Maximal Function and Riesz Transform Characterizations of Hardy Spaces Associated with Homogeneous Higher Order Elliptic Operators and Ball Quasi-Banach Function Spaces

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Abstract Let $L$ be a homogeneous divergence form higher order elliptic operator with complex bounded measurable coefficients on $\mathbb{R}^n$ and $X$ a ball quasi-Banach function space on $\mathbb{R}^n$ satisfying some mild assumptions. Denote by $H_{X,L}(\mathbb{R}^n)$ the Hardy space, associated with both $L$ and $X$, which is defined via the Lusin area function related to the semigroup generated by $L$. In this article, the authors establish both the maximal function and the Riesz transform characterizations of $H_{X,L}(\mathbb{R}^n)$. The results obtained in this article have a wide range of generality and can be applied to the weighted Hardy space, the variable Hardy space, the mixed-norm Hardy space, the Orlicz–Hardy space, the Orlicz-slice Hardy space, and the Morrey–Hardy space, associated with $L$. In particular, even when $L$ is a second order divergence form elliptic operator, both the maximal function and the Riesz transform characterizations of the mixed-norm Hardy space, the Orlicz-slice Hardy space, and the Morrey–Hardy space, associated with $L$, obtained in this article, are totally new.

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2020 Mathematics Subject Classification. Primary 42B30; Secondary 42B25, 47B06, 35J30, 42B35.
Key words and phrases. ball quasi-Banach function space, high order elliptic operator, Hardy space, maximal function, Riesz transform.

This project is partially supported by the National Key Research and Development Program of China (Grant No. 2020YFA0712900), the National Natural Science Foundation of China (Grant Nos. 11971058, 12071197, 12122102, 11871100, 11871254 and 12071431) and the Fundamental Research Funds for the Central Universities (Grant No. lzujbky-2021-ey18).

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1 Introduction

The real-variable theory of the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (0, 1]$, creatively initiated by Stein and Weiss [71] and then further seminally developed by Fefferman and Stein [35], plays a key role in both harmonic analysis and partial differential equations (see, for instance, [23, 35, 67] and the references therein). It is well known that, when $p \in (0, 1]$, the Hardy space $H^p(\mathbb{R}^n)$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ in the study of the boundedness of some classical operators. For instance, when $p \in (0, 1]$, the Riesz transform is bounded on $H^p(\mathbb{R}^n)$ but not on $L^p(\mathbb{R}^n)$. However, there exist many settings in which the real-variable theory of the Hardy space can not be applicable; for instance, the Riesz transform $\nabla L^{-1/2}$ may not be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $L := -\text{div}(A\nabla)$ is a second order divergence elliptic operator with complex bounded measurable coefficients (see, for instance, [46, 47]).

In recent years, there has been a lot of studies which pay attention to the real-variable theory of Hardy spaces associated with operators. Here, we give a brief overview of this research field. Auscher et al. [4] first introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated with an operator $L$ whose heat kernel satisfies a pointwise Gaussian upper bound estimates and obtained its molecular characterization. Moreover, Duong and Yan [33, 34] established the dual theory of the Hardy space $H^1_L(\mathbb{R}^n)$. Yan [73] further generalized these results to the Hardy space $H^p_L(\mathbb{R}^n)$ with certain $p \in (0, 1]$. Furthermore, Hardy spaces associated operators satisfying the Davies–Gaffney estimates which are weaker than the Gaussian upper bound estimates were studied in [5, 16, 17, 32, 45, 46, 47]. Meanwhile, the real-variable theory of various variants of Hardy spaces associated with operators were developed; see, for instance, [10, 12, 70] for weighted Hardy spaces associated with operators, [11, 15, 51, 52, 76] for (Musielak–)Orlicz–Hardy spaces associated with operators, and [77, 78] for variable Hardy spaces associated with operators. For more studies on Hardy spaces and other function spaces associated with operators, we refer the reader to [13, 14, 38, 39, 40, 55, 3, 69, 75, 68] and the references therein.

In particular, let $L$ be a homogeneous divergence form higher order elliptic operator with complex bounded measurable coefficients on $\mathbb{R}^n$ (see, for instance, [17] or Subsection 2.1 below for its detailed definition). For the Hardy space $H^p_L(\mathbb{R}^n)$ with $p \in (0, 1]$, Cao and the second author of this article [18] established several characterizations of $H^p_L(\mathbb{R}^n)$ by means of the molecule, the square function, and the Riesz transform. Furthermore, Deng et al. [29] also established the corresponding characterizations of the Hardy space $H^1_L(\mathbb{R}^n)$. Moreover, Cao et al. [17] obtained the various maximal function characterizations of $H^p_L(\mathbb{R}^n)$.

Let $X$ be a ball quasi-Banach function space on $\mathbb{R}^n$ (see, for instance, [66] or Subsection 2.2

5 Applications

5.1 Weighted Hardy Spaces

5.2 Variable Hardy Spaces

5.3 Mixed-norm Hardy Spaces

5.4 Orlicz–Hardy Spaces

5.5 Orlicz-slice Hardy Spaces

5.6 Morrey–Hardy Spaces
below for its detailed definition) and $L$ a homogeneous divergence form higher order elliptic operator with complex bounded measurable coefficients on $\mathbb{R}^n$. Sawano et al. [66] introduced the Hardy space $H_{X,L}(\mathbb{R}^n)$ associated with both $X$ and $L$, via the Lusin area function related to the semigroup generated by $L$, and established its molecular characterization. It is then quite natural to ask whether or not there exist the maximal function and the Riesz transform characterizations of $H_{X,L}(\mathbb{R}^n)$.

Recall that ball quasi-Banach function spaces were introduced in [66] to include more important function spaces than quasi-Banach function spaces originally introduced in [9, Definitions 1.1 and 1.3]. Indeed, the former includes (weighted) Lebesgue spaces, variable Lebesgue spaces, mixed-norm Lebesgue spaces, Orlicz spaces, Orlicz-slice spaces, and Morrey spaces which, except Lebesgue spaces, are usually not quasi-Banach function spaces (see, for instance, [66, 72, 79]).

The main purpose of this article is to answer the aforementioned question, namely, to establish both the maximal function and the Riesz transform characterizations of the Hardy space $H_{X,L}(\mathbb{R}^n)$ associated with both $X$ and $L$. More precisely, we obtain the radial and the non-tangential maximal function characterizations of the space $H_{X,L}(\mathbb{R}^n)$ under some mild assumptions on both $X$ and $L$. Moreover, we prove that the Riesz transform $\nabla^m L^{-1/2}$ associated with $L$ is bounded from $H_{X,L}(\mathbb{R}^n)$ to $H_X(\mathbb{R}^n)$ and further establish the Riesz transform characterization of $H_{X,L}(\mathbb{R}^n)$. Here, $H_X(\mathbb{R}^n)$ denotes the Hardy space associated with $X$ introduced in [66]. Furthermore, some applications of the main results obtained as above to some concrete function spaces are given. It is worth pointing out that, when $X$ is just the classical Lebesgue space, both the maximal function and the Riesz transform characterizations of $H_{X,L}(\mathbb{R}^n)$ were established in [17, 18]; however, when $L$ is a general homogeneous divergence form higher order elliptic operator with complex bounded measurable coefficients on $\mathbb{R}^n$ (except a second order divergence form elliptic operator) and $X$ is one of the weighted Lebesgue space, the variable Lebesgue space, the mixed-norm Lebesgue space, the Orlicz space, the Orlicz-slice space, or the Morrey space, both the maximal function and the Riesz transform characterizations of $H_{X,L}(\mathbb{R}^n)$ are totally new.

Compared with the second order divergence form elliptic operator, the argument arose in the homogeneous divergence form higher order elliptic operator $L$ is more complicated. Because of this and the deficiency of the explicit expression of the norm for the ball quasi-Banach function space $X$, the methods used in [17] are no longer applicable. To overcome these difficulties, in this article, by the Sobolev embedding theorem, we deal with the higher derivative caused by the higher order elliptic operator. Meanwhile, by taking full advantage of the divergence form elliptic operator, the Besicovitch covering lemma, and the Caccioppoli inequality, we first establish a good-$\lambda$ inequality concerning both the non-tangential maximal function and the truncated Lusin area function associated with the heat semigroup of $L$ in Lemma 3.6 below. Using this good-$\lambda$ inequality and borrowing some ideas from the proof of the extrapolation theorem in the scale of Banach function spaces (see, for instance, [26]), we then establish the maximal function characterization of the Hardy space $H_{X,L}(\mathbb{R}^n)$ associated with both $X$ and $L$. Moreover, to obtain the Riesz transform characterization of $H_{X,L}(\mathbb{R}^n)$, following an approach similar to that used in [47, 18], we need to introduce the homogeneous Hardy--Sobolev space associated with the ball quasi-Banach function space $X$ and establish its atomic characterization. When $X$ is just the Lebesgue space, the atomic characterization of the homogeneous Hardy--Sobolev space associated with $X$ can be obtained easily by the connection between the classical homogeneous Hardy--Sobolev space and the homogeneous Triebel--Lizorkin space. However, this is impossible when $X$ is a general ball.
quasi-Banach function space. Motivated by [57, 41], we directly establish the atomic decomposition of the homogeneous Hardy–Sobolev space associated with the ball quasi-Banach function space \( X \) via both the atomic decomposition of the X-tent space \( T_X(R^{n+1}_+) \) and the Calderón reproducing formula. Then, using the atomic characterization of the homogeneous Hardy–Sobolev space associated with \( X \), we finally obtain the Riesz transform characterization of \( H_{X, L}(R^n) \).

The remainder of this article is organized as follows.

In Section 2, we state some known results on the homogeneous divergence form higher order elliptic operator \( L \), the ball quasi-Banach function space \( X \), and the Hardy space \( H_{X, L}(R^n) \) associated with both \( X \) and \( L \).

In Section 3, we establish the radial and the non-tangential maximal function characterizations of the Hardy space \( H_{X, L}(R^n) \) (see Theorem 3.1 below). By making full use of the special structure of the divergence form elliptic operator, the Caccioppoli inequality, the Besicovitch covering lemma, and the Sobolev embedding theorem, we first establish a good-\( \lambda \) inequality concerning the non-tangential maximal function and the truncated Lusin area function associated with the heat semigroup generated by \( L \), which plays an important role in the proof of Theorem 3.1 (see Lemma 3.6 below). Moreover, to overcome the essential difficulty caused by the deficiency of the explicit expression of the norm of the ball quasi-Banach function space \( X \), we cleverly use the extrapolation theorem in the scale of ball Banach function spaces (see, for instance, [26] or Lemma 3.8 below).

Section 4 is devoted to establishing the Riesz characterization of the Hardy space \( H_{X, L}(R^n) \) (see Theorems 4.2 and 4.4 below). In Subsection 4.1, with the help of the molecular characterization of both the Hardy spaces \( H_X(R^n) \) and \( H_{X, L}(R^n) \), we prove that the Riesz transform \( \nabla^n L^{-1/2} \) is bounded from \( H_{X, L}(R^n) \) to \( H_X(R^n) \) under some mild assumptions on the ball quasi-Banach space \( X \) (see Theorem 4.2 below). Furthermore, in Subsection 4.2, motivated by [57, 41], we first introduce the homogeneous Hardy–Sobolev space \( H_{m, X}(R^n) \) associated with the ball quasi-Banach function space \( X \) (see Definition 4.12 below for the details). Then, using both the atomic decomposition of the X-tent space \( T_X(R^{n+1}_+) \) and the Calderón reproducing formula, we establish the atomic decomposition of the homogeneous Hardy–Sobolev space \( H_{m, X}(R^n) \), which is essential in the proof of Theorem 4.4. Applying the atomic decomposition of the homogeneous Hardy–Sobolev space \( H_{m, X}(R^n) \) and following the approach used in [47, 18], we then obtain the Riesz transform characterization of \( H_{X, L}(R^n) \).

In Section 5, we give some applications of both the maximal function and the Riesz transform characterizations of the space \( H_{X, L}(R^n) \) obtained in Sections 3 and 4, respectively, to the weighted Hardy space, the variable Hardy space, the mixed-norm Hardy space, the Orlicz–Hardy space, the Orlicz-slice Hardy space, and the Morrey–Hardy space, associated with the operator \( L \). It is worth pointing out that, when \( L \) is a general homogeneous divergence form higher order elliptic operator with complex bounded measurable coefficients on \( R^n \) (except a second order divergence form elliptic operator), both the maximal function and the Riesz transform characterizations of the weighted Hardy space, the variable Hardy space, the mixed-norm Hardy space, the Orlicz–Hardy space, the Orlicz-slice Hardy space, and the Morrey–Hardy space associated with the operator \( L \), obtained in this article, are new. In particular, even when \( L \) is just a second order divergence form elliptic operator, both the maximal function and the Riesz transform characterizations of the mixed-norm Hardy space, the Orlicz-slice Hardy space, and the Morrey–Hardy space associated with \( L \), established in this article, are totally new.
Due to the generality and the practicability, more applications of these main results of this article are predictable.

Finally, we make some conventions on notation. Let \( \mathbb{N} := \{1, 2, \ldots\} \), \( \mathbb{Z}_+ := \mathbb{N} \cup \{0\} \), and \( \mathbb{Z}_+^n := (\mathbb{Z}_+)^n \). We always denote by \( C \) a positive constant which is independent of the main parameters, but it may vary from line to line. We also use \( C_{(a, b, \ldots)} \) to denote a positive constant depending on the indicated parameters \( a, b, \ldots \). The symbol \( f \lesssim g \) means that \( f \leq Cg \). If \( f \leq g \) and \( g \leq f \), we then write \( f \sim g \). If \( f \leq Cg \) and \( g = h \) or \( g \leq h \), we then write \( f \lesssim g \sim h \) or \( f \lesssim g = h \) or \( f \lesssim g \leq h \). We use \( 0 \) to denote the origin of \( \mathbb{R}^n \). For any measurable subset \( E \) of \( \mathbb{R}^n \), we denote by \( 1_E \) its characteristic function. For any \( \alpha \in (0, \infty) \) and any ball \( B := B(x_B, r_B) \) in \( \mathbb{R}^n \), with \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \), let \( \alpha B := B(x_B, \alpha r_B) \). Denote by \( \mathcal{Q} \) the set of all the cubes having their edges parallel to the coordinate axes. For any \( j \in \mathbb{N} \) and any ball \( B \subset \mathbb{R}^n \), let \( U_j(B) := (2^{j+1}B) \setminus (2^jB) \) and \( U_0(B) := 2B \). Denote the differential operator \( \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \) simply by \( \partial^{\alpha} \), where \( \alpha := (\alpha_1, \ldots, \alpha_n) \) and \( |\alpha| := \alpha_1 + \cdots + \alpha_n \). Denote by \( S(\mathbb{R}^n) \) the space of all Schwartz functions, equipped with the well-known topology determined by a countable family of norms, and by \( S'(\mathbb{R}^n) \) its dual space, equipped with the weak-* topology (namely, the space of all tempered distributions). For any \( f \in S'(\mathbb{R}^n) \) and \( i \in \{1, \ldots, n\} \), denote \( \frac{\partial f}{\partial x_i} \) by \( \partial_i f \), and let
\[
\nabla f := (\partial_1 f, \ldots, \partial_n f).
\]

Let \( \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty) \). For any sets \( E, F \subset \mathbb{R}^n \), let
\[
\text{dist} (E, F) := \inf_{x \in E, y \in F} |x - y|.
\]

Finally, for any given \( q \in [1, \infty] \), we denote by \( q' \) its conjugate exponent, namely, \( 1/q + 1/q' = 1 \).

## 2 Preliminaries

In this section, we state some known concepts and facts on the homogeneous higher order elliptic operator \( L \), the ball quasi-Banach function space \( X \), and the Hardy space \( H_{X,L}(\mathbb{R}^n) \) associated with both \( X \) and \( L \).

### 2.1 Homogeneous Higher Order Elliptic Operators

In this subsection, we recall the definition and some properties of the homogeneous divergence form higher order elliptic operator \( L \).

Let \( m \in \mathbb{N} \) and \( W^{m,2}(\mathbb{R}^n) \) be the \( m \)-order homogeneous Sobolev space equipped with the norm
\[
\|f\|_{W^{m,2}(\mathbb{R}^n)} := \left[ \sum_{|\alpha| = m} \|\partial^{\alpha} f\|_{L^2(\mathbb{R}^n)}^2 \right]^{1/2} < \infty,
\]
where, for any given \( \alpha \in \mathbb{Z}_+^n \) with \( |\alpha| = m \), \( \partial^{\alpha} f \) denotes the \( m \)-order derivative in the sense of distributions. For any multi-indices \( \alpha, \beta \in \mathbb{Z}_+^n \) satisfying \( |\alpha| = m = |\beta| \), let \( a_{\alpha, \beta} \) be a complex
bounded measurable function on \( \mathbb{R}^n \). For any given \( f, g \in W^{m,2}(\mathbb{R}^n) \), define the sesquilinear form \( a_0 \), mapping \( W^{m,2}(\mathbb{R}^n) \times W^{m,2}(\mathbb{R}^n) \) into \( \mathbb{C} \), by setting

\[
a_0(f, g) := \sum_{|\alpha|=|\beta|=m} \int_{\mathbb{R}^n} a_{\alpha,\beta}(x) \partial^\alpha f(x) \overline{\partial^\beta g(x)} \, dx.
\]

The following both ellipticity conditions on \( a_0 \) imply Ellipticity Condition 2.1. How-ever, the equivalence between Ellipticity Condition 2.1 and Strong Ellipticity Condition 2.2 is only a specific feature of second order divergence form elliptic operators (see, for instance, [17, Remark 2.1]), which is no longer true for divergence form elliptic operators of order greater than two.

**Ellipticity Condition 2.1.** Let \( m \in \mathbb{N}_0 \). There exist constants \( 0 < \Lambda_0 < \infty \) such that, for any \( f, g \in W^{m,2}(\mathbb{R}^n) \),

\[
a_0(f, g) \leq \Lambda_0 \| \nabla^m f \|_{L^2(\mathbb{R}^n)} \| \nabla^m g \|_{L^2(\mathbb{R}^n)}
\]

and

\[
\Re(a_0(f, g)) \geq \lambda_0 \| \nabla^m f \|_{L^2(\mathbb{R}^n)}^2.
\]

Here and thereafter, for any \( z \in \mathbb{C} \), \( \Re z \) denotes the real part of \( z \) and

\[
\| \nabla^m f \|_{L^2(\mathbb{R}^n)} := \left[ \sum_{|\alpha|=m} \int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 \, dx \right]^{1/2}.
\]

**Strong Ellipticity Condition 2.2.** Let \( m \in \mathbb{N}_0 \). There exists a positive constant \( \lambda_1 \) such that, for any \( \xi := \{\xi_\alpha\}_{|\alpha|=m} \) with \( \xi_\alpha \in \mathbb{C} \), and for almost every \( x \in \mathbb{R}^n \),

\[
\Re \left\{ \sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}(x) \xi_\beta \overline{\xi_\alpha} \right\} \geq \lambda_1 |\xi|^2 = \lambda_1 \left\{ \sum_{|\alpha|=m} |\xi_\alpha|^2 \right\}.
\]

It is easy to prove that Strong Ellipticity Condition 2.2 implies Ellipticity Condition 2.1. However, the equivalence between Ellipticity Condition 2.1 and Strong Ellipticity Condition 2.2 is only a specific feature of second order divergence form elliptic operators (see, for instance, [17, Remark 2.1]), which is no longer true for divergence form elliptic operators of order greater than two.

Assume that, for any \( \alpha, \beta \in \mathbb{Z}_+^n \) satisfying \( |\alpha| = m = |\beta| \), \( a_{\alpha,\beta} \) is a complex bounded measurable function on \( \mathbb{R}^n \). Assume further that the sesquilinear form \( a_0 \) satisfies Ellipticity Condition 2.1. Then it is well known that there exists a densely defined operator \( L \) in \( L^2(\mathbb{R}^n) \) associated with \( a_0 \) (see, for instance, [17, p. 830]), which is formally written as

\[
L := \sum_{|\alpha|=m=|\beta|} (-1)^m \partial^\alpha (a_{\alpha,\beta} \overline{\partial^\beta}.
\]

Usually, we call \( L \) a homogeneous divergence form \( 2m \)-order elliptic operator on \( \mathbb{R}^n \).

Denote by \( (p_-, L), p_+(L) \) the interior of the maximal interval of exponents \( p \in [1, \infty] \), for which the family \( \{e^{-tL}\}_{t \in (0, \infty)} \) of operators is bounded on \( L^p(\mathbb{R}^n) \). Meanwhile, denote by \( (q_-, L), q_+(L) \) the interior of the maximal interval of exponents \( q \in [1, \infty] \) such that the family \( \{\sqrt[n]{n}^{m}e^{-tL}\}_{t \in (0, \infty)} \) of operators is bounded on \( L^q(\mathbb{R}^n) \). For the exponents \( p_-, L), p_+(L), q_-(L), \) and \( q_+(L) \), we have the following conclusions (see, for instance, [3, p. 67] and [17, Proposition 2.5]).
Proposition 2.3. Let \( n, m \in \mathbb{N} \) and \( L \) be a homogeneous divergence form \( 2m \)-order elliptic operator on \( \mathbb{R}^n \) in (2.1) satisfying Ellipticity Condition 2.1. Then the following conclusions hold true:

(i) It holds true that
\[
\left\{ (p_-(L), p_+(L)) \right\} = (1, \infty) \quad \text{if } n \leq 2m,
\]
\[
\left[ \frac{2n}{n + 2m}, \frac{2n}{n - 2m} \right] \subset (p_-(L), p_+(L)) \quad \text{if } n > 2m.
\]

(ii) \( q_-(L) = p_-(L) \) and \( q_+(L) \in (2, \infty) \).

(iii) For any \( k \in \mathbb{Z}_+ \) and \( p_-(L) < p \leq q \leq p_+(L) \), the family \( \{ (tL)^k e^{-tL} \}_{t \in (0, \infty)} \) of operators satisfies the following \( m - L^p - L^q \) off-diagonal estimates: There exist positive constants \( C_1 \) and \( C_2 \) such that, for any closed sets \( E \) and \( F \) of \( \mathbb{R}^n \), any \( t \in (0, \infty) \), and any \( f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \) supported in \( E \),
\[
\| (tL)^k e^{-tL} (f) \|_{L^q(F)} \leq C_1 t^m \left( \frac{1}{2} \right) \exp \left\{ -C_2 \frac{\dist (E, F)}{t^{\frac{1}{2m+1}}} \right\} \| f \|_{L^p(E)}.
\]

Furthermore, we also have the following Caccioppoli inequality which was obtained in [17, Proposition 3.2].

Proposition 2.4. Let \( m \in \mathbb{N} \) and \( L \) be a homogeneous divergence form \( 2m \)-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. For any \( f \in L^2(\mathbb{R}^n) \), \( x \in \mathbb{R}^n \), and \( t \in (0, \infty) \), let \( u(x, t) := e^{-tL} - (f)(x) \). Then there exists a positive constant \( C \), independent of \( f \), such that, for any \( x_0 \in \mathbb{R}^n \), \( r \in (0, \infty) \), and \( t_0 \in (3r, \infty) \),
\[
\int_{B(x_0, r)} \int_{B(x_0, r)} |\nabla^m u(x, t)|^2 \, dx \, dt \leq \frac{C}{r^{2m}} \int_{B(x_0, 2r)} \int_{B(x_0, 2r)} |u(x, t)|^2 \, dx \, dt.
\]

2.2 Ball Quasi-Banach Function Spaces

In this subsection, we recall some preliminaries on ball quasi-Banach function spaces introduced in [66]. Denote by the symbol \( \mathcal{M}(\mathbb{R}^n) \) the set of all measurable functions on \( \mathbb{R}^n \). For any given \( x \in \mathbb{R}^n \) and \( r \in (0, \infty) \), let \( B(x, r) := \{ y \in \mathbb{R}^n : |x - y| < r \} \) and
\[
(2.2) \quad \mathbb{B} := \{ B(x, r) : x \in \mathbb{R}^n \text{ and } r \in (0, \infty) \}.
\]

Definition 2.5. A quasi-Banach space \( X \subset \mathcal{M}(\mathbb{R}^n) \), equipped with a quasi-norm \( \| \cdot \|_X \) which makes sense for all functions in \( \mathcal{M}(\mathbb{R}^n) \), is called a ball quasi-Banach function space if it satisfies that

(i) for any \( f \in \mathcal{M}(\mathbb{R}^n) \), \( \| f \|_X = 0 \) implies that \( f = 0 \) almost everywhere;

(ii) for any \( f, g \in \mathcal{M}(\mathbb{R}^n) \), \( |g| \leq |f| \) almost everywhere implies that \( \| g \|_X \leq \| f \|_X \);

(iii) for any \( \{ f_m \}_{m \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n) \) and \( f \in \mathcal{M}(\mathbb{R}^n) \), \( 0 \leq f_m \uparrow f \) almost everywhere as \( m \to \infty \) implies that \( \| f_m \|_X \uparrow \| f \|_X \) as \( m \to \infty \).
(iv) \( B \in \mathbb{B} \) implies that \( \mathbf{1}_B \in X \), where \( \mathbb{B} \) is the same as in (2.2).

Moreover, a ball quasi-Banach function space \( X \) is called a ball Banach function space if the norm of \( X \) satisfies the triangle inequality: for any \( f, g \in X \),

\[
\|f + g\|_X \leq \|f\|_X + \|g\|_X.
\]

and that, for any \( B \in \mathbb{B} \), there exists a positive constant \( C_{(B)} \), depending on \( B \), such that, for any \( f \in X \),

\[
\int_B |f(x)| \, dx \leq C_{(B)} \|f\|_X.
\]

**Remark 2.6.**

(i) Let \( X \) be a ball quasi-Banach function space on \( \mathbb{R}^n \). By [74, Remark 2.6(i)], we conclude that, for any \( f \in \mathcal{M}(\mathbb{R}^n) \), \( \|f\|_X = 0 \) if and only if \( f = 0 \) almost everywhere.

(ii) As was mentioned in [74, Remark 2.6(ii)], we obtain an equivalent formulation of Definition 2.5 via replacing any ball \( B \) by any bounded measurable set \( E \) therein.

(iii) In Definition 2.5, if we replace any ball \( B \) by any measurable set \( E \) with \( |E| < \infty \), we obtain the definition of (quasi-)Banach function spaces originally introduced in [9, Definitions 1.1 and 1.3]. Thus, a (quasi-)Banach function space is always a ball (quasi-)Banach function space.

(iv) By [28, Theorem 2], we conclude that both (ii) and (iii) of Definition 2.5 imply that any ball quasi-Banach function space is complete.

It is known that Lebesgue spaces, Lorentz spaces, variable Lebesgue spaces, and Orlicz spaces are (quasi-)Banach function spaces (see, for instance, [66, 72, 79]). However, weighted Lebesgue spaces, mixed-norm Lebesgue spaces, Orlicz-slice spaces, and Morrey spaces are not necessary to be quasi-Banach function spaces (see, for instance [66, 72, 79, 80] for more details and examples). Therefore, in this sense, compared with the concept of ball (quasi-)Banach function spaces, the concept of (quasi-)Banach function spaces is more restrictive. Based on this, the ball (quasi-)Banach function spaces were originally introduced in [66] to extend (quasi-)Banach function spaces further so that weighted Lebesgue spaces, mixed-norm Lebesgue spaces, Orlicz-slice spaces, and Morrey spaces are also included in this more generalized framework.

The following concept of the associate space of a ball Banach function space can be found in [66, Section 2.1].

**Definition 2.7.** For any ball Banach function space \( X \), the associate space (also called the Köthe dual) \( X' \) is defined by setting

\[
X' := \left\{ f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{X'} := \sup_{g \in X : \|g\|_X = 1} \|fg\|_{L^1(\mathbb{R}^n)} < \infty \right\},
\]

where \( \| \cdot \|_{X'} \) is called the associate norm of \( \| \cdot \|_X \).

**Remark 2.8.** By [66, Proposition 2.3], we find that, if \( X \) is a ball Banach function space, then its associate space \( X' \) is also a ball Banach function space.
Lemma 2.13. Let \( X \) be a ball quasi-Banach function space and \( p \in (0, \infty) \). The \( p \)-convexification \( X^p \) of \( X \) is defined by setting \( X^p := \{ f \in \mathcal{M}(\mathbb{R}^n) : |f|^p \in X \} \) equipped with the quasi-norm \( \| f \|_{X^p} := \| |f|^p \|_X^{1/p} \).

Definition 2.10. Let \( X \) be a ball quasi-Banach function space. A function \( f \in X \) is said to have an absolutely continuous quasi-norm in \( X \) if \( \| f 1_{E_j} \|_X \downarrow 0 \) whenever \( \{E_j\}_{j=1}^\infty \) is a sequence of measurable sets that satisfy \( E_j \supset E_{j+1} \) for any \( j \in \mathbb{N} \) and \( \bigcap_{j=1}^\infty E_j = \emptyset \). Moreover, \( X \) is said to have an absolutely continuous quasi-norm if, for any \( f \in X \), \( f \) has an absolutely continuous quasi-norm in \( X \).

It is worth pointing out that the Lebesgue space \( L^q(\mathbb{R}^n) \) and the Morrey space \( M_q^p(\mathbb{R}^n) \) with \( 1 \leq q < p < \infty \) do not have an absolutely continuous norm (see, for instance, [66, p. 10]).

Denote by \( L^1_{\text{loc}}(\mathbb{R}^n) \) the set of all locally integrable functions on \( \mathbb{R}^n \). Recall that the Hardy–Littlewood maximal operator \( M \) is defined by setting, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
M(f)(x) := \sup_{r \in (0, \infty)} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y)| \, dy.
\]

(2.3)

For any \( \theta \in (0, \infty) \), the powered Hardy–Littlewood maximal operator \( M^{(\theta)} \) is defined by setting, for any \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \),

\[
M^{(\theta)}(f)(x) := \left[ M \left( |f|^\theta \right)(x) \right]^{\frac{1}{\theta}}.
\]

To study the Hardy space \( H_{X, L}(\mathbb{R}^n) \) (see Definition 2.14 below), we need the following assumption on \( X \).

Assumption 2.11. Let \( X \) be a ball quasi-Banach function space on \( \mathbb{R}^n \). Assume that there exist constants \( \theta, s \in (0, 1] \) and \( C \in (0, \infty) \) such that, for any \( \{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathbb{R}^n) \),

\[
\left\| \left\{ \sum_{j=1}^\infty \left[ M^{(\theta)}(f_j) \right]^1 \right\}^{1/1} \right\|_X \leq C \left\| \left\{ \sum_{j=1}^\infty |f_j|^s \right\}^{1/1} \right\|_X.
\]

Assumption 2.12. Let \( s \in (0, 1] \). Assume that \( X^{1/s} \) is a ball Banach function space on \( \mathbb{R}^n \) and there exists a positive constant \( q \in (1, \infty) \) such that, for any \( f \in ((X^{1/s})')^{1/(q/s)} \),

\[
\| M(f) \|_{((X^{1/s})')^{1/(q/s)}} \leq C \| f \|_{((X^{1/s})')^{1/(q/s)}},
\]

where \( C \) is a positive constant independent of \( f \).

By a proof similar to that of [66, Theorem 2.11], we have the following conclusion; we omit the details.

Lemma 2.13. Assume that \( X \) is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some \( \theta, s \in (0, 1] \) and \( q \in (1, \infty) \). Let \( \tau \in (n[1/\theta - 1/q], \infty) \), \( \{Q_j\}_{j=1}^\infty \subset Q \), and both \( \{m_j\}_{j=1}^\infty \subset L^q(\mathbb{R}^n) \) and \( \{\lambda_j\}_{j=1}^\infty \subset [0, \infty) \) satisfy that, for any \( j \in \mathbb{N} \) and \( k \in \mathbb{Z}_+ \),

\[
\| m_j 1_{Q_j} \|_{L^q(\mathbb{R}^n)} \leq 2^{-\tau k} \frac{|Q_j|^{1/q}}{1_{Q_j}}
\]
and
\[ \left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|1_Q\|_X} \right)^s 1_Q \right\} \right\|_X < \infty. \]

Then \( \sum_{j=1}^{\infty} \lambda_j m_j \) converges in \( S'(\mathbb{R}^n) \). \( \sum_{j=1}^{\infty} \lambda_j m_j \in X \), and there exists a positive constant \( C \), independent of \( f \), such that
\[ \left\| \sum_{j=1}^{\infty} \lambda_j m_j \right\|_X \leq C \left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|1_Q\|_X} \right)^s 1_Q \right\} \right\|_X. \]

2.3 Hardy Spaces \( H_{X,L}(\mathbb{R}^n) \)

In this subsection, we recall the definition of the Hardy space \( H_{X,L}(\mathbb{R}^n) \) which was introduced in [66, Section 6.1]. For any \( \alpha \in (0, \infty) \) and \( x \in \mathbb{R}^n \), let
\[ (2.4) \quad \Gamma_{\alpha}(x) := \left\{ (y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < \alpha t \right\}, \]

where \( \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty) \). When \( \alpha = 1 \), we denote \( \Gamma_{\alpha}(x) \) simply by \( \Gamma(x) \). Let \( m \in \mathbb{N} \) and \( L \) be a homogeneous divergence form 2m-order elliptic operator in (2.1). Then \( f \in L^2(\mathbb{R}^n) \), the Lusin area function \( S_L(f) \), associated with \( L \), is defined by setting, for any \( x \in \mathbb{R}^n \),
\[ S_L(f)(x) := \left[ \int_{\Gamma(x)} \left| 2m Le^{-\theta m L} f(y) \right|^2 \frac{dydt}{t^{n+1}} \right]^{1/2}. \]

**Definition 2.14.** Let \( m \in \mathbb{N} \), \( X \) be a ball quasi-Banach function space on \( \mathbb{R}^n \), and \( L \) a homogeneous divergence form 2m-order elliptic operator in (2.1). Then the **Hardy space** \( H_{X,L}(\mathbb{R}^n) \), associated with both \( X \) and \( L \), is defined as the completion of the set
\[ H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) : \|S_L(f)\|_X < \infty \right\} \]

with respect to the quasi-norm \( \|f\|_{H_{X,L}(\mathbb{R}^n)} := \|S_L(f)\|_X \).

**Definition 2.15.** Let \( m \in \mathbb{N} \), \( X \) be a ball quasi-Banach function space on \( \mathbb{R}^n \) satisfying Assumption 2.11 for some \( \theta, s \in (0, 1] \), and \( L \) a homogeneous divergence form 2m-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that \( M \in \mathbb{N} \), \( \epsilon \in (0, \infty) \), and \( q \in (p_-(L), q_+(L)) \). Denote by \( \mathcal{R}(L^M) \) the range of \( L^M \).

(i) A function \( \alpha \in L^q(\mathbb{R}^n) \) is called an \( (X, q, M, \epsilon)_{L^M} - \text{molecule} \) associated with the ball \( B := B(x_B, r_B) \subset \mathbb{R}^n \), with some \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \), if \( \alpha \in \mathcal{R}(L^M) \) and, for any \( k \in \{0, \ldots, M\} \) and \( j \in \mathbb{Z}_+ \), it holds true that
\[ \left\| \left( r_B^{2m L^{-1}} \right)^k \alpha \right\|_{L^q(U_j(B))} \leq 2^{-j\epsilon} \left\| 2^j B^{1/q} \right\|_X. \]

Moreover, if \( \alpha \) is an \( (X, q, M, \epsilon)_{L^M} - \text{molecule} \) for all \( q \in (p_-(L), q_+(L)) \), then \( \alpha \) is called an \( (X, M, \epsilon)_{L^M} - \text{molecule} \).
(ii) For any $f \in L^2(\mathbb{R}^n)$, $f = \sum_{j=1}^{\infty} \lambda_j \alpha_j$ is called a molecular $(X, q, M, \epsilon)$-representation of $f$ if, for any $j \in \mathbb{N}$, $\alpha_j$ is an $(X, q, M, \epsilon)_L$-molecule associated with the ball $B_j \subset \mathbb{R}^n$, the summation converges in $L^2(\mathbb{R}^n)$, and $\{\lambda_j\}_{j \in \mathbb{N}} \subset [0, \infty)$ satisfies

$$\Lambda(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}}) := \left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|1_{B_j}\|_X} \right)^{s} 1_{B_j} \right\}^{1/s} \right\|_X < \infty.$$  

Let

$$\widetilde{H}^{M,q,\epsilon}_{X,L}(\mathbb{R}^n) := \{ f \in L^2(\mathbb{R}^n) : f \text{ has a molecular } (X, q, M, \epsilon)\text{-representation} \}$$

equipped with the quasi-norm $\| \cdot \|_{H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)}$ given by setting, for any $f \in \widetilde{H}^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)$,

$$\|f\|_{H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)} := \inf \left\{ \Lambda(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}}) : f = \sum_{j=1}^{\infty} \lambda_j \alpha_j \right\},$$

where $\Lambda(\{\lambda_j \alpha_j\}_{j \in \mathbb{N}})$ is the same as in (2.5) and the infimum is taken over all the molecular $(X, q, M, \epsilon)$-representations of $f$ as above.

The molecular Hardy space $H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)$ is then defined as the completion of $\widetilde{H}^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)$ with respect to the quasi-norm $\| \cdot \|_{H^{M,q,\epsilon}_{X,L}(\mathbb{R}^n)}$.

The following molecular decomposition of $H_{X,L}(\mathbb{R}^n)$ can be found in [66, Proposition 6.11].

**Proposition 2.16.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that $X$ is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some $s, \theta \in (0, 1]$ and $q \in [2, p_+)(L)$. Let $\epsilon \in (n/\theta, \infty)$ and $M \in (\frac{n}{2m}, \infty) \cap \mathbb{N}$. Then, for any $f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, there exists a sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset [0, \infty)$ and a sequence $\{\alpha_j\}_{j \in \mathbb{N}}$ of $(X, M, \epsilon)_L$-molecules associated, respectively, with the balls $\{B_j\}_{j \in \mathbb{N}}$ such that $f = \sum_{j=1}^{\infty} \lambda_j \alpha_j$ in $L^2(\mathbb{R}^n)$. Moreover, there exists a positive constant $C$ such that, for any $f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|1_{B_j}\|_X} \right)^{s} 1_{B_j} \right\}^{1/s} \right\|_X \leq C \|f\|_{H_{X,L}(\mathbb{R}^n)}.$$  

**Remark 2.17.** We point out that Proposition 2.16 was obtained in [66, Proposition 6.11] under the additional assumption that $X$ has an absolutely continuous quasi-norm. However, by checking the proof of [66, Proposition 6.11] very carefully, we find that, under the assumption that $f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, the additional condition that $X$ has an absolutely continuous quasi-norm in [66, Proposition 6.11] is superfluous.
3 Maximal Function Characterizations of $H_{X,L}(\mathbb{R}^n)$

In this section, by making full use of the special structure of the divergence form elliptic operator and the extrapolation theorem in the scale of ball quasi-Banach function spaces, we establish the radial and the non-tangential maximal function characterizations of $H_{X,L}(\mathbb{R}^n)$.

Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1). We first recall the definitions of several maximal functions associated with $L$. For any given $\alpha \in (0, \infty)$ and for any $f \in L^2(\mathbb{R}^n)$, the radial maximal function $R_{h}^{(\alpha)}(f)$ and the non-tangential maximal function $N_{h}^{(\alpha)}(f)$, associated with the heat semigroup generated by $L$, are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$R_{h}^{(\alpha)}(f)(x) := \sup_{t \in (0,\infty)} \left[ \frac{1}{(\alpha t)^{\alpha}} \int_{B(x,\alpha t)} \left| e^{-\alpha x L} f(z) \right|^2 dz \right]^{1/2}$$

and

$$N_{h}^{(\alpha)}(f)(x) := \sup_{(y,t) \in \Gamma_{\alpha}(x)} \left[ \frac{1}{(\alpha t)^{\alpha}} \int_{B(y,\alpha t)} \left| e^{-\alpha x L} f(z) \right|^2 dz \right]^{1/2},$$

where $\Gamma_{\alpha}(x)$ is the same as in (2.4). For any given $\alpha \in (0, \infty)$ and for any $f \in L^2(\mathbb{R}^n)$, we also define the Lusin area function $S_{h}^{(\alpha)}(f)$, associated with the heat semigroup generated by $L$, by setting, for any $x \in \mathbb{R}^n$,

$$S_{h}^{(\alpha)}(f)(x) := \left[ \int_{\Gamma_{\alpha}(x)} \left| (\nabla)^{m} e^{-\alpha x L} f(y) \right|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2}.$$

In what follows, when $\alpha := 1$, we remove the superscript $\alpha$ for simplicity. In a similar way, the $S_{h}$-adapted, the $R_{h}$-adapted, and the $N_{h}$-adapted Hardy spaces, $H_{X,S_{h}}(\mathbb{R}^n)$, $H_{X,R_{h}}(\mathbb{R}^n)$, and $H_{X,N_{h}}(\mathbb{R}^n)$, are defined in the way same as $H_{X,L}(\mathbb{R}^n)$ with $S_{L}(f)$ replaced, respectively, by $S_{h}(f)$, $R_{h}(f)$, and $N_{h}(f)$.

The main result of this section is stated as follows.

**Theorem 3.1.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that $X$ is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some $\theta, s \in (0, 1]$ and $q \in (p_-(L), p_+(L))$. Then the spaces $H_{X,L}(\mathbb{R}^n)$, $H_{X,S_{h}}(\mathbb{R}^n)$, $H_{X,R_{h}}(\mathbb{R}^n)$, and $H_{X,N_{h}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

**Remark 3.2.** When $X := L^p(\mathbb{R}^n)$ with $p \in (0, p_+(L))$, Theorem 3.1 is just [17, Theorem 1.4].

To prove Theorem 3.1, we need several lemmas. Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1), and $\epsilon, R, \alpha \in (0, \infty)$ with $\epsilon < R$. For any $f \in L^2(\mathbb{R}^n)$, the truncated Lusin area function $S_{h}^{\epsilon, R, \alpha}(f)$, associated with the heat semigroup of $L$, is defined by setting, for any $x \in \mathbb{R}^n$,

$$S_{h}^{\epsilon, R, \alpha}(f)(x) := \left[ \int_{\Gamma_{\alpha}^{\epsilon, R}(x)} \left| (\nabla)^{m} e^{-\alpha x L} f(y) \right|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$
Lemma 3.3. Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, $\alpha \in (0, 1)$, and $\epsilon, R \in (0, \infty)$ with $\epsilon < R$. Then there exists a positive constant $C$, depending only on both $\alpha$ and $m$, such that, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

\[
S_h^\epsilon R,\alpha(f)(x) \leq C[1 + \ln(R/\epsilon)]^{1/2}N_h(f)(x).
\]

To show Lemma 3.3, we need the Besicovitch covering lemma. For any $(x, t), (z, \tau) \in \mathbb{R}^{n+1}$, let

\[
\rho((x, t), (z, \tau)) := \max\{|x - z|, |t - \tau|\}.
\]

It is obvious that $\rho$ is a metric on $\mathbb{R}^{n+1}$. In what follows, we denote by $B_{\rho}((z, \tau), r)$ the ball of $\mathbb{R}^{n+1}$ with center $(z, \tau)$ and radius $r$; namely

\[
B_{\rho}((z, \tau), r) := \left\{(x, t) \in \mathbb{R}^{n+1} : \rho((x, t), (z, \tau)) < r\right\}.
\]

The following Besicovitch covering lemma is a part of [43, Theorem 1.14].

Lemma 3.4. Let $\mathcal{F}$ be any collection of closed balls with uniformly bounded diameter in $(\mathbb{R}^{n+1}, \rho)$ and $\Lambda$ the set of centers of balls in $\mathcal{F}$. Then there exist $N_n$ countable collections $\{\mathcal{G}_i\}_{i=1}^{N_n}$ of disjoint balls in $\mathcal{F}$ such that

\[
A \subset \bigcup_{i=1}^{N_n} \bigcup_{B \in \mathcal{G}_i} B,
\]

where the positive integer $N_n$ depends only on $n$.

Now, we prove Lemma 3.3 by using Lemma 3.4.

Proof of Lemma 3.3. Let all the symbols be the same as in the present lemma. Fix an $f \in L^2(\mathbb{R}^n)$ and an $x \in \mathbb{R}^n$, let

\[
\mathcal{F} := \left\{B_{\rho}((z, \tau), \gamma \tau)\right\}_{(z, \tau) \in \Gamma^\epsilon R,\alpha(x)},
\]

where $\gamma \in (0, \min\{1/3, (1 - \alpha)/2\})$. Then, by Lemma 3.4, we find that there exists a collection $\{B_{\rho}((z_j, \tau_j), \gamma \tau_j)\}_{j \in \Lambda}$ of $\mathcal{F}$ such that

\[
\Gamma^\epsilon R,\alpha(x) \subset \bigcup_{j \in \Lambda} B_{\rho}((z_j, \tau_j), \gamma \tau_j) \quad \text{and} \quad \sum_{j \in \Lambda} 1_{B_{\rho}((z_j, \tau_j), \gamma \tau_j)} \leq N_n,
\]

where $N_n$ is the same as in Lemma 3.4. In the remainder of this proof, for the simplicity of the presentation, for any $j \in \Lambda$, we denote $B_{\rho}((z_j, \tau_j), \gamma \tau_j)$ simply by $E_j$.

Next, we show that, for any $j \in \Lambda$,

\[
E_j \subset \Gamma^{\epsilon/2, 2R}(x).
\]
Indeed, for any \((y, t) \in E_j\), we have

\[(3.3) \quad |z_j - y| < \gamma \tau_j \quad \text{and} \quad (1 - \gamma) \tau_j < t < (1 + \gamma) \tau_j.\]

Since \((z_j, \tau_j) \in \Gamma^r_0(x)\), it follows that

\[|x - z_j| < \alpha \tau_j \quad \text{and} \quad \epsilon < \tau_j < R.\]

Using this, (3.3), and \(\gamma \in (0, (1 - \alpha)/2)\), we find that, for any \((y, t) \in E_j\),

\[|x - y| \leq |x - z_j| + |z_j - y| < (\alpha + \gamma) \tau_j < (1 - \gamma) \tau_j < t\]

and

\[\epsilon/2 < (1 - \gamma) \epsilon < (1 - \gamma) \tau_j < t < (1 + \gamma) \tau_j < (1 + \gamma)R < 2R,\]

which implies that \((y, t) \in \Gamma^{r/2,2R}(x)\) and hence \(E_j \subset \Gamma^{r/2,2R}(x)\). That is, (3.2) holds true.

On the other hand, we have, for any \(j \in \Lambda\),

\[\int_{E_j} \frac{dy \, dt}{t^{\frac{n+1}{p}+1}} = \int_{(1+\gamma)\tau_j}^{(1+\gamma)\tau_j} \int_{B(z_j, \gamma \tau_j)} t^{-\frac{n+1}{p}} dy \, dt \sim 1.\]

Using this, (3.2), and (3.1), we conclude that

\[\text{card } (\Lambda) \sim \sum_{j \in \Lambda} \int_{E_j} \frac{dy \, dt}{t^{\frac{n+1}{p}+1}} \leq \int_{\Gamma^{r/2,2R}(x)} \sum_{j \in \Lambda} 1_{E_j}(y, t) \frac{dy \, dt}{t^{\frac{n+1}{p}+1}} \leq \int_{\Gamma^{r/2,2R}(x)} \frac{dy \, dt}{t^{\frac{n+1}{p}+1}} \sim 1 + \ln(R/\epsilon).\]

Here and thereafter, \text{card } (\Lambda) denotes the cardinality of the set \(\Lambda\). From this, (3.1), the definition of \(N_h(f)\), and Proposition 2.4 with \(x_0 := z_j\), \(r := \gamma \tau_j\), and \(t_0 := \tau_j\), it follows that

\[
\left[ S_{h, R, \alpha}^m(f)(x) \right]^2 = \int_{\Gamma^{r/2}(x)} |(\nabla y)^m e^{-\rho n L(f)(y)}|^2 \frac{dy \, dt}{t^{\frac{n+1}{p}+1}} \\
\leq \sum_{j \in \Lambda} \tau_j^{2m-n-1} \int_{(1+\gamma)\tau_j}^{(1+\gamma)\tau_j} \int_{B(z_j, \gamma \tau_j)} \left| \nabla y^m e^{-\rho n L(f)(y)} \right|^2 dy \, dt \\
\leq \sum_{j \in \Lambda} \tau_j^{-\frac{n}{p+1}} \int_{(1-2\gamma)\tau_j}^{(1-2\gamma)\tau_j} \int_{B(z_j, 2\gamma \tau_j)} \left| e^{-\rho n L(f)(y)} \right|^2 dy \, dt \\
\leq \sum_{j \in \Lambda} \tau_j^{-1} \int_{(1-2\gamma)\tau_j}^{(1+2\gamma)\tau_j} \int_{B(z_j, \gamma \tau_j)} \left| e^{-\rho n L(f)(y)} \right|^2 dy \, dt \\
\leq \text{card } (\Lambda) \left[ N_h(f)(x) \right]^2 \leq [1 + \ln(R/\epsilon)] \left[ N_h(f)(x) \right]^2.
\]

This finishes the proof of Lemma 3.3. \(\square\)

Now, we recall the concept of the Muckenhoupt weight class \(A_p(\mathbb{R}^n)\) (see, for instance, [42]).
**Definition 3.5.** An $A_p(\mathbb{R}^n)$ weight $\omega$, with $p \in [1, \infty)$, is a nonnegative locally integrable function on $\mathbb{R}^n$ satisfying that, when $p \in (1, \infty)$,

$$[\omega]_{A_p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \left\{ \frac{1}{|Q|} \int_Q \omega(x) \, dx \right\} \left\{ \frac{1}{|Q|} \int_Q [\omega(x)]^{1/p} \, dx \right\}^{p-1} < \infty$$

and

$$[\omega]_{A_1(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q \omega(x) \, dx \left\| \omega^{-1} \right\|_{L^\infty(Q)} < \infty,$$

where the suprema are taken over all cubes $Q \subset \mathbb{R}^n$. Moreover, let

$$A_\infty(\mathbb{R}^n) := \bigcup_{p \in [1, \infty)} A_p(\mathbb{R}^n).$$

Then we have the following good-$\lambda$ inequality concerning the non-tangential maximal function and the truncated Lusin area function associated with the heat semigroup of $L$, which plays a key role in the proof of Theorem 3.1.

**Lemma 3.6.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that $\omega \in A_1(\mathbb{R}^n)$ and $\epsilon, R \in (0, \infty)$ with $\epsilon < R$. Then there exist positive constants $e_0$ and $C$, depending only on both $[\omega]_{A_1(\mathbb{R}^n)}$ and $L$, such that, for any $\gamma \in (0, 1]$, $\lambda \in (0, \infty)$, and $f \in L^2(\mathbb{R}^n)$,

$$\omega \left( \left\{ x \in \mathbb{R}^n : S_{\epsilon, R}^{e,R,h} (f)(x) > 2\lambda, \, \mathcal{N}_h(f)(x) \leq \gamma \lambda \right\} \right) \leq C \gamma^{e_0} \omega \left( \left\{ x \in \mathbb{R}^n : S_{h}^{\epsilon,R,\frac{1}{2}} (f)(x) > \lambda \right\} \right).$$

**Proof.** Let all the symbols be the same as in the present lemma. Let $\gamma \in (0, 1]$, $\lambda \in (0, \infty)$, $f \in L^2(\mathbb{R}^n)$, and

$$O := \left\{ x \in \mathbb{R}^n : S_{h}^{\epsilon,R,\frac{1}{2}} (f)(x) > \lambda \right\}.$$

It is obvious that $O$ is an open set of $\mathbb{R}^n$. By the Whitney decomposition of $O$, we find that there exists a family $\{Q_j\}_{j \in \mathbb{N}}$ of dyadic cubes, respectively, with the edge-lengths $\{l_j\}_{j \in \mathbb{N}}$, satisfying that

(i) $O = \bigcup_{j \in \mathbb{N}} Q_j$ and $\{Q_j\}_{j \in \mathbb{N}}$ are disjoint;

(ii) $2Q_j \subset O$ and $4Q_j \cap O^C \neq \emptyset$ for any $j \in \mathbb{N}$.

By the fact that $\{Q_j\}_{j \in \mathbb{N}}$ are disjoint and the estimate that $S_{h}^{\epsilon,R,\frac{1}{2}} (f) \leq S_{h}^{\epsilon,R,\frac{1}{2}} (f)$, we conclude that, to prove (3.4), it suffices to show that, for any $j \in \mathbb{N}$,

$$\omega \left( \left\{ x \in Q_j \cap F : S_{h}^{\epsilon,R,\frac{1}{2}} (f)(x) > 2\lambda \right\} \right) \leq \gamma^{e_0} \omega(Q_j),$$

where

$$F := \{ x \in \mathbb{R}^n : \mathcal{N}_h(f)(x) \leq \gamma \lambda \}.$$
and

$$\omega(Q_j) := \int_{Q_j} \omega(x) \, dx.$$  

Using both the assumption that $\omega \in A_1(\mathbb{R}^n)$ and [42, Proposition 7.2.8], we find that, to prove (3.5), it suffices to show that, for any $j \in \mathbb{N},$

$$\left| \left\{ x \in Q_j \cap F : S^\epsilon, R, \frac{1}{10} (f(x) > 2\lambda) \right\} \right| \lesssim \gamma^2 |Q_j|.$$  

Let $j \in \mathbb{N}$. We first prove that, if $\epsilon \in [10 \sqrt{nl_j}, R)$, then, for any $x \in Q_j,$

$$S^{\epsilon, R, \frac{1}{10}}(f)(x) \leq \lambda.$$  

Indeed, let $x_j \in 4Q_j \cap O^C$ and $x \in Q_j$. Then, for any $(y, t) \in \Gamma^{\epsilon, R, 1/20}(x)$, we have

$$|x_j - y| \leq |x_j - x| + |x - y| < \frac{5}{2} \sqrt{nl_j} + \frac{t}{20} \leq \frac{1}{4} \epsilon + \frac{t}{20} < \frac{1}{2},$$

and hence $(y, t) \in \Gamma^{\epsilon, R, 1/20}(x_j).$ Thus, $\Gamma^{\epsilon, R, 1/20}(x) \subset \Gamma^{\epsilon, R, 1/20}(x_j).$ This, together with the fact that $x_j \in O^C,$ implies that

$$S^{\epsilon, R, \frac{1}{10}}(f)(x) \leq S^{\epsilon, R, \frac{1}{20}}(f)(x_j) \leq \lambda.$$  

This proves (3.7).

By (3.7), we conclude that (3.6) holds true if $\epsilon \in [10 \sqrt{nl_j}, R).$ Thus, in the remainder of this proof, we always assume that $\epsilon \in (0, 10 \sqrt{nl_j})$. We claim that, to prove (3.6) in this case, it suffices to show that

$$\left| \left\{ x \in Q_j \cap F : S^{\epsilon, 10 \sqrt{nl_j}, \frac{1}{10}}(f(x) > \lambda) \right\} \right| \lesssim \gamma^2 |Q_j|.$$  

Assume that (3.8) holds true for the moment. If $R \in (0, 10 \sqrt{nl_j}]$, by the estimate that

$$S^{\epsilon, R, \frac{1}{10}}(f)(x) \geq S^{\epsilon, R, \frac{1}{10}}(f)(x),$$

for any $x \in \mathbb{R}^n$, we find that (3.6) in this case holds true. If $R \in (10 \sqrt{nl_j}, \infty)$, applying an argument similar to that used in the estimation of (3.7), we conclude that, for any $x \in Q_j,$

$$S^{10 \sqrt{nl_j}, R, \frac{1}{10}}(f)(x) \leq \lambda;$$

using this and the estimate that

$$S^{\epsilon, R, \frac{1}{10}}(f)(x) \leq S^{\epsilon, 10 \sqrt{nl_j}, \frac{1}{10}}(f)(x) + S^{10 \sqrt{nl_j}, R, \frac{1}{10}}(f)(x)$$

for any $x \in \mathbb{R}^n$, we find that (3.6) in this case also holds true. Thus, (3.6) always holds true.
Now, we prove (3.8). By the Chebyshev inequality, we conclude that (3.8) can be deduced from

\[(3.9) \quad \int_{Q \cap F} \left[ S_{\epsilon, 10}^{\epsilon, 10} \right] dx \leq (\gamma t)^2 |Q|.
\]

Next, we prove (3.9). If \( \epsilon \in (5 \sqrt{m_1}, 10 \sqrt{m_1}) \), from Lemma 3.3, it follows that

\[
\int_{Q \cap F} \left[ S_{\epsilon, 10}^{\epsilon, 10} \right] dx \leq \int_{Q \cap F} \left[ N_{\epsilon}(f)(x) \right] dx \leq (\gamma t)^2 |Q|,
\]

which proves that (3.9) in this case holds true. In the remainder of this proof, we assume that \( \epsilon \in (0, 5 \sqrt{m_1}) \). For any \( y \in \mathbb{R}^n \), let

\[
\psi(y) := \text{dist} (y, Q_j \cap F),
\]

\[
G := \left\{ (y, t) \in \mathbb{R}^n \times (\epsilon, 10 \sqrt{m_1}) : \psi(y) < \frac{t}{20} \right\}
\]

and

\[
G_1 := \left\{ (y, t) \in \mathbb{R}^n \times (\epsilon/2, 20 \sqrt{m_1}) : \psi(y) < \frac{t}{10} \right\}.
\]

Denote by \( C_c^\infty(G_1) \) the set of all the infinitely differentiable functions on \( G_1 \) with compact support. Then there exists a function \( \eta \in C_c^\infty(G_1) \) such that \( \eta \equiv 1 \) on \( G \), \( 0 \leq \eta \leq 1 \), and, for any \( k \in \{1, \ldots, m\} \) and \( (x, t) \in G_1 \),

\[(3.10) \quad \sum_{|\alpha|=k} |\partial^\alpha \eta(x, t)|^2 \leq \frac{1}{t^{2k}} \quad \text{and} \quad \partial_t \eta(x, t) \leq \frac{1}{t}.
\]

Let \( u(y, t) := e^{-t^m L}(f)(y) \) for any \( (y, t) \in \mathbb{R}^{n+1}_+ \). In what follows, for any \( \gamma := (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n \) and any weakly differentiable functions \( h \) and \( g \) on \( \mathbb{R}^{n+1}_+ \), let \( \partial^\gamma (h g)(y, t) := \partial^\gamma (h(y, t)g(y, t)) \) for any \( (y, t) \in \mathbb{R}^{n+1}_+ \), where \( \partial^\gamma := (\partial_{\gamma_1})^{\gamma_1} \cdots (\partial_{\gamma_n})^{\gamma_n} \). By the Tonelli theorem, Strong Ellipticity Condition 2.2, the integral by parts, and the Leibniz rule, we obtain

\[
\int_{Q \cap F} \left[ S_{\epsilon, 10}^{\epsilon, 10} \right] dx \leq \int_{Q \cap F} \int_{1/20}^{10 \sqrt{m_1}} \left| (t \nabla)^m u(y, t) \right|^2 \frac{dy dt}{t^{n+1}} dx
\]
Let \( \delta \).

By Lemma 3.4, we find that there exists a collection \( y \) for any \((\cdot, \cdot)\) and both the integral by parts and (3.10), further implies that (3.13)

\[ (3.11) \]

where the positive constant \( C_{(\alpha, \tilde{\alpha})} \) depends only on both \( \alpha \) and \( \tilde{\alpha} \). This further implies that, to prove (3.9), it suffices to show that, for any \( k \in \{0, \ldots, m\}, \)

(3.11)

\[ J_k \leq (\gamma, 1)^2 |Q_j|. \]

We first estimate \( J_0 \). Since

\[ \frac{\partial u(y, t)}{\partial t} = -2mt^{2m-1} L(u)(y, t) \]

for any \((y, t) \in \mathbb{R}^{n+1}_+\), it follows that

\[ \frac{\partial |u(y, t)|^2}{\partial t} = -2mt^{2m-1} L(u)(y, t)\overline{u(y, t)} - 2mt^{2m-1} u(y, t) \overline{L(u)(y, t)} \]

\[ = -4mt^{2m-1} \Re L(u)(y, t)\overline{u(y, t)}, \]

which, together with both the integral by parts and (3.10), further implies that

(3.12)

\[ J_0 \sim \left| \mathbb{R} \left\{ \int_{G_1} t^{2m-1} \eta(y, t) L(u)(y, t) \overline{u(y, t)} dy dt \right\} \right| \]

\[ \sim \left| \int_{G_1} \frac{\partial |u(y, t)|^2}{\partial t} \eta(y, t) dy dt \right| \sim \int_{G_1} |u(y, t)|^2 \partial \eta(y, t) dy dt \]

\[ \leq \int_{G_1 \setminus \Gamma} \frac{|u(y, t)|^2}{t} dy dt. \]

Let \( \delta \in (0, \frac{1}{11}) \) and

\[ \mathcal{F} := \left\{ B_{\rho}((y, t), \delta \eta) \right\}_{(y, t) \in G_1 \setminus \Gamma}. \]

By Lemma 3.4, we find that there exists a collection \( \{B_{\rho}((y_i, t_i), \delta t_i)\}_{i \in \Lambda} \) of \( \mathcal{F} \) such that

(3.13)

\[ G_1 \setminus \Gamma \subset \bigcup_{i \in \Lambda} B_{\rho}((y_i, t_i), \delta t_i) \quad \text{and} \quad \sum_{i \in \Lambda} 1_{B_{\rho}((y_i, t_i), \delta t_i)} \leq N_\eta. \]
where $N_\mu$ is the same as in Lemma 3.4. For simplicity, let $E_i := B_\mu((y_i, t_i), \delta t_i)$ for any $i \in \Lambda$. From both the fact that
\[
G_1 \setminus G \subset \left\{(y, t) \in \mathbb{R}^n \times (\epsilon/2, 20 \sqrt{nl_j}) : \frac{t}{l_j} \leq \psi(y) < \frac{t}{10} \right\}
\cup \left\{(y, t) \in \mathbb{R}^n \times (\epsilon/2, 20 \sqrt{nl_j}) : \psi(y) < \frac{t}{10}, 10 \sqrt{nl_j} \leq t < 20 \sqrt{nl_j} \right\}
\cup \left\{(y, t) \in \mathbb{R}^n \times (\epsilon/5, 30 \sqrt{nl_j}) : \psi(y) < \frac{t}{2}, 5 \sqrt{nl_j} \leq t < 30 \sqrt{nl_j} \right\}
\]
and $\delta \in (0, \frac{1}{4\pi})$, we deduce that, for any $i \in \Lambda$,
\[
(3.14) \quad E_i \subset G_2,
\]
where
\[
G_2 := \left\{(y, t) \in \mathbb{R}^n \times (\epsilon/5, 30 \sqrt{nl_j}) : \frac{t}{40} \leq \psi(y) < \frac{t}{2} \right\}
\cup \left\{(y, t) \in \mathbb{R}^n \times (\epsilon/5, 30 \sqrt{nl_j}) : \psi(y) < \frac{t}{2}, 5 \sqrt{nl_j} \leq t < 30 \sqrt{nl_j} \right\}.
\]
It is easy to prove that, for any $i \in \Lambda$,
\[
(3.15) \quad \int_{E_i} |u(y, t)|^2 \, dy \, dt \leq \int_{E_i} |u(y, t)|^2 \, dy \, dt \leq (\gamma \lambda)^2 t_i^{n+1},
\]
where $E_i := B_\mu((y_i, t_i), 2\delta t_i)$. Indeed, by both $(y_i, t_i) \in G_1$ and $\delta \in (0, \frac{1}{4\pi})$, we conclude that there exists an $x_{i,j} \in Q_j \cap F$ such that
\[
|x_{i,j} - y_i| < \frac{t_i}{10} < (1 - 2\delta)t_i.
\]
From this and the definition of $N_\mu(f)$, it follows that
\[
\int_{E_i} |u(y, t)|^2 \, dy \, dt = \int_{(1-2\delta)t_i}^{(1+2\delta)t_i} \int_{B(y_{i,j}, 2\delta t_i)} |u(y, t)|^2 \, dy \, dt
\leq t_i^n \int_{(1-2\delta)t_i}^{(1+2\delta)t_i} \int_{B(y_{i,j})} |u(y, t)|^2 \, dy \, dt
\leq |N_\mu(f)(x_{i,j})|^2 t_i^n \int_{(1-2\delta)t_i}^{(1+2\delta)t_i} dt \leq (\gamma \lambda)^2 t_i^{n+1},
\]
which implies that (3.15) holds true. Then, using (3.13), (3.15), and (3.14), we find that
\[
(3.16) \quad \int_{G_1 \setminus G} \frac{|u(y, t)|^2}{t} \, dy \, dt \leq \sum_{i \in \Lambda} \int_{E_i} \frac{|u(y, t)|^2}{t} \, dy \, dt
\leq t_i^{-1} \sum_{i \in \Lambda} \int_{E_i} |u(y, t)|^2 \, dy \, dt \leq \sum_{i \in \Lambda} (\gamma \lambda)^2 t_i^n
\]
\[
\sim (\gamma \lambda)^2 \sum_{i \in \Lambda} \int_{E_i} r^{-1} \, dy \, dt \leq (\gamma \lambda)^2 \int_{G_2} r^{-1} \, dy \, dt
\]
\[
\leq (\gamma \lambda)^2 \int_{H_1} \left\{ \int_{\{\mathbf{2}^{(\delta)}(y)\}} \frac{dt}{t} + \int_{\mathbf{e}/5} \frac{dt}{t} + \int_{30 \sqrt{\mathbf{n}^j}} \frac{dt}{t} \right\} dy
\]
\[
\sim (\gamma \lambda)^2 |H_1|,
\]
where
\[
H_1 := \{ y \in \mathbb{R}^n : \text{there exists a } t \in (\epsilon/5, 30 \sqrt{n}^j) \text{ such that } (y, t) \in G_2 \}.
\]
By the definitions of both \( H_1 \) and \( G_2 \), we conclude that, for any \( y \in H_1 \), there exists a \( t \in (\epsilon/5, 30 \sqrt{n}^j) \) and an \( x \in Q_j \) such that
\[
|x - y| < \frac{t}{2} < 15 \sqrt{n}^j,
\]
which further implies that \( y \in 32Q_j \). Therefore, \( H_1 \subset 32Q_j \). From this, (3.16), and (3.12), we deduce that
\[
J_0 \lesssim (\gamma \lambda)^2 |Q_j|.
\]
Let \( k \in \{1, \ldots, m\} \). Next, we deal with \( J_k \). Via the integral by parts, (3.13), and the Hölder inequality, we find that
\[
J_k \lesssim \sum_{|\alpha|=|\beta|=m} \sum_{|\alpha|=k, \alpha \leq \alpha} \left| \int_{G_1 \setminus G} t^{2m-1} \partial^{\alpha} \eta(y, t) u(y, t) \partial^{\alpha - \alpha} (a_{\alpha, \beta} \partial^{\beta} u)(y, t) \, dy \, dt \right|
\]
\[
\sim \sum_{|\alpha|=|\beta|=m} \sum_{|\alpha|=k, \alpha \leq \alpha} \left| \int_{G_1 \setminus G} t^{2m-1} a_{\alpha, \beta}(y) \partial^{\beta} u(y, t) \partial^{\alpha - \alpha} \left( (\partial^{\alpha} \eta) u \right)(y, t) \, dy \, dt \right|
\]
\[
\lesssim \sum_{|\alpha|=|\beta|=m} \sum_{|\alpha|=k, \alpha \leq \alpha} \sum_{\Lambda} \int_{E_i} \left| t^{2m-1} a_{\alpha, \beta}(y) \partial^{\beta} u(y, t) \partial^{\alpha - \alpha} \left( (\partial^{\alpha} \eta) u \right)(y, t) \right| \, dy \, dt
\]
\[
\lesssim \sum_{|\alpha|=|\beta|=m} \sum_{|\alpha|=k, \alpha \leq \alpha} \sum_{\Lambda} \left\{ \int_{E_i} t^{2m-1} \left| \partial^{\alpha} u(y, t) \right|^2 \, dy \, dt \right\}^{1/2}
\times \left\{ \int_{E_i} t^{2m-1} \left| \left( (\partial^{\alpha} \eta) u \right)(y, t) \right|^2 \, dy \, dt \right\}^{1/2}
\]
\[
=: \sum_{|\alpha|=|\beta|=m} \sum_{|\alpha|=k, \alpha \leq \alpha} \sum_{\Lambda} J_{k, i, \alpha, \alpha, \beta}.
\]
Using both Proposition 2.4 and (3.15), we conclude that, for any \( i \in \Lambda \),
\[
\int_{E_i} t^{2m-1} \left| \partial^{\alpha} u(y, t) \right|^2 \, dy \, dt \leq \int_{E_i} \left| \nabla^m u(y, t) \right|^2 \, dy \, dt
\]
\[
\leq \int_{E_i} |u(y, t)|^2 \, dy \, dt \lesssim (\gamma \lambda)^2 r_i^p.
\]
On the other hand, from [1, Theorem 5.2] (with some slight modifications), it follows that, for any ball $B$, any $g \in W^{m,2}(B)$ (see, for instance, [1, Chapter 3] for the precise definition), and $\ell \in \{0,1,\ldots,m\}$,

$$\|\nabla^\ell g\|_{L^2(B)} \leq C_{(n,m)} \|\nabla^m g\|_{L^2(B)} \|g\|_{L^2(B)},$$

where the positive constant $C_{(n,m)}$ depends only on both $n$ and $m$. By this, (3.10), the Hölder inequality, Proposition 2.4, and (3.15), we find that

$$\int_{E_i} t^{2m-1-|\alpha\alpha\tilde{\alpha}|} \left| \partial^{\alpha\alpha\tilde{\alpha}} u(y, t) \right|^2 \, dy \, dt$$

$$\leq \int_{E_i} \sum_{|\alpha|=h, \tilde{\alpha} \geq |\alpha\alpha\tilde{\alpha}|} t^{2m-2|\alpha\alpha\tilde{\alpha}|-1} \left| \partial^{\alpha\alpha} u(y, t) \right|^2 \, dy \, dt$$

$$\leq \sum_{|\alpha|=h, \tilde{\alpha} \geq |\alpha\alpha\tilde{\alpha}|} t^{2m-2|\alpha\alpha\tilde{\alpha}|-1} \int_{E_i} \left\{ \int_{B(y, \delta t)} \left| \nabla^m u(y, t) \right|^2 \, dy \right\}^{\frac{\alpha}{m}}$$

$$\times \left\{ \int_{B(y, \delta t)} \left| u(y, t) \right|^2 \, dy \right\}^{1-\frac{\alpha}{m}} \, dt$$

$$\leq \sum_{|\alpha|=h, \tilde{\alpha} \geq |\alpha\alpha\tilde{\alpha}|} t^{2m-2|\alpha\alpha\tilde{\alpha}|-1} \left\{ \int_{E_i} \left| \nabla^m u(y, t) \right|^2 \, dy \right\}^{\frac{\alpha}{m}} \left\{ \int_{E_i} \left| u(y, t) \right|^2 \, dy \right\}^{1-\frac{\alpha}{m}}$$

$$\leq \sum_{|\alpha|=h, \tilde{\alpha} \geq |\alpha\alpha\tilde{\alpha}|} t^{2m-2|\alpha\alpha\tilde{\alpha}|-1} \left\{ \frac{1}{t^{2m}} \int_{E_i} \left| u(y, t) \right|^2 \, dy \right\}^{\frac{\alpha}{m}} \left\{ \int_{E_i} \left| u(y, t) \right|^2 \, dy \right\}^{1-\frac{\alpha}{m}}$$

$$\leq (\gamma \lambda)^2 t^{\alpha}.$$ 

This, combined with (3.19), (3.18), and the estimation of (3.16), implies that

$$J_k \leq (\gamma \lambda)^2 \sum_{i \in \Lambda} t^{\alpha} \leq (\gamma \lambda)^2 |H_1| \leq (\gamma \lambda)^2 |Q_j|,$$

which, together with (3.17), further implies that (3.11) holds true. This finishes the proof of Lemma 3.6. \(\square\)

For any $M \in \mathbb{Z}_+$, $f \in L^2(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, let

$$R_{h,M}(f)(x) := \sup_{t \in (0,\infty)} \left\{ \frac{1}{t^{2m}} \int_{B(x,t)} \left( t^{2m} L \right)^M e^{-t^{2m} L}(f)(y) \, dy \right\}^{1/2}.$$

By an argument similar to that used in the estimations of [46, (6.48) and (6.49)], we have the following conclusion; we omit the details here.

**Lemma 3.7.** Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, $q \in (p_-(L), p_+(L))$, and $M \in \mathbb{Z}_+$. Then there exists a positive constant $C$ such that, for any $f \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,

$$\|R_{h,M}(f)\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}.$$
The following extrapolation theorem is a slight variant of a special case of [26, Theorem 4.6] via replacing Banach function spaces by ball Banach function spaces; we omit the details here.

**Lemma 3.8.** Assume that $X$ is a ball quasi-Banach function space on $\mathbb{R}^n$ and $p_0 \in (0, \infty)$. Let $\mathcal{F}$ be the set of all pairs $(F, G)$ of nonnegative measurable functions such that, for any given $\omega \in A_1(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} [F(x)]^{p_0} \omega(x) \, dx \leq C_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n} [G(x)]^{p_0} \omega(x) \, dx,
$$

where $C_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})}$ is a positive constant independent of $(F, G)$, but depends on both $p_0$ and $[\omega]_{A_1(\mathbb{R}^n)}$. Assume further that $X^{1/p_0}$ is a ball Banach function space and $M$ is bounded on $(X^{1/p_0})'$, where $M$ is the same as in (2.3). Then there exists a positive constant $C_0$ such that, for any $(F, G) \in \mathcal{F}$,

$$
\|F\|_X \leq C_0 \|G\|_X.
$$

For any $\alpha \in (0, \infty)$, any measurable function $F$ on $\mathbb{R}^{n+1}$, and any $x \in \mathbb{R}^n$, let

$$
\mathcal{A}^{(\alpha)}(F)(x) := \left( \int_{\Gamma_\alpha(x)} |F(y, t)|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2},
$$

where $\Gamma_\alpha(x)$ is the same as in (2.4). We have the following lemma, which is a part of [19, Proposition 4.9].

**Lemma 3.9.** Let $\alpha, \beta \in (0, \infty)$, $p \in (0, 2]$, and $\omega \in A_1(\mathbb{R}^n)$. Then there exist positive constants $C_1$ and $C_2$, depending only on both $p$ and $[\omega]_{A_1(\mathbb{R}^n)}$, such that, for any measurable function $F$ on $\mathbb{R}^{n+1}$,

$$
C_1 \left\| \mathcal{A}^{(\beta)}(F) \right\|_{L^p_\omega(\mathbb{R}^n)} \leq \left\| \mathcal{A}^{(\alpha)}(F) \right\|_{L^p_\omega(\mathbb{R}^n)} \leq C_2 \left\| \mathcal{A}^{(\beta)}(F) \right\|_{L^p_\omega(\mathbb{R}^n)}.
$$

Now, we show Theorem 3.1 by using Lemmas 3.6, 3.7, 3.8, and 3.9.

**Proof of Theorem 3.1.** Let all the symbols be the same as in the present theorem. The proof of the present theorem is divided into the following four steps.

**Step 1.** In this step, we show that

$$
\left[ L^2(\mathbb{R}^n) \cap H_{X,S_h}(\mathbb{R}^n) \right] \subset \left[ L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n) \right].
$$

Let $p_0 := s$. By Lemma 3.8, to prove (3.21), it suffices to show that, for any $\omega \in A_1(\mathbb{R}^n)$, there exists a positive constant $C_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})}$, depending only on both $p_0$ and $[\omega]_{A_1(\mathbb{R}^n)}$, such that, for any $f \in L^2(\mathbb{R}^n) \cap H_{X,S_h}(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} [S^0_h(f)(x)]^{p_0} \omega(x) \, dx \leq C_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n} [S^0_h(f)(x)]^{p_0} \omega(x) \, dx.
$$

Recall that, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$
S^0_h(f)(x) := \left( \int_{\Gamma_2(x)} \left| (\nabla)^m e^{-2mL} f(y) \right|^2 \frac{dy \, dt}{t^{n+1}} \right)^{1/2}.
$$
and
\[ S^2_L(f)(x) := \left[ \int_{T_2(x)} \left| \int_{T_2(y)} L e^{-|x-y|^2 L}(f)(y) \right|^2 dy \, dt \right]^{1/2}. \]

By [17, (3.30)], we find that, for any \( f \in L^2(\mathbb{R}^n) \) and almost every \( x \in \mathbb{R}^n \),
\[ S_L(f)(x) \leq \left[ S^2_h(f)(x) \right]^{1/2} \left[ S^2_L(f)(x) \right]^{1/2}, \]
and hence, for any \( \epsilon \in (0, \infty) \),
\[ S_L(f)(x) \leq \frac{1}{\epsilon} S^2_h(f)(x) + \epsilon S^2_L(f)(x). \]

From this and Lemma 3.9, it follows that, for any \( \epsilon \in (0, \infty) \) and \( f \in L^2(\mathbb{R}^n) \cap H_{X,S_h}(\mathbb{R}^n) \),
\[
\| S_L(f) \|_{L^p_0(\mathbb{R}^n)} \leq \frac{1}{\epsilon} \| S^2_h(f) \|_{L^p_0(\mathbb{R}^n)} + \epsilon \| S^2_L(f) \|_{L^p_0(\mathbb{R}^n)} \sim \frac{1}{\epsilon} \| S_h(f) \|_{L^p_0(\mathbb{R}^n)} + \epsilon \| S_L(f) \|_{L^p_0(\mathbb{R}^n)}
\]
with the implicit positive constants independent of both \( f \) and \( \epsilon \). Letting \( \epsilon \) be sufficiently small in (3.23), we then obtain (3.22).

**Step 2.** In this step, we prove that
\[
\left[ L^2(\mathbb{R}^n) \cap H_{X,N_h}(\mathbb{R}^n) \right] \subset \left[ L^2(\mathbb{R}^n) \cap H_{X,S_h}(\mathbb{R}^n) \right].
\]

Let \( p_0 := s \) with \( s \) in the present theorem. We first show that, for any \( \omega \in A_1(\mathbb{R}^n) \), there exists a positive constant \( C_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})} \), depending only on both \( p_0 \) and \( [\omega]_{A_1(\mathbb{R}^n)} \), such that, for any \( f \in L^2(\mathbb{R}^n) \cap H_{X,N_h}(\mathbb{R}^n) \),
\[
\int_{\mathbb{R}^n} [S_h(f)(x)]^{p_0} \omega(x) \, dx \leq C_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n} [N_h(f)(x)]^{p_0} \omega(x) \, dx.
\]

Without loss of generality, we may assume that
\[
\int_{\mathbb{R}^n} [N_h(f)(x)]^{p_0} \omega(x) \, dx < \infty,
\]
otherwise (3.25) holds true automatically. By both the Fubini theorem and Lemma 3.6, we conclude that, for any \( \gamma \in (0, 1) \) and \( \epsilon, R \in (0, \infty) \) with \( \epsilon < R \),
\[
\int_{\mathbb{R}^n} \left[ S_{h}^{R, \frac{1}{2\epsilon}}(f)(x) \right]^{p_0} \omega(x) \, dx \sim \int_0^\infty t^{p_0 - 1} \omega \left( \left\{ x \in \mathbb{R}^n : S_{h}^{R, \frac{1}{2\epsilon}}(f)(x) > t \right\} \right) dt \\
\leq \int_0^\infty t^{p_0 - 1} \omega \left( \left\{ x \in \mathbb{R}^n : S_{h}^{R, \frac{1}{2\epsilon}}(f)(x) > t, \, N_h(f)(x) \leq \gamma t \right\} \right) dt
\]
\[ + \int_0^\infty t^{p_0 - 1} \omega \left( \{ x \in \mathbb{R}^n : N_h(f)(x) > \gamma t \} \right) dt \]
\[ \leq \gamma^{p_0} \int_0^\infty t^{p_0 - 1} \omega \left( \{ x \in \mathbb{R}^n : S_{\epsilon R} \frac{1}{2} (f)(x) > \frac{t}{2} \} \right) dt \]
\[ + \int_0^\infty t^{p_0 - 1} \omega \left( \{ x \in \mathbb{R}^n : N_h(f)(x) > \gamma t \} \right) dt \]
\[ \sim \gamma^{p_0} \int_{\mathbb{R}^n} \left[ S_{\epsilon R} \frac{1}{2} (f)(x) \right]^{p_0} \omega(x) dx + \frac{1}{\gamma} \int_{\mathbb{R}^n} [N_h(f)(x)]^{p_0} \omega(x) dx. \]

On the other hand, using both Lemma 3.3 and (3.26), we find that

\[ \int_{\mathbb{R}^n} \left[ S_{\epsilon R} \frac{1}{2} (f)(x) \right]^{p_0} \omega(x) dx \leq \int_{\mathbb{R}^n} [N_h(f)(x)]^{p_0} \omega(x) dx < \infty. \quad (3.28) \]

Moreover, from Lemma 3.9, we deduce that

\[ \int_{\mathbb{R}^n} \left[ S_{\epsilon R} \frac{1}{2} (f)(x) \right]^{p_0} \omega(x) dx \sim \int_{\mathbb{R}^n} \left[ S_{\epsilon R} \frac{1}{2} (f)(x) \right]^{p_0} \omega(x) dx \]
\[ \sim \int_{\mathbb{R}^n} \left[ S_{\epsilon R} \frac{1}{2} (f)(x) \right]^{p_0} \omega(x) dx \]

with the positive equivalence constants independent of \( f, \epsilon, \) and \( R. \) By this, (3.28), and (3.27) with \( \gamma \) sufficient small, we conclude that

\[ \int_{\mathbb{R}^n} \left[ S_{\epsilon R} \frac{1}{2} (f)(x) \right]^{p_0} \omega(x) dx \leq \int_{\mathbb{R}^n} [N_h(f)(x)]^{p_0} \omega(x) dx < \infty. \quad (3.29) \]

Letting both \( \epsilon \to 0 \) and \( R \to \infty \) in (3.29), we find that (3.25) holds true. Then, from Lemma 3.8 and (3.25), it follows that, for any \( f \in L^2(\mathbb{R}^n) \cap H_{X,N_0}(\mathbb{R}^n), \) \( f \in H_{X,S_0}(\mathbb{R}^n) \) and

\[ ||S_h(f)||_X \leq ||N_h(f)||_X, \]

which implies that (3.24) holds true.

**Step 3.** In this step, we show that

\[ \left[ L^2(\mathbb{R}^n) \cap H_{X,R_0}(\mathbb{R}^n) \right] \subset \left[ L^2(\mathbb{R}^n) \cap H_{X,N_0}(\mathbb{R}^n) \right]. \quad (3.30) \]

Let \( p_0 := s \) with \( s \) in the present theorem. By Lemma 3.8, to prove (3.30), it suffices to show that, for any \( \omega \in A_1(\mathbb{R}^n), \) there exists a positive constant \( \overline{C}_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})} \), depending only on both \( p_0 \) and \([\omega]_{A_1(\mathbb{R}^n)}, \) such that, for any \( f \in L^2(\mathbb{R}^n) \cap H_{X,R_0}(\mathbb{R}^n), \)

\[ \int_{\mathbb{R}^n} [N_h(f)(x)]^{p_0} \omega(x) dx \leq \overline{C}_{(p_0, [\omega]_{A_1(\mathbb{R}^n)})} \int_{\mathbb{R}^n} [R_h(f)(x)]^{p_0} \omega(x) dx. \quad (3.31) \]

It is obvious that, for any \( f \in L^2(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n, \)

\[ N_h^{(\frac{1}{2})} (f)(x) \leq 2^{n/2} R_h(f)(x). \quad (3.32) \]
On the other hand, applying an argument similar to that used in the proof of [46, Lemma 6.2], we conclude that, for any \( \alpha, \beta \in (0, \infty) \),
\[
\int_{\mathbb{R}^n} \left[ \mathcal{N}_h^{(\alpha)}(f(x)) \right]^{p_0} \omega(x) \, dx \sim \int_{\mathbb{R}^n} \left[ \mathcal{N}_h^{(\beta)}(f(x)) \right]^{p_0} \omega(x) \, dx,
\]
where the positive equivalence constants depend only on both \( p_0 \) and \( [\omega]_{A_1(\mathbb{R}^n)} \). From this and (3.32), we deduce that (3.31) holds true.

**Step 4.** In this step, we prove that
\[
\left[ L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n) \right] \subset \left[ L^2(\mathbb{R}^n) \cap H_{X,R_0}(\mathbb{R}^n) \right].
\]

Let \( f \in L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n) \), \( \epsilon \in (n/\theta, \infty) \), and \( M \in \mathbb{N} \) be sufficiently large. By Proposition 2.16, we conclude that there exist \( \{\lambda_i\}_{i \in \mathbb{N}} \subset [0, \infty) \) and a sequence \( \{m_i\}_{i \in \mathbb{N}} \) of \( (X, M, \epsilon) \)-molecules associated, respectively, with the balls \( \{B_i\}_{i \in \mathbb{N}} \) such that
\[
f = \sum_{i=1}^{\infty} \lambda_i m_i
\]
in \( L^2(\mathbb{R}^n) \), and
\[
\left\| \left\{ \sum_{i=1}^{\infty} \left( \frac{\lambda_i}{\|1_{B_i}\|_X} \right)^s 1_{B_i} \right\}^{\frac{1}{s}} \right\|_X \leq \|f\|_{H_{X,L}(\mathbb{R}^n)}.
\]

Next, we show that there exists a positive constant \( \beta \in (n/\theta - n/q, \infty) \) such that, for any \( j \in \mathbb{Z}_+ \) and any \( (X, M, \epsilon) \)-molecule \( b \) associated with the ball \( B := B(x_B, r_B) \subset \mathbb{R}^n \) for some \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \),
\[
\|\mathcal{R}_b(b)\|_{L^q(U_j(B))} \leq |B|^{1/q} \|1_B\|_X^{-1},
\]
where \( \theta \) is the same as in Assumption 2.11, \( U_0(B) := 2B \), and \( U_j(B) := (2^{j+1}B) \setminus (2^jB) \) for any \( j \in \mathbb{N} \).

When \( j = 0 \), from Lemma 3.7 and the assumption that \( b \) is a molecule, we deduce that
\[
\|\mathcal{R}_b(b)\|_{L^q(U_0(B))} \leq \|b\|_{L^q(\mathbb{R}^n)} \leq |B|^{1/q} \|1_B\|_X^{-1}.
\]

For any \( j \in \mathbb{N} \) and \( x \in U_j(B) \), we have
\[
\mathcal{R}_b(b)(x) \leq \left( \sup_{t \in (0, 2^{j-2}r_B)} + \sup_{t \in [2^{j-2}r_B, \infty)} \right) \left[ \frac{1}{|B(x,t)|} \int_{B(x,t)} \left| e^{-r^{2m}L}(b)(y) \right|^2 \, dy \right]^{1/2} =: I_j(x) + II_j(x).
\]

For any \( j \in \mathbb{N} \), let \( S_j(B) := (2^{j+3}B)/(2^{j-3}B) \), \( R_j(B) := (2^{j+5}B)/(2^{j-5}B) \), and \( E_j(B) := [R_j(B)]^C \). It is obvious that, for any \( j \in \mathbb{N} \), \( t \in (0, 2^{j-2}r_B) \), and \( x \in U_j(B) \), we have \( B(x,t) \subset S_j(B) \) and \( \text{dist}(S_j(B), E_j(B)) \sim 2^j r_B \). By this, Proposition 2.3(iii), and Definition 2.15(i), we conclude that
\[
\left\| \sup_{t \in (0, 2^{j-2}r_B)} \left[ \frac{1}{|B(x,t)|} \int_{B(x,t)} \left| e^{-r^{2m}L}(b1_{E_j(B)})(y) \right|^2 \, dy \right]^{1/2} \right\|_{L^q(U_j(B))}.
\]
By this and (3.37), we find that, for any
\[
\frac{1}{p_n} \int_{S_{j}(B)} \left| e^{-2nL} \left( b1_{E_j(B)} \right)(y) \right|^2 \, dy \right]^{1/2} \leq \frac{1}{r_n} \int_{B(x,t)} \left( t^{2nL} \right)^M \left( r_B^{-2M} L^{-M}(b) \right) \, dy \right]^{1/2}
\]
where \( N \) is a positive number satisfying \( N \in (2n/\theta - n/2, \infty) \). On the other hand, from Lemma 3.7 and Definition 2.15(i), it follows that
\[
\| I_j \|_{L^q(U_j(B))} \leq \left[ 2^{-j(\epsilon-n/q)} + 2^{-j(n/4+N/2-n/2)} \right] |B|^{1/q} \| 1_B \|^{-1}_X.
\]
Moreover, it is obvious that, for any \( j \in \mathbb{N} \) and \( x \in U_j(B) \),
\[
II_j(x) \leq 2^{-mMj} \sup_{r \in (2^{(j+1)}r_B, \infty)} \left[ \frac{1}{p_n} \int_{B(x,t)} \left( t^{2nL} \right)^M \left( r_B^{-2M} L^{-M}(b) \right) \, dy \right]^{1/2}
\]
From this, Lemma 3.7, and Definition 2.15(i), we deduce that
\[
\| II_j \|_{L^q(U_j(B))} \leq 2^{-mMj} \| R_{h,M} \left( r_B^{-2M} L^{-M}(b) \right) \|_{L^q(\mathbb{R}^n)} \leq 2^{-mMj} \| r_B^{-2M} L^{-M}(b) \|_{L^q(\mathbb{R}^n)} \leq 2^{-mMj} |B|^{1/q} \| 1_B \|^{-1}_X.
\]
By this and (3.38), we conclude that
\[
\| R_h(b) \|_{L^q(U_j(B))} \leq 2^{-jnmM} \| b \|_{L^q(\mathbb{R}^n)} \leq 2^{-j(\epsilon-n/q)} \| b \|_{L^q(\mathbb{R}^n)} + 2^{j(n/4+N/2-n/2)} \leq 2^{-j(\epsilon-n/q)} \| b \|_{L^q(\mathbb{R}^n)} + 2^{j(n/4+N/2-n/2)} \| b \|_{L^q(\mathbb{R}^n)}
\]
which implies that (3.36) holds true with \( \beta := \min \{ mM, \epsilon-n/q, n/4+N/2-n/2 \} \in (n/\theta - n/2, \infty) \).

Furthermore, from (3.34) and the fact that \( R_h \) is bounded on \( L^2(\mathbb{R}^n) \), it follows that, for almost every \( x \in \mathbb{R}^n \),
\[
|R_h(f)(x)| \leq \sum_{i=1}^\infty |A_i R_h(m_i)(x)| = \sum_{i=1}^\infty \sum_{j=0}^\infty |A_i R_h(m_i)(x)| 1_{U_j(B_i)}(x).
\]
Using this, (3.36) with \( b := m_i \) for \( i \in \mathbb{N} \), Lemma 2.13, and (3.35), we find that
\[
\| R_h(f) \|_X \leq \left\| \sum_{i=1}^\infty \sum_{j=0}^\infty |A_i R_h(m_i)| 1_{U_j(B_i)} \right\|_X \leq \left\| \left\{ \sum_{i=1}^\infty \frac{A_i}{|1_B|_X} \right\}^{1/2} \right\|_X \leq \| f \|_{H^{\theta,1}(\mathbb{R}^n)},
\]
which further implies that (3.33) holds true.

Then, by the conclusions obtained in Steps 1 though 4, we conclude that

\[ [L^2(\mathbb{R}^n) \cap H_{X,S}(\mathbb{R}^n)] = [L^2(\mathbb{R}^n) \cap H_{X,N}(\mathbb{R}^n)] = [L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n)] \]

with equivalent quasi-norms. From this and a density argument, we deduce that the spaces $H_{X,S}(\mathbb{R}^n)$, $H_{X,N}(\mathbb{R}^n)$, $H_{X,R}(\mathbb{R}^n)$, and $H_{X,L}(\mathbb{R}^n)$ coincide with the equivalent quasi-norms. This finishes the proof of Theorem 3.1.

\[ \square \]

4 Riesz Transform Characterization of $H_{X,L}(\mathbb{R}^n)$

In this section, we establish the Riesz transform characterization of $H_{X,L}(\mathbb{R}^n)$. In Subsection 4.1, we prove that the Riesz transform $\nabla^m L^{-1/2}$ is bounded from $H_{X,L}(\mathbb{R}^n)$ to $H_X(\mathbb{R}^n)$ by using the molecular characterization of both $H_{X,L}(\mathbb{R}^n)$ and $H_X(\mathbb{R}^n)$, where $H_X(\mathbb{R}^n)$ denotes the Hardy space associated with $X$ introduced in [66]. In Subsection 4.2, we first introduce the homogeneous Hardy–Sobolev space $H_{m,X}(\mathbb{R}^n)$ and then establish its atomic decomposition which plays a key role in the proof of the Riesz transform characterization of $H_{X,L}(\mathbb{R}^n)$. Then, using the atomic decomposition of $H_{m,X}(\mathbb{R}^n)$ and borrowing some ideas from the proof of [47, Proposition 5.17], we obtain the Riesz transform characterization of $H_{X,L}(\mathbb{R}^n)$.

Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Then the Riesz transform $\nabla^m L^{-1/2}$ is defined by setting, for any $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

\[ \nabla^m L^{-1/2}(f)(x) := \frac{1}{2\sqrt{\pi}} \int_0^\infty \nabla^m e^{-sL}(f)(x) \frac{ds}{\sqrt{s}}. \]

**Definition 4.1.** Let $X$ be a ball quasi-Banach function space on $\mathbb{R}^n$, $m \in \mathbb{N}$, and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1). The *Hardy space* $H_{X,L,\text{Riesz}}(\mathbb{R}^n)$, associated with $L$, is defined as the completion of the set

\[ \left\{ f \in L^2(\mathbb{R}^n) : \|f\|_{H_{X,L,\text{Riesz}}(\mathbb{R}^n)} := \left\| \nabla^m L^{-1/2}(f) \right\|_{H_X(\mathbb{R}^n)} < \infty \right\} \]

with respect to the quasi-norm $\| \cdot \|_{H_{X,L,\text{Riesz}}(\mathbb{R}^n)}$, where $H_X(\mathbb{R}^n)$ denotes the Hardy space associated with $X$ (see Definition 4.6 below for its definition).

The following theorems are the main results of this section.

**Theorem 4.2.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that $X$ is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some $\theta \in (n/(n+m), 1]$, $s \in (\theta, 1]$, and $q \in (p_-(L), \min(p_+(L), q_+(L)))$. Then there exists a positive constant $C$ such that, for any $f \in H_{X,L}(\mathbb{R}^n)$, $f \in H_{X,L,\text{Riesz}}(\mathbb{R}^n)$ and

\[ \|f\|_{H_{X,L,\text{Riesz}}(\mathbb{R}^n)} \leq C \|f\|_{H_{X,L}(\mathbb{R}^n)}. \]
Remark 4.3. When \( X := L^p(\mathbb{R}^n) \) with \( p \in (n/(n + m), 1) \), Theorem 4.2 is just [18, Theorem 6.2].

Theorem 4.4. Let \( m \in \mathbb{N}, L \) be a homogeneous divergence form \( 2m \)-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and the family \( \{ e^{-tL} \}_{t \in (0, \infty)} \) of operators satisfy \( m - L'((\mathbb{R}^n) - L^2(\mathbb{R}^n)) \) off-diagonal estimates with some \( r \in (1, 2] \). Assume that \( X \) is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some \( s \in \theta, \theta \), associated with \( X \), and \( q \leq \frac{n}{n + mr}, 1 \), and \( q \in [2, p_+(L)) \). Then there exists a positive constant \( C \) such that, for any \( h \in H_{X,L,Riesz}(\mathbb{R}^n), \ h \in H_{X,L}(\mathbb{R}^n) \) and
\[
\|h\|_{H_{X,L}(\mathbb{R}^n)} \leq C\|h\|_{H_{X,L,Riesz}(\mathbb{R}^n)}.
\]

Remark 4.5. When \( X := L^p(\mathbb{R}^n) \) with \( p \in (nr/(n + mr), 1) \), Theorem 4.4 is just [18, Proposition 6.6].

The proofs of Theorems 4.2 and 4.4 are given, respectively, in Subsections 4.1 and 4.2 below.

4.1 Proof of Theorem 4.2

In this subsection, we prove Theorem 4.2. To this end, we need the molecular characterization of the Hardy space \( H_X(\mathbb{R}^n) \) which was obtained in [66].

Let \( \Phi \in S(\mathbb{R}^n) \) satisfy that \( \int_{\mathbb{R}^n} \Phi(x) \, dx \neq 0 \) and \( \Phi_t(x) := t^n\Phi(x/t) \) for any \( x \in \mathbb{R}^n \) and \( t \in (0, \infty) \). For any \( f \in S'(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), let
\[
M_b^\star(f, \Phi)(x) := \sup_{(y,t) \in \mathbb{R}^n \times (0, \infty)} \frac{|(\Phi_t \ast f)(x - y)|}{(1 + t^{-1}|y|)^b}
\]
with \( b \in (0, \infty) \) sufficiently large, where, for any \( t \in (0, \infty) \), \( \Phi_t(\cdot) := \frac{1}{t^n}\Phi(\cdot/t) \) and \( \Phi_t \ast f \) denotes the convolution of \( \Phi_t \) and \( f \).

Definition 4.6. Let \( X \) be a ball quasi-Banach function space on \( \mathbb{R}^n \). Then the Hardy space \( H_X(\mathbb{R}^n) \) associated with \( X \) is defined as
\[
H_X(\mathbb{R}^n) := \{ f \in S'(\mathbb{R}^n) : \|f\|_{H_X(\mathbb{R}^n)} := \|M_b^\star(f, \Phi)\|_X < \infty \}.
\]

Definition 4.7. Let \( X \) be a ball quasi-Banach function space satisfying Assumption 2.11 for some \( \theta, s \in (0, 1), q \in [1, \infty), n \in \mathbb{N} \cap \{d_X, \infty\}, \) and \( \tau \in (0, \infty) \), where \( d_X := \lfloor n(1/\theta - 1) \rfloor \). A measurable function \( m \) on \( \mathbb{R}^n \) is called an \( (X, q, d, \tau) \)-molecule centered at a cube \( Q \in \mathcal{Q} \) if it satisfies that, for any \( j \in \mathbb{Z}_+ \),
\[
\|\mathbf{1}_{U_j(Q)}m\|_{L^q(\mathbb{R}^n)} \leq 2^{-j\tau} \frac{|Q|^{\frac{1}{d}}}{{\|1_Q\|}_X}
\]
and, for any \( \alpha \in \mathbb{Z}_+^n \) with \( |\alpha| \leq d \),
\[
\int_{\mathbb{R}^n} m(x)x^\alpha \, dx = 0.
\]
Similarly, in the above definition, if any \( Q \in \mathcal{Q} \) is replaced by any ball \( B \), then one obtains the definition of an \( (X, q, d, \tau) \)-molecule centered at a ball \( B \).
The following molecular characterization of the Hardy space \( H_X(\mathbb{R}^n) \) is just [66, Theorem 3.9].

**Lemma 4.8.** Assume that \( X \) is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some \( \theta \), \( s \in (0, 1] \) and \( q \in (1, \infty] \). Let \( \tau \in (\eta[1/\theta - 1/q], \infty) \). Then \( f \in H_X(\mathbb{R}^n) \) if and only if there exists a sequence \( \{m_j\}_{j=1}^{\infty} \) of \((X, q, d_X, \tau)\)-molecules centered, respectively, at the cubes \( \{Q_j\}_{j=1}^{\infty} \subset Q \) and \( \{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty) \) satisfying

\[
\left\| \left\{ m_j \left( \frac{\lambda_j}{\|1_{Q_j}\|_X} \right)^{s} 1_{Q_j} \right\} \right\|_{X} < \infty
\]

such that

\[
f = \sum_{j=1}^{\infty} \lambda_j m_j
\]

in \( S'(\mathbb{R}^n) \). Moreover,

\[
\|f\|_{H_X(\mathbb{R}^n)} \sim \left\| \left\{ m_j \left( \frac{\lambda_j}{\|1_{Q_j}\|_X} \right)^{s} 1_{Q_j} \right\} \right\|_{X},
\]

where the positive equivalence constants are independent of \( f \).

To prove Theorem 4.2, we need the following conclusion whose proof is a slight modification of the proof of [18, Lemma 6.1]; we omit the details here.

**Lemma 4.9.** Let \( m \in \mathbb{N} \), \( L \) be a homogeneous divergence form \( 2m \)-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, \( M \in \mathbb{N} \), and \( q \in (p_-(L), \min(p_+(L), q_+(L))) \). Then there exists a positive constant \( C \) such that, for any \( t \in (0, \infty) \), any closed subsets \( E, F \) of \( \mathbb{R}^n \) with \( \text{dist} (E, F) \in (0, \infty) \), and any \( f \in L_q(\mathbb{R}^n) \) with \( \text{supp}(f) \subset E \),

\[
\left\| \nabla^m L^{-1/2} \left( I - e^{-tL} \right)^M (f) \right\|_{L^q(F)} \leq C \left( \frac{t}{\text{dist} (E, F)^{2m}} \right)^M \|f\|_{L^q(E)}
\]

and

\[
\left\| \nabla^m L^{-1/2} \left( tLe^{-tL} \right)^M (f) \right\|_{L^q(F)} \leq C \left( \frac{t}{\text{dist} (E, F)^{2m}} \right)^M \|f\|_{L^q(E)}.
\]

Furthermore, we also need the following boundedness of \( \nabla^m L^{-1/2} \), which can be found in [3, p. 68].

**Lemma 4.10.** Let \( m \in \mathbb{N} \) and \( L \) be a homogeneous divergence form \( 2m \)-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that \( q \in (q_-(L), q_+(L)) \). Then the Riesz transform \( \nabla^m L^{-1/2} \) is bounded on \( L^q(\mathbb{R}^n) \).

Now, we prove Theorem 4.2 by using Lemmas 4.8, 4.9, and 4.10.
Proof of Theorem 4.2. Let all the symbols be the same as in the present theorem, \( f \in H_{X,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), \( \epsilon \in (\max\{\epsilon_0, n/q\}, \infty) \), and \( M \in \mathbb{N} \cap \{n[1/(2m\theta) - 1/(2mq)]\}, \infty) \). By Proposition 2.16, we find that there exists a sequence \( \{\lambda_j\} \in [0, \infty) \) and a sequence \( \{\alpha_j\} \in [\infty) \) of \( (X, M, \epsilon)_L \)-molecules associated, respectively, with the balls \( \{B_j\} \in \mathbb{N} \) such that

\[
(4.2) \quad f = \sum_{j=1}^{\infty} \lambda_j \alpha_j, \quad \text{in } L^2(\mathbb{R}^n),
\]

and

\[
\left\| \left( \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|B_j\|_X} \right)^s 1_{B_j} \right)^{\frac{1}{s}} \right\|_X \leq \| f \|_{H_{X,L}(\mathbb{R}^n)}. \tag{4.3}
\]

Then, from (4.2) and Lemma 4.10, it follows that

\[
\nabla^m L^{-1/2}(f) = \sum_{j=1}^{\infty} \lambda_j \nabla^m L^{-1/2}(\alpha_j)
\]

in \( L^2(\mathbb{R}^n) \).

Next, we show that, for any \( (X, M, \epsilon)_L \)-molecule \( b \) associated with the ball \( B := B(x_B, r_B) \subset \mathbb{R}^n \) with some \( x_B \in \mathbb{R}^n \) and \( r_B \in (0, \infty) \), and for any \( j \in \mathbb{Z}_+ \),

\[
\| \nabla^m L^{-1/2}(b) \|_{L^q(U_j(B))} \leq 2^{-j/q} |B|^{1/q} \|b\|_X^{-1}
\]

where both the implicit positive constant and \( \tau \in (n[1/\theta - 1/q], \infty) \) are independent of \( m \).

When \( j = 0 \), by Lemma 4.10, Definition 2.15(i), and \( \epsilon \in (n/q, \infty) \), we conclude that

\[
\| \nabla^m L^{-1/2}(b) \|_{L^q(U_0(B))} \leq \|b\|_{L^q(\mathbb{R}^n)} = \sum_{j=0}^{\infty} \|b\|_{L^q(U_j(B))} \leq \sum_{j=0}^{\infty} 2^{-j/q} |B|^{1/q} \|b\|_X^{-1} \leq |B|^{1/q} \|b\|_X^{-1}.
\]

Furthermore, for any \( j \in \mathbb{N} \), we have

\[
(4.4) \quad \| \nabla^m L^{-1/2}(b) \|_{L^q(U_j(B))} \leq \| \nabla^m L^{-1/2} \left( \frac{1 - e^{-2mL}}{2} \right)^M (b) \|_{L^q(U_j(B))} + \| \nabla^m L^{-1/2} \left( \frac{1 - e^{-2mL}}{2} \right)^M (b) \|_{L^q(U_j(B))} \leq \| \nabla^m L^{-1/2} \left( \frac{1 - e^{-2mL}}{2} \right)^M (b) \|_{L^q(U_j(B))} + \sum_{k=1}^{M} \| \nabla^m L^{-1/2} \left( \frac{k}{M} \frac{2mL - (k/M)r_2^M}{2} \right)^M (r_2^M L^{-1})^M (b) \|_{L^q(U_j(B))}
\]
which, together with the assumption that \( \theta \), combined with both (4.5) and (4.4), further implies that (4.3) holds true with 

\[
\| f \|_{H^s_x(L^p_{\mathbb{R}^n})} \leq \left\{ \sum_{j=1}^{\infty} \left( \frac{1}{\| b_j \|_\infty} \right) \right\}^{\frac{s}{d}} \| f \|_{L^p_{\mathbb{R}^n}}.
\]

This, combined with the fact that \( H_{X,1}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) is dense in \( H_{X,1}(\mathbb{R}^n) \), then finishes the proof of Theorem 4.2. \( \Box \)
4.2 Proof of Theorem 4.4

In this subsection, we prove Theorem 4.4. We first establish the atomic decomposition of the homogeneous Hardy–Sobolev space $\dot{H}_{m,X}(\mathbb{R}^n)$.

**Definition 4.11.** Let $X$ be a ball quasi-Banach function space on $\mathbb{R}^n$ and $m \in \mathbb{N}$. Then the homogeneous Hardy–Sobolev space $\dot{H}_{m,X}(\mathbb{R}^n)$, associated with $X$, is defined as

$$\dot{H}_{m,X}(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{H_{m,X}(\mathbb{R}^n)} := \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = m} \|\partial^\alpha f\|_{H_X(\mathbb{R}^n)} < \infty \right\}.$$ 

Let us recall the definition of weak derivatives of the locally integrable function. For any $\alpha \in \mathbb{Z}_+^n$ and any locally integrable functions $f$ and $g$, $g$ is called the $\alpha$-order weak derivative of $f$ if, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$ (the set of all the infinitely differentiable functions on $\mathbb{R}^n$ with compact support),

$$\int_{\mathbb{R}^n} f(x) \partial^\alpha \varphi(x) \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} g(x) \varphi(x) \, dx.$$ 

Moreover, we denote $g$ by $\partial^\alpha f$.

**Definition 4.12.** Let $X$ be a ball quasi-Banach function space, $m \in \mathbb{N}$, and $p \in (1, \infty)$. Then a function $a$ is called an $(H_{m,X}, p)$-atom if there exists a ball $B \subset \mathbb{R}^n$ such that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = m$,

(i) $\text{supp} (a) := \{ x \in \mathbb{R}^n : a(x) \neq 0 \} \subset B$;

(ii) $\|a\|_{L^p(\mathbb{R}^n)} \leq |B|^{1/p-m/n} \|1_B\|_X^{-1}$;

(iii) $\|\partial^\alpha a\|_{L^p(\mathbb{R}^n)} \leq |B|^{1/p} \|1_B\|_X^{-1}$.

Then we have the following atomic decomposition theorem for the space $\dot{H}_{m,X}(\mathbb{R}^n)$.

**Theorem 4.13.** Let $m \in \mathbb{N}$, $p \in (1, \infty)$, and $X$ be a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some $\theta, s \in (0, 1]$ and $q \in (1, \infty)$. Assume that $f \in \dot{H}_{m,X}(\mathbb{R}^n)$ and $\partial^\alpha f \in L^2(\mathbb{R}^n)$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = m$. Then there exists a sequence $\{a_j\}_{j=1}^\infty \subset [0, \infty)$ and a sequence $\{a_j\}_{j=1}^\infty$ of $(H_{m,X}, p)$-atoms associated, respectively, with the balls $\{B_j\}_{j=1}^\infty$ such that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| = m$,

$$\partial^\alpha f = \sum_{j=1}^\infty \lambda_j \partial^\alpha a_j,$$

in $L^2(\mathbb{R}^n)$, and

$$\left\| \left\{ \sum_{j=1}^\infty \left( \frac{\lambda_j}{\|1_{B_j}\|_X} \right)^s \right\}^{1/s} \right\|_X \leq \|f\|_{H_{m,X}(\mathbb{R}^n)},$$

where the implicit positive constant is independent of $f$. 

Furthermore, if a set of all the measurable functions $g$ that a measurable function $f \in L^p(\mathbb{R}^n)$, then $\|g\|_{L^p} \leq \|f\|_{L^p}$. Note that if $g \in L^p(\mathbb{R}^n)$, then $\|g\|_{L^p} \leq \|f\|_{L^p}$.

By definition, for an open set $O \subset \mathbb{R}^n$, define the tent $\hat{O}$ over $O$ by setting

$$\hat{O} := \{(x, t) \in \mathbb{R}_{++}^n : B(x, t) \subset O\}.$$

Coifman et al. [24] introduced the tent space $T^p(\mathbb{R}_+^n)$ for any given $p \in (0, \infty)$. Recall that a measurable function $g$ is said to belong to the tent space $T^p(\mathbb{R}_+^n)$, with $p \in (0, \infty)$, if $\|g\|_{T^p(\mathbb{R}_+^n)} := \|\mathcal{A}(g)\|_{L^p(\mathbb{R}^n)} < \infty$, where $\mathcal{A}$ is the same as in (3.30) with $\alpha := 1$.

For a given ball quasi-Banach function space $X$, the $X$-tent space $T_X(\mathbb{R}_+^n)$ is defined to be the set of all the measurable functions $g : \mathbb{R}_+^n \rightarrow \mathbb{C}$ with the finite quasi-norm

$$\|g\|_{T_X(\mathbb{R}_+^n)} := \|\mathcal{A}(g)\|_X$$

(see [66, p. 28]). We need the atomic decomposition of the $X$-tent space $T_X(\mathbb{R}_+^n)$, which is a part of [66, Theorem 3.19].

**Definition 4.14.** Let $X$ be a ball quasi-Banach function space and $p \in (1, \infty)$. A measurable function $a : \mathbb{R}_+^n \rightarrow \mathbb{C}$ is called a $(T_X, p)$-atom if there exists a ball $B \subset \mathbb{R}^n$ such that

(i) $\text{supp}(a) := \{(x, t) \in \mathbb{R}_+^n : a(x, t) \neq 0\} \subset B,$

(ii) $\|a\|_{T(\mathbb{R}_+^n)} \leq \frac{|B|^{1/p}}{\|B\|_X}.$

Furthermore, if $a$ is a $(T_X, p)$-atom for any $p \in (1, \infty)$, then $a$ is called a $(T_X, \infty)$-atom.

**Lemma 4.15.** Assume that $X$ is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some $\theta, s \in (0, 1]$ and $q \in (1, \infty)$. Let $f \in T_X(\mathbb{R}_+^n)$. Then there exists a sequence $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ and a sequence $\{a_j\}_{j=1}^\infty$ of $(T_X, \infty)$-atoms associated, respectively, with the balls $\{B_j\}_{j=1}^\infty$ such that, for almost every $(x, t) \in \mathbb{R}_+^n$,

$$f(x, t) = \sum_{j=1}^\infty \lambda_j a_j(x, t), \quad |f(x, t)| = \sum_{j=1}^\infty \lambda_j |a_j(x, t)|,$$

and

$$\left\|\left\{\sum_{j=1}^{\infty} \left(\frac{\lambda_j}{\|B_j\|_X}\right)^s 1_{B_j}\right\}^{1/s}\right\|_X \leq \|f\|_{T_X(\mathbb{R}_+^n)},$$

where the implicit positive constant is independent of $f$.

Let $\psi \in S(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x) \, dx = 0$. For any $g \in T^p(\mathbb{R}_+^n)$ with compact support and for any $x \in \mathbb{R}^n$, let

$$\pi_\phi(g)(x) := \int_0^\infty g(\cdot, t) * \psi(t) \frac{dt}{t}.$$  

By [24, Theorem 6(1)], we have the following lemma.
Lemma 4.16. Let $p \in (1, \infty)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x) \, dx = 0$, and $\pi_\psi$ be the same as in (4.8). Then the operator $\pi_\psi$ can extend to be a bounded linear operator from $T^p(\mathbb{R}^{n+1})$ to $L^p(\mathbb{R}^n)$.

Now, we show Theorem 4.13 via using Lemmas 4.15 and 4.16.

Proof of Theorem 4.13. Let all the symbols be the same as in the present theorem. By the proof of [37, Lemma 1.1], we conclude that there exists a radial function $\varphi \in C^\infty_c(\mathbb{R}^n)$ such that $\text{supp}(\varphi) \subset B(0, 1)$, $\int_{\mathbb{R}^n} \varphi(x) \, dx = 0$, and, for any $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\int_0^\infty t^{2m-1} |\xi|^2 m \left[ \hat{\varphi}(t\xi) \right]^2 \, dt = 1,$$

where $\hat{\varphi}$ denotes the Fourier transform of $\varphi$, namely, for any $\xi \in \mathbb{R}^n$,

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} \varphi(x)e^{-2\pi i x \cdot \xi} \, dx.$$

For any measurable function $h$ on $\mathbb{R}^n$ and any $t \in (0, \infty)$, let $h_t(\cdot) := t^{-n} h(\cdot/t)$. For any $(x, t) \in \mathbb{R}^{n+1}_+$, let

$$F(x, t) := \sum_{\alpha \in \mathbb{Z}^n, \text{ } |\alpha| = m} \frac{m!}{\alpha!} (\partial^\alpha f) * (\partial^\alpha \varphi)_t(x).$$

For any $g \in H_X(\mathbb{R}^n)$, any $\psi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x) \, dx = 0$, and any $x \in \mathbb{R}^n$, let

$$S^\alpha g(x) := \left\{ \left( \int_{T(x)} |g * \psi_t(y)|^2 \, dy \, dt \right)^{1/2} \right\}_{t \to 0^+}.$$

Let $\alpha \in \mathbb{Z}^n$ with $|\alpha| = m$. By the proof of [66, Theorem 3.21], we conclude that $S^\alpha \varphi$ is bounded from $H_X(\mathbb{R}^n)$ to $X$. Therefore,

$$\|S^\alpha \varphi (\partial^\alpha f)\|_X \leq \|\partial^\alpha f\|_{H_X(\mathbb{R}^n)},$$

which further implies that

$$\|F\|_{T_X(\mathbb{R}^{n+1})} \leq \|f\|_{H^m_X(\mathbb{R}^n)}.$$

Moreover, from Lemma 4.15, we deduce that there exists a sequence $\{\lambda_j\}_{j=1}^\infty \subset [0, \infty)$ and a sequence $\{b_j\}_{j=1}^\infty$ of $(T_X, \infty)$-atoms associated, respectively, with the balls $\{\mathcal{B}_j\}_{j=1}^\infty$ such that, for almost every $(x, t) \in \mathbb{R}^{n+1}_+$,

$$F(x, t) = \sum_{j=1}^\infty \lambda_j b_j(x, t), \quad |F(x, t)| = \sum_{j=1}^\infty \lambda_j |b_j(x, t)|,$$

and

$$\left\| \left\{ \sum_{j=1}^\infty \left( \frac{\lambda_j}{\|1_{\mathcal{B}_j}\|_X} \right)^s 1_{\mathcal{B}_j} \right\}^{1/s} \right\|_X \leq \|F\|_{T_X(\mathbb{R}^{n+1})}.$$
This, combined with (4.9), implies that (4.7) holds true.

For any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$, let

$$a_j(x) := \int_0^\infty b_j(\cdot, t) \ast \varphi_i(x) t^{m-1} \, dt.$$  

Next, we prove that, for any $j \in \mathbb{N}$, $a_j$ is an $(\mathcal{H}_m, p)$-atom up to a harmless constant multiple.

Let $j \in \mathbb{N}$. Since $b_j$ is supported in $\hat{B}_j$ and $\varphi$ is supported in $\mathcal{B}(0, 1)$, it follows that $\text{supp}(a_j) \subset B_j$. We first show that

$$\int_{\mathbb{R}^n} a_j(x) h(x) \, dx = \int_{\mathbb{R}^n} \int_0^\infty b_j(\cdot, t) \ast \varphi_i(x) t^{m-1} \, dth(x) \, dx$$

where the implicit positive constant is independent of $j$. By the Tonelli theorem, the H"{o}lder inequality, and the fact that $\text{supp}((a_j)) \subset B_j$, we find that, for any $h \in L^{p'}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} a_j(x) h(x) \, dx \leq \int_{\mathbb{R}^{n+1}} |h \ast \varphi_i(y) b_j(y, t)| \, dy \, dt \leq r_{B_j}^m \int_{\mathbb{R}^{n+1}} \mathbf{1}_{B(y, t)}(x) \, dx \int_{\mathbb{R}^n} |h \ast \varphi_i(y) b_j(y, t)| \, dy \, dt$$

$$\leq r_{B_j}^m \int_{\mathbb{R}^n} S_{\varphi}(h)(x) \mathcal{A}(b_j)(x) \, dx,$$

here and thereafter, $\frac{1}{p} + \frac{1}{p'} = 1$. Using this, the H"{o}lder inequality, and the facts that $S_{\varphi}$ is bounded on $L^p(\mathbb{R}^n)$ (see, for instance, [36, Theorem 7.8]) and that $b_j$ is a $(T_X, \infty)$-atom, we conclude that

$$\int_{\mathbb{R}^n} a_j(x) h(x) \, dx \leq r_{B_j}^m \|b_j\|_{T^p(\mathbb{R}^{n+1})} \|S_{\varphi}(h)\|_{L^p(\mathbb{R}^n)} \leq |B_j|^{1/p + m/n} \|1_{B_j}\|^{-1} \|h\|_{L^p(\mathbb{R}^n)},$$

which further implies that (4.10) holds true.

Then we prove that, for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$,

$$\partial^\alpha a_j = \pi_{\partial^\alpha \varphi}(b_j)$$

and

$$\left\|\partial^\alpha a_j\right\|_{L^p(\mathbb{R}^n)} \leq |B_j|^{1/p} \|1_{B_j}\|^{-1},$$

where the implicit positive constant is independent of $j$. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = m$ and $\pi_{\partial^\alpha \varphi}$ be the same as (4.8) with $\psi$ replaced by $\partial^\alpha \varphi$. Let $b_{j, k}(y, t) := b_j(y, t)1_{(1/k, k)}(t)$ for any $(y, t) \in \mathbb{R}^{n+1}$ and
\( k \in \mathbb{N} \). It is obvious that \( b_{j,k} \to b_j \) in \( T^p(\mathbb{R}^{n+1}_+) \) as \( k \to \infty \). By this and Lemma 4.16 with \( \psi \) replaced by \( \partial^\alpha \varphi \), we obtain that \( \pi_{\partial^\alpha \varphi}(b_{j,k}) \to \pi_{\partial^\alpha \varphi}(b_j) \) in \( L^p(\mathbb{R}^n) \) as \( k \to \infty \). This, together with the Hölder inequality, implies that, for any \( \psi \in C_c^\infty(\mathbb{R}^n) \),

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \pi_{\partial^\alpha \varphi}(b_{j,k})(x)\varphi(x) \, dx = \int_{\mathbb{R}^n} \pi_{\partial^\alpha \varphi}(b_j)(x)\varphi(x) \, dx.
\]

Using this, the Tonelli theorem, and the Fubini theorem, we find that, for any \( \psi \in C_c^\infty(\mathbb{R}^n) \),

\[
\int_{\mathbb{R}^n} a(x)\partial^\alpha \psi(x) \, dx = \int_0^\infty \int_{\mathbb{R}^n} b_j(\cdot, t) * \varphi(x)\partial^\alpha \psi(x) \, dx \, dt
\]

\[
= \lim_{k \to \infty} \int_{1/k}^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi_j(x-y) b_j(y,t) \, dy \partial^\alpha \psi(x) \, dx \, dt
\]

\[
= \lim_{k \to \infty} \int_{1/k}^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi_j(x-y) \partial^\alpha \psi(x) \, dx b_j(y,t) \, dy \, dt
\]

\[
= (-1)^m \lim_{k \to \infty} \int_{1/k}^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\partial^\alpha \varphi)_j(x-y) \psi(x) \, dx b_j(y,t) \, dy \, dt
\]

\[
= (-1)^m \lim_{k \to \infty} \int_{1/k}^k \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (\partial^\alpha \varphi)_j(x-y) b_j(y,t) \, dy \psi(x) \, dx \, dt
\]

\[
= (-1)^m \int_{\mathbb{R}^n} \pi_{\partial^\alpha \varphi}(b_j)(x)\varphi(x) \, dx.
\]

This implies that (4.11) holds true. Furthermore, applying Lemma 4.16 with \( \psi \) replaced by \( \partial^\alpha \varphi \) and the fact that \( b_j \) is a \((T_X, \infty)\)-atom, we find that

\[
\|\partial^\alpha a_j\|_{L^p(\mathbb{R}^n)} = \|\pi_{\partial^\alpha \varphi}(b_j)\|_{L^p(\mathbb{R}^n)} \lesssim \|b_j\|_{T^p(\mathbb{R}^{n+1}_+)} \lesssim |B_j|^{1/p} \|1_{B_j}\|_{X}.
\]

This proves (4.12). By (4.10) and (4.12), we conclude that \( a_j \) is an \((H_{m_X}, p)\)-atom up to a harmless constant multiple.

To complete the proof of Theorem 4.13, it suffices to show that (4.6) holds true. For any \( \alpha \in \mathbb{Z}^n_+ \) with \( |\alpha| = m \), since \( S_{\partial^\alpha \varphi} \) is bounded on \( L^2(\mathbb{R}^n) \) (see, for instance, [36, Theorem 7.8]), it follows that

\[
\|S_{\partial^\alpha \varphi} \partial^\alpha f\|_{L^2(\mathbb{R}^n)} \lesssim \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.
\]

Thus, \( F \in T^2(\mathbb{R}^{n+1}_+) \). Moreover, by [66, Lemma 6.9(i)], we conclude that

\[
F = \sum_{j=1}^{\infty} \lambda_j b_j
\]

in \( T^2(\mathbb{R}^{n+1}_+) \). Let \( \beta \in \mathbb{Z}^n_+ \) with \( |\beta| = m \). From (4.13), Lemma 4.16 with \( \psi \) replaced by \( \partial^\alpha \varphi \), and (4.11), we deduce that

\[
\pi_{\partial^\alpha \varphi}(F) = \sum_{j=1}^{\infty} \lambda_j \pi_{\partial^\alpha \varphi}(b_j) = \sum_{j=1}^{\infty} \lambda_j \partial^\alpha a_j
\]
in $L^2(\mathbb{R}^n)$. By an argument similar to that used in the proof of [37, Theorem 1.2], we find that

$$\lim_{k \to \infty} \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = m} \frac{m!}{\alpha!} \int_{1/k}^k \partial^\alpha f \ast (\partial^\alpha \varphi)_t \frac{dt}{t} = \partial^\alpha f$$

in $L^2(\mathbb{R}^n)$. On the other hand, applying an argument similar to that used in the estimation of (4.10), we obtain

$$\lim_{k \to \infty} \int_{1/k}^k F(\cdot, t) \ast (\partial^\alpha \varphi)_t \frac{dt}{t} = \pi_{\partial^\alpha \varphi}(F)$$

in $L^2(\mathbb{R}^n)$, which, together with the fact that

$$\int_{1/k}^k F(\cdot, t) \ast (\partial^\alpha \varphi)_t \frac{dt}{t} = \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = m} \frac{m!}{\alpha!} \int_{1/k}^k \partial^\alpha f \ast (\partial^\alpha \varphi)_t \frac{dt}{t}$$

$$= \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = m} \frac{m!}{\alpha!} \int_{1/k}^k \partial^\alpha f \ast (\partial^\alpha \varphi)_t \ast (\partial^\alpha \varphi)_t \frac{dt}{t},$$

further implies that

$$\lim_{k \to \infty} \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| = m} \frac{m!}{\alpha!} \int_{1/k}^k \partial^\alpha f \ast (\partial^\alpha \varphi)_t \ast (\partial^\alpha \varphi)_t \frac{dt}{t} = \pi_{\partial^\alpha \varphi}(F)$$

in $L^2(\mathbb{R}^n)$. By this and (4.15), we conclude that, for any $\beta \in \mathbb{Z}_+^n$ with $|\beta| = m$, $\partial^\beta f(x) = \pi_{\partial^\beta \varphi}(F)(x)$ for almost every $x \in \mathbb{R}^n$. From this and (4.14), it follows that (4.6) holds true. This finishes the proof of Theorem 4.13. □

By an argument similar to that used in the proof of [47, Lemma 2.26], we have the following lemma; we omit the details here.

**Lemma 4.17.** Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form 2m-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and $r \in [1, 2)$. Assume that the family $\{e^{-tL}\}_{t \in (0, \infty)}$ of operators satisfies the $m - L^r(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimate. Then the family $\{tLe^{-tL}\}_{t \in (0, \infty)}$ of operators also satisfies the $m - L^r(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimate and is bounded from $L^r(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ with norm bounded by $Ct^{\frac{n}{2}r(\frac{2}{r} - \frac{n}{2})}$, where $C$ is a positive constant independent of $t$.

For any $h \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$S_1(h)(x) := \left[ \int \int_{\Gamma(x)} \left| \frac{pL^{1/2}e^{-tL}h(y)}{p^{n+1}} \right|^2 dy dt \right]^{1/2}.$$

Then we have the following conclusion.

**Lemma 4.18.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form 2m-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that $X$ is a ball quasi-Banach function space satisfying both Assumptions 2.11 and 2.12 for some $\theta, s \in (0, 1)$ and $q \in [2, p_+(L))$. Then there exists a positive constant $C$ such that, for any $h \in L^2(\mathbb{R}^n)$ with $\|S_1(h)\|_X < \infty$,

$$\|h\|_{\mathcal{H}_{X,L}(\mathbb{R}^n)} \leq C\|S_1(h)\|_X.$$
Proof. Let all the symbols be the same as in the present lemma. Let \( h \in L^2(\mathbb{R}^n) \) with \( \|S_1(h)\|_X < \infty \), \( \epsilon \in (n/\theta, \infty) \), and \( M \in \mathbb{N} \) be sufficiently large. Then, repeating the proof of [66, Proposition 6.11], we conclude that there exists a sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \in [0, \infty) \) and a sequence \( \{\alpha_j\}_{j \in \mathbb{N}} \) of \( (X, M, \epsilon) \)-molecules associated, respectively, with the balls \( \{B_j\}_{j \in \mathbb{N}} \) such that \( h = \sum_{j=1}^{\infty} \lambda_j \alpha_j \) in \( L^2(\mathbb{R}^n) \). Moreover, there exists a positive constant \( C \) such that, for any \( h \in L^2(\mathbb{R}^n) \) with \( \|S_1(h)\|_X < \infty \),

\[
\left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|B_j\|_X} \right)^s 1_{B_j} \right\} \right\|_X \leq C \|S_1(h)\|_X.
\]

From this and [66, Theorem 6.12], we deduce that \( h \in H_{X,L}(\mathbb{R}^n) \) and

\[
\|h\|_{H_{X,L}(\mathbb{R}^n)} \leq \left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|B_j\|_X} \right)^s 1_{B_j} \right\} \right\|_X \leq \|S_1(h)\|_X.
\]

This finishes the proof of Lemma 4.18. \( \square \)

Now, we show Theorem 4.4 by using Theorems 4.13, 4.17, and 4.18.

**Proof of Theorem 4.4.** Let all the symbols be the same as in the present theorem. Assume that \( h \in L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n) \) and \( f := L^{-1/2}(h) \). By Lemma 4.18, we find that

\[
\|h\|_{H_{X,L}(\mathbb{R}^n)} \leq \|S_1(h)\|_X.
\]

From this, we deduce that, to prove (4.1), it suffices to show that

\[
\|S_1 \left( L^{1/2} f \right) \|_X \leq \|f\|_{H_{m,X}(\mathbb{R}^n)}.
\]

By both the boundedness of the Riesz transform \( \nabla^m L^{-1/2} \) on \( L^2(\mathbb{R}^n) \) (see Lemma 4.10) and the assumption that \( h \in L^2(\mathbb{R}^n) \cap H_{X,L}(\mathbb{R}^n) \), we conclude that \( f \in H_{m,X}(\mathbb{R}^n) \) and \( \partial^\alpha f \in L^2(\mathbb{R}^n) \) for any \( \alpha \in \mathbb{Z}_+^n \) with \( |\alpha| = m \). From this and Theorem 4.13, it follows that there exists a sequence \( \{\lambda_j\}_{j=1}^{\infty} \subset [0, \infty) \) and a sequence \( \{\alpha_j\}_{j=1}^{\infty} \) of \( (H_{m,X}, q) \)-atoms associated, respectively, with the balls \( \{B_j\}_{j=1}^{\infty} \) such that, for any \( \alpha \in \mathbb{Z}_+^n \) with \( |\alpha| = m \),

\[
\partial^\alpha f = \sum_{j=1}^{\infty} \lambda_j \partial^\alpha a_j,
\]

in \( L^2(\mathbb{R}^n) \), and

\[
\left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|B_j\|_X} \right)^s 1_{B_j} \right\} \right\|_X \leq \|f\|_{H_{m,X}(\mathbb{R}^n)}.
\]

For any ball \( B \subset \mathbb{R}^n \), let \( S_0(B) := B \) and \( S_i(B) := (2^i B) \setminus (2^{i-1} B) \) any \( i \in \mathbb{N} \). Next, we prove that, for any \((H_{m,X}, q)\)-atom \( a \) associated with the ball \( B \) and for any \( i \in \mathbb{Z}_+^n \),

\[
\left\| S_1 \left( L^{1/2} a \right) \right\|_{L^q(S_i(B))} \leq 2^{-i(1+n/q-\eta/q)} \|B\|^{1/q} \|1_B\|^{-1}_X,
\]

\[
\left\| S_1 \left( L^{1/2} a \right) \right\|_{L^q(S_i(B))} \leq 2^{-i(1+n/q-\eta/q)} \|B\|^{1/q} \|1_B\|^{-1}_X,
\]

\[
\left\| S_1 \left( L^{1/2} a \right) \right\|_{L^q(S_i(B))} \leq 2^{-i(1+n/q-\eta/q)} \|B\|^{1/q} \|1_B\|^{-1}_X,
\]
where the implicit positive constant is independent of $a$ and $j \in \mathbb{N}$.

When $i \in \{0, 1, 2\}$, by both the fact that $S_1$ is bounded in $L^q(\mathbb{R}^n)$ (see, for instance, [3, Corollary 6.10 and p. 67]) and the assumption that $a$ is an $(H_{m,x}, q)$-atom, we find that

\begin{equation}
(4.20) \quad \left\| S_1 \left( L^{1/2} a \right) \right\|_{L^q(S_i(B))} \leq \left\| L^{1/2} a \right\|_{L^q(\mathbb{R}^n)} \leq \| \nabla^m a \|_{L^q(\mathbb{R}^n)} \leq |B|^{1/q} \| 1_B \|^{-1}.
\end{equation}

Let $i \in \mathbb{N} \cap [3, \infty)$ and $R(S_i(B)) := \cup_{j \in S_i(B)} \Gamma(x)$. Then, from the Minkowski integral inequality, we deduce that

\begin{equation}
(4.21) \quad \left\| S_1 \left( L^{1/2} a \right) \right\|^2_{L^q(S_i(B))} \leq \int_{\mathbb{R}^n \setminus 2r_B} \int_0^{2r_B} \left| t^m L e^{-t^m L}(a)(y) \right|^2 t^{-n-1+\frac{2m}{q}} dy \, dt
\end{equation}

\begin{align*}
&\quad \leq \int_{\mathbb{R}^n \setminus 2r_B} \int_0^{2r_B} \left| t^m L e^{-t^m L}(a)(y) \right|^2 t^{-n-1+2m+\frac{2m}{q}} \, dt \, dy
\end{align*}

\begin{align*}
&\quad + \int_{\mathbb{R}^n \setminus 2r_B} \int_0^{\infty} \cdots \, dt \, dy + \int_{2r_B}^{\infty} \int_0^{\infty} \cdots \, dt \, dy
\end{align*}

\begin{align*}
&\quad =: \text{I} + \text{II} + \text{III}.
\end{align*}

By Lemma 4.17, the Hölder inequality, and the assumption that $a$ is an $(H_{m,x}, q)$-atom, we conclude that

\begin{equation}
(4.22) \quad \text{III} \leq \left\| a \right\|^2_{L^1(\mathbb{R}^n)} \int_{2r_B}^\infty t^{-\frac{2m}{q}-1+2m+\frac{2m}{q}} dt \leq (2r_B)^{-\frac{2m}{q}-2+2m+\frac{2m}{q}} |B|^{\frac{1}{2} - \frac{2}{q}} \left\| a \right\|^2_{L^1(\mathbb{R}^n)}
\end{equation}

\begin{align*}
&\quad \leq 2^{-2(\frac{2}{q} + m - \frac{2}{q})} |B|^{2/q} \| 1_B \|^{-2}.
\end{align*}

Moreover, using an argument similar to that used in the estimation of (4.22), we find that

\begin{align*}
\text{II} \leq 2^{-2(\frac{2}{q} + m - \frac{2}{q})} |B|^{2/q} \| 1_B \|^{-2}
\end{align*}

and

\begin{align*}
\text{I} \leq \left\| a \right\|^2_{L^1(\mathbb{R}^n)} \int_0^{2r_B} t^{-\frac{2m}{q}-1+2m+\frac{2m}{q}} e^{-t^{2m+\frac{2m}{q}}} dt \leq 2^{-2(\frac{2}{q} + m - \frac{2}{q})} |B|^{2/q} \| 1_B \|^{-2}.
\end{align*}

From this, (4.22), and (4.21), it follows that (4.19) holds trues.

Similarly to (4.20), we have

\begin{align*}
\left\| S_1 \left( L^{1/2} f \right) \right\|_{L^2(\mathbb{R}^n)} \leq \left\| L^{1/2} f \right\|_{L^2(\mathbb{R}^n)} \leq \| \nabla^m f \|_{L^2(\mathbb{R}^n)},
\end{align*}

which, together with (4.17), further implies that, for almost every $x \in \mathbb{R}^n$,

\begin{align*}
S_1 \left( L^{1/2} f \right)(x) \leq \sum_{j=1}^{\infty} \lambda_j S_1 \left( L^{1/2} a_j \right)(x).
\end{align*}

By this, (4.19), Lemma 2.13, and (4.18), we find that

\begin{align*}
\left\| S_1 \left( L^{1/2} f \right) \right\|_{X} \leq \left\| \sum_{j=1}^{\infty} \lambda_j S_1 \left( L^{1/2} a_j \right) \right\|_{X} \leq \left\{ \left\| \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\| 1_B \|_X} \right)^{1/3} 1_B \right\| \right\|_{X} \leq \| f \|_{H_{m,x}(\mathbb{R}^n)}.
\end{align*}

This finishes the proof of (4.16), and hence of Theorem 4.4.
5 Applications

In this section, we apply Theorems 3.1, 4.2, and 4.4, respectively, to weighted Hardy spaces associated with $L$ (see Subsection 5.1 below), variable Hardy spaces associated with $L$ (see Subsection 5.2 below), mixed-norm Hardy spaces associated with $L$ (see Subsection 5.3 below), Orlicz–Hardy spaces associated with $L$ (see Subsection 5.4 below), Orlicz-slice Hardy spaces associated with $L$ (see Subsection 5.5 below), and Morrey–Hardy spaces associated with $L$ (see Subsection 5.6 below). These applications explicitly indicate the generality and the flexibility of the main results of this article and more applications to new function spaces are obviously possible.

5.1 Weighted Hardy Spaces

In this subsection, we apply Theorems 3.1, 4.2, and 4.4 to the weighted Hardy space associated with $L$. We begin with recalling the definition of the weighted Lebesgue space.

**Definition 5.1.** Let $p \in (0, \infty)$ and $\omega \in A_{\infty}(\mathbb{R}^n)$. Then the weighted Lebesgue space $L^p_{\omega}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ such that

$$
|f|_{L^p_{\omega}(\mathbb{R}^n)}^p := \left[ \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right]^{1/p} < \infty.
$$

We point out that the space $L^p_{\omega}(\mathbb{R}^n)$ with $p \in (0, \infty)$ and $\omega \in A_{\infty}(\mathbb{R}^n)$ is a ball quasi-Banach function space, but it may not be a (quasi-)Banach function space (see, for instance, [66, Section 7.1]).

When $X := L^p_{\omega}(\mathbb{R}^n)$, the Hardy space $H_{X,L}(\mathbb{R}^n)$ is just the weighted Hardy space associated with $L$; in this case, we denote $H_{X,L}(\mathbb{R}^n)$ simply by $H^p_{L,\omega}(\mathbb{R}^n)$. Moreover, for any given $s \in (0, 1)$, $p \in (s, \infty)$, and $\omega \in A_{p/s}(\mathbb{R}^n)$, let

$$
\epsilon(p, s, \omega) := \frac{\log(\frac{1}{A_{p/s}(\mathbb{R}^n)}) - \log(\frac{1}{A_{p/s}(\mathbb{R}^n)} - 2^{-p})}{2n \log 2}.
$$

Then, applying Theorem 3.1 to the weighted Hardy space $H^p_{L,\omega}(\mathbb{R}^n)$, we have the following conclusion.

**Theorem 5.2.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that $s \in (0, 1]$, $p \in (s, \infty)$, and $\omega \in A_{p/s}(\mathbb{R}^n)$ satisfy that

$$
p < \frac{p_s(L)}{\epsilon(p, s, \omega)} + 1,
$$

where $\epsilon(p, s, \omega)$ is the same as in (5.1). Then the conclusion of Theorem 3.1 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^p_{L,\omega}(\mathbb{R}^n)$.

**Proof.** Let all the symbols be the same as in the present theorem. By Theorem 3.1, to show the present theorem, it suffices to prove that $X := L^p_{\omega}(\mathbb{R}^n)$ satisfies both Assumptions 2.11 and 2.12 for some $\theta \in (0, 1]$, $s \in (\theta, 1]$, and $q \in (p/L, p_s(L))$. 

Let $s$ be the same as in the present theorem, $\theta \in (0, s)$, and

$$q \in (\max(p_-(L), p(\epsilon(p,s,\omega) + 1)/\epsilon(p,s,\omega)), p_+(L)).$$

From the Fefferman–Stein vector-valued maximal inequality on the weighted Lebesgue space (see, for instance, [2, Theorem 3.1(b)]), we deduce that $X := L^p_{\omega}(\mathbb{R}^n)$ satisfies Assumption 2.11 for such a $\theta$ and an $s$. Moreover, it is easy to show that

$$
\left(\frac{X^+}{s}\right)^{\frac{1}{p'}} = L^{(p/s')/(q/s')}_{\omega^{1-(p/q')}}(\mathbb{R}^n),
$$

which, together with the assumption that $h < (p/s')/(q/s')$, further implies that $\omega^{1-(p/q')} \in A_h(\mathbb{R}^n) \subset A_{(p/s')/(q/s')}(\mathbb{R}^n)$. From this, (5.2), and the boundedness of the Hardy–Littlewood maximal operator $M$ on the weighted Lebesgue space (see, for instance, [42, Theorem 7.1.9]), we deduce that $X := L^p_{\omega}(\mathbb{R}^n)$ satisfies Assumption 2.12 for such an $s$ and a $q$. This finishes the proof of Theorem 5.2.

Moreover, by both Theorems 4.2 and 4.4, we have the following results; since their proofs are similar to that of Theorem 5.2, we omit the details here.

**Theorem 5.3.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that $s \in (n/(n + m), 1]$, $p \in (s, \infty)$, and $\omega \in A_{p/s}(\mathbb{R}^n)$ satisfy

$$p < \frac{\min(p_+(L), q_+(L))\epsilon(p,s,\omega)}{\epsilon(p,s,\omega) + 1},$$

where $\epsilon(p,s,\omega)$ is the same as in (5.1). Then the conclusion of Theorem 4.2 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^p_{L,\omega}(\mathbb{R}^n)$.

**Theorem 5.4.** Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and the family $\{e^{-tL}\}_{t \in (0,\infty)}$ of operators satisfy the $m-L^r(\mathbb{R}^n)-L^2(\mathbb{R}^n)$ off-diagonal estimate for some $r \in (1, 2]$. Assume that $s \in (nr/(n + mr), 1]$, $p \in (s, \infty)$, and $\omega \in A_{p/s}(\mathbb{R}^n)$ satisfy

$$p < \frac{p_+(L)\epsilon(p,s,\omega)}{\epsilon(p,s,\omega) + 1},$$

where $\epsilon(p,s,\omega)$ is the same as in (5.1). Then the conclusion of Theorem 4.4 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^p_{L,\omega}(\mathbb{R}^n)$.

**Remark 5.5.** To the best of our knowledge, the conclusions of Theorems 5.2, 5.3, and 5.4 are totally new.
5.2 Variable Hardy Spaces

In this subsection, we apply Theorems 3.1, 4.2, and 4.4 to the variable Hardy space associated with $L$. We first recall the definition of the variable Lebesgue space.

Let $r : \mathbb{R}^n \to (0, \infty)$ be a measurable function,

$$\bar{r}_- := \text{ess inf}_{x \in \mathbb{R}^n} r(x), \quad \text{and} \quad \bar{r}_+ := \text{ess sup}_{x \in \mathbb{R}^n} r(x).$$

A function $r : \mathbb{R}^n \to (0, \infty)$ is said to be globally log-Hölder continuous if there exists an $r_{\infty} \in \mathbb{R}$ and a positive constant $C$ such that, for any $x, y \in \mathbb{R}^n$,

$$|r(x) - r(y)| \leq \frac{C}{\log(e + |x - y|)} \quad \text{and} \quad |r(x) - r_{\infty}| \leq \frac{C}{\log(e + |x|)}.$$

The variable Lebesgue space $L^{r(\cdot)}(\mathbb{R}^n)$ associated with the function $r : \mathbb{R}^n \to (0, \infty)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ with the finite quasi-norm

$$\|f\|_{L^{r(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{\mathbb{R}^n} \left[ \frac{|f(x)|}{\lambda} \right]^{r(x)} \, dx \leq 1 \right\}.$$

Then it is known that $L^{r(\cdot)}(\mathbb{R}^n)$ is a ball quasi-Banach function space (see, for instance, [66, Section 7.8]). In particular, when $1 < \bar{r}_- \leq \bar{r}_+ < \infty$, $(L^{r(\cdot)}(\mathbb{R}^n), \| \cdot \|_{L^{r(\cdot)}(\mathbb{R}^n)})$ is a Banach function space in the terminology of Bennett and Sharpley [9] and hence also a ball Banach function space (see, for instance, [66, p. 94]). More results on variable Lebesgue spaces can be found in [60, 61, 59, 25, 27, 30] and the references therein.

When $X := L^{r(\cdot)}(\mathbb{R}^n)$, the Hardy space $H_{X,L}(\mathbb{R}^n)$ is just the variable Hardy space associated with $L$; in this case, we denote $H_{X,L}(\mathbb{R}^n)$ simply by $H^r_L(\mathbb{R}^n)$. Then, applying Theorem 3.1 to the variable Hardy space $H^r_L(\mathbb{R}^n)$, we have the following conclusion.

**Theorem 5.6.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that $r : \mathbb{R}^n \to (0, \infty)$ is globally log-Hölder continuous and $0 < \bar{r}_- \leq \bar{r}_+ < p_+(L)$. Then the conclusion of Theorem 3.1 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^r_L(\mathbb{R}^n)$.

**Proof.** Let all the symbols be the same as in the present theorem. By Theorem 3.1, to prove the present theorem, it suffices to show that $X := L^{r(\cdot)}(\mathbb{R}^n)$ satisfies both Assumptions 2.11 and 2.12 for some $\theta \in (0, 1]$, $s \in (\theta, 1)$, and $q \in (p_-(L), p_+(L))$.

Let $\theta \in (0, \bar{r}_-)$, $s \in (\theta, \min \{1, \bar{r}_-\})$, and $q \in \text{max} \{1, \bar{r}_+, p_-(L), p_+(L)\}$. Then, from the Fefferman–Stein vector-valued maximal inequality on variable Lebesgue spaces (see, for instance, [59, Lemma 2.4]), we deduce that $X := L^{r(\cdot)}(\mathbb{R}^n)$ satisfies Assumptions 2.11 for such a $\theta$ and an $s$.

Moreover, by the dual result on variable Lebesgue spaces (see, for instance, [25, Theorem 2.80]), we conclude that

$$\left[ \left( X^{\frac{1}{s'}} \right)^{q/s'} \right] = L^{r(\cdot)/s'(q/s')} (\mathbb{R}^n),$$

which, together with the assumption that $q \in \text{max} \{1, \bar{r}_+, p_-(L), p_+(L)\}$ and the boundedness of the Hardy–Littlewood maximal operator $M$ on variable Lebesgue spaces (see, for instance, [25,
Theorem 3.16), further implies that $X := L^{r'}(\mathbb{R}^n)$ satisfies Assumption 2.12 for such an $s$ and a $q$. This finishes the proof of Theorem 5.6. □

By both Theorems 4.2 and 4.4, we have the following conclusions; since their proofs are similar to that of Theorem 5.6, we omit the details here.

**Theorem 5.7.** Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and $r : \mathbb{R}^n \to (0, \infty)$ be globally log-Hölder continuous. Assume that $\frac{n}{m+r} < \bar{r}_- \leq \bar{r}_+ < \min(p_+L), q_+(L))$. Then the conclusion of Theorem 4.2 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^{(r)}_{L}(\mathbb{R}^n)$.

**Theorem 5.8.** Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and the family $\{e^{-L}\}_{t \in (0,\infty)}$ of operators satisfy the $m-L'(\mathbb{R}^n)-L^2(\mathbb{R}^n)$ off-diagonal estimate with some $r \in (1,2]$. Let $r : \mathbb{R}^n \to (0, \infty)$ be globally log-Hölder continuous. Assume that $\frac{n}{m+r} < \bar{r}_- \leq \bar{r}_+ < p_+(L)$. Then the conclusion of Theorem 4.4 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^{(r)}_{L}(\mathbb{R}^n)$.

**Remark 5.9.** When $m := 1$ and $\bar{r}_+ \in (0,1]$, Theorem 5.6 is just [78, Theorem 5.3]. Meanwhile, when $m := 1$ and $\frac{n}{n+1} < \bar{r}_- \leq \bar{r}_+ \leq 1$, Theorem 5.7 is just [78, Theorem 5.17]. Moreover, to the best of our knowledge, Theorem 5.8 is totally new even when $m := 1$.

### 5.3 Mixed-norm Hardy Spaces

In this subsection, we apply Theorems 3.1, 4.2, and 4.4 to the mixed-norm Hardy space associated with $L$. We begin with recalling the definition of the mixed-norm Lebesgue space.

For a given vector $\vec{r} := (r_1, \ldots, r_n) \in (0, \infty)^n$, the **mixed-norm Lebesgue space** $L^\vec{r}(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ with the finite quasi-norm

$$
\|f\|_{L^\vec{r}(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |f(x_1, \ldots, x_n)|^{r_1} dx_1 \cdots dx_n \right\}^{\frac{1}{\vec{r}}},
$$

where the usual modifications are made when $r_i := \infty$ for some $i \in \{1, \ldots, n\}$. Here and in the remainder of this subsection, let $r_- := \min\{r_1, \ldots, r_n\}$ and $r_+ := \max\{r_1, \ldots, r_n\}$.

It is worth pointing out that the space $L^\vec{r}(\mathbb{R}^n)$ with $\vec{r} \in (0, \infty)^n$ is a ball quasi-Banach function space; but $L^\vec{r}(\mathbb{R}^n)$ with $\vec{r} \in [1, \infty)^n$ may not be a Banach function space (see, for instance, [79, Remark 7.21]). The study of mixed-norm Lebesgue spaces can be traced back to Hörmander [49] and Benedek and Panzone [8]. More results on mixed-norm Lebesgue spaces and other mixed-norm function spaces can be found in [56, 20, 21, 22, 62, 63, 50] and the references therein.

In particular, when $X := L^\vec{r}(\mathbb{R}^n)$, the Hardy space $H_{X, L}(\mathbb{R}^n)$ is just the **mixed-norm Hardy space associated with $L$**; in this case, we denote $H_{X, L}(\mathbb{R}^n)$ simply by $H^\vec{r}_{L}(\mathbb{R}^n)$. Then, applying Theorem 3.1 to the mixed-norm Hardy space $H^\vec{r}_{L}(\mathbb{R}^n)$, we have the following conclusion.

**Theorem 5.10.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that $\vec{r} := (r_1, \ldots, r_n) \in (0, \infty)^n$ with $r_+ \in (0, p_+(L))$. Then the conclusion of Theorem 3.1 holds true with $H_{X, L}(\mathbb{R}^n)$ replaced by $H^\vec{r}_{L}(\mathbb{R}^n)$.
Proof. Let all the symbols be same as in the present theorem. By Theorem 3.1, to show the present theorem, it suffices to prove that $X := L^T(\mathbb{R}^n)$ satisfies both Assumptions 2.11 and 2.12 for some $\theta \in (0, 1]$, $s \in (\theta, 1]$, and $q \in (p_-(L), p_+(L))$.

Let $\theta \in (0, r_+)$, $s \in (\theta, \min\{1, r_+\})$, and $q \in (\max\{1, r_+, p_-(L)\}, p_+(L))$. Then, by the Fefferman–Stein vector-valued maximal inequality on mixed-norm Lebesgue spaces (see, for instance, [50, Lemma 3.7]), we find that $\vec{r}$ replaced by $H$.

Remark 5.13. To the best of our knowledge, Theorems 5.10, 5.11, and 5.12 are totally new.

Theorem 5.11. Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that $\vec{r} := (r_1, \ldots, r_n) \in (0, \infty)^n$ satisfies

\[ \frac{n}{n+m} < r_- \leq r_+ < \min\{p_+(L), q_+(L)\}. \]

Then the conclusion of Theorem 4.2 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^T_{X,L}(\mathbb{R}^n)$.

Theorem 5.12. Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and the family $\{e^{-L}\}_{t \in (0, \infty)}$ of operators satisfy the $m - L^T(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimate for some $r \in (1, 2]$. Assume that $\vec{r} := (r_1, \ldots, r_n) \in (0, \infty)^n$ satisfies

\[ \frac{n}{n+m} < r_- \leq r_+ < p_+(L). \]

Then the conclusion of Theorem 4.4 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H^T_{X,L}(\mathbb{R}^n)$.

Remark 5.13. To the best of our knowledge, Theorems 5.10, 5.11, and 5.12 are totally new.

### 5.4 Orlicz–Hardy Spaces

In this subsection, we apply Theorems 3.1, 4.2, and 4.4 to the Orlicz–Hardy space associated with $L$. We begin with recalling the definition of the Orlicz function.

A non-decreasing function $\Phi : [0, \infty) \to [0, \infty)$ is called an Orlicz function if $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0, \infty)$, and $\lim_{t \to \infty} \Phi(t) = \infty$. Moreover, an Orlicz function $\Phi$ is said to be of lower [resp., upper] type $r$ for some $r \in \mathbb{R}$ if there exists a positive constant $C_1(r)$ such that, for any $t \in [0, \infty)$ and $s \in (0, 1]$ [resp., $s \in [1, \infty)$],

\[ \Phi(st) \leq C_1(r)s^r \Phi(t). \]
In the remainder of this subsection, we always assume that \( \Phi : [0, \infty) \to [0, \infty) \) is an Orlicz function with positive lower type \( r_{\Phi}^\ast \) and positive upper type \( r_{\Phi}^\ast \). The Orlicz norm \( \|f\|_{L^\Phi(R^n)} \) of a measurable function \( f \) on \( R^n \) is then defined by setting

\[
\|f\|_{L^\Phi(R^n)} := \inf \left\{ \lambda \in (0, \infty) : \int_{R^n} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.
\]

Then the Orlicz space \( L^\Phi(R^n) \) is defined to be the set of all the measurable functions \( f \) on \( R^n \) with finite norm \( \|f\|_{L^\Phi(R^n)} \).

It is easy to prove that the Orlicz space \( L^\Phi(R^n) \) is a ball quasi-Banach function space (see, for instance, [66, Section 7.6]). In particular, if \( 1 \leq r_{\Phi}^\ast \leq r_{\Phi}^\ast < \infty \), then \( L^\Phi(R^n) \) is a Banach function space (see [65, p. 67, Theorem 10]), and hence is also a ball Banach function space. For more results on Orlicz spaces, we refer the reader to [54, 64, 79] and the references therein.

In particular, when \( X := L^\Phi(R^n) \), the Hardy space \( H_{\chi,L}(R^n) \) is just the Orlicz–Hardy space associated with \( L \); in this case, we denote \( H_{\chi,L}(R^n) \) simply by \( H_{\Phi,L}(R^n) \). Then, applying Theorem 3.1 to the Orlicz–Hardy space \( H_{\Phi,L}(R^n) \), we have the following conclusion.

**Theorem 5.14.** Let \( m \in \mathbb{N} \) and \( L \) be a homogeneous divergence form \( 2m \)-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that \( \Phi \) is an Orlicz function with positive lower type \( r_{\Phi}^\ast \) and positive upper type \( r_{\Phi}^\ast \in (0, p_+(L)) \). Then the conclusion of Theorem 3.1 holds true with \( H_{\chi,L}(R^n) \) replaced by \( H_{\Phi,L}(R^n) \).

**Proof.** Let all the symbols be the same as in the present theorem. By Theorem 3.1, to prove the present theorem, it suffices to show that \( X := L^\Phi(R^n) \) satisfies both Assumptions 2.11 and 2.12 for some \( \theta \in (0, 1] \), \( s \in (\theta, 1] \), and \( q \in (p_-(L), p_+(L)) \).

Let \( \theta \in (0, r_{\Phi}^\ast) \), \( s \in (\theta, \min(1, r_{\Phi}^\ast)) \), and \( q \in (\max(1, r_{\Phi}^\ast), p_+(L)) \). Then, from [66, Theorem 7.14(i)], we deduce that \( X := L^\Phi(R^n) \) satisfies Assumption 2.11 for such a \( \theta \) and \( s \). Furthermore, by the dual theorem on Orlicz spaces (see, for instance, [64, Theorem 13]), we find that, when \( X := L^\Phi(R^n) \),

\[
\left\|X^\frac{1}{s'}\right\|^{1-s'}_{1/s'} = L^\Psi(R^n),
\]

where, for any \( t \in [0, \infty) \),

\[
\Psi(t) := \sup_{h \in (0, \infty)} \left[ t^{1/(q/s') - 1} - \Phi \left( h^{1/s} \right) \right].
\]

Then, from [66, Proposition 7.8] and the assumption that \( q \in (\max(1, r_{\Phi}^\ast), p_-(L)) \), it follows that \( \Psi \) is an Orlicz function with positive lower type \( r_{\Phi}^\ast := (r_{\Phi}^\ast/s')/(q/s) < \infty \), which, combined with both (5.4) and the boundedness of the Hardy–Littlewood maximal operator \( M \) on Orlicz spaces (see, for instance, [66, Theorem 7.12]), further implies that \( X := L^\Phi(R^n) \) satisfies Assumption 2.12 for such an \( s \) and a \( q \). This finishes the proof of Theorem 5.14. \( \square \)

Moreover, by both Theorems 4.2 and 4.4, we have the following conclusions; since their proofs are similar to that of Theorem 5.14, we omit the details here.
Theorem 5.15. Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that $\Phi$ is an Orlicz function with positive lower type $r_\Phi$ and positive upper type $r_\Phi^+$. Assume further that $\frac{m}{r_\Phi} < r_\Phi^+ < r_\Phi < \min\{p_+(L), q_+(L)\}$. Then the conclusion of Theorem 4.2 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H_{\Phi,L}(\mathbb{R}^n)$.

Theorem 5.16. Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and the family $\{e^{-L}r\}_{r \in (0, \infty)}$ of operators satisfy the $m-L'(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimate for some $r \in (1, 2]$. Assume that $\Phi$ is an Orlicz function with positive lower type $r_\Phi$ and positive upper type $r_\Phi^+$. Assume further that $\frac{m-r}{n+m} < r_\Phi < r_\Phi^+ < p_+(L)$. Then the conclusion of Theorem 4.4 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $H_{\Phi,L}(\mathbb{R}^n)$.

Remark 5.17. When $m := 1$ and $r_\Phi^+ \in (0, 1]$, Theorem 5.14 was obtained in [51, Theorem 5.2]. Meanwhile, when $m := 1$ and $\frac{m}{r_\Phi} < r_\Phi^+ \leq 1$, Theorem 5.19 is just [51, Theorem 7.4]. Furthermore, to the best of our knowledge, Theorem 5.16 is totally new even when $m := 1$.

5.5 Orlicz-slice Hardy Spaces

In this subsection, we apply Theorems 3.1, 4.2, and 4.4 to the Orlicz-slice Hardy space associated with $L$. We first recall the definitions of Orlicz-slice spaces and then describe briefly some related facts. Throughout this subsection, we always assume that $\Phi : [0, \infty) \rightarrow [0, \infty)$ is an Orlicz function with positive lower type $r_\Phi$ and positive upper type $r_\Phi^+$. For any given $t, r \in (0, \infty)$, the Orlicz-slice space $(E_{\Phi,t})_r(\mathbb{R}^n)$ is defined to be the set of all the measurable functions $f$ on $\mathbb{R}^n$ with the finite quasi-norm

$$
\|f\|_{(E_{\Phi,t})_r(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} \frac{\|f1_{B(x,t)}\|_{L^r(\mathbb{R}^n)}}{\|1_{B(x,t)}\|_{L^s(\mathbb{R}^n)}}^r \, dx \right\}^{\frac{1}{r}}.
$$

Orlicz-slice spaces were introduced in [80] as a generalization of slice spaces in [6, 7] and Wiener amalgam spaces in [48, 53, 44]. Meanwhile, by [80, Lemma 2.28] and [79, Remark 7.41(i)], we find that the Orlicz-slice space $(E_{\Phi,t})_r(\mathbb{R}^n)$ is a ball Banach function space, but in general is not a Banach function space.

In particular, when $X := (E_{\Phi,t})_r(\mathbb{R}^n)$, the Hardy space $H_{X,L}(\mathbb{R}^n)$ is just the Orlicz-slice Hardy space associated with $L$; in this case, we denote $H_{X,L}(\mathbb{R}^n)$ simply by $(HE_{\Phi,t,L})_r(\mathbb{R}^n)$. Then, applying Theorem 3.1 to the Orlicz-slice Hardy space $(HE_{\Phi,t,L})_r(\mathbb{R}^n)$, we have the following conclusion.

Theorem 5.18. Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that $t \in (0, \infty)$, $r \in (0, p_+(L))$, and $\Phi$ is an Orlicz function with positive lower type $r_\Phi$ and positive upper type $r_\Phi^+ \in (0, p_+(L))$. Then the conclusion of Theorem 3.1 holds true with $H_{X,L}(\mathbb{R}^n)$ replaced by $(HE_{\Phi,t,L})_r(\mathbb{R}^n)$.

Proof. Let all the symbols be the same as in the present theorem. Assume further that $\theta \in (0, 1)$, $s \in (\theta, \min\{1, r_\Phi^{-1}, r\})$, and $q \in (\max\{r, r_\Phi^{-1}, p_-(L), p_+(L)\})$. It was proved in [80, Lemmas 4.3 and 4.4] that $X := (E_{\Phi,t})_r(\mathbb{R}^n)$ satisfies both Assumptions 2.11 and 2.12 for such a $\theta$, an $s$, and a $q$. From this and Theorem 3.1, it follows that the conclusion of Theorem 5.18 holds true.

Moreover, by both Theorems 4.2 and 4.4, we have the following results; since their proofs are similar to that of Theorem 5.18, we omit the details here.
**Theorem 5.19.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that $t \in (0, \infty)$,

$$r \in \left(\frac{n}{n + m}, \min\{p_\ast(L), q_\ast(L)\}\right),$$

and $\Phi$ is an Orlicz function with positive lower type $r^-_\Phi \in \left(\frac{n}{n + m}, p_\ast(L)\right)$ and positive upper type $r^+_\Phi \in \left(0, \min\{p_\ast(L), q_\ast(L)\}\right)$. Then the conclusion of Theorem 4.2 holds true with $H_{X,L}^r(\mathbb{R}^n)$ replaced by $(HE_{\Phi,L}^r)^r(\mathbb{R}^n)$.

**Theorem 5.20.** Let $m \in \mathbb{N}$, $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and the family $\{e^{-L}\}_{t \in [0, \infty)}$ of operators satisfy the $m - L'(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ off-diagonal estimate for some $r \in (1, 2]$. Assume that $t \in (0, \infty)$, $r \in \left(\frac{n}{n + m}, p_\ast(L)\right)$, and $\Phi$ is an Orlicz function with positive lower type $r^-_\Phi \in \left(\frac{n}{n + m}, p_\ast(L)\right)$ and positive upper type $r^+_\Phi \in \left(0, p_\ast(L)\right)$. Then the conclusion of Theorem 4.4 holds true with $H_{X,L}^r(\mathbb{R}^n)$ replaced by $(HE_{\Phi,L}^r)^r(\mathbb{R}^n)$.

**Remark 5.21.** To the best of our knowledge, Theorems 5.18, 5.19, and 5.20 are totally new even when $m := 1$.

### 5.6 Morrey–Hardy Spaces

In this subsection, we apply Theorems 3.1, 4.2, and 4.4 to the Morrey–Hardy space associated with $L$. We begin with recalling the definition of the Morrey space.

Let $0 < r \leq p \leq \infty$. Recall that the Morrey space $\mathcal{M}_r^p(\mathbb{R}^n)$ is defined to be the set of all the $f \in L^r_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}_r^p(\mathbb{R}^n)} := \sup_{B \subset \mathbb{R}^n} |B|^\frac{1}{r} \left(\int_B |f(y)|^r dy\right)^\frac{1}{r} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

The space $\mathcal{M}_r^p(\mathbb{R}^n)$ was introduced by Morrey [58]. It is known that $\mathcal{M}_r^p(\mathbb{R}^n)$ with $1 \leq r < p < \infty$ is not a Banach function space, but is a ball Banach function space (see, for instance, [66, Section 7.4]). Moreover, by the definition of $\mathcal{M}_r^p(\mathbb{R}^n)$, it is easy to find that $\mathcal{M}_r^p(\mathbb{R}^n)$ with $0 < r \leq p \leq \infty$ is a ball quasi-Banach space.

In particular, when $X := \mathcal{M}_r^p(\mathbb{R}^n)$, the Hardy space $H_{X,L}^r(\mathbb{R}^n)$ is just the Morrey–Hardy space associated with $L$; in this case, we denote $H_{X,L}^r(\mathbb{R}^n)$ simply by $HM_{r,L}^p(\mathbb{R}^n)$. Then, applying Theorem 3.1 to the Morrey–Hardy space $HM_{r,L}^p(\mathbb{R}^n)$, we have the following conclusion.

**Theorem 5.22.** Let $m \in \mathbb{N}$ and $L$ be a homogeneous divergence form $2m$-order elliptic operator in (2.1) satisfying Strong Ellipticity Condition 2.2. Assume that $p \in (0, p_\ast(L))$ and $r \in (0, p]$. Then the conclusion of Theorem 3.1 holds true with $H_{X,L}^r(\mathbb{R}^n)$ replaced by $HM_{r,L}^p(\mathbb{R}^n)$.

**Proof.** Let all the symbols be the same as in the present theorem. Assume further that $\theta \in (0, 1)$, $s \in (\theta, \min\{1, r\})$, and $q \in (\max\{p, p_\ast(L)\}, p_\ast(L))$. Then it is known that $X := \mathcal{M}_r^p(\mathbb{R}^n)$ satisfies both Assumptions 2.11 and 2.12 for such a $\theta$, an $s$, and a $q$ (see, for instance, [72, Remarks 2.4(e) and 2.7(e)]). By this and Theorem 3.1, we find that the conclusion of Theorem 5.22 holds true. □
Moreover, by both Theorems 4.2 and 4.4, we have the following results; since their proofs are similar to that of Theorem 5.22, we omit the details here.

**Theorem 5.23.** Let \( m \in \mathbb{N} \) and \( L \) be a homogeneous divergence form \( 2m \)-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1. Assume that \( p \in (\frac{n}{n+m}, \min(p_+, q_+)) \) and \( r \in (\frac{n}{n+m}, p) \). Then the conclusion of Theorem 4.2 holds true with \( H_{X,L}(\mathbb{R}^n) \) replaced by \( H_{M_p^r,L}(\mathbb{R}^n) \).

**Theorem 5.24.** Let \( m \in \mathbb{N} \), \( L \) be a homogeneous divergence form \( 2m \)-order elliptic operator in (2.1) satisfying Ellipticity Condition 2.1, and the family \( \{e^{-tL}\}_{t \in (0, \infty)} \) of operators satisfy the \( m - L^s(\mathbb{R}^n) - L^2(\mathbb{R}^n) \) off-diagonal estimate for some \( s \in (1, 2] \). Assume that \( p \in (\frac{n}{n+m}, p_+(L)) \) and \( r \in (\frac{n}{n+m}, p) \). Then the conclusion of Theorem 4.4 holds true with \( H_{X,L}(\mathbb{R}^n) \) replaced by \( H_{M_p^r,L}(\mathbb{R}^n) \).

**Remark 5.25.** To the best of our knowledge, Theorems 5.22, 5.23, and 5.24 are totally new even when \( m := 1 \).

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