HEAT KERNEL ESTIMATES FOR CRITICAL FRACTIONAL DIFFUSION OPERATOR

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Abstract. In this work we construct the heat kernel of the \( \frac{1}{2} \)-order Laplacian perturbed by the first-order gradient term in Hölder space and the zero-order potential term in generalized Kato’s class, and obtain sharp two-sided estimates as well as the gradient estimate of the heat kernel.

1. Introduction and Main Result

For \( \alpha \in (0, 2) \), let \( \Delta^\alpha \) be the fractional Laplacian in \( \mathbb{R}^d \) defined by

\[
\Delta^\alpha f(x) = \lim_{\epsilon \to 0} \int_{|y| > \epsilon} \frac{f(x + y) - f(x)}{|y|^{d+\alpha}} \, dy.
\]

It is well-known that the heat kernel \( \rho^{(\alpha)}(t, x) \) of \( \Delta^\alpha \) has the following estimate (e.g. see [9, 8]):

\[
\rho^{(\alpha)}(t, x) \asymp t^{-\frac{d}{\alpha}} |x|^{-\frac{d}{\alpha}} + |x|^{\frac{d}{\alpha}},
\]

where \( \asymp \) means that both sides are comparable up to some positive constants.

In [3], Bogdan and Jakubowski studied the following perturbation of \( \Delta^\alpha \) by gradient operator:

\[
\mathcal{L}^{(\alpha)}_b(x) := \Delta^\alpha + b(x) \cdot \nabla, \quad \alpha \in (1, 2),
\]

where \( b \) belongs to Kato’s class \( \mathcal{H}_d^{\alpha-1} \), i.e., for \( \gamma > 0 \),

\[
b \in \mathcal{H}_d^{\gamma} := \left\{ f \in L^1_{loc}(\mathbb{R}^d) : \limsup_{x \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \epsilon} \frac{|f(y)|}{|x-y|^{d-\gamma}} \, dy = 0 \right\}.
\]

Notice that by Hölder’s inequality, \( L^p(\mathbb{R}^d) \subset \mathcal{H}_d^{\gamma} \) provided \( p > \frac{d}{\gamma} \). The sharp two-sided heat kernel estimates of \( \mathcal{L}^{(\alpha)}_b \) like (1.1) were obtained therein. The reason of limiting \( \alpha \in (1, 2) \) lies in that the heat kernel \( p^{(\alpha)}_1(t, x) \) of \( \Delta^\alpha_1 \) is not comparable with \( \rho^{(\alpha)}(t, x) \) for \( \alpha \in (0, 1) \) (see [3]). In [13], Jakubowski and Szczypkowski considered the time-dependent perturbation of \( \Delta^\alpha \). In [11], Jakubowski established the global time estimate of heat kernel of \( \Delta^\alpha \) with small singular drift. In [6], Chen, Kim and Song obtained sharp two-sided estimates for the Dirichlet heat kernel of \( \mathcal{L}^{(\alpha)}_b \). Moreover, the Dirichlet heat kernel estimates for nonlocal operators under Feynman-Kac or Schrödinger type perturbations were also considered in [7]. Recently, in [22], Wang and the second named author extended Bogdan and Jakubowski’s results to the more general subordinated stable operator over Riemannian manifold and obtained sharp two-sided estimates as well as the gradient estimate.

However, in the critical case of \( \alpha = 1 \), the heat kernel estimate of \( \mathcal{L}^{(1)}_b \) is left open. It is noticed that the critical case has particular interest in physics and mathematics (see [5, 15, 14, 19, 20] and references therein). We first recall some related results. In [17], Maekawa and Miura obtained the upper bounds estimates for the fundamental solutions of general nonlocal diffusions with divergence free drift. Their proofs are based upon the classical Davies’ method. In [19] and [20], Silvestre established the Hölder regularity to the critical parabolic operator \( \mathcal{L}^{(1)}_b(x) \) with bounded measurable \( b \). In [18], Priola proved the pathwise uniqueness of SDEs.
with Hölder’s drifts and driven by Cauchy processes. In [25], the well-posedness of multidimensional critical Burgers’ equation was obtained (see [14] for the study of one dimensional critical Burgers’ equations).

In this paper we consider the following critical fractional diffusion operator

\[ \mathcal{L}(t, x) := \mathcal{L}_{a,b,c}(t, x) := a(t, x) \Delta^\beta + b(t, x) \cdot \nabla + c(t, x), \]

where \( a, c : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R} \) and \( b : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) are three measurable functions. We shall prove the following result in the present work.

**Theorem 1.1.** Assume that for some \( a_0, a_1 > 0, \)

\[ a_0 \leq a(t, x) \leq a_1, \]

and for some \( \beta \in (0, 1), \)

\[ a, b \in \mathbb{H}^\beta, \quad c \in \mathbb{K}_\beta, \]

where \( \mathbb{H}^\beta \) (resp. \( \mathbb{K}_\beta \)) is the Hölder space (resp. the generalized Kato’s class) defined in Definition [2.1]. Then there exists a continuous function \( p(t, x; s, y) \) such that:

(i) (C-K equation) For all \( 0 \leq s < r < t \) and \( x, y \in \mathbb{R}^d, \) the following Chapman-Kolmogorov’s equation holds:

\[ \int_{\mathbb{R}^d} p(t, x; r, z)p(r, z; s, y)dz = p(t, x; s, y). \]  

(ii) (Generator) For any bounded continuous function \( f, \) we have

\[ \lim_{t \downarrow s} P_{t,s}f(x) := \lim_{t \downarrow s} \int_{\mathbb{R}^d} p(t, x; s, y)f(y)dy = f(x), \]

and if \( a, b, c \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^d)), \) then for all \( f, g \in C_0^\infty(\mathbb{R}^d), \)

\[ \lim_{t \downarrow s} \frac{1}{t-s} \int_{\mathbb{R}^d} g(x)(P_{t,s}f(x) - f(x))dx = \int_{\mathbb{R}^d} g(x)\mathcal{L}(s, x)f(x)dx, \quad s > 0. \]

(iii) (Two-sided estimates) For any \( T > 0, \) there exist constants \( \kappa_1, \kappa_2 > 0 \) such that for all \( 0 \leq s < t \leq T \) and \( x, y \in \mathbb{R}^d, \)

\[ |p(t, x; s, y)| \leq \kappa_1(t-s)(|x-y| + (t-s))^{-d-1}, \]  

and in the case of \( a(t, x) = a(t) \) independent of \( x, \)

\[ p(t, x; s, y) \geq \kappa_2(t-s)(|x-y| + (t-s))^{-d-1}. \]

(iv) (Hölder’s estimate) Assume that \( c \in \mathbb{K}_d^{-\gamma} \) for some \( \gamma \in (0, 1). \) Then for any \( T > 0, \) there exists a constant \( \kappa_3 > 0 \) such that for all \( 0 \leq s < t \leq T \) and \( x, x', y \in \mathbb{R}^d, \)

\[ |p(t, x; s, y) - p(t, x'; s, y)| \leq \kappa_3(|x-x'|^\gamma + 1)|t-s|^{1-\gamma} \times \left( (|x-y| + (t-s))^{-d-1} + (|x'-y| + (t-s))^{-d-1} \right). \]  

(v) (Gradient estimate) If we further assume that \( c \in \mathbb{K}_d^\gamma \) for some \( \gamma \in (0, 1), \) then for any \( T > 0, \) there exists a constant \( \kappa_4 > 0 \) such that for all \( 0 \leq s < t \leq T \) and \( x, y \in \mathbb{R}^d, \)

\[ |\nabla_x p(t, x; s, y)| \leq \kappa_4(|x-y| + (t-s))^{-d-1}. \]

In order to prove this theorem, we shall use Levi’s method of freezing coefficients and Duhamel’s formula. Compared with the classical case of second order parabolic equations, the main difficulty of proving this theorem lies in the heavy tail property of Poisson’s kernel.
and the nonlocal property of $\Delta^{\frac{1}{2}}$. We mention that in the case of second order parabolic equation, the following property of Gaussian heat kernel plays a key role in the construction of Levi’s argument (cf. [10, 16]): for $\beta \in (0,1)$ and some $C > 0$,

$$t^{-\frac{1}{2}}|x|^\beta e^{-\frac{|x|^2}{t}} \leq t^{-\frac{\beta}{2}} e^{-\frac{|x|^2}{2t}}, \quad t > 0, x \in \mathbb{R}^d.$$ 

This means that the spatial Hölder regularity can compensate the time singularity. However, such type estimate does not hold for Poisson’s kernel in view of the heavy tail property. A suitable substitution is an analogue of the so called 3P-inequality (see Lemma 2.3 below).

This paper is organized as follows: In Section 2, we prepare some lemmas for later use. In Section 3, by using Levi’s method of constructing the fundamental solutions, we first construct the heat kernel of $\mathcal{L}_{a,b} = \mathcal{L}_{a,0}$. In Section 4, we prove Theorem 1.1 by using Duhamel’s formula.

We conclude this section by introducing the following conventions: The letter $C$ with or without subscripts will denote a positive constant, whose value is not important and may change in different places. We write $f(x) \lesssim g(x)$ to mean that there exists a constant $C_0 > 0$ such that $f(x) \leq C_0 g(x)$; and $f(x) \asymp g(x)$ to mean that there exist $C_1, C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$.

## 2. Preliminaries

For $\gamma, \beta \in \mathbb{R}$, we introduce the following function on $\mathbb{R}_+ \times \mathbb{R}^d$ for later use:

$$\gamma^\beta(t, x) := t^\beta(\beta_{\gamma} \wedge 1)(|x|^\beta + t^2)^{-\frac{d+1}{2}} \asymp t^\beta(\beta_{\gamma} \wedge 1)(|x| + t)^{-d-1}. \quad (2.1)$$

By simple calculations, there exists a constant $C_d > 0$ such that for all $\beta \in [0, \frac{1}{2}]$ and $\gamma \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \gamma^\beta(t, x)dx \leq C_d t^{\beta+\beta_0-1}. \quad (2.2)$$

Indeed, we have

$$\int_{\mathbb{R}^d} \frac{|x|^\beta}{(|x| + t)^{d+1}}dx \leq \int_0^{\infty} t^\beta + d-1 \frac{r^{\beta+d-1}}{(r + t)^{d+1}}dr = \left( \int_0^t + \int_t^{\infty} \right) \frac{r^{\beta+d-1}}{(r + t)^{d+1}}dr$$

$$\leq \frac{t^{\beta+d-1}}{d+1} + \int_t^{\infty} \frac{r^{\beta-1}}{r^{d+1}}dr = \frac{t^{\beta-1}}{d+1} + \frac{t^{\beta-1}}{1 - \beta},$$

which in turn implies (2.2). Notice that the following 3P-inequality holds (cf. [3, Lemma 2.1]):

$$\gamma^\beta_0(t, x) \gamma^\beta_0(s, y) \leq (\gamma^\beta_0(t, x) + \gamma^\beta_0(s, y)) \gamma^\beta_0(t + s, x + y). \quad (2.3)$$

We introduce the following classes of functions used in this paper.

**Definition 2.1. (Hölder’s space)** For $\beta \in (0, 1]$, define

$$H^\beta := \left\{ f \in \mathcal{B}(\mathbb{R} \times \mathbb{R}^d) : \|f\|_{H^\beta} := \sup_{r \in \mathbb{R}} \sup_{x \in \mathbb{R}^d} |f(t, x)| + \sup_{r \in \mathbb{R}} \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(t, x) - f(t, y)|}{|x - y|^\beta} < \infty \right\}.$$ 

(Generalized Kato’s class) For $\gamma > 0$, define

$$K_\gamma^\gamma := \left\{ f \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d) : \lim_{\varepsilon \downarrow 0} K^\gamma(\varepsilon) = 0 \right\},$$

where

$$K^\gamma(\varepsilon) := \sup_{(t, x) \in [0, \varepsilon] \times \mathbb{R}^d} \int_0^\varepsilon \int_{\mathbb{R}^d} \gamma^\gamma_0(s, y) |f(t \pm s, x - y)|dyds, \quad \varepsilon > 0.$$
A function \( f(t, x) \) on \([0, \infty) \times \mathbb{R}^d\) will be automatically extended to \( \mathbb{R} \times \mathbb{R}^d \) by letting \( f(t, \cdot) = 0 \) for \( t < 0 \). The following proposition gives a characterization for \( \mathbb{K}_d^\gamma \) (see [1, 24, 22] for more discussions).

**Proposition 2.2.** For \( \gamma > 0 \) and \( p, q \in [1, \infty) \) with \( \frac{d}{p} + \frac{1}{q} < \gamma \), we have
\[
L^q(\mathbb{R}; L^p(\mathbb{R}^d)) \subset \mathbb{K}_d^\gamma,
\]
and for \( \gamma \in (0, d) \),
\[
\mathcal{K}_d^\gamma \subset \mathbb{K}_d^\gamma.
\]

**Proof.** Noticing that
\[
\int_0^\infty \int_{\mathbb{R}^d} \varrho_0^\gamma(s, y)|f(t \pm s, x - y)|dyds = \int_0^\infty s^\gamma \int_{\mathbb{R}^d} \varrho_1^\gamma(s, y)|f(t \pm s, x - y)|dyds,
\]
by Hölder’s inequality, for the first inclusion, it is enough to prove
\[
\lim_{\varepsilon \downarrow 0} I(\varepsilon) = 0,
\]
where
\[
I(\varepsilon) := \int_0^\infty \left( \int_{\mathbb{R}^d} \varrho_0^\gamma(s, y) p \right)^{\frac{q^*}{p}} s^{(\gamma - 1)q^*} ds
\]
with \( q^* = \frac{q}{q - 1} \) and \( p^* = \frac{p}{p - 1} \). As in the proof of (2.2), we have
\[
\int_{\mathbb{R}^d} \varrho_0^\gamma(s, y) p \leq s^{d - dp^*},
\]
and so,
\[
I(\varepsilon) \leq \int_0^\infty \frac{d\varepsilon}{p} s^{d - dp^* + (\gamma - 1)q^*} ds \leq \varepsilon^{1 + \frac{d\varepsilon}{p} - d\gamma + (\gamma - 1)q^*},
\]
since \( \frac{d\varepsilon}{p} - d\gamma + (\gamma - 1)q^* < -1 \) by \( \frac{d}{p} + \frac{1}{q} < \gamma \), thus (2.4) holds.

Next we prove the second inclusion. Assume \( f \in \mathcal{K}_d^\gamma \). By definition, we have
\[
\sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} \varrho_0^\gamma(s, y)|f(x - y)|dyds \leq I_1(\varepsilon) + I_2(\varepsilon),
\]
where
\[
I_1(\varepsilon) := \sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{|y| \leq \varepsilon} \frac{s^\gamma |f(x - y)|}{(|y| + s)^{d+1}} dyds,
\]
\[
I_2(\varepsilon) := \sup_{x \in \mathbb{R}^d} \int_0^\infty \int_{|y| > \varepsilon} \frac{s^\gamma |f(x - y)|}{(|y| + s)^{d+1}} dyds.
\]
For \( I_1(\varepsilon) \), in view of \( \gamma < d \), we have
\[
I_1(\varepsilon) \leq \sup_{x \in \mathbb{R}^d} \int_{|y| \leq \varepsilon} |f(x - y)| \left( \int_{|y|}^\infty s^\gamma ds + |y|^{d-1} \int_0^{\gamma|y|} s^\gamma ds \right) dy
\]
\[
\leq \sup_{x \in \mathbb{R}^d} \int_{|y| \leq \varepsilon} |f(x - y)| \left( \frac{|y|^{d+\gamma}}{d - \gamma} + \frac{|y|^{d+\gamma}}{\gamma + 1} \right) dy \to 0, \quad \varepsilon \downarrow 0.
\]
For \( I_2(\varepsilon) \), we have
\[
I_2(\varepsilon) \leq \sup_{x \in \mathbb{R}^d} \int_{|y| > \varepsilon} |f(x - y)| \frac{dy}{|y|^{d+1}} \int_0^\infty s^\gamma ds = \frac{1}{\gamma + 1} \sup_{x \in \mathbb{R}^d} \int_{|y| > \varepsilon} \frac{e^{\gamma|y|} |f(x - y)|}{|y|^{d+1}} dy,
\]
which converges to zero by [3 Lemma 11] as \( \varepsilon \downarrow 0 \). \( \square \)
Set for $s < t$ and $x, y \in \mathbb{R}^d$,
\[ \mathcal{G}_\gamma^\beta(t, x; s, y) := \mathcal{G}_\gamma^\beta(t - s, x - y). \]
The following lemma is an analogue of 3P-inequality, which will play a crucial role in the sequel.

**Lemma 2.3.** For all $\beta_1, \beta_2 \in [0, \frac{1}{2}]$ and $y_1, y_2 \in \mathbb{R}$, we have
\[
\int_{\mathbb{R}^d} \mathcal{G}_{\gamma_1}^{\beta_1}(t, x; r, z) \mathcal{G}_{\gamma_2}^{\beta_2}(r, z; s, y) dz \leq C_d \left[ (t - r)^{\gamma_1 + \beta_1 + \beta_2 - 1} (r - s)^{\gamma_2 - \beta_1} \mathcal{G}_0^0(t, x; s, y) + (t - r)^{\gamma_1 + \beta_1 + \beta_2 - 1} \mathcal{G}_0^0(t, x; s, y) + (t - r)^{\gamma_2} (r - s)^{\gamma_2 + \beta_1 + \beta_2 - 1} \mathcal{G}_0^0(t, x; s, y) \right],
\]
where $C_d$ only depends on $d$; and if $\gamma_1 > -\beta_1$ and $\gamma_2 > -\beta_2$, then
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{G}_{\gamma_1}^{\beta_1}(t, x; r, z) \mathcal{G}_{\gamma_2}^{\beta_2}(r, z; s, y) dz dr \leq C_d \left[ \mathcal{G}_{\gamma_1 + \gamma_2 + \beta_1 + \beta_2}^0(t, x; s, y) \mathcal{B}(y_1 + \beta_1 + \beta_2 + 1 + \gamma_2) + \mathcal{G}_{\gamma_1 + \beta_1 + \beta_2}^0(t, x; s, y) \mathcal{B}(\gamma_2 + \beta_1 + \beta_2 + 1 + \gamma_1) \right],
\]
where $\mathcal{B}(y, \beta)$ is the usual Beta function defined by
\[ \mathcal{B}(y, \beta) := \int_0^1 (1 - s)^{y - 1} s^{\beta - 1} ds, \quad y, \beta > 0. \]
Moreover, there exist $p > 1$ and a constant $C > 0$ such that for all $0 \leq s < t \leq 1$ and $x \neq y \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \mathcal{G}_{\gamma_1}^{\beta_1}(t, x; r, z) \mathcal{G}_{\gamma_2}^{\beta_2}(r, z; s, y) dz \right)^p dr \leq \frac{C}{|x - y|^{(d + 1)p}}. \tag{2.7}
\]
**Proof.** First of all, in view of
\[
(|x - y|^2 + |t - s|^2) \frac{d^3}{d^4} \leq 2^d \left( (|x - z|^2 + |t - r|^2) \frac{d^3}{d^4} + (|z - y|^2 + |r - s|^2) \frac{d^3}{d^4} \right),
\]
we have
\[
\mathcal{G}_{\gamma_0}^0(t, x; r, z) \mathcal{G}_{\gamma_0}^0(r, z; s, y) \leq 2^d \left( \mathcal{G}_{\gamma_0}^0(t, x; r, z) + \mathcal{G}_{\gamma_0}^0(r, z; s, y) \right) \mathcal{G}_{\gamma_0}^0(t, x; s, y). \tag{2.8}
\]
Noticing that by $(a + b)^\beta \leq a^\beta + b^\beta$ for $\beta \in (0, 1)$,
\[
(|x - z|^{\beta_1} \land 1)(|y - z|^{\beta_2} \land 1) \leq (|x - z|^{\beta_1} \land 1)(|x - z|^{\beta_2} + |x - y|^{\beta_2} \land 1) \leq |x - z|^{\beta_1 + \beta_2} \land 1 + (|x - z|^{\beta_1} \land 1)(|x - y|^{\beta_2} \land 1),
\]
\[
(|x - z|^{\beta_1} \land 1)(|y - z|^{\beta_2} \land 1) \leq (|y - z|^{\beta_1} + |x - y|^{\beta_1} \land 1)(|y - z|^{\beta_2} \land 1) \leq |y - z|^{\beta_1 + \beta_2} \land 1 + (|y - z|^{\beta_1} \land 1)(|x - y|^{\beta_1} \land 1),
\]
we have
\[
\mathcal{G}_{\gamma_1}^{\beta_1}(t, x; r, z) \mathcal{G}_{\gamma_2}^{\beta_2}(r, z; s, y)
\]
\[
= |t - r|^{\gamma_1} |r - s|^{\gamma_2} (|x - z|^{\beta_1} \land 1)(|y - z|^{\beta_2} \land 1) \mathcal{G}_{\gamma_0}^0(t, x; r, z) \mathcal{G}_{\gamma_0}^0(r, z; s, y) \leq |t - r|^{\gamma_1} |r - s|^{\gamma_2} (|x - z|^{\beta_1} \land 1)(|x - y|^{\beta_2} \land 1) \mathcal{G}_{\gamma_0}^0(t, x; s, y)
\]
\[
	imes \mathcal{G}_{\gamma_0}^0(t, x; r, z) \mathcal{G}_{\gamma_0}^0(r, z; s, y)
\]
Then we have

\[ \begin{align*}
&\text{Lemma 2.4.} \\
&\quad\text{and} \\
&\quad\text{which is also called Poisson kernel (cf. \([21]\)).}
\end{align*} \]

By elementary calculations, one has

\[ \begin{align*}
&\text{We define} \\
&\quad\text{Let} \quad \gamma_x^0(t, z; s, y) = \mathcal{G}_0^0(t, x; s, y) \\
&\quad\text{by observing that for} \gamma, \beta > 0, \\
&\quad\text{Estimate (2.5) follows by (2.2), and estimate (2.6) follows by} \\
&\quad\text{As for estimate (2.7), it follows by (2.5) and (2.9).} \quad \square
\end{align*} \]

Let \( \rho(t, x) \) be the heat kernel of the Cauchy operator \( \Delta^\frac{1}{2} \), i.e.,

\[ \partial_t \rho(t, x) = \Delta^\frac{1}{2} \rho(t, x). \] (2.10)

It is well-known that

\[ \rho(t, x) = \pi^{-\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right) (|x|^2 + t^2)^{-\frac{d+1}{2}} = \pi^{-\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right) \gamma_x^0(t, x), \]

which is also called Poisson kernel (cf. \([21]\)). By elementary calculations, one has

\[ \begin{align*}
&\text{Let} \quad a : [0, \infty) \times \mathbb{R}^d \to (0, \infty) \text{ and } b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \text{ be two bounded measurable functions.} \\
&\text{We define} \\
&\quad p_0(t, x; s, y) := \rho \left( \int_s^t a(r, y) dr, x - y + \int_s^t b(r, y) dr \right), \\
&\text{and} \\
&\quad \mathcal{L}_{a,b}^x(t, y) := a(t, y) \Delta^\frac{1}{2} + b(t, y) \cdot \nabla_x. \\
&\text{By (2.10) and the Lebesgue differential theorem, we have for all} x, y \in \mathbb{R}^d \text{ and almost all} t > s, \\
&\quad \partial_t p_0(t, x; s, y) = \mathcal{L}_{a,b}^x(t, y) p_0(t, s, y)(x). \\
&\text{We prepare the following important estimates for later use.}
\end{align*} \]

**Lemma 2.4.** Suppose that for some \( a_0, a_1, b_1 > 0, \)

\[ a_0 \leq a(r, y) \leq a_1, \quad |b(r, y)| \leq b_1. \] (2.16)

Then we have

\[ p_0(t, x; s, y) = \mathcal{G}_1^0(t, x; s, y), \] (2.17)

and

\[ \begin{align*}
&\text{We have} \\
&\quad |\Delta^\frac{1}{2} p_0(t, x; s, y)| \leq (|x - y| + |t - s|)^{-d-1}, \\
&\quad |\nabla_x p_0(t, x; s, y)| \leq |t - s| (|x - y| + |t - s|)^{-d-2}, \\
&\quad |\partial_t p_0(t, x; s, y)| \leq (|x - y| + |t - s|)^{-d-1}, \\
&\quad |\nabla_x \Delta^\frac{1}{2} p_0(t, x; s, y)| \leq (|x - y| + |t - s|)^{-d-2}, \\
&\quad |\nabla^2_x p_0(t, x; s, y)| \leq (|x - y| + |t - s|)^{-d-2}.
\end{align*} \] (2.18) (2.19) (2.20) (2.21) (2.22)
Moreover, if we further assume that \(a, b \in \mathbb{H}_\beta\) for some \(\beta \in (0, 1)\), then

\[
\left| \int_{\mathbb{R}^d} \nabla_x p_0(t, x; s, y) dy \right| \leq (t - s)^{\beta - 1}, \tag{2.23}
\]

\[
\left| \int_{\mathbb{R}^d} \Delta_t^{\frac{1}{2}} p_0(t, x; s, y) dy \right| \leq (t - s)^{\beta - 1}, \tag{2.24}
\]

\[
\left| \int_{\mathbb{R}^d} \partial_t p_0(t, x; s, y) dy \right| \leq (t - s)^{\beta - 1}, \tag{2.25}
\]

\[
\lim_{t \to x} \int_{\mathbb{R}^d} p_0(t, x; s, y) dy = 1, \tag{2.26}
\]

and for all \(w \in \mathbb{R}^d\) and \(\gamma \in [0, \beta]\),

\[
\left| \int_{\mathbb{R}^d} (\nabla_x p_0(t, x + w; s, y) - \nabla_x p_0(t, x; s, y)) dy \right| \leq |w|^\gamma (t - s)^{\beta - \gamma - 1}. \tag{2.27}
\]

**Proof.** For the simplicity of notation, we write

\[ F_s^i(y) := \int_s^y a(r, y) dr, \quad G_s^i(y) := \int_s^y b(r, y) dr. \]

(1) By (2.16), we have

\[ F_s^i(y) \asymp t - s, \tag{2.28} \]

and for any \(|w| \leq |t - s|\),

\[
|x + w - y + G_s^i(y) dr| + |t - s| \asymp |x - y| + |t - s|. \tag{2.29}
\]

Estimate (2.17) follows by definition. For (2.18), by (2.10) we have

\[
\Delta_t^{\frac{1}{2}} p_0(t, x; s, y) = (\Delta_t^{\frac{1}{2}} \rho)(F_s^i(y), x - y + G_s^i(y)) = (\partial_t \rho) (F_s^i(y), x - y + G_s^i(y)).
\]

Estimate (2.18) follows by (2.11). Similarly, (2.19)-(2.22) follow by (2.11), (2.12) and (2.15).

(2) Define

\[
\xi(t, x; s, y; z) := \rho \left( \int_s^y a(r, z) dr, x - y + \int_s^y b(r, z) dr \right) = \rho \left( F_s^i(z), x - y + G_s^i(z) \right).
\]

Clearly, for any \(s < t\) and \(x, z \in \mathbb{R}^d\),

\[
\int_{\mathbb{R}^d} \xi(t, x; s, y; z) dy = \int_{\mathbb{R}^d} \rho \left( F_s^i(z), y \right) dy = 1
\]

and

\[
\int_{\mathbb{R}^d} \nabla_x \xi(t, x; s, y; z) dy = 0, \quad \int_{\mathbb{R}^d} \Delta_t^{\frac{1}{2}} \xi(t, x; s, y; z) dy = 0.
\]

Thus, for proving (2.23), it suffices to prove that

\[
\left| \int_{\mathbb{R}^d} \left( \nabla_x p_0(t, x; s, y) - \nabla_x \xi(t, x; s, y; z) \right) dy \right|_{z=x} \leq (t - s)^{\beta - 1}. \tag{2.30}
\]

By \(a, b \in \mathbb{H}_\beta\) and definitions of \(p_0\) and \(\xi\), one has

\[
|\nabla_x p_0(t, x; s, y) - \nabla_x \xi(t, x; s, y; z)|_{z=x} = |(\nabla_x \rho)(F_s^i(y), x - y + G_s^i(y)) - (\nabla_x \rho)(F_s^i(x), x - y + G_s^i(x))|
\]

\[
\leq \|a\|_{\mathbb{H}_\beta} |x - y|^\beta + 1)(t - s) \int_0^1 |\nabla_x \partial_\theta \rho| (\theta F_s^i(y) + (1 - \theta) F_s^i(x), x - y + G_s^i(y)) d\theta
\]
This estimate is important for the lower bound estimate of the heat kernel. Similarly, we can prove

\[
\left| \int_{\mathbb{R}^d} \left( \Delta^\frac{1}{2} p_0(t, x; s, y) - \Delta^\frac{1}{2} \xi(t, x; s, y) \right) dy \right|_{z=x} \leq (t-s)^{\beta-1},
\]

and

\[
\left| \int_{\mathbb{R}^d} \left( p_0(t, x; s, y) - \xi(t, x; s, y) \right) dy \right|_{z=x} \leq (t-s)^{\beta}.
\]

Thus, (2.24) and (2.26) follow. (3) Next, we prove (2.25). By (2.15), (2.18), (2.19), (2.23) and (2.24), we have

\[
\left| \int_{\mathbb{R}^d} \partial_t p_0(t, x; s, y) dy \right| = \left| \int_{\mathbb{R}^d} \left( a(t, y) \Delta^\frac{1}{2} p_0(t, x; s, y) + b(t, y) \cdot \nabla_x p_0(t, x; s, y) \right) dy \right|
\]

\[
\leq |a(t, x)| \left| \int_{\mathbb{R}^d} \Delta^\frac{1}{2} p_0(t, x; s, y) dy \right| + |b(t, x)| \left| \int_{\mathbb{R}^d} \nabla_x p_0(t, x; s, y) dy \right|
\]

\[
+ \int_{\mathbb{R}^d} |a(t, y) - a(t, x)| \cdot |\Delta^\frac{1}{2} p_0(t, x; s, y) dy|
\]

\[
+ \int_{\mathbb{R}^d} |b(t, y) - b(t, x)| \cdot |\nabla_x p_0(t, x; s, y) dy|
\]

\[
\leq (t-s)^{\beta-1} + \int_{\mathbb{R}^d} \gamma_0(t, x; s, y) dy \leq (t-s)^{\beta-1}.
\]

(4) Lastly, we prove (2.27). If \(|w| \leq |t-s|\), then

\[
|\nabla_x p_0(t, x+w; s, y) - \nabla_x \xi(t, x+w; s, y) |_{z=x} - (\nabla_x p_0(t, x; s, y) - \nabla_x \xi(t, x; s, y))|_{z=x}
\]

\[
= \left| w \cdot \int_0^1 \left[ (\nabla^2 \rho)(F_s^r(y), x + \theta w - y + G_s^r(y)) - (\nabla^2 \rho)(F_s^r(x), x + \theta w - y + F_s^r(x)) \right] dr \right|
\]

\[
\leq |w| \frac{(t-s)^{\gamma} |x-y|^{\gamma} \gamma_0(t, x; s, y) }{(|x-y| + (t-s))^{\gamma+3}} \leq |w|^{\gamma} (t-s)^{\gamma} \gamma_0(t, x; s, y),
\]

where we have used the same argument as in proving (2.31). Integrating both sides with respect to \(y\) and using (2.2), we obtain (2.27) for \(|w| \leq |t-s|\). If \(|w| > |t-s|\), it follows by (2.23).

**Remark 2.5.** By (2.19), we also have for any \(\gamma \in (0, 1)\),

\[
|\nabla_x p_0(t, x; s, y)| \leq |t-s|^{-\gamma} |x-y|^{-\gamma} \gamma_0(t, x; s, y).
\]

This estimate is important for the lower bound estimate of the heat kernel.

3. HEAT KERNEL OF \(L_{a,b} := a \Delta^\frac{1}{2} + b \cdot \nabla\)

Let \(L_{a,b} := L_{a,b}(t, x) = a(t, x) \Delta^\frac{1}{2} + b(t, x) \cdot \nabla_x\). Now we want to seek the heat kernel of \(L_{a,b}\) with the following form:

\[
p_{a,b}(t, x; s, y) = p_0(t, x; s, y) + \int_s^t \int_{\mathbb{R}^d} p_0(t, x; r, z) q(r, z; s, y) dz dr.
\]
The classical Levi’s continuity argument (see [16, 10]) suggests that \( q(t, x; s, y) \) must satisfy the following integral equation:

\[
q(t, x; s, y) = q_0(t, x; s, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t, x; r, z) q(r, z; s, y) \, dz \, dr,
\]

where

\[
q_0(t, x; s, y) := (a(t, x) - a(t, y)) \Delta^\frac{1}{2} p_0(t, x; s, y) + (b(t, x) - b(t, y)) \cdot \nabla_x p_0(t, x; s, y).
\]

In the remainder of this paper, we shall work on the time interval \([0, 1]\), and always assume

\[
0 \leq s < t \leq 1, \quad x \neq y \in \mathbb{R}^d,
\]

and for some \( \beta \in (0, 1) \),

\[
a, b \in \mathbb{R}^d \beta.
\]

Our first task is thus to solve the integral equation (3.2).

Let us now recursively define

\[
q_0(t, x; s, y) = \int_0^t \int_{\mathbb{R}^d} q_0(t, x; r, z) q_0(r, z; s, y) \, dz \, dr, \quad n \in \mathbb{N}.
\]

**Lemma 3.1.** For \( \beta \in (0, \frac{1}{4}] \), there exists a constant \( C_\beta > 0 \) such that for all \( n \in \mathbb{N} \),

\[
|q_n(t, x; s, y)| \leq \frac{(C_\beta \Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)} \left( e_{(n+1)\beta}(t, x; s, y) + e_{n\beta}(t, x; s, y) \right).
\]

Moreover, if \( a(t, x) = a(t) \) is independent of \( x \), then

\[
|q_n(t, x; s, y)| \leq \frac{(C_\beta \Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)} e_{(n+1)\beta}(t, x; s, y).
\]

**Proof.** First of all, by (3.4) and Lemma 2.3, we have

\[
|q_0(t, x; s, y)| \leq C_d e_0(t, x; s, y).
\]

Notice that

\( B(\gamma, \beta) \) is symmetric and non-increasing with respect to each variable \( \gamma \) and \( \beta \).

For \( n = 1 \), by Lemma 2.3, we have

\[
|q_1| \leq C_d B(2\beta, 1) e_0 + C_d B(\beta, 1) e_{2\beta} \leq C_d B(\beta, \beta) \left( e_0 + e_{2\beta} \right).
\]

Suppose now that

\[
|q_n| \leq \gamma_n \left( e_{(n+1)\beta} + e_{n\beta} \right),
\]

where \( \gamma_n > 0 \) will be determined below. By Lemma 2.3, we have

\[
|q_{n+1}| \leq C_d \gamma_n \left( B(\beta, 1 + (n + 1)\beta) + B((n + 2)\beta, 1) + B(2\beta, 1 + n\beta) \right) e_{(n+1)\beta}
\]

\[
+ C_d \gamma_n \left( B((n+1)\beta, 1) + B(\beta, 1 + n\beta) \right) e_{(n+1)\beta}
\]

\[
\leq C_d \gamma_n B(\beta, (n + 1)\beta) \left( e_{(n+2)\beta} + e_{(n+1)\beta} \right) =: \gamma_{n+1} \left( e_{(n+2)\beta} + e_{(n+1)\beta} \right),
\]

where

\[
\gamma_{n+1} = C_d \gamma_n B(\beta, (n + 1)\beta).
\]

Hence, by \( B(\gamma, \beta) = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)} \), we obtain

\[
\gamma_n = C_d^{n+1} B(\beta, \beta) B(\beta, 2\beta) \cdots B(\beta, n\beta) = \frac{(C_d \Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)},
\]

which gives (3.6).
\begin{equation}
|q_0(t, x; s, y)| \leq \varepsilon_\beta^0(t, x, s, y).
\end{equation}
Repeating the above proof, we obtain \eqref{3.7}.

We also need the following Hölder continuity of \( q_n \) with respect to \( x \).

**Lemma 3.2.** For all \( n \geq 0 \) and \( \gamma \in (0, \beta) \), we have
\begin{equation}
|q_0(t, x; s, y) - q_0(t, x'; s, y)| \leq (\varepsilon_\beta^0 + \varepsilon_{\gamma - \beta}^0)(t, x, s, y) \leq (\varepsilon_\gamma^0 + \varepsilon_{\gamma - \beta}^0)(t, x', s, y).
\end{equation}

**Proof.** Let us first prove the following estimate:
\begin{equation}
|q_0(t, x; s, y) - q_0(t, x'; s, y)| \leq (|x - x'|^{\beta - \gamma} + |1|)(\varepsilon_\gamma^0 + \varepsilon_{\gamma - \beta}^0)(t, x, s, y) \leq (\varepsilon_\gamma^0 + \varepsilon_{\gamma - \beta}^0)(t, x', s, y).
\end{equation}

In the case of \( |x - x'| > 1 \), we have
\begin{equation}
|q_0(t, x; s, y)| \leq (\varepsilon_\beta^0 + \varepsilon_{\gamma - \beta}^0)(t, x, s, y) \leq (\varepsilon_\gamma^0 + \varepsilon_{\gamma - \beta}^0)(t, x, s, y)
\end{equation}
and
\begin{equation}
|q_0(t, x'; s, y)| \leq (\varepsilon_\beta^0 + \varepsilon_{\gamma - \beta}^0)(t, x', s, y) \leq (\varepsilon_\gamma^0 + \varepsilon_{\gamma - \beta}^0)(t, x, s, y).
\end{equation}

In the case of \( 1 \geq |x - x'| > |t - s| \), by \eqref{2.18} and \eqref{2.19} we have
\begin{equation}
|q_0(t, x; s, y)| \leq \varepsilon_\beta^0(t, x, s, y) \leq (|x - x'|^{\beta - \gamma} \varepsilon_{\gamma - \beta}^0)(t, x, s, y) \leq |x - x'|^{\beta - \gamma} \varepsilon_{\gamma - \beta}^0(t, x; s, y),
\end{equation}
and also
\begin{equation}
|q_0(t, x'; s, y)| \leq |x - x'|^{\beta - \gamma} \varepsilon_{\gamma - \beta}^0(t, x'; s, y).
\end{equation}

Suppose now that
\begin{equation}
|x - x'| \leq |t - s|.
\end{equation}

We can write
\begin{equation}
|q_0(t, x; s, y) - q_0(t, x'; s, y)| \leq |a(t, x) - a(t, y)| \cdot |\Delta_\alpha^t p_0(t, x; s, y) - \Delta_\alpha^t p_0(t, x'; s, y)|
+ |a(t, x) - a(t, x')| \cdot |\Delta_\alpha^t p_0(t, x'; s, y)|
+ |b(t, x) - b(t, y)| \cdot |\nabla_x p_0(t, x; s, y) - \nabla_x p_0(t, x'; s, y)|
+ |b(t, x) - b(t, x')| \cdot |\nabla_x p_0(t, x'; s, y)|
=: I_1 + I_2 + I_3 + I_4.
\end{equation}

For \( I_1 \), by \eqref{2.21} and the mean value theorem, we have for some \( \theta \in [0, 1] \),
\begin{equation}
I_1 \leq (|x - x'|^{\beta - 1}) |x - x'| (|x + \theta(x' - x) - y| + |t - s|)^{-d - 2}.
\end{equation}

By \eqref{3.3}, we have
\begin{equation}
|x - y| + |t - s| \leq |x + \theta(x' - x) - y| + 2|t - s|.
\end{equation}

Hence,
\begin{equation}
I_1 \leq (|x - y|^{\beta - 1}) |x - x'| (|x - y| + |t - s|)^{-d - 1}
\leq |x - x'|^{\beta - 1} |t - s|^{1 + \gamma - \beta} (|x - y|^{\beta - 1} + |t - s|)^{-d - 1}
\leq |x - x'|^{\beta - 1} |t - s|^{\gamma} (|x - y| + |t - s|)^{-d - 1} = |x - x'|^{\beta - 1} \varepsilon_{\gamma - \beta}^0(t, x; s, y).
\end{equation}
By (2.19), we have
\[ I_2 \leq |x - x'|^{\beta}(|x' - y| + |t - s|)^{d-1} \leq |x - x'|^{\beta-\gamma} \mathcal{E}_\gamma^0(t, x'; s, y). \]

Similarly, we have
\[ I_3 \leq |x - x'|^{\beta-\gamma} \mathcal{E}_\gamma^0(t, x; s, y). \]
\[ I_4 \leq |x - x'|^{\beta-\gamma} \mathcal{E}_\gamma^0(t, x'; s, y). \]

Combining the above calculations, we obtain (3.8).

Now, by definition (3.5), (3.8) and Lemma 3.1, we have for \( n \in \mathbb{N} \),
\[ |q_n(t, x; s, y) - q_n(t, x'; s, y)| \leq \int_{n+1}^{\infty} \left( C_d \Gamma(\beta+1) \right)^n (|x - x'|^{\beta-\gamma} \wedge 1) \int_{n+1}^{\infty} \left( \left( \mathcal{E}_\gamma^0(t, x; s, y) + \mathcal{E}_\gamma^0(t, x'; s, y) \right) \cdot \left( \mathcal{E}_\gamma^0(r, z; s, y) + \mathcal{E}_\gamma^0(t, x'; s, y) \right) \right) dz dr, \]
which yields the result by Lemma 2.3

Basing on the above two lemmas, we have

**Theorem 3.3.** The function \( q(t, x; s, y) := \sum_{n=0}^{\infty} q_n(t, x; s, y) \) solves the integral equation (3.2). Moreover, \( q(t, x; s, y) \) has the following estimates:
\[ |q(t, x; s, y)| \leq \mathcal{E}_\gamma^0(t, x; s, y) + \mathcal{E}_\gamma^0(t, x; s, y) \]
and for any \( \gamma \in (0, \beta) \),
\[ |q(t, x; s, y) - q(t, x'; s, y)| \leq (|x - x'|^{\beta-\gamma} \wedge 1) \left( \mathcal{E}_\gamma^0(t, x; s, y) + \mathcal{E}_\gamma^0(t, x'; s, y) \right). \]

In the case of \( a(t, x) = a(t) \) independent of \( x \), we have
\[ |q(t, x; s, y)| \leq \mathcal{E}_\gamma^0(t, x; s, y). \]

**Proof.** By Lemma 3.1 one sees that
\[ \sum_{n=0}^{\infty} |q_n(t, x; s, y)| \leq \sum_{n=0}^{\infty} \left( C_d \Gamma(\beta+1) \right)^n \left( \mathcal{E}_\gamma^0(t, x; s, y) + \mathcal{E}_\gamma^0(t, x; s, y) \right) \]
\[ \leq \left\{ \sum_{n=0}^{\infty} \left( C_d \Gamma(\beta+1) \right)^n \right\} \left( \mathcal{E}_\gamma^0(t, x; s, y) + \mathcal{E}_\gamma^0(t, x; s, y) \right). \]

Since the series is convergent, we obtain (3.10). Similarly, estimate (3.11) follows by Lemma 3.2 Moreover, by (3.5), we have
\[ \sum_{n=0}^{m+1} q_n(t, x; s, y) = q_0(t, x; s, y) + \int_{n+1}^{\infty} \int_{n+1}^{\infty} q_0(t, x; s, y) \sum_{n=0}^{m} q_n(r, z; s, y) dz dr, \]
which yields (3.2) by taking limits \( m \to \infty \) for both sides.

In the case of \( a(t, x) = a(t) \), we use (3.7) to repeat the above proof, and obtain (3.12).
where the integral is taken in the generalized sense, i.e.,
\[ \phi(t, x, r) = \lim_{\varepsilon \to 0} \int_{|x - z| > \varepsilon} p_0(t, x; r, z)q(r, z; s, y)dz. \]
Notice that by (2.17), (3.10) and (2.5),
\[ |\phi_{x,y}(t, x, r)| \leq \int_{\mathbb{R}^d} p_0(t, x; r, z)|q(r, z; s, y)|dz \leq \int_{\mathbb{R}^d} \tilde{g}^0(t, x; r, z)(\tilde{g}^0_\beta + \tilde{g}^0_0)(r, z; s, y)dz \]
\[ \leq ((t - r)^\beta + (r - s)^\beta + (t - r)(r - s)^{\beta - 1})g_0^0(t, x; s, y) + g_0^0(t, x; s, y). \] (3.13)

Below we study the smoothness of \((t, x) \mapsto \varphi(t, x)\).

**Lemma 3.4.** For all \(x \neq y \in \mathbb{R}^d\) and almost all \(t > s\), we have
\[ \partial_t \varphi(t, x) = q(t, x; s, y) + \int_s^t \mathcal{L}^R_{\alpha, \beta}(t, z)p_0(t, z)q(r, z; s, y)dzdr. \] (3.14)

**Proof.** (Claim 1): For \(r \in (s, t)\), we have
\[ \partial_t \phi_{x,y}(t, x, r) = \int_{\mathbb{R}^d} \partial_t p_0(t, x; r, z)q(r, z; s, y)dz. \] (3.15)

**Proof of Claim 1:** Write
\[ \frac{\phi_{x,y}(t + \varepsilon, x, r) - \phi_{x,y}(t, x, r)}{\varepsilon} = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} (p_0(t + \varepsilon, x; r, z) - p_0(t, x; r, z))q(r, z; s, y)dz \]
\[ = \int_{\mathbb{R}^d} \left( \int_0^1 \partial_t p_0(t + \theta \varepsilon, x; r, z)d\theta \right)q(r, z; s, y)dz. \]
By (2.18) and (2.19), we have for \(|\varepsilon| < \frac{2}{r}\),
\[ |\partial_t p_0(t + \theta \varepsilon, x; r, z)| \leq (|x - z| + t + \theta \varepsilon - r)^{-d - 1} \leq (|x - z| + (t - r))^{-d - 1}, \]
which together with (3.10) yields
\[ |\partial_t p_0(t + \theta \varepsilon, x; r, z)q(r, z; s, y)| \leq g_0^0(t, x; r, z)(\tilde{g}^0_\beta + \tilde{g}^0_0)(r, z; s, y) =: g(z). \]
By (2.5), one sees that
\[ \int_{\mathbb{R}^d} g(z)dz < +\infty. \]
Hence, by the dominated convergence theorem, we have
\[ \lim_{\varepsilon \to 0} \frac{\phi_{x,y}(t + \varepsilon, x, r) - \phi_{x,y}(t, x, r)}{\varepsilon} = \int_{\mathbb{R}^d} \partial_t p_0(t, x; r, z)q(r, z; s, y)dz, \]
and (3.15) is proven.

(Claim 2): For \(x \neq y\), we have
\[ \int_s^t \int_s^{t'} |\partial_r \phi_{x,y}(t', x, r)|drdr' < +\infty. \] (3.16)

**Proof of Claim 2:** By (3.15), we have
\[ |\partial_r \phi_{x,y}(t', x, r)| \leq \int_{\mathbb{R}^d} |\partial_r p_0(t', x; r, z)| \cdot |q(r, z; s, y) - q(r, x; s, y)|dz \]
\[ + |q(r, x; s, y)| \int_{\mathbb{R}^d} |\partial_r p_0(t', x; r, z)|dz \]
\[ =: Q_{x,y}^{(1)}(t', x, r) + Q_{x,y}^{(2)}(t', x, r). \] (3.17)
For $Q_{s,y}^{(1)}(t', x, r)$, by (2.20) and (3.11), we have

\[
\int_s^t \int_s^{t'} Q_{s,y}^{(1)}(t', x, r) dr' dt' \leq \int_s^t \int_s^{t'} g_{0}^{\beta}(t', x, r, z)(g_{y}^{0} + g_{y}^{\beta})(r, x, s, y) dz dr' + \int_s^t \int_s^{t'} g_{0}^{\beta}(t', x, r, z)(g_{y}^{0} + g_{y}^{\beta})(r, z, s, y) dz dr'
\]
\[
\leq \int_s^t \int_s^{t'} (t' - r)^{\beta - 1} (g_{0}^{\beta} + g_{y}^{\beta})(r, x, s, y) dr' + \int_s^t \int_s^{t'} (g_{0}^{\beta} + g_{y}^{0})(t', x, s, y) dr'
\]
\[
\leq \frac{1}{|x - y|^{d+1}} \int_s^t \int_s^{t'} (t' - r)^{\beta - 1} ((r - s)^{\gamma} + (r - s)^{\beta}) dr' + \frac{1}{|x - y|^{d+1}} \int_s^t \int_s^{t'} ((t' - s)^{\gamma} + 1 + (t' - s)^{\beta}) dr' < +\infty. \quad (3.18)
\]

For $Q_{s,y}^{(2)}(t', x, r)$, by (2.25) and (3.10) we have

\[
\int_s^t \int_s^{t'} Q_{s,y}^{(2)}(t', x, r) dr' dt' \leq \int_s^t \int_s^{t'} (g_{0}^{\beta} + g_{0}^{\beta})(r, x, s, y) (t' - r)^{1 - \beta} dr' < +\infty. \quad (3.19)
\]

Combining (3.17)–(3.19), we obtain (3.16).

(Claim 3): For fixed $r, x, s, y$, we have

\[
\lim_{t \downarrow r} \phi_{s,y}(t, x, r) = q(r, x, s, y). \quad (3.20)
\]

Proof of Claim 3: By (2.26), it suffices to prove that

\[
\lim_{t \downarrow r} \left| \int_{\mathbb{R}^d} p_0(t, x; r, z) q(r, z; s, y) - q(r, x; s, y) dz \right| = 0.
\]

Notice that for any $\delta > 0$,

\[
\left| \int_{\mathbb{R}^d} p_0(t, x; r, z) q(r, z; s, y) - q(r, x; s, y) dz \right|
\]
\[
\leq \int_{|x - z| < \delta} p_0(t, x; r, z) q(r, z; s, y) - q(r, x; s, y) dz + \int_{|x - z| > \delta} p_0(t, x; r, z) q(r, z; s, y) - q(r, x; s, y) dz
\]
\[
=: J_1(\delta, t, r) + J_2(\delta, t, r).
\]

For any $\varepsilon > 0$, by (3.11), there exists a $\delta = \delta(r, x, s, y) > 0$ such that for all $|x - z| \leq \delta$,

\[
|q(r, z; s, y) - q(r, x; s, y)| \leq \varepsilon.
\]

Thus,

\[
J_1(\delta, t, r) \leq \varepsilon \int_{|x - z| < \delta} p_0(t, x; r, z) dz \leq \varepsilon \int_{\mathbb{R}^d} p_0(t, x; r, z) dz \leq \varepsilon \int_{\mathbb{R}^d} g_{0}^{\beta}(t, x; r, z) dz \leq \varepsilon.
\]

On the other hand, we have

\[
J_2(\delta, t, r) \leq (t - r) \int_{|x - z| > \delta} \frac{|q(r, z; s, y)| + |q(r, x; s, y)|}{|x - z|^{d+1}} dz
\]
\[
\leq \varepsilon (t - r) \int_{|x - z| > \delta} \frac{1}{|x - z|^{d+1}} dz < +\infty.
\]

Therefore,

\[
\lim_{t \downarrow r} \left| \int_{\mathbb{R}^d} p_0(t, x; r, z) q(r, z; s, y) - q(r, x; s, y) dz \right| = 0
\]

as desired.
Lemma 3.5. For all \( t \) which in turn implies (3.14) by the Lebesgue differential theorem.

Proof of Claim. By (3.20), we have

\[
\int_{|z| > \delta} \delta^{-d-1} \int_{\mathbb{R}^d} |q(r, z; s, y)|dz + |q(r, x; s, y)|dz \leq (t - r) \delta^{-d-1} \int_{|z| > \delta} |z|^{-d-1}dz,
\]

which, by (3.10) and (2.2), converges to zero as \( t \downarrow r \). The claim (3.20) is thus proved.

Now, by integration by parts formula and (3.20), we have

\[
\int_{\mathbb{R}^d} \partial_r \phi_{s,y}(r', x, r)dr' = \phi_{s,y}(t, x, r) - q(r, x; s, y).
\]

Integrating both sides with respect to \( r \) from \( s \) to \( t \), and then by (3.16) and Fubini’s theorem, we obtain

\[
\varphi(t, x) - \int_{s}^{t} q(r, x; s, y)dr = \int_{s}^{t} \int_{s}^{r'} \partial_r \phi_{s,y}(r', x, r)dr'dr \quad \text{(3.16)}
\]

\[
\overset{(3.15)(2.15)}{=} \int_{s}^{t} \int_{s}^{r} \int_{\mathbb{R}^d} \mathcal{L}_{a,y}(r', z)p_0(r', z; r, z)q(r, z; s, y)dzdr'dr',
\]

which in turn implies (3.14) by the Lebesgue differential theorem. \( \square \)

Lemma 3.5. For all \( t > s \) and \( x \neq y \), we have

\[
\nabla_s \varphi(t, x) = \int_{\mathbb{R}^d} \nabla_s p_0(t, x; r, z)q(r, z; s, y)dzdr,
\]

(3.21)

\[
\Delta_{s}^{\frac{1}{2}} \varphi(t, x) = \int_{\mathbb{R}^d} \Delta_{s}^{\frac{1}{2}} p_0(t, x; r, z)q(r, z; s, y)dzdr,
\]

(3.22)

where the integrals are understood in the sense of iterated integrals.

Proof. First of all, for fixed \( s < r < t \), since

\( (x, z) \mapsto p_0(t, x; r, z) \in C_{p}^{\infty}(\mathbb{R}^d \times \mathbb{R}^d) \)

and

\( z \mapsto q(r, z; s, y) \in C_{b}(\mathbb{R}^d) \),

by Lemma [2.4] it is easy to see that

\[
\nabla_s \phi_{s,y}(t, x, r) = \int_{\mathbb{R}^d} \nabla_s p_0(t, x; r, z)q(r, z; s, y)dz,
\]

(3.23)

and

\[
\Delta_{s}^{\frac{1}{2}} \phi_{s,y}(t, x, r) = \int_{\mathbb{R}^d} \Delta_{s}^{\frac{1}{2}} p_0(t, x; r, z)q(r, z; s, y)dz.
\]

(3.24)

(1) We prove the following claim: For any \( t > s \) and \( x \neq y \), there exists a \( p > 1 \) such that

\[
\sup_{|w| \leq |x - y|/2} I(p, w) < \infty, \quad \text{where} \quad I(p, w) := \int_{s}^{t} |\nabla_s \phi_{s,y}(t, x + w; r)|^pdr.
\]

(3.25)

Proof of Claim. By (3.23) and (2.23) we have

\[
I(p, w) \leq \int_{s}^{t} \left[ \int_{\mathbb{R}^d} \left| \nabla_s p_0(t, x + w; r, z)(q(r, z; s, y) - q(r, x + w; s, y))dz \right|^pdr 
\right. \\
+ \left. \int_{s}^{t} \left[ \int_{\mathbb{R}^d} \left| \nabla_s p_0(t, x + w; r, z)dz \right|^p |q(r, x + w; s, y)|^pdr 
\right] \right.
\]

\[
\overset{2.19(3.11)}{\leq} \int_{s}^{t} \left( \int_{\mathbb{R}^d} \mathcal{G}_0^\beta \gamma(t, x + w; r, z)(\mathcal{G}_0^\beta + \mathcal{G}_\gamma^\beta)(r, z; s, y)dz \right)^pdr
\]

14
\[
\begin{aligned}
&+ \int_s^t \left( \int_{\mathbb{R}^d} \nabla_0^\beta (t, x + w; r, z) (\nabla_0^\gamma + \nabla_{\gamma - \beta}) (r, x + w; s, y) dz \right)^p dr \\
&+ \int_s^t (t - r)^{p(\beta - 1)} (\nabla_0^\beta + \nabla_0^0) (r, x + w; s, y) dr \\
&=: I_1(p, w) + I_2(p, w) + I_3(p, w).
\end{aligned}
\]

For \( I_1(p, w) \), it follows by \((2.7)\) that for some \( p > 1 \),
\[
\sup_{|w| \leq |x - y|/2} I_1(p, w) < +\infty.
\]

For \( I_2(p, w) \), by definition \((2.1)\) and \((2.2)\), we have for all \(|w| \leq |x - y|/2\),
\[
I_2(p, w) \leq \int_s^t \left( \int_{\mathbb{R}^d} \nabla_0^\beta (t, x + w; r, z) dz \right)^p \left( \frac{(r - s)^\gamma}{|x + w - y|^{d+1}} + \frac{(r - s)^{\gamma - \beta}}{|x + w - y|^{d+1}} \right) dr \\
\leq \int_s^t (t - r)^{p(\beta - \gamma - 1)} \left( \frac{1}{|x - y|^{d+1}} + \frac{(r - s)^{\gamma - \beta}}{|x - y|^{d+1}} \right) dr < +\infty,
\]

provided \( p < \frac{1}{1+\gamma - \beta} \wedge \frac{1}{1-\beta} \). Similarly, for \( p < \frac{1}{1-\beta} \), we have
\[
\sup_{|w| \leq |x - y|/2} I_3(p, w) < +\infty.
\]

Thus, we obtain \((3.25)\).

Now, for any \( \epsilon_i = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^d \), we can write
\[
\frac{\varphi(t, x + \epsilon \epsilon_i) - \varphi(t, x)}{\epsilon} = \int_s^t \int_0^1 \partial_{x_i} \phi(t, x + \theta \epsilon \epsilon_i, r) d\theta dr.
\]

By \((3.25)\) one can take limits to get
\[
\partial_{x_i} \varphi(t, x) = \lim_{\epsilon \to 0} \frac{\varphi(t, x + \epsilon \epsilon_i) - \varphi(t, x)}{\epsilon} = \int_s^t \int_0^1 \lim_{\epsilon \to 0} \partial_{x_i} \phi(t, x + \theta \epsilon \epsilon_i, r) d\theta dr = \int_s^t \partial_{x_i} \phi(t, x, r) dr,
\]

and \((3.21)\) is proven.

(2) Next, we prove \((3.22)\). Recalling the definition of \( \phi_{x,y} \), we have
\[
\nabla_x \phi_{x,y}(t, x + w, r) - \nabla_x \phi_{x,y}(t, x, r)
\]

\[
= \int_{\mathbb{R}^d} \nabla_x p_0(t, x + w; r, z) - \nabla_x p_0(t, x; r, z) q(r, z; s, y) dz
\]

\[
= \int_{\mathbb{R}^d} \nabla_x p_0(t, x + w; r, z) (q(r, z; s, y) - q(r, x + w; s, y))
\]

\[
- \nabla_x p_0(t, x; r, z) (q(r, z; s, y) - q(r, x; s, y)) dz
\]

\[
+ \left\{ q(r, x + w; s, y) \int_{\mathbb{R}^d} \nabla_x p_0(t, x + w; r, z) dz
\right\}
\]

\[
- q(r, x; s, y) \int_{\mathbb{R}^d} \nabla_x p_0(t, x; r, z) dz
\]

\[
=: \int_{\mathbb{R}^d} Q(t, x; r, z; s, y; w) dz + R(t, x, w; s, y).
\]

We now prove the following claim: For any \( \gamma \in (0, \beta) \) and \( \sigma \in (0, \beta - \gamma) \),
\[
|Q(t, x; r, z; s, y; w)| \leq |w|^p \nabla_0^\beta (t, x + w; r, z) \left( \nabla_0^\gamma + \nabla_{\gamma - \beta} \right) (r, x + w; s, y) + (\nabla_0^\gamma + \nabla_0^0) (r, z; s, y)
\]

\[
= \int_{\mathbb{R}^d} Q(t, x; r, z; s, y; w) dz + R(t, x, w; s, y).
\]
Similarly, we have
\[
\begin{align*}
& + |w|^\sigma \mathcal{E}_{-\sigma}^\beta \gamma(t, x; r, z) \left( (\mathcal{E}_{\gamma}^\beta - \mathcal{E}_{\gamma}^0)(r, x; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, z; s, y) \right) \\
& + |w|^\sigma \mathcal{E}_{\mu-\gamma-\sigma}^0(t, x; r, z) \left( (\mathcal{E}_{\gamma}^\beta - \mathcal{E}_{\gamma}^0)(r, x + w; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, z; s, y) \right) \\
& + |w|^\sigma \mathcal{E}_{\mu-\gamma-\sigma}^0(t, x; r, z) \left( (\mathcal{E}_{\gamma}^\beta - \mathcal{E}_{\gamma}^0)(r, x + w; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, x; s, y) \right),
\end{align*}
\]
and for \( w \in \mathbb{R}^d \),
\[
R(t, t, x; s, y; w) \leq |w|^\gamma(t - r)^{\beta-\gamma-1} \left( (\mathcal{E}_{\gamma}^\beta + \mathcal{E}_{\gamma-\beta})(r, x + w; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, x; s, y) \right)
+ |w|^\gamma(t - r)^{\beta-\gamma-1} \left( (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, x, s, y). \right)
\]

**Proof of Claim:** Case 1: We assume
\[
|w| > |t - r|.
\]
By (3.11) we have
\[
|\nabla_x p_0(t, x; r, z) (q(r, z; s, y) - q(r, x; s, y))|
\leq \mathcal{E}_{0}(t, x; r, z)(|x - z|^{\beta-\gamma} + 1) \left( (\mathcal{E}_{\gamma}^\beta + \mathcal{E}_{\gamma}^0)(r, x; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, z; s, y) \right)
\leq |w|^\sigma \mathcal{E}_{-\sigma}^\beta \gamma(t, x; r, z) \left( (\mathcal{E}_{\gamma}^\beta + \mathcal{E}_{\gamma}^0)(r, x; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, z; s, y) \right),
\]
and also
\[
|\nabla_x p_0(t, x + w; r, z) (q(r, z; s, y) - q(r, x + w; s, y))|
\leq |w|^\sigma \mathcal{E}_{-\sigma}^\beta \gamma(t, x + w; r, z) \left( (\mathcal{E}_{\gamma}^\beta + \mathcal{E}_{\gamma}^0)(r, x + w; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, z; s, y) \right).
\]

Case 2: We assume
\[
|w| \leq |t - r|.
\]
Noticing that
\[
|x + w - z| \leq |x - z| + |w| \leq |x - z| + |t - r|
\]
and
\[
|x - z| \leq |x + w - z| + |w| \leq |x + w - z| + |t - r|,
\]
we have for any \( \theta_0 \in (0, 1) \),
\[
|w| \cdot |\nabla_x^2 p_0(t, x + \theta_0 w; r, z)| \cdot |x + w - z|^{\beta-\gamma} \leq |w|^\sigma \mathcal{E}_{\mu-\gamma-\sigma}^0(t, x; r, z).
\]
Hence, for some \( \theta_0 \in (0, 1) \),
\[
|\nabla_x p_0(t, x + w; r, z) - \nabla_x p_0(t, x; r, z) (q(r, z; s, y) - q(r, x + w; s, y))|
\leq |w| \cdot |\nabla_x^2 p_0(t, x + \theta_0 w; r, z)| \cdot |x + w - z|^{\beta-\gamma}
\times \left( (\mathcal{E}_{\gamma}^\beta + \mathcal{E}_{\gamma}^0)(r, x + w; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, z; s, y) \right)
\leq |w|^\sigma \mathcal{E}_{\mu-\gamma-\sigma}^0(t, x; r, z) \left( (\mathcal{E}_{\gamma}^\beta + \mathcal{E}_{\gamma}^0)(r, x + w; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, z; s, y) \right).
\]
Similarly, we have
\[
|\nabla_x p_0(t, x; r, z) (q(r, z; s, y) - q(r, x + w; s, y))|
\leq |w|^\sigma \mathcal{E}_{\mu-\gamma-\sigma}^0(t, x; r, z) \left( (\mathcal{E}_{\gamma}^\beta + \mathcal{E}_{\gamma}^0)(r, x + w; s, y) + (\mathcal{E}_{\gamma}^0 + \mathcal{E}_{\gamma}^0)(r, x; s, y) \right).
\]
Combining the above calculations, we obtain (3.27). As for (3.28), it follows by Lemma 2.4 and Theorem 3.3.
(3) Now, we are ready to prove the following claim: For any $t > s$ and $x \neq y$, there exists a $p > 1$ such that

$$
\sup_{|\epsilon| < |x - y|/2} J(p, \epsilon) < +\infty,
$$

where $J(p, \epsilon) := \int_s^\infty \left| \int_{|w| > \epsilon} \frac{\phi_{s,\lambda}(t, x + w, r) - \phi_{s,\lambda}(t, x, r)}{|w|^{d+1}} \, dw \right|^p \, dr. \tag{3.29}

**Proof of Claim:** Notice that

$$
J(p, \epsilon) \leq \int_s^\infty \left| \int_{|w| < |x - y|/2} \frac{\phi_{s,\lambda}(t, x + w, r) - \phi_{s,\lambda}(t, x, r) - w \cdot \nabla \phi_{s,\lambda}(t, x, r)}{|w|^{d+1}} \, dw \right|^p \, dr
$$

$$
+ \int_s^\infty \left| \int_{|w| > |x - y|/2} \frac{\phi_{s,\lambda}(t, x + w, r) - \phi_{s,\lambda}(t, x, r)}{|w|^{d+1}} \, dw \right|^p \, dr \leq J_1(p, \epsilon) + J_2(p).
$$

For $J_1(p, \epsilon)$, observe that

$$
J_1(p, \epsilon) = \int_s^\infty \left| \int_{|w| < |x - y|/2} \frac{w}{|w|^{d+1}} \cdot \left( \int_0^1 \left( \nabla \phi_{s,\lambda}(t, x + \theta w, r) - \nabla \phi_{s,\lambda}(t, x, r) \right) \, d\theta \right) \, dw \right|^p \, dr
$$

$$
\leq \int_s^\infty \left( \int_{|w| < |x - y|/2} \left| \int_0^1 \left( \nabla \phi_{s,\lambda}(t, x + \theta w, r) - \nabla \phi_{s,\lambda}(t, x, r) \right) \, d\theta \right| \, dw \right)^p \, dr
$$

$$
+ \int_s^\infty \left( \int_{|w| > |x - y|/2} \left| \int_0^1 \left( \nabla \phi_{s,\lambda}(t, x + \theta w, r) - \nabla \phi_{s,\lambda}(t, x, r) \right) \, d\theta \right| \, dw \right)^p \, dr.
$$

Using (3.27), (3.28) and by (2.7), as in proving (3.25), one has that for some $p > 1$,

$$
\sup_{|\epsilon| < |x - y|/2} J_1(p, \epsilon) < +\infty.
$$

For $J_2(p)$, we have by (3.13) that for some $p > 1$,

$$
J_2(p) \leq \int_s^\infty \left| \int_{|w| > |x - y|/2} \frac{|\phi_{s,\lambda}(t, x + w, r)| + |\phi_{s,\lambda}(t, x, r)|}{|w|^{d+1}} \, dw \right|^p \, dr < +\infty.
$$

Thus, (3.29) is proven.

(4) Now by (3.29), one has

$$
\Delta^\frac{1}{2}_t \varphi(t, x) = \lim_{\epsilon \downarrow 0} \int_{|w| > \epsilon} \int_s^\infty \frac{\phi_{s,\lambda}(t, x + w, r) - \phi_{s,\lambda}(t, x, r)}{|w|^{d+1}} \, dw \, dr
$$

$$
= \lim_{\epsilon \downarrow 0} \int_s^\infty \int_{|w| > \epsilon} \frac{\phi_{s,\lambda}(t, x + w, r) - \phi_{s,\lambda}(t, x, r)}{|w|^{d+1}} \, dw \, dr
$$

$$
= \int_s^\infty \lim_{\epsilon \downarrow 0} \int_{|w| > \epsilon} \frac{\phi_{s,\lambda}(t, x + w, r) - \phi_{s,\lambda}(t, x, r)}{|w|^{d+1}} \, dw \, dr = \int_s^\infty \Delta^\frac{1}{2}_t \phi_{s,\lambda}(t, x, r) \, dr,
$$

which together with (3.24) yields (3.22). \hfill \square

Now we prove the following main result of this section.

**Theorem 3.6.** Assume that $a, b \in \mathbb{R}^\beta$ for some $\beta \in (0, \frac{1}{4}]$ and satisfy (2.16). Then there exists a unique nonnegative continuous function $p_{a,b}(t, x; s, y)$ with

$$
\int_{\mathbb{R}^d} p_{a,b}(t, x; s, y) \, dy = 1, \tag{3.30}
$$

and satisfying that

(i) For all $x \neq y \in \mathbb{R}^d$ and almost all $t > s$,

$$
\partial_t p_{a,b}(t, x; s, y) = \mathcal{L}^x_{a,b}(t) p_{a,b}(t, s, y) = \mathcal{L}^y_{b,a}(s) p_{a,b}(t, s, y)(x). \tag{3.31}
$$
(ii) For any bounded continuous function \( f \),
\[
\lim_{t \downarrow s} \int_{\mathbb{R}^d} p_{a,b}(t, x; s, y) f(y) dy = f(x),
\]
and
\[
\lim_{t \downarrow s} \int_{\mathbb{R}^d} p_{a,b}(t, x; s, y) f(x) dx = f(y).
\]

(iii) For all \( 0 \leq s < t \leq 1 \) and \( x, y \in \mathbb{R}^d \),
\[
p_{a,b}(t, x; s, y) \leq p^0_{1-y}(t, x; s, y);
\]
and in the case of \( a(t, x) = a(t) \) independent of \( x \), we have
\[
p_{a,b}(t, x; s, y) \geq p^0_{1-y}(t, x; s, y).
\]

(iv) For any \( \gamma \in (0, 1) \),
\[
|p_{a,b}(t, x; s, y) - p_{a,b}(t, x'; s, y)| \leq (|x - x'|^p + 1)\left\{p^0_{1-y}(t, x; s, y) + p^0_{1-y}(t, x; s, y)\right\},
\]
and
\[
|\nabla_s p_{a,b}(t, x; s, y)| + |\Delta_s p_{a,b}(t, x; s, y)| \leq p^0_{1-y}(t, x; s, y).
\]

(v) If \( a, b \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^d)) \), then for all \( f, g \in C^\infty_0(\mathbb{R}^d) \),
\[
\lim_{t \downarrow s} \frac{1}{t-s} \int_{\mathbb{R}^d} g(x)(P_{t,s}^{a,b}(x) - f(x)) dx = \int_{\mathbb{R}^d} g(x) \mathcal{L}^{x}_{a,b}(s, x) f(x) dx, \quad s > 0,
\]
where
\[
P_{t,s}^{a,b}(x) := \int_{\mathbb{R}^d} p_{a,b}(t, x; s, y) f(y) dy.
\]

Proof. (i) First, we prove (3.31). By equation (3.31), we have that for all \( x, y \in \mathbb{R}^d \) and almost all \( t > s \),
\[
\partial_t p_{a,b}(t, x; s, y) = \partial_t p_0(t, x; s, y) + g(t, x; s, y) + \int_{\mathbb{R}^d} \mathcal{L}^{x}_{a,b}(t, y) p_0(t, t; r, z)(x) q(r, z; s, y) dz dr
\]
\[
= \mathcal{L}_{a,b}^x(t, x) p_0(t, t; s, y) + \int_{\mathbb{R}^d} q_0(t, x; r, z) q(r, z; s, y) dz dr
\]
Recalling that
\[
g_0(t, x; s, y) = \left(\mathcal{L}_{a,b}^x(t, x) - \mathcal{L}_{a,b}^x(t, y)\right) p_0(t, t; s, y)(x),
\]
we further have
\[
\partial_t p_{a,b}(t, x; s, y) = \mathcal{L}_{a,b}^x(t, x) p_0(t, t; s, y)(x) + \int_{\mathbb{R}^d} \mathcal{L}_{a,b}^x(t, x) p_0(t, t; r, z)(x) q(r, z; s, y) dz dr,
\]
which together with (3.21) and (3.22) yields (3.31).

(ii) We prove (3.32) and (3.33). As in proving (3.20), we can prove that for any bounded continuous function \( f \),
\[
\lim_{t \downarrow s} \int_{\mathbb{R}^d} p_0(t, x; s, y) f(y) dy = f(x),
\]
and
\[
\lim_{t \downarrow s} \int_{\mathbb{R}^d} p_0(t, x; s, y) f(x) dx = f(y).
\]
Moreover, by \((3.13)\) we also have
\[
\left| \int_{\mathbb{R}^d} \int_{s} \int_{s} p_0(t, x; r, z) q(r, z; s, y) f(y) dydrdz \right| + \left| \int_{\mathbb{R}^d} \int_{s} \int_{s} p_0(t, x; r, z) q(r, z; s, y) f(x) dydrdx \right| \\
\leq \int_{\mathbb{R}^d} \left( g_1^0(t, x; s, y) + g_1^{\beta}(t, x; s, y) \right) dy + dx \leq |t - s|^{\beta} \to 0, \ t \downarrow s.
\]

Thus, \((3.32)\) and \((3.33)\) are proven by equation \((3.1)\).

For proving \((3.30)\), if we set \(u_s(t, x) := \int_{\mathbb{R}^d} p_{u,b}(t, x; s, y) dy\), then by \((3.31)\) and \((3.32)\),
\[
\partial_t u_s(t, x) = \mathcal{L}_{u,b}(t, x) u_s(t, x), \quad \lim_{t \downarrow s} u_s(t, x) = 1.
\]

By the maximal principal of nonlocal equation (cf. [25] or [26, Theorem 2.3]), it follows that
\[
u_s(t, x) \equiv 1, \ t > s, \ x \in \mathbb{R}^d.
\]

(iii) By \((3.13)\), one has
\[
\int_{\mathbb{R}^d} p_0(t, x; r, z) q(r, z; s, y) dzdr \leq g_1^{\beta}(t, x; s, y) + g_1^{\alpha}(t, x; s, y) \leq g_1^{\beta}(t, x; s, y),
\]
which in turn gives estimate \((3.34)\) by equation \((3.1)\) and \((2.17)\).

In the case of \(a(t, x) = a(t)\), by \((3.1)\) and \((3.12)\), we have
\[
|p(t, x; s, y) - p_0(t, x; s, y)| \leq \int_{\mathbb{R}^d} p_0(t, x; r, z) q(r, z; s, y) dzdr \leq \int_{\mathbb{R}^d} g_1^{\beta}(t, x; s, y) dzdr \leq |t - s|^{\beta} p_0(t, x; s, y),
\]
where \(\Lambda > 0\) is a constant independent of \(t, s, x, y\). Choosing \(T_0 \in (0, 1)\) such that \(|t - s|^{\beta} < \frac{1}{2\Lambda}\) for all \(0 \leq s < t \leq T_0\), we obtain \((3.35)\) for small time by \((2.17)\). For the large time, it follows by a standard time shift argument (see [3] [22]).

(iv) As in proving \((3.8)\), we have for any \(\gamma \in (0, 1)\),
\[
|p_0(t, x; s, y) - p_0(t, x'; s, y)| \leq (|x - x'|^{\gamma} \wedge 1) \left( g_1^{\beta}(t, x; s, y) + g_1^{\alpha}(t, x; s, y) \right).
\]

Thus, by \((3.10)\) and Lemma 2.3 we have
\[
\int_{\mathbb{R}^d} p_0(t, x; r, z) dzdr - \int_{\mathbb{R}^d} p_0(t, x'; r, z) dzdr \leq (|x - x'|^{\gamma} \wedge 1) \left( g_1^{\beta}(t, x; s, y) + g_1^{\alpha}(t, x; s, y) \right) dzdr \leq (|x - x'|^{\gamma} \wedge 1) \left( g_1^{\beta}(t, x; s, y) + g_1^{\alpha}(t, x; s, y) \right) dzdr \\
\leq (|x - x'|^{\gamma} \wedge 1) \left( g_1^{\beta}(t, x; s, y) + g_1^{\alpha}(t, x; s, y) \right) dzdr \\
\leq (|x - x'|^{\gamma} \wedge 1) \left( g_1^{\beta}(t, x; s, y) + g_1^{\alpha}(t, x; s, y) \right),
\]
which together with equation \((3.1)\) yields \((3.36)\).

Next, we prove \((3.37)\). By \((3.21)\), we can write
\[
\nabla \varphi(t, x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla \varphi(t, x; r, z) q(r, z; s, y) - q(r, x; s, y) dzdr \\
+ \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \nabla \varphi(t, x; r, z) dz \right) q(r, x; s, y) dr.
\]
\[
\begin{aligned}
&\quad + \int_t^s \int_{\mathbb{R}^d} \nabla_x p_0(t, x; r, z)q(r, z; s, y)dzdr \\
&=: Q_1(t, x; s, y) + Q_2(t, x; s, y) + Q_3(t, x; s, y).
\end{aligned}
\]

For \(Q_1(t, x; s, y)\), by (2.19), (3.11) and Lemma 2.3 we have
\[
|Q_1(t, x; s, y)| \leq \int_t^s \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \phi_0^{\beta-\gamma}(t, x; r, z)dz \right) \left( \phi_0^\beta(r, x; s, y) + (\phi_0^\beta + \phi_0^\beta)(r, z; s, y) \right)dr \right)
\]
\[
\leq \int_t^s \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \phi_0^{\beta-\gamma}(t, x; r, z)dz \right) \left( \phi_0^\beta(r, x; s, y) + (\phi_0^\beta + \phi_0^\beta)(r, z; s, y) \right)dr \right)
\]
\[
+ \int_s^t \left( \int_{\mathbb{R}^d} \phi_0^{\beta-\gamma}(t, x; r, z)dz \right) \left( \phi_0^\beta(r, x; s, y) + (\phi_0^\beta + \phi_0^\beta)(r, z; s, y) \right)dr
\]
\[\leq \int_t^s (t-r)^{\beta-1} (1 + (r-s)^{\beta-\gamma}) \phi_0^\beta(r, x; s, y)dr \]
\[
+ \phi_0^\beta + \phi_0^\beta + \phi_0^\beta)(t, x; s, y) \leq \phi_0^\beta(t, x; s, y).
\]

For \(Q_2(t, x; s, y)\), we have
\[
|Q_2(t, x; s, y)| \leq \int_t^s (t-r)^{\beta-1} \phi_0^\beta(r, x; s, y)dr \leq \phi_0^\beta(t, x; s, y).
\]

For \(Q_3(t, x; s, y)\), we have
\[
|Q_3(t, x; s, y)| \leq \int_t^s \int_{\mathbb{R}^d} \phi_0^\beta(r, x; z; s, y) + \phi_0^\beta(r, z; s, y)dzdr \leq \phi_0^\beta(t, x; s, y).
\]

Combining the above calculations, we obtain
\[
|\nabla_x \varphi(t, x)| \leq \phi_0^\beta(t, x; s, y). \quad (3.41)
\]

Similarly,
\[
|\Delta_1^\frac{1}{2} \varphi(t, x)| \leq \phi_0^\beta(t, x; s, y). \quad (3.42)
\]

Estimate (3.37) then follows by equation (3.1), (2.18), (2.19) and (3.41), (3.42).

(v) For \(f \in C_0^\alpha(\mathbb{R}^d)\), by (3.31) and (3.32), we have
\[
\frac{P_{r,s}^{a,b}(f(x) - f(x))}{t-s} - \mathcal{L}_{a,b}^{\alpha}(s, x)f(x) = \frac{1}{t-s} \int_s^t (a(r, x) - a(s, x)) \Delta_1^\frac{1}{2} P_{r,s}^{a,b} f(x)dr
\]
\[
+ \frac{a(s, x)}{t-s} \int_s^t \Delta_1^\frac{1}{2} (P_{r,s}^{a,b} f(x) - f(x))dr
\]
\[
+ \frac{1}{t-s} \int_s^t (b(r, x) - b(s, x)) \cdot \nabla_x P_{r,s}^{a,b} f(x)dr
\]
\[
+ \frac{b(s, x)}{t-s} \cdot \int_s^t \nabla_x P_{r,s}^{a,b} f(x) - f(x)dr
\]
\[=: I_1(t, s, x) + I_2(t, s, x) + I_3(t, s, x) + I_4(t, s, x).
\]

We have the following claim: for \(t > s\),
\[
\sup_{r \in [s,t]} \|\Delta_1^\frac{1}{2} P_{r,s}^{a,b} f\|_\infty + \sup_{r \in [s,t]} \|\nabla_x P_{r,s}^{a,b} f\|_\infty < +\infty, \quad (3.43)
\]

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Proof of Claim: By Lemma 3.5 and (3.30), we have

\[ \Delta_t^\frac{1}{2} P_{r,s} \omega(x) = \int_{\mathbb{R}^d} \Delta_t^\frac{1}{2} \omega(r, x; s, y) f(y) dy = \int_{\mathbb{R}^d} \Delta_t^\frac{1}{2} \omega_a(r, x; s, y)(f(y) - f(x)) dy \]

and

\[ \nabla_x P_{r,s} \omega(x) = \int_{\mathbb{R}^d} \nabla_x \omega_a(r, x; s, y) f(y) dy = \int_{\mathbb{R}^d} \nabla_x \omega_a(r, x; s, y)(f(y) - f(x)) dy. \]

Hence, by (3.37) we have

\[ |\Delta_t^\frac{1}{2} P_{r,s} \omega(x)| + |
abla_x P_{r,s} \omega(x)| \leq C ||\omega||_{L^p} \int_{\mathbb{R}^d} g_0^1(r, s, y) dy \overset{2,2}{\leq} C, \]

which gives (3.43).

For \( g \in C_0^\infty(\mathbb{R}^d) \), since \( a, b \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^d)) \), by (3.43), we have

\[ \lim_{t \to s} \int_{\mathbb{R}^d} g(x)(I_1(t, s, x) + I_3(t, s, x)) dx = 0. \]

Let \( a_n(x) \in C_0^\infty(\mathbb{R}^d) \) with

\[ \lim_{n \to \infty} |a_n(x) - a(s, x)| = 0, \quad x \in \mathbb{R}^d. \]

Then, by (3.43), we have

\[ \lim_{t \to s} \left| \int_{\mathbb{R}^d} g(x)I_2(t, s, x) dx \right| \leq \lim_{n \to \infty} \left| \int_{\mathbb{R}^d} g(x) \frac{a_n(x) - a(s, x)}{t-s} \int_s^t \Delta_x^\frac{1}{2} (P_{r,s}^a f(x) - f(x)) dr dx \right| \]

\[ + \lim_{n \to \infty} \int_{\mathbb{R}^d} g(x) \frac{a_n(x)}{t-s} \int_s^t \Delta_x^\frac{1}{2} (P_{r,s}^a f(x) - f(x)) dr dx \]

\[ \leq \lim_{n \to \infty} \int_{\mathbb{R}^d} |g(x)| \cdot |a_n(x) - a(s, x)| dx \]

\[ + \lim_{n \to \infty} \left| \frac{1}{t-s} \int_s^t \int_{\mathbb{R}^d} \Delta_x^\frac{1}{2} (ga_n)(x)(P_{r,s}^a f(x) - f(x)) dx dr \right|, \]

which converges to zero by (3.32). Similarly, we also have

\[ \lim_{t \to s} \left| \int_{\mathbb{R}^d} g(x)I_4(t, s, x) dx \right| = 0. \]

Combining the above calculations, we obtain (3.38).

(vi) Lastly, we show the uniqueness. For \( f \in C_0^\infty(\mathbb{R}^d) \) and \( t \geq s \), set

\[ u_t^f(t, x) := \int_{\mathbb{R}^d} p_{a,b}(t, x; s, y) f(y) dy. \]

Let us first show the following claim: For any \( T > s \) and \( p > 1 \),

\[ \sup_{t \in [s, T]} \left( ||u_t^f(t)||_p + ||\nabla u_t^f(t)||_p \right) < +\infty. \quad (3.44) \]

Proof of Claim: By Young’s inequality for convolution, it follows that

\[ ||u_t^f(t)||_p \overset{3.34}{\leq} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} g_0^1(t, x; s, y)|f(y)| dy \right)^p dx \right)^{\frac{1}{p}} \overset{2,2}{\leq} ||f||_p. \]
We next prove \( \sup_{t \in [s, T]} \| \nabla u^f_s(t) \|_p < +\infty \). Let the support of \( f \) be contained in the ball \( \{ x \in \mathbb{R}^d : |x| \leq N \} \). We have

\[
\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \nabla_x p_{a,b}(t, x; s, y) f(y) dy \right|^p dx \leq I_1 + I_2,
\]

where

\[
I_1 := \int_{|x| \geq 2N} \left| \int_{|y| \leq N} \nabla_x p_{a,b}(t, x; s, y) f(y) dy \right|^p dx,
\]

\[
I_2 := \int_{|x| < 2N} \left| \int_{|y| \leq N} \nabla_x p_{a,b}(t, x; s, y) f(y) dy \right|^p dx.
\]

For \( I_1 \), we have

\[
I_1 \leq C_N \int_{|x| \geq 2N} \int_{|y| \leq N} \left| \nabla_x p_{a,b}(t, x; s, y) \right|^p f(y)^p dy dx \leq C_N \| f \|_\infty^p \int_{|y| \leq N} \int_{|x| \geq 2N} \left| \nabla_x p_{a,b}(t, x; s, y) \right|^p dx dy \leq C_N \| f \|_\infty^p \int_{|y| \leq N} \int_{|x| \geq 2N} \frac{1}{|x-y|^{(d+1)p}} dx dy < +\infty.
\]

For \( I_2 \), by (3.43), we have

\[
I_2 \leq \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} \nabla_x p_{a,b}(t, x; s, y) f(y) dy \right|^p \left( \int_{|x| < 2N} dx \right) < +\infty.
\]

The claim is proven.

Let \( p'_{a,b}(t, x; s, y) \) be another function satisfying (3.30)–(3.37). We want to prove that for any \( f \in C_0^\infty(\mathbb{R}^d) \) and \( t > s \),

\[
\tilde{u}^f_s(t, x) := \int_{\mathbb{R}^d} p_{a,b}(t, x; s, y) f(y) dy - \int_{\mathbb{R}^d} p'_{a,b}(t, x; s, y) f(y) dy = 0.
\]

In view of \( \lim_{t \downarrow s} u^f_s(t, x) = 0 \), by (3.31) we have

\[
\tilde{u}^f_s(t, x) = \int_s^t \mathcal{L}_{a,b}^f(r, x) \tilde{u}^f_s(r, \cdot)(x) dr.
\]

The uniqueness follows by (3.44) and [26] Lemma 3.1. \( \square \)

4. Proof of Theorem 1.1

By Duhamel’s formula, we construct the heat kernel \( p(t, x; s, y) \) of \( \mathcal{L}(t, x) \) by solving the following integral equation:

\[
p(t, x; s, y) = p_{a,b}(t, x; s, y) + \int_{s}^{t} \int_{\mathbb{R}^d} p_{a,b}(t, r; x; z) c(r, z) p(r, z; s, y) dz dr. \tag{4.1}
\]

For \( t > s \geq 0 \) and \( x, y \in \mathbb{R}^d \), set \( \Theta_0(t, x; s, y) := p_{a,b}(t, x; s, y) \), and define recursively for \( n \in \mathbb{N} \),

\[
\Theta_n(t, x; s, y) := \int_s^t \int_{\mathbb{R}^d} p_{a,b}(t, r; x; z) c(r, z) \Theta_{n-1}(r, z; s, y) dz dr. \tag{4.2}
\]

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For $\gamma \in (0, 1]$ and $c \in \mathbb{K}_d^\gamma$, define

$$
\ell^c_\gamma(c) := \sup_{(t, x) \in [0, \infty) \times \mathbb{R}^d} \int_0^c \int_{\mathbb{R}^d} \Theta^0_\gamma(s, z) \left( |c(t - s, x - z)| + |c(t + s, x + z)| \right) dz ds.
$$

**Lemma 4.1.** If $c \in \mathbb{K}_d^1$, then there exists a constant $\Lambda > 0$ such that for all $n \in \mathbb{N}$,

$$
|\Theta_n(t, x; s, y)| \leq \{\Lambda \ell^c_1(t - s)\}^n \varphi^0_1(t, x; s, y). \tag{4.3}
$$

If $c \in \mathbb{K}_d^{1-\gamma}$ for some $\gamma \in (0, 1)$, then there exists a constant $C_1 > 0$ such that for any $n \in \mathbb{N}$,

$$
|\Theta_n(t, x; s, y) - \Theta_n(t, x'; s, y)| \leq C_1 (|x - x'|^\gamma \wedge 1) \{\Lambda \ell^c_1(t - s)\}^{n-1} \ell^c_1(t - s) \times \left( \varphi^0_1(t, x; s, y) + \varphi^0_1(t, x'; s, y) \right). \tag{4.4}
$$

If $c \in \mathbb{H}^\gamma$ for some $\gamma \in (0, 1)$, then there exists a constant $C_2 > 0$ such that for any $n \in \mathbb{N}$,

$$
|\nabla_3 \Theta_n(t, x; s, y)| \leq C_2 \Lambda \{\Lambda \|c\|_{t-s}(t - s)\}^n \varphi^0_1(t, x; s, y). \tag{4.5}
$$

**Proof.** (1) First of all, by (3.34), we have for some $C_0 > 0$,

$$
p_{a, b}(t, x, s, y) \leq C_0 \ell^0_1(t, x, s, y).
$$

Now we use induction to prove (4.3). Suppose that (4.3) is true for $n \in \mathbb{N}$. Then

$$
|\Theta_{n+1}(t, x; s, y)| \leq \int \int \int p_{a, b}(t, x, s, y) \cdot |\Theta_n(r, z; s, y)| dz dr
$$

$$
\leq C_0 \Lambda \ell^c_1(t - s)^n \int \int \int \varphi^0_1(t, x; r, z) \varphi^0_1(r, z; s, y) c(r, z) dz dr
$$

$$
\leq \Lambda \{\Lambda \ell^c_1(t - s)\}^{n-1} \int \int \int \varphi^0_1(t, x; r, z) \varphi^0_1(r, z; s, y) c(r, z) dz dr
$$

$$
\leq \{\Lambda \ell^c_1(t - s)\}^{n-1} \ell^c_1(t, x; s, y).
$$

(2) By (4.2) and (3.36), we have

$$
|\Theta_n(t, x; s, y) - \Theta_n(t, x'; s, y)|
$$

$$
\leq (|x - x'|^\gamma \wedge 1) \int \int \int \left( \varphi^0_1(t, x; s, y) + \varphi^0_1(t, x'; s, y) \right) c(r, z) \cdot |\Theta_{n-1}(r, z; s, y)| dz dr
$$

$$
\leq (|x - x'|^\gamma \wedge 1) (|x - x'|^\gamma \wedge 1) \{\Lambda \ell^c_1(t - s)\}^{n-1}
$$

$$
\times \int \int \int \left( \varphi^0_1(t, x; r, z) + \varphi^0_1(t, x'; r, z) \right) c(r, z) \varphi^0_1(r, z; s, y) dz dr
$$

$$
\leq \{\Lambda \ell^c_1(t - s)\}^{n-1}
$$

$$
\times \left\{ \int \int \int (t - r)^{-\gamma} (r - s) \left( \varphi^0_0(t, x; r, z) + \varphi^0_0(r, z; s, y) \right) c(r, z) dz dr \right\}
$$

$$
+ \int \int \int (t - r)^{-\gamma} (r - s) \left( \varphi^0_0(t, x'; r, z) + \varphi^0_0(r, z; s, y) \right) c(r, z) dz dr
$$

$$
\leq C_1 (|x - x'|^\gamma \wedge 1) \{\Lambda \ell^c_1(t - s)\}^{n-1}
$$

$$
\times \left\{ \int \int \int \varphi^0_0(t, x; r, z) + \varphi^0_0(t, x'; r, z) c(r, z) dz dr \right\}
$$

$$
+ \int \int \int \varphi^0_0(t, x'; r, z) c(r, z) dz dr
$$
\[ C_1 \leq C_1 |x - x'|^\gamma \wedge 1 \{ \Lambda \ell_1(t - s) \}^{n-1} \ell_1^\gamma(t - s) \Phi_1^0(t, x, s, y) + \Phi_1^0(t, x', s, y) \],

and (4.4) holds.

(3) If \( c \) is bounded, by definition and (2.2), it is easy to see that for some \( C_1 > 0 \),

\[ \ell_1^\gamma(\varepsilon) \leq C_1 |c|_{\infty} \varepsilon^\gamma, \quad \varepsilon > 0. \]  

(4.6)

As in Lemma 3.5 one can prove

\[ \nabla_y \Theta_n(t, x; s, y) = \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)c(r, z)\Theta_{n-1}(r, z; s, y)dzdr. \]

By (3.30), we can write

\[ \nabla_y \Theta_n(t, x; s, y) = \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)c(r, z)\Theta_{n-1}(r, z; s, y)dzdr \]

\[ + \int_{\mathbb{R}^d} \Theta_{n-1}(r, x; s, y)dzdr \]

\[ = \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)c(r, z)(\Theta_{n-1}(r, z; s, y) - \Theta_{n-1}(r, x; s, y))dzdr \]

\[ + \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)c(r, z)dz \right) \Theta_{n-1}(r, x; s, y)dzdr \]

\[ = Q_1(t, x; s, y) + Q_2(t, x; s, y) + Q_3(t, x; s, y). \]

For \( Q_1(t, x; s, y) \), by (4.6) and (4.4), we have

\[ Q_1(t, x; s, y) \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n-1} \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)c(r, z)\Phi_1^0(r, z; s, y)dzdr \]

\[ \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n-1} \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)c(r, z)dz \Phi_1^0(r, x; s, y)dzdr \]

(2.5)(2.2)

\[ \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n-1} \Phi_1^0(t, x; s, y). \]

For \( Q_2(t, x; s, y) \), by (4.6) and (4.3), we have

\[ Q_2(t, x; s, y) \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n-1} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)dz \right) \Phi_1^0(t, x; s, y)dr \]

\[ \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n-1} \left( \int_{\mathbb{R}^d} (t - r)^{n-1}(r - s)dr \right) \Phi_1^0(t, x; s, y) \]

\[ \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n} \Phi_1^0(t, x; s, y). \]

For \( Q_3(t, x; s, y) \), we have

\[ Q_3(t, x; s, y) \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n-1} \int_{\mathbb{R}^d} \nabla_y p_{a,b}(t, x; r, z)\Phi_1^0(r, z; s, y)dzdr \]

\[ \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n-1} \left( \int_{\mathbb{R}^d} (r - s)dr \right) \Phi_1^0(t, x; s, y) \]

\[ \leq \{ \Lambda |c|_{\infty}(t - s) \}^{n} \Phi_1^0(t, x; s, y). \]

Combining the above calculations, we obtain (4.5). \( \square \)
Proof of Theorem 1.1. By the standard time shift technique, it suffices to prove the conclusions on a small time interval. We divide the proof in several steps.

(1) Define

\[ p(t, x; s, y) = p_{a,b}(t, x; s, y) + \sum_{n=1}^{\infty} \Theta_n(t, x; s, y). \]

By virtue of \( c \in \mathbb{R}^d \), we have

\[ \lim_{\varepsilon \to 0} \epsilon \ell_1^\prime(\varepsilon) = 0. \]

Hence, for any given \( \varepsilon \in (0, 1) \), one can choose \( T_\varepsilon \in (0, 1) \) small enough such that for all \( 0 \leq s < t \leq T_\varepsilon \),

\[ \ell_1^\prime(t - s) \leq \frac{\varepsilon}{\Lambda}. \]

Thus,

\[ |p(t, x; s, y) - p_{a,b}(t, x; s, y)| \leq \sum_{n=1}^{\infty} |\Theta_n(t, x; s, y)| \leq \frac{\Lambda \ell_1^\prime(t - s)}{1 - \Lambda \ell_1^\prime(t - s)} \ell_1^0(t, x; s, y) \]

\[ \leq \frac{\varepsilon}{1 - \varepsilon} \ell_1^0(t, x; s, y), \]

which together with (3.34) gives (1.5) for \( 0 \leq s < t \leq T_\varepsilon \). Moreover, noticing that

\[ \sum_{n=0}^{m} \Theta_n(t, x; s, y) = p_{a,b}(t, x; s, y) + \int_{s}^{t} p_{a,b}(t, x; r, z)c(r, z) \sum_{n=0}^{m} \Theta_n(r, z; s, y)dzdr, \]

by taking limits, we obtain equation (4.1). In the case of \( a(t, x) = a(t) \), by (3.35), if we let \( \varepsilon \) be small enough, we also have (1.6). Moreover, estimates (1.7) and (1.8) follow by (4.4), (3.36) and (4.3), (3.37).

(2) Define

\[ P_{t,s}f(x) := \int_{\mathbb{R}^d} p(t, x; s, y)f(y)dy \]

and

\[ P_{t,s}^{a,b}f(x) := \int_{\mathbb{R}^d} p_{a,b}(t, x; s, y)f(y)dy. \]

For proving (1.2), it suffices to prove that for any \( f \in C_0^\infty(\mathbb{R}^d) \),

\[ P_{t,s}f(x) = P_{t,r}P_{r,s}f(x), \quad s < r < t. \]  

(4.7)

By (4.1), we have

\[ P_{t,s}f(x) = P_{t,s}^{a,b}f(x) + \int_{s}^{t} P_{t,r}^{a,b}(c(r', \cdot)P_{r,s}^{a,b}f)(x)dr' \]

\[ = P_{t,r}P_{r,s}^{a,b}f(x) + \int_{s}^{t} P_{t,r}^{a,b}P_{r,s}^{a,b}(c(r', \cdot)P_{r,s}^{a,b}f)(x)dr' + \int_{r}^{t} P_{t,r}^{a,b}(c(r', \cdot)P_{r,s}^{a,b}f)(x)dr' \]

\[ = P_{t,r}P_{r,s}^{a,b}f(x) + \int_{r}^{t} P_{t,r}^{a,b}(c(r', \cdot)P_{r,s}^{a,b}f)(x)dr', \]

where we have used \( P_{t,s}^{a,b}f = P_{t,s}^{a,b}P_{r,s}^{a,b}f \), which follows by the uniqueness of Theorem 3.6. On the other hand, we also have

\[ P_{t,r}P_{r,s}^{a,b}f(x) = P_{t,r}^{a,b}P_{r,s}^{a,b}f(x) + \int_{r}^{t} P_{t,r}^{a,b}(c(r', \cdot)P_{r,s}^{a,b}f)(x)dr'. \]
Fix \( s < r \) and set
\[
 u_t(x) := P_{t,r}P_{r,s}f(x) - P_{t,s}f(x).
\]
Then, we have
\[
 u_t(x) = \int_{t}^{0} \int_{\mathbb{R}^d} p_{a,b}(t, x; r', y)c(r', y)u_{r'}(y)dydr'.
\]
By \((3.34)\), we have
\[
 \|u_t\|_\infty \leq \sup_{r' \in [t,r]} \|u_{r'}\|_\infty \int_{t}^{0} \int_{\mathbb{R}^d} q_{1}(t, x; r', y)|c(r', y)|dydr' = \ell_{1}(t - r) \sup_{r' \in [t,r]} \|u_{r'}\|_\infty,
\]
which implies that
\[
 \sup_{r' \in [t,r]} \|u_{r'}\|_\infty \leq \sup_{\varepsilon \in (0, r-t)} \ell_{1}(\varepsilon) \sup_{r' \in [t,r]} \|u_{r'}\|_\infty.
\]
In particular, if \( t - r \) is small enough (say less than \( \varepsilon_0 \)), then
\[
 \sup_{r' \in [t,r]} \|u_{r'}\|_\infty = 0.
\]
Thus, we obtain \((4.7)\) for \( t - r < \varepsilon_0 \). For general \( t \), it follows by repeatedly using \((4.7)\).

(3) We prove \((1.3)\). By \((4.1)\) and \((3.32)\), we only need to prove that for any \( f \in C_{b}(\mathbb{R}^d)\),
\[
 \lim_{t \downarrow s} \int_{\mathbb{R}^d} \int_{s}^{t} \int_{\mathbb{R}^d} p_{a,b}(t, x; r, z)c(r, z)p(r, z; s, y)f(y)dzdy = 0.
\]
This limit follows by noticing that
\[
 \left| \int_{\mathbb{R}^d} \int_{s}^{t} \int_{\mathbb{R}^d} p_{a,b}(t, x; r, z)c(r, z)p(r, z; s, y)f(y)dzdy \right|
\]
\[
 \leq \int_{\mathbb{R}^d} \int_{s}^{t} \int_{\mathbb{R}^d} q_{1}(t, x; r, z)c(r, z)p_{1}(r, z; s, y)f(y)dzdy
\]
\[
 \leq \int_{\mathbb{R}^d} \left( \int_{s}^{t} \int_{\mathbb{R}^d} q_{1}(t, x; r, z)c(r, z)p_{1}(r, z; s, y)dzdy \right)q_{1}(t, x; s, y)dy
\]
\[
 \leq \ell_{1}(t - s) \int_{\mathbb{R}^d} q_{1}(t, x; s, y)dy \leq C\ell_{1}(t - s) \rightarrow 0, \ t \downarrow s.
\]

(4) Let \( f, g \in C_{0}^{\infty}(\mathbb{R}^d) \). By definitions, we make the following decomposition:
\[
 \frac{P_{t,s}f(x) - f(x)}{t - s} = \mathcal{L} f(x) = \frac{1}{t - s} \int_{s}^{t} \left( P_{s,t}^{a,b}(c(r)P_{r,s}f)(x) - c(r, x)P_{r,s}f(x) \right)dr
\]
\[
 + \frac{1}{t - s} \int_{s}^{t} (c(r, x) - c(s, x))P_{r,s}f(x)dr
\]
\[
 + \frac{1}{t - s} \int_{s}^{t} c(s, x) \left( P_{r,s}f(x) - f(x) \right)dr
\]
\[
 + \left( \frac{P_{t,s}^{a,b}f(x) - f(x)}{t - s} - \mathcal{L}_{a,b}^{2}(s, x)f(x) \right)
\]
\[
 = I_{1}(t, s, x) + I_{2}(t, s, x) + I_{3}(t, s, x) + I_{4}(t, s, x).
\]
For \( I_{1}(t, s, x) \), if we write
\[
 (P_{t,r}^{a,b})^{*}g(y) := \int_{\mathbb{R}^d} p_{a,b}(t, x; r, y)g(x)dx,
\]
then
\[
\left| \int_{\mathbb{R}^d} g(x)I_1(t, s, x)dx \right| \leq \frac{1}{t-s} \int_{s}^{t} \int_{\mathbb{R}^d} \left| (P_{t,r}^{a,b})^r g(x) - (P_{t,r}^{a,b})^s 1(x) \cdot g(x) \right| c(r, x)P_{r,s}f(x)dx dr
\]
\[
\leq \frac{1}{t-s} \int_{s}^{t} \int_{\mathbb{R}^d} \left| (P_{t,r}^{a,b})^r 1 - 1 \right| (x) g(x) c(r, x) P_{r,s} f(x) dx dr
\]
\[
=: J_1(t, s) + J_2(t, s).
\]
For \(J_1(t, s)\), noticing that
\[
\left| (P_{t,r}^{a,b})^r g(y) - (P_{t,r}^{a,b})^s 1(y) \cdot g(y) \right| = \left| \int_{\mathbb{R}^d} p_{a,b}(t, x; r, y) (g(x) - g(y)) dx \right|
\]
\[
\leq C \|g\|_{\mathbb{E}^1} \int_{\mathbb{R}^d} g_s^0(t, x; r, y) \left| x - y \right| 1 dx
\]
\[
\leq C \|g\|_{\mathbb{E}^1} |t - r|,
\]
by definition of \(P_{r,s}f\) and \((1.5)\), we have
\[
J_1(t, s) \leq C \|g\|_{\mathbb{E}^1} \int_{s}^{t} \int_{\mathbb{R}^d} |c(r, x)| \cdot |P_{r,s}f(x)| dx dr
\]
\[
\leq C \|g\|_{\mathbb{E}^1} \int_{s}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |c(r, x)| g_s^0(r, s, y)|f(y)| dy dx dr
\]
\[
\leq C \|g\|_{\mathbb{E}^1} C_1(t - s) \int_{\mathbb{R}^d} |f(y)| dy \rightarrow 0, \quad t \downarrow s.
\]
For \(J_2(t, s)\), since \(c \in C((0, \infty); L_1^{1} (\mathbb{R}^d))\), by \((3.33)\) and the dominated convergence theorem, we have
\[
\lim_{t \downarrow s} J_2(t, s) = 0.
\]
It is the same reason that
\[
\lim_{t \downarrow s} \int_{\mathbb{R}^d} g(x)(I_2(t, s, x) + I_3(t, s, x))dx = 0.
\]
Moreover, if \(a, b \in C((0, \infty); L_1^{1} (\mathbb{R}^d))\), by \((3.38)\) we have
\[
\lim_{t \downarrow s} \int_{\mathbb{R}^d} g(x)I_4(t, s, x)dx = 0.
\]
Combining the above limits, we obtain \((1.4)\). The whole proof is complete. \(\square\)

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