LAPLACE’S FORMULA : AN APPROACH BY NONSTANDARD ANALYSIS

A PREPRINT

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January 28, 2020

ABSTRACT

Using nonstandard analysis (NSA), the proof of the Laplace’s formula is given. The usage of NSA reduces the intricacy of taking limit, and the crude line of the proof would be clearly seen, compared to the done with the rigorous classical calculus. We use very elementary tools of NSA.

Keywords Laplace’s formula · Laplace’s method · maximum term method · nonstandard analysis

1 Introduction

Laplace’s formula concerns the asymptotic behaviour of integral
\[ \int_a^b \varphi(x) \exp(n h(x)) \, dx \]
when \( n \) goes to infinity, where \( h \) takes maximum at only one point on the integral interval. Various application of the formula is well known. For example, if we take \( a = 0, b = \infty, \varphi(x) \equiv 1, \) and \( h(x) = \log(x) - x, \) asymptotic behaviour of \( \Gamma(n + 1)/n^{n+1} \) as \( n \to \infty, \) hence Stirling’s formula is obtained.

Before showing things correctly, let us make a rough observation. For the exact formulation and proof, see[2] Suppose \( h(x) \) takes its only maximum at \( \xi_0 \in (a, b) \). And now consider the integral
\[ \exp(-n h(\xi_0)) \int_a^b \varphi(x) \exp(n h(x)) \, dx = \int_a^b \varphi(x) \exp [n(h(x) - h(\xi_0))] \, dx. \] (1)

For some \( \varepsilon, \alpha > 0 \) we have \( |x - \xi_0| > \varepsilon \implies h(x) - h(\xi_0) < -\alpha, \) hence the term \( \exp [n(h(x) - h(\xi_0))] \) decays as \( n \to \infty. \) So the major portion of the integral is
\[ \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \varphi(x) \exp[n(h(x) - h(\xi_0))] \, dx. \]

For small \( \varepsilon, \varphi(x) \) can be approximated by \( \varphi(\xi_0) \) if \( \varphi \) is continuous. Hence we obtain
\[ \varphi(\xi_0) \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \exp[n(h(x) - h(\xi_0))] \, dx. \]

Now suppose \( h''(\xi_0) < 0. \) Then \( h(x) - h(\xi_0) \) is well approximated by \( \frac{h''(\xi_0)}{2}(x - \xi_0)^2. \) Thus, by change of variable, we obtain
\[ \varphi(\xi_0) \sqrt{\frac{2}{-n h''(\xi_0)}} \int_{|z| \leq R_{n,\varepsilon}} e^{-\varepsilon^2} \, dz, \]
where
\[ R_{n,\varepsilon} = \varepsilon \sqrt{\frac{-n h''(\xi_0)}{2}}. \]
Taking $n$ so large to cancell the smallness of $\varepsilon$, $R_{n,\varepsilon}$ is effectively approximated by $\infty$. Hence we obtain

$$\varphi(\xi_0) \sqrt{\frac{2}{-n h''(\xi_0)}} \int_{\mathbb{R}} e^{-z^2} \, dz \approx \varphi(\xi_0) \sqrt{\frac{2\pi}{-n h''(\xi_0)}}.$$ 

Finally we obtain:

$$\int_a^b \varphi(x) \exp(n h(x)) \, dx \sim \varphi(\xi_0) \exp(n h(\xi_0)) \sqrt{\frac{2\pi}{-n h''(\xi_0)}} \quad (n \to \infty).$$

Of course, above explanation cannot be accepted as rigorous proof, though it captures the essence of the phenomenon. The most significant defects in the above explanation is (i) the arbitrariness of $n$ between infinite and some big natural and (ii) vagueness of the size of $\varepsilon$.

To forge the above explanation into rigorous proof in the classical analysis, one must pay the cost of taking intricate limit as done in [3], which is somewhat difficult to follow. However, nonstandard analysis (NSA) can afford the rigorous proof of Laplace’s formula along the line of the above explanation, without any complication. For in NSA, we can treat “infinitely large natural number” and “infinitely small real number”, though it may feel contradictory at first.

In this note, we prove the Laplace’s formula using NSA, along the line of the above explanation. We need only elementary tools of NSA and basic calculus; we do not need neither advanced understanding of NSA, intricate limit, nor Big O notation. In sec 2 we show the exact statement of the problem and the proof. In sec 3 we state a generalizaiton of the formula and show the proof of them.

In what follows, basic familiarity with Nelson’s IST (Internal Set Theory [1]) is supposed. However, the reader with some knowledge on (ordinary) nonstandard analysis would follow the argument with little effort.

### 2 Exact Statement and the proof

**Theorem 1** (Laplace’s theorem). Let $[a, b] \subseteq \mathbb{R}$ be finite or infinite interval, $\varphi(x)$ and $h(x)$ be functions defined on the interval, and $\xi_0 \in (a, b)$, satisfying following conditions:

(C1) $\varphi(x) \exp(n h(x))$ is absolutely integrable over $[a, b]$, for $n = 0, 1, 2, \ldots$.

(C2) There is a neighbourhood of $\xi_0$ where $h''(x)$ exists and is continuous and $h''(\xi_0) < 0$.

(C3) For some $r_1 > 0$ and for any $\rho \in [0, r_1)$ we have

$$\xi_0 \geq x \geq \xi_0 - \rho \geq y \geq a \implies h(x) \geq h(y).$$

(C4) For some $r_2 > 0$ and for any $\rho \in [0, r_2)$ we have

$$\xi_0 \leq x \leq \xi_0 + \rho \leq y \leq b \implies h(x) \geq h(y).$$

(C5) $\varphi(x)$ is continuous at $x = \xi_0$ and $\varphi(\xi_0) \neq 0$.

Then the following asymptotic formula holds as $n \to \infty$.

$$\int_a^b \varphi(x) \exp(n h(x)) \, dx \sim \varphi(\xi_0) \exp(n h(\xi_0)) \sqrt{\frac{2\pi}{-n h''(\xi_0)}}.$$ 

**Proof.** It suffices to see

$$\lim_{n \to \infty} \sqrt{n} \int_a^b \varphi(x) \exp [n(h(x) - h(\xi_0))] \, dx = \varphi(\xi_0) \sqrt{\frac{2\pi}{h''(\xi_0)}}.$$ (2)

We assume $a, b, \varphi(x), h(x)$ and $\xi_0$ are standard. Then, desired relation is now equivalent to:

$$\forall \nu \in \mathbb{N}_\infty \left( \sqrt{\nu} \int_a^b \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \sim \varphi(\xi_0) \sqrt{\frac{-2\pi}{h''(\xi_0)}} \right).$$
Now, take arbitrary \( \nu \in \mathbb{N}_\infty \) and let \( \varepsilon := \nu^{-1/6} \). Since \( \varepsilon \) is positive infinitesimal, \( h''(x) \) exists and is less than 0 if \( |x - \xi_0| < \varepsilon \), by condition (C2). And since \( h''(\xi_0) \) is standard negative number, more precise estimation is given:

\[
|x - \xi_0| \leq \varepsilon \implies 0 < -\frac{1}{2} h''(\xi_0) \leq -h''(x) \leq -\frac{3}{2} h''(\xi_0). \tag{3}
\]

We separate the integral into three parts:

\[
\sqrt{\nu} \int_a^b \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx = \sqrt{\nu} \int_a^{\xi_0 - \varepsilon} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \\
+ \sqrt{\nu} \int_{\xi_0 - \varepsilon}^{\xi_0 + \varepsilon} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \\
+ \sqrt{\nu} \int_{\xi_0 + \varepsilon}^b \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx.
\]

From (C3) and (C4) \( \xi_0 \) is the local maximum point of \( h(x) \), and \( h'(\xi_0) = 0 \) by (C2).

Now, for any \( x \in [a, \xi_0 - \varepsilon] \), we have

\[
h(\xi_0) - h(x) = h(\xi_0) - h(\xi_0 - \varepsilon) + h(\xi_0 - \varepsilon) - h(x) \\
\geq h(\xi_0) - h(\xi_0 - \varepsilon) = -\int_{\xi_0 - \varepsilon}^{\xi_0} h'(\xi_0) - h'(t) \, dt \\
= \int_{\xi_0 - \varepsilon}^{\xi_0} t \left( \int_t^{\xi_0} -h''(u) \, du \right) \, dt \geq -\frac{h''(\xi_0)}{2} \int_{\xi_0 - \varepsilon}^{\xi_0} t \, dt \\
= -\frac{h''(\xi_0)}{4} \varepsilon^2 = -\frac{h''(\xi_0)}{4} \nu^{-1/3} (\varepsilon > 0),
\]

by condition (C3) and (3). Thus, \( \nu(h(x) - h(\xi_0)) \leq \frac{h''(\xi_0)}{4} \nu^{2/3} (\varepsilon) \) and, since \( \nu^{2/3} \) is infinitely large number,

\[
\sqrt{\nu} \exp [\nu(h(x) - h(\xi_0))] \leq \nu^{3/6} \exp \left( \frac{h''(\xi_0)}{4} \nu^{4/6} \right) \\
= \nu^{-1/6} \left( \nu^{4/6} \exp \left( \frac{h''(\xi_0)}{4} \nu^{4/6} \right) \right) \\
\leq C \nu^{-1/6}, \tag{4}
\]

by some positive constant \( C \). Hence

\[
\simeq 0.
\]

**case 1: \( a > -\infty \)** In this case, we simply have

\[
\int_a^{\xi_0 - \varepsilon} \sqrt{\nu} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \simeq 0.
\]

**case 2: \( a = -\infty \)** For any standard \( k = 1, 2, 3, \ldots \) we have:

\[
\int_{-\infty}^{\xi_0 - \varepsilon} \sqrt{\nu} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \simeq 0.
\]

By Robinson’s extension theorem, some \( \kappa \in \mathbb{N}_\infty \) exists and

\[
\int_{-\infty}^{\xi_0 - \varepsilon} \sqrt{\nu} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \simeq 0.
\]

From the integrability of \( |\varphi(x)| \) over \([-\infty, b]\) and (4) we have:

\[
\left| \int_{-\infty}^{\xi_0 - \kappa} \sqrt{\nu} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \right| \leq C \nu^{-1/6} \int_{-\infty}^{\xi_0 - \kappa} |\varphi(x)| \, dx \simeq 0.
\]
Hence, regardless of whether $a$ is $-\infty$ or not, we have
\[ \int_a^{\xi_0-\varepsilon} \sqrt{\nu} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \simeq 0. \quad (5) \]
By similar argument, we have:
\[ \int_{\xi_0+\varepsilon}^b \sqrt{\nu} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \simeq 0. \quad (6) \]
From (5),(6) we have:
\[ \sqrt{\nu} \int_a^{\xi_0-\varepsilon} \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \simeq \sqrt{\nu} \int_{\xi_0+\varepsilon}^b \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx. \quad (7) \]
For any $x \in [\xi_0 - \varepsilon, \xi_0 + \varepsilon]$, we set
\[ p(x) := h(x) - h(\xi_0), \quad q(x) := \frac{h''(\xi_0)}{2}(x - \xi_0)^2. \]
Obviously $p(x), q(x) \leq 0$. So
\[ |\exp(\nu p(x)) - \exp(\nu q(x))| \leq \frac{1}{\nu} \int_{q(x)}^{p(x)} |\exp(\nu t)| \, dt \leq \frac{|p(x) - q(x)|}{\nu}. \quad (8) \]
Since
\[ p(x) = h(x) - h(\xi_0) = \int_{\xi_0}^{x} \frac{d}{dt}(t - x) h'(t) \, dt = \int_{\xi_0}^{x} (x - t) h''(t) \, dt \]
and
\[ q(x) = \frac{h''(\xi_0)}{2}(x - \xi_0)^2 = \int_{\xi_0}^{x} (x - t) h''(\xi_0) \, dt, \quad (9) \]
we have
\[ |p(x) - q(x)| \leq \int_{\xi_0}^{x} |x - t| |h''(t) - h''(\xi_0)| \, dt \leq \frac{1}{2} |h''(\xi_0)| \int_{\xi_0}^{x} |x - t| \, dt \]
\[ = \frac{1}{4} |h''(\xi_0)||x - \xi_0|^2 \leq \frac{1}{4} |h''(\xi_0)|\varepsilon^2 = \frac{1}{4} |h''(\xi_0)|\varepsilon^{-1/3}, \]
by the estimate (4). Substituting this to (8), we obtain
\[ |\exp(\nu p(x)) - \exp(\nu q(x))| \leq \frac{1}{4} |h''(\xi_0)|\varepsilon^{-8/6}. \]
Hence
\[ \sqrt{\nu} \exp(\nu p(x)) \simeq \sqrt{\nu} \exp(\nu q(x)). \]
From the continuity of $\varphi(x)$ at $x = \xi_0$, we have $\varphi(x) \simeq \varphi(\xi_0)$. Thus,
\[ \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \sqrt{\nu} \varphi(x) \exp(\nu p(x)) \, dx \simeq \sqrt{\nu} \varphi(\xi_0) \exp(\nu q(x)) \, dx \]
\[ = \varphi(\xi_0) \sqrt{\nu} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \exp \left[ \nu \left( \frac{h''(\xi_0)}{2}(x - \xi_0)^2 \right) \right] \, dx \]
By letting $z := (x - \xi_0)\sqrt{-\nu \frac{h''(\xi_0)}{2}}$.
\[ = \varphi(\xi_0) \sqrt{\nu} \left( -\frac{2}{\nu h''(\xi_0)} \int_{|z| \leq R} e^{-z^2} \, dz \right) \]
\[ = \varphi(\xi_0) \sqrt{\frac{2}{h''(\xi_0)}} \int_{|z| \leq R} e^{-z^2} \, dz, \quad (*) \]
where
\[ R := \varepsilon \sqrt{\frac{-h''(\xi_0)}{2}} = \nu^{1/3} \sqrt{-\frac{h''(\xi_0)}{2}}. \]

Since \( R \) is infinitely large, the integral in (\( \text{C1} \)) is infinitely close to \( \int_{-\infty}^{\infty} e^{-z^2} \, dz = \sqrt{\pi} \). Therefore,
\[ (\text{C1}) \simeq \varphi(\xi_0) \sqrt{-\frac{2}{h''(\xi_0)}} \sqrt{\pi} = \varphi(\xi) \sqrt{-\frac{2\pi}{h''(\xi)}}. \]

By (\( \text{C1} \)) and above estimation, we obtain
\[ \sqrt{\nu} \int_a^b \varphi(x) \exp [\nu(h(x) - h(\xi_0))] \, dx \simeq \varphi(\xi_0) \sqrt{-\frac{2\pi}{h''(\xi_0)}}. \]

Since \( \nu \) was arbitrary, (\( \text{C2} \)) is shown for any standard \( a, b, \varphi(x), h(x) \) and \( \xi_0 \). By transfer, (\( \text{C2} \)) is true for any (possibly nonstandard) \( a, b, \varphi(x), h(x) \) and \( \xi_0 \). This completes the proof. \( \square \)

### 3 A Generalization

The approach used in proving Theorem 1 can be extended to show the generalized formula.

**Theorem 2** (generalized Laplace’s theorem). Let \([a, b] \subseteq \mathbb{R}\) be finite or infinite interval, \( \varphi(x) \) and \( h(x) \) be functions defined on the interval, and \( \xi_0 \in (a, b) \). And let \( m \) be the natural number \( \geq 1 \). Suppose these data satisfies (C1)(C3)(C4)(C5) of Theorem 1 and (C2') of the following:

- \( (\text{C2}') \) \( h^{(k)}(\xi_0) = 0 \) (\( 1 \leq k < 2m \)), \( h^{(2m)}(\xi_0) < 0 \).

Then the following asymptotic formula holds as \( n \to \infty \).
\[ \int_a^b \varphi(x) \exp(n \, h(x)) \, dx \sim \varphi(\xi_0) \exp(n \, h(\xi_0)) \frac{\Gamma \left( \frac{2m}{2m} \right)}{m} \left( -\frac{(2m)!}{h^{(2m)}(\xi_0)} \right)^{\frac{1}{2m}}. \]

**Proof.** Since the following is parallel to the proof of Theorem 1, sometimes we omit the detail. We assume \( a, b, \varphi(x), h(x) \) and \( \xi_0 \) are standard. So, desired relation is now equivalent to:
\[ \forall \nu \in \mathbb{N}_\infty \left( \nu^{\frac{1}{2m}} \int_a^b \varphi(x) \exp[\nu(h(x) - h(\xi_0))] \, dx \sim \varphi(\xi_0) \frac{\Gamma \left( \frac{2m}{2m} \right)}{m} \left( -\frac{(2m)!}{h^{(2m)}(\xi_0)} \right)^{\frac{1}{2m}} \right). \] \( (10) \)

Take arbitrary \( \nu \in \mathbb{N}_\infty \) and let \( \varepsilon := \nu^{-1/6m^2} \). Since \( h^{(2m)}(\xi_0) \) is standard negative number, following estimation holds:
\[ |x - \xi_0| \leq \varepsilon \implies 0 < -\frac{1}{2} h^{(2m)}(\xi_0) \leq h^{(2m)}(x) \leq -\frac{3}{2} h^{(2m)}(\xi_0). \] \( (11) \)

As before, we separate the integral into three parts:
\[ \nu^{\frac{1}{2m}} \int_a^b \varphi(x) \exp[\nu(h(x) - h(\xi_0))] \, dx = \nu^{\frac{1}{2m}} \int_a^{\xi_0 - \varepsilon} \varphi(x) \exp[\nu(h(x) - h(\xi_0))] \, dx \]
\[ + \nu^{\frac{1}{2m}} \int_{\xi_0 + \varepsilon}^{\xi_0 + \varepsilon} \varphi(x) \exp[\nu(h(x) - h(\xi_0))] \, dx \]
\[ + \nu^{\frac{1}{2m}} \int_{\xi_0 - \varepsilon}^{\xi_0 + \varepsilon} \varphi(x) \exp[\nu(h(x) - h(\xi_0))] \, dx. \]
Now, for any \( x \in [a, \xi_0 - \varepsilon] \), we have

\[
h(\xi_0) - h(x) \geq h(\xi_0) - h(\xi_0 - \varepsilon) = - \int_{\xi_0 - \varepsilon}^{\xi_0} h'(t) \, dt_1
\]

\[
= - \int_{\xi_0 - \varepsilon}^{\xi_0} \left( \int_{t_1}^{t_2} h''(t) \, dt_2 \right) \, dt_1
g(\xi_0 - \varepsilon) - \xi_0 \leq \frac{h(\xi_0 - \varepsilon)}{2(2m)!} \nu^{\frac{1}{2m}}
\]

Thus, \( \nu (h(x) - h(\xi_0)) \leq \frac{h(\xi_0 - \varepsilon)}{2(2m)!} \nu^{\frac{1}{2m}} \) \((< 0)\) and, since \( \nu^{1 - \frac{1}{2m}} \) is infinitely large number,

\[
\nu^{1 - \frac{1}{2m}} \exp [\nu (h(x) - h(\xi_0))] \leq \nu^{\frac{1}{2m}} \exp \left( \frac{h(\xi_0)}{2(2m)!} \nu^{1 - \frac{1}{2m}} \right)
\]

\[
= \nu^{\frac{1}{2m} - 1} \left( \nu^{1 - \frac{1}{2m}} \exp \left( \frac{h(\xi_0)}{2(2m)!} \nu^{1 - \frac{1}{2m}} \right) \right) \leq C \nu^{\frac{1}{2m} - 1}, \tag{12}
\]

by some positive constant \( C \). Hence

\[ \nu^{1 - \frac{1}{2m}} \exp [\nu (h(x) - h(\xi_0))] \leq C \nu^{\frac{1}{2m} - 1} \]

By similar argument in the proof of Theorem 1, we obtain

\[
\int_a^{\xi_0 - \varepsilon} \nu^{\frac{1}{2m}} \varphi(x) \exp [\nu (h(x) - h(\xi_0))] \, dx \simeq 0 \tag{13}
\]

and

\[
\int_{\xi_0 + \varepsilon}^{b} \nu^{\frac{1}{2m}} \varphi(x) \exp [\nu (h(x) - h(\xi_0))] \, dx \simeq 0. \tag{14}
\]

From (13), (14) we have:

\[
\nu^{\frac{1}{2m}} \int_a^{b} \varphi(x) \exp [\nu (h(x) - h(\xi_0))] \, dx \simeq \nu^{\frac{1}{2m}} \int_{\xi_0 - \varepsilon}^{\xi_0 + \varepsilon} \varphi(x) \exp [\nu (h(x) - h(\xi_0))] \, dx. \tag{15}
\]

For any \( x \in [\xi_0 - \varepsilon, \xi_0 + \varepsilon] \), we set

\[ p(x) := h(x) - h(\xi_0), \quad q(x) := \frac{h(\xi_0)}{(2m)!} (x - \xi_0)^{2m}. \]

Obviously \( p(x), q(x) \leq 0 \). So again we have

\[
|\exp(\nu p(x)) - \exp(\nu q(x))| \leq \frac{1}{\nu} \left| \int_{q(x)}^{p(x)} |\exp(\nu t)| \, dt \right| \leq \frac{|p(x) - q(x)|}{\nu}. \tag{16}
\]
As is easily verified,
\[ p(x) = \frac{1}{(2m-1)!} \int_{\xi_0}^{x} (x-t)^{2m-1} h^{(2m)}(t) dt \]
and
\[ q(x) = \frac{1}{(2m-1)!} \int_{\xi_0}^{x} (x-t)^{2m-1} h^{(2m)}(\xi_0) dt. \]

Thus we have
\[ |p(x) - q(x)| \leq \frac{1}{(2m-1)!} \int_{\xi_0}^{x} |x-t|^{2m-1} \left| h^{(2m)}(t) - h^{(2m)}(\xi_0) \right| dt \]
\[ \leq \frac{1}{2(2m-1)!} \left| h^{(2m)}(\xi_0) \right| \int_{\xi_0}^{x} |x-t|^{2m-1} dt \]
\[ = \frac{1}{2(2m)!} \left| h^{(2m)}(\xi_0) \right| |x-\xi_0|^{2m} \leq \frac{1}{2(2m)!} \left| h^{(2m)}(\xi_0) \right| \varepsilon^{2m} = \frac{1}{2(2m)!} \left| h^{(2m)}(\xi_0) \right| \nu^{-\frac{1}{m}}, \]
by the estimate (11). Substituting this to (16), we obtain
\[ |\exp(\nu x) - \exp(\nu q(x))| \leq \frac{1}{2(2m)!} \left| h^{(2m)}(\xi_0) \right| \nu^{-1-\frac{1}{m}}. \]

Hence
\[ \nu^{-\frac{1}{m}} \exp(\nu x) \simeq \nu^{-\frac{1}{m}} \exp(\nu q(x)). \]

From the continuity of \( \varphi(x) \) at \( x = \xi_0 \), we have \( \varphi(x) \simeq \varphi(\xi_0) \). Thus:
\[ \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \nu^\frac{1}{m} \varphi(x) \exp(\nu x) dx \simeq \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \nu^\frac{1}{m} \varphi(\xi_0) \exp(\nu x) dx \]
\[ = \varphi(\xi_0) \nu^\frac{1}{m} \int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} \exp \left[ \nu \left( \frac{h^{(2m)}(\xi_0)}{(2m)!} (x-\xi_0)^{2m} \right) \right] dx \]

By letting \( z := (x-\xi_0) \left( -\frac{h^{(2m)}(\xi_0)}{(2m)!} \right)^{\frac{1}{2m}} \),

\[ \varphi(\xi_0) \nu^\frac{1}{m} \left( \nu \frac{h^{(2m)}(\xi_0)}{(2m)!} \right)^{-\frac{1}{2m}} \int_{R'} e^{-z^2 m} dz \]
\[ = \varphi(\xi_0) \left( \frac{h^{(2m)}(\xi_0)}{(2m)!} \right)^{-\frac{1}{2m}} \int_{R'} e^{-z^2 m} dz, \]

where
\[ R' := \varepsilon \left( \nu \frac{h^{(2m)}(\xi_0)}{(2m)!} \right)^{\frac{1}{2m}} = \nu^{-\frac{1}{m}+\frac{1}{2m}} \left( \frac{h^{(2m)}(\xi_0)}{(2m)!} \right)^{\frac{1}{2m}}. \]

Since \( R' \) is infinitely large, the integral in (**) is infinitely close to \( \int_{-\infty}^{\infty} e^{-z^2 m} dz = \frac{1}{m} \Gamma \left( \frac{1}{2m} \right) \). Therefore,
\[ (**) \simeq \varphi(\xi_0) \frac{\Gamma \left( \frac{1}{2m} \right)}{m} \left( -\frac{(2m)!}{h^{(2m)}(\xi_0)} \right)^{\frac{1}{2m}}. \]

By (15) and above estimation, we obtain
\[ \sqrt{\nu} \int_{a}^{b} \varphi(x) \exp \left[ \nu (h(x) - h(\xi_0)) \right] dx \simeq \varphi(\xi_0) \frac{\Gamma \left( \frac{1}{2m} \right)}{m} \left( -\frac{(2m)!}{h^{(2m)}(\xi_0)} \right)^{\frac{1}{2m}}. \]

Since \( \nu \) was arbitrary, (10) is shown for any standard \( a, b, \varphi(x), h(x) \) and \( \xi_0 \). By transfer, (10) is true for any (possibly nonstandard) \( a, b, \varphi(x), h(x) \) and \( \xi_0 \). This completes the proof.
4 Note

The statement of the theorem here is taken and modified from the problem 201 of part two of [3]. To be more specific, the condition (C3) and (C4) are added to the premise. On the other hand, one condition is dropped from the original statement. It seems conditions (C3) and (C4) hold for typical cases of application of the formula.

Laplace’s theorem is already proved by nonstandard method in [2], based on wider point of view. The author of [2] uses general technique dealing with external numbers to show the theorem.

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