Relativistic dynamical friction in stellar systems

Caterina Chiari¹ and Pierfrancesco Di Cintio²,³,⁴

1 Dipartimento di Scienze Fisiche, Informatiche e Matematiche, Università di Modena e Reggio Emilia, Via Campi 213/A, 41125 Modena, Italy
e-mail: caterina.chiari@unimore.it
² CNR – ISC, Via Madonna del piano 10, 50019 Sesto Fiorentino, Italy
e-mail: pierfrancesco.dicintio@cnr.it
³ INAF – Osservatorio Astrofisico di Arcetri, Largo Enrico Fermi 5, 50125 Firenze, Italy
⁴ INFN – Sezione di Firenze, Via G. Sansone 1, 50019 Sesto Fiorentino, Italy

Received 28 November 2022 / Accepted 18 July 2023

ABSTRACT

Aims. We extend the classical formulation of the dynamical friction effect on a test star by Chandrasekhar to the case of relativistic velocities and velocity distributions, also accounting for post-Newtonian corrections to the gravitational force.

Methods. The original kinetic framework was revised and used to construct a special-relativistic dynamical friction formula where the relative velocity changes in subsequent encounters are added up with Lorentz transformation, and the velocity distribution of the field stars accounts for relativistic velocities. Furthermore, a simple expression is obtained for systems where the post-Newtonian correction on the gravitational forces become relevant even at non-relativistic particle velocities. Finally, using a linearized Lagrangian we derived another expression for the dynamical friction expression in a more compact form than previously used.

Results. Comparing our formulation with the classical one, we observe that a given test particle undergoes a slightly stronger drag when moving through a distribution of field stars with relativistic velocity distribution. Vice versa, a purely classical treatment of a system where post-Newtonian (PN) corrections should be included, overestimates the effect of dynamical friction at low test particle velocity, regardless of the form of velocity distribution. Finally, a first-order PN dynamical friction covariant formulation is weaker its classical counterpart at small velocities, but much higher for large velocities over a broad range of mass ratios.

Key words. methods: analytical – galaxies: kinematics and dynamics – stars: kinematics and dynamics – stars: black holes

1. Introduction

Dynamical friction (DF) is an important physical phenomenon that has several consequences in stellar dynamics (and in plasma physics). It can be qualitatively thought of as the slowing-down of a test particle of mass $M$, moving at $v_T$ in a background of field particles of mass $m$, mean number density $n$, and velocity distribution $f(v_T)$, due to the cumulative effect of their long-range gravitational (or Coulomb) interactions.

An analytical estimation of the DF was evaluated for the first time for stellar systems by Chandrasekhar (1943), who found that $M$ must experience a slowing down along its initial direction of propagation as

$$\frac{dv_T}{dt} = -4\pi G^2 n m (M + m) \log \Lambda \frac{\Xi(v_T)}{v_T^3}.$$  (1)

In this equation, $G$ is the gravitational constant, $\log \Lambda$ is the Coulomb logarithm (analogously with the same quantity in plasma physics, Spitzer 1965), of the ratio $\Lambda$ of the maximum and minimum impact parameters $b_{\text{max}}$ and $b_{\text{min}}$ and

$$\Xi(v_T) \equiv 4\pi \int_0^{v_T} f(v_T) v_T^2 dv_T$$  (2)

is the fractional velocity volume function.

The process of DF is crucial for the evolution of collisional systems (see e.g. Binney & Tremaine 1987), from the large scales of galaxies clusters (Ostriker & Tremaine 1975; Richstone 1976; Gunn & Tinsley 1976; Adhikari et al. 2016) to the smaller scales of dense stellar systems, for its consequences on the motion of supermassive black holes (SMBHs) in galactic cores (Antonini & Merritt 2012; Tremmel et al. 2018; Di Cintio et al. 2020; Ricarte et al. 2021; Chen et al. 2022), globular clusters (GCs) orbiting their host galaxies (Weinberg 1989; Colpi & Pallavicini 1998; Bertin et al. 2003; Arena et al. 2006; Arena & Bertin 2007), or exotic stellar objects, for example blue straggler stars (BSS, Ferraro et al. 1995) in GCs (see Paresce et al. 1992, Ferraro et al. 1995, 2001, 2009; Procter Sills et al. 1995; Ransom et al. 2005; Pooley & Hut 2006; Alessandri et al. 2014, 2016; Miocchi et al. 2015; Pasquato et al. 2018; Pasquato & Di Cintio 2020).

Since the pioneering work of Chandrasekhar, the DF formalism, initially conceived for a point-like particle in an infinitely extended background of scatterers, has been extended to the case of finite-sized objects (see e.g. Mulder 1983; Zel’nikov & Kuskov 2016) sinking in the host stellar system, flattened or spherical models with self-gravity (see e.g. Kalnajs 1971; Tremaine & Weinberg 1984), or spheroids with anisotropic velocity distribution (Binney 1977).

More recently, Ciotti & Binney (2004) and Nipoti et al. (2008) derived an expression for the DF formula in the case of modified Newtonian dynamics (MOND, Milgrom 1983; Bekenstein & Milgrom 1984), and performed N-body simulations of sinking satellites in MOND, while Ciotti (2010) considered the effect of a mass spectrum for the field particles and Silva et al. (2016) that of a non-thermal power-law-like velocity distribution. The main conclusion of these works is that, in general, using the original formulation of the DF for idealized...
infinite systems (see Eq. (1)), leads to a substantial underestimation (even by a factor of 10) of its effectiveness when applied to more realistic models.

Prompted by the detection of gravitational waves (GWs) from binary compact objects announced by the LIGO and VIRGO collaborations Abbott et al. (2016a,b), a renewed interest in relativistic stellar dynamics (Shapiro & Teukolsky 1985; Hamers et al. 2014) has recently emerged, in particular with respect to the formation and migration processes of single and binary black holes (BHs) in GCs (Samsing & D’Orazio 2018; Rodriguez et al. 2018; Antonini et al. 2019; Torniamenti et al. 2022), supermassive BHs in galactic cores (Merritt 2015; Fang & Huang 2020; Kelley et al. 2021; Liu & Lai 2022), or runaway objects from dense star clusters (Ryu et al. 2017; Farias et al. 2020; Bhat et al. 2022).

From the theoretical point of view, if on the one hand the distribution function-based approach to relativistic stellar dynamics has been widely explored since Fackerell (1968) made a first attempt (see e.g. Katz et al. 1975; Ellis et al. 1983; Israel & Kandrup 1984; Kandrup 1994, 1984; Kandrup & Morrison 1993; Chavanis 2020a,b and references therein), on the other hand much less has been done in the context of relativistic collisional systems in the original Chandrasekhar picture. Lee (1969) formally extended Eq. (1) accounting for the first post-Newtonian corrections to the gravitational force, though did not evaluate it for an explicit choice of the velocity distribution $f(v)$ or include the effects of relativistic velocities. Nevertheless, he concluded that DF in a dense stellar system, where strong deflections induced by small impact parameters may happen, is always enhanced when considering corrections to the Newtonian force of order $1/c^2$, where $c$ is the speed of light.

Syer (1994) derived an alternative expression for the DF in the case of a relativistic test particle crossing an isotropic medium of lighter particles with $m \ll M$ in the limit of small scattering angles. More recently, Barausse (2007), Katz et al. (2019), Traykova et al. (2021), Vicente & Cardoso (2022), and Correia (2022) explored the onset of DF in relativistic fluids where the test particle induces a wake, and Cashen et al. (2017) included the precession effect induced by the contribution of the gravitomagnetic effect (see e.g. Costa & Natário 2014 and references therein).

In this work we explore this matter further. Our aim is to formulate a general treatment of DF that can be applied in different regimes dominated by collisional effects, involving very large test particle masses and/or relativistic background systems.

The paper is structured as follows. In Sect. 2 we introduce the astrophysical systems where relativistic collisional processes are relevant. In Sect. 3 we revise the classical formalism of DF based on the hyperbolic two-body problem to introduce the fundamental quantities used in the following. In Sect. 4 we derive the (special) relativistic DF expression, complete with a relativistic velocity distribution. In Sect. 5 we compute the post-Newtonian $1/c^2$ corrections to the classical DF expression, and we show an alternative derivation of the relativistic post-Newtonian DF expression using the Darwin Lagrangian. Finally, in Sect. 6 we summarize our findings and outline the possible applications of this work.

2. Relativistic stellar dynamical systems

When discussing relativistic effects in stellar dynamics (in particular with respect to DF) three set-ups can be considered: (i) the test mass $M$ moves at high speed $v$ so that its relativistic (Lorentz) factor

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

becomes significantly larger than unity; (ii) the test mass $M$ is so large that some degree of post-Newtonian approximation should be used when resolving the close approaches with the background stars; (iii) the velocity distribution of the system $f(v)$ has non-negligible relativistic tails.

An important example of the first case is represented by hypervelocity stars (HVSs). These could be produced during multiple strong interactions between the central galactic SMBH and an infalling GC, which can accelerate many stars belonging to the GC to high velocities, and can even eject them in jets from the inner galactic region. This usually occurs as a result of three- or four-body interactions with the SMBH (or SMBH-binary). Alternatively, dynamical kicks due to supernova explosions or close encounters of a hard massive binary star and a single massive star could accelerate stars to extreme velocities. In addition, some galactic HVSs are thought to be the result of a merger event with another nearby galaxy with a high-velocity relative to the present galactic environment (see Fragione & Capuzzo-Dolcetta 2016).

Typically, HVSs observed in the galactic halo have velocities that can reach $\sim 1.2 \times 10^3$ km s$^{-1}$; the typical stellar velocity in the Galaxy is roughly $0.1 \times 10^3$ km s$^{-1}$. Therefore, the size of the relativistic corrections, quantified in $\gamma - 1$, on a HVS are of the order of $8 \times 10^{-6}$, while for an average star it is $5.6 \times 10^{-8}$.

The second case is exemplified by a massive black hole moving through a dense stellar system, for example a nuclear star cluster (NSCs) or a core-collapsed GC. BHs can be accelerated to large velocities, for instance during the final stage of SMBHs coalescence by the large recoil due to anisotropic emission of GWs, with $v_{\text{recoil}} \approx 10^3$ km s$^{-1}$ for the coalesced object, that may also be displaced from the minimum of the host galaxy potential-well, or even ejected Lena et al. (2014), Kim et al. (2017).

Nuclear star clusters have typical mass densities in the range $3 \times 10^4 - 2 \times 10^5$ M$_{\odot}$ pc$^{-3}$. Assuming an average stellar mass of about 0.5 M$_{\odot}$ implies that their average interstellar distance $r_{\text{int}}$ is of the order of $2 \times 10^2$ pc, while the Schwarzschild radius $r_s \approx 2GM_{\odot}/c^2$ for a $10^2 - 10^3$ M$_{\odot}$ massive BH is roughly $10^{-7} - 10^{-8}$ pc. Therefore, a close encounter with a star at about $10^{-7}$ pc from a BH can be effectively resolved, where relativistic corrections should be taken into account.

An example of the third case is the distribution of dark matter in proximity of a massive black hole. In general, in the Lambda-cold dark matter (ACDM) paradigm, dark matter is supposed to be in a non-relativistic and collisionless regime (see e.g. Binney & Tremaine 2008). However, in the presence of the deep gravitational potential of a BH, (dark) matter particles could reach relativistic velocities (Zamir 1993). Quantifying the relativistic corrections to DF in a dark matter cusp with a central BH is therefore worth exploring in the context of seeding mechanisms on primordial BHs (Kavanagh et al. 2020; Cole et al. 2023).

3. Dynamical friction: Classical case

Before tackling the problem of its relativistic generalization, it is instructive to revisit the classical treatment of the DF and rederive Eq. (1). As usual, we consider each single encounter between the test particle and the field particles as a hyperbolic two-body problem in the frame centred on the field particle. Let $(x_F, v_F)$
and \((x_F, v_F)\) be the positions and velocities of \(M\) and \(m\), respectively, and let
\[
r = x_F - x_e; \quad V = v_F - v_e\tag{4}
\]
be their relative position and velocity. We recall that the equation of motion for a fictitious particle of reduced mass \(\mu = mM/(M + m)\) moving in the Keplerian potential of the fixed body of mass \(M + m\), is
\[
\frac{mM}{m + M} \ddot{r} = -\frac{G M m}{r^2} \hat{r}.	ag{5}
\]
The energy conservation along the orbit of \(\mu\) for a given encounter with impact parameter \(b\), implies that the relative velocity vector \(V\) is deflected by an angle \(\pi - 2\psi\), in the orbital plane defined by
\[
\cos \psi = \frac{1}{\sqrt{1 + \frac{b^2 V^2}{G (M + m)^2}}}.	ag{6}
\]
Using finite differences, we can always express the relative velocity change as
\[
\Delta V = \Delta v_F - \Delta v_T, \tag{7}
\]
where \(\Delta v_F\) and \(\Delta v_T\) are the velocity variations of \(m\) and \(M\) during the encounter. Since the velocity of the centre of mass is constant (by definition) during the encounter, we have that
\[
m \Delta v_F + M \Delta v_T = 0. \tag{8}
\]
Eliminating \(\Delta v_F\) in the two equations above yields
\[
\Delta v_T = -\left(\frac{m}{m + M}\right) \Delta V. \tag{9}
\]
We must now evaluate \(\Delta V\) in order to find \(\Delta v_T\). The conserved angular momentum per unit mass of the reduced particle is \(L = bv\). Let us now label with \(\theta_{\text{def}}\) the deflection angle. The relation between the radius and azimuthal angle of a particle on a Keplerian orbit becomes
\[
\frac{1}{r} = C \cos(\psi - \psi_0) + \frac{G (M + m)}{b^2 V^2}, \tag{10}
\]
where the constant \(C\) and the phase angle \(\psi_0 = \psi(t = 0)\) are determined by the initial conditions. Deriving (10) with respect to the time we obtain
\[
\frac{dr}{dt} = C r^2 \sin(\psi - \psi_0) = C b V \sin(\psi - \psi_0), \tag{11}
\]
where the last term arises from \(L = r^2 \dot{\psi}\). If we impose that \(\psi = 0\) when \(t \to -\infty\), we obtain from (11)
\[
-\dot{V} = C b \sin(-\psi_0). \tag{12}
\]
Evaluating Eq. (10) we then have
\[
0 = C \cos(\psi_0) + \frac{G (M + m)}{b^2 V^2}, \tag{13}
\]
and eliminating \(C\) from the equations above we obtain
\[
\tan \psi_0 = -\frac{b V^2}{G (M + m)}. \tag{14}
\]
From Eqs. (10) and (11) we find that the point of closest approach is reached when \(\psi = \psi_0\) and, since the orbit is symmetrical about this point, the deflection angle is \(\theta_{\text{def}} = 2\psi_0 - \pi\).

Thanks to the conservation of energy, after the encounter, the modulus of the relative velocity, equals the modulus of the initial relative velocity, and therefore the components of \(\Delta V\) (perpendicular and parallel to the initial relative velocity vector \(V\): \(\Delta V_{\parallel}\))
\[
||\Delta V_{\perp}|| = \frac{2 b V^3}{G (M + m)} \left[1 + \frac{b^2 V^4}{G^2 (M + m)^2}\right]^{-1}; \tag{15}
\]
and
\[
||\Delta V_{\parallel}|| = 2 V \left[1 + \frac{b^2 V^4}{G^2 (M + m)^2}\right]^{-1}. \tag{16}
\]
In a homogeneous background of particles equal masses \(m\), all \(\Delta v_{\parallel}\), sum to zero by symmetry, (using the ‘Jeans swindle’; see e.g. Binney & Tremaine 1987), while the parallel velocity changes add up, and thus the mass \(M\) will experience a deceleration (see Eq. (9)) as a result of the DF. Therefore, it is sufficient to evaluate \(\Delta v_{\parallel}\) as
\[
||\Delta v_{\parallel}|| = \frac{2 m V}{M + m} \left[1 + \frac{b^2 V^4}{G^2 (M + m)^2}\right]^{-1}. \tag{17}
\]
In a system defined by the phase-space distribution function \(F = n f(\psi)\), where \(n\) is a constant number density and \(f(\psi)\) is the velocity distribution, the rate at which the mass \(M\) encounters stars with impact parameter between \(b\) and \(b + db\), and velocities between \(v_F\) and \(v_F + dv_F\), is
\[
n_{\text{enc}} = 2 \pi b db V n f(\psi) d^3 v_F, \tag{18}
\]
where \(d^3 v_F\) is the velocity-space element. The total change in velocity undergone by \(M\) is found by adding all the contributions of \(||\Delta v_{\parallel}||\) due to particles with impact parameters from \(b_{\text{min}}\) to \(b_{\text{max}}\), and then summing over all velocities of stars as
\[
\frac{\partial \Delta v_{\parallel}}{\partial b} = \int_{b_{\text{min}}}^{b_{\text{max}}} \left[\frac{2 m V}{M + m} \left[1 + \frac{b^2 V^4}{G^2 (M + m)^2}\right]^{-1}\right] 2 \pi b db.	ag{19}
\]
We first perform the integral over \(b\):
\[
\int_{b_{\text{min}}}^{b_{\text{max}}} \frac{2 m V}{M + m} \left[1 + \frac{b^2 V^4}{G^2 (M + m)^2}\right]^{-1} 2 \pi b db = 2 \pi G^2 (M + m) \log \left[1 + \frac{b_{\text{min}}^2 V^4}{G^2 (M + m)^2}\right] \left[1 + \frac{b_{\text{max}}^2 V^4}{G^2 (M + m)^2}\right]. \tag{20}
\]
We note that choosing the minimum and maximum impact parameters is a rather delicate step. When using the impulsive approximation (i.e. \(|\Delta V_{\parallel}|| = 2 G M m / b V\); see e.g. Ciotti 2010), the ‘ultraviolet divergence’ occurs when performing the integral over \(b\) in (19) and setting \(b_{\text{min}} = 0\). This divergence is actually artificial as it disappears when the full solution of the hyperbolic two-body problem is taken into account as in this discussion. Conversely, the ‘infrared divergence’ appearing for \(b \to \infty\) cannot be eliminated in an infinite system, and therefore an upper cutoff must be used with a suitable choice of \(b_{\text{max}}\).

Not surprisingly, this point is still a source of debate (see e.g. the discussion in Van Albada & Szomoru 2020 and
references therein). The problem lies in what should dominate between a few strong encounters with nearby stars (see Chandrasekhar 1941, 1942, 1943, see also Kandrup 1983) or many weak encounters with distant stars (see Spitzer 1987; Binney & Tremaine 1987). In the first interpretation \( b_{\text{max}} \) should be of the order of the average interparticle distance, while in the second, it should be of the order of the size of the system. Both views are in principle plausible, the former as it is more intuitive to think that the largest contribution must be due to the nearest stars and the latter because there is no screening in gravitational systems, at variance with (quasi-)neutral plasmas where charges of opposite signs are present. In this work we follow the Spitzer approach.

We note that, under most conditions of practical interest in astrophysics, the quantity \( \Lambda^2 = b^2 V^4 / G^2 (M + m)^2 \) is typically much greater than unity. For this reason, from now on we replace \( \log 1 + \Lambda^2 \) with \( 2 \log \Lambda \), recovering the widely used definition of \( \log 1 + b_{\text{max}}^2 V^4 / G^2 (M + m)^2 \).

In this approximation, the right-hand side of Eq. (20) becomes

\[
\frac{2 \pi G^2 (M + m)}{V^3} \log \left[ \frac{1 + \frac{b_{\text{max}}^2 V^4}{G^2 (M + m)^2}}{1 + \frac{b_{\text{max}}^2 V^4}{G^2 (M + m)^2}} \right] \approx \frac{4 \pi G^2 (M + m)}{V^3} \log \Lambda. \tag{21}
\]

Combining the expression above with Eq. (19) yields

\[
\frac{d\mathbf{r}}{dt} = -4 \pi G^2 n m (M + m) \log \Lambda \int f(\mathbf{v}_f) \frac{v_f}{||v_f - v_r||^3} d\mathbf{v}_f, \tag{22}
\]

where we have replaced \( \mathbf{V} \) with its definition (see Eq. (4)) and assumed \( \log \Lambda \) as the velocity averaged Coulomb logarithm (see e.g. the discussion in Ciotti 2021). The velocity integral in Eq. (22) is often referred to as the first Rosenbluth potential (see e.g. Rosenbluth et al. 1957).

Remarkably, the problem of computing the acceleration \( d\mathbf{r}/dt \) integrating over all field star velocities is formally equivalent to that of evaluating the gravitational field at \( v_r \) generated by the mass density: \( \rho(v_f) = 4 \pi \log G m (M + m) f(v_f) \). Assuming an isotropic (spherically symmetric) distribution in virtue of Newton’s second theorem (see e.g. Chandrasekhar 1995) we have that only the stars such that \( v_f < v_r \) contribute to the slowing down of \( M \), and hence

\[
\frac{d\mathbf{r}_r}{dt} = -16 \pi^2 G^2 n m (M + m) \log \Lambda \int_{\mathbf{3}v_r}^{\mathbf{3}v_r} \frac{v_f}{f(v_f) n_2} d\mathbf{v}_f. \tag{23}
\]

In the special case where \( f(v_f) \) is a Maxwellian with dispersion \( \sigma \),

\[
f(v_f) = \frac{v_f^2}{(2\pi\sigma^2)^{3/2}} \exp \left( -\frac{v_f^2}{2\sigma^2} \right), \tag{24}
\]

evaluating the velocity volume function integral in Eq. (23) yields

\[
\frac{d\mathbf{r}_r}{dt} = -4 \pi G^2 (M + m) \rho \log \Lambda \left[ \text{Erf} \left( \frac{v_r}{\sqrt{2}\sigma} \right) - \frac{2v_r e^{-v_r^2/(2\sigma^2)}}{\sqrt{2\pi}\sigma} \right] \frac{v_r}{v_r^2}, \tag{25}
\]

where we have condensed \( n m \) in \( \rho \) (the mean mass density of the field particles), and \( \text{Erf}(x) \) is the standard error function defined

\[\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{26}\]

4. Dynamical friction: Special relativistic generalization

As mentioned above, the Chandrasekhar DF formula was extended to relativistic velocities by Syer (1994), but only in the weak scattering limit (i.e. \( b \gg r_s \approx GM/c^2 \) and small deflection angle \( \theta_{\text{defl}} \)). We will now derive a more general expression for a generic \( \theta_{\text{defl}} \), therefore also accounting for the case of strong scattering, (i.e. \( \theta_{\text{defl}} \rightarrow \pi/2 \)).

In this derivation we keep the classical \( 1/r^2 \) Newtonian force and replace velocity composition with its relativistic counterpart. For this purpose we define an inertial frame \( S' \), in which the test star of mass \( M \) is stationary at the beginning of the encounter (i.e. the field star \( m \) is at infinity). In this frame, \( \theta_{\text{defl}} = \pi - 2\psi \) (with \( \psi \) given by Eq. (6)) is the deflection angle according to the classical unbound two-body problem. This assumption is motivated by the fact that in the astrophysically relevant case where \( M > m \), the relativistic scattering angle in an elastic collision has the same majorant as the classical case (see Landau & Lifshitz 1976, Chap. 2) given by \( \sin \theta_{\text{defl, max}} = m/M \) (see also Thornton & Marion 2004). In this approximation, however, we do not assume a small angle limit (i.e. weak scattering) as no assumption has been made on the relative angles between \( v_r \) and \( v_f \).

In the (special) relativistic encounter the relative velocity \( V \) becomes (see e.g. Landau & Lifshitz 1976)

\[
V^2 = \frac{||v_r - v_f||^2 - \frac{1}{4} (v_r \cdot v_f)^2}{(1 - \frac{v_r \cdot v_f}{c^2})^2}, \tag{27}
\]

for arbitrary choices of \( v_r \) and \( v_f \). We note that, in the relativistic case, the velocity \( \mathbf{v}_{\text{AB}} \) of a body \( A \) with respect to another \( B \) is not equal to \( -\mathbf{v}_{\text{BA}} \) of \( B \) with respect to \( A \). This loss of symmetry is related to the Thomas (1927) precessionootnote{In practice, two subsequent non-collinear boosts are equivalent to the composition of a rotation of the coordinate system and a boost.}, and the fact that two subsequent Lorentz transformations rotate the coordinate system (see Weinberg 1972). This rotation, however, conveniently has no effect on the magnitude of a vector, and hence the modulus of the relative velocity is symmetrical.

We let \( \mathbf{v}_{\gamma\nu} = \gamma_{\gamma\nu}(c, \mathbf{v}_{\gamma\mu}) \) and \( \mathbf{v}_{\gamma\nu} = \gamma_{\gamma\nu}(c, \mathbf{v}_{\gamma\nu}) \) be the four-velocities of the field and test stars, respectively, and where \( \gamma_{\gamma\nu} = \left(1 - v_r^2/c^2\right)^{-1/2} \) are the Lorentz factors of \( M \) and \( m \). We consider a single encounter in the ‘laboratory frame’ \( S \). This process can be expressed as a product of a Lorentz boost \( \Gamma \) in \( S' \), a rotation \( \mathcal{R}(\theta_{\text{defl}}) \) in the three-space and, finally, an inverse boost \( \Gamma^{-1} \), reverting back to \( S \).

Defining \( \mathbf{p}_{\gamma\nu} = m y_{\gamma\nu}(c, \mathbf{v}_{\gamma\nu}) \) as the four-momentum of \( m \) before the encounter, we have that \( \mathbf{p}_{\gamma\nu}^\mu = \Lambda^\mu_\nu \mathbf{p}_\nu^\mu \), where \( \Lambda = \Gamma^{-1} \mathcal{R}(\theta_{\text{defl}}) \Gamma \), and thus we formally obtain

\[
\Delta \mathbf{p}_\nu^\mu = (\Gamma^{-1} \mathcal{R}(\theta_{\text{defl}}) \Gamma - 1) \mathbf{p}_\nu^\mu. \tag{28}
\]

Since the motion is planar, as we are still dealing with a classical two-body problem, we can simplify the notation involving four vectors by using three vectors instead, where only two of
the space dimensions are maintained, one parallel to $v_T$ and one perpendicular. However, as argued before, by reasons of symmetry, only the parallel component of $V$ contributes to the DF. Denoting with $\phi$ the angle between $v_T$ and $v_F$, we can now write

$$\mathbf{p}^\mu_F = m\gamma v_T \begin{pmatrix} c \\ v_T \cos \phi \sin \theta \\ v_T \cos \phi \cos \theta \\ v_T \sin \phi \end{pmatrix}. \quad (29)$$

With this choice of $\Gamma$, $\mathcal{R}$ and $\Gamma^{-1}$ read

$$\mathcal{R}(\theta_{\text{def}}) = \begin{pmatrix} \gamma v_T & -\gamma v_T & 0 & 0 \\ \gamma v_T & \gamma v_T & 0 & 0 \\ 0 & 0 \cos \theta_{\text{def}} & -\sin \theta_{\text{def}} & 0 \\ 0 & 0 \sin \theta_{\text{def}} & \cos \theta_{\text{def}} & 0 \end{pmatrix}, \quad \Gamma^{-1} = \begin{pmatrix} \gamma v_T & \gamma v_T & 0 & 0 \\ -\gamma v_T & \gamma v_T & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (30)$$

After the encounter the four-momentum of the field star $\mathbf{p}_F^\mu$ changes by $\Delta \mathbf{p}_F^\mu$ defined as

$$\Delta \mathbf{p}_F^\mu = m\gamma v_T \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \quad (31)$$

where, respectively,

$$A = \gamma v_T^2 - \frac{\gamma v_T v_F}{c^2} \cos \phi + \frac{v_T}{c^2} v_T \gamma v_T (v_T \cos \phi - v_T),$$

$$B = \gamma v_T^2 \left( 1 - \frac{v_T v_F}{c^2} \cos \phi \right) + \gamma v_T (v_T \cos \phi - v_T),$$

$$C = \gamma v_T (v_T \cos \phi - v_T) \sin \theta_{\text{def}} + v_T \sin \phi \cos \theta_{\text{def}} - v_T \sin \phi,$$

and where

$$\theta_{\text{def}} = \pi - 2 \cos^{-1}\left( \frac{1}{\sqrt{1 + \frac{\gamma v_T^2}{G^2(M+m)^2}}} \right). \quad (33)$$

is the deflection angle. We now have to multiply $\Delta \mathbf{p}_F^\mu$ for the differential number of encounters $d\sigma_{\text{enc}} = 2\pi n V dtdv_F bdbd\sigma$ in the laboratory frame. This quantity, at variance with that given by Eq. (18), is a Lorentz-invariant quantity, where

$$V \equiv V(1-v_T \cdot v_F/c^2) = V(1-v_T v_F \cos \phi/c^2). \quad (34)$$

In integral form, the momentum variation of the field particle $m$ is now given by

$$\frac{d\mathbf{p}_F^\mu}{dt} = 2\pi \int b \Delta \mathbf{p}_F^\mu V f(v_T) db d^3v_F. \quad (35)$$

The momentum of the test particle $M$, $\mathbf{p}_M^\mu = M \gamma v_T (c, v_T)$, will undergo the opposite change

$$\frac{d\mathbf{p}_M^\mu}{dt} = -\frac{d\mathbf{p}_F^\mu}{dt}. \quad (36)$$

To evaluate Eq. (35) we need first to substitute the expression for the cosine and sine of the deflection angle Eq. (33) in Eqs. (30)–(32) that read

$$\cos \theta_{\text{def}} = \frac{\gamma v_T^2}{G^2(M+m)^2} - 1; \quad \sin \theta_{\text{def}} = \frac{\gamma v_T^2}{G^2(M+m)^2} + 1. \quad (37)$$

We note that for the derivation of the DF formula, only the parallel component of $v_T$ contributes, so it is sufficient to evaluate the following expression for the parallel component $p_T$:

$$\frac{dp_{T}}{dt} = 2\pi \int b \gamma v_T V f(v_T) \left( \gamma v_T^2 \cos \phi - \frac{\gamma v_T^2}{c^2} \cos \phi \right) \times \left( 1 - \frac{\gamma v_T^2}{G^2(M+m)^2} - 1 - \gamma v_T \sin \phi \right) db d^3v_T. \quad (38)$$

Following the classical derivation discussed in Sect. 3, we perform first the integral over the impact parameter $b$. As we are considering an isotropic $f(v_T)$, we assume $\phi$ such that all odd terms involving $\sin \phi$ zero out. We then apply Eq. (36) obtaining the deceleration on the momentum of particle $M$, and then we obtain the DF formula for the velocity $v_T$ dividing by $\gamma v_T M$ as

$$\frac{dv_T}{dt} = \frac{4\pi G^2 (M+m)^2}{M} \gamma v_T \log \Lambda \int \gamma v_T V f(v_T) (v_T - v_F) d^3v_F, \quad (39)$$

where again we made use of the limit $\log(1 + A^2) \to \log \Lambda$ for large values of $A$ and substituted $\rho$ for $mm$.

As we are accounting for relativistic velocities and transformations, when performing the velocity integral in Eq. (39), with $V$ given by Eq. (27), we need to use a velocity distribution function in covariant form. For example, when considering a thermalized relativistic gas, a natural choice is the Maxwell-Jüttner distribution (see Jüttner 1911; see also Fackerell 1968):

$$f(v_T) = \frac{\gamma v_T^5 v_T^2}{c^7 \Theta K_2(\Theta)} \exp(-\frac{\gamma v_T}{\Theta}). \quad (40)$$

In the expression above $\Theta = c^2/3c^2$ and $K_2(s)$ is the modified Bessel function of the second kind (see e.g. Arfken et al. 2012), often somewhat improperly dubbed the Neumann function.

In the limit of $c \to +\infty$, Eq. (39) becomes Eq. (22) with $(M+m)^2/M$ in lieu of $(M+m)$. This discrepancy with the classical case occurs because in the relativistic treatment we used the total relativistic momentum conservation of Eq. (36), whereas classically Eqs. (7)–(9) are used. The application of the latter would be in principle commute with the limit $c \to +\infty$ and the term $(M+m)$ at the denominator in Eq. (9) would elide with another identical factor in Eq. (39), while Eq. (36) loses meaning in such a limit, as the time component would diverge.

We note that Kahnajs (1972) also encounters a similar disagreement between the Chandrasekhar expression and the particle limit of his fluid formalism (i.e. the frictional force on the test mass is proportional to $M^2 m$ rather than $Mm(M+m)$). However, the reason for this difference is ascribed to the fluid picture where the force on $M$ is due to a continuous mass density distribution (see e.g. Kandrup 1983). We also note that, in the fully

---

3 Notably, due to the relativistic length contraction, the number density $n$ along the direction of $M$ would increase in the rest frame of $m$ (see Landau & Lifshitz 1976).
kinetic approach on DF based on the Fluctuation-Dissipation theorem pioneered by Bekenstein & Maoz (1992), among others, one recovers the original expression formulated by Chandrasekhar with the term $Mm(M + m)$.

Having established this, we estimated the relativistic DF for three cases of astrophysical interest: a $10^5 M_\odot$ BH in a star cluster with velocity dispersion $\sigma \sim 10$ km s$^{-1}$ and core density of $10^3 M_\odot$ pc$^{-3}$; a $10^7 M_\odot$ BH in a galactic core with $\sigma \sim 500$ km s$^{-1}$ and density of $4 \times 10^6 M_\odot$ pc$^{-3}$; and, finally, stellar mass BH in a relativistic dark matter cusp around a SMBH with $\sigma \sim 2 \times 10^6$ km s$^{-1}$ and mean density of $7 \times 10^3$ g cm$^{-3}$.

We always assumed an isotropic relativistic velocity distribution as given in Eq. (40) when solving (numerically) the integral in Eq. (39). In Fig. 1 (red solid lines) we compare it with the classical expression (Eq. (1), black dashed lines). In agreement with Syer (1994) we observe for all systems, even if $v < \sigma$, that the relativistic DF is augmented with respect to its classical counterpart, up to a factor 1.1, which is in general much smaller than the $16/3\pi$ found by Syer. For $v \geq \sigma$ the relativistic DF force is slightly smaller than the classical force, while at large $v_f$ increases again due to the diverging pre-factor $\gamma_v$, in the limit $v_f \rightarrow c$.

We note that, for relativistic power-law velocity distributions, Eq. (39) would become for a given $v_f$ considerably larger than (1), due to the contribution of large $v_f$ tails. We speculate that this could be relevant in the context of plasma physics where (multiple) power-law velocity distributions are often encountered (see e.g. Sandquist et al. 2006). The derivation of Eq. (39) can be carried out in a similar fashion for a charge $q_f$ deflected in a plasma, as the impact parameter and velocity integrals are the same, the only difference being the dimensional factor containing the masses and the gravitational constant. We also note that Eq. (39) was obtained under the assumption that the classical angle $\psi_0$ given in Eq. (14) holds even in the case of relativistic velocities. To be more rigorous, one should use instead its relativistic generalization given by

$$\tan \psi_0 = -\frac{L \sqrt{2E}}{G(M + m)}$$

where $L = \gamma_v Vb$ and $E = c^2(\gamma_v - 1)$ are the norm of the spatial part of the specific relativistic angular momentum and the specific relativistic kinetic energy, respectively, and $\gamma_v$ is the Lorentz factor of the relative velocity $V_f$.

With this choice, $\tan \psi_0$ and $\tan \psi_f$ differ by the multiplicative factor $\gamma_v \sqrt{2(\gamma_v - 1)/v}$.

The latter increases significantly the value of $\tan \psi_0$ only for $V_f \gg 0.5c$. In this limit, expanding the square roots arising from $\gamma_v$, the terms containing $b^2V^4/G^2(M + m)^2$ in Eqs. (38)–(37) in the derivation above are augmented by a factor $(1 + V^2/2c^2)^2$, so that Eq. (38) becomes

$$\frac{dp_f}{dt} = 2\pi nm \int dy v_f Vf(\psi_f) \left\{ \frac{1}{\gamma_v} \gamma_v \left(1 - \frac{v_f^2}{v_T^2} \cos \phi \right) + \gamma_v \left[ \gamma_v (\gamma_v \cos \phi - v_T^2) \right] \right\} \left[ \frac{2\sqrt{2} \sqrt{\gamma_v^2 + 1}}{\gamma_v^2 G^2(\gamma_v m)^2} + 1 \right] - v_f \cos \phi$$

The integration over the impact parameter $b$ can be carried out first in the same way as above, yielding a velocity averaged Coulomb logarithm now multiplied by the pre-factor $(1 + V^2/2c^2)^2$, so that Eq. (39) becomes

$$\frac{dv_f}{dt} = -4\pi G^2 (M + m)^2 \rho_v \log \Lambda \int \gamma_v Vf(\psi_f)(\psi_f - \psi) \frac{1}{(1 + V^2/2c^2)^2} d\psi_f.$$  

Equation (43) differs significantly from the simplified expression Eq. (39) only in the limit of large velocity dispersion (and for high velocities), as shown by the blue dot-dashed lines in the right panel of Fig. 1.

5. Post-Newtonian approximation

So far, we have derived a formal generalization of the dynamical friction formula in the limit of high test particle velocities or relativistic velocity distributions assuming classical Newtonian forces. We now carry out an alternative derivation involving strong gravitational scattering in the post-Newtonian regime.
Introduced by Einstein (1915; see also Weinberg 1972; Blanchet 2010 and references therein) to study the precession of the perihelion of Mercury, the post-Newtonian approximation consists of an expansion in orders of the parameter \(\nu/c\), such that at the zeroth order it reduces to Newtonian gravity, while at higher orders (\(n\)PN) the acceleration on the mass \(m\) due to the mass \(M\) is augmented by corrections of order \((\nu/c)^{2n}\) \((GM/rc^2)^n\).

### 5.1. Non-relativistic velocities

The cores of dense star clusters are often dominated by massive objects due to dynamical mass segregation; it is therefore interesting to evaluate the DF on a test particle in such an environment where strong scattering by larger masses are likely to happen, even though the velocity distribution might not be relativistic. To do so, we begin with a naive derivation of \(d\nu_{\text{ff}}/db\) in impulsive approximation keeping the Galilean transformations of velocities, but using the 1PN acceleration

\[
a_{1\text{PN}} = -\frac{G(M + m)}{r^2} \hat{r} + \frac{G(M + m)}{c^2 Vb} \left\{ \left[4(4 + 2\nu)\frac{G(M + m)}{b} + (1 + 3\nu)V^2 + \frac{3}{2}\nu V^2 \right] \hat{r} + (4 - 2\nu)\hat{V} \right\},
\]

\[
\Delta t_{\text{ff}} \sim \frac{\mu}{M} \frac{||\Delta V||^2}{2V^2} \nu.
\]

Defining \(r = r\hat{r} \sim b\hat{r}\) and \(V = \hat{V}\), we obtain the perpendicular relative velocity change as

\[
\Delta V_{\perp} \sim a_{1\text{PN}} \frac{2b}{V} \sim -\frac{2G(M + m)}{Vb} \hat{r} + \frac{2G(M + m)}{c^2 Vb} \left\{ \left[4(4 + 2\nu)\frac{G(M + m)}{b} + (1 + 3\nu)V^2 + \frac{3}{2}\nu V^2 \right] \hat{r} + (4 - 2\nu)\hat{V} \right\},
\]

and its square as

\[
||\Delta V_{\perp}||^2 \sim \frac{4G^2(M + m)^2}{Vb^2} - \frac{8G^2(M + m)^2}{c^2 Vb^2} \left\{ \left[4(4 + 2\nu)\frac{G(M + m)}{b} + 3V^2 - \frac{7}{2}\nu V^2 \right] \right\},
\]

where we assume that \(\hat{r} \cdot \hat{V} \sim 1\) and drop the terms proportional to \(1/c^4\). The parallel velocity change of the test particle becomes

\[
\Delta v_{\parallel} \sim \frac{\mu}{M} \frac{||\Delta V_{\parallel}||^2}{2V^2} V
\]

so that, assuming again as in Sect 2. that the number of encounters is given by Eq. (18), the finite differences velocity change of particle \(M\) is expressed as

\[
\Delta v_{\perp} = 2\pi b dh V n f(\nu_{\text{ff}}) d^3\nu_{\text{ff}} \left\{ -2mG^2(M + m) \frac{V}{b^2 V^4} + \frac{4G^2m(M + m)^2(4 + 2\nu)}{c^2 b^3 V^4} \right\}
\]

\[
+ \frac{4G^2m(M + m)^2(4 + 2\nu)}{c^2 b^3 V^2} \left\{ \frac{c^2 b^3 V^4}{V^2} \right\}.
\]

With the standard integration over the impact parameter \(b\), we easily obtain

\[
\Delta v_{\parallel} = -4\pi mn G^2(M + m) \frac{V}{\log \Lambda} \int \frac{f(\nu_{\text{ff}}) V}{V^3} d^3\nu_{\text{ff}}
\]

\[
+ \frac{16\pi mn G^2(M + m)^2(2 + \nu)}{b_{\text{min}}^2} \int \frac{f(\nu_{\text{ff}}) V}{V^3} d^3\nu_{\text{ff}}
\]

\[
+ \frac{8\pi mn G^2(M + m)(3 - \frac{7}{2}\nu)}{c^2} \log \Lambda \int \frac{f(\nu_{\text{ff}}) V}{V} d^3\nu_{\text{ff}},
\]

where the term \((b_{\text{max}} - b_{\text{min}})/b_{\text{min}}b_{\text{max}}\), arising from the integration in \(b\), is substituted in the second addendum with \(b_{\text{min}}^{-2}\). The latter is a \(a\) \(b\) \(c\) \(d\) \(e\) \(f\) for the most part of the classical case, independently of the specific choice of \(f(\nu_{\text{ff}})\) augments the Chandrasekhar expression of the DF (first line of Eq. (50)) of two additional terms. The first has a different dependence on the impact parameter but the same integral on \(V\), while the second contains the classical Coulomb logarithm but a different integral on \(V\). For the special case of an isotropic Maxwellian distribution with velocity dispersion \(\sigma\), the IPN DF expression can be easily integrated in the limit of \(\nu_{\text{ff}} < \sigma\) and rewritten as

\[
\Delta v_{\parallel} = -\frac{4\pi \nu G^2(M + m)}{\log \Lambda} \int \frac{f(\nu_{\text{ff}}) V}{\sqrt{V^3}} d^3\nu_{\text{ff}},
\]

where the corrective factor \(\zeta\), plotted in Fig. 2 as a function of \(\sigma/c\) is defined by

\[
\zeta = 1 - \frac{8 + 4\nu}{\log \Lambda} + 6 - 14\nu \frac{\sigma^2}{c^2}.
\]

Interestingly, the net effect of a simple IPN correction is to reduce the DF drag force with respect to its classical counterpart given by Eq. (1). This has the relevant consequence that a highly massive object, for which the GR corrections to its gravitational field cannot be neglected (e.g. a SMBH moving at non-relativistic speed in a star system), undergoes a less effective gravitational drag than what would be estimated using the Chandrasekhar formula. This implies in that specific case an even longer in-spiral timescale for the BHs in galactic cores. The reason for a lower DF effect in a simple IPN approximation is ascribed to the fact that the relativistic precession due to the velocity dependent force term in a two-body scatter always acts in the opposite direction with respect to the deflection caused by the \(1/r^2\) force term.

The behaviour of the IPN DF cannot be extrapolated for high (relativistic) velocities since for such high values of \(\nu\) the velocity composition should be made in terms of Lorentz transforms, as in Sect. 4.
5.2. Relativistic velocities

We now extend the DF formula to the case where the particle velocities are relativistic and the density is such that the force is to be evaluated at 1PN order during close encounters. Following Lee (1969), we consider a mass \(M\) moving at \(v_T\) in a uniform medium of field stars, of constant number density \(n\), with isotropic velocity distribution \(f(w_F)\), in an inertial frame \(S\). As usual, in a time interval \(\Delta t\), its velocity changes by an amount \(\sum \Delta v_T\) after \(n_{\text{enc}}\) encounters.

As the effect of General Relativity is accounted for only during the deflection of the test particle \(M\), its velocity is transformed according to the Lorentz transformations of Special Relativity.

As in the classical case, the isotropy of \(f(w_F)\) allows us to write

\[
\sum \Delta v_T \cdot v_T = 0,
\]

so it is sufficient to perform

\[
\sum \Delta v_T = \frac{\sum \Delta v_T \cdot v_T}{v_T}.
\]

Let us consider a single encounter of a test star with a field star: the velocity \(v_F\) becomes \(v'_F\), so we have

\[
\Delta v_T = \frac{(v'_F - v_F) \cdot v_T}{v_T}.
\]

It is now useful to introduce a second reference frame, \(S'\), in which the test star is, initially, at rest. Let \(w'_F\) be the velocity of the test star after the encounter, in the frame \(S'\). With our assumptions (the motion of all the stars is approximately described by straight line) we can write, without losing generality, that

\[
w'_F = \frac{(w'_F + v_F)}{1 + \frac{v'_F}{c^2}}.
\]

and therefore

\[
w'_F = \frac{(w'_F + v_F) \cdot v_T}{1 + \frac{w'_F}{c^2} v_T}.
\]

Noting that \(\Delta w_T = w'_F\), (i.e. the test star is initially at rest in \(S'\)) and keeping only terms of order \(1/c^2\), after some algebra Eq. (55) becomes

\[
\Delta w_T = \frac{1}{v_T} \left[ \frac{(w'_F + v_F) \cdot v_T}{1 + \frac{w'_F}{c^2} v_T} - v_T^2 \right] \approx \frac{1}{v_T} \left[ \Delta v_T \cdot v_T \gamma_{\text{tr}}^{-1} = \frac{(\Delta v_T \cdot v_T)^2}{c^2} \right].
\]

We must now sum Eq. (58) over all possible values of \(\Delta w_T\). We recall that \(v_T\) and \(w_T\) are the initial velocities of field stars in the frames \(S\) and \(S'\), related by the following Lorentz transformation (see e.g. Landau & Lifshitz 1976)

\[
w_T = \frac{1}{1 - \frac{v_T}{c^2}} \left[ v_T - v_F + \frac{1}{c^2} \gamma_{\text{tr}} (v_T - v_F) \gamma_{\text{tr}} + 1 \right].
\]

where \(v_F\) is the translational velocity of \(S'\) relative to \(S\); \(\gamma_{\text{tr}} = (1 - v_T^2/c^2)^{-1/2}\); and \(b\) is, as usual, the impact parameter of the encounter, defined in \(S'\). We also have that \(w_F \cdot b = w_T \cdot b \cos \varphi\), while \(\Delta w_T\), decomposed into its components parallel and perpendicular to \(w_T\), becomes

\[
\Delta w_T = \Delta w_{T\parallel} \frac{b}{b} + \Delta w_{T\perp} 1.
\]

The velocity distribution of field particles in \(S'\) is (see e.g. Landau & Lifshitz 1976)

\[
f'(w_F) = \gamma_{\text{tr}} f(w_T) \left( \frac{\gamma_{\text{tr}}}{\gamma_{\text{tr}}} \right)^4,
\]

where \(v_F = v_T(w_F, v_T)\). Let us denote with

\[
\Delta t' = \gamma_{\text{tr}}^{-1} \Delta t
\]

the transformed time interval during which the encounters with the field particles are summed.

It is important to note that \(f'(w_F)\) in the frame \(S'\) is only homogeneous, but no longer isotropic. As a consequence of this, the impact parameter \(b\) and the deflection angle \(\varphi\) are randomly distributed for a given value of \(w_F\). Integrating Eq. (58) over all values of \(\Delta w_T\), we obtain the formal result

\[
\frac{dw_T}{dt} = \int n f'(w_F) \gamma_{\text{tr}}^{-1} v_T d^3 w_T \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} 2\pi b db \left[ \Delta v_T \cdot v_T \gamma_{\text{tr}}^{-2} = \frac{(\Delta v_T \cdot v_T)^2}{c^2} \right] d\varphi.
\]

The quantity \(\Delta v_T(w_F, b)\) appearing in the equation above must be expressed as a function of \(w_F\) and \(b\), and can be computed with the help of the two-body problem 1PN-Lagrangian. For this purpose, it is convenient to introduce a third reference frame \(S''\), in which the centre of mass (c.o.m.) of the encounter is at rest at the origin of coordinates\(^4\), where we denote the particle velocities as \(u\).

\(^4\) As the location of the origin depends on the masses and velocities of both stars in \(S\), we must consider a new frame for each encounter.
At 1PN order the equations of motion for the two-body encounter are usually derived from the Einstein-Infeld-Hoffmann Lagrangian (EIH, see Einstein et al. 1938, see also Eddington & Clark 1938)

\[ \mathcal{L}_{\text{EIH}} = \frac{1}{2} \mu v^2 + \frac{1}{2} M u^2 + \frac{1}{8c^2} (\mu c^2 + M c^2) + \frac{GM}{r} \times \left( 1 - \frac{1}{2c^2} \{u_F \cdot u_T + (u_F \cdot n)(u_T \cdot n) - 3(u_F - u_T) \} - \frac{G(m + M)}{2c^2} \right) . \]  

(64)

Remarkably, qualitatively similar (but slightly simpler) equations of motion can be derived using the gravitational analogue of the Darwin Lagrangian of electrodynamics (see e.g. Jackson 1975; see also Essen 2007, 2014), often referred to as the Fock Lagrangian (see Fock 1964; Kennedy 1972; Deruelle & Uzan 2018):

\[ \mathcal{L}_{\text{Darwin}} = \frac{1}{2} \mu v^2 + \frac{1}{2} M u^2 + \frac{1}{8c^2} (\mu c^2 + M c^2) + \frac{GM}{r} \times \left( 1 - \frac{1}{2c^2} (u_F \cdot u_T + (u_F \cdot n)(u_T \cdot n)) \right) . \]  

(65)

In Eqs. (64) and (65), \( r = r_F - r_T \) is the (instantaneous) relative position vector and \( n = r/||r|| \) (see Fagundes et al. 1976; Damour & Deruelle 1985; Zürcher 2017, for details). In both cases the c.o.m. coordinates at the 1PN order are

\[ r_{\text{cm}} = - \frac{\mathcal{E}_T r_T + \mathcal{E}_F r_F}{\mathcal{E}_T + \mathcal{E}_F} , \]

where

\[ \mathcal{E}_T = Mc^2 + \frac{1}{2} M u^2 - \frac{1}{2} G M m / ||r_T - r_F|| , \]

\[ \mathcal{E}_F = mc^2 + \frac{1}{2} \mu v^2 - \frac{1}{2} G M m / ||r_T - r_F|| . \]

Since the c.o.m is by definition at rest at the origin of \( S' \) (i.e. \( r_{\text{cm}} = 0 \)), in terms of the relative velocity \( u \) we can always write

\[ u_T = \frac{m}{m + M} u + O(1/c^2) ; \quad u_F = - \frac{M}{m + M} u + O(1/c^2) , \]

(68)

so that

\[ \frac{1}{2} (M u^2 + \mu v^2) = \frac{1}{2} \left( \frac{m M}{M + m} u^2 + O(1/c^2) \right) . \]

(69)

At variance with Lee (1969), we will assume hereafter the Lagrangian (65), that in terms of the relative velocity \( u \), once conveniently transformed in polar coordinates, becomes

\[ \mathcal{L}_0 = \frac{1}{2} \frac{d}{dt} \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d \phi}{dt} \right)^2 + \Phi + \frac{1}{2c^2} \left( \frac{\mu}{\mu_3} \right)^3 \times \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d \phi}{dt} \right)^2 \right] + \Phi \frac{\mu}{2c^2 M} \left( \frac{d^2 r}{dt^2} \right)^2 + r^2 \left( \frac{d \phi}{dt} \right)^2 \]  

(70)

In the equation above, \( \mu \) is again the reduced mass, \( M = m + M, \mu_3 = (m^2 M^3/m^3 + M^3))^{1/3}, \) and \( \Phi = G M m / r \) is the Newtonian gravitational potential energy. With this choice, the equations of motion of the associated particles are obtained in explicit form at the order \( 1/c^2 \), while conversely, in the EIH case the force of particle \( M \) on particle \( m \) depends on the force of particle \( m \) on particle \( M \).

In analogy with the non-relativistic velocity case discussed above, we now obtain (see Appendix A for the explicit mathematical details) the DF formula in integral form as

\[ \frac{d v_T}{d t} = \int n f'(w_F) \gamma_{w_T}^{-1} w_T d^3 w_F \int_{b_{\text{max}}}^{b_{\text{min}}} 2 \pi n db \]

\[ \times \left( \int_0^{2\pi} \frac{1}{2} \left( \gamma_{w_T}^{-2} w_T \cdot w_F \right) (\Delta w_T) + \frac{1}{c^2} \left( \frac{\gamma_{w_T}^{-2} w_T \cdot w_F}{w_F} \right) (\Delta \omega_T) \right) \int_0^{2\pi} \frac{1}{2} \left( \gamma_{w_T}^{-2} w_T \cdot w_F \right) (\Delta w_T) \Delta \omega_T \cdot \hat{b} \]  

(71)

We note that all the linear terms in the equation above containing \( w_T \cdot b \) vanish when they are integrated in \( db \). Strictly speaking, \( \varphi \) would be the angle between \( w_F \) and \( b \). As in Lee (1969), we assume, however, that it is also the angle between \( v_T \) and \( b \), while the quadratic terms yield

\[ \int_0^{2\pi} \frac{1}{2} \left( \gamma_{w_T}^{-2} w_T \cdot w_F \right) b \]  

\[ \left( \frac{w_T \cdot b}{w_F} \right) \Delta w_T = \frac{1}{2} \left( \frac{w_T \cdot b}{w_F} \right) \frac{1}{2} \left( \frac{w_T \cdot b}{w_F} \right) \left( \Delta w_T \right)^2 \]  

(72)

In the equation above, we assume that \( w_F \cdot b = 0 \) (i.e. \( w_F \) is perpendicular to \( b \) and then \( \sin \varphi = 1 \)). We then rewrite Eq. (71) as

\[ \frac{d v_T}{d t} = \int n f'(w_F) \gamma_{w_T}^{-1} w_T d^3 w_F \int_{b_{\text{max}}}^{b_{\text{min}}} 2 \pi n db \]

\[ \times \left( \int_0^{2\pi} \frac{1}{2} \left( \gamma_{w_T}^{-2} w_T \cdot w_F \right) (\Delta w_T) - \frac{1}{2} \left( \frac{w_T \cdot b}{w_F} \right) \left( \frac{w_T \cdot b}{w_F} \right) (\Delta w_T)^2 \right) \]  

(73)

In order to perform the integration over the impact parameter, otherwise hardly feasible, we can neglect all the terms that decrease faster than \( 1/b \) as \( b \rightarrow +\infty \), using the ‘dominant approximation’. By doing so, dropping all terms proportional to \( (\Delta w_T)^2 \), we obtain

\[ \frac{d v_T}{d t} = \int n f'(w_F) \gamma_{w_T}^{-1} w_T d^3 w_F \int_{b_{\text{max}}}^{b_{\text{min}}} 2 \pi n db \]

\[ \times \left( \int_0^{2\pi} \frac{1}{2} \left( \gamma_{w_T}^{-2} w_T \cdot w_F \right) (\Delta w_T) - \frac{1}{2} \left( \frac{w_T \cdot b}{w_F} \right) (\Delta w_T)^2 \right) \]  

(74)

We now have to evaluate \( \Delta w_T \) and \( (\Delta w_T)^2 \) (the finite parallel and perpendicular velocity changes) as functions of the deflection angles (see Appendix B). We obtain at 1PN the following expressions:

\[ \Delta w_T \approx -2 \alpha_T \left( \frac{1}{1 + b^2 / R^3} + \frac{\alpha^2}{c^3} + \frac{1}{1 + b^2 / R^2} \right) + \frac{\alpha^2 b^2}{c^3 R} \frac{\mu}{\mu_3} \frac{1}{(1 + b^2 / R^2)^2} , \]

(75)

\[ (\Delta w_T)^2 \approx \alpha^4 \frac{4 b^2}{R(1 + b^2 / R^2)^2} . \]

(76)
Substituting Eq. (75) in Eq. (73) and performing the integral over \(b\) keeping only the terms yielding the Coulomb logarithm in

\[
\int_{b_{\text{min}}}^{b_{\text{max}}} 2\pi b \Delta \omega_{\text{TF}} db = -2 \pi u_T \int_{b_{\text{min}}}^{b_{\text{max}}} 2h \left[ \frac{1}{1 + \rho_{\text{eff}}^2/R^2} + \frac{\rho_{\text{eff}}^2}{c^2} \right] \times \frac{1}{1 + b^2/R^2} + \frac{u^2 b^2}{c^2 R^2} \left( \frac{\mu}{M} - \frac{\mu}{\mu^2} \right) \frac{1}{(1 + b^2/R^2)^2} \right] db
\]

\[
\approx -2 \pi u_T R^2 \log \left( \frac{1 + b_{\text{max}}^2 / R^2}{1 + b_{\text{min}}^2 / R^2} \right) \left[ 1 + \frac{u^2}{c^2} + \frac{u^2}{c^2} \left( \frac{\mu}{M} - \frac{\mu}{\mu^2} \right) \right],
\]

gives

\[
\int_{b_{\text{min}}}^{b_{\text{max}}} 2\pi b \Delta \omega_{\text{TF}} db \approx 4 \pi^2 R^2 \log \left( \frac{1 + b_{\text{max}}^2 / R^2}{1 + b_{\text{min}}^2 / R^2} \right).
\]

After some further algebraic manipulation, we finally obtain the 1PN dynamical friction formula in compact form as

\[
\frac{d\nu_T}{dt} = -4 \pi G^2 (M + m) \rho \log \Lambda \int \frac{f'(\nu_T)}{\gamma_{\nu_T}^3} \left[ \frac{1 + \frac{4M \mu^2}{(M+m)c^2} \nu_T^3}{1 + \frac{4M \mu^2}{(M+m)c^2} \nu_T^3} \right] d^3 \nu_T,
\]

where again we assume the velocity averaged Coulomb logarithm.

Equation (79), although highly simplified by all the assumptions made above, represents the most general relativistic derivation of the dynamical friction formula. In analogy with the treatment of the special relativistic DF formula, we investigated numerically the behaviour of Eq. (79) against its classical counterparts in some cases of astrophysical interest by solving the integral appearing in the equation above for the relativistic \(f(v)\) given again by Eq. (40).

In Fig. 3 we show the DF force acting on the test particle \(M\) as a function of its velocity for a 10\(^5\) \(M_\odot\) and a 10\(^7\) \(M_\odot\) black hole moving respectively in a dense star cluster and a galactic core. In both cases, for \(v_T < \sigma\) the classical expression slightly overestimates the drag force acting on \(M\), thus confirming the small \(v\) estimates of Sect. 5.1 (see Fig. 2). For high velocities the extra terms containing dot and cross products of the test particle velocity and the relative velocity \(\nu_T\) in \(S''\) become dominant, thus yielding a much higher dynamical friction with respect to the Chandrasekhar formula.

Finally, for reasons of completeness it is worth discussing its behaviour in its two fundamental limits (i.e. \(c \to \infty\) and \(M \gg m\)). In the limit \(c \to \infty\), Eq. (79) becomes

\[
\frac{d\nu_T}{dt} = -4 \pi G^2 (M + m) \rho \log \Lambda \int \frac{f(\nu_T)}{\nu_T^3} (\nu_T \cdot \nu_T) d^3 \nu_T,
\]

where we use Eq. (61) to re-obtain \(f(\nu_T)\) from \(f(\nu_T)\). In order to retrieve Eq. (22), we have to substitute \(\nu_{\text{TF}}\) with \(\nu_T\), by using Eq. (59) in its classical limit (i.e. the classical velocity summation rule \(\nu_{\text{TF}} = v_{\text{TF}} - v_r\) choosing \(v_r = v_T\). Once making explicit \(v_T \cdot \nu_T\) as \(v_T \cdot \nu_T \cos \phi\), the test particle velocity \(v_T\) cancels out with the same factor outside the integral and then, in the assumption of isotropy for \(f_{\text{TF}}\), the same considerations about the angle \(\phi\) between \(v_T\) and \(\nu_T\) used in Sect. 4 can be made to retrieve Eq. (1) in vectorial form.

In the limit of very large test particle mass \(M\) we obtain

\[
\frac{d\nu_T}{dt} = -4 \pi G^2 M \rho \log \Lambda \int \frac{f'(\nu_T)(\nu_T \cdot \nu_T)}{\nu_T^3} \left[ 1 + \frac{9 \mu^2}{2(M + m)c^2} + \frac{m^2}{(M + m)c^2} \right] d^3 \nu_T,
\]

where \(\xi = m^2 \mu^2 / M^2 c^2\) and all terms proportional to \((m/M)^2\) are discarded. Curiously, the multiplicative factor containing \(\xi\) in Eq. (81) is roughly unity when \(m\) is vanishingly small, which implies that, at least at low (relative) velocities \(\nu_{\text{TF}}\), the mass dependent terms in Eq. (79) can be neglected, while in other cases it still has to bear an explicit dependence on \(M\) and \(m\) inside the integral, differently from what one would obtain in the same limit for the classical expression (see the discussion in Binney & Tremaine 2008 and Ciotti 2010). In other words, in a relativistic set-up the dynamical friction acting on a massive particle \(M\) is slightly different in two systems with the same mass density \(\rho\) but different field particle mass \(m\). However, for high
$M/m$, as in the case for example of a star moving in a gas of relativistic dark matter particles with typical masses in the range of those of elementary particles, the specific value of $m$ is immaterial once $\rho$ is known.

Moreover, we note that Eq. (81) has a $1/\gamma^2_0$ dependence on the Lorentz factor, instead of $1/\gamma_v$, as in the weak scattering limit for a high test mass in a background of particles of infinitesimally low masses $m$ (see Eq. (2.26) of Syer 1994).

6. Summary and perspectives

In this preparatory work on the dynamical friction in relativistic systems, we have explored two relevant cases, one involving relativistic velocity distribution and classical forces between particles and the other involving strong scattering with and without relativistic velocities. We find that extending the asymptotic expression of Syer (1994), valid only for scattering angles to a generic $\theta_{\text{esc}}$ and for the case of a pure Newtonian $1/r^2$ force law, a particle moving through a medium with a relativistic velocity distribution undergoes a slightly greater drag with respect to that evaluated with the standard classical Chandrasekhar approach since all particles contribute in the slowing down of the test mass (not only those with $\gamma > \gamma_0$).

Remarkably, we also found that a naive generalization of the classical DF formula in the case of strong scattering in post-Newtonian approximation and non-relativistic velocity distribution gives a smaller drag on the test particle with respect to the original Chandrasekhar expression due to the competing effects of orbit deflection and relativistic precession. A more complete treatment of relativistic DF using the gravitational Darwin Lagrangian (in lieu of the more complex Einstein-Infeld-Hoffmann Lagrangian) appears to confirm this. However, for values of $\gamma_v$ higher than the peak value of the velocity distribution, the behaviour is reversed and the relativistic DF expression dominates over its parent classical formulation. This result appears to be confirmed by preliminary direct $N$-body simulations (to be published elsewhere; see Di Cintio et al., in prep.), where we have studied the orbital decay of a $10^5 M_\odot$ BH placed initially on a circular orbit in a star cluster with and without the post-Newtonian correction to the force law, finding a slightly greater in-spiraling time (by a factor 1.1) in the runs with the post-Newtonian corrections.

The results discussed in the present paper could be relevant for the dynamics of massive objects at the centre of dense star clusters where strong deflections and large velocity dispersion may both occur, and relevant for models of hot dark matter with relativistic velocities. It is worth mentioning that the formalism used in Sect. 4 could be also extended to treat relativistic charged particles (in the limit of negligible radiation losses) using the electrodynamical Darwin Lagrangian. In this respect, it should be noted again that the expression derived in Sect. 4 can also be used in plasma physics as the velocity and impact parameter integrals are exactly the same as in the DF friction formula in charged-particle systems. In particular, as already mentioned above, in these environments fat-tailed power-law velocity distributions are rather common. Therefore, due to the large contribution of such tails in the relativistic velocity integrals, we can expect relevant discrepancies with the classical predictions.

At this point it remains to be determined whether the inclusion of explicitly dissipative terms (i.e. the effect of momentum loss due to the emission of gravitational waves) would alter significantly the relativistic dynamical friction drag. To do so, in principle, we should include in Eq. (44) all terms up to order 2.5. Moreover, an additional generalization that seems to be feasible could be in the direction of systems with a mass spectrum with a mass-dependent average relativistic factor, in other words, where for example only the low-mass particles have relativistic velocities.

In this paper we have explored only infinitely extended systems in the spirit of the original treatment of the dynamical friction. As mentioned above, star cluster simulations are in the works. In the next paper in this series, we will investigate by means of post-Newtonian $N$-body simulations the collisional dynamics of compact objects kicked by gravitational wave emission in dense stellar systems and the relativistic corrections on their dynamical friction induced retention.

Acknowledgements. We thank Lapo Casetti, for the useful discussions at an early stage of this work. One of us (PDCF) wishes to acknowledge partial financial support from the MIUR-PRIN2017 project Grained description for non-equilibrium systems and transport phenomena (CO-NEST) n. 201798CZL.

References

Abbott, B. P., Abbott, R., Abbott, T. D., et al. (LIGO Scientific Collaboration and Virgo Collaboration) 2016a, Phys. Rev. Lett., 116, 061102
Abbott, B. P., Abbott, R., Abbott, T. D., et al. (LIGO Scientific Collaboration and Virgo Collaboration) 2016b, Phys. Rev. Lett., 115, 221101
Adhikari, S., Dalal, N., & Clandert, J. 2016, J. Cosmol. Astropart. Phys., 2016, 022
Alessandrini, E., Lanzoni, B., Micocchi, P., Ciotti, L., & Ferraro, F. R. 2014, ApJ, 795, 169
Alessandrini, E., Lanzoni, B., Ferraro, F. R., Micocchi, P., & Vesperini, E. 2016, ApJ, 833, 252
Antonini, F., & Merritt, D. 2012, ApJ, 745, 83
Antonini, F., Gieles, M., & Guandalini, A. 2019, MNRAS, 486, 5008
Arena, S. E., & Bertin, G. 2007, A&A, 463, 921
Arena, S. E., Bertin, G., Liseikina, T., & Pegoraro, F. 2006, A&A, 453, 9
Arfken, G. B., Weber, H. J., & Harris, F. E. 2012, Mathematical Methods for Physicists: A Comprehensive Guide (Elsevier Science Publishing Co Inc)
Barnes, E. T., Blandford, R. D., & Rasio, F. A. 2004, ApJ, 611, 735
Bartel, B. 2018, MNRAS, 479, 6
Barausse, E. 2007, MNRAS, 382, 826
Beklenstein, J., & Milgrom, M. 1984, ApJ, 286, 7
Beklenstein, J. D., & Maoz, E. 1992, ApJ, 390, 79
Bertin, G., Liseikina, T., & Pegoraro, J. 2003, A&A, 405, 73
Bhat, A., Iragang, A., & Heber, U. 2022, A&A, 663, A39
Binney, J. 1977, MNRAS, 181, 735
Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton Series in Astrophysics)
Binney, J., & Tremaine, S. 2008, Galactic Dynamics, Second Edition (Princeton Series in Astrophysics)
Blanchet, L. 2010, Post-Newtonian Theory and the Two-body Problem
Cashen, B., Aker, A., & Keesden, M. 2017, Phys. Rev. D, 95, 064014
Chandrasekhar, S. 1941, ApJ, 94, 511
Chandrasekhar, S. 1942, Principles of Stellar Dynamics (Dover Publications, Inc.)
Chandrasekhar, S. 1943, ApJ, 97, 255
Chandrasekhar, S. 1995, Newton’s Principia for the Common Reader (Clarendon Press)
Chavanis, P.-H. 2020a, Eur. Phys. J. Plus, 135, 290
Chavanis, P.-H. 2020b, Eur. Phys. J. Plus, 135, 310
Chen, N., Ni, Y., Tremmel, M., et al. 2022, MNRAS, 510, 531
Ciotti, L. 2010, Am. Inst. Phys. Conf. Ser., 1242, 117
Ciotti, L. 2019, MNRAS, 491, 407
Ciotti, L. 2021, in Introduction to Stellar Dynamics (Cambridge University Press)
Ciotti, L., & Binney, J. 2004, MNRAS, 351, 285
Cole, P. S., Coogan, A., Kavanagh, B. J., & Bertone, G. 2023, Phys. Rev. D, 107, 083006
Colpi, M., & Pallavicini, A. 1998, ApJ, 502, 150
Correia, M. 2022, Phys. Rev. D, 105, 084041
Costa, L. F. O., & Natário, J. 2014, Gen. Rel. Grav., 46, 1792
Dergachev, T., & Deruelle, N. 1985, Annales de l’I.H.P. Physique théorique, 43, 107
Deruelle, N., & Uzan, J.-P. 2018, Relativity in Modern Physics (Oxford University Press)
Di Cintio, P., Ciotti, L., & Nipoti, C. 2020, in Star Clusters: From the Milky Way to the Early Universe, eds. A. Bragaglia, M. Davies, A. Sills, & E. Vesperini, 351, 93
Eddington, A., & Clark, G. L. 1938, Proc. R. Soc. London Ser. A, 166, 465

A140, page 11 of 15
Appendix A: Post-Newtonian dynamical friction and the (gravitational) Darwin Lagrangian

When computing the deflection for an isolated encounter, we need to express (as done classically) the equation of motion for the reduced particle. Instead of deriving such equations by using the principle of least action, it is easier to find their first integrals as

\[ p_\theta = \frac{\partial L_D}{\partial (d\theta / dt)}; \quad E = p_\theta \frac{d\theta}{dt} + p_r \frac{dr}{dt} - L_D \equiv \mathcal{H}_D, \quad (A.1) \]

where \( \mathcal{H}_D \) is the Darwin Hamiltonian, in analogy with the electromagnetic case. At the lowest order (i.e. when \( L_{\text{Newton}} \equiv L_{\text{Darwin}} \)), we have

\[ p_\theta = \mu \frac{d\theta}{dt}, \quad E = \frac{1}{2} p_\theta^2 \left( \frac{d\theta}{dt} \right)^2 + \Phi \left( \frac{\mu^3}{\mu_3} + \frac{\mu}{M} \right). \quad (A.2) \]

These expressions will be used to eliminate \( dr/dt \) and \( d\theta/dt \) from \( p_\theta \) and \( E \) in all terms of order \( 1/c^2 \). This is possible because

\[ L_{\text{Darwin}} = L_{\text{Newton}} + 1/c^2(\ldots) \]

and all the terms in parentheses, which already contain a factor of order \( 1/c^2 \), are multiplied by another \( 1/c^2 \) factor out of parentheses, and therefore adding up to the terms in \( 1/c^2 \), that are neglected in the IPN approximation (for a more sophisticated proof of this argument, see e.g. Damour & Deruelle 1985).

At IPN we then find

\[ p_\theta = \mu \frac{d\theta}{dt} \left( 1 + \frac{1}{\mu} \left[ \frac{\mu^3}{\mu_3} + \frac{\mu}{M} \right] \right), \quad (A.3) \]

from which

\[ \frac{d\theta}{dt} = \frac{p_\theta}{\mu^2 \left( 1 + \ldots \right)} \approx \frac{p_\theta}{\mu^2} \left( 1 - \frac{1}{\mu} \left[ \frac{\mu^3}{\mu_3} + \frac{\mu}{M} \right] \right), \quad (A.4) \]

and

\[ p_r = \mu \frac{dr}{dt} \left( 1 + \frac{1}{\mu} \left[ \frac{\mu^3}{\mu_3} + \frac{\mu}{M} \right] \right). \quad (A.5) \]

In order to find \( \mathcal{H}_D \) as a function only of \( r \) and \( dr/d\theta \), we have to replace

\[ \frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} \quad \text{(A.6)} \]

in \( p_r \) and \( L_D \), while \( d\theta/dt \) is given by (A.4). Therefore, keeping only terms of order \( 1/c^2 \), we obtain

\[ L_D = \frac{1}{2} p_\theta^2 \left( \frac{d\theta}{dt} \right)^2 + \frac{1}{2} p_r^2 \left( \frac{dr}{dt} \right)^2 + \Phi \left( \frac{\mu^3}{\mu_3} + \frac{\mu}{M} \right), \quad (A.7) \]

and finally

\[ \mathcal{H}_D = p_\theta \frac{d\theta}{dt} + p_r \frac{dr}{dt} - L_D = \frac{1}{2} p_\theta^2 \left( \frac{d\theta}{dt} \right)^2 \left( 1 - \frac{2\Phi}{M \mathcal{C}^2} \right) \]

\[ + \frac{1}{2} p_r^2 \left( 1 - \frac{2\Phi}{M \mathcal{C}^2} \right) \left( 1 - \frac{2\mathcal{M}}{M \mathcal{C}^2} \right) + \Phi \times \left( \frac{\mu^3}{\mu_3} - \frac{\mu}{M} \right) \]

\[ \times \left( 1 - \frac{2\Phi}{M \mathcal{C}^2} \right) = E. \quad (A.8) \]

At this point we need to express the azimuthal angle, \( \theta \), in \( S' \). In order to do so, we manipulate Equation (A.8) by isolating the terms in \( d\theta \) and \( dr/r \), and obtain

\[ \left[ \frac{1}{2} p_\theta^2 \left( \frac{d\theta}{dt} \right)^2 + \frac{1}{2} p_r^2 \left( \frac{dr}{dt} \right)^2 \left( 1 - \frac{2\Phi}{M \mathcal{C}^2} \right) \right] = E + \Phi + \frac{1}{\mu c^2} \left[ \frac{1}{2} \frac{\mu^3}{\mu_3} + \frac{1}{2} \Phi \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right], \quad (A.9) \]

from which, with a little rearrangement of terms we then get

\[ \left[ \frac{1}{2} p_\theta^2 \left( \frac{d\theta}{dt} \right)^2 \right] = \left( \frac{E + \Phi}{(1 - \frac{2\Phi}{M \mathcal{C}^2})} - \frac{1}{2} \frac{\mu^3}{\mu_3} + \frac{1}{2} \Phi \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right) \times \]

\[ \left( 1 - \frac{2\mathcal{M}}{M \mathcal{C}^2} \right) \left( 1 + \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right) \]

\[ \times \left( 1 - \frac{2\Phi}{M \mathcal{C}^2} \right) \left( 1 + \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right) \] \quad (A.10)

The angle \( \theta \) is obtained in integral form as

\[ \theta = 2 \int_{r_c}^r \frac{1}{r} \left( \frac{2\mu^2}{p_\theta} - \frac{E + \Phi}{(1 - \frac{2\Phi}{M \mathcal{C}^2})} - \frac{1}{2} \frac{\mu^3}{\mu_3} + \frac{1}{2} \Phi \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right) \right] \frac{1}{r^2} dr. \quad (A.11) \]

where \( r_c \) is the distance of closest approach, which can be found by setting \( dr/d\theta = 0 \) in Eq. (A.10). It is now useful to make a change of variable by introducing \( x = r_c/r \), so that Equation (A.11) becomes

\[ \theta = 2 \int_0^1 \frac{dx}{x} \left[ \frac{2\mu^2}{p_\theta^2} - \frac{E + \Phi}{\mu^2 \mathcal{C}^2} \left( \frac{1}{1 - \frac{2\Phi}{M \mathcal{C}^2}} \right) - 1 \right. \]

\[ + \frac{2\mu^2}{p_\theta^2} \frac{1}{1 - \frac{2\Phi}{M \mathcal{C}^2}} \left( \frac{1}{1 - \frac{2\Phi}{M \mathcal{C}^2}} \right) \frac{1}{2} \frac{E \mu^3}{\mu_3} - \frac{E \mathcal{M} \mu^3}{2 \mu_3} \left( \frac{\mu_3}{\mu_3} - \frac{2\mu}{M} \right) + \frac{1}{2} \frac{E \mu^3}{\mu_3} - \frac{2\mu}{M} \right] \left( \frac{2\mu^2}{p_\theta^2} \right) \]

\[ \frac{1}{x^2} \left( \frac{2\mu^2}{p_\theta^2} \right) \left( 1 + \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right) \left( 1 - \frac{2\Phi}{M \mathcal{C}^2} \right) \left( 1 + \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right) \] \quad (A.12)

Furthermore, let \( G\mu/r_c c^2 = \delta/r_c \ll 1 \), so that we can perform the expansion \( (1 - 2\delta x/r_c)^{-1} \approx 1 + 2\delta x/r_c \) and neglect the terms proportional to \( 1/c^2 \), and we obtain

\[ \theta = 2 \int_0^1 \left[ -a_1 x^2 + a_2 r_c x + a_3 r_c^2 \right]^{1/2} dx, \quad (A.13) \]

where we define the following quantities to simplify the notation:

\[ a_1 = 1 - \frac{G^2 \mu^2 \mathcal{M}^2}{\mu_3^2} \left( \frac{\mu_3}{\mu_3} + 2\frac{\mu}{M} \right), \]

\[ a_2 = \frac{2G^2 \mu^2 \mathcal{M}^2}{p_\theta^2} \left( 1 + \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right), \]

\[ a_3 = \frac{2G^2 \mu^2 \mathcal{M}^2}{p_\theta^2} \left( 1 + \frac{\mu^3}{\mu_3} - \frac{2\mu}{M} \right), \quad (A.14) \]

Solving the elementary integral in Eq. (A.13) yields

\[ \theta = \frac{2}{\sqrt{a_1}} \left[ \sin^{-1} \left( \frac{a_2 r_c}{\sqrt{a_2 r_c^2 + 4a_1 a_3 r_c^2}} \right) - \sin^{-1} \left( \frac{a_2 r_c - 2a_1}{\sqrt{a_2 r_c^2 + 4a_1 a_3 r_c^2}} \right) \right]. \quad (A.15) \]
To eliminate the variable \( r_c \) from the expression above, we solve \([-a_i x^2 + a_2 r_c x + a_3 r_c^2]_{x=1} = 0 \) and substitute its positive root in Eq. (A.15), obtaining
\[
\vartheta = \frac{2}{\sqrt{a_1}} \left[ \sin^{-1} \left( 1 + \frac{4a_1 a_3}{a_2^2} \right)^{-1/2} + \frac{\pi}{2} \right]. 
\tag{A.16}
\]

This is the azimuthal angle of the collision in the frame \( S' \). In this form \( \vartheta \) is given as a function of the first integrals \( \rho_s \) and \( \mathcal{E} \). We now proceed to express it in terms of the impact parameter \( b \) and the asymptotic relative velocity of the stars in the frame \( S' \).

We let \( \mathbf{u}, \mathbf{u}_{T}, \) and \( \mathbf{u}_{T} \) be the asymptotic values of \( \mathbf{u}, \mathbf{u}_{T}, \) and \( \mathbf{u} \) in \( S' \) after the encounter. In this limit (i.e., \( r \to +\infty \)) we have that \( r^2 \mathbf{d} \rho/dt \to b \mathbf{u}, \Phi \to 0, \) and \( \mathcal{E} \to \mu u^2/2, \) so Equations (A.3, A.8) become
\[
\rho_s = \mu b u^2 \left( 1 + \frac{u^2}{2} \right), \quad E \approx \frac{1}{2} \mu c^2 \left( 1 + \frac{3}{4} \frac{u^2}{c^2} \frac{\mu^4}{\mu^3} \right).
\tag{A.17}
\]

Introducing the effective impact parameter for sharp deflections \( \mathcal{R} = GM/u^2 \), the factors \( a_i \) in Eq. (A.13) become
\[
a_1 = 1 - \frac{u^2 R^2}{b^2} \left( \frac{\mu^3}{\mu_3^2} + 2 \frac{\mu}{M} \right),
\]
\[
a_2 = \frac{2 R^2}{b^2} \left[ 1 + \frac{u^2}{2} \left( \frac{\mu}{\mu_3} - \frac{\mu^3}{\mu_3} \right) \right],
\]
\[
a_3 = \frac{1}{b^2}.
\tag{A.18}
\]

The azimuthal angle \( \vartheta \) is finally given as a function of \( b \) and \( u \) as
\[
\vartheta = \frac{2 \left( \sin^{-1} \left( \frac{\mu}{\mu_3} \left[ 1 + \frac{u^2}{2} \left( \frac{\mu}{\mu_3} - \frac{\mu^3}{\mu_3} \right) \right] \right) + \frac{\pi}{2} \right)}{\sqrt{1 - \frac{u^2}{2} \left( \frac{\mu}{\mu_3} + \frac{\mu^3}{\mu_3} \right)}} \approx \frac{2 \sin^{-1} \left( 1 + \frac{u^2}{2} \left( \frac{\mu}{\mu_3} + \frac{\mu^3}{\mu_3} \right) \right) + \pi}{\sqrt{1 - \frac{u^2}{2} \left( \frac{\mu}{\mu_3} + \frac{\mu^3}{\mu_3} \right)}},
\tag{A.19}
\]

from which we can recover the net deflection angle as \( \theta_{\text{defl}} = \vartheta - \pi \).

In order to switch back to \( S' \) and evaluate \( \Delta \mathbf{v}_{Tj} \), we let \( \mathbf{u}_{T} \) and \( \mathbf{u}_{f} \) be the initial and final velocity of the test star in \( S' \). By decomposing them into the components parallel and perpendicular to \( \mathbf{u}_{T} \), we have
\[
\mathbf{u}_{Tf} = \frac{\mathbf{u}_{f} \cdot \mathbf{u}_{T}}{|| \mathbf{u}_{T} ||} \mathbf{u}_{T},
\tag{A.20}
\]
\[
|| \mathbf{u}_{T} \cdot \mathbf{u}_{T} || = 0,
\tag{A.21}
\]
\[
\mathbf{u}_{Tf} = \frac{\mathbf{u}_{f} \cdot \mathbf{u}_{T}}{|| \mathbf{u}_{T} ||} \mathbf{u}_{T},
\tag{A.22}
\]
\[
\mathbf{u}_{Tf} = \frac{\mathbf{u}_{f} \cdot \mathbf{u}_{T}}{|| \mathbf{u}_{T} ||} \mathbf{u}_{T},
\tag{A.23}
\]

In the \( S' \) frame, \( \Delta \mathbf{v}_{T} \equiv \mathbf{w}_{f} - \mathbf{w}_{e} \), since \( \mathbf{w}_{e} = 0 \). Moreover, as the test star moves with velocity \( -\mathbf{u}_{T} \) in \( S' \), we can write in components
\[
\Delta \mathbf{v}_{T} = \frac{u_{T} \cos(\theta - \pi) - u_{T}}{1 - \frac{u_{T} \cos(\theta - \pi)}{c^2}} \approx \left[ u_{T} \cos(\theta - \pi) - u_{T} \right] \left( 1 + \frac{u_{T}^2}{c^2} \right),
\tag{A.24}
\]
and
\[
\Delta \mathbf{v}_{T} = \frac{u_{T} \sin(\theta - \pi) \sqrt{1 - \frac{u_{T}^2}{c^2}}}{1 - \frac{u_{T} \cos(\theta - \pi)}{c^2}} \approx \left[ u_{T} \sin(\theta - \pi) \right] \left( 1 - \frac{u_{T}^2}{2 c^2} \right) x \left( 1 + \frac{u_{T}^2}{c^2} \right) \approx \left[ u_{T} \sin(\theta - \pi) \right] \left( 1 + \frac{u_{T}^2}{c^2} \cos(\theta - \pi) - \frac{1}{2} \frac{u_{T}^2}{c^2} \right)
\tag{A.25}
\]

The two equations above should then be expressed in terms of \( \mathbf{w}_{f} \) instead of \( \mathbf{u}_{T} \) and \( \mathbf{u} \). We recall that \( \mathbf{u} \) is the relative velocity in \( S' \), as given by Eq. (27), where \( \mathbf{u}_{f} \) and \( \mathbf{u}_{T} \) are defined in Eq. (68).

Since we only need the relative velocity at 1PN approximation, expanding Eq. (27) and keeping the terms in \( 1/c^2 \) yields
\[
\mathbf{u} = \left( 1 + \frac{M m}{M_{c}^2} \right) \mathbf{w}_{f},
\tag{A.26}
\]
where we replaced \( \mathbf{u}_{T} \) and \( \mathbf{u}_{f} \) with their expression given by Eq. (68). At the first order we have \( \mathbf{u} = \mathbf{w}_{f} \) and the previous equation becomes
\[
\mathbf{u} = \left( 1 + \frac{M M_{c}^2}{M_{c}^2} \right) \mathbf{w}_{f}.
\tag{A.27}
\]

Making use of the definition of \( \mathbf{u}_{T} \) in the limit \( r \to \infty \) given in Damour & Deruelle (1985),
\[
\mathbf{u}_{T} = \frac{m}{M} \left( 1 + \frac{1}{2} \frac{M_{c}^2}{M_{c}^2} \right) \mathbf{w}_{f},
\tag{A.28}
\]
and substituting Equations (A.24) and (A.27) in Equation (63), we obtain the DF formula in 1PN approximation given in Eq. (71).

Appendix B: Post-Newtonian deflection angle

In order to recover the parallel and perpendicular components of the relative velocity during an encounter, we should first evaluate \( \cos(\theta - \pi) \) and \( \sin(\theta - \pi) \) in 1PN approximation. In this limit we have that \( \cos(1/c^2) \approx 1 \) and \( \sin(1/c^2) \approx (1/c^2) \). Using the standard trigonometry and some further algebraic manipulation, we define the angle \( \theta \) by its sine and cosine as
\[
\cos(\theta - \pi) \approx 1 - \frac{2}{1 + b^2/R^2} + \frac{2u^2}{c^2} \left( \frac{b}{b^2/R^2} + \frac{b}{b^2/R^2} \right)
\tag{B.1}
\]
and
\[
\sin(\theta - \pi) \approx 1 + b^2/R^2 - \frac{2u^2}{c^2} \left( \frac{b}{b^2/R^2} + \frac{b}{b^2/R^2} \right).
\tag{B.2}
\]

Chiari, C. and Di Cintio, P., A&A 677, A140 (2023)
and

\[
\sin(\theta - \pi) = \frac{2b}{R(1 + b^2/R^2)} + \frac{u^2}{c^2 R} \left[ \frac{\mathcal{R}^2}{b} \left( \frac{\mu^3}{\mu_3} + 2 \frac{\mu}{M} \right) \left( \frac{\mu}{M} - \frac{\mu^3}{\mu_3} \right) \left( 1 + \frac{b^2}{R^2} \right) \right]
\]

\[+ \left( 1 - \frac{2}{1 + b^2/R^2} \right) \left( \frac{u^2 \mathcal{R}^2}{c^2 b^2} \left( \frac{\mu^3}{\mu_3} + 2 \frac{\mu}{M} \right) \right) \left( \sin^{-1} (1 + b^2/R^2)^{-1/2} + \frac{\pi}{2} \right) \]