ON SEMISIMPLECTICITY OF QUANTUM COHOMOLOGY OF $\mathbb{P}^1$-ORBIFOLDS

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ABSTRACT. For a $\mathbb{P}^1$-orbifold $\mathcal{C}$, we prove that its big quantum cohomology is generically semisimple. As a corollary, we verify a conjecture of Dubrovin for orbicurves. We also show that the small quantum cohomology of $\mathcal{C}$ is generically semisimple iff $\mathcal{C}$ is Fano, i.e. it has positive orbifold Euler characteristic.

Keywords: quantum cohomology, orbicurve, Dubrovin’s conjecture.
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1. INTRODUCTION

Quantum cohomology stems from genus-zero Gromov-Witten theory, which concerns virtual counts of rational curves in target manifolds or orbifolds. One can naively view the quantum cohomology ring as a deformation of the ordinary cohomology ring. A fundamental problem in Gromov-Witten theory is to understand the topology and geometry of target spaces hidden behind the algebraic structure of their quantum cohomology.

Unlike the ordinary cohomology, the quantum cohomology can be semisimple for some targets, and deep structural results for semisimple Gromov-Witten theories are known (e.g. Givental-Teleman’s reconstruction theorem [15, 29]). It is important to understand the geometry of such semisimple target spaces. One of the most important conjectures in this direction was proposed by Dubrovin [13] in his ICM talk in 1998 (later made more precise in [5, 19]):

Conjecture 1.1. For a smooth projective variety $X$, the followings are equivalent:

1. The (even parity) big quantum cohomology $\text{QH}(X)$ is generically semisimple.
2. The bounded derived category of coherent sheaves $\text{D}^b(X)$ admits a full exceptional collection.

In the last two decades, only a few examples of smooth projective varieties with semisimple quantum cohomology are known, since it is difficult to check the semisimplicity. Such examples include projective toric manifolds [20], certain rational homogeneous spaces and hypersurfaces inside them [9, 10, 26], rational surfaces [5, 6], certain Fano threefolds [8], blow-up of $\mathbb{P}^3$ along a smooth rational curve [23], and blow-ups of such varieties at points [5]. To the knowledge of the author, all known semisimple examples admit full exceptional collections.

It is natural to generalize Dubrovin’s conjecture to orbifolds. In this article, we will prove this conjecture for orbi-curves. An orbi-curve $\mathcal{C}$ is a complex orbifold with trivial generic stabilizer, whose underlying space $|\mathcal{C}|$ is a compact Riemann surface (in the literature, an orbi-curve is also called an orbifold Riemann surface). If $|\mathcal{C}| \cong \mathbb{P}^1$, then we say that $\mathcal{C}$ is a $\mathbb{P}^1$-orbifold.

It was shown by Geigle-Lenzing in the 80’s that $\mathbb{P}^1$-orbifolds admit full exceptional collections [16]. This inspires us to prove the following proposition.

Proposition 1.2. Let $\mathcal{C}$ be a $\mathbb{P}^1$-orbifold. Then $\text{QH}(\mathcal{C})$ is generically semisimple.
As a corollary, we verify Dubrovin’s conjecture for orbi-curves.

**Corollary 1.3.** Let $\mathcal{C}$ be an orbi-curve. Then $QH(\mathcal{C})$ is generically semisimple iff $D^b(\mathcal{C})$ admits a full exceptional collection.

It was observed that, to formulate Dubrovin’s conjecture for smooth projective varieties, we need $QH(X)$ instead of $qH(X)$, since there are smooth projective varieties $X$ such that $QH(X)$ is generically semisimple while $qH(X)$ is not. Here $qH(X)$ is the (even parity) small quantum cohomology of $X$. The first known example of this kind is $IG(2, 6)$ [17], and up until now, only $IG(2, 2n)(n \geq 3)$ and $F_4/P_4$ are proved to have this pattern (see Theorem 4 in [26]). Note that all these examples have dimensions at least seven. In the category of orbifolds, our second main result shows that such phenomena appear in dimension one, and non-Fano $\mathbb{P}^1$-orbifolds give a new class of examples.

**Proposition 1.4.** Let $\mathcal{C}$ be an orbi-curve. Then $qH(\mathcal{C})$ is generically semisimple iff $\mathcal{C}$ is Fano, i.e. it has positive orbifold Euler characteristic.

Recall that the orbifold Euler characteristic of an orbi-curve $\mathcal{C}$ is

$$
\chi_{orb}(\mathcal{C}) = \chi_{orb}(\mathcal{C}) - \sum_{p \mid \mathcal{C}} (1 - \frac{1}{a_p}),
$$

where $a_p$ is the order of $p$. Note that $a_p$ is larger than 1 for only finitely many $p$. We say that $\mathcal{C}$ is Fano if $\chi_{orb}(\mathcal{C}) > 0$, $\mathcal{C}$ is Calabi-Yau if $\chi_{orb}(\mathcal{C}) = 0$, and $\mathcal{C}$ is of general type if $\chi_{orb}(\mathcal{C}) < 0$. One can check that $\mathcal{C}$ is Fano iff it is one of the followings:

$$
\mathbb{P}^1_{a_1,a_2}(a_1, a_2 \geq 1), \mathbb{P}^1_{2,2,2}(a \geq 2), \mathbb{P}^1_{2,3,4}(a = 3, 4, 5),
$$

and $\mathcal{C}$ is Calabi-Yau iff it is one of the followings:

- elliptic curves, $\mathbb{P}^1_{1,1,2,2,2}, \mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{2,4,4}, \mathbb{P}^1_{2,3,6}$.

Here for a Riemann surface $C$ and a tuple of positive integers $a = (a_1, \cdots, a_r)$, we use $C_a$ to denote an orbi-curve with underlying space $C$ and $r$ distinct (possibly trivial) orbifold points with local groups $\mu_{a_1}, \cdots, \mu_{a_r}$. It is well-known that every orbi-curve has this form.

To prove Proposition 1.2, our strategy is to show the invertibility of $e_q$, the quantum Euler class introduced by Abrams in [1], since the semisimplicity of the big quantum cohomology is equivalent to the invertibility of $e_q$ (Theorem 3.4 in [1]). For a $\mathbb{P}^1$-orbifold, we will show that the degree-zero part of $\det(e_q \star)$ vanishes, but the degree-one part is non-vanishing. Besides the dimension axiom and WDVV, a key ingredient in the computation of $\det(e_q \star)$ is a decomposition result of the degree-zero and degree-one parts of the genus-zero (primary) potential of orbi-curves (Proposition 3.1). This decomposition comes from the computation of the genus-zero potential by Rossi [27], who used Symplectic Field Theory (SFT) technique [14] to express the potential in terms of connected Hurwitz numbers and SFT invariants of orbifold caps. For Proposition 1.4, we use the dimension axiom to prove the “only if” part, and to prove the “if” part, we use the explicit presentation of the small quantum cohomology of Fano orbi-curves, which was obtained or implicitly known in [25, 27, 21].

We remark that the semisimplicity of $QH(\mathcal{C})$ was known only for $\mathcal{C} = \mathbb{P}^1_{a_1,a_2,a_3}$ and $\mathbb{P}^1_{2,2,2,2}$, and the semisimplicity of $qH(\mathcal{C})$ was known only for $\mathcal{C} = \mathbb{P}^1_{a_1,a_2}$ [25, 27, 24, 28]. The existing methods in the literature do not seem to work for general cases.

In this article, we only consider orbi-curves which are effective in the sense that the generic stabilizer is trivial. For an ineffective orbi-curve $\mathcal{C}$, we conjecture that its big quantum cohomology is generically semisimple iff its underlying space is $\mathbb{P}^1$, and its small
quantum cohomology is generically semisimple iff its rigidification [3, 7] is an effective Fano orbicurve. We hope to study this in the future.

The rest of the article is organized as follows. In Section 2, we briefly review some basic materials on quantum cohomology of orbicurves, and give several different but equivalent characterization of semisimplicity. In Section 3, we prove Proposition 1.2 and Corollary 1.3. In Section 4, we prove Proposition 1.4.

2. Preliminaries

In this section, we briefly review some basic materials on quantum cohomology of orbicurves and fix notations used throughout the rest of the article. We also give several different but equivalent characterization of semisimplicity of quantum cohomology.

2.1. Quantum cohomology of orbicurves. In this subsection, we assume the readers have some familiarity with orbifold quantum cohomology, and we refer interested readers to [4, 11, 12] for details.

We have the following decomposition of the inertia orbifold of $C_a$ into disjoint union of connected components:

$$\text{IC}_a = C_a \sqcup \bigoplus_{\alpha=1}^{r} B_{\mu_a}(i).$$

Here $B_{\mu_a}(i) \cong B_{\mu_a}$, which is the classifying stack of the group of $a_{\alpha}$-th roots of units. Then the (even parity) orbifold cohomology group of $C_a$ is

$$H_{\text{orb}}^{\text{even}}(C_a) = H_{\text{orb}}^{\text{even}}(\text{IC}_a) = H^0(C_a) \oplus H^2(C_a) \oplus \bigoplus_{\alpha=1}^{r} \bigoplus_{i=1}^{a_{\alpha}-1} H^0(B_{\mu_a}(i)).$$

Here “even” means we only consider classes of even topological degree.

Fix an index set

$$S = \{(0,0), (0,1)\} \sqcup \mathcal{T}$$

with

$$\mathcal{T} = \bigcup_{a=1}^{r} \{(\alpha, 1), \ldots, (\alpha, a_{\alpha} - 1)\},$$

and set

$$\phi_{00} := 1 \in H^0(C_a),$$
$$\phi_{01} := [\text{point}] \in H^2(C_a),$$
$$\phi_{ai} := 1 \in H^0(B_{\mu_a}(i)).$$

Then $B = \{\phi_i\}_{i \in S}$ is a basis of $H_{\text{orb}}^{\text{even}}(C_a)$, which is homogeneous with respect to the orbifold degree. Here the orbifold degrees of $\phi_{00}$ and $\phi_{01}$ are their topological degrees, and the orbifold degree of $\phi_{ai}$ is $\frac{a_{ai}}{a_{\alpha}}$.

In terms of classes in $B$, the orbifold Poincaré pairing of $C_a$ is given by

$$\langle \phi_{00}, \phi_{01} \rangle_{\text{orb}}^{C_a} = 1, \quad \langle \phi_{ai}, \phi_{a_{ai}^{-1}} \rangle_{\text{orb}}^{C_a} = \frac{1}{a_{ai}},$$

and $0$ otherwise.

Let $g_{x,x'} = \langle \phi_x, \phi_{x'} \rangle_{\text{orb}}^{C_a}$. Then the matrix $(g_{x,x'})$ is nonsingular, and we let $(g^{x'x}) = (g_{x,x'})^{-1}$. Let $(\phi^i)_{i \in S}$ be the dual basis of $B$ with respect to the orbifold Poincaré pairing. Then

$$\phi^{00} = \phi_{01}, \quad \phi^{01} = \phi_{00}, \quad \phi^{ai} = a_{ai} \phi_{a_{ai}^{-1}}.$$
The Chen-Ruan product \( \cup_{CR} \) on \( H^{even}_{orb}(C_a) \) satisfies

\[
\phi_{a_k} \cup_{CR} \cdots \cup_{CR} \phi_{a_k} = \begin{cases} 
\phi_{a_k}, & 1 \leq k \leq a_a - 1, \\
\frac{1}{a_a} \phi_{01}, & k = a_a, \\
0, & k \geq a_a + 1,
\end{cases}
\]

and

\[
\phi_{a_1} \cup_{CR} \phi_{a_2} = 0 (a_1 \neq a_2).
\]

We remark that \( \cup_{CR} \) also respects the orbifold degree.

The genus-zero potential of \( C_a \) is a formal function of \( t = \sum t^a \phi_a \in H^*_{orb}(C_a) \) given by

\[
F(t) = \sum_{d=0}^{\infty} \sum_{m=0}^{\infty} \left( \sum_{t \in S} t^a \phi_a \right) \frac{Q^d}{m!} = \sum_{d=0}^{\infty} \sum_{m=0}^{\infty} \langle \bigotimes_{t \in S} \phi_{a_0}^{m_{a_0}} C^*_{m_{a_0},d} \rangle \prod_{t \in S} (t^a)^m \chi^d,
\]

where \( \langle \bigotimes_{t \in S} \phi_{a_0}^{m_{a_0}} C^* \rangle \) is a Gromov-Witten invariant of \( C_a \) of genus-zero, degree-\( d \).

Write

\[
t = t^{00} + t^{01} + t^a = \sum_{a \in S} t^a \phi_a,
\]

Then from properties of Gromov-Witten invariants (dimension axiom, fundamental class axiom, divisor axiom), the potential has the following form:

\[
F = \frac{1}{2} (t^{00})^2 + \sum_{d=0}^{\infty} \frac{1}{2a_a} t^{01} t^{a_a-1} + A^a + \sum_{d=1}^{\infty} B^a_d(Qe^{\mu^d})^d,
\]

where

\[
A^a = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{t \in S} t^a \phi_a \right)^{m_{a_0},0} \in \mathbb{Q}[t^t | t \in T],
\]

\[
B^a_d = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{t \in S} t^a \phi_a \right)^{m_{a_0},d} \in \mathbb{Q}[t^t | t \in T].
\]

Set

\[
\deg t^{00} = 1, \quad \deg t^{01} = 0, \quad \deg t^{a_a-1} = 1 - \frac{i}{a_a}, \quad \deg Qe^{\mu^d} = \chi_{orb}(C_a).
\]

Then we can use the dimension axiom to show that \( F, A^a, B^a_d \) are weighted homogeneous with degree

\[
\deg F = \deg A^a = 2, \quad \deg B^a_d = 2 - d \cdot \chi_{orb}(C_a).
\]

The big quantum product is given by

\[
\phi_{a_1} \ast \phi_{a_2} = \sum_{t \in S} F_{a_1,a_2,t}(t) \phi^t,
\]

with coefficients in \( \mathbb{C}[t^t | t \in T][[Qe^{\mu^d}]] \). So the big quantum cohomology of \( C_a \)

\[
QH(C_a) = H^{even}_{orb}(C_a) \otimes_{\mathbb{C}} \mathbb{C}[t^t | t \in T][[Qe^{\mu^d}]].
\]
The big quantum product is clearly commutative, but it is a highly nontrivial fact that it is associative, which is due to the famous WDVV equations satisfied by the genus-zero potential. For $s_1, s_2, s_3, s_4 \in S$, the WDVV of type $(s_1, s_2; s_3, s_4)$ reads

$$
\sum_{s', s'' \in S} F_{s_1, s_2; s', s''} g^{s', s''} F_{s', s_3, s_4} = \sum_{s', s'' \in S} F_{s_1, s_3; s', s''} g^{s', s''} F_{s', s_2, s_4}.
$$

The small quantum product is clearly commutative, but it is a highly nontrivial fact that it is associative, which is due to the famous WDVV equations satisfied by the genus-zero potential.

Let $A$ be a $k$-algebra (not necessarily finite dimensional) which is also an integral domain, and let $B$ be an $A$-algebra which is freely finitely generated as an $A$-module. We say that $B$ is (generically) semisimple over $A$ if there exists a non-empty open subset $U$ of $\text{Spec}(A)$ such that, for every $\mathfrak{p} \in U$, $(\text{Spec}(B))_{\mathfrak{p}}$ is reduced, i.e., $B \otimes_A k(\mathfrak{p})$ is semisimple over $k(\mathfrak{p})$.

The following lemma is well-known.

**Lemma 2.1.** The followings are equivalent.

1. $B$ is generically semisimple over $A$.
2. There exists $\mathfrak{p} \in \text{Spec}(A)$ such that $(\text{Spec}(B))_{\mathfrak{p}}$ is semisimple over $k(\mathfrak{p})$.
3. $(\text{Spec}(B))_{\eta}$ is semisimple over $k(\eta)$, where $\eta = (0) \in \text{Spec}(A)$ is the generic point.

A direct corollary of the above lemma is the following.

**Corollary 2.2.** $B$ is generically semisimple over $A$ iff $B$ contains no nilpotent elements.

For quantum cohomology, we choose $k = \mathbb{C}, A_Q = \mathbb{C}[[t^1, \ldots, t^n]][Qe^{s_1}], B_Q = QH(C_A), A_\eta = \mathbb{C}[[Q]]$ and $B_\eta = qH(C_A)$. Then we have the following Cartesian diagram:

$$
\begin{array}{ccc}
\text{Spec}(B_\eta) & \longrightarrow & \text{Spec}(B_Q) \\
\downarrow & & \downarrow \\
\text{Spec}(A_\eta) & \longrightarrow & \text{Spec}(A_Q),
\end{array}
$$

where the base morphism is the inclusion given by modding out $t$ from $A_Q$. So the semisimplicity of $qH(C_A)$ implies that of $QH(C_A)$, but the converse is not true (Proposition 1.4).

We will use Lemma 2.1 and Corollary 2.2 to deal with the semisimplicity of the small quantum cohomology. For the big case, we need another ingredient. As in (3) of Lemma 2.1, let $\eta$ be the generic point of $A_Q$. Then $\tilde{B}_Q = B_Q \otimes_{A_Q} k(\eta)$ is a finite-dimensional
Frobenius algebra over $k(\eta)$, and it has a distinguished element called quantum Euler class, introduced by Abrams [1]:

$$e_q = \sum_{i \in \mathbb{S}} \phi_i \phi' \in B_q \subset \tilde{B}_q.$$ (6)

**Lemma 2.3.** $\tilde{B}_q$ is semisimple over $k(\eta)$ iff $\det(e_q \bullet i) \neq 0$.

**Proof.** This is a special case of Theorem 3.4 in [1].

We will use Lemma 2.1 and 2.3 to deal with the semisimplicity of the big quantum cohomology.

3. **Big Quantum Cohomology**

In this section, we prove Proposition 1.2 and Corollary 1.3.

3.1. **Genus-zero potential.** Let $a = (a_1, \cdots, a_t)$ be a tuple of positive integers, with $r \geq 1$ and each $a_i \geq 2$. From Lemma 2.1 and 2.3, to prove the semisimplicity of $QH(F^3_a)$, it suffices to show that $\det(e_q \bullet i) \neq 0$. Note that $\det(e_q \bullet i) \in \mathbb{C}[t^{r_i}][[Q_{e^{\mu^i}}]]$, and we will show that the coefficient of $\det(e_q \bullet i)$ at $(Q_{e^{\mu^i}})0$ is zero, but the coefficient at $(Q_{e^{\mu^i}})^1$ is nonvanishing. To this end, we need to understand the structure of the degree-zero and degree-one parts of the genus-zero potential of $\mathbb{P}^1_{a}$.

Recall that the genus-zero potential for $\mathbb{P}^1_{a}$ is

$$F = \frac{1}{2} (t^0)^2 t^0 + \sum_{a=1}^{r} \sum_{i=1}^{d_{a_i}} \frac{t^{d_{a_{i}}-i} t^0 P_{a_{i}}} {2a_i} + A^a + \sum_{d=1}^{\infty} B_d^a(Qe^{\mu^i})^d.$$ (7)

Rossi [27] used the Symplectic Field Theory (SFT) technique [14] to express the potential in terms of SFT invariants of orbifold caps and connected Hurwitz numbers. The result is (see formula (2) in [27])

$$\sum_{a=1}^{r} \sum_{i=1}^{d_{a_i}} \frac{t^{d_{a_{i}}-i} t^0 P_{a_{i}}} {2a_i} + A^a(t^0) = \sum_{a=1}^{r} F_{a_{i},0}(t^0, t^{\mu^i}, \cdots, t^{\mu_{a_{i}}-1}),$$ (8)

and

$$B_d^a(t^0) = \sum_{[\mu^1, \cdots, \mu^d]} H^0_{d,a}([\mu^1, \cdots, \mu^d]) \prod_{a=1}^{d} (\sum_{w=1}^{a_{i}} F_{a_{i},w}(t^{\mu^i}, \cdots, t^{\mu_{a_{i}}-1})w^d).$$ (9)

Here each $\mu^a = (1^w 2^{w_2} \cdots)$ is a partition of $d$ with $m^a_w = 0$ for $w > a$, and $H^0_{d,a}([\mu^1, \cdots, \mu^d])$ is the Hurwitz number of genus-zero, degree-$d$ connected coverings over $\mathbb{P}^1$ with ramification profile $\mu^1, \cdots, \mu^d$. Moreover, $F_{a_{i},w}(0 \leq w \leq a)$ come from the SFT potential of the orbifold cap $\mathbb{C}/\mu_a$:

$$F_{a} = \frac{1}{r} (F_{a_{i},0} + \sum_{w=1}^{a} F_{a_{i},w} \frac{p_w}{w}),$$

with

$$F_{a_{i},w} = \begin{cases} \sum_{j_{1},j_{2},\cdots,j_{a_{i}} \in \mathbb{Z}_{\geq 0}} A_{j_{1},j_{2},\cdots,j_{a_{i}}}^{a} \prod_{i=0}^{a_i-1} (t^{j_{i}})^{j_{i}}, & w = 0, \\
\sum_{j_{1},j_{2},\cdots,j_{a_{i}} \in \mathbb{Z}_{\geq 0}} B_{j_{1},j_{2},\cdots,j_{a_{i}}}^{a} \prod_{i=0}^{a_i-1} (t^{j_{i}})^{j_{i}}, & 1 \leq w \leq a. \end{cases}$$
Here $A^a_{\mu_1,\mu_2,\ldots,\mu_n}$'s and $B^a_{\mu_1,\mu_2,\ldots,\mu_n}$'s are SFT invariants of the orbifold cap $[\mathbb{C}/\mu_a]$. We refer interested readers to [14, 27] for detailed explanations of SFT techniques.

The following result follows directly from (7), (8), (9), which will be used in Section 3.3.

**Proposition 3.1.** We have the following decomposition for the degree-zero and degree-one parts of the genus-zero potential of $\mathbb{P}^1_a$:

$$A^a(t^i) = \sum_{a=1}^r A^{\mu_a}(t^{\mu_1}, \ldots, t^{\mu_n}), \quad B^a_1(t^i) = \prod_{a=1}^r B^a_{\mu_a}(t^{\mu_1}, \ldots, t^{\mu_n}).$$

**3.2. Special case: tear drops.** In this subsection, as a warmup, we prove the semisimplicity of big quantum cohomology of tear drops $\mathbb{P}^1_2(a \geq 2)$. To ease notations, throughout this subsection, for $1 \leq i \leq a-1$, we set

$$t^i := t_1^i, \quad \phi_i := \phi_1.$$

From the Riemann-Hurwitz formula, we have

$$H^0_{0,a}(\mu) = \begin{cases} 1, & \text{if } d = 1 \text{ and } \mu = (1), \\ 0, & \text{otherwise.} \end{cases}$$

So from (7), (8), (9), the genus-zero potential of $\mathbb{P}^1_2$ has the following simple form:

$$F = \frac{1}{2}(t^{(0)})^2 t^{\mu_1} + \frac{1}{2a} \sum_{i=1}^{a-1} t^i t^{i-1} + A^a + B^a_1 Q e^{\mu_1},$$

with $A^a, B^a_1 \in \mathbb{Q}[t^1, \ldots, t^{r-1}]$. So the big quantum product is given by

$$\phi_i \star t \phi_j = e_i J_{a-i} \phi_0 t + \sum_{k=1}^{a-1} A^a_{i,a-k} \cdot a \phi_k + Q e^{\mu_1} [(B^a_1)_{i,j} + \sum_{k=1}^{a-1} (B^a_1)_{i,a-k} \cdot a \phi_k].$$

$$\phi_i \star t \phi_0 = e_i J_{a-i} \phi_0 t + \sum_{k=1}^{a-1} (B^a_1)_{i,a-k} \cdot a \phi_k.$$

$$\phi_0 \star t \phi_0 = e_i J_{a-i} \phi_0 t + \sum_{k=1}^{a-1} (B^a_1)_{i,a-k} \cdot a \phi_k.$$

**Lemma 3.2.** $B^a_1 = t_1^1 + O(t^{r-1}).$

**Proof.** This follows from the dimension axiom and the fact that $(\phi_1)^{\mathbb{P}^1_2}_{0,1,1} = 1$. \hfill \Box

**Lemma 3.3.** For $i = 1, \ldots, a-1$, we have $(B^a_1)_{i,a-i} = 0.$

**Proof.** From (3) and (4), we have

$$\deg t^i = \frac{a-i}{a}, \quad \deg B^a_i = \frac{a-1}{a}.$$

Therefore,

$$\deg(B^a_1)_{i,a-i} = \frac{a-1}{a} - \frac{a-i}{a} \cdot \frac{i}{a} = \frac{-1}{a} < 0.$$

\hfill \Box

**Lemma 3.4.** The quantum Euler class has the form

$$e_q = (a+1)\phi_0 + a^2 \sum_{k=1}^{a-1} \sum_{i=1}^{a-1} A^a_{i,a-i,a-k} \phi_k.$$


Proof. From (6), we have
\[ e_q = (a + 1)\phi_{01} + a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}\phi_k + Qe^{\phi_1} \left( a \left( B^i_{i,a-i} \right)_{i=1}^{a-1} + a^2 \sum_{k=1}^{a-1} \sum_{i=1}^{a} (B^i_{i,a-a-i}k)_{i=1}^{a-1} \phi_k \right). \]

Now the required formula follows from Lemma 3.3. \qed

We observe that \( e_q \ast_0 \phi_{01} = O(Qe^{\phi_1}) \). Moreover, consider the coefficients of \( e_q \ast_0 \phi_{01} \) with respect to \( B \), we have the following observation.

**Lemma 3.5.** The coefficient of \( e_q \ast_0 \phi_{01} \) at \( \phi_{00} \) is \( 2B^i_{i}Qe^{\phi_1} \).

Proof. One can check that
\[ \langle e_q \ast_0 \phi_{01} \rangle = e^{\phi_1} \left( (a + 1)B^i_{i} + a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}(B^i_{i})_{k} \right) = \frac{-B^a_{i}}{a^2}. \]

Using WDVV (5) of type \( (i, a - i; 01), (01) \), we obtain
\[ \sum_{k=1}^{a-1} A^a_{i,a-i,a-k}(B^i_{i})_{k} = \frac{-B^a_{i}}{a^2}, \]
which implies the required result. \qed

Direct calculation gives
\[ e_q \ast_0 \phi_{01} = a \sum_{i=1}^{a-1} A^a_{i,a-i}\phi_{00} + a^3 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,a-a-k}\phi_{01} + O(Qe^{\phi_1}). \]

Now consider the matrix \( M \) of \( (e_q \ast_0 \phi) \) with respect to the basis \( B \):
\[ (e_q \ast_0 \phi)[\phi_{00}, \phi_{01}, \cdots, \phi_{a-1}] = [\phi_{00}, \phi_{01}, \cdots, \phi_{a-1}]M. \]

Here \( M \) has the form
\[
\begin{bmatrix}
0 & 2B^i_{i}Qe^{\phi_1} & O(Qe^{\phi_1}) & \cdots & O(Qe^{\phi_1}) \\
(a + 1)Qe^{\phi_1} & a \sum_{i=1}^{a-1} A^a_{i,a-i} + O(Qe^{\phi_1}) & \cdots & a \sum_{i=1}^{a-1} A^a_{i,a-i} + O(Qe^{\phi_1}) \\
a^2 \sum_{i=1}^{a-1} A^a_{i,a-a-i} & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k} + O(Qe^{\phi_1}) & \cdots & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k} + O(Qe^{\phi_1}) \\
\vdots & \vdots & \ddots & \vdots \\
a^2 \sum_{i=1}^{a-1} A^a_{i,a-i} & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k} + O(Qe^{\phi_1}) & \cdots & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k} + O(Qe^{\phi_1})
\end{bmatrix}
\]

Observe that the second column of \( M \) is \( O(Qe^{\phi_1}) \). Recall that det is a multilinear function on column vectors. So we can take out the common factor \( Qe^{\phi_1} \) in the second column to obtain
\[
\text{det}(e_q \ast_0 \phi) = \text{det} M = Qe^{\phi_1} \text{det} M_1 + o(Qe^{\phi_1}),
\]

where
\[
M_1 = \begin{bmatrix}
0 & 2B^i_{i} & 0 & \cdots & 0 \\
(a + 1) & a \sum_{i=1}^{a-1} A^a_{i,a-i,1} & \cdots & a \sum_{i=1}^{a-1} A^a_{i,a-i} \\
a^2 \sum_{i=1}^{a-1} A^a_{i,a-i,a-1} & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k} & \cdots & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k} \\
\vdots & \vdots & \ddots & \vdots \\
a^2 \sum_{i=1}^{a-1} A^a_{i,a-i,1} & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k} & \cdots & a^2 \sum_{i=1}^{a-1} \sum_{k=1}^{a} A^a_{i,a-i,a-k}A^a_{k,k-a-k}
\end{bmatrix}
\]
So
\[
(11) \quad \det(e_2 \ast_t) = -2B_1^2 Qe^{pl} \det M_2 + o(Qe^{pl}),
\]
where
\[
M_2 = \begin{bmatrix}
    a + 1 & a \sum_{i=1}^{a-1} A_{j,a-i-1}^a & \cdots & a \sum_{i=1}^{a-1} A_{j,a-i,a}^a \\
    a^2 \sum_{i=1}^{a-1} A_{j,a-i-1}^a & a^2 \sum_{i=1}^{a-1} A_{j,a-i,k}^a A_{k,1,a-1}^a & \cdots & a^2 \sum_{i=1}^{a-1} A_{j,a-i,k}^a A_{k,1,a-1}^a \\
    \vdots & \vdots & \ddots & \vdots \\
    a^2 \sum_{i=1}^{a-1} A_{j,a-i-1}^a & a^2 \sum_{i=1}^{a-1} A_{j,a-i,k}^a A_{k,1,a-1}^a & \cdots & a^2 \sum_{i=1}^{a-1} A_{j,a-i,k}^a A_{k,1,a-1}^a \\
\end{bmatrix}
\]

Lemma 3.6.
\[
\sum_{i=1}^{a-1} A_{j,a-i,a-k}^a = \begin{cases}
    -\frac{a-1}{a} t^1 + O(t^{r+1}), & k = 1, \\
    O(t^{r+1}), & 2 \leq k \leq a-1.
\end{cases}
\]

Proof. From (3) and (4), we have
\[
\deg A_{j,a-i,a-k} = 2 - \frac{a - l}{a} - \frac{i}{a} - \frac{k}{a} = \frac{a - k}{a}.
\]
So for $k = 2, \ldots, a - 1$, the polynomial $A_{j,a-i,a-k}$ lives in the ideal generated by $t^2, \ldots, t^{a-1}$. Moreover, from Lemma 3.2 and formula (10), we have
\[
\sum_{i=1}^{a-1} \sum_{k=1}^{a-1} A_{j,a-i,a-k}^a (B_1^a)_k = -\frac{a-1}{a^2} B_1^a = -\frac{a-1}{a^2} t^1 + O(t^{r+1}).
\]
Since $A_{j,a-i,a-k} \in O(t^{r+1})$ for $k = 2, \ldots, a - 1$, it follows that
\[
\sum_{i=1}^{a-1} A_{j,a-i,a-1}^a (B_1^a)_1 = -\frac{a-1}{a^2} t^1 + O(t^{r+1}).
\]
Now the required equality comes from $(B_1^a)_1 = 1 + O(t^{r+1})$, which is a result of Lemma 3.2.
\[
\square
\]

Lemma 3.7. For $j, l = 1, \cdots, a - 1$, we have
\[
A_{1,j,a-l}^a = \begin{cases}
    \frac{1}{a} + O(t^{r+1}), & l = j + 1, \\
    -\frac{l}{a} + O(t^{r+1}), & (j, l) = (a - 1, 1), \\
    O(t^{r+1}), & \text{else}.
\end{cases}
\]

Proof. From (3) and (4), we have
\[
\deg A_{1,j,a-l}^a = 2 - \frac{a - 1}{a} - \frac{a - j}{a} - \frac{l}{a} = \frac{j + 1 - l}{a}.
\]
So
\[
A_{1,j,a-l}^a \not\in O(t^{r+1}) \Rightarrow (a - 1)(j + 1 - l) \Rightarrow j + 1 - l = 0 \text{ or } a - 1.
\]
If $j + 1 - l = 0$, then
\[
A_{1,j,a-l}^a |_{x^i = 0} = (\phi_1, \phi_j, \phi_{a-j-1})^a_{1,3,0} = \frac{1}{a}.
\]
If $j + 1 - l = a - 1$, then $(j, a - l) = (a - 1, a - 1)$, and the required result follows from Corollary 3.34 and 3.35 in [22].
\[
\square
\]
From Lemma 3.6 and 3.7, we have

\[
M_2 = \begin{bmatrix}
 a + 1 & 0 & \cdots & 0 & - \frac{a-1}{a} t^1 \\
-(a-1)t^1 & 0 & \cdots & 0 & - \frac{a}{a^2} (t^1)^2 \\
0 & -(a-1)t^1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -(a-1)t^1 & 0 \\
\end{bmatrix} + O(t^{-1}).
\]

(12)

Now from Lemma 3.2 and formula (11), (12), we have

\[
\det(e_q \star t) = Qe^{p_{\alpha}} \left[ -f + \frac{1}{\alpha} e^{(t^1)^{\alpha-1}} + O(t^{-1}) \right] + o(Qe^{p_{\alpha}}) \neq 0.
\]

This shows the semisimplicity of \( QH(\mathbb{P}^1) \).

3. General case. In this subsection, we prove the semisimplicity of \( QH(\mathbb{P}^1) \), where \( a = (a_1, \cdots, a_r) \) is a tuple of positive integers with \( r \geq 1 \) and each \( a_i \geq 2 \). Some results in the last two subsections will be used.

From (7) and Corollary 3.1, the genus-zero potential of \( \mathbb{P}^1 \) has the form:

\[
F = \frac{1}{2} (t_{01}^1)^2 + \sum_{a=1}^{r} \phi_{a_{01}} \cdot \sum_{k=1}^{a_{01} - 1} A_{i_{a_{01}-k}}(t_{i_{a_{01}-k}}) + Qe^{p_{\alpha}} B^2_{1}(t^1_{\alpha}) + O(t_{\alpha}^1),
\]

with

\[
B^2_{1}(t^1_{\alpha}) = \prod_{a=1}^{r} B^2_{1}(t^1_{\alpha_1}, \cdots, t^1_{\alpha_{r-1}}).
\]

In particular, from Lemma 3.2, we have

\[
(13) \quad B^2_{1}(t^1_{\alpha}) = \prod_{a=1}^{r} t^1_{\alpha_1} + O(t_{\alpha_1}^{1}, \cdots, t_{\alpha_{r-1}}).
\]

So the big quantum product is given by

\[
\phi_{a_{01}} \star t \phi_{a_{ij}} = \delta_{j, a_{01}} \phi_{a_{01}} + \sum_{k=1}^{a_{01} - 1} A_{i_{a_{01}-k}}(t_{i_{a_{01}-k}}) + Qe^{p_{\alpha}} B^2_{1}(t^1_{\alpha_1}) + o(Qe^{p_{\alpha}}),
\]

\[
\phi_{a_{01}} \star t \phi_{\alpha_{ij}} = Qe^{p_{\alpha}} B^2_{1}(t^1_{\alpha_1}) + o(Qe^{p_{\alpha}}),
\]

\[
\phi_{a_{01}} \star t \phi_{\alpha_{01}} = Qe^{p_{\alpha}} B^2_{1}(t^1_{\alpha_1}) + o(Qe^{p_{\alpha}}),
\]

\[
\phi_{0_{a_{01}}} \star t \phi_{0_{01}} = Qe^{p_{\alpha}} B^2_{1}(t^1_{\alpha_1}) + o(Qe^{p_{\alpha}}).
\]

From (6), we can use Lemma 3.3 to check that the quantum Euler class has the form

\[
e_{q} = (2 + \sum_{a=1}^{r} (a_{a} - 1) \phi_{0_{a}}) + \sum_{a=1}^{r} \sum_{k=1}^{a_{a} - 1} A_{i_{a_{a}-k}}(t_{i_{a_{a}-k}}) + o(Qe^{p_{\alpha}}).
\]
We observe that \( e_{q} \star \phi_{01} = O(Qe^{\alpha}) \). Moreover, consider the coefficients of \( e_{q} \star \phi_{01} \) with respect to \( B \), we have the following observation.

**Lemma 3.8.** The coefficient of \( e_{q} \star \phi_{01} \) at \( \phi_{01} \) is \( 2B_{1}^{2}Qe^{\alpha} + o(Qe^{\alpha}) \).

**Proof.** One can check that

\[
\langle e_{q} \star \phi_{01}, \phi_{01} \rangle_{\mathcal{B}}^{21} = B_{1}^{2}Qe^{\alpha} \left( 2 + \sum_{a=1}^{r} (a_{a} - 1) + \sum_{a=1}^{r} \alpha_{a}^{2} \sum_{k=1}^{a_{a} - 1} \sum_{i=1}^{a_{a} - k} \frac{(B_{1})_{k}}{(B_{1})^{a_{a} - k}} \right) + o(Qe^{\alpha}).
\]

Now we can use (10) to conclude the required result. \( \square \)

Direct calculation gives

\[
e_{q} \star \phi_{aj} = a_{a} \sum_{k=1}^{a_{a} - 1} A_{a\le a_{a} - k} \phi_{01} + a_{a} \sum_{k=1}^{a_{a} - 1} \sum_{i=1}^{a_{a} - k} A_{a\le a_{a} - k} A_{k\le a_{a} - k} \phi_{01} + O(Qe^{\alpha}).
\]

For \( \alpha = 1, \ldots, r \), let \( \vec{\phi}_{\alpha} := [\phi_{01}, \ldots, \phi_{0,aa_{a} - 1}] \) be a row vector. Consider the matrix \( M \) of \( \langle e_{q} \star \rangle \) with respect to the basis \( \mathcal{B} \):

\[
\langle e_{q} \star \rangle [\phi_{00}, \phi_{01}, \ldots, \vec{\phi}_{r}] = [\phi_{00}, \phi_{01}, \ldots, \vec{\phi}_{r}] M.
\]

Then \( M \) is \( \mathbb{Q}[t^{1}, \ldots, t^{r-1}][[Qe^{\alpha}]] \)-valued, and the second column of \( M \) is \( O(Qe^{\alpha}) \). Recall that \( \det \) is a multilinear function on column vectors. So we can take out the common factor \( Qe^{\alpha} \) in the second column of \( M \) to obtain

\[
\det(e_{q} \star) = \det M = Qe^{\alpha} \det M_{1} + o(Qe^{\alpha}),
\]

with

\[
M_{1} = \begin{bmatrix}
0 & 2B_{1}^{2} & 0 & \cdots & 0 \\
2 + \sum_{a=1}^{r} (a_{a} - 1) & r_{1} & \cdots & r_{r} \\
\vec{c}_{1} & b_{1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vec{c}_{r} & 0 & \cdots & b_{r}
\end{bmatrix}
\]

Here for \( \alpha = 1, \ldots, r \),

\[
\begin{align*}
r_{a} &= \left[ a_{a} \sum_{i=1}^{a_{a} - 1} A_{a\le a_{a} - i, i}^{a_{a}}, \ldots, a_{a} \sum_{i=1}^{a_{a} - 1} A_{a\le a_{a} - a_{a} - 1}^{a_{a}} \right], \\
\vec{c}_{a} &= \left[ a_{a} \sum_{i=1}^{a_{a} - 1} A_{a\le a_{a} - i, i}^{a_{a}}, \ldots, a_{a} \sum_{i=1}^{a_{a} - 1} A_{a\le a_{a} - 1, 1}^{a_{a}} \right]^{T}, \\
b_{a} &= \left[ \begin{array}{cccc}
ad_{a}^{2} \sum_{k=1}^{a_{a} - 1} \sum_{i=1}^{a_{a} - k} A_{a\le a_{a} - k, i}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}}, & \cdots & \cdots & ad_{a}^{3} \sum_{k=1}^{a_{a} - 1} \sum_{i=1}^{a_{a} - k} A_{a\le a_{a} - k, i}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}} \\
\vdots & \vdots & \vdots & \vdots \\
ad_{a}^{3} \sum_{k=1}^{a_{a} - 1} \sum_{i=1}^{a_{a} - k} A_{a\le a_{a} - k, i}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}}, & \cdots & \cdots & ad_{a}^{3} \sum_{k=1}^{a_{a} - 1} \sum_{i=1}^{a_{a} - k} A_{a\le a_{a} - k, i}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}} A_{k\le a_{a} - k}^{a_{a}}
\end{array} \right].
\end{align*}
\]

So we have

\[
\det(e_{q} \star) = -2B_{1}^{2}Qe^{\alpha} \det M_{2} + o(Qe^{\alpha}),
\]
Proof of Corollary 1.3

This shows the semisimplicity of $QH^*(\mathbb{P}^1)^a$.

3.4. Proof of Corollary 1.3. In this subsection, we prove Corollary 1.3.

If $\mathcal{C}$ is a $\mathbb{P}^1$-orbifold, then the semisimplicity of $QH^*(\mathcal{C})$ comes from Proposition 1.2, and the existence of a full exceptional collection in $D^b(\mathcal{C})$ was proved by Geigle-Lenzing (Proposition 4.1 in [16]).

Now assume that $\mathcal{C}$ is not a $\mathbb{P}^1$-orbifold. On one hand, since the genus of $|\mathcal{C}|$ is positive, it follows that the quantum product of $\mathcal{C}$ is identical to the Chen-Ruan product, which implies that $QH^*(\mathcal{C})$ contains nilpotent elements, and hence not semisimple. On the other hand, since the odd cohomology group of $|\mathcal{C}|$ does not vanish, it follows that we can use the orbifold HKR isomorphism (see Section 1.15-1.17 in [2]) to conclude that $D^b(\mathcal{C})$ does not admit a full exceptional collection.

This finishes the proof of Corollary 1.3.

4. SMALL QUANTUM COHOMOLOGY

In this section, we prove Proposition 1.4. To show the “only if” part, from Corollary 2.2, it suffices to prove the following lemma.

Lemma 4.1. If $\chi_{\text{orb}}(\mathcal{C}) \leq 0$, then $QH^*(\mathcal{C})$ contains nilpotent elements.

Proof. For homogenous $x, y \in H^\text{even}_{\text{orb}}(\mathcal{C})$, we have

$$x \circ y = x \cup_{\text{CR}} y + \sum_{a,b \in S} \sum_{d > 0} \langle x, y, \phi_s \rangle_{0, 3, d}^\mathcal{C} Q^d \phi_s.$$

Note that $\deg_{\text{orb}}(x \cup_{\text{CR}} y) = \deg_{\text{orb}} x + \deg_{\text{orb}} y$. If for some $d > 0$ and $s \in S$ we have

$$\langle x, y, \phi_s \rangle_{0, 3, d}^\mathcal{C} \neq 0,$$

then the dimension constraint gives

$$\deg_{\text{orb}} x + \deg_{\text{orb}} y + \deg_{\text{orb}} \phi_s = 2 + 2d \cdot \chi_{\text{orb}}(\mathcal{C}) \leq 2$$

$$\Rightarrow \deg_{\text{orb}} \phi_s \geq \deg_{\text{orb}} x + \deg_{\text{orb}} y.$$
As a consequence, $x \circ y$ is a linear combination of \{\phi^t : x \in S, \text{ and } \deg_{\text{orb}} \phi^t \geq \deg_{\text{orb}} x + \deg_{\text{orb}} y\} with coefficients in $\mathbb{Q}[[Q]]$. In particular, if $\deg_{\text{orb}} x > 0$, then $x$ is nilpotent. \hfill \Box

To prove the “if” part of Proposition 1.4, we first use the dimension axiom to observe that for a Fano orbicurve $C$, the coefficients in the small quantum product take values in $\mathbb{C}[Q]$. So it suffices to show the generic semisimplicity of

$$\bar{q}H(C) := H^{\text{vir}}_{\text{orb}}(C_a) \otimes_{\mathbb{C}} \mathbb{C}[Q]$$

over $\mathbb{C}[Q]$, since we have the following Cartesian diagram

$$\begin{array}{c}
\text{Spec}(qH(C)) \\
\downarrow \\
\text{Spec}(\bar{q}H(C))
\end{array} \quad \begin{array}{c}
\text{Spec}(\mathbb{C}[Q]) \\
\downarrow \\
\text{Spec}(\bar{q}H(C))
\end{array}$$

where the base morphism is dominant, given by the natural injective map $\mathbb{C}[Q] \to \mathbb{C}[Q]$.

Recall that, from (1) and (2), the orbifold cohomology ring $H^{\text{vir}}_{\text{orb}}(C_a)$ is generated by $\phi_{a_1}$’s over $\mathbb{C}$, with relations

$$f_{agl} = \phi_{a_1} \oplus_{\mathbb{CR}} \phi_{g_{ij}} = 0, \quad g_{agl} = a_g \phi_{a_1} \oplus_{\mathbb{CR}} \phi_{a_g} - a_\beta \phi_{g_{ij}} \oplus_{\mathbb{CR}} \phi_{\beta_{ij}} = 0, \quad \alpha \neq \beta.$$ 

So $\bar{q}H(C)$ is generated by $\phi_{a_1}$’s over $\mathbb{C}[Q]$, with new relations $f'_{agl}, g'_{agl}(\alpha \neq \beta)$. Here the new relation $f'_{agl}$ (resp. $g'_{agl}$) is just the relation $f_{agl}$ (resp. $g_{agl}$) evaluated in the small quantum cohomology ring structure. In other words, we have the following presentation for $\bar{q}H(C)$:

$$\bar{q}H(C) \cong \mathbb{C}[Q][x_1, \ldots, x_r]/I,$$

where the ideal $I$ is generated by the relations $f'_{agl}, g'_{agl}(\alpha \neq \beta)$. Note that at $Q = 1$, $\bar{q}H(C)_{Q=1}$ is a $\mathbb{C}$-algebra of dimension $N = 2 + \sum_{a=1}^r (a_n - 1)$. Our strategy is to show that the ideal $I_{Q=1}$ determines exactly $N$ distinct points in $\mathbb{C}^r$, from which the semisimplicity of $\bar{q}H(C)_{Q=1}$ over $\mathbb{C}$ follows. From Lemma 2.1, this implies the generic semisimplicity of $\bar{q}H(C)$ over $\mathbb{C}[Q]$.

Now we check the “if” part of Proposition 1.4 case by case.

\(C = \mathbb{P}^{1}_{a_1, a_2}(a_1, a_2 \geq 1)\): We have the following presentation for $\bar{q}H(\mathbb{P}^{1}_{a_1, a_2})$ (see (4.32) in [25]):

$$\bar{q}H(\mathbb{P}^{1}_{a_1, a_2}) \cong \mathbb{C}[Q][x_1, x_2]/I,$$

where the ideal $I$ is generated by $x_1 x_2 - Q$ and $a_1 x_1^{a_2} - a_2 x_2^{a_1}$. Direct calculation gives the set of solutions of $I_{Q=1}$:

$$x_1 = \left(\frac{a_2}{a_1}\right)^{a_1} - e^{a_{1+2}}, \quad x_2 = \left(\frac{a_1}{a_2}\right)^{a_2} - e^{a_{1+2}}, \quad (1 \leq k \leq a_1 + a_2).$$

Here for a positive integer $N$, $\xi_N := e^{2\pi i/N}$.

**Remark 4.2.** The semisimplicity of $\bar{q}H(\mathbb{P}^{1}_{a_1, a_2})$ was also pointed out by Milanov-Tseng (see Section 4.4 in [25]).

\(C = \mathbb{P}^{1}_{2,2,\alpha}(\alpha \geq 2)\): We have the following presentation for $\bar{q}H(\mathbb{P}^{1}_{2,2,\alpha})$ (see Section 5 in [27]):

$$\bar{q}H(\mathbb{P}^{1}_{2,2,\alpha}) \cong \mathbb{C}[Q][x,y,z]/I,$$

where $\bar{q}H(\mathbb{P}^{1}_{2,2,\alpha})$.
where the ideal $I$ is generated by
\[ xy + a \sum_{k=0}^{[\frac{n}{2}]} (-1)^{k+1} \binom{a-1-k}{k} Q^{2k+1} \zeta^{a-1-2k}, \]
and $xz - 2Qy, \quad yz - 2Qx$.

Using formulae (2.3) and (2.4) in [18], we can solve the equations directly. For $a = 2m$, the set of solutions of $I_{Q=1}$ is
\[ (\pm a, \pm a, 2), \quad (\pm a, \mp a, -2), \quad (0, 0, 0), \quad (0, 0, \pm \sqrt{2 - 2 \cos \frac{k\pi}{m}}) \quad (1 \leq k \leq m - 1), \]
and for $a = 2m + 1$, the set of solutions of $I_{Q=1}$ is
\[ (\pm a, \pm a, 2), \quad (\pm a \sqrt{-1}, \mp a \sqrt{-1}, -2), \quad (0, 0, \pm \sqrt{2 - 2 \cos \frac{(2k+1)\pi}{2m+1}}) \quad (0 \leq k \leq m - 1). \]

$\mathcal{C} = \mathbb{P}^1_{2,3,3}$: Using the explicit formula for the genus-zero potential (see Appendix A.1 in [21]), we have the following presentation for $\tilde{q}H(\mathbb{P}^1_{2,3,3})$:
\[ \tilde{q}H(\mathbb{P}^1_{2,3,3}) \cong \mathbb{C}[Q][x, y, z]/I, \quad \text{with } \phi_{11} \mapsto x, \phi_{21} \mapsto y, \phi_{31} \mapsto z, \]
where the ideal $I$ is generated by
\[ xy = 3Qy^2 + 6Qz^2, \quad xz - 3Qy^2 + 6Qz^2, \quad yz - 2Qx - 4Q^4. \]

Direct calculation gives the set of solutions of $I_{Q=1}$:
\[ (-2, 0, 0), \quad (0, 2\xi_1^2, 2\xi_1^2), \quad (6, 4\xi_1^2, 4\xi_1^2) \quad (k = 0, 1, 2). \]

$\mathcal{C} = \mathbb{P}^1_{2,3,4}$: Using the explicit formula for the genus-zero potential (see Appendix A.5 in [21]), we have the following presentation for $\tilde{q}H(\mathbb{P}^1_{2,3,4})$:
\[ \tilde{q}H(\mathbb{P}^1_{2,3,4}) \cong \mathbb{C}[Q][x, y, z]/I, \quad \text{with } \phi_{11} \mapsto x, \phi_{21} \mapsto y, \phi_{31} \mapsto z, \]
where the ideal $I$ is generated by
\[ xy = 4Qz^3 + 28Qy^3z + 72Q^2z, \quad xz - 3Qy^2 + 8Q^2z^2 - 18Q^3y - 24Q^9, \quad yz - 2Qx - 4Q^4z. \]

We use MAPLE to get the set of solutions of $I_{Q=1}$:
\[ (0, -4, 0), \quad (0, -2, 0), \quad (\pm 4, 0, \mp 2), \quad (0, 4, 3 \pm \sqrt{2}), \quad (\pm 12, 8, \pm 6). \]

$\mathcal{C} = \mathbb{P}^1_{2,3,5}$: Using the explicit formula for the genus-zero potential (see Appendix A.9 in [21]), we have the following presentation for $\tilde{q}H(\mathbb{P}^1_{2,3,5})$:
\[ \tilde{q}H(\mathbb{P}^1_{2,3,5}) \cong \mathbb{C}[Q][x, y, z]/I, \quad \text{with } \phi_{11} \mapsto x, \phi_{21} \mapsto y, \phi_{31} \mapsto z, \]
where $I$ is the ideal generated by
\[ xy = 5Qz^4 - 129Q^{10}y^2 + 350Q^{11}z^2 - 2920Q^{15}x - 8140Q^{15}z^2 + 14130Q^{13}y + 20400Q^{19}z + 76080Q^{25}, \]
\[ xz = 3Qy^2 - 10Q^3z^3 + 72Q^6x + 205Q^{10}y - 510Q^{15}y - 1920Q^{21}, \]
\[ yz = 2Qx + 5Q^2z^2 - 12Q^6y - 20Q^{10}z - 60Q^{16}. \]

We use MAPLE to get the set of solutions of $I_{Q=1}$:
\[ (0, 0, -2), (6, -4, 0), (30, 20, 12), (0, \pm 5, \mp 2), (0, 10, 3 \pm 3 \sqrt{5}), (-10, 0, 2 \pm 2 \sqrt{5}). \]

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