Critical level spacing distribution in long-range hopping Hamiltonians

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Abstract. – The nearest level spacing distribution $P_c(s)$ of $d$-dimensional disordered models ($d = 1$ and 2) with long-range random hopping amplitudes is investigated numerically at criticality. We focus on both the weak ($b^d \gg 1$) and the strong ($b^d \ll 1$) coupling regime, where the parameter $b^{-d}$ plays the role of the coupling constant of the model. It is found that $P_c(s)$ has the asymptotic form $P_c(s) \sim \exp[-A_d s^\alpha]$ for $s \gg 1$, with the critical exponent $\alpha = 2 - a_d/b^d$ in the weak coupling limit and $\alpha = 1 + c_d b^d$ in the case of strong coupling.

It is well established that the statistical properties of spectra of disordered one-electron systems are closely related to the localization properties of the corresponding wavefunctions [1–3]. In the metallic phase, the large overlap of delocalized states, which are essentially structureless, induces correlations in the spectrum, leading to the well known level repulsion effect. If the system is invariant under rotation and under time-reversal symmetry (orthogonal symmetry), the normalized spacings $s$ follow Wigner-Dyson statistics at the infinite system size limit:

$$P_W(s) = \frac{\pi}{2} s \exp \left[ -\frac{\pi}{4} s^2 \right].$$

In contrast, in the localized regime, states with close energy levels are typically localized at different parts of space and have an exponentially small overlap. Their levels are therefore uncorrelated and the corresponding spacings are distributed according to the Poisson law

$$P_P(s) = \exp[-s].$$

It has been argued that the statistics of energy levels at the disorder-induced metal-insulator transition (MIT) is characterized by a third universal (i.e., independent of the system size and of the details of the Hamiltonian model) distribution $P_c(s)$, which is different from both Wigner-Dyson statistics and the Poisson statistics [2,4]. The asymptotic behavior of this distribution for $s \gg 1$ has been a controversial issue and still remains unresolved. On the one hand, the influence of the MIT on the spectral properties was studied in Refs. [2,4] by means of the impurity diagram technique combined with scaling assumptions. In these studies, it was conjectured that

$$P_c(s) \sim \exp[-\kappa s], \quad s \gg 1, \quad \kappa > 0.$$
with $\kappa \approx 3.3$, the reason for such behavior being that the Thouless energy at the transition point is of the order of the average level spacing ($E_c/\Delta \approx 1$), and so the levels’s repulsion is effective only for $s \lesssim 1$.

On the other hand, by mapping the energy level distribution onto the Gibbs distribution for a classical one-dimensional gas with a repulsive pairwise interaction, ref. [5] derived the following asymptotic form for $P_c(s)$:

$$P_c(s) \sim \exp[-A_d s^\alpha] , \quad s \gg 1 ,$$

where the coefficient $A_d$ depends only on the dimensionality, $d$, and where the critical exponent, $\alpha$, which ranges in the interval $1 < \alpha < 2$, is related to the correlation length exponent $\nu$ and to the dimensionality through $\alpha = 1 + (d\nu)^{-1}$.

As regards the numerical description of $P_c(s)$, there is also no consensus. The exponential decay, eq. (3), of $P_c(s)$ has been confirmed by most groups at different MITs (see Ref. [6] and references therein), while an exponent $\alpha \approx 1.2$ has been found in refs. [7,8] from a fit in the whole range of spacings to a distribution of the form $P_c(s) = B s \exp[-A s^\alpha]$ or, indirectly, from the two-point correlation function of the density of states [9]. Anyway, the behavior with some nontrivial $1 \leq \alpha \leq 2$ is what one would expect at the mobility edge.

It should be pointed out that MITs generically take place at strong disorder (conventional Anderson transition, quantum Hall transition, transition in $d = 2$ for electrons with strong spin-orbit coupling, etc.). In this regime, the predicted [5] exponent $\alpha = 1 + (d\nu)^{-1}$ slightly deviates from unity, making it relatively difficult to see on the numerically calculated tails of $P_c(s)$ (e.g., at the standard Anderson transition in 3D $\alpha \approx 1.2$). To overcome this problem, it is necessary to investigate transitions which occur at the opposite limit (weak coupling regime). This area has been left unexplored, but one would expect to find an exponent $\alpha$ far from unity and closer to the Wigner-Dyson value $\alpha = 2$.

In this work, we try to definitively solve the existing controversy about the large $s$ asymptotic form of $P_c(s)$. From results of detailed high-precision numerical investigations, we will show unambiguously that eq. (4) is indeed correct, while the validity of eq. (3) is limited to the case of very strong disorder (strictly at the limit of infinity coupling strength). In addition, we find that the exponent $\alpha$ in eq. (4) continuously varies between 1 and 2 as the coupling strength of the Hamiltonian model changes from 0 to $\infty$.

To this end, we performed numerical calculations of $P_c(s)$ on a generalization to $d$ dimensions of the power law random banded matrix (PRBM) model [10–19] (for closely related models see also Ref. [20]). The corresponding Hamiltonian, which describes non-interacting electrons on a disordered $d$-dimensional square lattice with random long-range hopping, is represented by real symmetric matrices, whose entries are randomly drawn from a normal distribution with zero mean, $\langle H_{ij} \rangle = 0$, and a variance which depends on the distance between the lattice sites $r_i$:

$$\langle |H_{ij}|^2 \rangle = \frac{1}{1 + (|r_i - r_j|/b)^{2d}} \times \left\{ \begin{array}{ll} \frac{1}{2}, & i \neq j, \\ 1, & i = j. \end{array} \right.$$

We refer the reader to Ref. [19] (and references therein) for the advantages of the present model with respect to Hamiltonians with short-range, off-diagonal matrix elements, and for the many real systems of interest that can be described by Hamiltonians [5].

The parameter $b^d$ in eq. (5) is an effective bandwidth that serves as a continuous control parameter over a whole line of criticality, i.e., for an exponent equal to $d$ in the hopping elements $H_{ij} \sim b^d$ [21]. Furthermore, it determines the critical dimensionless conductance in the same way as the dimensionality labels the different Anderson transitions.
Each regime is characterized by its respective coupling strength, which depends on the ratio \( \frac{\langle |H_{ii}|^2 \rangle}{\langle |H_{ij}|^2 \rangle} \), which results in a small energy window, containing about 10% of the states around the center of the spectral band. The number of random realizations is such that the number of critical levels included for each \( L \) is roughly \( 1.2 \times 10^8 \), except for the larger system size in 2D, for which this number is about \( 3 \times 10^7 \). In order to reduce edge effects, periodic boundary conditions are included.

For the computation of \( P_c(s) \), we unfold the spectrum in each case to a constant density, and rescale it so as to have the mean spacing equal to unity. In order to diminish the magnitude of the relative fluctuations and to analyze the asymptotic behavior in detail, it is more convenient to consider the cumulative level spacing distribution function \( I(s) = \int_{0}^{\infty} P(s')ds' \). Note that the integration does not change the asymptotic behavior of \( P(s) \). The Wigner surmise, eq. 1, and the Poisson distribution, eq. 2, yield \( I_W(s) = \exp[-\pi s^2 / 4] \) and \( I_P(s) = \exp[-s] \), respectively.

Figure 1 displays our results for the integrated probability \( I_c(s) \) of the 2D system for \( L = 60 \) at \( b^2 = 0.02, 0.04, 0.06, 0.08, 0.1, 0.4, 0.8, 1 \) and 10, which are depicted consecutively from bottom to top. Dotted and dashed lines, which correspond to \( I_W(s) \) and \( I_P(s) \), respectively, are given for comparison. A gradual crossover in the large \( s \) tail of \( I_c(s) \) from the Poisson to the Wigner-Dyson limiting forms as one increases the inverse coupling constant \( b^2 \) of the
model can clearly be seen. So, we can therefore expect an exponent $\alpha$ in eq. (4), which spans the interval $[1, 2]$, in agreement with ref. [5].

Next we consider the behavior of $I_c(s)$ with system size $L$. The results for $s$ large of the critical $I_c(s)$ for the 1D system at different values of $b$ are shown in a log-log scale in fig. 2 for different system sizes: $L = 1000$ (circles), 2000 (squares) and 4000 (diamonds). As in the 2D case (see fig. 1), one can visualize the crossover between the small-$b$ and large-$b$ asymptotics. Note that $I_c(s)$ is an $L$-independent universal scale-invariant function that interpolates, as previously mentioned, between Wigner and Poisson limits. This result confirms the existence of a critical distribution exactly at the transition. Dotted and dashed lines correspond to $I_W(s)$ and $I_P(s)$, respectively. We checked that the normalized variances of $P_c(s)$ are indeed scale-invariant at each critical point studied [23]. The straight line behavior of the data in such a plot at all values of $b$ considered is undoubtedly consistent with a $b$-dependent exponent $\alpha$ in eq. (4). The values of $b$ reported are 0.02, 0.1, 0.2, 0.5, 1.0, 1.5, 2 and 5, from bottom to top. The best fit to eq. (4) in the interval $2.5 \lesssim s \lesssim 5$ for small $b$ and $2.5 \lesssim s \lesssim 4$ for large $b$, yields $\alpha = 1.008, 1.040, 1.085, 1.201, 1.376, 1.600, 1.703$ and 1.866, respectively, thus confirming the result of [5]. Note that for the large energy ranges considered, where $I_c(s)$ vary by one to three orders of magnitude, the quality of the fits, which are represented as solid straight lines, is evident.

The disorder dependence of the critical exponent, $\alpha$, as obtained from the previous fits for the 1D (circles) and 2D (squares) systems is shown in fig. 3 in the broad range of the parameter $b^d$ of the PRBM model. For both dimensions, $d = 1$ and 2, it clearly changes...
continuously from 1 as $b^d \to 0$ to 2 as $b^d \to \infty$. In the two limiting cases of weak ($b^d \gg 1$) and strong ($b^d \ll 1$) disorder regimes it can be well fitted by

$$\alpha = \begin{cases} 2 - a_d/b^d, & b^d \gg 1, \\ 1 + c_d b^d, & b^d \ll 1, \end{cases}$$

respectively. These fits are shown as solid lines in fig. 4. The fitting parameters are $a_1 = 0.60$, $a_2 = 0.33$, $c_1 = 0.42$ and $c_2 = 0.76$. Note that the different values of these parameters reflect its dependence on the dimensionality. From eq. (6), the Poissonian tail of $P_c(s)$, eq. (3), is recovered for large spacings at the limit of very strong coupling $b^d \to 0$. So, we conclude that in the case of very strongly coupled Hamiltonians only, eq. (4) loses its validity and eq. (3) applies.

The observed $b^d$-dependence of $\alpha$ in eq. (6) is not surprising, since other critical properties, such as the spectral compressibility $\chi$, or the correlation dimension $d_2$ in 1D, for which analytical treatment is feasible, present a similar behavior towards $b$. Specifically, $\chi = 1/2\pi b$ ($b \gg 1$), $\chi = 1 - 4b$ ($b \ll 1$), $d_2 = 1 - 1/\pi b$ ($b \gg 1$), and $d_2 = 2b$ ($b \ll 1$) were derived in refs. [10,12,13]. We stress that eq. (6) is based on numerical results and at present it should be considered as a conjecture. So, further analytical work is needed to check this form of the critical exponent and its origin from the model (5).

Finally, we present the limiting behavior of $P_c(s)$ as $s \to 0$. From general considerations for the orthogonal symmetry $P_c(s) \sim s$ at small $s \ll 1$ [2,24]. The results for the 1D case at different values of $b$ for various system sizes, $L = 1000$ (circles), 2000 (squares) and 4000 (diamonds), are plotted in fig. 4. Dotted and dashed lines are the Wigner surmise, eq. (1), and the Poisson distribution, eq. (2), respectively. Here we find that $P_c(s) \sim s$ for all disorder regimes in accordance with the predictions of refs. [2,24]. The slopes of straight lines fitting the data in the intervals shown are 1.63, 6.07, and 11.30 at $b = 5$, 0.1 and 0.05, respectively.
We have checked that, for the 2D case, the same linear behavior of $P_c(s)$ towards $s$ is fulfilled in the whole range considered of the parameter $b^2$.

To summarize, we have investigated the critical level spacing distribution $P_c(s)$ of non-interacting electrons on a $d$-dimensional disordered system with long-range transfer terms in the whole range of the coupling constant $b^{-d}$. $P_c(s)$ is found to be scale independent at all values of $b^{-d}$. $P_c(s)$ is found to be scale independent at all values of $b^{-d}$. The large $s$ part of $P_c(s)$ obtained is shown to have an exp$[-A_d s^\alpha]$ decay with $1 \leq \alpha \leq 2$. We determined the disorder dependence of $\alpha$ in both the strong ($b^d \ll 1$) and the weak ($b^d \gg 1$) coupling regimes. At the limit of very strong disorder $b^d \to 0$, we found that $\alpha \to 1$ and so we obtain the expected results of the Poissonian decay predicted in refs. [2, 4].

The small-$s$ behavior of $P_c(s) \sim s$ is in agreement with the analytical predictions at all values of $b^d$.

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