Response to: “Limitations of the Method of Lagrangian Descriptors”

http://arxiv.org/abs/1510.04838

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Abstract

This Response is concerned with the recent Comment of Ruiz-Herrera, “Limitations of the Method of Lagrangian Descriptors” [http://arxiv.org/abs/1510.04838], criticising the method of Lagrangian Descriptors. In spite of the significant body of literature asserting the contrary, Ruiz-Herrera claims that the method fails to reveal the presence of stable and unstable manifolds of hyperbolic trajectories in incompressible systems and in almost all linear systems. He supports this claim by considering the method of Lagrangian descriptors applied to three specific examples. However in this response we show that Ruiz-Herrera does not understand the proper application and interpretation of the method and, when correctly applied, the method beautifully and unambiguously detects the stable and unstable manifolds of the hyperbolic trajectories in his examples.

1 Introduction

Before analyzing in detail the examples and claims of Ruiz-Herrera, we first describe the set-up for Lagrangian Descriptors as originally described in Madrid and Mancho (2009), and further developed in Mancho et al. (2013). We consider a vector field,

$$\frac{dx}{dt} = v(x, t), \quad x \in \mathbb{R}^n, \ t \in \mathbb{R}$$

(1)
where \( \mathbf{v}(\mathbf{x}, t) \in C^r \) \((r \geq 1)\) in \( \mathbf{x} \) and continuous in time. We denote a trajectory of this vector field by \( \mathbf{x}(t; \mathbf{x}_0) \), with initial condition \( \mathbf{x}(t_0; \mathbf{x}_0) = \mathbf{x}_0 \).

The classical definition of the Lagrangian Descriptor (LD, the function \( M \)), as introduced in Madrid and Mancho (2009); Mendoza and Mancho (2010), is:

\[
M(\mathbf{x}_0, t_0, \tau) = \int_{t_0-\tau}^{t_0+\tau} \| \dot{\mathbf{x}}(t; \mathbf{x}_0) \| \, dt
\]  

where \( \| \dot{\mathbf{x}}(t; \mathbf{x}_0) \| \equiv \sqrt{\langle \dot{\mathbf{x}}(t; \mathbf{x}_0), \dot{\mathbf{x}}(t; \mathbf{x}_0) \rangle} \). So the LD is a function depending on the initial condition of a trajectory \( \mathbf{x}(t_0; \mathbf{x}_0) = \mathbf{x}_0 \) and on a time interval \([t_0 - \tau, t_0 + \tau]\). Hence (2) is the arclength of trajectories over the time interval \([t_0 - \tau, t_0 + \tau]\). Phase space structure, e.g. stable and unstable manifolds of hyperbolic trajectories, is encoded in the properties of the function \( M(\mathbf{x}_0, t_0, \tau) \). Concerning this point, in his Comment Ruiz-Herrera states the following,

The method of Lagrangian descriptors says that the invariant manifolds of saddle points in (2.2) are given by “singular points” (i.e. “non-smooth points”) of the contour lines of \( M \), see [10, 4, 7].

However, no such assertion is made in any the references quoted by Ruiz-Herrera. In fact, it is not true, and it explains some of the confusion of Ruiz-Herrera and why, in his examples, he does not detect the stable and unstable manifolds of hyperbolic trajectories with his incorrect usage of LDs. The term “singularities” does not refer to properties of the contour lines of \( M(\mathbf{x}_0, t_0, \tau) \), but to points at which certain directional derivatives of \( M(\mathbf{x}_0, t_0, \tau) \) do not exist. While this is discussed, somewhat, in the references Mendoza and Mancho (2010); Mancho et al. (2013), it is made precise in Lopesino et al. (2015).

Henceforth we will consider his examples separately (and refer to specific equation numbers in his Comment when necessary).

2 First Example of Ruiz-Herrera

In his equation (2.3) Ruiz-Herrera considers the vector field:

\[
\begin{align*}
\dot{x} &= \lambda x, \\
\dot{y} &= -\lambda y, \quad \lambda > 0.
\end{align*}
\]

Note that the origin is a hyperbolic fixed point, with the \( x \)-axis corresponding to its unstable manifold and its \( y \)-axis corresponding to its stable manifold. In his figure 1 Ruiz-Herrera
plots the contours of $M$ (for $\lambda = 1$ and $\tau = 20$). For this case the contours of $M$ do appear to converge to the stable and unstable manifolds of the hyperbolic fixed point. He claims (without proof, and incorrectly as we will show) that this is a result of the equal expansion and contraction rates of the saddle.

From now until the end of the paper, the value of variables $t_0$ and $\tau$ will be prefixed for every analyzed example. Without loss of generality we can take $t_0 = 0$ since every example considered is an autonomous (time independent) system. As a consequence of this, the domain of the function $M(x_0, t_0, \tau) = M(x_0, y_0, t_0, \tau)$ will be the plane $\mathbb{R}^2$ of initial conditions $x_0 = (x_0, y_0)$. Now we also compute $M$ for $\lambda = 1$ and $\tau = 20$ as Ruiz-Herrera in his Comment. In the left-hand panel of Fig. 1 we show the contours of $\partial_{x_0} M = \partial M / \partial x_0$, the partial derivative of $M$ with respect to $x_0$. Here we can see that $\partial_{x_0} M$ changes in sign on the stable manifold $\{x_0 = 0\}$, exactly as expected. In the right-hand panel of Fig. 1 we show the contours of $\partial_{y_0} M = \partial M / \partial y_0$, the partial derivative of $M$ with respect to $y_0$. Similarly, $\partial_{y_0} M$ changes in sign on the unstable manifold $\{y_0 = 0\}$, as expected. These two special features highlight the two manifolds over a region of the hyperbolic point.

![Figure 1: The left-hand panel shows contours of $\partial_{x_0} M$ and the right-hand panel shows contours of $\partial_{y_0} M$. This figure should be compared with figure 1 of the Comment of Ruiz-Herrera.](image)

3 Second Example of Ruiz-Herrera

In his equation (3.5) Ruiz-Herrera considers the vector field:

\[
\begin{align*}
\dot{x} &= f(x), \\
\dot{y} &= -y \cdot f'(x),
\end{align*}
\]
where \( f(x) \) is given in a complicated expression in (3.4), which we will not reproduce here. One feature of the definition of \( f(x) \) is that the origin is a hyperbolic fixed point whose stable manifold is given by the \( y \)-axis and whose unstable manifold is given by the \( x \)-axis. The other feature is that it allows him to show that the contour lines of \( M \) are horizontal lines in a neighborhood of the stable manifold. He claims that this demonstrates that the method of LDs does not detect the stable and unstable manifolds. However, as in the previous example, we will show that this is not true by using \( M \) properly.

We have computed \( M \) for this example using \( \tau = 10 \). In the left-hand panel of Fig. 2 we show contours of \( \partial_{x_0} M \). Its sign changes on the stable manifold, exactly as expected. In the right-hand panel of Fig. 2 we show contours of \( \partial_{y_0} M \). Similarly to the left-hand panel, \( \partial_{y_0} M \) experiences a change in sign on the unstable manifold, although this change is more abrupt than in the previous example and this might be seen only as an anomaly of \( \partial_{y_0} M \).

Figure 2: The left-hand panel shows contours of \( \partial_{x_0} M \) and the right-hand panel shows contours of \( \partial_{y_0} M \). This figure should be compared with figure 2 of the Comment of Ruiz-Herrera.

We will not consider the third example of Ruiz-Herrera since it is a linear saddle point and illustrates exactly the same type of misunderstanding of LDs as his other examples. We will go directly to his fourth and final example.

4 Fourth Example of Ruiz-Herrera

In his equation (4.9) Ruiz-Herrera considers the vector field,
\[
\dot{x} = \lambda x, \\
\dot{y} = -\mu y, \quad \lambda \neq \mu > 0.
\]

He computes \( M \) and plots the contour lines of \( M \) for two cases: \( \lambda = 1, \mu = 2 \) (shown in the left-hand panel of his figure 4) and \( \lambda = 2, \mu = 1 \) (shown in the right-hand panel of his figure 4), both using \( \tau = 10 \). In both cases the contour lines are straight lines and fail to reveal the stable and unstable manifolds. However as we have pointed out in previous examples, this is not the correct way to rule out that \( M \) detects stable and unstable manifolds.

In panels a) and b) we plot contours of the the derivatives \( \partial_{x_0} M \) and \( \partial_{y_0} M \), respectively, for the case \( \lambda = 1, \mu = 2, \tau = 10 \). We see that the stable and unstable manifolds are clearly identified in these plots, just as in the earlier examples. In panels c) and d) we plot contours of the the derivatives \( \partial_{x_0} M \) and \( \partial_{y_0} M \), respectively, for the case \( \lambda = 2, \mu = 1, \tau = 10 \). Also here in this case we see that the stable and unstable manifolds are clearly identified in these plots.

Apparently Ruiz-Herrera’s reasoning behind his Example 1 and Example 4 was that differing expansion and contraction rates of the saddle could cause the method of Lagrangian Descriptors to fail. Clearly this is not true.

5 Conclusions of Ruiz-Herrera

Ruiz-Herrera ends his Comment with a section of conclusions based on his “counterexamples”. We have shown that, due to his lack of understanding of the singular properties of the LDs, all of his “counterexamples” and therefore all of his conclusions are wrong.

He ends his conclusions section by commenting on the recent development of LDs for discrete time systems described in Lopesino et al. (2015). He claims that the results of his Comment are applicable to the time 1 map of his examples and consequently this fact implies that the results of Lopesino et al. (2015) are incorrect. As we have seen, such an assertion is wrong.

Finally, we note that Ruiz-Herrera has recently published another collection of “counterexamples” to the method of Lagrangian Descriptors (Ruiz-Herrera (2015)). These “counterexamples” also suffer from an incorrect understanding of the singular properties of the of LDs, and are also wrong.

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Figure 3: In panels a) and c) the stable manifold is clearly identified and in panels b) and d) the unstable manifold is clearly identified. This figure should be compared with figure 4 of the Comment of Ruiz-Herrera.
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