Changes in post critical equilibrium and localization effects of elastic elongated plates

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Abstract. The influence of three hinged support variants on the lateral sides of linear elastic compressed elongated rectangular plate on its post critical equilibrium development is investigated. We consider cases of classical hinged support, rigid or elastic hinging, and also a variant of point hinged support. The latter is realized at the nodes during the FEM model of the plate. It was established (apparently for the first time) that within the last variant of the boundary conditions, at sufficiently strong longitudinal shortening of the plate, first a "pre-soliton" equilibrium appears (the appearance of lateral "half-dents" on the surface of the extreme half-waves) which turns into a single soliton-like ejection of bending deformations as a result of a load large decrease under conditions of kinematic loading. With the classical hinged support, the lateral sides of the plate remain rectilinear, as a result of which the soliton does not form; the "pre-soliton" equilibrium appears. If the plate has a rigid hinge fixation, the monotonic development of the post critical equilibrium will be interrupted by jumps of the load at the secondary symmetrical points and unstable bifurcations. In this case, no effects of localization and soliton formation were observed.

1. Introduction
It is well known that the longitudinally compressed elastic plates have a stable post critical equilibrium, which allows them to withstand loads of ≈ 1.5 ÷ 2 times greater than the bifurcation one [1-6]. At the same time, subsequent secondary bifurcations of the compressed curved equilibrium are accompanied by "snap-through" and reduce the effective stiffness of the plate.

An unpleasant feature of secondary bifurcation (under certain boundary conditions) is the development of a special localized form of stability loss in the formation of post critical equilibrium. The localization of buckling consists in the local (soliton-like) ejection formation of flexural deformation of large amplitude with comparatively small dents and bulges in the rest of the plate. The localization phenomenon of the elongated plates shape was studied experimentally in the works of Moxam [7]. Attempts to explain the condition of localization occurrence were made by V. Tvergard and A. Needleman [8, 9], and also by M. Potier-Ferry [10]. Clement and co-workers [11] examined the post critical equilibrium of a long longitudinally compressed rectangular plate-strip with a side ratio of 9. The side edges of the plate were hinged and the short (loaded) sides had a sliding fit.
In its essence, the localization of post critical equilibrium is the result of the stability loss of the primary post critical equilibrium, in the regular periodic set form of almost identical dents and bulges, characteristic of sufficiently elongated structures. Localization is observed during the compressed rods buckling on an elastic but softened base, with a secondary loss of compressed stability, continuous, non-sparring, elastoplastic beam, as a result of heating, when the railway track is buckled, etc. We note that in [8] the possibility of localization for a plate with a bilinear diagram of the material deformation was investigated. The authors of [8] believe that the phenomenon of localization occurs most often immediately after the maximum longitudinal compression load is reached (after passing the upper limit point by an elongated imperfect system when it is in an unstable equilibrium state). At the same time, no «snaps» were observed with any localization effects.

We investigate the influence of the hinged support variants of boundary conditions along the lateral sides on the appearance possibility of the snap-through post critical equilibrium changes and the formation of localized soliton-like bending deformations in this paper.

1. Classical hinged support with the straight form preservation of the plate unloaded sides;
2. Hinged support (rigid or elastic in the plane of the plate) along the sides;
3. "Spot" hinged support (or fixation in the plane of the plate perpendicular to the edge)

The last type of boundary conditions is created within the finite element method using, when the hinged support or fixing is carried out in the grid nodes of the finite element model. When it is loaded, this version of the support allows plate rectilinear edge to curve in its plane, which leads to the formation of localization special types - soliton-like outbursts of flexural deformations with significant overall longitudinal shortening of the plate.

As the model studied, a plate-strip made of an aluminum alloy was selected, corresponding in geometry to one of the 11 parts of the reinforced panel in the experiment of M. Stein [12, 13] (length \(L = 64.4\) cm, width \(b = 12\) cm, thickness \(\delta = 0.188\) cm, \(\xi = L / b = 5.38\), \(E = 7000\) kg / m\(^2\), \(\nu = 0.3\)). The finite element grid consisted of \(\approx 2000\) plate elements (quad4, NASTRAN). The compressive load was applied on short sides, mostly kinematically (end shortening), through a rigid element, which allowed to accurately calculate the amount of the compressive force. The boundary conditions along the short sides correspond to the hinged support or sealing on the unloaded short side and the hinged support or sliding fit along the loaded side (Figure 1)

![Figure 1](https://via.placeholder.com/150)

**Figure 1.** Plate-strip with fixed short edges (a), with all edges hinged (b).

In the case of hinged support along the edges, the primary bifurcation (loss of stability of a flat compressed plate) with a ratio of sides \(\xi = 5.38\) occurs in a 5-half wave form. In the case of fixed short sides the primary bifurcation also occurs in a 5-half-wave shape. The first few eigenforms of plate buckling with two types of boundary conditions obtained with the buckling option are shown on figure 2.

### 2. Classic hinged support on the sides

In order to create the boundary conditions for the classical hinged support of the plate, it was necessary to ensure the straightness of the lateral sides of the plate finite element model when it was
loaded in a state of post critical equilibrium. So, we used a rigid element, which ensured the same movements of all the model’s grid nodes, located on the long edges of the plate. We studied the loading process of the plate with initial geometric imperfection similar to the first 5 half-wave form (amplitude ≈0.01) in geometric nonlinear formulation, under the unlimited elasticity condition of the material. The curve of equilibrium states (shortening-load curve), as well as the flexural deformation forms that arose as a result of the rearrangements with snap-through are shown in figure 3. Analysis of the presented curve shows that, on the whole, stable load growth with the development of post critical longitudinal shortenings was interrupted by comparatively small jumps in load. The first jump (P = 31.5 kN → 29.1 kN at ΔL = 0.19 cm) transferred the 5 half-wave form to 7 half-wave ones.

Further, the equilibrium curve monotonically developed until the next jump of 7-half-wave equilibrium (with a more "trough-like" form of dents and bulges) in 11-half-wave equilibrium at a significantly higher load (P = 97.3 kN → 83.9 kN at ΔL≈1.06 cm). After that, the longitudinal stiffness of the plate decreased noticeably. However, at a load of ≈196.9 kN and ΔL≈3.68 cm, a "almost soft" change in the shape of the bulges occurred (the load decreased to 196.4 kN). At the same time, the number of half-waves (eleven) did not change, but lateral "half-dents" appeared on the side bulges (Figure 3). As a restructuring result of the equilibrium form, the above-mentioned "side half-dents" appeared to the next bulges towards the centre. With further plate compression, no fundamentally new changes in the shape of the plate deformation were observed (max ΔL≈10 cm).

For another initial imperfection (the sum of 6, 5, and 4 half-wave eigenmodes with an initial amplitude of 0.01δ, Figure 4 (a)), the kinematic compression (end shortening) showed approximately the same monotonic character of the development of the equilibrium curve, interrupted by "small snaps" (Figure 4 (b)).

The end loading of the same plate with a similar initial imperfection showed monotonous growth of the load, interrupted by small snap-through shortening. The six half-wave form of equilibrium under loads $P_{cr1} = 63.3$ kN passed with snap-through ($\Delta L=0.53$ cm → 0.64 cm) into 8-half-wave form, which at $P_{cr2} = 122.6$ kN was reconstructed ($\Delta L\approx 1.57$ cm → 1.75 cm) into a 10-half-wave shape. Later, this
10-half-wave form, when the load reaches the value $P_{cr3} = 173.2$ kN, was rearranged with snap-through to 12-half-wave ($\Delta L \approx 2.92 \text{ cm} \rightarrow 3.1 \text{ cm}$). In this case, the appearance of "side half-dents" was not observed, which significantly distinguishes force loading from the kinematic one (Figure 5).

**Figure 3.** The curve of equilibrium states (shortening-load curve) and deformation forms of plate with the classic hinged boundary conditions.

**Figure 4.** The form of initial imperfection (a), the curve of equilibrium states (shortening-load curve, b) and deformation forms of plate with the classic hinged boundary conditions (c).

A comparison of the kinematic results and force loading of the plate under consideration (Table 1) shows that the reorganization of post critical equilibrium forms occurs at sufficiently close values of loads and longitudinal shortenings.
3. "Spot" hinged support of the side edges of the plate

In this case, the boundary condition of the plate lateral sides the rigid element providing the rectilinear edge shape of the plate was removed, the long edges nodes of the plate were able to have different transverse displacements in the plane of the plate, and the edge line of the plate could be warped. The short (loaded) edges of the plate had also a sliding fit.

Table 1. Comparison of kinematic loading and loading with force.

| No | Change n→n+2 | Kinematic loading | Loading with force |
|----|--------------|-------------------|--------------------|
| 1  | 6→8         | P = 62 kN → 55.9 kN | ΔL = 0.54 cm | P = 63.3kN |
|    |             | ΔL = 0.53 cm → 0.64 cm |          |\n| 2  | 8→10        | P = 125.9k → 117.1kN | ΔL = 1.64 cm | P = 122.6kN |
|    |             | ΔL = 1.57 cm → 1.75 cm |          |\n| 3  | 10→10\textsuperscript{a} | P = 177kN → 175kN | ΔL = 3.03 cm | P = 173.2kN |
|    |             | ΔL = 2.92 cm → 3.1 cm |          |\n| 4  | 10'→10\textsuperscript{b} | P = 197.8kN → 197.9kN | ΔL = 3.6 cm |          |

\textsuperscript{a} the transition 10 → 10' corresponds to the appearance of "side half-dents", on the side bulges;  
\textsuperscript{b} in the 10' → 10' 'transition,' lateral half-dents' appear on intermediate bumps

Figure 5. The curve of equilibrium states (shortening-load curve) and deformed forms of plate with the classic hinged boundary conditions (end loading).

The development of the plate supercritical equilibrium with initial imperfection similar to the first 5-half-wave eigenform (the amplitude of the deflection of 0.01δ) under kinematic loading showed that the longitudinal shortening ΔL = 3.13 cm gives a developed 5-half-wave preclinical shape with "trough-like" dents and bulges. The load jump (P = 150.6 kN → 144.1 kN) transferred the plate to a new 5-half-wave shape, characterized by "half dents" formation on the lateral sides of the side bulges (Figure 6), while the lengths of these bulges increased. This equilibrium is pre-localization, since an insignificant increase in the load (up to 165.9 kN) has led to stability loss of this equilibrium. A secondary and stronger load jump occurred (P = 165.8 kN → 67.2 kN), which resulted in a soliton-
like ejection of plate flexural deformations at a distance of 1/3L with a shortening ΔL≈5 cm (≈0.4b,
b – width of the plate). Here, a strong curvature of the plate edge near the base of the soliton was
observed (as a result of the strong development of the lateral "half-dents" on its surface).

![Deformed model at shortening <3.13 cm](image1)

![“clap” loss of stability at shortening 3.13 cm](image2)

![second “clap” loss of stability at shortening 4.4 cm](image3)

End shortening of a plate with initial imperfection in the sum form of three first eigen modes with
amplitudes ≈0.01δ with fixed short edges and the plate with hinged support along the entire contour
also resulted in formation a soliton-like ejection of flexural deformations.

To prevent the solitons formation, one can either create classical conditions for the hinged support
of the unloaded sides of a longitudinally compressed plate, or by setting up rigid point anchors that
prevent the local curvature and displacement of the plate side edges in its plane.

Calculations with the second variant of boundary condition along the entire contour showed that
the monotonous development of post critical equilibrium under kinematic loading is interrupted by
small snap-through in the load at which the number of half-waves increases by one (n → n + 1; 3 → 4,
4 → 5, and etc. to the last calculated snap-through 8 → 9). Here, an attempt to determine the boundary
stiffness of the elastic fastening of lateral point hinges in the plate plane was made, after a soliton was
not formed. For this, springs of the same rigidity were attached to each node of the lateral edge of the
plate FE model (101 nodes on each side). The spring stiffness varied from 100 N/m to 1500 kN/m. It
turned out that a critical value of the longitudinal shortening exists, at which a soliton-like ejection
occurs for each value of this lateral stiffness (Figure 9, Table 2).

4. Conclusion
We present numerical study of the post critical equilibrium development of a longitudinally
compressed elastic rectangular plate showed that in the case of the boundary conditions correspond to
the classical hinge support or rigid "point" hinge fixation along the side edges, localization of bending
deformations does not take place in this paper. If along the mentioned lateral sides the boundary
conditions allow the bending of the edges and their local displacements in the plate plane (for
example, the conditions of hinged support at the nodes of plate finite element model), then, for
sufficiently large longitudinal shortenings, soliton-like outbursts of bending deformations are formed.

The shape of the soliton is characterized by significant lateral "half-dents" and local distortions of
the plate edges at its base. The height of the soliton "hump" can be of of 0.4-0.45 the order with width
of the plate. The location of the soliton along the length of the plate is not defined: it can be formed
both near the centre of the rectangular plate and close enough from its short edge. We note that the
results of the elastic localization investigation described in this article must be supplemented by studying the features of localization in the elastoplastic stage of the plate material deformation.

Table 2. Critical shortenings of the plate with springs on side edges.

| spring stiffness, (N/m) | Critical shortening (cm) | spring stiffness, (N/m) | Critical shortening (cm) |
|------------------------|--------------------------|------------------------|--------------------------|
| 100                    | 4.38                     | 60000                  | 5.41                     |
| 500                    | 4.39                     | 70000                  | 5.22                     |
| 1000                   | 4.40                     | 100000                 | 6.06                     |
| 10000                  | 4.56                     | 200000                 | 7.5                      |
| 50000                  | 4.90                     | 500000                 | 7.3                      |
| 55000                  | 4.96                     | 1500000                | 8.37                     |

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