb{C} : \text{Re} z > 0 \}$. By general semigroup theory (see, e.g., [32, Proposition 7.1.1]), this implies that the numerical range of $\mathcal{L}$ is contained in $(-\infty, 0]$. By the observation just made, this implies that $\mathcal{L}$ is self-adjoint on $L^2(E, \mu)$.

Not only does $i\mathcal{L}$ fail to generate a $C_0$-group on $L^p(E, \mu)$ unless $p = 2$ and $\mathcal{L}$ is self-adjoint, but the situation is in fact worse than that. As we will see shortly, for any given $\lambda > 0$ and $\alpha > 0$, the regularised operators

$$(\lambda - \mathcal{L})^{-\alpha} e^{it\mathcal{L}}$$

fail to extend to bounded operators on $L^p(E, \mu)$, unless $p = 2$ and $\mathcal{L}$ is self-adjoint. This result contrasts with the analogous situation for the Laplace operator: it is a classical result of Lanconelli [39] (see also Da Prato and Giusti [18] for integer values of $\alpha$) that the regularised Schrödinger operators $(\lambda - \Delta)^{-\alpha} e^{it\Delta}$ are bounded on $L^p(\mathbb{R}^d)$ for all $\alpha > n \left| \frac{1}{p} - \frac{1}{2} \right|$.

With regard to the rigorous statement of our result there is a small issue here in the non-self-adjoint case, for then it is not even clear how to define these operators.
for $p = 2$. We get around this in the following way. Any reasonable definition should respect the identity

$$e^{sL}[\lambda - \mathcal{L}]^{-\alpha} e^{it\mathcal{L}} = (\lambda - \mathcal{L})^{-\alpha} e^{(s + it)\mathcal{L}}, \quad s > 0.$$ 

More precisely, it should be true that the mapping $z \mapsto (\lambda - \mathcal{L})^{-\alpha} e^{z\mathcal{L}}$ is holomorphic in $\{\text{Re}z > 0\}$ and that the above identity holds. In the converse direction, if the mapping $z \mapsto (\lambda - \mathcal{L})^{-\alpha} e^{z\mathcal{L}}$ (which is well-defined and holomorphic on an open sector about the positive real axis) extends holomorphically to a function $F_\alpha$ on $\{\text{Re}z > 0\}$ which is bounded on every bounded subset of this half-plane, then by general principles the strong non-tangential limits $\lim_{s \downarrow 0} F_\alpha(s + it)$ exist for almost all $t \in \mathbb{R}$. For these $t$ we may define the operators $(\lambda - \mathcal{L})^{-\alpha} e^{it\mathcal{L}}$ to be this limit. In what follows, “boundedness of the operators $(\lambda - \mathcal{L})^{-\alpha} e^{it\mathcal{L}}$ in $L^p(E, \mu)$” will always be understood in this sense.

This procedure defines the operators for almost all $t \in \mathbb{R}$. As a side-remark we mention that this can be improved by using a version of the argument in [2, Proposition 9.16.5]. For $\beta \geq \alpha$ let $G_\beta$ be the set of full measure for which the non-tangential strong limits $\lim_{s \downarrow 0} F_\alpha(s + it)$ exist. We claim that $G_\beta = \mathbb{R}$ for all $\beta \geq 2\alpha$. To prove this, first observe that for all $\gamma' > \gamma > \alpha$ we have $G_\gamma \subseteq G_{\gamma'}$ and $G_{\gamma} + G_{\gamma'} \subseteq G_{\gamma + \gamma'}$. If the claim were wrong, then there would be a $t \in \mathbb{C} \setminus G_\beta$ for some $\beta \geq 2\alpha$. But then for any $t' \in G_{\frac{1}{2}\beta}$ we have $t - t' \in \mathbb{C} \setminus G_{\frac{1}{2}\beta}$, otherwise the identity $t = t' + (t - t')$ implies $t \in G_{\frac{3}{2}\beta} = G_{\frac{1}{2}\beta} \subseteq G_\beta$. This contradiction concludes the proof of the claim.

**Theorem 3.3.** Let Assumptions 2.1 and 2.3 hold and let $1 < p < \infty$. If, for some $\lambda > 0$ and $\alpha > 0$, the operators $(\lambda - \mathcal{L})^{-\alpha} e^{s\mathcal{L}}$, $t \in \mathbb{R}$, are bounded in $L^p(E, \mu)$, then $p = 2$ and $\mathcal{L}$ is self-adjoint.

**Proof.** For all $s > 0$, the operators $(\lambda - \mathcal{L})^\alpha e^{s\mathcal{L}}$ are bounded in $L^p(E, \mu)$ by the analyticity of the semigroup $(e^{t\mathcal{L}})_{t \geq 0}$. The assumptions of the theorem then imply that the operators

$$e^{(s + it)\mathcal{L}} = (\lambda - \mathcal{L})^\alpha e^{s\mathcal{L}} \circ (\lambda - \mathcal{L})^{-\alpha} e^{it\mathcal{L}}$$

are bounded on $L^p(E, \mu)$ for all $s > 0$ and $t \in \mathbb{R}$, in the sense that the right-hand side provides us with an analytic extension of $t \mapsto e^{t\mathcal{L}}$ to $\{\text{Re}z > 0\}$. But, as was observed in the proof of Theorem 3.1, for $p \neq 2$ and $B \neq B^*$ the optimal angle of holomorphy of this semigroup is strictly smaller than $\pi/2$. □

**Remark 3.4.** The ‘exponentially regularised’ operators $e^{s\mathcal{L}} e^{it\mathcal{L}}$ extend to $L^p(E, \mu)$ if $s + it$ belongs to the connected component of the domain of analyticity in $L^p(E, \mu)$ of $z \mapsto e^{z\mathcal{L}}$ which contains the positive real axis. For the standard Ornstein-Uhlenbeck operator in finite dimensions (see (5.1) for its definition), this is the Epperson region

$$E_p = \{x + iy \in \mathbb{C} : |\sin y| \leq \tan \theta_p \sinh x\},$$

where $\theta_p = \arccos |2/p - 1|$ [23, Theorem 3.1] (see also [27, Proposition 1.1]). It contains the right-half plane $\{z \in \mathbb{C} : \text{Re}z > s_p\}$ for a suitable abscissa $s_p > 0$.

Hence, for all $s > s_p$, the operators $e^{s\mathcal{L}} e^{it\mathcal{L}}$, $t \in \mathbb{R}$, extend to $L^p(E, \mu)$.

In the general case, a similar conclusion can be drawn in the presence of hypercontractivity (which holds if Assumption 5.3 below is satisfied, see [17]). In that case the operators $e^{s\mathcal{L}} e^{it\mathcal{L}}$ are bounded on $L^p(E, \infty)$ for all $s > s_p^*$ and $t \in \mathbb{R}$,
where $s^*_p > 0$ is the infimum of all $s > 0$ with the property that $e^{sF}$ maps $L^p(E, \mu)$ into $L^2(E, \mu)$ (if $1 < p < 2$), respectively $L^2(E, \mu)$ into $L^p(E, \mu)$ (if $2 < p < \infty$).

4. The $C_0$-groups generated by $i\sqrt{-\nabla^2}$ and $i\mathcal{D}$

Throughout this section, Assumptions 2.1 and 2.3 are in force. On the direct sum $L^p(E, \mu) \oplus L^p(E, \mu; H)$, $1 < p < \infty$, we introduce the 	extit{Hodge-Dirac operator}

$$\mathcal{D} := \begin{bmatrix} 0 & \nabla_H B \\ \nabla_H & 0 \end{bmatrix}.$$ 

Hodge-Dirac operators have their origins in Dirac’s desire to use first-order operators that square to the Laplacian. They are commonly used in Riemannian geometry, where they arise as $d + d^*$ for the exterior derivative $d$. In their influential paper [9], Axellson, Keith, and McIntosh have introduced a general operator theoretic framework that allows one to transfer ideas used in geometry to problems in harmonic analysis and PDE related to Riesz transform estimates. For Ornstein-Uhlenbeck operators, this perspective has been introduced in [42].

On various occasions we will use the fact (see [9]) that $\mathcal{D}$ is bisectorial on $L^2(E, \mu) \oplus L^2(E, \mu; H)$. We recall that a closed operator $A$ is called 	extit{bisectorial} if $i\mathbb{R} \setminus \{0\} \subseteq \rho(A)$ and

$$\sup_{t \neq 0} \| (I + itA)^{-1} \| < \infty.$$ 

For some background on bisectoriality we recommend the lecture notes [1] and Duelli’s Ph.D. thesis [22].

Note the formal identity

$$\frac{1}{2} \mathcal{D}^2 = \frac{1}{2} \begin{bmatrix} -\nabla_H^* B \nabla_H & 0 \\ 0 & -\nabla_H \nabla_H^* B \end{bmatrix} = \begin{bmatrix} -\mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{bmatrix}.$$ 

Here, the operator $\mathcal{L} := -\frac{1}{2} \nabla_H \nabla_H^* B$ is defined as follows. First, we define $\nabla_H \nabla_H^*$ on $L^2(E, \mu; H)$ by means of the form $(u, v) \mapsto \langle \nabla_H u, \nabla_H^* v \rangle$, and use this operator to define $-\frac{1}{2} \nabla_H \nabla_H^* B$ in the natural way on the domain $\text{Dom}(\mathcal{L}) = \{ u \in L^2(E, \mu; H) : Bu \in \text{Dom}(\nabla_H \nabla_H^*) \}$. The operator $\mathcal{L}$ generates a bounded analytic $C_0$-semigroup on $L^2(E, \mu; H)$ and we have

$$e^{t\mathcal{L}}\nabla_H = \nabla_H e^{t\mathcal{L}}.$$ 

This identity implies that $(e^{t\mathcal{L}})_{t \geq 0}$ restricts to a bounded analytic $C_0$-semigroup on $\text{R}(\nabla_H)$.

The situation for $1 < p < \infty$ is slightly more subtle. The semigroup $(e^{t\mathcal{L}})_{t \geq 0}$ on $\text{R}(\nabla_H)$ can be shown to extend to a bounded analytic $C_0$-semigroup on $\text{R}_p(\nabla_H)$. We then define $\mathcal{L}$ on $\text{R}_p(\nabla_H)$ as its generator. This suggests to consider the part of the Dirac operator $\mathcal{D}$ in $L^p(E, \mu) \oplus \text{R}(\nabla_H)$, and indeed it can be shown that this operator is bisectorial on $L^p(E, \mu) \oplus \text{R}(\nabla_H)$. The reader is referred to [42] for the details. If $\mathcal{L}$ has a bounded $H^\infty$-calculus on $\text{R}_p(\nabla_H)$ (this is the case if $E = H = \mathbb{R}^d$ and also if $\mathcal{L}$ is self-adjoint on $L^2(E, \mu)$), then it follows from the second part of [42, Theorem 2.5] that $\mathcal{D}$ is bisectorial on all of $L^p(E, \mu) \oplus L^p(E, \mu; H)$.

If $\mathcal{D}$ is self-adjoint on the direct sum $L^2(E, \mu) \oplus L^2(E, \mu; H)$, then $i\mathcal{D}$ generates a bounded $C_0$-group on this space by Stone’s theorem. In the non-self-adjoint case, one may ask whether it is still true that $i\mathcal{D}$ generates a $C_0$-group on $L^p(E, \mu) \oplus L^p(E, \mu; H)$ for certain exponents $1 < p < \infty$. In the light of the above discussion we
have to be a little cautious as to the precise meaning of this question; we ask whether the restriction of \((e^{it\mathcal{D}})_{t \in \mathbb{R}}\) to \([L^2(E,\mu) \oplus L^2(E,\mu;H)] \cap [L^p(E,\mu) \oplus L^p(E,\mu;H)]\) extends to a \(C_0\)-group on \(L^p(E,\mu) \oplus L^p(E,\mu;H)\). Alternatively, one may ask whether \(i\mathcal{D}\) generates a \(C_0\)-group on \(L^p(E,\mu) \oplus \overline{\mathbb{R}(\nabla_H)}\). In this formulation of the question one may interpret \(\mathcal{D}\) as the bisectorial operator on \(L^p(E,\mu) \oplus \overline{\mathbb{R}(\nabla_H)}\) as outlined above.

In the one-dimensional Euclidean situation, \((e^{it\mathcal{D}})_{t \in \mathbb{R}}\) can be expressed in terms of the translation group. This suggests that the answer to both questions for \(\mathcal{D}\) could be positive at least in dimension one. The following result shows however that the answer is always negative, except when \(p = 2\) and \(\mathcal{L}\) is self-adjoint.

**Theorem 4.1.** Let Assumptions 2.1 and 2.3 hold and let \(1 < p < \infty\). The following assertions are equivalent:

(i) the operator \(i\mathcal{D}\) generates a \(C_0\)-group on \(L^p(E,\mu) \oplus L^p(E,\mu;H)\);
(ii) the operator \(i\mathcal{D}\) generates a \(C_0\)-group on \(L^p(E,\mu) \oplus \mathbb{R}_p(\nabla_H)\);
(iii) the operator \(i\sqrt{-\mathcal{L}}\) generates a \(C_0\)-group on \(L^p(E,\mu)\);
(iv) the operator \(\mathcal{L}\) generates a \(C_0\)-cosine family on \(L^p(E,\mu)\);
(v) \(p = 2\) and \(\mathcal{L}\) is self-adjoint on \(L^2(E,\mu)\).

A thorough discussion of cosine families is presented in [2], which will serve as our standard reference. For the reader’s convenience we recall some relevant definitions. Let \(X\) be a Banach space. A strongly continuous function \(C : \mathbb{R} \to \mathcal{L}(X)\) is called a \(C_0\)-cosine family if \(C(0) = I\) and

\[
2C(t)C(s) = C(t + s) + C(t - s), \quad t, s \in \mathbb{R}.
\]

By an application of the uniform boundedness theorem, \(C_0\)-cosine functions are exponentially bounded; see [2, Lemma 3.14.3]. Denoting the exponential type of \(C\) by \(\omega\), by [2, Proposition 3.14.4] there exists a unique closed densely defined operator \(A\) on \(X\) such that for all \(\lambda > \omega\) we have \(\lambda^2 \in \sigma(A)\) and

\[
\lambda(\lambda^2 - A)^{-1}x = \int_0^{\infty} e^{-\lambda t}C(t)x\,dt, \quad x \in X.
\]

This operator \(A\) is called the generator of \(C\).

**Proof of Theorem 4.1.** (i) \(\Rightarrow\) (v) and (ii) \(\Rightarrow\) (v): By a well-known result from semigroup theory, if \(\mathcal{A}\) generates a \(C_0\)-group \(G\) on a Banach space \(X\), then \(\mathcal{A}^2\) generates an analytic \(C_0\)-semigroup \(T\) of angle \(\frac{1}{2}\pi\) on \(X\) given by the formula

\[
T(z)x = \frac{1}{\sqrt{2\pi z}} \int_{-\infty}^{\infty} e^{-t^2/4z}G(t)x\,dt, \quad \text{Re } z > 0.
\]

Suppose now that (i) or (ii) holds. By the observation just made \(-\mathcal{D}^2\) generates an analytic \(C_0\)-semigroup on \(L^p(E,\mu) \oplus L^p(E,\mu;H)\), respectively on \(L^p(E,\mu) \oplus \overline{\mathbb{R}_p(\nabla_H)}\), of angle \(\frac{1}{2}\pi\). In particular, by considering the first coordinate, \(\mathcal{L}\) generates an analytic \(C_0\)-semigroup on \(L^p(E,\mu)\) of angle \(\frac{1}{2}\pi\). As we have seen in the proof of Theorem 3.1, this implies that \(p = 2\) and that \(\mathcal{L}\) is self-adjoint.

(v) \(\Rightarrow\) (i) and (v) \(\Rightarrow\) (ii): For \(p = 2\), the self-adjointness of \(\mathcal{L}\) implies \(B = I_H\) and \(\mathcal{L} = \nabla_H^{1/2}\nabla_H^{1/2}\), and therefore the realisations of \(\mathcal{D}\) considered in (i) and (ii) are both self-adjoint. Now (i) and (ii) follows from Stone’s theorem.
(v)⇒(iii) and (v)⇒(iv): The group and cosine family may be defined through the Borel functional calculus of \(-\mathcal{L}\) by \(e^{i\sqrt{-\mathcal{L}}}\) and \(\cos(t\sqrt{-\mathcal{L}})\); it follows from (4.1) that \(\mathcal{L}\) is the generator of this cosine family.

(iv)⇒(iii)⇒(v): By a theorem of Fattorini [24] (see also [2, Theorem 3.16.7]) the operator \(i\sqrt{-\mathcal{L}}\) generates a \(C_0\)-group on \(L^p(E,\mu)\). Then by (4.2), its square \(\mathcal{L}\) generates an analytic \(C_0\)-semigroup on \(L^p(E,\mu)\) of angle \(\pi/2\), and we have already seen that this forces \(p = 2\) and self-adjointness of \(\mathcal{L}\).

Let \(L^p_0(E,\mu)\) be the codimension-one subspace of \(L^p(E,\mu)\) comprised of all functions \(f\) for which \(\mathcal{F} := \int_E f \, d\mu = 0.

**Lemma 4.2.** Let Assumptions 2.1 and 2.3 hold and let \(1 < p < \infty\). Then
\[
\mathcal{N}_p(\mathcal{L}) = \mathcal{N}_p(\sqrt{-\mathcal{L}}) = \mathcal{N}_p(\nabla H) = \mathbb{C} 1,
\]
\[
\mathcal{R}_p(\mathcal{L}) = \mathcal{R}_p(\sqrt{-\mathcal{L}}) = \mathcal{R}_p(\nabla H B) = L^p_0(E,\mu).
\]
On \(\mathcal{R}_p(\nabla H)\) we have
\[
\mathcal{N}_p(\mathcal{L}) = \mathcal{N}_p(\sqrt{-\mathcal{L}}) = \mathcal{N}_p(\nabla H B) = \{0\},
\]
\[
\mathcal{R}_p(\mathcal{L}) = \mathcal{R}_p(\sqrt{-\mathcal{L}}) = \mathcal{R}_p(\nabla H).
\]

**Proof.** All this is contained in [42, Proposition 9.5], with the exception of the identities \(\mathcal{R}_p(\mathcal{L}) = L^p_0(E,\mu)\) and the four equalities relating the kernels and closed ranges of \(\mathcal{L}\) and \(\mathcal{L}^2\) with those of their square roots.

Since \(\mathcal{L}\) is sectorial we have a direct sum decomposition \(L^p(E,\mu) = \mathcal{N}(\mathcal{L}) \oplus \mathcal{R}(\mathcal{L}) = \mathbb{C} 1 \oplus \mathcal{R}(\mathcal{L})\). If \(f\) is any \(C^1_0\)-cylindrical function belonging to \(\mathcal{D}(\mathcal{L})\), then \(\langle \mathcal{L} f, 1 \rangle = \langle B \nabla H f, \nabla H 1 \rangle = 0\). Since these functions \(f\) are dense in \(\mathcal{D}(\nabla H)\), and \(\mathcal{D}(\nabla H)\) is dense in \(\mathcal{D}(\mathcal{L})\), it follows from \(\langle \mathcal{L} f, 1 \rangle = 0\) that \(\mathcal{R}(\mathcal{L}) \subseteq L^p_0(E,\mu)\). Since both \(\mathcal{R}(\mathcal{L})\) and \(L^p_0(E,\mu)\) have codimension one, these spaces must in fact be equal.

The four equalities for the square roots follow from the general fact that if \(S\) is sectorial or bisectorial and \(S^2\) is sectorial, then \(\mathcal{N}(S) = \mathcal{N}(S^2)\) and \(\mathcal{R}(S) = \mathcal{R}(S^2)\).

**Remark 4.3.** Assumption 2.1 implies the identity
\[
\langle \mathcal{L} f, g \rangle + \langle f, \mathcal{L} g \rangle = -\frac{1}{2} \int_E \langle \nabla H f, \nabla H g \rangle_H \, d\mu.
\]
This establishes a connection with the theory of Dirichlet forms, and part of the above lemma could be deduced from it. A comprehensive treatment of this theory and its many ramifications is presented in the monograph [10].

In the remainder of this section we shall assume that \(p = 2\) and that \(\mathcal{L}\) is self-adjoint, and turn to the problem of representing the group generated by \(i\mathcal{L}\) in an explicit matrix form. Since \(\sqrt{-\mathcal{L}}\) is self-adjoint, \(i\sqrt{-\mathcal{L}}\) generates a unitary \(C_0\)-group on \(L^2(E,\mu)\) by Stone’s theorem. By the Borel functional calculus for self-adjoint operators, we have the identities
\[
C(t) := \cos(t\sqrt{-\mathcal{L}}) = \frac{1}{2}(e^{it\sqrt{-\mathcal{L}}} + e^{-it\sqrt{-\mathcal{L}}}),
\]
\[
S(t) := \sin(t\sqrt{-\mathcal{L}}) = \frac{1}{2i}(e^{it\sqrt{-\mathcal{L}}} - e^{-it\sqrt{-\mathcal{L}}}).
\]
Lemma 4.4. For all \( t \in \mathbb{R} \) the formulas
\[
\begin{align*}
\mathcal{C}(t)(\nabla_H f) &:= \nabla_H C(t)f, \\
\mathcal{S}(t)(\nabla_H f) &:= \nabla_H S(t)f,
\end{align*}
\]
define bounded operators \( \mathcal{C}(t) \) and \( \mathcal{S}(t) \) on \( \mathbb{R}(\nabla_H) \) of norms \( \|\mathcal{C}(t)\| \leq \|C(t)\| \) and \( \|\mathcal{S}(t)\| \leq \|S(t)\| \).

Proof. We will prove the statements for the cosines; the same proof works for the sines. In fact, all we use is that the operators \( C(t) \) and \( S(t) \) are bounded, map the constant function \( 1 \) to itself, and commute with \( \mathcal{L} \).

First note that the operators \( \mathcal{C}(t) \) are well-defined on the range of \( \nabla_H \). Indeed, if \( \nabla_H f = 0 \), then \( f = \overline{f} \in C1 \) by Lemma 4.2, where \( \overline{f} = \int_E f \, d\mu \). But \( C(t) \, \overline{1} = \overline{1} \) and therefore \( \nabla_H C(t)f = \overline{\nabla_H} \, \overline{C(t)} \, \overline{f} = \overline{\nabla_H} \, \overline{f} = 0 \).

From the representation \( \mathcal{L} = -\frac{1}{2} \nabla_H^2 \nabla_H \) we have \( \mathcal{D}(\sqrt{-\mathcal{L}}) = \mathcal{D}(\nabla_H) \) and \( \|\sqrt{-\mathcal{L}} f\|_2 = \frac{1}{\sqrt{2}} \|\nabla_H f\|_2 \) (see (2.2)). This gives, for \( f \in \mathcal{D}(\sqrt{-\mathcal{L}}) = \mathcal{D}(\nabla_H) \),
\[
\|\mathcal{C}(t)\nabla_H f\|_2 = \|\nabla_H C(t)f\|_2 = \sqrt{2} \|\sqrt{-\mathcal{L}} C(t)f\|_2 = \sqrt{2} \|C(t)\| \|\sqrt{-\mathcal{L}} f\|_2 = \|C(t)\| \|\nabla_H f\|_2.
\]

Via the \( H^\infty \)-functional calculus of the self-adjoint bisectorial operator \( \mathcal{D} \) on \( L^2(E, \mu) \oplus \mathbb{R}(\nabla_H) \) (see [9, 42]) we can define the bounded operator \( \text{sgn}(\mathcal{D}) \) on \( L^2(E, \mu) \oplus \mathbb{R}(\nabla_H) \). The fact that this operator encodes Riesz transforms gives the main motivation of [9]: to obtain functional calculus results for second-order differential operators together with the corresponding Riesz transforms estimates through the functional calculus of an appropriate first-order differential operator. We recall the link between \( \text{sgn}(\mathcal{D}) \) and Riesz transform in the next lemma. The constant \( 1/\sqrt{2} \) arising here is an artefact of the fact that we consider the operator \( -\mathcal{L} = \frac{1}{2} \nabla_H^2 \nabla_H \) (rather than \( \nabla_H^* \nabla_H \)).

Lemma 4.5. On \( L^2(E, \mu) \oplus \mathbb{R}(\nabla_H) \) we have
\[
\text{sgn}(\mathcal{D}) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix},
\]
where
\[
\begin{align*}
R : \sqrt{-\mathcal{L}} f \mapsto \nabla_H f & \quad \text{and} \quad R : \overline{1} \mapsto 0, \\
\overline{R} : \sqrt{-\mathcal{L}} g \mapsto \nabla_H^* g,
\end{align*}
\]
denote the Riesz transforms associated with \( -\mathcal{L} \) and \( -\mathcal{L} \), respectively.

Proof. Recall from Lemma 4.2 that \( L^2(E, \mu) = \mathbb{R}(\sqrt{-\mathcal{L}}) \oplus C1 \) and \( \mathbb{R}(\nabla_H) = \mathbb{R}(\sqrt{-\mathcal{L}}) \). Hence the above relations define \( R \) and \( \overline{R} \) uniquely.

By the convergence lemma for the \( H^\infty \)-calculus we have \( \text{sgn}(\mathcal{D}) = \lim_{n \to \infty} f_n(\mathcal{D}) \) strongly, where, for all \( z \not\in i\mathbb{R} \),
\[
f_n(z) = \frac{n z}{1 + n \sqrt{z^2}}.
\]
Here we take the branch of the square root that is holomorphic on \( \mathbb{C} \setminus (-\infty, 0] \). Hence,

\[
\begin{align*}
\text{sgn}(\mathcal{D}) &= \lim_{n \to \infty} n \mathcal{D}(I + n\sqrt{-\mathcal{D}})^{-1} \\
&= \lim_{n \to \infty} \mathcal{D} \begin{bmatrix} (n^{-1} + \sqrt{-2\mathcal{D}})^{-1} & 0 \\ 0 & (n^{-1} + \sqrt{2\mathcal{D}})^{-1} \end{bmatrix} \\
&= \lim_{n \to \infty} \begin{bmatrix} 0 & \nabla_H^* (n^{-1} + \sqrt{-2\mathcal{D}})^{-1} \\ \nabla_H (n^{-1} + \sqrt{-2\mathcal{D}})^{-1} & 0 \end{bmatrix}.
\end{align*}
\]

It is immediate from the above representation that \( \text{sgn}(\mathcal{D}) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \). Also,

\[
\text{sgn}(\mathcal{D}) \begin{bmatrix} \sqrt{-\mathcal{D}}f \\ \sqrt{-\mathcal{D}}g \end{bmatrix} = \lim_{n \to \infty} \begin{bmatrix} \nabla_H^* (n^{-1} + \sqrt{-2\mathcal{D}})^{-1} \sqrt{-\mathcal{D}}g \\ \nabla_H (n^{-1} + \sqrt{-2\mathcal{D}})^{-1} \sqrt{-\mathcal{D}}f \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \nabla_H g \\ \nabla_H f \end{bmatrix}.
\]

\( \square \)

**Lemma 4.6.** For all \( t \in \mathbb{R} \) we have

\[
RC(t) = C(t)R, \quad C(t)R = RC(t),
\]

\[
RS(t) = S(t)R, \quad S(t)R = RS(t).
\]

Furthermore, if \( f \in D(\sqrt{-\mathcal{D}}) \), then \( Rf \in D(\sqrt{-\mathcal{D}}) \) and

\[
\sqrt{-\mathcal{D}}Rf = R\sqrt{-\mathcal{D}}f.
\]

Likewise, if \( g \in D(\sqrt{-\mathcal{D}}) \), then \( Rg \in D(\sqrt{-\mathcal{D}}) \) and

\[
\sqrt{-\mathcal{D}}Rg = R\sqrt{-\mathcal{D}}g.
\]

Finally,

\[
\frac{1}{2}RR = P_{\mathcal{D}(\nabla_H)} = P_{\mathcal{L}_2(E, \mu)}, \quad \frac{1}{2}RR = P_{\mathcal{D}(\nabla_H)}
\]

where the right-hand sides denote the orthogonal projections onto the indicated subspaces.

**Proof.** We have, for \( f \in D(\sqrt{-\mathcal{D}}) = D(\nabla_H) \),

\[
RC(t)\sqrt{-\mathcal{D}}f = R\sqrt{-\mathcal{D}}C(t)f = \nabla_H C(t)f = C(t)\nabla_H f = C(t)R\sqrt{-\mathcal{D}}f.
\]

This gives the first identity on the range of \( \sqrt{-\mathcal{D}} \). On \( N(\sqrt{-\mathcal{D}}) = C1 \) (see Lemma 4.2) the identity is trivial since \( R1 = 0 \). Since \( L^2(E, \mu) = N(\sqrt{-\mathcal{D}}) \oplus R(\sqrt{-\mathcal{D}}) \) by the sectoriality of \( \sqrt{-\mathcal{D}} \), this proves the first identity. The corresponding identity for the sine function is proved similarly.

The identities \( R\sqrt{-\mathcal{D}} = \sqrt{-\mathcal{D}}R \) and \( R\sqrt{-\mathcal{D}} = \sqrt{-\mathcal{D}}R \) follow by differentiating the identities \( S(t)R = RS(t) \) and \( RS(t) = S(t)R \) at \( t = 0 \).

If \( \nabla_H f \in D(\sqrt{-\mathcal{D}}) = D(\nabla_H) \), then

\[
C(t)R\sqrt{-\mathcal{D}}\nabla_H f = C(t)\nabla_H^* \nabla_H f = -2C(t)\mathcal{L} f = -2\mathcal{L} C(t)f
\]

\[
= \nabla_H C(t)\nabla_H f = R\sqrt{-\mathcal{D}}C(t)\nabla_H f = RC(t)\sqrt{-\mathcal{D}}\nabla_H f.
\]

Noting that \( R(\nabla_H) \cap D(\sqrt{-\mathcal{D}}) \) is a core for \( D(\sqrt{-\mathcal{D}}) \) (it contains the \( e^{t\mathcal{L}} \)-invariant dense linear subspace \( \{e^{t\mathcal{L}}\nabla_H f : t > 0, f \in D(\nabla_H)\} \)), it follows that \( C(t)R\sqrt{-\mathcal{D}} = RC(t)\sqrt{-\mathcal{D}} \). This proves the second identity on the range of \( \sqrt{-\mathcal{D}} \). Since \( R(\sqrt{-\mathcal{D}}) = \)
this proves the identity \( C(t)\mathcal{R} = \mathcal{R}C(t) \). The corresponding sine identity is proved in the same way.

Finally, the last two identities follow from

\[
\begin{bmatrix}
\frac{P_{\mathcal{R}(\nabla_H)}}{0} \\
0 \\
\frac{P_{\mathcal{R}(\nabla_H)}}{0}
\end{bmatrix} = \mathcal{R}^2 = \text{sgn}^2(\mathcal{D}) = \frac{1}{2} \begin{bmatrix} 0 & \mathcal{R} \\
\mathcal{R} & 0 \\
\frac{\mathcal{R} \mathcal{R}}{0} & \frac{\mathcal{R} \mathcal{R}}{0} \\
\end{bmatrix},
\]

recalling that \( \mathcal{R}(\nabla_H^t) = L_0^E(E, \mu) \).

**Theorem 4.7.** Let Assumption 2.1 hold and suppose that \( \mathcal{L} \) is self-adjoint on \( L^2(E, \mu) \). The \( C_0 \)-group generated by \( \sqrt{2} \mathcal{D} \) on \( L^2(E, \mu) \oplus \mathcal{R}(\nabla_H) \) is given by

\[
e^{\sqrt{2} t \mathcal{D}} = \begin{bmatrix}
C(t) & \frac{i}{\sqrt{2}} RS(t) \\
\frac{i}{\sqrt{2}} RS(t) & C(t)
\end{bmatrix}, \quad t \in \mathbb{R}.
\]

By a scaling argument, this also gives a matrix representation for the group generated by \( i \mathcal{D} \).

**Proof.** On \( L_0^E(E, \mu) \oplus \mathcal{R}(\nabla_H) \) the group property follows by an easy computation using the lemmas and the addition formulas for \( C(t) \) and \( S(t) \) and their underscored relatives (see [2, Formula (3.95)]). On \( \mathcal{C}_1 \oplus \mathcal{R}(\nabla_H) \) we argue similarly.

Strong continuity and uniform boundedness are evident from the corresponding properties of the matrix entries. To see that its generator equals \( i \mathcal{D} \), we set \( G(t) := \begin{bmatrix}
C(t) & \frac{i}{\sqrt{2}} RS(t) \\
\frac{i}{\sqrt{2}} RS(t) & C(t)
\end{bmatrix} \) take \( f \in D(\mathcal{L}), g \in D(\mathcal{L}) \), and differentiate. Lemma 4.6 then gives us

\[
\lim_{t \to 0} \frac{1}{t} \begin{bmatrix} f \\
g \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & R
\end{bmatrix} \mathcal{D} \begin{bmatrix} f \\
g \end{bmatrix} = \frac{i}{\sqrt{2}} \begin{bmatrix}
0 & \nabla^*_H
\end{bmatrix} \begin{bmatrix} f \\
g \end{bmatrix} = \frac{i}{\sqrt{2}} \mathcal{D} \begin{bmatrix} f \\
g \end{bmatrix}.
\]

This shows that \( \frac{i}{\sqrt{2}} \mathcal{D} \) is an extension of the generator \( \mathcal{D} \) of \( (G(t))_{t \in \mathbb{R}} \). However, by the general theory of cosine families, the left-hand side limit exists if and only if \( f \in D(\sqrt{\mathcal{L}}) \) and \( g \in D(\sqrt{\mathcal{L}}) \). In view of the domain identifications \( D(\sqrt{\mathcal{L}}) = D(\nabla_H) \) and \( D(\sqrt{\mathcal{L}}) = D(\nabla_H^t) \) this precisely happens if and only if \( \begin{bmatrix} f \\
g \end{bmatrix} \in D(\mathcal{D}) \).

Therefore we actually have equality \( \mathcal{D} = \frac{i}{\sqrt{2}} \mathcal{D} \).

We proceed with an analogue of Theorem 3.3.

**Theorem 4.8.** Let Assumption 2.1 hold and let \( 1 < p < \infty \). If, for some \( \lambda > 0 \) and \( \alpha > 0 \), the operators

\[
(\lambda - \mathcal{L})^{-\alpha} \cos(t \sqrt{-\mathcal{L}}), \quad t \in \mathbb{R},
\]

extend to bounded operators on \( L^p(E, \mu) \), then \( p = 2 \) and \( \mathcal{L} \) is self-adjoint.

**Proof.** As the proof follows the ideas of that of Theorem 3.3, we only sketch the main lines and leave the details to the reader.
If the operators \((\lambda - \mathcal{L})^{-\alpha} \cos(t\sqrt{-\mathcal{L}}), t \in \mathbb{R}\), are bounded on \(L^p(E, \mu)\), then so are the operators
\[
(\lambda - \mathcal{L})^{-\alpha} \sin(t\sqrt{-\mathcal{L}}) = \int_0^t (\lambda - \mathcal{L})^{-\alpha} \sqrt{-\mathcal{L}} \cos(s\sqrt{-\mathcal{L}}) \, ds,
\]
as well as
\[
(\lambda - \mathcal{L})^{-\alpha} e^{it\sqrt{-\mathcal{L}}} := (\lambda - \mathcal{L})^{-\alpha} [\cos(t\sqrt{-\mathcal{L}}) + i \sin(t\sqrt{-\mathcal{L}})].
\]
Then also the operators
\[
e^{(\alpha+\beta)t\sqrt{-\mathcal{L}}} = (\lambda - \mathcal{L})^{(\alpha+\beta)} e^{s\sqrt{-\mathcal{L}}} \circ (\lambda - \mathcal{L})^{-\alpha} e^{it\sqrt{-\mathcal{L}}}
\]
are bounded on \(L^p(E, \mu)\), in the sense that the right-hand side defines a holomorphic extension of the semigroup \((e^{t\sqrt{-\mathcal{L}}})_{t \geq 0}\) to the right half-plane \(\{\text{Re} z > 0\}\). This means that \(\sqrt{-\mathcal{L}}\) is sectorial of angle zero. But \(-\mathcal{L}\) is sectorial as well, and, by the general theory of sectorial operators, the angles of sectoriality are related by \(\omega(\sqrt{-\mathcal{L}}) = \frac{1}{2} \omega(-\mathcal{L})\) (see, e.g., [38, Theorem 15.16]). It follows that \(-\mathcal{L}\) is sectorial of angle zero. As we have seen in the proof of Theorem 3.1, this is false unless \(p = 2\) and \(\mathcal{L}\) is self-adjoint.

Concerning exponential regularisation by \(e^{\alpha\mathcal{L}}\), analogous observations as in Remark 3.4 can be made. We leave this to the interested reader.

5. Speed of Propagation

It will be useful to make the natural identification
\[
L^2(E, \mu) \oplus L^2(E, \mu; H) = L^2(E, \mu; \mathbb{C} \oplus H).
\]
The support of an element \(u = (f, g) \in L^2(E, \mu) \oplus L^2(E, \mu; H)\) will always be understood as the support of the corresponding element in \(L^2(E, \mu; \mathbb{C} \oplus H)\). Thus,
\[
\text{supp}(u) = \text{supp}(f) \cup \text{supp}(g).
\]
Definition 5.1. Let \(\mathcal{H}\) be any Hilbert space. We say that a one-parameter family \((T_t)_{t \in \mathbb{R}}\) of bounded operators on \(L^2(E, \mu; \mathcal{H})\) has speed of propagation \(\kappa\) if the following holds. For all closed subsets \(K\) of \(E\), all \(u \in L^2(E, \mu; \mathcal{H})\), and all \(t \in \mathbb{R}\), we have
\[
\text{supp}(u) \subseteq K \implies \text{supp}(T_t u) \subseteq K_{\kappa|t|}
\]
where
\[
K_{\kappa|t|} := \{x \in E : \text{dist}(x, K) \leq \kappa|t|\}.
\]
The family \((T_t)_{t \in \mathbb{R}}\) is said to have infinite speed of propagation if it does not propagate at any finite speed.

In the above, \(\text{dist}(x, K) = \inf \{\|x - y\| : y \in K\}\). Note that \((T_t)_{t \in \mathbb{R}}\) has speed of propagation \(\kappa\) if and only if for all subsets \(K\) of \(E\) and all \(u, u' \in L^2(E, \mu; \mathcal{H})\) with supports in \(K\) and \(\mathcal{C} K_{\kappa|t|}\) respectively, we have
\[
\langle T_t u, u' \rangle = 0,
\]
the brackets denoting the inner product of \(L^2(E, \mu; \mathcal{H})\).

In the next proposition we consider the case \(E = \mathbb{R}^d = H\) and \(A = \frac{1}{2}J\). The resulting operator \(\mathcal{L}\) is called the classical Ornstein-Uhlenbeck operator and is given explicitly as
\[
(5.1) \quad \mathcal{L} = \frac{1}{2} \Delta - \frac{1}{2} x \cdot \nabla
\]
and the associated invariant measure is the standard Gaussian measure $\gamma$ on $\mathbb{R}^d$,
\[
d\gamma(x) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{1}{2} |x|^2\right) dx.
\]
The semigroup generated by $\mathcal{L}$ is given by
\[
e^{t\mathcal{L}} f(x) = \int_{\mathbb{R}^d} M_t(x, y) f(y) dy,
\]
where $M$ is the Mehler kernel,
\[
M_t(x, t) = \frac{1}{(2\pi)^{d/2}} (1 - e^{-2t})^{-d/2} \exp\left(-\frac{1}{2} \frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right).
\]

The following theorem is the Ornstein-Uhlenbeck analogue of the classical fact that the Schrödinger group $(e^{it\Delta})_{t \in \mathbb{R}}$ on $L^2(\mathbb{R}^d)$ has infinite speed of propagation.

**Theorem 5.2.** Let $\mathcal{L}$ be the classical Ornstein-Uhlenbeck operator on $L^2(\mathbb{R}^d, \gamma)$. The $C_0$-group $(e^{it\mathcal{L}})_{t \in \mathbb{R}}$ generated by $i\mathcal{L}$ has infinite speed of propogation.

**Proof.** It suffices to show that, for some given $t_0 > 0$, and any $R > 0$, there exist compactly supported functions $f, g \in L^2(\mathbb{R}^d, \gamma)$ whose supports are separated at least by a distance $R$, and which satisfy $(e^{it_0\mathcal{L}} f, g) \neq 0$.

We take $t_0 := \pi/2$. On the one hand, by [2, Proposition 3.9.1] we have $e^{it_0\mathcal{L}} f = \lim_{n \to 0} e^{(n+i\pi)\mathcal{L}} f$ in $L^2(\mathbb{R}^d, \gamma)$. On the other hand, for almost all $x \in \mathbb{R}^d$ we have
\[
e^{(s+i\pi)\mathcal{L}} f(x) = \int_{\mathbb{R}^d} M_{s+i\pi}(x, y) f(y) dy
\]
\[
= \frac{1}{(2\pi)^{d/2}} (1 + e^{-2s})^{-d/2} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2} \frac{|ie^{-s}x - y|^2}{1 + e^{-2s}}\right) f(y) dy
\]
by analytic continuation. For compactly supported $f$ we may use dominated convergence to pass to the limit for $s \downarrow 0$ and obtain, for almost all $x \in \mathbb{R}^d$,
\[
e^{i\pi \mathcal{L}} f(x) = (4\pi)^{-d/2} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{4} |ix - y|^2\right) f(y) dy
\]
Fix arbitrary $x_0, y_0$ in $\mathbb{R}^d$ satisfying $|x_0 - y_0| > R$ and let $f_m := \frac{1_{B(x_0, \frac{3}{4}R)}}{|B(x_0, \frac{1}{4}R)|}$ and $g_n := \frac{1_{B(y_0, \frac{3}{4}R)}}{|B(y_0, \frac{1}{4}R)|}$ for $m, n \in \mathbb{N}$. Then, by continuity,
\[
\lim_{m, n \to \infty} (e^{it_0\mathcal{L}} f_m, g_n) = \lim_{m, n \to \infty} (4\pi)^{-d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{4} |ix - y|^2\right) f_m(y) g_n(x) dy dx
\]
\[
= (4\pi)^{-d/2} \exp\left(-\frac{1}{4} |x_0 - y_0|^2\right) \neq 0.
\]
It follows, by taking $n, m$ large enough, that $(e^{it_0\mathcal{L}} f_m, g_n) \neq 0$, while the supports of $f_m$ and $g_n$ are separated by a distance $R \geq R$. \qed

Using the identity
\[
e^{z\sqrt{-\mathcal{L}}} f = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \exp\left(\frac{z^2 u}{4u}\right) f du, \quad \text{Re} z \geq 0,
\]
by a similar argument one shows that the $C_0$-group $(e^{it\sqrt{-\mathcal{L}}})_{t \in \mathbb{R}}$ generated by $i\sqrt{-\mathcal{L}}$ has infinite speed of propogation.
The main result of this section provides conditions under which the $C_0$-group $(e^{it\mathcal{D}})_{t \in \mathbb{R}}$ generated by $\mathcal{D}$ has finite speed of propagation; as an immediate corollary, the cosine family $(\cos(t\sqrt{-\mathcal{D}}))_{t \in \mathbb{R}}$ has finite speed of propagation. In addition to Assumption 2.1 we need an assumption on the reproducing kernel Hilbert space $H_\mu$ associated with the Gaussian measure $\mu$. Recall that this is the Hilbert space completion of $\mathbb{R}(Q_\mu)$, where $Q_\mu$ is the covariance operator of $\mu$, with respect to the norm
\begin{equation}
\|Q_\mu x^*\|_{H_\mu}^2 := \langle Q_\mu x^*, x^* \rangle \quad \forall x^* \in E^*.
\end{equation}
This completion embeds continuously into $E$. Denoting by $i_\mu : H_\mu \to E$ the embedding mapping, we have $i_\mu \circ i_\mu^* = Q_\mu$. For more information we refer the reader to [11, 52].

**Assumption 5.3.** $H_\mu$ is densely contained in $H$.

By an easy closed graph argument the inclusion mapping $i_{\mu,H} : H_\mu \to H$ is bounded.

The relevance of this Assumption lies in the fact that $H_\mu$ is contained in $H$ if and only if $\mathcal{L}$ has a spectral gap, which in turn is equivalent to the validity of the following Poincaré inequality: for some (equivalently, for all) $1 < p < \infty$ there is a constant $C_p$ such that for all $f \in D_p(\nabla H)$,
\[ \|f - \bar{f}\|_p \leq C_p \|\nabla_H f\|_p, \]
with $\bar{f} := \int_E f \, d\mu$ (see [17, 30, 53]).

If $H_\mu \subset H$, then the inclusion is dense if and only if $\nabla H$ is closable as a densely defined operator from $L^p(E, \mu)$ to $L^p(E, \mu; H)$ for some/all $1 \leq p < \infty$ [29, Corollary 4.2]. These equivalent conditions are satisfied if the semigroup $S$ restricts to a $C_0$-semigroup on $H$ [30, Theorem 3.5]; the latter is the case if Assumption 2.3 is satisfied [43, Theorem 3.3].

Now we are ready to state the main result of this section.

**Theorem 5.4.** Let Assumptions 2.1 and 5.3 hold and let $\mathcal{L}$ be self-adjoint. Then the group $(e^{it\mathcal{L}})_{t \in \mathbb{R}}$ on $L^2(E, \mu) \oplus L^2(E, \mu; H)$ propagates, at most, with speed $\|1_H\|_{\mathcal{L}(H,E)}$.

Our proof of this theorem follows an argument of Morris and McIntosh [49], which in turn is a group analogue of a similar resolvent argument in [9]. The main difficulty in carrying over the proof to the present situation is to prove that suitable Lipschitz functions belong to $D(\nabla H)$.

We begin with some lemmas. It will be understood that the assumptions of Theorem 5.4 are satisfied, although not all assumptions are needed in each lemma.

**Lemma 5.5.** For all real-valued $\eta \in D(\nabla H)$ satisfying $\nabla_H \eta \in L^\infty(E, \mu; H)$ and all $u = (f, g) \in D(\mathcal{D})$ we have $\eta u = (\eta f, \eta g) \in D(\mathcal{D})$ and the commutator $[\eta, \mathcal{D}] : u \mapsto \eta \mathcal{D} u - \mathcal{D}(\eta u)$ extends to a bounded operator on $L^2(E, \mu) \oplus L^2(E, \mu; H)$ with norm
\[ \|[\eta, \mathcal{D}]\| \leq \|\nabla_H \eta\|_\infty. \]

This operator is local, with support contained in the support of $\eta$, in the sense that $[\eta, \mathcal{D}] u = 0$ whenever $\text{supp}(u) \cap \text{supp}(\eta) = \emptyset$. Furthermore,
\[ [\eta, [\eta, \mathcal{D}]] = 0. \]
Proof. For all \( f \in D(\nabla_H) \) we have (by approximating \( \eta \) and \( f \) with cylindrical functions) \( \eta f \in D(\nabla_H) \) and \( \nabla_H(\eta f) = \eta \nabla_H f + (\nabla_H \eta)f \). Also, for all \( f \in D(\nabla_H) \) and \( g \in D(\nabla_H^*) \), we have
\[
\langle \nabla_H f, \eta g \rangle = \langle \nabla_H(\eta f), g \rangle = \langle \nabla_H(\eta f) - f \nabla_H \eta, g \rangle,
\]
where the brackets denote the inner product of \( L^2(E, \mu; H) \). It follows that \( \eta g \in D(\nabla_H^*) \) and \( \nabla_H^*(\eta g) = \eta \nabla_H g - \langle \nabla_H \eta, g \rangle_H \); here the brackets \( \langle \cdot, \cdot \rangle_H \) denote the (pointwise) inner product of \( H \). Hence, for \( u = \begin{bmatrix} f \\ g \end{bmatrix} \in D(\mathcal{D}) \) (that is, \( f \in D(\nabla_H) \) and \( g \in D(\nabla_H^*) \)),
\[
[\eta, \mathcal{D}]u = \begin{bmatrix} \eta \nabla_H^* g \\ \eta \nabla_H f \end{bmatrix} - \begin{bmatrix} \nabla_H(\eta g) \\ \nabla_H(\eta f) \end{bmatrix} = \begin{bmatrix} \langle \nabla_H \eta, g \rangle_H \\ -\langle \nabla_H \eta, f \rangle_H \end{bmatrix}.
\]
We infer that \([\eta, \mathcal{D}]\) is bounded and \( \| [\eta, \mathcal{D}] \| \leq \| \nabla_H \eta \|_{\infty} \). The locality assertion is an immediate consequence of the above representation of \([\eta, \mathcal{D}]\).

To prove that \([\eta, [\eta, \mathcal{D}]] = 0\), just note that
\[
[\eta, [\eta, \mathcal{D}]]u = \begin{bmatrix} \eta(\nabla_H \eta, g)H \\ -\eta(\nabla_H \eta, f)H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.
\]

As in [49] we deduce:

**Lemma 5.6.** Under the hypotheses of Theorem 5.4, the following commutator identity holds for all \( t \in \mathbb{R} \), \( \eta \in D(\nabla_H) \), and \( u \in L^2(E, \mu) \oplus L^2(E, \mu; H) \):
\[
[\eta, e^{it\mathcal{D}}]u = it \int_0^1 e^{is\mathcal{D}}[\eta, \mathcal{D}]e^{i(1-s)t\mathcal{D}} u \, ds.
\]

At the heart of the approach in [49] is the following lemma. We include its proof for the sake of completeness.

**Lemma 5.7** (McIntosh & Morris). Let \( u, v \in L^2(E, \mu) \oplus L^2(E, \mu; H) = L^2(E, \mu; C) \oplus H \) have disjoint supports and let \( \eta \in D(\nabla_H) \) be a real-valued function satisfying
\[
\eta u = u \quad \text{and} \quad \eta v = 0.
\]

Then for all \( t \in \mathbb{R} \) we have
\[
|\langle e^{it\mathcal{D}}u, v \rangle| \leq |t|\| \nabla_H \eta \|_{\infty} \| u \|_2 \| v \|_2.
\]

In particular, for \( |t| < 1/\| \nabla_H \eta \|_{\infty} \) it follows that \( \langle e^{it\mathcal{D}}u, v \rangle = 0 \).

**Proof.** To simplify the notation, let \( \delta \) be the derivation defined by \( \delta(S) = [\eta, S] \) and inductively write \( \delta^k(S) := \delta(\delta^{k-1}(S)) \) for the higher commutators, adopting the convention that \( \delta^0(S) := S \). Then, for all integers \( k \geq 1 \),
\[
\langle \delta^k(e^{it\mathcal{D}})u, v \rangle = \langle \eta \delta^{k-1}(e^{it\mathcal{D}})u - \delta^{k-1}(e^{it\mathcal{D}})\eta u, v \rangle = -\langle \delta^{k-1}(e^{it\mathcal{D}})u, v \rangle,
\]
using the assumptions that \( \eta u = u \) and \( \eta v = 0 \). Hence, by induction,
\[
\langle \delta^n(e^{it\mathcal{D}})u, v \rangle = (-1)^n\langle e^{it\mathcal{D}}u, v \rangle, \quad n \geq 1.
\]

On the other hand, using the identity \( \delta(ST) = \delta(S)T + S\delta(T) \), Lemma 5.6, and the fact, given by the second assertion in Lemma 5.5, that \( \delta([\eta, \mathcal{D}]) = [\eta, [\eta, \mathcal{D}]] = 0 \), we obtain
\[
\delta^{m+1}(e^{it\mathcal{D}}) = it \int_0^1 \sum_{k=0}^m \binom{m}{k} \delta^m \langle \eta \mathcal{D}, [\eta, \mathcal{D}] \rangle \delta^{m-k}(e^{i(1-s)\mathcal{D}}) u \, ds, \quad m \geq 0,
\]

Thus, \( \mathcal{D}E \subset C(E, \mu; H) \).
where \( \binom{m}{k} := \frac{m!}{k!(m-k)!} \). We now prove by induction that

\[
(5.5) \quad \| \delta^n(e^{itD}) \| \leq |t|^n \| [\eta, \mathcal{D}] \|^n, \quad m \geq 0.
\]

For \( n = 0 \), this follows from the fact that the operators \( e^{itD} \) are unitary. Now let \( m \geq 0 \) and suppose that (5.5) holds for all integers \( 0 \leq n \leq m \). We then use (5.4) to obtain

\[
\| \delta^{m+1}(e^{itD}) \| \leq |t|^{m+1} \| [\eta, \mathcal{D}] \|^{m+1} \int_0^1 \sum_{k=0}^m \binom{m}{k} s^{m-k}(1-s)^k \, ds
\]

\[
= |t|^{m+1} \| [\eta, \mathcal{D}] \|^{m+1} \int_0^1 (s+(1-s))^m \, ds
\]

\[
= |t|^{m+1} \| [\eta, \mathcal{D}] \|^{m+1}.
\]

This proves (5.5).

The lemma now follows by using the estimate (5.5) in (5.3) together with Lemma 5.5. \( \square \)

Proof of Theorem 5.4. What remains to be proven is that, given \( \varepsilon > 0 \), disjoint closed sets \( A \) and \( B \) in \( E \) can be ‘separated’ by an \( \eta \in D(\nabla_H) \), in the sense that \( \eta \equiv 1 \) on \( A \) and \( \eta \equiv 0 \) on \( B \), that satisfies \( \nabla_H \eta \in L^\infty(E,\mu;H) \) and

\[
\| \nabla_H \eta \|_\infty \leq (1 + \varepsilon) \| i_H \| / \text{dist}(A,B).
\]

It is clear that we can do the separation with bounded Lipschitz functions \( f \) whose Lipschitz constant \( L \) is at most \( (1 + \varepsilon)/\text{dist}(A,B) \). To complete the proof, we need to show that such functions do indeed belong to \( D(\nabla_H) \) and satisfy \( \| \nabla_H f \|_\infty \leq \| i_H \| L \). This last step is the most important technical difficulty that needs to be overcome in order to apply McIntosh and Morris’ approach to finite speed of propagation in the Ornstein-Uhlenbeck context. We prove it in Theorem 7.2 from the Appendix, as it is of independent interest. \( \square \)

6. OFF-DIAGONAL BOUNDS

The results of the previous sections will now be applied to obtain \( L^2 - L^2 \) off-diagonal bounds for Ornstein-Uhlenbeck operators. Such off-diagonal bounds can be seen as integrated versions of heat kernel bounds, and play a key role in the modern approach to spectral multiplier problems. As can be seen, e.g., in [9], such bounds are particularly useful when dealing with semigroups that do not have standard Calderón-Zygmund kernels, but still exhibit a diffusive behaviour. For more information on the role of Davies-Gaffney bounds and finite speed of propagation from the point of view of geometric heat kernel estimates, see e.g. [12]. For their use in spectral multiplier theory, see e.g. [13].

We begin with some general observations. If \( -iG \) generates a bounded \( C_0 \)-group \( U \) on a Banach space \( X \), for any \( \phi \in L^1(\mathbb{R}) \) we may define a bounded operator
\( \hat{\phi}(G) \) by means of the Weyl functional calculus (see, e.g., [37]):

\[
\hat{\phi}(G)x := \int_{-\infty}^{\infty} \phi(t)U(t)x \, dt, \quad x \in X.
\]

When \( X \) is a Hilbert space and \( G \) is self-adjoint, \( U \) is unitary and the definition of \( \hat{\phi}(G) \) agrees with the one obtained by the spectral theorem:

\[
\int_{-\infty}^{\infty} \phi(t)U(t)x \, dt = \int_{-\infty}^{\infty} \phi(t) \int_{\sigma(G)} e^{-it\lambda} \, dE(\lambda)x \, dt \\
= \int_{\sigma(G)} \int_{-\infty}^{\infty} \phi(t)e^{-it\lambda} \, dE(\lambda)x = \int_{\sigma(G)} \int_{-\infty}^{\infty} \hat{\phi}(\lambda) \, dE(\lambda)x.
\]

As an application of finite speed of propagation we prove, under the assumptions of Theorem 5.4, some off-diagonal bounds for the operators \( \hat{\phi}(\mathcal{D}) \) in the self-adjoint case, i.e., where \( \mathcal{D} = \begin{bmatrix} 0 & \nabla H \\ \nabla H & 0 \end{bmatrix} \). We have learnt this argument from Alan McIntosh. The main observation is the following. If \( u, v \in L^2(E, \mu) \oplus L^2(E, \mu; H) \) have supports separated by a distance \( R \), we apply the Weyl calculus to \( \mathcal{D} \) and note that \( \langle e^{-it\mathcal{D}}u, v \rangle = 0 \) for \( |t| \leq R \) since \( e^{-it\mathcal{D}} \) propagates with speed at most \( ||i_H|| \). As a consequence we obtain

\[
|\langle \hat{\phi}(\mathcal{D})u, v \rangle| = \left| \int_{|t| \geq R} \phi(t)\langle e^{-it\mathcal{D}}u, v \rangle \, dt \right| \leq \left( \int_{|t| \geq R} |\phi(t)| \, dt \right) ||u||_2 ||v||_2.
\]

We will work out two special cases where this leads to an interesting explicit estimate.

**Example 6.1.** Let \( R > 0 \).

1. If \( f \in L^2(E, \mu) \) and \( g \in L^2(E, \mu) \) have supports separated by a distance at least \( R ||i_H|| \), then

\[
|\langle e^{t\mathcal{D}}f, g \rangle| \leq \frac{2t}{\pi R^2} \exp\left( \frac{-R^2}{2t} \right) ||f||_2 ||g||_2.
\]

2. If \( f \in L^2(E, \mu) \) and \( g \in L^2(E, \mu; H) \) have supports separated by a distance at least \( R ||i_H|| \), then

\[
|\langle \nabla_H e^{t\mathcal{D}}f, g \rangle| \leq \sqrt{\frac{2}{\pi t}} \exp\left( \frac{-R^2}{2t} \right) ||f||_2 ||g||_2.
\]

The same estimate holds for \( e^{t\mathcal{D}}\nabla_H^* \).

**Proof.** By the Weyl calculus,

\[
\begin{bmatrix} e^{t\mathcal{D}} & 0 \\ 0 & e^{t\mathcal{D}} \end{bmatrix} = e^{-\frac{1}{2}t\mathcal{D}^2} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-s^2/2t} e^{-is\mathcal{D}} \, ds
\]

and

\[
\begin{bmatrix} 0 & e^{t\mathcal{D}} \nabla_H \\ \nabla_H e^{t\mathcal{D}} & 0 \end{bmatrix} = \mathcal{D} e^{-\frac{1}{2}t\mathcal{D}^2} = \frac{i}{\sqrt{2\pi t^3}} \int_{-\infty}^{\infty} se^{-s^2/2t} e^{-is\mathcal{D}} \, ds,
\]

where we used that \( \nabla_H e^{t\mathcal{D}} = \nabla_H e^{t\mathcal{D}^*} = e^{t\mathcal{D}} \nabla_H \).

The first assertion of the theorem now follows from the theorem via

\[
|\langle e^{t\mathcal{D}}f, g \rangle| = |\langle e^{-\frac{1}{2}t\mathcal{D}^2} \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} g \\ 0 \end{bmatrix} \rangle| \leq \frac{2}{\sqrt{2\pi t}} \int_{R}^{\infty} e^{-s^2/2t} ||f||_2 ||g||_2 \, ds
\]
where the last inequality follows from a standard estimate for the Gaussian distribution. Similarly,

$$|\langle \nabla H e^{tL} f, g \rangle| \leq \sqrt{2 \pi R^2 \exp\left(-\frac{R^2}{t}\right)} \|f\|_2 \|g\|_2,$$

The proof for $e^{tL} \nabla_H$ is similar. □

Similar results can be obtained by considering other functions $\phi$. For instance, off-diagonal bounds for $L e^{tL}$ may be obtained by taking $\phi(s) = s^2 e^{-s^2/2t}$ and using the identity

$$\left[ t L e^{tL} 0 
\begin{array}{c} 0 
\end{array} \right] = -\frac{1}{2} t L e^{-t L}.$$

We leave the details to the reader.

6.1. Off-diagonal bounds for resolvents in the non-self-adjoint case. In the non-self-adjoint case, we cannot make use of finite speed of propagation for an underlying group to prove off-diagonal bounds for $(e^{tL})_{t \geq 0}$. However, it is possible to use the direct approach from [9] (and its refinement in [5]) to obtain off-diagonal bounds for $((I + t^2 L)^{-1})_{t \in \mathbb{R}}$. We leave the investigation of possible other approaches for $(e^{tL})_{t \geq 0}$ in the non-self-adjoint case for future work.

We adopt Assumptions 2.1 and 5.3. We do not assume $L$ to be self-adjoint, so $iD$ may fail to generate a $C_0$-group on $L^2(E, \mu \odot C \oplus H)$. Nevertheless, $D$ does enjoy some good properties; for instance it is bisectorial on $L^2(E, \mu; C \oplus H)$ and therefore the quantity

$$(6.1) \quad M := \sup_{t \in \mathbb{R}} \| (I - itD)^{-1} \|$$

is finite. This follows from the general operator-theoretic framework presented in [9].

**Proposition 6.2.** Let Assumptions 2.1 and 2.3 hold. Suppose $u, v \in L^2(E, \mu; \mathbb{C} \oplus H)$ have disjoint supports at a distance greater than $R$. Then

$$|\langle (I + itD)^{-1} u, v \rangle| \leq C \exp(-\alpha R/|t|) \|u\|_2 \|v\|_2,$$

for some $\alpha, C > 0$ independent of $u, v$ and $R, t$.

**Proof:** The proof is a straightforward adaptation of [5, Proposition 5.1], and is included for the sake of completeness.

By the uniform boundedness of the operators $R_t := (I - itD)^{-1}$, $t \in \mathbb{R}$, it suffices to prove the estimate in the statement of the proposition for $|t| < \alpha R$, where $\alpha > 0$ is a positive constant to be chosen in a moment.
Let \( u \in L^2(E, \mu; \mathbb{C} \oplus H) \) be supported in a set \( B \subseteq E \) and let \( A \subseteq E \) be another set such that \( \text{dist}(A, B) \geq R \). Define
\[
\tilde{A} := \{ x \in E : \text{dist}(x, A) < \frac{1}{2} \text{dist}(x, B) \}.
\]

Note that \( \text{dist}(\tilde{A}, B) \geq \frac{1}{2} \text{dist}(A, B) \).

Let \( \varphi : E \to [0, 1] \) be a bounded Lipschitz function with support in \( \tilde{A} \) such that \( \varphi|_A \equiv 1 \), \( \varphi|_B \equiv 0 \), and whose Lipschitz constant is at most \( 4/R \). By Theorem 7.2, \( \varphi \in \mathcal{D}(\nabla_H) \) and \( \|\nabla_H \varphi\| \leq 4\|i_H\|/R \).

Set \( \eta := \exp(\alpha R \varphi/|t|) - 1 \). Then, for all \( x \in A \),
\[
\eta(x) = \exp(\alpha R/|t|) - 1 \geq \frac{1}{2} \exp(\alpha R/|t|)
\]
(recall the assumption \( |t| < \alpha R \) and \( \eta|_B \equiv 0 \). Hence,
\[
\frac{1}{2} \exp(\alpha R/|t|)\|R_t u\|_{L^2(A, \mu; \mathbb{C} \oplus H)} \leq \|\eta R_t u\|_2 = \|[\eta, R_t]u\|_2
\]
using that \( \eta u = 0 \) by the support properties of \( \eta \) and \( u \).

It is elementary to verify the commutator identity
\[
[\eta, R_t] = itR_t[\eta, \mathcal{D}]R_t.
\]
Moreover, using Leibniz rule (see the proof of Lemma 5.5), we have
\[
[\eta, \mathcal{D}]v = [\exp(\alpha R \varphi/|t|) - 1, \mathcal{D}]v = (\mathcal{D} \exp(\alpha R \varphi/|t|))v = m \exp(\alpha R \varphi/|t|)v,
\]
where \( m \) is supported on \( \tilde{A} \) and satisfies (cf. 5.5) \( \|m\|_{\infty} \leq C \alpha R \|\nabla_H \varphi\|/|t| \leq 4C \alpha \|i_H\|/|t| \).

Therefore,
\[
\|[\eta, R_t]u\|_{L^2(A, \mu; \mathbb{C} \oplus H)} = |t|\|R_t[\eta, \mathcal{D}]R_t u\|_2 \\
\leq 4MC\alpha \|i_H\| \|\exp(\alpha R \varphi/|t|)R_t u\|_2 \\
\leq 4MC\alpha \|i_H\| \left( \|\eta R_t u\|_{L^2(A, \mu; \mathbb{C} \oplus H)} + \|R_t u\|_2 \right)
\]
where \( M \) is defined by (6.1). The choice \( \alpha = (8MC\|i_H\|)^{-1} \), in combination with (6.2), gives
\[
\frac{1}{2} \exp(\alpha R/|t|)\|R_t u\|_{L^2(A, \mu; \mathbb{C} \oplus H)} \leq \|[\eta, R_t]u\|_2 \leq \|R_t u\|_2 \leq M\|u\|_2.
\]

\[\square\]

Remark 6.3. With the same proof, Proposition 6.2 holds in the more general context of elliptic divergence-form operators on abstract Wiener spaces considered in [42].

Since
\[
(I + it \mathcal{D})^{-1} + (I - it \mathcal{D})^{-1} = 2(I + t^2 \mathcal{D}^2)^{-1} = 2 \begin{bmatrix} (I - 2t^2 \mathcal{L})^{-1} & 0 \\ 0 & (I - 2t^2 \mathcal{L})^{-1} \end{bmatrix},
\]
we have the following corollary.

**Corollary 6.4.** Let Assumptions 2.1 and 2.3 hold. Suppose \( u, v \in L^2(E, \mu; \mathbb{C} \oplus H) \) have disjoint supports at a distance greater than \( R \). Then
\[
|\langle (I - t^2 \mathcal{L})^{-1}u, v \rangle| \leq C \exp(-\alpha R/|t|)\|u\|_2\|v\|_2,
\]
for some \( \alpha, C > 0 \) independent of \( u, v \) and \( R, t \).
7. Appendix: H-Lipschitz functions

It is assumed that Assumptions 2.1 and 5.3 hold. Our aim is to prove that under these conditions, bounded Lipschitz functions on $E$ (and more generally, bounded $H$-Lipschitz functions on $E$) belong to $D(\nabla_H)$ with a suitable bound; this result was needed in the proof of Theorem 5.4. We point out that this result becomes trivial in the case $E = \mathbb{R}^d = H$, which is the setting for studying the classical Ornstein-Uhlenbeck operator on $L^p(\mathbb{R}^d, \gamma)$ (see (5.1)). Readers whose main interests concern this particular case will therefore not need the result presented here.

We recall some further standard facts about reproducing kernel Hilbert spaces. The reader is referred to [11, 52] for the proofs and more details. Recall that $H_\mu$ denotes the reproducing kernel Hilbert space associated with the invariant measure $\mu$ (see (5.2)) and that $i_\mu : H_\mu \to E$ denotes the inclusion mapping. Since $\mu$ is Radon, the Hilbert space $H_\mu$ is separable. When no confusion can arise we will suppress the mapping $i_\mu$ from our notations and identify $H_\mu$ with its image in $E$.

The mapping
\[ \phi : i_\mu^* x^* \mapsto \langle \cdot, x^* \rangle \]
extends to an isometric embedding of $H_\mu$ into $L^2(E, \mu)$. In what follows we shall write $\phi_h := \phi h$ for the image in $L^2(E, \mu)$ of a vector $h \in H_\mu$. By the Karhunen-Loève decomposition (see [11, Corollary 3.5.11]), if $(h_n)_{n \geq 1}$ is an orthonormal basis for $H_\mu$, then for $\mu$-almost all $x \in E$ we have
\[ x = \sum_{n=1}^{\infty} \phi_{h_n}(x) h_n \]
with convergence both $\mu$-almost surely in $E$ and in the norm of $L^2(E, \mu)$. We may furthermore choose the vectors $h_n \in H_\mu$ in such a way that $h_n = i_\mu^* x_n^*$ for suitable $x_n^* \in E^*$. In doing so, this exhibits the function $x \mapsto x$ as the limit (for $N \to \infty$) of the cylindrical functions $\sum_{n=1}^{N} \langle x, x_n^* \rangle h_n$.

The next lemma relates functions which have pointwise directional derivatives in the direction of $H$ with functions in the domain of the directional gradient $\nabla_H$. To this end we recall that a function $f : E \to \mathbb{R}$ is said to be Gâteaux differentiable in the direction of $H$ at a point $x \in E$ if there exists an element $h(x) \in H$, the Gâteaux derivative of $f$ at the point $x$ such that for all $h \in H$ we have
\[ \lim_{t \downarrow 0} \frac{1}{t} (f(x + th) - f(x)) = \langle h, h(x) \rangle. \]
The function $f$ is said to be Gâteaux differentiable in the direction of $H$ if it is Gâteaux differentiable in the direction of $H$ at every point $x \in E$. The resulting function which assigns to each point $x \in E$ the Gâteaux derivative of $f$ at the point $x$ is denoted by $D_H f : E \to H$.

**Lemma 7.1.** If $f : E \to \mathbb{R}$ is uniformly bounded and Gâteaux differentiable in the direction of $H$, with bounded and strongly measurable derivative $D_H f$, then $f \in D(\nabla_H)$ and $\nabla_H f = D_H f$. 


Proof. Let \((h_n)_{n \geq 1}\) be a fixed orthonormal basis in \(H_\mu\), chosen in such way that \(h_n = i_\mu^* x_n^*\) for suitable \(x_n^* \in E^*\). For all \(N \geq 1\) we have, for \(\mu\)-almost all \(x \in E\),

\[
f_N(x) := f \left( \sum_{n=1}^{N} \phi_{h_n}(x) h_n \right) = \psi_N(\phi_{h_1}(x), \ldots, \phi_{h_N}(x)),
\]

where the function

\[
\psi_N(t_1, \ldots, t_N) = f \left( \sum_{n=1}^{N} t_n h_n \right)
\]

belongs to \(C^1_b(\mathbb{R}^N)\). Since we are assuming that \(h_n = i_\mu^* x_n^*\), \(f_N\) belongs to \(D(\nabla_H)\) and for \(\mu\)-almost all \(x \in E\) we have

\[
\nabla_H f_N(x) = \sum_{j=1}^{N} \frac{\partial \psi_N}{\partial t_j}(\phi_{h_1}(x), \ldots, \phi_{h_N}(x)) h_j = \sum_{j=1}^{N} \langle h_j, D_H f \left( \sum_{n=1}^{N} \phi_{h_n}(x) h_n \right) \rangle h_j
\]

noting that

\[
\frac{\partial \psi_N}{\partial t_j}(t_1, \ldots, t_N) = \lim_{\tau \to 0} \frac{1}{\tau} \left[ f \left( \sum_{n=1}^{N} (t_n + \delta_{jn} \tau) h_n \right) - f \left( \sum_{n=1}^{N} t_n h_n \right) \right]
\]

\[
= \langle h_j, D_H f \left( \sum_{n=1}^{N} t_n h_n \right) \rangle,
\]

with \(\delta_{jn}\) the Kronecker symbol.

To finish the proof we will show three things:

(i) \(\lim_{N \to \infty} f_N = f\) in \(L^2(E, \mu)\);

(ii) the sequence \((\nabla_H f_N)_{n \geq 1}\) is Cauchy in \(L^2(E, \mu; H)\);

(iii) \(\mu\)-almost everywhere we have \(\lim_{N \to \infty} \nabla_H f_N = D_H f\).

Once we have this, the closedness of \(\nabla_H\) will imply that \(f \in D(\nabla_H)\) and \(\nabla_H f = \lim_{N \to \infty} \nabla_H f_N = D_H f\).

(i): The first claim follows by dominated convergence.

(ii) and (iii): Fix integers \(M \geq N \geq 1\). Then,

\[
\left\| \sum_{j=1}^{M} \langle h_j, D_H f \left( \sum_{n=1}^{M} \phi_{h_n}(\cdot) h_n \right) \rangle h_j - \sum_{j=1}^{N} \langle h_j, D_H f \left( \sum_{n=1}^{N} \phi_{h_n}(\cdot) h_n \right) \rangle h_j \right\|_{L^2(E, \mu; H)}
\]

\[
\leq \left\| \sum_{j=1}^{M} \langle h_j, D_H f \left( \sum_{n=1}^{M} \phi_{h_n}(\cdot) h_n \right) - D_H f(\cdot) \rangle h_j \right\|_{L^2(E, \mu; H)}
\]

\[
+ \left\| \sum_{j=1}^{M} \langle h_j, D_H f(\cdot) \rangle h_j - \sum_{j=1}^{N} \langle h_j, D_H f(\cdot) \rangle h_j \right\|_{L^2(E, \mu; H)}
\]

\[
+ \left\| \sum_{j=1}^{N} \langle h_j, \left[ D_H f \left( \sum_{n=1}^{N} \phi_{h_n}(\cdot) h_n \right) - D_H f(\cdot) \right] \rangle h_j \right\|_{L^2(E, \mu; H)}
\]

\[
=: (I) + (II) + (III).
\]

To deal with (II) we note that from \(H_\mu \hookrightarrow H\) it follows that \(f\) has a bounded Gâteaux derivative in the direction of \(H_\mu\), given by \(D_{H_\mu} f = i_\mu^* H D_H f\), where \(i_\mu, H\)
is the embedding mapping of $H_\mu$ into $H$. Now, since $(h_n)_{n \geq 1}$ is an orthonormal basis in $H_\mu$, for $\mu$-almost all $x \in E$ we have

$$\lim_{K \to \infty} \sum_{j=1}^{K} (D_H f(x), h_j)_H h_j = \lim_{K \to \infty} \sum_{j=1}^{K} (i_{\mu,H}^* D_H f(x), h_j)_H h_j = \lim_{K \to \infty} \sum_{j=1}^{K} (D_{H_\mu} f(x), h_j)_{H_\mu} h_j = D_{H_\mu} f(x)$$

with convergence in $H_\mu$, hence in $H$. Convergence in $L^2(E, \mu; H)$ then follows by dominated convergence, noting that $D_{H_\mu} f$ is uniformly bounded as an $H_\mu$-valued function, hence also as an $H$-valued function.

Convergence of (I) and (III) follows in the same way, now using that

$$\left\| \sum_{j=1}^{K} \langle h_j, \left[ D_H f \left( \sum_{n=1}^{K} \phi_{h_n}(\cdot) h_n \right) - D_H f(\cdot) \right] \right\|_{L^2(E, \mu; H)}$$

$$\leq \| i_{\mu,H} \| \left\| \sum_{j=1}^{K} \left( \phi_{h_n}(\cdot) h_n \right) \right\|_{L^2(E, \mu; H_\mu)}$$

$$\leq \| i_{\mu,H} \| \left\| D_{H_\mu} f \left( \sum_{n=1}^{K} \phi_{h_n}(\cdot) h_n \right) - D_{H_\mu} f(\cdot) \right\|_{L^2(E, \mu; H_\mu)},$$

and the right-hand side tends to 0 as $K \to \infty$ by dominated convergence, since $\sum_{n=1}^{K} \phi_{h_n}(x) h_n \to x$ for $\mu$-almost all $x \in E$ and the function $D_{H_\mu} f = i_{\mu,H}^* D_H f$ is uniformly bounded.

We now define $\text{Lip}_H(E)$ as the vector space of all measurable functions that are Lipschitz continuous in the direction of $H$, i.e., for which there exists a finite constant $L_f(H)$ such that

$$\| f(x+h) - f(x) \| \leq L_f(H) \| h \|_H \quad \forall x \in E.$$

Note that we take norms in $H$ on the right-hand side. Obviously, every $f \in \text{Lip}(E)$ belongs to $\text{Lip}_H(E)$, since

$$\| f(x+h) - f(x) \| \leq L_f \| h \|_E \leq L_f \| i_H \|_{\mathcal{L}(H,E)} \| h \|_H.$$

Here, $L_f$ is the Lipschitz constant of $f$ and $i_H$ is the embedding of $H$ into $E$. It is also easy to see that if $f : E \to \mathbb{R}$ has a uniformly bounded Gâteaux derivative in the direction of $H$, then $f \in \text{Lip}_H(E)$ with constant $L_f(H) \leq \| D_H f \|_E$.

**Theorem 7.2.** Let Assumptions 2.1 and 5.3 hold. If $f \in \text{Lip}_H(E)$ is uniformly bounded and has $H$-Lipschitz constant $L_f(H)$, then $f \in \mathcal{D}(\nabla_H)$, $\nabla_H f \in L^\infty(E, \mu)$, and $\| \nabla_H f \|_\infty \leq L_f(H)$.

**Proof.** It follows from [11, Theorem 5.11.2] and the observation following it that $f$ is Gâteaux differentiable in the direction of $H \mu$-almost everywhere, with derivative satisfying $\| D_H f \| \leq L_f(H) \mu$-almost everywhere. This derivative is weakly measurable, as each $(D_H f, x^\ast)$ is the almost everywhere limit of continuous difference quotients. Since $H$ is separable, the Pettis Measurability theorem (see [21, Section 2]) implies that $D_H f$ is strongly measurable. Now the result follows from the previous lemma. \[\square\]
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