Further evidence of the absence of Replica Symmetry Breaking in Random Bond Potts Models

MARC-ANDRÉ LEWIS(*)

Laboratoire de Physique Théorique et des Hautes Energies,
Universités Pierre et Marie Curie (Paris VI) et Denis Diderot (Paris VII),
Boîte 126, Tour 16, 1er étage,
4 pl. Jussieu, 75251 Paris CEDEX 05, FRANCE

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Abstract. – In this short note, we present supporting evidence for the replica symmetric approach to the random bond $q$-state Potts models. The evidence is statistically strong enough to reject the applicability of the Parisi replica symmetry breaking scheme to this class of models. The test we use is a generalization of one formerly proposed by Dotsenko et al. [1] and consists in measuring scaling laws of disordered-averaged moments of the spin-spin correlation functions. Numerical results, obtained via Monte Carlo simulations for several values of $q$, are shown to be in fair agreement with the replica symmetric values computed by using perturbative CFT [1, 2] for the second and third moments of the $q = 3$ model. RSB effects, which should increase in strength with moment, are unobserved.

Since its first application to the study of glassy systems, the replica approach has been a useful tool in the disordered models research field. However, as was rapidly observed, its straightforward application (that is, assuming all replicas are identical) to physical systems can lead to serious problems, the most notorious being the negative entropy it gives for the spin glass model. These non-physical conclusions can be avoided by breaking the replica symmetry. The way in which the symmetry has to be broken in the spin glass problem was first explained by Parisi [3]. However, there are systems in which the symmetry is not broken and where the replica symmetric (RS) approach is valid. There is no straightforward arguments to decide whether a system studied using the replica method exhibits replica symmetry breaking (RSB) or not.

The aim of this short note is to implement a test that can reveal the presence of RSB in disordered local-interaction spin systems. We shall consider disordered Potts models, where disorder is introduced via randomness in bond strengths. This problem was studied perturbatively using the replica technique [4, 5]. There, critical exponents were computed assuming RS and the obtained values were shown to be in agreement with Monte Carlo data.
However, a possible RSB would not affect those quantities significantly and the apparent agreement couldn’t rule out its existence.

Recently, Dotsenko et al. [1] proposed a test of RSB for random bond Potts model. There, they showed that eventual RSB effects could be observed if one considered the disorder-averaged moments of spin-spin correlation functions. They studied the second moment and agreement was found with RS solutions, thus rejecting RSB-induced deviations that would have been greater than the obtained statistical resolution.

In a recent letter, we exposed perturbative CFT computations of the $p$-th moment in the replica symmetric case. This naturally proposed the search for signs of RSB in these correlators. Although we have not computed explicitly the deviations (this would involve a tensorial formulation of the Parisi scheme), one can easily convince himself that they should become more and more important as $p$ increases. Performing Monte-Carlo simulations, we computed the third moment for 3, 4 and 8-state Potts Model. For all these models, we observe scaling laws, thus showing that there is no RSB. The associated critical exponents are shown to be in fair agreement, for the 3-states model, with CFT predictions, although the perturbative expansion is not expected to be very precise for $q = 3$. This is the first validation of our formula, previously exposed [2], which goes a order further in perturbation than the one originally given by Ludwig [4].

**Perturbative CFT results.** – We shall not repeat here the renormalization group computations of higher moments, which can be found, although not in details, in references [1, 2]. We rather give a short overview, only stating relevant results.

The partition function of the nearly-critical $q$-states random bond Potts model, is well known to be of the form

$$Z(\beta) = \text{Tr} \exp\{-H_0 - H_1\},$$

where $H_0$ is the Hamiltonian of the conformal field theory corresponding to the $q$-states Potts model with coupling constant $J_0$ the same for each bond. The Hamiltonian $H_1$, being the deviation from the critical point induced by disorder is of the form

$$H_1 = \int d^2 x \tau(x)\epsilon(x),$$

where $\tau(x) \sim \beta J(x) - \beta_c J_0$ is the random temperature parameter. The theory is defined on the whole plane. We shall assume, for simplicity, that $\tau(x)$ has a gaussian distribution for each $x$, with

$$\tau(x) = \tau_0 = \frac{\beta - \beta_c}{\beta_c}$$

$$\frac{\tau(x) - \tau_0}{\tau(x') - \tau_0} = g_0 \delta(2)(x - x').$$

The usual way of averaging over disorder is to introduce replicas, that is, $n$ identical copies of the same model, for which:

$$(Z(\beta))^n = \text{Tr} \exp\left\{-\sum_{a=1}^n H_0^{(a)} - \int d^2 x \tau(x) \sum_{a=1}^n \epsilon_a(x)\right\}.$$  

Taking the average over disorder by performing gaussian integration, one gets

$$\overline{(Z(\beta))^n} = \text{Tr} \exp\left\{-\sum_{a=1}^n H_0^{(a)} - \tau_0 \int d^2 x \sum_{a=1}^n \epsilon_a(x) + g_0 \int d^2 x \sum_{a \neq b}^n \epsilon_a(x)\epsilon_b(x)\right\}.$$
This is a field theory of \( n \) coupled models with coupling action given by

\[
H_{\text{int}} = -g_0 \int d^2 x \sum_{a \neq b} \varepsilon_a(x)\varepsilon_b(x).
\] (7)

Only non-diagonal terms are kept since diagonal ones can be included in the Hamiltonian \( H_0 \). Moreover, they can be shown to have irrelevant contributions, since their OPE consist of the identity plus terms that are irrelevant at the pure fixed point. We now turn our attention to the \( p \)-th moment of the spin-spin correlation function \( \langle \sigma(0)\sigma(R) \rangle^p \). In terms of replicas, it can be written as

\[
\langle \sigma(0)\sigma(R) \rangle^p = \lim_{n \to 0} \frac{(n-p)!}{n! n^p!} \left( \sum_{a_1 \neq \cdots \neq a_p} \sigma_{a_1}(0) \cdots \sigma_{a_p}(0) \sum_{b_1 \neq \cdots \neq b_p} \sigma_{b_1}(R) \cdots \sigma_{b_p}(R) \right)
\] (8)

Thus, the operator to be renormalized is

\[
O_p(x) \equiv \sigma_{a_1}(x)\sigma_{a_2}(x) \cdots \sigma_{a_p}(x)
\] (9)

perturbed by the interaction term

\[
\tilde{O}_p(x) \equiv O_p \exp\{-H_{\text{int}}\} = O_p \left( 1 - H_{\text{int}} + \frac{1}{2}(H_{\text{int}})^2 - \cdots \right).
\] (10)

Renormalization group computations lead to the identification of a non-trivial fixed point, at which we are able to compute the correlation functions. Using scaling laws, we get

\[
\langle \sigma(0)\sigma(R) \rangle^p \sim \lim_{n \to 0} \frac{(n-p)!}{n! n^p!} \sum_{a_1 \neq \cdots \neq a_p} (Z(\xi_R))^2 \frac{1}{R^{2p\Delta_\sigma}}
\sim \frac{(Z(\xi_R))^2}{R^{2p\Delta'_\sigma}}.
\] (11)

The final result is obtained by using the fixed point value \( Z(\xi_R) \sim e^{\gamma_\ast \xi_R} = R^{\gamma_\ast} \). The RG study introduces a parameter \( \epsilon \), which can be seen as proportional to the central charge deviation of the pure model from the Ising value of 1/2. For the 3-state Potts model, \( \epsilon = 2/15 \). For generic \( \epsilon \), one gets (in [2], \( \alpha \) should be replaced by \(-\alpha\)):

\[
\langle \sigma(0)\sigma(R) \rangle^p \sim \frac{1}{ R^{2p\Delta'_\sigma} },
\] (12)

with

\[
\Delta'_\sigma = \Delta_\sigma - \gamma_\ast(p),
\] (13)

\[
\gamma_\ast(p) = \frac{9}{32}(p-1) \left( \frac{2}{3} \epsilon + \left( \frac{11}{12} \frac{2K}{3} + \frac{\alpha}{24}(p-2) \right) \epsilon^2 \right) + O(\epsilon^3),
\] (14)

and

\[
K = 6 \log 2 \quad \alpha = 33 - \frac{29\sqrt{3}\pi}{3}.
\] (15)
Thus, perturbed conformal field theory predicts, for the 3-state Potts models, the following values for the second and third moments:

\[ 2\Delta'_{\sigma^2} = \frac{4}{15} - 0.0314 = 0.235 \]  

(18)

\[ 2\Delta'_{\sigma^3} = \frac{4}{15} - 0.0466 = 0.220 \]  

(19)

Monte Carlo Simulations. – To search for signs of RSB, and, in the absence of it, to confirm RS values, we performed Monte Carlo simulations of the random bond \( q \)-Potts model for \( q = 2, 4, 8 \). The method used follows the one in [1]. To study scaling effects on the correlation functions, we studied square lattices of side \( L \) ranging from 10 to 500. Since we wanted to exhibit a possible break of the replica symmetry, the algorithm has to be chosen in such a way that it doesn’t assume the symmetry a priori. For this reason, we simulated three configurations of the \( q \)-Potts model with same disorder, but different initial conditions and independent thermalizations. We computed the products of magnetization

\[ Q_3 = \frac{1}{L^2} \sum_{i=1,L^2}^{} \langle \sigma^a_i \rangle \langle \sigma^b_i \rangle \langle \sigma^c_i \rangle, \]  

(20)

and

\[ Q_2 = \frac{1}{L^2} \sum_{i=1,L^2}^{} \langle \sigma^a_i \rangle \langle \sigma^b_i \rangle, \]  

(21)

with \( \langle \sigma^a_i \rangle \) being the thermal average of the local magnetization

\[ \sigma^a_i = \vec{\sigma}^a_i \cdot \vec{m}^a, \]  

(22)

where \( \vec{m}^a \) is the mean magnetization of lattice \( a \);

\[ \vec{m}^a = \frac{1}{L^2} \sum_{i=1,L^2}^{} \vec{\sigma}^a_i. \]  

(23)

It is rather obvious, since all lattices are thermalized independently, that \( Q_3 \) and \( Q_2 \) are indeed the same as

\[ Q_3 = \frac{1}{L^2} \sum_{i=1,L^2}^{} \langle \sigma^a_i \sigma^b_i \sigma^c_i \rangle \]  

(24)

\[ Q_2 = \frac{1}{L^2} \sum_{i=1,L^2}^{} \langle \sigma^a_i \sigma^b_i \rangle. \]  

(25)

Measurement were performed on square lattices with toroidal boundary conditions. The Hamiltonian of the simulated model is

\[ H = -\sum_{\{i,j\}} J_{ij} \left( \delta_{\sigma_i^a,\sigma_j^a} + \delta_{\sigma_i^b,\sigma_j^b} + \delta_{\sigma_i^c,\sigma_j^c} \right), \]  

(26)

where we took the coupling between nearest neighbours to be

\[ J_{ij} = J_0 \text{ or } J_1 \]  

(27)
with equal probabilities. This makes it possible to make the system self-dual by tuning the temperature so that the relation
\[
\frac{1 - e^{-\beta J_0}}{1 + (q - 1)e^{-\beta J_0}} = e^{-\beta J_1}
\]
(28)
is obeyed. We chose \(J_0/J_1 = 10\) for the simulations with \(q = 3, 4\), which is strong enough to avoid cross-over effects [3, 4]. For the \(q = 8\) model, we rather chose \(J_0/J_1 = 8.5\), again because this seems the appropriate value to avoid cross-over and minimize the spread of our data set.

Autocorrelation times were coarsely evaluated and the statistics adjusted in such a way that thermal fluctuations can be ignored (typically, for a single disorder configuration, thermalization period was at least 70 auto-correlation times long and at least 200 measures were taken (one every auto-correlation time). To average over disorder, we made measurements for 20,000 disorder configurations (100,000 for \(q = 8\)). Doing so, one can extract critical exponents straightforwardly:
\[
\overline{Q}_p = KL^{-p\delta_{\nu}},
\]
(29)
where \(K\) is a non-universal constant. The exponent can then be obtained by taking logarithms.

The results of our simulations, shown in Figures 1, 2 and 3, clearly support the RS scenario. In these figures, we present log-log plots of \(\overline{Q}_p L^p\) versus \(L\) \((p = 2, 3)\), for the three, four and eight-state Potts models. By taking the slopes of these graphs, we can extract \(2\delta_{\nu}(q)\) (which is minus the slope). None of the models presented show significant deviations from scaling which should arise if the replica symmetry was broken.

For the 3-state Potts model, the critical exponents associated to the scaling behaviour are in fair agreement with the values predicted by perturbative CFT computations. The deviations from the pure model behaviour are:

\[
\begin{align*}
2\gamma^*(2) &= 0.0387 & \text{Monte Carlo} \\
&= 0.0314 & \text{CFT prediction}
\end{align*}
\]
(30)

\[
\begin{align*}
2\gamma^*(3) &= 0.0648 & \text{Monte Carlo} \\
&= 0.0466 & \text{CFT prediction}
\end{align*}
\]
(31)

The numerical agreement is indeed quite surprising, especially for the third moment, where the perturbative expansion is near the end of its validity region [2]. Olson and Young [2] also computed spin-spin correlation functions moments, but in a different optic and with a different method. Our values for the exponents, presented in Table I, are in fair agreement with theirs, although they seem to be systematically lower. Using other methods, Palagyi, Chatelain et al. [1] obtain values that confirm this discrepancy. Our values are equal to theirs, within statistical errors.

Conclusion. – We believe that the presented evidence is enough to rule out the RSB scenario in random bond Potts models. If this symmetry was broken following Parisi’s scheme, deviations from the observed scaling laws would be, for the second moment, of the order of 10%, and thus would be easily observed. One can convince himself that the deviation should become more apparent for the third moment, something which is clearly not observed.

It will be interesting to see how more precise numerical methods, such as transfer matrices iterations [10], could give accurate values for moments via cumulant expansions (for integer and non-integer values of \(q\)).
Fig. 1. – Log-Log plot of $Q_p^{-2/p}$ with $p = 2$ (lower line) and $p = 3$ (higher line) for the random 3-state Potts model.

Fig. 2. – Log-Log plot of $Q_p^{-2/p}$ with $p = 2$ (lower line) and $p = 3$ (higher line) for the random 4-state Potts model.

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Fig. 3. – Log-Log plot of $Q_p^{2/p}$ with $p = 2$ (lower line) and $p = 3$ (higher line) for the random 8-state Potts model.

| $q$ | $2\Delta'_{\sigma^2}$ | $2\Delta'_{\sigma^3}$ |
|-----|---------------------|---------------------|
| 3   | 0.228(1)            | 0.202(2)            |
| 4   | 0.231(2)            | 0.198(2)            |
| 8   | 0.229(2)            | 0.184(3)            |

Table I. – The values of the spin exponents $2\Delta'_{\sigma^2}$ and $2\Delta'_{\sigma^3}$, for $q = 3, 4, 8$. The normalisation was chosen so that exponents for the same value of $q$ could be compared. The number in parentheses is the statistical error in the last decimal place.

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