ON THE GROWTH OF VON NEUMANN DIMENSION OF HARMONIC SPACES OF SEMIPOSITIVE LINE BUNDLES OVER COVERING MANIFOLDS

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ABSTRACT. We study the harmonic space of line bundle valued forms over a covering manifold with a discrete group action \( \Gamma \), and obtain an asymptotic estimate for the \( \Gamma \)-dimension of the harmonic space with respect to the tensor times \( k \) in the holomorphic line bundle \( L^k \otimes E \) and the type \( (n, q) \) of the differential form, when \( L \) is semipositive. In particular, we estimate the \( \Gamma \)-dimension of the corresponding reduced \( L^2 \)-Dolbeault cohomology group. Essentially, we obtain a local estimate of the pointwise norm of harmonic forms with valued in semipositive line bundles over Hermitian manifolds.

1. INTRODUCTION

The purpose of this paper is to study the growth of the von Neumann dimension of the space of harmonic forms with values in powers of an invariant semipositive line bundle over a Galois covering of a compact Hermitian manifold. The main technical tool will be an estimate of the Bergman kernel on a compact set of a Hermitian manifold, which generalizes a result of Berndtsson [3] for compact manifolds.

Let \( (X, \omega) \) be a Hermitian (paracompact) manifold of dimension \( n \) and \( (L, h^L) \) and \( (E, h^E) \) be Hermitian holomorphic line bundles over \( X \). For \( k \in \mathbb{N} \) we form the Hermitian line bundles \( L^k := L^\otimes k \) and \( L^k \otimes E \), the latter endowed with the metric \( h_k = (h^L)^{\otimes k} \otimes h^E \).

To the metrics \( \omega, h^L \) and \( h^E \) we associate the Kodaira Laplace operator \( \Box_k \) acting on forms with values in \( L^k \otimes E \) and also \( L^2 \) spaces of forms with values in \( L^k \otimes E \), and \( \Box_k \) has a (Gaffney) self-adjoint extension in the space of \( L^2 \)-forms, denoted by the same symbol.

The space \( \mathcal{H}^{p,q}(X, L^k \otimes E) \) of harmonic \( L^k \otimes E \)-valued \( (p, q) \)-forms is defined as the kernel of (the self-adjoint extension of) \( \Box_k \) acting on the \( L^2 \) space of \( (p, q) \)-forms.

In this paper we mainly work with \( (n, q) \)-forms. Since \( \mathcal{H}^{n,q}(X, L^k \otimes E) \) is separable, let \( \{s_j^k\}_{j \geq 1} \) be an orthonormal basis and denote by \( B_k^2 \) the Bergman density function defined by

\[
B_k^2(x) = \sum_{j=1}^{\infty} |s_j^k(x)|_{h_k^\omega}^2, \quad x \in X,
\]

where \(| \cdot |_{h_k^\omega} \) is the pointwise norm of a form. Definition (1.1) is independent of the choice of basis.

The first main result of this paper is a uniform estimate of the Bergman density function for semipositive line bundles in a neighborhood of a compact subset of a Hermitian manifold.

Theorem 1.1. Let \( (X, \omega) \) be a Hermitian manifold of dimension \( n \) and \( (L, h^L) \) and \( (E, h^E) \) be Hermitian holomorphic line bundles over \( X \). Let \( K \subset X \) be a compact subset and assume that \( (L, h^L) \) is semipositive on a neighborhood of \( K \).
Then there exists $C > 0$ depending on the compact set $K$, the metric $\omega$ and the bundles $(L_i, h_i^2)$ and $(E, h^2)$, such that for any $x \in K$, $k \geq 1$ and $q \geq 1$,
\begin{equation}
B_k^q(x) \leq Ck^{n-q},
\end{equation}
where $B_k^q(x)$ is the Bergman kernel function \[(1.1)\] of harmonic $(n, q)$-forms with values in $L^k \otimes E$.

For $X$ compact and $K = X$, Theorem \[1.1\] reduces to \[3\] Theorem 2.3. Theorem \[1.1\] will be used to obtain the following bounds for the von Neumann dimension of the harmonic spaces on covering manifolds.

**Theorem 1.2.** Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ on which a discrete group $\Gamma$ acts holomorphically, freely and properly such that $\omega$ is a $\Gamma$-invariant Hermitian metric and the quotient $X/\Gamma$ is compact. Let $(L_i, h_i^2)$ and $(E, h^2)$ be two $\Gamma$-invariant holomorphic Hermitian line bundles on $X$. Assume $(L_i, h_i^2)$ is semipositive on $X$. Then there exists $C > 0$ such that for any $q \geq 1$ and $k \geq 1$ we have
\begin{equation}
\dim_{\Gamma} \mathcal{H}^{n-q}(X, L^k \otimes E) \leq Ck^{n-q}, \quad \dim_{\Gamma} \mathcal{H}^{0,q}(X, L^k \otimes E) \leq Ck^{n-q}.
\end{equation}
The same estimate also holds for the reduced $L^2$-Dolbeault cohomology groups,
\begin{equation}
\dim_{\Gamma} H^{0,q}_{(2)}(X, L^k \otimes E) \leq Ck^{n-q}.
\end{equation}

For a compact manifold $X$, the growth of the dimension of the Dolbeault cohomology $H^{0,q}(X, L^k \otimes E)$ as $k \to \infty$ is of fundamental importance in algebraic and complex geometry and is linked to the structure of the manifold, cf. \[8, 9, 15\]. If $(L, h^2)$ is positive, then $H^{0,q}(X, L^k \otimes E) = 0$ for $q \geq 1$ and $k$ large enough, by the Kodaira-Serre vanishing theorem \[15\] Theorem 1.5.6. This reflects the fact that the remaining cohomology space $H^{0,0}(X, L^k \otimes E)$ is rich enough to provide a projective embedding of $X$, for large $k$.

Assume now that $(L, h^2)$ is semipositive. The solution of the Grauert-Riemenschneider conjecture by Demailly \[9\] and Siu \[17\] shows that $\dim H^{0,q}(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$ for $q \geq 1$. This can be used to show that $X$ is a Moishezon manifold, if $(L, h^2)$ is moreover positive at at least one point. Berndtsson \[3\] showed that we have actually $\dim H^{0,q}(X, L^k \otimes E) = O(k^{n-q})$ as $k \to \infty$ for $q \geq 1$. Note that the latter estimate can be proved by induction on the dimension if $X$ is projective, see \[10\] (6.7) Lemma. For the Bergman kernel $B_k^q$ on $(n, 0)$-forms with values in a semipositive line bundle, it was shown by Hsiao-Marinescu-Todor \[12\] Theorem 1.7 that it has an asymptotic expansion on the set where the curvature is strictly positive.

The study of $L^2$ cohomology spaces on coverings of compact manifolds has also interesting applications, cf. \[11, 14\]. The results are similar to the case of compact manifolds, but we have to use the reduced $L^2$ cohomology groups and von Neumann dimension instead of the usual dimension. For example, in the situation of Theorem \[1.2\] if the invariant line bundle $(L_i, h_i^2)$ is positive, the Andreotti-Vesentini vanishing theorem \[1\] shows that $\bar{H}^{0,q}_{(2)}(X, L^k \otimes E) \cong \mathcal{H}^{0,q}(X, L^k \otimes E) = 0$ for $q \geq 1$ and $k$ large enough. The holomorphic Morse inequalities of Demailly \[9\] generalized to coverings by Chiose-Marinescu-Todor \[16, 18\] (cf. also \[15\] (3.6.24)) and yield in the conditions of Theorem \[1.2\] that $\dim_{\Gamma} H^{0,q}_{(2)}(X, L^k \otimes E) = o(k^n)$ as $k \to \infty$ for $q \geq 1$. Hence Theorem \[1.2\] generalizes \[3\] to covering manifolds and refines the estimates obtained in \[16, 18\].

Note also that the magnitude $k^{n-q}$ in \[1.3\] cannot be improved in general \[3\] Proposition 4.2.
Our paper is organized in the following way. In the section 2, we introduce the notations and recall the necessary facts. In the section 3, we prove some properties of harmonic line bundle valued forms, including $\overline{\partial}\partial$-formulas on non-compact manifolds and submean-value formulas, which imply Theorem 1.1. In the section 4, we prove our main results and a corollary, and explain that Theorem 1.1 implies Theorem 1.2.

2. Preliminaries

We introduce here the notations and recall the necessary facts used in the paper. Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ and $(L, h^L)$ and $(F, h^F)$ be Hermitian holomorphic line bundles over $X$. Let $\Omega^{p,q}(X, F)$ be the space of smooth $(p, q)$-forms on $X$ with values in $F$ for $p, q \in \mathbb{N}$. The curvature of $(F, h^F)$ is defined by $R^F = \overline{\partial}\partial\log|s|^2_{h^F}$ for any local holomorphic frame $s$, then the Chern-Weil form of the first Chern character of $F$ is $c_1(L, h^L) = \frac{1}{2\pi} R^F$, which is a real $(1, 1)$-forms on $X$. The volume form is given by $dv_X = \omega_n$, where $\omega_n := \frac{\omega^\otimes n}{n!}$ for $1 \leq q \leq n$.

We will use several times the notion of positive $(p, p)$-form, for which we refer to [8], Chapter III, §1, (1.1) (1.2) (1.5) (1.7). If a $(p, p)$-form $T$ is positive, we write $T \geq 0$. Given two $(p, p)$-forms $T_1$, $T_2$ we say that $T_1 \geq T_2$, if $T_1 - T_2 \geq 0$. From definitions the following statement follows easily: if $T \geq 0$ is a $(p, p)$-form and $u \geq 0$ is a $(1, 1)$-form, then $T \wedge u \geq 0$.

The $L^2$-scalar product on $\Omega^{p,q}(X, F)$ is given by $\langle s_1, s_2 \rangle = \int_X \langle s_1(x), s_2(x) \rangle_{h^F, \omega} dv_X(x)$, where $\langle \cdot, \cdot \rangle_{h^F, \omega}$ is the pointwise Hermitian inner product induced by $\omega$ and $h^F$. We denote by $L^2_{p,q}(X, F)$, the $L^2$ completion of $\Omega^{0,0}_0(X, F)$, which is the subspace of $\Omega^{p,q}(X, F)$ consisting of elements with compact support.

Let $\overline{\partial}^F : \Omega^{p,q}_0(X, F) \to L^2_{p,q+1}(X, F)$ be the Dolbeault operator and let $\overline{\partial}^F_{\text{max}}$ be its maximal extension (see [15], Lemma 3.1.11). From now on we still denote the maximal extension by $\overline{\partial}^F := \overline{\partial}^F_{\text{max}}$ and the associated Hilbert space adjoint by $\overline{\partial}^{F^*} := (\overline{\partial}^F_{\text{max}})^*_{\text{H}}$ for simplifying the notations. Consider the complex of closed, densely defined operators $L^2_{p,q-1}(X, F) \xrightarrow{\overline{\partial}^F} L^2_{p,q}(X, F) \xrightarrow{\overline{\partial}^{F^*}} L^2_{p,q+1}(X, F)$, then $(\overline{\partial}^F)^2 = 0$.

By [15], Proposition 3.1.2, the operator defined by

$$\text{Dom}(\Box^F) = \{ s \in \text{Dom}(\overline{\partial}^F) \cap \text{Dom}(\overline{\partial}^{F^*}) : \overline{\partial}^F s \in \text{Dom}(\overline{\partial}^{F^*}), \overline{\partial}^{F^*} s \in \text{Dom}(\overline{\partial}^F) \},$$

(2.1) $\Box^F s = \overline{\partial}^F \overline{\partial}^{F^*} s + \overline{\partial}^{F^*} \overline{\partial}^F s$ for $s \in \text{Dom}(\Box^F)$,

is a positive, self-adjoint extension of Kodaira Laplacian, called the Gaffney extension.

**Definition 2.1.** The space of harmonic forms $\mathcal{H}^{p,q}(X, F)$ is defined by

$$\mathcal{H}^{p,q}(X, F) := \text{Ker}(\Box^F) = \{ s \in \text{Dom}(\Box^F) \cap L^2_{p,q}(X, F) : \Box^F s = 0 \}.$$

The $q$-th reduced $L^2$-Dolbeault cohomology is defined by

$$\overline{\mathcal{F}}^{0,q}_0(X, F) := \frac{\text{Ker}(\overline{\partial}^F) \cap L^2_{0,q}(X, F)}{[\text{Im}(\overline{\partial}^{F^*}) \cap L^2_{0,q}(X, F)]},$$

where $[V]$ denotes the closure of the space $V$.

According to the general regularity theorem of elliptic operators (also cf. [15], Theorem A.3.4), $s \in \mathcal{H}^{p,q}(X, F)$ implies $s \in \Omega^{p,q}(X, F)$. The Bergman kernel function defined in
(1.1) is well-defined by an adaptation of [7] Lemma 3.1 to form case. By weak Hodge decomposition (cf. [15] (3.1.21)(3.1.22)), we have a canonical isomorphism

\[ H^{0,q}(X, F) \cong \mathcal{H}^{0,q}(X, F) \]

for any \( q \in \mathbb{N} \), which associates to each cohomology class its unique harmonic representative.

A Hermitian manifold \((X, \omega)\) is called complete, if all geodesics are defined for all time for the underlying Riemannian manifold. By [15] Corollary 3.3.3, 3.3.4, if \( X \) is complete, then \( i\partial \bar{\partial} \Omega^{p,q}(X, F) \) is essentially self-adjoint.

3. Some properties of harmonic line bundle valued forms

In this section, we work under the following general setting, and later the covering manifold with a group action \( \Gamma \) will be treated as a special case in the section 4. Let \((X, \omega)\) be a Hermitian manifold of dimension \( n \) and \((F, h^F)\) be a holomorphic Hermitian line bundle on \( X \). For the Kodaira Laplacian \( \square := \square^F \) we denote still by \( \square \) its (Gaffney) self-adjoint extension.

3.1. The \( \square \)-Bochner formula for non-compact manifolds. The following \( \square \)-formula was obtained by B. Berndtsson in [3] and [4] for \( \square \)-closed, line bundle valued, \((n, q)\)-forms over compact manifolds. We can rephrase it for \( \square \)-closed, line bundle valued, \((n, q)\)-forms over any compact subset of a Hermitian (possibly non-compact) manifold. The proof is analogue to [3, Proposition 2.2] and [4, Proposition 6.2], thus we omit it here.

Let \( \alpha \in \mathcal{H}^{n,q}(X, F) \). We define now a positive \((n - q, n - q)\) form \( T_\alpha \) on \( X \) as follows. Let \( U \) be an open set such that \( F|_U \) is trivial and let \( e_F \) be a local holomorphic frame on \( U \) and set \( |e_F|^2_h = e^{-\psi} \). Write \( \alpha|_U = \xi \otimes e_F \), with \( \xi \in \Omega^{n,q}(U, \mathbb{C}) \). The \((n - q, n - q)\) form \( T_\alpha \) is defined locally by \( T_\alpha|_U := i^{(n-q)^2} (\ast \xi) \wedge (\ast \xi)^* h^{-\psi} \), where \( \ast : \Omega^{n,q}(U, \mathbb{C}) \to \Omega^{n-q,0}(U, \mathbb{C}) \) is the Hodge star operator associated to the metric \( \omega \) given by \( \xi \wedge \ast \xi = |\xi|^2_\omega \). It is easy to check that \( T_\alpha \) is well-defined globally.

We have a \( F \)-valued \((n - q, 0)\) form \( \gamma_\alpha \in \Omega^{n-q,0}(X, F) \) associated to \( \alpha \) defined locally by \( \gamma_\alpha|_U := (\ast \xi) \otimes e_F \), which is also well defined globally. Let \( L := \omega \wedge \cdot \) be the Lefschetz operator on \( \Omega^{p,q}(X, F) \) and let \( \Lambda \) be its dual operator defined by \( \langle \Lambda \gamma, \gamma \rangle_{h^F, \omega} = \langle \gamma, L\gamma \rangle_{h^F, \omega} \).

Theorem 3.1. Let \((F, h^F)\) be a holomorphic Hermitian line bundle over a Hermitian manifold \((X, \omega)\). Assume \( \alpha \in \mathcal{H}^{n,q}(X, F) \), \( q \geq 1 \), and \( K \subset X \) is a compact subset. Then there exist non-negative constants \( C_1 \) and \( C_2 \) depending on \( \omega \) and \( K \), such that

\[ \langle i\partial \bar{\partial} (T_\alpha \wedge \omega_{q-1}) \rangle \geq (\langle 2\pi c_1(F, h^F) \wedge \Lambda \alpha, \alpha \rangle_{h^F, \omega} - C_1 |\alpha|_{h^F, \omega}^2 + C_2 \bar{\partial} F \gamma_\alpha |\gamma_\alpha|_{h^F, \omega}^2 \rangle_{h^F, \omega} \]

\[ \geq (\langle 2\pi c_1(F, h^F) \wedge \Lambda \alpha, \alpha \rangle_{h^F, \omega} - C_1 |\alpha|_{h^F, \omega}^2) \omega_n \]

on \( K \). Here \( \langle c_1(F, h^F) \wedge \Lambda \alpha, \alpha \rangle_{h^F, \omega} \omega_n = c_1(F, h^F) \wedge T_\alpha \wedge \omega_{q-1} \) on \( X \). In particular, if \( X \) is Kählerian, then

\[ i\partial \bar{\partial} (T_\alpha \wedge \omega_{q-1}) = (i\partial \bar{\partial} T_\alpha) \wedge \omega_{q-1} \]

\[ = (\langle 2\pi c_1(F, h^F) \wedge \Lambda \alpha, \alpha \rangle_{h^F, \omega} + |\bar{\partial} F \gamma_\alpha |\gamma_\alpha|_{h^F, \omega}^2) \omega_n \]

on \( K \). Above \( \langle \cdot, \cdot \rangle_{h^F, \omega} \) and \( |\cdot|_{h^F, \omega} \) denote the pointwise Hermitian metric and norm on \( F \)-valued differential forms induced by \( \omega \) and \( h^F \).
Based on the same argument as in Theorem 3.1 and the \(\overline{\partial}\)-closed case in [3] Proposition 2.2, we have the following equality for Kähler manifolds, which generalizes both the above formula (3.2) and the Kähler case of [3] Proposition 2.2]. For compact Kähler manifolds, a general formula of this type can be found in [5].

**Corollary 3.2.** Let \((F, h^F)\) be a holomorphic Hermitian line bundle over a Kähler manifold \((X, \omega)\). Assume \(\alpha \in \Omega^{n, q}(X, F) \cap \text{Dom}(\overline{\partial}^F) \cap \text{Dom}(\overline{\partial}^{\overline{\partial}^F})\) such that \(\overline{\partial}^F \alpha = 0\) and \(q \geq 1\). Then,

\[
(3.3) \quad i\overline{\partial}(T_\alpha \wedge \omega_{q-1}) = (i\overline{\partial}T_\alpha) \wedge \omega_{q-1} = -2\text{Re}(\overline{\partial}^F \overline{\partial}^{\overline{\partial}^F} \alpha, \alpha)_{h^F, \omega} \omega_n + \langle 2\pi c_1(F, h^F) \wedge \Lambda \alpha, \alpha \rangle_{h^F, \omega_n} + |\overline{\partial}^{\overline{\partial}^F} \alpha|^2_{h^F, \omega_n} + |\overline{\partial} \gamma_n|^2_{h^F, \omega_n}
\]

where \(\langle \cdot, \cdot \rangle_{h^F, \omega}\) and \(| \cdot |^2_{h^F, \omega}\) are the pointwise Hermitian metric and norm on \(F\)-valued differential forms induced by \(\omega^F\) and \(h^F\).

**3.2. Submeanvalue formulas of harmonic forms in \(\mathcal{H}^{n,q}(X, L^k \otimes E)\).** Let \((L, h^L)\) and \((E, h^E)\) be Hermitian holomorphic line bundles over \(X\). For any compact subset \(K\) in \(X\), the interior of \(K\) is denoted by \(\mathring{K}\). Let \(K_1, K_2\) be compact subsets in \(X\), such that \(K_1 \subset K_2\). Then there exists a constant \(c_0 = c_0(\omega, K_1, K_2) > 0\) such that for any \(x_0 \in K_1\), the local exponential normal coordinate around \(x_0\) is \(V = W \subset \mathbb{C}^n\), where

\[
W := B(c_0) := \{z \in \mathbb{C}^n : |z| < c_0\}, \quad V := B(x_0, c_0) \subset \mathring{K}_2 \subset K_2,
\]

\[
z(x_0) = 0, \quad \text{and } \omega(z) = \sqrt{-1} \sum_{i,j} h_{ij}(z) dz_i \wedge d\overline{z}_j \text{ with } h_{ij}(0) = \frac{1}{2} \delta_{ij} \text{ and } dh_{ij}(0) = 0.
\]

**Lemma 3.3.** Let \((X, \omega)\) be a Hermitian manifold of dimension \(n\) and \((L, h^L)\) and \((E, h^E)\) be Hermitian holomorphic line bundles over \(X\). Let \(K_1\) and \(K_2\) be compact subsets in \(X\) such that \(K_1 \subset K_2\). Assume \(L \geq 0\) in \(K_2\) and \(q \geq 1\). Then there exists a constant \(C > 0\) depending on \(\omega, K_1, K_2\) and \((E, h^E)\), such that

\[
(3.4) \quad \int_{|z| < r} |\alpha|^2_{h_{|\omega|}} dV_X \leq Cr^2q \int_X |\alpha|^2_{h_{\omega \omega}} dV_X
\]

for any \(\alpha \in \mathcal{H}^{n,q}(X, L^k \otimes E)\) and \(0 < r < \frac{c_0}{2}\), where \(| \cdot |^2_{h_{\omega \omega}}\) is the pointwise Hermitian norm induced by \(\omega, h^L\) and \(h^E\).

**Proof.** For simplifying notations, we denote by \(\langle \cdot, \cdot \rangle_{h}\) and \(| \cdot |_h\) the associated pointwise Hermitian metrics and norms here, and their meaning will be clear from the context. For \(0 < t < c_0\), define \(\sigma(t) := \int_{|z| < t} |\alpha|^2_{h_{\omega \omega}} dV_X = \int_{|z| < t} T_{\alpha} \wedge \omega_q\). Assume \(|\alpha|^2_{2,q} := \int_X |\alpha|^2_{h_{\omega \omega}} dV_n = 1\). Then this lemma says that: There exists a constant \(C\), which is independent of the point \(x_0\) and \(k\) in \(L^k \otimes E\), such that

\[
(3.5) \quad \sigma(r) \leq C r^{2q},
\]

when \(0 < r < c_0/2^n\) (eventually we will use the special case \(r = \frac{2}{\sqrt{k}}\) as \(k \to \infty\)).

From the Theorem 3.1 for \(F = L^k \otimes E\), there exist \(C_3 = C_3(\omega, K_2, E, h^E) \geq 0\) such that

\[
\langle c_1(F, h^F) \wedge \Lambda \alpha, \alpha \rangle_{h^F, \omega_n} = c_1(F, h^F) \wedge T_{\alpha} \wedge \omega_{q-1} = (k c_1(L, h^L) + c_1(E, h^E)) \wedge T_{\alpha} \wedge \omega_{q-1} \\
\geq c_1(E, h^E) \wedge T_{\alpha} \wedge \omega_{q-1} = \langle c_1(E, h^E) \wedge \Lambda \alpha, \alpha \rangle_{h^F, \omega_n} \\
\geq -C_3 |\alpha|^2_{h^E \omega_n}
\]
on \( \hat{K}_2 \), since \( L \geq 0, T_\alpha \geq 0 \) and \( \omega \) is positive Hermitian \((1,1)\)-form on \( \hat{K}_2 \). Thus over \( \hat{K}_2 \), (3.1) becomes

\[
(3.6) \quad i \partial \bar{\partial} (T_\alpha \wedge \omega_{q-1}) \geq -C_4 |\alpha|^2_n \omega_n
\]

where \( C_4 = C_4(\omega, K_2, E, h_E) \geq 0 \). Then

\[
(3.7) \quad \int_{|z|<t} (t^2 - |z|^2) i \partial \bar{\partial} (T_\alpha \wedge \omega_{q-1}) \geq -C_4 t^2 \sigma(t).
\]

Denote \( \beta := \frac{x^{\perp}}{|x^{\perp}|^2} \partial \bar{\partial} |x|^2 = \frac{x^{\perp}}{2} \sum_j dz_j \wedge d \bar{z}_j \), and apply Stokes’ formula to the left hand side of (3.7), that is

\[
\int_{|z|<t} (t^2 - |z|^2) i \partial \bar{\partial} (T_\alpha \wedge \omega_{q-1}) = \int_{|z|=t} \partial |z|^2 \wedge i \partial (T_\alpha \wedge \omega_{q-1}).
\]

Thus

\[
(3.8) \quad 2 \int_{|z|<t} T_\alpha \wedge \omega_{q-1} \wedge \beta \leq \int_{|z|=t} -iT_\alpha \wedge \omega_{q-1} \wedge \partial |z|^2 + C_4 t^2 \sigma(t).
\]

By the choice of exponential normal coordinates,

\[
(3.9) \quad h_{ij}(z) = \frac{1}{2} \delta_{ij} + O(|z|^2).
\]

for any \( z \in B(c_0) \). In particular, for \( |z| = t \) with \( 0 \leq t < c_0 \), we can approximate the metric \( \omega \) on \( X \) by the standard one \( \beta \) on \( \mathbb{C}^n \) in the following sense

\[
(3.10) \quad (1 - R_1(t)) \beta \leq \omega(z) \leq (1 + R_1(t)) \beta
\]

by the smoothness of \( \omega \), where \( R_1(t) \geq 0 \) and \( R_1(t) = O(t^2) \) as \( t \to 0 \). (Trivially if \( X = \mathbb{C}^n \) then \( R_1(t) = 0 \)) Hence

\[
(3.11) \quad T_\alpha \wedge \omega_{q-1} \wedge \beta = q(1 - R_1(t)) |\alpha|^2_n \omega_n.
\]

Notice \( (1 + R_1(t))^{-n} \leq \frac{\omega^n}{\beta^n} \leq (1 + R_1(t))^n \) and

\[
(3.12) \quad \int_{|z|=t} -iT_\alpha \wedge \beta_{q-1} \wedge \partial |z|^2 \leq t \int_{|z|=t} |\alpha|^2_n dS
\]

where \( dS \) is the surface measure. Then (3.10), (3.11) and (3.12) imply

\[
(3.13) \quad \int_{|z|=t} -iT_\alpha \wedge \omega_{q-1} \wedge \partial |z|^2 \leq t(1 + R_2(t)) \sigma'(t)
\]

where \( \sigma'(t) = \int_{|z|=t} |\alpha|^2_n (\omega_n / \beta_n) dS \) by the definition of \( \sigma(t) \), \( R_2(t) \geq 0 \) and \( R_2(t) = O(t^2) \). Combining (3.8), (3.11) and (3.13), we have

\[
2q(1 - R_1(t)) \sigma(t) \leq t(1 + R_2(t)) \sigma'(t) + C_4 t^2 \sigma(t),
\]

that is, \( 2q(1 - R_3(t)) \sigma(t) \leq t(1 + R_2(t)) \sigma'(t) \) for any \( 0 \leq t < c_0 \), where \( R_3(t) \geq 0 \) and \( R_3(t) = O(t^2) \). Substituting \( s(t)^2 := \sigma(t) \geq 0 \) and dividing by \( 2ts(t) \), we obtain

\[
(3.14) \quad q(\frac{1}{t} - R_4(t)) s(t) \leq (1 + R_5(t)) s'(t)
\]

for \( q \geq 1 \) and any \( 0 \leq t < c_0 \), where \( R_4(t) \geq 0 \) and \( R_4(t) = O(t) \).

Now we only need to prove: There exists \( C \geq 0 \), such that for any \( 1 \leq q \leq n \) and \( 0 \leq t < \frac{c_0}{2n} \),

\[
(3.15) \quad s(t) \leq Ct^q,
\]
which is equivalent to (3.5). Next we fix \( q \geq 1 \), so we only need to prove that \( s(t) \leq Ct^m \) for any \( 0 \leq m \leq q \) and \( 0 \leq t \leq c_0/2^q \) by induction over \( m \). Firstly, for \( m = 0 \), \( s(t) := \sigma(t) := \int_{t < t} |\alpha|^2 \omega_n \leq 1 \) for \( 0 \leq t \leq c_0 \). Secondly, assume there exists a constant \( C_5 > 0 \) such that \( s(t) \leq C_5 t^m \) for \( 0 \leq m < q \) and \( 0 \leq t \leq c_0/2^m \). Thirdly, in particular, we consider \( 1 \leq m + 1 = q \) for \( 0 \leq t \leq c_0/2^{m+1} \), and thus (3.14) becomes

\[
(3.16) \quad s'(t) - (m + 1)\frac{s(t)}{t} \geq -R_2(t)s'(t) - (m + 1)R_4(t)s(t).
\]

By the second step of the induction, \( s(t) = O(t^m) \) with \( s(t) \geq 0 \) and \( s'(t) = O(t^{m-1}) \) with \( s'(t) \geq 0 \). And according to (3.16), for \( 0 \leq t < c_0/2^{m+1} \),

\[
(3.17) \quad \left( \frac{s(t)}{t^{m+1}} \right)' = \frac{1}{t^{m+1}} \left[ s'(t) - \frac{m + 1}{t} s(t) \right] \geq -R_5(t),
\]

where \( R_6(t) \geq 0 \) and \( R_6(t) = O(1) \). Integral (3.17) from \( r \) to \( c_0/2^{m+1} \), then

\[
(3.18) \quad \frac{s(\alpha^{(m+1)})}{\alpha^{(m+1)}} - \frac{s(\alpha)}{\alpha} = \int_r^{\alpha} \left( \frac{s(t)}{t^{m+1}} \right)' dt \geq -C_6(m + 1, \omega, K_1, K_2, E, h_B).
\]

Finally, for the fixed \( q \geq 1 \) and any \( 0 \leq r < c_0/2^q \), we have

\[
(3.19) \quad \frac{s(r)}{r^q} \leq \frac{C_5}{C_0} q + C_6(q, \omega, K_1, K_2, E, h_B).
\]

Let \( q \) run over \( \{1, ..., n\} \) in (3.19). Then there exists \( C = C(\omega, K_1, K_2, E, h_B) \geq 0 \) such that (3.15) verifies and also (3.4) and (3.5). \( \square \)

We will consider the following trivialization of holomorphic Hermitian line bundles in local charts. For any \( x_0 \in K_1 \subset K_2 \), we fix the exponential normal coordinates on \( V \equiv W \subset \mathbb{C}^n \) as before such that

\[
\omega(x_0) = \beta := \frac{\sqrt{-1}}{2} \sum dz_j \wedge d\bar{z}_j,
\]

which is the standard metric on \( \mathbb{C}^n \). Let \( L \geq 0 \) on \( K_2 \). Then we can choose the trivialization of \( L \) and \( E \) over \( V \) such that for any \( z \in B(c_0) \), \( |e_L(z)|_{h_L}^2 = e^{-\phi(z)} \) and \( |e_E(z)|_{h_E}^2 = e^{-\psi(z)} \) satisfying

\[
(3.20) \quad \phi(z) = \sum \lambda_i |z_i| + O(|z|^3), \quad \varphi(z) = \sum \mu_i |z_i| + O(|z|^3)
\]

and \( \lambda_i = \lambda_i(x_0) \geq 0 \). The induced Hermitian metric on \( F := L^k \otimes E \) is given by

\[
|e_F(z)|_{h_F}^2 = e^{-\psi(z)} \psi(z) := k\phi(z) + \varphi(z).
\]

The quadratic part of \( \phi \) is denoted by

\[
\phi_0(z) := \sum \lambda_i |z_i|^2.
\]

Assume \( \alpha \in \Omega^{p,q}(X, F) \), then it has the form \( \alpha = \xi \otimes e_F \) around \( x_0 \in K_1 \) where \( \xi = \sum f_{I,J} dz_I \wedge d\bar{z}_J \) is a local \((p, q)\)-form and \( f_{I,J} \) are smooth functions on \( W \subset \mathbb{C}^n \). The scaled functions and sections with respect to \( k \in \mathbb{N} \) are defined by

\[
(3.23) \quad \psi^{(k)}(z) := \psi(z/\sqrt{k}), \quad e_L^{(k)}(z) := e_L(z/\sqrt{k}), \quad e_E^{(k)}(z) := e_L(z/\sqrt{k}), \quad \text{for } z \in \sqrt{k}W = B(\sqrt{k}c_0),
\]
hence $|e^{(k)}_{β}|_{n,ω}^2 = e^{-ψ^{(k)}}$. The scaled forms are defined for $z \in \sqrt{k}W$ by

$$\omega^{(k)}(z) := \sqrt{-1} \sum h^{(k)}_{ij}(z)dz_i \wedge d\bar{z}_j := \sqrt{-1} \sum h_{ij}(z/\sqrt{k})dz_i \wedge d\bar{z}_j,$$

(3.24)

$$\xi^{(k)}(z) := f^{(k)}_{ij}(z)dz_i \wedge d\bar{z}_j := f_{ij}(z/\sqrt{k})dz_i \wedge d\bar{z}_j,$$

$$\alpha^{(k)}(z) := \xi^{(k)}(z) \otimes e^{(k)}_{β}(z).$$

Lemma 3.4. Let $(X, ω)$ be a Hermitian manifold of dimension $n$ and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. Let $K_1$ and $K_2$ be compact subsets in $X$ such that $K_1 \subset K_2$. Assume $L \geq 0$ on $K_2$ and $q \geq 1$. Then there exists a constant $C > 0$ depending on $ω$, $K_1$, $K_2$, $(L, h^L)$ and $(E, h^E)$, such that

$$|α(x_0)|_{h, ω}^2 \leq Ck^n \int_{|z| < \sqrt{k}} |α|^2_{h, ω} dv_X,$$

(3.25)

for any $x_0 \in K_1$, $α \in \mathcal{H}_q^n(X, L^k \otimes E)$ and $k$ sufficiently large, where $|α|^2_{h, ω}$ is the pointwise Hermitian norm induced by $ω$, $h^L$ and $h^E$.

Proof. Denote a ball centred at the origin in $\mathbb{C}^n$ with radius $r > 0$ by $B(r) := \{z \in \mathbb{C}^n : |z| < r\}$. Then, $B(r)$ is a subset of $W \cong V \subset X$ via the local chart, when $0 < r \leq c_0$. Thus the right side of (3.25) is well defined, when $k$ is large enough. Let $r_k := \frac{\log k}{\sqrt{k}}$ for $k \in \mathbb{N}$. Then $0 < r_k \leq 1$ and $r_k \to 0$ as $k \to \infty$.

Under the local representation of forms valued in $F := L^k \otimes E$, the Kodaira Laplacian can be represented by $\Box = \partial\partial^* + \partial^*\partial$ locally over $B(\frac{\log k}{\sqrt{k}})$ for $k$ large enough, where $\partial^* = \partial F^*$. Then the scaled Laplacian

$$\Box^{(k)} := \partial\partial^{*(ψ^{(k)})} + \partial^{*(ψ^{(k)})}\partial$$

(3.26)

is well defined on $B(\log k)$ for $k$ large enough.

Under our assumptions, we consider the harmonic $L^k \otimes E$-valued $(n, q)$-form $α = α|_{B(\frac{\log k}{\sqrt{k}})}$ on $B(\frac{\log k}{\sqrt{k}})$ for $k$ large enough, then the rescaled $α^{(k)}$ is a $L^k \otimes E$-valued $(n, q)$-forms on $B(\log k)$ by (3.24). By definitions,

$$\Box^{(k)}α^{(k)} = \frac{1}{k}(\Boxα)^{(k)} = 0$$

(3.27)

over $B(\log k)$. That is, the scaled forms are still harmonic with respect to the scaled Laplacian.

Next we introduce the following $L^2$-norms,

$$\|\cdot\|^2_{B(\frac{\log k}{\sqrt{k}})} := \int_{B(\frac{\log k}{\sqrt{k}})} |\cdot|^2 e^{-ψ}ω_n \quad \text{and} \quad \|\cdot\|^2_{φ_0 B(2)} := \int_{B(2)} |\cdot|^2 e^{-φ_0}β_n.$$

We claim that there exist $C(k) > 0$ bounded above and below for $k$ large enough (in fact, $C(k) \to 1$ as $k \to \infty$) such that

$$\|α^{(k)}\|^2_{φ_0 B(2)} = C(k)k^n \|α\|^2_{B(\frac{\log k}{\sqrt{k}})},$$

(3.28)

In fact, by (3.20)–(3.23), $ψ^{(k)}(z) - φ_0(z) = O(\frac{1}{\sqrt{k}})$ and thus

$$\lim_{k \to \infty} \sup_{|z| < \log k} |\partial N(ψ^{(k)} - φ_0)(z)| = 0,$$

(3.29)
which means the scaled Hermitian metric on $L^k \otimes E$ converges to a model metric on $B(\log k)$ with all derivatives. In particular, as $k \to \infty$, $\psi^{(k)}(z) \to \phi_0(z)$ uniformly over $B(\log k)$, and also $\omega^{(k)}(z) \to \beta$. Then (3.28) follows by

\[
\|\alpha^{(k)}\|_{\phi_0, B(2)}^2 = \int_{B^2} |\xi^{(k)}(z)|^2 e^{-\phi_0(z)} \beta_n, \quad k^n \|\alpha\|_{B(\frac{2}{k^2})}^2 = \int_{B^2} |\xi^{(k)}(z)|^2 e^{-\psi^{(k)}(z)} \omega_n(z).
\]

Finally, we apply [2] Lemma 3.1 and identify $\alpha^{(k)}$ with a form in $L^2(C^n, \phi_0)$ by extending with zero outside $B(\log k)$. Then there exists a constant $C_1 > 0$ independent of $k$ such that

\[
(3.30) \quad \sup_{z \in B(1)} |\alpha^{(k)}(z)|_{\beta, \phi_0, B(2)}^2 \leq C_1 \|\alpha\|_{\phi_0, B(2)}^2
\]

for $k$ large enough, where $|\cdot|_{\beta, \phi_0} := |\cdot| e^{-\phi_0}$. Combining (3.28) and (3.30), we get $|\alpha(x_0)|_{H^2, \omega}^2 = |\alpha^{(k)}(0)|_{\beta, \phi_0}^2 \leq 2C_1 k^n \|\alpha\|_{B(\frac{2}{k^2})}^2$ for $k$ large enough. Notice that here $C_1$ works for all points sufficiently close to $x_0$ by continuity. That is, there exists a constant $C_1 > 0$ and a neighbourhood $B(x_0, \epsilon)$ of $x_0$, such that $|\alpha(x)|_{H^2, \omega} \leq 2C_1 k^n \|\alpha\|_{B(\frac{2}{k^2})}$ for any $x \in B(x_0, \epsilon)$ and $k$ large enough. Since $K_1$ is compact, there exists a uniform constant $C > 0$ which works for all $x \in K_1$, and (3.25) follows.

To summarize, we have a local estimate of the pointwise norm of harmonic forms valued in semipositive line bundles, which is equivalent to Theorem [1.1]. We define

\[
S_k^q(x) := \sup \left\{ \frac{|\alpha(x)|_{H^2, \omega}^2}{\|\alpha\|_{L^2}^2} : \alpha \in \mathcal{H}^{n,q}(X, L^k \otimes E) \right\},
\]

where $(L, h^L)$ and $(E, h^E)$ are holomorphic Hermitian line bundles over a Hermitian manifold $(X, \omega)$ as before.

**Theorem 3.5.** Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. Let $K_1$ and $K_2$ be compact subsets in $X$ such that $K_1 \subset K_2$. Assume $L \geq 0$ on $K_2$ and $q \geq 1$. Then there exists $C > 0$ depending on $\omega$, $K_1$, $K_2$, $(L, h^L)$ and $(E, h^E)$ such that

\[
(3.31) \quad S_k^q(x) \leq C k^{n-q}.
\]

for any $x \in K_1$ and $k \geq 1$.

**Proof.** Combine (3.25) and the case $r = \frac{2}{\sqrt{k}}$ of (3.4). \qed

**Remark 3.6.** In particular, when $X$ is compact without boundary, and $K_1 = K_2 = X$, then (3.31) implies the case of $\lambda = 0$ in [3, Theorem 2.3].

4. **Proof of the Main Results and a Corollary**

At first, the following lemma is clear by the definitions of $S_k^q(x)$ and $B_k^q(x)$ in Theorem 3.5 and Theorem 1.1.

**Lemma 4.1.** $S_k^q(x) \leq B_k^q(x) \leq C_n S_k^q(x)$ on $X$.

By this lemma and the submeanvalue formulas of harmonic forms in $\mathcal{H}^{n,q}(X, L^k \otimes E)$ in the section 3.2 we can prove the first main result immediately.

\[9\]
Proof of Theorem 1.1. Assume $U$ is a neighbourhood of $K$ such that $L$ is semipositive on $U$. Then we choose $K_2$ such that $K_1 \subset K_2 \subset K_2 \subset U$, and apply Theorem 3.5 and Lemma 3.1.

Now we can prove the second main results on $\Gamma$-dimension and covering manifolds.

Proof of Theorem 1.2. Under the assumption of $X$, $\Gamma$ and $X/\Gamma$, there exists an open fundamental domain $U \subset X$ of the action $\Gamma$ on $X$ such that the closure $\overline{U}$ is compact.

Since $L$ and $E$ are $\Gamma$-invariant holomorphic Hermitian line bundles over $X$, the induced Hermitian line bundle $F := L^k \otimes E$ is also $\Gamma$-invariant and holomorphic. Then the Kodaira Laplacian $\Box := \Box^F$ is $\Gamma$-invariant. Thus $\Box$ is essentially self-adjoint(cf. [15] Corollary 3.3.4), and we denote still by $\Box$ its self-adjoint extension, which commutes to the action of $\Gamma$. According to ([15], Lemma C.3.1, Lemma 3.6.3), the space of harmonic $F$-valued $(n, q)$-forms $\mathcal{H}^{n,q}(X, F)$ is a $\Gamma$-module on which $\Gamma$-dimension is well-defined. By [15] (3.6.11), (3.6.17), we have

\[(4.1) \quad \dim_\Gamma \mathcal{H}^{n,q}(X, F) = \sum_i \int_U |s_i(x)|^2_{\mathcal{H}^q, \omega} dv_X(x),\]

where $\{s_i\}$ is an orthonormal basis of $\mathcal{H}^{n,q}(X, F)$ with respect to the scalar product in $L^2_{\mathcal{H}^q}(X, F)$. Using Theorem 1.1 we have

\[(4.2) \quad B^q_k(x) := \sum_i |s_i(x)|^2_{h_n, \omega} \leq Ck^{n-q}\]

for any $x \in U$. Then integrating $B^q_k(x)$ over $U$ and combining (4.2) and (4.1), we obtain $\dim_\Gamma \mathcal{H}^{n,q}(X, L^k \otimes E) \leq Ck^{n-q}$, that is the first asymptotic estimate in (1.3).

Let $H^{0,q}_{(2)}(X, L^k \otimes E)$ be the reduced $L^2$-Dolbeault cohomology group, which is canonically isomorphic to $\mathcal{H}^{0,q}(X, L^k \otimes E)$ as $\Gamma$-modules by the weak Hodge decomposition, thus $\dim_\Gamma H^{0,q}_{(2)}(X, L^k \otimes E) = \dim_\Gamma \mathcal{H}^{0,q}(X, L^k \otimes E)$. Substituting $E \otimes \Lambda^n(T^{1,0}X)$ for $E$ in the first estimate of (1.3), then the same asymptotic estimate also holds for the space of harmonic $L^k \otimes E$ valued $(0, q)$-forms and (1.4) follows.

Corollary 4.2. Let $(X, \omega)$ be a compact Hermitian manifold of dimension $n$ and $(E, h^E)$ be a semipositive holomorphic Hermitian vector bundle of rank $r$ (i.e. $L(E^*)^r \geq 0$). Then there exists $C > 0$ such that for any $q \geq 1$ and $k \geq 1$ we have

\[(4.3) \quad \dim H^q(X, S^k(E)) \leq Ck^{(n+r-1)-q}\]

where $S^k(E)$ is the $k$-th symmetric tensor power of $E$.

Proof. We assume $\Gamma$ and $E$ are trivial in (1.4), and notice the theorem of Le Potier (cf. [13] Chap.III §5 (5.7)), which relates vector bundle cohomology to line bundle cohomology, then $\dim H^q(X, S^k(E)) = \dim H^q(P(E^*), (L(E^*)^r)^k) \leq Ck^{(n+r-1)-q}$. □

Remark 4.3. Let $(X, \omega)$ be a complete Hermitian manifold of dimension $n$ and $(L, h^L)$ and $(E, h^E)$ be Hermitian holomorphic line bundles over $X$. Suppose $X$, $L$ and $E$ have bounded geometry and $L \geq 0$ over $X$. By Theorem 1.1 there exists a constant $C > 0$ such that the Bergman kernel function $B^q_k(x) \leq Ck^{n-q}$ for any $x \in X$, $k \geq 1$ and $q \geq 1$.

Remark 4.4. Let $(X, \omega)$ be a Hermitian manifold of dimension $n$ on which a discrete group $\Gamma$ acts holomorphically, freely and properly such that $\omega$ is a $\Gamma$-invariant Hermitian metric and the quotient $X/\Gamma$ is compact. Let $(L, h^L)$ be a $\Gamma$-invariant holomorphic Hermitian line bundle on $X$. Assume $L \leq 0$ is semi-negative (i.e. $L^* \geq 0$). According
to Serre duality (cf. [6, 3.15]) and Theorem 1.2, there exists $C > 0$ such that for any $q \leq n - 1$ and $k \geq 1$ we have
\[
\dim_T \mathcal{H}^q_{(2)}(X, L^k) = \dim_T \mathcal{H}^{n-q}_{(2)}(X, L^n) = \dim_T \mathcal{H}^{0,q}_{(2)}(X, \Lambda^n(T^{*1,0}X) \otimes L^k) \leq Ck^q.
\]
In particular, for all $k \in \mathbb{N}$, $\dim_T \mathcal{H}^{0,0}_{(2)}(X, L^k) \leq C$.

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