Abstract: We present a condition that guarantees the existence and uniqueness of fixed (or best proximity) points in complete metric space (or uniformly convex Banach spaces) for a wide class of cyclic maps, called $p$–cyclic summing maps. These results generalize some known results from fixed point theory. We find a priori and a posteriori error estimates of the fixed (or best proximity) point for the Picard iteration associated with the investigated class of maps, provided that the modulus of convexity of the underlying space is of power type. We illustrate the results with some applications and examples.

Keywords: fixed point; cyclical operator; contractive condition; best proximity point; uniformly convex Banach space; $p$–summing contraction

MSC: Primary 47H10; 58E30; 54H2

1. Introduction and Preliminaries

Banach contraction principle and its numerous generalizations turn out to be a powerful tool in mathematical research. A direction for a generalization of the Banach contraction principle is the concept of cyclical maps [1]. Fixed point theory is a widely applied technique, when trying to solve $Tx = x$, provided that $T : Z \to Z$, when $Z$ is a metric space. Due to the fact that a non-self mapping $T : Z \to Y$, $Z \cap Y = \emptyset$ do not have a fixed point, an approach can be to search for $x \in Z$ that is as close as possible to its image $Tx$ i.e., to try to solve $\min \{ \| x - Tx \| : x \in Z \}$. The last minimization problem, when $\min \{ \| x - Tx \| : x \in Z \} = 0$, coincides with $x = Tx$. Best proximity point results are applicable in this context. The concept of mentioned above points is initiated by Eldred and Veeramani in [2]. This definition is broader than that of cyclical maps because whenever the sets intersect the best proximity point reduces to a fixed point. A condition that guarantees the existence and uniqueness of best proximity points is presented in [2], provided that the underlying Banach space is uniformly convex. It is well known that a plentiful number of contractive-type maps that are known to have fixed points can be generalized to ensure the existence of best proximity points. The number of such generalizations is enormous and we could not mention even a small part of them. Some results of this kind are obtained in [3–9] and some very recent investigations [10–18]. It is curious that, in all the explored conditions for the presence of best proximity, the distances between the consecutive sets are equal. A condition that is completely different from the known ones and which warrants the existence and uniqueness of the best proximity points and for the cases when the distances between them are not equal is considered in [19]. These new types of maps were named $p$–cyclic summing contraction maps,
but the authors have investigated only the case of $p = 3$ there. A further investigation about different classes of $p$–cyclic summing contraction maps was presented in [20]. We fill the gaps from [19] by proving that the results from [19] can be generalized also for $p$–cyclic summing contraction maps. Some main tools for the proof are the results from [20].

Error estimates about fixed points for self (or cyclic) maps, starting with the classical Banach contraction principle, some resent results from this year e.g., [21,22] and the approximations of fixed points in [23,24], for example, are one of the greatest advantages in the applications of the fixed points technique. There have been a lack of such results about error estimates for best proximity points. This gap has been filled first for some kind of cyclic maps in [25] and later for other cyclic maps in [26–30].

We have obtained a priori error estimates and a posteriori error estimates for the $p$–cyclic summing contractions.

The structure of the paper is the following:

Preliminary results—We present the definitions and results, which we will need for the main theorem

Main Result—We define the notion of $p$–cyclic summing contraction map and we state and prove that any such map has a unique best proximity point and we obtain error estimates, when a sequence of successive iterations is used

Applications—We illustrate the main result, by applying it to the known $p$–cyclic maps, define in [5], and we extend the results from [5] by getting error estimates. We apply the main result in getting error estimates in the example presented in [19].

Conclusions—We discuss some open problems and possible future generalizations.

2. Preliminary Results

We will recall basic definitions and concepts which are related to our investigation. Let $(X, \rho)$ be a metric space. A distance between two subset $Y, Z \subset X$ is $\text{dist}(Y, Z) = \inf \{ \rho(y, z) : y \in Y, z \in Z \}$.

Let $\{A_i\}_{i=1}^{p}$ be non-empty subsets of a metric space $(X, \rho)$. A well-known agreement, just to simplify the notations, is $A_{p+i} = A_i$ for any $i \in \mathbb{N}$. A map $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called a $p$–cyclic map if $T(A_i) \subseteq A_{i+1}$ for every $i = 1, 2, \ldots, p$. A point $\xi \in A_i$ is called a best proximity point of $T$ in $A_i$ if $\rho(\xi, T\xi) = \text{dist}(A_i, A_{i+1})$, provided that $T$ is a cyclic map.

Most of the results about best proximity points utilize the norm-structure of the underlying space $X$. Everywhere in the article the distance between the elements of $(X, \| \cdot \|)$ will be the classical one $\rho(x, y) = \|x - y\|$. We will denote by $S_X$ and $B_X$ the unit sphere and the unit ball in $(X, \| \cdot \|)$, respectively.

The uniformly convex $(X, \| \cdot \|)$ assumption plays a decisive part in the proofs in most of the research about best proximity points.

Definition 1. ([31,32] p. 285) Let $(X, \| \cdot \|)$ be a Banach space. For every $\varepsilon \in (0, 2])$, we define the modulus of convexity of $\| \cdot \|$ by

$$
\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \varepsilon \right\}.
$$

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. The space $(X, \| \cdot \|)$ is then called uniformly convex space.

The modulus of convexity depends both on the space $X$ and its norm $\| \cdot \|$. Just to simplify the notations, we will use $\delta_{\| \cdot \|}$, when there is no risk of confusion.

The next lemmas, proved in [2], are key results that we will need.

Lemma 1. ([2]) Let $A$ be a non-empty closed, convex subset, and $B$ be a non-empty, closed subset of a uniformly convex Banach space. Let $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ be sequences in $A$ and $\{y_n\}_{n=1}^{\infty}$ be a sequence in $B$ satisfying:

1. $\lim_{n \to \infty} \|z_n - y_n\| = \text{dist}(A, B)$;
(2) For every $\varepsilon > 0$, there exists $N_{0} \in \mathbb{N}$, such that, for all $m > n \geq N_{0}$, $\|x_{m} - y_{n}\| \leq \text{dist}(A, B) + \varepsilon$, then, for every $\varepsilon > 0$, there exists $N_{1} \in \mathbb{N}$, such that, for all $m > n > N_{1}$, $\|x_{m} - z_{n}\| \leq \varepsilon$.

Lemma 2. ([2]) Let $A$ be a non-empty closed, convex subset, and $B$ be a non-empty, closed subset of a uniformly convex Banach space. Let $\{x_{n}\}_{n=1}^{\infty}$ and $\{z_{n}\}_{n=1}^{\infty}$ be sequences in $A$ and $\{y_{n}\}_{n=1}^{\infty}$ be a sequence in $B$ satisfying:

1. $\lim_{n \to \infty} \|x_{n} - y_{n}\| = \text{dist}(A, B)$;
2. $\lim_{n \to \infty} \|z_{n} - y_{n}\| = \text{dist}(A, B)$;

then $\lim_{n \to \infty} \|x_{n} - z_{n}\| = 0$.

The inequality

$$\left\| \frac{x + y}{2} - z \right\| \leq \left( 1 - \delta_{X} \left( \frac{r}{R} \right) \right) R$$

(1)

for any $x, y, z \in X, R > 0, r \in [0, 2R], \|x - z\| \leq R, \|y - z\| \leq R,$ and $\|x - y\| \geq r$ holds, provided that the Banach space $X$ is uniformly convex.

The modulus of convexity $\delta$ is a strictly increasing function in any uniformly convex Banach space and consequently there exists its inverse function, which we will denote by $\delta^{-1}$. The modulus of convexity $\delta$ is said to be of power type $q$ if the inequality $\delta_{X}(\|x\|) \geq C\|x\|^{q}$ holds for any $\varepsilon \in (0, 2]$ and some strictly positive constants $C$ and $q$ ([33], p. 154). It is well known that the inequality $\delta_{X}(\|x\|) \leq K\|x\|^{2}$ holds for any Banach space endowed with any norm $\| \cdot \|$; thus, if the modulus of convexity is of power type $q$, then $q \geq 2$.

A comprehensive presentation of the geometry of Banach spaces can be found, for example, in [32–35].

Let $\{A_{i}\}_{i=1}^{p}$ be non-empty subsets of the metric space $(X, \rho)$. We will use the notions $P = \sum_{i=1}^{p} \text{dist}(A_{i}, A_{i+1}), d_{i+1} = \text{dist}(A_{i}, A_{i+1})$ and

$$s_{p}(x_{1}, x_{2}, \ldots, x_{p}) = \sum_{j=1}^{p-1} \rho(x_{j}, x_{j+1}) + \rho(x_{p}, x_{1}),$$

(2)

where, if $x_{1} \in A_{i}$, then $x_{i+k} \in A_{i+k}$ for every $k = 1, 2, \ldots, p - 1$ (where we use assume that $A_{p+i} = A_{i}$, for every $i \in \{1, 2, \ldots, p\}$). Just for simplicity of the notations, we will denote

$$s_{p,n}(x) = s_{p}\left( T^{n}x, T^{n+1}x, T^{n+2}x, \ldots, T^{n+p-1}x \right)$$

for any $x \in \bigcup_{i=1}^{p} A_{i}$, where $T$ is a $p$–cyclic map.

Definition 2. ([20]) Let $A_{i}, i = 1, 2, \ldots, p$ be subsets of a metric space $(X, \rho)$. A map $T : \bigcup_{i=1}^{p} A_{i} \to \bigcup_{i=1}^{p} A_{i}$ is said to be a $p$–cyclic summing iterated contraction if it satisfies the next two conditions:

1. $T$ is a $p$–cyclic map;
2. there is a constant $k \in (0, 1)$, so that, for every $x \in \bigcup_{i=1}^{p} A_{i}$, the inequality

$$s_{p,1}(x) \leq ks_{p,0}(x) + (1 - k)P$$

(3)

holds.

We use in the sequel an equivalent form of (3)

$$s_{p,1}(x) - P \leq k(s_{p,0}(x) - P).$$

(4)

We will need some results from [20].
Definition 3. [20,36] Let $A_i$, $i = 1, 2, \ldots, p$ be non-empty subsets of a metric space and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a $p$–cyclic map. We say that $T$ satisfies the proximal property if whenever $\lim_{n \to \infty} x_n = x \in A_i$, $x_n \in A_i$, and $\lim_{n \to \infty} \rho(x_n, Tx_n) = \dist(A_i, A_{i+1})$ hold, then follows that $\rho(x, Tx) = \dist(A_i, A_{i+1})$ for all $i = 1, 2, \ldots p$.

Let us point out that the proximal property for two sets in normed spaces was defined in [36] and for $p$–sets in [20].

Theorem 1. (20) Let $(X, \| \cdot \|)$ be a uniformly convex Banach space and $A_i \subset X$, $i = 1, 2, \ldots, p$ be closed, convex sets and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a $p$–cyclic summing iterated contraction. Then, for every $x \in A_1$, the sequence $\{T^nx\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n \to \infty} T^nx$ and $T$ is continuous at $z$ or $T$ satisfies the proximal property, then $z \in A_1$ is a best proximity point of $T$ in $A_1$, $T^nz \in A_{i+1}$ is a best proximity point of $T$ in $A_{i+1}$ for $i = 1, 2, \ldots, p$.

Lemma 3. (20) Let $(X, \| \cdot \|)$ be a uniformly convex Banach space, $A_i \subset X$, $i = 1, 2, \ldots, p$ be closed, convex sets and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a $p$–cyclic summing iterated contraction. Then, $\lim_{n \to \infty} \|T^{n+k}x - T^{n+p+1}x\| = 0$, $k = 0, 1, 2, \ldots, p - 1$.

Lemma 4. (20) Let $(X, \rho)$ be a metric space, $A_i \subset X$, $i = 1, 2, \ldots, p$ be subsets and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a $p$–cyclic summing iterated contraction. Then, $s_{p,n}(x) - P \leq k^p(s_{p,0}(x) - P)$, $s_{p,n}(x) - P \leq k^l(s_{p,n-l}(x) - P)$, $\lim_{n \to \infty} s_{p,n}(x) = P$.

From Lemma 4, it is easy to observe that there holds the inequality

$$\|T^{n}x - T^{n+1}x\| - d_{i,l+1} \leq k^{n^p}(s_{p,0}(x) - P),$$

whenever $x \in A_i$.

3. Main Result

Definition 4. Let $A_i$, $i = 1, 2, \ldots, p$ be subsets of a metric space $(X, \rho)$. A map $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ will be called a $p$–cyclic summing contraction if it satisfies the next two assumptions:

1) $T$ is a $p$–cyclic map;
2) there is a constant $k \in (0, 1)$, so that the inequality

$$s_{p}(Tx_1, Tx_2, \ldots, Tx_p) \leq k^{p}(x_1, x_2, \ldots, x_p) + (1 - k)P$$

holds for every $x_i \in A_i$, $i = 1, 2, \ldots, p$.

By the fact that any $p$–cyclic summing contraction is a $p$–cyclic summing iterated contraction, it follows that we can apply Theorem 1 for $p$–cyclic summing contraction.

Theorem 2. Let $(X, \| \cdot \|)$ be a uniformly convex Banach space with modulus of convexity $\delta_{\| \cdot \|}(\varepsilon)$ and $A_i \subset X$, $i = 1, 2, \ldots, p$ be closed, convex sets and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a $p$–cyclic summing contraction.

Then, for every $x \in A_1$, the sequence $\{T^nx\}_{n=1}^{\infty}$ is convergent. If $z = \lim_{n \to \infty} T^nx$, then $z \in A_1$ is a best proximity point of $T$ in $A_1$, $T^nz \in A_{i+1}$ is a best proximity point of $T$ in $A_{i+1}$ for $i = 1, 2, \ldots, p - 1$ and $T^nz = z$.

If $(X, \| \cdot \|)$ is with modulus of convexity of power type with constants $C > 0$ and $q \geq 2$, then
• a priori error estimate holds

\[ ||\xi - T^{pm}x|| \leq s_{p,0}(x) \frac{\sqrt{s_{p,0}(x) - p}}{Cd_{1,2}} \cdot \left( \sqrt[2]{K} \right)^{pm} \frac{1}{1 - \sqrt[2]{K}}. \]  \tag{6}

• a posteriori error estimate holds

\[ ||T^{pm}x - \xi|| \leq s_{p,pm-1}(x) \frac{\sqrt{s_{p,pm-1}(x) - p}}{Cd_{1,2}} \cdot \sqrt[2]{K} \] \tag{7}

**Proof.** As far as \( T \) is a \( p \)-cyclic summing contraction, it follows that it is \( p \)-cyclic summing iterated contraction. There, for any arbitrary chosen \( x \in A_i \) from Theorem 1, we get that the iterated sequence \( \{T^{pm}x\}_{m=0}^{\infty} \) is convergent to a point \( z \in X \). From the assumption that \( A_i \) are closed subsets, it follows that \( z \in A_i \).

Without loss of generality we can assume that \( x \in A_1 \), indeed, we can enumerate the sets so that \( x \in A_1 \). This will simplify the notations.

By the continuity of the function \( \| \cdot - \cdot \| \), it follows that \( \|z - Tz\| = \lim_{n \to \infty} \|T^{pm}x - Tz\| \) and

\[ \lim_{n \to \infty} \|T^{pn-1}x - z\| = \lim_{n \to \infty} \|T^{pn-1}x - T^{pn}x\|. \] \tag{8}

We apply consecutively (8) and Lemma 4 to obtain the next chain of inequalities:

\[ s_p(z, Tz, T^2z, \ldots, T^{p-1}z) - P = \lim_{n \to \infty} s_p(T^{pn}x, Tz, T^2z, \ldots, T^{p-1}z) - P \]
\[ \leq k \left( \lim_{n \to \infty} s_p(T^{pn-1}x, Tz, T^2z, \ldots, T^{p-2}z) - P \right) \]
\[ = k \left( \lim_{n \to \infty} s_p(T^{pn-1}x, T^{pn}x, Tz, \ldots, T^{p-2}z) - P \right) \]
\[ \leq k^2 \left( \lim_{n \to \infty} s_p(T^{pn-2}x, T^{pn-1}x, Tz, \ldots, T^{p-3}z) - P \right) \]
\[ = k^2 \left( \lim_{n \to \infty} s_p(T^{pn-2}x, T^{pn-1}x, T^{pn}x, \ldots, T^{p-3}z) - P \right) \]
\[ \leq k^3 \left( \lim_{n \to \infty} s_p(T^{pn-3}x, T^{pn-2}x, T^{pn-1}x, T^{p-4}z, \ldots, T^{p-4}z) - P \right) \]
\[ \vdots \]
\[ \leq k^p \left( \lim_{n \to \infty} s_p(T^{pn}x, T^{pn}z, T^{pn}+x, T^{pn}+1x, \ldots, T^{pn}z) - P \right) \]
\[ \leq k^p \left( \lim_{n \to \infty} s_p(x, z, Tz, T^2z, \ldots, T^{p-1}z, z) - P \right) \]
\[ = k^p (P - P) = 0. \] \tag{9}

Since \( z \in A_1 \), it follows that \( Tz \in A_2 \), \( T^k z \in A_{1+k} \). From the inequalities \( \|T^k z - T^{k+1}z\| \geq d_{1+k,2+k}, \) \( k = 0, \ldots, p-2 \) and \( \|T^{p-1}z - z\| \geq d_{p,1} \) and (9), it follows that

\[ \|T^k z - T^{k+1}z\| - d_{1+k,2+k} \leq s_p(z, Tz, T^2z, \ldots, T^{p-1}z) - P = 0 \]
for \( k = 0, \ldots, p-2 \) and

\[ \|T^{p-1}z - z\| - d_{p,1} \leq s_p(z, Tz, T^2z, \ldots, T^{p-1}z) - P = 0. \]

Thus, \( T^k z - T^{k+1}z \leq d_{1+k,2+k} = \text{dist}(A_{i+k}, A_{i+k+1}) \) for \( k = 0, \ldots, p-2 \) and \( \|T^{p-1}z - z\| = d_{p,1} = \text{dist}(A_p, A_1) \). Therefore, \( z \) is a best proximity point of \( T \) in \( A_1 \) and \( T^k z \) is a best proximity point of \( T \) in \( A_{1+k} \), for \( k = 1, 2, \ldots, p-1 \).
From the inequality \( s_p(Tpz, Tz, T^2z, \ldots, T^{p-1}z) \leq ks_p(T^{p-1}z, z, Tz, \ldots, T^2z) = s_{p,0}(z) = P \), it follows that \( \| Tpz - Tz \| - d_{1,2} \leq s_{p,0}(z) - P = 0 \) and thus \( \| Tpz - Tz \| = d_{1,2} \). From the equality \( \| z - Tz \| = d_{1,2} \) and Lemma 1, it follows that \( Tpz = z \).

Now, we will prove the a priori error estimate. Let us assume now that \((X, \| \cdot \|)\) is uniformly convex with a modulus of convexity of power type with constants \( C > 0 \) and \( q \geq 2 \).

For any \( x \in A, n \in \mathbb{N} \) and \( l \leq 2n \) there holds the inequality
\[
\delta_l \left\| \frac{\| T^{2n}x - T^{2n+2}x \|}{d_{1,2} + k^l S_{2n-1,2n+1-1}(x)} \right\| \leq \frac{k^l S_{2n-1,2n+1-1-l}(x)}{d_{1,2} + k^l S_{2n-1,2n+1-1-l}(x)}.
\]

Indeed, let \( x \in A_l \) be arbitrarily chosen. Let us denote \( S_{p, pn-1}(x) = s_{p, pn-1}(x) - P \). From Lemma 4, we have the inequalities
\[
\| T^{pn}x - T^{pn+1}x \| \leq d_{1,2} + k^l(s_{p, pn-1}(x) - P),
\]
\[
\| T^{pn+p}x - T^{pn+1}x \| \leq d_{1,2} + k^{l+1}(s_{p, pn-1}(x) - P) < d_{1,2} + k^l(s_{p, pn-1}(x) - P)
\]
and
\[
\| T^{pn+p}x - T^{pn}x \| \leq \| T^{pn+p}x - T^{pn+1}x \| + \| T^{pn+1}x - T^{pn}x \| \leq 2 \left( d_{1,2} + k^l(s_{p, pn-1}(x) - P) \right).
\]

After a substitution in (1) with \( x = T^{pn}x, y = T^{pn+p}x, z = T^{pn+1}x, r = T^{pn+p}x - T^{pn}x \) and \( R = d_{1,2} + k^l(s_{p, pn-1}(x) - P) \) and, using the convexity of the set \( A \), we get the chain of inequalities
\[
d_{1,2} \leq \frac{\| T^{pn}x + T^{pn+p}x - T^{pn+1}x \|}{2} \leq \left( 1 - \delta_l \left\| \frac{\| T^{pn}x - T^{pn+p}x \|}{d_{1,2} + k^l(s_{p, pn-1}(x) - P)} \right\| \right) \left( d_{1,2} + k^l(s_{p, pn-1}(x) - P) \right).
\]
(10)

From (10), we obtain the inequality
\[
\delta_l \left\| \frac{\| T^{pn}x - T^{pn+p}x \|}{d_{1,2} + k^l(s_{p, pn-1}(x) - P)} \right\| \leq \frac{k^l(s_{p, pn-1}(x) - P)}{d_{1,2} + k^l(s_{p, pn-1}(x) - P)}.
\]
(11)

From the assumption that \( X \) is uniform convexity of \( X \), it follows that both \( \delta_l \) and its inverse function \( \delta^{-1}_l \) are strictly increasing functions. From (11), we get
\[
\| T^{pn}x - T^{pn+p}x \| \leq \left( d_{1,2} + k^l(s_{p, pn-1}(x) - P) \right) \delta^{-1}_l \left( \frac{k^l(s_{p, pn-1}(x) - P)}{d_{1,2} + k^l(s_{p, pn-1}(x) - P)} \right).
\]
(12)

It is easy to observe that
\[
k^l(s_{p, pn-1}(x) - P) \leq s_{p, pn-1}(x) - P \leq s_{p, pn-1}(x) - d_{1,2}
\]
i.e.,
\[
d_{1,2} \leq d_{1,2} + k^l(s_{p, pn-1}(x) - P) \leq s_{p, pn-1}(x).
\]
From the inequality \( \delta\|\| (t) \geq C t^l \), we get the inequality \( \delta^{-1} (t) \leq \left( \frac{1}{t} \right)^{1/q} \) and by the last inequality and (12), we obtain

\[
\| T^{p} x - T^{p+1} x \| \leq \left( d_{1,2} + k^{l} (s_{t, p-1}(x) - p) \right) \sqrt{\frac{k^{l} (s_{t, p-1}(x) - p)}{C (d_{1,2} + k^{l} (s_{t, p-1}(x) - p))}} \leq s_{t, p-1}(x) \sqrt{\frac{s_{t, p-1}(x) - p}{Cd_{1,2}}} \left( \sqrt{k} \right)^l.
\]

(13)

We have proven in the first part that there exists a unique \( \xi \in A_{t} \), such that \( \| \xi - T\xi \| = \text{dist}(A_{t}, A_{t+1}), T^{l} \xi = \xi \) and \( \xi \) is a limit of the sequence \( \{ T^{p} x \}_{n=1}^{\infty} \) for any \( x \in A_{t} \).

After substituting in (13) \( l \) with \( pn \), we obtain the inequality

\[
\sum_{n=1}^{\infty} \| T^{p} x - T^{p+n} x \| \leq s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \sum_{n=1}^{\infty} \left( \sqrt{k} \right)^{pn} = s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \cdot \sqrt{k}^{pn}
\]

and, consequently, the series \( \sum_{n=1}^{\infty} (T^{p} x - T^{p+n} x) \) is absolutely convergent. Therefore, for any \( m \in \mathbb{N} \), \( \xi = T^{p} x - \sum_{n=m}^{\infty} (T^{p} x - T^{p+n} x) \) holds and consequently we get the inequality

\[
\| \xi - T^{p} x \| \leq \sum_{n=m}^{\infty} \| T^{p} x - T^{p+n} x \| \leq s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \cdot \frac{\sqrt{k}^{pn}}{1 - \sqrt{k}^{pn}}.
\]

It remains to prove the "a posteriori" error estimate. After a substitution with \( l = 1 + pi \) in (13), we obtain

\[
\| T^{p} x - T^{p+1} x \| \leq s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \left( \sqrt{k} \right)^{1+pi}.
\]

(14)

From (14), we get that there holds the inequality

\[
\| T^{p} x - T^{p+m} x \| \leq \sum_{j=0}^{m-1} \| T^{p+j} x - T^{p+j+1} x \|
\leq \sum_{j=0}^{m-1} s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \left( \sqrt{k} \right)^{1+j}
= s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \sum_{j=0}^{m-1} \left( \sqrt{k} \right)^{1+j}
= s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \cdot \frac{1 - \left( \sqrt{k} \right)^{pn}}{1 - \sqrt{k}^{pn}} \cdot \sqrt{k}
\]

and, after letting \( m \to \infty \) in (15), we obtain the inequality

\[
\| T^{p} x - \xi \| \leq s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - p}{Cd_{1,2}}} \cdot \frac{\sqrt{k}}{1 - \sqrt{k}^{pn}}.
\]

\[\square\]

4. Applications

Let us recall the definition of \( p \)-cyclic contractions.

Definition 5. ([5]) Let \( A_{t}, i = 1, 2, \ldots, p \) be subsets of a metric space \((X, \rho)\). A map \( T : \bigcup_{i=1}^{p} A_{i} \to \bigcup_{i=1}^{p} A_{i} \) will be called a \( p \)-cyclic contraction if it satisfies the next two assumptions:
(1) T is a p–cyclic map;
(2) there is a constant \( k \in (0, 1) \), so that the inequality
\[
\rho(Tx, Ty) \leq kp(x, y) + (1 - k)\text{dist}(A_i, A_{i+1})
\]  
holds for every \( x \in A_i \) and every \( y \in A_{i+1}, i = 1, 2, \ldots, p \).

**Theorem 3.** Let \((X, \| \cdot \|)\) be a uniformly convex Banach space with a modulus of convexity \( \delta_{\| \cdot \|} \) and \( A_i \subset X, i = 1, 2, \ldots, p \) be closed, convex sets and \( T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i \) be a p–cyclic contraction.

Then, for every \( x \in A_1 \), the sequence \( \{T^{pn}x\}_{n=1}^{\infty} \) is convergent. If \( z = \lim_{n \to \infty} T^{pn}x \), then \( z \in A_1 \) is a best proximity point of \( T \) in \( A_1 \), \( T^iz \in A_{i+1} \) is a best proximity point of \( T \) in \( A_{i+1} \) for \( i = 1, 2, \ldots, p-1 \) and \( Tpz = z \).

If there exist \( C > 0 \) and \( q \geq 2 \), such that \( \delta_{\| \cdot \|}(\varepsilon) \geq Ce^q \), then

- a priori error estimate holds
\[
\|\xi - T^{pn}x\| \leq s_{p,0}(x) \sqrt{\frac{s_{p,0}(x) - P}{C_1}} \cdot \frac{(\sqrt{k})^{pn}}{1 - \sqrt{k}^p}.
\]  
(17)
- a posteriori error estimate holds
\[
\|T^{pn}x - \xi\| \leq s_{p,pn-1}(x) \sqrt{\frac{s_{p,pn-1}(x) - P}{C_1}} \cdot \frac{\sqrt{k}}{1 - \sqrt{k}^p}.
\]  
(18)

The first part of the theorem is proven in [5].

**Proof.** We will show how the above theorem follows from Theorem 2. Let us choose an arbitrary \( x_i \in A_i, i = 1, 2, \ldots, p \). Then, after summing the inequalities
\[
\|Tx_i - Tx_{i+1}\| \leq k\|x_i - x_{i+1}\| + (1 - k)\text{dist}(A_i, A_{i+1}), \quad i = 1, 2, \ldots, p - 1.
\]
and
\[
\|Tx_p - Tx_1\| \leq k\|x_p - x_1\| + (1 - k)\text{dist}(A_p, A_1),
\]
we get
\[
s_p(Tx_1, Tx_2, \ldots, Tx_p) \leq ks_p(x_1, x_2, \ldots, x_p) + (1 - k)P.
\]  
(19)
The proofs of the error estimates follow directly from (19). \( \square \)

We believe that similar results about the error estimates can be obtained, for example, for the classical p–cyclic Kannan maps investigated in [6] or for the proximal contractions; see, e.g., [37].

We will illustrate Theorem 2 with an example from [19].

It is well known that any Hilbert space with a norm generated by the scalar product there holds the inequality \( \delta_{\| \cdot \|}(\varepsilon) \geq \frac{\varepsilon^2}{\pi} \) [38].

**Example** ([19]) Let the underlying space \( X \) be three-dimensional space \((\mathbb{R}^3_2, \| \cdot \|_2)\), endowed with the Euclidian norm \( \|(x, y, z)\|_2 = \sqrt{x^2 + y^2 + z^2} \). Let \( A_1 \subset \mathbb{R}^3_2 \) be \( A_1 = \{(x, y, z) : x \in [4, 5], y, z = 0\} \),
A_2 \subset \mathbb{R}_2^3 be A_2 = \{(x,y,z) : y \in [1,2], x,z = 0\}, A_3 \subset \mathbb{R}_2^3 be A_3 = \{(x,y,z) : z \in [1,2], x,y = 0\}.

Define the 3-cyclic map \( T : A_i \to A_{i+1} \), for \( i = 1,2,3 \) and \( A_4 = A_1 \) by
\[
T(x,0,0) = \left( 0, \frac{y}{5} + \frac{1}{7}, 0 \right), \quad x \in [4,5]
\]
\[
T(0,y,0) = \left( 0, 0, \frac{y}{5} + \frac{7}{8} \right), \quad y \in [1,2]
\]
\[
T(0,0,z) = \left( \frac{z}{8} + \frac{31}{8}, 0, 0 \right), \quad z \in [1,2].
\]

It is shown in [20] that, for every \( x \in A_1, y \in A_2, z \in A_3 \), there holds the inequality:
\[
\|Tx - Ty\|_2 + \|Ty - Tz\|_2 + \|Tz - Tx\|_2 \leq \frac{1}{2} \left( \|x - y\|_2 + \|y - z\|_2 + \|z - x\|_2 \right) + \frac{1}{2} P,
\]
where \( P = \text{dist}(A_1, A_2) + \text{dist}(A_2, A_3) + \text{dist}(A_3, A_1) = 2\sqrt{17} + \sqrt{2} \). The distances between the three sets are different and \( \text{dist}(X,Y) = \sqrt{17} \). The map \( T \) is not a cyclical contraction in the sense of [5].

Let us start with an initial guess \( x = 5 \) in \( X \), see Tables 1–3.

| \( n \) | \( T_3^0 x \) | \( T_3^1 x \) | \( T_3^2 x \) | \( T_3^3 x \) |
|-------|-------------|-------------|-------------|-------------|
| 0     | 5.00        | 4.00195     | 4.00000381  | 4.00000000 |
| 1     | 1.12        | 1.00024     | 1.0000047   | 1.00000000 |
| 2     | 1.01        | 1.00003     | 1.0000006   | 1.00000000 |

Table 1. The values of the iterated sequence \( T^nx \) if started with \( x = 5 \) in \( X \).

| \( \varepsilon \) | \ 0.1 \ | \ 0.01 \ | \ 0.0001 \ | \ 0.000001 \ |
|------------------|--------|--------|--------|--------|
| \( n \)          | 6      | 9      | 13     | 17     |

Table 2. Number of the necessary iterations needed if started with \( x = 5 \) in \( X \) to get an \( \varepsilon \) a priori error estimate.

| \( \varepsilon \) | \ 0.1 \ | \ 0.01 \ | \ 0.0001 \ | \ 0.000001 \ |
|------------------|--------|--------|--------|--------|
| \( n \)          | 2      | 3      | 5      | 7      |

Table 3. Number of the necessary iterations needed if started with \( x = 5 \) in \( X \) to get an \( \varepsilon \) a posteriori error estimate.

5. Conclusions

Let us mention that we get a larger number of the iterations that are needed to get the desired error. It happens because we use the modulus of convexity, which is the infinum of \( \|1 - \frac{x+y}{2}\| \) among all \( x,y \in S_X \), such that \( \|x - y\| \geq \varepsilon \). A reason for this may be that the modulus of convexity is greater in the direction of the best proximity point \( \xi \) than in the other directions, but, for the estimation of the error, we do not use it. We would like to pose the following question of whether it possible to get better estimates if we use the directional modulus of convexity \( \delta_{\|\cdot\|}(x,\varepsilon) \) [39]? For the estimations, we use geometric progression and that is why we impose the condition for the modulus of convexity to be of power type ([33], p. 154). Is it possible to obtain error estimates if the modulus of convexity is not of power type? Results about best proximity points in modular function spaces are obtained in [40,41]. Is it possible to generalize the notion of best proximity points in modular function spaces for \( p \)-cyclic summing contractions and to get error estimates? Sufficient conditions for the existence
of best proximity points for weak $p$–cyclic Kannan contraction is obtained in [42]. It seems that the technique of obtaining error estimates could be possible to be applied for these class of maps.

**Author Contributions:** The listed authors have made equal contributions to the presented research. All authors have read and agreed to the published version of the manuscript.

**Funding:** The first author is thankful for the support of Shumen University through Scientific Research Grant RD-08-73/23.01.2020. The second author would like to thank for the support of National Program “Young Scientists and Postdoctoral Students”—second stage.

**Acknowledgments:** The authors wish to express their hearty thanks to the anonymous referees for their valuable suggestions and comments.

**Conflicts of Interest:** The authors declare no conflict of interest.

**References**

1. Kirk, W.; Srinivasan, P.; Veeramani, P. Fixed points for mappings satisfying cyclical contractive conditions. *Fixed Point Theory* **2003**, *4*, 79–189.

2. Eldred, A.; Veeramani, P. Existence and convergence of best proximity points. *J. Math. Anal. Appl.* **2006**, *323*, 1001–1006. [CrossRef]

3. Baria, C; Suzuki, T; Vetroa, C. Best proximity points for cyclic Meir—Keeler contractions. *Nonlinear Anal.* **2008**, *69*, 3790–3794. [CrossRef]

4. Karpagam, S.; Agrawal, S. Best Proximity Point Theorems for p-Cyclic Meir-Keeler Contractions. *Fixed Point Theory Appl.* **2009**, *2009*, 1–10. [CrossRef]

5. Karpagam, S.; Agrawal, S. Existence of best proximity Points of P–cyclic contractions. *Fixed Point Theory* **2012**, *13*, 99–105.

6. Petric, A. Best proximity point theorems for weak cyclic Kannan contractions. *Filomat* **2011**, *25*, 145–154. [CrossRef]

7. Horvat-Marc, A.; Petric, M. Examples of cyclical operators. *Carpathian J. Math.* **2016**, *32*, 331–338.

8. Saksirikun, W.; Berinde, V.; Petrot, N. Coincidence point theorems for cyclic multi-valued and hybrid contractive mappings. *Carpathian J. Math.* **2019**, *35*, 85–94.

9. Radenović, S.; Chandok, S.; Shatanawi, W. Some cyclic fixed point results for contractive mappings. *Univ. Thought Publ. Nat. Sci.* **2016**, *6*, 38–40. [CrossRef]

10. Iqbal, I.; Hussain, N.; Kutbi, M. A. Existence of the solution to variational inequality, optimization problem, and elliptic boundary value problem through revisited best proximity point results. *J. Comput. Appl. Math.* **2020**, *375*, 112804. [CrossRef]

11. Rohen, Y.; Mlaiki, N. Tripled best proximity point in complete metric spaces. *Open Math.* **2020**, *18*, 204–210. [CrossRef]

12. Gupta, A.; Rohilla, M. On coupled best proximity points and Ulam–Hyers stability. *J. Fixed Point Theory Appl.* **2020**, *22*, 28. [CrossRef]

13. Gabeleh, M.; Felicit, J.M.; Eldred, A.A. Edelstein’s Theorem for Cyclic Contractive Mappings in Strictly Convex Banach Spaces. *Numer. Funct. Anal. Optim.* **2020**, *41*, 1027–1044. [CrossRef]

14. Karapinar, E.; Abbas, M.; Farooq, S. A Discussion on the Existence of Best Proximity Points That Belong to the Zero Set. *Axioms* **2020**, *9*, 19. [CrossRef]

15. Saleem, N.; Vujaković, J.; Baloch, W.U.; Radenović, S. Coincidence Point Results for Multivalued Suzuki Type Mappings Using $\theta$-Contraction in $b$-Metric Spaces. *Mathematics* **2019**, *7*, 1017. [CrossRef]

16. Al-Sulami, H.H.; Hussain, N.; Ahmad, J. Best Proximity Results with Applications to Nonlinear Dynamical Systems. *Mathematics* **2019**, *7*, 900. [CrossRef]

17. Karapinar, E.; Chen, C.-M.; Lee, C.-T. Best Proximity Point Theorems for Two Weak Cyclic Contractions on Metric-Like Spaces. *Mathematics* **2019**, *7*, 349. [CrossRef]

18. Pragadeeswarar, V.; Gopi, R.; De La Sen, M.; Radenović, S. Proximally Compatible Mappings and Common Best Proximity Points. *Symmetry* **2020**, *12*, 353. [CrossRef]

19. Petric, M.A.; Zlatanov, B. Best proximity points and fixed points for p-summing maps. *Fixed Point Theory Appl.* **2012**, *2012*, 86. [CrossRef]
20. Petric, M.; Zlatanov, B. Best proximity points for p-cyclic summing iterated contractions. *Filomat* 2018, 32, 3275–3287. [CrossRef]

21. Berinde, V.; Păcurar, M. Approximating fixed points of enriched contractions in Banach spaces. *J. Fixed Point Theory Appl.* 2020, 22, 38. [CrossRef]

22. Vrahatis, M.N. Intermediate value theorem for simplices for simplicial approximation of fixed points and zeros. *Topol. Its Appl.* 2020, 275, 107036. [CrossRef]

23. Ćirić, L. Some Recent Results in Metric Fixed Point Theory; University of Belgrade: Beograd, Serbia, 2003.

24. Berinde, V. Iterative Approximation of Fixed Points; Springer: Berlin, Germany, 2007.

25. Zlatanov, B. Error estimates for approximating of best proximity points for cyclic contractive maps. *Carpathian J. Math.* 2016, 32, 265–270.

26. Ilchev, A. On an application of coupled best proximity points theorems for solving systems of linear equations. In Proceedings of the AIP Conference Proceedings, Sozopol, Bulgaria, 8–13 June 2018; AIP Publishing: Melville, NY, USA, 2018; Volume 2048, p. 050003.

27. Ilchev, A. Error estimates for approximating best proximity points for Kannan cyclic contractive maps. In Proceedings of the AIP Conference Proceedings, Sozopol, Bulgaria, 8–13 June 2018; AIP Publishing: Melville, NY, USA, 2018; Volume 2048, p. 050002.

28. Ilchev, A.; Zlatanov, B. Error estimates for approximation of coupled best proximity points for cyclic contractive maps. *Appl. Math. Comput.* 2016, 290, 412–425. [CrossRef]

29. Ilchev, A.; Zlatanov, B. Error estimates of best proximity points for Reich maps in uniformly convex Banach spaces. *Tom XIX C* 2018, 3–20.

30. Zlatanov, B. Coupled best proximity points for cyclic contractive maps and their applications. *Fixed Point Theory* (to appear).

31. Clarkson, J.A. Uniformly convex spaces. *Trans. Amer. Math. Soc.* 1936, 40, 396–414. [CrossRef]

32. Fabian, M.; Habala, P.; Hájek, P.; Montesinos, V.; Pelant, J.; Zizler, V. *Functional Analysis and Infinite–Dimensional Geometry*; Springer: New York, NY, USA, 2001.

33. Deville, R.; Godefroy, G.; Zizler, V. *Smoothness and Renormings in Banach Spaces*; Longman Scientific Technical: Harlow, UK; copublished in the United States with John Wiley Sons, Inc.: New York, NY, USA, 1993.

34. Beauzamy, B. *Introduction to Banach Spaces and their Geometry*; North-Holland Publishing Company: Amsterdam, The Netherlands, 1979.

35. Malkowski, E.; Rakočević, V. *Advanced Functional Analysis*; CRS Press, Taylor and Francis Group: Boca Raton, FL, USA, 2019.

36. Al-Thagafi, M.A.; Shahzad, N. Convergence and existence results for best proximity points. *Nonlinear Anal. Theory Methods Appl.* 2009, 70, 3665–3671. [CrossRef]

37. Gabeleh, M.; Vetro, P. A note on best proximity point theory using proximal contractions. *J. Fixed Point Theory Appl.* 2018, 20, 149. [CrossRef]

38. Meir, A. On the uniform convexity of $L_p$ spaces, $1 < p < 2$. *Illinois J. Math.* 1964, 28, 420–424.

39. Garkavi, A.L. The best possible net and the best possible cross-section of a set in a normed space. *Izv. Akad. Nauk SSSR Ser. Mat.* 1962, 26, 87–106; *Am. Math. Soc. Trans. Ser.* 2 1964, 39, 111–132.

40. Zlatanov, B. Best proximity points in modular function spaces. *Arab. J. Math.* 2015, 4, 215–227. [CrossRef]

41. Ilchev, A.; Zlatanov, B. Fixed and Best Proximity Points for Kannan Cyclic Contractions in Modular Function Spaces. *J. Fixed Point Theory Appl.* 2017, 19, 2873–2893. [CrossRef]

42. Petric, M. Fixed Points and Best Proximity Points Theorems for Cyclic Contractive Operators. Ph.D. Thesis, North University of Baia Mare, Baia Mare, Romania, 2011.

© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).