MINIMAL RATIONAL CURVES
ON THE MODULI SPACES OF
SYMPLECTIC AND ORTHOGONAL BUNDLES

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Abstract. Let \( C \) be an algebraic curve of genus \( g \) and \( L \) a line bundle over \( C \). Let \( \mathcal{M}_{SC}(n, L) \) and \( \mathcal{M}_{OC}(n, L) \) be the moduli spaces of \( L \)-valued symplectic and orthogonal bundles respectively, over \( C \) of rank \( n \). We construct rational curves of Hecke type on these moduli spaces which generalize the Hecke curves on the moduli space of vector bundles. As a main result, we show that these curves have the minimal degree among the rational curves passing through a general point of the moduli spaces. As its byproducts, we show the non-abelian Torelli theorem and compute the automorphism group of the moduli spaces.

1. Introduction

Let \( C \) be a smooth algebraic curve of genus \( g \geq 2 \) over \( \mathbb{C} \). For integers \( n \geq 2 \) and \( d \), the moduli space \( SU_C(n, d) \) of semistable bundles over \( C \) of rank \( n \) with a fixed determinant of degree \( d \) is known to be a Fano variety of Picard number one. Xiaotao Sun [17] proved that under the assumption that \( g \geq 3 \) (except the case when \( g = 3, n = 2, \) and \( d \) even), a rational curve passing through a general point of \( SU_C(n, d) \) has the minimal degree if and only if it is a so called Hecke curve. This readily yields a simple proof of nonabelian Torelli theorem and the description of the automorphism group of \( SU_C(n, d) \) ([17, Corollary 1.3 and 1.4]). The Hecke curves have been widely studied for \( n = 2 \), for which case the degree minimality was proven in [7, Proposition 8]. For introduction to Hecke curves for \( n > 2 \), see [8] and [9].

The goal of this paper is to establish a similar result for symplectic bundles and orthogonal bundles. Let us briefly explain the main results, pending the precise definition and backgrounds for symplectic and orthogonal bundles to \( \S \) 2.

Let \( \mathcal{M}_{SC}(n, L) \) (resp. \( \mathcal{M}_{OC}(n, L) \)) be the moduli space of \( L \)-valued symplectic (resp. orthogonal) bundles over \( C \) of rank \( n \) for a fixed line bundle \( L \) over

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We first construct variants of Hecke curves on these moduli spaces, called \textit{symplectic} and \textit{orthogonal Hecke curves}, which provide covering families of rational curves for \( \mathcal{M}_{SC}(n, L) \) and \( \mathcal{M}_{OC}(n, L) \) respectively. Interestingly, it turns out that the orthogonal Hecke curves correspond to rational curves having degree higher than that of the Hecke curves on \( SU_C(n, d) \), while the symplectic Hecke curves are just special type of Hecke curves on \( SU_C(n, d) \).

Since the Picard groups of \( \mathcal{M}_{SC}(n, L) \) and \( \mathcal{M}_{OC}(n, L) \) are infinitely cyclic (under the assumption on \( n \) below), the notion of minimality of the degree of a curve makes sense without a reference to a specific ample line bundle. The main result of this paper is as follows:

\textbf{Theorem 1.1.} Assume \( g \geq 3 \), \( n \geq 4 \) in the symplectic case and \( g \geq 5 \), \( n \geq 5 \) in the orthogonal case. Among the rational curves passing through a general point of \( \mathcal{M}_{SC}(n, L) \) (resp. \( \mathcal{M}_{OC}(n, L) \)), a curve has minimal degree if and only if it is a symplectic (resp. orthogonal) Hecke curve.

More detailed version of this statement is given in Theorem 5.2 and Theorem 5.3 for symplectic and orthogonal cases, respectively.

The reason here for the assumption on the rank \( n \) is as follows: A symplectic bundle must have even rank, and \( \mathcal{M}_{SC}(2, L) \cong SU_C(2, L) \) since every vector bundle \( V \) of rank 2 has a symplectic structure coming from the isomorphism \( V \cong V^* \otimes (\det V) \). Also orthogonal bundles of rank \( \leq 4 \) are constructed from line bundles and vector bundles of rank 2 (see [10, p.185] and [16, § 4, 5]).

The idea of proof of Theorem 1.1 is as follows. Once the relevant variants of Hecke curves are defined, we can follow [17, § 2] to compute the degree of rational curves. One additional ingredient is the Harder–Narasimhan filtration of symplectic and orthogonal bundles. In particular, we observe that a symplectic or orthogonal bundle over \( \mathbb{P}^1 \) must have a special kind of splitting type. This enables us to adapt the formulas in [17], originally for vector bundles, to the context of symplectic and orthogonal bundles.

To guarantee that the involved maps from \( \mathbb{P}^1 \) to the moduli spaces are (generically) injective, we need to handle a technical issue regarding generic isomorphisms between symplectic and orthogonal bundles. This can be done by a rather long but elementary argument based on dimension counting.

Theorem 1.1 suggests that the symplectic and orthogonal Hecke curves play the same role as the Hecke curves on \( SU_C(n, d) \) as in [8], [9] and [17]. As samples, we will discuss two consequences in § 6: Nonabelian Torelli theorem (Theorem 6.4) and the description of automorphisms of the moduli space (Theorem 6.5). These results are not new, but our approach provides another geometric view to solve these problems.
This paper is organized as follows. In § 2, the basic facts on symplectic and orthogonal bundles will be explained and their moduli spaces are described. In § 3, the description of the symplectic and orthogonal version of Harder-Narasimhan filtrations will be given in a concrete way. In § 4, we construct relevant variants of Hecke curves. We prove Theorem 1.1 in § 5 and discuss its applications in § 6.

2. MODULI SPACES OF SYMPLECTIC AND ORTHOGONAL BUNDLES

Let $L$ be a line bundle over a smooth algebraic curve $C$ of genus $g \geq 0$.

**Definition 2.1.** A vector bundle $V$ over $C$ of rank $n$ is called an $L$-valued symplectic bundle (resp. orthogonal bundle) if it is equipped with a bundle map $\omega: V \otimes V \to L$ whose restriction to each fiber is a non-degenerate skew-symmetric (resp. symmetric) bilinear form.

An $L$-valued symplectic bundle (resp. orthogonal bundle) can also be viewed as a principal $G_{P(n, C)}$-bundle (resp. $G_{O(n, C)}$-bundle); see [2] and [3]. From the isomorphism $V \cong V^* \otimes L$, we have $\text{det}(V) \cong L^n$ and so $\deg(V) = \frac{1}{2} n \ell$, where $\ell = \deg(L)$.

**Definition 2.2.** A subbundle $E$ of $V$ is said to be isotropic if $\omega|_{E \otimes E} \equiv 0$.

By linear algebra, the rank of an isotropic subbundle of $V$ cannot exceed the half of $\text{rk}(V)$.

**Definition 2.3.** As a symplectic or orthogonal bundle, $V$ is stable (resp. semistable) if $\mu(V) > \mu(E)$ (resp. $\mu(V) \geq \mu(E)$) for every nonzero isotropic subbundle $E$, where $\mu$ is the slope of the vector bundle defined by $\mu(V) = \frac{\text{deg}(V)}{\text{rk}(V)}$.

When $g \geq 2$, the semistable $L$-valued symplectic (resp. orthogonal) bundles over $C$ of rank $n$ form a moduli space denoted by $\mathcal{M}_{SC}(n, L)$ (resp. $\mathcal{M}_{OC}(n, L)$).

Note that every vector bundle $V$ of rank 2 has a $\text{det}(V)$-valued symplectic structure coming from the isomorphism $V \cong V^* \otimes \text{det}(V)$. Hence we have $\mathcal{M}_{SC}(2, L) \cong SU_C(2, \ell)$. In general, the value of the symplectic form $\omega: \wedge^2 V \to L$ determines $\text{det}(V)$ via the isomorphism given by the composition:

$$\text{det}(V) = \wedge^n V \hookrightarrow (\wedge^2 V)^{\otimes 2} \xrightarrow{\omega \otimes \cdots \otimes \omega} L \wedge^{n-2}.$$ 

It is known that $\mathcal{M}_{SC}(n, L)$ is irreducible of dimension $\frac{1}{2} n(n+1)(g-1)$.

On the other hand, $\mathcal{M}_{OC}(n, L)$ has several irreducible components: An $L$-valued orthogonal bundle $V$ has a determinant such that $\text{det}(V)^2 \cong L^n$, hence there are many components distinguished by $c_1(V)$. Also for a fixed $c_1(V)$, there is another topological invariant called the second Stiefel–Whitney class $w_2(V) \in \mathbb{Z}_2$ (see [15] for details). It is known that all the components of $\mathcal{M}_{SC}(n, L)$ have
dimension $\frac{1}{2}n(n-1)(g-1)$. For dimension of the moduli spaces, see [14, Theorem 5.9].

Since most of the arguments in this paper will go parallel for the symplectic and orthogonal case, we simply write $\mathcal{M}$ to denote either $\mathcal{MS}_C(n,L)$ or an irreducible component of $\mathcal{MO}_C(n,L)$, unless there is a possibility of confusion.

The notion of the minimality of the degree of a curve on $\mathcal{M}$ is independent of the choice of an ample line bundle on $\mathcal{M}$, due to the following fact:

**Lemma 2.4.** Assume $n \geq 4$ for symplectic case and $n \geq 5$ for orthogonal case. Then the Picard group of $\mathcal{M}$ is infinitely cyclic.

**Proof.** This is shown in [1]. We remark that this does not hold in the orthogonal case for $n = 4$ (see [16, § 4, 5]). □

In this paper, we choose the reference line bundle as follows. Let $f : \mathcal{M} \to SU_C(n, \frac{1}{2}n\ell)$ be the forgetful morphism sending an $L$-valued symplectic or orthogonal bundle to the underlying vector bundle of degree $\frac{1}{2}n\ell$, where $\ell = \deg(L)$. A priori, $f$ is a rational map defined only on the points represented by symplectic or orthogonal bundles whose underlying vector bundles are semistable. But it is known that every semistable symplectic or orthogonal bundle is semistable as a vector bundle (see Lemma 3.1). Furthermore, $f$ is a generically injective morphism, since a simple vector bundle $W$ cannot have more than one $L$-valued symplectic or orthogonal structure (giving an isomorphism $W \cong W^* \otimes L$).

Now the reference line bundle will be the pull-back of the anti-canonical line bundle on $SU_C(n, \frac{1}{2}n\ell)$. As an advantage of this choice, we can compare the degree of rational curves on the moduli spaces $\mathcal{M}$ and $SU_C(n, \frac{1}{2}n\ell)$.

**Definition 2.5.** Let $u : \mathbb{P}^1 \to \mathcal{M}$ be a generically injective map which gives a rational curve passing through a general point of $\mathcal{M}$. We define its *degree* as the degree of $(f \circ u)^*(-K)$, where $K$ is the canonical line bundle on $SU_C(n, \frac{1}{2}n\ell)$.

### 3. Harder–Narasimhan filtrations

In this section, we describe the Harder–Narasimhan filtration of symplectic and orthogonal bundles. Most of the results in this section can be found in the literature, written in the language of principal bundles in general. For our convenience later, we rewrite the results in the language of symplectic and orthogonal bundles.

The discussion in this section applies to any genus $g \geq 0$ and any rank $n \geq 2$.

First we recall:
Lemma 3.1. Let $V$ be an $L$-valued symplectic/orthogonal bundle with $\deg V = \frac{1}{2} n\ell$, where $\ell = \deg L$.

1. A symplectic/orthogonal bundle $V$ is semistable if and only if it is semistable as a vector bundle.
2. If a symplectic/orthogonal bundle $V$ is stable, then the underlying vector bundle is polystable: a direct sum of stable bundles of the same slope.

Hence to show the semistability of $V$, it suffices to check that there is no destabilizing isotropic subbundles.

Proof. The argument below is borrowed from [13, proposition 4.2], in which the orthogonal case was discussed. Assume $V$ is semistable as a symplectic/orthogonal bundle. Let $E$ be any subbundle of rank $r$. From the isomorphism $V/E \cong (E^\perp)^* \otimes L$, we observe:

\begin{equation}
\deg(E^\perp) = \deg(E) + \left(\frac{n}{2} - r\right)\ell.
\end{equation}

Let $M$ and $N$ be the subbundles generated by $E \cap E^\perp$ and $E + E^\perp$, respectively. If $M = 0$, then the map $E \oplus E^\perp \to V$ gives a subsheaf whose quotient is a torsion sheaf. Hence

\[2 \deg E + \left(\frac{n}{2} - r\right)\ell \leq \deg(V) = \frac{1}{2} n\ell,
\]

and so $\mu(E) = \frac{\deg(E)}{r} \leq \frac{\ell}{2} = \mu(V)$.

If $M$ has rank $k > 0$, there is an exact sequence

\[0 \to M \to E \oplus E^\perp \to N \to 0.
\]

Since the symplectic or orthogonal pairing $\langle E + E^\perp, E \cap E^\perp \rangle$ is zero, we have $M \subset N^\perp$. Since $M$ and $N^\perp$ have the same rank, they must coincide. Therefore, we have

\[\rk(E) + \rk(E^\perp) = \rk(M) + \rk(N) = \rk(M) + \rk(M^\perp).
\]

Applying (3.1) to both $E$ and $M$, we have

\begin{equation}
\deg(E) = \deg(M) + \frac{1}{2} (r - k)\ell.
\end{equation}

Since $M = E \cap E^\perp$ is isotropic,

\begin{equation}
\deg(M) \leq k\mu(V) = \frac{1}{2} k\ell.
\end{equation}

Hence $\deg(E) \leq \frac{1}{2} r\ell$ and $V$ is semistable as a vector bundle.

When we assume $V$ is a stable symplectic/orthogonal bundle, (3.3) becomes strict inequality. Hence if the equality $\mu(E) = \mu(V)$ holds, then $M = 0$ and in this case the map $E \oplus E^\perp \to V$ is an isomorphism by (3.1). This shows the second claim. □
The following gives the Harder–Narasimhan filtration of a symplectic/orthogonal bundle.

**Proposition 3.2.** For any $L$-valued symplectic (resp. orthogonal) bundle $V$, there is a chain of isotropic subbundles:

\[(3.4) \quad 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k \subset V \]

such that

\[(3.5) \quad \mu(V/E_k) < \mu(E_k/E_{k-1}) < \cdots < \mu(E_2/E_1) < \mu(E_1). \]

Moreover, $(E_i/E_{i-1}) \oplus (E_i^\perp/E_{i-1}^\perp)$ for $1 \leq i \leq k$ and $E_k^\perp/E_k$ are semistable $L$-valued symplectic (resp. orthogonal) bundles under the inherited symplectic (resp. orthogonal) structure.

As a consequence, the Harder–Narasimhan filtration of the underlying vector bundle of $V$ is given by:

\[(3.6) \quad 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k \subset E_k^\perp \subset E_{k-1}^\perp \subset \cdots \subset E_1^\perp \subset E_0^\perp = V, \]

where the middle term $E_k$ may or may not be equal to $E_k^\perp$.

**Proof.** If $V$ itself is semistable, there is nothing to prove. If $V$ is unstable, let $E_1$ be an isotropic subbundle of $V$ such that

- $\mu(E_1)$ is maximal among the isotropic subbundles and
- $\text{rk}(E_1)$ is maximal among them.

Since $E_1$ has maximal slope and every subbundle of $E_1$ is isotropic, it must be a semistable vector bundle.

We stop here if $V$ has no destabilizing isotropic subbundle $S$ containing $E_1$. Otherwise, let $E_2$ be an isotropic subbundle containing $E_1$ such that

- $\mu(E_2)$ is maximal among the isotropic subbundles containing $E_1$ and
- $\text{rk}(E_2)$ is maximal among them.

By the maximality of slope, the quotient $E_2/E_1$ is a semistable vector bundle. Also, since $\mu(E_2) < \mu(E_1)$, we have $\mu(E_2/E_1) < \mu(E_1)$.

Repeating this process, we have the chain

\[(3.7) \quad 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k \subset V \]

of isotropic subbundles satisfying (3.5) such that each $E_i/E_{i-1}$ is semistable for $i \leq k$. The quotient $V/E_k$ need not be semistable, but

\[(*) \quad \text{for any isotropic subbundle } S \subset V \text{ containing } E_k, \text{ we have } \mu(S/E_k) \leq \mu(V/E_k). \]

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1One can use the convenient fact that given a subbundle $A \subset B$, the values of $\mu(B) - \mu(A)$, $\mu(B/A) - \mu(B)$, and $\mu(B/A) - \mu(A)$ have the same sign.
It is easy to check that \((E_i/E_{i-1}) \oplus (E^\perp_{i-1}/E^\perp_i)\) and \(E^\perp_k/E_k\) have the induced symplectic/orthogonal structure by linear algebra. The semistability of \((E_i/E_{i-1}) \oplus (E^\perp_{i-1}/E^\perp_i)\) follows from \(E^\perp_{i-1}/E^\perp_i \cong (E_i/E_{i-1})^*\).

Now we show that \(E^\perp_k/E_k\) is semistable. Since it is a symplectic/orthogonal bundle, it suffices to check that there is no destabilizing isotropic subbundle. Let \(\tilde{S}\) be any isotropic subbundle of \(E^\perp_k/E_k\). Then \(\tilde{S} = S/E_k\), where \(S\) is an isotropic subbundle of \(E^\perp_k\) containing \(E_k\). To prove the claim that \(\mu(E^\perp_k/E_k) \geq \mu(\tilde{S})\), we note

\[\mu(E^\perp_k/E_k) - \mu(\tilde{S}) = (\mu(E^\perp_k/E_k) - \mu(S)) + (\mu(S) - \mu(\tilde{S})).\]

Since the above process ends up at \(E_k\), we have \(\mu(V) \geq \mu(S)\). From the isomorphism \(V/E_k \cong (E_k)^+ \otimes L\), we compute \(\mu(E^\perp_k/E_k) = \mu(V)\). This shows that the first term on the righthand side is nonnegative. Also the sign of the second term is same as that of \(\mu(E_k) - \mu(S)\), which is nonnegative from the choice of \(E_k\) with maximal slope. This shows the claim. \(\square\)

Note that the uniqueness of the Harder–Narasimhan filtration of the underlying vector bundle guarantees the uniqueness of the sequence (3.4) for a symplectic/orthogonal bundle \(V\).

In particular for \(g = 0\), the existence of a symplectic or orthogonal structure on a vector bundle \(V\) on \(\mathbb{P}^1\) forces its splitting type to be special. First note that the form \(\omega: V \otimes V \to \mathcal{O}(\ell)\) can be normalized: when \(\ell\) is even (resp. odd), we may assume that \(\ell = 0\) (resp. \(\ell = 1\)) by replacing \(V\) by \(V \otimes \mathcal{O}(\ell)\) (resp. \(V \otimes \mathcal{O}(\ell)\)).

**Corollary 3.3.** Let \(V\) be an \(\mathcal{O}(\ell)\)-valued symplectic or orthogonal bundle over \(\mathbb{P}^1\) of the splitting type

\[V \cong \mathcal{O}(a_1)^{r_1} \oplus \mathcal{O}(a_2)^{r_2} \oplus \cdots \oplus \mathcal{O}(a_m)^{r_m}\]

with \(a_1 > \cdots > a_m\). Then for each \(i\), we always have \(r_{m+1-i} = r_i\) and

\[a_{m+1-i} = -a_i \quad \text{for} \quad \ell = 0 \quad \text{and} \quad a_{m+1-i} = -a_i + 1 \quad \text{for} \quad \ell = 1.\]  

**Proof.** According to the type of Harder-Narasimhan filtration in (3.6), we have

\[r_1 + (2n - r_m) = 2n, \quad (r_1 + r_2) + (2n - (r_m + r_{m-1})) = 2n,\]

and so on. This shows \(r_{m+1-i} = r_i\) for \(1 \leq i \leq m\).

Also when \(\ell = 0\), we have \(\deg(V) = 0\) and \(\deg(E^\perp) = \deg(E)\), so the filtration (3.6) yields the equalities

\[a_1 r_1 = -a_m r_m, \quad a_1 r_1 + a_2 r_2 = -(a_m r_m + a_{m-1} r_{m-1}),\]

and so on. This inductively shows \(a_{m+1-i} = -a_i\) for \(1 \leq i \leq m\).

When \(\ell = 1\), we have \(\deg(V) = n\) and \(\deg(E^\perp) = \deg(E) + n - \text{rk}(E)\). From this, the equality \(a_{m+1-i} = -a_i + 1\) is obtained similarly. \(\square\)
4. HECKE CURVES

Throughout this section, we assume \( n \geq 4 \) for symplectic case and \( n \geq 5 \) for orthogonal case. In §4.1, we establish a result on the uniqueness of a generic isomorphism as a technical toolkit. And then we construct symplectic and orthogonal variants of Hecke curves in §4.2 and 4.3 respectively.

4.1. Uniqueness of a generic isomorphism. For a nonnegative integer \( \delta \), a symplectic or orthogonal bundle \( V \) is said to be \( \delta \)-stable if

\[
\frac{\deg(E) + \delta}{\text{rk}(E)} < \frac{\deg(V)}{n},
\]

for each nonzero isotropic subbundle \( E \). This is a notion adapted to the symplectic and orthogonal case, which is originally the \((\delta, \delta)\)-stability in [11, Definition 5.1]. Note that the 0-stability is equivalent to the stability. It is easy to see that the \( \delta \)-stability is an open condition in a family for each \( \delta \geq 0 \).

Lemma 4.1. Let \( M \) denote either \( \mathcal{M}_{SC}(n, L) \) or an irreducible component of \( \mathcal{M}_{OC}(n, L) \). Let \( \delta \) be a positive integer. A general symplectic or orthogonal bundle in \( M \) is \( \delta \)-stable if either \((g = \delta + 2 \text{ and } n > \delta + 1)\) or \((g > \delta + 2 \text{ and } n > \frac{1}{2}(\delta + 2))\).

Proof. A similar result for orthogonal case was proven in [3, Proposition 3.5], but the above claim is slightly stronger.

Let \( S \) be the sublocus of \( M \) consisting of bundles \( V \) admitting an isotropic subbundle \( E \) of rank \( r \) and degree \( d \) such that

\[
\frac{d + \delta}{r} \geq \frac{\ell}{2}, \quad \text{or equivalently } \quad d \geq \frac{1}{2}r\ell - \delta.
\]

It suffices to show that the dimension of (any irreducible component of) \( S \) is strictly smaller than \( \dim(M) \).

From the exact sequence

\[
0 \to E \to V \to (E^\perp)^* \otimes L \to 0,
\]

we have

\[
\deg(E^\perp) = d + (\frac{n}{2} - r)\ell.
\]

The inherited form on the quotient \( E^\perp/E \) is non-degenerate, so it is an \( L \)-valued symplectic/orthogonal bundle of rank \( n - 2r \). The above sequence can be put into the middle sequence of the following diagram:
Note that this diagram is symmetric with respect to the diagonal line (possibly with a change of sign). That is, the central term has the symmetry $V \cong V^* \otimes L$ and the right vertical sequence gives the bottom horizontal sequence after taking dual and tensoring $L$.

The middle extensions are parametrized by $\mathbb{P}H^1(C, \text{Hom}(E^\perp \otimes E \otimes L^*))$, which appears again in the long exact sequence associated to the bottom sequence (tensored by $E \otimes L^*$):

$$
\begin{align*}
(4.5) \quad & H^1(E \otimes E \otimes L^*) \to H^1(E^\perp \otimes E \otimes L^*) \to H^1((E^\perp/E) \otimes E \otimes L^*) \to 0.
\end{align*}
$$

Now we compute an upper bound on the dimension of those bundles $V$ fitting into this diagram. A relevant result for this dimension count is in [12, Proposition 2.6], which guarantees that for any given family of vector bundles, there is an irreducible family of bundles whose generic member is stable, and contains all the members of the given family. Since the arguments in [12, Proposition 2.6] can be adapted to the symplectic or orthogonal cases, we may assume the stability of the objects in the dimension count below.

Consider the diagram (4.4). For the right vertical sequence,

- $E$ moves in the moduli of vector bundles of rank $r$, which has dimension $r^2(g-1) + 1$.
- $E^\perp/E$ moves in a family of symplectic/orthogonal bundles of rank $n-2r$, which has dimension $\frac{1}{2}(n-2r)(n-2r \pm 1)(g-1)$, where the sign $\pm$ refers to $+$ in the symplectic case and $-$ in the orthogonal case (throughout the proof).
- The bundle $F := (E^\perp)^* \otimes L$ is obtained as an extension of $E^* \otimes L$ by $E^\perp/E$, which lies on $\mathbb{P}H^1(C, (E^\perp/E) \otimes E \otimes L^*)$. To count the dimension of deformations, we can assume that $E^\perp$ is stable, and so

$$
H^0(C, \text{Hom}(E \otimes L^*, E^\perp/E)) = 0.
$$

Since $(E^\perp/E) \otimes E \otimes L^*$ has rank $r(n-2r)$ and degree $(n-2r)(d - \frac{1}{2}r \ell)$, we have

$$
dim \mathbb{P}H^1((E^\perp/E) \otimes E \otimes L^*) = (n-2r)(\frac{1}{2}r \ell - d) + r(n-2r)(g-1) - 1.
$$
These information determine all the terms in the diagram by symmetry mentioned above, except the central term $V$. This is determined by the choice of a point in $\mathbb{P}H^1(C, E^\perp \otimes E \otimes L^*)$, which is in the middle of (4.5). Since a class in $H^1((E^\perp / E) \otimes E \otimes L^*)$ is already chosen, it suffices to choose a class in $H^1(E \otimes E \otimes L^*)$. By the (anti)-symmetry of the diagram (4.4), the extension class lies on $H^1((E^\perp / E) \otimes E \otimes L^*)$ for the symplectic and orthogonal case respectively (see [6, Criterion 2.1]), which has dimension $(r \pm 1)(\frac{1}{2}r\ell - d) + \frac{1}{2}r(r \pm 1)(g - 1)$.

The total sum of the above moduli gives the upper bound:

$$\dim(S) \leq (n - r \pm 1)(\frac{1}{2}r\ell - d) + \frac{1}{2}(n^2 - 2rn + 3r^2 \pm (n - r))(g - 1).$$

Since $\dim(M) = \frac{1}{2}n(n \pm 1)(g - 1)$, the codimension of $S$ in $M$ is positive if

$$(n - r \pm 1)(\frac{1}{2}r\ell - d) < \frac{r}{2}(2n - 3r \pm 1)(g - 1).$$

When $g = \delta + 2$, this holds by the inequality (4.1) if

$$(n - r \pm 1)\delta < \frac{r}{2}(2n - 3r \pm 1)(\delta + 1).$$

This holds for $r = 1$ if $n > \delta + 1$. Also if $2 \leq r \leq \frac{n}{2}$, then $n - r \pm 1 \leq 2n - 3r \pm 1$ from which the above inequality follows. Similarly, (4.6) holds if $g > \delta + 2$ and $n > \frac{1}{2}(\delta + 2)$. This shows that the dimension of the locus $S$ is strictly smaller than $\dim(M)$ under the assumptions on $g$ and $n$.

**Lemma 4.2.** Let $M$ denote either $\mathcal{M}SC(n, L)$ or an irreducible component of $\mathcal{M}OC(n, L)$. Suppose that $[V]$ is a general point of $M$ representing a symplectic or orthogonal bundle $V$. Let $\phi : W \to V$ be a generic isomorphism of vector bundles with $\deg(V) - \deg(W) = \delta > 0$. Then $\dim H^0(C, W^* \otimes V) = 1$ if either $(\delta = 1$ and $g \geq 3)$ or

$$(4.7) \quad g > \frac{3(\delta - 1)n}{n - 1} + 1, \quad n > \frac{1}{2}(\delta + 2).$$

**Remark 4.3.**

1. The argument below is a refinement of the proof of [11, Lemma 5.6], which was originally given for vector bundles.
2. Note that in each case the condition on $g$ guarantees that $V$ is $\delta$-stable by Lemma 4.1. It can be checked that for $\delta \geq 2$ and $n \geq 4$,

$$\delta + 2 < \frac{3(\delta - 1)n}{n - 1} + 1.$$

3. Later we will apply this result for $\delta \leq 4$, in which case the condition $n > \frac{1}{2}(\delta + 2)$ can be removed.

**Proof of Lemma 4.2.** First we show that if $\dim H^0(C, W^* \otimes V) > 1$, then there is a nonzero map $W \to V$ with a nontrivial kernel.
Choose a general \( y \in C \). Then \( y \notin \text{Supp}(D) \), where \( D \) is an effective divisor of degree \( \delta < g \) such that \((\det V) \otimes (\det W^*) \cong \mathcal{O}_C(D)\). Let \( \psi: W \to V \) be a map which is not a constant multiple of \( \phi \). Since \( \phi \) and \( \psi \) are linearly independent at a general point \( y \), some linear combination \( a\phi + b\psi \) is degenerate at \( y \). This map cannot be a generic isomorphism: if it were, the quotient of \( V \) by \((a\phi + b\psi)(W)\) is a torsion sheaf supported on a divisor containing \( y \), which is a contradiction. This shows that \( a\phi + b\psi: W \to V \) is a nonzero map with a nontrivial kernel.

Therefore, it suffices to show that those bundles \( V \) which admit a nonzero map \( \psi: W \to V \) with a nontrivial kernel are contained in a proper closed subset of \( \mathcal{M} \).

Now let \( V \) be a bundle in \( \mathcal{M} \) admitting a nonzero map \( \psi: W \to V \) with a nontrivial kernel \( S \) and the quotient \( E = W/S \) which is a subsheaf of \( V \) of rank \( r < n \). Let \( M := E \cap E^\perp \). If \( M \) is nonzero of rank \( r_M > 0 \), then from the \( \delta \)-stability (see Remark 4.3 (2)), we have
\[
\frac{\deg(M) + \delta}{r_M} < \frac{\ell}{2}.
\]
Hence as before in (3.2),
\[
\deg(E) = \deg(M) + (r - r_M) \cdot \frac{\ell}{2} < \frac{r\ell}{2} - \delta.
\]
Also, since \( S \) is a subsheaf of \( V \) via \( \phi \), we have \( \deg(S) < \frac{1}{2}(n - r)\ell \). Therefore,
\[
\deg(W) = \deg(S) + \deg E < \frac{n\ell}{2} - \delta = \deg(W),
\]
which is a contradiction.

This shows that \( E \cap E^\perp = 0 \) and hence the restriction of the form \( (V, \omega) \) to \( E \) is non-degenerate on a general fiber. Note that
\[
\deg(E) = \deg(W) - \deg(S) > \frac{n\ell}{2} - \delta - \frac{(n - r)\ell}{2} = \frac{r\ell}{2} - \delta.
\]
Since
\[
\deg(E^\perp) = \deg(E) + \frac{(n - 2r)\ell}{2} > \frac{(n - r)\ell}{2} - \delta,
\]
we have
\[
\deg(E) + \deg(E^\perp) \geq \deg(V) - 2(\delta - 1).
\]
Hence we get a sequence
\[
(4.9) \quad 0 \to E \oplus E^\perp \to V \to \tau_0 \to 0,
\]
where \( \tau_0 \) is a torsion sheaf of degree \( \leq 2(\delta - 1) \).

Now we count dimension of bundles \( V \in \mathcal{M} \) fitting into (4.9). First, we note that the restriction of the form \( \omega \) to \( E \) yields a generic isomorphism \( E \to E^* \otimes L \) such that the quotient \((E^* \otimes L)/E\) is a torsion sheaf of degree \(-2\deg(E) +\)
Given two bundles $E$ and $E^\perp$, those bundles $V$ fitting into the sequence of the form (4.9) have dimension bounded from above by

$$2(\delta - 1) \cdot \dim \mathbb{P}(E \oplus E^\perp) = 2(\delta - 1)n.$$ 

Summing up these three dimensions, we get

$$\frac{1}{2}(n^2 - 2rn + 2r^2 \pm n)(g - 1) + 3(\delta - 1)n =: D_0.$$ 

From the inequality $D_0 < \dim \mathcal{M} = \frac{1}{2}n(n \pm 1)(g - 1)$, we get

$$(rn - r^2)(g - 1) > 3(\delta - 1)n.$$ 

The lefthand side is a quadratic polynomial in $r$, which has minimum at $r = 1$. Thus the inequality reduces to: $(n - 1)(g - 1) > 3(\delta - 1)n$. Hence it can be seen that $\dim \mathcal{M} > D_0$ under the assumption that $g > \frac{3(\delta - 1)n}{n - 1} + 1$.

We conclude that a general bundle $V$ in $\mathcal{M}$ does not admit a generic isomorphism $\phi: W \to V$ together with a map $\psi: W \to V$ having a nontrivial kernel. Therefore, given a general $[V] \in \mathcal{M}$, a generic isomorphism $\phi: W \to V$ is unique up to a constant multiplication, if it exists. \hfill \Box

### 4.2. Symplectic Hecke curves on $\mathcal{M} = \mathcal{M}_{SC}(n, L)$

Let $(V, \omega)$ be a point of $\mathcal{M}$ representing an $L$-valued symplectic bundle $V$ of rank $n$. Now we construct symplectic Hecke curves, which will turn out to have minimal degree on $\mathcal{M}$. The construction closely follows that of Hecke curves on $SU_C(n, d)$ in [8] (and [9]). To construct a symplectic version of Hecke curves, we need to keep track of the symplectic forms in the process.

Given a linear functional $\theta \in V^*_|x$ for some $x \in C$, let $V^\theta$ be the kernel sheaf of the composition $V \to V|_x \xrightarrow{\theta} \mathbb{C}$. Then we have an extension

$$0 \to V^\theta \to V \to (V|_x/ \ker \theta) \otimes \mathbb{C}_x \to 0,$$  

(4.10)
where $C_x$ is the skyscraper sheaf supported at $x$. Taking dual of this Hecke transformation, we get

$$0 \to V^* \to (V^\theta)^* \to C_x \to 0,$$

which restricts to the fiber at $x$ as an exact sequence of vector spaces:

$$(4.11) \quad 0 \to \langle \theta \rangle \to V^* \mid_x \to (V^\theta)^* \mid_x \to C \to 0.$$ 

Then there is a sheaf injection from $(V^\theta)^*(-x) := (V^\theta)^* \otimes O_C(-x)$ into $V^*$. Consider the composition map:

$$V^\theta(-x) \xrightarrow{\alpha} V^* \xrightarrow{\bar{\omega}} V \otimes L^* \xrightarrow{\beta} V^\theta(x) \otimes L^*,$$

where $\alpha$ and $\beta$ are dual to each other and $\bar{\omega}$ is the skew-symmetric isomorphism associated to $\omega$. This gives a skew-symmetric map $(V^\theta)^* \to V^\theta \otimes L^*(2x)$, or equivalently an $L^*(2x)$-valued form

$$\Lambda^2(V^\theta)^* \to L^*(2x).$$

The restriction of this form to the fiber $\Lambda^2(V^\theta)^* \mid_x$ factors through $\Lambda^2(\text{Im}(\alpha_x))$, which is zero since $\dim \text{Im}(\alpha_x) = 1$. Hence we get an $L^*(x)$-valued skew-symmetric form on $(V^\theta)^*$, or equivalently a skew-symmetric map

$$\omega^x : (V^\theta)^* \to V^\theta \otimes L^*(x).$$

By computing the difference of degrees, we see that the subspace $\ker(\omega_x^\theta)$ has codimension two in $(V^\theta)^* \mid_x$.

For each linear functional $\lambda$ of $(V^\theta)^* \mid_x$, let $\tilde{V}^\lambda$ be the bundle obtained by the Hecke transformation

$$0 \to \tilde{V}^\lambda \to (V^\theta)^* \to ((V^\theta)^* \mid_x / \ker(\lambda)) \otimes C_x \to 0.$$

**Lemma 4.4.** The bundle $\tilde{V}^\lambda$ is equipped with an $L^*$-valued symplectic form $\tilde{\omega}^\lambda$ induced from $\omega^\theta$ if and only if $\ker(\omega_x^\theta) \subset \ker(\lambda)$.

**Proof.** Consider the composition map

$$\tilde{V}^\lambda \xrightarrow{\bar{\alpha}} (V^\theta)^* \xrightarrow{\omega^x} V^\theta \otimes L^*(x) \xrightarrow{\bar{\beta}} (\tilde{V}^\lambda)^* \otimes L^*(x),$$

where $\bar{\alpha}$ and $\bar{\beta}$ are dual to each other. This gives a form

$$\tilde{\omega}^\lambda : \Lambda^2\tilde{V}^\lambda \to L^*(x),$$

which is nondegenerate outside the fiber at $x$. This factors through $\Lambda^2 \text{Im}(\bar{\alpha})$, where $\text{Im}(\bar{\alpha}_x) = \ker(\lambda)$ has codimension one in $(V^\theta)^* \mid_x$. If $\ker(\omega_x^\theta) \subset \ker(\lambda)$, then $\tilde{\omega}_x^\lambda$ further factors through the quotient space $\Lambda^2 ((\ker(\lambda) / \ker(\omega_x^\theta)))$, which is zero. Therefore in this case, we have an $L^*$-valued form

$$\tilde{\omega}^\lambda : \Lambda^2\tilde{V}^\lambda \to L^*.$$

This is nondegenerate everywhere since $\tilde{V}^\lambda$ and $(\tilde{V}^\lambda)^* \otimes L^*$ have the same degree.
If $\ker(\omega^\theta_x) \not\subset \ker(\lambda)$, it is easy to see that the form $\tilde{\omega}^\lambda : \Lambda^2 \tilde{V}^\lambda \to L^*(x)$ does not identically vanish on the fiber at $x$, and it remains to be a partially degenerate $L^*(x)$-valued form.

Hence for each linear functional $\lambda$ of $(V^\theta)^*_x$ whose kernel contains $\ker(\omega^\theta_x)$, we get a bundle $(\tilde{V}^\lambda)^*$ equipped with an $L$-valued symplectic form $(\tilde{\omega}^\lambda)^*$. Since this is parameterized by $\lambda \in \mathbb{P}((V^\theta)^*_x/\ker(\omega^\theta_x)) \cong \mathbb{P}^1$, we get a family $\mathcal{C}^\theta[V]$ of $L$-valued symplectic bundles. In particular when $\ker(\lambda)$ coincides with the subspace $V^*_x/\langle \theta \rangle$ from the sequence (4.11), we get back $(V,\omega) \cong ((\tilde{V}^\lambda)^*,(\tilde{\omega}^\lambda)^*)$.

This shows that for each $\theta \in \mathbb{P}V^*$, the family $\mathcal{C}^\theta[V]$ gives a rational curve on $\mathcal{M}$ passing through the point $[V]$, provided that it is a nonconstant family and the symplectic bundles $\tilde{V}^\lambda$s are stable.

**Lemma 4.5.** Assume $g \geq 3$ and $n \geq 4$. Let $V$ be a general point of $\mathcal{M}$. Then

1. all the symplectic bundles $\tilde{V}^\lambda$ appearing in $\mathcal{C}^\theta[V]$ are stable, and
2. the map $\mathbb{P}^1 \to \mathcal{C}^\theta[V]$ given by $\lambda \mapsto ((\tilde{V}^\lambda)^*,(\tilde{\omega}^\lambda)^*)$ is generically injective.

**Proof.** By Lemma 4.1, we may assume that $V$ is 1-stable.

To show (1), for each $[\tilde{V}^\lambda]$, let $E$ be an isotropic subbundle of $\tilde{V}^\lambda$. Then by the construction, $V^*$ has a subsheaf $\tilde{E}$ defined by the kernel of the composition $(E \to \tilde{V}^\lambda \to (V^\theta)^* \to (V^\theta)^*/V^* \cong \mathbb{C}_x)$. Then in any case $\deg(\tilde{E}) \geq \deg(E) - 1$. By the 1-stability of $V^*$, we have

$$\frac{\deg(E)}{\text{rk}(E)} \leq \frac{\deg(\tilde{E}) + 1}{\text{rk}(\tilde{E})} < \frac{\deg(V^*)}{\text{rk}(V^*)} = \frac{\deg(\tilde{V}^\lambda)}{\text{rk}(\tilde{V}^\lambda)}.$$ 

To show (2), first we note that a general member $\tilde{V}^\lambda$ is 1-stable, since the 1-stability is an open condition and the family contains a 1-stable bundle $V^*$. Suppose that for $\lambda_1, \lambda_2 \in \mathbb{P}^1$, both $\tilde{V}^\lambda_1$ and $\tilde{V}^\lambda_2$ are 1-stable and $\tilde{V}^\lambda_1 \cong \tilde{V}^\lambda_2$. Then there are two linearly independent generic isomorphisms between $(V^\theta)^*$ and $\tilde{V}^\lambda_1$. By Lemma 4.2, this implies $\lambda_1 = \lambda_2$. Therefore, a general member $\tilde{V}^\mu$ is not isomorphic to any other member. \hfill $\Box$

This rational curve $\mathcal{C}^\theta[V]$ is called the **symplectic Hecke curve** on $\mathcal{M}$ associated to $\theta \in \mathbb{P}V$. It will be shown in § 5 that $\mathcal{C}^\theta[V]$ has degree $2n$.

### 4.3. Orthogonal Hecke curves on $\mathcal{MO}_C(n, L)$. The construction in the orthogonal case is similar to the symplectic case, but also there are some important differences. We follow the construction in § 4.2, pointing out the required modifications.

Let $\mathcal{M}$ be any irreducible component of $\mathcal{MO}_C(n, L)$. Let $(V,\omega)$ be a point of $\mathcal{M}$ representing an $L$-valued orthogonal bundle $V$ of rank $n$. For $x \in C$, let $IG(2, V|_x)$ be the Grassmannian of 2-dimensional isotropic subspaces of $V|_x$. Then
for each $\Theta \in IG(2, V|_x)$, we can associate a codimension 2 subspace $\Theta^\perp$ of $V|_x$. Let $V^\Theta$ be the kernel sheaf of the composition $V \to V|_x \to (V|_x)/(\Theta^\perp)$. This gives an extension

\[(4.12) \quad 0 \to V^\Theta \to V \to (V|_x/\Theta^\perp) \otimes \mathbb{C}_x \to 0,\]

Taking dual of this Hecke transformation, we get

\[(4.13) \quad 0 \to V^* \to (V^\Theta)^* \to C^\otimes 2 \to 0,\]

which restricts to the fiber at $x$ as an exact sequence of vector spaces

since there is a canonical isomorphism $(V|_x/\Theta^\perp)^* \cong \Theta$.

Then there is a sheaf injection from $(V^\Theta)^*(-x)$ into $V^*$. Consider the composition map

\[(V^\Theta)^*(-x) \xrightarrow{\alpha} V^* \xrightarrow{\tilde{\omega}} V \otimes L^* \xrightarrow{\beta} V^\Theta(x) \otimes L^*,\]

where $\tilde{\omega}$ is the symmetric isomorphism associated to $\omega$. This gives a symmetric form

\[\text{Sym}^2 (V^\Theta)^* \to L^*(2x).\]

By computing the difference of degrees, we see that the subspace $\ker(\omega^\Theta_x)$ has codimension four in $(V^\Theta)^*_x$. Then a subspace $\Lambda \subset (V^\Theta)^*_x$ of codimension two which is isotropic with respect to this form corresponds to a 2-dimensional isotropic subspace of the orthogonal vector space $(V^\Theta)^*/\ker(\omega^\Theta_x) \cong \mathbb{C}^4$, and vice versa.

For any subspace $\Lambda \cong \mathbb{C}^{n-2}$ of $(V^\Theta)^*_x$, let $\tilde{V}^\Lambda$ be the bundle obtained by the Hecke transformation

\[0 \to \tilde{V}^\Lambda \to (V^\Theta)^* \to ((V^\Theta)^*_x/\Lambda) \otimes \mathbb{C}_x \to 0.\]

**Lemma 4.6.** The bundle $\tilde{V}^\Lambda$ is equipped with an $L^*$-valued orthogonal form $\tilde{\omega}^\Lambda$ induced from $\omega^\Theta$ if and only if $\Lambda$ is an isotropic subspace containing $\ker(\omega^\Theta|_x)$.

**Proof.** We can argue in the same way as in the proof of Lemma 4.4. Consider the composition map

\[\tilde{V}^\Lambda \xrightarrow{\tilde{\alpha}} (V^\Theta)^* \xrightarrow{\tilde{\omega}^\Theta} V^\Theta \otimes L^*(x) \xrightarrow{\beta} (\tilde{V}^\Lambda)^* \otimes L^*(x),\]

where $\tilde{\alpha}$ and $\tilde{\beta}$ are dual to each other. This gives a form

\[\tilde{\omega}^\Lambda : \text{Sym}^2 \tilde{V}^\Lambda \to L^*(x).\]
Then the family

Lemma 4.7. Assume $V$ is a rational curve on $M$ and we get the generic injectivity. It is well-known that $IG$ is isomorphic to the isotropic Grassmannian $IG(2, \{((V^\theta)^*|_x/\ker(\omega^\theta|_x) = IG(2, 4)\). By Lemma 4.1, we get back the original bundle $(\tilde{V}_0, \tilde{\omega}_0) = (V, \omega)$.

Choose the $\mathbb{P}^1$-component of $IG(2, 4)$ containing $t_0$. Then the family

$$C^\theta[V] := \{(\tilde{V}_t, \tilde{\omega}_t) : t \in \mathbb{P}^1\}$$

is a rational curve on $M$ passing through the point $[V]$, provided that the bundles $\tilde{V}_t$’s are stable as orthogonal bundles.

**Lemma 4.7.** Assume $g \geq 5$ and $n \geq 5$. Let $V$ be a general point of $M$. Then

1. all the orthogonal bundles $\tilde{V}_t$ appearing in $C^\theta[V]$ are stable, and
2. the map $\mathbb{P}^1 \to C^\theta[V]$ given by $t \mapsto (\tilde{V}_t, \tilde{\omega}_t)$ is generically injective.

**Proof.** By Lemma 4.1, we may assume that $V$ is 2-stable.

The stability of $\tilde{V}_t$ is equivalent to the that of its dual: $\tilde{V}^\Lambda_t$. Let $E$ be an isotropic subbundle of $\tilde{V}^\Lambda_t$. Then by the construction, $V^*$ has a subsheaf $\tilde{E}$ defined by the kernel of the composition $(E \to \tilde{V}^\Lambda_t \to (V^\theta)^* \to \mathbb{C}^\oplus_2)$. Then in any case $\deg(\tilde{E}) \geq \deg(E) - 2$. By the stability condition of $V^*$, we have

$$\frac{\deg(E)}{\text{rk}(E)} \leq \frac{\deg(\tilde{E}) + 2}{\text{rk}(\tilde{E})} < \frac{\deg(V^*)}{\text{rk}(V^*)} = \frac{\deg(\tilde{V}^\Lambda_t)}{\text{rk}(\tilde{V}^\Lambda_t)}.$$  

To show (2), first we note that a general member $\tilde{V}_t$ is 2-stable, since the 2-stability is an open condition and the family contains a 2-stable bundle $V$. Suppose that for $s, t \in \mathbb{P}^1$, both $V_s$ and $V_t$ are 2-stable and $\tilde{V}_s \cong \tilde{V}_t$. Then there are two linearly independent generic isomorphisms between $V^\theta$ and $\tilde{V}_t$. By Lemma 4.2, this implies $s = t$. Therefore, a general member $\tilde{V}_t$ is not isomorphic to any other member, and we get the generic injectivity.

\[\text{We expect that another component (which does not contain } t_0 \text{) produces orthogonal bundles with 2nd Stiefel–Whitney class different from } w_2(V)\]
This rational curve $C^\Theta[V]$ is called the orthogonal Hecke curve on $\mathcal{M}$ associated to $\Theta \in IG(2, V)_x$. It will be shown in §5 that $C^\Theta[V]$ has degree $4n$.

**Remark 4.8.** In Lemma 4.5 and 4.7, we have seen that the symplectic/orthogonal Hecke curve is a generically injective image of $\mathbb{P}^1$. If we assume a higher genus bound, then we can guarantee the 2- or 4-stability of a general member of $\mathcal{M}$, and show the injectivity of the maps $\mathbb{P}^1 \to C^\Theta[V]$ or $\mathbb{P}^1 \to C^\Theta[V]$.

5. **MINIMAL RATIONAL CURVES**

Based on the results in §3 and §4, we can now compute the minimal degree of rational curves on the moduli spaces by adapting the computation in [17, §2] to our situation. In doing that, we need to additionally keep track of the form $\omega$.

5.1. **Degree formula of rational curves.** Let $\mathcal{M}$ be either $\mathcal{MS}_C(n, L)$ or an irreducible component of $\mathcal{MO}_C(n, L)$. Let $u : \mathbb{P}^1 \to \mathcal{M}$ be a generically injective morphism. Let $\mathcal{M}^{reg} \subset \mathcal{M}$ be the open sublocus consisting of those symplectic/orthogonal bundles whose underlying vector bundle is stable. Let $\pi_1 : C \times \mathbb{P}^1 \to C$ and $\pi_2 : C \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projections. Let $\mathcal{L} := \pi_1^*L$.

**Lemma 5.1.** If $u(\mathbb{P}^1) \subset \mathcal{M}^{reg}$, there is a vector bundle $\mathcal{V}$ over $C \times \mathbb{P}^1$ equipped with a form $\Omega : \mathcal{V} \otimes \mathcal{V} \to \mathcal{L} \otimes \pi_2^*O_{\mathbb{P}^1}(\tilde{d})$ for some $\tilde{d}$ such that for each $t \in \mathbb{P}^1$, the restriction of $(\mathcal{V}, \Omega)$ to $\pi_2^{-1}(t) = C \times \{t\}$ is isomorphic to the symplectic/orthogonal bundle represented by $u(t)$.

**Proof.** Let $V_t$ be the bundle represented by $u(t)$ for $t \in \mathbb{P}^1$. First we note that $\dim H^0(C, \mathcal{E}nd(V_t, V_t^* \otimes L)) = 1$ for each $t$ since $V_t \cong V_t^* \otimes L$ and $V_t$ is a stable vector bundle.

By composing with the generically injective morphism $f : \mathcal{M} \to SU_C(n, \frac{1}{2}n\ell)$, the curve $(f \circ u)(\mathbb{P}^1)$ lies in the moduli of stable vector bundles over $C$. By [17, Lemma 2.1], there is a vector bundle $\mathcal{V}$ over $C \times \mathbb{P}^1$ such that for each $t \in \mathbb{P}^1$, the restriction of $\mathcal{V}$ to $\pi_2^{-1}(t) = C \times \{t\}$ is isomorphic to the vector bundle $V_t$. From the above observation, we see that $(\pi_1)_*\mathcal{E}nd(\mathcal{V}, \mathcal{V}^* \otimes \mathcal{L})$ is a line bundle over $\mathbb{P}^1$, say $\mathcal{O}_{\mathbb{P}^1}(-\tilde{d})$ for some $\tilde{d}$. By adjunction formula we have

$$\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-\tilde{d}), (\pi_1)_*\mathcal{E}nd(\mathcal{V}, \mathcal{V}^* \otimes \mathcal{L})) \cong \text{Hom}((\pi_1)_*\mathcal{O}_{\mathbb{P}^1}(\tilde{d}), \mathcal{E}nd(\mathcal{V}, \mathcal{V}^* \otimes \mathcal{L})), $$

so there is a nonzero map

$$\mathcal{O}_{C \times \mathbb{P}^1} \to \mathcal{E}nd(\mathcal{V}, \mathcal{V}^* \otimes \mathcal{L}) \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(\tilde{d}).$$

This gives a map $\Omega : \mathcal{V} \to \mathcal{V}^* \otimes \mathcal{L} \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(\tilde{d})$ with the desired property. \qed

Note that the form $\Omega : \mathcal{V} \otimes \mathcal{V} \to \pi_1^*\mathcal{O}_{\mathbb{P}^1}(\tilde{d})$ can be normalized: When $\tilde{d}$ is even (resp. odd), we may assume that $\tilde{d} = 0$ (resp. $\tilde{d} = 1$) by replacing $\mathcal{V}$ by $\mathcal{V} \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(\frac{	ilde{d}}{2})$ (resp. $\mathcal{V} \otimes \pi_1^*\mathcal{O}_{\mathbb{P}^1}(\frac{\tilde{d} - 1}{2})$). The normalized family $\mathcal{V}$ can be viewed as
a family of $\mathcal{O}(d)$-valued symplectic/orthogonal bundles over $\mathbb{P}^1$ parametrized by $C$ for $d = 0, 1$. For $x \in C$, we write $\mathbb{P}^1_x := \pi^{-1}_1(x)$.

Under this normalization, the generic splitting type of $\mathcal{V}$ is given as in Corollary 3.3. Then there is a relative Harder–Narasimhan filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{V}$$

which restricts to a generic fiber $\mathbb{P}^1_x$ as $(\mathcal{E}_i/\mathcal{E}_{i-1})|_{\mathbb{P}^1_x} \cong \mathcal{O}_{\mathbb{P}^1}(a_i)^{\oplus r_i}$ for $1 \leq i \leq m$. If we let $\mathcal{F}_i = \mathcal{E}_i/\mathcal{E}_{i-1}$, the bundle $\mathcal{F}'_i := \mathcal{F}_i \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(-a_i)$ is torsion-free with generic splitting type $\mathcal{O}_{\mathbb{P}^1}^{r_i}$.

For a general $p \in \mathbb{P}^1$, the restriction of $\mathcal{V}$ to $\pi_2^{-1}(p) = C \times \{p\}$ is denoted simply by $\mathcal{V}$ and we write $\mathcal{E}_i := \mathcal{E}_i|_{C \times \{p\}}$, $\mathcal{F}_i := \mathcal{F}_i|_{C \times \{p\}}$. By the degree formula of [17, (2.2)], we have

$$\text{(5.1)} \quad \deg(u(\mathbb{P}^1)) = 2n \left( \sum_{i=1}^{m} c_2(\mathcal{F}'_i) + \sum_{i=1}^{m-1} (a_i - a_{i+1})(\mu(V) - \mu(E_i))r_i \right).$$

Furthermore, we observe:

- The bundles $\mathcal{E}_i$ are isotropic and $\mathcal{E}_i^\perp = E_{m-i}$ for $1 \leq i \leq [m/2]$.
- Hence the summation for $\lfloor \frac{m}{2} \rfloor \leq i \leq m$ can be computed from the summation for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$ by using the formula

$$\deg(E^\perp_i) - \deg(E_i) = (\mu(V) - \mu(E_i))r_i,$$

which comes from $\deg(E^\perp_i) = \deg(E_i) + (\frac{n}{2} - r_i)\ell$.
- In particular, (5.1) can be computed using the isotropic bundles only.

5.2. Minimality of symplectic and orthogonal Hecke curves. First consider the moduli space $\mathcal{M} = \mathcal{M}_{C}(n, L)$ of symplectic bundles.

**Theorem 5.2.** Assume $g \geq 3$ and $n \geq 4$. Suppose that $u(\mathbb{P}^1) \subset \mathcal{M}$ is a rational curve passing through a general point. Then $u(\mathbb{P}^1)$ has degree $\geq 2n$. Also if $u(\mathbb{P}^1)$ has degree $2n$, it is a symplectic Hecke curve.

**Proof.** We use the notations from § 5.1. By [17, Lemma 2.2], $c_2(\mathcal{F}'_i) \geq 0$ in (5.1) since the generic splitting type of $\mathcal{F}'_i$ is trivial. By Lemma 4.1, the curve $u : \mathbb{P}^1 \to \mathcal{M}$ passes through a point $[\mathcal{V}]$ corresponding to a 1-stable symplectic bundle. Hence we have $(\mu(V) - \mu(E_i))r_i > 1$. These together show that $\deg(u(\mathbb{P}^1)) \geq 2n$.

If the equality holds, then $m = 1$ and $c_2(\mathcal{V}) = 1$. In particular, $d$ must be even. Now we can apply the same argument as in [17]: The bundle $\mathcal{V}$ has exactly one jumping line

$$\mathcal{V}_{p_2} \cong \mathcal{O}(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}^{n-2}$$

([17, Lemma 2.5]) and $\mathcal{V}$ admits an elementary transformation

$$0 \to \pi_1^* W \to \mathcal{V} \to \mathcal{O}_{\mathbb{P}^1}(-1) \to 0$$
for some vector bundle $W$ on $C$ by [17, Lemma 2.2]. Therefore $V$ induces a symplectic Hecke curve as was defined in §4.2. (In §4.2, the dual family $V^*$ was constructed as a Hecke transformation of a fixed bundle and then its dual was taken at the final stage.)

Now consider the moduli space $MO_C(n, L)$ of orthogonal bundles. Let $\mathcal{M}$ be any irreducible component of $MO_C(n, L)$. We note that the degree bound is twice as large as that of symplectic case.

**Theorem 5.3.** Assume $g \geq 5$ for $n \geq 5$. Suppose that $u(\mathbb{P}^1) \subset \mathcal{M}$ is a rational curve passing through a general point. Then $u(\mathbb{P}^1)$ has degree $\geq 4n$. Also if it has degree $4n$, it is an orthogonal Hecke curve.

**Proof.** As before we have $c_2(F^*_i) \geq 0$. Since the curve $u: \mathbb{P}^1 \to \mathcal{M}$ passes through a point $[V]$ corresponding to a 2-stable orthogonal bundle by Lemma 4.1, we have $(\mu(V) - \mu(E_i))r_i > 2$. Therefore, $\deg(u(\mathbb{P}^1)) \geq 2n$. (Also we observe that if $\deg(u(\mathbb{P}^1)) \leq 4n$, then $m = 1$ and $c_2(V) = 1$. In particular, $\delta$ must be even.)

Now we exclude the possibility of degree $2n$. If $\deg(u(\mathbb{P}^1)) = 2n$, then again by applying [17, Lemma 2.2 and 2.5], we see that $V$ admits an elementary transformation of the type:

\begin{equation}
0 \to \pi_1^* W \to V \to O_{\mathbb{P}^1}(1) \to 0
\end{equation}

for some vector bundle $W$. If this defines a 1-parameter family of orthogonal bundles, then each $V|_{C \times \{p\}}$ corresponding to a 2-stable orthogonal bundle by Lemma 4.1, the form $\omega_V$ is independent of $p$. But when we take elementary transformations associated to a one-dimensional subspaces in $W|_x$, even if it is chosen as isotropic subspace, the induced symmetric forms of the resulting bundles $V|_{C \times \{p\}}$ do not vanish identically on the fiber at $x$ except two choices. (This is a consequence of the fact that the isotropic Grassmannian $IG(1, 2)$ consists of two points.) Hence the family $\{V|_{C \times \{p\}} : p \in \mathbb{P}^1\}$ in (5.2) do not lie inside $MO_C(n, L)$.

So far we have shown that $\deg(u(\mathbb{P}^1)) \geq 4n$. Now we characterize the curves of degree $4n$. If $\deg(u(\mathbb{P}^1)) = 4n$, there are three possible types of the associated elementary transformations: Namely,

\begin{equation}
0 \to \pi_1^* W \to V \to O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1) \to 0
\end{equation}

for $x_1 \neq x_2$ in $C$ or

\begin{equation}
0 \to \pi_1^* W \to V \to O_{\mathbb{P}^1}(-2) \to 0,
\end{equation}

or

\begin{equation}
0 \to \pi_1^* W \to V \to O_{\mathbb{P}^1}(-1)^\oplus 2 \to 0.
\end{equation}
We now rule out the first two cases. The case (5.3) is ruled out by the same reason as (5.2): since the elementary transformations are taken at two fibers separately, we get only $2 \cdot 2 = 4$ bundles in the family $\{V|_{C \times \{p\}} : p \in \mathbb{P}^1\}$ on which the induced $L(x_1 + x_2)$-valued symmetric form vanishes identically on the fibers at $x_1, x_2$.

The case (5.4) can be understood as a consecutive elementary transformation at the fiber $W|_x$ such that the first one is taken for a fixed choice of a point in $\mathbb{P}(W|_x)$ and the second one is taken for a line in $\mathbb{P}(W|_x)$. The same problem arise in this process as in (5.2).

In the remaining case (5.5), the sequence restricts to each $C \times \{p\}$ as an elementary transformation of the form

$$0 \rightarrow W \rightarrow V|_{C \times \{p\}} \rightarrow \mathbb{C}^{\oplus 2} \rightarrow 0.$$ 

Then we can see that the rational curve $\{V|_{C \times \{p\}} : p \in \mathbb{P}^1\}$ lies on a component $\mathcal{M}$ of $\mathcal{MO}_C(n, L)$ only if it is an orthogonal Hecke curve as was defined in §4.3. (In §4.3, we constructed the dual family $V^*$ first as a Hecke transformation of a fixed bundle and then took its dual at the final stage. This corresponds to the above construction in family.)

6. Applications

To discuss applications of symplectic and orthogonal Hecke curves, we need to check that the symplectic (resp. orthogonal) Hecke curves $C^\theta[V]$ (resp. $C^\Lambda[\Theta]$) are effectively parameterized by $\theta \in \mathbb{P}(V^*)$ (resp. by $\Theta \in IG(2, V)$).

**Lemma 6.1.**

1. For $n \geq 4$, assume $g > 4 + \frac{3}{n-1}$ and let $[V] \in \mathcal{MS}_C(n, L)$ be a general point. If $\theta_1 \neq \theta_2$ in $\mathbb{P}(V^*)$, then $C^{\theta_1}[V]$ and $C^{\theta_2}[V]$ are different symplectic Hecke curves.

2. For $n \geq 5$, assume $g > 10 + \frac{n}{n-1}$ and let $[V] \in \mathcal{MO}_C(n, L)$ be a general point. If $\Theta_1 \neq \Theta_2$ in $IG(2, V)$, then $C^{\Theta_1}[V]$ and $C^{\Theta_2}[V]$ are different orthogonal Hecke curves.

**Proof.** We exploit the notations from §4. Assume $\theta_1 \neq \theta_2$. We show that a general $[V_2] \in C^{\theta_2}[V]$ is not isomorphic to any $[V_1] \in C^{\theta_1}[V]$. Suppose there were an isomorphism $\psi: V_1 \cong V_2$. Let $V^{\theta_1, \theta_2}$ be the subsheaf of $V$ given by $V^{\theta_1, \theta_2} = V^{\theta_1} \cap V^{\theta_2}$, so that $\deg(V^{\theta_1, \theta_2}) = \deg(V) - 2$. Then there are two generic isomorphisms $V^{\theta_1, \theta_2} \rightarrow V_2$, one given by the composition

$$V^{\theta_1, \theta_2} \subset V^{\theta_1} \subset V_1 \overset{\psi}{\rightarrow} V_2$$

and another by the composition

$$V^{\theta_1, \theta_2} \subset V^{\theta_2} \subset V_2.$$
Since $[V_2]$ is general in $C^{θ_2}[V]$ and $[V] ∈ C^{θ_2}[V]$, the point $[V_2]$ is as much general as $[V]$, and so $V_2$ is 2-stable. Thus we may apply Lemma 4.2 to the map $V^{θ_1,θ_2} → V_2$ in place of $W → V$. By the assumption on $g$ and Lemma 4.2 for $δ = 2$, these two generic isomorphisms coincide.

Now consider the dual map $V^*_2 → (V^{θ_1,θ_2})^*$ of this generic isomorphism. From the above observations, the restriction of this dual map to the fiber at $x$ has the image inside the kernel of two maps for $i = 1, 2$:

$$(V^{θ_i})^*|_x → (V^{θ_1,θ_2})^*|_x.$$  

This shows $V_2 ∼ = V$, which is a contradiction.

The same argument shows (2). In this case, we apply Lemma 4.2 for $δ = 3, 4$ to show the coincidence of two generic isomorphisms. (In modifying the argument, the genus assumption should be adapted correspondingly).

**Remark 6.2.** To show that the spaces $P(V^*)$ and $IG(2, V)$ effectively parameterize the symplectic and orthogonal Hecke curves respectively, one may instead try to show the injectivity of the tangent maps

$$P(V^*) → PT|_V MS_C(n, L) \quad \text{and} \quad IG(2, V) → PT|_V MO_C(n, L)$$

sending a symplectic or orthogonal Hecke curve to its tangent at $[V]$. We believe this can be shown to be an embedding under certain assumptions on $g$ and $n$ as in [9, theorem 3.1], but we do not pursue it here.

By Theorems 5.2, 5.3, and Lemma 6.1, we get the following result.

**Corollary 6.3.** In the same notation as Lemma 6.1, the space $P(V^*)$ (resp. $IG(2, V)$) is the normalization of the Chow variety of rational curves passing through $[V]$ in $MS_C(n, L)$ (resp. an irreducible component of $MO_C(n, L)$).

Now we discuss the nonabelian Torelli theorem for the moduli of symplectic and orthogonal bundles. This was proven before in [5, Theorem 0.1] for principal bundles in general, and later reproved in [4, Theorem 4.3] for the symplectic case. The approach in [5] was to examine the strictly semistable locus and reduce to the classical Torelli theorem for the principally polarized Jacobians. On the other hand, both the proof of [4] and ours use the Hecke correspondence, focusing on the stable locus. But we get the wanted result directly from the minimal rational curves on $M$.

**Theorem 6.4.** Let $C_1$ and $C_2$ be smooth algebraic curves of genus $g$. Also let $L_1$ and $L_2$ be line bundles of the same degree $ℓ$ over $C_1$ and $C_2$, respectively.

1. Assume $g > 4 + \frac{3}{n}$ for $n ≥ 4$. If the moduli spaces $MS_{C_1}(n, L_1)$ and $MS_{C_2}(n, L_2)$ are isomorphic, then $C_1$ and $C_2$ are isomorphic.
(2) Assume \( g > 10 + \frac{9}{n-1} \) for \( n \geq 5 \). Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be any irreducible components of \( \mathcal{MO}_{C_1}(n,L_1) \) and \( \mathcal{MO}_{C_2}(n,L_2) \), respectively. It \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are isomorphic, then \( C_1 \) and \( C_2 \) are isomorphic.

Proof. (1) Given an isomorphism \( \Phi: \mathcal{MS}_{C_1}(n,L_1) \cong \mathcal{MS}_{C_2}(n,L_2) \), let \( V_1 \) be a symplectic bundle over \( C \) corresponding to a general point \([V_1]\) of \( \mathcal{MS}_{C_1}(n,L_1) \), and \( V_2 \) be the symplectic bundle over \( C_2 \) corresponding to the general point \( \Phi([V_1]) \) of \( \mathcal{MS}_{C_2}(n,L_2) \). Then \( \Phi \) sends the rational curves of degree 2 passing through \([V_1]\) in \( \mathcal{MS}_{C_1}(n,L_1) \) to those rational curves through \([V_2]\) in \( \mathcal{MS}_{C_2}(n,L_2) \). By Corollary 6.3, we get the induced isomorphism \( \mathbb{P}(V_1^*) \cong \mathbb{P}(V_2^*) \). From this isomorphism of rational fibrations, we get the induced isomorphism \( C_1 \cong C_2 \) on the base curves.

(2) In the same way, the isomorphism \( \Phi: \mathcal{M}_1 \cong \mathcal{M}_2 \) induces the isomorphism of isotropic Grassmannians \( IG(2,V_1) \) and \( IG(2,V_2) \), where \( \Phi([V_1]) = [V_2] \). From this, we get an isomorphism \( C_1 \cong C_2 \).

We can also classify the automorphisms on the moduli spaces of symplectic or orthogonal bundles. The symplectic case has been shown in [4, Theorem 6.1] by another method. Below we discuss the orthogonal case only.

**Theorem 6.5.** Let \( \mathcal{M}_0 \) be the component of \( \mathcal{MO}_C(n,O_C) \) containing the trivial orthogonal bundle \( O_C^{\oplus n} \). If \( n \geq 5 \) and \( g > 10 + \frac{9}{n-1} \), every automorphism of \( \mathcal{M}_0 \) is induced by an automorphism of \( C \) and a line bundle of order two.

Proof. Let \( \sigma \) be an automorphism of \( \mathcal{M}_0 \). Since the degree of rational curves is fixed under \( \sigma \), we get an induced isomorphism \( \tilde{\sigma} \) between isotropic Grassmannian bundles \( \pi: IG(2,V) \to C \) and \( \pi': IG(2,V') \to C \), where \( V' \) is the bundle represented by \( \sigma([V]) \). Note that \( \tilde{\sigma} \) sends a fiber to a fiber, since \( g \geq 2 \). Thus by composing with an automorphism \( \tau \) of \( C \), we may assume that \( \pi = \pi' \circ \tilde{\sigma} \).

Let \( \mathcal{U}^* \to IG(2,V) \) and \( (\mathcal{U}')^* \to IG(2,V') \) be the dual of the relative universal bundles. Then \( \tilde{\sigma}^*(\mathcal{U})^* \cong \mathcal{U}^* \otimes \pi^* N \) for some line bundle \( N \) over \( C \). By taking push-forward, we have

\[
\pi_* \tilde{\sigma}^*(\mathcal{U}')^* = \pi_* \tilde{\sigma}^*(\mathcal{U})^* \cong V'
\]

and also

\[
\pi_* \tilde{\sigma}^*(\mathcal{U}')^* \cong \pi_* (\mathcal{U}^* \otimes \pi^* N) \cong V \ot N.
\]

Hence \( V' \cong V \ot N \) for a line bundle \( N \), up to \( \tau \in \text{Aut}(C) \).

Now it remains to show that \( N \) is a line bundle of order two. Since both \( V \) and \( V' \) are \( O_C \)-valued symplectic or orthogonal bundles, we get

\[
V' \cong V \ot N \cong V^* \ot N \cong (V'^*)^* \ot N \cong V' \ot N^2.
\]
This shows that the morphism $[V] \mapsto [V \otimes N^2]$ is the identity map on $M_0$. Hence we can see that $N^2 \cong O_C$ by specializing $V$ to the trivial orthogonal bundle $V \cong O_C^{\oplus n}$.

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