THE RATIO OF TWO GENERAL CONTINUOUS-STATE BRANCHING PROCESSES WITH IMMIGRATION, AND ITS RELATION TO COALESCENT THEORY

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ABSTRACT. We study the ratio of two different continuous-state branching processes with immigration whose total mass is forced to be constant at a dense set of times. These lead to the definition of the $\Lambda$-asymmetric frequency process ($\Lambda$-AFP) as a solution of an SDE. We prove that this SDE has a unique strong solution which is a Feller process. We also calculate a large population limit when the mass tends to infinity and study the fluctuations of the process around its deterministic limit. Furthermore, we find conditions for the $\Lambda$-AFP to have a moment dual. The dual can be interpreted in terms of selection, (coordinated) mutation, pairwise branching (efficiency), coalescence, and a novel component that comes from the asymmetry between the reproduction mechanisms. A pair of equally distributed continuous-state branching processes has an associated $\Lambda$-AFP whose dual is a $\Lambda$-coalescent. The map that sends each continuous-state branching process to its associated $\Lambda$-coalescent (according to the former procedure) is a homeomorphism between metric spaces.

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1. INTRODUCTION

Heuristically, there is a strong relation between continuous-state branching processes (CB processes) and coalescents. However, providing this relation explicitly is a problem that has been subject to a notable history. In this paper, we construct an explicit homeomorphism between the two families of processes. This map is constructed by first defining a notion of frequency process associated with each CB process. The frequency process happens to have as moment dual the block counting process of a $\Lambda$-coalescent, which can be interpreted as the genealogy of the CB process. The homeomorphism sends each CB to its genealogy.

Our motivation is to study the changes of the genetic profile of a population consisting of two types of individuals that reproduce using radically different mechanisms, each of them modeled by a continuous-state branching processes with immigration (CBI) denoted by $X^{(1)}$ and $X^{(2)}$ respectively, which are assumed to be independent. For example, one type can reproduce by seldom big reproduction events while the other reproduces more often but only a few new offsprings are produced at each reproduction event. This study will give rise to a frequency process, which will, for obvious reasons, be asymmetric.

To this end we will characterize the evolution of the total size of the population as well as the frequency process associated to one of the types. Consider the process $Z = \{Z_t : t \geq 0\}$, describing the total mass of the population, defined by

$$Z_t = (X_t^{(1)} + X_t^{(2)}), \quad t \geq 0, \quad Z_0 = z,$$

(1.1)
where \( z := x^{(1)} + x^{(2)} := X^{(1)}_0 + X^{(2)}_0 \). In addition, we consider the frequency process of type 1 individuals, \( R = \{ R_t : t \geq 0 \} \), given by

\[
R_t = \frac{X^{(1)}_t}{X^{(1)}_t + X^{(2)}_t}, \quad t \geq 0, \quad R_0 = r,
\]

with \( r = x^{(1)}/(x^{(1)} + x^{(2)}) \).

It is important to note that the process \((R, Z)\) has the Markov property and we will show that it can be characterized as the solution to a martingale problem. However, the process \( R \) is not Markovian by itself, so it is not an autonomous frequency process. The main difficulty in obtaining a notion of a frequency processes is that to study only the frequency process \( R \) while preserving a Markovian structure, in general, the total mass should be constant in time (the case of \( \alpha \)-stable CB processes is an interesting special case). This restriction, which is a classic assumption in population genetics, seemed difficult to impose in the present case without losing properties of the CB processes \( X^{(1)} \) and \( X^{(2)} \). For example, naively conditioning the processes so that the total mass stays close to some value would inhibit big jumps. To overcome this difficulty, we will consider the dynamics of the frequency process \( R \) but at certain points in time we return the process \( Z \) to its original value \( z \). By taking the lengths of the intervals between these *culling times* tend to zero and speeding up time we derive as a scaling limit an autonomous Markov process \( R^{(z,r)} = \{ R^{(z,r)}_t : t \geq 0 \} \), that we call the \( \Lambda \)-asymmetric frequency process (AFP).

In many cases, our \( \Lambda \) asymmetric frequency processes have a moment dual which provides a notion of generalized ancestry in the spirit of the celebrated ancestral selection graph [31]. When there is no immigration and \( X^{(1)} \) and \( X^{(2)} \) have the same branching mechanisms, a relation between CB processes and \( \Lambda \)-coalescents is uncovered providing a homeomorphism between the metric spaces defined by these two classes of processes.

In the last decades, the relation between CB processes and coalescents has been a subject to great interest, and several important contributions in this area have been published. We will briefly describe those that inspired this paper. First, Etheridge and March in [11] and Perkins in [36] realized that the Fleming-Viot superprocess [14], which is the dual of the Kingman coalescent [28], can be obtained as a functional of two independent Dawson-Watanabe superprocesses (see [9] for an introduction to this class of processes). A similar result was found by Bertoin and Le Gall, who observed that the Bolthausen Szmitman coalescent describes the genealogy of Neveu’s CB process [3]. Finally, in a celebrated seven authors paper [5], the method of considering the ratio of two independent identically distributed \( \alpha \)-stable CB processes and time-changing it by using a functional of their total mass, reached its highest point.

Based on the ideas of Bertoin and Le Gall [3], Berestycki, Berestycki, and Limic [2] constructed a look-down coupling between \( \Lambda \)-coalescents and CB processes with the characteristic triplet \((0,0,y^{-2}\Lambda(y))\), where \( \Lambda \) is a finite measure in \([0,1]\) that characterizes the \( \Lambda \)-coalescent. Their construction works for small times and clarifies the relation between extinction of a CB process and coming down from infinity of a \( \Lambda \)-coalescent.

Recently Johnston and Lambert [24] studied the genealogy of general CB processes and discovered that, although the genealogy is not a Markovian object in general, it can be coupled to \( \Lambda \)-coalescents at small times. Their idea is to map each \( \Lambda \)-coalescent to the CB process with the triplet \((0,z^{-1}\Lambda(\{0\}),y^{-2}(T^{(z)})^{-1}(\Lambda-\Lambda(\{0\})\delta_0))\) (for a precise definition of the triplet characterizing a CB process see Section 2.1), where \( T^{(z)} : \mathcal{M}(0,\infty) \mapsto \mathcal{M}[0,1] \) (\( \mathcal{M}[0,\infty) \) and \( \mathcal{M}[0,1] \) denoting the space of measures in \([0,\infty)\) and \([0,1]\) respectively) be such that for every
measurable set $A \subset [0, 1]$ and $\nu \in \mathcal{M}[0, \infty)$, $T^{(\nu)}(\nu)(A) = \nu(T^{-1}_z(A))$ with $T_z : [0, \infty) \mapsto [0, 1]$ given by $T_z(w) = w/(w + z)$. The transformation $T_z$ is very useful and plays a central role in the present paper as well. For example, we will map each $\Lambda$ coalescent to $(0, z^{-1}\Lambda\{0\}, zy^{-2}(T^{(\nu)})^{-1}(\Lambda - \Lambda\{0\}\delta_0))$ and prove that this mapping is a homeomorphism between the metric spaces of $\Lambda$-coalescents and a particular subset of the class of CB processes. Note that our map differs slightly from the one obtained in [24]. This is because we use the map that arises naturally from the duality properties of the $\Lambda$-asymmetric frequency process introduced in the present paper.

In two innovative, although not very well known papers Gillespie [16, 17], introduces a stochastic differential equation (SDE) to study the probability of fixation of an allele in a scenario where two types compete. He assumes that both types reproduce for some time according to a Feller process with possibly different parameters, and then some external force, such as winter, discards individuals randomly to maintain the population size constant at each sampling time. His ideas reinforced our methods and as far as we are aware, Gillespie found the first known relation between Feller processes and frequency processes.

We summarise the main results and give an outline of the paper:

(1) The construction and the study of $\Lambda$-asymmetric frequency processes. This is the core of the paper and it is done in Sections 3 and 4. We first prove that the two-dimensional process $(R, Z)$ satisfies a martingale problem (Section 3). In Section 4 we introduce the $\Lambda$-asymmetric frequency process $R^{(z, r)}$ as the solution of an SDE; we prove that there exists a unique strong solution to this SDE and show that the solution is a Feller process. The culling procedure, discussed above, is a very important ingredient of the construction of the $\Lambda$-asymmetric frequency process and it is defined in Section 4.2, where we also prove that the scaling limit of the sequence of processes obtained by the culling procedure corresponds to the process $R^{(z, r)}$.

(2) In Section 5, we derive a large population limit for the $\Lambda$-asymmetric frequency process by making the total mass go to infinity. The limit is deterministic and consists of a logistic equation that provides a natural notion of the Malthusian in the context of competing populations with different branching mechanisms. Furthermore, we quantify the error of the deterministic approximation through a fluctuation result. We obtain explicitly the Gaussian process that characterizes the fluctuations of the process $R^{(z, r)}$ around the limiting logistic equation. This result besides being of mathematical interest can trigger research in the direction of statistical tests.

(3) In section 6 we study moment duality for $\Lambda$-asymmetric frequency processes. In particular Theorem 6.1 gives conditions for the $\Lambda$-asymmetric frequency process $R^{(z, r)}$ to have a moment dual. This gives a notion of generalized ancestry in presence of possibly skewed and asymmetric reproduction mechanisms, population dependent variance, mutation, coordinated mutation, and selection. It is important to note that as a particular case, in the absence of immigration and when the two independent CB processes have the same branching mechanism the dual becomes a $\Lambda$-coalescent.

(4) Section 7 is dedicated to the case of two equally distributed independent CB processes. The former procedure leads to a homeomorphism between the space of $\Lambda$-coalescents equipped with the Skorohod $J_1$-topology and a subspace of the CB processes equipped with the uniform Skorohod topology. This is the content of Theorem 7.1. The subspace of the CB processes homeomorphic to the $\Lambda$-coalescents can be
thought as the quotient space obtained by using the equivalence relation in which two CB processes are related if and only if they have the same diffusion term $c$ and the same Lévy measure $\nu$.

(5) To illustrate the main ideas of the paper, we give the simple example in Section 8 of the $\Lambda$-asymmetric Eldon-Wakely coalescent. With this example, we also show that the map sending a pair of CB processes to their dual is not continuous if one considers CB processes with different branching mechanisms.

(6) In Section 9 we provide some biological remarks related to the evolutionary forces that emerge from the asymmetry between the reproduction and immigration mechanisms of two competing CBI processes. These can be observed from the generator of the moment dual of the $\Lambda$-asymmetric frequency process.

Finally, it is important to note that our results bring back to the discussion an important biological observation made by Gillespie (see [16, 17]), which is that the variance of the reproduction mechanisms of competing phenotypes is also under natural selection. He introduced an asymmetric version of the Wright-Fisher diffusion that takes into account the difference in the variance in populations that grow by the action of frequent and small reproduction events. We extend this result to include seldom and big reproduction events and construct a reasonable model to study a vast spectrum of questions arising in biology and ecology.

2. Notations and Prerequisites

In order to introduce our main results we first need to recall a few facts about CBI’s and coalescent processes

2.1. Continuous-state branching processes with immigration. Continuous-state branching process with immigration $X := \{X_t : t \geq 0\}$ are $[0, \infty]$-valued strong Markov processes which are the continuous-time and state versions of Galton-Watson processes with immigration and were introduced by Kawazu and Watanabe in [27], where they show that they are described in terms of a branching mechanism $\psi$ of the form

$$\psi(\lambda) = b\lambda + c\lambda^2 + \int_{(0, \infty)} (e^{-\lambda x} - 1 + \lambda x 1_{(0,1)}(x))m(dx), \quad \lambda \geq 0,$$

where $b \in \mathbb{R}$, $c \geq 0$, and $m$ is a measure concentrated on $(0, \infty)$ which satisfies that $\int_{(0, \infty)} (1 \wedge x^2)m(dx) < \infty$, and a general immigration mechanism given by

$$\varphi(\lambda) = \eta \lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(dx), \quad \lambda \geq 0,$$

where $\eta \geq 0$ and $\nu$ is a measure on $(0, \infty)$ such that $\int_{(0, \infty)} (1 \wedge x)\nu(dx) < \infty$. More precisely, its semigroup is characterized by its Laplace transform as follows

$$\log \mathbb{E}_x [e^{-\lambda X_t}] = -x u_t(\lambda) - \int_0^t \varphi(u_{t-s}(\lambda))ds, \quad t, x, \lambda \geq 0,$$

where $u_t(\lambda)$ is the unique solution to the following evolution equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))ds = \lambda, \quad t, \lambda \geq 0,$$

with $u_0(\lambda) = \lambda$.

We recall that a branching mechanism is explosive if

$$\int_{0+} \frac{1}{|\psi(\xi)|}d\xi = \infty.$$
The probability that the process $X$ explodes in finite time is positive if and only the branching mechanism is explosive (see Caballero et al. [8] and Grey [22] for the case of no immigration).

In the case when there is no immigration (i.e. $\varphi \equiv 0$), the triplet $(b, c, m)$ completely characterizes the CB process $X$ and thus it will be referred to as the characteristic triplet of $X$.

### 2.2. Coalescents and frequency processes.

Given a finite measure $\Lambda$ on $[0, 1]$ the block counting process of a $\Lambda$-coalescent, $N = \{N_t : t \geq 0\}$, is an $\mathbb{N}$-valued decreasing process that goes from the state $n$ to the state $n-i+1$, for any $i \in \{2, ..., n\}$ at rate $\binom{n}{i} \lambda_{n,i}$, where

$$
\lambda_{n,i} := \int_0^1 y^i(1-y)^{n-i} \frac{\Lambda(dy)}{y^2}.
$$

These processes have a biological interpretation, they are related to the genealogy of a population (in a generalized Wright-Fisher model) and they are moment duals to frequency processes which are the solutions to the following class of SDE’s:

$$
R_t = \Lambda(\{0\}) \int_0^t \sqrt{R_s(1-R_s)} dB_s + \int_0^t \int_0^1 \int_0^1 y(1_{\{\theta < R_s\}} - R_s) N(ds,dy,d\theta), \quad t \geq 0,
$$

where $B = \{B_t : t \geq 0\}$ is a Brownian motion and $N(ds,dy,d\theta)$ is a Poisson random measure with state-space $\mathbb{R}^+ \times [0,1] \times [0,1]$ and intensity measure $dt \times \Lambda(dy) y^{-2} \times d\theta$. They are called frequency processes because they arise as scaling limits of the frequency of individuals of a certain type in a generalized Wright-Fisher model. In light of these facts, moment duality relates the genealogy of a population with the evolution of its genetic profile. To say that $R$ and $N$ are moment duals is equivalent to saying that for all $x \in [0, 1]$, $n \in \mathbb{N}$ and $t > 0$

$$
\mathbb{E}_x[R^n_t] = \mathbb{E}_n[x^{N_t}].
$$

Frequency processes are moment duals of block counting processes of coalescent processes. The simplest example is the duality between the Wright-Fisher diffusion and the Kingman coalescent [28], which was extended in many directions, for example, to include selection [31]. Indeed, many evolutionary forces rule the fate of populations and the shape of their genealogies, these can be included in a generalized Wright-Fisher model and lead to generalizations of coalescents and frequency processes. In many cases the duality property holds. We refer to [1] for further insight in coalescent theory, and to [23] for an introduction to moment duality.

### 3. FREQUENCY AND TOTAL SIZE OF A TWO POPULATION PROCESSES.

In this section we will consider two independent CBI’s as a model for two subpopulations of different type. We are interested in describing the total size of the population as well as the associated frequency process.

To this end, consider two independent CBI’s, $X^{(i)} = \{X^{(i)}_t : t \geq 0\}$ $i = 1, 2$ with branching mechanisms given by

$$
\psi^i(\lambda) = b^i \theta + c^i \lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x 1_{(0,1)}(x)) m^i(dx), \quad \lambda \geq 0, i = 1, 2,
$$

and immigration mechanisms

$$
\varphi^i(\lambda) = \eta^i \lambda + \int_0^\infty (e^{-\lambda x} - 1) \nu^i(dx), \quad \lambda \geq 0, i = 1, 2.
$$
For each $i = 1, 2$, let us consider a standard Brownian motion $B^{(i)} := \{B_t^{(i)} : t \geq 0\}$, a Poisson random measure $N^{(i)}(ds,dz,du)$ on $(0,\infty)^3$ with intensity measure $dsdm^i(dz)du$ and an independent subordinator $\xi^{(i)} = \{\xi^i_t : t \geq 0\}$ with Laplace exponent given by $\varphi^{(i)}$ as in (3.2). All these elements are assumed to be defined in the same complete probability space and are independent of each other. It is a well known fact (see for instance Section 9.5 in [35] or Proposition 4 in [7] for the case with no immigration) that for each $i = 1, 2$, the process $X^{(i)}$ can be seen as the solution to the following stochastic differential equation

$$X_t^{(i)} = x^{(i)} + \int_0^t \sqrt{2aX_s^{(i)}} dB_s^{(i)} - b^{(i)} \int_0^t X_s^{(i)} ds + \int_0^t \int_{[1,\infty]} \int_0^\infty \xi_t^{(i)} z N^{(i)}(ds,dz,du) \right)_{s \leq t} ds,dz,du$$

(3.3)

$$+ \int_0^t \int_{(0,1)} \int_0^\infty \xi_t^{(i)} (ds,dz,du) + \xi_t^{(i)}, \quad t \geq 0,$$

where $\tilde{N}^{(i)}(ds,dz,du) = N^{(i)}(ds,dz,du) - dsdm^i(dz)du$ denotes the compensated associated random measure.

Let us consider the process $(R,Z)$ given by (1.1) and (1.2). It is important to note that $(R,Z)$ is a Markov process, and as shown in the next result, it can be characterized as the solution to a martingale problem. We denote the law of the process $(R,Z)$ starting from the initial position $(r,z) \in [0,1] \times (0,\infty)$ by $\mathbb{P}_{(r,z)}$. Accordingly, we write $\mathbb{E}_{(r,z)}$ for the associated expectation operator.

For $\varepsilon \in (0,z)$ and $L > z$, we denote by $\tau^\varepsilon := \inf\{t \geq 0 : Z_t = \varepsilon\}$ and $\tau^+_L = \inf\{t \geq 0 : Z_t > L\}$, the first hitting time of $\varepsilon$ and the first passage time above the level $L > z$ for the process $Z$, respectively. The two-dimensional process $(R,Z)$ stopped at the time $\tau = \tau^\varepsilon \wedge \tau^+_L$, completely encodes the dynamics between the two subpopulations originally described by the processes $X^{(1)}$ and $X^{(2)}$ before the size of the population becomes relatively small or explodes.

The characterization of the dynamics of the process $(R,Z)$ until the stopping time $\tau$, as a solution of a martingale problem is provided in the next result.

**Proposition 3.1.** For any $f \in C^2([0,1] \times [0,\infty))$ the process

$$M_t := f(R_{t\wedge \tau}, Z_{t\wedge \tau}) - f(r,z) - \int_0^{t\wedge \tau} \mathcal{L}_f(R_s, Z_s) ds,$$

is a local martingale. Where

$$\mathcal{L}_f(r,z) = -b^1 \partial_1 f(r,z) + c^1 (\partial_2 f(r,z) \cdot r(1-r) + \partial_{22} f(r,z) r z)$$

$$- b^1 r z \partial_2 f(r,z) + c^1 \partial_{12} f(r,z) (1-r) r + \frac{c^1}{z} (\partial_{11} f(r,z) \cdot r(1-r)^2 - \partial_1 f(r,z) \cdot 2r(1-r))$$

$$+ r z \int_{(0,\infty)} \left[ f \left( r \left( 1 - \frac{w}{z+w} \right), \frac{w}{z+w}, z+w \right) - f(r,z) - w1_{(0,1)}(w) \left( \partial_1 f(r,z) \cdot \frac{1-r}{z} + \partial_2 f(r,z) \right) \right] m^1(dw)$$

$$+ \eta^1 \partial_1 f(r,z) \left( \frac{1-r}{z} \right) + \eta^1 \partial_2 f(r,z) + \int_{(0,\infty)} \left[ f \left( r \left( 1 - \frac{w}{z+w} \right), \frac{w}{z+w}, z+w \right) - f(r,z) \right] \nu^1(dw)$$

$$+ b^2 \partial_1 f(r,z) \cdot r(1-r) - b^2 (1-r) z \partial_2 f(r,z) - c^2 \partial_{12} f(r,z) r(1-r)$$

$$+ \frac{c^2}{z} (\partial_{11} f(r,z) \cdot r^2(1-r) + \partial_1 f(r,z) \cdot 2r(1-r) + c^2 (\partial_{21} f(r,z) \cdot r(1-r) + (1-r) z \partial_{22} f(r,z)))$$
\[ + (1 - r) z \int_{(0, \infty)} \left[ f \left( r \left( 1 - \frac{w}{z + w} \right), z + w \right) - f \left( r, z \right) - w \eta_1(x_1) \left( -\partial_1 f \left( r, z \right) \frac{r}{z} + \partial_2 f \left( r, z \right) \right) \right] m^2(dw) \\
- \eta^2 \partial_1 f \left( r, z \right) \frac{r}{z} + \eta^2 \partial_2 f \left( r, z \right) + \int_{(0, \infty)} \left[ f \left( r \left( 1 - \frac{w}{z + w} \right), z + w \right) - f \left( r, z \right) \right] \nu^2(dw). \]

**Proof.** Since \(X^{(1)}\) and \(X^{(2)}\) are semi-martingales and \(f\) is sufficiently smooth on \([0, 1] \times [0, \infty)\), we can use the change of variables/Meyer-Itô's formula (cf. Theorems II.31 and II.32 of [38]) to deduce that

\[
f \left( \frac{X^{(1)}_{t \wedge \tau}}{X^{(1)}_{t \wedge \tau} + X^{(2)}_{t \wedge \tau}} \right) = \frac{f \left( x^{(1)}_{t \wedge \tau} + x^{(2)}_{t \wedge \tau} \right)}{f \left( x^{(1)} \right) + f \left( x^{(2)} \right)} + \int_0^{t \wedge \tau} \left[ A^{(1)}(X^{(1)}_s, X^{(2)}_s) + A^{(2)}(X^{(1)}_s, X^{(2)}_s) + A^{(3)}(X^{(1)}_s, X^{(2)}_s) + A^{(4)}(X^{(1)}_s, X^{(2)}_s) \right] ds + M_{t \wedge \tau},
\]

where

\[
A^{(1)}(x, y) := -b^2 x \partial_1 f \left( \frac{x}{x + y}, x + y \right) \frac{y}{(x + y)^2} - b^2 y \partial_2 f \left( \frac{x}{x + y}, x + y \right) + c^1 x \partial_1 \partial_2 f \left( \frac{x}{x + y}, x + y \right) \frac{y}{(x + y)^2} \\
+ c^1 x \left( \partial_1^2 f \left( \frac{x}{x + y}, x + y \right) \frac{y^2}{(x + y)^2} - \partial_1 f \left( \frac{x}{x + y}, x + y \right) \frac{2y}{(x + y)^3} \right) \\
+ c^1 x \left( \partial_2 f \left( \frac{x}{x + y}, x + y \right) \frac{y}{(x + y)^2} + \partial_2 \partial_2 f \left( \frac{x}{x + y}, x + y \right) \right),
\]

\[
A^{(2)}(x, y) := x \int_{(0, \infty)} \left[ f \left( \frac{x + u}{x + u + y}, x + u + y \right) - f \left( \frac{x}{x + y}, x + y \right) \\
- u \eta_1(x_1) \left( \partial_1 f \left( \frac{x}{x + y}, x + y \right) \frac{y}{(x + y)^2} + \partial_2 f \left( \frac{x}{x + y}, x + y \right) \right) \right] m^1(du) \\
+ \eta^2 \partial_1 \partial_2 f \left( \frac{x}{x + y}, x + y \right) \frac{y}{(x + y)^2} + \eta^1 \partial_2 \partial_2 f \left( \frac{x}{x + y}, x + y \right) \\
+ \int_{(0, \infty)} \left[ f \left( \frac{x + u}{x + u + y}, x + u + y \right) - f \left( \frac{x}{x + y}, x + y \right) \right] \nu^1(du).
\]

Additionally

\[
A^{(3)}(x, y) = b^2 y \partial_1 f \left( \frac{x}{x + y}, x + y \right) \frac{x}{(x + y)^2} - b^2 y \partial_2 f \left( \frac{x}{x + y}, x + y \right) - c^2 y \partial_1 \partial_2 f \left( \frac{x}{x + y}, x + y \right) \frac{x}{(x + y)^2} \\
+ c^2 y \left( \partial_1^2 f \left( \frac{x}{x + y}, x + y \right) \frac{x^2}{(x + y)^4} + \partial_1 f \left( \frac{x}{x + y}, x + y \right) \frac{2x}{(x + y)^4} \right) \\
+ c^2 y \left( -\partial_2 f \left( \frac{x}{x + y}, x + y \right) \frac{x}{(x + y)^2} + \partial_2 \partial_2 f \left( \frac{x}{x + y}, x + y \right) \right),
\]
Using (1.1) together with (1.2) we can write (3.4) in terms of the process \((R, Z)\) as follows

\[
A^{(4)}(x, y) = y \int_{(0, \infty)} \left[ f \left( \frac{x}{x + u + y}, x + u + y \right) - f \left( \frac{x}{x + y}, x + y \right) - \eta^2 \partial_1 f \left( \frac{x}{x + y}, x + y \right) \frac{x}{(x + y)^2} + \eta^2 \partial_2 f \left( \frac{x}{x + y}, x + y \right) \right] m^2(du)
- \eta^2 \partial_1 f \left( \frac{x}{x + y}, x + y \right) \frac{x}{(x + y)^2} + \eta^2 \partial_2 f \left( \frac{x}{x + y}, x + y \right)
+ \int_{(0, \infty)} \left[ f \left( \frac{x}{x + u + y}, x + u + y \right) - f \left( \frac{x}{x + y}, x + y \right) \right] \nu^2(du).
\]

and \(M = \{M_t : t \geq 0\}\) is a local martingale.

Using (1.1) together with (1.2) we can write (3.4) in terms of the process \((R, Z)\) as follows

\[
f(R_{t \wedge \tau}, Z_{t \wedge \tau}) = f(r, z) + \int_0^{t \wedge \tau} \left[ B^{(1)}(R_s, Z_s) + B^{(2)}(R_s, Z_s) + B^{(3)}(R_s, Z_s) + B^{(4)}(R_s, Z_s) \right] ds + M_{t \wedge \tau},
\]

where

\[
B^{(1)}(r, z) := -b^1 \partial_1 f (r, z) r(1 - r) - b^1 r z \partial_2 f (r, z) + c^1 (\partial_2 f (r, z) r(1 - r) + \partial_2 f (r, z) r z)
+ c^1 \partial_2 f (r, z) (1 - r)r + c^1 \left( \partial_{11} f (r, z) r(1 - r)^2 - \partial_1 f (r, z) 2r(1 - r) \right),
\]

\[
B^{(2)}(r, z) := b^2 \partial_1 f (r, z) r(1 - r) - b^2 (1 - r) z \partial_2 f (r, z) - c^2 \partial_1 f (r, z) r(1 - r)
+ \frac{c^2}{z} \left( \partial_{11} f (r, z) r^2(1 - r) + \partial_1 f (r, z) 2r(1 - r) \right) + c^2 (\partial_2 f (r, z) r(1 - r) + (1 - r) z \partial_2 f (r, z)).
\]

In addition,

\[
B^{(3)}(x, y) := r z \int_{(0, \infty)} \left[ f \left( r \left( 1 - \frac{u}{z + u} \right) + \frac{u}{z + u}, z + u \right) \right] m^1(du) + \eta^1 \partial_1 f (r, z) \left( \frac{1 - r}{z} \right) + \eta^1 \partial_2 f (r, z)
+ \int_{(0, \infty)} \left[ f \left( r \left( 1 - \frac{u}{z + u} \right) + \frac{u}{z + u}, z + u \right) \right] \nu^1(du),
\]

\[
B^{(4)}(x, y) := (1 - r) z \int_{(0, \infty)} \left[ f \left( r \left( 1 - \frac{u}{z + u} \right) + \frac{u}{z + u}, z + u \right) \right] m^2(du)
- \eta^2 \partial_1 f (r, z) \frac{r}{z} + \eta^2 \partial_2 f (r, z) + \int_{(0, \infty)} \left[ f \left( r \left( 1 - \frac{u}{z + u} \right) + \frac{u}{z + u}, z + u \right) \right] \nu^2(du).
\]

Hence, noting that for \((r, z) \in [0, 1] \times [0, \infty)\)

\[
\mathcal{L} f(r, z) = B^{(1)}(r, z) + B^{(2)}(r, z) + B^{(3)}(r, z) + B^{(4)}(r, z),
\]
we obtain the result. □

4. Λ-ASYMMETRIC FREQUENCY PROCESSES AND CULLING OF POPULATION PROCESSES.

In Section 3 we characterized the dynamics of the two-dimensional process \((R, Z)\) as the solution of a martingale problem. The process \((R, Z)\) describes the frequency of one of the types and the total size of the population, respectively. Inspired by many models in population genetics we would like to have a description of the frequency process under the additional assumption that the total size of the population remains constant in time. To this end we will apply a sampling/culling procedure to the process \((R, Z)\) and obtain a new class of frequency process that we call Λ-asymmetric frequency processes.

We begin this section with a precise definition of Λ-asymmetric frequency processes.

4.1. Λ-asymmetric frequency processes. For \(z > 0\) (representing the population size) and \(r \in [0, 1]\), let us consider the process \(R^{(z,r)} = \{R_t^{(z,r)} : t \geq 0\}\) given as the solution to the following stochastic differential equation

\[
\begin{align*}
    dR_t^{(z,r)} &= R_t^{(z,r)}(1 - R_t^{(z,r)})1_{\{R_t^{(z,r)} \in [0,1]\}}(b^2 - b^1)dt \\
    &+ R_t^{(z,r)}(1 - R_t^{(z,r)})1_{\{R_t^{(z,r)} \in [0,1]\}} \left( \frac{2}{z}(c^2 - c^1) + \int_{(0,1)} \frac{w^2}{w + z} m^2(dw) - \int_{(0,1)} \frac{w^2}{w + z} m^1(dw) \right)dt \\
    &+ \eta^1 \bigg( \frac{1 - R_t^{(z,r)}}{z} \bigg) dt - \eta^2 \frac{R_t^{(z,r)}}{z} dt + \sqrt{\frac{1}{z} R_t^{(z,r)}(1 - R_t^{(z,r)})[c^1(1 - R_t^{(z,r)}) + c^2 R_t^{(z,r)}]} 1_{\{R_t^{(z,r)} \in [0,1]\}} dB_t \\
    &+ \int_{(0,1) \times (0,\infty)} g(z) \bigg( R_t^{(z,r)}, w, v \bigg) \tilde{N}_1(dt, dw, dv) + \int_{(0,1) \times (0,\infty)} h(z) \bigg( R_t^{(z,r)}, w, v \bigg) \tilde{N}_2(dt, dw, dv) \\
    &+ \int_{(1,\infty) \times (0,\infty)} g(z) \bigg( R_t^{(z,r)}, w, v \bigg) N_1(dt, dw, dv) + \int_{(1,\infty) \times (0,\infty)} h(z) \bigg( R_t^{(z,r)}, w, v \bigg) N_2(dt, dw, dv) \\
    &+ \int_{(0,\infty)} \tilde{g}(z) \bigg( R_t^{(z,r)}, w \bigg) N_3(dt, dw) + \int_{(0,\infty)} \tilde{h}(z) \bigg( R_t^{(z,r)}, w \bigg) N_4(dt, dw),
\end{align*}
\]

(4.1)

\[R_0^{(z,r)} = r.\]

where

(i) \(B = \{B_t : t \geq 0\}\) is a standard Brownian motion.

(ii) \(N_1(dt, dw, dv)\) is a Poisson random measure on \((0, \infty)^3\) with intensity measure \(dtm^1(dw)dv\).

(iii) \(N_2(dt, dw, dv)\) is a Poisson random measure on \((0, \infty)^3\) with intensity measure \(dtm^2(dw)dv\).

(iv) \(N_3(dt, dw)\) is a Poisson random measure on \((0, \infty)^2\) with intensity measure \(dtv^1(dw)\).

(v) \(N_4(dt, dw)\) is a Poisson random measure on \((0, \infty)^2\) with intensity measure \(dtv^2(dw)\).

And for each \(i = 1, 2\), \(\tilde{N}^{(i)}(ds, dw, dv) = N^{(i)}(ds, dw, dv) - ds dsm^i(dw)dv\) denotes the compensated associated random measure.
All the previous elements are assumed to be defined in the same complete probability space and are independent of each other. Additionally we have that

(i) For \( x, w, v \in (0, \infty)^3 \)

\[
g^{(z)}(x, w, v) := \frac{w}{z + w}(1 - x)1_{\{v \leq x\}}1_{\{x \in [0,1]\}}.
\]

(ii) For \( x, w, v \in (0, \infty)^3 \)

\[
h^{(z)}(x, w, v) := -\frac{w}{z + w}x1_{\{v \leq (1-x)z\}}1_{\{x \in [0,1]\}}.
\]

(iii) For \( x, w \in (0, \infty)^2 \)

\[
g^{(z)}(x, w) := \frac{w}{z + w}(1 - x)1_{\{x \in [0,1]\}}.
\]

(iv) For \( x, w \in (0, \infty)^2 \)

\[
h^{(z)}(x, w) := -\frac{w}{z + w}x1_{\{x \in [0,1]\}}.
\]

As we will see later in this section, the process \( R^{(z, r)} \) will be obtained through a sampling/culling procedure of the process \((R, Z)\) to describe the frequency of one of the types in the population under the assumption that the total size of the population is constant and is equal to \( z > 0 \).

**Remark 4.1.** In the case when the process \( R^{(z, r)} \) has no jump terms and \( \eta^1 = \eta^2 = 0 \), the resulting diffusion process was obtained by Gillespie in [17] via a sampling procedure, as a continuous-time approximation of a finite gametic-pool selection model. As noted by Gillespie the form of the drift term in (4.1) points to a new form of natural selection acting on the variance.

The same case was also studied by Lambert in [32] where \( R^{(z, r)} \) was obtained by conditioning the process \( R \) on the event \( \{Z_t = z, \text{ for all } t \geq 0\} \), where the processes \( R \) and \( Z \) are defined in (1.2) and (1.1) respectively.

As the first step in our construction, we will show that the process \( R^{(z, r)} \) is well-defined, which is given in the next result.

**Proposition 4.1.** There exists a unique strong solution \( R^{(z, r)} \) to (4.1) such that \( R^{(z, r)}_t \in [0, 1] \) for all \( t \geq 0 \) \( \mathbb{P} \)-a.s. Furthermore for any \( t > 0 \), there exists a constant \( C(t) > 0 \) such that

\[
\mathbb{E} \left[ |R^{(z, r)}_t - R^{(z, \tau)}_t| \right] \leq C(t)|r - \tau|, \quad r, \tau \in [0, 1].
\]

**Proof.** Step 1.- First we will prove that any solution \( R^{(z, r)} \) to (4.1) satisfies that \( R^{(z, r)}_t \in [0, 1] \) for all \( t \geq 0 \) \( \mathbb{P} \)-a.s. To this end let us denote for \( x \geq 0 \)

\[
b(x) := (b^2 - b^1)x(1 - x)1_{\{x \in [0,1]\}} + \frac{2}{z}(c^2 - c^1)x(1 - x)1_{\{x \in [0,1]\}} + \eta^1\left(\frac{1 - x}{z} - \eta^2\frac{x}{z}\right)
\]

\[
+ x(1 - x)1_{\{x \in [0,1]\}} \int_{(0,\infty)} \frac{w^2}{z + w}m^1(dw) - x(1 - x)1_{\{x \in [0,1]\}} \int_{(0,\infty)} \frac{w^2}{z + w}m^1(dw),
\]

\[
\sigma(x) := \sqrt{\frac{1}{z}x(1 - x)(c^1(1 - x) + c^2x)1_{\{x \in [0,1]\}}}.
\]

We note that the following conditions are satisfied:

(i) \( \sigma^1(x) = \sigma^2(x) = 0 \) for all \( x \in \mathbb{R} \setminus [0, 1] \).
(ii) For $x > 1$,
\[ b(x) = \eta^1 \frac{(1 - x)}{z} - \eta^2 \frac{x}{z} \leq 0, \]
and for $x < 0$,
\[ b(x) = \eta^1 \frac{(1 - x)}{z} - \eta^2 \frac{x}{z} \geq 0. \]

(iii) For $(w, v) \in (0, \infty)^2$
\[ 0 \leq h^{(z)}(x, w, v) + x = x - \frac{w}{z+w}x1_{\{v \leq (1-x)z\}} \leq 1, \quad \text{for } x \in [0, 1], \]
and $h^{(z)}(x, w, v) = 0$ for $x \in \mathbb{R} \setminus [0, 1]$.

(iv) For $w \in (0, \infty)$ observe that
\[ 0 \leq \tilde{h}^{(z)}(x, w) + x = x - \frac{w}{z+w}x \leq 1, \quad \text{for } x \in [0, 1]. \]
and $\tilde{h}^{(z)}(x, w) = 0$ for $x \in \mathbb{R} \setminus [0, 1]$.

(v) For $(w, v) \in (0, \infty)^2$
\[ 0 \leq g^{(z)}(x, w, v) + x = x + \frac{w}{z+w}(1-x)1_{\{v \leq xz\}} \leq 1, \quad \text{for } x \in [0, 1], \]
and $g^{(z)}(x, w, v) = 0$ for $x \in \mathbb{R} \setminus [0, 1]$.

(vi) For $w \in \times (0, \infty)$ observe that
\[ 0 \leq \tilde{g}^{(z)}(x, w) + x = x + \frac{w}{z+w}(1-x) \leq 1, \quad \text{for } x \in [0, 1]. \]
and $\tilde{g}^{(z)}(x, w) = 0$ for $x \in \mathbb{R} \setminus [0, 1]$.

Then by a modification of Proposition 2.1 in [15] we have that $\mathbb{P}(R^{(z,r)}_t \in [0, 1]$ for all $t \geq 0) = 1$.

**Step 2.** In order to prove the existence of a strong solution to (4.1), we first obtain the following estimations:

(i) For $x, y \in [0, 1]$ we obtain
\[
|b(x) - b(y)| \leq \left(2(|b^2 - b^1| + \frac{4}{z}e^2 - c^1) + \frac{\eta^1}{z} + \frac{\eta^2}{z} \right) + 2 \int_{(0, \infty)} \frac{w}{z+w}(m^1 + m^2)(dw) |x - y|.
\]

Additionally, for $x, y \in [0, 1]$
\[
\int_{(0, \infty)} |\tilde{g}^{(z)}(x, w) - \tilde{g}^{(z)}(y, w)| \nu^1(dw) = \int_{(0, \infty)} \frac{w}{z+w} \nu^1(dw) |x - y|,
\]
and
\[
\int_{(0, \infty)} |\tilde{h}^{(z)}(x, w) - \tilde{h}^{(z)}(y, w)| \nu^2(dw) = \int_{(0, \infty)} \frac{w}{z+w} \nu^2(dw) |x - y|.
\]

Now we note that for $x, y \in [0, 1]$ and $(w, v) \in (0, \infty)^2$
\[
|g^{(z)}(x, w, v) - g^{(z)}(y, w, v)| \leq \frac{w}{z+w} \left(|x - y|1_{\{v \leq z(x \wedge y)\}} + (1-x)1_{\{y \wedge v \leq xz\}} + (1-y)1_{\{xz \leq v \leq yz\}}\right),
\]
\[
|h^{(z)}(x, w, v) - h^{(z)}(y, w, v)| \leq \frac{w}{z+w} \left(|x - y|1_{\{v \leq (1-x) \wedge (1-y)\}} + x1_{\{(1-y)z \leq v \leq (1-x)z\}}
\]
\[
\quad + (1-y)1_{\{(1-x)z \leq v \leq (1-y)z\}}\right).
\]

(4.3)
Therefore for \( x, y \in [0, 1] \)
\[
\int_{(0, \infty)} \int_{[1, \infty]} |g^{(z)}(x, w, v) - g^{(z)}(y, w, v)| \mu^1(dw) dv \leq 3z \int_{[1, \infty]} \frac{w}{z + w} \mu^1(dw) |x - y|,
\]
(4.6) \[
\int_{(0, \infty)} \int_{[1, \infty]} |h^{(z)}(x, w, v) - h^{(z)}(y, w, v)| \mu^2(dw) dv \leq 3z \int_{[1, \infty]} \frac{w}{z + w} \mu^2(dw) |x - y|.
\]
Hence using (4.3), (4.4), and (4.6) we have that for every \( x, y \in [0, 1] \) there exists \( K_1 > 0 \) such that
\[
|b(x) - b(y)| + \int_{(0, \infty)} |g^{(z)}(x, w, v) - g^{(z)}(y, w, v)| \mu^1(dw) + \int_{(0, \infty)} |\tilde{h}^{(z)}(x, w) - \tilde{h}^{(z)}(y, w)| \mu^2(dw)
\]
\[
+ \int_{(0, \infty)} \int_{[1, \infty]} |g^{(z)}(x, w, v) - g^{(z)}(y, w, v)| \mu^1(dw) dv
\]
\[
+ \int_{(0, \infty)} \int_{[1, \infty]} |h^{(z)}(x, w, v) - h^{(z)}(y, w, v)| \mu^2(dw) dv \leq K_1 |x - y|.
\]
(4.7)

(ii) On the other hand for \( x, y \in [0, 1] \)
\[
|\sigma(x) - \sigma(y)|^2 \leq \frac{3(c_1 + c_2)}{z} |x - y|.
\]

In addition by (4.5) we have for \( x, y \in [0, 1] \)
\[
\int_{(0, \infty)} \int_{(0, 1)} |g^{(z)}(x, w, v) - g^{(z)}(y, w, v)|^2 \mu^1(dw) dv \leq 9z \int_{(0, \infty)} \frac{w^2}{(z + w)^2} \mu^1(dw) |x - y|,
\]
(4.8)
\[
\int_{(0, \infty)} \int_{(0, 1)} |h^{(z)}(x, w, v) - h^{(z)}(y, w, v)|^2 \mu^2(dw) dv \leq 9z \int_{(0, \infty)} \frac{w^2}{(z + w)^2} \mu^2(dw) |x - y|.
\]
The previous identities imply that there exists \( K_2 > 0 \)
\[
|\sigma(x) - \sigma(y)|^2 + \int_{(0, \infty)} \int_{(0, 1)} |g^{(z)}(x, w, v) - g^{(z)}(y, w, v)|^2 \mu^1(dw) dv
\]
\[
+ \int_{(0, \infty)} \int_{(0, 1)} |h^{(z)}(x, w, v) - h^{(z)}(y, w, v)|^2 \mu^2(dw) dv \leq K_2 |x - y|.
\]
(4.9)

(iii) We note that for \( x \in [0, 1] \) the mapping
\[
x \mapsto x + g^{(z)}(x, u, v) = x + \frac{w}{z + w} (1 - x) \[v \leq x\] = \begin{cases} x & \text{if } v > xz, \\ x \left(1 - \frac{w}{z + w}\right) + \frac{w}{z + w} & \text{if } v \leq xz. \end{cases}
\]
is non-decreasing for \( u, v \in (0, 1) \times (0, \infty) \). In addition, the mapping
\[
x \mapsto x + h^{(z)}(x, u, v) = x - \frac{w}{z + w} x \[v \leq (1 - x)z\] = \begin{cases} x & \text{if } v > (1 - x)z, \\ x \left(1 - \frac{w}{z + w}\right) & \text{if } v \leq (1 - x)z, \end{cases}
\]
is non-decreasing \( w, v \in (0, \infty)^2 \) as well.
Hence by an application of Gronwall’s inequality, we obtain (4.2).

\[ \square \]

Hence proceeding like in (4.7) and (4.9) we can find a constant \( K > 0 \) such that

\[ |\sigma(x)|^2 + |b(x)|^2 + \int_{(0,\infty)} \int_{(0,1)} |g^<(z)(x,w,v)|^2 \mu^1(dw)dv + \int_{(0,\infty)} \int_{(0,1)} |h^<(z)(x,w,v)|^2 \mu^2(dw)dv 
+ \int_{(0,\infty)} |g^<=1(z)(x, w, v)|^2 \nu^1(dw) + \int_{(0,\infty)} |\bar{h}^<=1(z)(x, w, v)|^2 \nu^2(dw) 
+ \left[ \int_{(0,\infty)} \int_{[1,\infty)} |g^<=1(z)(x, w, v)| \mu^1(dw)dv + \int_{(0,\infty)} \int_{[1,\infty)} |\bar{h}^<=1(z)(x, w, v)| \mu^2(dw)dv \right]^2 \leq K(z). \]

Hence the inequalities (4.7), (4.9), and (4.10) together with the fact the mappings \( x \mapsto x + g^<(z)(x, w, v) \) and \( x \mapsto x + h^<(z)(x, w, v) \) are non-decreasing for \( (x, w, v) \in [0,1] \times (0,\infty)^2 \) imply, using a slight modification of Theorem 5.1 in [39], that there exists a unique strong solution to (4.1).

Step 3.- In order to prove the last assertion in the statement, we obtain by the proof of Theorem 3.2 [39] together with identities (4.7), (4.9), and (4.10), and the fact that the mappings \( x \mapsto x + g^<(z)(x, u, v) \) and \( x \mapsto x + h^<(z)(x, w, v) \) are non-decreasing for \( (x, w, v) \in [0,1] \times (0,\infty)^2 \), that for \( r, \bar{r} \in [0,1] \)

\[ \mathbb{E} \left[ \left| R^{(r,z)}_t - R^{(\bar{r},z)}_t \right| \right] \leq |r - \bar{r}| + K \int_0^t \mathbb{E} \left[ \left| R^{(r,z)}_s - R^{(\bar{r},z)}_s \right| \right] ds, \ t \geq 0. \]

Hence by an application of Gronwall’s inequality, we obtain (4.2).

\[ \square \]

In the next result, we show that the process \( R^{(z,r)} \) is Feller and obtain its infinitesimal generator.
Proposition 4.2. For any \( z > 0 \), \( R^{(x,r)} \) is a Feller process and its infinitesimal generator is given for any \( f \in C^2([0,1]) \) by
\[
\mathcal{L}^{(z)} f(r) = f'(r) \left[ r(1-r)(b^2 - b^1) + \frac{2r(1-r)}{z}(c^2 - c^1) \right] + f''(r) \frac{r(1-r)}{z}(c^1(1-r) + c^2 r) \\
+ \frac{\eta^1}{z} f'(r)(1-r) + \int_{(0,1)} \left[ f(r(1-u) + u) - f(r) \right] \mathbf{T}^{(x)}(u^1)(du)
\]
\[
+ zr \int_{(0,1)} \left[ f(r(1-u) + u) - f(r) - \frac{u}{1-u} f'(r)(1-r)1_{(0,1)}(u) \right] \mathbf{T}^{(x)}(m^1)(du)
\]
\[
+ z(1-r) \int_{(0,1)} \left[ f(r(1-u)) - f(r) + \frac{u}{1-u} f'(r) r1_{(0,1)}(u) \right] \mathbf{T}^{(x)}(m^2)(du)
\]
(4.11)
\[
- \frac{\eta^2}{z} f'(r)r + \int_{(0,1)} \left[ f(r(1-u)) - f(r) \right] \mathbf{T}^{(x)}(u^2)(du).
\]

Proof. (i) Let us consider the semigroup \((\mathcal{T}_t)_{t \geq 0}\) of the process \( R^{(x,r)} \) given for any \( f \in C_0([0,1]) \) by \( \mathcal{T}_t f(r) = \mathbb{E} \left[ f(R^{(x,r)}_t) \right] \). For \( f \in C^1([0,1]) \) and \( r, \bar{r} \in [0,1] \) we obtain using (4.2)
\[
|\mathcal{T}_t f(r) - \mathcal{T}_t f(\bar{r})| \leq \mathbb{E} \left[ |f(R^{(x,r)}_t) - f(R^{(x,\bar{r})}_t)| \right] \leq ||f'||_{\infty} \mathbb{E} \left[ |R^{(x,r)}_t - R^{(x,\bar{r})}_t| \right] \leq C(t) ||f'||_{\infty}|r - \bar{r}|,
\]
which implies that the mapping \( r \mapsto \mathcal{T}_t f(r) \) is continuous. On the other hand, for any function \( g \in C_0([0,1]) \) we can find a sequence \( (f_n)_{n \geq 1} \subset C^1([0,1]) \) such that \( f_n \rightarrow g \) uniformly on \([0,1]\) as \( n \rightarrow \infty \). Therefore, \( \mathcal{T}_t f_n \rightarrow \mathcal{T}_t g \) uniformly on \([0,1]\) as \( n \rightarrow \infty \) as well. This implies that the mapping \( r \mapsto \mathcal{T}_t f g(r) \) is continuous, and therefore \( \mathcal{T}_t(C_0([0,1])) \subset C_0([0,1]) \).

(ii) Fix \( f \in C^2([0,1]) \). Using that \( R^{(x,r)} \) is a semi-martingale we can use the change of variables/Meyer-Itô’s formula (cf. Theorems II.31 and II.32 of [38]) to obtain for \( t \geq 0 \)
\[
(4.12) \quad f(R^{(x,r)}_{t\wedge T_n}) = f(r) + \int_{0}^{t\wedge T_n} \left[ C^{(1)}(R^{(x,r)}_s) + C^{(2)}(R^{(x,r)}_s) + C^{(3)}(R^{(x,r)}_s) \right] ds + M_{t\wedge T_n},
\]
where for \( x \in [0,1] \)
\[
C^{(1)}(x) = f'(x) \left[ x(1-x)(b^2 - b^1) + \frac{2x(1-x)}{z}(c^2 - c^1) + \frac{\eta^1}{z}(1-x) - \frac{\eta^2}{z} x \right]
+ f''(x) \left[ \frac{x(1-x)}{z} + c^1(1-x) + c^2 x \right],
\]
\[
C^{(2)}(x) = zx \int_{(0,\infty)} \left[ f\left(x + \frac{w}{w+z}(1-x)\right) - f(x) - w f'(x) \frac{1-x}{z} 1_{(0,1)}(w) \right] m^1(dw)
+ \int_{(0,\infty)} \left[ f\left(x + \frac{w}{w+z}(1-x)\right) - f(x) \right] \nu^1(dw)
= zx \int_{(0,1)} \left[ f(x(1-u) + u) - f(x) - \frac{u}{1-u} f'(x)(1-x)1_{(0,1)}(u) \right] \mathbf{T}^{(x)}(m^1)(du)
+ \int_{(0,1)} \left[ f(x(1-u) + u) - f(x) \right] \mathbf{T}^{(x)}(\nu^1)(du),
\]
and
\[ C^{(3)}(x) = z(1 - x) \int_{(0, \infty)} \left[ f \left( x - \frac{w}{w + z} \right) - f(x) + w f'(x) \frac{x}{z} 1_{(0,1)}(w) \right] m^2(dw) \]
\[ + \int_{(0, \infty)} \left[ f \left( x - \frac{w}{w + z} \right) - f(x) \right] \nu^2(dw) \]
\[ = z(1 - x) \int_{(0,1)} \left[ f(x(1 - u)) - f(x) + \frac{u}{1 - u} f'(x) x 1_{(0,1/1 + z)}(u) \right] T(x)(m^2)(du) \]
\[ + \int_{(0,1)} \left[ f(x(1 - u)) - f(x) \right] T(x)(\nu^2)(du), \]
and \( M = \{ M_t : t \geq 0 \} \) is a local martingale.

Now, the fact that \( f \in C^2([0, 1]) \) implies that we can find a constant \( K > 0 \) such that
\[ |C^{(1)}(x) + C^{(2)}(x) + C^{(3)}(x)| \leq K, \quad x \in [0, 1]. \]
Hence, using dominated convergence and taking \( n \uparrow \infty \) in (4.12) we obtain
\[ \mathbb{E} \left[ f(R_t^{(z,r)}) \right] - f(r) = \mathbb{E} \left[ \int_0^t \left[ C^{(1)}(R_s^{(z,r)}) + C^{(2)}(R_s^{(z,r)}) + C^{(3)}(R_s^{(z,r)}) \right] ds \right], \quad t \geq 0. \]
Therefore using (4.13) and (4.14) we obtain
\[ \sup_{r \in [0,1]} |\mathbb{E} \left[ f(R_t^{(z,r)}) \right] - f(r)| \leq Kt \to 0, \quad \text{as } t \to 0, \]
which implies that \( R^{(z,r)} \) is a Feller process.

(iii) Finally in order to obtain the infinitesimal generator of \( R^{(z,r)} \) we use (4.13) and (4.14) together with dominated convergence to obtain
\[ \lim_{t \to 0} \frac{\mathbb{E} \left[ f(R_t^{(z,r)}) \right] - f(r)}{t} = C^{(1)}(r) + C^{(2)}(r) + C^{(3)}(r). \]
Hence the result follows from Theorem 1.33 in [6]. \( \square \)

4.2. **Culling of the population process.** In Section 3 we obtained a two dimensional Markov process \( (R, Z) \) which describes the dynamics of two coexisting populations. The first component of this process describes the frequency of a specific type in the population, while the latter provides information on the total population size. One of our main interests in this paper is to study the role of natural selection on the within-generation variance in the offspring distribution.

Inspired by Gillespie’s model [17] we would like to maintain the total size of the population constant while allowing the frequency process \( R \) to evolve randomly, obtaining a one-dimensional stochastic process. To this end, throughout the rest of this section, we will use a sampling method in order to obtain a stochastic model of the frequency of a particular type in the population under the assumption that the total population size is constant.

Formally speaking, let us consider a fixed population size level \( z > 0 \), and consider a sequence of homogeneous Markov jump processes \( \{(R_t^{(z,n)})_{t \geq 0} : n \geq 1\} \). We denote the law of the process \( R_t^{(z,n)} \) by \( P_y \) when it starts
at the position \( r \in [0, 1] \). For each fixed \( n \geq 1 \), the Markov process \( R_{T_1^n}^{(z,n)} \) has jump times \( (T_m^n)_{m \geq 1} \) given by independent exponential variables with rate \( n \), and a transition kernel \( \{ \kappa^{(z,n)} : n \geq 1 \} \) defined by

\[
(4.15) \quad \kappa^{(z,n)}(y, A) = P_y(R_{T_1^n}^{(z,n)} \in A) := P_{(y,z)} \left( R_{1/n \land \tau}^{1, A} \in A, Z_{1/n \land \tau}^{1, \land \tau} \in \mathbb{R}_+ \right), \quad y \in [0, 1], \ A \in \mathcal{B}([0, 1]),
\]

where \( \tau = \tau_{\epsilon}^{-} \land \tau_{\epsilon}^{+} \) and the process \( (R, Z) \) is described in Section 3.

The infinitesimal generator \( \mathcal{L}^{(z,n)} \) of the process \( R_{T_1^n}^{(z,n)} \) is given for any \( f \in C([0, 1]) \) by

\[
(4.16) \quad \mathcal{L}^{(z,n)} f(r) = n \int_{[0,1]} (f(r) - f(y)) \kappa^{(z,n)}(r, dy), \quad r \in [0, 1].
\]

Intuitively for each fixed \( n \geq 1 \), we can think \( R_{T_1^n}^{(z,n)} \) as a sampling of the first coordinate of the process \( (R_{t\land \tau}, Z_{t\land \tau})_{t \geq 0} \) started at the position \( (r, z) \) at time \( t = 1/n \). Then, using the fact that the process \( (R, Z) \) is a homogenous Markov process, we restart the process \( (R_{t\land \tau}, Z_{t\land \tau})_{t \geq 0} \) at the initial position \( (R_{1/n \land \tau}^{1, \land \tau}, z) \) and we sample the process at time \( t = 1/n \) in order to define \( R_{T_1^n}^{(z,n)} \). By continuing this procedure we obtain the process \( R_{T_1^n}^{(z,n)} \).

From the previous construction, we note that in order to define the process \( R_{T_1^n}^{(z,n)} \) in the time interval \( [T_{m-1}^n \land \tau, T_m^n \land \tau) \) for each \( m = 1, \ldots, \) we consider the evolution of the process \( (R, Z) \) in the time interval \( [0, T_1^n \land \tau) \), starting from the state \( (R_{T_1^n}^{(z,n)}, z) \). Hence, the process \( R_{T_1^n}^{(z,n)} \) evolves as the first coordinate of the process \( (R, Z) \) but with the fluctuations of the total size process \( Z \) around \( z \) becoming smaller as we take \( n \to \infty \) as the jump times converge to zero.

In the next result, we will show that the sequence of Markov jump processes \( \{ R_{T_1^n}^{(z,n)} : n \geq 1 \} \) converges weakly to the process \( R_{T_1^n}^{(z,r)} \) given as the unique solution to (4.1), which by construction, can be understood as having the same dynamics of the first coordinate of the process \( (R, Z) \) but with total population size constant and equal to \( z > 0 \).

**Theorem 4.1.** For any fixed \( z > 0 \) and \( T > 0 \), \( R_{T_1^n}^{(z,n)} \to R_{T_1^n}^{(z,r)} \) as \( n \to \infty \) weakly in \( D([0, T], [0, 1]) \).

**Proof.** Observing that \( R_{T_1^n}^{(z,r)} \) is a Feller process and that all the processes involved take values in the compact interval \( [0, 1] \), following Theorem 17.28 in [25], we are only left with the task of proving

\[
(4.17) \quad \mathcal{L}^{(z,n)} f \to \mathcal{L}^{(z)} f, \quad \text{strongly as } n \to \infty,
\]

for every \( f \in C^2([0, 1]) \). To this end, we have by (4.15) together with (4.16)

\[
(4.18) \quad \mathcal{L}^{(z,n)} f(r) = n \left[ \mathbb{E}_{(r,z)} \left[ f(R_{n-1 \land \tau}) \right] - f(r) \right].
\]

In order to compute the limit in (4.17), we note that Proposition 3.1 implies that

\[
(4.19) \quad f(R_{n-1 \land \tau}) - f(r) = \int_0^{n-1 \land \tau} B(R_s, Z_s) ds + M_{n-1 \land \tau},
\]
where

\[ B(r, z) := -b^1 f'(r) r (1 - r) + \frac{c_1}{z} (f''(r) r (1 - r)^2 - f'(r) 2r (1 - r)) \]

\[ + b^2 f'(r) r (1 - r) + \frac{c_2}{z} (f''(r) r^2 (1 - r) + f'(r) 2r (1 - r)) \]

\[ + r z \int_{(0, \infty)} f \left( r \left( 1 - \frac{w}{z + w} \right) + \frac{w}{z + w} \right) - f (r) - w 1_{(0, 1)} (w) f' (r) \frac{(1 - r)}{z} \right] m^1 (d w) \]

\[ + \int_{(0, \infty)} f \left( r \left( 1 - \frac{w}{z + w} \right) + \frac{w}{z + w} \right) - f (r) \nu^1 (d w) + \eta^1 f'(r) \frac{(1 - r)}{z} \]

\[ + (1 - r) z \int_{(0, \infty)} f \left( r \left( 1 - \frac{w}{z + w} \right) \right) - f (r) + w 1_{(0, 1)} (w) f' (r) \frac{r}{z} \right] m^2 (d w) \]

\[ - \eta^2 f'(r) \frac{r}{z} + \int_{(0, \infty)} f \left( r \left( 1 - \frac{w}{z + w} \right) \right) - f (r) \nu^2 (d w) \} ds, \]

and \( M = \{ M_t : t \geq 0 \} \) is a local martingale.

Using the fact that \( f \in C^2 ([0, 1]) \) and that \((R_s, Z_s)\) takes values in \([0, 1] \times [\varepsilon, L]\) for \( s \in [0, \tau) \) we can find a constant \( K > 0 \) such that

\[ |B(R_s, Z_s)| \leq K, \quad \text{for } s \in [0, \tau) \, \mathbb{P}\text{-a.s.} \tag{4.20} \]

Therefore, by (4.19) and (4.20) we obtain that \((M_t)_{t \geq 0}\) is indeed a true martingale. Next, taking expectations in (4.19) we obtain

\[ n \left[ \mathbb{E}(r, z) \left[ f \left( R_n^{(1)} \right) \right] - f (r) \right] = \mathbb{E}(r, z) \left[ n \int_0^{n^{-1} \wedge \tau} B(R_s, Z_s) ds \right]. \tag{4.21} \]

On the other hand using (4.20) we have

\[ n \int_0^{n^{-1} \wedge \tau} B(R_s, Z_s) ds \leq K, \quad \text{for } s \in [0, \tau) \, \mathbb{P}\text{-a.s.} \]

Hence by dominated convergence together with identity (4.11)

\[ \lim_{n \to \infty} \mathbb{E}(r, z) \left[ n \int_0^{n^{-1} \wedge \tau} B(R_s, Z_s) ds \right] = B(r, z) = \mathcal{L}(z) f(r). \tag{4.22} \]

Therefore, using (4.18), (4.21) and (4.22)

\[ \lim_{n \to \infty} \mathcal{L}^{(z, n)} f(r) = \lim_{n \to \infty} n \left[ \mathbb{E}(r, z) \left[ f \left( R_n^{(1)} \right) \right] - f (r) \right] = \mathcal{L}(z) f(r). \tag{4.23} \]

In order to show the convergence in the strong sense of (4.23), we use Theorem 1.33 in [6] to obtain the result. \( \square \)

5. LARGE POPULATION ASYMPTOTICS OF \( \Lambda \)-ASYMMETRIC FREQUENCY PROCESSES.

In this section, we first obtain the large population limit of \( \Lambda \)-asymmetric frequency processes, while in the second part we study the fluctuations of the process around the large population limit obtained in the first part of this section.
5.1. Large population limit of $\Lambda$-asymmetric frequency processes. In this section we will study the asymptotic behavior of the $\Lambda$-asymmetric frequency process $R(z,r)$ as the size of the population becomes large. To this end we introduce the deterministic process $R_t^{(\infty,r)} = \{ R_t^{(\infty,r)} : t \geq 0 \}$ given by

$$R_t^{(\infty,r)} = \frac{re^{(\psi(2)r(0+) - \psi(1)r(0+))t}}{(1 - r) + re^{(\psi(2)r(0+) - \psi(1)r(0+))t}}, \quad t \geq 0.$$ 

The large population limit of the $\Lambda$-asymmetric frequency process $R_t^{(z,r)}$ is given in the following result.

**Theorem 5.1.** Fix $T > 0$ and assume that $\int_{(1,\infty)} w^m(i)(dw) < \infty$ for $i = 1, 2$. Then

$$\lim_{t \to T} \mathbb{E} \left[ \sup_{t \leq T} |R_t^{(z,r)} - R_t^{(\infty,r)}|^2 \right] = 0.$$ 

**Proof.** (i) First we note that $R_t^{(\infty,r)}$ is solution to the following ordinary differential equation

$$(5.1) \quad dR_t^{(\infty,r)} = R_t^{(\infty,r)} (1 - R_t^{(\infty,r)}) \left( b^2 - \int_{[1,\infty)} w^m^2(dw) - b^1 + \int_{[1,\infty)} w^m^1(dw) \right) dt, \quad t > 0,$$

with $R_0^{(\infty,r)} = r$.

Hence using (4.1) together with (5.1) we obtain that

$$R_t^{(z,r)} - R_t^{(\infty,r)} = A_t^{(1,z)} + A_t^{(2,z)} + A_t^{(3,z)}, \quad t \geq 0,$$

where for $t \geq 0$

$$A_t^{(1,z)} := \int_0^t \left[ \int_{(0,\infty)} \frac{1}{z} R_s^{(z,r)} (1 - R_s^{(z,r)}) [c^1 (1 - R_s^{(z,r)}) + c^2 R_s^{(z,r)}] dB_s ight.$$

$$+ \int_0^t \int_{(0,\infty)^2} g^1(z) (R_s^{(z,r)}, w, v) \tilde{N}_1(ds, dw, dv) + \int_0^t \int_{(0,\infty)^2} h^1(z) (R_s^{(z,r)}, w, v) \tilde{N}_2(ds, dw, dv)$$

$$+ \int_0^t \int_{(0,\infty)} \tilde{g}^1(z) (R_s^{(z,r)}, w, v) \tilde{N}_3(ds, dw) + \int_0^t \int_{(0,\infty)} \tilde{h}^1(z) (R_s^{(z,r)}, w, v) \tilde{N}_4(ds, dw)$$

$$A_t^{(2,z)} := \int_0^t \left[ R_s^{(z,r)} (1 - R_s^{(z,r)}) \left( \frac{2}{z} (c^2 - c^1) + \int_{(0,1)} \frac{w^2}{w + z} m^2(dw) - \int_{(0,1)} \frac{w^2}{w + z} m^1(dw) \right) \right] ds$$

$$+ \int_0^t \left[ \eta^1 (1 - R_s^{(z,r)}) - \eta^2 R_s^{(z,r)} \right] ds + \int_0^t \int_{(0,\infty)} \tilde{g}^2(z) (R_s^{(z,r)}, w) \nu^1(dw) ds$$

$$+ \int_0^t \int_{(0,\infty)} \tilde{h}^2(z) (R_s^{(z,r)}, w) \nu^2(dw) ds.$$

$$A_t^{(3,z)} = T.$$
and
\[ A_t^{(3, z)} = \int_0^t \int_{[1, \infty)} R_s(z, r) \frac{w}{w + z} m_1(dw)ds - \int_0^t \int_{[1, \infty)} R_s(\infty, r) \frac{w}{w + z} m_2(dw)ds + \int_0^t \int_{[1, \infty)} R_s(\infty, r) \frac{w}{w + z} m_2(dw)ds - \int_0^t \int_{[1, \infty)} R_s(z, r) \frac{w}{w + z} m_1(dw)ds \]
\[ + \int_0^t \int_{[1, \infty)} R_s(\infty, r) \frac{w}{w + z} m_1(dw)ds \]
\[ (5.4) \]

(ii) We will start by obtaining some estimations for the term \( A^{(1, z)} \), so using Doob’s inequality we have for \( t \in [0, T] \)
\[ \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \int_{(0, \infty)^2} g(z) (R_{s-}^{(z, r)}, w, v) \tilde{N}_1(ds, dw, dv) \right)^2 \right] \]
\[ \leq C_1 \mathbb{E} \left[ \int_0^T \left( R_s^{(z, r)} (1 - R_s^{(z, r)}) \right) \left[ c_1 (1 - R_s^{(z, r)}) + c_2 R_s^{(z, r)} \right] dB_s \right] \]
\[ \leq C_1 \left( c_1 + c_2 \right) T, \]
\[ (5.5) \]

for some constant \( C_1 > 0 \). Next, by an application of Doob’s inequality there exists a constant \( C_2 > 0 \) such that for \( t \in [0, T] \)
\[ \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \int_{(0, \infty)^2} h(z) (R_{s-}^{(z, r)}, w, v) \tilde{N}_2(ds, dw, dv) \right)^2 \right] \]
\[ \leq C_2 \left( \frac{1}{z} \int_{(0, 1]} w^2 m_1(dw) + \int_{[1, \infty)} \frac{w^2}{(w + z)^2} m_1(dw) \right) T, \]
\[ (5.6) \]

Proceeding as in (5.6) we have for \( t \in [0, T] \)
\[ \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \int_{(0, \infty)^2} \tilde{g}(z) (R_{s-}^{(z, r)}, w) \tilde{N}_3(ds, dw) \right)^2 \right] \]
\[ \leq C_3 \left( \frac{1}{z} \int_{(0, 1]} w^2 m_2(dw) + \int_{[1, \infty)} \frac{w^2}{(w + z)^2} m_2(dw) \right) T, \]
\[ (5.7) \]

and
\[ \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \int_{(0, \infty)} \tilde{h}(z) (R_{s-}^{(z, r)}, w, v) \tilde{N}_4(ds, dw) \right)^2 \right] \]
\[ \leq C_3 \mathbb{E} \left[ \int_0^T \int_{(0, \infty)} \frac{w^2}{(w + z)^2} (1 - R_s^{(z, r)})^2 \nu^1(dw)ds \right] + C_3 \mathbb{E} \left[ \int_0^T \int_{(0, \infty)} \frac{w^2}{(w + z)^2} (R_s^{(z, r)})^2 \nu^2(dw)ds \right] \]
\[ \leq C_3 T \left( \int_{(0, \infty)} \frac{w^2}{(w + z)^2} \nu^1(dw) + \int_{(0, \infty)} \frac{w^2}{(w + z)^2} \nu^2(dw) \right), \quad t \in [0, T], \]
\[ (5.8) \]
for some constant $C_3 > 0$.

Therefore using inequalities (5.6), (5.7), and (5.8) together with the fact that $\int_{[1,\infty)} w m^i(dw) < \infty$ for $i = 1, 2$, we can find a constant $K_1(T, z) > 0$ such that

$$
(5.9) \quad \mathbb{E} \left[ \left( \sup_{u \leq t} A_u^{(1, z)} \right)^2 \right] \leq K_1(T, z), \quad t \in [0, T],
$$

such that $\lim_{z \to \infty} K_1(T, z) = 0$.

(ii) Now for the term $A^{(2, z)}$ we note that for $t \in [0, T]$

$$
(5.10) \quad \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \left[ R_s^{(z, r)} (1 - R_s^{(z, r)}) \frac{2}{z} (c^2 - c^1) + \int_{(0,1)} \frac{w^2}{w + z} m^2(dw) - \int_{(0,1)} \frac{w^2}{w + z} m^1(dw) \\
+ \eta^1 (1 - R_s^{(z, r)}) \frac{\eta^2}{z} R_s^{(z, r)} ds - \eta^2 \frac{R_s^{(z, r)}}{z} ds \right) \right]^2 \right] \\
\leq \frac{T^2}{\pi} \left( 2c^2 - c^1 + \int_{(0,1)} w^2 m^2(dw) + \int_{(0,1)} w^2 m^1(dw) + \eta^1 + \eta^2 \right)^2,
$$

and

$$
(5.11) \quad \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \left( \tilde{g}(z) (R_s^{(z, r)}, u) \nu^1(dw)ds - \int_0^u \tilde{h}(z) (R_s^{(z, r)}, u) \nu^2(dw)ds \right) \right]^2 \right] \\
\leq T^2 \left( \int_{(0,\infty)} \frac{w}{w + z} \nu^1(dw) + \int_{(0,\infty)} \frac{w}{w + z} \nu^2(dw) \right)^2, \quad t \in [0, T].
$$

Hence by (5.10) and (5.11) there exists a constant $K_2(T, z) > 0$ such that

$$
(5.12) \quad \mathbb{E} \left[ \left( \sup_{u \leq t} A_u^{(2, z)} \right)^2 \right] \leq K_2(T, z), \quad t \in [0, T],
$$

such that $\lim_{z \to \infty} K_2(T, z) = 0$.

(ii) For the term $A^{(3, z)}$, we have for $t \in [0, T]$

$$
(5.13) \quad \mathbb{E} \left[ \sum_{i=1}^2 \left( \int_0^t \left( \int_{(1,\infty)} R_s^{(z, r)} (1 - R_s^{(z, r)}) \frac{w z}{w + z} m^i(dw)ds - \int_0^t \int_{(1,\infty)} R_s^{(\infty, r)} (1 - R_s^{(\infty, r)}) w m^i(dw)ds \right)^2 \right) \right] \\
\leq 9 \sum_{i=1}^2 \left( \int_{(1,\infty)} w m^i(dw) \right)^2 \mathbb{E} \left[ \int_0^t \sup_{u \leq s} |R_u^{(z, r)} - R_u^{(\infty, r)}|^2 du \right] + 9T^2 \sum_{i=1}^2 \left( \int_{(1,\infty)} \frac{w^2}{w + z} m^i(dw) \right)^2,
$$

and

$$
(5.14) \quad \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^t (b^2 - b^1) R_s^{(z, r)} (1 - R_s^{(z, r)}) ds - \int_0^t (b^2 - b^1) R_s^{(\infty, r)} (1 - R_s^{(\infty, r)}) ds \right)^2 \right] \\
\leq 9(b^2 - b^1)^2 \mathbb{E} \left[ \int_0^t \sup_{u \leq s} |R_u^{(z, r)} - R_u^{(\infty, r)}|^2 du \right], \quad t \in [0, T].
$$
Therefore using (5.13) together with (5.14) we can find constants $K_3(T, z)$, $K_4(T) > 0$ such that

\[
\mathbb{E} \left[ \left( \sup_{u \leq t} A_u^3 \right)^2 \right] \leq K_3(T, z) + K_4(T) \mathbb{E} \left[ \int_0^t \sup_{u \leq s} |R_u^{(z,r)} - R_u^{(\infty,r)}|^2 du \right], \quad t \in [0, T],
\]

such that $\lim_{z \to \infty} K_3(T, z) = 0$.

(iv) Using (5.9), (5.12) and (5.15) together with (5.2) we have that

\[
\mathbb{E} \left[ \sup_{u \leq t} |R_u^{(z,r)} - R_u^{(\infty,r)}|^2 \right] \leq \sum_{i=1}^3 K_i(T, z) + K_4(T) \int_0^t \mathbb{E} \left[ \sup_{u \leq s} |R_u^{(z,r)} - R_u^{(\infty,r)}|^2 \right] du, \quad t \in [0, T],
\]

Hence by an application of Gronwall’s inequality we obtain that for $T > 0$

\[
\mathbb{E} \left[ \sup_{u \leq T} |R_u^{(z,r)} - R_u^{(\infty,r)}|^2 \right] \leq \sum_{i=1}^3 K_i(T, z)e^{K_4(T)T} \to 0, \quad \text{as } z \to \infty.
\]

\[\square\]

5.2. Fluctuations of $\Lambda$-asymmetric frequency process. In this section, we will characterize the fluctuations of the process $R^{(z,r)}$ around the large population limit $R^{(\infty,r)}$. Throughout this section, we will make the following assumption.

**Assumption 5.1.** We assume that $\int_{(0,\infty)} w^2 m^i(dw) + \int_{[1,\infty)} w^i dw < \infty$ for $i = 1, 2$.

Let us denote for $t > 0$

\[
X_t^{(z)} = \sqrt{R_t^{(\infty,r)}(1 - R_t^{(\infty,r)})[c_1(1 - R_t^{(\infty,r)}) + c_2 R_t^{(\infty,r)}]} dB_t
\]

\[+ \sqrt{z} \int_{(0,\infty)^2} g(z)(R_t^{(\infty,r)}, w, v)\tilde{N}_1(dt, dw, dv) + \sqrt{z} \int_{(0,\infty)^2} h(z)(R_t^{(\infty,r)}, w, v)\tilde{N}_2(dt, dw, dv).
\]

Additionally let $X^{(\infty)}$ be a zero mean Gaussian process with covariance function $C^{X^{(\infty)}}$ given by

\[
C^{X^{(\infty)}}(s, t) = \int_0^{s \wedge t} R_t^{(\infty,r)}(1 - R_t^{(\infty,r)})[\sigma^2(1 - R_t^{(\infty,r)}) + \sigma^2 R_t^{(\infty,r)}] ds, \quad s, t \geq 0,
\]

where $\sigma^2 = c_1 + \int_{(0,\infty)} w^2 m^i(dw)$ $i = 1, 2$.

We are now ready to prove the following auxiliary result.

**Lemma 5.1.** Fix $T > 0$ and consider a sequence $(z_n)_{n \geq 1}$ such that $\lim_{n \to \infty} z_n = \infty$. Assume that $X^{(z_n)} \to Y^{(\infty)}$ as $n \to \infty$ weakly in $\mathbb{D}([0, T], \mathbb{R})$. Then $Y^{(\infty)} \overset{D}{=} X^{(\infty)}$ as elements of $\mathbb{D}([0, T], \mathbb{R})$.

**Proof.** We will characterize the finite-dimensional distributions of the weak limit $X^{\infty}$ by means of its characteristic function. To this end consider $0 \leq t_0 < t_1 < \cdots < t_n \leq T$ and $a_i \in \mathbb{R}$ for $i = 1, \ldots, n$, and denote

\[
\Phi(\lambda) = \mathbb{E} \left[ e^{i \lambda \sum_{i=1}^n a_i (X^{(z)}(t_i) - X^{(z)}(t_{i-1}))} \right], \quad \lambda \in \mathbb{R}.
\]

Using the fact that $B$, $\tilde{N}_1(dt, dw, dv)$, and $\tilde{N}_2(dt, dw, dv)$ are independent we obtain for $\lambda \in \mathbb{R}$

\[
\Phi(\lambda) = \prod_{i=1}^3 \Phi_i(\lambda),
\]

where
Hence, using that
\[
\Phi = \lim_{n \to \infty} \sum_{i=1}^{n} a_i \int_{t_{i-1}}^{t_i} \sqrt{R_t^{(\infty,r)} (1 - R_t^{(\infty,r)})(1 - R_t^{(\infty,r)})} \right) + c^2 R_t^{(\infty,r)} dB_t \right) ,
\]
\[
\Phi_2(\lambda, z) = \mathbb{E} \left[ \exp \left\{ i \lambda \sum_{i=1}^{n} a_i \int_{t_{i-1}}^{t_i} \int_{(0,\infty)^2} g(z)(R_t^{(\lambda,r)}, w, v) \mathcal{N}_1 (dt, dw, dv) \right\} ,
\]
\[
\Phi_3(\lambda, z) = \mathbb{E} \left[ \exp \left\{ i \lambda \sum_{i=1}^{n} a_i \int_{t_{i-1}}^{t_i} \int_{(0,\infty)^2} h(z)(R_t^{(\lambda,r)}, w, v) \mathcal{N}_2 (dt, dw, dv) \right\} .
\]
Using the fact that the process \( R_t^{(\infty,r)} \) is deterministic we obtain
\[
\Phi_1(\lambda) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ i \lambda a_i \int_{t_{i-1}}^{t_i} \int_{(0,\infty)^2} g(z)(R_s^{(\lambda,r)}, w, v) \mathcal{N}_1 (ds, dw, dv) \right\} ,
\]
\[
\Phi_2(\lambda, z) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ -\left( \frac{\lambda a_i^2}{2} \int_{t_{i-1}}^{t_i} (1 - R_s^{(\lambda,r)}) \right) \frac{w \sqrt{z}}{z + w} \right\} ,
\]
\[
\Phi_3(\lambda, z) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ -\left( \frac{\lambda a_i^2}{2} \int_{t_{i-1}}^{t_i} (1 - R_s^{(\lambda,r)}) \right) \frac{w \sqrt{z}}{z + w} \right\} .
\]
On the other hand, using the exponential formula for Poisson random measures
\[
\Phi_2(\lambda, z) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ -\left( \frac{\lambda a_i^2}{2} \int_{t_{i-1}}^{t_i} (1 - R_s^{(\lambda,r)}) \right) \frac{w \sqrt{z}}{z + w} \right\} .
\]
Hence, using that \( \int_{(0,\infty)} z^2 m^1(dz) < \infty \) we have
\[
\lim_{z \to \infty} \Phi_2(\lambda, z) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ -\left( \frac{\lambda a_i^2}{2} \int_{(0,\infty)} w \sqrt{z} \right) \right\} .
\]
Proceeding as in (5.19)
\[
\Phi_3(\lambda, z) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ -\left( \frac{\lambda a_i^2}{2} \int_{t_{i-1}}^{t_i} (1 - R_s^{(\lambda,r)}) \right) \frac{w \sqrt{z}}{z + w} \right\} .
\]
Next, using that \( \int_{(0,\infty)} z^2 m^2(dz) < \infty \) we obtain
\[
\lim_{z \to \infty} \Phi_3(\lambda, z) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ -\left( \frac{\lambda a_i^2}{2} \int_{(0,\infty)} w \sqrt{z} \right) \right\} .
\]
Therefore using (5.18), (5.19), and (5.20) in (5.17) we obtain that
\[
\mathbb{E} \left[ e^{i \lambda \sum_{i=1}^{n} a_i (\omega_{t_i} - \omega_{t_{i-1}})} \right] = \lim_{z \to \infty} \Phi(z, \lambda) = \prod_{i=1}^{n} \mathbb{E} \left[ \exp \left\{ -\left( \frac{\lambda a_i^2}{2} \int_{t_{i-1}}^{t_i} (1 - R_t^{(\lambda,r)}) \right) \frac{w \sqrt{z}}{z + w} \right\} .
\]
where \( \sigma^i = c^i + \int_{(0,\infty)} w^2m^i(dw) \) for \( i = 1, 2 \). Hence, (5.21) implies that the random vector \((X_{t_1}^{(\infty)}, X_{t_2}^{(\infty)} - X_{t_1}^{(\infty)}, \ldots, X_{t_{n-1}}^{(\infty)} - X_{t_n}^{(\infty)})\) has a Gaussian distribution, and hence \( X^{(\infty)} \) is a Gaussian process.

Now we prove the next auxiliary result. We now prove the next auxiliary result.

\[
\mathbb{E} \left[ e^{\lambda_1 X_{t_1}^{(\infty)} + \lambda_2 X_{t_2}^{(\infty)}} \right] = \exp \left\{ -\sum_{i=1}^{2} \frac{\lambda_i^2}{2} \int_{0}^{t_i} R_t^{(\infty,r)}(1 - R_t^{(\infty,r)})[\sigma^1(1 - R_t^{(\infty,r)}) + \sigma^2 R_t^{(\infty,r)}]dt \right. \\
\left. - \lambda_1 \lambda_2 \int_{0}^{t_1} R_t^{(\infty,r)}(1 - R_t^{(\infty,r)})[\sigma^1(1 - R_t^{(\infty,r)}) + \sigma^2 R_t^{(\infty,r)}]dt \right\}.
\]

Therefore the covariance function of the process \( X^{(\infty)} \) is given for \( s, t \geq 0 \)

\[
C_{X^{(\infty)}}(s, t) = \int_{0}^{s\wedge t} R_t^{(\infty,r)}(1 - R_t^{(\infty,r)})[\sigma^1(1 - R_t^{(\infty,r)}) + \sigma^2 R_t^{(\infty,r)}]ds.
\]

\[
\square
\]

We now prove the next auxiliary result.

**Lemma 5.2.** For any \( T > 0 \),

\[
\lim_{z \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| X_t^{(z)} - \sqrt{z}A_t^{(1,\infty)} \right|^2 \right] = 0.
\]

**Proof.** Using Doob’s inequality we obtain

\[
\begin{align*}
\mathbb{E} \left( \sup_{t \in [0,T]} \left| \int_{0}^{t} \sqrt{R_s^{(z,r)}(1 - R_s^{(z,r)})[c^1(1 - R_s^{(z,r)}) + c^2 R_s^{(z,r)}]} dB_s \right|^2 \right) \\
\leq \mathbb{E} \left[ \int_{0}^{T} \left( \sqrt{R_s^{(z,r)}(1 - R_s^{(z,r)})[c^1(1 - R_s^{(z,r)}) + c^2 R_s^{(z,r)}]} - \sqrt{R_s^{(\infty,r)}(1 - R_s^{(\infty,r)})[c^1(1 - R_s^{(\infty,r)}) + c^2 R_s^{(\infty,r)}]} \right)^2 ds \right] \\
\leq \mathbb{E} \left[ \int_{0}^{T} \left( R_s^{(z,r)}(1 - R_s^{(z,r)})[c^1(1 - R_s^{(z,r)}) + c^2 R_s^{(z,r)}] - R_s^{(\infty,r)}(1 - R_s^{(\infty,r)})[c^1(1 - R_s^{(\infty,r)}) + c^2 R_s^{(\infty,r)}] \right) ds \right] \\

to (5.22) \\
\leq 3(c^1 + c^2) \int_{0}^{T} \mathbb{E} \left[ \left| R_s^{(z,r)} - R_s^{(\infty,r)} \right|^2 \right] ds \leq 3(c^1 + c^2)T \mathbb{E} \left[ \sup_{t \in [0,T]} \left| R_t^{(z,r)} - R_t^{(\infty,r)} \right|^2 \right]^{1/2}.
\end{align*}
\]
In a similar way by Doob’s inequality

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_{(0,\infty)^2} \sqrt{z} g^{(z)}(R_{s^{-}}^{(z,r)}, w, v) \tilde{N}_1(ds, dw, dv) - \int_0^t \int_{(0,\infty)^2} \sqrt{z} g^{(z)}(R_{\infty}^{(z,r)}, w, v) \tilde{N}_1(ds, dw, dv) \right|^2 \right] \\
\leq \mathbb{E} \left[ \int_0^T \int_{(0,\infty)} |R_{s}^{(\infty,r)} - R_{s}^{(z,r)}|^2 \frac{w^2+2}{(z+w)^2} m^1(dw)ds \right] + \mathbb{E} \left[ \int_0^T \int_{(0,\infty)} |R_{\infty}^{(\infty,r)} - R_{\infty}^{(z,r)}|^2 \frac{w^2+2}{(z+w)^2} m^1(dw)ds \right]
\]

(5.23)

\[
\leq 2T \int_{(0,\infty)} w^2 m^1(dw) \mathbb{E} \left[ \sup_{t \in [0,T]} |R_{t}^{(z,r)} - R_{t}^{(\infty,r)}|^2 \right]^{1/2}.
\]

Proceeding like in (5.23) we obtain

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t \int_{(0,\infty)^2} \sqrt{z} h^{(z)}(R_{s^{-}}^{(z,r)}, w, v) \tilde{N}_2(ds, dw, dv) - \int_0^t \int_{(0,\infty)^2} \sqrt{z} h^{(z)}(R_{\infty}^{(z,r)}, w, v) \tilde{N}_2(ds, dw, dv) \right|^2 \right] \\
\leq 2T \int_{(0,\infty)} w^2 m^1(dw) \mathbb{E} \left[ \sup_{t \in [0,T]} |R_{t}^{(z,r)} - R_{t}^{(\infty,r)}|^2 \right]^{1/2}.
\]

(5.24)

Finally using (5.8) we obtain that

\[
\mathbb{E} \left[ \left( \sup_{t \in [0,T]} \int_0^t \int_{(0,\infty)} \sqrt{z} g^{(z)}(R_{s^{-}}^{(z,r)}, w) \tilde{N}_3(ds, dw) \right)^2 \right] + \mathbb{E} \left[ \left( \sup_{t \in [0,T]} \int_0^t \int_{(0,\infty)} \sqrt{z} h^{(z)}(R_{s^{-}}^{(z,r)}, w) \tilde{N}_4(ds, dw) \right)^2 \right] \\
\leq T \left( \int_{(0,\infty)} \frac{w^2}{(w+z)^2} \nu^1(dw) + \int_{(0,\infty)} \frac{w^2}{(w+z)^2} \nu^2(dw) \right).
\]

(5.25)

Hence using (5.25) and the fact that \( \int_{(0,\infty)} w^i(dw) < \infty \) for \( i = 1, 2 \) we can find a constant \( C(T, z) \) such that

\[
\mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \int_{(0,1)} \sqrt{z} g^{(z)}(R_{s^{-}}^{(z,r)}, u) \tilde{N}_3(ds, du) \right)^2 \right] + \mathbb{E} \left[ \left( \sup_{u \leq t} \int_0^u \int_{(0,1)} \sqrt{z} h^{(z)}(R_{s^{-}}^{(z,r)}, u, v) \tilde{N}_4(ds, du) \right)^2 \right] \leq C(T, z),
\]

(5.26)

and such that \( \lim_{z \to \infty} C(z, T) = 0 \).

Hence using (5.22), (5.23), (5.24), and (5.26) gives

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| X_{t}^{(z)} - \sqrt{z} A_1 \right|^2 \right] \leq C(z, T) + C_1(T) \mathbb{E} \left[ \sup_{t \in [0,T]} |R_{t}^{(z,r)} - R_{t}^{(\infty,r)}|^2 \right]^{1/2}.
\]

The result now follows from Theorem 5.1. \qed

Next we provide a tightness result for the fluctuations of the process \( R^{(z,r)} \) when the size of population becomes large.

**Proposition 5.1.** The family \( \{ \sqrt{z}(R^{(z,r)} - R^{(\infty,r)}) : z \geq 1 \} \) is tight in the space \( D(\mathbb{R}_+, \mathbb{R}) \).
Proof. Let $\delta \in (0,1)$ and $\theta \in [0,\delta]$. Let $T > 0$ and $(\tau_n)_{n \geq 1}$ be a sequence of stopping times such that $0 \leq \tau_n < T'$.

By (5.2) we have
\[
\sqrt{z}(R_t^{(z,r)} - R_t^{(\infty,r)}) = \sqrt{z}A_t^{(1,z)} + \sqrt{z}A_t^{(2,z)} + \sqrt{z}A_t^{(3,n)}, \quad t > 0,
\]
where $A^{(1,z)}$ is a local martingale and $A^{(2,z)}$ and $A^{(3,z)}$ are bounded variation processes.

(i) We now provide some estimates for the quadratic variation of the local martingale $A^{(1,z)}$. By (5.3) we have that the quadratic variation of $\sqrt{z}A^1$ is given by
\[
\left[ \sqrt{z}A^{(1,z)} \right]_t = \int_0^t R_t^{(z,r)} (1 - R_t^{(z,r)}) [c^1 (1 - R_t^{(z,r)}) + c^2 R_t^{(z,r)}] ds
+ \int_0^t \int_{(0,\infty)} \frac{w^2 z^2}{(z+w)^2} (1 - R_s^{(z,r)})^2 R_s^{(z,r)} m_1 (dw) ds
+ \int_0^t \int_{(0,\infty)} \frac{w^2 z^2}{(z+w)^2} (1 - R_s^{(z,r)}) (R_s^{(z,r)})^2 m_2 (dw) ds
+ \int_{(0,\infty)} \frac{w^2 z}{z+w} + (1 - R_s^{(z,r)})^2 \nu_1 (dw) ds + \int_{(0,\infty)} \frac{w^2 z}{z+w} (R_s^{(z,r)})^2 \nu_2 (dw) ds, \quad t \geq 0.
\]

Hence
\[
\sup_{z \geq 1} \sup_{\theta \in [0,\delta]} \mathbb{E} \left[ \left[ \sqrt{z}A^{(1,z)} \right]_{\tau_n + \theta} - \left[ \sqrt{z}A^{(1,z)} \right]_{\tau_n} \right] \leq \sum_{i=1}^2 \left( c^i + \int_{(0,\infty)} w^2 m_1 (dw) + \int_{(0,\infty)} w \nu_1 (dw) \right) \delta.
\]

(ii) Next proceeding as in (5.10) and (5.11) we have
\[
\sup_{z \geq 1} \sup_{\theta \in [0,\delta]} \mathbb{E} \left[ \left[ \sqrt{z}A^{(2,z)} \right]_{\tau_n + \theta} - \left[ \sqrt{z}A^{(2,z)} \right]_{\tau_n} \right] \leq \sum_{i=1}^2 \left( 2 c^i + \int_{(0,1)} w^2 m_1 (dw) + \eta^i + \int_{(0,\infty)} w \nu_1 (dw) \right) \delta.
\]

(iii) For the last term we obtain proceeding like in (5.13) and (5.14)
\[
\sup_{z \geq 1} \sup_{\theta \in [0,\delta]} \mathbb{E} \left[ \left[ \sqrt{z}A^{(3,z)} \right]_{\tau_n + \theta} - \left[ \sqrt{z}A^{(3,z)} \right]_{\tau_n} \right] \leq C_1 + C_2 \delta \sqrt{z} \mathbb{E} \left[ \sup_{t \in [0,T' + \delta]} \left| R_u^{(z,r)} - R_u^{(\infty,r)} \right| \right] \left( \int_{[1,\infty)} w m_1 (dw) \right)^{1/2},
\]
where $C_1$ and $C_2$ are positive constants not dependent on $z$. On the other hand using (5.16) we obtain that
\[
\sqrt{z} \mathbb{E} \left[ \sup_{t \in [0,T' + \delta]} \left| R_u^{(z,r)} - R_u^{(\infty,r)} \right| \right]^{1/2} \leq C_3 e^{C_4 (T' + \delta)^2},
\]
where $C_3$ and $C_4$ are positive constants not dependent on $z$. Therefore
\[
\sup_{z \geq 1} \sup_{\theta \in [0,\delta]} \mathbb{E} \left[ \left[ \sqrt{z}A^{(3,z)} \right]_{\tau_n + \theta} - \left[ \sqrt{z}A^{(3,z)} \right]_{\tau_n} \right] \leq \delta C_5,
\]
where $C_5 > 0$ is independent of $z$. 

THE RATIO OF TWO GENERAL CBI’S, AND ITS RELATION TO COALESCENT THEORY
(iv) For fixed \( t > 0 \), by proceeding like in (5.28) and (5.29) we have that there exists a constant \( C_6(t) > 0 \) independent of \( z \) such that
\[
\sup_{z \geq 1} \mathbb{E} \left[ \left( \sqrt{z} A_t^{(2,z)} + \sqrt{z} A_t^{(3,z)} \right) \right] \leq C_6(t).
\]
On the other hand by a slight modification of (5.27) we obtain
\[
\sup_{z \geq 1} \mathbb{E} \left[ \left( \sqrt{z} A_t^{(1,z)} \right) \right] \leq \sup_{z \geq 1} \mathbb{E} \left[ \left( \sqrt{z} A_t^{(1,z)} \right)^2 \right]^{1/2} \leq C_7(t),
\]
where \( C_7(t) > 0 \) is a constant independent of \( z \). Next, by Markov’s inequality we obtain for \( M > 0 \)
\[
\sup_{z \geq 1} \mathbb{P} \left( \sqrt{z} \left( R_t^{(z,r)} - R_t^{(\infty,r)} \right) > M \right) \leq \frac{1}{M} \mathbb{E} \left[ \left( \sqrt{z} A_t^{(1,z)} + \sqrt{z} A_t^{(2,z)} + \sqrt{z} A_t^{(3,z)} \right) \right] \leq C_8(t) \to 0, \quad \text{as } M \to \infty.
\]
Therefore for any \( t > 0 \) the random variable \( \sqrt{z} \left( R_t^{(z,r)} - R_t^{(\infty,r)} \right) \) is tight in \( \mathbb{R} \).

(v) The fact that \( \sqrt{z} \left( R_t^{(z,r)} - R_t^{(\infty,r)} \right) \) is tight for every \( t > 0 \), together with (5.27), (5.28), and (5.29) implies by The Aldous-Rebolledo criterion see [40], that the family \( \{ \sqrt{z} (R_t^{(z,r)} - R_t^{(\infty,r)}) : z \geq 1 \} \) is tight in the space of cadlag paths from \( \mathbb{R}_+ \) to \( \mathbb{R} \) with the Skorohod topology.

\[\Box\]

Now we are ready to state the main result in this section.

**Theorem 5.2.** Under Assumption 5.1 for any fixed \( T > 0 \), \( \sqrt{z} (R_t^{(z,r)} - R_t^{(\infty,r)}) \to X^\infty \) as \( z \to \infty \) weakly in \( \mathbb{D}([0,T], \mathbb{R}) \).

**Proof.** From Proposition 5.1, the family \( \{ \sqrt{z} (R_t^{(z,r)} - R_t^{(\infty,r)}) : z \geq 1 \} \) is relatively compact. Hence, there exists a subsequence \( \{ z_n \}_{n \geq 1} \) such that \( \{ (\sqrt{z_n} (R_t^{(z_n,r)} - R_t^{(\infty,r)}))_{t \geq 0} : n \geq 1 \} \) converges weakly to some \( (Y^{(\infty)}(t))_{t \geq 0} \) in \( \mathbb{D}(\mathbb{R}_+, \mathbb{R}) \). Therefore, it is enough to prove that there is a unique limit point for any convergent subsequence.

We define for \( t \geq 0 \)
\[
A_t^{(5,z_n)} := \sqrt{z_n} \left( R_t^{(z_n,r)} - R_t^{(\infty,r)} \right) + \frac{1}{\sqrt{z_n}} \int_0^t \left( b_2^1 - b_1 \right) \left( \sqrt{z_n} \left( R_s^{(z_n,r)} - R_s^{(\infty,r)} \right) - R_s^{(\infty,r)} \right) ds - \frac{1}{\sqrt{z_n}} \int_0^t \int_{(1,\infty)} \sqrt{z_n} \left( R_s^{(z_n,r)} - R_s^{(\infty,r)} \right) \left( R_s^{(z_n,r)} - R_s^{(\infty,r)} \right) \sqrt{z_n} ds \left( R_s^{(\infty,r)} - R_s^{(z_n,r)} \right) w_{s}^{(1)}(dw) ds
\]
\[
- \frac{1}{\sqrt{z_n}} \int_0^t \int_{(1,\infty)} \sqrt{z_n} \left( R_s^{(z_n,r)} - R_s^{(\infty,r)} \right) \left( R_s^{(z_n,r)} - R_s^{(\infty,r)} \right) \sqrt{z_n} ds \left( R_s^{(\infty,r)} - R_s^{(z_n,r)} \right) w_{s}^{(2)}(dw) ds.
\]

Using Skorohod representation theorem (see [4, Theorem 6.7]), the fact that \( \int_{(0,\infty)} w_{s}^{(i)}(dw) < \infty \) for \( i = 1, 2 \), and bounded convergence we have that
\[
\frac{1}{\sqrt{z_n}} \int_0^T \int_{(1,\infty)} \left( \sqrt{z_n} \left| R_s^{(z_n,r)} - R_s^{(\infty,r)} \right| + \sqrt{z_n} \left| R_s^{(\infty,r)} - R_s^{(z_n,r)} \right| \right) w_{s}^{(i)}(dw) ds \to 0, \quad \text{weakly as } n \to \infty,
\]
and hence the convergence holds in probability as well. Therefore for $\varepsilon > 0$ and $i = 1, 2$

$$\mathbb{P}\left( \sup_{t \in [0, T]} \left| \frac{1}{\sqrt{n}} \int_0^t \int_{(1, \infty)} \left( \sqrt{\frac{1}{n}} (R_s^{(z_n, r)} - R_s^{(\infty, r)}) (1 - R_s^{(z_n, r)} - R_s^{(\infty, r)} \sqrt{\frac{1}{n}} (R_s^{(\infty, r)} - R_s^{(z_n, r)})) \right) \mathrm{w} m_i (dw) \right| > \varepsilon \right) \leq \mathbb{P}\left( \frac{1}{\sqrt{n}} \int_0^T \int_{(1, \infty)} \left( \sqrt{\frac{1}{n}} |R_s^{(z_n, r)} - R_s^{(\infty, r)}| + \sqrt{\frac{1}{n}} |R_s^{(\infty, r)} - R_s^{(z_n, r)}| \right) \mathrm{w} m_i (dw) > \varepsilon \right) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.30)$$

By a similar argument we have that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \int_0^t (b^2 - b^1) \left( \sqrt{\frac{1}{n}} (R_s^{(z_n, r)} - R_s^{(\infty, r)})(1 - R_s^{(z_n, r)} - R_s^{(\infty, r)}) \sqrt{\frac{1}{n}} (R_s^{(\infty, r)} - R_s^{(z_n, r)}) \right) ds = 0, \quad \text{in probability.} \quad (5.31)$$

Hence (5.30) and (5.31) imply that

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |A_t^{(5, z_n)} - \sqrt{\frac{1}{n}} (R_t^{(z_n, r)} - R_t^{(\infty, r)})| = 0, \quad \text{in probability}. \quad (5.32)$$

Therefore (5.32) together with the fact that $\sqrt{\frac{1}{n}} (R_s^{(z_n, r)} - R_s^{(\infty, r)})$ converges weakly to $Y^{(\infty)}$ in $\mathbb{D}([0, T], \mathbb{R})$ as $n \rightarrow \infty$ implies

$$A_t^{(5, z_n)} \rightarrow Y^{(\infty)}, \quad \text{as } n \rightarrow \infty \text{ weakly in } \mathbb{D}([0, T], \mathbb{R}). \quad (5.33)$$

Now let

$$A_t^{(6, z_n)} := A_t^{(2, z_n)} + \int_0^t \int_{(1, \infty)} R_s^{(z_n, r)} (1 - R_s^{(z_n, r)}) \frac{w^2 m^1 (dw)}{w + z_n} ds - \int_0^t \int_{(1, \infty)} R_s^{(z_n, r)} (1 - R_s^{(z_n, r)}) \frac{w^2 m^2 (dw)}{w + z_n} ds.$$ 

Then proceeding like in (5.10) and (5.11) we obtain that

$$\mathbb{E} \left[ \left( \sup_{t \in [0, T]} \sqrt{\frac{1}{n}} A_t^{(6, z_n)} \right)^2 \right] \leq \sum_{i=1}^2 16 \frac{T}{z_n} \left( 2t^2 + \int_{(0,1)} w^2 m_i (dw) + \eta^2 + \int_{(0, \infty)} w \nu^i (dw) \right)$$

$$+ 16 \sum_{i=1}^2 \frac{T^2}{z_n} \left( \int_{(1, \infty)} w^2 m_i (dw) \right)^2.$$ 

Hence

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \sup_{t \in [0, T]} \sqrt{\frac{1}{n}} A_t^{(6, z_n)} \right)^2 \right] = 0. \quad (5.34)$$

Now by (5.2) we have for $t \geq 0$

$$\sqrt{z} X_t^{(z_n)} = A_t^{(n, 5)} - A_t^{(6, n)} + \sqrt{z} (A_t^{(1, z_n)} - X_t^{(z_n)}).$$

Therefore using (5.33), (5.34), and Lemma 5.2 we obtain that

$$\sqrt{z} X_t^{(z_n)} \rightarrow Y^{(\infty)}, \quad \text{as } n \rightarrow \infty \text{ weakly in } \mathbb{D}([0, T], \mathbb{R}). \quad (5.35)$$
Hence by an application of Lemma 5.1 we obtain that $Y(\infty) \overset{D}{=} X(\infty)$, which implies that the limit of any convergent subsequence of the family \( \{ \sqrt{z}(R(z,r) - R(\infty,r)) : z \geq 1 \} \) is equal in law to $X(\infty)$. Therefore

$$\sqrt{z}(R(z,r) - R(\infty,r)) \to X(\infty), \quad \text{as } z \to \infty,$$

weakly in $\mathbb{D}([0, T], \mathbb{R})$. \qed

6. Moment Duality for the $\Lambda$-Asymmetric Frequency Process.

This section is dedicated to studying the relation between the $\Lambda$-asymmetric frequency process $R(z,r)$, introduced in Section 4 and a particular class of branching-coalescent process.

This class consists of continuous time Markov chains taking values in $\mathbb{N}_0 \cup \{ \Delta \}$ (where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$). The point $\Delta$ is a cemetery state and we assume that $x^\Delta = 0$ for all $x \in [0, 1]$.

For each $i, k \in \mathbb{N}_0$ with $i \geq k$, and $v \in [0, 1]$ we define the following terms

$$\lambda_{i,k}^l(v) = \int_{(0,v)} (1 - u)^{i-k} u^k \mathbf{T}^l(m)(du),$$

$$\mu_{i,k}^l = \int_{(0,1)} (1 - u)^{i-k} u^k \mathbf{T}^l(\nu)(du), \quad l = 1, 2.$$

Now for each $i, j \in \mathbb{N}_0 \cup \{ \Delta \}$ let us consider the following set of real numbers

$$q_{ij}^\nu = \begin{cases} \overline{\mu}_{i,i}^1 & \text{if } i \in \mathbb{N} \text{ and } j = 0, \\ si + \sum_{k=2}^{i} \kappa_k \binom{i}{k} + \beta_i & \text{if } i \in \mathbb{N} \text{ and } j = i + 1, \\ \binom{i}{i-j} \overline{\mu}_{i,i-j}^1 + \binom{i}{i-j+1} \overline{\lambda}_{i,i-j+1} & \text{if } i \geq 2 \text{ and } j \in \{1, \ldots, i-1\}, \\ \alpha_i & \text{if } i \in \mathbb{N} \text{ and } j = \Delta, \\ 0 & \text{otherwise}, \end{cases}$$

where

- For $2 \leq k \leq i$,

$$\overline{\lambda}_{i,k}^1 = \int_{(0,1)} (1 - u)^{i-k} u^k u^{-2} \Lambda^1(du),$$

with $\Lambda^1(du) = \frac{n^2}{z} \delta_0(du) + zy^2 \mathbf{T}(m)(du)$.

- For $1 \leq k \leq i$,

$$\overline{\mu}_{i,k}^1 = \int_{(0,1)} (1 - u)^{i-k} u^k u^{-1} \Gamma^1(du),$$

with $\Gamma^1(du) = \frac{n^2}{z} \delta_0(du) + y \mathbf{T}(\nu)(du)$. 

In the case that \( N \) interested in the particular case of moment duality which follows from Proposition 6.1 by taking \( Y \) and \( M \) on Proposition 6.1. Let of Jansen and Kurt in [23].

\[
\begin{align*}
\kappa_k &= z \left[ k \left( \lambda_{1,k} \left( \frac{1}{1 + z} \right) - \lambda_{i,k} \left( \frac{1}{1 + z} \right) \right) + \left( \lambda_{1,k} (1) - \lambda_{i,k} (1) \right) \right] + \frac{2(c_1 - c_2)^2}{z} 1_{(k = 2)}.
\end{align*}
\]

For \( k \geq 1 \)

\[
\beta_k = -kz \left[ \left( \lambda_{1,1} (1) - \lambda_{k,1} (1) \right) - \left( \lambda_{k,1} \left( \frac{1}{1 + z} \right) - \lambda_{k,1} \left( \frac{1}{1 + z} \right) \right) \right].
\]

For \( k \geq 1 \)

\[
\alpha_k = \int_{[0,1]} (1 - (1 - u)^k) \rho^2 (du),
\]

with \( \rho^2 (du) = \frac{2}{z} \delta_0 (du) + y T(z)(\nu^2)(du) \).

In the case that \( q_{ij}^z \geq 0 \) for every \( i, j \in \mathbb{N}_0 \cup \{ \Delta \} \), we will define a \( \mathbb{N}_0 \cup \{ \Delta \} \)-valued continuous Markov chain \( N^{(z,n)} = \{ N_t^{(z,n)} : t \geq 0 \} \) starting from \( n \in \mathbb{N} \), whose generator is given by \( Q^{(z)} = (q_{ij}^z)_{i,j \in \mathbb{N}} \).

The rest of this section is devoted to showing that the moment dual of the frequency process \( R^{(z,r)} \) is the continuous-time Markov chain \( (N_t^{(z,n)})_{t \geq 0} \). An effective procedure to prove the moment duality is using their infinitesimal generators. The following proposition is a direct consequence of Theorem 4.11 in Ethier Kurtz [12] taking \( H \) bounded and continuous and \( \alpha = \beta = 0 \), and can also be seen as a small modification of Proposition 1.2 of Jansen and Kurt in [23].

**Proposition 6.1.** Let \( Y^{(1)} = \{ Y_t^{(1)} : t \geq 0 \} \) and \( Y^{(2)} = \{ Y_t^{(2)} : t \geq 0 \} \) be two Markov processes taking values on \( E_1 \) and \( E_2 \), respectively. Let \( H : E_1 \times E_2 \to \mathbb{R} \) be a bounded and continuous function and assume that there exist functions \( g_i : E_1 \times E_2 \to \mathbb{R} \), for \( i = 1, 2 \), such that for every \( n \in E_1, x \in E_2 \) and every \( T > 0 \), the processes \( M^{(1)} = \{ M_t^{(1)} : 0 \leq t \leq T \} \) and \( M^{(2)} = \{ M_t^{(2)} : 0 \leq t \leq T \} \), defined as follows

\[
\begin{align*}
M_t^{(1)} &= H(Y_t^{(1)}, x) - \int_0^t g_1(Y_s^{(1)}, x) ds \\
M_t^{(2)} &= H(n, Y_t^{(2)}) - \int_0^t g_2(n, Y_s^{(2)}) ds
\end{align*}
\]

are martingales with respect to the natural filtration of \( Y_t^{(1)} \) and \( Y_t^{(2)} \) respectively. Then, if \( g_1(n, x) = g_2(n, x) \) for all \( n \in \mathbb{N}, x \in \mathbb{N}_0 \), the processes \( Y^{(1)} \) and \( Y^{(2)} \) are dual with respect to \( H \).

The previous result provides a general duality relationship between two Markov processes; in our case we are interested in the particular case of moment duality which follows from Proposition 6.1 by taking \( H(n, x) := x^n \), for \( x \in [0, 1] \) and \( n \in \mathbb{N}_0 \).
Theorem 6.1. Assume that $q_{ij} \geq 0$ for every $i, j \in \mathbb{N}_0 \cup \{\Delta\}$. Then, for every $r \in [0, 1]$, $n \in \mathbb{N}_0 \cup \{\Delta\}$ and $t > 0$
\[ \mathbb{E}[(R_t^{(z,r)})^n] = \mathbb{E}[\mathbb{N}_t^{(z,n)}]. \]

Proof. We will consider $\mathbb{N}_0 \cup \{\Delta\}$ endowed with the discrete topology and $\mathbb{N}_0 \cup \{\Delta\} \times [0, 1]$ with the product topology. We recall that for every fixed $r \in [0, 1]$, $H(n, r) = r^n$ with $H(0, r) = 1$ and $H(\Delta, r) = 0$, which are bounded and continuous. In addition, for every fixed $k \in \mathbb{N}_0 \cup \{\Delta\}$, $H(k, r) = r^k$ is continuous, therefore we conclude that $H : \mathbb{N}_0 \cup \{\Delta\} \times [0, 1] \mapsto [0, 1]$ is continuous.

We observe that $H(\cdot, n)$ is a polynomial in $[0, 1]$ for fixed $n \in \mathbb{N}_0 \cup \{\Delta\}$, this fact clearly implies that $H(\cdot, n) \in C^2([0, 1])$ and hence it lies in the domain of the generator $\mathcal{L}^{(z)}$. Therefore the process
\[ H(n, R_t^{(z,r)}) - \int_0^t \mathcal{L}^{(z)} H(n, R_s^{(z,r)})ds \]
is a martingale.

Additionally, as in the proof of Lemma 2 in [21], we have that for fixed $r \in [0, 1]$ the function $H(\cdot, r)$ lies in the domain of the generator $Q^{(z)}$, which implies that the process
\[ H(N_t^{(z,n)}, r) - \int_0^t Q^{(z)} H(N_s^{(z,n)}, r)ds \]
is also a martingale. In view of Proposition 6.1 we will compute $\mathcal{L}^{(z)} H(n, r)$ for $r \in [0, 1]$ and $n \in \mathbb{N}$, hence using (4.11)

(6.4)
\[ \mathcal{L}^{(z)} H(n, r) = nr^{n-1} \left[ r(1-r)(b^2 - b^1) + \frac{2r(1-r)}{z}(c^2 - c^1) \right] + n(n-1)r^{n-1} \frac{(1-r)}{z}(c^1(1-r) + c^2r) \]
\[ + \frac{\eta^1}{z}nr^{n-1}(1-r) + \int_{(0,1)} [(r(1-u)+u)^n - r^n] T(z) T^{(z)}(\nu^1)(du) \]
\[ + z\gamma \int_{(0,1)} [(r(1-u)+u)^n - r^n - \frac{1}{1-u}nr^{n-1}(1-r)1_{(0,1/(1+z))}(u)] T(z) T^{(z)}(m^1)(du) \]
\[ + z(1-r) \int_{(0,1)} [(r(1-u))^n - r^n + \frac{u}{1-u}nr^{n-1}1_{(0,1/(1+z))}(u)] T(z) T^{(z)}(m^2)(du) \]
\[ - \frac{\eta^2}{z}nr^n + \int_{(0,1)} [(r(1-u))^n - r^n] T(z) T^{(z)}(\nu^2)(du). \]

(i) We note that for $r, u \in [0, 1]$ and $n \in \mathbb{N}$

\[ (r(1-u)+u)^n - r^n - nr^{n-1}u(1-r) = \sum_{k=0}^n \binom{n}{k} r^{n-k}(1-u)^n u^k - r^n - u(1-r)nr^{n-1} \]
\[ = \sum_{k=2}^n \binom{n}{k} r^{n-k}u^k(r^{n-k} - r^n) - nr^{n-1}u(1-r)(1 - (1-u)^n)^{-1} \]
\[ = \sum_{k=2}^n \binom{n}{k} r^{n-k}u^k(r^{n-k} - r^n) - n \sum_{k=1}^{n-1} \binom{n-1}{k} (1-u)^n u^{k+1} r^{n-1} - r^n). \]
Similarly for \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \)

\[
(r(1 - u) + u)^n - r^n = \sum_{k=1}^{n} \binom{n}{k} r^{n-k}(1-u)^{n-k}u^k - r^n(1-(1-u)^n) = \sum_{k=1}^{n} \binom{n}{k} (1-u)^{n-k}u^k (r^{n-k} - r^n).
\]

Therefore for \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \)

\[
zr \int_{(0,1)} \left[ (r(1 - u) + u)^n - r^n - \frac{u}{1-u} nr^{n-1}(1-r)1_{(0,1/(1+z))}(u) \right] T^{(2)}(m^1)(du)
\]

\[
= zr \int_{(0,1/(1+z))} \left[ (r(1 - u) + u)^n - r^n - unr^{n-1}(1-r) \right] T^{(2)}(m^1)(du)
\]

\[
- znr^n(1-r) \int_{(0,1/(1+z))} \frac{u^2}{1-u} T^{(2)}(m^1)(du) + zr \int_{[1/(1+z),1]} \left[ (r(1 - u) + u)^n - r^n \right] T^{(2)}(m^1)(du)
\]

\[
= \sum_{k=2}^{n} \binom{n}{k} z \lambda_{n,k}^1 \left( \frac{1}{1+z} \right) (r^{n+1-k} - r^n) + \sum_{k=1}^{n} \binom{n}{k} z \left[ \lambda_{n,k}^1(1) - \lambda_{n,k}^1 \left( \frac{1}{1+z} \right) \right] (r^{n-k} - r^n)
\]

\[
+ (r^{n+1} - r^n) \sum_{k=1}^{n} \binom{n}{k} z \lambda_{n,k+1}^1 \left( \frac{1}{1+z} \right) + zn \int_{(0,1/(1+z))} \frac{u^2}{1-u} T^{(2)}(m^1)(du)
\]

\[
- \sum_{k=2}^{n} \binom{n}{k} z \lambda_{n,k}^1 \left( \frac{1}{1+z} \right) - \sum_{k=1}^{n} \binom{n}{k} z \left[ \lambda_{n,k}^1(1) - \lambda_{n,k}^1 \left( \frac{1}{1+z} \right) \right]
\]

\[
= \sum_{k=2}^{n} \binom{n}{k} z \lambda_{n,k}^1(1) (r^{n+1-k} - r^n) + (r^{n+1} - r^n) \sum_{k=2}^{n} \binom{n}{k} z \left[ k \lambda_{n,k}^1 \left( \frac{1}{1+z} \right) - \lambda_{n,k}^1(1) \right]
\]

\[
+ zn \int_{(0,1/(1+z))} \frac{u^2}{1-u} T^{(2)}(m^1)(du) - n z \left[ \lambda_{n,1}^1(1) - \lambda_{n,1}^1 \left( \frac{1}{1+z} \right) \right].
\]

(ii) For \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \) we observe

\[
(r(1-u))^n - r^n + nu^n = nu^n(1-(1-u)^{n-1}) - r^n(1-(1-u)^n) - nu(1-u)^{n-1}
\]

\[
= r^n \left( n \sum_{k=1}^{n-1} \binom{n-1}{k} (1-u)^{n-k+1} - \sum_{k=2}^{n} \binom{n}{k} (1-u)^{n-k}u^k \right).
\]
Therefore, for \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \)

\[
z(1 - r) \int_{(0,1)} \left[ (r(1 - u)^n - r^n + \frac{u}{1-u}nr^n 1_{(0,1/(1+z))}(v)) \right] T(z)(m^2)(du)
\]

\[
= z(1 - r) \int_{(0,1/(1+z))} \frac{u^2}{1-u} T(z)(m^2)(du)
\]

\[+ z(1 - r) \int_{[1/(1+z), 1]} (r(1 - u)^n - r^n) T(z)(m^2)(du)
\]

\[= (r^{n+1} - r^n) \sum_{k=2}^{n} \binom{n}{k} z \lambda_{n,k}^2 \left( \frac{1}{1+z} \right) - \sum_{k=1}^{n-1} n \binom{n-1}{k} z \lambda_{n,k+1}^2 \left( \frac{1}{1+z} \right) - zn \int_{(0,1/(1+z))} \frac{u^2}{1-u} T(z)(m^2)(du)
\]

\[+ \sum_{k=1}^{n} \binom{n}{k} z \left[ \lambda_{n,k}^2 (1 - \lambda_{n,k}^2 \left( \frac{1}{1+z} \right)) \right]
\]

\[= (r^{n+1} - r^n) \sum_{k=2}^{n} \binom{n}{k} z \left[ \lambda_{n,k}^2 (1 - k \lambda_{n,k}^2 \left( \frac{1}{1+z} \right)) \right] - zn \int_{(0,1/(1+z))} \frac{u^2}{1-u} T(z)(m^2)(du)
\]

(6.6)

\[+ nz \left[ \lambda_{n,1}^2 (1 - \lambda_{n,1}^2 \left( \frac{1}{1+z} \right)) \right].
\]

(iii) For the jump terms due to immigration in the expression for \( L(z) H(n, x) \) given in (6.4), we obtain for \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \)

\[
(r(1 - u) + u)^n - r^n = \sum_{k=1}^{n} r^{n-k}(1 - u)^{n-k}u^k - r^n(1 - (1 - u)^n) = \sum_{k=1}^{n} \binom{n}{k} (1 - u)^{n-k}u^k (r^{n-k} - r^n).
\]

Hence for \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \)

(6.7)

\[
\int_{(0,1)} [(r(1 - u) + u)^n - r^n] T(z)(\nu^1)(du) = \sum_{k=1}^{n} \binom{n}{k} \mu_{n,k}^1 (r^{n-k} - r^n).
\]

Similar computations give for \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \)

(6.8)

\[
\int_{(0,1)} [(r(1 - u))^n - r^n] T(z)(\nu^2)(du) = -r^n \sum_{k=1}^{n} \binom{n}{k} \mu_{n,k}^2.
\]

(iv) Finally, for the terms due to the continuous part of the process \( R(z,r) \) in the expression for \( L(z) H(n, x) \) given in (6.4), we obtain for \( r, u \in [0, 1] \) and \( n \in \mathbb{N} \)

(6.9)

\[
nr^{n-1} \left[ r(1 - r)(b^2 - b^1) + \frac{2r(1 - r)}{z} (c^2 - c^1) \right] = nr^{n+1} - r^n \left[ (b^1 - b^2) + \frac{2}{z}(c^1 - c^2) \right],
\]

and for \( r, u \in [0, 1] \) and \( n \geq 2 \)

(6.10)

\[
n(n-1)r^{n-1} \frac{(1-r)}{z} (c^1(1-r) + c^2 r) = n(n-1)(r^{n+1} - r^n) \frac{(c^1 - c^2)}{z} + n(n-1) \frac{c^1}{z} (r^{n-1} - r^n).
\]
(v) So putting the pieces together we obtain using identities (6.5)-(6.10) in (6.4)

\[\mathcal{L}(z) H(n, r) = \sum_{k=3}^{n} \binom{n}{k} z^{k} \lambda_{n,k}(1)(r^{n+1-k} - r^{n}) + \sum_{k=1}^{n} \binom{n}{k} \mu_{n,k}^{1}(r^{n+k} - r^{n}) - r^{n} \left[\frac{\eta^{2}}{z} + \sum_{k=1}^{n} \binom{n}{k} \mu_{n,k}^{2}\right]
\]

\[+ (r^{n-1} - r^{n}) \left[n(n-1)\frac{c_{1}}{z} + \binom{n}{2} z \lambda_{n,2}(1) + n \frac{n^{1}}{z}\right] + (r^{n+1} - r^{n}) \left[n(n-1)\left(c_{1} - c_{2}\right)\right]
\]

\[+ \sum_{k=2}^{n} \binom{n}{k} z \left[k \lambda_{n,k}^{1} \left(\frac{1}{1+z}\right) - \lambda_{n,k}^{1}(1)\right] + zn \int_{(0,1/(1+z))} \frac{u^{2}}{1-u} T(z)(m^{1})(du)
\]

\[\left[-nz \left[\lambda_{n,1}^{1}(1) - \lambda_{n,1}^{1}(1)\frac{1}{1+z}\right] + \sum_{k=2}^{n} \binom{n}{k} z \left[\lambda_{n,k}^{2}(1) - k \lambda_{n,k}^{2} \left(\frac{1}{1+z}\right)\right]\right]
\]

\[\left[-zn \int_{(0,1/(1+z))} \frac{u^{2}}{1-u} T(z)(m^{2})(du) + nz \left[\lambda_{n,1}^{2}(1) - \lambda_{n,1}^{2}(1)\frac{1}{1+z}\right] + n \left(b^{1} - b^{2}\right) + \frac{2z}{z}(c_{1} - c_{2})\right].
\]

Further computations give for \(r \in [0, 1]\) and \(n \in \mathbb{N}\)

\[\mathcal{L}(z) H(n, r) = \sum_{k=3}^{n} \binom{n}{k} z^{k} \lambda_{n,k}(1) + \binom{n}{k} \mu_{n,k-1}^{1}(r^{n+1-k} - r^{n}) + \mu_{n,n}^{1}(1 - r^{n})
\]

\[+ n(r^{n+1} - r^{n}) \left[b^{1} - b^{2}\right] + \frac{2z}{z}(c_{1} - c_{2})\right] - r^{n} \left[\frac{\eta^{2}}{z} + \sum_{k=1}^{n} \binom{n}{k} \mu_{n,k}^{2}\right]
\]

\[+ (r^{n-1} - r^{n}) \left[n(n-1)\frac{c_{1}}{z} + \binom{n}{2} z \lambda_{n,2}(1) + n \frac{n^{1}}{z} + n \mu_{n,1}^{1}\right] + (r^{n+1} - r^{n}) \left(n(n-1)\left(c_{1} - c_{2}\right)\right]
\]

\[+ \sum_{k=2}^{n} \binom{n}{k} z \left[k \lambda_{n,k}^{1} \left(\frac{1}{1+z}\right) - \lambda_{n,k}^{1}(1)\right] - (\lambda_{n,k}^{1}(1) - \lambda_{n,k}^{2}(1))\]

\[\left[-nz \left[\lambda_{n,1}^{1}(1) - \lambda_{n,1}^{1}(1)\frac{1}{1+z}\right] - (\lambda_{n,1}^{1}(1) - \lambda_{n,1}^{2}(1)\frac{1}{1+z})\right]
\]

\[+ zn \int_{(0,1/(1+z))} \frac{u^{2}}{1-u} T(z)(m^{1})(du) - zn \int_{(0,1/(1+z))} \frac{u^{2}}{1-u} T(z)(m^{2})(du)\right]\]

\[= \sum_{k=2}^{n} \binom{n}{k} \lambda_{n,k}^{1} + \binom{n}{k-1} \mu_{n,k-1}^{1} (r^{n+1-k} - r^{n}) + \mu_{n,n}^{1}(1 - r^{n}) - \alpha_{n} r^{n}
\]

\[+ (r^{n+1} - r^{n}) \left[ns + \sum_{k=2}^{n} \binom{n}{k} \kappa_{k} + \beta_{n}\right] = Q(z) H(n, r).
\]

On the other hand, for the case \(n = 0\), we have that \(\mathcal{L}(z) H(0, r) = 0 = Q(z) H(0, r)\) for \(r \in [0, 1]\). Similarly we have that \(\mathcal{L}(z) H(\Delta, r) = 0 = Q(z) H(\Delta, r)\) for \(r \in [0, 1]\). Therefore the result follows from Proposition 6.1. \(\Box\)

7. THE SPACE OF CSBPs IS HOMEOMORPHIC TO THE SPACES OF LAMBDA COALESCENTS

In this section, we will assume that the processes \(X^{(1)}\) and \(X^{(2)}\) given in (3.3) correspond to equally distributed CB processes with characteristic triplet \((b^{1}, c^{1}, m^{1})\) (i.e. \(\xi^{(i)} = 0\) for \(i = 1, 2\)). After an application of the
culling procedure at the level \( z > 0 \), to the two-dimensional process \((R, Z)\) given by (1.1) and (1.2), we obtain, by Proposition 4.2, that the frequency process \( R^{(z,r)} \) given in (4.1) has an infinitesimal generator given for any \( f \in C^2([0, 1]) \) by

\[
\mathcal{L}^{(z)} f(r) = c^1 \frac{r(1 - r)}{z} f''(r) + z \int_{(0, 1)} \left[ rf(r(1 - u) + u) + (1 - r)f(r(1 - u)) - f(r) \right] T^{(z)}(m^1)(du),
\]

and therefore the process \( R^{(z,r)} \) corresponds to the classic \( \Lambda \)-frequency process, whose dual is the block counting process of a \( \Lambda \)-coalescent. Indeed, by Theorem 6.1 we have that the associated moment dual \( N^{(z,r)} \) has a generator \( Q^z = (q^z_{ij})_{i,j \in \mathbb{N}} \) given by

\[
q^z_{ij} = \begin{cases} 
(i - j + 1) \lambda^1_{i,i-j+1} & \text{if } i \geq 2 \text{ and } j \in \{1, \ldots, i-1\}, \\
0 & \text{otherwise},
\end{cases}
\]

where for \( 2 \leq k \leq i \),

\[
\lambda^1_{i,k} = \int_{(0,1)} [(1 - u)^{i-k}u^{k}] u^{-2} \lambda^1(du),
\]

with \( \lambda^1(du) = \frac{c^1}{z} \delta_0(du) + z y^2 T^{(z)}(m^1)(du) \).

Hence, using this procedure it is natural to map any CB process with the characteristic triplet \((b^1, c^1, m^1)\) to the \( \Lambda \)-coalescent with associated measure given by \( \Lambda^1 \), which can be understood as the genealogy of the CB process \( X_1 \). We observe that under this mapping all the CB processes with the same diffusion term and jump measure are mapped to the same \( \Lambda \)-coalescent. Therefore we will consider the previous mapping from the quotient space obtained by using the equivalence relation in which two CB processes are related if and only if they have the same diffusion term and the same Lévy measure to the space of \( \Lambda \)-coalescents. In this section, we will show that this mapping from the quotient space of CB processes described above, to the genealogy associated with each class is a homeomorphism.

Our strategy in this section is to first show that if the sequence of characteristic triplets associated with a sequence of CB processes converges to the characteristic triplet of some CB process suitably, then the sequence of CB processes converge. Then we show that if a sequence of finite measures on \([0, 1]\) converges to other such measure, then the sequence of their associated \( \Lambda \)-coalescents also converges. These two results induce an easy to check equivalent reformulation of our desired result: the map that sends characteristic triplets of CB processes to finite measures characterizing \( \Lambda \)-coalescents, induced by sending each CB process to its genealogy, is a homeomorphism. This is proved in the final step of the proof.

We will denote by \( \mathcal{LM}(\mathbb{R}) \) the space of Lévy measures, i.e. a positive measure \( m \) belongs to \( \mathcal{LM}(\mathbb{R}) \) if and only if it satisfies the condition \( \int_{\mathbb{R}} (1 \wedge x^2) m(dx) < \infty \).

Now, let us consider \( \Psi \) the space of CB processes, we have seen by (3.1), that each element \( Z \in \Psi \) can be characterized in terms of its branching mechanism \( \psi \), and therefore by its associated triplet \((b, c, m) \in \mathbb{R} \times \mathbb{R}^+ \times \mathcal{LM}(\mathbb{R}) \).

We now provide a criterion for the convergence of a sequence of CB processes in terms of the convergence of the associated sequence of characteristic triplets. Hence, following pg. 244 in [25], for each triplet \((b, c, m) \in \)
In the space of triplets $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{L}M(\mathbb{R})$ we define
\[
\tilde{b} : = b + \int_{\mathbb{R}\setminus\{0\}} \left( \frac{x}{x^2 + 1} - x1_{\{|x| \leq 1\}} \right) m(dx),
\]
\[
\tilde{m} : = c\delta_0(dx) + \frac{x^2}{x^2 + 1} m(dx).
\]
In the space of triplets $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{L}M(\mathbb{R})$ we introduce the following metric:
\[
d_\psi((b^1, c^1, m^1), (b^2, c^2, m^2)) : = |\tilde{b}^1 - \tilde{b}^2| + \rho(\tilde{m}^1, \tilde{m}^2),
\]
where $\rho$ denotes the Prohorov distance in the space of finite measures.

We recall the Skorohod topology on the space of cadlag functions from $\mathbb{R}_+$ to $\mathbb{R}_+$; a sequence $(f_n)_{n \geq 1}$ converges to $f$ in the Skorohod topology if there exists a sequence of homeomorphisms $(\lambda_n)_{n \geq 1}$ of $\mathbb{R}_+$ into itself such that
\[
f_n - f \circ \lambda_n \rightarrow_\lambda Id, \quad \lambda_n \rightarrow Id, \quad \text{uniformly on compact sets.}
\]
Additionally, we consider the uniform Skorohod topology introduced in [7]. Consider a distance $d$ on $[0, \infty]$ which makes it homeomorphic to $[0, 1]$. We say that a sequence $(f_n)_{n \geq 1}$ converges to $f$ in the uniform Skorohod topology if there exists a sequence of homeomorphisms $(\lambda_n)_{n \geq 1}$ of $\mathbb{R}_+$ into itself such that
\[
d(f_n, f \circ \lambda_n) \rightarrow 0 \quad \lambda_n \rightarrow Id, \quad \text{uniformly on } \mathbb{R}_+.
\]
We first provide some auxiliary results which will be needed in the proof of our main result.

**Proposition 7.1.** Let $\{Z^n\}$ be a sequence of continuous-state branching processes with the characteristic triplets $(b_n, c_n, m_n)$. Additionally, consider a continuous-state branching process $Z$ with the characteristic triplet $(b, c, m)$. Assume that
\[
\lim_{n \rightarrow \infty} d_\psi((b_n, c_n, m_n), (b, c, m)) = 0.
\]
Then $Z^n \rightarrow Z$ as $n \rightarrow \infty$, weakly on the space of cadlag paths from $\mathbb{R}_+$ to $[0, \infty]$ with the Skorohod topology if the branching mechanism $\psi$ of $Z$ is nonexplosive, and with the uniform Skorohod topology if $\psi$ is explosive.

**Proof.** For each $n \in \mathbb{N}$ consider a spectrally positive Lévy process $X^n$ with the characteristic triplet $(b_n, c_n, m_n)$; i.e. the Laplace exponent of $X^n$ is given by
\[
\log \mathbb{E}[e^{-\lambda T}] = b\lambda + c\lambda^2 + \int_{(0,\infty)} (e^{-\lambda z} - 1 + z\lambda x1_{(0,1)}(z))m(dz), \quad \lambda \geq 0.
\]
Then using Lemma 13.15 in [25] together with the fact that $\lim_{n \rightarrow \infty} d_\psi((b_n, c_n, m_n), (b, c, m)) = 0$, we obtain that
\[
X^n \Rightarrow X, \quad \text{as } n \rightarrow \infty,
\]
weakly in the space of cadlag paths form $\mathbb{R}_+$ to $\mathbb{R}_+$ endowed with the Skorohod topology, where $X$ is a Lévy process with the characteristic triplet $(b, c, m)$.

By the continuity of the Lamperti transform, as in Corollary 6 in [8], together with (7.1) we obtain that
\[
Z^n \Rightarrow Z, \quad \text{as } n \rightarrow \infty,
\]
weakly on the space of cadlag paths from $\mathbb{R}_+$ to $[0, \infty]$ endowed with the Skorohod topology if the branching mechanism $\psi$ of $Z$ is nonexplosive, and with the uniform Skorohod topology if $\psi$ is explosive. \qed
For our next result, we denote the space of finite measures on $[0, 1]$ by $\mathcal{M}_F([0, 1])$ and let $(\mathcal{P}, d)$ be the space of partitions of the natural numbers endowed with the distance $d$, defined for any two partitions $\pi, \pi' \in \mathcal{P}$ by

$$d(\pi, \pi') = M^{-1} \text{ if and only if } \pi|_M = \pi'|_M \text{ and } \pi|_{M+1} \neq \pi'|_{M+1}$$

where $\pi|_M$ is the restriction of $\pi$ to $[M] = \{1, 2, \ldots, M\}$. With this we mean that given $\pi, \pi'|_M$ is the partition of $[M] = \{1, 2, \ldots, M\}$ constructed by the rule $i, j \in [M]$ are in the same block on $\pi|_M$, i.e., $i \sim j$ in $\pi|_M$, if $i \sim j$ in $\pi$.

Observe that the neighbourhoods in $(\mathcal{P}, d)$ are characterized by the rule $\pi \in A^{1/M}$ if and only if there exists an element of $A$ whose restriction to $[M]$ agrees with the restriction of $\pi$ to $[M]$. Finally, denote $\mathcal{D}_M = \{\pi \in \mathcal{P} : \{M + 1, M + 2, \ldots\} \in \pi\}$ and note that $\mathcal{D} = \bigcup_{i=1}^\infty \mathcal{D}_i$ is a countable dense set. This implies that $(\mathcal{P}, d)$ is separable, which in turn implies that the Prohorov metric can be used to study weak convergence of stochastic processes with trajectories in $(\mathcal{P}, d)$.

**Proposition 7.2.** Let $\{\Pi_N\}_{N \in \mathbb{N}_0}$ be a sequence of $\Lambda$-coalescents with characteristic measures $\{\Lambda^N\}_{N \in \mathbb{N}_0} \subset \mathcal{M}_F([0, 1])$, such that $\Lambda^N \to \Lambda^0$ weakly as $N \to \infty$. Then

$$\Pi^N \to \Pi^0, \quad \text{as } N \to \infty$$

weakly in $\mathbb{D}(\mathbb{R}_+, (\mathcal{P}, d))$.

**Proof.** Note that if $\Pi$ has characteristic measure $\Lambda$, we have that when $\Pi$ is in the state $\pi$ it can jump to the state $\pi'$ if there exist $i \leq k := |\pi|$ such that $\pi'$ can be constructed by merging $i$ blocks of $\pi$. In this case, it jumps from $\pi$ to $\pi'$ at rate $\tilde{\lambda}_{n,n-j+1}$ where

$$\tilde{\lambda}_{k,i} = \int_0^1 u^i(1-u)^{k-i} \Lambda(du)/u^2, \quad \text{for } 2 \leq i \leq k.$$

As $u^{i-2}(1-u)^{k-i}$ is a bounded and continuous function for every $1 < i \leq k$, and $k > 1$, the fact that $\Lambda^N \to \Lambda^0$ weakly as $N \to \infty$, implies that the transitions of the processes $\{\{\Pi^N_t, t > 0\}\}_{N \in \mathbb{N}}$ converge to the transitions of the process $\{\Pi^0_t, t > 0\}$. Note that for every $M \in \mathbb{N}$, $\{\Pi_t|_{[M]}, t > 0\}$ is a continuous-time Markov chain with a finite state-space. Thus the convergence of their transitions implies that

$$\{\Pi^N_t|_{[M]}, t > 0\} \to \{\Pi^0_t|_{[M]}, t > 0\}$$

weakly as $N \to \infty$ in the space of cadlag paths from $\mathbb{R}_+$ to $(\mathcal{P}|_{[M]}, d)$ with the Skorohod topology. Using that the state space $\mathcal{P}|_{[M]}$ is finite, the convergence of the restricted processes and the Skorohod representation theorem, we see that in some probability space $\lim_{N \to \infty} P(\Pi^N_t|_{[M]} = \Pi^0_t|_{[M]}, \forall t \in [0, T]) = 1$. As $\{1, 2, \ldots, M\}$ is the only absorbing state and it is accessible, we can strengthen this to

$$\lim_{N \to \infty} P(\Pi^N_t|_{[M]} = \Pi^0_t|_{[M]} ) = 1.$$

Take $N$ such that $P(\Pi^N_t|_{[M]} = \Pi^0_t|_{[M]} ) > 1 - 1/M$. Then, for any measurable set $A \subset \mathbb{D}(\mathbb{R}_+, (\mathcal{P}, d))$

$$(7.2) \quad P(\Pi^N \in A) \leq P(\Pi^N|_{[M]} \in A|_{[M]}) \leq P(\Pi|_{[M]} \in A^{1/M}|_{[M]}) + 1/M = P(\Pi^N \in A^{1/M}) + 1/M$$

Where in the first inequality we used the containment of events $\{\Pi^N|_{[M]} \in A|_{[M]}\} \subset \{\Pi^N \in A\}$ and in the second, we used the characterization of the neighbourhoods in $(\mathcal{P}, d)$ discussed just before the statement of this result.
From Equation (7.2) we conclude that $\rho(\mathbb{P}(\Pi^N \in \cdot), \mathbb{P}(\Pi \in \cdot)) < 1/M$, where $\rho$ is the Prohorov metric and $M$ is arbitrary. Thus, the proof is complete. 

**Theorem 7.1.** Consider the metric space $L$ of $\lambda$-coalescents with no atom at $\{1\}$ equipped with the Prohorov distance over the space of probability measures defined on the space $\mathcal{D}(\mathbb{R}_+, (\mathcal{P}, d))$. In addition, for $r \in \mathbb{R}$, consider the space $\Psi_r \subset \Psi$ of CB processes with $\hat{b} = r$ equipped with the Prohorov distance over the space of probability measures defined on the space $\mathcal{D}([0, T], \mathbb{R}_+)$ endowed with the uniform Skorohod topology. Then $L$ and $\Psi_r$ are homeomorphic.

Furthermore, consider the mapping $H^{(z)} : \Psi_r \mapsto L$ such that a CB process with the triplet $(b, c, \nu)$ is mapped to the $\Lambda$-coalescent with the associate measure

$$H^{(z)}((b, c, \nu)) = \frac{c}{z} \delta_0 + z y^2 T^{(z)}(\nu).$$

Then, for every $z > 0$, $H^{(z)}$ is a homeomorphism, with inverse $H^{(z)^{-1}}$ sending a $\Lambda$-coalescent to the CB process with characteristic triplet

$$r - \int_{\mathbb{R} \setminus \{0\}\} \left( \frac{x}{x^2 + 1} - x 1_{\{|x| \leq 1\}} \right) \nu(dx), z \Lambda(\{0\}), (z y^2)^{-1} (T^{(z)})^{-1}(\Lambda - \Lambda(\{0\}) \delta_0)$$

**Proof.** As $H^{(z)}$ and its inverse are continuous, which is the content of Propositions 7.1 and 7.2, it remains to show that $H^{(z)}$ is one-to-one and onto.

1. **Onto**
   Chose an arbitrary finite measure $\Lambda$ and note that the branching process with triplet specified in (7.3) is mapped under $H^{(z)}$ to the coalescent with characteristic measure $\Lambda$.

2. **One-to-one**
   Assume that $H^{(z)}((b_1, c_1, \nu_1)) = H^{(z)}((b_2, c_2, \nu_2))$. Then,

   $$\frac{c_1}{z} = H^{(z)}((b_1, c_1, \nu_1))(\{0\}) = H^{(z)}((b_2, c_2, \nu_2))(\{0\}) = \frac{c_2}{z},$$

   which implies that $c_1 = c_2$. Now consider the measurable function $f : (0, \infty) \rightarrow \mathbb{R}$ and write $w = \frac{yz}{1-y}$. We observe that

   $$\int_{(0, \infty)} f(w)\nu_1(dw) = \int_{(0, 1)} f\left(\frac{yz}{1-y}\right) (z y^2)^{-1} z y^2 T^{(z)}(\nu_1)(dy)$$

   $$= \int_{(0, 1)} f\left(\frac{yz}{1-y}\right) (z y^2)^{-1} z y^2 T^{(z)}(\nu_2)(dy) = \int_{(0, \infty)} f(w)\nu_2(dw).$$

   So we conclude that $\nu_1 = \nu_2$. 

**8. The asymmetric Eldon-Wakely coalescent: A minimalistic example**

In order illustrate our results, we study a simple example heuristically. Fix parameters $z, v_1, v_2, y_1, y_2 > 0$, the simple $\Lambda$-asymmetric frequency process is the solution to the SDE

$$dR_t^{(z,r)} = \int_{(0, \infty)} y_1(1 - R_t^{(r)}) 1_{\{v < z R_t^{(r)}\}} N_1(dt, dv) - \int_{(0, \infty)} y_2 R_t^{(r)} 1_{\{v < z (1 - R_t^{(r)})\}} N_2(dt, dv),$$

(8.1) $R_0^{(z,r)} = r \in [0, 1].$

where for $i = 1, 2$, $N_i(dt, dv)$ are independent Poisson random measures on the space $[0, \infty) \times (0, 1)$ with intensity measures $z v_i ds \times dv$ for $i = 1, 2$. 

Let $X^{(1)}$ and $X^{(2)}$ be two CB processes such that the only transitions of $X^{(i)}$ are jumps of size $w_i > 0$ that occur at rate $x_v$ when the process is at the state $x$ for $i = 1, 2$. More formally, for each $i = 1, 2$, let $N^{(i)} = \{ N^{(i)}_t : t \geq 0 \}$ be a Poisson process with intensity parameters $v_i > 0$, and define $Y^{(i)}_t = w_i N^{(i)}_t$ for $t \geq 0$. Then, we define the CB process $X^{(i)}$ by means of the Lamperti transform i.e.

$$
X^{(i)}_t = Y^{(i)}_t \int_0^t X^{(i)}_s \, ds, \quad t \geq 0, \ i = 1, 2.
$$

**Figure 1.** A realization of the $\Lambda$-asymmetric Eldon-Wakeley frequency process, starting from two simple CB processes. In the upper left corner the process $X^{(1)}$ (dark grey) and the process $X^{(2)}$ (light gray) are depicted. $X^{(1)}$ performs jumps of size $w_1$ at rate $v_1$ (the first jump in the picture), while $X^{(2)}$ performs jumps of size $w_2$ at rate $v_2$ (the second jump). The total mass process $Z$ is the sum of the two CB processes. In the lower left corner we draw the ratio process. Note that at the first jump the ratio process makes a jump of size $y_1 = T_z(w_1) = w_1/(z + w_1)$, where $z = X^{(1)}_0 + X^{(2)}_0$. However, at the second jump the total mass is no longer $z$ and the jump of the ratio process is not $T_z(w_2)$. This is an indication that the ratio process is not a Markov process. In the right side of the figure we observe how the Gillespie’s culling procedure allow us to overcome this difficulty. At each sampling point the total mass is returned to $z$, while the ratio is unchanged. Sampling points occurs so often that with probability tending to one no more that one jump occurs between subsequent sampling times. Thus the jumps of the CB process are always pushed forward to jumps of the frequency process by means of the function $T_z$ and the frequency process is Markovian at the sampling times.

If we take $z = x^{(1)}_0 + x^{(2)}_0$ and $r = x^{(1)}_0/z$, we will show that the associated $\Lambda$-asymmetric frequency process $R^{(z,r)}$ is the limit of the culling procedure at level $z$ introduced in Section 4.2. To this end, let $y_i = w_i/(z + w_i)$ and note that, at the position $(z, r)$, $X^{(1)}$ jumps at rate $r z v_1$. At each jump of $X^{(1)}$ the associated frequency process $R$ as defined in (1.2), will jump to the level

$$
(x_1 + w_1)/(z + w_1) = \frac{x_1}{z}(1 - \frac{w_1}{z + w_1}) + \frac{w_1}{z + w_1} = r(1 - y_1) + y_1 = r + y_1(1 - r).
$$

On the other hand at level $(z, r)$, $X^{(2)}$ jumps at rate $(1 - r)z v_2$ and $R$ jumps to the state $r - y_2 r$. From the previous computations and applying the culling procedure in Section 4.2, it is not difficult to show that for $T > 0$

$$
\mathcal{R}^{n,z} \rightarrow R^{(z,r)}, \quad \text{as } n \rightarrow \infty \text{ weakly in } \mathcal{D}([0,T], [0,1]).
$$
where \( R^{(z,r)} \) is the \( \Lambda \)-asymmetric frequency process in (8.1) with parameters \( z, v_1, v_2, y_1, y_2 > 0 \), and \( \tilde{R}^{(n,z)} \) is the jump Markov process with generator (4.16) obtained by the culling procedure introduced in Section 4.2. This is a particular example of Theorem 4.1.

We are now interested in finding the moment dual of \( R^{(z,r)} \), i.e., we will construct the process \( N^{(z,n)} \) such that for every \( t > 0, r \in [0, 1] \) and \( n \in \mathbb{N} \)

\[
E[r N^{(z,n)}_t] = E[(R^{(z,r)}_t)^n].
\]

Let \( A(v_1, v_2, y_1, y_2) := v_2(1 - (1 - y_2)^n) - v_1(1 - (1 - y_1)^n) \) and assume \( A(v_1, v_2, y_1, y_2) > 0 \). We will call \( A(v_1, v_2, y_1, y_2) \) the difference between the total activities, for reasons that will become clear later. Our model will reveal that \( A(v_1, v_2, y_1, y_2) \) is, in some sense, the term under evolutionary selection.

The block counting process of the simple \( \Lambda \)-asymmetric coalescent with parameters \( z, v_1, v_2, y_1, y_2 \), is the asymmetric version of the Eldon-Wakely-Coalescent [13], which is the coalescent arising from reproduction events with constant size. It is the \( \mathbb{N} \) valued process \( N^{(z)} = \{N_t^{(z)} : t \geq 0\} \) with generator

\[
q_{ij} = \begin{cases} 
zv_1 & \text{if } i \geq 2 \text{ and } j \in \{1, \ldots, i - 1\}, \\
zA(v_1, v_2, y_1, y_2) & \text{if } i \in \mathbb{N} \text{ and } j = i + 1, \
0 & \text{otherwise.} \end{cases}
\]

It is not difficult to see that the previous transitions correspond to those given in (6.1) with \( \alpha_i = 0 \) for \( i \geq 1 \), \( \overline{\alpha}_{i,k} = 0 \) for \( 1 \leq k \leq i \), and

- For \( 2 \leq k \leq i \),
  \[
  \overline{\alpha}_{i,k} = zv_1(1 - y_1)^{i-k}y_1^k.
  \]
- \( s = v_2y_21_{(0,1/1+z)}(y_2) - v_1y_11_{(0,1/1+z)}(y_1). \)
- For \( k \geq 2 \)
  \[
  \kappa_k = z \left[ v_1(1 - y_1)^{i-k}y_1^k(1_{(0,1/1+z)}(y_1) - 1) - v_2(1 - y_2)^{i-k}y_2^k(1_{(0,1/1+z)}(y_2) - 1) \right].
  \]
- For \( k \geq 1 \)
  \[
  \beta_k = -kz \left[ v_1(1 - y_1)^{k-1}y_1(1 - 1_{(0,1/1+z)}(y_1)) - v_2(1 - y_2)^{k-1}y_2(1 - 1_{(0,1/1+z)}(y_2)) \right].
  \]

Note that in the first line we have the transitions of a \( \Lambda \)-coalescent with \( \Lambda = v_1 \delta_{y_1} \). Interestingly, only in the second line, we see the parameters \( v_2 \) and \( y_2 \). They are causing branching events that account for the asymmetry between the upper and lower jumps. If \( v_1 = v_2 \) and \( y_1 = y_2 \) the second line is zero and we are left with the Eldon Wakely coalescent with \( \Lambda = v_1 \delta_{y_1} \).

It is also surprising that the branching coefficient is in terms of \( A(v_1, v_2, y_1, y_2) \) and that \( v_1(1 - (1 - y_1)^n) \) is the rate at which one observes an event of any type in an Eldon Wakely coalescent with \( \Lambda = v_1 \delta_{y_1} \). The fact that branching is related to selection allows us to state, in the spirit of Gillespie, that reproduction mechanisms are more likely to go to fixation if they have a larger total activity.

It is possible to use standard techniques to show that \( N^{(z,n)} \) is the moment dual of \( R^{(z,r)} \). The generator of \( R^{(z,r)} \), applied to any \( f \in C^2([0, 1]) \) is given by

\[
\mathcal{L}^{(z)}f(r) = v_1 zr [f(r + y_1(1 - r)) - f(r)] + v_2 z(1 - r) [f(r - y_2r) - f(r)].
\]
Choosing as test function \( f_n(x) = x^n \) we observe that

\[
L^{(z)} f_n(r) = zv_1r[(r + y_1(1 - r))^n - r^n] + zv_2(1 - r)[(r - y_2r)^n - r^n]
\]
\[
= zv_1r[(r + y_1(1 - r))^n - v_1r^n + (1 - r)r^n[v_1 - v_2] + zv_2(1 - r)(r - y_2r)^n
\]
\[
= zv_1 \sum_{k=2}^{n} \binom{n}{k} (1 - y_1)^{n-k} y_1^k [r^{n-k+1} - r^n] + v_1(1 - y_1)^n[r^{n+1} - r^n]
\]
\[
+ zv_2(1 - r)r^n(1 - y_2)^n + (v_2 - v_1)[r^{n+1} - r^n]
\]
\[
= zv_1 \sum_{k=2}^{n} \binom{n}{k} (1 - y_1)^{n-k} y_1^k [r^{n-k+1} - r^n]
\]
\[
+ [zv_2(1 - (1 - y_2)^n) - zv_1(1 - (1 - y_1)^n)][r^{n+1} - r^n] = Q(z) f_r(n),
\]

where \( Q^{(z)} \) is the generator of \( N^{(z,n)} \) and \( f_x(n) = x^n \). This is a special case of Theorem 6.1.

If we take \( v = v_1 = v_2 \) and \( w = w_1 = w_2 = yz/(1 - y) \), we note that this implies \( y = w/(z + w) \). This confirms the fact that the culling procedure at level \( z \) and the duality relationship maps the CB process with the characteristic triplet \((0, 0, v\delta_w)\) as an element of \( \Psi_\tilde{w} \) where \( \tilde{w} = w/(w^2 + 1) - w1_{|w|\leq 1} \) to the \( \Lambda \)-coalescent with \( \Lambda = v\delta_y \). This confirms the result in Theorem 7.1 were we additionally showed that this is a homeomorphism of metric spaces.

However, this is not true if we don’t restrict ourselves to equally distributed CB processes. Indeed, let us consider \( s > 0 \) and the following CB processes in \( \Psi_0 \) given by

\[
X_t^{(i,\varepsilon)} := Y_t^{(i,\varepsilon)} f_n^{(i,\varepsilon)} ds, \quad t \geq 0, i = 1, 2.
\]

where \( Y^{(1,\varepsilon)} \) and \( Y^{(2,\varepsilon)} \) are Poisson process with generating triplets \((b^1, 0, z\varepsilon^{-2}\delta_\varepsilon)\) and \((b^2, 0, (z\varepsilon^{-2} + (s\varepsilon)^{-1})\delta_\varepsilon)\), respectively. Where

\[
b^1 := -z\varepsilon^{-2} \int_{(0,\infty)} \left( \frac{x}{x^2 + 1} - x1_{x \leq 1} \right) \delta_\varepsilon(dx) = \frac{z}{\varepsilon^2 \varepsilon^2 + 1},
\]
\[
b^2 := -\frac{sz - \varepsilon}{s\varepsilon^2} \int_{(0,\infty)} \left( \frac{x}{x^2 + 1} - x1_{x \leq 1} \right) \delta_\varepsilon(dx) = \frac{sz - \varepsilon}{s\varepsilon^2 \varepsilon^2 + 1}.
\]

Now, since

\[
\tilde{m}^1 := z\varepsilon^{-2} \frac{x^2}{x^2 + 1} \delta_\varepsilon(dx) \rightarrow z\delta_0,
\]
\[
\tilde{m}^2 := \frac{sz - \varepsilon}{s\varepsilon^2} \frac{x^2}{x^2 + 1} \delta_\varepsilon(dx) \rightarrow z\delta_0, \quad \text{weakly as } \varepsilon \rightarrow 0.
\]

we have by Proposition 7.1 that for \( i = 1, 2, X^{(i,\varepsilon)} \rightarrow X^{(i,0)} \) as \( \varepsilon \rightarrow 0 \) weakly in \( \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) \) where \( X^{(i,0)} \) is the solution to

\[
X_t^{(i,0)} = x^{(i)} + \int_0^t \sqrt{2zX_s^{(i,0)}} dB_s^{(i)}, \quad t \geq 0,
\]
and $B^{(i)} = \{B^{(i)}_t : t \geq 0\}$ are independent Brownian motions. By Theorem 6.1 the dual process of the associated $\Lambda$-asymmetric frequency process has generator $Q^{(z,0)}$ which satisfies

$$Q^{(z,0)} f_r(n) = \left(\frac{n}{2}\right) [r^{n-1} - r^n].$$

On the other hand let us denote by $R^{(z,r,\varepsilon)}$ the $\Lambda$-asymmetric frequency process associated to the couple of CB processes $(X^{(1,\varepsilon)}, X^{(2,\varepsilon)})$. Then by (8.2) we have that the generator $Q^{(z,\varepsilon)}$ of the dual process of $R^{(z,r,\varepsilon)}$ satisfies

$$Q^{(z,\varepsilon)} f_r(n) = \frac{z^2}{\varepsilon^2} \sum_{k=2}^{n} \binom{n}{k} \left(\frac{z}{z+\varepsilon}\right)^{n-k} \left(\frac{\varepsilon}{z+\varepsilon}\right)^k [r^{n-k+1} - r^n]$$

$$+ \left[ z(\varepsilon^{-2} + (s\varepsilon)^{-1}) \left(1 - \left(\frac{z}{z+\varepsilon}\right)^n\right) - \frac{z^2}{\varepsilon^2} \left(1 - \left(\frac{z}{z+\varepsilon}\right)^n\right) \right] [r^{n+1} - r^n].$$

Therefore

$$\lim_{\varepsilon \to 0} Q^{(z,\varepsilon)} f_r(n) = \left(\frac{n}{2}\right) [r^{n-1} - r^n] + \frac{n}{s} [r^{n+1} - r^n].$$

The fact that for $r \in [0,1]$, and $n \geq 1$, $\lim_{\varepsilon \to 0} Q^{(z,0)} f_r(n) \neq Q^{(z,\varepsilon)} f_r(n)$ implies that the mapping that sends a couple of CB process in $\Psi_0$ with different distributions to their associated $\Lambda$-asymmetric frequency process by the culling procedure and then by the duality relationship to the space of $\Lambda$-coalescent processes is in general not continuous.

9. BIOLOGICAL REMARKS

By studying the ratio of two general CB processes, we have characterized the evolutionary forces that emerge from the differences in the reproduction mechanisms of two competing species. These are:

(1) Selection: Classic selection is visible in the term $s$ of the generator of the dual process, given by (6.1), in the form of branching. We distinguish three sources of selection coming from the difference of the reproduction mechanisms. The first one is unsurprising, the difference between the drift terms $b_1 - b_2$. The second one is related to the difference between the diffusion terms and was first observed by Gillespie (see [16, 17]), having the form $2\varepsilon^{-1}(c_1 - c_2)$. The third one is new in the literature and comes from the difference of the terms associated with the compensation of the jump measures of the CB processes.

(2) Frequency-dependent selection: The term $\beta_i$ in (6.1) corresponds to frequency-dependent selection, and to our knowledge is new in the literature. This frequency-dependent selection term is related to the jump measures of the competing CB processes.

(3) Frequency-dependent variance: As observed by Gillespie in [16, 17], the difference between the diffusion terms $c_1$ and $c_2$ modifies the variance. To be precise Gillespie introduced the Gillespie-Wright-Fisher diffusion, which he obtained as the ratio process of two Feller processes and solves the following SDE:

$$dX_t = X_t(1 - X_t)[b_2 - b_1 + \frac{1}{z}(c_2^2 - c_1^2)]dt + \frac{1}{\sqrt{X_t(1 - X_t)(c_2^2 X_t + c_1^2(1 - X_t))}}dB_t, \quad t \geq 0.$$ 

In [20] a similar population dependent variance was obtained in the context of populations that require different amount of resources to reproduce (efficiency) and in [21] it was shown that the efficiency term has a dual term, which is pairwise branching. We further generalize this by observing that there is an
additional frequency dependent term modifying the variance associated with the jumps measures of the CB processes. These are the terms $\kappa_k$ in the transitions of the dual process in (6.1).

4) Coalescence: The terms $\lambda^1_{i,i-j+1}$ in the generator of the dual process given in (6.1) are associated to coalescence. A novel characterisation of the $\Lambda$-coalescent arises naturally from this work: those that can be obtained as functionals of CB processes in the sense of being dual to an asymmetric frequency process.

5) Mutation: If one allows immigration, then one obtain mutation. Mutation can be found in the terms $\alpha_i$ and $\mu^1_{i,i-j}$ in (6.1). If the immigration is discontinuous, modelling big immigration events, then one obtains coordinated mutation in the sense of [18].

It is a natural direction for future research to study and understand the mechanisms behind the appearance of each of the new terms (like the selection term arising from the jump measures or the frequency-dependent selection term) obtained due to the asymmetry in the dynamics of the CBI processes from which the $\Lambda$-asymmetric frequency process is constructed.

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