ON BOUNDS OF THE SINE AND COSINE ALONG A CIRCLE
ON THE COMPLEX PLANE

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Dedicated to people facing and fighting COVID-19

Abstract. In the paper, the author finds bounds of the sine and cosine along a circle on the complex plane in terms of two double inequalities for the norms of the sine and cosine along a circle on the complex plane.

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1. Motivations

In the theory of complex functions, the sine and cosine functions $\sin z$ and $\cos z$ on the complex plane $\mathbb{C}$ are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

respectively, where $z = x + iy$, $x, y \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. They have the least positive periodicity $2\pi$, that is,

$$\sin(z + 2k\pi) = \sin z \quad \text{and} \quad \cos(z + 2k\pi) = \cos z$$

for $k \in \mathbb{Z}$.

When restricting $z = x \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ become $\sin x$ and $\cos x$ and satisfy

$$0 \leq |\sin x| \leq 1 \quad \text{and} \quad 0 \leq |\cos x| \leq 1.$$ 

When restricting $z = iy$ for $y \in \mathbb{R}$, the sine and cosine functions $\sin z$ and $\cos z$ reduce to

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} = i \sinh y \to \pm i\infty$$

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and
\[ \cos(iy) = \frac{e^{-y} + e^{y}}{2} = \cosh y \to +\infty \]
as \( y \to \pm \infty \). These imply that the sine and cosine are bounded on the real \( x \)-axis, but unbounded on the imaginary \( y \)-axis.

In the textbook [8, p. 93], Exercise 6 states that, if \( z \in \mathbb{C} \) and \( |z| \leq R \), then
\[ |\sin z| \leq \cosh R \quad \text{and} \quad |\cos z| \leq \cosh R. \]

In [5], a criterion to justify a holomorphic function was discussed.

In [5], the author discussed and computed bounds of the sine and cosine functions \( \sin z \) and \( \cos z \) along straight lines on the complex plane \( \mathbb{C} \). The main results in the paper [5] can be recited as follows.

1. The complex functions \( \sin z \) and \( \cos z \) are bounded along straight lines parallel to the real \( x \)-axis on the complex plane \( \mathbb{C} \):
   - (a) along the horizontal straight line \( y = \alpha \) on the complex plane \( \mathbb{C} \),
     \[ |\sinh \alpha| \leq |\sin(x + i\alpha)| \leq \cosh \alpha \]  
   - and
     \[ |\sinh \alpha| \leq |\cos(x + i\alpha)| \leq \cosh \alpha, \]
   where \( \alpha \in \mathbb{R} \) is a constant and \( x \in \mathbb{R} \);
   - (b) the equalities in the left hand side of (1) and in the right hand side of (2) hold if and only if \( x = k\pi \) for \( k \in \mathbb{Z} \);
   - (c) the equalities in the right hand side of (1) and in the left hand side of (2) hold if and only if \( x = k\pi + \frac{\pi}{2} \) for \( k \in \mathbb{Z} \).

2. The complex functions \( \sin z \) and \( \cos z \) are unbounded along straight lines whose slopes are not horizontal:
   - (a) along the sloped straight line \( y = \alpha + \beta x \) on the complex plane \( \mathbb{C} \),
     \[ |\sin z| \geq |\sinh(\alpha + \beta x)| \quad \text{and} \quad |\cos z| \geq |\sinh(\alpha + \beta x)|, \]
     where \( \alpha \in \mathbb{R} \) and \( \beta \neq 0 \) are constants;
   - (b) along the vertical straight line \( x = \gamma \) on the complex plane \( \mathbb{C} \),
     \[ |\sin z| \geq |\sinh y| \quad \text{and} \quad |\cos z| \geq |\sinh y|, \]
     where \( \gamma \in \mathbb{R} \) is a constant.

In this paper, we find bounds of the sine and cosine functions \( \sin z \) and \( \cos z \) along a circle centered at the origin \( z = 0 \) of radius \( r \) on the complex plane \( \mathbb{C} \) in terms of double inequalities for their norms.

2. A DOUBLE INEQUALITY FOR THE NORM OF SINE ALONG A CIRCLE

In this section, we find a double inequality for the sine function.

**Theorem 2.1.** Let \( r > 0 \) be a constant and let \( C(0, r) : z = re^{i\theta} \) for \( \theta \in [0, 2\pi) \) denote a circle centered at the origin \( z = 0 \) of radius \( r \). Then
\[ |\sin r| \leq |\sin(re^{i\theta})| \leq \sinh r, \quad \theta \in [0, 2\pi). \]  
The left equality is valid if and only if \( \theta = 0, \pi \) while the right equality is valid if and only if \( \theta = \frac{\pi}{2}, \frac{3\pi}{2} \).
Proof. The circle $C(0, r)$ can be represented by

$$z = re^{i\theta}, \quad \theta \in [0, 2\pi].$$

It is not difficult to see that, for fixed $r > 0$, $|\sin(re^{i\theta})| = |\sin r|$ for $\theta = 0, \pi$, $|\sin(re^{i\theta})| = \sin r$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and $|\sin(re^{i\theta})|$ has a least positive periodicity $\pi$ with respect to the argument $\theta$.

Straightforward computation yields

$$\sin z = \sin(re^{i\theta}) = \sin(r \cos \theta + ir \sin \theta)$$

$$= e^{i(r \cos \theta + ir \sin \theta)} - e^{-i(r \cos \theta + ir \sin \theta)}$$

$$= \frac{2i}{2i} (e^{-r \sin \theta \cos \theta} - e^{r \sin \theta \cos \theta})$$

$$= \frac{2i}{2i} (e^{-r \sin \theta} \cos(r \cos \theta) + i \sin (r \cos \theta)) - e^{r \sin \theta} \cos(r \cos \theta) - i \sin (r \cos \theta))$$

$$= (e^{-r \sin \theta} - e^{r \sin \theta}) \cos(r \cos \theta) + i(\cos(r \cos \theta) + e^{-r \sin \theta} + e^{r \sin \theta}) \sin(r \cos \theta))$$

$$= \cosh(r \sin \theta) \cos(r \cos \theta) + i \sinh(r \sin \theta) \cos(r \cos \theta)$$

and

$$|\sin(re^{i\theta})| = \sqrt{\cosh(r \sin \theta)^2 \sin(r \cos \theta)^2 + \sinh(r \sin \theta) \cos(r \cos \theta)^2}.$$ 

In Figure 1, we plot the 3D graph of $|\sin(re^{i\theta})|$ for $r \in [0, 5]$ and $\theta \in [0, 2\pi)$. In Figure 2, we plot the graph of $|\sin(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$. These two figures are helpful for analyzing and understanding the behaviour of the sine function $\sin z$ along the circle $C(0, r)$ centered at the origin $z = 0$ of radius $r$.

From Figure 2, we can see that the norm $|\sin(\pi e^{i\theta})|$ has only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while it has only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$. 

**Figure 1.** The 3D graph of $|\sin(re^{i\theta})|$ for $r \in [0, 5]$ and $\theta \in [0, 2\pi)$
Figure 2. The graph of $|\sin(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$

Differentiating the square of $|\sin(re^{i\theta})|$ yields

$$\frac{d|\sin(re^{i\theta})|^2}{d\theta} = r[\cos \theta \sinh(2r \sin \theta) - \sin \theta \cos(2r \cos \theta)]$$

$$= r[\sinh(2r \sin \theta) - \tan \theta \sin(2r \cos \theta)] \cos \theta$$

$$= r[\cot \theta \sinh(2r \sin \theta) - \sin(2r \cos \theta)] \sin \theta$$

$$= r^2 \left[ \frac{\sinh(2r \sin \theta)}{2r \sin \theta} - \frac{\sin(2r \cos \theta)}{2r \cos \theta} \right] \sin(2\theta).$$

From the first three expressions above, we conclude that the derivative $\frac{d|\sin(re^{i\theta})|^2}{d\theta}$ is equal to 0 at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Considering the fourth expression above on the intervals $(k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ for $k = 0, 1, 2, 3$, in order that $\frac{d|\sin(re^{i\theta})|^2}{d\theta} \neq 0$ for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$, it is sufficient to find

$$\frac{\sinh(2r \sin \theta)}{2r \sin \theta} > 1$$

and

$$\frac{\sin(2r \cos \theta)}{2r \cos \theta} < 1$$

for $\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})$ and $r > 0$. Then, for fixed $r > 0$, the square $|\sin(re^{i\theta})|^2$ and the norm $|\sin(re^{i\theta})|$ have only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while they have only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$. At $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, the values of $|\sin(re^{i\theta})|$ are both $\sin r$; at $\theta = 0, \pi$, the values of $|\sin(re^{i\theta})|$ are both $|\sin r|$.

Considering the odevity of $\sin t$ and $\sin t$, we see that two inequalities in (4) and (5) are equivalent to

$$\frac{\sinh t}{t} > 1 \quad \text{and} \quad \frac{\sin t}{t} < 1$$

(6)
for $t \in (0, \infty)$. The first inequality in (6) follows from $\cosh x > 1$ for $x \neq 0$ and the Lazarević inequality
\[
\cosh x < \left( \frac{\sinh x}{x} \right)^3
\]  
(7)
in [2, p. 270, 3.6.9]. When $t \in (0, \frac{\pi}{2})$, the second inequality in (6) follows from the right hand side of the Jordan inequality
\[
\frac{\pi}{2} \leq \frac{\sin t}{t} < 1, \quad 0 < |t| \leq \frac{\pi}{2}
\]  
(8)
in [2, Section 2.3] and the papers [1, 3, 4, 7]. When $t > \frac{\pi}{2}$, the second inequality in (6) follows from $\sin t \leq 1$ on $(0, \infty)$ and standard argument. The double inequality (3) is thus proved. The proof of Theorem 2.1 is complete. □

3. A DOUBLE INEQUALITY FOR THE NORM OF COSINE ALONG A CIRCLE

In this section, we find a double inequality for the cosine function.

**Theorem 3.1.** Let $r > 0$ be a constant and let $C(0, r) : z = re^{i\theta}$ for $\theta \in [0, 2\pi)$ denote a circle centered at the origin $z = 0$ of radius $r$. Then
\[
|\cos r| \leq |\cos(re^{i\theta})| \leq \cosh r, \quad \theta \in [0, 2\pi).
\]  
(9)
The left equality is valid if and only if $\theta = 0, \pi$ while the right equality is valid if and only if $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

**Proof.** It is easy to see that, for fixed $r > 0$, $|\cos(re^{i\theta})| = |\cos r|$ for $\theta = 0, \pi$, $|\cos(re^{i\theta})| = \cosh r$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, and $|\cos(re^{i\theta})|$ has a least positive periodicity $\pi$ with respect to the argument $\theta$.

Direct calculation yields
\[
\cos z = \cos(re^{i\theta}) = \cos(r \cos \theta + ir \sin \theta) = \frac{e^{i(r \cos \theta + ir \sin \theta)} + e^{-i(r \cos \theta + ir \sin \theta)}}{2} = \frac{e^{-(r \sin \theta - ir \cos \theta)} + e^{r \sin \theta - ir \cos \theta}}{2} = \frac{e^{-r \sin \theta}[\cos(r \cos \theta) + i \sin(r \cos \theta)] + e^{r \sin \theta}[\cos(r \cos \theta) - i \sin(r \cos \theta)]}{2} = (e^{-r \sin \theta} + e^{r \sin \theta}) \cos(r \cos \theta) + i(e^{-r \sin \theta} - e^{r \sin \theta}) \sin(r \cos \theta)
\]
and
\[
|\cos(re^{i\theta})| = \sqrt{\cosh(r \sin \theta) \cos(r \cos \theta)^2 + \sinh(r \sin \theta) \sin(r \cos \theta)^2}.
\]

In Figure 3, we plot the 3D graph of $|\cos(re^{i\theta})|$ for $r \in [0, 5]$ and $\theta \in [0, 2\pi)$. In Figure 4, we plot the graph of $|\cos(re^{i\theta})|$ for $r = \pi$ and $\theta \in [0, 2\pi)$. These two figures are helpful for analyzing and understanding the behaviour of the cosine function $\cos z$ along the circle $C(0, r)$ centered at the origin $z = 0$ of radius $r$.

From Figure 4, we can see that the norm $|\cos(re^{i\theta})|$ has only two maximums at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, while it has only two minimums at $\theta = 0, \pi$ on the interval $[0, 2\pi)$. 

\[\text{PROPERTIES OF SINE AND COSINE ALONG A CIRCLE} \quad 5\]
Differentiating the square of $|\cos(re^{i\theta})|$ with respect to $\theta$ gives
\[
\frac{d}{d\theta} |\cos(re^{i\theta})|^2 = r[\sin \theta \sin(2r \cos \theta) + \cos \theta \sinh(2r \sin \theta)]
\]
\[
= r[\tan \theta \sin(2r \cos \theta) + \sinh(2r \sin \theta)] \cos \theta
\]
\[
= r[\sin(2r \cos \theta) + \cot \theta \sinh(2r \sin \theta)] \sin \theta
\]
\[
= r^2 \left[ \frac{\sin(2r \cos \theta)}{2r \cos \theta} + \frac{\sinh(2r \sin \theta)}{2r \sin \theta} \right] \sin(2\theta).
\]
From the first three expressions above, we conclude that the derivative $\frac{d}{d\theta} |\cos(re^{i\theta})|^2$ is equal to 0 at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Considering the fourth expression above on the
intervals \((k\frac{\pi}{2}, (k+1)\frac{\pi}{2})\) for \(k = 0, 1, 2, 3\), in order that \(\frac{d}{d\theta} \cos(re^{i\theta})^2 \neq 0\), it is sufficient to show

\[
\frac{\sinh(2r \sin \theta)}{2r \sin \theta} > 1
\]

and

\[
\frac{\sin(2r \cos \theta)}{2r \cos \theta} > -1
\]

for \(\theta \in (k\frac{\pi}{2}, (k+1)\frac{\pi}{2})\) and \(r > 0\). Then, for fixed \(r > 0\), the square \(|\cos(re^{i\theta})|^2\) and the norm \(|\cos(re^{i\theta})|\) have only two maximums at \(\theta = \frac{\pi}{2}, \frac{3\pi}{2}\), while they have only two minimums at \(\theta = 0, \pi\) on the interval \([0, 2\pi]\). At \(\theta = \frac{\pi}{2}, \frac{3\pi}{2}\), the values of \(|\cos(re^{i\theta})|\) are both \(\cosh r\); at \(\theta = 0, \pi\), the values of \(|\cos(re^{i\theta})|\) are both \(|\cos r|\).

Considering oddity of \(\sinh t\) and \(\sin t\), two inequalities in (10) and (11) are equivalent to

\[
\frac{\sinh t}{t} > 1 \quad \text{and} \quad \frac{\sin t}{t} > -1
\]

for \(t \in (0, \infty)\). The first inequality in (12) follows from \(\cosh x > 1\) for \(x \neq 0\) and the Lazarević inequality \((7)\). When \(t \in \left(0, \frac{\pi}{2}\right)\), the second inequality in (12) follows from the left hand side of the Jordan inequality \((8)\). When \(t > \frac{\pi}{2}\), the second inequality in (12) follows from \(\sin t \geq -1\) on \((0, \infty)\) and simple argument. The double inequality \((9)\) is thus proved. The proof of Theorem 3.1 is complete. □

4. Remarks

From Figures 1 and 3, it is not easy to see the difference between \(|\sin(re^{i\theta})|\) and \(|\cos(re^{i\theta})|\). In fact, the difference \(|\sin(re^{i\theta})| - |\cos(re^{i\theta})|\) for \(r \in [0, 2\pi]\) and \(\theta \in [0, 2\pi]\) can be showed by Figure 5.

![Figure 5](image)

**Figure 5.** The difference \(|\sin(re^{i\theta})| - |\cos(re^{i\theta})|\) for \(r, \theta \in [0, 2\pi]\)

From Figures 2 and 4, it is not easy to see the difference between \(|\sin(\pi e^{i\theta})|\) and \(|\cos(\pi e^{i\theta})|\). In fact, the difference \(|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|\) for \(\theta \in [0, 2\pi]\) can be demonstrated by Figure 6.
Figure 6. The difference $|\sin(\pi e^{i\theta})| - |\cos(\pi e^{i\theta})|$ for $\theta \in [0, 2\pi)$

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