A remark on a theorem of the Goldbach-Waring type

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Abstract

Let $p_i$, $2 \leq i \leq 5$ be prime numbers. It is proved that all but $\ll x^{23027/23040+\epsilon}$ even integers $N \leq x$ can be written as $N = p_1^2 + p_2^3 + p_3^4 + p_4^5$.

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1 Introduction and Statement of Results

In the thirties, I. M. Vinogradov and Hua established many fundamental theorems in additive prime number theory. Their methods were consecutively applied to various problems in additive number theory. Among others, Pracchar established in 1952, the following result:

There exists a constant $c > 0$ such that all but $\ll x(\log x)^{-c}$ even integers $N$ smaller than $x$ are representable as

$$N = p_1^2 + p_2^3 + p_3^4 + p_4^5$$

for prime numbers $p_i$.

In [1] and [2], this theorem was improved as follows:

All but $\ll x^{19193/19200+\epsilon}$ positive even integers smaller than $x$ can be represented as in (1.1).

Here we improve upon this result by showing the following theorem:

Theorem. All but $\ll x^{23027/23040+\epsilon}$ positive even integers smaller than $x$ can be represented as in (1.1).

2 Notation and structure of the proof

We will choose our notation similar as in [2]. By $k$ we will always denote an integer $k \in \{2, 3, 4, 5\}$, by $p$ we denote a prime number and $L$ denotes $\log x$. $c$ is an effective positive constant and $\epsilon$ will denote an arbitrarily small positive number; both of them may take different values at different occasions. For example, we may write $L^c L^\epsilon \ll L^c$, $x^\epsilon L^c \ll x^\epsilon$. 

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\(d(n)\) denotes the number of divisors of \(n\) and \([a_1, \ldots, a_n]\) denotes the least common multiple of the integers \(a_1, \ldots, a_n\). Be further

\[
r \sim R \iff R/2 < r \leq R, \quad \sum_{\chi \text{mod } q}^* \chi = \sum_{\chi \text{primitive}}^* \chi, \quad \sum_{1 \leq a \leq q}^* = \sum_{1 \leq a \leq q}^* \frac{q}{(a, q)}.\]

We set

\[
P = N^{\frac{1}{100}} - \epsilon, \quad Q = NP^{-1}L^{-E} \quad (E > 0 \text{ will be defined later}),
\]

and

\[
\mu = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - 1.
\]

We define for any characters \(\chi, \chi_j (mod \ q), q \leq P\) and a fixed integer \(N\):

\[
C_k(a, \chi) = \sum_{l=1}^{q} \chi(l) e\left(\frac{al^k}{q}\right), \quad C_k(a, \chi_0) = C_k(a, q).
\]

\[
Z(q, \chi_2, \chi_3, \chi_4, \chi_5) = \sum_{1 \leq h \leq q}^* e\left(-\frac{hN}{q}\right) \prod_{k=2}^{5} C_k(h, \chi_k),
\]

\[
Y(q) = Z(q, \chi_0, \chi_0, \chi_0, \chi_0), \quad A(q) = \frac{Y(q)}{\delta^4(q)}.
\]

When the variable \(N\) is fixed, we will always write \(A(q)\) and neglect the dependency of \(A(q)\) on \(N\). Otherwise, we will write \(A(q, n)\).

\[
s(p) = 1 + \sum_{\alpha \geq 1} A(p^\alpha), \quad S_k(\lambda) = \sum_{\psi/2^{k+1} < n \leq \psi/\lambda} \Lambda(n)e(n^k\lambda),
\]

\[
S_k(\lambda, \chi) = \sum_{\psi/2^{k+1} < n \leq \psi/\lambda} \Lambda(n)\chi(n)e(n^k\lambda), \quad T_k(\lambda) = \sum_{\psi/2^{k+1} \leq n \leq \psi/\lambda} e(n^k\lambda),
\]

\[
W_k(\lambda, \chi) = S_k(\lambda, \chi) - E_0T_k(\lambda, \chi), \quad E_0 = \begin{cases} 1, & \text{if } \chi = \chi_0, \\ 0, & \text{otherwise.} \end{cases}
\]

Using the circle method we define the major arcs \(M\) and minor arcs \(m\) as follows:

\[
M = \sum_{q \leq P} \sum_{1 \leq a \leq q}^* I(a, q), \quad I(a, q) = \left[\frac{a}{q} - \frac{1}{Qq}, \frac{a}{q} + \frac{1}{Qq}\right],
\]

\[
m = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \setminus M.
\]

Let

\[
R(N) = \sum_{\psi/2^{k+1} \leq n \leq \psi, \ k \in \{2, \ldots, 5\}}^* \Lambda(n_2) \ldots \Lambda(n_5).
\]
Then we find
\[ R(N) = \int_{\frac{1}{2}} e(-Na) \prod_{k=2}^{5} S_k(\alpha) d\alpha = \left( \int_{M} + \int_{m} \right) e(-Na) \prod_{k=2}^{5} S_k(\alpha) d\alpha =: R_1(N) + R_2(N). \tag{2.1} \]

Arguing as in [2], we see that
\[ I_2(N) \ll N^\mu L^{-A} \tag{2.2} \]
for any \( A > 0 \) and all but \( x^{1+2\epsilon} L^{-1/128} < x^{2^{3027/23040+3\epsilon}} \) even integers \( x/2 \leq N < x \). In the sections 3 and 4 we will show that for any given \( A > 0 \)

\[ R_1(N) = \frac{1}{120} P_0 \prod_{p \leq P} s(p) + O \left( x^\mu L^{-A} \right), \tag{2.3} \]

where
\[ x^\mu \ll P_0 := \sum_{m_1 + m_2 + m_3 + m_4 = N} \frac{1}{m_1} \ll x^\mu \quad \text{for} \quad N \in (x/2, x]. \tag{2.4} \]

Using that
\[ \prod_{p \leq P} s(p) \gg (\log P)^{-960}, \]
(see p. lemma 4.5 in [1]), the theorem follows from (2.1) - (2.4).

### 3 The major arcs

We will make use of the following lemmas:

**Lemma 3.1** Let \( f(x) \), \( g(x) \) and \( f'(x) \) be three real differentiable and monotonic functions in the interval \( [a, b] \). If \( |f'(x)| \leq \theta < 1 \), \( g(x), g'(x) \ll 1 \), then
\[ \sum_{a < n \leq b} g(n)e(f(n)) = \int_{a}^{b} g(x)e(f(x))dx + O \left( \frac{1}{1 - \theta} \right). \]

**Proof:** See lemma 4.8 in [1].

**Lemma 3.2** For primitive characters \( \chi_i \mod r_i \) \( i=1,2,3,4 \) and the principal character \( \chi_0 \mod q \) we have
\[ \sum_{\chi \cong \chi_i \mod r_i} \frac{|Z(q, \chi \chi_0, \chi_0 \chi_2, \chi_0 \chi_3, \chi_0 \chi_4)|}{\phi^4(q)} \ll r^{-1+\epsilon}(\log P)^{c}, \]
where \( r = [r_1, r_2, r_3, r_4] \).

**Proof:** This is lemma 3.3 in [2].

**Lemma 3.3**
\[ \sum_{q \leq x} |A(n, q)| \ll x^{-1+\epsilon}d(n). \]
Proof: The proof follows literally the proof of lemma of (4.12) in [6].

Lemma 3.4 For \( P \leq x^{13/80-\epsilon} \) there is

\[
\sum_{N \leq x} \prod_{p \leq P} s(p, N) - \sum_{q \leq P} A(q, N) \ll xP^{-1/3+\epsilon},
\]

which implies that

\[
\prod_{p \leq P} s(p, N) = \sum_{q \leq P} A(q, n) + O(x^{-\epsilon})
\]

for all but \( \ll x^{1+2\epsilon} P^{-1/3} \) even integers \( N \) with \( 1 \leq N \leq x \).

Proof: This theorem is stated in [3] for all \( P \leq x^{7/150-\epsilon} \). The proof shows however that it holds for \( P \leq x^{13/80-\epsilon} \) as well.

Splitting the summation over \( n \) in residue classes modulo \( q \), we obtain

\[
S_k \left( \frac{a}{q} + \lambda \right) = \frac{C_k(a, q)}{\phi(q)} T_k(\lambda) + \frac{1}{\phi(q)} \sum_{\chi \mod q} C_k(a, \chi) W_k(\lambda, \chi) + O(L^2).
\]

Thus we obtain from (2.1)

\[
R_1(N) = R_1^m(N) + R_1^r(N) + O(x^\mu L^{-A}) \quad \text{(for any } G > 0\text{),}
\]

where

\[
R_1^m(N) = \sum_{q \leq P} \frac{1}{\phi(q)} \sum_{1 \leq a \leq q} \ast \int_{-1/Qq}^{1/Qq} \prod_{k=2}^{5} C_k(a, q) T_k(\lambda) e\left( -\frac{a}{q} N \right) T_k(\lambda) e(-\lambda N) d\lambda,
\]

\[
R_1^r(N) = \sum_{k=2}^{5} \sum_{q \leq P} \frac{1}{\phi(q)} \sum_{1 \leq a \leq q} \ast \int_{-1/Qq}^{1/Qq} \prod_{l=2 \atop l \neq k}^{5} C_l(a, q) T_l(\lambda) \sum_{\chi \mod q} C_k(a, \chi) W_k(\lambda, \chi) e\left( -\frac{a}{q} N - \lambda N \right) d\lambda
\]

\[
+ \sum_{k, l \leq 2 \atop k \neq l} \frac{1}{\phi(q)} \sum_{1 \leq a \leq q} \ast \int_{-1/Qq}^{1/Qq} \prod_{m \in \{k, l\}} C_m(a, q) T_m(\lambda) \sum_{\chi \mod q} C_k(a, \chi) W_k(\lambda, \chi) e\left( -\frac{a}{q} N - \lambda N \right) d\lambda
\]

\[
+ \sum_{q \leq P} \frac{1}{\phi(q)} \sum_{1 \leq a \leq q} \ast \int_{-1/Qq}^{1/Qq} \prod_{k=2 \atop k \neq \ell}^{5} C_k(a, q) T_k(\lambda) \prod_{l=2 \atop l \neq k}^{5} C_l(a, q) W_l(\lambda, \chi) e\left( -\frac{a}{q} N - \lambda N \right) d\lambda
\]

\[
+ \sum_{q \leq P} \frac{1}{\phi(q)} \sum_{1 \leq a \leq q} \ast \int_{-1/Qq}^{1/Qq} \prod_{k=2}^{5} C_k(a, q) W_k(\lambda, \chi) e\left( -\frac{a}{q} N - \lambda N \right) d\lambda,
\]

\[= S_1 + S_2 + S_3 + S_4. \]
We first calculate \( R_n^m(N) \). Applying lemma 3.1 yields

\[
T_k(\lambda) = \int_{\frac{1}{x} \times 2^{k+1}} e(\lambda u^k)du + O(1) = \frac{1}{k} \int_{x/2^{k+1}}^x v^{k-1}e(\lambda v)dv + O(1)
\]

\[
= \frac{1}{k} \sum_{x/2^{k+1} < m \leq x} e(\lambda m) + O(1).
\]

Substituting this in \( R_n^m(N) \) we see

\[
R_n^m(N) = \frac{1}{120} \sum_{q \leq P} A(q) \int_{-1/4q}^{1/4q} \prod_{k=2}^{5} \left( \sum_{x/2^{k+1} < m \leq x} e(\lambda m) \right) e(-N\lambda)d\lambda
\]

\[
+ O \left( \max_{2 \leq l \leq 5} \sum_{q \leq P} A(q) \int_{1/4q}^{-1/4q} \prod_{k \neq l} \sum_{x/2^{k+1} < m \leq x} e(\lambda m) d\lambda \right).
\]

Using lemma 3.3 and the trivial bound

\[
\sum_{x/2^{k+1} < m \leq x} e(\lambda m) \ll \min \left( \frac{\sqrt{x}}{x^{3/4} |\lambda|}, \frac{1}{x^{1/2} |\lambda|} \right), \tag{3.2}
\]

we derive using lemma 3.4,

\[
R_n^m(N) = \frac{1}{120} \sum_{q \leq P} A(q) \int_{1/2}^{-1/2} \prod_{k=2}^{5} \sum_{x/2^{k+1} < m \leq x} e(\lambda m) e(-N\lambda)d\lambda
\]

\[
+ O \left( \sum_{q \leq P} |A(q)| \int_{1/4q}^{1/4q} \frac{1}{x^{3-\mu} |\lambda|^4} d\lambda \right) + O(x^\mu L^{-A})
\]

\[
= \frac{1}{120} P_0 \sum_{q \leq P} A(q) + O((PQ)^3 x^{\mu-3} L^r) + O(x^\mu L^{-A})
\]

\[
= \frac{1}{120} P_0 \prod_{p \geq 1} s(p) + O(x^\mu L^{-A}), \tag{3.3}
\]

for all but \( x^{1 + 2\epsilon} P^{-1/3} \) integers \( N \leq x \), where \( P_0 \) is defined as in (2.4) and \( E \) is chosen sufficiently large in \( Q = NP^{-1} L^{-E} \). In the sequel \( E = E(G) \) is fixed. Now we estimate the terms \( S_i, i = 1, 2, 3, 4 \). Using lemma 3.3 we can estimate \( S_4 \) in the following way:

\[
|S_4|
\]

\[
= \left| \sum_{q \leq P} \frac{1}{\varphi(q)} \sum_{\chi_2 \text{mod} q} \sum_{\chi_3 \text{mod} q} \sum_{\chi_4 \text{mod} q} \sum_{\chi_5 \text{mod} q} \sum_{\chi_6 \text{mod} q} Z(q, \chi_2, \chi_3, \chi_4, \chi_5) \int_{-1/4q}^{1/4q} \prod_{k=2}^{5} W_k(\lambda, \chi_j) e(-n\lambda)d\lambda \right|
\]

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Arguing similarly we obtain

\[
S_4 \ll \sum_{r_2 \leq r_3 \leq r_4 \leq r_5} r_2^{-\epsilon} \sum_{\chi \bmod r_k} \sum_{|\lambda| \leq 1/r_k} |W_2(\lambda, \chi)| \sum_{r_3 \leq P} r_3^{-1/13+2\epsilon} \sum_{\chi_3 \bmod r_3} \sum_{|\lambda| \leq 1/r_3} |W_3(\lambda, \chi)|
\]

\[
\times \sum_{r_4 \leq P} r_4^{-4/13+2\epsilon} \sum_{\chi_4 \bmod r_4} \left( \int_{-1/Qr_4}^{1/Qr_4} |W_4(\lambda, \chi_4)|^2 d\lambda \right)^{1/2}
\]

\[
\times \sum_{r_5 \leq P} r_5^{-8/13+2\epsilon} \sum_{\chi_5 \bmod r_5} \left( \int_{-1/Qr_5}^{1/Qr_5} |W_5(\lambda, \chi_5)|^2 d\lambda \right)^{1/2}
\]

\[
\ll L_c I_2 I_3 W_4 W_5,
\]

where

\[
I_k = \sum_{r \leq P} r^{-a_k} \sum_{\chi} \max_{|\lambda| \leq 1/rQ} |W_k(\lambda, \chi)|
\]

\[
W_k = \sum_{r \leq P} r^{-a_k} \sum_{\chi} \left( \int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi)|^2 d\lambda \right)^{1/2}
\]

\[
a_k = \begin{cases} 
\epsilon, & \text{for } k = 2, \\
\frac{1}{r^2} - 2\epsilon, & \text{for } k = 3, \\
\frac{1}{r^3} - 2\epsilon, & \text{for } k = 4, \\
\frac{1}{r^4} - 2\epsilon, & \text{for } k = 5.
\end{cases}
\]

Arguing similarly we obtain

\[
S_1 + S_2 + S_3 \ll L_c \max_{2 \leq k, j, m, n \leq 5} \max_{|\lambda| \leq 1/Q} |T_k(\lambda)| \max_{|\lambda| \leq 1/Q} |T_j(\lambda)| \left( \int_{-1/Q}^{1/Q} |T_m(\lambda)|^2 d\lambda \right)^{1/2} W_n
\]
\[ L^c \max_{2 \leq k, l, m, n \leq 5} \max_{|\lambda| \leq 1/Q} |T_k(\lambda)| \max_{|\lambda| \leq 1/Q} |T_l(\lambda)| W_m W_n \]
\[ + L^c \max_{2 \leq k, l, m, n \leq 5} \max_{|\lambda| \leq 1/Q} |T_k(\lambda)| I_l W_m W_n, \tag{3.5} \]

We have trivially
\[ \max_{|\lambda| \leq 1/Q} |T_k(\lambda)| \ll x^{1/k}. \]

Using (3.2) we obtain
\[ \left( \int_{-1/Q}^{1/Q} |T(\lambda)|^2 d\lambda \right)^{1/2} \ll x^{\frac{1}{k} - \frac{1}{4}}. \]

Thus we see from (3.1) and (3.3) - (3.5) that the proof of (2.3) reduces to the proof of the following two lemmas:

**Lemma 3.5** If \( P \leq x^{180^{-\epsilon}} \) and \( 2 \leq k \leq 5 \)
\[ W_k \ll B x^{1/k - 1/2} L^{-B}, \]
for any \( B > 0 \).

**Lemma 3.6** If \( P \leq x^{180^{-\epsilon}} \) and \( 2 \leq k \leq 5 \)
\[ I_k \ll x^{1/k} L^A \]
for a certain \( A > 0 \).

### 4 Proof of lemma 3.5

In order to prove the lemma it is enough to show that
\[ W_{k,R} \ll x^{\frac{1}{k} - \frac{1}{4}} R^{\alpha + 1} L^{-B}, \tag{4.1} \]
where
\[ W_{k,R} = \sum_{r=R}^{Q} \sum_{\chi}^* \left( \int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi)|^2 d\lambda \right)^{1/2} \]
for \( R \leq P/2 \). Applying lemma 1, \[ \text{[3]} \] we see
\[ \int_{-1/Qr}^{1/Qr} |W_k(\lambda, \chi)|^2 d\lambda \ll (QR)^{-2} \int_{x/2^{k+2}}^{x} \left| \sum_{t \leq m \leq x/2^{k+1}} \Lambda(m) \chi(m) - \sum_{t \leq m \leq x/2^{k+1}} \chi(m) \right|^2 dt. \]  

We set \( X = \max(x/2^{k+1}, t) \) and \( X + Y = \min(x, t + Qr) \). In the sequel we will treat the cases \( R > L^D \) and \( R \leq L^D \) for a sufficiently large constant \( D > 0 \) separately. In the first case we apply a slight modification of Heath-Brown’s identity \([3]\):

\[
- \frac{\zeta'}{\zeta}(s) = \sum_{j=1}^{K} \left( \frac{K}{j} \right) (-1)^{j-1} \zeta'(s) \zeta^{j-1}(s) M^j(s) - \frac{\zeta'}{\zeta}(s)(1 - \zeta(s)M(s))^K,
\]

with \( K = 5 \) and

\[
M(s) = \sum_{n \leq x^{1/5k}} \mu(n)
\]
to the sum

\[
\sum_{X < m \leq X+Y}.
\]

Arguing exactly as in part III, \([1]\) we find by applying Heath Brown’s identity and Perron’s summation formula (see \([9]\), Lemma 3.12) that the inner sum of (4.3) - where always \( E_0 = 0 \) because of \( R > L^D \) and the primitivity of the characters - is a linear combination of \( O(L^c) \) terms of the form

\[
S_{k,l_{a_1},\ldots,l_{a_{10}}} = \frac{1}{2\pi i} \int_{-T}^{T} F_k(\frac{1}{2} + it, \chi) \frac{(X + Y)^{\frac{1}{2} + it} - X^{\frac{1}{2} + it}}{\frac{1}{2} + it} \, du + O(T^{-1}x^{1/2 + \epsilon}),
\]

where \( 2 \leq T \leq x \),

\[
F_k(s, \chi) = \prod_{j=1}^{10} f_{k,j}(s, \chi), \quad f_{k,j}(s, \chi) = \sum_{n \in I_{k,j}} a_{k,j}(n) \chi n^{-s},
\]

\[
a_{k,j}(n) = \begin{cases} 
\log n \text{ or } 1, & j = 1, \\
1, & 1 < j \leq 5, \\
\mu(n), & 6 \leq j \leq 10, 
\end{cases} \quad I_j = (N_{k,j}, 2N_{k,j}], \quad 1 \leq j \leq 10,
\]

\[
\sqrt{x} \ll \prod_{j=1}^{10} N_{k,j} \ll \sqrt{x}, \quad N_{k,j} \leq x^{1/5k}, \quad 6 \leq j \leq 10.
\] (4.3)

Since

\[
\frac{(X + Y)^{\frac{1}{2} + it} - X^{\frac{1}{2} + it}}{\frac{1}{2} + it} \ll \min \left( QRx^{\frac{1}{2k} - 1}, x^{\frac{1}{2}}(|u| + 1)^{-1} \right)
\]

by taking \( T = x^{2\epsilon} P^2(1 + |\lambda|x) \) and \( T_0 = x(QR)^{-1} \), we conclude that \( S_{l_{a_1},\ldots,l_{a_{10}}} \) is bounded by

\[
\ll QRx^{\frac{1}{2k} - 1} \int_{-T_0}^{T_0} \left| F_k\left(\frac{1}{2} + it, \chi\right) \right| \, du + x^{\frac{1}{2}} \int_{T_0 \leq |u| \leq T} \left| F_k\left(\frac{1}{2} + it, \chi\right) \right| \, du |
\]

\[
+ x^{\frac{1}{2} + \epsilon} P^{-2},
\]

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Thus we derive from (4.2) that in order to prove (4.1) it is enough to show that
\[
\sum_{r \sim R} \sum_{\chi} \int_{0}^{T_0} |F_k \left( \frac{1}{2} + it, \chi \right)| \, dt \ll x^{1/2k} R^{a_k - \epsilon} L^{-B},
\]
(4.4)
\[
\sum_{r \sim R} \sum_{\chi} \int_{T_1}^{2T_1} |F_k \left( \frac{1}{2} + it, \chi \right)| \, dt \ll x^{1/2k - 1} Q R^{1 + a_k - \epsilon} T_1 L^{-B}, \quad T_0 < |T_1| \leq T.
\]
(4.5)

For the proof of (4.4) and (4.5) we will prove two propositions. We will need the estimate
\[
\sum_{n \leq x} d_k(n) \ll_k x L^{c(k)}.
\]
(4.6)

We now establish

**Proposition 1** If there exists \( N_{k,j_1} \) and \( N_{k,j_2} \) \((1 \leq j_1, j_2 \leq 5)\) such that \( N_{k,j_1} N_{k,j_2} \geq p^{2 - 2a_k + 3\epsilon} \) then (4.4) is true.

*Proof:* We suppose without loss of generality \( j_1 = 1 \), \( a_1(n) = \log n \) and \( j_2 = 2 \), \( a_2(n) = 1 \). Arguing exactly as in the proof of proposition 1 in [11], we find
\[
f_{k,1} \left( \frac{1}{2} + it, \chi \right) \ll L \left( \int_{x^{1/k}}^{x} \left| L' \left( \frac{1}{2} + iv, \chi \right) \right| \frac{dv}{1 + |v|} \right)^{1/4} + L,
\]
and so we find by using lemma 3.7:
\[
\sum_{r \sim R} \sum_{\chi} \int_{0}^{T_0} \left| f_{1} \left( \frac{1}{2} + it, \chi \right) \right|^4 \, dt \ll L^4 \int_{-x^{1/k}}^{x^{1/k}} \frac{dv}{1 + |v|} \sum_{r \sim R} \sum_{\chi} \int_{v}^{T_0 + v} \left| L' \left( \frac{1}{2} + it, \chi \right) \right|^4 \, dt + T_0 R^2 L^4
\]
\[
\ll L^5 \max_{|N| \leq x^{1/k}} \int_{N/2}^{N} \frac{dv}{1 + |v|} \sum_{r \sim R} \sum_{\chi} \int_{v}^{T_0 + v} \left| L' \left( \frac{1}{2} + it, \chi \right) \right|^4 \, dt + T_0 R^2 L^4
\]
\[
+ L^5 \max_{|N| \leq x^{1/k}} \int_{0}^{T_0} dt \sum_{r \sim R} \sum_{\chi} \sum_{N \leq x^{1/k}} \int_{\frac{N}{2} + t}^{N + t} \left| L' \left( \frac{1}{2} + iv, \chi \right) \right|^4 \, dv + T_0 R^2 L^4
\]
\[
\ll R^2 T_0 L^c,
\]
Using lemma 3.8, (4.6) and Hölder’s inequality we obtain
\[
\sum_{r \sim R} \sum_{\chi} \int_{0}^{T_0} \left| F_k \left( \frac{1}{2} + it, \chi \right) \right| \, dt
\]
Proposition by the definition of $T$ the case $R > L$.

Proof: Let

$$F_{k,i}(s, \chi) = \prod_{j \in J_i} f_{k,j}(s, \chi) = \sum_{n \leq N_i} b_i(n) \chi(n)n^{-s}, b_i(n) \ll d^e(n), \; i = 1, 2,$$

where $M_i = \prod_{j \in J_i} N_{k,j}, \; i = 1, 2$. Applying lemma 3.8, (4.3) and (4.6) we see

$$\sum_{r \sim R} \sum_{\chi}^* \int_0^{T_0} \left| F_k \left( \frac{1}{2} + it, \chi \right) \right| dt \ll \left( \sum_{r \sim R} \sum_{\chi}^* \int_0^{T_0} \left| F_{k,1} \left( \frac{1}{2} + it, \chi \right) \right| dt \right)^{1/2} \left( \sum_{r \sim R} \sum_{\chi}^* \int_0^{T_0} \left| F_{k,2} \left( \frac{1}{2} + it, \chi \right) \right| dt \right)^{1/2} \ll \left( R^2 T_0 + M_1 \right)^{1/2} \left( R^2 T_0 + M_2 \right)^{1/2} \ll R^2 T_0 + x^{\frac{e}{2} P^{-1+a_k - \frac{4e}{5}}} T_0^{1/2} + x^{1/2k} L^e.$$

This proves the proposition because of $R > L^D$. Using proposition 1 and 2, we can prove (4.4) in nearly the same way as (4.4) is proved in [2]. The only difference in the proof is that instead of assuming

$$N_{k,i} N_{k,j} \leq P^{12/7+3e} \leq x^{2/5k}, \; 1 \leq i, j \leq 5, \; i \neq j$$
as in [2], we assume in view of proposition 1 that

$$N_{k,i} N_{k,j} \leq P^{2-2a_k + 3e} \leq x^{2/5k}, \; 1 \leq i, j \leq 5, \; i \neq j.$$

The proof of (4.3) goes along the same lines. (4.1) is now proved in the case $R > L^D$. The case $R \leq L^D$ is treated exactly as in [2].
5 Proof of lemma 3.6

To prove the lemma it is enough to show that

\[
\max_{R \leq P/2} \sum_{r \sim R} \sum_{\chi} \max_{|\lambda| \leq 1/rQ} |W_k(\lambda, \chi_r)| \ll x^{1/k} R^{a_k} L^A.
\]

Arguing as in the section before - we do not have to apply Gallagher’s lemma here - we find

\[
W_k(\lambda, \chi) \ll L^c \max_{I_{a_1}, \ldots, I_{a_{2k+1}}} \left| \int_{-T}^{T} F_1(\frac{1}{2} + it, \chi) dt \int_{x/2^{k+1}}^{x} u^\frac{1}{2} e\left(\frac{t}{2k\pi} \log u + \lambda u\right) du \right| + x^{1/k} P^{-1},
\]

for \( T = P^3 \). Estimating the inner integral by lemma 3.2 we obtain

\[
\int_{x/2^{k+1}}^{x} u^\frac{1}{2} e\left(\frac{t}{2k\pi} \log u + \lambda u\right) du \ll \frac{x}{\sqrt{|t| + 1}} \min_{x/2^{k+1} < u \leq x} \frac{x}{|t + 2k\pi \lambda u|}.
\]

Taking \( T_0 = 4k\pi x(rQ)^{-1} \) we conclude that in order to prove the lemma it is enough to prove that for \( P \leq x^{15/14} - \epsilon \) and \( 2 \leq k \leq 5 \) there holds

\[
\sum_{r \sim R} \sum_{\chi} \int_{0}^{T_0} \left| F_k(\frac{1}{2} + it, \chi) \right| dt \ll x^{1/2k} R^{a_k} L^c, \tag{5.1}
\]

\[
\sum_{r \sim R} \sum_{\chi} \int_{T_1}^{2T_1} \left| F_k(\frac{1}{2} + it, \chi) \right| dt \ll x^{1/2k} R^{a_k} T_1 L^c, \quad T_0 < |T_1| \leq T. \tag{5.2}
\]

These estimates are shown in the same way as (4.4) and (4.5). Two propositions analogous to the propositions 1 and 2 are proved:

**Proposition 3** If there exist \( N_{k,j_1} \) and \( N_{k,j_2} \) (\( 1 \leq j_1, j_2 \leq 5 \)) such that \( N_{k,j_1} N_{k,j_2} \geq P^{2-2a_k+3\epsilon} \) then (5.1) is true.

**Proposition 4** Let \( J = \{1, \ldots, 10\} \). If \( J \) can be divided into two non overlapping subsets \( J_1 \) and \( J_2 \) such that

\[
\max \left( \prod_{j \in J_1} N_{k,j}, \prod_{j \in J_2} N_{k,j} \right) \ll x^k P^{-2+2a_k-3\epsilon}
\]

then (5.1) is true.

Remark: Here we do not need to treat the case \( R > L^D \) separately because we do not have to save a factor \( L^{-B} \).

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