What does Quantum Field Theory in Curved Spacetime Have to Say about the Dark Energy?

L.H. Ford

Institute of Cosmology, Department of Physics and Astronomy
Tufts University
Medford, Massachusetts 02155

Abstract

The issue of the vacuum energy of quantum fields is briefly reviewed. It is argued that this energy is normally either much too large or much too small to account for the dark energy. However, there are a few proposals in which it would be of the order needed to effect the dynamics of the present day universe. Backreaction models are reviewed, and the question of whether quantum effects can react against a cosmological constant is discussed.

1 Introduction

The subject of quantum field theory in curved spacetime has been extensively developed over the past thirty years. These developments include an understanding of cosmological and black hole particle creation, and a better understanding of the quantum stress tensor operator. Thus it is appropriate to ask if any of these insights help to understand the nature of the dark energy. One can phrase the question as follows: Is the dark energy of quantum origin, and if so, can it be understood without radical new physics?

2 The Expectation Value of the Quantum Stress Tensor

If quantum effects are to influence the large scale evolution of the universe, the simplest description is in terms of a semiclassical theory where the expectation value of the stress tensor, \( \langle T_{\mu\nu} \rangle \), acts as the source of gravity. This quantity is formally infinite, and hence needs to be modified before it can make physical sense. The simplest modification is a noncovariant frequency cutoff. For a massless field, such as the electromagnetic field, the energy density might then become

\[
\rho = \frac{1}{(2\pi)^3} \int d^3 k \omega e^{-\alpha \omega},
\]

\( \text{(1)} \)

1email: ford@cosmos.phy.tufts.edu
where $\alpha$ is an arbitrary cutoff parameter and the pressure is $P = \frac{1}{3} \rho$. The obvious objection to this modification is that it violates Lorentz invariance. However, in the context of cosmology, there is a preferred frame so this cutoff may not be quite as crazy as it seems. The energy density can be written as a function of $\alpha$ as

$$\rho = 10^{-30} \frac{\text{gm}}{\text{cm}^3} \left(\frac{10^{-2} \text{cm}}{\alpha}\right)^4. \quad (2)$$

There are several problems with this result. There would not seem to be any natural reason to select $\alpha \approx 10^{-2} \text{cm}$ as a cutoff. A much smaller value of $\alpha$ leads to a radiation dominated universe which recollapses in less than the age of the present universe. In any case, this model does not describe anything resembling the dark energy.

A better approach is to use a covariant cutoff, which preserves local Lorentz invariance. There are various covariant regularization methods which have been developed, including dimensional regularization, zeta-function regularization, and covariant point-splitting with direction averaging [1]. For our purposes, the details of these techniques are not important. The key result is that the divergent parts of $\langle T_{\mu\nu}\rangle$ may be written in terms of geometrical quantities as follows:

$$\langle T_{\mu\nu}\rangle_{\text{div}} = A \frac{g_{\mu\nu}}{\beta^4} + B \frac{G_{\mu\nu}}{\beta^2} + (C_1 H_{\mu\nu}^{(1)} + C_2 H_{\mu\nu}^{(2)}) \ln \beta. \quad (3)$$

Here $\beta$ is a cutoff parameter with the dimensions of length, $A$, $B$, $C_1$, and $C_2$ are constants, $G_{\mu\nu}$ is the Einstein tensor, and the $H_{\mu\nu}^{(1)}$ and $H_{\mu\nu}^{(2)}$ tensors are covariantly conserved tensors which are quadratic in the Riemann tensor. Specifically, they are the functional derivatives with respect to the metric tensor of the square of the scalar curvature and of the Ricci tensor, respectively:

$$H_{\mu\nu}^{(1)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} [\sqrt{-g} R^2] = 2 \nabla_\nu R - 2 g_{\mu\nu} \nabla_\rho R - \frac{1}{2} g_{\mu\nu} R^2 + 2 R R_{\mu\nu}, \quad (4)$$

and

$$H_{\mu\nu}^{(2)} \equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} [\sqrt{-g} R_{\alpha\beta} R^{\alpha\beta}] = 2 \nabla_\alpha \nabla_\nu R_{\mu\nu}^{\alpha} - \nabla_\rho \nabla^{\rho} R_{\mu\nu}^{\alpha} - \frac{1}{2} g_{\mu\nu} \nabla_\rho R - \frac{1}{2} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + 2 R_{\mu\nu}^{\rho} R_{\rho\nu}. \quad (5)$$

Let us focus our attention on the leading term, that proportional to $\beta^{-4} g_{\mu\nu}$. This is of the form of a cosmological constant, so that the corresponding equation of state is $P = -\rho$. Note that the price of a covariant regularization scheme is the breaking of conformal invariance, so massless fields no longer have traceless stress tensors. If
\( \beta \ll 10^{-2} \text{cm} \), then the resulting cosmological constant is too large to be consistent with the observed universe, and must be removed by renormalization.

All of the cutoff-dependent terms in \( \langle T_{\mu\nu} \rangle \) may be absorbed into redefinitions of the constants appearing in the gravitational action

\[
S_G = \frac{1}{16\pi G_0} \int d^4x \sqrt{-g} \left( R - 2\Lambda_0 + \alpha_0 R^2 + \beta_0 R_{\alpha\beta} R^{\alpha\beta} \right),
\]

(6)

We now include a matter action, \( S_M \), and vary the total action, \( S = S_G + S_M \), with respect to the metric. If we replace the classical stress tensor in the resulting equation by the quantum expectation value, \( \langle T_{\mu\nu} \rangle \), we obtain the semiclassical Einstein equation including the quadratic counterterms:

\[
G_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \alpha_0 H_{\mu\nu}^{(1)} + \beta_0 H_{\mu\nu}^{(2)} = 8\pi G_0 \langle T_{\mu\nu} \rangle.
\]

(7)

We may remove the divergent parts of \( \langle T_{\mu\nu} \rangle \) in redefinitions of the coupling constants \( G_0, \Lambda_0, \alpha_0 \), and \( \beta_0 \).

However, the renormalized values of these constants are free parameters which cannot be calculated by the theory. At this level, quantum field theory can no more calculate the cosmological constant \( \Lambda \) than it can find Newton’s constant or the mass of the electron. Thus, so far the answer to the question in the title of this article is “Nothing!”.

We might next inquire about the finite part of \( \langle T_{\mu\nu} \rangle \) which is not of the form of any of the counterterms used in renormalization. This part is unambiguous, and can be explicitly calculated for simple models, such as a massless scalar field in a Friedmann-Robertson-Walker universe \(^2\). Unfortunately, the results are typically of order

\[
\langle T_{\mu\nu} \rangle_{\text{fin}} \approx \frac{1}{t^4},
\]

(8)

where \( t \) is the present age of the universe. This is too small by a factor of about \( 10^{-120} \) to alter the dynamics of the universe at the present time. Conversely, if we use the cutoff dependent expressions Eqs. (1) or (3) with \( \beta \) of the order of the Planck scale, our answer is too large by a factor of about \( 10^{120} \). It is not clear how to find a result which is the geometric mean of these two extremes in a natural way. In other words, a cosmologically interesting energy density arises from a scale of the order of \( 10^{-2} \text{cm} \), which is about the geometric mean of the size of the observable universe and the Planck length. It is far from clear why such a length scale should arise.

There have been a number of ideas proposed for mechanisms which might solve this puzzle by providing a model for the finite part of \( \langle T_{\mu\nu} \rangle \) which is much larger than given in Eq. (8). Parker and Raval\(^3\) have suggested that there could be a large contribution from low mass scalar fields, leading to a limiting value of the scalar curvature. In this model, the scalar curvature could drop as in standard cosmology until this limiting value is approached. Then the universe would enter a phase of accelerated expansion. Another model, due to Sahni and Habib\(^4\) also postulates a low mass scalar field. In this model, the \( \langle T_{\mu\nu} \rangle \) due to created particle is large
and approximately of the form of a cosmological constant term. Schützhold[5] has proposed a model in which the QCD trace anomaly may produce a term in the stress tensor of the form

\[ \langle T_{\mu\nu} \rangle_{\text{fin}} \approx \frac{\Lambda_{QCD}^3}{t}, \]  

where \( \Lambda_{QCD} \approx 10^{2}\text{MeV} \) is the QCD scale. At the present age of the universe, \( t \), this term has about the right order of magnitude to begin to dominate the cosmological expansion. The three models discussed in this paragraph are all somewhat speculative, but indicate that there are possible ways to get cosmologically significant energy densities in the present epoch from quantum effects.

### 3 Backreaction Models

Now I wish to turn to a class of models which attempt to explain why the cosmological constant term is not enormously large today, and also possibly explain there may be an effective cosmological constant term which is large enough to alter the present expansion rate. These are backreaction models (or adjustment mechanisms). The basic idea is that some type of instability which causes a large value of \( \Lambda_{\text{eff}} \) in the early universe to decay naturally to a smaller value today. Ideally, one would like have a mechanism which act slowly enough to allow inflation to proceed. Thus if deSitter space is unstable, it should be so on a scale of more than about sixty horizon lengths, the minimum time needed for inflationary models to explain the horizon and flatness problems. One would also like a natural evolution to \( \Lambda_{\text{eff}} \leq 10^{-30}\text{g/cm}^3 \) today. In such a model, the effective cosmological constant is now very small compared to particle physics energy scales because the universe is very old compared to particle physics time scales.

No compelling mechanism which accomplishes both of these goals has yet been found, and doubts have been expressed as to whether such a mechanism can exist in principle [6]. Nonetheless, it is worth looking at some of the possibilities, as a successful backreaction model would be a great advance. It is also likely that any such model would rely upon quantum effects.

#### 3.1 Quantum Instability of deSitter Space?

One way in which backreaction against a cosmological term could manifest itself is through an instability of deSitter space, the solution which is the attractor in the set of solutions of Einstein’s equations with a positive cosmological constant. There is one example of a quantum instability in deSitter space, if not of deSitter space. This is the case of a free, massless, minimally coupled scalar field, and arises from the infrared behavior[7] of this theory. A massive scalar field has a well-defined deSitter invariant vacuum state. However, the two-point function diverges in the limit that
the mass \( m \) goes to zero:

\[
\langle \varphi(x)\varphi(x') \rangle \sim \frac{1}{m^2} \quad m \to 0.
\]  

(10)

There exist a class of quantum states which are free of the infrared divergence, but which all break deSitter invariance. In all of these states, \( \langle \varphi^2 \rangle \) becomes a function of time. In particular, in the representation of deSitter space as a spatially flat Robertson-Walker universe, all infrared-finite states lead to linear growth\[8\] in the comoving time, \( t \):

\[
\langle \varphi^2 \rangle \sim \frac{H^3 t}{4\pi^2} \quad t \to \infty,
\]  

(11)

where \( H \) is the inverse expansion time. However, this instability of the massless free field does not not lead to any backreaction on the rate of expansion. The reason is that the stress tensor for this field is bounded

\[
\langle T_{\mu\nu} \rangle \to \text{constant} \quad t \to \infty.
\]  

(12)

The derivatives of \( \varphi \) in the expression for \( T_{\mu\nu} \) remove the contribution of the long wavelength modes which cause \( \langle \varphi^2 \rangle \) to grow.

In the case of an interacting field theory involving a massless scalar field, it is possible for \( \langle T_{\mu\nu} \rangle \) to grow for a finite amount of time\[9, 10\]. Consider, for example, a self-coupled field with a \( \lambda \varphi^4 \) interaction. In this case, there will be a term in the stress tensor of the form of \( \lambda \langle \varphi^2 \rangle^2 g_{\mu\nu} \), which will initially grow as \( t^2 \). This growth will only last a finite time, however, before higher order contributions become important and act to stop the growth of \( \langle T_{\mu\nu} \rangle \). The field begins to acquire an effective mass, which suppresses the infrared instability driving the growth.

A more promising source of instability comes when we quantize the gravitational field on the deSitter background. Consider classical gravitational wave perturbations of a spatially flat Robertson-Walker universe. If we impose the transverse-tracefree gauge condition, which removes all gauge freedom and is the analog of the Coulomb gauge in electrodynamics, then the independent components of the perturbation are equivalent to a pair of massless scalar fields\[11, 12\]. This means that linearized quantum gravity on a deSitter background is subject to the same infrared instability as is the massless scalar field. In particular, the mean square of the perturbation will grow in time:

\[
\langle h_{\mu\nu}h^{\mu\nu} \rangle \sim t \quad t \to \infty.
\]  

(13)

However, at this one loop level, there is no backreaction on the expansion, just as for the case of scalar fields. This raises the question of what happens in the next, two loop, order. This question has been examined by Dolgov, \textit{et al} [13] and especially by Tsamis and Woodard [14], who find that the effective stress tensor of the gravitons does grow in this order, and will tend to react against the deSitter expansion. Unfortunately, the backreaction only begins to be significant at the point that the two-loop approximation becomes questionable, and higher loop corrections
cannot be ignored. This leaves us with the intriguing possibility that quantum gravity may be a source of backreaction, but no means of testing this possibility beyond two loop perturbative quantum gravity.

3.2 Dolgov-type Models

3.2.1 The Original Dolgov Model

In 1982, Dolgov [13] proposed a remarkably simple classical model for backreaction, based upon a massless, nonminimally coupled scalar field. This field has the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\alpha \varphi \partial^\alpha \varphi - \xi R \varphi^2),$$

(14)

where $R$ is the scalar curvature and $\xi$ is a negative constant. The associated equation for $\varphi$ is

$$\nabla_\alpha \nabla^\alpha \varphi + \xi R \varphi = 0,$$

(15)

and has growing solutions if $\xi < 0$ and $R > 0$. In deSitter space, where $R$ is a positive constant, the unstable solution grows exponentially:

$$\varphi(t) \sim e^{\gamma t},$$

(16)

where

$$\gamma = \frac{3}{2} H \left[ \left( 1 + \frac{16}{3} |\xi| \right)^{\frac{1}{2}} - 1 \right].$$

(17)

Here $t$ is the comoving time in the spatially flat Robertson-Walker coordinates. The stress tensor of this mode also grows exponentially,

$$\langle T_{\mu\nu} \rangle \sim e^{2\gamma t},$$

(18)

and causes the system to exit deSitter space on a time scale of the order of $1/\gamma$. This time scale is in turn determined by the coupling constant $\xi$, and can be long compared to the expansion time $1/H$ if $|\xi| \ll 1$.

At late times, the scalar field grows linearly in time

$$\varphi \sim \lambda t,$$

(19)

where $\lambda$ is a constant determined by $\xi$, and the scale factor grows as a power of time

$$a(t) \sim t^\alpha \quad \alpha = \frac{2|\xi| + 1}{4|\xi|}.$$

(20)

Most importantly, the scalar field stress tensor approaches the form of a cosmological constant term:

$$\langle T_{\mu\nu} \rangle \sim \frac{1}{8\pi} \Lambda_0 g_{\mu\nu} + O(t^{-2}),$$

(21)
where \( \Lambda_0 = 3H^2 \) is the effective value of the cosmological constant during the deSitter phase. The leading term in \( \langle T_{\mu\nu} \rangle \) cancels the effect of this cosmological constant. The remarkable feature of the backreaction is that it provides a natural cancellation of the original cosmological constant for any value of \( \Lambda_0 \) and to just the accuracy needed. The residual term of order \( t^{-2} \) is of just the magnitude needed to be cosmologically significant at the present. It should be noted that Dolgov’s model evades Weinberg’s “no-go” theorem [6], which attempted to rule out backreaction models. Weinberg’s theorem assumes that all fields are asymptotically constant in the future, which is not the case here, as can be seen from Eq. (19). The Dolgov model basically uses the kinetic energy of the growing \( \phi \) field to cancel the cosmological constant.

Unfortunately, this model also suffers from a fatal flaw, which was recognized in the original paper of Dolgov: The effective value of Newton’s constant is also driven to zero. This arises because the \( R \) term in Eq. (14) is of the form of the Lagrangian for gravity. This leads to an effective value of Newton’s constant of

\[
G_{\text{eff}} = \frac{G_0}{1 + 8\pi G_0 |\xi| \phi^2} \sim \frac{1}{t^2},
\]

where \( G_0 \) is the “bare” value of Newton’s constant when \( \phi = 0 \). There is a lesser flaw in the Dolgov model as well. In order to achieve adequate inflation, we want \( |\xi| \ll 1 \), which implies that \( \alpha \gg 1 \). In this case, inflation never really ends and we do not have a very realistic cosmology.

### 3.2.2 A Variant of the Dolgov Model

An alternative model [16] uses the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left[ \partial_{\alpha} \phi \partial^{\alpha} \phi - \xi_0 R \ln(R\ell^2) \phi^2 \right],
\]

where \( \ell \) is an arbitrary length scale. The motivation for the introduction of the \( \ln(R\ell^2) \) factor comes from quantum effects in curved spacetime. It can be shown from renormalization group arguments that \( \langle \phi^2 \rangle \) for a free field always acquires the term [17]

\[
\langle \phi^2 \rangle_R = \frac{\xi - \frac{1}{6}}{96\pi^2} R \ln(R\ell^2) .
\]

This means that one loop quantum corrections will cause a self coupled scalar field with a \( \lambda \phi^4 \) interaction to acquire a \( \ln(R\ell^2) \) term of the form of that in Eq. (23). The effect of this term is to cause \( \xi \) to become a running coupling constant which scales with curvature. The effective value of \( \xi \) is

\[
\xi_{\text{eff}} = \xi_0 \ln(R\ell^2) .
\]

The time dependence of \( \xi_{\text{eff}} \) is sufficiently weak that we can arrange that it has one constant value during the initial deSitter phase, and another approximately constant
value today. In particular, it is possible to have $|\xi_{\text{eff}}| \ll 1$ in the deSitter phase, but $|\xi_{\text{eff}}| \approx 1$ today. This solves the problem in the original model that one could not have both adequate inflation in the early universe, and noninflationary expansion today.

Unfortunately, the bigger problem of the vanishing of gravity on scales small compared to the horizon remains in this new model. However, we can see what might solve this problem. If one had a compelling reason to treat Eq. (23) only as an effective action for deriving the equation for $\varphi$, but not the Einstein equations, then the problem would disappear. In other words, one needs a model in which quantum corrections create a Dolgov-type model on cosmological scales, but which do not modify gravity on much smaller scales. It is not clear if such a model is possible, but this seems to be worth exploring.

4 Summary

We have seen that the cutoff-dependent vacuum energy of quantum fields, whether given by Eq. (1) or by Eq. (3), is much too large if the cutoff is dictated by any particle physics length scale. If the cutoff is at the Planck scale, then the answer is too large by a factor of about $10^{120}$. On the other hand, the renormalized energy density, such as that in Eq. (8), is too small to influence the present expansion of the universe by a factor of about $10^{-120}$. There are several models which give an intermediate value which is neither too large nor too small to be of interest. However, all of these models are somewhat speculative. It is not yet clear that there is a natural and compelling way to get a cosmologically interesting energy density from quantum effects.

A viable backreaction model would seem to be the best way both to resolve the fine tuning problem and to explain the dark energy. The outstanding question is whether any such model exists. The quantum instability of gravitons in deSitter space provides a possible route for the onset on an instability. The difficult problem is describe what happens next. The Dolgov model gives an intriguing picture of the form that late time backreaction might take. It remains to be seen if this sort of behavior can arise in a realistic theory.

Acknowledgement: This work was supported in part by the National Science Foundation under Grant PHY-9800965.

References

[1] See, for example, N.D. Birrell and P.C.W. Davies, Quantum Fields in Curved Space, (Cambridge University Press, 1982), Chap. 6.

[2] T.S. Bunch, J. Phys. A 12, 517 (1979).

[3] L. Parker and A. Raval, Phys. Rev. D 60, 123502 (1999), gr-qc/9908013.
[4] V. Sahni and S. Habib, Phys. Rev. Lett. 81, 1766 (1998), hep-ph/9808204.
[5] R. Schützhold, gr-qc/0204018.
[6] S. Weinberg, Rev. Mod. Phys. 61, 1 (1989).
[7] L.H. Ford and L. Parker, Phys. Rev. D 16, 245 (1977).
[8] A. Vilenkin and L.H. Ford, Phys. Rev. D 26, 1231 (1982).
[9] L.H. Ford, Phys. Rev. D 31, 710 (1985).
[10] L.R. Abramo and R.P. Woodard, Phys. Rev. D 65, 063516 (2001), astro-ph/0109273.
[11] E.M. Lifshitz, J. Phys. USSR 10, 116 (1946).
[12] L.H. Ford and L. Parker, Phys. Rev. D 16, 1601 (1977).
[13] A.D. Dolgov, M.B. Einhorn, and V.I. Zakharov, Phys. Rev. D 52, 717 (1995), gr-qc/9403056.
[14] N.C. Tsamis and R.P. Woodard, Nucl. Phys. B 474, 235 (1996), hep-ph/9602315; Ann. Phys. 253, 1 (1997), hep-ph/9602316; Phys. Rev. D 54, 2621 (1996), hep-ph/9602317.
[15] A.D. Dolgov, in The Very Early Universe, G.W. Gibbons, S.W. Hawking, and S.T.C. Siklos, eds. (Cambridge University Press, 1983).
[16] L.H. Ford, Phys. Rev. D 35, 2339 (1987).
[17] L.H. Ford and D.J. Toms, Phys. Rev. D 25, 1510 (1984).