SOME COMBINATORIAL PROBLEMS ON FINITE ABELIAN GROUPS AND THE RATIONAL DYCK PATHS

DONGCHUN HAN AND HANBIN ZHANG

ABSTRACT. In this paper, we study some objects from combinatorial number theory and relate them to the study of the rational Dyck paths. Let $G$ and $H$ be finite abelian groups with $\gcd(|G|, |H|) = 1$. For any positive integer $m$, let $M(G, m)$ be the set of all zero-sum sequences over $G$ of length $m$. Firstly, we provide bijections between $M(G, |H|)$, $M(H, |G|)$, and $D_{|G|, |H|}$, where $D_{|G|, |H|}$ is the set of all $|G|, |H|$-Dyck paths. Consequently we have

$$\#M(G, |H|) = \#M(H, |G|) = \text{Cat}_{|G|, |H|} := \frac{1}{{|G| + |H|}} {\frac{|G| + |H|}{|G|}}.$$

Secondly, we study the following counting problem. Assume that $|G| = n$. Let $a_1, \ldots, a_k$ be given positive integers with $k \leq n$. We consider the number of non-equivalent solutions of the following equation in $G$:

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = b,$$

where $b \in G$. If $\gcd(a_1 + \cdots + a_k, n) = 1$, then we show that the number of all non-equivalent solutions coincides with the Kreweras number which initially counts the number of some special rational Dyck paths. We provide both combinatorial and algebraic proofs of these results.

1. Introduction

For any positive integers $n, m$ with $\gcd(n, m) = 1$, the rational Catalan number is defined as

$$\text{Cat}_{n, m} = \frac{1}{n + m} \binom{n + m}{n},$$

which is a natural generalization of Catalan numbers $\text{Cat}_n := \frac{1}{2n+1} \binom{2n+1}{n}$ and is related to many problems in combinatorics, representation theory and geometry (see Section 2 for more discussion). A typical object counted by $\text{Cat}_{n, m}$ is the set $D_{n, m}$ of all $(n, m)$-Dyck paths which is defined as the number of lattice paths from $(0, 0)$ to $(n, m)$ which only go east or north and stay above the diagonal line $y = \frac{m}{n}x$. In this paper, we study some objects from combinatorial number theory and relate them to the study of the rational Dyck paths.

Let $G$ be an abelian group written additively. We call $S$ a sequence over $G$ if $S = g_1 \cdot \cdots \cdot g_m$ is a multiset of elements in $G$, where $m$ is called the length of $S$ and we denote it by $|S| = m$. We also denote $\sigma(S) = g_1 + \cdots + g_m$.

Let $k, m$ be positive integers with $k \leq n$. For any $g \in G$, let

$$N(G, k, g) = \{ S \text{ be a subset over } G \mid \sigma(S) = g \text{ and } |S| = k \}$$

and

$$M(G, m, g) = \{ S \text{ be a sequence over } G \mid \sigma(S) = g \text{ and } |S| = m \}. $$
In particular, we denote
\[ N(G, k) = N(G, k, 0) \text{ and } M(G, k) = M(G, k, 0). \]

Counting formulas for \( N(G, k, g) \) and \( M(G, m, g) \) for arbitrary abelian group \( G \) and any positive integers \( k \leq |G| \) and \( m \) were recently obtained in [29, 26, 35] via sieve method (known as Li-Wan sieve method) and generating function.

**Theorem 1.1.** ([29] [26] [35]) Let \( G = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r\mathbb{Z} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle \) be a finite abelian group with \( |G| = n = n_1 \cdots n_r \) and \( n_1 \cdots n_r = n \). Let \( g = g_1 e_1 + \cdots + g_r e_r \in G \), where \( g_i \in [0, n_i - 1] \) for every \( i \). Then for any \( k \leq n \) and any \( m \) we have
\[
\begin{align*}
(1) \quad \#N(G, k, g) &= \frac{1}{n} \sum_{d \mid (n, k)} \Phi_G(g, d)(-1)^{k + \frac{d}{k}}(\frac{n/d}{k/d}) \\
(2) \quad \#M(G, k, g) &= \frac{1}{n + m} \sum_{d \mid (n, m)} \Phi_G(g, d)(\frac{n/d + m/d}{n/d}),
\end{align*}
\]
where \( \Phi_G(g, d) = \sum_{\chi \in \hat{G}, \text{ord}(\chi) = d} \chi(g) = \sum_{d \mid (n, l), l \mid |g|} \mu(\frac{d}{l}) \prod_{i=1}^{r} (n_i, l). \) In particular, for any \( k \leq n \) and any \( m \), we have
\[
\begin{align*}
\#N(G, k) &= \frac{1}{n} \sum_{d \mid (n, k)} \varphi_G(d)(-1)^{k + \frac{d}{k}}(\frac{n/d}{k/d}) \\
\#M(G, m) &= \frac{1}{n + m} \sum_{d \mid (n, m)} \varphi_G(d)(\frac{n/d + m/d}{n/d}),
\end{align*}
\]
where \( \varphi_G(d) = \sum_{d \mid l} \mu(\frac{d}{l}) \prod_{i=1}^{r} (n_i, l). \)

While studying some combinatorial problems in the invariant theory, we found a very interesting paper of Panyushev [37] in which we discovered that the above results were already implicitly obtained much earlier before, as consequences of some more general works from invariant theory by Molien [34] in 1897 and Almkvist [2] in 1982. To the best of our knowledge, complete explicit formula for general abelian groups in Theorem 1.1 was firstly presented in the paper of Li and Wan [29]. While, in the paper of Panyushev [37], following the ideas of Molien [34] and Almkvist [2], he obtained an explicit formula ([37] formula (4.3)) for cyclic groups, and an almost explicitly formula for general abelian groups ([37] Section 4). In Section 3, we will provide more detailed explanations.

In 1975, Fredman [13] obtained the special case of \( M(G, m, g) \) in Theorem 1.1 when \( G \) is a cyclic groups. In fact, let
\[
M(C_n, m, i) = \{ S \text{ be a sequence over } \mathbb{Z}/n\mathbb{Z} \mid \sigma(S) \equiv i \pmod n \text{ and } |S| = m \}.
\]
Then using generating functions, Fredman proved that
\[
\#M(C_n, m, i) = \#M(C_m, n, i).
\]
He also provided a combinatorial explanation of this symmetric relationship using a necklace interpretation. In 1999, Elashvili, Jibladze and Pataraia [13] [14] rediscovered the same result but they employed the above mentioned ideas from invariant theory by Molien and Almkvist. Motivated by the classical Hermite reciprocity in the representation theory of Lie group [39], they called the above symmetric relationship the Hermite reciprocity for cyclic groups. Based on Theorem 1.1 it is natural to ask whether there is a similar reciprocity for general abelian group. The first main result in this paper is motivated by this problem, we also provide combinatorial interpretations which relate to the rational Dyck paths.
Theorem 1.2. Let $k, m, n$ be positive integers. Let $G$ and $H$ be two abelian groups. Then we have

1. If $\gcd(|G|, |H|) = 1$, there are bijections between $M(G, |H|)$, $M(H, |G|)$, and $D_{|G|, |H|}$.

Therefore

$$\#M(G, |H|) = \#M(H, |G|) = \text{Cat}_{|G|, |H|}.$$ (1.2)

2. For abelian groups $G = C_n^r$ and $H = C_m^r$ with $\gcd(n, m^r) = \gcd(m, n^r)$, we have

$$\#M(G, |H|) = \#M(H, |G|).$$ (1.3)

3. For $k \leq |G|$ with $\gcd(k, |G|) = 1$, there are bijections between $N(G, k)$, $N(G, |G| - k)$, and $D_{k, |G| - k}$.

Therefore

$$\#N(G, k) = \#N(G, |G| - k) = \frac{|G|!}{k!} \binom{|G|}{k} = \text{Cat}_{k, |G| - k}.$$ (1.4)

Note that in Theorem 1.2(1), the equality $\#M(G, |H|) = \#M(H, |G|)$ only depends on the condition that $\gcd(|G|, |H|) = 1$ (but not the group structures of $G$ and $H$). The special case of Theorem 1.2(1) when $G$ is a cyclic group was obtained recently by Johnson [25] Lemma 27) while proving a conjecture of Armstrong [6] concerning the average size of the simultaneous core partitions. We will provide detailed explanations in Section 3.

We also have a generalization of Theorem 1.2 based on results of Panyushev [37]. We briefly recall some definitions. Let $G$ be a finite group and $V$ a $G$-module. Let $(S(V) \otimes \wedge(V))_{G, \chi}$ denote the isotypic component in symmetric tensor exterior algebra of $V$ corresponding to an irreducible representation $\chi$. It is a bi-graded vector space and its Poincaré series is the formal power series

$$\mathcal{F}((S(V) \otimes \wedge(V))_{G, \chi}; s, t) = \sum_{p, m \geq 0} \dim (S^p(V) \otimes \wedge^m(V))_{G, \chi} s^p t^m.$$ (1.5)

Actually, Theorem 1.1 can be derived from this Poincaré series by methods of Molien [34] and Almkvist [2]. We denote $(S(V) \otimes \wedge(V))_G = (S(V) \otimes \wedge(V))_{G, \chi_0}$ if $\chi_0$ is the trivial representation. Let $C_{q+m}$ and $C_{p+m}$ be two cyclic groups of order $q + m$ and $p + m$. Let $R$ and $\bar{R}$ be the regular representations of $C_{q+m}$ and $C_{p+m}$ respectively. Panyushev [37] proved the following interesting generalization of (1.1)

$$(1.2) \quad \dim (S^p(R) \otimes \wedge^m(\bar{R}))_{C_{q+m}} = \dim (S^q(\bar{R}) \otimes \wedge^m(R))_{C_{p+m}}.$$ (1.6)

Panyushev also asked for a combinatorial interpretation of (1.2). In this paper, following Panyushev’s idea, we have the following generalization. We also provide a combinatorial interpretation in a special case using a necklace construction.

Theorem 1.3. Let $p, q, m$ be positive integers. Let $G_{q+m}$ and $H_{p+m}$ be two abelian groups of order $q + m$ and $p + m$. Let $R$ and $\bar{R}$ be the regular representations of $G_{q+m}$ and $H_{p+m}$ respectively. If $(p, q, m) = 1$, then we have

$$(1.3) \quad \dim (S^p(R) \otimes \wedge^m(\bar{R}))_{G_{q+m}} = \dim (S^q(\bar{R}) \otimes \wedge^m(R))_{H_{p+m}}.$$ (1.7)

Indeed, both dimensions are equal to $\frac{1}{p+q+m}(p+q+m)$. (1.8)
Our second aim of this paper is to consider a subset counting problem. Let $G$ be a finite abelian group of order $n$. Let $a_1, \ldots, a_k$ be positive integers with $k \leq n$. Let $\sum_{i=1}^{k} a_i = m$ and $c_0 = n - k$. For any $i \in [1, m]$, we define $c_i$ be the number of $a_j$’s such that $a_j = i$, i.e.,

$$a_1 \cdots a_k = 1^{[c_1]} \cdots m^{[c_m]}.$$ 

We may assume that $a_1 \leq \cdots \leq a_k$. For any $b \in G$, let $(g_1, \ldots, g_k) \in G^k$ be a solution of the following equation in $G$:

$$(1.4) \quad a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = b,$$

where $g_i \neq g_j$ for any $i \neq j$. We say that two solutions $(g_1, \ldots, g_k)$ and $(h_1, \ldots, h_k)$ are equivalent if $\{g_1, \ldots, g_{c_1}\} = \{h_1, \ldots, h_{c_1}\}$ and

$$\{g_{c_1} + \cdots + c_{i+1}, \ldots, g_{c_1} + \cdots + c_{i+1}\} = \{h_{c_1} + \cdots + c_{i+1}, \ldots, h_{c_1} + \cdots + c_{i+1}\}$$

holds for any $i \in [1, m-1]$. Let $N_G(a_1, \ldots, a_k, b)$ be the set of representatives of all non-equivalent solutions of (1.4). When $G$ is the additive subgroup of a finite field $F_q$, similar problem was recently tackled by Li and Yu [31] (we also refer to some discussion of related problems therein). In this paper, we provide a partial solution of the problem of finding a formula of $N_G(a_1, \ldots, a_k, b)$ for general abelian group.

**Theorem 1.4.** Let $G$ be a finite abelian group of order $n$. If $\gcd(a_1 + \cdots + a_k, n) = 1$, then for any $b \in G$ we have

$$\# N_G(a_1, \ldots, a_k, b) = \frac{1}{n} \left( \begin{array}{c} n \\ c_0, \ldots, c_m \end{array} \right),$$

which is called the Kreweras number which initially counts the number of $(n, m)$-Dyck paths in $D_{n,m}$ with $c_i$ vertical runs of length $i$.

We shall provide proofs of this result from two different perspectives. One is combinatorial by constructing bijections between the set of solutions and the special $(n, m)$-Dyck paths, and the other one is algebraic by employing Molien’s idea from invariant theory. We also obtain an explicit formula in the following case.

**Theorem 1.5.** Let $G = \mathbb{Z}/m_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r \mathbb{Z} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle$ be a finite abelian group with $|G| = n = n_1 \cdots n_r$ and $n_1 | \cdots | n_r$. Let $a_1$ be a positive integer and

$$b = b_1 e_1 + \cdots + b_r e_r \in G,$$

where $b_i \in [0, n_i - 1]$ for every $i$. Then we have the following explicit counting formula of $N_G(a_1, 1, \ldots, 1, b)$:

$$(-1)^{k-1} \sum_{d|(n, k+a_1-1)} \sum_{m \geq \frac{d}{k}} \Phi_G(b, d) \left( \frac{d}{k+a_1-1} - m \right) (-1)^{k+a_1-1-m},$$

where

$$\Phi_G(b, d) = \sum_{\chi \in \hat{G}, \text{ord}(\chi) = d} \chi(b) = \sum_{l|d, (n_i, l)|b_i} \mu \left( \frac{d}{l} \right) \prod_{i=1}^{r} (n_i, l).$$

The following sections are organized as follows. In Section 2, we shall introduce some notations and preliminary results. In Section 3, we will explain Theorem 1.4 from the perspective from the invariant theory and then we prove Theorems 1.2 and 1.3. In Section 4, we will study the problem of counting $N_G(a_1, \ldots, a_k, b)$ and prove Theorems 1.4 and 1.5.
2. Preliminaries

In this section, we will provide more rigorous definitions and notations. We also introduce some preliminary results that will be used repeatedly below.

Let $G$ be an additive finite abelian group. By the fundamental theorem of finite abelian groups we have

$$G \cong \mathbb{Z}/n_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r \mathbb{Z} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle$$

where $r = r(G) \in \mathbb{N}_0$ is the rank of $G$, $n_1 | \cdots | n_r \in \mathbb{N}$ are positive integers. Moreover, $n_1, \ldots, n_r$ are uniquely determined by $G$. We also use $C^n_r$ to denote the abelian group of the following form

$$\mathbb{Z}/n\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n\mathbb{Z}.$$  

We define a sequence over $G$ to be an element of the multiplicatively written free abelian monoid $(F(G), \cdot)$, see Chapter 5 of [18] for detailed explanation. Our notations of sequences follow the notations in the paper [17]. In particular, in order to avoid confusion between exponentiation of the group operation in $G$ and exponentiation of the sequence operation $\cdot$ in $F(G)$, we define:

$$g^{[k]} = g \cdot \ldots \cdot g \in F(G),$$

for $g \in G$ and $k \in \mathbb{N}_0$.

We write a sequence $S$ in the form

$$S = g_1 \cdot \ldots \cdot g_l \text{ where } g_i \in G.$$  

We call

- $|S| = l \in \mathbb{N}_0$ the length of $S$,
- $\sigma(S) = g_1 + \cdots + g_l \in G$ the sum of $S$,
- $S$ a zero-sum sequence if $\sigma(S) = 0$.

For the convenience of our proofs later, the following remark will be used throughout this paper.

Remark 2.1. We write a sequence $S$ over $\mathbb{Z}/n\mathbb{Z} = \{0, \ldots, n-1\}$ as a vector $(x_0, \ldots, x_{n-1}) \in \mathbb{N}^n$, where $x_i$ is the multiplicity that $i$ occurs in $S$, that is in our notation

$$S = \left[\begin{array}{c}
\underbrace{0, \ldots, 0}_x \cdot n-1^{(x_{n-1})}
\end{array}\right].$$

Similarly, we regard $G$ as

$$\mathbb{Z}/n_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r \mathbb{Z} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle,$$

where $n_1 | \cdots | n_r$ and $n_1 \cdots n_r = n$. Then every element in $G$ can be written uniquely as $a_1 e_1 + \cdots + a_r e_r$ for some positive $a_i \leq n_i - 1$ and $1 \leq i \leq r$. In order to attach a similar vector $(y_0, \ldots, y_{n-1}) \in \mathbb{N}^n$ to an arbitrary sequence over $G$, we introduce the following label of elements in $G$. Any integer $k \in [0, n_1 \cdots n_r - 1]$ can be written uniquely as

$$k = b_r n_1 \cdots n_{r-1} + b_{r-1} n_1 \cdots n_{r-2} + \cdots + b_2 n_1 + b_1$$

where $b_i \in [0, n_i - 1]$ for $1 \leq i \leq r$. Therefore, let $g = a_1 e_1 + \cdots + a_r e_r$, then $g$ will be attached the label

$$l_g = a_r n_1 \cdots n_{r-1} + \cdots + a_2 n_1 + a_1.$$
With this label, a vector \((y_0, \ldots, y_{n-1}) \in \mathbb{N}^n\) will corresponds to a sequence \(S\) over \(G\), where \(y_i\) is the multiplicity that \(g = a_1 e_1 + \cdots + a_r e_r\) (with \(l_g = i\)) occurs in \(S\). In our notation
\[
S = \prod_{g \in G} g^{[y_g]}.
\]

For any positive integers \(n, m\) with \(\gcd(n, m) = 1\), the rational Catalan number is defined as
\[
\text{Cat}_{n,m} = \frac{1}{n+m} \binom{n+m}{n},
\]
which is a natural generalization (take \(m = n + 1\)) of Catalan numbers \(\text{Cat}_n := \frac{1}{n+1} \binom{2n+1}{n}\) (see [23]). Rational Catalan numbers (and their \(q\) or \((q,t)\) analogs) appeared in many problems, such as simultaneously core partitions, rational Dyck paths, non-crossing partitions, parking functions, Hecke algebra, affine Springer varieties, compactified Jacobians of singular curves, see [3, 5, 6, 7, 8, 10, 16, 19, 20, 21, 33]. In particular, the rational Catalan numbers have the following interesting algebraic generalization. Let \(W\) be a Weyl group with root lattice \(Q\), degrees \(d_1 \leq d_2 \leq \cdots \leq d_l\), and Coxeter number \(h := d_l\). Let \(p \in \mathbb{N}\) with \((h,p) = 1\). Then Haiman [23] showed that the number of orbits in the action of \(W\) on the finite torus \(Q/pQ\) is
\[
\text{Cat}(W, p) = \prod_i \frac{p + d_i - 1}{d_i},
\]
which is called the rational Coxeter-Catalan number of \(W\) at parameter \(p\). In particular, let \(\mathfrak{S}_n\) and \(\mathfrak{S}_m\) be symmetric groups with \((n, m) = 1\), then we have
\[
\text{Cat}(\mathfrak{S}_n, m) = \text{Cat}(\mathfrak{S}_m, n) = \text{Cat}_{n,m}.
\]

A typical object counted by \(\text{Cat}_{n,m}\) is the set \(D_{n,m}\) of all \((n,m)\)-Dyck paths which is defined as the number of lattice paths from \((0,0)\) to \((n,m)\) which only go east or north and stay above the diagonal line \(y = \frac{m}{n} x\) (see [3]). We will also need to consider rational Dyck paths with a specified vertical run structure. For any \(n, m \in \mathbb{N}\) with \((n, m) = 1\) and \(c_0, \ldots, c_m \in \mathbb{N}\) such that \(\sum_{i=1}^{m} ic_i = m\) and \(\sum_{i=0}^{m} c_i = n\), the Kreweras number
\[
\frac{1}{n} \binom{n}{c_0, \ldots, c_m}
\]
is defined as the number of \((n,m)\)-Dyck paths in \(D_{n,m}\) with \(c_i\) vertical runs of length \(i\). We refer to [4, 8, 38] for some studies related to the Kreweras number and its algebraic generalizations.

3. Invariants and symmetric relationships in finite abelian groups

In this section, we will firstly present a proof of Theorem 1.1 from the perspective of invariant theory.

Let \(G\) be a finite group and \(V\) a finite dimensional representation of \(G\) over \(\mathbb{C}\). Let \((S(V) \otimes \Lambda(V))_{G,\chi}\) denote the isotypic component in symmetric tensor exterior algebra of \(V\) corresponding to an irreducible representation \(\chi\). It is a bi-graded vector space and its Poincaré series is the formal power series
\[
\mathcal{F}\left((S(V) \otimes \Lambda(V))_{G,\chi}, s, t\right) = \sum_{p,m \geq 0} \dim (S^p(V) \otimes \Lambda^m(V))_{G,\chi} s^p t^m.
\]
Based on a remarkable theorem of Molien ([34]. Section 2), Almkvist ([2] Theorem 1.33) proved the following formula

\[
F \left( (S(V) \otimes \wedge(V))_{G, \chi}; s, t \right) = \frac{1}{|G|} \sum_{g \in G} \frac{\det_V (E + gt)}{\det_V (E - gs)},
\]

where \( E \) is the identity matrix in \( GL(V) \). The idea used in the proof of (3.2) by Molien and Almkvist will be used implicitly in our solution of Stanley’s problem. Later, Panyushev obtained the following easy consequence of (3.2). The special cases when \( t = 0 \) in (3.4) was obtained in 1978 by Almkvist and Fossum ([3] V.1.8), when \( s = 0 \) see Section 4 of Panyushev’s paper.

**Lemma 3.1.** ([37] Lemma 3.1) Let \( G \) be a finite group and \( R \) the regular representation of \( G \), then we have

\[
F \left( (S(R) \otimes \wedge(R))_{G, \chi}; s, t \right) = \frac{\deg(\chi)}{|G|} \sum_{d \geq 1} \left( \sum_{g, \text{ord}(g) = d} \chi(g^{-1}) \cdot \left( \frac{1 - (-t)^d}{1 - s^d} \right)^{|G|/d} \right).
\]

In particular, if \( G \) is abelian, then we have

\[
F \left( (S(R) \otimes \wedge(R))_{G, \chi}; s, t \right) = \frac{1}{|G|} \sum_{d \geq 1} \left( \sum_{g, \text{ord}(g) = d} \chi(g^{-1}) \cdot \left( \frac{1 - (-t)^d}{1 - s^d} \right)^{|G|/d} \right)
\]

and

\[
F \left( (S(R) \otimes \wedge(R))_{G, \chi}; s, t \right) = \frac{1}{|G|} \sum_{d \geq 1} \left( \phi_G(d) \left( \frac{1 - (-t)^d}{1 - s^d} \right)^{|G|/d} \right)
\]

where \( \phi_G(d) \) is the number of elements in \( G \) of order \( d \).

It is easy to see that, using the isomorphism \( G \cong \hat{G} \), the formula [33] actually provide counting formulas for \( N(G, k, g) \) and \( M(G, m, g) \) simultaneously. The last minor step is just an explicit calculation of \( \sum_{g, \text{ord}(g) = d} \chi(g^{-1}) \) or, for our need, \( \sum_{\chi, \text{ord}(\chi) = d} \chi^{-1}(g) \). Let \( G = \mathbb{Z}/n_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r \mathbb{Z} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle \) be a finite abelian group with \( |G| = n \). Let \( g = g_1 e_1 + \cdots + g_r e_r \in G \), where \( g_i \in [0, n_i - 1] \) for every \( i \). By basic representation theory and the principle of inclusion-exclusion (for details we refer to [20]), one obtains the following

\[
\sum_{\chi, \text{ord}(\chi) = d} \chi^{-1}(g) = \sum_{l | d, (n_i, l) | g_i} \mu\left( \frac{d}{l} \right) \prod_{i=1}^{r} (n_i, l).
\]

In particular,

\[
\phi_G(d) = \sum_{l | d} \mu\left( \frac{d}{l} \right) \prod_{i=1}^{r} (n_i, l).
\]

With [33] \( - [36] \), extracting the coefficient of \( t^k s^m \), we get Theorem 1.1.
Remark 3.2. Note that Li and Wan [29] provided an elegant sieve method which is different from the above method. Li-Wan’s sieve method is very useful to study similar counting problems, we refer to their subsequent papers for detailed discussions [28, 30, 31]. Kosters [26] started from the following

\[ \sum_{i=0}^{n} \sum_{g \in G} N(G, i, g) gX^i = \prod_{h \in G} (1 + hX) \in \mathbb{C}[G][X]. \]

Then he used character theory to extract \( N(G, i, g) \). From the perspective of generating function, the proof of Kosters and the above proof of Molièr-Almkvist which used Poincaré series are similar. But as far as we think, the details in these proofs can be different. There are several ways to prove (3.2) and (3.3). For example, Molièr’s original idea ([42] Section 2) which we shall employ later; Almkvist’s method ([2] Theorem 1.33) and ([1] Example 4.6); Panyushev ([37], Lemma 3.1). For more detailed explanations of Molièr’s result, we refer to the Chapter 3 in [36].

Remark 3.3. As we have mentioned that (3.2) is a more general formula, we briefly introduce its applications in the more general setting (for non-abelian groups). Actually, the Poincaré series is a widely studied and used tool in the invariant theory. For example, it was used [40] to provide tight lower bounds for the Noether number of \( Q_8 \) and \( A_4 \), where the Noether number \( \beta(G) \) of a finite group \( G \) which is closely related to zero-sum theory is defined to be the maximal degree bound for the generators of the algebra of polynomial invariants. We refer to [11] for a survey of studies of the Noether number, also to [12, 24] for some recent results on the connection between zero-sum theory and Noether number.

With the help of Remark 2.1, we have the following lemma which is easy but crucial in our proof.

Lemma 3.4. A sequence \( T = (y_0, \ldots, y_{n-1}) \) over \( G \) satisfies \( \sigma(T) = 0 \) if and only if the following congruences

\[ \sum_{k=0}^{n-1} k \left( \sum_{0 \leq j_1 \leq n-1, j_1 \in [1, r] \setminus \{t\}} y_{j_1} n_1 \cdots n_{r-1} + \cdots + k n_1 \cdots n_{r-1} + \cdots + i_2 n_1 + i_1 \right) \equiv 0 \pmod{n_1} \]

hold simultaneously for \( t \in [1, r] \).

Proof. Considering the coefficients of \( e_t \) in \( \sigma(T) \) for any \( t \in [1, r] \), then we get the desired result. \( \square \)

Proof of the Theorem 1.2. (1) Let \( |G| = n \) and \( |H| = m \). Let \( S = (x_0, \ldots, x_{n-1}) \) be a zero-sum sequence over \( G \) of length \( m \), we have to construct a unique \((n, m)\)-Dyck path \( P \) using the vector \( (x_0, \ldots, x_{n-1}) \). The method here is essentially the same as the proof of Theorem 12.1 in [32]. The only difference is that, in [32], each path was associated with a vector of length \( n + m + 1 \) instead. We provide the complete proof here for the convenience of readers.

Firstly we construct a path (not necessarily Dyck path) \( Q \) from \((0, 0)\) to \((n, m)\) as follows. Let \( (i, \sum_{j=0}^{i-1} x_j) \) be the lowest lattice point in the \( i \)-th column of the path \( Q \) where \( i \in [1, n - 1] \). We associate \( Q \) with a vector \( (y_1, \ldots, y_{n-1}) \), where \( y_i = \sum_{j=0}^{i-1} x_j - \frac{m}{n} i \). Then it is obvious that \( Q \) is a Dyck path if and only if \( y_i > 0 \) holds for any \( i \in [1, n - 1] \).
As \((n, m) = 1\) and \(\sum_{j=0}^{i-1} x_j \leq m\) for \(i \in [1, n-1]\), it can be verified that \(y_i \neq y_j\) for any \(i \neq j\). Therefore we may denote the unique minimal element as \(y_\lambda\) for some \(\lambda \in [1, n-1]\). Moreover, we define \(Q^i\) as a path obtained from the vector
\[
(x_i, x_{i+1}, \ldots, x_{n-1}, x_0, \ldots, x_{l-1}),
\]
that is \((i, \sum_{j=0}^{l-1} x_{l+j})\) be the lowest lattice point in the \(i\)-th column of the path \(Q^i\) where \(i \in [1, n-1]\). Note that for any integer \(t\) with \(t \equiv i \pmod{n}\), we have \(x_t = x_i\) and \(y_t = y_i\). Therefore we have \(Q^n = Q\). We denote \(y_i^i = \sum_{j=0}^{l-1} x_{l+j} - \frac{n}{n} i\). Similarly, \(Q^i\) is a Dyck path if and only if
\[
y_i^i > 0\]
holds for any \(i \in [1, n-1]\).

By \((3.8)\) and \((3.9)\), we have \(y_i^i = y_{i+\lambda} - y_\lambda\). By the minimality of \(y_\lambda\), we have \(y_i^i > 0\) for any \(i \in [1, n-1]\). Therefore we obtain a unique \((n, m)\)-Dyck path \(Q^\lambda\) which corresponds to the vector \((x_0, \ldots, x_{n-1})\).

Conversely, let \(P\) be an \((n, m)\)-Dyck path from \((0, 0)\) to \((n, m)\), then it is clear that \(P\) corresponds to a vector \(P = (x_0, \ldots, x_{n-1})\) with \((i, \sum_{j=0}^{l-1} x_{l+j})\) be the highest lattice point in the \(i\)-th column of the path \(P\). We assume that
\[
\sum_{i=0}^{n-1} x_i \equiv \alpha \pmod{n},
\]
as \(\sum_{i=0}^{n-1} x_i = m\) and \((n, m) = 1\), it is easy to see that there is a unique cyclic shift \((x'_0, \ldots, x'_{n-1})\) of \((x_0, \ldots, x_{n-1})\) such that
\[
\sum_{i=0}^{n-1} x'_i \equiv 0 \pmod{n}.
\]
By the first part of the proof, we know that \(P\) is the unique Dyck path among other non-trivial cyclic shifts of \(P\). Therefore without lose of generality, we may assume that \(P\) satisfies
\[
\sum_{i=0}^{n-1} x_i \equiv 0 \pmod{n}. \tag{3.8}
\]
That is \(P = (x_0, \ldots, x_{n-1})\) corresponds to a zero-sum sequence over \(C_n\). As we mentioned \((2.2)\) that \((x_0, \ldots, x_{n-1})\) also corresponds to a sequence \(T\) over \(G\), though we not necessarily have \(\sigma(T) = 0\). We will prove that a cyclic shift of \((x_0, \ldots, x_{n-1})\) will corresponds to a zero-sum sequence \(T'\) over \(G\).

By \((3.8)\) and \(n_1|n\), it is easy to see that
\[
\sum_{j=0}^{n_1-1} j \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1, r] \setminus \{1\}} x_{i_j, n_1, \ldots, n_2-1, \ldots, i_2 n_1} + \right) \equiv 0 \pmod{n_1}. \tag{3.9}
\]
Therefore the case for \(n_1\) in \((3.7)\) holds immediately. Also, according to \((3.7)\), we assume that
\[
\sum_{k=0}^{n_2-1} k \left( \sum_{0 \leq i_k \leq n_k-1, j \in [1, r] \setminus \{2\}} x_{i_k, n_1, \ldots, n_2-1, \ldots, i_2 n_1} + \right) \equiv \alpha_2 \pmod{n_2} \tag{3.10}
\]
Let \(T^{2,l}\) be a sequence over \(G\) obtained by cyclically shifting \(T\) in the following way:
\[
T^{2,l} = (x_1, \ldots, x_{n-1}, x_0, x_1, \ldots, x_{l-1}).
\]
Then for the sequence $T^{2,l_{n_1}}$, (3.10) becomes
\begin{equation}
\sum_{k=0}^{n_2-1} k \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1,r] \setminus \{2\}} x_{i_{r,n_1\cdots n_{r-1}+\cdots+kn_1+i_1+l_{n_1}}} \right)
\equiv \sum_{k=0}^{n_2-1} (k + l - l) \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1,r] \setminus \{2\}} x_{i_{r,n_1\cdots n_{r-1}+\cdots+(k+l)n_1+i_1}} \right)
\equiv \alpha_2 - lm \pmod{n_2}.
\end{equation}

As $(n_2,m) = 1$, there exists $l_2 \in [0,n_2 - 1]$ such that $\alpha_2 - l_2m \equiv 0 \pmod{n_2}$. Meanwhile, it is easy to see that
\begin{equation}
\sum_{j=0}^{n_1-1} j \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1,r] \setminus \{1\}} x_{i_{r,n_1\cdots n_{r-1}+i_2n_1+j+l_{2n_1}}} \right)
\equiv \sum_{j=0}^{n_1-1} (j + l_2n_1 - l_2n_1) \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1,r] \setminus \{1\}} x_{i_{r,n_1\cdots n_{r-1}+i_2n_1+j+l_{2n_1}}} \right)
\equiv \sum_{j=0}^{n_1-1} (j + l_2n_1) \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1,r] \setminus \{1\}} x_{i_{r,n_1\cdots n_{r-1}+i_2n_1+j+l_{2n_1}}} \right) - l_2n_1m
\equiv 0 - l_2n_1 \equiv 0 \pmod{n_1},
\end{equation}
which means that the case for $n_1$ in (3.7) still holds for the sequence $T^{2,l_{n_1}}$. (This is the key idea in the proof). For simplicity, we still denote $T^{2,l_{n_1}}$ by $(x_0,\ldots,x_{n_1-1})$.

Next we consider
\begin{equation}
\sum_{k=0}^{n_3-1} k \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1,r] \setminus \{3\}} x_{i_{r,n_1\cdots n_{r-1}+\cdots+kn_1n_2+i_2n_1+i_3+l_{n_1n_2}}} \right)
\equiv \alpha_3 \pmod{n_3}
\end{equation}

Let $T^{3,l}$ be a sequence over $G$ obtained by cyclically shifting $T^{2,l_{2n_1}}$ in the following way:
\begin{equation}
T^{3,l} = (x_1,\ldots,x_{n_1-1},x_0,x_1,\ldots,x_{l_1}).
\end{equation}

Then for the sequence $T^{3,l_{n_1n_2}}$, (3.13) becomes
\begin{equation}
\sum_{k=0}^{n_3-1} (k + l - l) \left( \sum_{0 \leq i_j \leq n_j-1, j \in [1,r] \setminus \{3\}} x_{i_{r,n_1\cdots n_{r-1}+\cdots+(k+l)n_1n_2+i_2n_1+i_3+l_{n_1n_2}}} \right)
\equiv \alpha_3 - lm \pmod{n_3}.
\end{equation}

As $(n_3,m) = 1$, there exists $l_3 \in [0,n_3 - 1]$ such that $\alpha_3 - l_3m \equiv 0 \pmod{n_3}$. Meanwhile, similar to above, it is easy to see that, the cases for $n_1$ and $n_2$ in (3.7) still hold for the sequence $T^{3,l_{n_1n_2}}$. Continuing this process, we will obtain a unique sequence $T^{r,l_{r,n_1\cdots n_{r-1}}}$ such that (3.7) hold simultaneously for every $t \in [1, r]$. Therefore, $T^{r,l_{r,n_1\cdots n_{r-1}}}$ is a zero-sum sequence over $G$ of length $m$ which corresponds to $(n,m)$-Dyck path $P$. This completes the bijection between $M(G,m)$ and $D_{n,m}$.
For bijection between $M(G, m)$ and $M(H, n)$, we use the necklace interpretation. Let $S = (x_0, \ldots, x_{n-1})$ be a zero-sum sequence over $G$ of length $m$, then we define a necklace $\mathcal{L}_S$ associated to $S$ as follows. Let $\mathcal{L}_S$ be a necklace with $n + m$ beads, $n$ of them are colored by red ($R$) and the rest $m$ beads are colored by blue ($B$), and they are arranged in the following way
\[
(R, B, \ldots, B, R, B, \ldots, B, R, \ldots, R, B, \ldots, B).
\]
That is, $x_i$ ($i \in [1, n - 1]$) is the number of blue beads between two successive red beads (read $x_{n-1}$ cyclically). Let $y_i$ ($i \in [0, m - 1]$) be the number of red beads between two successive blue beads. Then we have a vector $T = (y_0, \ldots, y_{m-1})$ with $\sum_{i=0}^{n-1} y_i = n$. As we have mentioned, $T$ corresponds to a sequence over $H$ of length $n$, though we not necessarily have $\sigma(T) = 0$. Similar to the above, we will find a unique cyclic shift $T'$ of $(y_0, \ldots, y_{m-1})$ such that $T'$ corresponds to a zero-sum sequence over $H$. This completes the proof of (1).

(2) Let $G = C_n^r$ and $H = C_n^r$. Then for $d|n$ we have
\[
\varphi_G(d) = \sum_{l|d} \mu \left( \frac{d}{l} \right) l',
\]
and for $d|m$ we have
\[
\varphi_H(d) = \sum_{l|d} \mu \left( \frac{d}{l} \right) l'.
\]
Therefore, for any $m \geq 1$ we have
\[
M_G(m') = \frac{1}{n^r + m^r} \sum_{d|(n,m')} \varphi_G(d) \left( \frac{n^r + m^r}{d} \right) = M_H(n').
\]

(3) We assume that $|G| = n$. Let $S$ be a zero-sum subset of $G$ of cardinality $k$. Similar to the sequence, $S$ corresponds to a vector $(x_0, \ldots, x_{n-1})$ with $x_i \in \{0, 1\}$ and $\sum_{i=0}^{n} x_i = k$. We will construct a $(k, n-k)$-Dyck path corresponds to $S$. As $x_i \in \{0, 1\}$ for any $i \in [0, n - 1]$, we associate $S = (x_0, \ldots, x_{n-1})$ with a lattice path $L_S$ (not necessarily a Dyck path) as follows. Let 0 be a one-step lattice walk towards north and 1 towards east, then $S = (x_0, \ldots, x_{n-1})$ corresponds to a lattice path $L_S$ from $(0,0)$ to $(k, n-k)$ which only goes north and east. Similar to the previous method, it is easy to see that there is a unique $(k, n-k)$-Dyck path among the cyclic shifts of $(x_0, \ldots, x_{n-1})$.

Conversely, let $P = (x_0, \ldots, x_{n-1})$ be a $(k, n-k)$-Dyck path with $x_i \in \{0, 1\}$ which depends on the direction of the $(i+1)$-th step, 0 for the north and 1 for the east, where $i \in [0, n-1]$. Then by our previous construction, $(x_0, \ldots, x_{n-1})$ corresponds to a subset of $G$ of cardinality $k$. As $(k, n) = 1$, similar to the previous method, it is easy to see that there is a unique cyclic shift $S$ of $(x_0, \ldots, x_{n-1})$ such that $S$ is a zero-sum subset of $G$. This completes the proof.

\[\square\]

Remark 3.5. Let $(n, m) = 1$. In [4], Anderson provided, now well-known to experts, an elegant bijection between the set of $(n, m)$-core partitions and $\mathcal{D}_{n,m}$ using the abacus construction. Later in [25], while proving a conjecture of Armstrong [6]...
concerning the average size of the \((n,m)\)-core partitions, as a byproduct, Johnson obtained an interesting bijection between \(M(\mathbb{Z}/n\mathbb{Z}, m)\) and the set of \((n,m)\)-core partitions. He used the abacus construction and various coordinate changes from the perspective of the Ehrhart theory. Therefore, combining these results, although slightly complicated, one gets the bijection between \(M(\mathbb{Z}/n\mathbb{Z}, m)\) and \(D_{n,m}\). Our proof of (1) provide a direct bijection between \(M(G, m)\) and \(D_{n,m}\), without using the abacus construction.

Next, we are going to prove Theorem 1.3. We recall (3.3) in the following:

\[
\mathcal{F}\left((\mathcal{S}(R) \otimes \wedge(R))_{G}; s, t\right) = \frac{1}{|G|} \sum_{d \geq 1} \left( \varphi_{G}(d) \left( \frac{1 - (-t)^d}{1 - s^d} \right)^{|G|/d} \right).
\]

**Proof of the Theorem 1.3.** By (3.4) and after extracting the coefficient of \(s^pt^m\), we have

\[
\dim (\mathcal{S}^p(R) \otimes \wedge^m(R))_{G_{q+m}} = \frac{1}{p + q + m} \sum_{d \mid (p,q,m)} \varphi_{G_{q+m}}(d) \left( \frac{p + q + m}{d} \right) \left( \frac{p}{d} \right) \left( \frac{q}{d} \right) \left( \frac{m}{d} \right)
\]

and

\[
\dim (\mathcal{S}^q(R) \otimes \wedge^m(R))_{H_{p+m}} = \frac{1}{p + q + m} \sum_{d \mid (p,q,m)} \varphi_{H_{p+m}}(d) \left( \frac{p + q + m}{d} \right) \left( \frac{p}{d} \right) \left( \frac{q}{d} \right) \left( \frac{m}{d} \right).
\]

Under the assumption that \((p, q, m) = 1\), the desired result follows immediately. \(\square\)

**Remark 3.6.** As we have mentioned the problem proposed by Panyushev [37], we provide a combinatorial interpretation of Theorem 1.3 under a further condition that \((p, q + m) = (q, p + m) = 1\). In this case, by the definition of the symmetric tensor exterior algebra, it is easy to see that calculating

\[
\dim (\mathcal{S}^p(R) \otimes \wedge^m(R))_{G_{q+m}}
\]

is equivalent to counting the number the pairs

\[(3.15) \quad (A, B),\]

where \(A\) is a sequence over \(G_{q+m}\) of length \(p\) and \(B\) is a subset of \(G_{q+m}\) of cardinality \(m\) such that \(\sigma(A) + \sigma(B) = 0\) in \(G_{q+m}\). Note that

\[
\frac{1}{p + q + m} \left( \frac{p + q + m}{p} \right) \left( \frac{q + m}{m} \right) = \frac{1}{p + q + m} \left( \frac{p + q + m}{p} \right) \left( \frac{q + m}{m} \right).
\]

Suppose that \((S, T)\) is a pair satisfies the above condition (3.15). Now, we have to correspond \((S, T)\) to a pair \((U, V)\) where \(U\) is a sequence over \(H_{p+m}\) of length \(q\) and \(V\) is a subset of \(H_{p+m}\) of length \(m\), such that \(\sigma(U) + \sigma(V) = 0\).

Based on Remark 2.1, we may assume that \(S = (x_0, \ldots, x_{q+m-1})\) and \(T = (y_0, \ldots, y_{q+m-1})\). We will construct a necklace \(\mathcal{L}_{S,T}\) associated to \((S, T)\). Let \(\mathcal{L}_{S,T}\) be a necklace with \(p + q + m\) beads \((p\) red \((R)\) beads, \(m\) green \((G)\) beads and \(q\) blue \((B)\) beads). Fixing a bead \(\mathcal{D}\) among these \(p+q+m\) beads. Proceeding in the clockwise direction from \(\mathcal{D}\), let

\[x_0, \ldots, x_{q+m-1}\]

be the number of \(R\) beads between two successive \(G\) or \(B\) beads (read cyclically). Also proceeding in the clockwise direction from \(\mathcal{D}\), let \((y_0, \ldots, y_{q+m-1})\) be the positions of the \(G\) beads among the \(G\) and \(B\) beads. That is, there are \(q+m\) \(G\) or
\textbf{B} beams, \(m\) of them are \(\mathcal{G}\) which are located in places \(\{i_1, \ldots, i_m\} \subset \{1, \ldots, q + m\}\) (read cyclically), then \(y_i = 1\) if and only if \(i \in \{i_1, \ldots, i_m\}\).

Proceeding in the clockwise direction from \(\Omega\), let

\[U = (u_0, \ldots, u_{p+m-1}),\]

where the \(u_i\)'s are the number of \(\mathcal{B}\) beams between two successive \(\mathcal{R}\) or \(\mathcal{G}\) beams (read cyclically). Also proceeding in the clockwise direction from \(\Omega\), let \(V = (v_0, \ldots, v_{p+m-1})\) be the positions of the \(\mathcal{G}\) beams among the \(\mathcal{R}\) and \(\mathcal{G}\) beams. That is, there are \(p + m\) \(\mathcal{R}\) or \(\mathcal{G}\) beams, \(m\) of them are \(\mathcal{G}\) which are located in places \(\{i_1, \ldots, i_m\} \subset \{1, \ldots, p + m\}\) (read cyclically), then \(y_i = 1\) if and only if \(i \in \{i_1, \ldots, i_m\}\). Then \(V\) be a subset of \(H_{p+m}\) of cardinality \(m\). Consequently, similar to the above proofs, there is a unique cyclic shift \(U^l\) of \(U\) such that \(\sigma(U^l) + \sigma(V) = 0\), and \((U^l, V)\) is exactly what we want to find.

\section{On a subset counting problem}

In this section, we will prove Theorems 1.4 and 1.5. We will provide both combinatorial and algebraic proofs for Theorem 1.4.

\textbf{Proof of Theorem 1.4 (Combinatorial proof):} We will prove the case \(b = 0\) in 1.4, as it is easy to see the following proof also valid for any non-zero \(b \in \mathbb{G}\).

Let \((b_1, \ldots, b_k) \in \mathbb{G}^k\) be a solution of (1.4), i.e.,

\[\sum_{i=1}^{k} a_i b_i = 0.\]

We assume that

\[\{g_{i_1}, \ldots, g_{i_k}\} = \{b_1, \ldots, b_k\}\]

with \(i_1 < \cdots < i_k\) (see our discussion before Lemma 3.4).

We associate the solution \((b_1, \ldots, b_k)\) with a vector \((z_0, \ldots, z_{n-1})\) with

\[z_j = \begin{cases} a_j, & \text{if } j \in \{i_1, \ldots, i_k\}, \\ 0, & \text{otherwise}. \end{cases}\]

where \(0 \leq j \leq n - 1\). Since \(\sum_{j=0}^{n-1} z_j = m\), following the same way in the proof of Theorem 1.2 (1), \((z_0, \ldots, z_{n-1})\) will correspond to a lattice path \(Q\) from \((0, 0)\) to \((n, m)\) \((z_i\) is the length of the vertical run of the \((i+1)\)-th step). As \(\gcd(n, m) = 1\), also following the same way in the proof of Theorem 1.2 (1), there is a unique \((n, m)\)-Dyck path \(Q^l\) among all the cyclic shifts of \(Q\), where

\[Q^l = (z_l, z_{l+1}, \ldots, z_{n-1}, z_0, \ldots, z_{l-1}).\]

Moreover \(Q^l\) is a \((n, m)\)-Dyck paths with \(c_i\) vertical runs of length \(i\).

Conversely, let \(P\) be an \((n, m)\)-Dyck paths in \(\mathcal{D}_{n,m}\) with \(c_i\) vertical runs of length \(i\). Then we may associate \(P\) with a vector \(V_P = (d_0, \ldots, d_{n-1})\) such that \(d_i\) is the length of the vertical run on the line \(x = i\). Then \(a_1, \ldots, a_k\) are the only non-zero elements in \(\{d_0, \ldots, d_{n-1}\}\). By the discussion before Lemma 3.4, \(V_P\) corresponds to a sequence \(S\) over \(\mathbb{G}\) of length \(\sum_{j=0}^{n-1} d_j = \sum_{j=1}^{k} a_j = m\). For any \(l \in [0, n - 1]\), we define \(S^l = (d_l, \ldots, d_{n-1}, d_0, \ldots, d_{l-1})\) as a cyclic shift of \(S\). As \(\gcd(n, m) = 1\),
following the same way in the proof of Theorem 1.2 (1), we can find a unique \( t \in [1, n - 1] \) such that \( S^t = (d_1, \ldots, d_{n-1}, d_0, \ldots, d_t) \) satisfying the following

\[
\sum_{j=0}^{n-1} d_{t+j}g_j = 0.
\]

Recall that there are exactly \( k \) non-zero elements

\[
\{d_{t+i_1}, \ldots, d_{t+i_k}\} = \{a_1, \ldots, a_k\} \subset \{d_0, \ldots, d_{n-1}\}.
\]

Therefore, (4.1) now becomes

\[
\sum_{j=1}^{k} d_{t+i_j}g_j = 0.
\]

In another words, we get a solution (a rearrangement of \( g_{i_1}, \ldots, g_{i_k} \)) of (4.1).

With the above bijection, together with the definition and known result of the Kreweras number, we get the desired result.

**Algebraic proof:** Our method employ tools from the invariant theory (see for example [12] Section 2). This idea is essentially due to Molien.

Let \( V = (e_1) \oplus \cdots \oplus (e_n) \) be the regular representation of \( G \) over \( \mathbb{C} \) with \( g \cdot e_i = \chi_i(g)e_i \), where \( \chi_i \in \hat{G} \). For any \( b \in G \), let \( \hat{b} \) be the element in \( \hat{G} \) which corresponds to \( b \) under the isomorphism \( G \cong \hat{G} \). Let \( S(V) \) be the symmetric algebra of \( V \).

Let \( \Lambda^{a_1, \ldots, a_k}(V) \) be a subspace of \( S(V) \) generated by elements of the following forms

\[
e_{i_1} \otimes \cdots \otimes e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \cdots \otimes e_{i_k}
\]

where \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \). Recall that

\[
\{a_1, \ldots, a_k\} = \{1, \ldots, i_1, \ldots, i_k, m, \ldots, m\}
\]

and that \( c_0 \) is defined such that \( c_0 + \sum_{i=1}^{m} c_i = n \), then we have

\[
\dim \Lambda^{a_1, \ldots, a_k}(V) = \binom{n}{c_0, \ldots, c_m}.
\]

We define a set \( B(a_1, \ldots, a_k) \) consists of the k-tuples \( (i_1, \ldots, i_k) \) such that

\[
e_{i_1} \otimes \cdots \otimes e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \cdots \otimes e_{i_k}
\]

form the basis of \( \Lambda^{a_1, \ldots, a_k}(V) \).

The action of \( G \) on \( V \) can be induced on \( S(V) \) and further on \( \Lambda^{a_1, \ldots, a_k}(V) \) as follows:

\[
g \cdot (e_{i_1} \otimes \cdots \otimes e_{i_1} \otimes \cdots \otimes e_{i_k} \otimes \cdots \otimes e_{i_k}) = g \cdot e_{i_1} \otimes \cdots \otimes g \cdot e_i \otimes \cdots \otimes g \cdot e_{i_k} \otimes \cdots \otimes g \cdot e_{i_k} = \chi_{i_1}(g) \cdot e_{i_1} \otimes \cdots \otimes \chi_{i_1}(g) \cdot e_{i_1} \otimes \cdots \otimes \chi_{i_k}(g) \cdot e_{i_k} \otimes \cdots \otimes \chi_{i_k}(g) \cdot e_{i_k}
\]

Clearly, \( \Lambda^{a_1, \ldots, a_k}(V) \) is a representation space of \( G \). Let \( \psi \) denote the character of this representation and \( \chi_0 \) the character of trivial representation of \( G \).
Therefore we have

Finally, we have

By the definition of \( \psi \), we have

By our assumption that \( \gcd(a_1 + \cdots + a_k, n) = 1 \), for any non-identity \( g \), there exists \( \chi \in \widehat{G} \) such that \( \chi^{a_1 + \cdots + a_k} (g) \neq 1 \). Then we have

Therefore we have \( \sum_{(i_1, \ldots, i_k) \in B(a_1, \ldots, a_k)} \chi_{i_1}^{a_1} \cdots \chi_{i_k}^{a_k} (g) = 0 \) for any non-identity \( g \).

Finally, we have

This completes the proof.

**Elementary proof:** Firstly, it is easy to see that there are \( \binom{n}{c_0, \ldots, c_m} \) non-equivalent subsets of \( G \) with \( k \) elements. We will prove that for any \( g, h \in G \),

\[
\#N_G(a_1, \ldots, a_k, g) = \#N_G(a_1, \ldots, a_k, h).
\]

Let \( \{g_1, \ldots, g_k\} \) be a representative in \( N_G(a_1, \ldots, a_k, g) \), that is

\[
a_1g_1 + \cdots + a_kg_k = g.
\]

Assume that \( h = g + h' \). As \( \left( \sum_{i=1}^{k} a_i, n \right) = 1 \), we have \( \left( \sum_{i=1}^{k} a_i, \text{ord}(h') \right) = 1 \). Therefore, there exists \( \alpha \in [1, \text{ord}(h')] \) such that

\[
\left( \sum_{i=1}^{k} a_i \right) \alpha \equiv 1 \pmod{\text{ord}(h')}.
\]

Consequently, the following map

\[
\psi : N_G(a_1, \ldots, a_k, g) \to N_G(a_1, \ldots, a_k, h)
\]

\[
\{g_1, \ldots, g_k\} \mapsto \{g_1 + \alpha h', \ldots, g_k + \alpha h'\}
\]

is a bijection. This completes the proof. \( \square \)
Next, we are going to prove Theorem 4.3, i.e., find all the non-equivalent solutions of the following equation

$$a_1x_1 + x_2 + \cdots + x_k = b,$$

where $a_1 \in \mathbb{N}$ and $b \in G$. Let $\hat{b}$ be the element in $\hat{G}$ which corresponds to $b$ under the isomorphism $G \cong \hat{G}$.

**Proof of Theorem 4.3** Let $\Lambda^{k,a_1}(V)$ be a subspace of $S(V)$ generated by elements of the following forms

$$e_i \otimes \cdots \otimes e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}$$

where $1 \leq i \leq n$ and $\{i_1, \ldots, i_{k-1}\} \subset \{1, \ldots, n\} \setminus \{i\}$ for $1 \leq i \leq n$.

As the action of $G$ on $V$ can be induced on $S(V)$, therefore it can be induced on $\Lambda^{k,a_1}(V)$ as follows:

$$
\begin{align*}
    & g \cdot e_i \otimes \cdots \otimes e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_{k-1}} \\
    & = g \cdot e_i \otimes \cdots \otimes g \cdot e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_{k-1}} \\
    & = \chi_i(g)e_i \otimes \cdots \otimes \chi_i(g)e_i \otimes \chi_i(1)e_{i_1} \otimes \cdots \otimes \chi_{i_{k-1}}(g)e_{i_{k-1}} \\
    & = \chi_i(g)^{a_1} \chi_i(1)(g) \cdots \chi_{i_{k-1}}(g)e_i \otimes \cdots \otimes e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_{k-1}} \\
    & = \chi_i^2 \chi_{i_1} \cdots \chi_{i_{k-1}}(g)e_i \otimes \cdots \otimes e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_{k-1}}
\end{align*}
$$

Clearly, $\Lambda^{k,a_1}(V)$ is a representation space of $G$. Let $\psi$ denote the character of this representation.

Consider $\Lambda^{k,a_1}(V)_{G,\hat{b}}$, the isotypic component of $\Lambda^{k,a_1}(V)$ which corresponds to the character $\hat{b}$. Then by a basic result in the representation theory [41] (Theorem 4, P.16), we have the following.

$$
N_G(a_1, 1, \ldots, 1, b) = \dim \Lambda^{k,a_1}(V)_{G,\hat{b}} = \langle \psi, \hat{b} \rangle = \frac{1}{n} \sum_{g \in G} \psi(g) \hat{b}(g^{-1}).
$$

By the definition of $\psi$, we have

$$
\langle \psi, \hat{b} \rangle = \frac{1}{n} \sum_{g \in G} \psi(g) \hat{b}(g^{-1})
$$

$$
= \frac{1}{n} \sum_{g \in G} \left( \sum_{i=1}^{n} \left( \chi_i(g)^{a_1} \left( \sum_{\{i_1, \ldots, i_{k-1}\} \subset \{1, \ldots, n\} \setminus \{i\}} \chi_{i_1} \cdots \chi_{i_{k-1}}(g) \right) \right) \hat{b}(g^{-1}) \right)
$$
Consider the formal power series $\sum_{k=1}^{n} \dim \Lambda^{k,a_1}(V)_{G,b}(T)^{k+a_1-1}$. We have
\[
\sum_{k=1}^{n} \dim \Lambda^{k,a_1}(V)_{G,b}(T)^{k+a_1-1} = \frac{1}{n} \sum_{g \in G} \hat{b}(g^{-1}) \left( \sum_{i=1}^{n} \chi_i(g)^{a_1} \frac{1}{1 - \chi_i(g)T} \prod_{j=1}^{\dim \Lambda} (1 - \chi_j(g)T) \right)
\]
\[
= \frac{1}{n} \sum_{g \in G} \hat{b}(g^{-1}) \det(E - gT) \left( \sum_{u \geq a_1} \sum_{i=1}^{n} \chi_i(g)^{u}T^u \right)
\]
\[
= \frac{(-1)^{a_1}}{n} \sum_{d \mid n, g \cdot \text{ord}(g) = d} \hat{b}(g^{-1}) \left( \left(1 - T^d\right)^{\frac{n}{d}} \sum_{m \geq \frac{a}{m}} mT^{md} \right)
\]
\[
= \frac{(-1)^{a_1}}{n} \sum_{d \mid n, g \cdot \text{ord}(g) = d} \hat{b}(g) \left( \sum_{j=0}^{\frac{n}{d}} \left(\begin{array}{c} \frac{n}{d} \\ j \end{array}\right) \sum_{m \geq \frac{a}{d}} T^{jmd} \right)
\]
\[
= \frac{(-1)^{a_1}}{n} \sum_{d \mid n, g \cdot \text{ord}(g) = d} \hat{b}(g) \left( \sum_{m \geq \frac{a}{d}} \sum_{j=0}^{\frac{n}{d}} \left(\begin{array}{c} \frac{n}{d} \\ j \end{array}\right) (-1)^j T^{(j+m)d} \right)
\]

Therefore, we have
\[
N_{G}(a_1, 1, \ldots, 1, b) = \dim \Lambda^{k,a_1}(V)_{G,b}
\]
\[
= (-1)^{k-1} \sum_{d \mid (n,k+a_1-1), m \geq \frac{a}{d}, g \cdot \text{ord}(g) = d} \hat{b}(g) \left( \frac{n}{d} - m \right) (-1)^{k+a_1-1-m}
\]

Recall that
\[ G = \mathbb{Z}/n_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r \mathbb{Z} = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle \]
is a finite abelian group with $|G| = n$. Let
\[ b = b_1e_1 + \cdots + b_r e_r \in G, \]
where $b_i \in [0, n_i - 1]$ for every $i$. Similar to [10], one obtains the following
\[
\sum_{g \cdot \text{ord}(g) = d} \hat{b}(g) = \sum_{\chi \in G, \text{ord}(\chi) = d} \chi(b) = \sum_{l \mid d, (n_i, l) \mid b_i} \mu \left( \frac{d}{l} \right) \prod_{i=1}^{r} (n_i, l).
\]
This completes the proof. □

Acknowledgments. We learned about the problems in this paper after attending several interesting talks of Prof. Jiyou Li, we would like to thank him for helpful discussions. A part of this work was done during a visit by the second author to the Karl Franzens Universität Graz in the spring semester 2019, he would like to thank Prof. Alfred Geroldinger for invitation and providing him with a wonderful working environment, as well as many helpful comments on the manuscript. We thank Dr. Qinghai Zhong for suggesting the elementary proof of Theorem 14 and
many useful discussions. We are also grateful for Prof. Weidong Gao for his support and encouragement all the time.

\textbf{References}

1. G. Almkvist, \textit{Endomorphisms of finitely generated projective modules over a commutative ring}, Ark. Mat. 11 (1973) 263-301.
2. G. Almkvist, \textit{Some formulas in invariant theory}, J. Algebra 77 (1982) 338-359.
3. G. Almkvist and R. Fossum, \textit{Decomposition of exterior and symmetric powers of indecomposable \(\mathbb{Z}/p\mathbb{Z}\)-modules in characteristic \(p\) and relations to invariants}, In: Séminaire d’Algèbre P. Dubreil, Paris, 1976-1977. Lecture Notes in Math., vol. 641, pp. 1-111. Springer, Berlin (1978).
4. J. Anderson, \textit{Partitions which are simultaneously \(t_1\)- and \(t_2\)-core}, Discrete Math. 248 (2002) 237-243.
5. D. Armstrong, \textit{Generalized noncrossing partitions and combinatorics of Coxeter groups}, Mem. Amer. Math. Soc. 949, Amer. Math. Soc., Providence, RI (2009).
6. D. Armstrong, C. Hanusa and B. Jones, \textit{Results and conjectures on simultaneous core partitions}, European J. Combin. 41 (2014) 205-220.
7. D. Armstrong, N.A. Loehr and G.S. Warrington, \textit{Rational parking functions and Catalan numbers}, Ann. Combin. 12 (2016) 21-58.
8. D. Armstrong, B. Rhoades and N. Williams, \textit{Rational associahedra and noncrossing partitions}, Electron. J. Combin 20(3) (2013) #P 54.
9. M.T.L. Bizley, \textit{Derivation of a new formula for the number of minimal lattice paths from (0, 0) to \((km, kn)\) having just \(t\) contacts with the line \(my = nx\) and having no points above this line; and a proof of Grossman’s formula for the number of paths which may touch but do not rise above this line}, J. Inst. Actuar. 80 (1954) 55-62.
10. M. Bodnar and B. Rhoades, \textit{Cyclic sieving and rational Catalan theory}, Electron. J. Combin, 23(2) 2016 #P 4.
11. K. Cziszter, M. Domokos and A. Geroldinger, \textit{The interplay of invariant theory with multiplicative ideal theory and with arithmetic combinatorics}, \textit{Multiplicative Ideal Theory and Factorization Theory}, Springer Proceedings in Mathematics and Statistics, vol. 170, Springer, (2016) 43-95.
12. K. Cziszter, M. Domokos and I. Szöllösi, \textit{The Noether numbers and the Davenport constants of the groups of order less than 32}, J. Algebra, 510 (2018) 513-534.
13. A.G. Elashvili and M. Jibladze, \textit{Hermite reciprocity for the regular representations of cyclic groups}, Indag. Math. 9 (1998) 233-238.
14. A.G. Elashvili, M. Jibladze, D. Pataraa, \textit{Combinatorics of Necklaces and Hermite Reciprocity}, J. Algebraic Comb. 10 (1999) 173-188.
15. M. Fredman, \textit{A symmetry relationship for a class of partitions}, J. Combin. Theory Ser. A 18 (1975) 199-202.
16. A.M. Garsia and M. Haiman, \textit{A remarkable q, t-Catalan sequence and q-Lagrange inversion}, J. Algebraic Combin. 5(3) (1996) 191-244.
17. W.D. Gao and A. Geroldinger, \textit{Zero-sum problems in finite abelian groups: A survey}, Expo. Math. 24 (2006) 337-369.
18. A. Geroldinger and F. Halter-Koch, \textit{Non-unique factorizations}. Algebraic, Combinatorial and Analytic Theory, Pure Appl. Math., vol. 278, Chapman & Hall/CRC, 2006.
19. E. Gorsky and M. Mazin, \textit{Compactified Jacobians and q, t-Catalan numbers, I}, J. Combin. Theory Ser. A 120(1) (2013) 49-63.
20. E. Gorsky and M. Mazin, \textit{Compactified Jacobians and q, t-Catalan numbers, II}, J. Algebraic Combin. 39(1) (2014) 153-186.
21. E. Gorsky, M. Mazin and M. Vazirani, \textit{Affine permutations and rational slope parking functions}, Trans. Amer. Math. Soc. 368(12) (2016) 8403-8445.
22. D.J. Grynkiewicz, \textit{Structural additive theory}, Developments in Mathematics, 30. Springer, Cham, 2013.
23. M. Haiman, \textit{Conjectures on the quotient ring by diagonal invariants}, J. Algebraic Combin. 3 (1994) 17-76.
24. D.C. Han and H.B. Zhang, \textit{Erdös-Ginzburg-Ziv theorem and Noether number for \(C_m \ltimes \varphi C_{mn}\)}, J. Number Theory 198 (2019) 159-175.
25. P. Johnson, *Lattice points and simultaneous core partitions*, Electron. J. Combin 25(3) (2018) #P 3.47.
26. M. Kusters, *The subset sum problem for finite abelian groups*, J. Combin. Theory Ser. A 120(2013) 527-539.
27. G. Kreweras, *Sur les partitions non croisées d’un cycle*, Discrete Math. 1 (1972) 333-350.
28. J.Y. Li and D.Q. Wan, *On the subset problem over finite fields*, Finite Fields Appl. 14 (2008) 991-929.
29. J.Y. Li and D.Q. Wan, *Counting subset sums in finite abelian groups*, J. Combin. Theory Ser. A 119 (2012) 170-182.
30. J.Y. Li and D.Q. Wan, *Counting polynomial subset sums*, Ramanujan J. 47 (2018) 67-84.
31. J.Y. Li and X. Yu, *Number of distinct coordinate solutions to linear equations over finite fields*, arXiv: 1905.00306.
32. N.A. Loehr, *Bijective combinatorics*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL, 2011. xxii+590 pp.
33. G. Lusztig and J.M. Smelt, *Fixed point varieties on the space of lattices*, Bull. London Math. Soc. 23 (1991) 213-218.
34. T. Molein, *Über die Invarianten der linearen Substitutionsgruppe*, Sitzungsber. König. Preuss. Akad. Wiss. (1897), 1152-1156.
35. A. Muratović-Ribić and Q. Wang, *The multisubset sum problem for finite abelian groups*, Ars Math. Contemp. 8(2) (2015) 417-423.
36. M.D. Neusel and L. Smith, *Invariant theory of finite groups*, Mathematical Surveys and Monographs, 94
37. D.I. Panyushev, *Fredman’s reciprocity, invariants of abelian groups, and the permanent of the Cayley table*, J. Algebraic Combin. 33 (2011) 111-125.
38. V. Reiner and E. Sommer, *Weyl Group q-Kreweras Numbers and Cyclic Sieving*, Ann. Combin. 22 (2018) 819-874.
39. T.A. Springer, *Invariant theory*, Lecture Notes in Math., vol. 585, Springer, Berlin 1977.
40. B.J. Schmid, *Finite groups and invariant theory*, Topics in invariant theory, Lecture Notes in Math., vol. 1478, Springer, (2003) 35-66.
41. J.P. Serre, *Linear representation theory of finite groups*, Graduate Texts in Math. 42, Springer, New York 1977.
42. R. Stanley, *Invariants of finite groups and their applications to combinatorics*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 3, 475-511.
43. R. Stanley, *Catalan numbers*, Cambridge university press, 2015.

DEPARTMENT OF MATHEMATICS, SOUTHWEST JIAOTONG UNIVERSITY, CHENGDU 610000, P.R. CHINA

E-mail address: handongchun@swjtu.edu.cn

ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, P.R. CHINA

E-mail address: zhanghanbin@amss.ac.cn