A stabilized computational nonlocal poromechanics model for dynamic analysis of saturated porous media

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Abstract
In this article we formulate a stable computational nonlocal poromechanics model for dynamic analysis of saturated porous media. As a novelty, the stabilization formulation eliminates zero-energy modes associated with the original multiphase correspondence constitutive models in the coupled nonlocal poromechanics model. The two-phase stabilization scheme is formulated based on an energy method that incorporates inhomogeneous solid deformation and fluid flow. In this method, the nonlocal formulations of skeleton strain energy and fluid flow dissipation energy equate to their local formulations. The stable coupled nonlocal poromechanics model is solved for dynamic analysis by an implicit time integration scheme. As a new contribution, we validate the coupled stabilization formulation by comparing numerical results with analytical and finite element solutions for one-dimensional and two-dimensional dynamic problems in saturated porous media. Numerical examples of dynamic strain localization in saturated porous media are presented to demonstrate the efficacy of the stable coupled poromechanics framework for localized failure under dynamic loads.

KEYWORDS
coupled, dynamics, nonlocal, poromechanics, saturated porous media, stabilization

1 | INTRODUCTION

Dynamic behavior of deforming porous media is a significant problem in engineering and science. A fully coupled dynamic analysis of porous media is essential in earthquake engineering (soil liquefaction), geo-hazard engineering (dynamic landslide triggering), and offshore wind industry (wind turbine foundations). The integrity of civil infrastructure (e.g., dams and levee systems) can be seriously compromised by the poor performance of soils under dynamic loading conditions. Thus, numerous researchers have studied the dynamics and wave propagation characteristics of saturated porous media, for example References 5-9, among others. One typical failure of porous media under dynamic loads is dynamic strain localization (e.g., References 2,10). A topic has been insufficiently studied compared to the extensive research on strain localization of porous media under static or quasi-static loading conditions (see Reference 11 and 12 for a recent review of the latter). Over the past decades, viscoplasticity and gradient-dependent plasticity have been adopted to study dynamic strain localization of solid or porous media (see References 10,13-19, among others), as well as a means of regularizing the rate-independent problem so that the governing equation of dynamic problems remains hyperbolic. Heider et al. developed a coupled dynamic elasto-viscoplastic model for sands and found that inertial loads had a significant impact on dynamic strain localization. Shahbodagh et al. proposed an elasto-viscoplastic model for dynamic...
analysis of strain localization in fully saturated clay. Enriched finite element methods such as the extended finite element and multiscale models have been applied to model saturated porous media (e.g., References 20-23, among others). For example, Khoei\(^{21}\) proposed an enriched finite element technique for numerical simulation of interacting discontinuities in naturally fractured porous media. Note the above methods are based on classical local poromechanics. It has been generally recognized that nonlocal models are robust to investigate multiphysics failure mechanisms including the dynamic strain localization phenomenon in porous media (e.g., References 24-27 and many others). In this article, we propose a stabilized coupled nonlocal poromechanics model to study dynamic strain localization in saturated porous media.

In Reference 28, a computational periporomechanics model was formulated to model localized failure in unsaturated porous media under static conditions. The coupled nonlocal model in Reference 28 was formulated based on the peridynamic state concept,\(^{29}\) the effective force state concept and multiphase correspondence principle.\(^{30}\) We refer to the literature for other nonlocal models for porous media (see References 31-39, among others), which are formulated using peridynamics (PD) theory (i.e., the bond-based or ordinary state-based PD)\(^{29,32,40}\) and poroelasticity (see Reference 41). It was demonstrated that the coupled nonlocal model in Reference 30 is robust for modeling localized failure in unsaturated soils under static/quasi-static loading conditions. However, it is recognized that the PD model formulated based on the correspondence principle for the single-phase solid has stability issues under extreme large deformation and dynamic loading conditions (e.g., References 42-52, among others). Silling\(^{45}\) showed that the numerical oscillation in PD material models formulated via the single-phase correspondence principle is associated with material instability instead of merely an artifact of the meshless discretization.\(^{53}\) This study suggests that the primary causes of instability and numerical oscillations are (1) the weak dependence of the force density in a bond on its own deformation and (2) the loss of the nonuniform part of deformation due to the integration over the horizon of a material point. Note that the above studies and remedies for the instability were focused on modeling the large deformation or extreme damage of single-phase solid materials through the original correspondence principle.\(^{29}\) Here for the first time, we prove that the recently formulated multiphase correspondence principle for modeling unsaturated porous media inherits the aforementioned instability under large deformation and dynamic loading conditions. To circumvent the issue, we have formulated the coupled stabilization terms for solid deformation and fluid flow, respectively, in Section 2.

In this article, we formulate and implement a stabilized coupled nonlocal poromechanics model for dynamic strain localization in saturated porous media. As a novelty, the stabilization terms eliminate zero-energy modes associated with the multiphase correspondence constitutive models in the coupled nonlocal framework for modeling saturated porous media under dynamic loads. The stabilization terms are formulated based on the energy method in which the nonlocal formulations of skeleton strain energy and fluid flow dissipation energy equate to their local formulations in line with the classical poromechanics for saturated porous media. Specifically, the method can incorporate nonhomogeneous solid deformation and fluid flow around a material point. The stabilized coupled nonlocal poromechanics model is numerically solved using a hybrid Lagrangian–Eulerian meshless method with an implicit time integration scheme. Parallel computing is also adopted for computational efficiency. The coupled stabilized formulation is validated by comparing numerical results with analytical and finite element solutions for one-dimensional and two-dimensional dynamic problems in saturated porous media. Numerical examples of dynamic strain localization in saturated porous media are presented to demonstrate the efficacy of the stable coupled poromechanics framework for localized failure under dynamic loads. We note that the stabilized coupled nonlocal model for saturated porous media can be readily extended to model dynamic problems in unsaturated porous media.

The contribution of this article includes (1) a proof of the zero-energy modes associated with the original multiphase correspondence constitutive principle for modeling porous media through multiphase peridynamic states, (2) a remedy based on the energy method to remove the multiphase zero-energy modes for dynamic analysis, and (3) an implicit numerical implementation of the proposed stabilized dynamic nonlocal poromechanics model and its validation. For sign convention, the assumption in continuum mechanics is followed, that is, for the solid skeleton, tensile force/stress is positive and compression is negative, and for fluid compression is positive and tension is negative.

2 | STABILIZED NONLOCAL FORMULATION OF COUPLED DYNAMIC POROMECHANICS

For conciseness of notations, it is assumed that the peridynamic state variable without a prime means that the variable is evaluated at \(x\) on the associated bond \(x' - x\) and the peridynamic state variable with a prime means that the variable is evaluated at \(x'\) on the associated bond \(x - x'\), for example, \(T = T|x|(x' - x)\) and \(T' = T|x'||(x - x')\).
2.1 Dynamic saturated periporomechanics model

Periporomechanics is a fully coupled, nonlocal theory of porous media based on the peridynamic state concept.\textsuperscript{29} It is a reformulation of classical poromechanics through PD for modeling continuous or discontinuous deformation and physical processes in porous media.\textsuperscript{28,30} In periporomechanics, it is assumed that a porous media body is composed of material points which have two kinds of degree of freedom, that is, displacement and fluid pressure. A material point $x$ has poromechanical and physical interactions with any material point $x'$ within its neighborhood, $\mathcal{H}$. Here $\mathcal{H}$ is a spherical domain around $x$ with radius $\delta$, which is called the horizon for the porous medium. Let $\rho$ be the density of the two-phase mixture that is determined by

$$\rho = \rho_s (1 - \phi) + \rho_w \phi. \quad (1)$$

where $\rho_s$ and $\rho_w$ are the intrinsic density of the solid and the fluid, respectively, and $\phi$ is the porosity (i.e., the volume of pore space divided by the total volume). Figure 1 shows the schematic of the kinematics of two material points.

The relative position of material points $x$ and $x'$ in the reference configuration is denoted by

$$\xi = x' - x. \quad (2)$$

Let the displacements of $x$ and $x'$ be $u$ and $u'$, respectively. The relative displacement between $x$ and $x'$ in the deformed configuration is

$$\eta = u' - u. \quad (3)$$

The position vectors $y'$ and $y$ of $x'$ and $x$ in the deformed configuration are

$$y = x + u, \quad \text{and} \quad y' = x' + u'. \quad (4)$$

The deformation state of $x$ on $\xi$ is defined as

$$\mathcal{Y} \prec \xi \succ = y' - y = \xi + \eta. \quad (5)$$

The fluid potential state at $x$ imposed on $\xi$ is defined as

$$\Phi(\xi) = p'_w - p_w. \quad (6)$$

where $p'_w$ and $p_w$ are pore fluid pressures at $x'$ and $x$, respectively. The equation of motion in saturated periporomechanics reads

$$\int_{\mathcal{H}} [(T - T_{w}) - (T' - T'_{w})] d\mathcal{V}' + \rho g = \rho \ddot{u}. \quad (7)$$

\textbf{FIGURE 1} Schematic of the kinematics of two material points $x$ and $x'$.
where $\mathbf{T}$ and $\mathbf{T}_w$ are the effective force state and the fluid phase force state respectively at material point $\mathbf{x}$. $\mathbf{T}'$ and $\mathbf{T}'_w$ are the effective force state and the fluid phase force state respectively at material point $\mathbf{x}'$. $\mathbf{g}$ is the gravity acceleration, and $\mathbf{u}$ is the acceleration vector. Assuming an incompressible solid phase, the fluid mass balance equation incorporating the volume change of the solid skeleton reads

$$\frac{\phi}{K_w} \dot{p}_w + \int_{H} (\dot{\mathbf{Y}}_s - \dot{\mathbf{Y}}'_s) dV' + \frac{1}{\rho_w} \int_{H} (Q - Q') dV' = 0,$$

(8)

where $K_w$ is the bulk modulus of fluid, $\dot{\mathbf{Y}}_s$ and $\dot{\mathbf{Y}}'_s$ are the rate of volume change scalar states of the solid at $\mathbf{x}$ and $\mathbf{x}'$, respectively, and $Q$ and $Q'$ are fluid mass flow scalar states relative to the solid skeleton at $\mathbf{x}$ and $\mathbf{x}'$, respectively.

Through the multiphase correspondence principle, the effective force state and the fluid mass flow state can be determined by the peridynamic constitutive models via the deformation state of the solid skeleton and the fluid potential state, respectively. The momentum balance equation for saturated porous media in periporomechanics under dynamic loading reads

$$\int_{H} \left[ (\omega(\sigma - p_w \mathbf{1})) \mathbf{F}^{-T} K^{-1} \mathbf{r} - (\omega(\sigma' - p'_w \mathbf{1})) \mathbf{F}'^{-T} K^{-1} \mathbf{r}' \right] dV' + \rho g = \rho \dddot{u}.$$  

(9)

where $\omega$ is a scalar influence function, $\mathbf{F}$ is the nonlocal deformation gradient, $J$ is the determinant of $\mathbf{F}$, $\sigma$ is the effective stress tensor that can be determined by classical constitutive models given $\mathbf{F}$, $\mathbf{1}$ is the second-order identity tensor, and $K$ is the shape tensor. The nonlocal deformation gradient $\mathbf{F}$ is defined as

$$\mathbf{F} = \left( \int_{H} \omega \mathbf{Y} \otimes \mathbf{r} dV' \right) K^{-1}, \text{ with } K = \int_{H} \omega \mathbf{r} \otimes \mathbf{r} dV'.$$

(10)

It is assumed that the accelerations of the fluid and the solid are identical. Through the multiphase correspondence principle, the fluid mass balance equation accounting for the volume change of the solid skeleton reads

$$\frac{\phi}{K_w} \dot{p}_w + \int_{H} (\omega \dot{\mathbf{u}} K^{-1} \mathbf{r} - \omega \dot{\mathbf{u}}' K^{-1} \mathbf{r}') dV' + \int_{H} [\omega(q_w - k_w \mathbf{u})K^{-1} \mathbf{r} - \omega(q'_w - k'_w \mathbf{u}')K^{-1} \mathbf{r}'] dV' = 0,$$

(11)

where $K_w$, $q_w$, and $q'_w$ are the fluid flux vectors at $\mathbf{x}$ and $\mathbf{x}'$, respectively, and $k_w$ is the hydraulic conductivity. The fluid flux can be determined by Darcy’s law as

$$q_w = -k_w \nabla \Phi.$$

(12)

where $k_w$ $\nabla \Phi$ is the approximate nonlocal fluid pressure gradient

$$\nabla \Phi = \left( \int_{H} \omega \Phi \mathbf{r} dV' \right) K^{-1}.$$

(13)

### 2.2 Two-phase stabilization formulation

As stated in the multiphase correspondence principle, the nonlocal deformation gradient and fluid pressure gradient are approximate for the nonuniform deformation and fluid flow while they are exact for the uniform deformation and fluid flow, respectively. There can be increments $d\mathbf{Y}$ that may have no effect on the approximate deformation gradient, leading to zero-energy modes of deformation. Similarly, for fluid flow there can be increments $d\Phi_w$ that may have no effect on the approximate fluid pressure gradient, leading to zero-energy modes of fluid flow. It can be demonstrated as follows. The nonuniform part of the solid deformation state and fluid potential state can be defined as

$$\mathbf{R}^s(\mathbf{r}) = \mathbf{Y}(\mathbf{r}) - \mathbf{F} \mathbf{r},$$

(14)

$$\mathbf{R}^w(\mathbf{r}) = \Phi_w(\mathbf{r}) - \nabla \Phi_w \mathbf{r}.$$

(15)
Substituting Equations (14) and (15) into (10) and (13) generates
\[ \left( \int_{\Omega} \omega R^I \otimes \xi dV' \right) K^{-1} = \left( \int_{\Omega} \omega (Y - F_x) \otimes \xi dV' \right) K^{-1} \]
\[ = \left( \int_{\Omega} \omega Y \otimes \xi dV' \right) K^{-1} - \tilde{F} \left( \int_{\Omega} \omega \xi \otimes \xi dV' \right) K^{-1} \]
\[ = \tilde{F} - \tilde{FKK}^{-1} \]
\[ = 0. \]  
(16)

\[ \left( \int_{\Omega} \omega R^w \xi dV' \right) K^{-1} = \left( \int_{\Omega} \omega (\Phi_w - \tilde{\Phi}_w \xi) \xi dV' \right) K^{-1} \]
\[ = \left( \int_{\Omega} \omega \Phi_w \xi dV' \right) K^{-1} - \tilde{\Phi}_w \left( \int_{\Omega} \omega \xi \otimes \xi dV' \right) K^{-1} \]
\[ = \tilde{\Phi}_w - \tilde{\Phi}_w K K^{-1} \]
\[ = 0. \]  
(17)

For the solid skeleton, the strain energy density is assumed as
\[ \mathcal{W} = \mathcal{W}_c + \mathcal{W}_s, \]  
(18)
where \( \mathcal{W}_c \) is the stored energy from the standard correspondence material model and \( \mathcal{W}_s \) is the energy density related to the non-uniform deformation state. \( \mathcal{W}_s \) is assumed to take the form
\[ \mathcal{W}_s(Y) = \frac{1}{2} \left( \beta \mathcal{R}^s \right) \cdot \mathcal{R}^s = \frac{1}{2} \int_{\Omega} \beta \mathcal{R}^s \mathcal{R}^s dV', \]  
(19)
where \( \beta \) is a prescribed positive-valued scalar state. In what follows, \( \mathcal{W}_s \) will be used to derive a stabilized term for the effective force state. Given a small increment \( dY \), it follows from (14) and (19) that
\[ d\mathcal{W}_s = \int_{\Omega} \beta \mathcal{R}^s (dY_i - dF_{ij} \xi_j) dV' \]
\[ = \int_{\Omega} \beta \mathcal{R}^s dY_i dV' - \int_{\Omega} \beta \mathcal{R}^s dF_{ij} \xi_j dV' \]
\[ = \int_{\Omega} \beta \mathcal{R}^s dY_i dV' - \left( \int_{\Omega} \beta \mathcal{R}^s \xi_j dV' \right) \left( \int_{\Omega} \omega dY_i \xi_j dV' \right) K_{ij}^{-1} \]
\[ = \int_{\Omega} \left[ \beta \mathcal{R}^s - \left( \int_{\Omega} \beta \mathcal{R}^s \xi_j dV' \right) \omega \xi_j K_{ij}^{-1} \right] dY_i dV', \]  
(20)
where \( i, j, l = 1, 2, 3 \). It follows from (20) that the stabilized term for the effective force state accounting for the non-uniformed deformation state can be written as
\[ \overline{T}_s = \beta \mathcal{R}^s - \left( \int_{\Omega} \beta \mathcal{R}^s \xi_j dV' \right) \omega \xi_j K_{ij}^{-1}. \]  
(21)

Given \( dY \) it can be proved that \( d\mathcal{W}_s \) is always positive for a positive value of \( \beta \). It follows from (21) along with the original multiphase correspondence principle\(^{30}\) that the stabilized effective force state can be written as
\[ \overline{T} = \omega \overline{F}^T K^{-1} \xi + \beta \mathcal{R}^s - \left( \int_{\Omega} \beta \mathcal{R}^s \otimes \xi dV' \right) \omega K^{-1} \xi. \]  
(22)

To simplify (22), it is assumed that
\[ \beta = \frac{GC}{\omega_0} \omega, \]  
with \( \omega_0 = \int_{\Omega} \omega dV' \).  
(23)
where $G$ is a positive constant on the order of 1 and $C$ is a micromodulus which will be provided in the next section. With (23) and (16), it can be proved that the third term of (22) becomes zero. Thus, Equation (22) can be expressed as

$$\bar{T} = \omega \left( J\sigma F^{-1} K^{-1} \xi + \frac{GC}{\omega_0} R^i \right). \tag{24}$$

Similarly, for the fluid phase the energy dissipation accounting for the nonuniform fluid potential state can be written as

$$\mathcal{Y} = \mathcal{Y}_c + \mathcal{Y}_s, \tag{25}$$

where $\mathcal{Y}_c$ is the energy dissipation related to the uniform fluid potential through the corresponding Darcy’s law (i.e., (12)) and $\mathcal{Y}_s$ is the dissipation energy through the nonuniform fluid potential. It is assumed that $\mathcal{Y}_s$ takes the general form

$$\mathcal{Y}_s = \frac{1}{2} (\lambda w) \cdot R^w = \frac{1}{2} \int_{V'} \lambda w R^w dV', \tag{26}$$

where $\lambda$ is a positive-valued scalar state. In what follows, $\mathcal{Y}_s$ will be used to derive a stabilized term for the fluid flow state. Given a small increment $d\Phi$, it follows from (17) and (26) that

$$d\mathcal{Y}_s = \int_{V'} \lambda w (d\overline{\Phi} \cdot \xi) dV'$$

$$= \int_{V'} \lambda w d\Pi dV' - \int_{V'} \lambda w d(\overline{\Phi}) dV'$$

$$= \int_{V'} \lambda w d\Pi dV' - \left( \int_{V'} \lambda w \xi dV' \right) \left( \int_{V'} \overline{\Phi} \xi dV' \right) K^{-1}$$

$$= \int_{V'} \left[ \lambda w - \left( \int_{V'} \lambda w \xi dV' \right) \omega \xi K^{-1} \right] d\Pi dV', \tag{27}$$

where $i, j = 1, 2, 3$. It follows from (27) that the stabilized term for the fluid flow state accounting for the nonuniform fluid potential state can be written as

$$Q^s = \lambda w - \left( \int_{V'} \lambda w \xi dV' \right) \omega \xi K^{-1}. \tag{28}$$

Given $d\Phi$, it can be proved that $d\mathcal{Y}_s$ is always positive for a positive value of $\lambda$. From (28) and the original multiphase correspondence principle the stabilized fluid flow state can be written as

$$Q = \omega (q_w - k_u \bar{u}) K^{-1} \xi + \lambda w - \left( \int_{V'} \lambda w \xi dV' \right) \omega K^{-1} \xi. \tag{29}$$

To simplify (29), it is assumed that

$$\lambda = \frac{G K_p}{\omega_0} \omega, \tag{30}$$

where $G$ is a positive constant on the order of 1 as defined earlier and $K_p$ is a micro-conductivity which will be derived in the next section. With (30) and (17), it can be shown that the third term of (29) becomes zero. Thus, Equation (29) can be expressed as

$$Q = \omega \left[ (q_w - k_u \bar{u}) K^{-1} \xi + \frac{G K_p}{\omega_0} R^w \right]. \tag{31}$$

In this study the same value of $G$ is adopted in the stabilized terms of the effective force state and fluid flow state for simplicity.
With (24) and (31), Equations (9) and (11) with stabilization terms respectively can be rewritten as
\[
\int_{\Omega} \left\{ \omega (\ddot{\sigma} - p_{w} \mathbf{1}) \mathbf{F}^{-T} \mathbf{K}^{-1} \dddot{\xi} + \frac{GC}{\omega_{0}} \mathbf{\bar{R}} \right\} \, dV' \\
+ \rho g = \rho \dddot{\mathbf{u}}, \tag{32}
\]

\[
\frac{\phi}{K_{w}} \dddot{p}_{w} + \int_{\Omega} (\omega \ddot{\mathbf{u}} \mathbf{K}^{-1} \dddot{\xi} - \omega \dddot{\mathbf{u}} \mathbf{K}^{-1} \dddot{\xi}') \, dV' + \int_{\Omega} \left\{ \omega \left[ (\mathbf{q}_{w} - k_{w} \dddot{\mathbf{u}}) \mathbf{K}^{-1} \dddot{\xi} + \frac{GC_{p}}{\omega_{0}} \mathbf{\bar{R}}^{w} \right] \\
- \omega \left[ (\mathbf{q}_{w}' - k_{w} \dddot{\mathbf{u}}') \mathbf{K}^{-1} \dddot{\xi}' + \frac{GC_{p}}{\omega_{0}} \mathbf{\bar{R}}^{w}' \right] \right\} \, dV' = 0, \tag{33}
\]

### 2.3 Determination of stabilization parameters

In this part the parameters in the stabilized terms will be derived based on the energy equivalence between PD and classical theory for the solid skeleton and fluid phase. For the solid phase, it is postulated that the stored elastic energy in the solid skeleton from PD equals to the elastic energy from the classical poromechanics theory at the same material point \(x\).

For simplicity, a micro-elastic peridynamic model\(^5\) is adopted to determine the elastic energy in the solid skeleton at material point \(x\). In the micro-elastic material model, the effective pairwise force function \(f\) that material point \(x'\) imposes on material point \(x\) is determined from a micro-potential \(w\) as
\[
f(\eta, \xi) = \frac{\xi + \eta}{|\xi + \eta|} \frac{\partial w(\eta, \xi)}{\partial \eta}, \tag{34}
\]
where \(\eta = |\eta|\) and \(\xi = |\xi|\). The micro-potential function is a measure of the elastic strain energy stored in a single bond of the solid skeleton due to its deformation. The total strain energy density at point \(x\) is expressed as
\[
\mathcal{W} = \frac{1}{2} \int_{\Omega} w(\eta, \xi) \, dV', \tag{35}
\]
where the factor of 1/2 means that each endpoint of a bond between two solid material points owns only half the energy in this bond. Let \(f\) be the magnitude of \(f\) as
\[
f(\eta, \xi) = \frac{\partial w}{\partial \eta}. \tag{36}
\]

We further assume a homogeneous solid skeleton under isotropic extension. It follows
\[
\eta = \mathcal{E}_{1} \xi, \tag{37}
\]
where \(\mathcal{E}_{1}\) is a constant for all \(\xi\). Thus \(f\) can be written as
\[
f = C \mathcal{E}_{1} = C \eta/\xi, \tag{38}
\]
where \(C\) is the constant defined previously. It follows from (36) and (38) that
\[
w = C \eta^{2}/(2 \xi) = C \mathcal{E}_{1}^{2} \xi/2. \tag{39}
\]

Substituting (39) into (35) gives
\[
\mathcal{W} = \frac{1}{2} \int_{\Omega} \frac{1}{2} (C \mathcal{E}_{1}^{2} \xi) \mathbf{d}V' = \frac{1}{2} \int_{0}^{\delta} \frac{1}{2} (C \mathcal{E}_{1}^{2} \xi)(4 \pi \xi^{2}) \, d\xi = \frac{\pi C \delta^{4}}{4} \mathcal{E}_{1}^{2}. \tag{40}
\]
The elastic strain energy of the solid skeleton at material point \( x \) from the classical elastic theory under isotropic deformation is

\[
\mathcal{W} = \frac{1}{2} \mathbf{\varepsilon} : \mathbf{\varepsilon} = \frac{1}{2} K^e (\mathbf{\varepsilon}_v) (\mathbf{\varepsilon}_v) = \frac{1}{2} K^e (3 \mathbf{\varepsilon}_1^v)^2 = \frac{9 K^e}{2} \mathbf{\varepsilon}_1^2, \tag{41}
\]

where \( K^e \) is the classical elastic bulk modulus and \( \mathbf{\varepsilon}_v \) is the elastic volumetric strain. Combining (40) and (41) leads to an expression for \( C \) under three-dimensional condition as

\[
C_{3d} = \frac{18 K^e}{\pi \delta^4}. \tag{42}
\]

For the fluid phase, the pairwise fluid flow density \( f_w \) at material point \( x \) in the bond-based PD can be determined from a fluid dissipation micro-potential \( w' \) through

\[
f_w(x, x') = \frac{\partial w'}{\partial \Phi}(x, x'), \tag{43}
\]

where \( \Phi = p'_w - p_w \) is the fluid pressure difference between material points \( x' \) and \( x \). The dissipation micro-potential \( w' \) represents the dissipation potential along a bond between two material points and is a function of the fluid potential scalar state \( \Phi \) of that bond. Then the total dissipation potential at point \( x \) is a summation over all the micro-potentials in the family of this point.

\[
\mathcal{W}' = \frac{1}{2} \int_{\mathcal{H}} w'(x', x) dV', \tag{44}
\]

where similar to the solid phase the factor of \( 1/2 \) means that each endpoint of a bond between two solid material points owns only half the energy in this bond. The pairwise fluid flow density at \( x \) is assumed as

\[
f_w(x', x) = K_p \frac{\Phi}{|\xi|}, \tag{45}
\]

where \( K_p \) is the peridynamic hydraulic micro-conductivity. It follows from (43) and (45) that fluid flow dissipation micro-potential can be written as

\[
w' = \frac{1}{2} K_p \frac{\Phi^2}{|\xi|}. \tag{46}
\]

The peridynamic hydraulic micro-conductivity can be related to the classical hydraulic conductivity by equating the peridynamic fluid dissipation potential to the classical fluid dissipation potential at point \( x \). For simplicity, we assume a linear pressure field in a body, \( p_w = \mathcal{C}_2 (1 \cdot x) \) for a three-dimensional case. Thus, the fluid pressures at material points \( x \) and \( x' \) are written as

\[
p_w = \mathcal{C}_2 (1 \cdot x) \quad \text{and} \quad p'_w = \mathcal{C}_2 (1 \cdot x'). \tag{47}
\]

It follows that

\[
w' = K_p \frac{1}{2} \mathcal{C}_2^2 \xi. \tag{48}
\]

Substituting (47) into (46) and conducting integration in a sphere with radius \( \delta \) centered at \( x \) lead to

\[
\mathcal{W}' = \frac{1}{2} \int_0^\delta \left( \frac{1}{2} K_p \mathcal{C}_2^2 \xi \right) (4\pi \xi^2) d\xi = \frac{\pi K_p}{4} \delta^4 \mathcal{C}_2^2. \tag{49}
\]

Assuming a homogeneous body and isotropic fluid flow, the classical fluid dissipation energy through Darcy’s law at material point \( x \) is expressed as

\[
\mathcal{W}_c = \frac{1}{2} (\nabla p_w) k_w (\nabla p_w). \tag{50}
\]
where $k_w$ is the hydraulic conductivity of saturated porous media. It follows from (50) and (47) that the classical fluid dissipation energy at material point $x$ is

$$\mathcal{Y} = \frac{3}{2} k_w \mathcal{E}_2^2.$$

(51)

Combining (49) and (51) gives the hydraulic micro-conductivity under three-dimensional condition as

$$K_{p,3d} = \frac{6k_w}{\pi \delta^4}.$$

(52)

It is worth noting that the expressions for $C$ and $K_p$ in (42) and (52), respectively, are derived assuming the simple cases of solid deformation and fluid flow that may limit their general applicability. However, the two formulations are based on the general energy equivalence principle. Therefore, complex cases of solid deformation and fluid flow can be adopted to derive more general expressions for both parameters following the same lines in this section.

### 3 | NUMERICAL IMPLEMENTATION

#### 3.1 | Spatial discretization

The Equations (32) and (33) are spatially discretized by a hybrid Lagrangian–Eulerian meshfree scheme, as shown in Figure 2. In this method, a porous continuum material is discretized into a finite number of mixed material points (i.e., mixed solid skeleton and pore water material points). Each material point has two types of degree of freedom, the displacement and the pore water pressure. The uniform grid is used to spatially discretize the problem domain in which all material points have an identical size.

It is assumed that inertia loading has no impact on the fluid flow and water is incompressible. Let $n_{ij}$ be the number of material points in the horizon of a material point $i$. The spatially discretized equations at material point $i$ can be written as

$$T_i + T^s_i + \rho_i g = \rho_i \ddot{u}_i,$$

(53)

$$\dot{V}_i + Q_i + Q^s_i = 0,$$

(54)

where

$$T_i = \sum_{j=1}^{n_{ij}} \left[ \omega_i \sigma_i \bar{F}^{-T}(i) K_{ij}^{-1} \xi_{ij} - \omega_j \sigma_j \bar{F}^{-T}(j) K^{-1} \xi_{ij} \right] V_j,$$

(55)

$$T^s_i = \sum_{j=1}^{n_{ij}} \frac{GC}{\omega_0} \left[ R_{ij}^s - R_{ji}^s \right] V_j,$$

(56)

$$\dot{V}_i = \sum_{j=1}^{n_{ij}} \left[ \omega_i (\dot{u}_i - \ddot{u}_j) K_{ij}^{-1} \xi_{ij} \right] V_j,$$

(57)

\[ \text{F I G U R E 2} \quad \text{Schematic of a Lagrangian–Eulerian meshless spatial discretization scheme of material point} \ i \ \text{and its neighboring material points (solid points in red and fluid points in blue)} \]
\[ Q_i = \sum_{j=1}^{N} \left[ \omega q_i K^{-1}(i,j) \xi_{(i,j)} - \omega q_i R_{(i,j)} \right] V_j, \]  

(58)

\[ Q^I_i = \sum_{j=1}^{N} \frac{G K_i}{\delta_0} \left[ R^w_{(i,j)} - R^w_{(j,i)} \right] V_j. \]  

(59)

We define the linear assembly operator \( \mathcal{A} \). \(^{54}\) Let \( N \) be the number of the total material points. The global discretization form of the coupled equations can be written as

\[ \mathcal{A}_{i=1}^{N} \left( T_i + T^s_i + \rho \mathbf{g} \right) V_i = \mathcal{A}_{i=1}^{N} \left( \rho \ddot{u}_i \right) V_i, \]  

(60)

\[ \mathcal{A}_{i=1}^{N} \left( \dot{V}_i + Q_i + Q^I_i \right) V_i = \mathcal{A}_{i=1}^{N} \dot{0}, \]  

(61)

### 3.2 Time integration and linearization

We formulate an implicit scheme to integrate the coupled system of equations in time. \(^{54}\) At \( t_{n+1} \), the residual vector of the the coupled system is defined as

\[ r^u_{n+1} = \mathcal{A}_{i=1}^{N} \left( \rho \ddot{u}_i - T_i - T^s_i - \rho \mathbf{g} \right) V_i |_{n+1}, \]  

(62)

\[ r^p_{n+1} = \mathcal{A}_{i=1}^{N} \left( \dot{V}_i + Q_i + Q^I_i \right) V_i |_{n+1}. \]  

(63)

The Newmark method \(^{1,55}\) is adopted. In the temporal domain a second-order scheme is used to integrate the momentum balance equation and a first-order scheme is applied to integrate the mass balance equation. At time step \( n \), \( \mathbf{u}_n, \mathbf{u}, \mathbf{u}, \mathbf{p}_n, \) and \( \mathbf{p} \) are known. Let \( \Delta \mathbf{u}_{n+1} = \mathbf{u}_{n+1} - \mathbf{u}_n \) and \( \Delta \mathbf{p}_{n+1} = \mathbf{p}_{n+1} - \mathbf{p}_n \), the acceleration, velocity, displacement, and water pressure vectors at \( t_{n+1} \) can be written as follows,

\[ \ddot{u}_{n+1} = \ddot{u}_n + \Delta \ddot{u}_{n+1}, \]  

(64)

\[ \ddot{u}_n = \ddot{u}_n + \Delta \ddot{u}_n + \beta_2 \Delta t \Delta \ddot{u}_{n+1}, \]  

(65)

\[ \ddot{u}_n = \ddot{u}_n + \frac{(\Delta t)^2}{2} \ddot{u}_n + \beta_1 (\Delta t)^2 \Delta \ddot{u}_{n+1}, \]  

(66)

\[ \mathbf{p}_{n+1} = \mathbf{p}_n + \Delta \mathbf{p}_n + \beta_3 \Delta t \Delta \mathbf{p}_{n+1}, \]  

(67)

where \( \beta_1, \beta_2, \beta_3 \in [0,1] \) are numerical integration parameters. For unconditional stability,

\[ \beta_1 \geq \beta_2 \geq \frac{1}{2}, \quad \text{and} \quad \beta_3 \geq \frac{1}{2}. \]  

(68)

Substituting (64), (65), (66), and (67) into (62) and (63), \( \Delta \mathbf{u}_{n+1} \) and \( \Delta \mathbf{p}_{n+1} \), can be solved by Newton’s method as follows. Let \( k \) be the iteration number

\[ \left\{ \begin{array}{c} \Delta \mathbf{u}_{n+1}^k \\ \Delta \mathbf{p}_{n+1}^k \end{array} \right\} = \left\{ \begin{array}{c} \mathcal{A}^k \left\{ \Delta \mathbf{u}_{n+1}^{k+1} \right\} \\ \mathcal{A}^k \left\{ \Delta \mathbf{p}_{n+1}^{k+1} \right\} \end{array} \right\} \approx \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}, \]  

(69)

where

\[ \mathcal{A}^k = \left[ \begin{array}{cc} \frac{\partial \mathcal{A} \mathbf{u}}{\partial \mathbf{u}} & \frac{\partial \mathcal{A} \mathbf{p}}{\partial \mathbf{p}} \\ \frac{\partial \mathcal{A} \mathbf{u}}{\partial \mathbf{u}} & \frac{\partial \mathcal{A} \mathbf{p}}{\partial \mathbf{p}} \end{array} \right]_{n+1}^k. \]  

(70)

By solving (69), we have

\[ \left\{ \begin{array}{c} \Delta \mathbf{u}_{n+1}^{k+1} \\ \Delta \mathbf{p}_{n+1}^{k+1} \end{array} \right\} = -\mathcal{A}^{k,-1} \left\{ \begin{array}{c} \mathcal{A}^k \mathbf{u}_{n+1}^k \\ \mathcal{A}^k \mathbf{p}_{n+1}^k \end{array} \right\}. \]  

(71)
Finally, we have

\[
\begin{bmatrix}
\Delta \hat{\mathbf{u}}^{k+1} \\
\Delta \hat{\mathbf{p}}^{k+1}
\end{bmatrix}
= \begin{bmatrix}
\Delta \mathbf{u}^k \\
\Delta \mathbf{p}^k
\end{bmatrix}
+ \begin{bmatrix}
\delta \Delta \hat{\mathbf{u}}^{k+1} \\
\delta \Delta \hat{\mathbf{p}}^{k+1}
\end{bmatrix}.
\] (72)

### 3.3 | Tangent operator

Given the relationships in Equations (64)–(67), by chain rule the individual components of \( \mathbf{A}^k \) can be written as follows.

\[
\begin{align*}
\bar{K}_{uu} &= \frac{\partial \rho^u}{\partial \Delta \mathbf{u}} = \frac{\partial (\rho \mathbf{u})}{\partial \mathbf{u}}, \\
\bar{K}_{up} &= \frac{\partial \rho^u}{\partial \Delta \mathbf{p}} = \rho I_1 + \frac{1}{2} \beta_1 \Delta t^2 \left( \frac{\partial \rho}{\partial \mathbf{u}} (\mathbf{u} - \mathbf{g}) - \frac{\partial \mathbf{T}}{\partial \mathbf{u}} \right), \\
\bar{K}_{pu} &= \frac{\partial \rho^u}{\partial \Delta \mathbf{u}} = -\frac{\partial \mathbf{T}}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \Delta \mathbf{p}} = -\beta_2 \Delta t \frac{\partial \mathbf{T}}{\partial \mathbf{p}}, \\
\bar{K}_{pp} &= \frac{\partial \rho^u}{\partial \Delta \mathbf{p}} = \beta_3 \Delta t \left( \frac{\partial \mathbf{Q}}{\partial \mathbf{p}} \right).
\end{align*}
\] (73–76)

where \( \bar{K}_{uu} \) is the global solid tangent operator, \( \bar{K}_{up} \) and \( \bar{K}_{pu} \) are the global coupling matrices, \( \bar{K}_{pp} \) is the global fluid tangent operator, \( I_1 \) is the second-order identity tensor with the dimension of the number of total material points in the problem domain, and for brevity we define

\[
\begin{align*}
\hat{T} &= \mathbf{T} + \mathbf{T}^s, \\
\hat{Q} &= \mathbf{Q} + \mathbf{Q}^s.
\end{align*}
\] (77–78)

Inspired by the standard procedure in the finite element method, the stiffness matrix here will be constructed from the corresponding local stiffness matrices at material points. In this article, the stiffness matrix at a material point will be first computed and then the assembly operator will be utilized to construct the global stiffness matrix. In what follows, the derivation is focused on the stiffness matrices at one material point incorporating all the material points in its horizon.

We linearize the momentum balance equation at material point \( i \) by following the chain rule. The incremental forms of \( \mathbf{T}_i \) and \( \mathbf{T}^s_i \) at material point \( i \) can be written as

\[
\begin{align*}
\delta \mathbf{T}_i &= \sum_{j=1}^{N_i} \frac{\partial \mathbf{T}_i}{\partial Y_{lj}} \delta \mathbf{u}_j + \sum_{k=1}^{N_i} \frac{\partial \mathbf{T}_i}{\partial \mathbf{p}_l} \delta \mathbf{p}_k, \\
\delta \mathbf{T}_i^s &= \sum_{j=1}^{N_i} \frac{\partial \mathbf{T}_i^s}{\partial Y_{lj}} \delta \mathbf{u}_j + \sum_{k=1}^{N_i} \frac{\partial \mathbf{T}_i^s}{\partial \mathbf{p}_l} \delta \mathbf{p}_k.
\end{align*}
\] (79–80)

where \( Y_{lj} = y_j - y_l \).

It follows from (79) and (80) that we have

\[
\begin{align*}
\frac{\partial \mathbf{T}_i}{\partial \mathbf{u}_i} &= \sum_{j=1}^{N_i} \frac{\partial \mathbf{T}_i}{\partial Y_{lj}} \frac{\partial Y_{lj}}{\partial \mathbf{u}_i}, \\
\frac{\partial \mathbf{T}_i}{\partial \mathbf{u}_i} &= \sum_{j=1}^{N_i} \frac{\partial \mathbf{T}_i}{\partial Y_{lj}} \frac{\partial Y_{lj}}{\partial \mathbf{u}_i} = \sum_{j=1}^{N_i} \left[ \sum_{j=1}^{N_i} \frac{\partial \mathbf{T}_i}{\partial Y_{lj}} (\mathbf{T}_{lj} - \mathbf{T}_{lj}) V_j \right] \frac{\partial Y_{lj}}{\partial \mathbf{u}_i},
\end{align*}
\] (81–84)
where $\partial T_i / \partial Y^j$ and $\partial T_i / \partial Y^j$ can be determined from a peridynamic material model (e.g., ordinary or nonordinary). For brevity of notations, let us define

$$\frac{\partial T_i}{\partial u_i} = \frac{\partial T_i}{\partial u_i} + \frac{\partial T_i}{\partial u_i},$$  

(85)

$$\frac{\partial T_i}{\partial u_i} = \frac{\partial T_i}{\partial u_i} + \frac{\partial T_i}{\partial u_i},$$  

(86)

For the dynamic loading term, $\rho \dot{u}$, we define,

$$M_i = \rho_1 \frac{\beta_1 \Delta t^2}{2} \left( \frac{\partial \rho_i}{\partial u_i} (\dot{u}_i - g) \right),$$  

(87)

$$M_i = \frac{\beta_1 \Delta t^2}{2} \left( \frac{\partial \rho_i}{\partial u_i} (\dot{u}_i - g) \right),$$  

(88)

where

$$\frac{\partial \rho_i}{\partial u_i} = \rho_i (-\frac{\partial \phi_i}{\partial u_i}) + \rho_u \frac{\partial \phi_i}{\partial u_i},$$  

(89)

$$\frac{\partial \rho_i}{\partial u_i} = \rho_i (-\frac{\partial \phi_i}{\partial u_i}) + \rho_u \frac{\partial \phi_i}{\partial u_i}.$$  

(90)

Therefore, from (81) to (88), the solid tangent matrix $\overrightarrow{K_{uu}}$ at material point $i$ can be constructed as

$$\overrightarrow{K_{uu}} = \left[ M_i - \frac{\beta_1 \Delta t^2}{2} \frac{\partial T_i}{\partial u_i} \right] \left[ M_i - \frac{\beta_1 \Delta t^2}{2} \frac{\partial T_i}{\partial u_i} \right] \ldots \left[ M_i - \frac{\beta_1 \Delta t^2}{2} \frac{\partial T_i}{\partial u_i} \right],$$  

(91)

where $\overrightarrow{K_{uu}}$ is a matrix with dimensions of $3 \times 3(N_i + 1)$.

Similarly, it follows from (79) that we have

$$\frac{\partial T_i}{\partial p_i} = \sum_{j=1}^{N_i} \frac{\partial T_j}{\partial p_j} V_j,$$  

(92)

$$\frac{\partial T_i}{\partial p_j} = -\frac{\partial T_j}{\partial p_j} V_j.$$  

(93)

Thus, the coupling matrix $\overrightarrow{K_{up}}$ at material point $i$ can be constructed as

$$\overrightarrow{K_{up}} = -\beta_3 \Delta t \left[ \frac{\partial T_i}{\partial p_i} \frac{\partial T_i}{\partial p_j} \ldots \frac{\partial T_i}{\partial p_{N_i}} \right],$$  

(94)

where $\overrightarrow{K_{up}}$ is a matrix with dimensions of $3 \times (N_i + 1)$.

Next, we derive the tangent matrices associated with the mass balance equation at material point $i$. The linearization of $Q_i$ and $Q_i'$ at point $i$ can be written as

$$\delta Q_i = \sum_{i=1}^{N_i} \frac{\partial Q_i}{\partial p_i} \frac{\partial \Phi_i}{\partial p_i} \delta p_i + \sum_{i=1}^{N_i} \frac{\partial Q_i}{\partial p_i} \frac{\partial \Phi_i}{\partial p_i} \delta p_i.$$  

(95)

$$\delta Q_i' = \sum_{i=1}^{N_i} \frac{\partial Q_i'}{\partial p_i} \frac{\partial \Phi_i}{\partial p_i} \delta p_i + \sum_{i=1}^{N_i} \frac{\partial Q_i'}{\partial p_i} \frac{\partial \Phi_i}{\partial p_i} \delta p_i.$$  

(96)

where $\Phi_i = p_i - p_i$. From (95) and (96) we obtain

$$\frac{\partial Q_i}{\partial p_i} = \sum_{i=1}^{N_i} \frac{\partial Q_i}{\partial p_i} \frac{\partial \Phi_i}{\partial p_i} = \sum_{i=1}^{N_i} \left[ \sum_{i=1}^{N_i} \frac{\partial (\Phi_i - Q_i')}{\partial p_i} V_j \right] \frac{\partial \Phi_i}{\partial p_i},$$  

(97)
\[
\frac{\partial Q_i}{\partial p_i} = \frac{\partial Q_i}{\partial \Phi_{il}} \frac{\partial \Phi_{il}}{\partial p_i} = \left[ \sum_{j=1}^{N_i} \frac{\partial}{\partial \Phi_{il}} (Q_{ij} - Q_i) V_j \right] \frac{\partial \Phi_{il}}{\partial p_i},
\]
(98)
\[
\frac{\partial Q_i^s}{\partial p_i} = \sum_{i=1}^{N_i} \frac{\partial Q_i^s}{\partial \Phi_{il}} \frac{\partial \Phi_{il}}{\partial p_i} = \sum_{i=1}^{N_i} \left[ \sum_{j=1}^{N_i} G K_p \frac{\partial}{\partial \Phi_{il}} (R_{ij}^w - R_{ji}^w) V_j \right] \frac{\partial \Phi_{il}}{\partial p_i},
\]
(99)
\[
\frac{\partial Q_i^s}{\partial p_i} = \frac{\partial Q_i^s}{\partial \Phi_{il}} \frac{\partial \Phi_{il}}{\partial p_i} = \left[ \sum_{j=1}^{N_i} G K_p \frac{\partial}{\partial \Phi_{il}} (R_{ij}^w - R_{ji}^w) V_j \right] \frac{\partial \Phi_{il}}{\partial p_i}.
\]
(100)

For brevity of notations, let us define
\[
\frac{\partial Q_i}{\partial p_i} = \frac{\partial Q_i}{\partial p_i} + \frac{\partial Q_i^s}{\partial p_i},
\]
(101)
\[
\frac{\partial \dot{Q}_i}{\partial p_i} = \frac{\partial Q_i}{\partial p_i} + \frac{\partial Q_i^s}{\partial p_i}.
\]
(102)

It follows from (101) and (102) that the flow tangent matrix \( \bar{K}_{pp} \) at point \( i \) can be written as
\[
\bar{K}_{pp}^i = \beta_3 \Delta t \left[ \frac{\partial \dot{Q}_i}{\partial p_i} \frac{\partial \dot{Q}_i}{\partial p_i} \ldots \frac{\partial \dot{Q}_i}{\partial p_i} \right],
\]
(103)

where \( \bar{K}_{pp} \) is a row vector with dimensions of \((N_i + 1) \times 1\). For the rate of the solid volume change, it can be readily derived by chain rule that
\[
\frac{\partial V_i}{\partial u_i} = - \sum_{j=1}^{N_i} \left[ \alpha_1 K_p^{-1} \xi_{(ij)} \right] V_j,
\]
(104)
\[
\frac{\partial V_i}{\partial u_j} = \left[ \alpha_1 K_p^{-1} \xi_{(ij)} \right] V_j.
\]
(105)

Thus, the coupling tangent matrix \( \bar{K}_{pu} \) at material point \( i \) can be written as
\[
\bar{K}_{pu}^i = \beta_2 \Delta t \left[ \frac{\partial V_i}{\partial u_i} \frac{\partial V_i}{\partial u_1} \ldots \frac{\partial V_i}{\partial u_{N_i}} \right],
\]
(106)

where \( \bar{K}_{pu} \) is a vector with dimensions of \( 1 \times 3(N_i + 1) \).

Finally, the global tangent matrix can be constructed from (91), (94), (106), and (103) for all material points.
\[
\bar{K}_{uu} = \mathcal{A} \bar{K}_{uu}^i,
\]
(107)
\[
\bar{K}_{up} = \mathcal{A} \bar{K}_{up}^i,
\]
(108)
\[
\bar{K}_{pu} = \mathcal{A} \bar{K}_{pu}^i,
\]
(109)
\[
\bar{K}_{pp} = \mathcal{A} \bar{K}_{pp}^i,
\]
(110)

where \( \mathcal{A} \) is the assembly operator that constructs the global stiffness matrix from the tangent matrices at individual point. In the computer code, each material point and the material points in its horizon are assigned unique global identification (ID) numbers that are used by \( \mathcal{A} \) to construct the global tangent matrix. Indeed the assembly procedure adopted here is exactly the same as the global stiffness assembly operator in the finite element method. Algorithm 1 summarizes the general procedure for the computation of local tangent matrices at the material point level and the assembly of the global tangent matrix. The global tangent matrix \( \mathcal{A} \) is not symmetric in general. A nonsymmetric solver is used for the final system of equations. Note that a symmetrization scheme can be applied to symmetrize \( \mathcal{A} \). The computer program is written in C++. Parallel computing through Open MPI is exploited to increase the computational efficiency of the stabilized coupled periporomechanics model.
**Algorithm 1.** Compute local tangent matrices and assemble the global tangent matrix

1: procedure ASSEMBLE TANGENT MATRIX
2: Allocate and initialize the global tangent matrix.
3: for node $i \in \mathcal{P}$ do
4: for neighbor $j \in \mathcal{N}_i$ do
5: Compute $\frac{\partial \mathbf{T}_i}{\partial \mathbf{u}_i}$ and $\frac{\partial \mathbf{T}_i}{\partial \mathbf{u}_l}$ using (81) and (83).
6: Compute $\frac{\partial Q_i}{\partial p_i}$ and $\frac{\partial Q^i}{\partial p_l}$ using (97) and (99).
7: Compute $\frac{\partial \mathbf{F}_i}{\partial p_l}$ using (92).
8: Compute $\frac{\partial \mathbf{V}_i}{\partial \mathbf{u}_l}$ using (104).
9: Compute $\frac{\partial \mathbf{M}_i}{\partial p_l}$ using (88).
10: for neighbor $l \in \mathcal{N}_i$ do
11: Compute $\frac{\partial \mathbf{T}_i}{\partial \mathbf{u}_l}$ and $\frac{\partial \mathbf{T}_i}{\partial \mathbf{u}_l}$ using (82) and (84).
12: if $j == l$ then
13: Compute $\frac{\partial \mathbf{F}_i}{\partial p_l}$ using (93).
14: end if
15: Compute $\frac{\partial \mathbf{Q}_i}{\partial p_l}$ and $\frac{\partial \mathbf{Q}_i}{\partial p_l}$ using (98) and (100).
16: end for
17: end for
18: Compute $\mathbf{M}_i$ using (87).
19: Construct $\mathbf{K}_{i:uu}$, $\mathbf{K}_{i:up}$, $\mathbf{K}_{i:pu}$, and $\mathbf{K}_{i:pp}$.
20: end for
21: Assemble the global tangent matrix using (107)–(110).
22: end procedure

### 3.4 Linearization of correspondence material models

We present the material-point level linearization of correspondence solid and fluid models implemented in this study. First, the variation of the total force density at a material point $i$ with respect to a deformation state $\mathbf{Y}_{il}$ reads,

$$
\frac{\partial \mathbf{T}_i}{\partial \mathbf{Y}_{il}} = \sum_{j=1}^{N_i} \frac{\partial}{\partial \mathbf{Y}_{il}} (\mathbf{T}_{ij} - \mathbf{T}_{il}) \mathbf{V}_j = \sum_{j=1}^{N_i} \left[ \omega \left( \left\{ \frac{\partial \mathbf{F}_i(0)}{\partial \mathbf{Y}_l(0)} \delta \xi_{il} \mathbf{V}_l \right\} (\mathbf{K}_i(0))^{-1} \xi_{lj} \right) - \omega \left( \left\{ \frac{\partial \mathbf{F}_i(0)}{\partial \mathbf{Y}_l(0)} \delta \xi_{lj} \mathbf{V}_l \right\} (\mathbf{K}_i(0))^{-1} \xi_{il} \right) \right] \mathbf{V}_j, \tag{115}
$$

where $\frac{\partial \mathbf{F}_i(0)}{\partial \mathbf{Y}_l(0)}$ can be determined through the return mapping algorithm following the lines in computational plasticity (e.g., References 57-59, and others)

$$
\frac{\partial \mathbf{F}_i(0)}{\partial \mathbf{Y}_l(0)} = \omega \left[ \frac{\partial \mathbf{Y}_l(0)}{\partial \mathbf{Y}_l(0)} \otimes \xi_{il} \mathbf{V}_l \right] (\mathbf{K}_i(0))^{-1} = \omega [\mathbf{1} \otimes \xi_{il} \mathbf{V}_l] (\mathbf{K}_i(0))^{-1}, \tag{112}
$$

$$
\frac{\partial \mathbf{F}_i(0)}{\partial \mathbf{Y}_l(0)} = \omega \left[ \frac{\partial \mathbf{Y}_l(0)}{\partial \mathbf{Y}_l(0)} \otimes \xi_{lj} \mathbf{V}_l \right] (\mathbf{K}_i(0))^{-1} = \omega [-\mathbf{1} \otimes \xi_{lj} \mathbf{V}_l] (\mathbf{K}_i(0))^{-1}. \tag{113}
$$

The variation of the stabilization force state with respect to a deformation state $\mathbf{Y}_{il}$ reads,

$$
\frac{\partial \mathbf{T}_i}{\partial \mathbf{Y}_{il}} = \sum_{j=1}^{N_i} \omega \left[ \frac{\partial \mathbf{T}_i}{\partial \mathbf{Y}_{il}} \right] V_j, \tag{114}
$$
where $\frac{\partial \mathbf{\sigma}}{\partial \mathbf{x}}$ and $\frac{\partial \mathbf{\sigma}}{\partial \mathbf{y}}$ is nonzero only if $j = l$. Next, the variation of the momentum balance with respect to the fluid pressure reads,

$$\frac{\partial \mathbf{T}_i}{\partial p_i} = \sum_{j=1}^{J} \frac{\partial \mathbf{T}_{ij}}{\partial p_j} = \sum_{j=1}^{J} \omega(K_{ij})^{-1}\xi_{ij}V_j,$$

(118)

$$\frac{\partial \mathbf{T}_i}{\partial p_j} = -\frac{\partial \mathbf{T}_{ji}}{\partial p_j} = -\omega(K_{ji})^{-1}\xi_{ji}V_j.$$  

(105)

Similarly, for the mass balance, the variation of fluid flow density with respect to a pressure potential state reads,

$$\frac{\partial \mathbf{Q}_i}{\partial \Phi_{il}} = \sum_{j=1}^{J} \frac{\partial \mathbf{Q}_{il}}{\partial \Phi_{il}}(Q_{ij} - Q_{ji})V_j$$

$$= \sum_{j=1}^{J} \left[ \omega \left( \frac{\partial \mathbf{Q}_{iw}}{\partial \Phi_{il}} \frac{\partial \Phi_{il}}{\partial \Phi_{ij}} \right) (K_{ij})^{-1}\xi_{ij} - \omega \left( \frac{\partial \mathbf{Q}_{iw}}{\partial \Phi_{lj}} \frac{\partial \Phi_{lj}}{\partial \Phi_{il}} \right) (K_{ji})^{-1}\xi_{ji} \right] \xi_{il} V_j,$$

(117)

where

$$\frac{\partial \Phi_{il}}{\partial \Phi_{il}} = \left[ \omega \frac{\partial \Phi_{il}}{\partial \Phi_{il}} \xi_{il} V_l \right] (K_{ij})^{-1} = \omega \xi_{il} (K_{ij})^{-1} V_j,$$

(118)

$$\frac{\partial \Phi_{il}}{\partial \Phi_{il}} = \left[ \omega \frac{\partial \Phi_{il}}{\partial \Phi_{il}} \xi_{il} V_l \right] (K_{ji})^{-1} = -\omega \xi_{il} (K_{ji})^{-1} V_i.$$  

(119)

The variation of the stabilization flow density with respect to a pressure potential state $\Phi_{il}$ reads

$$\frac{\partial \mathbf{Q}_{il}}{\partial \Phi_{il}} = \sum_{j=1}^{J} \left[ \frac{G K_p}{\omega_{\theta_0}} \left( \frac{\partial \Phi_{il}}{\partial \Phi_{ij}} - \left( \frac{\partial \Phi_{il}}{\partial \Phi_{ij}} \xi_{ij} \right) \right) - \omega \frac{G K_p}{\omega_{\theta_0}} \left( \frac{\partial \Phi_{il}}{\partial \Phi_{il}} - \left( \frac{\partial \Phi_{il}}{\partial \Phi_{il}} \xi_{il} \right) \right) \right] V_j,$$

(120)

where $\frac{\partial \Phi_{il}}{\partial \Phi_{il}}$ and $\frac{\partial \Phi_{il}}{\partial \Phi_{il}}$ are nonzero only if $j = l$.

## 4 | NUMERICAL EXAMPLES

For the numerical examples in this section we adopt the time integration parameters in the literature that satisfy the criteria for unconditional stability and high frequency numerical dissipation.\(^1\,^2\) For all three examples in what follows we have $\beta_1 = 0.605$ and $\beta_2 = \beta_3 = 0.6$. All boundary conditions are imposed through a fictitious boundary layer (see References 29,32,40).

### 4.1 | Correspondence material models

For the material models, we implement two classical material models for the solid skeleton and the classical Darcy’s law for fluid flow through the extended correspondence principle with stabilization. Following the lines in continuum mechanics,\(^3\)\(^7\) the total strain $\mathbf{\varepsilon}$ can be determined from the nonlocal deformation gradient at a material point. In examples 1 and 2, the solid skeleton is modeled by an isotropic elastic constitutive relationship, which reads

$$\frac{\partial \mathbf{\Sigma}_{ij}}{\partial \mathbf{x}} = C_{ijkl}^e \mathbf{\varepsilon}_{kl} \varepsilon_{ij}^e,$$

(121)

where $i,j,k,l = 1,2,3$, $C_{ijkl}^e$ is the isotropic elastic tensor, and $\mathbf{\varepsilon}_{ij}^e$ is the elastic strain tensor. The isotropic elastic tensor is written as

$$C_{ijkl}^e = K^e \delta_{ij} \delta_{kl} + 2\mu^e (I_{ijkl} - 1/3 \delta_{ij} \delta_{kl}),$$

(122)

where $\mu^e$ is the shear modulus and $I_{ijkl}$ is the rank-four identity tensor.
In example 3, the critical-state-based elastoplastic model for saturated soils is adopted and numerically implemented through the celebrated return-mapping algorithm in computational plasticity. In this material model, the total strain tensor is additively decomposed into the elastic strain tensor $\varepsilon^e$ and the plastic strain tensor $\varepsilon^p$ as

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij},$$

(123)

Given the elastic strain tensor, the effective stress can be computed by Equation (121). The yield function $f$ is written as

$$f(\bar{p}, q, p_c) = (\bar{p} - p_c)\bar{p} + \frac{q}{M}^2 \leq 0,$$

(124)

where $\bar{p}$ is the effective mean stress, $q$ is the equivalent shear stress, $M$ is the slope of the critical state line, and $p_c$ is the preconsolidation pressure. As a hardening law, $p_c$ evolves with plastic volumetric strain,

$$\dot{p}_c = \frac{p_c}{\lambda - \kappa} (\dot{\varepsilon}^p_{11} + \dot{\varepsilon}^p_{22} + \dot{\varepsilon}^p_{33}),$$

(125)

where $\lambda$ and $\kappa$ are the compression index and the swelling index, respectively. The plastic strain is determined below by assuming the associative flow rule,

$$\dot{\varepsilon}^p_{ij} = \dot{\gamma} \frac{\partial f}{\partial \sigma_{ij}},$$

(126)

where $\dot{\gamma}$ is the nonnegative plastic multiplier. It is noted that advanced constitutive models for geomaterials can be applied in the formulated periporomechanics in this article by the recently proposed multiphase correspondence principle (see Reference 30).

4.2 One-dimensional dynamic consolidation problem

This example concerns the one-dimensional dynamic consolidation of a saturated soil specimen under instantaneous and sinusoidal loading conditions, respectively. Numerical results from the coupled PD model are compared with the analytical and finite element (FE) solutions in the literature.

Figure 3 depicts the geometry of the soil column. The load is imposed on the top boundary. The soil column is restricted to deform vertically while the bottom is fixed. For the fluid phase, the top boundary is drained and all other boundaries are impervious. The initial effective stress and water pressure are set zero as assumed in. The problem domain is discretized into 25,000 mixed material points. The center-to-center distance of two neighboring material points is $\Delta x = 0.04$ m.
4.2.1 Instantaneous load

Figure 4 plots the instantaneous load imposed on the top of the specimen. The simulation time $t = 0.3$ s and the time increment $\Delta t = 1 \times 10^{-4}$ s. The solid skeleton is modeled by a correspondence elastic model. The fluid flow is modeled by the correspondence Darcy’s law. The material parameters adopted from Reference 63 are: bulk modulus $K = 2.1 \times 10^5$ kPa, shear modulus $\mu_s = 9.8 \times 10^4$ kPa, $\rho_s = 1884$ kg/m$^3$, initial porosity $\phi_0 = 0.48$, $\rho_w = 1000$ kg/m$^3$, $k_w = 3.55 \times 10^{-5}$ m/s. The horizon $\delta = 2.05 \Delta x$. Note that in all numerical examples the porosity is updated through the following equation in classical poromechanics (e.g., References 58, 65),

$$\phi = 1 - (1 - \phi_0)/J.$$ (127)

Figure 5 plots the coupled PD solutions with different values of $G$ parameter and analytical solution of the vertical displacement at A (shown in Figure 3). Figure 6 compares the PD solutions with different values of $G$ parameter and analytical solution of the water pressure at B (shown in Figure 4).

It is shown from Figure 5 that the value of $G$ can have a significant effect on the predicted amplitude of the oscillations of the vertical displacement at point A. For $G = 0$, the PD results of vertical displacement are in good agreement with the analytical solution early in the simulation, with the triangular waveform of the analytical solution being largely preserved. These oscillations in amplitude diminish over time due to fluid viscous dissipation, and will eventually go to zero. However, the oscillations in the PD solution appear to dissipate more rapidly than the analytical solution. With $G > 0$, the PD solution approaches the analytical one, with $G = 1.0$ giving the best results. It is also apparent that increasing the value of $G > 1$, appears to have a negligible influence on the PD solutions, with practically no difference in the results obtained for $G = 1$ and $G = 2$.

The water pressure response at B as shown in Figure 6 takes the form of periodic square waves of constant amplitude. For $G = 0$, the PD solution has noticeable high-frequency oscillations at the peak and valley of the square waves. It is apparent that $G = 0.1$ almost eliminates high-frequency oscillations in the plot of water pressure. Increasing the value of $G$ above 0.1 makes no difference in the results.

![Graphical depiction of the instantaneous loading profile](image1)

![Comparison of the PD solution and analytical solution of the vertical displacement at point A](image2)
$G$ to 1 or 2 does not improve the accuracy of the PD solution with respect to the analytical one. Indeed, a value of $G = 2$ may decrease the period of the PD wave shown in the analytical wave profile.

4.2.2 Sinusoidal load

The sinusoidal loading profile as shown in Figure 7 is expressed as

$$f(t) = 160 \left[1 - \cos(\omega t)\right],$$  \hspace{1cm} (128)

where the angular frequency $\omega = 20\pi \text{ rad/s}$. The total simulation time $t = 0.5 \text{ s}$ and the time increment $\Delta t = 1 \times 10^{-3} \text{ s}$. The solid skeleton is modeled by a correspondence elastic model and the fluid flow is modeled by the correspondence Darcy’s law. The input material parameters adopted from Reference 66 are: $K = 1.22 \times 10^4 \text{ kPa}$, $\mu_s = 5.62 \times 10^3 \text{ kPa}$, $\rho_s = 2000 \text{ kg/m}^3$, $\phi_0 = 0.33$, $\rho_w = 1000 \text{ kg/m}^3$, and $k_w = 1.0 \times 10^{-2} \text{ m/s}$. The horizon $\delta = 2.05\Delta x$. The initial and boundary conditions are identical to the instantaneous loading scenario.

Figure 8 plots the vertical displacement at the point A in Figure 3 over time from the PD solution and the analytical and FE solutions. Figure 9 plots the water pressure at the point B in Figure 3 over time from the PD solution and the FE solution.

The results in Figure 8 show that the PD solution generally matches the analytical and FE solutions of the vertical displacement at point A. It is found that the value of $G$ slightly affects the PD solution in this case. With $G = 0$, the PD solution of the displacement is close to the analytical solution at the peaks of the displacement profile. However, the PD solution diverges from the analytical solution at the valleys of the displacement profile. With $G > 0$, the PD solution generated a uniform decrease in the amplitude of the oscillations of the displacement profile. Figure 9 shows that the PD solution with of the water pressure with $G = 1.0$ is close to the FE solution. There is no noticeable change in the PD solution with $G = 2.0$. 

![Comparison of the PD solution and analytical solution of the water pressure at B](image1)

![Graphical depiction of the sinusoidal loading profile](image2)
FIGURE 8 Comparison of the PD solution and analytical and FE solutions of the vertical displacement at point A

FIGURE 9 Comparison of the PD solution and analytical and FE solutions of the water pressure at point B

The influence of the stabilization parameter on the solid skeleton and pore fluid discussed in this example is consistent with the results obtained from the quasi-static analysis of the solid material. In the dynamic analysis of single-phase solids in Reference 45, it was suggested the value of $G$ should be in the order of 1 for high strain rates and large deformation. Note that the results presented here are preliminary. It is necessary to study some other cases of loading to reach a more robust conclusion.

4.3 Wave propagation in saturated soil

This example deals with a two-dimensional saturated soil specimen under dynamic loading through a strip footing. The numerical results from PD are compared with the FE solutions in the literature.

Figure 10 depicts the problem geometry and the boundary conditions. For the fluid phase, the fluid pressure at the top boundary is set to zero. The remaining boundaries are impermeable. The solid skeleton is modeled by a correspondence elastic model and the fluid flow is modeled by the correspondence Darcy’s law. The problem domain is discretized into 100,000 uniform mixed material points with $\Delta x = 0.05$ m. $\delta = 2.05\Delta x$. The input material parameters adopted from References 66-68 are: $K = 1.22 \times 10^4$ kPa, $\mu_s = 5.62 \times 10^3$ kPa, $\rho_s = 2000$ kg/m$^3$, $\phi_0 = 0.33$, $\rho_w = 1000$ kg/m$^3$, and $k_w = 1 \times 10^{-2}$ m/s.

As in References 66-68, the problem domain is prescribed with null initial effective stress and water pressure. The dynamic load $f(t)$ is imposed on a strip footing on the top surface. As shown in Figure 11

$$f(t) = 2500 \sin(25\pi t) H,$$

where $H$ is equal to 1 if $t \leq 0.04$ s, and is zero if $t > 0.04$ s. The simulation time $t = 0.2$ s and $\Delta t = 5 \times 10^{-4}$ s.
Figure 12 compares the PD solutions with different values of $G$ and FE solutions\textsuperscript{66} of the water pressure at the point A. With $G = 0$, the PD solution of the water pressure at point A slightly lags behind the FE solution and there are some oscillations at $t > 0.5$ s. The plot of water pressure with $G = 0.1$ is smooth and closely matches the FE solution. The PD solutions with larger values of $G$ seem to generate mild changes in the plot of water pressure. Figure 13 compares the PD solutions with the FE solutions of the vertical displacement (heave) versus the horizontal displacement at point B. The results show the expected elliptical motion at point A under the dynamic load on the top boundary.

Figure 14 plots the contours of the magnitude of displacement in the problem domain from the PD solution and the FE solution at $t = 0.05, 0.1, 0.15,$ and $0.2$ s, respectively. In the PD solution, it is assumed $G = 0.05$.

The results in Figure 14 demonstrate that the PD solution is consistent with the FE solution in Reference 66. Both the PD and FE solutions show two-dimensional wave propagation through the bulk of the poroelastic medium with the surface showing elliptic motion by the dynamic load. The amplitude of this wave decreases as it travels farther away from the strip footing. Given the spike load profile, the deformation energy slowly dissipates as the wave moves through the problem domain, which may reduce the amplitude.
We note that the coupled poromechanics model is a strong nonlocal formulation, while the analytical and FE solutions are based on a local theory assumption. The difference between the numerical solution of the stabilized formulation and analytical and FE solutions can be caused by the nonlocal and local assumptions and the stabilization scheme. Furthermore, the wave dispersion behavior of state-based PD as a strong nonlocal method for dynamic problems\textsuperscript{69} may be another reason that deserves further investigation.

### 4.4 Dynamic strain localization in a two-dimensional soil specimen

In this example we simulate dynamic strain localization in a two-dimensional saturated soil sample subjected to a vertical compression. The problem geometry adopted and boundary conditions applied are shown in Figure 15.
The problem domain is discretized into 20,000 mixed material points with $\Delta x = 0.3$ m and volume 0.027 m$^3$. The material parameters used are: $\rho_s = 2000$ kg/m$^3$, $K = 2.5 \times 10^4$ kPa, $\mu_s = 1.154 \times 10^4$ kPa, $\phi_0 = 0.3$, initial preconsolidation pressure $p_{c0} = -250.0$ kPa, $\kappa = 0.03$, $\lambda = 0.10$, $M = 1.0$, $\rho_w = 1000$ kg/m$^3$, $k_w = 3 \times 10^{-5}$ m/s. The horizon is set to $2.05 \Delta x$. The stabilization parameter $G = 0.025$ is chosen to avoid excessive influence on the postlocalization behavior.

For the initial state, the problem domain is assigned an isotropic effective stress $-100$ kPa and zero water pressure. The solid boundary conditions are depicted in Figure 15, where $\sigma_0$ is the lateral confining pressure of 100 kPa. For the fluid phase, all boundaries are impermeable. The velocity load as shown in Figure 16 is imposed on the top boundary. The simulation time $t = 3.5$ s and the time increment $\Delta t = 5 \times 10^{-3}$ s.

Figures 17, 18, and 19 draw the contours of the equivalent shear strain, plastic volume strain, and water pressure at $t = 1.0$, 2.0, and 3.0 s, respectively. The corresponding displacements on the top boundary are $u_y = 0.225$, 0.52, 0.83 m, respectively. Here the equivalent shear strain is the second invariant of the strain tensor $\epsilon$, that is, $\epsilon_s = \sqrt{\frac{2}{3} | \epsilon - \frac{1}{3} \text{tr}(\epsilon)I |}$.

The results show that the deformation and pressure have localized into symmetric banded zones. At $t = 1.0$ s, plastic deformation has initiated and propagated diagonally through the specimen (Figures 17 and 18(A)). We observe similar behavior in the fluid pressure field (Figure 19(A), where localization manifests as regions of decreased water pressure. Figure 17(B), (C) shows that plastic deformation progressively resolves into sharply defined zones of intense shear deformation. However, in the contours of plastic volume strain (see Figure 18(B), (C)) and water pressure (see Figure 19(B), (C)) the banded zones appear to be more diffusive. The plastic volume change in the banded deformation is positive.
denoting dilatation. As such, the increase in skeleton volume leads to a decrease in the water pressure inside the banded zone. The dilatation under dynamic loading can be expected for a moderately over-consolidated soil specimen. Due to the relatively large permeability chosen, the pore water can readily move into the zones of plastic dilatation, leading to more diffuse zones of water pressure.

The second-order work principle for saturated porous media\textsuperscript{70,71} is checked to see the extension of the shear bands in what follows. At the material point level, the second-order work $\mathcal{W}_2$ can be defined as

$$\mathcal{W}_2 = d\sigma : d\varepsilon.$$ (130)

Note that the second-order work in this study is a nonlocal variable. The contour of the second-order work at $t = 2$ s is presented in Figure 20(C). The results show that two conjugate bands have formed in the contour of the second-order work within which the value of $\mathcal{W}_2$ is negative. For comparison purposes, the contours of the equivalent plastic shear strain and plastic volume strain at $t = 2$ s are plotted in Figure 20(A) and (B), respectively. Comparison of Figure 20(A),(C) shows that the width of the bands of the second-order work is similar to that of the equivalent plastic shear strain in Figure 20(A). The bands of the plastic volume strain in Figure 20(B) seem much wider than the bands of the second-order work. This finding may imply that the second-order work can be useful to characterize the width of the shear bands.

### 4.4.1 Sensitivity to spatial discretization

In what follows, we present a discretization sensitivity analysis to demonstrate that the dynamic strain localization problem remains well-posed through the proposed nonlocal formulation. We rerun the numerical simulation with a fine spatial discretization. The fine spatial discretization consists of 43,000 mixed material points with $\Delta x = 0.2$ m. For comparison, all material parameters and conditions remain the same. Figures 21 and 22 compare the equivalent shear strain and water pressure, respectively, from the simulations with the coarse and fine discretizations at $u_y = 0.83$ m. The results in Figures 21 show that the contours of the equivalent shear strain from both simulations are almost identical. The same conclusion can be drawn from Figure 22 regarding the water pressure. It can be concluded that both the location and orientation of the shear band are independent of the spatial discretization.

In Figures 23 and 24 we plot the variation of equivalent plastic shear strain and water pressure along a horizontal line at 10 m above the specimen center at $t = 3.0$ s. The values of equivalent shear strain and water pressure are identical for both discretizations. It is known that in the dynamic strain localization analysis by the finite element method the mesh dependence of plastic strain and water pressure in the banded zone can be resolved by using a viscoplasticity model.\textsuperscript{10,18} However, the width of the banded zone still showed some sensitivity to the spatial discretization scheme (i.e., element size).

![Figure 20](image1.png)

**Figure 20** Contours of (A) the equivalent plastic shear strain, (B) the plastic volume strain, and (C) the second-order work at $t = 2.0$ s (magnification $= \times 2$)

![Figure 21](image2.png)

**Figure 21** Contours of equivalent shear strain ($\varepsilon_{xy}$) at $u_y = 0.83$ m for (A) coarse discretization and (B) fine discretization (magnification $= \times 2$)
4.4.2 | Influence of dynamic loading

We investigate the influence of dynamic loading rates on the coupled response during dynamic strain localization. The base simulation with $u_y = 0.3 \text{ m/s}$ is repeated with $u_y = 0.9 \text{ m/s}$ and $u_y = 1.5 \text{ m/s}$, respectively. All other material parameters and conditions remain the same for comparison. The results are compared at an identical displacement of the top boundary. Figures 25, 26 and 27 show the contours of the equivalent shear strain, plastic volume strain, and water pressure, respectively at $u_y = 0.5 \text{ m}$ on the top boundary for three different dynamic loading rates.

The results in these figures show that the dynamic loading rate impacts the formation of banded deformation and pressure zones. For $u_y = 0.3 \text{ m/s}$, it is apparent that the contour of plastic deformation as shown in Figures 25 and 26(A) and the contour of water pressure as shown in Figure 27(A) have localized into a single pair of two conjugate banded zones. For the simulations with two larger loading rates, $u_y = 0.9$ and $1.5 \text{ m/s}$, two distinct pairs of localized deformation
bands are formed in the solid skeleton as shown in Figures 25 and 26(B). Consistent with the skeleton deformation, the contour of water pressure also shows two pairs of banded zones as demonstrated in Figure 27(B). It can be concluded from the results in Figures 25, 26, and 27 that the dynamic loading rate impacts the number, location, and orientation of shear bands in saturated porous media.

Figure 28 plots the reaction force over the applied vertical displacement for the three different loading rates. The results show that the specimen under a higher dynamic loading rate shows a higher peak value in the loading capacity. The oscillations in the reaction force curves with larger loading rates may be correlated to a locally undrained condition under these loading conditions and thus less dilatation in the specimen.

**FIGURE 25** Contours of equivalent shear strain ($\varepsilon_s$) at $u_y = 0.5$ m from simulations with three loading rates: (A) $\dot{u}_y = 0.3$ m/s, (B) $\dot{u}_y = 0.9$ m/s, and (C) $\dot{u}_y = 1.5$ m/s (magnification = x2)

**FIGURE 26** Contours of plastic volume strain at $u_y = 0.5$ m from simulations with three loading rates: (A) $\dot{u}_y = 0.3$ m/s, (B) $\dot{u}_y = 0.9$ m/s, and (C) $\dot{u}_y = 1.5$ m/s (magnification = x2)

**FIGURE 27** Contours of water pressure (kPa) at $u_y = 0.5$ m from simulations with three loading rates: (A) $\dot{u}_y = 0.3$ m/s, (B) $\dot{u}_y = 0.9$ m/s, and (C) $\dot{u}_y = 1.5$ m/s (magnification = x2)

**FIGURE 28** Plot of reaction force versus the vertical displacement at the top boundary for simulations with three loading rates.
5 | CLOSURE

In this article, we propose a stabilized computational nonlocal poromechanics model for dynamic strain localization in saturated porous media. The stabilized coupled nonlocal model is solved using a Lagrangian–Eulerian meshless method with an implicit time integration scheme. Parallel computing is adopted for computing efficiency. As a new contribution, we present a theoretical proof of the zero-energy modes associated with the multiphase correspondence principle. We propose a remedy based on the energy method to circumvent zero-energy modes in solid deformation and fluid flow. We present an energy-based method to determine the stabilization parameters for both the solid deformation and fluid flow process. The stabilized coupled nonlocal model for saturated porous media under dynamic loading can be readily extended to model dynamic problems in unsaturated porous media. We have validated the coupled stabilized nonlocal formulation by comparing numerical results with analytical and finite element solutions for dynamic problems in saturated porous media. Numerical examples are conducted to demonstrate the robustness of the coupled nonlocal dynamic model for dynamic strain localization analysis of saturated porous media.

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DATA AVAILABILITY STATEMENT

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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REFERENCES

1. Zienkiewicz OC, Chan A, Pastor M, Schrefler B, Shiomi T. Computational Geomechanics. Vol 613. Chichester: Wiley; 1999.
2. Lewis RW, Schrefler BA. The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media. Hoboken, NJ: John Wiley; 1998.
3. Zienkiewicz OC, Chan AH, Pastor M, Paul DK, Shiomi T. Static and dynamic behaviour of soils: a rational approach to quantitative solutions. I. fully saturated problems. Proc Royal Soc Lond A Math Phys Sci. 1990;429(1877):285-309.
4. Alonso EE. Triggering and motion of landslides. Géotechnique. 2021;71(1):3-59.
5. Prevost JH. Wave propagation in fluid-saturated porous media: an efficient finite element procedure. Int J Soil Dyn Earthq Eng. 1985;4(4):183-202. https://doi.org/10.1016/0261-7277(85)90038-5
6. Vardoulakis I. Dynamic stability analysis of undrained simple shear on water-saturated granular soils. Int J Numer Anal Methods Geomech. 1986;10(2):177-190.
7. Diebels S, Ehlers W. Dynamic analysis of a fully saturated porous medium accounting for geometrical and material non-linearities. Int J Numer Methods Eng. 1996;39(1):81-97.
8. Alonso E, Gens A, Delahaye C. Influence of rainfall on the deformation and stability of a slope in overconsolidated clays: a case study. Hydrogeol J. 2003;11(1):174-192.
9. Popescu R, Prevost JH, Deodatis G, Chakrabortty P. Dynamics of nonlinear porous media with applications to soil liquefaction. Soil Dyn Earthq Eng. 2006;26(6-7):648-665.
10. Loret B, Prevost JH. Dynamic strain localization in fluid-saturated porous media. J Eng Mech. 1991;117(4):907-922.
11. Song X, Ye M, Wang K. Strain localization in a solid-water-air system with random heterogeneity via stabilized mixed finite elements. Int J Numer Methods Eng. 2017;112(13):1926-1950.
12. Wang K, Song X. Strain localization in non-isothermal unsaturated porous media considering material heterogeneity with stabilized mixed finite elements. Comput Methods Appl Mech Eng. 2020;359:112770.
13. Needleman A. Material rate dependence and mesh sensitivity in localization problems. Comput Methods Appl Mech Eng. 1988;67(1):69-85.
14. De Borst R, Mühlhaus HB. Gradient-dependent plasticity: formulation and algorithmic aspects. Int J Numer Methods Eng. 1992;35(3):521-539.
15. Schrefler B, Sanavia L, Majorana C. A multiphase medium model for localisation and postlocalisation simulation in geomaterials. Mech Cohesive-Frictional Mater Int J Exper Modell Comput Mater Struct. 1996;1(1):95-114.
16. Zhang HW, Schrefler BA. Gradient-dependent plasticity model and dynamic strain localisation analysis of saturated and partially saturated porous media: one dimensional model. Eur J Mech-A/Solids. 2000;19(3):503-524.
17. Heider Y, Avci O, Markert B, Ehlers W. The dynamic response of fluid-saturated porous materials with application to seismically induced soil liquefaction. Soil Dyn Earthq Eng. 2014;63:120-137.
18. Shahbodagh KB, Mirjalili M, Kimoto S, Oka F. Dynamic analysis of strain localization in water-saturated clay using a cyclic elasto-viscoplastic model. *Int J Numer Anal Methods Geomech*. 2014;38(8):771-793.
19. Oka F, Shahbodagh B, Kimoto S. A computational model for dynamic strain localization in unsaturated elasto-viscoplastic soils. *Int J Numer Anal Methods Geomech*. 2019;43(1):138-165.
20. Hajiahadi MR, Khoei AR. A bridge between dual porosity and multiscale models of heterogeneous deformable porous media. *Int J Numer Anal Methods Geomech*. 2019;43(1):212-238.
21. Khoei A, Valab M, Hirmand M. An enriched–fem technique for numerical simulation of interacting discontinuities in naturally fractured porous media. *Comput Methods Appl Mech Eng*. 2018;331:197-231.
22. Khoei AR. *Extended Finite Element Method: Theory and Applications*. Hoboken, NJ: John Wiley & Sons; 2014.
23. Khoei A, Hajiahadi M. Fully coupled hydromechanical multiscale model with microdynamic effects. *Int J Numer Methods Eng*. 2018;115(3):293-327.
24. Cosserat E, Cosserat F. *Théorie des corps déformables*; Paris: A. Hermann et fils; 1909.
25. Eringen AC, Suhubi E. Nonlinear theory of simple micro-elastic solids—I. *Int J Eng Sci*. 1964;2(2):189-203.
26. Kröner E. Elasticity theory of materials with long range cohesive forces. *Int J Solids Struct*. 1967;3(5):731-742.
27. De Borst R. Simulation of strain localization: a reappraisal of the Cosserat continuum. *Eng Comput*. 1991;8(4):317-332.
28. Menon S, Song X. A computational periporomechanics model for localized failure in unsaturated porous media; 2020; arXiv preprint arXiv:201015793.
29. Silling SA, Epton M, Weckner O, Xu J, Askari E. Peridynamic states and constitutive modeling. *J Elast*. 2007;88(2):151-184.
30. Song X, Silling SA. On the peridynamic effective force state and multiphase constitutive correspondence principle. *J Mech Phys Solids*. 2020;145:104161.
31. Turner DZ. A non-local model for fluid-structure interaction with applications in hydraulic fracturing. *Int J Comput Methods Eng Sci Mech*. 2013;14(5):391-400.
32. Madenci E, Oterkus E. *Peridynamic Theory and Its Applications*. New York, NY: Springer; 2014.
33. Jabakhanji R, Mohtar RH. A peridynamic model of flow in porous media. *Adv Water Resour*. 2015;78:22-35.
34. Ouchi H, Katiyar A, York J, Foster JT, Sharma MM. A fully coupled porous flow and geomechanics model for fluid driven cracks: a peridynamics approach. *Comput Mech*. 2015;55(3):561-576.
35. Oterkus S, Madenci E, Oterkus E. Fully coupled poreoelastic peridynamic formulation for fluid-filled fractures. *Eng Geol*. 2017;225:19-28.
36. Song X, Menon S. Modeling of chemo-hydromechanical behavior of unsaturated porous media: a nonlocal approach based on integral equations. *Acta Geotech*. 2019;14(3):727-747.
37. Menon S, Song X. Coupled analysis of desiccation cracking in unsaturated soils through a non-local mathematical formulation. *Geosciences*. 2019;9(10):428.
38. Zhang H, Li H, Ye H, Zheng Y. A coupling peridynamic approach for the consolidation and dynamic analysis of saturated porous media. *Comput Mech*. 2019;64(4):1097-1113.
39. Ni T, Pesavento F, Zaccariotto M, Galvanetto U, Zhu QZ, Schrefler BA. Hybrid fem and peridynamic simulation of hydraulic fracture propagation in saturated porous media. *Comput Methods Appl Mech Eng*. 2020;366:113101.
40. Silling S. Reformulation of elasticity theory for discontinuities and long-range forces. *J Mech Phys Solids*. 2000;48(1):175-209.
41. Cheng AHD. *Poroelasticity*. Vol 27. New York, NY: Springer; 2016.
42. Littlewood DJ. A nonlocal approach to modeling crack nucleation in AA 7075-t651. Paper presented at: Proceedings of the ASME 2011 International Mechanical Engineering Congress and Exposition American Society of Mechanical Engineers Digital Collection, Denver, Colorado, USA; 2011:567-576.
43. Breitenfeld M, Geubelle PH, Weckner O, Silling S. Non-ordinary state-based peridynamic analysis of stationary crack problems. *Comput Methods Appl Mech Eng*. 2014;272:233-250.
44. Tupek MR, Radovitzky R. An extended constitutive correspondence formulation of peridynamics based on nonlinear bond-strain measures. *J Mech Phys Solids*. 2014;65(1):82-92.
45. Silling SA. Stability of peridynamic correspondence material models and their particle discretizations. *Comput Methods Appl Mech Eng*. 2017;322:42-57.
46. Bobaru F, Foster JT, Geubelle PH, Silling SA. *Handbook of Peridynamic Modeling*. Boca Raton, FL: CRC Press; 2016.
47. Li P, Tao ZM, Zhen WQ. A stabilized non-ordinary state-based peridynamic model. *Comput Methods Appl Mech Eng*. 2018;339:262-280.
48. Gu X, Madenci E, Zhang Q. Revisit of non-ordinary state-based peridynamics. *Eng Fract Mech*. 2018;190:31-52.
49. Wu CT, Ren B. A stabilized non-ordinary state-based peridynamics for the nonlocal ductile material failure analysis in metal machining process. *Comput Methods Appl Mech Eng*. 2015;291:197-215.
50. Chen H. Bond-associated deformation gradients for peridynamic correspondence model. *Mech Res Commun*. 2018;90:34-41.
51. Roy CS, Roy P, Reddy JN. A modified peridynamics correspondence principle: removal of zero-energy deformation and other implications. *Comput Methods Appl Mech Eng*. 2019;346:530-549.
52. Hashim NA, Coombs W, Augarde C, Battori G. An implicit non-ordinary state-based peridynamics with stabilised correspondence material model for finite deformation analysis. *Comput Methods Appl Mech Eng*. 2020;371:113304.
53. Silling SA, Askari E. A meshfree method based on the peridynamic model of solid mechanics. *Comput Struct*. 2005;83(17-18):1526-1535.
54. Hughes TJ. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Mineola, NY: Courier Corporation; 2000.
55. Newmark NM. A method of computation for structural dynamics. *J Eng Mech Div*. 1959;85(3):67-94.
56. Gabriel E, Fagg GE, Bosilca G, et al. Open MPI: goals, concept, and design of a next generation MPI implementation. Paper presented at: Proceedings of the European Parallel Virtual Machine/Message Passing Interface Users’ Group Meeting; 2004:97-104; Springer, New York, NY.

57. Simo JC, Hughes TJ. Computational Inelasticity. Vol 7. Berlin, Germany; Springer Science & Business Media; 1998.

58. Song X, Borja RI. Mathematical framework for unsaturated flow in the finite deformation range. Int J Numer Methods Eng. 2014;97(9):658-682.

59. Borja RI, Song X, Rechenmacher AL, Abedi S, Wu W. Shear band in sand with spatially varying density. J Mech Phys Solids. 2013;61(1):219-234.

60. Schofield A, Wroth P. Critical State Soil Mechanics. New York, NY: McGraw-Hill; 1968.

61. Wood DM. Soil Behaviour and Critical State Soil Mechanics. Cambridge, MA: Cambridge University Press; 1990.

62. Borja RI. Plasticity: Modeling & Computation. Berlin, Germany: Springer Science & Business Media; 2013.

63. Schanz M, Cheng AD. Transient wave propagation in a one-dimensional poroelastic column. Acta Mech. 2000;145(1-4):1-18.

64. de Boer R, Ehlers W, Liu Z. One-dimensional transient wave propagation in fluid-saturated incompressible porous media. Arch Appl Mech. 1993;63(1):59-72.

65. Song X. Transient bifurcation condition of partially saturated porous media at finite strain. Int J Numer Anal Methods Geomech. 2017;41(1):135-156.

66. Markert B, Heider Y, Ehlers W. Comparison of monolithic and splitting solution schemes for dynamic porous media problems. Int J Numer Methods Eng. 2010;82(11):1341-1383.

67. Pastor M, Li T, Liu X, Zienkiewicz O, Quecedo M. A fractional step algorithm allowing equal order of interpolation for coupled analysis of saturated soil problems. Mech Cohesive-Frictional Mater Int J Exper Modell Comput Mater Struct. 2000;5(7):511-534.

68. Breuer S. Quasi-static and dynamic behavior of saturated porous media with incompressible constituents. Porous Media: Theory and Experiments. New York, NY: Springer; 1999:285-303.

69. Bažant ZP, Luo W, Chau VT, Bessa MA. Wave dispersion and basic concepts of peridynamics compared to classical nonlocal damage models. J Appl Mech. 2016;83(11):1-16.

70. Mikaeili E, Schrefler B. XFEM, strong discontinuities and second-order work in shear band modeling of saturated porous media. Acta Geotech. 2018;13(6):1249-1264.

71. Hill R. A general theory of uniqueness and stability in elastic-plastic solids. J Mech Phys Solids. 1958;6(3):236-249.

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