NON-SPLITTING FLAGS, ITERATED CIRCUITS, $g$-MATRICES AND CAYLEY CONFIGURATIONS

EDUARDO CATTANI AND ALICIA DICKENSTEIN

Dedicated to Bernd Sturmfels on the occasion of his 60th birthday

Abstract. We explore four approaches to the question of defectivity for a complex projective toric variety $X_A$ associated with an integral configuration $A$. The explicit tropicalization of the dual variety $X_A^\vee$ due to Dickenstein, Feichtner, and Sturmfels allows for the computation of the defect in terms of an affine combinatorial invariant $\rho(A)$. We express $\rho(A)$ in terms of affine invariants $\iota(A)$ associated to Esterov’s iterated circuits and $\lambda(A)$, an invariant defined by Curran and Cattani in terms of a Gale dual of $A$. Thus we obtain formulae for the dual defect in terms of iterated circuits and Gale duals. An alternative expression for the dual defect of $X_A$ is given by Furukawa-Ito in terms of Cayley decompositions of $A$. We give a Gale dual interpretation of these decompositions and apply it to the study of defective configurations.

1. Introduction

Given a complex projective variety $X$, its dual $X^\vee$ is defined as the closure in the dual projective space of all hyperplanes tangent to $X$ at a smooth point. It is classically known that generically $X^\vee$ is a hypersurface ([11], Corollary 1.2). If $\text{codim} X^\vee > 1$, $X$ is said to be defective and the quantity

$$\text{def}(X) := \text{codim} X^\vee - 1$$

is called the dual defect of $X$. If $X$ is irreducible and non-defective, then $X^\vee$ is irreducible and the polynomial defining $X^\vee$ is known as the discriminant of $X$.

Suppose now that $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^e$ is a configuration (of not necessarily distinct points) which is homogenous in the sense that the $n$-tuple $(1, \ldots, 1)$ lies in the row span of $A$ viewed as the $e \times n$ matrix whose columns are the $a_i$’s. Then $A$ defines a projective toric variety $X_A = \mathbb{P}^{n-1}$ rationally parametrized by monomials with exponents in $A$. The dimension $\dim(X_A) = \text{rank}(A) - 1$, which is equal to the affine dimension $d(A)$ of $A$.

We will also assume that $A$ is not a pyramid; i.e. no affine hyperplane contains all points of $A$ except one or, equivalently, the dual variety $X_A^\vee$ is not contained in a hyperplane. We discuss at the end of Section 3 the extension of our definitions and results to the pyramidal case.

The dual defect of $X_A$ is an affine invariant of the configuration $A$ which has been studied from various points of view. In this paper we will discuss four such

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approaches to the dual defect of projective toric varieties. Let us briefly describe them postponing for a moment the detailed definitions:

i) In [5] Dickenstein, Feichtner, and Sturmfels described the tropicalization of the dual variety $X_A^\vee$. This leads to the computation of the dual defect of $X_A$, explicitly, in terms of certain $n(A) \times n(A)$ matrices $M_\sigma(A)$, where $n = n(A)$ is the cardinality of $A$. These matrices are constructed from the matrix $A$ and a maximal chain $\sigma$ of support of vectors in the kernel of $A$ (see Definition 2.3). Indeed [5, Corollary 4.5] gives the equality

$$\text{def}(X_A) = n(A) - 1 - \rho(A),$$

where

$$\rho(A) = \max_\sigma \text{rank}(M_\sigma(A)).$$

We refer to $M_\sigma(A)$ as a $\sigma$-matrix. We point out that a key ingredient in the computation of the dual defect is the Horn-Kapranov parametrization map, also used by Forsgård [9], who applies it to real configurations.

ii) Recall that a point configuration $Z$ is called a circuit if it is minimally affinely dependent i.e. it is affinely dependent but every proper subset is affinely independent. It is not hard to show that if $A$ contains a circuit $Z$ with $d(Z) = d(A)$ then $X_A$ is non-defective. In [8], Esterov shows that if $\text{def}(X_A) = 0$ then $A$ might not contain a circuit of affine dimension $d(A)$, but it will necessarily contain what he calls an iterated circuit with this maximal possible affine dimension. We generalize this notion in Definition 2.1.

iii) We can associate to the configuration $A$ a Gale dual configuration $B$. Let $B \in \mathbb{Q}^{n \times m}$, where $m = m(A) = \dim \ker Q(A)$, be a matrix whose columns are a basis of $\ker Q A \subset \mathbb{Q}^n$. The rows of $B$ define a linear matroid which we also denote by $B$ and which gives a choice of Gale dual configuration (see Section 3). If $A$ is homogeneous, then $B$ satisfies a dual homogeneity condition:

$$\sum_{b \in B} b = 0.$$

Associated to the matroid $B$ is its lattice of flats (see Section 3). We say that a chain of flats

$$F_1 \subset F_2 \subset \cdots \subset F_\ell; \quad \dim(F_j) = j,$$

is non-splitting if $\sum_{b \in F_1} b \neq 0$ and $\sum_{b \in F_{j+1}} b \notin L(F_j), \ j \geq 1$, where $L(F_i)$ denotes the linear span of the vectors in $F_i$. We call $\ell$ the length of the flag and denote by $\lambda(B) = \lambda(A)$ the maximal length of a non-splitting flag of flats in $B$. It is shown in [3] that $X_A$ is defective if and only if $\lambda(B) < m - 1$. This generalizes the work of Dickenstein and Sturmfels [4] in the case $m = 2$.

iv) Every defective configuration admits a Cayley decomposition (see Definition 4.1) but this is not enough to characterize defectivity. However, Furukawa and Ito [10] show that among all possible Cayley decompositions of $A$ there is a distinguished one, which we will call the FI-decomposition, and it is then possible to compute $\text{def}(X_A)$ in terms of simple invariants of this
decomposition (see Section 4). Their approach is fundamentally different to
the tropical approach used in [5].

The purpose of this paper is twofold. First of all, we extend the computation of
def(X_A) to the settings described in ii) and iii). In Section 2 we extend Esterov’s
notion of iterated circuit to the defective case and define an invariant ι(A) which is
the maximal rank of an iterated circuit I ⊂ A. We show that

\[ n(A) - 1 - \rho(A) = d(A) - \iota(A). \]

This allows us to compute the defect of X_A in the context of Esterov’s work. On
the other hand, in Section 3 we prove that

\[ n(A) - 1 - \rho(A) = m(A) - 1 - \lambda(A), \]

and, consequently, the right-hand side computes the dual defect of X_A in terms of
the Gale dual B.

We should point out that, though inspired by the geometric case of integral con-
figurations and the associated toric varieties over \( \mathbb{C} \), the proof of these results is
linear-algebraic in nature and the above equalities hold for finite configurations
A = \{a_1, \ldots, a_n\} \subset \mathbb{K}^e, defined over an arbitrary field \( \mathbb{K} \) of characteristic zero.
Thus, we will work in this generality.

We give all the definitions about iterated circuits and \( \sigma \)-matrices in Section 2
where we prove equality (1.2) in Theorem 2.7. In Section 3 we introduce necessary
concepts about Gale duality and homogenizations and we prove equality (1.3) in
Theorem 3.8. The expression (1.1) for def(X_A) given in [5] is valid provided A is
not a pyramid. At the end of Section 3 we include a brief discussion of how our
results can be applied in the pyramidal case.

Finally, Section 4 should be considered as the second part of this paper. We begin
by describing the results of Furukawa and Ito in [10] which allow us to compute the
dual defect of complex toric varieties X_A in terms of the family of Cayley decompo-
sitions of A. We then translate this to the Gale dual setting and use this formulation
to obtain results in low codimension and for configurations with “large” defect.

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2. Iterated Circuits and \( \sigma \)-Matrices: Proof of (1.2)

In this section we generalize Esterov’s notion of iterated circuits (see Definition
2.1) to include the case of defective configurations and prove identity (1.2)
which allows us to compute the dual defect of a toric variety in terms of iterated
circuits. This is the content of the main result in this section, Theorem 2.7, which
follows from Theorem 2.6 where we make explicit the relationship between iterated
circuits and chains of supports.
We begin by establishing the notation that we will use throughout the paper. We will also give careful definitions of the objects mentioned in the Introduction. As has been noted already, we will work with a finite configuration \( A = \{a_1, \ldots, a_n\} \subset \mathbb{K}^e \), defined over an arbitrary field \( \mathbb{K} \) of characteristic zero.

There are three basic affine quantities associated with \( A \), namely, its cardinality \( n(A) \), its affine dimension \( d(A) \) over \( \mathbb{K} \) (that is, the \( \mathbb{K} \)-dimension of its affine span \( L_{\text{aff}}(A) \)), and the dimension \( m(A) = n(A) - (d(A) + 1) \) of the space \( R_{\text{aff}}(A) \) of \( \mathbb{K} \)-affine relations among the points in \( A \). We also refer to \( m(A) \) as the codimension of \( A \). If there is no possibility of confusion, we will simply write \( n, d, \) and \( m \) for the quantities \( n(A), d(A) \) and \( m(A) \).

As before, we may identify \( A \) with the \( e \times n \) matrix whose columns are the \( a_i \)'s. We will say that \( A \) is homogeneous if the \( n \)-tuple \((1, \ldots, 1)\) lies in the row span of \( A \). Up to a linear transformation, this is equivalent to saying that \( A \) is contained in an affine hyperplane not passing through the origin. If \( A \) is homogeneous then \( R_{\text{aff}}(A) = \ker \mathbb{K}(A) \).

In what follows we also need to consider non-homogeneous configurations. If \( A \) is not homogeneous we associate to it two homogeneous configurations: in case \( A \) is not homogeneous, we denote by \( \overline{A} \) the homogeneous configuration
\[
\overline{A} = \{(1, a_1), \ldots, (1, a_n)\} \subset \mathbb{K}^{e+1},
\]
which is affinely equivalent to \( A \). We also set
\[
A^h = \{(1,0,\ldots,0)\} \cup \overline{A} = (0,\ldots,0) \cup \overline{A} \subset \mathbb{K}^{e+1}.
\]
We will refer to \( A^h \) as the homogenization of \( A \). When \( A \) is homogeneous, it is enough to take \( \overline{A} = A \).

Remark 1. We note that if \( Z \) is a circuit, that is a minimally affinely dependent set of points, then \( Z \) is not a pyramid and \( n(Z) = d(Z) + 2 \) (or, equivalently, \( m(Z) = 1 \)). In particular, a zero-dimensional circuit is a configuration of cardinality two with a repeating point. Observe also that if \( W \subset \mathbb{K}^e \) is minimally linearly dependent over \( \mathbb{K} \), then it is a circuit provided it is homogeneous and, otherwise, its homogenization \( W^h \) is a circuit.

We will denote the \( \mathbb{K} \)-linear span of a subset \( S \) in a \( \mathbb{K} \)-vector space by \( L(S) \).

The following is a generalization of the definition of iterated circuit given by Esterov [7, 8].

**Definition 2.1.** Let \( A \) be a non-pyramidal homogeneous point configuration over \( \mathbb{K} \) of maximal rank. An **iterated circuit of length** \( p \) in \( A \) is a subconfiguration \( I \subset A \) together with a partition into nonempty subconfigurations
\[
I = I_1 \cup \cdots \cup I_p,
\]
such that:

i) \( I_1 \) is a circuit;

ii) For every \( j, 1 \leq j \leq p - 1 \), write \( L_j = L(I_1 \cup \cdots \cup I_j) \), and consider the natural projection \( \pi_j: L(I) \to L(I)/L_j \). Then, either \( \pi_j(I_{j+1}) \) is a circuit with affine dimension 0, or \( \pi_j \) acts injectively on \( I_{j+1} \). Moreover, in the
latter case, the configuration $\pi_j(I_{j+1})$ is minimally linearly dependent (cf. Remark 1).

Given an iterated circuit $I = I_1 \cup \cdots \cup I_p \subset A$, let

$$\eta(I) = \sum_{j=1}^{p} d(I_j).$$

Note that $\eta(I)$ equals $d(I)$ minus the number of indices $j = 1, \ldots, p-1$ for which the projection $\pi_j(I_{j+1})$ is homogeneous.

We now define

$$\iota(A) = \max\{\eta(I), I \subset A \text{ iterated circuit}\}.$$

We point out that in [8] Esterov reserves the name iterated circuit for iterated circuits $I$ with $\eta(I) = d(A)$. In this case $\eta(I) = d(I)$ and none of the configurations $\pi_j(I_{j+1}), j = 1, \ldots, p-1$, may be homogeneous. For a non-homogeneous configuration $A$ we set $\iota(A) = \iota(\bar{A})$.

**Example 2.2.** We present here two simple examples to illustrate Definition 2.1.

The figures below depict non-homogeneous configurations $A'$ and we take $A = A'$.

**Figure 1.** Octahedron (left) and prism (right) configurations

In the octahedron configuration, we have

$$A' = \{(1,0,0), (-1,0,0), (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\}.$$

It is clear that the four points in each of the coordinate planes define two-dimensional circuits. Setting $I_1 = \{(1,1,0,0), (1,-1,0,0), (1,0,1,0), (1,0,-1,0)\}$ and $I_2 = \{(1,0,0,1), (1,0,0,-1)\}$ we obtain a maximal dimension iterated circuit in $A$. Indeed, $I_1$ is a homogeneous circuit of dimension 2 and $\pi_1(I_2)$ consists of two different points in dimension 1, which is a minimally linearly dependent configuration and, consequently, its homogenization is a circuit as noted in Remark 1. Clearly, $\eta(I) = 2 + 1 = 3$ and $A$ is non-defective.

Consider next the prism:

$$A' = \{(1,0,0), (1,0,1), (0,1,0), (0,1,1), (0,0,0), (0,0,1)\}.$$
Once again, there are three, two-dimensional circuits, but none of them may be extended to a three-dimensional iterated circuit in $A$. For example, if we start with the circuit $I_1 = \{(1, 0, 1, 0), (1, 0, 1, 1), (1, 0, 0, 0), (1, 0, 0, 1)\}$ then $\pi_1(A \setminus I_1)$ consists of two equal points and thus, is a minimally linearly dependent configuration which is a homogeneous zero-dimensional circuit. Thus $I = I_1 \cup (A \setminus I_1)$ is an iterated circuit but $\eta(I) = 2$. Indeed, $\nu(A) = 2$, and the configuration $A$ is defective.

We now recall the invariant introduced by Dickenstein, Feichtner, and Sturmfels in [5], which in the case of integral configurations yields the dual defect of the associated complex toric variety.

**Definition 2.3.** Given a homogeneous non-pyramidal configuration $A$ over $\mathbb{K}$, we denote by $S(A) \subset \{0, 1\}^{n(A)}$ the geometric lattice whose elements are the supports, ordered by inclusion, of the vectors in $R_{\text{aff}}(A) = \ker \mathbb{K}(A)$. Given a maximal chain of supports

$$\sigma = \{\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_{m(A)}\}$$

in $S(A)$ we define the $n(A) \times n(A)$ matrix

$$M_{\sigma}(A) = \begin{pmatrix} A \\ \sigma_1 \\ \vdots \\ \sigma_{m(A)} \end{pmatrix}.$$ 

We will refer to $M_{\sigma}(A)$ as a $\sigma$-matrix associated with $A$ and set:

$$\rho(A) = \max_{\sigma} \text{rank}(M_{\sigma}(A)),$$

where $\sigma$ runs over all maximal chains of elements in $S(A)$.

For a non-homogeneous configuration $A$ we set $\rho(A) = \rho(\tilde{A})$.

The implicit assertion in Definition 2.3 that the length of a maximal chain of supports is $m(A)$ follows from the fact that $\mathbb{K}$ is infinite, since we assume that $\text{char}(\mathbb{K}) = 0$. Hence, given any two vectors $v_1, v_2 \in \mathbb{K}^e$, there exists a $\mathbb{K}$ linear combination $v = \lambda_1 v_1 + \lambda_2 v_2$ such that the support of $v$ is the union of the supports of $v_1$ and $v_2$. Thus, the length of a maximal chain of supports in $\ker \mathbb{K}(A)$ equals $\text{dim}(\ker \mathbb{K}(A)) = m(A)$.

Before stating Theorem 2.7, we prove some preliminary results on iterated circuits and $\sigma$-matrices.

Given an iterated circuit $I = I_1 \cup \cdots \cup I_p$, an iterated circuit $I' = I'_1 \cup \cdots \cup I'_{p'}$ is called an extension of $I$ if $p \leq p'$ and $I'_j = I_j$, for $j = 1, \ldots, p$. Clearly, if $I'$ is an extension of $I$ then $\eta(I') \geq \eta(I)$. We have:

**Lemma 2.4.** Let $A$ be a homogeneous non-pyramidal configuration over $\mathbb{K}$. Then any iterated circuit $I = I_1 \cup \cdots \cup I_p$ has an extension $I'$ such that $d(I') = d(A)$.

**Proof.** Let $I'$ be an extension of $I$, maximal with respect to affine dimension. If $L(I') \neq L(A)$, consider the natural projection $\pi: L(A) \rightarrow L(A)/L(I')$. Then $\pi(A \setminus L(I'))$ or its homogenization contain no circuits. This means that $\pi(A \setminus L(I'))$ consists of linearly independent vectors, contradicting the fact that $A$ is non-pyramidal. $\square$
We now introduce some notation which will help us to better understand the relationship between iterated circuits and maximal chains of supports in \( \ker_K(A) \).

Denote by \([n]\) the set \(\{1, \ldots, n\}\). We identify the set \(\{0, 1\}^n\) with the set of functions \(\sigma: [n] \to \{0, 1\}\) and consider the partial order defined by
\[
\sigma \prec \sigma' \iff \sigma(i) \leq \sigma'(i) \quad \text{for all } i = 1, \ldots, n.
\]
Whenever convenient we will identify an element \(\sigma \in \{0, 1\}^n\) with the subset of \([n]\):
\[
S_\sigma = \{i \in [n] : \sigma(i) = 1\}.
\]
Note that a chain \(\sigma = \{\sigma_1 \prec \cdots \prec \sigma_s\} \in \{0, 1\}^n\) is equivalent to a chain \(S = \{S_1 \subset \cdots \subset S_s\}\) of subsets of \([n]\), strictly ordered by inclusion and where we simply write \(S_j\) for \(S_{\sigma_j}\).

**Definition 2.5.** A chain \(S = \{S_1 \subset \cdots \subset S_s\}\) of subsets of \([n]\) is called a chain of \(A\)-supports if and only if for each \(j = 1, \ldots, s\) there exists a vector \(v^j \in \ker_K(A)\) such that
\[
S_j = \{i \in [n] : v^j_i \neq 0\}.
\]
A chain \(S\) may be alternatively described by the collection of mutually disjoint differences: \(\overline{D} = \{D_1, \ldots, D_s\}\), where \(D_1 = S_1\) and, for \(j \geq 2\), \(D_j = S_j \setminus S_{j-1}\).

Again, we think of \(D_j\) as given by the 0, 1 vector with coordinates 1 in those indices belonging to \(D_j\). When the chain \(S\) comes from a maximal chain of \(A\)-supports, we have that \(s = m(A)\). Moreover, as the fact that \(A\) is not a pyramid is equivalent to the fact that \(\ker_K(A)\) is not contained in a coordinate hyperplane, the family \(\overline{D}\) is a partition of \([n(A)]\). A partition of \([n(A)]\) arises from a maximal chain of \(A\)-supports if its associated chain \(S\) does.

Recall that given a homogeneous configuration \(A\) and a maximal chain \(\sigma\) of \(A\)-supports, we have defined the matrix \(M_\sigma(A)\) by:
\[
M_\sigma(A) = \begin{pmatrix} A \\ \Sigma \end{pmatrix} ; \quad \Sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{pmatrix}.
\]
Note that the matrix \(\Sigma\) is row-equivalent to the matrix defined by the partition of differences:
\[
\begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix},
\]
where the row \(D_i\) indicates a row with ones in the positions indicated by \(D_i\) and zeroes elsewhere, because the \(i\)-th row of this matrix is equal to the difference \(\sigma_i - \sigma_{i-1}\) for any \(i > 1\).

Given a homogeneous non-pyramidal configuration \(A\) over \(K\), we will denote by \(\mathcal{I}\) the collection of all iterated circuits \(I \subset A\) with \(d(I) = d(A)\) and by \(\mathcal{M}\) the collection of maximal chains \(\sigma\) of \(A\)-supports. We then have:

**Theorem 2.6.** For any homogeneous non-pyramidal configuration \(A\) over \(K\), there are maps
\[
\varphi: \mathcal{I} \to \mathcal{M}, \quad \psi: \mathcal{M} \to \mathcal{I},
\]
such that:
i) For $I \in \mathcal{I}$,

\[(2.8)\quad d(I) - \eta(I) = n(A) - 1 - \text{rank}(M_{\varphi(I)}(A)),\]

ii) For $\sigma \in \mathcal{M}$,

\[(2.9)\quad d(\psi(\sigma)) - \eta(\psi(\sigma)) = n(A) - 1 - \text{rank}(M_{\sigma}(A)).\]

Proof. For simplicity of notation we write $d = d(A)$, $n = n(A)$ and $m = m(A)$. Suppose $I = I_1 \cup \cdots \cup I_p$ is an iterated circuit with $d(I) = d$. We may assume without loss of generality that $I = \{a_1, \ldots, a_r\}$.

We now define $\varphi(I)$ as the chain of $A$-supports whose differences are defined as follows. Consider the partition $D_j = \{i \in [n] : a_i \in I_j\}, j = 1, \ldots, p, D_{p+k} = \{r+k\}, k = 1, \ldots, n-r$.

We need to show that this partition comes as the differences of a chain of $A$-supports as in Definition 2.5 and to compute the rank of the corresponding matrix $M_{\varphi(I)}(A)$.

Note that after reordering, we may assume that the matrix defined by the configuration $I$ is affinely equivalent to a matrix of the form

\[(2.10)\quad \begin{pmatrix} I_1 & * & \cdots & * \\ 0 & I_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_p \end{pmatrix},\]

where for $j \geq 2$, $I_j$ corresponds to the columns “containing” $\bar{I}_j$.

Since $A$ is homogeneous, $I_1$ is a homogeneous circuit and, consequently, the set $D_1$ is the support of an element in $\ker G(A)$, and this support is minimal. Similarly, since the elements in $\bar{I}_j$ are a minimally linearly dependent set in $L(A)/L_{j-1}$, it follows that $D_1 \cup \cdots \cup D_j$ is the support of an element in $\ker G(A)$.

For $k = 1, \ldots, n-r$, we have that $D_{p+k} = \{r+k\}$. Note that the assumption $d(I) = d(A)$ implies that $a_{r+k}$ is a linear combination of elements in $I_1 \cup \cdots \cup I_p$ for any $k$. As the union of supports is a support, we have that $D_1 \cup \cdots \cup D_p \cup D_{p+1} \cdots \cup D_{p+k}$ is the support of some element in $\ker G(A)$ for any $k = 1, \ldots, n-r$. Then, $\varphi(I) \in \mathcal{M}$.

The matrix $M_{\varphi(I)}(A)$ is row equivalent to a block-upper triangular matrix of the form

\[(2.11)\quad \begin{pmatrix} I_1 & * & \cdots & * & * \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & I_2 & \cdots & * & * \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \bar{I}_p & * \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & I_{n-r} \end{pmatrix},\]

where the symbol 1 represents a row of 1’s of the appropriate size.
Clearly \( \text{rank}(M_\sigma(A)) \) equals \( (n-1) \) minus the number of homogeneous configurations \( \bar{I}_j \), for \( j = 2, \ldots, p \), which in turn equals \( d(I) - \eta(I) \). This completes the proof of (2.8).

We will now define the map \( \psi : \mathcal{M} \rightarrow \mathcal{I} \) and prove (2.9). Let \( \sigma \) be a maximal proper chain of \( A \)-supports and let us denote by

\[ \mathcal{D} = \{D_1, \ldots, D_m\} \]

the associated difference sets. We may assume – up to reordering of \( A \) – that the subsets \( D_j \) are coherently ordered in the sense that for any \( k \leq m \), the sets \( \{D_1, \ldots, D_k\} \) are a partition of \( \{1, \ldots, |D_1 \cup \cdots \cup D_k|\} \).

Denote by \( D_{i_1}, \ldots, D_{i_p} \) the difference sets with cardinality greater than 1. We must have \( D_{i_1} = D_1 \) since \( A \) is homogeneous. Let \( I_j \subset A \) be the subsets indexed by \( D_{i_j} \). We claim that \( I = I_1 \cup \cdots \cup I_p \) is an iterated circuit.

It is clear that \( D_1 \) equals the non-zero support of \( \sigma_1 \) and, hence, is a minimally dependent set of \( A \), i.e., a (homogenous) circuit. We note next that if \( D_s = \{\ell\} \) is a singleton then \( a_\ell \in L(\bigcup_{1 \leq j \leq s} D_j) \).

Let \( \pi_j : L(A) \rightarrow L(A)/L_j \) be the projection where, as before, \( L_j = L(I_1 \cup \cdots \cup I_j) \). Since \( D_{j+1} \) is a minimal \( A \)-support modulo \( L_j \), we have that the elements in \( \pi_j(I_{j+1}) \) are minimally linearly dependent. Moreover, because of the maximality of the chain \( \sigma \), if the affine dimension of \( \pi_j(I_{j+1}) \) is positive, the map \( \pi_j \) must be injective on the subset \( I_{j+1} \). Hence, \( I \) is an iterated circuit.

Finally, we note that the proof of (2.9) is completely analogous to that of (2.8) since after row operation and column and row permutations the matrix \( M_\sigma(A) \) will be equal to the matrix (2.11), so that the previous computation of its rank applies. \( \Box \)

**Remark 2.** Note that the composition \( \psi \circ \varphi \) of the maps in the statement of Theorem 2.6 is the identity, while the composition \( \varphi \circ \psi \) is the identity up to possible reordering the singletons in the differences of the supports.

We are now ready to prove equality (1.2).

**Theorem 2.7.** Let \( A \subset \mathbb{K}^{d+1} \) be a homogeneous non-pyramidal configuration with cardinality \( n(A) \) and affine dimension \( d(A) \). Let \( \iota(A) \) and \( \rho(A) \) be as in (2.7) and (2.8), respectively. Then, as asserted by (1.2),

\[ n(A) - 1 - \rho(A) = d(A) - \iota(A). \]

**Proof.** Let \( I \subset A \) be an iterated circuit with \( \iota(A) = \eta(I) \). We may assume without loss of generality that \( d(I) = d \). Indeed, by Lemma 2.4 there exists an extension \( I' \) of \( I \) with \( d(I') = d \) but, \( \eta(I') \geq \eta(I) \) and, consequently, \( \iota(A) = \eta(I') \) as well.

Then, we deduce from (2.8) in Theorem 2.6 that

\[ n - 1 - d + \iota(A) = \text{rank}(M_{\varphi(I)}(A)) \leq \rho(A). \]

Conversely, let \( \sigma = \{\sigma_1 \prec \cdots \prec \sigma_m\} \) be a maximal chain of \( A \)-supports and assume that \( \text{rank}(M_\sigma(A)) = \rho(A) \). Then, using (2.9) in Theorem 2.6 we have:

\[ \rho(A) = n - 1 - (d(\varphi(\sigma)) - \eta(\varphi(\sigma))) = n - 1 - d + \eta(I) \leq n - 1 - d + \iota(A), \]

which ends the proof. \( \Box \)
3. Gale Dual: Proof of (1.3)

In this section we will recall the construction of the \( \lambda \)-invariant introduced in [3] and prove identity (1.3) in Theorem 3.8. This gives a formula for the dual defect of a complex toric variety \( X_A \) in terms of a Gale dual of \( A \). In the final subsection, we describe the behavior of all the invariants in case \( A \) is a pyramid. As in the previous section, we will work with configurations defined over an arbitrary field \( K \) of characteristic zero.

Let \( A = \{a_1, \ldots, a_n\} \subset K^{d+1} \) be a configuration such that \( \kappa(A) = n-d-1 \) is the dimension of \( W = \ker K(A) \). Note that if \( d(A) = d \), in case \( A \) is homogeneous \( \kappa(A) \) coincides with \( m(A) \), which is the dimension of the space of affine relations of \( A \), and \( \kappa(A) = m(A) - 1 \) otherwise.

We have a short exact sequence

\[
0 \longrightarrow W \overset{\iota}{\longrightarrow} K^n \overset{\alpha}{\longrightarrow} K^{d+1} \longrightarrow 0,
\]

where \( \alpha \) maps the standard basis element \( e_i \in K^n \) to \( a_i \in \mathbb{Z}^{d+1} \). Dualizing (3.1) we get

\[
0 \longrightarrow (K^{d+1})^* \overset{\alpha^*}{\longrightarrow} (K^n)^* \overset{\iota^*}{\longrightarrow} W^* \longrightarrow 0.
\]

Choosing a basis we may identify \( W^* \cong K^{\kappa(A)} \) and we denote by \( b_i := \beta(\xi_i) \in K^{\kappa(A)} \), where \( \xi_i \) is the standard dual basis of \( (K^n)^* \). We denote by \( B \) the \( n \times \kappa(A) \) matrix whose rows are the \( b_i \)'s and note that the columns of \( B \) are a basis of \( \ker K A \). We will refer to the configuration \( \{b_1, \ldots, b_n\} \) as a Gale dual of \( A \).

In what follows it will be useful to consider the linear matroid of rank \( \kappa(A) \) defined by the configuration \( \{b_1, \ldots, b_n\} \) on \( \{1, \ldots, n\} \). Note that \( A \) is non-pyramidal if and only if none of the \( b_j = 0 \); i.e. if and only if the matroid \( B \) has no loops.

Given any subset \( C \subset B \) we denote

\[
\mathfrak{s}(C) = \sum_{b \in C} b.
\]

The configuration \( A \) is homogeneous if and only if any Gale dual \( B \) satisfies the dual homogeneity condition

\[
\mathfrak{s}(B) = \sum_{b \in B} b = 0.
\]

In this case we refer to \( B \) as a dual-homogeneous configuration.

Given a configuration \( A \) we have defined in (2.1) and (2.2) two associated homogeneous configurations \( \bar{A} \) and \( A^h \). It is easy to see that their corresponding Gale duals may be obtained from a Gale dual \( B \) of \( A \) as follows: A Gale dual \( B \) of \( \bar{A} \) is the \( m(A) \)-dimensional configuration obtained by projecting \( B \) to the quotient \( L(B)/L(\mathfrak{s}(B)) \), and a Gale dual of \( A^h \) is the dual-homogeneous configuration of rank \( m(A) + 1 \)

\[
B^H = \{-\mathfrak{s}(B)\} \cup B.
\]

A subset \( F \subset B \) is called a flat if and only if \( F = L(F) \cap B \). The rank of a flat \( F \) is defined by \( \text{rank}(F) = \dim L(F) \). A flag in \( B \) is a collection of flats
We say that a flag $\mathcal{F} = \{F_1 \subset \cdots \subset F_\ell\}$, with rank $F_j = j$ for every $j = 1, \ldots, \ell$. We call $\ell$ the length of the flag.

**Definition 3.1.** We say that a flag $\mathcal{F} = \{F_1 \subset \cdots \subset F_\ell\}$ in $B$ is **non-splitting** if

$$s(F_1) \neq 0, \text{ and } s(F_j) \notin L(F_{j-1}), \text{ for any } 2 \leq j \leq \ell.$$  

Given a vector configuration $B$ defined over $\mathbb{K}$ we denote by $\lambda(B)$ the maximal length of a non-splitting flag $\mathcal{F}$ in $B$. We carry the notion to the $A$-side by setting $\lambda(A) = \lambda(B)$ for any Gale dual $B$ of $A$.

Since two Gale duals of a given homogeneous configuration $A$ differ by the action of an element in $GL(m, \mathbb{K})$, it is clear that $\lambda(A)$ does not depend on the choice of a Gale dual. Note that if $A$ is homogeneous and $B$ is a Gale dual of $A$ then $s(B) = 0$ and therefore $\lambda(A) \leq m(A) - 1$.

Before stating our main result in this section Theorem 3.8 we collect some preliminary results.

**Lemma 3.2.** Let $A = \{a_1, \ldots, a_n\}$ be a homogeneous, non-pyramidal configuration over $\mathbb{K}$ and suppose that $Z \subset A$ is a circuit. Set $A_1 = (L(Z) \cap A) \setminus Z$ and $A_2 = A \setminus (Z \cup A_1)$. Let $C = \{b_i, a_i \in Z\}, B_1 = \{b_i, a_i \in A_1\}, B_2 = \{b_i, a_i \in A_2\}$ denote the corresponding collections in a Gale dual configuration $B = \{b_1, \ldots, b_n\}$ of $A$. Then

i) The set $B \setminus C = B_1 \cup B_2$ is a codimension-one flat in $B$.

ii) The elements in $B_1$ are linearly independent.

iii) The set $B_2$ is a flat of codimension $n(A_1) + 1$.

iv) $L(B_1 \cup B_2) = L(B_1) \oplus L(B_2)$.

**Proof.** Since $Z$ is a circuit, its Gale dual is one-dimensional. But the Gale dual of $Z$ is the projection of $C$ to the quotient $L(B)/L(B_1 \cup B_2)$. Hence, $L(B_1 \cup B_2)$ has codimension one. Moreover, since $Z$ is a circuit it is non-pyramidal and hence no element of $Z$ projects to zero on the quotient. Hence $B_1 \cup B_2$ is a flat.

Since $d(Z) = d(Z \cup A_1)$ we must have

$$\dim(L(B_1 \cup B_2)) = \dim(L(B_2)) + n(A_1).$$

Hence the elements in $B_1$ are linearly independent.

Similarly, $L(B_2)$ is a subspace of codimension equal to the dimension of the quotient $L(B)/L(B_2)$. This is the underlying space of a Gale dual of $Z \cup A_1$, and therefore has codimension $n(A_1) + 1$. Moreover since $L(Z) = L(Z \cup A_1)$ and $Z$ is not pyramidal, then $A_1$ is not a pyramid either and, consequently, $B_2$ is a flat. Item iv) follows from items ii) and iii). □

**Lemma 3.3.** Let $B = \{b_1, \ldots, b_n\}$ be a dual-homogeneous configuration over $\mathbb{K}$ and let $\lambda$ be the maximal length of a non-splitting flag in $B$. Let $F_1 \subset \cdots \subset F_\lambda \subset B$ be a non-splitting flag. Set $\tilde{B} = \Pi(B \setminus F_1)$, where $\Pi: L(B) \to L(B)/L(F_1)$ is the canonical projection. Let $\tilde{\lambda}$ denote the maximal length of a non-splitting flag in $\tilde{B}$. Then $\tilde{\lambda} = \lambda - 1$.

**Proof.** Suppose $H_1 \subset \cdots \subset H_\tilde{\lambda}$ is a maximal non-splitting flag in the dual-homogeneous configuration $\tilde{B}$. For $i = 2, \ldots, \tilde{\lambda} + 1$, let $G_i$ denote the flat of $B$ defined by
\( \{ b \in B : \Pi(b) \in H_{i-1} \} \). It follows that \( F_1 \subset G_2 \subset \cdots \subset G_{\lambda+1} \) is a non-splitting flag in \( B \). Hence \( \lambda \geq \lambda + 1 \).

On the other hand, it is clear that the projections \( \Pi(F_2) \subset \cdots \subset \Pi(F_\lambda) \) define a non-splitting flag in \( B \) and therefore \( \lambda \geq \lambda - 1 \).

We now recall for later use the statement of Lemma 22 in \cite{3}.

**Lemma 3.4.** Let \( B \) be a dual-homogeneous configuration and \( \Lambda \subset L(B) \) a line. If \( B \) has a non-splitting flag \( F \) of length \( \ell \), then \( B \) has a non-splitting flag \( G \) of length \( \ell \) such that \( \Lambda \cap L(G_i) = \{0\} \).

Next we will consider the case of maximal non-splitting flags in non-homogeneous configurations.

**Lemma 3.5.** Let \( B = \{b_1, \ldots, b_n\} \) be a configuration and suppose \( s(B) \neq \emptyset \). Let \( B^H \) denote its homogenization; i.e. the configuration \( B^H = \{-s(B), b_1, \ldots, b_n\} \). Then

\[
\lambda(B) = \lambda(B^H) + 1,
\]

where \( \lambda(B) \) and \( \lambda(B^H) \) are the maximal lengths of non-splitting flags in \( B \) and \( B^H \).

**Proof.** Let \( \lambda = \lambda(B) \) and \( \lambda^H = \lambda(B^H) \). Suppose \( \hat{F}_1 \subset \cdots \subset \hat{F}_{\lambda^H} \) is a maximal non-splitting flag in \( B^H \). By Lemma 3.4, we may assume that \( s(B) \not\in L(\hat{F}_{\lambda^H}) \) and therefore all \( \hat{F}_i \) are flats in \( B \). We symbolize this by dropping the \( \hat{\ } \) from the notation.

Suppose now that \( F_1 \subset \cdots \subset F_{\lambda^H} \) may not be extended to a longer non-splitting flag in \( B \). Let \( G_1, \ldots, G_r \) denote the flats of rank \( \lambda^H + 1 \) in \( B \) containing \( F_{\lambda^H} \). Then, for each \( G_i \) we must have \( s(G_i) \in L(F_{\lambda^H}) \). But this implies

\[
s(B) = \sum_{i=1}^r s(G_i) - (r - 1)s(F_{\lambda^H}) \in L(F_{\lambda^H}),
\]

which is a contradiction. Hence, \( \lambda(B) \geq \lambda(B^H) + 1 \).

To prove the opposite inequality, we begin by observing that given a maximal non-splitting flag \( F = \{F_1 \subset \cdots \subset F_{\lambda}\} \) in \( B \), there exists \( j = 1, \ldots, \lambda \) such that \( s(B) \in L(F_j) \). Otherwise, we could regard \( F \) as a non-splitting flag of length \( \lambda \) in \( B^H \) and, consequently, \( \lambda(B^H) \geq \lambda(B) \) which would contradict the inequality already proven. Thus, let \( j = 1, \ldots, \lambda \) be such that

\[
s(B) \in L(F_j) \setminus L(F_{j-1}).
\]

If \( j = \lambda \), then \( F_1 \subset \cdots \subset F_{\lambda-1} \) may be regarded as a non-splitting flag in \( B^H \) implying that \( \lambda^H \geq \lambda - 1 \) as desired.

Thus, it suffices to show that we can always find a non-splitting flag \( G \) of length \( \lambda \) in \( B \) so that

\[
s(B) \in L(G_\lambda) \setminus L(G_{\lambda-1}).
\]

Suppose then that \( s(B) \in L(F_j) \setminus L(F_{j-1}) \) and \( j < \lambda \). Let \( F_j, H_1, \ldots, H_r \) denote the distinct \( B \)-flats of rank \( j \) in \( F_{j+1} \) containing \( F_{j-1} \). There must exist an index \( i = 1, \ldots, r \), such that

\[
s(H_i) \not\in L(F_{j-1})
\]
since, otherwise,
\[ s(F_{j+1}) = s(F_j) + r s(F_{j-1}) \in L(F_j) \]
which is not possible since \( F \) is non-splitting.

We next claim that the index \( i = 1, \ldots, r \) may be chosen so that (3.6) is satisfied
and \( s(F_{j+1}) \not\in L(H_i) \). Indeed, suppose there exists a unique flat \( H_i \) satisfying (3.6).
Then
\[ s(F_{j+1}) \equiv s(F_j) + r s(H_i) \mod L(F_j) \]
but then \( s(F_{j+1}) \not\in L(H_i) \) since \( s(F_j) \) and \( s(H_i) \) are linearly independent modulo \( L(F_j-1) \).

Thus, we can construct a new non-splitting flag \( G \) replacing \( F_j \) by \( H_i \) so that
\( s(B) \in L(G_{j+1}) - L(G_j) \). Continuing inductively we construct a flag satifying (3.5). \( \square \)

The following simple example illustrates these invariants in the non-homogeneous case.

**Example 3.6.** Let \( A \) be the non-homogeneous configuration
\[ A = [1 \ 2 \ 3] \subset \mathbb{Q}^{1 \times 3}. \]
Then a Gale dual \( B \subset \mathbb{Q}^2 \) and its homogenization \( B^H \) are given by the rows of the matrices:
\[ B = \begin{pmatrix} -2 & 0 \\ 1 & -3 \\ 0 & 2 \end{pmatrix}; \quad B^H = \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 1 & -3 \\ 0 & 2 \end{pmatrix}. \]
Clearly, \( \lambda(B) = 2 \) and \( \lambda(B^H) = 1 \). Note that a Gale dual of \( \tilde{A} \) is the one-dimensional configuration \( \tilde{B} = \{1, -2, 1\} \subset \mathbb{Q} \) and \( \lambda(\tilde{B}) = 0 \).

Finally, we need a result, somewhat analogous to the previous Lemma, about the behavior of the \( \rho \)-invariant under homogenization.

**Lemma 3.7.** Let \( A \) be a non-pyramidal, non-homogeneous configuration with cardinality \( n \) and let \( A^h \) be as in (2.2). Suppose \( \sigma = \{\sigma_1 < \cdots < \sigma_{m+1}\} \) is a maximal chain of supports of elements in \( \text{ker}_\mathbb{K}(A) \). Let \( \Sigma \) be the matrix whose rows are the elements \( \sigma_1, \ldots, \sigma_{m+1} \). Then
\[ \rho(A^h) \geq \text{rank} \left( \begin{array}{c} A \\ \Sigma \end{array} \right). \]

**Proof.** For each \( j = 1, \ldots, m + 1 \), let \( v_j \in \text{ker}_\mathbb{K}(A) \) be such that \( \text{supp}(v_j) = \sigma_j \). Then the vector \( v_j^h = (-\sum_i v^j_i, v_j) \in \text{ker}_\mathbb{K}(A^h) \), where \( v^j_i \) are the components of \( v_j \).
As \( A \) is non-homogeneous, there is a minimal index \( j \) for which \( \sigma(v_j) \neq 0 \). After adding a multiple of \( v_j \) to any \( v_k \) with \( k > j \), if necessary, we can assume that the supports \( \sigma_j^h = \text{supp}(v_j^h), j = 1, \ldots, m + 1 = m(A^h) \) define a maximal chain \( \sigma^h \) of \( A^h \)-supports.
Note that $A^h$ is not a pyramid because $A$ is not a pyramid and it is non-homogeneous. Denote by $\Sigma^h$ the matrix with rows the elements of $\sigma^h$. Then,

$$
\begin{pmatrix}
A^h \\
\Sigma^h
\end{pmatrix} =
\begin{pmatrix}
1 & 1_n \\
0 & A \\
* & \Sigma
\end{pmatrix}.
$$

Hence

$$
\rho(A^h) \geq \text{rank} \left( \begin{pmatrix} A^h \\ \Sigma^h \end{pmatrix} \right) \geq \text{rank} \left( \begin{pmatrix} 1_n \\ A \\ \Sigma \end{pmatrix} \right) = \text{rank} \left( \begin{pmatrix} A \\ \Sigma \end{pmatrix} \right),
$$

where the last equality follows from the fact that $\sigma^h_{m(A^h)} = 1_{n+1}$ is the all 1-vector since $A^h$ is homogeneous.

We are now ready to prove equality 1.3.

**Theorem 3.8.** Let $A \subset \mathbb{K}^{d+1}$ be a homogeneous, non-pyramidal, configuration with cardinality $n$ and affine dimension $d$. Let $m = n - d - 1$ and let $\rho(A)$ be as in (2.7). Then, as asserted by (1.3)

$$
n - 1 - \rho(A) = m - 1 - \lambda(A).
$$

**Proof.** We proceed by induction on $m$. If $m = 1$ then $\lambda(A) = 0$ and $m - 1 - \lambda(A) = 0$. On the other hand, since $A$ is a circuit, there are no proper supports in $S(A)$ and therefore $\rho(A) = n - 1$. Thus, (1.3) holds.

We assume that the equality holds for configurations of codimension less than $m$ and begin by showing that

$$
(3.7) \quad n - \rho(A) \geq m - \lambda(A).
$$

Let $\sigma = \{\sigma_1 < \cdots < \sigma_m\}$ be a maximal chain of $A$-supports such that $\rho(A) = \text{rank}(M_{\sigma^h}(A))$. We may assume without loss of generality that

$$
\sigma_1 = \left\{1, \ldots, 1, 0, \ldots, 0\right\}.
$$

Then, since $\sigma_1$ is a minimal support in $\ker_{\mathbb{K}}(A)$, $Z = \{a_1, \ldots, a_s\}$ is a (homogeneous) circuit in $A$. In particular, $d(Z) = s - 2$ and, after affine transformation, we may assume that $L(Z)$ is the subspace of $\mathbb{K}^{d+1}$ spanned by $e_1, \ldots, e_{s-1}$.

Let $A_1 = (A \cap L(Z)) \setminus Z$ and assume that (if non-empty) $A_1 = \{a_{s+1}, \ldots, a_{s+k}\}$. Thus, we may write

$$
A = \begin{pmatrix}
Z & A_1 \\
0 & A_2
\end{pmatrix},
$$

where no column of $A_2$ is zero. By Lemma 3.2 we have that the vectors $\{b_{s+1}, \ldots, b_n\}$ define a codimension-one flat in $B$, while the vectors $\{b_{s+1}, \ldots, b_{s+k}\}$ are linearly independent. Moreover, $L(\{b_{s+1}, \ldots, b_n\}) = L(\{b_{s+1}, \ldots, b_{s+k}\}) \oplus L(\{b_{s+k+1}, \ldots, b_n\})$.

Thus, $B$ can be assumed to be of the form

$$
B = \begin{pmatrix}
v & * & * \\
0 & E_k & 0 \\
0 & 0 & B_2
\end{pmatrix},
$$

where

$$
(\frac{1}{v} - 1_n) E_k = \begin{pmatrix}
1 & 1_n \\
0 & A \\
* & \Sigma
\end{pmatrix} - \begin{pmatrix}
1 & 1_n \\
0 & A \\
* & \Sigma
\end{pmatrix},
$$

where the last equality follows from the fact that $\sigma^h_{m(A^h)} = 1_{n+1}$ is the all 1-vector since $A^h$ is homogeneous.
where $v$ is a column vector of length $s$ with support $\sigma_1$ (a generator of $\ker K(Z)$), $E_k$ is a $k \times k$ identity matrix, and the columns of $B_2$ are a basis of $\ker K(A_2)$. Note that since $A$ is non-pyramidal, no row of $B_2$ can be zero and that the last $n - s - k$ coordinates of any vector in $\ker K(A)$ is a vector in $\ker K(A_2)$.

Now, for any $j \in \{1, \ldots, k\}$, the element $a_{s+j}$ is in the span of $Z$, and therefore there exists a vector $v_j \in \ker K(A)$ with support $\{1, \ldots, s\} \cup \{s + j\}$. Hence any maximal chain $\sigma$ should satisfy that there exists an index $\ell(j)$ such that the difference of the supports $\supp(\sigma_{\ell(j)}) \setminus \supp(\sigma_{\ell(j)-1})$ equals $\{s + j\}$. Otherwise, we would be able to add another support to the chain, contradicting the maximality.

Then, $M_\sigma(A)$ is then row equivalent to a matrix of the form

$$
\begin{pmatrix}
Z & A_1 & * \\
1_s & 0 & 0 \\
0 & E_k & 0 \\
0 & 0 & A_2 \\
0 & 0 & \tilde{\Sigma}
\end{pmatrix},
$$

where as before $1_s$ is a row vector of length $s$ whose entries are all 1, and $\tilde{\Sigma}$ is a matrix whose rows define a maximal chain of $A_2$-supports. Therefore, since $Z$ is homogeneous we have that

$$
(3.8) \quad \rho(A) = \text{rank}(M_\sigma(A)) = (s - 1) + k + \text{rank} \left( \begin{array}{c}
A_2 \\
\tilde{\Sigma}
\end{array} \right).
$$

We now consider two cases. Assume first that $A_2$ is homogeneous and so the rows of $B_2$ form a dual-homogeneous configuration which is a Gale dual of $A_2$. Then,

$$
\text{rank} \left( \begin{array}{c}
A_2 \\
\tilde{\Sigma}
\end{array} \right) \leq \rho(A_2),
$$

and consequently

$$
\rho(A) \leq (s - 1) + k + \rho(A_2).
$$

We have $n(A_2) = n - s - k$, $m(A_2) = m - 1 - k$, and, by inductive hypothesis,

$$
n(A_2) - \rho(A_2) = m(A_2) - \lambda(A_2).
$$

Hence

$$
n - \rho(A) \geq n - s - k + 1 - \rho(A_2) = n(A_2) - \rho(A_2) + 1 = m(A_2) - \lambda(A_2) + 1 = m - (\lambda(A_2) + k) \geq m - \lambda(A),
$$

where the inequality $k + \lambda(A_2) \leq \lambda(A)$ follows from the fact that every non-splitting flag of length $\ell$ in $B_2$ may be extended to a non-splitting flag of length $\ell + k$ by adjoining the $k$ linearly independent elements of $B_1$ one at a time.

Suppose now that $s(B_2) \neq 0$. Let $B_2^H$ be the homogenization of $B_2$. Then, $B_2^H$ is a Gale dual of $A_2^h$ and

$$
m(A_2^h) = m(A_2), \quad n(A_2^h) = n(A_2) + 1 = n - s - k + 1.
$$
It follows from Lemma 3.7 that

\[ \rho(A_2^h) \geq \text{rank} \left( \begin{array}{c} A_2 \\ \Sigma \end{array} \right) \]

and therefore we may argue as in the previous case and have:

\[
\begin{align*}
n - \rho(A) & \geq n - s - k + 1 - \rho(A_2^h) = n(A_2^h) - \rho(A_2^h) \\
& = m(A_2^h) - \lambda(A_2^h) = (m - 1 - k) - \lambda(A_2) \\
& = m - (\lambda(A_2) + k + 1) \geq m - (\lambda(A_2) + k) \\
& \geq m - \lambda(A),
\end{align*}
\]

where the next to last inequality follows from Lemma 3.5 and the last inequality follows, as above, from the fact that every non-splitting flag of length \( \ell \) in \( B_2 \) may be extended to a non-splitting flag of length \( \ell + k \) by adjoining the \( k \) linearly independent elements of \( B_1 \) one at a time. This completes the proof of (3.7).

To complete the proof of Theorem 3.8 we now show that

\[ (3.9) \quad n - \rho(A) \leq m - \lambda(A). \]

Set \( \lambda = \lambda(A) \) and suppose that \( F_1 \subset \cdots \subset F_\lambda \) is a maximal non-splitting flag in \( B \). Let \( \tilde{B} \) denote the projection of \( B \setminus F_1 \) to the quotient \( L(B)/L(F_1) \). Then \( \tilde{B} \) may be seen as the Gale dual of the (homogeneous) configuration \( A_1 = \{ a_i \in A : b_i \notin F_1 \} \).

We assume without loss of generality that \( |F_1| = s \), and that \( F_1 \) consists of the last \( s \) vectors in \( B \). So, \( A_1 \) consists of the first \( n - s \) vectors in \( A \). As \( m(A_1) = m - 1 \) and \( n(A_1) = n - s \), we have that \( d(A_1) + 1 = n - s - m + 1 \). We can assume that \( A \) is of the form

\[
A = \left( \begin{array}{cc} A_1' & * \\ 0 & Z \end{array} \right),
\]

where \( Z \in \mathbb{K}^{(s-1)\times s} \).

From Lemma 3.3 we have that \( \lambda(\tilde{B}) = \lambda(B) - 1 \). Thus, by inductive assumption we have that \( n - s - \rho(A_1') = (m - 1) - \lambda(\tilde{B}_1) = m - \lambda(A) \), and therefore

\[ (3.10) \quad \rho(A_1') + s = n - m + \lambda(A). \]

Suppose now that \( \tilde{\sigma}_1 \prec \cdots \prec \tilde{\sigma}_{m-1} \) is a maximal chain of \( \ker K(A'_1) \)-supports computing \( \rho(A'_1) \). Set

\[
\sigma_j = (\tilde{\sigma}_j, 0, \ldots, 0); \quad j = 1, \ldots, m - 1,
\]

and \( \sigma_m = (1, \ldots, 1) = 1_n \). Then, \( \sigma_1 \prec \cdots \prec \sigma_m \) is a maximal chain of supports for \( \ker K(A) \) and we have \( \rho(A) \geq \text{rank}(M_A) \). But \( M_A(A) \) is row-equivalent to the matrix

\[
\left( \begin{array}{cc} A_1' & * \\ \Sigma & 0 \\ 0 & Z \end{array} \right)
\]
whose rank is $\rho(A'_1) + s$ since $F_1$ is a non-splitting flat which implies that $Z$ is not homogeneous; i.e. $1_s$ is not in the rowspan of $Z$. Hence, we deduce from (3.10) the inequality

$$\rho(A) \geq \rho(A'_1) + s = n - m + \lambda(A),$$

as wanted. \hfill \square

3.1. Pyramids. Let us consider the invariants defined in the previous sections in the pyramidal case.

**Definition 3.9.** Let $A$ be a configuration over a field $\mathbb{K}$ of characteristic zero. We say that $A$ is a pyramid if all points in $A$ but one lie on an (affine) hyperplane. When $A$ is not a pyramid, we define $p(A) = 0$. If $A$ is a pyramid, let $a \in A$ such that the points in $A \setminus \{a\}$ lie in a hyperplane not containing $a$. Then, $p(A)$ is defined, inductively, as $p(A \setminus \{a\}) + 1$. We will call $p(A)$ the pyramidal index of $A$.

Alternatively, we may describe the quantity $p(\bar{A}) = p(A)$ as the number of indices $i$ with $b_i = 0$ in a Gale dual configuration $B$ of $\bar{A}$.

Assume $A$ is a pyramid, that is $p(A) \geq 1$. The matrix $A$ has the following shape, after left multiplication by an invertible matrix and probably renumbering the elements of $A$:

$$\begin{pmatrix} E_{p(A)} & 0 \\ 0 & A' \end{pmatrix},$$

where $E_{p(A)}$ is the $p(A) \times p(A)$ identity matrix and $p(A') = 0$. Note that $A$ is homogeneous if and only if $A'$ is homogeneous.

The following summarizes the behavior of the invariants studied above in the pyramidal case. The proofs are straightforward and are left to the reader:

**Theorem 3.10.** Let $A$ be a configuration over $\mathbb{K}$ with pyramidal index $p(A)$. Let $A'$ be as in (3.11). Then

i) $n(A) = n(A') + p(A);$

ii) $d(A) = d(A') + p(A);$

iii) $m(A) = m(A');$

iv) $\iota(A) = \iota(A');$

v) $\rho(A) = \rho(A') + p(A);$

vi) $\lambda(A) = \lambda(A').$

It is then clear that both (1.2) and (1.3) fail if $p(A) > 0$. On the other hand, in the geometric case, we have that

$$\text{def}(X_A) = \text{def}(X'_A) + p(A).$$

Thus we see that only the quantity

$$d(A) - \iota(A)$$

computes the geometric defect; while the $m(A) - 1 - \lambda(A)$ and $n(A) - 1 - \rho(A)$ invariants only give the right answer if $A$ is non-pyramidal.
4. Defect and Cayley Decompositions

Furukawa and Ito introduced in [10] an alternative method for computing the dual defect of a projective toric variety $X_A$ over $\mathbb{C}$ in terms of Cayley decompositions of the configuration $A$. In this section, we translate their results to the Gale dual setting and describe some applications. We begin with a brief description of Furukawa and Ito’s approach, referring to [10] for details and proofs. Our translation to the Gale dual setting starts with Proposition 4.7 in § 4.1 and yields Theorem 4.16, which is a vast improvement over [3, Theorem 25]. This makes it possible to give a detailed description of dual defective toric varieties. We conclude this section by sketching some of its consequences in § 4.2.

Since in this section we are interested in the geometric setting, we will assume from the start that $A \subset \mathbb{Z}^{d+1}$ is a homogeneous configuration, which arises as $A = A'$ with $A' \subset \mathbb{Z}^d$ as in (2.1). Following [10], we denote by $\langle A' - A' \rangle$ the affine span of $A'$ over $\mathbb{Z}$ and assume that $\langle A' - A' \rangle = \mathbb{Z}^d$ and, consequently, $d(A) = d$. As before, let $n = n(A)$ denote the cardinality of $A$ and $m = m(A) = n - d - 1$ the rank of a Gale dual $B$ of $A$.

**Definition 4.1.** We say that a homogeneous integer configuration $A$ has a Cayley decomposition of length $r$ if $A$ is equivalent, up to a $\mathbb{Z}$-linear isomorphism, to a configuration of the form:

$$
\{e_0\} \times A_0 \cup \{e_1\} \times A_1 \cup \ldots \cup \{e_r\} \times A_r,
$$

where $A_0, \ldots, A_r \subset \mathbb{Z}^{d-r}$ are non-empty finite configurations and $e_0, e_1, \ldots, e_r$ is the canonical basis in $\mathbb{Z}^{r+1}$. We write

$$
A = A_0 \ast \cdots \ast A_r.
$$

Note that we may view any homogeneous configuration $A$ as admitting a Cayley decomposition of length 0.

If $A = A_0 \ast \cdots \ast A_r$ then, up to left multiplying by an invertible integer matrix and permuting the columns, we can write the associated matrix $A$ in the form:

$$
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
A_0 & A_1 & \cdots & A_r
\end{pmatrix},
$$

where $1$ (resp. $0$) is used to denote row vectors (of appropriate lengths) with all entries equal to 1 (resp. 0).

The first link between dual defect and Cayley decompositions appears in the work of Di Rocco [6], where it is shown that if $X_A$ is a defective smooth toric variety, then $A$ admits a Cayley decomposition of length $r = (d + n)/2$ and such that the convex hull of the $A_i$’s have the same normal fan. Similar results were obtained by Casagrande and Di Rocco [2] for normal $\mathbb{Q}$-factorial varieties. More generally, it is shown in [3, 7] that every dual defective toric variety admits a Cayley decomposition of positive length. This result was further extended by Ito [12], who proved that...
every dual defective toric variety admits a Cayley decomposition of length at least $\text{def}(X_A)$.

Before we can state the main result of [10] computing the dual defect in terms of Cayley decompositions, we need to introduce a further invariant of a Cayley decomposition.

**Definition 4.2.** We say that a Cayley decomposition as above is a *join* if the map $(m_0,\ldots,m_r) \mapsto m_0 + \ldots m_r$ defines an isomorphism

$$\langle A_0 - A_0 \rangle \oplus \cdots \oplus \langle A_r - A_r \rangle \to \mathbb{Z}^{d-r}.$$ 

The following is a restatement of Theorem 1.3 and Corollary 1.5 in [10]:

**Theorem 4.3 (Furukawa-Ito).** Let $A = A' \subset \mathbb{Z}^{d+1}$ be a homogeneous configuration such that $\langle A' - A' \rangle = \mathbb{Z}^d$. Then, there exist

- An integer $r$, $0 \leq r \leq d$, and a Cayley decomposition $A = A_0 * \cdots * A_r$,
- An integer $c$, $0 \leq c \leq d-r$, and a subgroup $S \subset \mathbb{Z}^{d+1}$ with $\text{rank}(S) = c$,

such that if $\pi: \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1}/S \simeq \mathbb{Z}^{d+1-c}$ is the natural projection, then

i) $\pi(A) = \pi(A_0) * \cdots * \pi(A_r)$ is a join (with $d(\pi(A)) = d - c$) and

ii) $\text{def}(X_A) = r - c$.

This Cayley decomposition is unique in the following sense: given any other Cayley decomposition $A = A'_0 * \cdots * A'_r$ and subgroup $S'$ satisfying i) and ii) above, we have $S \subset S'$ and for each $j = 0,\ldots,r$ we have

$$A_j = A'_{i_1} * \cdots * A'_{i_s}$$

for some $A'_{i_k}$. In particular, $r \leq r'$ and $c \leq c'$.

It is convenient to introduce the affine invariant $\theta(A)$.

**Definition 4.4.** Given a Cayley decomposition $A = A_0 * A_1 * \cdots * A_r$ and a subgroup $S$ of rank $c$ verifying item i) in Theorem 4.3, we call $\theta(A)$ the maximum value of the differences $r - c$ as we run over all Cayley decompositions and subgroups.

We say that a given Cayley decomposition and subgroup is a $\theta$-decomposition of $A$ if $r - c = \theta(A)$. Furthermore, we say that it is an FL-decomposition of $A$ if it is a $\theta$-decomposition and its length $r$ is the smallest possible for a $\theta$-decomposition.

Note that there might be different Cayley decompositions of a configuration $A$ which are incomparable (see e.g. the prism configuration in Example 4.9). Theorem 4.3 asserts that given an integer configuration $A$ as above, there exist a unique FI-decomposition with $r - c = \theta(A)$ with $r$ minimal among all $\theta$-decompositions of $A$.

Moreover, in the geometric case, we have that $\text{def}(X_A) = \theta(A)$. On the other hand, it follows from (1.1) that if $A$ is a homogeneous, non-pyramidal, configuration $A = A' \subset \mathbb{Z}^{d+1}$ satisfying $\langle A' - A' \rangle = \mathbb{Z}^d$, then $\text{def}(X_A) = n(A) - 1 - \rho(A)$. Hence, together with Theorems 2.7 and 3.8 we have that for such a configuration $A$:

$$(4.4) \quad \theta(A) = m(A) - 1 - \lambda(A) = n(A) - 1 - \rho(A) = d(A) - \nu(A).$$

We will illustrate all these invariants in Example 4.10 below.
By abuse of notation we will refer to $\theta(A)$ as the defect of the configuration $A$ of a Gale dual $B$. As every homogeneous configuration $A$ may be viewed as defining a Cayley decomposition with $r = 0$ and conditions i) and ii) in Theorem 4.3 are satisfied taking $S = \{0\}$, we have that $\theta(A) \geq 0$ for all $A$. If $\theta(A) > 0$ we will say that $A$ (or $B$) is defective.

Remark 3. Since all the above invariants are defined purely in terms of the configuration $A$ it should be possible to prove the leftmost equality in (4.4) with purely linear algebraic arguments, as we did for the other equalities in Sections 2 and 3. However we have only been able to show, with such direct methods, that $\theta(A) \leq m(A) - 1 - \lambda(A)$ (cf. Proposition 4.15). We point out, however, that the following example shows that (4.4) does not necessarily hold for configurations defined over fields of positive characteristic.

**Example 4.5.** Let $\mathbb{K} = \mathbb{F}_2$ be the field with two elements and consider the following three-dimensional configuration $A \subset \mathbb{K}^7$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$  

A Gale dual $B = \{b_1, \ldots, b_7\}$ of $A$ over $\mathbb{K}$ consists of all non-zero vectors in $\mathbb{K}^3$, suitably ordered. All rank one flats in $B$ are non-splitting but, since $\text{char}(\mathbb{K}) = 2$, all rank two flats are splitting. Thus, over $\mathbb{K}$, $\lambda(A) = 1$ and $m(A) - 1 - \lambda(A) = 1$. Indeed, the associated matroid is known as the Fano matroid and it contains exactly 7 lines containing 3 vectors adding-up to 0. Each one of these lines $L$ gives rise to a Cayley decomposition $B = B_0 \cup B_1$ where $B_0$ has cardinality 3 (and $L(B_0) = L$). Via a linear change of coordinates, we can assume that $B_0 = \{b_1, b_2, b_3\}$ and $B_1 = \{b_4, b_5, b_6, b_7\}$, which gives the corresponding Cayley decomposition of $A$ which is visible in (4.5), where $r = 1$, $c = 2$ and $r - c = -1$. Thus, the FI-decomposition is just $A$ itself, with $r = 0$, $c = 0$, $r - c = 0 = \theta(A) < m(a) - 1 - \lambda(A)$, and the leftmost equality in (4.4) does not hold.

On the other hand, if we consider the same matrix $A$ but over $\mathbb{Z}$ (or over any field with characteristic different from 2), a Gale dual of $A$ is given by the following vectors in $\mathbb{Z}^3$:

$$\{-e_2 - e_3, -e_1 - e_3, e_1 + e_2 + 2e_3, -e_1 - e_2 - e_3, e_1, e_2, e_3\}.$$  

It is then clear that $\{e_1\} \subset \{e_1, e_2\}$ defines a non-splitting flag of length 2. It is still true that the FI-decomposition is again given by $A$ itself, with $r = 0$, $c = 0$ and $r - c = \theta(A) = m(a) - 1 - \lambda(A) = 3 - 1 - 2 = 0$. Hence, $X_A$ is non-defective over $\mathbb{C}$.

### 4.1. The Gale dual setting

We will now translate these concepts to the Gale dual setting for an arbitrary homogeneous configuration.

Let $A \subset \mathbb{K}^{d+1}$ be a non-pyramidal homogeneous configuration such that $d(A) = d$. Suppose $A = A_0 \ast \cdots \ast A_r$ and a $\mathbb{K}$-linear subspace $S$ of $\mathbb{K}^{d+1}$ of dimension $c$ define a decomposition satisfying i) in Theorem 4.3 (with $\mathbb{Z}$ replaced by $\mathbb{K}$). In particular,
up to left multiplication by an invertible matrix and permutation of columns, the associated matrix $A$ may be written as

$$(4.6) \quad A = \begin{pmatrix} A'_0 & 0 & \cdots & 0 \\ 0 & A'_1 & \cdots & 0 \\ 0 & 0 & \cdots & A'_r \\ C_0 & C_1 & \cdots & C_r \end{pmatrix},$$

where the $A'_i$ are homogeneous configurations in $K^{d_i + 1}$ with $d(A'_i) = d_i$, $\sum_{i=0}^r d_i = d + 1 - c$, and the lower matrix $C = (C_0C_1 \ldots C_r) \in K^{c \times n}$ has rank $c$.

The following result will be needed below:

**Lemma 4.6.** With notation and hypotheses as above, assume moreover that $S$ is minimal in the sense that we cannot find a subspace $S'$ properly contained in $S$ so that i) in Theorem 4.3 remains valid. Then, for any index $j$, $j = 0, \ldots, r$, we have that

$$(4.7) \quad \text{rank} \begin{pmatrix} C_0 & \cdots & C_{j-1} & C_{j+1} & \cdots & C_r \end{pmatrix} = c.$$ 

**Proof.** Assume that $j = 0$ and that the matrix $C' = (C_1 \ldots C_r)$ has rank $c' < c$. We may identify $S$ with the span of the last $c$ vectors in the chosen basis of $K^{d+1}$. Then, the span $S'$ of the columns of $C'$ has rank $c' < c$, and modding out by $S'$ results in a configuration of join type of the form

$$(4.8) \quad \begin{pmatrix} A'_0 & 0 & \cdots & 0 \\ 0 & A'_1 & \cdots & 0 \\ 0 & 0 & \cdots & A'_r \\ C'_0 & 0 & \cdots & 0 \end{pmatrix},$$

with $C'_0$ of rank $c - c' > 0$. This contradicts the minimality of $S$. \qed

Given a Gale dual of a configuration $B \subset K^m$ of $A = A_0 \ast \cdots \ast A_r$, the elements $b \in B$, corresponding to the $A_j$-columns of $A$

$$(4.9) \quad B_j = \{b_i \in B : a_i \in \{e_j\} \times A_j\},$$

define a dual-homogeneous subconfiguration $B_j$ and we get a decomposition:

$$(4.10) \quad B = B_0 \cup B_1 \cup \cdots B_r.$$ 

Conversely, any decomposition of a Gale dual $B$ of $A$ into dual-homogeneous subconfigurations as in (4.10) defines a corresponding Cayley decomposition $A = A_0 \ast A_1 \ast \cdots \ast A_r$ of length $r$. Moreover, $A$ is equivalent, possible after renumbering, to a configuration of the form (4.6), where the row span of $C$ gives the affine relations among the elements in $B$ which involve elements from at least two distinct $B_j$.

This may be described in more intrinsic terms as follows. Assume $A = A_0 \ast \cdots \ast A_r$ and $S \subset K^{d+1}$ is a subspace of dimension $c$ which define a decomposition of $A$ satisfying i) of Theorem 4.3 (replacing $\mathbb{Z}$ with $K$) with $S$ minimal. Let $\pi : \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1}/S$. Consider the short exact sequences as in (3.1) and the commutative
diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & W & \overset{i}{\rightarrow} & \mathbb{K}^n & \overset{\alpha}{\rightarrow} & \mathbb{K}^{d+1} & \rightarrow & 0 \\
\downarrow{\rho} & \quad & \downarrow{\text{id}} & \quad & \downarrow{\pi} & \quad & \downarrow{\text{id}} & \quad & \downarrow{\rho}
\end{array}
\]

where \( W = \ker(K(A)) \) and \( \hat{W} = \ker(K\pi(A)) \). Dualizing, we get

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (\mathbb{K}^{d+1}/S)^* & \overset{\overline{\alpha}^*}{\rightarrow} & (\mathbb{K}^n)^* & \overset{i^*}{\rightarrow} & \hat{W}^* & \rightarrow & 0 \\
\downarrow{\pi^*} & \quad & \downarrow{\text{id}^*} & \quad & \downarrow{\rho^*} & \quad & \downarrow{\text{id}^*} & \quad & \downarrow{\rho^*}
\end{array}
\]

The map \( \rho^* \) is surjective and its kernel is isomorphic to the row span of the matrix \( C \) in (4.6) via the map:

\[
(u_1, \ldots, u_n) \in \text{rowspan}(C) \mapsto \sum_{i=1}^{n} u_i i^*(\xi_i),
\]

where as before \( \xi_1, \ldots, \xi_n \) denotes the dual of the standard basis of \( \mathbb{K}^n \).

Since \( \pi(A) \) is a join, a Gale dual \( \hat{B} \) of \( \pi(A) \) may be written as \( \hat{B} = \hat{B}_0 \cup \cdots \cup \hat{B}_r \), where \( \hat{B}_j \) is a Gale dual of the configuration \( A'_j \) depicted in (4.8) and \( L(\hat{B}) = L(\hat{B}_0) \oplus \cdots \oplus L(\hat{B}_r) \). In particular, since \( \dim L(\hat{B}) = \dim L(B) + c \), we have

\[
c = \sum_{j=0}^{r} \dim(L(\hat{B}_j)) - \dim L(B).
\]

The following result is now a consequence of (4.13) together with Lemma 4.6

**Proposition 4.7.** Suppose \( A \) is a non-pyramidal homogeneous Cayley configuration \( A = A_0 \ast \cdots \ast A_r \), \( B \) is a Gale dual of \( A \), and \( B = B_0 \cup \cdots \cup B_r \) is the corresponding Cayley decomposition as in (4.9). Let \( c \) be the maximal dimension of a subspace \( S \subset \mathbb{K}^{d+1} \) such that under the projection to the quotient by \( S \) the image \( \pi(A) = \pi(A_0) \ast \cdots \ast \pi(A_r) \) is a join. Then, for each \( j = 0, \ldots, r \), \( B_j \) is dual-homogeneous and

\[
c = \sum_{j=0}^{r} \dim(L(B_j)) - \dim L(B).
\]

**Proof.** Because of (4.13) we only need to show that \( \dim L(B_j) = \dim L(\hat{B}_j) \) for \( j = 0, \ldots, r \). But this follows from the fact that for each \( j = 0, \ldots, r \), \( \rho^*(\hat{B}_j) = B_j \) and the map

\[
\rho^*: L(\hat{B}_j) \rightarrow L(B)
\]

is injective. In fact, take \( j = 0 \), and assume that \( L(\hat{B}_0) \) is spanned by \( \hat{\beta}(\xi_1), \ldots, \hat{\beta}(\xi_{n_0}) \). Suppose there exists an element \( u_1 i^*(\xi_1) + \cdots + u_{n_0} i^*(\xi_{n_0}) \in \ker(\rho^*) \), where \( n_0 = n(A_0) \). Then \( (u_1, \ldots, u_{n_0}, 0, \ldots, 0) \in \text{rowspan}(C) \), implying that \( \text{rank}(C_1, \ldots, C_r) < c \).
and thus contradicting Lemma 4.6.

The following examples illustrate the constructions above.

**Example 4.8.** Consider the configuration

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]

A Gale dual of \(A\) is given by the configuration \(B \subset \mathbb{Q}^3\):

\[
B = \{b_1, \ldots, b_6\} = \{e_1, e_2, -e_1 - e_2, e_3, e_1 + e_2, -e_1 - 2e_3\}.
\]

\(B\) has only two proper dual-homogeneous subconfigurations: \(B_0 = \{b_1, b_2, b_3\}\) and \(B_1 = \{b_4, b_5, b_6\}\). Hence the only possible Cayley decompositions of \(B\) are \(B_0\) and \(B_0 \cup B_1\). Since \(c(B_0, B_1) = 1\), both Cayley decompositions are \(\theta\)-decompositions and \(B\) itself is the FI-decomposition. Note that the matrix \(A\) is in the form \([4.6]\) and the bottom row gives the relation:

\[
b_1 + b_4 - b_5 = 0.
\]

**Example 4.9.** We return to Example 2.2. For the case of the octahedron \(A = \overline{A}\), where \(A'\) is described in \([2.5]\), a Gale dual \(B\) is given by

\[
B = \{b_1, \ldots, b_6\} = \{e_1, e_1, e_2, -e_1 - e_2, -e_1 - e_2\}.
\]

The only proper dual-homogeneous subconfigurations correspond to the choice of one of each of \(\{e_1, e_2, -e_1 - e_2\}\). Thus we can obtain decompositions \(B = B_0 \cup B_1\) with \(r = 1\) and \(c = 2\). The FI-decomposition is then \(B\) itself and \(B\) is non-defective.

On the other hand in the case of the prism \(A = \overline{A}\) with \(A'\) is described in \([2.6]\), a Gale dual \(B\) is given by the configuration \(B \subset \mathbb{Q}^2\):

\[
B = \{b_1, \ldots, b_6\} = \{e_1, e_2, -e_1 - e_2, -e_1 - e_2, e_1 + e_2\}.
\]

There are eight proper dual-homogeneous subconfigurations of \(B\):

\[
B_0 = \{b_1, b_2, b_3\}; \quad B_1 = \{b_4, b_5, b_6\}; \quad C_0 = \{b_1, b_4\}; \quad C_1 = \{b_2, b_5\}; \quad C_0 = \{b_3, b_6\},
\]

together with \(D_{ij} = C_i \cup C_j\) for \(i \neq j\). Thus, the possible Cayley decompositions of \(B\) are

\[
B, \quad B_0 \cup B_1, \quad C_0 \cup C_1 \cup C_2, \quad D_{ij} \cup C_k,
\]

for distinct indices \(i, j, k\). It is now easy to see that \(\theta(B) = 0\) and

\[
\theta(B_0, B_1) = 1 - 2 = -1; \quad \theta(C_0, C_1, C_2) = 2 - 1 = 1; \quad \theta(D_{ij}, C_k) = 1 - 1 = 0.
\]

Thus, \(B = C_0 \cup C_1 \cup C_2\) is the only \(\theta\)-decomposition and therefore it is the FI-decomposition. We have \(\theta(B) = 1\) and, since all rank-one flats are splitting, \(\lambda(B) = 0\). We then have that \(\theta(B) = m(B) - 1 - \lambda(B) = 1\).

We next consider Example 5.8 in [10]:

**Example 4.10.** The configuration \(A = A_0 * A_1 * A_2 * A_3 \subset \mathbb{Z}^6\) is the Cayley configuration of length 3 associated with the lattice configurations in \(\mathbb{Z}^2\)

\[
A_0 = \{0, (1,0), (2,0)\}, \quad A_1 = \{0, (0,1), (0,2)\}, \quad A_3 = A_4 = \{0, (1,0), (0,1), (1,1)\}.
\]
We will use the Gale dual formulation to show that this Cayley decomposition is the FI-decomposition of \( A \). The configuration

\[(4.15) \quad B = B_0 \cup B_1 \cup B_2 \cup B_3,\]

where, in terms of the canonical basis \( \{e_1, \ldots, e_8\} \) of \( \mathbb{Q}^8 \):

\[
B_0 = \{e_1 - e_8, -2e_1 + e_8, e_1\}; \quad B_1 = \{e_2 - e_7 - e_8, -2e_2 + e_7 + e_8, e_2\};
\]

\[
B_2 = \{e_3 + e_5 + e_6 + e_8, -e_3 - e_5, -e_3 - e_6, e_3 - e_8\};
\]

\[
B_3 = \{e_4 - e_5 - e_6 + e_7, -e_4 + e_5, -e_4 + e_6 - e_7, e_4\}
\]
is a Gale dual of \( A \) and this decomposition is associated to the given Cayley decomposition of \( A \). We denote the elements of \( B \) by \( \{b_1, \ldots, b_{14}\} \) according to the listing above. We claim that the only dual-homogeneous flats in \( B \) must be unions of the \( B_i \)'s. Let \( H \) be a dual-homogeneous flat, its projection to \( L(f_1) \) must be dual-homogeneous and is contained in the projection of \( B \) which equals \( \{e_1, -2e_1, e_1, 0, \ldots, 0\} \), where the non-zero terms come from the projection of \( B_0 \). Thus, if \( H \) contains any element in \( B_0 \) then it must contain all of \( B_0 \). Similarly, considering the projections to \( L(e_2), L(\{e_3, e_5, e_6, e_8\}) \) and \( L(\{e_4, e_5, e_6, e_7\}) \) we see that for any \( i = 1, 2, 3 \), \( H \cap B_i \neq \emptyset \) implies \( B_i \subset H \). This proves our claim. Now, we may verify that for all \( i \neq j \), \( c(B_i, B_j) = 0 \) and therefore \( \theta(B_i \cup B_j) \geq 1 \). Therefore, by Corollary 4.12 if \( i \neq j \), \( B_i \cup B_j \) may not be a term in an FI-decomposition. Similarly we may check that

\[
c(B_0, B_1, B_2) = c(B_0, B_1, B_3) = 0; \quad c(B_0, B_2, B_3) = c(B_1, B_2, B_3) = 1.
\]

Hence, any union of three distinct \( B_i \) is defective and cannot be a term in an FI-decomposition. Thus, the only dual-homogeneous flats that can be part of an FI-decomposition are the \( B_i \)'s themselves. This implies that the Cayley decomposition \((4.15)\) is the (unique) FI-decomposition. Note that by Proposition 4.7 we have \( c = \sum_{j=0}^{3} \dim L(B_j) - \dim L(B) = 10 - 8 = 2 \). Thus, \( \theta(A) = 3 - 2 = 1 \). From (4.4) it follows that \( \iota(A) = 4 \), \( \lambda(A) = 6 \), and \( \rho(A) = 12 \).

The subset \( Z = \{a_1, a_2, a_4, a_5, a_7, a_8\} \) defines a 4-dimensional circuit, and thus an iterated circuit of maximal dimension. More generally we make a four-dimensional circuit by choosing any two points from \( A_0 \), any two points from \( A_1 \) and any two points from either \( A_2 \) or \( A_3 \).

We can construct a non-splitting flag of length 6 as follows: Note that

\[
L(Z) \cap A = Z \cup \{a_3, a_6, a_9, a_{10}\}
\]

and therefore it follows from (i) in Lemma 3.2—as well as by direct computation—that the corresponding elements in \( B \) are linearly independent. Thus, we can take any non-splitting flag of length 2 in \( B_3 \) such as \( \{b_{11}\} \subset \{b_{11}, b_{12}\} \), and add one by one the linearly independent elements \( b_4, b_6, b_9, b_{10} \) to produce a non-splitting flag of length 6.

For reasons of economy of space we will not write a matrix \( M_\sigma(A) \) with rank equal to \( \rho(A) = 12 \), but will instead observe that we can easily construct the chain \( \sigma \) from the description of the maximal dimensional circuit \( Z \) as in Section 2.
We will now apply the Gale dual formulation of the Furukawa-Ito construction to obtain some basic properties of $\theta$ and FI-decompositions. We begin by showing that these properties are inherited by sub-decompositions. More precisely,

**Proposition 4.11.** Let $B = B_0 \cup \cdots \cup B_r$ be a $\theta$-decomposition (resp. an FI-decomposition). Then for every subset $I = \{i_0, \ldots, i_\ell\} \subset \{0, \ldots, r\}$, the decomposition

$$B_I = B_{i_0} \cup \cdots \cup B_{i_\ell}$$

is a $\theta$ (resp. FI)-decomposition.

**Proof.** Assume that $B = B_0 \cup \cdots \cup B_r$ is a $\theta$-decomposition, $m = \dim(L(B))$, and that $I = \{0, \ldots, \ell\}$. If $B_I = B_0 \cup \cdots \cup B_\ell$ is not a $\theta$-decomposition then there exists another Cayley decomposition: $B_I = C_0 \cup \cdots \cup C_k$ such that

$$k - c(C_0, \ldots, C_k) > \ell - c(B_0, \ldots, B_\ell).$$

Consider the Cayley decomposition of length $r' = k + r - \ell$:

$$B = C_0 \cup \cdots \cup C_k \cup B_{\ell+1} \cup \cdots \cup B_r,$$

which we will just write as $B = C_0 \cup \cdots \cup B_r$ and set $c' = c(C_0, \ldots, B_r)$. Then,

$$r' - c' = r' - \sum_{j=0}^k \dim(L(C_j)) - \sum_{j=\ell+1}^r \dim(L(B_j)) + m$$

$$= (r - \ell) + (k - c(C_0, \ldots, C_k) + \dim(L(B_I))) - \sum_{j=\ell+1}^r \dim(L(B_j)) + m$$

$$> (r - \ell) + (\ell - c(B_0, \ldots, B_\ell) + \dim(L(B_I))) - \sum_{j=\ell+1}^r \dim(L(B_j)) + m$$

$$= r - \sum_{j=0}^\ell \dim(L(B_j)) - \sum_{j=\ell+1}^r \dim(L(B_j)) + m = r - c(B_0, \ldots, B_r),$$

contradicting the assumption that $B = B_0 \cup \cdots \cup B_r$ is a $\theta$-decomposition.

Finally, if we assume that $B = B_0 \cup \cdots \cup B_r$ is an FI-decomposition, then we already know that $B_I = B_0 \cup \cdots \cup B_\ell$ is a $\theta$-decomposition. If $B_I$ admitted a $\theta$-decomposition of length $k < \ell$, then this would yield a new decomposition which, by an argument completely analogous to the one above, is seen to be a $\theta$-decomposition. But this $\theta$-decomposition would have fewer than $r$ terms which is impossible. \hfill $\Box$

Applying Proposition 4.11 to the case $I = \{j\}$ we get:

**Corollary 4.12.** If $B = B_0 \cup \cdots \cup B_r$ is a $\theta$-decomposition, then for every $j = 0, \ldots, r$, $\theta(B_j) = 0$.

Moreover we may also deduce the following
From Lemma 3.3 we have that \( \lambda \) that the result holds for \( \dim L \).

Proof. If \( \dim L \geq 2 \), we may assume that \( F \). Let \( \pi \) be the natural projection and \( \tilde{B} = \pi(B \setminus F) \). Then

\[
\sum_{i \in I} \dim(L(B_i)) - \dim(L(B_I)) \leq |I| - 1. \tag{4.16}
\]

Moreover, if \( B = B_0 \cup \cdots \cup B_r \) is the FI-decomposition then the inequality above is strict and, in this case

i) For each \( 0 \leq i, j \leq r \), \( i \neq j \), \( L(B_i) \cap L(B_j) = 0 \).

ii) For every \( j = 0, \ldots, r \), \( B_j \subset B \) is a flat.

Proof. The first statement follows from the fact that if \( B = B_0 \cup \cdots \cup B_r \) is a \( \theta \)-decomposition then so is \( B_I = \bigcup_{i \in I} B_i \) and, therefore, this decomposition computes \( \theta(B_I) \geq 0 \) which yields inequality (4.16). If \( B = B_0 \cup \cdots \cup B_r \) is the FI-decomposition then \( \theta(B_I) > 0 \) and this gives the strict inequality.

Item i) follows from taking \( I = \{i, j\} \). Finally, suppose \( B_i \) is not a flat for some \( i = 0, \ldots, r \). Then, there exist an element \( b \in B \) such that \( b \in L(B_i) \setminus B_i \). But then \( b \in B_j \) with \( j \neq i \) implying that \( L(B_i) \cap L(B_j) \neq \{0\} \), contradicting the previous statement. \( \square \)

Remark 4. Note that Corollary 4.13 gives necessary conditions for a Cayley decomposition to be a \( \theta \)-decomposition or the FI-decomposition. We expect those conditions to be sufficient as well. We also point out that it follows from the uniqueness statement in Theorem 4.3 that the FI-decomposition is unique up to reordering and that the passage from a \( \theta \)-decomposition to the FI-decomposition is accomplished by collecting together sub-decompositions \( B_I \) with \( \theta(B_I) = 0 \).

Lemma 4.14. Let \( B \) be a configuration and \( F \subset B \) a non-splitting rank-one flat. Let \( \pi: L(B) \to L(B)/L(F) \) be the natural projection and \( \tilde{B} = \pi(B \setminus F) \). Then

\[
\theta(B) \leq \theta(\tilde{B}).
\]

Proof. Let \( B = B_0 \cup \cdots \cup B_r \) be the FI-decomposition of \( B \). Then by Corollary 4.13 we may assume that \( F \subset C_0 \) and \( L(F) \cap (C_j) = \{0\} \) for \( j \geq 1 \). Note that \( F \neq C_0 \) because \( \mathfrak{s}(F) \neq 0 \). Then, \( \tilde{B} = \pi(B_0 \setminus F) \cup \pi(B_1) \cup \cdots \cup \pi(B_r) \) is a Cayley decomposition of \( \tilde{B} \) with the same invariants as the decomposition of \( B \).

Hence, \( \theta(\tilde{B}) \geq r - c(B_0, \ldots, B_r) = \theta(B) \). \( \square \)

We may now use Lemma 4.14 to show the following inequality.

Proposition 4.15. Let \( B \) be a dual-homogeneous configuration (as always, we assume \( 0 \notin B \)) over any field \( \mathbb{K} \) of characteristic zero. Then

\[
\theta(B) \leq \dim(L(B)) - 1 - \lambda(B). \tag{4.17}
\]

Proof. If \( \dim L(B) = 1 \) then both sides of the inequality equal zero. Suppose now that the result holds for \( \dim L(B) = m - 1 \) and let \( B \) be a configuration of rank \( m \).

Let \( \lambda = \lambda(B) \) and \( F_1 \subset F_\lambda \) a maximal non-splitting flag in \( B \). Let \( \pi: L(B) \to L(B)/L(F_1) \) be the natural projection and \( \tilde{B} = \pi(B \setminus F_1) \) the projected configuration. From Lemma 3.3 we have that \( \lambda(\tilde{B}) = \lambda(B) - 1 \).
Now, by inductive hypothesis we may assume that \( \theta(B) \leq \dim(L(B)) - 1 - \lambda(B) \). But \( \dim(L(\tilde{B})) = \dim(L(B)) - 1 \) and by Lemma 4.14 we have \( \theta(B) \geq \theta(\tilde{B}) \). Hence, \( \theta(B) \leq \dim(L(B)) - 1 - \lambda(B) \), as asserted. \( \square \)

We next show how to compute the invariant \( \lambda \) (the maximal length of a non-splitting flag) from a \( \theta \)-decomposition of \( B \).

**Theorem 4.16.** Let \( A \subset \mathbb{Z}^{d+1} \) be a homogeneous, non-pyramidal configuration of affine dimension \( d \) and \( B \) a Gale dual of \( A \). If \( B = B_0 \cup B_1 \cup \cdots \cup B_r \) is any \( \theta \)-decomposition, we have

\[
\lambda(A) = \sum_{j=0}^{r} \left( \dim(L(B_j)) - 1 \right).
\]

**Proof.** We have by the leftmost equality in (4.4) that \( \lambda = \lambda(A) = m - 1 - \theta(A) \), with \( m = m(A) = \dim(L(B)) \). Then, by Proposition 4.7 we get

\[
\lambda = m - 1 - r + c = -1 - r + \sum_{j=0}^{r} \left( \dim(L(B_j)) - 1 \right) = \sum_{j=0}^{r} \left( \dim(L(B_j)) - 1 \right),
\]

as claimed. \( \square \)

4.2. **Consequences of Theorem 4.16.** Recall from [3, Definition 5] that a configuration \( B \subset \mathbb{Q}^m \) is called irreducible if any two elements in \( B \) are linearly independent. If \( B \) is non-pyramidal and irreducible then the linear matroid \( B \) is simple. We carry this terminology to a configuration \( A \subset \mathbb{Z}^{d+1} \) and say that it is irreducible if and only if any (every) Gale dual of \( A \) is irreducible. Recall that a rank-one flat \( F \subset B \) is called splitting if \( s(F) = 0 \) and non-splitting otherwise. Given \( B \subset \mathbb{Q}^d \), we denote by \( B^{\text{red}} \) the irreducible configuration obtained by eliminating all elements lying in splitting lines and replacing those in non-splitting lines \( F \) by \( s(F) \). We call \( B^{\text{red}} \) the reduction of \( B \). It is easy to see that \( \lambda(B) = \lambda(B^{\text{red}}) \) and therefore it follows from (4.4) that:

\[
\theta(B) = \theta(B^{\text{red}}) + \left( \dim(L(B)) - \dim(L(B^{\text{red}})) \right).
\]

Thus, in order to understand the dual defect of toric varieties, it suffices to consider irreducible configurations.

**Remark 5.** We point out that even though, for simplicity, we have deduced (4.19) from (4.4), it can also be deduced directly with linear-algebraic arguments.

The following result bounds the dual defect of irreducible configurations.

**Proposition 4.17.** Let \( A \subset \mathbb{Z}^{d+1} \) be a homogeneous irreducible configuration with \( d(A) = d \). Then \( \theta(A) \leq (m(A) - 2)/2 \). Moreover for the cases of maximal possible defect we have:

- If \( m(A) = 2k \) is even and \( \theta(A) = k - 1 \), then \( A \) is affinely equivalent to a join of \( k \) non-defective configurations \( A_0, \ldots, A_{k-1} \) with \( m(A_j) = 2 \).
If $m(A) = 2k+1$ is odd and $\theta(A) = k-1$, then either $A$ is affinely equivalent to a join of $k$ non-defective configurations $A_0, \ldots, A_{k-1}$ with $m(A_0) = 3$ and $m(A_j) = 2$, $j \geq 1$, or to a Cayley configuration $A_0 \ast \cdots \ast A_k$ of $k+1$ configurations of codimension 2 with $c(A_0, \ldots, A_k) = 1$.

Proof. Let $B \subset \mathbb{Q}^m$ be a Gale dual of $A$ and $B = B_0 \cup \cdots B_r$ the FI-decomposition. Setting $\lambda_i = \dim L(B_i) - 1$, we have by Theorem 4.16 that

\begin{equation}
\lambda = \lambda_0 + \cdots + \lambda_r.
\end{equation}

Moreover, the irreducibility of $A$ implies that $\lambda_i \geq 1$ for all $i$.

Now, since $\theta(A) = r - c \leq r$ we have

\begin{equation}
\theta(A) + 1 \leq r + 1 \leq \sum_{j=0}^r \lambda_j = \lambda = m - 1 - \theta(A).
\end{equation}

Thus $2\theta(A) \leq m - 2$ and the result follows.

In particular we have, if $\theta > 0$ that

\begin{equation}
1 \leq r \leq \lambda - 1 = m - \theta - 2,
\end{equation}

where the left inequality follows from Corollary 4.12 that $r \geq 1$.

Suppose now that $m = 2k$ and $\theta(A) = k - 1$. Then $\lambda = m - 1 - \theta = k$ and therefore $r \leq k - 1$. Hence $\theta(A) = k - 1$ if and only if $r = k - 1$, which implies $\lambda_i = 1$ for all $i = 0, \ldots, r - 1$, and $c = 0$. Hence $A$ is equivalent to a join as asserted.

Finally, suppose $m = 2k + 1$ and $\theta(A) = k - 1$. Then $\lambda = 2k - k + 1 = k + 1$ and there are two possible partitions of $\lambda$ of length at least $k$:

\begin{align*}
\lambda_0 = & \cdots = \lambda_k = 1; \text{ and } \\
\lambda_0 = & 2, \lambda_1 = \cdots = \lambda_{k-1} = 1.
\end{align*}

In the first case we must have $c = 1$ and in the second $c = 0$. This yields the result. \hfill \square

The following consequences of Proposition 4.17 are expressed in terms of Gale duality to facilitate the comparison with results in [3].

Example 4.18. If $m = 3$, then $\theta < 1$ and therefore, every irreducible configuration with $m(A) = 3$ is non-defective. This is Theorem 20 in [3].

Example 4.19. If $m = 4$ then $\theta \leq 1$ and, therefore by Proposition 4.17, if $B$ is defective and irreducible then $B = B_0 \cup B_1$, where $\dim L(B_0) = \dim L(B_1)$ and $L(B_0) \cap L(B_1) = \{0\}$. This is Theorem 21 in [3].

Example 4.20. If $m = 5$ then, again $\theta \leq 1$ but now there are two non-equivalent irreducible defective configurations:

- $B = B_0 \cup B_1$, $\dim L(B_0) = 3$, $\dim L(B_1) = 2$.
- $B = B_0 \cup B_1 \cup B_2$, $\dim L(B_i) = 2$, for all $i$ and $L(B_i) \cap L(B_j) = \{0\}$ for $i \neq j$.

We point out that, although possible, obtaining this result with the methods in [3] would involve very long and awkward arguments.

It follows from Proposition 4.17 that if $\theta(A) > (m(A) - 2)/2$ then $A$ may not be irreducible. We can give a more precise statement.
Proposition 4.21. Let $A \subset \mathbb{Z}^{d+1}$ be a homogeneous configuration of maximal rank and suppose that $\theta(A) > (m(A) - 2)/2$. Then,

- If $m(A) = 2k$ and $\theta(A) = k - 1 + \ell$, then
  $$\dim L(B) - \dim L(B^{\text{red}}) \geq 2\ell.$$ 
- If $m(A) = 2k + 1$ and $\theta(A) = k - 1 + \ell$, then
  $$\dim L(B) - \dim L(B^{\text{red}}) \geq 2\ell - 1.$$ 

Proof. We prove the even case and leave the analogous odd case to the reader. Suppose $m = 2k$ and $\theta(B) = k - 1 + \ell$. Let $m_1 = \dim L(B^{\text{red}})$, then

$$k - 1 + \ell = \theta(B) = \theta(B^{\text{red}}) + (m - m_1)$$

and therefore $m - m_1 \geq 2\ell$ as asserted.

Given a configuration $A$ we have $\theta(A) \leq m(A) - 1$. If we have equality, then $\lambda(A) = 0$ and $\dim X_A = \dim X_A^\vee$ and it follows that every rank-one flat in $B$ must be splitting or, equivalently, that $\dim L(B^{\text{red}}) = 0$ (cf. [1]). Moreover, as shown in [1, Theorem 3.3], if $X_A$ is equivariantly embedded then $X_A$ is self-dual; i.e. $X_A \cong X_A^\vee$.

Proposition 4.22. Let $A$ be a configuration with $\theta(A) = m(A) - 2$ and let $B$ be a Gale dual. Then $B^{\text{red}}$ is a non-defective configuration of rank 2. Similarly, if $\theta(A) = m(A) - 3$, then either $B^{\text{red}}$ is a non-defective configuration of rank 3 or it is a rank 4 configuration with defect equal to 1.

Proof. In the first case we must have $\lambda = 1$ therefore the FI-decomposition of $B$ must be of the form $B = \Lambda_1 \cup \cdots \cup \Lambda_s \cup B_0$, where $\Lambda_i$ are dual-homogeneous rank-one flats and $m(B_0) = 2$, $\lambda(B_0) = 1$. Clearly $B^{\text{red}} = B_0^{\text{red}}$.

If $\theta(A) = m(A) - 3$ then $\lambda(A) = 2$ and therefore there are two possibilities for the FI-decomposition of $B$: 

$$B = \Lambda_1 \cup \cdots \cup \Lambda_s \cup B_0, \quad \text{or} \quad B = \Lambda_1 \cup \cdots \cup \Lambda_s \cup B_1 \cup B_2,$$

with $\dim L(\Lambda_i) = 1$, $\dim L(B_0) = 3$ and $\dim L(B_1) = \dim L(B_2) = 2$. Since in the first case $B^{\text{red}} = B_0^{\text{red}}$ and in the second $B^{\text{red}} = (B_1 \cup B_2)^{\text{red}}$, the result follows.

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Department of Mathematics and Statistics, University of Massachusetts Amherst, Amherst MA 01003-9305, USA

Email address: cattani@math.umass.edu

Dpto. de Matemática, FCEN, Universidad de Buenos Aires, and IMAS (UBA-CONICET), Ciudad Universitaria, Pab. I, C1428EGA Buenos Aires, Argentina

Email address: alidick@dm.uba.ar

URL: http://mate.dm.uba.ar/~alidick