Non-extremal Stringy Black Hole

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Abstract

We construct a four-dimensional BPS saturated heterotic string solution from the Taub-NUT solution. It is a non-extremal black hole solution since its Euler number is non-zero. We evaluate its black hole entropy semiclassically. We discuss the relation between the black hole entropy and the degeneracy of string states. The entropy of our string solution can be understood as the microscopic entropy which counts the elementary string states without any complications.

PACS numbers : 04.20.Gz, 04.70.Dy, 11.25.Mj
keywords : entropy, BPS, heterotic string, extremal, Euler number
1 Introduction

The Bekenstein-Hawking black hole entropy has been studied semiclassically [1, 2, 3, 4, 5, 6, 7, 8]. This entropy has recently been interpreted microscopically in string theory where the black holes can be identified with elementary string excitations [4]. It was shown that stringy effects could correct the Bekenstein-Hawking formula for the black hole entropy in such a way that it correctly reproduces the logarithm of the density of elementary string states. Strominger and Vafa [10] have applied this idea to the entropy of the 5-dimensional Reissner-Nordström-type (RN-type) extremal black holes, which carry axion charges and electric charges. As a result, they found that the entropy of their black holes counts the degeneracy of the BPS soliton bound states. Callan and Maldacena [11] have also studied the case of near-extremal black holes.

On the other hand, Hawking and Horowitz [12] have argued that the extremal RN-type black holes have no entropy because of their topology $\mathbb{R}^1 \times S^1 \times S^2$. This point is also discussed recently by Das et al. [13]. Gibbons and Kallosh [14] have discussed this idea with the Euler number. According to their arguments, the entropy of a black hole vanishes if the Euler number of the black hole is zero in four-dimensions. In two dimensions, clearly the Einstein-Hilbert action is proportional to the Euler number. In contrast, in four-dimensions there is no simple relation between the Einstein-Hilbert action and the Euler number. However a relation between the entropy and the Euler number of the black hole exists when we consider the diagonal metric. Using this relation the Euler number is found to vanish if the black hole is extremal. Therefore we call the black hole whose Euler number is zero extremal.

The entropy of black holes can be interpreted as the statistical entropy that counts the degeneracy of the microscopic states in string theories. Strominger and Vafa have applied this idea to the extremal black holes in their paper [10]. However the entropies of the extremal black hole solutions are zero according to Hawking’s argument. Hence it is not simple to discuss the interpretation of black hole entropy in string theory using the extremal RN-black holes. In order to discuss the issue without these complications, we need to find the non-extremal black hole solutions.

Simultaneously, we need to take into account that the quantum correction affects the states of the black hole solution and disturbs the counting of states.
if we use the ordinary black hole solutions, for instance the Schwarzschild one. In contrast, the quantum correction is small for the BPS saturated states. The extremal RN black holes are the BPS saturated solution. However these solutions may have no entropy as mentioned above. Therefore in order to discuss the relation between the entropy of black holes and the degeneracy of elementary string states, the solutions we need are non-extremal and the BPS saturated states.

In order to find the BPS saturated solutions, the four-dimensional effective heterotic string action was investigated by Sen \[15, 9, 16\]. These solutions were led from Schwarzschild and Kerr solutions using the $O(7, 23)$ transformation. However they are extremal because their Euler numbers are zero as it will be shown later. Therefore his solutions have no entropy, and we need to find another solution for the issue.

The purpose of this paper is to derive a new solution, which is BPS saturated and non-extremal. We construct this solution from the Taub-NUT solution by using Sen’s argument. Using this solution, we discuss the relation between the black hole entropy and the degeneracy of string states.

The organization of this paper is as follows. In section 2 we review the relation between the entropy of black holes and the Gauss-Bonnet action which counts the Euler number. We consider the relation between the Euler number and the extremal black hole. From this consideration, we show that Sen’s solutions have no entropy because of its Euler numbers. In section 3 we study the low energy effective action of four-dimensional heterotic string and its symmetry. We subsequently derive a new solution by using the Sen’s argument. In section 4 we evaluate the entropy of this solution semiclassically. We then discuss the relation between this black hole entropy and the degeneracy of string states using the concept of the stretched horizon. We present our conclusions in section 5.

2 Gauss-Bonnet Action and Extremal Black Hole

In this section we review a relation between the black hole entropy and the Euler number. This relation leads to the conclusion that the black hole entropy is zero if its Euler number vanishes. We then consider the extremal
U(1) dilaton solution \[20\] as an example. We conclude this section by finding that Sen’s metric \[15, 9, 16\] has no entropy because of its Euler number.

Firstly we note the relation between the Euler number and the entropy for the spherically symmetric metric \[14, 18, 19\]. We consider the following metric:

\[
ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} dr^2 + R(r)^2 d\Omega^2.
\] (1)

This metric corresponds to the vielbein forms

\[
\hat{\theta}^0 = e^{U(r)} dt, \quad \hat{\theta}^1 = e^{-U(r)} dr, \\
\hat{\theta}^2 = R(r) d\theta, \quad \hat{\theta}^3 = R(r) \sin \theta d\phi.
\]

The entropy is

\[
S_{\text{ent}} = \beta E - \ln Z
\]

where \(E\) is the energy of this system. \(Z\) is the partition function

\[
Z = e^{-I_E},
\]

where \(I_E\) is the Euclidean on-shell action. It is explicitly given by

\[
I_E = \frac{1}{16\pi} \int_W (-R) + \frac{1}{8\pi} \int_{\partial W} [K] d\Sigma,
\]

where \(W\) is the Euclidean manifold defined from the original black hole manifold with \(t \to i t\). \(\partial W\) is the boundary of \(W\). \([K]\) is the extrinsic curvature. \(d\Sigma\) is

\[
d\Sigma = \hat{\theta}^0 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3
\]

This action has only one boundary, at spatial infinity \(r \to \infty\). The reason is that the event horizon is not present in the Euclidean action.

On the other hand, \(\beta E\) is represented by

\[
\beta E = I_{E,h} = \frac{1}{16\pi} \int_M (-R) + \frac{1}{8\pi} \int_{\partial M} [K] d\Sigma,
\]

where \(M\) is the original manifold. This action has two boundaries, namely, event horizon and spatial infinity because the time translation Killing vector
\( \frac{\partial}{\partial r} \) is zero at horizon if the manifold is non-extremal. Therefore we need to take account of the contribution to \( E \) from the horizon as well as from the infinity. Therefore the entropy is

\[
S_{\text{ent}} = I_{E,h} - I_E
\]

\[
= -\frac{1}{8\pi} (\int_{\partial W} - \int_{\partial M}) [K] d\Sigma = -\frac{1}{8\pi} (\int_{\partial W} - \int_{\partial M}) [K] \hat{\theta}^0 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3
\]

\[
= \frac{1}{8\pi} \frac{1}{\sqrt{g_{rr}}} \frac{\partial}{\partial r} (\int_{\partial W} - \int_{\partial M}) \hat{\theta}^0 \wedge \hat{\theta}^2 \wedge \hat{\theta}^3
\]

\[
= \frac{1}{2} (\int_{\partial W} - \int_{\partial M}) R(2 \partial_r U + 2 \partial_r R)e^{2U} dt.
\]

Secondly we study the Gauss-Bonnet action with boundary term, which was led by Chern [17]. We define the Gauss-Bonnet action of the four-dimensional Riemannian manifold \( M^n \) \((n=4)\):

\[
S_{\text{GB}} = S_{\text{GB}}^{\text{vol}} + S_{\text{GB}}^{\text{boun}} = \chi,
\]

where \( \chi \) is Euler number. \( S_{\text{GB}}^{\text{vol}} \) is the volume term

\[
S_{\text{GB}}^{\text{vol}} = \frac{1}{32\pi} \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd},
\]

and \( S_{\text{GB}}^{\text{boun}} \) is the boundary term

\[
S_{\text{GB}}^{\text{boun}} = -\frac{1}{32\pi^2} \int_{\partial M} \epsilon_{abcd} (2 \theta^{ab} \wedge R^{cd} + \frac{4}{3} \theta^{ab} \wedge \theta^c \wedge \theta^{cd}),
\]

where \( \partial M \) is the boundary of \( M \). In these equations the curvature two-form is defined as

\[
R^a_b \equiv d\omega^a_b + \omega^a_c \wedge \omega^c_b,
\]

where \( \omega^a_b \) is a spin connection one-form. The second fundamental form of the boundary is

\[
\theta^{ab} \equiv \omega^{ab} - (\omega^{ab})_0,
\]

where \( (\omega^{ab})_0 \) is spin connections at the boundary \( r = r_0 \).
For the spherically symmetric metric (1) (which was discussed by Gibbons and Kallosh [14, 18, 19]), the spin connections are
\[
\omega^{01} = \frac{1}{2} \partial_r (e^{2U}) dt, \quad \omega^{21} = e^U \partial_r R d\theta,
\]
\[
\omega^{31} = e^U \partial_r R \sin \theta d\phi, \quad \omega^{32} = \cos \theta d\phi, \tag{6}
\]
\[
(else = 0).
\]

For this metric, the range of integration in \( t \) is infinite in the Gauss-Bonnet action (3). However if we use the Riemannian version of the metric (1),

\[
ds^2 = e^{2U(r)} d\tau^2 + e^{-2U(r)} dr^2 + R(r)^2 d\Omega^2, \tag{7}
\]
where \( \tau \) is a periodic coordinate, the range of \( \tau \) integration is constrained to be from 0 to \( \beta \) by the standard requirements where

\[
\beta = \frac{2\pi}{\kappa}.
\]

In this equation, the surface gravity \( \kappa \) is given by

\[
\kappa \equiv \frac{1}{2} \frac{\partial_r g_{tt}}{\sqrt{-g_{rr} g_{tt}}} \bigg|_{r=r_H} = (\omega^{01})_t = \frac{\partial_r (e^{2U})}{2} \bigg|_{r=r_H}, \tag{8}
\]

where \( r_H \) is a radius of the black hole event horizon, which satisfies that

\[
(e^{2U}) \bigg|_{r=r_H} = 0
\]

The Gauss-Bonnet volume integral can be calculated using the values of the Riemann tensors for the metric (1). As a result, the Gauss-Bonnet integrand in the volume term is a total derivative. The part of the volume term (1) which does not contain \( \theta^{01} \) is completely cancelled by the boundary term (5). Then, the Gauss-Bonnet action for the Riemannian version of this
metric is
\[ S_{GB} = S_{GB}^{vol} + S_{GB}^{boun} \]
\[ = \frac{1}{4\pi^2} \left( \int_{\partial V} - \int_{\partial M} \right) \omega^{01} \wedge R^{23} \]
\[ = \frac{1}{2\pi} \left( \int_{\partial V} - \int_{\partial M} \right) \partial_r (e^{2U}) (1 - e^{2U} (\partial_r R)^2) dt, \quad (9) \]

where \( M^{2n-1} \) is a \((2n-1)\)-dimensional manifold defined from an original \( n \)-dimensional manifold \( M^n \) whose extra dimensions are formed by the unit tangent vectors. \( V \) is an \( n \)-dimensional submanifold of \( M^{2n-1} \). Chern has shown that the integrand in the Gauss-Bonnet action is equal to the exterior derivative of a differential form \( \Phi \) in \( M^{2n-1} \). According to Stokes’ theorem, the volume integral is equal to the integral over the boundaries of \( V \). The boundaries of \( V \) correspond to the non-isolated singular points of the tangent vector field defined from \( M^n \). For example, the Schwarzschild manifold has two boundaries at its horizon and infinity. The manifold \( V \) defined from the original manifold has only a single boundary at infinity because the horizon of the manifold corresponds to the isolated singular point of \( V \).

This expression provides an exact cancellation when the first term and the second term are the same, namely the boundaries of the manifold \( M^n \) are the same as the boundaries of the submanifold \( V \) of the manifold \( M^{2n-1} \). In this case the Euler number is zero.

The boundary \( \partial V \) is the same as \( \partial W \) defined in (2). So we find the relation between the black hole entropy \((2)\) and the Euler number \((9)\) :

\[
S_{\text{ent}} = 2\pi \chi (e^{2U})'^{-1} (1 - e^{2U} R^2)^{-1} R \left[ \left( (U' R + 2R') e^{2U} \right) \bigg|_{r=r_H} \right]
\]
\[
= \pi \chi [(e^{2U})' - R^2 e^{2U} (e^{2U})']^{-1} R \left[ \left( \frac{R}{2} (e^{2U})' + 2R' e^{2U} \right) \bigg|_{r=r_H} \right]
\]

By defining \( r_h \) in such a way that \( e^{2U} \bigg|_{r=r_H} = 0 \), we find :

\[
S_{\text{ent}} = \frac{\pi \chi R^2(r_H)}{2} = \frac{\chi (4\pi R^2(r_H))}{8} = \frac{\chi A}{8} = \frac{\chi A}{8}.
\]
Therefore the entropy of the black hole is zero if the Euler number of the black hole vanishes.

Thirdly we define the extremal black holes in an ordinary way. We define the temperature of the black hole \( T \) as,

\[
T \equiv 2\pi \kappa = \pi \partial_r (e^{2U}) \bigg|_{r=r_H}. \tag{10}
\]

Using this quantity, the extremal black hole is defined by

\[
T = 0. \tag{11}
\]

As an example, we consider the extremal U(1) dilaton solution \cite{20}:

\[
ds^2 = -(1 - \frac{r_+}{r})^{\frac{2}{1+a^2}}dt^2 + (1 - \frac{r_+}{r})^{-\frac{2}{1+a^2}}dr^2 + r^2 (1 - \frac{r_+}{r})^{\frac{2a^2}{1+a^2}}d\Omega^2 \tag{12}
\]

From the above definition, the surface gravity is

\[
k = \frac{1}{2} \partial_r [(1 - \frac{r_+}{r})^{\frac{2}{1+a^2}}] \big|_{r=r_+} = \frac{2}{1 + a^2} (1 - \frac{r_+}{r})^{\frac{1-a^2}{1+a^2}} \frac{r_+}{r^2} \big|_{r=r_+} = \begin{cases} 0 & (0 \leq a < 1) \\ \frac{1}{2r_+} & (a \to 1). \end{cases} \tag{13}
\]

Then for \( 0 \leq a < 1 \), this solution is extremal from the definition \( \ref{11} \). For these solutions, the Gauss-Bonnet actions \( \ref{9} \) are evaluated to be

\[
S_{GB} = \frac{1}{4\pi^2} (\int_{\partial V} - \int_{\partial M}) \omega^{01} (1 - (\frac{a^2}{1+a^2})^2) d\Omega, \tag{14}
\]

where

\[
\frac{1}{2\pi} \int_{r_H} \omega^{01} = \frac{1}{2\pi} \int_0^\beta dt (\omega^{01})_t \bigg|_{r_H} = \frac{1}{4\pi} \beta (e^{2U}) \bigg|_{r_H} = 1, \tag{15}
\]

\[ \text{8} \]
and $r_h$ is the horizon radius of $V$ (or $M$). From the knowledge of the topology, the Euler number should be an integer. Then in the case $0 < a \leq 1$, we are forced to conclude that $\partial V = \partial M$, and their Euler numbers are zero. In the case $a = 0$ (which is the extremal R-N black hole) it seems that there is no problem to find $\chi = 2$. However in $r - \tau$ space ($\tau$ is equal to the imaginary time coordinate $it$) the vector field $\frac{\partial}{\partial \tau}$ has no fixed points because the surface gravity is zero. Then the topology of this space is $\mathbb{R}^1 \times S^1$, and its Euler number is also zero. Therefore we find that the Euler numbers are zero for $0 \leq a \leq 1$. From these considerations, we conclude that the extremal black holes are defined by:

$$T = 0$$

and (or)

$$S_{GB} = \chi = 0, \quad (\partial V = \partial M)$$

Finally we consider the Sen’s solution [9]:

$$ds_E^2 = -\Delta^{-1/2} r dt^2 + \Delta^{1/2} r^{-1} dr^2 + \Delta^{1/2} r (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\Delta = (r^2 + 2m_0 r \cosh \alpha + m_0^2)$$

We can calculate the Euler number of this solution using the formula (3). The result is that

$$S_{GB} = \frac{1}{4\pi^2} \left( \int_{\partial V} - \int_{\partial M} \right) \frac{3}{4} \omega^{01} d\Omega.$$  

From this formula, we conclude that Sen’s solution is extremal since its Euler number vanishes.

### 3 Heterotic string and Symmetry

In this section we derive a new solution for the heterotic string action. This solution is non-extremal and could possess non zero entropy.

We recall the heterotic string theory that is compactified on a six- dimensional torus [13, 9, 16]. The bosonic part of the effective field theory is given by

$$S = \frac{1}{32\pi} \int d^4 x \sqrt{-G} e^{-\Phi} \left[ R_G + G_{\mu\nu} \partial_{\mu} \Phi \partial_{\nu} \Phi - \frac{1}{12} G^{\mu\lambda \nu \kappa} G_{\mu \nu}^{\rho \sigma} H_{\rho \sigma}^{\lambda \kappa} H_{\lambda \kappa}^{\rho \sigma} 

- G^{\mu \lambda} G^{\nu \kappa} F_{\mu \nu}^{(a)} (LML)_{ab} F_{\lambda \kappa}^{(b)} + \frac{1}{8} G^{\mu \nu} Tr(\partial_{\mu} M L \partial_{\nu} M L) \right].$$

(21)
where
\[ F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)}, \]
\[ H_{\mu\nu\rho} = (\partial_\mu B_{\nu\rho} - 2A^{(a)}_{\mu} L_{ab} F_{\nu\rho}^{(b)}) + (\mu, \nu, \rho : \text{cyclic}). \]  

The massless fields we consider are gravitational fields \( G_{\mu\nu} \), the anti-symmetric tensor fields \( B_{\mu\nu} \), twenty-eight \( U(1) \) gauge fields \( A_\mu^{(a)} (1 \leq a \leq 28) \), the scalar dilaton field \( \Phi \), and a \( 28 \times 28 \) matrix valued scalar field \( M \) satisfying,

\[ MLM^T = L, \quad L = \begin{pmatrix} -I_{22} & 0 \\ 0 & I_6 \end{pmatrix}. \]  

This action (21) is invariant under the \( O(6,22) \) transformation,

\[ M \rightarrow \Omega M \Omega^T, \quad A_\mu^{(a)} \rightarrow \Omega_{ab} A_\mu^{(b)}, \quad \Phi \rightarrow \Phi, \quad G_{\mu\nu} \rightarrow G_{\mu\nu}, \quad B_{\mu\nu} \rightarrow B_{\mu\nu}. \]  

where \( \Omega \) is a \( 28 \times 28 \) matrix satisfying

\[ \Omega L \Omega^T = L. \]  

In addition to that, the action (21) is expected to have an \( O(7,23) \) symmetry if backgrounds are independent of the time coordinate \( t \). To see how this appears, we define new variables as follows :

\[
\bar{M} = \begin{pmatrix}
M + 4(G_{tt})^{-1}A_t A_t^T & -2(G_{tt})^{-1}A_t & 2MLA_t \\
-2(G_{tt})^{-1}A_t^T & (G_{tt})^{-1} & -2(G_{tt})^{-1}A_t^T L A_t \\
2A_t^T LM + 4(G_{tt})^{-1}A_t^T (A_t^T L A_t) & -2(G_{tt})^{-1}A_t^T L A_t & G_{tt} + 4A_t^T LMLA_t + 4(G_{tt})^{-1}(A_t^T L A_t)^2
\end{pmatrix}
\]

\[ \bar{A}_i^{(a)} = A_i^{(a)} - (G_{tt})^{-1}G_{ti} A_t^{(a)}, \quad \bar{A}_i^{(29)} = \frac{1}{2} (G_{tt})^{-1}G_{ti}, \]
\[ \bar{A}_i^{(30)} = \frac{1}{2} B_{ti} + A_t^{(a)} L_{ab} A_t^{(b)}, \quad \bar{G}_{ij} = G_{ij} - (G_{tt})^{-1}G_{ti} G_{tj}, \]
\[ \bar{B}_{ij} = B_{ij} + (G_{tt})^{-1}(G_{ti} A_j^{(a)} - t_j A_i^{(a)}) L_{ab} A_t^{(b)} + \frac{1}{2} (G_{tt})^{-1}(B_{ti} G_{tj} - B_{tj} G_{ti}), \]
\[ \Phi = \Phi - \frac{1}{2} \ln(-G_{tt}), \quad \bar{L} = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (1 \leq a \leq 28, 1 \leq i \leq 3). \]  

(26)
Using these variables, the action (21) is rewritten as

\[
S = \frac{1}{32\pi} \int dtd^3x \sqrt{G} e^{-\Phi} [R_G + G^{ij} \partial_i \Phi \partial_j \Phi - \frac{1}{12} G^{il} G^{jm} G^{kn} \tilde{H}_{ij} \tilde{H}_{lmn} - G^{il} G^{jm} \tilde{F}^{(a)}_{ij} (\bar{\Lambda} \bar{M} \bar{L})_{ab} \tilde{F}^{(b)}_{tm} + \frac{1}{8} G^{ij} Tr(\partial_i \bar{M} \partial_j \bar{M} \bar{L})],
\]  

(27)

where

\[
\tilde{F}^{(a)}_{ij} = \partial_i \bar{A}^{(a)}_j - \partial_j \bar{A}^{(a)}_i,
\]
\[
\tilde{H}_{ij} = (\partial_i \bar{B}_{jk} - 2\bar{A}^{(a)}_i \bar{L}^{(b)}_{ab} \tilde{F}^{(b)}_{jk}) + (i, j, k : \text{cyclic}).
\]  

(28)

It is obvious that this action (27) has the \( O(7, 23) \) symmetry:

\[
\bar{M} \to \bar{\Omega} \bar{M} \bar{\Omega}^T, \quad \bar{A}^{(a)}_i \to \bar{\Omega} \bar{a}^{(a)}_i, \quad \bar{\Phi} \to \bar{\Phi}, \quad \bar{G}_{ij} \to \bar{G}_{ij}, \quad \bar{B}_{ij} \to \bar{B}_{ij},
\]  

(29)

where \( \bar{\Omega} \) is a 30 \( \times \) 30 matrix satisfying

\[
\bar{\Omega} \bar{L} \bar{\Omega}^T = \bar{L}.
\]  

(30)

In order to simplify \( \bar{\Omega} \), we use the orthogonal matrix \( U \) that diagonalizes \( \bar{L} \),

\[
U = \begin{pmatrix}
I_{28} & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{pmatrix}.
\]  

(31)

Then

\[
U \bar{L} U^T \equiv \bar{L}_d = \begin{pmatrix}
-I_{22} & 0 & 0 & 0 \\
0 & I_6 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]  

(32)

From the elements of \( \bar{L}_d \), we see that \( \bar{\Omega} \) is equal to \( O(6, 1) \times O(22, 1) \), which are the subgroups of \( O(7, 23) \). In other word, the \( O(6, 1) \) subgroup acts on the 23rd-28th and 30th index, whereas the \( O(22, 1) \) subgroup acts on the 1st-22nd, and 29th index. Then we can write

\[
U \bar{\Omega}_1 U^T = \begin{pmatrix}
I_{21} & 0 & 0 & 0 & 0 & 0 \\
0 & \cosh \alpha & 0 & 0 & \sinh \alpha & 0 \\
0 & 0 & I_5 & 0 & 0 & 0 \\
0 & 0 & 0 & \cosh \beta & 0 & \sinh \beta \\
0 & \sinh \alpha & 0 & 0 & \cosh \alpha & 0 \\
0 & 0 & 0 & \sinh \beta & 0 & \cosh \beta
\end{pmatrix}.
\]  

(33)
In the action (27) we consider the solution with the following asymptotic flat space time forms for various fields:

\[ M_{as} = I_{28}, \quad \Phi_{as} = 0, \quad (A^{(a)})_{as} = 0, \quad (G_{\mu\nu})_{as} = \eta_{\mu\nu}, \quad (B_{\mu\nu})_{as} = 0. \] (34)

Using these fields,

\[ \bar{M}_{as} = \begin{pmatrix} I_{28} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \] (35)

With these asymptotic fields the action (27) is invariant under an \( O(6) \times O(22) \) transformation, and \( \bar{\Omega} \) is parametrized by the coset \( (O(6, 1) \times O(22, 1))/(O(6) \times O(22)). \)

It is because the whole \( \Omega \) is written as \( \bar{\Omega} = \Omega_2 \Omega_1 \), where \( \Omega_2 \) is

\[ \tilde{\Omega}_2 = \begin{pmatrix} R_{22}(\vec{n}) & 0 & 0 \\ 0 & R_6(\vec{p}) & 0 \\ 0 & 0 & I_2 \end{pmatrix}. \] (36)

In equation (36) \( \vec{n}, \vec{p} \) are arbitrary 22- and six-dimensional unit vectors, and \( R_N(\vec{k}) \) denotes an \( N \)-dimensional rotation matrix that rotates an \( N \)-dimensional column vector with only the \( N \)-th component non-zero and equal to 1 into an arbitrary \( N \)-dimensional unit vector \( \vec{k} \).

We can derive the new solutions of the action (27) with the following method:

1) Firstly we consider the time-independent solution of general relativity such as Kerr or Schwarzschild solutions for \( G_{\mu\nu} \), and other fields \( M, \Phi, A_{\mu}^{(a)}, B_{\mu\nu} \) are the same as (34).

2) Secondly we apply the transformation (26) to the above solution. As a result, we obtain a new solution of the action (27).

3) Thirdly we apply the transformation \( \bar{\Omega} \) to 2). This solution is also the solution of the action (27).

4) Finally we apply the inverse of the transformation (26) to 3), and then we get a new solution of the action (27).

We use the Taub-NUT solution, as the solution 1):

\[ ds^2 \equiv G_{\mu\nu} dx^\mu dx^\nu = -\frac{r-n}{r+n} (dt + 2i n \cos \theta d\phi)^2 + (r^2 - n^2) (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r+n}{r-n} dr^2 \] (37)
The Riemannian version of this metric is self-dual. As the result of above transformations 2),3),4), we obtain the following solution:

\[ \text{ds}^2 \equiv e^{-\Phi} G_{\mu\nu} dx^\mu dx^\nu = -\frac{r-n}{\sqrt{\Delta}} \left( dt + in \cos \theta (\cosh \alpha + \cosh \beta) d\phi \right)^2 \]

\[ +(r-n)\sqrt{\Delta} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{\sqrt{\Delta}}{r-n} dr^2 \]

(38)

where

\[ \Delta = (r+n)^2 + 2n(r+n)(\cosh \alpha \cosh \beta - 1) + n^2(\cosh \alpha - \cosh \beta)^2 \]

\[ A_t^{(a)} = -\frac{n^{(a)}}{\sqrt{2\Delta}} n \sinh \alpha \left[(r+n) \cosh \beta + n(\cosh \alpha - \cosh \beta)\right] \quad (1 \leq a \leq 22) \]

\[ A_t^{(a)} = -\frac{p^{(a-22)}}{\sqrt{2\Delta}} n \sinh \beta \left[(r+n) \cosh \alpha + n(\cosh \beta - \cosh \alpha)\right] \quad (23 \leq a) \]

(39)

This solution is not included in [21] because it has Taub-NUT charge.

From the above solution, the ADM mass \(m_0\), the surface gravity \(\kappa\), electric charge \(Q^{(a)}\) are found respectively as,

\[ m_0 = n(1 + \cosh \alpha \cosh \beta), \quad \kappa = \frac{1}{2n(\cosh \alpha + \cosh \beta)}, \]

\[ Q^{(a)} = \left\{ \begin{array}{ll}
\frac{n}{\sqrt{2}} \sinh \alpha \cosh \beta n^{(a)} & (1 \leq a \leq 22) \\
\frac{n}{\sqrt{2}} \sinh \beta \cosh \alpha p^{(a-22)} & (22 \leq a)
\end{array} \right. \]

(40)

We define that

\[ Q_L^{(a)} = \frac{1}{2} (I_{28} - L)_{ab} Q^{(b)}, \quad Q_R^{(a)} = \frac{1}{2} (I_{28} + L)_{ab} Q^{(b)}. \]

(41)

In order to obtain the BPS saturated state, we consider the limit

\[ n \to 0, \quad \cosh \alpha \to \infty, \quad n \cosh \alpha = n_0 : \text{fix.} \]

(42)

Then the solution takes the form:

\[ \text{ds}^2 \equiv G_{\mu\nu} dx^\mu dx^\nu = -\frac{r}{\sqrt{\Delta'}} \left( dt + in_0 \cos \theta d\phi \right)^2 \]
\[ + r \sqrt{\Delta'} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{\sqrt{\Delta'}}{r} dr^2, \]

\[ \Delta' = r^2 + 2n_0 r \cosh \beta + n_0^2, \quad m_0 = n_0 \cosh \beta, \quad \kappa = \frac{1}{2n_0}, \]

\[ Q_L^{(a)} = \frac{n_0}{\sqrt{2}} \sinh \beta n^{(a)} \quad (1 \leq a \leq 22), \]

\[ Q_R^{(a)} = \frac{n_0}{\sqrt{2}} \cosh \beta p^{(a-22)} \quad (22 < a), \quad (\text{else } = 0). \]

From this solution we get the following relations between \( m_0, Q_L, Q_R : \)

\[ m_0^2 = 2\overline{Q}_R^2, \quad m_0 = \sqrt{2|Q_L|} \tanh \beta, \quad n_0 = \sqrt{m_0^2 - 2\overline{Q}_L^2} \quad (43) \]

Finally we calculate the Euler number of this solution for \( \beta \to 0. \) For this calculation, we use (4) directly because (9) is the equation for the diagonal metric. The result is that

\[ \chi = \frac{1}{32 \pi} \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd} = 1. \quad (44) \]

For the extremal metric, we need to use both the volume term(4) and the boundary term(5) in order to get the integer number as the Euler number. In contrast, our solution only need the volume term(4), namely \( \partial V \neq \partial M. \) Therefore we conclude that this solution is non-extremal.

4 Black Hole Entropy

In this section we evaluate the entropy of the black hole solution we have found in the previous section semiclassically. We calculate the entropy of the black hole using the following method instead of the ordinary equation \( S_{\text{ent}} = \frac{A_{\text{ren}}}{4}. \) We use the metric

\[ ds^2 = e^{2U(r)} (dt^2 + m_0 \cos \theta d\phi)^2 + e^{-2U(r)} dr^2 + R(r)^2 d\Omega^2 \quad (45) \]

The vierbeins are
\[
\begin{align*}
\dot{\theta}^0 &= e^U (dt + m_0 \cos \theta d\phi), \\
\dot{\theta}^1 &= e^{-U} dr, \\
\dot{\theta}^2 &= Rd\theta, \\
\dot{\theta}^3 &= R \sin \theta d\phi.
\end{align*}
\]

Then the entropy of the black hole is written as \[14, 18, 19\]:

\[
S_{\text{ent}} = -\frac{1}{8\pi} \left( \int_{\partial V} - \int_{\partial M} \right) [K] d\Sigma = -\frac{1}{8\pi} \left( \int_{\partial V} - \int_{\partial M} \right) [K] \dot{\theta}^0 \wedge \dot{\theta}^2 \wedge \dot{\theta}^3
\]

\[
= \frac{1}{8\pi} \frac{1}{\sqrt{g_{rr}}} \frac{\partial}{\partial r} \int_{\partial M} \dot{\theta}^t \wedge \dot{\theta}^\theta \wedge \dot{\theta}^\phi \bigg|_{r=r_H}
\]

\[
= \frac{\beta R}{2} \left( R \partial_r U + 2 \partial_r R e^{2U} \right) \bigg|_{r=r_H}
\]

\[
= \text{Area} \frac{4}{4} + \frac{\beta R}{2} \left[ e^{2U} 2 \partial_r R \right] \bigg|_{r=r_H}
\]

Here the second term in the last line vanishes when we consider the solution like the Schwarzschild or RN solutions.

It is not clear whether the explicit relation between the entropy and the Euler number exists for the Taub-NUT type metric unlike the spherically symmetric solutions which are discussed in the section 2. However the important point is that the entropy inevitably vanishes when \(\partial V = \partial M\) in (46). For our solution, \(\partial V\) does not coincide \(\partial M\) as it is discussed in section 3. Therefore this solution is non-extremal and it can possess the non-zero entropy.

In contrast, for the extremal black holes it is necessary that \(\partial V\) must coincide with \(\partial M\) because of their Euler numbers as is discussed in section 2. Therefore their entropies are zero even if their areas of the event horizons are non-vanishing.

For our solution, we calculate the entropy of black hole by this formula. Its area vanishes because \(r_H = 0\). However this is not the essential problem unlike the vanishing Euler number. In order to get a non-zero area, we introduce the stretched horizon \(\overline{r}\) as

\[
r_H = C : \text{constant.}
\]
In contrast, it is not enough just to introduce the stretched horizon if we consider the solutions whose Euler numbers are zero. If we introduce stretched horizons to both $\partial V$ and $\partial M$, the entropy still vanishes. We believe that the stringy effects can be neglected because we study the BPS saturated states. Therefore it is possible to use our solution in order to discuss the relation between the entropy and degeneracy of the string states.

We clarify the difference between the Sen’s solution and our solution. The Sen’s solution is extremal because its Euler number is zero (as we evaluated it in the previous section). According to Hawking’s argument, Sen’s solution has no entropy. On the other hand our solution is not extremal because its Euler number is non-zero. Hence it can possess non-zero entropy. Therefore we think that the entropy of our solution can be understood as the microscopic entropy which counts the elementary string states without any complications.

In this way we obtain the entropy of our solution:

$$S_{\text{ent}} = 3\pi C n_0 = 3\pi C \sqrt{m_0^2 - 2\vec{Q}_L^2}$$ (48)

We compare this entropy with the statistical entropy which counts the degeneracy of elementary string states. The mass formula for heterotic string is:

$$m_0^2 = \frac{\vec{Q}_R^2}{8g^2} = \frac{g^2}{8} (\frac{\vec{Q}_L^2}{g^4} + 2N_L - 2) \quad (N_R = \frac{1}{2})$$ (49)

where $N_L$ is the total oscillator contribution to the squared mass from the left moving oscillator. $g =< e^{\phi} >_{\infty}$ is the vacuum expectation value of the dilaton field at infinity. The degeneracy of the states is given by

$$d_{E.S} \simeq \exp(4\pi \sqrt{N_L}).$$ (50)

Thus, the entropy, which is calculated from the elementary string spectrum, is given by

$$S_{E.S} \equiv \ln d_{E.S} \simeq 4\pi \sqrt{N_L} \simeq \frac{8\pi}{g} \sqrt{m_0^2 - \frac{\vec{Q}_L^2}{8g^2}}.$$ (51)

If we make a choice that $g = \frac{1}{4}$ and $C = \frac{32}{3}$, then we obtain that $S_{E.S} = S_{\text{ent}}$.  

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5 Conclusion

We have derived the new solution of the effective heterotic string action, which is non-extremal and is BPS saturated. This solution has non-zero entropy because its Euler number is non-zero. That is consistent with the Hawking’s argument that the extremal black holes don’t have the entropy. Furthermore this solution can be treated semiclassically because it is the BPS saturated state. Consequently we can interpret the entropy of this black hole microscopically by the degeneracy of string states as Sen did. In comparison to Sen’s solution which is extremal, our solution is non-extremal and it can possess non-vanishing entropy. Therefore the entropy of our solution can be understood as the microscopic entropy which counts the elementary string states in a straightforward way. We expect that we can find other interesting solutions of string theory from the classical solution of the Einstein equation in an analogous method.

Acknowledgements

We thank Y. Kitazawa for discussions and for carefully reading the manuscript and suggesting various improvements.

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