BOSONIZATION AND DUALITY IN 2+1-DIMENSIONS:
APPLICATIONS IN GAUGED MASSIVE THIRRING MODEL

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Abstract:

Bosonization of the gauged, massive Thirring model in 2+1-dimensions produces a Maxwell-Chern-Simons gauge theory, coupled to a dynamical, massive vector field. Exploiting the Master Lagrangian formalism, two dual theories are constructed, one of them being a gauge theory. The full two-point functions of both the interacting fields are computed in the path integral quantization scheme. Furthermore, some new dual models, derived from the original master lagrangian and valid for different regimes of coupling parameters, are constructed and analysed in details.

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Introduction:

The notion of duality has played a key role in the common goal of unification in physics. Broadly, two particular models are said to be dual to each other if in some sense they are equivalent to each other as regards to, *e.g.* effective action, Green’s function, etc. One way of obtaining a pair of dual systems is to consider the so called Master lagrangian [1] of a model of larger phase space, from which each of the dual models are derivable. The presence of the Master lagrangian assures the equivalence of the said (dual) models. The other more direct way is to derive one model from another one - such as Bosonization (in 1+1-dimensions), where a fermionic model, *e.g.* the Massive Thirring Model (MTM) was shown to be dual to the (bosonic) sine-Gordon model [2].

Recently there has been a marriage of sorts between these two approaches in higher (2+1) dimensions. Early works [3] in 2+1-dimensional bosonization were not completely satisfactory, as the fermion-boson mapping was plagued with non-locality. However, the programme received an impetus from later works [4], where a local bosonic action for the MTM is obtained at the cost of truncating the Seelay expansion of the fermion determinant at one loop. The small parameter in the expansion is the inverse of fermion mass.

In a recent paper [5], we have been able to bosonize the Gauged Massive Thirring Model (GMTM). The spectrum of the bosonic model, a gauge theory, was obtained in a classical setup [5]. The present work is devoted to a quantum analysis of the bosonized GMTM. Exact two-point functions of the GMTM are derived. The concept of duality is utilised in two ways: On the one hand to generate the full propagators of GMTM and on the other hand, to establish the equivalence between several (manifestly) gauge invariant models with non-invariant ones. The latter phenomenon is reminiscent of the duality between abelian self-dual and Maxwell-Chern-Simons models [6], and non-abelian self-dual and Yang-Mills-Chern-Simons models [7].

The rest of this section is devoted to a brief elaboration of the computational scheme adopted.

In section II, the GMTM and its bosonized version is stated from previous works [5]. The lagrangian of GMTM, $L_F(\psi, A_\mu, m, e, g)$ consists of the following parameters: the fermion ($\psi$) mass $m$, the gauge field ($A_\mu$) coupling $e$ and the Thirring coupling $g$. The bosonized version has $L_B(B_\mu, A_\mu, m, e, g)$ where $B_\mu$ is an auxiliary vector field introduced to linearize the Thirring interaction. The latter model is our Master Lagrangian (ML). Note that the duality between $L_F$ and $L_B$ is not exact, for reasons mentioned earlier.

In section III we derive the dual partition functions $Z(A_\mu)$ and $Z(B_\mu)$, via integrating out $B_\mu$ and $A_\mu$ (in the Lorentz gauge) from ML respectively. This produces an exact duality between $Z(A_\mu)$, a gauge invariant theory, and $Z(B_\mu)$, a non-invariant one.

The dual system helps us to compute the full propagators $<A_\mu A_\nu>$ and $<B_\mu B_\nu>$ of ML. Limiting values of the parameters are studied to show the consistency of the above framework. This is the content of section IV.

Section V comprises of some more dual models, one of the pairs (of a dual system) being a gauge theory. All these models are derived from the starting ML, the distinction being the relative strengths of the couplings $e$ and $g$. Once again Green’s functions of these models are derived.

We end the paper with a brief conclusion in section VI.
II. Bosonized model

Our starting point is the gauged massive Thirring model,

\[ L_F = \bar{\psi} i \gamma^\mu (\partial_\mu - ieA_\mu) \psi - m \bar{\psi} \psi + \frac{g}{2} \bar{\psi} \gamma^\mu \psi \bar{\psi} \gamma_\mu \psi. \]  

(1)

Here \( A_\mu \) is an external, abelian gauge field and \( g \) is the Thirring coupling constant. The system has abelian gauge invariance under,

\[ \psi \rightarrow \exp(i \alpha), \quad A_\mu \rightarrow A_\mu + \partial_\mu \alpha. \]

Notice that no kinetic terms for \( A_\mu \), (e.g. Maxwell or Chern-Simons term), are present in (1) to give dynamics to the photon. However this is no restriction since we will study the bosonized version of (1), where the kinetic terms corresponding to \( A_\mu \) will be automatically generated in the Seelay expansion of the fermion determinant. Hence the coefficients of the starting kinetic terms, (if present), will get renormalized anyway after bosonization.

We now introduce an auxiliary field \( B_\mu \) to linearize the Thirring interaction,

\[ L_F = \bar{\psi} i \gamma^\mu [\partial_\mu - i(B_\mu + eA_\mu)] \psi - m \bar{\psi} \psi - \frac{1}{2g} B_\mu B_\mu. \]  

(2)

The partition function to be evaluated is

\[ Z_B = \int \mathcal{D}(\psi) \exp(i \int L_F). \]  

(3)

Unlike in 1+1-dimensions, the fermion determinant is non-local and conventionally one computes the one loop expression in a power series with inverse fermion mass, (i.e. \( \frac{1}{m} \)), as the small parameter. Keeping the \( m \)-independent and \( O(\frac{1}{m}) \) terms only we obtain,

\[ L_B = -\frac{a}{4} (B + eA)_{\mu\nu} (B + eA)^{\mu\nu} + \frac{\alpha}{2} \epsilon_{\mu\nu\lambda} (B + eA)^{\mu}(B + eA)^{\nu\lambda} - \frac{1}{2g} B_\mu B^\mu, \]  

(4)

where \( \alpha = 1/(4\pi) \) and \( a = -1/(24\pi m) \). In the present work, we treat the above \( L_B \) in (4) as our ML and the analysis actually begins here. The ML has an invariance under local gauge transformation of \( A_\mu \). The partition function corresponding to (4) in Lorentz gauge is

\[ Z = \int \mathcal{D}(A, B) \delta(\partial_\mu A_\mu) \exp[-\frac{a}{4} (B + eA)_{\mu\nu} (B + eA)^{\mu\nu} \]

\[ + \frac{\alpha}{2} \epsilon_{\mu\nu\lambda} (B + eA)^{\mu}(B + eA)^{\nu\lambda} - \frac{1}{2g} B_\mu B^\mu + J_\mu A_\mu + L_\mu B_\mu]. \]  

(5)

The source current \( J_\mu \) is conserved (\( \partial_\mu J^\mu = 0 \)), but there is no such restriction on the source current \( L_\mu \). The classical analysis including the spectrum of this model has already been discussed in [4]. At present we are interested in the quantum Green Functions (GF) \( < A_\mu(x) A_\nu(y) > \) and \( < B_\mu(x) B_\nu(y) > \) of the model. This will be achieved by the introduction of two dual lagrangians, obtained by selective integration of the \( A_\mu \) and \( B_\mu \) fields respectively.
### III. Dual lagrangians

Our objective is to integrate out $A_\mu$ and $B_\mu$ separately from the ML. Since $A_\mu$ is a gauge field, the computations have been carried out in a particular (Lorentz) gauge, so that

$$\partial_\mu A^\mu = 0.$$ 

The effective action is rewritten as,

$$Z = \int \mathcal{D}(B) \exp \left[ -\frac{a}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\alpha}{2} \epsilon_{\mu\nu\lambda} B^\mu B^\nu B^\lambda - \frac{1}{2g} B_\mu B^\mu + B_\mu L^\mu \right]$$

$$\int \mathcal{D}(A) \exp \left[ e^2 A^\mu D^\mu A^\nu + e H_\mu A^\mu \right]$$

where,

$$D_{\mu\nu} = e^2 \left[ \frac{a \partial^2}{2} g_{\mu\nu} + (\zeta - \frac{a}{2}) \partial_\mu \partial_\nu + (-\alpha) \epsilon_{\mu\nu\lambda} \partial^\lambda \right]$$

$$= e^2 (P g_{\mu\nu} + Q \partial_\mu \partial_\nu + R \epsilon_{\mu\nu\lambda} \partial^\lambda),$$

$$H_\mu = e a \partial^\nu B_{\nu\mu} + e \alpha \epsilon_{\mu\nu\lambda} B^\nu B^\lambda + J_\mu.$$ 

$\zeta$ is the gauge parameter. Let us define,

$$D^{-1}_{\mu\nu} = \frac{1}{e^2} \left[ p g_{\mu\nu} + q \partial_\mu \partial_\nu + r \epsilon_{\mu\nu\lambda} \partial^\lambda \right].$$

Demanding $D_{\mu\nu} D^{-1}_{\nu\lambda} = g^\mu_\lambda$, we solve for $p$, $q$ and $r$ and get

$$e^2 p = \frac{P}{P^2 + R^2 \partial^2} = \frac{2a \partial^2}{a^2 (\partial^2)^2 + 4\alpha^2 \partial^2},$$

$$e^2 r = -\frac{R}{P^2 + R^2 \partial^2} = \frac{4\alpha}{a^2 (\partial^2)^2 + 4\alpha^2 \partial^2},$$

$$e^2 q = \frac{R^2 - Q P}{(P^2 + R^2 \partial^2)(P + Q \partial^2)} = \frac{4\alpha^2 - a(2\zeta - a) \partial^2}{\zeta \partial^2 (a^2 (\partial^2)^2 + 4\alpha^2 \partial^2)}.$$ 

After integration of $A_\mu$ we are left with,

$$Z_B(B_\mu) = \int \mathcal{D}(B_\mu) \exp \left[ \int \left( -\frac{a}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\alpha}{2} \epsilon_{\mu\nu\lambda} B^\mu B^\nu B^\lambda - \frac{1}{2g} B_\mu B^\mu + B_\mu L^\mu \right) 

+ \left( -\frac{i}{4} \right)(H^\mu D_{\mu\nu} D^{-1}_{\nu\lambda} H^\lambda) \right].$$

After simplification we arrive at

$$Z(B_\mu) = \int \mathcal{D}(B) \exp \left( \frac{i}{4} \left[ -\frac{2}{g} B_\mu B^\mu + 4 B_\mu \left( \frac{1}{e} J^\mu - L^\mu \right) + \frac{2a}{e^2} J_\mu \frac{1}{a \partial^2 + 4\alpha^2} J^\mu 

- \frac{4\alpha}{e^2} J_\mu \frac{1}{\partial^2 (a^2 \partial^2 + 4\alpha^2)} \epsilon_{\mu\nu\lambda} \partial^\nu J^\lambda \right] \right)$$
Even though the source free expression of \( Z(B_\mu) \) looks trivial with only a contact term in \( B_\mu \), the Green functions are quite involved, as we will demonstrate in the next section.

In a similar way, we now carry out the \( B_\mu \) integration. No gauge fixing is required here. The starting partition function is reexpressed as

\[
Z_B(A_\mu) = \int \mathcal{D}(A) \delta(\partial_\mu A_\mu) \exp[-\frac{ae^2}{4} A^{\mu\nu} A_{\mu\nu} + \frac{\alpha e^2}{2} \epsilon_{\mu\nu\lambda} A^{\nu\lambda} + J_\mu A^\mu] \\
\int \mathcal{D}(B) \exp[-\frac{a}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\alpha}{2} \epsilon_{\mu\nu\lambda} B^{\nu\lambda} - \frac{1}{2g} B_\mu B^\mu - \frac{ae}{2} B_{\mu\nu} A^{\mu\nu} + \alpha \epsilon_{\mu\nu\lambda} B^{\nu\lambda} A_\lambda + L_\mu B^\mu].
\]

Once again we define the \( B_\mu \) part as

\[
\int \mathcal{D}(B) \exp[B^\mu K_{\mu\nu} B^{\nu} + h_\mu B^\mu]
\]

\[
K_{\mu\nu} = \left[ (\frac{a\partial^2}{2} - \frac{1}{2g}) g_{\mu\nu} + (-\frac{a}{2}) \partial_\mu \partial_\nu + (-\alpha) \epsilon_{\mu\nu\lambda} \partial^\lambda \right] = Pg_{\mu\nu} + Q\partial_\mu \partial_\nu + R\epsilon_{\mu\nu\lambda} \partial^\lambda
\]

\[
h_\mu = e\alpha \partial^\nu A_{\nu\mu} + e\alpha \epsilon_{\mu\nu\lambda} A^{\nu\lambda} + L_\mu.
\]

Proceeding exactly as in the previous case, the required inverse operator is,

\[
K_{\mu\nu}^{-1} = [pg_{\mu\nu} + q\partial_\nu \partial_\mu + r\epsilon_{\mu\nu\lambda} \partial^\lambda].
\]

\[
p = \frac{P}{P^2 + R^2\partial^2} = \frac{2g(a\alpha \partial^2 - 1)}{(a\alpha \partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2},
\]

\[
q = \frac{R^2 - QP}{(P^2 + R^2\partial^2)(P^2 + Q\partial^2)} = -\frac{2g^2(4\alpha^2 g + a^2 \alpha \partial^2)}{(a\alpha \partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2}
\]

\[
r = -\frac{R}{P^2 + R^2\partial^2} = \frac{4\alpha g^2}{(a\alpha \partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2}.
\]

The integration of \( B_\mu \) leads to

\[
Z_B(A_\mu) = \int \mathcal{D}(A_\mu) \delta(\partial_\mu A_\mu) \exp[-\frac{ae^2}{4} A^{\mu\nu} A_{\mu\nu} + \frac{\alpha e^2}{2} \epsilon_{\mu\nu\lambda} A^{\nu\lambda} + J_\mu A^\mu]
\]

\[
+ \left(-\frac{i}{4}\right) (h^\mu K_{\mu\nu}^{-1} h^\nu).
\]

The final result is the gauge invariant action,

\[
Z(A_\mu) = \int \mathcal{D}\delta(\partial_\mu A_\mu) \exp(-i/4)[2e^2 A_\mu \frac{4\alpha^2 g - a + a^2 \alpha \partial^2}{(a\alpha \partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2} (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu
\]

\[
+ e^2 A_\mu \frac{-4\alpha + 8g(1 - \alpha) \partial^2(2\alpha^2 g - a + a^2 \alpha \partial^2)}{(a\alpha \partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda
\]
\[
+4\epsilon g A_\mu \frac{4\alpha^2 g - a + a^2 g \partial^2}{(ag\partial^2 - 1)^2} + 4\alpha^2 g^2 \partial^2 (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) L_\nu \\
-8\epsilon g A_\mu \frac{1}{(ag\partial^2 - 1)^2} (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A^\lambda \\
-2g^2 L^{\mu}_{\nu} \partial^\mu (4\alpha^2 g - a + a^2 g \partial^2) \frac{1}{(ag\partial^2 - 1)^2} \partial^\nu L^{\nu}_\mu \\
-4\epsilon g^2 L^{\mu}_{\mu} \frac{1}{(ag\partial^2 - 1)^2} (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) L^{\lambda}_\lambda + 2g L^{\mu}_{\mu} (ag\partial^2 - 1) \frac{1}{(ag\partial^2 - 1)^2} \partial^\mu L^{\mu}_\mu - 4J_\mu A^\mu].
\] (15)

The $\partial A$ terms in (3) have been displayed to show the manifest gauge invariance but they are dropped later since Lorentz gauge has been adopted. Thus we have established an exact duality between the complicated gauge invariant effective action in (15) and the more simpler looking one in (4). It should be kept in mind that the duality regarding the original fermion model in (4) and its bosonized version in (3), as mentioned before, was not exact. On the other hand, the just exposed duality is exact since they both originate from the same master lagrangian in (3) and no further approximation are involved so far. The next task is to compute the $A_\mu$ and $B_\mu$ GF’s by exploiting the latter duality.

IV. Green’s functions

The sources present in (3) allow us to formally define the GF’s,

\[
\left(\frac{1}{i}\right)^2 \frac{\delta^2 Z_B}{\delta J_\nu(y) \delta J_\mu(x)} \equiv < A^\mu(x), A^\nu(y) >, \quad (16)
\]

\[
\left(\frac{1}{i}\right)^2 \frac{\delta^2 Z_B}{\delta L_\nu(y) \delta L_\mu(x)} \equiv < B^\mu(x), B^\nu(y) >. \quad (17)
\]

But thanks to the duality we can replace $Z_B$ by $Z_B(A_\mu)$ or $Z_B(B_\mu)$. Using $Z_B(B_\mu)$ in (10) we find,

\[
e^2 < A^\mu(x) A^\nu(y) >= < B^\mu(x) B^\nu(y) > + \frac{i}{\alpha^2 \partial^2 + 4\alpha^2} (4\alpha^2 g - a + a^2 g \partial^2) \delta^\mu(x-y) + \frac{2\alpha}{\partial^2} \epsilon^{\mu\nu\lambda} \partial^\nu \delta(x-y). \quad (18)
\]

In a similar way, performing the variation \((\frac{1}{i})^2 \delta^2 Z_B(A_\mu)/(\delta L_\nu(x) \delta L_\nu(y))\) on (17), we obtain the relation,

\[
< B_\mu(x) B_\nu(y) >= [\epsilon g \frac{(4\alpha^2 g - a + a^2 g \partial^2)}{(ag\partial^2 - 1)^2} + 4\alpha^2 g^2 \partial^2 \partial^2 A_\mu(x)] \frac{1}{(ag\partial^2 - 1)^2} \epsilon^{\mu\nu\beta} \partial^{\nu} A^{\beta}(y)
\]

\[-2\epsilon g \frac{1}{(ag\partial^2 - 1)^2} + 4\alpha^2 g^2 \partial^2 \partial^2 A_\mu(y)]
\]

\[+ i \left[ g^2 \frac{(4\alpha^2 g - a + a^2 g \partial^2)}{(ag\partial^2 - 1)^2} + 4\alpha^2 g^2 \partial^2 \partial^\mu \partial^\nu \delta(x-y) + 2\alpha g^2 \frac{1}{(ag\partial^2 - 1)^2} + 4\alpha^2 g^2 \partial^2 \epsilon^{\nu\mu\lambda} \partial^\lambda \delta(x-y)\right].
\]
$$+ g(a g^2 - 1) \frac{1}{(a g^2 - 1)^2 + 4 \alpha^2 g^2 \partial^2} g_{\mu \nu} \delta(x - y).$$

(19)

From the above coupled Green functions, we can derive Green function identities containing exclusively $A_{\mu}$ or $B_{\mu}$ fields, which are given below in a compact form.

$$e^2 < A_{\mu}(x) A_\nu(y) > = e^2 g^2 M^{\mu \lambda}(x) M_{\nu \sigma}(y) < A_{\lambda}(x) A_\sigma(y) >$$

$$+ i \{ \frac{a}{a^2 \partial^2 + 4 \alpha^2} + g(a g^2 - 1) \phi \} g^{\mu \nu} \delta(x - y)$$

$$- g^2 (4 \alpha^2 g - a + a^2 g \partial^2) \phi \partial^\mu \partial^\nu \delta(x - y) + \{ \frac{2 \alpha}{\partial^2 (a^2 \partial^2 + 4 \alpha^2)} + 2 \alpha g^2 \phi \} \epsilon^{\mu \nu \lambda} \partial_\lambda \delta(x - y),$$

$$< B_{\mu}(x) B_\nu(y) > = g^2 M^{\mu \lambda}(x) M_{\nu \sigma}(y) < B_{\lambda}(x) B_\sigma(y) >$$

$$+ i \{ \frac{a g^2 \partial^2}{a^2 \partial^2 + 4 \alpha^2} (\frac{m_1^2}{\partial^2} + m_3^2) + g(a g^2 - 1) \phi \} g^{\mu \nu} \delta(x - y)$$

$$- \{ \frac{a g^2}{a^2 \partial^2 + 4 \alpha^2} (\frac{m_1^2}{\partial^2} + m_3^2) + g^2 (4 \alpha^2 g - a + a^2 g \partial^2) \phi \} \partial^\mu \partial^\nu \delta(x - y)$$

$$+ \{ \frac{2 \alpha g^2}{a^2 \partial^2 + 4 \alpha^2} (\frac{m_1^2}{\partial^2} + m_3^2) + 2 \alpha g^2 \phi \} \epsilon^{\mu \nu \lambda} \partial_\lambda \delta(x - y).$$

(20)

(21)

The operator $M^{\mu \lambda}(x)$ acts on $x$ and is defined below,

$$M^{\mu \lambda} = m_1 g^{\mu \lambda} + m_2 \partial^\mu \partial^\lambda + m_3 \epsilon^{\mu \nu \lambda} \partial_\nu$$

$$= (4 \alpha^2 g - a + a^2 g \partial^2) \phi [\partial^2 g^{\mu \lambda} - \partial^\mu \partial^\lambda] - 2 \alpha \phi \epsilon^{\mu \nu \lambda} \partial_\nu,$$

$$\phi = [(a g^2 - 1)^2 + 4 \alpha^2 g^2 \partial^2]^{-1}.$$

(22)

For the sake of comparison, we also write down the propagators for the Maxwell-Chern-Simons model,

$$L_{MCS} = -\frac{a}{4} A_{\mu \nu} A^{\mu \nu} + \alpha \epsilon^{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda + J_\mu A^\mu, \quad \partial . J = 0,$$

(23)

and the Maxwell-Chern-Simons-Proca model,

$$L_{MCSP} = -\frac{a}{4} B_{\mu \nu} B^{\mu \nu} + \alpha \epsilon^{\mu \nu \lambda} B_\mu \partial_\nu B_\lambda - \frac{1}{2 g} B_{\mu \nu} B^{\mu \nu} + L_\mu B^\mu.$$

(24)

The propagators are respectively,

$$< A^\mu(x) A^\nu(y) >_{MCS} = i \{ \frac{a}{a^2 \partial^2 + 4 \alpha^2} g^{\mu \nu} \delta(x - y) + \frac{2 \alpha}{\partial^2 (a^2 \partial^2 + 4 \alpha^2)} \epsilon^{\mu \nu \lambda} \partial_\lambda \delta(x - y) \},$$

(25)

$$< B^\mu(x) B^\nu(y) >_{MCSP} = \frac{i}{4 \phi} [2 g (a g^2 - 1) g^{\mu \nu} \delta(x - y) - 2 g^2 (4 \alpha^2 g + a^2 g \partial^2 - a) \partial^\mu \partial^\nu \delta(x - y)$$

$$+ 4 \alpha g^2 \epsilon^{\mu \nu \lambda} \partial_\lambda \delta(x - y)],$$

(26)

It is imperative to check the consistency of the above procedure through a qualitative analysis of (20) and (21), by considering limiting values of the parameters involved, specifically the fermion mass $m$ and the Thirring coupling $g$. 

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The bosonization result was obtained in the large $m$ limit. In keeping conformity with that, it is natural to keep only $O(m^{-1})$ terms and hence $a^2 \approx 0$. Indeed, this choice is by no means mandatory, since the bosonized ML is an independent field theory by itself. First we consider the large $g$ limit and keep terms of $O(a, g^{-1})$ only in the Green’s functions.

$$\frac{ae^2}{2\alpha^2 g} < A^\mu(x) A^\nu(y) > = e^2 (\frac{a}{2\alpha^2 g} - 1) < \frac{\partial^\mu \partial.}{\partial^2} A(x) \frac{\partial^\nu \partial.}{\partial^2} A(y)$$

$$+ A^\mu(x) \frac{\partial^\nu \partial.}{\partial^2} A(y) + \frac{\partial^\mu \partial.}{\partial^2} A(x) A^\nu(y) >$$

$$+ \frac{e^2}{2\alpha g} < [- A^\mu(x) \frac{\epsilon^{\nu\rho\sigma}}{2} \partial. A_\sigma(y) - A^\nu(y) \frac{\epsilon^{\mu\rho\sigma}}{2} \partial. A_\sigma(x) + \frac{\partial \partial.}{\partial^2} A(x) \frac{\epsilon^{\nu\rho\sigma}}{2} \partial. b_\sigma(y)$$

$$+ \frac{\epsilon^{\mu\rho\sigma}}{2} A_\sigma(x) \frac{\partial^\nu \partial.}{\partial^2} A(y)] >$$

$$+ i\left( \frac{a}{4\alpha^2} - \frac{1}{4\alpha^2 g \partial^2} \right) g^\mu\nu \delta(x - y) - \left( \frac{a}{4\alpha^2} + g \right) \frac{\partial^\mu \partial.}{\partial^2} \delta(x - y) + \frac{1}{\alpha^2} \epsilon^{\mu\nu\lambda} \partial. \delta(x - y). \right) \right)$$

$$$$

$$\left[ \frac{\partial^\nu \partial.}{\partial^2} B(y) + B^\mu(x) \frac{\partial^\nu \partial.}{\partial^2} B(y) + \left( \frac{\partial \partial.}{\partial^2} B(x) B^\nu(y) >$$

$$+ \frac{1}{2\alpha g} < [- B^\mu(x) \frac{\epsilon^{\nu\rho\sigma}}{2} \partial. B_\sigma(y) - B^\nu(y) \frac{\epsilon^{\mu\rho\sigma}}{2} \partial. B_\sigma(x) + \frac{\partial \partial.}{\partial^2} B(x) \frac{\epsilon^{\nu\rho\sigma}}{2} \partial. B_\sigma(y)$$

$$+ \frac{\epsilon^{\mu\rho\sigma}}{2} B_\sigma(x) \frac{\partial^\nu \partial.}{\partial^2} B(y)] >$$

$$+ i\left( \frac{a}{2\alpha^2} - \frac{1}{4\alpha^2 g \partial^2} \right) g^{\mu\nu} \delta(x - y) - \left( \frac{a}{2\alpha^2} + g \right) \frac{\partial^\mu \partial.}{\partial^2} \delta(x - y) + \frac{1}{2\alpha^2} \left( 1 + \frac{a}{2\alpha^2 g} \right) \epsilon^{\mu\nu\lambda} \partial. \delta(x - y). \right) \right) \right)$$

If we consider $g$-independent terms only, the $A_\mu$ and $B_\mu$ propagators become identical, but for an overall factor of $\frac{1}{2}$ in the $c$-number expressions. This is also the same as the $L_{MCS}$ propagator in (25). This is consistent with the fact that in the master lagrangian in (3), the difference between $A_\mu$ and $B_\mu$ disappears as $g \to \infty$ and the model tends to $L_{MCS}$ in the composite field $(B_\mu + eA_\mu)$.

In case of small $g$, we have $\phi \approx 1$, $m_1 \approx -2\alpha$ and $m_3 \approx -2\alpha$. This reproduces the following approximate propagators,

$$< B^\mu(x) B^\nu(y) > \approx -ig^{\mu\nu} \delta(x - y),$$

$$e^2 < A^\mu(x) A^\nu(y) > \approx i\left[ \frac{a}{a^2 \partial^2 + 4\alpha^2 g} g^{\mu\nu} \delta(x - y) + \frac{2\alpha}{\partial^2 (a^2 \partial^2 + 4\alpha^2)} \epsilon^{\mu\nu\lambda} \partial. \delta(x - y). \right) \right)$$

Notice that the $A_\mu$-propagator in (24) is identical to the MCS propagator in (24). This behaviour is justified in the limit of small $g$ where the mass term for $B_\mu$ in (1) becomes infinite and hence the $B_\mu$ field effectively decouples. This is also the reason for trivial form of the $B_\mu$ propagator in (25).

V: New dualities
We now consider two different ML’s, which can be thought of as large or small $e$-the electromagnetic coupling limit of the original ML in (4) considered in the beginning.

(i). Small $e$ limit:
This procedure produces a different ML where we drop $O(e^2)$ terms from starting ML in (4) and integrate $B_\mu$ first. The resulting partition function is

$$Z_1 = \int \mathcal{D}(A) \delta(\partial.A) e^{\pi(J_\mu A^\mu)} \int \mathcal{D}(B)[ -\frac{ae}{2} B_{\mu\nu} A^{\mu\nu} + e\alpha \epsilon_{\mu\nu\lambda} B^\mu A^\nu A^\lambda + J_\mu A^\mu - \frac{a}{4} B_{\mu\nu} B^{\mu\nu} + \alpha \epsilon_{\mu\nu\lambda} B^\mu B^\nu - \frac{1}{2g} B_\mu B^\mu + B_\mu L^\mu].$$

with $\partial.J = 0$. The result is

$$Z_1(A_\mu) = \int \mathcal{D}(A) \delta(\partial.A) e^{\pi(\tilde{J}_\mu A^\mu)} \int \mathcal{D}(B)[ -4J_\mu A^\mu + e^2 A_\mu \{2a^2 g(\partial^2)^2 (ag\partial^2 - 1)
+ 4\alpha^2 g^2 (ag\partial^2 + 1)\phi \} (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu
+ e^2 A_\mu \{4g^2 (aa^2 g^2 - 2a - 4a^2 g)\} \phi \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda
+ 4eg(A_\mu \{4a^2 g - a + a^2 g^2\} \phi (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu
- 8\alpha e g L^\mu \epsilon_{\mu\nu\lambda} \frac{1}{(ag\partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2 \partial^\nu A^\lambda}
- 2g^2 L^\mu \partial_\mu (4a^2 g - a + a^2 g^2) \frac{1}{(ag\partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2 \partial_\nu L^\nu
- 4\alpha g^2 L^\mu \frac{1}{(ag\partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2 \epsilon^{\mu\nu\lambda} \partial_\nu L_\lambda + 2g L_\mu (ag\partial^2 - 1) \frac{1}{(ag\partial^2 - 1)^2 + 4\alpha^2 g^2 \partial^2 L^\mu}].$$

In case of $A_\mu$ integration, the gauge fixing is really not necessary since $A_\mu$ appears only linearly in the action. The $A_\mu$ equation of motion produces the relation

$$- e\alpha \epsilon^{\mu\nu\lambda} B_{\nu\lambda} + e\alpha \epsilon_{\mu\nu\lambda} B^{\nu\lambda} + J_\mu = 0.$$  

Substituting the above relation in the action, the result is

$$Z_1(B_\mu) = \int \mathcal{D}(B) e^{\pi(-\frac{a}{2} B_{\mu\nu} B^{\mu\nu} - \frac{1}{2g} B_\mu B^\mu - \frac{1}{2e} B_\mu J_\mu + B_\mu L^\mu]}.$$

Without the sources, this is the Proca Model. However, the Green’s functions are,

$$4e^2 < A^\mu(x) A^\nu(y)> = e^2 g^2 M^{\mu\lambda}(x) M^{\nu\sigma}(y) < A_\lambda(x) A_\sigma(y) >
+ i \{g(\partial x) \delta(x - y) - g^2 (4a^2 g - a + a^2 g\partial^2) \phi \partial^{\nu\lambda} \partial^\lambda(x - y) + 2\alpha g^2 \phi \epsilon^{\mu\nu\lambda} \partial_\nu \delta(x - y)\},$$

where the operator $M^{\mu\nu}(x)$ is the same as in (22). The $B_\mu$ Green function is obtained from the above one by replacing $A_\mu$ by $B_\mu/(2e)$. The reason is that the variation by $-i\delta/\delta J_\mu$ reproduces $A_\mu$ and $-B_\mu/(2e)$ when operated upon $Z_1$ and $Z_1(B_\mu)$ respectively.

(ii) Large $e$ limit:
We drop the $e$-independent quadratic $B$-terms from the original ML.

$$Z_2 = \int \mathcal{D}(A) \delta(\partial.A) e^{\pi(-\frac{ae^2}{4} A^{\mu\nu} A_{\mu\nu} + \frac{ae^2}{2} \epsilon_{\mu\nu\lambda} A^\mu A^\nu A^\lambda + J_\mu A^\mu]}$$
\[
\int \mathcal{D}(B) e^{i\chi} \left[ -\frac{1}{2g} B_\mu B^\mu - \frac{ae}{2} B_{\mu\nu} A^{\mu\nu} + \alpha \epsilon^{\mu\nu\lambda} B_\mu A_\nu + L_\mu B^\mu \right].
\]

After \( B_\mu \) integration, the result is
\[
Z_2(A_\mu) = \int \mathcal{D}(A) \delta(\partial_\mu A_\mu) e^{i\chi} \left[ e^2 A_\mu \left( \frac{a}{2} + \frac{a^2 g}{2} \partial^2 - 2\alpha^2 g \right) \partial^\mu \partial^\nu A_\nu + 2\alpha g e^2 A_\mu \partial^2 \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \right.
\]
\[
+ eag L_\mu \partial_\nu A^{\mu\nu} + \frac{g}{2} L_\mu L^\mu + 2\alpha e g L^\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \right].
\]

Integrating out the \( A_\mu \) field in the Lorentz gauge gives,
\[
Z_2(B_\mu) = \int \mathcal{D}(B) e^{i\chi} \left[ \frac{a}{2} (B_\mu \partial^2 B^\mu - B_\mu \partial^\mu \partial^\nu B_\nu) + \frac{1}{2g} B_\mu B^\mu + B_\mu \left( \frac{1}{e} J^\mu - L^\mu \right) + \frac{a}{2e^2} J_\mu \frac{1}{\alpha^2 + 4\alpha^2} J^\mu \right.
\]
\[
- \frac{\alpha}{e^2} J_\mu \frac{1}{\alpha^2 + 4\alpha^2} \epsilon^{\mu\nu\lambda} \partial^\nu J^\lambda \right].
\]

Without the sources, this is the Chern-Simons-Proca model. The Green’s functions in this case are
\[
\langle A^{\mu}(x) A^\nu(y) \rangle = \frac{1}{e^2} N^{\mu\beta}(x) N^{\nu\lambda}(y) \langle A_\beta(x) A_\lambda(y) \rangle
\]
\[
+ \frac{i}{e^2} \left\{ \left( g + \frac{a}{a^2 \partial^2 + 4\alpha^2} \right) g^{\mu\nu} \delta(x - y) - \frac{2\alpha}{\partial^2(a^2 \partial^2 + 4\alpha^2)} \epsilon^{\mu\nu\lambda} \partial_\lambda \delta(x - y) \right\},
\]
\[
\langle B_\mu(x) B^\nu(y) \rangle = \frac{1}{e^2} N^{\mu\beta}(x) N^{\nu\lambda}(y) \langle B_\beta(x) B_\lambda(y) \rangle
\]
\[
+ i \left\{ g g^{\mu\nu} \delta(x - y) + \frac{1}{e^2} N^{\mu\beta}(x) N^{\nu\lambda}(y) \left( \frac{a}{a^2 \partial^2 + 4\alpha^2} g^{\beta\lambda} \delta(x - y) + \frac{2\alpha}{\partial^2(a^2 \partial^2 + 4\alpha^2)} \epsilon^{\beta\lambda\sigma} \partial^\sigma \delta(x - y) \right) \right\},
\]

The operator \( N^{\mu\nu} \) is defined as
\[
N^{\mu\nu} = e a g \partial^2 g^{\mu\nu} - e a g \partial^\mu \partial^\nu - 2\alpha g \epsilon^{\mu\nu\lambda} \partial_\lambda.
\]

**VI: Conclusions**

Bosonized version of the gauged, massive Thirring model is analysed exhaustively at the level of effective action. Duality invariance is used to construct exactly equivalent gauge invariant and non-invariant models. The correlation functions of the bosonized models are studied. Validity of this new scheme to achieve this is tested in different coupling regimes. Depending on the relative strengths of the electromagnetic and Thirring couplings, some new dual models, including their propagators, are also presented.

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