A New Approach for Analytic Amplitude Calculations

Cong-Feng Qiao

Department of Physics, Graduate School of Chinese Academy of Sciences,
YuQuan Road 19A, 100039 Beijing, China
and
The Abdus Salam International Centre for Theoretical Physics (ICTP),
Strada Costiera, 11-34014 Trieste, Italy

Abstract

We present a method for symbolic calculation of Feynman amplitudes for processes involving both massless and massive fermions. With this approach fermion strings in a specific amplitude can be easily evaluated and expressed as basic Lorentz scalars. The new approach renders the symbolic calculation of some complicated physical processes more feasible and easier, especially with the assistance of algebra manipulating code for computer.

PACS number(s) : 13.85.Qk, 14.70.Dj, 14.70.Hp
1 Introduction

In high energy collisions, generally, multi-particle final states are produced. To calculate a practical physical process with several fermions in the final state, conventionally, is a tedious work to implement, even at the tree level. One needs to square the Feynman amplitude, sum over fermion polarizations, and to take trace for each possible fermion string loops. The non-Abelian nature of the Standard Model makes the number of independent Feynman diagrams grow very rapidly as the external particles increasing. Fortunately, due to the recent development in computer technology, a lot of time consuming and tedious Dirac algebra and symbolic simplification work can be carried out on an ordinary personal computer. Nevertheless, even with the up-to-date algebra manipulating programs for computer, many a multi-particle process tend to be still cumbersome to calculate. On the other hand, another drawback of conventional amplitude squaring technique is that it loses the information on spin correlations, which is more and more attainable in present experimental measurements.

An alternative technique, the so-called helicity method, has been developed pretty applicable in practice in recent years. With this method, the independent fermion strings are contracted and reduced to analytic expressions of four-vector products, such that the matrix-element squaring becomes a trivial work. The development of helicity method has experienced a bit long history. Some early efforts on this can be seen in Refs. [1, 2, 3]. In recent development, a bunch of approaches for calculating the helicity amplitudes are proposed in Refs. [4-14], and it should be noted the references given here are far from complete on this subject.

In practice, most of physical processes involve fermion production and decays, which give a number of independent fermion lines within a Feynman amplitude. The generic form of a fermion line involving a pair of fermions can be expressed as

\[ \bar{u}(p_1, \lambda_1) \ G_1 \ G_2 \cdots \ G_n u(p_2, \lambda_2), \quad (1) \]

where \((p_1, \lambda_1)\) and \((p_2, \lambda_2)\) stand for the momenta and polarizations of the external fermions; and in between of the two spinors there is a string of Dirac matrices. Since real physical processes are described by the matrix element square, in conventional method of calculating an interaction process, one squares the amplitude, which contains more or less \(\mathbf{1}\)-like fermion lines when there are external fermions involved, and sums over the fermion polarizations to get fermion loop(s). Using the nature of spinor projection operators, one then is confronted with couple of traces. As aforementioned, with the growth of the number of external fermions one need to evaluate a vast number of fermion loop traces. Even with the help of computer algebra manipulation, quite often one finds it is still a cumbersome work to perform the calculation.

Helicity method enables one to evaluate physical amplitude directly, resulting in a limited number of four-vector scalars. In literatures, there are several kinds of approaches in evaluating the fermion strings analytically. One of them is carried out by choosing a specific representation for the Dirac matrices \([9]\), and then get the explicit forms of all the related matrices. Another kind of popular helicity method \([10]\) \([11]\) is performed by introducing extra lightlike auxiliary momenta and expressing the massive particle spinors by the corresponding massless ones.
In this work, we use the maximum credits of Refs. [10, 11] to generalize a straightforward method for the evaluation of fermion strings. We believe this approach, which leads to a covariant form for the helicity amplitude, is a relatively simple and fast way in practical computations.

2 Formalism

In the conventional method one computes unpolarized cross sections by squaring the eq. (1)-like amplitude while summing over fermion polarizations, and obtains

$$\text{tr}[\not{p}_1 \pm m_1 \mathcal{G}_1 \mathcal{G}_2 \cdots \mathcal{G}_n(\not{p}_2 \pm m_2) \mathcal{G}_n \cdots \mathcal{G}_2 \mathcal{G}_1],$$

(2)

where $m_1$ and $m_2$ are the masses of $p_1$ and $p_2$, and the plus and minus signs before them correspond to particle and anti-particle, respectively. Taking the above trace in principle is not a tedious work, especially with the help of computer. However, with the number of external fermions increasing, in conventional amplitude modulus square approach one encounters a prohibitive number of eq.(2)-like traces to evaluate, which makes the physical computations very time consuming, even for the machine.

Noticing that the fermion string (1) can be rewritten as a trace form, like

$$\text{tr}[\mathcal{G}_1 \mathcal{G}_2 \cdots \mathcal{G}_n u(p_2, \lambda_2) \otimes \bar{u}(p_1, \lambda_1)],$$

(3)

therefore, now, the question of how to evaluate the helicity amplitude becomes how to re-express $u(p_2, \lambda_2) \otimes \bar{u}(p_1, \lambda_1)$ by basic Dirac matrices. In Ref. [13], Vega and Wudka obtained one kind of expression for it by making use of the Bouchiat-Michel identity [15]. While in principle their result is applicable for practical calculation, we still feel it not so convenient for programming, since the main result of (9) in Ref. [13] is not expressed in a convenient and covariant way for different helicities.

The helicity method we will present is a straightforward generalization of the results of Refs. [10, 11]. By choosing two auxiliary four-vectors, $k_0$ and $k_1$, one can fix up the basic spinors $u(k_0, \lambda)$ by the equations:

$$u(k_0, \lambda)\bar{u}(k_0, \lambda) = 1 + \lambda \gamma_5 \frac{k_0}{2},$$

(4)

and

$$u(k_0, \lambda) = \lambda \not{k}_1 u(k_0, -\lambda).$$

(5)

Here,

$$k_0 \cdot k_0 = 0, \; k_1 \cdot k_1 = -1, \; k_0 \cdot k_1 = 0.$$

(6)

With the above defined basic spinors, the wavefunction of a massive fermion (particle or anti-particle) can be expressed as

$$u(p, \lambda) = \frac{\not{p} \pm m}{\sqrt{2p \cdot k_0}} u(k_0, -\lambda).$$

(7)
It is easy to check that these spinors, and their conjugations defined by the normal way, satisfy Dirac equations; they are also the eigenstates of $\gamma_5 \not{p}$ with eigenvalues of $\pm 1$. Here, $n$ is the fermion polarization vector. Using Eqs. (4), (5), and (7), one can readily get the desired spinor products in (3), like

$$u(p_2, \lambda_2) \otimes \bar{u}(p_1, \lambda_1) = \frac{p_2 \pm m_2}{2\sqrt{p_2 \cdot k_0}} u(k_0, -\lambda_2) \bar{u}(k_0, -\lambda_1) \frac{p_1 \pm m_1}{2\sqrt{p_1 \cdot k_0}}$$

$$= \frac{p_2 \pm m_2}{2\sqrt{p_2 \cdot k_0}} \left\{ ((1 + \lambda_1 \lambda_2) - (\lambda_1 + \lambda_2)\gamma_5 \right\} k_0 \frac{p_1 \pm m_1}{2\sqrt{p_1 \cdot k_0}}.$$  (8)

In fact (8) was obtained in [11], but it was not realized that (8) can be directly applied to the calculation of helicity amplitude, rather, there it was used for evaluating the segments of a splitted fermion string. This finding makes the computation of practical processes much simplified from the method of Ref. [11].

3 Applications and Concluding Remarks

In arriving at eq. (8) we have taken no special provisos on the fermion mass, hence, it will remain valid for massless case. From (8) we can easily obtain the known identities of the inner product of two massless spinors [16], like

$$\bar{u}(p, -)u(p', +) = [\bar{u}(p', +)u(p, -)]^*, \quad (9)$$

$$\bar{u}(p, +)u(p', -) = -\bar{u}(p', +)u(p, -), \quad (10)$$

and

$$|\bar{u}(p, +)u(p', -)|^2 = 2 \cdot p \cdot p'.$$  (11)

When taking a specific choice for auxiliary vector $k_0$, that is

$$k_0^0 = \frac{\alpha}{m^2} (p^0 - |p|), \quad k_0 = \frac{\alpha}{m^2} (|p| - p^0) \hat{p},$$

(12)

where $\alpha$ is an arbitrary parameter and $\hat{p}$ the unit vector of space components of momentum $p$, the polarization vector now becomes the conventional helicity vector:

$$n = (\frac{P}{m}, \frac{p_0}{m} \hat{p}).$$  (13)

Then, from (8) one easily finds that at helicity basis, the spinors $u(p, \lambda)$ satisfies the usual projection relation

$$u(p, \lambda)\bar{u}(p, \lambda) = \frac{(1 + \lambda \gamma_5 \not{p})}{2}(\not{p} \pm m),$$  (14)
As the last example, we compute the helicity amplitude for fermion line with a single vector current insertion. That is

\[ A_\mu(p, \lambda; p', \lambda') = \bar{u}(p, \lambda) \gamma_\mu u(p', \lambda') . \] (15)

This kind of amplitude appears quite often in fermion production and decay processes, e.g., \( e^+ e^- \rightarrow f \bar{f} \). In the center-of-mass (CM) system, with choosing

\[ p = (p_0, 0, 0, p_3) , \quad p' = (p_0, 0, 0, -p_3) , \quad k_0 = (1, 0, 0, -1) , \quad k_1 = (0, 1, 0, 0) \] (16)

and using (8) the helicity amplitude \( A_\mu \) can be be evaluated as

\[ A_\mu(p, \lambda; p', \lambda') = \frac{1}{2m} \left\{ \frac{(1 + \lambda' \lambda)}{2} \left[ (s \delta_{\mu 0} - (s - 4m^2) \delta_{\mu 3} - sk_0) \delta_{\mu 0} \right] + (\lambda - \lambda')m \sqrt{s - 4m^2} \delta_{\mu 1} - I(1 - \lambda \lambda') m \sqrt{s} \delta_{\mu 2} \right\} . \] (17)

Here, \( m \) is the fermion mass, \( s \) is the CM energy square. From (17) we have

\[ A_\mu(p, \lambda; p', \lambda') = 2m \delta_{\mu 3} \quad \text{for} \quad \lambda = \lambda' , \] (18)

\[ A_\mu(p, \lambda; p', \lambda') = -\lambda \sqrt{s - 4m^2} \delta_{\mu 1} + I \sqrt{s} \delta_{\mu 2} \quad \text{for} \quad \lambda = -\lambda' , \] (19)

which are exactly the same as given in eq. (23) of Ref. [13] after taking a specific momentum direction there.

In summary, we provide a new approach for evaluating Feynman diagrams symbolically, which is a straightforward and efficient promotion of the methods developed in Refs.[10, 11]. We have presented examples in displaying the simpleness of this approach, and the new method is applicable to both massive and massless fermion amplitude calculations. Especially, since there is no restriction on the working frame, or in other words the fermion momentum directions, by this approach, it is quite convenient to encode the helicity amplitudes into a symbolic manipulating program for computer. Another advantage of the method is that the expression (8) is Lorentz index covariant, although one needs to specify the auxiliary vectors in the end for one’s convenience, which makes the intermediate steps of calculation not so tedious. In the end, we would like to mention that most phenomenological applications of massive fermion amplitudes can be improved by also including the decay of the fermions. It is often possible to factorize the problem into production and decay amplitudes, and the new technique could be used for both in principle. However, to do this much care should be paid to the helicity state conventions.

**Acknowledgments**

The author would like to thank the ICTP for hospitality, while part of this work was carried out.
References

[1] J.L. Powell, Phys. Rev. 75, 32 (1949).

[2] M. Jacob and G. Wick, Ann. Phys. (N.Y.) 7, 404 (1959).

[3] J. Bjorken and M. Chen, Phys. Rev. 154, 1335 (1966).

[4] P. De Causmaecker, R. Gastmans, W. Troosts and T.T. Wu, Phys. Lett. B105, 215 (1981); Nucl. Phys. B206, 53 (1982).

[5] J.F. Gunion and Z. Kunszt, Phys. Lett. B161 333 (1985).

[6] G. Farar and F. Neri, Phys. Lett. B130 109 (1983).

[7] M. Caffo and E. Remiddi, Helv. Phys. Acta. 55 (1982) 339.

[8] G. Passarino, Phys. Rev. D28 (1983) 2867, Nucl. Phys. B237 (1984) 249.

[9] K. Hagiwara and D. Zeppenfeld, Nucl. Phys. B274 (1986) 1.

[10] F.A. Berends, P.H. Daverveldt and R. Kleiss Nucl. Phys. B253 (1985) 441, R. Kleiss and W.J. Stirling, Nucl. Phys. B262 (1985) 235.

[11] A. Ballestrero and E. Maina Phys. Lett. B350, 225 (1995).

[12] Z. Xu, Da-Hua Zhang and L. Chang Nucl. Phys. B291 (1987) 392.

[13] R. Vega and J. Wudka, Phys. Rev. D53, 5286 (1996).

[14] C. Mana and M. Martinez, Nucl. Phys. B287 (1987) 601.

[15] C. Bouchiat and L. Michel, Nucl. Phys. 5 (1958) 416.

[16] M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley, USA, 1995).