Abstract

We extend the notion of “quantum blob” studied in previous work to excited states of the generalized harmonic oscillator in \( n \) dimensions. This extension is made possible by Fermi’s observation in 1930 that the state of a quantum system may be defined in two different (but equivalent) ways, namely by its wavefunction \( \Psi \) or by a certain function \( g_F \) on phase space canonically associated with \( \Psi \). We study Fermi’s function when \( \Psi \) is a Gaussian (generalized coherent state). A striking result is that we can use the Ekeland–Hofer symplectic capacities to characterize the Fermi functions of the excited states of the generalized harmonic oscillator, leading to new insight on the relationship between symplectic topology and quantum mechanics.

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1 Introduction

1.1 What We Want to Do

We address the question whether it is possible to represent geometrically a function $\psi$ of the variables $x = (x_1, x_2, ..., x_n)$. The problem is in fact easy to answer if $\psi$ is a Gaussian function because then its Wigner transform is proportional to a Gaussian $e^{-\frac{1}{2}S^T S z \cdot x}$ where $S$ is a symplectic matrix uniquely determined by $\psi$. It follows that there is a one-to-one correspondence between Gaussians and the sets $S^T S z \cdot x \leq \hbar$. We have called these sets ”quantum blobs” in [12, 13, 14, 15, 16, 17, 18]; the interest of these quantum blobs comes from the fact that they represent minimum uncertainty sets in phase space.

The Gaussian function

$$\Psi_0(x) = e^{-x^2/2\hbar},$$

is the (unnormalized) ground state of the one-dimensional harmonic oscillator with mass and frequency equal to one: $\hat{H}\Psi_0 = E_0\Psi_0$ where $E_0 = \frac{1}{2}\hbar$ and

$$\hat{H} = \frac{1}{2} \left( -\hbar^2 \frac{d^2}{dx^2} + x^2 \right)$$

This operator is the quantization of the classical oscillator Hamiltonian

$$H(x, p) = \frac{1}{2}(p^2 + x^2)$$

The set $\Omega_0$ defined by the inequality $H \leq E_0$ is the interior of the energy hypersurface $H \leq E_0$; it is the disk $p^2 + x^2 \leq \hbar$ with radius $R_0 = \sqrt{\hbar}$. Let us now consider the $N$-th excited state of the operator $\hat{H}$; it is the (unnormalized) Hermite function

$$\Psi_N(x) = e^{-x^2/2\hbar} H_N(x/\sqrt{\hbar})$$

where

$$H_N(x) = (-1)^n x^2 \frac{d^n}{dx^n} e^{-x^2}$$

is the $N$-th Hermite polynomial. It is a solution of $\hat{H}\Psi_N = (N + \frac{1}{2})\hbar \Psi_N$ and the set $\Omega_N$ defined by the inequality $H \leq E_N = (2N + 1)\hbar$ is again a disk, but this time with radius $R_N = \sqrt{(N + \frac{1}{2})\hbar}$.

In this paper we introduce a non-trivial extension of the notion of “quantum blob” we defined and studied in previous work. Quantum blobs are
deformations of the phase space ball $|x|^2 + |p|^2 \leq \hbar$ by translations and linear canonical transformations. Their interest come from the fact that they provide us with a coarse-graining of phase space different from the usual coarse graining by cubes with volume $\sim h^n$ commonly used in statistical mechanics. They appear as space units of minimum uncertainty in one-to-one correspondence with the generalized coherent states familiar from quantum optics, and have allowed us to recover the exact ground states of generalized harmonic oscillators, as well as the semiclassical energy levels of quantum systems with completely integrable Hamiltonian function, and to explain them in terms of the topological notion of symplectic capacity [24, 30] originating in Gromov's [23] non-squeezing theorem (alias “the principle of the symplectic camel”). Quantum blobs, do not, however, allow a characterization of excited states; for instance there is no obvious relation between them and the Hermite functions. Why this does not work is easy to understand: quantum blobs correspond to the states saturating the Schrödinger–Robertson inequalities

$$\bigl(\Delta X_j\bigr)^2 (\Delta P_j)^2 \geq \Delta(X_j, P_j)^2 + \frac{1}{4} \hbar^2, \quad 1 \leq j \leq n; \quad (6)$$

as is well-known [21] the quantum states for which all these inequalities become equalities are Gaussians, in this case precisely those who are themselves the ground states of generalized harmonic oscillators. As soon as one consider the excited states the corresponding eigenfunctions are Hermite functions and for these the inequalities (6) are strict. The way out of this difficulty is to define new phase space objects, the “Fermi blobs” of the title of this paper. Such an approach should certainly be welcome in times where phase space is beginning to be taken seriously (see the recent review paper [7]).

1.2 How We Will Do It

We will show that a complete geometric picture of excited states can be given using an idea of the physicist Enrico Fermi in a largely forgotten paper [8] from 1930. Fermi associates to every quantum state $\Psi$ a certain hypersurface $g_F(x, p) = 0$ in phase space. The underlying idea is actually surprisingly simple. It consists in observing that any complex twice continuously differentiable function $\Psi(x) = R(x)e^{i\Phi(x)/\hbar}$ ($R(x) \geq 0$ and $\Phi(x)$ real) defined on $\mathbb{R}^n$ satisfies the partial differential equation

$$\left[ (-i\hbar \nabla_x - \nabla_x \Phi)^2 + h^2 \frac{\nabla^2 R}{R} \right] \Psi = 0. \quad (7)$$
where $\nabla_x^2$ is the Laplace operator in the variables $x_1, \ldots, x_n$ (it is assumed that $R(x) \neq 0$ for $x$ in some subset of $\mathbb{R}^n$). Performing the gauge transformation $-i\hbar \nabla_x \rightarrow -i\hbar \nabla_x - \nabla_x \Phi$, this equation is in fact equivalent to the trivial equation

$$\left( -\hbar^2 \nabla_x^2 + \frac{\hbar^2 R^2}{R} \right) R = 0.$$  \hfill (8)

The operator

$$\hat{g}_F = (-i\hbar \nabla_x - \nabla_x \Phi)^2 + \hbar^2 \nabla_x^2 \frac{R^2}{R}$$  \hfill (9)

appearing in the left-hand side of Eqn. (7) is the quantisation (in every reasonable physical quantisation scheme) of the real observable

$$g_F(x,p) = (p - \nabla_x \Phi)^2 + \hbar^2 \nabla_x^2 \frac{R^2}{R}$$  \hfill (10)

and the equation $g_F(x,p) = 0$ in general determines a hypersurface $\mathcal{H}_F$ in phase space $\mathbb{R}^{2n}_{x,p}$ which Fermi ultimately identifies with the state $\Psi$ itself. The remarkable thing with this construction is that it shows that to an arbitrary function $\Psi$ it associates a Hamiltonian function of the classical type

$$H = (p - \nabla_x \Phi)^2 + V$$  \hfill (11)

even if $\Psi$ is the solution of another partial (or pseudo-differential) equation. We notice that when $\Psi$ is an eigenstate of the operator $\hat{H} \Psi = E \Psi$ then $g_F = H - E$ and $\mathcal{H}_F$ is just the energy hypersurface $H(x,p) = E$.

Of course, Fermi’s analysis was very heuristic and its mathematical rigour borders the unacceptable (at least by modern standards). Fermi’s paper has recently been rediscovered by Benenti [2] and Benenti and Strini [3], who study its relationship with the level sets of the Wigner transform of $\Psi$.

**Notation 1** The points in configuration and momentum space are written $x = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$ respectively; in formulas $x$ an $p$ are viewed as column vectors. We will also use the collective notation $z = (x, p)$ for the phase space variable. The matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (0 and I the $n \times n$ zero and identity matrices) defines the standard symplectic form on the phase space $\mathbb{R}^{2n}_x$ via the formula $\sigma(z, z') = Jz \cdot z' = p \cdot x' - p' \cdot x$. We write $\hbar = h/2\pi$, $h$ being Planck’s constant. The symplectic group is denoted by $\text{Sp}(2n, \mathbb{R})$; it is the multiplicative group of all real $2n \times 2n$ matrices $S$ such that $\sigma(Sz, Sz') = \sigma(z, z')$ for all $z, z'$. 

4
2 Symplectic Capacities and Quantum Blobs

To generalize the discussion above to the multi-dimensional case we have to introduce some concepts from symplectic topology. For a review of these notions see de Gosson and Luef [22].

2.1 Symplectic Capacities

Intrinsic symplectic capacities

An *intrinsic* symplectic capacity assigns a non-negative number (or $+\infty$) $c(\Omega)$ to every subset $\Omega$ of phase space $\mathbb{R}^{2n}$; this assignment is subjected to the following properties:

- **Monotonicity:** If $\Omega \subset \Omega'$ then $c(\Omega) \leq c(\Omega')$;
- **Symplectic invariance:** If $f$ is a canonical transformation (linear, or not) then $c(f(\Omega)) = c(\Omega)$;
- **Conformality:** If $\lambda$ is a real number then $c(\lambda \Omega) = \lambda^2 c(\Omega)$; here $\lambda \Omega$ is the set of all points $\lambda z$ when $z \in \Omega$;
- **Normalization:** We have

$$c(B^{2n}(R)) = \pi R^2 = c(Z^{2n}_j(R));$$

(12)

here $B^{2n}(R)$ is the phase-space ball $|x|^2 + |p|^2 \leq R^2$ and $Z^{2n}_j(R)$ the phase-space cylinder $x_j^2 + p_j^2 \leq R^2$.

Let $c$ be a symplectic capacity on the phase plane $\mathbb{R}^2$. We have $c(\Omega) = \text{Area}(\Omega)$ when $\Omega$ is a connected and simply connected surface. In the general case there exist infinitely many intrinsic symplectic capacities, but they all agree on phase space ellipsoids as we will see below. The smallest symplectic capacity is denoted by $c_{\text{min}}$ ("Gromov width"): by definition $c_{\text{min}}(\Omega)$ is the supremum of all numbers $\pi R^2$ such that there exists a canonical transformation such that $f(B^{2n}(R)) \subset \Omega$. The fact that $c_{\text{min}}$ really is a symplectic capacity follows from a deep and difficult topological result, Gromov’s symplectic non-squeezing theorem, alias the principle of the symplectic camel. (For a discussion of Gromov’s theorem from the point of view of Physics see de Gosson [17], de Gosson and Luef [22].) Another useful example is provided by the Hofer–Zehnder [24] capacity $c^{HZ}$. It has the property
that it is given by the integral of the action form \( p dx = p_1 dx_1 + \cdots + p_n dx_n \) along a certain curve:

\[
c^{HZ}(\Omega) = \oint_{\gamma_{\text{min}}} p dx
\]  

(13)

when \( \Omega \) is a compact convex set in phase space; here \( \gamma_{\text{min}} \) is the shortest (positively oriented) Hamiltonian periodic orbit carried by the boundary \( \partial \Omega \) of \( \Omega \). This formula agrees with the usual notion of area in the case \( n = 1 \).

It turns out that all intrinsic symplectic capacities agree on phase space ellipsoids, and are calculated as follows (see e.g. [16, 22, 24]). Let \( M \) be a \( 2n \times 2n \) positive-definite matrix and consider the ellipsoid:

\[
\Omega_{M,z_0} : M(z - z_0)^2 \leq 1.
\]  

(14)

Then, for every intrinsic symplectic capacity \( c \) we have

\[
c(\Omega_{M,z_0}) = \pi / \lambda_{\text{max}}^\sigma
\]  

(15)

where \( \lambda_{\text{max}}^\sigma = \) is the largest symplectic eigenvalue of \( M \). The symplectic eigenvalues of a positive definite matrix are defined as follows: the matrix \( J M \) (\( J \) the standard symplectic matrix) is equivalent to the antisymmetric matrix \( M^{1/2} J M^{1/2} \) hence its \( 2n \) eigenvalues are of the type \( \pm i \lambda_1^\sigma, \ldots, \pm i \lambda_n^\sigma \) where \( \lambda_j^\sigma > 0 \). The positive numbers \( \lambda_1^\sigma, \ldots, \lambda_n^\sigma \) are called the *symplectic eigenvalues* of the matrix \( M \).

In particular, if \( X \) and \( Y \) are real symmetric \( n \times n \) matrices, then the symplectic capacity of the ellipsoid

\[
\Omega_{(A,B)} : X x^2 + Y p^2 \leq 1
\]  

(16)

is given by

\[
c(\Omega_{(A,B)}) = \pi / \sqrt{\lambda_{\text{max}}}
\]  

(17)

where \( \lambda_{\text{max}} \) is the largest eigenvalue of \( AB \).

**Extrinsic symplectic capacities**

The definition of an extrinsic symplectic capacity is similar to that of an intrinsic capacity, but one weakens the normalization condition [12] by only requiring that:

- **Nontriviality:** \( c(B^{2n}(R)) < +\infty \) and \( c(Z_j^{2n}(R)) < +\infty \).
In [6] Ekeland and Hofer defined a sequence $c^\text{EH}_1, c^\text{EH}_2, \ldots, c^\text{EH}_k, \ldots$ of extrinsic symplectic capacities satisfying the nontriviality properties

$$c^\text{EH}_k(B^{2n}(R)) = \left[\frac{k + n - 1}{n}\right] \pi R^2, \quad c^\text{EH}_k(Z^2_{2n}(R)) = k\pi R^2. \quad (18)$$

Of course $c^\text{EH}_1$ is an intrinsic capacity; in fact it coincides with the Hofer–Zehnder capacity on bounded convex sets $\Omega$. We have

$$c^\text{EH}_1(\Omega) \leq c^\text{EH}_2(\Omega) \leq \cdots \leq c^\text{EH}_k(\Omega) \leq \cdots \quad (19)$$

The Ekeland–Hofer capacities have the property that for each $k$ there exists an integer $N \geq 0$ and a closed characteristic $\gamma$ of $\partial\Omega$ such that

$$c^\text{EH}_k(\Omega) = N \left| \oint_\gamma pdx \right| \quad (20)$$

(in other words, $c^\text{EH}_k(\Omega)$ is a value of the action spectrum [5] of the boundary $\partial\Omega$ of $\Omega$; this formula shows that $c^\text{EH}_k(\Omega)$ is solely determined by $\partial\Omega$; therefore the notation $c^\text{EH}_k(\partial\Omega)$ is often used in the literature. The Ekeland–Hofer capacities $c^\text{EH}_k$ allow us to classify phase-space ellipsoids. In fact, the non-decreasing sequence of numbers $c^\text{EH}_k(\Omega_M)$ is determined as follows for an ellipsoid $\Omega : Mz \cdot z \leq 1$ ($M$ symmetric and positive-definite): let $(\lambda_1^\sigma, \ldots, \lambda_n^\sigma)$ be the symplectic eigenvalues of $M$; then

$$\{c^\text{EH}_k(\Omega) : k = 1, 2, \ldots\} = \{N\pi\lambda_j^\sigma : j = 1, \ldots, n; N = 0, 1, 2, \ldots\}. \quad (21)$$

Equivalently, the increasing sequence $c^\text{EH}_1(\Omega) \leq c^\text{EH}_2(\Omega) \leq \cdots$ is obtained by writing the numbers $N\pi\lambda_j^\sigma$ in increasing order with repetitions if a number occurs more than once.

### 2.2 Quantum Blobs

By definition a quantum blob $QB^{2n}(z_0, S)$ is the image of the phase space ball $B^{2n}(S^{-1}z_0, \sqrt{\hbar}) : |z - S^{-1}z_0| \leq \sqrt{\hbar}$ by a linear canonical transformation (identified with a symplectic matrix $S$). A quantum blob is thus a phase space ellipsoid with symplectic capacity $\pi\hbar = \frac{1}{2}\hbar$, but it is not true that, conversely, an arbitrary phase space ellipsoid with symplectic capacity $\frac{1}{2}\hbar$ is a quantum blob. One can however show (de Gosson [14, 15, 16], de Gosson and Luef [22]) that such an ellipsoid contains a unique quantum blob. One proves (ibid.) that a quantum blob $QB^{2n}(z_0, S)$ is characterized by the two following equivalent properties:
• The intersection of the ellipsoid $\mathcal{Q}B^{2n}(z_0, S)$ with a plane passing through $z_0$ and parallel to any of the plane of canonically conjugate coordinates $x_j, p_j$ in $\mathbb{R}^{2n}$ is an ellipse with area $\frac{1}{2}\hbar$

• The supremum of the set of all numbers $\pi R^2$ such that the ball $B^{2n}(\sqrt{R}) : |z| \leq R$ can be embedded into $\mathcal{Q}B^{2n}(z_0, S)$ using canonical transformations (linear, or not) is $\frac{1}{2}\hbar$. Hence no phase space ball with radius $R > \sqrt{\hbar}$ can be “squeezed” inside $\mathcal{Q}B^{2n}(z_0, S)$ using only canonical transformations.

It turns out (de Gosson [16]) that in the first of these conditions one can replace the plane of conjugate coordinates with any symplectic plane (a symplectic plane is a two-dimensional subspace of $\mathbb{R}^{2n}$ on which the restriction of the symplectic form $\sigma$ is again a symplectic form). There is a natural action

$$\text{Sp}(2n, \mathbb{R}) \times \mathcal{Q}B(2n, \mathbb{R}) \rightarrow \mathcal{Q}B(2n, \mathbb{R})$$

of the symplectic group on quantum blobs.

3 Generalized Coherent States

3.1 The Fermi Function of a Gaussian

We next consider arbitrary (normalized) generalized coherent states

$$\Psi_{X,Y}(x) = \left( \frac{1}{\pi\hbar} \right)^{n/4} (\det X)^{1/4} \exp \left[ -\frac{1}{2\hbar} (X + iY)x \cdot x \right]$$  (22)

where $X$ and $Y$ are real symmetric $n \times n$ matrices, and $X$ is positive definite. Setting $\Phi(x) = -\frac{1}{2}Yx \cdot x$ and $R(x) = \exp \left( -\frac{1}{2\hbar}Xx \cdot x \right)$ we have

$$\nabla_x \Phi(x) = -Yx , \quad \frac{\nabla^2_x R(x)}{R(x)} = -\frac{1}{\hbar} \text{Tr} X + \frac{1}{\hbar^2}X^2x \cdot x$$  (23)

hence the Fermi function of $\Psi_{X,Y}$ is the quadratic form

$$g_F(x, p) = (p + Yx)^2 + X^2x \cdot x - \hbar \text{Tr} X.$$  (24)

We can rewrite this formula as

$$g_F(x, p) = M_{\mathcal{F}z} \cdot z - \hbar \text{Tr} X$$  (25)

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\((z = (x, p))\) where \(M_F\) is the symmetric matrix

\[
M_F = \begin{pmatrix} X^2 + Y^2 & Y \\ Y & I \end{pmatrix}
\]  

(26)

A straightforward calculation shows that we have the factorization

\[
M_F = S^T \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} S
\]

(27)

where \(S\) is the symplectic matrix

\[
S = \begin{pmatrix} X^{1/2} & 0 \\ X^{-1/2}Y & X^{-1/2} \end{pmatrix}
\]

(28)

It turns out –and this is really a striking fact!– that \(M_F\) is closely related to the Wigner transform

\[
W \Psi_{X,Y}(z) = \left( \frac{1}{2\pi \hbar} \right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p \cdot y} \Psi_{X,Y}(x + \frac{1}{2} y)\Psi^*_X(x - \frac{1}{2} y) dy
\]

(29)

of the state \(\Psi_{X,Y}\) because we have

\[
W \Psi_{X,Y}(z) = \left( \frac{1}{\pi \hbar} \right)^n \exp \left( -\frac{1}{\hbar} Gz \cdot z \right)
\]

(30)

where \(G\) is the symplectic matrix

\[
G = S^T S = \begin{pmatrix} X + YX^{-1}Y & YX^{-1} \\ X^{-1}Y & X^{-1} \end{pmatrix}
\]

(31)

(see e.g. [16, 27]). When \(n = 1\) and \(\Psi_{X,Y}(x) = \Psi_0(x)\) the fiducial coherent state \((1)\) we have \(S^{-1} D^{-1/2} S = I\) and \(\text{Tr} X = 1\) hence the formula

\[
W \Psi_0(z) = \left( \frac{1}{\pi \hbar} \right)^{1/4} \frac{1}{e} \exp \left[ -\frac{1}{\hbar} M_F z \cdot z \right]
\]

already observed by Benenti and Strini in [3].

### 3.2 Geometric Interpretation

Recall (formula (15)) that the symplectic capacity \(c(\Omega)\) of an ellipsoid \(Mz \cdot z \leq 1\) \((M\) a symmetric positive definite \(2n \times 2n\) matrix) is given by

\[
c(\Omega) = \pi / \lambda_{\text{max}}^2
\]

(32)
where \( \lambda_{\text{max}} = \max \{\lambda_1^\sigma, \ldots, \lambda_n^\sigma\} \), the \( \lambda_j^\sigma \) being the symplectic eigenvalues of \( M \). We denote by \( \Omega_F \) the phase space ellipsoid defined by \( g_F(x, p) \leq 0 \), that is:

\[
\Omega_F : M_F z \cdot z \leq \hbar \text{Tr} X;
\]

it is the ellipsoid bounded by the Fermi hypersurface \( \mathcal{H}_F \) corresponding to the generalized coherent state \( \Psi_{X,Y} \). Let us perform the symplectic change of variables \( z' = S z \); in the new coordinates the ellipsoid \( \Omega_F \) is represented by the inequality

\[
X x' \cdot x' + X p' \cdot p' \leq \hbar \text{Tr} X
\]

hence \( c(\Omega_F) \) equals the symplectic capacity of the ellipsoid (33). Applying the rule above we thus have to find the symplectic eigenvalues of the block-diagonal matrix \( \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \); a straightforward calculation shows that these are just the eigenvalues \( \omega_1, \ldots, \omega_n \) of \( X \) and hence

\[
c(\Omega_F) = \pi \hbar \text{Tr} X / \omega_{\text{max}}
\]

where \( \omega_{\text{max}} = \max \{\omega_1, \ldots, \omega_n\} \). In view of the trivial inequality

\[
\omega_{\text{max}} \leq \text{Tr} X = \sum_{j=1}^{n} \omega_j \leq n \lambda \omega_{\text{max}}
\]

we have

\[
\frac{1}{2} \hbar \leq c(\Omega_F) \leq \frac{n \hbar}{2}.
\]

An immediate consequence of the inequality \( \frac{1}{2} \hbar \leq c(\Omega_F) \) is that the Fermi ellipsoid \( \Omega_F \) of a generalized coherent state always contains a quantum blob; this is of course consistent with the uncertainty principle.

Notice that when all the eigenvalues \( \omega_j \) are equal to a number \( \omega \) then \( c(\Omega_F) = n \hbar/2 \); in particular when \( n = 1 \) we have \( c(\Omega_F) = \hbar/2 \) which is exactly the action calculated along the trajectory corresponding to the ground state. This observation leads us to the following question: what is the precise geometric meaning of formula (33)? Let us come back to the interpretation of the ellipsoid defined by the inequality (33). We have seen that the symplectic eigenvalues of the matrix \( \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \) are precisely the eigenvalues \( \omega_j, 1 \leq j \leq n \), of the positive-definite matrix \( X \). It follows that there exist linear symplectic coordinates \( (x''^j, p''^j) \) in which the equation of the ellipsoid \( \Omega_F \) takes the normal form

\[
\sum_{j=1}^{n} \omega_j (x''^j)^2 + (p''^j)^2 \leq \sum_{j=1}^{n} \hbar \omega_j
\]

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whose quantum-mechanical interpretation is clear: dividing both sides by two we get the energy shell of the anisotropic harmonic oscillator in its ground state. Consider now the planes $P_1, P_2, \ldots, P_n$ of conjugate coordinates $(x_1, p_1), (x_2, p_2), \ldots, (x_n, p_n)$. The intersection of the ellipsoid $\Omega_F$ with these planes are the circles

$$C_1 : \omega_1(x_1''^2 + p_1''^2) \leq \sum_{j=1}^{n} \hbar \omega_j$$

$$C_2 : \omega_2(x_2''^2 + p_2''^2) \leq \sum_{j=1}^{n} \hbar \omega_j$$

$$\ldots \ldots \ldots \ldots$$

$$C_n : \omega_n(x_n''^2 + p_n''^2) \leq \sum_{j=1}^{n} \hbar \omega_j.$$

Formula (34) says that $c(\Omega_F)$ is the area of the circle $C_j$ with smallest radius, and this corresponds to the index $j$ such that $\omega_j = \omega_{\text{max}}$. This is of course perfectly in accordance with the definition of the Hofer–Zehnder capacity $c^{HZ}(\Omega_F)$ since all symplectic capacities agree on ellipsoids. We are now led to another question: is there any way to describe topologically Fermi’s ellipsoid in such a way that the areas of every circle $C_j$ becomes apparent? The problem with the standard capacity of an ellipsoid is that it only “sees” the smallest cut of that ellipsoid by a plane of conjugate coordinate. The way out of this difficult lies in the use of the Ekeland–Hofer capacities $c^{EH}_j$ discussed above. To illustrate the idea, let us first consider the case $n = 2$; it is no restriction to assume $\omega_1 \leq \omega_2$. If $\omega_1 = \omega_2$ then the ellipsoid

$$\omega_1(x_1''^2 + p_1''^2) + \omega_2(x_2''^2 + p_2''^2) \leq \hbar \omega_1 + \hbar \omega_2$$

is just the ball $B^2(\sqrt{2\hbar})$ whose symplectic capacity is $2\pi \hbar = \hbar$. Suppose now that $\omega_1 < \omega_2$. Then the Ekeland–Hofer capacities are the numbers

$$\frac{\pi \hbar}{\omega_2}(\omega_1 + \omega_2), \frac{\pi \hbar}{\omega_1}(\omega_1 + \omega_2), \frac{2\pi \hbar}{\omega_2}(\omega_1 + \omega_2), \frac{2\pi \hbar}{\omega_1}(\omega_1 + \omega_2), \ldots.$$  \hspace{1cm} (39)

and hence

$$c^{EH}_1(\Omega_F) = c(\Omega_F) = \frac{\pi \hbar}{\omega_2}(\omega_1 + \omega_2).$$

What about $c^{EH}_2(\Omega_F)$? A first glance at the sequence (39) suggests that we have

$$c^{EH}_2(\Omega_F) = \frac{\pi \hbar}{\omega_1}(\omega_1 + \omega_2).$$
but this is only true if $\omega_1 < \omega_2 \leq 2 \omega_1$ because if $2 \omega_1 < \omega_2$ then $(\omega_1 + \omega_2)/\omega_2 < (\omega_1 + \omega_2)/\omega_1$ so that in this case

$$c_{EH}^2(\Omega_F) = \frac{\pi \hbar}{\omega_2}(\omega_1 + \omega_2) = c_{EH}^1(\Omega_F).$$

The Ekeland–Hofer capacities thus allow a geometrical classification of the eigenstates.

4 Fermi Function and Excited States

The generalized coherent states can be viewed as the ground states of a generalized harmonic oscillator, with Hamiltonian function a homogeneous quadratic polynomial in the position and momentum coordinates:

$$H(x, p) = \sum_{i,j} a_{ij} p_i p_j + b_{ij} p_i x_j + c_{ij} x_i x_j.$$ 

Such a function can always be put in the form

$$H(z) = \frac{1}{2} M z \cdot z$$

where $M$ is a symmetric matrix (the Hessian matrix, i.e. the matrix of second derivatives, of $H$). We will assume for simplicity that $M$ is positive-definite; we can then always bring it into the normal form

$$K(z) = \sum_{j=1}^n \frac{\omega_j}{2} (x_j^2 + p_j^2)$$

using a linear symplectic transformation of the coordinates (symplectic diagonalization): there exists a symplectic matrix $S$ (depending on $M$) such that

$$S^T MS = D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

where $\Omega$ is a diagonal matrix whose diagonal entries consist of the symplectic spectrum $\omega_1, ..., \omega_n$ of $M$. Thus, we have $K(z) = H(Sz)$, or, equivalently,

$$H(z) = K(S^{-1}z)$$

The ground state of each one-dimensional quantum oscillator

$$\hat{K}_j = \frac{\omega_j}{2} \left( x_j^2 - \hbar^2 \frac{\partial^2}{\partial x_j} \right)$$

$$\hat{K}_j = \frac{\omega_j}{2} \left( x_j^2 - \hbar^2 \frac{\partial^2}{\partial x_j} \right)$$

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is the solution of $\hat{K}_j \Psi = \frac{1}{2} \hbar \omega_j \Psi$, it is thus the one-dimensional fiducial coherent state $(\pi \hbar)^{-1/4} e^{-x^2/2\hbar}$. It follows that the ground $\Psi_0$ state of $\hat{K} = \sum_j \hat{K}_j$ is the tensor product of $n$ such states, that is $\Psi_0(x) = (\pi \hbar)^{-n/4} e^{-|x|^2/2\hbar}$, the fiducial coherent state. Returning to the initial Hamiltonian $H$ we note that the corresponding Weyl quantisation $\hat{H}$ satisfies, in view of Eqn. (42) the symplectic covariance formula $\hat{H} = \hat{S} \hat{K} \hat{S}^{-1}$ where $\hat{S}$ is any of the two metaplectic operators corresponding to the symplectic matrix $S$ (see the Appendix). It follows that the ground state of $\hat{H}$ is given by the formula $\Psi = \hat{S} \Psi_0$.

The case of the excited states is treated similarly. The solutions of the one-dimensional eigenfunction problem $\hat{K}_j \Psi = E \Psi$ are given by the Hermite functions

$$\Psi_N(x) = e^{-x^2/2\hbar} H_N(x/\sqrt{\hbar})$$

with corresponding eigenvalues $E_N = (N + \frac{1}{2}) \hbar \omega_j$. It follows that the solutions of the $n$-dimensional problem $\hat{K} \Psi = E \Psi$ are the tensor products

$$\Psi_{(N)} = \Psi_{N_1} \otimes \Psi_{N_2} \otimes \cdots \otimes \Psi_{N_n}$$

where $(N) = (N_1, N_2, ..., N_n)$ is a sequence of non-negative integers, and the corresponding energy level is

$$E_{(N)} = \sum_{j=1}^n (N_j + \frac{1}{2}) \hbar \omega_j.$$ 

This allows us to give a geometric description of all eigenfunctions of the generalized harmonic oscillator, corresponding to a quadratic Hamiltonian. We claim that:

**Let $\Psi$ be an eigenfunction of the operator**

$$\hat{H} = (x, -i \hbar \nabla_x) M (x, -i \hbar \nabla_x)^T.$$ 

**The symplectic capacity of the corresponding Fermi blob $\Omega_F$ is**

$$c(\Omega_F) = \sum_{j=1}^n (N_j + \frac{1}{2}) \hbar$$

**where the numbers $N_1, N_2, ..., N_n$ are the non-negative integers corresponding to the state**

$$(1)$$

**of the diagonalized operator $\hat{K} = \sum_{j=1}^n \hat{K}_j$.**

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APPENDIX: The Metaplectic Group

The symplectic group $\text{Sp}(2n, \mathbb{R})$ has a covering group of order two, the metaplectic group $\text{Mp}(2n, \mathbb{R})$. That group consists of unitary operators (the metaplectic operators) acting on $L^2(\mathbb{R}^n)$. There are several equivalent ways to describe the metaplectic operators. For our purposes the most tractable is the following: assume that $S \in \text{Sp}(2n, \mathbb{R})$ has the block-matrix form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with} \quad \det B \neq 0. \quad (A1)$$

The condition $\det B \neq 0$ is not very restrictive, because one shows (de Gosson [10, 16, 19], Littlejohn [27]) that every $S \in \text{Sp}(2n, \mathbb{R})$ can be written (non uniquely) as the product of two symplectic matrices of the type above; moreover the symplectic matrices arising as Jacobian matrices of Hamiltonian flows determined by physical Hamiltonians of the type “kinetic energy plus potential” are of this type for almost every time $t$. To the matrix (A1) we associate the following quantities (de Gosson [10, 16]):

- A quadratic form
  $$W(x, x') = \frac{1}{2}DB^{-1}x \cdot x - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax' \cdot x'; \quad (A2)$$
  the matrices $DB^{-1}$ and $B^{-1}A$ are symmetric because $S$ is symplectic;

- The complex number $\Delta(W) = i^m \sqrt{|\det B^{-1}|}$ where $m$ (“Maslov index”) is chosen in the following way: $m = 0$ or $2$ if $\det B^{-1} > 0$ and $m = 1$ or $3$ if $\det B^{-1} < 0$.

The two metaplectic operators associated to $S$ are then given by

$$\hat{S}\Psi(x) = \left(\frac{1}{2\pi \hbar}\right)^{n/2} \Delta(W) \int e^{i\hbar W(x, x')}\Psi(x')d^n x'. \quad (A3)$$

The fact that we have two possible choices for the Maslov index is directly related the fact that $\text{Mp}(2n, \mathbb{R})$ is a two-fold covering group of the symplectic group $\text{Sp}(2n, \mathbb{R})$ [10] [16] [9].

The main interest of the metaplectic group in quantization questions comes from the two following (related) “symplectic covariance” properties:

- Let $\Psi$ be a square integrable function (or, more generally, a tempered distribution), and $S$ a symplectic matrix. We have
  $$W(\Psi(S^{-1}z)) = W(\hat{S}\Psi)(z) \quad (A4)$$
  where $\hat{S}$ is any of the two metaplectic operators corresponding to $S$;
Let $\hat{H}$ be the Weyl quantisation of the symbol (= observable) $H$. Let $S$ be a symplectic matrix. Then the quantisation of $K(z) = H(Sz)$ is $\hat{K} = \hat{S}^{-1}\hat{H}\hat{S}$ where $\hat{S}$ is again defined as above.

References

[1] Arnold, V.I.: Mathematical Methods of Classical Mechanics. Graduate Texts in Mathematics, second edition, Springer-Verlag (1989)

[2] Benenti, G.: Gaussian wave packets in phase space: The Fermi $g_F$ function, Am. J. Phys. 77(6), 546–551 (2009)

[3] Benenti, G., Strini, G.: Quantum mechanics in phase space: first order comparison between the Wigner and the Fermi function, Eur. Phys. J. D 57, 117–121 (2010)

[4] Bohm, D., Hiley, B.: The Undivided Universe: An Ontological Interpretation of Quantum Theory, London & New York: Routledge (1993)

[5] Cielibak, K., Hofer, H., Latschev, J., Schlenk, F.: Quantitative symplectic geometry. Recent Progress in Dynamics, MSRI Publications 54 (2007); arXiv:math/0506191v1 [math.SG]

[6] Ekeland, I., Hofer, H.: Symplectic topology and Hamiltonian dynamics, II. Math. Zeit. 203, 553–567 (1990)

[7] Gefter, A.: Beyond space-time: Welcome to phase space. New Scientist, 2824, 08 August 2011

[8] Fermi, E.: Rend. Lincei 11, 980 (1930); reprinted in Nuovo Cimento 7, 361 (1930)

[9] Folland, G.B.: Harmonic Analysis in Phase space, Annals of Mathematics studies, Princeton University Press, Princeton, N.J. (1981)

[10] de Gosson, M.: The Principles of Newtonian and Quantum Mechanics: The need for Planck’s constant, $h$. With a foreword by Basil Hiley. Imperial College Press, London (2001)

[11] de Gosson, M.: Maslov Classes, Metaplectic Representation and Lagrangian Quantization. Research Notes in Mathematics 95, Wiley–VCH, Berlin (1997)
[12] de Gosson, M.: The “symplectic camel principle” and semiclassical mechanics. J. Phys. A: Math. Gen. 35(32), 6825–6851 (2002)

[13] de Gosson, M.: Phase Space Quantization and the Uncertainty Principle. Phys. Lett. A, 317/5-6 365–369 (2003)

[14] de Gosson, M.: The optimal pure Gaussian state canonically associated to a Gaussian quantum state. Phys. Lett. A, 330:3–4, 161–167 (2004)

[15] de Gosson, M.: Cellules quantiques symplectiques et fonctions de Husimi–Wigner. Bull. Sci. Math. 129 211–226 (2005)

[16] de Gosson, M.: Symplectic Geometry and Quantum Mechanics, Birkhäuser, Basel, series “Operator Theory: Advances and Applications” (subseries: “Advances in Partial Differential Equations”), Vol. 166 (2006)

[17] de Gosson, M.: The Symplectic Camel and the Uncertainty Principle: The Tip of an Iceberg? Found. Phys. 99, 194–214 (2009)

[18] de Gosson, M.: On the use of minimum volume ellipsoids and symplectic capacities for studying classical uncertainties for joint position–momentum measurements, J. Stat. Mech. (2010)

[19] de Gosson, M.: Symplectic Methods in Harmonic Analysis and in Mathematical Physics. Birkhäuser; Springer Basel (2011)

[20] de Gosson, M.: Quantum Blobs. Found. Phys., Published online: 29 February 2012

[21] de Gosson, M.: On the partial saturation of the uncertainty relations of a mixed Gaussian state. J. Phys. A: Math. Theor. 45 415301 (2012)

[22] de Gosson, M., Luef, F.: Symplectic Capacities and the Geometry of Uncertainty: the Irruption of Symplectic Topology in Classical and Quantum Mechanics. Physics Reports 484, 131–179 (2009), DOI 10.1016/j.physrep.2009.08.001

[23] Gromov, M.: Pseudoholomorphic curves in symplectic manifolds, Invent. Math., 82, 307–347 (1985)

[24] Hofer H. and Zehnder E.: Symplectic Invariants and Hamiltonian Dynamics, Birkhäuser Advanced texts, (Basler Lehrbücher, Birkhäuser Verlag, (1994)
[25] Keller, J.B.: Semiclassical Mechanics. SIAM Review, 27(4) 485–504 (1985)

[26] Leray, J.: Lagrangian Analysis and Quantum Mechanics, a mathematical structure related to asymptotic expansions and the Maslov index, MIT Press, Cambridge, Mass. (1981)

[27] Littlejohn, R.G.: The semiclassical evolution of wave packets, Physics Reports 138(4–5), 193–291 (1986)

[28] Maslov, V.P.: Théorie des Perturbations et Méthodes Asymptotiques. Dunod, Paris, 1972; translated from Russian [original Russian edition 1965]

[29] Maslov, V.P., Fedoriuk, M.V.: Semi-Classical Approximations in Quantum Mechanics. Reidel, Boston (1981)

[30] Polterovich, L.: The Geometry of the Group of Symplectic Diffeomorphisms, Lectures in Mathematics, Birkhäuser, (2001)

[31] Seip, K.: Curves of maximum modulus in coherent state representations. Annales de l’institut Henri Poincaré (A) Physique théorique, 51(4), 335–350 (1989)