Representation of non-special curves of genus 5 as plane sextic curves and its application to finding curves with many rational points

Momonari Kudo* and Shushi Harashita†

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Abstract

In algebraic geometry, it is important to provide effective parametrizations for families of curves, both in theory and in practice. In this paper, we present such an effective parametrization for the moduli of genus-5 curves that are neither hyperelliptic nor trigonal. Subsequently, we construct an algorithm for a complete enumeration of non-special genus-5 curves having more rational points than a specified bound, where “non-special curve” means that the curve is non-hyperelliptic and non-trigonal with mild singularities of the associated sextic model that we propose. As a practical application, we implement this algorithm using the computer algebra system MAGMA, specifically for curves over the prime field of characteristic 3.

1. Introduction

Let $K$ be a field and let $\mathbb{P}^n_K$ denote the projective $n$-space over $K$. Parameterizing the space of curves over $K$ of given genus is of significance in algebraic geometry, number theory, and arithmetic geometry. For the hyperelliptic case, it is well-known that any hyperelliptic curve over $K$ of genus $g$ is the normalization of $y^2 = f(x)$ for a square-free polynomial $f(x) \in K[x]$ of degree $2g + 1$ or $2g + 2$, if the characteristic of $K$ is odd. Over the algebraic closure $\overline{K}$, we can make $f(x)$ monic with $\deg f = 2g + 1$, eliminate one coefficient and normalize another to yield $2g - 1$ parameters. The dimension of the moduli space is $2g - 1$, resulting in effective parametrizations. However, for a family of non-hyperelliptic curves, it is not easy in general to find their defining equations with the fewest possible parameters, by choosing a suitable model of the family. Note that for $g \geq 2$, a curve is non-hyperelliptic if and only if the canonical sheaf is very ample (cf. [6, Chap. IV, Prop. 5.2]). For a non-hyperelliptic curve, its image of the canonical embedding is called a canonical curve.

For genus 3, a suitable parametrization is reported in [13]. For genus 4, a canonically embedded curve is the complete intersection of a quadric and a cubic in $\mathbb{P}^3_K$. In [7] and [10], we discussed

\*Faculty of Information Engineering, Fukuoka Institute of Technology, Japan
†Graduate School of Environment and Information Sciences, Yokohama National University, Japan

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reductions of the space of pairs of quadratic forms and cubic forms, where reduction means representing the space by as few parameters as possible up to the action of the projective general linear group \( \text{PGL}_4(K) \) of degree 4.

The next target is the case of \( g = 5 \). In this case, it is known that there are two types of non-hyperelliptic curves; trigonal or non-trigonal. For the trigonal case, we previously studied a quintic model in \( \mathbb{P}^2_K \) of a trigonal curve of genus 5 in order to enumerate superspecial ones over small finite fields \([9]\). Under some assumptions, we presented reductions of quintic forms defining trigonal curves of genus 5; see \([9, \text{Section 3}] \) for details. The remaining case considers that the curve is non-hyperelliptic and non-trigonal, and it is known that the curve is realized as the complete intersection of three quadratic hypersurfaces in \( \mathbb{P}^4_K \) (cf. \([6, \text{Chap. IV, Example 5.5.3 and Exercises 5.5}] \)). This case is significantly more difficult than the hyperelliptic and trigonal cases, since we need many parameters to define three quadratic forms in five variables. Although it is natural to reduce the parameters by using the natural action by \( \text{PGL}_5(K) \), it would be extremely hard to give an efficient reduction in this manner, since the group \( \text{PGL}_5(K) \) is large and complicated.

In this paper, we propose an effective parametrization for the space of non-hyperelliptic and non-trigonal curves of genus 5. Specifically, it will be proved that any non-hyperelliptic and non-trigonal curve \( C \) of genus 5 is bi-rational to a sextic \( C' = V(F) \) in \( \mathbb{P}^2_K \). Note that experimentally in most cases, \( C' \) has five double points. In such a case, we say that \( C \) is \emph{non-special}; see Definition 2.2.3 (and Lemma 2.2.2) for the rigorous definition. We also show that the dimension of the space of non-special curves with fixed singularities is at most 12, which is precisely the dimension of the moduli space of curves of genus 5. Specifically, the coefficients of \( F \) have linear expressions in 12 parameters, and the expressions can all be computed.

As an application of this parametrization, we present an algorithm to enumerate non-special curves \( C \) over a finite field \( K \) of genus 5 with a prescribed number of \( L \)-rational points, where \( L \) is a finite extension field of \( K \). Executing the algorithm for \( K = \mathbb{F}_3 \) and \( L = \mathbb{F}_9 \) on MAGMA, we obtain the following theorem:

**Theorem 1.** The maximal number of \( \mathbb{F}_9 \)-rational points on a non-special curve \( C \) of genus 5 defined over \( \mathbb{F}_3 \) (and not over \( \mathbb{F}_9 \)) is 32. Moreover, there are exactly four \( \mathbb{F}_9 \)-isogeny classes of Jacobian varieties of non-special curves \( C \) of genus 5 over \( \mathbb{F}_3 \) with 32 \( \mathbb{F}_9 \)-rational points, whose Weil polynomials are

1. \((t^2 + 2t + 9)(t^2 + 5t + 9)^4\
2. \((t + 3)^2(t^4 + 8t^3 + 32t^2 + 72t + 81)^2\
3. \((t + 3)^4(t^2 + 2t + 9)(t^2 + 4t + 9)^2\
4. \((t + 3)^6(t^2 + 2t + 9)^2\

In Section 4.2, examples of non-special curves \( C \) over \( \mathbb{F}_3 \) with \( \#C(\mathbb{F}_9) = 32 \) will be given.

As in the website manypoints.org \([5]\), the maximal number of \( \#C(\mathbb{F}_9) \) of curves \( C \) of genus 5 over \( \mathbb{F}_9 \) is unknown, but is known to belong between 32 and 35 (this upper bound is due to Lauter \([12]\)). On the website, three examples of \( C \) with 32 \( \mathbb{F}_9 \)-rational points are listed. The above theorem gives at least one new example. More concretely, the Weil polynomial of Fischer’s example \((x^4 + 1)y^4 + 2x^3y^3 + y^2 + 2xy + x^4 + x^2 = 0 \) is \((t^2 + 2t + 9)(t^2 + 5t + 9)^4\). In fact, this curve appears in our computation, since by dividing the example by \( x^4y^4 \) we obtain the sextic form
$1 + X^4 + 2XY + X^4Y^2 + 2X^3Y^3 + Y^4 + X^2Y^4$ (having distinct 5 singular points) with $X := 1/x$ and $Y := 1/y$. The example of [14] (submitted by Ritzenthaler to the site) $y^8 = a^2x^2(x^2 + a^7)$ with $a^2 + a + 2 = 0$ has the Weil polynomial $(t + 3)(t^2 + 2t + 9)^2$. This curve is defined over $\mathbb{F}_9$, but the above theorem finds a curve over the prime field $\mathbb{F}_3$ with the same Weil polynomial. From this theorem, we see that if one wants to find curves of genus 5 over $\mathbb{F}_9$ with $\#C(\mathbb{F}_9) > 32$, one needs to search those not defined over $\mathbb{F}_3$ or curves whose sextic models have more complex singularities.

The explicit parametrization and algorithm presented in this paper have the potential to yield valuable applications in both theoretical understanding and computational exploration. One possible application is the classification of non-hyperelliptic and non-trigonal curves of genus 5 with specific invariants (Hasse-Witt rank and Ekedahl-Oort type, etc.). Some open problems will be summarized in Section 5. One of our future works aims to enumerate superspecial non-special curves of genus 5 over finite fields.

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2. Non-hyperelliptic and non-trigonal curves of genus 5

In this section, we study genus-5 curves which are neither hyperelliptic nor trigonal. In order to parameterize these curves, we propose using the realization of these curves as (singular) plane sextic curves. This method is much more effective than using the realization by the complete intersection of three quadratic hypersurfaces in $\mathbb{P}^4$. Here we give an explicit construction of the sextic model for our purpose. Although one may find a more conceptional way to construct such a sextic model in a general curve in [1, Chap. VI, Exercises, F-24 on p. 275], it requires some assumptions and does not explain all of this section.

2.1. Sextic models

The canonical model of a non-hyperelliptic and non-trigonal curve of genus 5 is the intersection of three quadrics in $\mathbb{P}^4$. Let $C$ be such a curve, say $V(\varphi_1, \varphi_2, \varphi_3)$ in $\mathbb{P}^4$ for three quadratic forms $\varphi_1$, $\varphi_2$, and $\varphi_3$ in $x_0, x_1, x_2, x_3, x_4$. A sextic model associated to $C$ is obtained by additional data: two points $P$ and $Q$ on $C$. We find a short explanation of this construction in [6, Chap. IV, Example 5.5.3], but that appears away from the context of looking at the space of curves. The sextic model is defined by the scheme-theoretic image of the birational map defined by the (incomplete) linear system defined by the three-dimensional subspace consisting of elements of $H^0(C, \Omega_C)$ vanishing at $P$ and $Q$, but here we give an explicit construction toward parameterizing the space of these curves.
By a linear transformation, we may assume that
\[ P = (1 : 0 : 0 : 0 : 0) \quad \text{and} \quad Q = (0 : 0 : 0 : 0 : 1). \]
Since \( \varphi_i \) vanishes at \( P \) and \( Q \), the quadratic forms \( \varphi_i \) \((i = 1, 2, 3)\) must be of the form
\[ \varphi_i = a_i \cdot x_0 x_4 + f_i \cdot x_0 + g_i \cdot x_4 + h_i \] (2.1.1)
with \( a_i \in K \), where \( f_i \) and \( g_i \) are linear forms in \( x_1, x_2, x_3 \), and where \( h_i \) is a quadratic form in \( x_1, x_2, x_3 \). We shall find an equation only in \( x_1, x_2, x_3 \) from \( \varphi_1 = \varphi_2 = \varphi_3 = 0 \).

Put
\[ (v_1, v_2, v_3) := -(h_1, h_2, h_3) \cdot \Delta_A, \]
where \( \Delta_A \) is the adjugate matrix of
\[ A := \begin{pmatrix} a_1 & a_2 & a_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}. \]

Note that \( v_i \) \((i = 1, 2, 3)\) are polynomials in \( x_1, x_2, x_3 \). Since
\[ (v_1, v_2, v_3) = \det(A)(x_0 x_4, x_0, x_4) \] (2.1.2)
on \( C \), we have the sextic equation
\[ \det(A) \cdot v_1 - v_2 v_3 = 0 \] (2.1.3)
in \( x_1, x_2, x_3 \). Let \( C' \) be the curve defined by (2.1.3) in \( \mathbb{P}^2 = \text{Proj } K[x_1, x_2, x_3] \). We claim that this constructs a birational map from \( C \) to \( C' \). It suffices to see that \( \det A \neq 0 \) holds generically, since \( x_0 \) and \( x_4 \) are recovered from \( v_1, v_2, v_3 \) where \( \det A \neq 0 \) by (2.1.2). If \( \det A \) were identically zero, then from (2.1.1) we have \((3 - \text{rank} A)\) quadratic forms only in \( x_1, x_2, x_3 \). If \( \text{rank} A < 2 \), then this contradicts that \( C \) is irreducible. If \( \text{rank} A = 2 \), then let \( \psi \) be the quadratic form in \( x_1, x_2, x_3 \); then there is a dominant morphism \( C \to V(\psi) \subset \mathbb{P}^2 \), which turns out to be of degree 2. This contradicts the assumption that \( C \) is not hyperelliptic.

This construction of the sextic \( C' \) from \( C \) with \( P \) and \( Q \) has the following properties.

**Proposition 2.1.1.**
1. Suppose \( \langle \varphi_1, \varphi_2, \varphi_3 \rangle = \langle \phi_1, \phi_2, \phi_3 \rangle \) as a linear space over a field defining \( \varphi_i \) and \( \phi_i \) for all \( i = 1, 2, 3 \). Then \( C' \) obtained from \( \varphi_1, \varphi_2, \varphi_3 \) is the same as that obtained from \( \phi_1, \phi_2, \phi_3 \).
2. The coordinate change
\[ (x_0, x_1, x_2, x_3, x_4) \mapsto (x_4, x_1, x_2, x_3, x_0) \]
does not change the sextic.
3. The coordinate change
\[ (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0 + \phi, x_1, x_2, x_3, x_4 + \psi) \]
with any linear \( \phi, \psi \) in \( x_1, x_2, x_3 \) does not change the sextic.
Proof. (1) Let \( B \) be the square matrix of size 3 such that \((\varphi_1, \varphi_2, \varphi_3)B = (\phi_1, \phi_2, \phi_3)\). Let \( v'_i \) be \( v_i \) associated to \( \phi_1, \phi_2, \phi_3 \). Then

\[
(v'_1, v'_2, v'_3) = -(h_1, h_2, h_3)B \cdot \det(AB)(AB)^{-1} = \det(B)(v_1, v_2, v_3).
\]

Hence we have \( \det(AB)v'_1 - v'_2v'_3 = \det(B)(\det(A)v_1 - v_2v_3) \).

(2) The matrix \( A \) for the new coordinate is

\[
A' := \begin{pmatrix} a_1 & a_2 & a_3 \\ g_1 & g_2 & g_3 \\ f_1 & f_2 & f_3 \end{pmatrix}.
\]

Let \( v'_i \) be \( v_i \) associated to the new coordinate. It is straightforward to see \((v'_1, v'_2, v'_3) = -(v_1, v_3, v_2)\). Thus the sextic does not change.

(3) Let \( A', h'_i, \) and \( v'_i \) be the matrix \( A \), the quadratic form \( h_i \), and \( v_i \) for the new coordinate. Then we have

\[
A' = \begin{pmatrix} 1 & 0 & 0 \\ \psi & 1 & 0 \\ \phi & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}
\]

and \( h'_i = h_i + a_i\psi + f_i\phi + g_i\psi \). The equation \( \det(A')v'_1 - v'_2v'_3 = \det(A)v_1 - v_2v_3 \) is derived from a straightforward computation, which is tedious but has some beautiful cancellations. \( \square \)

From Proposition 2.1.1 (1) and (2), we have the following corollary:

**Corollary 2.1.2.** Let \( K \) be a field. If \( C \) and the divisor \( P + Q \) on \( C \) for distinct two points \( P \) and \( Q \) are defined over \( K \), then the associated sextic \( C' \) is defined over \( K \).

This corollary suggests us to use the sextic realization to find and/or enumerate curves over \( \mathbb{F}_q \) of genus 5 with many \( \mathbb{F}_{q^2} \)-rational points, since the assumption that \( P + Q \) is defined over \( \mathbb{F}_q \) is satisfied if \( C \) has many (a few) \( \mathbb{F}_{q^2} \)-rational points. The case we mainly treat in this paper is the case of \( q = 3 \). The maximal number \#\( C(\mathbb{F}_9) \) for curves of genus 5 over \( \mathbb{F}_9 \) is unknown, as mentioned in Section 1. We give an implementation of our algorithm, restricting ourselves to the case where \( C \) is defined over \( \mathbb{F}_3 \).

2.2. Non-special curves of genus 5

Let \( C, P, \) and \( Q \) be as in Subsection 2.1, especially \( C \) is non-hyperelliptic and non-trigonal. Let \( C' \) be the sextic associated to \((C, P + Q)\) obtained in Subsection 2.1. First we see that \( C' \) has singular points, because the arithmetic genus \( g(C') \) of \( C' \) is 10, but \( g(C) \) is 5. Note that \( C \) is isomorphic to the desingularization of \( C' \). In general, we have the following formula:

\[
g(C) \leq g(C') - \sum_{P' \in C'} \frac{m_{P'}(m_{P'} - 1)}{2}, \tag{2.2.1}
\]

where \( m_{P'} \) is the multiplicity of \( C' \) at \( P' \).

**Lemma 2.2.1.** The sextic \( C' \) does not have any singular point of multiplicity 3, i.e., \( m_{P'} \neq 3 \).
Proposition 2.2.5. If distinct four elements of \( \{P_1, P_2, P_3, P_4\} \) are contained in a line, then \( C' \) is geometrically reducible (i.e., \( C'_{\overline{K}} := C \times_{\text{Spec}(K)} \text{Spec}(\overline{K}) \) is reducible, where \( \overline{K} \) is the algebraic closure of \( K \)).

Proof. Suppose that \( P_1, \ldots, P_4 \) are contained in a line. By a linear transformation over \( \overline{K} \), we may assume

\[
\begin{align*}
P_1 &= (0 : 0 : 1), & P_2 &= (1 : 0 : 1), & P_3 &= (c : 0 : 1), & P_4 &= (d : 0 : 1),
\end{align*}
\]

where 0, 1, \( c \), and \( d \) are mutually distinct. Let \( f \) be the sextic (in \( x \) and \( y \)) of \( C' \) obtained by substituting 1 for \( z \). Since \( P_1 \) is a singular point, the smallest degree of the non-zero terms of \( f \) is two. Let \( a_i \) be the \( x^iy^0 \)-coefficient of \( F \) for \( i = 2, 3, 4, 5, 6 \). Then the constant term and the
degree-one term of the Taylor expansion of \( f \) at \((x, y) = (b, 0)\) are \( \sum_{i=2}^{6} b^{i}a_i \) and \( \sum_{i=2}^{6} i(b-1)^{i-1}a_i \) for \( b \in \{1, c, d\} \). These are zero, as \( C \) is singular at \( P_i \) for \( i = 1, \ldots, 5 \). Since

\[
\det \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
c^2 & c^3 & c^4 & c^5 & c^6 \\
2c & 3c^2 & 4c^3 & 5c^4 & 6c^5 \\
d^2 & d^3 & d^4 & d^5 & d^6 \\
2d & 3d^2 & 4d^3 & 5d^4 & 6d^5 \\
\end{pmatrix} = c^4d^4(c-1)^2(d-1)^2(c-d)^4,
\]

we have that \( a_i \) are zero for all \( i = 2, \ldots, 6 \). This says that \( f \) is divided by \( y \), in particular \( C' \) is geometrically reducible.

Let us study the space of \( C' \) having five singular points with multiplicity 2. The number of monomials of degree 6 in three variables is 28. For each singular point, we have three linear equations which assure that the point is singular. Considering a scalar multiplication to the whole sextic, the number of free parameters is \( 28 - 5 \times 3 - 1 = 12 \), see Proposition 2.2.6 below for the linear independence of \( 5 \times 3 \) linear equations. This is precisely the dimension of the moduli space of curves of genus 5. Note that both of the dimension of the space of two points on \( C \) and that of five points on \( \mathbb{P}^2 \) modulo the action of \( \text{PGL}_3 \) are equal to two. This says that the parametrization by the sextic models is very effective.

**Proposition 2.2.6.** Let \( \{P_1, \ldots, P_3\} \) be distinct five points of \( \mathbb{P}^2(\mathbb{K}) \). Assume that any distinct four points in \( \{P_1, \ldots, P_3\} \) are not contained in a line. Then the space of sextics with double points at \( P_1, \ldots, P_5 \) up to scalar multiplications is of dimension 12.

**Proof.** Renumbering the subscripts of the five points if necessary, we may assume that \( P_1, P_2, \) and \( P_3 \) are not in a line. By a linear transformation by an element of \( \text{PGL}_3(\mathbb{K}) \), we may also assume that \( P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0), \) and \( P_3 = (0 : 0 : 1) \). Considering the permutation of \( \{x, y, z\} \) by the symmetric group of degree 3, we may assume that \( P_4 = (b : c : 1) \) and \( P_5 = (d : e : 1) \), by the assumption of four points. Then, any sextic with \( z = 1 \) having singularity at \( P_1, P_2, \) and \( P_3 \) is of the form

\[
F = a_1x^4y^2 + a_2x^4y + a_3x^4 + a_4x^3y^3 + a_5x^3y^2 + a_6x^3y + a_7x^3 + a_8x^2y^4 + a_9x^2y^3 + a_{10}x^2y^2 + a_{11}x^2y + a_{12}x^2 + a_{13}xy^4 + a_{14}xy^3 + a_{15}xy^2 + a_{16}xy + a_{17}y^4 + a_{18}y^3 + a_{19}y^2.
\]

The condition that this sextic is singular at \( P_4 \) (resp. \( P_5 \)) is described by three linear equations in \( a_{11}, a_{19} \) obtained from the condition that \( F, \frac{\partial F}{\partial x}, \) and \( \frac{\partial F}{\partial y} \) are zero at \( P_4 \) (resp. \( P_5 \)). The coefficients of these six linear equation make a \( 6 \times 19 \) matrix \( M \). It suffices to show that \( M \) is of rank 6. By a direct computation, the determinant of the minor (square matrix of size 6) corresponding to the coefficients of \( a_7, a_{11}, a_{12}, a_{15}, a_{18}, a_{19} \) is \( (be - cd)^5(b + cd) \), that of \( a_6, a_7, a_9, a_{10}, a_{11}, a_{12} \) is \( b^5d^5(c - e)^5 \), and that of \( a_5, a_{10}, a_{14}, a_{15}, a_{18}, a_{19} \) is \( c^6e^6(b - d)^5 \). If the rank of \( M \) were less than 6, then the equalities \( (be - cd)^5(b + cd) = 0, b^5d^5(c - e)^5 = 0, \) and \( c^6e^6(b - d)^5 = 0 \) have to hold. However, this contradicts the assumption of four points. Indeed, if \( b = 0 \), then we have \( c = 0 \) or \( d = 0 \) by the first equation, and then \( P_1 = P_4 \) holds or \( P_1, P_3, P_4, \) and \( P_5 \) is contained in the line \( y = 0 \). Thus \( b \neq 0 \). Similarly \( c, d, \) and \( e \) are not zero. Then we have \( b = d \) and \( c = e \), which says \( P_3 = P_4 \). This is absurd. Hence \( M \) has to be of rank 6.
Remark 2.2.7. Let $\mathcal{M}_g$ be the moduli space of curves of genus $g$. Let $S$ be the subvariety of $\mathcal{M}_5$ consisting of non-hyperelliptic and non-trigonal curves. Let $C' \to S$ be the family of sextic curves $C'$ in $\mathbb{P}^2$ whose desingularization $C \to C'$ satisfies (2.2.2). The family $C \to S$ obtained by the fiberwise desingularization of $C' \to S$ defines $f : S \to S$. It follows from the construction that $f$ is surjective over every algebraically closed field, in other words $C \to S$ is geometrically surjective in the sense of [13, Definition 2.1]. The fiber of $f$ is related to the space of choices of two points on $C$. It would be interesting to give a precise description of the fiber.

The next remark explains what (2.2.2) means.

Remark 2.2.8. We claim that if the desingularization $C \to C'$ satisfies (2.2.2), then, at each singular point $P \in C'$, the desingularization of $C'$ at $P$ is obtained by the one-time blow-up centered at $P$. Indeed, by a linear coordinate change, the structure ring around $P$ is locally described as $\mathbb{F}_q[X,Y]/(G)$ with $G \in \mathbb{F}_q[X,Y]$, where $P$ is the origin. Write $G = G_m + G_{m+1} + \cdots$, where $m = m_P$ is the multiplicity at $P$, and where $G_i$ is the homogenous part of degree $i$. By a linear coordinate change again, we may assume that the $Y^m$-coefficient of $G_m$ is not zero if we assume $q - 1 \geq m$ (which is satisfied in our case: $q = 9$ and $m \leq 3$). Then, the morphism from the strict transform of the blow-up to $C'$ is locally described by the ring homomorphism

$$\mathbb{F}_q[X,Y]/(G) \to \mathbb{F}_q[X,Z]/(\tilde{G})$$

with $\tilde{G}(X,Z) = G(X,ZX)X^{-m}$ sending $(X,Y)$ to $(X,XZ)$. Its cokernel is the $\frac{m(m-1)}{2}$-dimensional $\mathbb{F}_q$-vector space generated by $Z^iX^j$ for $1 \leq i \leq m - 1$ and $0 \leq j \leq i - 1$. Thus, we have the claim.

Finally, we give a formula on the number of rational points on a non-special curve $C$ over a finite field $\mathbb{F}_q$.

Proposition 2.2.9. We have

$$\# C(\mathbb{F}_q) = \# C'(\mathbb{F}_q) + \sum_{P \in C'(\mathbb{F}_q)} \left( \# V(h_P)(\mathbb{F}_q) - 1 \right), \quad (2.2.3)$$

where $h_P$ is the $G_m$ as in Remark 2.2.8 for $P$, i.e., the homogeneous part of the least degree (i.e., $m_P$) of the Taylor expansion at $P$ of the defining polynomial of an affine model containing $P$ of $C'$, and where $V(h_P)$ is the closed subscheme of $\mathbb{P}^1$ defined by the ideal $(h_P)$. Note that $\# V(h_P)(\mathbb{F}_q) - 1 = 0$ if $C'$ is nonsingular at $P$.

Proof. Let $\pi : C \to C'$ be the desingularization of $C'$. As $\pi$ maps $C(\mathbb{F}_q)$ to $C'(\mathbb{F}_q)$, it suffices to count the $\mathbb{F}_q$-rational points on the fiber of each point $P \in C'(\mathbb{F}_q)$. As in Remark 2.2.8, by a linear coordinate change, we locally describe the structure ring around $P$ as $\mathbb{F}_q[X,Y]/(G)$ with $G \in \mathbb{F}_q[X,Y]$, where $P$ is the origin. By the construction of the desingularization using blow-ups, $\pi^{-1}(P)$ is the scheme obtained by glueing in $\mathbb{P}^1$ the subscheme of $\mathbb{P}^1$ defined by $h_P(1,Z) = 0$ for $(1 : Z) \in \mathbb{P}^1$, where $h_P(1,Z) = G_m(Z,X)X^{-m}$ and the subscheme of $\mathbb{P}^1$ defined by $h_P(W,1) = 0$ for $(W : 1) \in \mathbb{P}^1$, where $h_P(W,1) = G_m(WY,Y)Y^{-m}$. (Note that the fiber of $P$ does not depend on the higher-degree part of $G$.) Hence $\pi^{-1}(P)$ is isomorphic to $V(h_P)$ over $\mathbb{F}_q$. Clearly the proposition follows from this fact. \hfill $\Box$

If $h_P$ in Proposition 2.2.9 is quadratic, then $\# V(h_P)(\mathbb{F}_q) - 1$ is equal to 1 if the discriminant $\Delta(h_P)$ of $h_P$ is a nonzero square and to $-1$ if $\Delta(h_P)$ is a nonzero non-square and to 0 if $\Delta(h_P) = 0$. 
From a computational point of view, it may have a little advantage to use the following fact: If \( q = p^2 \) and if \( \Delta(h_P) \) belongs to \( \mathbb{F}_p \), then \( \Delta(h_P) \neq 0 \) is equivalent to that \( \Delta(h_P) \) is a nonzero square in \( \mathbb{F}_p^2 \).

**Remark 2.2.10.** The case of other kinds of singularities (i.e., the number of the singular points of \( C' \) is less than 5) is one of our future works, see Section 5. As in the non-special case, one could also construct an algorithm to find a reduced equation defining \( C' \) even in other cases, once the type of the singularities is known. However, the number of singular points of \( C' \) can take any of 5, 4, 3, 2, and 1. For example, consider \( C \) defined by

\[
\begin{align*}
\varphi_1 &= x_0x_1 + x_3x_4, \\
\varphi_2 &= x_0x_3 + x_2x_4 + x_1^2, \\
\varphi_3 &= x_0x_4 + x_2^2 + x_2x_3
\end{align*}
\]

in characteristic 2. Then \( C' \) is defined by

\[x_1^5x_3 + x_1^2x_4^2 + x_1^2x_2x_3^2 + x_2^2x_3^2 + x_2x_3^5.\]

Then \( C' \) has a single singular point \((0 : 1 : 0)\), whose multiplicity is two. This means that there are various kinds of singularities, and thus it would require much effort to classify their possible types.

## 3. A concrete algorithm

Let \( K \) be a finite field with characteristic \( p \), and \( K' \) a finite extension field of \( K \). As it was shown in Subsection 2.1, every non-hyperelliptic and non-trigonal curve of genus 5 over \( K \) is realized as the normalization of a sextic in \( \mathbb{P}^2 \). In particular, such a curve \( C \) is said to be non-special if it satisfies the condition stated in Definition 2.2.3 of Subsection 2.2.

In this section, we first present a concrete algorithm (Algorithm 3.1.1 below) for enumerating all non-special curves \( C \) of genus 5 over \( K \) satisfying \( \#C(K') \geq N \), where \( N \) is a given positive integer. After presenting the algorithm, we also give some remarks for implementation.

### 3.1. Algorithm

Let \( L \) be a finite extension field of \( K \), and let \( \{P_1, P_2, P_3, P_4, P_5\} \) be a set of distinct five points in \( \mathbb{P}^2(L) \). Assume that \( \text{Gal}(L/K) \) stabilizes the set \( \{P_1, P_2, P_3, P_4, P_5\} \). Given \( \{P_1, P_2, P_3, P_4, P_5\} \) and an integer \( N \geq 1 \), Algorithm 3.1.1 presented below enumerates all sextic forms \( F \) in \( K[x, y, z] \) such that the projective scheme \( C' : F = 0 \) is a singular (irreducible) curve of geometric genus 5 in \( \mathbb{P}^2 \) with \( \text{Sing}(C') = \{P_1, P_2, P_3, P_4, P_5\} \) and \( \#C(K') \geq N \), where \( C \) is the normalization of \( C' \).

**Algorithm 3.1.1.** Input. \( \{P_1, P_2, P_3, P_4, P_5\}, K' \), and \( N \geq 1 \).

Output. A set \( \mathcal{F} \) of sextic forms \( F \) in \( K[x, y, z] \) such that \( C' : F = 0 \) is a singular (irreducible) curve of geometric genus 5 in \( \mathbb{P}^2 \) with \( \text{Sing}(C') = \{P_1, P_2, P_3, P_4, P_5\} \) and \( \#C(K') \geq N \), where \( C \) is the normalization of \( C' \).

1. Set \( \mathcal{F} := \emptyset \).
2. Construct \( F = \sum_{i=1}^{28} \frac{a_i}{x^4}x^{\alpha^{(i)}}y^{\beta^{(i)}}z^{\gamma^{(i)}} \in K[a_1, \ldots, a_{28}][x, y, z] \) with
   \[
   \{(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)}), \ldots, (\alpha_1^{(28)}, \alpha_2^{(28)}, \alpha_3^{(28)})\} \in \{(\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{Z}_{\geq 0})^3 : \alpha_1 + \alpha_2 + \alpha_3 = 6\},
   \]
   where \( a_1, \ldots, a_{28} \) are indeterminates.
(3) For each $\ell \in \{1, \ldots, 5\}$:

(a) Write $P_\ell = (p_1^{(\ell)} : p_2^{(\ell)} : p_3^{(\ell)})$ for $p_1^{(\ell)}, p_2^{(\ell)}, p_3^{(\ell)} \in L$.

(b) Let $k$ be the minimal element in $\{1, 2, 3\}$ such that $p_k^{(\ell)} \neq 0$, and put $z_k = p_k^{(\ell)}$, $z_i = X$, and $z_j = Y$ for $i, j \in \{1, 2, 3\} \setminus \{k\}$ with $i < j$.

(c) Compute $F_\ell := F(z_1, z_2, z_3) \in L[a_1, \ldots, a_{28}][X, Y]$. Let $f_{3\ell-2}$ and $f_{3\ell-1}$ be the coefficients of $X$ and $Y$ in $F_\ell$ respectively, and $f_{3\ell}$ the constant term of $F_\ell$ as a polynomial in $X, Y$.

(4) Compute a basis $\{b_1, \ldots, b_d\} \subset K^{\oplus 28}$ for the null-space of the linear system over $L$ defined by $f_t(a_1, \ldots, a_{28}) = 0$ with $1 \leq t \leq 15$, where $d$ denotes the dimension of the null-space. By Proposition 2.2.6 we have $d = 13$.

(5) For each $(v_1, \ldots, v_d) \in K^{\oplus d} \setminus \{(0, \ldots, 0)\}$ with $v_1 \in \{0, 1\}$:

(a) Compute $c := \sum_{j=1}^d v_j b_j \in K^{\oplus 28} \setminus \{(0, \ldots, 0)\}$. For each $1 \leq i \leq 28$, we denote by $c_i$ the $i$-th entry of $c$.

(b) If $\sum_{i=1}^{28} a_{i,1} x^{a_{i,1}}_y y^{a_{i,2}} z^{a_{i,3}}$ is irreducible, and if $\sum_{i=1}^{28} c_i x^{a_{i,1}}_y y^{a_{i,2}} z^{a_{i,3}} = 0$ in $\mathbb{P}^2$ has geometric genus 5:

(i) Set $F_c := \sum_{i=1}^{28} c_i x^{a_{i,1}}_y y^{a_{i,2}} z^{a_{i,3}}$, and let $C'$ be the plane curve in $\mathbb{P}^2$ defined by $F_c = 0$.

(ii) Compute $\#C(K')$ by the formula (2.2.3) given in Subsection 2.2, where $C$ is the normalization of $C'$.

(iii) If $\#C(K') \geq N$, replace $\mathcal{F}$ by $\mathcal{F} \cup \{F_c\}$.

(6) Output $\mathcal{F}$.

**Remark 3.1.2.** (1) In Step (4), we can take a basis $\{b_1, \ldots, b_d\}$ so that $b_i \in K^{\oplus 28}$ for all $1 \leq i \leq d$, by the following general fact: Let $L/K$ be a separable extension of fields, and $E$ the Galois closure of $L$ over $K$. If a linear system over $L$ is Gal($E/K$)-stable, then each entry in the Echelon form of the coefficient matrix of the system belongs to $K$.

(2) In Step (5)(b)(ii), we need to compute the discriminant $\Delta_\ell$ of the degree-2 part of the Taylor expansion of $F_c$ at each $P_\ell$. To achieve this, we can use $F_c$ computed in Step (3) as follows: Let $D_\ell \in L[a_1, \ldots, a_{28}]$ denote the discriminant of the degree-2 part of $F_c$ as a polynomial in $X, Y$. Then clearly we have $\Delta_\ell = D_\ell(c)$.

### 3.2. Correctness of our algorithm and remarks for implementation

The correctness of Algorithm 3.1.1 follows mainly from the definition of multiplicity for a singular point on a projective plane curve: Indeed, each polynomial $F_\ell$ computed in Step (3) is equal to the Taylor expansion of $F$ at $P_\ell$. Since any vector $c$ defining $F_c$ in Step (5)(b)(i) is a root of the system $f_{3\ell-2} = f_{3\ell-1} = f_{3\ell} = 0$ for all $1 \leq \ell \leq 5$, it follows that $P_\ell$ is a singular point of $C'$: $F_c = 0$ with multiplicity 2 for each $1 \leq \ell \leq 5$. Conversely, suppose $c$ is a vector in $K^{\oplus 28} \setminus \{(0, \ldots, 0)\}$ such that $F_c = 0$ defines an irreducible plane curve of geometric genus 5 with multiplicity 2 at the points $P_1, \ldots, P_5$. By the definition of multiplicity for a singularity, the vector $c$ is a root of the system $f_{3\ell-2} = f_{3\ell-1} = f_{3\ell} = 0$ for all $1 \leq \ell \leq 5$. As it was described in Remark 3.1.2 (1), each entry in the Echelon form of the coefficient matrix of the system belongs to $K$, and thus $c$ is a root of a
linear system over $K$ whose null-space over $L$ is the same as that of $f_{3\ell-2} = f_{3\ell-1} = f_{3\ell} = 0$ for all $1 \leq \ell \leq 5$. Therefore, $\mathbf{c}$ can be expressed as $\mathbf{c} = \sum_{j=1}^{d} v_j \mathbf{b}_j$, where $(v_1, \ldots, v_d) \in K^{\oplus d} \setminus \{(0, \ldots, 0)\}$. Note that it suffices to compute the part with $v_1 \in \{0, 1\}$, since scalar multiplication does not change the isomorphism class of the sextic.

We implemented Algorithm 3.1.1 over MAGMA V2.25-8 [3], [4] in its 64-bit version. Details of our computational environment will be provided in Section 4.2. For Step (5)(b), our implementation makes use of MAGMA’s built-in functions such as IsIrreducible, GeometricGenus, and Variety. In particular, the function Variety is employed to compute $\#C'(K') = \#V(F_c)$.

4. Computational results in characteristic 3

This section presents our computational results in characteristic $p = 3$ obtained by applying Algorithm 3.1.1. For this, we first explicitly determine possible positions in $\mathbb{P}^2$ of singular points of the sextic model associated to a non-special curve of genus 5 over $K = \mathbb{F}_3$, considering the action by $\text{PGL}_3(\mathbb{F}_3)$. With the classification, we execute Algorithm 3.1.1 for each position type, and then obtain non-special curves of genus 5 over $\mathbb{F}_3$ with many $\mathbb{F}_9$-rational points.

4.1. Position analysis of singular points in the case where $K = \mathbb{F}_3$

In this subsection, we classify positions of five points $\{P_1, \ldots, P_5\}$ in $\mathbb{P}^2$ which can be singular points of our sextic model $C' : F = 0$ associated to a non-special curve of genus 5 over $\mathbb{F}_3$, where we identify two positions $\{P_1, \ldots, P_5\}$ and $\{P'_1, \ldots, P'_5\}$ if there exists an element $g$ of $\text{PGL}_3(\mathbb{F}_3)$ that sends $\{P_1, \ldots, P_5\}$ to $\{P'_1, \ldots, P'_5\}$. This is the problem to enumerate the orbits of an action of a finite group on a finite set. We solved it by using MAGMA. In the following, we state only the results.

The Frobenius map $\sigma$ over $\mathbb{F}_3$ (defined as the map raising each entry to the third power) makes a permutation of $\{P_1, \ldots, P_5\}$, i.e.,

$$\{P_1, \ldots, P_5\} = \{\sigma(P_1), \sigma(P_2), \sigma(P_3), \sigma(P_4), \sigma(P_5)\}.$$ 

The pattern of the Frobenius orbits in $\{P_1, \ldots, P_5\}$ is either of (1, 1, 1, 1, 1), (1, 1, 1, 2), (1, 2, 2), (1, 1, 3), (2, 3), (1, 4), and (5), where for example (1, 2, 2) means that $\{P_1, \ldots, P_5\}$ consists of three Frobenius orbits each of which has cardinality 1, 2, and 2 respectively.

Case (1,1,1,1,1) This is the case where every singular point is $\mathbb{F}_3$-rational. By a linear transformation by an element of $\text{PGL}_3(\mathbb{F}_3)$, we may assume

$$P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (0 : 0 : 1).$$

A computation shows that every position of $\{P_1, \ldots, P_5\}$ such that any four points among them are not contained in a line is equivalent by the linear transformation by a diagonal matrix and a permutation of $\{x, y, z\}$ to either of the following two cases:

(1) $P_4 = (1 : 1 : 0)$ and $P_5 = (0 : 1 : 1), \quad (2) P_4 = (1 : 1 : 0)$ and $P_5 = (1 : 2 : 1)$. 
**Example 4.1.1.** In the case of characteristic 3, the defining equation of any sextic having singular points \( P_1 = (0 : 0 : 1), P_2 = (0 : 1 : 0), P_3 = (1 : 0 : 0), P_4 = (1 : 1 : 0), P_5 = (0 : 1 : 1) \) is

\[
F = a_1 x^4 y^2 + a_2 x^4 y z + a_3 x^4 z^2 + a_4 x^3 y^3 + a_5 x^3 y^2 z + a_6 x^3 y z^2 + a_7 x^3 z^3 + a_8 x^2 y^4 + a_9 x^2 y^3 z + a_{10} x^2 y^2 z^2 + a_{11} x^2 y z^3 + a_{12} x^2 z^4 + a_{13} x y^4 z + a_{14} x y^3 z^2 + a_{15} x y^2 z^3 + a_{16} x y z^4 + a_{17} y^4 z^2 + a_{18} y^3 z^3 + a_{19} y^2 z^4
\]

with

\[
\begin{align*}
& a_1 + a_4 + a_8 = 0, & a_{17} + a_{18} + a_{19} = 0, \\
& 2a_1 + a_8 = 0, & a_{13} + a_{14} + a_{15} + a_{16} = 0, \\
& a_2 + a_5 + a_9 + a_{13} = 0, & a_{17} + 2a_{19} = 0.
\end{align*}
\]

The quadratic form \( h_p(= G_2) \) with the notation of Remark 2.2.8 and Proposition 2.2.9 associated to the singular point \( P_i \) (of multiplicity 2) is as follows:

\[
\begin{align*}
& h_{P_1} = a_{12} x^2 + a_{16} x y + a_{19} y^2, \\
& h_{P_2} = a_8 x^2 + a_{13} x z + a_{17} z^2, \\
& h_{P_3} = a_1 y^2 + a_2 y z + a_5 z^2, \\
& h_{P_4} = a_1 (y - x)^2 + (a_2 + 2a_5 + a_{13})(y - x)z + (a_3 + a_6 + a_{10} + a_{14} + a_{17})z^2, \\
& h_{P_5} = (a_8 + a_9 + a_{10} + a_{11} + a_{12})x^2 + (a_{13} + 2a_{14} + a_{16})x(y - z) + a_{17}(y - z)^2
\end{align*}
\]

respectively.

**Case (1,1,1,2) with linearly independent** \( P_1, P_2, P_3 \) We consider the case where \( \{P_1, \ldots, P_5\} \) contains three \( \mathbb{F}_3 \)-rational points, say \( P_1, P_2, P_3 \), where \( P_1, P_2 \), and \( P_3 \) are linearly independent, and the other points are defined over \( \mathbb{F}_3 \) (not over \( \mathbb{F}_3 \)) and are conjugate to each other, i.e., \( (P_4, P_5) = (\sigma(P_3), \sigma(P_4)) \). By a transformation by an element of \( \text{PGL}_3(\mathbb{F}_3) \), we may assume

\[
P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (0 : 0 : 1).
\]

Let \( \zeta \) be a primitive element of \( \mathbb{F}_3 \). A computation shows that every position of \( \{P_1, \ldots, P_3\} \) such that any four points among them are not contained in a line is equivalent by a linear transformation by a diagonal matrix and a permutation of \( \{x, y, z\} \) to either of the three cases:

1. \( P_4 = (1 : \zeta^5 : \zeta^7) \),
2. \( P_4 = (1 : \zeta^7 : 1) \),
3. \( P_4 = (1 : \zeta^2 : \zeta^2) \)

with \( P_5 = \sigma(P_4) \).

**Case (1,1,1,2) with linearly dependent** \( P_1, P_2, P_3 \) We consider the case where \( \{P_1, P_2, P_3, P_4, P_5\} \) contains three \( \mathbb{F}_3 \)-rational points, say \( P_1, P_2, P_3 \), where \( P_1, P_2 \), and \( P_3 \) are linearly dependent, and the other points are defined over \( \mathbb{F}_3 \) and are conjugate to each other. By a transformation by an element of \( \text{PGL}_3(\mathbb{F}_3) \), we may assume

\[
P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0), \quad P_3 = (1 : 1 : 0).
\]

Let \( \zeta \) be a primitive element of \( \mathbb{F}_3 \). As in the case (1,1,1,2) with linearly independent \( P_1, P_2, P_3 \), we have three equivalent classes:

1. \( P_4 = (1 : \zeta^5 : \zeta^7) \),
2. \( P_4 = (1 : 1 : \zeta^7) \),
3. \( P_4 = (1 : \zeta^2 : \zeta^2) \)

with \( P_5 = \sigma(P_4) \).
Case (1,2,2) We consider the case where \( \{P_1, P_2, P_3, P_4, P_5\} \) contains one \( \mathbb{F}_3 \)-rational point \( P_1 \) and two pairs \((P_2, P_3)\) and \((P_4, P_5)\) of \( \mathbb{F}_{32} \)-rational points, where \( P_3 = \sigma(P_2) \) and \( P_5 = \sigma(P_4) \). By a transformation by an element of \( \text{PGL}_3(\mathbb{F}_3) \), we may assume

\[ P_1 = (1 : 0 : 0). \]

Let \( \zeta \) be a primitive element of \( \mathbb{F}_{32} \). We have five equivalent classes:

\[(1) \ (P_2, P_4) = ((1 : 1 : \zeta^2), (0 : 1 : \zeta^6)), \quad (2) \ (P_2, P_4) = ((1 : 2 : \zeta^5), (1 : \zeta^2 : \zeta^7)), \]
\[(3) \ (P_2, P_4) = ((1 : \zeta : 1), (1 : \zeta^7 : \zeta^7)), \quad (4) \ (P_2, P_4) = ((1 : \zeta^2 : \zeta^6), (1 : 0 : \zeta^5)), \]
\[(5) \ (P_2, P_4) = ((1 : 0 : \zeta^2), (1 : 1 : \zeta^5)). \]

Case (1,1,3) We consider the case where \( \{P_1, P_2, P_3, P_4, P_5\} \) contains two \( \mathbb{F}_3 \)-rational points \( P_1, P_2 \) and conjugate three points \((P_3, P_4, P_5)\) of \( \mathbb{F}_{32} \)-rational points, where

\[(P_3, P_4, P_5) = (P_3, \sigma(P_3), \sigma^2(P_3)). \]

By a transformation by an element of \( \text{PGL}_3(\mathbb{F}_3) \), we may assume

\[ P_1 = (1 : 0 : 0), \quad P_2 = (0 : 1 : 0). \]

Let \( \zeta \) be a primitive element of \( \mathbb{F}_{32} \). We have four cases:

\[(1) \ P_3 = (1 : 2 : \zeta^5), \quad (2) \ P_3 = (1 : \zeta^6 : \zeta^{25}), \quad (3) \ P_3 = (1 : \zeta^{17} : \zeta^2), \quad (4) \ P_3 = (1 : 2 : \zeta^{10}). \]

Case (2,3) We consider the case where \( \{P_1, P_2, P_3, P_4, P_5\} \) contains conjugate two points \((P_1, P_2)\) of \( \mathbb{F}_{32} \)-rational points with \((P_1, P_2) = (P_1, \sigma(P_1))\) and conjugate three points \((P_3, P_4, P_5)\) of \( \mathbb{F}_{32} \)-rational points, where \((P_3, P_4, P_5) = (P_3, \sigma(P_3), \sigma^2(P_3))\). By a transformation by an element of \( \text{PGL}_3(\mathbb{F}_3) \), we may assume

\[ P_1 = (1 : \xi : 0), \]

where \( \xi \) is a primitive element of \( \mathbb{F}_{32} \). Let \( \zeta \) be a primitive element of \( \mathbb{F}_{32} \). We have three cases:

\[(1) \ P_3 = (1 : 2 : \zeta^5), \quad (2) \ P_3 = (1 : \zeta^{22} : 2), \quad (3) \ P_3 = (1 : \zeta^{17} : \zeta^2). \]

Case (1,4) We consider the case where \( \{P_1, P_2, P_3, P_4, P_5\} \) contains one \( \mathbb{F}_3 \)-rational point \( P_1 \) and the other 4 points are over \( \mathbb{F}_{34} \) and are conjugate to each other, say

\[(P_2, P_3, P_4, P_5) = (P_2, \sigma(P_2), \sigma^2(P_2), \sigma^3(P_2)). \]

By a transformation by an element of \( \text{PGL}_3(\mathbb{F}_3) \), we may assume \( P_1 = (1 : 0 : 0) \). Let \( \zeta \) be a primitive element of \( \mathbb{F}_{34} \). We have five equivalent classes:

\[(1) \ P_2 = (1 : \zeta^{75} : \zeta^{49}), \quad (2) \ P_2 = (1 : \zeta^8 : \zeta^{70}), \quad (3) \ P_2 = (1 : \zeta^{50} : \zeta^{53}), \]
\[(4) \ P_2 = (1 : \zeta^{72} : \zeta^{29}), \quad (5) \ P_2 = (1 : \zeta^5 : \zeta^{75}). \]
Case (5) This is the case where singular points on $C'$ are defined over $\mathbb{F}_3^2$ (but not over $\mathbb{F}_3$). Then $\{P_1, \ldots, P_5\}$ consists of a single Frobenius orbit, namely
$$\{P_1, P_2, P_3, P_4, P_5\} = \{P_1, \sigma(P_1), \sigma^2(P_1), \sigma^3(P_1), \sigma^4(P_1)\}.$$ 
In this case, the five points are determined only by $P_1$. A computation says that there are two cases:

1. $P_1 = (1 : \zeta^{127} : \zeta^{143})$, 
2. $P_1 = (1 : \zeta^{218} : \zeta^{72})$

for a primitive element $\zeta$ of $\mathbb{F}_3^2$.

### 4.2. Non-special curves of genus 5 over $\mathbb{F}_3$ with many $\mathbb{F}_9$-rational points

For $K = \mathbb{F}_p$ and $K' = \mathbb{F}_p^2$ with $p = 3$, we executed Algorithm 3.1.1 over MAGMA V2.25-8 in each case given in Section 4.1, in order to prove Theorem 1. In our computations, we choose $N = 32$ as the input, since 32 is the maximal number among the known numbers of $\mathbb{F}_9$-rational points of genus-five curves over $\mathbb{F}_9$. Our implementations over MAGMA were conducted on a PC equipped with an Ubuntu 18.04.5 LTS OS, utilizing a 3.50GHz quad-core CPU (Intel(R) Xeon(R) E-2224G) and 64GB of RAM. It took about 58 hours in total to execute the algorithm for obtaining Theorem 1. For curves $C$ with $\#C(\mathbb{F}_3^2) \geq 32$ listed up, we also computed their Weil polynomials over MAGMA. We have also obtained explicit equations for non-special curves of genus 5 over $\mathbb{F}_3$ having 32 $\mathbb{F}_9$-rational points. Here, we show some examples:

1. Case (1, 1, 1, 1, 1) with $P_5 = (1 : 2 : 1)$. The normalization $C$ of the sextic $C' : F = 0$ with

$$F = x^4 y^2 + x^3 y^3 + x^2 y^4 + x^2 y z + x^2 y^2 z + 2 x y^2 z + 2 x y^2 z^2 + x^2 y^2 z^3 + x^2 y^3 z^2 + 2 x y^2 z^3 + x^2 y^3 z^2 + 2 x y^2 z^4$$

has 32 $\mathbb{F}_9$-rational points with Weil polynomial $(t + 3)^6(t^2 + 2t + 9)^2$.

2. Case (1, 1, 1, 2) with linearly independent $P_1, P_2, P_3$ where $P_4 = (1 : \zeta^2 : \zeta^2)$ for a primitive element $\zeta$ of $\mathbb{F}_3^2$. The normalization $C$ of the sextic $C' : F = 0$ with

$$F = x^2 y^4 + x^4 y z + 2 y^4 z^2 + x^2 z^4 + 2 y^2 z^4$$

has 32 $\mathbb{F}_9$-rational points with Weil polynomial $(t^2 + 2t + 9)(t^2 + 5t + 9)^4$.

3. Case (1, 1, 1, 2) with linearly dependent $P_1, P_2, P_3$ where $P_4 = (1 : \zeta^5 : \zeta^7)$ for a primitive element $\zeta$ of $\mathbb{F}_3^2$. The normalization $C$ of the sextic $C' : F = 0$ with

$$F = x^4 y^2 + x^3 y^3 + x^2 y^4 + 2 x^3 y^2 z + x y^4 z + x^2 y^2 z^2 + 2 x y^2 z + 2 x y^2 z^2 + 2 x^3 z^3 + 2 y^2 z^3 + x^2 z^4 + 2 x y z^4 + 2 y^2 z^4 + 2 y z^6$$

has 32 $\mathbb{F}_9$-rational points with Weil polynomial $(t + 3)^4(t^2 + 2t + 9)(t^2 + 4t + 9)^2$.

4. Case (1, 2, 2) with $P_2 = (1 : 2 : \zeta^5)$ and $P_4 = (1 : \zeta^2 : \zeta^7)$, where $\zeta$ is a primitive element of $\mathbb{F}_3^2$. The normalization $C$ of the sextic $C' : F = 0$ with

$$F = x^4 y^2 + 2 x^3 y^3 + 2 x^2 y^5 + 2 y^6 + x^2 y^3 z + 2 y^5 z + 2 x^4 z + 2 x^3 y z^2 + x y^3 z^2 + 2 x^3 z^3 + 2 x^2 y z^3 + x y z^4 + y^2 z^4 + 2 x z^5 + 2 y z^5 + z^6$$

has 32 $\mathbb{F}_9$-rational points with Weil polynomial $(t + 3)^2(t^4 + 8t^3 + 32t^2 + 72t + 81)^2$. 

(5) Case (1,1,3) with $P_3 = (1 : 2 : \zeta^5)$, where $\zeta$ is a primitive element of $\mathbb{F}_{33}$. The normalization $C$ of the sextic $C' : F = 0$ with

$$F = x^4 y^2 + x^2 y^4 + x^4 yz + 2x y^4 z + 2x^4 z^2 + 2x^3 yz^2 + x^2 y^2 z^2 + 2x^3 z^2 + 2y^4 z^2 + 2x^3 y^2 z^2 + 2x^2 y^3 z + 2x^2 y^2 z^3 + y^3 z^3 + 2x^2 z^4 + xyz^4 + 2y^2 z^4 + xz^5 + 2yz^5 + 2z^6$$

has $32 \mathbb{F}_9$-rational points with Weil polynomial $(t^2 + 2t + 9)(t^2 + 5t + 9)^4$.

(6) Case (2,3) with $P_1 = (1 : \xi : 0)$ and $P_3 = (1 : 2 : \zeta^5)$, where $\xi$ is a primitive element of $\mathbb{F}_{32}$ and where $\zeta$ is a primitive element of $\mathbb{F}_{33}$. The normalization $C$ of the sextic $C' : F = 0$ with

$$F = x^5 y + x^4 y^2 + 2x^2 y^4 + 2y^6 + x^5 z + x^4 yz + 2x^3 y^2 z + 2x^3 yz^2 + x^2 y^2 z^2 + x^3 z^3 + 2x^2 yz^3 + x^2 y^2 z^3 + x^2 z^4 + xyz^4 + 2y^2 z^4 + xz^5 + z^6$$

has $32 \mathbb{F}_9$-rational points with Weil polynomial $(t^2 + 2t + 9)(t^2 + 5t + 9)^4$.

5. Concluding remarks with some open problems

We provided a new effective parametrization for the space of curves of genus 5 that are neither hyperelliptic nor trigonal. We realized such a curve as the normalization of a sextic in $\mathbb{P}^2$, and termed a non-special curve if the associated sextic had five double points. It was also proved that the dimension of the space of non-special curves of genus 5 does not exceed 12, where the value 12 is equal to the dimension of the moduli space of curves of genus 5. As an application, we also presented an algorithm for enumerating non-special curves of genus 5 over finite fields with more rational points than a prescribed bound. By executing the algorithm on MAGMA, we showed that the maximal number of $\mathbb{F}_9$-rational points of non-special curves of genus 5 over $\mathbb{F}_3$ is 32.

Our explicit parametrization and the algorithm presented in this paper may find fruitful applications both in theory and in computation. One of such applications is the classification of possible invariants (Hasse-Witt rank and so on) for non-hyperelliptic and non-trigonal curves of genus 5. Finally, we list some considerable open problems:

(a) Extend the parametrization to the case where curves have more complex singularities. Concretely, present a parametrization for non-hyperelliptic and non-trigonal curves $C$ of genus 5, such that the equality (2.2.2) in Subsection 2.2 does not hold (cf. Remark 2.2.10). A more general problem is to give an explicit model in $\mathbb{P}^2$ for non-hyperelliptic curves of the other genus $\geq 4$. Cf. a parametrization of generic curves of genus 3 is presented in [13], where an equation with 7 parameters (the moduli dimension is 6) is given. We also refer to [2], where the authors give representative families for all strata by automorphism group (except for the stratum associated with automorphism group $\mathbb{Z}/5\mathbb{Z}$) of smooth plane curves of genus 6.

(b) Present methods for computing invariants of non-special curves of genus 5, such as Cartier-Manin and Hasse-Witt matrices. Computing Cartier-Manin and Hasse-Witt matrices enables us to determine whether given curves are superspecial or not, as in [7], [8], [9], and [10]. To compute Cartier-Manin matrices, a method provided in [15] for a plane curve can be applied.

(c) Construct an algorithm for determining whether given two non-special curves of genus 5 are isomorphic or not. With the algorithm, we might present a characterization of the moduli space of curves of genus 5, employing the concept of representative families as described in [13].
(cf. Remark 2.2.7). Computing the automorphism group of such a curve is also an interesting problem. Cf. in the case of non-hyperelliptic curves of genus 4, the authors (resp. the authors and Senda) presented an algorithm for the isomorphism test (resp. for computing automorphism groups) in [10] (resp. [11]).

(d) Improve the efficiency of Algorithm 3.1.1, and then enumerate non-special curves of genus 5 having many rational points for $p > 3$. To accomplish this, it is important to reduce the number of curves in the search space.

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E-mail address of the first author: m-kudo@fit.ac.jp
E-mail address of the second author: harasita@ynu.ac.jp