Effective Field calculations of the Energy Spectrum of the
\(\mathcal{P}\mathcal{T}\)-Symmetric \((-x^4)\) Potential

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Abstract

In this work, we show that the traditional effective field approach can be applied to the \(\mathcal{P}\mathcal{T}\)-symmetric wrong sign \((-x^4)\) quartic potential. The importance of this work lies in the possibility of its extension to the more important \(\mathcal{P}\mathcal{T}\)-symmetric quantum field theory while the other approaches which use complex contours are not willing to be applicable. We calculated the effective potential of the massless \(-x^4\) theory as well as the full spectrum of the theory. Although the calculations are carried out up to first order in the coupling, the predicted spectrum is very close to the exact one taken from other works. The most important result of this work is that the effective potential obtained, which is equivalent to the Gaussian effective potential, is bounded from below while the classical potential is bounded from above. This explains the stability of the vacuum of the theory.

The obtained quasi-particle Hamiltonian is non-Hermitian but \(\mathcal{P}\mathcal{T}\)-symmetric and we showed that the calculation of the metric operator can go perturbatively. In fact, the calculation of the metric operator can be done even for higher dimensions (quantum field theory) which, up till now, can not be calculated in the other approaches either perturbatively or in a closed form due to the possible appearance of field radicals. Moreover, we argued that the effective theory is perturbative for the whole range of the coupling constant and the perturbation series is expected to converge rapidly (the effective coupling \(g_{\text{eff}} = \frac{1}{6}\)).

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The pioneering article of Carl Bender and Stefan Boettcher [1] puts the pseudo Hermitian theories with real spectra on a track in which both theories, Hermitian and pseudo Hermitian, are in the same footing in regard to the physical acceptability. For pseudo Hermitian theories, the calculation of a positive definite metric operator, or the $C$ operator, is indispensable. An exception is noticed for theories in which the $Q$ operator represents a gauge transformation [2]. For such theories, the $Q$ operator disappears from the physical calculations. However, even for quantum mechanical theories in which the metric operator is not a gauge transformation, there exists known successful algorithms for the calculations of the metric or the $C$ operator. On the other hand, for bounded from above scalar field theories like the $-\phi^4$ theory, it seems that the conventional algorithms used in the quantum mechanical case are inapplicable. For instance, the metric operator has not been obtained yet either perturbatively or in a closed form. Besides, Physical amplitudes have not been calculated in any regime for the $-\phi^4$ scalar field theory. Accordingly, we need a successful as well as applicable algorithm for the investigation of this theory which bears the interesting asymptotic-freedom property.

In the regime of $\mathcal{PT}$-symmetric theories, the quantum mechanical, bounded from above, potentials are always treated in a complex contour which seems to be inapplicable for quantum field cases because of the probable appearance of a root of the field. Moreover, the theory is non-perturbative and the perturbative calculations for the $Q$ operator, for instance, leaded to a trivial $C$ operator. However, the traditional effective field technique, as a non-perturbative tool, has not been advocated seriously in the regime of $\mathcal{PT}$-symmetric theories. In fact, in previous works, we have used effective field calculations to obtain the one particle irreducible (1PI) amplitudes for the $-\phi^4$ theory in $1 + 1$ and $2 + 1$ dimensions [8, 4, 5]. The results of Ref. [4] has reproduced the exponential vanishing of the vacuum condensate as the coupling constant goes to zero as predicted by a different technique used in Ref. [6] which assures the success of the algorithm in non-Hermitian quantum field theories. What makes this trend very impressive is that we keep an eye on the fruitful features of the $\mathcal{PT}$—symmetric field theory to play the role of the Higgs mechanism in the standard model. However, in this work, we aim to test the validity of the algorithm in a quantitative manner and since there exist rigorous numerical results for the quantum mechanical $-x^4$ model, we will test the validity of the effective field approach for the massless quantum mechanical
Hamiltonian of the from;

\[ H = \frac{1}{2} p^2 - \frac{g}{2} x^4, \]  

(1)

or in a quantum field language one may use the notations;

\[ H(x) = \frac{1}{2} \left( (\nabla \phi)^2 + \pi^2 \right) - \frac{g}{2} \phi^4, \]

for the Hamiltonian density \( H(x) \) with the calculation proceeds in 0+1 dimensions (quantum mechanics). The effective field studies start by applying the canonical transformation of the form;

\[ \phi = \psi + B, \quad \pi = \Pi = \dot{\psi}, \]

where \( B \) is the vacuum condensate and the fields \( \psi \) and \( \Pi \) follow from the relations;

\[
\psi(x) = \int \frac{d \vec{k}}{(2\pi)^d} \sqrt{2w} \left\{ a(\vec{k}) \exp \left( i \vec{k} \cdot \vec{x} - iwx_o \right) + a^\dagger(\vec{k}) \exp \left( -i \vec{k} \cdot \vec{x} + iwx_o \right) \right\},
\]

\[
\Pi(x) = \frac{1}{i} \int \frac{d \vec{k}}{(2\pi)^d} \sqrt{\frac{w}{2}} \left\{ a(\vec{k}) \exp \left( i \vec{k} \cdot \vec{x} - iwx_o \right) - a^\dagger(\vec{k}) \exp \left( -i \vec{k} \cdot \vec{x} + iwx_o \right) \right\},
\]

\[
w(k) = \sqrt{\vec{k}^2 + M^2}, \quad \left[ a(\vec{k}), a^\dagger(\vec{k}') \right] = \delta(\vec{k}, \vec{k}').
\]

(2)

Also, \( M \) is the mass of the field \( \psi \) (\( \phi \) is massless) and \( d \) is the dimension of the position space. Accordingly, the Hamiltonian density transforms as

\[
H(x) = \frac{1}{2} \left( (\nabla \psi)^2 + \Pi^2 \right) - \frac{g}{2} (\psi + B)^4,
\]

\[
= \frac{1}{2} \left( (\nabla \psi)^2 + \Pi^2 + M^2 \psi^2 \right) - \frac{g}{2} (\psi^4 + 4B\psi^3 + (6B^2) \psi^2 + (4B^3) \psi + B^4) - \frac{1}{2} M^2 \psi^2,
\]

\[
= \frac{1}{2} \left( (\nabla \psi)^2 + \Pi^2 + M^2 \psi^2 \right) - \frac{g}{2} (\psi^4 + 4B\psi^3)
\]

\[
+ \left( -\frac{1}{2} M^2 - 3gB^2 \right) \psi^2 - 2gB^3\psi - \frac{g}{2} B^4
\]

\[
= H_0 + H_I - \frac{g}{2} B^4,
\]

(3)

where

\[
H_0 = \frac{1}{2} \left( (\nabla \psi)^2 + \Pi^2 + M^2 \psi^2 \right),
\]

\[
H_I = -\frac{g}{2} (\psi^4 + 4B\psi^3) + \left( -\frac{1}{2} M^2 - 3gB^2 \right) \psi^2 - 2gB^3\psi.
\]

(5)
The effective potential is defined as the vacuum energy \( V \). Before we go into the calculation of the effective potential, we need to explain the nature of the algorithm we use and its relation to other algorithms. Regarding this, it is well known that an amplitude calculated with respect to the true vacuum is converted into a calculation with respect to the free vacuum via the insertion of time evolution operator \( g^2 \psi^4 \) in the effective Hamiltonian in Eq. (3) is Hermitian and thus the theory is a real line one. Moreover, through the expansion of the time evolution operator, one obtains the Feynman diagrams of different orders in the perturbation series. With this in mind, the effective potential is generated as the expectation value of the potential term \( H_I - \frac{g}{2} \psi^4 \) in Eq. (3) plus the expectation value of the kinetic term. Through the expansion of the time evolution operator one obtains

\[
U(t, t_0) = 1 + (-i) \frac{1}{c} \int_{t_0}^{t} dt H_I(t_1) + \ldots,
\]

where \( U(t, t_0) \) is the time evolution operator. Accordingly, keeping only the first term (first order of \( V_{eff} \)) we get

\[
V_{eff} = \langle 0 | H | 0 \rangle,
\]

which has the same form as the Gaussian effective potential (GEP) studied in Ref. [8]. However, the parameters \( B \) and \( M \) are fixed using the fact that the effective potential is the generating functional of the one particle irreducible amplitudes and thus we have the relations:

\[
\frac{\partial V_{eff}}{\partial B} = \frac{\partial E_0}{\partial B} = 0, \quad \frac{\partial^2 V_{eff}}{\partial B^2} = \frac{\partial^2 E_0}{\partial B^2} = M^2.
\]

Now, up to first order in the coupling \( g \), one have the Feynman diagrams shown in Fig. 1. The two diagrams will contribute \( (D = d + 0 = 1) \);

Diagram (a) \( \equiv \frac{(-i)^2}{2} \left( -\frac{M^2}{2} - 3gB^2 \right) \left( \frac{1}{(2\pi)^Dp} \right)^2 \frac{1}{p^2 - M^2} \)

\[
= -\frac{1}{2M} \left( 3gB^2 + \frac{M^2}{2} \right),
\]

Diagram (b) \( \equiv (-i)^2 \frac{12ig}{8} \left( \int \frac{d^Dp}{(2\pi)^D} \frac{1}{p^2 - M^2} \right)^2 = \frac{-3g}{8 |M|^2} \)
to the effective potential. Thus the ground state energy can be obtained as

\[ E_0 = \frac{1}{2} M - \frac{1}{2} g \frac{3}{4M^2} + \left( -\frac{1}{2} M^2 - 3gB^2 \right) \frac{1}{2M} + \left( -\frac{1}{2} B^4 g \right). \]  

(8)

The regime of the effective potential relates the derivatives of the effective potential with respect to the vacuum condensate \( B \) to the one particle irreducible amplitudes (1PI) [7]. In other words, up to first order in the coupling, the effective potential is constrained by the following two conditions;

\[ \frac{\partial V_{\text{eff}}}{\partial B} = \frac{\partial E_0}{\partial B} = \frac{\partial}{\partial B} \left( \frac{1}{2} M - \frac{1}{2} g \left( \frac{3}{4M^2} \right) + \left( -\frac{1}{2} M^2 - 3gB^2 \right) \frac{1}{2M} + \left( -\frac{1}{2} B^4 g \right) \right) = 0, \]

\[ \frac{\partial^2 V_{\text{eff}}}{\partial B^2} = \frac{\partial^2 E_0}{\partial B^2} = \frac{\partial^2}{\partial B^2} \left( \frac{1}{2} M - \frac{1}{2} g \left( \frac{3}{4M^2} \right) + \left( -\frac{1}{2} M^2 - 3gB^2 \right) \frac{1}{2M} + \left( -\frac{1}{2} B^4 g \right) \right) = M^2, \]  

(9)

or equivalently

\[ (-2g) B^3 + \left( -\frac{3}{M} g \right) B = 0, \]

\[ (-6g) B^2 - \frac{3}{M} g = M^2. \]  

(10)

Note that, \( E_0 \) obtained here is equivalent to the GEP with the condition \( \frac{\partial^2 E_0}{\partial B^2} \) is equivalent to the condition of minimal sensitivity \( \frac{\partial E_0}{\partial M} \) used in Ref. [8]. However, the conditions used here, \( \frac{\partial E_0}{\partial B} \) and \( \frac{\partial^2 E_0}{\partial B^2} \), are more illuminating as they are constraining the predicted \( B \) and \( M \) values to represent a minimum of the GEP while the classical potential is bounded from above. To show that \( E_0 \) is in fact bounded from below, we note that for \( B \neq 0 \), one can get the parametrization;

\[ B = -\sqrt{\frac{M^2}{-4g}}, \]

\[ M = \frac{3}{\sqrt{6g}}. \]  

(11)

Accordingly, we get the relation \( M = \frac{3}{2B^2} \) and thus

\[ E_0 = \frac{1}{24B^2} \left( 8B^6 g - 9 \right), \]

or in terms of a real parameter \( b = \frac{B}{i} \);

\[ E_0 = \frac{1}{24b^2} \left( 8gb^6 + 9 \right), \]

5
which is bounded from below and positive (see Fig. [2]).

For \( g \) positive, the condensate \( B \) is pure imaginary and thus the quasi-particle Hamiltonian in Eq. (3) is \( \mathcal{P}\mathcal{T} \)–symmetric and up to first order one can get the metric operator in a simple fashion. Indeed, the existence of a positive definite metric operator assures the reality of the spectrum. We will stress this point later in this work.

Let us test the vacuum energy in comparison with its exact (numerical) and WKP predictions from Ref. [1] (table I). The calculations are carried out at \( g = 1 \) and the value in the table is \( 2E_0 \) which is equivalent to \( E \) in Ref. [1].

| Exact  | our prediction (first order) | WKB  |
|--------|-------------------------------|------|
| 1.4771 | 1.3628                       | 1.3765 |

**TABLE I:** The first order \( 2E_0 \) calculated at \( g = 1 \) and compared to the exact and WKP results from Ref. [1].

One can realize that the effective field calculations though simple are reasonable, taking into account that this is the first order calculations and one can refine it by taking Feynman diagrams from higher orders into account. Moreover, the effective potential is the generating functional from which one can predict all the 1PI amplitudes. For instance at \( g = \frac{1}{2} \), we get the one point function \( \langle 0|\phi|0 \rangle = B = -1.0198i \) compared to the exact value \(-0.97347i\) in Ref. [6] which again assures the reliability of the effective field calculation for the \( \mathcal{P}\mathcal{T} \)-symmetric \((-\frac{2}{4}\phi^4)\) theory.

The success of the effective field theory to predict the ground state energy and the one point function of the \((-\phi^4)\) potential may be thought as an accidental result and more tests are needed to support the validity of the method as a whole. To do that, one may calculate the whole spectrum using the quasi-particle Hamiltonian in Eq. (3). This can be easily obtained as

\[
E_n = \langle n|H|n \rangle = \left( n + \frac{1}{2} \right) M - \frac{1}{2} g \left( 3 \left( \frac{4 (n + \frac{1}{2})^2 + 1}{8M^2} \right) \right) + \left( \frac{1}{2} M^2 - 3gB^2 \right) \frac{\left( n + \frac{1}{2} \right)}{M} + \left( -\frac{1}{2} B^4 g \right).
\]

For \( g = 1 \), the values of the first four levels of \( 2E_n \) are listed in table II.
The above calculations are carried out in the non-Hermitian representation. While the energy levels are the same in any representation, the 1PI amplitudes do differ from representation to another. In fact, the physical 1PI amplitudes have to be obtained by endowing the Hilbert space with the inner product $\langle 0 | \hat{O} | 0 \rangle_\eta^+$, where $\eta^+$ is the positive definite metric operator and $\hat{O}$ is the operator representing the amplitude \[9, 10\]. To obtain $\eta^+$, let us rewrite the Hamiltonian in the form

$$H = H_0 - \frac{g}{2} B^4 + \epsilon H_I,$$

where $\epsilon$ is used for book keeping purposes which will be sent to 1 at the end of the calculations. Note that, the presence of the $-\psi^4$ in the effective Hamiltonian in Eq.\[3\] may lead to the conclusion that this term is non-Hermitian. However, quantum field calculations uses a free field representation or in a quantum mechanical language we are working in a space of harmonic oscillator basis which are square integrable on the real axis. Accordingly, the effective field theory in Eq.\[3\] is a real line problem and thus the operator $-\psi^4$ is Hermitian. Moreover, the Hamiltonian in Eq.\[3\] is in the quantum mechanical form:

$$\frac{p^2}{2} + \rho x + \alpha x^2 + \beta x^3 + \gamma x^4,$$

where the case of $\gamma < 0$ has been investigated in a pure real line study \[12\]. In this study, it has been shown that the Hamiltonian form is a quasi-exactly solvable potential because the Hamiltonian can be written in a Lie algebraic form \[13, 14\]:

$$H = \sum_{a,b} C_{a,b} J^a J^b + \sum_a C_a J^a,$$  \[12\]

where $J^i$ is a set of first order differential operators which generate a finite-dimensional Lie algebra. Accordingly, the first few levels can be obtained exactly. Moreover, the study in

| n | $2E_n$ (Exact) | $E_n$ (our prediction) | $E_n$ (WKB) |
|---|---|---|---|
| 0 | 1.4771 | 1.3628 | 1.3765 |
| 1 | 6.0033 | 6.3335 | 5.9558 |
| 2 | 11.8023 | 12.584 | 11.7689 |
| 3 | 18.4590 | 19.739 | 18.4321 |

TABLE II: $2E_n$ at $g = 1$ compared to the exact values and WKP predictions from Ref.\[1\].
Ref. [12], showed that the ground state function is square integrable provided that \( \beta \) is pure imaginary as in our case. In other words, rather than the original theory in Eq.(1) which is a non-real line problem, the study supports our prediction that the Hamiltonian in Eq.(3) is a real line problem. In fact, one can show that

\[
J = -\frac{1}{2}i - 2\sqrt{g}B^3 \frac{i}{i},
\]

and since \( J \) (called the spin, \( J^a, J^b \) are characterized by \( J = 0, \frac{1}{2}, 1, \ldots \)) should be positive real then \( B \) should be negative and imaginary. This explains our choice of the negative root for \( B \) in Eq.(11). In fact, a detailed analysis of the spectrum and wave functions of the theory using the quasi-exactly solvable technique will follow in another paper (in progress).

Now, since \( \eta_+ H \eta_+^{-1} = H^\dagger \) where \( \eta_+ = \exp(-Q) \) and \( Q = Q_0 + \epsilon Q_1 + \epsilon^2 Q_2 + \ldots \), we can get

\[
H^\dagger = \exp(-Q)H \exp(Q) = H + [-Q, H] + \frac{[-Q, [-Q, H]]}{2!} + \frac{[-Q, [-Q, [-Q, H]]]}{3!} + \ldots
\]

Up to first order in \( \epsilon \) we get \( (H^\dagger_1 - H_1) = \frac{1}{2}[-Q_1, H_0] \Rightarrow [Q_1, H_0] = 2i \text{Im} H_1 \), where \( \text{Im} H_1 \) is the imaginary part of \( H_1 \). This leads to the first order \( Q \) operator for the \( \mathcal{PT} \)-symmetric quasi-particle Hamiltonian in Eq.(3) of the form:

\[
Q_1 = -\frac{4}{3M^4} \frac{\Psi^4 B}{\Pi^3} - \frac{2}{M^2} \frac{\Psi^4 B}{3} \left( \Pi \psi^2 + \psi \Pi \psi + \psi^2 \Pi \right) - \frac{2(g \Pi B^3)}{M^2} \Pi.
\]

In fact, one can go beyond this order in a systematic way but it is out of the scope of this work. What we need to clarify is that the traditional quantum field effective theory can be applied to the \( \mathcal{PT} \)-symmetric theories without the need to the choice of a complex contour and try to investigate the theory on this contour. In fact, this is an interesting result especially for people who keep an eye on the Higgs mechanism for which the traditional effective field theory is popular while the method of choosing a complex contour is not willing to work in higher dimensions (quantum field theory).

To test the range of the coupling for which the method is applicable to the case under investigations, we plotted the effective field mass \( M \), the vacuum condensate squared \( B^2 \) and the ground state energy in Figs.3, 4 and 5 respectively. In fact, the effective coupling
of the theory is \( \frac{g}{\sqrt{x^4}} = \frac{1}{6} \) \[11\]. Accordingly, the method is perturbative for the whole range of the coupling constant. In Figs.\[3\] we can realize that the effective field mass parameter goes to zero as \( g \to 0^+ \) as expected. For the vacuum condensate, Fig. \[4\] we plotted \( B^2 \) for positive values of the coupling \( g \). In this figure, \( B^2 \) is negative and thus the effective field theory is non-Hermitian and \( \mathcal{PT} \)-symmetric for the \( -\phi^4 \) theory. Moreover, the condensate squared blows down as \( g \to 0^+ \), reflecting the full indeterminacy of position for the free particle. To make this point clear, we consider the Hamiltonian form in Eq.(3) as \( g \to 0^+ \). At this limit, \( M \to 0, 2gB = g\sqrt{\frac{M^2}{g}} = \sqrt{-gM^2} \to 0 \) and \( gB^3 = -\frac{3}{8}g^2\sqrt{-\frac{1}{3g}} \to 0 \). Accordingly, the Hamiltonian \( H \) goes to \( \frac{1}{2}\Pi^2 \) which describes a free particle and the corresponding wave solution is a plane wave. Hence, the uncertainty in the position \( x \) is \( \Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle} = \sqrt{\frac{1}{2M}} = \sqrt{\frac{B^2}{\frac{1}{2}x^4}} \), which is known to be infinity for a plane wave and thus explains the big value of the condensation as \( g \to 0^+ \). Moreover, \( \Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle} = \sqrt{\frac{M}{2}} = \sqrt{-\frac{1}{8}x^4} \to 0 \) as \( g \to 0^+ \) with the relation \( \Delta x \Delta p = \hbar/2 \) (\( \hbar = 1 \)) is satisfied. Accordingly, in passing the tests of the axioms of quantum mechanics, our calculations are proved to be valid.

In conclusion, we used an effective field theory formulation to calculate the energy spectrum for the massless \( -x^4 \). The calculations showed that the effective potential, up to first order in the coupling, is equivalent to GEP. Moreover, the effective potential is bounded from below (Fig.\[2\]) and thus explains the stability of the energy spectrum which is against the naive classical analysis of concluding that the theory is unstable as the classical potential is bounded from above.

From the vacuum energy (effective potential) one is able to predict all the \( 1PI \) amplitudes. We compared our results with known exact numerical calculations and though we have used only first order calculations of the method, our results are in good agreement with the numerical calculations. Moreover, we argued that the effective theory is perturbative for the whole range of the coupling constant and the perturbation series is expected to converge rapidly (the effective coupling \( g_{\text{eff}} = \frac{1}{6} \)).

The effective field theory introduced in Eq.(3) is a general form in the sense that it can be applied for any space time dimensions. What makes this form interesting is that one can use it to calculate the positive definite metric operator for the \( \mathcal{PT} \)-symmetric \( -\phi^4 \) \[15\] which is a forward step toward a concrete formulation of a \( \mathcal{PT} \)-symmetric Higgs mechanism.

At the quantum mechanical level, the effective field theory introduced in Eq.(3) is a
quasi-exactly solvable theory and thus one can obtain a number of energy levels and wave functions exactly with the parameters $M$ and $B$ are obtained from the effective potential. Such kind of calculations are interesting and will follow in another paper.

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FIG. 1: The Feynman diagrams contributing to the first order effective potential.
FIG. 2: The ground state energy as a function of the one pint function $B$ measured in units of $i$ for $g = 0.5$ for the $PT$-symmetric $-x^4$ potential.

FIG. 3: The effective mass versus the coupling $g$ for the $PT$-symmetric $-x^4$ potential.
FIG. 4: The one point function $\langle 0|x|0 \rangle$ squared versus the coupling $g$ for the $\mathcal{PT}$-symmetric $-x^4$ potential.

FIG. 5: The ground state energy versus the coupling $g$ for the $\mathcal{PT}$-symmetric $-x^4$ potential.