Characterization of hyperchaotic states in the parameter-space of a modified Lorenz system

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Abstract. In this paper we report a new four-dimensional autonomous system, constructed from a Lorenz system by introducing an adequate feedback controller to the third equation. We use a numerical method that considers the second largest Lyapunov exponent value as a measure of hyperchaotic motion, to construct a two-dimensional parameter-space color plot for this system. Different levels of hyperchaos are represented in this plot by a continuously changing yellow-red scale. Practical applications of this plot includes, by instance, walking in the parameter-space of hyperchaotic systems along suitable paths.

1. Introduction
Historically, hyperchaos was first presented by Rössler [1] to characterize a chaotic system with at least two positive Lyapunov exponents. Some years after, the existence of hyperchaos was experimentally confirmed [2, 3, 4]. The dynamics of a hyperchaotic system is expanded in two or more directions simultaneously, resulting in a more complex attractor than a chaotic attractor, which has only one positive Lyapunov exponent. This expansion of the dynamics, happening at the same time in at least two directions makes hyperchaotic systems to have better performance in many chaos based fields, including technological applications, when compared to chaotic systems. For instance, hyperchaotic systems, due to the much more complicated structure of the attractors, can be used to improve the security in chaotic communication systems, where a chaotic signal is used to mask the message to be transmitted. It was shown that a message masked by a chaotic system is not always secure [5].

Hyperchaotic systems are common in many fields such as nonlinear circuits [6, 7], secure communications [8, 9], lasers [10, 11], colpitts oscillators [12], control [13, 14, 15], synchronization [16, 17, 18], and in a quantum cellular neural network [19]. Hyperchaos is generally present in high-dimensional nonlinear dynamical systems [20], and its occurrence is related with the loss of shadowing of chaotic trajectories on account of a strong form of non-hyperbolicity called unstable dimension variability [21]. A standard way to investigate the dynamics of a hyperchaotic system is by modeling them with differential equations. If we prefer to use autonomous first order ordinary differential equations to model a hyperchaotic system, we need to consider at least four these equations. In a hyperchaotic four-dimensional dissipative system, there is only one possibility to the Lyapunov exponents spectrum: two are positive, one is null, and one is negative.
In this paper we consider the magnitude of the second largest Lyapunov exponent to numerically characterize the points with hyperchaotic behavior in a two-dimensional parameter-space of a dynamical system modeled by a set of four nonlinear autonomous first order ordinary differential equations. Each point of this parameter-space is painted with a color that indicates the level of hyperchaos of the point. Here we report specific results obtained for a system constructed by us, from a Lorenz system, by introducing an adequate feedback controller to the third equation. This new controlled system is given by

\[
\begin{align*}
\dot{x} &= -\sigma(x - y), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - bz + w, \\
\dot{w} &= -a(x^2 + w),
\end{align*}
\] (1)

where \(x, y, z, w\) represent dynamical variables, and \(\sigma > 0, r > 0, b > 0,\) and \(a > 0\), are parameters.

The paper is organized as follows. In Sec. 2 we discuss some basic properties of this new system, and show that the trivial equilibrium point can be unstable. In Sec. 3 numerical results involving Lyapunov exponents spectrum and phase-space portraits are presented. Finally, the paper is summarized in Sec. 4.

2. Some Properties of the System (1)

The divergence of the vector field (1) is given by

\[
\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{w}}{\partial w} = -(\sigma + b + 1 + a),
\] (2)

from where we conclude that the system (1) is always dissipative and, therefore, that the phase-space contracts volumes as the time increases. As a consequence, all the bounded system trajectories finally settle onto an attractor in a four-dimensional phase-space.

The equilibrium points of the system (1) are calculated by doing \(\dot{x} = \dot{y} = \dot{z} = \dot{w} = 0\), that is, by solving the set of coupled equations

\[
\begin{align*}
-\sigma(x - y) &= 0, \\
rx - y - xz &= 0, \\
xy - bz + w &= 0, \\
-a(x^2 + w) &= 0,
\end{align*}
\] (3)

for \(x, y, z,\) and \(w\). the origin \(P_0 = (0, 0, 0, 0)\) is an equilibrium point. The Jacobian matrix for system (1) at \(P_0\), denoted by \(J_0\), is given by

\[
J_0 = \begin{pmatrix}
-\sigma & 0 & 0 & 0 \\
r & -1 & 0 & 0 \\
0 & 0 & -b & 1 \\
0 & 0 & 0 & -a
\end{pmatrix},
\]

and the characteristic equation, calculated using \(\det(J_0 - mI) = 0\), where \(m\) represents the eigenvalues and \(I\) is the standard \(4 \times 4\) identity matrix, is

\[
(m + a)(m + b)(m^2 + (1 + \sigma)m + (1 - r)\sigma) = 0,
\] (4)
from where we obtain the eigenvalues

\[ m_1 = -a, \quad m_2 = -b, \quad \text{and} \quad m_\pm = \frac{-\sigma - 1 \pm \sqrt{(\sigma - 1)^2 + 4r\sigma}}{2}. \]

The origin \( P_0 \) is a stable equilibrium point if the real part of its corresponding eigenvalues is negative. Take into account that \( a > 0, \ b > 0, \) and \( \sigma > 0, \) we conclude that \( m_1, \ m_2, \) and \( m_+ \) are always less than zero. Otherwise, for \( r > 1, \) \( m_+ = \frac{-(\sigma - 1) + \sqrt{(\sigma - 1)^2 + 4r\sigma}}{2} \) is greater than zero. Therefore, \( P_0 \) is an unstable saddle-node point for \( r > 1, \) a necessary condition to the occurrence of chaos, hyperchaos, and limit cycles in system (1).

3. Some Numerical Results

Figure 1 shows two parameter-space plots displaying different dynamical behaviors for system (1). Both plots were obtained by computing Lyapunov exponents \( \lambda \) on a \( 500 \times 500 \) mesh.

**Figure 1.** Regions of different dynamical behaviors in \((a,r)\) parameter-space of system (1), for \( b = 8/3, \ \sigma = 10. \) (a) The largest Lyapunov exponent. (b) The second largest Lyapunov exponent. In each plot colors represent the magnitude of the Lyapunov exponent, as shown in the respective column at right. About the points \( \beta, \ \gamma, \ \chi, \) and \( \delta, \) see the text.
of parameters \((a,r)\). In Fig. 1(a) is considered the largest Lyapunov exponent, while Fig. 1(b) considers the second largest Lyapunov exponent, for \(b = 8/3\), \(\sigma = 10\), \(1 < a < 6\), and \(6 < r < 30\). In these and further plots, system (1) was always integrated with a fourth-order Runge-Kutta algorithm with a fixed step size equal to \(10^{-3}\), and considering \(5 \times 10^5\) steps to compute each exponent. Furthermore, with respect to the initial conditions, every orbit in the phase-space, for each one of the \(2.5 \times 10^5\) pairs \((a,r)\), was started from the same \((x_0,y_0,z_0,w_0) = (10,10,10,10)\), that is, we do not follow the attractor.

Colors are associated with the magnitude of the Lyapunov exponent. Blue-white for more negative, black for zero, and red for more positive. Indeed, a positive exponent is indicated by a continuously changing yellow-red scale, while a negative exponent is indicated by a continuously changing blue-white scale. Note that the color scale of each one diagram was redefined in each individual plot: in plot shown in Fig. 1(a) \(-0.5 < \lambda < 2.5\), while in Fig. 1(b) \(-9 < \lambda < 1\). A negative largest Lyapunov exponent indicates a stable equilibrium point, a zero largest Lyapunov exponent indicates a stable periodic attractor (or a quasiperiodic attractor, when the second largest Lyapunov exponent is also equal to zero), and a chaotic attractor has a positive largest Lyapunov exponent.

Now we concentrate our attention in the last case, that is, when the largest Lyapunov exponent is positive. In consequence, there are only two possibilities to the second largest Lyapunov exponent (the third largest Lyapunov exponent is null, and the minor is negative): a positive value, which indicates hyperchaotic motion with two positive Lyapunov exponents, and a zero value indicating chaotic motion. Therefore, from above said, we conclude that the points pertaining to the yellow-red region in parameter-space of Fig. 1(b), are characterized by hyperchaotic motion. In other words, we numerically characterize hyperchaotic behavior in the parameter-space of the set of four nonlinear autonomous first order ordinary differential equations (1), by looking for the positive second largest Lyapunov exponent. For all set of parameters \((a,r)\) that this exponent is greater than zero, the motion is hyperchaotic. Note that the yellow-red region in Fig. 1(b) is a subset of the region with same shading in Fig. 1(a). It is a consequence of the fact that if the second largest Lyapunov exponent is greater than zero, then the largest Lyapunov exponent also is greater than zero.

**Figure 2.** Regions of different dynamical behaviors in \((a,r)\) parameter-space of system (1), for \(b = 8/3\), \(\sigma = 10\), when the third largest Lyapunov exponent is considered. About the points \(\beta\), \(\gamma\), \(\chi\), and \(\delta\), see the text.
Figure 2 shows the parameter-space plot obtained when the third largest Lyapunov exponent is considered. It is useful, together with plots in Fig. 1, to characterize all possible dynamical behaviors presented by system (1), namely equilibrium, periodicity, quasiperiodicity, chaos, and hyperchaos. For instance, a two-torus region (quasiperiodic region), for which the two largest Lyapunov exponents are null and the third is negative, is concerned with a common region painted in black in plots of Fig. 1, and painted in blue-white in plot of Fig. 2.

In Fig. 3 one sees four two-dimensional projections of typical phase-space trajectories

\[ \begin{align*}
  &\text{(a) Point } \chi, \text{ inside a periodic region.} \\
  &\text{(b) Point } \gamma, \text{ inside a quasiperiodic region.} \\
  &\text{(c) Point } \delta, \text{ inside a chaotic region.} \\
  &\text{(d) Point } \beta, \text{ inside a hyperchaotic region.}
\end{align*} \]

\[ \begin{align*}
  &\text{(attractors) in } w - x \text{ plane, which are result of numerical simulations for the four points, } \\
  &\chi, \gamma, \delta, \text{ and } \beta, \text{ shown in the } a - r \text{ parameter-spaces of Figs. 1 and 2, and whose coordinates are } \\
  &(a,r)=(4.0,20.750), (a,r)=(3.675,22.78125), (a,r)=(4.5,17.625), \text{ and } (a,r)=(3.625,23.09375), \\
  &\text{respectively. Shown are a cycle-3 periodic orbit in Fig. 3(a), a quasiperiodic orbit in Fig. 3(b),} \\
  &\text{a chaotic orbit in Fig. 3(c), and a hyperchaotic orbit in Fig. 3(d). Our motivation to include} \\
  &\text{these diagrams here is that they give us an opportunity to observe the structure of each type} \\
  &\text{of orbit. All attractors were constructed with } 20 \times 10^3 \text{ points, and from the initial condition} \\
  &(x_0,y_0,z_0,w_0)=(10,10,10,10). \\
  &\text{The complete Lyapunov exponents spectrum for points } \chi, \gamma, \delta, \text{ and } \beta \text{ is given in Table 1,} \\
  &\text{where one sees that this results corroborate all said above.}
\end{align*} \]
| Point | Attractor       | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|-------|----------------|-------------|-------------|-------------|-------------|
| $\chi$ | Periodic       | 0.017       | -1.403      | -1.562      | -22.414     |
| $\gamma$ | Quasiperiodic  | 0.014       | 0.005       | -0.446      | -24.598     |
| $\delta$ | Chaotic       | 1.848       | 0.012       | -0.880      | -27.185     |
| $\beta$    | Hyperchaotic   | 0.729       | 0.310       | 0.014       | -25.961     |

Table 1. Lyapunov exponents spectrum for points $\chi$, $\gamma$, $\delta$, and $\beta$.

4. Summary
In summary, in this paper we have reported the usage of the second largest Lyapunov exponent value as a measure of hyperchaotic motion, to construct two-dimensional parameter-space color plots for a system modeled by a set of four nonlinear autonomous first order ordinary differential equations, namely a system constructed from the paradigmatic Lorenz system by introducing an adequate feedback controller to the third equation. The method consists in to associate colors to the magnitude of the above-mentioned Lyapunov exponent. More specifically, we have used here a continuously changing yellow-red scale to indicate a positive second largest Lyapunov exponent. It should be pointed out that is not necessary knowledge of the model equations. In the case of an experimental time series, for instance, the Lyapunov exponents spectrum can be estimated by applying techniques for nonlinear time series analysis and, in consequence, the parameter-space can be constructed in the same manner.

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