THE PIERI RULE FOR $GL_n$ OVER FINITE FIELDS

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ABSTRACT. The Pieri rule gives an explicit formula for the decomposition of the tensor product of irreducible representation of the complex general linear group $GL_n(\mathbb{C})$ with a symmetric power of the standard representation on $\mathbb{C}^n$. It is an important and long understood special case of the Littlewood-Richardson rule for decomposing general tensor products of representations of $GL_n(\mathbb{C})$.

In our recent work [Gurevich-Howe17, Gurevich-Howe19] on the organization of representations of the general linear group over a finite field $\mathbb{F}_q$ using small representations, we used a generalization of the Pieri rule to the context of this latter group.

In this note, we demonstrate how to derive the Pieri rule for $GL_n(\mathbb{F}_q)$. This is done in two steps; the first, reduces the task to the case of the symmetric group $S_n$, using the natural relation between the representations of $S_n$ and the spherical principal series representations of $GL_n(\mathbb{F}_q)$; while in the second step, inspired by a remark of Nolan Wallach, the rule is obtained for $S_n$ invoking the $S_l$-$GL_n(\mathbb{C})$ Schur duality.

Along the way, we advertise an approach to the representation theory of the symmetric group which emphasizes the central role played by the dominance order on Young diagrams. The ideas leading to this approach seem to appear first, without proofs, in [Howe-Moy86].

Dedicated to the memory of Ronald Douglas

0. Introduction

Two basic tasks in the representation theory of a finite group $G$ are: the parameterization of its set $\hat{G}$ (of isomorphism classes) of irreducible representations (irreps); and the decomposition into direct sum of irreps of certain of its naturally arising representations.

The Pieri rule that we formulate and prove in this note addresses a particular instance of the second task mentioned above, for the case of the general linear group $GL_n = GL_n(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$. It can be used to give a recursive solution to the general problem of decomposing the permutation actions of $GL_n$ on functions on flag manifolds.

The Pieri rule can be useful in other ways. Indeed, in [Gurevich-Howe17, Gurevich-Howe19] we developed a precise notion of ”size” for irreps of $GL_n$, called ”tensor rank”. This is an integer $0 \leq k \leq n$ that is naturally attached to an irreducible representation (irrep) and helps to compute important analytic properties such as its dimension and character values on certain elements of interest. In particular, in loc. cit. the Pieri rule for $GL_n$ enabled us to give an effective formula for the irreps of $GL_n$ of a given tensor rank $k$.

We proceed to consider the subgroups involved in the construction of representations involved in the formulation of the Pieri rule.
0.1. Young Diagrams and Parabolic Subgroups. The representations we are interested in are naturally realized on spaces constructed using standard parabolic subgroups \[\text{Borel69}\] of general linear groups, as we will now describe.

Fix an integer \(0 \leq k \leq n\), and denote by \(\mathcal{Y}_k\) the collection of Young diagrams of size \(k\) \[\text{Fulton97}\]. In more detail, by a Young diagram (or partition) \(D \in \mathcal{Y}_k\), we mean an ordered list of non-negative integers

\[
D = (d_1 \geq \ldots \geq d_r), \text{ with } d_1 + \ldots + d_r = k. \tag{0.1}
\]

It is common to visualize—see Figure 1 for illustration—the diagram \(D\) with the help of a drawing of \(r\) rows of square boxes, each row one on top of the other, starting at the left upper corner, in such a way that the \(i\)-th row contains \(d_i\) boxes.

![Figure 1. The Young diagram \(D = (3, 1) \in \mathcal{Y}_4\).](image)

To the diagram \(D\) \[0.1\] we can attach the following increasing sequence \(F_D\) of subspaces of the \(k\)-dimensional vector space \(\mathbb{F}_q^k\):

\[
F_D : 0 \subset \mathbb{F}_q^{d_1} \subset \mathbb{F}_q^{d_1+d_2} \subset \ldots \subset \mathbb{F}_q^k, \tag{0.2}
\]

and call it the standard flag attached to \(D\). In particular, having \(D\) we can form—see Figure 2 for illustration\(^\text{1}\)—the stabilizer subgroup

\[
P_D = \text{Stab}_{\text{GL}_k}(F_D) \subset \text{GL}_k, \tag{0.3}
\]

that we will call the standard parabolic subgroup attached to \(D\).

![Figure 2. The parabolic \(P_D \subset \text{GL}_4\), \(D = (3, 1)\), has \(A_3 \in \text{GL}_3\), \(A_1 \in \text{GL}_1\), \(U_{3,1} \in M_{3,1}\).](image)

\(^{1}\text{We denote } M_{k,n} = M_{k,n}(F) \text{ the space of } k \times n \text{ matrices over a field } F.\)
Probably the most important example from this class of parabolic subgroups is the Borel subgroup \( B \) of upper triangular matrices in \( GL_k \), which is just \( P_D \) with

\[
D = \begin{bmatrix}
& & \\
& & \\
& & \\
\end{bmatrix}
k \text{ times.}
\]

Next, we describe the specific type of representations that the Pieri rule attempts to decompose.

0.2. The Pieri Problem. Take \( D \in \mathcal{Y}_k \), denote by \( 1 \) the trivial representation of \( P_D \), and consider the induced representation

\[
I_D = \text{Ind}_{P_D}^{GL_k}(1),
\]

which is given by the space of complex valued functions on \( GL_k/P_D \), equipped with the standard left action of \( GL_k \) on it.

In the case when \( P_D = B \) is the Borel subgroup, the set \( GL_k/B \) is the flag variety, and we will call the collection of irreps that appear inside \( \text{Ind}_B^{GL_k}(1) \) the spherical principal series (SPS).

There is a natural recipe (that we will recall in detail below) to parametrize the SPS by Young diagrams. Note that for each Young diagram \( D \in \mathcal{Y}_k \), we have \( I_D < \text{Ind}_B^{GL_k}(1) \), where \(<\) denotes subrepresentation. Interestingly, each \( I_D \) contains (with multiplicity one) a well defined ”largest” irreducible subrepresentation \( \rho_D \). We will leave the details of that story for the body of the note, but the collection \( \{ \rho_D; D \in \mathcal{Y}_k \} \) realizes the totality of SPS representations of \( GL_k \).

We proceed to formulate the Pieri problem.

Fix \( 0 \leq k \leq n \), and denote by \( P_{k,n-k} \subset GL_n \) the standard parabolic fixing the first \( k \) coordinate subspace of \( \mathbb{F}_q^n \). There is a natural surjective map \( P_{k,n-k} \twoheadrightarrow GL_k \times GL_{n-k} \). Take an SPS representation \( \rho_D \) of \( GL_k \), and denote by \( 1_{n-k} \) the trivial representation of \( GL_{n-k} \). Pull back the representation \( \rho_D \otimes 1_{n-k} \) from \( GL_k \times GL_{n-k} \) to \( P_{k,n-k} \) and form the induced representation

\[
I_{\rho_D} = \text{Ind}_{P_{k,n-k}}^{GL_n}(\rho_D \otimes 1_{n-k}). \tag{0.4}
\]

Now we can write down the natural,

**Problem 0.2.1** (Pieri problem). Decompose the representation \( I_{\rho_D} \) into irreducibles.

It is easy to see that the components of \( I_{\rho_D} \) are SPS representations of \( GL_n \), so we are looking for a solution to Problem 0.2.1 in terms of Young diagrams, i.e., members of \( \mathcal{Y}_n \).

In this note we present a solution to the Pieri problem for \( GL_n \) in two steps. First we explain why it is enough to solve the analogous problem for the representations of the symmetric group \( S_n \). Then, in the second step, we demonstrate that the Pieri rule holds for \( S_n \), invoking the Schur (a.k.a. Schur-Weyl) duality for \( S_t GL_n(\mathbb{C}) \), and a use of the classical Pieri rule for \( GL_n(\mathbb{C}) \) \cite{Howe92, Pieri1893, Weyman89}. We note that in \cite{Ceccherini-Silberstein-Scarabotti-Tolli10}, Section 3.5, there is a proof of the Pieri rule for \( S_n \) based on a quite different approach.
We proceed to a short acknowledgements paragraph, after that give the table of contents, and start the body of the note.

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1. Representations of $S_n$

The standard parametrization of the irreps of $S_n$ is done using Young diagrams [Sagan91]. We will discuss various aspects of the construction leading to this parametrization, emphasizing the
role played by the dominance relation on the set \( \mathcal{Y}_n \) of Young diagrams. We will follow closely ideas formulated (without proofs) in Appendix 2 of [Howe-Moy86].

1.1. The Young Modules. Recall that partitioning the set \( \{1, \ldots, n\} \) into \( r \) disjoint subsets of size \( d_i \) each, and assigning these numbers, respectively, to the rows of the Young diagram \( D = (d_1 \geq \ldots \geq d_r) \in \mathcal{Y}_n \), gives rise to a Young tabloid \([Fulton97]\). Let us denote by \( \mathcal{T}_D \) the collection of all Young tabloids that one can make using \( D \). The natural action of the group \( S_n \) on \( \mathcal{T}_D \) is transitive. Moreover, we can identify

\[
\mathcal{T}_D = S_n / S_D,
\]

where \( S_D \subset S_n \) is the stabilizer subgroup

\[
S_D = \text{Stab}_{S_n}(T_D),
\]

of the tabloid \( T_D \) that obtained by assigning to the first row of \( D \) the numbers \( 1, \ldots, d_1 \), to the second \( d_1 + 1, \ldots, d_1 + d_2 \), etc. The group \( S_D \) is naturally isomorphic to the product \( S_{d_1} \times \ldots \times S_{d_r} \) embedded in \( S_n \) in the usual way.

Now, we consider the induced representation, called the Young module associated to \( D \),

\[
Y_D = \text{Ind}_{S_D}^{S_n}(1),
\]

where \( 1 \) stands for the trivial representation of \( S_D \). It is naturally realized as the permutation representation of \( S_n \) on the space of functions on \( \mathcal{T}_D \).

1.2. Properties of the Young Modules. We derive basic properties of the family of Young modules (1.2). They give, in particular, as a corollary the standard classification of the irreps of \( S_n \), and, as we mentioned earlier, they can be effectively understood using the important dominance relation \( \preceq \) on the set \( \mathcal{Y}_n \) of Young diagrams, which we recall now.

Suppose—see Figure 3 for illustration—we have a Young diagram \( D \) which is obtained from another diagram by moving one of the boxes of \( D' \) to a (perhaps new) lower row, then we write

\[
D \preceq D',
\]

and \( \preceq \) on \( \mathcal{Y}_n \) is the order generated from all the inequalities of the form (1.3).

\[
\begin{array}{cccc}
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array} & \overset{\preceq}{\longrightarrow} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array} & \overset{\preceq}{\longrightarrow} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array} & \overset{\preceq}{\longrightarrow} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array} & \overset{\preceq}{\longrightarrow} & \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array}
\end{array}
\]

**Figure 3.** The set \( \mathcal{Y}_4 \) is totally ordered by \( \preceq \) (this is not true for \( \mathcal{Y}_n, n \geq 6 \)).
Now, using the terminology afforded by the dominance relation, we can formulate the main technical results concerning the Young modules.

For two representations \( \pi \) and \( \tau \) of a finite group \( G \), let us denote by \( \langle \pi, \tau \rangle \) their intertwining number [Serre77]

\[
\langle \pi, \tau \rangle = \dim \text{Hom}(\pi, \tau).
\] (1.4)

In addition, we denote the sign representation of \( S_n \) by \( sgn \), and introduce the twisted Young module \( Y_E(sgn) = \text{Ind}_{S_E}^{S_n}(sgn) \) attached to \( E \in \mathcal{Y}_n \). Finally, let us denote by \( D^t \) the diagram in \( \mathcal{Y}_n \) which is transpose to \( D \). That is, \( D^t \) is gotten from \( D \) by reflecting across the downward diagonal from the top left box; in other words, the columns of \( D \) become the rows of \( D^t \). Then,

**Theorem 1.2.1.** For any two Young diagrams \( D, E \in \mathcal{Y}_n \), we have,

1. **Intertwinity:** \( \langle Y_E(sgn), Y_D \rangle = \begin{cases} 0, & \text{iff } E \not\succeq D^t; \\ 1, & \text{if } E = D^t. \end{cases} \)

2. **Monotonicity:** \( D \preceq E \) if and only if \( Y_E \preceq Y_D \).

For a proof of Theorem 1.2.1 see Appendix A.1.1.

### 1.3. The Irreducible Representations of \( S_n \)

Part (1) of Theorem 1.2.1 produces the standard classification, by Young diagrams, of the unitary dual (i.e., the set of irreps) \( \hat{S}_n \) of \( S_n \), due to Frobenius and others [Frobenius68]. Indeed, for each \( D \in \mathcal{Y}_n \), let us denote by \( \sigma_D \) the unique joint component of \( Y_{D^t}(sgn) \) and \( Y_D \). Then,

**Corollary 1.3.1** (Classification). The irreps

\[
\sigma_D, \quad D \in \mathcal{Y}_n,
\] (1.5)

are pairwise non-isomorphic and exhaust \( \hat{S}_n \).

For a proof of Corollary 1.3.1 see A.1.2.

### 1.4. The Grothendieck Group of \( S_n \)

In Section 2 we will draw certain conclusions for the representation theory of the general linear group \( GL_n = GL_n(\mathbb{F}_q) \), using the properties obtained in this section for the representations of \( S_n \). An effective way to formulate this passage from \( S_n \) to \( GL_n \), is to use the formalism of the Grothendieck group of representations, and in particular to describe consequences of Theorem 1.2.1 to the structure of this group in the case of \( S_n \).

Given a finite group \( G \), we can consider the Abelian group \( K(G) \) generated from the set \( \hat{G} \) of isomorphism classes of irreps of \( G \) using the direct sum operation \( \oplus \). Note that \( K(G) \) has a natural partial order \( < \) given by the sub-representation relation, and it comes equipped with a bilinear form \( \langle \cdot, \cdot \rangle \), giving any two representations \( \pi, \tau \), their intertwining number \( \langle \pi, \tau \rangle \) (1.4).

In particular, \( K(S_n) \) is a free \( \mathbb{Z} \)-module with basis \( \hat{S}_n = \{ \sigma_D, \quad D \in \mathcal{Y}_n \} \), where \( \sigma_D \) are the irreps (1.5). However, \( K(S_n) \) has another natural \( \mathbb{Z} \)-basis, i.e.,

**Proposition 1.4.1.** The collection of Young modules \( Y_D, \quad D \in \mathcal{Y}_n \), forms a \( \mathbb{Z} \)-basis for \( K(S_n) \).
Proposition 1.4.1 follows from the following two consequences of Theorem 1.2.1:

Scholium 1.4.2. The following hold,

1. **Spectrum:** The irrep $\sigma_E$ (1.5), appears in the Young module $Y_D$ if and only if $D \leq E$.
2. **Characterization:** The irrep $\sigma_D$ (1.5) is the only irrep that appears in $Y_D$ but not in $Y_E$ for every $D \nleq E$.

In particular, from Part (2) of Scholium 1.4.2 we deduce that the collection of Young modules is a minimal generating set of $K(S_n)$, confirming Proposition 1.4.1.

We proceed to describe a class of irreps of $GL_n$, that in a formal sense behave as if they also form $K(S_n)$.

### 2. Spherical Principal Series Representations of $GL_n$

In this section we want first to construct/classify the spherical principal series representations, and second, to recast certain properties of this collection. Both tasks involve, as in the case of $S_n$, the dominance relation on the set $\mathcal{Y}_n$, of Young diagrams with $n$ boxes.

#### 2.1. The Spherical Principal Series

Inside $GL_n = GL_n(F_q)$, consider the Borel subgroup $B$ of upper triangular matrices

$$B = \begin{pmatrix} * & \ldots & * \\ \vdots & \ddots & \vdots \\ & & * \end{pmatrix}.$$  

Recall, see Section 0.2, that by definition an irreducible representation $\rho$ of $GL_n$ belongs to the spherical principal series (SPS) if it appears inside the induced representation $\text{Ind}_{B}^{GL_n}(1)$, where $1$ denotes the trivial representation of $B$.

The construction of the SPS representations, and the verification of some of their properties can be done intrinsically (e.g., see in Section 10.5. of [Gurevich-Howe17]), without the relation to the representation theory of $S_n$. However, for purposes of this note, we prefer to get all the information from what was obtained already for $S_n$ in Section 1. This, in particular, will enable us to derive the Pieri rule for $GL_n$ from that of $S_n$.

#### 2.2. The Grothendieck Group of the Spherical Principal Series

Let us denote by $K_B(GL_n)$ the Abelian group generated, using the operation of direct sum $\oplus$, from the SPS representations. The notion of subrepresentation induces a partial order $<$ on $K_B(GL_n)$ and the intertwining number pairing $\langle , \rangle$ (1.4) gives on it an inner product structure.

We proceed to give an effective description of $K_B(GL_n)$.

Recall, see Section 0.2, that the group $K_B(GL_n)$ has a distinguished collection of members in the form of induced representations that are associated to Young diagrams. Indeed, to a Young diagram $D \in \mathcal{Y}_n$ one attaches in a natural a way a flag $F_D$ in $F_q^n$, see Equation 0.2, and a
corresponding parabolic subgroup \( P_D = \text{Stab}_{GL_n}(P_D) \subset GL_n \). Then, we can consider the trivial representation \( 1 \) of \( P_D \), and induce to obtain

\[
I_D = \text{Ind}_{P_D}^{GL_n}(1) .
\] (2.1)

Of course each \( I_D \) sits inside \( \text{Ind}_{GL_n}^{GL_n}(1) \), but we can say much more on the relation between the various \( I_D \)'s. Indeed, for a given Young diagram \( D = (d_1 \geq \ldots \geq d_r) \in \mathcal{Y}_n \), we have defined in (1.1) the subgroup \( S_D \cong S_{d_1} \times \cdots \times S_{d_r} \subset S_n \) and the corresponding Young module \( Y_D = \text{Ind}_{S_D}^{S_n}(1) \).

Then, the Bruhat decomposition \([\text{Borel69, Bruhat56}]\) gives a bijection between the double cosets

\[
P_D \backslash GL_n / P_E \text{ and } S_D \backslash S_n / S_E ,
\] (2.2)

for every \( D, E \in \mathcal{Y}_n \).

But, the cardinalities of the double cosets in (2.2) are exactly the dimensions of, respectively, the intertwining spaces \( \text{Hom}_{GL_n}(I_D, I_E) \) and \( \text{Hom}_{S_n}(Y_D, Y_E) \), so we conclude that,

**Proposition 2.2.1** (Bruhat decomposition). For any two Young diagrams \( D, E \in \mathcal{Y}_n \), we have,

\[
\langle I_D, I_E \rangle = \langle Y_D, Y_E \rangle .
\] (2.3)

One way to interpret identity (2.3) is as follows:

**Corollary 2.2.2.** The correspondence

\[
Y_D \mapsto \hat{I}_D, \quad D \in \mathcal{Y}_n ,
\] (2.4)

induces an order preserving isometry

\[
\iota : K(S_n) \cong K_B(GL_n) .
\] (2.5)

On how to deduce Corollary 2.2.2 from Proposition 2.2.1, see the next section.

2.3. The Grothendieck Groups of \( S_n \) and of the Spherical Principal Series. We confirm Corollary 2.2.2 and along the way construct the SPS representations, and deduce various other facts on this collection.

Consider the map \( \iota \) (2.5), extended by (integral) linearity from the correspondence (2.4). Denote by

\[
\rho_D = \iota(\sigma_D), \quad D \in \mathcal{Y}_n ,
\] (2.6)

the element of \( K_B(GL_n) \) corresponding to the irrep (1.5) of \( S_n \). Note that,

- \( \rho_D \prec I_D \); and,
- \( \langle \rho_D, I_D \rangle = 1 \),

so, in particular, \( \rho_D \) is irreducible. In fact, the corresponding properties for \( S_n \) imply that

- \( \langle \rho_D, \text{Ind}_{B}^{GL_n}(1) \rangle = \dim(\sigma_D) \); and,
- we have,

\[
\{ \rho_D \} = \hat{I}_D \setminus \bigcup_{D \not\geq E} \hat{I}_E ,
\] (2.7)
i.e., $\rho_D$ is the unique irrep that sits in $I_D$ (we denote by $\hat{I}_D$ the set of irreps inside $I_D$) but not in $I_E$, for any Young diagram $E \in \mathcal{Y}_n$ that strictly dominates $D$.

**Remark 2.3.1.** In fact, Property (2.7) characterizes the representation $\rho_D$, and is useful, e.g., you can compute out of it explicitly the dimension of $\rho_D$ and find that (we use bold-face letters to denote the corresponding algebraic groups [Borel69]) it is equal to $\dim(\rho_D) = q^{\dim(\text{GL}_n/P_D)} + o(...)$, as $q \to \infty$, a fact that in turn characterizes (again, asymptotically) $\rho_D$ uniquely among all irreps in $I_D$.

How do we know we get all the SPS?

A possible answer is that, as we already said, each $I_D$ has a unique irrep that does not occur in the induced module $I_E$ corresponding to any strictly dominating diagram $E \succcurlyeq D$, namely, $\rho_D = \iota(\sigma_D)$. On the $S_n$ side, the irreps $\sigma_D$, $D \in \mathcal{Y}_n$, completely decompose each of the induced representations. By Bruhat, this transfers to $\text{GL}_n$, so we get complete decompositions over there also. In particular, we get a complete decomposition of $\text{Ind}_{\text{B}}^{\text{GL}_n}(1) = I_{(1,...,1)}$, the constituents of which are exactly the SPS representations.

Finally, the above discussion also validates Corollary 2.2.2.

Having at our disposal the understanding that the SPS representations and the representations of $S_n$ are in some formal sense the same thing, we can proceed to discuss the Pieri rule.

### 3. The Pieri Rule

Fix $0 \leq k \leq n$, and denote by $P_{k,n-k} \subset \text{GL}_n$ the parabolic subgroup fixing the first $k$ coordinate subspace of $\mathbb{F}_q^n$. There is a natural surjective map $P_{k,n-k} \twoheadrightarrow GL_k \times GL_{n-k}$. Take a Young diagram $D \in \mathcal{Y}_k$, and consider the irreducible SPS representation $\rho_D$ of $GL_k$ defined by (2.6). Denote by $1_{n-k}$ the trivial representation of $GL_{n-k}$. Pull back the representation $\rho_D \otimes 1_{n-k}$ from $GL_k \times GL_{n-k}$ to $P_{k,n-k}$ and form the induced representation

$$I_{\rho_D} = \text{Ind}_{P_{k,n-k}}^{\text{GL}_n}(\rho_D \otimes 1_{n-k}).$$

(3.1)

Recall (see Problem 0.2.1 in Section 0.2) that, the narrative of the story we are telling in this note is that, we are seeking to compute the decomposition of $I_{\rho_D}$ (3.1) into irreps. Moreover, it is easy to see that all constituents of the representation $I_{\rho_D}$ are SPS, so we are seeking an answer to the decomposition problem in term of Young diagrams.

To arrive at our goal, after introducing some needed terminology, we will

(a) State the Pieri rule for the long established case of the complex general linear group $\text{GL}_n(\mathbb{C})$.

(b) Recall Schur duality.

(c) State and prove the Pieri rule for representations of $S_n$.

Our proof was suggested by a remark of Nolan Wallach, and uses the Schur (a.k.a. Schur-Weyl) duality, to deduce the result from the Pieri rule for $\text{GL}_n(\mathbb{C})$. 
We note that, recently, in [Ceccherini-Silberstein-Scarabotti-Tolli10], the authors gave a different proof of the Pieri rule for $S_n$. Their treatment uses the Okounkov-Vershik approach [Okounkov-Vershik05] through the branching rule from $S_n$ to $S_{n-1}$ and Gelfand-Tsetlin basis.

We would also like to remark that, nowadays the Pieri rule for $S_n$ can be understood as a particular case of the celebrated Littlewood-Richardson rule [Littlewood-Richardson34, Macdonald79], but was known [Pieri1893] a long time before this general result.

(d) Derive the Pieri rule for the Group $GL_n = GL_n(\mathbb{F}_q)$, using the equivalence, discussed in Section 2.2 between the representation theory of the spherical principal series, and that of $S_n$.

3.1. Skew-Diagrams and Horizontal Strips. The various formulations we present of the Pieri rule use the notions of skew diagram and horizontal strip, that we recall here.

Suppose we have Young diagrams $E \in \mathcal{Y}_n$ and $D \in \mathcal{Y}_k$ such that $E$ contains $D$, denoted $E \supseteq D$, i.e., each row of $E$ is at least as long as the corresponding row of $D$. Then, by removing from $E$ all the boxes belonging to $D$, we obtain a configuration, denoted $E - D$, called skew-diagram [Macdonald79]. If, in addition—see Figure 4 for illustration, each column of $E$ is at most one box longer than the corresponding column of $D$, then we call $E - D$ a horizontal strip (or horizontal $m$-strip if $E - D$ has $m$ boxes).

Figure 4. In $\mathcal{Y}_2$: $(2, 1, 1), (3, 1)$, contain $(1, 1) \in \mathcal{Y}_2$, with difference a horizontal 2-strip.

Remark 3.1.1. In [Ceccherini-Silberstein-Scarabotti-Tolli10] the term that is being used for ”horizontal strip” is ”totally disconnected skew diagram”.

3.2. The Pieri Rule for $GL_n(\mathbb{C})$. The Pieri rule for $GL_n(\mathbb{C})$ is a (very) special case of the general Littlewood-Richardson rule [Howe-Lee12, Littlewood-Richardson34, Macdonald79] for decomposing the tensor product of any pair of irreducible finite dimensional representations of $GL_n(\mathbb{C})$. The Pieri rule has been known since the 19th century [Pieri1893], and is relatively easy to establish [Fulton-Harris91, Howe92, Weyman89].

There is a standard way to label the irreducible representations of $GL_n(\mathbb{C})$. It is by their highest weights (see, for example [Fulton-Harris91, Howe92, Weyl46]). A highest weight for $GL_n(\mathbb{C})$ is specified by a decreasing sequence

$$d_1 \geq ... \geq d_n,$$

of integers.
When all the $d_j$ are non-negative, the above sequence can be thought of as specifying a Young diagram $D$, with $j$-th row having length $d_j$. The number of boxes in $D$ can be arbitrarily large, but the number of rows is bounded by $n$. Irreps of $GL_n(\mathbb{C})$ corresponding to sequences with all $d_j$ non-negative are called polynomial representations. These are exactly all the irreps of $GL_n(\mathbb{C})$ that appear in the tensor powers $(\mathbb{C}^n)^{\otimes l}$ of $\mathbb{C}^n$ for some $l \geq 0$. (Any irrep of $GL_n(\mathbb{C})$ is isomorphic to a twist by a power of determinant of a polynomial representation.). We will denote by

$$\pi_n^D, \quad D = (d_1 \geq ... \geq d_n \geq 0), \quad (3.2)$$

the polynomial representation of $GL_n(\mathbb{C})$ whose highest weight corresponds to the diagram $D$.

The one-rowed diagrams, given by $(d_1, 0, ..., 0)$ correspond to the symmetric powers $S^{d_1}(\mathbb{C}^n)$.

**Proposition 3.2.1** (Pieri rule for $GL_n(\mathbb{C})$). The representation $\pi_n^D \otimes S^d(\mathbb{C}^n)$ is multiplicity free. Moreover, we have,

$$\pi_n^D \otimes S^d(\mathbb{C}^n) \simeq \sum_E \pi_n^E, \quad (3.3)$$

where $E$ runs through all diagrams such that

1. $D \subset E$; and,
2. $E - D$ is an horizontal $d$-strip.

### 3.3. Schur-Weyl Duality

The group $GL_n(\mathbb{C})$ is defined in terms of its action on $\mathbb{C}^n$. By taking tensor products, this action gives rise naturally to an action on the $l$-fold tensor product $(\mathbb{C}^n)^{\otimes l}$ of $\mathbb{C}^n$ with itself (a.k.a., the $l$-th tensor power of $\mathbb{C}^n$). Clearly, the permutation group $S_l$ also acts on $(\mathbb{C}^n)^{\otimes l}$ by permuting the factors of the product. This action of $S_l$ clearly commutes with the action of $GL_n(\mathbb{C})$. Schur-Weyl duality [Howe92, Schur27, Weyl46] says that

**Proposition 3.3.1** (Schur-Weyl duality - non-explicit form). The actions of $S_l$ and $GL_n(\mathbb{C})$ on $(\mathbb{C}^n)^{\otimes l}$ generate mutual commutants of each other.

From this, Burnside’s double commutant theorem [Burnside1905, Weyl46] lets us conclude that, as an $S_l \times GL_n(\mathbb{C})$-module, we have a decomposition

$$(\mathbb{C}^n)^{\otimes l} \simeq \sum_D \sigma_l^D \otimes \tau_n^D, \quad (3.4)$$

where $D \in \mathcal{Y}_l$ runs through diagrams with $l$ boxes, $\sigma_l^D$ are the associated irreps (1.5) of $S_l$, and the $\tau_n^D$ are appropriate irreps of $GL_n(\mathbb{C})$. Some computation then shows that, remarkably, $\tau_n^D$ is equal to the representation $\pi_n^D$ (provided of course that $D$ does not have more than $n$ rows; otherwise $\tau_n^D = 0$) given by Equation (3.2). Thus, we can rewrite (3.4), and obtain

**Proposition 3.3.2** (Schur-Weyl duality - explicit form). As an $S_l \times GL_n(\mathbb{C})$-module, we have the decomposition

$$(\mathbb{C}^n)^{\otimes l} \simeq \sum_D \sigma_l^D \otimes \pi_n^D, \quad (3.5)$$
where $D$ runs over all diagrams in $\mathcal{Y}_l$ with at most $n$ rows.

3.4. **The Pieri Rule for $S_n$.** With the usual notation, consider $k < n$, and $S_k \subset S_n$, in the standard way, as the group that fixes the last $n-k$ letters on which $S_n$ acts. Then the symmetric group on these letters is $S_{n-k}$, and we have the product $S_k \times S_{n-k} \subset S_n$.

Take a partition/Young diagram $D$ of size $k$, and let $\sigma_D$ be the associated irreducible representation (1.5) of $S_k$. Let $1_{n-k}$ be the trivial representation of $S_n$. Form the induced representation

$$I_{\sigma_D} = \text{Ind}_{S_k \times S_{n-k}}^{S_n} (\sigma_D \otimes 1_{n-k}),$$

(3.6)

of $S_n$.

The Pieri Rule for $S_n$ describes the decomposition of this induced representation into irreducible subrepresentations.

**Theorem 3.4.1 (Pieri rule for $S_n$).** The representation $I_{\sigma_D}$ (3.6) is multiplicity-free. It consists of one copy of each representation $\sigma_E$ of $S_n$, for diagrams $E \in \mathcal{Y}_n$, such that

1. $D \subset E$; and,
2. $E - D$ is an horizontal $(n-k)$-strip.

In Appendix A.2.1 we give our proof of Theorem 3.4.1 demonstrating how it follows from the Pieri rule for $GL_n(\mathbb{C})$, invoking the Schur-Weyl duality.

**Remark 3.4.2 (Description of the Young Module).** Theorem 3.4.1 can be used to give a recursive description of the Young module $Y_D$ (1.2).

Given a Young diagram $D \in \mathcal{Y}_n$ with $n$ boxes, let $D_s$ be the diagram consisting of the first $s$ rows of $D$, and let $k_s$ be the number of boxes in $D_s$. Suppose that in $D$ there are $r$ rows in all, so that $k_r = n$. Suppose we know how to decompose $Y_{D_s}$. Then, if we apply the Pieri rule to each component of $Y_{D_s}$ and $k_{s+1}$ is the number of boxes in $D_{s+1}$, we learn how to decompose $Y_{D_{s+1}}$. Starting with $s = 1$, we can successively decompose the $Y_{D_s}$ for all $s$ up to $r$, at which point we will have found the decomposition of $Y_D$.

For example, the above method provides us with the following combinatorial description of the multiplicity of the irrep $\sigma_E$, $E \in \mathcal{Y}_n$, in $Y_D$: it is the number of ways to fill $E$ with a nested family of sub-diagrams $E_s$, such that

- $E_s \subset E_{s+1}$; and,
- $E_{s+1} - E_s$ is a horizontal strip with $k_{s+1} - k_s$ boxes.

From this, we can see, again, that the multiplicity of $\sigma_D$ in $Y_D$ is 1.

3.5. **The Pieri Rule for $GL_n(\mathbb{F}_q)$.** Now we can finish our story, and deliver the answer to the introduction’s motivating problem of decomposing $I_{\rho_D}$ (3.1).

Note that the isomorphism $\iota$ (2.5), between the representation groups $K(S_n)$ and $K_{B}(GL_n)$, sends $I_{\sigma_D}$ to $I_{\rho_D}$. So the Pieri rule for $S_n$, implies the Pieri rule for $GL_n(\mathbb{F}_q)$, i.e., the same description as in Theorem 3.4.1 just replace there, $S_n$ by $GL_n(\mathbb{F}_q)$, and $\sigma_D$, $\sigma_E$, by $\rho_D$, $\rho_E$, respectively.
Remark 3.5.1 (Decomposing permutation representation on flag variety). Replacing the Young module $Y_D$, in Remark 3.4.2, by the induced representation $I_D = \text{Ind}_{P_D}^{GL_n}(1)$, which is the space of functions on the flag variety $GL_n/P_D$. We get a recursive formula for the decomposition into irreps of the permutation representation of $GL_n$ on functions on a fairly general flag variety.

Appendix A. Proofs

A.1. Proofs for Section II

A.1.1. Proof of Theorem 1.2.1 For a set $X$ let us denote by $L(X)$ the space of complex valued functions on $X$. We also use this notation to denote the standard permutation representation of a group $G$, in case it acts on $X$.

Now we can proceed to give the proof.

Proof. Part (1). Let us analyze the space of intertwiners $\text{Hom}(Y_E(sgn), Y_D)$. This has a ”geometric” description from which the information we are after can be read.

- First, recall that we can realize $Y_D$ as the permutation representation $L(T_D)$ associated with the action of $S_n$ on the set $T_D$ of all tabloids that one can make out of $D$ (see Section 1.1). In the same way, $Y_E(sgn)$ can be realized on the space $L(T_E)$ with the permutation action of $S_n$ on it twisted by $sgn$.

- Second, using the bases of delta functions of $L(T_E)$ and $L(T_D)$, we can associate to every intertwiner $\text{Hom}(Y_E(sgn), Y_D)$ a kernel function (i.e., a matrix) $K$ on $T_D \times T_E$ that satisfies

$$K(s(T_D), s(T_E)) = sgn(s)K(T_D, T_E),$$

for every $s \in S_n$, $T_D \in T_D$ and $T_E \in T_E$.

Let us denote by $L(T_D \times T_E)^{1 \otimes sgn}$ the collection of all $K$ satisfying Identity (A.1).

In summary, we obtained,

$$\text{Hom}(Y_E(sgn), Y_D) = L(T_D \times T_E)^{1 \otimes sgn}. \quad \text{(A.2)}$$

This is a geometric description of the space of intertwiners.

Now suppose we have $K$ from (A.2), and suppose there are $T_D \in T_D$ and $T_E \in T_E$ with rows, one of $T_D$ and one of $T_E$, that share two numbers $i, j \in \{1, \ldots, n\}$. Then, the permutation that transposes $i$ and $j$ must preserve $K(T_D, T_E)$, and also must change its sign. Therefore, $K(T_D, T_E) = -K(T_D, T_E)$, so $K(T_D, T_E) = 0$. In other words $K(T_D, T_E) \neq 0$ only if

- each number from the first row of $E$, should sit in a different row of $D$, so, $E_1 \leq D_1^t$.
  i.e., the length $E_1$ of the first row of $E$ is not more than that of the first row of $D^t$.
  and
- each number from the second row of $E$ should sit in a different row of $D$, so we also have, $E_1 + E_2 \leq D_1^t + D_2^t$. 

Namely, for the space $\text{(A.2)}$ to be non-trivial, it is necessary to have

\[(\ast) \quad E \preceq D^t.\]

Next, assuming $E = D^t$, we want to show that the intertwining space $\text{(A.2)}$ is one dimensional. Let us first give one orbit in $T_D \times T_{D^t}$ that supports a non-trivial intertwiner:

- Take the Young diagram $D$ and fill each box of it with numbers from $\{1, \ldots, n\}$. The object we obtained in this way is called Young tableau [Fulton97]. From it we can make in a natural way a Young tabloid $T_D$ by grouping together the numbers in each line of the tableau.
- We could also first ”transpose” the filled $D$ to obtain a tableau associated with $D^t$, and then, in the same way as above, form the corresponding tabloid $T_{D^t}$.

It is clear that, any two rows, one of $T_D$ and one of $T_{D^t}$, share no more than one number in common. Hence, the group $S_n$ acts freely on the orbit $O_{T_D, T_{D^t}} \subset T_D \times T_{D^t}$ (A.3) of $(T_D, T_{D^t})$, and, in particular, there exists an intertwiner $K$ from $\text{(A.2)}$ which is supported on it.

Now, let us show that (A.3) is the only orbit that supports such $K$. Indeed, take $s \in S_n$, such that $s(T_{D^t}) \neq T_{D^t}$. But then, there are rows, one of $s(T_{D^t})$, and one of $T_D$, that share two numbers in common, then, as was explained earlier, the orbit of $(T_D, s(T_{D^t}))$ does not support an intertwiner.

Finally, let us show that if $E \preceq D^t$, then the space $\text{(A.2)}$ is non-zero, i.e., the condition $(\ast)$ is also sufficient. It is enough to examine the case when $E = (D^t)^o$ obtained from $D^t$, by moving one box down to form a new lower row. Now, look at the tabloids $T_D$ and $T_{D^t}$, that we used in the paragraph just above, and the natural tabloid $T_{(D^t)^o}$ one obtains from the filling of $D^t$ by numbers as we did above in order to create $T_{D^t}$. Then, as we argued above, the orbit of $(T_D, T_{(D^t)^o})$ supports a non-trivial intertwiner.

**Part (2).** If $D$ is not dominated by $E$, then from Part (1) we see that $\langle Y_{E^t} (\text{sgn}), Y_D \rangle = 0$, and in particular, again by Part (1), $Y_E$ cannot be a subrepresentation of $Y_D$.

On the other hand, let us assume that $D$ is strictly dominated by $E$, and show that $Y_E \nleq Y_D$. First, we realize the space of intertwiners between $Y_E$ and $Y_D$ geometrically,

$$Hom(Y_E, Y_D) = L(T_D \times T_E)^{S_n},$$

where on the right hand side of the equality we have, the space of $S_n$-invariant kernels $K$ on $T_D \times T_E$, or equivalently the space of functions on the set of orbits $S_n \setminus (T_D \times T_E)$.

Second, we can parametrize the above set of orbits as follows. Take $T_D \in T_D$, and $T_E \in T_E$, and denote by $R_i(T_D)$ and $R_j(T_E)$, the $i$-th row of $T_D$, and $j$-th row of $T_E$, respectively. Then, we can
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Define the intersection matrix
\[
R_{R,T} = (r_{ij}), \quad r_{ij} = \# ( R_i(T_D) \cap R_j(T_E) ) \tag{A.4}
\]
i.e., \( r_{ij} \) is the number of elements common to both rows. It is clear that \( R_{R,T} \) is an invariant of the orbit. Moreover, it gives a complete invariant. Indeed, it is not difficult to see that if \( R_{R,T} = R_{R',T'} \), then there exists \( s \in S_n \) such that \( s(T_D) = T'_D \), and \( s(T_E) = T'_E \).

A direct computation, using the parametrization \((A.4)\), reveals that,

**Claim A.1.1.** Consider the Young diagrams \( D_{n-k,k} = (n-k,k) \) and \( D_{n-k',k'} = (n-k',k') \), where \( 0 \leq k, k' \leq n/2 \). Then,
\[
\langle Y_{D_{n-k,k}}, Y_{D_{n-k',k'}} \rangle = \min\{k + 1, k' + 1\}.
\]

So \( Y_{D_{n,0}} \) contains 1 representation - the trivial representation. Then \( Y_{D_{n-1,1}} \) contains two representations, one of which is the trivial representation. Since \( Y_{D_{n-2,2}} \) has intertwining number 1 with \( Y_{D_{n,0}} \) and 2 with \( Y_{D_{n-1,1}} \), it must contain the two representations of \( Y_{D_{n-1,1}} \) with multiplicity 1 each. Since its self intertwining number is 3, it contains 3 representations, each with multiplicity 1. Then we can continue like that: \( Y_{D_{n-3,3}} \) contains each of the representations of \( Y_{D_{n-2,2}} \) with multiplicity 1, and then one new representation, and so on. So in particular:

\((**)\) \( Y_{D_{n-k-1,k+1}} \) contains \( Y_{D_{n-k,k}} \) when \( k + 1 \leq n/2 \).

Now take any diagram \( D \), containing two rows \( R \) and \( R' \), with \( R' \) (which might be of length equal to 0) at least two boxes shorter than \( R \). Then we can form \( Y_D \) by first forming the representation \( Y_{D_{R,R'}} \) of \( S_{R+R'} \), and then extending to be trivial on the stabilizers of the other rows, and then inducing up to \( S_n \). So if we replace \( R \) and \( R' \) with \( R - 1 \) and \( R' + 1 \), we will get a larger representation, using Fact \((**)\). This completes the verification of Part (2), and of Theorem 1.2.1 \[\square\]

### A.1.2. Proof of Corollary 1.3.1

**Proof.** Note that the dominance order on \( \mathcal{Y}_n \) is a partial order, and in particular, is anti-symmetric, i.e., for every \( E, D \in \mathcal{Y}_n \), if \( E \preceq D \) and \( E \succeq D \), then \( E = D \). But, if \( \sigma_E \simeq \sigma_D \), then, by the "iff" of Part (1) of Theorem 1.2.1 \( E \preceq D \) and \( E \succeq D \), so the Corollary follows. \[\square\]

### A.2. Proofs for Section 3

#### A.2.1. Proof of Theorem 3.4.1

**Proof.** Schur duality for \( S_k \times GL_n(\mathbb{C}) \) on the \( k \)-fold tensor product \( (\mathbb{C}^n)^\otimes k \) says (Proposition 3.3.2 Equation (3.5)) that we have
\[
(\mathbb{C}^n)^\otimes k \simeq \sum_{D \in \mathcal{Y}_k} \sigma_k^D \otimes \pi_n^D,
\]
where \( \pi_n^D \) is the irrep of \( GL_n(\mathbb{C}) \) with highest weight corresponding to the diagram \( D \).

We can also apply Schur duality to the action of \( S_{n-k} \times GL_n(\mathbb{C}) \) on \( (\mathbb{C}^n)^\otimes (n-k) \). Then the space of fixed vectors for \( S_{n-k} \) is the \( S_{n-k} \times GL_n(\mathbb{C}) \)-module \( 1_{n-k} \otimes \pi_n^{(n-k)} \), corresponding to the diagram
with one row of length \( n - k \). This is just the \((n - k)\)-th symmetric power of the standard action on \( \mathbb{C}^n \).

Now consider,
\[
(\mathbb{C}^n)^\otimes^n \simeq (\mathbb{C}^n)^\otimes^k \otimes (\mathbb{C}^n)^\otimes^{n-k},
\]
as an \( S_n \)-module, again with \( S_n \) acting by permutation of the factors. If we take the isotypic component of \( \sigma_D \) inside \((\mathbb{C}^n)^\otimes^k\), and the space of fixed vectors for \( S_{n-k} \) inside \((\mathbb{C}^n)^\otimes^{n-k}\), their tensor product will be the isotypic component inside \((\mathbb{C}^n)^\otimes^n\) of the representation \( \sigma_D \otimes 1_{n-k} \) of \( S_k \times S_{n-k} \).

On the other hand, the action of \( GL_n(\mathbb{C}) \) on the indicated tensor product is described by (a multiple of) the tensor product \( \pi^D_n \otimes \pi_{(n-k)}^n \). By the Pieri rule for \( GL_n(\mathbb{C}) \) (see Proposition \ref{prop:3.2.1}) this decomposes into a multiplicity free sum of irreps for \( GL_n(\mathbb{C}) \) whose highest weights are given by diagrams \( E \) having the form indicated in the statement of the proposition: \( E \) has \( n \) boxes, contains \( D \), and \( E - D \) consists of a horizontal \((n - k)\)-strip. Thus, the \( S_k \times S_{n-k} \) isotypic component for \( \sigma_D \otimes 1_{n-k} \) of \((\mathbb{C}^n)^\otimes^n\) has the structure
\[
\sum_E \sigma_D \otimes 1_{n-k} \otimes \pi^E_n,
\]
as \( S_k \times S_{n-k} \times GL_n(\mathbb{C}) \)-module, where \( E \) runs over the diagrams specified in the statement of the proposition.

Now consider the representation of \( S_n \) generated by this space. By Schur duality for \( S_n \), it will be
\[
\sum_E \sigma_E \otimes \pi^E_n
\]
as \( S_n \times GL_n(\mathbb{C}) \)-module. Comparing this with Formula \((A.5)\), we conclude that each representation \( \sigma_E \) of \( S_n \) contains one copy of the representation \( \sigma_D \otimes 1_{n-k} \) when restricted to \( S_k \times S_{n-k} \), and that these are the only representations of \( S_n \) that do contain \( \sigma_D \otimes 1_{n-k} \). By Frobenius reciprocity, this is equivalent to the statement of Theorem \ref{thm:3.4.1}. The proof is complete. \( \square \)

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