Antiferromagnetism and singlet formation in underdoped high-Tc cuprates: Implications for superconducting pairing

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The extended $t-J$ model is theoretically studied, in the context of hole underdoped cuprates. Based on results obtained by recent numerical studies, we identify the mean field state having both the antiferromagnetic and staggered flux resonating valence bond orders. The random-phase approximation is employed to analyze all the possible collective modes in this mean field state. In the static (Bardeen Cooper Schrieffer) limit justified in the weak coupling regime, we obtain the effective superconducting interaction between the doped holes at the small pockets located around $k = (\pm \pi/2, \pm \pi/2)$. In contrast to the spin-bag theory, which takes into account only the antiferromagnetic order, this effective force is pair breaking for the pairing without the nodes in each of the small hole pocket, and is canceled out to be very small for the $d_{x^2-y^2}$ pairing with nodes which is realized in the real cuprates. Therefore we conclude that no superconducting instability can occur when only the magnetic mechanism is considered. The relations of our work with other approaches are also discussed.

I. INTRODUCTION

Since the discovery of high temperature superconducting cuprates\textsuperscript{1} it has been established that the strong Coulomb repulsion between electrons plays an essential role in the physics there\textsuperscript{2} and the magnetic mechanism of the superconductivity has been studied intensively. There are two streams of thoughts; one is the antiferromagnetic (AF) spin fluctuation exchange based on the (nearly) AF ordered state, while the other is the resonating valence bond (RVB) mechanism with the focus being put on the spin singlet formation by the kinetic exchange interaction $J$. The representative of the former is the spin-bag theory\textsuperscript{3} where the doped carriers into the spin density wave (SDW) state form small hole pockets, and exchange the AF spin fluctuation to result in the $d_{xy}$ superconductivity. The $z$-component of the spin fluctuation, i.e., $\chi^{zz}$, gives the dominant contribution. The RVB picture, on the other hand, puts more weight on the spin singlet formation and takes into account the order parameter defined on the bond but usually does not consider the antiferromagnetic long range order (AFLRO).\textsuperscript{4} These two scenarios have been studied rather separately thus far, and the relation between them remains unclear. Hsu was the first to take into account both the AFLRO and the RVB correlation at half-filling in terms of the Gutzwiller approximation.\textsuperscript{5} His picture is that the AFLRO occurs on top of the $d$-wave RVB or equivalently the flux state. It is also supported by the variational Monte Carlo study on Heisenberg model, which shows that the Gutzwiller-projected wavefunction $\Psi_{RVB,SDW}$ starting from the coexisting $d$-wave RVB and SDW mean field state gives an excellent agreement with the exact diagonalization concerning the ground state energy and staggered moment.\textsuperscript{6} The variational wavefunction projecting the simple SDW state, on the other hand, gives higher energy. These results mean that both aspects, i.e., SDW and RVB, coexist in the Mott insulator. At finite hole doping concentration $x$, the AFLRO is rapidly suppressed and the superconductivity emerges. Here an important question still remains, namely short range AF fluctuation dominates or the RVB correlation is more important. Because there is no AFLRO and no distinction between $\chi^{zz}$ and $\chi^{z\pm}$, this question might appear an academic one with the reality being somewhere inbetween. Well-known results from the variational Monte Carlo studies are that the variational wavefunction $\Psi_{RVB,SDW}$ has the lowest energy therefore RVB (superconductivity) and SDW coexist also at small doping level.\textsuperscript{7} From the viewpoint of the particle-hole SU(2) symmetry, this means that the degeneracy, i.e., the SU(2) gauge symmetry, between $d$-wave pairing and $\pi$-flux state is lifted at finite $x$, and the former has lower energy.

However this accepted view has been challenged by recent studies based on the exact diagonalization\textsuperscript{8} and variational Monte Carlo method.\textsuperscript{9} These works found that the energetically most favorable state with small pockets at small hole doping levels is not superconducting. This nonsuperconducting state is consistent with the doped SDW+$\pi$-flux state with the hole pockets near $k = (\pi/2, \pm \pi/2)$. This means that the mean field picture is not so reliable at small $x$ because the phase fluctuation of the superconductivity is huge at small $x$ where the charge density, which is the canonical conjugate operator to the phase, is suppressed. Once the superconductivity is destroyed by this quantum fluctuation, the only order surviving is the AFLRO, and the system remains nonsuperconducting. Considering that the AFLRO state at
x = 0 is well described by the SDW+(d-wave RVB or \(\pi\)-flux) state, the relevant mean field state at small but finite x is SDW+\(\pi\)-flux state with small hole pockets.

From the experimental side, recent ARPES datas on Na-CCOC found the small "Fermi arc" near \(\bm{k} = (\pi/2, \pm \pi/2)\), and the \(\bm{k}\)-dependence of the "pseudo-gap" is quite different from that of \((\cos k_x - \cos k_y)\) expected for the \(d_{x^2-y^2}\) pairing.\(^{10,11}\) This result strongly suggests that the pseudo-gap is distinct from the superconducting gap, and the superconductivity comes from some other interaction(s) different from \(J\). Although only the "arc" is observed experimentally, the Fermi surface can not terminate at some \(\bm{k}\)-points inside of the Brillouin zone (BZ). Considering that the Fermi surface disappears with large pseudogap at the anti-nodal direction, i.e., near \(\bm{k} = (\pi, 0)\) and \((0, \pi)\), it is natural to assume that the small hole pocket is formed, along half of which the intensity is small and/or too broad to be observed experimentally.\(^{14}\) Then the question of the pairing symmetry arises because \(d_{x^2-y^2}\) requires the nodes at the small hole pockets, which usually reduces the condensation energy and is energetically unfavorable. The natural symmetry appears to be \(d_{xy}\) without the nodes at the hole pockets, as has been claimed by the original spin-liquid (RVB)\(^{3,4}\). These two aspects might have crucial influence on the superconductivity in the underdoped region.

We have employed the 1/\(N\)-expansion, or random-phase approximation (RPA), to derive the effective interaction between the quasi-particles along the small hole pockets and the pairing force derived from it.

The plan of this paper follows. In Section II, we discuss the model, formulation, and its mean field treatment. The Gaussian (second order) fluctuation around the mean field saddle point is treated and the effective interactions between the quasi-particles are studied in Section III. Section IV is devoted to discussion and conclusions.

II. MODEL AND MEAN FIELD THEORY

A. Formulation of the microscopic model

The microscopic model of cuprates we consider for the study is the well-known \(t-J\) model.\(^{2,11}\) The second and third nearest neighbor hopping terms are taken into account as they are necessary to describe the ARPES experiments measurements.\(^{14}\) We use the slave bosons representation\(^{15,17}\) of the electronic operators

\[
c_i^{\uparrow} = f_i^{\uparrow} b_i, \tag{1}
\]

with the constraint

\[
b_i^{\uparrow} b_i + \sum_{\sigma=\uparrow,\downarrow} f_i^{\uparrow} f_i^{\downarrow} = 1. \tag{2}
\]

In terms of this formalism, the double occupancy of each site is excluded.\(^{18,19,20}\) The operators \(f_i^{\uparrow}\), \(f_i^{\downarrow}\) are the fermionic ones while \(b_i^{\uparrow}\), \(b_i^{\downarrow}\) are bosonic ones (slave bosons)\(^{19,22}\). In all what follows we will exclusively work at zero temperature, and all the slave bosons will be assumed to be condensed. (In actual calculation, we take an extremely low temperature by technical reason. The results, however, are saturated and can be regarded as those at zero temperature.)

We consider the following Hamiltonian on a square lattice with the lattice constant put to be unity and containing \(N_s\) sites

\[
\mathcal{H} = - \sum_{\langle i,j \rangle} t_{ij} n_i + \sum_{\langle i,j \rangle} t_{ij}^\prime n_i n_j + \sum_{\langle i,j \rangle} t_{ij}^{\prime\prime} n_i n_j + J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \tag{3}
\]

where \(\langle \rangle, \langle \prime \rangle, \langle \prime\prime \rangle\) denote nearest neighbor, second nearest neighbor and third nearest neighbor pair, respectively. We assume that the hopping contribution is uniform: \(t_{ij} = t\), \(t_{ij}^\prime = t^\prime\), \(t_{ij}^{\prime\prime} = t''\). For \(t^\prime < 0\), \(t'' > 0\) the previous Hamiltonian deals with hole doped (p-type) cuprates, while the case \(t^\prime > 0\), \(t'' < 0\) concerns electron doped (n-type) cuprates. The unity of energy is: \(t \equiv 0.4\) eV = 1, and we will only study the hole doped case. Following Ref.\(^{15}\) we assume \(t = -0.3\), \(t^\prime = 0.2\) for the extended hopping terms, and \(J = 0.3\) for the Heisenberg interaction. We have checked that slight variations in the numerical values of these parameters do not induce any significant modification of our results, as discussed in Section III C.

We construct the mean field theory to study this Hamiltonian. It has been recognized that the validity of the mean field theory is much less trivial in the theory with constraint. The simple comparison of energy does not offer the criterion because the mean field theory violates the constraint, and hence can lower the energy in the physically forbidden Hilbert space. Therefore the choice of the mean field theory requires a physical intuition, and we employ the nonsuperconducting saddle point according to the reasons explained in the Introduction. In order to study simultaneously the influences of antiferromagnetism and flux in the resonating valence bond state, we introduce the two mean field parameters. The first one is a staggered magnetization:\(^{15}\)

\[
m^{\uparrow}_{i\sigma} = \langle S^{\uparrow}_{i\sigma} \rangle, \quad S^{\uparrow}_{i\sigma} = \frac{1}{2} \sum_{\sigma',\sigma''} f_i^{\sigma'} \sigma_{\sigma',\sigma''} f_i^{\sigma''}, \tag{4}
\]
The second one is the flux phase parameter magnetization on the square lattice. Fig. 1 shows the pattern of the flux and the staggered

We rewrite the model Hamiltonian (3) as

\[ H = \sum_{i,j} \left( \sum_{\sigma=+1,\uparrow,\downarrow} f_{i,\sigma}^\dagger f_{j,\sigma} \right) + \lambda \sum_{\sigma} \chi_{\sigma} \cdot \exp \left[ \pm i \phi (1-1)^i \right], \]

\[ \chi_{i,i+x} = \chi \cdot \exp \left[ \pm i \phi (1-1)^i \right], \]

\[ \chi_{i,i+y} = \chi \cdot \exp \left[ \pm i \phi (1-1)^i \right]. \] (5)

Fig. 1 shows the pattern of the flux and the staggered magnetization on the square lattice.

We rewrite the model Hamiltonian (4) as

\[ H = H^t + H^m + H^x, \] (6)

with

\[ H^t = - \sum_{i,j} t - \sum_{(i,j)'} t' - \sum_{(i,j)''} t'' \]

\[ \times \sum_{\sigma} \left[ b_i b_j^\dagger f_{i,\sigma} f_{j,\sigma} + b_j b_i^\dagger f_{j,\sigma} f_{i,\sigma} \right]. \]

The mean field Hamiltonian for each parameter is given by

\[ H_{MF}^m = \alpha J \sum_{(i,j), s_m=x,y,z} \left[ m_j^m S_i^m + m_i^m S_j^m \right], \]

\[ H_{MF}^x = - \frac{1}{2} \sum_{(i,j)} \chi_{ij} f_{i,\sigma}^\dagger f_{j,\sigma} + \chi_{i,j} f_{i,\sigma}^\dagger f_{j,\sigma}. \] (7)

The mean field Hamiltonian for each parameter is given by

\[ H_{MF}^m = \alpha J \sum_{(i,j), s_m=x,y,z} \left[ m_j^m S_i^m + m_i^m S_j^m \right], \]

\[ H_{MF}^x = - \frac{1}{2} \sum_{(i,j)} \chi_{ij} f_{i,\sigma}^\dagger f_{j,\sigma} + \chi_{i,j} f_{i,\sigma}^\dagger f_{j,\sigma}. \] (7)

In the previous expressions a parameter \( \alpha \) has been introduced to divide the Heisenberg exchange interaction into AF part and flux part. Hence \( \alpha \) should be regarded as a variational parameter, but we will keep it constant \( \alpha = 0.301127 \), as explained in Section II C. The partition function is written

\[ Z = \int D\Psi D\chi Dm Db \exp \left( - \int d\tau \mathcal{L}(\tau) \right), \] (8)

with \( \beta = 1/k_B T \) the thermal factor and the Lagrangian

\[ \mathcal{L}(\tau) = - \left( \sum_{i,j} t + \sum_{(i,j)'} t' + \sum_{(i,j)''} t'' \right) \sum_{\sigma} \left[ b_i(\tau) b_j^\dagger(\tau) \Psi_{i,\sigma}(\tau) \Psi_{j,\sigma}(\tau) + b_j(\tau) b_i^\dagger(\tau) \Psi_{j,\sigma}(\tau) \Psi_{i,\sigma}(\tau) \right] \]

\[ + \sum_{i,\sigma} \Psi_{i,\sigma}(\tau) \left[ \partial_\tau - i \lambda_i(\tau) - \mu \right] \Psi_{i,\sigma}(\tau) + \sum_{i} b_i(\tau) \left[ \partial_\tau - i \lambda_i(\tau) \right] b_i(\tau) \]

\[ + \alpha J \sum_{i,j,\sigma,\sigma'} \sum_{x=y,z} \left[ m_j^x(\tau) \Psi_{i,\sigma}(\tau) \tilde{\sigma}_{\sigma',\sigma}(\tau) \Psi_{j,\sigma'}(\tau) + m_i^x(\tau) \Psi_{j,\sigma}(\tau) \tilde{\sigma}_{\sigma',\sigma}(\tau) \Psi_{i,\sigma'}(\tau) \right] \]

\[ - \frac{1}{2} \sum_{i,j,\sigma} \chi_{ij}(\tau) \Psi_{j,\sigma}(\tau) \Psi_{i,\sigma}(\tau) + \chi_{ij}^*(\tau) \Psi_{i,\sigma}(\tau) \Psi_{j,\sigma}(\tau) \]

\[ - \alpha J \sum_{i,j} \left[ m_j^x(\tau) m_i^y(\tau) + m_i^x(\tau) m_j^y(\tau) + m_i^x(\tau) m_j^x(\tau) \right] + \frac{1}{2} \sum_{i,j} \chi_{ij}(\tau) \chi_{ij}^*(\tau). \] (9)

In Eq. (4) \( \Psi_{i,\sigma}, \Psi_{i,\sigma} \) are the Grassmann variables associated with the \( f_{i,\sigma}, f_{i,\sigma} \) operators, respectively. The Lagrange multipliers \( (i\lambda_i) \) assure that the constraint of no double occupancy (2) is satisfied. The chemical potential \( \mu \) associated with the fermions controls the electron density \( n_f \), and hence the hole doping level \( x \).
B. Saddle point hypothesis and derivation of the mean field Hamiltonian

To identify the saddle point solution we consider the following assumptions. As we work at zero temperature, all the holons are supposed to be condensed, i.e., \((b_i)_0 = b_0\) with \((b_0)^2 = x\) where \(x\) is the hole doping concentration. Therefore the \(f\) operators can be simply viewed as the renormalized electron operators. The Lagrange multipliers are considered independent of the sites, which signifies that the constraint is imposed on average at the mean field level

\[
(i\lambda_0)_0 = \lambda_0. \tag{10}
\]

For the antiferromagnetic part we impose an AF order polarized in \(z\)-direction

\[
(m_z^x)_0 = 0, \quad (m_z^y)_0 = 0, \quad (m_z^t)_0 = (-1)^i m. \tag{11}
\]

Concerning the flux contribution we impose at half-filling (\(x = 0\)) a \(\pi\)-flux scheme \((\phi = \pi)\), which is energetically the most favorable case for a square lattice.

\[
(\chi(i,i+x))_0 = (\chi(i,y))_0 = \chi \cdot \exp \left[ +i\pi \frac{4}{\pi} (-1)^i \right], \quad (\chi(i,i+y))_0 = \chi \cdot \exp \left[ -i\pi \frac{4}{\pi} (-1)^i \right]. \tag{12}
\]

To write the mean field Hamiltonian \(\mathcal{H}_{MF}\), we note that the wavevector of the AF and \(\pi\)-flux is \(Q = (\pi, \pi)\). Therefore the summation over the momentum is done in the reduced magnetic first BZ. The borders of the magnetic (or reduced) BZ are given by the Fermi surface of the Hubbard model at half filling. The reduced BZ is characterized by the nesting property under the vector \(Q = (\pi, \pi)\). \(\mathcal{H}_{MF}\) can be expressed in an appropriate spinor basis of the momentum space

\[
\begin{bmatrix}
  f_{k,\sigma} \\
  f_{k+Q,\sigma}
\end{bmatrix}.
\tag{13}
\]

We obtain the mean field \(t - J\) Hamiltonian written in a matrix form

\[
\begin{align*}
\mathcal{H}_{MF} &= \sum_k \sum_{\sigma} \left[ f_{k,\sigma}^\dagger f_{k+Q,\sigma} \left( -\left( \Delta_{k,\sigma}^m \right)^* - \left( \Delta_{k+Q,\sigma}^\chi \right) \right) f_{k,\sigma}^\dagger f_{k+Q,\sigma} \right].
\end{align*}
\tag{14}
\]

The summation \(\sum_k \) in \(k\)-space is over the reduced magnetic BZ, and we have defined the following energies

\[
\begin{align*}
\epsilon_k &= -2t \left[ \cos(k_x) + \cos(k_y) \right] - 4t' \left[ \cos(k_x) \cos(k_y) \right], \\
\xi_k &= \epsilon_k - \lambda_0 + \mu, \\
\xi_{k+Q} &= \epsilon_k + \mu \cos(\phi/4) \cos(k_x) + \epsilon_k + \mu \cos(\phi/4) \cos(k_y), \\
\end{align*}
\tag{15}
\]

The order parameters associated with the staggered magnetization and \(\phi\)-flux are respectively

\[
\begin{align*}
\Delta_{k,\sigma}^m &= 2\alpha J m \sigma, \\
\Delta_{k,\sigma}^\chi &= i(1 - \alpha) J \chi \sin(\phi/4) \left[ \cos(k_x) - \cos(k_y) \right], \\
\end{align*}
\tag{17}
\]

for \(\sigma = \pm 1\). \(\mathcal{H}_{MF}\) can be diagonalized as

\[
\begin{align*}
\mathcal{H}_{MF} &= \sum_k \sum_{\sigma} \left[ E_{k,\sigma}^{up} \gamma_{1k,\sigma}^\dagger \gamma_{1k,\sigma} + E_{k,\sigma}^{low} \gamma_{2k,\sigma}^\dagger \gamma_{2k,\sigma} \right], \\
\end{align*}
\tag{18}
\]

by a Bogoliubov-Valatin unitary transformation

\[
\begin{bmatrix}
  f_{k,\sigma} \\
  f_{k+Q,\sigma}
\end{bmatrix} = \begin{bmatrix}
  u_{k,\sigma} & v_k \\
  -v_{k,\sigma} & u_{k,\sigma}
\end{bmatrix} \begin{bmatrix}
  \gamma_{1k,\sigma}^\dagger \\
  \gamma_{2k,\sigma}^\dagger
\end{bmatrix},
\tag{18}
\]

In the previous expression \(\gamma_{1k,\sigma}^\dagger, \gamma_{1k,\sigma}, \gamma_{2k,\sigma}^\dagger, \gamma_{2k,\sigma}\) are the creation and annihilation operators of the upper (lower) Hubbard band, respectively, with the corresponding energy eigenvalues (see Fig. 2)

\[
\begin{align*}
E_{k,\sigma}^{up} &= -1 + \frac{\xi_k - \xi_{k+Q}}{2}, \\
E_{k,\sigma}^{low} &= -1 + \frac{\xi_k - \xi_{k+Q}}{2}. \\
\end{align*}
\tag{19}
\]

The elements of the unitary matrix are given by

\[
\begin{align*}
u_{k,\sigma} &= \cos(\theta_k), \quad e^{i\phi_{k,\sigma}}, \quad u_k = \sin(\theta_k), \\
\end{align*}
\tag{20}
\]

with the trigonometric factors

\[
\begin{align*}
\cos(\theta_k) &= \frac{1}{2} \left( 1 + \frac{\xi_k - \xi_{k+Q}}{\sqrt{(\xi_k - \xi_{k+Q})^2 + 4(\Delta_{k,\sigma}^m \gamma_{1k,\sigma}^\dagger \gamma_{2k,\sigma}^\dagger)^2}} \right), \\
\sin(\theta_k) &= \frac{1}{2} \left( 1 - \frac{\xi_k - \xi_{k+Q}}{\sqrt{(\xi_k - \xi_{k+Q})^2 + 4(\Delta_{k,\sigma}^m \gamma_{1k,\sigma}^\dagger \gamma_{2k,\sigma}^\dagger)^2}} \right),
\end{align*}
\tag{21}
\]

\[
\begin{align*}
&\chi(i,i+x)_0 = \chi(i,y)_0 = \chi \cdot \exp \left[ +i\pi \frac{4}{\pi} (-1)^i \right], \\
&\chi(i,i+y)_0 = \chi \cdot \exp \left[ -i\pi \frac{4}{\pi} (-1)^i \right]. \tag{12}
\end{align*}
\]
free Green's functions' matrix \( \omega \) with \( i \) contains the complex phase factor \( e^{i\phi_{k,\sigma}} \), which becomes important when we solve the BCS equation in Section III. In the field-theory terminology, this offers an example of "parity anomaly" in \((2+1)D\).

**FIG. 2:** (Color online) Upper and lower Hubbard bands associated with the eigenvalues \( E_{k}^{up} \) and \( E_{k}^{low} \) of \( \mathcal{H}_{MF} \).

\[
\cos(\phi_{k,\sigma}) = \frac{\Delta_{k,\sigma}^{m}}{\sqrt{|\Delta_{k,\sigma}^{n}|^2 + |\Delta_{k,\sigma}^{\chi}|^2}} ,
\]

\[
\sin(\phi_{k,\sigma}) = \frac{-i\Delta_{k,\sigma}^{\chi}}{\sqrt{|\Delta_{k,\sigma}^{n}|^2 + |\Delta_{k,\sigma}^{\chi}|^2}} .
\]

It is noted here that the unitary transformation contains the complex phase factor \( e^{i\phi_{k,\sigma}} \), which becomes important when we solve the BCS equation in Section III. In the field-theory terminology, this offers an example of "parity anomaly" in \((2+1)D\).

**C. Saddle point solution and mean field equations**

In the path integral language, the saddle point action is

\[
S_{0} = -\sum_{k} \sum_{\sigma} \sum_{\omega_{n}} \left[ \Psi_{k,\sigma}(\omega_{n}) \bar{\Psi}_{k+Q,\sigma}(\omega_{n}) \right] \times \tilde{G}_{0}^{-1}(k, \sigma, \omega_{n}) \times \left[ \Psi_{k,\sigma}(\omega_{n}) \bar{\Psi}_{k+Q,\sigma}(\omega_{n}) \right] ,
\]

with \( \omega_{n} \) the fermionic Matsubara frequencies and \( \tilde{G}_{0} \) the free Green's functions' matrix

\[
\tilde{G}_{0}(k, \sigma, \omega_{n}) = \frac{1}{\omega_{n}^2 + i\omega_{n}(\xi_{k} + \xi_{k} + Q) - \xi_{k} \xi_{k} + Q + |\Delta_{k,\sigma}^{m}|^2 + |\Delta_{k,\sigma}^{\chi}|^2} \times \left[ -i(\omega_{n} + \xi_{k} + Q) \Delta_{k,\sigma}^{m} + \Delta_{k,\sigma}^{\chi} \right] \Delta_{k,\sigma}^{m} - \Delta_{k,\sigma}^{\chi} \right] .
\]

Now we expand the action up to the second order with respect to the deviation from the saddle point solution. At the first order we can derive the self-consistent mean field equations. The details are given in Appendix A and we give here the final results

\[
1 - \frac{4}{N_{s}} \sum_{k} \sum_{\sigma} \left\{ \frac{1}{E_{k}^{up} - E_{k}^{low}} \right\} = 0 ,
\]

\[
\frac{1}{N_{s}} \sum_{k} \sum_{\sigma} \left\{ \frac{[\cos(k_{x}) + \cos(k_{y})] \{\xi_{k+Q} - \xi_{k}\cos(\phi/4)}{E_{k}^{up} - E_{k}^{low}} 
- \frac{2i[\cos(k_{x}) - \cos(k_{y})] \Delta_{k,\sigma}^{\chi}\sin(\phi/4)}{E_{k}^{up} - E_{k}^{low}} \right\} = \chi ,
\]

\[
\frac{1}{N_{s}} \sum_{k} \sum_{\sigma} \left\{ \frac{[\cos(k_{x}) + \cos(k_{y})] \{\xi_{k+Q} - \xi_{k}\}}{E_{k}^{up} - E_{k}^{low}} \right\} = \chi \cos(\phi/4) ,
\]

\[
\frac{4}{N_{s}} \sum_{k} \sum_{\sigma} \left\{ \frac{\bar{i}_{k} \{\xi_{k+Q} - \xi_{k}\}}{E_{k}^{up} - E_{k}^{low}} \right\} + \frac{\bar{i}_{k} \{\xi_{k+Q} + \xi_{k} - 2E_{k}^{low}\}}{E_{k}^{up} - E_{k}^{low}} \right\} = -\lambda_{0} ,
\]

where the quantities \( \xi_{k} \), \( \xi_{k+Q} \), \( \Delta_{k,\sigma}^{\chi} \), \( E_{k}^{up} \), \( E_{k}^{low} \), \( \bar{i}_{k} \), \( \bar{i}_{k}' \) and \( \bar{i}_{k}'' \) are given in Eqs. \((15)\), \((16)\), \((17)\), \((18)\), \((19)\), \((20)\), \((A1)\), \((A2)\), \((A3)\) and \((A4)\), respectively.

We adjust the electron number \( n_{f} \) by the chemical potential \( \mu \). At zero temperature the upper Hubbard band is empty, while the lower band is partially filled as

\[
\frac{1}{N_{s}} \sum_{k} \exp[\beta E_{k}^{low}] + 1 = 1 - x ,
\]
where \((\sum_k)\) is extended over the first BZ.

We have solved numerically these equations by discretizing the reduced BZ in 2 millions of points, providing us a precision on the obtained values better than \(10^{-5}\). In particular at half-filling \((x = 0)\) we have found

\[
\alpha = 0.301127,
\]

by imposing the relation: \(m = 0.5\chi\) previously obtained by Hsu. Away from half-filling this relation between \(m\) and \(\chi\) will be changed but the value of \(\alpha\) \((29)\) is assumed to be the same on all the range of doping. This assumption means that the weight assigned to each of the decoupling terms of the Heisenberg exchange interaction is kept constant as a function of the doping. This assumption does not change the essential features of our results presented below.

We present the numerically obtained values of the mean field parameters as a function of the doping \(x\) in Figs. 3 and 4. We see that the Néel state disappears at a very small value of the doping, typically 1.5 \%, which is in agreement with the experimental phase diagram of hole doped cuprates. This is in sharp contrast to the previous studies of the \(t - J\) model, which found that the AF state could remain until doping values of around 15 \%. The flux \(\phi\) decreases relatively linearly from \(\pi\) at half-filling, and reaches \(\phi \approx 2\) for a doping of 10 \%; this result is similar to the data obtained by Hsu et al.\(^{32}\).

We also show the shape of the Fermi surface for three doping cases: \(x = 0.01, 0.05, 0.1\) in Fig. 5, where the Fermi surface is composed of arcs which delimit the hole pockets located around the four nodes \(\mathbf{k} = (\pm \pi/2, \pm \pi/2)\).

In the next section we will consider the quantum fluctuation around this mean field solution. The second order Gaussian fluctuation corresponds to the RPA. We will calculate the \(\text{"à la BCS}\)^{33,34} pairing potential in terms of the exchange of these fluctuations, to see the effects of the coexistence of antiferromagnetism and the staggered flux on the pairing force in underdoped region.

III. GAUSSIAN FLUCTUATION AND BCS PAIRING INTERACTION BETWEEN QUASI-PARTICLES

A. Second order action and correlation functions

In this Section we expand the action with respect to the fluctuations of the mean field parameters up to second order starting from the saddle point solution for the \(t - J\)
model [11]. By analogy with a diagrammatic language such a treatment is equivalent to taking into account all the bubbles associated with the fluctuating bosonic fields. This approach allows us to calculate the different correlation functions (or susceptibilities, both being linked via the fluctuation-dissipation theorem [26]) between those fields.

Integrating over the fermionic Grassman variables and expanding the action with respect to the fluctuations of the bosonic modes up to the second order, we obtain

\[
S_2 = \int_0^\beta d\tau \left\{- \sum_i (b_0)^2 \left[2 \cdot i \delta \lambda_i \cdot \delta b_i + \lambda_0 \cdot (\delta b_i)^2\right] - \alpha J \sum_{i,j} \sum_{s,m=+,-,g,z} \left[\delta m^s_m \cdot \delta m^s_m\right] + \frac{(1 - \alpha)}{2} J \sum_{i,j} \left[\delta X_{ij} \cdot \delta X_{ij}^*\right] \right. \\
\left. - \left(\sum_{(i,j)} t + \sum_{(i,j)'} t' + \sum_{(i,j)''} t''\right) (b_0)^2 \times \sum_{\sigma} \delta b_i \cdot \delta b_j \left[\langle \bar{\Psi}_{i,\sigma} \Psi_{j,\sigma}\rangle + \langle \bar{\Psi}_{j,\sigma} \Psi_{i,\sigma}\rangle\right] \right) + \frac{1}{2} \text{Tr} \left[\tilde{G}_0 \tilde{V}_1 \tilde{G}_0 \tilde{V}_1\right].
\]

(Eq. 30)

Evaluating the bubbles \(\frac{1}{2} \text{Tr} \left[\tilde{G}_0 \tilde{V}_1 \tilde{G}_0 \tilde{V}_1\right]\) as explained in Appendix [13] the quadratic action is given by

\[
S_2 = \sum_{q} \sum_{q_1, q_2 = q, q + Q} \sum_{i, j} \sum_{\omega} \delta X_i (q, \omega) \left(\mathcal{M}_{i,j}(q, q_1, q_2, \omega) + \frac{1}{2} \Pi_{i,j}(q, q_1, q_2, \omega)\right) \delta X_j (-q_2, -\omega),
\]

(Eq. 31)

where the first order fluctuations of the bosonic fields \(\delta X_i\) are defined in Eq. (31), and the matrix elements \(\mathcal{M}_{i,j}\) and \(\Pi_{i,j}\) are detailed in Eqs. (32) and (33), respectively. In the previous formula, \(\omega\) denotes the bosonic Matsubara frequencies.

In a path-integral formalism, the second order term of the effective action gives easy access to the different correlation functions between bosonic fields renormalized at a RPA level [25].

\[
\tilde{\mathcal{C}}(q, q_1, -q_2, \omega) = \left[\tilde{C}_{i,j}(q, q_1, -q_2, \omega)\right]_{1 \leq i,j \leq 9} = \left[\langle X_i (q, q_1, -q_2, \omega) X_j (-q_2, -\omega)\rangle_{\text{RPA}}\right]_{1 \leq i,j \leq 9} = \left(\tilde{\mathcal{M}}(q, q_1, q_2, \omega) + \frac{1}{2} \tilde{\Pi}(q, q_1, q_2, \omega)\right)^{-1}.
\]

(Eq. 32)

The behavior of the obtained transverse spin-spin correlation function \(\chi^\pm\) is in agreement with well-known results concerning the Heisenberg antiferromagnets, as discussed in Appendix [13].
B. Derivation of the effective action

We can now explicitly calculate the "à la BCS" pairing potential exchanging all the collective modes described in the previous section. Following Schrieffer et al. we adopt their approach to the framework of the $t - J$ model. We neglect the retardation effects specific to the Eliashberg theory and build a pairing potential by assuming the static limit ($i\omega_n = 0$). Therefore the frequency dependence will not be mentioned from now on. This is justified in the weak coupling region, where the adiabatic approximation $k_BT_c, \Delta_{SC} << h\omega_D$ is satisfied ($T_c$: transition temperature, $\Delta_{SC}$: superconducting order parameter, $\omega_D$: frequency of the exchange bosons). This approach is not sufficient to describe the superconducting state in underdoped cuprates where the features of the strong coupling effect are observed experimentally. However, as will be shown below, the magnetic mechanism based on the generalized spin-bag theory gives only a very small pairing force or is pair breaking. Therefore this weak coupling approximation is justified a posteriori, even though it does not describe the real cuprates.

We start with the linear interaction between the fermions and the bosonic fields \[ S^{fe}_f = \sum_{k,q} \sum_{\sigma',\sigma} \left[ \bar{\Psi}_{k+q,\sigma'} \Psi_{k+q+Q,\sigma} \right] \times \tilde{V}_1(k+q,\sigma';k,\sigma) \times \left[ \bar{\Psi}_{k,\sigma} \Psi_{k+Q,\sigma} \right]. \] (33)

Using the notations defined in Appendix B the interaction matrix $\tilde{V}_1$ can be rewritten as

\[ \tilde{V}_1(k+q,\sigma';k,\sigma) = \sum_{q_1=q+Q} \sum_{i=1}^9 C^{-1}_{i,j}(q_1-q) \cdot \delta X_i(q_1), \]

where $\delta X_i, c_{i,a}, \tilde{s}_{i,a'}$ are given by Eqs. (15), (15), and (10), respectively.

We remember that $\bar{\Psi}, \Psi$ are the Grassmann variables associated with the $f^\dagger, f$ spinon operators. As we are interested in the interactions between two $\gamma_2$ fermions of the lower Hubbard band, we introduce $\bar{\Phi}, \Phi$ the Grassmann variables associated with the $\gamma_2, \gamma_2$ operators. By neglecting the contribution of the upper Hubbard band we obtain from the diagonalization of the mean-field Hamiltonian

\[ \Psi_{k,\sigma} = \sin(\theta_k)\Phi_{k,\sigma}, \]

\[ \Psi_{k+Q,\sigma} = e^{-i\phi_{k+q,\sigma}}\cos(\theta_k)\Phi_{k,\sigma}. \] (35)

The first order action related to the $\gamma_2$ operators is given by

\[ S^{fe}_2 = \sum_{k,q} \sum_{\sigma',\sigma} \bar{\Phi}_{k+q,\sigma'} \times \left[ \sin(\theta_k+q) \times e^{i\phi_{k+q,\sigma'}}\cos(\theta_{k+q}) \right] \times \left\{ \sum_{q_1=q+Q} \sum_{i=1}^9 \left( \sum_{a'=1,2} c_{i,a'}^{\sigma'}(k_1) \tilde{s}_{i,a'}(q_1) \right) \times \delta X_i(q_1) \right\} \times \bar{\Phi}_{k,\sigma}. \] (36)

To incorporate the interaction effects at a RPA level, we build an effective action by adding to $S^{fe}$ the second order in $\delta X_i$ contribution $S_2$, i.e. using Eqs. (41) and (32),

\[ S_2 = \sum_q \sum_{q_1,q_2=q+Q} \sum_{i,j=1}^9 \delta X_i(q_1) \times C^{-1}_{i,j}(q_1) \times \delta X_j(-q_2), \]

then the effective action $S^{eff}$ is given by

\[ S^{eff} = S^{fe}_1 + S_2. \] (38)

In order to obtain the pairing potential, we integrate out the bosonic fields $\delta X_i$ in $S^{eff}$. For convenience we define $\tilde{\phi}(q_1)$ for $q_1 = q, q + Q$ as

\[ \tilde{\phi}(q_1) = \left[ \phi_i(q_1) \right]_{1 \leq i \leq 9}, \]

where $\phi_i(q_1) = 1/2 \sum_{k_1} \sum_{\sigma'_i,\sigma_i} \bar{\Phi}_{k_1+q,\sigma'_i} \sin(\theta_{k_1+q}) e^{i\phi_{k_1+q,\sigma'_i}} \cos(\theta_{k_1+q}) \times \left[ \sum_{a'_i=1,2} c_{i,a'_i}^{\sigma'_i}(k_1, q_1) \tilde{s}_{i,a'_i}(q_1) \right] \times \Phi_{k_1,\sigma_i}. \] (39)
After integrating out the bosonic fields we have

\[ S^{\text{eff}} = -\sum_{q} \sum_{q_1, q_2 = q, q + Q} \tilde{\phi}(q_2) \tilde{c}(q, q_1, -q_2) \tilde{\phi}(q_1), \]

or equivalently from Eq. 39

\[
S^{\text{eff}} = -\frac{1}{4} \sum_{q} \sum_{q_1, q_2 = q, q + Q} \sum_{i,j=1}^{9} \sum_{k_1, k_2} \sum_{\sigma', \sigma_1, \sigma_2} \Phi_{k_2-q, \sigma_2} \Phi_{k_1+q, \sigma_1} \Phi_{k_1, \sigma_1} \times \left\{ \left[ \sin(\theta_{k_2-q}) e^{i\phi_{k_2-q, \sigma_2}} \cos(\theta_{k_2-q}) \right] \left( \sum_{a_2=1,2} c_{i, a_2}^{\sigma_2} (k_2, -q_2) \tilde{s}_{i, a_2} (-q_2) \right) \left[ \frac{\sin(\theta_{k_2})}{e^{-i\phi_{k_2, \sigma_2}} \cos(\theta_{k_2})} \right] \right\} \times \left\{ \sin(\theta_{k_1+q}) e^{i\phi_{k_1+q, \sigma_1}} \cos(\theta_{k_1+q}) \right\} \times \left\{ \sum_{a_1=1,2} c_{j, a_1}^{\sigma_1} (k_1, q_1) \tilde{s}_{j, a_1} (q_1) \right\} \left[ \frac{\sin(\theta_{k_1})}{e^{-i\phi_{k_1, \sigma_1}} \cos(\theta_{k_1})} \right], \tag{40} \]

C. Calculation of the BCS pairing interaction

To have a BCS form in the previous effective action we impose the following relations

* \( k_1 = k, k_2 = -k, \ q = k' - k, \)
  \( \sigma_1' = \uparrow, \ \sigma_2' = \downarrow, \ \sigma_2 = \downarrow, \ \sigma_1 = \uparrow. \) \tag{41}

* \( k_1 = -k, k_2 = k, \ q = k - k', \)
  \( \sigma_1' = \downarrow, \ \sigma_2' = \uparrow, \ \sigma_2 = \uparrow, \ \sigma_1 = \downarrow. \) \tag{42}

* \( k_1 = -k, k_2 = k, \ q = k' + k, \)
  \( \sigma_1' = \uparrow, \ \sigma_2' = \downarrow, \ \sigma_2 = \uparrow, \ \sigma_1 = \downarrow. \) \tag{43}

* \( k_1 = k, k_2 = -k, \ q = -k' - k, \)
  \( \sigma_1' = \downarrow, \ \sigma_2' = \uparrow, \ \sigma_2 = \downarrow, \ \sigma_1 = \uparrow. \) \tag{44}

The configurations \( \text{(11)} \) and \( \text{(12)} \) are related to the \( z \) \( z \) contribution, while the forms \( \text{(13)} \) and \( \text{(14)} \) give the \( \pm \) contribution. Therefore the BCS effective action is

\[
S^{\text{eff}}_{\text{BCS}} = \sum_{k, k'} \left\{ V_{\text{BCS}}(k, k') \right\} \tag{45}
\]

where

\[
V_{\text{BCS}}(k, k') = V^{\pm}_{\text{BCS}}(k, k') + V^{\mp}_{\text{BCS}}(k, k') \tag{46}
\]

The \( z \) \( z \) BCS - type pairing potential \( V^{\pm}_{\text{BCS}} \) is obtained by putting the configurations \( \text{(11)} \) and \( \text{(12)} \) in the effective action \( \text{(40)} \)

\[
V^{\pm}_{\text{BCS}}(k, k') = \sum_{q_1, q_2 = k' - k, k - k + Q} \sum_{i,j=1}^{9} \left( -\frac{1}{4} \right) \times \left\{ \sin(\theta_{-k'}) e^{i\phi_{-k', \sigma_2}} \cos(\theta_{-k'}) \right\} \left( \sum_{a_2=1,2} c_{i, a_2}^{\sigma_2} (-k, -q_2) \tilde{s}_{i, a_2} (-q_2) \right) \left[ \frac{\sin(\theta_{-k})}{e^{-i\phi_{-k, \sigma_2}} \cos(\theta_{-k})} \right] C_{i,j}(k' - k, q_1, -q_2) \times \left\{ \sin(\theta_k') e^{i\phi_{k', \sigma_1}} \cos(\theta_k') \right\} \left( \sum_{a_1=1,2} c_{j, a_1}^{\sigma_1} (k, q_1) \tilde{s}_{j, a_1} (q_1) \right) \left[ \frac{\sin(\theta_k)}{e^{-i\phi_{k, \sigma_1}} \cos(\theta_k)} \right] C_{i,j}(k - k', -q_1, q_2) \times \left\{ \sin(\theta_{-k'}) e^{i\phi_{-k', \sigma_2}} \cos(\theta_{-k'}) \right\} \left( \sum_{a_2=1,2} c_{j, a_2}^{\sigma_2} (-k, -q_1) \tilde{s}_{j, a_2} (-q_1) \right) \left[ \frac{\sin(\theta_{-k})}{e^{-i\phi_{-k, \sigma_2}} \cos(\theta_{-k})} \right] \times \left\{ \sin(\theta_k') e^{i\phi_{k', \sigma_1}} \cos(\theta_k') \right\} \left( \sum_{a_1=1,2} c_{i, a_1}^{\sigma_1} (k, q_2) \tilde{s}_{i, a_1} (q_2) \right) \left[ \frac{\sin(\theta_k)}{e^{-i\phi_{k, \sigma_1}} \cos(\theta_k)} \right] C_{i,j}(k - k', -q_1, q_2). \tag{47} \]
By imposing the configurations (43) and (44) in the effective action, we get the ±BCS-type pairing potential

\[
V_{\text{BCS}}^{\pm}(k, k') = \sum_{q_1, q_2 = k + k'} \sum_{i,j=1}^{9} \left( \frac{1}{4} \right) \times \left\{ \sin(\theta_{-k'}) e^{i\phi_{-k}} \cos(\theta_{-k'}) \left[ \sum_{a_1=1,2} c_{i,a_1} (k, -q_2) \tilde{s}_{i,a_1} (-q_2) \right] \left[ \frac{\sin(\theta_k)}{e^{-i\phi_{-k}} \cos(\theta_k)} \right] C_{i,j} (k' + k, q_1, -q_2) \right.
\]

\[
\times \left[ \sin(\theta_{k'}) e^{i\phi_{k'}} \cos(\theta_{k'}) \left[ \sum_{a'_1=1,2} c_{i',a'_1} (k, q_1) \tilde{s}_{i',a'_1} (q_1) \right] \left[ \frac{\sin(\theta_{-k})}{e^{-i\phi_{-k}} \cos(\theta_{-k})} \right] \right]
\]

\[
+ \left[ \sin(\theta_{k'}) e^{i\phi_{k'}} \cos(\theta_{k'}) \left[ \sum_{a_2=1,2} c_{j,a_2} (-k, q_2) \tilde{s}_{j,a_2} (q_2) \right] \left[ \frac{\sin(\theta_{-k})}{e^{-i\phi_{-k}} \cos(\theta_{-k})} \right] C_{i,j} (-k - k', -q_1, q_2) \right. \\
\times \left[ \sin(\theta_{-k'}) e^{i\phi_{-k'}} \cos(\theta_{-k'}) \left[ \sum_{a'_2=1,2} c_{j',a'_2} (-k, -q_1) \tilde{s}_{j',a'_2} (-q_1) \right] \left[ \frac{\sin(\theta_{k})}{e^{-i\phi_{k}} \cos(\theta_{k})} \right] \right} \\
\] (48)

The correlation functions and pairing interactions have been numerically calculated, the results of which are presented in Figs. 6 and 7. These figures show the effective interactions between two $\gamma_2$ fermions, one located at the fixed position $k = (-\pi/2, -\pi/2)$, while the other one is at $k'$ moving inside the magnetic BZ. The intensities of the effective interactions are plotted as a function of $k'$, for example the values given for $k' = (\pi/2, \pi/2)$ corres-

![Diagram](image-url)
sets of numerical values for the parameters in the range: 0 ≤ J ≤ 0.3. We have also considered several additional sets of numerical values for the parameters in the range: 0 ≤ J ≤ 0.3. The robustness of these results with respect to slight variations of the parameter values, and therefore can explain the robustness of the results. In the next section, we will discuss the implications of the interaction V_{BCS} on the possible superconductivity in this model.

IV. DISCUSSION AND CONCLUSIONS

In previous sections, we have calculated the effective interactions between the two holes in the background of the AF and RVB orders. There are four half-pockets of holes in the reduced first BZ as shown in Fig. 7, or the two hole pockets in the shifted BZ. These small hole pockets around k = (±π/2, ±π/2) are different from those of the simple SDW state as assumed in the original spin-bag theory. Namely the description of the antiferromagnetic state includes the singlet formation via the RVB order parameter in addition to the Néel order, and correspondingly the wavefunctions of the doped holes are distinct from the SDW state. Already there opens the gap near k = (±π, 0), (0, ±π) due to the staggered flux, and the superconducting pairings near these points are irrelevant. However the pairing force is still dominated by the zz-component of the spin susceptibility, and is repulsive for the momentum transfer q_d = (π, π). When the su-
perconducting gaps at \( \mathbf{k} = (\pm \pi, 0), (0, \pm \pi) \) are dominant, this leads to the \( d_{x^2-y^2} \) pairing as discussed by several authors since the sign of the order parameter is opposite between \( \mathbf{k} = (\pm \pi, 0) \) and \( \mathbf{k} = (0, \pm \pi) \). However in the present case, only the states near \( \mathbf{k} = (\pm \pi/2, \pm \pi/2) \) could contribute to the pairing. One of the main predictions of the spin-bag theory is that the superconductivity occurs through the pairing of holes in the small pockets, which are delimited by the Fermi surface as we can see in Fig. 3. The present study shows that the total pairing potential \( V_{BCS} \) is clearly repulsive, or pair breaking, on all the range of doping for a wavevector \( \mathbf{q} \) in the (0, 0), \((\pi, 0) \) and \((0, \pi) \) areas of the magnetic BZ. An attractive behaviour is also found when the momentum transfer is around \((\pi, \pi) \). It corresponds to relatively large values of \( \mathbf{q} \). Namely, the repulsive interaction (dark contribution) near \( \mathbf{k} = (-\pi/2, -\pi/2) \) is dominating over the attractive (light) region with large \( \mathbf{q} \). Therefore as long as one considers the pairing form which is constant over each of the small hole pocket, there occurs no superconducting instability because the intra-pocket pair breaking force is larger than the inter-pocket force independently of the relative sign of the order parameters between different pockets.

On the other hand, one can consider the pairing with the nodes in the small hole pockets as in the case of \( d_{x^2-y^2} \) pairing realized in real cuprates. In this case, however, most of the interaction cancel within the small pocket due to the sign change of the order parameter. We could not determine the sign of the residual pairing interaction, but we can safely conclude that it is very weak even though pair creating. Therefore we conclude that the doped staggered flux state is stable against the superconducting instability when only the magnetic mechanism is considered. This is in accordance with the recent exact diagonalization studies on the \( t-J \) model at small doping. Furthermore this suggests that the other mechanism of superconductivity is active in cuprates at least for small hole doping region. Also a related work has been done by Singh and Tešanović where the quantum correction to the spin-bag mean field theory is studied up to one loop level and also the doped case is considered. This is along the same line as the present study, and they also obtained the repulsive interaction between two holes. The new aspect introduced in our study is the RVB correlation represented by the flux order parameter. Combining these two works, the pair breaking nature of the magnetic interactions seems to be rather robust in the low hole doping limit.

There are two possible routes leading to the superconductivity. One possibility is that the starting mean field ansatz becomes inappropriate for optimal doping case. According to the SU(2) formulation of the RVB state, the staggered flux state is the quantum mechanical mixture of the flux state we discussed above and the \( d_{x^2-y^2} \) pairing state, and the latter one is more and more weighted as the doping proceeds. Therefore our formulation is valid only for the small doping region and can not capture the crossover to the superconducting state since it is based on the perturbative method around the AF + flux state. On the other hand, starting from the d-wave superconducting state in the SU(2) formalism, the staggered flux fluctuation has been studied. This is another language describing the instability towards the AF ordering, which is represented by the chiral symmetry breaking in the gauge theory. It is also found the self-energy due to the staggered flux fluctuation leads to the quasi-particle damping strongly anisotropic in the momentum space. These works are the approach from the superconducting side, and are complementary to ours.

Taking the view that the magnetic interaction is pair breaking in the low hole doping limit, we should look for other forces which are active in the cuprates. One of the most promising candidates for this is the electron-phonon interaction which already manifests itself in the angle-resolved photoemission spectroscopy. However, much more work is needed to establish this scenario. Especially the interplay between the strong correlation and the electron-phonon interaction remains an important issue to be studied.

In conclusion, we have studied the extended spin-bag scenario taking into account the staggered flux resonating valence bond order in addition to the antiferromagnetic order. The pairing potential between the quasi-particles has been calculated at the Gaussian level, and it is found that the magnetically mediated interaction is pair breaking or very weak at the small hole doping limit, suggesting the other mechanisms such as electron-phonon interaction are active.

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**APPENDIX A: DERIVATION OF THE SELF-CONSISTENT EQUATIONS**

We allow the different fields to fluctuate at the first order around their saddle point values \( \chi^{(1)} \), \( m^{(1)} \) and \( \delta \chi^{(2)} \).

\[
\begin{align*}
 b_i &= b_0 (1 + \delta b_i) ; \\
 i\lambda_i &= \lambda_0 + i\delta \lambda_i ; \\
 m^{*m}_i &= (m^{*m}_i)_0 + \delta m^{*m}_i , \quad s_m = x, y, z ; \\
 \chi_{i+s}^x &= \chi_s^x = (\chi_s^x)_0 + \delta \chi_s^x , \\
 \delta \chi_s^x &= \delta \chi_s^x + i\delta \chi_s^x + i\delta \chi_s^x , \quad s = x, y .
\end{align*}
\]

By expanding the fluctuations at the first order we obtain
\[
S_1 = \int_0^\beta d\tau \left\{ - \sum_i (b_0)^2 (i \delta \lambda_i + 2 \lambda_0 \cdot \delta b_i) - \alpha J \sum_{(i,j)} [(-1)^i m \cdot \delta m_j^z + (-1)^j m \cdot \delta m_i^z] \\
+ \frac{(1 - \alpha)}{2} \sum_{(i,j)} \left[ (\chi_{ij})_0 \cdot \delta \chi_{ij}^* + (\chi_{ij})_0^* \cdot \delta \chi_{ij} \right] - \sum_{i,\sigma} [i \delta \lambda_i \cdot \bar{\Psi}_{i,\sigma} \Psi_{i,\sigma}]
\right. \\
- \left. \left( \sum_{(i,j)} t + \sum_{(i,j)'} t' + \sum_{(i,j)''} t'' \right) \sum_{\sigma} (b_0)^2 (\delta b_i + \delta b_j) \left[ \bar{\Psi}_{i,\sigma} \Psi_{j,\sigma} + \bar{\Psi}_{j,\sigma} \Psi_{i,\sigma} \right]
\right)
\]

\[
+ \frac{\alpha}{2} J \sum_{(i,j)} \sum_{\sigma,\sigma'} \sum_{s_m=x,y,z} \left[ \delta m_j^{s_m} (\bar{\Psi}_{i,\sigma} \bar{s}_{\sigma,\sigma'}^{s_m} \Psi_{i,\sigma'}) + \delta m_i^{s_m} (\bar{\Psi}_{j,\sigma} \bar{s}_{\sigma,\sigma'}^{s_m} \Psi_{j,\sigma'}) \right] \\
- \left. \frac{(1 - \alpha)}{2} J \sum_{(i,j),\sigma} \left[ \delta \chi_{i,j} \cdot \bar{\Psi}_{j,\sigma} \Psi_{i,\sigma} + \delta \chi_{i,j}^* \cdot \bar{\Psi}_{i,\sigma} \Psi_{j,\sigma} \right] \right\}. 
\]

(A1)

In the preceding formula, like in those which will follow
in the appendices, the imaginary time dependence of the
Grassmann variables and first order bosonic fluctuations
follows the Lagrangian \( \mathcal{L} \) and is implicit. For clarity
in the following calculations we decompose \( S_1 \) into two
parts

\[
S_1 = S_1^{b_0} + S_1^{f_0}.
\]

(A2)

\( S_1^{b_0} \) contains terms with exclusively auxiliary bosonic
fields and no Grassmann variables. It corresponds to the
three first terms of Eq. (A1)

\[
S_1^{b_0} = \int_0^\beta d\tau \left\{ - \sum_i (b_0)^2 (i \delta \lambda_i + 2 \lambda_0 \cdot \delta b_i) \right.
\]

\[
- \left. \alpha J \sum_{(i,j)} [(-1)^i m \cdot \delta m_j^z + (-1)^j m \cdot \delta m_i^z] \\
+ \left. \frac{(1 - \alpha)}{2} J \sum_{(i,j)} \left[ (\chi_{ij})_0 \cdot \delta \chi_{ij}^* + (\chi_{ij})_0^* \cdot \delta \chi_{ij} \right] \right\}. 
\]

(A3)

\( S_1^{f_0} \) includes all the terms containing fermionic variables
\( \bar{\Psi}, \Psi \). It takes the four last terms of Eq. (A1)

\[
S_1^{f_0} = \int_0^\beta d\tau \left\{ - \sum_{i,\sigma} [i \delta \lambda_i \cdot \bar{\Psi}_{i,\sigma} \Psi_{i,\sigma}] \\
- \left( \sum_{(i,j)} t + \sum_{(i,j)'} t' + \sum_{(i,j)''} t'' \right) (b_0)^2 \right.
\]

\[
\times \left. \sum_{\sigma} (\delta b_i + \delta b_j) \left[ \bar{\Psi}_{i,\sigma} \Psi_{j,\sigma} + \bar{\Psi}_{j,\sigma} \Psi_{i,\sigma} \right] \right\}. 
\]

(A4)

We start by focusing on the bosonic part \( S_1^{b_0} \) of the
action \( S_1 \) expanded at the first order in fluctuations. By passing in the space of moments and Matsubara frequencies we obtain

\[
S_1^{b_0} = \sqrt{\beta N}\left[ -(b_0)^2 i \delta \lambda_q = 0 (i \omega_\ell = 0) \right.
\]

\[
-2(b_0)^2 \lambda_0 \cdot \delta b_q = 0 (i \omega_\ell = 0) \\
+4 \alpha J \cdot m \cdot \delta m_q = 0 (i \omega_\ell = 0)
\]

\[
+ (1 - \alpha) J \sum_{q,\omega_\ell} \left[ \chi_q^y (i \omega_\ell) \cdot \delta \chi_q^x (i \omega_\ell) \\
+ \chi_q^x (i \omega_\ell) \cdot \delta \chi_q^y (i \omega_\ell) \right], 
\]

(A5)

where \( i \omega_\ell \) labels the bosonic Matsubara frequencies and
\( \sum_q \) is expanded over the first BZ.

We study now the fermionic contribution \( S_1^{f_0} \) and de-
compose it into four contributions

\[
S_1^{f_0} = S_1^t + S_1^t + S_1^m + S_1^t. 
\]

(A6)

By passing in the space of moments and Matsubara fre-
quencies we obtain
In the previous formulas $\sigma^0_{\sigma',\sigma}$ is an element of the $(2 \times 2)$ identity matrix, $\imath \omega_m$ is (like $\imath \omega_n$) a fermionic Matsubara frequency and we have defined the following quantities

$$\hat{t}_k = t [\cos(k_x) + \cos(k_y)] \, , \quad (A11)$$

$$\hat{t}'_k = t' [2 \cos(k_x) \cos(k_y)] \, , \quad (A12)$$

By using the Pauli matrices, we can put the different contributions together, and rewrite $S^f_{1''}$ as

$$S^{f_{1''}}_{1} = \sum_{k,q} \sum_{\sigma',\sigma} \left[ \sum_{\imath \omega_m,\imath \omega_n} \sigma^0_{\sigma',\sigma} \left[ \Psi_{k+q,\sigma'}(\imath \omega_m) \Psi_{k+q+Q,\sigma'}(\imath \omega_m) \right] \hat{V}_1 (k+q,\sigma';\imath \omega_m; k,\sigma,\imath \omega_n) \right] \left[ \Psi_{k,\sigma}(\imath \omega_m) \right] \left[ \Psi_{k+Q,\sigma}(\imath \omega_n) \right] \, , \quad (A15)$$
with $\tilde{V}_1$ the interaction matrix

$$\sqrt{\beta N_s} \tilde{V}_1(k + q, \sigma', \iota w_m; k, \sigma, \iota w_n) = \sum_{s_x = x, y} \left\{ \left[ - (1 - \alpha) J \cdot \cos(k_{s_x} + q_{s_x}/2) \cdot \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta' \chi^s_{q}(\iota w_m - \iota w_n) 
+ \left[ - (1 - \alpha) J \cdot \sin(k_{s_x} + q_{s_x}/2) \cdot \sigma'^0_{\sigma', \sigma} \cdot i\tilde{\sigma}^y \right] \delta' \chi^z_{q}(\iota w_m - \iota w_n) 
+ \left[ - (1 - \alpha) J \cdot \sin(k_{s_x} + q_{s_x}/2) \cdot \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta' \chi^z_{q}(\iota w_m - \iota w_n) 
+ \left[ + (1 - \alpha) J \cdot \sin(k_{s_x} + q_{s_x}/2) \cdot \sigma'^0_{\sigma', \sigma} \cdot i\tilde{\sigma}^y \right] \delta' \chi^s_{q}(\iota w_m - \iota w_n) \right\} + \sum_{s_m = x, y, z} \left\{ \left[ \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta m^{s_m}_{q}(\iota w_m - \iota w_n) + \left[ \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta m^{s_m}_{q+Q}(\iota w_m - \iota w_n) \right\} + \left\{ \left[ - \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta m^{s_m}_{q}(\iota w_m - \iota w_n) + \left[ - \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta m^{s_m}_{q+Q}(\iota w_m - \iota w_n) \right\} + \left\{ \left[ \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta m^{s_m}_{q}(\iota w_m - \iota w_n) + \left[ \sigma'^0_{\sigma', \sigma} \cdot \tilde{\sigma}^z \right] \delta m^{s_m}_{Q}(\iota w_m - \iota w_n) \right\}.
$$

Finally the mean field equations are obtained after having integrated out the fermionic fields by checking the saddle point property

$$\text{Tr} [\tilde{G}_0 \tilde{V}_1] = 0,$$

which gives by using the previously obtained expressions $^{28}$, $^{29}$, $^{30}$, $^{31}$ the set of coupled mean field equations $^{24} - ^{27}$.

**APPENDIX B: Derivation of the Gaussian Action**

We decompose the second order effective action $S_2$ $^{30}$ into two parts

$$S_2 = S_2^{bo} + S_2^{bub}. \quad (B1)$$

$S_2^{bo}$ contains only contributions due to auxiliary bosonic fields. It includes all the terms of Eq. $^{30}$ with the exception of the trace

$$S_2^{bo} = \int_{0}^{\beta} d\tau \left\{ \sum_{i} (b_0)^2 \left[ 2 \cdot i \delta \lambda_i \cdot \delta b_i + \lambda_0 \cdot (\delta b_i)^2 \right] - \alpha J \sum_{(i,j)} \sum_{s_m = x, y, z} \left[ \delta m^{s_{im}} \cdot \delta m^{s_{jm}} \right] + \frac{(1 - \alpha)}{2} J \sum_{(i,j)} \left[ \delta \chi_{ij} \cdot \delta \chi_{ij}^{*} \right] \right\} - \left( \sum_{(i,j)} + \sum_{(i,j)'} + \sum_{(i,j)''} \right) \sum_{\sigma} (b_0)^2 \delta b_{\sigma} \cdot \delta b_{\sigma} \left[ \langle \bar{\Psi}_{i,\sigma} \Psi_{j,\sigma} \rangle + \langle \bar{\Psi}_{j,\sigma} \Psi_{i,\sigma} \rangle \right]. \quad (B2)$$

$S_2^{bub}$ takes into account the “bubble” contributions

$$S_2^{bub} = \frac{1}{2} \text{Tr} [\tilde{G}_0 \tilde{V}_1 \tilde{G}_0 \tilde{V}_1]. \quad (B3)$$

We firstly evaluate $S_2^{bub}$ $^{32}$ by passing in the space of moments and Matsubara frequencies...
\[ S_{2b}^{bo} = -(b_0)^2 \sum_q \sum_{i,\omega} \left\{ 2i[\delta \lambda_q(\omega_k) \cdot \delta b_{-q}(-i\omega_k) + \delta \lambda_{q+Q}(i\omega_k) \cdot \delta b_{-q-Q}(-i\omega_k)] \\
+ \lambda_0 \left[ \delta b_q(i\omega_k) \cdot \delta b_{-q}(-i\omega_k) + \delta b_{q+Q}(i\omega_k) \cdot \delta b_{-q-Q}(-i\omega_k) \right] \right\} \]

\[
-\alpha \sum_q \sum_{i,\omega} \left( \sum_{s_{t,\omega}} \sum_{s_{t,\omega}} \left[ \delta m_{s_{t,\omega}}(\omega_k) \cdot \delta m_{s_{t,\omega}}(-i\omega_k) - \delta m_{s_{t,\omega}}(\omega_k) \cdot \delta m_{s_{t,\omega}}(-i\omega_k) \right] \right) \sum_{s_{t,\omega} = x, y, z} \left[ \delta \chi_{s_{t,\omega}}(i\omega_k) \cdot \delta \chi_{s_{t,\omega}}(-i\omega_k) + \delta \chi_{s_{t,\omega}}^*(i\omega_k) \cdot \delta \chi_{s_{t,\omega}}^*(-i\omega_k) \right] \]

\[
+ \left( \frac{1 - \alpha}{2} \right) \sum_q \sum_{i,\omega} \left( \sum_{s_{t,\omega} = x, y, z} \left[ \delta \chi_{s_{t,\omega}}(i\omega_k) \cdot \delta \chi_{s_{t,\omega}}(-i\omega_k) + \delta \chi_{s_{t,\omega}}^*(i\omega_k) \cdot \delta \chi_{s_{t,\omega}}^*(-i\omega_k) \right] \right) \]

\[-4(b_0)^2 \sum_q \sum_{i,\omega} \left( t \cdot t' \left[ \cos(q_x) + \cos(q_y) \right] + 2t' \cdot t'' \cos(q_x) \cdot \cos(q_y) + t'' \cdot t'' \left[ \cos(2q_x) + \cos(2q_y) \right] \right) \]

\[
\times \left[ \delta b_q(i\omega_k) \cdot \delta b_{-q}(-i\omega_k) \right] \right) + \left( \right. \]

\[
\times \delta b_q + Q(i\omega_k) \cdot \delta b_{-q-Q}(-i\omega_k) \right) \right), \quad (B4) \]

where \( \tilde{J}_q \) is given by Eq. \[ \text{[A11]} \] and we have defined for \( s_t, s_{t'} = x, y \)

\[
I_{1,s}^s = \langle \tilde{\Psi}_i, \tilde{\Psi}_{i+s_t, \sigma} \rangle = \frac{1}{N_q} \sum_k \left( \frac{\xi_{k+Q} - \xi_k}{E_{k+Q} - E_k} \right) \cos(k_{s_t}), \]

\[
I_{2,s_{t},s_{t'}} = \langle \tilde{\Psi}_i, \tilde{\Psi}_{i+s_{t'} + s_t, \sigma} \rangle = \frac{1}{N_q} \sum_k \left( \frac{\xi_{k+Q} - \xi_k}{E_{k+Q} - E_k} \right) \cos(k_{s_t + k_{s_{t'}}}). \]

In order to write \( S_{2b}^{bab} \) and \( S_{2b}^{bo} \) in a compact way which will be useful in the following calculations we define the vector of nine components \( \delta \bar{X} \) containing the first order fluctuations of the bosonic fields

\[
\delta \bar{X}(q, i\omega_k) = \left[ \delta X_i(q, i\omega_k) \right]_{1 \leq i \leq 9} = \begin{bmatrix}
\delta \chi_{q, i\omega_k} \\
\delta \chi_{q, i\omega_k}^* \\
\delta \chi_{q, i\omega_k} \cdot \delta m_{s_{t,\omega}}(i\omega_k) \\
\delta \chi_{q, i\omega_k} \cdot \delta m_{s_{t,\omega}}^*(i\omega_k) \\
\delta \chi_{q, i\omega_k} \cdot \delta \lambda_{q, i\omega_k} \\
\delta \chi_{q, i\omega_k} \cdot \delta b_{q, i\omega_k}
\end{bmatrix}, \quad \delta \bar{X}(q + Q, i\omega_k) = \left[ \delta X_i(q + Q, i\omega_k) \right]_{1 \leq i \leq 9} = \begin{bmatrix}
\delta \chi_{q+Q, i\omega_k} \\
\delta \chi_{q+Q, i\omega_k}^* \\
\delta \chi_{q+Q, i\omega_k} \cdot \delta m_{s_{t,\omega}}(i\omega_k) \\
\delta \chi_{q+Q, i\omega_k} \cdot \delta m_{s_{t,\omega}}^*(i\omega_k) \\
\delta \chi_{q+Q, i\omega_k} \cdot \delta \lambda_{q+Q, i\omega_k} \\
\delta \chi_{q+Q, i\omega_k} \cdot \delta b_{q+Q, i\omega_k}
\end{bmatrix} \quad (B5) \]

Using Eq. \[ \text{[B3]} \] we define the matrix \( \hat{M} \) so that

\[
\hat{M}(q_1, q_2, i\omega_k) = \left[ M_{i,j}(q_1, q_2, i\omega_k) \right]_{1 \leq i,j \leq 9},
\]

We now tackle the calculation of the “bubble” part \( S_{2b}^{bab} \) \[ \text{[B3]} \]. We have\[ \text{[B3]} \]

\[
S_{2b}^{bo} = \sum_q \sum_{q_1, q_2 = q+Q} \sum_{i,\omega} \sum_{i,\omega} \delta X_i(q_1, i\omega_k) \times M_{i,j}(q_1, q_2, i\omega_k) \cdot \delta X_j(-q_2, -i\omega_k). \quad (B6)
\]
where the terms \( \tilde{v}_k, \tilde{v}_k', \tilde{v}_k'' \) and \( \tilde{J}_q \) are defined in Eqs. (A11), (A12), (A13) and (A14), respectively. In the same manner we define the vector \( \tilde{s} \) to associate each fluctuation element \( \delta X_i \) to its corresponding Pauli matrix appearing in the expression of the \( \tilde{V}_1 \) matrix (A16)

\[
\tilde{c}^\sigma,\sigma(k, q) = [c^\sigma,\sigma(k, q)]_{1 \leq i \leq 9} = \begin{bmatrix}
(1-\alpha)J\cos(k_x + q_x/2)\sigma^0,\sigma \\
(1-\alpha)J\cos(k_y + q_y/2)\sigma^0,\sigma \\
(1-\alpha)J\sin(k_x + q_x/2)\sigma^0,\sigma \\
(1-\alpha)J\sin(k_y + q_y/2)\sigma^0,\sigma \\
\alpha\tilde{J}_q\sigma^z,\sigma \\
\alpha\tilde{J}_q\sigma^y,\sigma \\
\alpha\tilde{J}_q\sigma^x,\sigma \\
-\sigma^0,\sigma \\
-2(b_0)^2(\tilde{v}_k + \tilde{v}_k + \tilde{v}_k + \tilde{v}_k)\sigma^0,\sigma \\
-2(b_0)^2(\tilde{v}_k + \tilde{v}_k + \tilde{v}_k + \tilde{v}_k)\sigma^0,\sigma \\
\end{bmatrix}, \quad (B8)
\]

\[
\tilde{c}^\sigma,\sigma(k, q + Q) = [c^\sigma,\sigma(k, q + Q)]_{1 \leq i \leq 9} = \begin{bmatrix}
-i(1-\alpha)J\sin(k_x + q_x/2)\sigma^0,\sigma \\
-i(1-\alpha)J\sin(k_y + q_y/2)\sigma^0,\sigma \\
i(1-\alpha)J\cos(k_x + q_x/2)\sigma^0,\sigma \\
i(1-\alpha)J\cos(k_y + q_y/2)\sigma^0,\sigma \\
-\sigma^0,\sigma \\
-\sigma^0,\sigma \\
-\sigma^0,\sigma \\
-\sigma^0,\sigma \\
2i(b_0)^2(\tilde{v}_k + \tilde{v}_k + \tilde{v}_k + \tilde{v}_k)\sigma^0,\sigma \\
\end{bmatrix}, \quad (B9)
\]

The summation over the fermionic Matsubara frequencies \( i\omega_n \) is incorporated in

\[
F_{\tilde{s},\sigma,\sigma}(q_1, \tilde{s}_j, q_2)(k, q, i\omega) = \frac{1}{\beta} \sum_{\omega_n} \text{tr} \left[ \tilde{G}_0(k + q, \sigma', \omega_n + i\omega) \tilde{s}_{i, a'}(q_1) \tilde{G}_0(k, \sigma, \omega_n) \tilde{s}_{j, a'}(q_2) \right], \quad (B11)
\]

with \( q_1, q_2 = q, q + Q \). Considering the identity matrix and the three \( SU(2) \) Pauli matrices we get from Eq. (B11) sixteen different thermal integrals which are detailed in Appendix C.
Using the preceding expressions (135), (138), (139), (141), and the formulas associated with the $F$-integrals, we define the matrix $\hat{H}$ which explicitly incorporates at a RPA ("bubble") level the contributions of the different fields fluctuations to the correlation functions

$$\hat{H}(q,q_1,q_2,i\omega) = \left[ \Pi_{i,j}(q,q_1,q_2,i\omega) \right]_{1 \leq i,j \leq 9}, \quad (B12)$$

with the matricial elements

$$\Pi_{i,j}(q,q_1,q_2,i\omega) = \frac{1}{N_{\omega}} \sum_{k} \sum_{\sigma',\sigma} \sum_{\alpha',\alpha''=1} c_{\alpha',\sigma}^{\pi '}(k,q_1)$$

$$\times F_{\tilde{\omega} ; \tilde{\omega} ' ; \sigma}^{\pi '}(q_1,\tilde{\omega}) \tilde{\omega} ',(q_2,\tilde{\omega}) c_{\alpha',\sigma}^{\pi '}(k,q_2).$$

From Eqs. (134), (138) and (139) we get

$$\text{Tr}\left[ \hat{G}_0 \hat{\Pi}_0 \hat{G}_0 \hat{\Pi}_0 \right] = \sum_{q} \sum_{q_1,q_2=q,q+Q} \sum_{i\omega} \sum_{i,j=1}^{9} \delta X_i(q_1,i\omega)$$

$$\times \Pi_{i,j}(q,q_1,q_2,i\omega) \delta X_j(-q_2,-i\omega).$$

With Eqs. (131), (136), (138) and (139) we finally obtain $S_2$ given by Eq. (11).

**APPENDIX C: THERMAL INTEGRALS EXPRESSIONS**

We give here the analytic expressions of the sixteen different $F$ integrals (111) useful to calculate at a RPA level the two (bosonic) fields correlation functions (12). We define

$$E_{4,1} = \left[ E_{4,k+q}^{up} - (i\omega_n + i\omega) \right] \left[ (i\omega_n + i\omega) - E_{4,k+q}^{low} \right]$$

$$\times \left[ E_{4,k+q}^{up} - i\omega_n \right] \left[ i\omega_n - E_{4,k+q}^{low} \right] ,$$

$$g_{0,0} = 2(i\omega_n)^2 - i\omega(\xi_k + \xi_k + q)$$

$$+ i\omega \left[ 2i\omega - (\xi_k + \xi_k + q + \xi_k + q + q) \right] ,$$

$$g_{0,x} = -2i\omega_n(\Delta_{k+k+q}^{m} - 2i\omega(\xi_k + \xi_k + q + \xi_k + q + q)$$

$$+ \Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + \xi_k + q + q) + \Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + q) ,$$

$$g_{0,y} = -2i\omega_n(\Delta_{k+k+q}^{m} + i\Delta_{k+k+q}^{m}) - 2i\omega(\xi_k + \xi_k + q + \xi_k + q + q)$$

$$+ i\Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + \xi_k + q + q) + i\Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + q) ,$$

$$g_{0,z} = i\omega_n(\xi_k - \xi_k + q + \xi_k + q + \xi_k + q + q)$$

$$+ i\omega (\xi_k - \xi_k + q + \xi_k + q + q) ,$$

$$g_{x,y} = i\omega_n(-i\xi_k + i\xi_k + q + i\xi_k + q - \xi_k + q + q)$$

$$- i\omega (i\xi_k - i\xi_k + q + \xi_k + q + q) ,$$

$$g_{x,z} = 2i\omega_n(\Delta_{k+k+q}^{m} + 2i\omega(\xi_k + \xi_k + q + \xi_k + q + q)$$

$$- \Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + \xi_k + q + q) + \Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + q) ,$$

$$g_{y,z} = 2i\omega_n(\Delta_{k+k+q}^{m} - 2i\omega(\xi_k + \xi_k + q + \xi_k + q + q)$$

$$- \Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + \xi_k + q + q) + \Delta_{k+k+q}^{m}(\xi_k + \xi_k + q + q) ,$$

and we obtain

$$F_{\sigma \sigma,\sigma}^{\pi,\pi}(k,q,i\omega) = \frac{1}{\beta} \sum_{i\omega} \frac{1}{E_{4,k+q,\sigma}^{u,\pi}} \left\{ g_{0,0}$$

$$+ (\xi_k + \xi_k + q + \xi_k + q + q)$$

$$+ 2(\Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}) \right\} ,$$

$$F_{\sigma \sigma,\sigma}^{\pi,\pi}(k,q,i\omega) = \frac{1}{\beta} \sum_{i\omega} \frac{1}{E_{4,k+q,\sigma}^{u,\pi}} \left\{ g_{0,0}$$

$$+ (\xi_k + \xi_k + q + \xi_k + q + q)$$

$$+ 2(\Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}) \right\} ,$$

$$F_{\sigma \sigma,\sigma}^{\pi,\pi}(k,q,i\omega) = \frac{1}{\beta} \sum_{i\omega} \frac{1}{E_{4,k+q,\sigma}^{u,\pi}} \left\{ g_{0,0}$$

$$+ (\xi_k + \xi_k + q + \xi_k + q + q)$$

$$- 2(\Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}) \right\} ,$$

$$F_{\sigma \sigma,\sigma}^{\pi,\pi}(k,q,i\omega) = \frac{1}{\beta} \sum_{i\omega} \frac{1}{E_{4,k+q,\sigma}^{u,\pi}} \left\{ g_{0,0}$$

$$- \xi_k$$

$$+ \xi_k + \xi_k + \xi_k + q + \xi_k + q + q$$

$$+ \Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}$$

$$+ (\xi_k + \xi_k + q + \xi_k + q + q)$$

$$- 2(\Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}) \right\} ,$$

$$F_{\sigma \sigma,\sigma}^{\pi,\pi}(k,q,i\omega) = \frac{1}{\beta} \sum_{i\omega} \frac{1}{E_{4,k+q,\sigma}^{u,\pi}} \left\{ g_{0,0}$$

$$- \xi_k$$

$$+ \xi_k + \xi_k + \xi_k + q + \xi_k + q + q$$

$$- \Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}$$

$$+ (\xi_k + \xi_k + q + \xi_k + q + q)$$

$$- 2(\Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}) \right\} ,$$

$$F_{\sigma \sigma,\sigma}^{\pi,\pi}(k,q,i\omega) = \frac{1}{\beta} \sum_{i\omega} \frac{1}{E_{4,k+q,\sigma}^{u,\pi}} \left\{ g_{0,0}$$

$$- \xi_k$$

$$+ \xi_k + \xi_k + \xi_k + q + \xi_k + q + q$$

$$- \Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}$$

$$+ (\xi_k + \xi_k + q + \xi_k + q + q)$$

$$- 2(\Delta_{k+k+q}^{m} - \Delta_{k+k+q}^{m}) \right\} ;$$
By using Eqs. (B6) and (B12) we get in the simplified

\[ F_{\sigma', \sigma}(k, q, i\omega_l) = \frac{1}{\beta} \sum_{i\omega_n} \frac{1}{E_{a,l}} \left\{ g_{0,y} \right. \]

\[ + i\Delta^m_{k,\sigma}(\xi_{k+q} - \xi_{k+q+Q}) - i\Delta^m_{k,q,\sigma'}(\xi_k - \xi_{k+Q}) \left. \right\}, \]

\[ F_{\sigma', \sigma}(k, q, i\omega_l) = \frac{1}{\beta} \sum_{i\omega_n} \frac{1}{E_{a,l}} \left\{ g_{0,z} \right. \]

\[ + 2(\Delta^m_{k,\sigma}\Delta^\chi_{k+q,\sigma'} - \Delta^\chi_{k,\sigma}\Delta^m_{k+q,\sigma'}) \left. \right\}, \]

\[ F_{\sigma', \sigma}(k, q, i\omega_l) = \frac{1}{\beta} \sum_{i\omega_n} \frac{1}{E_{a,l}} \left\{ g_{0,z} \right. \]

\[ - 2(\Delta^m_{k,\sigma}\Delta^\chi_{k+q,\sigma'} - \Delta^\chi_{k,\sigma}\Delta^m_{k+q,\sigma'}) \left. \right\}, \]

\[ F_{\sigma', \sigma}(k, q, i\omega_l) = \frac{1}{\beta} \sum_{i\omega_n} \frac{1}{E_{a,l}} \left\{ g_{x,y} \right. \]

\[ + 2i(\Delta^m_{k,\sigma}\Delta^\chi_{k+q,\sigma'} + \Delta^\chi_{k,\sigma}\Delta^m_{k+q,\sigma'}) \left. \right\}, \]

\[ F_{\sigma', \sigma}(k, q, i\omega_l) = \frac{1}{\beta} \sum_{i\omega_n} \frac{1}{E_{a,l}} \left\{ - g_{x,y} \right. \]

\[ + 2i(\Delta^m_{k,\sigma}\Delta^\chi_{k+q,\sigma'} + \Delta^\chi_{k,\sigma}\Delta^m_{k+q,\sigma'}) \left. \right\}, \]

\[ F_{\sigma', \sigma}(k, q, i\omega_l) = \frac{1}{\beta} \sum_{i\omega_n} \frac{1}{E_{a,l}} \left\{ g_{x,z} \right. \]

\[ - \Delta^m_{k,\sigma}(\xi_{k+q} - \xi_{k+q+Q}) - \Delta^m_{k,q,\sigma'}(\xi_k - \xi_{k+Q}) \left. \right\}. \]

APPENDIX D: EXISTENCE OF A GOLDSTONE MODE

In this appendix we examine the contribution to the second order action \( S_2 \) given by \( m^x_{q+Q} \) and \( m^y_{q+Q} \) in the half filled, uniform and static limit

\[ x = 0, \quad q = 0, \quad i\omega_l = 0. \] (D1)

We write it as a part of \( S_2 \) \[ . \]

\[
S^{m x,y}_{2} = \left( M_{5, 5}(q = 0, Q, Q, i\omega_l = 0) + \frac{1}{2} \Pi_{5, 5}(q = 0, Q, Q, i\omega_l = 0) \right) \left[ \delta m^x_Q(\omega_l = 0) \right]^2 \\
+ \left( M_{6, 6}(q = 0, Q, Q, i\omega_l = 0) + \frac{1}{2} \Pi_{6, 6}(q = 0, Q, Q, i\omega_l = 0) \right) \left[ \delta m^y_Q(\omega_l = 0) \right]^2 \\
= \alpha \tilde{J}_{q = 0} \left( \delta m^x_Q \right)^2 + \alpha \tilde{J}_{q = 0} \left( \delta m^y_Q \right)^2 + \frac{1}{2N_s} \sum_k \sum_{\sigma, \sigma'} \left[ - \alpha \tilde{J}_{q = 0} \sigma^x_{\sigma, \sigma'} \right] F_{\sigma', \sigma} \left[ - \alpha \tilde{J}_{q = 0} \sigma^x_{\sigma, \sigma'} \right] \left( \delta m^x_Q \right)^2 \\
+ \frac{1}{2N_s} \sum_k \sum_{\sigma, \sigma'} \left[ - \alpha \tilde{J}_{q = 0} \sigma^y_{\sigma, \sigma'} \right] F_{\sigma', \sigma} \left[ - \alpha \tilde{J}_{q = 0} \sigma^y_{\sigma, \sigma'} \right] \left( \delta m^y_Q \right)^2. \tag{D2} \]

By using Eqs. (E6) and (E12) we get in the simplified

\[
\sigma_{2}^{m x,y} = \left( 2\alpha J - \frac{1}{N_s} \sum_k \frac{8\alpha^2 J^2}{E_{k, up} - E_{k, low}} \right) \times \left[ \left( \delta m^x_Q \right)^2 + \left( \delta m^y_Q \right)^2 \right]. \tag{D3} \]

The mean field equation \[ related to the magnetization gives

\[ \frac{1}{N_s} \sum_k \frac{8\alpha^2 J^2}{E_{k, up} - E_{k, low}} = 2\alpha J, \]

then with Eq. (D5) we have finally: \( S^{m x,y}_{2} = 0 \).
The behaviour of the second order action concerning the \( x \) and \( y \) directions of the magnetization implies that the transverse spin-spin correlation function \( \chi^\pm \) contains a gapless pole. It is a consequence of the Goldstone theorem\(^2\) which has to be applied in the present case because of the rotational symmetry breaking in spin space due to the imposed AF order. It is in agreement with the effective field theory of quantum antiferromagnets, which was built by Haldane for one-dimensional chains\(^3\) and extended later to square lattice systems\(^4\). The existence of this Goldstone mode has already been observed in the original spin-bag approach\(^4\).

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