Improved Product Structure for Graphs on Surfaces*

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Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [J. ACM 2020] proved that for every graph $G$ with Euler genus $g$ there is a graph $H$ with treewidth at most 4 and a path $P$ such that $G \subseteq H \boxtimes P \boxtimes K_{\max(2g,3)}$. We improve this result by replacing "4" by "3" and with $H$ planar. We in fact prove a more general result in terms of so-called framed graphs. This implies that every $(g, d)$-map graph is contained in $H \boxtimes P \boxtimes K_{\ell}$, for some planar graph $H$ with treewidth 3, where $\ell = \max\{2g\lfloor\frac{d}{2}\rfloor, d + 3\lceil\frac{d}{2}\rceil - 3\}$. It also implies that every $(g, 1)$-planar graph (that is, graphs that can be drawn in a surface of Euler genus $g$ with at most one crossing per edge) is contained in $H \boxtimes P \boxtimes K_{\max(4g,7)}$, for some planar graph $H$ with treewidth 3.

Keywords: product structure, graphs on surfaces

The motivation for this work is the following question: what is the global structure for graphs embeddable in a fixed surface? Dujmović et al. (2020b) answered this question for planar graphs in terms of products of graphs of bounded treewidth.

Theorem 1 ((Dujmović et al., 2020b)). Every planar graph is contained in $H \boxtimes P \boxtimes K_3$ for some planar graph $H$ with treewidth 3 and for some path $P$.

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(i) A plane graph is a graph embedded in the plane with no crossings. A plane triangulation is a plane graph in which every face is bounded by a triangle (that is, has length 3). A plane near-triangulation is a plane graph, where the outer-face is a cycle, and every internal face is a triangle.

(ii) For two graphs $G$ and $H$, the strong product $G \boxtimes H$ is the graph with vertex-set $V(G) \times V(H)$ and an edge between two vertices $(v, w)$ and $(v', w')$ if and only if $v = v'$ and $ww' \in E(H)$, or $w = w'$ and $vv' \in E(G)$, or $vv' \in E(G)$ and $ww' \in E(H)$.

(iii) A tree-decomposition of a graph $G$ is a collection $(B_x \subseteq V(G) : x \in V(T))$ of subsets of $V(G)$ (called bags) indexed by the nodes of a tree $T$, such that (a) for every edge $uv \in E(G)$, some bag $B_x$ contains both $u$ and $v$, and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty subtree of $T$. The width of a tree-decomposition is the size of a largest bag minus 1. The treewidth of a graph $G$, denoted by $tw(G)$, is the minimum width of a tree-decomposition of $G$.

(iv) A graph $G$ is contained in a graph $X$ if $G$ is isomorphic to a subgraph of $X$. A multigraph $G$ is contained in a graph $X$ if the simple graph underlying $G$ is contained in $X$. 

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This result, now known as the Planar Graph Product Structure Theorem, has been the key tool in solving several open problems regarding queue layouts (Dujmović et al., 2020b), non-repetitive colourings (Dujmović et al., 2020a), centred colourings (Dębski et al., 2021), clustered colourings (Dujmović et al., 2022), adjacency labellings (Bonamy et al., 2020; Dujmović et al., 2021; Esperet et al., 2020), vertex rankings (Bose et al., 2020), twin-width (Bonnet et al., 2022; Bekos et al., 2022) and infinite graphs (Huynh et al., 2021).

Dujmović et al. (2020b) generalised Theorem 1 for graphs embeddable in any fixed surface (v) as follows. A graph $H$ is apex if $H - v$ is planar for some vertex $v$ of $H$.

**Theorem 2** ((Dujmović et al., 2020b)). Every graph with Euler genus $g$ is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some apex graph $H$ with treewidth 4 and for some path $P$.

This paper improves this bound on the treewidth of $H$ from 4 to 3.

**Theorem 3.** Every graph with Euler genus $g$ is contained in $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some planar graph $H$ with treewidth 3 and for some path $P$.

The bound on the treewidth of $H$ in Theorem 3 is optimal since Dujmović et al. (2020b) showed that for every integer $\ell \geq 0$ there is a planar graph $G$ such that if $G$ is contained in $H \boxtimes P \boxtimes K_\ell$, then $H$ has treewidth at least 3.

We in fact prove a more general result in terms of so-called framed graphs. Let $G$ be a multigraph embedded in a surface $\Sigma$ without crossings, where each face is bounded by a cycle. For any integer $d \geq 3$, let $G^{(d)}$ be the multigraph embedded in $\Sigma$ obtained from $G$ as follows: for each face $F$ of $G$ bounded by a cycle $C$ of length at most $d$, for all distinct non-adjacent vertices $v, w$ in $C$, add an edge $vw$ across $F$ to $G^{(d)}$. We say that $G^{(d)}$ is a $(\Sigma, d)$-framed multigraph with frame $G$. If $\Sigma$ has Euler genus at most $g$, then $G^{(d)}$ is a $(g, d)$-framed multigraph.

We prove the following theorem.

**Theorem 4.** For all integers $g \geq 0$ and $d \geq 3$, every $(g, d)$-framed multigraph is contained in $H \boxtimes P \boxtimes K_\ell$ for some planar graph $H$ with treewidth 3 and for some path $P$, where $\ell = \max\{2g\frac{d}{2}, d + 3\frac{d}{2} - 3\}$.

Framed graphs (for $g = 0$) were introduced by Bekos et al. (2020) and are useful because they include several interesting graph classes, as shown by the following three examples.

First, every graph with Euler genus $g$ is a subgraph of a $(g, 3)$-framed multigraph. Thus Theorem 4 with $d = 3$ implies Theorem 3.

Now consider map graphs. Start with a graph $G$ embedded in a surface $\Sigma$ without crossings, with each face labelled a ‘nation’ or a ‘lake’, where each vertex of $G$ is incident with at most $d$ nations. Let $M$ be the graph whose vertices are the nations of $G$, where two vertices are adjacent in $G$ if the corresponding faces in $G$ share a vertex. Then $M$ is called a $(\Sigma, d)$-map graph. If $\Sigma$ has Euler genus at most $g$, then $M$ is called a $(g, d)$-map graph. Graphs embeddable in $\Sigma$ are precisely the $(\Sigma, 3)$-map graphs (Dujmović et al., 2017). So map graphs are a natural generalisation of graphs embeddable in surfaces.

We show that every $(\Sigma, d)$-map graph is a spanning subgraph of $G^{(d)}$ for some multigraph $G$ embedded in $\Sigma$ without crossings; see Lemma 11. Thus Theorem 4 implies that $(g, d)$-map graphs have the following product structure.

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(v) The Euler genus of a surface with $h$ handles and $c$ cross-caps is $2h + c$. The Euler genus of a graph $G$ is the minimum integer $g \geq 0$ such that there is an embedding of $G$ in a surface of Euler genus $g$; see Mohar and Thomassen (2001) for more about graph embeddings in surfaces. A triangulation of a surface $\Sigma$ is a graph embedded in $\Sigma$ with no crossings, such that every face is a triangle.
Theorem 5. Every \((g, d)\)-map graph is contained in \(H \boxtimes P \boxtimes K_\ell\) for some planar graph \(H\) with treewidth 3 and for some path \(P\), where \(\ell = \max\{2g, \lfloor d/2 \rfloor, d + 3\lfloor d/2 \rfloor - 3\}\).

A graph is \(k\)-planar if it has an embedding in the plane where each edge is involved in at most \(k\) crossings. This definition has a natural extension for other surfaces \(\Sigma\). A graph is \((\Sigma, k)\)-planar if it has an embedding in \(\Sigma\) where each edge is involved in at most \(k\) crossings. A graph is \((g, k)\)-planar if it is \((\Sigma, k)\)-planar for some surface \(\Sigma\) with Euler genus at most \(g\). In the planar setting \((g = 0)\), these graphs have been extensively studied; see Kobourov et al. (2017); Didimo et al. (2019) for surveys.

We show that every \((\Sigma, 1)\)-planar graph is contained in \(G^{(4)}\) for some multigraph \(G\) embedded in \(\Sigma\) without crossings; see Lemma 12. Thus Theorem 4 implies the following product structure theorem.

Theorem 6. Every \((g, 1)\)-planar graph is contained in \(H \boxtimes P \boxtimes K_{\max\{4g, 7\}}\) for some planar graph \(H\) with treewidth 3 and for some path \(P\).

Dujmovi\'c et al. (2019) proved that every \((g, k)\)-planar graph is contained in \(H \boxtimes P \boxtimes K_\ell\), for some graph \(H\) with treewidth \(\left(\frac{k+4}{3}\right)-1\) where \(\ell = \max\{2g, 3\}(6k^2 + 16k + 10)\). Hickingbotham and Wood (2021) improved \(\ell\) to \(2\max\{2g, 3\}(k+1)^2\). In the \(k = 1\) case, Theorem 6 is significantly stronger than both these results since \(H\) has treewidth 3 instead of treewidth 9. As mentioned above, treewidth 3 is best possible, even for planar graphs (Dujmovi\'c et al., 2020b). Note that Dujmovi\'c et al. (2019) previously proved Theorem 6 in the planar case \((g = 0)\), and a similar result was independently obtained by Bekos et al. (2022).

1 Proofs

Undefined terms and notation can be found in Diestel’s text (Diestel, 2018). A partition of a graph \(G\) is a set \(\mathcal{P}\) of non-empty sets of vertices in \(G\) such that each vertex of \(G\) is in exactly one element of \(\mathcal{P}\). Each element of \(\mathcal{P}\) is called a part. The quotient of \(\mathcal{P}\) is the graph, denoted by \(G/\mathcal{P}\), with vertex set \(\mathcal{P}\) where distinct parts \(A, B \in \mathcal{P}\) are adjacent in \(G/\mathcal{P}\) if and only if some vertex in \(A\) is adjacent in \(G\) to some vertex in \(B\). An \(H\)-partition of \(G\) is a partition \(\mathcal{P} = (A_x : x \in V(H))\) where \(H \cong G/\mathcal{P}\). For simplicity, we sometimes abuse notation and say \(J \in \mathcal{P}\) where \(J\) is a subgraph of \(G\) with \(V(J) \in \mathcal{P}\).

If \(T\) is a tree rooted at a vertex \(r\), then a non-empty path \(P\) in \(T\) is vertical if the vertex of \(P\) closest to \(r\) in \(T\) is an end-vertex of \(P\). If \(T\) is a rooted spanning tree in a graph \(G\), then a tripod in \(G\) (with respect to \(T\)) consists of up to three pairwise vertex-disjoint vertical paths in \(T\) whose lower end-vertices form a clique in \(G\).

A layering of a graph \(G\) is an ordered partition \(\mathcal{L} := (L_0, L_1, \ldots)\) of \(V(G)\) such that for every edge \(vw \in E(G)\), if \(v \in L_i\) and \(w \in L_j\), then \(|i - j| \leq 1\). A layered partition \((\mathcal{P}, \mathcal{L})\) of a graph \(G\) consists of a partition \(\mathcal{P}\) and a layering \(\mathcal{L}\) of \(G\). If \(\mathcal{P} = (A_x : x \in V(H))\) is an \(H\)-partition, then \((\mathcal{P}, \mathcal{L})\) is a layered \(H\)-partition with width \(\max\{|A_x \cap L| : x \in V(H), L \in \mathcal{L}\}\). Layered partitions were introduced by Dujmovi\'c et al. (2020b) who observed the following connection to strong products (which follows directly from the definitions).

Observation 7 (Dujmovi\'c et al. (2020b)). For all graphs \(G\) and \(H\), \(G\) is contained in \(H \boxtimes P \boxtimes K_\ell\) for some path \(P\) if and only if \(G\) has a layered \(H\)-partition \((\mathcal{P}, \mathcal{L})\) with width at most \(\ell\).

We need the following lemma of Dujmovi\'c et al. (2019), which is a special case of their Lemma 24 (which is an extension of Lemma 17 from (Dujmovi\'c et al., 2020b)).

Lemma 8 ((Dujmovi\'c et al., 2019)). Let \(G^+\) be a plane multigraph in which each face of \(G^+\) is bounded by a cycle with length in \(\{3, \ldots, d\}\). Let \(T\) be a spanning tree of \(G^+\) rooted at some vertex \(r\) on the
boundary of the outer-face of \( G^+ \). Assume there is a vertical path \( P \) in \( T \) with end-vertices \( p_1 \) and \( p_2 \) such that the cycle \( C \) obtained from \( P \) by adding the edge \( p_1p_2 \) is a subgraph of \( G^+ - r \). Let \( G \) be the plane graph consisting of all the vertices and edges of \( G^+ \) contained in \( C \) and the interior of \( C \). Then \( G^{(d)} \) has an \( H \)-partition \( P \) such that \( P \in P \) and each part \( S_i \in P \setminus \{P\} \) has a partition \( \{X, Y\} \) where \( |X| \leq d - 3 \) and \( Y \) is the union of at most three vertical paths in \( T \), and \( H \) is planar with treewidth at most 3.

The next lemma is the heart of our proof.

**Lemma 9.** Let \( G \) be a connected multigraph embedded in a surface of Euler genus \( g \) without crossings, where each face of \( G \) is bounded by a cycle. Then for every spanning tree \( T \) of \( G \) and every integer \( d \geq 3 \), \( G^{(d)} \) has an \( H \)-partition \( P \) such that one part \( Z \in P \) is the union of at most 2g vertical paths in \( T \) and each part \( S_i \in P \setminus \{Z\} \) has a partition \( \{X, Y\} \) where \( |X| \leq d - 3 \) and \( Y \) is the union of at most three vertical paths in \( T \), and \( H \) is planar with treewidth at most 3.

**Proof:** We start by following the proof of (Dujmović et al., 2020b, Lemma 21), which is the heart of the proof of Theorem 2. Near the end of our proof we follow a different strategy to obtain the stronger result.

If \( g = 0 \), then the claim follows from Lemma 8 by considering an appropriate supergraph \( G^+ \) of \( G \). Now assume that \( g \geq 1 \). Say \( G \) has \( n \) vertices, \( m \) edges, and \( f \) faces. By Euler’s formula, \( n - m + f = 2 - g \). Let \( D \) be the multigraph with vertex-set the set of faces in \( G \), where for each edge \( e \) of \( E(G) \setminus E(T) \), if \( f_1 \) and \( f_2 \) are the faces of \( G \) with \( e \) on their boundary, then there is an edge joining \( f_1 \) and \( f_2 \) in \( D \). (Think of \( D \) as the spanning subgraph of the dual graph consisting of those edges that do not cross edges in \( T \).)

Note that \( |V(D)| = f = 2 - g - n + m \) and \( |E(D)| = m - (n - 1) = |V(D)| - 1 + g \). Since \( T \) is a tree, \( D \) is connected; see (Dujmović et al., 2017, Lemma 11) for a proof. Let \( T^* \) be a spanning tree of \( D \). Thus \( |E(D) \setminus E(T^*)| = g \). Let \( Q = \{a_1b_1, a_2b_2, \ldots, a_gb_g\} \) be the set of edges in \( G \) dual to the edges in \( E(D) \setminus E(T^*) \). Let \( r \) be the root of \( T \), and for \( i \in \{1, 2, \ldots, g\} \), let \( Z_i \) be the union of the \( a_ib_i \)-path and the \( b_ir \)-path in \( T \), plus the edge \( a_ib_i \). Let \( Z := Z_1 \cup Z_2 \cup \cdots \cup Z_g \). By construction, \( Z \) is a connected subgraph of \( G \); see Figure 1 for an example. In fact, since \( r \) is contained in each of the \( 2g \) vertical paths, \( T[V(Z)] \) is connected. Say \( Z \) has \( p \) vertices and \( q \) edges. Since \( Z \) consists of a subtree of \( T \) plus the \( g \) edges in \( Q \), we have \( q = p - 1 + g \).

We now describe how to ‘cut’ along the edges of \( Z \) to obtain a new embedded graph \( \tilde{G} \); see Figure 2. First, each edge \( e \) of \( Z \) is replaced by two edges \( e' \) and \( e'' \) in \( \tilde{G} \). Each vertex of \( G \) that is not contained in \( V(Z) \) is untouched. Consider a vertex \( v \in V(Z) \) incident with edges \( e_1, e_2, \ldots, e_d \) in \( Z \) in clockwise order. In \( \tilde{G} \) replace \( v \) by new vertices \( v_1, v_2, \ldots, v_d \), where \( v_i \) is incident with \( e_i \), \( e_{i+1}' \), and all the edges incident with \( v \) clockwise from \( e_i \) to \( e_{i+1} \) (exclusive). Here \( e_{d+1} \) means \( e_1 \) and \( e_{d+1}' \) means \( e''_1 \). This operation defines a cyclic ordering of the edges in \( G \) incident with each vertex (where \( e_{d+i}' \) is followed by \( e_i \) in the cyclic order at \( v_i \)). This in turn defines an embedding of \( \tilde{G} \) in some orientable surface\(^{(v)} \). Let \( Z' \) be the set of vertices introduced in \( \tilde{G} \) by cutting through vertices in \( Z \).

We now show that \( \tilde{G} \) is connected. Consider vertices \( x_1 \) and \( x_2 \) of \( \tilde{G} \). Select faces \( f_1 \) and \( f_2 \) of \( \tilde{G} \) respectively incident to \( x_1 \) and \( x_2 \) that are also faces of \( G \). Let \( P \) be a path joining \( f_1 \) and \( f_2 \) in the dual tree \( T^* \). Then the edges of \( G \) dual to the edges in \( P \) were not split in the construction of \( \tilde{G} \). Therefore an \( x_1 x_2 \)-walk in \( \tilde{G} \) can be obtained by following the boundaries of the faces corresponding to vertices in \( P \). Hence \( \tilde{G} \) is connected.

\(^{(v)} \) If \( G \) is embedded in a non-orientable surface, then the edge signatures for \( G \) are ignored in the embedding of \( \tilde{G} \).
Say $\tilde{G}$ has $n'$ vertices and $m'$ edges, and the embedding of $\tilde{G}$ has $f'$ faces and Euler genus $g'$. Each vertex with degree $d$ in $Z$ is replaced by $d$ vertices in $\tilde{G}$. Each edge in $Z$ is replaced by two edges in $\tilde{G}$, while each edge of $E(G) - E(Z)$ is maintained in $\tilde{G}$. Thus

$$n' = n - p + \sum_{v \in V(Z)} \deg_G(v) = n + 2q - p = n + 2(p - 1 + g) - p = n + p - 2 + 2g$$

and $m' = m + q = m + p - 1 + g$. Each face of $G$ is preserved in $\tilde{G}$. Say $s$ new faces are created by the cutting. Thus $f' = f + s$. Since $\tilde{G}$ is connected, $n' - m' + f' = 2 - g'$ by Euler’s formula. Thus $(n + p - 2 + 2g) - (m + p - 1 + g) + (f + s) = 2 - g'$, implying $(n - m + f) - 1 + g + s = 2 - g'$. Hence $(2 - g) - 1 + g + s = 2 - g'$, implying $g' = 1 - s$. Since $g' \geq 0$, we have $s \leq 1$. Since $g \geq 1$, by construction, $s \geq 1$. Thus $s = 1$ and $g' = 0$. Hence $\tilde{G}$ is plane and all the vertices in $Z'$ are on the boundary of a single face, $F$, of $\tilde{G}$. Moreover, the boundary of $F$ is a cycle $C_F$ and $V(C_F) = Z'$. Consider $F$ to be the outer-face of $\tilde{G}$.

Now construct a supergraph $G^+$ of $\tilde{G}$ by adding a vertex $r^+$ in $F$ and edges from $r^+$ to each vertex in $Z'$. Then $G^+$ is a plane multigraph where each face of $G^+$ is bounded by a cycle.

We now depart from the proof of Dujmovič et al. (2020b, Lemma 21). Let $P^+$ be an arbitrary path such that $V(P^+) = V(C_F)$ and let $v^+ \in V(P^+)$ be an end-vertex of $P^+$. Let $T^+$ be the following spanning tree of $G^+$ rooted at $r^+$. Initialize $T^+$ to be the path $P^+$ plus the edge $r^+v^+$. Let $E' := \{vw \in E(T) : v \in Z, w \in V(G) \setminus V(Z)\}$ and $h := |E'|$. Observe that $T - V(Z)$ is a forest with $h$ components. For each edge $vw \in E'$, $w$ is adjacent to exactly one vertex $v_i \in V(Z')$ introduced when cutting $v$. Add the edge $v_iz$ to $T^+$. Finally, add the induced forest $T - V(Z)$ to $T^+$; see Figure 3. Then $T^+$ is connected since each component of $T - V(Z)$ is adjacent in $T^+$ to some vertex in $V(P^+)$. Furthermore, since $|V(T^+)| = |V(P^+) + |V(G) \setminus V(Z)|$ and $|E(T^+)| = |E(P^+) + h + (|V(G) \setminus V(Z)| - h) = |V(P^+) + |V(G) \setminus V(Z)| - 1$, it follows that $T^+$ is indeed a spanning tree of $G^+$. Consider each component of $T - V(Z)$ to be a subtree of $T^+$. 

![Fig. 1: Example of the construction in the proof of Lemma 9, where brown edges are in $T$, red edges are in $Q$, and blue edges are in $T$ and in $Z - E(Q)$.](image)
Now every vertical path in $T^+$ contained in $V(G) \setminus V(Z)$ corresponds to a vertical path in $T$. Every maximal vertical path in $T^+$ consists of the edge $r^+ v^+$, a subpath of $P^+$, some edge $v_i w$ (where $w \in V(G) \setminus V(Z)$), followed by a path in $T - V(Z)$ from $w$ to a leaf in $T$. Since every vertical path $P$ in $T^+$ is contained in some maximal vertical path in $T^+$, it follows that $P \cap (V(G) \setminus V(Z))$ is a vertical path in $T$. Thus every vertical path in $T^+$ that is contained in $V(G) \setminus V(Z)$ is a vertical path in $T$.

Triangulate every face in $G^+$ whose facial cycle has length greater than $d$. Since $r^+$ is on the boundary of the outer-face of $G^+$, $V(P^+) = V(C_F)$, every facial cycle has length in $\{3, \ldots, d\}$ and $P^+$ is a vertical path.
path of $T^+$, Lemma 8 is applicable. Let $P'$ be the $H$-partition of $G^{(d)}$ given by Lemma 8. Therefore, $H$ is planar with treewidth at most 3, where $P^+ \in P'$ and each part in $S_i \in P' \setminus \{P^+\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leq d - 3$ and $Y_i$ is the union of at most three vertical paths in $T'$. Let $P$ be the partition of $G^{(d)}$ obtained by replacing $P^+$ by $Z$. Since $V(P^+)$ is a subdivision of the faces of $G$ and all the split vertices of $G$ are in $Z$, we have $G^{(d)}/P \cong G^{(d)}/P' \cong H$. Hence $P$ is also an $H$-partition where $H$ is planar with treewidth at most 3. In addition, since each vertical path in $T^+$ that is disjoint from $V(Z' \cup \{r^+\}$ is a vertical path in $T$, each part $S_i \in P \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leq d - 3$ and $Y_i$ is the union of at most three vertical paths in $T$, as required.

Theorem 4 is an immediate consequence of Observation 7 and the next lemma.

**Lemma 10.** Let $G$ be a multigraph embedded in a surface of Euler genus $g$ without crossings, where each face is bounded by a cycle. Then $G^{(d)}$ has a layered $H$-partition $(P, L)$ with width at most $\max\{2g\lfloor \frac{d}{2}\rfloor, d + 3\lfloor \frac{d}{2}\rfloor - 3\}$, such that $H$ is planar with treewidth at most 3.

**Proof:** Since each face of $G$ is bounded by a cycle, $G$ is connected. Let $T$ be a BFS-spanning tree of $G$ with corresponding BFS-layering\(^{(vii)}\) $(V_0, V_1, \ldots)$. By Lemma 9, $G^{(d)}$ has an $H$-partition $P$ such that one part $Z \in P$ is the union of at most $2g$ vertical paths in $T$ and each part $S_i \in P \setminus \{Z\}$ has a partition $\{X_i, Y_i\}$ where $|X_i| \leq d - 3$ and $Y_i$ is the union of at most three vertical paths in $T$, and $H$ is planar with treewidth at most 3. It remains to adjust the layering of $G$ to obtain a layering of $G^{(d)}$. If $uv \in E(G^{(d)})$ then $\text{dist}_G(u, v) \leq \lfloor \frac{d}{2}\rfloor$. Thus if $u \in V_i$ and $v \in V_j$ then $|i - j| \leq \lfloor \frac{d}{2}\rfloor$. For each $j \in \mathbb{N}$, let $L_j = V_{\lfloor \frac{d}{2}\rfloor + 1} \cup \ldots \cup V_{(j+1)(\lfloor \frac{d}{2}\rfloor - 1)}$. Then $(P, L = (L_0, L_1, \ldots))$ is a layered $H$-partition of $G^{(d)}$ with width at most $\max\{2g\lfloor \frac{d}{2}\rfloor, d + 3\lfloor \frac{d}{2}\rfloor - 3\}$, as required.

We conclude by showing that $(\Sigma, d)$-map graphs and $(\Sigma, 1)$-planar graphs are contained in framed graphs.

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\(^{(vii)}\) If $G$ is a connected graph and $T$ is a spanning tree of $G$ rooted at vertex $r$, then $T$ is **BFS** if $\text{dist}_T(v, r) = \text{dist}_G(v, r)$ for every $v \in V(G)$. A layering $(L_0, L_1, \ldots)$ of a graph $G$ is **BFS** if $L_0 = \{r\}$ for some root vertex $r \in V(G)$ and $L_i = \{v \in V(G) : \text{dist}_G(v, r) = i\}$ for all $i \geq 1$. 
Dujmović et al. (2019) proved the following result in the case of plane map graphs (and similar results were previously known in the literature (Chen et al., 2006; Brandenburg, 2019, 2020)). An analogous proof works for arbitrary surfaces, which we include for completeness. Together with Theorem 4, this implies Theorem 5.

**Lemma 11.** For every surface $\Sigma$ and integer $d \geq 3$, every $(\Sigma, d)$-map graph is a subgraph of $G^{(d)}$ for some multigraph $G$ embedded in $\Sigma$ without crossings, where each face of $G$ is bounded by a cycle.

**Proof:** Let $G_0$ be a graph embedded in $\Sigma$, with each face labelled a nation or a lake, and where each vertex of $G_0$ is incident with at most $d$ nations. Let $M$ be the corresponding map graph.

If $G_0$ has a face $F$ of length 2, then add a new vertex inside $F$ adjacent to both vertices on the boundary of $F$, which creates two new triangular faces $F_1$ and $F_2$. If $F$ is a lake, then make $F_1$ and $F_2$ lakes. If $F$ is a nation, then make $F_1$ a nation and make $F_2$ a lake. The resulting map graph is still $M$. So we may assume that $G_0$ is an edge-maximal multigraph embedded in $\Sigma$ with no face of length 2 (and with each face labelled a nation or a lake), such that $M$ is the corresponding map graph. This is well-defined since the assumption of having no face of length 2 implies that $|E(G_0)| \leq 3(|V(G)| + g - 2)$, where $g$ is the Euler genus of $\Sigma$.

Suppose that some face $f$ of $G_0$ has a disconnected boundary. Let $v$ and $w$ be vertices in distinct components of the boundary of $f$. Add the edge $vw$ to $G_0$ across $f$. The corresponding map graph is unchanged, which contradicts the edge-maximality of $G_0$. Thus each face of $G_0$ has a connected boundary. Suppose that some face $f$ of $G_0$ has a repeated vertex $v$ in the boundary walk of $f$. Let $u, v, w$ be consecutive vertices on the boundary of $f$. So $u, v, w$ are distinct. Add the edge $uw$ inside $f$ so that $uvw$ bounds a disk. Label the resulting face $uvw$ as a lake. Since $v$ appears elsewhere in the boundary of $f$, the corresponding map graph is unchanged, which contradicts the edge-maximality of $G_0$. Thus no facial walk of $G_0$ has a repeated vertex. Since each facial walk is connected, every face of $G_0$ is bounded by a cycle.

Let $G^*_0$ be the dual multigraph of $G_0$. So the vertices of $G^*_0$ correspond to faces of $G_0$, and each vertex of $G^*_0$ is a nation vertex or a lake vertex. Since every face of $G_0$ is bounded by a cycle, every face of $G^*_0$ is bounded by a cycle.

Let $x$ be a vertex of $G_0$, let $F_x$ be the corresponding face of $G^*_0$, and let $(v_1, \ldots, v_s)$ be the facial cycle of $F_x$. Let $C_x := (w_1, \ldots, w_r)$ be the circular subsequence of $(v_1, \ldots, v_s)$ consisting of only the nation vertices. Since $x$ is incident to at most $d$ nations, $r \leq d$.

Let $G$ be the supergraph of $G^*_0$ obtained by adding an edge between each pair of consecutive vertices in $C_x = (w_1, \ldots, w_r)$ for each vertex $x$ of $G_0$. The graph consisting of $C_x$ plus these added edges is called the nation cycle (of $x$). Note that if $r = 1$ then the nation cycle has no edges, and if $r = 2$ then the nation cycle has one edge. Since every face of $G^*_0$ is bounded by a cycle, every face of $G$ is bounded by a cycle. Moreover, each nation cycle of length at least 3 is now a facial cycle of $G$ with length at most $d$. By construction, $G$ embeds in $\Sigma$ with no crossings. Let $G^{(d)}$ be the $d$-framed graph whose frame is $G$.

By definition, $V(M) \subseteq V(G^{(d)})$. To prove the claim, it suffices to show that $E(M) \subseteq E(G^{(d)})$. Indeed, if $vw \in E(M)$ then the nation faces corresponding to $v$ and $w$ have a common vertex $x$ on their boundary. The vertex $x$ corresponds to a face $F_x$ in $G^*_0$ and the facial cycle of $F_x$ contains $v$ and $w$. Therefore, the nation cycle $C_x$ of $F_x$ contains $v$ and $w$. If $C_x$ has length 2 then $vw \in E(G) \subseteq E(G^{(d)})$. If $C_x$ has length at least 3 then it has length at most $d$ and it bounds a face in $G$. So $vw \in E(G^{(d)})$.

Dujmović et al. (2019) proved the following result in the case of 1-planar graphs (and similar results were previously known in the literature (Chen et al., 2006; Bekos et al., 2020; Brandenburg, 2019, 2020)).
An analogous proof works for arbitrary surfaces, which we include for completeness. Together with Theorem 4, this implies Theorem 6.

**Lemma 12.** Every $(\Sigma, 1)$-planar graph $G$ with at least three vertices is contained in $G_0^{(4)}$ for some multigraph $G_0$ embedded in $\Sigma$ with no crossings where each face of $G_0$ is bounded by a cycle.

**Proof:** We may assume that $G$ is embedded in $\Sigma$ with at most one crossing on each edge, such that no two edges of $G$ incident to a common vertex cross, since such a crossing can be removed by a local modification to obtain an embedding of $G$ in which the two edges do not cross.

Initialise $G' := G$. Add edges to $G'$ to obtain an edge-maximal multigraph embedded in $\Sigma$ such that each edge is in at most one crossing, no two edges incident to a common vertex cross, and no face is bounded by two parallel edges. The final condition ensures that $G'$ is well-defined, since it follows from Euler’s formula that if $G$ has $k$ crossings, then $|E(G')| \leq 3(|V(G)| + k + q - 2) - 2k$.

Consider crossing edges $e_1 = vw$ and $e_2 = xy$ in $G'$. So $v, w, x, y$ are distinct. Since $e_1$ is the only edge that crosses $e_2$ and $e_2$ is the only edge that crosses $e_1$, by the edge-maximality of $G'$, there is a cycle $C = (v, x, w, y)$ in $G'$ that bounds a disc whose interior intersects no edge of $G'$ except $e_1$ and $e_2$.

Let $G_0$ be the embedded multigraph obtained from $G'$ by deleting each pair of crossing edges. Thus the above-defined cycle $C$ bounds a face of $G_0$. By the edge-maximality of $G'$, every other face of $G_0$ (that is, not arising from a pair of deleted crossing edges) is a triangular face of $G'$. Thus, $G_0$ is a multigraph embedded in $\Sigma$ with no crossings, such that each face of $G_0$ is bounded by a 3-cycle or a 4-cycle, and $G$ is contained in $G_0^{(4)}$.

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