Loop Quantum Mechanics and the Fractal Structure of Quantum Spacetime

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Abstract

We discuss the relation between string quantization based on the Schild path integral and the Nambu–Goto path integral. The equivalence between the two approaches at the classical level is extended to the quantum level by a saddle-point evaluation of the corresponding path integrals. A possible relationship between *M–Theory* and the quantum mechanics of string loops is pointed out. Then, within the framework of “loop quantum mechanics”, we confront the difficult question as to what exactly gives rise to the structure of spacetime. We argue that the large scale properties of the string condensate are responsible for the effective Riemannian geometry of classical spacetime. On the other hand, near the Planck scale the condensate “evaporates”, and what is left behind is a “vacuum” characterized by an effective fractal geometry.

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I. INTRODUCTION

At its most fundamental level, research in theoretical high energy physics means research about the nature of mass and energy, and ultimately about the structure of space and time. It may even be argued that the whole history of physics, to a large extent, represents the history of the ever changing notion of space and time in response to our ability to probe infinitesimally small distance scales as well as larger and larger cosmological distances. The “flow chart” in Figure 1 summarizes the dialectic process which has led, through nearly twenty five hundred years of philosophical speculation and scientific inquiry, to the current theoretical efforts in search of a supersynthesis of the two conflicting paradigms of 20th century physics, namely, the Theory of General Relativity and Quantum Theory. In that Hegelian perspective of the history of physics, such a supersynthesis is regarded by many as the “holy grail” of contemporary high energy physics. However, the story of the many efforts towards the formulation of that synthesis, from supergravity to superbranes, constitutes, in itself, a fascinating page in the history of theoretical physics at the threshold of the new millennium. The early ‘80s excitement about string theory ("The First String Revolution") followed from the prediction that only the gauge groups $SO(32)$ and $E(8) \otimes E(8)$ provide a quantum mechanically consistent, i.e., anomaly free, unified theory which includes gravity, and yet is capable, at least in principle, of reproducing the standard electro–weak theory below the GUT scale. However, several fundamental questions were left unanswered. Perhaps, the most prominent one regards the choice of the compactification scheme required to bridge the gap between the multi–dimensional, near–Planckian string–world, and the low energy, four dimensional universe we live in. Some related problems, such as the vanishing of the cosmological constant (is it really vanishing, after all?) and the breaking of supersymmetry were also left without a satisfactory answer. The common feature of all these unsolved problems is their intrinsically non–perturbative character. More or less ten years after the First String Revolution, the second one, which is still in progress, has offered a second important clue into the nature of the superworld. The diagram in Figure 1 encapsulates the essential pieces of a vast mosaic out of which the final theory of the superworld will eventually emerge. Among those pieces, the six surviving viable supermodels known at present, initially thought to be candidates for the role of a fundamental Theory of Everything, are now regarded as different asymptotic realizations, linked by a web of dualities, of a unique and fundamentally new paradigm of physics which goes under the name of M–Theory. The essential components of this underlying matrix theory appear to be string–like objects as well as other types of extendons, e.g., P–branes, D–branes, ..., (any letter)–branes. Moreover, a new computational approach is taking shape which is based on the idea of trading off the strongly coupled regime of a supermodel with the weakly coupled regime of a different model through a systematic use of dualities. Having said that, the fact remains that M–theory, at present, is little more than a name for a mysterious supertheory yet to be fully formulated. In particular, we have no clue as to what radical modification it will bring to the notion of spacetime in the short distance regime. In the meantime, it seems reasonable to attempt to isolate the essential elements of such non–perturbative approach to the dynamics of extended objects. One such approach that we have developed over the last few years is a refinement of an early formulation of quantum string theory by T.Eguchi, elaborated by following a formal analogy with a
Jacobi–type formulation of the canonical quantization of gravity. Thus, our immediate objective, in the following Section, is to illustrate the precise meaning of that analogy. In Sections III and IV we discuss our quantum mechanical elaboration of Eguchi’s approach in terms of “areal” string variables, string propagators and string wave functionals. This discussion, which can be easily extended to p–branes of higher dimensionality, enables us to exemplify a possible relationship between M–theory and the quantum mechanics of string loops. Section V is divided into two subsections where we discuss the functional Schrödinger equation of “loop quantum mechanics” and its solutions in order to derive the Uncertainty Principle for strings as well as its principal consequence, namely, the fractalization of quantum spacetime. We then conclude our discussion of the structure of spacetime in terms of an effective lagrangian based on a covariant, functional extension of the Ginzburg–Landau model of superconductivity.

II. GRAVITY/STRING QUANTIZATION SCHEMES

There is an intriguing similarity between the problem of quantizing gravity, as described by General Relativity, and that of quantizing a relativistic string, or any higher dimensional relativistic extended object. In either case, one can follow one of two main routes: i) a quantum field theory inspired–covariant quantization, or ii) a canonical quantization of the Schrödinger type. The basic idea underlying the covariant approach is to consider the metric tensor $g_{\mu\nu}(x)$ as an ordinary matter field and follow the standard quantization procedure, namely, Fourier analyze small fluctuation around a classical background configuration and give the Fourier coefficients the meaning of creation/annihilation operators of the gravitational field quanta, the gravitons. In the same fashion, quantization of the string world–sheet fluctuations leads to a whole spectrum of particles with different values of mass and spin. Put briefly,

$$g_{\mu\nu}(x) = \text{background} + \text{“graviton”}$$

$$X^\mu(\tau,s) = \text{zero–mode} + \text{particle spectrum}.$$ 

Against this background, one may elect to forgo the full covariance of the quantum theory of gravity in favor of the more restricted symmetry under transformations preserving the “canonical spacetime slicing” into a one–parameter family of space–like three–surfaces. This splitting of space and time amounts to selecting the spatial components of the metric, modulo three–space reparametrizations, as the gravitational degrees of freedom to be quantized. This approach focuses on the quantum mechanical description of the space itself, rather than the corpuscular content of the gravitational field. Then, the quantum state of the spatial 3–geometry is controlled by the Wheeler–DeWitt equation

$$[\text{Wheeler–DeWitt operator}] \Psi[G_3] = 0$$  \hspace{1cm} (1)$$

and the wave functional $\Psi[G_3]$, the wave function of the universe, assigns a probability amplitude to each allowed three geometry. The relation between the two quantization schemes is akin to the relationship in particle dynamics between first quantization, formulated in terms of single particle wave functions along with the corresponding Schrödinger equation,
and the *second quantization* expressed in terms of creation/annihilation operators along with the corresponding field equations. Thus, covariant quantum gravity is, *conceptually*, a second quantization framework for calculating amplitudes, cross sections, mean life, etc., for any physical process involving graviton exchange. Canonical quantum gravity, on the other hand, is a Schrödinger–type first quantization framework, which assigns a probability amplitude for any allowed geometric configuration of three dimensional physical space. It must be emphasized that there is no immediate relationship between the graviton field and the wave function of the universe. Indeed, even if one elevates $\Psi[G_3]$ to the role of *field operator*, it would create or destroy *entire three surfaces* instead of single gravitons. In a more pictorial language, the wave functional $\Psi[G_3]$ becomes a quantum operator creating/destroying full universes! Thus, in order to avoid confusion with quantum gravity as the “theory of gravitons”, the quantum field theory of universes is referred to as the *third quantization scheme*, and has been investigated some years ago mostly in connection with the cosmological constant problem.

Of course, as far as gravity is concerned, any quantization scheme is affected by severe problems: perturbative covariant quantization of General Relativity is not renormalizable, while the intrinsically non–perturbative Wheeler–DeWitt equations can be solved only under extreme simplification such as the mini–superspace approximation. These shortcomings provided the impetus toward the formulation of superstring theory as the only consistent quantization scheme which accommodates the graviton in its (second quantized) particle spectrum. Thus, according to the prevalent way of thinking, there is no compelling reason, nor clear cut procedure to formulate a first quantized quantum mechanics of relativistic extended objects. In the case of strings, this attitude is deeply rooted in the conventional interpretation of the world–sheet coordinates $X^\mu(\tau,s)$ as a “multiplet of scalar fields” defined over a two–dimensional manifold covered by the $\{\tau,s\}$ coordinate mesh. According to this point of view, quantizing a relativistic string is formally equivalent to quantizing a two–dimensional field theory, bypassing a preliminary quantum mechanical formulation. However, there are at least two objections against this kind of reasoning. The first follows from the analogy between the canonical formulation of General Relativity and 3–brane dynamics, and the second objection follows from the “Schrödinger representation” of quantum field theory. More specifically: i) the Wheeler–DeWitt equation can be interpreted, in a modern perspective, as the wave equation for the orbit of a relativistic 3–brane. In this perspective, then, why not conceive of a similar equation for a 1–brane? ii) the functional Schrödinger representation of quantum field theory assigns a probability amplitude to each field configuration over a space–like slice $t = \text{const.}$, and the corresponding wave function obeys a functional Schrödinger–type equation.

Pushing the above arguments to their natural conclusion, we are led to entertaining the interesting possibility of formulating a *functional* quantum mechanics for strings and other $p$–branes. This approach has received scant attention in the mainstream work about quantum string theory, presumably because it requires an explicit breaking of the celebrated reparametrization invariance, which is the distinctive symmetry of relativistic extended objects.

All of the above reasoning leads us to the central question that we wish to analyze, namely: is there any way to formulate a reparametrization invariant string quantum mechanics? As a matter of fact, a possible answer was suggested by T. Eguchi as early as 1980 [7], and
our own elaboration of that quantization scheme \cite{5} is the topic of Section III.

III. EGUCHI’S AREAL QUANTIZATION SCHEME

A. The original formulation

Eguchi’s approach to string quantization follows closely the point–particle quantiza-
tion along the guidelines of the Feynman–Schwinger method. The essential point is that reparametrization invariance is not assumed as an original symmetry of the classical action; rather, it is a symmetry of the physical Green functions to be obtained at the very end of the calculations by means of an averaging procedure over the string manifold parameters. More explicitly, the basic action is not the Nambu–Goto proper area of the string world–sheet, but the “square” of it, i.e. the Schild Lagrangian \cite{8}

\[
L_{\text{Schild}} = \frac{1}{4} \left[ \frac{\partial (x^\mu, x^\nu)}{\partial (\tau, \sigma)} \right]^2, \quad \frac{\partial (x^\mu, x^\nu)}{\partial (\tau, \sigma)} \equiv \partial_\tau x^\mu \wedge \partial_\sigma x^\nu. \tag{2}
\]

The corresponding (Schild) action is invariant under area preserving transformations only, i.e.

\[
(\tau, \sigma) \rightarrow (\tau', \sigma'); \quad \frac{\partial (\tau', \sigma')}{\partial (\tau, \sigma)} = 1. \tag{3}
\]

Such a restricted symmetry requirement leads to a new, Jacobi–type, canonical formalism in which the world–sheet proper area of the string manifold plays the role of evolution parameter. In other words, the “proper time” is neither \(\tau\) or \(x^0\), but the invariant combination of target and internal space coordinates \(x^\mu\) and \(\sigma^a = (\tau, \sigma)\) provided by

\[
A = \int d^2 \sigma \sqrt{-\gamma}, \quad \gamma \equiv \det \partial_\alpha x^\mu \partial_\beta x^\nu. \tag{4}
\]

Once committed to this unconventional definition of time, the quantum amplitude for the transition from an initial vanishing configuration to a final non–vanishing string configuration after a lapse of areal time \(A\), is provided by the kernel \(G(x(s); A)\) which satisfies the following diffusion–like equation, or imaginary area Schrödinger equation

\[
\frac{1}{2} \delta x^\mu(s) \delta x_\mu(s) G(x(s); A) = \frac{\partial}{\partial A} G(x(s); A). \tag{5}
\]

Here, \(x^\mu(s) = x^\mu(\tau(s), \sigma(s))\) represents the physical string coordinate, i.e. the only spacelike boundary of the world–sheet of area \(A\). It may be worth emphasizing at this point that in quantum mechanics of point particles the “time” \(t\) is \emph{not a measurable quantity but an arbitrary parameter}, since there does not exist a self–adjoint quantum operator with eigenvalues \(t\). Similarly, since there is no self–adjoint operator corresponding to the world–sheet area, \(G(x(s); A)\) turns out to be explicitly dependent on the arbitrary parameter \(A\), and cannot have an immediate physical meaning. However, the Laplace transformed Green function is \(A\)–independent and corresponds to the Feynman propagator
\[
G(x(s); M^2) \equiv \int_0^\infty dA G(x(s); A) \exp(-M^2A/2) \\
= -\frac{1}{2(2\pi)^{3/2}} \int \frac{dA}{A^{3/2}} \exp \left( -\frac{F}{2A} - \frac{1}{2}M^2A \right) \\
F = \frac{1}{4} (F^{\mu\nu} \pm * F^{\mu\nu})^2, \quad F^{\mu\nu}[C] = \int_C x^\mu dx^\nu
\] (6)

where \( F \) stands for the self–dual (anti self–dual) area element.

Evidently, this approach is quite different from the “normal mode quantization” based on the Nambu–Goto action or the path integral formulation à la’ Polyakov, and our immediate purpose, in the next subsection, is to establish a connection with the conventional path integral quantization of a relativistic string. Later, in Section IV, we speculate about a possible connection between our functional approach and a recently formulated matrix model of Type IIB superstring.

B. Quantum Mechanics in Loop Space

The Eguchi quantization program is essentially a sort of quantum mechanics formulated in a space of string loops, i.e., a space in which each point represents a possible geometrical configuration of a closed string. To establish a connection with the Nambu–Goto, or Polyakov path integral, it is advantageous to start from the quantum kernel

\[
K[C, C_0; A] = \int_{C_0}^C \int_{\gamma_0}^\gamma D[\mu(\sigma)] \exp \left( iS[x, \xi, p, \pi, N] \right) \\
D[\mu(\sigma)] \equiv D[x(\sigma)]D[\xi(\sigma)]D[P(\sigma)]D[\pi(\sigma)]D[N(\sigma)]
\] (8)

where the histories connecting the initial string \( C_0 \) to the final one \( C \) are weighted by the exponential of the reparametrized Schild action

\[
S[X, P, \xi, \pi, N] = \frac{1}{2} \int_X P_{\mu\nu} dx^\mu \wedge dx^\nu + \frac{1}{2} \int_\Sigma \pi_{ab} d\xi^a \wedge d\xi^b \ola{-}\frac{1}{2} \int_\Sigma d^2 \sigma N^{ab}(\sigma) [\pi_{ab} - \epsilon_{ab}H(P)].
\] (10)

The “dictionary” used in the above equation is as follows: i) \( P_{\mu\nu} \) is the momentum canonically conjugated to the world–sheet area element; ii) \( H(P) = P_{\mu\nu} P_{\mu\nu}/4m^4 \) is the Schild Hamiltonian density, and iii) \( m^2 = 1/4\pi\alpha' \) is the string tension. Furthermore, the original world–sheet coordinates \( \xi^a \) have been promoted to the role of dynamical variables, i.e., they now represent fields \( \xi^a(\sigma) \) defined over the string manifold,

\[
\xi^a \rightarrow \xi^a(\sigma^m), \quad m = 0, 1
\]

and \( \pi_{ab} \), the momentum conjugate to \( \xi^a \), has been introduced into the Hamiltonian form of the action. The relevant dynamical quantities in loop space are listed in Table [1].

The enlargement of the canonical phase space endows the Schild action with the full reparametrization invariance under the transformation \( \sigma^m \rightarrow \zeta^m(\sigma) \), while preserving the
polynomial structure in the dynamical variables, which is a necessary condition to solve the path integral. The regained reparametrization invariance forces the new (extended) hamiltonian to be weakly vanishing, i.e., \( H(P) - e^{ab} \pi_{ab}/2 \approx 0 \). The quantum implementation of this condition is carried out by means of the Lagrange multiplier \( N^{ab}(\sigma) \).

By integrating out \( \pi_{ab} \) and \( \xi^a \) one obtains

\[
K[C, C_0; A] = \int_0^\infty dE e^{iE A} \int_C D[x(\sigma)] D[P(\sigma)] D[N(\sigma)] \times 
\]
\[
\times \exp \left\{ i \int_X P_{\mu\nu} dx^\mu \wedge dx^\nu - i 2 \epsilon_{ab} \int_\Sigma d^2 \sigma N^{ab}(\sigma) [E - H(P)] \right\} 
\equiv 2im^2 \int_0^\infty dE e^{iE A} G[C, C_0; E] 
\]

\[
G[C, C_0; E] = \int_{C_0}^C D[x(\sigma)] D[N(\sigma)] \exp \left\{ -i \int_\Sigma d^2 \sigma \left[ -\frac{m^2}{4N(\sigma)} \dot{x}^{\mu\nu} \dot{x}_{\mu\nu} + N(\sigma) \mathcal{E} \right] \right\} 
\]
\[
= \int_{C_0}^C D[x(\sigma)] D[N(\sigma)] \exp \left\{ -i S_{\text{Schild}}[x, N] \right\} . 
\]

The above expressions show the explicit relation between the fixed area string propagator \( K[C, C_0; A] \), and the fixed “energy” string propagator \( G[C, C_0; E] \) without recourse to any ad hoc averaging prescription in order to eliminate the \( A \) parameter dependence. Moreover, the saddle point value of the string propagator (13) is evaluated to be

\[
G[C, C_0; E] \simeq \int_{C_0}^C D[x(\sigma)] \exp \left\{ -i \sqrt{m^2 E} \int_\Sigma d^2 \sigma \sqrt{-\dot{x}^{\mu\nu} \dot{x}_{\mu\nu}} \right\} . 
\]

Since \( \mathcal{E} \) has dimension of inverse length square, in natural units, we can set the string tension equal to \( m^2 \), and then (14) reproduces exactly the Nambu–Goto path integral. This result allows us to establish the following facts:

i) Eguchi’s approach corresponds to quantizing a string by keeping fixed the area of the string histories in the path integral, and then taking the average over the string tension values;

ii) the Nambu–Goto approach, on the other hand, corresponds to quantizing a string by keeping fixed the string tension and then taking the average over the world–sheet areas;

iii) the two quantization schemes are equivalent in the saddle point approximation.

Finally, since the Schild propagator \( K[C, C_0; A] \) can be computed exactly

\[
K[C, C_0; A] = \left( \frac{m^2}{2i\pi A} \right)^{3/2} \exp \left[ \frac{i m^2}{4A} \left( \sigma^{\mu\nu}(C) - \sigma^{\mu\nu}(C_0) \right)^2 \right] , \quad \sigma^{\mu\nu}(C) \equiv \int_C x^\mu dx^\nu 
\]

we obtain through Eqs.(12), (13) a non–perturbative definition of the Nambu–Goto propagator (14):

\[
G[C, C_0; m^2] = \frac{1}{2im^2} \int_0^\infty dA e^{-im^2 A} K[C, C_0; A] 
\]

where \( K[C, C_0; A] \) is given by Eq.(15).
One of the most enlightening features of the Eguchi approach is the formal correspondence it establishes between the quantum mechanics of point-particles and string loops. Such a relationship is summarized in the translation code displayed in Table [II]. Instrumental to this correspondence is the replacement of the canonical string coordinates $x^\mu(s)$ with the reparametrization invariant area elements $\sigma^{\mu\nu}[C]$. We shall refer to these areal variables as Plücker coordinates \[9\]. Surprising as it may appear at first sight, the new coordinates $\sigma^{\mu\nu}[C]$ are just as “natural” as the old $x^\mu(s)$ for the purpose of defining the string “position”. A naive argument to support this claim goes as follows. In the Jacobi-type formulation of particle dynamics, the position of the physical object is provided by the instantaneous end-point of its own world-line

$$x^\mu(P) \equiv x^\mu(T) = \int_{-\infty}^{T} d\tau \frac{dx^\mu}{d\tau} = \text{world-line end point.} \tag{17}$$

In other words, the instantaneous position of the particle is given by the line integral of the tangent vector up to the chosen final value $T$. Similarly, then, it seems natural to define the “string position” as the surface integral of the tangent bi-vector up to the final boundary of the world-sheet

$$\sigma^{\mu\nu}[C] = \int_{0}^{s_0} d\sigma \int_{-\infty}^{T} d\tau \partial_{\sigma} x^\mu \wedge \partial_{\sigma} x^\nu = \text{world-surface boundary} \tag{18}$$

which is nothing but the loop area element appearing in Eq.(15). A geometric interpretation of the new “matrix”-coordinate assignment to the loop $C$ is that they represent the areas of the loop projected shadows onto the coordinate planes. In this connection, note that the $\sigma$-tensor satisfies the constraint

$$\epsilon^{\lambda\mu\rho} \sigma_{\lambda}^{\mu} \sigma^{\nu\rho} = 0 \tag{19}$$

which ensures that there is a one-to-one correspondence between a given set of areas $\sigma^{\mu\nu}$ and a loop $C$. Thus, to summarize, the Plücker coordinates refer to the boundary of the world sheet, and enter the string wave functional as an appropriate set of position coordinates. Then, it is not surprising that in such a formulation homogeneity requires a timelike coordinate with area dimension as well \[10\].

Finally, we note for the record, that the Plücker coordinates provide a formal correspondence between string theory \[11\], and a certain class of electromagnetic field configurations \[11\]. As a matter of fact, a classical gauge field theory of relativistic strings was proposed

\[1\]Note the “Mach-ian flavor” of this new definition of position and time: spacetime coordinates are not defined by themselves but only in terms of objects located at a given point. This operative definition of coordinates as labels of some material stuff seems even more appropriate in M-Theory where spacetime itself is in some way built out of fundamental strings, or branes, or matrices, or... something else.
several years ago [12], but only recently it was extended to generic p–branes including their coupling to p + 1–forms and gravity [13].

Now that we have established the connection between areal quantization and the path integral formulation of quantum strings, it seems almost compelling to ponder about the relationship, if any, between the $\sigma^{\mu\nu}[C]$ matrix coordinates and the matrix coordinates which, presumably, lie at the heart of the M–Theory formulation of superstrings. Since the general framework of M–Theory is yet to be discovered, it seems reasonable to focus on a specific matrix model recently proposed for Type IIB super strings [15]. The dynamics of this model is encoded into a simple Yang–Mills type action

$$S_{IKKT} = -\frac{\alpha}{4} \text{Tr}[A^\mu, A^\nu]^2 + \beta \text{Tr} I + \text{fermionic part} \tag{20}$$

where the $A^\mu$ variables are represented by $N \times N$ hermitian matrices, and $I$ is the unit matrix. The novelty of this formulation is that it identifies the ordinary spacetime coordinates with the eigenvalues of the non–commuting Yang–Mills matrices. In such a framework, the emergence of classical spacetime occurs in the large–$N$ limit, i.e., when the commutator goes into a Poisson bracket [14] and the matrix trace operation turns into a double continuous sum over the row and column indices, which amounts to a two dimensional invariant integration. Put briefly,

$$\lim_{N \to \infty} \text{"Tr"} \to -i \int d\tau d\sigma \sqrt{\gamma} \tag{21}$$
$$-i \lim_{N \to \infty} [A^\mu, A^\nu] \to \{x^\mu, x^\nu\} \tag{22}$$
$$\sqrt{\gamma}\{x^\mu, x^\nu\} \equiv \dot{x}^{\mu\nu} \equiv \partial_\tau x^\mu \wedge \partial_\sigma x^\nu. \tag{23}$$

What interests us is that, in such a classical limit, the IKKT action (20) turns into the Schild action in Eq. (13) once we make the identifications

$$\alpha \longleftrightarrow -m^2, \quad \beta \longleftrightarrow E, \quad N(\tau, \sigma) \longleftrightarrow \sqrt{\gamma}, \tag{24}$$

while the trace of the Yang–Mills commutator turns into the oriented surface element

$$\lim_{N \to \infty} \text{Tr} [A^\mu, A^\nu] \to \int_{\Sigma} d\tau d\sigma \partial_\tau x^\mu \wedge \partial_\sigma x^\nu \equiv \sigma^{\mu\nu}(\partial_\Sigma). \tag{25}$$

This formal relationship can be clarified by considering a definite case. As an example let us consider a static $D$–string configuration both in the classical Schild formulation and in the corresponding matrix description. It is straightforward to prove that a length $L$ static string stretched along the $x^1$ direction, i.e.

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2 A similar matrix action for the Type IIA model has been conjectured in [14].

3 This is not the canonical Poisson bracket which is replaced by the quantum mechanical commutator. Rather, it is the world sheet symplectic structure which is replaced by the (classical) matrix commutator for finite $N$. 

9
\[ x^\mu = \tau T \delta^{\mu \nu} + \frac{L \sigma}{2\pi} \delta^{\mu \sigma} \], \quad 0 \leq \tau \leq 1, \quad 0 \leq \sigma \leq 2\pi \quad (26) \\
\[ x^\mu = 0, \quad \mu \neq 0, 1 \quad (27) \]
solves the classical equations of motion

\[ \{ x_\mu, \{ x^\mu, x^\nu \} \} = 0. \quad (28) \]

During a time lapse \( T \) the string sweeps a time–like world–sheet in the \((0,1)\)–plane characterized by an area tensor

\[ \sigma^{\mu \nu}(L,T) = \int_0^1 d\tau \int_0^{2\pi} d\sigma \sqrt{\gamma} \{ x^\mu, x^\nu \} = TL \delta^{\mu \nu} \delta^1 \delta^0 \quad (29) \]

Eq. (29) gives both the area and the orientation of the rectangular loop which is the boundary of the string world–sheet. The corresponding matrix solution, on the other hand, must satisfy the equation

\[ [ A_\mu, [ A^\mu, A^\nu ] ] = 0. \quad (30) \]

Consider, then, two hermitian, \( N \times N \) matrices \( \hat{q}, \hat{p} \) with an approximate c–number commutation relation \([ \hat{q}, \hat{p} ] = i \), when \( N >> 1 \). Then, a solution of the classical equation of motion (30), corresponding to a solitonic state in string theory, can be written as

\[ A^\mu = T \delta^{\mu \nu} \hat{q} + \frac{L}{2\pi} \delta^1 \delta^\nu \hat{p} \quad (31) \]

In the large–\( N \) limit we find

\[ -i [ A^\mu, A^\nu ] = -i \frac{LT}{2\pi} \delta^1 \delta^\nu \delta^\mu \hat{q} \approx \frac{LT}{2\pi} \delta^1 \delta^\mu \delta^\nu \hat{p} \quad (32) \]

and

\[ -i Tr [ A^\mu, A^\nu ] \approx \int_0^1 d\tau \int_0^{2\pi} \sqrt{\gamma} \{ x^\mu, x^\nu \} = \sigma^{\mu \nu}(L,T). \quad (33) \]

These results, specific as they are, point to a deeper connection between the loop space description of string dynamics and matrix models of superstrings which, in our opinion, deserves a more detailed investigation. Presently, we shall limit ourselves to take a closer look at the functional quantum mechanics of string loops with an eye on its implications about the structure of spacetime in the short distance regime.

V. LOOP QUANTUM MECHANICS

A. Correspondence Principle, Uncertainty Principle and the Fractalization of Quantum Spacetime

If history of physics is any guide, conflicting scientific paradigms, as outlined in Figure 1, generally lead to broader and more predictive theories. Thus, one would expect that
a synthesis of general relativity and quantum theory will provide, among other things, a deeper insight into the nature and structure of space and time. Thus, reflecting on those two major revolutions in physics of this century, Edward Witten writes [16], “Contemporary developments in theoretical physics suggest that another revolution may be in progress, through which a new source of “fuzziness” may enter physics, and spacetime itself may be reinterpreted as an approximate, derived concept.”.

If spacetime is a derived concept, then it seems natural to ask, “what is the main property of the fuzzy stuff, let us call it quantum spacetime, that replaces the smoothness of the classical spacetime manifold, and what is the scale of distance at which the transition takes place?”.

Remarkably, the celebrated Planck length represents a very near miss as far as the scale of distance is concerned. The new source of fuzziness comes from string theory, specifically from the introduction of the new fundamental constant which determines the tension of the string. Thus, at scales comparable to (\(\alpha'\)^1/2, spacetime becomes fuzzy, even in the absence of conventional quantum effects (\(h = 0\)). While the exact nature of this fuzziness is unclear, it manifests itself in a new form of Heisenberg’s principle, which now depends on both \(\alpha'\) and \(h\). Thus, in Witten’s words, while “a proper theoretical framework for the uncertainty principle has not yet emerged,.........the natural framework of the [string] theory may eventually prove to be inherently quantum mechanical.”.

That new quantum mechanical framework may well constitute the core of the yet undiscovered M–Theory, and the non perturbative functional quantum mechanics of string loops that we have developed in recent years may well represent a first step on the long road toward a matrix formulation of it. If this is the case, a challenging testing ground is provided by the central issue of the structure of quantum spacetime. This question was analyzed in Ref. [6] and we limit ourselves, in the remainder of this subsection, to a brief elaboration of the arguments presented there.

The main point to keep in mind, is the already mentioned analogy between “loop quantum mechanics” and the ordinary quantum mechanics of point particles. That analogy is especially evident in terms of the new areal variables, namely, the spacelike area enclosed by the string loop, given by Eq. (18), and the timelike, proper area of the string manifold, given by Eq. (4). With that choice of dynamical variables, the reparametrized formulation of the Schild action principle leads to the classical energy per unit length conservation \(H = E\).

Then, the loop wave equation can be immediately written down by translating this conservation law in the quantum language through the Correspondence Principle

\[
P_{\mu\nu}(s) \longrightarrow \frac{i}{\sqrt{x'^2(s)}} \frac{\delta}{\delta \sigma^{\mu\nu}(s)} \quad (34)
\]

\[
H \longrightarrow -i \frac{\partial}{\partial A} \quad (35)
\]

Thus, we obtain the Schrödinger equation anticipated in the Introduction,

\[
\frac{1}{4m^2c^2} \int_C d\mu(s) \frac{\delta^2 \Psi[\sigma ; A]}{\delta \sigma^{\mu\nu}(s) \delta \sigma_{\mu\nu}(s)} = i \frac{\partial}{\partial A} \Psi[\sigma ; A] \quad (36)
\]

where we have introduced the string wave functional \(\Psi[\sigma ; A]\) as the amplitude to find the loop \(C\) with area elements \(\sigma^{\mu\nu}[C]\) as the only boundary of a two–surface of internal area \(A\). When no \(A\)–dependent potential is present in loop space, the wave functional factors out as
\[ \Psi[\sigma ; A] = \psi[\sigma] e^{-iA\mathcal{E}} \]  

and Eq.(36) takes the Wheeler–DeWitt form

\[ \frac{1}{4} \oint_C d\mu(s) \frac{\delta^2 \psi[\sigma]}{\delta \sigma^{\mu\nu}(s) \delta \sigma_{\mu\nu}(s)} - m^2 \mathcal{E} \psi[\sigma] = 0. \]  

(38)

Alternatively, one can exchange the area derivatives with the more familiar functional variations of the string embedding through the tangential projection

\[ x'\mu(s) \frac{\delta}{\delta \sigma^{\mu\nu}(s)} = \frac{\delta}{\delta x\nu(s)}, \]  

(39)

where \( x'\mu(s) \) is the tangent vector to the string loop. As a consistency check on our functional equation, note that, if one further identifies the energy per unit length \( \mathcal{E} \) with the string tension by setting \( \mathcal{E} = m^2 \), then, Eq.(38) reads

\[ \frac{1}{l_C} \int_0^1 ds \frac{\delta^2 \psi[\sigma]}{\sqrt{x'^2(s)} \delta x\nu(s) \delta x\mu(s)} = m^4 \psi[\sigma] \]  

(40)

which is the string field equation proposed several years ago by Marshall and Ramond \[17\]. Note also that the Schrödinger equation is a “free” wave equation describing the random drifting of the string representative “point” in loop space. Perhaps, it is worth emphasizing that this “free motion” in loop space is quite different from the free motion of the string in physical space, not only because the physical string is subject to its own tension, i.e. elastic forces are acting on it, but also because drifting from point to point in loop space corresponds physically to quantum mechanically jumping from string shape to string shape. Any such “shape shifting” process, random as it is, is subject to an extended form of the Uncertainty Principle which forbids the exact, simultaneous knowledge of the string shape and its area conjugate momentum. The main consequence of the Shape Uncertainty Principle is the “fractalization” of the string orbit in spacetime. The degree of fuzziness of the string worldsheet is measured by its Hausdorff dimension, whose limiting value we found to be \( D_H = 3 \). In order to reproduce this result, we need the gaussian form of a string wave–packet, which was constructed in Ref. \[3\] as an explicit solution of the functional Schrödinger equation for loops. For our purposes Eq.(38) is quite appropriate: rather than Fourier expanding the string coordinates \( x\mu(s) \) and decomposing the functional wave equation into an infinite set of ordinary differential equations, we insist in maintaining the “wholeness” of the string and consider exact solutions in loop space, or adopt a minisuperspace approximation quantizing only one, or few oscillation modes, freezing all the other (infinite) ones. In the first case, it is possible to get exact “free” solutions, such as the plane wave

\[ \Psi[\sigma ; A] \propto \exp i \left\{ \mathcal{E} A - \oint_C x\mu dx\nu P_{\mu\nu}(x) \right\} \]  

(41)

which is a simultaneous eigenstate of both the area momentum and Hamiltonian operators. This wave functional describes a completely unlocalized state: any loop shape is equally likely to occur and therefore the string has no definite shape. A physical state of definite
shape is obtained by superposition of the fundamental plane wave solutions (41). The quantum analogue of a classical string is the Gaussian wave packet:

$$\Psi[\sigma, A] = \left[ \frac{1}{2\pi(\Delta\sigma)^2} \right]^{3/4} \left( 1 + \frac{iA}{m^2(\Delta\sigma)^2} \right)^{-3/2} \times$$

$$\times \exp \left\{ \frac{1}{1 + (iA/m^2(\Delta\sigma)^2)} \left[ -\frac{\sigma^2}{4(\Delta\sigma)^2} + i \oint_C x^\mu dx^\nu P_{\mu\nu}(x) \right. \right.$$

$$\left. - \frac{iA}{4m^2l_C} \oint_C d\mu(s) P_{\mu\nu} P^{\mu\nu} \right\}$$

(42)

where the width $\Delta\sigma$ of the packet at $A = 0$ represents the area uncertainty. From the solution (42) one can derive some interesting results. First, we note that the center of the wave packet drifts through loop space according to the stationary phase principle

$$\sigma^{\mu\nu}[C] - \frac{A}{m^2} P^{\mu\nu}[C] = 0$$

(43)

and spreads as $A$ increases

$$\Delta\sigma \rightarrow \Delta\sigma(A) = \Delta\sigma \left( 1 + \frac{A^2}{4m^4(\Delta\sigma)^4} \right)^{1/2}.$$  

(44)

Thus, as the areal time $A$ increases, the string “decays”, in the sense that it loses its sharply defined initial shape, but in a way which is controlled by the loop space uncertainty principle

$$\frac{1}{2} \Delta\sigma^{\mu\nu}[C] \Delta P_{\mu\nu}[C] \gtrsim 1,$$  

in natural units  

(45)

involving the uncertainty in the loop area, and the rate of area variation.

The central result that follows from the above equations, is that the classical world–sheet of a string, a smooth manifold of topological dimension two, turns into a fractal object with Hausdorff dimension three as a consequence of the quantum areal fluctuations of the string loop [6]. With historical hindsight, this result is not too surprising. Indeed, Abbott and Wise, following an earlier insight by Feynman and Hibbs, have shown in Ref. [18] that the quantum trajectory of a point-like particle is a fractal of Hausdorff dimension two. Accordingly, one would expect that the limiting Hausdorff dimension of the world–sheet of a string increases by one unit since one is dealing with a one parameter family of one–dimensional quantum trajectories. Next, we try to quantify the transition from the classical, or smooth phase, to the quantum, or fractal phase. Use of the Shape Uncertainty Principle, and of the explicit form of the loop wave–packet, enables us to identify the control parameter of the transition with the De Broglie area characteristic of the loop. As a matter of fact, the Gaussian wave packet (12) allows us to extend the Abbott and Wise calculation to the string case. By introducing the analogue of the “De Broglie wavelength $\Lambda$”as the inverse modulus of the loop momentum

$$4[\Lambda] = \text{length}^2.$$  Accordingly, the loop wavelength is strictly given by $\sqrt{\Lambda}$.  

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\[ \frac{1}{2} P^{\mu \nu} P_{\mu \nu} = \Lambda^{-2} \]  

(46)

one finds:

i) at low resolution, i.e., when the area uncertainty \( \Delta \sigma \gg \Lambda \), the Hausdorff dimension matches the topological value, i.e., \( D_H = 2 \);

ii) at high resolution, i.e., when the area uncertainty \( \Delta \sigma \ll \Lambda \), the Hausdorff dimension increases by one unit, i.e., \( D_H = 3 \).

Hence, quantum string dynamics can be described in terms of a fluctuating Riemannian two–surface only when the observing apparatus is characterized by a low resolution power. As smaller and smaller areas are approached, the graininess of the world–sheet becomes manifest. Then a sort of de–compactification occurs, in the sense that the thickness of the string history comes into play, and the “world–surface” is literally fuzzy to the extent that its Hausdorff dimension can be anything between its topological value of two and its limiting fractal value of three.

**B. Superconductivity and Quantum Spacetime**

Quantum strings, or more generally branes of various kind, are currently viewed as the fundamental constituents of everything: not only every matter particle or gauge boson must be derived from the string vibration spectrum, but spacetime itself is built out of them. If spacetime is no longer preassigned, then logical consistency demands that a matrix representation of p–brane dynamics cannot be formulated in any given background spacetime. The exact mechanism by which M–Theory is supposed to break this circularity is not known at present, but “loop quantum mechanics” points to a possible resolution of that paradox. If one wishes to discuss quantum strings on the same footing with point–particles and other p–branes, then their dynamics is best formulated in loop space rather than in physical spacetime. As we have repeatedly stated throughout this paper, our emphasis on string shapes rather than on the string constituent points, represents a departure from the canonical formulation and requires an appropriate choice of dynamical variables, namely the string configuration tensor and the areal time. Then, at the loop space level, where each “point” is representative of a particular loop configuration, our formulation is purely quantum mechanical, and there is no reference to the background spacetime. At the same time, the functional approach leads to a precise interpretation of the fuzziness of the underlying quantum spacetime in the following sense: when the resolution of the detecting apparatus is smaller than a particle De Broglie wavelength, then the particle quantum trajectory behaves as a fractal curve of Hausdorff dimension two. Similarly we have concluded on the basis of the “shape uncertainty principle” that the Hausdorff dimension of a quantum string world–sheet is three, and that two distinct phases (smooth and fractal phase) exist above and below the loop De Broglie area. Now, if particle world–lines and string world–sheets behave as fractal objects at small scales of distance, so does the world–history of a generic p–brane including spacetime itself [19], and we are led to the general expectation that a new kind of fractal geometry may provide an effective dynamical arena for physical phenomena near the string or Planck scale in the same way that a smooth Riemannian geometry provides an effective dynamical arena for physical phenomena at large distance scales.
Once committed to that point of view, one may naturally ask, “what kind of physical mechanism can be invoked in the framework of loop quantum mechanics to account for the transition from the fractal to the smooth geometric phase of spacetime?” A possible answer consists in the phenomenon of p–brane condensation. In order to illustrate its meaning, let us focus, once again, on string loops. Then, we suggest that what we call “classical spacetime” emerges as a condensate, or string vacuum similar to the ground state of a superconductor. The large scale properties of such a state are described by an “effective Riemannian geometry”. At a distance scale of order \((\alpha')^{1/2}\), the condensate “evaporates”, and with it, the very notion of Riemannian spacetime. What is left behind, is the fuzzy stuff of quantum spacetime.

Clearly, the above scenario is rooted in the functional quantum mechanics of string loops discussed in the previous sections, but is best understood in terms of a model which mimics the Ginzburg–Landau theory of superconductivity. Let us recall once again that one of the main results of the functional approach to quantum strings is that it is possible to describe the evolution of the system without giving up reparametrization invariance. In that approach, the clock that times the evolution of a closed bosonic string is the internal area, i.e., the area measured in the string parameter space \(A \equiv \frac{1}{2} \epsilon_{ab} \int_D d^2 \xi \wedge d^2 \eta\), of any surface subtended by the string loop. The choice of such a surface is arbitrary, corresponding to the freedom of choosing the initial instant of time, i.e., a fiducial reference area. Then one can take advantage of this arbitrariness in the following way. In particle field theory an arbitrary lapse of euclidean, or Wick rotated, imaginary time between initial and final field configurations is usually given the meaning of inverse temperature

\[i \Delta t \rightarrow \tau \equiv \frac{1}{\kappa_B T}\] (47)

and the resulting euclidean field theory provides a finite temperature statistical description of vacuum fluctuations.

Following the same procedure, we suggest to analytically extend \(A\) to imaginary values, \(iA \rightarrow a\), on the assumption that the resulting finite area loop quantum mechanics will provide a statistical description of the string vacuum fluctuations. This leads us to consider the following effective (euclidean) lagrangian of the Ginzburg–Landau type,

\[L(\Psi, \Psi^*) = \Psi^* \frac{\partial}{\partial a} \Psi - \frac{1}{4m^2} \left( \oint_C dl(s) \right)^{-1} \oint_C dl(s) \left( \frac{\delta}{\delta \sigma^{\mu\nu}(s)} - igA_{\mu\nu}(x) \right) \Psi^2 + \]

\[-V(|\Psi|^2) - \frac{1}{2 \cdot 3!} H^{\lambda\mu\nu} H_{\lambda\mu\nu}\]

\[V(|\Psi|^2) \equiv \rho_0^2 \left( \frac{a_c}{a} - 1 \right) |\Psi|^2 + \frac{\lambda}{4} |\Psi|^4\] (48)

\[H_{\lambda\mu\nu} = \partial_{[\lambda} A_{\mu\nu]}\].

(50)

Here, the string field is coupled to a Kalb–Ramond gauge potential \(A_{\mu\nu}(x)\) and \(a_c\) represents a critical loop area such that, when \(a \leq a_c\) the potential energy is minimized by the ordinary vacuum \(\Psi[C] = 0\), while for \(a > a_c\) strings condense into a superconducting vacuum. In other words,
\[ |\Psi[C]|^2 = \begin{cases} 0 & \text{if } ia \leq a_c \\ \frac{a^2}{\lambda} \left(1 - a_c/a\right) & \text{if } a > a_c \end{cases} \] (51)

Evidently, we are thinking of the string condensate as the large scale, background spacetime. On the other hand, as one approaches distances of the order \((\alpha')^{1/2}\) strings undergo more and more shape–shifting transitions which destroy the long range correlation of the string condensate. As we have discussed earlier, this signals the transition from the smooth to the fractal phase of the string world–surface. On the other hand, the quantum mechanical approach discussed in this paper is in no way restricted to string–like objects. In principle, it can be extended to any quantum p–brane, and we anticipate that the limiting value of the corresponding fractal dimension would be \(D_H = p + 2\). Then, if the above over all picture is correct, spacetime fuzziness acquires a well defined meaning. Far from being a smooth, four–dimensional manifold assigned “ab initio”, spacetime is, rather, a “process in the making”, showing an ever changing fractal structure which responds dynamically to the resolving power of the detecting apparatus. At a distance scale of the order of Planck’s length, i.e., when

\[ a_c = G_N \] (52)

then the whole of spacetime boils over and no trace is left of the large scale condensate of either strings or p–branes.

As a final remark, it is interesting to note that since the original paper by A.D. Sakarov about gravity as spacetime elasticity, \(G_N\) has been interpreted as a phenomenological parameter describing the large scale properties of the gravitational vacuum. Eq.(52) provides a deeper insight into the meaning of \(G_N\) as the critical value corresponding to the transition point between an “elastic” Riemannian–type condensate of extended objects and a quantum phase which is just a Planckian foam of fractal objects.
TABLE I. Loop Space functionals and boundary fields

\[ H[C] = (4m^2l_C)^{-1} \oint_C d\mu(s) P_{\mu\nu}(s) \]
\[ H(s) = (4m^2l_C)^{-1} P_{\mu\nu}(s) \]
\[ d\mu(s) \equiv \sqrt{x'^2(s)x'^\mu} x' = dx^\mu/ds \]
\[ l_C \equiv \oint_C d\mu(s) \]
\[ P_{\mu\nu}(s) = m^2 \epsilon^{\mu\nu} \partial_m x_\mu \partial_n x_\nu \]
\[ P_{\mu\nu}[C] \equiv l_C^{-1} \oint_C d\mu(s) P_{\mu\nu}(s) \]

(Schild) Loop Hamiltonian
(Schild) String Hamiltonian
loop invariant measure
loop proper length
area momentum density
loop momentum

TABLE II. The Particle/String “Dictionary”

| Physical object                           | Mathematical model            | Topological meaning          | Coordinates | Trajectory                                | Evolution parameter | Translations generators | Evolution generator | Topological dimension | Distance                           | Linear Momentum                        | Hamiltonian                        |
|------------------------------------------|-------------------------------|------------------------------|-------------|-------------------------------------------|--------------------|-------------------------|----------------------|------------------------|-------------------------------------|---------------------------------------|---------------------------|
| massive point–particle                   | point \( P \) in \( R^3 \)   | boundary (=endpoint) of a line → boundary of an open surface. | \( \{ x^1, x^2, x^3 \} \) → area element \( \sigma^{\mu\nu}[C] = \oint_C x^\mu dx^\nu \) | 1–parameter family of points \( \{ \vec{x}(t) \} \) → \( \{ x^\mu(s; A) \} \) 1–parameter family of loops. | “time” \( t \) → area \( A \) of the string manifold. | spatial shifts: \( \frac{\delta}{\delta x^\nu} \) → shape deformations: \( \frac{\delta}{\delta \sigma^{\mu\nu}[s]} \) | time shifts: \( \frac{\delta}{\delta t} \) → proper area variations: \( \frac{\delta}{\delta A} \) | particle trajectory \( D = 1 \) → string trajectory \( D = 2 \). | \( (\vec{x} - \vec{x}_0)^2 \) → \( (\sigma^{\mu\nu}[C] - \sigma^{\mu\nu}[C_0])^2 \) | rate of change of spatial position → rate of change of string shape. | time conjugate canonical variable → area conjugate canonical variable. |
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FIG. 1. History of physics shows that conflicting theories eventually merge into a broader and deeper synthesis. Will M-Theory lead to a UNIQUE supersynthesis of quantum theory, gravity theory and supersymmetry?