Extended $F_4$-buildings and the Baby Monster

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Abstract

Let $\Theta$ be the Baby Monster graph which is the graph on the set of $\{3, 4\}$-transpositions in the Baby Monster group $B$ in which two such transpositions are adjacent if their product is a central involution in $B$. Then $\Theta$ is locally the commuting graph of central (root) involutions in $^2E_6(2)$. The graph $\Theta$ contains a family of cliques of size 120. With respect to the incidence relation defined via inclusion these cliques and the non-empty intersections of two or more of them form a geometry $\mathcal{E}(B)$ with diagram $c.F_4(t)$ for $t = 4$ and the action of $B$ on $\mathcal{E}(B)$ is flag-transitive. We show that $\mathcal{E}(B)$ contains subgeometries $\mathcal{E}(^2E_6(2))$ and $\mathcal{E}(Fi_{22})$ with diagrams $c.F_4(2)$ and $c.F_4(1)$. The stabilizers in $B$ of these subgeometries induce on them flag-transitive actions of $^2E_6(2) : 2$ and $Fi_{22} : 2$, respectively. The geometries $\mathcal{E}(B)$, $\mathcal{E}(^2E_6(2))$ and $\mathcal{E}(Fi_{22})$ possess the following properties: (a) any two elements of type 1 are incident to at most one common element of type 2 and (b) three elements of type 1 are pairwise incident to common elements of type 2 if and only if they are incident to a common element of type 5. The paper addresses the classification problem of $c.F_4(t)$-geometries satisfying (a) and (b). We construct three further examples for $t = 2$ with flag-transitive automorphism groups isomorphic to $3^2E_2(2) : 2$, $E_6(2) : 2$ and $^2E_6(2)$ and one for $t = 1$ with flag-transitive automorphism group $3^2F_i_{22} : 2$. We also study the graph of an arbitrary (non-necessary flag-transitive) $c.F_4(t)$-geometry satisfying (a) and (b) and obtain a complete list of possibilities for the isomorphism type of subgraph induced by the common neighbours of a pair of vertices at distance 2. Finally, we prove that $\mathcal{E}(B)$ is the only $c.F_4(4)$-geometry, satisfying (a) and (b).

1 Introduction

The paper contributes to the geometric theory of sporadic simple groups. Our notation and terminology is mostly standard (see [Pasi94] and [I99]). On a diagram the types of elements increase rightwards from 1 to the rank of the geometry. If an element of type 2 in a geometry $\mathcal{G}$ is incident to exactly two elements of type 1 then in the graph $\Gamma(\mathcal{G})$ of $\mathcal{G}$ the vertices and edges are the elements of type 1 and 2 in $\mathcal{G}$ subject to the natural incidence relation (in general $\Gamma(\mathcal{G})$ might contain multiple edges). Recall that if $\Gamma$ is a graph and $x \in \Gamma$ is a vertex, then $\Gamma_i(x)$ is the set of vertices at distance $i$ from $x$ in $\Gamma$. Sometimes we write $\Gamma(x)$ instead of $\Gamma_1(x)$. If $\Sigma$ is a subset of the vertex-set of $\Gamma$, then $\Sigma$ also denotes the subgraph in $\Gamma$ induced by $\Sigma$. If for every $x \in \Gamma$ the subgraph in $\Gamma(x)$ is isomorphic

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to a fixed graph $\Delta$ then $\Gamma$ is said to be locally $\Delta$. A graph $\Gamma$ whose vertices are involutions in a group $G$ is said to be the commuting graph if $a, b \in \Gamma$ are adjacent if and only if the order $o(ab)$ of the product of $a$ and $b$ is 2 (equivalently if $a$ and $b$ commute). A graph $\Gamma$ is said to be a $m$-clique extension of a graph $\Delta$ if there is a mapping $\psi$ of the vertex-set of $\Gamma$ onto the vertex-set of $\Delta$ such that $|\psi^{-1}(x)| = m$ for every $x \in \Delta$ and two distinct vertices $u, v$ in $\Gamma$ are adjacent if and only if the images $\psi(u)$ and $\psi(v)$ are either equal or adjacent in $\Delta$. A graph $\Gamma$ is said to be the distance 1-or-2 graph of $\Delta$ if there is a bijection $\chi$ of the vertex-set of $\Gamma$ onto the vertex-set of $\Delta$ with the following property: two vertices $x$ and $y$ in $\Gamma$ are adjacent if and only if the vertices $\chi(x)$ and $\chi(y)$ are adjacent or at distance 2 in $\Delta$. By a 2-path in a graph $\Gamma$ we mean an ordered sequence $(x, y, z)$ of vertices, such that $x, z \in \Gamma(y)$ and $z \in \Gamma_2(x)$. If $G$ is a group, acting by permutations on a set $X$, then for $x \in X$ and $Y \subseteq X$ by $G(x)$ and $G[Y]$ we denote the stabilizer of $x$ and (the setwise) stabilizer of $Y$ in $G$ (the upper and lower indexes will have various different meanings).

The sporadic simple group $B$ known as the Baby Monster contains a conjugacy class $\Theta$ of involutions (with centralizers of the form $2 \cdot 2^{1+22}.Co_2$) such that for any two involutions $a, b \in \Theta$ the order $o(ab)$ of their product is at most 4. This is to say, $\Theta$ is a class of $\{3, 4\}$-transpositions in $B$. Let $\Theta$ denote also the graph on this set, in which two transpositions are adjacent whenever their product belongs to the class of central involutions in $B$ (with centralizers of the form $2^{1+22}.Co_2$). We call $\Theta$ the Baby Monster graph. The suborbit diagram of $\Theta$ with respect to the action of $B$ is given on Fig 1.

Let $a \in \Theta$ and $B(a) = C_B(a)$ be the stabilizer of $a$ in the action of $B$ on $\Theta$ by conjugation.
diameter of $\Theta$ is 3; $B(a)$ acts transitively on $\Theta_1(a)$; has two orbits $\Theta_2^m(a)$ and $\Theta_2^n(a)$ on $\Theta_2(a)$, where
\[ \Theta_2^m(a) = \{ b \mid b \in \Theta_2(a), o(ab) = m \}; \]
a commutes with every involution from $\Theta_3(a)$ and $B(a)$ acts transitively on this set. This implies particularly that $a$ (considered as an element of $B$) fixes $\{ a \} \cup \Theta_1(a) \cup \Theta_3(a)$ and acts on $\Theta_2(a)$ with orbits of length 2.

Locally $\Theta$ is the commuting graph $\Delta$ of central (root) involutions in $B(a)/\langle a \rangle \cong 2.E_6(2) : 2$. In terms of the $F_4$-building $F$ with the digram

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      c
   2-2-2-4-4
     1
```

the vertices of $\Delta$ are the elements of type 1 with two of them adjacent if incident to a common element of type 4.

The group $B$ contains a maximal 2-local subgroup $S \sim 2^{2+16}.Sp_8(2)$. The centre of $O_2(S)$ contains exactly 120 $\{ 3, 4 \}$-transpositions which induce in $\Theta$ a clique (a maximal complete subgraph) $Q$. Let $E(B)$ be an incidence system formed by the set $Q$ of images of $Q$ under $B$ together with the non-empty intersections of two or more cliques from $Q$; the incidence relation is via inclusion and the type of an element is determined by its size. Then $E(B)$ is a geometry with the diagram

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c.F_4(t) :
  1-2-2-2-2
    c
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for $t = 4$ mentioned in [SS0] and $B$ induces on $E(B)$ a flag-transitive action. The elements of type 1 are the vertices of $\Theta$ and the residue of such a vertex $a$ is isomorphic to the building $F$.

In [H92] within the characterization proof for the rank 5 Petersen geometry of $B$ the following result has been established.

**Proposition 1.1** Let $\Gamma$ be a graph which is locally the commuting graph of central involutions in $2.E_6(2)$. Suppose that $\Gamma$ possesses an automorphism group $G$ such that for every $x \in \Gamma$ the stabilizer $G(x)$ of $x$ in $G$ induces on $\Gamma(x)$ an action containing $2.E_6(2)$. Then $\Gamma$ is isomorphic to the Baby Monster graph $\Theta$. □

The main goal of this paper is to prove the assertion in Proposition 1.1 without assuming anything about the automorphism group of $\Gamma$ (see Theorem 1 below). One of the intended applications of Theorem 1 is using it as an identification tool in the classification of (non-necessary flag-transitive) $P$-geometries of rank 5, as a continuation of the project started in [HS00].

Let $a \in \Theta$ and $b \in \Theta_3(a)$. Then the maximal intersection with $\Theta_3(a)$ (resp. with $\Theta_3(a) \cap \Theta_3(b)$) of a clique from $Q$ is of size 64 (resp. 36). Let $E(E_6(2))$ and $E(F_{22})$ be the incidence systems formed by the maximal intersections of the sets $\Theta_3(a)$ and $\Theta_3(a) \cap \Theta_3(b)$, respectively with the cliques from $Q$ together with the elements of $E(B)$ contained in these sets. The incidence relation and type function are as in $E(B)$.

**Proposition 1.2**  
(i) $E(E_6(2))$ is a $c.F_4(2)$-geometry, on which $B(a)/\langle a \rangle \cong 2.E_6(2) : 2$ induces a flag-transitive action;

(ii) $E(F_{22})$ is a $c.F_4(1)$-geometry, on which $B(a) \cap B(b) \cong F_{22} : 2$ induces a flag-transitive action.
By the construction $\mathcal{E}(2E_6(2))$ is a subgeometry in $\mathcal{E}(B)$ and $\mathcal{E}(Fi_{22})$ is a subgeometry in $\mathcal{E}(2E_6(2))$. By \cite{1} the geometry $\mathcal{E}(B)$ is simply connected which is not the case for $\mathcal{E}(2E_6(2))$ and $\mathcal{E}(Fi_{22})$. Let $\mathcal{A}$ be the amalgam of maximal parabolics associated with the action of $2E_6(2) : 2$ on $\mathcal{E}(2E_6(2))$. Then $\mathcal{A}$ is embedded into the unique non-split extension $3 \cdot 2E_6(2) : 2$ and hence there is a (3-fold) covering
\[ \varphi : \mathcal{E}(3 \cdot 2E_6(2)) \to \mathcal{E}(2E_6(2)) \]
of geometries. The preimage of $\mathcal{E}(Fi_{22})$ with respect to $\varphi$ is connected and it is a flag-transitive triple cover $\mathcal{E}(3 \cdot Fi_{22})$ of $\mathcal{E}(Fi_{22})$.

The subdegrees of the action of $E_6(2)$ on the cosets of $F_4(2)$ calculated in \cite{93} show that one of the orbitals of this action is the graph of a flag-transitive $c.F_4(2)$-geometry $\mathcal{E}(E_6(2))$. In a similar way the orbit lengths of $F_4(2)$ on the vectors of a 26-dimensional $GF(2)$-module calculated in \cite{CC88} prove existence of a $c.F_4(2)$-geometry $\mathcal{E}(2^{26}.F_4(2))$.

We are focussed on the $c$-extensions $\mathcal{E}$ of buildings with diagrams $c.F_4(t)$, satisfying the following two conditions:

(a) any two elements of type 1 are incident to at most one common element of type 2;

(b) three elements of type 1 are pairwise incident to common elements of type 2 if and only if they are incident to a common element of type 5.

Let $\Gamma = \Gamma(\mathcal{E})$ be the graph of $\mathcal{E}$. Then the above conditions are equivalent, respectively, to the following ones:

(a') $\Gamma$ has no multiple edges;

(b') for an element $z$ of type 5 the vertices in $\Gamma$ incident to $z$ induce a complete graph $Q(z)$ and every triangle in $\Gamma$ is contained in $Q(z)$ for some $z$.

The following theorems constitute the main result of the paper.

**Theorem 1** Let $\Gamma$ be a graph which is locally the commuting graph of the central involutions in $2E_6(2)$. Then $\Gamma$ is isomorphic to the Baby Monster graph $\Theta$.

**Theorem 2** Let $\mathcal{E}$ be a $c.F_4(4)$-geometry satisfying the conditions (a) and (b). Then $\mathcal{E}$ is the Baby Monster geometry $\mathcal{E}(B)$.

We emphasise again that in Theorems 1 and 2 no automorphisms are assumed a priori. The condition (a) excludes some possible degenerate cases and it holds in every flag-transitive $c.F_4(t)$-geometry. On the other hand the examples presented at the end of Section 4 show that the class of $c.F_4(t)$-geometries which do not satisfy (b) is rather large and even in the flag-transitive case it is difficult to expect the complete classification.

The paper is organized as follows. In Section 2 we recall some basic properties of the $F_4$-buildings with 3 points per a line. In Section 3 we discuss some general properties of $c.F_4(t)$-geometries and their graphs. In Section 4 we present and discuss some examples of $c.F_4(t)$-geometries. In Section 5 we construct an important class of subgraphs in a graph of $c.F_4(t)$-geometry. In Sections 6 and 7 we characterize the $\mu$-graphs which are subgraphs induced by the common neighbours of a pair of vertices at distance 2. Finally in Section 8 (which consists of a few subsections) we restrict ourselves to the case $t = 4$ and eventually prove the main theorems.

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2 Some $F_t$-buildings

Let $\mathcal{F} = ^t\mathcal{F}$ be a building with diagram

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 2 2 t t
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By the classification \[174\] of spherical buildings we obtain that $t = 4, 2$ or 1 and the (flag-transitive) automorphism group $^t\mathcal{F}$ of $\mathcal{F}$ is isomorphic to

\[2E_6(2) : S_3, \quad F_4(2) \text{ or } \Omega_8^+(2) : S_3,\]

respectively. The elements of type 1, 2, 3 and 4 in $\mathcal{F}$ will also be called points, lines, planes and symplecta, respectively. The points of $\mathcal{F}$ are the central (root) involutions in $^t\mathcal{F}$ for $t = 1$ and 4 and one of the two classes of such involutions for $t = 2$. Let $^t\Psi$ be the collinearity graph of $\mathcal{F}$, that is the graph on points in which two of them are adjacent if incident to a common line. The following result is well known (\cite{Coh83} and \cite{Coo83}).

**Lemma 2.1** The suborbit diagram of $\Psi$ is the one given on Fig. 2, where $\{p\}$, $\Psi_1(p)$, $\Psi_2^2(p)$, $\Psi_2^4(p)$ and $\Psi_3(p)$ are the orbits of $^tF(p)$ on $\Psi$ ($p$ is a point). Furthermore

(i) $^tF(p) \cong 2_t^{1+20} : U_6(2).S_3, 2_{t}^{1+6+8} : Sp_6(2)$ and $2_{t}^{1+8} : (S_3 \wr S_3)$ for $t = 4, 2$ and 1;

(ii) $q \in \Psi_1(p)$ if and only if $q \in O_2(^tF(p));$

(iii) $q \in \Psi_2^2(p)$ if and only if $q \in ^tF(p) \setminus O_2(^tF(p))$, in which case there is a unique symplecton incident to $p$ and $q$;

(iv) $q \in \Psi_2^4(p)$ if and only if $o(pq) = 4$, in which case $(pq)^2$ is the unique point collinear to both $p$ and $q$;

(v) $q \in \Psi_3(p)$ if and only if $o(pq) = 3$, in which case $^tF(p) \cap ^tF(q)$ is a complement to $O_2(^tF(p))$ in $^tF(p)$, if $\{p, r_1, r_2\}$ is a line containing $p$, then up to renumbering $r_1 \in \Psi_2^2(q), r_2 \in \Psi_3(q);$

(vi) the orbits of $O_2(^tF(p))$ on $\Psi_1(p), \Psi_2^2(p), \Psi_2^4(p)$ and $\Psi_3(p)$ are of length $2, 2^4 \cdot t, 2^5 \cdot t^3$ and $2^9 \cdot t^6 = |\Psi_3(p)|$, respectively;

(vii) the orbits of $O_2(^tF(p))$ on $\Psi_1(p)$ and $\Psi_2^2(p)$ naturally correspond to the lines incident to $p$ while the orbits of $O_2(^tF(p))$ on $\Psi_2^4(p)$ naturally correspond to the symplecta incident to $p$.  

The points lines, planes and symplecta in $\mathcal{F}$ can be identified with the sets (of size 1, 3, 7 and $28t + 7$, respectively) of points incident to these elements, so that the incidence relation is via inclusion. The geometry $\mathcal{F}$ can be reconstructed from $\Psi$ in the following way. If $q \in \Psi_2^2(p)$, then the unique symplecton containing $p$ and $q$ is the minimal geodetically closed subgraph in $\Psi$ containing $p$ and $q$. The remaining elements of $\mathcal{F}$ are the non-empty intersections of two or more symplecta.

There are the following embeddings among the three geometries:

\[1^t\mathcal{F} \subset 2^t\mathcal{F} \subset 4^t\mathcal{F}\]

which can be seen as follows. The group $^tF$ contains an (outer) involution $x$ such that $C_{^tF}(x) \cong (x) \rtimes ^tF$. Furthermore there is a conjugate $y$ of $x$ in $^tF$ such that $\langle x, y \rangle \cong S_3$ and $C_{^tF}(\langle x, y \rangle) \cong ^tF$. The subgraph in $^t\Psi$ induced by the vertices fixed by $x$ is isomorphic to $2^t\Psi$ and the subgraph induced by the vertices fixed by $\langle x, y \rangle$ (equivalently by $xy$) is isomorphic to $4^t\Psi$.  

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Let \( \hat{\Psi} \) be a graph on the point-set of \( F \) in which two points \( p \) and \( q \) are adjacent if they are incident to a common symplecton but not to a common line, equivalently if \( q \in \Psi_2(p) \). The suborbit diagram of \( \hat{\Psi} \) computed from that of \( \Psi \) is presented on Fig. 3.

Let \( \Delta = ^t \Delta \) be the graph on the point-set of \( F \) in which two points are adjacent whenever they are incident to a common symplecton. This means that two points \( p \) and \( q \) are adjacent in \( \Delta \) if \( q \in \Psi_1(p) \cup \Psi_2(p) \), so that the edge-set of \( \Delta \) is the union of the edge-sets of \( \Psi \) and \( \hat{\Psi} \). By (2.1) \( \Delta \) is the commuting graph on points of \( F \) (considered as involutions in \( F \)).

The next two lemmas follow directly from the diagrams of \( \Psi \) and \( \hat{\Psi} \).

**Lemma 2.2** If \( (p, r, q) \) is a 2-path in \( \Delta \) such that \( q \in \Psi_3(p) \), then \( p, q \in \Psi_2(r) \). \( \square \)

**Lemma 2.3** Let \( \{p, q\} \) be an edge of \( \Delta \). Then exactly one of the following holds:

(i) \(|\Delta(p) \cap \Delta(q)| = (24t + 4)(t^2 + t + 1) + 1 \) (which is 2101, 365 and 85 for \( t = 4, 2 \) and 1) and \( \{p, q\} \) is an edge of \( \Psi \);

(ii) \(|\Delta(p) \cap \Delta(q)| = 8t^4 + 12t^3 + 4t^2 + 28t + 5 \) (which is 2997, 391 and 57 for \( t = 4, 2 \) and 1) and \( \{p, q\} \) is an edge of \( \hat{\Psi} \). \( \square \)
The subgraph in $\Psi$ induced by (the points incident to) a symplecton has the following suborbit diagram.

Comparing this diagram with that of $\Psi$ and $\hat{\Psi}$ we obtain the following

**Lemma 2.4** Let $p \in \Delta$, $q \in \Psi_2(p)$ and let $\Upsilon$ be the subgraph in $\Psi$ induced by (the points incident to) the unique symplecton containing $p$ and $q$. Then

$$\Upsilon_1(p) = \Delta(q) \cap \Psi_1(p) \quad \text{and} \quad \Upsilon_2(p) = \{r \mid r \in \Psi_2(p), \Delta(r) \cap \Psi_1(p) = \Upsilon_1(p)\}.$$ 

**Lemma 2.5** The automorphism group of $\Delta = \Delta$ is isomorphic to $\mathfrak{F}$ (which is $2E_6(2) : S_3$, $F_4(2)$ or $\Omega_+^+(2) : S_3$ for $t = 4$, 2 and 1, respectively).

**Proof.** By (2.3), the automorphism group of $\Delta$ does not mix the edges of $\Psi$ with the edges of $\hat{\Psi}$ and by (2.4) the set of symplecta of $\mathcal{F}$ can be reconstructed from $\Delta$. The lines and planes of $\mathcal{F}$ can...
be reconstructed as the non-empty intersections of two or more symplecta; the incidence relation in \( F \) is via inclusion. Hence every automorphism of \( \Delta \) induces a uniquely determined automorphism of \( F \) and the result follows.

## 3 \( c \)-extensions of \( F \)

Thoughout this section \( E =^t\mathcal{E} \) is a geometry with diagram \( c.F_4(t) \), where \( t = 4, 2 \) or 1, such that the residue of an element of type 1 is isomorphic to the building \( ^tF \) and the conditions \((a)\), \((b)\) formulated in Section 1 are satisfied. Since these conditions are equivalent to conditions \((a')\) and \((b')\), we have the following

**Lemma 3.1** The graph \( \Gamma(\mathcal{E}) \) is locally \( \Delta \).

We will see that a converse of this lemma holds.

Let \( \Delta = ^t\Delta \) for \( t = 4, 2, 1 \) and let \( \Gamma \) be a graph which is locally \( \Delta \). For every \( x \in \Gamma \) let us fix a bijection \( i_x : \Gamma(x) \to \Delta \) which induces an isomorphism of the subgraph in \( \Gamma \) induced by \( \Gamma(x) \) onto \( \Delta \) (so that if \( y \in \Gamma(x) \) then \( i_x(y) \) is an involution in \( ^tF \)). The following lemma is a direct consequence of \((2.3)\).

**Lemma 3.2** Let \( \{x_1, x_2, x_3\} \) be a triangle in \( \Gamma \). Then one of the following holds:

(i) \( i_{x_j}(x_k) \in \Psi_1(i_{x_j}(x_l)) \) for all \( \{j, k, l\} = \{1, 2, 3\} \);

(ii) \( i_{x_j}(x_k) \in \Psi_2(i_{x_j}(x_l)) \) for all \( \{j, k, l\} = \{1, 2, 3\} \).

The triangles in \( \Gamma \) as in \((3.2 \ (i))\) will be called short while those in \((3.2 \ (ii))\) will be called long.

**Lemma 3.3** [7] Let \( \Gamma \) be a graph which is locally \( \Delta \). Then \( \Gamma = \Gamma(\mathcal{E}) \), for a \( c.F_4(t) \)-geometry \( E \) as above.

**Proof.** Let \( x \) be a vertex of \( \Gamma \) and \( P \) be the point-set of a symplecton in \( F \). Then the set

\[
Q_{x,P} = \{x\} \cup \{y \in \Gamma(x) \mid i_x(y) \in P\}
\]

is a complete subgraph of size \( 28t + 8 \) in \( \Gamma \). It is easy to deduce using \((2.3)\) and \((3.2)\) that for every \( y \in Q_{x,P} \) we have \( Q_{x,R} = Q_{y,R} \) for a symplecton \( R \) in \( F \).

Define \( \mathcal{E} \) to be a rank 5 geometry in which the elements of type 5 are the subgraphs \( Q_{x,P} \) taken for all \( x \in \Gamma \) and all symplecta \( P \) in \( F \); the remaining elements of \( \mathcal{E} \) are the non-empty intersections of two or more such subgraphs; the elements of type 1, 2, 3 and 4 are the intersections of size 1, 2, 4 and 8, respectively. The incidence is via inclusion. Then it is immediately seen that \( \mathcal{E} \) has the required properties and that \( \Gamma \) is the graph of \( \mathcal{E} \).

In what follows the elements of type 1, 2, 3, 4 and 5 in \( \mathcal{E} \) will be identified with the corresponding complete subgraphs in \( \Gamma = \Gamma(\mathcal{E}) \) as in \((3.3)\) of size 1, 2, 4, 8 and \( 28t + 8 \), respectively.

By \((3.3)\) it is clear that \( \mathcal{E} \) satisfies the intersection property (cf. [Pas94]) and hence by \([CP92]\) the residue \( \mathcal{H} \) of an element of type 5 in \( \mathcal{E} \) is a standard quotient of an affine polar space of type \( C_4 \). On the other hand by condition \((b)\) in the considered situation \( \mathcal{H} \) is a 1-point extension of its \( C_3 \)-residue. This gives the following.
Lemma 3.4 Let $^1\mathcal{H}$ be the residue of an element of type 5 in $\mathcal{E}$. Then $^1\mathcal{H}$ is flag-transitive with the automorphism group isomorphic to
\[ Sp_6(2), \ 2^6 : Sp_6(2), \ Sp_6(2) \]
for $t = 4, 2, 1$, respectively, and acting doubly transitively on the set of elements of type 1 in $^1\mathcal{H}$. Furthermore $^1\mathcal{H}$ is a subgeometry in $^2\mathcal{H}$ and the latter is a subgeometry in $^4\mathcal{H}$. □

Lemma 3.5 Let $L = \{x, z, x_1, z_1\}$ be an element of type 3 in $\mathcal{E}$ and $y$ be an element of type 1, not in $L$, but incident with $L$ to a common element $Q$ of type 5 and $N = L \cup \{y\}$. Then exactly one of the following holds:

(i) $N$ is contained in an element of type 4 and every triangle in $N$ is short;

(ii) $N$ is not contained in an element of type 4, there are exactly two short triangles in $N$, say $T_1$ and $T_2$, which contain $y$ and $N = T_1 \cup T_2$.

Proof. Consider $F = \text{res}_E(x)$. Then $M := L \setminus \{x\}$ is a line and $y$ is a point, both contained in the symplecton $Q \setminus \{x\}$. Then by the basic property of polar spaces, either $y$ is collinear to every point in $M$, in which case $M \cup \{y\}$ is contained in a plane of $F$ and we have (i); or $y$ is collinear to a unique point, say $z$ in $M$, in which case $(y, x, z)$ is short and the remaining two triangles in $N$ containing $\{y, x\}$ are long. Since a similar assertion holds in $\text{res}_E(x_1)$, the triangle $\{y, x_1, z_1\}$ must also be short. □

From (3.3) we deduce that the short triangles contained in an element of type 5 define a 2-graph, which is equivalent to the following

Corollary 3.6 If $N$ is a 4-element subset of (the set of elements of type 1 in) an element of type 5 in $F$, then the number of short triangles in $N$ is even. □

The following lemma is nothing but a more detailed version of (3.6).

Lemma 3.7 Let $\{x, z\}$ and $\{u, v\}$ be disjoint elements of type 2 contained in an element of type 5. Then exactly one of the following holds:

(i) $\{x, z, u, v\}$ is contained in an element of type 4;

(ii) both $\{x, z, u\}$ and $\{x, z, v\}$ are short and both $\{x, u, v\}$ and $\{z, u, v\}$ are long;

(iii) exactly one of $\{x, z, u\}$ and $\{x, z, v\}$ is short and exactly one of $\{x, u, v\}$ and $\{z, u, v\}$ is short;

(iv) both $\{x, z, u\}$ and $\{x, z, v\}$ are long and $\{x, u, v\}$ and $\{z, u, v\}$ are of the same type (both short or both long). □

The following result can alternatively be checked in the residue $^1\mathcal{H}$, known by (3.4).

Lemma 3.8 Let $X = \{x, z, u_1, u_2\}$ and $Y = \{x, z, v_1, v_2\}$ be distinct elements of type 3 in $\mathcal{E}$ incident to the common element $\{x, z\}$ of type 2 and contained in a common element of type 5. Then the symmetric difference $Z = \{u_1, u_2, v_1, v_2\}$ of $X$ and $Y$ is also an element of type 3 in $\mathcal{E}$.

Proof. If $X$ and $Y$ are contained in an element $N$ of type 4, then the result is immediate from the structure of $\text{res}_E(N)$, which is the 3-dimensional $GF(2)$-space. Suppose that $X$ and $Y$ are not in an element of type 4. Then applying (3.5) to $\{x, u_1, u_2, v_1\}$, we observe that the triangle $\{u_1, u_2, v_1\}$ is short. By the obvious symmetry every triangle in $Z$ is short and hence $Z$ is an element of type 3. □
4 The examples

By (3.1) and (3.3) there is a natural bijection between the $c.F_4(t)$-geometries satisfying (a) and (b) and the graphs which are locally $^t\Delta$. Thus in order to construct a $c.F_4(t)$-geometry it is sufficient to construct a graph which is locally $^t\Delta$.

It is known (cf. Lemma 5.10.6 in [99]) that the Baby Monster graph is locally $^4\Delta$ which gives the geometry $\mathcal{E}(B)$ as in [BS85].

For $t = 4, 2, 1$ put $n(t) = |^t\Delta|$. The following well-known result comes, for instance, from information on the maximal 2-locals in $^tF$ (compare [ATLAS]).

**Lemma 4.1** Up to conjugation in the automorphism group of $^tF$ every transitive action of $^tF$ of degree $n(t)$ is similar to the action on the vertex-set of $^t\Delta$.

Consider the action of $^2E_6(2) : 2$ on the cosets of $F_4(2) \times 2$. The subdegrees of this action are calculated in [L93]. In fact this action is similar to the action of $O^2(B(a))$ on $\Theta_3(a)$ where $B$ is the Baby Monster and $\Theta$ is the Baby Monster graph. This observation enables one to deduce the subdegrees from the structure constants of the centralizer algebra of $B$ acting on $\Theta$ (cf. p. 128 in [PS97]).

**Lemma 4.2** The action of $^2E_6(2) : 2$ on the cosets of $F_4(2) \times 2$ has rank 4 with subdegrees 1, $n(2) = (2^4 + 1)(2^{12} - 1), 2^4(2^{8} - 1)(2^{12} - 1)$ and $2^{12}(2^4 + 1)(2^8 + 2^4 + 1)$.

Consider the orbital $\Gamma$ of valency $n(2)$ of the action in (4.2). Comparing the subdegrees in (4.2) with the subdegrees of the action of $F_4(2)$ on $^2\Delta$ and since the action in (4.2) is by conjugation on a class of outer involutions in $^2E_6(2) : 2$ it is easy to see that $\Gamma$ is locally $^2\Delta$ which gives the geometry $\mathcal{E}(^2E_6(2))$.

Similarly [L93] gives the following.

**Lemma 4.3** The action of $E_6(2) : 2$ on $F_4(2) \times 2$ has rank 6 with subdegrees 1, $n(2) = (2^4 + 1)(2^{12} - 1), 2^4(2^{8} - 1)(2^{12} - 1), 2^{12}(2^{12} - 1)$ and $2^{12}(2^4 - 1)(2^8 - 1)$ (twice).

Again we observe that the orbital of valency $n(2)$ is the graph of a $c.F_4(2)$-geometry $\mathcal{E}(E_6(2))$.

The following result follows from Table 2 in [CC88].

**Lemma 4.4** Let $V$ be a 26-dimensional $GF(2)$-module of $F \cong F_4(2)$. Then the orbit lengths of $F$ on the set of vectors of $V$ are the following: 1, $n(2) = (2^4 + 1)(2^{12} - 1), 2^4(2^{8} + 2^4 + 1), 2^4(2^{12} - 1)(2^8 - 1), 2^8(2^{12} - 1)(2^4 + 1), 2^{12}(2^8 - 1)(2^4 - 1)$ and $2^{12}(2^{12} - 1)$.

The orbit lengths in (4.4) are the subdegrees of the natural action on $V$ of the semidirect product $V : F \cong 2^{26} : F_4(2)$ and similarly to the above case we observe that the orbital of valency $n(2)$ of this action is locally $^2\Delta$ and hence it is the graph of a $c.F_4(2)$-geometry $\mathcal{E}(2^{26} : F_4(2))$.

From Lemma 2.17.1 in [ILLSS] or p. 112 in [PS97] we obtain the suborbits of the action of $Fi_{22} : 2$ on the cosets of $\Omega^+_8(2).S_3 \times 2$. Notice, that this action is similar to the action of $B(a) \cap B(b)$ on $\Theta_3(a) \cap \Theta_3(b)$ for $b \in \Theta_2(a)$ (compare the intersection number 61 776 in the top matrix on p. 128 in [PS97]).

**Lemma 4.5** The action of $Fi_{22} : 2$ on the cosets of $\Omega^+_8(2).S_3 \times 2$ has rank 4 with subdegrees 1, $n(1) = 1 575, 22 400, 37 800$. 

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It is immediate from the structure constants of the action from \([13]\) given in \([LLSS]\) that the orbital of valency \(n(1)\) is locally \(\Delta\) and hence it is the graph of a \(cF_4(1)\)-geometry \(\mathcal{E}(F_{122})\).

Let \(2^1H \cong 2^2E_6(2) : 2, 3^1H \cong Fi_{22} : 2\) and \(\Omega\) be the set of cosets of \(2^1K \cong F_4(2) \times 2\) and \(3^1K \cong \Omega_2^+ \times 2\), respectively. Then the elements of \(\Omega\) are identified with a conjugacy class of outer involutions in \(3^1H\) so that the action is by conjugation. Let \(d \in \Omega\) and \(\{d\}, \Omega_1(d), \Omega_2(d), \Omega_3(d)\) be the orbits of \(3^1K\) on \(\Omega\) (compare (4.2) and (4.3)). Then under a suitable numbering of the orbits we have \(o(de) = 2, 3, 4\) for \(e \in \Omega_1(d), \Omega_2(d), \Omega_3(d)\), respectively (so that \(3^1K = C_{3^1H}(d)\) and \(|\Omega_1(d)| = n(t)\)). For \(t = 2, 1\) let \(\tilde{3}^1H\) be the unique non-split extension of \(\tilde{3}^1H\) by a normal subgroup \(Z\) of order 3 (recall that the 3-part of the Schur multiplier of \(O^2(\tilde{3}^1H)\) is of order 3). Let \(\tilde{d}\) be an involution in the preimage of \(\langle d \rangle\) in \(\tilde{3}^1H\). Then \(\tilde{d}\) inverts \(Z\) and hence \(C_{\tilde{3}^1H}(\tilde{d})\) maps isomorphically onto (and will be identified with) \(3^1K\). Let \(\tilde{\Omega}\) be the set of cosets of \(3^1K\) in \(\tilde{3}^1H\). Notice that \(\tilde{\Omega}\) is a conjugacy class of involutions in \(\tilde{3}^1H\) containing \(\tilde{d}\). It is easy to see that the preimage of \(\Omega_1(d)\) in \(\tilde{\Omega}\) splits into two \(3^1K\)-orbits with lengths \(n(t)\) and \(2 \cdot n(t)\). Let \(\tilde{\Gamma}\) be the orbital of valency \(n(t)\). We claim that \(\tilde{\Gamma}\) is locally \(\tilde{\Delta}\). Let \(\tilde{e}, \tilde{f} \in \tilde{\Gamma}(\tilde{d}), e, f\) be their respective images in \(\tilde{\Omega}\) and suppose that \(o(ef) = 2\). Since \(\tilde{d}\) is an involution which commutes with \(\tilde{e}\) and \(\tilde{f}\), we have \(o(\tilde{e}f) = o(\tilde{e}(\tilde{d})\tilde{f}(\tilde{d}))\). Since \(Z\) commutes with the product of any two involutions from \(\tilde{\Omega}\) the claim follows and we have the following.

**Lemma 4.6** The geometries \(\mathcal{E}(2^2E_6(2))\) and \(\mathcal{E}(Fi_{22})\) possess 3-fold covers \(\mathcal{E}(3^2E_6(2))\) and \(\mathcal{E}(3 \cdot Fi_{22})\), respectively. \(\square\)

The subdegrees of \(3^2E_6(2)\) on the cosets of \(F_4(2)\) are given in Table 3 in \([L93]\).

To the end of this section we discuss some \(cF_4(t)\)-geometries which do not satisfy condition (b). One class of such examples can be constructed as follows. Consider the action of the Baby Monster \(B\) on the cosets of a subgroup \(2^2E_6(2)\). Then one of the orbitals of valency \(n(4)\) with respect to the action (the graph \(\Delta_2\) on p. 98 in \([194]\)) is the graph of a \(cF_4(4)\)-geometry which is a double cover of \(\mathcal{E}(B)\). The residue of an element of type 5 in this cover is a double cover of the complete graph on 120 vertices. The preimages of \(\mathcal{E}(3^2E_6(2))\)- and \(\mathcal{E}(Fi_{22})\)-subgeometries in \(\mathcal{E}(B)\) are also proper double covers. One can construct analogous double covers of \(\mathcal{E}(3 \cdot Fi_{22})\) and \(\mathcal{E}(3^2E_6(2))\).

Let \(\mathcal{F} = \mathcal{F}\) be an \(F_4\)-building with 3 points per a line and let \((R, \varphi)\) be a representation of \(\mathcal{F}\), which means that \(R\) is a group and \(\varphi\) is the a mapping \(\varphi : p \mapsto z_p\) of the point-set of \(\mathcal{F}\) into \(R\) such that (i) \(R\) is generated by the image of \(\varphi\); (ii) \(z_p^2 = 1\) for every point \(p\); (iii) \(z_p z_q z_r = 1\) whenever \(\{p, q, r\}\) is a line. For an arbitrary element \(u \in \mathcal{F}\) define \(\varphi(u)\) to be the subgroup of \(R\) generated by the elements \(z_p\) taken for all points \(p\) incident to \(u\). Suppose that \((R, \varphi)\) is separable which means that the mapping \(u \mapsto \varphi(u)\) of \(\mathcal{F}\) into the set of subgroups of \(R\) is injective. Then there is a standard procedure which enables one to construct a \(c\)-extension \(\mathcal{E}\) of \(\mathcal{F}\): the elements of \(\mathcal{E}\) are the elements of \(R\) and all cosets of the subgroups \(\varphi(u)\) taken for all \(u \in \mathcal{F}\); the incidence is via inclusion. Notice that \(\mathcal{E}(2^{26} : F_4(2))\) can be constructed along these lines.

If \(F\) is the automorphism group of \(\mathcal{F}\) then \((F^\infty_\psi)\) is a separable representation of \(\mathcal{F}\) where \(\psi\) is the identity mapping. In addition for each \(t = 4, 2\) and 1 the geometry \(\mathcal{F}\) possesses an abelian representation. By this reason the classification of all (even flag-transitive) \(cF_4(t)\)-geometries seems to be far too difficult.

## 5 A subgraph \(\Xi\)

We follow the notation introduced in Section 3 so that \(\mathcal{E}\) is a \(cF_4(t)\)-geometry for \(t = 4, 2\) and 1 whose elements of type 1, 2, 3, 4 and 5 are identified with the corresponding complete subgraphs of size 1,
2, 4, 8 and 28t + 8 in the graph \( \Gamma = \Gamma(E) \), which is locally \( t \Delta \).

Let \( \tilde{\Xi} = t\Xi \) be the graph on the set of elements of type 2 in \( E \) in which two such elements are adjacent if they are incident to a common element of type 3 but not to a common element of type 1 (equivalently if their union is an element of type 3). Let \( e = \{x, y\} \) be an element of type 2 in \( E \) (which is an edge of \( \Gamma \) and a vertex of \( \tilde{\Xi} \)). Then the set \( \tilde{\Xi}(e) \) of neighbours of \( e \) in \( \tilde{\Xi} \) is in the natural bijection with the set of elements of type 3 incident to \( e \) via the correspondence

\[
\psi : e_1 \rightarrow e \cup e_1
\]

for \( e_1 \in \tilde{\Xi}(e) \). Let \( \Lambda \) be the graph on the set of elements of type 3 incident to \( e \) in which two of them are adjacent if incident to a common element of type 4. Then \( \Lambda \) is the collinearity graph of the residual dual polar space \( \text{res}_2^3(E) \) with the diagram \( 2 \circ t \circ t \circ t \) and the suborbit diagram of \( \Lambda \) is

\[
\begin{array}{cccccc}
1 & 2(t^2 + t + 1) & 2(t + 1) & t + 1 & 2t^2 & t^2 + t + 1 \\
& 1 & t & 4t(t^2 + t + 1) & 2t^2 & 8t^3 \\
\end{array}
\]

Let \( e_1, e_2 \in \tilde{\Xi}(e) \), where \( e_1 = \{x_1, y_1\} \) and \( e_2 = \{x_2, y_2\} \). If \( \psi(e_1) \) and \( \psi(e_2) \) are at distance 1 or 2 in \( \Lambda \), then they are incident to a common element of type 5 in \( E \) and by \( \{8, 8\} \) \( e_1 \cup e_2 \) is an element of type 3 which means that \( e_1 \) and \( e_2 \) are adjacent in \( \tilde{\Xi} \). On the other hand, if \( \psi(e_1), \psi(e_2) \) are at distance 3 in \( \Lambda \) then \( i_x(x_2) \in \Psi_2^3(i_x(x_1)) \), in which case \( e_1 \) and \( e_2 \) are certainly not adjacent in \( \tilde{\Xi} \). This implies the following.

**Lemma 5.1** Locally \( \tilde{\Xi} \) is the distance 1-or-2 graph of \( \Lambda \). \( \square \)

Now by Theorem 1.3 (vi), (vii), (ix) in [C99] we obtain the following.

**Proposition 5.2** Let \( e \) be an element of type 2 in \( E \) and \( \tilde{\Xi}^e \) be the connected component of \( \tilde{\Xi} \) containing \( e \). Then

(i) \( \tilde{\Xi}^e \) has 40 vertices, its automorphism group is isomorphic to \( U_4(2) : 2 \) and the suborbit diagram is

\[
\begin{array}{cccccc}
1 & 27 & 1 & 27 & 8 & 12 \\
| & \downarrow & | & \downarrow & \downarrow & \downarrow \\
S_3 \wr S_3 & 2^3 S_3 & 3^{1+2} 2^2 & S_3 & 120 \\
\end{array}
\]

(ii) \( \tilde{\Xi}^e \) has 256 vertices, its automorphism group is isomorphic to \( 2^8 : Sp_6(2) \) and the suborbit diagram is

\[
\begin{array}{cccccc}
1 & 135 & 1 & 135 & 64 & 63 \\
| & \downarrow & | & \downarrow & \downarrow & \downarrow \\
Sp_6(2) & 2^{4+3}.L_3(2) & U_3(3) : 2 & U_3(3) : 2 \\
\end{array}
\]

(iii) \( \tilde{\Xi}^e \) has 2300 vertices, its automorphism group is isomorphic to \( Co_2 \) and the suborbit diagram is
Furthermore, $^1\Xi^e$ is a subgraph in $^2\Xi^e$ and the latter is a subgraph in $^4\Xi^e$. □

Let $\Xi^e$ be the subgraph in $\Gamma$ induced by the elements of type 1 in $E$ incident to the elements of type 2 in $\Xi$. Since the diameter of $\Xi^e$ is 2, in view of the paragraph before (5.1), every vertex in $\Xi^e$ is incident to exactly one element of type 2 in $\Xi^e$ which gives the following.

**Lemma 5.3** The subgraph $\Xi^e$ is the 2-clique extension of $\Xi^e$. □

### 6 $\mu$-graphs of $D_8$-type

In this and next sections for a pair $\{x, y\}$ of vertices at distance 2 in $\Gamma$ we analyze the subgraph $\Gamma(x, y)$ induced in $\Gamma$ by the common neighbours of $x$ and $y$ (the subgraphs $\Gamma(x, y)$ are commonly known as $\mu$-graphs).

First we introduce some terminology. Recall that $\pi = (x, z, y)$ is a 2-path in $\Gamma$ if $x, y \in \Gamma(z)$ and $y \in \Gamma_2(x)$. Such a 2-path $\pi$ is said to be of $D_6$- or $D_8$-type if

$$i_z(y) \in \Psi_3(i_z(x)) \text{ or } i_z(y) \in \Psi_2^4(i_z(x)),$$

(equivalently if $\langle i_z(x), i_z(y) \rangle \cong D_6$ or $D_8$), respectively. Clearly the paths $(x, z, y)$ and $(y, z, x)$ have the same type.

For $j = 3$ or 4 and a vertex $x$ of $\Gamma$ let $\Gamma_2^j(x)$ denote the set of vertices $y$ at distance 2 from $x$ in $\Gamma$ such that there is a path of $D_{2j}$-type joining $x$ and $y$. Notice that $\Gamma_2(x) = \Gamma_3^2(x) \cup \Gamma_4^2(x)$ and a priori the sets $\Gamma_3^2(x)$ and $\Gamma_4^2(x)$ might intersect.

The next lemma follows from the fact that $\Gamma$ is locally $\Delta$ and from the suborbit diagrams of $\Psi$ and $\hat{\Psi}$.

**Lemma 6.1** Let $\pi = (x, z, y)$ be a 2-path in $\Gamma$. Then

(i) if $\pi$ is of $D_8$-type, then the valency of $z$ in $\Gamma(x, y)$ is $(2t + 6)(t^2 + t + 1) + 1$ (which is 295, 71 and 25 for $t = 4$, 2 and 1);

(ii) if $\pi$ is of $D_6$-type, then the valency of $z$ in $\Gamma(x, y)$ is $(2t^2 + 1)(t^2 + t + 1)$ (which is 693, 63 and 9 for $t = 4$, 2 and 1). □

**Lemma 6.2** Let $\pi = (x, z, y)$ be a 2-path of $D_8$-type. Then

(i) there is an element $e$ of type 2 in $E$ incident to $x$ such that $y, z \in \Xi^e$;

(ii) every 2-path contained in $\Xi^e$ is of $D_8$-type;

(iii) the connected component of $\Gamma(x, y)$ containing $z$ is contained in $\Xi^e$. 

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Proof. Since $\pi$ is of $D_8$-type by (2.3 (iv)) there is a (unique) vertex $u$ adjacent to $z$, such that $i_z(x), i_z(y) \in \Psi_1(i_z(u))$. Then clearly $\pi \in \Xi^e$ where $e = \{z, u\}$ and (i) follows. Now (ii) follows from the paragraph before (5.1) and the fact that the diameter of $\Xi^e$ is 2. In order to prove (iii) it is sufficient to show that the valency $k_1$ of $z$ in $\Gamma(x, y) \cap \Xi^e$ is equal to the valency of $z$ in $\Gamma(x, y)$ given in (6.1 (i)). By (5.1) the graph $\Xi$ is locally the distance 1-or-2 graph of $\Lambda$. The latter is a near hexagon with classical quads. This shows that for $f \in \Xi, g \in \Xi_2(f)$ the subgraph in $\Xi$ induced by $\Xi(f) \cap \Xi(g)$ has valency $k_2 = (t + 3)(t^2 + t + 1)$. Since $\Xi^e$ is the 2-clique extension of $\Xi^e$ we have $k_1 = 2 \times k_2 + 1$, hence (iii) follows.

Lemma 6.3 Let $\pi = (x, z, y)$ and $\sigma = (x, u, y)$ be 2-paths in $\Gamma$ such that $z$ and $u$ are in the same connected component of $\Gamma(x, y)$ and $\delta = (z, v, w)$ be a 2-path contained in $\Gamma(x, y)$. Then $\pi$, $\sigma$ and $\delta$ have the same type.

Proof. If one of the paths $\pi$, $\sigma$, $\delta$ is of $D_8$-type then by (6.2) all three paths are of $D_8$-type and hence the result.

Let $y \in \Gamma_1^+(x)$, $e = \{x, z\}$ be an element of type 2 in $E$ such that $y \in \Xi^e$ (compare (6.2 (i)), $p = i_x(z)$, $\Omega = \{i_x(v) \mid v \in \Gamma(x, y) \cap \Xi^e\}$ and $I$ be the setwise stabilizer of $\Omega$ in $F = E/F$. It is clear that $I \leq F(p)$.

Lemma 6.4

(i) $I$ contains $O_2(F(p))$;

(ii) for every element $\alpha$ in $O_2(F(p))$ there is an automorphism $\beta$ of $\Xi^e$ stabilizing $e$ such that the action of $\alpha$ on the set of elements of type 3 in $E$ incident to $e$ is similar to the action of $\beta$ on $\Xi(e)$;

(iii) $I \cong 2^{1+20} : U_4(3) : 2^2 \cdot 2^1 + 6 + 8 : U_3(3) : 2$ and $2^{1+8} \cdot 3^{1+2} \cdot 2^2$ for $t = 4, 2$ and 1, respectively.

Proof. An element $e_1$ of type 2 in $E$ is an orbit of $O_2(F(p))$ on $\Psi_1(p)$ if and only if $e_1$ is the image under $i_x$ of an element from $\Xi(e)$, we obtain (i). Now (ii) and (iii) are immediate from (5.1).

With $\Omega$ and $I$ as above put $\mu_1 = |\Omega|$. Our next goal is to describe the orbits of $I$ on $\Delta$.

Lemma 6.5 Let $L$ and $S$ be the set of lines and the set of symplecta in $F$ incident to $p$, so that $|L| = 3(2t^2 + 1)(2t + 1)$ and $|S| = (2t^2 + 1)(t^2 + t + 1)$. Then

(i) $I$ has two orbits, $L_1$ and $L_2$ on $L$ with lengths $\mu_1/2$ and $|L| - \mu_1/2$, respectively;

(ii) if $t = 1$ or 2 then $I$ acts transitively on $S$;

(iii) if $t = 4$ then $I$ has two orbits $S_1$ and $S_2$ on $S$ with lengths 126 and 567, furthermore,

(a) the symplectas incident to a line from $L_1$ are contained in $S_2$;

(b) a line from $L_2$ is incident to 6 symplectas from $S_1$ and to 15 symplectas from $S_2$.

Proof. The result follows from well-known properties of $F(p)/O_2(F(p))$. For the case $t = 4$ see Lemma 5.10.15 in [9].

The next result is a direct consequence of (2.3 (vii)), (6.4 (i)) and (6.3).

Lemma 6.6 The following assertions hold:
(i) I has two orbits $\Omega_1 = \Omega$ and $\Omega_2$ on $\Psi_1(p)$;
(ii) if $t = 1, 2$ then $I$ acts transitively on $\Psi_2^t(p)$ while if $t = 4$, $I$ has two orbits $\Omega_3$ and $\Omega_4$ on $\Psi_2^4(p)$ with lengths $2^6 \cdot 126$ and $2^6 \cdot 567$, respectively;
(iii) $I$ has two orbits $\Omega_5$ and $\Omega_6$ on $\Psi_2^4(p)$ with lengths $16t^3 \cdot |\Omega_1|$ and $16t^3 \cdot |\Omega_2|$;
(iv) $I$ acts transitively on $\Omega_7 = \Psi_3(p)$.

Notice that in terms of (6.3) the orbits $\Omega_1$ and $\Omega_5$ correspond to $L_1$, $\Omega_2$ and $\Omega_6$ correspond to $L_2$, $\Omega_3$ corresponds to $S_1$ and $\Omega_4$ corresponds to $S_2$.

Lemma 6.7 Let $u \in \Psi_1(p) \cup \Psi_2^2(p) \cup \Psi_2^4(p)$ for $t = 1, 2$ and $u \in \Psi_1(p) \cup \Omega_4 \cup \Psi_2^4(p)$ for $t = 4$. Then $u$ is adjacent in $\Delta$ to a vertex from $\Omega$.

Proof. If $u \in \Psi_1(p)$, then the claim follows from the fact that the subgraph in $\Delta$ induced by $\Psi_1(p)$ is connected. A vertex from $\Omega$ is clearly adjacent in $\Delta$ to vertices in $\Psi_2^2(p)$. Thus the claim follows for $t = 1, 2$ and $u \in \Psi_2^2(p)$ as well as for $t = 4$ and $u \in \Omega_4$ (compare (1.3 (a))). Let $u \in \Psi_2^4(p)$ and $v$ be the unique vertex in $\Psi_1(p) \cap \Psi_1(u)$ (compare the diagram of $\Psi$). Let $A$ and $B$ be the orbits of $O_2(F(p))$ containing $u$ and $v$, respectively (so that by (2.1 (vi)) we have $|A| = 2^5 \cdot t^3$ and $|B| = 2$). Then by (2.1 (vii)) the stabilizer of $A$ in $F(p)$ coincides with the stabilizer of $B$ in $F(p)$ and it is the stabilizer of the unique line $l$ of $F$ containing $u$ and $v$. Let $C$ be the set of orbits of $O_2(F(p))$ on $\Psi_1(p)$ containing vertices adjacent to $u$ in $\Delta$. Since $|\Delta(p) \cap \Delta(u)| = 1 + 2(t^2 + t + 1)$ while the suborbits of the action of $F(p)$ on the set of lines incident to $p$ are $1, 2(t^2 + t + 1), 4t^2(t^2 + t + 1)$, $8t^3$ (compare the diagram of $\Delta$ before (6.1)), we conclude that $|C| = 1 + 2(t^2 + t + 1)$ and $C \setminus B$ correspond to the lines incident with $l$ to a common plane. Since every vertex from $\Psi_1(p)$ is adjacent in $\Psi$ to a vertex from $\Omega$, this completes the proof.

As a consequence of the above proof in view of (2.1 (v)) and (6.3 (iii) (a)) we obtain the following.

Corollary 6.8 If $u \in \Psi_3(p)$ for $t = 1, 2$ and $u \in \Omega_3 \cup \Psi_3(p)$ for $t = 4$, then no vertex in $\Omega$ is adjacent to $u$ in $\Delta$.

Lemma 6.9 In terms introduced before (1.4) if $w \in \Gamma(x, y) \setminus \Xi^c$, then $(x, w, y)$ is of $D_6$-type and either $t = 4$ or $i_2(w) \in \Psi_3(p)$.

Proof. By (6.3 (iii)) $\Gamma(x, y) \cap \Xi^c$ is a connected component of $\Gamma(x, y)$ and hence by (6.7) $i_2(w) \in \Psi_3(p)$ for $t = 1, 2$ and $i_2(w) \in \Psi_3(p) \cup \Omega_3$ for $t = 4$. Suppose that $(x, w, y)$ is of $D_6$-type. By (6.2 (i), (iii)) there is an element $f = \{x, v\}$ of type 2 in $E$ such that $y, w, x \in \Xi^f$ and $\Gamma(x, y) \cap \Xi^f$ is another connected component of $\Gamma(x, y)$. Put $q = i_2(v)$ and $\Omega' = \{i_2(s) \mid s \in \Gamma(x, y) \cap \Xi^f\}$. Recall that $\Omega'$ is the union of some $\mu_1/2$ lines containing $q$ with $q$ removed. It is immediate from the diagram of $\Psi$ that a line intersecting $\Psi_3(p)$ contains two points from $\Psi_3(p)$ and one point from $\Psi_2^2(p)$. Hence, assuming that $\Omega' \cap \Psi_3(p) \neq \emptyset$, we must have $q \in \Psi_2^2(p)$. In this case let $r$ be the unique vertex in $\Psi_1(p) \cap \Psi_1(q)$. Then by (6.7) $r$ is adjacent in $\Delta$ to a vertex from $\Omega'$ and by (2.2) $r$ is not adjacent to any vertex from $\Psi_3(p)$. Hence $\Omega' \cap \Omega_3 \neq \emptyset$, in particular $t = 4$. Let $\{q, s_1, s_2\}$ be a line such that $s_1, s_2 \in \Omega'$ and $s_1 \in \Omega_3$. Then one can see from Fig. 2 that $s_2 \notin \Psi_3(p)$ and hence $s_2 \in \Omega_3$. Comparing the diagram of $\Psi$ and the diagram of the subgraph in $\Psi$ induced by a symplecton given before (2.3) we observe that $s_1$ and $s_2$ are contained in the same symplecton, containing $p$ and $q \in \Psi_1(p)$, in particular $\Omega' \cap \Psi_3(p) = \emptyset$. By (6.3 (iii) (b)) $q$ is contained in exactly 6 symplecta intersecting $\Omega_3$ and each of these symplecta contains just 16 lines containing $q$ and intersecting $\Psi_2^2(p)$. Since $6 \times 16 < 324$, we have a contradiction, hence $(x, w, y)$ must be of $D_6$-type.

The results in (5.2), (6.2) and (6.9) can be summarized as follows.
Proposition 6.10 Let \( y \in \Gamma_2^4(x) \). Then

(i) there is a unique element \( e \) of type 2 in \( E \) incident to \( x \) such that \( y \in \Xi^e \);

(ii) for any two elements \( e \) and \( f \) of type 2 in \( E \) the intersection \( \Xi^e \cap \Xi^f \) induces in \( \Gamma \) a complete subgraph;

(iii) \( |\Gamma_2^4(x)| = |\Delta| \cdot |\Psi_2^4(p)|/\mu_1 \) where \( \mu_1 = 648, 144 \) and 36 for \( t = 4, 2 \) and 1.

\( \square \)

7 \( \mu \)-graphs of \( D_6 \)-type

In this section we analyze the common neighbours of \( x \) and \( y \in \Gamma_3^2(x) \). Let \( (x, z, y) \) be a 2-path of \( D_6 \)-type, \( p = i_z(x) \), \( q = i_z(y) \) and \( \Upsilon = \Upsilon = \{ i_z(u) \mid u \in \Gamma(x, y) \cap \Gamma(z) \} \). Then \( q \in \Psi_3^2(p) \) and by (2.1 (v)) \( H = H := F(p) \cap F(q) \) is a complement to \( O_2(F(p)) \) in \( F(p) \) isomorphic to \( U_6(2).S_3, Sp_6(2) \) and \( S_3 \) for \( t = 4, 2 \) and 1.

Lemma 7.1 The following assertions hold:

(i) \( |\Upsilon| = (2t^2 + 1)(t^2 + t + 1) \);

(ii) \( \Upsilon \) is the unique class of central involutions in \( H \) of size \( |\Upsilon| \);

(iii) \( \Upsilon \) is a class of 3-transpositions in \( H \), so that

(a) \( ^1\Upsilon \) is the complete 3-partite graph \( K_{3 \times 3} \) with the suborbit diagram

(b) \( ^2\Upsilon \) is the collinearity graph of the polar space \( \mathcal{P}(Sp_6(2)) \) of \( 2H \cong Sp_6(2) \) with the diagram

and the suborbit diagram of \( ^2\Upsilon \) is

(c) \( ^4\Upsilon \) is the collinearity graph of the polar space \( \mathcal{P}(U_6(2)) \) of \( 4H \cong U_6(2).S_3 \) with the diagram

and the suborbit diagram of \( ^4\Upsilon \) is

Proof. The assertion (i) is just (6.1 (ii)). By the definition, \( \Upsilon \) consists of all the involutions in \( \Delta \) which commute with \( p \) and \( q \). Hence \( \Upsilon \) is contained in \( H \) and it is closed under conjugation by the elements of \( H \). Hence \( \Upsilon \) is a union of some conjugacy classes of involutions in \( H \) and since \( |\Upsilon| \) is odd, at least one of these classes consists of central involutions. Now (ii) and (iii) follows from the well-known properties of \( H \) (cf. [ATLAS]).

\( \square \)
In terms introduced before (7.1) let \( \Pi \) be the connected component of \( \Gamma(x,y) \) containing \( z \) and \( \Sigma = \{ i_x(u) \mid u \in \Pi \} \).

**Lemma 7.2** Let \( u, v \in \Sigma \). Then

(i) \( v \in \Psi^2(u) \) if \( u \) and \( v \) are adjacent in \( \Sigma \) and \( v \in \Psi_3(u) \) if \( u \) and \( v \) are at distance 2 in \( \Sigma \);

(ii) the graph \( \Sigma \) is locally \( \Upsilon \).

**Proof.** The assertion (ii) follows directly from the definition of \( \Sigma \) and \( \Upsilon \) while (i) is immediate from (2.2) and (6.3). \( \square \)

**Lemma 7.3**

(i) \( 1\Sigma \) is the complete 4-partite graph \( K_{4 \times 3} \) with the automorphism group \( S_3 \wr S_4 \) and the suborbit diagram

\[
\begin{array}{cccccc}
1 & 9 & 1 & 6 & 2 & 9 \\
\hline
(S_3 \wr S_3) \times 2 & (S_3 \wr S_2) \times 2^2 & S_3 \wr S_3
\end{array}
\]

(ii) \( 2\Sigma \) is isomorphic to one of the following three graphs:

(a) the commuting graph \( 2\Sigma^a \) of 3-transpositions in \( \Omega^{-}_8(2) : 2 \) with the suborbit diagram

\[
\begin{array}{cccccc}
1 & 63 & 1 & 30 & 28 & 35 \\
\hline
Sp_6(2) \times 2 & 2^6 : Sp_4(2) & \Omega^+_6(2) : 2
\end{array}
\]

(b) the commuting graph \( 2\Sigma^b \) of 3-transpositions in \( \Omega^+_8(2) : 2 \) with the suborbit diagram

\[
\begin{array}{cccccc}
1 & 63 & 1 & 30 & 36 & 27 \\
\hline
Sp_6(2) \times 2 & 2^6 : Sp_4(2) & \Omega^-_6(2) : 2
\end{array}
\]

(c) the 2-fold antipodal cover \( 2\Sigma^c \) of the complete graph with the automorphism group \( 2^7 : Sp_6(2) \) and the suborbit diagram

\[
\begin{array}{cccccc}
1 & 63 & 1 & 30 & 32 & 32 & 30 & 63 & 1 \\
\hline
Sp_6(2) & 2^5 : Sp_4(2) & 2^5 Sp_4(2) & Sp_6(2)
\end{array}
\]

(iii) \( 4\Sigma \) is the commuting graph of 3-transpositions in the Fischer group \( Fi_{22} \) with the suborbit diagram

\[
\begin{array}{cccccc}
1 & 693 & 1 & 180 & 180 & 512 & 126 & 567 & 2816 \\
\hline
2 \cdot U_6(2) & 2^{2+8} U_4(2) & U_4(3) : 2
\end{array}
\]
Proof. By (7.2 (ii)) $\Sigma$ is locally $\Upsilon$. A connected graph which is locally the complete multipartite graph $K_{t,t,m}$ with $t \geq 2$ parts of size $m \geq 2$ each, is isomorphic to $K_{(t+1)\times m}$ (cf. Proposition 1.1.5 in [BCN89]) and hence (i) follows. For $t = 2$ and 4 let $P$ be the polar space with diagram

```
  i ---- i ---- i
     |    |    |
     1   2
```

as in (7.1 (iii) (b), (c)) so that $\Upsilon$ is the collinearity graph of $P$. Let $R$ be the geometry of rank 4, whose elements of type 4 are the maximal complete subgraphs (of size $t^2 + t + 2$) in $\Sigma$; the elements of type 1 are the vertices, the elements of type 2 are the edges and the elements of type 3 are the complete subgraphs of size $t + 2$ contained in more than one maximal complete subgraph; the incidence relation is via inclusion. It is a standard fact that $R$ is a $c$-extension of $P$ with the diagram

```
  1 ---- i ---- i ---- 2
```

If $t = 2$ then by [CP92] $R$ is a standard quotient of an affinization of the polar space of $S_{p o}(2)$. It is well known that $P$ contains three classes of hyperplanes and we obtain the possibilities (a), (b) and (c) in (ii) (notice that the antipodal quotient of $2\Sigma^c$ is the complete graph which is not locally $2\Upsilon$). Finally the isomorphism type of $4\Sigma$ follows from [Pase94] and [Pase95]. □

Let us match the possibilities in (7.3) with the examples we know. Let $\Gamma = \Gamma(\xi)$ where $\xi$ is one of the seven examples of $c.F_4(1)$-geometries in Section 4. If $\xi = \xi(3 \cdot F_{22})$ then $\Gamma(x, y) \cong 1\Sigma$ and if $\xi = \xi(F_{22})$ then $\Gamma(x, y)$ consists of three connected components, each isomorphic to $1\Sigma$. If $\xi = \xi(\xi(2^2 E_6(2)))$ then $\Gamma(x, y) \cong 3\Sigma^2$ and if $\xi = \xi(2^2 E_6(2))$ then $\Gamma(x, y)$ consists of three connected components, each isomorphic to $3\Sigma^2$. If $\xi = \xi(2^2 E_6(2))$ then $\Gamma(x, y) \cong 2\Sigma^2$ while if $\xi = \xi(2^2 E_6(2))$ then $\Gamma(x, y) \cong 2\Sigma^2$. Finally if $\xi = \xi(B)$ then $\Gamma(x, y) \cong 4\Sigma$.

Let $\Sigma$ be $1\Sigma$, $2\Sigma^2$, $2\Sigma^b$ or $4\Sigma$ as in (7.3) (not $2\Sigma^c$) and $T$ be isomorphic to $S_3 \times S_3 \times S_3 \times S_3$, $\Omega_5^c(2) : 2$, $\Omega_5^b(2) : 2$ or $F_{22}$, respectively, so that $\Sigma$ is isomorphic to the commuting graph of a conjugacy class of 3-transpositions in $T$. This means that there is a bijection

$$j: \Sigma \rightarrow T,$$

such that $o(j(u)j(v)) = 2$ if $v \in \Sigma(u)$ and $o(j(u)j(v)) = 3$ if $v \in \Sigma_2(u)$. Notice that by (7.2 (ii)) $u$ and $v$ (considered as involutions in $F$) satisfy $o(uv) = 2$ if $v \in \Sigma(u)$ and $o(uv) = 3$ if $v \in \Sigma_2(u)$. For $t \in T$ and $x \in \Sigma$ the action $x^t = j^{-1}j(x)^t$ turns $T$ into an automorphism group of $\Sigma$.

Lemma 7.4 In the above terms (i.e. with $\Sigma \not\cong 2\Sigma^c$)

(i) whenever $u, v \in \Sigma$ and $w = uvw$, we have $j(w) = j(u)j(v)j(u)$;

(ii) if $u \in \Sigma$, then $u$ (considered as an automorphism of $\Delta$) stabilizes $\Sigma$ as a whole.

Proof. Let us proceed with (i) (which will immediately imply (ii)). If $u$ and $v$ are adjacent then $w = v$, $j(w) = j(v)$ and the assertion is obvious. If $v \in \Sigma_2(u)$ we define $x$ so that $j(x) = j(u)j(v)j(u)$ (this is possible since $j(\Sigma)$ is a conjugacy class in $T$) and all we have to show is that $w := uvw$ coincides with $x$. Let $\Sigma(u, v)$ be the $\mu$-graph in $\Sigma$ (the subgraph induced by $\Sigma_1(u) \cap \Sigma_1(v)$). Since $j(u)$ fixes $\Sigma_1(u)$ elementwise and maps $v$ onto $x$, we observe that $\Sigma(u, x) = \Sigma(u, v)$. Furthermore, $j(\Sigma(u, v)) = j(\Sigma) \cap T(u) \cap T(v)$. In view of the suborbit diagrams in (7.3) this shows, particularly, that $\Sigma(u, v)$ contains a pair of non-adjacent vertices, say $p$ and $q$. Then $\{u, v, x\} \subset \Sigma_1(p) \cap \Sigma_1(q)$ and $q \in \Sigma_2(p)$. On the one hand $\Sigma(p, q)$ is isomorphic to the commuting graph of involutions in $j(\Sigma) \cap (T(p) \cap T(q))$ and on the other hand $\Sigma(p, q)$ is a subgraph in the commuting graph $\Delta(p, q)$ of
involutions in $F(p) \cap F(q)$. Thus the restriction $\sigma$ of $j^{-1}$ to $\Sigma(p, q)$ induces an isomorphic embedding of the commuting graph $\Pi$ of involutions in $j(\Sigma(p, q))$ into the commuting graph $\Upsilon$ (as in (7.1)) of involutions in $\Delta(p, q)$ and it remains to show that the embedding $\sigma$ is $S_3$-consistent in the sense that it maps the set $\{u, v, x\}$ of involutions in an $S_3$-subgroup in $K := T(p) \cap T(q)$ onto the set of involutions in an $S_3$-subgroup in $H = F(p) \cap F(q)$. If $t = 1$ then $|\Pi| = |\Upsilon| = 9$, $\sigma$ is an isomorphism and a triple of vertices in $\Pi$ or in $\Upsilon$ generate an $S_3$-subgroup if and only if these three vertices are pairwise non-adjacent, the result is trivial in this case.

In the case $t = 2$ the graph $\Pi$ is the commuting graph of $3$-transpositions in $K \cong \Omega_6^+(2) : 2 \cong S_8$ with the suborbit diagram

$$
\begin{array}{ccc}
1 & 15 & 6 \\
S_6 \times 2 & S_4 \times 2^2 & S_5 \cong \Omega_7^+(2) : 2
\end{array}
$$

or the commuting graph of $3$-transpositions in $K \cong \Omega_6^- (2) : 2 \cong U_4(2) : 2$ with the suborbit diagram

$$
\begin{array}{ccc}
1 & 15 & 6 \\
S_6 \times 2 & S_4 \times 2^2 & (S_3 \times S_3) \cdot 2 \cong \Omega_4^+(2) : 2
\end{array}
$$

By (7.1 (iii) (b)) $\Upsilon$ is the collinearity graph of the polar space $\mathcal{P} = \mathcal{P}(S_{p_6}(2))$. The lines of this space can be identified with a class of triangles in $\Upsilon$ so that every edge is contained in a unique line and a vertex outside a line is adjacent to 1 or 3 vertices on the line. It is easy to see that for every triangle in $\Pi$ there is a vertex adjacent to 2 vertices in the triangle. Hence $\sigma(\Pi)$ does not contain lines of $\mathcal{P}$ and since the valency of $\Pi$ (which is 15) equals to the number of lines containing a given point, every line intersects $\sigma(\Pi)$ in 2 or 0 vertices. Hence $\sigma(\Pi)$ is the complement to a hyperplane in $\mathcal{P}$. All hyperplanes in $\mathcal{P}$ are known, in particular every hyperplane with complement of size 28, respectively 36 has stabilizer in $H \cong S_{p_6}(2)$ isomorphic to $\Omega_6^+(2) : 2$ or $\Omega_6^- (2) : 2$, which shows that $\sigma$ is $S_3$-consistent.

In the case $t = 4$ $\Pi$ is the commuting graph of $3$-transpositions in $K \cong U_4(3) : 2$ with the suborbit diagram

$$
\begin{array}{ccc}
1 & 45 & 12 \\
U_4(2) \times 2 & 2^2+4 \cdot 3^2 \cdot 2 & 3^{1+2} \cdot 2^{1+2} \cdot 3
\end{array}
$$

and locally it is the point graph of the generalized quadrangle of order $(4, 2)$ associated with $U_4(2)$. This shows that the vertices, edges and maximal cliques (of size 6) with respect to inclusion form a geometry $\mathcal{G}$ with the diagram $\begin{array}{ccc} 1 & 6 & 2 \end{array}$. Recall that in the considered situation $\Upsilon$ is the collinearity graph of the polar space $\mathcal{P} = \mathcal{P}(U_6(2))$ so that the lines and planes in $\mathcal{P}$ can be identified with certain complete subgraphs of size 5 and the maximal complete subgraphs (of size 21) in $\Upsilon$. We extend $\sigma$ to a morphism $\delta$ of $\mathcal{G}$ into $\mathcal{P}$ as follows. For $t \in \Pi$ put $\delta(t) = \sigma(t)$; for an edge $e$ of $\Pi$ define $\delta(e)$ to be the unique line in $\mathcal{P}$ containing $\sigma(e)$ and for a 6-clique $X$ in $\Pi$ define $\delta(X)$ to be the unique plane in $\mathcal{P}$ containing $\sigma(X)$. In the polar space $\mathcal{P}$ a point outside a line is adjacent to 1 or 5 points of the line, while in $\Pi$ for every triangle there is a vertex adjacent to exactly 2 vertices in the triangle. It is an easy combinatorial exercise to check, using this observation, that $\delta$ is an isomorphic embedding. The distribution diagram of the graph on the elements of type 3 in $\mathcal{G}$ in
which two such elements are adjacent if they are incident to a common element of type \(2\) (cf. p. 202 in \[99\]) shows, that every edge in this graph is contained in a unique triangle. Hence the image \(\overline{\delta}\) of the set of elements of type \(3\) from \(\mathcal{G}\) is a hyperplane in the dual polar space associated with \(P\). Now by Proposition 2.1 in \[94\] the stabilizer of \(\overline{\delta}(\mathcal{G})\) in \(H \cong U_6(2) : S_3\) is isomorphic to \(K \cong U_4(3) : 2\) and hence \(\sigma\) is \(S_3\)-consistent. 

Let us try to identify \(\Sigma\) as a subgraph in \(\Delta\). We proceed to this via identifying the stabilizer \(S\) of \(\Sigma\) in \(F\). By the definition, \(S\) is the normalizer in \(F\) of \(\Sigma\) where the latter is considered as a set of central involutions in \(F\). The following result is a consequence of (7.4 (ii)).

**Corollary 7.5** Let \(U\) be the subgroup in \(F\) generated by the involutions from \(\Sigma\). If \(\Sigma \neq \Sigma^\varepsilon\) then \(S = N_F(U)\) and \(U\) is isomorphic to the group \(T\) as in the proof of (7.4). 

In order to identify \(S\) precisely we treat different values of \(t\) separately.

In the case \(t = 1\) let \(p \in \Delta\), \(q \in \Psi_3(p)\) and \(r\) be the image of \(q\) under conjugation by \(p\). Then by the proof of (7.4), up to conjugation in \(F\) we have \(\Sigma = \{p, q, r\} \cup (\Delta(p) \cap \Delta(q))\). This specifies \(\Sigma\) and in view of the list of maximal subgroups in \(\Gamma_4\) gives the following.

**Lemma 7.6** The stabilizer \(\Sigma\) of \(\Sigma\) in \(F \cong \Omega_8^+\) is isomorphic to \(S_3 \setminus S_4\). Furthermore \(\Sigma\) is a subgroup of index \(3\) in a maximal subgroup of \(\Gamma_4\).

Let us turn to the case \(t = 2\). Recall that \(\Gamma_2 = F_4(2)\) contains two conjugacy classes of subgroups isomorphic to \(Sp_8(2)\). The representatives \(P_1\) and \(P_2\) of these classes can be chosen in such way that the centralizer of a central involution in \(P_1\) (isomorphic to \(Sp_6(2)\)) stabilizes a point in \(\mathcal{F}\) while the similar centralizer in \(P_2\) stabilizes a symplecton. Also \(\Gamma_2\) has two irreducible 26-dimensional \(GF(2)\)-modules \(V_1\) and \(V_2\) such that \(P_i\) fixes a vector in \(V_i\) and acts irreducibly on \(V_3 - i\). The classes containing \(P_1\) and \(P_2\) as well as the modules \(V_1\) and \(V_2\) are permuted by the outer (diagram) automorphism of \(\Gamma_2\). Under the choice of the classes, the class \(M\) of 255 central involutions in \(P_1\) (which is a class of 3-transpositions) is a subset of \(\Delta\) and the subgraph induced by \(M\) is clearly the commuting graph. For \(\varepsilon = +\) or \(−\) let \(Q^\varepsilon\) be (the unique up to conjugation) subgroup in \(P_1\) isomorphic to \(\Omega_8^\varepsilon\). Then the involutions from \(M\) contained in \(Q^\varepsilon\) induce a subgraph isomorphic to \(2\Sigma^a\) or \(2\Sigma^b\) if \(\varepsilon = +\) or \(−\), respectively.

**Lemma 7.7** For \(a = a\) or \(b\) the stabilizer \(2\Sigma^a\) of \(\Sigma\) in \(\Gamma_4\) is a conjugate of \(Q^\varepsilon\) for \(\varepsilon = +\) or \(−\), respectively.

**Proof.** By Theorem 4 in \[LS98\] every subgroup in \(F_4(2)\) isomorphic to \(\Omega_8^\varepsilon\) stabilizes a vector either in \(V_1\) or in \(V_2\) and by Table 2 in \[CCS8\] such a subgroup is contained either in a conjugate of \(P_1\) or in a conjugate of \(P_2\). 

Notice that \(N_{\Gamma_4}(O_2(Q^\varepsilon)) \cong \Omega_8^\varepsilon\) and that for \(u \in M\) the subgraph in \(\Delta\) induced by the involutions from \(M\) which do not commute with \(u\) is isomorphic to \(2\Sigma^\varepsilon\).

Finally consider the case \(t = 4\). It has been shown by B. Fischer (cf. \[Coo83\]) that \(E_6(2)\) contains at least three conjugacy classes of subgroups isomorphic to \(Fi_{22}\) and that these three classes are fused in \(\Gamma_4 \cong E_6(2) : S_3\). By \[N92\] the subgroups in \(\Gamma_4\) isomorphic to \(Fi_{22}\) are all conjugate.

**Lemma 7.8** All subgroups in \(\Gamma_4\) isomorphic to \(Fi_{22}\) are conjugate and the stabilizer \(\Gamma_4^\varepsilon\) of \(\Sigma\) is the normalizer of such a subgroup, isomorphic to \(Fi_{22} : 2\).
Lemma 7.9 Let \( S = \Sigma \) in \( F = \Sigma \) in \( F \). Then

(i) \( S \), acting on \( \Delta \) has four orbits: \( \Omega_i = \Sigma, \Omega_2, \Omega_3, \) and \( \Omega_4; \)

(ii) \( |\Omega_i| = 3510, 142155, 694980 \) and \( 3127410, \) for \( i = 1, 2, 3, \) and \( 4 \), respectively;

(iii) if \( w_i \in \Omega_i, \) then \( S(w_i) \) is isomorphic, to \( 2 : U_6(2).2, 2^{10}.M_{22}.2, 2^7 : Sp_6(2) \) and \( 2.2^9 : P\Sigma L_3(4) \) for \( i = 1, 2, 3, \) and \( 4 \), respectively;

(iv) for \( i = 2, 3, \) and \( 4 \) the vertex \( w_i \) is adjacent in \( \Delta \), respectively, to \( 22, 126 \), and \( 22 \) vertices from \( \Sigma; \)

(v) \( S \) acts transitively on the set of maximal cliques (of size \( 22 \)) in \( \Sigma; \);

(vi) \( \Delta(w_2) \cap \Sigma \) and \( \Delta(w_4) \cap \Sigma \) are such cliques and \( S(w_2) \cong 2^{10}.M_{22}.2 \) is the stabilizer of \( \Delta(w_2) \cap \Sigma \) in \( S; \)

Next proposition is an immediate consequence of (7.9), (6.3) and (7.3 (iii)).

Lemma 7.10 In terms of (7.3), the subgroup \( L = \langle \Delta(w_2) \cap \Sigma \rangle = O_2(S(w_2)) \) is contained in exactly two conjugates of \( S \) in \( F \), namely, in \( S \) and in \( w_2S\Sigma w_2 \), moreover, \( S \cap w_2S\Sigma w_2 = S(w_2) \cong 2^{10}.M_{22}.2 \).

Proof. By (7.3 (v)) \( N_F(L) \) acts transitively on the set of conjugates of \( S \) in \( F \), containing \( L \). By (7.3 (vi)) \( w_2 \) is the unique vertex of \( \Delta \) such that \( L \leq O_2(F(w_2)) \) and hence \( N_F(L) \cong F(w_2) \). Since \( O_2(F(w_2)) \) is extraspecial of order \( 2^{21}, N_{O_2(F(w_2))}(L) = O_2(F(w_2)) \). We have

\[ N_5(L)O_2(F(w_2))/O_2(F(w_2)) \cong M_{22}.2 \]

and by the list of maximal subgroups in \( U_6(2) \) \( \text{ATLAS} \), the inclusion \( M_{22}.2 < N \leq U_6(2).S_3 \) implies \( N \cong U_6(2).2 \). Hence \( N_F(L) = N_5(L)(w_2) \) and the result follows.

The next proposition is an immediate consequence of (7.4), (6.3) and (7.3 (iii)).

Proposition 7.11 Suppose that \( t = 4 \) and let \( y \in \Gamma_3^2(x) \). Then

(i) the subgraph \( \Gamma(x, y) \) is connected with 3510 vertices;

(ii) if \( u \in \Gamma(y) \cap \Gamma(x, y) \), then \( u \in \Gamma_2(x) \) and \( u \) is adjacent in \( \Gamma \) to a vertex in \( \Gamma(x, y); \)

(iii) \( \Gamma_3^2(x) \cap \Gamma_3^2(x) \); is \( 0; \)

(iv) \( |\Gamma_3^2(x)| = |\Delta| \cdot |\Psi_3(x)| / 3510. \)

The following lemma provides some further details on (6.10 (ii)).

Lemma 7.12 Let \( e, f \) be distinct elements of type 2 in \( \mathcal{E} \), such that \( \Pi := \Xi \cap \Xi' \neq \emptyset \). Let \( \Pi \) be the image of \( \Pi \) under the natural mapping of \( \Xi \) onto \( \Xi' \). Assume without loss that \( e = \{x, y\}, f = \{x, z\} \). Then the following assertions hold:

(i) if \( i_x(y) \in \Psi_1(i_x(y)) \), then \( |\Pi| = 2 \cdot |\Pi| = 4(t^2 + t + 2) \) and \( z \in \Xi \);
(ii) if $i_x(z) \in \Psi_3(i_x(y))$ then $|\Pi| = |\bar{\Pi}| = 3(2t + 1) + 1$ and $|\Gamma(z) \cap \Xi^e| = 6(2t + 1) + 2$;

(iii) if $i_x(z) \in \Psi_2(i_x(y))$ then $|\Pi| = |\bar{\Pi}| = 2$ and $|\Gamma(z) \cap \Xi^e| = 2(t^2 + t + 1) + 1$;

(iv) if $i_x(z) \in \Psi_3(i_x(y))$ then $|\Pi| = |\bar{\Pi}| = 1$ and $|\Gamma(z) \cap \Xi^e| = 0$; if in addition $t = 4$, then

\[
\Gamma(z) \cap \Xi^e = \{x\}.
\]

In particular if $t = 4$ then $\Xi^e$ is geodetically closed.

**Proof.** By (6.7) and (7.11) if $z$ is adjacent to a vertex from $\Xi^e \setminus \{x\} \cup \Gamma(x)$, then either $z \in \Xi^e$, or \( t \neq 4 \) and $i_x(z) \in \Psi_3(i_x(y))$. Now the result can be easily deduced from Fig. 2, Fig. 3, the fact that

\[
\Gamma(x) \cap \Xi^e = \{y\} \cup \{u \mid i_x(u) \in \Psi_1(i_x(y))\}
\]

and the definition of $\Xi^e$. \hfill \Box

Notice that if $t = 4$, $^4G \cong C_{O_2}$ is the automorphism group of $\Xi^e$, $\bar{\Pi}$ is as in (7.12 (i)) and if $M$ is the stabilizer of $\bar{\Pi}$ in $^4G$, then $M \cong 2^{10}.M_{22}.2$, where $O_2(M)$ is the Golay code module for $M/O_2(M)$.

8 A characterization of the Baby Monster graph

In this section (which consists of a few subsections) we prove Theorems 8.1 and 8.2 by showing that $\Gamma(\mathcal{E}) \cong \Theta$ and that $\mathcal{E} \cong \mathcal{E}(B)$. Throughout the section we assume that $t = 4$.

8.1 x-equivalence on $\Gamma_2(x)$

Let $x \in \Gamma$, $y \in \Gamma_2(x)$ for $\alpha = 3$ or 4, $\Gamma(x, y)$ be the corresponding $\mu$-graph, $\Sigma^y = \{i_x(u) \mid u \in \Gamma(x, y)\}$ and $S^y$ be the stabilizer of $\Sigma^y$ in $F \cong 2E_6(2).S_3$. Then by (6.4), (6.6), (7.8), (7.9), $S^y$ is specified by $\alpha$ up to conjugation in $F$ and $S^y$ determines $\Sigma^y$ uniquely. We summarize this in the following proposition.

**Proposition 8.1** The following assertions hold:

(i) if $y \in \Gamma_3(x)$, then $S^y \cong 2^{1+20} : U_4(3).2^2$ is contained in $F(p) \cong 2^{1+20} : U_6(2).S_3$ for a unique point $p$ of $\mathcal{F}$ and $\Sigma^y$ is the unique orbit of length 648 of $S^y$ on $\Delta$ (contained in $\Psi_1(p)$);

(ii) if $y \in \Gamma_2(x)$, then $S^y \cong F_{i22} : 2$ and $\Sigma^y$ is the unique orbit of length 3510 of $S^y$ on $\Delta$;

(iii) in either case if $H^y = S^yF^\infty$, then $H^y \cong 2E_6(2) : 2$ is a subgroup of index 3 in $F$. \hfill \Box

In the next subsection we show that $H^y$ is independent on the choice of $y \in \Gamma_2(x)$. We need another preliminary result.

**Lemma 8.2** Let $F(p) \cong 2^{1+20} : U_6(2).S_3$ be the stabilizer in $F$ of a point $p$ of $\mathcal{F}$, $Z = \langle p \rangle$, $Q = O_2(F(p))$, $I = F(p)^\infty$. Then $I := I/Z \cong 2^{20} : U_6(2)$ has four classes $\mathcal{U}_i$, $0 \leq i \leq 4$, of complements to $Q/Z$, such that $F(p)/I \cong S_3$ normalizes $U_6$ and permutes transitively the remaining classes; the preimage in $I$ of a subgroup from $\mathcal{U}_i$ splits over $Z$ if and only if $i = 0$. \hfill \Box

Let $p \in F$, $M_1$, $M_2$, $M_3$ be representatives of the conjugacy classes of $\mathcal{F}_{i22}$-subgroups in $F^\infty \cong 2E_6(2)$ (compare the paragraph before (7.8)) such that $p \in M_i$ for $1 \leq i \leq 3$ and $q \in \Psi_3(p)$. Then (within a suitable ordering) we have $(F(p) \cap F(q))^\infty \in \mathcal{U}_0$, $M_i(p)/\langle p \rangle \in \mathcal{U}_i$ for $1 \leq i \leq 3$. 22
Lemma 8.3 For every $y \in \Gamma_2(x)$ there is exactly one vertex $y' \in \Gamma_2(x) \setminus \{y\}$ such that $\Sigma y = \Sigma y'$. Furthermore, $i_z(y') = i_z(x)i_z(y)i_z(x)$ for every $z \in \Gamma(x,y)$.

Proof. Suppose first that $y, y' \in \Gamma_2(x)$, $y \neq y'$, $\Sigma y = \Sigma y'$ and $z \in \Gamma(x,y)$. Put $p = i_z(x)$, $q = i_z(y)$, $q' = i_z(y')$. We claim that $q' = pqp$. The subgroup $U := O^\infty(F(p) \cap F(q)) \cong U_0(2)$ is generated by the involutions in $\Delta(p) \cap \Delta(q) = \Delta(p) \cap \Delta(q')$. Hence $U$ commutes with the unique element $r \in O_2(F(p))$ which maps $q$ onto $q'$. Since $U$ acts irreducibly on $O_2(F(p))/\langle q \rangle$ and $q \neq q'$, we have $r = p$, hence the claim follows and proves the uniqueness statement in the lemma. Now let $y$ and $z$ be as above, and let $y'$ be such that $i_z(y') = i_z(x)i_z(y)i_z(x)$. We claim that $\Sigma y = \Sigma y'$ (then (8.4) (ii)) will imply the equality $\Sigma y = \Sigma y'$). The subgroup $K$ of $F$ generated by

$$\{i_x(u) \mid u \in \{z\} \cup (\Gamma(x) \cap \Gamma(y) \cap \Gamma(z))\}$$

is isomorphic to $2 \cdot U_0(2)$ (a non-split extension) and $K = O^2(\Sigma y(r)) = O^2(\Sigma y'(r))$, where $r = i_z(z)$. By (7.8) there is $f \in F$ which conjugates $\Sigma y$ onto $\Sigma y'$. Since $\Sigma y$ acts transitively on the set of vertices of $\Sigma y'$ and $r$ is one of these vertices, we can choose $f$ to normalize $K$. By (7.9) $r$ is the unique point in $\mathcal{F}$ stabilized by $K$. This implies that $N_F(K) \leq F(r)$. Since $K$ does not split over its centre, by (8.2) $N_F(\zeta(K)) \cong 2 \cdot U_0(2).2 \cong NSy(K)$. Hence $\Sigma y = \Sigma y'$.\qed

As a consequence of the proof of (8.3) and in view of the fact that different pairs of vertices at distance 2 in $\Xi$ have different $\mu$-graphs, we obtain the following.

Lemma 8.4 Suppose that $y_1, y_2 \in \Gamma_2(x)$ and for some $z \in (\Gamma(x,y_1) \cap \Gamma(y_1,y_2) \cap \Gamma(z)) = \Gamma(x) \cap \Gamma(y_2) \cap \Gamma(z)$. Then $\Gamma(x,y_1) = \Gamma(x,y_2)$.\qed

As a direct consequence of (8.4) we obtain the following result (which can also be checked in $\Delta$ directly).

Lemma 8.5 Let $p \in \Delta$, $q, q' \in \Delta_2(p)$ and suppose that $\Delta(p) \cap \Delta(q) = \Delta(p) \cap \Delta(q')$. Then either $q' = q$ or $q' = pqp$.\qed

To the end of this subsection we introduce some important notions.

Definition 8.6 Let $x \in \Gamma$. Two vertices $y, y' \in \Gamma_2(x)$ are said to be $x$-equivalent if $\Gamma(x,y) = \Gamma(x,y')$. By $\pi_x$ we denote the permutation of the vertex-set of $\Gamma$ which fixes every vertex in $\Gamma \setminus \Gamma_2(x)$ and swaps the pairs of $x$-equivalent vertices in $\Gamma_2(x)$.

By (8.3) each class of $x$-equivalent vertices is of size 2, so that $\pi_x$ is well defined. In the Baby Monster graph $\Theta$ two vertices $b, b' \in \Theta_2(a)$ are $a$-equivalent if $\langle a, b \rangle = \langle a, b' \rangle$ (which happens exactly when $b' = aba$). Furthermore, $a, b \in \Theta$ do not commute if and only if $b \in \Theta_2(a)$, so that $\pi_a$ is nothing, but the conjugation by $a$, and clearly it is an automorphism of $\Theta$.

Our ultimate goal is to prove the following.

Proposition 8.7 For every $x \in \Gamma$ the permutation $\pi_x$ is an automorphism of $\Gamma$.\qed
The proof of (8.7) will be achieved by showing that restrictions of \( \pi_x \) to various subsets of vertices of \( \Gamma \) are automorphisms of the subgraphs induced by that subsets. Our first result of this type follows directly from (8.3).

**Lemma 8.8** If \( z \in \Gamma(x) \) and \( y \in \Gamma(z) \), then \( i_z(y^{\pi_x}) = i_z(x)i_z(y)i_z(x) \), in particular the restriction \( \pi_x \) of \( \pi_x \) to \( \Gamma(z) \) is an automorphism of the subgraph in \( \Gamma \) induced by \( \Gamma(z) \).

The permutation \( \pi_x \) fixes every vertex in \( \{x\} \cup \Gamma(x) \) and permutes the pairs of \( x \)-equivalent vertices in \( \Gamma_2(x) \). Hence by (8.8) we also have the following.

**Lemma 8.9** For every \( z \in \Gamma(x) \) the restriction of \( \pi_x \) to the set \( \Gamma(x) \cup \Gamma(z) \) is an automorphism of the subgraph induced by this set. \( \square \)

### 8.2 The class of \( \mu \)-graphs

In this subsection we specify the family
\[
\mathcal{S} = \{ \Sigma^u \mid u \in \Gamma_2(x) \}
\]
of subgraphs in \( \Delta \) up to simultaneous conjugation by automorphisms of \( \Delta \).

For \( y \in \Gamma_2^2(x) \) let us adopt the notation introduced in (8.1) and in the paragraph before that lemma. By (8.3) there are exactly \( |\Gamma_2^2(x)|/2 \) distinct \( \mu \)-graphs \( \Gamma(x,u) \) for \( u \in \Gamma_2^2(x) \) and by (6.10), (7.11), (8.1) and (8.3) we have
\[
|\Gamma_2^2(x)|/2 = [F : S^y]/3 = [H^y : S^y].
\]

Thus exactly one third of the images of \( \Sigma^y \) under the elements \( f \in F \) is contained in \( \mathcal{S} \) and we are going to show that such an image is contained in \( \mathcal{S} \) exactly when \( f \in H^y \). First of all by the above equality we obtain the following.

**Lemma 8.10** For \( \alpha = 3 \) or \( 4 \) let \( y \in \Gamma_2^2(x) \). Then the following two statements are equivalent
\begin{enumerate}
  \item \( H^u = H^y \) for every \( u \in \Gamma_2^2(x) \);
  \item for every \( h \in H^y \) the image of \( \Sigma^y \) under \( h \) coincides with \( \Sigma^u \) for some \( u \in \Gamma_2^2(x) \).
\end{enumerate}

**Lemma 8.11** The equivalent statements (i) and (ii) in (8.10) hold.

**Proof.** Suppose first that \( \alpha = 3 \). For a vertex \( z \in \Gamma(x,y) \) let \( p = i_z(x), q = i_z(y) \) (so that \( q \in \Psi_3(p) \)), \( r \in \Psi_3(p) \cap \Psi_1(q) \) and \( u = i_z^{-1}(r) \). Then \( u \in \Gamma_3^3(x) \cap \Gamma(y) \). By (2.1) \( F(p) \cap F(q) \cap F(r) \cong 2^9.L_3(4).S_3 \) is a maximal parabolic in \( F(p) \cap F(q) \cong U_6(2).S_3 \). Hence \( \Delta(p) \cap \Delta(q) \cap \Delta(r) \) consists of 21 3-transpositions from the set \( \Delta(p) \cap \Delta(q) \) of 693 3-transpositions of \( (F(p) \cap F(q))^{\infty} \cong U_6(2) \). This means that \( \Gamma(x) \cap \Gamma(y) \cap \Gamma(u) \) contains a set of 22 pairwise adjacent vertices, namely
\[
\{ z \} \cup i_z^{-1}(\Delta(p) \cap \Delta(q) \cap \Delta(r)).
\]

On the one hand by (8.8) \( \Gamma(x,y) \neq \Gamma(x,u) \) (equivalently \( S^y \neq S^u \)) and on the other hand by (7.3) (v) and (7.10) \( S^y \cap S^u \cong 2^{10}.M_{22}.2 \) and \( S^y \) is conjugate to \( S^u \) in \( F^{\infty} \), which implies \( H^u = H^y \), since \( H^y = F^{\infty}S^y \). By the construction for every 22-vertex complete subgraph \( \Upsilon \) in \( \Gamma(x,y) \) there is (a unique) \( u \in \Gamma_2^2(x) \cap \Gamma(y) \) such that \( \Gamma(x,y) \cap \Gamma(u) = \Upsilon \). Since a \( Fi_{22} \)-subgroup is maximal in
Now let \( y \in \Gamma_4^2(x) \), so that \( y \in \Xi \) for some \( e = \{x, z\} \) and \( S^y \leq F(p) \) where \( p = i_x(z) \). The stabilizer of \( e \) in the automorphism group \( Co_2 \) of \( \Xi \) is isomorphic to \( U_6(2) : 2 \) (cf. (5.2 (iii))) and hence there is a unique subgroup \( D^e \) of index 3 in \( F \) such that for every \( d \in D^e(p) \) the action of \( d \) on \( \Psi_1(p) \) can be extended to an automorphism of \( \Xi \). This shows that \( H^y = D^e \) for every \( y \in \Xi \cap \Gamma_4^2(x) \) and it remains to show that \( D^e \) is independent on \( x \). Let \( y \in \Gamma(x) \) be such that \( i_{y}(v) \) is contained in \( \Psi_1(p) \) and put \( f = \{x, v\} \). Then by (7.12 (i)) \( \Pi := \Xi \cap \Xi^f \) is a complete graph on 88 vertices. If \( R \) is the stabilizer of \( i_{x}(\Pi \setminus \{x\}) \) in \( F \), then \( R/O_2(R) \cong L_3(4).S_3 \). In view of the remark after (7.12) for an element \( r \in R \) its action on \( \Psi_1(p) \cap \Psi_1(q) \) can be extended to an automorphism of \( \Xi \) if and only if it can be extended to an automorphism of \( \Xi^f \). Hence \( D^e = D^v \) and the result follows by the connectivity of \( \Psi \). □

**Lemma 8.12** In terms of (7.9) let \( u_i \in \Gamma(x) \) be such that \( i_{x}(u_i) = w_i \) for \( i = 2, 3 \) and 4. Then \( u_2, u_3 \in \Gamma_3^2(y) \) and \( u_4 \in \Gamma_4^2(y) \).

**Proof.** For \( i = 2 \) and 4, in view of (7.9) and (7.10), the claim comes as a bi-product of the proof of (8.11) for the case \( \alpha = 3 \). By (7.9 (iv)) the subgraph induced by \( \Gamma(x) \cap \Gamma(y) \cap \Gamma(u_3) \) has 126 vertices and it is not a complete graph by (7.9 (v)). By (7.12 (v)) this can not happen when \( u_3 \in \Gamma_4^2(y) \). □

We need the following result.

**Lemma 8.13** Let \( \tilde{\Xi} \) be the graph on 2300 vertices for \( ^4G \cong Co_2 \) as in (5.2), \( f \) be at distance 2 from \( e \), \( \Phi \) be the subgraph on 324 vertices induced by the common neighbours of \( e \) and \( f \), \( \Pi \) be a complete 22-vertex subgraph in \( \Phi \) and \( D = ^4G(e) \cap ^4G(f) \cong U_4(3).2^2 \). Then the setwise stabilizer of \( \Pi \) in \( D \) induces on \( \Pi \) an action of \( P\Sigma L_3(4) \).

**Proof.** Notice that if \( \Lambda \) is as in (5.1), then \( \Phi \) is the subgraph in the distance 1-or-2 graph of \( \Lambda \) induced by the orbit \( X \) of length 324 of \( D \). It is easy to see, using for instance (5.4) that the subgraph in \( \Lambda \) itself induced by \( X \) has the following intersection diagram:

```
1 21 16 1
21 20 16
4 105 16 14
5 21 16
6 15 7
56 15
120 7
```

By (6.4 (i)), or otherwise we know that the valency of \( \Phi \) is 147 and 147 = 105 + 21 + 21 is the only way to present the valency as a sum of suborbits. Now it is a standard fact about dual polar spaces that if \( Y \) is a maximal clique in the distance 1-or-2 graph of \( \Lambda \), then either \( |Y| = 27 \) and \( Y \) is a quad, or \( |Y| = 43 \) and \( Y \) is a vertex together with its neighbours in \( \Lambda \). By (6.5 (iii)) a quad
intersects \( X \) in at most 12 vertices, which implies that the clique \( \Pi \) is a vertex together with its neighbours in \( \Lambda \) which are contained in \( \Phi \). Hence the stabilizer of \( \Pi \) coincides with the stabilizer in \( D \) of a vertex of \( \Phi \) and the result follows. \( \square \)

We have proved in \((8.11)\) that for \( \alpha \in \{3, 4\} \) \( H^y \) is independent on the choice of \( y \in \Gamma_2^y(x) \) and it remains to establish the following.

**Lemma 8.14** There are \( y \in \Gamma_2^y(x) \) and \( u \in \Gamma_2^y(x) \) such that \( H^y = H^u \).

**Proof.** By \((8.12)\) we can choose \( y \in \Gamma_2^y(x) \) and \( u \in \Gamma_2^y(x) \) such that \( \Pi := \Sigma^y \cap \Sigma^u \) contains a complete subgraph on 22 vertices. It is sufficient to show that \( S^u(\Pi) \cap S^y(\Pi) \) is not contained in \( F^{\infty} \). By the proof of \((8.10)\) we have \( F[\Pi] \cong 2 \times 2^{10}.M_{22},2, S^u[\Pi] \cong 2^{10}.M_{22}.2 \) and \( F^{\infty}[\Pi] \cong 2 \times 2^{10}.M_{22} \). So it is sufficient to show that \( S^u[\Pi] \) induces in \( \Pi \) the natural action of \( P\Sigma L_3(4) \). Since \( \Pi \subset \Sigma^y \), for any two distinct vertices \( p, q \) in \( \Pi \) we have \( q \in \Psi_2^y(p) \) and hence \( \Pi \) maps bijectively onto its image in \( \Xi^e \), where \( e = \{x, z\} \) is the edge of \( \Gamma \) such that \( u \in \Xi^e \). Hence the claim follows directly from \((8.13)\). \( \square \)

### 8.3 The second neighbourhood of a vertex

In this subsection for a given \( x \in \Gamma \) we analyze the adjacencies between the vertices in \( \Gamma_2(x) \). In view of \((2.4)\) the results established in the previous subsection can be summarized as follows.

**Proposition 8.15** Let \( x \) be a vertex of \( \Gamma \), \( F_x \cong 2^2E_6(2).S_3 \) be the automorphism group of the subgraph in \( \Gamma \) induced by \( \Gamma(x) \). Then there is a unique subgroup \( H_x \cong 2^2E_6(2) : 2 \) of index 3 in \( F_x \) such that for \( \alpha = 3 \) and \( 4 \) \( H_x \) acts transitively on the set

\[
M^\alpha = \{ \Gamma(x, y) \mid y \in \Gamma_2^y(x) \}
\]

of \( \mu \)-graphs as on the cosets of subgroups \( F_{22,2} \) and \( 2^{1+20}.U_4(3).2^2 \), respectively. If \( X \in M^\alpha \) and \( z \in \Gamma(x) \), then \( z \in \Gamma \) if and only if

\[
H_x(z) \cap H_x[X] \cong 2 \cdot U_6(2) : 2 \quad \text{and} \quad H_x(z) \cap H_x[X] \cong [2^{20}].P\Sigma L_3(4),
\]

for \( \alpha = 3 \) and \( 4 \), respectively. \( \square \)

Notice that for \( z \in \Gamma(x) \) the action \( \pi^z_x \) of \( \pi_x \) on \( \Gamma(x) \) is an automorphism of the subgraph in \( \Gamma \) induced by \( \Gamma(x) \). Furthermore, \( \pi^z_x \in H_x \), particularly \( \pi^z_x \in H_x \).

For \( \alpha = 3 \) or \( 4 \) and \( X \in M^\alpha \) the subgroup \( H_x(z) \cap H_x[X] \) in \((8.15)\) is determined uniquely up to conjugation in \( H_x(z) \cong 2^{1+20}.U_6(2).2 \). For \( X \in M^3 \) the subgroup \( (H_x(z) \cap H_x[X])/Z(H_x(z)) \cong U_6(2) \) is a complement to \( O_2(H_x(z))/Z(H_x(z)) \) in \( H_x(z)/Z(H_x(z)) \) and \( H_x(z) \) can be characterized as the stabilizer in \( F_x(z) \) of the conjugacy class of \( (H_x(z) \cap H_x[X])/Z(H_x(z)) \) in \( F_x(z)^\infty \) (compare \((8.12)\)). If \( e = \{x, z\} \) and \( \Xi^e \) is the 4600-vertex subgraph as in Section 8.2 then \( H_x(z) \) (resp. \( H_x(X) \)) consists of those elements of \( F_x(z) \) (resp. of \( F_x(X) \)) whose actions on \( \Xi^e \cap \Gamma(x) \) (resp. on \( \Xi^e \cap \Gamma(z) \)) can be extended to automorphisms of \( \Xi^e \). Furthermore, \( H_x = H_x(z)F_x^\infty \). This gives an alternative way to define the subgroups \( H_u \) consistently for all \( u \in \Gamma \).

We need another preliminary result.

**Lemma 8.16** Let \( p \) be a vertex of \( \Psi \), \( \Pi \) be the subgraph in \( \Psi \) induced by \( \Psi_1(p) \cup \Psi_2(p) \) and \( J \) be the automorphism group of \( \Pi \). Then

\[
2^{20} : U_6(2) \cong \bar{I} < J \leq \text{Aut} \, \bar{I} \cong 2^{20+2} : U_6(2) : S_3
\]

where \( \bar{I} \) is as in \((8.2)\), so that \( \text{Out} \, \bar{I} \cong S_4 \) acts naturally on the classes of complements in \( \bar{I} \) to \( O_2(\bar{I}) \).
Proof. By (2.4) the stabilizer of \( p \) in \( F \) induces on \( \Pi \) an action of \( 2^{20} : U_6(2).S_3 \), so to prove the lemma it is sufficient to show that \( J \) is contained in \( \text{Aut} \tilde{I} \). Let \( T \) be a symplecton containing \( p \) and \( A \) be the action induced by \( J[T] \) on \( T \). Then \( A \) is contained in the stabilizer of \( p \) in \( \text{Aut} \tilde{T} \cong \Omega_8^{−}(2).2 \) and \( J \) is contained in the action induced on \( T \) by its stabilizer in \( F(p) \). Hence \( A \cong 2^6Ω_6^{−}(2).2 \) and \( J \) acts faithfully on \( T \) containing \( \Psi_1(p) \). By (2.4) \( T \) is uniquely determined by its intersection with \( \Psi_1(p) \) and hence \( J \) acts faithfully on \( \Psi_1(p) \). Let \( V \) be the kernel of the action of \( J \) on the set \( L \) of 891 lines containing \( p \) and \( \tilde{J} = J/V \). In every symplecton containing \( p \) there are exactly 27 lines containing \( p \). The collection of 693 such 27-element subsets of \( L \) defines on \( L \) the structure \( D \) of dual polar space of \( U_6(2) \). This structure is preserved by \( J \) and hence \( \tilde{J} \leq U_6(2).S_3 \). Since every line in \( L \) contains \( p \) exactly two points, the kernel \( V \) is an elementary abelian 2-group. On every line from \( L \) the group \( V \) induces an action of order at most 2 and on the 27 lines in a symplecton \( T \) it induces an action of order at most \( 2^6 = |O_2(A)| \). Hence on a plane containing \( p \) the subgroup \( V \) induces an action of order at most \( 2^2 \). This shows that the dual of \( V \) is a representation module of \( D \) and the latter was proved in [Y94] to be of order \( 2^2 \). Hence \( J^\infty \cong \tilde{I} \) and the result follows. \( \square \)

For \( z \in \Gamma(x) \) put \( \Omega = (\Gamma(x) \cap \Gamma(z)) \setminus \{x, z\} \). The mappings \( i_x \) and \( i_z \) restricted to \( \Omega \) are bijections onto the vertex set of the graph \( \Pi \) as in (8.10). If \( u \in \Omega \), then by (3.2) the type of \( \{x, z, u\} \) is well defined and we can put

\[
\begin{align*}
\Omega_1 &= \{u \in \Omega \mid \{x, z, u\} \text{ is short}\} \\
\Omega_2 &= \{u \in \Omega \mid \{x, z, u\} \text{ is long}\}.
\end{align*}
\]

If \( e = \{u, v\} \) is an edge of \( \Gamma \) contained in \( \Omega \), then the triangles \( \{x, u, v\} \) and \( \{z, u, v\} \) might or might not be of the same type. Define \( (\tau_x, \tau_z) \) to be the type of \( e \), where

\[
\tau_x = s \text{ or } l \text{ if } \{x, u, v\} \text{ is short or long, respectively}
\]

and \( \tau_z \) is defined similarly. By (3.4) we have the following.

Lemma 8.17 Let \( e = \{u, v\} \) be an edge of \( \Gamma \) contained in \( \Omega \) and \( (\tau_x, \tau_z) \) be the type of \( e \). Then

(i) if \( e \subset \Omega_1 \) or \( e \subset \Omega_2 \), then \( \tau_x = \tau_z \);

(ii) if \( u \in \Omega_1 \) and \( v \in \Omega_2 \), then \( \tau_x \neq \tau_z \). \( \square \)

Lemma 8.18 Let \( K \) be the group of automorphisms of the subgraph in \( \Gamma \) induced by \( \Omega \), preserving the above defined types \( (\tau_x, \tau_z) \) of edges. Then \( K \cong J = \text{Aut} \Pi \), where \( \Pi \) is as in (8.10).

Proof. For \( \gamma = x \) or \( z \) let \( \Omega^\gamma \) be the graph on \( \Omega \), whose edges are the edges of \( \Gamma \) contained in \( \Omega \) for which \( \tau_y = s \). Then \( i_x \) induces an isomorphism of \( \Omega^x \) onto the graph \( \Pi \). In view of (8.17) every automorphism of \( \Pi \) can be realized as a type-preserving automorphism of the subgraph induced by \( \Omega \). Hence the result. \( \square \)

Let \( z \) be the set of \( x \)-equivalence classes of vertices in \( \Gamma_2(x) \). By (8.15) we can define the action of \( H_x \) on \( z \) by the following rule: if \( \{y, y\}' \in z \), with \( \Gamma(x, y) = \Gamma(x, y)' = X \), then \( \{y_1, y_1\}' = \{y, y\}' \). This action is well defined.

For \( z \in \Gamma(x) \) let \( H_x(z) \) be the stabilizer of \( z \) in \( H_x \) and \( z \) be the set of classes from \( z \) contained in \( \Gamma(z) \). Since \( \{y, y\}' \in z \) whenever \( z \in \Gamma(x, y) \), \( H_x(z) \) stabilizes \( z \) as a whole and we can consider the action of \( H_x(z) \) on \( z \).

Lemma 8.19 The above defined action of \( H_x(z) \) on \( z \) coincides with the natural action of \( H_x(x) \) on \( z \).
Proof. With $\Omega = (\Gamma(x) \cap \Gamma(z)) \setminus \{x, z\}$ as above, let $\{y, y'\} \in^x \mathcal{Y}_z$ and $X = \Gamma(x, y)$. By (8.4) and (8.5) the stabilizer of $\{y, y'\}$ in $H_z(x)$ coincides with the setwise stabilize of $\Gamma(y) \cap \Omega$ and the stabilizer of $X$ in $H_z(\Omega)$ coincides with the stabilizer of $X \cap \Omega$. Furthermore $\Gamma(y) \cap \Omega = X \cap \Omega$, which implies that the actions of $H_z(x)$ and $H_z(\Omega)$ on $^x \mathcal{Y}_z$ are determined by their natural actions on $\Omega$. Thus all we have to show is that the action $A_z$ of $H_z(x)$ on $\Omega$ coincides with the action $A_x$ of $H_z(x)$ on $\Omega$ this set. Notice that $A_x \cong A_z \cong 2^{20}: U_0(2).2$ and both $A_x$ and $A_z$ are subgroups in the group $K$ of type-preserving automorphisms of the subgraph in $\Gamma$ induced by $\Omega$ (compare (8.18)).

Let $t_0 \in \Gamma(x)$ with $i_3(t_0) \in \Psi_3(i_3(z))$, $t_1 \in \Gamma(z)$ with $i_3(t_1) \in \Psi_3(i_3(x))$. For $i = 0$ and $1$ let $U_i$ be the stabilizer of $\Gamma(t_i) \cap \Omega$ in $A_x$. Then $U_0 \cong U_1 \cong U_0(2).2$ and $U_0$ and $U_1$ belong to different classes of complements to $O_2(A_x)$. By the paragraph after (8.19) $A_x$ is characterized as the stabilizer in $J \cong 2^{20} : U_0(2).S_4$ of the classes in $J^\infty$ containing $U_0'$ and $U_1'$. Since $A_z$ is isomorphic to $A_x$ and possesses the same properties, the result follows.

Now we can establish some further properties of the permutation $\pi_x$.

Lemma 8.20 If $s, w \in \Gamma(x)$ and $s \in \Gamma(w)$ then the restriction of $\pi_x$ to the set $\Gamma(x) \cup \Gamma(s) \cup \Gamma(w)$ is an automorphism of the subgraph in $\Gamma$ induced by this set.

Proof. In terms of (8.19) let $d \in \Gamma(x) \cap \Gamma(z)$. Then the action of $\pi_d$ on $\Gamma(x)$ is an element in $H_z(x)$ while its action on $\Gamma(z)$ is an element of $H_z(\Omega)$. Hence the result follows from (8.19).

By now with every vertex $z \in \Gamma$ we have associated various actions on $\Gamma$. On the one hand in (8.4) we have defined the permutation $\pi_z$ of the vertex-set of $\Gamma$. On the other hand, if $x \in \Gamma(z)$, then the action $\pi_x^z$ of $\pi_z$ on the set $\Gamma(x)$ is an element of $H_z$ and an action of $\pi_x^z$ on the set $^x \mathcal{Y}_z$ of pairs of $x$-equivalent vertices in $\Gamma_z$ is defined in the paragraph before (8.19). The natural question is whether these actions are consistent. A partial affirmative answer is given in the following lemma.

Lemma 8.21 Let $x \in \Gamma$, $z \in \Gamma(x)$, $y \in \Gamma_2(x)$ and suppose that $z$ is adjacent in $\Gamma$ to a vertex $u \in \Gamma(x, y)$. Then $\Gamma(y^{\pi_z}, x)$ is the image of $\Gamma(x, y)$ under $\pi_x^z$, i.e. $\Gamma(x, y)^{\pi_x^z} = \Gamma(y^{\pi_z}, x)$.

Proof. Under the hypothesis $x, y, z \in \Gamma(u)$. Hence $y^{\pi_z}$ coincides with the image of $y$ under $\pi_x^z$ and $\Gamma(y^{\pi_z}, x) \cap \Gamma(u)$ is the image of $\Gamma(y, x) \cap \Gamma(u)$ under $\pi_x^z$ (equivalently under $\pi_y^z$). By (8.4) $\Gamma(y^{\pi_z}, x)$ is uniquely determined by $\Gamma(y^{\pi_z}, x) \cap \Gamma(u)$ and hence the result.

8.4 $\pi_x$ is an automorphism of $\Gamma$

In this subsection we prove Proposition 8.7. We need to show that whenever $\{y, u\}$ is an edge of $\Gamma$, its image under $\pi_x$ is again an edge of $\Gamma$. Since $\pi_x$ fixes every vertex in $\Gamma \setminus \Gamma_2(x)$, we can assume that at least one of the vertices on the edge, say $y$, is in $\Gamma_2(x)$. Then $y^{\pi_x} = y'$ is $x$-equivalent to $y$ in $\Gamma_2(x)$. If there is $z \in \Gamma(x)$ such that $\{y, u\} \subset \{z\} \cup \Gamma(z)$, then $\{y^{\pi_x}, u^{\pi_x}\}$ is an edge by (8.8). Hence we assume that $\Gamma(x) \cap \Gamma(y) \cap \Gamma(u) = \emptyset$.

By (7.11) (ii)) this implies that $y \notin \Gamma_2^\alpha(x)$ and so $y \in \Gamma_4^\alpha(x)$. Furthermore by the proof of (8.3) for $\alpha = 4$ we conclude that $x \in \Xi^e$ for the edge $e = \{y, y'\}$.

From now on we follow notation introduced in the paragraph after the proof of (6.3) (with the roles of $x$ and $y$ interchanged). As usual $i_y$ is a fixed isomorphism of the subgraph in $\Gamma$ induced by $\Gamma(y)$ onto the graph $\Delta$. Let

$$\Omega = \{i_y(v) \mid v \in \Gamma(x, y)\},$$

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By Proposition 8.7 for every the proof of (8.7).

Now we turn to the final case, when \( w \in \Gamma(x) \). We follow notation from the proof of (8.23), so that \( w \in \Gamma(x) \), \( u \in \Gamma_3(w) \), \( z \in \Gamma(x) \cap \Gamma(w, u) \). Then \( x, y, z \in \Gamma(w) \) and \( u \in \Gamma_2(x) \). Hence \( \pi^x_w \) is an element of \( H_w, y \in \Gamma(u, w) \) and \( x \) is adjacent to \( z \in \Gamma(u, w) \). Then by (8.24) \( \Gamma(u, w) \) is the image of \( \Gamma(u, w) \) under \( \pi^w_z \), in particular it contains \( y^{\pi^x_w} \). Hence \( \{ y^{\pi^x_w}, u^{\pi^x_w} \} \) is an edge of \( \Gamma \). This completes the proof of (8.7).

8.5 Proof of Theorems 1 and 2

By Proposition 8.3 for every \( x \in \Gamma \) the permutation \( \pi_x \) is an automorphism of \( \Gamma \). The actions \( \pi^x_z \) of the automorphisms \( \pi_x \) on \( \Gamma(x) \) taken for all \( z \in \Gamma(x) \) generate \( F_\infty^x \cong E_6(2) \), hence Theorem 1 follows from Proposition 1.1. Now Theorem 2 follows from (3.1) and Theorem 3.
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