ON FACTORIALITY OF NODAL THREEFOLDS

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Abstract. We prove the \( \mathbb{Q} \)-factoriality of a nodal hypersurface in \( \mathbb{P}^4 \) of degree \( n \) with at most \( \frac{(n-1)^2}{4} \) nodes and the \( \mathbb{Q} \)-factoriality of a double cover of \( \mathbb{P}^3 \) branched over a nodal surface of degree \( 2r \) with at most \( \frac{(2r-1)r}{3} \) nodes.

1. Introduction.

Nodal 3-folds\(^1\) arise naturally in many different topics of algebraic geometry. For example, the non-rationality of many smooth rationally connected 3-folds (see \[49\]) is proved via the degeneration to nodal 3-folds (see \[5\], \[70\], \[14\], \[15\]). Obviously, nodal 3-folds are the simplest degenerations of smooth ones. However, the geometry can be very different in smooth and nodal cases.

Every surface in a smooth hypersurface in \( \mathbb{P}^4 \) is a complete intersection by the Lefschetz theorem (see \[3\], \[9\], \[74\]), which is not the case if the hypersurface is nodal. The birational automorphisms of a smooth quartic 3-fold in \( \mathbb{P}^4 \) form a finite group consisting of projective automorphisms (see \[40\]), but for any non-smooth nodal quartic 3-fold this group is always infinite (see \[56\], \[23\], \[52\]). Every smooth quartic 3-fold and every smooth cubic 3-fold in \( \mathbb{P}^4 \) are non-rational (see \[40\], \[21\]), but all singular nodal cubic 3-folds are rational and there are rational nodal quartic 3-folds (see \[51\]). Every smooth double cover of \( \mathbb{P}^3 \) ramified in a sextic or quartic surface is non-rational (see \[39\], \[67\], \[68\], \[69\], \[20\]), but there are rational nodal ones (see \[51\], \[17\]).

The simplest examples of nodal 3-folds are nodal hypersurfaces in \( \mathbb{P}^4 \) and double covers of \( \mathbb{P}^3 \) branched over a nodal surfaces. The latter are called double solids. These 3-folds were studied in \[19\], \[63\], \[70\], \[32\], \[33\], \[23\], \[18\], \[66\], \[54\], \[24\], \[25\], \[26\], \[52\], \[18\], \[17\].

For a given nodal 3-fold, it is one of substantial questions whether it is \( \mathbb{Q} \)-factorial\(^2\) or not. This global topological property has very simple geometrical description. Namely, a three-dimensional ordinary double point admits two small resolutions that differs by a simple flop (see \[14\], \[76\], \[17\]). In particular, a nodal 3-fold with \( k \) nodes has \( 2^k \) small resolutions. Therefore the \( \mathbb{Q} \)-factoriality of a nodal 3-fold implies that it has no projective small resolutions.

Remark 1. The \( \mathbb{Q} \)-factoriality of a nodal 3-fold imposes a very strong geometrical restriction on its birational geometry. For example, \( \mathbb{Q} \)-factorial nodal quartic 3-folds and nodal sextic double solids are non-rational (see \[52\], \[17\]). On the other hand, there are rational non-\( \mathbb{Q} \)-factorial nodal quartic 3-folds and nodal sextic double solids (see \[54\], \[30\], \[17\]).
Consider a double cover \( \pi : X \to \mathbb{P}^3 \) branched over a nodal hypersurface \( S \subset \mathbb{P}^3 \) of degree \( 2r \) and a nodal hypersurface \( V \subset \mathbb{P}^4 \) of degree \( n \). The proof of the following result is due to [19], [76], [28], [25], [20].

**Proposition 2.** The 3-folds \( X \) and \( V \) are \( \mathbb{Q} \)-factorial if and only if the nodes of \( S \subset \mathbb{P}^3 \) and \( V \subset \mathbb{P}^4 \) impose independent linear conditions on homogeneous forms of degree \( 3r - 4 \) and \( 2n - 5 \) respectively.

In particular, \( X \) and \( V \) are \( \mathbb{Q} \)-factorial if \( |\text{Sing}(X)| \leq 3r - 3 \) and \( |\text{Sing}(V)| \leq 2n - 4 \) respectively. The main purpose of this paper is to prove the following two results.

**Theorem 3.** Suppose that \(|\text{Sing}(X)| \leq \frac{(2r-1)r}{3} \). Then \( \text{Cl}(X) \) and \( \text{Pic}(X) \) are generated by the class of \( \pi^*(H) \), where \( H \) is a hyperplane in \( \mathbb{P}^3 \). In particular, \( X \) is \( \mathbb{Q} \)-factorial.

**Theorem 4.** Suppose that \(|\text{Sing}(V)| \leq \frac{(n-1)^2}{4} \). Then \( \text{Cl}(V) \) and \( \text{Pic}(V) \) are generated by the class of a hyperplane section. In particular, \( V \) is \( \mathbb{Q} \)-factorial.

**Remark 5.** The statements of Theorems 3 and 4 are equivalent to the \( \mathbb{Q} \)-factoriality of the 3-folds \( X \) and \( V \) respectively. Indeed, the \( \mathbb{Q} \)-factoriality of \( X \) and \( V \) implies

\[
\text{Cl}(X) \otimes \mathbb{Q} \cong \text{Pic}(X) \otimes \mathbb{Q} \cong \text{Cl}(V) \otimes \mathbb{Q} \cong \text{Pic}(V) \otimes \mathbb{Q} \cong \mathbb{Q}
\]

due to the Lefschetz theorem and [19]. Moreover, the groups \( \text{Pic}(X) \) and \( \text{Pic}(V) \) have no torsion due to the Lefschetz theorem and [19]. On the other hand, the local class group of an ordinary double point is \( \mathbb{Z} \) (see [53]). Therefore, the groups \( \text{Cl}(X) \) and \( \text{Cl}(V) \) have no torsion as well (cf. [19]), which implies the equivalences.

Actually, the bounds for nodes in Theorems 3 and 4 are not sharp. For example, in the case \( r = 3 \) the 3-fold \( X \) is \( \mathbb{Q} \)-factorial if \( |\text{Sing}(X)| \leq 14 \) due to [17], and in the case \( n = 4 \) the 3-fold \( V \) is \( \mathbb{Q} \)-factorial if \( |\text{Sing}(V)| \leq 8 \) due to [15].

**Example 6.** Consider a hypersurface \( X \subset \mathbb{P}(1^4, r) \) given by the equation

\[
u^2 = g^3(x, y, z, t) + h_1(x, y, z, t)f_{2r-1}(x, y, z, t) \subset \mathbb{P}(1^4, r) \cong \text{Proj}(\mathbb{C}[x, y, z, t, u]),
\]

where \( g_i, h_i, \) and \( f_i \) are sufficiently general polynomials of degree \( i \). Let \( \pi : X \to \mathbb{P}^3 \) be a restriction of the natural projection \( \mathbb{P}(1^4, r) \to \mathbb{P}^3 \), induced by an embedding of the graded algebras \( \mathbb{C}[x_0, \ldots, x_{2n}] \subset \mathbb{C}[x_0, \ldots, x_{2n}, y] \). Then \( \pi : X \to \mathbb{P}^3 \) is a double cover branched over a nodal hypersurface

\[
g^3_r(x, y, z, t) + h_1(x, y, z, t)f_{2r-1}(x, y, z, t) = 0
\]

of degree \( 2r \) and \( |\text{Sing}(X)| = (2r-1)r \). Moreover, the 3-fold \( X \) is not \( \mathbb{Q} \)-factorial, because the divisor \( h_1 = 0 \) splits into two non-\( \mathbb{Q} \)-Cartier divisors.

**Example 7.** Let \( V \subset \mathbb{P}^4 \) be a hypersurface

\[
xg_{n-1}(x, y, z, t, w) + yf_{n-1}(x, y, z, t, w) \subset \mathbb{P}^4 \cong \text{Proj}(\mathbb{C}[x, y, z, t, w]),
\]

where \( g_{n-1} \) and \( f_{n-1} \) are general polynomials of degree \( n-1 \). Then \( V \) is nodal and contains the plane \( x = y = 0 \). Hence, the 3-fold \( V \) is not \( \mathbb{Q} \)-factorial and \(|\text{Sing}(V)| = (n-1)^2 \).

Therefore, asymptotically the bounds for nodes in Theorems 3 and 4 are not very far from being sharp. On the other hand, the following result is proved in [18].

**Proposition 8.** Every smooth surface on \( V \) is a Cartier divisor if \( \text{Sing}(V) < (n - 1)^2 \).

Hence, one can expect the following to be true.
Conjecture 9. The inequalities $|\text{Sing}(X)| < (2r - 1)r$ and $|\text{Sing}(V)| < (n - 1)^2$ imply the $\mathbb{Q}$-factoriality of the 3-folds $X$ and $V$ respectively.

The claim of Conjecture 9 is proved for $r \leq 3$ and $n \leq 4$ (see [33], [17], [15]). Unfortunately, we are unable to prove Conjecture 9 in any other case. However, for every given $r$ and $n$ we always can slightly improve the bounds in Theorem 3 and 4. For example, we prove the following result.

Proposition 10. Let $r = 4$ and $n = 5$, i.e. $X$ and $V$ are nodal Calabi-Yau 3-folds, and suppose that $|\text{Sing}(X)| \leq 25$ and $|\text{Sing}(V)| \leq 14$. Then $X$ and $V$ are $\mathbb{Q}$-factorial.

The following result is proved in [18].

Theorem 11. Suppose that the subset $\text{Sing}(V) \subset \mathbb{P}^4$ is a set-theoretic intersection of hypersurfaces of degree $l < \frac{9}{2}$ and $|\text{Sing}(V)| < \frac{32}{n}$. Then $V$ is $\mathbb{Q}$-factorial.

The saturated ideal of a set of $k$ points in general position in $\mathbb{P}^4$ is generated by polynomials of degree at most $\frac{k}{4}$ when $k < (n - 1)^2$ and $n > 72$ by [35]. Therefore, Theorem 11 implies the $\mathbb{Q}$-factoriality of the 3-fold $V$ having less than $\frac{k}{4}(n - 1)^2$ nodes in addition to the assumption that the nodes are in general position in $\mathbb{P}^4$. However, the latter generality condition implicitly assumes that the nodes of $V$ impose independent linear conditions on homogeneous forms of degree $2n - 5$ (see [35]), which implies the $\mathbb{Q}$-factoriality of the 3-fold $V$ due to Proposition 2. We prove the following generalization of Theorem 11.

Theorem 12. Let $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^3}(k)|$ and $\mathcal{D} \subset |\mathcal{O}_{\mathbb{P}^4}(l)|$ be linear subsystems of hypersurfaces vanishing at $\text{Sing}(S)$ and $\text{Sing}(V)$ respectively. Put $\mathcal{H} = \mathcal{H}|_S$ and $\mathcal{D} = \mathcal{D}|_V$. Suppose that inequalities $k < r$ and $l < \frac{9}{2}$ hold. Then $\dim(\text{Bs}(\mathcal{H})) = 0$ implies the $\mathbb{Q}$-factoriality of the 3-fold $X$, and $\dim(\text{Bs}(\mathcal{D})) = 0$ implies the $\mathbb{Q}$-factoriality of the 3-fold $V$.

Corollary 13. Suppose $\text{Sing}(S) \subset \mathbb{P}^3$ and $\text{Sing}(V) \subset \mathbb{P}^4$ are set-theoretic intersections of hypersurfaces of degree $k < r$ and $l < \frac{9}{2}$ respectively. Then $X$ and $V$ are $\mathbb{Q}$-factorial.

From the point of view of birational geometry the most important application of Theorems 3 and 11 is the $\mathbb{Q}$-factoriality condition for a nodal quartic 3-fold and a sextic double solid, i.e. the cases $r = 3$ and $n = 4$ respectively, because in these cases the $\mathbb{Q}$-factoriality implies the non-rationality (see [52], [17]). However, it is possible to apply Theorems 3 and 4 to certain higher-dimensional problems in birational algebraic geometry.

Theorem 14. Let $\tau : U \to \mathbb{P}^s$ be a double cover branched over a hypersurface $F$ of degree $2r$ and $D$ be a hyperplane in $\mathbb{P}^s$ such that $D_1 \cap \cdots \cap D_{s-3}$ is a $\mathbb{Q}$-factorial nodal 3-fold, where $D_i$ is a general divisor in $|\tau^*(D)|$. Then $\text{Cl}(U)$ and $\text{Pic}(U)$ are generated by $\tau^*(D)$.

Theorem 15. Let $W \subset \mathbb{P}^r$ be a hypersurface of degree $n$ such that $H_1 \cap \cdots \cap H_{r-4}$ is a $\mathbb{Q}$-factorial nodal 3-fold, where $H_i$ is a general enough hyperplane section of $W$. Then the groups $\text{Cl}(W)$ and $\text{Pic}(W)$ are generated by the class of a hyperplane section of $W \subset \mathbb{P}^r$.

A priori Theorems 14 and 15 can be used to prove the non-rationality of certain singular hypersurfaces of degree $r$ in $\mathbb{P}^r$ and double covers of $\mathbb{P}^4$ branched over singular hypersurfaces of degree $2s$ (see [53], [57], [58], [12], [60], [31], [16]). However, in the former case the problem can be very hard in general, but in the latter case the application of Theorems 14 can be very effective. For example, we prove the following result.
Proposition 16. Let \( \xi : Y \to \mathbb{P}^4 \) be a double cover branched over a hypersurface \( F \subset \mathbb{P}^4 \) of degree 8 such that \( F \) is smooth outside of a smooth curve \( C \subset F \), the singularity of the hypersurface \( F \) in sufficiently general point of \( C \) is locally isomorphic to the singularity
\[
x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]
the singularities of \( F \) in other points of \( C \) are locally isomorphic to the singularity
\[
x_1^2 + x_2^2 + x_3^2x_4 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]
and a general 3-fold in the linear system \( | - K_Y | \) is \( \mathbb{Q} \)-factorial. Then \( Y \) is a birationally rigid\(^3\) terminal \( \mathbb{Q} \)-factorial Fano 4-fold with \( \text{Pic}(Y) \cong \mathbb{Z} \) and \( \text{Bir}(Y) \) is a finite group consisting of biregular automorphisms. In particular, the 4-fold \( Y \) is non-rational.

Example 17. Let \( Y \subset \mathbb{P}(1^5, 4) \) be a hypersurface
\[
u^2 = \sum_{i=1}^{3} f_i(x, y, z, t, w)g_i^2(x, y, z, t, w) \subset \mathbb{P}(1^5, 4) \cong \text{Proj}(\mathbb{C}[x, y, z, t, w, u]),
\]
where \( f_i \) and \( g_i \) are sufficiently general non-constant homogeneous polynomials such that \( \deg(f_i) + 2\deg(g_i) = 8 \). Then the natural projection \( \mathbb{P}(1^5, 4) \longrightarrow \mathbb{P}^4 \) induces a double cover \( \tau : Y \to \mathbb{P}^4 \) branched over a hypersurface \( F \subset \mathbb{P}^4 \), whose equation is \( \sum_{i=1}^{3} f_ig_i^2 = 0 \) and which is smooth outside of a curve \( g_1 = g_2 = g_3 = 0 \). Therefore, the 4-fold \( X \) is not rational due to Proposition 16 and Theorems 8 and 12.

It is natural to ask how many nodes can \( X \) and \( V \) have? The best known upper bounds are due to [71]. Namely, \( |\text{Sing}(X)| \leq \text{A}_3(2r) \) and \( |\text{Sing}(V)| \leq \text{A}_4(n) \), where \( \text{A}_i(j) \) is a number of points \((a_1, \ldots, a_i) \subset \mathbb{Z}^i \) such that
\[
(i - 2)\frac{j}{2} + 1 < \sum_{i=1}^{i} a_t \leq \frac{ij}{2}
\]
and all \( a_t \in (0, j) \). Hence, \( |\text{Sing}(X)| \) does not exceed 68, 180 and 375 when \( r = 3, 4 \) and 5 respectively, and \( |\text{Sing}(V)| \) does not exceed 45, 135 and 320 when \( n = 4, 5 \) and 6 respectively. In the case \( n = 4 \) this bound is sharp (see [34]). Moreover, there is only one nodal quartic 3-fold with 45 nodes (see [11]), so-called Burkhardt quartic, which is rational and determinantal (see [54]). In the case \( r = 3 \) there is a better bound \( |\text{Sing}(X)| \leq 65 \) which is sharp (see [64, 10, 4, 38, 71]). In the case \( n = 5 \) there are no known examples of nodal quintic hypersurfaces in \( \mathbb{P}^4 \) with more than 130 nodes (see [66]).

2. Preliminaries.

Let \( X \) be a variety and \( B_X \) be a boundary\(^4\) on \( X \), i.e. \( B_X = \sum_{i=1}^{k} a_iB_i \), where \( B_i \) is a prime divisor on \( X \) and \( a_i \in \mathbb{Q} \). Basic notions, notations and results related to the log pair \((X, B_X)\) are contained in [16], [50], [18]. The log pair \((X, B_X)\) is called movable when every component \( B_i \) is a linear system on \( X \) without fixed components. Basic properties of movable log pairs are described in [1, 22], [23], [52], [11], [13]. In the following we assume that \( K_X \) and \( B_X \) are \( \mathbb{Q} \)-Cartier divisors.

\(^3\)Namely, the 4-fold \( Y \) is a unique Mori fibration birational to \( Y \) (see [23]).

\(^4\) Usually boundaries are assumed to be effective (see [10]), but we do not assume this.
Suppose that Theorem 22.

Let $\Lambda$ be a base locus of the linear system $D = K_X + B_X + H$ is Cartier. Then $H^i(X, I(X, B_X) \otimes O_X(D)) = 0$ for $i > 0$.

Proof. Let $f : W \to X$ be a log resolution of $(X, B_X)$. Then for $i > 0$

$$R^i f_* (f^* (O_X(D)) \otimes O_W([-B^W])) = 0$$

by the relative Kawamata-Viehweg vanishing (see [13], [73]). The equality

$$R^i f_* (f^* (O_X(D)) \otimes O_W([-B^W])) = I(X, B_X) \otimes O_X(D)$$

and the degeneration of local-to-global spectral sequence imply that for all $i$

$$H^i(X, I(X, B_X) \otimes O_X(D)) = H^i(W, f^*(O_X(D)) \otimes O_W([-B^W])),$$

but $H^i(W, f^*(O_X(D)) \otimes O_W([-B^W])) = 0$ for $i > 0$ by Kawamata-Viehweg vanishing. 

Consider the following application of Theorem 22 (cf. [29], [15]).

Lemma 23. Let $\Sigma \subset \mathbb{P}^n$ be a finite subset, $\mathcal{M}$ be a linear system of hypersurfaces of degree $k$ passing through all points of the set $\Sigma$. Suppose that the base locus of the linear system $\mathcal{M}$ is zero-dimensional. Then the points of the set $\Sigma$ impose independent linear conditions on the homogeneous forms on $\mathbb{P}^n$ of degree $n(k - 1)$.

Proof. Let $\Lambda \subset \mathbb{P}^n$ be a base locus of the linear system $\mathcal{M}$. Then $\Sigma \subseteq \Lambda$ and $\Lambda$ is a finite subset in $\mathbb{P}^n$. Now consider sufficiently general different divisors $H_1, \ldots, H_s$ in the linear system $\mathcal{M}$ for $s \gg 0$. Let $X = \mathbb{P}^n$ and $B_X = \sum_{i=1}^s H_i$. Then $\text{Supp}(\mathcal{L}(X, B_X)) = \Lambda$.

To prove the claim it is enough to prove that for every point $P \in \Sigma$ there is a hypersurface in $\mathbb{P}^n$ of degree $n(k - 1)$ that passes through all the points in the set $\Sigma \setminus P$ and does
not pass through the point $P$. Let $\Sigma \setminus P = \{P_1, \ldots, P_k\}$, where $P_i$ is a point of $X = \mathbb{P}^n$, and let $f : V \to X$ be a blow up at the points of $\Sigma \setminus P$. Then

$$K_V + (B_V + \sum_{i=1}^{k} (\text{mult}_{P_i}(B_X) - n)E_i) + f^*(H) = f^*(n(k-1)H) - \sum_{i=1}^{k} E_i,$$

where $E_i = f^{-1}(P_i)$, $B_V = f^{-1}(B_X)$ and $H$ is a hyperplane in $\mathbb{P}^n$. By construction

$$\text{mult}_{P_i}(B_X) = n\text{mult}_{P_i}(\mathcal{M}) \geq n$$

and the boundary $\hat{B}_V = B_V + \sum_{i=1}^{k} (\text{mult}_{P_i}(B_X) - n)E_i$ is effective.

Let $P = f^{-1}(P)$. Then $P \in \text{LCS}(V, \hat{B}_V)$ and $P$ is an isolated center of log canonical singularities of the log pair $(V, \hat{B}_V)$, because in the neighborhood of the point $P$ the birational morphism $f : V \to X$ is an isomorphism. On the other hand, the map

$$H^0(\mathcal{O}_V(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i)) \to H^0(\mathcal{O}_{\text{LCS}(V, \hat{B}_V)} \otimes \mathcal{O}_V(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i))$$

is surjective by Theorem 22. However, in the neighborhood of the point $\bar{P}$ the support of the subscheme $\mathcal{L}(V, \hat{B}_V)$ consists just of the point $\bar{P}$. The latter implies the existence of a divisor $D \subset |f^*(n(k-1)H) - \sum_{i=1}^{k} E_i|$ that does not pass through $\bar{P}$. Thus, $f(D)$ is a hypersurface in $\mathbb{P}^n$ of degree $n(k-1)$ that passes through the points of $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. 

Actually, the proof of Lemma 23 implies Theorem 12.

**Proof of Theorem 12.** We have a double cover $\pi : X \to \mathbb{P}^3$ branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree $2r$, a linear subsystem $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^3}(k)|$ of hypersurfaces vanishing at the points of the set $\text{Sing}(S)$ for $k < r$, a nodal hypersurface $V \subset \mathbb{P}^4$ of degree $n$, a linear subsystem $\mathcal{D} \subset |\mathcal{O}_{\mathbb{P}^4}(l)|$ of hypersurfaces vanishing at $\text{Sing}(V)$ for $l < \frac{n}{2}$, and

$$\dim(\text{Bs}(\mathcal{H})) = \dim(\text{Bs}(\mathcal{D})) = 0,$$

where $\mathcal{H} = \mathcal{H}|_S$ and $\mathcal{D} = \mathcal{D}|_V$. Due to Proposition 2 we must show that the nodes of the surface $S \subset \mathbb{P}^3$ and the nodes of the hypersurface $X \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ and $2n - 5$ respectively.

Suppose that the stronger condition

$$\dim(\text{Bs}(\mathcal{H})) = \dim(\text{Bs}(\mathcal{D})) = 0$$

holds, which is enough for Corollary 13. Then Lemma 23 immediately implies that the nodes of $S$ and the nodes of $X$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ and $2n - 5$ respectively. In the general case we can repeat the proof of Lemma 23 interchanging the boundary $\frac{2}{s} \sum_{i=1}^{s} H_i$ with the boundary $S + \frac{1}{s} \sum_{i=1}^{s} H_i$ for the proof of the $\mathbb{Q}$-factoriality of $X$ and interchanging the boundary $\frac{4}{s} \sum_{i=1}^{s} H_i$ with the boundary $V + \frac{2}{s} \sum_{i=1}^{s} H_i$ for the proof of the $\mathbb{Q}$-factoriality of the 3-fold $V$. 

The following result is Theorem 17.4 in 50 and Theorem 7.4 in 48.

**Theorem 24.** Let $g : X \to Z$ be a contraction such that $g_*(\mathcal{O}_X) = \mathcal{O}_Z$, $h : V \to X$ be a log resolution of the log pair $(X, B_X)$. Suppose the divisor $-(K_X + B_X)$ is $g$-nef and $g$-big, and $\text{codim}(g(B_i) \subset Z) \geq 2$ whenever $a_i < 0$. Define $a_E \in \mathbb{Q}$ by means of the equivalence

$$K_V \sim_{Q} f^*(K_X + B_X) + \sum_{E \subset V} a_E E,$$
where $E \subset V$ is a not necessary $h$-exceptional divisor. Then the locus $\cup_{a \leq -1} E$ is connected in a neighborhood of every fiber of the morphism $g \circ h$.

The following result is a corollary of Theorem 24 (see Theorem 17.6 in [50]).

**Theorem 25.** Suppose that $B_X$ is effective and $|B_X| = \emptyset$. Let $S \subset X$ be an effective irreducible divisor such that the divisor $K_X + S + B_X$ is $\mathbb{Q}$-Cartier. Then $(X, S + B_X)$ is purely log terminal if and only if $(S, \text{Diff}_S(B_X))$ is Kawamata log terminal.

**Definition 26.** A proper irreducible subvariety $Y \subset X$ is called a center of canonical singularities of $(X, B_X)$ if there is a birational morphism $f : W \to X$ and an $f$-exceptional divisor $E \subset W$ such that the discrepancy $a(X, B_X, E) \leq 0$ and $f(E) = Y$. The set of all centers of canonical singularities of the log pair $(X, B_X)$ is denoted by $\mathcal{CS}(X, B_X)$.

**Corollary 27.** Let $H$ be an effective Cartier divisor on $X$ and $Z \in \mathcal{CS}(X, B_X)$, suppose that $X$ and $H$ are smooth in the generic point of $Z$, $Z \subset H$, $H \not\subset \text{Supp}(B_X)$ and $B_X$ is an effective boundary. Then $\text{LCS}(H, B_X|_H) \neq \emptyset$.

The following result is Corollary 7.3 in [59], which holds even over fields of positive characteristic and implicitly goes back to [10] (see [22], [23], [42]).

**Theorem 28.** Suppose that $X$ is smooth, $\dim(X) \geq 3$, the boundary $B_X$ is effective and movable, and the set $\mathcal{CS}(X, M_X)$ contains a closed point $O \in X$. Then $\text{mult}_O(B_X^2) \geq 4$ and the equality implies $\text{mult}_O(B_X) = 2$ and $\dim(X) = 3$.

The following result is implied by Theorem 3.10 in [23] and Theorem 29.

**Theorem 29.** Suppose that $\dim(X) \geq 3$, $B_X$ is effective, and the set $\mathcal{CS}(X, B_X)$ contains an ordinary double point $O$ of $X$. Then $\text{mult}_O(B_X) \geq 1$, where $\text{mult}_O(B_X)$ is defined by means of the regular blow up of $O$. Moreover, $\text{mult}_O(B_X) = 1$ implies $\dim(X) = 3$.

The following result is an easy modification of Theorem 29.

**Proposition 30.** Suppose that $\dim(X) = 3$, $B_X$ is effective, and the set $\mathcal{CS}(X, B_X)$ contains an isolated singular point $O$ of the variety $X$, which is locally isomorphic to the singularity $y^3 = \sum_{i=1}^3 x_i^2$. Let $f : W \to X$ be a blow up of $O$ and $\text{mult}_O(B_X)$ be a rational number defined by means of the equivalence

$$f^{-1}(B_X) \sim_\mathbb{Q} f^{-1}(B_X) + \text{mult}_O(B_X)E,$$

where $E$ is an $f$-exceptional divisor. Then $\text{mult}_O(B_X) \geq \frac{1}{2}$.

**Proof.** The $3$-fold $W$ is smooth, $E$ is isomorphic to a cone in $\mathbb{P}^3$ over a smooth conic, the restriction $-E|_E$ is rationally equivalent to a hyperplane section of $E \subset \mathbb{P}^3$, and

$$K_W + B_W \sim_\mathbb{Q} f^*(K_X + B_X) + (1 - \text{mult}_O(B_X))E,$$

where $B_W = f^{-1}(B_X)$. Suppose that $\text{mult}_O(B_X) < \frac{1}{2}$. Then there is a proper irreducible subvariety $Z \subset E$ such that $Z \in \mathcal{CS}(W, B_W)$. Hence, $\text{LCS}(E, B_W|_E) \neq \emptyset$ by Theorem 25.

Let $B_E = B_W|_E$. Then $\text{LCS}(E, B_E)$ does not contains curves on $E$, because otherwise the intersection of $B_E$ with the ruling of $E$ is greater than $\frac{1}{2}$, which is impossible due to our assumption $\text{mult}_O(B_X) < \frac{1}{2}$. Therefore, $\dim(\text{Supp}(\mathcal{L}(E, B_E))) = 0$.

Let $H$ be a hyperplane of $E \subset \mathbb{P}^3$. Then

$$K_E + B_E + (1 - \text{mult}_O(B_X))H \sim_\mathbb{Q} -H.$$
and $H^0(O_E(-H)) = 0$. On the other hand, Theorem 22 implies surjectivity

$$H^0(O_E(-H)) \to H^0(O_{E_{B,E}}) \to 0,$$

which is a contradiction. □

The following result is due to 22 (see 59, 13).

**Theorem 31.** Let $X$ be a Fano variety with $\text{Pic}(X) \cong \mathbb{Z}$ with terminal $\mathbb{Q}$-factorial singularities such that either $X$ is not birationally rigid or $\text{Bir}(X) \neq \text{Aut}(X)$. Then there are a linear system $\mathcal{M}$ on $X$ having no fixed components and $\mu \in \mathbb{Q}_{>0}$ such that the singularities of the movable log pair $(X, \mu \mathcal{M})$ are not canonical and $\mu \mathcal{M} \sim -K_X$.

The following result is proved in 10 using the vanishing theorem for 2-connected divisors on algebraic surfaces in 61 in a way similar to 7 and 72.

**Theorem 32.** Let $\pi : Y \to \mathbb{P}^2$ be the blow up at points $P_1, \ldots, P_s$ on $\mathbb{P}^2$, $s \leq \frac{d^2+9d+10}{6}$, at most $k(d+3-k) - 2$ of the points $P_i$ lie on a curve of degree $k \leq \frac{d+3}{2}$, where $d \geq 3$ is a natural number. Then the linear system $|\pi^*(O_{\mathbb{P}^2}(d)) - \sum_{i=1}^s E_i|$ is free, where $E_i = \pi^{-1}(P_i)$.

**Corollary 33.** Let $\Sigma \subset \mathbb{P}^2$ be a finite subset such that the inequality $|\Sigma| \leq \frac{d^2+9d+16}{6}$ holds and at most $k(d+3-k) - 2$ points of $\Sigma$ lie on a curve of degree $k \leq \frac{d+3}{2}$, where $d \geq 3$ is a natural number. Then for every point $P \in \Sigma$ there is a curve $C \subset \mathbb{P}^2$ of degree $d$ that passes through all the points in $\Sigma \setminus P$ and does not pass through the point $P$.

In the case $d = 3$ the claim of Theorem 32 is nothing but the freeness of the anticanonical linear system of a weak del Pezzo surface of degree $9 - s \geq 2$ (see 27).

### 3. Double solids.

In this section we prove Theorem 3. Let $\pi : X \to \mathbb{P}^3$ be a double cover branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree $2r$ such that $|\text{Sing}(S)| \leq \frac{(2r-1)r}{3}$. We must show that the nodes of $S \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree $3r-4$ on $\mathbb{P}^3$ due to Proposition 2 and Remark 5. Moreover, we may assume $r \geq 3$, because in the case $r \leq 2$ the required claim is trivial.

**Definition 34.** The points of a subset $\Gamma \subset \mathbb{P}^3$ satisfy the property $\nabla$ if at most $k(2r-1)$ points of the set $\Gamma$ can lie on a curve in $\mathbb{P}^3$ of degree $k \in \mathbb{N}$.

Let $\Sigma = \text{Sing}(S) \subset \mathbb{P}^3$.

**Proposition 35.** The points of the subset $\Sigma \subset \mathbb{P}^3$ satisfy the property $\nabla$.

**Proof.** Let $L \subset \mathbb{P}^3$ be a line and $\Pi \subset \mathbb{P}^3$ be a sufficiently general hyperplane passing through the line $L$. Then $\Pi \not\subset S$ and $\Pi \cap S = L \cup Z$, where $Z \subset \Pi$ is a plane curve of degree $2r - 1$. Moreover, we have

$$\Sigma \cap L = \text{Sing}(S) \cap L \subset L \cap Z,$$

but $|L \cap Z| \leq 2r - 1$. Thus, at most $2r - 1$ points of $\Sigma$ can lie on a line.

Let $C \subset \mathbb{P}^3$ be a curve of degree $k > 1$. We must show that at most $k(2r-1)$ points of $\Sigma$ can lie on $C$. We may assume that $C$ is irreducible and reduced. Consider a general cone $Y \subset \mathbb{P}^3$ over the curve $C$. Then $Y \not\subset S$ and $Y \cap S = C \cup R$, where $R$ is an irreducible reduced curve of degree $k(2r-1)$. As above we have the inclusion

$$\Sigma \cap C = \text{Sing}(S) \cap C \subset C \cap R,$$
but in the set-theoretic sense $|C \cap R| \leq (2r - 1)k$. Hence, at most $k(2r - 1)$ points of the subset $\Sigma \subset \mathbb{P}^3$ can lie on the irreducible reduced curve $C \subset \mathbb{P}^3$ of degree $k$. \hfill \Box

Fix a point $P \in \Sigma$. To prove that the points of $\Sigma \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree $3r - 4$ it is enough to construct a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$.

**Lemma 36.** Suppose $\Sigma \subset \Pi$ for some hyperplane $\Pi \subset \mathbb{P}^3$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$.

**Proof.** Let us apply Corollary 33 to $\Sigma \subset \Pi$ and $d = 3r - 4 \geq 5$. We must check that all the conditions of Corollary 33 are satisfied, which is easy but not obvious. First of all

$$|\Sigma| \leq \frac{(2r - 1)r}{3} \Rightarrow |\Sigma| \leq \frac{d^2 + 9d + 16}{6}$$

and at most $d = 3r - 4$ points of $\Sigma$ can lie on a line in $\Pi$ because $r \geq 3$ and the points of the subset $\Sigma \subset \Pi$ satisfy the property $\nabla$ due to Proposition 35. 

Now we must prove that at most $k(3r - 1 - k) - 2$ points of $\Sigma$ can lie on a curve of degree $k \leq \frac{3r - 1}{2}$. The case $k = 1$ is already done. Moreover, at most $k(2r - 1)$ points of the set $\Sigma$ can lie on a curve of degree $k$ by Proposition 35. Thus, we must show that

$$k(3r - 1 - k) - 2 \geq k(2r - 1)$$

for all $k \leq \frac{3r - 1}{2}$. Moreover, we must prove the latter inequality only for such $k > 1$ that the inequality $k(3r - 1 - k) - 2 < |\Sigma|$ holds, because otherwise the corresponding condition on the points of the set $\Sigma$ is vacuous. Moreover, we have

$$k(3r - 1 - k) - 2 \geq k(2r - 1) \iff r > k,$$

because $k > 1$. Suppose that the inequality $r \leq k$ holds for some natural number $k$ such that $k \leq \frac{3r - 1}{2}$ and $k(3r - 1 - k) - 2 < |\Sigma|$. Let $g(x) = x(3r - 1 - x) - 2$. Then $g(x)$ is increasing for $x < \frac{3r - 1}{2}$. Thus, we have $g(k) \geq g(r)$, because $\frac{3r - 1}{2} \geq k \geq r$. Hence,

$$\frac{(2r - 1)r}{3} \geq \frac{|\Sigma|}{g(k)} \geq g(r) = r(2r - 1) - 2,$$

which is impossible when $r \geq 3$.

Therefore, there is a curve $C \subset \Pi$ of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through $P$ by Corollary 33. Let $Y \subset \mathbb{P}^3$ be a sufficiently general cone over the curve $C \subset \Pi \cong \mathbb{P}^2$. Then $Y \subset \mathbb{P}^3$ is a hypersurface of degree $3r - 4$ that passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. \hfill \Box

Take a sufficiently general hyperplane $\Pi \subset \mathbb{P}^3$. Let $\psi : \mathbb{P}^3 \dashrightarrow \Pi$ be a projection from a sufficiently general point $O \in \mathbb{P}^3$, $\Sigma' = \psi(\Sigma) \subset \Pi \cong \mathbb{P}^2$ and $P = \psi(P) \in \Sigma'$.

**Lemma 37.** Suppose that the points of $\Sigma' \subset \Pi$ satisfy the property $\nabla$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ containing $\Sigma \setminus P$ and not passing through $P$.

**Proof.** The proof of the claim of Lemma 33 implies the existence of a curve $C \subset \Pi$ of degree $3r - 4$ that passes through $\Sigma' \setminus P$ and does not pass through $P$. Let $Y \subset \mathbb{P}^3$ be a cone over the curve $C$ with the vertex $O$. Then $Y \subset \mathbb{P}^3$ is a hypersurface of degree $3r - 4$ that passes through $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

It seems to us that the points of the subset $\Sigma' \subset \Pi$ always satisfy the property $\nabla$ due to the generality in the choice of the projection $\psi : \mathbb{P}^3 \dashrightarrow \Pi$. Unfortunately, we are unable
to prove it. Hence, we may assume that the points of the subset $\Sigma' \subset \Pi \cong \mathbb{P}^2$ do not satisfy the property $\nabla$. Let us clarify this assumption.

**Definition 38.** The points of a subset $\Gamma \subset \mathbb{P}^s$ satisfy the property $\nabla_k$ if at most $i(2r - 1)$ points of the set $\Gamma$ can lie on a curve in $\mathbb{P}^s$ of degree $i \in \mathbb{N}$ for all $i \leq k$.

Therefore, there is a smallest $k \in \mathbb{N}$ such that the points of $\Sigma' \subset \Pi$ do not satisfy the property $\nabla_k$. Namely, there is a subset $\Lambda^1_k \subset \Sigma$ such that $|\Lambda^1_k| > k(2r - 1)$ and all the points of the set

$$\tilde{\Lambda}^1_k = \psi(\Lambda^1_k) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on a curve $C \subset \Pi$ of degree $k$. Moreover, the curve $C$ is irreducible and reduced due to the minimality of $k$. In the case when the points of the subset $\Sigma' \setminus \tilde{\Lambda}^1_k \subset \Pi$ does not satisfy the property $\nabla_k$ we can find subset $\Lambda^2_k \subset \Sigma \setminus \Lambda^1_k$ such that $|\Lambda^2_k| > k(2r - 1)$ and all the points of the set $\tilde{\Lambda}^2_k = \psi(\Lambda^2_k)$ lie on an irreducible curve of degree $k$. Thus, we can iterate this construction $c_k$ times and get $c_k > 0$ disjoint subsets

$$\Lambda^i_k \subset \Sigma \setminus \bigcup_{j=1}^{i-1} \Lambda^j_k \subsetneq \Sigma$$

such that $|\Lambda^i_k| > k(2r - 1)$, all the points of the subset $\tilde{\Lambda}^i_k = \psi(\Lambda^i_k) \subset \Sigma'$ lie on an irreducible reduced curve on $\Pi$ of degree $k$, and all the points of the subset

$$\Sigma' \setminus \bigcup_{i=1}^{c_k} \tilde{\Lambda}^i_k \subset \Pi \cong \mathbb{P}^2$$

satisfy the property $\nabla_k$. Now we can repeat this construction for the property $\nabla_{k+1}$ and find $c_{k+1} \geq 0$ disjoint subsets

$$\Lambda^i_{k+1} \subset \big( \Sigma \setminus \bigcup_{i=1}^{c_k} \Lambda^i_k \big) \setminus \bigcup_{j=1}^{i-1} \Lambda^j_{k+1} \subset \Sigma \setminus \bigcup_{i=1}^{c_k} \Lambda^i_k \subsetneq \Sigma$$

such that $|\Lambda^i_{k+1}| > (k+1)(2r - 1)$, the points of $\tilde{\Lambda}^i_{k+1} = \psi(\Lambda^i_{k+1}) \subset \Sigma'$ lie on an irreducible reduced curve on $\Pi$ of degree $k + 1$, and the points of the subset

$$\Sigma' \setminus \bigcup_{j=1}^{k+1} \tilde{\Lambda}^i_j \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

satisfy the property $\nabla_{k+1}$. Now we can iterate this construction for $\nabla_{k+2}, \ldots, \nabla_l$ and get disjoint subsets $\Lambda^j_k \subset \Sigma$ for $j = k, \ldots, l \geq k$ such that $|\Lambda^j_k| > j(2r - 1)$, all the points of the subset $\tilde{\Lambda}^j_k = \psi(\Lambda^j_k) \subset \Sigma'$ lie on an irreducible reduced curve of degree $j$ in $\Pi$, and all the points in the subset

$$\tilde{\Sigma} = \Sigma' \setminus \bigcup_{j=k}^{l} \tilde{\Lambda}^j_k \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

satisfy the property $\nabla$, where $c_j \geq 0$ is a number of subsets $\tilde{\Lambda}^j_k$. Note, that $c_k > 0$.

**Remark 39.** The subset $\Lambda^1_k \subset \Sigma$ is non-empty. However, every subset $\Lambda^j_k \subset \Sigma$ a priori can be empty when $j \neq k$ or $i \neq 1$. Moreover, the subset $\tilde{\Sigma} \subset \Sigma'$ can be empty as well.

**Remark 40.** The inequality $|\tilde{\Sigma}| < \frac{(2r-1)r}{3} - \sum_{i=k}^{l} c_i(2r - 1)i = \frac{(2r-1)}{3}(r - 3 \sum_{i=k}^{l} ic_i)$ holds.
Corollary 41. The inequality $\sum_{i=1}^{l} \sum_{k=1}^{n} c_i < {r \over 3}$ holds.

In particular, $\Lambda_j \neq \emptyset$ implies $j < {r \over 3}$.

Lemma 42. Suppose that $\Lambda_j \neq \emptyset$. Let $M$ be a linear system of hypersurfaces of degree $j$ in $\mathbb{P}^3$ passing through all the points in $\Lambda_j$. Then the base locus of $M$ is zero-dimensional.

Proof. By the construction of the set $\Lambda_j$ all the points of the subset

$$\tilde{\Lambda}_j \subset \psi(\Lambda_j) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on an irreducible reduced curve $C \subset \Pi$ of degree $j$. Let $Y \subset \mathbb{P}^3$ be a cone over $C$ with the vertex $O$. Then $Y$ is a hypersurface in $\mathbb{P}^3$ of degree $j$ that contains all the points of the set $\Lambda_j$. Therefore, $Y \in M$.

Suppose that the base locus of the linear system $M$ contains an irreducible reduced curve $Z \subset \mathbb{P}^3$. Then $Z \subset Y$ and $\psi(Z) = C$. Moreover, $\Lambda_j \subset Z$, because $\Lambda_j \not\subset Z$ implies that $\Lambda_j \not\subset C$ due to the generality of $\psi$. Finally, the restriction $\psi|_Z : Z \to C$ is a birational morphism, because the projection $\psi$ is general. Hence, $\deg(Z) = j$ and $Z$ contains at least $|\Lambda_j| > j(2r - 1)$ points of $\Sigma$. The latter contradicts Proposition 33.

Corollary 43. The inequality $k \geq 2$ holds.

For every $\Lambda_j \neq \emptyset$ let $\Xi_j \subset \mathbb{P}^3$ be a base locus of the linear system of hypersurfaces of degree $j$ in $\mathbb{P}^3$ passing through all the points in $\Lambda_j$. For $\Lambda_j = \emptyset$ put $\Xi_j = \emptyset$. Then $\Xi_j$ is a finite set by Lemma 33 and $\Lambda_j \subset \Xi_j$ by construction.

Lemma 44. Suppose that $\Xi_j \neq \emptyset$. Then the points of the subset $\Xi_j \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3(j - 1)$.

Proof. The claim follows from Lemma 23.

Corollary 45. Suppose that $\Lambda_j \neq \emptyset$. Then the points of the subset $\Lambda_j \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^3$ of degree $3(j - 1)$.

Lemma 46. Suppose that $\Sigma = \emptyset$. Then there is a hypersurface in $\mathbb{P}^3$ of degree $3r - 4$ containing $\Sigma \setminus P$ and not passing through the point $P$.

Proof. The set $\Sigma$ is a disjoint union $\cup_{j=k}^{l} \cup_{i=1}^{n} \Lambda_i$ and there is a unique set $\Lambda^b \subset \Pi$ containing the point $P$. In particular, $P \in \Xi^b$. On the other hand, the union $\cup_{j=k}^{l} \cup_{i=1}^{n} \Xi^i$ is not necessary disjoint. Thus, a priori the point $P$ can be contained in many sets $\Xi^i$.

For every $\Xi^i \neq \emptyset$ containing $P$ there is a hypersurface of degree $3(j - 1)$ that passes through $\Xi^i \setminus P$ and does not pass through $P$ by Lemma 33. For every $\Xi^i \neq \emptyset$ containing the point $P$ there is a hypersurface of degree $j$ that passes through $\Xi^i$ and $P$, but does not pass through the point $P$ by the definition of the set $\Xi^i$. Moreover, $j < 3(j - 1)$, because $k \geq 2$ by Corollary 43. Therefore, for every $\Xi^i \neq \emptyset$ there is a hypersurface $F^i \subset \mathbb{P}^3$ of degree $3(j - 1)$ that passes through $\Xi^i \setminus P$ and does not pass through the point $P$. Let

$$F = \bigcup_{j=k}^{l} \bigcup_{i=1}^{n} F^i \subset \mathbb{P}^3$$
be a possibly reducible hypersurface of degree $\sum_{i=k}^{l} 3(i-1)c_i$. Then $F$ passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point $P$. Moreover, we have

$$\deg(F) = \sum_{i=k}^{l} 3(i-1)c_i < \sum_{i=k}^{l} 3ic_i < r < 3r - 4$$

by Corollary $\Box$ which implies the claim.

Let $\hat{\Sigma} = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_i} \Lambda_j$ and $\Sigma = \Sigma \setminus \hat{\Sigma}$. Then $\Sigma = \hat{\Sigma} \cup \Sigma$ and $\psi(\Sigma) = \hat{\Sigma} \subset \Pi$. Therefore, we proved Theorem $\Box$ in the extreme cases: $\hat{\Sigma} = \emptyset$ and $\Sigma = \emptyset$. Now we must combine the proofs of the Lemmas $37$ and $46$ to prove Theorem $\Box$ in the case $\hat{\Sigma} \neq \emptyset$ and $\Sigma \neq \emptyset$.

Remark $47$. The proof of Lemma $46$ implies the existence of a hypersurface $F \subset \mathbb{P}^3$ of degree $\sum_{i=k}^{l} 3(i-1)c_i$ that passes through all the points of the subset $\hat{\Sigma} \setminus P \subset \Sigma$ and does not pass through the point $P \in \Sigma$.

Put $d = 3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i$. Let us check that the subset $\hat{\Sigma} \subset \Pi \cong \mathbb{P}^2$ and the number $d$ satisfy all the conditions of Theorem $32$. We may assume that $\emptyset \neq \hat{\Sigma} \subset \Sigma$.

Lemma $48$. The inequality $d \geq 6$ holds.

Proof. The inequality

$$|\Sigma| < \left(\frac{2r-1}{3}\right) - \sum_{i=k}^{l} c_i(2r-1)i = \left(\frac{2r-1}{3}\right)(r - 3 \sum_{i=k}^{l} ic_i)$$

implies $\sum_{i=k}^{l} 3ic_i < r$. Thus, $d = 3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i > 2r - 4 + 3c_k \geq 2r - 1 \geq 5$. $\Box$

Lemma $49$. The inequality $|\Sigma| \leq \frac{d^2+9d+10}{6}$ holds.

Proof. By construction $|\Sigma| < \left(\frac{2r-1}{3}\right)(r - 3 \sum_{i=k}^{l} ic_i)$. Thus, we must show that

$$2(2r-1)(r - 3 \sum_{i=k}^{l} ic_i) \leq (3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i)^2 + 9(3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i) + 10,$$

where $c_k \geq 1$ and $\sum_{i=k}^{l} 3ic_i < r$. However, we have

$$(3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i)^2 + 9(3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i) + 10 > (2r - 4 + 3c_k)^2 + 9(2r - 4 + 3c_k) + 10$$

and

$$(2r - 4 + 3c_k)^2 + 9(2r - 4 + 3c_k) + 10 \geq (2r - 1)^2 + 9(2r - 1) + 10 = 4r^2 + 14r + 2,$$

which implies $4r^2 + 14r + 2 > 4r^2 - 2r > 2(2r - 1)(r - 3 \sum_{i=k}^{l} ic_i)$. $\Box$

Lemma $50$. At most $d$ of the points of the subset $\hat{\Sigma} \subset \mathbb{P}^2$ lie on a line in $\mathbb{P}^2$.

Proof. The points of the subset $\hat{\Sigma} \subset \mathbb{P}^2$ satisfy the property $\nabla$. In particular, no more than $2r - 1$ of the points of $\hat{\Sigma}$ lie on a line in $\mathbb{P}^2$. On the other hand, we have

$$d = 3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i > 2r - 4 + 3c_k \geq 2r - 1,$$

which implies the claim. $\Box$
Lemma 51. At most \( k(d + 3 - k) - 2 \) points of \( \Sigma \) lie on a curve in \( \mathbb{P}^2 \) of degree \( k \leq \frac{d+3}{2} \).

Proof. We may assume that \( k > 1 \) due to Lemma 49. The points of the subset \( \Sigma \subset \mathbb{P}^2 \) satisfy the property \( \nabla \). Thus, at most \((2r-1)k\) of the points of \( \Sigma \) lie on a curve in \( \mathbb{P}^2 \) of degree \( k \). Therefore, to conclude the proof it is enough to show that the inequality

\[ k(d + 3 - k) - 2 \geq (2r - 1)k \]

holds for all \( k \leq \frac{d+3}{2} \). Moreover, it is enough to prove the latter inequality only for such natural number \( k > 1 \) that the inequality \( k(d + 3 - k) - 2 < |\Sigma| \) holds, because otherwise the corresponding condition on the points of the set \( \Sigma \) is vacuous.

Now we have

\[ k(d + 3 - k) - 2 \geq k(2r - 1) \iff k\left( r - \sum_{i=k}^{l} (3(i-1)c_i - k) \right) \geq 2 \iff r - \sum_{i=k}^{l} 3(i-1)c_i > k, \]

because \( k > 1 \). We may assume that the inequalities \( r - \sum_{i=k}^{l} 3(i-1)c_i \leq k \leq \frac{d+3}{2} \) and

\[ k(d + 3 - k) - 2 < |\Sigma| \]

hold. Let \( g(x) = x(d + 3 - x) - 2 \). Then \( g(x) \) is increasing for \( x < \frac{d+3}{2} \). Thus, we have

\[ g(k) \geq g(r - \sum_{i=k}^{l} 3(i-1)c_i), \]

because \( \frac{d+3}{2} \geq k \geq r - \sum_{i=k}^{l} 3(i-1)c_i \). Hence, we have

\[ \frac{(2r - 1)}{3}(r - 3 \sum_{i=k}^{l} ic_i) > |\Sigma| > g(k) \geq (r - \sum_{i=k}^{l} 3(i-1)c_i)(2r - 1) - 2 \]

and \((2r - 1)(6 \sum_{i=k}^{l} ic_i - 2r) + 6 - 9 \sum_{i=k}^{l} c_i(2r - 1) > 0\). Now \( \sum_{i=k}^{l} ic_i < \frac{r}{3} \) implies

\[ (2r - 1)(6 \sum_{i=k}^{l} ic_i - 2r) + 6 - 9 \sum_{i=k}^{l} c_i(2r - 1) < 6 - 9 \sum_{i=k}^{l} c_i(2r - 1) < 6 - 9c_k(2r - 1) < 0, \]

which is contradiction.

Therefore, we can apply Theorem 32 to the blow up of the hyperplane \( \Pi \) at the points of the set \( \hat{\Sigma} \setminus \hat{P} \subset \Pi \) due to Lemmas 48, 49 and 51. The application of Theorem 32 gives a curve \( C \subset \Pi \cong \mathbb{P}^2 \) of degree \( 3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i \) that passes through all the points of the set \( \hat{\Sigma} \setminus \hat{P} \) and does not pass through the point \( \hat{P} = \psi(P) \). It should be pointed out that the subset \( \hat{\Sigma} \subset \Sigma' \) may not contain \( \hat{P} \in \Sigma' \). Namely, \( \hat{P} \in \hat{\Sigma} \) if and only if \( P \in \Sigma \).

Let \( G \subset \mathbb{P}^3 \) be a cone over the curve \( C \) with the vertex \( O \), where \( O \in \mathbb{P}^3 \) is the center of the projection \( \psi: \mathbb{P}^3 \longrightarrow \Pi \). Then \( G \) is a hypersurface of degree \( 3r - 4 - \sum_{i=k}^{l} 3(i-1)c_i \) that passes through the points of \( \hat{\Sigma} \setminus \hat{P} \) and does not pass through \( P \). On the other hand, we already have the hypersurface \( F \subset \mathbb{P}^3 \) of degree \( \sum_{i=k}^{l} 3(i-1)c_i \) that passes through the points of \( \hat{\Sigma} \setminus \hat{P} \) and does not pass through \( P \). Therefore, \( F \cup G \subset \mathbb{P}^3 \) is a hypersurface of degree \( 3r - 4 \) that passes through all the points of the set \( \Sigma \setminus P \) and does not pass through the point \( P \in \Sigma \). Hence, we proved Theorem 3.
4. Hypersurfaces in $\mathbb{P}^4$.

In this section we prove Theorem \textcolor{red}{4}. Let $V \subset \mathbb{P}^4$ be a nodal hypersurface of degree $n$ such that $|\text{Sing}(V)| \leq \frac{(n-1)^2}{4}$. In order to prove Theorem \textcolor{red}{4} it is enough to show that the nodes of the hypersurface $V$ impose independent linear conditions on homogeneous forms of degree $2n - 5$ on $\mathbb{P}^4$ due to Proposition \textcolor{red}{2} and Remark \textcolor{red}{3}. Moreover, we always may assume that $n \geq 4$, because in the case $n \leq 3$ the required claim is trivial.

**Definition 52.** The points of a subset $\Gamma \subset \mathbb{P}^r$ satisfy the property $\star$ if at most $k(n-1)$ points of the set $\Gamma$ can lie on a curve in $\mathbb{P}^r$ of degree $k \in \mathbb{N}$.

Let $\Sigma = \text{Sing}(V) \subset \mathbb{P}^4$.

**Proposition 53.** The points of the subset $\Sigma \subset \mathbb{P}^4$ satisfy the property $\star$.

**Proof.** Let $C \subset \mathbb{P}^4$ be an irreducible and reduced curve of degree $k$. Consider a general cone $Y \subset \mathbb{P}^4$ over the curve $C$. Then $Y \not\subset V$ and $Y \cap V = C \cup Z$, where $Z$ is an irreducible reduced curve of degree $k(n-1)$. Moreover, we have the inclusion

$$\Sigma \cap C = \text{Sing}(V) \cap C \subseteq C \cap Z,$$

but in the set-theoretic sense $|C \cap Z| \leq k(n-1)$. Hence, at most $k(n-1)$ points of the subset $\Sigma \subset \mathbb{P}^4$ can lie on the curve $C \subset \mathbb{P}^4$ of degree $k$. The latter implies the claim. \hfill $\square$

Fix a point $P \in \Sigma$. To prove that the points of $\Sigma \subset \mathbb{P}^4$ impose independent linear conditions on homogeneous forms on $\mathbb{P}^4$ of degree $2n - 5$ it is enough to construct a hypersurface in $\mathbb{P}^4$ of degree $2n - 5$ that passes through the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

**Lemma 54.** Suppose that the subset $\Sigma \subset \mathbb{P}^4$ is contained in some two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$. Then there is a hypersurface in $\mathbb{P}^4$ of degree $2n - 5$ that passes through the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

**Proof.** Let us apply Corollary \textcolor{red}{32} to $\Sigma \subset \Pi$ and the number $d = 2n - 5 \geq 3$. We must check that all the conditions of Corollary \textcolor{red}{32} are satisfied. It is clear that

$$|\Sigma| \leq \frac{(n-1)^2}{4} \Rightarrow |\Sigma| \leq \frac{d^2 + 9d + 16}{6}$$

and at most $d = 2n - 5$ points of $\Sigma$ can lie on a line in $\Pi$ because $n \geq 4$ and the points of the subset $\Sigma \subset \Pi$ satisfy the property $\star$ due to Proposition \textcolor{red}{32}.

Now we must prove that at most $k(2n - 2 - k) - 2$ points of $\Sigma$ can lie on a curve of degree $k \leq n - 1$. The case $k = 1$ is already done. Moreover, at most $k(n-1)$ points of the set $\Sigma$ can lie on a curve of degree $k$ by Proposition \textcolor{red}{32}. Thus, we must show that

$$k(2n - 2 - k) - 2 \geq k(n-1)$$

for all $k \leq n - 1$. Moreover, we must prove the latter inequality only for such $k > 1$ that the inequality $k(2n - 2 - k) - 2 < |\Sigma|$ holds, because otherwise the corresponding condition on the points of the set $\Sigma$ is vacuous. Moreover,

$$k(2n - 2 - k) - 2 \geq k(n-1) \iff n - 1 > k,$$

because $k > 1$. So, we may assume that $k = n - 1$, but in this case

$$k(2n - 2 - k) - 2 = (n-1)^2 > \frac{(n-1)^2}{4} \geq |\Sigma|.$$
Therefore, there is a curve $C \subset \Pi$ of degree $2n - 5$ that passes through $\Sigma \setminus P$ and does not pass through $P$ by Corollary 53. Let $Y \subset \mathbb{P}^4$ be a three-dimensional cone over $C$ with the vertex in a general line in $\mathbb{P}^4$. Then $Y \subset \mathbb{P}^4$ is a hypersurface of degree $2n - 5$ that passes through the points of $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$. □

Fix a general two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$. Let $\psi : \mathbb{P}^4 \longrightarrow \Pi$ be a projection from a general line $L \subset \mathbb{P}^4$, $\Sigma' = \psi(\Sigma)$ and $\hat{P} = \psi(P)$. Then $\psi|_{\Sigma} : \Sigma \rightarrow \Sigma'$ is a bijection.

**Lemma 55.** Suppose that the points in $\Sigma' \subset \Pi$ satisfy the property $\star$. Then there is a hypersurface in $\mathbb{P}^4$ of degree $2n - 5$ containing $\Sigma \setminus P$ and not passing through $P \in \Sigma$.

**Proof.** The proof of Lemma 54 implies the existence of a curve $C \subset \Pi$ of degree $2n - 5$ that passes through $\Sigma' \setminus \hat{P}$ and does not pass through $\hat{P}$. Let $Y \subset \mathbb{P}^4$ be a three-dimensional cone over the curve $C$ with the vertex $L \subset \mathbb{P}^4$. Then $Y \subset \mathbb{P}^4$ is the required hypersurface. □

**Remark 56.** It seems to us that the points of the set $\Sigma' \subset \Pi \cong \mathbb{P}^2$ always satisfy the property $\star$ due to the generality in the choice of the projection $\psi : \mathbb{P}^4 \longrightarrow \Pi$, but we fail to prove it. In the case when $\Sigma' \subset \Pi \cong \mathbb{P}^2$ satisfy the property $\star$ the proof of Lemma 55 implies a stronger result than Theorem 4.

We may assume that the points of $\Sigma' \subset \Pi \cong \mathbb{P}^2$ do not satisfy the property $\star$ and, in particular, there is a subset $\Lambda^1_k \subset \Sigma$ such that $|\Lambda^1_k| > k(n - 1)$ and all the points of

$$\bar{\Lambda}^1_k = \psi(\Lambda^1_k) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on a curve $C \subset \Pi$ of degree $k$. We always may choose $k$ to be the smallest natural number having such a property. The latter implies that the curve $C \subset \Pi$ is irreducible and reduced. We can iterate the construction of the subset $\Lambda^1_k \subset \Sigma$ in the same way as in the proof of the Theorem 4 to get disjoint subsets $\Lambda^j_k \subset \Sigma$ for $j = k, \ldots, l \geq k$ such that the inequality $|\Lambda^j_k| > j(n - 1)$ holds, all the points of the subset $\bar{\Lambda}^j_k = \psi(\Lambda^j_k) \subset \Sigma'$ lie on an irreducible reduced curve in $\Pi \cong \mathbb{P}^2$ of degree $j$, and all the points in the subset

$$\bar{\Sigma} = \Sigma' \setminus \bigcup_{j=k}^{l} c_j \bar{\Lambda}^j_k \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

satisfy the property $\star$, where $c_j \geq 0$ is a number of subsets $\bar{\Lambda}^j_k$ and $c_k > 0$. In particular,

$$0 \leq |\bar{\Sigma}| < \frac{(n - 1)^2}{4} - \sum_{i=k}^{l} c_i(n - 1)i = \frac{n - 1}{4}(n - 1 - 4 \sum_{i=k}^{l} i c_i).$$

**Corollary 57.** The inequality $\sum_{i=k}^{l} i c_i < \frac{n-1}{4}$ holds.

In particular, $\Lambda^j_k \neq \emptyset$ implies $j < \frac{n-1}{4}$.

**Lemma 58.** Suppose that $\Lambda^j_k \neq \emptyset$. Let $\mathcal{M}$ be a linear system of hypersurfaces of degree $j$ in $\mathbb{P}^4$ passing through all the points in $\Lambda^j_k$. Then the base locus of $\mathcal{M}$ is zero-dimensional.

**Proof.** By the construction of the set $\Lambda^j_k$ all the points of the subset

$$\bar{\Lambda}^j_k = \psi(\Lambda^j_k) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie on an irreducible reduced curve $C \subset \Pi$ of degree $j$. Let $Y \subset \mathbb{P}^4$ be a three-dimensional cone over the curve $C$ whose vertex is the line $L \subset \mathbb{P}^4$. Then $Y$ is a hypersurface in $\mathbb{P}^4$ of degree $j$ that contains all the points of the set $\Lambda^j_k$. Therefore, $Y \in \mathcal{M}$. 15
Suppose that the base locus of the linear system $\mathcal{M}$ contains an irreducible reduced curve $Z \subset \mathbb{P}^3$. Then $Z \subset Y$. The curves $Z$ and $C$ are irreducible and the projection $\psi$ is sufficiently general. Therefore, $\psi(Z) = C$, $\Lambda_j^i \subset Z$ and $\psi|_Z : Z \to C$ is a birational morphism. In particular, $\deg(Z) = j$ and $Z$ contains at least $|\Lambda_j^i| > j(n - 1)$ points of the subset $\Sigma \subset \mathbb{P}^4$. The latter contradicts Proposition 53.

**Corollary 59.** The inequality $k \geq 2$ holds.

For every $\Lambda_j^i \neq \emptyset$ let $\Xi_j^i \subset \mathbb{P}^4$ be a base locus of the linear system of hypersurfaces of degree $j$ in $\mathbb{P}^4$ passing through all the points in $\Lambda_j^i$, otherwise put $\Xi_j^i = \emptyset$. Then $\Xi_j^i$ is a finite set by Lemma 53 and $\Lambda_j^i \subset \Xi_j^i$ by definition of the subset $\Xi_j^i \subset \mathbb{P}^4$. Therefore, the points of the set $\Xi_j^i \subset \mathbb{P}^4$ impose independent linear conditions on the homogeneous forms on $\mathbb{P}^4$ of degree $4(j - 1)$ by Lemma 23. In particular, the points of the set $\Lambda_j^i$ impose independent linear conditions on the homogeneous forms on $\mathbb{P}^4$ of degree $4(j - 1)$.

**Lemma 60.** Suppose that $\Sigma = \emptyset$. Then there is a hypersurface in $\mathbb{P}^4$ of degree $2n - 5$ containing all the points of the set $\Sigma \setminus P$ and not containing the point $P \in \Sigma$.

*Proof.* The points of the set $\Xi_j^i$ impose independent linear conditions on the homogeneous forms on $\mathbb{P}^4$ of degree $4(j - 1)$. Therefore, for every $\Xi_j^i \neq \emptyset$ containing $P$ there is a hypersurface in $\mathbb{P}^4$ of degree $4(j - 1)$ that passes through the points of the set $\Xi_j^i \setminus P$ and does not pass through the point $P$. On the other hand, for every set $\Xi_j^i \neq \emptyset$ not containing the point $P$ there is a hypersurface in $\mathbb{P}^4$ of degree $j$ that passes through $\Xi_j^i$ and does not pass through $P$ by the definition of the set $\Xi_j^i$. Moreover, $j < 4(j - 1)$, because $j \geq k \geq 2$ due to Corollary 59. Thus, for every non-empty set $\Xi_j^i$ there is a hypersurface $F_j^i \subset \mathbb{P}^4$ of degree $3(j - 1)$ that passes through $\Xi_j^i \setminus P$ and does not pass through $P$. Let

$$F = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} F_j^i \subset \mathbb{P}^4$$

be a possibly reducible hypersurface of degree $\sum_{i=k}^{l} 4(i - 1)c_i$. Then $F$ passes through all the points of the set $\Sigma \setminus P$ and does not pass through the point $P$. Moreover, we have

$$\deg(F) = \sum_{i=k}^{l} 4(i - 1)c_i < n - 1 \leq 2n - 5$$

by Corollary 57, which implies the claim. □

Let $\hat{\Sigma} = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i$ and $\hat{\Sigma} = \Sigma \setminus \hat{\Sigma}$. Then $\Sigma = \hat{\Sigma} \cup \hat{\Sigma}$ and $\psi(\hat{\Sigma}) = \hat{\Sigma} \subset \Pi$.

**Remark 61.** The proof of Lemma 60 implies the existence of a hypersurface $F \subset \mathbb{P}^4$ of degree $\sum_{i=k}^{l} 4(i - 1)c_i$ that passes through all the points of the subset $\hat{\Sigma} \setminus P \subset \Sigma$ and does not pass through the point $P \in \Sigma$.

Put $d = 2n - 5 - \sum_{i=k}^{l} 4(i - 1)c_i$. Let us check that the subset $\hat{\Sigma} \subset \Pi \cong \mathbb{P}^2$ and the number $d$ satisfy all the conditions of Theorem 52. We may assume $\hat{\Sigma} \neq \emptyset$ and $\hat{\Sigma} \neq \emptyset$.

**Lemma 62.** The inequality $d \geq 5$ holds.

*Proof.* We have $\sum_{i=k}^{l} 4ic_i < n - 1$ by Corollary 57. Thus, $d > n - 4 + 4c_k \geq n \geq 4$. □

**Lemma 63.** The inequality $|\hat{\Sigma}| \leq \frac{d^2 + 9d + 10}{6}$ holds.
Proof. Suppose that $|\Sigma| > \frac{d^2+9d+10}{6}$. Then

$$3(n-1)(n-1-4\sum_{i=k}^{l} ic_i) > 2(2n-5 - \sum_{i=k}^{l} 4(i-1)c_i)^2 + 18(2n-5 - \sum_{i=k}^{l} 4(i-1)c_i) + 20,$$

because $|\Sigma| < \frac{n-1}{4}(n-1-4\sum_{i=k}^{l} ic_i)$. Let $A = \sum_{i=k}^{l} ic_i$ and $B = \sum_{i=k}^{l} c_i$. Then

$$3(n-1)^2 - 12(n-1)A > 2(2n-1)^2 - 16A(2n-1) + 32A^2 + 18(2n-1) - 72A + 20,$$

because $B \geq c_k \geq 1$. Thus, for $n \geq 4$ we have

$$3(n-1)^2 > 8n^2 + 28n + 4 + 32A^2 - A(20n + 68) > 5n^2 + 12n + 23 > 3(n-1)^2,$$

because $A < \frac{n-1}{4}$ by Corollary 57. □

Lemma 64. At most $d$ of the points of the subset $\bar{\Sigma} \subset \mathbb{P}^2$ lie on a line in $\mathbb{P}^2$.

Proof. The points of the subset $\bar{\Sigma} \subset \mathbb{P}^2$ satisfy the property $\star$. In particular, no more than $n-1$ of the points of $\bar{\Sigma}$ lie on a line in $\mathbb{P}^2$. On the other hand, we have

$$d = 2n - 5 - \sum_{i=k}^{l} 4(i-1)c_i > n - 4 + 4c_k > n - 1$$

due to Corollary 57 which implies the claim. □

Lemma 65. At most $k(d+3-k) - 2$ points of $\bar{\Sigma}$ lie on a curve in $\mathbb{P}^2$ of degree $k \leq \frac{d+3}{2}$.

Proof. We may assume that $k > 1$ due to Lemma 64. The points of the subset $\bar{\Sigma} \subset \mathbb{P}^2$ satisfy the property $\star$. Thus, at most $k(n-1)$ of the points of $\bar{\Sigma}$ lie on a curve in $\mathbb{P}^2$ of degree $k$. Therefore, to conclude the proof it is enough to show that the inequality

$$k(d+3-k) - 2 \geq k(n-1)$$

holds for all $k \leq \frac{d+3}{2}$. Moreover, it is enough to prove the latter inequality only for such natural numbers $k > 1$ that the inequality $k(d+3-k) - 2 < |\Sigma|$ holds, because otherwise the corresponding condition on the points of the set $\bar{\Sigma}$ is vacuous.

Now we have

$$k(d+3-k) - 2 \geq k(n-1) \iff n - 1 - \sum_{i=k}^{l} 4(i-1)c_i > k,$$

because $k > 1$. Thus, we may assume that the inequalities

$$n - 1 - \sum_{i=k}^{l} 4(i-1)c_i \leq k \leq \frac{d+3}{2} \quad \text{and} \quad k(d+3-k) - 2 < |\Sigma|$$

hold. Let $g(x) = x(d+3-x) - 2$. Then $g(x)$ is increasing for $x < \frac{d+3}{2}$. Thus, we have

$$\frac{(n-1)}{4}(n-1 - 4\sum_{i=k}^{l} ic_i) > |\Sigma| > g(k) \geq g(n - 1 - \sum_{i=k}^{l} 4(i-1)c_i) = g(n-1 - 4A) \geq 4(n-1 - 4A + 4B)(n-1) - 2.$$

Let $A = \sum_{i=k}^{l} ic_i$ and $B = \sum_{i=k}^{l} c_i$. Then the inequality

$$\frac{(n-1)}{4}(n-1 - 4A) > 4(n-1 - 4A + 4B)(n-1) - 2$$

holds. Therefore, we have

$$n - 1 - 4A > 4(n-1 - 16A + 16B - 1) > 4(n-1) - 16A,$$
because $B \geq c_k \geq 1$. Thus, $4A > n - 1$, but $A < \frac{n-1}{4}$ by Corollary 57. □

Therefore, we can apply Theorem 32 to the blow up of the two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$ at the points of $\Sigma \setminus \hat{P} \subset \Pi$ due to Lemmas 62, 63 and 65. The latter gives a curve $C \subset \Pi$ of degree $2n - 5 - \sum_{i=k}^{l} 4(i-1)c_i$ that passes through all the points of the subset $\Sigma \setminus \hat{P} \subset \Pi \cong \mathbb{P}^2$ and does not pass through the point $\hat{P} \subset \Sigma'$.

Let $G \subset \mathbb{P}^4$ be a cone over the curve $C$ with the vertex $L \subset \mathbb{P}^4$, where $L$ is a center of the projection $\psi : \mathbb{P}^4 \dashrightarrow \Pi$. Then $G \subset \mathbb{P}^4$ is a hypersurface of degree $2n - 5 - \sum_{i=k}^{l} 4(i-1)c_i$ that passes through $\Sigma \setminus P$ and does not pass through $P$. However, we already have the hypersurface $F \subset \mathbb{P}^4$ of degree $\sum_{i=k}^{l} 4(i-1)c_i$ that passes through $\Sigma \setminus P$ and does not pass through $P$. Therefore, $F \cup G \subset \mathbb{P}^4$ is a hypersurface of degree $2n - 5$ that passes through $\Sigma \setminus P$ and does not pass through $P \in \Sigma$. Thus, Theorem 4 is proved.

5. Calabi-Yau 3-folds.

In this section we prove Proposition 10. Let $\pi : X \to \mathbb{P}^3$ be a double cover branched over a nodal hypersurface $S \subset \mathbb{P}^3$ of degree 8 such that $|\text{Sing}(S)| \leq 25$, and $V \subset \mathbb{P}^4$ be a nodal hypersurface of degree 5 such that $|\text{Sing}(V)| \leq 14$. Due to Proposition 2 it is enough to prove that the nodes of the surface $S \subset \mathbb{P}^3$ impose independent linear conditions on homogeneous forms of degree 8 on $\mathbb{P}^3$ and the nodes of the hypersurface $V \subset \mathbb{P}^4$ impose independent linear conditions on homogeneous forms of degree 5 on $\mathbb{P}^4$.

Let $\Sigma = \text{Sing}(S) \subset \mathbb{P}^3$ and $\Lambda = \text{Sing}(V) \subset \mathbb{P}^4$.

Lemma 66. No more than $7k$ points of the subset $\Sigma \subset \mathbb{P}^3$ and no more than $4k$ points of the subset $\Lambda \subset \mathbb{P}^4$ can lie on a curve of degree $k = 1, 2, 3$.

Proof. See the proof of Propositions 55 and 58. □

Fix a point $P \in \Sigma$ and a point $Q \in \Lambda$. To prove the claim of Proposition 10 we must construct a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through the points of $\Sigma \setminus P$ and does not pass through the point $P$ and a hypersurface in $\mathbb{P}^4$ of degree 5 that passes through the points of the set $\Lambda \setminus Q$ and does not pass through the point $Q$.

Take a general two-dimensional linear subspaces $\Pi \subset \mathbb{P}^3$ and $\Omega \subset \mathbb{P}^4$. Let $\psi : \mathbb{P}^3 \dashrightarrow \Pi$ be a projection from a general point $P \in \mathbb{P}^3$, and $\xi : \mathbb{P}^4 \dashrightarrow \Omega$ be a projection from a general line $L \subset \mathbb{P}^4$. Put $\Sigma' = \psi(\Sigma)$, $\hat{P} = \psi(P)$, $\Lambda' = \xi(\Lambda)$ and $\hat{Q} = \xi(Q)$.

Lemma 67. No more than 7 points of the subset $\Sigma' \subset \Pi$ and no more than 5 points of the subset $\Lambda' \subset \Omega$ can lie on a line.

Proof. Suppose there is subset $\Theta \subset \Sigma$ such that $|\Theta| > 7$ and the points of $\psi(\Theta) \subset \Sigma'$ are contained in a line. Let $\mathcal{H}$ be a linear system of hyperplanes in $\mathbb{P}^3$ passing through the points of $\Theta$. Then the base locus of $\mathcal{H}$ is zero-dimensional by Lemma 12. The latter is possible only when $|\Theta| = 1$, which is a contradiction.

Suppose there is subset $\Phi \subset \Lambda$ such that $|\Lambda| > 5$ and the points of $\xi(\Phi) \subset \Lambda'$ are contained in a line. Let $\mathcal{D}$ be a linear system of hyperplanes in $\mathbb{P}^4$ passing through the points of $\Phi$. Then the base locus of $\mathcal{D}$ is zero-dimensional by Lemma 58. The latter is possible only when $|\Phi| = 1$, which is a contradiction. □

Lemma 68. No more than 14 points of the subset $\Sigma' \subset \Pi$ and no more than 10 points of the subset $\Lambda' \subset \Omega$ can lie on a conic.
Proof. Suppose there is subset $\Theta \subset \Sigma$ such that $|\Theta| > 14$ and the points of $\psi(\Theta)$ are contained in a conic $C \subset \Pi \cong \mathbb{P}^2$. Then $C$ is irreducible due to Lemma 67. Let $\mathcal{H}$ be a linear system of quadrics in $\mathbb{P}^3$ passing through the points of $\Theta$. Then the base locus of the linear system $\mathcal{H}$ is zero-dimensional by Lemma 42. Take a cone $Y \subset \mathbb{P}^3$ over $C$ with the vertex $B$. Then $\Theta \subset Y$, $\Theta \subset Bs(\mathcal{H}|_Y)$ and the linear system $\mathcal{H}|_Y$ is free from base components. Let $H_1$ and $H_2$ be general enough curves in $\mathcal{H}|_Y$. Then $H_i$ is contained in the smooth locus of the cone $Y$ and on the surface $Y$ we have

$$8 = H_1 \cdot H_2 \geq \sum_{\omega \in \Theta} \text{mult}_\omega(H_1)\text{mult}_\omega(H_2) \geq |\Theta| > 14,$$

which is a contradiction.

Let $\Phi \subset \Lambda$ be a subset such that $|\Phi| > 10$. Consider the projection $\xi$ as a composition of a projection $\alpha : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$ from some point $A \in L$ and a projection $\beta : \mathbb{P}^3 \dashrightarrow \Omega$ from the point $B = \alpha(L)$. The generality in the choice of the line $L$ implies the generality of the projections $\alpha$ and $\beta$. We claim that the points of the sets $\alpha(\Phi)$ and $\xi(\Phi)$ do not lie on a conic in $\mathbb{P}^3$ and $\Omega \cong \mathbb{P}^2$ respectively.

Suppose that the points of $\alpha(\Phi)$ lie on a conic $C \subset \mathbb{P}^3$. Then conic $C$ is irreducible due to Lemma 67. Let $\mathcal{D}$ be a linear system of quadric hypersurfaces in $\mathbb{P}^4$ passing through the points of $\Phi$. The proof of Lemma 68 implies that the base locus of $\mathcal{D}$ is zero-dimensional, because the points of $\Phi \subset \mathbb{P}^4$ do not lie on a conic in $\mathbb{P}^4$. Take a cone $W \subset \mathbb{P}^4$ over the conic $C$ with the vertex $A$. Then $\Phi \subset W$. Moreover, we have

$$\Phi \subset Bs(\mathcal{D}|_W)$$

and $\mathcal{D}|_W$ has no base components. Let $D_1$ and $D_2$ be general curves in $\mathcal{D}|_W$. Then

$$8 = D_1 \cdot D_2 \geq \sum_{\omega \in \Phi} \text{mult}_\omega(D_1)\text{mult}_\omega(D_2) \geq |\Phi| > 10,$$

which is a contradiction. Therefore, the points of $\alpha(\Phi)$ do not lie on a conic in $\mathbb{P}^3$.

Suppose that the points of $\xi(\Phi)$ lie on a conic $C \subset \Pi \cong \mathbb{P}^3$. Then conic $C$ is irreducible due to Lemma 67. Let $\mathcal{B}$ be a linear system of quadrics in $\mathbb{P}^3$ passing through the points of the set $\alpha(\Phi)$. The proof of Lemma 68 implies that the base locus of $\mathcal{B}$ is zero-dimensional, because the points of $\alpha(\Phi)$ do not lie on a conic. Take a cone $U \subset \mathbb{P}^3$ over $C$ with the vertex $B$. Then $\alpha(\Phi) \subset U$. Then $\alpha(\Phi) \subset Bs(\mathcal{B}|_U)$ and the restriction $\mathcal{B}|_U$ has no base components. Let $B_1$ and $B_2$ be general curves in $\mathcal{B}|_U$. Then

$$8 = B_1 \cdot B_2 \geq \sum_{\omega \in \alpha(\Phi)} \text{mult}_\omega(B_1)\text{mult}_\omega(B_2) \geq |\alpha(\Phi)| = |\Phi| > 10,$$

which is a contradiction. \hfill $\Box$

Lemma 69. There is a hypersurface in $\mathbb{P}^4$ of degree 5 that passes through the points of the set $\Lambda \setminus Q$ and does not pass through the point $Q \in \Lambda$.

Proof. Put $s = |\Lambda \setminus \hat{Q}|$. Then $s \leq 13$. Let $\pi : Y \rightarrow \Omega \cong \mathbb{P}^2$ be a blow up of the points of the set $\Lambda \setminus \hat{Q}$. Then Lemmas 67 and 68 and Theorem 32 imply the freeness of the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(5)) - \sum_{i=1}^{s} E_i|$, where $E_i$ is a $\pi$-exceptional curve. Let $C \subset Y$ be a general curve in the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(5)) - \sum_{i=1}^{s} E_i|$. Then $\pi(C) \subset \Omega$ is a plane quintic curve passing through the points of the set $\Lambda \setminus \hat{Q}$ and not passing through the point $Q$. The cone in $\mathbb{P}^4$ over $\pi(C)$ with the vertex $L$ is the required hypersurface. \hfill $\Box$
Lemma 70. Suppose that at most 22 points of the subset $\Sigma' \subset \Pi$ can lie on a cubic curve in $\Pi \cong \mathbb{P}^2$. Then there is a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through the points of the set $\Sigma \setminus P$ and does not pass through the point $P \in \Sigma$.

Proof. Let $\pi : Y \to \Pi \cong \mathbb{P}^2$ be the blow up at the points $\{P_1, \ldots, P_s\} = \Sigma' \setminus \hat{P}$ for $s \leq 24$ and $E_i = \pi^{-1}(P_i)$. Then Lemmas 67 and 67 and Theorem 42 imply the freeness of the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(8)) - \sum_{i=1}^s E_i|$. Let $C \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(8)) - \sum_{i=1}^k E_i|$ be a general enough curve. Then $\pi(C) \subset \Pi$ is a plane octic curve that passes through the points of $\Sigma' \setminus \hat{P}$ and does not pass through the point $\hat{P}$. Hence, the cone in $\mathbb{P}^3$ over the curve $\pi(C) \subset \Pi$ with the vertex $O$ is the required hypersurface. □

Lemma 71. Suppose that there is a subset $\Upsilon \subset \Sigma$ such that $|\Upsilon| > 22$ and all the points of the set $\psi(\Upsilon)$ lie on a cubic curve in $\Pi \cong \mathbb{P}^2$. Then there is a hypersurface in $\mathbb{P}^3$ of degree 8 that passes through the points of $\Sigma \setminus \Upsilon$ and does not pass through the point $P$.

Proof. Let $\mathcal{H}$ be a linear system of cubic hypersurfaces in $\mathbb{P}^3$ passing through the points of the set $\Upsilon$. Then the base locus of $\mathcal{H}$ is zero-dimensional by Lemma 42.

Suppose $P \in \Upsilon$. Then there is a hypersurface $F \subset \mathbb{P}^3$ of degree 6 that passes through the points of $\Upsilon \setminus P$ and does not pass through the point $P$ by Lemma 23. On the other hand, the subset $\Sigma \setminus \Upsilon \subset \mathbb{P}^3$ contains at most 2 points. Hence, there is a quadric $G \subset \mathbb{P}^3$ that passes through the points of $\Sigma \setminus \Upsilon$ and does not pass through $P$. Thus, $F \cup G$ is the required hypersurface.

In the case when $P \not\in \Upsilon$ and $P \in \text{Bs}(\mathcal{H})$ we can repeat every step of the proof of the previous case. In the case when $P \not\in \Upsilon$ and $P \not\in \text{Bs}(\mathcal{H})$ there is a cubic hypersurface in $\mathbb{P}^3$ that passes through the points of $\Upsilon$ and does not pass through the point $P$, which easily implies the existence of the required hypersurface. □

Hence, Proposition 10 is proved. It seems to us that the bounds for nodes in Proposition 10 can be improved using the methods of [12, 29, 45] instead of Theorem 32.

6. Non-isolated singularities.

In this section we prove Theorems 14 and 15. Let $\tau : U \to \mathbb{P}^4$ be a double cover branched over a hypersurface $F$ of degree $2r$ such that $D_1 \cap \cdots \cap D_{s-3}$ is a $\mathbb{Q}$-factorial nodal 3-fold, where $D_i$ is a general divisor in $|\tau^*(\mathcal{O}_{\mathbb{P}^4}(1))|$. Let $W \subset \mathbb{P}^r$ be a hypersurface of degree $n$ such that $H_1 \cap \cdots \cap H_{r-4}$ is a $\mathbb{Q}$-factorial nodal 3-fold, where $H_i$ is a general hyperplane section of $W$. We may assume that $s \geq 4$ and $r \geq 5$. We must show that the group $\text{Cl}(U)$ is generated by $D_1$ and the group $\text{Cl}(W)$ is generated by $H_1$.

Remark 72. The varieties $U$ and $W$ are normal (see Proposition 8.23 in [36]).

Let $D$ be a general divisor in $|\tau^*(\mathcal{O}_{\mathbb{P}^4}(1))|$ and $H$ be a general hyperplane section of $W$.

Lemma 73. The groups $H^1(\mathcal{O}_U(-nD))$ and $H^1(\mathcal{O}_W(-nH))$ vanish for every $n > 0$.

Proof. In the case when the varieties $U$ and $W$ have mild singularities the claim is implies by the Kawamata-Viehweg vanishing (see [13, 73]). In general let us prove the claim by induction on $s$ and $r$. We consider only the vanishing of $H^1(\mathcal{O}_U(-nD))$, because the proof of the vanishing of the cohomology group $H^1(\mathcal{O}_W(-nH))$ is identical.

Suppose that $s = 4$. Then we have an exact sequence of sheaves

$$0 \to \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0$$
for any \( n \in \mathbb{Z} \). Therefore, we have an exact sequence of the cohomology groups
\[
0 \to H^1(\mathcal{O}_U(-(n+1)D)) \to H^1(\mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots
\]
for \( n > 0 \). However, the 3-fold \( D \) is nodal by assumption. Thus, the group \( H^1(\mathcal{O}_D(-nD)) \) vanishes by the Kawamata-Viehweg vanishing. Hence, we have
\[
H^1(\mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(-2D)) \cong \cdots \cong H^1(\mathcal{O}_U(-nD))
\]
for every \( n > 0 \). On the other hand, the group \( H^1(\mathcal{O}_U(-nD)) \) vanishes for \( n \gg 0 \) by the lemma of Enriques-Severi-Zariski (see \[77\] or Corollary 7.8 in \[36\]).

Suppose that \( s > 4 \). Then we have an exact sequence of sheaves
\[
0 \to \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0
\]
for any \( n \in \mathbb{N} \). Therefore, we have an exact sequence of the cohomology groups
\[
0 \to H^1(\mathcal{O}_U(-(n+1)D)) \to H^1(\mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots
\]
for \( n > 0 \). However, the group \( H^1(\mathcal{O}_D(-nD)) \) vanishes by the induction. Hence,
\[
H^1(\mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(-2D)) \cong \cdots \cong H^1(\mathcal{O}_U(-nD))
\]
for \( n > 0 \), but \( H^1(\mathcal{O}_U(-nD)) = 0 \) for \( n \gg 0 \) by the lemma of Enriques-Severi-Zariski. \( \Box \)

Consider a Weil divisor \( G \) on \( U \). Let us prove by the induction on \( s \) that \( G \sim kD \) for some \( k \in \mathbb{Z} \). Suppose that \( s = 4 \). Then the 3-fold \( D \) is nodal and \( \mathbb{Q} \)-factorial by assumption. Moreover, the group \( \text{Cl}(D) \) is generated by the class of the divisor \( R|_D \) due to Remark \[6\] where \( R \) is a general divisor in the linear system \( |D| \). Thus, there is an integer \( k \) such that \( G|_D \sim kR|_D \). Let \( \Delta = G - kR \). Then the sequence of sheaves
\[
0 \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D) \to \mathcal{O}_U(\Delta) \to \mathcal{O}_D \to 0
\]
is exact, because \( \mathcal{O}_U(\Delta) \) is locally free in the neighborhood of \( D \). Hence, the sequence
\[
0 \to H^0(\mathcal{O}_U(\Delta)) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D))
\]
is exact.

**Lemma 74.** The group \( H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD)) \) vanishes for every \( n > 0 \).

**Proof.** The sheaf \( \mathcal{O}_U(\Delta) \) is reflexive (see \[37\]). Thus, there is an exact sequence of sheaves
\[
0 \to \mathcal{O}_U(\Delta) \to \mathcal{E} \to \mathcal{F} \to 0
\]
where \( \mathcal{E} \) is a locally free sheaf and \( \mathcal{F} \) is a torsion free sheaf. Hence, the sequence of groups
\[
H^0(\mathcal{F} \otimes \mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_D(-nD)) \to H^1(\mathcal{E} \otimes \mathcal{O}_U(-nD))
\]
is exact. However, for \( n \gg 0 \) the cohomology group \( H^0(\mathcal{F} \otimes \mathcal{O}_U(-nD)) \) vanishes because the sheaf \( \mathcal{F} \) is torsion free, and the cohomology group \( H^1(\mathcal{E} \otimes \mathcal{O}_U(-nD)) \) vanishes by the lemma of Enriques-Severi-Zariski (see \[77\] or Corollary 7.8 in \[36\]). Therefore, the cohomology group \( H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD)) \) vanishes for \( n \gg 0 \).

Now consider an exact sequence of sheaves
\[
0 \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-(n+1)D) \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD) \to \mathcal{O}_D(-nD) \to 0
\]
and the induced sequence of the cohomology groups
\[
0 \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}(-(n+1)D)) \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD)) \to H^1(\mathcal{O}_D(-nD)) \to \cdots
\]
for \( n > 0 \). Then the group \( H^1(\mathcal{O}_D(-nD)) \) vanishes by Lemma \[73\]. Hence, we have
\[
H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D)) \cong H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-2D)) \cong \cdots \cong H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-nD))
\]
for \( n > 0 \), but we already proved that \( H^1(\mathcal{O}_U(-nD)) \) vanishes for \( n \gg 0 \). \( \square \)

Therefore, \( H^0(\mathcal{O}_U(\Delta)) \cong \mathbb{C} \). Similarly \( H^0(\mathcal{O}_U(-\Delta)) \cong \mathbb{C} \). Thus, the Weil divisor \( \Delta \) is rationally equivalent to zero and \( G \sim kD \) in the case \( s = 4 \).

Suppose that \( s > 4 \). By the induction we may assume that the group \( \text{Cl}(D) \) is generated by the class of the divisor \( R|D \), where \( R \) is a general divisor in \( |D| \). Thus, there is an integer \( k \) such that \( G|D \sim kR|D \). Put \( \Delta = G - kR \). Then the sequence of sheaves

\[
0 \to \mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D) \to \mathcal{O}_U(\Delta) \to \mathcal{O}_D \to 0
\]

is exact, because \( \mathcal{O}_U(\Delta) \) is locally free in the neighborhood of \( D \). Therefore, the sequence

\[
0 \to H^0(\mathcal{O}_U(\Delta)) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D))
\]

is exact. However, the proof of the Lemma 74 holds for \( s > 4 \). Thus, the cohomology group \( H^1(\mathcal{O}_U(\Delta) \otimes \mathcal{O}_U(-D)) \) vanishes. Hence, \( H^0(\mathcal{O}_U(\Delta)) \cong \mathbb{C} \). Same arguments prove that \( H^0(\mathcal{O}_U(-\Delta)) \cong \mathbb{C} \). Therefore, the Weil divisor \( \Delta \) is rationally equivalent to zero and \( G \sim kD \). Thus, we proved Theorem 14. We omit the proof of Theorem 15 because it is identical to the proof of the Theorem 14.

7. Birational rigidity.

In this section we prove Proposition 16. Let \( \xi : Y \to \mathbb{P}^4 \) be a double cover branched over a hypersurface \( F \subset \mathbb{P}^4 \) of degree 8 such that the hypersurface \( F \) is smooth outside of a smooth curve \( C \subset F \), the singularity of the hypersurface \( F \) in a sufficiently general point of the curve \( C \) is locally isomorphic to the singularity

\[
x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]

the singularities of \( F \) in other points of \( C \) are locally isomorphic to the singularity

\[
x_1^2 + x_2^2 + x_3^2x_4 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]

and a general 3-fold in \( |-K_Y| \) is \( \mathbb{Q} \)-factorial. Then \( Y \) is a Fano 4-fold with terminal singularities and \( -K_Y \sim \xi^*(\mathcal{O}_{\mathbb{P}^4}(1)) \). Moreover, \( \text{Cl}(Y) \) and \( \text{Pic}(Y) \) are generated by the divisor \( -K_Y \) by Theorem 14. Hence, \( Y \) is a Mori fibration (see 14). We must prove that the 4-fold \( Y \) is a unique Mori fibration birational to \( Y \) and \( \text{Bir}(Y) = \text{Aut}(Y) \). It is well known that the latter implies the finiteness of the group \( \text{Bir}(Y) \).

Suppose that either \( Y \) is not birationally rigid or \( \text{Bir}(Y) \neq \text{Aut}(Y) \). Then Theorem 51 imply the existence of a linear system \( \mathcal{M} \) on \( Y \) such that \( \mathcal{M} \) has no fixed components and the singularities of \( (X, \frac{1}{n}\mathcal{M}) \) are not canonical, where \( \mathcal{M} \sim -nK_Y \). Thus, there is a rational number \( \mu < \frac{1}{n} \) such that \( (X, \mu\mathcal{M}) \) is not canonical, i.e. \( \text{CS}(Y, \mu\mathcal{M}) \neq \emptyset \).

Let \( Z \) be an element of the set \( \text{CS}(Y, \mu\mathcal{M}) \). Then \( \text{mult}_Z(\mathcal{M}) > n \).

**Lemma 75.** The subvariety \( Z \subset Y \) is not a smooth point of \( Y \).

**Proof.** Suppose that \( Z \) is a smooth point of \( Y \). Then

\[
\text{mult}_Z(\mathcal{M}^2) > 4n^2
\]

by Theorem 28. Take general divisors \( H_1 \) and \( H_2 \) in \( |-K_Y| \) containing \( Z \). Then

\[
2n^2 = \mathcal{M}^2 \cdot H_1 \cdot H_2 \geq \text{mult}_Z(\mathcal{M}^2)\text{mult}_Z(H_1)\text{mult}_Z(H_2) > 4n^2
\]

which is a contradiction. \( \square \)

**Lemma 76.** The subvariety \( Z \subset Y \) is not a singular point of \( Y \).
Proof. Let \( \xi(Z) = O \). Then \( O \) is a singular point of the hypersurface \( F \subset \mathbb{P}^4 \). Therefore, the point \( O \) is contained in the curve \( C \subset F \) by assumption. There are two possible cases, i.e. either the singularity of \( F \) in the point \( O \) is locally isomorphic to the singularity
\[
    x_1^2 + x_2^2 + x_3^2 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]
or the singularity of \( F \) in the point \( O \) is locally isomorphic to the singularity
\[
    x_1^2 + x_2^2 + x_3^2 x_4 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]),
\]
where \( x_1 = x_2 = x_3 \) are local equations of the curve \( C \subset F \). Let us call the former case ordinary and the latter case non-ordinary.

Let \( X \) be a sufficiently general divisor in the linear system \( | - K_Y | \) passing through the point \( Z \). Then the double cover \( \xi \) induces the double cover \( \tau : X \rightarrow \mathbb{P}^3 \) ramified in an octic surface. The singularities of \( X \setminus Z \) are ordinary double points. Moreover, \( Z \) is an ordinary double point of \( X \) in the ordinary case. In the non-ordinary case the singularity of the 3-fold \( X \) at the point \( Z \) is locally isomorphic to
\[
    x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0 \subset \mathbb{C}^4 \cong \text{Spec}(\mathbb{C}[x_1, x_2, x_3, x_4]).
\]

Let \( \mathcal{D} = \mathcal{M}|_X \) and \( H = -K_Y|_X \). Then the linear system \( \mathcal{D} \) has no fixed components and \( \mathcal{D} \sim nH \). Moreover, \( Z \in \mathcal{LCS}(X, \mu \mathcal{D}) \) by Theorem 24. In particular, \( Z \in \mathcal{CS}(X, \mu \mathcal{D}) \).

Let \( f : V \rightarrow X \) be a blow up of \( Z \), \( E = f^{-1}(Z) \) and \( \mathcal{H} \) be a proper transform of the linear system \( \mathcal{D} \) on \( V \). Then \( V \) is smooth in the neighborhood of \( E \) and \( E \) is isomorphic to a quadric surface in \( \mathbb{P}^3 \). In the ordinary case \( E \) is smooth. In the non-ordinary case the quadric surface \( E \) has one singular point \( P \in E \), i.e. the surface \( E \) is isomorphic to a quadric cone in \( \mathbb{P}^3 \). Note, that \( K_V \sim E \).

Let \( \text{mult}_Z(\mathcal{D}) \in \mathbb{N} \) such that \( \mathcal{H} \sim f^*(nH) - \text{mult}_Z(\mathcal{D})E \). Then \( \text{mult}_Z(\mathcal{D}) > n \) in the ordinary case by Theorem 29. On the other hand, in the non-ordinary case we have the inequality \( \text{mult}_Z(\mathcal{D}) > \frac{n}{2} \) due to Proposition 30.

By construction the linear system \( |f^*(H) - E| \) is free and gives a morphism \( \psi : V \rightarrow \mathbb{P}^2 \) such that \( \psi = \phi \circ \tau \circ f \), where \( \phi : \mathbb{P}^3 \rightarrow \mathbb{P}^2 \) is a projection from the point \( O \). Moreover, the restriction \( \psi|_E : E \rightarrow \mathbb{P}^2 \) is a double cover. Let \( L \) be a sufficiently general fiber of the morphism \( \psi \). Then \( L \) is a smooth curve of genus 2 and \( L \cdot E = L \cdot f^*(H) = 2 \). Thus,
\[
    L \cdot \mathcal{H} = L \cdot f^*(nH) - \text{mult}_Z(\mathcal{D})L \cdot E = 2n - 2\text{mult}_Z(\mathcal{D}) \geq 0,
\]
because \( \mathcal{H} \) has no base components. Hence, \( \text{mult}_Z(\mathcal{D}) \leq n \). In particular, the ordinary case is impossible and it remains to eliminate the non-ordinary case.

The inequalities \( \text{mult}_Z(\mathcal{D}) \leq n \) and \( \mu < \frac{1}{n} \), the equivalence
\[
    K_V + \mu \mathcal{H} \sim f^*(K_X + \mu \mathcal{D}) + (1 - \mu \text{mult}_Z(\mathcal{D}))E
\]
and \( Z \in \mathcal{CS}(X, \mu \mathcal{D}) \) imply the existence of a proper irreducible subvariety \( S \subset E \) such that \( S \in \mathcal{CS}(V, \mu \mathcal{H}) \). In particular, \( S \in \mathcal{CS}(V, \mu \mathcal{H}) \).

Suppose that \( S \) is a curve. Then \( \text{mult}_S(\mathcal{H}) > n \). Let \( L_\omega \) be a fiber of \( \psi \) passing through a general point \( \omega \in S \). Then \( L_\omega \) spans a divisor in \( V \) when we vary \( \omega \) on \( C \). Hence,
\[
    L \cdot \mathcal{H} = L \cdot f^*(nH) - \text{mult}_Z(\mathcal{D})L \cdot E = 2n - 2\text{mult}_Z(\mathcal{D}) \geq \text{mult}_\omega(L_\omega)\text{mult}_S(\mathcal{H}) > n,
\]
which contradicts the inequality \( \text{mult}_Z(\mathcal{D}) > \frac{n}{2} \).

Therefore, \( S \) is a point on \( E \). Then \( \text{mult}_S(\mathcal{H}) > n \) and \( \text{mult}_S(\mathcal{H}^2) > 4n^2 \) by Theorem 28 because \( S \) is smooth on \( V \). It is easy to see that the point \( S \) is not a vertex \( P \) of the quadric cone \( E \), because the numerical intersection of a general ruling of \( E \) with a general divisor in \( \mathcal{H} \) is equal to \( \text{mult}_Z(\mathcal{D}) \leq n \). Let \( \Gamma \) be a fiber of the morphism \( \psi \) that passes
through the point $S$ and $D$ be a general divisor in the linear system $|f^*(H) - E|$ that passes through the point $S$. Then $\Gamma \subset D$. Note, that $\Gamma$ may be reducible and singular, but we always have the inequality $\text{mult}_S(\Gamma) \leq 2$, because $\tau \circ f(\Gamma)$ is a line passing through the point $O$ and $\tau|_{f(\Gamma)}$ is a double cover.

Suppose that $\Gamma$ is irreducible. Let

$$\mathcal{H}^2 = \lambda \Gamma + T,$$

where $\lambda \in \mathbb{Q}$ and $T$ is a one-cycle such that $\Gamma \not\subset \text{Supp}(T)$. Then the inequalities

$$\text{mult}_S(T) > 4n^2 - \lambda \text{mult}_S(\Gamma) \geq 4n^2 - 2\lambda$$

hold. On the other hand, the inequalities

$$\text{mult}_S(T) \leq \text{mult}_S(T)\text{mult}_S(D) \leq T \cdot D = \mathcal{H}^2 \cdot D = 2n^2 - \text{mult}_Z^2(D) < \frac{7}{4} n^2$$

holds. Thus, we have $\lambda > \frac{9}{8}n^2$. Let $\tilde{D}$ be a general divisor in $|f^*(H)|$. Then

$$2n^2 = \tilde{D} \cdot \mathcal{H}^2 \geq \lambda \Gamma \cdot \tilde{D} = 2\lambda > \frac{9}{4} n^2,$$

which is a contradiction.

Therefore, the fiber $\Gamma$ is reducible. Then $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_i$ is a smooth rational curve such that $\tau \circ f(\Gamma_1) = \tau \circ f(\Gamma_2)$ is a line in $\mathbb{P}^3$ containing point $O$. Let

$$\mathcal{H}^2 = \lambda_1 \Gamma_1 + \lambda_2 \Gamma_2 + T,$$

where $\lambda_i \in \mathbb{Q}$ and $T$ is a one-cycle such that $\Gamma_i \not\subset \text{Supp}(T)$. Then the inequalities

$$\frac{7}{4} n^2 > 2n^2 - \text{mult}_Z^2(D) \geq T \cdot D \geq \text{mult}_S(T) > 4n^2 - \lambda_1 - \lambda_2$$

hold. Thus, $\lambda_1 + \lambda_2 > \frac{9}{4} n^2$. Hence, we have

$$2n^2 = \tilde{D} \cdot \mathcal{H}^2 \geq \lambda_1 \Gamma_1 \cdot \tilde{D} + \lambda_2 \Gamma_2 \cdot \tilde{D} = \lambda_1 + \lambda_2 > \frac{9}{4} n^2$$

for a general divisor $\tilde{D} \in |f^*(H)|$, which is a contradiction. \qed

**Lemma 77.** The subvariety $Z \subset Y$ is not a curve.

**Proof.** Suppose $Z$ is a curve. Let $X$ be a general divisor in $|-K_Y|$ and $P$ be a point in the intersection $Z \cap X$. Then $X$ is a nodal Calabi-Yau 3-fold. The point $P$ is smooth on the 3-fold $X$ if and only if $Z \not\subset \text{Sing}(X)$. In the case $Z \subset \text{Sing}(X)$ the point $P$ is an ordinary double point on $X$. Moreover, $P \in \mathcal{CS}(X, \mu D)$, where $D = M|_X$. In the case when the point $P$ is smooth on $X$ we can proceed as in the proof of Lemma 76 to get a contradiction. In the case when the point $P$ is an ordinary double point on $X$ we can proceed as in the proof of Lemma 76 to get a contradiction. \qed

**Lemma 78.** The subvariety $Z \subset Y$ is not a surface.

**Proof.** Suppose $Z$ is a surface. Then $\text{mult}_Z(M) > n$. Let $V$ be a general divisor in the linear system $|-K_Y|$, $S = Z \cap V$ and $D = M|_V$. Then $V$ is a nodal Calabi-Yau 3-fold, the linear system $\mathcal{D}$ has no base components, $S \subset V$ is an irreducible reduced curve and $\text{mult}_S(\mathcal{D}) > n$. The double cover $\xi$ induces a double cover $\tau : V \to \mathbb{P}^3$ ramified in a nodal hypersurface $G \subset \mathbb{P}^3$ of degree 8.

Take a sufficiently general divisor $H$ in $|\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$. Then

$$2n^2 = \mathcal{D}^2 \cdot H \geq \text{mult}_S^2(\mathcal{D})S \cdot H > n^2 S \cdot H,$$
which implies \( S \cdot H = 1 \). Hence, \( \tau(S) \) is a line in \( \mathbb{P}^3 \) and \( \tau|_S \) is an isomorphism.

Suppose that \( \tau(S) \not\subset G \). Then there is a smooth rational curve \( \tilde{S} \subset V \) such that \( S \neq \tilde{S} \) and \( \tau(S) = \tau(\tilde{S}) \). Take a sufficiently general surface \( D \in |\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))| \) passing through the curve \( S \). Then \( D \) is smooth outside of \( S \cap \tilde{S} \). Moreover, the surface \( D \) is smooth in every point of \( S \cap \tilde{S} \) that is smooth on \( V \), and \( D \) has an ordinary double point in every point of \( S \cap \tilde{S} \) that is an ordinary double point on \( V \). On the other hand, at most 4 nodes of the hypersurface \( G \subset \mathbb{P}^3 \) can lie on the line \( \tau(S) \), i.e. \( |\text{Sing}(D)| \leq 4 \). The sub-adjunction formula (see [10], [50]) implies

\[
(K_D + \tilde{S})|_S = K_S + \text{Diff}_S(0)
\]

and \( \deg(\text{Diff}_S(0)) = \frac{k}{2} \), where \( k = |\text{Sing}(D)| \). Thus, the self-intersection \( \tilde{S}^2 \) is negative on the surface \( D \), because \( K_D \cdot \tilde{S} = 1 \). Put \( \mathcal{H} = D|_P \). A priori the linear system \( \mathcal{H} \) can have a base component. However, the generality in the choice of \( D \) implies

\[
\mathcal{H} = \text{mult}_S(\mathcal{D})S + \text{mult}_{\tilde{S}}(\mathcal{D})\tilde{S} + \mathcal{B}
\]

where \( \mathcal{B} \) is a linear system on \( D \) having no base components. Moreover, the equivalence

\[
(n - \text{mult}_{\tilde{S}}(\mathcal{D}))\tilde{S} \sim_Q (\text{mult}_S(\mathcal{D}) - n)S + \mathcal{B}
\]

and \( \tilde{S}^2 < 0 \) imply \( \text{mult}_{\tilde{S}}(\mathcal{D}) > n \). Take a general divisor \( H \) in \( |\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))| \). Then

\[
2n^2 = \mathcal{D} \cdot H \geq \text{mult}_S(\mathcal{D})S \cdot H + \text{mult}_{\tilde{S}}(\mathcal{D})\tilde{S} \cdot H > n^2S \cdot H + n^2\tilde{S} \cdot H = 2n^2,
\]

which is a contradiction.

Therefore, we have \( \tau(S) \subset G \). Let \( O \) be a general point on \( \tau(S) \) and \( \Pi \) be a hyperplane in \( \mathbb{P}^3 \) that tangents \( G \) at the point \( O \). Consider a sufficiently general line \( L \subset \Pi \) passing through \( O \). Let \( \hat{L} = \tau^{-1}(L) \) and \( \hat{O} = \tau^{-1}(O) \). Then \( \hat{L} \) is singular at \( \hat{O} \). Therefore, the curve \( \hat{L} \) is contained in the base locus of the linear system \( \mathcal{D} \), because otherwise\n
\[
2n = \hat{L} \cdot \mathcal{D} \geq \text{mult}_O(\hat{L})\text{mult}_O(\mathcal{D}) \geq 2\text{mult}_S(\mathcal{D}) > 2n
\]

which is impossible. On the other hand, the curve \( \hat{L} \) spans a divisor in \( V \) when we vary the line \( L \) in \( \Pi \). The latter is impossible, because \( \mathcal{D} \) has no base components.

Therefore, Proposition [10] is proved.

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