Abstract—This paper revisits the distributed $H_\infty$ robustness controller design for the platooning of connected vehicles. Recently, the relevant result subjected to the undirected topology has been studied, in the light of the symmetry of Laplace matrix. It is well known that the same problem is more challenging for the directed topology, since the Laplace matrix ceases to be symmetric. In this paper, the problem is solved by introducing more weighting parameters and setting suitable values for them. Then we show that the introduced weighting parameters lead to a positive effect on robustness, and solve the problem of feedback high gain. Finally, two numerical simulations and a practical simulation based on Next Generation Simulation (NGSIM) dataset are applied to illustrate the effectiveness of our method.

Index Terms—Platoon control, connected vehicles, directed topology, robustness analysis.

I. INTRODUCTION

The platooning of connected vehicles has been widely investigated for the past decades. It gets increasing attention in the control community for its applications, such as automated highway systems (AHS), intelligent transportation system (ITS), etc. For a comprehensive literature review, readers may refer to some recent survey papers [1], [2], [3], [4], [5] and references therein.

In the platooning of connected vehicles, a fundamental problem is inter-vehicle distance control. In this scenario, the platoon control can ensure that vehicles are driving with the pre-specified space among them, at the desired speeds [6], [7]. However, in the real world, the dynamics of a platoon system is inevitably affected by the environment (e.g., aerodynamic drag, frictional drag and slope, etc.). The cases will change the dynamic of the platoon system, even may well result in the system be unstable under the original controller.

In a word, the robustness problem of platoon control systems becomes particularly important. However such a problem has been solved by some existing works [8], [9], [10]. The paper [8] synthesises an $H_\infty$ controller to achieve the string stability for the platooning with linear dynamics, by solving a linear inequality matrix. The paper [9] designs the distributed $H_\infty$ control and analyzes the robustness in the frequency domain, addressing the dynamics with an uncertainty model in a platoon system. While the paper [10] deals with the scenario where the platoon system is with the external disturbance in the time domain. However both of them address the case that the interaction topologies of information are undirected among vehicles. This case is relatively easy to be solved, since the undirected topology’s Laplace matrix is symmetric. In other words, it is diagonalizable. If the matrix involving to graph is diagonalizable is decisive to analyze the robustness problem.

The platooning of vehicles over the undirected topology is more robust than the directed case in general, but it is at the expense of more sensors and network bandwidth in practice. Intuitively, it implies that a system is more robust if it can get more useful information. However, the platoon problem over directed graph is more important and more consistent with reality, taking into account the cost of devices and network bandwidth. At present papers [11] and [12] have been devoted to solving it. However all works subjected to directed topology are usually coupling, thus the corresponding robustness analysis is always a great challenge.

Motivated by the aforementioned work [10], this paper aims to propose a novel distributed $H_\infty$ controller by introducing more weighting parameters, which not only solves the inter-vehicle distance control problem over the directed topology, with the external disturbance, but also makes the robustness analysis in the time domain possible. More specifically, we intend to change the weights of the graph in disguise and make the collected system decoupled by designing suitable parameters. Compared with the work in [10], the contributions of our work are listed as follows:

1) The robustness analysis is solved for the directed topology scenario. The difficulty lies in that the Laplace matrix ceases to be diagonalizable (it is hard to be decoupled). As far as we know, no related results yet emerge at present;
2) It is shown that the robustness of platoon control is also dependent on the introduced weighting parameters, except for the number of the vehicles $N$ and the topological structure;
3) The feedback high gain problem is circumvented by our method. In general, the control signal is bounded and it may not be infinitely large. If the coupling strength is too large, it is difficult to implement in practice.

The rest of this paper is organized as follows. Section II formulates the problem to be solved in this paper. The robustness analysis with the directed topology is provided in Section III. The distributed $H_\infty$ synthesis for the platooning is shown in Section IV. Numerical examples in Section V illustrate the effectiveness of the proposed controller. Section VI concludes the paper.

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II. Problem Statement

Here we consider the case as the same as that in [10]. The platoon problem of connected $N+1$ homogeneous vehicles is studied, with a leading node indexed by 0. The remaining vehicles are indexed from 1 to $N$, which are referred to following nodes. Our objective is to analyze the robustness and synthesize a distributed $H_{\infty}$ controller to ensure that all vehicles move at the desired speed, meanwhile maintain the specific distances. The only difference is that the information flow among vehicles is directed. It is more challenging for this specific case, which is not yet solved up to now.

Here, we use the following dynamic model to express the acceleration response of vehicle dynamics

$$\tau \ddot{x}_i(t) + a_i(t) = u_i(t) + w_i(t),$$

where $a_i(t)$ denotes the acceleration of vehicle $i$; $\tau$ is the time delay parameter; $u_i$ is the control input, and $w_i(t)$ is the disturbance from the exosystem, such as the aerodynamic drag, frictional drag and slope, etc.

Letting $p_i(t)$, $v_i(t)$ and $a_i(t)$ to denote respectively each vehicle’s position, velocity and acceleration, a state-space system of a vehicle dynamics is written as

$$\dot{x}_i(t) = Ax_i(t) + B_1u_i(t) + B_2w_i(t),$$

where

$$x_i = \begin{bmatrix} p_i(t) \\ v_i(t) \\ a_i(t) \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix}.$$

The dynamic system (2) is a simplified model of vehicle with parasitic time delays and lags (see [13], [14] and [15], etc.).

In a platooning, the allowable communication connections subjected to directed graphs among vehicles is a more general case.

First of all, a neighbor set of node $i$ is defined as

$$N_i = \{ j \in \mathcal{V}_N | a_{ij} = 1 \},$$

where $\mathcal{V}_N$ is a set of $N$ nodes, i.e., $\mathcal{V}_N = \{1, 2, \ldots, N\}$; $a_{ij} = 1$ is the weight from neighbour node $j$ to node $i$ in a directed graph.

To model the communications from the leader to the followers, we define an augmented directed graph $G_{N+1} = (\mathcal{V}_{N+1}, \mathcal{E}_{N+1})$, with a set of $N+1$ vertices ($\mathcal{V}_{N+1} = \{0, 1, \ldots, N\}$), which includes both the leader and the followers in the platooning; $\mathcal{E}_{N+1}$ is a set of edges, which belongs to $\mathcal{V}_{N+1} \times \mathcal{V}_{N+1}$. We use a pinning matrix $P = \text{diag}\{g_1, g_2, \ldots, g_N\}$ to denote how each follower connects to the leader: $g_i = 1$ (or a positive constant) if $(0, i) \in \mathcal{E}_{N+1}$, otherwise $g_i = 0$. Note that the interactive information is directed among nodes.

Assumption 1 The graph of a platooning $G_{N+1}$ is a star topology [10].

The assumption means the leader can send its information to every follower directly. Note that the commonly used topologies in the platooning belongs to the star topology, (see Fig.1 in [12]).

Objective: The platoon control makes vehicles achieve the same speed and the desired spacing between the front and the rear vehicles:

$$\lim_{t \to \infty} \|v_i(t) - v_0(t)\| = 0;$$

$$\lim_{t \to \infty} \|p_i(t) - p_{i-1}(t) - d_{i,i-1}\| = 0: i = 1, \ldots, N,$$

where $d_{i,i-1}$ is the desired spacing between vehicles $i$ and $i-1$. $v_0$ is the leader’s velocity.

In this paper, to analyze the robustness of a platooning over the directed graph, we modify the local controller (8) in [10] as

$$u_i = -ck^T(g_i \dot{x}_i + \sum_{j \in N_i}(d_i \dot{x}_i - d_{ij} \dot{x}_j)), i = 1, \ldots, N,$$

where $k = [k_p \ k_v \ k_a]^T$, $c$ is the coupling strength, and $g_i, d_i, d_{ij} > 0$ are the weighting parameters, which are introduced deliberately to strengthen the self signal (by $g_i, d_i$), and weaken the neighborhood information (by $d_{ij}$). The aim is to implement the decoupling controller. $\dot{x}_i = [\dot{p}_i \ \dot{v}_i \ \dot{a}_i]^T$ is the tracking error for vehicle $i$, that is,

$$\dot{p}_i = p_i(t) - p_0(t) - d_{i,0},$$

$$\dot{v}_i = v_i(t) - v_0(t),$$

$$\dot{a}_i = a_i(t) - a_0(t),$$

where, $\dot{p}_i$ is the spacing error between the vehicles $i$ and 0 with $d_{i,0}$ which is the desired spacing between the vehicles $i$ and 0. $\dot{v}_i$ and $\dot{a}_i$ are respectively the velocity error and acceleration error between the vehicles $i$ and 0.

Remark 1 Note that in the controller (4), we use the absolute error of neighbours’ information by installing some amplifiers with the gain $d_i$ and $d_{ij}$, rather than a relative error. However the controller with relative error is more widely used in the platoon control community. For this purpose, we can rewrite the controller (4) as

$$u_i = -ck^T(g_i \tilde{x}_i + \sum_{j \in N_i} \tilde{d}_i \tilde{x}_i + \sum_{j \in N_i} d_{ij} \tilde{e}_{ij}), i = 1, \ldots, N,$$

where $\tilde{e}_{ij} = \tilde{x}_i - \tilde{x}_j$, and $\tilde{d}_i = d_i - d_{ij}$. Obviously, a vehicle only needs its own feedback, as the relative error can be obtained. In fact, the controller (4) with absolute error can be unified as the relative one (5).

The collective formulation of the controller (4) is

$$U = -cM \otimes k^T X,$$

where $X = [\tilde{x}_1^T \ldots \tilde{x}_N^T]^T$, $U = [u_1 \ldots u_N]^T$, $M = \mathcal{L}_d + P$, $D = D - A$, $D = \text{diag}\{d_i \sum_{j=1}^N a_{ij}\}$ and $A = \{d_{ij}a_{ij}\}_{N \times N}$.

Then, we can write the closed-loop dynamics of the platooning as

$$\dot{X} = A_e X + BW,$$

$$Y = CX,$$

where $W = [w_1 \ldots w_N]^T$, $A_e = I_N \otimes A - cM \otimes B_1 k^T$, $B = I_N \otimes B_2$, $C = I_N \otimes C_1$, $C_1 = [1, 0, 0]$, and $Y$ is the
tracking error of positions (i.e., $Y = [\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_N]^T$), which denotes the output of a platooning. Under the zero initial tracking errors, the transfer function from the disturbance $W$ to the position $Y$ is

$$G(s) = C(sI_{3N} - A_e)^{-1}B,$$

$$= [I_N(\tau s^3 + s^2) + c(\mathcal{L}_d + \mathcal{P})(k_p + k_c s + k_u s^2)]^{-1}.$$ 

(8)

Generally, for a directed graph it ceases to be trivial to analytically get the $H_\infty$ norms. The inverse operation in (8) is the greatest challenge, especially simultaneously involving the factor $\mathcal{L}_d + \mathcal{P}$ (It ceases to be symmetric as $\mathcal{L} + \mathcal{P}$ in (10)). In the next section, we will focus on the analysis and synthesis of the distributed robustness controller with a guaranteed performance by the novel controller (4).

III. MAIN RESULTS ON ROBUSTNESS ANALYSIS OF PLATOONING WITH DIRECTED GRAPH

In the light of the idea in (10), if only $\mathcal{M}$ is diagonalizable, the robustness analysis and controller synthesis are feasible. In the following lemma, we give a sufficient condition that the matrix $\mathcal{M}$ can be diagonalized.

**Lemma 1** For $\mathcal{M}$, if a sequence $\{\kappa_1, \cdots, \kappa_N\} = N$ satisfies

$$o_{\kappa_i} > r_{\kappa_i}, \quad o_{\kappa_{i+1}} - o_{\kappa_i} > r_{\kappa_{i+1}} + r_{\kappa_i}, \quad i = 1, \ldots, N - 1,$$

(9)

where $o_i = g_i + d_i \sum_{j=1}^{N} a_{ij}$ and $r_i = \sum_{j=1}^{N} d_{ij} a_{ij}$, then $\mathcal{M}$ is diagonalizable.

**Proof 1** We know that the diagonal entry of the $i$-th row of $\mathcal{M}$ is $o_i$ and the sum of the absolute values of the non-diagonal entries in the $i$-th row is $r_i = \sum_{j=1}^{N} d_{ij} a_{ij}$. It can define a Geršgorin disc centred at $o_i$ with radius $r_i$, denoted as $D(o_i, r_i)$.

Under the condition (9), the Geršgorin discs are $D(o_{\kappa_i}, r_{\kappa_i})$ in the order of $i = 1, \ldots, N$, from the left to the right, and they do not intersect; see Fig. 1. Therefore, by Geršgorin theorem, all eigenvalues of $\mathcal{M}$ are distinct positive real numbers. In other words, $\mathcal{M}$ is diagonalizable.

![Fig. 1. Non-intersect Geršgorin discs.](image)

**Remark 2** The proof of Lemma 1 shows that if the center of the leftmost Geršgorin disc is sufficiently large with constant radius, then the diagonal entries of $\mathcal{M}$ are sufficiently large relative to the non-diagonal ones. In brief, we use the idea of diagonal dominance to make the diagonal elements sufficiently large, or the non-diagonal elements sufficiently small, then the matrix $\mathcal{M}$ approximates to a diagonal matrix. This property will be used in the proof of the following theorem.

**Theorem 1** Consider a platooning, which consists of homogeneous vehicles, over the directed graph given by (3). Under the conditions in Lemma 1 using any stabilizing feedback gains, the robustness measure $\gamma$-gain satisfies

$$\gamma \leq \inf_{\omega} \left\{ \frac{1}{\lambda_{\min}(\gamma^* V^T V)} \sqrt{\frac{\lambda_{\max}(\gamma V^T V)}{\lambda_{\min}(\gamma V^T V)}} \right\},$$

where $\lambda_{\min}$ denotes the minimum eigenvalue of $\mathcal{M} := \mathcal{L}_d + \mathcal{P}$.

**Proof 2** Here, for convenience in robustness analysis, we assume that the coupling strength $c = 1$ in the controller (4).

Under the conditions in Lemma 1 the eigenvalues of $\mathcal{L}_d + \mathcal{P}$ are different from each other. There exists a nonsingular matrix $V \in \mathbb{R}^{N \times N}$, $V V^{-1} = I_N$, such that

$$\mathcal{L}_d + \mathcal{P} = V \Lambda V^{-1},$$

(10)

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, and $\lambda_i$ is the $i$-th real eigenvalue of $\mathcal{L}_d + \mathcal{P}$.

Inserting (10) into (8), we can get

$$G(s) = V \begin{bmatrix} G_1(s) & G_2(s) & \cdots & G_N(s) \end{bmatrix} V^{-1}$$

where $G_i(s) = \frac{1}{\tau s^3 + (1 + \lambda_i k_s) s^2 + \lambda_i k_c s + \lambda_i k_p}$, $i = 1, \ldots, N$.

In the light of

$$\gamma = \|G(s)\|_{H_\infty} = \sup_{\omega} |\sigma_{\max}(G(j\omega))|,$$

and letting $\tilde{G} = \text{diag}\{G_i(s)\}_{i=1}^N$, we have that

$$\gamma = \|G(s)\|_{H_\infty} = \sup_{\omega} \sqrt{\lambda_{\max}(\gamma V^T V)} \sup_i \sqrt{V^{-1} G_i^*(j\omega) G_i(j\omega) V^{-1}}$$

$$\leq \sqrt{\lambda_{\max}(\gamma V^T V)} \sup_i \max_{\omega} \sqrt{V^{-1} G_i^*(j\omega) G_i(j\omega) V^{-1}}$$

$$\leq \max_i \|G_i(s)\|_{H_\infty} \sqrt{\frac{\lambda_{\max}(\gamma V^T V)}{\lambda_{\min}(\gamma V^T V)}},$$

where $\cdot^*$ means the complex conjugate operation. Furthermore, according to $G_i(s)$, we can obtain

$$\gamma \leq \frac{1}{\lambda_{\min}(\gamma V^T V)} \sqrt{\frac{\lambda_{\max}(\gamma V^T V)}{\lambda_{\min}(\gamma V^T V)}}.$$

For the detail, see the proof of Theorem 1 in (10). And then,

$$\gamma \leq \gamma_u := \inf_{\omega} \left\{ \frac{1}{\lambda_{\min}(\gamma V^T V)} \sqrt{\frac{\lambda_{\max}(\gamma V^T V)}{\lambda_{\min}(\gamma V^T V)}} \right\},$$

(11)
From Remark 2, we know that the matrix $M$ can approximate to a diagonal matrix, if the parameters $d_i$ and $d_{i,j}$ are taken properly. The matrix $V$ approximates to $\bar{k}I$ ($\bar{k}$ is a constant). Further, we have
\[
\gamma \leq \gamma_u := \inf_{g, d_i, d_{i,j}} \left\{ \frac{1}{\lambda_{\text{min}} k_p} \right\}.
\]

Remark 3 Theorem 4 shows an implicit relation between the proposed robustness measure, i.e. $\gamma$-gain, and the introduced weighting parameters. In the proof, the key point lies in the diagonalization of $M$ matrix, thus it results in the decoupling of the transfer function $G(s)$. Therefore, it only needs to analyze the $H_\infty$ norm of a single transfer function, and this transfer function can be modified by adjusting the eigenvalue in the light of the parameters $g_i, d_i$ and $d_{i,j}$ in the controller 1.

Remark 4 To achieve the better robustness performance for a mass of platoons, (7) implies that we must select appropriate controller parameters $g_i, d_i$ and $d_{i,j}$ associated with a larger $\lambda_{\text{min}}$. A practical choice is to make the center of the leftmost Geršgorin disc be far away from the origin as possible, and then its radius as small as possible (See Fig. 1). It means that the minimum eigenvalue $\lambda_{\text{min}}$ is enlarged.

Different from the results in [10] (see Corollaries 1 and 2), Remark 3 shows that under the presented controller 4, the robustness performance for a mass of platoons does depend on the controller parameters $g_i, d_i$ and $d_{i,j}$, and is no longer limited to the total number $N$ of vehicles and the number of followers that are connected to the leader (structure). It will be a more flexible way to improve the robustness performance for a mass of platoons without changing the structure of the platoon and the number of vehicles.

IV. DISTRIBUTED $H_\infty$ CONTROLLER SYNTHESIS OF THE PLATOONING

Here, we introduce a modified distributed controller for a platooning with the guaranteed $H_\infty$ performance. To satisfy $\gamma = \|G(s)\|_{H_\infty} < \gamma_d$ with a given desired $\gamma_d$-gain, the feedback gains $((k = [k_p, k_y, k_v]^T))$ and coupling strength $c$ need to be designed.

Based on the decoupling technique in the proof of Theorem 1 the distributed $H_\infty$ control problem is converted into a set of $H_\infty$ control of independent systems sharing the same dimension with a single vehicle (See Theorem 3 in [16]).

The collective behavior of a platoon system (7) is decoupled into a set of $N$ individual subsystems (12)
\[
\begin{align*}
\dot{x}_i &= (A - c\lambda_i B_1 k^T)x_i + B_2 \bar{w}_i, \\
\bar{y}_i &= C_1 \bar{x}_i, & i = 1, \ldots, N,
\end{align*}
\] (12)
in the light of Theorem 3 in [16]. The distributed $H_\infty$ control problem of a platooning (7) is equivalently reduced to a set of $H_\infty$ control problems of subsystems (a single vehicle) with the same dimension $3 \times 3$. The computational complexity is dramatically reduced due to the less dimension of systems, for the controller synthesis.

Next we synthesize the distributed $H_\infty$ controller by the following theorem.

Theorem 2 [10] Consider a homogeneous platooning with the directed topology. For any desired $\gamma_d > 0$, we have $\|G(s)\|_{H_\infty} < \gamma_d$, if the feedback gains are chosen as $k^T = \frac{1}{2} B_1^T Q^{-1}$, and the coupling strength satisfies
\[
c \geq \frac{\sqrt{\alpha}}{\lambda_{\text{min}}},
\] (13)
where $Q > 0$ and $\alpha > 0$ are the feasible solutions to the linear matrix inequality (14)
\[
\begin{bmatrix}
A Q + Q A^T - \alpha B_1 B_1^T & B_2 & Q C_2^T \\
B_2^T & -\gamma_d^2 & 0 \\
C_1 Q & 0 & -1
\end{bmatrix} < 0.
\] (14)

To design a distributed $H_\infty$ controller, the feedback gains $k^T$ can be obtained by solving LMI (14), and the coupling strength $c$ can be adjusted to satisfy the condition (13). In addition, it retains the advantage of the computational complexity in [10].

Remark 5 It is worth mentioning that Theorem 2 implies that for a platoon system, a distributed controller (4) can be synthesized to satisfy any given $H_\infty$ performance. The new controller can circumvent the high-gain case mentioned in [10], if only the matrix $L_d + P$ associated with the directed graph has a larger $\lambda_{\text{min}}$. In fact, we can do it by moving the leftmost Geršgorin disc away from the origin as far as possible, in the light of adjusting the controller parameters $g_i, d_i$ and $d_{i,j}$. It not only improves the robustness performance for a given controller (See the expression (17)), but also reduces the value of coupling strength $c$ for a given $H_\infty$ performance (See the expression (13)).

V. NUMERICAL SIMULATION

In this section, one numerical experiment with passenger cars are used to verify the effectiveness of the proposed method for directed graphs. Next the synthesis of a distributed $H_\infty$ controller is also illustrated. Here we consider that the generalized directed topology with 8 vehicles and one leader vehicle 0 which is shown in Fig. 2.

![Fig. 2. Directed communication topology with 8 vehicles and one leader vehicle](image-url)
A. Validation of superiority

We can design a distributed $H_\infty$ controller in the light of Theorem 2 using the same parameters as that in [10] (i.e., $\tau = 0.5s$, $\alpha = 1.968$ and $k = [2.122, 3.425, 2.501]^T$ under the desired performance $\gamma_d = 1$). We also implement these controllers for the platooning, like the scenario in [10]: the zero initial tracking state errors for the platooning, a constant reference speed for the leader ($v_0 = 20m/s$), and the same external disturbances for each node:

$$w_i(t) = \begin{cases} 
0 & 0 < t < 5s \\
Q \sin\left(\frac{2\pi}{3}(t-5)\right) & 5s \leq t < 10s \\
0 & 10s \leq t 
\end{cases}$$

where $Q = 10$. The disturbance $w_i$ is a generalized expression for all possible disturbance (such as aerodynamic drag, frictional drag and slope, etc.), hence its physical significance hinges on the application scenarios.

Under the controller 4, the matrix $\mathcal{M}$ corresponding to the topology in Fig. 2 is

$$\mathcal{M} = \begin{bmatrix}
\bar{g}_1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & \bar{g}_2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & \bar{g}_3 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & \bar{g}_4 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & g_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \bar{g}_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & \bar{g}_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{g}_8 
\end{bmatrix},$$

with $d_{ij} = 1$ ($j \in \mathbb{N}_i$), $\bar{g}_1 = 2d_1 + g_1$, $\bar{g}_3 = 3d_3 + g_3$, $\bar{g}_i = d_i + g_i$, $i = \{2, 4, 7, 8\}$.

First of all, we chose the parameters of Test (a) in Table I for the $H_\infty$ controller. And then we can calculate numerically $\lambda_{\min}(\mathcal{L}_d + \mathcal{P})$ and the coupling strength $c$, the error amplification $\|Y(t)\|_{\mathcal{L}_2}$ (i.e., $\gamma$-gain $\gamma$) in time-domain, see Test (a) in Table I. The curves of spacing errors are shown in Fig. 3. It shows that the platooning is able to regain stability after receiving the disturbance from 5s to 10s, under the $H_\infty$ controller which is synthesized by Theorem 2. This simulation verifies the effectiveness of our controller for the directed graph. While the work [10] can not deal with this case for the directed graph.

Next we change the parameters as Test (b) shown in Table I. The corresponding results are obtained in the same way by the numerical calculation, as Test (b) shown in Table I. Meanwhile, the corresponding profiles of spacing errors are shown in Fig. 4. Obviously, the worst spacing error became smaller under the same external disturbances. Thus it has the better robustness with the larger $\lambda_{\min}(\mathcal{L}_d + \mathcal{P})$. In addition, the coupling strength $c$ does not have the high gain, and the robustness is also improved, compared with that in [10]. Note that the sign ‘–’ means that the data is not provided in [10].

| Test | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ | $g_5$ | $g_6$ | $g_i (i \neq 5, 6)$ | $d_{ij}$ |
|------|------|------|------|------|------|------|------|------|----------------|--------|
| (a)  | 4    | 6    | 1    | 5    | 3    | 2    | 12, 10| 0.1  | 1              |        |
| (b)  | 24   | 24   | 12   | 20   | 7    | 14   | 12, 10| 0.1  | 1              |        |
TABLE II
CALCULATING COUPLING STRENGTH AND SOME PERFORMANCE INDICES

| Test | $\lambda_{\text{min}}(L_d + P)$ | $c$ | $\gamma$ |
|------|-------------------------------|-----|---------|
| (a)  | 2.1                           | 0.6680 | 0.4501 |
| (b)  | 5.1                           | 0.2751 | 0.2996  |
| [10] | 0.0557                        | 35.33     |        |

B. Influence of disturbance for the platoon system

The spacing errors of the platoon are also mainly dependent on the amplitude (strength) of disturbance $w_i$ under the given $H_\infty$ controllers. It is well-known that the errors will generally become larger as the amplitude of $w_i$ grows. In the following, we present the simulation examples to verify this point in the light of the controller which is used in Fig. 3. We take $Q = 30$, $5s \leq t \leq 10s$, then the spacing errors lie in the interval $(0.4, 6.4)$ (meters), which are shown in Fig. 5.

![Fig. 5. Spacing errors for platooning under the same controllers in Fig. 3 with $Q = 30$.](image)

Obviously, the errors are larger than ones in Fig. 3 where the errors are in the interval $(0.3, 2.9)$ (meters). Thus we verify that the errors will become large with the increasing of strength of disturbance.

C. A simulation based on NGSIM dataset

Here, the Next Generation Simulation (NGSIM) dataset is used to verify the effectiveness of our method. In the dataset, the real-world trajectory information (including lane-specific location, velocity and acceleration, collected every one-tenth of a second) of the entire traffic flow were available. We screen out the information (time, longitudinal location, velocity and acceleration, referring to [17]) of vehicle (No.4) in the I-80 dataset as the state of the leading car (the node 0 in Fig. 2).

Due to the NGSIM trajectory data exhibit errors and noises that need to be filtered out, we adopt the locally weighted regression method [18] to get their smooth and continuous curve, see Fig. 6.

![Fig. 6. The smoothed curve of the raw data.](image)

Next, we use the designed controller in Test (a) of Table I to steer the vehicles to reach a consensus on speed and keep the desired spacing ($|d_{i,i-1}| = 4.5m$) over the communication topology in Fig. 2. To verify the robustness of the vehicle platoon under the controller, we exert a disturbance (aerodynamic drag $c_2 v_i^2$, where $c_2 = 0.5$) into the platoon system in interval $(57s, 63s)$.

From Fig. 7, except for the time interval $(57s, 71s)$, the effect of tracking is very good. From 57s to 63s, the velocity of the following cars slow down due to the effect of aerodynamic drag. And then, after 71s, it reaches a consensus on $v_0$ again. The phase from 63s to 71s is the transient process.

![Fig. 7. The trajectories of velocity tracking.](image)

From Fig. 8, the spacing error of every following vehicle has changed when the disturbance is exerted. The vehicles 3 and 8 are the most obvious ($|\hat{p}_3| = 5.3m$ and $|\hat{p}_8| = 4.9m$).
When the disturbance vanishes, the spacing errors go to zero again, that is, the platoon system is back to the desired state.

Above result is also verified by the trajectories of longitudinal location in Fig. 9, the speed of vehicles 3 and 8 drop off notably when the aerodynamic drag occurs. And then the vehicle 3 nearly has a collision with the vehicle 4. It is not what we would like to see, since there is a high probability of traffic accidents. Note that although the spacing error of the vehicle 3 is more than the desired spacing (4.5 m), they do not collide, since the vehicle 4 also slows down by the effect of the aerodynamic drag.

To avoid the collision between the front vehicle and the rear one, the robustness of the platoon system is improved by our method, in the light of the controller parameters in Test (b) of Table I. Apparently, the worst of spacing error is improved, from Fig. 10 that is, the spacing error is reduced to 3.9 m from 5.3 m compared with that in Fig. 8. Moreover, Fig. 11 shows that no vehicle is at risk of collision when the disturbance occurs. The above simulation verifies the effectiveness of our method for improving the robustness of platoon system by NGSIM data.

VI. Conclusion

In this paper, a new control structure is presented to study the robustness and distributed $H_{\infty}$ controller synthesis for a platooning of connected vehicles with the directed topology. Compared with the problem of the conventional undirected topology [10], we solve the problem on the directed topology, which is more challenging. This paper features itself in two aspects: first, the robustness performance of the platoon control can be improved by choosing the suitable controller parameters, and it no longer depends on the number of vehicles $N$ and information topology; second, we circumvent the high-gain problem for the synthesis of distributed $H_{\infty}$ controller. Namely the coupling strength $c$ is not very large such that the new control structure is practical in implementation. In the future, we will try to study the string stability problem by the
proposed controller. We think it is feasible, since our method has generality for platoon systems.

REFERENCES

[1] K.-K. Oh, M.-C. Park, and H.-S. Ahn, “A survey of multi-agent formation control,” *Automatica*, vol. 53, pp. 424 – 440, 2015.

[2] C. Bergenhem, H. Pettersson, E. Coelingh, C. Englund, S. Shladover, and S. Tsugawa, “Overview of platooning systems,” in *2012 19th ITS World Congress*, 2012.

[3] S. Tsugawa, “Inter-vehicle communications and their applications to intelligent vehicles: an overview,” in * Intelligent Vehicle Symposium*, 2002. *IEEE*, vol. 2, June 2002, pp. 564–569 Vol.2.

[4] Z. Wang, Y. Bian, S. E. Shladover, G. Wu, S. E. Li, and M. J. Barth, “A survey on cooperative longitudinal motion control of multiple connected and automated vehicles,” *IEEE Intelligent Transportation Systems Magazine*, 2019, DOI: 10.1109/MITS.2019.2953862.

[5] T. L. Willke, P. Tientrakool, and N. F. Maxemchuk, “A survey of inter-vehicle communication protocols and their applications,” *IEEE Communications Surveys Tutorials*, vol. 11, no. 2, pp. 3–20, Second 2009.

[6] J. Guo, Y. Luo, K. Li, and L. Guo, “Adaptive dynamic surface longitudinal tracking control of autonomous vehicles,” *IET Intelligent Transport Systems*, vol. 13, no. 8, pp. 1272–1280, 2019.

[7] S. Monduri, P. Pagilla, and S. Darbha, “Vehicle platooning with multiple vehicle look-ahead information,” *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 5768 – 5773, 2017, 20th IFAC World Congress.

[8] J. Ploeg, D. P. Shukla, N. van de Wouw, and H. Nijmeijer, “Controller synthesis for string stability of vehicle platoons,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 15, no. 2, pp. 854–865, 2014.

[9] S. E. Li, F. Gao, K. Li, L. Wang, K. You, and D. Cao, “Robust longitudinal control of multi-vehicle systems- A distributed h-infinity method,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 9, pp. 2779–2788, Sep. 2018.

[10] Y. Zheng, S. E. Li, K. Li, and W. Ren, “Platooning of connected vehicles with undirected topologies: Robustness analysis and distributed h-infinity controller synthesis,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 19, no. 5, pp. 1353–1364, 2018.

[11] Y. Zheng, S. E. Li, K. Li, F. Borrelli, and J. K. Hedrick, “Distributed model predictive control for heterogeneous vehicle platoons under unidirectional topologies,” *IEEE Transactions on Control Systems Technology*, vol. 25, no. 3, pp. 899–910, 2017.

[12] Y. Zheng, S. Eben Li, J. Wang, D. Cao, and K. Li, “Stability and scalability of homogeneous vehicular platoon: Study on the influence of information flow topologies,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 17, no. 1, pp. 14–26, 2016.

[13] S. Darbha, *String stability of interconnected systems: An application to platooning in automated highway systems*. Ph.D. dissertation, Univ. California, Berkeley, 1994.

[14] L. Xiao and F. Gao, “Practical string stability of platoon of adaptive cruise control vehicles,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 12, no. 4, pp. 1184–1194, 2011.

[15] D. Swaroop, J. Hedrick, C. C. Chiou, and P. Ioannou, “A comparison of spacing and headway control laws for automatically controlled vehicles: I,” *Vehicle System Dynamics*, vol. 23, no. 1, pp. 597–625, 1994.

[16] Z. Li, Z. Duan, and G. Chen, “On $H_{\infty}$ and $H_{2}$ performance regions of multi-agent systems.” *Automatica*, vol. 47, no. 4, pp. 797 – 803, 2011.

[17] Z. Sun, X. Yao, Z. Qin, P. Zhang, and Z. Yang, “Modeling car-following heterogeneities by considering leader–follower compositions and driving style differences,” *Transportation Research Record*.

[18] T. Toledo, H. N. Koutsopoulos, and K. I. Ahmed, “Estimation of vehicle trajectories with locally weighted regression,” *Transportation Research Record*, vol. 1999, no. 1, pp. 161–169, 2007.