Formation of rogue waves and modulational instability with zero-wavenumber gain in multi-component systems with coherent coupling

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It is known that rogue waves (RWs) are generated by the modulational instability (MI) of the baseband type. Starting with the Bers-Kaup-Reiman system for three-wave resonant interactions, we identify a specific RW-building mechanism based on MI which includes zero wavenumber in the gain band. An essential finding is that this mechanism works solely under a linear relation between the MI gain and a vanishingly small wavenumber of the modulational perturbation. The same mechanism leads to the creation of RWs by MI in other multi-component systems — in particular, in the massive Thirring model.

Introduction. The modulational instability (MI) of constant-amplitude continuous waves (CWs) against long-wavelength perturbations plays a fundamental role in understanding nonlinear-wave dynamics \(^1,4\). MI has been predicted and observed in deep water \(^1,2,3\), plasmas \(^4,5\), electric transmission lines \(^6,10\), optics \(^11–22\), matter waves \(^23–35\), and other physical media \(^37–52\). In the generation of RWs is driven by MI, not every kind of MI leads to this outcome \(^51,74\). Two generic types of MI are baseband and passband ones \(^51,74\). In the former case, the CW background is unstable against perturbations with infinitesimal wavenumbers \(Q\) and vanishingly small gain \(\eta\), while in the latter case MI is absent at \(|Q| < Q_{\text{min}}\) with finite \(Q_{\text{min}}\). It was found that RWs can be generated solely by MI of the baseband type.

The fact that the gain of the baseband MI vanishes at \(Q = 0\) suggests a question if a physically meaningful system can give rise to MI with nonzero gain at \(Q = 0\), and whether MI of this type leads to RW formation. Here, using a system for the three-wave resonant interaction, we demonstrate that such zero-wavenumber-gain (ZWG) MI exists, and indeed leads to RW formation, under the condition that an asymptotically linear relation between the MI gain and wavenumber \(Q\) holds, see Eq. \((11)\) below.

The three-wave resonant-interaction system and RW existence condition. We consider the Bers-Kaup-Reiman (BKR) system of equations for three waves \(E_{1,2,3}(x,t)\) coupled by the saturable quadratic interaction, which models the resonant three-wave coupling in hydrodynamics, optics, microwaves, and plasmas \(76–84\):

\[
(E_n)_t + V_n \cdot (E_n)_x = - \frac{\sigma_n E_n^2 E_n^*}{1 + \epsilon |\sigma_n E_n|^2} \quad (n=1,2,3). 
\]  

Here \(\{n,k,l\}\) are sets of \(1,2,3\) and their transpositions, \(V_1 > V_2 > V_3 \equiv 0\) are group velocities, \(*\) is complex conjugate, and \(\sigma_j\) are signs of the interactions, which represent the stimulated-backscattering (\(\sigma_1 = \sigma_2 = -\sigma_3 = 1\)), soliton-exchange (\(\sigma_1 = -\sigma_2 = \sigma_3 = 1\)), and explosive (\(\sigma_1 = \sigma_2 = \sigma_3 = 1\)) regimes. In the latter case, a complicating factor is that in the system is vulnerable to the onset of blowup, therefore it includes the saturation represented by the term \(\epsilon \geq 0\) in Eq. \((11)\). The original system \((\epsilon = 0)\) gives rise to CW solution \((3)\) written below, with amplitudes \(a_n\), whose MI and the emerging RWs are the same as in the saturable system \((\epsilon > 0)\), making it possible to produce exact RW solutions via the Hirota method \((32,35)\).

\[
E_j = a_j \frac{(\xi + \theta_j^2)(\xi^* - \theta_j^2) + \eta_0 \epsilon^{i\phi_j}}{|\xi|^2 + \eta_0}, \quad (j = 1,2), \\
E_3 = ia_3 \frac{(\xi - \theta_1 - \theta_2)(\xi^* + \theta_1^* + \theta_2^*) + \eta_0 \epsilon^{-i(\phi_1 + \phi_2)}}{|\xi|^2 + \eta_0}
\]  

where \(a_{1,2,3}\) are nonzero real constants, and \(\xi = (\alpha - \beta)x - (V_2 \alpha - V_1 \beta)t, \alpha = \frac{-\gamma_1}{(V_1 - V_2)p_0^2}, \beta = \frac{-\gamma_2}{(V_1 - V_2)(p_0 - \eta_0)^2}, \phi_1 = c_1x + d_1t, \phi_2 = c_2t, \phi_2 = 2\gamma_2 \frac{2\gamma_2 + \gamma_3}{4\gamma_3}, \theta_1 = \frac{1}{p_0 - \eta_0}, \theta_2 = -\frac{1}{\eta_0}, \eta_0 = \frac{1}{(p_0 + \eta_0)^2}, \gamma_1 = \sigma_1 a_1 a_3 / a_2, \gamma_2 = \sigma_2 a_1 a_2 / a_3, \gamma_3 = \sigma_3 a_1 a_2 / a_3, and \) \(p_0\) taken as a non-imaginary root of a quartic equation,

\[
\gamma_3(V_1 - V_2)p_0^2(p - i)^2 - \gamma_1 V_2(p - i)^2 + \gamma_2 V_1 p_0^2 = 0. \quad (3)
\]

To make parameters \(a_1, a_2, a_3, V_1\) and \(V_2\) satisfying the condition that the root of Eq. \((3)\) must be non-imaginary.
must be subject to constraint $\Delta < 0$. For the BKR system of the soliton-exchange type, the constraint securing the existence of the RWs is, instead, $\Delta \geq 0$.

**ZWG-MI and the general mechanism for the RW formation.** Equation (1) admits CW solutions

$$ E_j = a_j \exp \left[ i (c_j x + d_j t) \right], \\
E_3 = ia_3 \exp \left[ -i \left( (c_1 + c_2) x + (d_1 + d_2) t \right) \right], $$

(5)

where $c_1 = -\left[ \sigma_1 \sigma_3 a_3^2 + d_1 (d_1 + d_2) \right]/[V_1 (d_1 + d_2)]$, $c_2 = -\left[ \sigma_1 \sigma_3 a_3^2 + d_2 (d_1 + d_2) \right]/[V_2 (d_1 + d_2)]$, $a_3 = \sigma_3 a_1 a_2/(d_1 + d_2)$ with free real parameters $a_1$ and $d_j$ representing CW amplitudes and frequencies, respectively. Using invariances of Eq. (1), we fix $a_{1,2}$ to be real, and set $d_1 = d_2 = \sigma_3 a_1 a_2/(2a_3)$. Thus, $a_{1,2,3}$ control the CW. Actually, CW (3) is the background supporting the RW states (4). For the same set of $a_3$, the CW solution and the results for its MI, following below, remain fully valid for $\epsilon > 0$ in Eq. (4).

To address MI, the perturbed CW is written as $\tilde{E}_n = E_n(1 + p_n(x,t)/a_n)$, where

$$ p_n(x,t) \equiv \eta_n,1(t)e^{iQx} + \eta_n,2(t)e^{-iQx} $$

(6)

are small perturbations with wavenumber $Q$. The linearized equations for the perturbations amount to a 6 x 6 system, $d\eta_1/dt = iM\eta_1$, with $\eta = (\eta_{1,1}, \eta_{1,2}, \eta_{2,1}, \eta_{2,2}, \eta_{3,1}, \eta_{3,2})^T$, nonzero matrix elements of $M$ being $M_{11} = \sigma_3 a_2 a_3/a_1 - V_1 Q$, $M_{22} = -\sigma_2 a_2 a_3/a_1 - V_1 Q$, $M_{33} = \sigma_3 a_1 a_2/a_1 - V_2 Q$, $M_{44} = -\sigma_4 a_1 a_2/a_1 - V_2 Q$, $M_{55} = -M_{66} = \sigma_3 a_1 a_2/a_1$, $M_{41} = -M_{52} = \sigma_3 a_3$, $M_{23} = -M_{14} = \sigma_3 a_3$, $M_{61} = -M_{62} = \sigma_3 a_2$, $M_{25} = -M_{16} = \sigma_3 a_2$, $M_{36} = -M_{34} = \sigma_3 a_1$, and $M_{54} = -M_{53} = \sigma_3 a_1$.

The stability of $\tilde{E}_n$ is determined by eigenvalues $\Omega$ of $M$, which are roots of the following characteristic polynomial,

$$ B(\Omega) = \Omega^6 + \lambda_5 \Omega^5 + \lambda_4 \Omega^4 + \lambda_3 \Omega^3 + \lambda_2 \Omega^2 + \lambda_1 \Omega + \lambda_0, $$

(7)

where $\lambda_0 = -V_1^2 V_2^2 \gamma_3 Q^2$, $\lambda_1 = 2V_1 V_2 \gamma_3 [V_2 (\gamma_1 - \gamma_3) + V_1 (\gamma_2 - \gamma_3)]$, $\lambda_2 = \{V_1 V_2 [V_1 V_2 Q^2 + 6\gamma_3 (\gamma_1 + \gamma_2 - \gamma_3)] - [V_2 (\gamma_1 - \gamma_3) - V_1 (\gamma_2 - \gamma_3)]^2\}Q^2$, $\lambda_3 = 2\{V_1 (V_2 V_3 Q^2 + 3\gamma_3 (\gamma_1 + \gamma_2 - \gamma_3) + \gamma_1^2 + V_2 (\gamma_1 - \gamma_3) - V_1 (\gamma_2 - \gamma_3))\}Q$, $\lambda_4 = (V_3^2 + V_2^2 + V_1 V_2)Q^2 - (\gamma_1 + \gamma_2 - \gamma_3)^2 + \gamma_1^2 + \gamma_2^2$, and $\lambda_5 = 2(V_1 + V_2)Q$.

The six roots of (7) are either real ones or complex-conjugate pairs. The MI emerges in the latter case, being accounted for by the roots with Im$(\Omega) < 0$. There are three different types of the MI:

- **Baseband-MI**: Im$(\Omega) < 0$ at $|Q| > 0$ and Im$(\Omega) = 0$ at $Q = 0$, i.e., the MI band includes small wavenumbers $Q$ but not $Q = 0$.

- **Passband-MI**: Im$(\Omega) < 0$ at $|Q| > Q_{\text{min}} > 0$ with a nonzero boundary $Q_{\text{min}}$ of the MI band, which separates it from $Q = 0$.

- **ZWG-MI**: Im$(\Omega) < 0$ at $|Q| < Q_{\text{max}}$ with $Q_{\text{max}} > 0$, i.e., the MI band includes zero wavenumber, $Q = 0$. This situation implies that the mechanical system with three degrees of freedom, which corresponds to Eq. (1) with $x$-independent fields, is itself unstable, as it represents an amplifying setup.

To address the ZWG-MI, we set $Q = 0$ in Eq. (7), obtaining nonzero roots $\Omega = \pm \sqrt{\Omega_0^2}$, with

$$ \Omega_0^2 = (\gamma_1 + \gamma_2 - \gamma_3)^2 - 4\gamma_1 \gamma_2. $$

(8)

Thus, the ZWG-MI exists for $\Omega_0^2 < 0$, as Eq. (7) has two mutually conjugate imaginary roots $\Omega = Q = 0$. On the other hand, if $\Omega_0^2 \geq 0$, Eq. (7) has no imaginary roots at $Q = 0$, hence only the baseband/passband MI is possible. A conclusion is that the ZWG-MI occurs if all $\sigma_a$ in Eq. (1) have the same sign, i.e., solely in the case of the explosive three-wave system. Unless mentioned otherwise, we set $\sigma_1 = \sigma_2 = \sigma_3 = 1$ below.

Subsequently, we focus on the MI in the crucially important limit of $Q \rightarrow 0$. Accordingly, if Eq. (7) yields $\Omega_0^2 \neq 0$, we approximate $\Omega$ as $B(Q\Omega) = Q^4 b^{(1)}(Q)$, hence Eq. (8) amounts to

$$ b^{(1)}(Q) = -\Omega_0^2 Q^4 + b_3 Q^2 + b_2 Q^2 + b_1 Q + b_0 = 0, $$

(9)

where $b_0 = -V_1^2 V_2^2 \gamma_3^2$, $b_1 = 2V_1 V_2 \gamma_3 [V_2 (\gamma_1 - \gamma_3) + V_1 (\gamma_2 - \gamma_3)]$, $b_2 = 6V_1 V_2 \gamma_3 (\gamma_1 + \gamma_2 - \gamma_3) - [V_2 (\gamma_1 - \gamma_3) - V_1 (\gamma_2 - \gamma_3)]^2$, and $b_3 = 2\{V_1 (V_2 V_3 Q^2 + 3\gamma_3 (\gamma_1 + \gamma_2 - \gamma_3) + \gamma_1^2 + V_2 (\gamma_1 - \gamma_3) - V_1 (\gamma_2 - \gamma_3))\}$

Equation (9) with $b_0 < 0$ yields, at least, two simple real roots when $\Omega_0^2 < 0$. If $\Omega_0^2 > 0$, Eq. (9) has two simple real roots at all values of parameters. Because the discriminant of quartic equation (9) coincides with that of Eq. (8), i.e., $\Delta$ [see Eq. (4)], the RW existence condition, $\Delta < 0$, can be obtained from the discriminant of Eq. (8).

Thus, for $\Omega_0^2 \neq 0$, two cases are possible. (i) If $\Delta \geq 0$, all roots of Eq. (9) are real, and no baseband-MI occurs. Specifically, if $\Delta \geq 0$ and $\Omega_0^2 < 0$, there exists a ZWG-MI region; if $\Delta \geq 0$ and $\Omega_0^2 > 0$, there is passband-MI or no MI takes place. (ii) If $\Delta < 0$, Eq. (9) produces two complex-conjugate roots, and there exists a baseband-MI region at $\Omega_0^2 > 0$, or a ZWG region at $\Omega_0^2 < 0$. Figures 1 and 2 display the predicted characteristics of the MI and RW existence range. Figures 1a,b show that the MI of the baseband, ZWG, and passband (or maybe no-MI) types exists, respectively, in the regions of $0 < a_3 < 4/5, 4/5 < a_3 \leq 4/3$, and $a_3 > 4/3$. Based on the sign of $\Delta$ [see Eq. (4)], as shown in Fig. 1c, it is seen that the RW existence condition is $a_3 < 1.247$. Namely, when $0 < a_3 < 1.247$, the MI of the baseband or ZWG types occurs and the RWs exist, but when $1.247 < a_3 < 4/3$, the ZWG-MI occurs too, while RWs do not exist. Figures 2a,b show that the passband-MI (or maybe no-MI) is present when $0 < a_2 < 0.5$, while the ZWG-MI
occurs at $a_2 > 0.5$. Figure 2(c) demonstrates that the RW existence condition is $a_2 > 0.5$.

Figure 3 shows an example of a fundamental dark-bright-dark RW in the BKR system, as given by the solution (2), with the same parameters as in Fig. 2 and $a_2 = 1$. Such RWs emerge in the ZWG-MI region. Virtually the same RW is produced by direct simulations. In the generic case, multi-RW structures are produced by simulations of Eq. (1) initiated by a chaotically perturbed CW background, as shown in Fig. 4. Following the pattern of Ref. 56, an individual RW selected in the figure is compared to the analytical solution in Fig. 3 of Supplement.

The above results are established for $\Omega_0^2 \neq 0$. When $\Omega_0^2 = 0$, Eq. (9) is replaced by $B(Q^{3/2}) = Q^2 b(2)(\Omega)$, and

$$b(2)(\Omega) = \Omega^6 + b_3 \Omega^3 = 0.$$  \hspace{1cm} (10)

If $b_3 \neq 0$, there are two complex conjugate roots of Eq. (11), and MI is of the baseband type. If $\Omega_0^2 = b_3 = 0$, Eq. (10) is replaced by $B(\sqrt{Q} \Omega) = Q^3 b_{3}(\Omega)$ and $b(3)(\Omega) = \Omega^6 + b_2 \Omega^2 = 0$. We thus infer that, with $b_2 \neq 0$ ($b_2$, $b_3$ and $\Omega_0^2$ cannot all be equal to zero), there are at least two complex-conjugate roots, MI being of the baseband type. Therefore, while the baseband-MI occurs at $\Omega_0^2 = 0$, in the case of $\Delta \geq 0$ RWs are absent. Thus, a new feature of the present setting is that RWs may be absent in the baseband-MI region. This situation was not reported before, it being believed that the presence of baseband-MI always leads to the creation of RWs.

Thus we arrive at the following conclusions: (i) ZWG-MI generates RWs at $\Delta < 0$, which implies that there exist complex roots $\Omega$ of Eq. (7) satisfying

$$\text{Im}(\Omega) = O(Q)$$  \hspace{1cm} (11)

(an asymptotically linear dependence) at $Q \to 0$; (ii) the baseband-MI (when $\Omega_0^2 = 0$) cannot generate RWs at $\Delta > 0$, which implies that there are no complex roots of Eq. (7) satisfying relation (11); (iii) the baseband-MI (at $\Omega_0^2 \neq 0$) can generate RWs as it satisfies Eq. (11). Therefore, in the regions of MI of the baseband and ZWG types the crucial difference between the presence and absence of RWs is the existence or absence of complex roots of Eq. (9), rather than those of Eq. (7). These facts demonstrate that RWs are generated only when Eq. (11) is valid. Thus, the above analysis implies that solely the MI of the baseband and ZWG types, satisfying condition
leads to the formation of RWs. This criterion was not reported previously.

When $Q = 0$, Eq. (7) produces four zero roots and two other ones, $\Omega = \pm \sqrt{\Omega_0^2}$. Condition (11), which produces the asymptotically linear condition of the existence of the rational RW solutions, implies that RWs are related, at $Q = 0$, only to the set of the zero eigenvalues. This fact implies the rational growth of the MI of the respective CW background.

In Fig. (5), we summarize results of the MI analysis produced by varying $V_1$, while $a_0$ are fixed so as to have $\Omega_0^2 = 0$. As shown in Fig. (5a), the respective MI is of the baseband type, while RWs exist only in the interval of $0.1 < V_1 < 2.15$. Figures (5b,5c) show the MI gain, $|\text{Im}(\Omega)|$, as produced by all complex roots of (7) at $V_1 = 1$ and $V_1 = 3$. It is seen, in particular, that Eq. (11) holds for $V_1 = 1$, but not for $V_1 = 3$.

The predicted mechanism of the RW creation can be experimentally realized in amplified three-wave optical, microwave, and hydrodynamic systems. A suitable experimental setup in optics is based on a semiconductor amplifier, providing the generation of light beams with power $\sim 1$ W at the standard wavelength, 1.55 $\mu$m. For microwave systems, amplifiers using Josephson junctions make it possible to implement the interaction between waves with frequencies $\sim 10$ GHz. Experiments with water waves can be performed in the frequency range $15 - 30$ Hz, using an apparatus of size $30 \times 30$ cm. The boundary conditions which are used to initiate the required wave dynamics are specified in Supplement B.

Lastly, we present results obtained for the MI and RWs in other integrable systems, that fully agree with the above conclusions.

(i) For the BKR system of the soliton-exchange and stimulated-backscattering types, for which condition $\Omega_0^2 > 0$ holds, RWs exist if and only if Eq. (11) has complex roots. Table 1 in Supplement A shows the relationship between all possible MI types and RW existence conditions for all types of the BKR system. The interpretation of the ZWG-MI in terms of the three-wave mixing, which underlies the BKR system, is additionally considered in Supplement C.

(ii) In the two-component massive Thirring model, RWs are absent in the case of the ZWG-MI, as Eq. (11) does not hold in that case; RWs do or do not exist in the case of the baseband MI if, respectively, Eq. (11) does or does not hold, as shown in detail analytically and numerically in Supplement D.

(iii) For other integrable equations which do not give rise to the ZWG-MI, the results concerning the existence of RWs in the case of the baseband-MI amount to a particular case of the above analysis, as Eq. (7) is then the same as Eq. (9), provided that Eq. (11) holds.

**Conclusion.** The present work reveals the mechanism for the formation of RWs in multi-component systems with coherent coupling, i.e., energy exchange between the components. In the framework of this mechanism, the three-wave BKR system creates RWs in the case of the ZWG-MI, i.e., MI whose gain band includes zero wavenumber. An important finding is that, in both cases of the ZWG and baseband types of MI, the system creates RWs only under the condition of the asymptotically linear relation (11) between the MI gain and small perturbation wavenumber. The same analysis predicts the existence or absence of RWs in other coherently-coupled multi-component systems.

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deep water, J. Fluid Mech. 462, 1 (2002).
[6] T. Taniuti and H. Washimi, Self-trapping and instability of hydromagnetic waves along the magnetic field in a cold plasma, Phys. Rev. Lett. 21, 209 (1968).
[7] S. Watanabe, Self-modulation of a nonlinear ion wave packet, J. Plasma Phys. 17, 487 (1977).
[8] H. Bailung and Y. Nakamura, Observation of modulational instability in a multi-component plasma with negative ions, J. Plasma Phys. 50, 231 (1993).
[9] P. Marquie, J. M. Bilbault, and M. Remoissenet, Generation of envelope and hole solitons in an experimental transmission line, Phys. Rev. E 49, 828 (1994).
[10] E. Kengne, W.-M. Liu, L. Q. English, and B. A. Malomed, Ginzburg-Landau models of nonlinear electric transmission networks, Phys. Rep. 982, 1 (2022).
[11] A. Hasegawa, Generation of a train of soliton pulses by induced modulational instability in optical fibers, Opt. Lett. 9, 288 (1984).
[12] K. Tai, A. Hasegawa, and A. Tomita, Observation of Modulational Instability in Optical Fibers, Phys. Rev. Lett. 56, 135 (1986).
[13] S. Trillo and S. Wabnitz, Dynamics of the nonlinear modulational instability in optical fibers, Opt. Lett. 16, 986-988 (1991).
[14] M. Yu, C. J. McKinzie, and G. P. Agrawal, Modulational instability in dispersion-flattened fibers, Phys. Rev. E 52, 1072 (1995).
[15] S. Coen and M. Haelterman, Modulational instability induced by cavity boundary conditions in a normally dispersive optical fiber, Phys. Rev. Lett. 79, 4139 (1997).
[16] D. V. Petrov, L. Torner, J. Martorell, R. Vilaseca, J. P. Torres, and C. Cojocaru, Observation of azimuthal modulational instability and formation of patterns of optical solitons in a quadratic nonlinear crystal, Opt. Lett. 23, 1444 (1998).
[17] S. Pitois and G. Millot, Experimental observation of a new modulational instability spectral window induced by fourth-order dispersion in a normally dispersive single-mode optical fiber, Opt. Commun. 226, 415 (2003).
[18] W. Krollkowski, O. Bang, N. I. Nikolov, D. Neshev, J. Wyller, J. J. Rasmussen, and D. Edmundson, Modulational instability, solitons and beam propagation in spatially nonlocal nonlinear media, J. Opt. Soc. Am. B 6, S288-S294 (2004).
[19] M. Peccianti, C. Conti, G. Assanto, A. De Luca, and C. Umeton, Routing of anisotropic spatial solitons and modulational instability in liquid crystals, Nature 432, 733 (2004).
[20] J. Meier, G. I. Stegeman, D. N. Christodoulides, Y. Silberberg, R. Morandotti, H. Yang, G. Salamo, M. Sorel, and J. S. Aitchison, Experimental observation of discrete modulational instability, Phys. Rev. Lett. 92, 163902 (2004).
[21] L. Wang, J. H. Zhang, Z. Q. Wang, C. Liu, M. Li, F. H. Qi, and R. Guo, Breathers to soliton transitions, nonlinear wave interactions, and modulational instability in a higher-order generalized nonlinear Schrödinger equation, Phys. Rev. E 93, 12214 (2016).
[22] Y. V. Kartashov and D. V. Skryabin, Modulational instability and solitary waves in polariton topological insulators, Optica 3, 1228 (2016).
[23] V. V. Konotop and M. Salerno, Modulational instability in Bose-Einstein condensates in optical lattices, Phys. Rev. A 65, 021602 (2002).
[24] L. Salasnich, A. Parola, and L. Reatto, Modulational instability and complex dynamics of confined matter-wave solitons, Phys. Rev. Lett. 91, 080405 (2003).
[25] G. Theocharis, Z. Rapti, P. G. Kevrekidis, D. J. Frantzeskakis, and V. V. Konotop, Modulational instability of Gross-Pitaevskii-type equations in 1 + 1 dimensions, Phys. Rev. A 67, 063610 (2003).
[26] L. D. Carr and J. Brand, Spontaneous soliton formation and modulational instability in Bose-Einstein condensates, Phys. Rev. Lett. 92, 040401 (2004).
[27] P. G. Kevrekidis and D. J. Frantzeskakis, Pattern forming dynamical instabilities of Bose-Einstein condensates, Mod. Phys. Lett. B 18, 173 (2004).
[28] S. Rojas-Rojas, R. A. Vicencio, M. I. Molina, and F. Kh. Abdullaev, Nonlinear localized modes in dipolar Bose-Einstein condensates in optical lattices, Phys. Rev. A 84, 033621 (2011).
[29] J. H. V. Nguyen, D. Luo, and R. G. Hulet, Formation of matter-wave soliton trains by modulational instability, Science 356, 422 (2017).
[30] P. J. Everitt, M. A. Sooriyabandara, M. Guasoni, P. B. Wigley, C. H. Wei, G. D. McDonald, K. S. Hardman, P. Manju, J. D. Close, C. C. N. Kuhn, S. S. Szigi, Y. S. Kivshar, and N. P. Robins, Observation of a modulational instability in Bose-Einstein condensates, Phys. Rev. A 96, 041601(R) (2017).
[31] T. Mithun, A. Maluckov, K. Kasamatsu, B. Malomed, and A. Khare, Inter-component asymmetry and formation of quantum droplets in quasi-one-dimensional binary Bose gases, Symmetry 12, 174 (2020).
[32] S. Bhuvaneswari, K. Nithyanandan, P. Muruganandan, and K. Porsezian, Modulation instability in quasi-two-dimensional spin-orbit coupled Bose-Einstein condensates, J. Phys. B: At. Mol. Opt. Phys. 49, 24530 (2016).
[33] T. Congy, A. M. Kamchatnov, and N. Pavloff, Nonlinear waves in coherently coupled Bose-Einstein condensates, Phys. Rev. A 93, 043613 (2016).
[34] T. Mithun and K. Kasamatsu, Modulational instability associated nonlinear dynamics of spin-orbit coupled Bose-Einstein condensates, J. Phys. B: At. Mol. Opt. Phys. 52, 045301 (2019).
[35] C. B. Tabi, S. Veni, and T. C. Kofané, Generation of matter waves in Bose-Bose mixtures with helicoidal spin-orbit coupling, Phys. Rev. A 104, 033225 (2021).
[36] A. Cidrim, L. Salasnich, and T. Macrì, Soliton trains after interaction quenches in Bose mixtures, New J. Phys. 23, 023022 (2021).
[37] M. J. Lighthill, Contribution to the Theory of Waves in non-linear dispersive systems, J. Inst. Math. Appl. 1, 269 (1965).
[38] E. Mjolhus, Modulational instability of hydromagnetic waves parallel to magnetic field, J. Plasma Phys. 16, 321-334 (1976).
[39] Y. S. Kivshar and M. Peyrard, Modulational instabilities in discrete lattices, Phys. Rev. A 46, 3198-3205 (1992).
[40] A. M. Kamchatnov, New approach to periodic solutions of integrable equations and nonlinear theory of modulational instability, Phys. Rep. 286, 199 (1997).
[41] C. Khare and E. Pelinovsky, Physical mechanisms of the rogue wave phenomenon, Eur. J. Mech. B – Fluids 22, 603-634 (2003).
[42] M. Marklund and P. K. Shukla, Nonlinear collective effects in photon-photon and photon-plasma interactions, Phys. Rep. 78, 591-640 (2006).
tial stage of spray generation in strong winds, Izvestiya, Atmospheric and Oceanic Physics 57, 180-191 (2021).

[85] B. Yang and J. Yang, General rogue waves in the three-wave resonant interaction systems, IMA J. Appl. Math. 86, 378 (2021).

[86] P. J. Manning and D. A. O. Davies, Three-wavelength device for all-optical signal processing, Opt. Lett. 19, 989-991 (1994).

[87] N. Roch, E. Flurin, F. Nguyen, P. Morfin, P. Campagne-Ibarcq, M. H. Devoret, and B. Huard, Widely tunable, nondegenerate three-wave mixing microwave device operating near the quantum limit, Phys. Rev. Lett. 108, 147701 (2012).

[88] B. Abdo, A. Kamal, and M. Devoret, Nondegenerate three-wave mixing with the Josephson ring modulator, Phys. Rev. B 87, 014508 (2013).

[89] A. P. Abella and M. N. Soriano, Detection and visualization of water surface three-wave resonance via a synthetic Schlieren method, Phys. Scr. 94, 034006 (2019).