Comparison on the criticality parameters for two supercritical branching processes in random environments

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Abstract

Let \( \{Z_{1,n}, n \geq 0\} \) and \( \{Z_{2,n}, n \geq 0\} \) be two supercritical branching processes in different random environments, with criticality parameters \( \mu_1 \) and \( \mu_2 \) respectively. It is known that with certain conditions, \( \frac{1}{n} \ln Z_{1,n} \to \mu_1 \) and \( \frac{1}{m} \ln Z_{2,m} \to \mu_2 \) in probability as \( m, n \to \infty \). In this paper, we are interested in the comparison on the two criticality parameters, which can be regarded as two-sample U-statistic. To this end, we prove a non-uniform Berry-Esseen’s bound and Cramér’s moderate deviations for \( \frac{1}{n} \ln Z_{1,n} - \frac{1}{m} \ln Z_{2,m} \) as \( m, n \to \infty \). An application is also given for constructing confidence intervals of \( \mu_1 - \mu_2 \).

Keywords: Branching processes; Random environment; Berry-Esseen’s bound; Cramér’s moderate deviations

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1. Introduction

The branching process in a random environment (BPRE) was first introduced by Smith and Wilkinson [26] to model the growth of a population in an independent and identically distributed (iid) random environment. Various limit theorems for BPRE have been obtained: basic results for extinction probability and conditional limit theorems: see, for instance, Vatutin [28], Afanasyev et al. [1], Vatutin and Zheng [29] and Bansaye and Vatutin [5]. While, for supercritical BPRE, a number of researches have studied limit theorems, see, for instance, Gao et al. [15], Hong and Zhang [21], Gao [16] and Li et al. [22]. For moderate and large deviations, we refer to Böinghoff and Kersting [8], Bansaye and Böinghoff [4], Huang and Liu [19], Nakashima [24], Böinghoff [7], Grama et al. [17] and [14]. See also Wang and Liu [30] and Huang et al. [20] for BPRE with immigrations.

Let’s introduce, first of all, the classical model of BPRE. Suppose \( \xi = (\xi_n)_{n \geq 0} \) is a sequence of iid random variables. Usually, \( \xi \) is called environment process and is related to the environment condition

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of the system, which may be captured by a random process. Consider a particle system placed in a random environment $\xi$. In the system, the population size of particles at each generation is denoted by a discrete-time random process $(Z_n)_{n \geq 0}$, which evolves as follows. Given an environment process $\xi$, at generation 0, assume that there is only one particle $0$ and $Z_0 = 1$; at generation 1, the particle 0 splits and is replaced by a group of $Z_1$ new particles; and those new particles constitute the population in the first generation; at generation 2, every individual, say the individual $i$, in the generation 1 splits independently with the others and is replaced by a group of $X_{1,i}$ new particles and the new particles produced from the individuals in the first generation constitute the population in the second generation, and so forth. More generally, in the $n$-th $(n \geq 1)$ generation, there are $Z_n$ particles in the system and in the $(n+1)$-th generation, each particle, say $i$, splits independently and is replaced by a group of $X_{n,i}$ new particles. In mathematical words, a discrete-time random process $(Z_n)_{n \geq 0}$ is called BPRE in the environment $\xi$, if it satisfies the following recursive relation:

$$Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad n \geq 0,$$

where $X_{n,i}$ represents the number of particles produced by the $i$-th individual in the $n$-th generation. The distribution of $X_{n,i}$, which is also called the offspring distribution, depending on the environment $\xi_n$ is denoted by $p(\xi_n) = \{p_k(\xi_n) = P(X_{n,i} = k|\xi_n) : k \in \mathbb{N}\}$. Suppose that given $\xi_n$, $(X_{n,i})_{i \geq 1}$ is a sequence of iid random variables; moreover, $(X_{n,i})_{i \geq 1}$ is independent of $(Z_1, \ldots, Z_n)$. Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment $\xi$ is given. The state space of the environment process $\xi$ is denoted by $\Theta$ and the total probability space can be regarded as the product space $(\Theta^\mathbb{N} \times \Gamma, \mathbb{P})$, where $\mathbb{P}(dx, d\xi) = \mathbb{P}_\xi(dx)\tau(d\xi)$. That is, for any measurable positive function $g$ defined on $\Theta^\mathbb{N} \times \Gamma$, we have

$$\int g(x, \xi)\mathbb{P}(dx, d\xi) = \int \int g(x, \xi)\mathbb{P}_\xi(dx)\tau(d\xi),$$

where $\tau$ is the distribution law of the environment process $\xi$. And $\mathbb{P}_\xi$ can be regarded as the conditional probability when the environment $\xi$ is given. Usually, the conditional probability $\mathbb{P}_\xi$ is called quenched law, while the total probability $\mathbb{P}$ is called annealed law.

In this paper, we are interested in two branching processes placed separately in two different environments $\xi_1$ and $\xi_2$. More precisely, let $(\xi_1, \xi_2)^T := ((\xi_{1,n}, \xi_{2,n})^T)_{n \geq 0}$ be a sequence of iid random vectors, where $T$ is the transport operator and $(\xi_{1,n}, \xi_{2,n})^T \in \mathbb{R}^2$ stands for the random environment at generation $n$. Thus $(\xi_{1,n}, \xi_{2,n})^T_{n \geq 0}$ are independent random vectors, but for given $n$, $\xi_{1,n}$ and $\xi_{2,n}$ may not be independent. For any $n \in \mathbb{N}$, each realization of $\xi_{1,n}$ corresponds to a probability law $p(\xi_{1,n}) = \{p_i(\xi_{1,n}) : i \in \mathbb{N}\}$, that is $p_i(\xi_{1,n}) \geq 0$ and $\sum_{i=0}^{\infty} p_i(\xi_{1,n}) = 1$. Similarly, each realization of $\xi_{2,n}$ corresponds to a probability law $p(\xi_{2,n})$. Let $\{Z_{1,n}\}_{n \geq 0}$ and $\{Z_{2,n}\}_{n \geq 0}$ be two branching processes in the random environment $\xi_1$ and $\xi_2$, respectively. Then $\{Z_{1,n}\}_{n \geq 0}$ and $\{Z_{2,n}\}_{n \geq 0}$ can be described as follows: for $n \geq 0$,

$$Z_{1,0} = 1, \quad Z_{1,n+1} = \sum_{i=1}^{Z_{1,n}} X_{1,n,i}, \quad Z_{2,0} = 1, \quad Z_{2,n+1} = \sum_{i=1}^{Z_{2,n}} X_{2,n,i},$$

where $X_{1,n,i}$ and $X_{2,n,i}$ are the number of particles produced by the $i$-th individual in generation $n$ in the environments $\xi_1$ and $\xi_2$, respectively. Moreover, we assume that given $(\xi_{1,n}, \xi_{2,n})^T$, the random
variables \( \{X_{1,n,i}, X_{2,n,i}, i \geq 1\} \) are independent, and they are also independent of \( \{Z_{1,k}, Z_{2,k}, 0 \leq k \leq n\} \). Denote \( \mathbb{P}_{\xi_1,\xi_2} \) the conditional probability when the environment \((\xi_1, \xi_2)^T\) is given, and \( \tau \) the joint distribution law of the environment \((\xi_1, \xi_2)^T\). Then

\[
\mathbb{P}(dx_1, dx_2, dy_1, dy_2) = \mathbb{P}_{\xi_1,\xi_2}(dx_1, dx_2)\tau(dy_1, dy_2)
\]

is the joint annealed law of the two branching processes placed separately in two different environments.

In particular, if \( \xi_1 \) and \( \xi_2 \) are independent, then we have \( \tau(dy_1, dy_2) = \tau_1(dy_1)\tau_2(dy_2) \), where \( \tau_1 \) and \( \tau_2 \) are the marginal distributions of \( \xi_1 \) and \( \xi_2 \) respectively. In the sequel, \( \mathbb{E}_{\xi_1,\xi_2} \) and \( \mathbb{E} \) denote the expectations with respect to \( \mathbb{P}_{\xi_1,\xi_2} \) and \( \mathbb{P} \), respectively. For any \( n \geq 1 \), set

\[
m_{1,n}^{(p)} = \sum_{k=0}^{\infty} k^p p_k(\xi_{1,n}), \quad m_{2,n}^{(p)} = \sum_{k=0}^{\infty} k^p p_k(\xi_{2,n}),
\]

\[
\Pi_{1,n} = \prod_{i=0}^{n-1} m_{1,i}, \quad \Pi_{2,n} = \prod_{i=0}^{n-1} m_{2,i},
\]

with the convention that \( \Pi_{1,0} = \Pi_{2,0} = 1 \). Clearly, \( \{m_{1,n}^{(p)}\}_{n \geq 0} \) and \( \{m_{2,n}^{(p)}\}_{n \geq 0} \) are two sequences of iid random variables. For simplicity of notations, write

\[
m_{1,n} = m_{1,n}^{(1)} \quad \text{and} \quad m_{2,n} = m_{2,n}^{(1)},
\]

and denote

\[
M_{1,n} = \ln m_{1,n}, \quad M_{2,n} = \ln m_{2,n}, \quad \mu_1 = \mathbb{E}M_{1,0}, \quad \mu_2 = \mathbb{E}M_{2,0},
\]

\[
\sigma_1^2 = \text{Var}(M_{1,0}), \quad \sigma_2^2 = \text{Var}(M_{2,0}),
\]

where \( \mu_1 \) and \( \mu_2 \) are known as the criticality parameters for BPREs \( \{Z_{1,n}\}_{n \geq 0} \) and \( \{Z_{2,n}\}_{n \geq 0} \), respectively. To avoid the environments \( \xi_1 \) and \( \xi_2 \) are degenerated, assume that \( 0 < \sigma_1, \sigma_2 < \infty \). Denote

\[
\rho = \frac{\text{Cov}(M_{1,0}, M_{2,0})}{\sigma_1 \sigma_2}
\]

the correlation coefficient between \( M_{1,0} \) and \( M_{2,0} \). In particular, if \( \xi_1 \) and \( \xi_2 \) are independent, we have \( \rho = 0 \). Write \( \ln^+ x = \max\{\ln x, 0\} \). Throughout the paper, assume the following conditions used in Grama et al. [17]:

\[
\mathbb{E} \left[ \frac{Z_{1,1}}{m_{1,0}} \ln^+ Z_{1,1} + \frac{Z_{2,1}}{m_{2,0}} \ln^+ Z_{2,1} \right] < \infty \quad (1.1)
\]

and

\[
p_0(\xi_{1,0}) = p_0(\xi_{2,0}) = 0, \quad \text{a.s.} \quad (1.2)
\]

The assumption (1.2) implies that each individual has at least one offspring. The assumptions (1.1) and (1.2) together imply that the processes \( \{Z_{1,n}\}_{n \geq 0} \) and \( \{Z_{2,n}\}_{n \geq 0} \) are both supercritical, and satisfy that \( \mu_1, \mu_2 > 0 \) and \( \mathbb{P}(Z_{1,n} \to \infty) = \mathbb{P}(Z_{2,n} \to \infty) = 1 \). See Athreya and Karlin [3] and Tanny [27].
For the case of a single supercritical BPRE, say \( \{Z_{1,n}\}_{n \geq 0} \), the normal approximation has been well studied. With the additional conditions \( \mathbb{E}(Z_{1,n})^p < \infty \) for a constant \( p > 1 \) and \( \mathbb{E}M_{1,0}^{2+\rho} < \infty \) for a constant \( \rho \in (0,1) \), Grama et al. [17] have established the following Berry-Esseen bound for \( \ln Z_{1,n} \):

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{\ln Z_{1,n} - n\mu_1}{\sigma_1 \sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{n^{\rho/2}}. \tag{1.3}
\]

Assume \( \mathbb{E}Z_{1,n}^{2+\rho} < \infty \) for a constant \( p > 1 \) and \( \mathbb{E}e^{\lambda_0 M_{1,0}} < \infty \) for a constant \( \lambda_0 > 0 \), Grama et al. [17] have also established the following Cramér moderate deviation expansion: for \( 0 \leq x = o(1) \) as \( n \to \infty \),

\[
\left| \ln \mathbb{P}\left( \frac{\ln Z_{1,n} - n\mu_1}{\sigma_1 \sqrt{n}} \geq x \right) / (1 - \Phi(x)) \right| \leq C \frac{1 + x^3}{\sqrt{n}} , \tag{1.4}
\]

where \( C \) is a positive constant. Such type of inequality implies the following result about the equivalence to the normal tail, that is

\[
\mathbb{P}\left( \frac{\ln Z_{1,n} - n\mu_1}{\sigma_1 \sqrt{n}} \geq x \right) / (1 - \Phi(x)) = 1 + o(1) \tag{1.5}
\]

uniformly for \( x \in [0, o(n^{1/6})] \) as \( n \to \infty \). The results (1.3)-(1.5) are interesting both in theory and in applications. For example, when the parameter \( \sigma_1 \) is known, they can be applied to construct confidence intervals for estimating the criticality parameter \( \mu_1 \) in terms of the observation \( Z_{1,n} \) and the generation \( n \).

Despite the fact that the limit theorems for one supercritical BPRE are well studied, there is no result for comparison on the criticality parameters for two supercritical BPREs. The objective of the paper is to fit up this gap. Consider the following common hypothesis testing:

\[
H_0 : \mu_1 - \mu_2 = 0 \quad \text{versus} \quad H_1 : \mu_1 - \mu_2 \neq 0.
\]

When \( \mu_1 \) and \( \mu_2 \) are means of two independent populations, such type of hypothesis testing has been considered by Chang et al. [9], and Cramér type moderate deviations has been established therein. In this paper, we are interested in the case \( \mu_1 \) and \( \mu_2 \) are two criticality parameters of BPREs. Notice that \( \ln Z_{i,n} = \sum_{k=1}^{n} M_{i,k} + \ln(Z_{i,n}/\Pi_{i,n}), i = 1, 2 \). With some mild conditions, we have \( \frac{1}{n} \ln(Z_{i,n}/\Pi_{i,n}) \to 0 \) in probability as \( n \to \infty \). As \( \sum_{k=1}^{n} M_{i,k}, i = 1, 2 \), are both sums of iid random variables, by the law of large numbers, we have \( \frac{1}{n} \ln Z_{1,n} \to \mu_1 \) and \( \frac{1}{m} \ln Z_{2,m} \to \mu_2 \) in probability as \( m \wedge n \to \infty \). Thus, to test the hypothesis, it is crucial to estimate the asymptotic distribution of the random variable \( \frac{1}{n} \ln Z_{1,n} - \frac{1}{m} \ln Z_{2,m} \), which is the main purpose of this paper. Notice that \( \frac{1}{n} \ln Z_{1,n} - \frac{1}{m} \ln Z_{2,m} \) has an asymptotic distribution as \( \frac{1}{n} \sum_{k=1}^{n} M_{1,k} - \frac{1}{m} \sum_{k=1}^{m} M_{2,k} \). When \( \xi_1 \) and \( \xi_2 \) are independent, \( \sum_{k=1}^{n} M_{1,k} \) and \( \sum_{k=1}^{m} M_{2,k} \) are both sums of iid random variables and then the hypothesis testing can be regarded as a two-sample \( U \)-statistic problem.

Define

\[
R_{m,n} = \frac{\frac{1}{n} \ln Z_{1,n} - \frac{1}{m} \ln Z_{2,m} - (\mu_1 - \mu_2)}{\sqrt{\frac{1}{n} \sigma_1^2 + \frac{1}{m} \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \frac{m/n}{m+m/n}}}, \quad n, m \in \mathbb{N}.
\]
Throughout the paper, we also assume either
\[ \rho \in [-1, 1] \quad \text{or} \quad \rho = 1 \quad \text{but} \quad \sigma_1 \neq \sigma_2. \]
The last condition ensures that \( \frac{1}{n} \sigma_1^2 + \frac{1}{m} \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \frac{m \wedge n}{m n} \) is in order of \( \frac{1}{m \wedge n} \) as \( m \wedge n \to \infty \). Indeed, if \( m \leq n \), it is easy to see that
\[ \frac{1}{n} \sigma_1^2 + \frac{1}{m} \sigma_2^2 - 2 \rho \sigma_1 \sigma_2 \frac{m \wedge n}{m n} = \left( \frac{1}{m} - \frac{1}{n} \right) \sigma_2^2 + \frac{\sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2}{n} \times \frac{1}{m}. \]
Let us introduce our main results briefly. Firstly, Theorem 2.1 presents the central limit theorem (CLT) for \( R_{m,n} \): for all \( x \in \mathbb{R} \), it holds
\[ \lim_{m \wedge n \to \infty} \mathbb{P}(R_{m,n} \leq x) = \Phi(x), \quad (1.6) \]
where \( \Phi(x) \) is the standard normal distribution function. Secondly, under the moment condition
\[ \mathbb{E}[M_1^{2+\delta} + M_2^{2+\delta}] < \infty \quad \text{with} \quad \delta \in (0, 1] \quad \text{and the condition} \quad \mathbb{E}\left[\frac{Z_{1,1}}{m_{1,0}} + \frac{Z_{2,1}}{m_{2,0}}\right] < \infty \quad \text{with} \quad p > 1, \]
Theorem 2.2 gives a non-uniform Berry-Esseen bound for \( R_{m,n} \): for any constant \( \delta' \in (0, \delta) \) and all \( x \in \mathbb{R} \),
\[ \left| \mathbb{P}(R_{m,n} \leq x) - \Phi(x) \right| \leq \frac{C}{(m \wedge n)^{\delta/2}} \frac{1}{1 + |x|^{1+\delta'}}. \quad (1.7) \]
By Lemma 4.3 and (4.1) in the paper, it seems that \( R_{m,n} \) only has a finite moment of order \( 1 + \delta' \), \( \delta' \in (0, \delta) \), under the stated conditions, which explains why the non-uniform Berry-Esseen bound is of order \( |x|^{-1-\delta'} \) instead of order \( |x|^{-2-\delta} \) as \( x \to \infty \). In particular, when \( m \to \infty \), we have \( \frac{1}{m} \ln Z_{2,m} \to \mu_2 \) in probability, which leads to \( R_{m,n} \to \frac{\ln Z_{1,n} - \mu_1}{\sigma_1 \sqrt{n}} \) in probability. Thus inequality (1.7) implies that
\[ \left| \mathbb{P}\left(\frac{\ln Z_{1,n} - \mu_1}{\sigma_1 \sqrt{n}} \leq x\right) - \Phi(x) \right| \leq \frac{C}{(m \wedge n)^{\delta/2}} \frac{1}{1 + |x|^{1+\delta'}}, \]
which improves the Berry-Esseen bound (1.3) by adding a factor \( \frac{1}{1 + |x|^{1+\delta'}} \). Moreover, using this non-uniform Berry-Esseen bound, we can obtain an optimal convergence rate in the Wasserstein-1 distance. Thirdly, we establish Cramér’s moderate deviations. Assuming Cramér’s condition
\[ \mathbb{E}[e^{\lambda_0 M_{1,0}} + e^{\lambda_0 M_{2,0}}] < \infty \quad \text{for a constant} \quad \lambda_0 > 0 \quad \text{and} \quad \mathbb{E}\left[\frac{Z_{1,1}}{m_{1,0}} + \frac{Z_{2,1}}{m_{2,0}}\right] < \infty \quad \text{for a constant} \quad p > 1, \]
Theorem 2.3 shows that for all \( 0 \leq x \leq C^{-1} \sqrt{m \wedge n} \),
\[ \ln \frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)} \leq \frac{C}{\sqrt{m \wedge n}} \frac{1 + x^3}{\sqrt{m \wedge n}}. \quad (1.8) \]
From (1.8), we obtain the following equivalence to the normal tail: it holds
\[ \frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)} = 1 + o(1) \quad \text{uniformly for} \quad 0 \leq x = o((m \wedge n)^{1/6}) \quad \text{as} \quad m \wedge n \to \infty. \]
When \( m \to \infty \), it is easy to see that (1.8) and (1.9) remain valid when \( R_{m,n} \) is replaced by \( \frac{\ln Z_{1,n} - \mu_1}{\sigma_1 \sqrt{n}} \). Thus our results recover Cramér’s moderate
deviations (1.4) and (1.5) established by Grama et al. [17]. Finally, as an application of our results, we discuss the construction of confidence intervals for $\mu_1 - \mu_2$.

The paper is organized as follows. Our main results are stated and discussed in Section 2. In Section 3, an application of our results to construction of confidence intervals for $\mu_1 - \mu_2$ is demonstrated. The proofs of the main results are given in Section 4.

Throughout the paper, $c$ and $C$, probably supplied with some indices, denote respectively a small positive constant and a large positive constant. Their values may vary from line to line. For two sequences of positive numbers $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, we write $a_n \asymp b_n$ if there exists a positive constant $C$ such that for all $n$, it holds $C^{-1}b_n \leq a_n \leq Cb_n$.

2. Main results

In the sequel, we will need the following conditions.

**A1** There exists a constant $\delta \in (0, 1]$ such that
$$\mathbb{E}[M_{1,0}^{2+\delta} + M_{2,0}^{2+\delta}] < \infty.$$  

**A2** There exists a constant $p > 1$ such that
$$\mathbb{E} \left[ \frac{Z_{1,1}^p}{m_{1,0}^p} + \frac{Z_{2,1}^p}{m_{2,0}^p} \right] < \infty.$$  

Denote $\Phi(x)$ the standard normal distribution function. Let
$$V_{m,n,\rho} = \sqrt{\frac{1}{n} \sigma_1^2 + \frac{1}{m} \sigma_2^2 - 2\rho \sigma_1 \sigma_2 \frac{m \wedge n}{mn}},$$
and recall the definition
$$R_{m,n} = \frac{\frac{1}{n} \ln Z_{1,n} - \frac{1}{m} \ln Z_{2,m} - (\mu_1 - \mu_2)}{V_{m,n,\rho}}, \quad n, m \in \mathbb{N}.$$  

We have the following CLT for $R_{m,n}$.

**Theorem 2.1.** For all $x \in \mathbb{R}$, it holds
$$\lim_{m \wedge n \to \infty} \mathbb{P}(R_{m,n} \leq x) = \Phi(x).$$

The following theorem gives a non-uniform Berry-Esseen bound for $R_{m,n}$.

**Theorem 2.2.** Assume that the conditions **A1** and **A2** are satisfied. Let $\delta'$ be a constant such that $\delta' \in (0, \delta)$. Then for all $x \in \mathbb{R}$,
$$\left| \mathbb{P}(R_{m,n} \leq x) - \Phi(x) \right| \leq \frac{C}{(m \wedge n)^{\delta/2}} \frac{1}{1 + |x|^{1+\delta'}}.$$  

(2.2)
Remark 2.1. By (4.1) and Lemma 4.3, under the conditions A1 and A2, \( R_{m,n} \) has a finite moment of order \( 1 + \delta' \), \( \delta' \in (0, \delta) \), which explains why the non-uniform Berry-Esseen bound (2.2) is of order \( |x|^{-1-\delta'} \) instead of order \( |x|^{-2-\delta} \) as \( x \to \infty \).

The following corollary is a direct consequence of Theorem 2.2, which gives a convergence rate of \( R_{m,n} \) to the standard normal random variable in the Wasserstein-1 distance. We first recall the definition of the Wasserstein-1 distance. The Wasserstein-1 distance between two distributions \( \mu \) and \( \nu \) is defined as follows:

\[
W_1(\mu, \nu) = \sup \left\{ \| \mathbb{E}[f(X)] - \mathbb{E}[f(Y)] \| : (X, Y) \in \mathcal{L}(\mu, \nu), \ f \text{ is 1-Lipschitz} \right\},
\]

where \( \mathcal{L}(\mu, \nu) \) is the collection of all pairs of random variables whose marginal distributions are \( \mu \) and \( \nu \) respectively. In particular, if \( \mu_M \) is the distribution of a random variable \( M \) and \( \nu \) is the standard normal distribution, then we have

\[
W_1(\mu_M, \nu) = d_w(M) := \int_{-\infty}^{+\infty} \left| \mathbb{P}(M \leq x) - \Phi(x) \right| \, dx,
\]

see Röllin [25].

Corollary 2.1. Assume that the conditions A1 and A2 are satisfied. Then

\[
d_w(\lambda_{m,n}) \leq \frac{C (m \land n)^{\delta/2}}{\sqrt{n}^2}.
\]

By Theorem 2.2, we also have the following Berry-Esseen bounds for \( R_{m,n} \).

Corollary 2.2. Assume that the conditions A1 and A2 are satisfied. Then

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\lambda_{m,n} \leq x) - \Phi(x) \right| \leq \frac{C (m \land n)^{\delta/2}}{\sqrt{n}^2}, \tag{2.3}
\]

Notice that \( \frac{1}{m} \ln Z_2, m \to \mu_2 \) in probability as \( m \to \infty \), and thus

\[
R_{\infty,n} := \lim_{m \to \infty} R_{m,n} = \frac{\ln Z_{1,n} - n\mu_1}{\sigma_1 \sqrt{n}}
\]

in probability. Then when \( m \to \infty \), Corollary 2.2 recovers the Berry-Esseen bound established by Grama et al. [17], that is,

\[
\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{\ln Z_{1,n} - n\mu_1}{\sigma_1 \sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{\sqrt{n}^{\delta/2}}.
\]

It is known that the convergence rate of the last Berry-Esseen bound coincides the best possible one for iid random variables with finite moments of order \( 2 + \delta \).

Next, we are going to establish Cramér’s moderate deviations for \( R_{m,n} \). To this end, we need the following conditions.

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The random variables $M_{1,0}$ and $M_{2,0}$ have exponential moments, i.e. there exists a constant $\lambda_0 > 0$ such that

$$\mathbb{E}[e^{\lambda_0 M_{1,0}} + e^{\lambda_0 M_{2,0}}] < \infty.$$  

There exists a constant $p > 1$ such that

$$\mathbb{E}\left[\frac{Z_{1,1}^p}{m_{1,0}} + \frac{Z_{2,1}^p}{m_{2,0}}\right] < \infty.$$ 

By the definition of $M_{1,0}$ and $M_{2,0}$, condition A3 is equivalent to $\mathbb{E}[m_{1,0}^{\lambda_0} + m_{2,0}^{\lambda_0}] < \infty$.

We have the following Cramér’s moderate deviations for $R_{m,n}$.

**Theorem 2.3.** Assume that the conditions A3 and A4 are satisfied. Then for all $0 \leq x \leq c \sqrt{m \wedge n}$,

$$\left| \ln \frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)} \right| \leq C \frac{1 + x^3}{\sqrt{m \wedge n}}.$$  

(2.4)

Thanks to the symmetry between $m$ and $n$, Theorems 2.1, 2.2 and 2.3 remain valid when $R_{m,n}$ is replaced by $-R_{m,n}$.

By an argument similar to the proof of Corollary 2.2 in [13], it is easy to see that Theorem 2.3 implies the following moderate deviation principle (MDP) result for $R_{m,n}$.

**Corollary 2.3.** Assume that the conditions A3 and A4 are satisfied. Let $a_n$ be any sequence of real numbers satisfying $a_n \to \infty$ and $a_n/\sqrt{m \wedge n} \to 0$ as $m \wedge n \to \infty$. Then, for each Borel set $B$, the following inequalities hold

$$-\inf_{x \in B^o} \frac{x^2}{2} \leq \lim_{n \to \infty} \frac{1}{a_n^2} \mathbb{P}\left(\frac{R_{m,n}}{a_n} \in B\right) \leq \limsup_{n \to \infty} \frac{1}{a_n^2} \ln \mathbb{P}\left(\frac{R_{m,n}}{a_n} \in B\right) \leq -\inf_{x \in \overline{B}} \frac{x^2}{2}.$$  

(2.5)

where $B^o$ and $\overline{B}$ denote the interior and the closure of $B$, respectively.

From Theorem 2.3, using the inequality $|e^y - 1| \leq e^C|y|$ valid for $|y| \leq C$, we obtain the following result about the uniform equivalence to the normal tail.

**Corollary 2.4.** Assume that the conditions A3 and A4 are satisfied. Then it holds

$$\frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)} = 1 + o(1)$$  

uniformly for $x \in [0, o((m \wedge n)^{1/6}))$ as $m \wedge n \to \infty$. The result remains valid when $\frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)}$ is replaced by $\frac{\mathbb{P}(R_{m,n} \leq -x)}{\Phi(-x)}$. 


Notice that when \( \{Z_{2,n}, n \geq 0\} \) is an independent copy of \( \{Z_{1,n}, n \geq 0\} \), we have \( \mu_1 = \mu_2, \sigma_1 = \sigma_2 \) and \( \rho = 0 \). Then, for \( m = n \), it holds \( R_{n,n} = \frac{\ln Z_{1,n} - \ln Z_{2,n}}{\sqrt{2n}\sigma_1} \). Consequently, by Theorem 2.2, under the conditions A1 and A2, it holds for all \( x \in \mathbb{R} \),

\[
\left| \mathbb{P}\left( \frac{\ln Z_{1,n} - \ln Z_{2,n}}{\sqrt{2n}\sigma_1} \leq x \right) - \Phi(x) \right| \leq \frac{C}{n^{\delta/2}} \frac{1}{1 + |x|^{1+\delta}}.
\]

By Corollary 2.4, under the conditions A3 and A4, we have

\[
\frac{\mathbb{P}\left( \frac{\ln Z_{1,n} - \ln Z_{2,n}}{\sqrt{2n}\sigma_1} \geq x \right)}{1 - \Phi(x)} = 1 + o(1)
\]

uniformly for \( x \in [0, o(n^{1/6})] \) as \( n \to \infty \).

3. Applications to construction of confidence intervals

In this section, we are interested in constructing confidence intervals for \( \mu_1 - \mu_2 \). This problem is proposed by Behrens in 1929, and it is also known as Behrens-Fisher’s problem because Fisher also has discussed this problem. When parameters \( \sigma_1, \sigma_2, \) and \( \rho \) are known, we can apply Theorems 2.2 and 2.3 to construct confidence intervals for \( \mu_1 - \mu_2 \).

**Theorem 3.1.** Let \( \kappa_{m,n} \in (0,1) \). Consider the following two groups of conditions:

**B1** The conditions of Theorem 2.2 hold and

\[ |\ln \kappa_{m,n}| = o\left( \ln(m \land n) \right), \quad \text{as } m \land n \to \infty. \]

**B2** The conditions of Theorem 2.3 hold and

\[ |\ln \kappa_{m,n}| = o\left( (m \land n)^{1/3} \right), \quad \text{as } m \land n \to \infty. \]

If either B1 or B2 holds, then for \( n \) large enough, \([A_{m,n}, B_{m,n}]\) is the confidence interval of \( \mu_1 - \mu_2 \) with confidence level \( 1 - \kappa_{m,n} \), where

\[
A_{m,n} = \frac{1}{n} \ln Z_{1,n} - \frac{1}{m} \ln Z_{2,m} - V_{m,n,\rho} \Phi^{-1} \left( 1 - \frac{\kappa_{m,n}}{2} \right)
\]

and

\[
B_{m,n} = \frac{1}{n} \ln Z_{1,n} - \frac{1}{m} \ln Z_{2,m} + V_{m,n,\rho} \Phi^{-1} \left( 1 - \frac{\kappa_{m,n}}{2} \right).
\]

**Proof.** Assume that the condition B1 is satisfied. By Theorem 2.2, as \( m \land n \to \infty \), we have

\[
\frac{\mathbb{P}\left( R_{m,n} > x \right)}{1 - \Phi(x)} = 1 + o(1) \quad \text{and} \quad \frac{\mathbb{P}\left( R_{m,n} < -x \right)}{\Phi(-x)} = 1 + o(1) \quad (3.1)
\]
uniformly for \(0 \leq x = o\left(\sqrt{\ln(m \wedge n)}\right)\). For \(p \searrow 0\), the quantile function of the standard normal distribution has the following asymptotic expansion

\[
\Phi^{-1}(p) = -\sqrt{\ln \frac{1}{p^2} - \ln \frac{1}{p^2} - \ln(2\pi)} + o(1).
\]

In particular, when \(\kappa_{m,n}\) satisfies the condition \(B1\), the upper \((1 - \kappa_{m,n}/2)\)-th quantile of standard normal distribution satisfies

\[
\Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right) = -\Phi^{-1}\left(\frac{\kappa_{m,n}}{2}\right) = O\left(\sqrt{\ln \kappa_{m,n}}\right),
\]

which is of order \(o\left(\sqrt{\ln(m \wedge n)}\right)\). Then, applying the last equality to (3.1), we have as \(m \wedge n \to \infty\),

\[
\mathbb{P}\left(R_{m,n} > \Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right)\right) \sim \frac{\kappa_{m,n}}{2}
\]

and

\[
\mathbb{P}\left(R_{m,n} < -\Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right)\right) \sim \frac{\kappa_{m,n}}{2}.
\]

Therefore, as \(m \wedge n \to \infty\),

\[
\mathbb{P}\left(-\Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right) \leq R_{m,n} \leq -\Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right)\right) \sim 1 - \kappa_{m,n},
\]

which implies \(\mu_1 - \mu_2 \in [A_{m,n}, B_{m,n}]\) with probability \(1 - \kappa_{m,n}\) for \(m \wedge n\) large enough.

Now, assume that the condition \(B2\) holds. By Theorem 2.3, as \(m \wedge n \to \infty\), we have

\[
\frac{\mathbb{P}(R_{m,n} > x)}{1 - \Phi(x)} = 1 + o(1) \quad \text{and} \quad \frac{\mathbb{P}(R_{m,n} < -x)}{\Phi(-x)} = 1 + o(1)
\]

uniformly for \(0 \leq x = o((m \wedge n)^{1/6})\). When \(\kappa_{m,n}\) satisfies the condition \(B2\), the upper \((1 - \kappa_{m,n}/2)\)-th quantile of the standard normal distribution satisfies

\[
\Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right) = -\Phi^{-1}\left(\frac{\kappa_{m,n}}{2}\right) = O\left(\sqrt{\ln \kappa_{m,n}}\right),
\]

which is of order \(o\left((m \wedge n)^{1/6}\right)\). Then by (3.2), we get, as \(m \wedge n \to \infty\),

\[
\mathbb{P}\left(-\Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right) \leq R_{m,n} \leq -\Phi^{-1}\left(1 - \frac{\kappa_{m,n}}{2}\right)\right) \sim 1 - \kappa_{m,n}.
\]

From the equality above, the claim of Theorem 3.1 still holds. \(\square\)

When \(\{Z_{2,n}, n \geq 0\}\) is an independent copy of \(\{Z_{1,n}, n \geq 0\}\), we can apply Theorems 2.2 and 2.3 to construct confidence intervals for \(\sigma_1\).
Theorem 3.2. Let $\kappa_{n,n} \in (0,1)$. If either $B1$ or $B2$ holds, then for $n$ large enough, $[A_n, B_n]$ is the confidence interval of $\sigma_1^2$ with confidence level $1 - \kappa_{n,n}$, where

$$A_n = \frac{(\ln Z_{1,n} - \ln Z_{2,n})^2}{2n\chi^2_{1-\frac{1}{2}\kappa_{n,n}}(1)} \quad \text{and} \quad B_n = \frac{(\ln Z_{1,n} - \ln Z_{2,n})^2}{2n\chi^2_{1-\frac{1}{2}\kappa_{n,n}}(1)}$$

with $\chi^2_q(1)$ the $q$-quantiles for chi-squared distribution with one degree of freedom.

Proof. Assume that the condition $B1$ holds. By Theorem 2.2, as $n \to \infty$, we have

$$\mathbb{P}\left(\frac{(\ln Z_{1,n} - \ln Z_{2,n})^2}{2n\sigma^2} > x\right) = 1 + o(1) \quad (3.3)$$

uniformly for $0 \leq x = o(\sqrt{n})$. Then, applying the last equality to (3.3), we have, as $n \to \infty$,

$$\mathbb{P}\left(\chi^2_{1-\frac{1}{2}\kappa_{n,n}}(1) \leq \frac{(\ln Z_{1,n} - \ln Z_{2,n})^2}{2n\sigma^2} \leq \chi^2_{1-\frac{1}{2}\kappa_{n,n}}(1)\right) \sim 1 - \kappa_{n,n},$$

which implies $\sigma_1^2 \in [A_n, B_n]$ with probability $1 - \kappa_{n,n}$ for $n$ large enough.

Assume that the condition $B2$ holds. We can use the similar argument and complete the proof of Theorem 3.2. □

4. Proofs of Theorems

For $l = 1, 2$, denote the normalized population size

$$W_{l,n} = \frac{Z_{l,n}}{\Pi_{l,n}}, \quad n \geq 0.$$ 

Then $(W_{1,n})_{n \geq 0}$ and $(W_{2,n})_{n \geq 0}$ are both non-negative martingales under the annealed law $\mathbb{P}$, with respect to the natural filtration

$$\mathcal{F}_0 = \sigma\{\xi_1, \xi_2\}, \quad \mathcal{F}_n = \sigma\{\xi_1, \xi_2, M_{1,k,i}, M_{2,k,i}, 0 \leq k \leq n-1, i \geq 1\}, \quad n \geq 1.$$ 

Then, by Doob’s martingale convergence theorem, the limit

$$W_{l,\infty} = \lim_{n \to \infty} W_{l,n}$$

exists $\mathbb{P}$-a.s. and, by Fatou’s lemma, it satisfies $\mathbb{E}W_{l,\infty} \leq 1$. The conditions (1.1) and (1.2) together imply that $\mathbb{P}(W_{l,n} > 0) = \mathbb{P}(Z_{l,n} \to \infty) = \lim_{n \to \infty} \mathbb{P}(Z_{l,n} > 0) = 1$, and that the martingale $\{W_{l,n}\}_{n \geq 1}$ converges to $W_{l,\infty}$ in $L^1(\mathbb{P})$ (see Athreya and Karlin [3] and also Tanny [27]).

For simplicity of notations, without loss of generality, we assume that $m \leq n$. In the sequel, denote

$$\eta_{m,n,i} = \frac{M_{1,i-1} - \mu_1}{nV_{m,n,\rho}}, \quad i = 1, ..., n, \quad \text{and} \quad \eta_{m,n,n+j} = -\frac{M_{2,j-1} - \mu_2}{mV_{m,n,\rho}}, \quad j = 1, ..., m.$$
Then $R_{m,n}$ can be rewritten in the following form

$$R_{m,n} = \sum_{i=1}^{n+m} \eta_{m,n,i} + \frac{\ln W_{1,n}}{n V_{m,n,\rho}} - \frac{\ln W_{2,m}}{m V_{m,n,\rho}}.$$

(4.1)

Set

$$Y_i = \eta_{m,n,i} + \eta_{m,n,n+i}, \quad i = 1, \ldots, m, \text{ and } Y_i = \eta_{m,n,i}, \quad i = m + 1, \ldots, n.$$

Then $(Y_i)_{1 \leq i \leq n}$ is a finite sequence of centered and independent random variables, and satisfies

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n+m} \eta_{m,n,i} \quad \text{and} \quad \sum_{i=1}^{n} E Y_i^2 = 1.$$

Moreover, we have

$$\text{Var}(Y_i) \asymp \frac{1}{m}, \quad i = 1, \ldots, m, \quad \text{and} \quad \text{Var}(Y_i) \asymp \frac{m}{n^2}, \quad i = m + 1, \ldots, n,$$

as $m \to \infty$. From (4.1), we get

$$R_{m,n} = \sum_{i=1}^{n} Y_i + \frac{\ln W_{1,n}}{n V_{m,n,\rho}} - \frac{\ln W_{2,m}}{m V_{m,n,\rho}}.$$

(4.2)

4.1. Proof of Theorem 2.1

Without loss of generality, we assume that $m \leq n$. By the CLT for independent random variables, we have $\sum_{i=1}^{n} Y_i$ converges in distribution to the standard normal random variable as $m \to \infty$. Recall that for $l = 1, 2$, $W_{l,n}$ converges to $W_{l,\infty}$ in $L^1(\mathbb{P})$ as $n \to \infty$. By the fact that $V_{m,n,\rho} \asymp \frac{1}{\sqrt{m}}$, the random variables $\frac{\ln W_{1,n}}{n V_{m,n,\rho}}$ and $\frac{\ln W_{2,m}}{m V_{m,n,\rho}}$ both converge in probability to 0 as $m \to \infty$. Hence, by (4.2), we have $R_{m,n}$ converges in distribution to the normal random variable as $m \to \infty$. This completes the proof of Theorem 2.1.

4.2. Preliminary Lemmas for Theorem 2.2

In the proof of Theorem 2.2, we need the following non-uniform Berry-Esseen bound of Bikellis [6]. See also Chen and Shao [10] for more general results.

Lemma 4.1. Let $(Y_i)_{1 \leq i \leq n}$ be independent random variables satisfying $EY_i = 0$ and $E |Y_i|^{2+\delta} < \infty$ for some positive constant $\delta \in (0, 1]$ and all $1 \leq i \leq n$. Assume that $\sum_{i=1}^{n} E Y_i^2 = 1$. Then for all $x \in \mathbb{R}$,

$$\left| \mathbb{P}\left( \sum_{i=1}^{n} Y_i \leq x \right) - \Phi(x) \right| \leq \frac{C}{1 + |x|^{2+\delta}} \sum_{i=1}^{n} E |Y_i|^{2+\delta}.$$

Consider the Laplace transforms of $W_{1,\infty}$ and $W_{2,\infty}$ as follows: for all $t \geq 0$,

$$\phi_{i,\xi}(t) = \mathbb{E}_\xi e^{-t W_{i,\infty}} \quad \text{and} \quad \phi_i(t) = \mathbb{E} \phi_{i,\xi}(t) = \mathbb{E} e^{-t W_{i,\infty}}, \quad i = 1, 2.$$

Clearly, as $W_{i,\infty} \geq 0$ $\mathbb{P}$-a.s., we have $\phi_i(t) \in (0, 1], \quad i = 1, 2$. Moreover, we have the following bounds for $\phi_i(t), \quad i = 1, 2$, as $t \to \infty$. 

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Lemma 4.2. Assume that the conditions A1 and A2 are satisfied. Then for \( i = 1, 2 \), it holds for all \( t > 0 \),

\[
\phi_i(t) \leq \frac{C}{1 + (\ln^+ t)^{1+\delta}}.
\]

Proof. Set \( i = 1, 2 \). Define \( T^n \) the shift operator by \( T^n (\xi_{i,0}, \xi_{i,1}, \ldots) = (\xi_{i,n}, \xi_{i,n+1}, \ldots) \), for \( n \geq 1 \). Then we get for a fixed \( k \geq 0 \),

\[
\Pi_{i,n} \left( T^k \xi_i \right) = m_{i,k} m_{i,k+1} \cdots m_{i,k+n-1}.
\]

In particular, we have \( \Pi_{i,n} = \Pi_{i,n} (T^0 \xi_i) \). Next, we use the method of Grama et al. [18] to complete the proof. From (3.15) in [18], it is proven that for all \( t > 0 \) and \( n \geq 1 \),

\[
\phi_i(t) \leq \mathbb{E} \left[ \phi_i \left( \frac{t}{\Pi_{i,n}} \right) \prod_{j=0}^{n-1} \left( p_1(\xi_{i,j}) + (1 - p_1(\xi_{i,j})) \beta_K \right) \right] + \frac{1}{K} \mathbb{E} \left[ \phi_i \left( \frac{t}{\Pi_{i,n}} \right) \left( \mathbb{E}_{T^n \xi_i W_i}^{T^n \xi_i} \right) \right] + \mathbb{P} \left( \frac{t}{\Pi_{i,n}} < t_K \right),
\]

(4.3)

where \( t_K := (CK)^{-1/(p-1)} \), \( C \) and \( K \) are positive constants, such that

\[
\beta_K := 1 - (1 - 1/p)t_K \in (0,1).
\]

From (3.18) in [18], it is proven that for all \( t > 0 \) and \( n \geq 1 \),

\[
\mathbb{E} \left[ \phi_i \left( \frac{t}{\Pi_{i,n}} \right) \mathbb{E}_{T^n \xi_i W_i}^{T^n \xi_i} \right]
\]

\[
\leq \mathbb{E} \phi_i \left( \frac{t}{\Pi_{i,n}} \right) + \sum_{k=0}^{\infty} \mathbb{E} \frac{\phi_i \left( \frac{t}{\Pi_{i,n}} \frac{1}{\Pi_{i,k}^{p-1}(T^n \xi_i)} \right)}{m_{i,k}^{(p)}(T^n \xi_i)} m_{i,k}^{(p)}(T^n \xi_i).
\]

(4.4)

Let \( p \in (1,2) \). Given \( n, K \), let \( \tilde{N}_{i,n,K} \) be a random positive variable whose distribution is defined by

\[
\mathbb{E} g(\tilde{N}_{i,n,K}) = \frac{1}{q_{i,n,K}} \left[ \mathbb{E} g \left( \frac{1}{\Pi_{i,n}} \prod_{j=0}^{n-1} \left( p_1(\xi_{i,j}) + (1 - p_1(\xi_{i,j})) \beta_K \right) \right) + \frac{1}{K} \mathbb{E} g \left( \frac{1}{\Pi_{i,n}} \right) + \frac{1}{K} \sum_{k=0}^{\infty} \mathbb{E} g \left( \frac{1}{\Pi_{i,k}^{p-1}(T^n \xi_i)} \right) \frac{m_{i,k}^{(p)}(T^n \xi_i)}{m_{i,k}^{(p)}(T^n \xi_i)} \right],
\]

(4.5)

for any bounded and measurable function \( g \), where \( q_{i,n,K} \) is the normalizing constant (to make \( \mathbb{E} g(\tilde{N}_{i,n,K}) = 1 \), when \( g = 1 \) defined by

\[
q_{i,n,K} = \mathbb{E} \left[ \prod_{j=0}^{n-1} \left( p_1(\xi_{i,j}) + (1 - p_1(\xi_{i,j})) \beta_K \right) \right] + \frac{1}{K} \left[ 1 + \sum_{k=0}^{\infty} \mathbb{E} \frac{m_{i,k}^{(p)}}{m_{i,k}^{(p)}(T^n \xi_i)} \right].
\]

(4.6)
The last line follows by (3.21) in [18]. Combining (4.3)-(4.5) together, we have

$$\phi_i(t) \leq q_{i,n,K} E \phi_i(\bar{N}_{i,n,K} t) + \mathbb{P} \left( \frac{t}{\Pi_{i,n}} < t_K \right).$$  \hfill (4.7)

Choose $A_i$ such that $\ln A_i > \mu_i$. Clearly, we have $A_i > 1$. When $A_i^{n+1} \leq t < A_i^{n+2}$, let $K = K_{i,n} = ((n + 1) \ln A_i)^{1+\delta}$, then we have $\lim_{n \to \infty} \sqrt{t_{K_{i,n}}} = 1$. By Fuk-Nagaev’s inequality (see Corollary 2.5 in [12] with $p = 2 + \delta$, $V^2 = O(n)$ and $C_p = O(n)$), we have for $n$ large enough,

$$\begin{align*}
\mathbb{P} \left( \frac{t}{\Pi_{i,n}} < t_{K_{i,n}} \right) &= \mathbb{P} \left( \Pi_{i,n} > \frac{t}{t_{K_{i,n}}} \right) \\
&\leq \mathbb{P} \left( S_{i,n} - n\mu_i > n \left( \ln \frac{A_i}{\sqrt{t_{K_{i,n}}}} - \mu_i \right) \right) \\
&\leq \exp \left\{ -c n c_i^2 \right\} + C_1 \frac{n c_i^{2+\delta}}{n^{2+\delta}} \\
&\leq \frac{C_2}{n^{1+\delta}},
\end{align*}$$  \hfill (4.8)

where $S_{i,n} = \sum_{j=1}^{n} M_{i,j-1}$ and $c_i = \ln \frac{A_i}{\sqrt{t_{K_{i,n}}}} - \mu_i$, $i = 1, 2$. When $t \geq A_i^2$, let $n$ be a positive integer such that $A_i^{n+1} \leq t < A_i^{n+2}$, so we get

$$n > \frac{\ln t}{\ln A_i} - 2.$$  

Thus, by (4.8), for all $A_i^{n+1} \leq t < A_i^{n+2},$

$$\begin{align*}
\mathbb{P} \left( \frac{t}{\Pi_{i,n}} < t_{K_{i,n}} \right) &\leq \frac{C}{n^{1+\delta}} < C \left( \frac{\ln t}{\ln A_i} - 2 \right)^{-1-\delta} \\
&\leq \frac{C}{(\ln t)^{1+\delta}}.
\end{align*}$$  \hfill (4.9)

Notice that $0 \leq \phi_i(t) \leq 1$ ($t > 0$). By (4.7) and (4.9), we have

$$\phi_i(t) \leq \sum_{n=1}^{\infty} q_{i,n,K,n} 1_{\{A_i^{n+1} \leq t < A_i^{n+2}\}} + \frac{C}{1 + (\ln t)^{1+\delta}}. \hfill (4.10)$$

From the definition (4.6) of $q_{i,n,K,n}$, when $A_i^{n+1} \leq t < A_i^{n+2}$, we have

$$q_{i,n,K,n} \leq \frac{C}{1 + (\ln t)^{1+\delta}}. \hfill (4.11)$$

Finally, by (4.10) and (4.11), we obtain for all $t > 0$,

$$\phi_i(t) \leq \frac{C}{1 + (\ln t)^{1+\delta}}.$$

This completes the proof of Lemma 4.2. \hfill \Box

Grama et al. [17] (see Theorem 3.1 therein) have established a bound $\varphi(t) \leq C t^{-\alpha}, t > 0$, where $\alpha$ is a positive constant. Their bound is better than the one in Lemma 4.2. However, Theorem 3.1 of Grama et al. [17] requires condition A3 which is stronger than condition A1.

We have the following result for the $L_p(\mathbb{P})$ moments of $\ln W_{i,\infty}$ and $\ln W_{i,n}$. The same result with $q \in (1, 1 + \delta/2)$ has been established by Grama et al. [17] (cf. Lemma 2.3 therein). The following result is an improvement on their result by replacing $q \in (1, 1 + \delta/2)$ with $q \in (1, 1 + \delta)$.  

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Lemma 4.3. Assume that the conditions A1 and A2 are satisfied. Then for \( i = 1, 2 \) and \( q \in (1, 1 + \delta) \), the following two inequalities hold

\[
\mathbb{E}|\ln W_{i,\infty}|^q < \infty \tag{4.12}
\]

and

\[
\sup_{n \in \mathbb{N}} \mathbb{E}|\ln W_{i,n}|^q < \infty. \tag{4.13}
\]

Proof. Set \( i = 1, 2 \). Consider the following truncation

\[
\mathbb{E}|\ln W_{i,\infty}|^q = \mathbb{E}|\ln W_{i,\infty}|^q 1_{\{W_{i,\infty} > 1\}} + \mathbb{E}|\ln W_{i,\infty}|^q 1_{\{W_{i,\infty} \leq 1\}}. \tag{4.14}
\]

For the first term in the right-hand side of the equality above, we have

\[
\mathbb{E}|\ln W_{i,\infty}|^q 1_{\{W_{i,\infty} > 1\}} \leq C \mathbb{E}W_{i,\infty} < \infty. \tag{4.15}
\]

For the second term, we have

\[
\mathbb{E}|\ln W_{i,\infty}|^q 1_{\{W_{i,\infty} \leq 1\}} = q \int_1^\infty \frac{1}{t} (\ln t)^{q-1} \mathbb{P}(W_{i,\infty} \leq t^{-1}) \, dt \\
\leq q e \int_1^\infty \frac{\phi_i(t)}{t} (\ln t)^{q-1} \, dt \\
= q e \left( \int_e^\infty \frac{\phi_i(t)}{t} (\ln t)^{q-1} \, dt + \int_1^\infty \frac{\phi_i(t)}{t} (\ln t)^{q-1} \, dt \right). \tag{4.16}
\]

The last inequality above can be obtained by Markov’s inequality, i.e.,

\[
\mathbb{P}(W_{i,\infty} \leq t^{-1}) \leq e \mathbb{E}e^{-tW_{i,\infty}} = e \phi_i(t).
\]

Clearly, it holds

\[
\int_e^\infty \frac{\phi_i(t)}{t} (\ln t)^{q-1} \, dt < \infty. \tag{4.17}
\]

From Lemma 4.2 and \( q < 1 + \delta \), we have

\[
\int_e^\infty \frac{\phi_i(t)}{t} (\ln t)^{q-1} \, dt \leq C \int_e^\infty \frac{1}{t(\ln t)^{2+\delta-q}} \, dt < \infty. \tag{4.18}
\]

Substituting (4.17) and (4.18) into (4.16), we get

\[
\mathbb{E}|\ln W_{i,\infty}|^q 1_{\{W_{i,\infty} \leq 1\}} < \infty. \tag{4.19}
\]

Therefore, by (4.14), (4.15) and (4.19), we get (4.12).

Next, we give a proof for (4.13). Since \( x \mapsto |\ln^q(x)| 1_{\{x \leq 1\}} \), \( q > 1 \), is a non-negative and convex function, by Lemma 2.1 in [19], we have

\[
\sup_{n \in \mathbb{N}} \mathbb{E}|\ln W_{i,n}|^q 1_{\{W_{i,n} \leq 1\}} = \mathbb{E}|\ln W_{i,\infty}|^q 1_{\{W_{i,\infty} \leq 1\}}.
\]
With the similar truncation as \( E \left| \ln W_{i,\infty} \right|^q \), by (4.19), we get
\[
\sup_{n \in \mathbb{N}} E \left| \ln W_{i,n} \right|^q = \sup_{n \in \mathbb{N}} \left( E \left| \ln W_{i,n} \right|^q 1_{\{W_{i,n} > 1\}} + E \left| \ln W_{i,n} \right|^q 1_{\{W_{i,n} \leq 1\}} \right)
\leq \sup_{n \in \mathbb{N}} E \left| \ln W_{i,n} \right|^q 1_{\{W_{i,n} > 1\}} + \sup_{n \in \mathbb{N}} E \left| \ln W_{i,n} \right|^q 1_{\{W_{i,n} \leq 1\}}
\leq C E W_{i,\infty} + E \left| \ln W_{i,\infty} \right|^q 1_{\{W_{i,\infty} \leq 1\}} < \infty.
\]
This completes the proof of Lemma 4.3. □

The following lemma is a simple consequence of Lemma 2.4 of Grama et al. [17].

**Lemma 4.4.** Assume that the conditions \( A1 \) and \( A2 \) are satisfied. Then there exists a constant \( \gamma \in (0, 1) \), such that
\[
E \left| \ln W_{1,n} - \ln W_{1,\infty} \right| + E \left| \ln W_{2,m} - \ln W_{2,\infty} \right| \leq C \gamma^{m \wedge n}.
\]

In the proof of Theorem 2.2, the following lemma plays an important role.

**Lemma 4.5.** Assume that the conditions \( A1 \) and \( A2 \) are satisfied. Let \( \delta' \) be a constant such that \( \delta' \in (0, \delta) \). Then for all \( x \in \mathbb{R} \),
\[
P \left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \leq \frac{C}{(m \wedge n)^{\delta/2}} \frac{1}{1 + |x|^{1+\delta'}}
\]
and
\[
P \left( R_{m,n} \geq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \leq x \right) \leq \frac{C}{(m \wedge n)^{\delta/2}} \frac{1}{1 + |x|^{1+\delta'}}.
\]

**Proof.** We only give a proof for inequality (4.20), as inequality (4.21) can be proved in the same way. Without loss of generality, we may assume that \( m \leq n \).

First, we show that inequality (4.20) holds for \( x \leq -Cm^{1/2} \) with some positive constant \( C \). Recall
\[
R_{m,n} = \frac{\ln Z_{1,n} - n \mu_1}{nv_{m,n,\rho}} - \frac{\ln Z_{2,m} - m \mu_2}{mv_{m,n,\rho}}, \quad m, n \in \mathbb{N}.
\]
Then we have for all \( x \in \mathbb{R} \),
\[
P \left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \leq P_1 + P_2,
\]
where
\[
P_1 = P \left( \frac{\ln Z_{1,n} - n \mu_1}{nv_{m,n,\rho}} \leq \frac{x}{2} \right) \quad \text{and} \quad P_2 = P \left( -\frac{\ln Z_{2,m} - m \mu_2}{mv_{m,n,\rho}} \leq \frac{x}{2} \right).
\]
Since \( Z_{1,n} \geq 1 \) \( \mathbb{P} \)-a.s. and \( V_{m,n,\rho} \sim m^{-1/2} \) as \( m \to \infty \), there exists a positive constant \( C \) such that
\[
\frac{\ln Z_{1,n} - n \mu_1}{nv_{m,n,\rho}} > -\frac{\mu_1}{V_{m,n,\rho}} > -\frac{1}{2} C m^{1/2} \quad \mathbb{P} \text{-a.s.},
\]
and
\[
\frac{\ln Z_{2,m} - m \mu_2}{mv_{m,n,\rho}} \leq -\frac{1}{2} C m^{1/2} \quad \mathbb{P} \text{-a.s.}
\]
and thus $P_1 = 0$ for all $x \leq -Cm^{1/2}$. For $P_2$, by Lemma 4.1, Markov’s inequality and the fact $\mathbb{E} W_{2,m} = 1$, we have for all $x \leq -Cm^{1/2}$,

$$P_2 \leq \mathbb{P}\left( \sum_{j=1}^{m} \eta_{m,n,n+j} \leq -\frac{|x|}{4} \right) + \mathbb{P}\left( \frac{\ln W_{2,m}}{m V_{m,n,\rho}} \geq \frac{|x|}{4} \right)$$

$$\leq \mathbb{P}\left( \sum_{j=1}^{m} \eta_{m,n,n+j} \leq -\frac{|x|}{4} \right) + \exp\left\{ -\frac{|x|}{4} m V_{m,n,\rho} \right\} \mathbb{E} W_{2,m}$$

$$\leq \frac{C}{m^{\delta/2} 1 + |x|^{2+\delta}}.$$ 

Hence, inequality (4.20) holds for all $x \leq -Cm^{1/2}$.

Next, we show that inequality (4.20) holds for all $x \geq Cm^{1/2}$. For all $x \geq 0$, we have

$$\mathbb{P}\left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \leq \mathbb{P}\left( \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right).$$

Applying Lemma 4.1 to the right-hand side of the last inequality, we have for all $x \geq 0$,

$$\mathbb{P}\left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \leq 1 - \Phi(x) + \frac{C_1}{1 + |x|^{2+\delta}} \left( \sum_{i=1}^{m} \mathbb{E}|\eta_{m,n,i} + \eta_{m,n,n+i}|^{2+\delta} + \sum_{i=m+1}^{n} \mathbb{E}|\eta_{m,n,i}|^{2+\delta} \right).$$

Using the inequality

$$(a + b)^{2+\delta} \leq 2^{1+\delta}(|a|^{2+\delta} + |b|^{2+\delta}), \quad a, b \in \mathbb{R},$$

we deduce that

$$\sum_{i=1}^{m} \mathbb{E}|\eta_{m,n,i} + \eta_{m,n,n+i}|^{2+\delta} + \sum_{i=m+1}^{n} \mathbb{E}|\eta_{m,n,i}|^{2+\delta} \leq C_1 \left( \sum_{i=1}^{n} \mathbb{E}|\eta_{m,n,i}|^{2+\delta} + \sum_{i=1}^{m} \mathbb{E}|\eta_{m,n,n+i}|^{2+\delta} \right).$$

Hence, we get for all $x \geq 0$,

$$\mathbb{P}\left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \leq 1 - \Phi(x) + \frac{C_1}{1 + |x|^{2+\delta}} \left( \sum_{i=1}^{n} \mathbb{E}|\eta_{m,n,i}|^{2+\delta} + \sum_{i=1}^{m} \mathbb{E}|\eta_{m,n,n+i}|^{2+\delta} \right).$$

Notice that $V_{m,n,\rho} \asymp m^{-\delta/2}$ as $m \to \infty$. Then, by the inequalities

$$\frac{1}{\sqrt{2\pi(1 + x)}} e^{-x^2/2} \leq 1 - \Phi(x) \leq \frac{1}{\sqrt{\pi(1 + x)}} e^{-x^2/2}, \quad x \geq 0,$$

(4.23)
we have for all $x \geq Cm^{1/2}$,

$$
\mathbb{P}\left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \leq \frac{C_2}{m^{\delta/2}} \frac{1}{1 + |x|^{2+\delta}} + \frac{C_3}{1 + |x|^{2+\delta}} \left( \frac{n}{n^{2+\delta} m^{-1-\delta/2}} + \frac{m}{m^{2+\delta} m^{-1-\delta/2}} \right) \\
\leq \frac{C_4}{m^{\delta/2}} \frac{1}{1 + |x|^{2+\delta}}.
$$

Thus, inequality (4.20) holds for all $x \geq Cm^{1/2}$.

To end the proof of lemma, we only need to show that (4.20) holds for all $|x| < Cm^{1/2}$. Consider the following notations for all $0 \leq k \leq m - 1$,

$$
Y_{m,n,k} = \sum_{i=k+1}^{n} \eta_{m,n,i} + \sum_{j=k+1}^{m} \eta_{m,n,n+j}, \quad (4.24)
$$

$$
\tilde{Y}_{m,n,k} = Y_{m,n,0} - Y_{m,n,k}, \quad (4.25)
$$

$$
H_{m,n,k} = \frac{\ln \text{W}_1,k}{n V_{m,n,\rho}} - \frac{\ln \text{W}_2,k}{m V_{m,n,\rho}}, \quad (4.26)
$$

$$
D_{m,n,k} = \frac{\ln \text{W}_1,n}{n V_{m,n,\rho}} - \frac{\ln \text{W}_2,m}{m V_{m,n,\rho}} - H_{m,n,k}. \quad (4.27)
$$

Set $\alpha_m = m^{-\delta/2}$ and $k = \lfloor m^{1-\delta/2} \rfloor$, where $\lfloor t \rfloor$ stands for the largest integer less than $t$. From (4.1), we deduce that for all $x \in \mathbb{R}$,

$$
\mathbb{P}\left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \\
= \mathbb{P}\left( Y_{m,n,0} + H_{m,n,k} + D_{m,n,k} \leq x, Y_{m,n,0} \geq x \right) \\
\leq \mathbb{P}\left( Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x \right) + \mathbb{P}\left( |D_{m,n,k}| \geq \alpha_m \right). \quad (4.28)
$$

For the tail probability $\mathbb{P}(|D_{m,n,k}| \geq \alpha_m)$, by Markov’s inequality and Lemma 4.4, there exists a
constant $\gamma \in (0, 1)$ such that for all $-m < x < m$,

$$
P(|D_{m,n,k}| > \alpha_m) \leq \frac{\mathbb{E}|D_{m,n,k}|}{\alpha_m}$$

$$= \frac{m^{\delta/2}}{V_{m,n,\rho}} \mathbb{E}\left[\left(\frac{\ln W_{1,n}}{n} - \frac{\ln W_{1,\infty}}{n}\right) - \left(\frac{\ln W_{2,m}}{m} - \frac{\ln W_{2,\infty}}{m}\right)\right]$$

$$- \left[\left(\frac{\ln W_{1,k}}{n} - \frac{\ln W_{1,\infty}}{n}\right) - \left(\frac{\ln W_{2,k}}{m} - \frac{\ln W_{2,\infty}}{m}\right)\right] \mathbb{E}\left|\frac{\ln W_{1,n}}{n} - \frac{\ln W_{1,\infty}}{n}\right| + \mathbb{E}\left|\frac{\ln W_{2,m}}{m} - \frac{\ln W_{2,\infty}}{m}\right|$$

$$\leq \frac{m^{\delta/2}}{V_{m,n,\rho}} \left(\mathbb{E}\left|\frac{\ln W_{1,n}}{n} - \frac{\ln W_{1,\infty}}{n}\right| + \mathbb{E}\left|\frac{\ln W_{2,m}}{m} - \frac{\ln W_{2,\infty}}{m}\right|\right)$$

$$\leq C_1 m^{(1+\delta)/2} \left(\frac{1}{n} \gamma^n + \frac{1}{m} \gamma^m + \frac{1}{n} \gamma^k + \frac{1}{m} \gamma^k\right)$$

$$\leq \frac{C_2}{m^{\delta/2} 1 + |x|^{2+\delta}}. \quad (4.29)$$

Next, we give an estimation for the first term at the right-hand side of (4.28). Let

$$G_{m,n,k}(x) = \mathbb{P}(Y_{m,n,k} \leq x) \quad \text{and} \quad v_k(ds, dt) = \mathbb{P}\left(\tilde{Y}_{m,n,k} \in ds, H_{m,n,k} \in dt\right). \quad (4.30)$$

Since $Y_{m,n,k}$ and $(\tilde{Y}_{m,n,k}, H_{m,n,k})$ are independent, we have

$$\mathbb{P}\left(Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x\right)$$

$$= \mathbb{P}\left(Y_{m,n,k} + \tilde{Y}_{m,n,k} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,k} + \tilde{Y}_{m,n,k} \geq x\right)$$

$$= \iint \mathbb{P}(Y_{m,n,k} + s + t \leq x + \alpha_m, Y_{m,n,k} + s \geq x) v_k(ds, dt)$$

$$= \iint 1_{\{t \leq \alpha_m\}} \left(G_{m,n,k}(x - s - t + \alpha_m) - G_{m,n,k}(x - s)\right) v_k(ds, dt). \quad (4.31)$$

Denote $C_{m,n,k}^2 = \text{Var}(Y_{m,n,k})$, then it holds $C_{m,n,k} = 1 + O(k/n) \not\rightarrow 1$ as $m \to \infty$. By Lemma 4.1 and
(4.22), we obtain for all \( x \in \mathbb{R} \),
\[
\left| \mathbb{P}\left( \frac{Y_{m,n,k}}{C_{m,n,k}} \leq \frac{x}{C_{m,n,k}} \right) - \Phi \left( \frac{x}{C_{m,n,k}} \right) \right|
\leq C_1 \frac{1}{1 + |x|^2 + \delta} \left( \sum_{j=k+1}^{m} \mathbb{E} \left| \frac{\eta_{m,n,i}}{C_{m,n,k}} + \frac{\eta_{m,n,n+i}}{C_{m,n,k}} \right|^{2+\delta} + \sum_{i=m+1}^{n} \mathbb{E} \left| \frac{\eta_{m,n,i}}{C_{m,n,k}} \right|^{2+\delta} \right)
\leq C_2 \frac{1}{1 + |x|^2 + \delta} \left( \sum_{j=k+1}^{m} \mathbb{E} \left| \frac{M_{2,j-1} - \mu_2}{mV_{m,n}} \right|^{2+\delta} + \sum_{i=k+1}^{n} \mathbb{E} \left| \frac{M_{1,i-1} - \mu_1}{nV_{m,n}} \right|^{2+\delta} \right)
\leq C_3 \frac{1}{1 + |x|^2 + \delta} \left( \frac{m}{m^{2+\delta}m^{-\delta/2}} + \frac{n}{n^{2+\delta}m^{-\delta/2}} \right)
\leq C_4 \frac{1}{m^{\delta/2} + |x|^2 + \delta}.
\]
By the last inequality, we deduce that for all \( x \in \mathbb{R} \),
\[
|G_{m,n,k}(x) - \Phi(x)| \leq |\mathbb{P}\left( \frac{Y_{m,n,k}}{C_{m,n,k}} \leq \frac{x}{C_{m,n,k}} \right) - \Phi \left( \frac{x}{C_{m,n,k}} \right)| + |\Phi \left( \frac{x}{C_{m,n,k}} \right) - \Phi(x)|
\leq C \frac{1}{m^{\delta/2} + |x|^2 + \delta} + \exp \left\{ \frac{-x^2}{2} \right\} \left| \frac{x}{C_{m,n,k}} - x \right|
\leq C \frac{1}{m^{\delta/2} + |x|^2 + \delta} + C \frac{k}{n} \frac{1}{1 + |x|^2 + \delta}
\leq C \frac{1}{m^{\delta/2} + |x|^2 + \delta}.
\]
Therefore, we have for all \( x \in \mathbb{R} \),
\[
\mathbb{P}\left( Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x \right) \leq J_1 + J_2 + J_3, \tag{4.32}
\]
where
\[
J_1 = \int_{t \leq \alpha_m} 1_{t \leq \alpha_m} \left| \Phi(x - s - t + \alpha_m) - \Phi(x - s) \right| v_k(ds, dt),
J_2 = \frac{C}{m^{\delta/2}} \int_{t \leq \alpha_m} 1_{t \leq \alpha_m} \frac{1}{1 + |x - s|^2 + \delta} v_k(ds, dt)
\]
and
\[
J_3 = \frac{C}{m^{\delta/2}} \int_{t \leq \alpha_m} 1_{t \leq \alpha_m} \frac{1}{1 + |x - s - t|^2 + \delta} v_k(ds, dt).
\]
For \( J_1 \), by the mean value theorem, we have for all \( x \in \mathbb{R} \),
\[
1_{t \leq \alpha_m} \left| \Phi(x - s - t + \alpha_m) - \Phi(x - s) \right|
\leq C|\alpha_m - t| \exp \left\{ \frac{-x^2}{8} \right\} + |\alpha_m - t| 1_{|s| \geq 1 + \frac{1}{4}|x|} + |\alpha_m - t| 1_{|t| \geq 1 + \frac{1}{4}|x|}
\]
20
and

\[ J_1 \leq J_{11} + J_{12} + J_{13}, \]  

(4.33)

where

\[ J_{11} = C \int \int |\alpha_m - t| \exp \left\{ -\frac{x^2}{8} \right\} v_k(ds, dt), \]

\[ J_{12} = \int \int |\alpha_m - t| 1_{\{|s| \geq 1 + \frac{1}{4}|x|\}} v_k(ds, dt) \]

and

\[ J_{13} = \int \int |\alpha_m - t| 1_{\{|t| \geq 1 + \frac{1}{4}|x|\}} v_k(ds, dt). \]

By Lemma 4.3, it is obvious that for all \( x \in \mathbb{R} \),

\[ J_{11} \leq C_1 \exp \left\{ -\frac{x^2}{8} \right\} \left( \alpha_m + \mathbb{E}|H_{m,n,k}| \right) \leq \frac{C_2}{m^{\delta/2}} \frac{1}{1 + |x|^{2+\delta}}. \]

For \( J_{12} \), we have the following estimation for all \( x \in \mathbb{R} \),

\[ J_{12} \leq \alpha_m \mathbb{P}\left( |\tilde{Y}_{m,n,k}| \geq 1 + \frac{1}{4}|x| \right) + \mathbb{E}|H_{m,n,k}| 1_{\{|\tilde{Y}_{m,n,k}| \geq 1 + \frac{1}{4}|x|\}}. \]

Denote \( \tilde{C}_{m,n,k}^2 = \text{Var}(\tilde{Y}_{m,n,k}) \), then we have \( \tilde{C}_{m,n,k}^2 \sim \frac{1}{m^{\delta/2}} \). Let \( \delta' \in (0, \delta) \). By Lemma 4.1, we deduce that for all \( x \in \mathbb{R} \),

\[ \mathbb{P}\left( |\tilde{Y}_{m,n,k}| \geq 1 + \frac{1}{4}|x| \right) \leq 1 - \Phi \left( \frac{1 + |x|/4}{\tilde{C}_{m,n,k}} \right) + \Phi \left( -\frac{1 + |x|/4}{\tilde{C}_{m,n,k}} \right) + \frac{C}{1 + |1+|x|/4|^{2+\delta}} \sum_{i=1}^{k} \mathbb{E}\left| \eta_{m,n,i} + \eta_{m,n,n+i} \right|^{2+\delta} \tilde{C}_{m,n,k} \]

\[ \leq \frac{C_1}{1 + |x|^{2+\delta}} \frac{1}{m^{\delta/2}} \frac{1}{k^{\delta/2}} \]

\[ \leq \frac{C_2}{1 + |x|^{2+\delta}} \frac{1}{m^{\delta}}. \]  

(4.34)
and, by Hölder’s inequality with \( \tau = 1 + \frac{\delta + \delta'}{2 + 2\delta - \delta'} \) and \( \iota \) satisfying \( \frac{1}{\tau} + \frac{1}{\iota} = 1 \), it holds for all \( |x| \leq Cm^{1/2} \),

\[
\mathbb{E}|H_{m,n,k}| \mathbb{1}_{\{|\tilde{Y}_{m,n,k}| \geq 1 + \frac{1}{4}|x|\}} \leq \left( \mathbb{E}|H_{m,n,k}|^\tau \right)^{1/\tau} \mathbb{P}\left( |\tilde{Y}_{m,n,k}| \geq 1 + \frac{1}{4}|x| \right)^{1/\iota} \leq C \frac{1}{m^{1/2}} \left( \frac{C_1}{1 + |x|^{2+\delta} m^\delta} \right)^{1/\iota} \leq C \frac{1}{m^{\delta/2} 1 + |x|^{1+\delta'}}.
\]

Hence, we have for all \( |x| \leq Cm^{1/2} \),

\[
J_{12} \leq \frac{C_3}{m^{\delta/2} 1 + |x|^{1+\delta'}}.
\]

For \( J_{13} \), we have for all \( x \in \mathbb{R} \),

\[
J_{13} \leq \alpha_m \mathbb{P}\left( |H_{m,n,k}| \geq 1 + \frac{1}{4}|x| \right) + \mathbb{E}|H_{m,n,k}| \mathbb{1}_{\{|H_{m,n,k}| \geq 1 + \frac{1}{4}|x|\}}.
\]

By Lemma 4.3 with \( p' = 1 + \delta/2 \) and Markov’s inequality, we deduce that for all \( |x| \leq Cm^{1/2} \),

\[
\mathbb{P}\left( |H_{m,n,k}| \geq 1 + \frac{1}{4}|x| \right) \leq \frac{4^{p'}}{1 + |x|^{2+p'}} \mathbb{E}|H_{m,n,k}|^{p'} \leq \frac{C}{1 + |x|^{2+p'}} \frac{1}{m^{p'/2}} \leq \frac{C}{1 + |x|^{2+\delta}},
\]

and, by Lemma 4.3 with \( p'' = \frac{1}{2}(\delta + \delta') \),

\[
\mathbb{E}|H_{m,n,k}| \mathbb{1}_{\{|H_{m,n,k}| \geq 1 + \frac{1}{4}|x|\}} \leq \frac{C_1}{1 + |x|^{2+p''}} \mathbb{E}|H_{m,n,k}|^{1+p''} \leq \frac{C_2}{1 + |x|^{2+p''} m^{(1+p'')/2}} \leq \frac{C_3}{m^{\delta/2} 1 + |x|^{1+\delta'}}.
\]

Hence, we have for all \( |x| \leq Cm^{1/2} \),

\[
J_{13} \leq \frac{C}{m^{\delta/2} 1 + |x|^{1+\delta'}}.
\]

Returning to (4.33), we get for all \( |x| \leq Cm^{1/2} \),

\[
J_1 \leq \frac{C}{m^{\delta/2} 1 + |x|^{1+\delta'}}.
\]
Next, we consider $J_2$. By an argument similar to the proof of (4.34), we have for all $x \in \mathbb{R}$,

$$
J_2 = \frac{C_1}{m^{\delta/2}} \int \int 1_{\{|t| \leq \alpha_m\}} \frac{1}{1 + |x - s|^{2+\delta}} v_k(ds, dt)
$$

$$
\leq \frac{C_1}{m^{\delta/2}} \left( \int_{|s| < 1+|x|/2} \frac{1}{1 + |x - s|^{2+\delta}} v_k(ds) + \int_{|s| \geq 1+|x|/2} \frac{1}{1 + |x - s|^{2+\delta}} v_k(ds) \right)
$$

$$
\leq \frac{C_2}{m^{\delta/2}} \left[ \frac{1}{1 + |x/2|^{2+\delta}} + \mathbb{P}\left( \frac{\tilde{Y}_{m,n,k}}{C_{m,n,k}} > \frac{1 + |x|/4}{C_{m,n,k}} \right) \right]
$$

$$
\leq \frac{C_3}{m^{\delta/2}} \frac{1}{1 + |x|^{2+\delta}}.
$$

(4.37)

For $J_3$, by some arguments similar to that of (4.34) and (4.35), we have for all $|x| \leq C^{1/2}$,

$$
J_3 = \frac{C_1}{m^{\delta/2}} \int \int 1_{\{|t| \leq \alpha_m\}} \frac{1}{1 + |x - s - t|^{2+\delta}} v_k(ds, dt)
$$

$$
\leq \frac{C_1}{m^{\delta/2}} \left( \int \int_{|s+t| \leq 2+|x|/4} \frac{1}{1 + |x/2|^{2+\delta}} v_k(ds, dt) \right)
$$

$$
+ \int \int_{|s| > 1+|x|/4} v_k(ds, dt) + \int \int_{|t| > 1+|x|/4} v_k(ds, dt) \right)
$$

$$
\leq \frac{C_2}{m^{\delta/2}} \left[ \frac{1}{1 + |x/2|^{2+\delta}} + \mathbb{P}\left( \frac{\tilde{Y}_{m,n,k}}{C_{m,n,k}} > \frac{1 + |x|/4}{C_{m,n,k}} \right) + \mathbb{P}\left( |H_{m,n,k}| > 1 + \frac{|x|}{4} \right) \right]
$$

$$
\leq \frac{C_3}{m^{\delta/2}} \frac{1}{1 + |x|^{2+\delta}}.
$$

(4.38)

Applying the inequalities (4.36)-(4.38) to (4.32), we get for all $|x| \leq C^{1/2}$,

$$
\mathbb{P}\left( Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x \right) \leq \frac{C}{m^{\delta/2}} \frac{1}{1 + |x|^{1+\delta}}.
$$

(4.39)

Combining (4.28), (4.29) and (4.39) together, we get (4.20) for all $|x| \leq C^{1/2}$. This completes the proof of Lemma 4.5.  \[\square\]
4.3. Proof of Theorem 2.2

We are now in a position to end the proof of Theorem 2.2. By Lemma 4.1 and the fact $V_{m,n,\rho} \propto \sqrt{m^{-1} + n^{-1}}$, we have for all $x \in \mathbb{R}$,

\[
|\mathbb{P}(\sum_{i=1}^{n+m} \eta_{m,n,i} \leq x) - \Phi(x)|
\leq \frac{C_1}{1 + |x|^{2+\delta}} \left( \sum_{i=1}^{m} \mathbb{E}|\eta_{m,n,i} + \eta_{m,n,n+i}|^{2+\delta} + \sum_{i=m+1}^{n} \mathbb{E}|\eta_{m,n,i}|^{2+\delta} \right)
\leq \frac{C_2}{1 + |x|^{2+\delta}} \sum_{i=1}^{m+n} \mathbb{E}|\eta_{m,n,i}|^{2+\delta}
\leq \frac{C_3}{1 + |x|^{2+\delta}} \left( \frac{n}{n^2+\delta} \left( \frac{1}{n} + \frac{1}{m} \right)^{(2+\delta)/2} + \frac{m}{m^2+\delta} \left( \frac{1}{n} + \frac{1}{m} \right)^{(2+\delta)/2} \right)
\leq \frac{C}{(m \land n)^{\delta/2} 1 + |x|^{2+\delta}}.
\tag{4.40}
\]

Notice that

\[
\mathbb{P}(R_{m,n} \leq x) = \mathbb{P}(R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \leq x) + \mathbb{P}(R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} > x)
= \mathbb{P}\left(\sum_{i=1}^{n+m} \eta_{m,n,i} \leq x\right) - \mathbb{P}(R_{m,n} > x, \sum_{i=1}^{n+m} \eta_{m,n,i} \leq x)
+ \mathbb{P}(R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} > x).
\]

Applying (4.40) to the last equality, we deduce that for all $x \in \mathbb{R}$,

\[
\mathbb{P}(R_{m,n} \leq x) - \Phi(x) \leq \frac{C}{(m \land n)^{\delta/2} 1 + |x|^{2+\delta}} + \mathbb{P}(R_{m,n} > x, \sum_{i=1}^{n+m} \eta_{m,n,i} \leq x)
+ \mathbb{P}(R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} > x).
\]

By Lemma 4.5, it follows that for all $x \in \mathbb{R}$,

\[
\left|\mathbb{P}(R_{m,n} \leq x) - \Phi(x)\right| \leq \frac{C}{(m \land n)^{\delta/2} 1 + |x|^{1+\delta}}.
\]

This completes the proof of Theorem 2.2. \qed
4.4. Preliminary Lemmas for Theorem 2.3

To prove Theorem 2.3, we shall make use of the following lemma (see Theorem 3.1 of Grama et al. [17]). The lemma shows that the conditions \( A_3 \) and \( A_4 \) imply the existence of a harmonic moment of positive order \( \alpha > 0 \).

**Lemma 4.6.** Assume that the conditions \( A_3 \) and \( A_4 \) are satisfied. There exists a constant \( a_0 > 0 \) such that for all \( \alpha \in (0, a_0) \), the following inequalities hold

\[
\mathbb{E}W_{1,\infty}^{-\alpha} + \mathbb{E}W_{2,\infty}^{-\alpha} < \infty \quad (4.41)
\]

and

\[
\sup_{n \in \mathbb{N}} (\mathbb{E}W_{1,n}^{-\alpha} + \mathbb{E}W_{2,n}^{-\alpha}) < \infty. \quad (4.42)
\]

**Proof.** We give an alternative proof for Theorem 3.1 of Grama et al. [17]. Let \( i = 1, 2 \). By the fact that

\[
W_{i,\infty}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-t W_{i,\infty}^{-\alpha}} t^{\alpha-1} dt,
\]

we have

\[
\mathbb{E}W_{i,\infty}^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \phi_i(t) t^{\alpha-1} dt = \frac{1}{\Gamma(\alpha)} \left( \int_0^1 \phi_i(t) t^{\alpha-1} dt + \int_1^{\infty} \phi_i(t) t^{\alpha-1} dt \right), \quad (4.43)
\]

where \( \Gamma \) is the gamma function. For the first term in the above bracket, since \( 0 \leq \phi_i(t) \leq 1 \) \((t \geq 0)\), then we have for all \( a_0 > 0 \),

\[
\int_0^1 \phi_i(t) t^{a_0-1} dt \leq C \int_0^1 t^{a_0-1} dt < \infty. \quad (4.44)
\]

For the second term, it suffices that there exists a positive constant \( a_0 \) such that for all \( t > 0 \),

\[
\phi_i(t) \leq \frac{C}{1 + t a_0}. \quad (4.45)
\]

To this end, we use the method of Grama et al. [18]. Let \( \bar{N}_{i,n,K} \) be defined as in (4.5). By (4.7), we have for all \( t > 0 \),

\[
\phi_i(t) \leq q_{i,n,K} \mathbb{E} \phi_i(\bar{N}_{i,n,K} t) + \mathbb{P} \left( \frac{t}{\Pi_i,n} < t_K \right). \quad (4.46)
\]

Let \( p \in (1, 2] \). By (4.5), we have

\[
q_{i,n,K} \mathbb{E} \bar{N}_{i,n,K}^{-\alpha} = \mathbb{E} \left[ \prod_{i,n}^{a} \prod_{j=0}^{n-1} \left( p_1(\xi_{i,j}) + (1 - p_1(\xi_{i,j})) \beta K \right) \right] + \frac{1}{K} \mathbb{E} \Pi_i,n
\]

\[+ \frac{1}{K} \mathbb{E} \left[ \Pi_i,n(\xi_i) \sum_{k=0}^{\infty} \frac{\Pi_{i,k}^a(T^n \xi_i)}{\Pi_{i,k}^p \Pi_{i,k}^n} \frac{m_{i,k}^p(T^n \xi_i)}{m_{i,k}^p(T^n \xi_i)} \right]. \]
Since $\Pi_{i,n}(\xi_i)$ is independent of $\Pi_{i,k+1}(T^m\xi_i)$ and $\Pi_{i,k}(T^n\xi_i)$ under $\mathbb{P}$, for any $k \geq 0$, we have

$$q_{i,n,K}^n \mathbb{E} \tilde{N}_{i,n,K}^{-a} = \mathbb{E} \left[ \Pi_{k,n}^a \prod_{j=0}^{n-1} \left( p_1(\xi_{i,j}) + (1 - p_1(\xi_{i,j}))\beta_K \right) \right] + \frac{1}{K} \mathbb{E} \Pi_{i,n}^a \mathbb{E} \left[ 1 + \sum_{k=0}^{\infty} \frac{\Pi_{i,k+1}^a(T^m\xi_i)}{\Pi_{i,k}^a(T^n\xi_i)} \left( \frac{m_{i,k}^{(p)}}{m_{i,k}^{(p)}} (T^m\xi_i) \right) \right]$$

$$= \mathbb{E} \prod_{j=0}^{n-1} \left[ m_{i,j}^a (p_1(\xi_{i,j}) + (1 - p_1(\xi_{i,j}))\beta_K) \right] + \frac{1}{K} \mathbb{E} \Pi_{i,n}^a \mathbb{E} \left[ 1 + \sum_{k=0}^{\infty} \frac{\Pi_{i,k+1}^a(T^m\xi_i)}{\Pi_{i,k}^a(T^n\xi_i)} \left( \frac{m_{i,k}^{(p)}}{m_{i,k}^{(p)}} (T^m\xi_i) \right) \right]$$

$$= \left\{ \mathbb{E} \left[ m_{i,0}^a (p_1(\xi_{i,0}) + (1 - p_1(\xi_{i,0}))\beta_K) \right] \right\}^n + \frac{\mathbb{E} m_{i,0}^{a} (1 - \mathbb{E} m_{i,0}^{a-1} \mathbb{E} (m_{i,0}^{p} m_{i,0}^{p}))}{K} \left[ 1 + \frac{1}{1 - \mathbb{E} m_{i,0}^{a-1} (\mathbb{E} (m_{i,0}^{p} m_{i,0}^{p}))} \right].$$

(4.47)

Notice that $0 \leq m_{i,0}^a (p_1(\xi_{i,0}) + (1 - p_1(\xi_{i,0}))\beta_K) \leq m_{i,0}^a$. By condition A3, it holds $\mathbb{E} m_{i,0}^{\lambda_0} < \infty$. Thus by the dominated convergence theorem, we have

$$\lim_{a \to 0} \left\{ \mathbb{E} \left[ m_{i,0}^a (p_1(\xi_{i,0}) + (1 - p_1(\xi_{i,0}))\beta_K) \right] \right\}^n = \left\{ \mathbb{E} [p_1(\xi_{i,0}) + (1 - p_1(\xi_{i,0}))\beta_K] \right\}^n,$$

and

$$\lim_{a \to 0} (\mathbb{E} m_{i,0}^{a})^n = 1.$$

Since $m_{i,0}^a \geq 1$ a.s. and $p > 1$, we have $\lim_{a \to 0} \mathbb{E} m_{i,0}^{a-1} = \mathbb{E} m_{i,0}^{-(p-1)}$. When $a \in (0, p - 1]$, we have

$$0 \leq \frac{m_{i,0}^{a-1}}{m_{i,0}^a} \leq \frac{m_{i,0}^{a-1}}{m_{i,0}^a}.$$ Condition A4 implies that

$$\mathbb{E} \frac{m_{i,0}^{(p)}}{m_{i,0}^a} = \mathbb{E} Z_{i,0}^{p} = \mathbb{E} Z_{i,0}^{\frac{p}{a}} < \infty.$$

Thus by the dominated convergence theorem, we have

$$\lim_{a \to 0} \mathbb{E} \left( \frac{m_{i,0}^{a-1}}{m_{i,0}^a} \right) = \mathbb{E} \left( \frac{m_{i,0}^{(p)}}{m_{i,0}^a} \right).$$

Hence, from (4.47), we get

$$q_{i,n,K}^n \mathbb{E} \tilde{N}_{i,n,K}^{-a} \xrightarrow{a=0} q_{i,n,K} = \left\{ \mathbb{E} [p_1(\xi_{i,0}) + (1 - p_1(\xi_{i,0}))\beta_K] \right\}^n + \frac{1}{K} \left[ 1 + \frac{1}{1 - \mathbb{E} m_{i,0}^{a-1} \mathbb{E} (m_{i,0}^{p} m_{i,0}^{p})} \right].$$
As $\beta_K \in (0,1)$, we have $E[p_1(\xi_{i,0}) + (1 - p_1(\xi_{i,0}))\beta_K] < 1$, which leads to

$$\lim_{n \to \infty} \left\{ E[p_1(\xi_{i,0}) + (1 - p_1(\xi_{i,0}))\beta_K] \right\}^n = 0.$$ 

Therefore, it holds

$$q_{i,n,K} \xrightarrow{n \to \infty} \frac{1}{K} \left[ 1 + \frac{1}{1 - E m_{i,0}^{-p}} \right] \xrightarrow{K \to \infty} 0.$$ 

In conclusion, we have

$$q_{i,n,K} \xrightarrow{n \to \infty} a_{i,n,K} \xrightarrow{a_{i,n,K} \to a_0} 0.$$ 

Then, we take $n_0, K_0$ and $a_0 \in (0, \lambda_0)$ small enough such that $q_{i,n_0,K_0} \xrightarrow{n \to \infty} 1$. By (4.46) and Markov's inequality, we can get

$$\phi_\xi(t) \leq q_{i,n_0,K_0} E \left( \bar{N}_{i,n_0,K_0} t \right) + \mathbb{P} \left( \frac{t}{\Pi_{n_0,K_0}} < t_{K_0} \right) \leq q_{i,n_0,K_0} E \left( \bar{N}_{i,n_0,K_0} t \right) + C_{n_0,K_0} \xrightarrow{t_{a_0}} 0.$$ (4.48)

Notice that $0 \leq \phi_\xi(t) \leq 1 (t > 0)$. Finally, by (4.48) and Lemma 4.1 in [23], we have for all $t > 0$,

$$\phi_\xi(t) \leq \frac{C}{1 + t_{a_0}}.$$ 

Let $0 < \alpha < a_0$, we have

$$\int_1^\infty \phi_\xi(t) t^{\alpha - 1} dt \leq C \int_1^\infty t^{\alpha - a_0 - 1} dt < \infty.$$ (4.49)

By (4.43), (4.44) and (4.49), inequality (4.41) holds.

It remains to prove (4.42) now. Since $x \mapsto x^{-\alpha}$ $(\alpha > 0, x > 0)$ is a non-negative convex function. Then by Lemma 2.1 in [19], we have

$$\sup_{n \in \mathbb{N}} E W_{i,n}^{-\alpha} = E W_{i,\infty}^{-\alpha} < \infty.$$ 

This completes the proof of Lemma 4.6. □

Under the conditions A3 and A4, we have the following analogue to Lemma 4.5.

**Lemma 4.7.** Assume that the conditions A3 and A4 are satisfied. Then for all $|x| \leq \sqrt{\ln(m \wedge n)}$,

$$P \left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right) \leq C \frac{1 + x^2}{\sqrt{m \wedge n}} \exp \left\{ - \frac{1}{2} x^2 \right\}$$ (4.50)

and

$$P \left( R_{m,n} \geq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \leq x \right) \leq C \frac{1 + x^2}{\sqrt{m \wedge n}} \exp \left\{ - \frac{1}{2} x^2 \right\}.$$ (4.51)
**Proof.** As the conditions \( A_3 \) and \( A_4 \) together imply the conditions \( A_1 \) and \( A_2 \), when \( |x| \leq 1 \), the inequalities (4.50) and (4.51) are simple consequences of Lemma 4.5. Thus we only need to prove the inequalities (4.50) and (4.51) for all \( 1 \leq |x| \leq \sqrt{\ln(m \land n)} \). In the sequel, we only give a proof for inequality (4.50) with \( 1 \leq |x| \leq \sqrt{\ln(m \land n)} \), as inequality (4.51) with \( 1 \leq |x| \leq \sqrt{\ln(m \land n)} \) can be proved in the same way.

Without loss of generality, we may assume that \( m \leq n \). Recall the notations \( Y_{m,n,k}, \tilde{Y}_{m,n,k}, H_{m,n,k} \) and \( D_{m,n,k} \) defined by (4.24), (4.25), (4.26) and (4.27), respectively. Set \( \alpha_m = m^{-1/2} \) and \( k = [m^{1/2}] \). From (4.28), it holds for all \( x \in \mathbb{R} \),

\[
P\left( R_{m,n} \leq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \right)
\leq P\left( Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x \right) + P\left( |D_{m,n,k}| \geq \alpha_m \right). \tag{4.52}
\]

For \( P\left( |D_{m,n,k}| \geq \alpha_m \right) \), by an argument similar to (4.29), we have for all \( 1 \leq |x| \leq \sqrt{\ln m} \),

\[
P\left( |D_{m,n,k}| > \alpha_m \right) \leq C \frac{x^2}{\sqrt{m}} \exp \left\{ - \frac{1}{2} x^2 \right\}. \tag{4.53}
\]

Next, we give an estimation for the first term of bound (4.52). Recall the notations \( G_{m,n,k}(x), v_k(ds, dt) \) defined by (4.30). Recall \( C_{m,n,k}^2 = \text{Var}(Y_{m,n,k}) \). Then we have

\[
P\left( Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x \right)
= \int \int \mathbf{1}_{\{t \leq \alpha_m\}} \left( G_{m,n,k} \left( x - s - t + \alpha_m \right) - G_{m,n,k}(x - s) \right) v_k(ds, dt)
\]

and \( C_{m,n,k} = 1 + O(1/\sqrt{m}) \rightarrow 1 \) as \( m \rightarrow \infty \). Using Cramér’s moderate deviations (for \( |x| \leq m^{1/6} \)) and Bernstein’s inequality (for \( |x| > m^{1/6} \)) for independent random variables, we have the following non-uniform Berry-Esseen’s bound: for all \( x \in \mathbb{R} \),

\[
\left| P\left( \frac{Y_{m,n,k}}{C_{m,n,k}} \leq \frac{x}{C_{m,n,k}} \right) - \Phi\left( \frac{x}{C_{m,n,k}} \right) \right|
\leq C_1 \exp \left\{ - \frac{x^2}{2(1 + \frac{C_{m,n,k}}{\sqrt{m}} |x|)} \right\} \left( 1 + \left( \frac{|x|}{C_{m,n,k}} \right)^2 \right)
\times \left( \sum_{j=k+1}^{m} \mathbb{E} \left| \eta_{m,n,i} \right|^3 \left| C_{m,n,k} \right| + \mathbb{E} \left( \eta_{m,n,n+i} \right)^3 \left| C_{m,n,k} \right|^3 \right)
\leq C_2 \frac{1 + |x|^2}{\sqrt{m}} \exp \left\{ - \frac{x^2}{2(1 + \frac{C_{m,n,k}}{\sqrt{m}} |x|)} \right\}.
\]

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By the last inequality, we deduce that for all $x \in \mathbb{R}$,

$$
|G_{m,n,k}(x) - \Phi(x)|
\leq \left| \Pr \left( \frac{Y_{m,n,k}}{C_{m,n,k}} \leq \frac{x}{C_{m,n,k}} \right) - \Phi \left( \frac{x}{C_{m,n,k}} \right) + \Phi \left( \frac{x}{C_{m,n,k}} \right) - \Phi (x) \right|
\leq C_2 \sqrt{\frac{x^2}{m}} \exp \left\{ -\frac{x^2}{2(1 + \frac{C}{\sqrt{m}}|x|)} \right\} + \exp \left\{ -\frac{x^2}{2} \right\} \left| \frac{x}{C_{m,n,k}} - x \right|
\leq C_2 \sqrt{\frac{x^2}{m}} \exp \left\{ -\frac{x^2}{2(1 + \frac{C}{\sqrt{m}}|x|)} \right\} + C_3 \frac{|x|}{\sqrt{m}} \exp \left\{ -\frac{x^2}{2} \right\}
\leq C_4 \frac{1 + x^2}{\sqrt{m}} \exp \left\{ -\frac{x^2}{2(1 + \frac{C}{\sqrt{m}}|x|)} \right\}.
$$

Therefore, we have for all $x \in \mathbb{R}$,

$$
P\left( Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x \right) \leq J_1 + J_2 + J_3,
$$

where

$$
J_1 = \int \int 1_{\{t \leq \alpha_m\}} |\Phi(x-s-t+\alpha_m) - \Phi(x-s)| v_k(ds,dt),
$$

$$
J_2 = C \int \int 1_{\{t \leq \alpha_m\}} \frac{1 + |x-s|^2}{\sqrt{m}} \exp \left\{ -\frac{(x-s)^2}{2(1 + \frac{C}{\sqrt{m}}|x-s|)} \right\} v_k(ds,dt)
$$

and

$$
J_3 = C \int \int 1_{\{t \leq \alpha_m\}} \frac{1 + |x-s-t|^2}{\sqrt{m}} \exp \left\{ -\frac{(x-s-t)^2}{2(1 + \frac{C}{\sqrt{m}}|x-s-t|)} \right\} v_k(ds,dt).
$$

Denote $\tilde{C}_{m,n,k}^2 = \text{Var}(\tilde{Y}_{m,n,k})$, then it holds $\tilde{C}_{m,n,k}^2 = O(1/\sqrt{m})$ as $m \to \infty$. For the upper bound of $J_1$, by the mean value theorem, we have for all $1 \leq |x| \leq \sqrt{\ln m}$,

$$
1_{\{t \leq \alpha_m\}} |\Phi(x-s-t+\alpha_m) - \Phi(x-s)|
\leq C|\alpha_m - t| \exp \left\{ -\frac{x^2}{2(1 + \frac{C}{\sqrt{m}}|x|)} \right\} + |\alpha_m - t| \Phi(1_{\{|s| \geq 2|x|\tilde{C}_{m,n,k}\}}) + |\alpha_m - t| 1_{\{|t| \geq C_0|x|\tilde{C}_{m,n,k}\}},
$$

which leads to

$$
J_1 \leq J_{11} + J_{12} + J_{13},
$$

where

$$
J_{11} = C \int \int |\alpha_m - t| \exp \left\{ -\frac{x^2}{2(1 + \frac{C}{\sqrt{m}}|x|)} \right\} v_k(ds,dt),
$$

$$
J_{12} = \int \int |\alpha_m - t| 1_{\{|s| \geq 2|x|\tilde{C}_{m,n,k}\}} v_k(ds,dt)
$$

$$
J_{13} = \int \int |\alpha_m - t| 1_{\{|t| \geq C_0|x|\tilde{C}_{m,n,k}\}} v_k(ds,dt)
$$
and
\[ J_{13} = \int \int |\alpha_m - t|1_{\{|t| \geq C_0|\tilde{c}_{m,n,k}|\}} v_k(ds, dt). \]

By Lemma 4.3, it is obvious that for all \(1 \leq |x| \leq \sqrt{\ln m}\),
\[
J_{11} \leq C_1 \left( \alpha_m + \mathbb{E}|H_{m,n,k}| \right) \exp \left\{ - \frac{x^2}{2(1 + \frac{C_0}{\sqrt{m}}|x|)} \right\}
\leq \frac{C_2}{\sqrt{m}} \exp \left\{ - \frac{x^2}{2(1 + \frac{C_0}{\sqrt{m}}|x|)} \right\}.
\]

For \(J_{12}\), we have the following estimation for all \(1 \leq |x| \leq \sqrt{\ln m}\),
\[
J_{12} \leq \alpha_m \mathbb{P}\left( |\tilde{Y}_{m,n,k}| \geq 2|x|\tilde{C}_{m,n,k} \right) + \mathbb{E}|H_{m,n,k}|1_{\{|\tilde{Y}_{m,n,k}| \geq 2|x|\tilde{C}_{m,n,k}\}}.
\]

By Bernstein’s inequality, we deduce that for all \(x \in \mathbb{R}\),
\[
\mathbb{P}\left( |\tilde{Y}_{m,n,k}| \geq 2|x|\tilde{C}_{m,n,k} \right) = \mathbb{P}\left( \frac{\tilde{Y}_{m,n,k}}{\tilde{C}_{m,n,k}} \geq 2|x| \right) + \mathbb{P}\left( \frac{\tilde{Y}_{m,n,k}}{\tilde{C}_{m,n,k}} \leq -2|x| \right)
\leq 2 \exp \left\{ - \frac{(2x)^2}{2(1 + \frac{C_0}{\sqrt{m}}|x|)} \right\}.
\]

By Cauchy-Schwartz’s inequality, we have for all \(x \in \mathbb{R}\),
\[
\mathbb{E}|H_{m,n,k}|1_{\{|\tilde{Y}_{m,n,k}| \geq 2|x|\tilde{C}_{m,n,k}\}} \leq \left( \mathbb{E}|H_{m,n,k}|^2 \right)^{1/2} \mathbb{P}\left( |\tilde{Y}_{m,n,k}| \geq 2|x|\tilde{C}_{m,n,k} \right)^{1/2}
\leq \frac{C}{\sqrt{m}} \left( 2 \exp \left\{ - \frac{(2x)^2}{2(1 + \frac{C_0}{\sqrt{m}}|x|)} \right\} \right)^{1/2}
\leq \frac{C_1}{\sqrt{m}} \exp \left\{ - \frac{x^2}{2(1 + \frac{C_0}{\sqrt{m}}|x|)} \right\}.
\]

Hence, we get for all \(1 \leq |x| \leq \sqrt{\ln m}\),
\[
J_{12} \leq \frac{C}{\sqrt{m}} \exp \left\{ - \frac{x^2}{2(1 + \frac{C_0}{\sqrt{m}}|x|)} \right\}.
\]

For \(J_{13}\), we have for all \(1 \leq |x| \leq \sqrt{\ln m}\),
\[
J_{13} \leq \alpha_m \mathbb{P}\left( |H_{m,n,k}| \geq C_0|x|\tilde{C}_{m,n,k} \right) + \mathbb{E}\left[ |H_{m,n,k}|1_{\{|H_{m,n,k}| \geq C_0|x|\tilde{C}_{m,n,k}\}} \right].
\]

Notice that \(V_{m,n,\rho} \asymp \frac{1}{\sqrt{m}}\) and \(\tilde{C}_{m,n,k} \asymp \frac{1}{m^{1/7}}\). It is easy to see that for all \(1 \leq |x| \leq \sqrt{\ln m}\),
\[
\mathbb{P}\left( |H_{m,n,k}| \geq C_0|x|\tilde{C}_{m,n,k} \right) \leq T_1 + T_2,
\]

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where
\[ T_1 = \mathbb{P}\left( \left| \frac{\ln W_{1,k}}{n V_{m,n,\rho}} \right| \geq \frac{1}{2} C_0 |x| \tilde{C}_{m,n,k} \right) \quad \text{and} \quad T_2 = \mathbb{P}\left( \left| \frac{\ln W_{2,k}}{m V_{m,n,\rho}} \right| \geq \frac{1}{2} C_0 |x| \tilde{C}_{m,n,k} \right). \]

By Lemma 4.6 and Markov’s inequality, we have for all \( 1 \leq |x| \leq \sqrt{\ln m} \),
\[
T_1 \leq \mathbb{P}\left( \frac{\ln W_{1,k}}{n V_{m,n,\rho}} \geq \frac{1}{2} C_0 |x| \tilde{C}_{m,n,k} \right) + \mathbb{P}\left( \frac{\ln W_{1,k}}{n V_{m,n,\rho}} \leq -\frac{1}{2} C_0 |x| \tilde{C}_{m,n,k} \right)
\leq \mathbb{P}\left( W_{1,k} \geq \exp\left\{ \frac{1}{2} C_0 |x| n V_{m,n,\rho} \tilde{C}_{m,n,k} \right\} \right) + \mathbb{P}\left( W_{1,k}^{-1} \geq \exp\left\{ -\frac{1}{2} C_0 |x| n V_{m,n,\rho} \tilde{C}_{m,n,k} \right\} \right)
\leq \mathbb{E}[W_{1,k}] \exp\left\{ -\frac{1}{2} C_0 |x| n V_{m,n,\rho} \tilde{C}_{m,n,k} \right\} + \mathbb{E}\left[ W_{1,k}^{-\alpha} \right] \exp\left\{ -\frac{1}{2} \alpha C_0 |x| n V_{m,n,\rho} \tilde{C}_{m,n,k} \right\}
\leq C \exp\left\{ -\frac{1}{2} x^2 \right\},
\]
with \( C_0 \) large enough. Similarly, we have for all \( 1 \leq |x| \leq \sqrt{\ln m} \),
\[
T_2 \leq C \exp\left\{ -\frac{1}{2} x^2 \right\}.
\]
Hence, we get for all \( 1 \leq |x| \leq \sqrt{\ln m} \),
\[
\mathbb{P}\left( |H_{m,n,k}| \geq C_0 |x| \tilde{C}_{m,n,k} \right) \leq C \exp\left\{ -\frac{1}{2} x^2 \right\}.
\]
Clearly, by Lemma 4.6 and the inequality \( |\ln x|^2 \leq C_\alpha (x + x^{-\alpha}) \) for all \( \alpha, x > 0 \), it holds
\[
\mathbb{E}[H_{m,n,k}]^2 \leq \frac{C_1}{m} \left( \mathbb{E}[W_{1,k} + \mathbb{E}[W_{1,k}^{-\alpha}] + \mathbb{E}[W_{2,k} + \mathbb{E}[W_{2,k}^{-\alpha}]) \leq \frac{C_2}{m}.
\]
By Markov’s inequality, Cauchy-Schwartz’s inequality and Lemma 4.6, we deduce that for all \( 1 \leq
\[ |x| \leq \sqrt{\ln m}, \]

\[
\mathbb{E} \left[ |H_{m,n,k}| \mathbb{1}_{\{|H_{m,n,k}| \geq C_0 |x| \tilde{c}_{m,n,k}\}} \right] \\
\leq \mathbb{E} \left[ |H_{m,n,k}| \mathbb{1}_{\{W_{1,k} \geq \exp \left( \frac{1}{2} C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k}\right)\}} \right] \\
+ \mathbb{E} \left[ |H_{m,n,k}| \mathbb{1}_{\{W_{2,k} \geq \exp \left( \frac{1}{2} C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k}\right)\}} \right] \\
+ \mathbb{E} \left[ |H_{m,n,k}| \mathbb{1}_{\{W_{1,k}^{-1} \geq \exp \left( \frac{1}{2} C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k}\right)\}} \right] \\
+ \mathbb{E} \left[ |H_{m,n,k}| \mathbb{1}_{\{W_{2,k}^{-1} \geq \exp \left( \frac{1}{2} C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k}\right)\}} \right] \\
\leq \exp \left\{ -\frac{1}{4} C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \mathbb{E} \left[ W_{1,k}^{1/2} |H_{m,n,k}| \right] \\
+ \exp \left\{ -\frac{1}{4} C_0 |x| m \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \mathbb{E} \left[ W_{2,k}^{1/2} |H_{m,n,k}| \right] \\
+ \exp \left\{ -\frac{1}{4} \alpha C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \mathbb{E} \left[ W_{1,k}^{-\alpha/2} |H_{m,n,k}| \right] \\
+ \exp \left\{ -\frac{1}{4} \alpha C_0 |x| m \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \mathbb{E} \left[ W_{2,k}^{-\alpha/2} |H_{m,n,k}| \right] \\
\leq \exp \left\{ -\frac{1}{4} C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \left( \mathbb{E} W_{1,k} \right)^{1/2} \left( \mathbb{E} |H_{m,n,k}|^2 \right)^{1/2} \\
+ \exp \left\{ -\frac{1}{4} C_0 |x| m \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \left( \mathbb{E} W_{2,k} \right)^{1/2} \left( \mathbb{E} |H_{m,n,k}|^2 \right)^{1/2} \\
+ \exp \left\{ -\frac{1}{4} \alpha C_0 |x| n \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \left( \mathbb{E} W_{1,k}^{-\alpha} \right)^{1/2} \left( \mathbb{E} |H_{m,n,k}|^2 \right)^{1/2} \\
+ \exp \left\{ -\frac{1}{4} \alpha C_0 |x| m \tilde{v}_{m,n,\rho} \tilde{c}_{m,n,k} \right\} \left( \mathbb{E} W_{2,k}^{-\alpha} \right)^{1/2} \left( \mathbb{E} |H_{m,n,k}|^2 \right)^{1/2} \\
\leq \frac{C}{\sqrt{m}} \exp \left\{ -\frac{1}{2} x^2 \right\},
\]

where the last inequality holds with \( C_0 \) large enough. Hence, we have for all \( 1 \leq |x| \leq \sqrt{\ln m} \),

\[
J_{13} \leq \frac{C}{\sqrt{m}} \exp \left\{ -\frac{1}{2} x^2 \right\}.
\]

Returning to (4.55), we get for all \( 1 \leq |x| \leq \sqrt{\ln m} \),

\[
J_1 \leq \frac{C}{\sqrt{m}} \exp \left\{ -\frac{x^2}{2(1 + \frac{C}{\sqrt{m}} |x|)} \right\}. \quad (4.56)
\]
For $J_2$, by an argument similar to the proof of (4.56), we have for all $1 \leq |x| \leq \sqrt{\ln m}$,
\[
J_2 = C_1 \int_1^1 1_{\{t \leq \alpha_m\}} \frac{1 + |x - s|^2}{\sqrt{m}} \exp \left\{ - \frac{(x-s)^2}{2(1 + \frac{C_2}{\sqrt{m}}|x-s|)} \right\} v_k(ds, dt)
\leq \frac{C_3}{\sqrt{m}} \left( \int_1^1 (1 + x^2) \exp \left\{ - \frac{x^2}{2(1 + \frac{C_2}{\sqrt{m}}|x|)} \right\} v_k(ds) + \int_1^1 \frac{1 + x^2}{\sqrt{m}} v_k(ds) \right)
\leq \frac{C_3}{\sqrt{m}} (1 + x^2) \exp \left\{ - \frac{x^2}{2(1 + \frac{C_2}{\sqrt{m}}|x|)} \right\}.
\tag{4.57}
\]
Similarly, for $J_3$, we have for all $1 \leq |x| \leq \sqrt{\ln m}$,
\[
J_3 = C_1 \int_1^1 1_{\{t \leq \alpha_m\}} \frac{1 + |x - s - t|^2}{\sqrt{m}} \exp \left\{ - \frac{(x-s-t)^2}{2(1 + \frac{C_2}{\sqrt{m}}|x-s-t|)} \right\} v_k(ds, dt)
\leq \frac{C_2}{\sqrt{m}} \left( \int_1^1 (1 + x^2) \exp \left\{ - \frac{x^2}{2(1 + \frac{C_2}{\sqrt{m}}|x|)} \right\} v_k(ds, dt)
\quad + \int_1^1 |s| > |x| \tilde{C}_{m,n,k} v_k(ds, dt) + \int_1^1 |t| > C_0 |x| \tilde{C}_{m,n,k} v_k(ds, dt) \right)
\leq \frac{C_2}{\sqrt{m}} \left( (1 + x^2) \exp \left\{ - \frac{x^2}{2(1 + \frac{C_2}{\sqrt{m}}|x|)} \right\} + \mathbb{P} \left( \frac{\tilde{Y}_{m,n,k}}{\tilde{C}_{m,n,k}} > |x| \right)\right)
\quad + \mathbb{P} \left( |H_{m,n,k}| > C_0 |x| \tilde{C}_{m,n,k} \right)
\leq \frac{C_4}{\sqrt{m}} (1 + x^2) \exp \left\{ - \frac{x^2}{2(1 + \frac{C_2}{\sqrt{m}}|x|)} \right\}.
\tag{4.58}
\]
Applying the inequalities (4.56)-(4.58) to (4.54), we get for all $1 \leq |x| \leq \sqrt{\ln m}$,
\[
\mathbb{P} \left( Y_{m,n,0} + H_{m,n,k} \leq x + \alpha_m, Y_{m,n,0} \geq x \right)
\leq \frac{C_1}{\sqrt{m}} (1 + x^2) \exp \left\{ - \frac{x^2}{2(1 + \frac{C_2}{\sqrt{m}}|x|)} \right\}
\leq \frac{C_1}{\sqrt{m}} (1 + x^2)(1 + \frac{C_2}{\sqrt{m}}|x|^3) \exp \left\{ - \frac{x^2}{2} \right\}
\leq \frac{C_3}{\sqrt{m}} (1 + x^2) \exp \left\{ - \frac{x^2}{2} \right\}.
\tag{4.59}
\]
Combining (4.52), (4.53) and (4.59) together, we get (4.50) for all $1 \leq |x| \leq \sqrt{\ln m}$. This completes the proof of Lemma 4.7. \qed
4.5. Proof of Theorem 2.3

We give a proof of Theorem 2.3 for the case of $\frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)}$, $x \geq 0$. Thanks to the symmetry between $m$ and $n$, the case of $\frac{\mathbb{P}(-R_{m,n} \geq x)}{\Phi(-x)}$ can be proved in the similar way. To prove Theorem 2.3, we start with the proofs of Lemmas 4.8 and 4.9, and conclude Theorem 2.3 by combining Lemmas 4.8 and 4.9 together. To avoid trivial case, we assume that $m \wedge n \geq 2$.

The following lemma gives the upper bound in Theorem 2.3.

**Lemma 4.8.** Assume that the conditions A3 and A4 are satisfied. Then it holds for all $0 \leq x \leq c \sqrt{m \wedge n}$,

$$\ln \frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)} \leq C \frac{1 + x^3}{\sqrt{m \wedge n}}. \quad (4.60)$$

**Proof.** First, we consider the case $0 \leq x \leq \sqrt{\ln(m \wedge n)}$. Notice that

$$\mathbb{P}(R_{m,n} \geq x) = \mathbb{P}(R_{m,n} \geq x, \sum_{i=1}^{n+m} \eta_{m,n,i} \geq x) + \mathbb{P}(R_{m,n} \geq x, \sum_{i=1}^{n+m} \eta_{m,n,i} < x)$$

$$\leq \mathbb{P}(\sum_{i=1}^{n+m} \eta_{m,n,i} \geq x) + \mathbb{P}(R_{m,n} \geq x, \sum_{i=1}^{n+m} \eta_{m,n,i} < x).$$

Applying Cramér’s moderate deviations for independent random variables (cf. inequality (1) in [11]) to the last inequality, we deduce that for all $0 \leq x \leq c \sqrt{m \wedge n}$,

$$\mathbb{P}(R_{m,n} \geq x) \leq (1 - \Phi(x)) \left(1 + C \frac{1 + x^3}{\sqrt{m \wedge n}}\right) + \mathbb{P}(R_{m,n} \geq x, \sum_{i=1}^{n+m} \eta_{m,n,i} < x).$$

By Lemma 4.7 and (4.23), it follows that for all $0 \leq x \leq \sqrt{\ln(m \wedge n)},$

$$\mathbb{P}(R_{m,n} \geq x) \leq (1 - \Phi(x)) \left(1 + C \frac{1 + x^3}{\sqrt{m \wedge n}}\right).$$

By the last inequality and the inequality $\ln(1+x) \leq x, x \geq 0$, we get (4.60) for all $0 \leq x \leq \sqrt{\ln(m \wedge n)}$.

Next, we consider the case $\sqrt{\ln(m \wedge n)} \leq x \leq c \sqrt{m \wedge n}$. Clearly, it holds for all $x \in \mathbb{R}$,

$$\mathbb{P}(R_{m,n} \geq x) = \mathbb{P}\left(\sum_{i=1}^{n+m} \eta_{m,n,i} + \frac{\ln W_{1,n}}{n V_{m,n,\rho}} - \frac{\ln W_{2,m}}{m V_{m,n,\rho}} \geq x\right)$$

$$\leq I_1 + I_2 + I_3, \quad (4.61)$$

where

$$I_1 = \mathbb{P}\left(\sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \left(1 - \frac{1}{n} \frac{1}{m \alpha} x \right)^m\right),$$

$$I_2 = \mathbb{P}\left(\frac{\ln W_{1,n}}{n V_{m,n,\rho}} \geq \frac{x^2}{n V_{m,n,\rho}}\right) \quad \text{and} \quad I_3 = \mathbb{P}\left(- \frac{\ln W_{2,m}}{m V_{m,n,\rho}} \geq \frac{x^2}{m \alpha V_{m,n,\rho}}\right),$$

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with \( \alpha \) given by Lemma 4.6. Next, we give some estimations for \( I_1, I_2 \) and \( I_3 \). By condition A3, \( \sum_{i=1}^{n+m} \eta_{m,n,i} \) is a sum of independent random variables with finite moment generating functions. By Cramér’s moderate deviations for independent random variables (cf. [11]), we obtain for all \( 1 \leq x \leq c \sqrt{m \wedge n} \),

\[
I_1 \leq \left( 1 - \Phi \left( x \left( 1 - \frac{x}{V_{m,n,\rho}} \right) \right) \right) \exp \left\{ \frac{C}{\sqrt{m+n}} \left( x \left( 1 - \frac{m+n}{1/m} \right) \right)^3 \right\} \\
\leq \left( 1 - \Phi \left( x \left( 1 - \frac{m+n}{1/m} \right) \right) \right) \exp \left\{ C \frac{x^3}{\sqrt{m \wedge n}} \right\}.
\]

Using (4.23), we deduce that for all \( x \geq 1 \) and \( \varepsilon_n \in (0, \frac{1}{2}] \),

\[
1 - \Phi \left( x (1 - \varepsilon_n) \right) \leq 1 + \frac{\int_{x(1-x)}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt}{1 - \Phi (x)} \\
\leq 1 + \frac{1}{\sqrt{2\pi}} e^{-x^2(1-\varepsilon_n)^2/2} e^{x \varepsilon_n} \\
\leq 1 + C x^2 \varepsilon_n \exp \left\{ C x^2 \varepsilon_n \right\} \\
\leq \exp \left\{ 2 C x^2 \varepsilon_n \right\}. \tag{4.62}
\]

Hence, by the fact \( V_{m,n,\rho} \asymp \frac{1}{\sqrt{m}} \), it holds for all \( 1 \leq x \leq c \sqrt{m \wedge n} \),

\[
I_1 \leq \left( 1 - \Phi (x) \right) \exp \left\{ C \frac{x^3}{\sqrt{m \wedge n}} \right\}. \tag{4.63}
\]

By Markov’s inequality, it is easy to see that for all \( x \geq \sqrt{\ln(m \wedge n)} \),

\[
I_2 = P \left( W_{1,n} \geq x^2 \right) \leq \exp \left\{ - x^2 \right\} \mathbb{E} W_{1,n} = \exp \left\{ - x^2 \right\} \\
\leq C \frac{1+x}{\sqrt{m \wedge n}} \left( 1 - \Phi (x) \right) \tag{4.64}
\]

and

\[
I_3 = P \left( W_{2,m} \leq - \frac{\alpha x^2}{\sqrt{m \wedge n}} \right) \leq \exp \left\{ - x^2 \right\} \mathbb{E} W_{2,m}^{\alpha} = \exp \left\{ - x^2 \right\} \\
\leq C \frac{1+x}{\sqrt{m \wedge n}} \left( 1 - \Phi (x) \right) \tag{4.65}
\]

Combining (4.63)-(4.65) together, we obtain for all \( \sqrt{\ln(m \wedge n)} \leq x \leq c \sqrt{m \wedge n} \),

\[
P \left( R_{m,n} \geq x \right) \leq \left( 1 - \Phi (x) \right) \exp \left\{ C_1 \frac{x^3}{\sqrt{m \wedge n}} \right\} + C_2 \frac{1+x}{\sqrt{m \wedge n}} \left( 1 - \Phi (x) \right) \\
\leq \left( 1 - \Phi (x) \right) \exp \left\{ C_3 \frac{x^3}{\sqrt{m \wedge n}} \right\},
\]

which implies the desired inequality for all \( \sqrt{\ln(m \wedge n)} \leq x \leq c \sqrt{m \wedge n} \). \( \square \)

The following lemma gives the lower bound in Theorem 2.3.

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Lemma 4.9. Assume that the conditions $A3$ and $A4$ are satisfied. Then it holds for all $0 \leq x \leq c \sqrt{m \wedge n}$,
\[
\ln \frac{\mathbb{P}(R_{m,n} \geq x)}{1 - \Phi(x)} \geq -C \frac{1 + x^3}{\sqrt{m \wedge n}}.
\] (4.66)

Proof. The proof for the lower bound is similar to that of upper bound. For instance, to prove (4.66) for all $\sqrt{\ln(m \wedge n)} \leq x \leq c \sqrt{m \wedge n}$, we only need to notice that
\[
\mathbb{P}(R_{m,n} \geq x) = \mathbb{P}\left(\sum_{i=1}^{n+m} \eta_{m,n,i} + \frac{\ln W_{1,n}}{n V_{m,n,\rho}} - \frac{\ln W_{2,m}}{m V_{m,n,\rho}} \geq x\right) \geq I_4 - I_5 - I_6,
\]
where
\[
I_4 = \mathbb{P}\left(\sum_{i=1}^{n+m} \eta_{m,n,i} \geq x \left(1 + \frac{1}{n\alpha} + \frac{1}{m}x\right)\right),
\]
\[
I_5 = \mathbb{P}\left(-\frac{\ln W_{1,n}}{n V_{m,n,\rho}} \geq \frac{x^2}{n\alpha V_{m,n,\rho}}\right) \quad \text{and} \quad I_6 = \mathbb{P}\left(\frac{\ln W_{2,m}}{m V_{m,n,\rho}} \geq \frac{x^2}{m V_{m,n,\rho}}\right),
\]
with $\alpha$ given by Lemma 4.6. For the remaining of the proof, we can use the argument similar to the proof of Lemma 4.8. \qed

Conflict of interest

The authors declared that they have no conflict of interest.

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References

[1] Afanasyev, V.I., Böinghoff, C., Kersting, G. and Vatutin, V.A. (2014). Conditional limit theorems for intermediately subcritical branching processes in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.* 50(2): 602–627.

[2] Athreya, K.B. and Karlin, S. (1971). On branching processes with random environments: I: Extinction probabilities. *Ann. Math. Stat.* 42(5): 1499–1520.

[3] Athreya, K.B. and Karlin, S. (1971). Branching processes with random environments: II: Limit theorems. *Ann. Math. Stat.* 42(6), 1843–1858.
[4] Bansaye, V. and Böinghoff, C. (2011). Upper large deviations for branching processes in random environment with heavy tails. *Electron. J. Probab.* **16**(69): 1900–1933.

[5] Bansaye, V. and Vatutin, V. (2017). On the survival probability for a class of subcritical branching processes in random environment. *Bernoulli* **23**(1): 58–88.

[6] Bikelis, A. (1966). On estimates of the remainder term in the central limit theorem. *Lith. Math. J.* **6**(3): 323–346.

[7] Böinghoff, C. (2014). Limit theorems for strongly and intermediately supercritical branching processes in random environment with linear fractional offspring distributions. *Stochastic Process. Appl.* **124**(11): 3553–3577.

[8] Böinghoff, C. and Kersting, C. (2010). Upper large deviations of branching processes in a random environment for offspring distributions with geometrically bounded tails. *Stochastic Process. Appl.* **120**(10): 2064–2077.

[9] Chang, J., Shao Q.M. and Zhou W.X. (2016). Cramér-type moderate deviations for Studentized two-sample *U*-statistics with applications. *Ann. Stat.* **44**(5): 1931–1956.

[10] Chen, L.H.Y. and Shao, Q.M. (2001). A non-uniform Berry-Esseen bound via Stein’s method. *Probab. Theory Relat. Fields* **120**(2): 236-254.

[11] Fan, X., Grama, I. and Liu, Q. (2013). Cramér large deviation expansions for martingales under Bernstein’s condition. *Stochastic Process. Appl.* **123**: 3919–3942.

[12] Fan, X., Grama, I. and Liu, Q. (2017). Deviation inequalities for martingales with applications *J. Math. Anal. Appl.* **448**(1): 538–566.

[13] Fan, X., Grama, I., Liu, Q., Shao, Q.M. (2019). Self-normalized Cramér type moderate deviations for martingales. *Bernoulli* **25**(4A): 2793–2823.

[14] Fan, X., Hu, H. and Liu, Q. (2020). Uniform Cramér moderate deviations and Berry-Esseen bounds for a supercritical branching process in a random environment. *Front. Math. China* **15**(5): 891–914.

[15] Gao, Z.Q., Liu, Q., Wang, H. (2014). Central limit theorems for a branching random walk with a random environment in time. *Acta Math. Sci.* **34**(2): 501–512.

[16] Gao, Z.Q. (2021). Exact convergence rate in the central limit theorem for a branching process in a random environment. *Stat. Probab. Letters* **178**: 109194.

[17] Grama, I., Liu, Q. and Miqueu, E. (2017). Berry-Esseen bound and Cramér large deviation expansion for a supercritical branching process in a random environment. *Stochastic Process. Appl.* **127**(4): 1255–1281.

[18] Grama, I., Liu, Q. and Miqueu, E. (2021). Asymptotic of the distribution and harmonic moments for a supercritical branching process in a random environment. (hal-03416307).
[19] Huang, C. and Liu, Q. (2012). Moments, moderate and large deviations for a branching process in a random environment. *Stochastic Process. Appl.* **122**(2): 522–545.

[20] Huang, C., Wang, C. and Wang, X. (2022). Moments and large deviations for supercritical branching processes with immigration in random environments. *Acta Math. Sci.* **42**(1): 49-72.

[21] Hong, W., Zhang, X. (2019). Asymptotic behaviour of heavy-tailed branching processes in random environments. *Electr. J. Probab.* **24**: 1–17.

[22] Li, Y., Huang, X., Peng, Z. (2022). Central limit theorem and convergence rates for a supercritical branching process with immigration in a random environment. *Acta Math. Sci.* **42.3**: 957–974.

[23] Liu, Q. (1999). Asymptotic properties of supercritical age-dependent branching processes and homogeneous branching random walks. *Stochastic Process. Appl.* **82**(1): 61–87.

[24] Nakashima, M. (2013). Lower deviations of branching processes in random environment with geometrical offspring distributions. *Stochastic Process. Appl.* **123**(9): 3560–3587.

[25] Röllin, A. (2018). On quantitative bounds in the mean martingale central limit theorem. *Statist. Probab. Lett.* **138**: 171–176.

[26] Smith, W.L. and Wilkinson, W.E. (1969). On branching processes in random environments. *Ann. Math. Stat.* **40**(3): 814–827.

[27] Tanny, D. (1988). A necessary and sufficient condition for a branching process in a random environment to grow like the product of its means. *Stochastic Process. Appl.* **28**(1): 123–139.

[28] Vatutin, V. A. A refinement of limit theorems for the critical branching processes in random environment, in: Workshop on Branching Processes and their Applications. Lect Notes Stat Proc 197 (pp. 3-19), Springer, Berlin, 2010.

[29] Vatutin, V. and Zheng, X. (2012). Subcritical branching processes in random environment without Cramer condition. *Stochastic Process. Appl.* **122**: 2594–2609.

[30] Wang, Y. and Liu, Q. (2017). Limit theorems for a supercritical branching process with immigration in a random environment. *Sci. China Math.* **60**(12): 2481–2502.