Model updating in flexible-link multibody systems

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Abstract. The dynamic response of flexible-link multibody systems (FLMSs) can be predicted through nonlinear models based on finite elements, to describe the coupling between rigid-body and elastic behaviour. Their accuracy should be as high as possible to synthesize controllers and observers. Model updating based on experimental measurements is hence necessary.

By taking advantage of the experimental modal analysis, this work proposes a model updating procedure for FLMSs and applies it experimentally to a planar robot. Indeed, several peculiarities of the model of FLMS should be carefully tackled.

On the one hand, nonlinear models of a FLMS should be linearized about static equilibrium configurations. On the other, the experimental mode shapes should be corrected to be consistent with the elastic displacements represented in the model, which are defined with respect to a fictitious moving reference (the equivalent rigid link system). Then, since rotational degrees of freedom are also represented in the model, interpolation of the experimental data should be performed to match the model displacement vector.

Model updating has been finally cast as an optimization problem in the presence of bounds on the feasible values, by also adopting methods to improve the numerical conditioning and to compute meaningful updated inertial and elastic parameters.

1. Introduction

Over the last years, the interest towards lightweight multibody systems, such as mechanisms and robots, has considerably risen, and attention has been often paid to dynamic modelling of such systems. Indeed, the use of reliable dynamic models is becoming an even more important tool for model-based design and control, as well as for numerical simulations or for the synthesis of state observers. Hence accurate analytical models are required to predict the response of the physical system. To accomplish such a task, two issues should be tackled. At first, a correct model formulation should be adopted to represent the coupled relations between the large displacements of the mechanism, which can be thought of as rigid displacements, and the small elastic deformations. Secondly, correct model parameters must be determined, either through direct physical measurement or using experimental parameter identification techniques.

Attention in the literature on flexible link multibody systems (FLMS) is usually focused on the first issue, and several modeling techniques are proposed, thus allowing fast numerical simulations ([1], [2]). Among such techniques, finite elements (FEs) are mainly employed to represent the elastic
behavior, by assuming suitable moving frames of reference, to represent the large displacements of the mechanism.

In contrast, it is often neglected the issue of experimental identification of the model physical parameters, although it is a critical issue. As a matter of fact, FE methods are not able to predict the dynamic responses of systems with adequate accuracy whenever the inertial and elastic properties of the link, assumed in the model, are not correctly tuned. The above process of correcting the FE model parameters to feature a set of experimental measurement, is known in the field of structural dynamics as model updating [3].

Model updating techniques are usually classified into two categories: direct techniques (also denoted non-iterative or one-step) and iterative techniques (or parametric) ([3], [4]). Direct techniques find the solution to model updating problem in just a single step, usually through analytical solutions. However they often produce results without physical meaning, as well as they are usually not robust to measurement noise. In contrast, iterative techniques compute the updated parameters through an objective function that represents the difference between analytical and experimental results. Iterative techniques are usually computationally more expensive than direct techniques and may be affected by the presence of local optimal solutions if the updating problem is formulated as a non-convex optimization problem. Nonetheless, iterative methods are becoming however more popular, since they preserve physical meaning of the updated parameters and their solution can take advantage of the advanced numerical techniques and toolbox for optimization ([5], [6]).

A wide literature has been developed for model updating in structures or simple mechanical systems. In contrast, the issue of model updating in multibody systems is often neglected, and in particular experimental and systematic approaches are rarely addressed. The problem is indeed very challenging. On the one hand, the dynamic model representing FLMSs is nonlinear. On the other hand, fictitious and notional moving references are usually adopted to formulate the model, in order to represent the abovementioned dynamic coupling between the elastic and the “rigid-body” dynamics, while simplifying the mathematical development of the FE model [7].

In the light of this open issue, this work develops and validates a model updating approach for dynamic models representing FLMSs. Experimental application is proposed to a lightweight planar robot, by taking advantage of the experimental modal analysis. The proposed technique starts from the model linearization, in order to apply linear modal analysis, by carefully tackling several peculiarities of the model of FLMSs. Afterwards, a numerical technique based on convex optimization is proposed, to ensure convergence to global optimal solution while accounting for bounds on the feasible values of the updated inertial and elastic parameters.

2. Model of a flexible link mechanism

2.1. The nonlinear model

Let us consider an arbitrary planar multibody system with holonomic and scleronomous constraints, such as lightweight mechanisms or robots, and whose links are flexible. The total motion of each link can be separated into the large rigid-body motion of an equivalent rigid-link system (ERLS) and the small superimposed elastic displacements of the link with respect to the ERLS itself [7]. In practice the ERLS is the moving reference configuration from which elastic displacements are measured (see figure 1), by taking advantage of FE representations. Clearly, this subdivision is notional. However it allows an easy formulation and solution of the nonlinear differential equations governing the system motion. As a matter of the fact, once that the coupling with the elastic behavior is correctly modeled, the ERLS can be studied through the kinematics of rigid body systems. On the other hand, a suitable definition of the ERLS allows placing the moving reference very close to the actual deformed system, and hence ensures correctness of the small displacements assumption.

Let \( \mathbf{q} \) be the vector of the ERLS generalized coordinates, and \( \mathbf{u} \) the vector of the elastic displacements of the nodes of the FE model with respect to the ERLS. With reference to an arbitrary node \( i \), the sum of the entries of \( \mathbf{u} \) related to such a node (denoted \( \mathbf{u}_i \)) with the vector of the positions
of such a node in the ERLS (denoted \( r_i \)) provides the global absolute motion of such a node. By means of this representation, as it is shown in [8], it is possible to get this final expression of the system equations of motion through the following set of nonlinear ordinary differential equations (ODEs):

\[
\begin{align*}
\begin{bmatrix}
\mathcal{M}_c & \mathcal{M}_c \mathbf{S} \\
\mathbf{S}' \mathcal{M}_c & \mathbf{S}' \mathcal{M}_c \mathbf{S}
\end{bmatrix} & \begin{bmatrix}
\dot{\mathbf{u}}(t) \\
\dot{\mathbf{q}}(t)
\end{bmatrix} + \\
\begin{bmatrix}
\kappa_c & \mathcal{K}_c \\
\mathcal{K}_c & \mathcal{K}_c \mathbf{S}
\end{bmatrix} & \begin{bmatrix}
\mathbf{u}(t) \\
\mathbf{q}(t)
\end{bmatrix} \\
\mathbf{S}' & \left(2\mathcal{M}_c + \alpha \mathcal{M}_c \right) \mathbf{S}' \mathcal{M}_c \mathbf{S} \\
\mathbf{S}' & \left(2\mathcal{M}_c + \alpha \mathcal{M}_c \right) \mathbf{S}' \mathcal{M}_c \mathbf{S}
\end{bmatrix} & \begin{bmatrix}
\dot{\mathbf{u}}(t) \\
\dot{\mathbf{q}}(t)
\end{bmatrix}
\end{align*}
\]

(1)

In equation (1), \( \mathbf{g} \) is the gravity acceleration vector, \( \alpha \) and \( \beta \) are the Rayleigh coefficients employed to represent proportional damping, \( \mathbf{v} \) is the vector of the concentrated external nodal forces and torques and \( \mathbf{I} \) is the identity matrix. Matrix \( \mathbf{S} = \mathbf{S}(\mathbf{q}) \) is the ERLS sensitivity coefficient matrix for all the nodes, which relates the velocities of the ERLS generalized coordinates (\( \dot{\mathbf{q}} \)) to the velocities of all the nodes of the ERLS (\( \dot{\mathbf{r}} \)):

\[
\dot{\mathbf{r}} = \mathbf{S}(\mathbf{q}) \dot{\mathbf{q}}
\]

(2)

Matrix \( \mathcal{M}_c = \mathcal{M}_c(\mathbf{q}, \dot{\mathbf{q}}) \) represents the centrifugal and Coriolis effects of all the elements, while \( \mathcal{M} = \mathcal{M}(\mathbf{q}) \) and \( \mathcal{K} = \mathcal{K}(\mathbf{q}) \) are the matrices obtained assembling the consistent mass and stiffness matrices of all the FEs. Assembly is obtained by defining, at first, local reference frames following the motion of the ERLS for each element. Transformation from the local reference frame to the global one is then obtained through the ERLS rigid-body kinematic.

The model can accurately reproduce the system dynamics and accounts for the inertial coupling between rigid-body motion and vibrations. Hence, the dynamic behavior of the ERLS is not assumed independent from vibration and vice-versa, and the ERLS position is set by the elastic behavior of the mechanism.

Figure 1. ERLS definitions.

2.2. The linearized model

Although the dynamics of flexible link mechanisms and manipulators is nonlinear, it is widely recognized that in the case of small deformations the accuracy of linearized models about operating points is usually very satisfactory to make their use successfully in the synthesis of effective and stable control schemes [8]. On the other hand, linearized models, represented through ODEs, allow applying modal analysis. Hence, the model tuning made through the linearized model is an effective mean for tuning the non-linear one. It is worth noticing that several modeling approaches of MBS in the
literature do not feature this possibility, since they are formulated through differential algebraic
equations (DAE), where the second-order ordinary differential equations governing the dynamics of
the system are coupled to a set of algebraic constraint equations. More complicate and computational
challenging methods should be adopted to handle this kind of mathematical formulation [9].

By considering small displacements about a static equilibrium configuration, which is set by the
equilibrium configuration of the ERLS \( q_e \), the following linear model is obtained:

\[
\begin{bmatrix}
M_e & M_S \\
S^T M_e & S^T M_S
\end{bmatrix}
\begin{bmatrix}
\dot{u}(t) \\
\dot{q}(t)
\end{bmatrix}
+ \begin{bmatrix}
\alpha M_e + \beta K_e & 0 \\
\alpha S^T M_e & 0
\end{bmatrix}
\begin{bmatrix}
\dot{u}(t) \\
\dot{q}(t)
\end{bmatrix}
+ \begin{bmatrix}
K_e & 0 \\
0 & \left( \frac{d(S^T M_e)}{dq} \otimes g + \frac{dS^T}{dq} \otimes v_e \right)
\end{bmatrix}
\begin{bmatrix}
u(t) \\
q(t)
\end{bmatrix}
= \begin{bmatrix} 1 \\
S^T 
\end{bmatrix} \{v\}
\]

In equation (3), the matrices \( M_e, S, K_e \) and the derivatives \( \frac{d(S^T M_e)}{dq} \) and \( \frac{dS^T}{dq} \) are computed
about the equilibrium configuration. The term \( \left( \frac{d(S^T M_e)}{dq} \otimes v_e \right) \) represents the inner product of matrix
\( \left[ \frac{\partial S_{i,j}}{\partial q_j} \cdots \frac{\partial S_{i,j}}{\partial q_i} \right] \) with vector \( v_e \), for all the subscripts \( i, j \) \( (\text{obvious extension to} \left( \frac{d(S^T M_e)}{dq} \otimes g \right) \)).

Model updating should be performed by investigating the system in a stable equilibrium
configuration. Therefore, gravity forces should be compensated whenever the mechanism lies in the
vertical plane. One of the possible approach is to balance gravity force through tuned external springs,
which cause asymptotic stability of the equilibrium point (if properly chosen in the design). Clearly,
springs should be then included in the stiffness matrix of the model. In contrast the use of brakes, such
as those of the actuators, modifies the boundary conditions by making the mechanism behave as a
structure, and deletes in the model all the terms representing the coupling between the ERLS and the
elastic displacements. Therefore model updating may lead to less reliable results. On the other hand,
the absence of a stable equilibrium would make the mechanism diverge from the initial configuration
after the excitation, and therefore the system dynamics do not meet the hypothesis of the linearized
model.

3. The model updating technique

3.1. Size compatibility between FE model and measurements

As it often happens in model updating, the measured dofs are usually less than those included in the
FE model of the system, due to the limited number of sensors used in the experimental measurement
or to the presence of some inaccessible locations. Additionally, rotational degrees of freedom (dofs)
cannot be easily measured. Thus, the displacement vector of the FE model of the FLMS is not
compatible with the experimental data. In contrast, model updating requires one-to-one

correspondence between the displacement vector of the FLMS model and the measured eigenvectors.
This correspondence can be achieved by either reducing the FE model, through model reduction
techniques, or by expanding the experimental results.

In this work, the use of coordinate expansion techniques is adopted. In particular, least-square
fitting is employed to estimate the unmeasured dofs, while filtering measurement noise. Fitting
is based on the measurement of a redundant set of measurable translational dofs, and on the least-
square regression of the measured displacements along the links, by taking advantage of the
polynomial interpolation shape functions. Once that the coefficients of these interpolation polynomials
have been computed, the estimation of the rotational dofs for the model nodes of interest is trivial.

This fitting procedure can applied to each single link of the mechanisms, as well as to just some
parts of the links whenever the number of measurements allows fitting in a narrow scale.
Data representation from the physical reference frame to the ERLS reference frame

A second relevant issue that should be accounted for to perform correct updating in FLMS is the transformation of the measured eigenvectors from the reference adopted to measure the mode shapes, which is physical, to the fictitious one adopted in the ERLS-based model. Indeed, as discussed in Section 2.1, the displacement vector of the FLMS model \( \mathbf{u} \), and hence its eigenvectors, includes the elastic displacements with respect to the ERLS, and the displacement of the ERLS itself. In contrast, experimental measurements are defined with respect to the mechanism in its initial static configuration. Transformation should be therefore performed through the ERLS kinematic constraint equations:

\[
\eta_i = \tilde{\eta}_i + \Phi(q_e, \Delta q)
\]  

(4)

\( \eta_i \) is the \( i \)-th eigenvector represented in the FLM model, \( \tilde{\eta}_i \) is the related eigenvector in the physical frame (as measured and then expanded), and \( \Phi(q_e, \Delta q) \) is the transformation representing the kinematic constraint equations. \( \Phi \) is a function of the ERLS equilibrium configuration assumed for linearizing the model, \( q_e \), and of the displacement of the ERLS when the system moves accordingly with such an eigenvector, denoted \( \Delta q \). It is worth noticing that \( \Delta q \) is, in turn, related to \( \tilde{\eta}_i \), on the basis of the definition assumed for placing the ERLS with respect to the actual deformed mechanism.

Representation of the updating parameters

Model updating is performed in this work on just the inertial and elastic parameters and through the undamped model. In contrast, damping is neglected. Identification of the Rayleigh coefficients can be performed separately, through the well established methods. This simplifies the formulation of the eigenvalue problem by casting it as a first order problem and by using real eigenvectors and eigenvalues \( (\omega_i, \eta_i) \), \( i = 1, \ldots, n \), where \( n \leq N \) is the number of measured experimental modes \((N\) is the number of model dofs, including both rigid and elastic coordinates), \( \omega_i \) is the natural frequency of the mode whose shape is \( \eta_i \).

The system linearized model in equation (3) can be written in the usual form of an undamped vibrating system, represented through the mass and the stiffness matrices. The updating of the model matrices is represented through additive correction matrices denoted, respectively, \( \Delta M \in \mathbb{R}^{N \times N} \) and \( \Delta K \in \mathbb{R}^{N \times N} \), as often done in the literature on model updating. Hence, the mass and stiffness matrices of the \( N \)-dimensional updated system are \( M^0 + \Delta M \) and \( K^0 + \Delta K \), where \( M^0 \in \mathbb{R}^{N \times N} \) and \( K^0 \in \mathbb{R}^{N \times N} \) represent the original (nominal) system, synthesized through nominal or theoretical inertial and elastic parameters. The topology of the updating matrices \( \Delta M \) and \( \Delta K \) is defined so as to correctly represent the effect of the updated parameters on the different model dofs. Hence it depends on the topology of \( M^0 \) and \( K^0 \). Additionally, they should represent the model parameters that are affected by uncertainty, and hence should be updated (the so called updating parameters). In accordance with these requirements, the following representations are adopted in this work for \( \Delta M \) and \( \Delta K \):

\[
\Delta M = \sum_{j=1}^{N} \Delta M_j = \sum_{j=1}^{N} \left[ \frac{\partial M^0}{\partial m^0_j} \right] \Delta m_j \quad \Delta K = \sum_{k=1}^{N} \Delta K_k = \sum_{k=1}^{N} \left[ \frac{\partial K^0}{\partial k^0_k} \right] \Delta k_k
\]  

(5)

where \( \Delta M_j \) and \( \Delta K_k \) are the updating sub-matrices related to each updating parameter \( m^0_j \) or \( k^0_k \), to be modified through the additive corrections \( \Delta m_j \) and \( \Delta k_k \) respectively. In practice, all the uncertain parameters of the actual system model are represented as the sum of the value of the nominal one and the modification:

\[
m_j = m^0_j + \Delta m_j \quad k_k = k^0_k + \Delta k_k
\]  

(6)
These matrices define the location and the type of model uncertainties, and therefore are obtained as the first derivative of the nominal matrices with respect to the modifiable model parameters. Finally, \( N_m \) and \( N_k \) denote the number of parameters that can be updated in, respectively, the mass and stiffness matrix.

In order to improve the numerical conditioning of the problem, dimensionless corrections are employed to represent the percentage modifications of the parameters of the nominal model:

\[
\bar{m}_j = \frac{\Delta m_j}{m_j^0} \quad \bar{k}_h = \frac{\Delta k_h}{k_h^0}
\]

For those \( m_j^0 \) or \( k_h^0 \) whose value is zero, an arbitrary, though reasonable, reference number can be assumed. As a typical choice, these reference values can be defined as the upper bounds of their allowed modifications.

All the quantities \( \Delta m_j \) and \( \Delta k_h \) are collected in the \( (N_m + N_k) \)-dimensional unknown vector \( \chi = \{ \bar{m}_j, \bar{k}_h \} \), which gathers all the updating parameters. Hence, the corrective matrices can be finally defined as follows, where the sub-matrices \( A_j \) and \( B_h \) have been introduced for brevity of notation:

\[
\Delta M = \sum_{j=1}^{N_m} m_j^0 \left[ \frac{\partial M^0}{\partial m_j} \right] \bar{m}_j \quad \Delta K = \sum_{h=1}^{N_k} k_h^0 \left[ \frac{\partial K^0}{\partial k_h} \right] \bar{k}_h
\]

The derivatives are constant whenever Young’s modulus, mass density, nodal masses, nodal inertias, sectional area or lumped springs should be updated. In contrast, if some \( \bar{m}_j \) or \( \bar{k}_h \) represent the length of the beams, the derivatives are not constant, and iterations should be done in model updating. However, it is often reasonable assuming the beam length as a known and exact parameter. It should be pointed out that the selection of the updating parameters should be always done carefully. Indeed, assuming too many parameters for the updating often causes numerical problems, and might lead to local minima in the problem solution, unless the model updating problem is formulated as a convex problem.

The correction of the model parameters should be, in practice, bounded. On the one hand, physical meaning should be always ensured. On the other hand, unrealistic values due to the mathematical solution, although physically feasible, should be avoided. Hence, both lower and upper bounds should be defined, by setting a feasible set \( \Gamma \).

### 3.4. Model updating problem formulation

On the basis of the model assumed for representing the updating matrices, the following equation representing the eigenvalue problem must hold for any measured eigenpair:

\[
0 = \omega^2 \left[ M^0 + \Delta M \right] \eta_i - \left[ K^0 + \Delta K \right] \eta_i
\]

By pre-multiplying by \( \eta_i \), in order to obtain a scalar equation for any eigenpair, and by introducing the definitions of the updating matrices given in equations (8), the eigenvalue problem can be written as follows:

\[
0 = \eta_i^T \left[ \omega^2 M^0 - K^0 \right] \eta_i + \omega^2 \sum_{j=1}^{N_m} \eta_i^T A_j \bar{m}_j - \sum_{h=1}^{N_k} \eta_i^T B_h \bar{k}_h
\]

Equation (10) represents a linear problem in the unknown \( \{ \bar{m}_j, \bar{k}_h \} \), which can be represented in the usual compact form:

\[
0 = a_i \chi - b_i
\]

where the known term is defined as \( b_i = -\eta_i^T \left[ \omega^2 M^0 - K^0 \right] \eta_i \), while \( a_i \) denotes the vector collecting all the known coefficients.
Since the parameters collected in $\chi$ are constrained and more conditions like the one in equation (11) should be simultaneously satisfied, the problem should be approximated as a norm minimization problem, where the residual of each eigenvalue problem can be weighed through the positive and scalar weight $w_i$: 

$$\min_{\chi} \left| \begin{array}{c} w_1 a_1 \\ \vdots \\ w_n a_n \end{array} \right| - \left| \begin{array}{c} w_1 b_1 \\ \vdots \\ w_n b_n \end{array} \right|$$

Weighing the equations has several justifications when dealing with FLMS. Besides giving less importance to those eigenpairs whose measurements are less reliable, the weights should take into account the frequency range of interest for the analysis as well as the importance of each mode.

Finally, regularization can be adopted to weigh the components of $\chi$ suitably, i.e. to penalize the modifications of the model parameter selectively [10]. As a matter of fact, some parameters are much more uncertain than others, and therefore it is more desirable that the updating procedure would modify them. Hence, the term $\lambda \Omega \chi$ is added to the norm minimization problem. The scalar positive value $\lambda$ is the regularization parameter, trading-off between the cost of missing the eigenpair specifications and the cost of using large values of the design variables. The positive-definite matrix $\Omega$ is the regularization operator, and defines the relative weight between the updates of the different parameters. The model updating problem is therefore finally represented as the following constrained minimization problem, where it is defined vector $b$, for shortness of notation $b = [w_1 a_1]^\dagger, \ldots, [w_n b_n]^\dagger$ (where the superscript $^\dagger$ denotes the pseudo-inverse matrix):

$$\min_{\chi \in \mathcal{I}} \left| \begin{array}{c} 1 \\ \lambda \Omega \end{array} \right| \chi - \left| \begin{array}{c} b \\ 0 \end{array} \right|$$

The problem solution is straightforward through the reliable and efficient numerical algorithms for quadratic programming optimization problems, which are available in commercial software for numerical computing.

Finally, it is worth noticing that the proposed formulation does not require a specific normalization of the measured modes.

4. Experimental application

4.1. Mechanism description

The method proposed in this paper has been applied to a very challenging test, that is the experimental model updating in a six-bar linkage mechanism, including five flexible links (slender rods with circular cross-sections made of Aluminum) and a rigid frame. The rigid-body mechanism has three dofs, which are actuated through three servomotors. The system has a hybrid kinematic topology: links 1, 2, 3 and 4 are in a closed-chain kinematic configuration, while links 4 and 5 are connected in an open kinematic chain. The FEs adopted are Euler-Bernoulli beam elements, with two nodes and six dofs. Lumped masses and nodal inertias are adopted to represent the joints and the motors. A lumped spring model has been adopted to represent the balancing springs, which are employed in the mechanism to reduce the static torque required by the servomotors. Figure 2 shows a picture of the system. The three ERLS coordinates are the two absolute joint rotations of nodes A and B, and the relative rotation of joint E with respect to link 4. In practice, these variables represent the actuator rotations, and hence have also a physical meaning. The rigid moving reference is then defined by forcing the ERLS to coincide with the actual flexible mechanism at the rotation of these three joints. Hence, three elastic displacements with respect to the ERLS are forced to zero in the FLMS model, in order to place correctly the moving frame defined by the equivalent rigid mechanism.
Model updating has been done by investigating the system in the stable equilibrium configuration represented in figure 2, where gravity forces are compensated by the balancing force exerted by the two external springs, connecting link 2 to the frame.

4.2. Experimental procedure
Experimental modal analysis has been performed through impact tests. A series of frequency response functions (FRF) has been measured at various geometric locations, using an instrumented impact hammer to exert the input force, while the responses have been measured with accelerometers in the orthogonal (Y) and tangential (X) directions (defined in the local reference frames). The set of measurements is represented in figure 3.

The mode shapes have been estimated through the software LMS Test.Lab 11B, which is based on the so-called pole-residual model to extract the mode shapes [11]. Each frequency response function (FRF) has been computed by averaging three measurements. The number of measurement locations has been selected to allow a reliable representation of the mode shapes in the frequency range of interest, and to perform interpolation for expanding the experimental results while smoothing measurement noise and uncertainty. Coherently, the manipulator geometry has been created in the analysis software through the tool Test.Lab Impact Testing 11B Geometry, by defining the “drive points” related to each measurement point (see figure 4).

4.3. Statement of the model updating problem
The updating parameters have been selected as those more affected by uncertainty, such as the mass density of the links (which is assumed equal among all the links), the Young’s modulus (which has been instead treated separately for each link, to compensate for local stiffening due to the kinematic joints), the spring lumped stiffness, the nodal inertias and masses. The feasible values have been constrained by reasonable bounds, which are not reported here for brevity. Length of the links are instead assumed as exactly known. The modifications has been weighed through the regularization operator $\Omega$ by reducing the cost of modifying the nodal inertia, which have been set to zero in the nominal model and thus are more uncertain.

4.4. Method application and results
Evaluation of the results is carried by comparing both the eigenfrequencies and the mode shapes, for both the original nominal model (synthesized through nominal parameters of materials and of the components) and the updated one. As usual in the field of model updating, in order to obtain a concise and clear evaluation of the results, it is computed the percentage frequency error, $(\omega_i - \hat{\omega}_i)/\omega_i$, and
the Modal Assurance Criterion \( MAC = \left( \hat{\eta}^T \hat{\eta} \right) / \left( ||\hat{\eta}|| \cdot ||\hat{\eta}|| \right) \), where \( \hat{\omega} \) and \( \hat{\eta} \) denote, respectively, the eigenfrequency and the eigenvector estimated through the models. The results obtained for both the numerical models are listed in Table 1, with reference to the four lowest frequency modes. They clearly show that model updating has reduced remarkably the model discrepancy with the experimental measurements. The major improvement is obtained in terms of the eigenfrequencies, whose values are significantly missed in the nominal model. Indeed, the percentage frequency error for the updated model is smaller than 1% in the first three modes. As for the mode shapes, even if the mode shape specifications are accurately met also by the original modes, the average MAC increases from 0.89 to 0.91 after model updating. The results obtained are highly satisfactory, in terms of both the frequency and the mode shape. Indeed the test case is very challenging, given the complexity of the system investigated (whose actual dynamic is nonlinear), the simultaneous presence of more modes to be represented, the unavoidable uncertainty in performing accurate measurement in a mechanism (which has “rigid-body” dofs) and the presence of tight constraints bounding the feasible updating parameters.

The capability to represent the mode shapes is clearly confirmed in Figure 4, where the four mode shapes are depicted, by comparing the experimental ones and those of the updated model. It is evident that the model delivers very similar mode shapes, as expected by the MAC analysis.

![Mode shapes of the first four modes: experimental modes (left) and updated model (right).](image)

**Figure 4.** Mode shapes of the first four modes: experimental modes (left) and updated model (right).

| Frequency (Hz) | % Frequency error | MAC          |
|---------------|-------------------|--------------|
|               | Experimental     | Nominal model | Updated model | Nominal model | Updated model | Nominal model | Updated model |
| 13.39 Hz      | 13.72             | 13.29         | 2.46          | 0.74          | 0.822         | 0.865         |
| 43.57 Hz      | 43.88             | 43.30         | 0.71          | 0.62          | 0.976         | 0.989         |
| 64.65 Hz      | 66.96             | 64.78         | 3.58          | 0.20          | 0.960         | 0.968         |
| 112.99 Hz     | 127.25            | 124.24        | 12.62         | 9.95          | 0.820         | 0.829         |
5. Conclusions
This paper proposes a methodology for model updating in multibody systems with flexible links. The topic has some critical issues due to the dynamic behavior of this kind of mechanical systems, and in particular due to the coupling between the small elastic deformations and the large motion, which can be thought of as a rigid-body motion. To account for this prominent feature, a multibody models based on fictitious moving reference, denoted ERLS, has been synthesized. The use of such a reference allows simplifying the model development and solution, while describing correctly the system dynamics, as long as the model updating is properly addressed to tune inertial and elastic parameters.

The methods takes advantage of the linearization of the nonlinear model to update the mass and stiffness matrices (which represent both the rigid and the elastic behaviors, and their mutual coupling) and to the experimental modal analysis in a stable equilibrium configuration. Once that the experimental eigenvectors are represented in the moving reference frame defined by the ERLS, by means of a kinematic transformation and through coordinate expansion, the method is cast as a constrained inverse eigenvalue problem and solved as a quadratic programming optimization problem. The method has been applied to a very challenging test, which is the updating of the model of a six-bar linkage mechanism with five flexible links. The preliminary experimental results obtained demonstrate the effectiveness of the method.

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