Spiked eigenvalues of high-dimensional sample autocovariance matrices: CLT and applications

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Abstract

High-dimensional autocovariance matrices play an important role in dimension reduction for high-dimensional time series. In this article, we establish the central limit theorem (CLT) for spiked eigenvalues of high-dimensional sample autocovariance matrices, which are developed under general conditions. The spiked eigenvalues are allowed to go to infinity in a flexible way without restrictions in divergence order. Moreover, the number of spiked eigenvalues and the time lag of the autocovariance matrix under this study could be either fixed or tending to infinity when the dimension $p$ and the time length $T$ go to infinity together. As a further statistical application, a novel autocovariance test is proposed to detect the equivalence of spiked eigenvalues for two high-dimensional time series. Various simulation studies are illustrated to justify the theoretical findings. Furthermore, a hierarchical clustering approach based on the autocovariance test is constructed and applied to clustering mortality data from multiple countries.

Keywords: high dimensional sample autocovariance matrices; spiked eigenvalues; central limit theorem; autocovariance test; hierarchical clustering

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1 Introduction

Advances in modern technology have facilitated the collection and analysis of high-dimensional data. A major challenge of statistical inference on high-dimensional data is the well-known “curse of dimensionality” phenomenon [13]. Dimension reduction, which projects high-dimensional data into a low-dimensional subspace, is a natural idea to overcome the large-dimensional disaster. Principal component analysis (PCA) is a commonly-used dimension-reduction technique for high-dimensional independent and identically distributed (i.i.d) data, which pursues the low-dimensional subspace that keeps the most variation of the original data. A significant and intrinsic difference between i.i.d data and time series lies in the perception that time series have temporal dependence along the sample observations. As informed in earlier literature, Box and Tiao [10], Pena and Box [23], Tiao and Tsay [26], identifying the low-dimensional representation or common factors that drive the temporal dependence of original time series is the major purpose of dimension reduction for high-dimensional time series. Lam and Yao [17] conduct the eigen-decomposition of autocovariance matrices and justify that, the eigenvectors corresponding to the largest eigenvalues span a subspace where the projection of the original time series reserves the most temporal covariance. In view of such a close connection, there is a need to explore the spectral properties for high-dimensional autocovariance matrices.

The major contribution of this paper is to establish the asymptotic distribution of spiked eigenvalues for high-dimensional sample autocovariance matrices. Similar to the spiked covariance model raised in Johnstone [16], we consider a spiked autocovariance model in which the population autocovariance matrix has a few large eigenvalues, called spiked eigenvalues, that are detached from the bulk spectrum. The spiked autocovariance model could be expressed via a factor model, like Lam and Yao [17], in which all temporal dependence is absorbed in low-dimensional common-factor time series. Intuitively speaking, spiked eigenvalues are equal to autocovariances of common-factor time series. In view of this point, spiked eigenvalues from high-dimensional autocovariance matrices
could quantify the temporal dependence reserved in low-dimensional projected time series or common-factor time series. We will work under this factor model and investigate spiked eigenvalues from a symmetrized sample autocovariance matrix, which is the product of the autocovariance matrix and its transpose.

In the context of high-dimensional sample autocovariance matrix analysis, fundamental asymptotic properties for spiked eigenvalues under moderately high-dimensional settings are available in the literature. Although Lam and Yao [17] and Li et al. [22] both focus on a ratio-based selection criterion for the number of factors, they essentially contribute to asymptotic properties of spiked and nonspiked eigenvalues of high-dimensional sample autocovariance matrices. In the case of strong spikiness, that is the spiked eigenvalues tending to infinity, Lam and Yao [17] provide the rate of convergence for spiked and nonspiked eigenvalues. To be more sophisticated, Li et al. [22] investigate the exact phase transition that distinguishes the factor part and the noise part. So their proposed ratio-based estimator is also applicable in weak spiked cases when the spiked eigenvalues are of constant order. More result for unspiked eigenvalues are derived in Li et al. [21], Wang and Yao [28] and Bose and Bhattacharjee [7]. Recently, Yao and Yuan [31], Bose and Hachem [8] develop asymptotic properties of the smallest eigenvalues for large dimensional autocovariance matrices and their variants.

As the first main contribution of this paper, under general conditions, we establish the asymptotic normality of $\lambda_i$, the $i$-th largest spiked eigenvalue of the matrix $\hat{\Sigma}_y(\tau)\hat{\Sigma}_y^T(\tau)$, where $\hat{\Sigma}_y(\tau)$ is the lag $\tau$ sample autocovariance matrix of high-dimensional time series $\{y_t, t = 1, 2, \ldots, T\}$ under study. We assume that the spiked population eigenvalues $\{\mu_i\}_{i \leq K}$ diverge as $T \to \infty$ without restrictions on the diverging rates, which relaxes the specific rates used in Lam et al. [18] and Lam and Yao [17]. Additionally, we also allow the number of factor $K$ to be either fixed or diverging as $T \to \infty$. This type of assumption has been made in the literature for large covariance matrices such as Cai et al. [12], but has not yet been incorporated into the factor model for high-dimensional time series. Furthermore, the lag $\tau$ in the autocovariance matrix $\hat{\Sigma}_y(\tau)$ is allowed to be either
fixed or diverging. Our results show that the scalings for CLTs are not of the same order. In particular, if one is interested in the eigenvalues of $\hat{\Sigma}_y(\tau)\hat{\Sigma}_y(\tau)^\top$ for a moderately large $\tau$, the CLT in the regime where $K \to \infty$ might provide a more accurate result than the case for a fixed $\tau$. Since we are working in a regime with less restricted assumptions on the number of factors, the lag of autocovariance, and the spikiness, as a natural trade-off, some difficulties arising in our work are worth to be noted.

A major source of difficulty in our setting comes from less restriction on the rate of divergence of spiked population eigenvalues $\{\mu_i, i = 1, 2, \ldots, K\}$. We argue that the specification of the diverging speed of $\mu_i$ such as ones used in Lam et al. [18] and Lam and Yao [17] entirely reduce the analysis of a high-dimensional factor model to the study of low-dimensional common-factor time series (see the remarks below Theorem 3.1). While this aligns with the goals of dimension reductions in Lam et al. [18] and Lam and Yao [17], it obfuscates some interesting features otherwise seen in high-dimensional models. Without such restriction, the idiosyncratic noise is no longer negligible and we obtain a clearer picture of how the high-dimensional noise part accumulates and affects the location of spiked eigenvalues. More specifically, even though $\lambda_i$ is close to $\mu_i$ asymptotically, the convergence rate of $\lambda_i - \mu_i$ (after appropriate scaling) is in general slower than $T^{-1/2}$. In other words, we will not be able to obtain a CLT using $\mu_i$ as the centering term. What happens here is that the bias of $\lambda_i$ decays too slowly to obtain a CLT and a more accurate centering is needed. In our work, this centering term will be defined implicitly as the solution to an established equation. The phenomenon described above is common in large random matrix literature where, however, there is less emphasis on reducing high-dimensional models into low-dimensional ones [see, 12, for example].

Besides, instead of working with the autocovariance matrix $\hat{\Sigma}_y(\tau)$, we are dealing with the symmetrized version $\hat{\Sigma}_y(\tau)^\top\hat{\Sigma}_y(\tau)^\top$ in our analysis. From the technical aspect, the matrix $\hat{\Sigma}_y(\tau)\hat{\Sigma}_y(\tau)^\top$ could not be decomposed into a matrix with independent entries like the covariance matrix $\hat{\Sigma}$ does. Therefore, the common ideas and regular techniques of some existing works in large random matrix theory such as [3] and [12] are not applicable.
directly in our work. Consequently, as the first work on CLT for large autocovariance matrices under less restricted assumptions, we need a new approach to establish the asymptotic normality for the empirical eigenvalues \( \{ \lambda_i \} \). Moreover, the approach we develop here could potentially be applied to other types of products of covariance-type matrices.

Another important contribution of this paper is a novel autocovariance test which is built on the developed CLT for \( \{ \lambda_i \} \). It is well known that when the data dimension \( p \) increases with sample size \( T \), directly comparing and testing the equivalence of two autocovariance matrices is infeasible due to the “curse of dimensionality”. The major idea of the proposed so-called autocovariance test is to compare the autocovariance of the low-dimensional common-factor time series. It is equivalent to testing whether spiked population eigenvalues of two high-dimensional autocovariance matrices are the same. It is worth mentioning that, as the CLT involves some unknown parameters, we propose an AR-sieve bootstrap to derive a feasible test statistic. Furthermore, the proposed test statistic is powerful under some local alternative hypotheses, which are demonstrated via theoretical results and various simulation designs.

This autocovariance test is not only in its interest but also motivates other statistical inferences such as statistical clustering analysis on multi-population high-dimensional time series. In this paper, we construct a new hierarchical clustering approach based on the autocovariance test. It is applied to multi-country mortality data, for which we group those countries with similar low-dimensional autocovariances. The clustering results are consistent with findings in common literature on mortality studies.

The rest of this article is organized as follows. Section 2 introduces the setting and assumptions of our work, sets up the relevant notations, and presents some preliminary results. The theoretical results of our work are given in Section 3. In Section 3.1 we investigate the asymptotic location of empirical eigenvalues and construct an accurate centering for these eigenvalues. The CLT for empirical eigenvalues, which is the main result of our work, is given in Section 3.2. The proof of the CLT is quite involved and
is thus divided into a series of intermediate results collected in Appendix A and B, and technical lemmas are collected in Appendixes C and D. We give a summary of the strategy of the proof in Section 3.3 and explain how the intermediate results are used to obtain the CLT. Lastly, a novel auto-covariance test is illustrated in Section 4 as a statistical application of the proposed CLT where numerical results including simulation studies and real applications on mortality data are also provided.

2 Model Setting

In this work, we study a high-dimensional time series arising from a factor model considered in [17, 18, 22]. Suppose \((y_t)_{t=1,\ldots,T} \subseteq \mathbb{R}^{K+p}\) is a \((K+p)\)-dimensional stationary time series observed over a time period of length \(T\). Here the choice of writing \(K+p\) for the dimension of the time series is purely for notational convenience in our exposition and proofs. The time series \(\{y_t, t = 1, 2, \ldots, T\}\) follows a factor model below

\[
y_t = L f_t + \epsilon_t, \quad t = 1, \ldots, T,
\]

where the \(K \times T\) matrix \((f_t)_{t=1,\ldots,T}\) contains \(K\) independent factors, each assumed to be a stationary times series. The matrix \(L\) is the \((p+K) \times K\) factor loading matrix and \(\epsilon_t\) is a \((K+p)\)-dimensional idiosyncratic noise to be specified later.

To identify the factors \(f_t\) and factor loading \(L\), we assume that \(L^\top L\) is a diagonal matrix and all factors are standardized, i.e. \(\mathbb{E}[f_{it}] = 0\) and \(\mathbb{E}[f_{it}^2] = 1\) for all \(i = 1, \ldots, K\) and \(t = 1, \ldots, T\).

We work in a high-dimensional setting where \(p\) and \(T\) diverge simultaneously and the ratio \(p/T\) tends to a constant \(c > 0\) as \(T \to \infty\). The number of factors \(K\) is allowed to be either fixed or diverging as \(T \to \infty\), but the speed of its divergence is limited to be small in comparison to the dimension \(p\) (see Assumption 2.2 and Assumption 2.3).

Each factor \((f_{it})_t\) is assumed to be a stationary time series of the form

\[
f_{it} = \sum_{l=0}^{\infty} \phi_{il} z_{i,t-l}, \quad i = 1, \ldots, K, \quad t = 1, \ldots, T,
\]

where the \(K \times T\) matrix \((f_t)_{t=1,\ldots,T}\) contains \(K\) independent factors, each assumed to be a stationary times series. The matrix \(L\) is the \((p+K) \times K\) factor loading matrix and \(\epsilon_t\) is a \((K+p)\)-dimensional idiosyncratic noise to be specified later.
where the random variables \((z_{it})\) are i.i.d. with zero mean, unit variance and finite \((4 + \epsilon)\)-th moment for some small \(\epsilon > 0\). Under this setup, the constraint \(\text{Var}(f_{it}) = 1\) mentioned above directly translates to the constraint \(\|\phi_i\|_{\ell_2} = 1\) where \(\phi_i := (\phi_{il})_l\) is the vector of coefficients for the \(i\)-th factor and \(\|\cdot\|_{\ell_2}\) is the \(\ell_2\) norm on sequence spaces. Write \(\gamma_i(\tau) := \mathbb{E}[f_{i,1}f_{i,\tau+1}]\) for the lag-\(\tau\) population auto-covariance of the \(i\)-th factor time series \(f_i\). In terms of the representation (2.2), clearly \(\gamma_i(\tau)\) can be written as

\[
\gamma_i(\tau) := \mathbb{E}[f_{i,1}f_{i,\tau+1}] = \sum_{l=0}^{\infty} \phi_{il} \phi_{i,l+\tau}.
\] (2.3)

In general, the loading matrix \(L\) is important in the analysis of the factor model as it appears in the (population) covariance and auto-covariance matrices of \(y_t\). However, the recent work [22] makes an important observation that under additional Gaussian assumptions on the error time series \(\epsilon_t\), the factor model can be reduced to a canonical form where \(L = \begin{pmatrix} I_K & 0_{K \times p} \end{pmatrix}^T\). The authors of [22] are able to obtain explicit results on the phase transition of spiked eigenvalues under this assumption. As previously mentioned, for notational conveniences we employ a slightly different normalization for the matrix \(L\). Nevertheless, next we argue that under Gaussian assumptions on the error \(\epsilon_t\), the factor model could be reduced to a canonical form where \(L\) takes the form

\[
L = \begin{pmatrix} \text{diag}(\sigma_1, \ldots, \sigma_K) \\ 0_{p \times K} \end{pmatrix},
\] (2.4)

where \((\sigma_1, \ldots, \sigma_K)\) is a sequence of positive real numbers.

For the completeness of our exposition, we give a detailed explanation of the simplification (2.4). Clearly the \((p + K) \times K\) matrix \(\overline{L} := L \text{ diag}(\sigma_1^{-1}, \ldots, \sigma_K^{-1})\) satisfies \(\overline{L}^T \overline{L} = I_K\), thus there exists a \((p + K) \times p\) matrix \(\underline{L}\) with orthogonal columns such that \(\overline{L} := (\underline{L}, \underline{L})\) is an orthogonal matrix. Recall from (2.1) that \(y_t = L f_t + \epsilon_t\). Define

\[
z_t := \overline{L}^T y_t = \begin{pmatrix} \overline{L}^T \\ \underline{L}^T \end{pmatrix} L f_t + \overline{L}^T \epsilon_t = \begin{pmatrix} \overline{L}^T \\ \underline{L}^T \end{pmatrix} \text{ diag}(\sigma_1, \ldots, \sigma_K) f_t + \overline{L}^T \epsilon_t.
\]
By definition we clearly have $L^\top L = I_K$ and $L^\top \tilde{L} = 0_p$, therefore

$$z_t = \tilde{L}^\top y_t = \begin{pmatrix} \text{diag}(\sigma_1, \ldots, \sigma_K) \\ 0_{p\times K} \end{pmatrix} f_t + \tilde{L}^\top \epsilon_t. \quad (2.5)$$

Note that $z_t$ is simply the original data $y_t$ subjected to an orthogonal transformation, roughly speaking, the sample auto-covariance matrix of $(z_t)$ contains the same temporal information as that of $(y_t)$. More precisely, define the sample auto-covariance matrices

$$\Sigma_y(\tau) := \frac{1}{T} \sum_{t=1}^{T-\tau} y_{t+\tau}^\top y_t, \quad \Sigma_z(\tau) := \frac{1}{T} \sum_{t=1}^{T-\tau} z_{t+\tau}^\top z_t = \tilde{L}^\top \Sigma_y(\tau) \tilde{L}.$$  

It is easy to see that the spectrum of $\Sigma_y(\tau) \Sigma_y^\top(\tau)$ coincides with that of $\Sigma_z(\tau) \Sigma_z^\top(\tau)$. Indeed, we have

$$\Sigma_z(\tau) \Sigma_z^\top(\tau) = \tilde{L}^\top \Sigma_y(\tau) \tilde{L} \tilde{L}^\top \Sigma_y(\tau) \tilde{L} = \tilde{L}^\top \Sigma_y(\tau) \Sigma_y^\top(\tau) \tilde{L},$$

where $\tilde{L}$ is orthogonal so a conjugation by $\tilde{L}$ does not affect the spectrum $\Sigma_y(\tau) \Sigma_y^\top(\tau)$.

Recall that the main goal of our work is to establish the asymptotic distribution of the spiked eigenvalues of $\Sigma_y(\tau) \Sigma_y^\top(\tau)$. The eigenvalues of $\Sigma_z(\tau) \Sigma_z^\top(\tau)$ is the same as those of the matrix $\Sigma_y(\tau) \Sigma_y^\top(\tau)$. By the above arguments, it suffices to consider $\Sigma_z(\tau) \Sigma_z^\top(\tau)$ instead of $\Sigma_y(\tau) \Sigma_y^\top(\tau)$, that is, we may without any loss of generality assume that

$$y_t = \begin{pmatrix} \text{diag}(\sigma_1, \ldots, \sigma_K) \\ 0_{p\times K} \end{pmatrix} f_t + \tilde{L}^\top \epsilon_t.$$

Finally, when $\epsilon_t$ is assumed to be standard Gaussian and hence unitarily invariant, the transformed error $\tilde{L}^\top \epsilon_t$ is equal in distribution to $\epsilon_t$. Under this assumption, we have

$$y_t \overset{\text{dist}}{=} \begin{pmatrix} \text{diag}(\sigma_1, \ldots, \sigma_K) \\ 0_{p\times K} \end{pmatrix} f_t + \epsilon_t \quad (2.6)$$

and we may take this as the canonical form of the factor model (2.1). Motivated by these observations, we will work under the canonical form (2.6).
2.1 Assumptions

The asymptotic properties of empirical spiked eigenvalues $\lambda_i$ with $i = 1, 2, \ldots, K$ mainly depend on five criteria below

(1). **Factor strength**: $\sigma_i^2, i = 1, 2, \ldots, K$, which are the variances of factors (before normalization).

(2). **Spikeness**: $\mu_i, i = 1, 2, \ldots, K$, which are spiked eigenvalues of the population matrix $\Sigma_y \Sigma_y^\top$.

(3). **Time lag**: $\tau$, which is allowed to be fixed or tend to infinity.

(4). **The number of factors**: $K$, which could be fixed or tending to infinity.

(5). **The dimension and sample size**: $p$ and $T$, which tend to infinity simultaneously.

Next, we impose some conditions related to these five criteria. Some discussions and justification are also presented.

Observe that under the canonical representation (2.6), $\sigma_1, \ldots, \sigma_K$ are in fact the standard deviations of the original factors. Similar to [17], we assume that all $\sigma_i \to \infty, i = 1, 2, \ldots, K$ as $p \to \infty$, which indicate strong factors. It is noteworthy that we allow $\sigma_i$ to diverge at any rate while [17] restrict that each $\sigma_i$ diverges at the specific rate $p^{1-\delta}$ where $\delta \in [0, 1]$ is fixed. For technical simplicity, all factors are assumed to be asymptotically equal in strength, i.e. there exists a constant $C > 0$ such that $\sigma_i/\sigma_j \leq C$ for any $i, j = 1, \ldots, K$ and $T > 0$.

Spiked eigenvalues $\mu_{i,\tau}, i = 1, 2, \ldots, K$ of the population matrix $\Sigma_y(\tau) \Sigma_y^\top(\tau)$ are closely related to the factor strength and temporal dependence of the common factors. Recall $\gamma_i(\tau) := \mathbb{E}[f_{i,1}f_{i,\tau+1}]$ from (2.3). Under the canonical form (2.6), the (population) lag-$\tau$ auto-covariance function for each time series ($y_{it}$)$_t$ can be written as

$$
\mu_{i,\tau} := (\mathbb{E}[y_{it}y_{i,t+\tau}])^2 = \sigma_i^4 \gamma_i(\tau)^2, \quad i = 1, \ldots, K, \quad \tau \geq 0.
$$

(2.7)
We will consider two different types of asymptotic regimes on $\mu_{i,\tau}$ as $T \to \infty$. In the first case, we assume $\tau$ is a fixed integer for all $T$; in the second case, we allow $\tau$ to vary with $T$ and assume $\tau \to \infty$ as $T \to \infty$. In the case where $\tau$ is fixed, we will assume without any loss of generality that the sequence $(\mu_{i,\tau})_i$ is arranged in decreasing order. Furthermore, we assume that $\{\mu_{i,\tau}\}$ is well separated, i.e. there exists $\epsilon > 0$ such that $\mu_{i,\tau} / \mu_{i+1,\tau} > 1 + \epsilon$ for all $i$ and $T$. This assumption is standard (see e.g. [12]) and ensures that the empirical eigenvalues are separated asymptotically.

In the case where $\tau$ is allowed to vary with $T$, it is too restrictive to assume that such an ordering on $\mu_{i,\tau}$ exists for all $\tau \geq 0$. For example, suppose the $(y_{1t})_t$ has a large variance $\sigma_1^2$ but a very rapidly decaying auto-covariance function $\gamma_1(\cdot)$, while $(y_{2t})_t$ has a smaller variance but a slow decaying auto-covariance function. Then we can easily have $\mu_{1,1} > \mu_{2,1}$ as well as $\mu_{1,\tau} < \mu_{2,\tau}$ for a larger $\tau$ so the assumption $\mu_{1,\tau} > \mu_{2,\tau}$ for all $\tau$ is unrealistic. Instead, we will assume that the sequence $(\mu_{i,\tau})_i$ is well separated only asymptotically, i.e. we assume there exists $\tau_0$ large enough and $\epsilon > 0$ such that

$$\mu_{i,\tau} / \mu_{i+1,\tau} > 1 + \epsilon, \quad \forall \tau > \tau_0, \quad i = 1, \ldots, K.$$ 

For simplicity, we will assume that all $\gamma_i(\tau), i = 1, 2, \ldots, K$ decay at the same speed asymptotically, i.e. $\gamma_i(\tau) / \gamma_j(\tau) < C_1$ for $i, j = 1, \ldots, K$ and some constant $C_1$. This implies that the $\mu_{i,\tau}$’s are of the same order as well and a comparison between them is more reasonable.

For clarity and the convenience of the reader we summarize our settings into the following sets of conditions which will be referred to in later parts of the paper.

**Assumptions 2.1.**

a) $p, T \to \infty$ and $p/T \to c \in (0, +\infty)$.

b) $\sigma_i \to \infty$ and there exists $C > 0$ such that $\sigma_i / \sigma_j < C$ for all $i, j = 1, \ldots, K$.

c) $(z_{it})_{1 \leq i \leq K, 1 \leq t \leq T+1}$ is independent, identically distributed with $E[z_{it}] = 0, E[z_{it}^2] = 1$ and uniformly bounded $(4 + \epsilon)$-th moment for some $\epsilon > 0$.

d) $(\epsilon_{it})_{1 \leq i \leq p, 1 \leq t \leq T+1}$ is i.i.d. standard Gaussian.
e) \( \sup_i \| \phi_i \|_{\ell_1} < \infty. \)

We note that (a) and (b) of Assumption 2.1 capture our asymptotic regime where \( p \) diverges at the same rate as \( T \) and the strength of all factors diverges at comparable rates. The requirement on the same divergence rate for all factor strength is commonly used in factor model analysis literature [1]. Moment conditions such as (c) of Assumption 2.1 are standard in the literature; see for instance [4, 12, 21, 29, 30]. The normality assumption in (d) is solely for the purpose of reducing the model to a canonical form, as discussed in the previous section. Finally, condition (e) is very standard in the time series literature, see [11]. For instance, condition (e) is satisfied by an auto-regressive moving average process written in the form (2.2).

The following two sets of assumptions encapsulate the two asymptotic schemes discussed above. Most of our main results hold under either set of assumptions.

**Assumptions 2.2.**  

a) \( \tau \) is a fixed, positive integer.

b) \( K = o \left( T^{1/16} \right) \) and \( K = o \left( \sigma_1^2 \right) \) as \( T \to \infty. \)

c) the sequence \((\mu_{1,\tau}, \ldots, \mu_{K,\tau})\) is arranged in decreasing order and there exists \( \epsilon > 0 \) such that \( \mu_{i,\tau}/\mu_{i+1,\tau} > 1 + \epsilon \) for all \( i = 1, \ldots, K - 1. \)

**Assumptions 2.3.**  

a) \( \tau \) is a positive integer and \( \tau \to \infty \) as \( T \to \infty. \)

b) \( K = o \left( T^{1/16} \gamma_1(\tau)^{1/2} \right) \) and \( K = o \left( \sigma_1^2 \gamma_1(\tau)^3 \right) \) as \( T \to \infty. \)

c) there exists \( C_1 > 0 \) such that \( \mu_{i,\tau}/\mu_{j,\tau} \leq C_1 \) for all \( i, j = 1, \ldots, K \) and \( \tau > 0. \)

d) there exists \( \tau_0 \) large enough and some \( \epsilon > 0 \) such that \( \mu_{i,\tau}/\mu_{i+1,\tau} > 1 + \epsilon \) for all \( i = 1, \ldots, K - 1 \) and \( \tau > \tau_0. \)

Assumption 2.2 describes the asymptotic regime where \( \tau \) is a fixed integer and Assumption 2.3 allows \( \tau \) to diverge along with \( T \). We note that condition (b) of both of the assumptions above is trivially satisfied when the number of factors \( K \) is assumed to be finite. Under (b) of Assumption 2.1, condition (c) of Assumption 2.3 ensures that the
spiked eigenvalues are comparable when $\tau \to \infty$. Finally, (c) of Assumption 2.2 and (d) of Assumption 2.3 are standard and ensure that the empirical eigenvalues are separated from each other asymptotically, see for instance see e.g. [12].

2.2 Notations and Preliminaries

In our exposition and proofs, we will often encounter various resolvent matrices, which capture the spectral information of the random matrices we are studying. Since we are constantly dealing with many different matrices, assigning to each a different letter will easily exhaust the alphabet. Instead, we adopt some non-standard notations for matrices and sub-matrices. Write $(a_{ij})$ for a matrix where the $(i,j)$-th entry is equal to $a_{ij}$. For such a matrix $(a_{ij})$, we will write

$$a_{[i:j],[k:l]} := \begin{pmatrix} a_{ik} & \cdots & a_{il} \\ \vdots & \ddots & \vdots \\ a_{jk} & \cdots & a_{jl} \end{pmatrix}$$

for a specified sub-matrix. Similarly we will write $a_{i,[j:k]}$ and $a_{[i:j],k}$ for the column vectors $(a_{ij},\ldots,a_{ik})^\top$ and $(a_{ik},\ldots,a_{jk})^\top$ respectively.

First, we introduce notations for some of the more important random matrices in our study. We denote

$$x_{it} = \sigma_i f_{it} + \epsilon_{it}, \ i = 1,\ldots,K, \ t = 1,\ldots,T \quad (2.8)$$

and write $x = (x_{it})$,

$$X_0 := \frac{1}{\sqrt{T}} x_{[1:K],[1:T-\tau]}, \quad X_\tau := \frac{1}{\sqrt{T}} x_{[1:K],[\tau+1:T]}, \quad (2.9)$$

$$E_0 := \frac{1}{\sqrt{T}} \epsilon_{[K+1:K+p],[1:T-\tau]}, \quad E_\tau := \frac{1}{\sqrt{T}} \epsilon_{[K+1:K+p],[\tau+1:T]},$$

for matrices used later that contain the factors and noises in our model. We will also write

$$Y_0 := \frac{1}{\sqrt{T}} y_{[1:p+K],[1:T-\tau]}, \quad Y_\tau := \frac{1}{\sqrt{T}} y_{[1:p+K],[\tau+1:T]}, \quad (2.10)$$
i.e. we have $Y_0 = (X_0^\top, E_0^\top)^\top$ and $Y_\tau = (X_\tau^\top, E_\tau^\top)^\top$. For an integer $\tau > 0$, the lag-$\tau$ sample auto-covariance matrix of $y_t$ can then be written as

\[
\hat{\Sigma}_\tau := \frac{1}{T} \sum_{t=1}^{T-\tau} y_{t+\tau} y_t^\top = \begin{pmatrix} X_{\tau}^\top & X_0^\top \\ E_{\tau}^\top & E_0^\top \end{pmatrix}^\top \begin{pmatrix} X_{\tau} X_0^\top & X_\tau E_0^\top \\ E_{\tau} X_0^\top & E_\tau E_0^\top \end{pmatrix}.
\]

Next, we introduce resolvent matrices which are central to the study of spectral properties of random matrices. Most of our results rely on certain bilinear forms formed using the resolvents. For $a \in \mathbb{R}$ outside of the spectrum of the matrix $E_\tau^\top E_\tau E_0^\top E_0$ write

\[
R(a) := (I_{T-\tau} - a^{-1} E_\tau^\top E_\tau E_0^\top E_0)^{-1} = a(a - E_\tau^\top E_\tau E_0^\top E_0)^{-1} \quad (2.11)
\]

for the (scaled) resolvent of $E_\tau^\top E_\tau E_0^\top E_0$ at $a$. The resolvent $R(a)$ satisfies

\[
R(a) = I_{T-\tau} + a^{-1} R(a) E_\tau^\top E_\tau E_0^\top E_0, \quad (2.12)
\]

which follows from rearranging $R(a)(I_{T-\tau} - a^{-1} E_\tau^\top E_\tau E_\tau^\top) = I_{T-\tau}$. Using the identity

\[
A(\lambda I - BA)^{-1} = (\lambda I - AB)^{-1} A \quad (2.13)
\]

we may also obtain the following identities

\[
R(a) E_\tau^\top E_\tau = E_\tau^\top E_\tau R(a)^\top, \quad E_0^\top E_0 R(a) = R(a)^\top E_0^\top E_0. \quad (2.14)
\]

In our analysis we will constantly be dealing with certain quadratic forms involving matrices $X_0, X_\tau, E_0, E_\tau$ and the resolvent $R(a)$. To simplify notations we will write

\[
A(a) := \frac{1}{\sqrt{a}} X_0 R(a) X_\tau^\top, \quad B(a) := \frac{1}{a} X_\tau E_0^\top E_0 R(a) X_\tau^\top, \quad (2.15)
\]

\[
\overline{Q}(a) := I_K - a^{-1} X_\tau E_0^\top E_0 R(a) X_\tau^\top, \quad Q(a) := I_K - a^{-1} X_0 R(a) E_\tau^\top E_\tau X_\tau^\top. \quad (2.16)
\]

For any $a$ outside the spectrum of the matrix $X_0 R(a) E_\tau^\top E_\tau X_\tau^\top$, the matrix $Q(a)$ defined above is invertible and similar to (2.12), we have

\[
Q(a)^{-1} = I_K + \frac{1}{a} Q(a)^{-1} X_0 R(a) E_\tau^\top E_\tau X_\tau^\top. \quad (2.17)
\]
For two sequences of positive numbers \((a_n)\) and \((b_n)\), we write \(a_n \lesssim b_n\) if there exists a constant \(c > 0\) such that \(a_n \leq cb_n\). We write \(a_n \asymp b_n\) if \(a_n \lesssim b_n\) and \(b_n \lesssim a_n\) hold simultaneously. A sequence of events \((F_n)\) is said to hold with high probability if there exists constants \(c, C > 0\) such that \(\mathbb{P}(F_n) \leq Cn^{-c}\). The operator and Hilbert-Schmidt norms of a matrix \(M\) are denoted by \(\|M\|\) and \(\|M\|_F\) respectively, and we write \(\|(a_n)\|_{\ell^p}\) for the \(\ell^p\) norm of a sequence \((a_n)\). We will write \((e_i)_{i=1}^n\) for the standard orthonormal basis of Euclidean space \(\mathbb{R}^n\), often without specifying the dimension \(n\).

We will use the usual \(o_p\) and \(O_p\) notations for convergence in probability and stochastic compactness. For \(p \geq 1\), we will write \(o_{L^p}\) and \(O_{L^p}\) for convergence to zero and boundedness in \(L^p\), i.e. for a sequence of random variables \((X_n)_n\) and real numbers \((a_n)_n\), we write \(X_n = O_{L^p}(a_n)\) if \(\mathbb{E}|X_n/a_n| = O(1)\) and \(X_n = o_{L^p}(a_n)\) if \(\mathbb{E}|X_n/a_n| = o(1)\). For matrices \((A_n)\) we will write \(A_n = O_{p,\|(a_n)\|}(a_n)\) if \(\|A_n\| = O_p(a_n)\).

Throughout the paper we will make use of certain events of high probability. Define

\[ B_0 := \left\{ \|E_0^T E_0\| + \|E_{\tau}^T E_{\tau}\| \leq 4 \left( 1 + \frac{p}{T} \right) \right\}, \]

\[ B_1 := \left\{ \|X_0^T X_0\| + \|X_\tau^T X_{\tau}\| \leq 2 \sum_{i=1}^K \sigma_i^2 \right\} \tag{2.18} \]

and \(B_2 := B_0 \cap B_1\). We first state a preliminary result showing that these events happen with high probability as \(T \to \infty\). The proof will be given in Appendix A.

**Lemma 2.1.** Under Assumption 2.1 and either Assumption 2.2 or 2.3, we have

a) \(B_0\) holds with probability \(\mathbb{P}(B_0) = 1 - o(T^{-l})\) for any \(l \in \mathbb{N}^+\) as \(T \to \infty\).

b) For \(k = 1, 2\), \(B_k\) holds with probability \(\mathbb{P}(B_k) = 1 - O(KT^{-1})\) as \(T \to \infty\).

As an immediate consequence of this lemma and (b) of Assumption 2.1, we have

\[ \|E_0^T E_0\| + \|E_{\tau}^T E_{\tau}\| = O_p(1), \quad \|X_0^T X_0\| + \|X_{\tau}^T X_{\tau}\| = O_p(K^2 \sigma_1^2). \tag{2.19} \]

Furthermore, we observe that under the event \(B_0\), for any sequence \((a_T)_T\) such that \(a_T \to \infty\), the matrix \(I_{T-\tau} - a_T^{-1} E_\tau^T E_{\tau} E_0^T E_0\) is eventually invertible. Moreover, we
note that under $\mathcal{B}_0$ we have $\|a_T^{-1}E^\top E\tau E_0^\top E_01_{\mathcal{B}_0}\| \leq 4a_T^{-1}(1 + p/T) = O(a_T^{-1})$, which is a non-random upper-bound. By the reverse triangle inequality we immediately have $\|I_{T-\tau} - a_T^{-1}E^\top E\tau E_0^\top E_01_{\mathcal{B}_0}\| \geq 1 - O(a_T^{-1})$ and therefore
\[
\|R(a_T)1_{\mathcal{B}_0}\| = 1 + o(1), \quad \|R(a_T)\| = 1 + o_p(1),
\]
(2.20)
where the definition of $R(\cdot)$ is in (2.11).

Similarly, under the event $\mathcal{B}_2$ the matrix $Q(a_T)$ is eventually invertible as $a_T \to \infty$ and
\[
\|Q(a_T)^{-1}1_{\mathcal{B}_2}\| = 1 + o(1), \quad \|Q(a_T)^{-1}\| = 1 + o_p(1).
\]
(2.21)
Finally, let $\mathcal{F}_p$ be the $\sigma$-algebra generated by the noise time series $(\epsilon_t)$, i.e.
\[
\mathcal{F}_p := \sigma(\{\epsilon_{it}, i = K + 1, \ldots, K + p, t = 1, \ldots, T\}).
\]
(2.22)
We will often take expectations conditional on the noise series, in which case we shall write
\[
\mathbb{E}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_p].
\]
(2.23)

## 3 Main results

Write $\lambda_{n,\tau}$ for the $n$-th largest spiked eigenvalue of the symmetrized lag-$\tau$ sample autocovariance matrix $\hat{\Sigma}_\tau \hat{\Sigma}_\tau^\top$. The main goal of our work is to establish the asymptotic normality of $\lambda_{n,\tau}$ for $n \leq K$ after appropriate centering and scaling. We will first in Section 3.1 establish the asymptotic location of the eigenvalue $\lambda_{n,\tau}$ as well as identify the correct centering for $\lambda_{n,\tau}$ in order to obtain a central limit theorem. The proof will be presented in Appendix A. The central limit theorem itself, which is the main result of our work, is stated in Theorem 3.3 of Section 3.2.

Due to its length, the proof of Theorem 3.3 will be divided into a series of propositions and technical lemmas, which are collected in Appendixes B to D. For the convenience of the reader, we will summarize the strategy of the proof of Theorem 3.3 and explain how the intermediate results are used in Section 3.3.
3.1 Location of Spiked Eigenvalues

We first show that the spiked eigenvalue \( \lambda_{n,\tau} \) is close to its population counterpart \( \mu_{n,\tau} \) asymptotically. This will in particular give the asymptotic order of \( \lambda_{n,\tau} \) as \( T \to \infty \).

**Theorem 3.1.** Under Assumption 2.1 and either Assumption 2.2 or 2.3, we have

\[
\frac{\lambda_{n,\tau}}{\mu_{n,\tau}} - 1 = O_p \left( \frac{1}{\gamma_n(\tau)\sqrt{T}} \right) + O_p \left( \frac{K}{\sigma_n^2 \gamma_n(\tau)^2} \right), \quad n = 1, \ldots, K.
\]

where \( \mu_{n,\tau} \) and \( \gamma_n(\tau) \) are defined in (2.7) and (2.3) respectively.

**Remark 3.1.** A closer inspection on the convergence rate in Theorem 3.1 shows that \( \mu_{n,\tau} \) is not the appropriate centering constant for \( \lambda_{n,\tau} \) for the purpose of obtaining a CLT. The first term on the right-hand side of (3.1) can indeed be shown to be asymptotically normal at a scaling of \( \gamma_n(\tau)\sqrt{T} \), which is the same scaling as our main result in Theorem 3.3. However, the second term in (3.1) is in general not negligible after scaling by \( \gamma_n(\tau)\sqrt{T} \) unless some restrictions on the rate of divergence of \( \mu_{n,\tau} \) are imposed.

If we were to impose stronger assumptions on the rate of \( \mu_{n,\tau} \), for example assuming the rate \( \mu_{n,\tau} \asymp p^{1-\delta} \) required in [17], then the second term in (3.1) indeed becomes negligible. Under such assumptions the \( K \) spiked eigenvalues of the \( (p + K) \times (p + K) \) dimensional matrix \( \hat{\Sigma}_\tau \hat{\Sigma}_\tau^\top \) are extremely close to the eigenvalues of the \( K \times K \) matrix \( X_\tau X_0^\top X_0 X_\tau^\top \), as can be deduced from the proof of Theorem 3.1. The analysis of the matrix \( \hat{\Sigma}_\tau \hat{\Sigma}_\tau^\top \) reduces to the analysis of the much simpler matrix \( X_\tau X_0^\top X_0 X_\tau^\top \), which is essentially a low-dimensional problem. In this case, the derivation of a CLT is much easier.

As can be seen from the proof of Theorem 3.1, the second term in (3.1) represents the bias incurred when estimating \( \mu_{n,\tau} \) using \( \lambda_{n,\tau} \). In order to obtain a CLT, we need a more accurate centering term for \( \lambda_{n,\tau} \) to remove or reduce this bias. This centering term, which we write as \( \theta_{n,\tau} \), will be defined implicitly as the unique solution to the equation

\[
1 = \mathbb{E}[B(\theta_{n,\tau})_{nn}1_{G_0}] - \mathbb{E}[A(\theta_{n,\tau})_{nn}1_{G_0}]^2 \mathbb{E}[Q(\theta_{n,\tau})_{nn}^{-1}1_{G_2}], \quad (3.2)
\]

where the matrices \( B(a) \), \( A(a) \) and \( Q(a) \) are defined in (2.15) and (2.16) for \( a \in \mathbb{R} \).
To make this definition rigorous, we start with Proposition 3.2 which shows that (3.2) indeed has a unique solution for $T$ large enough. Furthermore, this solution is shown to exist in some small interval containing $\mu_{n,\tau} = \sigma_n^4 \gamma_n(\tau)^2$. This in particular establishes the asymptotic order of $\theta_{n,\tau}$.

**Proposition 3.2.** Suppose Assumption 2.1 and either Assumption 2.2 or Assumption 2.3 hold. Fix $n \in \{1, \ldots, K\}$ and let $\epsilon \in (0, 1)$ be an arbitrary constant not related to $p, T$. Then there exists $T_0$ large enough such that for $T > T_0$, the function

$$a \mapsto g(a) = 1 - \mathbb{E}[B(a)_{nn}1_{B_0}] - \mathbb{E}[A(a)_{nn}1_{B_0}]^2 \mathbb{E}[Q(a)_{nn}^{-1}1_{B_2}]$$

has a unique root in the interval

$$\sigma_n^4 \gamma_n(\tau)^2[1 - \epsilon, 1 + \epsilon]. \quad (3.3)$$

### 3.2 Central Limit Theorem for Spiked Eigenvalues

The constant $\theta_{n,\tau}$ defined in (3.2) turns out to be the appropriate centering constant for $\lambda_{n,\tau}$, in the sense that the second term in (3.1) becomes negligible after centering by $\theta_{n,\tau}$. We are ready to state the main result of our work. Define

$$\delta_{n,\tau} := \frac{\lambda_{n,\tau} - \theta_{n,\tau}}{\theta_{n,\tau}} = \frac{\lambda_{n,\tau}}{\theta_{n,\tau}} - 1.$$

**Theorem 3.3.** Under Assumption 2.1 and either Assumption 2.2 or 2.3, we have

$$\sqrt{T} \frac{\gamma_n(\tau)}{2v_{n,\tau}} \delta_{n,\tau} \Rightarrow N(0, 1), \quad n = 1, 2, \ldots, K$$

where $v_{n,\tau}$ is defined by

$$v_{n,\tau}^2 := \frac{1}{T} \text{Var}(f_{n0}^\top f_{n\tau}) = \sum_{|k| < T-\tau} \left(1 - \frac{|k|}{T-\tau}\right) u_k, \quad (3.4)$$

and $(u_k)_{|k| < T-\tau}$ is a sequence of constants given by

$$u_k := u_{nk} := \gamma_n(k)^2 + \gamma_n(k + \tau)\gamma_n(k - \tau) + (\mathbb{E}[z_{11}^4] - 3) \sum_{l=0}^{\infty} \phi_{n,l}\phi_{n,l+\tau}\phi_{n,l+k}\phi_{n,l+k+\tau}.$$
Remark 3.2. We remark that for generality as well as the tidiness of presentation we choose to formulate Theorem 3.3 in a form that holds under either one of Assumption 2.2 and 2.3. A closer inspection shows that the two cases are quite different. In the case where \( \tau \to \infty \), we observe that \( \gamma_n(\tau) \to 0 \) while the term \( \nu_{n,\tau}^2 \) defined by (3.4) can easily be shown to be bounded from zero. This implies that the scalings of CLT in the two cases are not of the same order. In the case where \( \tau \) is fixed, the variance of \( \sqrt{T}\delta_{n,\tau} \), which is equal to \( 4\nu_{n,\tau}\gamma_n(\tau)^{-2} \), is bounded both from above and away from zero from below. On the other hand, when \( \tau \to \infty \), the variance of \( \sqrt{T}\delta_{n,\tau} \) tends to infinity at a speed of \( \gamma_n(\tau)^{-1} \) while \( \nu_{n,\tau} \) remains bounded.

This result might seem surprising since \( \delta_{n,\tau} = (\lambda_{n,\tau} - \theta_{n,\tau})/\theta_{n,\tau} \) is already normalized in an obvious way so one might expect \( \delta_{n,\tau} \) to be of order \( T^{-1/2} \). One might be tempted to draw the conclusion that \( \lambda_{n,\tau} \) is less accurate of an estimator of \( \theta_{n,\tau} \) for larger values of \( \tau \), since the variance of \( \delta_{n,\tau} \) increases with \( \tau \). However, the exact opposite is true here. Since \( \theta_{n,\tau} \asymp \sigma_n^4 \gamma_n(\tau)^2 \) by Proposition 3.2, this implies that in fact \( \lambda_{n,\tau} - \theta_{n,\tau} = O_p(\sigma_n^4 \gamma_n(\tau) T^{-1/2}) \), which is faster than the rate \( \lambda_{n,\tau} - \theta_{n,\tau} = O_p(\sigma_n^4 T^{-1/2}) \) obtained in the case where \( \tau \) is fixed. In practical situations where we deal with the auto-covariance matrix with a larger \( \tau \), the CLT under Assumption 2.3 provides a much more accurate convergence speed and asymptotic variance than using fixed \( \tau \) results.

3.3 Proof strategy

To prove our main result Theorem 3.3, we need a collection of preliminary results and technical lemmas which are collected in the Appendix. Due to the convoluted structure of the proofs, for the convenience of the reader, we provide a summary of the key ideas in the whole proofs. We will first make some simplifications to commonly used notations.

Throughout the rest of the work, we will without loss of generality deal with the \( n \)-th largest spiked eigenvalue \( \lambda_{n,\tau} \) of \( \hat{\Sigma}_\tau \hat{\Sigma}_\tau^\top \) for a chosen \( n \in \{1, \ldots, K\} \). To avoid using too many layers of subscripts, we will routinely suppress the subscripts \( n, \tau \) and write \( \lambda := \lambda_{n,\tau} \). Recall the scaled resolvent \( R(a) \) of the matrix \( E_\tau^\top E_\tau E_0^\top E_0 \) evaluated at \( a \), as
defined in (2.11). When we evaluate $R(\cdot)$ at the values $\lambda_{n,\tau}$ and $\theta_{n,\tau}$, we will simply write $R_\lambda := R_{\lambda_{n,\tau}}$ and $R := R_{\theta_{n,\tau}}$. Similarly, we will write $Q_\lambda, \overline{Q}_\lambda, Q$ and $\overline{Q}$ for $Q_{\lambda_{n,\tau}}, \overline{Q}_{\lambda_{n,\tau}}, Q_{\theta_{n,\tau}}$ and $\overline{Q}_{\theta_{n,\tau}}$ respectively. Using these notations we define the matrix $(M_{ij})_{1 \leq i,j \leq K}$ by

$$M := I_K - \frac{1}{\theta} X_\tau E_0^T E_0 R X_\tau^T - \frac{1}{\theta} X_\tau R^T X_0^T Q^{-1} X_0 R X_\tau^T, \quad (3.5)$$

which will turn out to be the central object of our study.

The initial step in our analysis is to derive an expression for the eigenvalue $\lambda = \lambda_{n,\tau}$ and the quantity $\delta = \delta_{n,\tau}$. This is necessary because the eigenvalue $\lambda$ of the matrix $\hat{\Sigma}_\tau \hat{\Sigma}_\tau^T$, in general, depends on its entries in complicated and non-linear ways. We take an approach commonly seen in the random matrix literature (e.g. [3, 12, 22]) and express $\delta$ as the solution to an equation involving the determinant of certain random matrices. This is established in Proposition B.1 in which we express $\delta$ as the solution to the equation

$$\det \left( M + \frac{\delta}{\theta} X_\tau X_0^T X_0 X_\tau^T + \delta o_{p,||\cdot||}(1) \right) = 0. \quad (3.6)$$

The main idea is then to apply Leibniz’s formula to compute this determinant. Doing so will express $\delta$ as a polynomial function of the entries of the matrices $M$ and $\theta^{-1} X_\tau X_0^T X_0 X_\tau^T$ plus many higher order terms. After estimating the terms in this polynomial, it can be shown that the asymptotic normality of the ratio $\delta$ eventually follows from the asymptotic normality of the $n$-th diagonal entry $M$. Specifically, as shown in the proof of Theorem 3.3, to establish the CLT, it suffices to (a) show

$$\sqrt{T} \frac{M_{nn}}{2\gamma_{n,\tau} v_{n,\tau}} \Rightarrow N(0,1), \quad M_{ii} = 1 + o_p(1), \quad \forall i \neq n,$$

then (b) establish a uniform bound of sufficient sharpness on the off-diagonals of $M$, and (c) identify the limits in probability of the entries of $\theta^{-1} X_\tau X_0^T X_0 X_\tau^T$. From this, it is clear that $M$ and hence the resolvents $R$ and $Q^{-1}$ appearing in the definition of $M$ are the central objects of our analysis. To deal with the expression (3.5), we first construct an approximation to $M$ in Proposition B.2 that preserves its asymptotic distribution.

We now briefly discuss why this approximation in Proposition B.2 is constructed in a seemingly unusual way. Observe that since $\theta$ diverges, the resolvents $R$ and $Q^{-1}$ defined
in (2.11) and (2.16), respectively, are very close to identity matrices for large $T$, a fact used frequently in our proofs. However, one cannot simply replace them with identities to simplify (3.5). Indeed, it is easy to show that $R - I_{T - \tau} = O_{p, \|\cdot\|}(\theta^{-1})$, which converges to zero but not fast enough for obtaining a CLT after scaling by $\sqrt{T}$. This is because we allow $\theta$ to diverge at any rate and not as a specified function of $T$.

It can be shown however that this approximation error of order $\theta^{-1}$ appears only in the mean of the asymptotic distribution, see for instance (B.33). That is, we can essentially use identity matrices to approximate $R$ and $Q^{-1}$ in (B.2) as long as we include an appropriate centering step to adjust the expectation of $M$ before scaling by $\sqrt{T}$. This centering step for the matrix $M$ results in the equation in (3.2) from which the centering $\theta_{n, \tau}$ for $\lambda_{n, \tau}$ is defined. Recall from our discussion following the statement of Theorem 3.1 that $\theta$ is a more accurate centering term for $\lambda$ than $\mu$ is. From the discussion above, it can be seen that this is essentially due to the fact that $R$ and $Q^{-1}$ are not close enough (in spectral norm) to identity matrices to allow for the scaling of $\sqrt{T}$.

Instead of identity matrices, we, therefore, use more accurate approximations to $R$ and $Q^{-1}$, which in our case turn out to be their expectations under certain events of high probability. To show that these expectations are close enough to the resolvents themselves, we establish the concentration of $R$ around its expectation in Lemma D.1, the concentration of $Q^{-1}$ around a certain conditional expectation in Lemma D.2, and estimates on the differences between conditional and unconditional expectations in D.3.

Using these tools, we can show in Proposition B.2 that after centering by a certain conditional expectation (which is later replaced by an unconditional one using Lemma D.3), the asymptotic distribution of $M$ can be obtained from the asymptotic distribution of the bilinear form $X_0 R X_\tau^\top$, up to adjustments in the expectations.

It, therefore, remains to establish the asymptotics of $X_0 R X_\tau^\top$. Using tools developed in Lemma C.2-C.5, we study the bilinear form $X_0 R X_\tau^\top$ and establish its concentration around some conditional expectation. Using these results we show in the proof of Proposition B.4 that the asymptotic normality for $X_0 R X_\tau^\top$ follows from the asymptotic normality
of the much simpler auto-covariance matrix $X_0X_\tau^\top$, again up to adjustments in the expectations. The CLT for this matrix $X_0X_\tau^\top$ is established in Proposition B.3. Finally, Proposition B.4 gives the CLT for diagonals of the matrix $M$ and required estimates for the off-diagonals. The proof of Theorem 3.3 can then be assembled from the above pieces, we present the proof at the end of Appendix B.

To summarize, the quantity of interest $\delta$ is first shown to satisfy equation (B.1). Through a series of approximations, we establish the asymptotic normality of the diagonals of the matrix $M$ appearing in B.1. The off-diagonals of $M$ are bounded in probability and we establish the limit in probability of the matrix $X_\tau X_0^\top X_\tau X_0^\top$ appearing in (B.1). Leibniz’s formula is then applied to compute the determinant in (B.1), and our main result Theorem 3.3 follows.

### 3.4 Outline of the proof

We finally include a brief outline of the proofs to help the readers navigate.

Firstly, in Proposition B.1, the quantity of interest $\delta$ is expressed as the solution to equation (3.6). As will be shown in the proof of Theorem 3.3, the asymptotic normality of $\delta$ follows from the asymptotic normality of the matrix $M$ appearing in B.1.

The asymptotic normality of $M$ is established in Proposition B.2-B.4. We first identify an appropriate centering for the matrix $M$ in (B.11). Using this centering, we show in Proposition B.2 that the diagonal elements of $M$ can be approximated by certain bilinear forms defined in (B.10), at the scale of $o(T^{-1/2})$. Proposition B.4 obtains the asymptotic normality of $M$ by further reducing this approximation into a much simpler bilinear form, the asymptotic distribution of which is established in Proposition B.3.

The results described above all rely on a collection of technical lemmas. Throughout the proofs, we often encounter bilinear forms involving resolvent matrices $R$ and $Q^{-1}$. We routinely approximate these resolvent matrices and the bilinear forms by certain expectations. In Lemma D.1-D.3 we establish the concentration of $R$ and $Q^{-1}$ around certain expectations, and in Lemma C.2-C.5 we establish the concentration of bilinear
forms involving $R$ and $Q^{-1}$.

4 Statistical application: autocovariance test

In this section, a novel test, called the autocovariance test, is proposed to detect the equivalence of spikiness for two high-dimensional time series. As analyzed in Lam and Yao [17], the eigenvectors corresponding to the $K$ spiked eigenvalues of $\Sigma_y(\tau)\Sigma_y(\tau)^\top$ span a $K$-dimensional linear subspace, where the projection of the original high-dimensional time series holds all the temporal dependence. For easy reference, we call this subspace as $K$-dimensional temporal subspace and denote it as $\mathcal{M}_K$. When two high-dimensional time series share the same $K$-dimensional temporal subspace, the proposed test is equivalent to checking whether the two projected time series have the same autocovariance. As a further application of the proposed autocovariance test, new hierarchical clustering analysis is constructed to cluster a large set of high-dimensional time series, where the dissimilarity between two populations is measured via the p-value of the proposed autocovariance test. The major aim of this clustering analysis is to group high-dimensional time series with similarly projected autocovariances.

To explain the idea of the autocovariance test and its application on the hierarchical clustering in detail, we will simply revisit the factor structures for high-dimensional time series and introduce the proposed test statistic with its asymptotic properties in Section 4.1. Section 4.2 describes how the hypothesis test can be implemented in practice where a flow chart is also provided to clarify the essential idea of the test procedure. We then use numerical simulations to investigate the empirical sizes and powers of the proposed test in various scenarios, where the results are presented in Section 5. Finally, the proposed test and the hierarchical clustering method using p-values as the measure of dissimilarities are applied to mortality data from multiple countries in Section 6.
4.1 Hypotheses and test statistic

Consider now for two high-dimensional time series \( \{y^{(1)}_t \in \mathbb{R}^{K_1+p_1}, t = 1, 2, \ldots, T \} \) and \( \{y^{(2)}_t \in \mathbb{R}^{K_2+p_2}, t = 1, 2, \ldots, T \} \) following the factor model in (2.1), that is we have

\[
y^{(m)}_t = L^{(m)} f^{(m)}_t + \epsilon^{(m)}_t, \quad t = 1, \ldots, T, \quad m = 1, 2,
\]

where \( \{f^{(m)}_t \in \mathbb{R}^{K_m}, t = 1, 2, \ldots, T \} \) are stationary factor time series with \( K_m \ll p_m \), and \( L^{(m)} \) is a \( (p_m + K_m) \times K_m \) factor loading matrix with a normalization condition \( L^{(m)\top} L^{(m)} = I_{K_m} \). To simplify the notations, we also let \( N_m := (p_m + K_m) \) be the dimension of \( \{y^{(m)}_t\} \).

For high-dimensional time series \( \{y^{(m)}_t\} \) following factor models such as (4.1), \( L^{(m)} \) is the time invariant factor loading matrix. As discussed in Lam et al. [18] and Lam and Yao [17], when \( \{\epsilon^{(m)}_t\} \) are i.i.d, the temporal dependence of \( \{y^{(m)}_t\} \) is fully captured by \( \{f^{(m)}_t\} \). Denote by \( \mu^{(m)}_{1,\tau} \) the eigenvalues of \( \Sigma^{(m)}_y(\tau)\Sigma^{(m)}_y(\tau)^\top \). We then consider in this section the setting where there are \( K \) spiked eigenvalues in \( \Sigma^{(m)}_y(\tau)\Sigma^{(m)}_y(\tau)^\top \) and \( \mu^{(m)}_{1,\tau} > \mu^{(m)}_{2,\tau} > \ldots > \mu^{(m)}_{K_{m,\tau}} \) tend to infinity with \( T, p \), while \( \mu^{(m)}_{K_{m+1,\tau}} = \mu^{(m)}_{K_{m+2,\tau}} = \ldots = \mu^{(m)}_{N_{m,\tau}} = 0 \) for any \( \tau \geq 1 \). With this definition of spiked eigenvalues, we can show that the columns of \( L^{(m)} \) are the eigenvectors of \( \Sigma^{(m)}_y(\tau)\Sigma^{(m)}_y(\tau)^\top \) corresponding to the spiked eigenvalues, as follow.

Write \( M^{(m)} \) for a \( N_m \times p_m \) matrix where \( (L^{(m)}, W^{(m)}) \) forms a \( N_m \times N_m \) orthogonal matrix so that \( L^{(m)\top} W^{(m)} = 0 \) and \( W^{(m)\top} W^{(m)} = I_{p_m} \). It then follows that \( \Sigma^{(m)}_y(\tau)\Sigma^{(m)}_y(\tau)^\top W^{(m)} = 0 \), which means the columns of \( W^{(m)} \) are precisely the eigenvectors associated with zero-eigenvalues. In other words, the columns of \( L^{(m)} \) are the \( K_m \) eigenvectors of \( \Sigma^{(m)}_y(\tau)\Sigma^{(m)}_y(\tau)^\top \) corresponding to those non-zero eigenvalues, and those non-zero eigenvalues of \( \Sigma^{(m)}_y(\tau)\Sigma^{(m)}_y(\tau)^\top \) are precisely the eigenvalues of \( \Sigma^{(m)}_f(\tau)\Sigma^{(m)}_f(\tau)^\top \) since \( L^{(m)\top} L^{(m)} = I_{K_m} \).

Consequently, on one hand, \( M(L^{(m)}) \) is the temporal subspace spanned by the columns of \( L^{(m)} \), which are also the eigenvectors corresponding to the spiked eigenvalues of the symmetrized autocovariance matrix of \( \{y^{(m)}_t\} \). On the other hand, the eigenvalues
of the symmetrized autocovariance matrix of \( \{ f_t^{(m)} \} \), which summarize the information contained in the autocovariance matrix of \( \{ f_t^{(m)} \} \), are precisely the spikied eigenvalues of the symmetrized autocovariance matrix of \( \{ Y_t^{(m)} \} \). Therefore, if \( M(L^{(1)}) = M(L^{(2)}) \), we can build a test statistic based on the difference between spikied eigenvalues of the symmetrized lag-\( \tau \) sample autocovariance matrices of two high-dimensional time series \( \{ Y_t^{(1)} \} \) and \( \{ Y_t^{(2)} \} \). This test statistic is to detect the equivalence of autocovariance structure for two projected time series in the temporal subspace.

In this section, it is worth noting that we typically focus on testing the equivalence of spiked eigenvalues, but not the eigenspace of autocovariance matrices for two high-dimensional time series \( \{ Y_t^{(1)} \} \) and \( \{ Y_t^{(2)} \} \). Consequently, when \( M(L^{(1)}) = M(L^{(2)}) \) and \( K_1 = K_2 = K \), the null and alternative hypotheses of the autocovariance test for \( \{ Y_t^{(1)} \} \) and \( \{ Y_t^{(2)} \} \) with a finite time lag \( \tau \) can be summarized as

**Hypothesis Test. Autocovariance test for two high-dimensional time series \( \{ Y_t^{(1)} \} \) and \( \{ Y_t^{(2)} \} \)**

\[
H_0: \mu_{i,\tau}^{(1)} = \mu_{i,\tau}^{(2)} \text{ for all } i = 1, 2, \ldots, K \\
H_1: \mu_{i,\tau}^{(1)} \neq \mu_{i,\tau}^{(2)} \text{ for at least one } i, i = 1, 2, \ldots, K
\]

Recall that for factor models in canonical form (2.4), we can write \( \gamma_{i,\tau}^{(m)} := \mathbb{E}(f_{i,1}^{(m)} f_{i,\tau+1}^{(m)}) \) and \( (\nu_{i,\tau}^{(m)})^2 := \frac{1}{T-\tau} \text{Var}(\sum_{t=1}^{T-\tau} f_{i,t}^{(m)} f_{i,\tau+t}^{(m)}) \) for a finite time lag \( \tau \), \( i = 1, 2, \ldots, K \) and \( m = 1, 2 \). Denote by \( \lambda_{i,\tau}^{(m)} \) the \( i \)-th largest spiked eigenvalue of the symmetrized lag-\( \tau \) sample autocovariance matrix \( \mathbf{\Sigma}_Y^{(m)}(\tau) \mathbf{\Sigma}_Y^{(m)}(\tau)^\top \), where \( \mathbf{\Sigma}_Y^{(m)}(\tau) = \frac{1}{T-\tau-1} \sum_{t=1}^{T-\tau} (y_t^{(m)} - \mathbf{\bar{y}}_T^{(m)})(y_{t+\tau}^{(m)} - \mathbf{\bar{y}}_T^{(m)})^\top \), for \( m = 1, 2 \). Then, for \( i = 1, 2, \ldots, K \) and some finite \( \tau \), the test statistic is given by

\[
Z_{i,\tau} = \sqrt{T} \frac{\gamma_{i,\tau}}{2\sqrt{2} v_{i,\tau}} \frac{\lambda_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)}}{\theta_{i,\tau}}, \quad i = 1, 2, \ldots, K, \tag{4.2}
\]

where

\[
\theta_{i,\tau} = \frac{\theta_{i,\tau}^{(1)} + \theta_{i,\tau}^{(2)}}{2}, \quad v_{i,\tau} = \frac{\nu_{i,\tau}^{(1)} + \nu_{i,\tau}^{(2)}}{2}, \quad \text{and} \quad \gamma_{i,\tau} = \frac{\gamma_{i,\tau}^{(1)} + \gamma_{i,\tau}^{(2)}}{2}. \tag{4.3}
\]
and $\theta_{i,\tau}^{(m)}$ is the asymptotic centering of $\lambda_{i,\tau}^{(m)}$ defined in Proposition 3.2. It is then clearly that $|Z_{i,\tau}|$ is generally large if $\{y_{t}^{(1)}\}$ and $\{y_{t}^{(2)}\}$ follow different factor models where the $i$-th largest eigenvalues of the symmetrized lag-$\tau$ sample autocovariance matrix for two factor models are different. We name this test by autocovariance test since the idea behind is testing whether two independent high-dimensional time series observations share the same spiked eigenvalues of the autocovariance matrices.

For simplicity, we assume the idiosyncratic components $\{\epsilon_{t}^{(m)} \in \mathbb{R}^{N_{m}}, t = 1, 2, \ldots, T\}$ are independent of the factors $\{f_{t}^{(m)}\}$, with $\mathbb{E}(\epsilon_{j,t}^{(m)}) = 0$ for all $j = 1, 2, \ldots, N_{m}$, and $\mathbb{E}(\epsilon_{t}^{(m)})^2 = \Sigma_{\epsilon}^{(m)} = \text{diag}\left(\sigma_{\epsilon,1}^{(m)}^2, \sigma_{\epsilon,2}^{(m)}^2, \ldots, \sigma_{\epsilon,N_{m}}^{(m)}^2\right)$. Without loss of generality, we can again work on standardized factor models in canonical form, where the variance of $\epsilon_{j,t}^{(m)}$ is normalized to one, i.e. $(\sigma_{\epsilon,j}^{(m)})^2 = 1$. This standardization is nothing but a transformation on $\{y_{t}^{(1)}\}$ and $\{y_{t}^{(2)}\}$ so that they can be transformed to fulfill the same canonical form if they share the same number of factors. For factor models with $(\sigma_{\epsilon,j}^{(m)})^2 \neq 1$, we can simply standardize them by dividing $\sigma_{\epsilon,j}^{(m)}$. In this section, we only consider the case for a fixed time lag $\tau$ and follow Assumptions 2.2 to simplify the factor models into canonical form as

$$
\mathbf{y}_{t}^{(m)} = L^{(m)} \mathbf{f}_{t}^{(m)} + \mathbf{\epsilon}_{t}^{(m)} = \begin{pmatrix}
\sigma_{1}^{(m)} f_{1,t}^{(m)} \\
\vdots \\
\sigma_{K}^{(m)} f_{p,t}^{(m)} \\
0_{p \times K}
\end{pmatrix} + \mathbf{\epsilon}_{t}^{(m)},
$$

where the factor loading matrix is in canonical form

$$
L^{(m)} = \begin{pmatrix}
\text{diag}(\sigma_{1}^{(m)}, \ldots, \sigma_{K}^{(m)}) \\
0_{p \times K}
\end{pmatrix},
$$

and $\{\sigma_{i}^{(m)}, i = 1, 2, \ldots, K; m = 1, 2\}$ are positive real numbers representing (cross-sectional) factor strengths (we refer to Lam et al. [18] for the definition of factor strengths in this section).
In summary, we consider factor models in canonical form (4.4), where the loading matrix $L^{(m)}$ is defined by (4.5) and the variances of $\{f_{i,t}^{(m)}\}$ and $\{\epsilon_{j,t}^{(m)}\}$ are normalized to one. In addition, we assume the data $\{y_{t}^{(m)}\}$ comes from strong factor models where $\sigma_{i}^{(m)}$ is divergent as $N \to \infty$ for $i = 1, 2, \ldots, K$ and $m = 1, 2$. Besides, for a general strong factor model that is not in the canonical form (4.5), it can be normalized by standardizing the variance of $\{\epsilon_{j,t}^{(m)}\}$ to one first and then rotating the original data such that the loading matrix $L^{(m)}$ is in the canonical form (4.5). Moreover, recall that for a finite time lag $\tau$, $\gamma_{i,\tau}^{(m)} := \mathbb{E}\left(f_{i,1}^{(m)} f_{i,\tau+1}^{(m)}\right)$ is the population lag-$\tau$ autocovariance of the $i$-th factor time series $\{f_{i,t}^{(m)}\}$. Following (2.3), (2.7) and (3.4), $\gamma_{i,\tau}^{(m)}$ can be written as

$$\gamma_{i,\tau}^{(m)} = \mathbb{E}\left(f_{i,1}^{(m)} f_{i,\tau+1}^{(m)}\right) = \sum_{l=0}^{\infty} \phi_{i,l}^{(m)} \phi_{i,l+\tau}^{(m)},$$

with the constraint $\|\phi_i\|_{\ell_2} = 1$, the population lag-$\tau$ autocovariance can be defined as

$$\mu_{i,\tau}^{(m)} := \mathbb{E}\left(y_{i,t}^{(m)} y_{i,t+\tau}^{(m)}\right) = \left(\sigma_{i}^{(m)}\right)^2 \gamma_{i,\tau}^{(m)},$$

and $\left(v_{i,\tau}^{(m)}\right)^2$ is defined as

$$\left(v_{i,\tau}^{(m)}\right)^2 = \frac{1}{T-\tau} \text{Var}\left(\sum_{t=1}^{T-\tau} f_{i,t}^{(m)} f_{i,t+\tau}^{(m)}\right),$$

for a finite positive time lag $\tau$, $i = 1, 2, \ldots, K$, and $m = 1, 2$.

If $\{y_{t}^{(1)}\}$ and $\{y_{t}^{(2)}\}$ are assumed following the same canonical factor model under Assumptions 2.1 and 2.2, independently, it is clearly that $\lambda_{i,\tau}^{(1)}$ and $\lambda_{i,\tau}^{(2)}$ share the same asymptotic distribution as shown in Theorem 3.3, independently. Therefore, to test whether $\{y_{t}^{(1)}\}$ and $\{y_{t}^{(2)}\}$ share the same spiked eigenvalues of the autocovariance matrices, it is natural to create the test statistic (4.2) based on the difference between $\lambda_{i,\tau}^{(1)}$ and $\lambda_{i,\tau}^{(2)}$. When $\{y_{t}^{(1)}\}$ and $\{y_{t}^{(2)}\}$ follow the same factor model in the canonical form (4.4), we have the following CLT on the difference between $\lambda_{i,\tau}^{(1)}$ and $\lambda_{i,\tau}^{(2)}$.

**Theorem 4.1.** Under Assumptions 2.1 and 2.2, for two independent high-dimensional time series $\{y_{t}^{(1)}\}$ and $\{y_{t}^{(2)}\}$ following the same factors in canonical form (4.4), we have

$$Z_{i,\tau} = \sqrt{T} \frac{\gamma_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)}}{2\sqrt{2v_{i,\tau}} \theta_{i,\tau}} \to \mathcal{N}(0, 1), \quad (4.6)$$

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as \( T, p \to \infty \), where \( \theta_{i,\tau}, v_{i,\tau} \) and \( \gamma_{i,\tau} \) are defined in (4.3).

Theorem 4.1 is a direct result of Theorem 3.3, since an asymptotic distribution of
\[ \frac{\lambda_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)}}{\theta_{i,\tau}} \]
can be derived using the independence between \( \lambda_{i,\tau}^{(1)} \) and \( \lambda_{i,\tau}^{(2)} \). Consequently, under the null hypothesis of the autocovariance test, the test statistic \( Z_{i,\tau} \) converges weakly to a standard normal random variable when \( T, p \to \infty \). Nonetheless, under certain alternative hypotheses such as \( K_1 = K_2 = K, \gamma_{i,\tau}^{(1)} = \gamma_{i,\tau}^{(2)}, v_{i,\tau}^{(1)} = v_{i,\tau}^{(2)} \), but \( \left( \sigma_i^{(1)} \right)^2 \neq \left( \sigma_i^{(2)} \right)^2 \) and \( \theta_{i,\tau}^{(1)} \neq \theta_{i,\tau}^{(2)} \), it can be shown in the next theorem that, under a local alternative hypothesis, the power of the autocovariance test converges to 1 as \( T, p \to \infty \).

**Theorem 4.2.** Under Assumptions 2.1 and 2.2, if we additionally assume two independent high-dimensional time series \( \{y_i^{(1)}\} \) and \( \{y_i^{(2)}\} \) follow two different canonical factor models (4.4) such that
\[ K_1 = K_2 = K, \gamma_{i,\tau}^{(1)} = \gamma_{i,\tau}^{(2)}, v_{i,\tau}^{(1)} = v_{i,\tau}^{(2)}, \) and \( \theta_{i,\tau}^{(1)} = (1 + c)\theta_{i,\tau}^{(2)} \).

Then, for any \( c \) such that \( \sqrt{Tc} \to \infty \) as \( T, p \to \infty \), it holds that
\[ \text{Pr} \left( |Z_{i,\tau}| > z_\alpha |H_1 \right) \to 1, \quad (4.7) \]
for \( T, p \to \infty \), where \( z_\alpha \) is the \( \alpha \)-th quantile of the standard normal distribution.

**Remark 4.1.** The condition \( \sqrt{T} \frac{2c}{2+c} \to \infty \) as \( T, p \to \infty \) in Theorem 4.2 indicates a local alternative hypothesis. It implies that for \( T, p \to \infty \), the power of the test converges to 1 not only for a constant \( c \), but also for some \( c \to 0 \) as long as \( \sqrt{Tc} \to \infty \). In other words, this test even works asymptotically for a local alternative hypothesis where the difference between \( \theta_{i,\tau}^{(1)} \) and \( \theta_{i,\tau}^{(2)} \) tends to 0, but slower than \( 1/\sqrt{T} \).

### 4.2 Implementation of testing procedure

The test statistic \( Z_{i,\tau} \) is an infeasible statistic in practice as it involves some unknown parameters \( \gamma_{i,\tau}, v_{i,\tau} \) and \( \theta_{i,\tau} \). In this part, we will propose a practical procedure for the autocovariance test.
For two high-dimensional time series, the test procedure can be summarized into
four steps. Firstly, estimates of the factor models for both populations should be con-
ducted, where the number of factors needs to be determined. Secondly, the original
high-dimensional observations and the factor models’ estimates need to be standard-
ized to fulfill the canonical factor model (4.4). Thirdly, the quantities required to compute
the feasible test statistic $\tilde{Z}_{i,\tau}$ should be estimated from both populations. Furthermore,
we can compute the feasible test statistic $\tilde{Z}_{i,\tau}$ and its corresponding $p$-values for testing
the equivalence of eigenvalues. The details of the estimation and testing procedures are
illustrated and discussed as follows.

**Step 1:** Estimation of factor models.

For de-meaned high-dimensional time series observations $\{y_{it}^{(m)}\}$ with $m = 1, 2$, we
first compute the symmetrized lag-$\tau$ sample autocovariance matrix $\tilde{\Sigma}_y^{(m)}(\tau)\tilde{\Sigma}_y^{(m)}(\tau)^\top$,
where $\tilde{\Sigma}_y^{(m)}(\tau) = \frac{1}{T-\tau-1} \sum_{t=1}^{T-\tau} y_{it}^{(m)} y_{i,t+\tau}^{(m)^\top}$ is the lag-$\tau$ sample autocovariance matrix
of $\{y_{it}^{(m)}\}$. By applying spectral (eigenvalue) decomposition on $\tilde{\Sigma}_y^{(m)}(\tau)\tilde{\Sigma}_y^{(m)}(\tau)^\top$,
we can obtain an estimate of the factor loading matrix as $\hat{L}_i^{(m)} = (\hat{L}_1^{(m)}, \hat{L}_2^{(m)}, \ldots, \hat{L}_p^{(m)})$
with $\hat{L}_i^{(m)}$ the eigenvector of $\tilde{\Sigma}_y^{(m)}(\tau)\tilde{\Sigma}_y^{(m)}(\tau)^\top$ corresponding to the $i$-th largest
eigenvalue $\hat{\lambda}_i^{(m)}$. To determine the number of factors, we adopt the idea in Lam
et al. [18], and use a ratio-based estimator $\hat{K}_m = \arg\min_{1 \leq j \leq R} \hat{\lambda}_j^{(m)} / \hat{\lambda}_{j+1}^{(m)}$ where
$\hat{\lambda}_1^{(m)} \geq \hat{\lambda}_2^{(m)} \geq \cdots \geq \hat{\lambda}_{N_m}^{(m)}$ and $R$ is an integer satisfying $K_m \leq R < N_m$.

With $\hat{L}_i^{(m)}$, the factors can then be estimated by $\tilde{f}_i^{(m)} = \hat{L}_i^{(m)^\top} y_{it}^{(m)}$ and the high-
dimensional time series can be recovered by $\tilde{y}_{it}^{(m)} = \hat{L}_i^{(m)} \tilde{f}_i^{(m)}$. Hence we have estimates of the factor model that is not in the canonical form (4.4) and the residuals
can be estimated by

$$\tilde{e}_{it}^{(m)} = y_{it}^{(m)} - \tilde{f}_i^{(m)} \tilde{f}_i^{(m)^\top}.$$  (4.8)

Moreover, to standardize the estimated factor model into canonical form (4.4), we
need to find an estimate of $\Sigma_{\tilde{e}^{(m)}}$, the covariance of $\tilde{e}_{it}^{(m)}$. To achieve that, we can
obtain an estimate of the variance of \( \epsilon_{j,t}^{(m)} \) as

\[
\left( \hat{\sigma}_{\epsilon,j}^{(m)} \right)^2 = \frac{1}{T-1} \sum_{t=1}^{T} \left( \hat{\epsilon}_{j,t}^{(m)} - \bar{\epsilon}_{j,t}^{(m)} \right)^2.
\]

And \( \hat{\Sigma}_{\epsilon}^{(m)} \) can then be estimated by

\[
\hat{\Sigma}_{\epsilon}^{(m)} = \text{diag} \left( \left( \hat{\sigma}_{\epsilon,1}^{(m)} \right)^2, \left( \hat{\sigma}_{\epsilon,2}^{(m)} \right)^2, \ldots, \left( \hat{\sigma}_{\epsilon,N_{m}}^{(m)} \right)^2 \right).
\]

**Remark 4.2.** It is clear that for two high-dimensional time series where the estimated numbers of factors are different, i.e., \( \hat{K}_1 \neq \hat{K}_2 \), one can conclude that the two high-dimensional data follow different factor models where \( \mathcal{M} (L^{(1)}) \neq \mathcal{M} (L^{(2)}) \) and the numbers of spiked eigenvalues for their autocovariance matrices are different. However, if we are interested in testing the equivalence for the particular spiked eigenvalue of the autocovariance matrices for two high-dimensional data, it is still possible to perform the autocovariance test even if \( \hat{K}_1 \neq \hat{K}_2 \). The intuition is to test whether the low-dimensional representations of both high-dimensional time series have the same autocovariance in a certain principal direction, though the data cannot be fully projected into the same temporal subspace.

**Step 2:** Standardizing factor models to the canonical form.

With \( \hat{L}_\tau^{(m)} \) and \( \hat{\Sigma}_{\epsilon}^{(m)} \), we can now standardize the estimated factor models (4.8) to fulfill the canonical form. Firstly, we define a \( N_{m} \times N_{m} \) matrix \( M_{\tau}^{(m)} = \left( \hat{L}_\tau^{(m)}, \mathbf{0}_{p_{m} + K_{m} - \hat{K}_{m}} \right) \).

Then we can define \( \tilde{y}_{t}^{(m)} := \left( \hat{\Sigma}_{\epsilon}^{(m)} \right)^{-1/2} M_{\tau}^{(m)\top} y_{t}^{(m)} \) for the normalized data and \( \tilde{\epsilon}_{t}^{(m)} := \left( \hat{\Sigma}_{\epsilon}^{(m)} \right)^{-1/2} \epsilon_{t}^{(m)} \) for the normalized residuals. By left multiplying \( \left( \hat{\Sigma}_{\epsilon}^{(m)} \right)^{-1/2} M_{\tau}^{(m)\top} \), the estimated factor model is reduced to

\[
\tilde{y}_{t}^{(m)} = \left( \hat{\Sigma}_{\epsilon}^{(m)} \right)^{-1/2} M_{\tau}^{(m)\top} \hat{L}_\tau^{(m)} \hat{f}_{t}^{(m)} + \tilde{\epsilon}_{t}^{(m)},
\]

where note that

\[
M_{\tau}^{(m)\top} \hat{L}_\tau^{(m)} = \begin{pmatrix} I_{K_{m}} \\ \mathbf{0}_{(p_{m} + K_{m} - \hat{K}_{m}) \times \hat{K}_{m}} \end{pmatrix}.
\]
Secondly, to normalize $\tilde{f}_i^{(m)}$, we can estimate the variance of $\tilde{f}_i^{(m)}$ by $(\tilde{\sigma}_i^{(m)})^2 = \frac{1}{T-1} \sum_{t=1}^{T} (\tilde{f}_{i,t}^{(m)} - \tilde{f}_i^{(m)})^2$, for $i = 1, 2, \ldots, K_m$. Hence the covariance of $\tilde{f}_i^{(m)}$ can be obtained as

$$\tilde{\Sigma}_f^{(m)} = \text{diag} \left( (\tilde{\sigma}_1^{(m)})^2, (\tilde{\sigma}_2^{(m)})^2, \ldots, (\tilde{\sigma}_{K_m}^{(m)})^2 \right).$$

Write $\tilde{f}_i^{(m)} = \left( \Sigma_{\tilde{y}}^{(m)} \right)^{-1/2} \tilde{\Sigma}_f^{(m)} \left( \Sigma_{\tilde{y}}^{(m)} \right)^{-1/2}$ for the normalized estimates of factors, and $\tilde{L}_r^{(m)} = \left( \text{diag}(\tilde{\sigma}_1^{(m)}, \ldots, \tilde{\sigma}_{K_m}^{(m)}) \right)$ for the estimates of loading matrices. Then we have standardized the estimated factor models to

$$\tilde{y}_i^{(m)} = \tilde{L}_r^{(m)} \tilde{f}_i^{(m)} + \tilde{\epsilon}_i^{(m)}, \quad (4.9)$$

which follows the canonical form defined by (4.4).

**Step 3:** Estimation of unknown parameters in the test statistic.

For standardized data $\{\tilde{y}_i^{(m)}\}$ following the estimated factor model (4.9), $\lambda_{i,\tau}^{(m)}$ can be computed as the $i$-th largest eigenvalue of the symmetrized lag-$\tau$ sample autocovariance matrix $\tilde{\Sigma}_y^{(m)}(\tau)\tilde{\Sigma}_y^{(m)}(\tau)^\top$, where $\tilde{\Sigma}_y^{(m)}(\tau) = \frac{1}{T-\tau-1} \sum_{t=1}^{T-\tau} \tilde{y}_i^{(m)} \tilde{y}_i^{(m)}\top$ and $\gamma_{i,\tau}^{(m)}$ can be estimated from the sample lag-$\tau$ autocovariance of the $i$-th estimated factor $\{\tilde{f}_{i,t}^{(m)}\}$. Besides, we also need to estimate the quantities $\psi_{i,\tau}^{(m)}$ and $\theta_{i,\tau}^{(m)}$, as defined in the autocovariance test, for each sample to compute the test statistic. However, since $(\psi_{i,\tau}^{(m)})^2 = \frac{1}{T-\tau} \text{Var} \left( \sum_{t=1}^{T-\tau} f_{i,t}^{(m)} f_{i,t+\tau}^{(m)} \right)$ depends on the variance of $\sum_{t=1}^{T-\tau} f_{i,t}^{(m)} f_{i,t+\tau}^{(m)}$ and $\theta_{i,\tau}^{(m)}$ is the asymptotic centering of $\lambda_{i,\tau}^{(m)}$, they cannot be directly estimated from original sample observations. Instead, we can use bootstrap to estimate both quantities. It is worth noting that since the bootstrap is conducted on the estimated low-dimensional factor time series $\{\tilde{f}_i^{(m)}\}$, the bootstrap estimators are not affected by the increasing dimensions.

Therefore, bootstrap methods for time series such as the sieve bootstrap can be conducted on the estimated factors for estimating $\psi_{i,\tau}^{(m)}$ and $\theta_{i,\tau}^{(m)}$. Next, we will apply the AR-sieve bootstrap method in [6] to get a bootstrap estimation for the
unknown parameters. In specific, an AR($p$) model can be fitted for each estimated factor $\tilde{f}^{(m)}_i$ and the residuals can be taken as

$$\tilde{u}^{(m)}_{i,t} = \tilde{f}^{(m)}_i - \sum_{l=1}^{p} \tilde{\phi}^{(m)}_{i,l} \tilde{f}^{(m)}_{i,t-l},$$

where $\{\tilde{\phi}^{(m)}_{i,l}, l = 1, 2, ..., \tilde{K}_m\}$ are the AR coefficients. Then by resampling from the empirical distribution of the centralized residual $(\tilde{u}^{(m)}_{i,t} - \bar{u}^{(m)}_i)$, the bootstrap factors can be generated as

$$f^{(m)}_{i,t} = \sum_{l=1}^{p} \tilde{\phi}^{(m)}_{i,l} f^{(m)}_{i,t-l} + u^{(m)}_{i,t},$$

where $b = 1, 2, ..., B$ for $B$ bootstrap samples of $\{f^{(m)}_{i,t}\}$ and $u^{(m)}_{i,t}$ is the bootstrap residual. Hence, we can estimate $v^{(m)}_{i,\tau}$ by

$$\tilde{v}^{(m)*}_{i,\tau} = \sqrt{\frac{1}{T-\tau} \left( \frac{1}{B-1} \sum_{b=1}^{B} \left( \frac{\sum_{t=1}^{T-\tau} f^{(m)}_{i,t} f^{(m)}_{i,t+\tau}}{T} - \frac{1}{B} \sum_{b=1}^{B} \left( \frac{\sum_{t=1}^{T-\tau} f^{(m)}_{i,t} f^{(m)}_{i,t+\tau}}{T} \right) \right)^2 \right)}.$$

In addition, since $\tilde{\theta}^{(m)}_{i,\tau}$ is an estimate of the asymptotic centering of $\lambda^{(m)}_{i,\tau}$, we can also bootstrap $\{\tilde{y}^{(m)}_t\}$ by

$$y^{(m)}_{t} = \tilde{L}_{\tau} f^{(m)}_i,$$

for $B$ times and estimate $\theta^{(m)}_{i,\tau}$ by

$$\tilde{\theta}^{(m)*}_{i,\tau} = \frac{1}{B} \sum_{b=1}^{B} \lambda^{(m)}_{i,\tau},$$

where $\lambda^{(m)}_{i,\tau}$ is the $i$-th largest eigenvalue of the symmetrized lag-$\tau$ sample autocovariance matrices of $\{y^{(m)}_t\}$. Meanwhile, since the sieve bootstrap is conducted to estimate $\gamma^{(m)}_{i,\tau}$ and $\theta^{(m)}_{i,\tau}$, an alternative estimate of $\gamma^{(m)}_{i,\tau}$ can also be computed based on $B$ bootstrap samples, as

$$\tilde{\gamma}^{(m)*}_{i,\tau} = \frac{1}{B} \sum_{b=1}^{B} \left( \frac{1}{T-\tau-1} \sum_{t=1}^{T-\tau} \left( f^{(m)}_{i,t} - \frac{1}{T} \sum_{t=1}^{T} f^{(m)}_{i,t} \right) \left( f^{(m)}_{i,t+1} - \frac{1}{T} \sum_{t=1}^{T} f^{(m)}_{i,t} \right) \right).$$
Step 4: Computing the test statistic and p-value.

When the first three steps have been conducted on both high-dimensional times series \( \{y_t^{(1)}\} \) and \( \{y_t^{(2)}\} \), we can estimate the unknown parameters in (4.2) by

\[
\tilde{\theta}_{i,\tau}^* := \frac{T_1\bar{\theta}_{i,\tau}^{(1)*} + T_2\bar{\theta}_{i,\tau}^{(2)*}}{T_1 + T_2}, \quad \tilde{\nu}_{i,\tau}^* := \frac{T_1\bar{\nu}_{i,\tau}^{(1)*} + T_2\bar{\nu}_{i,\tau}^{(2)*}}{T_1 + T_2}, \quad \tilde{\gamma}_{i,\tau}^* := \frac{T_1\bar{\gamma}_{i,\tau}^{(1)*} + T_2\bar{\gamma}_{i,\tau}^{(2)*}}{T_1 + T_2},
\]

where \( \bar{\theta}_{i,\tau}^{(m)*} \), \( \bar{\nu}_{i,\tau}^{(m)*} \) and \( \bar{\gamma}_{i,\tau}^{(m)*} \) are computed from two high-dimensional time series following the procedure in Step 3. Then, the test statistic can be computed as

\[
\tilde{Z}_{i,\tau} := \left( \lambda_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)} \right) \sqrt{\frac{T_1 T_2}{T_1 + T_2}} \frac{\tilde{\gamma}_{i,\tau}^*}{2\tilde{\nu}_{i,\tau}^* \tilde{\theta}_{i,\tau}^*},
\]

(4.10)

where \( \lambda_{i,\tau}^{(1)} \) and \( \lambda_{i,\tau}^{(2)} \) are the \( i \)-th (\( 1 \leq i \leq \hat{K} \)) largest eigenvalues of the symmetrized lag-\( \tau \) sample autocovariance matrices for the standardized data \( \{\tilde{y}_t^{(1)}\} \) and \( \{\tilde{y}_t^{(2)}\} \), respectively. At last, the p-values of the test statistic \( \tilde{Z}_{i,\tau} \) are computed as \( \Pr(z > |\tilde{Z}_{i,\tau}|) = 2\left(1 - \Phi\left(|\tilde{Z}_{i,\tau}|\right)\right) \) for a two-sided test, and \( \Pr(z > \tilde{Z}_{i,\tau}) = 1 - \Phi\left(\tilde{Z}_{i,\tau}\right) \) or \( \Pr(z < \tilde{Z}_{i,\tau}) = \Phi\left(\tilde{Z}_{i,\tau}\right) \) for one-sided tests, where \( \Phi(\cdot) \) denotes the cumulative distribution function (CDF) of a standard normal random variable.

To clarify the four steps mentioned above, a flow chart is provided in Figure 1, which summarizes the basic logic and procedure for the autocovariance test.

5 Simulation studies

This section uses numerical simulations to investigate the empirical sizes and powers of the proposed autocovariance test in various scenarios.

To start, we first of all explore the empirical sizes of the autocovariance test under various scenarios, including various orders of factor strength and ratios between the sample size and the data dimension. In this section of simulation studies, we again write \( N_m := K_m + p_m \) as the data dimension of \( \{y_t^{(m)}\} \) and consider only the case that \( N_1 = N_2 := N \) for simplicity. We assume the high-dimensional observations \( \{y_t^{(1)}\} \) and \( \{y_t^{(2)}\} \) are generated
High-dimensional time series $y_t$

Apply spectral decomposition on $\tilde{\Sigma}(m)y(\tau)\tilde{\Sigma}(m)y(\tau)^\top$ to find the estimated number of factors $\hat{K}_m$.

Estimate factor models as $\hat{y}_t^{(m)} = \hat{L}_t^{(m)}\hat{f}_t^{(m)} + \hat{\epsilon}_t^{(m)}$ and obtain $\hat{\Sigma}_e$.

Standardize estimated factor models to the canonical form $\tilde{y}_t^{(m)} = \tilde{L}_t^{(m)}\tilde{f}_t^{(m)} + \tilde{\epsilon}_t^{(m)}$.

Obtain $\lambda_{i,\tau}^{(m)}$ and compute $\tilde{\nu}_{i,\tau}^{(m)*}$, $\tilde{\theta}_{i,\tau}^{(m)*}$, and $\tilde{\gamma}_{i,\tau}^{(m)*}$ via AR-sieve bootstrap on $\tilde{f}_t^{(m)}$.

Find the test statistic $\tilde{Z}_{i,\tau}$ and $p$-value.

Figure 1: Flow chart for the autocovariance test

from the one-factor model $y_t^{(m)} = L_t^{(m)}f_t^{(m)} + \epsilon_t^{(m)}$ in the canonical form (4.4). Moreover, we assume that the factors $\{f_t^{(m)}\}$ for both time series follow AR(1) models with zero means, AR coefficients $\phi_t^{(m)} = 0.5$ and variances equal to 1. In other words, the factors for both time series are generated by

$$f_t^{(m)} = \phi_t^{(m)}f_{t-1}^{(m)} + z_{1,t}^{(m)}, \; m = 1, 2,$$

where $\phi_t^{(m)} = 0.5$ and $\{z_{1,t}^{(m)}\}$ are i.i.d normally distributed $\mathcal{N}\left(0, (\sigma_{z}^{(m)})^2\right)$ with $\left(\sigma_{z}^{(m)}\right)^2 = 1/\left(1 - (\phi_t^{(m)})^2\right) = 3/4$, so that $Var\left(f_{1,t}^{(m)}\right) = 1$. As discussed for the canonical form of factor models, the variances $\left\{(\sigma_i^{(m)})^2\right\}$ of unnormalized factors are contained in the loading matrix $L^{(m)}$. To study the empirical sizes of the autocovariance test under various factor strengths, we consider the case $\left(\sigma_1^{(m)}\right)^2 \asymp N^{1-\delta}$ for $\delta \in [0, 1]$ utilized in Lam et al. [18]. Using this definition, $\delta = 0$ refers to the strongest factors with the pervasiveness, and factor strengths drop when $\delta$ increases from 0 to 1. In this section, we consider four different cases for factor strengths, where $\delta = 0, 0.1, 0.3,$ and 0.5. Specifically, $\left(\sigma_1^{(m)}\right)^2$ in
the loading matrix \( L^{(m)} \) that follows canonical form (4.5) is assumed to be \( N, N^{0.9}, N^{0.7}, \) and \( N^{0.5} \), respectively, and \( \{ \epsilon_{j,t}^{(m)} \} \) are assumed to be i.i.d. \( \mathcal{N}(0,1) \).

In summary, both \( N \)-dimensional time series observations are generated by

\[
y^{(m)}_t = \left( \sigma^{(m)}_1, 0_{N-1} \right) f^{(m)}_{1,t} + \epsilon_t^{(m)}, \quad m = 1, 2, \tag{5.2}
\]

where \( \sigma_1^{(m)} = N^{1-\delta} \), \( \{ \epsilon_{j,t} \} \) are i.i.d. \( \mathcal{N}(0,1) \), and \( \{ f_{1,t}^{(m)} \} \) are generated by (5.1).

To explore the impact of ratios between sample size \( T \) and data dimension \( N \), we generate data with \( T = 400, 800 \) and \( N = 100, 200, 400, 800, 1600 \). To compute the empirical sizes, for each combination of \( T, N \) and \( \delta \), the observations of two high-dimensional time series are first of all generated as \( \{ y_1^{(1)} \} \) and \( \{ y_1^{(2)} \} \). Then, by utilizing the estimation and testing procedures in Section 4.2, the test statistic \( \tilde{Z}_{i,\tau} \) can be computed by (4.10), where \( B = 500 \) bootstrap samples are generated to compute \( \tilde{\theta}_{i,\tau}^{(m)*}, \tilde{\nu}_{i,\tau}^{(m)*} \) and \( \tilde{\gamma}_{i,\tau}^{(m)*} \), and the numbers of factors are assumed to be known (i.e., \( \tilde{K}_m = 1 \)) for both samples.

The empirical sizes of a one-sided autocovariance test for \( i = 1, \tau = 1 \), and significant level \( \alpha = 0.1 \) under various combinations of \( T, N \) and \( \delta \) are computed as the empirical probabilities that \( \tilde{Z}_{1,1} \) is less than \( z_\alpha \) or greater than \( z_{1-\alpha} \), i.e.,

\[
\frac{1}{M} \sum_{m=1}^{M} 1\{ \tilde{Z}_{1,1}(m) < z_\alpha \}, \quad \text{or} \quad \frac{1}{M} \sum_{m=1}^{M} 1\{ \tilde{Z}_{1,1}(m) > z_{1-\alpha} \},
\]

for \( M = 500 \) Monte Carlo simulations, where \( \tilde{Z}_{1,1}(m) \) is the test statistic computed from the \( m \)-th simulation.

As presented in Figure 2, despite some minor fluctuations, the empirical sizes of the autocovariance test are close to the nominal significant level \( \alpha = 0.1 \) for all choices of \( N, T \) and \( \delta \). That is, when the numbers of factors are known or can be correctly estimated, the nominal type-I errors of the autocovariance test can be verified via empirical simulation studies for \( \delta = 0, 0.1, 0.3, 0.5 \), \( T = 400, 800 \), and \( N = 100, 200, 400, 800, 1600 \). The choice of \( \tau = 1 \) for the autocovariance test is to acquire the most information on temporal dependence of \( \{ y_t^{(m)} \} \) and to achieve the best accuracy on corresponding estimators.
Figure 2: Empirical sizes of the autocovariance test in the first scenario with $T = 400, 800$, $N = 100, 200, 400, 800, 1600$, and $\delta = 0, 0.1, 0.3, 0.5$.

$\tilde{\theta}_{i,\tau}^{(m)*}, \tilde{v}_{i,\tau}^{(m)*}$ and $\tilde{\gamma}_{i,\tau}^{(m)*}$, while other choices of finite $\tau$ may be considered with cautions as the temporal correlation $\gamma_{i,\tau}^{(m)}$ tends to 0 when $\tau$ increases.

To study the empirical powers of the autocovariance test, we notice that for two high-dimensional time series following factor models that are normalized to the canonical form (4.9), the difference between spiked eigenvalues $\mu_{i,\tau}^{(m)}$, $m = 1, 2$ may arise from either the difference between factor strength $(\sigma_i^{(m)})^2$, $m = 1, 2$ or the difference between temporal autocorrelation $\gamma_{i,\tau}^{(m)}$, $m = 1, 2$. Therefore, to empirically investigate the autocovariance test’s power, we study two typical scenarios where either variances or autocorrelations of factors are different between two factor models. We are particularly interested in whether the autocovariance test’s empirical power grows with the difference between variances or autocorrelations for two high-dimensional time series.

Specifically, to explore the impacts of $\delta$, $N$ and $T$ on empirical powers, we again generate observations from two populations with $T = 400, 800$, $N = 200, 400, 800$, and $\delta = 0, 0.1, 0.3, 0.5$. The data in the first population is generated by (5.2), which is precisely the same as we study the empirical sizes, while the data in the second population is generated with a different $\sigma_1^{(2)}$ or $\phi_1^{(2)}$ in the factor model. We consider the following two scenarios.

**Scenario 1: Different Factor Strengths.**
We keep the same temporal autocorrelation, i.e. AR coefficient $\phi_1^{(2)}$ is the same as $\phi_1^{(1)}$ (i.e. $\phi_1^{(2)} = \phi_1^{(1)} = 0.5$), and set
\[
\left(\sigma_1^{(2)}\right)^2 = 1.1 (\sigma_1^{(1)})^2, 1.3 (\sigma_1^{(1)})^2, 1.5 (\sigma_1^{(1)})^2, 1.7 (\sigma_1^{(1)})^2, 1.9 (\sigma_1^{(1)})^2,
\]
respectively.

**Scenario 2: Different Temporal Autocorrelations.**

Let $\left(\sigma_1^{(2)}\right)^2 = \left(\sigma_1^{(1)}\right)^2 = N^{1-\delta}$, but set the AR coefficients for the second population to be
\[
\phi_1^{(2)} = 0.9 \phi_1^{(1)} , 0.8 \phi_1^{(1)} , 0.7 \phi_1^{(1)} , 0.6 \phi_1^{(1)} , 0.5 \phi_1^{(1)},
\]
respectively.

By doing that, we can investigate how the empirical powers of the autocovariance test are affected by the difference between autocorrelations of factors in two factor models. Moreover, it is worth to mention that when generating $\{f_{i,t}^{(1)}\}$ and $\{f_{i,t}^{(2)}\}$, $\{z_{1,t}^{(1)}\}$ are i.i.d. $\mathcal{N} \left(0, \left(\sigma_{z}^{(1)}\right)^2\right)$ with $\left(\sigma_z^{(1)}\right)^2 = 1/ \left(1 - \left(\phi_1^{(1)}\right)^2\right)$, whereas $\{z_{1,t}^{(2)}\}$ are i.i.d. $\mathcal{N} \left(0, \left(\sigma_{z}^{(2)}\right)^2\right)$ with $\left(\sigma_z^{(2)}\right)^2 = 1/ \left(1 - \left(\phi_1^{(2)}\right)^2\right)$.

To compute the empirical powers, for each combination of $T, N$ and $\delta$, two high-dimensional time series observations are generated as $\{y_{t}^{(1)}\}$ and $\{y_{t}^{(2)}\}$ first. Then, we can follow the estimation and testing procedures in Section 4.2 and compute the test statistic $\tilde{Z}_{i,\tau}$ by (4.10), where again $B = 500$ bootstrap samples are generated to find $\tilde{\theta}_{i,\tau}^{(m)*}$, $\tilde{\gamma}_{i,\tau}^{(m)*}$, and $\tilde{\gamma}_{i,\tau}^{(m)*}$ for both samples with the number of factors assumed to be known (i.e., $\tilde{K}_m = 1$). Lastly, based on $M = 500$ Monte Carlo simulations, the empirical powers of a one-sided autocovariance test for $i = 1$, $\tau = 1$, and $\alpha = 0.1$ can be estimated by the empirical probability that $\tilde{Z}_{1,1}$ is less than $z_\alpha$, i.e.,
\[
\frac{1}{M} \sum_{m=1}^{M} 1\{\tilde{Z}_{1,1}(m) < z_\alpha\},
\]
for the first scenario, and the empirical probability that $\tilde{Z}_{1,1}$ is greater than $z_{1-\alpha}$, i.e.,
\[
\frac{1}{M} \sum_{m=1}^{M} 1\{\tilde{Z}_{1,1}(m) > z_{1-\alpha}\},
\]
for the second scenario, where we have assumed $\mu^{(1)}_{1,1} < \mu^{(2)}_{1,1}$ for various choices of $\left(\sigma^{(2)}_1\right)^2$ in the first scenario, and $\mu^{(1)}_{1,1} > \mu^{(2)}_{1,1}$ for various choices of $\phi^{(2)}_1$ in the second scenario.

Figure 3: Empirical powers of the autocovariance test in the first scenario with $T = 400$, $N = 200, 400, 800$, and $\delta = 0, 0.1, 0.3, 0.5$.

Figure 4: Empirical powers of the autocovariance test in the first scenario with $T = 800$, $N = 200, 400, 800$, and $\delta = 0, 0.1, 0.3, 0.5$.

Empirical powers of the autocovariance test in both scenarios with various choices of $N$, $T$, and $\delta$ are presented in Figures 3 to 6. As shown in Figures 3 and 4, it is clear
that for all combinations of $N$ and $T$, empirical powers in the first scenario increase towards 1 when $(\sigma_1^{(2)})^2$ increases from $1.1(\sigma_1^{(1)})^2$ to $1.9(\sigma_1^{(1)})^2$. Therefore, numerical results in Figure 3 and 4 suggest that the autocovariance test can correctly reject the null hypothesis when two high-dimensional time series follow different factor models with $(\sigma_1^{(2)})^2 \neq (\sigma_1^{(1)})^2$. Besides, despite the common temporal autocorrelation, for the same difference between $(\sigma_1^{(2)})^2$ and $(\sigma_1^{(1)})^2$, the empirical powers of one-sided autocovariance tests for $T = 800$ are generally higher than those associated with $T = 400$, which can be justified by the order $\sqrt{T}$ in (4.2). In detail, a larger value of $T$ could incur a larger power. Also, the powers of stronger factor models with smaller $\delta$ are slightly higher than those of weaker factor models with larger $\delta$, especially for $T = 400$.

Figure 5: Empirical powers of the autocovariance test in the second scenario with $T = 400$, $N = 200, 400, 800$, and $\delta = 0, 0.1, 0.3, 0.5$.

Similarly, as presented in Figure 5 and 6, for all ratios of $N$ and $T$, empirical powers in the second scenario also increase towards 1, while $\phi_1^{(2)}$ drops from $0.9\phi_1^{(1)}$ to $0.5\phi_1^{(1)}$. As a consequence, Figure 5 and 6 suggest that the autocovariance test can correctly reject the null hypothesis when two high-dimensional time series have different temporal autocorrelations $\phi_1^{(2)} \neq \phi_1^{(1)}$. However, unlike the first scenario, empirical powers of the one-sided autocovariance test for relatively weak factor models with large $\delta$, especially
Figure 6: Empirical powers of the autocovariance test in the second scenario with $T = 800$, $N = 200, 400, 800$, and $\delta = 0, 0.1, 0.3, 0.5$.

$\delta = 0.5$, are slightly lower than those of relatively strong factor models with small $\delta$. In other words, compared with strong factor models (i.e. high factor strength), the autocovariance test for weak factor models is slightly less potent in detecting the same proportional changes in autocorrelations of factors for two different factor models.

6 Hierarchical Clustering for Multi-country Mortality Data

To incorporate the proposed autocovariance test into hierarchical clustering analysis on real-world data, we study age-specific mortality rates from countries worldwide. In specific, we test whether the mortality rates for different countries share the same spiked eigenvalues of their autocovariance matrices and use the p-values as the measure of dissimilarities across all countries to implement the hierarchical clustering on the mortality rates across countries. In the past century, age-specific mortality rates have received massive attention, especially by insurance companies and governments, as accurate forecasting of mortality rates is crucial for the pricing of life insurance products and is highly
related to social and economic policies. Among many works on forecasting age-specific mortality rates, the Lee-Carter model [19] is prevalent and has been used globally. Despite some extensions on the original model [see, e.g., 15, 20], one drawback of the Lee-Carter model is that it only focuses on the death rates of a single country, therefore may produce quite different long-run forecasts of mortality rates from different countries. Recently, joint modeling of mortality rates for multiple countries has become more attractive since the common features extracted from multiple populations can further improve forecasting accuracy. In this sense, correctly classifying countries with similar patterns of mortality rates into the same group for joint modeling and combined statistical analysis becomes critical. In addition to the traditional grouping methods based on socioeconomic status or ethnic group, Tang et al. [25] emphasize the use of statistical clustering methods on determining the grouping of countries.

This section uses the proposed autocovariance test to explore multiple countries’ mortality data, especially the spiked eigenvalues of the autocovariance matrices and proposes a novel hierarchical clustering method for mortality data from different countries. To achieve this, we collect the total death rates for various countries from the Human Mortality Database [27]. The plots of log mortality rates for Australia are shown in Figure 7 as an example. For the best quality of data, we choose the death rates from age 0 to 90 and require each country’s sample size to be relatively large. For some countries such as the Republic of Korea and Chile, the data are only available for a short period, while for some other countries, the data quality cannot be guaranteed due to some historical reasons. As a result, we only study selected countries with total death rates available from 1957 to 2017. Besides, as seen in the first graph of Figure 7, the age-specific mortality rates are not stationary for most ages; therefore, they have been pre-processed by taking the logarithm and then differenced, since our method is developed for stationary time series. The second graph of Figure 7 illustrates the pattern of age-specific mortality rates for Australia and a similar pattern is commonly seen in many other developed countries. However, for countries such as Norway and Iceland, there are quite a few zero death rates
for young children due to the relatively small population; hence they are excluded from our study. For all the other countries, zero death rates are replaced by the averages of death rates in adjacent years. In summary, the data we study has dimension \( N = 91 \) and sample size \( T = 60 \) for each country.

![Observed time series of log death rates in Australia](image)

Figure 7: Observed time series of log death rates in Australia

According to the estimation and testing procedure in Section 4.2, factor models in canonical form (4.4) are firstly estimated and normalized from the differenced log death rates for each country. In the meantime, the number of factors in the factor model for each country is estimated and compared. As shown in Table 1, for most countries, there is only one factor estimated from the differenced log death rates, while there are some exceptions where two, three, and five factors are estimated. For countries with the same number of factors, we can compute the test statistic \( \tilde{Z}_{i,\tau} \) to test the equivalence of the temporal covariance in the temporal subspace. For the best accuracy in estimating the number of factors and temporal dependence among death rates, the autocovariance test is performed based on \( \tau = 1 \) throughout this section.

For countries with one factor, the test statistic \( \tilde{Z}_{1,1} \) for each pair of countries can be computed by following the procedure in Section 4.2. For all other countries with one factor, the \( p \)-values associated with all test statistics are computed. As illustrated in Figure 8, the spiked eigenvalues of the autocovariance matrices in the majority of European countries are similar as most \( p \)-values of test statistics between two European countries.
Table 1: Estimated number of factors in the factor model for each country

| Estimated number of factors | Countries                                                                 |
|-----------------------------|---------------------------------------------------------------------------|
| 1                           | Australia, Belgium, Bulgaria, Czechia, Finland, Greece, Hungary, Japan, Netherlands, Sweden, Switzerland, U.K. |
| 2                           | Denmark                                                                   |
| 3                           | Canada, France, Italy, Portugal                                            |
| 5                           | Poland                                                                    |

countries are greater than 0.1. However, the $p$-values between Finland and Bulgaria, the U.K., and Finland are relatively small. Following the results of the autocovariance test between each pair of countries with the same number of factors, a hierarchical clustering method can be proposed where the dissimilarity can be measured using the $p$-value, such as $(1-p)$-value or $(1/p)$-value. For the analysis of mortality data, we define the dissimilarity for all countries with one factor as $(1-p)$-value, and the result of hierarchical clustering using average linkage for all countries with one factor is presented in Figure 9.

For countries with three factors, to test on the equivalence of autocovariances through factor models, test statistics between each pair of countries are computed for all three factors as $\tilde{Z}_{1,1}, \tilde{Z}_{2,1}$ and $\tilde{Z}_{3,1}$. As depicted in Figure 10, the $p$-values for $\tilde{Z}_{1,1}$ and $\tilde{Z}_{2,1}$ between all pairs of countries are relatively large, which suggests that the differences of the first two factors between each pair of countries are not significant (at $\alpha = 0.1$). Nonetheless, $p$-values for $\tilde{Z}_{1,3}$ are relatively small between Canada and France, Canada and Italy, and Italy and Portugal. To measure the dissimilarity of mortality data between two countries with more than one factors, the overall distance between two countries can be defined as a weighted average of the distances for all factors. In specific, for each pair of countries, we can define the distance for the $i$-th factor as $dist_i = 1 - p_i$, where $p_i$ is the $p$-value of the autocovariance test computed using the $i$-th factor of both countries. It is then straightforward to compute the overall distance between this pair of countries.
Figure 8: $p$-values of the autocovariance test for each pair of countries that have one factor in the estimated factor model

Figure 9: Cluster dendrogram for countries that have one factor in the estimated factor model
as
\[ dist = \sum_{i=1}^{K} w_i \cdot \text{dist}_i, \]
where \( w_i \) is a weight on \( \text{dist}_i \). Practically, we suggest that the weight \( w_i \) is related to the magnitude of each singular value of the autocovariance matrix (or equivalently the squared root of the eigenvalues of symmetrized autocovariance matrix), since the singular values are related to the autocovariance explained by each factor. Based on this idea, we compute \( w_i \) as
\[ w_i = \left( w_i^{(1)} + w_i^{(2)} \right) / 2, \]
where \( w_i^{(1)} = \sqrt{\hat{\lambda}_{i,\tau}^{(1)}} / \left( \sum_{i=1}^{K} \sqrt{\hat{\lambda}_{i,\tau}^{(1)}} \right) \) and \( w_i^{(2)} = \sqrt{\hat{\lambda}_{i,\tau}^{(2)}} / \left( \sum_{i=1}^{K} \sqrt{\hat{\lambda}_{i,\tau}^{(2)}} \right) \). The result of hierarchical clustering using average linkage for all countries with three factors is presented in Figure 11.

![Figure 10: p-values of the autocovariance test for each pair of countries that have three factors in the estimated factor model](image)

However, as discussed in Remark 4.2, even the number of factors across some countries are different, it may still be of interest to perform the test for the first factor only. The idea behind is also very straightforward, that is we can study whether the mortality rates across all countries have the same low-dimensional representations in the eigenspace spanned by the first eigenvector shared by all countries. In this sense, we perform autocovariance tests and the hierarchical clustering analysis on the first factor for all countries regardless of the estimated total number of factors. The result of \( p \)-values computed for the first
Figure 11: Cluster dendrogram for countries that have three factors in the estimated factor model.

Factors are illustrated in Figure 12 and the result of hierarchical clustering using average linkage for the first factor is presented in Figure 13. As seen in Figure 12, in addition to what has been discussed for those countries with only one factor in their factor models, the first factor of Portugal and France are also different from the first factor of Bulgaria and Finland, respectively. Consequently, despite the differences between the estimated numbers of factors for Canada, Denmark, Italy, Poland, and all other countries, the total death rates projected in the eigenspace spanned by the first common eigenvector are not significantly different across these countries.
Figure 12: \( p \)-values of the autocovariance test of the first factor for all countries
Appendices

A Proofs of Lemma 2.1, Theorem 3.1 and Proposition 3.2

Proof of Lemma 2.1. (a) of the lemma can be found in [12], here we give a proof for (b). Since $X_0^\top X_0$ is symmetric and positive definite, we have

$$\|X_0^\top X_0\| \leq \text{tr}(X_0^\top X_0) = \frac{1}{T} \sum_{i=1}^K \sum_{t=1}^{T-\tau} x_{it}^2 \leq \sum_{i=1}^K \frac{1}{T} \sum_{t=1}^T x_{it}^2.$$  

By (a) of Lemma C.4 (whose proof does not depend the current lemma) we have

$$\mathbb{E} \left[ \frac{1}{T} \sum_{t=1}^T x_{it}^2 \right] = \sigma_i^2 + 1, \quad \text{Var} \left( \frac{1}{T} \sum_{t=1}^T x_{it}^2 \right) = O \left( \frac{\sigma_i^4}{T} \right).$$  

(A.1)
Taking a union bound we obtain
\[
P\left(\|X_0^\top X_0\| > 2 \sum_{i=1}^K \sigma_i^2 \right) \leq P\left(\sum_{i=1}^K \frac{1}{T} \sum_{t=1}^T x_{it}^2 > 2 \sum_{i=1}^K \sigma_i^2 \right) \leq \sum_{i=1}^K P\left(\frac{1}{T} \sum_{t=1}^T x_{it}^2 > 2 \sigma_i^2 \right) = \sum_{i=1}^K \frac{1}{T} \sum_{t=1}^T x_{it}^2 - (\sigma_i^2 + 1) > 2 \sigma_i^2 - 1 \right).
\]

Finally by Chebyshev’s inequality and (A.1) we have
\[
P\left(\|X_0^\top X_0\| > 2 \sum_{i=1}^K \sigma_i^2 \right) = O_p\left(\frac{K\sigma_1^4}{(\sigma_i^2 - 1)^2 T} \right) = O\left(\frac{K}{T} \right).
\]
and the proof is complete.

**Proof of Theorem 3.1.** We shall write \( \Lambda_n(A) \) for the \( n \)-th largest eigenvalue of a matrix \( A \). Note that the non-zero eigenvalues of \( \hat{\Sigma}_{\tau} \hat{\Sigma}_{\tau}^\top = Y_{\tau} Y_{\tau}^\top Y_0 Y_0^\top \) coincide with those of the matrix \( Y_0^\top Y_0 Y_\tau^\top Y_\tau = (X_0^\top X_0 + E_0^\top E_0)(X_\tau^\top X_\tau + E_\tau^\top E_\tau) \). We first show that the eigenvalue \( \Lambda_n(\hat{\Sigma}_{\tau} \hat{\Sigma}_{\tau}^\top) \) is close to \( \Lambda_n(X_0^\top X_0 X_\tau^\top X_\tau) \). By Weyl’s inequality (Lemma B.1 of [14]) we have
\[
\left| \Lambda_n(\hat{\Sigma}_{\tau} \hat{\Sigma}_{\tau}^\top) - \Lambda_n(X_0^\top X_0 X_\tau^\top X_\tau) \right| = \left| \Lambda_n(Y_\tau^\top Y_\tau Y_0^\top Y_0) - \Lambda_n(X_0^\top X_0 X_\tau^\top X_\tau) \right| \
\leq \|X_0^\top X_0 E_\tau^\top E_\tau + E_0^\top E_0 X_\tau^\top X_\tau + E_\tau^\top E_\tau E_\tau^\top E_\tau\| = O_p(K\sigma_1^2),
\]
where the last equality follows from (2.19). Dividing by \( \mu_{n,\tau} = \sigma_1^4 \gamma_n(\tau)^2 \) we have
\[
\frac{\Lambda_n(\hat{\Sigma}_{\tau} \hat{\Sigma}_{\tau}^\top) - \Lambda_n(X_0^\top X_0 X_\tau^\top X_\tau)}{\sigma_1^4 \gamma_n(\tau)^2} = O_p\left(\frac{K\sigma_1^2}{\sigma_1^4 \gamma_n(\tau)^2} \right). \tag{A.2}
\]
Next we compute \( \Lambda_n(X_\tau^\top X_0 X_0 X_\tau^\top) \) in more details. It is shown in Lemma C.4 that
\[
(X_0 X_\tau^\top)_{ij} = \mathbb{E}[(X_0 X_\tau^\top)_{ij}] + O_{L^2}(\sigma_i \sigma_j T^{-1/2}), \tag{A.3}
\]
where from equation (C.6) we know \( \mathbb{E}[(X_0 X_\tau^\top)_{ij}] = 1_{i=j} \sigma_i^2 \gamma_i(\tau) \). Therefore for any \( i \neq j \),
the off-diagonal elements of $X_\tau X_0^\top X_0 X_\tau^\top$ can be written into

$$(X_\tau X_0^\top X_0 X_\tau^\top)_{ij} = \sum_{k=1}^K (X_0 X_\tau^\top)_{ki} (X_0 X_\tau^\top)_{kj}$$

$$= (X_0 X_\tau^\top)_{ii} (X_0 X_\tau^\top)_{ij} + (X_0 X_\tau^\top)_{ji} (X_0 X_\tau^\top)_{jj} + \sum_{k \neq i,j} (X_0 X_\tau^\top)_{ki} (X_0 X_\tau^\top)_{kj}$$

$$= (\sigma_i^2 \gamma_i(\tau) + \sigma_j^2 \gamma_j(\tau)) O_{L^2}(\sigma_i \sigma_j T^{-1/2}) + O_{L^1}(\sigma_i^4 KT^{-1})$$

$$= O_{L^2}(\sigma_i^4 \gamma_1(\tau) T^{-1/2}) + O_{L^1}(\sigma_i^4 \gamma_1(\tau)) = O_{L^1} \left( \frac{\sigma_i^4 \gamma_1(\tau)}{\sqrt{T}} \right), \quad (A.4)$$

where the equality in the second last line follows from (A.3) and the last line follows from Assumptions 2.2 and 2.3.

Similarly, the diagonal elements of $X_\tau X_0^\top X_0 X_\tau^\top$ satisfy

$$(X_\tau X_0^\top X_0 X_\tau^\top)_{ii} = (X_0 X_\tau^\top)_{ii}^2 + \sum_{k \neq i} (X_0 X_\tau^\top)_{ki}^2$$

$$= (\sigma_i^2 \gamma_i(\tau) + O_{L^2}(\sigma_i^2 T^{-1/2}))^2 + O_{L^1}(KT^{-1}) = \mu_{i,\tau} + O_{L^1} \left( \frac{\sigma_i^4 \gamma_1(\tau)}{\sqrt{T}} \right). \quad (A.5)$$

Using (A.4), (A.5) and taking a union bound over $i,j$ we finally obtain

$$\|X_\tau X_0^\top X_0 X_\tau^\top - \text{diag}(\mu_{i,\tau})\|_\infty = O_p \left( \frac{K^2 \sigma_i^4 \gamma_1(\tau)}{\sqrt{T}} \right) \quad (A.6)$$

or equivalently we may write

$$X_\tau X_0^\top X_0 X_\tau^\top \text{diag}(\mu_{i,\tau}^{-1}) = I_K + O_{p,\| \cdot \|_\infty} \left( \frac{K^2}{\sqrt{T} \gamma_1(\tau)} \right). \quad (A.7)$$

Let $\omega_1, \ldots, \omega_K$ be the eigenvalues of $X_\tau X_0^\top X_0 X_\tau^\top$ arranged in decreasing order. Let $\omega$ be one of these eigenvalues. Define the function

$$G(\omega) := (X_\tau X_0^\top X_0 X_\tau^\top - \omega I_K) \text{diag}(\mu_{i,\tau}^{-1}),$$

then clearly we have $0 = \|X_\tau X_0^\top X_0 X_\tau^\top - \omega I_K\| = |G(\omega)|$. From (A.7) we get

$$0 = |G(\omega)| = \left| I_K + O_{p,\| \cdot \|_\infty} \left( \frac{K^2}{\sqrt{T} \gamma_1(\tau)} \right) - \omega \text{diag}(\mu_{i,\tau}^{-1}) \right|

= \left| I_K - \text{diag}(\omega \mu_{i,\tau}^{-1}) + O_{p,\| \cdot \|_\infty} \left( \frac{K^2}{\sqrt{T} \gamma_1(\tau)} \right) \right|,$$
and using Leibniz’s formula analogous to the derivation of (B.42) we obtain

\[ 0 = |G(\omega)| = \prod_{i=1}^{K} G(\omega)_{ii} + O_p \left( \frac{K^6}{\gamma_1(\tau)^2T} \right). \quad (A.8) \]

Since \( \prod_{i} G(\omega)_{ii} = o_p(1) \), there is at least one \( i \in \{1, \ldots, K\} \) such that \( G(\omega)_{ii} = o_p(1) \).

Now we show that in fact there can be only one such \( i \). For any \( i \neq j \), we have

\[ G(\omega)_{ii} - G(\omega)_{jj} = \omega(\mu_i^{-1} - \mu_j^{-1}) \geq \omega \mu_i^{-1}(1 + \epsilon) \quad (A.9) \]

for some \( \epsilon > 0 \), where the last inequality follows from either Assumption 2.2 or 2.3. We first observe that \( \omega \mu_i^{-1} \neq o_p(1) \) for any \( i \) as \( T \to \infty \). Indeed, suppose for a contradiction that \( \omega \mu_i^{-1} = o_p(1) \), since \( \mu_i \asymp \mu_j \) for any \( i = j \), we easily see that \( G(\omega)_{ii} = 1 + o_p(1) \) for every \( i \), which makes (A.8) impossible. Substituting back into (A.9) we see that \( G(\omega)_{ii} - G(\omega)_{jj} \gtrsim 1 + \epsilon \) for any \( i \neq j \). Clearly, if \( G(\omega)_{ii} = o_p(1) \), then \( G(\omega)_{jj} \gtrsim 1 + \epsilon + o_p(1) \) for any \( j \neq i \), i.e. there can be only one \( i \) such that \( G(\omega)_{ii} = o_p(1) \).

Therefore, for (A.8) to hold, there must exist some \( i \in \{1, \ldots, K\} \) such that

\[ 0 = G(\omega)_{ii} + O_p \left( \frac{K^6}{\gamma_1(\tau)^2T} \right), \]

which gives one solution to \( |G(\omega)| = 0 \). On the other hand, note that \( |G(w)| = 0 \) has \( K \) solutions in total. By the above arguments, it should be clear that each solution then corresponds to a particular \( i \in \{1, \ldots, K\} \), i.e. the \( K \) solutions to \( |G(\omega)| = 0 \) satisfy the system of equations

\[ 0 = G(\omega)_{ii} + O_p \left( \frac{K^6}{\gamma_1(\tau)^2T} \right), \quad i = 1, \ldots, K. \]

Using (A.5), we see that each \( G(\omega)_{ii} \) satisfies

\[ G(\omega)_{ii} = \frac{(X_\tau X_0^\top X_0 X_\tau^\top)_{ii} - \omega}{\mu_{i,\tau}} = 1 - \frac{\omega}{\mu_{i,\tau}} + O_p \left( \frac{1}{\sqrt{T}\gamma_1(\tau)} \right), \]

which implies that the \( K \) solutions to \( |G(\omega)| = 0 \) satisfy the system of equations

\[ \frac{\omega}{\mu_{i,\tau}} - 1 = O_p \left( \frac{1}{\gamma_1(\tau)\sqrt{T}} \right), \quad i = 1, \ldots, K. \quad (A.10) \]
Note that by definition there are $K$ possible choices of $\omega$, which are the order eigenvalues of $X_\tau X_0^\top X_0 X_\tau^\top$. Since $\{\mu_{i,\tau}\}$ are ordered under Assumption 2.2 or asymptotically ordered under Assumption 2.3, we can easily conclude that

$$\frac{\Lambda_i(X_\tau X_0^\top X_0 X_\tau^\top)}{\mu_{i,\tau}} - 1 = O_p\left(\frac{1}{\gamma_1(\tau)\sqrt{T}}\right).$$

Combining this result with (A.2) we get

$$\frac{\Lambda_i(\hat{\Sigma}_\tau \hat{\Sigma}_\tau^\top)}{\mu_{i,\tau}} - 1 = O_p\left(\frac{1}{\sqrt{T}\gamma_1(\tau)} + \frac{K}{\sigma_i^2 \gamma_i(\tau)^2}\right)$$

which completes the proof. \hfill \Box

**proof of Proposition 3.2.** We first consider the invertibility of the matrix $Q(a)$ defined in (2.16). Recall the matrix $R(a)$ from (2.11) and the event $B_2$ from (2.18). Since $a \sim \sigma_n^4 \gamma_n(\tau)^2 \to \infty$ and $\|E_\tau^\top E_\tau E_0^\top E_0 1_{B_2}\|$ is bounded by definition of $B_2$, the matrix $I - a^{-1}E_\tau^\top E_\tau E_0^\top E_0$ is invertible under $B_2$ and we have $\|R(a)\|1_{B_2} = O(1)$. Therefore we have $\|a^{-1}X_0 R_a E_\tau^\top E_\tau X_0^\top\|1_{B_2} = O(\sigma_n^{-2} \gamma_n(\tau)^{-2}) = o(1)$ and thus $Q(a)$ is invertible under $B_2$ for $T$ large enough.

It will be shown in Lemma C.5 that

$$\mathbb{E}[A(a)_{nn} 1_{B_0}] = \frac{\sigma_n^2 \gamma_n(\tau)}{\sqrt{a}} + o(1), \quad \mathbb{E}[B(a)_{nn} 1_{B_0}] = o(1), \quad \mathbb{E}[Q(a)^{-1}_{nn} 1_{B_2}] = 1 + o(1).$$

From (3.3) we have $a^{-1/2} \sigma_n^2 \gamma_n(\tau) \sim O(1)$, using which we can obtain

$$g(a) = 1 - \mathbb{E}[A(a) 1_{B_0}]^2_{nn} \mathbb{E}[Q(a)^{-1}_{nn} 1_{B_2}] + o(1) = 1 - \frac{\sigma_n^4 \gamma_n(\tau)^2}{a} + o(1).$$

Substituting the endpoints of the interval (3.3) into the function $g$, we have

$$g((1 \pm \epsilon)\sigma_n^4 \gamma_n(\tau)^2) = 1 - \frac{1}{1 \pm \epsilon} + o(1) = \frac{-\epsilon}{1 \pm \epsilon} + o(1).$$

For $T$ large enough, the signs of $g$ differ at the two endpoints of the interval (3.3) and therefore $g$ has a root inside the interval. It is not difficult to observe that $g$ is a monotone function in $a$ for $T$ large enough which implies the root is unique. \hfill \Box
B Proof of Theorem 3.3

We begin with the statements and proofs of the four propositions described in Section 3 of
the paper. The proof of our main result Theorem 3.3 is given at the end of this appendix.

We first give an expression for \( \delta := \delta_{n, \tau} \). Recall the matrix \( M \) from (3.5)
\[
M := I_K - \frac{1}{\theta} X_\tau E_0^\top E_0 R X_\tau^\top - \frac{1}{\theta} X_\tau R^\top X_0^\top Q^{-1} X_0 R X_\tau^\top.
\]

Proposition B.1. Suppose Assumption 2.1 and either Assumption 2.2 or 2.3 hold. Then
the ratio \( \delta \) is the solution to the following equation
\[
\det \left( M + \frac{\delta}{\theta} X_\tau X_0^\top X_\tau^\top + \delta o_\rho,\|1\right) = 0. \tag{B.1}
\]

Proof. Suppose \( \lambda \) is an eigenvalue of \( \hat{\Sigma}_\tau \hat{\Sigma}_\tau^\top \), then \( \sqrt{\lambda} \) is a singular value of the matrix \( \hat{\Sigma}_\tau \),
or equivalently an eigenvalue of the \((2p + 2K) \times (2p + 2K)\) matrix
\[
\begin{pmatrix}
0 & \hat{\Sigma}_\tau \\
\hat{\Sigma}_\tau^\top & 0
\end{pmatrix} =
\begin{pmatrix}
0 & X_\tau X_0^\top & X_\tau E_0^\top \\
0 & E_\tau X_0^\top & E_\tau E_0^\top \\
X_0 X_\tau^\top & X_0 E_\tau^\top & 0 & 0 \\
E_0 X_\tau^\top & E_0 E_\tau^\top & 0 & 0
\end{pmatrix}.
\]

By definition the eigenvalue \( \lambda \) satisfies
\[
0 = \begin{vmatrix}
\sqrt{\lambda} I_K & 0 \\
0 & \sqrt{\lambda} I_K
\end{vmatrix} - \begin{vmatrix}
0 & \hat{\Sigma}_\tau \\
\hat{\Sigma}_\tau^\top & 0
\end{vmatrix} =
\begin{vmatrix}
\sqrt{\lambda} I_K & 0 & -X_\tau X_0^\top & -X_\tau E_0^\top \\
0 & \sqrt{\lambda} I_p & -E_\tau X_0^\top & -E_\tau E_0^\top \\
-X_\tau X_\tau^\top & -X_\tau E_\tau^\top & \sqrt{\lambda} I_K & 0 \\
-E_\tau X_\tau^\top & -E_\tau E_\tau^\top & 0 & \sqrt{\lambda} I_p
\end{vmatrix},
\]
which, after interchanging the columns and rows, becomes
\[
0 = \begin{vmatrix}
\sqrt{\lambda} I_K & -X_\tau X_0^\top & 0 & -X_\tau E_0^\top \\
-X_\tau X_\tau^\top & \sqrt{\lambda} I_K & -X_\tau E_\tau^\top & 0 \\
0 & -E_\tau X_0^\top & \sqrt{\lambda} I_p & -E_\tau E_0^\top \\
-E_\tau X_\tau^\top & 0 & -E_\tau E_\tau^\top & \sqrt{\lambda} I_p
\end{vmatrix}. \tag{B.2}
\]
From Theorem 3.1 we know that the spiked eigenvalue $\lambda \to \infty$ as $T \to \infty$. From Lemma 2.1 we recall that the spectral norm of $E_T E_0^\top$ is bounded with probability tending to 1 as $T \to \infty$. Therefore the bottom right sub-matrix $\left( \frac{\sqrt{\lambda I_p}}{\sqrt{\lambda I_p}} - E_T E_0^\top \right)$ is invertible with probability tending to 1. Using the matrix identity

$$
\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}
$$

we can compute the inverse of the submatrix $\left( \frac{\sqrt{\lambda I_p}}{\sqrt{\lambda I_p}} - E_T E_0^\top \right)$ and get

$$
\begin{align*}
\left( \frac{\sqrt{\lambda I_p}}{\sqrt{\lambda I_p}} - E_T E_0^\top \right)^{-1} &= \begin{pmatrix} \frac{1}{\sqrt{\lambda}} E_0^\top \left( \sqrt{\lambda} I_p - \frac{1}{\sqrt{\lambda}} E_T E_0^\top E_0 E_T^\top \right)^{-1} \frac{1}{\sqrt{\lambda}} E_T E_0^\top \left( \sqrt{\lambda} I_p - \frac{1}{\sqrt{\lambda}} E_0 E_T^\top E_T E_0^\top \right)^{-1} \\ \frac{1}{\sqrt{\lambda}} E_0^\top \left( \sqrt{\lambda} I_p - \frac{1}{\sqrt{\lambda}} E_T E_0^\top E_0 E_T^\top \right)^{-1} \frac{1}{\sqrt{\lambda}} E_T E_0^\top \left( \sqrt{\lambda} I_p - \frac{1}{\sqrt{\lambda}} E_0 E_T^\top E_T E_0^\top \right)^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\lambda} \left( \lambda I_p - E_T E_0^\top E_0 E_T^\top \right)^{-1} E_T E_0^\top \left( \lambda I_p - E_0 E_T^\top E_T E_0^\top \right)^{-1} & \sqrt{\lambda} \left( \lambda I_p - E_T E_0^\top E_0 E_T^\top \right)^{-1} E_T E_0^\top \left( \lambda I_p - E_0 E_T^\top E_T E_0^\top \right)^{-1} \\
E_0 E_T^\top \left( \lambda I_p - E_T E_0^\top E_0 E_T^\top \right)^{-1} & E_0 E_T^\top \left( \lambda I_p - E_T E_0^\top E_0 E_T^\top \right)^{-1} \end{pmatrix}.
\end{align*}
$$

Observe that

$$
\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & \beta^\top \\ \alpha^\top & 0 \end{pmatrix} = \begin{pmatrix} \alpha D \alpha^\top & \alpha C \beta^\top \\ \beta B \alpha^\top & \beta A \beta^\top \end{pmatrix}.
$$

Substituting the above computations back into (B.2) we have

$$
0 = \begin{bmatrix} \sqrt{\lambda I_K} & -X_T X_0^\top & 0 & -E_T E_0^\top & \sqrt{\lambda I_K} \\ -X_0 X_T^\top & \sqrt{\lambda I_K} & X_0 E_T^\top E_0^\top \left( \lambda I_p - E_0 E_T^\top E_T E_0^\top \right)^{-1} E_0 X_0^\top & -E_0 E_T^\top \left( \lambda I_p - E_T E_0^\top E_T E_0^\top \right)^{-1} E_0 X_0^\top \\ -X_0 X_T^\top & \sqrt{\lambda I_K} & X_0 E_T^\top E_0^\top \left( \lambda I_p - E_0 E_T^\top E_T E_0^\top \right)^{-1} E_0 X_0^\top & X_0 E_T^\top E_0^\top \left( \lambda I_p - E_T E_0^\top E_T E_0^\top \right)^{-1} E_0 X_0^\top \\ -X_0 X_T^\top & \sqrt{\lambda I_K} & X_0 E_T^\top E_0^\top \left( \lambda I_p - E_0 E_T^\top E_T E_0^\top \right)^{-1} E_0 X_0^\top & X_0 E_T^\top E_0^\top \left( \lambda I_p - E_T E_0^\top E_T E_0^\top \right)^{-1} E_0 X_0^\top \end{bmatrix}
$$

where the last equality holds by (2.13). Recalling the notations we introduced in Section 2 and identity (2.12), we obtain

$$
0 = \begin{bmatrix} \sqrt{\lambda} Q_\lambda & -X_T R_\lambda^\top X_0^\top \\ -X_0 R_\lambda X^\top_T & \sqrt{\lambda} Q_\lambda \end{bmatrix} = \begin{bmatrix} Q_\lambda - \lambda^{-1} X_T R_\lambda^\top X_0^\top Q_\lambda^{-1} X_0 R_\lambda X^\top_T \end{bmatrix}.
$$

(B.3)
Next, we center $\lambda$ around the quantity $\theta$ defined in (3.2). Since $\lambda$ and $\theta$ diverge, they are outside of the spectrum of $E_r^T E_r E_0^T E_0$ with probability tending to 1. Then

$$
\frac{1}{\lambda} R_\lambda - \frac{1}{\theta} R = (\lambda I_{T-r} - E_r^T E_r E_0^T E_0)^{-1} - (\theta I_{T-r} - E_r^T E_r E_0^T E_0)^{-1}
$$

$$
= (\theta - \lambda)(\lambda I_{T-r} - E_r^T E_r E_0^T E_0)^{-1}(\theta I_{T-r} - E_r^T E_r E_0^T E_0)^{-1} = -\frac{\delta}{\lambda} R_\lambda R.
$$

Substituting back into itself, we obtain

$$
\frac{1}{\lambda} R_\lambda = \frac{1}{\theta} R - \frac{\delta}{\theta} R^2 + \delta o_p(|\lambda|^{-1}).
$$

Using the bounds in (2.19) and (2.20) we have

$$
R - I_{T-r} = \frac{1}{\theta} E_r^T E_r E_0^T E_0 R = O_p(\|r\|^{-1}), \quad R^2 = I_{T-r} + O_p(\|r\|^{-1}),
$$

where the second equation follows from expanding $(R - I)^2$. By Theorem 3.1 we have $\delta = o_p(1)$. Substituting back into (B.4) we get

$$
\frac{1}{\lambda} R_\lambda = \frac{1}{\theta} R - \frac{\delta}{\theta} R^2 + \delta o_p(|\lambda|^{-1}) = \frac{1}{\theta} R - \frac{\delta}{\theta} I_{T-r} + \delta o_p(|\lambda|^{-1}).
$$

Using this we can get

$$
\overline{Q} - Q = (I_K - X_r E_0^T E_0 \lambda^{-1} R_\lambda X_r^T) - (I_K - X_r E_0^T E_0 \theta^{-1} R X_r^T)
$$

$$
= X_r E_0^T E_0 (\theta^{-1} R - \lambda^{-1} R_\lambda) X_r = X_r E_0^T E_0 \left[ \frac{\delta}{\theta} I_{T-r} + \delta o_p(|\lambda|^{-1}) \right] X_r.
$$

From (2.19) we recall that $\|X_r\|^2 = O_p(K \sigma_r^2)$. Using $\delta = o_p(1)$ again we get

$$
\overline{Q} = Q + \frac{\delta}{\theta} X_r E_0^T E_0 X_r + \delta o_p(|\lambda|^{-1}) = \overline{Q} + \delta o_p(|\lambda|^{-1}),
$$

and similarly $Q = Q + \delta o_p(|\lambda|^{-1})$. Finally, since $\|Q^{-1}\| = O_p(1)$, we have

$$
Q^{-1}_\lambda - Q^{-1} = Q^{-1}(Q - Q_\lambda)Q^{-1} = o_p(|\lambda|^{-1}).
$$

Next we consider the matrix $X_0 R_\lambda X_r^T$ appearing in (B.3). From (B.4) we have

$$
\sqrt{\frac{\theta}{\lambda}} X_0 R_\lambda X_r^T = \frac{1}{\sqrt{\theta}} X_0 R X_r^T - \frac{\delta}{\sqrt{\theta}} X_0 R^2 X_r^T + \frac{\delta^2}{\lambda} X_0 R_\lambda R^2 X_r^T.
$$
For the second term on the right hand side of (B.8), using (B.5) and (2.19) we have
\[
\frac{\delta}{\sqrt{\theta}} X_0 R^2 X_r^T = \frac{\delta}{\sqrt{\theta}} X_0 X_r^T + \frac{\delta}{\sqrt{\theta}} X_0(R^2 - I)X_r^T \\
= \frac{\delta}{\sqrt{\theta}} X_0 X_r^T + \delta O_p,\| \| \left( \frac{K\sigma_1^2}{\theta^{3/2}} \right) = \frac{\delta}{\sqrt{\theta}} X_0 X_r^T + \delta O_p,\| \| (1).
\]
Similarly the last term in (B.8) satisfies \( \frac{\delta^2\sqrt{\theta}}{\lambda} X_0 R \lambda^2 R^2 X_r^T = \delta O_p,\| \| (1) \). Therefore
\[
\frac{\sqrt{\theta}}{\lambda} X_0 R_\lambda X_r^T = \frac{1}{\sqrt{\theta}} X_0 R X_r^T - \frac{\delta}{\sqrt{\theta}} X_0 X_r^T + \delta O_p,\| \| (1). \tag{B.9}
\]
To deal with the second term appearing in the determinant in (B.3), we first make the following computations. Using (B.8)-(B.9) as well as (B.7) we have
\[
\frac{\theta}{\lambda^2} X_r R_\lambda X_o^T Q_\lambda^{-1} X_0 R_\lambda X_r^T = \left( \frac{1}{\sqrt{\theta}} X_r R^T X_o^T - \frac{\delta}{\sqrt{\theta}} X_r X_o^T + \delta O_p,\| \| (1) \right) \\
\times \left( Q^{-1} + \delta O_p,\| \| (1) \right) \left( \frac{1}{\sqrt{\theta}} X_0 R X_r^T - \frac{\delta}{\sqrt{\theta}} X_0 X_r^T + \delta O_p,\| \| (1) \right) \\
= \left( \frac{1}{\sqrt{\theta}} X_r R^T X_o^T - \frac{\delta}{\sqrt{\theta}} X_r X_o^T \right) Q^{-1} \left( \frac{1}{\sqrt{\theta}} X_0 R X_r^T - \frac{\delta}{\sqrt{\theta}} X_0 X_r^T \right) + \delta O_p,\| \| (1) \tag{B.9}.
\]
Expanding the expression above and using (B.8)-(B.9) again we obtain
\[
\frac{\theta}{\lambda^2} X_r R_\lambda X_o^T Q_\lambda^{-1} X_0 R_\lambda X_r^T = \frac{1}{\theta} X_r R^T X_o^T Q^{-1} X_o R X_r^T \\
- \frac{\delta}{\theta} \left( X_r R^T X_o^T Q^{-1} X_o X_r^T + X_r X_o^T Q^{-1} X_o R X_r^T \right) + \delta O_p,\| \| (1) \tag{B.9}.
\]
Finally, recalling \( \lambda/\theta = 1 + \delta \), we can conclude
\[
\frac{1}{\lambda} X_r R_\lambda X_o^T Q_\lambda^{-1} X_0 R_\lambda X_r^T = (1 + \delta) \frac{\theta}{\lambda^2} X_r R^T X_o^T Q^{-1} X_0 R_\lambda X_r^T \\
= \frac{1}{\theta} X_r R^T X_o^T Q^{-1} X_0 R X_r^T - \frac{2\delta}{\theta} X_r X_o^T Q^{-1} X_0 X_r^T \\
+ \delta \left( \frac{1}{\theta} X_r R^T X_o^T Q^{-1} X_0 R X_r^T - \frac{2\delta}{\theta} X_r X_o^T Q^{-1} X_0 X_r^T \right) + \delta O_p,\| \| (1) \tag{B.9}.
\]
where in the last line we have used (B.5)-(B.9) again. To conclude, we have shown
\[
\overline{Q}_\lambda = I_K - \frac{1}{\theta} X_r E_n^T E_o R X_r^T + \delta O_p,\| \| (1),
\]
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for the first term in the right hand side of (B.3) and 
\[
\frac{1}{\lambda} X_t R_\tau^T X_0^T Q_\lambda^{-1} X_0 R_\lambda X_\tau^T = \frac{1}{\theta} X_t R_\tau^T X_0^T Q^{-1} X_0 R X_\tau^T - \frac{\delta}{\theta} X_t X_\tau^T X_0 X_\tau^T + \delta o_p(1),
\]
for the second term. The claim then follows. \(\square\)

We now work towards establishing the the asymptotic distribution of the matrix \(M\) from (3.5) with the help of Lemma C.4-D.3. For notational convenience we define
\[
A := \frac{1}{\sqrt{\theta}} X_0 R X_\tau^T, \quad B := \frac{1}{\theta} X_t E_0^T E_0 R X_\tau^T,
\]
so that \(M = I_K - B - A^T Q^{-1} A\). For each \(i = 1, \ldots, K\), define
\[
\overline{M}_{ii} := 1 - E[B_{ii} 1_{B_0}] - E[A_{ii} 1_{B_0}]^2 E[Q_{ii}^{-1} 1_{B_2}]
\]
which serves as a deterministic centering for the \(i\)-th diagonal entry of \(M\). We first give an approximation for \(M_{ii} - \overline{M}_{ii}\) up to the scaling of \(T^{-1/2}\).

**Proposition B.2.** Under Assumption 2.1 and either Assumption 2.2 or 2.3, we have
\[
M_{ii} - \overline{M}_{ii} = -2(A_{ii} - E[A_{ii}]) E[A_{ii} 1_{B_0}] E[Q_{ii}^{-1} 1_{B_2}] + o_p \left( \frac{1}{\sqrt{T}} \right),
\]
for all \(i = 1, \ldots, K\), where \(E[ \cdot ]\) is defined in (2.23). Furthermore,
\[
\max_{i \neq j} |M_{ij}| = O_p \left( \frac{K^3}{\gamma_1(\tau)^2 \sqrt{T}} \right).
\]

**Proof.** We first recall from Lemma C.5 and Assumption 2.1 that
\[
E[A_{ij} 1_{B_0}] = 1_{i=j} \left( \frac{\sigma^2_i \gamma_i(\tau)}{\theta^{1/2}} + o(1) \right), \quad \text{Var}(A_{ij} 1_{B_0}) = O \left( \frac{\sigma^2_i \sigma^2_j}{\theta T} \right),
\]
We also recall from Lemma D.2 that
\[
Q_{kk}^{-1} 1_{B_2} = E[Q_{kk}^{-1} 1_{B_2}] + o_{L^1}(T^{-1/2}), \quad Q_{ij}^{-1} 1_{B_2} = o_{L^2}(T^{-1}).
\]
Recall that \(M = I_K - B - A^T Q^{-1} A\). We first consider the \(i\)-th diagonal of \(A^T Q^{-1} A\) and show that it is close to \(A_{ii}^2 E[Q_{ii}^{-1} 1_{B_2}]\) under the event \(B_2\). Note that we can write
\[
(A^T Q^{-1} A)_{ii} = \sum_{m,n} A_{mi} A_{mn} Q_{mn}^{-1} = \sum_{m,n \neq i} A_{mi} A_{mn} Q_{mn}^{-1} + A_{ii} \left( \sum_{n \neq i} A_{ni} Q_{ii}^{-1} + \sum_{m \neq i} A_{mi} Q_{mi}^{-1} \right).
\]

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We will consider each term in (B.15) separately. Recall from (2.21) that \( \| Q^{-1}1_{B_2} \| = 1 + o(1) \) which implies \( Q_{ij}^{-1}1_{B_2} = 1 + o(1) \) for all \( i, j \leq K \). Using the triangle inequality followed by the Cauchy Schwarz inequality we have

\[
\mathbb{E} \left| \sum_{m,n \neq i} A_{mi} A_{ni} Q_{mn}^{-1} 1_{B_2} \right| \lesssim \sum_{m,n \neq i} \mathbb{E}[A_{mi}^2 1_{B_2}]^{1/2} \mathbb{E}[A_{ni}^2 1_{B_2}]^{1/2} \\
\lesssim \sum_{m,n \neq i} \mathbb{E}[A_{mi}^2 1_{B_0}]^{1/2} \mathbb{E}[A_{ni}^2 1_{B_0}]^{1/2} = O \left( \frac{K^2 \sigma_i^2 \sigma_m \sigma_n}{\theta T} \right),
\]

where the second inequality follows since \( 1_{B_2} \leq 1_{B_0} \) by definition and the last equality follows from (B.13).

By Proposition 3.2 we have \( \sigma_i^2 \sigma_m \sigma_n / \theta = O(\gamma_1(\tau)^{-2}) \). From either (b) of Assumption 2.2 or (b) 2.3 we know that \( K^2 \gamma_1^{-2}(\tau) = o(T^{-1/2}) \), therefore

\[
\sum_{m,n \neq i} A_{mi} A_{ni} Q_{mn}^{-1} 1_{B_2} = o_L \left( \frac{1}{\sqrt{T}} \right). \tag{B.16}
\]

For the last term in (B.15), we note that \( 1_{B_2} = 1_{B_2} 1_{B_0} \) by definition. Using \( \| Q^{-1}1_{B_2} \| = 1 + o(1) \) and the triangle inequality we have

\[
\mathbb{E} \left| \sum_{n \neq i} A_{ii} A_{ni} Q_{in}^{-1} 1_{B_2} \right| \lesssim \sum_{n \neq i} \mathbb{E} \left| (A_{ii} 1_{B_0} - \mathbb{E}[A_{ii} 1_{B_0}]) A_{ni} 1_{B_2} \right| \\
+ \mathbb{E} |A_{ii} 1_{B_0}| \sum_{n \neq i} \mathbb{E} |A_{ni} Q_{in}^{-1} 1_{B_2}|.
\]

By the Cauchy Schwarz inequality and (B.13) we have

\[
\mathbb{E} \left| A_{ii} \sum_{n \neq i} A_{ni} Q_{in}^{-1} 1_{B_2} \right| \lesssim \sum_{n \neq i} \text{Var}(A_{ii} 1_{B_0})^{1/2} \mathbb{E}[A_{ni}^2 1_{B_0}]^{1/2} \\
+ \mathbb{E}[A_{ii}^2 1_{B_0}]^{1/2} \left( \sum_{n \neq i} \mathbb{E}[A_{ni}^2 1_{B_0}]^{1/2} \mathbb{E}[(Q^{-1})_{in}^2 1_{B_2}]^{1/2} \right) = o \left( \frac{1}{\sqrt{T}} \right).
\]

Combining with (B.16), substituting back into (B.15) and applying (B.14) we obtain

\[
(A^\top Q^{-1} A)_{ii} 1_{B_2} = A_{ii}^2 1_{B_2} \mathbb{E}[Q_{ii}^{-1} 1_{B_2}] + o_L \left( \frac{1}{\sqrt{T}} \right).
\]

From Lemma 2.1 we know that \( 1_{B_2} = 1 + o(1) \), therefore

\[
(A^\top Q^{-1} A)_{ii} = A_{ii}^2 \mathbb{E}[Q_{ii}^{-1} 1_{B_2}] + o_p \left( \frac{1}{\sqrt{T}} \right). \tag{B.17}
\]
Next, we expand $A_{ii}^2$ around the conditional mean $\mathbb{E}[A_{ii}1_{B_0}]^2$. Note that

$$A_{ii}^21_{B_0} = \mathbb{E}[A_{ii}1_{B_0}]^2 + 2\mathbb{E}[A_{ii}1_{B_0}](A_{ii}1_{B_0} - \mathbb{E}[A_{ii}1_{B_0}]) + (A_{ii}1_{B_0} - \mathbb{E}[A_{ii}1_{B_0}])^2. \quad (B.18)$$

Using (c) of Lemma C.4 and either (b) of Assumption 2.2 or (b) of Assumption 2.3, the last term satisfies

$$(A_{ii}1_{B_0} - \mathbb{E}[A_{ii}1_{B_0}])^2 = O_{L^1}(\sigma_i^4) = o_{L^1}(1)\sqrt{T}.$$ 

Note that by definition of $\mathbb{E}$ and $B_0$, we have

$$A_{ii}1_{B_0} - \mathbb{E}[A_{ii}1_{B_0}] = (A_{ii} - \mathbb{E}[A_{ii}])1_{B_0} = A_{ii} - \mathbb{E}[A_{ii}] + o_p\left(\frac{1}{\sqrt{T}}\right),$$

where the last equality follows from Lemma 2.1. Therefore from (B.18) we may obtain

$$A_{ii}^2 = \mathbb{E}[A_{ii}1_{B_0}]^2 + 2\mathbb{E}[A_{ii}1_{B_0}](A_{ii} - \mathbb{E}[A_{ii}]) + o_p\left(\frac{1}{\sqrt{T}}\right).$$

Substituting back into (B.17) we have

$$(A^\top Q^{-1}A)_{ii} = \mathbb{E}[A_{ii}1_{B_0}]^2\mathbb{E}[Q_{ii}^{-1}1_{B_2}] + 2\mathbb{E}[A_{ii}1_{B_0}]\mathbb{E}[Q_{ii}^{-1}1_{B_2}](A_{ii} - \mathbb{E}[A_{ii}]) + o_p\left(\frac{1}{\sqrt{T}}\right)$$

$$= \mathbb{E}[A_{ii}1_{B_0}]^2\mathbb{E}[Q_{ii}^{-1}1_{B_2}] + 2\mathbb{E}[A_{ii}1_{B_0}]\mathbb{E}[Q_{ii}^{-1}1_{B_2}](A_{ii} - \mathbb{E}[A_{ii}]) + o_p\left(\frac{1}{\sqrt{T}}\right),$$

where in the last equality, Lemma D.3 is used to replace the conditional expectations with the unconditional ones (except for the centering of $A_{ii}$ where the conditional expectation is intentionally kept).

Finally, we recall $M_{ii} = 1 - B_{ii} - (A^\top Q^{-1}A)_{ii}$, so it remains to consider the matrix $B = \frac{1}{\theta^2}X_rE_0^\top E_0RX_r^\top$ in the same manner as above. By Lemma C.4, we have

$$\mathbb{E}|B_{ij}1_{B_0} - 1_{i=j}\mathbb{E}[B_{ii}1_{B_0}]|^2 \lesssim \frac{1}{\theta^2 T^2}O(\sigma_i^2 \sigma_j^2 T) = o\left(\frac{1}{\sqrt{T}}\right). \quad (B.19)$$

Using Lemma D.3 to replace $\mathbb{E}[B_{ii}1_{B_0}]$ with $\mathbb{E}[B_{ii}1_{B_0}]$ and $1_{B_0} = 1 - o_p(1)$, we get

$$B_{ij} = 1_{i=j}\mathbb{E}[B_{ii}1_{B_0}] + o_p\left(\frac{1}{\sqrt{T}}\right).$$
Combining the above computations, we get

\[ M_{ii} = M_{ii} - 2\mathbb{E}[A_{ii}1_{B_0}]\mathbb{E}[Q_{ii}^{-1}1_{B_2}](A_{ii} - \mathbb{E}[A_{ii}]) + o_p\left(\frac{1}{\sqrt{T}}\right) \]

and the first claim follows.

For the off-diagonal elements, write

\[
(A^T Q^{-1} A)_{ij} = \sum_{m,n} A_{mi} A_{nj} Q_{mn}^{-1} = A_{ii} A_{jj} Q_{ij}^{-1} + A_{ij} A_{ji} Q_{ij}^{-1} + A_{ij} A_{ji} Q_{jj}^{-1} + \sum_{m \neq i,n \neq j, m \neq n} A_{mi} A_{nj} Q_{mn}^{-1} + A_{ii} \sum_{n \neq i,j} A_{nj} Q_{in}^{-1} + A_{jj} \sum_{m \neq i,j} A_{mi} Q_{mj}^{-1}.
\]

Observe that by definition of \( A_{ii}, Q_{ii} \) and the event \( B_2 \) we have

\[
A_{ii} 1_{B_2} = O\left(\frac{K \sigma_1^2}{\sqrt{T}}\right) = O\left(\frac{K}{\gamma_1(\tau)}\right), \quad Q_{ij}^{-1} 1_{B_2} = O(1).
\]

Recall from (B.13), Lemma D.2 and Lemma C.5 that

\[
A_{ij} 1_{B_2} = O_{L^2}\left(\frac{K}{\gamma_1(\tau)\sqrt{T}}\right), \quad Q_{ij}^{-1} 1_{B_2} = O_{L^2}\left(\frac{1}{\gamma_1(\tau)^2 \sigma_1^2 \sqrt{T}}\right), \quad \forall i \neq j.
\]

Substituting (B.21) and (B.22) back into the terms in (B.20) we have

\[
A_{ii} A_{jj} Q_{ij}^{-1} 1_{B_2} = O_{L^1}\left(\frac{K^4}{\gamma_1(\tau)^4 \sigma_1^4 T}\right), \quad A_{ii} A_{ij} Q_{ii}^{-1} 1_{B_2} = O_{L^1}\left(\frac{K^2}{\gamma_1(\tau)^2 \sqrt{T}}\right).
\]

With similar computation, the rest of (B.20) satisfy

\[
\sum_{m \neq i,n \neq j, m \neq n} A_{mi} A_{nj} Q_{mn}^{-1} = O_{L^1}\left(\frac{K^4}{\gamma_1(\tau)^2 T}\right), \quad A_{ii} \sum_{n \neq i,j} A_{nj} Q_{in}^{-1} = O_{L^1}\left(\frac{K^2}{\gamma_1(\tau)^2 \sigma_1^2 T}\right),
\]

both of which are of order \( O_{L^1}(T^{-1/2}) \) by either (b) of Assumption 2.2 or (b) of Assumption 2.3. Substituting the above four estimates back into (B.20) we get

\[
(A^T Q^{-1} A)_{ij} = O_{L^1}\left(\frac{K}{\gamma_1(\tau)^2 \sqrt{T}}\right),
\]

which is uniform in \( i,j \). Taking a union bound we obtain

\[
\mathbb{P}\left(\max_{i \neq j} (A^T Q^{-1} A)_{ij} > \epsilon\right) \leq \frac{1}{\epsilon} \sum_{i \neq j} \mathbb{E}|(A^T Q^{-1} A)_{ij}| = \frac{1}{\epsilon} O\left(\frac{K^3}{\gamma_1(\tau)^2 \sqrt{T}}\right),
\]

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or in other words we have the bound

\[
\max_{i \neq j} |(A^\top Q^{-1}A)_{ij}| = O_p \left( \frac{K^3}{\gamma_1(\tau)^2 \sqrt{T}} \right)
\]

Lastly, since \( M = I_K - B - A^\top Q^{-1}A \), it remains to bound the off-diagonals of \( B \). Routine computations similar to (B.13) and the union bound above show that \( B_{ij} \) is of high order compared to \((A^\top Q^{-1}A)_{ij}\) and is thus negligible. This completes the proof. \( \square \)

From Proposition B.2 we can conclude that the CLT of \( M_{ii} \) is given by the CLT of \( A_{ii} \), up to centering and scaling. This is what we compute next.

**Proposition B.3.** Suppose Assumption 2.1 and either Assumption 2.2 or 2.3 hold. For any \( i = 1, \ldots, K \) and \( \tau \geq 0 \), define

\[
v_{i,\tau}^2 := \frac{1}{T - \tau} \text{Var} \left( \sum_{t=1}^{T-\tau} f_{i,t} f_{i,t+\tau} \right).
\]

a) For any \( i \) and \( \tau \), the quantity \( v_{i,\tau}^2 \) satisfies

\[
v_{i,\tau}^2 = \sum_{|k| < T-\tau} \left( 1 - \frac{|k|}{T - \tau} \right) u_{i,k}(\tau), \quad (B.23)
\]

where \( (u_{i,k}(\tau))_k \) is given by

\[
u_{i,k}(\tau) := \gamma_i(k)^2 + \gamma_i(k + \tau)\gamma_i(k - \tau) + (\mathbb{E}[z_{11}^4] - 3) \sum_{l=0}^{\infty} \phi_{i,l}\phi_{i,l+\tau}\phi_{i,l+k}\phi_{i,l+k+\tau}.
\]

b) As \( T \to \infty \), the sequence \( (v_{i,\tau}^2) \) tends to a limit

\[
\lim_{T \to \infty} v_{i,\tau}^2 = (\mathbb{E}[z_{11}^4] - 3)\gamma_i(\tau)^2 + \sum_{k \in \mathbb{Z}} (\gamma_i(k)^2 + \gamma_i(k + \tau)\gamma_i(k - \tau))
\]

in the case where \( \tau \) is a fixed constant, and

\[
\lim_{T \to \infty} v_{i,\tau}^2 = \sum_{k \in \mathbb{Z}} \gamma_i(k)^2
\]

in the case where \( \tau = \tau_T \to \infty \) as \( T \to \infty \).
c) In both cases where \( \tau \) is fixed and where \( \tau \to \infty \), we have
\[
\frac{1}{\sqrt{T v_{i,\tau}}} \left( \sum_{t=1}^{T-\tau} f_{i,t} f_{i,t+\tau} - \mathbb{E} \left[ \sum_{t=1}^{T-\tau} f_{i,t} f_{i,t+\tau} \right] \right) \Rightarrow N(0,1), \quad T \to \infty.
\]

**Proof.** In the case where \( \tau \) is fixed, the proof can be adapted from the arguments in section 7.3 of [11] so it remains to consider the case where \( \tau \to \infty \). For concreteness, the following proof covers both the case where \( \tau \) is finite and fixed and where \( \tau \) is diverging as \( T \to \infty \). For brevity of notation we will drop the subscript \( i \) (denoting the \( i \)-th factor) within the proof and write for instance \( f_t := f_{it} \) and \( \phi_l := \phi_{il} \).

With some adaptations to the computations in page 226-227 of [11], we may obtain
\[
v_T^2 := \frac{1}{T-\tau} \text{Var} \left( \sum_{t=1}^{T-\tau} f_{t} f_{t+\tau} \right) = \frac{1}{T-\tau} \mathbb{E} \left[ \sum_{t=1}^{T-\tau} \sum_{s=1}^{T-\tau} f_{t} f_{t+\tau} f_{s} f_{s+\tau} \right] - (T-\tau) \gamma(\tau)^2 = \sum_{|k|<T-\tau} \left( 1 - \frac{|k|}{T-\tau} \right) u_k(\tau), \tag{B.24}
\]
where \((u_k(\tau))_k\) is given by
\[
u_k(\tau) := \gamma(k)^2 + \gamma(k + \tau)\gamma(k - \tau) + (\mathbb{E}[z_{11}^4] - 3) \sum_{l=0}^{\infty} \phi_l \phi_{l+\tau} \phi_{l+k} \phi_{l+k+\tau}.
\]
Note that the sequence \((\phi_l)_l\) is summable and so is the sequence \((u_k(\tau))_k\). Taking the limit of (B.24) and invoking the dominated convergence theorem we conclude
\[
v^2 := \lim_{T \to \infty} v_T^2 = \sum_{k \in \mathbb{Z}} \lim_{T \to \infty} \left( 1 - \frac{|k|}{T-\tau} \right) u_k(\tau).
\]
In the case where \( \tau \) is a fixed constant, we have, as in Proposition 7.3.1 of [11],
\[
v^2 = (\mathbb{E}[z_{11}^4] - 3) \gamma(\tau)^2 + \sum_{k \in \mathbb{Z}} \left( \gamma(k)^2 + \gamma(k + \tau)\gamma(k - \tau) \right), \tag{B.25}
\]
and in the case where \( \tau \) is diverging, i.e. \( \gamma(\tau) \to 0 \) as \( T \to \infty \), we easily see that
\[
v^2 = \lim_{T \to \infty} v_T^2 = \sum_{k \in \mathbb{Z}} \gamma(k)^2. \tag{B.26}
\]
This settles the first two claims of the proposition.
We first prove a version of the CLT for a truncated version of the factor \( f_t \). The truncation will be justified further below. Fix \( L > 0 \) and define \( (f^{(L)}_t)_{t=1, \ldots, T} \) by

\[
f^{(L)}_t := \sum_{l=0}^{L} \phi_l z_{t-l},
\]

Consider the stochastic process \( (f^{(L)}_t f^{(L)}_{t+\tau})_{t=1, \ldots, T-\tau} \). Clearly \( f^{(L)}_t f^{(L)}_{t+\tau} \) is an \((L + \tau)\)-dependent process, i.e. \( f^{(L)}_t f^{(L)}_{t+\tau} \) is independent from \( f^{(L)}_s f^{(L)}_{s+\tau} \) whenever \(|s-t| > L + \tau\).

The mean is given by

\[
E[f^{(L)}_t f^{(L)}_{t+\tau}] = \gamma^{(L)}(\tau),
\]

where \( \gamma^{(L)}(\tau) \) is the auto-covariance function of the truncated process \( (f^{(L)}_t) \). Similar to (B.24)-(B.26) we may compute

\[
v^2_{T,(L)} := \frac{1}{T-\tau} \text{Var} \left( \sum_{t=1}^{T-\tau} f^{(L)}_t f^{(L)}_{t+\tau} \right),
\]

which has limits, in the case where \( \tau \) is fixed:

\[
v^2_{(L)} := \lim_{T \to \infty} v^2_{T,(L)} = (E[z_{11}^4] - 3)\gamma^{(L)}(\tau)^2 + \sum_{k\in\mathbb{Z}} \left( \gamma^{(L)}(k)^2 + \gamma^{(L)}(k+\tau)\gamma^{(L)}(k-\tau) \right)
\]

and in the case where \( \tau \to \infty \):

\[
v^2_{(L)} = \lim_{T \to \infty} v^2_{T,(L)} = \sum_{k\in\mathbb{Z}} \gamma^{(L)}(k)^2.
\]

Note that in either case \( V^{(L)} \) is a non-zero constant. It can easily be checked that, under Assumption 2.1 and either Assumption 2.2 or 2.3, the process \( (f^{(L)}_t f^{(L)}_{t+\tau})_{t=1, \ldots, T-\tau} \) (after centering) satisfies the conditions in [5], whose main theorem can be applied here to obtain

\[
\left( \sqrt{T} - \tau \right) \left( \frac{1}{T-\tau} \sum_{t=1}^{T-\tau} f^{(L)}_t f^{(L)}_{t+\tau} - \gamma^{(L)}(\tau) \right) \Rightarrow N(0, v^2_{(L)}), \quad T \to \infty.
\]

We now justify the truncation. Since \( (\phi_t)_t \in \ell_1 \) and \( (z_t)_t \) is uniformly bounded in \( L^4 \), it is easy to conclude that, for each fixed \( T \), we have

\[
\left\| \sum_{t=1}^{T-\tau} f^{(L)}_t f^{(L)}_{t+\tau} - \sum_{t=1}^{T-\tau} f_t f_{t+\tau} \right\|_{L^2} \to 0, \quad L \to \infty.
\]

Consequently, we may conclude that \( \gamma^{(L)}(\tau) \to \gamma(\tau) \) and \( v^2_{(L)} \to v^2 \) as \( L \to \infty \), since they are the first and second moments of the sums in (B.27). We may then follow the
arguments on page 229 of [11] and apply Proposition 6.3.9 of [11] to obtain
\[(\sqrt{T} - \tau) \left( \frac{1}{T - \tau} \sum_{t=1}^{T-\tau} f_t f_{t+\tau} - \gamma(\tau) \right) \Rightarrow N(0, \nu^2), \quad T \to \infty. \quad (B.28)\]

Finally, using (B.25), (B.26) and \(T/\tau \to 1\), by Slutsky’s theorem we may conclude that
\[\frac{1}{\sqrt{T}v_\tau} \left( \sum_{t=1}^{T-\tau} f_t f_{t+\tau} - (T - \tau)\gamma_L(\tau) \right) \Rightarrow N(0, 1).\]

It remains to observe that \((T - \tau)\gamma_L(\tau)\) is exactly the expectation of \(\sum f_t f_{t+\tau}\) and the last claim of the proposition follows. \(\square\)

**Proposition B.4.** Under Assumption 2.1 and either Assumption 2.2 or 2.3, we have
\[\sqrt{T} \frac{\theta}{2\sigma_i^4 \gamma_i(\tau) v_{i,\tau}} \left( M_{ii} - \overline{M}_{ii} \right) \Rightarrow N(0, 1), \quad i = 1, \ldots, K, \quad (B.29)\]

where \(\overline{M}_{ii}\) is as defined in (B.11) and \(v_{i,\tau}\) is defined as in (3.4).

**Proof.** For simplicity, within the current proof we will denote \(x_{i_0} := x_{i,[1:T-\tau]}, x_{i\tau} := x_{i,[\tau+1:T]}\) and similarly \(f_{i_0} := f_{i,[1:T-\tau]}, f_{i\tau} := f_{i,[\tau+1:T]}\); \(\epsilon_{i_0} := \epsilon_{i,[1:T-\tau]}, \epsilon_{i\tau} := \epsilon_{i,[\tau+1:T]}\).

Observe from (B.12) that the asymptotic distribution of \(M\) depends crucially on that of \(A\). We first give the asymptotic distribution of \(A_{ii}\). Recall from (2.12) that \(R - I_{T-\tau} = \theta^{-1} E_\tau^T E_\tau E_0^T E_0 R\), using which we can write
\[A_{ii} = \frac{1}{\sqrt{\theta T}} x_{i_0}^T x_{i\tau} + \frac{1}{\sqrt{\theta T}} x_{i_0}^T (R - I_{T-\tau}) x_{i\tau} = \frac{1}{\sqrt{\theta T}} x_{i_0}^T x_{i\tau} + \frac{1}{\theta^{3/2} T} x_{i_0}^T E_\tau^T E_\tau E_0^T E_0 R x_{i\tau}. \quad (B.30)\]

Applying Lemma C.4 to the last term in (B.30) we have
\[\frac{1}{\theta^{3/2} T} x_{i_0}^T E_\tau^T E_\tau E_0^T E_0 R x_{i\tau} 1_{B_0} = \frac{1}{\theta^{3/2} T} \mathbb{E}[x_{i_0}^T E_\tau^T E_\tau E_0^T E_0 R x_{i\tau} 1_{B_0}] = O_L^2 \left( \frac{\sigma_i^2}{\theta^{3/2} \sqrt{T}} \right). \]

Recalling from Theorem 3.1 that \(\theta \asymp \sigma_i^4 \gamma_i(\tau)^2\), we get
\[A_{ii} 1_{B_0} - \mathbb{E}[A_{ii} 1_{B_0}] = \frac{1}{\sqrt{\theta T}} \left( x_{i_0}^T x_{i\tau} 1_{B_0} - \mathbb{E}[x_{i_0}^T x_{i\tau} 1_{B_0}] \right) + O_L^2 \left( \frac{\sigma_i^2}{\theta^{3/2} \sqrt{T}} \right) = \frac{1}{\sqrt{\theta T}} \left( x_{i_0}^T x_{i\tau} - \mathbb{E}[x_{i_0}^T x_{i\tau}] \right) 1_{B_0} + O_L^2 \left( \frac{1}{\sigma_i^4 \gamma_i(\tau)^3 \sqrt{T}} \right). \quad (B.31)\]
Next, we recall that \( x_{i0} = \frac{1}{\sqrt{\theta T}}(\sigma_i f_{i0} + \epsilon_{i0}) \) and \( x_{i\tau} = \frac{1}{\sqrt{\theta T}}(\sigma_i f_{i\tau} + \epsilon_{i\tau}) \) so that

\[
x_{i0}^\top x_{i\tau} = \sigma_i^2 f_{i0}^\top f_{i\tau} + (\sigma_i f_{i0}^\top \epsilon_{i\tau} + \sigma_i f_{i\tau}^\top \epsilon_{i0} + \epsilon_{i0}^\top \epsilon_{i\tau}).
\]

Applying Lemma C.3 to the three terms in parenthesis on the right hand we get

\[
x_{i0}^\top x_{i\tau} - \mathbb{E}[x_{i0}^\top x_{i\tau}] = \sigma_i^2 f_{i0}^\top f_{i\tau} - \sigma_i^2 \mathbb{E}[f_{i0}^\top f_{i\tau}] + O_{L^2}(\sigma_i \sqrt{T}). \tag{B.32}
\]

Substituting back into (B.31) and using \( \theta \approx \sigma_i^4 \gamma_1(\tau)^2 \) again we obtain

\[
A_{ii}1_{B_0} - \mathbb{E}[A_{ii}1_{B_0}] = \left( \frac{\sigma_i^2}{\sqrt{\theta T}} (f_{i0}^\top f_{i\tau} - \mathbb{E}[f_{i0}^\top f_{i\tau}]) 1_{B_0} + O_{L^2} \left( \frac{\sigma_i}{\sqrt{\theta T}} \right) \right) + O_{L^2} \left( \frac{1}{\sigma_i^4 \sqrt{T} \gamma_1(\tau)^3} \right) = \left( \frac{\sigma_i^2}{\sqrt{\theta T}} (f_{i0}^\top f_{i\tau} - \mathbb{E}[f_{i0}^\top f_{i\tau}]) 1_{B_0} + O_{L^2} \left( \frac{1}{\sigma_i \gamma_1(\tau) \sqrt{T}} \right) \right). \tag{B.33}
\]

Rescaling and recalling that \( 1_{B_0} = 1 - o_p(T^{-l}) \) for any \( l \geq 1 \), we have

\[
\sqrt{T} \frac{\sqrt{\theta}}{\sigma_i^2} (A_{ii} - \mathbb{E}[A_{ii}]) = \frac{1}{\sqrt{\theta}} (f_{i0}^\top f_{i\tau} - \mathbb{E}[f_{i0}^\top f_{i\tau}]) + O_p(\sigma_i^{-1}). \tag{B.34}
\]

From this we observe that we can obtain a CLT for \( A_{ii} \) from a CLT for the auto-covariance function of \( f_i \). Indeed, by Proposition B.3 we have

\[
\sqrt{T} \frac{\sqrt{\theta}}{\sigma_i^2 \nu_{i\tau}} (A_{ii} - \mathbb{E}[A_{ii}]) \Rightarrow N(0, 1), \quad p, T \to \infty, \tag{B.34}
\]

where \( \nu_{i\tau} \) is specified in the statement of Proposition B.3.

Finally, we recall from Proposition B.2 that

\[
M_{ii} - \overline{M}_{ii} = -2(A_{ii} - \mathbb{E}[A_{ii}]) \mathbb{E}[A_{ii}1_{B_0}] \mathbb{E}[Q_{ii}^{-1}1_{B_2}] + o_p \left( \frac{1}{\sqrt{T}} \right). \tag{B.35}
\]

In order to apply the CLT in (B.34) to (B.35), we need to divide (B.35) by the coefficient of \( A_{ii} - \mathbb{E}[A_{ii}] \), which requires it to be bounded away from zero. Indeed, we recall from (2.21) that \( Q_{ii}^{-1}1_{B_2} = 1 + o(1) \). Furthermore, from Lemma C.5 we have

\[
\mathbb{E}[A_{ii}1_{B_0}] = \frac{\sigma_i^2 \gamma_i(\tau)}{\sqrt{\theta}} + o(1) \tag{B.36}
\]

which is bounded from below as well. Therefore from (B.35) we get

\[
\frac{\sqrt{\theta}}{-2\sigma_i^2 \gamma_i(\tau)} (M_{ii} - \overline{M}_{ii}) = (A_{ii} - \mathbb{E}[A_{ii}]) (1 + o(1)) + o_p \left( \frac{1}{\sqrt{T}} \right),
\]

and the claim follows then from the CLT in (B.34).
The asymptotic distribution of $M_{ii}$ proved in Proposition B.4 is the last piece of ingredient we need to prove the main result of the paper, which we present below.

**Proof of Theorem 3.3.** Recall from Section 3 that up to now we have dealt with, without loss of generality, the $n$-th largest eigenvalue $\lambda := \lambda_n$ and the corresponding $\theta := \theta_n$ and $\delta := \delta_n := \lambda_n/\theta_n - 1$. Recall from Proposition B.1 that $\delta_n$ satisfies

$$\det \left( M + \frac{\delta_n}{\theta_n} X_\tau X_0^T X_0 X_\tau^T + \delta o_p(1) \right) = 0 \quad (B.37)$$

We first consider the asymptotic properties of the elements of the matrix $M$. From Proposition 3.2 and the definition of $M_{ii}$ in (B.11), clearly we see $M_{nn} = 0$. From Theorem 3.1 and Proposition 3.2 we also recall that $\theta_n/\left( \sigma_n^4 \gamma_n(\tau)^2 \right) = \theta_n/\mu_{n,\tau}^2 = 1 + o(1)$.

Then, using Proposition B.4 we immediately have

$$\sqrt{T} \frac{\gamma_n(\tau)}{2\nu_{n,\tau}} M_{nn} \Rightarrow N(0, 1), \quad T \to \infty.$$ 

For $i \neq n$, we recall from (B.11) and Lemma C.5 that

$$\overline{M}_{ii} = 1 - \frac{\sigma_i^4 \gamma_i(\tau)^2}{\theta_n} + o(1) = 1 - \frac{\mu_{i,\tau}^2}{\mu_{n,\tau}^2} + o(1) \approx 1,$$

where the last equality is due to Assumption 2.1. Using Proposition B.2 we have

$$M_{ii} \approx 1 + o_p(1), \quad \forall i \neq n, \quad \max_{i \neq j} |M_{ij}| = O_p \left( \frac{K^3}{\gamma_1(\tau)^2 \sqrt{T}} \right).$$

Next, recall $\delta = o_p(1)$ from Theorem 3.1. From (A.6) recall that

$$\| \theta^{-1} X_\tau X_0^T X_0 X_\tau^T - \theta^{-1} \text{diag}(\sigma_i^4 \gamma_i(\tau)^2) \|_\infty = O_p \left( \frac{K^2 \sigma_i^4 \gamma_1(\tau)}{\theta \sqrt{T}} \right) = o_p(1).$$

This in particular implies $\delta \theta^{-1} (X_\tau X_0^T X_0 X_\tau^T)_{ii} = \delta \theta^{-1} \sigma_i^4 \gamma_i(\tau)^2 + o_p(1) = o_p(1)$ and

$$\frac{\delta}{\theta} (X_\tau X_0^T X_0 X_\tau^T)_{ij} = o_p(\delta), \quad \forall i \neq j.$$

Combining the above, equation (B.37) becomes $\det(Q) = 0$, where $Q$ is a matrix satisfying

$$Q_{nn} = M_{nn} + \frac{\delta_n}{\theta_n} \frac{\sigma_n^4 \gamma_n(\tau)^2}{\theta_n} + \delta_n o_p(1),$$
for its $n$-th diagonal element, $Q_{ii} \asymp 1 + o_p(1), \ \forall i \neq n$ and
\[
\sup_{ij} Q_{ij} = O_p \left( \frac{K^3}{\gamma_1(\tau)^2 \sqrt{T}} \right) + \delta o_p(1). \tag{B.38}
\]
Using Leibniz’s formula to compute $\det(Q)$, we have
\[
0 = \det(Q) = \sum_{\pi \in S_K} \text{sgn}(\pi) \prod_{i=1}^{K} Q_{i,\pi(i)}, \tag{B.39}
\]
where $\text{sgn}(\pi)$ is the sign of a permutation $\pi$ in the symmetry group $S_K$. Next we show that $\prod_i Q_{ii}$ is the leading term in the sum in (B.39). Write $S_{K,k}$ for the subgroup of permutations that has exactly $K - k$ fixed points, i.e.
\[
S_{K,k} = \{ \pi \in S_K, i = \pi(i) \text{ for exactly } K - k \text{ such } i's \}.
\]
Using this notation we can rewrite (B.39) into
\[
0 = \det(Q) = \sum_{k=0}^{K} \sum_{\pi \in S_{K,k}} \text{sgn}(\pi) \prod_{i=1}^{K} Q_{i,\pi(i)}. \tag{B.40}
\]
We recall that the order of $S_{K,k}$ is given by the rencontres numbers (see [24])
\[
|S_{K,k}| = D_{K,K-k} := \frac{K!}{(K-k)!} \sum_{i=0}^{k} \frac{(-1)^i}{i!}.
\]
Observe that $|S_{K,0}| = 1$ since $S_{K,0}$ contains only the identity permutation and $|S_{K,1}| = 0$ since for any non-identity permutation $\pi$, there exists at least two indices $i, j \in \{1, \ldots, K\}$, $i \neq j$ such that $i \neq \pi(i)$ and $j \neq \pi(j)$. Therefore (B.40) becomes
\[
0 = \det(Q) = \prod_{i=1}^{K} Q_{ii} + \sum_{k=2}^{K} \sum_{\pi \in S_{K,k}} \text{sgn}(\pi) \prod_{i=1}^{K} Q_{i,\pi(i)}. \tag{B.41}
\]
Note that for any $k \geq 2$ and any permutation $\pi \in S_{K,k}$, the product $\prod_{i=1}^{K} Q_{i,\pi(i)}$ contains exactly $k$ off-diagonal elements of $Q$. By (B.38) we have the estimate
\[
\prod_{i=1}^{K} Q_{i,\pi(i)} = \left( O_p \left( \frac{K^3}{\gamma_1(\tau)^2 \sqrt{T}} \right) + \delta o_p(1) \right)^k.
\]
Finally, after substituting back into (B.41) and a lengthy computation we have

\[ 0 = \det(Q) = \prod_{i=1}^{K} Q_{ii} + \delta o_p(1) + o_p(T^{-1/2}), \] (B.42)

which shows that the product \( \prod_{i=1}^{K} Q_{ii} \) is the leading term of \( \det(Q) \).

Next, using (B.38) again we see that \( \prod_{i=1}^{K} Q_{ii} \) can be written into

\[ \prod_{i=1}^{K} Q_{ii} = \left( M_{nn} + \frac{\sigma_n^4 \gamma_n(\tau)^2}{\theta_n} + \delta_n o_p(1) \right) (1 + o_p(1)) = (M_{nn} + \delta_n) (1 + o_p(1)), \]

which can be substituted back into (B.42) to obtain

\[ 0 = M_{nn} (1 + o_p(1)) + \delta_n (1 + o_p(1)) + \delta_n o_p(1) + o_p(T^{-1/2}). \]

Rearranging (and recalling \( \theta_n \asymp \sigma_n^4 \gamma_n(\tau)^2 \) by Proposition 3.2), we finally get

\[ -\sqrt{T \gamma_n(\tau)} \frac{\delta_n}{2v_{n,\tau}} (1 + o_p(1)) = \sqrt{T} \frac{\theta_n}{2\sigma_n^4 \gamma_n(\tau) v_{n,\tau}} M_{nn} (1 + o_p(1)) + o_p(1). \]

Applying Proposition B.4 we immediately have

\[ \sqrt{T \gamma_n(\tau)} \frac{\delta_n}{2v_{n,\tau}} \Rightarrow N(0,1), \quad T \to \infty \]

and the proof is complete.

\[ \square \]

### C Estimates on bilinear forms

The following linear algebraic result will be useful throughout.

**Lemma C.1** (Sherman-Morrison formula). Suppose \( A \) and \( B \) are invertible matrices of the same dimension, such that \( A - B \) is of rank one. Then

\[ A^{-1} - B^{-1} = -\frac{B^{-1}(A - B)B^{-1}}{1 + \text{tr}(B^{-1}(A - B))}. \] (C.1)

Further more, if \( A - B = uv^T \), then

\[ A^{-1}u = \frac{B^{-1}u}{1 + v^TB^{-1}u}, \quad v^TA^{-1} = \frac{v^TB^{-1}}{1 + v^TB^{-1}u}. \] (C.2)
We first establish some concentration inequalities for quadratic forms of the random vector \( x \). To do so we will need to introduce some notations. We recall from (2.2) and (2.8) that

\[
x_{it} = \sigma_i f_{it} + \epsilon_{it} = \sigma_i \sum_{l=0}^{\infty} \phi_{il} z_{i,t-l} + \epsilon_{it}, \quad i = 1, \ldots, K, \quad t = 1, \ldots, T.
\]

We truncate the series and define an approximation

\[
x_{it}^{(L)} := \sigma_i f_{it}^{(L)} + \epsilon_{it} := \sigma_i \sum_{l=0}^{L} \phi_{il} z_{i,t-l} + \epsilon_{it}, \quad L \geq 1,
\]

and write \( x_{i,[1:T]}^{(L)} \), \( f_{i,[1:T]}^{(L)} \) for \( (x_{it}^{(L)})_{t=1,\ldots,T} \) and \( (f_{it}^{(L)})_{t=1,\ldots,T} \). For each \( L \), We write

\[
\Phi_i := \left( \phi_{i0} \Phi_{i0}, \cdots, \phi_{iT} \Phi_{iT} \right) \in \mathbb{R}^{T \times T + L}.
\]

Let \( S \) be the right-shift operator on \( \mathbb{R}^{T + L} \), i.e. \( Se_i = e_{i+1} \). Define

\[
\Phi_i := \left( \phi_{i0}, \cdots, \phi_{iT-1} \right) \in \mathbb{R}^{(T + L) \times T},
\]

then clearly we can write the approximation \( f_{i,[1:T]}^{(L)} \) as

\[
f_{i,[1:T]}^{(L)} = z_{i,[1-L:T]}^\top \Phi_i.
\]

We note that the spaces of \( n \times n \) matrices equipped with the Frobenius norm is isometrically isomorphic to \( \mathbb{R}^{n \times n} \) with the Euclidean norm. For each \( 1 \leq i, j \leq K \), we define linear operators \( \Psi_{n}^{ij}, n = 0, 1, 2, \)

\[
\Psi_{0}^{ij} : \mathbb{R}^{(T - \tau) \times (T - \tau)} \rightarrow \mathbb{R}^{(2T + L) \times (2T + L)}
\]

by sending a \( (T - \tau) \times (T - \tau) \) matrix \( B \) to the \( (2T + L) \times (2T + L) \) matrices

\[
\Psi_{0}^{ij} B := \left( \begin{array}{c} \sigma_i \Phi_i \\ I_T \end{array} \right) \begin{pmatrix} I_{T-\tau} & 0_{\tau \times (T-\tau)} \end{pmatrix} B \begin{pmatrix} I_{T-\tau} \\ 0_{\tau \times (T-\tau)} \end{pmatrix} \left( \begin{array}{c} \sigma_j \Phi_j \end{array} \right)^\top,
\]

\[
\Psi_{1}^{ij} B := \left( \begin{array}{c} \sigma_i \Phi_i \\ I_T \end{array} \right) \begin{pmatrix} I_{T-\tau} & 0_{\tau \times (T-\tau)} \end{pmatrix} B \begin{pmatrix} 0_{\tau \times (T-\tau)} \\ I_{T-\tau} \end{pmatrix} \left( \begin{array}{c} \sigma_j \Phi_j \end{array} \right)^\top,
\]

\[
\Psi_{2}^{ij} B := \left( \begin{array}{c} \sigma_i \Phi_i \\ I_T \end{array} \right) \begin{pmatrix} 0_{\tau \times (T-\tau)} \\ I_{T-\tau} \end{pmatrix} B \begin{pmatrix} 0_{\tau \times (T-\tau)} \\ I_{T-\tau} \end{pmatrix} \left( \begin{array}{c} \sigma_j \Phi_j \end{array} \right)^\top.
\]
where $\Phi_i := (\Phi_i, S\Phi_i, \ldots, S^{T-1}\Phi_i) \in \mathbb{R}^{(T+L) \times T}$ is as defined in (C.3c). We first give some estimates on the operators $\Psi_{ij}^n$.

**Lemma C.2.** The following estimates hold uniformly in $L \in \mathbb{N}$.

a) The matrix $\Phi_i^\top \Phi_i$ is symmetric and (banded) Toeplitz with

$$\sup_i \|\Phi_i^\top \Phi_i\| \leq 1 + \sup_i \|\phi_i\|_2^2 = O(1).$$

b) For $n = 0, 1, 2$, the operator norms of $\Psi_{ij}^n$ be bounded by

$$\|\Psi_{ij}^n\| \leq (1 + \sigma_i^2\|\phi_i\|_2^2)(1 + \sigma_j^2\|\phi_j\|_2^2) = O(\sigma_i^2\sigma_j^2).$$

c) For any $B \in \mathbb{R}^{T \times T}$, the trace of $\Psi_{ii}^n B$ can be bounded by

$$\left|\text{tr}(\Psi_{ii}^n B)\right| \leq (T - \tau)(1 + \sigma_i^2\|\phi_i\|_2^2)\|B\| = O(\sigma_i^2(T - \tau)\|B\|).$$

**Proof.** a) From the definitions (C.3b) and (C.3c) we immediately have

$$(\Phi_i^\top \Phi_i)_{s,t} = 1_{|s-t| \leq L}\Phi_i^\top S^{s-t}\Phi_i = 1_{|s-t| = k \leq L} \sum_{l=0}^{L-k} \phi_{i,l+k}\phi_{i,l}.$$  

It is clear that $\Phi_i^\top \Phi_i$ is a banded, symmetric Toeplitz matrix. The operator norm of $\Phi_i^\top \Phi_i$ is controlled by the supremum of its symbol over $\mathbb{C}$ (see [9]) and we have

$$\|\Phi_i^\top \Phi_i\| \leq \sup_{\lambda \in \mathbb{C}} \left| \sum_{|k|=0}^{L} \phi_i^\top S^{|k|} e^{i\lambda k} \right| \leq \|\phi_i\|_2^2 + \sum_{k=1}^{L} \sum_{l=0}^{L-k} |\phi_{i,l+k}\phi_{i,l}| \leq 1 + \|\phi_i\|_2^2,$$

which is bounded uniformly in $i = 1, \ldots, K$, due to Assumption (2.1).

b) By the cyclic property of the trace and Cauchy-Schwarz inequality we get

$$\|\Psi_{ij}^n B\|_2^2 = \text{tr}((\Psi_{ij}^n B)(\Psi_{ij}^n B)^\top)$$

$$= \text{tr}((I_T + \sigma_i^2\Phi_i^\top \Phi_i)(I_{T-\tau}, 0)^\top B(0, I_{T-\tau}) + (I_T + \sigma_j^2\Phi_j^\top \Phi_j)(0, I_{T-\tau})^\top B^\top(I_{T-\tau}, 0)),$$

$$\leq \|(I_T + \sigma_i^2\Phi_i^\top \Phi_i)(I_{T-\tau}, 0)^\top B(I_{T-\tau}, 0)\|_F \|(I_T + \sigma_j^2\Phi_j^\top \Phi_j)(0, I_{T-\tau})^\top B^\top(I_{T-\tau}, 0)\|_F.$$
Since \( \|AB\|_F \leq \|A\|\|B\|_F \), we have
\[
\|\Psi_{ij}^{1j}B\|_F^2 \leq \|I_T + \sigma_i^2 \Phi_i^\top \Phi_i\|\|I_T + \sigma_j^2 \Phi_j^\top \Phi_j\|\|B\|_F^2,
\]
where \( \|I_T + \sigma_i^2 \Phi_i^\top \Phi_i\| \leq 1 + \sigma_i^2 \|\Phi_i\|_2^2 \) by the first claim of the Lemma. By identifying \( \Psi_{ij}^{1j} \) as an operator between spaces of matrices equipped with the Frobenius norm, this translates to a bound on its spectral norm. The case of \( \Psi_0 \) and \( \Psi_2 \) hold analogously.

c) For the last bound, similar computations give
\[
|\text{tr}(\Psi_{ii}^{0j}B)| = |\text{tr}((I_T + \sigma_i^2 \Phi_i^\top \Phi_i)(I_{T-\tau}, 0)^\top B(I_{T-\tau}, 0))|
\leq \|I_T + \sigma_i^2 \Phi_i^\top \Phi_i\|\|B\|_F
\leq (T-\tau)(1 + \sigma_i^2 \|\Phi_i\|_2^2)\|B\|
\]
The rest of the claims hold similarly. \( \square \)

Next, we state an easy extension to Lemma 2.7 of \cite{2} suited to our needs.

**Lemma C.3.** Let \( z = (z_1^\top, z_2^\top)^\top \), where \( z_1 = (z_1, \ldots, z_m) \) and \( z_2 = (\tilde{z}_1, \ldots, \tilde{z}_n) \) are independent random vectors each with i.i.d. entries satisfying \( \mathbb{E}[z_1] = \mathbb{E}[\tilde{z}_1] = 0, \mathbb{E}[z_1^2] = \mathbb{E}[\tilde{z}_2^2] = 1, \nu_q := \mathbb{E}|z_1|^q < \infty \) and \( \tilde{\nu}_q := \mathbb{E}|	ilde{z}_1|^q < \infty \) for some \( q \in [1, \infty) \).

a) Let \( C \) be a deterministic \( m \times n \) matrix, then
\[
z_1^\top Cz_2 = O_{L^q}(\|C\|_F),
\]
where the constant in the estimate depends only on \( q \) and \( \nu_q, \tilde{\nu}_q \).

b) Let \( M \) be a deterministic \((m+n) \times (m+n)\) matrix, then
\[
z_1^\top Mz - \text{tr}M = O_{L^q}(\|M\|_F),
\]
where the constant in the estimate depends only on \( q \) and \( \nu_k, \tilde{\nu}_k \) for \( k \leq 2q \).

**Proof.** a) By Lemma 2.2 and Lemma 2.3 of \cite{2} we have
\[
\mathbb{E}|z_1^\top Cz_2|^q = \mathbb{E}\left|\sum_{i,j} z_i \tilde{z}_j C_{ij}\right|^q \lesssim \mathbb{E}\left|\sum_{i,j} z_i^2 \tilde{z}_j^2 C_{ij}^2\right|^{q/2}
\lesssim \left( \sum_{i,j} \mathbb{E}[z_i^2 \tilde{z}_j^2 C_{ij}^2]\right)^{q/2} + \sum_{i,j} \mathbb{E}|z_i|^q|\tilde{z}_j|^q|C_{ij}|^q
= \left( \sum_{i,j} M_{ij}^2\right)^{q/2} + \nu_q \tilde{\nu}_q \sum_{i,j} |C_{ij}|^q \leq (1 + \nu_q \tilde{\nu}_q)\|C\|_F^q,
\]
where the last inequality holds since $\sum |C_{ij}|^q \leq (\sum |C_{ij}|^2)^{q/2}$ for $q \geq 2$.

b) Write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A, B, C, D$ are of dimensions such that

$$z^\top Mz = z_1^\top Az_1 + z_1^\top Bz_2 + z_2^\top Cz_1 + z_2^\top Dz_2.$$  

By Lemma 2.7 of Bai and Silverstein [2] we have

$$\mathbb{E} |z_1^\top Az_1 - \text{tr} A|^q \lesssim (\nu_4^{q/2} + \nu_2q) \text{tr}(A)^q/2 \leq (\nu_4^{q/2} + \nu_2q) \|M\|_F^q,$$

$$\mathbb{E} |z_2^\top Dz_2 - \text{tr} D|^q \lesssim (\tilde{\nu}_4^{q/2} + \tilde{\nu}_2q) \text{tr}(D)^q/2 \leq (\tilde{\nu}_4^{q/2} + \tilde{\nu}_2q) \|M\|_F^q.$$  

Then we can write

$$\mathbb{E} |z^\top Mz - \text{tr} M|^q \lesssim \mathbb{E} |z_1^\top Az_1 - \text{tr} A|^q + \mathbb{E} |z_2^\top Bz_2 - \text{tr} B|^q$$

$$+ \mathbb{E} |z_1^\top Cz_2|^q + \mathbb{E} |z_2^\top Dz_1|^q.$$

and the claim follows from (a) of the lemma. \qed

Using Lemma C.2 and Lemma C.3 we can derive the following concentration inequalities for quadratic forms involving certain high probability events.

**Lemma C.4.** Let $\mathbf{x}, \mathbf{f}, \mathbf{e}$ be defined as in (2.9) and $q \leq 2$. Under Assumptions 2.1 and either Assumptions 2.2 or 2.3 we have

a) For any (deterministic) square matrix $B$ of size $T - \tau$, we have

$$\mathbf{x}_{i,[1:T-\tau]}^\top B \mathbf{x}_{j,[\tau+1:T]} - \mathbb{E}[\mathbf{x}_{i,[1:T-\tau]}^\top B \mathbf{x}_{j,[\tau+1:T]}] = O_L\left(\sigma_i \sigma_j \sqrt{T} \|B\|\right),$$

where the expectation is satisfies

$$\mathbb{E}[\mathbf{x}_{i,[1:T-\tau]}^\top B \mathbf{x}_{j,[\tau+1:T]}] = \mathbf{1}_{i=j} \text{tr}(\Psi_{ii}^1(B)) = \mathbf{1}_{i=j} O(\sigma_i^2 T \|B\|).$$

b) For all $i, j$ we have $\mathbb{E}[\mathbf{f}_{i,[1:T-\tau]}^\top B \mathbf{e}_{j,[\tau+1:T]}] = 0$ and

$$\mathbf{f}_{i,[1:T-\tau]}^\top B \mathbf{e}_{j,[\tau+1:T]} = O_L\left(\sigma_i \sqrt{T} \|B\|\right).$$
Suppose \( n \in \{1, \ldots, K\} \) and \( c_1, c_2 \) are positive constants with \( c_1 < c_2 \). Pick any
\[
a \in [c_1, c_2] \mu_{n, \tau}^2.
\]

Recall from (2.11) the resolvent \( R(a) := (I_{T-\tau} - a^{-1} E_\tau^T E_\tau E^T)^{-1} \), then
\[
x_i^{(L)}^T x_j^{(L)} = E_i^{(L)} + \epsilon_i^{(L)} \left( I_{T-\tau} \right) \left( 0 \right) B \left( I_{T-\tau} \right) (\sigma_j^{(L)} + \epsilon_j^{(L)})
\]
for all \( k \in \mathbb{N} \), where \( E_i^{(L)} \) is defined in (2.23). In particular,
\[
x_i^{(L)}^T x_j^{(L)} = \left( z_i^{(L)} + \epsilon_i^{(L)} \right) (\Psi_i^{ij}) (z_j^{(L)} + \epsilon_j^{(L)})
\]

Proof. a) We apply the truncation procedure as described in (C.3a). Recalling (C.3a), (C.3d) and (C.4) we may write
\[
\left( z_i^{(L)} + \epsilon_i^{(L)} \right) (\Psi_i^{ij}) (z_j^{(L)} + \epsilon_j^{(L)})
\]
Applying (b) of Lemma C.3 to the above quadratic form gives
\[
\mathbb{E} \left[ x_i^{(L)}^T x_j^{(L)} - \mathbb{E} \left[ x_i^{(L)}^T x_j^{(L)} \right] \right]^q \lesssim \left\| \Psi_i^{ij} \right\|_F^q,
\]
where \( \mathbb{E} \left[ x_i^{(L)}^T x_j^{(L)} \right] = 1_{i=j} \text{tr}(\Psi_i^{ij}) \). Using Lemma C.2 we see that
\[
\left\| \Psi_i^{ij} \right\|_F^q = O \left( \sigma_i^2 \sigma_j^2 \right) \left\| B \right\|_F^q = (T-\tau)^{q/2} O \left( \sigma_i^2 \sigma_j^2 \right) \left\| B \right\|_q^q,
\]
and
\[
\mathbb{E} \left[ x_i^{(L)}^T x_j^{(L)} \right] = 1_{i=j} O(\sigma_i^2 (T-\tau)) \left\| B \right\| = 1_{i=j} O(\sigma_i^2 T) \left\| B \right\|,
\]
both of which are uniform in \( L \).

Since \( (\phi_{\alpha})_i \) is summable and \( (z_{\alpha})_i \) have uniformly bounded 4-th moments, it is clear that \( x_i^{(L)}/\sigma_i \) converges to \( x_i/\sigma_i \) in \( L^4 \) as \( L \to \infty \), for each fixed \( T \). By the dominated
convergence theorem with (C.5) as an upper-bound, we can take the limit as $L \to \infty$ inside the expectation in (C.5) and the claim follows.

b) follows from similar computations as in (a) and is omitted.

c) Note that $E_r^T E_r E^T E$ has bounded operator norm under the event $B_0$ defined in (2.18). Since $a \asymp \sigma^4_n \gamma_n(\tau)^2$ diverges as $T \to \infty$, the resolvent $R(a)$ is well-defined under $B_0$ and $\|R(a)^k 1_{B_0}\| = O(1)$. After conditioning on the $\sigma$-algebra $F$ defined in (2.22), we can then apply (a) of the Lemma and get

$$E \left| x_{i,[1:T-\tau]} R(a)^k x_{j,[\tau+1:T]} 1_{B_0} - E \left[ x_{i,[1:T-\tau]} R(a)^k x_{j,[\tau+1:T]} 1_{B_0} \right] \right|^q \lesssim T^{q/2} O(\sigma^q \sigma_j^q).$$

Taking expectations again to remove the conditioning, we obtain

$$E \left| x_{i,[1:T-\tau]} R(a)^k x_{j,[\tau+1:T]} 1_{B_0} - E \left[ x_{i,[1:T-\tau]} R(a)^k x_{j,[\tau+1:T]} 1_{B_0} \right] \right|^q \lesssim T^{q/2} O(\sigma^q \sigma_j^q).$$

Note that $E [x_{i,[1:T-\tau]} R(a)^k x_{j,[\tau+1:T]} 1_{B_0}] = 0$ for all $i \neq j$ by (a) of the Lemma. So

$$x_{i,[1:T-\tau]} R(a)^k x_{j,[\tau+1:T]} 1_{B_0} = 1_{i=j} E [x_{i,[1:T-\tau]} R(a)^k x_{i,[\tau+1:T]} 1_{B_0}] + O_L (\sigma_2 \sigma_j \sqrt{T}).$$

By Lemma 2.1 we have $1_{B_0} = 1 - o_p(1)$, from which the last claim follows.

d) follows from similar computations to the above and is omitted. 

Note that the expectations appearing in the previous lemma are conditional on the noise series $\epsilon$. The following lemma gives a preliminary computation on the unconditional moments of certain quadratic forms. Recall matrices $B(a)$, $A(a)$ and $Q(a)$:

$$A(a) := \frac{1}{\sqrt{a}} X_0 R(a) X_r^\top, \quad B(a) := \frac{1}{a} X_r E_0^\top E_0 R(a) X_r^\top, \quad Q(a) := I_K - a^{-1} X_0 R_a E_r^\top E_r X_0^\top.$$

**Lemma C.5.** Under the same setting as (c) of Lemma C.4, we have

$$E[A(a)_{ij} 1_{B_0}] = 1_{i=j} \left( \frac{\sigma_i^2 \gamma_i(\tau)}{a^{1/2}} + o(1) \right),$$

$$\text{Var}(A(a)_{ij} 1_{B_0}) = O \left( \frac{\sigma_i^2 \sigma_j^2}{a T} \right),$$

$$E[B(a)_{ij} 1_{B_0}] = 1_{i=j} o(1), \quad E[Q(a)^{-1}_{ij} 1_{B_2}] = 1_{i=j} + o(1).$$
Proof. Since $x_i = \sigma_i f_i + \epsilon_i$, we first observe that
\begin{equation}
\frac{1}{\sqrt{aT}} \mathbb{E}[x_{i,[1,T-\tau]}^T x_{j,[\tau+1:T]}] = \frac{1}{\sqrt{aT}} \mathbb{E}[\sigma_i^2 f_{i,[1,T-\tau]}^T f_{j,[\tau+1:T]}] = 1_{i=j} \frac{\sigma_i^2 \gamma_i(\tau)}{\sqrt{a}}.
\end{equation}
(C.6)

By definition, the event $B_0$ is independent from the vector $x$. Therefore
\begin{align*}
\mathbb{E}[A(a)_{ij} 1_{B_0}] &= \frac{1}{\sqrt{aT}} \mathbb{E}[x_{i,[1,T-\tau]}^T x_{j,[\tau+1:T]} 1_{B_0}] + \frac{1}{\sqrt{aT}} \mathbb{E}[x_{i,[1,T-\tau]}^T (R(a) - I) 1_{B_0} x_{j,[\tau+1:T]}] \\
&= 1_{i=j} \frac{\sigma_i^2 \gamma_i(\tau)}{\sqrt{a}} \mathbb{P}(B_0) + \frac{1}{\sqrt{aT}} \mathbb{E}[x_{i,[1,T-\tau]}^T (R(a) - I) 1_{B_0} x_{j,[\tau+1:T]}] \\
&= 1_{i=j} \left( \frac{\sigma_i^2 \gamma_i(\tau)}{\sqrt{a}} + o(1) \right) + \frac{1}{\sqrt{aT}} \mathbb{E}[x_{i,[1,T-\tau]}^T (R(a) - I) 1_{B_0} x_{j,[\tau+1:T]}],
\end{align*}
where the last equality follows since $\mathbb{P}(B_0) = 1 + o(1)$ by Lemma 2.1. It remains to compute the last expectation above. Recall from (2.12) that the resolvent $R(a)$ satisfies $R(a) - I = a^{-1} E_r^T E_r E^T E R(a)$. By definition of $B_0$ we have $\|E_r^T E_r E^T E 1_{B_0}\| = O(1)$ and $\|R(a) 1_{B_0}\| = O(1)$. Therefore
\begin{equation}
(R(a) - I) 1_{B_0} = O(\|a^{-1}\|).
\end{equation}
(C.7)

Using (C.7) and (a) of Lemma C.4 and taking iterated expectations we obtain
\begin{align*}
\frac{1}{\sqrt{aT}} \mathbb{E}[x_{i,[1,T-\tau]}^T (R(a) - I) 1_{B_0} x_{j,[\tau+1:T]}] &= \frac{1}{\sqrt{aT}} \mathbb{E}[\mathbb{E}[x_{i,[1,T-\tau]}^T (R(a) - I) 1_{B_0} x_{j,[\tau+1:T]]}]] \\
&= 1_{i=j} \frac{1}{\sqrt{aT}} O(\sigma_i^2 T) \mathbb{E}[\|R(a) - I\| 1_{B_0}] = 1_{i=j} o(1).
\end{align*}

For the second moment, using $(a - b)^2 = (a - c)^2 + (c - b)^2 + 2(a - c)(c - b)$, we write
\begin{align*}
(A(a)_{ij} 1_{B_0} - \mathbb{E}[A(a)_{ij} 1_{B_0}])^2
&= (A(a)_{ij} 1_{B_0} - \mathbb{E}[A(a)_{ij} 1_{B_0}])^2 + (\mathbb{E}[A(a)_{ij} 1_{B_0}] - \mathbb{E}[A(a)_{ij} 1_{B_0}])^2 \\
&\quad + 2(A(a)_{ij} 1_{B_0} - \mathbb{E}[A(a)_{ij} 1_{B_0}])(\mathbb{E}[A(a)_{ij} 1_{B_0}] - \mathbb{E}[A_{ij} 1_{B_0}]).
\end{align*}

where by (c) of Lemma C.4 we have
\begin{equation}
\mathbb{E}[(A(a)_{ij} 1_{B_0} - \mathbb{E}[A(a)_{ij} 1_{B_0}])^2] = \frac{1}{aT^2} O(\sigma_i^2 \sigma_j^2 T) = O\left(\frac{\sigma_i^2 \sigma_j^2}{aT}\right).
\end{equation}
and from Lemma D.3 (whose proof does not depend on the current lemma) we recall
\[
\mathbb{E} \left[ (\mathbb{E}[A(a)_{ij}1_{B_0}] - \mathbb{E}[A(a)_{ij}1_{B_0}])^2 \right] = O \left( \frac{1}{aT} \right).
\]
Taking expectation of (C.8) and using the Cauchy Schwarz inequality we have
\[
\mathbb{E}[ (A(a)_{ij}1_{B_0} - \mathbb{E}[A(a)_{ij}1_{B_0}])^2 ] = O \left( \frac{\sigma_i^2 \sigma_j^2}{aT} \right).
\]
The expectation of \( B(a) \) can be computed based on the same ideas and is omitted.

Lastly, under the event \( B_2 \), the matrix \( Q(a) \) is invertible with \( \|Q(a)1_{B_2}\| = O(1) \). We recall from (2.17) that the inverse of \( Q(a) \) satisfies
\[
Q(a)^{-1} = I_K + \frac{1}{a} Q(a)^{-1} X_0 R(a) E_\tau E_\tau^T X_0^T.
\]
By definition of \( B_2 \) we know \( 1_{B_2} Q(a)^{-1} X_0 R(a) E_\tau E_\tau^T X_0^T = O_{\|\cdot\|}(\sigma_i^2) \) and therefore
\[
Q(a)^{-1} 1_{B_2} = 1_{B_2} I_K + o_{\|\cdot\|}(1)
\]
and the last claim follows after taking expectations.

\[\square\]

### D Estimates on resolvents

Define the following families of \( \sigma \)-algebras \( \{\mathcal{F}_i\}_{i=1}^p \) and \( \{\mathcal{E}_i\}_{i=1}^K \) by
\[
\mathcal{F}_i := \sigma(\epsilon_{[K+1,K+p],[1:T]}), \quad \mathcal{E}_i := \sigma(x_{[1:i],[1:T]}, \epsilon_{[K+1,K+p],[1:T]}),
\]
i.e. \( \mathcal{F}_i \) is the \( \sigma \)-algebra generated by first \( i \) coordinates of the noise series \( \epsilon \) and \( \mathcal{E}_i \) is generated by all \( p \) coordinates of \( \epsilon \) plus the first \( i \) coordinates of the series \( x \).

Throughout the appendix we will write
\[
\mathbb{E}_i[\cdot] := \mathbb{E}_i[\cdot|\mathcal{F}_i], \quad \mathbb{E}_i[\cdot] := \mathbb{E}_i[\cdot|\mathcal{E}_i]. \tag{D.1}
\]
Note that by definition \( \mathbb{E}_0[\cdot] = \mathbb{E}[\cdot] \) and \( \mathbb{E}_p[\cdot] = \mathbb{E}_0[\cdot] = \mathbb{E}[\cdot] \).

We first develop a concentration inequality for normalized traces of the resolvent \( R \).
Lemma D.1. For any matrix $B$ with $T - \tau$ columns, we have

\[
\frac{1}{T} \text{tr}(B(R1_{B_0} - \mathbb{E}[R1_{B_0}]abyrin)) = O_{L^2}\left(\frac{\|B\|}{\theta \sqrt{T}}\right),
\]

(a)

\[
\frac{1}{T} \text{tr}(B(E_0^T E_0R1_{B_0} - \mathbb{E}[E_0^T E_0R1_{B_0}])) = O_{L^2}\left(\frac{\|B\|}{\theta \sqrt{T}}\right),
\]

(b)

Proof. a) Similar to Lemma D.2 the proof is based on a martingale difference decomposition of $R1_{B_0} - \mathbb{E}[R1_{B_0}]$. We first setup the necessary notations and carry out some preliminary computations.

Recall that the $k$-th row of $E_0$ is equal to $T^{-1/2}\epsilon_{K+k,[1:T-\tau]}^T$. For brevity of notation we will adopt the following notation

\[
\epsilon_{k0} := \epsilon_{K+k,[1:T-\tau]}, \quad \epsilon_{kr} := \epsilon_{K+k,[r+1:T]}.
\]  

(D.2)

Let $E_{k0}$ and $E_{kr}$ be the matrices $E_0$ and $E_\tau$ with the $k$-th row replaced by zeros, i.e. $E_{k0} := E_0 - T^{-1/2}e_k\epsilon_{k0}$ and $E_{kr} := E_0 - T^{-1/2}e_k\epsilon_{kr}$. Define

\[
R_k := (I_T - \frac{1}{\theta} E_{kr}^T E_{k0} E_{k0}^T E_0)^{-1}, \quad R_k := (I_T - \frac{1}{\theta} E_{kr}^T E_{k0} E_{k0}^T E_0)^{-1},
\]

where $R_k$ is not to be confused with $R_\theta$ and $R_\lambda$ defined previously. Then

\[
E_0^T E_0 - E_{k0}^T E_{k0} = \frac{1}{T} \epsilon_{k0}^T \epsilon_{k0}, \quad E_\tau^T E_{\tau} - E_{kr}^T E_{kr} = \frac{1}{T} \epsilon_{kr}^T \epsilon_{kr},
\]

from which we can compute

\[
R_{k}^{-1} - R_{k}^{-1} = \frac{1}{\theta} (E_\tau^t E_\tau - E_{kr}^t E_{k0}) E_{k0}^T E_0 = \frac{1}{\theta T} \epsilon_{k0}^T \epsilon_{k0} E_0^T E_0
\]

\[
R_{k}^{-1} - R_{k}^{-1} = \frac{1}{\theta} E_{kr}^T E_{k0} (E_0^T E_0 - E_{k0}^T E_{k0}) = \frac{1}{\theta T} \epsilon_{kr}^T \epsilon_{k0} E_0^T E_0
\]

We furthermore define scalars

\[
\beta_{k} = \frac{1}{1 + \text{tr}(R_k (R_{k}^{-1} - R_{k}^{-1}))} = \frac{1}{1 - \frac{1}{T} \epsilon_{k0}^T E_0^T E_0 R_k \epsilon_{k0}},
\]

\[
\beta_{k} = \frac{1}{1 + \text{tr}(R_k (R_{k}^{-1} - R_{k}^{-1}))} = \frac{1}{1 - \frac{1}{T} \epsilon_{k0}^T E_0^T E_0 R_k \epsilon_{k0}},
\]

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both of which are clearly of order $1 + o(1)$ under the event $\mathcal{B}_0$. Using (C.1) we get
\begin{align}
R - R_k &= -\beta_k R_k (R^{-1} - R_k^{-1}) R_k = \frac{\beta_k}{\theta T} R_k \epsilon_{kT} \epsilon_{kT}^T E_0^T E_0 R_k, \quad \text{(D.3a)} \\
R_k - R_k &= -\beta_k R_k (R_k^{-1} - R_k^{-1}) R_k = \frac{\beta_k}{\theta T} R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k. \quad \text{(D.3b)}
\end{align}
Substituting (D.3b) back into (D.3a) we get
\[ R - R_k = \frac{\beta_k}{\theta T} (R_k + \frac{\beta_k}{\theta T} R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k) \epsilon_{kT} \epsilon_{kT}^T E_0^T E_0 \left( R_k + \frac{\beta_k}{\theta T} R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k \right), \]
and so we have
\[ R - R_k = (R_k - R_k) + (R - R_k) =: U_1 + U_2 + U_3 + U_4 + U_5, \quad \text{(D.4)} \]
where we have defined
\begin{align}
U_1 &:= \frac{\beta_k}{\theta T} R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k, \quad U_2 := \frac{\beta_k}{\theta T} R_k \epsilon_{kT} \epsilon_{kT}^T E_0^T E_0 R_k, \\
U_3 &:= \frac{\beta_k}{\theta^2 T^2} R_k \epsilon_{kT} \epsilon_{kT}^T E_0^T E_0 R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k, \\
U_4 &:= \frac{\beta_k}{\theta^2 T^2} R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k \epsilon_{kT} \epsilon_{kT}^T E_0^T E_0 R_k, \\
U_5 &:= \frac{\beta_k^2}{\theta^3 T^3} R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k \epsilon_{kT} \epsilon_{kT}^T E_0^T E_0 R_k E_{kT}^T E_{kT} \epsilon_{k0} \epsilon_{k0}^T R_k.
\end{align}
Recall the event $\mathcal{B}_0$ from (2.1). Define
\[ \mathcal{B}_0^k := \left\{ \| E_{k0}^T E_{k0} \| + \| E_{kT}^T E_{kT} \| \leq 4 \left( 1 + \frac{p}{T} \right) \right\}, \quad k = 1, \ldots, p. \quad \text{(D.6)} \]
Clearly $\| E_{k0}^T E_{k0} \| \leq \| E_0^T E_0 \|$ which implies $\mathcal{B}_0 \subseteq \mathcal{B}_0^k$ and so $1_{\mathcal{B}_0} \leq 1_{\mathcal{B}_0^k}$. Recall the family of conditional expectations $\mathbb{E}_n[\cdot]$ defined in (D.1). Then
\begin{align}
\frac{1}{T} \text{tr}(B(R1_{\mathcal{B}_0} - \mathbb{E}[R1_{\mathcal{B}_0}])) &= \frac{1}{T} \sum_{k=1}^{p} (\mathbb{E}_k - \mathbb{E}_{k-1}) \text{tr}(BR_k) \\
&= \frac{1}{T} \sum_{k=1}^{p} (\mathbb{E}_k - \mathbb{E}_{k-1}) \left( \text{tr}(BR_k) - \text{tr}(BR_{k-1}) \right) \\
&= \frac{1}{T} \sum_{k=1}^{p} (\mathbb{E}_k - \mathbb{E}_{k-1}) \text{tr}(B(R - R_k)1_{\mathcal{B}_0}) - \frac{1}{T} \sum_{k=1}^{p} (\mathbb{E}_k - \mathbb{E}_{k-1}) \text{tr}(BR_k(1_{\mathcal{B}_0^k} - 1_{\mathcal{B}_0})) \\
&=: I_1 + I_2, \quad \text{(D.7)}
\end{align}
where the second equality holds since \( \mathbb{E}_k[\text{tr}(BR_k 1_{B_0^k})] = \mathbb{E}_{k-1}[\text{tr}(BR_k 1_{B_0^k})] \) and the third equality is purely algebraic computations. We first deal with the second term in (D.7). Using \( \text{tr}(BR_k) \leq p\|BR_k\| \) and \( \|BR_k 1_{B_0^k}\| = O(\|B\|) \) we have

\[
\mathbb{E}|I_2|^2 = \frac{1}{T^2} \sum_{k=1}^{p} \mathbb{E}\left[| (E_k - E_{k-1}) \text{tr}(BR_k(1_{B_0^k} - 1_{B_0}))|^2 \right] \leq \frac{4p^2}{T^2} \sum_{k=1}^{p} \mathbb{E}\left[ \|BR_k\|(1_{B_0^k} - 1_{B_0}) \right]^2
\]

\[
= O\left( \frac{p^2}{T^2} \|B\|^2 \right) \sum_{k=1}^{p} \mathbb{E}\left[ 1_{B_0^k} - 1_{B_0} \right]^2 = O\left( \frac{p^2}{T^2} \|B\|^2 \right) \sum_{k=1}^{p} \mathbb{P}(B_0^k) = o(T^{-1}\|B\|^2),
\]

for any \( l \in \mathbb{N} \) by Lemma 2.1. For the first term in (D.7), since \( I_1 \) is a sum of a martingale difference sequence, using (D.4) and \( B_0 \subseteq B_0^k \) we have

\[
\mathbb{E}|I_1|^2 \leq \frac{1}{T^2} \sum_{k=1}^{p} \mathbb{E}\left| (E_k - E_{k-1}) \text{tr}(B(R - R_k)1_{B_0^k}) \right|^2
\]

\[
\leq \frac{4}{T^2} \sum_{k=1}^{p} \mathbb{E}\left| \text{tr}(B(R - R_k)1_{B_0^k}) \right|^2 \leq \frac{20}{T^2} \sum_{k=1}^{p} \sum_{n=1}^{5} \mathbb{E}\left| \text{tr}(BU_n 1_{B_0^k}) \right|^2,
\]

and it remains to bound the second moment of each \( \text{tr}(BU_n 1_{B_0^k}) \). Since \( \{\epsilon_{kt}\} \) are assumed to be i.i.d. standard Gaussian, we have the following moment estimate

\[
\mathbb{E}\left[ \|\epsilon_{k0}\|^n \right] = \mathbb{E}\left[ \left( \sum_{t=1}^{T-\tau} \epsilon_{kt}^2 \right)^{n/2} \right] \lesssim (T - \tau)^{n/2-1} \sum_{t=1}^{T-\tau} \mathbb{E}|\epsilon_{kt}|^n = O(T^{n/2}). \tag{D.8}
\]

Using \( \beta_k 1_{B_0^k} = 1 + o(1) \) and the trivial inequality \( x^\top Ax \leq \|x\|^2\|A\| \) we obtain

\[
\mathbb{E}\left| \text{tr}(BU_1 1_{B_0^k}) \right|^2 \lesssim \frac{1}{\theta^2 T^2} \mathbb{E}\left[ (\epsilon_{k0}^\top R_k B R_k E_{k\tau}^\top E_{k\tau} \epsilon_{k0})^2 1_{B_0^k} \right] \tag{D.9a}
\]

\[
\leq \frac{1}{\theta^2 T^2} \mathbb{E}\left[ \|\epsilon_{k0}\|^4\|R_k\|^4\|E_{k\tau}^\top E_{k\tau}\|^2 1_{B_0^k} \right] \|B\|^2 \lesssim \frac{1}{\theta^2} \|B\|^2.
\]

The second term \( U_2 \) can be dealt in exactly the same way to obtain

\[
\mathbb{E}\left| \text{tr}(BU_2 1_{B_0^k}) \right|^2 \lesssim \frac{1}{\theta^2} \|B\|^2, \tag{D.9b}
\]

and we omit the details. For \( U_3 \), similar computations gives

\[
\mathbb{E}\left| \text{tr}(BU_3 1_{B_0^k}) \right|^2 \lesssim \frac{1}{\theta^4 T^4} \mathbb{E}\left[ (\epsilon_{k0}^\top R_k B R_k \epsilon_{k\tau}^\top E_{0}^\top E_{0} R_k E_{k\tau}^\top E_{k\tau} \epsilon_{k0})^2 1_{B_0^k} \right]
\]

\[
\leq \frac{1}{\theta^4 T^4} \mathbb{E}\left[ \|\epsilon_{k0}\|^4\|\epsilon_{k\tau}\|^4\|R_k\|^4\|E_{0}^\top E_{0} R_k E_{k\tau}^\top E_{k\tau}\|^2 1_{B_0^k} \right] \|B\|^2,
\]

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since $x^\top y \leq \|x\|\|y\|\|A\|$. Therefore
\[
\mathbb{E}\left|\text{tr}(BU_3)\right|^2 \lesssim \frac{1}{\theta^4 T^4} \mathbb{E}\left[\|\varepsilon_{k0}\|^8\right]^{1/2} \mathbb{E}\left[\|\varepsilon_{kT}\|^8\right]^{1/2} \lesssim \frac{1}{\theta^4} \|B\|^2.
\] (D.9c)

Once again $U_4$ can be bounded in the same way to obtain
\[
\mathbb{E}\left|\text{tr}(BU_4)1_{B_0^k}\right|^2 \leq \frac{1}{\theta^4} \|B\|^2.
\] (D.9d)

With the same approach but more laborious computations we can obtain
\[
\mathbb{E}\left|\text{tr}(BU_5)1_{B_0^k}\right|^2 \lesssim \frac{1}{\theta^6} \|B\|^2.
\] (D.9e)

Note that the estimates (D.9a)-(D.9e) are uniform in $k = 1, \ldots, p$. We then conclude
\[
\mathbb{E}\left|\frac{1}{T}\text{tr}(B(R1_{B_0^k} - \mathbb{E}[R1_{B_0^k}]))\right|^2 = O\left(\frac{p}{T^2 \theta^2}\right) \|B\|^2 = O\left(\frac{1}{T \theta^2}\right) \|B\|^2,
\]
and the conclusion follows.

b) Similar to (a), via a martingale difference decomposition we obtain
\[
\mathbb{E}\left|\frac{1}{T}\text{tr}(B(E_0^T E_0 R1_{B_0^k} - \mathbb{E}[E_0^T E_0 R1_{B_0^k}]))\right|^2 \lesssim \frac{1}{T^2} \sum_{k=1}^{p} \mathbb{E}\left|\text{tr}(B(E_0^T E_0 R - E_{k0}^T E_{k0} R_k))1_{B_0^k}\right|^2,
\]
where, recalling the $U_n$’s defined in the proof of (a), we have
\[
E_0^T E_0 R - E_{k0}^T E_{k0} R_k = \frac{1}{T} \varepsilon_{k0}^\top \varepsilon_{k0} R_k + \frac{1}{T} \varepsilon_{k0}^\top \varepsilon_{k0} (R - R_k) + E_{k0}^T E_{k0} (R - R_k)
\] (D.10)
\[
= \frac{1}{T} \varepsilon_{k0}^\top \varepsilon_{k0} R_k + \frac{1}{T} \sum_{n=1}^{5} \varepsilon_{k0}^\top \varepsilon_{k0} U_n + \sum_{n=1}^{5} E_{k0}^\top E_{k0} U_n.
\]

We deal with the first two term in (D.10) to illustrate the ideas of the proof, the other terms can be dealt with similarly. Using (D.8) and $p \asymp T$, clearly we have
\[
\frac{1}{T^2} \sum_{k=1}^{p} \mathbb{E}\left|\frac{1}{T}\text{tr}(B\varepsilon_{k0}^\top \varepsilon_{k0} R_k)1_{B_0^k}\right|^2 \lesssim \frac{1}{T^4} \sum_{k=1}^{p} T^2 \mathbb{E}[\|BR_k1_{B_0^k}\|^2] = O\left(\frac{1}{T}\right) \|B\|^2.
\] (D.11)

Similar to the computations in (D.9a), we can get
\[
\mathbb{E}\left|\frac{1}{T}\text{tr}(B(\varepsilon_{k0}^\top U_1))1_{B_0^k}\right|^2 \lesssim \frac{1}{\theta^2 T^2} \frac{1}{T^2} \mathbb{E}[\varepsilon_{k0}^\top R_k B \varepsilon_{k0}^\top \varepsilon_{k0} R_k E_{kT}^\top E_{kT}^\top \varepsilon_{k0}^\top 1_{B_0^k}^2] \\
\quad \leq \frac{1}{\theta^2 T^4} \mathbb{E}\left[\|\varepsilon_{k0}\|^8 \|R_k\|^4 \|E_{kT}^\top E_{kT}\|^2 1_{B_0^k}\right] \|B\|^2 \lesssim \frac{1}{\theta^2} \|B\|^2,
\]

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which immediately gives

\[
\frac{1}{T^2} \sum_{k=1}^{p} \mathbb{E} \left[ \frac{1}{T} \text{tr}(B(e_k e_0^\top U_1))1_B (e_k e_0^\top U_1) \right]^2 = O \left( \frac{p}{\theta^2 T^2} \right) \|B\|^2 = O \left( \frac{1}{\theta^2 T} \right) \|B\|^2.
\]

Note that this term is negligible in comparison to (D.11). Using the same ideas, it is routine to check that the other 9 terms in (D.10) are negligible as well, and we omit the details. The bound therefore follows from (D.11).

Next recall that

\[
Q = I_K - \frac{1}{\theta} X_0 R E_r^\top E_r X_0^\top.
\]

We now state a concentration inequality for entries of the matrix $Q^{-1}$, under the event $B_2$.

**Lemma D.2.** Write $Q_{ij}^{-1} := (Q^{-1})_{ij}$. Then

a) For all $k = 1, \ldots, K$, we have

\[
Q_{kk}^{-1} 1_{B_2} - \mathbb{E}[Q_{kk}^{-1} 1_{B_2}] = o_{L^1} \left( \frac{1}{\sqrt{T}} \right).
\]

b) The off-diagonal elements of $Q^{-1}$ satisfies

\[
Q_{ij}^{-1} 1_{B_2} = O_{L^2} \left( \frac{1}{\gamma_1(\tau)^2 \sigma_i^2 \sqrt{T}} \right)
\]

uniformly in $i, j = 1, \ldots, K$, $i \neq j$.

**Proof.** a) Recalling the event $B_2$, we note that the matrix $Q$ is invertible with probability tending to 1. The proof relies on expressing $Q_{kk}^{-1} 1_{B_2} - \mathbb{E}[Q_{kk}^{-1} 1_{B_2}]$ as a sum of martingale differences. We first setup the notations necessary.

Let $T^{-1/2} x_i := T^{-1/2} x_{i:[1:T-\tau]}$ be the (column vector) of the $i$-th row of $X_0$, i.e. we can write $X_0 = T^{-1/2} \sum_{i=1}^{K} e_i x_i^\top$. Define $X_{i0} := X_0 - \frac{1}{\sqrt{T}} e_i x_i^\top$, and

\[
Q_{(i)} := I_K - \frac{1}{\theta} X_{i0} R E_r^\top E_r X_{i0}^\top,
\]

from which we can immediately compute

\[
Q - Q_{(i)} = -\frac{1}{\theta \sqrt{T}} e_i x_i^\top R E_r^\top E_r X_0^\top,
\]

\[
Q_{(i)} - Q_{(ii)} = -\frac{1}{\theta \sqrt{T}} X_{i0} R E_r^\top E_r x_i x_i^\top.
\]

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Note that all elements on the $i$-th row of $Q(i)$ are equal to zero except for the diagonal which is equal to 1, i.e. $Q(i)$ is equal to the identity when restricted to the $i$-th coordinate. Then the inverse $Q^{-1}(i)$, whenever it exists, must also equal to the identity when restricted to the $i$-th coordinate. A similar observation can be made for the matrix $Q(ii)$ and it is not hard to observe that

$$e_i^\top (Q(ii)^{-1})e_i = 1, \quad e_i^\top (Q(i)^{-1})e_j = 0, \quad \forall j \neq i,$$

$$e_j^\top (Q(ii)^{-1})e_j = e_j^\top (Q(i)^{-1})e_i = 0, \quad \forall j \neq i.$$

To compute the difference $Q^{-1} - Q^{-1}(i)$, which will turn out to be the central focus of the proof, we first define the following scalars

$$b_i := \frac{1}{1 + \text{tr}(Q^{-1}(i)(Q - Q(i)))} = \frac{1}{1 - \frac{1}{\theta \sqrt{T}} x_i^\top RE_\tau^\top E_\tau x_i Q^{-1}(i) e_i}$$

$$b_{ii} := \frac{1}{1 + \text{tr}(Q^{-1}(ii)(Q(i) - Q(ii)))} = \frac{1}{1 - \frac{1}{\theta \sqrt{T}} e_i^\top Q^{-1}(ii) X_0 RE_\tau^\top E_\tau x_i} = 1,$$

where the last equality holds by (D.12). Then using the identity (C.1) we have

$$Q^{-1} - Q^{-1}(i) = \frac{b_i}{\theta \sqrt{T}} Q^{-1}(i) e_i x_i^\top RE_\tau^\top E_\tau x_i Q^{-1}(i),$$

$$Q^{-1}(i) - Q^{-1}(ii) = \frac{1}{\theta \sqrt{T}} Q^{-1}(ii) X_0 RE_\tau^\top E_\tau x_i e_i^\top Q^{-1}(ii).$$

We observe that the matrices $Q^{-1}(i)$ and $Q^{-1}(ii)$ differ only on off-diagonal elements on the $i$-th column. Indeed, from (D.12) and (D.14b), if $n \neq i$ or if $n = m = i$ then

$$e_m^\top (Q^{-1}(i) - Q^{-1}(ii)) e_n = \frac{1}{\theta \sqrt{T}} e_m^\top Q^{-1}(i) X_0 RE_\tau^\top E_\tau x_i e_i^\top Q^{-1}(ii) e_n = 0.$$
Then, substituting (D.14b) back into (D.14a) we obtain

\[ e_k^\top (Q^{-1} - Q_{(i)}^{-1}) e_k = \frac{b_i}{\theta \sqrt{T}} e_k^\top Q_{(i)}^{-1} e_k x_i^\top R E_{\tau}^\top E_{\tau} X_0^\top Q_{(ii)}^{-1} e_k \]

\[ = \frac{b_i}{\theta \sqrt{T}} e_k^\top Q_{(i)}^{-1} e_k x_i^\top R E_{\tau}^\top E_{\tau} X_0^\top Q_{(ii)}^{-1} e_k + \frac{b_i}{\theta^2 T} e_k^\top Q_{(i)}^{-1} e_k x_i^\top R E_{\tau}^\top E_{\tau} X_0^\top Q_{(ii)}^{-1} x_{(i)0} R E_{\tau}^\top E_{\tau} x_i e_i^\top Q_{(ii)}^{-1} e_k \]

\[ = \frac{b_i}{\theta \sqrt{T}} e_k^\top Q_{(i)}^{-1} e_k x_i^\top R E_{\tau}^\top E_{\tau} X_0^\top Q_{(ii)}^{-1} e_k + \frac{b_i}{\theta^2 T} e_k^\top Q_{(i)}^{-1} e_k x_i^\top R E_{\tau}^\top E_{\tau} x_{(i)0} R E_{\tau}^\top E_{\tau} x_i e_i^\top Q_{(ii)}^{-1} e_k \]

\[ + \frac{b_i}{\theta^2 T} e_k^\top Q_{(i)}^{-1} e_k x_{(i)0} R E_{\tau}^\top E_{\tau} x_i e_i^\top Q_{(ii)}^{-1} e_k R E_{\tau}^\top E_{\tau} X_0^\top Q_{(ii)}^{-1} e_k \]

\[ =: I_1 + I_2 + I_3. \quad \text{(D.16)} \]

To simplify this expression further, define the following quadratic forms

\[ \xi_i := \frac{1}{\theta T} x_i^\top R E_{\tau}^\top E_{\tau} x_i, \quad \eta_i := \frac{1}{\theta^2 T} x_i^\top R E_{\tau}^\top E_{\tau} x_{(i)0}^\top Q_{(ii)}^{-1} x_{(i)0} R E_{\tau}^\top E_{\tau} x_i, \]

\[ \zeta_{ik} := \frac{1}{\theta^2 T} x_i^\top R E_{\tau}^\top E_{\tau} x_i^\top Q_{(ii)}^{-1} e_k e_k^\top Q_{(ii)}^{-1} x_{(i)0} R E_{\tau}^\top E_{\tau} x_i, \quad \text{(D.17)} \]

then using (D.12), we can easily write \( I_1, I_2 \) and \( I_3 \) into

\[ I_1 = 1_{i=k} \frac{b_k}{\theta \sqrt{T}} x_k^\top R E_{\tau}^\top E_{\tau} X_0^\top Q_{(kk)}^{-1} e_k = 1_{i=k} b_k \xi_k, \quad \text{(D.18)} \]

\[ I_2 = 1_{i=k} \frac{b_k}{\theta^2 T} x_k^\top R E_{\tau}^\top E_{\tau} x_i^\top Q_{(ii)}^{-1} x_{(i)0} R E_{\tau}^\top E_{\tau} x_k = 1_{i=k} b_k \eta_k, \]

\[ I_3 = 1_{i \neq k} \frac{b_i}{\theta^2 T} e_k^\top Q_{(ii)}^{-1} x_{(i)0} R E_{\tau}^\top E_{\tau} x_i e_i^\top Q_{(ii)}^{-1} x_{(i)0} R E_{\tau}^\top E_{\tau} x_k = 1_{i \neq k} b_i \zeta_{ik}. \]

We first state some estimates on \( \xi \) and \( \eta \) under the appropriate events. Recall from (2.18) the event \( B_1 := \left\{ \|X_0^\top X_0\| \leq 2 \sum_{i=1}^{K} \sigma_i^2 \right\} \). Define the event

\[ B_{1i} := \left\{ \|X_{(i)0}^\top X_{(i)0}\| \leq 2 \sum_{i=1}^{K} \sigma_i^2 \right\}, \quad i = 1, \ldots, K \quad \text{(D.19)} \]

and write \( B_2 := B_0 \cap B_1 \). Then clearly \( B_2 \subseteq B_2 \). Define

\[ \bar{\xi}_i := \frac{1}{\theta T} \text{tr}(\Psi_i^0(R E_{\tau}^\top E_{\tau})), \quad \bar{\eta}_i := \frac{1}{\theta^2 T} \text{tr}(\Psi_i^0(R E_{\tau}^\top E_{\tau} X_{(i)0}^\top Q_{(ii)}^{-1} X_{(i)0} R E_{\tau}^\top E_{\tau})), \]

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where $\Psi_0^{ii}$ is defined in (C.4). Write
\[
\xi_i := \xi_i - \bar{\xi}_i, \quad \eta_i := \eta_i - \bar{\eta}_i. \tag{D.20}
\]
Using Lemma C.4 and taking iterated expectations we have
\[
\mathbb{E}[\xi_i^2 1_{B_2}] = \mathbb{E}[\mathbb{E}[\xi_i^2 1_{B_2}]] = \frac{1}{\theta^2 T} O(\sigma_i^4 T) \mathbb{E}[\|RE_{\tau} E_1 1_{B_2}\|^2] = O\left(\frac{\sigma_i^4}{\theta^2 T}\right),
\]
\[
\mathbb{E}[\eta_i^2 1_{B_2}] = \frac{1}{\theta^4 T^2} O(\sigma_i^4 T) \mathbb{E}[\|RE_{\tau} E_{\tau} X_{i0}^T Q_{(ii)}^{-1} X_{i0} RE_{\tau} E_{\tau} 1_{B_2}\|^2] = O\left(\frac{\sigma_i^4 \sum_{j=1}^{K} \sigma_j^4}{\theta^4 T}\right).
\]
Using $\theta = \theta_k \sim \sigma_k^4 \gamma_k(\tau)^2$ from Proposition 3.2, Lemma C.2 to deal with $\Psi_0^{ii}$ and Assumptions 2.1 to compare the different speeds, we conclude
\[
\mathbb{E}[\xi_i^2 1_{B_2}] = O\left(\frac{1}{\gamma_1(\tau)^2 \theta T}\right), \quad \mathbb{E}[\eta_i^2 1_{B_2}] = O\left(\frac{K}{\gamma_1(\tau)^4 \theta^2 T}\right), \tag{D.21}
\]
\[
\xi_i 1_{B_2} = O(\sigma_i^2 \theta^{-1}), \quad \eta_i 1_{B_2} = O(K \sigma_i^4 \theta^{-2}).
\]
We then consider the scalar $b_i$ defined in (D.13). From (D.12) and (D.14b) we observe
\[
\frac{1}{\theta \sqrt{T}} x_i^T R E_{\tau} E_{\tau} X_{i0}^T Q_{(i)}^{-1} e_i = \frac{1}{\theta \sqrt{T}} x_i^T R E_{\tau} E_{\tau} X_{i0}^T Q_{(i)}^{-1} e_i
\]
\[
+ \frac{1}{\theta^2 T} x_i^T R E_{\tau} E_{\tau} X_{i0}^T Q_{(i)}^{-1} X_{i0} RE_{\tau} E_{\tau} x_i^T Q_{(ii)}^{-1} e_i = \xi_i + \eta_i.
\]
Substituting back into (D.13) we can simplify to obtain
\[
b_i = (1 - \xi_i - \eta_i)^{-1}. \tag{D.22}
\]
Define $\bar{b}_i = (1 - \xi_i - \eta_i)^{-1}$ so that subtracting the two we get
\[
b_i = (1 - \xi_i - \eta_i)^{-1} = \bar{b}_i - b_i (\xi_i + \eta_i). \tag{D.23}
\]
Finally, from the expression (D.22) and the bounds (D.21) we clearly have
\[
b_i 1_{B_2} = 1 + o(1), \quad \bar{b}_i 1_{B_2} = 1 + o(1). \tag{D.24}
\]
We can now carry out the main idea of the proof. Recall notations $\mathbb{E}[\cdot]$ and $\mathbb{E}^i[\cdot]$ from (D.1). By definition of $Q_{(ii)}$ and $B_2^i$ we have
\[
e^T_k (Q^{-1} 1_{B_2} - \mathbb{E}[Q^{-1} 1_{B_2}]) e_k = \sum_{i=1}^{K} \left(\mathbb{E}_i - \mathbb{E}_{i-1}\right) \left(e^T_k Q^{-1} 1_{B_2} e_k - e^T_k Q_{(ii)}^{-1} 1_{B_2^i} e_k\right)
\]
\[
= \sum_{i=1}^{K} \left(\mathbb{E}_i - \mathbb{E}_{i-1}\right) \left(e^T_k Q^{-1} 1_{B_2} e_k - e^T_k Q_{(ii)}^{-1} 1_{B_2^i} e_k\right),
\]

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where the last equality follows from (D.15). Similar to how we dealt with the second term in (D.7) in the proof of Lemma D.1, using Lemma 2.1 we may obtain
\[
e_k^\top (Q^{-1}1_{B_2} - \mathbb{E}[Q^{-1}1_{B_2}]) e_k = \sum_{i=1}^{K} (\mathbb{E}_i - \mathbb{E}_{i-1}) e_k \left( Q^{-1} - Q^{-1}_i \right) 1_{B_2} e_k + o_{L^2} \left( \frac{1}{\sqrt{T}} \right)
\]
\[
= \sum_{i=1}^{K} (\mathbb{E}_i - \mathbb{E}_{i-1})(I_1 + I_2 + I_3)1_{B_2} + o_{L^2} \left( \frac{1}{\sqrt{T}} \right),
\]
(D.25)

where the second equality holds by (D.15).

As will be shown, the term involving $I_1$ is the leading term of (D.25), this is what we consider now. Using the identity (D.18) we simply have
\[
\sum_{i=1}^{K} (\mathbb{E}_i - \mathbb{E}_{i-1})I_11_{B_2^i} = (\mathbb{E}_k - \mathbb{E}_{k-1})b_k \xi_k 1_{B_2},
\]
which, recalling (D.20) and using (D.23), can be written into
\[
(\mathbb{E}_k - \mathbb{E}_{k-1})b_k \xi_k 1_{B_2} = (\mathbb{E}_k - \mathbb{E}_{k-1}) \left( \bar{b}_k - b_k \bar{b}_k(\xi_k + \eta_k) \right) \left( \bar{\xi}_k + \xi_k \right) 1_{B_2}
\]
\[
= (\mathbb{E}_k - \mathbb{E}_{k-1}) \left[ b_k \xi_k + b_k \bar{b}_k(\xi_k + \eta_k) \bar{\xi}_k + b_k \xi_k(\xi_k + \eta_k) \right] 1_{B_2}.
\]
(D.26)

We consider the three terms in the square bracket in (D.26) separately. For the first term, we note that $(\mathbb{E}_k - \mathbb{E}_{k-1})\bar{b}_k \xi_k 1_{B_2^i} = 0$ by definition of $\bar{b}_k \xi_k$ and $B_2^i$. Using this, we have

\[
(\mathbb{E}_k - \mathbb{E}_{k-1})\bar{b}_k \xi_k 1_{B_2} = 0 - (\mathbb{E}_k - \mathbb{E}_{k-1})\bar{b}_k \xi_k 1_{B_2}.
\]

Recalling (D.21) and (D.24) and using Assumptions 2.1 we have
\[
\mathbb{E}|(\mathbb{E}_k - \mathbb{E}_{k-1})\bar{b}_k \xi_k 1_{B_2}| \leq 2\mathbb{E}|\bar{b}_k \xi_k 1_{B_2} - 1_{B_2}| = O(\sigma_i^2 \theta^{-1})\mathbb{E}|1_{B_2} - 1_{B_2}| = o(T^{-1}),
\]
where the last equality follows from the fact that $B_2 \subseteq B_2^k$ and Lemma 2.1. For the second term in (D.26), using $B_2 \subseteq B_2^k$, (D.24) and (D.21) we have
\[
\mathbb{E}|(\mathbb{E}_k - \mathbb{E}_{k-1})\bar{b}_k \xi_k 1_{B_2}|^2 \lesssim 4\mathbb{E}|\bar{b}_k \xi_k 1_{B_2}|^2 = o(T^{-1/2}).
\]

Similarly the third term of (D.26) is bounded by
\[
\mathbb{E}|(\mathbb{E}_k - \mathbb{E}_{k-1})[b_k \bar{b}_k(\xi_k + \eta_k)(\bar{\xi}_k + \xi_k)1_{B_2}]| \lesssim 2\mathbb{E}|(\xi_k + \eta_k)(\bar{\xi}_k + \xi_k)1_{B_2}|.
\]
Expanding, applying the Cauchy-Schwarz inequality and using (D.24) and (D.21), we may obtain a bound of order \( o_{L^1}(T^{-1/2}) \); we omit the repetitive details. Substituting the above bounds back into equation (D.26) we obtain
\[
\mathbb{E} \left| \sum_{i=1}^{K} (\mathbb{E}_i - \mathbb{E}_{i-1}) I_i \right| = \mathbb{E} \left| (\mathbb{E}_K - \mathbb{E}_{K-1})b_k\xi_k \right| = o \left( \frac{1}{\sqrt{T}} \right).
\]
The cases of \( I_2 \) and \( I_3 \) can be dealt with with similar approaches and we omit the details.

In fact, from the definitions in (D.18) it is not difficult to see that \( \eta \) and \( \zeta \) are higher order terms relative to \( \xi \) under the event \( B_2 \). It can therefore be shown that the term involving \( I_1 \) is the leading term in (D.25) and the claim follows.

b) Define \( \overline{Q} := I_T - \theta^{-1}X_0^\top X_0RE_r^\top E_r \) so that similar to (2.17) we have
\[
Q^{-1} - I_K = \frac{1}{\theta}X_0RE_r^\top E_rX_0^\top(I_K - \theta^{-1}X_0RE_r^\top E_rX_0^\top)^{-1} = \frac{1}{\theta}X_0RE_r^\top E_r\overline{Q}^{-1}X_0^\top.
\]
Recall that we have \( X_0^\top X_0 = T^{-1}\sum_{i=1}^{K} x_ix_i^\top \), define the matrices
\[
\overline{Q}(j) := I_T - \frac{1}{\theta T} \sum_{k\neq j} x_k x_k^\top RE_r^\top E_r, \quad \overline{Q}(ij) := I_T - \frac{1}{\theta T} \sum_{k\neq i,j} x_k x_k^\top RE_r^\top E_r,
\]
so that \( \overline{Q} - \overline{Q}(j) = -\frac{1}{\theta T} x_j x_j^\top RE_r^\top E_r \) and \( \overline{Q}(j) - \overline{Q}(ij) = -\frac{1}{\theta T} x_j x_j^\top RE_r^\top E_r \). Let
\[
a_j := \frac{1}{1 + \text{tr}(\overline{Q}(j)(\overline{Q} - \overline{Q}(j)))} = \frac{1}{1 - \frac{1}{\theta T} x_j x_j^\top RE_r^\top E_r \overline{Q}^{-1}(j)x_j},
\]
\[
a_{ij} := \frac{1}{1 + \text{tr}(\overline{Q}^{-1}(ij)(\overline{Q}(j) - \overline{Q}(ij)))} = \frac{1}{1 - \frac{1}{\theta T} x_i x_i^\top RE_r^\top E_r \overline{Q}^{-1}(ij)x_i},
\]
then by (C.2) we have
\[
\overline{Q}^{-1} x_j = a_j \overline{Q}^{-1}(j)x_j, \quad x_i^\top RE_r^\top E_r \overline{Q}^{-1}(ij) = a_{ij} x_i^\top RE_r^\top E_r \overline{Q}^{-1}(ij).
\]
We can therefore write
\[
Q_{ij}^{-1} = \frac{1}{\theta T} x_i^\top RE_r^\top E_r \overline{Q}^{-1} x_j = \frac{a_j a_{ij}}{\theta T} x_i^\top RE_r^\top E_r \overline{Q}^{-1}(ij) x_j.
\]
Now define \( X_{ij0} := X_0 - T^{-1/2}(e_i x_i^\top + e_j x_j) \) and events \( B^{ij}_2 \) and \( B^{ij}_3 \) analogous to (D.19) with \( X_{i0} \) replaced by \( X_{ij0} \). Similar to (a) of the Lemma we have \( a_j = 1 + o(1) \) and \( a_{ij} = 1 + o(1) \) under the event \( B_2 \). Therefore we have
\[
\mathbb{E} \left| Q_{ij}^{-1} 1_{B_2} \right|^2 \lesssim \frac{1}{\theta^2 T^2} \mathbb{E} \left| x_i^\top RE_r^\top E_r \overline{Q}^{-1}(ij) x_j 1_{B_2} \right|^2 \lesssim \frac{1}{\theta^2 T^2} \mathbb{E} \left| x_i^\top RE_r^\top E_r \overline{Q}^{-1}(ij) x_j 1_{B_3} \right|^2.
\]
By Lemma C.4 and Assumptions 2.1 we have
\[ E[\|Q^{-1}_i B_2\|^2] = \frac{1}{\theta^2 T^2} O(\sigma_i^2 \sigma_j^2 T) = O\left(\frac{1}{\gamma_1^4 \sigma_i^4 T}\right). \]

Finally we remark that the uniformity of this bound in \(i, j = 1, \ldots, K\) should be obvious from the proof since all \(\sigma_i\)'s are of the same order.

We finally show that the conditional expectations of diagonal elements of \(A, B\) and \(Q^{-1}\), defined in (2.15), are sufficiently close to the unconditional expectation.

**Lemma D.3.** For each \(i = 1, \ldots, K\), we have
\[ E[A_{ii} 1_{B_0}] - E[A_{ii} 1_{B_0}] = O_L^2 \left(\frac{1}{\sqrt{T}}\right), \quad E[B_{ii} 1_{B_0}] - E[B_{ii} 1_{B_0}] = O_L^2 \left(\frac{1}{\sqrt{T}}\right), \]
\[ E[Q_{ii}^{-1} 1_{B_2}] - E[Q_{ii}^{-1} 1_{B_2}] = o_L \left(\frac{1}{\sqrt{T}}\right). \]

**Proof.** From (a) of Lemma C.4 we recall that
\[ E[A_{ii} 1_{B_0}] = \frac{1}{\sqrt{T}} E[\|x_{i,[1:T-\tau]}^\top R 1_{B_0} x_{i,[\tau+1:T]}\|] = \frac{1}{\sqrt{T}} \text{tr}(\Psi_{ii}^T(R)) 1_{B_0} \] (D.27)
where, using (C.4) and the cyclic property of the trace, we have
\[ \text{tr}(\Psi_{ii}^T(R)) = \text{tr}\left((0, I_{T-\tau})(\sigma_i^2 \Phi_i^\top \Phi_i + I_T)(I_{T-\tau}, 0)^\top R\right) =: \text{tr}(GR). \] (D.28)

Furthermore, using (a) of Lemma C.2, we see that
\[ G := (0, I_{T-\tau})(\sigma_i^2 \Phi_i^\top \Phi_i + I_T)(I_{T-\tau}, 0)^\top = O_{\|\cdot\|}(\sigma_i^2). \] (D.29)

From (D.27) we have \(E[A_{ii} 1_{B_0}] = E[E[A_{ii} 1_{B_0}]] = \frac{1}{\sqrt{T}} E[\text{tr}(\Psi_{ii}^T(R)) 1_{B_0}]\) and so
\[ E[A_{ii} 1_{B_0}] - E[A_{ii} 1_{B_0}] = \frac{1}{\sqrt{T}} \left(\text{tr}(\Psi_{ii}^T(R)) 1_{B_0} - E[\text{tr}(\Psi_{ii}^T(R)) 1_{B_0}]\right) \]
\[ = \frac{1}{\sqrt{T}} \left(\text{tr}(GR) 1_{B_0} - E[\text{tr}(GR) 1_{B_0}]\right) = \frac{1}{\sqrt{T}} \text{tr}\left(G(R 1_{B_0} - E[R 1_{B_0}])\right), \]
by linearity of the expectation and the trace. By (a) of Lemma D.1 we have
\[ E[A_{ii} 1_{B_0}] - E[A_{ii} 1_{B_0}] = \frac{1}{\sqrt{T}} O_L^2 \left(\frac{\|G\|}{\theta \sqrt{T}}\right) = O_L^2 \left(\frac{1}{\sqrt{T}}\right), \]
where the last equality follows from (D.29) and Assumption 2.1. For the case of $B$, similar computations and (b) of Lemma D.1 give

$$\mathbb{E}[B_{ii}1_{B_0}] - \mathbb{E}[B_{ii}1_{B_0}] = \frac{1}{\theta T} \left( \text{tr}(\Psi^{ii}_1(E_0^T E_0 R))1_{B_0} - \mathbb{E}[\text{tr}(\Psi^{ii}_1(E_0^T E_0 R))1_{B_0}] \right)$$

$$= \frac{1}{\theta T} \text{tr} \left( G(E_0^T E_0 R 1_{B_0}) - \mathbb{E}[E_0^T E_0 R 1_{B_0}] \right) = \frac{1}{\theta} O_L \left( \frac{\|G\|}{\sqrt{T}} \right) \approx O_L \left( \frac{1}{\sqrt{\theta T}} \right).$$

It remains to consider $\mathbb{E}[Q_{ii}^{-1}]$. We recall from (2.17) that

$$Q := I_K - \frac{1}{\theta} X_0 RE_r^T E_r X_0^T.$$

(D.30)

The strategy of the proof, similar to that of Lemma D.1 and Lemma D.2, is express $\mathbb{E}[Q_{ii}^{-1}1_{B_2}] - \mathbb{E}[Q_{ii}^{-1}1_{B_2}]$ into a sum of martingale difference sequence. We first introduce the necessary notations and carry out some algebraic computations.

Similar to (D.2), we will define $\epsilon_{k0} := \epsilon_{K+k,\lfloor 1:T-\tau \rfloor}$ and $\epsilon_{k\tau} := \epsilon_{K+k,\lfloor \tau+1:T \rfloor}$. Recall from (D.4) that $R - R_k = \sum_{n=1}^5 U_n$, where the $U_n$’s are defined in (D.5). Similar to the computations in (D.10), we may obtain

$$RE_r^T E_r - R_k E_k^T E_k = \frac{1}{T} R_k \epsilon_{k\tau} \epsilon_{k\tau}^T + \sum_{n=1}^5 U_n E_n^T E_n =: V + W,$$

where we defined

$$V := \frac{1}{T} R_k \epsilon_{k\tau} \epsilon_{k\tau}^T + (U_2 + U_3) E_r^T E_r,$$

$$W := (U_1 + U_4 + U_5) E_r^T E_r.$$

Define matrices $V_1, V_2, V_3, W_1, W_2, W_3$ by,

$$V_1 := I_T, \quad V_2 := \frac{\beta_k}{\theta} E_0^T E_0 R_k E_r^T E_r, \quad V_3 := \frac{\beta_k}{\theta^2 T^2} E_0^T E_0 R_k E_{k\tau}^T E_k \epsilon_{k0}^\top \epsilon_{k0} E_k^T E_r$$

$$W_1 := \beta_k R_k E_r^T E_r, \quad W_2 := \frac{\beta_k}{\theta^2 T^2} R_k \epsilon_{k\tau} \epsilon_{k\tau}^T E_0^T E_0 R_k E_r^T E_r,$$

$$W_3 := \frac{\beta_k^2}{\theta^4 T^4} R_k \epsilon_{k\tau} \epsilon_{k\tau}^T E_0^T E_0 R_k E_{k\tau}^T E_k \epsilon_{k0}^\top \epsilon_{k0} R_k E_r^T E_r,$$

so that using (D.5) we can decompose $V$ and $W$ into

$$V = \frac{1}{T} R_k \epsilon_{k\tau} \epsilon_{k\tau}^T (V_1 + V_2 + V_3) \quad (D.31a)$$

$$W = \frac{1}{\theta T} R_k E_r^T E_r \epsilon_{k0} \epsilon_{k0}^T (W_1 + W_2 + W_3). \quad (D.31b)$$

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It is clear that $V$ and $W$ are matrices of rank one. We define
\[
Q_{(k)} := I_K - \frac{1}{\theta} X_0 (R E^\top \tau E^\tau - V) X_0 \top,
\]
\[
Q_{(kk)} := I_K - \frac{1}{\theta} X_0 (R E^\top \tau E^\tau - V - W) X_0 \top,
\]
then from (D.30) we can write $Q - Q_{(k)} = -\theta^{-1} X_0 V X_0 \top$ and $Q - Q_{(kk)} = -\theta^{-1} X_0 W X_0 \top$.

Define the following scalars quantities
\[
\alpha_k := \frac{1}{1 - \theta^{-1} \text{tr}(Q_{(k)}^{-1} X_0 V X_0 \top)}; \quad \alpha_{kk} := \frac{1}{1 - \theta^{-1} \text{tr}(Q_{(kk)}^{-1} X_0 W X_0 \top)},
\]
then using (C.1) we obtain
\[
Q^{-1} = Q_{(k)}^{-1} + \frac{1}{\theta} Q_{(k)}^{-1} X_0 V X_0 \top Q_{(k)}^{-1}, \quad Q_{(k)}^{-1} = Q_{(kk)}^{-1} + \frac{1}{\theta} Q_{(kk)}^{-1} X_0 W X_0 \top Q_{(kk)}^{-1}.
\]

Substituting the second identity into the first gives
\[
Q^{-1} - Q_{(kk)}^{-1} = \frac{1}{\theta} Q_{(k)}^{-1} X_0 V X_0 \top Q_{(k)}^{-1} + \frac{1}{\theta} (Q_{(kk)}^{-1} + \frac{1}{\theta} Q_{(kk)}^{-1} X_0 W X_0 \top Q_{(kk)}^{-1}) \left( Q_{(k)}^{-1} + \frac{1}{\theta} Q_{(kk)}^{-1} X_0 W X_0 \top Q_{(kk)}^{-1} \right),
\]
which after simplifying becomes
\[
Q^{-1} - Q_{(kk)}^{-1} = \frac{1}{\theta} Q_{(k)}^{-1} X_0 V X_0 \top Q_{(k)}^{-1} + \frac{1}{\theta} Q_{(kk)}^{-1} X_0 W X_0 \top Q_{(kk)}^{-1} + \frac{1}{\theta^2} Q_{(kk)}^{-1} X_0 V X_0 \top Q_{(kk)}^{-1} X_0 W X_0 \top Q_{(kk)}^{-1}.
\]

Before we proceed with the proof we first prove some moment estimates for the terms in (D.32). We start with some informal observations. By comparing (D.31a) and (D.31a), we see that the matrix $W$ is smaller in magnitude in comparison to $V$ by a factor of $\theta^{-1}$. This suggests that the first term in (D.32) is the leading term while the rest are high order terms in comparison and we will therefore only deal with first term in detail below.

The same arguments can be applied to the rest of (D.32) to make the above argument rigorous, but we omit the repetitive details.

Recall the family of event \( \{ B^k_0, k = 1, \ldots, p \} \) from (D.6) and define \( B^k_2 := B^k_0 \cap B_1 \). From definition we note that \( B_2 \subseteq B^k_2 \). Furthermore, from Lemma 2.1 we have
\[
1_{B_2^c} - 1_{B_2} \leq 1 - 1_{B_2} = o_p(T^{-l}), \quad \forall l \in \mathbb{N}.
\]
where, recalling (D.31a), we have interchangeably below without further justifications. Note that under the event \( B \), we can easily see that

\[
\left\|\frac{1}{\sqrt{T}} \sum_{i=1}^{3} \langle x_i^\top \rangle V_n x_m \right\| \ll 1,
\]

so that \( 1 \) \( \mbox{tr} \) \( 2 \sum_{n=1}^{K} \sum_{m=1}^{K} \sum_{l=1}^{K} \sum_{i=1}^{3} \left| \mathbb{E} \left[ e_i^\top (x_i^\top V_n x_m) e_m Q\frac{-1}{kk} e_i \right] \right| \]

where, recalling (D.31a), we have

In the computations below, we will often substitute \( 1_{B_2} \) with \( 1_{B_2} \) and vice versa in expectations. Whenever we do so, we may use (D.33) and a similar argument to how we dealt with (D.7) to show that the error term of such a substitution is negligible for the purpose of the proof. Hence from now on we will use the two indicators \( 1_{B_2} \) and \( 1_{B_2} \) interchangeably below without further justifications.

Since we can write \( X_0^\top = \frac{1}{\sqrt{T}} \sum_{l=1}^{K} x_l e_i^\top \), the first term in (D.32) can be expressed as

\[
\frac{1}{\sqrt{T}} e_i^\top Q^{-1}_{kk} X_0 V X_0^\top Q^{-1}_{kk} e_i = \frac{1}{\sqrt{T}} \sum_{l=1}^{K} \sum_{m=1}^{K} e_i^\top Q^{-1}_{kk} e_l (x_i^\top V x_m) e_m Q^{-1}_{kk} e_i,
\]

where, recalling (D.31a), we have

\[
x_i^\top V x_m = \frac{1}{T} \sum_{n=1}^{3} x_i^\top R_k e_{k\tau}^\top e_{k\tau} V_n x_m.
\]

Using (D.34)-(D.35) and the inequality \( (\sum_{i=1}^{n} x_i)^p \ll n^{p-1} \sum_{i=1}^{n} x_i^p \) we have

\[
\mathbb{E} \left[ \left| \frac{1}{\sqrt{T}} e_i^\top Q^{-1}_{kk} X_0 V X_0^\top Q^{-1}_{kk} e_i \right| \right] ^2 \ll \frac{K^2}{\sqrt{T}^2} \sum_{l=1}^{K} \sum_{m=1}^{K} \mathbb{E} \left[ e_i^\top Q^{-1}_{kk} e_l (x_i^\top V x_m) e_m Q^{-1}_{kk} e_i \right] ^2
\]

\[
\ll \frac{3 K^2}{\sqrt{T}^4} \sum_{l=1}^{K} \sum_{m=1}^{K} \sum_{n=1}^{3} \mathbb{E} \left[ x_i^\top R_k e_{k\tau}^\top e_{k\tau} V_n x_m \right] ^2 \ll 1_{B_2} ^2.
\]

Note that under the event \( B_2 \), we can easily see that \( \| Q^{-1}_{kk} 1_{B_2} \| = O(1) \). Therefore by the Cauchy Schwarz inequality we can obtain

\[
(D.36) \ll \frac{K^2}{\sqrt{T}^4} \sum_{l=1}^{K} \sum_{m=1}^{K} \sum_{n=1}^{3} \mathbb{E} \left[ (x_i^\top R_k e_{k\tau} )^4 1_{B_2} \right] ^{1/2} \mathbb{E} \left[ (e_{k\tau}^\top V_n x_m )^4 1_{B_2} \right] ^{1/2}.
\]

Note that \( B_2 = B_1 \cap B_0 \subseteq B_0 \subseteq B_0^k \) so that \( 1_{B_2} \ll 1_{B_2} \). We can then condition on \( R_k \) and apply (a) of Lemma C.3 to the first quadratic form in (D.36) to get

\[
\mathbb{E} \left[ x_i^\top R_k e_{k\tau} 1_{B_2} \right] ^4 \leq \mathbb{E} \left[ \left( z_{l,[1:T]}^\top \sigma_{l\tau}^2 \Phi_l I_T \right) 1_{B_2} \right] ^4 \ll \mathbb{E} \left[ \text{tr} \left( R_k^2 (\sigma_{l\tau}^2 \Phi_l I_T )^2 1_{B_2} \right) \right] = O(\sigma_{l\tau}^4 T^2) \]

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where the last equality follows from using \( \text{tr}(R) \leq T\|R\| \) and applying Lemma C.2. Similarly, for the quadratic involving \( V_1 \) in (D.37), we have

\[
\mathbb{E} |\mathbf{e}_{k\tau}^\top V_1 x_m |^4 = \mathbb{E} |\mathbf{e}_{k\tau}^\top x_m |^4 \lesssim \text{tr}((\sigma_m^2 \Phi_m^\top \Phi_m + I_T)^2) = O(\sigma_m^4 T^2).
\]

We observe here that the matrices \( V_2 \) and \( V_3 \) are smaller in magnitude in comparison to \( V_1 \) by a factor of \( \theta^{-1} \) under the event \( B_2 \). Hence it is to be expected that the quadratic forms involving \( V_2 \) and \( V_3 \) in (D.37) should be negligible in comparison to the one involving \( V_1 \). To be more concrete, we sketch here how bound the quadratic form involving \( V_2 \); the case of \( V_3 \) can be dealt with in a similar manner. Recall that the matrix \( E_0^\top E_0 \) can be written as

\[
E_0^\top E_0 = E_{k0}^\top E_{k0} + \frac{1}{T} \mathbf{e}_{k0} \mathbf{e}_{k0}^\top.
\]

Then we can write

\[
\mathbb{E} |\mathbf{e}_{k\tau}^\top V_2 x_m |^4 = \frac{1}{\theta^4} \mathbb{E} |\mathbf{e}_{k\tau}^\top E_0 R_k E_{k\tau}^\top E_0 E_{k\tau} x_m 1_{B_2} |^4
\]

\[
\lesssim \frac{1}{\theta^4} \mathbb{E} |\mathbf{e}_{k\tau}^\top E_0 R_k E_{k\tau}^\top E_0 E_{k\tau} x_m 1_{B_2} |^4 + \frac{1}{\theta^4} \mathbb{E} |\mathbf{e}_{k\tau}^\top E_0 R_k \mathbf{e}_{k0} \mathbf{e}_{k0}^\top x_m 1_{B_2} |^4
\]

\[
+ \frac{1}{\theta^4} \mathbb{E} |\mathbf{e}_{k\tau}^\top E_0 R_k \mathbf{e}_{k0} \mathbf{e}_{k0}^\top R_k E_{k0}^\top E_{k0} x_m 1_{B_2} |^4.
\]

At this point we recognize that the four terms above has a similar structure as the case of \( V_1 \). Namely they all involve quadratic forms where the matrix in the middle is independent from the vectors on each side. Using the same approach as we did in the case of \( V_1 \) we can indeed show that this is a negligible term in comparison. The case of \( V_3 \) is similar albeit more tedious, and we omit the details.

After the above arguments, we may conclude that

\[
\mathbb{E} |\mathbf{e}_i^\top (Q^{-1} - Q^{-1}_{(kk)}) e_i 1_{B_2} |^2 \lesssim \frac{K^2}{\theta^2 T^4} \sum_{l=1}^{K} \sum_{m=1}^{K} \sigma_l^2 \sigma_m^2 T^2 = o \left( \frac{1}{\sqrt{\theta T^2}} \right).
\] (D.38)

The same strategy described above can then be repeated for each of the remaining three terms in (D.32) to show that they are negligible (c.f. the remark right below (D.32)). We may therefore conclude that

\[
\mathbb{E} |\mathbf{e}_i^\top (Q^{-1} - Q^{-1}_{(kk)}) e_i 1_{B_2} |^2 = o \left( \frac{1}{\sqrt{\theta T^2}} \right).
\] (D.39)
Finally, we can decompose $\mathbb{E}[e_i^\top Q^{-1}e_i1_{B_2}] - \mathbb{E}[e_i^\top Q^{-1}e_i1_{B_2}]$ into

$$\mathbb{E}[e_i^\top Q^{-1}e_i1_{B_2}] - \mathbb{E}[e_i^\top Q^{-1}e_i1_{B_2}] = \sum_{k=1}^{p}(E_i - E_{i-1})e_i^\top Q^{-1}e_i1_{B_2}$$

$$= \sum_{k=1}^{p}(E_i - E_{i-1})e_i^\top(Q^{-1} - Q_{(kk)}^{-1})e_i1_{B_2},$$

where for the last equality we refer to (D.33) and the remark immediately below it. Using the bound (D.39) we immediately have

$$\mathbb{E}[\mathbb{E}[e_i^\top Q^{-1}e_i1_{B_2}] - \mathbb{E}[e_i^\top Q^{-1}e_i1_{B_2}]]^2 \leq 4 \sum_{k=1}^{p} \mathbb{E}[e_i^\top(Q^{-1} - Q_{(kk)}^{-1})e_i1_{B_2}] = o\left(\frac{p}{\sqrt{T}}\right),$$

from which the claim follows.

\[\square\]

### E Proof of Theorem 4.2

**Proof of Theorem 4.2.** Without loss of generality, we only consider the case for $Z_{i,\tau} > 0$ since the case for $Z_{i,\tau} < 0$ can be considered in precisely the same way. For a constant significant level $\alpha$, to see $Pr(Z_{i,\tau} > z_\alpha | H_1) \to 1$ as $T, p \to \infty$, it is sufficient to show that $Z_{i,\tau} \to \infty$ as $T, p \to \infty$.

To start, we firstly notice that for any $i \in \{1, 2, ..., K\}$ and a finite time lag $\tau$, $\gamma_{i,\tau}$ does not divergent with $T$ and $p$, since both $\gamma_{i,\tau}$ and $v_{i,\tau}$ are some constants when $T, p \to \infty$. It then suffices to show $\sqrt{T} \lambda_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)} \to \infty$ when $T, p \to \infty$. Note that by the definition of $\theta_{i,\tau}$ in (4.3), we can show that

$$\frac{\lambda_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)}}{\theta_{i,\tau}} = \frac{\lambda_{i,\tau}^{(1)} \theta_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)} \theta_{i,\tau}^{(2)}}{\theta_{i,\tau}^{(1)} \theta_{i,\tau}^{(2)} \theta_{i,\tau}} = \frac{\lambda_{i,\tau}^{(1)} 2 + 2c - \lambda_{i,\tau}^{(2)} 2 + 2c}{\theta_{i,\tau}^{(1)} 2 + c},$$

where the second equation follows from the fact that $\theta_{i,\tau}^{(1)} = (1+c)\theta_{i,\tau}^{(2)}$ and $\theta_{i,\tau} = \frac{\theta_{i,\tau}^{(1)} + \theta_{i,\tau}^{(2)}}{2} = \frac{2+c}{2}\theta_{i,\tau}$. Moreover, under Assumptions 2.1 and 2.2, we know from Theorem 3.3 that for $m = 1$ and 2,

$$\sqrt{T} \frac{\gamma_{i,\tau}^{(m)} \lambda_{i,\tau}^{(m)} - \theta_{i,\tau}^{(m)}}{2v_{i,\tau}^{(m)} \theta_{i,\tau}^{(m)}} \Rightarrow \mathcal{N}(0, 1),$$

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as $T, p \to \infty$ where $\theta_{i,\tau}^{(m)}$ is the asymptotic centering of $\lambda_{i,\tau}^{(m)}$. As a result,

$$\frac{\lambda_{i,\tau}^{(m)}}{\theta_{i,\tau}^{(m)}} = 1 + o_P \left( \frac{1}{\sqrt{T}} \right),$$

as $T, p \to \infty$, where we stress the fact that $\gamma_{i,\tau}^{(m)}$ and $v_{i,\tau}^{(m)}$ are constant when $T, p \to \infty$.

Therefore, (E.1) reduces to

$$\frac{\lambda_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)}}{\theta_{i,\tau}} = \frac{2 + 2c}{2 + c} \left( 1 + o_P \left( \frac{1}{\sqrt{T}} \right) \right) - \frac{2}{2 + c} \left( 1 + o_P \left( \frac{1}{\sqrt{T}} \right) \right) = \frac{2c}{2 + c} + o_P \left( \frac{1}{\sqrt{T}} \frac{2c}{2 + c} \right),$$

for $T, p \to \infty$, and we conclude that

$$\sqrt{T} \frac{\lambda_{i,\tau}^{(1)} - \lambda_{i,\tau}^{(2)}}{\theta_{i,\tau}} = \sqrt{T} \frac{2c}{2 + c} + o_P \left( \frac{2c}{2 + c} \right),$$

when $T, p \to \infty$.

Consequently, when $T, p \to \infty$, $Z_{i,\tau} \to \infty$ as long as $\sqrt{T} \frac{2c}{2 + c} \to \infty$ and $\lambda_{i,\tau}^{(1)} \neq \lambda_{i,\tau}^{(2)}$. And it is sufficient to show the assertion in this theorem.

\[
\]

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