RECURSIVE FORMULAS FOR THE CHARACTERISTIC NUMBERS OF RATIONAL PLANE CURVES

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Abstract. We derive recursive equations for the characteristic numbers of rational nodal plane curves with at most one cusp, subject to point conditions, tangent conditions and flag conditions, developing techniques akin to quantum cohomology on a moduli space of stable lifts.

1. Introduction

Although of recent vintage, Kontsevich’s recursive formula

\[ N_d = \sum_{d_1 + d_2 = d} N_{d_1} N_{d_2} \left[ \frac{d_1^2 d_2^2 (3d - 4)}{3d_1 - 2} - d_1^3 d_2 \frac{3d - 4}{3d_1 - 1} \right] \]

for the number \( N_d \) of rational plane curves of degree \( d \) through \( 3d - 1 \) general points is already a celebrated result. The formula is wonderfully simple: it determines all such characteristic numbers, beginning with the triviality \( N_1 = 1 \).

In his proof, Kontsevich uses a compactification \( \overline{M}_{0,n}(\mathbb{P}^2, d) \) of the space of \( n \)-pointed maps from \( \mathbb{P}^1 \) to the projective plane, called the space of stable maps. The key fact about these spaces is that (for \( n \geq 4 \)) they carry three linearly equivalent divisors, called the special boundary divisors, which are the inverse images of the special points 0, 1, and \( \infty \), under a natural map from \( \overline{M}_{0,n}(\mathbb{P}^2, d) \) to the space of stable rational curves with four distinguished points. (As is well-known, the latter space is isomorphic to \( \mathbb{P}^1 \).) This linear equivalence readily yields Kontsevich’s identity. Furthermore, each divisor is a union of components, each of which is isomorphic to a fiber product

\[ \overline{M}_{0,n_1+1}(\mathbb{P}^2, d_1) \times_{\mathbb{P}^2} \overline{M}_{0,n_2+1}(\mathbb{P}^2, d_2); \]

as a consequence, one can interpret Kontsevich’s formula as the assertion that a certain new product on the cohomology group of \( \mathbb{P}^2 \) (tensored with a certain power series ring) is associative. The resulting associative ring is called the quantum cohomology.

In this paper we show that Kontsevich’s compactifications can also be used to derive recursive formulas for these other classical characteristic numbers:

\[ N_d(a, b, c) = \text{the number of rational plane curves of degree } d \text{ through } a \text{ general points, tangent to } b \text{ general lines, and tangent to } c \text{ general lines at a specified general point on each line} \]

(\( \text{where } a + b + 2c = 3d - 1 \)).

In examining questions of tangency, it is natural to work with the incidence correspondence \( I \) of points and lines in \( \mathbb{P}^2 \). However, the space of stable maps to \( I \) is too large; instead we need to consider only those stable maps which can be lifted from maps to \( \mathbb{P}^2 \), and degenerations of such maps. Thus we will define a space \( \overline{M}_{0,n}(\mathbb{P}^2, d) \) of stable lifts. The superscript 1 indicates that \( I \) parametrizes first-order
jets of curves in \( \mathbf{P}^2 \); similar definitions for higher order jet bundles will be considered in another paper. As a subspace of the space of stable maps to the incidence correspondence, \( \mathcal{M}_{0,n}(\mathbf{P}^2, d) \) inherits the special boundary divisors. A chief part of our project is to describe the components of these divisors. We show, for example, that for many components the general point represents a map from a curve with three components, with the central component mapping to a fiber of \( I \) over \( \mathbf{P}^2 \).

From the linear equivalence of the special boundary divisors, we extract many recursive formulas. We show, in an admittedly ad hoc fashion, that these formulas suffice to determine all the characteristic numbers \( N_d(a, b, c) \), starting with the four trivial cases in degree 1. At the same time we obtain recursive formulas which determine, in all cases, three other types of characteristic numbers:

\[
C_d(a, b, c; 1) = \text{the number of rational plane curves of degree } d, \text{ with one cusp, through } a \text{ points, tangent to } b \text{ lines, and tangent to } c \text{ lines at } c \text{ specified points (where } a + b + 2c = 3d - 2);
\]

\[
C_d(a, b, c; h) = \text{the number of such curves having the cusp on a specified line (where } a + b + 2c = 3d - 3);
\]

\[
C_d(a, b, c; h^2) = \text{the number of such curves having the cusp at a specified point (where } a + b + 2c = 3d - 4).
\]

In fact, it is easier to determine these numbers than to avoid them, since the special boundary divisors on the space of stable lifts include, in addition to the components already described, other components corresponding to cuspidal rational curves.

Because of the occurrence of stable maps with automorphism, it is most convenient to utilize the framework of algebraic stacks and the corresponding intersection theory (with rational coefficients) as defined by Vistoli [Vis89]. For automorphism-free stable maps the intersection theory of the stack coincides with the usual intersection theory of the corresponding moduli scheme. However, on the stack \( \mathcal{M}_{0,n}(\mathbf{P}^2, d) \) of stable lifts there are components of the special boundary divisors for which the general member has automorphism group of order two. Therefore we will encounter the fractional intersection number 1/2. Another possibility would be to work with equivariant Chow groups as defined by Edidin and Graham [EG96].

Here is an outline of the paper. In §2 we recall Kontsevich’s notion of stable maps, the definition of the Gromov-Witten invariants, and the basic linear equivalence which leads to Kontsevich’s recursive formula. As we have said, this linear equivalence implies the associativity of the quantum product, but—since in our project we do not use any analogous “higher-order” product—we refrain from discussing this point. In §3 we introduce spaces of stable lifts for \( \mathbf{P}^2 \), by associating to a general stable map its lift to the projectivized tangent bundle of \( \mathbf{P}^2 \). In §4 we use these spaces to define first-order Gromov-Witten invariants, and discuss their significance in enumerative geometry. In §5 we describe the structure of the special boundary divisors on the space of stable lifts. In §6 we use the physicists’ method of quantum potentials to write down generating functions for the Gromov-Witten invariants, and derive the basic identity (6.6), analogous to the associativity identity for the quantum product. In §7 we derive, from selected cases of the basic identity, recursive equations for the characteristic numbers; we also present tables of characteristic numbers through degree 5.

We believe that these methods will work for a large class of surfaces. We also believe that they will enable us to calculate the higher-order characteristic numbers of curves, defined by conditions of higher-order tangency to lines or other specified curves. Here the appropriate parameter spaces are, we believe, those originally defined by Semple (later investigated in [Col88] and [CK91]). We intend to pursue this matter with Susan Colley. A theory for more general varieties and higher genus seems possible as a consequence of the results of Behrend [Beh96] and those of Li and Tian [LT96].
However, the enumerative significance of the Gromov-Witten invariants is, in this more general set-up, not obvious.

In recent papers, Pandharipande has constructed an algorithm for computing the numbers $N_d(a, b, 0)$ and Francesco and Itzykson have found a recursive relation that determines the same numbers, with the restriction $a \geq 3$, given characteristic numbers of lower degree. This project was inspired by W. Fulton’s lecture series at the AMS summer institute at Santa Cruz 1995. We have benefited greatly from discussions with R. Pandharipande, and from his preprints. We also wish to thank P. Aluffi, C. Ban, S. Colley, B. Crauder, D. Edidin, S. Kleiman, D. Laksov, L. McEwan, and S. Yokura for their help during the preparation of this paper.

2. Stables maps and Gromov-Witten invariants

As general references for the material in this section we suggest [BM95], [Ful95], [Kon95], [KM94], [LT95] and [Pan94]. In this section we will utilize the intersection theory of the coarse moduli space of stable maps. Generic members of the special boundary divisors of the space of stable maps have no automorphisms. Therefore, for our purposes, the intersection theory of the stack of stable maps is the same as that of the coarse moduli space. Throughout this paper, we will be working over an algebraically closed field of characteristic zero. Given a variety or stack $X$, homology $A_*(X)$ and cohomology $A^*(X)$ will denote the rational equivalence groups with coefficients in $\mathbb{Q}$.

Suppose that $C$ is a connected reduced curve of arithmetic genus zero whose singularities are at worst nodes. Then $C$ must be a tree of $\mathbb{P}^1$’s. We will call the nodes points of attachment. Suppose that on $C$ we have $n$ distinct points $p_1, \ldots, p_n$, none of which is a point of attachment; we call these points markings. A special point is a point of attachment or a marking.

Now suppose that $X$ is a nonsingular projective variety. Following Kontsevich, we define an $n$-pointed stable (genus 0) map to $X$ to be a map $\mu: C \to X$ from such a curve, subject to the following condition: if the restriction of $\mu$ to a component is constant, then that component contains at least three special points. (Since we never consider stable maps from curves of higher arithmetic genus, we will generally omit the parenthesized phrase.) A family of stable maps consists of a flat, proper map $\pi: C \to S$, a map $\mu: C \to X$, and $n$ sections $\{p_i\}_{i=1,\ldots,n}$ of $\pi$, such that, for each geometric fiber $C(s)$ of $\pi$, the restriction of $\mu$ to this fiber, with the markings $p_i(s)$, is an $n$-pointed stable map. A morphism over $S$ from one such family $(\pi: C \to S, \mu, \{p_i\})$ to another $(\pi': C' \to S, \mu', \{p'_i\})$ is a morphism $\tau: C \to C'$ over $S$ such that $\mu = \mu' \circ \tau$ and $p_i \circ \tau = p'_i$.

Suppose that $\varphi$ is a specified class in $A_1(X)$. Kontsevich showed that there is a coarse moduli space $\overline{M}_{0,n}(X, \varphi)$ for isomorphism classes of stable maps whose image in $X$ represents $\varphi$ [KM94, 2.4.1], [Kon93, §1, p. 336], [Pan94, Theorems 1 and 2]. We call $\overline{M}_{0,n}(X, \varphi)$ the space of stable (genus 0) maps. There are $n$ evaluation maps $e_t: \overline{M}_{0,n}(X, \varphi) \to X$ which associate to the point representing a stable map $\mu$ the images $p(t)$ of the markings. If $f: X \to Y$ is a morphism, then there is a morphism $Mf: \overline{M}_{0,n}(X, \varphi) \to \overline{M}_{0,n}(Y, f_*s(\varphi))$ which associates to the point representing $\mu$ the point representing the stable map obtained from $f \circ \mu$ by contracting any components which have become unstable; if $f_*s(\varphi) = 0$, we must assume that $n \geq 3$. In particular, under this hypothesis there is a “forgetful” morphism from $\overline{M}_{0,n}(X, \varphi)$ to $\overline{M}_{0,n}$, the space of stable rational $n$-pointed curves.

If $n \geq 4$ then we can compose the forgetful morphism with another sort of forgetful morphism $\overline{M}_{0,n} \to \overline{M}_{0,4}$, this forgetting about all markings except the first four. The space $\overline{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$. It has a distinguished point $P(12 | 34)$ representing the two-component curve having the first two markings on one component and the latter two on the other; similarly there are two other distinguished points $P(13 | 24)$ and $P(14 | 23)$. Their inverse images on $\overline{M}_{0,n}(X, \varphi)$ are three linearly equivalent divisors $D(12 | 34)$, $D(13 | 24)$, and $D(14 | 23)$, called the special boundary divisors.

We allow the index set to be an arbitrary finite set $A$ rather than just $\{1, \ldots, n\}$; we then write $\overline{M}_{0,A}(X, \varphi)$ for the moduli space. Suppose that $A_1 \cup A_2$ is a partition of $\{1, \ldots, n\}$ and that $\varphi_1, \varphi_2$
are classes in $A_1(X)$ whose sum is $\varphi$. Suppose that $\{\ast\}$ is a single-element set. Then the fiber product

$$D(A_1, A_2, \varphi_1, \varphi_2) = \mathcal{M}_{0, A_1 \cup \{\ast\}}(X, \varphi_1) \times_X \mathcal{M}_{0, A_2 \cup \{\ast\}}(X, \varphi_2)$$

is naturally a subspace of $\mathcal{M}_{0,A}(X, \varphi)$; the typical point represents a map from a curve with two components, with the point of attachment corresponding to the point indexed by $\ast$. In Figure 2.1 the attachment point and the image thereof are marked $\bullet$.

![Figure 2.1 Stable map.](image)

If $X$ is a product of homogeneous spaces (for example, if $X$ is a projective space or a flag variety) then the dimension of $\mathcal{M}_{0, n}(X, \varphi)$ (if it is nonempty) is $\dim X + \int_\varphi c_1(T_X) + n - 3$ [Kon95, Ful95, Proposition 1]. Furthermore the divisor $D(12 \mid 34)$ is the sum

$$D(12 \mid 34) = \sum D(A_1, A_2, \varphi_1, \varphi_2)$$

over all partitions in which 1 and 2 belong to $A_1$, and 3 and 4 belong to $A_2$.

Suppose that $\gamma_1, \ldots, \gamma_n$ are elements of $A^*X \otimes \mathbb{Q}$. Then the Gromov-Witten invariant associated to these classes and to $\varphi$ is the number

$$N_\varphi(\gamma_1 \cdots \gamma_n) = \int e_1^n(\gamma_1) \cup \cdots \cup e_n^n(\gamma_n) \cap [\mathcal{M}_{0, n}(X, \varphi)],$$

the degree of the top-dimensional component. If the classes $\gamma_1, \ldots, \gamma_n$ are homogeneous, then the Gromov-Witten invariant is nonzero only if the sum of their codimensions is the dimension of the moduli space. (If the moduli space is empty, we declare the Gromov-Witten invariant to be zero.)

We now restrict our attention to the case $X = \mathbb{P}^2$. The class $\varphi$ of a curve must be some multiple $dh$ of the class $h$ of a line. We will write $\mathcal{M}_{0, n}(\mathbb{P}^2, d)$ for the space of $n$-pointed stable maps; its dimension is $3d - 1 + n$ (unless $d = 0$ and $n \leq 2$, in which case the space is empty). We define

$$N_d = N_{dh}(h^2 \cdots h^2).$$

By standard transversality arguments, one can show that $N_d$ is the number of rational plane curves of degree $d$ through $3d - 1$ general points.

The linear equivalence of the special boundary divisors $D(12 \mid 34)$ and $D(13 \mid 24)$ implies, for each choice of $\varphi$ and of $\gamma_1, \ldots, \gamma_n$, the numerical equality

$$\int e_1^n(\gamma_1) \cup \cdots \cup e_n^n(\gamma_n) \cap [D(12 \mid 34)] = \int e_1^n(\gamma_1) \cup \cdots \cup e_n^n(\gamma_n) \cap [D(13 \mid 24)].$$

By (2.1) and the similar decomposition of the divisor $D(13 \mid 24)$, this is an equation among various values of $N_d$. For example, if $n = 3d + 1$, $\varphi = dh$, $\gamma_1 = \gamma_2 = h$, and $\gamma_i = h^2$ for $i \geq 3$, then (2.2) is Kontsevich’s recursive formula!

Equation (2.2) can also be interpreted as an equation among formal power series. Let $T_0 = 1$, $T_1 = h$, and $T_2 = h^2$, so that an arbitrary class $\gamma$ is a linear combination $y_0T_0 + y_1T_1 + y_2T_2$. Define the potential by

$$P = \sum_{n \geq 0} \frac{1}{n!} \sum_\varphi N_\varphi(\gamma^n),$$

which one can show is equal to

$$\frac{1}{2} (y_0^2y_2 + y_0y_1^2) + \sum_{d \geq 1} N_{d \varphi} e^{dy_1} \frac{y_2^{d-1}}{(3d - 1)!}.$$
The first term, called the \textit{classical potential}, encodes the intersection product on $\mathbb{P}^2$; the second term is called the \textit{quantum potential}. Using these potentials, we can translate the special case

$$
\int e_1^*(T_i) \cup e_2^*(T_j) \cup e_3^*(T_k) \cup e_4^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap [D(12 \mid 34)]
$$

$$
= \int e_1^*(T_i) \cup e_2^*(T_j) \cup e_3^*(T_k) \cup e_4^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap [D(13 \mid 24)]
$$

of (2.4) into the partial differential equation

$$
(2.4) \quad \sum_{s=0}^{2} \frac{\partial^3 \mathcal{P}}{\partial y_i \partial y_j \partial y_s} = \sum_{s=0}^{2} \frac{\partial^3 \mathcal{P}}{\partial y_i \partial y_j \partial y_{2-s}}.
$$

The case $i = j = 2$, $k = l = 1$ is equivalent to Kontsevich’s formula.

3. \textsc{lifts of stable maps}

Denote by $I = \mathbb{P}(T_{\mathbb{P}^2})$ the projectivized tangent bundle of the projective plane; it is the variety of point-line incidence in $\mathbb{P}^2 \times \mathbb{P}^2$. Suppose that $\mu: (\mathbb{P}^1, p_1, \ldots, p_n) \to \mathbb{P}^2$ is a nonconstant map with $n$ distinct marked points. For a general point $x$ in $\mathbb{P}^1$ define $\tilde{\mu}: \mathbb{P}^1 \to I$ by $\tilde{\mu}(x) = (\mu(x), \mu'(x))$, where $\mu'(x)$ is the tangent line to $\mu(\mathbb{P}^1)$ at $\mu(x)$. The construction extends to all of $\mathbb{P}^1$: even at points where the map $\mu$ is singular there is a unique tangent direction associated to the branch of $\mu(\mathbb{P}^1)$ at $x$. The map $\tilde{\mu}: (\mathbb{P}^1, p_1, \ldots, p_n) \to I$ is called the \textit{strict lift} of $\mu$. (By symmetry, we may also define the strict lift of a nonconstant map to $\mathbb{P}^2$.)

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1}
\caption{The strict lift of a map from $\mathbb{P}^1$ to $\mathbb{P}^2$.}
\end{figure}

Strict lifts do not behave nicely in families of maps. For example if $\mu$ is an immersion of degree $d$ and its image is a nodal curve, then the number of nodes is $\delta = (d-1)(d-2)/2$ and the class of the curve is

$$
\delta = d(d-1) - 2\delta = 2d - 2.
$$

Thus the homology class of $\tilde{\mu}(\mathbb{P}^1)$ in $I$ is $d$ times the strict lift of a line plus $2d-2$ times the strict lift of a dual line. We will say that it has \textit{bidegree} $(d, 2d-2)$. If instead $\mu(\mathbb{P}^1)$ is a rational curve with $(d-1)(d-2)/2 - 1$ nodes and one cusp then the bidegree of $\tilde{\mu}(\mathbb{P}^1)$ is $(d, 2d-3)$. Thus in a family degenerating a node to a cusp, the strict lifts of the members in the family do not piece together.

Suppose that we have a family of stable maps $(\pi: C \to S, \mu: C \to I, \{p_i\})$ whose general member is the strict lift of an immersion $\mathbb{P}^1 \to \mathbb{P}^2$. Then we say that the members of this family are \textit{stable lifts} of the corresponding members of the family of maps to $\mathbb{P}^2$. Note that a stable lift of a map of degree $d$ has bidegree $(d, 2d-2)$; for example (supposing there are no markings) the stable lift of a $d$-fold branched cover of a line consists of the strict lift together with maps to the fibers of $I$ over the $2d-2$ branch points. In Figure 3.2 we illustrate the case $d = 3$. Ramification and branch points are marked by $\times$. 

The stable lift of a three-cover of a line in $\mathbb{P}^2$ with ramification at four points.

For simplicity of notation, we will write $\overline{M}_{0,n}(I, (d, 2d - 2))$ for the stack of stable (genus 0) maps representing the class of a curve of bidegree $(d, 2d - 2)$. Let $\overline{M}_{0,n}^1(\mathbb{P}^2, d)$ be the closed substack of $\overline{M}_{0,n}(I, (d, 2d - 2))$ which represents stable lifts; we call it the *stack of stable lifts*. We continue to use the notations $e_1, \ldots, e_n$ for the evaluation maps $\overline{M}_{0,n}^1(\mathbb{P}^2, d) \to I$. Note that the map $\overline{M}_{0,n}^1(\mathbb{P}^2, d) \to \overline{M}_{0,n}(\mathbb{P}^2, d)$ (inclusion followed by projection) is a birational morphism, whose inverse is the *lifting map* $\lambda$ which associates to each immersion its strict lift.

**Proposition 3.1.** Let $C$ be a curve consisting of two irreducible component, each isomorphic to $\mathbb{P}^1$ and attached at a single point $p$. Let $\mu: C \to \mathbb{P}^2$ be a map defined by mapping one component of $C$ to an irreducible curve of degree $d_1 \geq 1$ and mapping the other component to an irreducible curve of degree $d_2 \geq 1$, so that $\mu$ is represented by a point of $\overline{M}_{0,0}(\mathbb{P}^2, d_1 + d_2)$. Assume that both maps are immersions and that they are transverse at $p$.

Then the stable lift of $\mu$ is unique. It is a map $\tilde{\mu}$ from a curve $\tilde{C}$ consisting of three components, each isomorphic to $\mathbb{P}^1$. On the central component $\tilde{\mu}$ is a map of degree 2 to the pencil of tangent directions at $p$, with the points of attachment mapping to the tangent directions of $C$; the map is ramified at these attachment points. On each peripheral component $\tilde{\mu}$ is the strict lift of a component of $\mu$.

See Figure 3.3.

![Figure 3.3](image-url) Stable lift of Proposition 3.1.

**Proof.** The versal deformation theory of a plane node is given by the local equation $xy = \epsilon$. Thus any family degenerating to $\mu$ will be the pullback of a family which, in appropriate local coordinates, has total space isomorphic to a neighborhood of the origin, with the fibers of the family being the curves $xy = \epsilon$ and with the map being the identity. Locally around the origin, $I$ is a trivial $\mathbb{P}^1$-bundle; in coordinates the family of stable lifts is given by

$$(x, y) \mapsto (x, y, [x : -y]).$$
According to [Pan94, section 3.2], there is a unique extension of the family of lifts, obtained by blowing up the points of indeterminacy, possibly after a base change. We choose to make the base change which has the effect of replacing our original family by the family $xy = \epsilon^2$. (In fact, one can show that this base change is unavoidable.) Then we blow up the origin on this singular surface. In the resulting family, the fiber over $\epsilon = 0$ is reduced and has a new component mapping to the pencil of directions at the origin; the map, which is of degree 2, is ramified at the points of attachment to the strict transforms of the two original components.

But in general a stable map may have more than one stable lift. Figure 3.4 illustrates a stable lift of a degree three cover of a line with double ramification at two points. The two maps into the fibers over the branch points are of degree two, ramified at the attachment point and at one other point. The latter point may appear anywhere on the fiber. Thus the stable lift in this situation is not unique; there is a two dimensional family of lifts corresponding to a single stable map.

![Figure 3.4](image)

**Figure 3.4** A stable lift of a map with double ramification is not unique.

**Proposition 3.2.** Let $\tilde{\mu} : \tilde{C} \to I$ be an $n$-pointed stable lift. Let $\mu : C \to P^2$ be the stable map obtained by composing $\mu$ with the projection $I \to P^2$, forgetting about all markings, and (if necessary) contracting components which have become unstable. Then, for each irreducible component $\tilde{C}_i$ of $\tilde{C}$, the restricted map $\tilde{\mu}|_{\tilde{C}_i} : \tilde{C}_i \to I$ is one of the following:

1. the strict lift of a map obtained by restricting $\mu$ to a component of $C$,
2. a map into a fiber of $I$ over some point $x$ in $P^2$, where $x$ is either the image of an attachment point of $C$ or a singularity of the restriction of $\mu$ to some component of $C$,
3. a constant map.

**Proof.** We may assume that $n = 0$. It then suffices to prove that each component is of type (1) or (2).

By definition $\tilde{\mu}$ is a member of a family of stable maps from $\tilde{C}$ to $I$ over a base $S$, where the generic member of the family is the strict lift of a map to $P^2$. We may and will assume that the base $S$ is a nonsingular curve. Consider the family of maps $C$ to $P^2$ over the same base $S$ obtained by composing with the projection $I \to P^2$, forgetting about all markings, and (if necessary) contracting components which have become unstable. We will next reconstruct the family of maps to $I$, and conclude the facts about the special member $\tilde{\mu}$. Consider the open set $U$ of $C$ over $S$ that is the complement of the singular points (the nodes or attachment points) of curves in the family. Then there is a map $T_{U/S} \to T_{P^2}$ from the relative tangent bundle of $U$ over $S$ to the tangent bundle of $P^2$. Composing with the map $T_{P^2} \to$ zero section $\to P(T_{P^2}) = I$

we get a rational map from $P(T_{U/S}) = U$ and hence from $C$ to $I$, which is indeterminate exactly at the attachment points and singularities of the map $\mu$ on components of the special member $C$. For generic members of $C$ the rational map is clearly the lifting map. Finally, by [Pan94, Prop.3.3] the rational map may be extended by blowing up the points of indeterminacy. Furthermore by [Pan94, Prop.3.2] the
extension is unique and thus we recover the map \( \hat{\mu} \) as the special member after sufficient blowing-up. The result now follows from what we have concluded about the points of indeterminacy.

4. Characteristic numbers

The rational cohomology of \( I \) is given by

\[
A^*(I) = \mathbb{Q}[h, \hat{h}]/(h^3, h^2 + \hat{h} - \hat{h}),
\]

where \( h \) is the first Chern class of the pullback of the line bundle \( \mathcal{O}_{P^2}(1) \) on \( P^2 \) and \( \hat{h} \) is the pullback of the line bundle \( \mathcal{O}_{\hat{P}^2}(1) \) on \( \hat{P}^2 \). Note that the fundamental class of the strict lift of a line is \( \hat{h}^2 \), and that the class of the strict lift of a dual line is \( h^2 \). We fix the following basis for \( A^*(I) \):

\[
\{ T_0, T_1, T_2, T_3, T_4 \} = \{ 1, h, \hat{h}, h^2, \hat{h}^2, h^3 \}.
\]

With respect to this basis the fundamental class of the diagonal \( \Delta \) in \( I \times I \) has the simple decomposition

\[
[\Delta] = \sum_{s=0}^{5} [T_s] \times [T_{5-s}].
\]

Suppose that \( d \) is a positive integer, and that \( \gamma_1, \ldots, \gamma_n \) are elements of \( A^*(I) \otimes \mathbb{Q} \). We define the first-order Gromov-Witten invariant by

\[
N_d(\gamma_1 \cdots \gamma_n) = \int e_1^* (\gamma_1) \cup \cdots \cup e_n^* (\gamma_n) \cap [\overline{\mathcal{M}}_{0,n}(P^2, d)].
\]

Suppose that \( a \) of the \( \gamma_i \)'s equal the class \( \hat{h}^2 \), that \( b \) of them equal the class \( \hat{h}^2 \), and that the remaining \( c \) of them equal \( h^2 \hat{h} \), where \( a + b + 2c = 3d - 1 \). In this case we denote the Gromov-Witten invariant by \( N_d(a, b, c) \).

**Theorem 4.2.** \( N_d(a, b, c) \) is the number of rational plane curves of degree \( d \) through \( a \) general points, tangent to \( b \) general lines, and tangent to \( c \) general lines at a specified general point on each line.

(We will say that such a curve is incident to \( c \) specified flags.)

**Proof.** Associated to each class \( e_i^* h \) and \( e_i^* \hat{h} \) there is a complete basepoint-free linear system parametrized by lines in \( P^2 \) or by lines in \( \hat{P}^2 \). Applying the Kleiman-Bertini Theorem repeatedly \([Kle74, 5, \text{p.291}]\), we find that the intersection of general members of the linear systems is regular away from the singular locus of \( \overline{\mathcal{M}}_{0,n}(P^2, d) \). By a dimension count it is a set of reduced points of \( \overline{\mathcal{M}}_{0,n}(P^2, d) \). By the same dimension count, each point of the intersection corresponds to the strict lift of an immersion \( P^1 \rightarrow P^2 \). Thus no point of the singular locus is in the intersection.

The following Proposition is analogous to \([KM94, 2.2.3 \text{and} 2.2.4]\) and \([Ful95, \text{p.9}]\).

**Proposition 4.3.** (1) For all \( d \geq 1 \) and all \( \gamma_1, \ldots, \gamma_{n-1} \),

\[
N_d(\gamma_1 \cdots \gamma_{n-1}, 1) = 0.
\]

(2) If \( \gamma_n \) is a divisor class in \( A^1(I) \), then for all \( d \geq 1 \) and all \( \gamma_1, \ldots, \gamma_{n-1} \),

\[
N_d(\gamma_1 \cdots \gamma_n) = N_d(\gamma_1 \cdots \gamma_{n-1}) \int \gamma_n \cap [C],
\]

where \( [C] = dh^2 + (2d - 2)h^3 \) is the class of the strict lift of a rational nodal plane curve of degree \( d \).
For $d \geq 3$ let $C_{0,n,(\ast)}(\mathbf{P}^2, d)$ be the open substack of $\overline{M}_{0,n+1}(\mathbf{P}^2, d)$ consisting of maps $\mu: \mathbf{P}^1 \to \mathbf{P}^2$ for which $\mu(\mathbf{P}^1)$ has $(d - 1)(d - 2)/2 - 1$ nodes and exactly one cusp marked $\ast$, coinciding with the $(n + 1)$st marking. The strict lift of any such map has bidegree $(d, 2d - 3)$. Thus there is a lifting map
\[
\lambda_{C}: C_{0,n,(\ast)}(\mathbf{P}^2, d) \to \overline{M}_{0,n+1}(I, (d, 2d - 3)).
\]
The closure of $C_{0,n,(\ast)}(\mathbf{P}^2, d)$ will be denoted by $\overline{C}_{0,n,(\ast)}(\mathbf{P}^2, d)$; its dimension is $3d - 2 + n$. We define the stack of cuspidal stable lifts $\overline{C}_{0,n,(\ast)}(\mathbf{P}^2, d)$ to be the closure of $\lambda_C(C_{0,n+1}(\mathbf{P}^2, d))$ in $\overline{M}_{0,n}(\mathbf{P}^2, (d, 2d - 3))$; its dimension is likewise $3d - 2 + n$. Let $e_1, \ldots, e_n$ and $e_C$ be the evaluation maps $\overline{C}_{0,n,(\ast)}(\mathbf{P}^2, d) \to I$. For each $d \geq 3$ and each $\gamma_1, \ldots, \gamma_n, \gamma_C \in A^*(I)$ we define the cuspidal first-order Gromov-Witten invariant by
\[
C_d(\gamma_1 \cdots \gamma_n; \gamma_C) = \int e_1^\ast(\gamma_1) \cup \cdots \cup e_n^\ast(\gamma_n) \cup e_C^\ast(\gamma_C) \cap [\overline{C}_{0,n,(\ast)}(\mathbf{P}^2, d)].
\]
Suppose that among $\gamma_1, \ldots, \gamma_n$ there are $a$ occurrences of the class $h^2$, also $b$ occurrences of $h^2$, and $c$ occurrences of $h^2 h$, where $a + b + 2c = 3d - 1 - \text{codim}(\gamma_C)$. In this case we denote the cuspidal Gromov-Witten invariant by $C_d(a, b, c; \gamma_C)$.

**Theorem 4.4.**

1. $C_d(a, b, c; 1)$ is the number of rational plane curves of degree $d$, with one cusp, through a general points, tangent to general lines, and incident to $c$ general flags.
2. $C_d(a, b, c; h)$ is the number of such curves having the cusp on a specified general line.
3. $C_d(a, b, c; h^2)$ is the number of such curves having the cusp at a specified point.
4. $C_d(a, b, c; h^2)$ is the number of such curves for which the cusp tangent line passes through a specified point.
5. $C_d(a, b, c; h^2 h)$ is the number of such curves for which the cusp is a specified point and the cusp tangent is a specified line through that point.

**Proof.** The proof is almost identical to the proof of Theorem 4.3. \qed

**Proposition 4.5.**

1. For all $d \geq 3$ and all $\gamma_1, \ldots, \gamma_{n-1}$,
   \[
   C_d(\gamma_1 \cdots \gamma_{n-1}; 1; \gamma_C) = 0.
   \]
2. If $\gamma_n$ is a divisor class in $A^1(I)$, then for all $d \geq 3$ and all $\gamma_1, \ldots, \gamma_{n-1}$,
   \[
   C_d(\gamma_1 \cdots \gamma_n; \gamma_C) = C_d(\gamma_1 \cdots \gamma_{n-1}; \gamma_C) \int (\gamma_n \cap [C]),
   \]
   where $|C| = d h^2 + (2d - 3) h^2$ is the class of a the strict lift of a rational plane curve of degree $d$ with one cusp and $(d - 1)(d - 2)/2 - 1$ nodes.

5. **Special boundary divisors on the space of stable lifts**

   The components of the special boundary divisors of $\overline{M}_{0,n}(I, (d, 2d - 2))$ are indexed by the set of 4-tuples $(A_1, A_2, (d_1, c_1), (d_2, c_2))$ where $d_1 + d_2 = d$, $c_1 + c_2 = 2d - 2$, and $A_1 \cup A_2 = \{1, \ldots, n\}$ is a partition in which two of the four numbers $1, 2, 3, 4$ belong to $A_1$ and the other two to $A_2$. The general member of the corresponding divisor is a stable map with two components, one with markings indexed by $A_1$ and of bidegree $(d_1, c_1)$, and the other with markings indexed by $A_2$ and of bidegree $(d_2, c_2)$ [Pan94]. Since the general stable map of $\overline{M}_{0,n}(\mathbf{P}^2, d)$ is not contained in this divisor, its restriction to $\overline{M}_{0,n}(\mathbf{P}^2, d)$ is a divisor on the space of stable lifts, which we will denote by $D(A_1, A_2, (d_1, c_1), (d_2, c_2))$. This divisor may be empty; for example, $D(A_1, A_2, (d, 0), (0, 2d - 2))$ is empty for all $A_1$ and $A_2$. 


The linear equivalence of the special boundary divisors on $\overline{M}_{0,n}(I, (d, 2d - 2))$ passes to their restrictions:
\[
D(12 \mid 34) \cong D(13 \mid 24) \cong D(14 \mid 23),
\]
where
\[
D(12 \mid 34) = \sum D(A_1, A_2, (d_1, c_1), (d_2, c_2)),
\]
the sum over all partitions in which $1, 2 \in A_1$ and $3, 4 \in A_2$; the other two divisors have similar decompositions. Thus for all choices of cohomology classes $\gamma_1, \ldots, \gamma_n$ we have a numerical equality
\[
\int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cap [D(12 \mid 34)] = \int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cap [D(14 \mid 23)].
\]

To obtain these equations in an explicit form, we must identify the components of $(12 \mid 34)$ and $D(14 \mid 23)$, at least to the extent that they affect our calculations. We say that a divisor $D$ on $\overline{M}_{0,n}(P^2, d)$ is irrelevant if for any $\gamma_1, \ldots, \gamma_n$ in $A^*(I)$ we have
\[
\int e_1^*(\gamma_1) \cup \cdots \cup e_n^*(\gamma_n) \cap [D] = 0.
\]

Other divisors are said to be relevant.

Let $\tau : \overline{M}_{0,n}(P^2, d) \to \overline{M}_{0,0}(P^2, d)$ be the map which composes stable maps $\mu : C \to I$ with $I \to P^2$, forgets all markings and contracts any components which have become unstable.

**Proposition 5.4.** A component $D$ of $D(A_1, A_2, (d_1, c_1), (d_2, c_2))$ is irrelevant unless $\text{codim}(\tau(D)) \leq 1$ in $\overline{M}_{0,0}(P^2, d)$.

**Proof.** It suffices to establish (5.3) when $\gamma_1, \ldots, \gamma_n$ are elements of the basis $\{T_0, T_1, T_2, T_3, T_4\}$. Assume first that one of the classes, say $\gamma_n = 1$ is the identity. Then by the projection formula applied to the morphism $\pi : \overline{M}_{0,n}(P^2, d) \to \overline{M}_{0,n-1}(P^2, d)$ which forgets the $n$th marking, it follows that the above degree is equal to
\[
\int e_1^*(\gamma_1) \cup \cdots \cup e_{n-1}^*(\gamma_{n-1}) \cap \pi_*[D].
\]
However, $\pi_*[D]$ vanishes unless $\dim(\pi(D)) = \dim(D) = \dim(\overline{M}_{0,n-1}(P^2, d))$, so $\pi(D) = \overline{M}_{0,n-1}(P^2, d)$ and thus $\text{codim}(\tau(D)) = 0$.

It remains to consider the case when none of the $\gamma_i$’s is the identity. We will count dimensions. The dimension of $\overline{M}_{0,3}(P^2, d)$ is $3d - 1 + n$, and
\[
\text{codim}(e_1^*(h)) = \text{codim}(e_1^*(\hat{h})) = 1, \quad \text{codim}(e_1^*(h^2)) = \text{codim}(e_1^*(\hat{h}^2)) = 2, \quad \text{codim}(e_1^*(h^2 \hat{h})) = 3.
\]
Let $a$ be the number of pullbacks of $h^2$, $b$ the number of pullbacks of $\hat{h}^2$ and $c$ the number of pullbacks of $h^2 \hat{h}$. Then the class $e_1^*(\gamma_1) \cap \cdots \cap e_n^*(\gamma_n)$ has dimension 1 if $a + b + 2c = 3d - 2$. The number of pullbacks of the divisor classes $h$ and $\hat{h}$ does not influence the vanishing nor the dimension of the class.

The effect of a divisor class on the degree of the class, evaluated on $[D]$ is multiplication by $d$ in the case of $h$, and multiplication by $2d - 2$ in the case of $\hat{h}$, by Proposition 4.3.

Assume $\text{codim}(\tau(D)) \geq 2$; then $\dim(\tau(D)) \leq 3d - 3$, and a dimension count checks that there are no stable maps in $\tau(D)$ through $a$ general points, tangent to $b$ lines and incident to $c$ flags with $a + b + 2c = 3d - 2$. Hence one must put at least one condition on a map to a fiber of $I \to P^2$ contracted by $f$. However, because of Proposition 5.2 this will not bring down the count $3d - 2$ of conditions on the maps in $\tau(D)$.

Indeed, putting one tangency condition on a map to a fiber will, in case of a fiber over the image of an attachment point, induce an extra point condition on the maps in $\tau(D)$, and in the case of a fiber over a singularity of maps in $\tau(D)$, induce the condition that a singularity should
be on a line; also a codimension 1 condition. Similarly, putting two tangent conditions or one flag condition on a map to a fiber will induce two extra point conditions or the condition that a singularity of the map should be at a fixed point, respectively. These are both codimension 2 conditions. It is clearly impossible to put point conditions on the fiber maps, so one concludes that the evaluation is zero unless $\text{codim}(\tau(D)) \leq 1$.

\textbf{Corollary 5.5.} A component $D$ of $D(A_1, A_2, (d_1, c_1), (d_2, c_2))$ is irrelevant unless the bidegrees $(d_1, c_1)$ and $(d_2, c_2)$ are either

1. $(d_1, 2d_1 - 2)$ and $(d_2, 2d_2)$ with $d_1 \geq 1$, $d_2 \geq 0$ and $d_1 + d_2 = d$, or
2. $(d, 2d - 3)$ and $(0, 1)$ with $d \geq 2$,

or vice versa.

\textit{Proof.} First case: assume that $d_1 = d$ and $d_2 = 0$ or vice versa. The locus of curves in $\overline{M}_{0,0}(\mathbb{P}^2, d)$ of degree $d$ and class $c$ is dense if $c = 2d - 2$, has codimension 1 if $c = 2d - 3$ and codimension at least 2 for all other values of $c$. (The case $d = 2$ is special. Here $\tau(D)$ could parametrize degree two covers of lines in $\mathbb{P}^2$.)

Second case: $d_1 \geq 1$ and $d_2 \geq 1$. We have that $\tau(D)$ is a subset of the irreducible divisor $D(d_1, d_2)$ on $\overline{M}_{0,0}(\mathbb{P}^2, d)$. Therefore $D$ is irrelevant unless $\tau(D) = D(d_1, d_2)$. Now, by Proposition 3.1 the stable lift of a generic member of $D(d_1, d_2)$ is a map from a curve with three components and the bidegrees of the map, restricted to the three components, are $(d_1, 2d_1 - 2)$, $(0, 2)$ and $(d_2, 2d_2 - 2)$. Thus, if $D$ is relevant then the bidegrees must be $(d_1, 2d_1 - 2)$ and $(d_2, 2d_2)$, or vice versa.

We will next give a detailed description of the relevant divisors. To accomplish this we need to introduce some more special stacks of stable maps.

The stack $M_{0,n+2}(I, (0, 2))$ parametrizes maps of degree 2 from $\mathbb{P}^1$ to a fiber of $I$ over $\mathbb{P}^2$. Let $M_{0,n+2, \{\ast, \diamond\}}(I, (0, 2))$ be the substack in which we demand that the last two markings $\ast$ and $\diamond$ are the ramification points; let $\overline{M}_{0,n, \{\ast, \diamond\}}(I, (0, 2))$ be its closure in $\overline{M}_{0,n+2}(I, (0, 2))$. In the same way, for maps of degree 2 to a fiber of $I$ over $\mathbb{P}^2$, we define $\overline{M}_{0,n, \{\ast, \diamond\}}(I, (2, 0))$.

We define

\begin{equation}
\overline{M}_{0,n, \{\ast\}}(\mathbb{P}^2, (d, 2d)) = \bigcup_{A_1 \cup A_2 = \{1, \ldots, n\}} \overline{M}_{0, A_1 \cup \{\diamond\}}(\mathbb{P}^2, d) \times I \overline{M}_{0, A_2, \{\ast\}}(I, (0, 2))
\end{equation}

where each fiber product is defined by using the evaluation maps corresponding to $\diamond$; it is a closed substack of $\overline{M}_{0,n+1}(I, (d, 2d))$. The general member of each component of $\overline{M}_{0,n, \{\ast\}}(\mathbb{P}^2, (d, 2d))$ represents a map from a curve with two components. The restriction of the map to one component is the strict lift of a map to a rational nodal curve; the restriction to the other component is as described in the previous paragraph. Similarly, we define a closed substack of $\overline{M}_{0,n+1}(I, (2, 1))$ by

\begin{equation}
\overline{M}_{0,n, \{\ast\}}(\mathbb{P}^2, 2) = \bigcup_{A_1 \cup A_2 = \{1, \ldots, n\}} \overline{M}_{0, A_1, \{\ast, \diamond\}}(I, (2, 0)) \times I \overline{M}_{0, A_2 \cup \{\diamond\}}(I, (0, 1)).
\end{equation}

The general member of each component of $\overline{M}_{0,n, \{\ast\}}(\mathbb{P}^2, 2)$ represents a map from a curve with two components. The restriction of the map to one component is a degree 2 cover of a line of $\mathbb{P}^2$; the restriction to the other component is a map of degree 1 to the pencil of all tangent directions through the ramification point $\diamond$ of the first component. The other ramification point is specially marked $\ast$, and there are $n$ additional markings. The reason for the notation is that the bidegree $(2, 1)$ follows the pattern of bidegrees $(d, 2d - 3)$ for cuspidal curves.
Proposition 5.8. Ignoring irrelevant components, we have the following isomorphisms of divisors in $\overline{M}_{0,n}(\mathbb{P}^2, d)$:

1. $D(A_1, A_2, (d, 2d - 2), (0, 0)) \cong \overline{M}^3_{0, A_1 \cup \{\ast\}}(\mathbb{P}^2, d) \times_I \overline{M}_{0, A_2 \cup \{\ast\}}(I, (0, 0))$ for $d \geq 1$.
2. $D(A_1, A_2, (d, 2d - 3), (0, 1)) \cong \overline{C}^3_{0, A_1 \cup \{\ast\}}(\mathbb{P}^2, d) \times_I \overline{M}_{0, A_2 \cup \{\ast\}}(I, (0, 1))$ for $d \geq 2$.
3. $D(A_1, A_2, (d_1, 2d_1 - 2), (d_2, 2d_2)) \cong \overline{M}^3_{0, A_1 \cup \{\ast\}}(\mathbb{P}^2, d_1) \times_I \overline{M}^3_{0, A_2 \cup \{\ast\}}(\mathbb{P}^2, d_2)$ for $d_1, d_2 > 0$ with $d_1 + d_2 = d$.

In particular each relevant component occurs with multiplicity one.

Proof. (1) Note that on the right side we have a fiber product of smooth morphisms, hence an irreducible stack. The forgetful morphism

$$D(A_1, A_2, (d, 2d - 2), (0, 0)) \to \overline{M}^3_{0, A_1 \cup \{\ast\}}(\mathbb{P}^2, d)$$

is the restriction of the projection

$$\overline{M}^3_{0, A_1 \cup \{\ast\}}(I, (d, 2d - 2)) \times_I \overline{M}_{0, A_2 \cup \{\ast\}}(I, (0, 0))$$

onto its first factor. Comparing dimensions, we conclude that we have the desired isomorphism of stacks.

To see that the multiplicity is one, we remark that the lifting map $\lambda$ extends to general points of our divisor. Indeed, a general point represents the lift of a stable map from a curve with two components, consisting of an immersion onto a rational degree $d$ curve together with a constant map to some nonsingular point. The unique stable lift consists of the strict lift of the immersion together with the constant map to the unique tangent direction at the point. (See Figure 5.1.) Since $\lambda$ is a birational morphism, and since $D(A_1, A_2, d, 0)$ is a reduced divisor on $\overline{M}_{0,n}(\mathbb{P}^2, d)$, our divisor is likewise reduced.

(2) For $d \geq 3$ we again observe that on the right side we have a fiber product of smooth morphisms, hence an irreducible stack. By Proposition 5.4, the image $\tau(D)$ of a relevant component $D$ of $D(A_1, A_2, (d, 2d - 3), (0, 1))$ must have codimension at most 1. But a point of $\tau(D)$ cannot represent an immersion onto a nodal curve, since the lift of such a curve has bidegree $(d, 2d - 2)$. Nor can it represent a map to a reducible curve, since by Proposition 3.1 the stable lift of a general map of this type has a component of bidegree $(0, 2)$. Hence $\tau(D)$ must be the divisor on $\overline{M}_{0,0}(\mathbb{P}^2, d)$ representing cuspidal curves. By Proposition 3.2, the stable lift of a general cuspidal curve consists of its strict lift together with a map to the pencil of tangent directions at the cusp. Hence we have the desired isomorphism of stacks (ignoring irrelevant components).

To see that the multiplicity is one, we first consider the case when $A_2$ is empty. By a local calculation we will verify that the lifting map $\lambda$ extends to general points of our divisor. The deformation theory

Figure 5.1 Stable lift of $D(A_1, A_2, (d, 2d - 2), (0, 0))$, with two markings in $A_2$ shown.
of a node degenerating into a cusp is given in local coordinates by \(x^3 + \epsilon x^2 = y^2\). We may parametrize this family by
\[
(\epsilon, t) \mapsto (t^2 - \epsilon, t^3 - t\epsilon).
\]
Then the family of strict lifts is given by
\[
(\epsilon, t) \mapsto \left((t^2 - \epsilon, t^3 - t\epsilon), [2t : 3t^2 - \epsilon]\right)
\]
for \(\epsilon \neq 0\). This map is not determined at \(\epsilon = 0, t = 0\). Blowing up the point of indeterminacy by introducing projective coordinates \([\epsilon : t]\) satisfying \(\epsilon t = t\epsilon\), we extend the map to
\[
(\epsilon, t), [\epsilon : t] \mapsto \left((t^2 - \epsilon, t^3 - t\epsilon), [2t : 3t^2 - \epsilon]t\right).
\]
The special map for \(\epsilon = 0\) is (locally) the stable lift of \(D\). It has two components: the strict lift of the cuspidal curve, with bidegree \((d, 2d - 3)\), together with a map to the pencil of tangent directions at the cusp, with bidegree \((0, 1)\). (See Figure 5.2.) Thus, as claimed, the lifting map \(\lambda\) is defined at the point \(D(A_1, \emptyset, (d, 2d - 3), (0, 1)))\) representing the cuspidal curve. And thus, by the same argument as in part (1), this divisor is reduced.

Figure 5.2 Stable lift of \(D(A_1, A_2, (d, 2d - 3), (0, 1)), d \geq 3\).

Interpreting this calculation in a different way, we have shown that, for the unique stable lift of a stable map to a cuspidal curve, we can build a one-parameter family for which the general member is the strict lift of an immersion and for which the special member is the specified cuspidal stable lift; since the special member appears in the family with multiplicity one, the divisor \(D(A_1, \emptyset, (d, 2d - 3), (0, 1))\) must be reduced. Similarly, we can build such a family when \(A_2\) is nonempty. We assume we are at a general point of the divisor \(D(A_1, A_2, (d, 2d - 3), (0, 1))\), so that the corresponding map has just two components, with the markings indexed by \(A_2\) lying on the \(\mathbb{P}^1\) mapping to the pencil of directions at the cusp. Using the same family of maps as above, we let \(L_1, L_2, \ldots\) be lines through the origin in the \(\epsilon-t\) plane whose directions correspond to these markings, and let the markings on nearby members of the family be the points of intersection with these lines. Then blowing up the origin, as above, creates a family whose special member is the specified cuspidal stable lift, with the specified markings. Since this member appears in the family with multiplicity one, the divisor \(D(A_1, A_2, (d, 2d - 3), (0, 1))\) must be reduced.

We now consider the case \(d = 2\). The image \(\tau(D)\) of a relevant component \(D\) of \(D(A_1, A_2, (2, 1), (0, 1))\) must have codimension at most 1. But a point of \(\tau(D)\) cannot represent a map to a nonsingular conic, since the lift of such a curve has bidegree \((2, 2)\), nor a map to a pair of distinct lines, since by Proposition 3.1 the stable lift of such a map has a component of bidegree \((0, 2)\). Hence \(\tau(D)\) must be the stack of degree 2 covers of lines in \(\mathbb{P}^2\). By Proposition 3.2, we have the desired isomorphism of stacks (again ignoring irrelevant components).
To see that the multiplicity is one, we consider a general point of the divisor. This point represents the stable lift of a degree 2 cover of a line. A deformation of such a map is given by

\[(\epsilon, t) \mapsto (\epsilon t, t^2),\]

and the family of lifts is given by

\[(\epsilon, t) \mapsto \left((\epsilon t^2), [\epsilon : 2t]\right).\]

Blowing up the point of indeterminacy \(\epsilon = t = 0\) by introducing projective coordinates \([\epsilon_1 : t_1]\) satisfying \(\epsilon t_1 = t \epsilon_1\), we extend the family of lifts to

\[\left((\epsilon, t), [\epsilon_1 : t_1]\right) \mapsto \left((\epsilon t^2), [\epsilon_1 : 2t_1]\right).\]

We do a similar blowup at the other ramification point (located at \(t = \infty\)). Then the special member \(\epsilon = 0\) has one component that is the strict lift of the cover of the line and two components mapping into the pencils of tangent directions at the ramification points. (See Figure 5.3.) As in the case \(d \geq 3\), we can arrange for the markings on the special member to appear in any specified configuration. Since the special member appears in the family with multiplicity 1, the divisor \(D(A_1, A_2, (2, 1), (0, 1))\) must be reduced.

![Figure 5.3](image-url)

**Figure 5.3** Stable lift of \(D(A_1, A_2, (d, 2d - 3), (0, 1)), d = 2\).

(3) The image \(\tau(D)\) of a relevant component \(D\) of \(D(A_1, A_2, (d_1, 2d_1 - 2), (d_2, 2d_2))\) must have codimension at most 1. A point of \(\tau(D)\) cannot represent a map to an irreducible curve of degree \(d\). (Nor, in case \(d = 2\), can it represent a double cover of a line.) Hence \(\tau(D)\) must be a special boundary divisor on \(\overline{M}_{0,0}(\mathbb{P}^2, d)\): \(\tau(D) = \overline{M}_{0,\{\ast\}}(\mathbb{P}^2, d_1) \times_{\mathbb{P}^2} \overline{M}_{0,\{\ast\}}(\mathbb{P}^2, d_2)\) for some \(d_1\) and \(d_2\). A general point of \(D\) represents a stable lift of the sort of map described in Proposition 3.1, which tells us that the unique stable lift has three components of bidegrees \((d_1, 2d_1 - 2), (0, 2)\) and \((d_2, 2d_2 - 2)\), and thus shows that we have the desired isomorphism of stacks. (Refer again to Figure 3.3.)

To see that the divisor is reduced, we use the family of maps described in the proof of Proposition 3.1, with total space the surface \(xy = \epsilon^2\). The special member \(\epsilon = 0\) occurs with multiplicity 1; thus the divisor must be reduced. Furthermore we can specify markings on the nearby members so as to obtain any configuration of specified markings on the central component of the special member, as follows. We introduce projective coordinates \([x_1 : y_1 : \epsilon_1]\), so that the central component is the curve \(x_1y_1 = \epsilon_1^2\). Then to obtain \([x_1 : y_1 : \epsilon_1]\) as a marking on the central component, we use \((\frac{\partial}{\partial \epsilon_1}, \frac{\partial}{\partial \epsilon_1} \epsilon, \epsilon)\) as a marking on the general member. \(\square\)
6. Quantum potentials

We now introduce potential functions for each of the seven different kinds of stable maps we are considering. We are not concerned with questions of convergence, and all of these potentials should be interpreted as formal power series in the indeterminates $y_0, \ldots, y_5$, where

$$\gamma = y_0 T_0 + \cdots + y_5 T_5$$

is an arbitrary element of $A^*(I) \otimes \mathbb{Q}$ (or in some cases as formal power series in two sets of indeterminates). Our definitions are inspired by Proposition 5.8 and the auxiliary equations (5.6), (5.7).

We define $P$ to be the classical potential of the incidence correspondence:

$$P = \sum_{n \geq 3} \frac{1}{n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap \overline{\mathcal{M}}_{0,n}(I, (0, 0)).$$

We begin the summation at $n = 3$ since otherwise the moduli stacks are empty. In fact the only nonzero term is the first one, which encodes the intersection product:

$$P = y_2 y_5^2 + y_0 y_1 y_4 + y_0 y_2 y_3 + y_1^2 y_3^2.$$

Similarly, we define a quantum potential

$$\mathcal{N} = \sum_{n \geq 0} \frac{1}{n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap \overline{\mathcal{M}}_{0,n}(P^2, d).$$

By Proposition 4.3 we have

$$\mathcal{N} = \sum_{d \geq 1} \sum_{a+b+c+2d=3d-1 \atop a, b, c \geq 0} N_d(a, b, c) y_2^a y_4^b y_5^c \exp(dy_1 + (2d - 2)y_3).$$

To define the quantum potential for cuspidal stable lifts we need to introduce a second arbitrary cohomology class

$$\delta = z_0 T_0 + \cdots + z_5 T_5.$$

The potential is

$$C = \sum_{n \geq 0} \frac{1}{n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cup e_1^*(\delta) \cap \overline{\mathcal{M}}_{0,n,\{\ast\}}(P^2, d)$$

$$= \sum_{d \geq 2} \sum_{a+b+c+2d=3d-2-\dim(T_s) \atop a, b, c \geq 0} z_s C_d(a, b, c; T_s) y_2^a y_4^b y_5^c \exp(dy_1 + (2d - 3)y_3).$$

We will also need three potentials corresponding to maps to a projective line. The potential for stable maps of degree 1 to a fiber of $I$ over $P^2$ is

$$\mathcal{F} = \sum_{n \geq 0} \frac{1}{n!} \int e_1^*(\gamma) \cup \cdots \cup e_n^*(\gamma) \cap \overline{\mathcal{M}}_{0,n}(I, (0, 1))$$

$$= \left( \frac{y_1^2}{2} + y_5 \right) \exp(y_3).$$
The potential for stable maps of degree 2 to such a fiber, with the ramification at two specially marked points, is

\[ R = \sum_{n \geq 0} \frac{1}{2n!} \int e_1^* (\gamma) \cup \cdots \cup e_n^* (\gamma) \cup e_{n+1}^* (\delta) \cup \cup e_{n+2}^* (\delta) \cap [\overline{M}_{0,n,\{\ast, \ast\}} (I, (0, 2))]. \]

Here it is important, in case \( n = 0 \), that we use the stack of stable maps of this type, since every such map has a nontrivial automorphism. To write \( R \) in an explicit way, we use the Gromov-Witten invariants for such maps:

\[ R(a, b, c; \delta_1 \cdot \delta_2) = \int e_1^* (\gamma_1) \cup \cdots \cup e_n^* (\gamma_1) \cup e_{n+1}^* (\delta_1) \cup e_{n+2}^* (\delta_2) \cap [\overline{M}_{0,n,\{\ast, \ast\}} (I, (0, 2))], \]

where \( a \) of the \( \gamma_i \)'s equal the class \( h^2 \), \( b \) of them equal the class \( \bar{h}^2 \), and the remaining \( c \) of them equal \( h^2 \bar{h} \). Unless \( n = 0 \) we may interpret these as the number of maps satisfying \( a \) point conditions, \( b \) tangency conditions and \( c \) flag conditions, plus the conditions \( \delta_1 \) and \( \delta_2 \) at the two ramification points; if \( n = 0 \), however, the invariant is a fraction:

\[
\begin{align*}
R(0, 2, 0; T_3 \cdot T_3) &= 2, \\
R(0, 0, 1; T_3 \cdot T_3) &= 1, \\
R(0, 1, 0; T_3 \cdot T_4) &= 1, \\
R(0, 0, 0; T_4 \cdot T_4) &= \frac{1}{2}.
\end{align*}
\]

Thus

\[ R = \left\{ \frac{z_2^2}{2} (y_2^2 + y_5) + z_3 z_4 y_4 + \frac{z_4 z_5}{2} + \frac{z_5^2}{4} \right\} \exp(2y_3). \]

Similarly, the potential for stable maps of degree 2 to a fiber of \( I \) over the dual projective plane \( \bar{\mathbb{P}}^2 \), with the ramification at two specially marked points, is

\[ \mathcal{L} = \sum_{n \geq 0} \frac{1}{2n!} \int e_1^* (\gamma) \cup \cdots \cup e_n^* (\gamma) \cup e_{n+1}^* (\delta) \cup \cup e_{n+2}^* (\delta) \cap [\overline{M}_{0,n,\{\ast, \ast\}} (I, (2, 0))]. \]

\[ = \left\{ \frac{z_2^2}{2} (y_2^2 + y_5) + z_1 z_2 y_2 + \frac{z_1 z_5}{2} + \frac{z_5^2}{4} \right\} \exp(2y_1). \]

Finally we have the potential \( \mathcal{E} \) for maps represented by the stacks \( \overline{M}_{0,n,\{\ast, \ast\}} (\mathbb{P}^2, (d, 2d)) \):

\[ \mathcal{E} = \sum_{n \geq 0} \frac{1}{n!} \int e_1^* (\gamma) \cup \cdots \cup e_n^* (\gamma) \cup e_{n+1}^* (\delta) \cap [\overline{M}_{0,n,\{\ast\}} (\mathbb{P}^2, (d, 2d))]. \]

\[ = \sum_{d \geq 1} \sum_{a+b+c=3d+1-\dim(T_x)} \sum_{a,b,c \geq 0 \atop s=0,\ldots,5} \frac{z_5 E_d(a, b, c; T_x) y_5^3}{a! b! c!} \exp(dy_1 + (2d-3)y_5), \]

where \( E_d(a, b, c; \delta) \) denotes the Gromov-Witten invariant for such maps subject to \( a \) point conditions, \( b \) tangency conditions and \( c \) flag conditions, and subject to the condition \( \delta \) at the ramification point.

Each of the divisors described in Proposition 5.8 is a fiber product \( M_1 \times_I M_2 \) inside the moduli stack \( \overline{M}_{0,n}(I, (d, 2d - 2)) \), and each therefore fits into a fiber square

\[
\begin{array}{ccc}
M_1 \times_I M_2 & \overset{1}{\to} & M_1 \times M_2 \\
\downarrow & & \downarrow \\
I^{n+1} & \overset{\Delta}{\to} & I^{n+2}.
\end{array}
\]
in which $\Delta$ is the diagonal inclusion that repeats the last factor. The components of the vertical
morphism on the right are the various evaluation maps to $I$; the last two $\epsilon_{n+1}$ and $\epsilon_{n+2}$ become equal
when restricted to $M_1 \times I M_2$. By the decomposition (5.3) of the diagonal class, we have

$$\nu_* \left( \epsilon_1^* (\gamma_1) \cup \cdots \cup \epsilon_n^* (\gamma_n) \right) = \sum_{s=0}^5 \epsilon_1^* (\gamma_1) \cup \cdots \cup \epsilon_n^* (\gamma_n) \cup \epsilon_{n+1}^* (T_s) \cup \epsilon_{n+2}^* (T_{5-s}).$$

There is a partition $A_1 \cup A_2 = \{1, \ldots, n\}$, so that the evaluation maps $\epsilon_t$ indexed by $t \in A_1$ factor
through $M_1$ and those indexed by $A_2$ factor through $M_2$. Taking degrees in (6.1), we obtain the
following Proposition. (For details of this computation, see [Ful93, Lemma, p.15].)

**Proposition 6.2.** In this situation

$$\int e_1^* (\gamma_1) \cup \cdots \cup e_n^* (\gamma_n) \cap [M_1 \times I M_2]$$

$$= \sum_{s=0}^5 \int \bigcup_{t \in A_1} e_1^* (\gamma_1) \cup e_{n+1}^* (T_s) \cap [M_1] \cdot \int \bigcup_{t \in A_2} e_1^* (\gamma_1) \cup e_{n+2}^* (T_{5-s}) \cap [M_2].$$

To use Proposition 6.2, we need the following elementary observation, which is a consequence of the
chain rule.

**Proposition 6.3.** Let $\gamma = y_0 T_0 + y_1 T_1 + \cdots + y_5 T_5$ and $\delta = z_0 T_0 + z_1 T_1 + \cdots + z_5 T_5$. Suppose that $\mathcal{K}$
is a power series in $y_0, \ldots, y_5, z_0, \ldots, z_5$ which can be written in the following way:

$$\mathcal{K} = \sum_{n \geq 0} K(n; \gamma^m) n! m!.$$

Let $k_1, \ldots, k_r$ and $l_1, \ldots, l_s$ have values in $\{0, \ldots, 5\}$. Then

$$\frac{\partial \mathcal{K}}{\partial y_{k_1} \cdots \partial y_{k_r} \partial z_{l_1} \cdots \partial z_{l_s}} = \sum_{n \geq 0} K(n; \gamma^m) \frac{\partial^r \gamma^m}{\partial y_{k_1} \cdots \partial y_{k_r}} \frac{\partial^s \gamma^m}{\partial z_{l_1} \cdots \partial z_{l_s}}.$$

**Proposition 6.4.** $\mathcal{E} = \sum_{s=0}^5 \frac{\partial N}{\partial y_s} \frac{\partial R}{\partial z_{5-s}}$.

**Proof.** Set $\gamma_1 = \cdots = \gamma_n = \gamma = y_0 T_0 + y_1 T_1 + \cdots + y_5 T_5$. Apply Proposition 6.2 to each component of
$\mathcal{M}_{0,n,\{+\}}(\mathbb{P}^2, (d, 2d))$, as listed in (5.3). Sum this identity over all components, over all $d \geq 1$, and over
all $n \geq 1$. Then apply Proposition 6.3.

Let $C_2$ be the leading term $(d = 2)$ of the potential $C$. Then a similar argument, applied to (5.4),
yields the following identity.

**Proposition 6.5.** $C_2 = \sum_{s=0}^5 \frac{\partial L}{\partial z_s} \frac{\partial F}{\partial y_{5-s}}$.

The same sort of argument yields the basic identity from which we will derive recursive equations
for characteristic numbers. Let $i, j, k,$ and $l$ be integers between 1 and 5. Let

$$\mathcal{G}(ij \mid kl) = \sum_{n \geq 4} \int e_1^* (T_i) \cup e_2^* (T_j) \cup e_3^* (T_k) \cup e_4^* (T_l) \cup e_5^* (\gamma) \cup \cdots \cup e_n^* (\gamma) \cap [D(12 \mid 34)].$$
find that the only nonzero cuspidal characteristic numbers in degree $C$

calculate the corresponding potential $E$

Then, by (5.1), Corollary 5.5 and Proposition 5.8,

The linear equivalence of the divisors $D(12 | 34)$ and $D(13 | 24)$ immediately implies the basic identity, which we record as a Theorem.

**Theorem 6.6.** For each $i, j, k$ and $l$ in $\{1, 2, 3, 4, 5\}$, there is an identity

\[ G(ij \mid kl) = G(il \mid jk). \]

### 7. Recursive Formulas for Characteristic Numbers

In this section we will derive recursive formulas for the characteristic numbers $N_d(a, b, c)$, $C_d(a, b, c; 1)$, $C_d(a, b, c; h)$, and $C_d(a, b, c; h^2)$. For characteristic numbers of the first type, the base cases are

\[ N_1(2, 0, 0) = N_1(0, 0, 1) = 1 \quad \text{and} \quad N_1(0, 2, 0) = 0. \]

The cuspidal characteristic numbers make sense only for $d \geq 2$. In the case $d = 2$ they are defined using the stack $\overline{M}_{0,n,1}(\mathbb{P}^2, 2)$ whose decomposition is shown in (7.7). Proposition 5.3 tells us how to calculate the corresponding potential $C_2$ from the known potentials $L$ and $F$. Equating coefficients, we find that the only nonzero cuspidal characteristic numbers in degree $d = 2$ are

\[
\begin{align*}
C_2(2, 1, 0; T_1) & = 2, \quad C_2(0, 1, 1; T_1) = 1, \quad C_2(1, 2, 0; T_1) = 1, \quad C_2(1, 0, 1; T_1) = 1, \\
C_2(1, 1, 0; T_2) & = 1, \quad C_2(0, 2, 0; T_2) = \frac{1}{2}, \quad C_2(0, 0, 1; T_2) = \frac{1}{2}, \quad C_2(0, 1, 0; T_5) = \frac{1}{2}.
\end{align*}
\]

The fact that some of these characteristic numbers are fractions reflects the fact that the generic map of this type has a nontrivial automorphism. One could also calculate these characteristic numbers directly, again taking into account the automorphisms.

In addition to the desired characteristic numbers, we will also calculate the characteristic numbers $E_d(a, b, c; T_s)$ for maps represented by the stacks $\overline{M}_{0,n,1}(\mathbb{P}^2, (d, 2d))$, which appear as coefficients in the potential $E$. We are not particularly interested in these numbers, but using them simplifies many of the formulas.

**Proposition 7.1.**

\[
\begin{align*}
E_d(a, b, c; \hat{h}) & = dbN_d(a, b - 1, c) + b(b - 1)N_d(a + 1, b - 2, c) + cN_d(a + 1, b, c - 1) \quad \text{for} \ a + b + 2c = 3d; \\
E_d(a, b, c; h^2) & = \frac{d}{2}N_d(a, b, c) + bN_d(a + 1, b - 1, c) \quad \text{for} \ a + b + 2c = 3d - 1; \\
E_d(a, b, c; h^2 \hat{h}) & = \frac{1}{2}N_d(a + 1, b, c) \quad \text{for} \ a + b + 2c = 3d - 2; \\
E_d(a, b, c; T_s) & = 0 \quad \text{if} \ s = 0, 1 \ \text{or} \ 2.
\end{align*}
\]

**Proof.** Use the identity of Proposition 5.3, equate the coefficients. \hfill \square

**Theorem 7.2.** There is a recursive algorithm, based on Theorem 6.4, for calculating all the characteristic numbers $N_d(a, b, c)$, $C_d(a, b, c; 1)$, $C_d(a, b, c; h)$, $C_d(a, b, c; h^2)$, $E_d(a, b, c; \hat{h})$, $E_d(a, b, c; h)$, and $E_d(a, b, c; h^2 \hat{h})$. 
Pandharipande [Pan95, Proposition 4] proved that there is an explicit algorithm for calculating all the numbers \(N_d(a, b, 0)\) and gave explicit formulas for the numbers \(C_d(3d-2, 0, 0; 1)\) [Pan95, Proposition 5].

**Proof.** The proof will use induction on \(d\). Assume that all characteristic numbers have been determined for degrees less than \(d\). Note that the last three types of characteristic numbers (the \(E_d\)'s) are determined by the other types; hence we need only to worry about the \(N_d\)'s and \(C_d\)'s. To simplify the argument we will denote by \([d-1]\) any expression involving characteristic numbers for curves of degree \(d-1\) or less.

First we list various cases of Theorem 6.6 which immediately determine a characteristic number in degree \(d\); each of these identities can be written in the form

\[N_d(a, b, c) = [d-1].\]

Equation 1122 (meaning this case of the identity in Theorem 5.6) determines \(N_d(a, b, c)\) for \(a \geq 3\).
Equation 1224 determines \(N_d(a, b, c)\) for \(a \geq 2\) and \(c \geq 1\).
Equation 1155 determines \(N_d(a, b, c)\) for \(a \geq 1\) and \(c \geq 2\).
Equation 1455 determines \(N_d(a, b, c)\) for \(c \geq 3\).
This leaves only six of the numbers \(N_d(a, b, c)\) undetermined, namely

\[N_d(2, 3d-3, 0), \quad N_d(1, 3d-4, 1), \quad N_d(0, 3d-5, 2), \quad N_d(1, 3d-2, 0), \quad N_d(0, 3d-3, 1), \quad N_d(0, 3d-1, 0).\]

Similarly (for \(d \geq 3\)) there are equations of the form:

\[C_d(a, b, c; h^2) = [d-1].\]

Equation 2445 determines \(C_d(a, b, c; h^2)\) for \(a \geq 1\) and \(c \geq 1\).
Equation 4455 determines \(C_d(a, b, c; h^2)\) for \(a \geq 2\) and \(c \geq 1\).
Equation 2244 determines \(C_d(a, b, c; h^2)\) for \(a \geq 2\).
This leaves

\[C_d(1, 3d-5, 0; h^2), \quad C_d(0, 3d-6, 1; h^2), \quad C_d(0, 3d-4, 0; h^2)\]

undetermined.

Next, we exhibit in matrix form 8 independent equations involving the undetermined characteristic numbers listed above, except \(N_d(0, 3d-1, 0)\).

\[
\begin{pmatrix}
2(3d-3) & -d & 0 & 0 & 0 & 0 & 0 & 0 \\
2(d-2) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3d-4 & -2d & 0 & 0 & 0 & 0 & 0 \\
0 & 3d-5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3d-3 & -d+4 & 0 & (3d-4)(3d-3) & 0 & 0 & (3d-4)d \\
0 & d-2 & 0 & 0 & 0 & -d(3d-3) & 0 & d^2 \\
0 & 0 & 0 & 0 & -d(3d-4) & d^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
N_d(2, 3d-3, 0) \\
N_d(1, 3d-4, 1) \\
N_d(0, 3d-5, 2) \\
N_d(1, 3d-2, 0) \\
N_d(0, 3d-3, 1) \\
C_d(1, 3d-5, 0, h^2) \\
C_d(0, 3d-6, 1, h^2) \\
C_d(0, 3d-4, 0, h^2)
\end{pmatrix} = [d-1]
\]

The rows of this equation correspond to the following equations: 1124, 1145, 1123, 1445, 2345, 1345, 1135, and 1135 again (for a different choice of \(a, b, \) and \(c\)). The determinant of the matrix is

\[-12d^6(3d-5)(d-1)(3d-4)(3d-2),\]

which is nonzero for all integers \(d \geq 2\). Thus these equations determine the eight degree \(d\) characteristic numbers listed in the column vector.
Next, the equation 1134 can be written
\[ C_d(a, b, c; h) = -C_d(a, b - 1, c; h^2)b + \frac{1}{d^2} \left\{ N_d(a, b, c + 1)d(2d - 2) + N_d(a + 1, b + 1, c)(2 - d) + N_d(a, b + 2, c)d \right\} + [d - 1]. \]

It determines all the numbers \( C_d(a, b, c; h) \) except \( C_d(0, 3d - 3, 0; h) \).

Equation 1133 can be written
\[ C_d(a, b, c; 1) = -C_d(a, b - 1, c; h)b - C_d(a, b, c - 1; h^2)c - C_d(a, b - 2, c; h^2)^{\left(\begin{array}{c} b \\ 2 \end{array}\right)} + \frac{1}{d^2} \left\{ 4N_d(a + 1, b, c)(d - 1) + N_d(a, b + 1, c)(3d^2 - 4d) \right\} + [d - 1] \]

It determines all the numbers \( C_d(a, b, c; 1) \) except \( C_d(0, 3d - 2, 0; 1) \).

It remains to determine the three numbers \( N_d(0, 3d - 1, 0), C_d(0, 3d - 3; 0; h) \) and \( C_d(0, 3d - 2; 0; 1) \).

The two equations 1133 and 1134 just listed give two equations. Another is given by 3344. To show that they are linearly independent we write the equations in matrix form and compute the determinant.

\[
\begin{pmatrix}
(3d - 1)d(4 - 3d) \\
-(3d - 2)(3d - 1)d \\
(3d - 3)(3d - 2)(3d - 1)
\end{pmatrix}
\begin{pmatrix}
d^2 \\
d^2 \\
(3d - 3)(7d + 2)
\end{pmatrix}
\begin{pmatrix}
N_d(0, 3d - 1, 0) \\
C_d(0, 3d - 3, 0; h) \\
C_d(0, 3d - 2, 0; 1)
\end{pmatrix} = \begin{pmatrix}
\text{known} \\
\text{known} \\
\text{known}
\end{pmatrix}
\]

Here, “known” means expressions involving already determined numbers of degree \( d \) or less. The determinant of the matrix is
\[ 6d(d - 1)(3d - 1)(3d - 2)(d^2 - 4d - 1), \]

which is nonzero for all integers \( d \geq 2 \). This shows that we can find the characteristic numbers in degree \( d \) from those in characteristic degree \( d - 1 \).

We will next give an explicit recursive algorithm, developed and implemented by the first author, which determines all the characteristic numbers of Theorem 7.2. The algorithm will use 7 equations compared to the 17 used in the proof of Theorem 7.2. A drawback is, however, that the recursion is not as transparent as the recursion of the proof. Some degree \( d + 1 \) characteristic numbers must be computed before all numbers in degree \( d \) may be computed.

Using equation 1122 we derive the following formula, which we call equation 1122a. We must assume that \( a \geq 3 \). In all sums \( d_1, d_2 \geq 0 \) and \( a_1, a_2, b_1, b_2, c_1, c_2 \geq 0 \).

\[ N_d(a, b, c) = \sum_{d_1 + d_2 = d}^{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; h^2) \left[ 2d_1^2d_2 \left( a_1 - 1 \right) - d_1^2d_2 \left( a_1 \right) - d_1d_2^2 \left( a_1 - 1 \right) \right] \left( b_1 \right) \left( c_1 \right) \]

\[ + \sum_{d_1 + d_2 = d}^{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; h) \left[ 2d_1d_2 \left( a_1 - 2 \right) - d_1^2 \left( a_1 \right) - d_2^2 \left( a_1 - 2 \right) \right] \left( b_1 \right) \left( c_1 \right) \]

If we put \( a = 3d - 1 \) and \( b = c = 0 \) we recover Kontsevich’s formula [KM94, Claim 5.2.1]. If we put \( c = 0 \) we recover the recursive formula of Francesco and C. Itzykson [FI95, 2.95, p.104].
The next equation 1122b is a rewrite of 1122a in a very specific case. We use 1122a with \(d\) replaced by \(d+1\), \(a = 3\), \(b = 3d - 1\) and \(c = 0\). Then we solve for \(N_d(0, b, 0)\), which can only occur in the first sum of the right hand side of 1122a. There are two cases:

\[
(d_1, a_1, b_1, c_1) = (d, 0, 0, b), \quad (d_2, a_2, b_2, c_2) = (1, 2, 0, 0),
\]

and the same values with the indexes 1 and 2 interchanged. Using Proposition 7.1 we find

\[
E_1(2, 0, 0; \hat{h}^2) = \frac{1}{2}, \quad E_d(0, b, 0; \hat{h}^2) = \frac{d}{2} N_d(0, b, 0) + b N_d(1, b - 1, 0).
\]

Then we have the following equation called 1122b:

\[
N_d(0, b, 0) = \frac{1}{d^3} \left\{ - N_{d+1}(3, b, 0) - d^2 b N_d(1, b - 1, 0) \right. \\
+ \sum_{d_1 + d_2 = d+1} N_{d_1}(a_1, b_1, 0) E_{d_2}(a_2, b_2, 0; \hat{h}^2) \left[ 2d_1^2 d_2 \left( \frac{0}{a_1 - 1} \right) - d_1^3 d_2 \left( \frac{0}{a_1} \right) - d_1 d_2^2 \left( \frac{0}{a_1 - 1} \right) \right] \left( \frac{b}{b_1} \right) \\
+ \sum_{d_1 + d_2 = d+1} N_{d_1}(a_1, b_1, c_1) E_{d_2}(a_2, b_2, c_2; \hat{h}) \left[ 2d_1 d_2 \left( \frac{0}{a_1 - 2} \right) - d_1^2 \left( \frac{0}{a_1 - 1} \right) - d_2^2 \left( \frac{0}{a_1 - 2} \right) \right] \left( \frac{b}{b_1} \right)
\]

Equation 1155: If \(a \geq 1\) and \(c \geq 2\) then

\[
N_d(a, b, c) = \\
\sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1) E_{d_2}(a_2, b_2, c_2; \hat{h}^2) \left[ 2d_1^2 d_2 \left( \frac{c - 2}{c_1 - 1} \right) - 2d_1^3 d_2 \left( \frac{c - 2}{c_1 - 2} \right) - d_1^3 \left( \frac{c - 2}{c_1} \right) \right] \left( \frac{a - 1}{a_1} \right) \left( \frac{b}{b_1} \right) \\
+ \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1) E_{d_2}(a_2, b_2, c_2; \hat{h}) \left[ 2d_1 d_2 \left( \frac{c - 2}{c_1 - 1} \right) - d_1^2 \left( \frac{c - 2}{c_1 - 2} \right) - d_2^2 \left( \frac{c - 2}{c_1 - 2} \right) \right] \left( \frac{a - 1}{a_1} \right) \left( \frac{b}{b_1} \right)
\]

Equation 1123a: If \(c \geq 1\) then

\[
N_d(a, b, c) = \frac{1}{d^2} \left\{ N_d(a + 1, b + 1, c - 1) - N_d(a + 2, b, c - 1)(d - 2) \right. \\
+ \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1) E_{d_2}(a_2, b_2, c_2; \hat{h}^2) \left[ 2d_1^2 d_2 \left( \frac{a}{a_1 - 1} \right) - 2d_1^3 d_2 \left( \frac{a}{a_1} \right) \right] \left( \frac{b}{b_1} \right) \left( \frac{c - 1}{c_1} \right) \\
+ \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1) E_{d_2}(a_2, b_2, c_2; \hat{h}) \left[ 2d_2^2 \left( \frac{a}{a_1 - 2} \right) - 2d_1 d_2 \left( \frac{a}{a_1 - 1} \right) \right] \left( \frac{b}{b_1} \right) \left( \frac{c - 1}{c_1} \right) \left\} 
\]
Then we solve for two cases:

To get a recursion equation, we rewrite Equation 2245 with

\[ N_d(a, b, c) = (d - 2)N_d(a + 1, b - 1, c) + d^2 N_d(a - 1, b - 1, c + 1) \]

\[ + \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; \hat{d}) \left[ 2d_1^2 d_2 \left( \frac{a - 1}{a_1} \right) - 2d_1 d_2^2 \left( \frac{a - 1}{a_1} \right) \right] \left( \frac{b - 1}{b_1} \right) \left( \frac{c}{c_1} \right) \]

\[ \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; \hat{d}) \left[ 2d_1 d_2 \left( \frac{a - 1}{a_1} \right) - 2d_2 d_1^2 \left( \frac{a - 1}{a_1} \right) \right] \left( \frac{b - 1}{b_1} \right) \left( \frac{c}{c_1} \right) \]

Equation 1123b: This is just a rewrite of 1123a. If \( a \geq 1 \) and \( b \geq 1 \) then

\[ N_d(a, b, c) = (d - 2)N_d(a + 1, b - 1, c) + d^2 N_d(a - 1, b - 1, c + 1) \]

\[ + \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; \hat{d}) \left[ 2d_1^2 d_2 \left( \frac{a - 1}{a_1} \right) - 2d_1 d_2^2 \left( \frac{a - 1}{a_1} \right) \right] \left( \frac{b - 1}{b_1} \right) \left( \frac{c}{c_1} \right) \]

Equation 2245: This is not an obvious recursion equation. Given \( a \geq 2, b \geq 1, c \geq 1 \) and \( d \) with \( a + b + 2c = 3d - 2 \), we have that

\[ 0 = \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; \hat{d})d_1 \left[ \left( \frac{a - 2}{a_1 - 2} \right) \left( \frac{b - 1}{b_1} \right) \left( \frac{c - 1}{c_1 - 1} \right) + \left( \frac{a - 2}{a_1} \right) \left( \frac{b - 1}{b_1 - 1} \right) \left( \frac{c - 1}{c_1 - 1} \right) \right] \]

\[ + \sum_{d_1 + d_2 = d} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; \hat{d})d_1 \left[ \left( \frac{a - 2}{a_1 - 3} \right) \left( \frac{b - 1}{b_1} \right) \left( \frac{c - 1}{c_1} \right) + \left( \frac{a - 2}{a_1} \right) \left( \frac{b - 1}{b_1 - 1} \right) \left( \frac{c - 1}{c_1} \right) \right] \]

To get a recursion equation, we rewrite Equation 2245 with \( d \) replaced by \( d + 1 \), \( a = 2 \) and \( b + c = 3d - 1 \). Then we solve for \( N_d(0, b, c) \), which can only occur in the first sum of the right hand side of 2245. There are two cases:

\[ (d_1, a_1, b_1, c_1) = (d, 0, b, c), \quad (d_2, a_2, b_2, c_2) = (1, 2, 0, 0), \]

and the same values with the indexes 1 and 2 interchanged. Using Proposition 7.4, we find

\[ E_1(2, 0, 0; \hat{d}) = \frac{1}{2}, \quad E_d(0, b, c; \hat{d}) = \frac{d}{2} N_d(0, b, c) + bN_d(1, b - 1, c). \]
We have, for \( b \geq 1, c \geq 1 \), equation 2245:

\[
N_d(0, b, c) = \frac{1}{d!} \left\{ -bN_d(1, b - 1, c) - \sum_{d_1 + d_2 = d + 1} N_d(a_1, b_1, c_1)E_d(a_2, b_2, c_2; h^2)d_1 \left[ \binom{0}{a_1 - 2}\binom{0}{b_1}\binom{0}{c_1} + \binom{0}{a_1}\binom{b - 1}{b_1}\binom{c - 1}{c_1}\right] \right. \\
- \left. \binom{0}{a_1 - 2}\binom{b - 1}{b_1}\binom{c - 1}{c_1}\right\}
\]

Now we are ready to present the algorithm for calculating the numbers \( N_d(a, b, c) \) with \( a + b + 2c = 3d - 1 \). The inputs to the algorithm consist of the trivial degree 1 numbers, which are zero except for \( N_1(2, 0, 0) = 1 \) and \( N_1(0, 0, 1) = 1 \). It is also understood that the instruction “use equation \( ijkl \)” means to use this equation and then to replace all occurrences of the \( E_d \)'s by means of Proposition 7.1.

\[
N_d(a, b, c) := \text{if } (d = 1) \text{ then if } (a = 2 \text{ and } b = 0 \text{ and } c = 0) \text{ then } 1 \text{ else if } (a = 0 \text{ and } b = 0 \text{ and } c = 1) \text{ then } 1 \text{ else 0} \text{ else if } ((a \geq 4) \text{ or } (a = 3 \text{ and } c \geq 1)) \text{ then use equation 1122a} \text{ else if } (a \geq 1 \text{ and } c \geq 2) \text{ then use equation 1155} \text{ else if } (c \geq 3) \text{ then use equation 1123a} \text{ else if } ((a = 3 \text{ and } c = 0) \text{ or } (a = 2 \text{ and } c = 1) \text{ or } (a = 1 \text{ and } c = 1) \text{ or } (a = 2 \text{ and } c = 0) \text{ or } (a = 1 \text{ and } c = 0)) \text{ then use equation 1123b} \text{ else if } ((a = 0 \text{ and } c = 2) \text{ or } (a = 0 \text{ and } c = 1)) \text{ then use equation 2245} \text{ else use equation 1122b.}
\]

We finally present tables of characteristic numbers for \( d \leq 5 \). In the tables of characteristic numbers \( N_d(a, b, c) \), columns have the same \( a \), rows have the same \( c \), and \( b \) is given by \( 3d - 1 - a - 2c \).

| \( c \) | \( N_2(a, b, c) \) |
|---|---|
| 2 | 1 1 |
| 1 | 1 2 2 1 |
| 0 | 1 2 4 4 2 1 |
| 0 \( a \rightarrow \) | 0 \( 1 \) 2 3 4 5 |

Table 7.1 Characteristic numbers for plane conics.
Table 7.2 Characteristic numbers for plane rational nodal cubics.

The numbers for nodal cubics were calculated by Zeuthen [Zeu72], Mâillard [Mai71], Schubert [Sch79], Sacchiero [Sac85], Kleiman and Speiser [KS87], Aluffi [Alu91] and Pandharipande [Pan95].

| c ↓ | 4 | 4 |
|-----|---|---|
| 3   | 16 | 12 | 6 |
| 2   | 56 | 56 | 40 | 20 | 8 |
| 1   | 148 | 200 | 196 | 136 | 68 | 28 | 10 |
| 0   | 400 | 600 | 756 | 712 | 480 | 240 | 100 | 36 | 12 |
| a → | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Table 7.3 Characteristic numbers for plane rational nodal quartics.

The numbers for nodal quartics with \( c = 0 \) were calculated by Pandharipande [Pan95].

| c ↓ | 5 | 120 | 60 |
|-----|---|-----|----|
| 4   | 816 | 528 | 264 | 108 |
| 3   | 5040 | 3960 | 2472 | 1224 | 504 | 180 |
| 2   | 26408 | 25352 | 19424 | 11840 | 5816 | 2408 | 872 | 284 |
| 1   | 124592 | 140912 | 130824 | 97496 | 58208 | 28392 | 11792 | 4304 | 1416 | 428 |
| 0   | 581904 | 728160 | 783584 | 699216 | 505320 | 295544 | 143040 | 72000 | 2184 | 620 |
| a → | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Table 7.4 Characteristic numbers for plane rational nodal quintics.

The numbers with \( c = 0 \) and \( a \geq 3 \) were calculated by Francesco and Itzykson [FI95, 2.97, p.104].

We use the same kind of equations to derive the characteristic numbers \( C_d(a, b, c; 1) \), \( C_d(a, b, c; h) \) and \( C_d(a, b, c; h^2) \).
Equation 1144: For any $a$, $b$ and $c$ with $a+b+2c=3d-4$ we have that

$$C_d(a, b, c; h^2) = \frac{1}{d^2} \left\{ 2dN_d(a, b+1, c+1) - N_d(a+1, b+2, c) \right\}$$

$$+ \sum_{d_1+d_2=d \atop a_1+a_2=a \atop b_1+b_2=b+2 \atop c_1+c_2=c} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; h^2) \left[ 2d_1^2d_2 \left( \frac{b}{b_1} - 1 \right) - d_1d_2^2 \left( \frac{b}{b_1} - 2 \right) - d_1^3 \left( \frac{b}{b_1} \right) \right] \left( \frac{a}{a_1} \right) \left( \frac{c}{c_1} \right)$$

$$+ \sum_{d_1+d_2=d \atop a_1+a_2=a+1 \atop b_1+b_2=b+2 \atop c_1+c_2=c} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; h) \left[ 2d_1d_2 \left( \frac{b}{b_1} - 1 \right) - d_1^2 \left( \frac{b}{b_1} \right) - d_2^2 \left( \frac{b}{b_1} - 2 \right) \right] \left( \frac{a}{a_1} \right) \left( \frac{c}{c_1} \right)$$

In the tables of the numbers $C_d(a, b, c; h^2)$ the number of tangent conditions $b$ is given by $3d-4-a-2c$.

\begin{table}[h]
\centering
\begin{tabular}{|c|ccc|}
\hline
$c$ & $C_3(a, b, c; h^2)$ \\
\hline
2 & 4 & 2 \\
1 & 14 & 12 & 6 & 2 \\
0 & 32 & 41 & 38 & 20 & 8 & 2 \\
\hline
$a \to$ & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\end{tabular}
\caption{Characteristic numbers for plane rational 1-cuspidal cubics; cusp at a specified point.}
\end{table}

The numbers for cubics were calculated by Schubert \cite{Sch79}, by Kleiman and Speiser \cite{KS86}, and by Aluffi \cite{Alu91}.

\begin{table}[h]
\centering
\begin{tabular}{|c|ccc|}
\hline
$c$ & $C_4(a, b, c; h^2)$ \\
\hline
4 & 18 \\
3 & 132 & 72 & 30 \\
2 & 816 & 580 & 312 & 132 & 46 \\
1 & 4084 & 3760 & 2636 & 1420 & 616 & 224 & 70 \\
0 & 17444 & 19912 & 17904 & 13292 & 6700 & 2964 & 1112 & 364 & 102 \\
\hline
$a \to$ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\end{tabular}
\caption{Characteristic numbers for plane rational 1-cuspidal quartics; cusp at a specified point.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|ccc|}
\hline
$c$ & $C_5(a, b, c; h^2)$ \\
\hline
5 & 1080 & 480 \\
4 & 10728 & 5808 & 2592 & 984 \\
3 & 95760 & 61872 & 32988 & 14736 & 5664 & 1920 \\
2 & 747952 & 575864 & 366256 & 193744 & 86864 & 33832 \\
1 & 5169728 & 4692096 & 354408 & 2224088 & 1170192 & 526608 \\
0 & 33071072 & 34336864 & 30314016 & 22428704 & 13882384 & 7264872 \\
\hline
$a \to$ & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\end{tabular}
\caption{Characteristic numbers for plane rational 1-cuspidal quintics; cusp at a specified point.}
\end{table}
Equation 1134: For any $a$, $b$ and $c$ with $a + b + 2c = 3d - 3$ we have that

\[
C_d(a, b, c; h) = -bC_d(a, b - 1, c; h^2) + \frac{1}{d^2} \left\{ d(2d - 2)N_d(a, b, c + 1) + (2 - d)N_d(a + 1, b + 1, c) \right\}
\]

\[+ dN_d(a, b + 2, c) + \sum_{\substack{d_1 + d_2 = d \\
 a_1 + a_2 = a \\
b_1 + b_2 = b+1 \\
c_1 + c_2 = c}} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; h^2) \left[ 2d_1d_2 \left( \frac{b}{b_1 - 1} \right) - 2d_2^2d_2 \left( \frac{b}{b_1} \right) \right] \left( \frac{a}{a_1} \right) \left( \frac{c}{c_1} \right) \]

\[+ \sum_{\substack{d_1 + d_2 = d \\
 a_1 + a_2 = a+1 \\
b_1 + b_2 = b+1 \\
c_1 + c_2 = c}} N_{d_1}(a_1, b_1, c_1)E_{d_2}(a_2, b_2, c_2; h) \left[ 2d_2 \left( \frac{b}{b_1 - 1} \right) - 2d_2^2d_2 \left( \frac{b}{b_1} \right) \right] \left( \frac{a}{a_1 - 1} \right) \left( \frac{c}{c_1} \right) \}

In the tables of the numbers $C_d(a, b, c; h)$ the number of tangent conditions $b$ is given by $3d-3-a-2c$.

| $c \downarrow$ | $C_3(a, b, c; h)$ |
|----------------|------------------|
| 3              | 6                |
| 2              | 20               | 18 | 10 |
| 1              | 42               | 60 | 54 | 30 | 12 |
| 0              | 72               | 132| 186| 168| 96 | 42 | 12 |
| $a \rightarrow$| 0                | 1  | 2  | 3  | 4  | 5  | 6  |

Table 7.8 Characteristic numbers for plane rational 1-cuspidal cubics; cusp on a specified line.

The numbers for cubics were calculated by Schubert [Sch79], and Aluffi [Alu91].

| $c \downarrow$ | $C_4(a, b, c; h)$ |
|----------------|------------------|
| 4              | 210              | 120 |
| 3              | 1200             | 900 | 510 | 228 |
| 2              | 5640             | 5404| 3968| 2232| 1006| 380 |
| 1              | 21844            | 26344| 24812| 17956| 10084| 4604| 1774| 592 |
| 0              | 81324            | 111776| 128992| 84284| 47284| 21816| 8552| 2926| 864 |
| $a \rightarrow$| 0                | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |

Table 7.9 Characteristic numbers for plane rational 1-cuspidal quartics; cusp on a specified line.
Table 7.10 Characteristic numbers for plane rational 1-cuspidal quintics; cusp on a specified line.

Equation 1133: For any $a$, $b$ and $c$ with $a + b + 2c = 3d - 2$ we have that
\[
C_d(a, b, c; 1) = -bC_d(a, b - 1, c; h) - cC_d(a, b, c - 1; h^2) - \left(\frac{b}{2}\right)C_d(a, b - 2, c; h^2)
\]
\[
+ \frac{1}{d^2}\left\{4(d - 1)N_d(a + 1, b, c) + (3d^2 - 4d)N_d(a, b + 1, c) - 4\sum_{a_1 + a_2 = a\atop b_1 + b_2 = b\atop c_1 + c_2 = c}N_{d_1}(a_1, b_1, c_1)E_{d_1}(a_2, b_2, c_2; h)_{d_1}d_2\left(a\atop a_1\right)\left(b\atop b_1\right)\left(c\atop c_1\right)\right\}
\]
\[
- 4\sum_{a_1 + a_2 = a + 1\atop b_1 + b_2 = b\atop c_1 + c_2 = c}N_{d_1}(a_1, b_1, c_1)E_{d_1}(a_2, b_2, c_2; \tilde{h})_{d_1}d_2\left(a\atop a_1 - 1\right)\left(b\atop b_1\right)\left(c\atop c_1\right)\}
\]

In the tables of the numbers $C_d(a, b, c; 1)$ the number of tangent conditions $b$ is given by $3d - 2 - a - 2c$.

| $c$ | $C_3(a, b, c; 1)$ |
|-----|-------------------|
| 3   | 6                 |
| 2   | 12 18             |
| 1   | 30 54 36          |
| 0   | 60 114 68         |
| $a$ | 24 45 18          |

Table 7.11 Characteristic numbers for plane rational 1-cuspidal cubics.

The numbers for cubics were calculated by Schubert [Sch79], by Kleiman and Speiser [KS86], and by Aluffi [Alu91]. Note how the numbers reflect the fact that the dual of a cuspidal cubic is also a cuspidal cubic, with duality between point and tangent conditions.

Given a plane cuspidal cubic $C$, under the morphism $C \to \tilde{C}$ the image of the cusp of $C$ is the point of flex tangency on $\tilde{C}$ and vice versa. Therefore we have the following alternative enumerative significance: $C_3(a, b, c; h^2)$ (listed in Table 7.5) is equal to the number of cuspidal cubics with given flex, through $b$ points, tangent to $a$ lines and incident to $c$ flags, and $C_3(a, b, c; h)$ (listed in Table 7.8) is equal to the number of cuspidal cubics with flex tangent through a given point, through $b$ points, tangent to $a$ lines and incident to $c$ flags.

| $c$ | $C_4(a, b, c; 1)$ |
|-----|-------------------|
| 5   | 90                |
| 6   | 450 360 216       |
| 3   | 2016 2016 1566 936 450 |
| 2   | 7344 9228 9096 6948 4134 2004 828 |
| 1   | 24012 35668 43462 42012 31644 18792 9198 3852 1122 |
| 0   | 75924 126720 180288 212976 201132 149364 88560 43668 18486 6912 2304 |
| $a$ | 0 1 2 3 4 5 6 7 8 9 10 |

Table 7.12 Characteristic numbers for plane rational 1-cuspidal quartics.
Table 7.13 Characteristic numbers for plane rational 1-cuspidal quintics.

The four series of characteristic numbers of Theorem 7.2 have been calculated up to degree 10 and are available upon request to the authors. The Maple source code for the calculation may also be requested.
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