UNIFORMIZING GROMOV HYPERBOLIC SPACES WITH BUSEMANN FUNCTIONS

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Abstract. Given a complete Gromov hyperbolic space $X$ that is roughly starlike from a point $\omega$ in its Gromov boundary $\partial G_X$, we use a Busemann function based at $\omega$ to construct an incomplete unbounded uniform metric space $X_\varepsilon$ whose boundary $\partial X_\varepsilon$ can be canonically identified with the Gromov boundary $\partial X$ of $X$ relative to $\omega$. This uniformization construction generalizes the procedure used to obtain the Euclidean upper half plane from the hyperbolic plane. Furthermore we show, for an arbitrary metric space $Z$, that there is a hyperbolic filling $X$ of $Z$ that can be uniformized in such a way that the boundary $\partial X_\varepsilon$ has a biLipschitz identification with the completion $\bar{Z}$ of $Z$.

1. Introduction

The goal of this paper is to construct an unbounded analogue of the uniformizations of Gromov hyperbolic spaces built by Bonk, Heinonen and Koskela in their extensive study of a number of problems in conformal analysis [4]. The most familiar special case of our procedure is the construction of the upper half-space $\{(x, y) : y > 0\}$ in $\mathbb{R}^2$ from the hyperbolic plane $\mathbb{H}^2$, which is discussed in Example 1.4. The guiding example in [4], by comparison, is the relationship between $\mathbb{H}^2$ and the Euclidean unit disk $\{(x, y) : x^2 + y^2 < 1\}$. As can be seen from these examples, the input for uniformization is a geodesic Gromov hyperbolic space $X$ and the output is an incomplete metric space $\Omega$ obtained from a conformal deformation of $X$ that is uniform in the sense of Definition 1.1 below. The density used for uniformizing a Gromov hyperbolic space $X$ in [4] is exponential in the distance to a fixed point $z$ of $X$. In contrast we will be using a density that is exponential in a Busemann function associated to a particular point of the Gromov boundary of $X$. This choice of density is natural as Busemann functions are often interpreted as distance functions “from infinity” and can themselves be used to define a boundary of the space $X$ [1, §3].

Our principal application of this uniformization construction will be to hyperbolic fillings $X$ of a metric space $Z$, with a particular focus on the case in which $Z$ is unbounded. When $Z$ is bounded a hyperbolic filling $X$ of $Z$ can be thought of as a Gromov hyperbolic graph whose Gromov boundary is canonically identified with $Z$; in the case that $Z$ is unbounded there are some additional subtleties to this notion owing to the fact that the Gromov boundary of a Gromov hyperbolic space is always bounded. We refer to the discussion prior to Theorem 1.7 for further information on this, as well as the contents of Section 5. Our use of Busemann functions in this setting is inspired by the hyperbolic filling construction of Buyalo and Schroeder [8, Chapter 6] for arbitrary metric spaces $Z$.

Our uniformization construction for hyperbolic fillings will be used in a followup work in order to establish a correspondence between Newton-Sobolev classes of functions on the hyperbolic filling of $Z$ and Besov classes of functions on $Z$ in the special case that $Z$ carries a doubling measure. This is heavily inspired by work of A. Björn, J. Björn, and Shanmugalingam [3] that establishes the corresponding result in the case that $Z$ is bounded.
In that same paper we will also generalize to our setting their results \cite{2} on how local Poincaré inequalities transform under the uniformization in \cite{4}. This will yield some interesting new examples of uniform metric spaces satisfying Poincaré inequalities. There are a number of known variants on the correspondence between function spaces on the hyperbolic filling and function spaces on $Z$, see for instance \cite{7}, \cite{5}, \cite{6}. Such correspondences were one of the original motivating factors in the use of hyperbolic fillings in analysis on metric spaces. For applications to trace theorems on Ahlfors regular metric spaces that demonstrate the power of these correspondences we refer to \cite{12}.

Stating our main theorem requires two key preliminary definitions. For a metric space $(X, d)$ and a curve $\gamma: I \to X$, $I \subseteq \mathbb{R}$, we write $\ell(\gamma)$ for the length of $\gamma$ measured in $X$. We say that $\gamma$ is rectifiable if $\ell(\gamma) < \infty$. The curve $\gamma$ is a geodesic if it is isometric as a mapping of $I$ into $X$. We will follow the standard practice of using $\gamma$ to denote both the parametrization of the curve and the image of the curve in $X$. We say that $X$ is rectifiably connected if any two of its points can be joined by a rectifiable curve, and we say that $X$ is geodesic if any two points can be joined by a geodesic. We will use the following distance notation for distance from a point $x$ to a set $E$ in any metric space $(X, d)$,

$$\text{dist}(x, E) = \inf_{y \in E} d(x, y),$$

and in particular will write $\text{dist}(x, \gamma)$ for the distance of a point $x \in X$ to (the image of) a curve $\gamma$. Lastly, for reference later, a mapping $f : (X, d) \to (X', d')$ between metric spaces is $C$-Lipschitz for a constant $C \geq 0$ if $d'(f(x), f(y)) \leq Cd(x, y)$ for all $x, y \in X$. We will usually be applying this notion in the case $X' = \mathbb{R}$ with $d'$ being the Euclidean metric on $\mathbb{R}$.

We now define uniform metric spaces. We start with an incomplete metric space $(\Omega, d)$ that is rectifiably connected. We denote the boundary of $\Omega$ in its completion $\bar{\Omega}$ by $\partial \Omega = \bar{\Omega} \setminus \Omega$. We continue to write $d$ for the canonical extension of the metric on $\Omega$ to $\bar{\Omega}$. For $x \in \Omega$ we write $d_{\Omega}(x) := \text{dist}(x, \partial \Omega)$ for the distance from $x$ to the boundary $\partial \Omega$.

**Definition 1.1.** For a constant $A \geq 1$ and a compact interval $I \subset \mathbb{R}$, a curve $\gamma: I \to \Omega$ with endpoints $x, y \in \Omega$ is $A$-uniform if

$$\ell(\gamma) \leq Ad(x, y),$$

and if for every $t \in I$ we have

$$\min\{\ell(\gamma|\{s \in I : s \leq t\}), \ell(\gamma|\{s \in I : s \geq t\})\} \leq Ad_{\Omega}(\gamma(t)).$$

We say that the metric space $\Omega$ is $A$-uniform if any two points in $\Omega$ can be joined by an $A$-uniform curve.

We note that the standard definition of a uniform metric space also requires local compactness. We drop this requirement from the definition because the output metric space of our uniformization procedure need not be locally compact in all cases.

For a continuous function $\rho : (0, \infty) \to X$ we write

$$\ell_\rho(\gamma) = \int_\gamma \rho \, ds,$$

for the line integral of $\rho$ along $\gamma$. We refer to \cite{4} Appendix for a detailed discussion of line integrals in our context.
Definition 1.2. Let \((X, d)\) be a rectifiably connected metric space and let \(\rho : X \to (0, \infty)\) be continuous. The \textit{conformal deformation} of \(X\) with conformal factor \(\rho\) is the metric space \(X_\rho = (X, d_\rho)\) with metric
\[
d_\rho(x, y) = \inf \ell_\rho(\gamma),
\]
with the infimum taken over all curves \(\gamma\) joining \(x\) to \(y\).

If \(X\) is geodesic then we say further that the density \(\rho\) is \textit{admissible} for \(X\) with constant \(M \geq 1\) if for any \(x, y \in X\) and any geodesic \(\gamma\) joining \(x\) to \(y\) we have
\[
\ell_\rho(\gamma) \leq M d_\rho(x, y).
\]

(1.3)

We can now state our main theorem. Most formal definitions regarding Gromov hyperbolicity and the Gromov boundary are postponed to Section 2 as they can be found in any standard reference such as [8, 10]. A geodesic metric space \(X\) is \textit{Gromov hyperbolic} if there is a \(\delta \geq 0\) such that all geodesic triangles are \(\delta\)-\textit{thin}, meaning that for any geodesic triangle \(\Delta\) each edge of \(\Delta\) is contained in a \(\delta\)-neighborhood of the other two edges of \(\Delta\). In this case we will also say that \(X\) is \(\delta\)-\textit{hyperbolic}. We write \(\partial_G X\) for the Gromov boundary of \(X\), to be defined in Section 2.

We consider a complete geodesic \(\delta\)-hyperbolic space \(X\) and a geodesic ray \(\gamma : [0, \infty) \to X\). The \textit{Busemann function} \(b_\gamma : X \to \mathbb{R}\) associated to \(\gamma\) is defined by the limit
\[
b_\gamma(x) = \lim_{t \to \infty} d(\gamma(t), x) - t.
\]

(1.4)

Using the triangle inequality and the fact that \(d(\gamma(t), \gamma(0)) = t\), it’s easy to check that the right side is increasing in \(t\) and bounded above by \(d(\gamma(0), x)\), so this limit exists. It’s also easily verified that \(b_\gamma\) is 1-Lipschitz, thus in particular is continuous. For \(\varepsilon > 0\) we can therefore define a positive continuous density \(\rho_{\varepsilon, \gamma}\) on \(X\) by
\[
\rho_{\varepsilon, \gamma}(x) = e^{-\varepsilon b_\gamma(x)}.
\]

We write \(X_{\varepsilon, \gamma} = (X, d_{\varepsilon, \gamma})\) for the conformal deformation of \(X\) with conformal factor \(\rho_{\varepsilon, \gamma}\).

We write \(\omega = [\gamma] \in \partial_G X\) for the point on the Gromov boundary defined by this geodesic ray.

Theorem 1.3. Let \(X\) be a complete geodesic \(\delta\)-hyperbolic space and let \(\gamma : [0, \infty) \to X\) be a geodesic ray in \(X\). We say that \(X\) is \(K\)-\textit{roughly startlike} from the endpoint \(\omega \in \partial_G X\) of \(\gamma\) for some \(K \geq 0\). Let \(\varepsilon > 0\) be given such that \(\rho_{\varepsilon, \gamma}\) is admissible for \(X\) with constant \(M\).

Then bounded geodesics in \(X\) are \(A\)-\textit{uniform} curves in \(X_{\varepsilon, \gamma}\), with \(A = A(\delta, K, \varepsilon, M)\) depending only on \(\delta, K, \varepsilon, M\) and the admissibility constant \(M\). Consequently \(X_{\varepsilon, \gamma}\) is an \(A\)-\textit{uniform} metric space.

We describe the motivating example for this theorem below.

Example 1.4. Let \(\mathbb{U}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}\) be the upper half space in \(\mathbb{R}^2\) equipped with the Euclidean metric, which is easily seen to be a uniform metric space. Let \(\mathbb{H}^2\) denote the upper half plane model of the hyperbolic plane, which is \(\mathbb{U}^2\) equipped with the Riemannian metric \(ds^2 = \frac{dx^2 + dy^2}{y^2}\). Define \(\gamma : [0, \infty) \to \mathbb{H}^2\) by \(\gamma(t) = (0, e^t)\). Then \(\gamma\) is a geodesic ray in \(\mathbb{H}^2\).

From explicit formulas for the hyperbolic distance in this model (see for instance [8, A.3]) it is straightforward to calculate that the associated Busemann function is given by \(b_\gamma(x, y) = -\log y\). Setting \(\varepsilon = 1\), the density \(\rho_{1, \gamma}\) is thus simply given by \(\rho_{1, \gamma}(x, y) = y\).

Therefore the uniformized metric space \(\mathbb{H}^2_{1, \gamma}\) is isometric to \(\mathbb{U}^2\).
Returning to the setting of Theorem 1.3 since \( b_\gamma(t) = -t \) for \( t \in [0, \infty) \), the claim that \( X_{\varepsilon, \gamma} \) is unbounded follows from an easy calculation of the restriction of the metric \( d_{\varepsilon, \gamma} \) to \( \gamma \) using the admissibility inequality (1.3). See Lemma 4.3.

Rough starlikeness is a technical condition on geodesic lines \( \sigma : \mathbb{R} \to X \) starting from \( \omega \). For a given \( K \geq 0 \) it requires that any point in \( X \) is within distance \( K \) of a geodesic line starting from \( \omega \), and it also requires that any point \( \xi \in \partial_G X \setminus \{ \omega \} \) is the endpoint of a geodesic line starting from \( \omega \). We refer to Definition 2.1 for the full details. This rough starlikeness condition will be satisfied for \( K = \frac{1}{2} \) in our application of Theorem 1.3 to hyperbolic fillings in Theorem 1.7. The rough starlikeness condition is not particularly restrictive; it is satisfied for \( K = 0 \) in the setting of simply connected Riemannian manifolds with pinched negative curvature, for instance, by considering the gradient flow associated to the Busemann function \( b_\gamma \).

In contrast to [4, Theorem 1.1], we do not impose any local compactness hypotheses on the space \( X \). It turns out that the rough starlikeness hypothesis on its own is sufficient to prove that \( X_{\varepsilon, \gamma} \) is a uniform metric space. The requirement that \( X \) is complete is necessary in order to ensure the correspondence between \( \partial_G X \setminus \{ \omega \} \) and \( \partial X_{\varepsilon, \gamma} \) that we discuss below.

The admissibility inequality (1.3) axiomatizes a key property of the density \( \rho \) that is referred to in [4, Chapter 5] as the Gehring-Hayman theorem for Gromov hyperbolic spaces. More precisely, they show that if a continuous positive density \( \rho \) on \( X \) satisfies a Harnack type inequality (see inequality (1.2) later in the paper) then for \( \varepsilon > 0 \) sufficiently small the property (1.3) will hold for the density \( \rho' \). We refer to [4, Chapter 5] for a history of this property, dating back to its analogue for simply connected hyperbolic domains in the complex plane [9]. Since \( b_\gamma \) is 1-Lipschitz, the density \( \rho_{\varepsilon, \gamma} \) satisfies the exact same Harnack type inequality as was used for the densities in [4, Chapter 4]. Consequently by [4, Theorem 5.1] the same \( \varepsilon_0 = \varepsilon_0(\delta) \) used as an upper threshold for uniformization in [4, Theorem 1.1] works in our context as well. In particular we have for all \( 0 < \varepsilon \leq \varepsilon_0(\delta) \) that \( \rho_{\varepsilon, \gamma} \) is admissible for \( X \) with admissibility constant \( M = M(\delta) \) depending only on \( \delta \). Thus we can always satisfy the admissibility condition of Theorem 1.3 by taking \( \varepsilon \) sufficiently small, depending only on \( \delta \). We’ve made inequality (1.3) part of the hypotheses of Theorem 1.3 because we will need to go outside the range \( 0 < \varepsilon \leq \varepsilon_0(\delta) \) in the proof of Theorem 1.7.

We write \( \partial_\omega X = \partial_G X \setminus \{ \omega \} \) for the complement of a point \( \omega \in \partial_G X \) in the Gromov boundary of a \( \delta \)-hyperbolic space \( X \). We will refer to \( \partial_\omega X \) as the Gromov boundary relative to \( \omega \) for reasons that will be explained in Proposition 2.8. As part of the proof of Theorem 1.3 we will show that there is a canonical identification \( \iota : \partial_\omega X \to \partial X_{\varepsilon, \gamma} \) between the Gromov boundary of \( X \) relative to \( \omega \) and the boundary of \( X_{\varepsilon, \gamma} \) as a metric space. The most important property of this identification is summarized in Theorem 1.5 below.

The Busemann function \( b_\gamma \) can be used to define a natural class of metrics on \( \partial_\omega X \) known as visual metrics based at \( \omega \) (see [8, Chapter 3] as well as Section 2.3 below). These visual metrics have an associated parameter \( q > 0 \). While one may not be able to find a visual metric \( \alpha \) on \( \partial_\omega X \) for every parameter \( q \), visual metrics with parameter \( q \) always exist when \( q \) is sufficiently small, with the upper bound depending only on \( \delta \). We recall that a map \( f : (X, d) \to (X', d') \) between metric spaces is \( L \)-biLipschitz for a constant \( L \geq 1 \) if for all \( x, y \in X \) we have

\[
L^{-1}d(x, y) \leq d'(f(x), f(y)) \leq Ld(x, y).
\]

**Theorem 1.5.** Let \( X \) be a complete geodesic \( \delta \)-hyperbolic space that is \( K \)-roughly starlike from \( \omega \in \partial_G X \), let \( \gamma \) be a geodesic ray in \( X \), and let \( \varepsilon > 0 \) be given such that \( \rho_{\varepsilon, \gamma} \) is admissible for \( X \) with constant \( M \). Let \( \alpha \) be a visual metric on \( \partial_\omega X \) based at \( \omega \) with parameter \( 0 < q \leq \varepsilon \). Then there is a canonical identification \( \iota : \partial_\omega X \to \partial X_{\varepsilon, \gamma} \) that induces
an $L$-biLipschitz map

$$\iota : (\partial_\omega X, \alpha) \to (\partial X_{\varepsilon, \gamma}, d_{\varepsilon, \gamma}^2),$$

with $L = L(\delta, K, \varepsilon, M, q)$.

For a precise description of the identification $\iota$ we refer to Lemma 1.6. Observe in particular that when $q = \varepsilon$ we obtain a biLipschitz identification of $(\partial_\omega X, \alpha)$ with $(\partial X_{\varepsilon, \gamma}, d_{\varepsilon, \gamma})$. This biLipschitz identification is a key component of the trace theorems in [3].

**Remark 1.6.** It is possible to deduce the uniformization results of [4] directly from our results through a basic construction. We let $(X, d)$ be a complete geodesic $\delta$-hyperbolic space and write for $x, z \in X$, $\varepsilon > 0$,

$$\rho_{\varepsilon, z}(x) = e^{-\varepsilon d(x, z)}.$$

We write $X_{\varepsilon, z}$ for the conformal deformation of $X$ with conformal factor $\rho_{\varepsilon, z}$. We say that $X$ is $K$-roughly starlike from $z$ if each point of the Gromov boundary $\partial G X$ is the endpoint of a geodesic ray starting from $z$ and if for each $x \in X$ there is a geodesic ray $\gamma$ starting at $z$ such that $\text{dist}(x, \gamma) \leq K$. We will assume this condition on $X$ in what follows. We will also be assuming that $\rho_{\varepsilon, z}$ is admissible for $X$ with constant $M$.

Let $Y = X \cup_{z \sim_0} [0, \infty)$ be the complete geodesic metric space obtained by identifying $z$ with $0 \in [0, \infty)$. Let $\gamma : [0, \infty) \to Y$ be the geodesic ray parametrizing the copy of $[0, \infty)$ that we glued to $X$. It is easy to compute that $b_{\gamma}(x) = d(x, z)$ for $x \in X$. Consequently the density $\rho_{\varepsilon, \gamma}$ on $Y$ coincides with $\rho_{\varepsilon, z}$ on $X \subset Y$. It’s straightforward to show from this that we have an isometric inclusion $X_{\varepsilon, z} \subset Y_{\varepsilon, \gamma}$ induced by the isometric inclusion of $X$ into $Y$.

One then checks directly that $Y$ is also $\delta$-hyperbolic, that $Y$ is $K$-roughly starlike from the point $\omega \in \partial G Y$ corresponding to the geodesic ray $\gamma$, and that $\rho_{\varepsilon, \gamma}$ is admissible for $Y$ with the same constant $M$. Furthermore we have an identification $\partial G X = \partial Y$ of the Gromov boundary of $X$ with the Gromov boundary of $Y$ relative to $X$. The main uniformization claims of [4] Chapter 4 then follow by applying Theorems 3.13 and 3.15 to $Y$ and restricting to $X_{\varepsilon, z} \subset Y_{\varepsilon, \gamma}$.

We will apply our uniformization results to hyperbolic fillings of an arbitrary metric space $(Z, d)$. We briefly describe the construction of the hyperbolic filling here, with further details in Section 5 including proofs for the claims made here. Our construction will depend in part on two parameters $0 < a < 1$ and $\tau > 1$. For an $r > 0$ we say that a subset $S \subset Z$ is $r$-separated if for each $x, y \in S$ we have $d(x, y) \geq r$. Given a parameter $0 < a < 1$, we choose for each $n \in \mathbb{Z}$ a maximal $a^n$-separated subset $S_n$ of $Z$. For $n \in \mathbb{Z}$ we write $V_n = \{(z, n) : z \in S_n\}$ and set $V = \bigcup_{n \in \mathbb{Z}} V_n$. The set $V$ will serve as the vertex set for $X$.

We associate to each vertex $v = (z, n) \in V$ the ball $B(v) = B(z, \tau a^n)$ of radius $\tau a^n$ centered at $z$. We place an edge between vertices $v, w \in V$ if and only if $|h(v) - h(w)| \leq 1$ and $B(v) \cap B(w) \neq \emptyset$. We write $X$ for the resulting graph and call this a **hyperbolic filling** of $Z$. If $\tau$ is sufficiently large in relation to $(1 - a)^{-1}$ (see inequality (5.1)) then $X$ will be a connected graph by Proposition 5.6. We make $X$ into a geodesic metric space by declaring all edges to have unit length.

As a metric space $X$ is $\delta$-hyperbolic with $\delta = \delta(a, \tau)$ depending only on the parameters $a$ and $\tau$. There is a distinguished point $\omega \in \partial G X$ in the Gromov boundary that can be thought of as an ideal point at infinity for $Z$. We have an identification $\partial_\omega X \cong \bar{Z}$ of the Gromov boundary relative to $\omega$ with the completion $\bar{Z}$ of $Z$. Under this identification the extension of the metric $d$ to $\bar{Z}$ defines a visual metric based at $\omega$ on $\partial_\omega X$ with parameter $q = -\log a$. 
We define the height function $h : X \to \mathbb{R}$ by setting $h(v) = n$ for a vertex $v = (z, n)$ of $X$ and linearly interpolating the values of $h$ from the vertices to the edges of $X$. The height function $h$ is $1$-Lipschitz and, by Lemma 5.13 below, there is a Busemann function $b$ based at $\omega$ such that $|h(x) - h(x)| \leq 3$ for all $x \in X$.

We write $\rho_{c}(x) = e^{-ch(x)}$ and let $X_{c}$ be the metric space obtained by conformally deforming $X$ by the conformal factor $\rho_{c}$. Observe that none of the hypotheses or conclusions of Theorems 1.3 and 1.4 are affected if we use the conformal factor $\rho_{c}$ instead of the conformal factor $\rho_{c}^{*}(x) = e^{-c\rho(x)}$, since the estimate $|h - b| \leq 3$ implies that $\rho_{c}$ is comparable to $\rho_{c}^{*}$ with factor $e^{3\epsilon}$. Thus $\rho_{c}$ is admissible if and only if $\rho_{c}^{*}$ is admissible (with comparable constants) and the deformation $X_{c}$ is $L$-biLipschitz to $X_{c}^{*}$ with $L$ depending only on $\epsilon$. We will thus be using the height function in place of a Busemann function below.

**Theorem 1.7.** Let $Z$ be a metric space and let $X$ be a hyperbolic filling of $Z$ with parameters $0 < a < 1$ and $\tau > \min\{3, (1 - a)^{-1}\}$. Then $X$ is $\frac{1}{\tau}$-roughly starlike from $\omega$ and for each $0 < \epsilon \leq -\log a$ the density $\rho_{c}$ is admissible for $X$ with constant $M = M(a, \tau, \epsilon)$.

Thus the conclusions of Theorems 1.3 and 1.4 hold for $X_{c}$ for each $0 < \epsilon \leq -\log a$. In particular for $\epsilon = -\log a$ we have a canonical $L$-biLipschitz identification of $\partial X_{c}^{*}$ and $Z$, with $L = L(a, \tau)$.

We compare our results to those of [3] in Remark 6.7.

We provide here an outline of the contents of the rest of the paper. In Section 2 we review several key notions in the setting of Gromov hyperbolic spaces. Section 3 establishes some basic properties of geodesic triangles in Gromov hyperbolic spaces with vertices on the Gromov boundary and gives a rough formula for evaluating Busemann functions on their edges. Section 4 is occupied by the proofs of Theorems 1.3 and 1.5. In Section 5 we construct the hyperbolic fillings of metric spaces that we use in Theorem 1.7 and establish their basic properties. Lastly Theorem 1.7 is proved in Section 6.

We are very grateful to Nageswari Shanmugalingam for providing multiple drafts of the work [3] on which a significant part of this paper is based.

## 2. Hyperbolic metric spaces

### 2.1. Definitions

Let $X$ be a set and let $f$, $g$ be real-valued functions defined on $X$. For $c \geq 0$ we will write $f \simeq_{c} g$ if

$$|f(x) - g(x)| \leq c,$$

for all $x \in X$. If the exact value of the constant $c$ is not important or implied by context we will often just write $f \simeq g$. We will sometimes refer to the relation $f \simeq g$ as a rough equality between $f$ and $g$.

If $C \geq 1$ and $f$ and $g$ both take values in $(0, \infty)$, we will write $f \asymp_{C} g$ if

$$C^{-1}g(x) \leq f(x) \leq Cg(x).$$

We will similarly write $f \asymp g$ if the value of $C$ is not important or implied by context. Note that if $f \asymp_{c} g$ then $e^{t} \asymp_{c} e^{g}$, and similarly if $f \asymp_{C} g$ then $\log f \asymp_{\log C} \log g$. We will stick to a convention of using lowercase $c$ for additive constants and uppercase $C$ for multiplicative constants. When this constant $c$ is determined by other parameters $\delta$, $K$, etc. under discussion we will write $c = c(\delta, K)$, while continuing to use the shorthand $c$ where it is not ambiguous.

For a metric space $(X, d)$ we write $B(x, r) = \{y : d(x, y) < r\}$ for the open ball of radius $r > 0$ centered at $x$. A map $f : (X, d) \to (X', d')$ between metric spaces $X$ and $X'$ is isometric if $d'(f(x), f(y)) = d(x, y)$ for $x, y \in X$. If furthermore $f$ is surjective then we
say that it is an isometry and that \( X \) and \( X' \) are isometric. For a constant \( c \geq 0 \) a map \( f : X \to X' \) is defined to be \( c \)-roughly isometric if \( d(f(x), f(y)) \approx_c d(x, y) \). As usual we will not mention the exact value of the constants if they are unimportant.

When dealing with Gromov hyperbolic spaces \( X \) in this paper we will use the generic distance notation \( |xy| := d(x, y) \) for the distance between \( x \) and \( y \) in \( X \), except for cases where this could cause confusion. We will often use the generic notation \( xy \) for a geodesic connecting two points \( x, y \in X \), even when this geodesic is not unique; in these cases there will be no ambiguity regarding the geodesic that we are referring to. A geodesic triangle \( \Delta \) in \( X \) is a collection of three points \( x, y, z \in X \) together with geodesics \( xy, xz, \) and \( yz \) joining these points, which we will refer to as the edges of \( \Delta \). We will also alternatively write \( xyz = \Delta \) for a geodesic triangle with vertices \( x, y \) and \( z \).

For \( x, y, z \in X \) the Gromov product of \( x \) and \( y \) based at \( z \) is defined by

\[
(\langle x|y\rangle)_z = \frac{1}{2}(\langle xz\rangle + \langle yz\rangle - \langle xy\rangle).
\]

We note the basepoint change inequality for \( x, y, z, p \in X \),

\[
||\langle x|y\rangle_z - \langle x|y\rangle_p| \leq |zp|,
\]

which follows from the triangle inequality.

By [10] Chapitre 2, Proposition 21 we have two key consequences of \( \delta \)-hyperbolicity for a metric space \( X \) regarding Gromov products. The first is that for every \( x, y, z, p \in X \) we have

\[
(\langle x|z\rangle)_p \geq \min\{(\langle x|y\rangle)_p, (\langle y|z\rangle)_p\} - 4\delta.
\]

We refer to (2.3) as the \( 4\delta \)-inequality.

The second is that for any geodesic triangle \( xyz \) in \( X \) we have that if \( p \in xy, q \in xz \) are points with \( |xp| = |xq| \leq (\langle y|z\rangle)_x \) then \( |pq| \leq 4\delta \). Here \( xy \) and \( xz \) are referring to the corresponding geodesics in the triangle \( \Delta \). We will refer to this as the \( 4\delta \)-tripod condition.

Both inequality (2.3) and the tripod condition can be taken as equivalent definitions of hyperbolicity. By [10] Chapitre 2, Proposition 21 all of these definitions are equivalent up to a factor of 4. We note that the definition using inequality (2.3) does not use the fact that \( X \) is geodesic, and is therefore used as a definition of \( \delta \)-hyperbolicity for general metric spaces. We will be citing several basic results from [8] in which inequality (2.3) is used as the definition of \( \delta \)-hyperbolicity (with \( \delta \) in place of \( 4\delta \)). Wherever necessary we have multiplied the constants used in their results by 4 in order to account for this discrepancy.

Let \( X \) be a geodesic Gromov hyperbolic space and fix \( p \in X \). A sequence \( \{x_n\} \) converges to infinity if we have \( (x_n|x_m)_p \to \infty \) as \( m, n \to \infty \). The Gromov boundary \( \partial G \) of a Gromov hyperbolic space \( X \) is defined to be the set of all equivalence classes of sequences \( \{x_n\} \subset X \) converging to infinity, with the equivalence relation \( \{x_n\} \sim \{y_n\} \) if \( (x_n|y_n)_p \to \infty \) as \( n \to \infty \).

Inequality (2.2) shows that these notions do not depend on the choice of basepoint \( p \).

A second boundary that we can associate to \( X \) is the geodesic boundary \( \partial^g \) of \( X \), which is defined as equivalence classes of geodesic rays \( \gamma : [0, \infty) \to X \), with two geodesic rays \( \gamma \) and \( \sigma \) being equivalent if there is a constant \( c \geq 0 \) such that \( |\gamma(t)\sigma(t)| \leq c \) for \( t \geq 0 \). There is a natural inclusion \( \partial^g X \subseteq \partial G X \) given by sending a geodesic ray \( \gamma \) to the sequence \( \{\gamma(n)\}_{n \in \mathbb{N}} \). This inclusion need not be surjective in general. However, it is always surjective if \( X \) is proper, meaning that the closed balls in \( X \) are compact.

For a point \( \omega \in \partial G \) and a sequence \( \{x_n\} \) converging to infinity we will write \( \{x_n\} \in \omega \) or \( x_n \to \omega \) if \( \{x_n\} \) belongs to the equivalence class of \( \omega \), and for a geodesic ray \( \gamma : [0, \infty) \to X \) and a point \( \omega \in \partial G \) we will write \( \gamma \in \omega \) if \( \{\gamma(n)\} \in \omega \). We will sometimes also consider
geodesic rays \( \gamma : (-\infty, 0] \to X \) with a reversed oriented parametrization, for which we write \( \gamma \in \omega \) if \( \{ \gamma(-n) \} \in \omega \).

For the rest of this paper we will be using the standard notation \( \partial X := \partial_G X \) for the Gromov boundary of a Gromov hyperbolic space \( X \). While this notation does conflict with the notation \( \partial \Omega = \Omega \backslash \Omega \) introduced prior to Definition \([11]\), the meaning of the notation will always be clear from context since we will never use it in the sense of \([11]\) in the context of Gromov hyperbolic spaces.

We can now formally define rough starlikeness from a point of \( \partial X \). We recall that for \( \omega \in \partial X \) we write \( \partial_\omega X = \partial X \backslash \{ \omega \} \) for the Gromov boundary of \( X \) relative to \( \omega \).

**Definition 2.1.** Let \( X \) be a geodesic Gromov hyperbolic space. Let \( \omega \in \partial X \) and \( K \geq 0 \) be given. We say that \( X \) is \( K \)-roughly starlike from \( \omega \) if

1. For each \( x \in X \) there is a geodesic line \( \gamma : \mathbb{R} \to X \) such that \( \text{dist}(x, \gamma) \leq K \) and the geodesic ray \( \gamma|_{[-\infty, 0]} \) satisfies \( \gamma|_{[-\infty, 0]} \in \omega \).
2. For each \( \xi \in \partial_\omega X \) there is a geodesic line \( \gamma : \mathbb{R} \to X \) such that \( \gamma|_{[0, \infty)} \in \xi \) and \( \gamma|_{[-\infty, 0]} \in \omega \).

Part (1) of Definition \([2.1]\) is the analogue, for points in the Gromov boundary \( \partial X \), of the rough starlikeness property required in the main theorem of \([4]\). Part (2) of Definition \([2.1]\) implies that \( \partial^p X = \partial_G X \), i.e., the geodesic boundary and the Gromov boundary coincide. It will be used as a replacement for the properness hypothesis in the main theorem of \([4]\). Recall that a metric space \( X \) is proper if its closed balls are compact. We note that Property (2) of Definition \([2.1]\) automatically holds for any \( \omega \in \partial X \) when \( X \) is proper, since in this case any two points of \( \partial X \) can be joined by a geodesic line \( \gamma : \mathbb{R} \to X \).

We now extend some notions regarding geodesic triangles to the Gromov boundary. For a point \( x \in X \) and a point \( \xi \in \partial X \) we will often write \( x\xi \) for a geodesic ray \( \gamma : [0, \infty) \to X \) with \( \gamma(0) = x \) and \( \gamma \in \xi \), provided such a geodesic ray exists. Similarly, for \( \zeta, \xi \in \partial X \) we will write \( \zeta\xi \) for a geodesic line \( \gamma : \mathbb{R} \to X \) with \( \gamma|_{[-\infty, 0]} \in \zeta \) and \( \gamma|_{[0, \infty)} \in \xi \), provided such a geodesic line exists. Such geodesic lines and rays always exist when \( X \) is proper. We extend the definition of geodesic triangles \( \Delta \) in \( X \) to allow for vertices in \( \partial X \): a geodesic triangle \( xyz = \Delta \) in \( X \) is a collection of three points \( x, y, z \in X \cup \partial X \) together with geodesics \( xy, xz, yz \) connecting them in the sense described above.

**Remark 2.2.** It is easy to see from the definitions that there is no geodesic \( \gamma : \mathbb{R} \to X \) such that \( \gamma|_{[0, \infty)} \) and \( \gamma|_{(-\infty, 0]} \) belong to the same equivalence class in the Gromov boundary \( \partial X \). Hence, for a geodesic triangle \( \Delta \), all vertices of \( \Delta \) on \( \partial X \) must be distinct.

Gromov products based at points \( p \in X \) can be extended to points of \( \partial X \) by defining the Gromov product of equivalence classes \( \xi, \zeta \in \partial X \) based at \( p \) to be

\[
(\xi|\zeta)_p = \inf_{n \to \infty} \liminf_{n \to \infty}(x_n|y_n)_p, 
\]

with the infimum taken over all sequences \( \{x_n\} \in \xi, \{y_n\} \in \zeta \). By \([8]\) Lemma 2.2.2, if \( X \) is \( \delta \)-hyperbolic then for any choices of sequences \( \{x_n\} \in \xi, \{y_n\} \in \zeta \) we have

\[
(\xi|\zeta)_p \leq \liminf_{n \to \infty}(x_n|y_n)_p \leq \limsup_{n \to \infty}(x_n|y_n)_p \leq (\xi|\zeta)_p + 8\delta.
\]

We also have the \( 4\delta \)-inequality for \( \xi, \zeta, \omega \in \partial X \) and \( p \in X \),

\[
(\xi|\omega)_p \geq \min\{(\xi|\zeta)_p, (\zeta|\omega)_p\} - 4\delta.
\]

For \( x \in X, \xi \in \partial X \) the Gromov product is defined analogously as

\[
(x|\xi)_p = \inf_{n \to \infty} \liminf_{n \to \infty}(x_n|x_n)_p
\]
with the infimum taken over \( \{ x_n \} \in \xi \), and the analogues of (2.3) and (2.5) hold as well.

We next note that geodesic triangles \( \Delta \) with vertices in \( X \cup \partial X \) are \( 10\delta \)-thin, in the precise sense that if \( u \in \Delta \) is any given point then there is a point \( v \in \Delta \) satisfying \( |uv| \leq 10\delta \) that does not belong to the same edge of \( \Delta \) as \( u \). When \( X \) is proper this can be easily deduced from the \( \delta \)-thin triangles property for triangles in \( X \) by a limiting argument. This is surely also known without the properness hypothesis, however we lack a reference so we provide a proof here.

**Lemma 2.3.** Let \( \Delta \) be a geodesic triangle in \( X \) with vertices in \( X \cup \partial X \). Then \( \Delta \) is \( 10\delta \)-thin.

**Proof.** Let \( x, y, z \in X \cup \partial X \) be the vertices of \( \Delta \). Let \( u \in \Delta \) be given. Since \( X \) has \( \delta \)-thin triangles, we may assume that \( \Delta \) has at least one vertex on \( \partial X \). We first consider the case in which \( \Delta \) has exactly one vertex on \( \partial X \), which by relabeling we can assume is \( z \). We first assume that \( u \in xy \). Let \( \{ z_n \} \subset xz \) and \( \{ z'_n \} \subset yz \) be sequences such that \( z_n \to z \) and \( z'_n \to z \). For each \( n \) we let \( \Delta_n = xyz_n \) be the geodesic triangle sharing the edge \( xy \) with \( \Delta \), having a second edge be the subsegment \( xz_n \) of \( xz \), and having a third edge be any choice of geodesic \( yz_n \). Then \( \Delta_n \) is \( \delta \)-thin, so we have for each \( n \) that either \( \text{dist}(u, xz_n) \leq \delta \) or \( \text{dist}(u, yz_n) \leq \delta \) (or both). In the first case we are done since \( xz_n \subset xz \), so we can assume that \( \text{dist}(u, yz_n) \leq \delta \). Let \( v_n \in yz_n \) be such that \( |uv_n| \leq \delta \). Then \( |v_n y| \leq \delta + |uy| \).

Since both \( z_n \) and \( z'_n \) converge to \( z \), for sufficiently large \( n \) we will have \((z_n|z'_n)y \geq \delta + |uy|\), which implies in particular that \( |z_n|y| \geq |v_n y| \). The \( 4\delta \)-tripod condition applied to \( y, z_n \), and \( z'_n \) then implies that if \( w_n \in yz_n \) is the unique point such that \( |yw_n| = |yv_n| \) then \( |v_n w_n| \leq 4\delta \), from which it follows that \( |w_n v_n| \leq 5\delta \) for all sufficiently large \( n \). Since \( w_n \in yz \) this completes the proof of this case.

The other cases are \( u \in xz \) and \( u \in yz \). By symmetry it suffices to prove the case \( u \in xz \). We define the sequences \( \{ z_n \} \) and \( \{ z'_n \} \) and the triangle \( \Delta_n \), as before. As in the first case \( u \in xy \) we can assume that \( \text{dist}(u, yz_n) \leq \delta \) for all \( n \), as otherwise by the \( \delta \)-thin triangles property we have \( \text{dist}(u, xy) \leq \delta \) and we are done. We let \( v_n \in yz_n \) be such that \( |uv_n| \leq \delta \), note that \( |v_n y| \leq \delta + |uy| \) as before, and choose \( n \) large enough that \((z_n|z'_n)y \geq \delta + |uy| \). As before the \( 4\delta \)-tripod condition then supplies a point \( w_n \in yz_n \) such that \( |w_n v_n| \leq 4\delta \) and we conclude that \( \text{dist}(u, yz_n) \leq 5\delta \).

We can now handle the case in which potentially two or three vertices of \( \Delta \) belong to \( \partial X \). By symmetry it suffices to show for a point \( u \in xy \) that \( u \) is \( 10\delta \)-close to either \( xz \) or \( yz \). Let \( \{ x_n \} \subset xz \) and \( \{ y_n \} \subset yz \) be sequences such that \( x_n \to x \) and \( y_n \to y \); if \( x \in X \) then we set \( x_n = x \) for all \( n \) and similarly if \( y \in Y \) then we set \( y_n = y \) for all \( n \). Let \( \Delta_n = x_n y_n z \) be a geodesic triangle with one edge the subsegment \( x_n y_n \) of \( xy \). Then \( \Delta_n \) has at most one vertex \( z \) on \( \partial X \). We conclude from the previous case that \( u \) is \( 5\delta \)-close to either \( x_n z \) or \( y_n z \). By switching the roles of \( x \) and \( y \) if necessary, we can then assume that there is \( v \in x_n z \) such that \( |uw| \leq 5\delta \). If \( x \in X \) then \( x_n = x \) and we are done. Thus we can assume that \( x \in \partial X \).

Fix any point \( w \in xz \) and let \( x'_n \in uw \) be defined such that \( |ux'_n| = |uw| \). Since the geodesic rays \( uw \) and \( ux \) define the same point \( x \) of the Gromov boundary, there is a constant \( c \geq 0 \) such that \( |x_n x'_n| \leq c \) for all \( n \). We apply the previous case again to a triangle \( \Delta'_n = x_n x'_n z \) with edges the segment \( x'_n z \), the segment \( x_n z \), and a choice of geodesic \( x_n x'_n \), obtaining that \( v \) is \( 5\delta \)-close to either \( x'_n z \) or \( x_n x'_n \). If \( v \) is \( 5\delta \)-close to \( x_n x'_n \) for all \( n \) then

\[
|x_n u| \leq |uv| + |x_n x'_n| + |x_n x'_n| \leq 10\delta + c,\
\]

contradicting that \( |x_n u| \to \infty \) as \( n \to \infty \). We conclude that \( v \) is \( 5\delta \)-close to \( x'_n z \subset xz \) for all sufficiently large \( n \), which implies that \( \text{dist}(u, xz) \leq 10\delta \) as desired. \( \square \)
2.2. Busemann functions. In this section we closely follow [8] Chapter 3. Throughout much of the paper we will need to work with Gromov products based at a point \( \omega \in \partial X \). These will be defined through the use of Busemann functions. In order to use the results from [8] Chapter 3 we have to show, for a geodesic ray \( \gamma \in \omega \), that \( b_\gamma \) is a Busemann function based at \( \omega \) in their sense. The definition of a Busemann function given there starts with the function

\[
(2.6) \quad b_{\omega, p}(x) = (\omega|p)_x - (\omega|x)_p,
\]

for \( x, p \in X \) and \( \omega \in \partial X \) and defines a Busemann function based at \( \omega \) to be any function \( b : X \to \mathbb{R} \) satisfying \( b \equiv_{8\delta} b_{\omega, p} + c \) for some \( p \in X \) and \( c \in \mathbb{R} \) (recall that we are multiplying all of their constants by 4 due to differing definitions of hyperbolicity). Note that this alternative definition (2.6) makes sense even for points in the Gromov boundary that do not belong to the geodesic boundary.

**Lemma 2.4.** Let \( \omega \in \partial X \), let \( p \in X \), and let \( \gamma \in \omega \) be a geodesic ray with \( \gamma(0) = p \). Then we have \( b_{\omega, p} \equiv_{24\delta} b_\gamma \).

**Proof.** By [8] Example 3.1.4] we have for all \( x \in X \) that

\[
b_{\omega, p}(x) \equiv_{8\delta} |xp| - (\omega|x)_p.
\]

By inequality (2.4) we have \( (\gamma(n)|x)_p \equiv_{8\delta} (\omega|x)_p \) for \( n \in \mathbb{N} \) sufficiently large. Since \( p = \gamma(0) \) we have

\[
(\gamma(n)|x)_p = \frac{1}{2}(n + |xp| - |\gamma(n)x|).
\]

Then

\[
(2.7) \quad |xp| - 2(\omega|x)_p \equiv_{16\delta} |\gamma(n)x| - n.
\]

Since the right side converges to \( b_\gamma(x) \) as \( n \to \infty \), the result follows. \( \square \)

In this paper, for a point \( \omega \in \partial^0 X \) in the geodesic boundary we define a Busemann function based at \( \omega \) to be any Busemann function \( b_\gamma \) associated to a geodesic ray \( \gamma \in \omega \). Thus \( b_\gamma \) is a Busemann function in the sense of [8] Chapter 3 as well, provided that we use a cutoff of \( b \equiv_{24\delta} b_{\omega, p} + c \) instead of the \( 8\delta \)-cutoff used there. This only has the effect of further multiplying constants by 3 in the claims of that chapter. An easy consequence of Lemma 2.4 is the following.

**Lemma 2.5.** Let \( \omega \in \partial X \) and let \( \gamma, \sigma : [0, \infty) \to X \) be geodesic rays with \( \gamma, \sigma \in \omega \). Then there is a constant \( c \in \mathbb{R} \) such that \( b_\gamma \equiv_{72\delta} b_\sigma + c \). The constant \( c \) depends only on the starting points \( \gamma(0) \) and \( \sigma(0) \) of the rays. In particular \( c = 0 \) if \( \gamma(0) = \sigma(0) \).

**Proof.** By [8] Lemma 3.1.2], for each \( p, q, x \in X \) we have

\[
b_{\omega, p}(x) \equiv_{24\delta} b_{\omega, q}(x) + b_{\omega, q}(p).
\]

Setting \( p = \gamma(0) \), \( q = \sigma(0) \), and applying Lemma 2.4 gives

\[
b_\gamma(x) \equiv_{72\delta} b_\sigma(x) + b_{\omega, \sigma(0)}(\gamma(0)).
\]

This gives the claim of the lemma with \( c = b_{\omega, \sigma(0)}(\gamma(0)) \). The last claim follows from the fact that \( b_{\omega, p}(p) = 0 \) for any \( p \in X \). \( \square \)

We will usually use the following lemma to perform computations with Busemann functions in practice. Note that the geodesics are parametrized as starting from the basepoint \( \omega \in \partial X \) instead of ending there.
Lemma 2.6. Let $b$ be a Busemann function on $X$ based at $\omega \in \partial X$. Let $a \in \mathbb{R} \cup \{\infty\}$ and let $\gamma : (-\infty, a] \to X$ be a geodesic with $\gamma(t) \to \omega$ as $t \to -\infty$.

(1) For any $s \in (-\infty, a]$ (or any $s \in \mathbb{R}$ in the case $a = \infty$) we have
\begin{equation}
\label{eqn:1.4}
b(\gamma(t)) - b(\gamma(s)) \geq 144\delta \ t - s.
\end{equation}

(2) For any constant $u \in \mathbb{R}$ there is an arclength reparametrization $\tilde{\gamma} : (-\infty, a] \to X$ of $\gamma$ such that $b(\tilde{\gamma}(t)) \geq 144\delta \ t + u$ for $t \in (-\infty, a]$.

Proof. Let $\sigma : [-a, \infty) \to X$ be defined by $\sigma(t) = \gamma(s - t)$. It’s easily checked from the definition \eqref{eqn:1.4} that $b_\sigma(\sigma(t)) = -t$ for $t \in [-a, \infty)$. Lemma 2.5 shows that there is a constant $c \in \mathbb{R}$ such that $b = 72\delta \ b_\sigma + c$. It follows that
\begin{equation}
b(\gamma(t)) - b(\gamma(s)) \geq 144\delta \ b_\sigma(s - t) - b_\sigma(\sigma(0)) = t - s,
\end{equation}
for $t \in [-a, \infty)$. This proves (1).

For the second claim we fix an $s \in (-\infty, a)$ and define $\tilde{\gamma}(t) = \gamma(t - b(\gamma(s)) + s + u)$ for $t \in (-\infty, a]$, $\tilde{a} = a - s + b(\gamma(s)) - u$ (if $a = \infty$ we take $\tilde{a} = \infty$). Then by \eqref{eqn:1.4},
\begin{align*}
b(\tilde{\gamma}(t)) &= b(\gamma(t - b(\gamma(s)) + s + u)) \\
&\geq 144\delta \ (t - b(\gamma(s)) + s + u) - s + b(\gamma(s)) \\
&= t + u.
\end{align*}

\hfill \Box

For $x, y \in X$ and a Busemann function $b$ based at $\omega \in \partial X$ the Gromov product based at $b$ is defined by
\begin{equation}
\langle x|y \rangle_b = \frac{1}{2} (b(x) + b(y) - |xy|).
\end{equation}
Since $b$ is 1-Lipschitz, we have the useful inequality
\begin{equation}
\label{eqn:2.9}
\langle x|y \rangle_b \leq \min\{b(x), b(y)\}.
\end{equation}

The Gromov product based at $b$ is extended to $\partial X$ by, for $(\xi, \zeta) \not= (\omega, \omega)$,
\begin{equation}
\langle \xi|\zeta \rangle_b = \inf_{n \to \infty} \inf_{x_n, y_n} \langle x_n|y_n \rangle_b
\end{equation}
with the infimum taken over $\{x_n\} \in \xi$, $\{y_n\} \in \zeta$ as before, and similarly for $x \in X$ and $\xi \in \partial X$ we define
\begin{equation}
\langle x|\xi \rangle_b = \inf_{n \to \infty} \inf_{x_n} \langle x_n|x \rangle_b,
\end{equation}
with the infimum taken over $\{x_n\} \in \xi$. The following lemma extends the $4\delta$-inequality to Gromov products based at $b$. It follows from [8] Lemma 3.2.4. Recall that we have multiplied their additive constants by a total of 12 due to the differing definition of hyperbolicity and larger cutoff in defining Busemann functions; we then rounded up to $600\delta$ afterward. The corresponding additive constant in [8] Lemma 3.2.4 below is $44\delta$.

Lemma 2.7. Let $b$ be a Busemann function based at $\omega \in \partial X$. Then

(1) For any $\xi, \zeta \in \partial X \setminus \{\omega\}$ and any $\{x_n\} \in \xi$, $\{y_n\} \in \zeta$ we have
\begin{equation}
\langle \xi|\zeta \rangle_b \leq \liminf_{n \to \infty} \langle x_n|y_n \rangle_b \leq \limsup_{n \to \infty} \langle x_n|y_n \rangle_b \leq \langle \xi|\zeta \rangle_b + 600\delta,
\end{equation}
and the same holds if we replace $\zeta$ with $x \in X$.

(2) For any $\xi, \zeta, \lambda \in X \cup \partial X \setminus \{\omega\}$ we have
\begin{equation}
\langle \xi|\lambda \rangle_b \geq \min\{\langle \xi|\zeta \rangle_b, \langle \zeta|\lambda \rangle_b\} - 600\delta.
\end{equation}
Combining (1) of Lemma 2.7 with inequality \(2.9\) gives for all \(x, y \in X \cup \partial X\) with \((x, y) \neq (\omega, \omega)\),

\[
(2.10) \quad (x|y)_b \leq \min\{b(x), b(y)\} + 600\delta,
\]

where we set \(b(\omega) = -\infty\) and \(b(\xi) = \infty\) for \(\xi \in \partial X \setminus \{\omega\}\).

For a point \(\omega \in \partial^q X\) belonging to the geodesic boundary, a sequence \(\{x_n\}\) converges to infinity with respect to \(\omega\) if for some (hence any) Busemann function \(b\) based at \(\omega\) we have \(\lim_{m,n \to \infty} (x_m|x_n)_b = \infty\). Two sequences \(\{x_n\}, \{y_n\}\) converging to infinity with respect to \(\omega\) are equivalent with respect to \(\omega\) if \(\lim_{n \to \infty} (x_n|y_n)_b = \infty\). One then defines the boundary with respect to \(\omega\) as the set of all equivalence classes of sequences converging to infinity with respect to \(\omega\). We will denote this by \(\partial_\omega X\). As our past use of the notation \(\partial_\omega X = \partial X \setminus \{\omega\}\) suggests, we have the following, which is [8 Proposition 3.4.1].

**Proposition 2.8.** A sequence \(\{x_n\}\) converges to infinity with respect to \(\omega\) if and only if it converges to a point \(\xi \in \partial X \setminus \{\omega\}\). This correspondence defines a canonical identification between \(\partial_\omega X\) and \(\partial X \setminus \{\omega\}\).

### 2.3. Visual metrics.

Let \(K \geq 1\) and let \(Z\) be a set. A function \(\alpha : Z \times Z \to [0, \infty)\) is a \(K\)-quasi-metric if

1. \(\alpha(z, z') = 0\) if and only if \(z = z'\),
2. \(\alpha(z, z') = \sigma(z, z')\),
3. \(\alpha(z, z'') \leq K \max\{\sigma(z, z'), \sigma(z', z'')\}\).

By a standard construction (see [8 Lemma 2.2.5]) a \(K\)-quasi-metric with \(K \leq 2\) is always 4-biLipschitz to a metric on \(Z\). Since for \(\varepsilon > 0\) we have that \(\alpha^\varepsilon\) is a \(K^\varepsilon\) quasi-metric if \(\alpha\) is a \(K\)-quasi-metric, for any quasi-metric \(\alpha\) we always have that \(\alpha^\varepsilon\) is 4-biLipschitz to a metric \(d\) on \(Z\) whenever \(\varepsilon\) is small enough that \(K^\varepsilon \leq 2\). We give \(Z\) the topology induced by this metric \(d\).

Let \(X\) be a \(\delta\)-hyperbolic space. For \(x \in X\) and \(q > 0\) we define for \(\xi, \zeta \in \partial X\),

\[
\alpha_{x,q}(\xi, \zeta) = e^{-q(\xi|\zeta)_x}.\]

By (2.5) the function \(\alpha_{x,q}\) defines an \(e^{8q}\)-quasi-metric on \(\partial X\). We refer to any metric \(\alpha\) on \(\partial X\) that is bi-Lipschitz to \(\alpha_{x,q}\) as a visual metric on \(\partial X\) with parameter \(q\). A visual metric always exists once \(q\) is small enough that \(e^{8q} \leq 2\). We give \(\partial X\) the topology induced by any visual metric. Equipped with a visual metric with respect to any basepoint \(p \in X\) and any parameter \(\alpha > 0\) the set \(\partial X\) is a complete bounded metric space. The basepoint change inequality (2.2) combined with inequality (2.4) shows that the notion of a visual metric does not depend on the choice of basepoint \(x \in X\).

Let \(\omega \in \partial^q X\) and let \(b\) be a Busemann function based at \(\omega\). We define for \(q > 0\) and \(\xi, \zeta \in \partial_\omega X\),

\[
(2.11) \quad \alpha_{b,q}(\xi, \zeta) = e^{-q(\xi|\zeta)}.\]

Then \(\alpha_{b,q}\) defines an \(e^{600\delta q}\)-quasi-metric on \(\partial_\omega X\) by Lemma 2.7. A visual metric based at \(\omega\) with parameter \(q\) is defined to be any metric \(\alpha\) on \(\partial_\omega X\) that is biLipschitz to \(\alpha_{b,q}\). Since all Busemann functions associated to \(\omega\) differ from each other by a constant, up to a bounded error (by Lemma 2.8), the notion of a visual metric based at \(\omega\) does not depend on the choice of Busemann function \(b\) based at \(\omega\). Equipped with any visual metric based at \(\omega\) the metric space \(\partial_\omega X\) is complete. It is bounded if and only if \(\omega\) is an isolated point in \(\partial X\).

**Remark 2.9.** The definition of a visual metric allows for any size of biLipschitz constant comparing to the quasi-metric \(\alpha_{b,q}\). To avoid introducing this additional comparison constant in
the proofs (in particular in the proof of Theorem 1.5), we will constrain our visual metrics to have comparison constants depending only on the parameters under consideration. This is sufficient to handle the key application in Theorem 1.7.

3. Equiradial points and Tripod maps

In this section we let \( X \) be a geodesic \( \delta \)-hyperbolic space for a given parameter \( \delta \geq 0 \). We will be establishing some standard claims regarding geodesic triangles in \( X \) that have vertices on the Gromov boundary \( \partial X \). When \( X \) is proper these claims can be obtained via limiting arguments from the corresponding claims for geodesic triangles in [10, Chapitre 2]. They are likely known without the properness hypothesis as well, however we lack a detailed reference for this case that provides the estimates that we need. We will thus provide proofs for this more general case, since we will be relying heavily on the claims from this section in the subsequent sections. We will then use these claims regarding geodesic triangles to evaluate Busemann functions on geodesics in \( X \) in Lemmas 3.9 and 3.11.

We start with a definition. The terminology is taken from [8, Chapter 2]. Compare [10, Chapitre 2, Définition 18].

**Definition 3.1.** Let \( \Delta \) be a geodesic triangle in \( X \) with vertices \( x, y, z \in X \cup \partial X \) and let \( \chi \geq 0 \) be given. A collection of points \( \hat{x} \in yz, \hat{y} \in xz, \hat{z} \in xy \) is \( \chi \)-equiradial if

\[
\text{diam}\{\hat{x}, \hat{y}, \hat{z}\} = \max\{|\hat{x}\hat{y}|, |\hat{y}\hat{z}|, |\hat{z}\hat{x}|\} \leq \chi.
\]

We then refer to \( \hat{x}, \hat{y}, \hat{z} \) as \( \chi \)-equiradial points for \( \Delta \).

**Remark 3.2.** For \( x, y, z \in X \) Definition 3.1 makes sense in any geodesic metric space \( X \). Taking \( \chi = \delta \) gives yet another equivalent definition of \( \delta \)-hyperbolicity for \( X \). See [10, Chapitre 2, Proposition 21].

We let \( \Upsilon \) be the tripod geodesic metric space composed of three copies \( L_1, L_2, \) and \( L_3 \) of the closed half-line \([0, \infty)\) identified at 0. For convenience we will refer to this identification point as \( o \). The space \( \Upsilon \) is clearly 0-hyperbolic. The Gromov boundary \( \partial \Upsilon \) consists of three points \( \zeta_i, i = 1,2,3 \), corresponding to the half-lines \( L_i \) thought of as geodesic rays starting from \( o \).

For a geodesic triangle \( \Delta \) with an ordered triple of \( \chi \)-equiradial points \( (\hat{x}, \hat{y}, \hat{z}) \) as in Definition 3.1 we define the associated **tripod map** \( T : \Delta \to \Upsilon \) to be the map that sends the sides \( xz, yz, \) and \( xy \) isometrically into \( L_1 \cup L_3, L_2 \cup L_3, \) and \( L_1 \cup L_2 \) respectively in the unique way that satisfies \( T(x) \in L_1 \cup \{\zeta_1\}, T(y) \in L_2 \cup \{\zeta_2\}, T(z) \in L_3 \cup \{\zeta_3\} \), and \( T(\hat{x}) = T(\hat{y}) = T(\hat{z}) = o \). To be more precise for boundary points, when \( x \in \partial X \) we mean here that \( T(x) = \zeta_1 \), i.e., \( T \) maps the geodesic rays \( \hat{y}x \) and \( \hat{z}x \) isometrically onto \( L_1 \). A choice of ordering of the equiradial points is required to define the map \( T \) but is not important, as changing the ordering simply corresponds to permuting the lines \( L_i \) in \( \Upsilon \) while keeping the origin \( o \) fixed.

When \( x, y, z \in X \), the \( 4\delta \)-tripod condition directly provides us with a set of \( 4\delta \)-equiradial points \( \hat{x}, \hat{y}, \hat{z} \) defined by the system of equalities \( |x\hat{y}| = |x\hat{z}| = (y|z)_x, |y\hat{x}| = |y\hat{z}| = (x|z)_y, \) and \( |z\hat{x}| = |z\hat{y}| = (x|y)_z \). We will typically refer to these points as the **canonical** equiradial points for \( \Delta \), since they are uniquely determined. The \( 4\delta \)-tripod condition directly implies that the associated tripod map \( T \) for these canonical equiradial points is \( 8\delta \)-roughly isometric.

The following definition encodes a convenient hypothesis to make on equiradial points of a geodesic triangle \( \Delta \) that partially generalizes the notion of canonical equiradial points to the case that some of the vertices of \( \Delta \) belong to \( \partial X \).
Definition 3.3. Let $\Delta$ be a geodesic triangle in $X$ with vertices $x, y, z \in X \cup \partial X$, let $\chi \geq 0$ be given, and let $(\hat{x}, \hat{y}, \hat{z})$ be a collection of $\chi$-equiradial points for $\Delta$. We say that this collection is calibrated if we have $|\hat{x}z| = |\hat{y}z|$, $|\hat{y}x| = |\hat{x}x|$, and $|\hat{z}y| = |\hat{xy}|$.

Note that this condition is trivially satisfied when all vertices of $\Delta$ belong to $\partial X$, since all of the subsegments involved have infinite length.

We first obtain the following direct consequence of Lemma 2.3.

Lemma 3.4. Let $\Delta$ be a geodesic triangle with vertices $x, y, z \in X \cup \partial X$. Then there is a calibrated 60$\delta$-equiradial collection of points $\hat{x} \in yz$, $\hat{y} \in xz$, $\hat{z} \in xy$.

Proof. If all vertices of $\Delta$ belong to $X$ then the canonical equiradial points give a calibrated 4$\delta$-equiradial collection for $\Delta$, so we can assume that at least one vertex of $\Delta$ belongs to $\partial X$. Thus we can assume without loss of generality that $z \in \partial X$.

Parametrize the side $xy$ by arclength as $\gamma : I \rightarrow X$ for an interval $I \subset \mathbb{R}$, oriented from $x$ to $y$. Let $E_x \subset I$ be the collection of times $t$ such that $\text{dist}(\gamma(t), xz) \leq 10\delta$ and $E_y \subset I$ the collection of times $t$ such that $\text{dist}(\gamma(t), yz) \leq 10\delta$. Each of the sets $E_x$ and $E_y$ are closed, and we have $E_x \cup E_y = I$ by Lemma 2.3. We can then assume without loss of generality that $E_x$ is nonempty. If $E_y$ is also nonempty then by the connectedness of $I$ we must have $E_x \cap E_y \neq \emptyset$. Letting $s \in E_x \cap E_y$, setting $w := \gamma(s)$, and selecting points $u \in xz$, $v \in yz$ such that $|uw| \leq 10\delta$ and $|wv| \leq 10\delta$, we conclude that $\{w, u, v\}$ is a 20$\delta$-equiradial collection of points for $\Delta$.

We must still show that $E_y$ is always nonempty. Since this is obvious when $y \in X$, we can assume that $y \in \partial X$. If $E_y = \emptyset$ then for each $t \in I$ we let $x_t \in xz$ be a point such that $|x_t \gamma(t)| \leq 10\delta$. For $n \in \mathbb{N}$ the sequence $\{\gamma(n)\}$ converges to $y$, which implies that the sequence $\{x_n\}$ converges to $y$ since these sequences are a bounded distance from one another. However, any sequence of points converging to infinity in $xz$ can only possibly converge to $x$ or $z$, which is a contradiction. Thus $E_y$ is nonempty.

Lastly we need to produce a calibrated collection of equiradial points from the collection $\{w, u, v\}$. If all vertices of $\Delta$ belong to $\partial X$ then the collection is trivially calibrated, so we can assume at least one vertex of $\Delta$ belongs to $X$. By relabeling the vertices we can then assume that either $x \in X$ and $y \in \partial X$ or $x \in X$ and $y \in X$. In both cases we can find $u' \in xz$ such that $|xu'| = |zx|$ since $|xz| = \infty$. Then

$$|uw| \leq ||xu| - |xu'||| = ||xu| - |xw|| \leq |uw| \leq 20\delta.$$  

It follows that the collection $\{w, u', v\}$ is 40$\delta$-equiradial. If $y \in \partial X$ then this collection is also calibrated and we are done.

If $y \in X$ then we repeat the argument again, using the fact that $|yz| = \infty$ to find $v' \in yz$ such that $|yu'| = |yu|$. The same calculation shows that $|vv'| \leq 20\delta$, so we can then conclude that the collection $\{w, u', v\}$ is calibrated and 60$\delta$-equiradial, as desired. \hfill \square

Our next goal will be to prove that the tripod map $T : \Delta \rightarrow \Upsilon$ associated to the calibrated collection of equiradial points produced by Lemma 3.4 is roughly isometric. We will require the following simple lemma.

Lemma 3.5. Let $X$ be a geodesic metric space and let $x, y, z \in X$ with $|xz| \leq |yz|$. Let $w \in xz$, $v \in yz$ be given points that satisfy $|xu| = |yz|$ and let $w \in yz$ be the unique point satisfying $|zw| = |zu|$. Then $w \in vz$ and $|vw| \leq |xz|$.

Proof. The point $w$ must belong to the subsegment $vz$ of $yz$, as if $w \in yv$ and $w \neq v$ then

$$|yz| = |yw| + |zw| - |uv| = |xu| + |uz| - |vw| = |xz| - |vw| < |xz|,$$
contradicting that $|yz| \geq |xz|$. Since $w \in v$ we then have
\[ |yz| = |yw| + |vw| + |wz| = |xz| + |vu| + |uz| = |xz| + |vw|, \]
which implies by the triangle inequality that $|vw| \leq |xy|$. 

We now apply Lemma 3.5 to the setting of a $\delta$-hyperbolic space $X$.

**Lemma 3.6.** Let $x, y \in X$, let $z \in X \cup \partial X$, and let $\bar{x} \in xz$, $\bar{y} \in yz$ satisfy $|\bar{x}\bar{y}| = |yy|$. Then we have
\[(3.1) \quad |\bar{x}\bar{y}| \leq 3|xy| + 8\delta.\]

**Proof.** Set $t = |\bar{x}\bar{y}|$. If $t \leq |xy|$ then
\[|\bar{x}\bar{y}| \leq |\bar{x}x| + |xy| + |\bar{y}y| \leq 3|xy|,\]
which verifies inequality (3.1). We can thus assume that $t > |xy|$.

We first assume that $z \in X$. We can then assume without loss of generality that $|xz| \leq |yz|$. We consider a geodesic triangle $\Delta = xyz$ with sides the given geodesics $xz$ and $yz$, as well as a geodesic $xy$ from $x$ to $y$. Let $w \in yz$ be the unique point such that $|wz| = |\bar{x}z|$. Lemma 3.5 shows that $w \in \bar{y}z$ and $|w\bar{y}| \leq |xy|$.

Let $x' \in xz$ and $y' \in yz$ be the canonical equiradial points for $\Delta$ on these edges. These points must satisfy $\max\{|x'x|, |y'y|\} \leq |xy|$ since $xy$ is an edge of $\Delta$. The assumption $t > |xy|$ then implies that $\bar{x} \in x'z$ and $\bar{y} \in y'z$. Thus $w \in y'z$. The $4\delta$-tripod condition then implies that $|w\bar{x}| \leq 4\delta$, from which it follows that $|\bar{x}\bar{y}| \leq |xy| + 4\delta$. This proves (3.1) in this case.

We now consider the case $z \in \partial X$. For each $s \geq 0$ we define $x_s \in xz$, $y_s \in yz$ to be the points such that $|xx_s| = s$ and $|yy_s| = s$. Since the geodesics $xz$ and $yz$ have the same endpoint $z \in \partial X$, we must have $(x_s|y_s)y \to \infty$ as $s \to \infty$ and the same for $(x_s|y_s)x$. We choose $s$ large enough that $(x_s|y_s)x \geq t$ and $(x_s|y_s)y \geq t$. We consider a geodesic triangle $\Delta_1 = xx_sy_s$ with edges the subsegment $xx_s$ of the given geodesic $xz$ as well as geodesics $x_sy_s$ and $y_sx$, and a triangle $\Delta_2 = yyy_s$, with edges the subsegment $yy_s$ of the given geodesic $yz$, the edge $xy_s$ of $\Delta_1$, and a geodesic $xy$. Then $\bar{x} \in xx_s$ and $\bar{y} \in yy_s$ by our choice of $s$.

Since $(x_s|y_s)x \geq t$, we must have $|yy_s| \geq t$. Therefore there is a unique point $w \in yy_s$ such that $|wv| = |\bar{x}x| = t$. The $4\delta$-tripod condition applied to the triangle $\Delta_1$ then implies that $|\bar{x}w| \leq 4\delta$. If $|xx_s| \leq |yy_s|$ then we let $u \in yy_s$ be the unique point such that $|uy_s| = |uy_s|$. By applying Lemma 3.5 we then conclude that $u \in \bar{y}y_s$ and $|uw| \leq |xy|$. Since $xy$ is an edge of the triangle $\Delta_2$, the canonical equiradial points of this triangle on the edges $xy_s$ and $yy_s$ can be at most a distance $|xy| \leq t$ from the vertices $x$ and $y$ respectively. We thus conclude from the $4\delta$-tripod condition that $|uw| \leq 4\delta$. Combining these inequalities together gives
\[(3.2) \quad |\bar{x}\bar{y}| \leq |\bar{x}w| + |uw| + |w\bar{y}| \leq |xy| + 8\delta,\]
which proves (3.1). The case $|xx_s| \geq |yy_s|$ is similar: we let $v \in xy_s$ be the point such that $|vy_s| = |\bar{y}y_s|$, apply Lemma 3.5 to obtain $|wv| \leq |xy|$ and $v \in wy_s$, then apply the $4\delta$-tripod condition to obtain $|v\bar{y}| \leq 4\delta$. This gives inequality (3.1) through the same calculation as (3.2). 

We will use Lemma 3.6 to show that the tripod map associated to a collection of calibrated equiradial points for a geodesic triangle $\Delta$ is roughly isometric.

**Proposition 3.7.** Let $x, y, z \in X \cup \partial X$ be given vertices of a geodesic triangle $\Delta$ in $X$. Let $\bar{x} \in xz$, $\bar{y} \in xz$, $\bar{z} \in xy$ be points such that $(\bar{x}, \bar{y}, \bar{z})$ is a calibrated ordered triple of
\( \chi \)-equiradial points for \( \Delta \) for a given \( \chi \geq 0 \). Let \( T : \Delta \to Y \) be the tripod map associated to this triple. Then \( T \) is \((6 \chi + 16 \delta)\)-roughly isometric.

In particular if \((\hat{x}, \hat{y}, \hat{z})\) is the calibrated \(60\delta\)-equiradial triple produced in Lemma 3.4 then \( T \) is \(400\delta\)-equiradial.

**Proof.** By symmetry (permuting the vertices \( x, y, \) and \( z \)), to estimate \(|T(p)T(q)|\) for \( p, q \in \Delta \) it suffices to restrict to the case \( p \in \hat{y}z \) and then consider the possible locations of \( q \). By construction we have \(|T(p)T(q)| = |pq|\) if \( p \) and \( q \) belong to the same edge of \( \Delta \), since the tripod map is isometric on the edges of \( \Delta \). This handles the case that \( q \) belongs to the same edge as \( p \), i.e., that \( q \in xz \).

We next consider the case \( q \in yz \). Since \(|\hat{y}z| = |\hat{x}z|\), we can find a point \( u \in \hat{x}z \) such that \(|\hat{x}u| = |\hat{y}p|\). Then \(|T(p)T(q)| = |upq|\). Applying Lemma 3.6 yields

\[
|up| \leq 3|\hat{x}z| + 8\delta \leq 3\chi + 8\delta,
\]

so that

\[
||upq| - |pq|| \leq |up| \leq 3\chi + 8\delta,
\]

which gives the desired estimate in this case.

Lastly we must consider the case \( q \in xy \), which we subdivide into the cases \( q \in x\hat{z} \) and \( q \in \hat{y} \). When \( q \in x\hat{z} \) we can use the condition \(|x\hat{z}| = |xy|\) to find a point \( v \in xy \) such that \(|q\hat{z}| = |v\hat{y}|\). Then \(|T(p)T(q)| = |vpq|\). Similarly to the previous case, Lemma 3.6 gives us the estimate \(|vp| \leq 3\chi + 8\delta\) which implies that

\[
||vpq| - |pq|| \leq |vp| \leq 3\chi + 8\delta,
\]

as desired. When \( q \in \hat{y} \) we use the equality \(|\hat{y}z| = |\hat{x}y|\) to find \( w \in \hat{x}y \) such that \(|\hat{z}q| = |\hat{x}w|\), and we use the equality \(|\hat{x}z| = |\hat{y}z|\) to find \( s \in \hat{x}z \) such that \(|s\hat{x}| = |\hat{y}w|\). Then \(|T(p)T(q)| = |spq|\). Lemma 3.6 gives us the estimate

\[
\max\{|sp|, |wq|\} \leq 3\chi + 8\delta,
\]

which implies by the triangle inequality that

\[
||sp| - |pq|| \leq ||sp| - |wp|| + ||wp| - |qp|| \leq |sp| + |wq| \leq 6\chi + 16\delta.
\]

This completes the proof of the main claim. The final assertion follows by plugging in \( \chi = 60\delta \) and rounding up. \( \square \)

**Remark 3.8.** Throughout this paper we will often suppress the exact choice of calibrated equiradial points used to define a tripod map \( T : \Delta \to Y \). To make this more formal, for a geodesic triangle \( \Delta \) in \( X \) we will refer to a tripod map \( T : \Delta \to Y \) associated to \( \Delta \) as being any tripod map \( T \) for \( \Delta \) associated to an ordered triple of calibrated \(60\delta\)-equiradial points for \( \Delta \) obtained from Lemma 3.4.

We recall that the Gromov boundary \( \partial Y \) of the tripod \( Y \) is a disjoint union of three points \( \zeta_i \), \( i = 1, 2, 3 \), corresponding to the geodesic rays \( \gamma_i : [0, \infty) \to Y \) that parametrize the half-lines \( L_i \) starting from \( o \) for \( i = 1, 2, 3 \). Set \( b_\gamma := b_{\gamma_1} \) to be the Busemann function associated to the geodesic ray \( \gamma_1 \). A straightforward calculation shows that \( b_\gamma \) is given by \( b_\gamma(s) = -s \) for \( s \in L_1 \) and \( b_\gamma(s) = s \) for \( s \in L_2 \) or \( s \in L_3 \), when we consider each of these rays as identified with \((0, \infty)\).

In this next proposition we consider a geodesic triangle \( \Delta \) in \( X \) with a distinguished vertex \( \omega \in \partial X \) together with a Busemann function \( b \) based at \( \omega \). We will not keep track of exact constants in the proof of this lemma so we will not produce an explicit value for \( \kappa = \kappa(\delta) \) below. If one does careful bookkeeping in the proof it is possible to show that \( \kappa = 2000\delta \) works.
Proposition 3.9. Let $\Delta = \omega xy$ be a geodesic triangle in $X$ with $\omega \in \partial X$ and $x, y \in X \cup \partial_\omega X$. There is a constant $\kappa = \kappa(\delta)$ such that the following holds: let $\hat{\omega} \in xy$, $\hat{x} \in \omega y$, and $\hat{y} \in \omega x$ be a calibrated set of $60\delta$-equiradial points on $\Delta$ provided by Lemma 3.4 and let $b$ be a Busemann function based at $\omega$. Let $T : \Delta \to \Upsilon$ be the tripod map associated to the triple $(\hat{x}, \hat{y}, \hat{\omega})$. Then applying (3.5), we have

\begin{equation}
 b(p) \doteq_{\kappa} b_T(T(p)) + (x|y)_b. \tag{3.3}
\end{equation}

Consequently we have $b(p) \doteq_{\kappa} (x|y)_b$ for $p \in \{\hat{\omega}, \hat{x}, \hat{y}\}$ and

\begin{equation}
 (x|y)_b \doteq_{\kappa} \inf_{p \in \xy} b(p). \tag{3.4}
\end{equation}

Proof. Since we will not be keeping track of the exact value of the final constant $\kappa = \kappa(\delta)$ in the proof, we will let $\doteq$ denote any equality up to an additive error depending only on $\delta$. Set $u = b(\hat{\omega})$. Then $u \doteq b(\hat{x})$ and $u \doteq b(\hat{y})$ since $b$ is 1-Lipschitz. We will prove the rough equality (3.3) with $u$ in place of $(x|y)_b$ and use this to deduce that $u \doteq (x|y)_b$. Thus we will first show that for $p \in \Delta$ we have

\begin{equation}
 b(p) \doteq b_T(T(p)) + u. \tag{3.5}
\end{equation}

We first handle the case in which $p \in \omega x$ or $p \in \omega y$. Since the roles of $x$ and $y$ are symmetric, we can assume without loss of generality that $p \in \omega x$. Let $\gamma : (\infty, a] \to X$ be an arclength parametrization of $\omega x$ with $\gamma(0) = \hat{x}$ and $\gamma(t) \to \omega$ as $t \to \infty$. If we define $s \in (-\infty, u]$ such that $\gamma(s) = p$ then it follows from the construction of the tripod map that $b_T(T(\gamma(s))) = s$. Applying Lemma 3.6 gives

\begin{equation*}
 b(p) - b(\hat{x}) \doteq s = b_T(T(\gamma(s))),
\end{equation*}

which gives (3.5) since $b(\hat{x}) \doteq u$.

The remaining case is when $p \in xy$. By the symmetric roles of $x$ and $y$ we can assume that $p \in x\hat{\omega}$. As in the proof of Proposition 3.7 since $|x\hat{\omega}| = |x\hat{y}|$ we can find $q \in x\hat{y}$ such that $|p\hat{\omega}| = |q\hat{y}|$. Then by Lemma 3.6 we have

$$|pq| \leq 3|\hat{\omega}| + 10\delta \leq 190\delta.$$ 

Thus $b(p) \doteq b(q)$ since $b$ is 1-Lipschitz. It then follows, from the rough equality (3.5) for $q \in \omega x$ that we established above, that

$$b(p) \doteq b(q) \doteq |q\hat{y}| + u = |p\hat{\omega}| + u,$$

which gives (3.5) in this case.

We next show that $u \doteq (x|y)_b$. By Lemma 2.7 we have for any sequences $x_n \to x$ and $y_n \to y$ that $(x_n|y_n)_b \doteq_{600\delta} (x|y)_b$ for sufficiently large $n$; if $x \in X$ then we can just set $x_n = x$ for all $n$ and the same goes for $y$. We choose sequences $\{x_n\}$ and $\{y_n\}$ that belong to $xy$ and consider only those $n$ large enough that $(x_n|y_n)_b \doteq_{600\delta} (x|y)_b$ and $x_n \in \hat{\omega}x$, $y_n \in \hat{\omega}y$. Then applying (3.5),

\begin{equation*}
 (x|y)_b \doteq \doteq (x_n|y_n)_b \\
 = \frac{1}{2}(b(x_n) + b(y_n) - |x_ny_n|) \\
 \doteq \frac{1}{2}(|x_n\hat{\omega}| + |y_n\hat{\omega}| + 2u - |x_ny_n|) \\
 = u.
\end{equation*}

Thus we can substitute in $(x|y)_b$ for $u$ in (3.5) at the cost of an additional additive constant depending only on $\delta$. The main claim (3.3) follows. The assertion that $b(p) \doteq_{\kappa} (x|y)_b$ for
Lemma 3.11. \( p \in \{ \hat{\omega}, \hat{x}, \hat{y} \} \) follows from (3.3) since each point of \( \{ \hat{\omega}, \hat{x}, \hat{y} \} \) has image \( o \in \Upsilon \) under the tripod map \( T \) and \( b_T(o) = 0 \). The rough equality (3.3) also follows directly from (3.3) since the image of \( xy \) under \( T \) is contained in \( L_2 \cup L_3 \) and \( b_T \) is nonnegative on this subset of \( \Upsilon \). \( \square \)

Proposition 3.9 leads to the following important definition, which is useful for calculations.

**Definition 3.10.** Let \( X \) be a geodesic \( \delta \)-hyperbolic space, let \( \omega \in \partial X \), and let \( x, y \in X \cup \partial_x X \), with \( x \neq y \) if both belong to \( \partial X \). Let \( b \) be a Busemann function based at \( \omega \) and let \( c \geq 0 \) be a given constant. Suppose that \( xy \) is a geodesic joining \( x \) to \( y \). We say that a parametrization \( \eta : I \to X, I \subseteq \mathbb{R}, \) is \( c \)-adapted to \( b \) if \( 0 \in I \) and

\[
(3.6) \quad b(\eta(t)) = c |t| + (x|y)_b,
\]

for \( t \in I \).

When the value of \( c \) is implied by context we will shorten this to just saying that the parametrization \( \eta \) is adapted to \( b \). The inclusion of 0 in the domain of \( \eta \) will be vital for our applications. For a geodesic triangle \( \Delta = \omega xy \) the existence of a parametrization of the side \( xy \) that is \( c \)-adapted to \( b \) is just a reformulation of the conclusions of Lemma 3.9 with the constant \( c = \kappa \). To be more precise, one obtains this parametrization by inverting the restriction of the tripod map to the edge \( xy \) of \( \Delta \) and identifying the corresponding pair of half-lines \( L_2 \cup L_3 \) with \( \mathbb{R} \) in the appropriate orientation, sending \( o \) to the origin in \( \mathbb{R} \).

We conclude this section by constructing adapted parametrizations under the rough star-likeness hypothesis of Theorem 1.3. We emphasize that the points \( x \) and \( y \) in the lemma need not be the vertices of a geodesic triangle \( \Delta \) with a third vertex at \( \omega \).

**Lemma 3.11.** Let \( X \) be a geodesic \( \delta \)-hyperbolic space that is \( K \)-roughly starlike from a point \( \omega \in \partial X \). Let \( b \) be a Busemann function based at \( \omega \) and let \( x, \xi \in X \cup \partial X \) with \( x \neq \xi \) if both belong to \( \partial X \). Let \( xy \) be a given geodesic from \( x \) to \( y \). Then there is a constant \( c = c(\delta, K) \) depending only on \( \delta \) and \( K \) such that there is a parametrization \( \eta : I \to X \) of \( xy \) that is \( c \)-adapted to \( b \).

**Proof.** If \( x \) and \( y \) both belong to \( \partial X \) then (2) of Definition 2.1 supplies geodesics \( x\omega \) and \( y\omega \) from \( \omega \) to \( x \) and \( y \) respectively. The existence of the desired parametrization then follows by applying Proposition 3.9 to the geodesic triangle \( \Delta = \omega xy \) made up of these geodesics and the given geodesic \( xy \).

If \( x \) and \( y \) both belong to \( X \) then (1) of Definition 2.1 implies that we can find points \( \xi, \zeta \in \partial X \) and geodesics \( \omega \xi, \omega \zeta \) such that \( \text{dist}(x, \omega \xi) \leq K \) and \( \text{dist}(y, \omega \zeta) \leq K \). Let \( x' \in \omega \xi \), \( y' \in \omega \zeta \) be points such that \( |xx'| \leq K \) and \( |yy'| \leq K \). We then form a geodesic triangle \( \Delta = \omega x'y' \) out of the geodesics \( \omega x', \omega y' \), and a choice of geodesic \( x'y' \) from \( x' \) to \( y' \). We apply Proposition 3.9 to derive a \( \kappa \)-adapted parametrization \( \eta' : I' \to X \) of \( x'y' \) oriented from \( x' \) to \( y' \), \( I' = [t'_-, t'_+] \) with \( \kappa = \kappa(\delta) \). Since \( 0 \in I' \) we have \( t'_- \leq 0 \) and \( t'_+ \geq 0 \).

Let \( \eta : I \to X, I = [t'_-, t'_+] \), be the unique arclength parametrization of \( xy \) that is oriented from \( x \) to \( y \) and starts from the same time parameter \( t'_- \) as \( \eta' \). The piecewise geodesic curve \( xx' \cup x'y' \cup yy' \) joining \( x \) to \( y \) can be parametrized as a \( 4K \)-roughly isometric map \( \sigma : J \to X \) for an appropriate interval \( J \subseteq \mathbb{R} \). By the stability of geodesics in Gromov hyperbolic spaces [8] Theorem 1.3.2] this implies that there is a constant \( c' = c'(\delta, K) \) such that the given geodesic \( xy \) is contained in a \( c' \)-neighborhood of the curve \( \sigma \).

Now let \( t \in I \) be given and let \( s \in I' \) be such that \( |\eta(t)\eta'(s)| \leq c' \). Since \( b \) is 1-Lipschitz it follows that

\[
b(\eta(t)) \leq c' b(\eta'(s)) \leq c |s| + (x|y)_b.
\]
Thus it suffices to show that \( t \approx_{c''} s \) for a constant \( c'' = c''(\delta, K) \). Since \( t - t'' = |\eta(t)x| \) and \( s - t'' = |\eta(s)x'| \), we have
\[
|t - s| = |(t - t'') - (s - t'')| \\
= |\eta(t)x| - |\eta(s)x'| \\
\leq |\eta(t)x| - |\eta(t)x'| + |\eta(t)x'| - |\eta(s)x'| \\
\leq |x| + |\eta(t)\eta'(s)| \\
\leq K + c',
\]
so that we can set \( c'' = K + c' \). It follows that \( \eta \) satisfies (3.6) with constant \( c = c(\delta, K) \) depending only on \( \delta \) and \( K \).

If \( 0 \in I \) then \( \eta \) gives a parametrization of \( xy \) that is \( c \)-adapted to \( b \) and we are done. We can therefore assume that \( 0 \notin I \) which implies that \( t_+ \leq 0 \) since \( t'_+ \leq 0 \). We then note that \( |x'y'| \approx_{2K} |xy| \) and \( t'_+ - t'_- = |x'y'| \), \( t_+ - t'_- = |xy| \), which implies that \( t_+ \approx_{2K} t'_+ \).

Since \( t'_+ \geq 0 \) and \( t_+ \leq 0 \), we conclude that \( |t'_+| \leq 2K \). We set \( I'' = [t'_-, t_+, 0] \) and set \( \eta''(t) = \eta(t + t'_+) \) for \( t \in I'' \). The parametrization \( \eta'' \) still satisfies (3.6) with \( c = c(\delta, K) \) since \( b \) is 1-Lipschitz, and \( 0 \in I'' \) by construction. Thus \( \eta'' \) gives the desired adapted parametrization.

Lastly we consider the case in which \( x \) or \( y \) belong to \( \partial X \), but not both. Without loss of generality we can assume that \( x \in X \) and \( y \in \partial X \). Let \( \{y_n\} \subset xy \) be the sequence of points with \( |xy_n| = n \) for each \( n \in \mathbb{N} \). Let \( \eta_n : I_n \to X \) be the arclength parametrizations of \( xy_n \) for each \( n \) that were constructed in the previous case, \( I_n = [s_n, t_n] \). Since \( 0 \in I_n \) for each \( n \) we have \( s_n \leq 0 \) for each \( n \). Since \( \eta_n(s_n) = x \) for each \( n \), we have from the condition that \( \eta_n \) is \( c \)-adapted to \( b \),
\[
b(x) \approx_{c} |s_n| + (x|y)_b = -s_n + (x|y)_b
\]
with \( c = c(\delta, K) \). It follows that \( s_m \approx_{c} s_n \) for each \( m, n \in \mathbb{N} \). Thus, by replacing \( \eta_n \) with the parametrization \( \eta_n' \) defined by \( \eta_n'(t) = \eta_n(t - s_1 + s_n) \) on the domain \( I_n' = [s_1, t_n + s_1 - s_n] \),

we can assume that \( s_n = s_1 := s \) for all \( n \in \mathbb{N} \). Note also that since \( t_n \to \infty \) as \( n \to \infty \) and \( s \leq 0 \), we have \( 0 \in I_n \) for all large enough \( n \). It follows that the resulting parametrization \( \eta_n \) will be \( c \)-adapted to \( b \) for \( n \) large enough since \( b \) is 1-Lipschitz, with \( c = c(\delta, K) \).

With these modifications the parametrizations \( \eta_n \) now have the same starting point \( s \leq 0 \). Since these are parametrizations of \( xy_n \) by arclength and the sequence \( \{y_n\} \) defines progressively longer subsegments \( xy_n \) of \( xy \) that exhaust \( xy \), the maps \( \eta_n \) coincide wherever their domains overlap and can therefore be used to define a parametrization \( \eta : [s, \infty) \to X \) of \( xy \) that is \( c \)-adapted to \( b \) by construction.

\[\square\]

4. Uniformization

Our task in this section will be to prove Theorems 4.3 and 4.5. Let \( X \) be a complete geodesic \( \delta \)-hyperbolic space, let \( \omega \in \partial X \), and suppose that \( X \) is \( K \)-roughly starlike from \( \omega \). We let \( b_\gamma \) be a Busemann function associated to a geodesic ray \( \gamma \) in the equivalence class of \( \omega \). As in the previous section we will drop the geodesic ray \( \gamma \) from the notation and write \( b := b_\gamma \) for a Busemann function based at \( \omega \), write
\[
\rho_\varepsilon(x) = e^{-c_b(x)},
\]
for the density used in Theorem 4.3 and write \( X_\varepsilon := X_{\varepsilon, \gamma} \) for the uniformization for \( \varepsilon > 0 \). We write \( d_\varepsilon := d_{\varepsilon, \gamma} \) for the metric on \( X_\varepsilon \) and \( \ell_\varepsilon(\eta) := \ell_{\varepsilon, \gamma}(\eta) \) for the length of a curve.
\( \eta : I \to X_\varepsilon \) measured in this metric. The hypothesis that \( X \) is complete will not be used until Lemma 4.6.

Remark 4.1. Throughout the remainder of this paper we will be using [4 Proposition A.7], which for a geodesic metric space \( X \) and a continuous function \( \rho : X \to (0, \infty) \) allows us to compute the lengths \( \ell_\rho(\gamma) \) in the conformal deformation \( X_\rho \) of curves \( \gamma : I \to X \) parametrized by arclength in \( X \) as

\[
\ell_\rho(\gamma) = \int_I \rho \circ \gamma \, ds,
\]

with \( ds \) denoting the standard length element in \( \mathbb{R} \).

We suppose that \( \varepsilon > 0 \) is chosen such that the density \( \rho_\varepsilon \) is admissible for \( X \) and let \( M \) be the associated admissibility constant. Since \( b \) is 1-Lipschitz we have the Harnack type inequality for \( x, y \in X \),

\[
e^{-|xy|} \leq \frac{\rho_\varepsilon(x)}{\rho_\varepsilon(y)} \leq e^{|xy|},
\]

which corresponds to [3 (4.4)]. As remarked in the introduction, this inequality is sufficient to apply the results of [3, Chapter 5] to the density \( \rho_\varepsilon \), which produces the same threshold \( \varepsilon \leq \varepsilon_0(\varepsilon) \) for admissibility with the same constant \( M = M(\varepsilon) \) as was used there.

The metric spaces \( X_\varepsilon \) and \( X \) are biLipschitz on bounded subsets of \( X \) by inequality (4.2). It follows that \( X_\varepsilon \) is rectifiably connected since \( X \) since \( X \) is geodesic. We next show that \( X_\varepsilon \) is incomplete, and perform a useful calculation in the process of doing so. We note that this particular claim does not require \( \rho_\varepsilon \) to be admissible for \( X \). The existence of a geodesic line \( \gamma \) from \( \omega \) to a given point \( \xi \in \partial X \) follows from the rough starlikeness hypothesis, and the claimed parametrization for \( \gamma \) in the statement of the lemma follows from (2) of Lemma 2.6 with \( u = 0 \).

**Lemma 4.2.** Let \( \gamma : \mathbb{R} \to X \) be a geodesic line starting at \( \omega \) and ending at \( \xi \in \partial X \), parametrized by arclength such that \( b(\gamma(t)) =1_{444} \) \( t \). Then for each \( s \in \mathbb{R} \) we have

\[
\ell_\varepsilon(\gamma|_{[s, \infty)}) \asymp_{C(\delta, \varepsilon)} e^{-1} e^{-\varepsilon s}.
\]

Consequently any sequence \( \{t_n\} \subset \mathbb{R} \) with \( t_n \to \infty \) defines a Cauchy sequence \( \{\gamma(t_n)\} \subset X_\varepsilon \) in the metric \( d_\varepsilon \) that does not converge in \( X_\varepsilon \). In particular \( X_\varepsilon \) is incomplete. If \( \rho_\varepsilon \) is admissible for \( X \) then \( X_\varepsilon \) is also unbounded.

**Proof.** The comparison (4.3) is a consequence of a straightforward calculation,

\[
\ell_\varepsilon(\gamma|_{[s, \infty)}) = \int_s^\infty e^{-b(\gamma(t))} \, dt \\
= \varepsilon^{-1} e^{-\varepsilon t} \int_s^\infty e^{-\varepsilon t} \, dt \\
= \varepsilon^{-1} e^{-\varepsilon s},
\]

with \( C(\delta, \varepsilon) = e^{1444\varepsilon} \). It’s clear from (4.3) that \( \{\gamma(t_n)\} \) is a Cauchy sequence in \( X_\varepsilon \) if \( t_n \to \infty \). We claim that this sequence does not converge in \( X_\varepsilon \). If, to the contrary, there was a point \( z \in X_\varepsilon \) such that \( \gamma(t_n) \to z \) in \( X_\varepsilon \) then we would also have to have \( \gamma(t_n) \to z \) in \( X \) since the metrics on \( X \) and \( X_\varepsilon \) are biLipschitz on the unit ball \( B(z, 1) \) centered at \( z \) in \( X \). But this directly contradicts the fact that \( \{\gamma(t_n)\} \) converges to infinity in \( X \). Thus the Cauchy sequence \( \{\gamma(t_n)\} \) in \( X_\varepsilon \) does not converge in \( X_\varepsilon \). It follows that \( X_\varepsilon \) is incomplete.
For the final assertion we assume that \( \varepsilon \) is admissible and repeat the calculation above for \( s \leq 0 \) on the domain \([s, 0]\) instead to obtain
\[
\ell_\varepsilon(\gamma|_{[s,0]}) = \int_s^0 e^{-\varepsilon \beta(\gamma(t))} \, dt
\approx C(\delta, \varepsilon) \int_s^0 e^{-\varepsilon t} \, dt
= \varepsilon^{-1} (e^{-\varepsilon s} - 1),
\]
Thus \( \ell_\varepsilon(\gamma|_{[s,0]}) \to \infty \) as \( s \to -\infty \). The admissibility inequality (1.3) then implies that \( d_\varepsilon(\gamma(s), \gamma(0)) \to \infty \) as \( s \to -\infty \). Consequently \( X_\varepsilon \) is unbounded. \( \square \)

**Remark 4.3.** In inequality (1.3) and all subsequent inequalities in this section up until the final two propositions, when we indicate that a constant \( C(\ldots, \varepsilon) \geq 1 \) depends on the parameter \( \varepsilon \) it will always be the case that this constant depends monotonically on \( \varepsilon \) so that we have \( C(\ldots, \varepsilon_1) \leq C(\ldots, \varepsilon_2) \) for \( \varepsilon_1 \leq \varepsilon_2 \). Thus one can remove the dependence of these constants on \( \varepsilon \) by imposing an a priori choice of upper bound; in [4] a bound \( \varepsilon \leq \varepsilon_0(\delta) \) depending only on \( \delta \) is chosen.

As in the introduction we write \( \bar{X}_\varepsilon \) for the completion of the uniformization \( X_\varepsilon \) and \( \partial X_\varepsilon = X_\varepsilon \setminus X_\varepsilon \) for the boundary of this completion. We write \( d_\varepsilon(x) = \text{dist}(x, \partial X_\varepsilon) \) for the distance to the boundary; as shown in Lemma 4.2 the metric space \( X_\varepsilon \) is always incomplete and so the boundary \( \partial X_\varepsilon \) is always nonempty.

We next have a lemma that follows directly from the Harnack inequality (4.2) and the admissibility inequality (1.3) corresponding to the parameter \( \varepsilon \).

**Lemma 4.4.** For any \( x, y \in X \) we have
\[
M^{-1} \rho_\varepsilon(x)e^{-1}(1 - e^{-\varepsilon|x|y|}) \leq d_\varepsilon(x, y) \leq \rho_\varepsilon(x)e^{-1}(e^{\varepsilon|x|y|} - 1).
\]

**Proof.** Let \( xy \) be a geodesic joining \( x \) to \( y \). Then, using (4.2),
\[
d_\varepsilon(x, y) \leq \int_{xy} \rho_\varepsilon \, dt
\leq \rho_\varepsilon(x) \int_0^{|xy|} e^{\varepsilon t} \, dt
= \rho_\varepsilon(x)e^{-1}(e^{\varepsilon|x|y|} - 1).
\]
For the lower bound, we apply (1.3) to the geodesic \( xy \) together with (4.2) to obtain
\[
d_\varepsilon(x, y) \geq M^{-1} \int_{xy} \rho_\varepsilon \, dt
\geq M^{-1} \rho_\varepsilon(x) \int_0^{|xy|} e^{-\varepsilon t} \, dt
= M^{-1} \rho_\varepsilon(x)e^{-1}(1 - e^{-\varepsilon|x|y|}).
\]
\( \square \)

We can refine Lemma 4.4 based on the size of the quantity \( \varepsilon|x|y| \). For brevity, in the rest of this section we write \( \tilde{=} \) for equality up to an additive that depends only on \( \delta, K, \varepsilon, \) and \( M \), and write \( \approx \) for equality up to a multiplicative constant that depends only on those same parameters. We write \( c \geq 0 \) and \( C \geq 1 \) for additive and multiplicative constants depending only on these parameters. Remark 4.3 will always be in effect for these constants.
Lemma 4.5. For any $x, y \in X$ we have

\begin{equation}
\ell \epsilon(x, y) \asymp e^{-\epsilon(x|y)_{b}}|xy|,
\end{equation}

if $\epsilon|xy| \leq 1$ and

\begin{equation}
\ell \epsilon(x, y) \asymp \epsilon^{-1}e^{-\epsilon(x|y)_{b}},
\end{equation}

if $\epsilon|xy| \geq 1$.

Proof. Let $x, y \in X$ be given and let $xy$ be a geodesic joining $x$ to $y$. Let $\eta : I \to X$ be a parametrization of $xy$ that is $c$-adapted to $b$ as in Lemma 3.11 with $c = c(\delta, K)$ depending only on $\delta$ and $K$. We assume that $\eta$ is oriented from $x$ to $y$. Let $w = \eta(0)$. By (3.6) we have $b(w) \equiv (x|y)_{b}$.

We first assume that $\epsilon|xy| \leq 1$. Since $|zw| \leq |xy|$ for all $z \in xy$, we have $\epsilon|zw| \leq 1$ for all $z \in xy$ and inequality (4.2) implies that

\begin{equation}
\rho_{\epsilon}(z) \asymp \rho_{\epsilon}(w),
\end{equation}

Integrating the comparison (4.6) over $\eta$, we obtain

\begin{equation}
\ell \epsilon(\eta) \asymp \rho_{\epsilon}(w)|xy| \asymp e^{-\epsilon(x|y)_{b}}|xy|.
\end{equation}

The estimate (4.4) then follows upon applying inequality (4.3) to $\eta$.

Now assume that $\epsilon|xy| \geq 1$. Let $\eta_{1} : [-|zw|, 0] \to X$ and $\eta_{2} : [0, |yw|] \to X$ denote the parametrizations of the subsegments of $\eta$ from $x$ to $w$ and from $w$ to $y$ respectively. Then, using (3.6) and $b(w) \equiv (x|y)_{b}$, we have

\begin{equation}
\ell \epsilon(\eta_{1}) + \ell \epsilon(\eta_{2}) = \int_{\eta_{1}} \rho_{\epsilon} \, dt + \int_{\eta_{2}} \rho_{\epsilon} \, dt \asymp e^{-\epsilon(x|y)_{b}} \left( \int_{0}^{|zw|} e^{-\epsilon t} \, dt + \int_{0}^{|yw|} e^{-\epsilon t} \, dt \right) = e^{-\epsilon(x|y)_{b}} e^{-1}(2 - e^{-\epsilon|yw|} - e^{-\epsilon|yw|}).
\end{equation}

It follows immediately that

\begin{equation}
\ell \epsilon(x, y) \leq \ell \epsilon(\eta) \leq C e^{-1} e^{-\epsilon(x|y)_{b}},
\end{equation}

which gives the upper bound. Since $\epsilon|xy| \geq 1$ and $|zw| + |yw| = |xy|$, we must have

\begin{equation}
\epsilon \min \{|zw|, |yw|\} \geq \frac{1}{2}.
\end{equation}

Therefore

\begin{equation}
e^{-1} e^{-\epsilon(x|y)_{b}}(2 - e^{-\epsilon|zw|} - e^{-\epsilon|yw|}) \geq \epsilon^{-1} e^{-\epsilon(x|y)_{b}}(1 - e^{-\frac{1}{2}}).
\end{equation}

Combining this with inequality (4.3) gives the lower bound on $d_{\epsilon}(x, y)$. \hfill \square

Recall that $\partial_{\omega} X = \partial X \setminus \{\omega\}$ denotes the complement of $\omega$ in $\partial X$. The next task is to identify $\partial X_{\omega}$ with $\partial_{\omega} X$. We will construct an identification $\iota : \partial_{\omega} X \to \partial X_{\omega}$ using Lemma 4.2. For $\xi \in \partial_{\omega} X$ we let $\gamma : \mathbb{R} \to X$ be a geodesic line from $\omega$ to $\xi$ given by the rough starlikeness hypothesis. We let $\{t_{n}\} \subset \mathbb{R}$ be a sequence with $t_{n} \to \infty$. Then $\{\gamma(t_{n})\}$ defines a Cauchy sequence in $X_{\omega}$ by Lemma 4.2 that converges to a point $z \in \partial X_{\omega}$. We then set $\iota(\xi) := z$. It’s clear from (4.3) that the point $z$ does not depend on the choice of sequence $\{t_{n}\}$, since any such sequence must eventually belong to $[s, \infty)$ for a given $s \in \mathbb{R}$ and $\ell \epsilon(z)_{[s, \infty)} \to 0$ as $s \to \infty$. The point $z$ also does not depend on the choice of geodesic
line \(\gamma\) from \(\omega\) to \(\xi\): if \(\sigma: \mathbb{R} \to X\) is any other geodesic line from \(\omega\) to \(\xi\) then Lemma 3.6 shows that for \(n \in \mathbb{N}\) we have

\[
|\gamma(n)\sigma(n)| \leq 3|\gamma(0)\sigma(0)| + 10\delta.
\]

Thus \(|\gamma(n)\sigma(n)|\) is uniformly bounded independently of \(n\). Since \(\rho_{\varepsilon}(\gamma(n)) \to 0\) as \(n \to \infty\), Lemma 1.3 shows that \(d_{\varepsilon}(\gamma(n),\sigma(n)) \to 0\) as \(n \to \infty\). It follows that \(\{\gamma(n)\}\) and \(\{\sigma(n)\}\) have the same limit in \(\partial X_\varepsilon\).

**Lemma 4.6.** The map \(\iota: \partial_\omega X \to \partial X_\varepsilon\) is a bijection.

**Proof.** We first show that \(\iota\) is injective. Let \(\xi \neq \zeta \in \partial_\omega X\) and let \(\gamma, \sigma: \mathbb{R} \to X\) be corresponding geodesic lines from \(\omega\) to \(\xi, \zeta\) respectively. For each \(n \in \mathbb{N}\) we set \(x_n = \gamma(n)\) and \(y_n = \sigma(n)\). Then \(|x_n y_n| \to \infty\) as \(n \to \infty\) since the sequences \(\{x_n\}\) and \(\{y_n\}\) are converging to different points of \(\partial X\). We also have that \((x_n|y_n)_b = 600\delta (\xi|\zeta)_b\) for \(n\) large enough by Lemma 2.7. Lemma 1.3 then implies that for sufficiently large \(n\) we have

\[
d_{\varepsilon}(x_n, y_n) \asymp \varepsilon^{-1}e^{-c|x_n|y_n)}_b \asymp \varepsilon^{-1}e^{-c(x|y)_b}.
\]

In particular \(d_{\varepsilon}(x_n, y_n)\) is bounded away from 0 independently of \(n\). It follows that \(\{x_n\}\) and \(\{y_n\}\) converge to distinct points of \(\partial X_\varepsilon\), so that \(\iota(\xi) \neq \iota(\zeta)\).

It remains to show that \(\iota\) is surjective, which is more difficult. Let \(\{x_n\}\) be a Cauchy sequence in \(X_\varepsilon\) that converges to a point \(z \in \partial \partial X_\varepsilon\). We claim that the sequence \(\{x_n\}\) cannot belong to a bounded subset of \(X\). If it did then for a fixed \(p \in X\) there would be an \(r > 0\) such that \(\{x_n\} \subset B(p,r)\) for all \(n\), with \(B(p,r)\) denoting the ball of radius \(r\) centered at \(p\) in \(X\). The Harnack inequality (1.2) shows that the metrics on \(X\) and \(X_\varepsilon\) are biLipschitz to one another on \(B(p,2r)\), which implies that \(\{x_n\}\) is also a Cauchy sequence in \(X\). Since \(X\) is complete this Cauchy sequence must converge in \(X\) to a point \(y \in B(p,2r)\). However this means that \(\{x_n\}\) also converges to \(y\) in \(X_\varepsilon\), contradicting that \(\{x_n\}\) converges to a point of \(\partial X_\varepsilon\).

Thus, by passing to a subsequence if necessary, we can assume that \(\varepsilon|x_n x_m| \geq 1\) for \(m \neq n\). As in the proof of injectivity, it then follows that for \(m \neq n\),

\[
d_{\varepsilon}(x_m, x_n) \asymp \varepsilon^{-1}e^{-c|x_n x_m)_b}.
\]

Since \(d_{\varepsilon}(x_n, x_m) \to 0\) as \(m, n \to \infty\), we conclude that \((x_n|y_n)_b \to \infty\) as \(m, n \to \infty\). Thus \(\{x_n\}\) converges to infinity with respect to \(\omega\). Proposition 2.8 then shows that \(\{x_n\}\) defines a point \(\xi \in \partial_\omega X\) of the Gromov boundary of \(X\) relative to \(\omega\).

The rough starlikeness hypothesis implies that there is a geodesic line \(\gamma: \mathbb{R} \to X\) from \(\omega\) to \(\xi\). We claim that \(\iota(\xi) = z\). Combining the inequalities of Lemma 1.3 shows that

\[
d_{\varepsilon}(x_n, \gamma(n)) \leq C\varepsilon^{-1}e^{-c(x_n|\gamma(n))_b} \min\{1, c|x_n|\gamma(n)|}\b\]

\[
\leq C\varepsilon^{-1}e^{-c(x_n|\gamma(n))_b}.
\]

Since the sequences \(\{x_n\}\) and \(\{\gamma(n)\}\) define the same point of \(\partial_\omega X\) they must be equivalent with respect to \(\omega\), i.e., we must have \((x_n|\gamma(n))_b \to \infty\) as \(n \to \infty\). This implies that \(d_{\varepsilon}(x_n, \gamma(n)) \to 0\) as \(n \to \infty\), which implies that \(\iota(\xi) = z\).

The next proposition is the key step in finishing the proofs of the main theorems in this section.

**Proposition 4.7.** For \(x \in X\) we have

\[
d_{\varepsilon}(x) \asymp \varepsilon^{-1}\rho_{\varepsilon}(x).
\]
Proof. Let $x \in X$ be given. We first compute the upper bound in (4.7). By the rough-starlikeness condition we can find a geodesic line $\gamma : \mathbb{R} \to X$ starting at $\omega$ and ending at some $\xi \in \partial X$ such that there is an $s \in \mathbb{R}$ with $|x\gamma(s)| \leq K$. Then by (4.3) and Lemma 4.4 we have

$$d_\varepsilon(x, \xi) \leq d_\varepsilon(x, \gamma(s)) + d_\varepsilon(\gamma(s), \xi) \leq \varepsilon^{-1} \rho_\varepsilon(x) \left(e^{\varepsilon |x\gamma|} - 1 \right) + \ell_\varepsilon(\gamma)|_{\mathbb{R}, \infty} \leq C \varepsilon^{-1} \rho_\varepsilon(x).$$

Since $d_\varepsilon(x) \leq d_\varepsilon(x, \xi)$ the upper bound follows.

For the lower bound we let $\xi \in \partial X$ be a given point, which we can think of as a point in $\partial_\omega X$ using Lemma 4.6. By rough starlikeness we can then find a geodesic line $\gamma : \mathbb{R} \to X$ starting at $\omega$ and ending at $\xi$. For $n \in \mathbb{N}$ we note that $|x\gamma(n)| \to \infty$ as $n \to \infty$, so we will have $\varepsilon |x\gamma(n)| \geq 1$ for all sufficiently large $n$. For sufficiently large $n$ we can then apply (4.5) and Lemma 2.7 to obtain

$$d_\varepsilon(x, \gamma(n)) \approx \varepsilon^{-1} e^{-\varepsilon |x\gamma(n)| \delta} \approx \varepsilon^{-1} e^{-\varepsilon |x\xi| \delta}.$$ 

By (2.10) we have $(x|\xi)_b \leq b(x) + 600 \delta$. By combining this with the above we obtain that

$$d_\varepsilon(x, \gamma(n)) \geq C \varepsilon^{-1} \rho_\varepsilon(x).$$

This gives the lower bound since $d_\varepsilon(x, \gamma(n)) \to d_\varepsilon(x, \xi)$ as $n \to \infty$. \hfill \Box

We can now finish the proof of Theorem 1.5. For a metric on $\partial_\omega X$ we choose a visual metric $\alpha$ on $\partial_\omega X$ for a sufficiently small parameter $0 < q \leq \varepsilon$ as in Section 2.3. By the discussion in Section 2.3 we can choose $\sigma$ so that it is biLipschitz to the quasi-metric $\alpha_{b,q}$ (2.11) with biLipschitz constant depending only on $q$ and $\delta$.

We will require the following lemma. We do not claim any quantitative dependence of $N$ below on the other parameters.

Lemma 4.8. Let $\xi_1 \in \partial X$, $i = 1, 2$ be given with $\xi_1 \neq \xi_2$ and let $\gamma_i : \mathbb{R} \to X$ be geodesic lines starting at $\omega$ and ending at $\xi_i$, $i = 1, 2$. For each $n \in \mathbb{N}$ let $\beta_n : I_n \to X$ be a parametrization of a geodesic from $\gamma_1(n)$ to $\gamma_2(n)$ that is $c$-adapted to $b$, $c = c(\delta, K)$. Then there is an $N \in \mathbb{N}$ such that for $n \geq N$ we have

$$d_\varepsilon(\xi_1, \xi_2) \approx \sup_{t \in I_n} d_\varepsilon(\beta_n(t)).$$

Furthermore we have $d_\varepsilon(\beta_n(0)) \approx d_\varepsilon(\xi_1, \xi_2)$ and if $s \in I_n$ satisfies $\beta_n(s) = \sup_{t \in I_n} d_\varepsilon(\beta_n(t))$ then $|s| \leq c$, $c = c(\delta, K, \varepsilon, M)$.

Proof. Using (5.6) and Lemma 4.7 we have for $t \in I_n$,

$$d_\varepsilon(\beta_n(t)) \approx \varepsilon^{-1} \rho_\varepsilon(\beta_n(t)) \approx e^{-\varepsilon |\gamma_1(n)| \delta} e^{-\varepsilon |\gamma_2(n)| \delta}.$$ 

Since $\xi_1 \neq \xi_2$ we have $|\gamma_1(n)\gamma_2(n)| \to \infty$ as $n \to \infty$, so there is an $N \in \mathbb{N}$ such that for $n \geq N$ we have $\varepsilon |\gamma_1(n)\gamma_2(n)| \geq 1$. By (4.5) we then have

$$d_\varepsilon(\beta_n(t)) \approx d_\varepsilon(\gamma_1(n), \gamma_2(n)) e^{-\varepsilon |t|}.$$ 

Since $d_\varepsilon(\gamma_1(n), \gamma_2(n)) \to d_\varepsilon(\xi_1, \xi_2)$ as $n \to \infty$, by increasing $N$ if necessary we can assume that for $n \geq N$ we have $d_\varepsilon(\gamma_1(n), \gamma_2(n)) \approx 2 d_\varepsilon(\xi_1, \xi_2)$. We conclude that

$$d_\varepsilon(\beta_n(t)) \approx d_\varepsilon(\xi_1, \xi_2) e^{-\varepsilon |t|}.$$ 

Because $0 \in I_n$, the right side of (4.5) is maximized for $t \in I_n$ at $t = 0$. This gives (4.8) as well as the claim that $d_\varepsilon(\beta_n(0)) \approx d_\varepsilon(\xi_1, \xi_2)$. 


For the final claim, let \( s \in I_n \) be such that \( d_\varepsilon(\beta_n(t)) \leq d_\varepsilon(\beta_n(s)) \) for all \( t \in I_n \). Applying the comparison (4.9) with \( t = 0 \) and \( t = s \) implies that
\[
d_\varepsilon(\xi_1, \xi_2)e^{-\varepsilon |s|} \geq C^{-1}d_\varepsilon(\xi_1, \xi_2),
\]
with \( C = C(\delta, K, \varepsilon, M) \). Since \( d_\varepsilon(\xi_1, \xi_2) > 0 \) this implies that \( e^{-\varepsilon |s|} \geq C^{-1} \), which implies after rearrangement that \( |s| \leq c, c = c(\delta, K, \varepsilon, M) \).

**Remark 4.9.** If \( \xi_1 \) is joined to \( \xi_2 \) by a geodesic line \( \beta : \mathbb{R} \to X \) then we can find a parametrization of this line that is \( c \)-adapted to \( b, c = c(\delta, K) \), by Lemma 3.11 as well. Applying the arguments of Lemma 4.8 to the curve \( \beta \) instead, replacing \( \gamma_1(n) \) with \( \beta(-n) \) and \( \gamma_2(n) \) with \( \beta(n) \) and noting that \( (\beta(-n)|\beta(n))_b \equiv_{600\delta} (\xi|\zeta)_b \) for \( n \) sufficiently large by Lemma 2.7, we conclude that
\[
d_\varepsilon(\xi_1, \xi_2) \asymp \sup_{t \in \mathbb{R}} d_\varepsilon(\beta(t)),
\]
that \( d_\varepsilon(\beta(0)) \asymp d_\varepsilon(\xi_1, \xi_2) \), and that if \( s \in \mathbb{R} \) satisfies \( \beta(s) = \sup_{t \in \mathbb{R}} d_\varepsilon(\beta(t)) \) then \( |s| \leq c, c = c(\delta, K, \varepsilon, M) \).

The constant \( L \) in Proposition 4.10 below is an exception to our convention that constants depend monotonically on \( \varepsilon \).

**Proposition 4.10.** The identification \( \iota : (\partial_\omega X, \sigma) \to (\partial_\varepsilon X, d_\varepsilon^\hat{X}) \) is \( L \)-biLipschitz with \( L = L(\delta, K, \varepsilon, M, q) \).

**Proof.** For this proof only we will extend our convention about generic additive and multiplicative constants \( c, C \geq 0 \) to allow them to also depend on the parameter \( q \) of the visual metric \( \alpha \), and we will drop the requirement that these constants depend monotonically on \( \varepsilon \). Then we have \( \alpha \asymp \alpha_{b, \omega} \) as discussed prior to Lemma 4.8. It thus suffices to prove that for any \( \xi_1, \xi_2 \in \partial_\omega X \) there is a constant \( L = L(\delta, K, \varepsilon, M, q) \) such that
\[
d_\varepsilon(\xi_1, \xi_2) \asymp L \alpha_{b, \omega}(\xi_1, \xi_2),
\]
or equivalently (for a potentially different constant \( L \)),
\[
d_\varepsilon(\xi_1, \xi_2) \asymp L e^{-\varepsilon(\xi_1, \xi_2)}.
\]

For \( i = 1, 2 \) we let \( \gamma_i : \mathbb{R} \to X \) be geodesic lines from \( \omega \) to \( \xi_i \) given by rough starlikeness, parametrized using Lemma 2.6 such that \( b(\gamma_i(t)) \equiv 1445 \delta \). Let \( N \in \mathbb{N} \) be large enough that for \( n \geq N \) the conclusions of Lemma 4.9 apply to these lines paired together. We also assume \( N \) is large enough that for \( n \geq N \) we have \( \varepsilon|\gamma_1(n)|\gamma_2(n)| \geq 1 \). Let \( \beta_n : I_n \to X \) be adapted parametrizations of geodesics from \( \gamma_1(n) \) to \( \gamma_2(n) \) as considered in Lemma 4.8.

Using Lemma 2.7 we fix \( n \geq N \) large enough that \( (\xi_1|\xi_2)_b \equiv_{600\delta} (\gamma_1(n)|\gamma_2(n))_b \). Lemma 4.8 implies that \( d_\varepsilon(\xi_1, \xi_2) \asymp d_\varepsilon(\beta_n(0)) \). Combining this with Proposition 4.7, the adapted condition 3.10, and our application of Lemma 2.7, we conclude that
\[
d_\varepsilon(\xi_1, \xi_2) \asymp C d_\varepsilon(\beta_n(0))
\[
\asymp C e^{-1} \rho_\varepsilon(\beta_n(0))
\]
\[
\asymp C e^{-1} e^{-(\gamma_1(n)|\gamma_2(n))_b}
\]
\[
\asymp C e^{-1} e^{-(\xi_1|\xi_2)_b},
\]
with \( C = C(\delta, K, \varepsilon, M, a) \). We can therefore set \( L = C\varepsilon^{-1} \). \( \square \)
Proposition 4.10 completes the proof of Theorem 1.5. We conclude this section by completing the proof of Theorem 1.3. As in Proposition 4.10 we drop the condition that constants depend monotonically on $\varepsilon$ in the proof.

**Proposition 4.11.** There is an $A = A(\delta, K, \varepsilon, M)$ such that any geodesic joining two points of $X$ is an $A$-uniform curve in $X_{\varepsilon}$. Consequently $X_{\varepsilon}$ is $A$-uniform.

**Proof.** Let $\eta : I \to X$ be a geodesic joining two points $x, y \in X$ with a parametrization that is $c$-adapted to $b$, $c = c(\delta, K)$, as in Lemma 3.11. Inequality (1.3) implies (1.4) for $\eta$ with $A = M$. Thus we only need to verify inequality (1.2). We write $I = [t_-, t_+]$ and let $s \in I$ be given. It suffices to verify inequality (1.2) in the case that $s \in [0, t_+]$, since we can deduce the case $s \in [t_-, 0]$ from this by reversing the roles of $x$ and $y$. We thus assume that $s \in [0, t_+]$. A straightforward calculation with (3.6) gives us that

$$
\ell_{\varepsilon}(\eta|[s, t_+]) \leq e^{-c(x|y)s} \int_s^{t_+} e^{-c t} dt \leq e^{-1} e^{-c(x|y)s} \varepsilon^{c s}.
$$

Since $s + (x|y)s = b(\eta(s))$, it then follows from (4.10) and Proposition 4.7 that

$$
\ell_{\varepsilon}(\eta|[s, t_+]) \leq C\varepsilon^{-1} \rho_{\varepsilon}(\eta(s)) \leq C d_{\varepsilon}(\eta(s)),
$$

with $C = C(\delta, K, \varepsilon, M)$. The proposition follows. \hfill \square

5. Hyperbolic fillings

Let $(Z, d)$ be a metric space and let $0 < a < 1$ and $\tau > 1$ be given parameters. We recall the construction of a hyperbolic filling $X$ of $Z$ with these parameters described prior to Theorem 1.7. For each $n \in Z$ we select a maximal $a^n$-separated subset $S_n$ of $Z$. Then for each $n \in Z$ the balls $B(z, a^n)$, $z \in S_n$, cover $Z$.

The vertex set of $X$ has the form

$$
V = \bigcup_{n \in \mathbb{Z}} V_n, \quad V_n = \{(x, n) : x \in S_n\}.
$$

To each vertex $v = (x, n)$ we associate the dilated ball $B(v) = B(x, \tau a^n)$. We will often use $v$ to denote both a vertex in $X$ and its associated point in $Z$. We also define the height function $h : V \to \mathbb{Z}$ by $h(x, n) = n$. We note that, by construction, for each $z \in Z$ there is an $v \in V_n$ such that $\rho(v, z) < a^n$.

We place an edge in $X$ between distinct vertices $v$ and $w$ if and only if $|h(v) - h(w)| \leq 1$ and $B(v) \cap B(w) \neq \emptyset$. Thus there is an edge between vertices if they are of the same or adjacent height and there is a nonempty intersection of their associated balls. For vertices $v, w$ we write $v \sim w$ if there is an edge between $v$ and $w$. Edges between vertices of the same height are referred to as horizontal, and edges between vertices of different heights are called vertical. We say that an edge path between two vertices is vertical if it is composed exclusively of vertical edges.

We give each connected component of $X$ the unique geodesic metric in which all edges have unit length. We will see later in Proposition 5.10 that $X$ is actually connected and is therefore a geodesic metric space itself. Since edges can only connect vertices of the same or adjacent heights, all vertical edge paths are geodesics in $X$. We will refer to these vertical paths as vertical geodesics. We will use the generic distance notation $|xy|$ for the distance between $x, y \in X$. Thus for $v = (x, n), w = (y, n) \in V$ we will denote their distance in $X$ by $|vw|$ and their distance in $Z$ by $d(v, w) := d(x, y)$.

Identifying an edge $g$ from a vertex $v$ to a vertex $w$ isometrically with $[0, 1]$, we extend the height function $h$ to $g$ by $h(s) = sh(v) + (1 - s)h(w)$. Then $h$ defines a function $h : X \to \mathbb{R}$
that is 1-Lipschitz on the connected components of $X$. For any vertical geodesic $\gamma : I \to X$ we can always reparametrize $\gamma$ such that we have $h(\gamma(t)) = t$ (possibly changing the domain $I$ in the process).

While we will allow any choice of $a$ satisfying $0 < a < 1$, we will need to place some constraints on the values of the parameter $\tau$ based on $a$. We will require that

$$\tau > \max \left\{ 3, \frac{1}{1-a} \right\} .$$

**Remark 5.1.** We do not know whether the constraint (5.1) can be relaxed while preserving the properties of $X$ described below. In particular we do not know whether $X$ is always Gromov hyperbolic or even connected for all $\tau > 1$. However, by applying Lemma 5.3 below it is easy to see that $X$ is connected for any $\tau > 1$ when $Z$ is bounded. The arguments of [3] then imply that $X$ is Gromov hyperbolic for all $\tau > 1$; while their construction is slightly different from ours, their proofs can be easily adapted to our setting.

**Remark 5.2.** Our hyperbolic filling incorporates the principal innovation of [3], which is to allow all parameter values $0 < a < 1$ by allowing the dilation factor $\tau$ to increase as $a \to 1$. Our parameter $a$ corresponds to their parameter $\alpha$ through $a = \alpha^{-1}$. While their construction works only for bounded metric spaces, it also works for all values $0 < a < 1$ and $\tau > 1$ smaller than what are used here. However they impose a constraint similar to (5.1) in order to deduce their analogue of Theorem 1.3. We note that one cannot take $\tau = 1$ in the construction, as it is possible for the resulting graph to fail to be Gromov hyperbolic even in the bounded case [3].

We begin with a simple lemma.

**Lemma 5.3.** Let $v, w \in V$ with $h(v) \neq h(w)$ and $B(v) \cap B(w) \neq \emptyset$. Then there is a vertical edge path from $v$ to $w$.

**Proof.** Let $v = (x, m)$, $w = (y, n)$, and let $z \in B(v) \cap B(w)$. We can assume without loss of generality that $m < n$. For each integer $m \leq k \leq n$ we can find a vertex $v_k \in V_k$ with $z \in B(v_k)$; we set $v_m = v$ and $v_n = w$. Then $v_k \sim v_{k+1}$ for each $m \leq k < n$ by the construction of the graph $X$. It follows that $v$ is connected to $w$ by a vertical edge path passing through the vertices $v_k$. \hfill \square

The next lemma estimates the distance in $Z$ between vertices in $X$ that are connected by a vertical edge path.

**Lemma 5.4.** Let $v, w \in V$. Suppose that $v$ is joined to $w$ by a vertical edge path and $h(v) \leq h(w)$. Then

$$d(v, w) \leq \frac{2\tau a^{h(v)}}{1-a} .$$

**Proof.** We first derive a sharper inequality in the case $h(w) = h(v) + 1$. Set $h(v) = m$. Let $x \in B(v) \cap B(w)$. Then

$$d(v, w) \leq d(x, v) + d(x, w) < \tau a^m + \tau a^{m+1} < 2\tau a^m .$$

Now let $h(v) = m$, $h(w) = n$. For each $m \leq k \leq n$ we let $v_k \in V_k$ be the vertex at this height in the vertical edge path joining $v$ to $w$. Then by the “$h(w) = h(v) + 1$” case we have

$$d(v, w) \leq \sum_{k=m}^{n-1} d(v_k, v_{k+1}) \leq 2\tau a^m \sum_{k=m}^{n-1} a^k \leq \frac{2\tau a^m}{1-a} ,$$

with the final inequality following by summing the geometric series in $a$. \hfill \square
Following the hyperbolic filling construction in [8], we define a cone point \( u \in V \) for a pair of vertices \( \{v, w\} \subseteq V \) to be a vertex that can be joined to both \( v \) and \( w \) by vertical geodesics and that satisfies \( h(u) \leq \min\{h(v), h(w)\} \). A branch point for \( \{v, w\} \) is defined to be a cone point of maximal height. A branch point for \( \{v, w\} \) always exists as long as there is at least one cone point for \( \{v, w\} \).

**Lemma 5.5.** Let \( v, w \in V_n \) be distinct vertices with \( v \sim w \). Then there is a branch point \( u \in V_{n-1} \) for the set \( \{v, w\} \).

**Proof.** The assumptions imply that \( B(v) \cap B(w) \neq \emptyset \). Let \( z \in B(v) \cap B(w) \) be a point in this intersection. Since \( V_{n-1} \) is a maximal \( a^{n-1} \)-separated set in \( Z \) we can find \( u \in V_{n-1} \) such that \( d(u, z) < a^{n-1} \). We compute

\[
 d(v, u) \leq d(v, z) + d(z, u) < \tau a^n + a^{n-1} < \tau a^{n-1},
\]

by inequality (5.1), noting that the final inequality here is equivalent to

\[
 \tau a + 1 < \tau,
\]

which is equivalent to (5.1). It follows that \( v \in B(u) \) and therefore \( B(v) \cap B(u) \neq \emptyset \). Thus \( v \) is joined to \( u \) by a vertical edge. Since the roles of \( v \) and \( w \) are symmetric, we conclude by the same calculation that \( B(w) \cap B(u) \neq \emptyset \), i.e., \( w \) is also joined to \( u \) by a vertical edge. Thus \( u \) is a cone point for \( \{v, w\} \). Since a cone point on an adjacent level is trivially maximal, we conclude that \( u \) is a branch point for \( \{v, w\} \). \( \square \)

We can now show that the graph \( X \) is connected.

**Proposition 5.6.** For each \( v, w \in V \) there is a branch point \( u \) for the set \( \{v, w\} \) that satisfies

\[
 a^{h(u)} \preceq_C (a, \tau) d(v, w) + a^{\min\{h(v), h(w)\}}.
\]

Consequently the graph \( X \) is connected.

**Proof.** Let \( v \in V_m, w \in V_n \) be given. We can assume without loss of generality that \( m \leq n \). We let \( k \in \mathbb{Z} \) be any integer satisfying \( a^k > d(v, w) \) and \( k \leq m \); note that such an integer always exists since \( a^k \to \infty \) as \( k \to -\infty \). Let \( p \in V_k \) be a vertex such that \( d(v, p) < a^k \) and let \( q \in V_k \) be a vertex such that \( d(q, w) < a^k \). Then

\[
 d(p, q) \leq d(v, p) + d(v, w) + d(w, q) < 3a^k < \tau a^k,
\]

by (5.1). Thus \( q \in B(p) \), so \( B(p) \cap B(q) \neq \emptyset \). We conclude that \( p \sim q \). By Lemma 5.3 we can then find a cone point \( x \in V_{k-1} \) for the set \( \{p, q\} \). Since \( B(p) \cap B(v) \neq \emptyset \) and \( B(q) \cap B(w) \neq \emptyset \), Lemma 5.3 shows that \( p \) and \( q \) are connected to \( v \) and \( w \) respectively by vertical edge paths, and the requirement \( k \leq m \) implies that \( h(p) \leq \min\{h(v), h(w)\} \) as well as the same for \( h(q) \). Since \( p \) and \( q \) are each connected to \( x \) by a vertical edge, we conclude that \( x \) is a cone point for the set \( \{v, w\} \).

It follows that there is a branch point \( u \) for the set \( \{v, w\} \). Since \( u \) is joined to \( v \) and \( w \) by vertical edge paths, the triangle inequality and Lemma 5.4 implies that

\[
 d(v, w) \leq 2 \max\{d(v, u), d(w, u)\} \leq C(a, \tau) a^{h(u)}.
\]

Since \( h(u) \leq m \), we have \( a^m \leq a^{h(u)} \) and therefore

\[
 d(v, w) + a^m \leq C(a, \tau) a^{h(u)} + a^m \leq C(a, \tau) a^{h(u)},
\]

which gives one side of the comparison (5.2).
For the other side, we split into two cases. The first case is that in which \( v \) can be joined to \( w \) by a vertical edge path. In this case \( v \) is a branch point for the set \( \{v, w\} \) and the inequality

\[
d(v, w) + a_v h(v) \geq a_w h(w),
\]

holds trivially for \( u = v \). The second case is that in which \( v \) cannot be joined to \( w \) by a vertical edge path. By Lemma 5.3 we must then have \( B(v) \cap B(w) = \emptyset \), and in particular we must have \( w \notin B(v) \). Thus \( d(v, w) \geq \tau a_v^m > 0 \). Let \( k \in \mathbb{Z} \) be the maximal integer such that \( k \leq m \) and \( a_k > d(v, w) \). Then either \( k = m \) or \( d(v, w) \geq a_k^{+1} \). Since \( a_k > d(v, w) \) and \( d(v, w) \geq \tau a_v^m \), we conclude in both cases that \( d(v, w) \geq_C (a, \tau) a_k \). Making this choice of \( k \) in the construction of \( x \) above, we can thus construct a cone point \( x \) for the set \( \{v, w\} \) with \( h(x) = k - 1 \) and therefore

\[
a_v h(x) \geq_C (a, \tau) d(v, w).
\]

Since the branch point \( u \) satisfies \( h(u) \geq h(x) \), it follows that

\[
a_v h(u) \leq C(a, \tau) d(v, w) \leq C(a, \tau)(d(v, w) + a^m).
\]

The comparison (5.2) follows.

Lastly, since we can connect \( v \) to \( w \) through the branch point \( u \), it follows that \( v \) and \( w \) can be connected by an edge path in the graph \( X \). Since \( v \) and \( w \) were arbitrary, it follows that \( X \) is connected.

Now that we’ve shown \( X \) is connected, the metrics we put on its connected components give it the structure of a geodesic metric space in which all edges of \( X \) have unit length. The height function then defines a 1-Lipschitz function \( h : X \to \mathbb{R} \). We formally define the Gromov product based at \( h \) by, for \( x, y \in X \),

\[
(x|y)_h = \frac{1}{2}(h(x) + h(y) - |xy|).
\]

Since \( h \) is 1-Lipschitz we have

(5.3)

\[
(x|y)_h \leq \min\{h(x), h(y)\}.
\]

Our next lemma gives a key relation of the Gromov product based at \( h \) to branch points.

**Lemma 5.7.** Let \( v, w \in V \) and let \( u \) be a branch point for \( \{v, w\} \). Then

\[
h(u) \geq_C (a, \tau) (v|w)_h,
\]

and therefore

\[
a_v (v|w)_h \geq_C (a, \tau) d(v, w) + a^{\min\{h(v), h(w)\}}
\]

**Proof.** Proposition 5.6 gives the existence of a branch point \( u \) for \( \{v, w\} \) satisfying (5.2). The vertical edge path from \( v \) to \( u \) followed by the vertical edge path from \( u \) to \( w \) gives an edge path from \( v \) to \( w \), which shows that

\[
|vw| \leq |vu| + |uw| = h(v) - h(u) + h(w) - h(u) = h(v) + h(w) - 2h(u).
\]

Rearranging this we obtain

\[
h(u) \leq \frac{1}{2}(h(v) + h(w) - |vw|) = (v|w)_h.
\]

To get a bound in the other direction, let \( v = v_0, v_1, \ldots, v_k = w \) be a sequence of vertices joined by edges that gives a geodesic \( \gamma \) from \( v \) to \( w \). Then \( |vw| = k - 1 \). For \( 1 \leq i \leq k \) we have \( B(v_{i-1}) \cap B(v_i) \neq \emptyset \) and therefore, using \( |h(v_{i-1}) - h(v_i)| \leq 1 \),

\[
d(v_{i-1}, v_i) < 2\tau a^{\min\{h(v_{i-1}), h(v_i)\}} \leq 2\tau a^{h(v_{i-1})-1}.
\]
We can run the same argument viewing $\gamma$ as a geodesic from $w$ to $v$ instead, setting $w_i = v_{k-i}$ for $0 \leq i \leq k$. We see from this that we also have
\[
d(w_{i-1}, w_i) < 2\tau a^{h(w_{i-1})-1},
\]
for $1 \leq i \leq k$. For each $1 \leq l \leq k$ we thus obtain an estimate (using $h(v_{i-1}) \geq h(v) - i + 1$ and $h(w_{i-1}) \geq h(w) - i + 1$),
\[
d(v, w) \leq \sum_{i=1}^{k} d(v_{i-1}, v_i)
= \sum_{i=1}^{l} d(v_{i-1}, v_i) + \sum_{i=1}^{k-l} d(v_{k-i+1}, v_{k-i})
= \sum_{i=1}^{l} d(v_{i-1}, v_i) + \sum_{i=1}^{k-l} d(w_{i-1}, w_i)
< 2\tau a^{h(v)} \sum_{i=1}^{l} a^{-i} + 2\tau a^{h(w)} \sum_{i=1}^{k-l} a^{-i}
\leq \frac{2\tau a^{-1}}{a^{-1} - 1} (a^{h(v)} (a^{-l} - 1) + a^{h(w)} (a^{k-l} - 1))
\leq \frac{2\tau}{1-a} (a^{h(v)-l} + a^{h(w)+k-l}).
\]
We set $l = \lceil \frac{1}{2} (k - h(w) + h(v)) \rceil$ (the least integer greater than this quantity), observing that $1 \leq l \leq k$ since $|h(v) - h(w)| \leq k$. This gives, after some simplification,
\[
d(v, w) \leq \frac{4\tau a^{-2}}{1-a} a^{\frac{1}{2} (h(v) + h(w) - k + 1)}.
\]
Recalling that $|vw| = k - 1$, we conclude that
\[
d(v, w) \leq C(a, \tau) a^{(\tau|w|)_h}.
\]
By Proposition 5.8 and inequality \eqref{4.3}, we then have
\[
a^{h(u)} \leq C(a, \tau) a^{(\tau|w|)_h},
\]
which implies upon taking logarithms that
\[
h(u) \geq (\tau|w|)_h - c(a, \tau).
\]
This gives the desired lower bound of the first approximate equality of the lemma. The second comparison inequality follows upon exponentiating each side and using Proposition 5.6 again.

We now prove an inequality similar to the $4\delta$-inequality \eqref{2.3} for our formal Gromov products based at $h$.

**Lemma 5.8.** Let $u, v, w \in V$. Then
\[
(u|w)_h \geq \min\{ (u|v)_h, (v|w)_h \} - c(a, \tau).
\]

**Proof.** Let $u, v, w \in V$ be vertices. By the triangle inequality in $Z$ we have
\[
d(u, w) + a^{\min\{h(u), h(w)\}} \leq d(u, v) + a^{\min\{h(v), h(w)\}} + d(v, w) + a^{\min\{h(v), h(w)\}},
\]
for $0 \leq i \leq k$. For each $1 \leq l \leq k$ we thus obtain an estimate (using $h(v_{i-1}) \geq h(v) - i + 1$ and $h(w_{i-1}) \geq h(w) - i + 1$),
which becomes, upon applying Lemma 2.4,
\[ a^{(u|w)_h} \leq C(a,\tau)(a^{(u|v)_h} + a^{(v|w)_h}) \leq C(a,\tau)a^{\min\{u|v)_h, (v|w)_h\}}. \]

Taking logarithms of each side gives the desired inequality. \(\square\)

We can now show that \(X\) is Gromov hyperbolic. For this we use some terminology from \[8\] Chapter 2: a \(\delta\)-triple for \(\delta > 0\) is a triple \((a,b,c)\) of real numbers such that the two smallest numbers differ by at most \(\delta\). We observe that \((a,b,c)\) is a \(\delta\)-triple if and only if the inequality
\[ c \geq \min\{a,b\} - \delta, \]
holds for all permutations of the roles of \(a, b,\) and \(c\). We will also need the following standard claim \[8\] Lemma 2.1.4 which is referred to as the Tetrahedron lemma.

**Lemma 5.9.** Let \(d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}\) be six numbers such that the four triples \((d_{23},d_{24},d_{34}), (d_{13},d_{14},d_{34}), (d_{12},d_{14},d_{24}), \) and \((d_{12},d_{13},d_{23})\) are \(\delta\)-triples. Then
\[ (d_{12} + d_{34}, d_{13} + d_{24}, d_{14} + d_{23}) \]
is a \(2\delta\)-triple.

**Proposition 5.10.** The space \(X\) is \(\delta\)-hyperbolic with \(\delta = \delta(a,\tau)\).

**Proof.** We will use the cross-difference triple defined in \[8\] Chapter 2.4. For a quadruple of points \(Q = (x,y,z,u) \in X\) and a fixed basepoint \(o \in X\) this triple is defined by
\[ A_o(Q) = ((x|y)_o + (z|u)_o, (x|z)_o + (y|u)_o, (x|u)_o + (y|z)_o). \]
The triple \(A_o(Q)\) has the same differences among its members as the triple
\[ A_h(Q) = ((x|y)_h + (z|u)_h, (x|z)_h + (y|u)_h, (x|u)_h + (y|z)_h), \]
as a routine calculation shows for instance that
\[ (x|y)_o + (z|u)_o - (x|z)_o - (y|u)_o = (x|y)_h + (z|u)_h - (x|z)_h - (y|u)_h, \]
with both expressions being equal to
\[ \frac{1}{2}(-|xy| - |zu| + |xz| + |yu|). \]
Similar calculations give equality for the other differences. Thus \(A_o(Q)\) is a \(\delta\)-triple for a given \(\delta > 0\) if and only if \(A_h(Q)\) is a \(\delta\)-triple.

Using Lemma 5.8 we conclude that the six numbers \((x|y)_h, (z|u)_h, (x|z)_h, (y|u)_h, (x|u)_h, \) and \((y|z)_h\) together satisfy the hypotheses of Lemma 5.9 with parameter \(\delta = \delta(a,\tau)\). This implies that \(A_h(Q)\) is a \(2\delta\)-triple and therefore that \(A_o(Q)\) is a \(2\delta\)-triple. By \[8\] Proposition 2.4.1 this implies that inequality (2.3) holds for Gromov products based at \(o\) in \(X\) (with \(2\delta\) replacing \(4\delta\)). By the discussion \[10\] Chapitre 2, Proposition 21 this implies that geodesic triangles in \(X\) are \(8\delta\)-thin, i.e., \(X\) is \(8\delta\)-hyperbolic. \(\square\)

We next show that any vertex in \(V\) is part of a vertical geodesic line. We will in fact show something stronger. We let \(\tilde{Z}\) denote the completion of \(Z\), and continue to write \(d\) for the canonical extension of the metric on \(Z\) to its completion. For \(r \geq 0\) and a point \(z \in \tilde{Z}\) we will write \(B'(z, r)\) for the open ball of radius \(r\) centered at \(z\) in the completion \(\tilde{Z}\).
Lemma 5.11. Let \( z \in \tilde{Z} \). Then there is a vertical geodesic \( \gamma : \mathbb{R} \to X \) with \( h(\gamma(t)) = t \) for \( t \in \mathbb{R} \) such that, writing \( \gamma(n) = (z_n, n) \) for \( n \in \mathbb{Z} \), we have \( z \in B'(z_n, \frac{3}{4}a^n) \) for each \( n \in \mathbb{Z} \). Furthermore if \( v = (z, m) \) is a vertex of \( V \) for some \( m \in \mathbb{Z} \) then we can construct \( \gamma \) such that \( \gamma(m) = v \).

Proof. Since \( \frac{3}{4} > 1 \) by (5.1) and for each \( n \in \mathbb{Z} \) the balls \( B(y, a^n) \) cover \( Z \) for \( y \in S_n \), it follows from the fact that \( Z \) is dense in \( \tilde{Z} \) that the balls \( B'(y, \frac{3}{4}a^n) \) for \( y \in A_n \) cover \( \tilde{Z} \).

Thus, given \( z \in \tilde{Z} \), for each \( n \in \mathbb{Z} \) we can find \( z_n \in S_n \) such that \( z \in B'(z_n, \frac{3}{4}a^n) \).

Let \( v_n = (z_n, n) \) be the associated vertex in \( V \). We claim that for each \( n \in \mathbb{Z} \) we have \( B(v_n) \cap B(v_{n+1}) \neq \emptyset \). Since \( Z \) is dense in \( \tilde{Z} \) we can find \( y \in \mathbb{Z} \) such that \( d(y, z) < \frac{7}{3}a^{n+1} \). Then

\[
\frac{d(y, z)}{3} + d(z, z_{n+1}) < \frac{7}{3}a^{n+1} + \frac{7}{3}a^{n+1} < \tau a^{n+1},
\]

which implies that \( y \in B(v_{n+1}) \). A similar calculation shows that \( y \in B(v_n) \) since \( a^{n+1} < a^n \). Thus \( B(v_n) \cap B(v_{n+1}) \neq \emptyset \) and therefore \( v_n \sim v_{n+1} \). We can therefore find a vertical geodesic \( \gamma : \mathbb{R} \to X \) through the sequence of vertices \( \{v_n\}_{n \in \mathbb{Z}} \), and that can be parametrized such that \( h(\gamma(t)) = t \) for \( t \in \mathbb{R} \). Finally, if \( v = (z, m) \) is a vertex of \( V \) then we can choose \( z_m = z \) in our construction since we trivially have \( z \in B(z, \frac{3}{4}a^m) \).

A descending geodesic ray \( \gamma : [0, \infty) \to X \) is a vertical geodesic ray with \( h(\gamma(t)) \to -\infty \) as \( t \to \infty \). In this case we have \( h(\gamma(t)) = h(\gamma(0)) - t \) for each \( t \geq 0 \). A descending geodesic ray \( \gamma \) is anchored at \( z \in \tilde{Z} \) if for each vertex \((z_m, m)\) belonging to \( \gamma \) we have \( z \in B'(z_m, \frac{3}{4}a^m) \); when the point \( z \) does not need to be referenced we will just say that \( \gamma \) is anchored. Reversing the orientation of the geodesic line constructed in Lemma 5.11 shows that for any vertex \( v = (z, n) \) there is a descending geodesic ray \( \gamma \) starting at \( v \) that is anchored at \( z \).

Lemma 5.12. Let \( \gamma, \sigma : [0, \infty) \to X \) be two descending geodesic rays in \( X \) starting at vertices \( v = \gamma(0) \) and \( w = \sigma(0) \) of \( X \) respectively and anchored at \( y, z \in \tilde{Z} \) respectively. Let \( k \in \mathbb{Z} \) be such that \( k \leq \min\{h(v), h(w)\} \) and \( \frac{7}{3}a^k > d(y, z) \). Let \( v_k \in \gamma \cap V_k, w_k \in \sigma \cap V_k \) be the vertices on these geodesics at the height \( k \). Then \( v_k \sim w_k \).

Proof. By the anchoring condition we have \( d(v_k, y) < \frac{7}{3}a^k \) and \( d(w_k, z) < \frac{7}{3}a^k \). Hence

\[
d(v_k, y) + d(y, z) + d(z, w_k) < \tau a^k.
\]

Thus \( w_k \in B(v_k) \) and therefore \( B(v_k) \cap B(w_k) \neq \emptyset \), which implies that \( v_k \sim w_k \).

The Busemann functions of anchored descending geodesic rays have a particularly simple form.

Lemma 5.13. Let \( \gamma \) be an anchored descending geodesic ray in \( X \). Then for all \( x \in X \) we have

\[
(5.5) \quad b_\gamma(x) = \frac{3}{4} h(x) + h(\gamma(0)).
\]

Proof. Since both \( b_\gamma \) and \( h \) are 1-Lipschitz and the edges of \( X \) have unit length, it suffices to prove the estimate (5.13) on the vertices of \( X \) with the constant \( 1 \) instead of \( 3 \). Let \( z \in \tilde{Z} \) be the anchoring point for \( \gamma \), let \( v = \gamma(0) \), let \( k = h(v) \) and let \( v_n = k \) be the sequence of vertices on \( \gamma : [0, \infty) \to X \) with \( h(v_n) = n \). Let \( w \in V_m \) be an arbitrary vertex at height \( m \in \mathbb{Z} \). Let \( \sigma : [0, \infty) \to X \) be a descending geodesic ray with \( \sigma(0) = w \), is anchored at the point \( y \in Z \) associated to \( w \), as was constructed in Lemma 5.11. Let \( w_n = k \) be the sequence of vertices on \( \sigma \) with \( h(w_n) = n \). Let \( n \in \mathbb{Z} \) be small enough that \( n \leq \min\{k, m\} \) and \( \frac{7}{3}a^m > d(y, z) \). Then by Lemma 5.12 we have \( v_n \sim w_n \). Since \( w_n \) is joined to \( w \) by a
vertical edge path with \( h(w) - h(w_n) = m - n \) edges, it follows immediately from this and the fact that \( h \) is 1-Lipschitz that
\[
m - n \leq |v_n w| \leq m - n + 1,
\]
and therefore
\[
|v_n w| + n \geq 1 m.
\]
Since \( v_n = \gamma(-n+k) \), this implies that
\[
|\gamma(-n+k)|w| - (-n+k) \geq 1 m + k.
\]
Letting \( n \to \infty \) and recalling that \( m = h(w) \) and \( k = h(v) \), we conclude that
\[
b_\gamma(w) \geq 1 h(w) + h(v),
\]
which gives the desired result.

□

In particular, for an anchored descending geodesic ray \( \gamma \) with \( h(\gamma(0)) = 0 \), Lemma 5.13 shows that \( b_\gamma \approx_3 h \). We fix such a descending geodesic ray \( \gamma \) for the remainder of this section and write \( b := b_\gamma \) for the associated Busemann function. We note that \( (x|y)_b \approx_3 (x|y)_h \) for all \( x, y \in X \) as well, so that in particular the conclusions of Lemma 5.6 hold with \( b \) replacing \( h \) and \( (v|w)_b \) replacing \( (v|w)_h \) everywhere, at the cost of adding 6 to the constant \( C(a, \tau) \) there and multiplying the constant \( C(a, \tau) \) by \( e^3 \). We will use this observation without further comment below.

Let \( \omega \in \partial X \) be the point corresponding to the equivalence class of \( \gamma \) in the Gromov boundary of \( X \); note that Lemma 5.12 shows that all anchored descending geodesic rays belong to the equivalence class \( \omega \) defined by \( \gamma \). Our next goal is to show that the boundary \( \partial_\omega X \) of \( X \) relative to \( \omega \) can be canonically identified with the completion \( \bar{Z} \) of \( Z \) in such a way that the extension of the metric \( d \) to \( \bar{Z} \) is a visual metric on \( \partial_\omega X \) based at \( \omega \) with parameter \( -\log a \).

**Proposition 5.14.** The boundary \( \partial_\omega X \) of \( X \) with respect to \( \omega \) canonically identifies with the completion \( \bar{Z} \) of \( Z \). The extension of the metric \( d \) to \( \bar{Z} \) defines a visual metric on \( \partial_\omega X \) with parameter \( -\log a \).

**Proof.** We recall from Proposition 2.5 that \( \partial_\omega X \) can be defined as equivalence classes of sequences \( \{x_n\} \) in \( X \) such that \( (x_m|x_n)_b \to \infty \) as \( m, n \to \infty \), with two sequences \( \{x_n\}, \{y_n\} \) being equivalent if \( (x_n|y_n)_b \to \infty \) as \( n \to \infty \). Since \( b \) is 1-Lipschitz, it is easy to see that we can always choose these sequences to consist of vertices in \( X \) by replacing \( x_n \) with a nearest vertex \( v_n \).

Thus let \( \{v_n\} \) be a sequence of vertices defining a point of \( \partial_\omega X \). Let \( \{z_n\} \) be the associated sequence of points in \( Z \). By Lemma 5.7 we have
\[
a^{(v_n|v_m)_b} \geq_C (z_n, z_m) + a^{\min(b(v_n), b(v_m))}.
\]
Since \( (v_n|v_m)_b \to \infty \) and \( b(v_n) \to \infty \), it follows immediately that \( \{z_n\} \) is a Cauchy sequence in \( Z \) and therefore defines a point of \( \bar{Z} \). If \( \{w_n\} \) is another sequence of vertices in \( X \), with underlying points \( \{y_n\} \) in \( Z \), that satisfies \( (v_n|w_n)_b \to \infty \) as \( n \to \infty \) then the same estimate from Lemma 5.7 shows that the associated Cauchy sequences \( \{z_n\} \) and \( \{y_n\} \) are equivalent, and conversely if these Cauchy sequences are equivalent then the associated sequences of vertices \( \{v_n\} \) and \( \{w_n\} \) must also be equivalent. This gives us an injective map \( \partial_\omega X \to \bar{Z} \).

To show that this map is surjective, let \( z \in \bar{Z} \) be given. Let \( \sigma : \mathbb{R} \to X \) be the geodesic line constructed using Lemma 5.11 such that the vertices \( \sigma(n) = (z_n, n) \) satisfy \( z \in B'(z_n, \frac{1}{n}a^n) \) for \( n \in \mathbb{Z} \). Then \( d(z, z_n) < \frac{1}{n}a^n \) and therefore \( z_n \to z \) as \( n \to \infty \). It thus suffices to show that
the sequence \( \{v_n\}_{n \in \mathbb{N}} \) defines a point of \( \partial_\infty X \), i.e., that \( (v_n|v_m)_b \to \infty \) as \( n, m \to \infty \). But this follows easily from Lemma 5.7 as the comparison in (5.6) shows us that \( (v_n|v_m)_b \to \infty \) if \( d(z_n, z_m) \to 0 \) and \( b(v_n) \to \infty \) as \( n, m \to \infty \).

Our final task is to show that the extension of \( \rho \) to \( \bar{Z} \) defines a visual metric on \( \partial_\infty X \) with parameter \( -\log a \). This is just a straightforward consequence of Lemma 5.7 two sequences \( \{v_n\} \) and \( \{w_n\} \) of vertices defining points in \( \partial_\infty X \) with underlying sequences of points \( \{x_n\} \) and \( \{y_n\} \) in \( Z \) have the estimate

\[
(a^{(v_n|w_n)}_b + a^{\min\{b(v_n), b(w_n)\}}_e) \sim C(a, \tau) \ d(x_n, y_n).
\]

Let \( x, y \in \bar{Z} \) be the points such that \( x_n \to x \) and \( y_n \to y \). We can assume that \( x \neq y \), since the claimed comparison to a visual metric trivially holds if \( x = y \). Then \( d(x, y) > 0 \), so since \( \min\{b(v_n), b(w_n)\} \to \infty \) as \( n \to \infty \), for \( n \) large enough we will have \( a^{\min\{b(v_n), b(w_n)\}}_e \leq 1/2 d(x, y) \) and \( |d(x, y) - d(x_n, y_n)| < 1/4 d(x, y) \). For such \( n \) we then have

\[
d(x_n, y_n) + a^{\min\{b(v_n), b(w_n)\}}_e \sim C(a, \tau) \ d(x, y).
\]

Combining this with (5.7) and letting \( n \to \infty \) gives the claim. \( \square \)

6. Uniformizing the filling

The rest of this paper is devoted to proving Theorem 1.7. We let \( (Z, d) \) be a metric space and let \( X \) be a hyperbolic filling of \( Z \) with parameters \( 0 < a < 1 \) and \( \tau > \min\{3, (1-a)^{-1}\} \) as in the previous section. We let \( h : X \to \mathbb{R} \) be the height function and set \( \rho_{0\tau}(x) = e^{-\alpha h(x)} \).

We write \( X_\tau \) for the conformal deformation of \( X \) with conformal factor \( \rho_{0\tau} \), \( d_\tau \) for the metric on \( X_\tau \), and \( \ell_\tau \) for lengths of curves measured in the metric \( d_\tau \).

For Theorem 1.7 we need to check that the hypotheses of Theorem 1.3 are satisfied. Clearly \( X \) is geodesic and complete, and Proposition 5.14 shows that \( X \) is \( \delta \)-hyperbolic with \( \delta = \delta(a, \tau) \). We next look at rough starlikeness from \( \omega \).

Lemma 6.1. The metric space \( X \) is \( \frac{1}{\delta} \)-roughly starlike from \( \omega \).

Proof. Let \( v \in V \) be a vertex of \( X \) with associated point \( z \in \bar{Z} \). Let \( \gamma : \mathbb{R} \to X \) be a vertical geodesic line through \( v \) as constructed in Lemma 5.11 oriented in the direction of increasing height and parametrized such that \( \gamma(0) = v \). Put \( \tilde{\gamma}(t) = \gamma(-t) \). Then \( \tilde{\gamma}|_{[0, \infty)} \) is an anchored descending geodesic ray and therefore belongs to the equivalence class \( \omega \) by Lemma 5.12.

This shows that any vertex of \( X \) lies on a geodesic line starting at \( \omega \). Since any point in \( X \) is within distance \( \frac{1}{\delta} \) of some vertex, condition (1) of Definition 2.1 follows.

For condition (2) we use the identification of \( \partial_\infty X \) with \( \bar{Z} \) from Proposition 5.14. Let \( z \in \bar{Z} \) be given and let \( \gamma : \mathbb{R} \to X \) be a vertical geodesic line constructed as in Lemma 5.11 and parametrized such that \( h(\gamma(t)) = t \), so that for each vertex \( \gamma(n) = (z_n, n) \) on this line we have \( z \in B'(z_n, \frac{1}{\delta} a^n) \). Putting \( \tilde{\gamma}(t) = \gamma(-t) \) as above, we have that \( \tilde{\gamma}|_{[0, \infty)} \) is a descending geodesic ray anchored at \( z \) and therefore belongs to the equivalence class of \( \omega \) by Lemma 5.12. We clearly have \( z_n \to z \) as \( n \to \infty \), so the argument in the proof of Proposition 5.14 shows that \( \{\gamma(n)\} \) converges to \( z \) considered as a point of \( \partial_\infty X \). It follows that \( \gamma|_{[0, \infty)} \) has \( z \) as its endpoint at infinity. Since \( z \in \bar{Z} \) was arbitrary, condition (2) follows. \( \square \)

We next need to show that all densities \( \rho_\varepsilon \) for \( 0 < \varepsilon \leq -\log a \) are admissible for \( X \) with admissibility constant \( M = M(a, \tau, \varepsilon) \) depending only on \( a, \tau, \varepsilon \). We first obtain a loose description of geodesics between vertices in \( X \) using Proposition 3.7. A finite sequence of vertices \( \{v_k\}_{k=0}^n \) in \( X \) is vertical if \( v_k \sim v_{k+1} \) for each \( 0 \leq k \leq n - 1 \) and either \( h(v_{k+1}) = h(v_k) + 1 \) for each \( 0 \leq k \leq n - 1 \) or \( h(v_{k+1}) = h(v_k) - 1 \) for each \( 0 \leq k \leq n - 1 \).
Proposition 6.2. Let $v, w \in V$ be vertices and let $\gamma$ be a geodesic from $v$ to $w$. Let $\{v_i\}_{i=0}^l$ be the sequence of vertices encountered on $\gamma$ going from $v$ to $w$, $l = |vw|$. Then there is an index $k$ such that there are vertical sequences of vertices $\{x_i\}_{i=0}^k$ and $\{y_i\}_{i=0}^{l-k}$, with $x_0 = v$ and $y_0 = w$, for which we have

\begin{equation}
|x_i v_i| \leq c(a, \tau),
\end{equation}

for $0 \leq i \leq k$ and

\begin{equation}
|y_{m-i}| \leq c(a, \tau),
\end{equation}

for $0 \leq i \leq l - k$. Furthermore we have

\begin{equation}
h(x_k) \approx_{c(a, \tau)} h(y_{l-k}) \approx_{c(a, \tau)} (v|w)_h.
\end{equation}

Proof. Let $v, w \in V$ be vertices and let $\gamma$ be a geodesic from $v$ to $w$. By Lemma 5.11 we can find vertical geodesics $\alpha$ and $\beta$ starting from $\omega$ and ending at $v$ and $w$ respectively. Taken together with $\gamma$, these form a geodesic triangle $\Delta$ to which we can apply Proposition 3.7. Let $T : \Delta \to \Upsilon$ be a tripod map associated to $\Delta$ as in Remark 3.8 which is $c(a, \tau)$-roughly isometric since $\delta = \delta(a, \tau)$. Let $u$ be a vertex on $\gamma$ such that $T(u)$ minimizes distance to the core $o$ of $\Upsilon$ among all vertices on $\gamma$; we note that $|T(u) o| \leq c(a, \tau)$ since there is a point on $\gamma$ that is mapped to $o$ within distance $\frac{1}{2}$ of some vertex of $\gamma$. We let $k$ be the index such that $u = v_k$.

We divide $\gamma$ into geodesics $\gamma_1$ from $v$ to $u$ and $\gamma_2$ from $w$ to $u$ (considering $\gamma_2$ with orientation reversed from $\gamma$). We let $\sigma_1$ and $\sigma_2$ be the segments of $\alpha$ and $\beta$ starting at $v$ and $w$ that are of the same length $l_1$ and $l_2$ as $\gamma_1$ and $\gamma_2$ respectively. Since the tripod map $T$ is $c(a, \tau)$-roughly isometric, we conclude in particular that

\begin{equation}
|\sigma_i(n)\gamma_i(n)| \leq c(a, \tau),
\end{equation}

for all integers $n$ satisfying $0 \leq n \leq l_i$, when we consider these geodesics parametrized as starting from $0$ at $v$ and $w$ respectively. This gives the first conclusion of the proposition. The second conclusion follows from Lemma 5.9 and Lemma 5.13.

We next estimate the distance $d_x$ between points at sufficiently large scales on $X$.

Lemma 6.3. Let $x, y \in X$ with $|xy| \geq 2$. For $0 < \varepsilon \leq -\log a$ we have

\begin{equation}
d_x(x, y) \approx_{C(a, \tau, \varepsilon)} e^{-\varepsilon |xy|}.
\end{equation}

Proof. We set $\beta = -\frac{\varepsilon}{\log a}$, noting that $0 < \beta \leq 1$ by hypothesis. Observe that for an edge $g$ of $X$, considered as a path between its endpoints $v$ and $w$ and assuming the orientation in which $h(v) \leq h(w)$, when $h(v) = h(w) = k$ we have

\begin{equation}
\ell_x(g) = e^{-ck} a^{\beta k}.
\end{equation}

On the other hand, since $B(v) \cap B(w) \neq \emptyset$ we have $d(v, w) < 2\tau a^k$. It follows that

\begin{equation}
\ell_x(g) > C(a, \tau, \varepsilon)^{-1} d(v, w)^\beta
\end{equation}

Similarly, when $h(v) = k$ and $h(w) = k + 1$ we have

\begin{equation}
\ell_x(g) = e^{-1} (e^{-ck} e^{-\varepsilon(k+1)}) = e^{-1} (1 - e^{-\varepsilon}) a^{\beta k},
\end{equation}

while $B(v) \cap B(w) \neq \emptyset$ implies again that $d(v, w) < 2\tau a^k$. Thus in this case we also have

\begin{equation}
\ell_x(g) > C(a, \tau, \varepsilon)^{-1} d(v, w)^\beta.
\end{equation}

Now let $\gamma$ be a rectifiable curve joining $x$ to $y$. Let $v$ be the first vertex on $\gamma$ met traveling from $x$ to $y$ and let $w$ be the first vertex on $\gamma$ met traveling from $y$ to $x$. The assumption
$|xy| \geq 2$ implies that we must have $v \neq w$. Let $\gamma$ be the subcurve of $\gamma$ from $v$ to $w$ starting from this first occurrence of $v$ and ending at this last occurrence of $w$. Let $\{v_i\}_{i=0}^l$ be the sequence of vertices encountered along the path $\sigma$. Then from our calculations above we have

$$\ell_{\varepsilon}(\gamma) \geq C(a, \tau, \varepsilon)^{-1} \sum_{i=0}^{l-1} d(v_i, v_{i+1})^\beta.$$

Since $0 < \beta \leq 1$, we can apply Hölder’s inequality to conclude that

$$\ell_{\varepsilon}(\sigma) \geq C(a, \tau, \varepsilon)^{-1} \left( \sum_{i=0}^{l-1} d(v_i, v_{i+1}) \right)^\beta \geq C(a, \tau, \varepsilon)^{-1} d(v, w)^\beta. \tag{6.6}$$

On the other hand, since $v \neq w$ the curve $\sigma$ must contain at least one full edge of $X$ with one vertex being $v$ and at least one full edge with one vertex being $w$ (these may be the same edge). Then it follows from (6.4) and (6.5) applied to those edges that

$$\ell_{\varepsilon}(\sigma) \geq C(a, \tau, \varepsilon)^{-1} a^{\beta \min\{h(v), h(w)\}}. \tag{6.7}$$

Combining (6.6) and (6.7), using Hölder’s inequality again, and using Lemma 5.7, we conclude that

$$\ell_{\varepsilon}(\sigma) \geq C(a, \tau, \varepsilon)^{-1} a^{\beta \min\{h(v), h(w)\}} = C(a, \tau, \varepsilon)^{-1} e^{-\varepsilon(v|w)_h}.$$ 

Since $\sigma$ is a subcurve of $\gamma$ it follows that this inequality holds for $\gamma$ as well. Minimizing over all possible paths $\gamma$ from $v$ to $w$ then gives

$$d_{\varepsilon}(x, y) \geq C(a, \tau, \varepsilon)^{-1} e^{-\varepsilon(v|w)_h} \geq C(a, \tau, \varepsilon)^{-1} e^{-\varepsilon(x|y)_h},$$

with the second inequality following from the fact that $h$ is 1-Lipschitz and $|xy| \leq 1$, $|yw| \leq 1$.

To get a bound on $d_{\varepsilon}(x, y)$ from above, let $v$ and $w$ be nearest vertices to $x$ and $y$ respectively as in the previous argument. Let $u \in V$ be a branch point for the set $\{v, w\}$ as in Lemma 5.7. Let $\gamma$ be the path from $v$ to $w$ consisting of a vertical geodesic $\gamma_1$ from $v$ to $u$ followed by a vertical geodesic $\gamma_2$ from $u$ back down to $w$. Applying the equality (6.5) to each of the edges in $\gamma_1$, we see that we have a telescoping sum that implies that

$$\ell_{\varepsilon}(\gamma_1) = \varepsilon^{-1} \left( e^{-\varepsilon h(u)} - e^{-\varepsilon h(v)} \right) \leq \varepsilon^1 e^{-\varepsilon h(u)}$$

and similarly,

$$\ell_{\varepsilon}(\gamma_2) = \varepsilon^{-1} \left( e^{-\varepsilon h(u)} - e^{-\varepsilon h(w)} \right) \leq \varepsilon^1 e^{-\varepsilon h(u)}.$$

Thus, by Lemma 5.7

$$\ell_{\varepsilon}(\gamma) \leq 2\varepsilon^{-1} e^{-\varepsilon h(u)} \leq C(a, \tau, \varepsilon)e^{-\varepsilon(v|w)_h}.$$

Let $\eta$ be the path from $x$ to $y$ consisting of the geodesic from $x$ to $v$, followed by the geodesic $\gamma$, then followed by the geodesic from $w$ to $y$. The geodesic from $x$ to $v$ lies within a single edge of $X$ that has $V$ as a vertex, and the same is true for the geodesic from $w$ to $y$. The estimates (6.1) and (6.3) then show that

$$\ell_{\varepsilon}(\eta) \leq \ell_{\varepsilon}(\gamma) + C(a, \tau, \varepsilon)e^{-\varepsilon h(v)} + C(a, \tau, \varepsilon)e^{-\varepsilon h(w)} \leq C(a, \tau, \varepsilon)(e^{-\varepsilon(v|w)_h} + e^{-\varepsilon h(v)} + e^{-\varepsilon h(w)}) \leq C(a, \tau, \varepsilon)e^{-\varepsilon(x|y)_h}.$$
Lemma 6.5. If \( \rho \) is a full edge of \( \gamma \) encountered on \( |x| \leq 2 \) implies that \( h(x) \leq 2 (x|y)_h \). Thus it is enough to show that
\[
d_{\varepsilon}(x, y) \approx C(\varepsilon) e^{-\varepsilon h(x)|xy|}.
\]

Proof. We first observe that the hypothesis \(|xy| \leq 2 \) implies that \( h(x) \leq 2 (x|y)_h \). Thus it is enough to show that
\[
d_{\varepsilon}(x, y) \approx C(\varepsilon) e^{-\varepsilon h(x)|xy|}.
\]

For the upper bound, let \( \gamma \) be a geodesic joining \( x \) to \( y \). Since \(|xy| \leq 2 \) the Harnack inequality implies that we have \( \rho_{\varepsilon}(p) \approx C(\varepsilon) e^{-\varepsilon h(x)} \) for \( p \in \gamma \). Integrating this over \( \gamma \) gives
\[
d_{\varepsilon}(x, y) \leq \ell_{\varepsilon}(\gamma) \approx C(\varepsilon) e^{-\varepsilon h(x)|xy|},
\]
as desired.

For the lower bound, let \( \gamma \) be a rectifiable curve joining \( x \) to \( y \). Suppose first that \( \gamma \) contains a full edge \( g \) of \( X \). Then \( \gamma \) must contain a full edge of \( X \) that has a vertex \( v \) on it with \(|xv| \leq 1 \). The calculations \( \text{Eq. \ref{eq:6.4}} \) and \( \text{Eq. \ref{eq:6.5}} \) show that we must have
\[
\ell_{\varepsilon}(\gamma) \geq C(\varepsilon)^{-1} e^{-\varepsilon h(v)} \geq C(\varepsilon)^{-1} e^{-\varepsilon h(x)}.
\]
Using that \( \frac{1}{2}|xy| \leq 1 \), this implies that
\[
\ell_{\varepsilon}(\gamma) \geq C(\varepsilon)^{-1} e^{-\varepsilon h(x)|xy|},
\]
giving the desired lower bound.

Now suppose that \( \gamma \) does not contain a full edge \( g \) of \( X \). Then there must be a vertex \( v \) of \( X \) such that \( \gamma \) is contained in the union of all edges having \( v \) as a vertex. In particular \(|xv| \leq 2 \) for all \( p \in \gamma \). As in the upper bound the Harnack inequality then implies that \( \rho_{\varepsilon}(p) \approx C(\varepsilon) e^{-\varepsilon h(x)} \) for all \( p \in \gamma \). Integrating this over \( \gamma \) and writing \( \ell(\gamma) \) for the length of \( \gamma \) measured in \( X \), we have
\[
\ell_{\varepsilon}(\gamma) \geq C(\varepsilon)^{-1} \geq e^{-\varepsilon h(x)} \ell(\gamma) \geq e^{-\varepsilon h(x)}|xy|.
\]
This again gives the desired lower bound. Minimizing these bounds over all curves \( \gamma \) joining \( x \) to \( y \) gives the result. \( \square \)

We can now prove admissibility of the density \( \rho_{\varepsilon} \) in the parameter range \( 0 < \varepsilon \leq -\log a \).

Lemma 6.5. If \( 0 < \varepsilon \leq -\log a \) then there is a constant \( M = M(a, \tau, \varepsilon) \) such that, for any \( x, y \in X \), if \( \gamma \) is any geodesic in \( X \) from \( x \) to \( y \) then
\[
\ell_{\varepsilon}(\gamma) \leq Md_{\varepsilon}(x, y).
\]

Proof. When \(|xy| \leq 2 \) the claimed inequality follows from the proof of Lemma \( \text{Eq. \ref{eq:6.3}} \) so we can assume that \(|xy| \geq 2 \). Let \( \gamma \) be a given geodesic from \( x \) to \( y \). Let \( v \in V \) be the first vertex encountered on \( \gamma \) traveling from \( x \) to \( y \), and let \( w \in V \) be the first vertex encountered on \( \gamma \) traveling from \( y \) to \( x \), noting that \( v \neq w \) since \(|xy| \geq 2 \). Let \( \sigma \) be the subgeodesic of \( \gamma \) from \( v \) to \( w \).

Let \( \{v_n\}_{n=0}^{\ell} \) be the sequence of vertices encountered on \( \sigma \) traveling from \( v \) to \( w \). Let \( u = v_k \) be the distinguished vertex obtained from Lemma \( \text{Eq. \ref{eq:6.2}} \) and let \( \{x_i\}_{i=0}^{\ell-k} \) and \( \{y_i\}_{i=0}^{\ell-k} \)
be the vertical sequences of vertices obtained from that lemma. We let \( \sigma_1 \) be the segment of \( \sigma \) from \( v \) to \( v_k \) and let \( \sigma_2 \) be the segment of \( \sigma \) from \( v_k \) to \( w \).

Let \( f_i \) be the vertical edge joining \( x_i \) to \( x_{i+1} \) for \( 0 \leq i \leq k-1 \) and let \( g_i \) be the edge joining \( v_i \) to \( v_{i+1} \) for \( 0 \leq i \leq k-1 \). Combining inequality \( 6.1 \) with the Harnack inequality \( 4.2 \) implies that

\[
\ell_\varepsilon(g_i) \approx_{C(a,\tau,\varepsilon)} \ell_\varepsilon(f_i)
\]

We thus conclude, by summing the telescoping series coming from the estimate \( 6.5 \) as in the concluding arguments of Lemma \( 6.3 \) and using \( 6.9 \) as well as \( 6.3 \),

\[
\ell_\varepsilon(\sigma_1) \leq C(a,\tau,\varepsilon)e^{-\varepsilon h(u)} \leq C(a,\tau,\varepsilon)e^{-\varepsilon (v|w)h}.
\]

Repeating this argument for the segment \( \sigma_2 \) gives the same estimate, so we conclude that

\[
\ell_\varepsilon(\sigma) \leq C(a,\tau,\varepsilon)e^{-\varepsilon (v|w)h}.
\]

We apply the calculation of Lemma \( 6.4 \) to the segments of \( \gamma \) from \( x \) to \( v \) and from \( w \) to \( y \). This implies that

\[
\ell_\varepsilon(\gamma) \leq C(a,\tau,\varepsilon) (e^{-\varepsilon (v|w)h} + e^{-\varepsilon h(v)}|xv| + e^{-\varepsilon h(w)}|yw|).
\]

Since \( |xv| \leq 1 \), \( |yw| \leq 1 \), and \( \min\{h(v),h(w)\} \geq (v|w)h \), this implies that

\[
\ell_\varepsilon(\gamma) \leq C(a,\tau,\varepsilon) e^{-\varepsilon (v|w)h} \leq C(a,\tau,\varepsilon) e^{-\varepsilon (x|y)h}.
\]

The bound \( 6.8 \) then follows from Lemma \( 6.3 \). \( \square \)

This completes the proof of Theorem \( 1.7 \) aside from the final assertion in the case \( \varepsilon = -\log a \). We conclude the paper by proving that in the case \( \varepsilon = -\log a \) the identification of \( \partial_2 X \) with \( \tilde{Z} \) given by the combination of Lemma \( 1.6 \) and Proposition \( 5.14 \) is actually biLipschitz, which completes the proof of Theorem \( 1.7 \).

**Proposition 6.6.** The identification \( \partial_2 X \cong \tilde{Z} \) is biLipschitz when \( \varepsilon = -\log a \), with biLipschitz constant \( L = L(a,\tau) \) depending only on \( a \) and \( \tau \).

**Proof.** We consider \( \partial_2 X \) as equipped with the visual metric with parameter \(-\log a\) defined by Proposition \( 5.14 \) which coincides with the extension of the metric \( d \) on \( Z \) to the completion \( \bar{Z} \) under the identification of that proposition. By Theorem \( 1.3 \) applied with this visual metric and \( \varepsilon = -\log a \), the identification \( \iota : \partial_2 X \rightarrow \partial X_\varepsilon \) is biLipschitz. Hence the induced identification \( \tilde{Z} \cong \partial_2 X \) is also biLipschitz. Furthermore all of the parameters involved in the biLipschitz constant can be taken to depend only on \( a \) and \( \tau \) once we set \( \varepsilon = -\log a \), by the results in this section. \( \square \)

**Remark 6.7.** For \( k \in \mathbb{Z} \) there is a canonical correspondence between hyperbolic fillings with fixed parameters \( 0 < a < 1 \) and \( \tau > 1 \) of the metric spaces \((Z,d)\) and \((Z,a^kd)\) given by considering \( a^n \)-separated sets in \((Z,d)\) as \( a^{n+k} \)-separated sets in \((Z,a^kd)\). Thus when \( Z \) is bounded there is no harm in assuming that \( \text{diam} Z < 1 \) by multiplying the metric by \( a^k \) for \( k \) sufficiently large. The hyperbolic filling can then be written as \( X = X_{\geq 0} \cup X_{\leq 0} \), where \( X_{\geq 0} = h^{-1}([0,\infty)) \) is the set of all points of nonnegative height and \( X_{\leq 0} = h^{-1}((\infty,0]) \) is the set of all points of nonpositive height. The condition \( \text{diam} Z < 1 \) implies that the vertex sets \( V_n = \{v_n\} \) for \( n \leq 0 \) consist only of a single point, and in particular \( X_{\leq 0} \) is simply a geodesic ray starting from \( v_0 \).

The graph \( X_{\geq 0} \) is essentially the hyperbolic filling of \( Z \) constructed in \( 3 \), with the exception that they have a stricter condition for the placement of vertical edges. They uniformize this filling using the density \( \rho_\varepsilon,v_0(x) = e^{-\varepsilon |xv_0|} \) for \( 0 < \varepsilon \leq -\log a \), for which it
is not hard to show that $\rho_{\varepsilon, v_0} \asymp c(\varepsilon) \rho_{\varepsilon} |_{X \geq 0}$, where $\rho_{\varepsilon}(x) = e^{-\varepsilon h(x)}$ and $h : X \to \mathbb{R}$ is the height function. Thus their results can be deduced from ours once $\tau$ satisfies (5.1); they impose the constraint $\tau \geq \frac{1+\alpha}{\alpha}$ instead, which is weaker than our constraint when $\alpha$ is close to 0.

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