Harmonic analysis/Functional analysis

Hypergroupoids and C*-algebras

Hypergroupoïdes et C*-algèbres

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Abstract

Let G be a locally compact groupoid. If X is a free and proper G-space, then \((X \ast X)/G\) is a groupoid equivalent to G. We consider the situation where X is proper, but no longer free. The formalism of groupoid C*-algebras and their representations is suitable to attach C*-algebras to this new object.

Résumé

Soit G un groupoïde localement compact. Si X est un G-espace qui est libre et propre, alors \((X \ast X)/G\) est un groupoïde équivalent à G. On considère la situation où X est seulement propre. Le formalisme des C*-algèbres de groupoïdes permet d’associer des C*-algèbres à ce nouvel objet.
Théorème 0.1. On suppose G à base dénombrable d’ouverts. Les représentations de l’algèbre involutive $C_c(G)$ qui sont non dégénérées et continues pour la topologie limite inductive se prolongent au triplet $(C_c(G), C_c(X), C_c((X \ast X)/G))$.

Comme dans [7], la démonstration repose sur le théorème de désintégration des représentations. On obtient la norme pleine et la norme réduite de $C_c((X \ast X)/G)$ en prenant respectivement la représentation universelle et la représentation régulière de $C_c(G)$. On définit les $C^*$-algèbres $C^*((X \ast X)/G)$ et $C_c^*((X \ast X)/G)$ de l’hypergroupoïde $(X \ast X)/G$ comme les $C^*$-complétions relatives à ces normes.

Les paires $(G, K)$, où $K$ est un sous-groupe compact d’un groupe localement compact $G$ fournissent des exemples classiques d’hypergroupes qui rentrent dans le cadre ci-dessus (avec $G = K \ast K$ et $\alpha$ la mesure invariante). On obtient les $C^*$-algèbres $C^*(K/GK)$ et $C_c^*(K/GK)$. Si $(G, K)$ est la complétion de Schlichting d’une paire de Hecke $(\Gamma, \Gamma_0)$ comme dans [8], $C_c^*(K/GK)$ s’identifie à la $C^*$-algèbre de Hecke de ce triplet.

Considérons maintenant une paire $(G, K)$ où $K$ est un sous-groupe fermé d’un groupe localement compact. On suppose, de plus, que $H^{(0)} = G^{(0)}$, $K$ est propre, l’application $r : G/K \rightarrow G^{(0)}$ admet un système de mesures invariant $\alpha$ et $G$ possède un système de Haar $\lambda$. Alors $(G = K/K, \alpha)$ est un $G$-espace propre mesuré. Le théorème ci-dessus permet de définir les $C^*$-algèbres pleine et réduite de l’hypergroupoïde $(X \ast X)/G = K\backslash G/K$. Dans la section 1 de [6], les auteurs, motivés par la construction de $C^*$-algèbres de Hecke, considèrent le cas où $G = \Gamma \times Y$ est le groupoïde de l’action d’un groupe $\Gamma$ sur un espace $Y$ et $H = \Lambda \times Y$ où $\Lambda$ est un sous-groupe de $\Gamma$ qui agit proprement sur $Y$. L’article [5], qui propose une définition d’une paire de Gelfand dans le cadre des groupoïdes, considère aussi l’algèbre de convolution $C_c(K\backslash G/K)$ dans le cas où $K$ est un sous-groupoïde compact d’un groupoïde localement compact $G$.

1. Introduction

This note stems from the elementary observation that the $C^*$-category of a groupoid $G$ defined in [7] can be extended from principal $G$-spaces to proper $G$-spaces. When $X$ is a principal locally compact $G$-space with invariant $r$-system $\alpha$, one can construct the $*$-algebra $(\alpha, \alpha)_c$ and its $C^*$-completion $(\alpha, \alpha)$; it is the $C^*$-algebra of the locally compact groupoid $(X \ast X)/G$ equipped with the Haar system induced by $\alpha$. When $X$ is only proper, the same formulas define the $*$-algebra $(\alpha, \alpha)_c$ and its $C^*$-completion $(\alpha, \alpha)$; however, $(X \ast X)/G$ is no longer a groupoid, but a hypergroupoid. Objects like $(X \ast X)/G$ generalize both hypergroupoids (when the $G$-space $X$ is transitive) and groupoids (when $X$ is free). While convolution algebras of measures are commonly associated with hypergroupoids, our construction gives convolution algebras of functions and $C^*$-algebras. It also covers the construction of $C^*$-algebras from Hecke pairs as in [2,8]. In fact, the observation that $(X \ast X)/G$ is no longer a groupoid when $X$ is not a free $G$-space but that its convolution algebra can still be defined appears in this context (see [6,3]). There, it is usual to introduce the reduced norm, while the existence of a maximal norm is problematic. Our framework provides natural maximal and reduced norms on the hypergroupoids we consider.

2. The $C^*$-category of a groupoid

We review the framework and the main results of [7], but assuming that the $G$-spaces are proper and no longer free. For the sake of simplicity, we consider here an untwisted groupoid $G$. Given a topological groupoid $G$ (with unit space $G^{(0)}$ and range and source maps $r$ and $s$), a left $G$-space is a topological space $X$ endowed with a continuous map $r_X : X \rightarrow G^{(0)}$, assumed to be open and onto, and a continuous action map $G \times X \rightarrow X$, where $G \times X$ is the subspace of composable pairs, i.e. $(y, x) \in G \times X$ such that $s(y) = r(x)$, sending $(y, x)$ to $y x$ in such a way that $(y y') x = y (y' x)$ for all composable triples $(y, y', x)$. The convolution product is given by:

$$f \ast g(x, z) = \int f[x, y]g[y, z] d\beta^{r(x)}(y)$$

(2)
In this formula, a representative $(x, z)$ has been fixed and $[x, z]$ denotes its class. The integration is over a compact set because the map $\varphi^x : Y^x(\lambda) \to (X * Y)/G$ defined by $\varphi^x(y) = [x, y]$ is proper. The resulting integral depends on $[x, z]$ only because of the invariance of $\beta$. One also defines:

$$(\alpha, f, \beta)^* = (\beta, f^*, \alpha)$$

where the involution is given by $f^*[y, x] = f[y, x]$.

**Lemma 2.1.** (Cf. [7, Lemma 3.1].) These operations are well defined and turn $C_c(G)$ into a $*$-category.

The next step is to define a $C^*$-norm on the $*$-category $C_c(G)$. A unitary representation of $G$ is a pair $(m, H)$ where $m$ is a transverse measure class [1, Definition A.1.19] and $H$ is a Borel $G$-Hilbert bundle. We recall that $m$ associates with $(X, \alpha)$ a measure class $m(\alpha)$ on $X/G$ in a coherent fashion. A unitary representation of $G$ defines by integration a representation of $C_c(G)$, that is, a functor into the $W^*$-category of Hilbert spaces. It associates to the object $(X, \alpha)$ the Hilbert space $H(\alpha) = L^2(X/G, m(\alpha), X * H/G)$ and to the arrow $(\alpha, f, \beta) : H(\beta) \to H(\alpha)$ defined by:

$$\langle \xi \sqrt{\mu}, (L(\alpha, f, \beta)\eta) \sqrt{v} \rangle = \int f(x, y) \langle \xi[x], \eta[y] \rangle \sqrt{(\mu \circ \hat{\beta}_1)(v \circ \hat{\alpha}_2)[x, y]}$$

where the sections $\xi, \sqrt{\mu} \in H(\alpha)$ and $\eta, \sqrt{v} \in H(\beta)$ are written as half-densities: $\mu$ [resp. $v$] is a measure on $X/G$ [resp. $Y/G$] in $m(\alpha)$ [resp. $m(\beta)$]. The systems of measures $\hat{\beta}_1$ and $\hat{\alpha}_2$ are induced by $\beta$ and $\alpha$ respectively as in [7] or [1, Lemma A.1.3] for the proper case. For example, one has $\int f \, dm(\alpha) = \int f(x, y) \, dm(\alpha)(x, y)$. By definition, the measures $m_1 = \mu \circ \hat{\beta}_1$ and $m_2 = v \circ \hat{\alpha}_2$ are equivalent; their geometric mean is the measure $(dm_1/dm_2)^{1/2} \, dm_2$. Note that by Cauchy–Schwarz inequality,

$$\|L(\alpha, f, \beta)\| \leq \max \left( \sup_x \left\| \int f(x, y) \, dm(\alpha)(x, y) \right\|, \sup_y \left\| \int f(x, y) \, dm(\beta)(x, y) \right\| \right)$$

The $1$-norm of $f$ is defined as the right-hand side. Just as in [7], we have:

**Theorem 2.2.** (Cf. [7, Proposition 3.5, Theorem 4.1].)

1. Let $(m, H)$ be a unitary representation of a locally compact groupoid $G$. Then the above formulas define a representation $L$ of the $*$-category $C_c(G)$, called the integrated representation, which is continuous for the inductive limit topology and bounded for the $1$-norm.

2. Let $(G, \lambda)$ be a second countable locally compact groupoid with Haar system. Every representation of the $*$-algebra $C_c(G, \lambda)$ in a separable Hilbert space that is non-degenerate and continuous for the inductive limit topology is equivalent to an integrated representation.

We deduce from this theorem that, given a locally compact groupoid with the Haar system $(G, \lambda)$, the $*$-category $C_c(G)$ can be completed into a $C^*$-category by defining the full $C^*$-norm $\|L(\alpha, f, \beta)\|$ as the supremum of $\|L(\alpha, f, \beta)\|$ over all unitary representations of $G$ in separable Hilbert bundles. In particular, if $(X, \alpha)$ is a measured proper $G$-space, this defines the $C^*$-algebra $C_c(\alpha, \alpha)$. If, moreover, $X$ is a free $G$-space, $(X * X)/G$ is a groupoid equivalent to $G$; the algebra $(\alpha, \alpha)$ is the full $C^*$-algebra of this groupoid (endowed with the Haar system induced by $\alpha$) and is Morita equivalent to $C^*_r(G, \lambda) = (\lambda, \lambda)$. If $X$ is not free, $(X * X)/G$ is a hypergroupoid (the multiplication law is defined on its subsets rather than on its elements). It is still true that $(\lambda, \alpha)$ is a full $C^*$-module over $(\alpha, \alpha)$, but its algebra of compact operators is only an ideal of $C^*_r(G, \lambda)$. One has similar results with the regular representation and the reduced norm. If we identify $(G * X)/G = X$ through the map $(y, x) \mapsto y^{-1}x$, we obtain the various incarnations (1) of the formula (2).

### 3. Examples

1. Let $K$ be a compact subgroup of a locally compact group $G$. The homogeneous space $X = G/K$ is a proper $G$-space equipped with an invariant measure $\alpha$. Then, $(X * X)/G$ is the double coset hypergroup $K'G/K$. The full and the regular representations of $G$ yield respectively the full and the reduced $C^*$-algebras of this hypergroup.

2. Let $\Gamma_0$ be an almost normal subgroup of a discrete group $\Gamma$ as in [2,8]. We equip $\Gamma_G$ with the counting measure. Since $\Gamma_0$ acts on $\Gamma/\Gamma_0$ with finite orbits, the convolution product is well defined on $C_c(\Gamma_0 \setminus \Gamma/\Gamma_0)$, which becomes the Hecke algebra $H(\Gamma_0)$. Let $(G, K)$ be the Schlichting completion of $(\Gamma, \Gamma_0)$. Then $H(\Gamma, \Gamma_0)$ can be identified with $C_c(K \setminus G/K)$ and we are in the situation of the first example.

3. A particular case of the next example, which generalizes the first example, is given in [6, Section 1]. Let $(G, \lambda)$ be a locally compact groupoid with the Haar system and $K$ a closed subgroupoid with $K^0 = G^0$. Assume that $K$ is a proper groupoid and that the map $r : G/K \to G^0$ has a $G$-invariant system of measures $\alpha$. Then $(X = G/K, \alpha)$ is a measured proper $G$-space. Thus we can construct the hypergroupoid $(X * X)/G = K'G/K$ and its full and its reduced $C^*$-algebras. If $K$ is principal, $(X * X)/G$ is a groupoid equivalent to $G$. The situation considered in [6] is the case of a semi-direct groupoid.
\[ G = \Gamma \ltimes Y \] where a group \( \Gamma \) acts on a space \( Y \) and \( H = \Lambda \ltimes Y \), where \( \Lambda \) is a subgroup of \( \Gamma \) acting properly on \( Y \). The convolution algebra \( C_c(K \backslash G / K) \) also appears in [5] (with \( K \) compact), where the authors give a groupoid version of a Gelfand pair.

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