On the integrability of a generalized variable-coefficient Kadomtsev–Petviashvili equation

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Abstract
By considering the inhomogeneities of media, a generalized variable-coefficient Kadomtsev–Petviashvili (vc-KP) equation is investigated, which can be used to describe many nonlinear phenomena in fluid dynamics and plasma physics. In this paper, we systematically investigate the complete integrability of the generalized vc-KP equation under an integrable constraint condition. With the aid of generalized Bell’s polynomials, its bilinear formalism, bilinear Bäcklund transformations, Lax pairs and Darboux covariant Lax pairs are succinctly constructed, which can be reduced to the ones of several integrable equations such as KdV, cylindrical KdV, KP, cylindrical KP, generalized cylindrical KP, non-isospectral KP equations, etc. Moreover, the infinite conservation laws of the equation are found by using its Lax equations. All conserved densities and fluxes are given with explicit recursion formulas. Furthermore, an extra auxiliary variable is introduced to obtain the bilinear formalism, based on which, the soliton solutions and Riemann theta function periodic wave solutions are presented. The influence of inhomogeneity coefficients on solitonic structures and interaction properties are discussed for physical interest and possible applications by some graphic analysis. Finally, a limiting procedure is presented to analyze in detail the asymptotic behavior of the periodic waves and the relations between the periodic wave solutions and soliton solutions.

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(Some figures may appear in colour only in the online journal)
1. Introduction

It is important to investigate the integrability of the nonlinear evolution equation (NLEE), which can be regarded as a pretest and the first step of its exact solvability. There are many significant properties, such as bilinear form, Lax pairs, infinite conservation laws, infinite symmetries, Hamiltonian structure, Painlevé test and bilinear Bäcklund transformation, that can characterize the integrability of nonlinear equations. Although there have been many methods proposed to deal with the NLEEs, e.g., inverse scattering transformation [1], Darboux transformation [2], Bäcklund transformation (BT) [3], Hirota method [4] and so on, by using the bilinear form for a given NLEE, one can not only construct its multisoliton solutions, but also derive the bilinear BT and some other properties [4–7]. Unfortunately, one of the key steps of this method is to replace the given NLEE by some more tractable bilinear equations for new Hirota’s variables. There is no general rule to find the transformations, nor for choice or application of some essential formulas (such as exchange formulas). In the early 1930s, Bell proposed the classical Bell polynomials, which are specified by a generating function and exhibiting some important properties [8]. Since then the Bell polynomials have been exploited in combinatorics, statistics and other fields [11–13]. However, in recent years Lambert and co-workers have proposed an alternative procedure based on the use of the Bell polynomials to obtain parameter families of bilinear Bäcklund transformation and lax pairs for soliton equations in a lucid and systematic way [8–10]. The Bell polynomials are found to play an important role in the characterization of bilinearizable equations and the relationship between the integrability of a nonlinear equation and the Bell polynomials.

Recently, there has been growing interest in studying the variable-coefficient NLEEs, which are often considered to be more realistic than their constant-coefficient counterparts in modeling a variety of complex nonlinear phenomena under different physical backgrounds [14]. Since those variable-coefficient NLEEs are of practical importance, it is meaningful to systematically investigate completely integrable properties such as bilinear form, Lax pairs, infinite conservation laws, infinite symmetries, Hamiltonian structure, Painlevé test, bilinear Bäcklund transformation, symmetry algebra and construct various exact analytic solutions, including the soliton solutions and periodic solutions. For describing the propagation of solitonic waves in inhomogeneous media, the variable-coefficient KP-type equations have been derived from many physical applications in plasma physics, fluid dynamics and other fields [15, 16].

In this paper, we focus on a generalized variable-coefficient Kadomtsev–Petviashvili (vc-KP) equation with nonlinearity, dispersion and perturbed term,

\[
[u_t + h_1(y,t)u_{xx} + h_2(y,t)u_{uu}]_x + h_3(y,t)u_{2x} + h_4(y,t)u_{xy} + h_5(y,t)u_{2y} + h_6(y,t)u_x + h_7(y,t)u_y = 0, \tag{1.1}
\]

where \( u \) is a differentiable function of \( x, y \) and \( t \), and \( h_i(y, t), \ i = 1, \ldots, 7, \) are all analytic, sufficiently differentiable functions, which may provide a more realistic model equation in several physical situations, e.g. in the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying depth and width and nonvanishing vorticity. Equation (1.1) can reduce to a series of integrable models or describe such physical phenomena as the electrostatic wave potential in plasma physics, the amplitude of the shallow-water wave and/or surface wave in fluid dynamics, etc [16–19]. Obviously, equation (1.1) contains quite a number of variable-coefficient KP models arising from various branches of physics, e.g. the KdV, cylindrical KdV, KP, cylindrical KP, generalized cylindrical KP and non-isospectral KP equations etc. Some currently important examples are given below.
• The celebrated historic Korteweg–de Vries (KdV) equation [1, 20]

\[ u_t + 6uu_x + u_{xxx} = 0 \]  

(1.2)

has been found to model many physical, mechanical and engineering phenomena, such as ion-acoustic waves, geophysical fluid dynamics, lattice dynamics, the jams in congested traffic, etc.

• The Kadomtsev–Petviashvili (KP) equation [21]

\[ (u_t + 6uu_x + u_{xx})_x + \sigma_0 u_{xy} = 0, \]  

(1.3)

where \( \sigma_0 = \pm 1 \), has been discovered to describe the evolution of long water waves, small-amplitude surface waves with weak nonlinearity, weak dispersion and weak perturbation in the \( y \) direction, weakly relativistic soliton interactions in the magnetized plasma and some other nonlinear models.

• The cylindrical KdV equation [22, 23]

\[ u_t + 6uu_x + u_{xx} + \frac{1}{2\sigma} u_x = 0 \]  

(1.4)

was first proposed by Maxon and Viecelli in 1974 when they studied propagation of radially ingoing acoustic waves. It has a \((2+1)\) dimensional counterpart, the cylindrical KP equation [24, 25] and generalized cylindrical KP equation [17, 26],

\[ (u_t + 6uu_x + u_{xx})_x + \frac{\sigma_0^2}{I^2} u_{xy} + \frac{1}{2\sigma} u_x = 0, \]  

(1.5)

\[ (u_t + h_2(t) uu_x + h_1(t) u_{xx})_x + [f(t) + yg(t)] u_{xx} + r(t) u_{xy} + \frac{3\sigma_0^2}{I^2} u_{xy} + \frac{1}{2\sigma} u_x = 0, \]  

(1.6)

with \( \sigma_0^2 = \pm 1 \), have also been constructed to describe the nearly straight wave propagation which varies in a very small angular region [17, 24–26].

• The KP equation with time-dependent coefficients [18],

\[ (u_t + uu_x + u_{xx})_x + \mu_3(t) u_x + \mu_4(t) u_{xy} = 0, \]  

(1.7)

models the propagation of small-amplitude surface waves in straits or large channels of slowly varying depth and width and nonvanishing vorticity.

• Jacobi elliptic function solutions and integrability property for the variable-coefficient KP equation

\[ (u_t + h_1(t) uu_x + h_2(t) u_{xx})_x + h_3(t) u_{xx} + 6h_4(t) u_x = 0 \]  

(1.8)

have been presented in [27].

• The equation

\[ (u_t + h_1(t) uu_x + h_2(t) u_{xx})_x + h_3(t) u_{xx} + h_4(t) u_{xy} = 0 \]  

(1.9)

can be used to describe nonlinear waves with a weakly diffracted wave beam, internal waves propagating along the interface of two fluid layers, etc [19].

• Non-isospectral and variable-coefficient KP equations read [28]

\[ (u_t + uu_x + u_{xx})_x + au_x + bu_y + cu_{xy} + du_{yy} + eu_{xx} = 0, \]  

(1.10)

\[ u_t + h_1(u_{xx} + 6uu_x + 3\sigma^2 \partial_x^{-1} u_{yy}) + h_2(u_x - \sigma xu_y - 2\sigma \partial_x^{-1} u_y) - h_3(xu_x + 2u + 2yu_y) = 0, \]  

(1.11)

where \( a, b, c, d \) and \( e \) are functions of \( y \) and \( t \), and \( h_i \ (i = 1, 2, 3) \) are functions of \( t \). Bilinear representations, bilinear Bäcklund transformations and Lax pairs for non-isospectral KP equations (1.10) and (1.11) are systematically investigated, respectively, in [28].
As is well known, the KdV, cylindrical KdV, KP, cylindrical KP, generalized cylindrical KP and non-isospectral KP equations belong to the integrable hierarchy of the KP equation. In recent years, a large number of papers have focused on the Painlevé property, dromion-like structures and various exact solutions of the NLEE [29–48]. But their integrability, to the best of our knowledge, has not been studied in detail. The existence of infinite conservation laws can be considered one of the many remarkable properties deemed to characterize soliton equations. Under certain constraint conditions, the variable-coefficient models may be proved to be integrable given explicit analytic solutions. The corresponding constraint conditions on equation (1.1) in this paper, which can be naturally found in the procedure of applying the Bell polynomials, will be

\[ h_2 = c_0 h_1 e^{h_4 u}, \quad \partial_t h_4 = h_6 + \partial_t \ln h_1 h_2^{-1}, \quad h_5 = 3\alpha^2 h_1, \quad \partial_t h_1 = \partial_y h_2 = h_7 = 0, \]

(1.12)

where \( c_0 \) and \( \alpha \) are the arbitrary parameters.

The main purpose of this paper is to extend the binary Bell polynomial approach to systematically construct the bilinear formalism, bilinear Bäcklund transformations, Lax pairs and Darboux covariant Lax pairs of the generalized vc-KP equation (1.1) under conditions (1.12). To our knowledge, there have been no discussions about equation (1.1) under conditions (1.12). Based on its Lax equations, the infinite conservation laws of the equation will be constructed. By using the bilinear formula, the soliton solutions and Riemann theta function periodic wave solutions are also presented.

The structure of this paper is as follows. By virtue of the properties of the binary Bell polynomials, we systematically construct the bilinear representation, Bäcklund transformation, Lax pair and Darboux covariant Lax pairs of the generalized vc-KP equation (1.1) in sections 2–4, respectively. By means of its Lax equation, in section 5, the infinite conservation laws of the equation are also constructed. In section 6, based on the bilinear formula and the recent results in [51, 52], we present the soliton solutions and Riemann theta function periodic wave solutions of the generalized vc-KP equation (1.1) under conditions (1.12) with \( c_0 = 6 \). We also discuss the influence of inhomogeneity coefficients on solitonic structures and interaction properties for physical interest and possible applications by some graphic analysis. Finally, a limiting procedure is presented to analyze in detail the relations between the periodic wave solutions and soliton solutions. Some introduction of multidimensional Bell polynomials and Riemann theta function wave is given in appendices A and B, respectively.

2. Bilinear representation

In this section, we construct the bilinear representation of equation (1.1) by using an extra auxiliary variable instead of the exchange formulas.

**Theorem 2.1.** Using the transformation

\[ u = 12h_1 h_2^{-1} (\ln f)_{xx}, \]

(2.1)

the generalized vc-KP equation (1.1) can be bilinearized into

\[ \mathcal{D}(D_x, D_y, D_z) \equiv [D_x D_z + h_1 D_x^2 + h_2 D_z^2 + h_3 D_x D_y + h_4 D_y^2 + (h_6 + \partial_t \ln h_1 h_2^{-1}) \partial_z + h_7 \delta - \delta] f \cdot f = 0, \]

(2.2)

where \( \partial_t f \cdot f \equiv \partial_x f^2 = 2f f_x, \partial_y f \cdot f \equiv \partial_y f^2 = 2f f_y, \partial_z f \cdot f \equiv \delta f^2, \) and \( \delta = \delta(y, t) \) is a constant of integration.
Proof. In order to detect the existence of linearizable representation of equation (1.1), we introduce a potential field $q$ by setting
\[ u = c(t)q_{2x}, \tag{2.3} \]
with $c = c(t)$ being a free function to be the appropriate choice such that equation (1.1) connects with $\mathcal{P}$-polynomials. Substituting transformation (2.3) into equation (1.1), we can write the resulting equation in the form
\[ q_{2x,t} + h_1q_{5x} + ch_2q_{2x}q_{3x} + h_3q_{3x} + h_4q_{2x,xy} + h_5q_{x,y} + (h_6 + \partial_t \ln c)q_{2x} + h_7q_{xy} = 0, \tag{2.4} \]
where we will see that such decomposition is necessary to obtain the bilinear form of equation (1.1). Moreover integrating equation (2.4) with respect to $x$ yields
\[ E(q) = q_{xt} + h_1(q_{4x} + 3q_{2x}^2) + h_3q_{3x} + h_5q_{xy} + (h_6 + \partial_t \ln h_1h_2^{-1})q_x + h_7q_y = \delta \tag{2.5} \]
if one chooses the function $c(t) = 6h_1h_2^{-1}$ and uses formula (A.7), where $\delta = \delta(x,t)$ is a constant of integration. Equation (2.5) can be cast into a combination form of $\mathcal{P}$-polynomials by using formula (A.7),
\[ E(q) = P_0(q) + h_1P_1(q) + h_2P_2(q) + h_3P_3(q) + h_4P_4(q) + (h_6 + \partial_t \ln h_1h_2^{-1})q_x + h_7q_y = \delta. \tag{2.6} \]
Finally, according to property (A.9), under the change of the dependent variable,
\[ q = 2\ln f \quad \iff \quad u = c(t)q_{2x} = 12h_1h_2^{-1}(\ln f)_{xx}, \tag{2.7} \]
equation (2.6) produces the same bilinear representation $\mathcal{D}$ (2.2) of the generalized vc-KP equation (1.1).

Formula (2.2) is a new bilinear form, which can also reduce to the ones obtained in [4, 7, 21, 24, 25, 49, 50] by choosing the appropriate coefficients $h_i$ (i = 1, . . . , 7).

(i) If $h_i = 0$ (i = 3, 4, 5, 6, 7), $h_1 = 1$ and $h_2 = 6$, equation (1.1) becomes the constant-coefficient KdV equation. The corresponding bilinear form (2.2) reduces to
\[ [D_xD_t + D_x^2]f \cdot f = 0, \tag{2.8} \]
which is also obtained in [4, 7, 49, 50].

(ii) In the case of $h_i = 0$ (i = 3, 4, 5, 6, 7), $h_1 = 1$, $h_2 = 6$ and $h_5 = \pm 1$, equation (1.1) reduces to a general KP equation. The corresponding bilinear form (2.2) becomes
\[ [D_xD_t + D_x^2 \pm D_y^2]f \cdot f = 0, \tag{2.9} \]
which is also researched in [4, 21, 49].

(iii) Assuming that $h_i = 0$ (i = 3, 4, 7), $h_5 = 3 \sigma_0^2/\tau^2$ and $h_6 = 1/2\tau$, equation (1.1) becomes the cylindrical KP model [24, 25]. The corresponding bilinear form (2.2) reduces to
\[ [D_xD_t + h_1D_x^3 + 3\sigma_0^2/\tau^2D_x^2 + (h_6 + \partial_t \ln h_1h_2^{-1})\partial_x]f \cdot f = 0, \tag{2.10} \]
with $\sigma_0$ an arbitrary constant, which is a new bilinear formalism for the cylindrical KP model.
3. The bilinear Bäcklund transformation and associated Lax pair

In this section, we construct the bilinear Bäcklund transformation and the Lax pair of the generalized vc-KP equation (1.1). The bilinear Bäcklund transformation is useful in constructing solutions and also serves as a characteristic of integrability for a given system.

In the following, we derive a bilinear Bäcklund for the generalized vc-KP equation (1.1) by using the use of binary Bell polynomials.

**Theorem 3.1.** Suppose that $f$ is a solution of the bilinear equation (2.2) under conditions (1.12), i.e. the coefficients $h_i$ ($i = 1, 2, 5, 6, 7$) satisfy $h_2 = c_0 h_1 e^{\alpha x}$, $h_5 = 3\alpha^2 h_1$, $h_7 = 0$; then $g$ satisfying

$$
(D_x^2 + \alpha D_y - \lambda) f \cdot g = 0,
$$

$$
(D_x + h_1 (D_x^3 - 3\alpha D_x D_y + 3\lambda D_y) + h_3 D_y + h_6 D_y + \gamma) f \cdot g = 0
$$

(3.1)

is another solution of equation (2.2), where $c_0$ and $\alpha$ are arbitrary parameters and $\gamma = \gamma(y, t)$ is an arbitrary function. So the system (3.1) is called a bilinear Bäcklund transformation for the generalized vc-KP equation (1.1).

**Proof.** In order to obtain the bilinear Bäcklund transformation of the generalized vc-KP equation (1.1), let

$$
q = 2 \ln g, \quad q' = 2 \ln f
$$

(3.2)

be two different solutions of equation (2.5), respectively. We associate equation (3.2) with the two-field condition from equation (2.5)

$$
E(q') - E(q) = \frac{(q' - q)_{xy} + h_1 (q' - q)_{4x} + 3h_1 (q' + q)_{2x} + h_3 (q' - q)_{2x}}{x}
$$

$$
+ h_4 (q' - q)_{xy} + h_5 (q' - q)_{2y} \left( (h_6 + \alpha \ln h_2) (q' - q)_{x} + h_7 (q' - q)_{y} \right) = 0.
$$

(3.3)

This two-field condition can be regarded as an ansatz for a bilinear Bäcklund transformation and may produce the required transformation under appropriate additional constraints.

To find such constraints, we introduce the new auxiliary variables

$$
u = \frac{(q' - q)}{2} = \ln(f/g), \quad \omega = \frac{(q' + q)}{2} = \ln(g/f),
$$

(3.4)

and then rewrite condition (3.3) as the form

$$
E(q') - E(q) = E(\omega + \nu) - E(\omega - \nu) = \nu_{xy} + h_1 (\nu_{4x} + 6\omega \nu_{2x}) + h_3 \nu_{2x} + h_4 \nu_{xy}
$$

$$
+ h_5 \nu_{2y} + (h_6 + \alpha \ln h_2) \nu_x + h_7 \nu_y
$$

$$
= \partial_x [\mathcal{R}(\nu) + h_1 \mathcal{B}_x(\nu, \omega)] + \mathcal{R}(\nu, \omega) = 0,
$$

(3.5)

with

$$
\mathcal{R}(\nu, \omega) = 3h_1 \text{Wronskian}[\mathcal{B}_x(\nu, \omega), \mathcal{B}_y(\nu)] + h_3 \nu_{2x} + h_4 \nu_{xy} + h_5 \nu_{2y}
$$

$$
+ (h_6 + \alpha \ln h_2) \nu_x + h_7 \nu_y.
$$

In order to express $\mathcal{R}(\nu, \omega)$ as the form of the $x$-derivative of a combination of $\mathcal{B}$-polynomials and to decouple the two-field condition (3.5) into a pair of constraints, we choose a constraint

$$
\mathcal{B}_x(\nu, \omega) + \alpha \mathcal{B}_y(\nu, \omega) = \lambda,
$$

(3.6)
where $\alpha = \alpha(t)$ is to be determined and $\lambda$ is an arbitrary parameter. On account of equation (3.6), $\mathcal{R}(\nu, \omega)$ can be obtained as follows:

$$
\mathcal{R}(\nu, \omega) = 3h_1\lambda \nu_{2x} - \alpha^{-1}[h_5 \omega_{2x, y} + (2h_5 - 3\alpha^2 h_1) \nu_1 \nu_{1,y} + 3\alpha^2 h_1 \nu_{2x, y} + h_3 \nu_{2x} + h_4 \nu_{1,y}]
+ (h_6 + \partial_t \ln h_1 h^{-1}_2) \nu_t + h_7 \nu_y,
$$

which can also be rewritten in the form

$$
\mathcal{R}(\nu, \omega) = \partial_t [(3h_1 \lambda + h_3) \mathcal{I}_y(\nu) - 3\alpha h_1 \mathcal{I}_y(\nu, \omega) + h_4 \mathcal{I}_y(\nu)]
$$

if we take simple constraints

$$
h_5 = 2h_5 - 3\alpha^2 h_1 = 3\alpha^2 h_1, \quad h_6 + \partial_t \ln h_1 h^{-1}_2 = 0, \quad h_7 = 0,
$$

namely

$$
h_2 = c_0 h_1 e^{\rho h_0 \nu}, \quad h_5 = 3\alpha^2 h_1, \quad h_7 = 0.
$$

Then, combining relations (3.6)–(3.8), we deduce a coupled system of $\mathcal{I}$-polynomials:

$$
\mathcal{I}_{2x}(\nu, \omega) + \alpha \mathcal{I}_y(\nu, \omega) - \lambda = 0,
\partial_x \mathcal{I}_y(\nu) + \partial_t [h_1 \mathcal{I}_{2x}(\nu, \omega) - 3\alpha \mathcal{I}_y(\nu, \omega) + 3\lambda \mathcal{I}_y(\nu)] + h_3 \mathcal{I}_y(\nu) + h_4 \mathcal{I}_y(\nu) = 0.
$$

(3.10)

By the application of identity (A.6), the system (3.10) immediately leads to the bilinear Bäcklund transformation (3.1) where $\gamma = \gamma(t)$ is an arbitrary function.

The Bäcklund transformation (3.1) can be applied to construct exact solutions for the generalized vc-KP equation (1.1). Next, using the system (3.10), we will derive Lax pairs of equation (1.1).

**Theorem 3.2.** Under conditions (1.12) and $c_0 = 6$, the generalized vc-KP equation (1.1) admits a Lax pair

$$
(\mathcal{Z}_1 + \alpha \mathcal{Z}_2) \psi \equiv \psi_{2x} + \alpha \psi_y + (u e^{\rho h_0 \nu} - \lambda) \psi = 0,
$$

(3.11a)

$$
(\partial_t + \mathcal{Z}_2) \psi \equiv \psi_t + 4h_1 \psi_{3x} - h_4 \alpha^{-1} \psi_{2x} + (6h_1 u e^{\rho h_0 \nu} + 3h_1 \lambda + h_3) \psi_x + (3h_1 u_e e^{\rho h_0 \nu} - 3h_1 \alpha^{-1} u e^{\rho h_0 \nu} + h_4 \alpha^{-1} \lambda) \psi = 0,
$$

(3.11b)

where $u$ is a solution of equation (1.1).

**Proof.** In order to linearize the Bell systems (3.10) into a Lax pair, we make the Hopf–Cole transformation $\nu = \ln \psi$. It follows from formulas (A.8) and (A.9) that

$$
\mathcal{I}_x(\nu) = \psi_x/\psi, \quad \mathcal{I}_{2x}(\nu, \omega) = q_{2x} + \psi_{2x}/\psi, \quad \mathcal{I}_y(\nu, \omega) = q_{3y} + \psi_{3y}/\psi, \quad \mathcal{I}_y(\nu) = \psi_y/\psi, \quad \mathcal{I}_y(\nu) = \psi_y/\psi, \quad \mathcal{I}_{3x}(\nu, \omega) = 3q_{2x} \psi_x/\psi + \psi_{3x}/\psi,
$$

on account of which the system (3.10) is then linearized into a system with double parameters $\lambda$ and $\gamma$,

$$
(\mathcal{Z}_1 + \alpha \mathcal{Z}_2) \psi \equiv \psi_{2x} + \alpha \psi_y + (q_{2x} - \lambda) \psi = 0,
$$

(3.12a)

$$
(\partial_t + \mathcal{Z}_2) \psi \equiv \psi_t + 4h_1 \psi_{3x} - h_4 \alpha^{-1} \psi_{2x} + (6h_1 q_{2x} + 3h_1 \lambda + h_3) \psi_x
+ (3h_1 q_{3x} - 3h_1 \alpha q_{3y} - h_4 \alpha^{-1} q_{2x} + h_4 \alpha^{-1} \lambda) \psi = 0,
$$

(3.12b)

which is equivalent to the Lax pair (3.11a) and (3.11b), respectively, by replacing $q_{2x}$ with $u e^{\rho h_0 \nu}$. □
Corollary 3.3. Using conditions (1.12) and $c_0 = 6$, the Lax pair (3.11a) and (3.11b) of the generalized vc-KP equation (1.1) is equivalent to the following Lax pair:

\[(\Box_1 + \alpha a_3)\psi \equiv \psi_{2x} + \alpha^2 \psi_y + (u \alpha \cdot h_0 \omega - \lambda) \psi = 0, \quad (3.13a)\]

\[(\partial_t + \Box_2)\psi \equiv \psi_t - 4h_1 \alpha \psi_{xy} - (h_1 u \alpha \cdot h_0 \omega - 7h_1 \lambda - h_3) \psi_x + h_2 \psi_y - (h_1 u \alpha \cdot h_0 \omega + 3h_1 \alpha a_3^{-1} u \alpha \cdot h_0 \omega) \psi = 0, \quad (3.13b)\]

where $u$ is a solution of equation (1.1).

Formulas (3.1) and (3.11a), (3.11b) are the new bilinear Bäcklund transformation and Lax pair, respectively, which can also reduce to the ones obtained in [1, 4, 17–20, 24–27, 29, 30] by choosing the appropriate coefficients $h_i$ $(i = 1, \ldots, 7)$. Without loss of generality, take $c_0 = 6$; then $c(t) = e^{-h_0 \omega} dt$.

(i) Assuming that $\alpha = h_1 = 0$ $(i = 3, 4, 5, 6, 7)$, and $h_1 = 1$, $h_2 = 6$, equation (1.1) becomes the general KdV model. The corresponding Bäcklund transformation (3.1) reduces to

\[
(D_3^2 - \lambda) f \cdot g = 0,
\]

\[
[D_1 + D_3^2 + 3\lambda D_3] f \cdot g = 0,
\]

which is studied in [4, 30]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

\[(\Box_1 + \alpha a_3)\psi \equiv \psi_{2x} + (\alpha^2 \psi_y \equiv 0, \quad (3.15a)\]

\[(\partial_t + \Box_2)\psi \equiv \psi_t + 4 \psi_{3x} + (2 \alpha + \lambda) \psi_x + 3u_x \psi = 0, \quad (3.15b)\]

where $u$ is a solution of equation (1.1). The Lax pair (3.15a) and (3.15b) is investigated by Lax, Ablowitz and co-workers in [1, 20], respectively.

(ii) For $h_1 = 0$ $(i = 3, 4, 7)$, and $h_1 = 1/2$, $h_2 = 6/2$, $h_3 = 3\sigma_0^2/12$, $h_6 = 1/2$, equation (1.1) becomes the cylindrical KP equation [24, 25]. The corresponding formula (3.1) reduces to

\[(D_3^2 - \lambda) f \cdot g = 0,
\]

\[
[D_1 + 1/2 (D_3^2 - 3\sigma_0 D_3) + \gamma] f \cdot g = 0,
\]

which is a new one and not obtained in [24, 25]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

\[(\Box_1 + \alpha a_3)\psi \equiv \psi_{2x} + \sigma_0 \psi_y + (u \sqrt{\gamma} - \lambda) \psi = 0, \quad (3.17a)\]

\[(\partial_t + \Box_2)\psi \equiv \psi_t + 4 \psi_{3x} + (6u \sqrt{\gamma} \psi_{3x} + 3\sqrt{\gamma} \psi_3^x + (3u_x \sqrt{\gamma} \psi_{3x} - 3\sigma_0 a_3^{-1} u_x \sqrt{\gamma}) \psi = 0, \quad (3.17b)\]

where $u$ is a solution of equation (1.1). The Lax pair (3.17a) and (3.17b) is a new one, which is not studied in [24, 25].

(iii) In the case of $h_1 = 1/2$, $h_2 = 6/2$, $h_3 = f(t) + yg(t)$, $h_4 = r(t)$, $h_5 = 3\sigma_0^2/12$, $h_6 = 1/2$, $h_7 = 0$, equation (1.1) becomes a generalized cylindrical KP equation [17, 26]. The corresponding formula (3.1) reduces to

\[(D_3^2 - \lambda) f \cdot g = 0,
\]

\[
[D_1 + 1/2 (D_3^2 - 3\sigma_0 D_3) + (f + yg) D_3 + rD_y + \gamma] f \cdot g = 0, \quad (3.18)\]
which is also a new one and not obtained in [17, 26]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

\[(\partial_t + \alpha \partial_x) \psi \equiv \psi_{2x} + \sigma_0 \psi_x + (u \sqrt{t} - \lambda) \psi = 0, \quad (3.19a)\]

\[(\partial_t + \partial_x) \psi \equiv \psi_t + 4/t^2 \psi_{3x} - \sigma_0^{-1} r(t) \psi_{2x} + [6u \sqrt{t}/t^2 + 3 \lambda/t^2 + (f(t) + yg(t))] \psi_x
+ [3u \sqrt{t}/t^2 - 3\sigma_0 \partial_x^{-1} u \sqrt{t}/t^2 - \sigma_0^{-1} r(t)u \sqrt{t} + \sigma_0^{-1} r(t) \lambda] \psi = 0, \quad (3.19b)\]

where \(u\) is a solution of equation (1.1). The lax pair (3.19a) and (3.19b) is a new one, which is not obtained in [17, 26].

(iv) If \(h_1 = f_2(t)\), \(h_2 = f_1(t)\), \(h_5 = g^2(t)\), \(h_6 = 6f(t)\), \(h_7 = 0\) \((i = 3, 4, 7)\), equation (1.1) becomes a variable-coefficient KP equation [27]. The corresponding formula (3.1) reduces to

\[(D_t^2 + \sigma_0 D_x - \lambda) f \cdot g = 0, \quad (3.20)\]

which is also a new one and not studied in [27]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

\[(\partial_t + \alpha \partial_x) \psi \equiv \psi_{2x} + [g(t)]/\sqrt{3f_2(t)} \psi_x + (u e^{f(t)} \psi - \lambda) \psi = 0, \quad (3.21a)\]

\[(\partial_t + \partial_x) \psi \equiv \psi_t + 4f_2(t) \psi_{3x} + (6f_2(t)u e^{f(t)} \psi + 3f_2(t) \lambda) \psi_x + (3f_2(t)u e^{f(t)} \psi
- 5f_2(t)g(t)]/\sqrt{3f_2(t)} \psi_x + (u e^{f(t)} \psi - \lambda) \psi = 0, \quad (3.21b)\]

where \(u\) is a solution of equation (1.1). The lax pair (3.21a) and (3.21b) is a new one, which is not obtained in [27].

(v) Suppose \(h_i = h_i(t)\) \((i = 1, 2, 3, 5)\), \(h_4 = 0\) \((j = 4, 6, 7)\), equation (1.1) becomes a generalized variable coefficient KP equation [18, 19, 29]. The corresponding formula (3.1) reduces to

\[(D_t^2 + \alpha D_x - \lambda) f \cdot g = 0, \quad (3.22)\]

which is also a new one and not obtained in [18, 19, 29]. The corresponding Lax pair (3.11a) and (3.11b) reduces to

\[(\partial_t + \alpha \partial_x) \psi \equiv \psi_{2x} + \sqrt{h_5}/3h_1 \psi_x + (u - \lambda) \psi = 0, \quad (3.23a)\]

\[(\partial_t + \partial_x) \psi \equiv \psi_t + 4h_1 \psi_{3x} + (6h_1 u + 3h_1 \lambda + h_1) \psi_x
+ (3h_1 u - 3h_1 \sqrt{h_5}/3h_1 \alpha^{-1} u) \psi = 0, \quad (3.23b)\]

where \(u\) is a solution of equation (1.1). The lax pair (3.23a) and (3.23b) is a new one, which is not obtained in [18, 19, 29].

Starting from Lax pairs and Darboux transformation, the soliton-like solutions of the generalized vc-KP equation (1.1) can be established.
4. Darboux covariant Lax pair

**Theorem 4.1.** Using the associated Lax pair (3.12a), (3.12b) and assuming that the parameter \( \lambda \) is independent of the variables \( x, y \) and \( t \), the generalized vc-KP equation (1.1) admits a kind of Darboux covariant Lax pair as follows:

\[
(\hat{\mathcal{L}}_1 + \alpha \partial_y) \phi = \lambda \phi, \quad \hat{\mathcal{L}}_1 = \partial_x^2 + q_{2x}, \tag{4.1a}
\]

\[
(\partial_t + \hat{\mathcal{L}}_{2,\text{cov}}) \phi = 0, \quad \hat{\mathcal{L}}_{2,\text{cov}} = 4h_1 \partial_x^3 - h_4 \alpha^{-1} \partial_x^2 + (6h_1 q_{2x} + h_3) \partial_t + 3h_1 q_{3x}
\]

whose form is Darboux covariant, namely

\[
T(\hat{\mathcal{L}}_1 + \alpha \partial_y)(q) T^{-1} = (\hat{\mathcal{L}}_1 + \alpha \partial_y)(\hat{q}), \tag{4.2a}
\]

\[
T(\partial_t + \hat{\mathcal{L}}_{2,\text{cov}})(q) T^{-1} = (\partial_t + \hat{\mathcal{L}}_{2,\text{cov}})(\hat{q}), \tag{4.2b}
\]

with \( \hat{q} = q + 2 \ln \phi \), under a certain gauge transformation

\[
T = \phi \partial_x \phi^{-1} = \partial_x - \sigma, \quad \sigma = \partial_t \ln \phi. \tag{4.3}
\]

The integrability condition of the Darboux covariant Lax pair (4.1a) and (4.1b) precisely gives rise to equation (1.1) in Lax representation,

\[
[\partial_t + \hat{\mathcal{L}}_{2,\text{cov}}, \hat{\mathcal{L}}_1 + \alpha \partial_y] = [q_{2x}, + h_1(q_{4x} + 3q_{22x}) + h_2 q_{2x} + h_3 q_{3x} + h_4 q_{23x}]_t = 0 \tag{4.4}
\]

if one chooses \( \lambda_h = h_5 + \partial_t \ln h_1 h_2^{-1}, \lambda_h = h_7 = 0 \). Equation (4.4) is equivalent to equation (2.5), which implies that Lax equations (4.1a) and (4.1b) are also a Lax pair for the generalized vc-KP equation (1.1).

**Proof.** Suppose that \( \phi \) is a solution of the Lax pair (3.12a). It is well known that the gauge transformation (4.3) maps the operator \( \hat{\mathcal{L}}_1(q) + \alpha \partial_y - \lambda \) onto a similar operator

\[
T(\hat{\mathcal{L}}_1(q) + \alpha \partial_y - \lambda) T^{-1} = \hat{\mathcal{L}}_1(\hat{q}) + \alpha \partial_y - \lambda, \tag{4.5}
\]

which satisfies the covariance condition

\[
\hat{\mathcal{L}}_1(\hat{q}) = \hat{\mathcal{L}}_1(q = \Delta q), \quad \text{with} \quad \Delta q = 2 \ln \phi. \tag{4.6}
\]

The next step is to find another third-order operator \( \hat{\mathcal{L}}_{2,\text{cov}}(q) \) with appropriate coefficients by gauge transformation (4.3), such that \( \partial_t + \hat{\mathcal{L}}_{2,\text{cov}}(q) \) can be mapped onto a similar operator \( \partial_t + \hat{\mathcal{L}}_{2,\text{cov}}(\hat{q}) \) which satisfies the covariance condition

\[
\hat{\mathcal{L}}_{2,\text{cov}}(\hat{q}) = \hat{\mathcal{L}}_{2,\text{cov}}(q = \Delta q). \tag{4.7}
\]

Assume that \( \phi \) is a solution of the following Lax pair:

\[
(\mathcal{L}_1 + \alpha \partial_y) \phi = \lambda \phi, \quad \mathcal{L}_1 = \partial_x^2 + q_{2x}, \tag{4.8a}
\]

\[
(\partial_t + \mathcal{L}_{2,\text{cov}}) \phi = 0, \quad \mathcal{L}_{2,\text{cov}} = 4h_1 \partial_x^3 + b_1 \partial_x^2 + b_2 \partial_x + b_3, \tag{4.8b}
\]
where \( b_j \) (\( i = 1, 2, 3 \)) are functions to be determined. It suffices to verify that such transformation (4.3) maps \( \hat{\sigma} + \hat{\mathcal{L}}_{2, \text{COV}} \) onto a similar one,

\[
T(\hat{\sigma} + \hat{\mathcal{L}}_{2, \text{COV}})T^{-1} = \hat{\sigma} + \hat{\mathcal{L}}_{2, \text{COV}}, \quad \hat{\mathcal{L}}_{2, \text{COV}} = 4h_1 \hat{\alpha}_1^3 + \hat{b}_1 \hat{\alpha}_1^2 + \hat{b}_2 \hat{\alpha}_1 + \hat{b}_3, \tag{4.9}
\]

where \( \hat{b}_j \) (\( j = 1, 2, 3 \)) and \( \hat{\mathcal{L}}_{2, \text{COV}} \) satisfy the covariant conditions

\[
\hat{b}_j = b_j(q) + \Delta b_j = b_j(q + \Delta q), \quad j = 1, 2, 3. \tag{4.10}
\]

Combining (4.3) and (4.9) yields

\[
\Delta b_1 = 0, \quad \Delta b_2 = 12h_1 \sigma_3 + b_{1,1} + \sigma b_{1,1},
\]

\[
\Delta b_3 = 12h_1 \sigma_2 + 12h_1 \sigma \sigma_3 + \sigma b_{1,1} + b_{2,1} + 2 \sigma \hat{\sigma}. \tag{4.11}
\]

According to relation (4.10), it remains to determine \( \hat{b}_i \), \( i = 1, 2, 3 \), in the form of polynomial expressions in terms of the derivatives of \( q \).

\[
\hat{b}_j = \mathcal{C}_j(q, q_x, q_y, q_{2x}, q_{2y}, \ldots), \quad j = 1, 2, 3, \tag{4.12}
\]

such that

\[
\Delta \mathcal{C}_j = \mathcal{C}_j(q + \Delta q, q_x + \Delta q_x, q_y + \Delta q_y, \ldots) - \mathcal{C}_j(q, q_x, q_y, \ldots) = \Delta b_j, \tag{4.13}
\]

with \( \Delta q_{i1, i2} = 2 \alpha_1^{i1, i2} q^{x_1, x_2} \ln q, n_1, n_2 = 1, 2, \ldots, \) and \( \Delta b_j \) being determined by relations (4.11).

A direct calculation shows that

\[
\hat{b}_1 = c_1(y, t), \tag{4.14}
\]

by using equations (4.11)–(4.13), with \( c_1(y, t) \) being an arbitrary function about \( y \) and \( t \).

By expanding the left-hand of equation (4.13), one can obtain

\[
\Delta b_2 = \Delta \mathcal{C}_2 = \mathcal{C}_2(q_x + \Delta q_x, q_y + \Delta q_y, \ldots) - \mathcal{C}_2(q, q_x, q_y, \ldots) = 12h_1 \sigma_3 = 6h_1 \Delta q_{2x}, \tag{4.15}
\]

which implies that we can determine \( \hat{b}_2 \) up to an arbitrary constant \( c_2(y, t) \), namely

\[
\hat{b}_2 = \mathcal{C}_2(q_{2x}) = 6h_1 q_{2x} + c_2(y, t), \tag{4.16}
\]

with \( c_2(y, t) \) being an arbitrary function about \( y \) and \( t \).

By means of the eigenvalue equation (4.8a), one can find the following relation:

\[
q_{3x} = -\alpha \sigma v - (\sigma x + \sigma^2) x. \tag{4.17}
\]

Substituting equations (4.14), (4.16) and (4.17) into the system (4.11) yields

\[
\Delta b_3 = 6h_1 \sigma_{2x} - 6h_1 \alpha \sigma v + 2c_1 \sigma_3 = 3h_1 \Delta q_{3x} - 3h_1 \alpha \Delta q_{x} + c_1 \Delta q_{2x}, \tag{4.18}
\]

which can verify that the third condition

\[
\Delta \mathcal{C}_3 = \mathcal{C}_3(q_x + \Delta q_x, q_y + \Delta q_y, \ldots) = 3h_1 q_{3x} - 3h_1 \alpha q_{xy} + c_1(q, t)q_{2x} + c_3(y, t), \tag{4.19}
\]

can be satisfied if one chooses

\[
\hat{b}_3 = \mathcal{C}_3(q, q_x, q_y, q_{2x}, q_{2y}, q_{3x}, \ldots) = 3h_1 q_{3x} - 3h_1 \alpha q_{xy} + c_1(y, t)q_{2x} + c_3(y, t), \tag{4.20}
\]

where \( c_3(y, t) \) is an arbitrary function of \( y \) and \( t \).

Setting \( c_1(y, t) = -\alpha^{-1} h_2 \), \( c_2(y, t) = h_3 \), \( c_3(y, t) = 0 \) in equations (4.14), (4.16) and (4.20), it follows from (4.8a) and (4.8b) that we find the Darboux covariant evolution equation (4.1b).

Through a tedious calculation of the Lie bracket [\( \hat{\sigma} + \hat{\mathcal{L}}_{2, \text{COV}}, \hat{\mathcal{L}}_1 + \alpha \partial_y \)], one obtains equation (4.4) if one chooses \( \hat{\sigma}, h_4 = h_0 + h_1 \ln h_1, \hat{\sigma}, h_1 = h_2 = 0 \).

From the above, the higher operators can be obtained in a similar way step by step,

\[
\hat{\mathcal{L}}_{m, \text{COV}}(\hat{q}) = 4h_1 \hat{\alpha}_1^m + \hat{b}_1 \hat{\alpha}_1^{m-1} + \cdots + \hat{b}_s, \quad s = 5, 6, 7, \ldots, \tag{4.21}
\]

which are Darboux covariant with respect to \( \hat{\mathcal{L}}_1 \), so as to obtain higher order members of equation (1.1).
5. Infinite conservation laws

In this section, we derive the infinite conservation laws for the generalized vc-KP equation (1.1) by using the binary Bell polynomials.

**Theorem 5.1.** Under conditions (1.12), the generalized vc-KP equation (1.1) admits an infinite conservation law

$$J_{n,t} + [J_{n,x} + J_{n,y}] = 0, \quad n = 1, 2, \ldots.$$  \hspace{1cm} (5.1)

The conversed densities $J_n$'s are given by the recursion formulas

$$J_1 = -\frac{1}{2} q_{2x} = -\frac{1}{2} e^h h c u,$$

$$J_2 = \frac{1}{4} q_{3x} + \frac{1}{4} \alpha q_{xy} = \frac{1}{4} e^h h \partial x (\partial_x^{-1} u_y + u_{2x}),$$

$$J_{n+1} = -\frac{1}{2} \left( J_{n,x} + \alpha \partial_x^{-1} J_{n,y} + \sum_{i=1}^{n} \partial_x^{-1} J_{n-i} \right), \quad n = 2, 3, \ldots.$$  \hspace{1cm} (5.2)

and the first fluxes $F_n$ are given by

$$F_1 = h_1 A_{2x} - 6 h_1 \alpha A_{3x} + 2 x A_1 - 2 h_1 A_2,$$

$$F_2 = h_1 A_{2x} - 6 h_1 \alpha A_{3x} + 2 x A_1 - 2 h_1 A_2 - 12 h_1 A_6 + 2 h_3 A_2,$$

$$F_n = h_1 \left( h_1 A_{2x} - 6 \sum_{k=1}^{n} \partial_x A_{n+1-k} - 2 \sum_{k_1+k_2+k_3=n} \partial_x A_{k_1} \partial_x A_{k_2} \right)$$

$$- 6 h_1 \alpha \left( \partial_x^{-1} A_{n+1} - \sum_{k=1}^{n} \partial_x^{-1} A_{n-k} \right) + 2 h_3 A_n, \quad n = 3, 4, \ldots.$$  \hspace{1cm} (5.3)

and the second fluxes $G_n$ are given by

$$G_1 = 6 h_1 \alpha A_1 + h_2 A_1 + h_3 A_2,$$

$$G_2 = 3 h_1 \alpha A_1^2 + 6 h_1 \alpha A_2 + 2 h_3 A_1 + h_4 A_1 + h_5 A_2,$$

$$G_n = 3 h_1 \alpha \sum_{k=1}^{n} \partial_x A_{n-k} + 6 h_1 \alpha A_{n+1} + h_4 A_n + h_5 A_{n-1}, \quad n = 2, 3, \ldots.$$  \hspace{1cm} (5.4)

**Proof.** To redecompose the two-field condition (3.3) into the x- and y-derivative of $\partial x$-polynomials, we return to revisit $\mathcal{R}(u, \omega)$ in the two-field condition (3.5) and write it in another form,

$$\mathcal{R}(u, \omega) = [(3 h_1 \lambda + h_3) u_{3x} - 3 h_1 \alpha u_1 u_{3x}] + [-3 h_1 \alpha u_{2x} + h_4 u_1],$$  \hspace{1cm} (5.5)

The conservation laws have actually been hinted in the two-field constraint system (3.10), which can be rewritten in the conserved form

$$\omega_{2x} + u_2 x + \alpha v_y - \lambda = 0,$$

$$\partial_y [u_1] + \partial_y [h_1 u_{2x} + 3 h_1 u_1 \omega_{2x} + h_1 u_{3x} + \lambda (h_1 + h_3) u_{3x} - 3 h_1 \alpha u_1 u_{3x}]$$

$$+ \partial_y [3 h_1 \alpha u_1^2 + h_4 u_1 + h_5 u_2 - 3 h_1 \alpha \lambda] = 0,$$  \hspace{1cm} (5.6)

by applying the relation $\partial_t (u_t) = \partial_t (u_x) = u_{xt}$. By introducing a new potential function

$$\eta = (q_x - q_x)/2,$$  \hspace{1cm} (5.7)
it follows from relation (3.4) that
\[ u_t = \eta, \quad \omega_x = q_x + \eta. \] (5.8)
Substituting (5.8) into (5.6), we decompose the two-field condition (3.5) into a Riccati-type equation
\[ q_{2x} + \eta_t + \eta^2 + \alpha \partial_x^{-1} \eta_x - \varepsilon^2 = 0. \] (5.9)
which is a new potential function about \( q \), and a divergence-type equation
\[ \eta_t + \partial_x \left[ h_1 \left( \eta_{2x} - 2\eta^3 - 6\varepsilon \eta \partial_x^{-1} \eta_x + 6\varepsilon^2 \eta \right) + h_3 \eta \right] \\
+ \partial_x \left[ 3h_1 \alpha \eta^2 + h_2 \eta + h_3 \partial_x^{-1} \eta_x - 3h_1 \alpha \varepsilon^2 \right] = 0. \] (5.10)
where we obtain equation (5.10) by virtue of equation (5.9) and set \( \lambda = \varepsilon^2 \).

By inserting the expansion
\[ \eta = \varepsilon + \sum_{n=1}^{\infty} \mathcal{G}_n(q_x, q_{2x}, \ldots) \varepsilon^n \] (5.11)
into equation (5.9) and equating the coefficients for the power of \( \varepsilon \), we explicitly obtain the recursion relations (5.2) for the conserved densities \( \mathcal{G}_n \).

In addition, substituting expansion (5.11) into the divergence-type equation (5.10) leads to
\[ \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} + \partial_x \left\{ h_1 \left( \sum_{n=1}^{\infty} \mathcal{G}_{2n} \varepsilon^{-n} - 2 \sum_{n=1}^{\infty} \mathcal{G}_{n} \varepsilon^{-n} \right) - 6\varepsilon \left( \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} \right)^2 + 4\varepsilon^3 \right\} \\
+ h_3 \left( \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} + \varepsilon \right) - 6h_1 \alpha \left[ \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} \right] \partial_x^{-1} \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} \right] \right] \\
- 6h_1 \alpha \varepsilon \partial_x^{-1} \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} \right] + \partial_x \left\{ 3h_1 \alpha \left[ \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} \right] \partial_x^{-1} \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} + 2\varepsilon \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} \right] \right] \\
+ h_2 \left( \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} + \varepsilon \right) + h_3 \left( \partial_x^{-1} \sum_{n=1}^{\infty} \mathcal{G}_n \varepsilon^{-n} + \varepsilon \partial_x \right) \right\} = 0, \] (5.12)
which provides us the infinite conservation laws (5.1)
\[ \mathcal{G}_n + \mathcal{G}_{n,xx} + \mathcal{G}_{n,xy} = 0, \quad n = 1, 2, \ldots. \]
In equation (5.1), the conserved densities \( \mathcal{G}_n \)s are given by recursion formulas (5.2), and the first fluxes \( \mathcal{G}_{n} \)s and the second fluxes \( \mathcal{G}_{n,xy} \)s, respectively, are obtained by (5.3) and (5.4), respectively, through a cumbersome calculation.

From the above, one concludes that the first fluxes \( \mathcal{G}_{n} \)s (5.3) and the second fluxes \( \mathcal{G}_{n,xy} \)s (5.4) can be obtained from the solution \( u \) by algebraic and differential manipulation, and the conservation law (5.1) implies that \( \{ \mathcal{G}_n, n = 1, 2, \ldots \} \) constitute infinite conserved densities of the generalized vc-KP equation (1.1). We present the recursion formulas (5.2)–(5.4) for generating an infinite sequence of conservation laws for each equation; the first few conserved density and the associated first and second fluxes are explicitly given. The first equation of conservation law (5.1) is exactly the generalized vc-KP equation (1.1).

6. Soliton solution and Riemann theta function periodic wave solution

Under conditions (1.12) and \( c_0 = 6 \), we can discuss the solutions of the generalized vc-KP equation (1.1) by using the bilinear form (2.2). The following subsections are independent of each other, and the parameters are also independent.
6.1. Soliton solution

**Theorem 6.1.** Assuming \( \delta = 0 \), under conditions (1.12) and \( c_0 = 6 \), the generalized vc-KP equation (1.1) admits an \( N \)-soliton solution as follows:

\[
 u = 12h_1h_2^{-1}(\ln f)_{xx},
\]

\[
f = \sum_{\rho = 0}^{N} \exp \left( \sum_{j=1}^{N} \rho_j \eta_j + \sum_{1 \leq j < i \leq N} \rho_i \rho_j A_{ij} \right),
\]

(6.1)

where \( \eta_j = \mu_j x + v_j y - (h_1 \mu_j^3 + h_3 \mu_j + h_4 v_j + h_5 \mu_j^{-1} v_j^3) t + c_j \) and \( \exp(A_{ij}) = \frac{e^{\rho_{ij}(\mu_j^3 - \mu_i^3) t - h_3 (\mu_j - \mu_i) t + h_4 (\mu_j - \mu_i) v_j + h_5 (\mu_j - \mu_i)^{-1} v_j^3}}{e^{\rho_{ij}(\mu_j^3 - \mu_i^3) t - h_3 (\mu_j - \mu_i) t + h_4 (\mu_j - \mu_i) v_j + h_5 (\mu_j - \mu_i)^{-1} v_j^3}} \) (1 \( \leq j < i \leq N \), while \( \mu_j \) and \( v_j \) are the parameters characterizing the \( j \)-th soliton, \( \sum_{1 \leq j < i \leq N} \) is the summation over all possible pairs chosen from \( N \) elements under the condition \( 1 \leq j < i \leq N \) and \( \sum_{\rho = 0}^{N} \) denotes the summation over all possible combinations of \( \rho_i, \rho_j = 0, 1 \) \( (i, j = 1, 2, \ldots, N) \).

**Proof.** Substituting (6.1) into the bilinear form (2.2) yields

\[
\sum_{\rho = 0}^{N} \exp \left( \sum_{j=1}^{N} \rho_j \eta_j + \sum_{1 \leq j < i \leq N} \rho_i \rho_j A_{ij} \right)
\]

\[
\sum_{j=1}^{N} (\rho_j - \rho_j') v_j \exp \left( \sum_{j=1}^{N} (\rho_j + \rho_j') \eta_j + \sum_{1 \leq j < i \leq N} (\rho \rho_j + \rho' \rho_j') A_{ij} \right) = 0,
\]

(6.2)

in which the bilinear operator \( \mathcal{D} \) is given by equation (2.2) with \( \delta = 0 \). Let the coefficient of the factor

\[
\exp \left( \sum_{j=1}^{m} \eta_j + 2 \sum_{j=m+1}^{n} \eta_j \right)
\]

on the left-hand side of (6.2) be \( \mathcal{D} \); it follows that

\[
\mathcal{D} = \sum_{\rho = 0}^{N} \sum_{\rho' = 0}^{N} \mathcal{C}(\rho, \rho') \mathcal{D} \left( - \sum_{j=1}^{N} (\rho_j - \rho_j') (h_1 \mu_j^3 + h_3 \mu_j + h_4 v_j + h_5 \mu_j^{-1} v_j^3), \right.
\]

\[
\sum_{j=1}^{N} (\rho_j - \rho_j') \mu_j, \sum_{j=1}^{N} (\rho_j - \rho_j') v_j \right) \exp \left( \sum_{1 \leq j < i \leq N} (\rho \rho_j + \rho' \rho_j') A_{ij} \right) = 0,
\]

(6.4)

where the coefficient \( \mathcal{C}(\rho, \rho') \) denotes that the summations over \( \rho \) and \( \rho' \) performed under the following conditions:

\[
\rho_j = \begin{cases} 
1 - \rho_j', & \text{if } 1 \leq j \leq m, \\
0, & \text{if } m + 1 \leq j \leq n, \\
1, & \text{if } n + 1 \leq j \leq N.
\end{cases}
\]

(6.5)

By introducing a new variable

\[
\sigma_j = \rho_j - \rho_j',
\]

(6.6)
one obtains the following equality:

$$\exp \left( \sum_{1 \leq j < i \leq N} (\rho_i \rho_j + \rho_j \rho_i) A_{ij} \right) = \sum_{1 \leq j < i \leq N} \frac{1}{2} (1 + \sigma_j \sigma_j) A_{ij} + \sum_{i=1}^{m} \sum_{j=m+1}^{n} A_{ij}$$

(6.7)

On account of $\sigma_j$, $\sigma_j = \pm 1$ and the relations

$$\mathcal{A}(h_1 \mu_3^3 + h_3 \mu_j + h_4 v_j + h_5 \mu_j^{-1} v_j, \mu_j, v_j) = \mathcal{A}(-h_1 \mu_3^3 - h_3 \mu_j - h_4 v_j - h_5 \mu_j^{-1} v_j, -\mu_j, -v_j),$$

$$\exp (A_{ij}) = \frac{\mathcal{A}(h_1 (\mu_j^3 - \mu_j^3) + h_3 (\mu_i - \mu_j) + h_4 (v_i - v_j) + h_5 (\mu_i^{-1} v_i - \mu_j^{-1} v_j), \mu_j - \mu_i, v_j - v_i)}{\mathcal{A}(-h_1 (\mu_j^3 + \mu_j^3) - h_3 (\mu_i + \mu_j) - h_4 (v_i + v_j) - h_5 (\mu_i^{-1} v_i + \mu_j^{-1} v_j), \mu_i + \mu_j, v_i + v_j)} \times \sigma_i \sigma_j,$$

one obtains

$$\sum_{1 \leq j < i \leq N} \frac{1}{2} (1 + \sigma_j \sigma_j) A_{ij} =$$

$$= \sum_{j=1}^{N} \mathcal{A}(-h_1 (\mu_j^3 + \mu_j^3) + h_3 (\mu_i - \mu_j) + h_4 (v_i - v_j) + h_5 (\mu_i^{-1} v_i - \mu_j^{-1} v_j), \mu_j - \mu_i, v_j - v_i)$$

$$- \mathcal{A}(-h_1 (\mu_j^3 - \mu_j^3) - h_3 (\mu_i + \mu_j) - h_4 (v_i + v_j) - h_5 (\mu_i^{-1} v_i + \mu_j^{-1} v_j), \mu_i + \mu_j, v_i + v_j) \times \sigma_i \sigma_j = 0,$$

(6.9)

Substituting equations (6.6)–(6.9) into equation (6.4) yields

$$\mathcal{F} = \mathcal{A} \sum_{i=\pm 1} \mathcal{A} \left( - \sum_{j=1}^{N} \sigma_j (h_1 \mu_j^3 + h_3 \mu_j + h_4 v_j + h_5 \mu_j^{-1} v_j), \sum_{j=1}^{N} \sigma_j \mu_j, \sum_{j=1}^{N} \sigma_j v_j \right)$$

$$\times \prod_{j < i} \mathcal{A}(-h_1 (\mu_j^3 + \mu_j^3) + h_3 (\mu_i - \mu_j) + h_4 (v_i - v_j)$$

$$+ h_5 (\mu_i^{-1} v_i - \mu_j^{-1} v_j), \mu_j - \mu_i, v_j - v_i), \sigma_i \sigma_j = 0,$$

(6.10)

where $\mathcal{F} = \mathcal{F}(\exp(A_{ij}))$ is independent of the summation indices $\sigma_j$ ($i = 1, 2, \ldots, N$). If we can verify identity (6.10) for $\mathcal{F} \equiv 1, N = 1, 2, \ldots$, then (6.1) is the solution of equation (1.1). Using the bilinear form (2.2), one can rewrite (6.10) as follows:

$$\tilde{\mathcal{F}}_{N}(\mu_1, v_1, \mu_2, v_2, \ldots, \mu_N, v_N)$$

$$\equiv \mathcal{A} \sum_{i=\pm 1} \left\{ - \sum_{i,j=1}^{N} \sigma_i \sigma_j (h_1 \mu_i^3 + h_4 v_i + h_5 \mu_i^{-1} v_i) \mu_j$$

$$+ h_1 \left( \sum_{j=1}^{N} \sigma_j v_j \right)^4 + h_3 \left( \sum_{j=1}^{N} \sigma_j \mu_j \right)^2 + h_4 \sum_{j=1}^{N} \sigma_j \sigma_j \mu_i v_j$$

$$+ h_5 \left( \sum_{j=1}^{N} \sigma_j v_j \right)^2 \prod_{j < i} \left[ 3h_1 \mu_i^2 \mu_i^2 (\sigma_j \mu_i - \sigma_j \mu_j)^2 - h_5 (\mu_i v_j - \mu_j v_i) \right] = 0.$$
\[ \hat{F}(\mu_1, v_1, \mu_2, v_2, \ldots, \mu_N, v_N) \]

is a symmetric and homogeneous polynomial and is also an even function of \( \mu_j, v_j \) \((j = 1, 2, \ldots, N)\). Suppose \((\mu_1, v_1) = (\pm \mu_2, \pm v_2)\); then we have the following relationship:

\[
\hat{F}(\mu_1, v_1, \ldots, \mu_N, v_N) = 8(3h_1\mu_1^2 - h_5\mu_1^2v_1^2) \prod_{j=3}^{N} [3h_1\mu_1^2\mu_j^2(\mu_j^2 - \mu_1^2)]^4
\]

\[+ h_5(\mu_1^2v_1^2 - \mu_j^2v_j^2) \prod_{j=3}^{N} \hat{F}(\mu_3, \ldots, \mu_N, v_N). \tag{6.12} \]

For \( \alpha \equiv 1, n = 1, 2 \), identity (6.11) is easily verified. Let us assume that the identity hold for \( N = 2 \); utilizing relationship (6.12), it is seen that \( \hat{F}(\mu_1, \mu_2, \ldots, \mu_N) \) can be the factor by a symmetric homogeneous polynomial as follows:

\[
\hat{F}(\mu_1, v_1, \ldots, \mu_N, v_N) = \prod_{i=1}^{N} (3h_1\mu_1^2 - h_5\mu_i^2v_i^2) \prod_{j=3}^{N} [3h_1\mu_1^2\mu_j^2(\mu_j^2 - \mu_1^2)]^4
\]

\[+ h_5(\mu_1^2v_1^2 - \mu_j^2v_j^2) \prod_{j=3}^{N} \hat{F}(\mu_1, v_1, \ldots, \mu_N, v_N). \tag{6.13} \]

According to the degrees of equations (6.11) and (6.13), \( \hat{F}(\mu_1, v_1, \ldots, \mu_N, v_N) \) must be zero for \( \alpha \equiv 1, n \geq 2 \), and the identity is proved. Hence, expression (6.1) is the \( N \)-soliton solution of the generalized vc-KP equation (1.1). \( \square \)

Based on theorem 6.1, one can easily obtain the following corollary.

**Corollary 6.2.** For the case \( N = 1 \), the one-soliton solution of the generalized vc-KP equation (1.1) can be written as follows:

\[
u = 12h_1h_2^{-1}[\ln(1 + e^{\eta})]_{x_0}, \tag{6.14} \]

where \( \eta = \mu x + vy - (h_1\mu_1^3 + h_3\mu_1 + h_4v_1 + h_5\mu_1^{-1}v_1^2)t + c. \) For the case \( N = 2 \), the expression

\[
u = 12h_1h_2^{-1}[\ln(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}})]_{x_0}, \tag{6.15} \]

with \( \eta_1 = \mu x + vy - (h_1\mu_1^3 + h_3\mu_1 + h_4v_1 + h_5\mu_1^{-1}v_1^2)t + c_1 \) \( i = 1, 2 \), \( e^{\eta_1} = \exp[3h_1\mu_1^2\mu_j^2(\mu_1^2 - \mu_j^2) - h_5(\mu_1^2v_1^2 - \mu_j^2v_j^2)] \), describes the two-soliton solution for equation (1.1). 

Based on the soliton solutions obtained by the Hirota method, we present some figures to describe the propagation situations of the solitary waves. Figures 1 and 2 show the pulse...
Figure 2. Propagation of the solitary wave for the generalized vc-KP equation (1.1) via expression (6.14) with the parameters $h_1 = y^2$, $h_2 = -\text{sech}^2(t)$, $h_3 = t$, $h_4 = y$, $h_5 = 2$, $\mu = 1$, $\nu = 2$ and $c = -1$. (a) The perspective view of the wave. (b) The overhead view of the wave. (c) The corresponding contour plot.

Figure 3. Evolution plots of the two solitary waves for the generalized vc-KP equation (1.1) via expression (6.15) with the parameters $h_1 = 1$, $h_2 = \text{sech}^2(t)$, $h_3 = 1$, $h_4 = -t$, $h_5 = t$, $\mu_1 = 1$, $\nu_1 = 3$, $\mu_2 = 2$, $\nu_2 = 4$ and $c_1 = c_2 = 0$. (a) The perspective view of the wave. (b) The overhead view of the wave. (c) The corresponding contour plot.

Figure 4. Evolution plots of the two solitary waves for the generalized vc-KP equation (1.1) via expression (6.15) with the parameters $h_1 = 1$, $h_2 = \text{sech}^2(t)$, $h_3 = 1$, $h_4 = -1$, $h_5 = t$, $\mu_1 = 1$, $\nu_1 = 2$, $\mu_2 = 2$, $\nu_2 = -2$ and $c_1 = c_2 = 0$. (a) The perspective view of the wave. (b) The overhead view of the wave. (c) The corresponding contour plot.

propagation of the fundamental soliton along the distance $(x, y)$-surface with suitable choice of the parameters in equation (6.14). In figures 3 and 4, we choose the same value of $\mu_1$ and $\mu_2$ but different $\nu_1$ and $\nu_2$. In this case, the phases of the two solitons are the same and two sets of parallel solitons are obtained via equation (6.15).
6.2. Riemann theta function periodic wave solution

Using a multidimensional Riemann theta function, in [51, 52] we proposed two key theorems to systematically construct Riemann theta function periodic wave solutions for nonlinear equations and discrete soliton equations, respectively. Using the results in [51], we can directly obtain some periodic wave solutions for the generalized vc-KP equation (1.1) (see detail in appendix B).

Considering conditions (1.12), we consider the following bilinear form when $\delta$ is a nonzero constant in equation (2.2):

$$
\mathcal{L}(D_x, D_y, D_t) f \cdot f \equiv (D_x D_y + h_1 D_x^2 + h_3 D_y^2 + h_2 D_x D_y + h_3^2 - \delta) f \cdot f = 0. \tag{6.16}
$$

Let us now consider the Riemann theta function

$$
\vartheta(\xi, \tau) = \sum_{\mathbf{n} \in \mathbb{Z}^N} e^{i(\tau, \mathbf{n}) + 2\pi i (\xi, \mathbf{n})}, \tag{6.17}
$$

where the integer value vector $\mathbf{n} = (n_1, n_2, \ldots, n_N) \in \mathbb{Z}^N$, the complex phase variables $\xi = (\xi_1, \xi_2, \ldots, \xi_N)^T \in \mathbb{C}^N$ and $-i\tau$ is a positive definite and real-valued symmetric $N \times N$ matrix.

**Theorem 6.3.** Assuming that $\vartheta(\xi, \tau)$ is a Riemann theta function for $N = 1$ with $\xi = kx + ly + ot + \varepsilon$, the generalized vc-KP equation (1.1) admits a one-periodic wave solution as follows:

$$
u = 12h_1h_2^{-1} \frac{1}{\tau} \ln \vartheta(\xi, \tau), \tag{6.18}
$$

where

$$
\omega = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad \delta = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}, \tag{6.19}
$$

with

$$
\varphi = e^{i\tau}, \quad a_{11} = \sum_{n=-\infty}^{+\infty} 16n^2 \pi^2 k_\tau^2, \quad a_{12} = \sum_{n=-\infty}^{+\infty} \varphi^{2n^2},
$$

$$
a_{21} = \sum_{n=-\infty}^{+\infty} 4\pi^2 (2n - 1)^2 k_\tau^2 \varphi^{2n^2 + 2n + 1},
$$

$$
a_{22} = \sum_{n=-\infty}^{+\infty} \varphi^{2n^2 - 2n + 1},
$$

$$
b_1 = \sum_{n=-\infty}^{+\infty} (256h_1n^4 \pi^4 k^4 - 16h_3n^2 \pi^2 k^2 - 16h_4n^2 \pi^2 kl - 16h_5n^2 \pi^2 l^2) \varphi^{2n^2},
$$

$$
b_2 = \sum_{n=-\infty}^{+\infty} (16h_1 \pi^4 (2n - 1)^4 k^4 - 4h_3 \pi^2 (2n - 1)^2 k^2 - 4h_4 \pi^2 (2n - 1)^2 kl
$$

$$
- 4h_5 \pi^2 (2n - 1)^2 l^2) \varphi^{2n^2 - 2n + 1}, \tag{6.20}
$$

and the other parameters $k$, $l$, $\tau$ and $\varepsilon$ are free.

**Proof.** In order to obtain the one-periodic wave solutions of equation (1.1), we consider the one-Riemann theta function $\vartheta(\xi, \tau)$ as $N = 1$,

$$
\vartheta(\xi, \tau) = \sum_{n=-\infty}^{+\infty} e^{i\pi n^2 \tau + 2\pi i n \xi}, \tag{6.21}
$$
where the phase variable \( \xi = kx + ly + \omega t + \epsilon \) and the parameter \( \text{Im} \tau > 0 \). According to theorem B.1 in appendix B (see details in [51]), \( k, l, \omega \) and \( \epsilon \) satisfy the following system:

$$
\sum_{n=-\infty}^{\infty} \zeta(4\pi n^2 k, 4\pi n^2 l, 4\pi n^2 \omega) e^{2\pi^2 n^2 \tau} = 0, \quad (6.22a)
$$

$$
\sum_{n=-\infty}^{\infty} \zeta(2\pi n(2n-1)k, 2\pi n(2n-1)l, 2\pi n(2n-1)\omega) e^{(2\pi^2 + 2\pi^2) n^2 \tau} = 0. \quad (6.22b)
$$

Substituting the bilinear form \( \zeta \) (6.16) into the system (6.22a), (6.22b) yields

$$
\sum_{n=-\infty}^{\infty} (16n^2 \pi^2 k^2 \omega - 256h_1 n^4 \pi^2 k^4 + 16h_3 n^2 \pi^2 k^2 + 16h_4 n^2 \pi^2 k l)
$$

$$
+ 16h_5 n^2 \pi^2 l^2 + \delta) e^{2\pi^2 n^2 \tau} = 0,
$$

$$
\sum_{n=-\infty}^{\infty} (4\pi^2 (2n-1)^2 k^2 \omega - 16h_1 n^4 \pi^2 (2n-1)^4 k^4 + 4h_3 n^2 \pi^2 (2n-1)^2 k^2 + 4h_4 n^2 \pi^2 (2n-1)^2 k l)
$$

$$
+ 4h_5 \pi^2 (2n-1)^4 l^2 + \delta) e^{(2\pi^2 + 2\pi^2) n^2 \tau} = 0. \quad (6.23b)
$$

The notations are the same as the system (6.20); the system (6.23a), (6.23b) is simplified into a linear system for the frequency \( \omega \) and the integration constant \( \delta \), namely

$$
\begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
  \omega \\
  \delta
\end{pmatrix}
= \begin{pmatrix}
  b_1 \\
  b_2
\end{pmatrix}. \quad (6.24)
$$

Now solving this system, we obtain a one-periodic wave solution of equation (1.1),

$$
u = 12h_1 h_2^{-1} \delta^2 \ln \vartheta (\xi, \tau),
$$

which provided the vector \( (\omega, \delta)^T \). It solves the system (6.24) with the theta function \( \vartheta (\xi, \tau) \) given by equation (6.21). The other parameters \( k, l, \omega \) and \( \epsilon \) are free. \( \square \)

**Theorem 6.4.** Assuming that \( \vartheta (\xi_1, \xi_2, \tau) \) is a Riemann theta function for \( N = 2 \) with \( \xi_i = k_i x + l_i y + \omega_i t + \epsilon_i \) \((i = 1, 2)\), the generalized vc-KP equation (1.1) admits a two-periodic wave solution as follows:

$$
u = \nu_0 + 12h_1 h_2^{-1} \delta^2 \ln \vartheta (\xi_1, \xi_2, \tau), \quad (6.25)
$$

where the parameters \( \omega_1, \omega_2, \nu_0 \) and \( \delta \) satisfy the linear system

$$
H(\omega_1, \omega_2, \nu_0, \delta)^T = b, \quad (6.26)
$$

with

\begin{align*}
H &= (h_{ij})_{4 \times 4}, \quad b = (b_1, b_2, b_3, b_4)^T, \quad h_{i1} = \sum_{(n_1, n_2) \in \mathbb{Z}^2} 4\pi^2 (2n - \theta_i k)^2 (2n_1 - \theta_i k^2) \vartheta (n), \\
h_{i2} &= \sum_{(n_1, n_2) \in \mathbb{Z}^2} 4\pi^2 (2n - \theta_i k)^2 (2n_2 - \theta_i k^2) \vartheta (n), \\
h_{i3} &= -\sum_{(n_1, n_2) \in \mathbb{Z}^2} 16h_1 \pi^2 (2n - \theta_i k)^4 \vartheta (n), \\
h_{i4} &= -\sum_{(n_1, n_2) \in \mathbb{Z}^2} \vartheta (n), \quad b_i = \sum_{(n_1, n_2) \in \mathbb{Z}^2} \left[ 16h_1 \pi^4 (2n - \theta_i k)^4 - 4h_3 \pi^2 (2n - \theta_i k)^2 ight. \\
&\left. - 4h_5 \pi^2 (2n - \theta_i k)^2 (2n - \theta_i l) - 16h_5 \pi^2 (2n - \theta_i l)^2 \right] \vartheta (n), \\
\vartheta (n) &= \vartheta_1^{n_1 + (n_1 - \theta_i k)^2} \vartheta_2^{n_2 + (n_2 - \theta_i k)^2} \vartheta_3^{n_1 + (n_1 - \theta_i l)^2} \vartheta_4^{n_2 + (n_2 - \theta_i l)^2}, \quad \vartheta_1 = e^{\pi it_1}, \quad \vartheta_2 = e^{\pi it_2}, \quad i = 1, 2, 3, 4, \quad (6.27)
\end{align*}
and \( \theta_i = (\theta_i^1, \theta_i^2)^T \), \( \theta_1 = (0, 0)^T \), \( \theta_2 = (1, 0)^T \), \( \theta_3 = (0, 1)^T \), \( \theta_4 = (1, 1)^T \), \( i = 1, 2, 3, 4 \); the other parameters \( k_i, l_i, \tau_{ij} \) and \( \epsilon_i (i, j = 1, 2) \) are free.

**Proof.** To obtain two-periodic wave solutions of equation (1.1), we consider the two-Riemann theta function \( \vartheta(\xi_1, \xi_2, \tau) \) as \( N = 2 \),

\[
\vartheta(\xi_1, \xi_2, \tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i (n \cdot \xi + 2 \pi i \langle n, \epsilon \rangle)},
\]

(6.28)

where the phase variable \( \xi = (\xi_1, \xi_2)^T \in \mathbb{C}^2 \), \( \xi_i = k_i x + b_i y + \omega_i t + \epsilon_i \), \( i = 1, 2 \), \( n = (n_1, n_2)^T \in \mathbb{Z}^2 \), and \( -\pi \tau \) is a positive definite and real-valued symmetric \( 2 \times 2 \) matrix which can take the form

\[
\begin{pmatrix}
\tau_{11} & \tau_{12} \\
\tau_{21} & \tau_{22}
\end{pmatrix}, \quad \text{Im}(\tau_{11}) > 0, \quad \text{Im}(\tau_{22}) > 0, \quad \tau_{11} \tau_{22} - \tau_{12}^2 < 0.
\]

(6.29)

By considering a variable transformation

\[ u = u_0 + 12 h_1 h_2^{-1} \hat{\alpha}_i \ln \vartheta(\xi_1, \xi_2, \tau), \]

(6.30)

and integrating with respect to \( x \), \( \mathcal{Z} \) becomes the following bilinear form:

\[
\mathcal{Z}(D_{x_1}, D_{x_2}, D_t) \varphi \cdot \varphi' \equiv (D_{x_1} D_{x_2} + h_1 u_0 D_{x_1}^2 + h_2 u_0 D_{x_2}^2 + h_3 D_{x_1}^2 + h_4 D_{x_2} + h_5 D_{x_2}^2 - \delta) \varphi \cdot \varphi' = 0.
\]

(6.31)

According to theorem B.2 in appendix B (see details in [51]), \( k_i, \omega_i \) and \( \epsilon_i \) \((i = 1, 2)\) satisfy the following system:

\[
\sum_{n \in \mathbb{Z}^2} \mathcal{Z}(2 \pi i (2n - \theta_i, k), 2 \pi i (2n - \theta_i, l), 2 \pi i (2n - \theta_i, \omega)) e^{\pi i [\langle (n \cdot \theta) + \langle n, \epsilon \rangle \rangle]} = 0,
\]

(6.32)

where \( \theta_i = (\theta_i^1, \theta_i^2)^T \), \( \theta_1 = (0, 0)^T \), \( \theta_2 = (1, 0)^T \), \( \theta_3 = (0, 1)^T \), \( \theta_4 = (1, 1)^T \), \( i = 1, 2, 3, 4 \).

Substituting the bilinear form \( \mathcal{Z} \) (6.31) into the system (6.32) yields

\[
\sum_{n \in \mathbb{Z}^2} \left[4 \pi^2 (2n - \theta_i, k) (2n - \theta_i, \omega) - 16 h_1 \pi^4 (2n - \theta_i, k)^4 - 16 h_4 u_0 \pi^4 (2n - \theta_i, k)^4 + 4 h_3 \pi^2 (2n - \theta_i, k)^2 + 4 h_4 \pi^2 (2n - \theta_i, k) (2n - \theta_i, l) + 4 h_5 \pi^2 (2n - \theta_i, l)^2 + \delta e^{\pi i [\langle (n \cdot \theta) + \langle n, \epsilon \rangle \rangle]} \right] = 0, \quad i = 1, 2, 3, 4.
\]

(6.33)

The notations are the same as the system (6.27); equations (6.33) can be written as a linear system about the frequency \( \omega_1, \omega_2, u_0 \) and the integration constant \( \delta \), namely

\[
\begin{pmatrix}
\hat{h}_{11} & \hat{h}_{12} & \hat{h}_{13} & \hat{h}_{14} \\
\hat{h}_{21} & \hat{h}_{22} & \hat{h}_{23} & \hat{h}_{24} \\
\hat{h}_{31} & \hat{h}_{32} & \hat{h}_{33} & \hat{h}_{34} \\
\hat{h}_{41} & \hat{h}_{42} & \hat{h}_{43} & \hat{h}_{44}
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
u_0 \\
\delta
\end{pmatrix}
= \begin{pmatrix}
\hat{b}_1 \\
\hat{b}_2 \\
\hat{b}_3 \\
\hat{b}_4
\end{pmatrix}.
\]

(6.34)

Now solving this system, we obtain a two-periodic wave solution of equation (1.1),

\[ u = u_0 + 12 h_1 h_2^{-1} \hat{\alpha}_i \ln \vartheta(\xi_1, \xi_2, \tau), \]

which provided the vector \( (\omega_1, \omega_2, u_0, \delta)^T \). It solves the system (6.34) with the theta function \( \vartheta(\xi_1, \xi_2, \tau) \) given by equation (6.28). The other parameters \( k_i, l_i, \tau_{ij} \) and \( \epsilon_i (i, j = 1, 2) \) are free. \( \Box \)

We now present some figures to describe the propagation situations of the periodic waves. Figure 5 shows the propagation of the one-periodic wave via solution (6.18). Figure 6 shows the propagation of the degenerate two-periodic wave via solution (6.25). Figures 7 and 8 show the propagation of the asymmetric and symmetric two-periodic waves via solution (6.25).
6.3. Asymptotic property of Riemann theta function periodic waves

Based on the results of [51], the relation between the one- and two-periodic wave solutions (6.18), (6.25) and the one- and two-soliton solutions (6.14), (6.15) can be directly established as follows.

**Theorem 6.5.** If the vector \((\omega, \delta)^T\) is a solution of the system (6.24) for the one-periodic wave solution (6.18), we let

\[
\begin{align*}
\delta & \to 0, \\
2\pi i \xi & \to \eta + \pi \tau, \\
\vartheta (\xi, \tau) & \to 1 + e^\eta,
\end{align*}
\]

(6.36)

when \(\wp \to 0\). It implies that the one-periodic solution (6.18) converges to the one-soliton solution (6.14) under a small amplitude limit, that is, \((u, \wp) \to (u_1, 0)\).

**Proof.** By using the system (6.20), \(a_{ij}, b_i, i, j = 1, 2\), can be rewritten as the series about \(\wp\):

\[
\begin{align*}
a_{11} & = 32\pi^2 k (\varphi^2 + 4\wp^4 + 9\wp^8 + \cdots + n^2 3^n \varphi^{2n^2} + \cdots), \\
a_{12} & = 1 + 2 (\varphi^2 + \wp^8 + \varphi^{18} + \cdots + \wp^{2n^2} + \cdots), \\
a_{21} & = 8\pi^2 k (\varphi + 9\wp^6 + 25\wp^{10} + \cdots + (2n - 1)^2 3^n \wp^{2n^2-2n+1} + \cdots),
\end{align*}
\]
With the aid of proposition B.3 in appendix B, we have

\[ a_{22} = 2(\rho + g_3 + g_4 + \cdots + g_{2n+1} + \cdots), \]
\[ b_1 = 32\pi^2[(16h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2)\rho^2 + (256h_1\pi^2k^4 - 4h_3k^2 - 4h_4kl - 4h_5l^2)\rho^3 + \cdots + (16h_1\pi^2k^4 - h_3n^2k^2 - h_4n^2kl - h_5n^2l^2)\rho^3 + \cdots], \]
\[ b_2 = 8\pi^2[(4h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2)\rho + (324h_1\pi^2k^4 - 9h_3k^2 - 9h_4kl - 9h_5l^2)\rho^2 + \cdots + (4h_1(2n+1)\pi^2k^4 - h_3(2n+1)^2k^2 - h_4(2n+1)^2kl - h_5(2n+1)^2l^2)\rho^2 + \cdots]. \]

With the aid of proposition B.3 in appendix B, we have

\[ A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 8\pi^2k & 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 32\pi^2k & 2 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 72\pi^2k & 2 \end{pmatrix}, \]
\[ A_4 = A_4 = 0, \ldots, \]
\[ B_1 = \begin{pmatrix} 0 & 0 \\ 8\pi^2\Delta_1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 32\pi^2\Delta_2 \\ 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 \\ 72\pi^2\Delta_3 \end{pmatrix}, \quad B_4 = B_3 = B_4 = 0, \ldots, \]

where \( \Delta_1 = 4h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2 \), \( \Delta_2 = 16h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2 \) and \( \Delta_3 = 36h_1\pi^2k^4 - h_3k^2 - h_4kl - h_5l^2 \).

Substituting the system (6.38) into formulas (B.14), one can obtain

\[ X_0 = \begin{pmatrix} -k^{-1}\Delta_1 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} k\Delta_1 \\ 32\pi^2\Delta_1 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 89k^{-1}\Delta_1 + 9k^{-1}\Delta_3 \\ 320\pi^2\Delta_1 \end{pmatrix}, \]
\[ X_1 = X_3 = 0, \ldots. \]
Figure 7. An asymmetric two-periodic wave of the generalized vc-KP equation (1.1) via expression (6.25) with the parameters $h_1 = -1$, $h_2 = 2$, $h_3 = 4$, $h_4 = 6$, $h_5 = 8$, $k_1 = 0.1$, $l_1 = 1$, $k_2 = l_2 = 0.3$, $r_{11} = 1$, $r_{12} = 0.5i$, $r_{25} = 2i$ and $r_1 = r_2 = 0$. The asymmetric two-periodic wave is spatially periodic in three directions, but it need not to be periodic in either the $x$, $y$ or $t$ directions. (a) The perspective view of the real part of the periodic wave $\text{Re}(u)$. (b) The overhead view of the wave; the green points represent crests and the red points troughs. (c) The corresponding contour plot. (d) The wave propagation pattern of the wave along the $x$ axis. (e) The wave propagation pattern of the wave along the $y$ axis. (f) The wave propagation pattern of the wave along the $t$ axis.

From (B.9), one then has

$$\omega = -k^{-1} \Delta_1 + 8k^{-1} \Delta_1 \delta^2 - (89k^{-1} \Delta_1 + 9k^{-1} \Delta_3) \delta^2 + o(\delta^2),$$

$$\delta = 32\pi^2 \Delta_1 \delta - 320\pi^2 \Delta_1 \delta^3 + 0(\delta^3),$$

which implies by using relation (6.35) that

$$\delta \to 0, \quad 2\pi i \omega \to -(h_1\mu^3 + h_3\mu + h_4v + h_5\mu^{-1}v^2), \quad \text{when} \quad \varphi \to 0.$$  

(6.41)

In order to show that the one-periodic wave (6.18) degenerates to the one-soliton solution (6.14) under the limit $\varphi \to 0$, we first expand the periodic function $\vartheta(\xi, \tau)$ in the form of

$$\vartheta(\xi, \tau) = 1 + (e^{2\pi i \xi} + e^{-2\pi i \xi}) \varphi + (e^{4\pi i \xi} + e^{-4\pi i \xi}) \varphi^3 + \ldots.$$  

(6.42)

Using the transformation (6.35), one has

$$\widetilde{\vartheta}(\xi, \tau) = 1 + e^{\xi} + (e^{-\xi} + e^{2\xi}) \varphi^2 + (e^{-2\xi} + e^{3\xi}) \varphi^4 + \ldots \to 1 + e^{\xi}, \quad \text{when} \quad \varphi \to 0,$$

$$\xi = 2\pi i \xi - \pi \tau = \mu x + \nu y + 2\pi i \omega \varphi + \kappa.$$  

(6.43)

Combining equations (6.41) and (6.43), one deduces that

$$\xi \to \mu x + \nu y - (h_1\mu^3 + h_3\mu + h_4v + h_5\mu^{-1}v^2) \mu + \kappa, \quad \text{when} \quad \varphi \to 0,$$

$$2\pi i \xi \to \eta + \pi \tau, \quad \text{when} \quad \varphi \to 0.$$  

(6.44)

With the aid of equations (6.43) and (6.44), one can obtain

$$\vartheta(\xi) \to 1 + e^{\eta}, \quad \text{when} \quad \varphi \to 0.$$  

(6.45)
Figure 8. An symmetric two-periodic wave of the generalized vc-KP equation (1.1) via expression (6.25) with the parameters $h_1 = -1$, $h_2 = 2$, $h_3 = 4$, $h_4 = 6$, $h_5 = 8$, $k_1 = 1$, $l_1 = 2$, $k_2 = 3$, $l_2 = 4$, $\tau_{11} = 1$, $\tau_{12} = 2\tau$, $\tau_{22} = 2i$ and $\epsilon_1 = \epsilon_2 = 0$. The symmetric two-periodic wave is periodic in three directions. (a) The perspective view of the real part of the periodic wave $\text{Re}(u)$. (b) The overhead view of the wave; the green points represent crests and the red points troughs. (c) The corresponding contour plot. (d) The wave propagation pattern of the wave along the $x$ axis. (e) The wave propagation pattern of the wave along the $y$ axis. (f) The wave propagation pattern of the wave along the $t$ axis.

From the above, we conclude that the one-periodic solution (6.18) just converges to the one-soliton solution (6.14) as the amplitude $\wp \to 0$. □

**Theorem 6.6.** If $(\omega_1, \omega_2, u_0, \delta)^T$ is a solution of the system (6.26) for the two-periodic wave solution (6.25), we take

$$k_i = \frac{\mu_i}{2\pi i}, \quad l_i = \frac{v_i}{2\pi i}, \quad \epsilon_i = \frac{c_i + \pi \tau_{ij}}{2\pi i}, \quad \tau_{12} = \frac{A_{12}}{2\pi i}, \quad i = 1, 2, \quad (6.46)$$

where $\mu_i$, $v_i$, $c_i$, $i = 1, 2$, and $A_{12}$ are given in equation (6.15). Then we have the following asymptotic relations:

$$u_0 \to 0, \quad \delta \to 0, \quad 2\pi i \xi_i \to \eta_i + \pi \tau_{ij}, \quad i = 1, 2, \quad \vartheta(\xi_1, \xi_2, \tau) \to 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}, \quad \text{when} \quad \wp_1, \wp_2 \to 0. \quad (6.47)$$

It implies that the two-periodic solution (6.25) converges to the two-soliton solution (6.15) under a small amplitude limit, that is, $(u, \wp_1, \wp_2) \to (u_1, 0, 0)$. □

**Proof.** The proof is similar to the one of theorem 6.5. □
7. Conclusions and discussions

In this paper, under conditions (1.12), we have systematically researched the integrability features of the generalized vc-KP equation (1.1), which is an important model of various nonlinear real situations in hydrodynamics, plasma physics and some other nonlinear science when the inhomogeneities of media and nonuniformities of boundaries are taken into consideration. Using the properties of the binary Bell polynomials, we systematically construct the bilinear representation, Bäcklund transformation, Lax pair and Darboux covariant Lax pair, respectively, which can be reduced to the ones of several integrable equations such as KdV (1.2), KP (1.3), cylindrical KdV (1.4), cylindrical KP and generalized cylindrical KP (1.5) equations, etc. Based on its Lax equation, the infinite conservation laws of the equation can also be constructed. Using the bilinear formula and the recent results in [51, 52], we have presented the soliton solutions and Riemann theta function periodic wave solutions of the vc-KP equation (1.1). We are also able to choose different parameters and functions to obtain some solutions and also analyze their graphics in figures 1–4 and 5–8, respectively.

Finally, a limiting procedure is presented to analyze in detail the relations between the periodic wave solutions and soliton solutions. In conclusion, the generalized vc-KP equation (1.1) is completely integrable under conditions (1.12) in the sense that it admits the bilinear Bäcklund transformation, Lax pair and infinite conservation laws. The integrable constraint conditions (1.12) on the variable coefficients can be naturally found in the procedure of applying binary Bell polynomials. The results presented in this paper may provide further evidence of structures and complete integrability of these equations.

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Appendix A. Multidimensional Bell polynomials

In the following, we simply recall some necessary notations on multidimensional binary Bell polynomials; for details refer, for instance, to Lembert and Gilson’s work [8–10].

Suppose \( f = f(x_1, x_2, \ldots, x_n) \) to be a \( \mathbb{C}^\infty \) function with multi-variables; the polynomial

\[
Y_{n_1, \ldots, n_r}(f) \equiv Y_{n_1, \ldots, n_r}(f_1, \ldots, f_r) = e^{-f} \partial_{i_1}^{n_1} \cdots \partial_{i_r}^{n_r} e^f
\]

(A.1)

is called the multi-dimensional Bell polynomial, in which \( f_{i_1, \ldots, i_r} = \partial_{i_1}^{n_1} \cdots \partial_{i_r}^{n_r} (0 \leq i_1 \leq n_1, \ldots, 0 \leq i_r \leq n_r) \), \( i = 1, 2, \ldots, r \). Taking \( n = 1 \), the Bell polynomials are presented as follows:

\[
Y_1(f) = f, \quad Y_2(f) = f_x + f^2, \quad Y_3(f) = f_x + 3f_x f + f_x^3, \ldots
\]

(A.2)
To make the link between the Bell polynomials and the Hirota D-operator, the multidimensional binary Bell polynomials can be defined as follows [9]:

\[ Y_{n_1, \ldots, n_N}(\nu, \omega) = Y_{n_1, \ldots, n_N}(f) \bigg|_{f_{i_1, \ldots, i_N} = \left\{ \begin{array}{ll} 1 & \text{if } i_1 + \cdots + i_r \text{ is odd}, \\ \omega & \text{if } i_1 + \cdots + i_r \text{ is even}, \end{array} \right. } \]  

(A.3)

The key property of the multidimensional Bell polynomials

\[ Y_{n_1, \ldots, n_N}(\nu, \omega) = Y_{n_1, \ldots, n_N}(\nu) \bigg|_{\nu = \ln F/G, \omega = \ln FG} = (FG)^{-1}D^0_{n_1} \cdots D^0_{n_N} F \cdot G, \]  

(A.5)

where \( F \) and \( G \) are both the functions of \( x \) and \( t \). In the particular case when \( F = G \), identity

\[ F^{-2}D^0_{n_1} \cdots D^0_{n_N} F \cdot F = Y(0, q = 2 \ln F) = \begin{cases} 0, & n_1 + \cdots + n_r \text{ is odd}, \\ P_{n_1, \ldots, n_N}(q), & n_1 + \cdots + n_r \text{ is even}, \end{cases} \]  

(A.6)

where the \( P \)-polynomials can be characterized by an equally recognizable even-partitioned structure,

\[ P_{2s}(q) = q^{2s}, \quad P_{2s+1}(q) = q^{2s+1}, \quad P_{4s+2}(q) = q^{4s} + 3q^{2s+1}, \quad P_{4s+3}(q) = q^{4s} + 15q^{2s+1}. \]  

(A.7)

The binary Bell polynomials

\[ Y_{n_1, \ldots, n_N}(\nu, \omega) \]  

can be separated into \( P \)-polynomials and \( Y \)-polynomials:

\[ (FG)^{-1}D^0_{n_1} \cdots D^0_{n_N} F \cdot G = (FG)^{-1}Y_{n_1, \ldots, n_N}(\nu) \bigg|_{\nu = \ln F/G, \omega = \ln FG} = \begin{cases} 0, & n_1 + \cdots + n_r \text{ is odd}, \\ P_{n_1, \ldots, n_N}(q), & n_1 + \cdots + n_r \text{ is even}, \end{cases} \]  

(A.8)

The key property of the multidimensional Bell polynomials

\[ Y_{n_1, \ldots, n_N}(\nu) \bigg|_{\nu = \ln \psi} = Y_{n_1, \ldots, n_N}/\psi \]  

(A.9)

implies that the binary Bell polynomials

\[ Y_{n_1, \ldots, n_N}(\nu, \omega) \]  

can still be linearized by means of the Hopf–Cole transformation \( \nu = \ln \psi \), that is, \( \psi = F/G \). Formulas (A.8) and (A.9) will then provide the shortest way to the associated Lax system of the nonlinear equations.

Appendix B. Riemann theta function periodic wave

Based on the results in [51], we consider the one-periodic wave solutions of the NLEE. Then the Riemann theta function reduces the following Fourier series in \( n \):

\[ \vartheta(\xi, \tau) = \sum_{n=-\infty}^{\infty} e^{i(n\nu + 2\pi n)\tau + 2\pi i n\xi}, \]  

(B.1)

where the phase variable \( \xi = kx_1 + lx_2 + \cdots + nx_N + \omega t + \varepsilon \) and the parameter \( \text{Im}(\tau) > 0 \).
\textbf{Theorem B.1} ([51]). Assuming that $\vartheta(\xi, \tau)$ is a Riemann theta function for $N = 1$ with $\xi = k_1 l + l_2 + \cdots + \rho_N x_N + \omega t + \epsilon$ and $k, l, \ldots, \rho, \omega, \epsilon$ satisfy the system

$$
\sum_{n=-\infty}^{\infty} \mathcal{L}(4n\pi ik, 4n\pi il, \ldots, 4n\pi i\rho, 4n\pi i\omega) e^{2n^2\pi i\tau} = 0, \quad (B.2a)
$$

$$
\sum_{n=-\infty}^{\infty} \mathcal{L}(2\pi i(2n-1)k, 2\pi i(2n-1)l, \ldots, 2\pi i(2n-1)\rho), 2\pi i(2n-1)\omega) e^{(2n^2-2n+1)\pi i\tau} = 0, \quad (B.2b)
$$

the expression

$$u = u_0 + a \partial_x^N \ln \vartheta(\xi) \quad (B.3)$$

is the one-periodic wave solution of the NLEE.

Let us now consider the case when $N = 2$; the Riemann theta function takes the form of

$$
\vartheta(\xi, \tau) = \vartheta(\xi_1, \xi_2, \tau) = \sum_{n \in \mathbb{Z}^2} e^{\pi i (\tau n n^T) + 2\pi i (\xi, \eta)}, \quad (B.4)
$$

where $n = (n_1, n_2)^T \in \mathbb{Z}^2$, $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, $\xi_i = k_i l_1 + l_i x_2 + \cdots + \rho_i x_N + \omega_i t + \epsilon_i$, $i = 1, 2$, and $-i\tau$ is a positive definite whose real-valued symmetric $2 \times 2$ matrix is

$$
\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}, \quad \text{Im}(\tau_{11}) > 0, \quad \text{Im}(\tau_{22}) > 0, \quad \tau_{11}\tau_{22} - \tau_{12}^2 < 0. \quad (B.5)
$$

\textbf{Theorem B.2} ([51]). Assuming that $\vartheta(\xi_1, \xi_2, \tau)$ is one Riemann theta function with $N = 2$, $\xi_i = k_i l_1 + l_i x_2 + \cdots + \rho_i x_N + \omega_i t + \epsilon_i$, $i = 1, 2$, and $k_i, l_i, \ldots, \rho_i, \omega_i, \epsilon_i(i = 1, 2)$ satisfy the following system

$$
\sum_{n \in \mathbb{Z}^2} \mathcal{L}(2\pi i(2n - \theta, k), 2\pi i(2n - \theta, l), \ldots, 2\pi i(2n - \theta, \rho), 2\pi i(2n - \theta, \omega)) \times e^{\pi i (\tau (n-n_0) + (n-n_0)^T)} = 0, \quad (B.6)
$$

where $\theta = (\theta_1, \theta_2)^T$, $\theta_1 = (0, 0)^T$, $\theta_2 = (1, 0)^T$, $\theta_3 = (0, 1)^T$, $\theta_4 = (1, 1)^T$, $i = 1, 2, 3, 4$, the expression

$$u = u_0 + a \partial_x^N \ln \vartheta(\xi_1, \xi_2) \quad (B.7)$$

is the two-periodic wave solution of the NLEE.

Finally, we present a key proposition to investigate the asymptotic property of periodic waves. We write the system (6.24) into power series of

$$
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A_0 + A_1 \varphi + A_2 \varphi^2 + \cdots, \quad (B.8)
$$

$$
\begin{pmatrix} a \omega \\ c \end{pmatrix} = X_0 + X_1 \varphi + X_2 \varphi^2 + \cdots, \quad (B.9)
$$

$$
\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = B_0 + B_1 \varphi + B_2 \varphi^2 + \cdots, \quad (B.10)
$$

Substituting equations (B.8)–(B.10) into equation (6.24) leads to the following recursion relations:

$$A_0 X_0 = B_0, \quad A_0 X_n + A_1 X_{n-1} + \cdots + A_n X_0 = B_n, \quad n \geq 1, n \in \mathbb{N}, \quad (B.11)$$

form which we then recursively obtain each vector $X_i, i = 0, 1, \ldots$. 

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Proposition B.3 ([51]). Assuming that the matrix $A_0$ is reversible, we can obtain

$$X_0 = A_0^{-1}B_0, \quad X_n = A_0^{-1} \left( B_n - \sum_{i=1}^{n} A_i B_{n-i} \right), \quad n \geq 1, n \in \mathbb{N}. \quad (B.12)$$

If the matrices $A_0$ and $A_1$ are not inverse,

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -8\pi^2 k & 2 \end{pmatrix}, \quad (B.13)$$

we can obtain

$$X_0 = \left( \frac{2B_0^{(1)} - B_0^{(2)}}{8\pi^2 k}, B_0^{(1)} \right)^T, \quad X_1 = \left( \frac{2B_1^{(1)} - (B_2 - A_2 X_0)^{(2)}}{8\pi^2 k} B_1^{(1)} \right)^T, \ldots,$$

$$X_n = \left( \frac{2(B_{n+1} - \sum_{i=2}^{n} A_i X_{n-i})^{(1)} - (B_{n+1} - \sum_{i=2}^{n} A_i X_{n-i})^{(2)}}{8\pi^2 k}, \quad (B_{n+1} - \sum_{i=2}^{n} A_i X_{n-i})^{(1)} \right)^T,$$

$$n \geq 2, \quad n \in \mathbb{N}, \quad (B.14)$$

where $\alpha^{(1)}$ and $\alpha^{(2)}$ denote the first and second components of a two-dimensional vector $\alpha$, respectively.

References

[1] Ablowitz M J and Clarkson P A 1991 Solitons, Nonlinear Evolution Equations and Inverse Scattering (New York: Cambridge University Press)

Wadati M, Konno K and Ichikawa Y H 1979 J. Phys. Soc. Japan 46 1965

[2] Matveev V B and Salle M A 1991 Darboux Transformation and Solitons (Berlin: Springer)

Dubrovin V B and Konopelchenko B G 1994 J. Phys. A: Math. Gen. 27 4619

[3] Wadati M 1975 J. Phys. Soc. Japan 38 673

[4] Hirota R 1991 The Direct Method in Soliton Theory (New York: Cambridge University Press)

Matsuno Y 1984 Bilinear Transformation Method (Academic)

[5] Hu X B and Clarkson P A 1995 J. Phys. A: Math. Gen. 28 5009

Hu X B, Li C X, Nimmo J J C, Yu G F 2005 J. Phys. A: Math. Gen. 38 195

[6] Zhang D J and Chen D Y 2003 J. Phys. Soc. Japan 72 448

Zhang D J 2002 J. Phys. Soc. Japan 71 2649

[7] Ma W X and You Y C 2005 Trans. Am. Math. Soc. 357 1753

[8] Bell E T 1834 Ann. Math. 35 258

[9] Gilson C, Lambert F, Nimmo J and Willox R 1996 Proc. R. Soc. A 452 223

[10] Lambert F, Loris I and Springael J 2001 Inverse Problems 17 1067

Lambert F and Springael J 2008 Acta Appl. Math. 102 147

[11] Abramowitz M and Stegun I A (ed) 1972 Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables (New York: Dover)

[12] Riordan J 1996 Combinatorial Identities (New York: Wiley)

Comtet L 1974 Advanced Combinatorics (Dordrecht: Reidel)

[13] Howard F T 1980 Math. Comput. 35 977

Noschese S and Ricci P E 2003 J. Comput. Anal. Appl. 5 (3) 333

Bernardini A, Natalini P and Ricci P E 2005 Comput. Math. Appl. 50 1697

Paris R B 2009 J. Comput. Appl. Math. 232 216

[14] Coffey M 1996 Phys. Rev. B 54 1279

Das G and Sarma J 1999 Phys. Plasmas 6 4394

Tian B, Shan W R, Zhang C Y, Wei G M and Gao Y T 2005 Eur. Phys. J. B 47 329

[15] Anders I and Boutet de Monvel A 2000 J. Nonlinear Math. Phys. 7 284

Shen S F, Zhang J, Ye C E and Pan Z L 2005 Phys. Lett. A 337 101

[16] Fan E G 2000 J. Phys. A: Math. Gen. 33 6925 (arXiv:1008.4194v1)

Yan Z Y and Zhang H Q 2001 J. Phys. A: Math. Gen. 34 1785

Dai C Q and Zhang J F 2006 J. Phys. A: Math. Gen. 39 723
Zhang J F, Zhu Y J and Lin J 1995 Commun. Theor. Phys. 24 69
Wang Q and Cheng Y 2006 Appl. Math. Comput. 181 48

[17] David D, Levi D and Winternitz P 1987 Stud. Appl. Math. 76 133
David D, Levi D and Winternitz P 1986 Phys. Lett. A 118 390
Levi D and Winternitz P 1988 Phys. Lett. A 129 165

[18] Clarkson P A 1990 IMA J. Appl. Math. 44 27
Ouel W and Steeb W H 1984 Phys. Lett. A 103 239
Güngör F and Winternitz P 2002 J. Math. Anal. Appl. 276 314

[19] Lou S Y and Tang X Y 2004 J. Math. Phys. 45 1020
Chen Y and Liu P L-F 1995 Appl. Math. Comput. 181 48

[20] Lax P D 1968 Commun. Pure Appl. Math. 21 467

[21] Kadomtsev B B and Petviashvili V I 1970 Sov. Phys. Dokl. 15 539

[22] Hon Y C and Fan E G 2011 IMA J. Appl. Math. in press

[23] Tian S F and Zhang H Q 2010 J. Nonlinear Math. Phys. 17 491
Hereman W and Zhuang W 1995 Acta Appl. Math. 39 361

[24] Lax P D 1968 Commun. Pure Appl. Math. 21 467

[25] Nakamura A 1988 Prog. Theor. Phys. Suppl. 94 195

[26] Li J B and Zhang Y 2009 Nonlinear Anal.: Real World Appl. 10 2502

[27] Zhu Z N 1993 Phys. Lett. A 182 277

[28] Hon Y C and Fan E G 2011 IMA J. Appl. Math. in press

[29] Lou S Y and Hu X B 1997 J. Math. Phys. 38 6401

[30] Fan E G and Hon Y C 2008 Phys. Rev. E 78 036607

[31] Ma W X, Zhou R G and Gao L 2009 J. Math. Phys. 40 3948

[32] Zhou Z X 1996 Inverse Problems 12 89

[33] He J S, Li Y S and Cheng Y 2003 J. Math. Phys. 44 3928

[34] Daif H and Li Y S 2005 J. Phys. A: Math. Gen. 38 L685

[35] Zhou Z X 1996 J. Phys. A: Math. Gen. 29 5035

[36] Zeng Y B, Shao Y J and Xue W M 2003 J. Math. Anal. Appl. 288 326

[37] Zhang Y F and Zhang H Q 2002 J. Math. Phys. 43 466

[38] Wang D S and Zhang Z F 2009 J. Phys. A: Math. Theor. 42 035209

[39] Weiss J, Tabor M and Carnevale G 1983 J. Math. Phys. 24 522

[40] Tian S F, Wang Z and Zhang H Q 2010 J. Math. Anal. Appl. 366 646

[41] Liu S Y and Hu X B 1997 J. Math. Phys. 38 6401

[42] Geng X G and Cao C W 2004 Chaos Solitons Fractals 22 683

[43] Xia T C and Fan E G 2005 J. Math. Phys. 46 043510

[44] Yu G F and Tam H W 2006 J. Phys. A: Math. Gen. 39 3367

[45] Qu C Z and Shen S F 2009 J. Math. Phys. 50 103522

[46] Weiss J, Tabor M and Carnevale G 1983 J. Math. Phys. 24 522

[47] Tian S F, Wang Z and Zhang H Q 2010 J. Nonlinear Math. Phys. 17 491

[48] Tian S F and Zhang H Q 2010 J. Nonlinear Math. Phys. 17 491

[49] Hereman W and Zhuang W 1995 Acta Appl. Math. 39 361

[50] Hirota R, Hu X B and Tang X Y 2003 J. Math. Anal. Appl. 288 326

[51] Tian S F and Zhang H Q 2010 J. Math. Anal. Appl. 371 585

[52] Tian S F and Zhang H Q 2011 Commun. Nonlinear Sci. Numer. Simul. 16 173