RECONSTRUCTION IN THE PARTIAL DATA CALDERÓN PROBLEM ON ADMISSIBLE MANIFOLDS

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ABSTRACT. We consider the problem of developing a method to reconstruct a potential $q$ from the partial data Dirichlet-to-Neumann map for the Schrödinger equation $(-\Delta + q)u = 0$ on a fixed admissible manifold $(M, g)$. If the part of the boundary that is inaccessible for measurements satisfies a flatness condition in one direction, then we reconstruct the local attenuated geodesic ray transform of the one-dimensional Fourier transform of the potential $q$. This allows us to reconstruct $q$ locally, if the local (unattenuated) geodesic ray transform is constructively invertible. We also reconstruct $q$ globally, if $M$ satisfies certain concavity condition and if the global geodesic ray transform can be inverted constructively. These are reconstruction procedures for the corresponding uniqueness results given by Kenig and Salo [8]. Moreover, the global reconstruction extends and improves the constructive proof of Nachman and Street [15] in Euclidean setting. We derive a certain boundary integral equation which involves the given partial data and describes the traces of complex geometrical optics solutions. For construction of complex geometrical optics solutions, following [15] and improving their arguments, we use a new family of Green’s functions for the Laplace-Beltrami operator and the corresponding single layer potentials. The constructive inversion problem for local or global geodesic ray transforms is one of the major topics of interest in integral geometry.

1. INTRODUCTION

In 1980, Alberto Calderón [2] proposed the problem whether one can determine the electrical conductivity of a medium from voltage and current measurements at the boundary. In the mathematical literature, this problem is known as Calderón’s inverse conductivity problem.

The Calderón’s problem can be reduced to the problem of determining electric potential $q$ from the Dirichlet-to-Neumann map associated to the Schrödinger operator $-\Delta + q$. We will first discuss the case of Euclidean space in dimension $n \geq 3$. In the fundamental paper by Sylvester and Uhlmann [20] it was shown that bounded potential in a bounded domain of Euclidean space can be uniquely determined from the knowledge of the Dirichlet-to-Neumann map. Since then, substantial progress has been achieved on Calderón’s problem. Then corresponding reconstruction procedure was given by Nachman [14] and independently by Novikov [16]. The reader is referred to recent expository paper by Uhlmann [22] for a survey of progress made on Calderón’s problem.
In the current paper we are interested in the case when the Dirichlet-to-Neumann map is known only on part of the boundary. Let \( \Gamma^+ \) and \( \Gamma^- \) be the open subsets of the boundary where Dirichlet data inputs are prescribed and Neumann data measurements are made. The first result is due to Bukhgeim and Uhlmann [1]. They prove unique determination result if \( \Gamma^+ \) and \( \Gamma^- \) are roughly complementary and slightly more than half of the boundary. This result has been improved significantly by Kenig, Sjöstrand and Uhlmann [10] where they show that bounded potential can be uniquely recovered if \( \Gamma^- \) possibly very small open subset of the boundary, but \( \Gamma^+ \) must be slightly larger than the complement of \( \Gamma^- \) in the boundary. Constructive proof of this result is given by Nachman and Street [15]. For recent results on Calderón’s inverse problem with partial data, see [7]. The approaches of [1, 10, 15] are based on Carleman estimates with boundary terms.

There is a result by Isakov [6] where he gives uniqueness result when \( \Gamma^- = \Gamma^+ = \Gamma \) and the inaccessible part of the boundary for measurements is either part of a hyperplane or part of a sphere. This work is based on a reflection argument.

In the current paper we consider partial data Calderón’s problem on manifolds. The methods of [10, 6] were unified and extended to so-called admissible manifolds (which will be described below) by Kenig and Salo [8] obtaining improved results. To appreciate these improvements, the reader is referred to [8, Section 3] for detailed corresponding discussions. The goal of this paper, is to give the reconstruction procedures to the corresponding results of [8].

Let us give the precise mathematical formulation of the problem. Let \( (M,g) \) be a compact oriented Riemannian manifold with boundary. Following Bukhgeim and Uhlmann [1], we work with the following Hilbert space which is the largest domain of the Laplace-Beltrami operator \( \Delta_g \):

\[
H_{\Delta_g}(M) = \{ u \in L^2(M) : \Delta_g u \in L^2(M) \}.
\]

The trace maps \( \text{tr}(u) = u|_{\partial M} \) and \( \text{tr}_\nu(u) = \frac{\partial u}{\partial n}|_{\partial M} \) defined on \( C^\infty(M) \) have extensions to a bounded operators \( H_{\Delta_g}(M) \to H^{-1/2}(\partial M) \) and \( H_{\Delta_g}(M) \to H^{-3/2}(\partial M) \), respectively; see Proposition 2.1.

Now we introduce the following space on the boundary \( \partial M \):

\[
\mathcal{H}_g(\partial M) = \{ \text{tr}(u) : u \in H_{\Delta_g}(M) \} \subset H^{-1/2}(\partial M).
\]

The topology on \( \mathcal{H}_g(\partial M) \) is defined in Section 2, right before Proposition 2.4. Under this topology, the operator \( \text{tr} : H_{\Delta_g}(M) \to \mathcal{H}_g(\partial M) \) is bounded.

We consider the closely related problem for Schrödinger operators. Suppose that \( q \in L^\infty(M) \) such that 0 is not a Dirichlet eigenvalue of \( -\Delta_g + q \) in \( M \). Then for \( f \in \mathcal{H}_g(\partial M) \), the Dirichlet problem has a unique solution \( u \in H_{\Delta_g}(M) \)

\[
(-\Delta_g + q)u = 0 \quad \text{in} \quad M,
\]

\[
\text{tr}(u) = f \quad \text{on} \quad \partial M.
\]

The Dirichlet-to-Neumann map is defined by

\[
\Lambda_{g,q}(f) = \text{tr}_\nu(u).
\]
By the results of Section 2, $\Lambda_{g,q}$ is a bounded linear operator $\Lambda_{g,q} : H_g(\partial M) \to H^{-3/2}(\partial M)$. Given two open subsets $\Gamma_-, \Gamma_+ \subseteq \partial M$. The partial data inverse problem is to determine $q$ from the knowledge of $\Lambda_{g,q}f$ on $\Gamma_-$ for all $f \in H_g(\partial M)$ supported in $\Gamma_+$.

We need to introduce the notion of admissible manifolds.

**Definition.** A compact Riemannian manifold $(M, g)$ with boundary of dimension $n \geq 3$, is said to be admissible if it is conformal to a submanifold with boundary of $\mathbb{R} \times (M_0, g_0)$ where $(M_0, g_0)$ is simple $(n - 1)$-dimensional manifold. By simplicity of $(M_0, g_0)$ we mean that the boundary $\partial M_0$ is strictly convex, and for any point $x \in M_0$ the exponential map $\exp_x$ is a diffeomorphism from its maximal domain in $T_x M_0$ onto $M_0$.

Compact submanifolds of Euclidean space, the sphere minus a point and of hyperbolic space are all examples of admissible manifolds.

If $(M, g)$ is admissible, points of $M$ can written as $x = (x_1, x')$, where $x_1$ is the Euclidean coordinate. We define

$$
\partial M_\pm = \{ x \in \partial M : \pm \partial_{x_1} \varphi(x) > 0 \},
$$

$$
\partial M_{\text{tan}} = \{ x \in \partial M : \partial_{x_1} \varphi(x) = 0 \},
$$

where $\varphi(x) = x_1$. The function $\varphi$ is a natural limiting Carleman weight in $(M, g)$; see [3]. In the results below we assume that there is a part which inaccessible for measurements $\Gamma_1 \subset \partial M_{\text{tan}}$, and the accessible part will be denoted by $\Gamma_a = \partial M_{\text{tan}} \setminus \Gamma_1$.

We say that a unit speed geodesic $\gamma : [0, T] \to M_0$, on transversal simple manifold $(M_0, g_0)$, is nontangential if $\gamma(0), \gamma(T) \in \partial M_0$ and $\gamma(t) \in M_0^{\text{int}}$ if $t \in (0, T)$.

The first main result of our paper, says that one can reconstruct the local attenuated geodesic ray transform of the one-dimensional Fourier transform (with respect to $x_1$-variable) of the potential $q$ from the partial knowledge of the Dirichlet-to-Neumann map with $\Gamma_+ \supset \partial M_+ \cup \Gamma_a$ and $\Gamma_- \supset \partial M_- \cup \Gamma_a$.

**Theorem 1.1.** Let $(M, g)$ be an admissible manifold, and suppose that $q \in C(M)$ such that $0$ is not a Dirichlet eigenvalue of $-\Delta_g + q$. Let $\Gamma_a \subset \partial M_{\text{tan}}$ be closed such that for some open $E \subset \partial M_0$ one has

$$
\Gamma_1 \subset \mathbb{R} \times (\partial M_0 \setminus E).
$$

Let $\Gamma_a = \partial M_{\text{tan}} \setminus \Gamma_1$ and let $\Gamma_+ \subset \partial M$ be a neighborhood of $\partial M_+ \cup \Gamma_a$. Then for any given nontangential geodesic $\gamma : [0, T] \to M_0$ with endpoints on $E$ and for any $\lambda \in \mathbb{R}$, the integral

$$
\int_0^T e^{-2\lambda \langle cq \rangle(2\lambda, \gamma(t))} \, dt
$$

can be constructively recovered from the knowledge of $\Lambda_{g,q}(f)$ on $\Gamma_-$ for all $f \in H_g(\partial M)$ supported in $\Gamma_+$. Here $q$ is extended outside of $M$ by zero, and $\langle cq \rangle$ is the one-dimensional Fourier transform of $q$ with respect to $x_1$-variable.
This is a constructive version of the corresponding uniqueness result by Kenig and Salo [8, Theorem 2.1].

In the next result, we consider the local geodesic ray transform $I_O$ in an open subset $O$ of the transversal simple manifold $(M_0, g_0)$ which is defined for $f \in C(M_0)$ as

$$I_O f(\gamma) := \int_{\gamma} f(\gamma(t)) \, dt,$$

$\gamma$ is a nontangential geodesic contained in $O$.

We say that $I_O$ is constructively invertible in $O$, if any $f \in C(M_0)$ can be recovered in $O$ from the knowledge of $I_O f$.

Using Theorem 1.1 one can constructively recover potentials in the set where the local geodesic ray transform is invertible.

**Theorem 1.2.** Suppose that $(M, g)$, $q \in C(M)$, $E \subset \partial M_0$ and $\Gamma_{\pm}$ are as in Theorem 1.1. Assume that $O \subset M_0$ is open such that $O \cap \partial M_0 \subset E$ and the local ray transform is constructively invertible on $O$. Then $q$ can be constructively determined in $M \cap (\mathbb{R} \times M_0)$ from the knowledge of $\Lambda_{g, q}(f)$ on $\Gamma_{-}$ for all $f \in H_{g}(\partial M)$ supported in $\Gamma_{+}$.

This result gives a constructive proof of the corresponding uniqueness result by Kenig and Salo [8, Theorem 2.2]; the latter is the above mentioned generalization of the result of Isakov [6].

Constructive invertibility of the local ray transform, to the best of author’s knowledge, is known in the following case: if $M_0$ has dimension $n \geq 3$ and if $p \in \partial M_0$ is such that $\partial M_0$ is strictly convex near $p$, then there is an open $O \subset M_0$ containing $p$ on which $I_O$ is constructively invertible; this result is due to Uhlmann and Vasy [23].

In two dimensions, no such result is known. Even injectivity of the local geodesic ray transform is an open question.

If $\partial M_{\text{tan}}$ has zero measure in $\partial M$, we give the reconstruction procedure to determine potentials globally. The problem is reduced to the constructive invertibility of the global geodesic ray transform on the transversal simple manifold $M_0$.

**Theorem 1.3.** Let $(M, g)$ be an admissible manifold, and suppose that $q \in C(M)$ such that $0$ is not a Dirichlet eigenvalue of $-\Delta_g + q$. Suppose that $\partial M_{\text{tan}}$ is of zero measure in $\partial M$. If the global geodesic ray transform is constructively invertible in $M_0$, then $q$ can be constructively determined in $M$ from the knowledge of $\Lambda_{g, q}(f)$ on $\partial M_{-}$ for all $f \in H_{g}(\partial M)$ supported in $\partial M_{+}$.

This is a generalization with refinements to admissible manifolds of the corresponding result by Nachman and Street [15] in Euclidean setting. More precisely, comparing to [15], we do not assume that the subsets of Dirichlet data inputs overlap with the subsets of Neumann data measurements. So our reconstruction procedure is new even in Euclidean space. The version of Theorem 1.3 was given by Kenig, Salo and Uhlmann [9] for full data case on admissible manifolds of dimension three.

Constructive invertibility of the global ray transform is known in the following cases:
• \((M_0, g_0) = (\Omega, e)\) where \(\Omega \subset \mathbb{R}^n\) is open and bounded with \(C^\infty\) boundary, and \(e\) is the Euclidean metric. In this case inversion formula is given in the book of Sharafutdinov [19, Section 2.12].

• \((M_0, g_0)\) of dimension \(n \geq 3\), have strictly convex boundary and is globally foliated by strictly convex hypersurfaces. For such case, there is a layer stripping type algorithm for reconstruction developed by Uhlmann and Vasy [23].

• \((M_0, g_0)\) is a simple surface. In this case, there is a Fredholm type inversion formula which was derived by Pestov and Uhlmann [17]; see also the article of Krishnan [11].

The problem of constructive inversion of local or global geodesic ray transforms is of independent interest in integral geometry.

The structure of the paper is as follows. In Section 2 we give some preliminaries about trace operators and Green’s identity for the space \(H_{\Delta_g}(M)\). We also consider the well-posedness of the Dirichlet problem for the Schrödinger equation \((-\Delta_g + q)u = 0\) with boundary condition in \(H_g(\partial M)\). Section 3, following the arguments of [15] and modifying them, is devoted to the construction of the new Green’s operators for the Laplace-Beltrami operator, and in Section 4 the corresponding single layer potentials are constructed. The solvability of the required boundary integral equation is given in Section 5. Then we construct complex geometrical optics solutions in Section 6, and we use these solutions to give reconstruction procedures in Section 7.

2. Trace operators and the Dirichlet-to-Neumann map

Let \((M, g)\) be a compact Riemannian manifold with boundary. We use the notation \(d\text{Vol}_g\) for the volume form of \((M, g)\) and \(d(\partial M)_g\) for the induced volume form on the boundary \(\partial M\). For any two functions \(u, v\) on \(M\), define an inner product

\[
(u|v)_{L^2(M)} := \int_M u(x)v(x) d\text{Vol}_g(x),
\]

and the corresponding norm will be denoted by \(\| \cdot \|_{L^2(M)}\). For any two functions \(f, h\) on \(\Gamma \subset \partial M\), define an inner product

\[
(f|h)_{\Gamma} := \int_{\Gamma} f(x)\overline{h(x)} d\sigma_{\partial M}(x),
\]

and by \(\| \cdot \|_{\Gamma}\) will be denoted the corresponding norm. We also write for short

\[
\|\nabla u\|_{L^2(M)} = \left(\int_M |\nabla u(x)|^2 d\text{Vol}_g(x)\right)^{1/2}.
\]

Following Bukhgeim and Uhlmann [1], we work with the following Hilbert space which is the largest domain of the Laplace-Beltrami operator \(\Delta_g\):

\[
H_{\Delta_g}(M) = \{u \in L^2(M) : \Delta_g u \in L^2(M)\}.
\]
The norm on $H_{\Delta_g}(M)$ is
\[ \|u\|_{H_{\Delta_g}(M)}^2 = \|u\|_{L^2(M)}^2 + \|\Delta_g u\|_{L^2(M)}^2. \]

The proof of the following result is essentially the same as in [1] (see also, for example [12]). We include it here for the completeness and accuracy of the exposition.

**Proposition 2.1.** The trace maps $\text{tr}(u) = u|_{\partial M}$ and $\text{tr}_\nu(u) = \frac{\partial u}{\partial \nu}|_{\partial M}$ defined on $C^\infty(M)$ have extensions to a bounded operators $H_{\Delta_g}(M) \rightarrow H^{-1/2}(\partial M)$ and $H_{\Delta_g}(M) \rightarrow H^{-3/2}(\partial M)$, respectively. Moreover, if $u \in H_{\Delta_g}(M)$ and $\text{tr}(u) \in H^{3/2}(\partial M)$, then $u \in H^2(M)$ and $\text{tr}_\nu(u) \in H^{1/2}(\partial M)$.

**Proof.** First, we show that the trace map $\text{tr}$ has an extension to a bounded operator $H_{\Delta_g}(M) \rightarrow H^{-1/2}(\partial M)$. Let $u \in C^\infty(M)$ and $w \in H^{1/2}(\partial M)$. By the surjectivity of the trace map on $H^2(M)$, there is $v \in H^2(M)$ such that
\[ \text{tr}(v) = 0, \quad \text{tr}_\nu(v) = \frac{\partial v}{\partial \nu}|_{\partial M} = w, \quad \|v\|_{H^2(M)} \leq C\|w\|_{H^{1/2}(\partial M)}. \]

Using Green’s formula, we get
\[ (\text{tr}(u)|w)_{\partial M} = \int_{\partial M} \text{tr}(u) \overline{w} \, d(\partial M)_g \]
\[ = \int_{\partial M} \text{tr}(u) \text{tr}_\nu(v) \, d(\partial M)_g = \int_M (u \Delta_g v - \nabla \Delta_g u) \, d\text{Vol}_g. \]

Therefore,
\[ |(\text{tr}(u)|w)_{\partial M}| \leq \|u\|_{H_{\Delta_g}(M)} \|v\|_{H^2(M)} \leq C\|u\|_{H_{\Delta_g}(M)} \|w\|_{H^{1/2}(\partial M)}. \]

This proves that the map $\text{tr} : C^\infty(M) \rightarrow H^{-1/2}(\partial M)$ is bounded and controlled by the $H_{\Delta_g}(M)$-norm. Since $C^\infty(M)$ is dense in $H_{\Delta_g}(M)$, we can extend $\text{tr}$ to a bounded linear map $H_{\Delta_g}(M) \rightarrow H^{-1/2}(\partial M)$.

Next, we show that the trace map $\text{tr}_\nu$ has an extension to a bounded operator $H_{\Delta_g}(M) \rightarrow H^{-3/2}(\partial M)$. Let $u \in C^\infty(M)$ and $w \in H^{3/2}(\partial M)$. By the surjectivity of the trace map on $H^2(M)$, there is $v \in H^2(M)$ such that
\[ \text{tr}(v) = w, \quad \text{tr}_\nu(v) = \frac{\partial v}{\partial \nu}|_{\partial M} = 0, \quad \|v\|_{H^2(M)} \leq C\|w\|_{H^{3/2}(\partial M)}. \]

Using Green’s formula, we get
\[ (\text{tr}_\nu(u)|w)_{\partial M} = \int_{\partial M} \text{tr}_\nu(u) \overline{w} \, d(\partial M)_g \]
\[ = \int_{\partial M} \text{tr}_\nu(u) \text{tr}_\nu(v) \, d(\partial M)_g = \int_M (\nabla \Delta_g u - u \Delta_g v) \, d\text{Vol}_g. \]

Therefore,
\[ |(\text{tr}_\nu(u)|w)_{\partial M}| \leq \|u\|_{H_{\Delta_g}(M)} \|v\|_{H^2(M)} \leq C\|u\|_{H_{\Delta_g}(M)} \|w\|_{H^{3/2}(\partial M)}. \]
This proves that the map $\text{tr}_\nu : C^\infty(M) \to H^{-3/2}(\partial M)$ is bounded and controlled by the $H_{\Delta_q}(M)$-norm. Since $C^\infty(M)$ is dense in $H_{\Delta_q}(M)$, we can extend $\text{tr}_\nu$ to a bounded linear map $H_{\Delta_q}(M) \to H^{-3/2}(\partial M)$.

Now, we give the proof of the last statement. First, we consider the case when $\text{tr}(u) = 0$. Let $u \in C^\infty(M)$ with $\text{tr}(u) = 0$. Using, Green’s identity, we have

$$
\|u\|_{H^1(M)}^2 = \|u\|_{L^2(M)}^2 + \|\nabla u\|_{L^2(M)}^2 \\
= \|u\|_{L^2(M)}^2 - (u, \Delta_g u)_{L^2(M)} \\
\leq \|u\|_{L^2(M)}^2 + \frac{1}{2} \left( \|u\|_{L^2(M)}^2 + \|\Delta_g u\|_{L^2(M)}^2 \right) \\
\leq C\|u\|_{H_{\Delta_q}(M)}^2,
$$

for some constant $C > 0$. By [21, Theorem 1.3] in Chapter 5, we have

$$
\|u\|_{H^2(M)}^2 \leq C\|\Delta_g u\|_{L^2(M)}^2 + C\|u\|_{H^1(M)}^2,
$$

for some another constant $C > 0$. Combining this with (2.1), we obtain

$$
\|u\|_{H^2(M)}^2 \leq C\|u\|_{H_{\Delta_q}(M)}^2, \quad u \in C^\infty(M), \quad \text{tr}(u) = 0.
$$

By density arguments, we obtain

$$
\|u\|_{H^2(M)}^2 \leq C\|u\|_{H_{\Delta_q}(M)}^2, \quad u \in H_{\Delta_q}(M), \quad \text{tr}(u) = 0.
$$

This proves the last statement for the case when $\text{tr}(u) = 0$.

Suppose now that $u \in H_{\Delta_q}(M)$ with $\text{tr}(u) \in H^{3/2}(\partial M)$. By the surjectivity of the trace operator, there is $v \in H^2(M)$ such that $\text{tr}(v) = \text{tr}(u)$. Set $w := u - v$, then $w \in H_{\Delta_q}(M)$ with $\text{tr}(w) = 0$. By what we have proved above, $w \in H^2(M)$, and hence $u \in H^2(M)$.

The proof of Proposition 2.1 gives the following.

**Corollary 2.2.** For $u \in H_{\Delta_q}(M)$ and $v \in H^2(M)$ we have the generalized Green’s identity

$$
(u|(-\Delta_g)v)_{L^2(M)} - ((-\Delta_g)u|v)_{L^2(M)} = \langle \text{tr}_\nu(u), \text{tr}_\nu(v) \rangle_{H^{-3/2,3/2}(\partial M)} - \langle \text{tr}(u), \text{tr}(v) \rangle_{H^{-1/2,1/2}(\partial M)},
$$

where $\langle \cdot, \cdot \rangle_{H^{-s,s}(\partial M)}$ is the duality between $H^{-s}(\partial M)$ and $H^s(\partial M)$.

Now we introduce the following space on the boundary $\partial M$:

$$
\mathcal{H}_g(\partial M) = \{ \text{tr}(u) : u \in H_{\Delta_q}(M) \} \subset H^{-1/2}(\partial M).
$$

Assume that $q \in L^\infty(M)$ and let us introduce the Bergman space $b_q(M)$ as follows

$$
b_q(M) = \{ u \in L^2(M) : (-\Delta_g + q)u = 0 \} \subset H_{\Delta_q}(M).
$$

The topology on this space is a subspace topology in $L^2(M)$. It is not difficult to check that $b_q(M)$ is a closed subspace of $L^2(M)$.

We need the following result to define a topology on $\mathcal{H}_g(\partial M)$:
Proposition 2.3. If \( q \in L^\infty(M) \) and 0 is not a Dirichlet eigenvalue of \(-\Delta_g + q\) in \( M \), then \( \text{tr} : b_q(M) \to \mathcal{H}_g(\partial M) \) is one-to-one and onto.

Proof. Let \( u, v \in b_q(M) \) is such that \( \text{tr}(u) = \text{tr}(v) \). Set \( w = u - v \), then \( w \in b_q(M) \) and \( \text{tr}(w) = 0 \). By the last statement of Proposition 2.1, \( w \in H^2(M) \). By assumption, 0 is not a Dirichlet eigenvalue of \(-\Delta_g + q\) in \( M \). Therefore, \((-\Delta_g + q)w = 0\) with \( w|_{\partial M} = 0 \) imply that \( w = 0 \).

Let \( h \in \mathcal{H}_g(\partial M) \). By definition of \( \mathcal{H}_g(\partial M) \), there is \( u \in H\Delta_q(M) \) such that \( \text{tr}(u) = h \). Take \( v \in H_0^1(M) \) being the solution to the Dirichlet problem \((-\Delta_g + q)v = (-\Delta_g + q)u, v|_{\partial M} = 0 \). Set \( w = u - v \). Then \( w \in H\Delta_q(M), (-\Delta_g + q)w = 0 \) and \( \text{tr}(w) = 0 \). In other words, \( w \in b_q(M) \) with \( \text{tr}(w) = h \). □

Let \( P_q \) denote the inverse of \( \text{tr} : b_q(M) \to \mathcal{H}_g(\partial M) \). We define the norm on \( \mathcal{H}_g(\partial M) \) as

\[
\|f\|_{\mathcal{H}_g(\partial M)} = \|P_q f\|_{L^2(M)}.
\]

In particular, by Proposition 2.3, this implies that \( \text{tr} : b_0 \to \mathcal{H}_g(\partial M) \) as well as \( P_0 : \mathcal{H}_g(\partial M) \to b_0 \) are bounded. Next, we give the following solvability result of the Dirichlet problem with boundary data in \( \mathcal{H}_g(\partial M) \):

Proposition 2.4. The operator \( \text{tr} : H\Delta_q(M) \to \mathcal{H}_g(\partial M) \) is bounded. If \( q \in L^\infty(M) \) and 0 is not a Dirichlet eigenvalue of \(-\Delta_g + q\) in \( M \), then \( \text{tr} : b_q(M) \to \mathcal{H}_g(\partial M) \) is a homeomorphism.

Proof. Let \( u \in H\Delta_q(M) \). Consider \( v \in H_0^1(M) \) being the solution to the Dirichlet problem \((-\Delta_g)v = (-\Delta_g)u, v|_{\partial M} = 0 \). Set \( w = u - v \). Note that \( u \mapsto v \) is bounded \( H\Delta_q(M) \to L^2(M) \), and hence \( u \mapsto w \) is bounded \( H\Delta_q(M) \to b_0(M) \) as well, since \((-\Delta_g)w = 0 \). Since \( \text{tr} : b_0 \to \mathcal{H}_g(\partial M) \) is bounded and since \( \text{tr}(w) = \text{tr}(u) \), we can conclude that the map \( u \mapsto \text{tr}(u) \) is bounded \( H\Delta_q(M) \to \mathcal{H}_g(\partial M) \).

Since the inclusion \( b_0 \hookrightarrow H\Delta_q(M) \) is bounded, by the first part of the proposition, the map \( \text{tr} : b_q \to \mathcal{H}_g(\partial M) \) is bounded. Bijectivity of the latter map, which follows from Proposition 2.3, together with Open Mapping Theorem, implies the last statement. □

We also extend the domain of the Dirichlet-to-Neumann map to \( \mathcal{H}_g(\partial M) \):

Proposition 2.5. Suppose that \( q \in L^\infty(M) \) and 0 is not a Dirichlet eigenvalue of \(-\Delta_g + q\) in \( M \). Then \( (\Lambda_q - \Lambda_0)|_{\mathcal{H}(\partial M)} \) is a bounded operator \( \mathcal{H}_g(\partial M) \to (\mathcal{H}_g(\partial M))^* \). Moreover, the following integral identity holds

\[
(h, (\Lambda_q - \Lambda_0)u)_{H^{-1/2,1/2}(\partial M)} = (P_0(h)qP_q(f))_{L^2(M)}, \tag{2.2}
\]

for all \( f, h \in \mathcal{H}_g(\partial M) \).

Proof. Suppose that \( f, h \in \mathcal{H}_g(\partial M) \). Let \( u \in H\Delta_q(M) \) be the unique solution to the boundary value problem

\((-\Delta_g + q)u = 0 \in \Omega, \quad \text{tr}(u) = f, \)

and let \( u_0 \) be the unique solution to the boundary value problem

\((-\Delta_g)u_0 = 0 \in \Omega, \quad \text{tr}(u_0) = f. \)
Set \( w := u - u_0 \), then we have
\[
(-\Delta_g)w = -qu \quad \text{in } \Omega, \quad \text{tr}(w) = 0.
\]
By the last statement of Proposition 2.1, we can conclude that \( w \in H^2(M) \). Note that by Proposition 2.4, there is \( v_h \in H_{\Delta_q}(M) \) such that \( (-\Delta_g)v_h = 0 \) and \( \text{tr}(v_h) = h \). Now, we can apply Corollary 2.2 and get
\[
(v_h|qu)_{L^2(M)} = -(((-\Delta_g)v_h|w)_{L^2(M)} + \langle h, \text{tr}_\nu(w) \rangle_{H^{-1/2,1/2}(\partial M)}).
\]
Since \( (-\Delta_g)v_h = 0 \) and \( \text{tr}_\nu(w) = \text{tr}_\nu(u-u_0) = (\Lambda_{g,q} - \Lambda_{g,0})f \), we obtain
\[
\langle h, (\Lambda_{g,q} - \Lambda_{g,0})f \rangle_{H^{-1/2,1/2}(\partial M)} = (v_h|qu)_{L^2(M)}.
\] (2.3)
The right-hand side depends continuously on \( f, h \in H_{\partial M} \). Hence, so does the left-hand-side and this together with (2.3) implies that the result. \( \square \)

3. The Green's operators

Let \( (M, g) \) be an admissible manifold and let \( q \in L^\infty(M) \). Let us introduce certain notations which will be used throughout the paper. For \( \tau \in \mathbb{R} \), we consider the following disjoint decomposition \( \partial M = S^+_\tau \cup S^-_\tau \), where
\[
S^+_\tau := \{ x \in \partial M : \text{sgn}(\tau)\partial_\nu \varphi(x) \geq |\tau|^{-1} \}, \quad S^-_\tau := \partial M \setminus S^+_\tau.
\]
For \( \delta > 0 \), we can write \( S^-_{\tau,\delta} = S^-_{\tau,-\delta} \cup S^0_{\tau,\delta} \), where
\[
S^-_{\tau,-\delta} := \{ x \in \partial M : \text{sgn}(\tau)\partial_\nu \varphi(x) \leq -\delta \}, \quad S^0_{\tau,\delta} := \{ x \in \partial M : -\delta < \text{sgn}(\tau)\partial_\nu \varphi(x) < (3|\tau|)^{-1} \}.
\]
Constructions of Green's operators and the corresponding single layer potentials, as well as construction of complex geometrical optics solutions are based on the following Carleman estimates with boundary terms for the conjugated operator
\[
e^{\tau x_1}(-\Delta_g + q)e^{-\tau x_1}.
\]

**Proposition 3.1.** Let \( (M, g) \) be an admissible manifold and let \( q \in L^\infty(M) \). There are constants \( C_0, \tau_0 > 0 \) such that for all \( \tau \in \mathbb{R} \) with \( |\tau| \geq \tau_0 \) and \( \delta > 0 \), we have
\[
(\delta|\tau|)^{1/2}\|\partial_\nu u\|_{S^-_{\tau,\delta}} + \|\partial_\nu u\|_{S^0_{\tau,\delta}} + |\tau|\|u\|_{L^2(M)} + \|\nabla u\|_{L^2(M)} \leq C_0\|e^{\tau x_1}(-\Delta_g + q)e^{-\tau x_1}u\|_{L^2(M)} + C_0|\tau|^{1/2}\|\partial_\nu u\|_{S^0_{\tau,\delta}}
\] (3.1)
for all \( u \in C^\infty(M) \) with \( u|_{\partial M} = 0 \).

**Proof.** This estimate was proven by Kenig and Salo; see [8, Proposition 4.2]. \( \square \)

Define
\[
D^\pm = \{ u \in C^\infty(M) : u|_{\partial M} = \text{tr}_\nu(u)|_{S^\pm} = 0 \}.
\]
The aim of this section is to prove the following result.
Theorem 3.2. Let \((M, g)\) be an admissible manifold. There is a constant \(\tau_0 > 0\) such that for all \(\tau \in \mathbb{R}\) with \(|\tau| \geq \tau_0\), there is a linear operator
\[
G_\tau : L^2(M) \to L^2(M)
\]
such that
\[
e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} G_\tau v = v, \quad v \in L^2(M)
\]
and
\[
G_\tau e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} u = u, \quad u \in D_\tau^+.
\]
This operator satisfies
\[
\|G_\tau f\|_{L^2(M)} \leq \frac{C_0}{|\tau|} \|f\|_{L^2(M)}, \quad f \in L^2(M),
\]
where \(C_0 > 0\) is independent of \(\tau\). Moreover, \(G_\tau : L^2(M) \to e^{\tau x_1} H_{\Delta_g}(M)\) and for all \(v \in L^2(M)\) support of \(\text{tr}(G_\tau v)\) is in \(S_\tau^+\).

Let \(\pi_\tau\) be the orthogonal projection onto \(L_\tau\) the closure of \(e^{\tau x_1}(-\Delta_g)e^{\tau x_1}D_\tau^+\) in \(L^2(M)\).

Lemma 3.3. Let \(\pi_\tau^+ := \text{Id} - \pi_\tau\). Then \(\pi_\tau^+\) is the orthogonal projection onto
\[
A_\tau = \{ u \in L^2(M) : e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u = 0, \text{ supported in } S_\tau^+ \}.
\]
Proof. It is enough to show that \(u\) is orthogonal to \(e^{\tau x_1}(-\Delta_g)e^{\tau x_1}D_\tau^+\) if and only if \(u\) is in \(A_\tau\). Suppose that \(u \in A_\tau\). Then for \(v \in D_\tau^+\), we have
\[
(u|e^{\tau x_1}(-\Delta_g)e^{\tau x_1} v)_{L^2(M)} = (e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u|v)_{L^2(M)} - (\text{tr}(u), \text{tr}_e(v))_{H^{-1/2, 1/2}(\partial M)}.
\]
Since \(\text{tr}(u)\) supported in \(S_\tau^+\), \(\text{tr}_e(v) = 0\) in \(S_\tau^+\) and \(e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u = 0\), we obtain
\[
(u|e^{\tau x_1}(-\Delta_g)e^{\tau x_1} v)_{L^2(M)} = 0,
\]
which means that \(u\) is orthogonal to \(e^{\tau x_1}(-\Delta_g)e^{\tau x_1}D_\tau^+\). Converse is as in [15, Lemma 3.3].

Proposition 3.4. Let \((M, g)\) be an admissible manifold. There is \(\tau_0 > 0\) such that for all \(\tau \in \mathbb{R}\) with \(|\tau| \geq \tau_0\) and for a given \(v \in L^2(M)\), there is a unique solution \(u \in L^2(M)\) of the equation
\[
e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u = v \quad \text{in } M
\]
such that \(\text{tr}(u)\) is supported in \(S_\tau^+\), \(\pi_\tau u = u\) and \(\|u\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|} \|v\|_{L^2(M)}\) with constant \(C_0 > 0\) independent of \(\tau\).

Proof. First, we show the existence. Define a linear functional \(L\) on \(e^{\tau x_1}(-\Delta_g)e^{\tau x_1}D_\tau^+\) by
\[
L(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1} w) = (v|w)_{L^2(M)}, \quad w \in D_\tau^+.
\]
Then we have
\[
|L(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1} w)| \leq \|v\|_{L^2(M)} \|w\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|} \|v\|_{L^2(M)} \|e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} w\|_{L^2(M)},
\]
where in the last step we have used the Carleman estimate (3.1). By the Hahn-Banach theorem, we may extend $L$ to a linear continuous functional $\tilde{L}$ on $L_\tau$. On the orthogonal complement of $L_\tau$ in $L^2(M)$ we define $\tilde{L}$ to be zero. By the Riesz representation theorem, there exists $u \in L^2(M)$ such that
\[
\tilde{L}(f) = (u|f)_{L^2(M)}, \quad f \in L^2(M).
\]
In particular,
\[
(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w)_{L^2(M)} = \tilde{L}(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w) = (v|w)_{L^2(M)}, \quad w \in D_\tau^+.
\]
If we take $w \in C_0^\infty(M^{\text{int}})$ in the above equation, we obtain $e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}u = v$. Moreover,
\[
\|u\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|}\|v\|_{L^2(M)}.
\]
Since $\tilde{L} \equiv 0$ on the orthogonal complement of $L_\tau$ in $L^2(M)$, we have that $u \in L_\tau$ and hence $\pi_\tau u = u$.

To finish the proof, we need to show that $\operatorname{tr}(u)$ is supported in $S_\tau^+$. For arbitrary $w \in D_\tau^+$, using the generalized Green’s identity from Corollary 2.2, we get
\[
(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}w)_{L^2(M)} = (\operatorname{tr}(u), \operatorname{tr}(w))_{H^{-1/2,1/2}(\partial M)} + (v|w)_{L^2(M)}.
\]
According to (3.3), we have $(\operatorname{tr}(w), \operatorname{tr}(u))_{\partial M} = 0$. Since $w \in D_\tau^+$ was arbitrary, we can conclude that $\operatorname{tr}(u)$ is supported in $S_\tau^+$.

Now, we prove uniqueness. Suppose that $u' \in L^2(M)$ is another solution of the equation $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u' = v$ satisfying all the conditions of the proposition. Then $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}(u - u') = 0$, $\operatorname{tr}(u - u')$ is supported in $S_\tau^+$, and $\pi_\tau(u - u') = u - u'$. However, by Lemma 3.3, $\pi_\tau(u - u') = 0$. Thus, we obtain $u - u' = 0$ which finishes the proof.

Let $H_\tau : L^2(M) \to L^2(M)$ be the solution operator obtained in the previous result. In other words, the operator $H_\tau$ is defined by $H_\tau v = u$, where $u$ and $v$ are as in Proposition 3.4. The following is an immediate corollary of the preceding result.

**Corollary 3.5.** Let $(M,g)$ be an admissible manifold. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, there is a linear operator
\[
H_\tau : L^2(M) \to L^2(M)
\]
such that
\[
e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}H_\tau v = v, \quad v \in L^2(M)
\]
and $\pi_\tau H_\tau = H_\tau$. This operator satisfies
\[
\|H_\tau f\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|}\|f\|_{L^2(M)}
\]
where $C_0 > 0$ is independent of $\tau$. Moreover, $H_\tau : L^2(M) \to e^{\tau x_1}H_\Delta_g(M)$ and for all $v \in L^2(M)$ support of $\operatorname{tr}(H_\tau v)$ is in $S_\tau^+$. 

Thus, the operator $H_\tau$ satisfies Theorem 3.2 except (3.2). We shall accordingly modify $H_\tau$ to obtain (3.2). We need the technical result.

**Lemma 3.6.** Let $T_\tau := H_\tau \pi_{-\tau}$. Then $T_\tau^* = T_{-\tau}$.

*Proof.* Note that $T_\tau^* \pi_{-\tau} = \pi_{-\tau} H_\tau^* (\text{Id} - \pi_{\tau}) = \pi_{-\tau} H_\tau^* - \pi_{-\tau} H_\tau^* \pi_{\tau} = 0$, where in the first step we have used the fact that $\pi_{-\tau} = \pi_{-\tau}$ (since $\pi_{-\tau}$ is projection) and in the last step we have used that $H_\tau^* \pi_{\tau} = H_\tau^*$ (this follows from $\pi_{\tau} H_\tau = H_\tau$ which is true by Corollary 3.5). Also note that $T_{-\tau} \pi_{\tau}^* = H_{-\tau} \pi_{-\tau} \pi_{\tau}^* = 0$. Thus, $T_\tau^* \pi_{-\tau} = T_{-\tau} \pi_{\tau}^* = 0$, and hence, to prove the lemma it is sufficient to show that

$$T_\tau^* e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v = H_{-\tau} e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v$$

for all $D_{\tau}^+$. Observe that $\pi_{-\tau} T_\tau^* = \pi_{-\tau} H_\tau^* = \pi_{-\tau} H_\tau = T_{-\tau}$ (since $\pi_{-\tau} = \pi_{-\tau}$).

Therefore, it is enough to show that

$$(e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | T_\tau^* e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | H_{-\tau} e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)},$$

for all $w \in D_{\tau}^+$ and for all $v \in D_{\tau}^+$. We have

$$(e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | T_\tau^* e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | \pi_{-\tau} H_\tau^* e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (H_{-\tau} \pi_{-\tau} e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | e^{\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (H_{-\tau} e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | e^{\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)}.$$

Since by Corollary 3.5 we know that $\text{tr}(H_{-\tau} e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w)$ is supported in $S_{-\tau}^+$, we can use Green’s identity and the fact that $v \in D_{\tau}^+$ to get

$$(e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | T_\tau^* e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} H_{-\tau} e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | v)_{L^2(M)}$$

Since $e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} H_{-\tau} = \text{Id}$ by Corollary 3.5, we obtain

$$(e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | T_\tau^* e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | v)_{L^2(M)} = (w | e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)}.$$

Here, in the last step we used the Green’s identity and that $w|_{\partial M} = v|_{\partial M} = 0$.

Using that $\text{tr}(H_{-\tau} e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)$ is supported in $S_{-\tau}^+$, $w \in D_{-\tau}$ and the Green’s identity, we obtain

$$(e^{\tau x_1} (-\Delta_g) e^{-\tau x_1} w | H_{-\tau} e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (w | e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} H_{-\tau} e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)} = (w | e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} v)_{L^2(M)}.$$

In the last step we used the fact that $e^{-\tau x_1} (-\Delta_g) e^{\tau x_1} H_{-\tau} = \text{Id}$ (by Corollary 3.5). The proof of the lemma is thus complete. □
Proof of Theorem 3.2. Define $G_\tau = H_\tau + \pi_\tau^+ H_{-\tau}^*$. By Corollary 3.5 and Lemma 3.3, it follows that

$$e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}G_\tau v = v, \quad v \in L^2(M),$$

$G_\tau : L^2(M) \to e^{\tau x_1}H\Delta_g(M)$ and that for all $v \in L^2(M)$ support of $\text{tr}(G_\tau v)$ is in $S_+^\tau$.

It is left to prove (3.2). For this, we need first to show that $G_\tau^* = G_{-\tau}$. Using Lemma 3.6, we can show

$$G_\tau^* = H_\tau^* + H_{-\tau} - \pi_\tau^+ = (H_{-\tau} + T_\tau)^* + H_{-\tau} - T_\tau = H_{-\tau} + \pi_{-\tau}^+ H_{-\tau}^* = G_{-\tau}.$$ (4.1)

Using this, for $f \in L^2(M)$ and $u \in D_\tau$, we have

$$\langle f | G_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u \rangle_{L^2(M)} = \langle (G_{-\tau} f) e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u \rangle_{L^2(M)}.$$ (2.1)

We have shown that $\text{tr}(G_{-\tau} f)$ is supported in $S_{-\tau}^\tau$. This fact together with $u \in D_\tau$ allows us to use the generalized Green’s identity from Corollary 2.2 and get

$$\langle f | G_\tau e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u \rangle_{L^2(M)} = \langle e^{\tau x_1}(-\Delta_g)e^{\tau x_1}G_{-\tau} f | u \rangle_{L^2(M)}.$$ (2.1)

Here, in the last step we used the already proven fact that $e^{-\tau x_1}(-\Delta_g)e^{-\tau x_1}G_{-\tau} = \text{Id}$. This finishes the proof. □

4. Single layer operators

The aim of this section is to construct the single layer operators $S_\tau$ corresponding to the Green’s operators $G_\tau$ constructed in the previous section.

Let $\tau_0 > 0$ be as in Theorem 3.2. For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, consider the operator

$$(\text{tr} \circ G_\tau)^* : e^{-\tau x_1}(H\tau_g(\partial M))^* \to L^2(M).$$

In other words, $(\text{tr} \circ G_\tau)^*$ defined for $h \in e^{-\tau x_1}(H\tau_g(\partial M))^*$ by

$$\langle f | (\text{tr} \circ G_\tau)^* h \rangle_{L^2(M)} = \langle (\text{tr} \circ G_\tau) f, h \rangle_{H^{-1/2,1/2}(\partial M)}, \quad f \in L^2(M).$$ (2.1)

Proposition 4.1. For all $h \in e^{-\tau x_1}(H\tau_g(\partial M))^*$ we have

$$e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}(\text{tr} \circ G_\tau)^* h = 0$$

and the support of $\text{tr}((\text{tr} \circ G_\tau)^* h)$ is in $S_{-\tau}^\tau$. Moreover, suppose that $B$ is a neighborhood of $S_{-\tau}^\tau$ such that $\overline{B} \subset \partial M_{\text{alg}(\tau)}$, and that the support of $h$ is in $\partial M \setminus B$. Then $(\text{tr} \circ G_\tau)^* h = 0$.

Proof. Let $h \in e^{-\tau x_1}(H\tau_g(\partial M))^*$. Then for all $f \in D_\tau$, we obtain

$$\langle e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}f | (\text{tr} \circ G_\tau)^* h \rangle_{L^2(M)} = \langle (\text{tr} \circ G_\tau) e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}f, h \rangle_{H^{-1/2,1/2}(\partial M)} \quad (4.1)$$

$$= \langle \text{tr}(f), h \rangle_{H^{-1/2,1/2}(\partial M)} = 0.$$ (2.1)
Here, we have used (3.2) and $\text{tr}(f) = 0$. If we take $f \in C^\infty_0(M)$ in (4.1) and use the generalized Green’s identity in Corollary 2.2, we can show that

$$(f|e^{-r\tau_1}(-\Delta_g)e^{r\tau_1}(\text{tr} \circ G_r)^*h)_{L^2(M)} = (e^{r\tau_1}(-\Delta_g)e^{-r\tau_1}f|\text{tr} \circ G_r)^*h)_{L^2(M)} = 0.$$  

Hence, we obtain $e^{-r\tau_1}(-\Delta_g)e^{r\tau_1}(\text{tr} \circ G_r)^*h = 0$ for all $h \in e^{-r\tau_1}(\mathcal{H}_g(\partial M))^*$.  

Let us now show that the support of $\text{tr}((\text{tr} \circ G_r)^*h)$ is in $S^+_r$. For arbitrary $f \in D^+_r$, using the generalized Green’s identity from Corollary 2.2, we get

$$(\text{tr}((\text{tr} \circ G_r)^*h), \text{tr}_\nu(f))_{H^{-1/2,1/2}(\partial M)} = (\text{tr}((\text{tr} \circ G_r)^*h), \text{tr}_\nu(f))_{H^{-3/2,1/2}(\partial M)} - (\text{tr}_\nu((\text{tr} \circ G_r)^*h), \text{tr}(f))_{H^{-3/2,1/2}(\partial M)} = (e^{-r\tau_1}(-\Delta_g)e^{r\tau_1}(\text{tr} \circ G_r)^*h,f)_{L^2(M)} = -(\text{tr} \circ G_r)^*h|e^{r\tau_1}(-\Delta_g)e^{-r\tau_1}f)_{L^2(M)} = 0,$$

where in the last step we have used (4.1).

Now, we prove the last statement of the proposition. If $h$ is supported in $\partial M \setminus B$, then for all $f \in L^2(M)$ we have

$$(f|\text{tr} \circ G_r)^*h)_{L^2(M)} = (\text{tr} \circ G_r)^*f, h)_{H^{-1/2,1/2}(S^+_r)} = 0.$$

This is because by the last statement of Theorem 3.2, $(\text{tr} \circ G_r)^*f$ is supported in $S^+_r$. The proof of the proposition is thus complete. □

For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, define the operator $S_\tau$ for $h \in (\mathcal{H}_g(\partial M))^*$ by

$$S_\tau h = e^{-r\tau_1}(\text{tr} \circ (\text{tr} \circ G_r)^*)^*(e^{r\tau_1}h).$$

**Proposition 4.2.** For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, the operator $S_\tau$ is bounded $(\mathcal{H}_g(\partial M))^* \to \mathcal{H}_g(\partial M)$, and for $h \in (\mathcal{H}(\partial M))^*$, $S_\tau h$ depends only on $h|_{\partial M - \text{sgn}(\tau)}$ and supported in $B$.

**Proof.** By Proposition 4.1, we have that the operator $(\text{tr} \circ G_r)^* : e^{-r\tau_1}(\mathcal{H}_g(\partial M))^* \to e^{-r\tau_1}H\Delta_\rho(M)$ is bounded, and hence the operator $\text{tr} \circ (\text{tr} \circ G_r)^* : e^{-r\tau_1}(\mathcal{H}_g(\partial M))^* \to e^{-r\tau_1}\mathcal{H}_g(\partial M)$ is bounded as well. This implies the boundedness of $(\text{tr} \circ (\text{tr} \circ G_r)^*)^* : e^{r\tau_1}(\mathcal{H}_g(\partial M))^* \to e^{r\tau_1}\mathcal{H}_g(\partial M)$. Therefore, $S_\tau$ is a bounded operator $(\mathcal{H}_g(\partial M))^* \to \mathcal{H}_g(\partial M)$.

We have by Proposition 4.1 that $\text{tr} \circ (\text{tr} \circ G_r)^*(e^{-r\tau_1}h)$ is supported in $S^+_r$ if $\tilde{h} \in (\mathcal{H}_g(\partial M))^*$. By duality, for any $h \in (\mathcal{H}_g(\partial M))^*$, $S_\tau h = e^{-r\tau_1}(\text{tr} \circ (\text{tr} \circ G_r)^*)^*(e^{r\tau_1}h)$ depends only on $h|_{\partial M - \text{sgn}(\tau)}$.

By the last statement of Proposition 4.1, if $\tilde{h} \in (\mathcal{H}_g(\partial M))^*$ is supported in $\partial M \setminus B$ then $\text{tr}((\text{tr} \circ G_r)^*(e^{-r\tau_1}h)) = 0$. By duality, for any $h \in (\mathcal{H}_g(\partial M))^*$, $e^{-r\tau_1}(\text{tr} \circ (\text{tr} \circ G_r)^*)^*(e^{r\tau_1}h)$ supported in $B$. □
5. Boundary integral equation

In the present section, we prove the solvability of the following boundary integral equation: for $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$

$$ (\text{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0})) h = f, \quad f, h \in \mathcal{H}_g(\partial M). \tag{5.1} $$

To prove the solvability of (5.1), we need the following result on basic properties of the operator $S_\tau(\Lambda_{g,q} - \Lambda_{g,0})$.

**Proposition 5.1.** Suppose that $q \in L^\infty(M)$ and 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in $M$. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, the operator $S_\tau(\Lambda_{g,q} - \Lambda_{g,0})$ is a bounded operator $\mathcal{H}_g(\partial M) \to \mathcal{H}_g(\partial M)$, and for $f \in \mathcal{H}_g(\partial M)$, $S_\tau(\Lambda_{g,q} - \Lambda_{g,0}) f$ is supported in $B$ and can be computed from the knowledge of $\Lambda_g f|_{\partial M_{\text{sgn}(\tau)}}$. Moreover, the following factorization identity holds

$$ S_\tau(\Lambda_{g,q} - \Lambda_{g,0}) = \text{tr} \circ e^{-\tau\xi_1}G_\tau e^{\tau\xi_1}qP_q. $$

**Proof.** First part of the proposition is a consequence of Proposition 2.5 and Proposition 4.2. To prove the last statement, consider $h \in (\mathcal{H}_g(\partial M))^*$ and $f \in \mathcal{H}_g(\partial M)$. Then

$$ \langle h, \text{tr} \circ e^{-\tau\xi_1}G_\tau e^{\tau\xi_1}qP_q(f) \rangle_{H^{-1/2,1/2}(\partial M)} = (e^{\tau\xi_1}(\text{tr} \circ G_\tau)^*(e^{-\tau\xi_1}h)|qP_q(f))_{L^2(M)}. $$

By Proposition 4.1, $e^{\tau\xi_1}(\text{tr} \circ G_\tau)^*(e^{-\tau\xi_1}h)$ is in $b_0$. Using (2.2), we show that

$$ (e^{\tau\xi_1}(\text{tr} \circ G_\tau)^*(e^{-\tau\xi_1}h)|qP_q(f))_{L^2(M)} = \langle h, S_\tau(\Lambda_{g,q} - \Lambda_{g,0}) f \rangle_{H^{-1/2,1/2}(\partial M)}. $$

The proof is thus finished. □

The following result shows that the boundary integral equation is equivalent to the certain integral equation; compare with [9, Proposition 3.2].

**Proposition 5.2.** Suppose that $q \in L^\infty(M)$ and 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in $M$. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$ and for all $f, h \in \mathcal{H}_g(\partial M)$, $(\text{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0})) h = f$ holds if and only if $(\text{Id} + e^{-\tau\xi_1}G_\tau e^{\tau\xi_1}q) P_q(h) = P_0(f)$.

**Proof.** Suppose that $f, h \in \mathcal{H}_g(\partial M)$ satisfies $(\text{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0})) h = f$. Note that by Theorem 3.2, we can show that

$$ \Delta_g (\text{Id} + e^{-\tau\xi_1}G_\tau e^{\tau\xi_1}q) P_q(h) = qP_q(h) - qP_q(h) = 0. $$

Therefore, it is enough to prove that

$$ \text{tr} \left( (\text{Id} + e^{-\tau\xi_1}G_\tau e^{\tau\xi_1}q) P_q(h) \right) = f, $$

or equivalently

$$ h + \text{tr} \left( e^{-\tau\xi_1}G_\tau e^{\tau\xi_1}qP_q(h) \right) = f. $$

Using the factorization identity in Proposition 5.1, we can see that the left hand-side is $(\text{Id} + S_\tau(\Lambda_{g,q} - \Lambda_{g,0})) h$, which is equal to $f$ by assumption.
The converse direction can be shown by applying $\text{tr}$ to the both sides of the identity
\[(\text{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1}) P_q(h) = P_0(f)\]
and using the factorization identity in Proposition 5.1.

**Corollary 5.3.** Suppose that $q \in L^\infty(M)$ and 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in $M$. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, the operator
\[
\text{Id} + S_\tau(\Delta_g + q - \Delta_g) : H_g(\partial M) \rightarrow H_g(\partial M)
\]
is an isomorphism if and only if so is the operator
\[
\text{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} : b_q(M) \rightarrow b_0(M).
\]
The following proposition combined together with the above two results implies the solvability of the boundary integral equation (5.1).

**Proposition 5.4.** Suppose that $q \in L^\infty(M)$ and 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in $M$. There is $\tau_0 > 0$ such that for all $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, the operator
\[
\text{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} : b_q(M) \rightarrow b_0(M)
\]
is an isomorphism.

**Proof.** Since $\|G_\tau\|_{L^2(M) \rightarrow L^2(M)} \leq \mathcal{O}(\tau^{-1})$ by Theorem 3.2, the operator $\text{Id} + G_\tau q : L^2(M) \rightarrow L^2(M)$ is an isomorphism for big enough $|\tau| \gg 1$. Then for such $\tau$, the operator $\text{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1} q : L^2(M) \rightarrow L^2(M)$ is an isomorphism whose inverse is $e^{-\tau x_1}(\text{Id} + G_\tau q)^{-1} e^{\tau x_1}$. Let $u \in b_0$ and $w = e^{-\tau x_1}(\text{Id} + G_\tau q)^{-1} e^{\tau x_1} u$. We need to show that $w \in b_q$. Applying $\text{Id} + e^{-\tau x_1} G_\tau e^{\tau x_1}$ to $w$, we get that
\[
w + e^{-\tau x_1} G_\tau e^{\tau x_1} q w = u.
\]
Since $e^{\tau x_1}(-\Delta_g) e^{-\tau x_1} G_\tau = \text{Id}$ (by Theorem 3.2), we get $(-\Delta_g) e^{-\tau x_1} G_\tau e^{\tau x_1} = \text{Id}$ and hence $(-\Delta_g + q)w = 0$. \qed

6. Complex geometrical optics solutions

Let $q \in L^\infty(M)$ be such that 0 is not a Dirichlet eigenvalue of $-\Delta_g + q$ in $M$, and let $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$. In this section we construct the complex geometrical optics solutions for the Schrödinger equation $(-\Delta_g + q)u = 0$ in $M$ whose trace is supported in $\Gamma_{\text{sgn}(\tau)}$.

6.1. Solution operator. To construct the complex geometrical optics solutions, we need to generalize Proposition 3.4 to the case when the solution is determined on $S^-\tau$.

Set $\mathcal{D} = \{ \psi \in C^\infty(M) : \text{tr}(\psi) = 0 \}$ and define
\[
M_\tau = \{ (e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi, \text{tr}_\tau(\psi)|_{S^-\tau}) : \psi \in \mathcal{D} \} \subset L^2(M) \times L^2(S^+_\tau).
\]

**Lemma 6.1.** For $\tau \in \mathbb{R}$ with $|\tau| > 0$, let $(u, u^\perp) \in L^2(M) \times L^2(S^+_\tau)$. Then $(u, u^\perp)$ is in orthogonal to the closure of $M_\tau$ if and only if $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u = 0$, $\text{tr}(u)|_{S^-\tau} = 0$ and $\text{tr}(u)|_{S^+_\tau} = u^\perp$.

**Proof.** Suppose that $(u, u^\perp)$ is orthogonal to $M_\tau$. Then for $\psi \in \mathcal{D}$, we have
\[
(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} + (u^\perp|\text{tr}_\tau(\psi)|_{S^+_\tau})_{S^+_\tau} = 0.
\]
Taking $\psi \in C^\infty_0(M^\text{int})$, this gives $e^{\tau x_1}(-\Delta_g)e^{-\tau x_1} u = 0.$
Now, consider arbitrary \( \psi \in \mathcal{D} \). Using the generalized Green’s identity from Corollary 2.2, we get

\[
(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} = -\langle \text{tr}(u),\text{tr}_\nu(\psi) \rangle_{H^{-1/2,1/2}(\partial M)}.
\]

Combining this together with the previous equality gives that \( \text{tr}(u)|_{S_\tau^-} = 0 \) and \( \text{tr}(u)|_{S_\tau^+} = u_\tau^+ \).

To prove the converse, suppose that \( (u,u_\tau^+) \) is such that \( e^{-\tau x_1}(-\Delta_g)e^{-\tau x_1}u = 0 \), \( \text{tr}(u)|_{S_\tau^-} = 0 \) and \( \text{tr}(u)|_{S_\tau^+} = u_\tau^+ \). Then for \( \psi \in \mathcal{D} \), we have

\[
(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} = \langle \text{tr}(u),\text{tr}_\nu(\psi) \rangle_{H^{-1/2,1/2}(\partial M)} = -\langle u_\tau^+|\text{tr}_\nu(\psi) \rangle_{S_\tau^+},
\]

which means that \( (u,u_\tau^+) \) is orthogonal to \( M_\tau \).

Let us denote by \( m_\tau \) the operator of orthogonal projection onto the closure of \( M_\tau \) in \( L^2(M) \times L^2(S_\tau^+) \).

**Proposition 6.2.** Let \( (M,g) \) be an admissible manifold. There are constants \( C_0, \tau_0 > 0 \) such that for all \( \tau \in \mathbb{R} \) with \( |\tau| \geq \tau_0, \delta > 0 \) and for given \( f \in L^2(M) \) and \( f_\tau^- \in L^2(S_\tau^-) \), there exists a unique solution \( u \in L^2(M) \) of the equation

\[
e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = f \quad \text{in} \quad M
\]

such that \( \text{tr}(u)|_{S_\tau^-} = f_\tau^-, m_\tau(u,\text{tr}(u)|_{S_\tau^+}) = (u,\text{tr}(u)|_{S_\tau^+}) \) and

\[
\|u\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|}\|f\|_{L^2(M)} + C_0 \frac{1}{(\delta|\tau|)^{1/2}}\|f_\tau^-\|_{L^2(S_\tau^-)} + C_0\|f_\tau^-\|_{L^2(S_\tau^+)}.\]

**Proof.** Define a linear functional \( l : M_\tau \to \mathbb{C} \) by

\[
l(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi,\text{tr}_\nu(\psi)|_{S_\tau^+}) = (f|\psi)_{L^2(M)} - (f_\tau^-|\text{tr}_\nu(\psi))_{S_\tau^-}.
\]

On the orthogonal complement of \( M_\tau \) we define \( l \) to be zero. By the Carleman estimate (3.1), we have

\[
|l(e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi,\text{tr}_\nu(\psi)|_{S_\tau^+})| \leq C_0 \left( \frac{1}{|\tau|}\|f\|_{L^2(M)} + \|f_\tau^-\|_{L^2(S_\tau^-)} + \|f_\tau^-\|_{S_\tau^-^2} \right) \times \left( \|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi\|_{L^2(M)} + |\tau|^{1/2}\|\text{tr}_\nu(\psi)\|_{S_\tau^+^2} \right).
\]

By Riesz representation theorem, there is \( (u,u_\tau^+) \in L^2(M) \times L^2(S_\tau^+) \) such that

\[
l(w|w) = (u|w)_{L^2(M)} + (u_\tau^+|w_\tau^+)_{S_\tau^+},
\]

for \( (w,w_\tau^+) \in L^2(M) \times L^2(S_\tau^+) \). In particular, for \( \psi \in \mathcal{D} \), we have

\[
(u|e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)_{L^2(M)} + (u_\tau^+|\text{tr}_\nu(\psi))_{S_\tau^+} = (f|\psi)_{L^2(M)} - (f_\tau^-|\text{tr}_\nu(\psi))_{S_\tau^-}.\]
Taking \( \psi \in C_0^\infty(M^{\text{int}}) \), gives that \( e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u = f \). Moreover,  
\[ \|u\|_{L^2(M)} \leq C_0 \frac{1}{|\tau|} \|f\|_{L^2(M)} + C_0 \frac{1}{(|\tau|)^{1/2}} \|F^-\|_{L^2(S^-)} + C_0 \|F^-\|_{L^2(S_0^\partial)} \].
Since \( l \equiv 0 \) on the orthogonal complement of \( M_\tau \) in \( L^2(M) \times L^2(S^\partial) \), we have that 
\( (u, \text{tr}(u)|_{S^\partial_+}) \) is in the closure of \( M_\tau \) and hence \( m_\tau' - m_\tau(u, \text{tr}(u)|_{S^\partial_+}) \).
For arbitrary \( \psi \in D \), using the generalized Green’s identity from Corollary 2.2, we get  
\[ (u e^{-\tau x_1}(-\Delta_g)e^{\tau x_1}\psi)|_{L^2(M)} + (\text{tr}(u), \text{tr}(\psi))_{H^{-1/2,1/2}(\partial M)} = (f|\psi)_{L^2(M)}. \]
Comparing this with the previous equality, this gives that \( \text{tr}(u)|_{S^\partial_-} = f^-_\tau \) and \( \text{tr}(u)|_{S^\partial_+} = u^+_\tau \).

Now, we prove uniqueness. Suppose that \( u' \in L^2(M) \) is another solution of the equation \( e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}u' = f \) satisfying all the conditions of the proposition. Then \( e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}(u - u') = 0 \), \( \text{tr}(u - u')|_{S^\partial_-} = 0 \), \( \text{tr}(u - u')|_{S^\partial_+} = u^+_\tau - u'^+_\tau \), and \( (u - u', u^+_\tau - u'^+_\tau) \) is in the closure of \( M_\tau \). However, by Lemma 6.1, \( (u - u', u^+_\tau - u'^+_\tau) \) is orthogonal to the closure of \( M_\tau \). Thus, we obtain \( u - u' = 0 \) which finishes the proof. 

Let \( R_\tau : L^2(M) \times L^2(S^-) \to L^2(M) \) be the solution operator obtained in the previous result. In other words, the operator \( R_\tau \) is defined by \( R_\tau(f, f^-) = u \), where \( u, f, f^- \) are as in Proposition 6.2.

6.2. Construction of complex geometrical optics solutions. Now, we are ready to construct complex geometrical optics solutions whose traces are supported in \( \Gamma_{\text{sgn}(\tau)} \). These are the solutions of the form
\[ u = e^{-\tau x_1}(a + r_0), \]
where \( r_0 \) is a correction term and \( a \) is an amplitude.

6.2.1. Construction in the case \( q = 0 \). Recall that the transversal manifold \((M_0, g_0)\) is assumed to be simple. Let \( \gamma : [0, T] \to M_0 \) be the given geodesic in \((M_0, g_0)\). Choose another simple manifold \((\tilde{M}_0, g_0)\) such that \((M_0, g_0) \subset \subset (\tilde{M}_0^{\text{int}}, g_0)\) and extend the geodesic \( \gamma \) in \( \tilde{M}_0 \). Choose \( \varepsilon > 0 \) such that \( \gamma(t) \in \tilde{M}_0 \setminus M_0 \) for all \( t \in (-2\varepsilon, 0) \cup (T, 2\varepsilon) \) and set \( p = \gamma(-\varepsilon) \) which is in \( \tilde{M}_0 \setminus M_0 \). Simplicity of \((\tilde{M}_0, g_0)\) implies that there are globally defined polar coordinates \((r, \theta)\) centered at \( p \). In these polar coordinates \( \gamma \) corresponds to \( r \mapsto (r, \theta_0) \) for some \( \theta_0 \in S^{n-2} \). Following [3, Section 5.2], we choose the following specific \( a \):
\[ a(x_1, r, \theta) = e^{-\mu r} |g|^{-1/4} e^{1/2 e^{\lambda(x_1 + ir)}} b(\theta), \]
where \( \lambda \in \mathbb{R} \) and \( b \in C^\infty(S^{n-2}) \) is fixed such that \( b \) is supported near \( \theta_0 \) so that \( a = 0 \) near \( \partial M_0 \setminus E \).

Assume now that \( u \) has the required form (6.1). Then the equation \((-\Delta_g)u_0 = 0\) is equivalent to
\[ e^{\tau x_1}(-\Delta_g)e^{-\tau x_1}r_0 = f, \]
where \( f := e^{rx_1} \Delta g e^{-rx_1} a \). Set \( \Phi = x_1 + ir \). Then a straightforward calculation shows that

\[
f = -e^{irr} e^{\tau \Phi} (-\Delta g) e^{-\tau \Phi} |g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)
\]

\[
= -e^{irr} [-\tau^2 \langle d\Phi, d\Phi \rangle_g + \tau (2\langle d\Phi, d\Phi \rangle_g + \Delta_g \Phi) - \Delta_g] (|g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)).
\]

Here the Riemannian inner product \( \langle \cdot, \cdot \rangle_g \) was extended as a complex bilinear form acting on complex valued 1-forms. It was shown in [3, Section 5] that \( \langle d\Phi, d\Phi \rangle_g = 0 \) and \( (2\langle d\Phi, d\Phi \rangle_g + \Delta_g \Phi)(|g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)) = 0 \). Hence, we get

\[
f = -e^{irr} (-\Delta_g) (|g|^{-1/4} e^{i\lambda(x_1 + ir)} b(\theta)). \tag{6.3}
\]

This shows that \( \|f\|_{L^2(M)} \lesssim 1 \) as \( \tau \to \infty \).

We want to ensure that \( \text{tr}(u_0) \) is supported in \( \Gamma_{\text{sgn}(\tau)} \) where \( \Gamma_{\text{sgn}(\tau)} \supset \partial M_{\text{sgn}(\tau)} \cup \Gamma_a \).

To achieve this, following [8], we take a small parameter \( \delta > 0 \) to be chosen later, and define the following sets

\[
V^{\tau,\delta} := \{ x \in S^- : \text{dist}_{\partial M}(x, \Gamma_i) < \delta \text{ or } x \in \partial M_{\text{sgn}(\tau)} \}, \quad \Gamma_a^{\tau,\delta} := S^- \setminus V^{\tau,\delta}.
\]

Note that \( \partial M_{\text{sgn}(\tau)} \cup \partial M_{\tan} \subset (S^-)^{\text{int}} \). For the boundary condition, we set

\[
f_{\tau,\delta} = \begin{cases} -a & \text{on } V^{\tau,\delta}, \\ 0 & \text{on } \Gamma_a^{\tau,\delta}. \end{cases}
\]

Defining \( f_{\tau,\delta}^- \) in such a way, we have \( f_{\tau,\delta}^- |_{V^{\tau,\delta} \setminus \partial M_{\tan}} = 0 \). Recall that \( \Gamma_i \subset \mathbb{R} \times (\partial M_0 \setminus E) \) and \( a \) was chosen in a way to satisfy \( a = 0 \) near \( \partial M_0 \setminus E \). Therefore, \( f_{\tau,\delta}^- |_{V^{\tau,\delta} \setminus \partial M_{\tan}} = 0 \), and hence we have

\[
f_{\tau,\delta}^- |_{\partial M_{\tan}} = 0.
\]

Since \( \|f_{\tau,\delta}^-\|_{L^\infty(S^-)} \lesssim 1 \), we obtain the following estimates

\[
\|f_{\tau,\delta}^-\|_{L^2(S^-_{\tau,\delta})} \lesssim \sigma_{\partial M}(S^-_{\tau,\delta})
\]

and

\[
\|f_{\tau,\delta}^-\|_{L^2(S^0_{\tau,\delta})} \lesssim \sigma_{M} \left( \{ x \in \partial M : -\delta < \text{sgn}(\tau) \partial_\nu \varphi(x) < 0 \} \right)
\]

\[
+ \sigma_{M} \left( \{ x \in \partial M : 0 < \text{sgn}(\tau) \partial_\nu \varphi(x) < (3|\tau|^{-1}) \} \right).
\]

If we set

\[
r_0 = R_{\tau}(f, f_{\tau,\delta}^-),
\]

then, by Proposition 6.2, \( r_0 \) solves (6.2) with \( \text{tr}(r_0)|_{S^-} = f_{\tau,\delta}^- \) and satisfies

\[
\|r_0\|_{L^2(M)} \lesssim \frac{1}{|\tau|} + \frac{1}{(3|\tau|)^{1/2}} \sigma_{\partial M}(S^-_{\tau,\delta})
\]

\[
+ \sigma_{M} \left( \{ x \in \partial M : -\delta < \text{sgn}(\tau) \partial_\nu \varphi(x) < 0 \} \right)
\]

\[
+ \sigma_{M} \left( \{ x \in \partial M : 0 < \text{sgn}(\tau) \partial_\nu \varphi(x) < (3|\tau|^{-1}) \} \right).
\]
Thus, there is constant $C_0 > 0$ such that
\[
\|r_0\|_{L^2(M)} \leq C_0 \left( \frac{1}{|\tau|} + \frac{1}{(|\delta|_\tau)^{1/2}} + a_{\tau \to \infty}(1) + o_{\delta \to 0}(1) \right).
\]
We choose $\delta$ such that $C_0a_{\delta \to 0}(1) \leq \varepsilon / 2$. Then we take $|\tau| \geq \tau_0$ large enough so that
\[
C_0 \left( \frac{1}{|\tau|} + \frac{1}{(|\delta|_\tau)^{1/2}} + a_{\tau \to \infty}(1) \right) \leq \varepsilon / 2.
\]
Therefore, we get $\|r_0\|_{L^2(M)} \to 0$ as $\tau \to \infty$. This will give the complex geometrical optics solution $u_0 = e^{-\tau x_1}(a + r_0)$ to $(-\Delta_g)u_0 = 0$ whose trace is supported in $\Gamma_{\sgn(\tau)}$. Thus, we have proved the following proposition.

**Proposition 6.3.** Let $(M, g)$ be an admissible manifold. Suppose that $\gamma : [0, T] \to M_0$ is a given nontangential geodesic in $(M_0, g_0)$, and let $\theta_0 \in S^{n-2}$ be as in the beginning of Section 6.2.1. For $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$ and $\delta > 0$, for any $\lambda \in \mathbb{R}$ and for any $b \in C^\infty(S^{n-2})$ supported sufficiently close to $\theta_0$, there is a solution $u_0 \in H_{\Delta_g}(M)$ to the equation $(-\Delta_g)u_0 = 0$ of the form
\[
u u_0 = e^{-\tau x_1}(a + r_0),
\]
and satisfying
\[
supp(\text{tr}(u_0)) \subset \Gamma_{\sgn(\tau)}
\]
where
\[
a = e^{-i\tau r}\infinitesimal{g} \frac{1}{4} e^{1/2} e^{i\lambda(x_1 + ir)} b(\theta),
\]
and $\|r_0\|_{L^2(M)} \to 0$ as $\tau \to \infty$.

**Remark 6.4.** Modifying the above arguments in appropriate places, one can construct complex geometrical optics solutions whose traces are supported in $\partial M_{\sgn(\tau)}$ if $\partial M_{\text{tan}}$ has zero measure in $\partial M$. Let us indicate these modifications. Up to (6.3) everything is same except that we do not put any restrictions on $b$, so that we do not require $a$ to vanish on any part of the boundary. In order to ensure that $supp(\text{tr}(u)) \subset \partial M_{\sgn(\tau)}$, for fixed $\delta > 0$, we set
\[
f_{\tau, \delta} = -a.
\]
Since $\|f_{\tau, \delta}\|_{L^\infty(S^-)} \lesssim 1$ and $\sigma_{\partial M}(\partial M_{\text{tan}}) = 0$, we obtain the following estimates
\[
\|f_{\tau, \delta}\|_{L^2(S^-)} \lesssim \sigma_{\partial M}(S^-_{\tau, \delta})
\]
and
\[
\|f_{\tau, \delta}\|_{L^2(S^+_{\tau, \delta})} \lesssim \sigma_{\partial M}(\{x \in \partial M : -\delta < \sgn(\tau)\partial(\varphi(x) < 0)\})
\]
\[
+ \sigma_{\partial M}(\{x \in \partial M : 0 < \sgn(\tau)\partial(\varphi(x) < 3|\tau|^{-1})\}).
\]
We use Proposition 6.2 to solve (6.2) for $r_0$ with $\text{tr}(r_0)|_{S^-} = f_{\tau, \delta}$ and to show that $r_0$ satisfies the same estimate as before for some $C_0 > 0$ constant:
\[
\|r_0\|_{L^2(M)} \leq C_0 \left( \frac{1}{|\tau|} + \frac{1}{(|\delta|_\tau)^{1/2}} + a_{\tau \to \infty}(1) + o_{\delta \to 0}(1) \right).
\]
Thus, we have constructed the complex geometrical optics solution $u_0 \in H_{\Delta_g}(M)$ to $(-\Delta_g)u_0 = 0$ of the form

$$u_0 = e^{-\tau x_1}(a + r_0)$$

whose trace is supported in $\partial M_{\text{sgn}(\tau)}$ and $\|r_0\|_{L^2(M)} \to 0$ as $\tau \to \infty$.

6.2.2. Construction for general $q$. Next, we construct complex geometrical optics solutions for the Schrödinger equation $(-\Delta_g + q)u = 0$ in $M$ with $q \in L^\infty(M)$ such that $\text{supp}(\text{tr}(u)) \subset \Gamma_{\text{sgn}(\tau)}$.

**Proposition 6.5.** Let $(M, g)$ be an admissible manifold and let $q \in L^\infty(M)$ be such that $0$ is not a Dirichlet eigenvalue of $-\Delta_g + q$ in $M$. Suppose that $\gamma : [0, T] \to M_0$ is a given nontangential geodesic in $(M_0, g_0)$. For any $\tau \in \mathbb{R}$ with $|\tau| \geq \tau_0$, there is a solution $u \in L^2(M)$ to the equation $(-\Delta_g + q)u = 0$ of the form

$$u = u_0 + e^{-\tau x_1}r_1,$$

where $u_0$ is as in Proposition 6.3 and one has

$$(\text{Id} + G_{\tau} \circ q)r_1 = -G_{\tau}qe^{\tau x_1}u_0,$$

and $\|r_1\|_{L^2(M)} \lesssim \frac{1}{|\tau|}$ as $\tau \to \infty$. Moreover, $\text{tr}(u)$ is supported in $\Gamma_{\text{sgn}(\tau)}$.

**Proof.** Consider the following integral equation

$$(\text{Id} + G_{\tau} \circ q)r_1 = -G_{\tau}qe^{\tau x_1}u_0. \quad (6.4)$$

Since $q \in L^\infty(M)$ and $\|G_{\tau}\|_{L^2(M) \to L^2(M)} \lesssim \frac{1}{|\tau|}$ by Theorem 3.2, for sufficiently large this integral equation has a unique solution $r_1 = -(\text{Id} + G_{\tau} \circ q)^{-1}G_{\tau}qe^{\tau x_1}u_0$ in terms of the convergent Neumann series. Then $\|G_{\tau}\|_{L^2(M) \to L^2(M)} \lesssim \frac{1}{|\tau|}$ implies that $\|r_1\|_{L^2(M)} \lesssim \frac{1}{|\tau|}$. Using the fact that $(-\Delta_g)e^{-\tau x_1}G_{\tau} = e^{-\tau x_1}$ (by Theorem 3.2) and that $(-\Delta_g)u_0 = 0$, and using (6.4), we can show

$$(-\Delta_g + q)u = (-\Delta_g + q)u_0 + (-\Delta_g + q)e^{-\tau x_1}r_1$$

$$= qu_0 + (-\Delta_g + q)(-e^{-\tau x_1}G_{\tau}qr_1 - e^{-\tau x_1}G_{\tau}qe^{\tau x_1}u_0)$$

$$= qu_0 - e^{-\tau x_1}qr_1 - qu_0 - qe^{-\tau x_1}G_{\tau}qr_1 - qe^{-\tau x_1}G_{\tau}qe^{\tau x_1}u_0$$

$$= -e^{-\tau x_1}q(\text{Id} + G_{\tau} \circ q)r_1 - qe^{-\tau x_1}G_{\tau}qe^{\tau x_1}u_0$$

$$= e^{-\tau x_1}qG_{\tau}qe^{\tau x_1}u_0 - qe^{-\tau x_1}G_{\tau}qe^{\tau x_1}u_0$$

$$= 0.$$ 

Let us now prove the last part of the proposition. By Proposition 6.3, we have that $\text{tr}(u_0)$ is supported in $\Gamma_{\text{sgn}(\tau)}$. Note that Theorem 3.2 implies that $\text{tr}(G_{\tau}qr_1)$ is supported in $\Sigma^+$. These, together with (6.4) imply that the trace of $u = u_0 + e^{-\tau x_1}r_1$ is supported in $\Gamma_{\text{sgn}(\tau)}$. \hfill $\Box$

**Remark 6.6.** If $\partial M_{\text{tan}}$ has zero measure in $\partial M$, one can replace $u_0$ in the above proposition with the one obtained in Remark 6.4. Then the proof of Proposition 6.5 shows that so-obtained complex geometrical optics solution $u \in H_{\Delta_g}(M)$ to $(-\Delta_g + q)u_0 = 0$ of the form

$$u = u_0 + e^{-\tau x_1}r_1$$
has \( \text{supp}(\text{tr}(u)) \subset \partial M_{\text{sgn}(\tau)} \) and \( \|r_1\|_{L^2(M)} \lesssim \frac{1}{|\tau|} \) as \( \tau \to \infty \).

7. Proofs of the main results

**Proof of Theorem 1.1.** Suppose that \( q \in C(M) \) such that 0 is not a Dirichlet eigenvalue of \(-\Delta_g + q\). Assume the knowledge of \((M, g)\) and \(\Lambda_{g,q}f\) on \(\Gamma_-\) for all \(f \in \mathcal{H}(\partial M)\) supported in \(\Gamma_+\). Then by Proposition 2.5, the following integral identity holds

\[
\langle \text{tr}(u_2), (\Lambda_{g,q} - \Lambda_{g,0})\text{tr}(u_1) \rangle_{H^{-1/2, 1/2}(\partial M)} = \langle u_2|q u_1 \rangle_{L^2(M)},
\]

(7.1)

where \(u_1 \in H_{\Delta_g}(M)\) is a solution of \((-\Delta_g + q)u_1 = 0\) in \(M\) with \(\text{tr}(u_1)\) supported in \(\Gamma_+\), and \(u_2 \in H_{\Delta_g}(M)\) is a solution of \((-\Delta_g)u_2 = 0\) in \(M\) with \(\text{tr}(u_2)\) supported in \(\Gamma_-\).

Let \(\tau \geq \tau_0\). By Proposition 6.5, there is \(u_1 \in H_{\Delta_g}(M)\) solving \((-\Delta_g + q)u_1 = 0\) in \(M\) with \(\text{tr}(u_1)\) supported in \(\Gamma_+\), and having the form

\[
u_1 = e^{-\tau x_1}(e^{-irr}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) + r' + r_0) = u_1' + e^{-\tau x_1}r_0
\]

where \(\|r_0\|_{L^2(M)} \lesssim \frac{1}{|\tau|}\) and \(\|r'\|_{L^2(M)} \to 0\) as \(\tau \to +\infty\) (here \(u_1'\) is a solution to \((-\Delta_g)u_1' = 0\) as in Proposition 6.3).

By Proposition 6.3, there is a \(\nu_2 \in H_{\Delta_g}(M)\) solving \((-\Delta_g)\nu_2 = 0\) in \(M\) with \(\text{tr}(\nu_2)\) supported in \(\Gamma_-\), and having the form

\[
u_2 = e^{\tau x_1}(e^{irr}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)} + r'')
\]

where \(\|r''\|_{L^2(M)} \to 0\) as \(\tau \to +\infty\). Then \(u_2\) will be the complex geometrical optics solutions to \((-\Delta_g)u_2 = 0\) in \(M\) with \(\text{tr}(u_2)\) supported in \(\Gamma_-\), and having the form

\[
u_2 = e^{\tau x_1}(e^{-irr}|g|^{-1/4}c^{1/2}e^{-i\lambda(x_1-ir)} + r''')
\]

The important thing to note is that \(u_1'\) as well as \(u_2\) depend only on \((M, g)\), i.e. independent on \(q\). Since \(\text{tr}(u_1)\) is supported in \(\Gamma_+\) and \(\text{tr}(u_2)\) is supported in \(\Gamma_-\), the left hand-side of (7.1) requires only the given partial data of \(\Lambda_{g,q}\).

Now, we show that \(\text{tr}(u_1)\) can be reconstructed from the above mentioned partial knowledge of \(\Lambda_{g,q}\). By Proposition 6.5 and Proposition 5.2, one can check that \(\text{tr}(u_1)\) satisfies the following boundary integral equation

\[(\text{Id} + S_{\tau}(\Lambda_{g,q} - \Lambda_{g,0}))\text{tr}(u_1) = \text{tr}(u_1').\]

Then Corollary 5.3 and Proposition 5.4 imply solvability of the above boundary integral equation for sufficiently large \(\tau\). Substituting this solution \(\text{tr}(u_1)\) into the left hand-side of (7.1), we can determine

\[
\langle u_2|q u_1 \rangle_{L^2(M)}
\]

for all complex geometrical optics solutions \(u_1, u_2\) of the above form.

Using the decay properties of \(r', r'', r \) and taking limit as \(\tau \to \infty\), we can reconstruct

\[
\int_M cq |g|^{-1/2}c^{1/2}(x_1+ir)\,b(\theta)\,d\text{Vol}_g.
\]
Now we extend $q$ as zero to $\mathbb{R} \times M_0$. Since $d\text{Vol}_g = |g|^{1/2}dx_1\,dr\,d\theta$, the above expression becomes
\[
\int_{S^{n-2}} \int_{0}^{\infty} e^{-2\lambda r} \left( \int_{-\infty}^{\infty} e^{2i\lambda x_1} (cq)(x_1, r, \theta) \,dx_1 \right) \,dr\,d\theta.
\]
Varying $b \in C^\infty(S^{n-2})$ so that the support of $b$ is sufficiently close to $\theta_0$ and noting that the term in the brackets is the one-dimensional Fourier transform of $q$ with respect to the $x_1$-variable, which we denote by $\hat{q}$, we determine
\[
\int_{0}^{\infty} e^{-2\lambda r} (\hat{cq})(2\lambda, r, \theta_0) \,dr.
\]
Recalling that $r \mapsto (r, \theta_0)$ corresponds to the given nontangential geodesic $\gamma : [0, T] \to M$, we finish the proof.

**Proof of Theorem 1.2.** Assume that $O \subset M_0$ is open such that $O \cap \partial M_0 \subset E$ and the local geodesic ray transform is invertible on $O$. According to Theorem 1.1, we can constructively determine
\[
\int_{0}^{T} e^{-2\lambda t} (\hat{cq})(2\lambda, \gamma(t)) \,dt \quad (7.2)
\]
for all nontangential geodesics $\gamma : [0, T] \to O$ with $\gamma(0), \gamma(T) \in E$. This is the local attenuated geodesic ray transform of $(\hat{cq})(2\lambda, \cdot)$ in $O$, with attenuation $-2\lambda$. Setting $\lambda = 0$, we determine an unattenuated local geodesic ray transform of $(\hat{cq})(0, \cdot)$ in $O$. Then using the constructive invertibility assumption for the local geodesic ray transform, we recover $(\hat{cq})(0, \cdot)$ in $O$.

Now, we go back to (7.2) and differentiate it with respect to $\lambda$ at $\lambda = 0$. Since we have reconstructed $(\hat{cq})(0, \cdot)$, we constructively determine the local geodesic ray transform of $(\frac{\partial}{\partial \lambda} (\hat{cq}))(0, \cdot)$ in $O$. Using the invertibility assumption for the local geodesic ray transform again, we obtain $(\frac{\partial}{\partial \lambda} (\hat{cq}))(0, \cdot)$ in $O$.

Using this argument iteratively by taking higher derivatives of (7.2) with respect to $\lambda$, we can reconstruct
\[
\left( \frac{\partial^k}{\partial \lambda^k} (\hat{cq}) \right)(0, \cdot) \text{ in } O \text{ for all integers } k \geq 0.
\]
Since $q$ is compactly supported in $x_1$-variable, its Fourier transform $(\hat{cq})(\lambda, \cdot)$ is analytic with respect to $\lambda$. Therefore, we have reconstructed the Taylor series expansion of $(\hat{cq})(\lambda, \cdot)$ in $O$. Then we determine $q$ in $M \cap (\mathbb{R} \times O)$ by inverting the one-dimensional Fourier transform of $cq$ with respect to the $x_1$-variable.

**Proof of Theorem 1.3.** Let $(M, g)$ be a known admissible manifold such that $\partial M_{\tan}$ is of measure zero in $\partial M$. Suppose that $q \in C(M)$ such that $0$ is not a Dirichlet eigenvalue of $-\Delta_g + q$. Assume the knowledge of $\Lambda_{g,q}f$ on $\partial M_-$ for all $f \in H(\partial M)$ supported in $\partial M_+$. 

Now, we go back to (7.2) and differentiate it with respect to $\lambda$ at $\lambda = 0$. Since we have reconstructed $(\hat{cq})(0, \cdot)$, we constructively determine the local geodesic ray transform of $(\frac{\partial}{\partial \lambda} (\hat{cq}))(0, \cdot)$ in $O$. Using the invertibility assumption for the local geodesic ray transform again, we obtain $(\frac{\partial}{\partial \lambda} (\hat{cq}))(0, \cdot)$ in $O$.

Using this argument iteratively by taking higher derivatives of (7.2) with respect to $\lambda$, we can reconstruct
\[
\left( \frac{\partial^k}{\partial \lambda^k} (\hat{cq}) \right)(0, \cdot) \text{ in } O \text{ for all integers } k \geq 0.
\]
Since $q$ is compactly supported in $x_1$-variable, its Fourier transform $(\hat{cq})(\lambda, \cdot)$ is analytic with respect to $\lambda$. Therefore, we have reconstructed the Taylor series expansion of $(\hat{cq})(\lambda, \cdot)$ in $O$. Then we determine $q$ in $M \cap (\mathbb{R} \times O)$ by inverting the one-dimensional Fourier transform of $cq$ with respect to the $x_1$-variable.
Using Remark 6.4 and Remark 6.6, as in the proof of Theorem 1.1, we can construct $u_1 \in H_{\Delta_g}(M)$ and $u_2 \in H_{\Delta_g}(M)$ solving $(-\Delta_g + q)u_1 = 0$ in $M$ with $\text{tr}(u_1)$ supported in $\partial M_+$ and solving $(-\Delta_g)u_2 = 0$ in $M$ with $\text{tr}(u_2)$ supported in $\partial M_-$, respectively, and having the forms

$$u_1 = e^{-\tau x_1}(e^{-ir\gamma}|g|^{-1/4}c^{1/2}e^{i\lambda(x_1+ir)}b(\theta) + r' + r_0) = u'_1 + e^{-\tau x_1}r_0,$$

$$u_2 = e^{\tau x_1}(e^{-ir\gamma}|g|^{-1/4}c^{1/2}e^{-i\lambda(x_1-ir)} + r')$$

where $\|r_0\|_{L^2(M)} \lesssim \frac{1}{|\gamma|}$, $\|r''\|_{L^2(M)} \to 0$ and $\|r''|\|_{L^2(M)} \to 0$ as $\tau \to +\infty$ (here $u'_1$ is a solution to $(-\Delta_g)u'_1 = 0$ as in Remark 6.4).

Continuing as in the proof of Theorem 1.1, but replacing $\Gamma_{\pm}$ by $\partial M_{\pm}$, for any $\lambda \in \mathbb{R}$, we can constructively determine

$$\int_0^Te^{-2\lambda t}(c\gamma)(2\lambda, \gamma(t)) \, dt$$

for all the nontangential geodesics $\gamma : [0, T] \to M_0$ in $(M_0, g_0)$. Using the constructive invertibility assumption for the global geodesic ray transform, we reconstruct $q$ in $M$ via similar steps as in the proof of Theorem 1.2.

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