General massive gauge theory

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Abstract

The concept of perturbative gauge invariance formulated exclusively by means of asymptotic fields is used to construct massive gauge theories. We consider the interactions of \( r \) massive and \( s \) massless gauge fields together with \((r+s)\) fermionic ghost and anti-ghost fields. First order gauge invariance requires the introduction of unphysical scalars (Goldstone bosons) and fixes their trilinear couplings. At second order additional physical scalars (Higgs fields) are necessary, their coupling is further restricted at third order. In case of one physical scalar all couplings are determined by gauge invariance, including the Higgs potential. For three massive and one massless gauge field the \( SU(2) \times U(1) \) electroweak theory comes out as the unique solution.

PACS. 11.15.-q Gauge field theories, 11.15.Bt General properties of perturbation theory

Keywords: Massive gauge theories, electroweak theory
1 Introduction

In gauge theories with massive gauge bosons the masses are conventionally generated by the Higgs mechanism [1]. One introduces scalar fields into the theory which have asymmetric self-interactions so that some physical scalar field gets a symmetry breaking vacuum expectation value (Higgs field). Then the gauge symmetry is spontaneously broken and the gauge fields can acquire mass. Here the notion "gauge symmetry" refers to the symmetry of the total action.

For various reasons there are still considerable doubts whether the above picture is really fundamental, one being the ad-hoc character of the construction. However, it is possible to consider massive gauge theories from a quite different point of view. If one takes the adiabatically switched S-matrix $S(g)$ ($g(x)$ a Schwartz test function) as the basic object, defined by the perturbation series [2]

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n T_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n), \quad (1.1)$$

then one would like to formulate gauge invariance in terms of the time-ordered products $T_n$. Since the latter are expressed by the asymptotic free fields, it is a priori not clear whether such a perturbative definition of gauge invariance is possible. We have found that this is indeed the case [3], no matter if the gauge fields are massless or massive [4]. The definition of perturbative gauge invariance reads as follows

$$d_Q T_n \overset{\text{def}}{=} [Q, T_n] = i \sum_{l=1}^{n} \frac{\partial}{\partial x_{l}} T_{n/l}^\mu(x_1, \ldots x_l \ldots x_n). \quad (1.2)$$

Here $Q$ is the nilpotent gauge charge, first introduced by Kugo and Ojima [5], and the $T_{n/l}^\mu$ are time-ordered products with a so-called $Q$-vertex at $x_l$. These quantities are defined in the next section and in sect.3.

The idea of the paper is to start from a general ansatz for $T_1(x)$ and to use perturbative gauge invariance (1.2) to determine the coupling parameters in $T_1$. This is a straightforward generalization of [4] with the merit that in the more general framework the discussion is simpler and more transparent. The general ansatz contains massless and massive gauge fields and
ghosts, as well as unphysical (Goldstone bosons) and physical scalar (Higgs) fields. In contrast to standard theory of spontaneous symmetry breaking where the scalar fields are members of some multiplet, we treat the unphysical and physical scalars completely free and independent. This turns out to be natural because their couplings come out quite different: the coupling of the unphysical scalars is (up to mass dependent factors) given by the structure constants $f_{abc}$ of the gauge group Lie algebra (sect.3), whereas the Higgs couplings are of a different diagonal type (sect.4). Nevertheless, in the case of one physical scalar the resulting couplings are in agreement with the usual theory, including the asymmetric Higgs potential (sect.5). For more than one Higgs field their couplings are not completely determined by gauge invariance.

As a consequence of perturbative gauge invariance we find many relations between the masses of the gauge fields and the structure constants $f_{abc}$. As an application we consider in sect.6 the physical case of three massive gauge fields and one massless (photon) field and ask the question: what are the possible gauge theories? The relations of gauge invariance enables us to calculate the $f_{abc}$ in terms of the masses. The unique result is the usual $SU(2) \times U(1)$ electroweak theory. In this way the standard theory looses its ad-hoc character.

The same problem has recently been considered by D.R. Grigore using a different definition of gauge invariance. Most of his results are in agreement with ours, only his treatment of the Higgs fields is misleading.

2 A general massive gauge theory

We consider $r$ massive and $s$ massless gauge fields $A_{a}^{\mu}$, $a = 1, \ldots, r + s$ together with $(r + s)$ fermionic ghost and anti-ghost fields $u_{a}$, $\tilde{u}_{a}$. These free asymptotic fields are quantized as follows

$$(\Box + m_{a}^{2})A_{a}^{\mu}(x) = 0, \quad [A_{a}^{\mu}(x), A_{b}^{\nu}(y)]_{-} = i\delta_{ab}g^{\mu\nu}D_{ma}(x - y),$$

$$(\Box + m_{a}^{2})u_{a}(x) = 0 = (\Box + m_{a}^{2})\tilde{u}_{a}(x)$$

$$\{u_{a}(x), \tilde{u}_{b}(y)\}_{+} = -i\delta_{ab}D_{ma}(x - y),$$

$$\Box + m_{a}^{2}u_{a}(x)$$
all other commutators vanish, \( D_m \) are the Jordan-Pauli distributions. The masses of a gauge field and the corresponding ghost and anti-ghost fields must be equal, otherwise perturbative gauge invariance cannot be achieved. We have \( m_a = 0 \) for \( a > r \).

In order to get a gauge charge \( Q \) which is nilpotent

\[
Q^2 = 0,
\]

we have to introduce for every massive gauge vector field \( A^\mu_a(x), a \leq r \), a scalar partner \( \Phi_a(x) \) with the same mass \( m_a \). The scalar fields are quantized according to

\[
(\Box + m^2_a)\Phi_a(x) = 0, \quad [\Phi_a(x), \Phi_b(y)] = -i\delta_{ab}D_m(x - y).
\]

Then the gauge charge \( Q \) is defined by

\[
Q \overset{\text{def}}{=} \int d^3x (\partial_\nu A^\nu_a + m_a\Phi_a) \partial_0 u_a.
\]

Calculating \( Q^2 \) as one half of the anticommutator \( \{Q, Q\} \) one easily verifies the nilpotency (2.4).

The scalar and ghost fields appearing in \( Q \) (2.6) are all unphysical because their excitations do not belong to the physical subspace \( \mathcal{H}_{\text{phys}} = \text{Ker}Q/\text{Ran}Q \).

To discuss this in detail it is necessary to introduce a concrete representation of the various asymptotic fields in Fock space. We want to avoid that to stress the fact that our definition of gauge invariance refers to a structural property independent of representation. Then we simply call a field unphysical if it appears in \( Q \) (2.6), otherwise it is physical. For the gauge fields that means \( \partial_\nu A^\nu \) is unphysical. Second order gauge invariance will force us to introduce additional physical scalar fields \( \varphi_p, p = 1, \ldots, t \), called Higgs fields, with arbitrary masses \( \mu_p \). We shall use indices \( p, q, \ldots = 1, \ldots t \) from the end of the alphabet to number the Higgs fields, letters \( h, j, k, l, \ldots = 1, \ldots r \) from the middle denote the other massive fields and \( a, b, c, d, e, f, \ldots = 1, \ldots r + s \) is used for unrestricted 'color' indices.
With this field content we are going to analyse the following trilinear couplings:

\[ T_1(x) = T_1^0 + T_1^1 + \ldots + T_1^{11} \quad (2.8) \]

where

\[ T_1^0 = ig f_{abc} (A_{\mu a} A_{\nu b} \partial^\nu A_\mu^c - A_{\mu a} u_b \partial^\mu \tilde{u}_c) \quad (2.9) \]

\[ T_1^1 = ig f_{ahj}^1 A_\mu^a (\Phi_h \partial_\mu \Phi_j - \Phi_j \partial_\mu \Phi_h), \quad f_{ahj}^1 = -f_{ajh}^1 \quad (2.10) \]

\[ T_1^2 = ig f_{abh}^2 A_\mu^a A_\sigma^b \Phi_h, \quad f_{abh}^2 = f_{bha}^2 \quad (2.11) \]

\[ T_1^3 = ig f_{hjk}^3 \tilde{u}_a u_b \Phi_h \quad (2.12) \]

\[ T_1^4 = ig f_{hjk}^4 \Phi_h \Phi_j \Phi_k \quad (2.13) \]

where \( f_{hjk}^4 \) is totally symmetric in \( h, j, k \) and \( g \) is a coupling constant. All \( f \)'s are real because \( T_1 \) must be skew-adjoint. For reasons of economy we assume the pure Yang-Mills coupling \( f_{abc} \) in (2.9) to be totally antisymmetric. If one starts with the most general ansatz, one must repeat the discussion in [8] to derive the antisymmetry. The Jacobi identity need not be assumed, it follows explicitly below in second order (Sect.4.1). In \( T_1^1 \) we have only considered the antisymmetric combination because the symmetric one can be expressed by a divergence

\[ A_\mu^a (\Phi_h \partial_\mu \Phi_j + \Phi_j \partial_\mu \Phi_h) = \partial_\mu (A_\mu^a \Phi_h \Phi_j) - \partial_\mu A_\mu^a \Phi_h \Phi_j. \]

The remaining \( \partial_\mu A_\mu^a \) term is a coboundary \( d_Q(\tilde{u}_a \Phi_h \Phi_j) \) plus terms of the form \( T_1^3, T_1^4 \). But divergence and coboundary couplings can always be skipped in the discussion of perturbative gauge invariance [9].

The Higgs couplings are obtained by replacing the scalar fields in (2.10-13) by Higgs fields:

\[ T_1^5 = ig f_{abp}^5 A_\mu^a (\Phi_h \partial_\mu \varphi_p - \varphi_p \partial_\mu \Phi_h) \quad (2.14) \]

\[ T_1^6 = ig f_{apq}^6 A_\mu^a (\varphi_p \partial_\mu \varphi_q - \varphi_q \partial_\mu \varphi_p), \quad f_{apq}^6 = -f_{apq}^6 \quad (2.15) \]

\[ T_1^7 = ig f_{abp}^7 A_\mu^a A_\sigma^b \varphi_p, \quad f_{abp}^7 = f_{bap}^7 \quad (2.16) \]

\[ T_1^8 = ig f_{abp}^8 \tilde{u}_a u_b \varphi_p \quad (2.17) \]
\[ T_1^0 = igf^0_{hjp} \Phi h \Phi \varphi_p, \quad f^0_{hjp} = f^0_{jhp} \]

\[ T_1^{10} = igf^{10}_{hpq} \Phi_h \varphi_p \varphi_q, \quad f^{10}_{hpq} = f^{10}_{hqp} \]

\[ T_1^{11} = igf^{11}_{pq} \varphi_p \varphi_q \varphi_u, \]

where \( f^{11} \) is totally symmetric. All products of field operators throughout are normally ordered \((Wick)\) products of free fields. Interacting fields do not appear at all.

3 \hspace{0.5cm} \textbf{First order gauge invariance}

The gauge charge \( Q \) (2.6) defines a gauge variation according to

\[ d_Q F = F Q - (-1)^{n_F} F Q, \]

where \( n_F \) is the number of ghost plus anti-ghost fields in the Wick monomial \( F \). We get the following gauge variations of the fundamental fields

\[ d_Q A_\mu^a(x) = i \partial^\mu u_a(x), \quad d_Q \Phi_h(x) = i m_h u_h(x) \]

\[ d_Q u_a(x) = 0, \quad d_Q \bar{u}_a(x) = -i (\partial_\mu A_\mu^a(x) + m_a \Phi_a(x)) \]

\[ d_Q \varphi_p = 0. \]

These infinitesimal gauge transformations have some similarity with the BRST transformations [10], but we emphasize the following differences. The BRST transformations are defined for interacting fields, whereas we work with asymptotic free fields only and establish gauge invariance order by order. BRST invariance only holds if the quadratic free Lagrangian, the gauge fixing term and the quartic term in the action are also transformed. We have no such terms in \( T_1 \) (2.8) so that the compensations of terms in the gauge variations are totally different.

We now calculate the gauge variation of all terms in \( T_1 \) and transform the result to a divergence form

\[ d_Q T_1 = i \partial_\mu T_1^\mu. \]
In this way we find:

\[ dQ T^0_1 = g f_{abc} \left\{ \partial_\mu [A_{\nu a} u_b (\partial^\nu A^\mu_c - \partial^\mu A^\nu_c) + \frac{1}{2} u_a u_b \partial^\mu \bar{u}_c] \right\} \]
\[ -m^2_c A_{\nu a} u_b A^\nu_c + \frac{1}{2} m^2_c u_a u_b \bar{u}_c + m_c A_{\nu a} u_b \partial^\nu \Phi_c \} \]  \hspace{1cm} (3.6)

\[ dQ T^1_1 = -g f^1_{abh} \left\{ \partial_\mu [u_a (\Phi_h \partial_\mu \Phi_j - \Phi_j \partial_\mu \Phi_h)] \right\} + m_j A^\mu_a \Phi_h u_j - m_h A^\mu_a \Phi_j u_h + (m_j^2 - m_h^2) u_a \Phi_h \Phi_j \]  \hspace{1cm} (3.7)

\[ dQ T^2 = -g f^2_{abh} \left\{ \partial_\mu (u_a A_\mu^a + A_{\mu a} u_b) \Phi_h - u_a \partial_\mu A^\mu_b \Phi_h - u_a A^\mu_b \partial_\mu \Phi_h \right\} \]
\[ -u_b \partial_\mu A^\mu_a \Phi_h - u_b A^\mu_a \partial_\mu \Phi_h + m_h A^\mu_a A^\nu_b u_h \} \]  \hspace{1cm} (3.8)

\[ dQ T^3 = g f^3_{abc} \left\{ (\partial_\mu A^\mu_a + m_a \Phi_a) u_b \Phi_h - m_h \bar{u}_a u_b u_h \right\} \]  \hspace{1cm} (3.9)

\[ dQ T^4_1 = -g f^4_{abh} \left\{ m_h u_h \Phi_j \Phi_k - m_j \Phi_j u_j \Phi_k + m_k \Phi_k \Phi_j u_k \right\} \]  \hspace{1cm} (3.10)

\[ dQ T^5 = -g f^5_{abh} \left\{ \partial_\mu [u_a (\Phi_h \partial_\mu \varphi_p - \varphi_p \partial_\mu \Phi_h) - m_h A^\mu_a \varphi_p u_h] \right\} + (m_h^2 - m^2_p) u_a \Phi_h \varphi_p + 2 m_h A^\mu_a u_b \partial_\mu \varphi_p + m_h \partial_\mu A^\mu_a u_h \varphi_p \right\} \]  \hspace{1cm} (3.11)

\[ dQ T^6 = -g f^6_{apq} \left\{ \partial_\mu [u_a (\varphi_p \partial_\mu \varphi_q - \varphi_q \partial_\mu \varphi_p)] + (m^2 - m^2_p) u_a \varphi_p \varphi_q \right\} \]  \hspace{1cm} (3.12)

\[ dQ T^7 = -g f^7_{apb} \left\{ \partial_\mu [(u_a A_\mu b + u_b A_\mu a) \varphi_p] \right\} \]
\[ -u_a \partial_\mu A^\mu_b + u_b \partial_\mu A^\mu_a \varphi_p - (u_a A_\mu^a + u_b A_\mu^b) \partial_\mu \varphi_p \} \]  \hspace{1cm} (3.13)

\[ dQ T^8 = g f^8_{apb} (\partial_\mu A^\mu_a + m_a \Phi_a) u_b \varphi_p \]  \hspace{1cm} (3.14)

\[ dQ T^9 = -g f^9_{hpj} (m_h u_h \Phi_j + m_j u_j \Phi_h) \varphi_p \]  \hspace{1cm} (3.15)

\[ dQ T^{10} = -g f^{10}_{hpq} m_h u_h \varphi_p \varphi_q. \]  \hspace{1cm} (3.16)
We have given this long list in detail because a lot of information can directly be read off.

The divergence terms give the Q-vertex

\[
T_{1/1}^\mu = g f_{abc} \left[ A_{\nu a} u_b (\partial^\nu A_\mu^c - \partial^\mu A_\nu^c) + \frac{1}{2} u_a u_b \partial^\mu \tilde{u}_c \right] 
\]  
(3.17.1)

\[
- g f_{ahj}^1 \left[ 2 u_a \Phi_h \partial^\mu \Phi_j + m_j A_{\mu a}^c \Phi_h u_j - m_h A_{\mu a}^c \Phi_j u_h \right] 
\]  
(3.17.2)

\[
- g f_{abh}^2 (u_a A_{\mu b}^c + u_b A_{\mu a}^c) \Phi_h 
\]  
(3.17.3)

\[
- g f_{ahp}^5 \left[ u_a (\Phi_h \partial^\mu \varphi_p - \varphi_p \partial^\mu \Phi_h) - m_h A_{\mu a}^c \varphi_p \right] 
\]  
(3.17.4)

\[
- 2 g f_{apq}^6 u_a \varphi_p \partial^\mu \varphi_q 
\]  
(3.17.5)

\[
- g f_{ahp}^7 (u_a A_{\mu b}^c + u_b A_{\mu a}^c) \varphi_p. 
\]  
(3.17.6)

The remaining terms must cancel out. Collecting the terms \( \sim u_b A_{\mu a} A_{\nu c}^\mu \) we get the relation

\[
2 m_b f_{acb}^2 = (m_a^2 - m_c^2) f_{abc}. 
\]  
(3.18)

Hence, if \( m_b = 0 \) and \( f_{abc} \neq 0 \) we must have

\[
m_a = m_c. 
\]  
(3.19)

For \( m_b, m_h \neq 0 \) we find

\[
f_{abh}^2 = \frac{m_h^2 - m_a^2}{2m_h} f_{abh}. 
\]  
(3.20)

Then, collecting terms \( \sim A_{\mu a} u_h \partial^\mu \Phi_j \) we get

\[
f_{ahj}^1 = \frac{m_j^2 + m_h^2 - m_a^2}{4m_h m_j} f_{ahj}. 
\]  
(3.21)

Using all these results in the equation \( \sim \partial_\mu A_{\nu a}^\mu \Phi_h u_j \) we arrive at

\[
f_{ahj}^3 = \frac{m_j^2 - m_h^2 + m_a^2}{2m_j} f_{ahj}, 
\]  
(3.22)

and then from \( u_h \Phi_j \Phi_k \) we obtain

\[
f_{hjk}^4 = 0. 
\]  
(3.23)
We have succeeded in expressing all couplings so far by \( f_{abc} \). With these results all remaining terms without Higgs couplings cancel.

We next turn to the Higgs couplings. From \( A_\mu^a u_b \partial_\mu \varphi_p \) we find

\[
f_7^{abp} = m_b f_5^{abp}, \quad f_7^{abp} = 0 \quad \text{for} \quad a > r \quad \text{or} \quad b > r,
\]

and from \( \partial_\mu A_\mu^a u_h \varphi_p \) we get

\[
f_8^{abp} = -m_b f_5^{abp}
\]

and \( = 0 \) for \( b > r \). Finally the terms \( \sim u_a \Phi_h \varphi_p \) give

\[
f_9^{ahp} = -\frac{\mu^2}{2m_a} f_5^{ahp}, \quad a \leq r
\]

and zero for \( a > r \). The terms \( \sim u_a \varphi_p \varphi_q \) lead to

\[
f_{10}^{apq} = \frac{\mu_2^2 - \mu_3^2}{m_a^2} f_6^{apq}, \quad a \leq r
\]

and zero for \( a > r \). We see that the Higgs couplings are not completely fixed by first order gauge invariance. So far the Higgs couplings could be set equal to zero, but then we would find a breakdown of gauge invariance at second order.

### 4 Second order gauge invariance

Following the inductive construction of Epstein and Glaser in the case of \( T_2 \), we have first to calculate the causal distribution

\[
D_2(x, y) = T_1(x)T_1(y) - T_1(y)T_1(x).
\]

It has a causal support \(( \subset \{ (x-y)^2 \geq 0 \} \) and must be decomposed into a retarded and advanced part: \( D_2 = R_2 - A_2 \), \( \text{supp} R_2 \subset V^+ \), \( \text{supp} A_2 \subset V^- \). For diagrams with singular order \( \omega \geq 0 \) this distribution splitting is not unique. There are undetermined local terms

\[
\sum_{|a| = 0}^{\omega} C_a D^n \delta(x-y) : O(x,y) : \quad a = (a_\mu)
\]
in $R_2$ which are called normalization terms (or finite renormalization terms in the old terminology). $D^a = \prod_\mu \partial_\mu^a$ is a partial differential operator and $: O(x,y) :$ is a Wick monomial. Finally, we obtain $T_2 = R_2 - R'_2,$ where $R'_2(x,y) \overset{\text{def}}{=} - T_1(x)T_1(y)$.

The main problem is whether gauge invariance can be preserved in the distribution splitting. Obviously, $D_2$ (4.1) is gauge invariant:

$$d_Q D_2(x,y) = [d_Q T_1(x), T_2(y)] + [T_1(x), d_Q T_2(y)] = 0.$$ (4.2)

Since the retarded part $R_2$ agrees with $D_2$ on the forward light cone $x \in x + (V^+ \setminus \{0\})$ and similarly for $R_{2/1}^\mu$, $R_{2/2}^\mu$, gauge invariance of $R_2$ can only be violated by local terms $\sim D^a \delta(x-y)$.

But such local terms also appear as normalization terms in the distribution splitting if the singular order is $\geq 0$. If the normalization terms $N_2, N_{2/1}^\mu, N_{2/2}^\mu$ can be chosen in such a way that

$$d_Q (R_2 + N_2) = \partial_\mu [R_{2/1}^\mu + N_{2/1}^\mu] + \partial_\mu [R_{2/2}^\mu + N_{2/2}^\mu] \quad (4.3)$$

holds, then the theory is gauge invariant to second order. Note that the distribution $T_2 = R_2 + N_2 - R'_2$ then fulfills (4.3), too, because $R'_2$ is clearly gauge invariant for the same reason as in (4.2). The local terms on the right-hand side of (4.3), which come from the causal splitting, are called "anomalies". The ordinary axial anomalies are of the same kind, they appear in the third order triangle diagrams with axial vector couplings to fermions (see [12], Sect.4). The difference is that the axial anomalies cannot be removed by finite renormalizations.

To prove (4.3) we only have to consider its local part. We concentrate on the tree graphs because gauge invariance is not a serious problem for second order loop graphs. Let $R_2$ be the splitting solution of $D_2$ obtained by replacing $D_m(x-y)$ by $D_{m}^{\text{ret}}(x-y)$. Since $d_Q$ operates only on the field operators, the local part on the left-hand side of (4.3) is only due to $d_Q N_2$. To calculate the anomalies on the right-hand side of (4.3) we start from

$$D_{2/1}^\mu \overset{\text{def}}{=} [T_{1/1}^\mu(x), T_1(y)]. \quad (4.4)$$
The anomalies come from those terms in $T_{1/1}^\mu$ (3.17) which contain a derivative $\partial^\mu$. These are the second and third term in (3.17.1), the first term in (3.17.2), the first two in (3.17.4) and the first in (3.17.5). We shall abbreviate these terms by $17.1/2...17.5/1$ in the following. Commuting the factors with derivative $\partial^\mu$ in these terms with all terms in $T_1(y)$ (2.9-20) we get tree-graph contributions with four external legs (sectors) which we now have to examine.

4.1 Sector $uA\tilde{u}u$:

These field operators come out if we commute the second term in (3.17.1) with the second one in (2.9)

$$(17.1/2) - (2.9/2) = i f_{abc} f_{def} A_{\nu a} u_b [\partial^\mu A_\nu^c(x), A_{\lambda d}(y)] u_e \partial^\lambda \tilde{u}_f, \quad (4.5)$$

where we set the coupling constant $g = 1$ from now on. This gives a result $\sim \partial^\mu D(x - y)$. After splitting this causal distribution we get the retarded part $\partial^\mu D_{\text{ret}}(x - y)$. If now the derivative $\partial^\nu_x$ of (4.2) is applied

$$\partial_\mu \partial^\mu_x D_{\text{ret}}(x - y) = -m^2 D_{\text{ret}} + \delta(x - y) \quad (4.6)$$

we get a local term

$$A_1 = -f_{abc} f_{cef} A_{\nu a} u_b u_e \partial^\nu \tilde{u}_f \delta(x - y) \quad (4.7)$$

which is the anomaly. The second term in (4.2) with $x$ and $y$ interchanged gives the same contribution so that we notice the short rule

$$\partial^\mu D(x - y) \longrightarrow 2\delta(x - y) \quad (4.8)$$

for the following. Proceeding in the same way with the third term in (3.17.1) commuted with the second one in (2.9) we get

$$(17.1/3) - (2.9/2) = f_{abc} f_{def} u_a u_b \partial^\nu \tilde{u}_f A_{\nu d} \delta(x - y). \quad (4.9)$$

There are no further contributions in this sector so that (4.7) must cancel against (4.9) in order to have gauge invariance. We interchange the indices of summation $b$ and $e$ in (4.7)

$$2A_1 = (-f_{abc} f_{cef} + f_{ace} f_{bdf}) u_b u_e \partial^\nu \tilde{u}_f A_{\nu a}$$
and add (4.9), then the total anomaly becomes

\[-f_{abc} f_{ceg} + f_{aeg} f_{cbf} - f_{ebc} f_{acg} \nu \partial_v \tilde{u}_f A_{(v)\alpha}. \] (4.10)

Taking the total asymmetry of \( f_{abc} \) into account the bracket vanishes iff the Jacobi identity is satisfied.

### 4.2 Sector uAAA:

As the foregoing one this is a pure Yang-Mills sector. From the commutator between (3.17.1/2) and (2.9/1) we get three contributions

\[(17.1/2) - (2.9/1) = f_{abc} A_{(v)\alpha} u_b(x) \{ f_{ceg} A_{\alpha e} \partial^\mu A_{\beta f}^\mu + f_{dcg} A_{\lambda d} \partial^\mu A_{\lambda f}^\mu \} \] (4.11)

\[+ f_{dec} A_{\nu}(y) A_{\alpha e} \partial^\mu \partial^\nu D(x - y) \}.

Here in the last term we have a new situation because the distribution \( \partial^\mu \partial^\nu D \) has singular order 0. Consequently its retarded part \( \partial^\mu \partial^\nu D \) contains a free normalization term which is part of \( \partial^\nu N_2/1 \) in (4.3), \( \alpha_1 \) is a free parameter. Applying the external derivative \( \partial_\mu \) we get local terms of the following form

\[G(x) F(y) \partial^\alpha \delta(x - y) + \alpha_1 \partial^\alpha [G(x) F(y) \delta]. \] (4.12)

In the \( \partial \delta \)-term we use the identity

\[F(x) G(y) \partial^\alpha \delta(x - y) + F(y) G(x) \partial^\alpha \delta(x - y) = \]

\[= F(x) (\partial^\alpha G)(x) \delta(x - y) - (\partial^\alpha F)(x) G(x) \delta(x - y). \] (4.13)

Here we have added the other anomaly with \( x \) and \( y \) interchanged which comes from \( \partial^\mu R_{2/2}^\mu \). Similarly, the normalization term of divergence form in (4.12) will be transformed with help of the relation

\[\partial^\alpha [F(x) G(y) \delta(x - y)] + \partial^\alpha [F(y) G(x) \delta(x - y)] = \]

\[= (\partial^\alpha F)(x) G(x) \delta(x - y) + F(x) (\partial^\alpha G)(x) \delta(x - y). \] (4.14)
Summing up, we have the following short rule for the calculation of this type of local terms:

\[ G(x)F(y)\partial^\mu \partial^\nu D(x - y) \rightarrow [(\alpha_1 + 1)\partial^\alpha GF + (\alpha_1 - 1)G\partial^\alpha F]\delta(x - y). \]  

(4.15)

Using this in (4.11) we get the following total result for the local terms

\[ = \int ABC A^\nu u_b(x)\left\{ f_{cef}A_{ae}\partial^\alpha A^\nu_f + f_{def}A_{\lambda d}\partial^\alpha A^\lambda_f \right\} 2\delta + \int ABC A^\nu (\alpha_1 + 1)(\partial_\alpha u_b A^\nu_{va} + u_b \partial_\alpha A^\nu_{va}) A^\nu_d A^\alpha_e \\
+ (\alpha_1 - 1)u_b A^\nu_{va}(\partial_\alpha A^\nu_d A^\alpha_e + A^\nu_d \partial_\alpha A^\alpha_e) \} \delta. \]  

(4.16)

From the vanishing of the term \( \sim u_b A^\alpha_{va} A^\nu_d \partial_\alpha A^\alpha_e \) we conclude

\[ (\alpha_1 - 1)f_{abc}f_{dec} = 0, \]  

(4.17)

which implies \( \alpha_1 = 1 \). Then the terms \( \sim u_b A^\alpha_{va} A^\nu_d \partial_\alpha A^\alpha_e \) cancel due to the Jacobi identity. But the term \( \sim \partial_\alpha u_b A^\nu_{va} A^\nu_d A^\alpha_e \) does not vanish

\[ (\alpha_1 + 1)f_{abc}f_{dec} = 4\beta_1. \]  

(4.18)

Here a normalization term \( N_2 \) in (4.3) is necessary. In fact, the 4-boson coupling

\[ N_1 = -i\beta_1 A^\nu_{va} A^\nu_d A^\alpha_e \delta(x - y) \]  

(4.19)

with the gauge variation

\[ d_Q N_1 = 4\beta_1 \partial_\alpha u_b A^\nu_{va} A^\alpha_e \delta \]  

(4.20)

gives just the desired local term. Such a normalization term (4.19) is indeed possible because the first term in (2.9) commuted with itself gives the following second order tree graph contribution

\[ D_2 = -f_{abc}f_{def}A^\mu A^\nu_{va}\partial^\nu A^\mu_{ce}(x), \partial^\alpha A^\lambda_f(y)]A^\lambda_d A^\alpha_e. \]  

The commutator \( \sim \partial^\nu \partial^\alpha D(x - y) \) has singular order 0 again, which allows the normalization term (4.19). \( \alpha_1 = 1 \) in (4.18) fixes \( \beta_1 \):

\[ N_1 = -\frac{i}{2} f_{abc}f_{dec} A^\nu_{va} A^\nu_d A^\alpha_e \delta(x - y). \]  

(4.21)
This is the mechanism how additional couplings are generated by gauge invariance. Note that in (4.17) no normalization term is possible.

For later use we list the form of all possible normalization terms. They come from second order tree graphs with two derivatives on the inner line:

\[(2.9) - (2.9) : AAAAA, (2.10) - (2.10) : A\Phi A\Phi, (2.10) - (2.14) : A\Phi A\varphi,\]
\[(2.14) - (2.14) : A\Phi A\Phi, A\varphi A\varphi, (2.14) - (2.15) : A\Phi A\varphi, (2.15) - (2.15) : A\varphi A\varphi. \quad (4.22)\]

In addition we shall need three further normalization terms

\[\Phi\Phi\Phi\Phi, \Phi\Phi\varphi\varphi, \varphi\varphi\varphi\varphi.\]

They are produced by fourth order box diagrams with all derivatives on inner lines.

4.3 **Sector** \(uA\Phi\varphi: \)

Now we have the tools to discuss all cases of compensation of local terms. For \(u_a\varphi q A_d^\nu \partial_\nu \Phi_k\) we find the relation

\[
2(\alpha_2 + 1)f_{akj}^1 f_{djq}^5 - 2f_{dac} f_{ckq}^5 - 2(\alpha_3 + 1)f_{akp}^5 f_{dqp}^6
\]
\[
-2(\alpha_1 - 3)f_{ajq}^5 f_{djk}^1 - 2(\alpha_4 - 3)f_{aqp}^6 f_{dkp}^5 = 0.
\]
\[
(4.23)
\]

For \(u_a\varphi q A_d^\nu \partial_\nu \Phi_k\) we have

\[
2(\alpha_2 - 3)f_{akj}^1 f_{djq}^5 + 2f_{dac} f_{ckq}^5 - 2(\alpha_3 - 3)f_{akp}^5 f_{dqp}^6
\]
\[
-2(\alpha_1 + 1)f_{ajq}^5 f_{djk}^1 - 2(\alpha_4 + 1)f_{aqp}^6 f_{dkp}^5 = 0.
\]
\[
(4.24)
\]

For \(\partial_\nu u_a\varphi q A_d^\nu \Phi_k\) we find

\[
2(\alpha_2 + 1)f_{akj}^1 f_{djq}^5 - 2(\alpha_3 + 1)f_{akp}^5 f_{dqp}^6 - 2(\alpha_1 + 1)f_{ajq}^5 f_{djk}^1
\]
\[
-2(\alpha_4 + 1)f_{aqp}^6 f_{dkp}^5 = \beta_2.
\]
\[
(4.25)
\]

For \(u_a\varphi q \partial_\nu A_d^\nu \Phi_k\) we get

\[
2(\alpha_2 - 1)f_{akj}^1 f_{djq}^5 - 2(\alpha_3 - 1)f_{akp}^5 f_{dqp}^6 - 2(\alpha_1 - 1)f_{ajq}^5 f_{djk}^1
\]
\[-2(\alpha_4 - 1)f^6_{aqp}f^5_{dkp} = 0. \] (4.26)

In choosing different parameters \(\alpha_1, \ldots, \alpha_4\) we have split every tree graph contribution separately. If we sum the terms with the same field operators before splitting, we have one \(\alpha\) only, but the results remain the same as we are now going to show.

Subtracting (4.24) from (4.23) and (4.26) from (4.23) and subtracting the two resulting equations we obtain

\[f^6_{aqp}f^5_{dkp} = 0.\]

Here the sum goes over \(p = 1, \ldots, t\) and can be regarded as a scalar product of two vectors in \(\mathbb{R}^t\). We will see below (see (4.44)) that the vectors \((f^5_{dk})_p\) are non-zero. We make the weak assumption that there are \(t\) linear independent vectors \((f^5_{dk})_p\) for different \(d, k\), then it follows \(f^6 = 0\). Subtracting (4.23) from (4.25) we conclude

\[\beta_2(a,d,k,q) = 2f_{dac}f^5_{ckq} - 8f_{ajq}f^4_{dkj}. \] (4.27)

This will be simplified below if we have more information about \(f^5\). \(\beta_2\) belongs to the normalization term

\[N_2 = -i\beta_2(a,d,k,q)A^\nu_a A^\nu_d \Phi_k \varphi_q \delta \] (4.28)

with

\[dQ N_2 = \beta_2 \partial_\nu u_a A^\nu_d \Phi_k \varphi_q \delta + \beta_2 \frac{m_k}{2} u_k A^\nu_a A^\nu_d \varphi_q \delta. \] (4.29)

The last term herein couples this sector to the sector \(uAA\varphi\).

### 4.4 Sector \(uu\tilde{u}\varphi\):

In this sector we have only one combination of external legs, namely \(u_a u_b \tilde{u}_d \varphi_p\). The corresponding relation is

\[2f_{abc}f^8_{dcp} - 2f^5_{ajp}f^3_{dbj} + 2f^5_{bjp}f^3_{daj} + 4f^6_{apq}f^8_{dbq} = 0.\]

The origin of the terms is clear from the upper indices. Since \(f^6 = 0\) we have

\[f^3_{dbj}f^5_{ajp} - f^3_{daj}f^5_{bjp} + m_j f_{abj}f^5_{dpj} = 0, \] (4.30)
We specialize to $d = a$ and insert (3.22):

$$\sum_{j=1}^{r} \frac{3m_j^2 - m_d^2 + m_a^2}{2m_j} f_{abj} f_{ajp}^5 = 0. \quad (4.31)$$

If we write a summation symbol then only the indicated index is summed over. For $a, b, j$ all different, $f_{abj}$ defines a non-singular matrix and the mass-dependent factor does not alter that. Consequently $f^5$ vanishes for different indices, only $f^5_{jjp}$, $j = 1, \ldots r$ are different from 0. That means the Higgs couplings are diagonal, in contrast to the couplings of the unphysical scalars which are non-diagonal.

Now (4.30) can be simplified

$$f_{dba}^3 f_{aap}^5 - f_{dab}^3 f_{bp}^5 + m_d f_{ad} f_{dp}^5 = 0 \quad (4.32)$$

without summation. Interchanging $a$ with $d$ and $b$ with $d$, we get a homogeneous linear system for $f^5$ where $p$ is a dummy index

$$m_a f_{dba} f_{aap}^5 - f_{dab}^3 f_{bp}^5 + f_{ad}^3 f_{dp}^5 = 0 \quad (4.33)$$

$$f_{bda}^3 f_{aap}^5 + m_b f_{adb} f_{bap}^5 - f_{bad}^3 f_{dp}^5 = 0. \quad (4.34)$$

Using (3.22) it is easy to check that the $3 \times 3$ determinant vanishes so that we get a non-trivial solution. The latter is very simple

$$f_{aap}^5 = \frac{m_a}{m_d} f_{dpp}^5, \quad (4.35)$$

in particular, $f_{aap}^5 = 0$ for $a > r$.

With help of (4.35) we can simplify the previous result (4.27) for the normalization factor

$$\beta_2(a, d, k, q) = \frac{2m_d^2 - m_a^2}{m_d m_k} f_{dak} f_{dqq}^5 = \frac{2m_a^2 - m_d^2}{m_k^2} f_{kda} f_{kkq}^5. \quad (4.36)$$

By (4.35) this is symmetric in $a, d$ as it must be (4.28). Furthermore, by means of (4.35) it is easy to check that all remaining relations in the sector $uA\Phi\varphi$ are satisfied. We have still to show that $f^5 \neq 0$. This follows from the following sector.
4.5 Sector $uA\Phi\Phi$:

From $u_\alpha \partial_\nu \Phi_j \Phi_h A_d^\nu$ we get

$$4 f_{d ac} f_{chj}^1 - 4(\alpha_1 - 3) f_{ahk}^1 f_{dj k}^1 - 4(\alpha_1 + 1) f_{ajp}^1 f_{d j k}^1$$

$$- (\alpha_2 + 1) f_{ajp}^5 f_{d j p}^5 - (\alpha_2 - 3) f_{ahp}^5 f_{d j p}^5 = 0,$$  \hspace{1cm} (4.37)

and, assuming $j \neq h$, $u_\alpha \Phi_j \Phi_h \partial_\nu A_d^\nu$ gives

$$(\alpha_2 - 1)(f_{ajp}^5 f_{d j p}^5 + f_{ahp}^5 f_{d j p}^5) + 4(\alpha_1 - 1)(f_{ajk}^1 f_{d j k}^1 + f_{ahk}^1 f_{d j k}^1) = 0.$$  \hspace{1cm} (4.38)

Finally $\partial_\nu u_\alpha \Phi_j \Phi_h A_d^\nu$ gives

$$- (\alpha_2 + 1)(f_{ajp}^5 f_{d j p}^5 + f_{ahp}^5 f_{d j p}^5) - 4(\alpha_1 + 1)(f_{ajk}^1 f_{d j k}^1 + f_{ahk}^1 f_{d j k}^1) = 2\beta_3,$$  \hspace{1cm} (4.39)

with

$$N_3 = -i \beta_3(a, d, j, h) A_{\nu a} A_d^\nu \Phi_j \Phi_h \delta$$  \hspace{1cm} (4.40)

$$\partial Q N_3 = \beta_3 \partial_\nu u_\alpha A_d^\nu \Phi_j \Phi_h \delta + \beta_3 m_j u_j A_{\nu a} A_d^\nu \Phi_h \delta.$$  \hspace{1cm} (4.41)

Subtracting (4.37) from (4.39) we find

$$\beta_3(a, d, j, h) = -2 f_{d ac} f_{chj}^1 - 8 f_{ahk}^1 f_{d j k}^1 - 2 f_{ahp}^5 f_{d j p}^5$$  \hspace{1cm} (4.42)

where the first term does not contribute to (4.40). The result (4.42) remains valid for $j = h$.

Subtracting now (4.37) and (4.38) and using previous results it follows

$$f_{ajp}^5 f_{d j p}^5 - f_{ahp}^5 f_{d j p}^5 = \frac{m_j^2 + m_h^2 - m_c^2}{2m_h m_j} f_{d ac} f_{chj}$$

$$- \frac{m_j^2 + m_h^2 - m_a^2}{m_j m_k} f_{ajk} - \frac{m_h^2 + m_j^2 - m_d^2}{4m_h m_k} f_{d h k}$$

$$+ \frac{m_k^2 + m_h^2 - m_a^2}{m_h m_k} f_{ahk} - \frac{m_j^2 + m_j^2 - m_d^2}{4m_j m_k} f_{d j k}.$$  \hspace{1cm} (4.43)

In the special case $a = j$ and $d = h$ ($j \neq h$) we have

$$\sum_{p=1}^{l} f_{j j p}^5 f_{j j p}^5 = \frac{1}{2m_h} \left\{ \sum_{c=1}^{r+s} (m_j^2 + m_h^2 - m_c^2) f_{j h c} f_{j h c} \right\}$$
\[- \sum_{k=1}^{r} \frac{m_k^4 - (m_j^2 - m_k^2)^2}{2m_k^2} f_{jkk} f_{jkh} \}.

(4.44)

The r.h.s. is known and generally different from 0, consequently \( f^5 \) must be also different from 0. In case of only one Higgs field \( t = 1 \), the Higgs coupling \( f^5 \) can be calculated from (4.44) as a square root. For \( t > 1 \) the Higgs couplings are no longer uniquely determined by gauge invariance. For fixed \( j \) equation (4.44) holds for all \( h \neq j \) and gives the same value on the l.h.s. This implies relations between the masses and the Yang-Mills couplings (see sect.6).

4.6 Sector \( uAA\Phi \):

In this sector there is only one Wick monomial \( u_a A_{\nu b} A^\nu_c \Phi_h \) which for \( b \neq c \) gives the relation

\[
4(f_{bad} f_{dch}^2 + f_{cad} f_{dbh}^2) - 4(f_{ahj} f_{bcj}^2 + f_{ahj} f_{cbj}^2) - 2(f_{ahp} f_{hbp}^2 + f_{ahp} f_{hbp}^2) = m_a \left( \beta_3(b, c, a, h) + \beta_3(c, b, a, h) \right),
\]

(4.45)

where (4.41) has been taken into account. Substituting (4.42) and previous results we obtain

\[
2m_a f_{bhj} f_{caj}^2 = \sum_{d > r} f_{abd} f_{bde}
\]

(4.46)

In the case \( h = b \neq c \) this leads to

\[
m_a m_b f_{aap} f_{bhp} \delta_{ac} = -m_c^2 \sum_{d > r} f_{abd} f_{bde}
\]

(4.47)

For \( b \neq h \neq c \) we find

\[
\sum_{d > r} (m_c^2 f_{bad} f_{chd} + m_b^2 f_{cad} f_{bhd}) =
\]

\[
= \sum_{j=1}^{r} \frac{1}{4m_j^2} \{ f_{bhj} f_{caj} [(m_j^2 - m_h^2)(3m_j^2 - m_a^2 + m_b^2) - m_h^2(m_j^2 + m_a^2 - m_b^2)]
\]

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\[ + f_{baj} f_{chj} [(m_j^2 - m_c^2)(3m_j^2 - m_a^2 + m_b^2) - m_k^2 (m_j^2 + m_a^2 - m_b^2)] \]

\[ + 2f_{bcj} f_{ahj} (m_j^2 + m_h^2 - m_a^2)(m_b^2 - m_c^2) \} \right] . \] (4.48)

In the remaining case \( b = c \) we have

\[ 2m_a f_{bap}^5 f_{bp}^5 - 2m_b f_{ahp}^5 f_{bhp}^5 = \]

\[ = - \frac{(m_k^2 + m_a^2 - m_b^2)(m_k^2 + m_h^2 - m_b^2)}{2m_a m_h} f_{bkh} - 2 \frac{m_k^2 - m_b^2}{m_h} f_{bad} f_{dbh}. \] (4.49)

For \( a \neq h \neq b \) this gives

\[ m_b f_{aap}^5 f_{bhp}^5 = \frac{m_b^2}{m_a} \sum_{d > r} (f_{bad})^2 \]

\[ + \sum_k \frac{(f_{bak})^2}{4m_a m_k^2} [(m_k^2 - m_b^2)(-3m_k^2 + m_a^2 + 2m_a^2) + m_a^4] \] (4.50)

and for \( h \neq a \neq b = c \neq h \) we get

\[ m_b^2 \sum_{d > r} f_{bdh} f_{kbd} = - \sum_{k=1}^r \sum_k f_{bkh} \frac{1}{4m_k^2} \]

\[ \times [(m_k^2 - m_b^2)(-3m_k^2 + m_a^2 - m_b^2 + m_a^2) + m_a^2 m_h^2]. \] (4.51)

4.7 Sector \( u A \varphi \varphi \):

From \( u_a \varphi_p \partial_p \varphi_q A_b^\nu \) we get

\[ 4f_{bac} f_{cpq}^6 - (\alpha_1 + 1) f_{ajq}^5 f_{bjp}^5 - (\alpha_1 - 3) f_{ajp}^5 f_{bjq}^5 \]

\[ - 4(\alpha_2 + 1) f_{aqv}^6 f_{bqv}^6 - 4(\alpha_2 - 3) f_{apv}^6 f_{bqv}^6 = 0, \] (4.52)

and, assuming \( p \neq q \), \( u_a \varphi_p \varphi_q \partial_p A_b^\nu \) gives

\[ (\alpha_1 - 1)(f_{ajp}^5 f_{bjq}^5 + f_{ajq}^5 f_{bjp}^5) + 4(\alpha_2 - 1)(f_{aqv}^6 f_{bqv}^6 + f_{apv}^6 f_{bqv}^6) = 0. \] (4.53)

Finally \( \partial_p u_a \varphi_p \varphi_q A_b^\nu \) gives

\[ -(\alpha_1 + 1)(f_{ajp}^5 f_{bjq}^5 + f_{ajq}^5 f_{bjp}^5) - 4(\alpha_2 + 1)(f_{aqv}^6 f_{bqv}^6 + f_{apv}^6 f_{bqv}^6) = 2\beta_4, \] (4.54)
with
\[ N_4 = -\frac{i}{2} \beta_4(a,b,p,q) A_{\alpha a} A^\nu_b \varphi_p \varphi_q \delta. \] (4.55)

Adding (4.53) and (4.54) and using previous results we get
\[ \beta_4(a,b,p,q) = -2 f^5_{aap} f^5_{aap} \delta_{ab} \] (4.56)
where no summation is involved. The same result remains valid for \( p = q \). One easily checks that all other relations in this sector are fulfilled.

### 4.8 Remaining sectors:

In the sector \( uAA\varphi \) we get another expression for the normalization factor \( \beta_2 \) in (4.29) which is consistent with (4.36). The sector \( u\Phi\Phi\varphi \) vanishes identically because \( f^4 = f^6 = f^{10} = 0 \). In the sector \( u\tilde{u}u\Phi \) we obtain the relation
\[
m_a f^5_{bjp} f^5_{dab} - m_b f^5_{ajp} f^5_{dbp} = \frac{m_j^2 - m_a^2 + m_d^2}{2m_j} f_{abc} f_{dej}
\]
\[ + f_{ajk} f_{dak} \frac{m_k^2 + m_j^2 - m_a^2}{4m_j m_k} (m_k^2 - m_a^2 + m_d^2) - f_{bjk} f_{dak} \frac{m_k^2 + m_j^2 - m_b^2}{4m_j m_k} (m_k^2 - m_a^2 + m_d^2). \] (4.57)
The sector \( u\varphi\varphi\varphi \) vanishes identically. In the sector \( u\Phi\Phi\Phi \) we find the following normalization term
\[ N_5 = -\frac{i}{2} \beta_5(l,j) \Phi^2_l \Phi^2_j \delta \] (4.58)
with
\[ \beta_5(l,j) = \sum_{p} \frac{\mu_p^2}{2m_l m_j} f^5_{ulp} f^5_{ljp}. \] (4.59)
By (4.35) this is independent of \( l,j \):
\[ \beta_5 = \sum_{p} \frac{\mu_p^2}{2} \left( \frac{f^5_{aap}}{m_a} \right)^2 \] (4.60)
with \( a < r \) arbitrary. Finally, in the sector \( u\Phi\varphi\varphi \) we obtain another normalization term
\[ N_6 = -\frac{i}{2} \beta_6(h,p,q) \Phi^2_h \varphi_p \varphi_q \delta \] (4.61)
where

\begin{align*}
\beta_6(h, p, q) &= \frac{1}{m_h} \left[-f_{hhp}^5 f_{hhq}^5 \frac{\mu_p^2 + \mu_q^2}{m_h} - 6 \sum_u f_{hhu}^5 f_{pq}^{11} \right] \\
&= - (\mu_p^2 + \mu_q^2) \frac{f_{aap}^3 f_{aaq}^3}{m_a} - 6 \sum_u f_{aau}^3 f_{pq}^{11} (4.62)
\end{align*}

is independent of $h$. The pure Higgs coupling $f_{pq}^{11}$ (2.20) is still completely free, it will be restricted at third order. In addition we shall need a pure Higgs normalization term of the form

\begin{align*}
N_7 = - \frac{i}{2} \beta_7(p, q, u, v) \varphi_p \varphi_q \varphi_u \varphi_v. 
\end{align*}

(4.63)

5 Third order gauge invariance

Instead of (4.4) we now have to look for local terms $\sim \delta^8(x - z, y - z)$ in

\begin{align*}
D_{3/1}^\mu(x, y, z) &= [T_{1/1}^\mu(x), T_2(y, z)] + [T_1(y), T_{2/1}^\mu(x, z)] + [T_1(z), \tilde{T}_{2/1}^\mu(x, y)] \\
D_{3/2}^\mu(x, y, z) &= [T_{1/1}^\mu(y), T_2(x, z)] + [T_1(x), T_{2/2}^\mu(y, z)] + [T_1(z), \tilde{T}_{2/2}^\mu(x, y)] \\
D_{3/3}^\mu(x, y, z) &= [T_1(x), T_{2/2}^\mu(y, z)] + [T_1(y), T_{2/2}^\mu(x, z)] + [T_{1/1}^\mu(z), \tilde{T}_2(x, y)]
\end{align*}

(5.1) (5.2) (5.3)

where $\tilde{T}_2$ refers to the inverse $S$-matrix [1]. The first term in (5.1) produces a local term if the second term in (3.17.1) is commuted with the second order normalization term (4.21). The latter contains $\delta(y - z)$ and the commutator $\sim \partial^\mu D(x - y)$ gives another $\delta(x - y)$ by the usual mechanism (4.8). The result is

\begin{align*}
(17.1/2) - (4.21) &= -2 f_{abc} f_{cd\ell} f_{a'\ell'd'} u_b A_{\mu a} A_{\nu a'} A_{\lambda b} A_{\delta d'} \delta(x - y) \delta(y - z). 
\end{align*}

(5.4)

To examine the second term in (5.1) we use the fact that $\tilde{T}_{2/1}^\mu = -T_{2/1}^\mu + \ldots$ plus terms which give to local contribution. From (4.19) we have

\begin{align*}
\partial^\mu T_{2/1}^\mu(x, z)|_{loc} = 2 f_{abc} f_{d\ell e} \partial_\lambda u_b A_\nu A_{\ell a} A^\lambda_\delta \delta(x - z).
\end{align*}

(5.5)
If this is commuted with the second term in (2.9), the anti-ghost - ghost contraction has two derivatives so that the resulting C-number distribution has $\omega = 0$ and, after splitting, allows a normalization term

$$(2.9/2) - (5.5) = -2i\alpha f_{a'b'b}f_{abc}f_{dec}u_{\alpha'b'}A_{\lambda a'}A_{\nu a}A_{\nu b}^e\delta(y - x)\delta(x - z). \quad (5.6)$$

After renaming the summation indices this has the same form as (5.4). However, the contributions from the second and third member in (5.1) cancel each other and similarly in (5.2). But in (5.3) these normalization terms survive and after suitable choice of $\alpha$ in (5.6) compensate the anomaly (5.4). Then the sector $uAAAA$ is gauge invariant. The situation is the same in the other sectors $uAA\Phi\Phi$, $uAA\Phi\varphi$ and $uAA\varphi\varphi$ containing $A$'s. Here, instead of $N_1$ the normalization terms $N_2, N_3$ and $N_4$ come into play.

Next we turn to the sector $u\Phi^3\varphi$ where we get two anomalies

$$(3.17.4/1) - (4.61) = f_{abp}^5\beta_6(j, p, u)u_a\Phi_h\Phi_j^2\varphi_u$$

$$(3.17.4/2) - (4.58) = -2f_{abp}^5\beta_5(h, j)u_a\Phi_h\Phi_j^2\varphi_p. \quad (5.7)$$

They must cancel each other because no normalization term is possible. This leads to the relation

$$\sum_{q=1}^t f_{aqp}^5\beta_6(j, q, p) = 2f_{aqp}^5\beta_5(a, j). \quad (5.8)$$

In case of one physical scalar ($t = 1$) this allows to determine the pure Higgs coupling $f^{11}$ in $\beta_6$ (4.62). Similarly, in the sector $u\Phi^3\varphi$ we find the relation

$$2\sum_{v=1}^t f_{aqv}^5\beta_7(v, p, q, u) = f_{aqp}^5\beta_6(a, q, u), \quad (5.9)$$

which, for $t = 1$, determines the quartic Higgs coupling (4.63). The remaining sectors $u\Phi^4$ and $u\Phi^2\varphi^2$ are automatically gauge invariant due to the facts that $\beta_5$ (4.60) is constant and $\beta_6(h, p, q)$ (4.62) is independent of $h$.

It is instructive to discuss the important special case $t = 1$ of one physical scalar in detail. Then (5.8) can be simplified as follows

$$\beta_6(j, 1, 1) = 2\beta_5(a, j),$$

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which, by (4.60) and (4.62), leads to

\[ f_{ppp}^{11} = -\frac{\mu_p^2}{2m_a}f_{aa1}^5. \]  

(5.10)

Here \( \mu_p \) is the Higgs mass and \( a \) is arbitrary. From (5.9) we get

\[ \beta_7(1, 1, 1, 1) = \frac{1}{2} \beta_0(a, 1, 1) = \frac{\mu_p^2}{2} \left( \frac{f_{aa1}^5}{m_a} \right)^2. \]  

(5.11)

Let us now collect all trilinear purely scalar coupling terms

\[ V_1 = i \left( f_{hj}^0 \Phi_h \Phi_j \varphi - f_{111}^{11} \varphi^3 \right) = 
- i \frac{\mu_p^2}{2m_a} f_{aa1}^5 \varphi \left( \sum_j \Phi_j^2 + \varphi^2 \right) \]  

(5.12)

and the quartic terms \( N_5, N_6 \) and \( N_7 \)

\[ V_2 = - i \left( \sum_{lj} \beta_5(l, j) \Phi_l^2 \Phi_j^2 + \beta_6 \varphi^2 \sum_j \Phi_j^2 + \beta_7 \varphi^4 \right) = 
- i \frac{\mu_p^2}{2} \left( \frac{f_{aa1}^5}{m_a} \right)^2 \left( \varphi^2 + \sum_j \Phi_j^2 \right)^2. \]  

(5.13)

Introducing the coupling constant \( g \) again, we must multiply (5.12) by \( g \) and (5.13) by \( g^2/2! \) because this is the second order contribution. Then the total scalar potential is equal to

\[ V_{\varphi} = -ig^2 \frac{\mu_p^2}{8m_a^2} (f_{aa1}^5)^2 \left[ \left( \varphi^2 + \sum_j \Phi_j^2 \right)^2 + \frac{4m_a}{g f_{aa1}^5} \varphi \left( \varphi^2 + \sum_j \Phi_j^2 \right) \right]. \]  

(5.14)

Completing the square inside the square bracket just amounts to addition of a mass term for the Higgs field

\[ V(\varphi) = V_{\varphi} - \frac{i}{2} \mu_p^2 \varphi^2 = 
- ig^2 \frac{\mu_p^2}{8m_a^2} (f_{aa1}^5)^2 \left[ \varphi^2 + \sum_j \Phi_j^2 + \frac{2m_a}{g f_{aa1}^5} \varphi \right]^2. \]  

(5.15)

This is the asymmetric Higgs potential. In fact, introducing the shifted Higgs field

\[ \tilde{\varphi} = \varphi + a, \quad a = \frac{m_a}{g f_{aa1}^5}, \]  

(5.16)
the Higgs potential (5.15) assumes a symmetric double-well form

\[ V \sim (\tilde{\phi}^2 + \sum_j \Phi_j^2 - a^2)^2. \]

The shifted Higgs field then has a non-vanishing vacuum expectation value \(a\) (5.16), so that we have recovered (i.e. actually deduced) the usual Higgs mechanism.

## 6 Derivation of the electroweak gauge theory

Let us seek all gauge theories with three massive gauge fields \(m_1, m_2, m_3 \neq 0\) and one massless photon field \(m_4 = 0\). There are many 4-dimensional Lie algebras, but we will see that gauge invariance is strong enough to fix the \(f_{abc}\) uniquely.

We put \(a = 4, d = 2, j = 1, h = 2\) in (4.43)

\[ 0 = f_{243}f_{321} \left( \frac{m_1^2 + m_2^2 - m_3^2}{2m_1m_2} \right) + f_{423}f_{213} \left( \frac{m_2^2 + m_3^2}{m_2m_3} \right) \left( \frac{m_2^2 + m_1^2 - m_2^2}{4m_1m_3} \right) \]

\[ = \frac{f_{243}f_{321}}{4m_1m_2m_3^2} (m_3^2m_1^2 + 2m_3^2m_2^2 - 3m_4^2 - m_1^2m_2^2 + m_4^2). \quad (6.1) \]

Since the bracket is different from zero, we must either have \(f_{243} = 0\) or \(f_{321} = 0\). We shall verify below that the second alternative leads to the trivial solution \(f = 0\) so we concentrate on the first case. For \(a = 4, d = 1, h = 2, j = 1\) we find from (4.43)

\[ 0 = \frac{f_{143}f_{321}}{4m_1m_2m_3^2} [m_3^2(m_2^2 + 2m_1^2 - 3m_3^2) + m_1^2(m_1^2 - m_2^2)], \quad (6.2) \]

which implies \(f_{143} = 0\). Next we put \(j = 1, h = 2\) in (4.44) and also \(j = 1, h = 3\):

\[ \sum_p (f_{11p}^5)^2 = \frac{1}{2m_2^2} \left[ (f_{123})^2(m_1^2 + m_2^2 - m_3^2) - \right. \]

\[ - (f_{123})^2 \frac{m_3^4 - (m_1^2 - m_2^2)^2}{2m_3^2} \]

\[ = \frac{1}{2m_3^2} \left[ (f_{132})^2(m_1^2 + m_3^2 - m_2^2) - (f_{132})^2 \frac{m_3^4 - (m_1^2 - m_3^2)^2}{2m_3^2} \right]. \quad (6.3) \]

This implies

\[ \left( \frac{f_{124}}{f_{123}} \right)^2 = 2 \frac{m_3^2(m_3^2 - m_1^2) + m_2^2(m_1^2 - m_2^2)}{m_3^2(m_1^2 + m_2^2)}. \quad (6.4) \]
If the r.h.s. is different from zero we have \( f_{124} \neq 0 \), otherwise the solution would be trivial. Then it follows from (3.19) that \( m_1 = m_2 \) which is the equal mass of the W-bosons. This simplifies (6.4) as follows

\[
\left( \frac{f_{124}}{f_{123}} \right)^2 = \frac{m_3^2}{m_1^2} - 1,
\]

(6.5)

which implies \( m_3 > m_1 \). Defining the weak mixing angle \( \Theta \) by

\[
\frac{m_1}{m_3} = \cos \Theta,
\]

(6.6)

we have

\[
\left( \frac{f_{124}}{f_{123}} \right)^2 = \tan^2 \Theta.
\]

Since a common factor in the \( f \)'s can be absorbed in the coupling constant \( g \), we end up with

\[
f_{124} = -\sin \theta, \quad f_{123} = -\cos \Theta
\]

(6.7)

in agreement with the Weinberg-Salam model [13, 14]. All other structure constants follow by asymmetry. The signs in (6.7) have been chosen according to standard convention, as well as \( m_3 \) for \( m_Z \). Of course any permutation of the indices 1, 2, 3 is possible, but the solution remains the same.

It remains to discuss the possibility \( f_{123} = 0 \). Then it follows from (6.3) that \( m_1 = m_2 = m_3 \) and \( |f_{124}| = |f_{134}| \). If we now put \( a = h = 4 \) and \( d = j = 1 \) in (4.43) we arrive at

\[
0 = \frac{1}{2} (f_{142} f_{241} + f_{143} f_{341}) - \frac{1}{4} (f_{412} f_{142} + f_{413} f_{143})
\]

\[
= -\frac{1}{4} ((f_{142})^2 + (f_{143})^2).
\]

Hence, all \( f \)'s vanish in this case.

It is not hard to verify that for the unique non-trivial solution (6.7) all conditions for gauge invariance are satisfied. By means of (6.7) all couplings can be calculated in the case of one Higgs field and are in complete agreement with the standard electroweak theory [15]. But we could use any number \( t \geq 1 \) of Higgs fields. Only for \( t = 1 \) the couplings are completely determined by gauge invariance.
In the same way one can construct the gauge theory with only two massive gauge fields $m_1, m_2 \neq 0$ and one massless field $m_3 = 0$. This is not the $SU(2)$ Higgs-Kibble model often discussed in the literature which has three massive fields. It turns out that $m_1 = m_2$ must be equal, so that this theory is a hypothetical electroweak theory without neutral currents. Therefore, the gauge principle cannot explain why there are neutral currents in nature.

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