MULTIPLE SOLUTIONS TO THE PLANAR PLATEAU PROBLEM

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Abstract. We give existence and nonuniqueness results for simple planar curves with prescribed geodesic curvature.

1. Introduction

We are interested in the planar Plateau problem: Given two points \(p_1\) and \(p_2\) in the plane and a smooth function \(k : \mathbb{R}^2 \times [0,1] \to \mathbb{R}\), find an immersed curve \(\gamma \in C^2([0,1],\mathbb{R}^2)\), such that \(\gamma(0) = p_1, \gamma(1) = p_2\), and for every \(t \in [0,1]\) the (signed) geodesic curvature \(k_\gamma(t)\) of \(\gamma\) at \(t\),

\[
k_\gamma(t) := |\dot{\gamma}(t)|^{-3} \langle \ddot{\gamma}(t), J\dot{\gamma}(t) \rangle,
\]

is given by \(k(\gamma(t), t)\), where \(J\) denotes the rotation by \(\pi/2\). We choose the orientation, such that the circle of radius \(r\) with counterclockwise parameterization has positive curvature \(r^{-1}\). Without loss of generality after a rotation and a translation we may assume that \(p_1 = (a,0)\) and \(p_2 = (-a,0)\) for some \(a > 0\). Then the planar Plateau problem is equivalent to the following ordinary differential equation

\[
\ddot{\gamma}(t) = |\dot{\gamma}(t)|^{-3} \langle \ddot{\gamma}(t), J\dot{\gamma}(t) \rangle,
\]

\[
\gamma(0) = (a,0), \gamma(1) = (-a,0),
\]

If the function \(k \equiv k_0\) is constant, by elementary geometry, the planar Plateau problem is only solvable for \(|k_0| \leq a^{-1}\); the solutions in this case are given by subarcs connecting \((a,0)\) and \((-a,0)\) of \(n\)-fold iterates of a circle of radius \(|k_0|\) with clockwise or counterclockwise parameterization depending on the sign of \(k_0\). If the analysis is restricted to simple solutions, then there are 2 solutions if \(|k_0| < a^{-1}\), the small and the large solution corresponding to the subarcs subtending an angle strictly smaller or strictly larger than \(\pi\). If \(k_0 = \pm a^{-1}\) then the unique simple solution is given by the half circle lying above or below the \(x\)-axis depending on the sign of \(k_0\). We will be mainly interested in the case when \(k\) is a positive function.

If the prescribed curvature function is independent of the variable \(t\), then the planar Plateau problem is ‘geometric’, in the sense that the set of solutions is invariant under reparameterizations. If in this case the function \(k\)

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satisfies $\|k\|_\infty < a^{-1}$, then from [1] there exists a stable solution $\gamma_s$ to $(P)$. We refer to $\gamma_s$ as a small solution. In the higher dimensional case and in the context of $H$-surfaces analogous results are given in [10,11]. For closed curves with prescribed curvature we refer to [4,6,16,17].

Concerning the existence of a second, large solution for non-constant functions $k$ there are only perturbative results, i.e. the function $k$ is assumed to be close to a constant $k_0$, see [4]. Concerning the existence of a large $H$-surface we refer to [3,18,19], if $H$ is constant, and to [2,7,14,15,20,21] for non-constant functions $H$.

We give existence criteria for a large solution, that are non-perturbative.

**Theorem 1.1.** Let $a > 0$ and $k \in C(\mathbb{R}^2 \times [0,1], \mathbb{R})$ be given, such that

$$0 < \inf_{\mathbb{R}^2 \times [0,1]} k \leq \sup_{\mathbb{R}^2 \times [0,1]} k < a^{-1},$$

then there is a simple curve that solves (1.1). If, moreover,

$$\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} k(x,t) \leq a + 1 < \inf_{(x,t) \in \mathbb{R}^2 \times [0,1]} k(x,t),$$

then equation (1.1) possesses at least two simple solutions.

To illustrate the pinching condition (1.2) we note that the assumptions of Theorem 1.1 are satisfied, if

$$\frac{1}{2}a^{-1} < \inf_{\mathbb{R}^2 \times [0,1]} k \quad \text{and} \quad \sup_{\mathbb{R}^2 \times [0,1]} k < a^{-1}.$$  

The small solution is found in the set

$$M_{\text{small}} := \{ \gamma \in C^2([0,1], \mathbb{R}^2) : \gamma(0) = (a,0), \gamma(1) = (-a,0),$$

$\gamma \oplus [-a,a]$ is simple, $|\dot{\gamma}(0)|^{-1}\dot{\gamma}(0) \in \{e^{i\theta} : \pi/2 < \theta < \pi\}$,

$|\dot{\gamma}(1)|^{-1}\dot{\gamma}(1) \in \{e^{i\theta} : \pi < \theta < 3\pi/2\}\},$$

whereas the large solution belongs to

$$M_{\text{large}} := \left\{ \gamma \in C^2([0,1], \mathbb{R}^2) : \gamma(0) = (a,0), \gamma(1) = (-a,0),$$

$\gamma \oplus [-a,a]$ is simple, $\frac{\dot{\gamma}(0)}{|\gamma(0)|} \in \{e^{i\theta} : -\pi/2 < \theta < \pi\}$,

$\frac{\dot{\gamma}(1)}{|\gamma(1)|} \in \{e^{i\theta} : \pi < \theta < 5\pi/2\}$, and

$$\left(\frac{\dot{\gamma}(0)}{|\gamma(0)|}, \frac{\dot{\gamma}(1)}{|\gamma(1)|}\right) \in \{e^{i\theta} : -\pi/2 < \theta < \pi/2\} \text{ or}$$

$$\left(\frac{\dot{\gamma}(0)}{|\gamma(0)|}, \frac{\dot{\gamma}(1)}{|\gamma(1)|}\right) \in \{e^{i\theta} : 3/2\pi < \theta < 5\pi/2\}\right\},$$
where we define for a curve \( \gamma \in C^0([0, L], \mathbb{R}^2) \) connecting \((a, 0)\) and \((-a, 0)\) the closed curve \( \gamma \oplus [-a, a] \in C([0, L + 2a], \mathbb{R}^2) \) by
\[
\gamma \oplus [-a, a](t) := \begin{cases} 
\gamma(t) & 0 \leq t \leq L \\
(-a + t - L, 0) & L \leq t \leq L + 2a.
\end{cases}
\]
The existence result is proved by using the Leray-Schauder degree and suitable apriori estimates, i.e. we show that the degree of (1.1) with respect to \( M_{\text{small}} \) equals 1 and is given by −1, when computed in the set \( M_{\text{large}} \).

The existence result then follows, since a non vanishing degree gives rise to a solution. The degree approach is interesting in itself and yields the flexibility to deal with functions \( k \) that depend on \( x \) and \( t \), for instance if \( k \) does only depend on \( t \), then the existence result shows that in contrast to the four vertex theorem for simple closed curves of prescribed curvature (see [8, 9]) there is no additional condition on \( k \) besides the \( L^\infty \)-bound for the corresponding boundary value problem. Moreover, the degree argument gives the perspective to be applied to the higher dimensional case as well, e.g. to surfaces in \( \mathbb{R}^3 \) with prescribed mean curvature.

2. Apriori estimates

**Lemma 2.1.** Let \( \gamma \in C^2([0, L], \mathbb{R}^2) \) be a unit speed curve with positive geodesic curvature connecting \((a, 0)\) and \((-a, 0)\), such that the closed curve \( \gamma \oplus [-a, a] \in C([0, L + 2a], \mathbb{R}^2) \) is simple. If
\[
\dot{\gamma}(0) = e^{i\theta_0} \quad \text{for some } \pi/2 \leq \theta_0 < \pi \quad \text{and}
\]
\[
\dot{\gamma}(L) = e^{i\theta_L} \quad \text{for some } \pi < \theta_L \leq \frac{3}{2}\pi,
\]
then \( \gamma \) is a graph over the \( x_1 \)-axis and there is a strictly decreasing \( C^2 \)-function \( \theta : [0, L] \to [\theta_L, \theta_0] \) such that
\[
\dot{\gamma}(t) = e^{i\theta(t)}.
\]

If
\[
\dot{\gamma}(0) = e^{i\theta_0} \quad \text{for some } -1/2\pi \leq \theta_0 < \pi \quad \text{and}
\]
\[
\dot{\gamma}(L) = e^{i\theta_L} \quad \text{for some } \pi < \theta_L \leq \frac{5}{2}\pi,
\]
then there are a strictly increasing $C^2$-function $\theta : [0, L] \to [\theta_0, \theta_L]$ such that

$$\dot{\gamma}(t) = e^{i\theta(t)}$$

and $0 \leq t_0 < t_1 \leq L$ such that $\gamma$ restricted to $[0, t_0]$, $[t_0, t_1]$, and $[t_1, L]$ is a graph over the $x_1$-axis.

Proof. We define the tangent angle $\theta : [0, L] \to \mathbb{R}$ of $\gamma$ as the unique continuous map such that $\theta(0) = \theta_0$ and

$$\dot{\gamma}(t) = e^{i\theta(t)}$$

for all $t \in [0, L]$.

Since the curvature of $\gamma$ is positive, the tangent angle $\theta$ is strictly increasing. We apply Hopf’s rotation angle theorem [12,13] to the simple positive oriented curve $\gamma \oplus [-a, a]$ and find that the rotation angle of $\gamma \oplus [-a, a]$ is exactly $2\pi$. Consequently,

$$2\pi = \theta(L) + (2\pi - \theta_L),$$

such that $\theta(L) = \theta_L$. The curve $\gamma$ fails to be a graph over the $x_1$-axis, if $\theta(t)$ crosses $\pi/2$ or $3\pi/2$. Since $\theta$ is strictly increasing, this can happen at most two times in the interval $(0, L)$. This yields the claim. \hfill \Box

Lemma 2.2. Let $\gamma \in C^2([0, L], \mathbb{R}^2)$ be a unit speed curve with positive geodesic curvature connecting $(a, 0)$ and $(-a, b)$, such that

$$\dot{\gamma}(t) = e^{i\theta(t)},$$

for some strictly increasing function $\theta \in C^0([0, L], \mathbb{R})$ satisfying $\pi/2 \leq \theta(0) < \pi$ and $\pi < \theta(L) \leq 3\pi/2$. Then

$$\min\{k_\gamma(t) : t \in [0, L]\} \leq a^{-1}.$$

Proof. Consider the upper half of the ball centered at $(0, 0)$ and radius $a$

$$B^+_a := \{(x, y) \in \mathbb{R}^2 : |x| \leq a, y \geq 0, x^2 + y^2 \leq a^2\},$$

$$C^+_a := \{(x, \sqrt{a^2 - x^2}) \in \mathbb{R}^2 : |x| \leq a\},$$

and

$$s_1 := \sup\{s \in \mathbb{R} : (0, s) + \gamma \cap B^+_a \neq \emptyset\}.$$

Obviously, there holds $s_1 \geq \max\{0, -b\}$. If $s_1 > \max\{0, -b\}$, then $s_1 + \gamma$ and $B^+_a$ intersect in a point $(s_1, 0) + \gamma(t_0)$ with $t_0 \in (0, L)$ and $s_1 + \gamma$ lies above $B^+_a$. From the maximum principle the curvature of $\gamma$ at $\gamma(t_0)$ is smaller than $a^{-1}$. If $s_1 = 0$, then $\gamma$ lies above $B^+_a$ and $\theta(0)$ has to be $\pi/2$, such that the slope of $\gamma$ and $C^+_a$ coincide at $(a, 0)$. Writing $\gamma$ and $C^+_a$ as graphs over the $x_2$-axis the maximum principle shows that the curvature of $\gamma$ at $(a, 0)$ is smaller than $a^{-1}$. If $s_0 = -b > 0$ then $\theta(L) = 3\pi/2$ and as above we deduce $k_\gamma(L) \leq a^{-1}$. \hfill \Box
Lemma 2.3. Let $\gamma \in C^2([0, L], \mathbb{R}^2)$ be a unit speed curve with positive geodesic curvature connecting $(a, 0)$ and $(-a, b)$, such that
\[ \dot{\gamma}(t) = e^{i\theta(t)}, \]
for some strictly increasing function $\theta \in C^0([0, L], \mathbb{R})$ satisfying $\pi/2 = \theta(0)$. Moreover, if $b > 0$, we assume that $\theta(L) = 3\pi/2$, and if $b \leq 0$, we assume that $\pi < \theta(L) \leq 3\pi/2$. Then
\[ \max\{k(\gamma(t)) : t \in [0, L]\} \geq a^{-1}. \]

Proof. The curve $\gamma$ may be written as a graph over the interval $[-a, a]$ for some function $g \in C^0([-a, a], \mathbb{R}) \cap C^2((-a, a), \mathbb{R})$. Let $G$ be set defined by
\[ G := \{(x, y) \in \mathbb{R}^2: -a \leq x \leq a, y \leq g(x)\}. \]
Due to the positive curvature of $\gamma$ the set $G$ is convex and
\[ G \cap \{(x, y) \in \mathbb{R}^2 : x \in \{\pm a\}, y > g(x)\} = \emptyset. \]
As in the proof of Lemma 2.2 we consider $C^+_a$ and
\[ s_0 := \sup\{s \in \mathbb{R} : (0, s) + C^+_a \cap G \neq \emptyset\}, \]
Obviously, there holds $s_0 \geq \max\{0, b\}$. If $s_0 > \max\{0, b\}$, then $(0, s_0) + C^+_a$ and $G$ intersect in a point $(t_0, g(t_0))$ with $|t_0| < a$ and $(0, s_0) + C^+_a$ lies above $G$. From the maximum principle the curvature of $\gamma$ at $(t_0, g(t_0))$ is bigger than $a^{-1}$. If $s_0 = 0$, then $C^+_a$ lies above $G$. Since $\theta(0) = \pi/2$ the slope of $\gamma$ and $C^+_a$ coincide at $(a, 0)$. From the maximum principle we deduce that $k(\gamma(0)) \geq a^{-1}$. If $s_0 = b > 0$ then the slope of $\gamma$ and $(0, b) + C^+_a$ coincide at $(-a, b)$ and the maximum principle shows that $k(\gamma(L)) \geq a^{-1}$. \qed

Lemma 2.4. Let $\gamma \in C^2([0, L], \mathbb{R}^2)$ be a unit speed curve with positive geodesic curvature connecting $(a, 0)$ and $(-a, 0)$, such that the closed curve $\gamma \oplus [-a, a]$ is simple and $\dot{\gamma}(L) \in \{e^{i\theta} : \pi < \theta \leq 5/2\pi\}$. If $\dot{\gamma}(0) = e^{-i\pi/2}$, then the maximum $k_{\max}$ and the minimum $k_{\min}$ of the geodesic curvature of $\gamma$ satisfy
\[ k_{\min} \leq \frac{k_{\max}}{k_{\max} a + 1}. \]

Proof. We apply Lemma 2.1 write
\[ \dot{\gamma}(t) = e^{i\theta(t)}, -\pi/2 < \theta(t) \leq 5/2\pi, \]
and denote by $t_0$ the point such that
\[ t_0 := \sup\{t \in [0, L] : \theta(s) \leq \pi/2 \text{ for all } 0 \leq s \leq t\}. \]
By Lemma 2.1 there holds $t_0 < L$, $\theta(t_0) = \pi/2$, and $\theta(\cdot)$ is strictly increasing. Consequently, after a rotation by $\pi$, we may apply Lemma 2.3 and deduce that $\gamma(t_0) = (x_0, y_0)$ with $x_0 \geq a + 2k_{\max}^{-1}$.
We denote by $t_1$ the point
\[ t_1 := \sup\{t \in [t_0, L] : \theta(s) < 3/2\pi \text{ for all } t_0 \leq s \leq t\}. \]
Since $\gamma \oplus [a, -a]$ is simple, we have $\gamma(t_1) = (x_1, y_1)$ for some $x_1 \leq -a$ (if $t_1 < L$, then $x_1 < -a$). From Lemma 2.2 applied to $\gamma$ restricted to $[t_0, t_1]$ we see that
\[
k_{\text{min}} \leq (a + k_{\text{max}})^{-1},
\]
which yields the claim. \hfill \Box

We define for a given curvature function $k \in C(\mathbb{R}^2 \times [0, 1], \mathbb{R})$ the set of small solutions $L_{\text{small}}(k)$ and large solutions $L_{\text{large}}(k)$ by
\[
L_{\text{small}}(k) := \{ \gamma \in M_{\text{small}} : \gamma \text{ solves } (1.1) \},
\]
\[
L_{\text{large}}(k) := \{ \gamma \in M_{\text{large}} : \gamma \text{ solves } (1.1) \}.
\]

**Lemma 2.5.** Let $\{k_s \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^+) : s \in [0, 1] \}$ be a continuous family of prescribed curvature function, such that
\[
\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} k_s(x,t) a < 1,
\]
\[
\inf_{(x,t) \in \mathbb{R}^2 \times [0,1]} k_s(x,t) > 0
\]
Then the set
\[
L_{\text{small}} := \{ \gamma \in M_{\text{small}} : \gamma \text{ solves } (1.1) \text{ for some } k \in \{k_s\} \}
\]
is compact in $C^2([0, 1], \mathbb{R}^2)$. If, moreover, for all $s \in [0, 1]$
\[
\frac{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_s(x,t)\}}{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_s(x,t)\} a + 1} < \inf_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_s(x,t)\}
\]
then
\[
L_{\text{large}} := \{ \gamma \in M_{\text{large}} : \gamma \text{ solves } (1.1) \text{ for some } k \in \{k_s\} \}
\]
is compact in $C^2([0, 1], \mathbb{R}^2)$.

**Proof.** We first show that that $L_{\text{large}}$ and $L_{\text{small}}$ are closed. To this end we observe that any $\gamma \in L_{\text{large}} \cup L_{\text{small}}$ is parameterized proportional to its arclength.

Let $(\gamma_n)$ be a sequence in $L_{\text{small}}$ converging to $\gamma_0$ in $C^2([0, 1], \mathbb{R}^2)$. Choosing a subsequence, we may assume that $\gamma_n$ is a solution to (1.1) with $k = k_{s_n}$ for some sequence $(s_n)$ converging to $s_0 \in [0, 1]$. Thus, $\gamma_0$ solves (1.1) with $k = k_{s_0}$. Using the maximum principle and the positive curvature of $\gamma_0$ it is easy to see that the curve $\gamma_0$ cannot touch itself or the straight line $[-a, a]$ tangentially, such that $\gamma_0 \oplus [-a, a]$ remains simple as a limit of simple curves and
\[
|\gamma_0(0)|^{-1}\dot{\gamma}_0(0) \in \{e^{i\theta} : 1/2\pi \leq \theta < \pi \},
\]
\[
|\gamma_0(1)|^{-1}\dot{\gamma}_0(1) \in \{e^{i\theta} : \pi < \theta \leq 3/2\pi \}.
\]
Since
\[
\sup \{k_{s_0}(x,t) : (x,t) \in \mathbb{R}^3\} a < 1
\]
by Lemma 2.3 it is impossible that

$$|\gamma_0(0)|^{-1} \gamma_0(0) = e^{i\pi/2} \text{ or } |\gamma_0(1)|^{-1} \gamma_0(1) = e^{i3\pi/2}.$$ 

Consequently, $\gamma_0$ is contained in $L_{small}$.

Let $(\gamma_n)$ be a sequence in $L_{large}$ converging to $\gamma_0$ in $C^2([0,1], \mathbb{R}^2)$. As above, we may deduce that $\gamma_0$ is a solution to (1.1) with $k = k_{s_0}$ for some $s_0 \in [0,1]$, $\gamma_0 \oplus [-a,a]$ is simple, and satisfies

$$|\gamma_0(0)|^{-1} \gamma_0(0) \in \{e^{i\theta} : -\pi/2 \leq \theta < \pi\},$$

$$|\gamma_0(1)|^{-1} \gamma_0(1) \in \{e^{i\theta} : \pi < \theta \leq 5/2\pi\},$$

and at least one of the following two conditions holds

$$\dot{\gamma}(0) |\dot{\gamma}(0)|^{-1} \in \{e^{i\theta} : -\pi/2 \leq \theta \leq \pi/2\},$$

$$\dot{\gamma}(1) |\dot{\gamma}(1)|^{-1} \in \{e^{i\theta} : 3/2\pi \leq \theta \leq 5\pi/2\}$$

Using Lemma 2.4 and the fact that

$$\frac{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_{s_0}(x,t)\}}{\sup_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_{s_0}(x,t)\} a + 1} < \frac{\inf_{(x,t) \in \mathbb{R}^2 \times [0,1]} \{k_{s_0}(x,t)\}}{\{k_{s_0}(x,t)\}}$$

we exclude the possibility that

$$\dot{\gamma}(0) |\dot{\gamma}(0)|^{-1} = e^{-i\pi/2} \text{ or } \dot{\gamma}(1) |\dot{\gamma}(1)|^{-1} = e^{i5\pi/2}.$$ 

If neither

$$\dot{\gamma}(0) |\dot{\gamma}(0)|^{-1} \in \{e^{i\theta} : \theta < \pi/2\}$$

nor

$$\dot{\gamma}(1) |\dot{\gamma}(1)|^{-1} \in \{e^{i\theta} : 3/2\pi < \theta\}$$

then Lemma 2.3 leads to a contradiction. Thus, $\gamma_0$ belongs to $L_{large}$.

To show the compactness of $L_{large}$ and $L_{small}$ we fix a sequence $(\gamma_n)$ of solutions in $L_{large} \cup L_{small}$. Since $\gamma_n \oplus [-a,a]$ is simple we may apply the Gauß-Bonnet formula and get

$$2\pi = \alpha_{1,n} + \alpha_{2,n} + \int_{\gamma_n} k_{\gamma_n},$$

where $\alpha_{1,n}, \alpha_{2,n} \in (-\pi/2, \pi)$ are the outward angles at $t = 0$ and $t = 1$ of the piecewise $C^2$ curve $\gamma_n \oplus [-a,a]$. Consequently,

$$L(\gamma_n) \inf_{(x,t,s) \in \mathbb{R}^2 \times [0,1]} \{k_s(x,t)\} \leq \int_{\gamma_n} k_{\gamma_n} \leq 3\pi,$$

where $L(\gamma_n)$ denotes the length of $\gamma_n$. Hence, $L(\gamma_n)$ is uniformly bounded, which yields a uniform $C^1$-bound of $\gamma_n$. Using the equation (1.1) and the Arzela-Ascoli theorem we may extract a subsequence of $(\gamma_n)$, which converges in $C^2([0,1], \mathbb{R}^2)$. This finishes the proof. \qed
3. The Leray-Schauder degree

For \( a > 0 \) we consider the affine space

\[
C^2_{a,-a}([0,1], \mathbb{R}^2) := \{ \gamma \in C^2([0,1], \mathbb{R}^2) : \gamma(0) = \left(\begin{array}{c} a \\ 0 \end{array}\right) \text{ and } \gamma(1) = \left(\begin{array}{c} -a \\ 0 \end{array}\right) \}.
\]

The operator \( L_k \) is defined by

\[
L_k : C^2_{a,a}([0,1], \mathbb{R}^2) \to C^2_{a,a}([0,1], \mathbb{R}^2)
\]

\[
L_k(\gamma) := ( - D^2_{\gamma} )^{-1} \left( - \dot{\gamma} + |\dot{\gamma}(\cdot)| k(\gamma(\cdot), \cdot) J(\dot{\gamma}(\cdot)) \right),
\]

where the operator \( D^2_{\gamma} \) is considered as an isomorphism

\[
D^2_{\gamma} : C^2_{a,a}([0,1], \mathbb{R}^2) \to C^0([0,1], \mathbb{R}^2).
\]

Since

\[
|\dot{\gamma}(\cdot)| k(\gamma(\cdot), \cdot) J(\dot{\gamma}(\cdot)) \in C^0([0,1], \mathbb{R}^2)
\]

depends only on \( \gamma \) and \( \dot{\gamma} \), the map

\[
\gamma \mapsto ( - D^2_{\gamma} )^{-1} \left( |\dot{\gamma}(\cdot)| k(\gamma(\cdot), \cdot) J(\dot{\gamma}(\cdot)) \right)
\]

is compact from \( C^2_{a,a}([0,1], \mathbb{R}^2) \) to itself. Thus \( L_k \) is of the form \( \text{Id} \) – compact and the Leray-Schauder degree of \( L_k \) is defined.

Fix \( a > 0 \) and a function \( k \in C(\mathbb{R}^2 \times [0, 1], \mathbb{R}) \) satisfying

\[
0 < \inf_{\mathbb{R}^2 \times [0, 1]} k \leq \sup_{\mathbb{R}^2 \times [0, 1]} k < a^{-1},
\]

\[
\frac{\sup{(x,t)\in\mathbb{R}^2\times[0,1]} k(x,t)}{\sup{(x,t)\in\mathbb{R}^2\times[0,1]} k(x,t)a + 1} < \inf_{(x,t)\in\mathbb{R}^2\times[0,1]} k(x,t).
\]

We define for \( s \in [0, 1] \) the function \( k_s \in C^0(\mathbb{R}^2 \times [0, 1], \mathbb{R}) \) by

\[
k_s(x,t) := (1 - s) \left( \frac{\sup{(x,t)\in\mathbb{R}^2\times[0,1]} k(x,t)}{\inf_{(x,t)\in\mathbb{R}^2\times[0,1]} k(x,t)} \right) + s k(x,t).
\]

Then the family \( \{ k_s : s \in [0, 1] \} \) satisfies the assumptions of Lemma 2.5 and the sets \( L_{\text{large}} \) and \( L_{\text{small}} \) are compact. Thus, there is \( R > 0 \) such that

\[
L_{\text{large}} \cup L_{\text{small}} \subset \{ \lambda \in C^2([0, 1], \mathbb{R}^2) : \|\lambda\|_{C^2([0, 1], \mathbb{R}^2)} < R \}.
\]

Consequently, if we define the open sets

\[
M_{\text{small}, R} := \{ \lambda \in M_{\text{small}} : \|\lambda\|_{C^2([0, 1], \mathbb{R}^2)} < R \},
\]

\[
M_{\text{large}, R} := \{ \lambda \in M_{\text{large}} : \|\lambda\|_{C^2([0, 1], \mathbb{R}^2)} < R \},
\]

then from the homotopy invariance of the degree

\[
\deg(L_k, M_{\text{small}, R}, 0) = \deg(L_{k_0}, M_{\text{small}, R}, 0),
\]

\[
\deg(L_k, M_{\text{large}, R}, 0) = \deg(L_{k_0}, M_{\text{large}, R}, 0).
\]

(3.1)

To compute the degree of \( L_{k_0} \), we note that solutions to (1.1) with a constant function \( k_0 \) are given by curves with constant geodesic curvature \( k_0 \), i.e.
subarcs of a $n$-fold iterate of a circle with radius $k_0^{-1}$. Thus the required simplicity and the bounds on the slope yields

\[
L_{small}(k_0) = \{ \gamma_s(t) := k_0^{-1} e^{i(\alpha_0 + \omega_s t)} - ik_0^{-1} \sin(\alpha_0) \},
\]
\[
L_{large}(k_0) = \{ \gamma_b(t) := k_0^{-1} e^{i(-\alpha_0 + \omega_b t)} + ik_0^{-1} \sin(\alpha_0) \},
\]

where

\[
\alpha_0 := \arccos(k_0 a) \in (0, \pi/2),
\]
\[
\omega_s := \pi - 2\alpha_0 \in (0, \pi),
\]
\[
\omega_b := \pi + 2\alpha_0 \in (\pi, 2\pi).
\]

Consequently, we have

\[
\deg(L_k, M_{small,R}, 0) = \deg_{loc}(DL_{k_0}|_{\gamma_s}, 0),
\]
\[
\deg(L_k, M_{large,R}, 0) = \deg_{loc}(DL_{k_0}|_{\gamma_b}, 0).
\] (3.2)

To compute the local degree’s we note for $V \in C^2_{0,0}([0, 1], \mathbb{R}^2)$ and $* \in \{s, b\}$

\[
DL_{k_0}|_{\gamma_*}(V) = (-D^2_t)^{-1} \left( -\dot{V} + (\gamma_* \dot{V}) |_{\gamma_*}^{-1} k_0 J(\dot{\gamma}_*) + |\dot{\gamma}_*| k_0 J(\dot{V}) \right)
\]
\[
= (-D^2_t)^{-1} \left( -\dot{V} - \omega_s (ie^{i(\alpha_0 + \omega_* t)} \dot{V}) e^{i(\alpha_0 + \omega_* t)} \right)
\]
\[
+ \omega_s J(\dot{V}) \right).
\]

For $\lambda \in [-1, 1]$ we consider the family of operators $A_\lambda : C^2_{0,0}([0, 1], \mathbb{R}^2) \to C^2_{0,0}([0, 1], \mathbb{R}^2)$ defined by

\[
A_\lambda(V) := (-D^2_t)^{-1} \left( -\dot{V} - (1 - \lambda) \omega_s (ie^{i(\alpha_0 + \omega_* t)} \dot{V}) e^{i(\alpha_0 + \omega_* t)} \right)
\]
\[
+ (1 + \lambda) \omega_s J(\dot{V}) \right).
\]

Writing

\[
V(t) = \alpha(t)e^{i(\alpha_0 + \omega_* t)} + \beta(t)ie^{i(\alpha_0 + \omega_* t)},
\] (3.3)

for some $\alpha, \beta \in C^2_{0,0}([0, 1], \mathbb{R})$ we find

\[
A_{\lambda,*}(V) = (-D^2_t)^{-1} \left( (\gamma(t) - \omega^2_s \alpha(t)) e^{i(\alpha_0 + \omega_* t)} \right)
\]
\[
+ \left( -\beta(t) - (1 - \lambda) \omega_s \dot{\alpha}(t) - \lambda \omega^2_s \beta(t) \right) ie^{i(\alpha_0 + \omega_* t)} \right) \]

The eigenvalues of the problem

\[
\hat{\varphi}(t) = \lambda \varphi(t) \quad \text{for} \quad t \in [0, 1] \quad \text{and} \quad \varphi(0) = \varphi(1) = 0
\]

are given by

\[
\{ \pi^2 n^2 : n \in \mathbb{N} \}. \] (3.4)
Since \( \omega_s < \pi \), each \( A_{\lambda,s} \) is injective and due to its form, identity-compact, \( A_{\lambda,s} \) is invertible for each \( \lambda \in [0,1] \). By the homotopy invariance of the degree we obtain

\[
\deg_{\text{loc}}(DL_{k_0}\vert_{\gamma_s}, 0) = \deg_{\text{loc}}(A_{1,s}, 0) = \deg_{\text{loc}}(id, 0) = 1, \tag{3.5}
\]

where we used for the second equality the admissible homotopy \( \{B_\sigma : \sigma \in [0,1]\} \) given by

\[
B_\sigma(V) := (-D^2_t)^{-1}(-\ddot{V} + 2(1-\sigma)\omega_s J(\dot{V})).
\]

To compute the degree of \( DL_{k_0}\vert_{\gamma_b} \) we note that by the above analysis and the homotopy property we may replace \( k_0 \) by some constant \( k_1 \) close to \( a \) without changing the degree, such that we may assume

\[
\pi < \omega_b < \sqrt{2}\pi. \tag{3.6}
\]

Moreover, using the homotopy \( \{A_{\lambda,b} : \lambda \in [-1,0]\} \), we see that

\[
\deg_{\text{loc}}(DL_{k_0}\vert_{\gamma_b}, 0) = \deg_{\text{loc}}(A_{-1,b}, 0).
\]

To compute \( \deg_{\text{loc}}(A_{-1,b}, 0) \) we consider the decomposition

\[
C^2_{0,0}([0,1], \mathbb{R}^2) = U_1 \oplus U_2,
\]

where

\[
U_1 := \{V \in C^2_{0,0}([0,1], \mathbb{R}^2) : \int_0^1 V(t) \cdot (\sin(\pi t)e^{i(\alpha_0 + \omega_b t)}) dt = 0\},
\]

\[
U_2 := \text{span}(\sin(\pi t)e^{i(\alpha_0 + \omega_b t)}).
\]

Using the decomposition in (3.3) we fix \( V_1 \in U_1 \setminus \{0\} \) and \( V_2 \in U_2 \setminus \{0\} \),

\[
V_1(t) = \alpha(t)e^{i(\alpha_0 + \omega_b t)} + \beta(t)e^{i(\alpha_0 + \omega_b t)},
\]

\[
V_2(t) = \lambda \sin(\pi t)e^{i(\alpha_0 + \omega_b t)}.
\]

From (3.4) and (3.6) we obtain

\[
\langle D_t A_{-1,b}(V_1), D_t V_1 \rangle_{L^2([0,1], \mathbb{R}^2)}
\]

\[
= \langle -(D_t)^2 A_{-1,b}(V_1), V_1 \rangle_{L^2([0,1], \mathbb{R}^2)}
\]

\[
= \int_0^1 (-\ddot{\alpha}(t) - \omega_b^2 \alpha(t))\alpha(t) + (-\ddot{\beta}(t) - 2\omega_b \dot{\alpha}(t) + \omega_b^2 \beta(t))\beta(t) dt
\]

\[
= \int_0^1 (\ddot{\alpha}(t))^2 - 2\omega_b^2 (\alpha(t))^2 + (\ddot{\beta}(t) - \omega_b \alpha(t))^2 + \omega_b^2 (\beta(t))^2 dt
\]

\[
\geq (4\pi^2 - 2\omega_b^2)(\alpha(t))^2 + \omega_b^2 (\beta(t))^2
\]

\[
> 0,
\]

\[
\langle D_t A_{-1,b}(V_2), D_t V_2 \rangle_{L^2([0,1], \mathbb{R}^2)} = \lambda^2 \int_0^1 (\pi^2 - \omega_b^2)(\sin(\pi t))^2 dt
\]

\[
= \frac{1}{2} \lambda^2 (\pi^2 - \omega_b^2) < 0,
\]
and
\[ \langle D_t A^{-1,\beta}(V_1), D_t V_2 \rangle_{L^2([0,1],\mathbb{R}^2)} = \lambda \int_0^1 \left( -\ddot{\alpha}(t) - \omega_b^2 \alpha(t) \right) \sin(\pi t) dt \]
\[ = \lambda (\pi^2 - \omega_b^2) \int_0^1 \alpha(t) \sin(\pi t) dt = 0. \]
Thus the following homotopy is admissible
\[ [0, 1] \ni \sigma \mapsto \sigma C + (1 - \sigma)A_{-1,b}, \]
where \( C \in \mathcal{L}(C^2_{0,0}([0,1],\mathbb{R}^2), C^2_{0,0}([0,1],\mathbb{R}^2)) \) is given in the decomposition \( U_1 \oplus U_2 \) by
\[ C := \begin{pmatrix} \text{id} & 0 \\ 0 & -1 \end{pmatrix}. \]
From the above computations we finally see that
\[ \text{deg}_{loc}(DL_{k_0}|_{\gamma_b}, 0) = \text{deg}_{loc}(C, 0) = -1, \]
which yields together with (3.2) the proof of Theorem 1.1 announced in the introduction.

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