MULTI-SCALE PROJECTIVE COORDINATES VIA PERSISTENT COHOMOLOGY OF SPARSE FILTRATIONS

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ABSTRACT. We present in this paper a framework for mapping data onto real and complex projective spaces. The resulting projective coordinates provide a multi-scale representation of the data, and capture low dimensional underlying topological features. An initial map is obtained in two steps: First, the persistent cohomology of a sparse filtration is used to compute systems of transition functions for (real and complex) line bundles over neighborhoods of the data. Next, the transition functions are used to produce explicit classifying maps for the induced bundles. A framework for dimensionality reduction in projective space (Principal Projective Components) is also developed, aimed at decreasing the target dimension of the original map. Several examples are provided as well as theorems addressing choices in the construction.

1. Introduction

Algebraic topology has emerged in the last few years as a powerful framework for analysing complex high-dimensional data [2, 3]. In this setting, a data set is interpreted as a finite subset \( X \) of an ambient metric space \((M, d)\); for instance \( M \) can be Euclidean space, a manifold or a simplicial complex. If \( X \) has been sampled from/around a “continuous” object \( X \subset M \), the topology inference problem asks whether topological features of \( X \) can be inferred from \( X \). This is relevant because several data science questions are reinterpretations of topological tests. To name a few: clustering is akin to finding the connected components of a space [4, 5]; coverage in a sensor network relates to the existence of holes [17, 10]; periodicity and quasiperiodicity are linked to non-trivial 1-cycles in time delay reconstructions [32, 30]. Moreover, a concrete description of \( X \) – e.g. via equations or as a quotient space – yields a (geometric) model for the data, which can then be used for simulations, predictions and hypothesis testing [24, 31].

Persistent (co)homology is one tool for approaching the topology inference problem [37, 11]; it provides a multiscale description of topological features as well as several inference theorems [29, 7, 9]. In practice, however, it is non-trivial to go from a persistent homology computation to actionable knowledge about the initial data set. Typical questions are: how are the persistent homology features reflected on the data? if the homology suggests a candidate underlying space \( X \), what is a concrete realization (e.g., via equations or as a quotient)? how does the data fit on/around the realization?

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At least three approaches have been proposed in the literature to address these questions: localization, homological coordinatization and circular coordinates. For a relevant homology class, the idea behind localization is to find an appropriate representative cycle. This strategy is successful in low dimensions [14, 15], but in general even reasonable heuristics lead to NP-hard problems which are NP-hard to approximate [8]. Homological coordinatization [35] attempts to map a simplicial complex on the data to a simplicial complex with prescribed homology – a model suggested by a persistent homology computation. The map is selected by examining the chain homotopy class of chain maps between the complex on the data and the model. The selection process, however, involves combinatorial optimizations which are difficult to solve in practice. Circular coordinates [12] combines two facts: the model. The selection process, however, involves combinatorial optimizations which are difficult to solve in practice. Circular coordinates [12] combines two facts: the

The circular coordinates approach has been used successfully – for instance to parameterize periodic dynamics [13] – and is part of a bigger picture: if $G$ is an abelian group, $n \in \mathbb{Z}_{\geq 0}$ and $K(G, n)$ is an Eilenberg-MacLane space (i.e. its $n$-th homotopy group is $G$ and the others are trivial) then [25, Chapter 22, sec. 2]

$$H^n(B; G) \cong [B, K(G, n)]$$

In particular $K(\mathbb{Z}, 1) = S^1$. Since the bijection above can be chosen so that it commutes with morphisms induced by maps of spaces (i.e. natural), using maps to other Eilenberg-MacLane spaces emerges as an avenue for interpreting persistent cohomology computations, and also to produce representations of data in spaces with interesting geometry/topology. One quickly runs into difficulties as the next candidate spaces $K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$, $K(\mathbb{Z}/p, 1) = S^\infty/(\mathbb{Z}/p)$ (action via scalar multiplication on $S^\infty \subset \mathbb{C}^\infty$ by the $p$-th roots of unity) and $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ are infinite. The purpose of this paper is to address the cases $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$.

1.1. **Approach.** If $\mathbb{F} = \mathbb{R}, \mathbb{C}$ then $\mathbb{F}P^\infty$ is the Grassmannian of 1-planes in $\mathbb{F}^\infty$; hence $[B, \mathbb{F}P^\infty]$ can be naturally identified with the set of isomorphism classes of $\mathbb{F}$-line bundles over $B$ (Thm 2.1). The isomorphism type of an $\mathbb{R}$-line bundle is uniquely determined by its first Stiefel-Whitney class $w_1 \in H^1(B; \mathbb{Z}/2)$, and the isomorphism type of a $\mathbb{C}$-line bundle is classified by its first Chern class $c_1 \in H^2(B; \mathbb{Z})$. These classes can be identified as elements of appropriate sheaf cohomology groups, and a classifying map $f : B \rightarrow \mathbb{F}P^\infty$ can be described explicitly from a Čech cocycle representative (Thm 3.2). We compute these representatives using the persistent cohomology of a sparse filtration (Thm 7.4). Recall that if $B$ has the homotopy type of a finite CW-complex then the cellular approximation theorem implies that $f$ can be deformed to a map $\tilde{f} : B \rightarrow \mathbb{F}P^d$. We propose a dimensionality reduction framework in projective space (Section 5), which has a similar effect for point cloud data.

1.2. **Motivation and road map.** Let us use an example: let $\mathbb{X}$ be the collection of intensity-centered $7 \times 7$ grey-scale images depicting a line segment of fixed width. Each $x \in \mathbb{X}$ is then a 7-by-7 matrix with entries between -1 (black) and 1 (white), whose sum equals zero; figure 1 shows some examples.
We regard $X$ as a subset of $\mathbb{R}^{49}$ via column concatenation, endow it with the restriction of the Euclidean distance, and generate a finite data set $X \subset \mathbb{R}^{49}$ by sampling $|X| = 1,682$ points. The thing to notice is that even when the ambient space for $X$ is $\mathbb{R}^{49}$, the intrinsic dimensionality is low. Indeed, each image can be described with two numbers: the angle of the line segment with the horizontal, and the distance from the segment to the center of the patch. This suggests that $X$ is locally (at most) 2-dimensional, and we show in Figure 1.2 the variance, from $X$, recovered by Principal Component Analysis (PCA) [20] and ISOMAP\textsuperscript{1} [36].

Given the low intrinsic dimensionality, it follows from the PCA plot that the original embedding $X \hookrightarrow \mathbb{R}^{49}$ is highly non-linear. Moreover, the ISOMAP plot implies that even after accounting for the way in which $X$ sits in $\mathbb{R}^{49}$ the data has intrinsic complexity that prevents it from being recovered in $\mathbb{R}^2$ or $\mathbb{R}^3$. That is, the data is locally simple (e.g. each $x \in X$ is described by angle and displacement) but globally complex. One possible source of said complexity is whether or not $X$ is orientable. When $X$ is a manifold this can be measured from $X$ as follows: First, construct a covering $\{U_r\}$ of $X$; next, on each $U_r$ apply Multi-Dimensional Scaling (MDS) [22] to get local Euclidean coordinates; and finally, compute the determinant $\omega_{rt} = \pm 1$ associated to the change of local Euclidean coordinates on each $U_r \cap U_t$. If there is global agreement of local orientations (e.g., $\omega_{rt} = 1$ always), or local orientations can be reversed in the appropriate $U_r$’s so that the result is globally consistent (i.e. $\{\omega_{rt}\}$ is a coboundary: there exist $\nu_t = \pm 1$ so that $\omega_{rt} = \nu_t / \nu_r$), then $X$ would be estimated to be orientable.

\textsuperscript{1} using a $k$-th nearest neighbor graph, with $k = 7$. 

\footnotesize

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{plot.png}
\caption{Explained variance, from $X$, versus embedding dimension.}
\end{figure}
The cover for $X$ will be a collection of open balls centered at landmark data points selected through \texttt{maxmin} (also known as farthest point) sampling. That is, first one chooses an arbitrary landmark $\ell_0 \in X$, and if $\ell_0, \ldots, \ell_r$ have been determined then $\ell_{r+1} \in X$ is given by

$$
\ell_{r+1} = \arg\max_{x \in X} \left( \min \{ d(x, \ell_0), \ldots, d(x, \ell_r) \} \right).
$$

Here $d$ is the geodesic distance estimate from the ISOMAP calculation. This results in a landmark set $\{\ell_0, \ldots, \ell_d\}$ which tends to be well distributed and well separated across the data. For the current example we used $d = 14$. Let

$$
\mathcal{U} = \{U_0, \ldots, U_d\} \text{ where } U_r = \{ x \in X : d(x, \ell_r) < \epsilon_r \}
$$

and $\epsilon_0 = \cdots = \epsilon_d = 9.3^2$. To put this number in perspective, the distance between distinct landmarks ranges from 4.5 to 14.6, and the mean pairwise distance is 9.5.

Now the determinant for the change of local coordinates. If $U_r \cap U_t \neq \emptyset$ then let

$$
f_r : U_r \rightarrow \mathbb{R}^2 \text{ and } f_t : U_t \rightarrow \mathbb{R}^2
$$

be the maps obtained from applying MDS on $\left( U_r, d|_{U_r} \right)$ and $\left( U_t, d|_{U_t} \right)$, respectively. If $\mathbb{O}(2)$ denotes the set of orthogonal $2 \times 2$ real matrices, then the solution to the orthogonal Procrustes problem

$$
(\Omega_{rt}, v_{rt}) = \arg\min_{\Omega \in \mathbb{O}(2), v \in \mathbb{R}^2} \sum_{x \in U_r \cap U_t} \| f_t(x) - (\Omega \cdot f_r(x) + v) \|^2
$$

computed following [33], yields the best linear approximation to an isometric change of local ISOMAP coordinates. We let $\omega_{rt} := \det(\Omega_{rt})$. The result is a covering $\mathcal{U} = \{U_0, \ldots, U_d\}$ for $X$ and a collection of numbers $\omega_{rt} \in \{-1, 1\}$ which, at least for this example, satisfy the \textbf{cocycle condition}: for all $0 \leq r \leq d$ we have $\omega_{rr} = 1$ and if $U_r \cap U_s \cap U_t \neq \emptyset$ then $\omega_{rt} \cdot \omega_{sr} = \omega_{st}$. In particular, one can check that $\{\omega_{rt}\}$ is not a coboundary and hence $X$ is estimated to be non-orientable.

What we will see now is that this type of cohomological features can be further leveraged to produce useful coordinates. Indeed (Theorem 3.2 and Corollary 3.4):

**Theorem.** For $\lambda \in \mathbb{R}$ let $|\lambda|_+ = \max\{\lambda, 0\}$, let $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and let $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$. For a metric space $(\mathbb{M}, d)$ let $\{\ell_0, \ldots, \ell_d\} \subset \mathbb{M}$ and fix positive real numbers $\epsilon_0, \ldots, \epsilon_d$. If we let $\mathcal{B} = \{B_r\}$ with $B_r = \{ b \in \mathbb{M} : d(b, \ell_r) < \epsilon_r \}$, $B = \bigcup \mathcal{B}$, and there is a collection $\omega = \{\omega_{rt} : B_r \cap B_t \rightarrow \mathbb{F}^\times\}$ of continuous maps satisfying the cocycle condition (3), then $f_\omega : B \rightarrow \mathbb{F}P^d$ given in homogeneous coordinates by

$$
f_\omega(b) = \left[ \omega_{0j}(b) \cdot |\epsilon_0 - d(b, \ell_0)|_+ : \cdots : \omega_{dj}(b) \cdot |\epsilon_d - d(b, \ell_d)|_+ \right] , \quad b \in B_j \quad (1)
$$

is well-defined (i.e. the value $f_\omega(b)$ is independent of the $j$ for which $b \in B_j$) and classifies the $\mathbb{F}$-line bundle on $B$ induced by $(\mathcal{B}, \omega)$.

The preliminaries needed to understand this theorem will be covered in Section 2. The map $f_\omega$ encodes in a global manner the local interactions captured by $\omega$, and the explicit formula allows us to map our data set $X \subset \mathbb{R}^{49}$ into $\mathbb{R}P^{14}$ using the computed landmarks and determinants of local changes of coordinates. If we compute the principal projective components for $f_\omega(X) \subset \mathbb{R}P^{14}$ (this will be developed in Section 5 as a natural extension to PCA in Euclidean space and of

\footnote{Determined experimentally using a persistent cohomology computation.}
Principal Nested Spheres Analysis [21]), the profile of recovered variance shown in Figure 3 emerges:

![Figure 3. Recovered variance, from $f_\omega(X) \subset \mathbb{RP}^{14}$, when projected onto principal projective subspace of given dimension.](image)

From this plot we conclude that 2-dimensional projective space provides an appropriate reduction for $f_\omega(X)$. We show in Figure 4 said representation; that is, each image is placed in the $\mathbb{RP}^2$ coordinate computed via principal projective component analysis on $f_\omega(X)$.

![Figure 4. Some elements from $X$ placed on their computed $\mathbb{RP}^2$-coordinates.](image)

As the figure shows, the resulting coordinates recover the variables which we identified as describing $X$: the radial coordinate in $\mathbb{RP}^2$ corresponds to distance from the line segment to the center of the patch, and the angular coordinate captures orientation. Also, it indicates how $\mathbb{RP}^2 \cong X$ parameterizes the original data set $X$. 
Though the strategy employed in this example (i.e. local MDS + determinant of local change of coordinates) was successful, one cannot assume in general that the data under analysis has been sampled from/around a manifold. That said, the result from formula (1) only requires a covering $B$ via open balls, and a collection of $\mathbb{F}^x$-valued continuous functions $\omega = \{\omega_{rt}\}$ satisfying the cocycle condition. Given a finite subset $X$ of an ambient metric space $(M, d)$, one can always use maxmin sampling to produce a covering. We will show that any 1-dimensional (resp. 2-dimensional) $\mathbb{Z}/2$-cocyle (resp. $\mathbb{Z}$-cocycle) of the nerve complex $\mathcal{N}(B)$, for $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{C}$), yields one such $\omega$ (Proposition 4.3 and Corollary 4.8). We also show that in dimension 1 (i.e., $\mathbb{F} = \mathbb{R}$) cohomologous cocycles yield equivalent projective coordinates, while in dimension 2 (i.e., $\mathbb{F} = \mathbb{C}$) the harmonic cocycle is needed (see Section 6).

It is entirely possible that a cohomology class reflecting sampling artifacts is chosen, as opposed to one associated to robust topological features of a continuous space $X \subset M$ underlying $X$. Here is where persistent cohomology comes in. Indeed, under mild connectivity conditions of $B$, distinct cohomology classes yield maps to projective space with distinct homotopy types. Moreover, the maps resulting from a persistent class across its lifetime are compatible up to homotopy (Theorem 7.4 and Proposition 7.6). Hence, the result is a multi-scale family of compatible maps which, for classes with long persistence, are more likely to reflect robust features of (neighborhoods around) $X$.

The strategy outlined here is in fact a two-way street. One can use persistent cohomology to compute multi-scale compatible projective coordinates, but the reserve is also useful: The resulting coordinates can be used to interpret the distinct persistent cohomology (= persistent homology) features of neighborhoods of the data, at least in cohomological dimensions 1 (with $\mathbb{Z}/2$ coefficients) and 2 (with $\mathbb{Z}/p$ coefficients for appropriate primes $p$).

2. Preliminaries

Vector Bundles. For a more thorough review please refer to [26]. Let $E$ and $B$ be topological spaces, and let $p : E \to B$ be a surjective continuous map. The triple $\zeta = (E, B, p)$ is said to be a rank $k \in \mathbb{N}$ vector bundle over a field $\mathbb{F}$ (i.e. an $\mathbb{F}$-vector bundle) if each fiber $p^{-1}(b)$ is an $\mathbb{F}$-vector space of dimension $k$, and $\zeta$ is locally trivial. That is, for every $b_0 \in B$ there exist an open neighborhood $U \subset B$ and a homeomorphism $\rho_U : U \times \mathbb{F}^k \to p^{-1}(U)$, called a local trivialization around $b_0$, satisfying:

1. $p(\rho_U(b, \mathbf{v})) = b$ for every $(b, \mathbf{v}) \in U \times \mathbb{F}^k$
2. $\rho_U(b, \cdot) : \mathbb{F}^k \to p^{-1}(b)$ is an isomorphism of $\mathbb{F}$-vector spaces for each $b \in U$

$E$ and $B$ are referred to as the total and base space of the bundle, and the function $p : E \to B$ is called the projection map. Two vector bundles $\zeta = (E, B, p)$ and $\zeta' = (E', B, p')$ are said to be isomorphic, $\zeta \cong \zeta'$, if there exists a homeomorphism $T : E \to E'$ so that $p' \circ T = p$ and for which each restriction $T|_{p^{-1}(b)}$, $b \in B$, is a linear isomorphism.

The collection of isomorphism classes of $\mathbb{F}$-vector bundles of rank $k$ over $B$ is denoted $\text{Vect}_k^\mathbb{F}(B)$. An $\mathbb{F}$-vector bundle of rank 1 is called an $\mathbb{F}$-line bundle, and the set $\text{Vect}_1^\mathbb{F}(B)$ is an abelian group with respect to fiberwise tensor product of $\mathbb{F}$-vector spaces.
Examples:

- **The trivial bundle** $B \times \mathbb{F}^k$: Fix $k \in \mathbb{N}$ and let
  \[ p : B \times \mathbb{F}^k \rightarrow B \]
  \[ (b, v) \mapsto b \]

  It follows that $\varepsilon_k = (B \times \mathbb{F}^k, B, p)$ is an $\mathbb{F}$-vector bundle over $B$ of rank $k$. $\varepsilon_k$ is referred to as the trivial bundle.

- **The Moebius band**: Let $\sim$ be the relation on $\mathbb{R} \times \mathbb{R}$ given by $(x, u) \sim (y, v)$ if and only if $x - y \in \mathbb{Z}$ and $u = (-1)^{x-y}v$. It follows that $\sim$ is an equivalence relation, and if $E = \mathbb{R} \times \mathbb{R}/ \sim$ then $\tilde{p} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $\tilde{p}(x, u) = x$ descends to a continuous surjective map $p : E \rightarrow \mathbb{R}/\mathbb{Z}$. Hence $\gamma^1 = (E, \mathbb{R}/\mathbb{Z}, p)$ is an $\mathbb{R}$-line bundle over the circle $\mathbb{R}/\mathbb{Z}$, whose total space is a model for the Moebius band. Since $E$ is nonorientable, it follows that $\gamma^1$ is not isomorphic to the trivial line bundle $\varepsilon_1$.

- **Grassmann manifolds and their tautological bundles**: Let $F = \mathbb{R}, \mathbb{C}$. Given $k \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, with $m \geq k$, let $\text{Gr}_k(\mathbb{F}^m)$ be the collection of $k$-dimensional linear subspaces of $\mathbb{F}^m$. This set is in fact a manifold, referred to as the Grassmannian of $k$-planes in $\mathbb{F}^m$. The tautological bundle over $\text{Gr}_k(\mathbb{F}^m)$, denoted $\gamma^k_{\mathbb{F}^m}$, has total space
  \[ E(\gamma^k_{\mathbb{F}^m}) = \{(V, u) \in \text{Gr}_k(\mathbb{F}^m) \times \mathbb{F}^m : u \in V\} \]

  and projection $p : E(\gamma^k_{\mathbb{F}^m}) \rightarrow \text{Gr}_k(\mathbb{F}^m)$ given by $p(V, u) = V$. In particular one has that $\text{Gr}_1(\mathbb{F}^{m+1}) = \mathbb{P}^m$, which shows that each projective space $\mathbb{P}^m$ can be endowed with a tautological line bundle $\gamma^1_{\mathbb{F}^m}$.

- **Pullbacks**: Let $B$ and $B'$ be topological spaces, let $\zeta = (E, B, p)$ be a vector bundle and let $f : B' \rightarrow B$ be a continuous map. The pullback of $\zeta$ through $f$, denoted $f^* \zeta$, is the vector bundle over $B'$ with total space
  \[ E(f^* \zeta) = \{(b, e) \in B' \times E : f(b) = p(e)\} \]

  and projection $p' : E(f^* \zeta) \rightarrow B'$ given by $p'(b, e) = b$.

**Theorem 2.1** ([26], 5.6 and 5.7). Let $B$ be a paracompact topological space and let $\zeta$ be an $\mathbb{F}$-vector bundle of rank $k$ over $B$. Then there exists a continuous map

\[ f_\zeta : B \rightarrow \text{Gr}_k(\mathbb{F}^\infty) \]

unique up to homotopy for which $f_\zeta^* \gamma^k_{\mathbb{F}^\infty} \cong \zeta$.

The previous theorem can be rephrased as follows: For $B$ paracompact, the function

\[ \text{Vect}_\mathbb{F}^k(B) \rightarrow [B, \text{Gr}_k(\mathbb{F}^\infty)] \]

\[ [\zeta] \mapsto [f_\zeta] \] \hspace{1cm} (2)

is a bijection. Any such $f_\zeta$ is referred to as a **classifying map** for the bundle $\zeta$.

**Transition Functions.** If $\rho_U : U \times \mathbb{F}^k \rightarrow p^{-1}(U)$ and $\rho_V : V \times \mathbb{F}^k \rightarrow p^{-1}(V)$ are local trivializations around a point $b_0 \in U \cap V$, then given $b \in U \cap V$ the composition

\[ \mathbb{F}^k \xrightarrow{\rho_V(b, \cdot)} p^{-1}(b) \xrightarrow{\rho_U(b, \cdot)^{-1}} \mathbb{F}^k \]
defines an element $\rho_{UV}(b)$ in the general linear group $\text{GL}_k(\mathbb{F})$. The resulting function $\rho_{UV} : U \cap V \to \text{GL}_k(\mathbb{F})$ is a continuous map, uniquely determined by

$$\rho_{UV}^{-1} \circ \rho_{V}(b,v) = (b, \rho_{UV}(b)(v)) \quad \text{for every} \quad (b,v) \in (U \cap V) \times \mathbb{F}^k.$$

This characterization of $\rho_{UV}$ readily implies that the set $\{\rho_{UV}\}$ satisfies:

**The Cocycle Condition**

- $\rho_{UU}(b)$ is the identity linear transformation of $\mathbb{F}^k$ for every $b \in U$.
- $\rho_{UV}(b) = \rho_{UW}(b) \circ \rho_{VW}(b)$ for every $b \in U \cap V \cap W$.

Each $\rho_{UV} : U \cap V \to \text{GL}_k(\mathbb{F})$ is called a *transition function* for the bundle $\zeta$, and the collection $\{\rho_{UV}\}$ is the system of transition functions associated to the system of local trivializations $\{\rho_U\}$. More importantly, this construction can be reversed: If $U = \{U_r\}$ is a covering of $B$ and $\omega = \{\omega_{rt} : U_r \cap U_t \to \text{GL}_k(\mathbb{F})\}$ is a collection of continuous functions satisfying the cocycle condition, then one can form the quotient space

$$E(\omega) = \left( \bigcup_r (U_r \times \{r\} \times \mathbb{F}^k) \right) / \sim$$

where $(b,r,v) \sim (b,t,\omega_{rt}(b)^{-1}(v))$ for $b \in U_r \cap U_t$. Moreover, if $p_\omega : E(\omega) \to B$ is projection onto the first coordinate, then $\zeta_\omega = (E(\omega), B, p_\omega)$ is an $\mathbb{F}$-vector bundle of rank $k$ over $B$. It follows that each composition

$$\rho_r : U_r \times \mathbb{F}^k \to U_r \times \{r\} \times \mathbb{F}^k \to p_\omega^{-1}(U_r)$$

is a local trivialization for $\zeta_\omega$, and that $\omega$ is the associated system of transition functions. We say that $\zeta_\omega$ is the vector bundle induced by $(U,\omega)$.

**Preshaves and their Čech Cohomology.** For a more detailed introduction please refer to [27]. A presheaf $\mathcal{F}$ of abelian groups over a topological space $B$ is a collection of abelian groups $\mathcal{F}(U)$, one for each open set $U \subset B$, and group homomorphisms $\eta^U_{V} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each pair $V \subset U$ of open subsets of $B$, called restrictions, so that:

1. $\mathcal{F}(\emptyset)$ is the group with one element
2. $\eta^U_{U}$ is the identity homomorphism
3. $\eta^V_{V} = \eta^V_{W} \circ \eta^W_{V}$ for every triple $W \subset V \subset U$

Furthermore, a presheaf $\mathcal{F}$ is said to be a sheaf if it satisfies the gluing axiom:

4. If $U \subset B$ is open, $\{U_j\}_{j \in J}$ is an open covering of $U$ and there are elements $\{s_j \in \mathcal{F}(U_j) : j \in J\}$ so that

$$\eta^U_{U_j \cap U_l}(s_j) = \eta^U_{U_l \cap U_j}(s_l)$$

for every non-empty intersection $U_j \cap U_\ell \neq \emptyset$, with $j, \ell \in J$, then there exists a unique $s \in \mathcal{F}(U)$ so that $\eta^U_{U_j}(s) = s_j$ for every $j \in J$.

**Examples:**
• **Presheaves of constant functions:** Let $G$ be an abelian group and for each $\emptyset \neq U \subset B$ open, let $P_G(U)$ be the set of constant functions from $U$ to $G$. Let $P_G(\emptyset) = \{0\} \subset G$. If for $V \subset U \subset B$ we let $\eta_{UV}^G : P_G(U) \rightarrow P_G(V)$ be the restriction map $f \mapsto f|_V$, then it follows that $P_G$ is a presheaf over $B$. It is not in general a sheaf since it does not always satisfy the gluing axiom: for if $U, V \subset B$ are disjoint nonempty open sets and $|G| \geq 2$, then $f \in P_G(U)$ and $g \in P_G(V)$ taking distinct values cannot be realized as restrictions of a constant function $h : U \cup V \rightarrow G$.

• **Sheaves of locally constant functions:** Let $G$ be an abelian group and for each $\emptyset \neq U \subset B$ open, let $G(U)$ be the set of functions $f : U \rightarrow G$ for which there exists $\emptyset \neq V \subset U$ open so that the restriction $f|_V : V \rightarrow G$ is a constant function. Define $G(\emptyset)$ and $\eta_{UV}^G$ as in the presheaf of constant functions. It follows that $G$ is a sheaf over $B$.

• **Sheaves of continuous functions:** Let $G$ be a topological abelian group and for $\emptyset \neq U \subset B$ open, let $\mathcal{C}_G(U)$ be the set of continuous functions from $U$ to $G$. If $\mathcal{C}_G(\emptyset)$ and $\eta_{UV}^G$ are as above then $\mathcal{C}_G$ is a sheaf over $B$. Moreover, $G$ is a subsheaf of $\mathcal{C}_G$ in that $G(U) \subset \mathcal{C}_G(U)$ for every $U \subset B$ open. Similarly, if $R$ is a commutative topological ring with unity, and $R^\times$ denotes its (multiplicative) group of units then $\mathcal{C}_R^\times := \mathcal{C}_{R^\times}$ is also a sheaf over $B$.

Let $n \geq 0$ be an integer, $\mathcal{U} = \{U_j\}$ an open cover of $B$ and let $\mathcal{F}$ be a presheaf over $B$. The group of Čech $n$-cochains is defined as

$$\check{C}^n(\mathcal{U}; \mathcal{F}) = \prod_{(j_0, \ldots, j_n)} \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_n})$$

and element in $\check{C}^n(\mathcal{U}; \mathcal{F})$ denoted $\{f_{j_0, \ldots, j_n}\}$, if $f_{j_0, \ldots, j_n} \in \mathcal{F}(U_{j_0} \cap \cdots \cap U_{j_n})$. For $0 \leq r \leq n$ let $(j_0, \ldots, \hat{j}_r, \ldots, j_n)$ denote the $n$-tuple obtained by removing $j_r$ from the $(n+1)$-tuple $(j_0, \ldots, j_n)$, let $U_{j_0, \ldots, j_n} = U_{j_0} \cap \cdots \cap U_{j_n}$, and let

$$\eta_{jr} : \mathcal{F}(U_{j_0, \ldots, j_r, \ldots, j_n}) \rightarrow \mathcal{F}(U_{j_0, \ldots, j_n})$$

denote the associated restriction homomorphism. The coboundary homomorphism

$$\delta^n : \check{C}^n(\mathcal{U}; \mathcal{F}) \rightarrow \check{C}^{n+1}(\mathcal{U}; \mathcal{F})$$

is given by $\delta^n(\{f_{j_0, \ldots, j_n}\}) = \{g_{k_0, \ldots, k_{n+1}}\}$ where

$$g_{k_0, \ldots, k_{n+1}} = \sum_{r=0}^{n+1} (-1)^r \eta_{kr} \left( f_{k_0, \ldots, \hat{k}_r, \ldots, k_{n+1}} \right)$$

One can check that $\delta^{n+1} \circ \delta^n = 0$. The group of Čech $n$-cocycles $\check{Z}^n(\mathcal{U}; \mathcal{F})$ is the kernel of $\delta^n$, the group of Čech $n$-boundaries $\check{B}^n(\mathcal{U}; \mathcal{F}) \subset \check{Z}^n(\mathcal{U}; \mathcal{F})$ is the image of $\delta^{n-1}$, and the $n$-th Čech cohomology group of $\mathcal{F}$ with respect to the covering $\mathcal{U}$ is given by the quotient of abelian groups

$$\check{H}^n(\mathcal{U}; \mathcal{F}) = \check{Z}^n(\mathcal{U}; \mathcal{F})/\check{B}^n(\mathcal{U}; \mathcal{F}).$$

**Persistent Cohomology of Filtered Complexes.** Given a nonempty set $S$, an abstract simplicial complex $K$ with vertices in $S$ is a set

$$K \subset \{\sigma \subset S : \sigma \text{ is finite and } \sigma \neq \emptyset\}$$

for which $\emptyset \neq \tau \subset \sigma \in K$ always implies $\tau \in K$. An element $\sigma \in K$ with cardinality $|\sigma| = n + 1$ is called an $n$-simplex of $K$, and a 0-simplex is referred to as a vertex.
Examples:

- **The Rips Complex**: Let $(\mathbb{M}, d)$ be a metric space, let $X \subset \mathbb{M}$ and $\epsilon \geq 0$. The Rips complex at scale $\epsilon$ and vertex set $X$, denoted $R_\epsilon(X)$, is the collection of finite nonempty subsets of $X$ with diameter less than $2\epsilon$.

- **The Čech Complex**: With $\mathbb{M}$, $d$, $X$, $\epsilon$ as above, the (ambient) Čech complex at scale $\epsilon$ and vertices in $X$ is the set

$$
\check{C}_\epsilon(X) = \{ \{ s_0, \ldots, s_n \} \subset X : B_\epsilon(s_0) \cap \cdots \cap B_\epsilon(s_n) \neq \emptyset, \ n \in \mathbb{Z}_{\geq 0} \}
$$

where $B_\epsilon(s)$ denotes the open ball in $\mathbb{M}$ of radius $\epsilon$ centered at $s \in X$. It follows that $\check{C}_\epsilon(X) \subset R_\epsilon(X) \subset \check{C}_{2\epsilon}(X)$ for all $\epsilon > 0$.

For each $n \in \mathbb{Z}_{\geq 0}$ let $K^{(n)}$ be the set of $n$-simplices of $K$. If $G$ is an abelian group, the set of functions $\varphi : K^{(n)} \rightarrow G$ which evaluate to zero in all but finitely many $n$-simplices form an abelian group denoted $C^n(K; G)$, and referred to as the group of $n$-cochains of $K$ with coefficients in $G$. The coboundary of an $n$-cochain $\varphi \in C^n(K; G)$ is the element $\delta^n(\varphi) \in C^{n+1}(K; G)$ which operates on each $n+1$ simplex $\sigma = \{s_0, \ldots, s_{n+1} \}$ as

$$
\delta^n(\varphi)(\sigma) = \sum_{j=0}^{n+1} (-1)^j \varphi(\sigma \setminus \{s_j\})
$$

This defines a homomorphism $\delta^n : C^n(K; G) \rightarrow C^{n+1}(K; G)$ that, as can be checked, satisfies $\delta^{n+1} \circ \delta^n = 0$ for all $n \in \mathbb{Z}_{\geq 0}$. The group of $n$-cocycles $Z^n(K; G)$ is the kernel of $\delta^n$, the group of $n$-coboundaries $B^n(K; G) = \text{Img}(\delta^{n-1})$ is therefore a subgroup of $Z^n(K; G)$, and the $n$-th cohomology group of $K$ with coefficients in $G$ is defined as the quotient

$$
H^n(K; G) = Z^n(K; G) / B^n(K; G).
$$

If $K$ is a field then $H^n(K; \mathbb{K})$ is in fact a vector space over $\mathbb{K}$.

A filtered simplicial complex is a collection $K = \{ K_\ell \}_{\ell \geq 0}$ of abstract simplicial complexes so that $K_0 = \emptyset$ and $K_\ell \subset K_{\ell+1}$ whenever $\ell \leq \ell'$. If $\epsilon_0 = \epsilon_1 < \cdots$ is a discretization of $\mathbb{R}_{\geq 0}$, then for each field $\mathbb{K}$ and $n \in \mathbb{Z}_{\geq 0}$ one obtains the diagram of $\mathbb{K}$-vector spaces and linear transformations

$$
H^n(K_{\epsilon_\ell}; \mathbb{K}) \xleftarrow{T_{\epsilon_\ell}} H^n(K_{\epsilon_{\ell+1}}; \mathbb{K}) \xleftarrow{T_{\epsilon_{\ell+1}}} \cdots \xleftarrow{T_{\epsilon_0}} H^n(K_{\epsilon_0}; \mathbb{K}) \xleftarrow{T_{\epsilon_0}} \cdots \quad (4)
$$

where $T_\ell : H^n(K_{\epsilon_\ell}; \mathbb{K}) \rightarrow H^n(K_{\epsilon_{\ell+1}}; \mathbb{K})$ is given by

$$
T_\ell ([\varphi]) = [\varphi|_{K_{\epsilon_{\ell+1}}^{(n)}}].
$$

If each $H^n(K_{\epsilon_\ell}; \mathbb{K})$ is finite dimensional and for all $\ell$ large enough $T_\ell$ is an isomorphism, we say that (4) is of finite type.

The Basis Lemma [16, Section 3.4] implies that when (4) is of finite type one can choose a basis $V_\ell = \{ v_1^\ell, \ldots, v_d^\ell \}$ for each $H^n(K_{\epsilon_\ell}; \mathbb{K})$ so that the following compatibility condition holds: $T_\ell(V_\ell^r) \subset V_{\ell+1}^{r-1} \cup \{0\}$ for all $r \in \mathbb{Z}_{\geq 0}$, and if $T_\ell(v_{\ell}^r) = T_{\ell+1}(v_{\ell+1}^r) \neq 0$ then $\ell = m$. The set

$$
V = \bigcup_{r \in \mathbb{Z}_{\geq 0}} V_\ell^r
$$

can be endowed with a partial order $\preceq$ where $v_{m}^j \preceq v_{m}^j$ if and only if $r \geq j$ and $v_{m}^j = T_{j+1} \circ \cdots \circ T_\ell(v_{\ell}^r)$. The maximal chains in $(V, \preceq)$ are pairwise disjoint, and...
hence represent independent cohomological features of the complexes \( K_\epsilon \) which are stable with respect to changes in \( \epsilon \). These are called persistent cohomology classes. A maximal chain of finite length

\[
v^j_m \preceq v^{j+1}_k \preceq \cdots \preceq v^r_\ell
\]

yields the interval \([\epsilon_j, \epsilon_r]\), while an infinite maximal chain \( v^j_m \preceq v^{j+1}_k \preceq \cdots \) yields the interval \([\epsilon_j, \infty)\). This is meant to signify that there is a class which starts (is born) at the cohomology group corresponding to the right end-point of the interval, here \( \epsilon_r \) or \( \infty \). This class, in turn, is mapped to zero (it dies) leaving the cohomology group for the left end-point, here \( \epsilon_j \), but not before. The multi-set of all such intervals (as several chains might yield the same interval) is independent of the choice of compatible bases \( V^r \), and can be visualized as a barcode:

![Exemplary barcode](image)

**Figure 5.** Exemplary barcode

Details on the computation of persistent cohomology, and its advantages over persistent homology, can be found in [11].

3. **Explicit Classifying Maps**

The goal of this section is to derive equation (1), which is in fact an specialization of the proof of Theorem 2.1 to the case of line bundles over metric spaces with finite trivializing covers. When a metric is given, the partition of unity involved in the argument can be described explicitly in terms of bump functions supported on metric balls. Moreover, the local trivializations used in the proof can be replaced by transition functions which – as we will see in sections 4 and 7 – can be calculated in a robust multi-scale manner from the persistent cohomology of an appropriate sparse filtration. From this point on, all topological spaces are assumed to be paracompact and Hausdorff.

**Classifying maps in terms of local trivializations.** Let us sketch the proof of existence in Theorem 2.1 when \( B \) has a finite trivializing cover. Starting with local trivializations

\[
\rho_r : U_r \times \mathbb{P}^k \xrightarrow{\cong} p^{-1}(U_r) \quad r = 0, \ldots, d
\]

for the vector bundle \( \zeta = (E, B, p) \), let \( \mu_r : p^{-1}(U_r) \to \mathbb{P}^k \) be \( \mu_r(\rho_r(b, v)) = v \) for all \((b, v) \in U_r \times \mathbb{P}^k\).
Definition 3.1. A collection of continuous maps $\varphi_r : U_r \to \mathbb{R}_{\geq 0}$ is called a partition of unity dominated by $\mathcal{U} = \{U_r\}$ if
\[
\sum_r \varphi_r = 1 \quad \text{and} \quad \text{support}(\varphi_r) \subset \text{closure}(U_r)
\]

Notice that this notion differs from the usual partition of unity subordinated to a cover in that supports need not be contained in the open sets. However, this is enough for our purposes. Notice that since for paracompact spaces there is always a partition of unity subordinated to a given cover, the same is true in the dominated case.

Let $\psi : [0, 1] \to [0, 1]$ be any homeomorphism so that $\psi(0) = 0$ and $\psi(1) = 1$. If $\{\varphi_r\}$ is a partition of unity dominated by the trivializing cover from equation (5) and we let $\psi_r = \psi \circ \varphi_r$, then each $\mu_r : p^{-1}(U_r) \to \mathbb{F}^k$ yields a fiberwise linear map $\bar{\mu}_r : E \to \mathbb{F}^k$ given by
\[
\bar{\mu}_r(e) = \begin{cases} 
\psi_r(p(e)) \cdot \mu_r(e) & \text{if } p(e) \in U_r \\
0 & \text{if } p(e) \notin U_r
\end{cases}
\]

Thus one has a continuous map
\[
\bar{\mu} : E \to \mathbb{F}^k \oplus \cdots \oplus \mathbb{F}^k \\
e \mapsto [\bar{\mu}_0(e), \ldots, \bar{\mu}_d(e)]
\]
which, as can be checked, is linear and injective on each fiber. It follows from [26, Lemma 3.1] that any such fiber-wise linear embedding induces a continuous map
\[
f_\zeta : B \to \text{Gr}_k (\mathbb{F}^{k(d+1)}) \\
b \mapsto \bar{\mu}(p^{-1}(b))
\]
which satisfies $f_\zeta(\gamma_{\bar{\mu}_m}) \cong \zeta$, if $m = k(d + 1)$. This completes the sketch of the proof, but let us now describe $f_\zeta$ more explicitly and in terms of objects that can be computed.

Classifying maps from transition functions. Fix $b \in B$ and let $0 \leq j \leq k$ be so that $b \in U_j$. If $\{v_1, \ldots, v_k\}$ is a basis for $\mathbb{F}^k$ then $\{\rho_j(b, v_s) \mid s = 1, \ldots, k\}$ is a basis for $p^{-1}(b)$ and therefore
\[
f_\zeta(b) = \text{Span}_{\mathbb{F}} \{\bar{\mu}(\rho_j(b, v_s)) : s = 1, \ldots, k\} \tag{6}
\]
where
\[
\bar{\mu}(\rho_j(b, v_s)) = [\bar{\mu}_0(\rho_j(b, v_s)), \ldots, \bar{\mu}_d(\rho_j(b, v_s))] \in \mathbb{F}^{k(d+1)}
\]
If $\{\omega_{rl} : U_r \cap U_l \to \text{GL}_k(\mathbb{F})\}$ is the collection of transition functions for $\zeta$ associated to the system of local trivializations $\{\rho_r\}$, then whenever $U_r \cap U_l \neq \emptyset$ we have the commutative diagram
\[
(U_r \cap U_l) \times \mathbb{F}^k \xrightarrow{(b, v)} (b \cdot \omega_{rl}(b)^{-1}(v)) \\
\xrightarrow{\rho_r} p^{-1}(U_r \cap U_l) \xrightarrow{\rho_l} (U_l \cap U_i) \times \mathbb{F}^k
\]
Let $0 \leq l \leq d$. If $b \in U_j \cap U_l$ then $\bar{\mu}_l(\rho_l(b, v_s)) = 0$, otherwise $b \in U_j \cap U_l$ and
\[
\bar{\mu}_l(\rho_l(b, v_s)) = \bar{\mu}_l(\rho_l(b, \omega_{jl}(b)^{-1}(v_s))) \\
= \psi_l(b) \cdot \omega_{jl}(b)(v_s)
\]
Putting this calculation together with Equation (6), it follows that for \( b \in U_j \)
\[
 f_\xi(b) = \text{Span}_F \left \{ \left[ \psi_0(b) \cdot \omega_{0j}(b)(v_s), \ldots, \psi_d(b) \cdot \omega_{dj}(b)(v_s) \right] : s = 1, \ldots, k \right \} \quad (7)
\]

**The case of line bundles.** If \( k = 1 \) we can take \( v_1 = 1 \in F \), and abuse notation by writing \( \omega_{rt}(b) \in \mathbb{F}^\times \) instead of \( \omega_{rt}(b)(1) \). Moreover, in this case we have \( \text{Gr}_k(\mathbb{F}^{(d+1)}) = \mathbb{F}P^d \), and if we use homogeneous coordinates and \( \psi(x) = \sqrt{x} \) then \( f_\xi : B \rightarrow \mathbb{F}P^d \) can be expressed locally (i.e. on each \( U_j \)) as
\[
 f_\xi(b) = \left[ \omega_{0j}(b) \cdot \sqrt{\varphi_0(b)} : \cdots : \omega_{dj}(b) \cdot \sqrt{\varphi_d(b)} \right] , \quad b \in U_j
\]
The choice \( \psi(x) = \sqrt{x} \) is so that when the transition functions \( \omega_{rt} \) are unitary, i.e. \( |\omega_{rt}(b)| = 1 \), then the formula above without homogeneous coordinates produces a representative of \( f_\xi(b) \) on the unit sphere of \( \mathbb{F}^{d+1} \). We summarize the results thus far in the following theorem:

**Theorem 3.2.** Let \( B \) be a topological space and let \( \mathcal{U} = \{ U_r \}_{r=0}^d \) be an open cover. If \( \{ \varphi_r \} \) is a partition of unity dominated by \( \mathcal{U} \), \( F = \mathbb{R}, \mathbb{C} \), and
\[
 \omega = \{ \omega_{rt} : U_r \cap U_t \rightarrow \mathbb{F}^\times \}
\]
is a collection of continuous maps satisfying the cocycle condition (3), then the map \( f_\omega : B \rightarrow \mathbb{F}P^d \) given in homogenous coordinates by
\[
 f_\omega(b) = \left[ \omega_{0j}(b) \cdot \sqrt{\varphi_0(b)} : \cdots : \omega_{dj}(b) \cdot \sqrt{\varphi_d(b)} \right] , \quad b \in U_j \quad (8)
\]
is well defined and classifies the \( \mathbb{F} \)-line bundle \( \zeta_\omega \) induced by \( (\mathcal{U}, \omega) \).

**Line bundles over metric spaces.** When \( B \) comes equipped with a metric \( d \), equation (8) can be further specialized to a covering via open balls, and a dominated partition of unity constructed from bump functions supported on the closure of each ball. Indeed, let
\[
 B_{r, \ell} = \{ b \in B : d(b, \ell_r) < \epsilon_r \} , \quad r = 0, \ldots, d
\]
for some collection \( \{ \ell_0, \ldots, \ell_d \} \subset B \) and radii \( \epsilon_r > 0 \).

**Proposition 3.3.** Let \( (B, d) \) be a metric space and let \( \mathcal{B} = \{ B_{r, \ell} \}_{r=0}^d \) be an open cover. If \( \phi : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0} \) is a continuous map so that \( \phi(0) = \mathbb{R}_{\leq 0} \), and \( \lambda_0, \ldots, \lambda_d \in \mathbb{R}_{\geq 0} \) is a set of weights, then
\[
 \varphi_r(b) = \frac{\lambda_r \cdot \phi \left( 1 - \frac{d(b, \ell_r)}{\epsilon_r} \right)}{\sum_{i=0}^d \lambda_i \cdot \phi \left( 1 - \frac{d(b, \ell_i)}{\epsilon_i} \right)} , \quad r = 0, \ldots, d
\]
is a partition of unity for \( B \) dominated by \( \mathcal{B} \).

Due to the shape of its graph, the map \( b \mapsto \lambda_r \cdot \phi \left( 1 - \frac{d(b, \ell_r)}{\epsilon_r} \right) \) is often referred to as a bump function supported on \( B_r \). The height of the bump is controlled by the weight \( \lambda_r \), while its overall shape is captured by the function \( \phi \). Of course one can choose different functions \( \phi_r \) on each ball, for instance to capture local density if \( B \) comes equipped with a measure. Some examples of bump-shapes are:

- **Triangular:** The positive part of \( x \in \mathbb{R} \) is defined as \( |x|_+ = \max\{x, 0\} \), and \( b \mapsto \lambda \cdot \left| 1 - \frac{d(b, \ell)}{\epsilon} \right|_+ \) is the associated triangular bump supported on \( B_r \).
• **Polynomial:** The polynomial bump with exponent $p > 0$ is induced by the function $\phi(x) = |x|^p$. The triangular bump is recovered when $p = 1$, while $p = 2$ yields the quadratic bump.

• **Gaussian:** The Gaussian bump is induced by the function $\phi(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$

• **Logarithmic:** Is the one associated to $\phi(x) = \log(1 + |x|)$

The next figure shows some of these bump functions, with weight $\lambda = 1$, for $B_1(0)$.

![Figure 6. Examples of bump functions supported on $B_1(0) \subset \mathbb{R}$](image)

Choosing $\phi(x) = |x|^p$ and the weights as $\lambda_r = \epsilon_r^2$ simplifies Theorem 3.2 to:

**Corollary 3.4.** Let $(B, d)$ be a metric space and $B = \{ B_r = B_{\epsilon_r}(\ell_r) \}_{r=0}^d$ a covering. If $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and $\omega = \{ \omega_{r,t} : B_r \cap B_t \to \mathbb{F}^x \}$ are continuous maps satisfying the cocycle condition, then $f_\omega : B \to \mathbb{F}P^d$ given in homogeneous coordinates by

$$f_\omega(b) = \left[ \omega_{0,j}(b) \cdot |\ell_0 - d(b, \ell_0)|^+ \cdots : \omega_{d,j}(b) \cdot |\ell_d - d(b, \ell_d)|^+ \right] , \quad b \in B_j$$

is well defined and classifies the line bundle $\zeta_\omega$ induced by $(B, \omega)$.

**Geometric Interpretation.** Let us clarify equation (8) for the case of constant transition functions and $\mathbb{F} = \mathbb{R}$. If $\mathcal{U} = \{ U_0, \ldots, U_d \}$ is a cover of $B$, then the nerve of $\mathcal{U}$, denoted $\mathcal{N}(\mathcal{U})$, is the abstract simplicial complex with one vertex for each open set $U_r \in \mathcal{U}$, and a simplex $\{ r_0, \ldots, r_k \}$ for each collection $U_{r_0}, \ldots, U_{r_k} \in \mathcal{U}$ such that $U_{r_0} \cap \cdots \cap U_{r_k} \neq \emptyset$. Given a geometric realization $|\mathcal{N}(\mathcal{U})| \subset \mathbb{R}^d$ let $v_r \in |\mathcal{N}(\mathcal{U})|$ be the point corresponding to the vertex $r \in \mathcal{N}(\mathcal{U})$. Each $x \in |\mathcal{N}(\mathcal{U})|$ is then uniquely determined by (and uniquely determines) its barycentric coordinates: a sequence $\{ x_r \}$ of real numbers between 0 and 1, one for each open set $U_r \in \mathcal{U}$, so that

$$\sum_r x_r = 1 \quad \text{and} \quad \sum_r x_r v_r = x$$

To see this, notice that given a non-vertex $x \in |\mathcal{N}(\mathcal{U})|$ there exists a unique maximal geometric simplex $\sigma$ of $|\mathcal{N}(\mathcal{U})|$ so that $x$ is in the interior of $\sigma$. If $v_{r_0}, \ldots, v_{r_k} \in |\mathcal{N}(\mathcal{U})|$ are the vertices of $\sigma$, then $x$ can be expressed uniquely as a convex combination of $v_{r_0}, \ldots, v_{r_k}$ which determines $x_{r_0}, \ldots, x_{r_k}$. If $r \neq r_0, \ldots, r_k$ then we let $x_r = 0$. 

Please refer to an online version for colors.
A partition of unity \( \{ \varphi_r \} \) dominated by \( \mathcal{U} \) induces a continuous map

\[
\varphi : B \rightarrow |N(\mathcal{U})| \\
\quad b \mapsto \sum_r \varphi_r(b)v_r
\]

That is, \( \varphi \) sends \( b \) to the point \( \varphi(b) \) with barycentric coordinates \( \varphi_r(b) \). Moreover, if \( \{ \omega_{rt} : U_r \cap U_t \rightarrow \{-1, 1\} \} \) is a collection of constant functions satisfying the cocycle condition, then the associated classifying map \( f_\omega : B \rightarrow \mathbb{R}P^d \) from Theorem 3.2 can be decomposed as

\[
B \xrightarrow{\varphi} |N(\mathcal{U})| \xrightarrow{F_\omega} \mathbb{R}P^d
\]

where \( F_\omega : |N(\mathcal{U})| \rightarrow \mathbb{R}P^d \), in barycentric coordinates, is given on the open star of a vertex \( v_j \in |N(\mathcal{U})| \) as

\[
F_\omega(x_0, \ldots, x_d) = [\omega_{0j} \sqrt{x_0} : \cdots : \omega_{dj} \sqrt{x_d}]
\]

Example: Let \( B = S^1 \), the unit circle, and let \( \mathcal{U} = \{U_0, U_1, U_2\} \) be the open covering depicted in figure 3(left).

![Diagram](image)

Figure 7. An open covering of the circle and the resulting nerve complex.

Define \( \omega_{rt} = 1 \) for \( r, t = 0, 1, 2 \); let \( \omega_{02} = -1 \) and let \( \omega_{rt} = 1 \) for all \( (r, t) \neq (0, 2), (2, 0) \). Let \( (x_0, x_1, x_2) \) denote the barycentric coordinates of a point in \( |N(\mathcal{U})| \). For instance, the vertex labeled as 0 has coordinates \((1/2, 1/2, 0)\) and the midpoint of the edge \( \{0, 1\} \) has coordinates \((1/2, 0, 0)\). Then

\[
F_\omega(x_0, x_1, x_2) = [\sqrt{x_0} : \sqrt{x_1} : \sqrt{x_2}]
\]

and for each \( 0 \leq x \leq 1 \)

\[
F_\omega(x, 1-x, 0) = [\sqrt{x} : \sqrt{1-x} : 0] \\
F_\omega(x, 0, 1-x) = [\sqrt{x} : 0 : -\sqrt{1-x}] \\
F_\omega(0, x, 1-x) = [0 : \sqrt{x} : \sqrt{1-x}]
\]

We show in figure 3 how \( F_\omega \) and \( F_{\omega'} \) map \( |N(\mathcal{U})| \) to \( \mathbb{R}P^2 = S^2/\{x \sim -x\} \).

If \( d \) is the geodesic distance on \( S^1 \) then each arc \( U_r \) is an open ball \( B_{\epsilon_r}(\ell_r) \) for some \( (\epsilon_r, \ell_r) \in \mathbb{R}^+ \times S^1 \). When \( \varphi : S^1 \rightarrow |N(\mathcal{U})| \) is induced by the partition of unity from the triangular bumps \( \phi(x) = |x| \), then \( \varphi \) maps each arc \( \{\ell_r, \ell_t\} \) linearly onto the edge \( |\{r, t\}| \). It is not hard to see that the \( \mathbb{R} \)-line bundle induced by \( \{\omega_{rt}\} \)
is trivial, while the one induced by \( \{ \omega'_r \} \) is the nontrivial bundle on \( S^1 \) having the Moebius band as total space. This is captured by \( f_\omega = F_\omega \circ \varphi : S^1 \to \mathbb{RP}^2 \) being null-homotopic and \( f_{\omega'} = F_{\omega'} \circ \varphi \) representing the non-trivial element in the fundamental group \( \pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2 \).

4. Transition Functions from Simplicial Cohomology

Let \( \mathcal{U} = \{ U_r \} \) be a cover for \( B \). We have shown thus far that given a collection \( \omega \) of continuous maps \( \omega_{rs} : U_r \cap U_t \to GL_1(\mathbb{F}) \cong \mathbb{F}^\times \), \( U_r \cap U_s \neq \emptyset \) satisfying the cocycle condition, one can explicitly write down (given a dominated partition of unity) a classifying map \( f_\omega : B \to \mathbb{FP}^d \) for the associated \( \mathbb{F} \)-line bundle \( \zeta_\omega \). What we will see next is that determining such transition functions can be reduced to a computation in simplicial cohomology.

Formulation in Terms of Sheaf Cohomology. If \( \mathcal{C}_\mathbb{F}^\times \) denotes the sheaf of continuous \( \mathbb{F}^\times \)-valued functions on \( B \) then \( \omega \in \check{\mathcal{C}}^1(\mathcal{U}; \mathcal{C}_\mathbb{F}^\times) \). Moreover, since \( \omega \) satisfies the cocycle condition then \( \omega \in \check{Z}^1(\mathcal{U}; \mathcal{C}_\mathbb{F}^\times) \), and hence we can consider the cohomology class \( [\omega] \in \check{H}^1(\mathcal{U}; \mathcal{C}_\mathbb{F}^\times) \). If \( \mathcal{V} \) is another covering of \( B \), we say that \( \mathcal{V} \) is a refinement of \( \mathcal{U} \), denoted \( \mathcal{V} \prec \mathcal{U} \), if for every \( V \in \mathcal{V} \) there exists \( U \in \mathcal{U} \) so that \( V \subset U \). A standard result (Exercise 6.2, [1]) is the following:

**Theorem 4.1.** If \( \omega, \omega' \in \check{Z}^1(\mathcal{U}; \mathcal{C}_\mathbb{F}^\times) \) are cohomologous then \( \zeta_\omega \cong \zeta_{\omega'} \). Moreover, the function

\[
\check{H}^1(\mathcal{U}; \mathcal{C}_\mathbb{F}^\times) \to \text{Vec}^1_\mathbb{F}(B)
\]

\[
[\omega] \mapsto [\zeta_\omega]
\]

is an injective homomorphism, which is natural with respect to refinements of \( \mathcal{U} \).

Here natural means that if \( H^d_{\mathcal{U}} : \check{H}^1(\mathcal{U}; \mathcal{C}_\mathbb{F}^\times) \to \check{H}^1(\mathcal{V}; \mathcal{C}_\mathbb{F}^\times) \) is the homomorphism induced by the refinement \( \mathcal{V} \prec \mathcal{U} \) (see [27], Chapter IX, Lemma 3.10) then the diagram

\[
\begin{array}{ccc}
\check{H}^1(\mathcal{U}; \mathcal{C}_\mathbb{F}^\times) & \longrightarrow & \text{Vec}^1_\mathbb{F}(B) \\
\downarrow & & \\
\check{H}^1(\mathcal{V}; \mathcal{C}_\mathbb{F}^\times) & \longrightarrow & \text{Vec}^1_\mathbb{F}(B)
\end{array}
\]

is commutative. Combining this with equation (2) yields
Corollary 4.2. Let $\mathcal{U}$ be an open covering of $B$. The function
\[
\Gamma_{\mathcal{U}} : \check{H}^1(\mathcal{U}; \mathbb{C}^\times) \rightarrow [B, \mathbb{F}\mathbb{P}^\infty]
\]
\[
[\omega] \mapsto [f_\omega]
\]
is injective, and natural with respect to refinements of $\mathcal{U}$.

The main point of this section is that the sheaf cohomology group $\check{H}^1(\mathcal{U}; \mathbb{C}^\times)$ can be replaced, under suitable conditions, by appropriate simplicial cohomology groups: $H^1(N(\mathcal{U}); \mathbb{Z}/2)$ when $\mathbb{F} = \mathbb{R}$ and $H^2(N(\mathcal{U}); \mathbb{Z})$ when $\mathbb{F} = \mathbb{C}$. We will not assume that $\mathcal{U}$ is a good cover (i.e. that each finite intersection $U_{r_0} \cap \ldots \cap U_{r_k}$ is either empty or contractible), but rather will phrase the reduction theorems in terms of the relevant connectivity conditions.

**Reduction to Simplicial Cohomology.** If $\tau \in Z^1(N(\mathcal{U}); \mathbb{Z}/2)$ then for each $U_r \cap U_t \neq \emptyset$ we have $\tau(\{r, t\}) = \tau_{rt} \in \{0, 1\} = \mathbb{Z}/2$. Let $\phi^\tau = \{\phi^\tau_{rt}\}$ be the collection of constant functions $\phi^\tau_{rt} : U_r \cap U_t \rightarrow \mathbb{R}^\times$.

Therefore each $\phi^\tau_{rt}$ is continuous, so $\phi^\tau \in \check{C}^1(\mathcal{U}; \mathbb{C}^\times)$, and since $\tau$ is a cocycle it follows that $\phi^\tau \in Z^1(\mathcal{U}; \mathbb{C}^\times)$. Moreover, the association $\tau \mapsto \phi^\tau$ induces the homomorphism
\[
\Phi_{\mathcal{U}} : H^1(N(\mathcal{U}); \mathbb{Z}/2) \rightarrow \check{H}^1(\mathcal{U}; \mathbb{C}^\times)
\]
\[
[\tau] \mapsto [\phi^\tau]
\]
which is well-defined and satisfies:

**Proposition 4.3.**
\[
\Phi_{\mathcal{U}} : H^1(N(\mathcal{U}); \mathbb{Z}/2) \rightarrow \check{H}^1(\mathcal{U}; \mathbb{C}^\times)
\]
is natural with respect to refinements. Moreover, if each $U_r$ is connected then $\Phi_{\mathcal{U}}$ is injective.

**Proof.** Fix $\tau \in Z^1(N(\mathcal{U}); \mathbb{Z}/2)$ and assume that there is a collection of continuous maps $\{f_r : U_r \rightarrow \mathbb{R}^\times\}$ for which $(-1)^{\tau_{rt}} = \frac{1 - \text{sign}(f_r)}{2}$ on $U_r \cap U_t \neq \emptyset$. If each element of $\mathcal{U}$ is connected then the $f_r$’s have constat sign (either $+1$ or $-1$) in their domains, and hence we can define $\nu \in C^0(N(\mathcal{U}); \mathbb{Z}/2)$ as
\[
\nu(\{r\}) = \frac{1 - \text{sign}(f_r)}{2}
\]
Therefore $\tau(\{r, t\}) = \delta(\nu)(\{r, t\})$ and the result follows. \hfill $\Box$

Define $w^\mathcal{U}$ as the composition
\[
w^\mathcal{U} : H^1(N(\mathcal{U}); \mathbb{Z}/2) \xrightarrow{\Phi_{\mathcal{U}}} \check{H}^1(\mathcal{U}; \mathbb{C}^\times) \xrightarrow{\Gamma_{\mathcal{U}}} [B, \mathbb{R}\mathbb{P}^\infty]
\]

**Corollary 4.4.** If each $U_r$ is connected then
\[
w^\mathcal{U}_1 : H^1(N(\mathcal{U}); \mathbb{Z}/2) \rightarrow [B, \mathbb{R}\mathbb{P}^\infty]
\]
is natural and injective.
Now the complex case. Let $\sigma \in Z^2(N(U); \mathbb{Z})$ and let $\psi^\sigma = \{\psi^\sigma_{rst}\}$ be the collection of constant functions

$$\psi^\sigma_{rst} : U_r \cap U_s \cap U_t \rightarrow \mathbb{Z} \quad y \mapsto \sigma_{rst}$$

Each $\psi^\sigma_{rst}$ is in particular locally constant, and therefore $\psi^\sigma \in \check{Z}^2(U; \mathbb{Z})$. It follows that the association $\sigma \mapsto \psi^\sigma$ induces a homomorphism

$$\Psi_U : H^2(N(U); \mathbb{Z}) \rightarrow \check{H}^2(U; \mathbb{Z})$$

$$[\sigma] \mapsto [\psi^\sigma]$$

which is well defined and satisfies:

**Proposition 4.5.**

$$\Psi_U : H^2(N(U); \mathbb{Z}) \rightarrow \check{H}^2(U; \mathbb{Z})$$

is natural with respect to refinements. Moreover, if each $U_r \cap U_t$ is either empty or connected then $\Psi_U$ is injective.

**Proof.** The result is deduced from the following observation: if $U_r \cap U_t$ is connected, then any function $\mu_{rt} : U_r \cap U_t \rightarrow \mathbb{Z}$ which is locally constant is in fact constant. $\square$

We will now link $\check{H}^2(U; \mathbb{Z})$ and $\check{H}^1(U; \mathbb{C}_C^\times)$ using the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C}_C \xrightarrow{\exp} \mathbb{C}_C^\times \rightarrow 0$$

which is given at the level of open sets $U \in \mathcal{U}$ by

$$\mathbb{Z}(U) \rightarrow \mathbb{C}_C(U) \xrightarrow{\exp} \mathbb{C}_C^\times(U)$$

$$\eta \mapsto \eta \quad f \mapsto \exp\{2\pi if\}$$

If $\Im(\exp)$ denotes the image presheaf

$$\Im(\exp)(U) = \text{Img} \left\{ \mathbb{C}_C(U) \xrightarrow{\exp} \mathbb{C}_C^\times(U) \right\}$$

then

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C}_C \xrightarrow{\exp} \Im(\exp) \rightarrow 0$$

is a short exact sequence of presheaves (i.e. exact for every open set), and hence we get a long exact sequence in Čech cohomology [34, Section 24]

$$\cdots \rightarrow \check{H}^k(U; \mathbb{C}_C) \rightarrow \check{H}^k(U; \Im(\exp)) \xrightarrow{\Delta} \check{H}^{k+1}(U; \mathbb{Z}) \rightarrow \check{H}^{k+1}(U; \mathbb{C}_C) \rightarrow \cdots$$

Since $\mathbb{C}_C$ admits partitions of unity (i.e. it is a fine sheaf) then

**Lemma 4.6.** $\check{H}^k(U; \mathbb{C}_C) = 0$ for every $k \geq 1$.

It follows that $\Delta : \check{H}^1(U; \Im(\exp)) \rightarrow \check{H}^2(U; \mathbb{Z})$ is an isomorphism. Moreover,

**Theorem 4.7.** Let $\{\varphi_t\}$ be a continuous partition of unity dominated by $\mathcal{U}$. If $\eta = \{n_{rst}\} \in Z^2(N(U); \mathbb{Z})$ then

$$\omega_{rs} = \exp \left( 2\pi i \sum_t \varphi_t \cdot n_{rst} \right), \quad U_r \cap U_s \neq \emptyset$$
defines an element \( \omega = \{ \omega_{rs} \} \in \check{C}^1(U; \mathfrak{Im}(\exp)) \). Moreover, \( \omega \) is a \( \check{C} \)ech cocycle, and the composition

\[
H^2(N(U); \mathbb{Z}) \xrightarrow{\Psi_U} \check{H}^2(U; \mathbb{Z}) \xrightarrow{\Delta^{-1}} \check{H}^1(U; \mathfrak{Im}(\exp))
\]
satisfies \( \Delta^{-1} \circ \Psi_U([\eta]) = [\omega] \).

**Proof.** First we check that \( \omega \) is a \( \check{C} \)ech cocycle:

\[
\omega_{rs} \cdot \omega_{st} = \exp \left( 2\pi i \sum \phi_{t} \cdot (n_{r_{st}} + n_{r_{tt}}) \right)
\]

\[
= \exp \left( 2\pi i \sum \phi_{t} \cdot (n_{r_{tt}} + n_{r_{st}}) \right)
\]

\[
= \exp \left( 2\pi i \cdot n_{r_{st}} + \sum \phi_{t} \cdot n_{r_{tt}} \right)
\]

\[
= \omega_{rt}
\]

In order to see that \( \Delta([\omega]) = \Psi_U([\eta]) \), we use the definition of the connecting homomorphism \( \Delta \). First, we let \( g = \{ g_{rs} \} \in \check{C}^1(U; \mathfrak{C}) \) be the collection of functions

\[
g_{rs}(b) = \sum_{t} \phi_{t}(b) \cdot n_{r_{st}} \quad b \in U_{r} \cap U_{s}
\]

It follows that \( \omega = \exp(g) \) and therefore \( \Delta([\omega]) = [\delta^1(g)] \). The coboundary \( \delta^1(g) \) can be computed as

\[
\delta^1(g)_{rst} = g_{rs} - g_{rt} + g_{st}
\]

\[
= \sum_{t} \phi_{t} \cdot (n_{r_{st}} - n_{r_{tt}} + n_{s_{tt}})
\]

\[
= \sum_{t} \phi_{t} \cdot n_{r_{st}}
\]

\[
= n_{r_{st}}
\]

and therefore \( \Delta([\omega]) = \Psi_U([\eta]) \). \( \square \)

**Corollary 4.8.** If each \( U_{r} \cap U_{t} \) is either empty or connected then

\[
\Delta^{-1} \circ \Psi_U : H^2(N(U); \mathbb{Z}) \rightarrow H^1(U; \mathfrak{Im}(\exp)) \quad [\{n_{rst}\}] \mapsto [\{\omega_{rs}\}]
\]

is injective, where \( \omega_{rs} = \exp \left( 2\pi i \sum \phi_{r} \cdot n_{r_{st}} \right) \).

When going from \( \check{H}^1(U; \mathfrak{Im}(\exp)) \) to \( \check{H}^1(U; \mathfrak{C}) \) one considers the inclusion of presheaves \( j : \mathfrak{Im}(\exp) \rightarrow \mathfrak{C} \) and its induced homomorphism in cohomology

\[
j_\mathfrak{C} : \check{H}^1(U; \mathfrak{Im}(\exp)) \rightarrow \check{H}^1(U; \mathfrak{C})
\]

After taking direct limits over refinements of \( \mathcal{U} \), the resulting homomorphism is in fact an isomorphism [34, Proposition 7, section 25]. That is, each element in \( \ker(j_\mathfrak{C}) \) is also in the kernel of \( \check{H}^1(U; \mathfrak{Im}(\exp)) \rightarrow \check{H}^1(U; \mathfrak{C}) \) for some refinement \( \mathcal{V} \) of \( \mathcal{U} \); and for every element in \( \check{H}^1(U; \mathfrak{C}) \) there exists a refinement \( \mathcal{W} \) of \( \mathcal{U} \) so that the image of said element via \( \check{H}^1(U; \mathfrak{C}) \rightarrow \check{H}^1(U; \mathfrak{C}) \) is also in the image of \( \check{H}^1(U; \mathfrak{Im}(\exp)) \rightarrow \check{H}^1(U; \mathfrak{C}) \).
The situation is sometimes simpler. Recall that a topological space is said to be simply connected if it is path-connected and its fundamental group is trivial. In addition, it is said to be locally path-connected if each point has a path-connected open neighborhood.

**Lemma 4.9.** Let \( \mathcal{U} = \{ U_j \} \) be an open covering of \( B \) such that each \( U_r \) is locally path-connected and simply connected. Then

\[
\mathcal{J}_\mathcal{U} : \hat{H}^1(\mathcal{U}; \mathfrak{m}(\exp)) \rightarrow \hat{H}^1(\mathcal{U}; \mathbb{C}_\times^\times)
\]

is injective.

**Proof.** Let \( \{ \omega_{rt} \} \in \hat{H}^1(\mathcal{U}; \mathfrak{m}(\exp)) \) be an element in the kernel of \( \mathcal{J}_\mathcal{U} \). Then there exists a collection of continuous maps \( \nu_r : U_r \rightarrow \mathbb{C}_\times^\times \) so that \( \omega_{rt} = \nu_t \nu_r \) on \( U_r \cap U_t \neq \emptyset \). If we let

\[
p : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}_\times^\times \\
(\rho, \theta) \mapsto \rho \cdot e^{2\pi i \cdot \theta}
\]

then it follows that \( (\mathbb{R}_+ \times \mathbb{R}, p) \) is the universal cover for \( \mathbb{C}_\times^\times \). Moreover, since each \( U_r \) is locally path-connected and simply connected, then each \( \nu_r \) has a lift [18, Proposition 1.33]

\[
\tilde{\nu}_r : U_r \rightarrow \mathbb{R}_+ \times \mathbb{R} \\
b \mapsto (\rho_r(b), \theta_r(b))
\]

That is \( p \circ \tilde{\nu}_r(b) = \nu_r(b) \) for all \( b \in U_r \). Let \( \phi_r : U_r \rightarrow \mathbb{C} \) be defined as

\[
\phi_r(b) = \theta_r(b) - i \frac{\ln(\rho_r(b))}{2\pi}
\]

It follows that \( \{ \phi_r \} \in \hat{C}^0(\mathcal{U}; \mathbb{C}_\times^\times) \) and that for all \( b \in U_r \)

\[
\exp\left(2\pi i \cdot \phi_r(b)\right) = \exp\left(\ln(\rho_r(b)) + 2\pi i \cdot \theta_r(b)\right) = \rho_r(b) \cdot e^{2\pi i \cdot \theta_r(b)} = \nu_r(b)
\]

Therefore \( \nu_r = \exp(2\pi i \cdot \phi_r) \) and \( \{ \nu_r \} \in \hat{C}^0(\mathcal{U}; \mathfrak{m}(\exp)) \), which implies \( \{ \omega_{rt} \} = 0 \) in \( \hat{H}^1(\mathcal{U}; \mathfrak{m}(\exp)) \) as claimed. \( \square \)

In summary, given an open cover \( \mathcal{U} \) of \( B \) we get the function

\[
c^1_\ell : H^2(\mathcal{N}(\mathcal{U}); \mathbb{Z}) \xrightarrow{\Delta^{-1} \circ \Psi_{\mathcal{U}}} \hat{H}^1(\mathcal{U}; \mathfrak{m}(\exp)) \xrightarrow{\mathcal{J}_\mathcal{U}} \hat{H}^1(\mathcal{U}; \mathbb{C}_\times^\times) \xrightarrow{\Gamma_{\mathcal{U}}} [B, \mathbb{C}P^\infty]
\]

which is natural with respect to refinements and satisfies:

**Corollary 4.10.** Let \( \mathcal{U} = \{ U_j \} \) be an open cover of \( B \) such that each \( U_r \) is locally path-connected and simply connected, and each \( U_r \cap U_t \) is either empty or connected. Then

\[
c^1_\ell : H^2(\mathcal{N}(\mathcal{U}); \mathbb{Z}) \rightarrow [B, \mathbb{C}P^\infty]
\]

is injective.

We summarize the results of this section in the following lemma:
Lemma 4.11. Let $U = \{U_0, \ldots, U_d\}$ be an open cover of $B$, and let $\{\varphi_r\}$ be a partition of unity dominated by $U$. Then we have functions
\[
w^d_r : H^1(\mathcal{N}(U); \mathbb{Z}/2) \rightarrow \mathbb{R}P^d, \quad c^d_r : H^2(\mathcal{N}(U); \mathbb{Z}) \rightarrow \mathbb{C}P^d
\]
for each $r \in \{r_1, \ldots, r_d\}$. Moreover, if each $U_r$ is either empty or connected then $w^d_r$ is well defined. Moreover, if each $U_r$ is locally path-connected and simply connected, and each $U_r \cap U_t$ is either empty or connected then $c^d_r$ is injective.

5. Dimensionality Reduction in $\mathbb{F}P^d$ via Principal Projective Coordinates

Let $V \subset \mathbb{F}^{d+1}$ be a linear subspace with $\dim(V) \geq 1$. If $\sim$ is the equivalence relation on $\mathbb{F}^{d+1} \setminus \{0\}$ given by $\mathbf{u} \sim \mathbf{v}$ if and only if $\mathbf{u} = \lambda \mathbf{v}$ for some $\lambda \in \mathbb{F}$, then it follows that $\sim$ is also an equivalence relation on $V \setminus \{0\}$ and hence we can define
\[
\mathbb{F}P^\dim(V)-1 := (V \setminus \{0\}) / \sim
\]
Notice that $\mathbb{F}P^\dim(U)-1 = \mathbb{F}P^\dim(V)-1$ if and only if $U = V$, and that $\mathbb{F}P^\dim(V)-1$ is a subset of $\mathbb{F}P^d$. Recall that $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, and for $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{d+1}$ let
\[
\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{r=0}^d u_r \cdot v_r
\]
be their inner product. If $d_g$ denotes the geodesic distance in $\mathbb{F}P^d$ induced by the Fubini-Study metric, then one has that
\[
d_g([\mathbf{u}], [\mathbf{v}]) = \arccos \left( \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\| \mathbf{u} \| \cdot \| \mathbf{v} \|} \right)
\]
and it follows that $\mathbb{F}P^\dim(V)-1$ is an isometric copy of $\mathbb{F}P^\dim(V)-1$ inside $\mathbb{F}P^d$.

If $1 \leq \dim(V) \leq d$ and $V^\perp = \{\mathbf{u} \in \mathbb{F}^{d+1} : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{v} \in V\}$ then the orthogonal projection $p_V : \mathbb{F}^{d+1} \rightarrow V$ descends to a continuous map
\[
P_V : \mathbb{F}P^d \setminus \mathbb{F}P^\dim(V) \rightarrow \mathbb{F}P^\dim(V)-1
\]
with $\| u \| \cdot \| v \|$ replaced by $\| \cdot \|$.

Recall that $p_V$ sends each $\mathbf{u} \in \mathbb{F}^{d+1}$ to its closest point in $V$ with respect to the distance induced by $\| \cdot \|$. A similar property is inherited by $P_V$:

Proposition 5.1. Let $[\mathbf{w}] \in \mathbb{F}P^d \setminus \mathbb{F}P^\dim(V)$ be arbitrary. Then $P_V([\mathbf{w}])$ is the point in $\mathbb{F}P^\dim(V)-1$ which is closest to $[\mathbf{w}]$ with respect to $d_g$. 

Proof. Let \([u] \in \mathbb{F}P_{\dim(V)}^{d-1}\). Since \([w] \notin \mathbb{F}P_{\dim(V)}^{d-1}\) then \(w - p_V(w) \in V^\perp\) with \(p_V(w) \neq 0\). Therefore \(\langle u, w - p_U(w) \rangle = 0\) and by the Cauchy-Schwartz inequality
\[
|\langle w, u \rangle| = |\langle p_U(w), u \rangle| \leq \|p_U(w)\| \cdot \|u\|
\]
Hence
\[
\frac{|\langle w, u \rangle|}{\|w\| \cdot \|u\|} \leq \frac{|\langle p_U(w), u \rangle|}{\|w\| \cdot \|p_U(w)\|} = \frac{\langle w, p_U(w) \rangle}{\|w\| \cdot \|p_U(w)\|}
\]
and since \(\arccos(\alpha)\) is decreasing then \(d_g([w], p_U([w])) \leq d_g([w], [u]).\)
\(\square\)

Therefore, we can think of \(P_V\) as the projection onto \(\mathbb{F}P_{\dim(V)}^{d-1}\). Moreover, let \(j_V : \mathbb{F}P_{\dim(V)}^{d-1} \hookrightarrow \mathbb{F}P^d \setminus \mathbb{F}P_{\dim(V)}^{d-1}\) be the inclusion map.

**Proposition 5.2.** \(j_V \circ P_V\) is a deformation retraction.

Proof. Since \(p_V\) is surjective and satisfies \(p_V \circ p_V(w) = p_V(w)\) for all \(w \in \mathbb{F}^{d+1}\), it follows that \(P_V\) is a retraction. Let \(h : \mathbb{F}^{d+1} \times [0, 1] \rightarrow \mathbb{F}^{d+1}\) be given by \(h(w, t) = (1 - t) \cdot w + t \cdot p_V(w)\). Since \(h(w, t) = 0\) implies that \(w \in V^\perp\), then \(h\) induces a continuous map
\[
\left( \mathbb{F}P^d \setminus \mathbb{F}P_{\dim(V)}^{d-1} \right) \times [0, 1] \rightarrow \mathbb{F}P^d \setminus \mathbb{F}P_{\dim(V)}^{d-1}
\]
\[
([w], t) \mapsto [h(w, t)]
\]
which is a homotopy between the identity of \(\mathbb{F}P^d \setminus \mathbb{F}P_{\dim(V)}^{d-1}\) and \(j_V \circ P_V\). \(\square\)

Notice that \([u] = \text{Span}(u)\) for \(u \in \mathbb{F}^{d+1} \setminus \{0\}\). The previous proposition yields

**Corollary 5.3.** Let \(f : B \rightarrow \mathbb{F}P^d\) be a continuous map which is not surjective. If \([u] \notin f(B)\) then \(f\) is homotopic to \(P_{[u]} \circ f : B \rightarrow \mathbb{F}P_{[u]}^{d-1} \hookrightarrow \mathbb{F}P^d\).

In summary, if \(f : B \rightarrow \mathbb{F}P^d\) is not surjective then it can be continuously deformed so that its image lies in \(\mathbb{F}P_{[u]}^{d-1} \subset \mathbb{F}P^d\), where \([u] \notin f(B)\). Moreover, the deformation is obtained by sending each \(f(b) \in \mathbb{F}P^d\) to its closest point in \(\mathbb{F}P_{[u]}^{d-1}\) with respect to \(d_g\), along a shortest path in \(\mathbb{F}P^d\). In particular, this implies that the topological properties encoded by \(f\) are preserved by the dimensionality reduction step if \(f(B) \neq \mathbb{F}P^d\).

Given a finite set \(Y \subset \mathbb{F}P^d\), we will show that \(u\) can be chosen so that \(\mathbb{F}P_{[u]}^{d-1}\) is the best \((d - 1)\)-dimensional approximation. Indeed, given
\[
Y = \{[y_1], \ldots, [y_N]\} \subset \mathbb{F}P^d
\]
the goal is to find \(u^* \in \mathbb{F}^{d+1}\) so that
\[
u^* = \arg\min_{u \in \mathbb{F}^{d+1}} \sum_{n=1}^{N} d_g \left( [y_n], \mathbb{F}P_{[u]}^{d-1} \right)^2
\]
Since $d_g ([y_n], F_{[u]^+} P_{[u]^+} ([y_n])) = d_g ([y_n], P_{[u]^+} ([y_n]) = d_g ([y_n], [y_n - (y_n, u) u])$ then

$$u^* = \arg\min_{u \in \mathbb{F}^{d+1}} \sum_{n=1}^{N} \arccos \left( \frac{\langle y_n, y_n - (y_n, u) u \rangle}{\|y_n\| \cdot \|y_n - (y_n, u) u\|} \right)^2$$

$$= \arg\min_{u \in \mathbb{F}^{d+1}} \sum_{n=1}^{N} \arccos \left( \frac{\|y_n - (y_n, u) u\|}{\|y_n\|} \right)^2$$

$$= \arg\min_{u \in \mathbb{F}^{d+1}} \sum_{n=1}^{N} \left( \frac{\pi}{2} - \arccos \left( \frac{\|y_n, u\|}{\|y_n\|} \right) \right)^2 \quad (10)$$

This nonlinear least squares problem – in a nonlinear domain – can be solved approximately using linearization; the reduction, in turn, has a closed form solution. Indeed, the Taylor series expansion for $\arccos(\alpha)$ around 0 is given by

$$\arccos(\alpha) = \frac{\pi}{2} - \left( \alpha + \sum_{n=1}^{\infty} \frac{(2n)!}{4^n n!^2} \frac{\alpha^{2n+1}}{2n+1} \right), \quad |\alpha| < 1$$

and therefore $\left| \frac{\pi}{2} - \arccos(\alpha) \right| \approx |\alpha|$ is an order 3 approximation. It follows that

$$u^* \approx \arg\min_{u \in \mathbb{F}^{d+1}} \sum_{n=1}^{N} \frac{\langle y_n, u \rangle^2}{\|y_n\|^2} \quad (11)$$

which is a linear least squares problem, and a solution is the eigenvector of the $(d + 1)$-by-$(d + 1)$ uncentered covariance matrix

$$\text{Cov} \left( \frac{y_1}{\|y_1\|}, \ldots, \frac{y_N}{\|y_N\|} \right) = \left[ \begin{array}{ccc} \frac{y_1}{\|y_1\|} & \cdots & \frac{y_N}{\|y_N\|} \\ \frac{y_1}{\|y_1\|} & \cdots & \frac{y_N}{\|y_N\|} \end{array} \right]$$

corresponding to the smallest eigenvalue. Notice that if $a_1, \ldots, a_N \in \mathbb{F}$ satisfy $|a_n| = 1$ for each $n = 1, \ldots, N$ then

$$\text{Cov} \left( \frac{y_1}{\|y_1\|}, \ldots, \frac{y_N}{\|y_N\|} \right) = \text{Cov} \left( a_1 \frac{y_1}{\|y_1\|}, \ldots, a_N \frac{y_N}{\|y_N\|} \right)$$

and hence we can write $\text{Cov}(Y)$ for the unique uncentered covariance matrix associated to $Y = \{ [y_1], \ldots, [y_N] \} \subset \mathbb{F}^d$. If $u \in \mathbb{F}^{d+1}$ is an eigenvector for the smallest eigenvalue then we use the notation $[u] = \text{LastProjComp}(Y, \mathbb{F}^d)$, with the understanding that $[u]$ is unique only if the smallest eigenvalue of $\text{Cov}(Y)$ has multiplicity one. If not, the choice is arbitrary.

**Principal Projective Coordinates.** First we define, inductively, the Principal Projective Components of $Y$. Starting with $[v_d] = \text{LastProjComp}(Y, \mathbb{F}^d)$, assume that for $1 \leq k \leq d - 1$ the components $[v_{k+1}], \ldots, [v_d] \in \mathbb{F}^d$ have been determined.
and let us define $[v_k]$. To this end, let $\{u_0, \ldots, u_k\}$ be an orthonormal basis for $V^k = \text{Span} (v_{k+1}, \ldots, v_d)^\perp$, let

$$A_k = \begin{bmatrix} u_0 & \cdots & u_k \end{bmatrix}$$

and let $A_k^\dagger$ be its conjugate transpose. If $A_k^\dagger \cdot Y = \{ [A_k^\dagger y_1], \ldots, [A_k^\dagger y_N] \}$, define

$$[v_k] := A_k \cdot \text{LastProjComp} (A_k^\dagger \cdot Y, F^k)$$

This is well defined as the following proposition shows.

**Proposition 5.4.** The class $[v_k] = A_k \cdot \text{LastProjComp} (A_k^\dagger \cdot Y, F^k)$ is independent of the choice of orthonormal basis $\{u_0, \ldots, u_k\}$.

**Proof.** Let $\{w_0, \ldots, w_k\}$ be another orthonormal basis for $V^k$ and let

$$B_k = \begin{bmatrix} | & | \\ w_0 & \cdots & w_k \end{bmatrix}$$

It follows that $B_k^\dagger B_k = A_k^\dagger A_k = I_{k+1}$, the $(k+1)$-by-$(k+1)$ identity matrix, and that $B_k B_k^\dagger = A_k A_k^\dagger$ is the matrix (with respect to the standard basis of $F^{d+1}$) of the orthogonal projection $p_{V^k} : F^{d+1} \rightarrow V^k$. Therefore

$$\begin{align*}
(B_k^\dagger A_k) (A_k^\dagger B_k) &= B_k^\dagger (A_k A_k^\dagger) B_k \\
&= B_k^\dagger (B_k B_k^\dagger) B_k \\
&= I_{r+1}
\end{align*}$$

which shows that $A_k^\dagger B_k$ is an orthogonal matrix. Since

$$\| A_k^\dagger y \|^2 = \langle y, A_k A_k^\dagger y \rangle = \| B_k^\dagger y \|^2$$

for every $y \in F^{d+1}$, then

$$\begin{align*}
\text{Cov} (A_k^\dagger Y) &= \text{Cov} \left( A_k^\dagger y_1, \ldots, A_k^\dagger y_N \right) \\
&= \text{Cov} \left( A_k^\dagger B_k \frac{B_k^\dagger y_1}{\| B_k^\dagger y_1 \|}, \ldots, A_k^\dagger B_k \frac{B_k^\dagger y_N}{\| B_k^\dagger y_N \|} \right) \\
&= A_k^\dagger B_k \cdot \text{Cov} (B_k^\dagger Y) \cdot B_k A_k
\end{align*}$$

and thus $\text{Cov} (A_k^\dagger Y)$ and $\text{Cov} (B_k^\dagger Y)$ have the same spectrum. Moreover, $u$ is an eigenvector of $\text{Cov} (A_k^\dagger Y)$ corresponding to the smallest eigenvalue $\lambda$ if and only if $u = A_k^\dagger B_k w$ for a unique eigenvector $w$ of $\text{Cov} (B_k^\dagger Y)$ with eigenvalue $\lambda$. Since $B_k w \in V^k$ and $A_k A_k^\dagger$ is the matrix of $p_{V^k}$, then

$$A_k u = A_k A_k^\dagger B_k w = B_k w$$
which shows that
\[ A_k \cdot \text{LastProjComp}\left( A_k^\dagger \cdot Y, FP^k \right) = B_k \cdot \text{LastProjComp}\left( B_k^\dagger \cdot Y, FP^k \right) \]
as claimed.

This inductive procedure defines \([v_1], \ldots, [v_d] \in FP^d\), and we let \(v_0 \in FP^{d+1}\) with \(||v_0|| = 1\) be so that \(\text{Span}(v_0) = \text{Span}(v_1, \ldots, v_d)\). We will use the notation
\[ \text{PrinProjComps}(Y) = \{[v_0], \ldots, [v_d]\} \]
for the principal projective components of \(Y\) computed in this fashion. Each choice of unitary (i.e. having norm 1) representatives \(u_0 \in [v_0], \ldots, u_d \in [v_d]\) yields an orthonormal basis \(\{u_0, \ldots, u_d\}\) for \(FP^{d+1}\), and each \(y \in FP^{d+1}\) can be represented in terms of its vector of coefficients
\[ \text{coeff}_U(y) = \begin{bmatrix} \langle y, u_0 \rangle \\ \vdots \\ \langle y, u_d \rangle \end{bmatrix} \]

Notice that if \(\bar{U}\) is another set of unitary representatives for \(\text{PrinProjComps}(Y)\) then there exists a \((d + 1)\)-by-\((d + 1)\) diagonal matrix \(\Lambda\), with entries in the unit circle in \(F\), and so that \(\text{coeff}_{\bar{U}}(y) = \Lambda \cdot \text{coeff}_U(y)\). That is, the resulting principal projective coordinates \(\text{coeff}_{\bar{U}}(y) \in FP^d\) are unique up to a diagonal isometry.

**Visualizing the Reduction.** Fix a set of unitary representatives \(v_0, \ldots, v_d\) for \(\text{PrinProjComps}(Y)\), and let \(V^k = \text{Span}(v_0, \ldots, v_k)\) for \(1 \leq k \leq d\). It is often useful to visualize \(P_{V^k}(Y) \subset FP_{V^k}\) for \(k\) small, specially in \(\mathbb{RP}^1, \mathbb{RP}^2, \mathbb{RP}^3\) and \(\mathbb{CP}^1\). We do this using the principal projective coordinates of \(Y\). For the real case (i.e. \(\mathbb{RP}^1, \mathbb{RP}^2\) and \(\mathbb{RP}^3\)) we consider the set
\[ \begin{cases} x_n \in S^k \\ ||x_n|| \end{cases} \quad x_n = \text{sign} (\langle y_n, v_0 \rangle) \begin{bmatrix} \langle y_n, v_0 \rangle \\ \vdots \\ \langle y_n, v_k \rangle \end{bmatrix}, \quad n = 1, \ldots, N \]
and its image through the stereographic projection \(S^k \setminus \{-e_1\} \longrightarrow D^k\) with respect to \(-e_1\), where \(e_1\) is the first standard basis vector \(e_1 \in \mathbb{R}^{k+1}\). That is, we visualize \(P_{V^k}(Y)\) in the \(k\)-disk \(D^k \subset \mathbb{R}^k\) with the understanding that antipodal points on the boundary are identified. For the complex case (i.e. \(\mathbb{CP}^1\)) we consider the set
\[ \begin{cases} z_n \in \mathbb{C}^2 \\ ||z_n|| \end{cases} \quad z_n = \begin{bmatrix} \langle y_n, v_0 \rangle \\ \langle y_n, v_1 \rangle \end{bmatrix}, \quad n = 1, \ldots, N \]
and its image through the H"{o}pf map
\[ H : S^3 \subset \mathbb{C}^2 \longrightarrow S^2 \subset \mathbb{C} \times \mathbb{R} \]
\[ [z_1, z_2] \longmapsto (z_1 \overline{z_2}, |z_1|^2 - |z_2|^2) \]
which is exactly the composition of \(S^3 \subset \mathbb{C}^2 \longrightarrow \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}\), sending \([z_1, z_2]\) to \(z_1/z_2\), and the isometry \(\mathbb{C}_\infty \approx S^2 \subset \mathbb{C} \times \mathbb{R}\) given by the inverse of the north-pole stereographic projection.
Choosing the Target Dimension. Given $1 \leq k \leq d$, the cumulative variance recovered by $P_{V^k}(Y) \subset \mathbb{F}P_{V^k}$ is given by the expression

$$\text{var}_Y(k) = \frac{1}{N} \sum_{\ell=1}^{k} \sum_{n=1}^{N} d_g \left( P_{V^\ell}([y_n]), \mathbb{F}P_{V^{\ell-1}} \right)^2$$

$$= \frac{1}{N} \sum_{\ell=1}^{k} \sum_{n=1}^{N} \left( \frac{\pi}{2} - d_g \left( P_{V^{\ell-1}}([y_n]), [v_\ell] \right) \right)^2$$

(12)

Define the percentage of cumulative variance as

$$p \cdot \text{var}_Y(k) = \frac{\text{var}_Y(k)}{\text{var}_Y(d)}$$

(13)

A common rule of thumb for choosing the target dimension is identifying the smallest value of $k$ so that $p \cdot \text{var}_Y$ exhibits a prominent reduction in growth rate. Visually, this creates an “elbow” in the graph of $p \cdot \text{var}_Y$ at $k$ (see figure 1.2(b), $k = 5$). The target dimension can also be chosen as the smallest $k \geq 1$ so that $p \cdot \text{var}_Y(k)$ is greater than a predetermined threshold, e.g. 0.7 (see figure 3, $k = 2$).

Examples. Let us illustrate the inner workings of the framework we have developed thus far.

The Projective Plane $\mathbb{R}P^2$: Will be realized as the quotient $S^2/(u \sim -u)$, and will be endowed with the geodesic distance $d_g(u, v) = \arccos(|\langle u, v \rangle|)$. We begin by selecting six landmark points $\ell_0, \ldots, \ell_5 \in \mathbb{R}P^2$ as shown in Figure 9(Left). If for each landmark $\ell_j$ we let $r_j = \min \{d_g(\ell_j, \ell_r) : r \neq j \}$ and let $\epsilon_j = 0.95 + r_j$, then $U = \{B_{\epsilon_j}(\ell_j)\}$ is a covering for $\mathbb{R}P^2$ and the corresponding nerve complex $\mathcal{N}(U)$ is shown in Figure 9(Left).

![Figure 9. Left: Landmark points on $\mathbb{R}P^2$. Right: Induced nerve complex $\mathcal{N}(U)$.](image)

Let $\mathbb{I}_{\{i,j\}} : \mathcal{N}(U)^{(1)} \rightarrow \mathbb{Z}/2$ be the indicator function $\mathbb{I}_{\{i,j\}}\{\{r,k\}\} = 1$ if $\{i,j\} = \{r,k\}$ and 0 otherwise. Then $\tau = \mathbb{I}_{\{0,1\}} + \mathbb{I}_{\{1,2\}} + \mathbb{I}_{\{0,2\}} + \mathbb{I}_{\{0,4\}} + \mathbb{I}_{\{2,5\}} + \mathbb{I}_{\{1,3\}}$ is a 1-cocycle, and its cohomology class $[\tau]$ is the non-zero element in $H^1(\mathcal{N}(U); \mathbb{Z}/2) \cong \mathbb{Z}/2$. Using the formula from Theorem 4.11, the cocycle $\tau$ above, and the quadratic bump functions with weights $\lambda_j = \epsilon_j^2$, we get the corresponding map $f_\tau : \mathbb{R}P^2 \rightarrow \mathbb{R}P^5$. For instance, if $b \in B_{\epsilon_0}(\ell_0)$ then

$$f_\tau(b) = \left[ |\epsilon_0 - d_g(b, \ell_0)_+| - |\epsilon_1 - d_g(b, \ell_1)_+| - |\epsilon_2 - d_g(b, \ell_2)_+| : |\epsilon_3 - d_g(b, \ell_3)_+| - |\epsilon_4 - d_g(b, \ell_4)_+| - |\epsilon_5 - d_g(b, \ell_5)_+| \right]$$
Let $X \subset \mathbb{R}P^2$ be a uniform random sample with 10,000 points. After computing the principal projective components of $f_\tau(X) \subset \mathbb{R}P^5$ and the percentage of cumulative variance $\text{p.var}_Y(k)$ (see equation (13)) for $k = 1, \ldots, 5$ we obtain the following:

![Graph showing the percentage of cumulative variance](image)

**Figure 10.** Percentage of cumulative variance

This profile of cumulative variance suggests that dimension 2 is appropriate for representing $f_\tau(X)$: both “the elbow” and the “70% of recovered variance” happen at around $k = 2$. Below in Figure 11 we show the original sample $X \subset \mathbb{R}P^2$ as well as the point cloud $P_{V^2}(f_\tau(X))$ resulting from projecting $f_\tau(X)$ onto $\mathbb{R}P^2_{V^2} \subset \mathbb{R}P^5$.

Recall that $P_{V^2}(f_\tau(X))$ is visualized on the unit disk $D^2 = \{u \in \mathbb{R}^2 : \|u\| \leq 1\}$, with the understanding that points in the boundary $\partial D^2 = S^1$ are identified with their antipodal.

![Visualizations of original sample and projective coordinates](image)

**Figure 11.** Left: Original sample $X \subset \mathbb{R}P^2$, Right: Visualization of resulting projective coordinates. Please refer to an electronic version for colors.

These results are consistent with the fact that any $f : \mathbb{R}P^2 \rightarrow \mathbb{R}P^\infty$ which classifies the nontrivial bundle over $\mathbb{R}P^2$, must be homotopic to the inclusion $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^\infty$. So not only did we get the right homotopy-type, but also the global geometry and the metric information were recovered to a large extent.

**The Klein Bottle $K$:** Will be modeled as the quotient of the unit square $[0, 1] \times [0, 1]$ by the relation $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$. The quotient $K$ is endowed with the induced flat metric, which we denote by $d$, and nine landmark points $\ell_0, \ldots, \ell_8 \in K$ are selected as shown in Figure 12(Left). If for each landmark $\ell_j$ we let $\epsilon_j = \min\{d(\ell_j, \ell_r) : j \neq r\}$ then $\mathcal{U} = \{B_{\epsilon_j}(\ell_j)\}$ is a covering for $K$ and the resulting nerve complex $\mathcal{N}(\mathcal{U})$ is shown in Figure 12(Right).
It follows that the 1-skeleton of $\mathcal{N}(\mathcal{U})$ is the complete graph on nine vertices, and that there are thirty-six 2-simplices and nine 3-simplices. Let us define the 1-chains $\tau_{\text{diag}}, \tau_{\text{horz}}, \tau_{\text{vert}} \in C^1(\mathcal{N}(\mathcal{U}); \mathbb{Z}/2)$ as follows: $\tau_{\text{diag}}$ will be the sum of indicator functions on the diagonal edges, $\tau_{\text{horz}}$ is the sum of indicator functions on the horizontal edges, and $\tau_{\text{vert}}$ will be the sum of indicator functions on the vertical edges. One can check that

$$\tau = \tau_{\text{diag}} + \tau_{\text{horz}} \quad \text{and} \quad \tau' = \tau_{\text{diag}} + \tau_{\text{vert}}$$

are cocycles and that their cohomology classes generate $H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Let $X \subset K$ be a random sample with 10,000 points. The formula from Theorem 4.11 yields classifying maps

$$f_\tau, f_{\tau'}, f_{\tau+\tau'} : K \to \mathbb{R}P^8$$

and we obtain the point clouds $f_\tau(X), f_{\tau'}(X), f_{\tau+\tau'}(X) \subset \mathbb{R}P^8$ of which we will compute their principal projective coordinates. Starting with $f_\tau(X)$ we get the profile of recovered variance shown in Figure 13.

The figure suggests that dimension 3 provides an appropriate representation of $f_\tau(X) \subset \mathbb{R}P^8$. As described above, we visualize $P_{V^3}(f_\tau(X)) \subset \mathbb{R}P^3_{V^3}$ in the 3-dimensional unit disk $D^3 = \{ u \in \mathbb{R}^3 : \|u\| \leq 1 \}$ with the understanding that points on the boundary $\partial D^3 = S^2$ are identified with their antipodal. The results are summarized in Figure 14.
This example highlights the following point: when representing data sampled from complicated spaces, e.g. the Klein bottle, it is advantageous to use target spaces with similar properties. In particular, the representation for \( X \subset K \) we recover here is much simpler than those obtained with traditional dimensionality reduction methods. We now transition to the 2-dimensional reduction \( P_{V^2}(f_\tau(X)) \subset \mathbb{R}P^2_{V^2} \). As before we visualize the representation in the 2-dimensional unit disk \( D^2 \) with the understanding that points on the boundary \( \partial D^2 = S^1 \) are identified with their antipodal.

We conclude this example by examining the \( \mathbb{R}P^2 \) coordinates induced by \( \tau' \) and \( \tau + \tau' \) (Figure 16). For completeness we include the one for \( \tau \) and also add figures with coloring by the vertical direction in \( K \).
Figure 16. **Columns:** $\mathbb{RP}^2$ coordinates for $X \subset K$ induced by the cocycles $\tau = \tau_{\text{diag}} + \tau_{\text{horz}}$, $\tau' = \tau_{\text{diag}} + \tau_{\text{vert}}$ and $\tau + \tau' = \tau_{\text{horz}} + \tau_{\text{vert}}$, respectively. **Rows:** Color schemes of the computed coordinates according to the horizontal and vertical directions in $K$. Please refer to an electronic version for colors.

6. **CHOOSING COCYCLE REPRESENTATIVES**

We now describe how the maps $f_{\tau} : B \to \mathbb{RP}^d$ and $f_{\eta} : B \to \mathbb{CP}^d$ depend on the choice of cocycle representatives $\tau = \{\tau_{rt}\} \in Z^1(N(U); \mathbb{Z}/2)$ and $\eta = \{n_{rst}\} \in Z^2(N(U); \mathbb{Z})$, respectively. We know that any two such choices yield homotopic maps (Lemma 4.11), but intricate geometries can negatively impact the dimensionality reduction step. Given $X = \{x_1, \ldots, x_N\} \subset B$, the goal is to elucidate the effects on the principal projective coordinates of $f_{\tau}(X) \subset \mathbb{RP}^d$ and $f_{\eta}(X) \subset \mathbb{CP}^d$. The results are: the real case is essentially independent of the cocycle representative; while the complex case requires the harmonic cocycle.

We begin with a simple observation. Let $Y = \{[y_1], \ldots, [y_N]\} \subset \mathbb{FP}^d$, let $A$ be a $(d+1) \times (d+1)$ orthogonal matrix with entries in $\mathbb{F}$, that is $A \in O(d+1, \mathbb{F})$, and let $A \cdot Y$ denote the set $\{[A \cdot y_1], \ldots, [A \cdot y_N]\}$.

**Proposition 6.1.** Let $A \in O(d+1, \mathbb{F})$ and let $Y \subset \mathbb{FP}^d$ be finite. Then

$$\text{PrinProjComps}(A \cdot Y) = A \cdot \text{PrinProjComps}(Y)$$

**Proof.** Since $A$ is an orthogonal matrix then

$$\text{Cov}(A \cdot Y) = A \cdot \text{Cov}(Y) \cdot A^\dagger$$

Hence, if $u$ is an eigenvector of $\text{Cov}(Y)$ then $A \cdot u$ is an eigenvector of $\text{Cov}(A \cdot Y)$ with the same eigenvalue and therefore, if $\text{LastProjComp}(Y, \mathbb{FP}^d) = [v_d]$ then

$$\text{LastProjComp}(A \cdot Y, \mathbb{FP}^d) = [A \cdot v_d]$$

Since the remaining principal projective components are computed in the same fashion, after the appropriate orthogonal projections, the result follows. □
The Real Case is Independent of the Cocycle Representative. Let $\alpha = \{\alpha_r\} \in C^0(\mathcal{N}(U); \mathbb{Z}/2)$ and let $\tau = \tau + \delta^0(\alpha)$. It follows that for $b \in U_j$

$$f_{\tau}(b) = \left( (-1)^{\tau_0} \sqrt{\phi_0(b)} : \cdots : (-1)^{\tau_d} \sqrt{\phi_d(b)} \right)$$

$$= \left( (-1)^{\tau_0 + \alpha_0} \sqrt{\phi_0(b)} : \cdots : (-1)^{\tau_0 + \alpha_d} \sqrt{\phi_d(b)} \right)$$

Hence, if $f_{\tau}(X) = \{[y_1], \ldots, [y_N]\}$ and

$$A_\alpha = \begin{bmatrix} (-1)^{\alpha_0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & (-1)^{\alpha_d} \end{bmatrix}$$

then $f_{\tau}(X) = \{[A_\alpha \cdot y_1], \ldots, [A_\alpha \cdot y_N]\} = A_\alpha \cdot f_{\tau}(X)$. This shows that

$$\text{PrinProjComps}(f_{\tau}(X)) = A_\alpha \cdot \text{PrinProjComps}(f_{\tau}(X))$$

which implies, in particular, that the profiles of cumulative variance for $f_{\tau}(X)$ and $f_{\tau}(X)$ are identical. Moreover, the resulting projective coordinates for both point-clouds differ by the isometry of FP$^d$ induced by $A_\alpha$.

The Harmonic Representative is Required for the Complex Case. Just as we did in section 3 (Geometric Interpretation), given $\eta = \{n_{rst}\} \in \mathbb{Z}^2(\mathcal{N}(U); \mathbb{Z})$ we can express $f_\eta : B \rightarrow \mathbb{CP}^d$ as

$$f_\eta : B \xrightarrow{\varphi} |\mathcal{N}(U)| \xrightarrow{F_\eta} \mathbb{CP}^d$$

where $\varphi$ is defined in equation (9) and $F_\eta : |\mathcal{N}(U)| \rightarrow \mathbb{CP}^d$ is given (in barycentric coordinates) on the open star of a vertex $v_j$ by

$$F_\eta(x_0, \ldots, x_d) = \left[ \sqrt{x_0} \cdot e^{2\pi i \sum_{r} (x_1 \cdot n_{0jr})} : \cdots : \sqrt{x_d} \cdot e^{2\pi i \sum_{r} (x_d \cdot n_{drj})} \right]$$

Let us describe the local behavior of $F_\eta$ when restricted to the 2-skeleton of $|\mathcal{N}(U)|$. To this end, let $\sigma$ be the 2-simplex of $|\mathcal{N}(U)|$ spanned by the vertices $v_r, v_s, v_t$, with $0 \leq r < s < t \leq d$. It follows that $F_\eta : \sigma \rightarrow \mathbb{CP}^d$ can be written as

$$F_\eta(x_r, x_s, x_t) = \left[ \sqrt{x_r} \cdot e^{-2\pi i \cdot n_{rst} \cdot x_r} : \sqrt{x_s} \cdot e^{2\pi i \cdot n_{rst} \cdot x_s} \right]$$

(14)

with the understanding that only the potentially-nonzero entries appear. Furthermore, if we fix $0 < c < 1$ and consider the straight line in $\sigma$ given by

$$L_c = \left\{ (1-c, x, c-x) : 0 \leq x \leq c \right\}$$

then $F_\eta : L_c \rightarrow \mathbb{CP}^d$ can be written as

$$F_\eta(x) = \left[ \sqrt{1-c} : \sqrt{x} \cdot e^{2\pi i (x-c) \cdot n_{rst}} : e^{c-x} \cdot e^{2\pi i \cdot n_{rst}} \right]$$

$$= \left[ \sqrt{1-c} \cdot e^{-2\pi i \cdot n_{rst}} : e^{2\pi i \cdot n_{rst}} \cdot (c-x) \right]$$

which parametrizes a spiral with radius $\sqrt{1-c}$ and winding number $|c \cdot n_{rst}|$. Hence, as each $|n_{rst}|$ gets larger, $F_\eta$ becomes increasingly highly-nonlinear on the 2-simplices of $|\mathcal{N}(U)|$. As a consequence, the dimensionality reduction scheme furnished by principal projective components is less likely to work as it relies on a (global) linear approximation. Let us illustrate this phenomenon via an example.
Example: Let $B = S^2$ be the unit sphere in $\mathbb{R}^3$, and for $r \in \{0, 1, 2, 3\}$ let $U_r \subset S^2$ be the geodesic open ball of radius $\arccos(-1/3)$ centered at

$$\ell_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \ell_1 = \frac{1}{3} \begin{bmatrix} 2\sqrt{2} \\ 0 \\ -1 \end{bmatrix}, \quad \ell_2 = \frac{1}{3} \begin{bmatrix} \sqrt{6} \\ -\sqrt{2} \\ -1 \end{bmatrix}, \quad \ell_3 = \frac{1}{3} \begin{bmatrix} -\sqrt{2} \\ \sqrt{6} \\ -1 \end{bmatrix}$$

respectively. It follows that $U = \{U_0, U_1, U_2, U_3\}$ is an open cover of $S^2$, and that $N(U)$ is the boundary of the 3-simplex. Therefore, if

$$\sigma_0 = \{0, 1, 2\}, \quad \sigma_1 = \{0, 2, 3\}, \quad \sigma_2 = \{0, 1, 3\}, \quad \sigma_3 = \{1, 2, 3\}$$

denote the 2-simplices of $N(U)$ and $\{\eta^0, \eta^1, \eta^2, \eta^3\}$ is the basis for $C^2(N(U); \mathbb{Z})$ of indicator functions, then each $\eta^r$ is a cocycle whose cohomology class generates

$$H^2(N(U); \mathbb{Z}) \cong \mathbb{Z}$$

Moreover, $\{\eta^0 - \eta^1, \eta^0 + \eta^2, \eta^0 + \eta^3\}$ is a basis for $B^2(N(U); \mathbb{Z})$ and therefore

$$[\eta^0] = [\eta^1] = -[\eta^2] = -[\eta^3]$$

Consider the map $f_\eta : S^2 \to \mathbb{CP}^3$ associated to $\eta = \eta^0$; the results will be similar for the other $\eta^r$’s. We show in Figure 17 the computed $\mathbb{CP}^1$ coordinates of $f_\eta(X)$ for a random sample $X \subset S^2$ with 10,000 points. As one can see, the homotopy type of the resulting map is correct, but the distances are completely distorted.

![Figure 17. $\mathbb{CP}^1$ coordinates for points on the 2-sphere using the map $f_\eta$ associated to the integer cocycle $\eta = \eta^0$. This shows the inadequacy of the integer cocycle; see Figure 18 for comparison. Please refer to an electronic version for colors.](image)

The main difference between the real and complex cases is that the former is locally linear, while the latter has local nonlinearities arising from the terms

$$\exp\left(2\pi i \cdot n_{rst} \cdot x_t\right).$$

The tempting conclusion would be then to choose the cocycle representative $\eta = \{n_{rst}\} \in Z^2(N(U); \mathbb{Z})$ which makes $F_\eta$ as locally linear as possible. This can be achieved by making each $|n_{rst}|$ small. The problem, as in the sphere example, is that since $n_{rst} \in \mathbb{Z}$ then even this choice is inadequate. What we will see now is that the integer constraint can be relaxed via Hodge theory (see, for instance, Section 2 of [23]).
Harmonic Smoothing. Let $K$ be a finite simplicial complex. Then for each $n \geq 0$ $C^n(K; \mathbb{R})$ is a finite dimensional vector space over $\mathbb{R}$, and hence can be endowed with an inner product. A common choice is

$$\langle \beta_1, \beta_2 \rangle_n = \sum_{\sigma} \beta_1(\sigma)\beta_2(\sigma)$$

where $\beta_1, \beta_2 \in C^n(K; \mathbb{R})$ and the sum ranges over all $n$-simplices $\sigma$ of $K$. In particular we have the induced norm

$$\|\beta\|^2 = \sum_{\sigma} |\beta(\sigma)|^2$$

Each boundary map $\delta^n : C^n(K; \mathbb{R}) \to C^{n+1}(K; \mathbb{R})$ is therefore a linear transformation between inner-product spaces, and hence has an associated dual map $d_{n+1} : C^{n+1}(K; \mathbb{R}) \to C^n(K; \mathbb{R})$ uniquely determined by the identity

$$\langle \delta^n(\nu), \beta \rangle_{n+1} = \langle \nu, d_{n+1}(\beta) \rangle_n$$

for all $\nu \in C^n(K; \mathbb{R})$ and all $\beta \in C^{n+1}(K; \mathbb{R})$. The Hodge Laplacian $\Delta_n$ is the endomorphism of $C^n(K; \mathbb{R})$ defined by the formula

$$\Delta_n = d_{n+1} \circ \delta^n + \delta^{n-1} \circ d_n$$

and a cochain $\theta \in C^n(K; \mathbb{R})$ is said to be harmonic if $\Delta_n(\theta) = 0$. A simple linear algebra argument shows that Harmonic cochains can be characterized as follows:

**Proposition 6.2.** $\theta \in C^n(K; \mathbb{R})$ is harmonic if and only if

$$d_{n}(\theta) = 0 \quad \text{and} \quad \delta^n(\theta) = 0.$$

That is, harmonic cochains are in particular harmonic cocycles. Moreover

**Theorem 6.3.** Every cohomology class $[\beta] \in H^n(K; \mathbb{R})$ is represented by a unique harmonic cocycle $\theta \in Z^n(K; \mathbb{R})$ satisfying $\theta = \beta - \delta^{n-1}(\nu^*)$, where $\nu^* = \arg\min \{ \|\beta - \delta^{n-1}(\nu)\| : \nu \in C^{n-1}(K; \mathbb{R}) \}$

In other words, given $\beta \in Z^n(K; \mathbb{R})$, $\theta$ is obtained by projecting $\beta$ orthogonally onto the orthogonal complement of $B^n(K; \mathbb{R})$ in $Z^n(K; \mathbb{R})$.

Let us now go back to our original set up: A covering $\mathcal{U} = \{U_r\}$ for a space $B$, a partition of unity $\{\varphi_r\}$ dominated by $\mathcal{U}$ and a class $[\eta] \in H^2(\mathcal{N}(\mathcal{U}); \mathbb{Z})$. The inclusion $j : \mathbb{Z} \to \mathbb{R}$ induces a homomorphism

$$j^*: H^2(\mathcal{N}(\mathcal{U}); \mathbb{Z}) \to H^2(\mathcal{N}(\mathcal{U}); \mathbb{R})$$

and if $\beta \in j^*([\eta])$ then there exists $\nu \in C^1(\mathcal{N}(\mathcal{U}); \mathbb{R})$ so that $j^*(\eta) = \beta + \delta^1(\nu)$.

**Lemma 6.4.** Let $\omega = \{\omega_{rs}\}$ and $\bar{\omega} = \{\bar{\omega}_{rs}\}$ be the sets of functions

$$\omega_{rs} : U_r \cap U_s \to \mathbb{C}^\times$$

$$\bar{\omega}_{rs} : U_r \cap U_s \to \mathbb{C}^\times$$

$$b \mapsto \exp\left\{2\pi i \sum_t \varphi_t(b)n_{rst}\right\}$$

$$b \mapsto \exp\left\{2\pi i (\nu_{rs} + \sum_t \varphi_t(b)\beta_{rst})\right\}$$

then $\omega, \bar{\omega} \in C^1(\mathcal{U}; \mathcal{C}_o^\times)$ are cohomologous Čech cocycles.
Proof. Since \( \omega \) is a cocycle, it is enough that check that \( \omega \) and \( \bar{\omega} \) are cohomologous. To this end let \( \mu = \{ \mu_r \} \in \tilde{C}^0(U; C^*_2) \), where

\[
\mu_r(b) = \exp \left\{ 2\pi i \sum_t \varphi_t(b) \cdot \nu_{rst} \right\} \quad b \in U_r
\]

Since for every \( U_r \cap U_t \neq \emptyset \)

\[
\sum_t \varphi_t \cdot n_{rst} = \sum_t \varphi_t \cdot (\beta_{rst} + \delta^1(\nu)_{rst})
= \sum_t \varphi_t \cdot (\beta_{rst} + \nu_{rs} - \nu_{rt} + \nu_{st})
= \nu_{rs} + \sum_t \varphi_t \cdot \beta_{rst} + \sum_t \varphi_t \cdot \nu_{st} - \sum_t \varphi_t \cdot \nu_{rt}
\]

then

\[
\omega_{rs} = \bar{\omega}_{rs} \cdot \frac{\mu_s}{\mu_r} = \bar{\omega}_{rs} \cdot \delta^0(\mu)_{rs}
\]

and the result follows. \( \square \)

Let \( \theta \in Z^2(\mathcal{N}(\mathcal{U}); \mathbb{R}) \) be the harmonic cocycle representing the class

\[
j^*([\eta]) \in H^2(\mathcal{N}(\mathcal{U}); \mathbb{R})
\]

let \( \nu \in C^1(\mathcal{N}(\mathcal{U}); \mathbb{R}) \) be so that \( j^\#(\eta) - \theta = \delta^1(\nu) \) and let \( f_{\theta, \nu} : B \rightarrow \mathbb{C}^d \) be given on \( b \in U_j \)

\[
f_{\theta, \nu}(b) = \left[ \sqrt{\varphi_0(b)} \cdot e^{2\pi i (\nu_{0j} + \sum_t \varphi_t(b) \delta_{01})} ; \cdots ; \sqrt{\varphi_d(b)} \cdot e^{2\pi i (\nu_{dj} + \sum_t \varphi_t(b) \delta_{d1})} \right] \quad (15)
\]

It follows that \( f_\eta \) and \( f_{\theta, \nu} \) are homotopic, \( f_{\theta, \nu} \) is as locally linear as possible, and for different choices of \( \nu \) the resulting principal projective coordinates of \( f_{\theta, \nu}(X) \) differ by a linear (diagonal) isometry.

We now revisit the 2-sphere example. One can check that

\[
\theta = j^\#(\eta^0 + \eta^1 - \eta^2 - \eta^3) / 4
\]

is the harmonic cocycle representing the cohomology class \( j^*([\eta^0]) \in H^2(\mathcal{N}(\mathcal{U}); \mathbb{R}) \). Let \( \nu \in C^1(\mathcal{N}(\mathcal{U}); \mathbb{R}) \) be so that \( \theta = j^\#(\eta^0) - \delta^1(\nu) \) and let \( f_{\theta, \nu} : S^2 \rightarrow \mathbb{C}P^3 \) be as in equation 15. We show in Figure 18 the computed \( \mathbb{C}P^3 \) coordinates of \( f_{\theta, \nu}(X) \) for the finite random sample \( X \subset S^2 \).

7. Multi-Scale Projective Coordinates via Persistent Cohomology of Sparse Filtrations

The goal of this section is to show how one can use persistent cohomology to construct multi-scale projective coordinates.

Greedy Permutations. Let \( k = \{ 0, \ldots, k \} \) for \( k \in \mathbb{Z}_{\geq 0} \), let \( (\mathbb{M}, \mathbf{d}) \) be a metric space and let \( X \subset \mathbb{M} \) be a finite subset with \( n + 1 \) elements. A greedy permutation on \( X \) is a bijection \( \sigma_g : \overline{\mathbb{N}} \rightarrow X \) which satisfies

\[
\sigma_g(s + 1) = \arg\max_{x \in X} \mathbf{d}(x, \sigma_g(s)) \quad s = 0, \ldots, n - 1
\]
Sparse Filtrations. Given a greedy permutation $\sigma_g : \mathbb{N} \rightarrow X$, let $x_s = \sigma_g(s)$ and $X_s = \sigma_g([s])$. The insertion radius of $x_s$, denoted $\lambda_s$, is defined as

$$\lambda_s = \begin{cases} \infty & \text{if } s = 0 \\ d(x_s, X_{s-1}) & \text{if } s > 0 \end{cases}$$

If follows that $\infty = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_n$. Fix $0 < \epsilon < 1$ and for $\alpha \geq 0$ define

$$r_s(\alpha) = \begin{cases} \alpha & \text{if } \alpha < \lambda_s(1 + \epsilon)/\epsilon \\ \lambda_s(1 + \epsilon)/\epsilon & \text{if } \lambda_s(1 + \epsilon)/\epsilon \leq \alpha \leq \lambda_s(1 + \epsilon)^2/\epsilon \\ 0 & \text{if } \alpha > \lambda_s(1 + \epsilon)^2/\epsilon \end{cases} \quad (16)$$

In particular $r_0(\alpha) = \alpha$ for all $\alpha \geq 0$, and for $s \geq 1$ the graph of $r_s$ is shown below.

**Definition 7.1.** For $\alpha \geq 0$ let

$$B_s^\alpha = \{ b \in \mathbb{M} : d(b, x_s) < r_s(\alpha) \}$$

It follows that $S_\alpha = \{ s \in \mathbb{N} : \lambda_s \geq \epsilon\alpha/(1 + \epsilon)^2 \}$ is the collection of indices $s$ for which $B_s^\alpha \neq \emptyset$. Moreover,

$$B^\alpha = \{ B_s^\alpha : s \in S_\alpha \}$$
satisfies \( X \subset \bigcup B^\alpha \) for each \( \alpha > 0 \), and it is a sparse covering in the sense that as \( \alpha \) increases there are fewer balls in \( B^\alpha \), but of larger radii. Moreover, it is a \((1 + \epsilon)\)-approximation of the \( \alpha \)-offset \( X^\alpha = \{ b \in M : d(b, X) < \alpha \} \):

**Proposition 7.2** (Corollary 2, [6]). If \( \beta \geq (1 + \epsilon)\alpha \), then

\[
\bigcup B^\alpha \subset X^\alpha \subset \bigcup B^\beta
\]

Let

\[
U_s^\alpha = \bigcup_{0 \leq \lambda \leq \alpha} (B^\lambda_s \times \{ \lambda \}) \quad \text{and} \quad U^\alpha = \{ U_s^\alpha : s \in \mathbb{N} \}
\]

It follows that \( N(U^\alpha) \subset N(U^\beta) \) whenever \( \alpha \leq \beta \).

**Definition 7.3.** The sparse Čech filtration, with sparsity parameter \( 0 < \epsilon < 1 \), induced by the greedy permutation \( \sigma_g : n \rightarrow X \) is the filtered simplicial complex

\[
\hat{\mathcal{C}}(\sigma_g, \epsilon) = \{ \hat{C}_\alpha(\sigma_g, \epsilon) : \alpha \geq 0 \}
\]

where

\[
\hat{C}_\alpha(\sigma_g, \epsilon) = N(U^\alpha)
\]

**Multi-Scale Projective Coordinates.** We will show now how the persistent cohomology of \( \hat{\mathcal{C}}(\sigma_g, \epsilon) \) can be used to compute multi-scale compatible classifying maps. The first thing to notice is that projection onto the first coordinate

\[
\bigcup U^\alpha \rightarrow \bigcup B^\alpha \\
(b, \lambda) \mapsto b
\]

is a deformation retraction if one regards \( B^\alpha_s \) as a subset of \( U^\alpha_s \) via the inclusion

\[
B^\alpha_s \ni b \mapsto (b, \alpha)
\]

\[
(17)
\]

**Theorem 7.4.** Let \((M, d)\) be a metric space and let \( X \subset M \) be a subset with \( n + 1 \) points. Given a greedy permutation \( \sigma_g : n \rightarrow X \) and a sparsity parameter \( 0 < \epsilon < 1 \), let \( r_s(\alpha) \) for \( \alpha \geq 0 \) be as in (16).

If \( \hat{\mathcal{C}}(\sigma_g, \epsilon) \) is the resulting sparse Čech filtration and

\[
\varphi_s^\alpha(b) = \frac{|r_s(\alpha) - d(b, x_s)|^2}{\sum_{t \in n} |r_t(\alpha) - d(b, x_t)|^2} \quad \text{for} \quad s \in \mathbb{N}, \ b \in \bigcup B^\alpha
\]

then we have well-defined maps

\[
w^\alpha_\tau : H^1(\hat{\mathcal{C}}_\tau(\sigma_g, \epsilon); \mathbb{Z}/2) \rightarrow [\bigcup B^\alpha, \mathbb{R}P^n]
\]

\[
[f^\alpha_\tau]
\]

\[
f^\tau_r : \bigcup B^\alpha \rightarrow \mathbb{R}P^n
\]

\[
B^\alpha_j \ni b \mapsto \left[ (-1)^{\tau_0} \sqrt{\varphi^\alpha_0(b)} : \cdots : (-1)^{\tau_n} \sqrt{\varphi^\alpha_n(b)} \right]
\]
and

\[ c_1^\alpha : H^2 (\hat{\mathcal{C}}_\alpha (\sigma, \epsilon); \mathbb{Z}) \longrightarrow \bigcup \mathcal{B}^\alpha, \mathbb{CP}^\infty \]

\[ f_{\theta, \nu}^\alpha : \bigcup \mathcal{B}^\alpha \longrightarrow \mathbb{CP}^n \]

where \( \theta = \{ \theta_{\tau, \ell} \} \in \mathbb{Z}^2 (\hat{\mathcal{C}}_\alpha (\sigma, \epsilon); \mathbb{R}) \) is the harmonic cocycle representing \( f^\ast (\{ \eta \}) \in H^2 (\hat{\mathcal{C}}_\alpha (\sigma, \epsilon); \mathbb{R}) \) and \( \nu = \{ \nu_{\tau, \ell} \} \in C^1 (\hat{\mathcal{C}}_\alpha (\sigma, \epsilon); \mathbb{R}) \) is so that \( \theta = f^\# (\eta) - \delta^1 (\nu) \).

Moreover, if each \( B^\alpha_{\tau} \in \mathcal{B}^\alpha \) is connected then \( w^0_1 \) is injective; if in addition each \( B^\alpha_{\tau} \) is locally path-connected and simply connected, and each \( B^\alpha_{\tau} \cap B^\alpha_{\tau'} \) is either empty or connected then \( c^1_1 \) is injective.

**Proof.** The first thing to notice is that the collection of continuous maps

\[ \varphi_s : \bigcup \mathcal{U}^\alpha \longrightarrow \mathbb{R} \]

\[ (b, \lambda) \quad \mapsto \quad \varphi^\lambda_s (b), \quad s \in \mathbb{N} \]

is a partition of unity dominated by \( \mathcal{U}^\alpha \). The Theorem follows from combining Lemma 4.11 and the following two facts: the inclusion \( \bigcup \mathcal{B}^\alpha \hookrightarrow \bigcup \mathcal{U}^\alpha \) from Equation 17 induces a bijection

\[ \left[ \bigcup \mathcal{U}^\alpha, \mathbb{FP}^\infty \right] \longrightarrow \left[ \bigcup \mathcal{B}^\alpha, \mathbb{FP}^\infty \right] \]

and the necessary connectedness conditions are satisfied by \( \mathcal{U}^\alpha \) if they are satisfied by \( \mathcal{B}^\alpha \). \( \square \)

**Remark 7.5.** As \( \alpha \) increases, the number of potentially nontrivial dimensions in the images of \( f^0_\tau \) and \( f^\theta_{\theta, \nu} \) decrease. Indeed, since \( \varphi^\alpha_s \) is identically zero if and only if \( B^\alpha_s = \emptyset \), it follows that for any \( b \in \bigcup \mathcal{B}^\alpha \) the only potentially non-zero entries in either \( f^\alpha_\tau (b) \) or \( f^\theta_{\theta, \nu} (b) \) correspond to the indices in \( \mathcal{S}_\alpha = \{ s \in \mathbb{N} : \lambda_s \geq \epsilon \alpha / (1 + \epsilon)^2 \} \). The observation follows from the fact that the sequence \( \{ \lambda_s \}_{s \in \mathbb{N}} \) is non-increasing, and monotonically decreasing for generic \( X \).

**Proposition 7.6.** Let \( \alpha \leq \beta \), then the diagrams

\[ H^1 (\hat{\mathcal{C}}_\alpha (\sigma, \epsilon); \mathbb{Z}/2) \xrightarrow{w^1_\alpha} \left[ \bigcup \mathcal{B}^\alpha, \mathbb{RP}^\infty \right] \xrightarrow{e^1_\alpha} \left[ \bigcup \mathcal{B}^\alpha, \mathbb{CP}^\infty \right] \]

\[ H^1 (\hat{\mathcal{C}}_\beta (\sigma, \epsilon); \mathbb{Z}/2) \xrightarrow{w^1_\beta} \left[ \bigcup \mathcal{B}^\beta, \mathbb{RP}^\infty \right] \xrightarrow{e^1_\beta} \left[ \bigcup \mathcal{B}^\beta, \mathbb{CP}^\infty \right] \]

are commutative.

As a consequence, if for \( \alpha = \alpha_0 < \cdots < \alpha_{\ell-1} < \alpha_\ell = \beta \) one has classes

\[ H^2 (\hat{\mathcal{C}}_{\alpha_0} (\sigma, \epsilon); \mathbb{Z}) \longrightarrow H^2 (\hat{\mathcal{C}}_{\alpha_{\ell-1}} (\sigma, \epsilon); \mathbb{Z}) \longrightarrow \cdots \longrightarrow H^2 (\hat{\mathcal{C}}_{\alpha_\ell} (\sigma, \epsilon); \mathbb{Z}) \]

\[ [\eta] \quad \longrightarrow \quad [\eta_{\ell-1}] \quad \longrightarrow \cdots \longrightarrow \quad [\eta_0] \]
then the diagram

\[
\bigcup B^{α_0} \xrightarrow{f^{α_0}} \bigcup B^{α_1} \xrightarrow{f^{α_1}} \cdots \xrightarrow{f^{α_ℓ}} \bigcup B^{α_ℓ} \xrightarrow{f^{α_ℓ}} \mathbb{C} \mathbb{P}^n
\]

commutes up to a homotopy which perhaps takes place in a higher dimensional projective space. The same is true in dimension one with \( \mathbb{Z}/2 \) coefficients. The persistent cohomology of the sparse Čech filtration \( \check{C}(σ_g, ϵ) \) now becomes relevant:

over \( \mathbb{Z}/2 \), a 1-dimensional cohomology class with nonzero persistence yields a multiscale system of compatible (up to homotopy) \( \mathbb{RP}^n \) coordinates. Constructing multiscale \( \mathbb{CP}^n \) coordinates from a persistent cohomology computation for \( k = 2 \) requires a bit more work, as the barcode decomposition is not valid for integer coefficients. Let \( p \) be a prime and consider the short exact sequence of abelian groups

\[
0 \rightarrow \mathbb{Z} \xrightarrow{x_p} \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0
\]

The induced homomorphism

\[
H^2 (\check{C}_α(σ_g, ϵ); \mathbb{Z}) \rightarrow H^2 (\check{C}_α(σ_g, ϵ); \mathbb{Z}/p)
\]

will be an epimorphism whenever \( H^3 (\check{C}_α(σ_g, ϵ); \mathbb{Z}) \) has no \( p \)-torsion. The universal coefficient theorem implies the following

**Proposition 7.7.** Let \( p \) be a prime not dividing the order of the torsion subgroup of \( H_3(\check{C}_α(σ_g, ϵ); \mathbb{Z}) \). Then the homomorphism

\[
H^2 (\check{C}_α(σ_g, ϵ); \mathbb{Z}) \rightarrow H^2 (\check{C}_α(σ_g, ϵ); \mathbb{Z}/p)
\]

is surjective.

Now one can follow the strategy in [12, Sections 2.4 and 2.5] for choosing \( p \), lifting to integer coefficients and constructing the harmonic representative. The solution to the harmonic representative problem is plugged into equation (15).

**Definition 7.8.** The sparse Rips filtration, with sparsity parameter \( 0 < ϵ < 1 \), induced by the greedy permutation \( σ_g : n \rightarrow X \) is the filtered simplicial complex

\[
\mathcal{R}(σ_g, ϵ) = \{ R_α(σ_g, ϵ) : α ≥ 0 \}
\]

where

\[
R_α(σ_g, ϵ) = \{ \{ s_0, \ldots, s_k \} \subset n : U_{s_t}, U_{s_r}^α \neq \emptyset \text{ for all } 0 ≤ r, t ≤ k \}
\]

**Remark 7.9.** It follows that for all \( β ≥ 0 \)

\[
\check{C}_β(σ_g, ϵ) \subset R_β(σ_g, ϵ)
\]

and if \( α \) is small enough (as the cones \( U_{s_r}^α \) stop growing) we also get the inclusion \( R_α(σ_g, ϵ) \subset \check{C}_{2α}(σ_g, ϵ) \). Then for each abelian group \( G \) and integer \( k ≥ 0 \) we get a commutative diagram

\[
\begin{align*}
H^k (R_β(σ_g, ϵ); G) & \rightarrow H^k (R_{α/4}(σ_g, ϵ); G) \\
\downarrow & \\
H^k (\check{C}_β(σ_g, ϵ); G) & \rightarrow H^k (\check{C}_{α/2}(σ_g, ϵ); G)
\end{align*}
\]
which shows that cohomology classes in the sparse Rips filtration with long enough persistence, and small enough death time, yield non-trivial persistent cohomology classes in the sparse Čech filtration. This is useful because the persistent cohomology of the sparse Rips filtration is easier to compute in practice.

**Example:** Let $X$ be a uniform random sample with 2,500 points from the 2-dimensional torus $S^1 \times S^1 \subset \mathbb{C}^2$, endowed with the metric $d$ given by

$$d((z_1, w_1), (z_2, w_2)) = \sqrt{|\arccos(z_1, z_2)|^2 + |\arccos(w_1, w_2)|^2}$$

There are two things we would like to illustrate with this example: First, that one does not need the entire data set $X$ to compute appropriate classifying maps $f_{\alpha \tau}$, in fact a small subsample suffices; and second, that one can use the sparse Rips filtration instead of the Čech filtration, which simplifies computations. Indeed, let $n = 34$ and let $X = \{x_0, \ldots, x_n\} \subset X$ be obtained through maxmin sampling. Notice that $X$ is $1.4\%$ of the total size of $X$ and that $\sigma_g : n \rightarrow X$ given by $\sigma_g(s) = x_s$ is a greedy permutation on $X$. We let $\epsilon = 0.01$ since the sample is already sparse. Computing the 1-dimensional persistent cohomology with coefficients in $\mathbb{Z}/2$ for the sparse Rips filtration $R(\sigma_g, \epsilon)$, yields the barcode shown in Figure 20(left).

**Figure 20.** $\mathbb{RP}^2$ coordinates for $X \subset S^1 \times S^1$, from the 1-dimensional $\mathbb{Z}/2$-persistent cohomology of a sparse rips filtration. **Left:** Computed barcode, **Right:** resulting $\mathbb{RP}^2$ coordinates induced by classes with large persistence. In both cases, the $\mathbb{RP}^2$ coordinates of a point $(z, w) \in X$ are colored according to $\arg(z) \in [0, 2\pi)$. Please refer to an electronic version for colors.

For this calculation we first determine the birth-times of the edges as in [6, Algorithm 3], and input them as a distance matrix into Dionysus’ persistent cohomology algorithm [28]. After selecting the two classes with the longest persistence, Dionysus outputs cocycle representatives $\mu_1$ and $\mu_2$ at cohomological birth $\alpha_1, \alpha_2 \approx 1.18$. Now, using the fact that $C_1(\sigma_g, \epsilon) \subset R_\alpha(\sigma_g, \epsilon)$ for all $\alpha \geq 0$, we have that the induced homomorphism

$$C^1(R_\alpha(\sigma_g, \epsilon); \mathbb{Z}/2) \rightarrow C^1(\tilde{C}_\alpha(\sigma_g, \epsilon); \mathbb{Z}/2)$$
sends $\mu_1$ and $\mu_2$ to $\tau_1$ and $\tau_2$, respectively. Moreover, since $R_\alpha(\sigma_\epsilon, \epsilon)$ is connected at $\alpha = \min\{\alpha_1, \alpha_2\}$ it follows that $X \subset \bigcup B_{\min\{\alpha_1, \alpha_2\}}$, and using the formula from Theorem 7.4 we get the point clouds $f_{\tau_1}^{\alpha_1}(X), f_{\tau_2}^{\alpha_2}(X) \subset \mathbb{RP}^2$. The result of computing their $\mathbb{RP}^2$ coordinates via principal projective components is shown in Figure 20(right).

8. Discussion

We have shown in this paper how 1-dimensional (resp. 2-dimensional) persistent cohomology classes with $\mathbb{Z}/2$ coefficients (resp. $\mathbb{Z}/p$ coefficients for appropriate primes $p$) can be used to produce multi-scale projective coordinates for data. The main ingredients were: interpreting a given cohomology class as the characteristic class corresponding to a unique isomorphism type of line bundle, and constructing explicit classifying maps from Čech cocycle representatives. In addition, we develop a dimensionality reduction step in projective space in order to lower the target dimension of the original classifying map.

Some questions/directions suggested by the current approach are the following: The case $H^3(B; \mathbb{Z})$ has a similar flavor to the bundle perspective presented here, and can perhaps be addressed using gerbes [19]. On the other hand, since Principal Projective Components is essentially a global fitting procedure, it would be valuable to investigate what local nonlinear dimensionality reduction techniques can be adapted to projective space.

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