Energy conditions outside a dielectric ball

Noah Graham∗

Department of Physics, Middlebury College, Middlebury, VT 05753

Ken D. Olum† and Delia Schwartz-Perlov‡

Institute of Cosmology, Department of Physics and Astronomy,
Tufts University, Medford, MA 02155

Abstract

We show analytically that the vacuum electromagnetic stress-energy tensor outside a ball with constant dielectric constant and permeability always obeys the weak, null, dominant, and strong energy conditions. There are still no known examples in quantum field theory in which the averaged null energy condition in flat spacetime is violated.

PACS numbers: 03.65.Nk 04.20.Gz

∗Electronic address: ngraham@middlebury.edu
†Electronic address: kdo@cosmos.phy.tufts.edu
‡Electronic address: Delia.Perlov@tufts.edu
I. INTRODUCTION

A longstanding puzzle of general relativity is whether or not it is possible to forbid the presence of exotic phenomena, such as the construction of closed timelike curves \(^1\) and superluminal travel \(^2\). A variety of energy conditions exist that would prevent these possibilities. The weak energy condition (WEC) and null energy condition (NEC) respectively require that \(T_{\mu\nu}V^\mu V^\nu \geq 0\) for any timelike and null vector \(V^\mu\), but these conditions have well-known violations even in free field theory in Minkowski space.

The conditions may be weakened by requiring that they hold only when averaged over a complete geodesic, and those averaged energy conditions are still sufficient to rule out exotic phenomena. The averaged weak energy condition (AWEC) is violated in the standard electromagnetic Casimir energy between parallel, perfectly conducting planes, and it appears that the violation occurs even with more realistic materials \(^3\). The system does not violate the averaged null energy condition (ANEC), however, because geodesics in directions which violate NEC all pass through the conducting planes. Assuming any realistic material conductor in these places, the positive stress-energy there more than cancels the negative energy between the planes. In \(^4\) it was shown that for a simple model of scalar fields, a static domain wall violates WEC, NEC and AWEC, even when the domain wall energy is included, but again ANEC is obeyed, for the same reason as above.

ANEC is violated in curved space-time \(^5, 6, 7, 8\), but it is not clear that such violations can be used to produce exotic phenomenon. They are typically overwhelmed by ordinary gravitational energy densities, for objects that are not at the Planck scale. No violation of ANEC is known in a consistent quantum field theory in flat spacetime.

One can try to avoid the problem of passing through the material in the examples above by considering a geodesic passing outside a ball, where one can safely ignore the contribution to the energy density from the material of the ball. For the case of minimally coupled scalar fields, outside a perfectly reflecting spherical shell, one finds the same results as in the domain wall case: WEC and NEC are violated, but ANEC is obeyed \(^9\).

In this paper, we analyze Casimir energy densities in the electromagnetic field outside a ball with fixed dielectric constant and permeability. The results also extend to the case of a perfectly conducting sphere.\(^1\) In contrast to the scalar field case, all energy conditions, including NEC (and therefore ANEC) are obeyed everywhere outside the ball. Thus there is still no counterexample to the conjecture that all quantum field theories in flat spacetime obey ANEC.

The generality of our result — depending only on the requirement that the index of refraction \(n\) is bigger than 1, which is necessary to maintain causality in classical electromagnetism without absorption — suggests that it might be possible to extend it to other cases in QED or other conformal theories.\(^2\).

A number of related calculations have been carried out in other contexts. The total energy

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\(^1\) In the case of finite, nonzero conductivity, we have absorption, coupling the incoming waves to modes within the material. The effects of this interaction might depend in detail on the internal dynamics of the material, so we do not consider it here.

\(^2\) In the case of a conformal scalar field, using the techniques of \(^8\) one finds that outside a perfectly reflecting spherical boundary, NEC is also always obeyed. Unlike the electromagnetic case, however, this result does not hold in each channel individually; for the \(\ell \neq 0\) partial wave contributions the NEC is not always obeyed, but when one includes all \(\ell\) it is obeyed everywhere.
of a dielectric ball has been studied, in particular with applications to the controversial question of its relevance to sonoluminescence [10, 11]. However, it remains difficult to formulate precisely because the energy is sensitive to the details of the dynamics within the material. These potential divergences were studied in detail in [12]; see also reviews [13, 14, 15], and references therein. The case of the Casimir energy density outside a conducting sphere is a standard calculation, originating with the work of Boyer [16] and reviewed in [17]. Milton [18] calculates the total energy including the interior and the net force on a dielectric sphere, and in [19, 20] uses a comparison of the the stress density inside and outside the sphere to find a Casimir contribution to the total stress on the sphere.

In the next section, we give the energy density and the radial and tangential pressures in terms of the electromagnetic field components. In Section III we decompose the fields in vector spherical harmonics, and in Section IV we derive the boundary conditions for the ball with constant dielectric constant and permeability. In section V we quantize our fields, and in Section VI we compute the energy density and pressures in the vacuum outside the ball. In Section VII we prove that all the energy conditions are obeyed, and we conclude in Section VIII.

II. THE ELECTROMAGNETIC STRESS-ENERGY TENSOR

For the electromagnetic field outside a ball of radius $R$, with constant dielectric constant and permeability, we will calculate the vacuum expectation value of the energy density, radial and tangential pressures. We will use Heaviside-Lorentz units throughout.

The symmetric electromagnetic stress-energy tensor is
\[ T^{\mu\nu} = g^{\mu\alpha} F_{\alpha\lambda} F^{\lambda\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\lambda} F^{\alpha\lambda} \] (1)

The 00 component is the electromagnetic energy density,
\[ T^{00} = \frac{1}{2} (E^2 + B^2) \] (2)
and the stresses are
\[ T^{ij} = -(E^i E^j + B^i B^j - \frac{1}{2} \delta^{ij} (E^2 + B^2)) \] (3)
where $i, j$ are Cartesian coordinates in three space. We can rewrite the diagonal components in terms of a spherical coordinate system, which we will decompose into radial and tangential components, denoted by subscripts $r$ and $t$ respectively. Let
\[ F_r = \frac{1}{2} (E_r^2 + B_r^2) \] (4)
\[ F_t = \frac{1}{2} (E_t^2 + B_t^2) . \] (5)

Then the energy density and the radial and tangential pressures are respectively given by
\[ \rho = T^{00} = F_t + F_r \] (6a)
\[ p_r = T^{rr} = F_t - F_r \] (6b)
\[ p_t = T^{\phi\phi} = F_r . \] (6c)
III. CLASSICAL FIELDS

We will work with the electromagnetic potential $A^\mu(x, t)$. In empty space we can work in the radiation gauge, where $\partial_\mu A^\mu = 0$ and $A^0 = 0$, so that

$$E = -\frac{\partial A}{\partial t}$$

and

$$B = \nabla \times A.$$  \hfill (8)

We separate the two polarization channels, so that we can write the vector potential as a sum of classical modes,

$$A(x, t) = \sum_{\ell, m} \int_0^\infty \frac{dk}{\sqrt{2\omega}} (A_{k\ell m}^+ e^{-i\omega t} a_{k\ell m}^+ + A_{k\ell m}^- e^{-i\omega t} a_{k\ell m}^- + \text{complex conjugate})$$ \hfill (9)

where $\omega = k$ outside the ball since the field is massless, the sum over $\ell$ and $m$ gives the partial wave expansion in the 3 spatial dimensions, and we define

$$A_{k\ell m}^+(r, \Omega) \equiv X_{\ell m}(\Omega) \psi_{k\ell}^+(r)$$

$$A_{k\ell m}^-(r, \Omega) \equiv \frac{1}{k} \nabla \times (X_{\ell m}(\Omega) \psi_{k\ell}^-(r))$$ \hfill (10-11)

in terms of the vector spherical harmonics

$$X_{\ell m}(\Omega) \equiv -i(r \times \nabla) Y_{\ell m}(\Omega) \sqrt{\ell(\ell + 1)}.$$ \hfill (12)

The modes $A_{k\ell m}^+$ are the magnetic multipoles, also known as the transverse electric (TE) modes, and $A_{k\ell m}^-$ are the electric multipoles, or transverse magnetic (TM) modes. The radial functions $\psi_{k\ell}^+(r)$ and $\psi_{k\ell}^-(r)$ satisfy the same equation of motion as a scalar field,

$$\left( -\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{\ell(\ell + 1)}{r^2} \right) \psi_{k\ell}^\pm(r) = k^2 \psi_{k\ell}^\pm(r).$$ \hfill (13)

For any function $g_{k\ell}(r)$ obeying this equation, we have the following useful identities:

$$\hat{r} \cdot X_{\ell m} g_{k\ell}(r) = 0$$ \hfill (14)

$$\hat{r} \cdot (\nabla \times (X_{\ell m} g_{k\ell}(r))) = \frac{i}{r} \sqrt{\ell(\ell + 1)} g_{k\ell}(r) Y_{\ell m}$$ \hfill (15)

$$\nabla \times (\nabla \times (X_{\ell m} g_{k\ell}(r))) = -\nabla^2 (X_{\ell m} g_{k\ell}(r)) = k^2 X_{\ell m} g_{k\ell}(r)$$ \hfill (16)

$$\hat{r} \times (\nabla \times (X_{\ell m} g_{k\ell}(r))) = -\frac{1}{r} \frac{\partial}{\partial r}(r g_{k\ell}(r)) X_{\ell m}.$$ \hfill (17)

The $E$ and $B$ fields, which are related to $A$ via Eqs. (7) and (8) can then be expressed

$$E(x, t) = \sum_{\ell, m} \int_0^\infty \frac{dk}{\sqrt{2\omega}} (E_{k\ell m}^+ e^{-i\omega t} a_{k\ell m}^+ + E_{k\ell m}^- e^{-i\omega t} a_{k\ell m}^- + \text{complex conjugate})$$ \hfill (18)
and likewise for \( \mathbf{B} \), with

\[
\mathbf{E}_{k\ell m}^+ = i\omega \mathbf{X}_{\ell m} \psi_{k\ell}^+(r) \quad \text{(19)}
\]

\[
\mathbf{B}_{k\ell m}^- = \nabla \times \left( \mathbf{X}_{\ell m} \psi_{k\ell}^+(r) \right) \quad \text{(20)}
\]

\[
\mathbf{E}_{k\ell m}^- = i\frac{1}{k} \nabla \times \left( \mathbf{X}_{\ell m} \psi_{k\ell}^+(r) \right) \quad \text{(21)}
\]

\[
\mathbf{B}_{k\ell m}^+ = \nabla \times \left( \frac{1}{k} \nabla \times \left( \mathbf{X}_{\ell m} \psi_{k\ell}^+(r) \right) \right) = k \mathbf{X}_{\ell m} \psi_{k\ell}^-(r). \quad \text{(22)}
\]

Using the identities above, we can now decompose the modes separately into radial and tangential components,

\[
E_{k\ell m, r}^+ = 0 \quad \text{(23)}
\]

\[
E_{k\ell m, t}^+ = i\omega \mathbf{X}_{\ell m} \psi_{k\ell}^+(r) \quad \text{(24)}
\]

\[
B_{k\ell m, r}^+ = \frac{i}{r} \sqrt{\ell(\ell + 1)} Y_{\ell m} \psi_{k\ell}^+(r) \quad \text{(25)}
\]

\[
B_{k\ell m, t}^+ = \frac{1}{r} \frac{\partial}{\partial r} \left( r\psi_{k\ell}^+(r) \right) \hat{\mathbf{r}} \times \mathbf{X}_{\ell m} \quad \text{(26)}
\]

and

\[
E_{k\ell m, r}^- = -\frac{\omega}{kr} \sqrt{\ell(\ell + 1)} Y_{\ell m} \psi_{k\ell}^-(r) \quad \text{(27)}
\]

\[
E_{k\ell m, t}^- = \frac{i\omega}{kr} \frac{\partial}{\partial r} \left( r\psi_{k\ell}^-(r) \right) \hat{\mathbf{r}} \times \mathbf{X}_{\ell m} \quad \text{(28)}
\]

\[
B_{k\ell m, r}^- = 0 \quad \text{(29)}
\]

\[
B_{k\ell m, t}^- = k \mathbf{X}_{\ell m} \psi_{k\ell}^-(r). \quad \text{(30)}
\]

IV. BOUNDARY CONDITIONS FOR THE UNIFORM BALL

The general solution to the radial equation of motion, Eq. (13), can be written

\[
\psi_{k\ell}^\pm(r) = \frac{k}{\sqrt{2\pi}} \left[ e^{2i\delta_\ell^\pm(k)} h_\ell^{(1)}(z) + h_\ell^{(2)}(z) \right] \quad \text{(31)}
\]

where \( z = kr \), \( \delta_\ell^\pm(k) \) is the scattering phase shift, and

\[
h_\ell^{(1,2)}(z) = \sqrt{\frac{\pi}{2z}} H_{\ell+1/2}^{(1,2)}(z) \quad \text{(32)}
\]

are the spherical Hankel functions.

We now consider a uniform ball of radius \( R \), dielectric constant \( \epsilon \), permeability \( \mu \), and zero conductivity. Nonzero and finite conductivity leads to absorption, in which case the Casimir energy may depend in detail on the properties of the conducting material, so we will not consider that possibility. The case of perfect conductivity, however, can be obtained as the limit where \( \epsilon \to \infty \).

Within the ball, we have the same decomposition of the fields, except there we have \( k = n\omega \), where \( n = \sqrt{\epsilon\mu} \) is the index of refraction. We will consider only cases with index
of refraction \( n \geq 1 \), since a frequency-independent index of refraction below 1 would violate causality.\(^3\) We can also scale the wavefunction inside the ball by an overall constant in each channel. At the surface, \( \varepsilon E, E_t, B_r, \) and \( \frac{1}{\mu} B_t \) are continuous, giving two independent conditions for the two unknown quantities: the phase shift and the normalization constant of the interior wavefunction. Solving for the TE phase shift, we have \(^21\)

\[
e^{2i\delta_+^T(k)} = -\frac{\alpha \hat{j}_\ell'(nx) \hat{h}_\ell^{(2)}(x) - \hat{j}_\ell(nx) \hat{h}_\ell^{(2)'}(x)}{\alpha \hat{j}_\ell'(nx) \hat{h}_\ell^{(1)}(x) - \hat{j}_\ell(nx) \hat{h}_\ell^{(1)'}(x)}
\]  

(33)

where prime denotes differentiation with respect to the function’s argument, and we have defined \( \alpha = \sqrt{\varepsilon/\mu}, x = kR \), and the Riccati-Bessel and Riccati-Hankel functions

\[
\hat{j}_\ell(z) = z j_\ell(z) = \sqrt{\frac{\pi z}{2}} J_{\ell+1/2}(z)
\]  

(34)

\[
\hat{h}_\ell^{(1,2)}(z) = z h_\ell^{(1,2)}(z) = \sqrt{\frac{\pi z}{2}} H_{\ell+1/2}^{(1,2)}(z). 
\]  

(35)

Defining

\[
 s(x) = \frac{\hat{j}_\ell'(x)}{\hat{j}_\ell(x)}
\]  

(36)

we can write

\[
e^{2i\delta_+^T(k)} = -\frac{\alpha s(nx) \hat{h}_\ell^{(2)}(x) - \hat{j}_\ell(x)}{\alpha s(nx) \hat{h}_\ell^{(1)}(x) - \hat{j}_\ell'(x)}
\]  

(37)

and thus

\[
e^{2i\delta_+^T(k)} - 1 = -2\frac{\alpha s(nx) \hat{j}_\ell'(x)}{\alpha s(nx) \hat{h}_\ell^{(1)}(x) - \hat{j}_\ell'(x)} \delta_\ell^+(k) \rightarrow \delta_\ell^-(k).
\]  

(38)

The TM mode is the same with \( \alpha \rightarrow 1/\alpha \), and \( \delta_\ell^+(k) \rightarrow \delta_\ell^-(k) \).

V. QUANTIZATION

To quantize the electromagnetic field we simply declare that the coefficients \( a_{k\ell m}^\pm \) in Eq. \(^9\) are operators and take the Hermitian conjugate of those operators in the complex conjugate term. For the operators to have the usual relations in which all commutators vanish except

\[
 [a_{k\ell m}, a_{k'\ell' m'}^\dag] = \delta(k - k') \delta_{\ell\ell'} \delta_{mm'}.
\]  

(39)

We require the radial wave functions be normalized by

\[
\int_0^\infty r^2 \psi_{k\ell}^\pm(r)^* \psi_{k'\ell}^\pm(r) dr = \delta(k - k')
\]  

(40)

\(^3\) A real material can have \( n < 1 \) at some frequencies, but such materials necessarily have absorption as well.
which is satisfied by Eq. (31). The vector and scalar spherical harmonics are normalized by the completeness relations

$$\sum_m |X_{\ell m}|^2 = \sum_m |Y_{\ell m}|^2 = \frac{2\ell + 1}{4\pi}.$$ (41)

The quantum field $A(x, t)$ then obeys the conventional equal-time commutation relations in radiation gauge,

$$[A_i(x, t), E_j(y, t)] = -i \left( \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta(x - y).$$ (42)

Since we are interested in computing the vacuum expectation values of the components of the electromagnetic field, from this point on, notations such as $E^r_\ell$ and $F_r$ will denote the quantum mechanical vacuum expectation values of those quantities.

For the TE mode we find

$$E^2_{+r} = 0$$ (43)

$$E^2_{+t} = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{8\pi} \int_0^{\infty} dk \omega \left| \psi^\pm_{k\ell}(r) \right|^2$$ (44)

$$B^2_{+r} = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{8\pi} \frac{\ell(\ell + 1)}{r^2} \int_0^{\infty} \frac{dk}{\omega} \left| \psi^\pm_{k\ell}(r) \right|^2$$ (45)

$$B^2_{+t} = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{8\pi} \int_0^{\infty} \frac{dk}{\omega} \left| \frac{1}{r} \frac{\partial}{\partial r} \left( r\psi^\pm_{k\ell}(r) \right) \right|^2.$$ (46)

The renormalized contribution is found by subtracting the free wave contribution, $\psi^{\pm(0)}_{k\ell}(r)$, yielding

$$E^2_{+r} = 0$$ (47)

$$E^2_{+t} = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{8\pi} \int_0^{\infty} dk \omega \left( \left| \psi^\pm_{k\ell}(r) \right|^2 - \left| \psi^{\pm(0)}_{k\ell}(r) \right|^2 \right)$$ (48)

$$B^2_{+r} = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{8\pi} \frac{\ell(\ell + 1)}{r^2} \int_0^{\infty} \frac{dk}{\omega} \left( \left| \psi^\pm_{k\ell}(r) \right|^2 - \left| \psi^{\pm(0)}_{k\ell}(r) \right|^2 \right)$$ (49)

$$B^2_{+t} = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{8\pi} \int_0^{\infty} \frac{dk}{\omega} \left( \left| \frac{1}{r} \frac{\partial}{\partial r} \left( r\psi^\pm_{k\ell}(r) \right) \right|^2 - \left| \frac{1}{r} \frac{\partial}{\partial r} \left( r\psi^{\pm(0)}_{k\ell}(r) \right) \right|^2 \right).$$ (50)

Using the general form of the wavefunction, Eq. (31), and subtracting the free wavefunctions given by Eq. (31) with $\delta_\ell = 0$, we find

$$\left| \psi^\pm_{k\ell}(r) \right|^2 - \left| \psi^{\pm(0)}_{k\ell}(r) \right|^2 = \frac{k^2}{2\pi} \left[ (e^{2i\delta^\pm_\ell} - 1) h^{(1)}_{\ell}(z)^2 + (e^{-2i\delta^\pm_\ell} - 1) h^{(2)}_{\ell}(z)^2 \right].$$ (51)

and

$$\left| \frac{1}{r} \frac{\partial}{\partial r} \left( r\psi^\pm_{k\ell}(r) \right) \right|^2 - \left| \frac{1}{r} \frac{\partial}{\partial r} \left( r\psi^{\pm(0)}_{k\ell}(r) \right) \right|^2 = \frac{k^2}{2\pi r^2} \left[ (e^{2i\delta^\pm_\ell} - 1) \tilde{h}^{(1)}_{\ell}(z)^2 + (e^{-2i\delta^\pm_\ell} - 1) \tilde{h}^{(2)}_{\ell}(z)^2 \right].$$ (52)
where prime denotes differentiation with respect to $z = kr$, and we write $\delta_\ell$ instead of $\delta_\ell(k)$ for simplicity. The Appendix of [9] showed that the second term of Eq. (51) is just the first term with the replacement $k \rightarrow -k + i\epsilon$ and likewise for Eq. (52). Thus we can drop the second terms in both equations and extend the range of integration over $k$ to $-\infty$, with the understanding that $k$ is to be taken above any branch cut on the negative real axis,

$$E_{\pm r}^2 = 0$$ (53a)  
$$E_{\pm t}^2 = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{16\pi^2} \int_{-\infty}^{\infty} dk k^2 \left( e^{2i\delta_\ell^+} - 1 \right) h_\ell^{(1)}(z)^2$$ (53b)  
$$B_{\pm r}^2 = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{16\pi^2} \ell(\ell + 1) \int_{-\infty}^{\infty} \frac{dk}{\omega} k^2 \left( e^{2i\delta_\ell^+} - 1 \right) h_\ell^{(1)}(z)^2$$ (53c)  
$$B_{\pm t}^2 = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{16\pi^2} \int_{-\infty}^{\infty} \frac{dk}{\omega} k^2 \left( e^{2i\delta_\ell^+} - 1 \right) \hat{h}_\ell^{(1)'}(z)^2$$ (53d)

The calculation for the TM modes is exactly analogous,

$$E_{\pm r}^2 = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{16\pi^2} \ell(\ell + 1) \int_{-\infty}^{\infty} dk k^2 \left( e^{2i\delta_\ell^-} - 1 \right) h_\ell^{(1)}(z)^2$$ (54a)  
$$E_{\pm t}^2 = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{16\pi^2} \int_{-\infty}^{\infty} \frac{dk}{\omega} k^2 \left( e^{2i\delta_\ell^-} - 1 \right) \hat{h}_\ell^{(1)'}(z)^2$$ (54b)  
$$B_{\pm r}^2 = 0$$ (54c)  
$$B_{\pm t}^2 = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{16\pi^2} \int_{-\infty}^{\infty} dk k^2 \left( e^{2i\delta_\ell^-} - 1 \right) h_\ell^{(1)}(z)^2$$ (54d)

VI. ENERGY DENSITY AND PRESSURE

Substituting Eqs. (53) and (54) into Eqs. (11) and (12) we find

$$F_r = \frac{1}{32\pi^2 r^2} \sum_{\ell=1}^{\infty} (2\ell + 1) \ell(\ell + 1) \int_{-\infty}^{\infty} dk T_k \frac{k^2}{\omega} h_\ell^{(1)}(z)^2$$ (55)  
$$F_t = \frac{1}{32\pi^2 r^2} \sum_{\ell=1}^{\infty} (2\ell + 1) \int_{-\infty}^{\infty} dk T_k \left\{ \omega \hat{h}_\ell^{(1)}(z)^2 + \frac{k^2}{\omega} \hat{h}_\ell^{(1)'}(z)^2 \right\}$$ (56)

where

$$T_k = \left( e^{2i\delta_\ell^+(k)} - 1 \right) + \left( e^{2i\delta_\ell^-(k)} - 1 \right)$$ (57)

Together, these expressions reproduce the energy density found in [17].

Following the methods used in [4] and [9], we now convert this expression to a contour integral, which we close in the upper half plane. Since our interaction has finite range and no bound states, the $S$-matrix in Eq. (57) is analytic everywhere in the upper half-plane [21]. Thus the only contribution to the integral comes from the branch cut along the imaginary
k axis coming from \( \omega = \sqrt{k^2 + i\epsilon} \). To the right \( \omega = \sqrt{k^2} = k \), but to the left \( \omega = -k \), so with \( k = i\kappa \), we obtain

\[
F_r = -\frac{1}{4\pi^3 r^2} \sum_{\ell=1}^{\infty} (2\ell + 1) \ell (\ell + 1) \int_0^{\infty} d\kappa \tilde{T}_\kappa k_\ell(\zeta)^2
\]

(58)

\[
F_t = \frac{1}{4\pi^3 r^2} \sum_{\ell=1}^{\infty} (2\ell + 1) \int_0^{\infty} d\kappa \tilde{T}_\kappa \kappa \{ \tilde{k}_\ell(\zeta)^2 - \tilde{k}_\ell'(\zeta)^2 \}
\]

(59)

where \( \zeta = \kappa r \),

\[
\tilde{k}_\ell(\zeta) = \zeta k_\ell(\zeta) = \sqrt{\frac{\pi}{2}} K_{\ell+1/2}(\zeta)
\]

(60)

is the modified Riccati-Bessel function of the third kind, and

\[
\tilde{T}_\kappa = \frac{(-)^\ell}{\pi} \left[ \left( e^{2i\delta^+_\ell(i\kappa)} - 1 \right) + \left( e^{2i\delta^-_\ell(i\kappa)} - 1 \right) \right].
\]

(61)

From Eq. (58), we get

\[
\tilde{T}_\kappa = \frac{\alpha f(n\chi) \tilde{i}_\ell(\chi) - \tilde{i}'_\ell(\chi)}{\alpha f(n\chi) \tilde{k}_\ell(\chi) - \tilde{k}'_\ell(\chi)} + \{ \alpha \to 1/\alpha \}
\]

(62)

where \( \chi = \kappa R \),

\[
f(\chi) = s(i\chi) = \frac{\tilde{i}'_\ell(\chi)}{\tilde{i}_\ell(\chi)}
\]

(63)

and

\[
\tilde{i}_\ell(\zeta) = \zeta i_\ell(\zeta) = \sqrt{\frac{\pi}{2}} I_{\ell+1/2}(\zeta)
\]

(64)

is the modified Riccati-Bessel function of the first kind.

**VII. POSITIVITY**

We would like to know the sign of \( F_r \) and \( F_t \). First consider \( F_r \). Everything on the right hand side of Eq. (58) is manifestly positive, except for \( \tilde{T}_\kappa \) and the overall negative sign, so \( F_r \) has the opposite sign from \( \tilde{T}_\kappa \).

Now consider \( F_t \). The properties of modified Riccati-Bessel functions are discussed in Appendix A. From Eq. (A12) we find

\[
\tilde{k}'_\ell(\zeta) < -\tilde{k}_\ell(\zeta)
\]

(65)

Since both sides are negative,

\[
\tilde{k}'_\ell(\zeta)^2 > \tilde{k}_\ell(\zeta)^2
\]

(66)

so the term in braces in Eq. (59) is negative, and \( F_t \) also has the opposite sign from \( \tilde{T}_\kappa \).

Let us now determine the sign of \( \tilde{T}_\kappa \). We can write \( \tilde{T}_\kappa \) in the form

\[
\alpha a - b \quad \frac{\alpha^{-1} a - b}{\alpha^{-1} c - d} \quad \frac{2ac + 2bd - (\alpha + \alpha^{-1}) (ad + bc)}{c^2 + d^2 - (\alpha + \alpha^{-1}) cd}
\]

(67)
with

\[ a = f(n\chi) \hat{\nu}(\chi) \]  
\[ b = \hat{\nu}'(\chi) \]  
\[ c = f(n\chi)\hat{k}_\ell(\chi) \]  
\[ d = \hat{k}_\ell'(\chi) \]  

(68) (69) (70) (71)

The functions \( \hat{\nu}_\ell \) and \( \hat{k}_\ell \) are positive, and \( \hat{\nu}_\ell \) is increasing, so \( f \) is positive and \( a, b, \) and \( c \) are positive. On the other hand, \( \hat{k}_\ell \) is a decreasing function, so \( d \) is negative. Thus each term in the denominator of Eq. (67) contributes positively.

We can write the numerator of Eq. (67) as

\[ 2(a - b)(c - d) - (\alpha + \alpha^{-1} - 2) (ad + bc) \]  

(72)

and we have

\[ ad + bc = f(n\chi) \left[ \hat{\nu}(\chi)\hat{k}_\ell'(\chi) + \hat{k}_\ell(\chi)\hat{\nu}'(\chi) \right] \]  

(73)

By multiplying Eq. (A14) by the positive quantity \( \hat{\nu}(\chi)\hat{k}_\ell(\chi) \) we find that the term in brackets in Eq. (73) is positive. Since \( f(n\chi) > 0 \), and \( (\alpha + \alpha^{-1} - 2) = (\alpha - 1)^2/\alpha > 0 \), the second term in Eq. (72) contributes negatively.

Now \( c - d \) is manifestly positive, and

\[ a - b = \hat{\nu}_\ell(\chi) [f(n\chi) - f(\chi)] \]  

(74)

Since \( n > 1 \) by assumption, Eq. (A6) tells us that the term in brackets is negative, so the first term in Eq. (72) is negative. Thus \( \tilde{T}_\kappa < 0 \), and both \( F_r \) and \( F_t \) are positive.

From Eqs. (6), we see that \( \rho > p_t > 0 \) and \( \rho > |p_r| \). Thus the weak, null, dominant, and strong energy conditions are satisfied at each point, and therefore the corresponding averaged energy conditions are satisfied on every geodesic that lies outside the ball.

**VIII. DISCUSSION**

We have demonstrated, without resorting to numerical calculations, that the electromagnetic Casimir energy density is positive outside a ball with constant dielectric constant and permeability, and that all the standard energy conditions are obeyed. We needed to assume only that the index of refraction is greater than 1, which is necessary to maintain causality. We did not consider the cases of absorption and dispersion. We expect, however, that our results would hold in such cases, and furthermore, we expect them to remain valid in the case of material properties that vary with radius.

The electromagnetic situation discussed here gives a simpler result than the case of a scalar field. For the minimally coupled scalar field \( \overline{3} \), the pointwise energy conditions are not obeyed, although the average null energy condition nevertheless holds for every geodesic outside the ball. For the conformally coupled case the null energy condition does hold at each point, but only when all angular momentum modes are considered. In the electromagnetic case, the energy conditions hold at each point separately for each \( k \) and \( l \).
IX. ACKNOWLEDGMENTS

We would like to thank Larry Ford and Robert L. Jaffe for helpful conversations. K. D. O. was supported in part by the National Science Foundation (NSF). N. G. was supported in part by the NSF through the Vermont Experimental Program to Stimulate Competitive Research (VT-EPSCoR).

APPENDIX A: RICCATI-BESSEL FUNCTION INEQUALITIES

We let $\hat{\mu}_\ell$ and $\hat{k}_\ell$ be modified Riccati-Bessel functions as in Eqs. (64) and (60), and define their logarithmic derivatives,

$$f(x) = \frac{\hat{\mu}_\ell'(x)}{\hat{\mu}_\ell(x)}$$  \hspace{1cm} (A1)

as in Eq. (63) and

$$g(x) = \frac{\hat{k}_\ell'(x)}{\hat{k}_\ell(x)}$$  \hspace{1cm} (A2)

Now the modified Riccati-Bessel functions satisfy the differential equation

$$x^2 F''_\ell - (x^2 + \ell(\ell + 1)) F_\ell = 0$$  \hspace{1cm} (A3)

with $F = \hat{\mu}$ or $\hat{k}$, so the derivatives satisfy a generalized Riccati equation [22],

$$h'(x) = \frac{F''_\ell(x)}{F_\ell(x)} - \frac{F'_\ell(x)^2}{F_\ell(x)^2} = \frac{\ell(\ell + 1)}{x^2} + 1 - h(x)^2$$  \hspace{1cm} (A4)

for $h = f$ or $g$.

Differentiating Eq. (A4) gives

$$h''(x) = -2\frac{\ell(\ell + 1)}{x^3} - 2h(x)h'(x)$$  \hspace{1cm} (A5)

from which we conclude that if $h'(x) = 0$ for some $x$ then $h''(x) < 0$ there. Thus $h'(x)$ cannot increase through 0, and so if $h'(x)$ is negative for some $x$, then it is negative for all larger $x$, while if it is positive for some $x$, then it must be positive for all smaller $x$.

Now for $x \to 0$, $f(x) \to (l + 1)/x$, and so for small $x$, $f'(x)$ is negative, so

$$f'(x) < 0$$  \hspace{1cm} (A6)

for all $x$. As $x \to \infty$, $f(x) \to 1$, so

$$f(x) > 1$$  \hspace{1cm} (A7)

for all $x$.

For $x \to \infty$,

$$\hat{k}_\ell(x) \to \frac{\pi}{2} e^{-x} \left(1 + \frac{\ell(\ell + 1)}{2x} + O(x^{-2})\right)$$  \hspace{1cm} (A8)

so

$$\hat{k}'_\ell(x) \to -\hat{k}_\ell(x) - \frac{\pi}{2} e^{-x} \left(\frac{\ell(\ell + 1)}{2x^2} + O(x^{-3})\right)$$  \hspace{1cm} (A9)
therefore
\[ g(x) \to -1 - \frac{\ell(\ell + 1)}{2x^2} + O(x^{-3}) \quad (\text{A10}) \]

Since \( g'(x) > 0 \) for large \( x \), we have
\[ g'(x) > 0 \quad (\text{A11}) \]
for all \( x \), and thus
\[ g(x) < -1 \quad (\text{A12}) \]
for all \( x \).

Now by subtracting Eq. (A4) for \( f \) from the same equation for \( g \), we find
\[ g'(x) - f'(x) = f(x)^2 - g(x)^2 \quad (\text{A13}) \]
From Eqs. (A5) and (A11) this quantity is positive, so \( f(x)^2 > g(x)^2 \). Since \( g(x) < 0 \) while \( f(x) > 0 \), we find
\[ f(x) + g(x) > 0 \quad (\text{A14}) \]