EXISTENCE OF THE EHRESMANN CONNECTION ON A MANIFOLD FOLIATED BY THE LOCALLY FREE ACTION OF A COMMUTATIVE LIE GROUP

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Abstract. In this paper the author determines necessary and sufficient conditions for existence of the Ehresmann connection on a manifold foliated by locally free action of the commutative Lie group. Also here we describe structure of $C_0(M)|_L$ for a leaf $L \subset M$ in case such a connection exists. Finally we give some results on structure of the spectrum of the family of Schrödinger operators related to the leaves of the foliation.

Introduction

In this paper the author considers some special structures on foliated manifolds [11, 16]. The main problem with foliated manifolds is that they generally do not possess a good transversal set (analogue to the base of the fibre bundle [8]). Nevertheless one can find a set whose features are close to that of the base of the fibre bundle. For example we can consider a so-called Ehresmann connection on the foliated manifold [2]. It is not clear though when we have such a structure on the manifold. The aim of this paper is to find out the necessary and sufficient conditions on the foliated manifold for the Ehresmann connection to exist. Note that here we consider only manifolds with the foliation given by the locally free action of the commutative group. Note also that once this Ehresmann connection exists one can reach some interesting results (e.g. [2, 1, 3, 17]). The existence question is then important and is considered in the first part of the paper. Clearly the

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existence of the total connected transversal on \((M, F)\) is necessary for the existence of the Ehresmann connection. The open question is then if this condition is sufficient. We consider here mostly foliations of codimension 1 though there are some notes on the general case. It is shown here that if the foliated manifold possess a compact connected transversal then we can construct an invariant Ehresmann connection on it. So in this situation existence of a total connected transversal is also a sufficient condition. In case the total connected transversal is not compact we also get some more or less satisfactory results on the possibility of existence of the Ehresmann connection.

So let the foliation \(F\) on the manifold \(M\) be generated by a locally free action of a commutative Lie group \(H\). Once we have an invariant Ehresmann connection on \((M, F)\) we can prove that there exists an almost everywhere continuous bijection of our manifold to the product \(P \times S\) where \(P\) is the transversal manifold and \(S \subset H\) is a fundamental set for a factor-group \(H' = H/H_1\) \((7\), Statement 1 and Corollary 1). Using this fact we arrive to the considerations in the rest of the paper. Namely, first we consider density of the intersection of the leaf \(L\) with the invariant transversal \(P\) and get a structure of the \(C_0(M)|_L\) in general situation. The question on the structure of this algebra was inspired by the paper \([4]\), where the author considered the case of the compact group \(H\). In the third section we consider a spectral problem for the set of operators parameterized by points on \(P\) with potentials — functions of \(C_0(M)|_L\).

1. Existence of the Ehresmann connection.

1.1. Construction of the Ehresmann connection invariant with respect to the group action. Let \(M\) be a manifold with the foliation \(F\) of codimension 1 generated by the locally free action of the commutative Lie group \(\mathbb{R}^n\). Let \(\text{codim}(F) = 1\). Let us consider closed connected transversal \(P \subset M\).
Condition (*). Assume the existence of a leaf $L \in F$, which has more than one intersection point with the manifold $P$. Consider a continuous bijection $p : \mathbb{R} \to P$. Suppose for any $h_i \in H_x = \{ h \in \mathcal{H} | hx \in P \}$ the existence of a differentiable mapping $h_i : \mathbb{R} \to H$ such that $h_i(t_x) = e \in H$, $h(t)p(t) \in P$, here $p(t_x) = x$. Note that locally — in a sufficiently small neighbourhood of the point $x$ — this mapping always exists. Let $M$ possess a riemannian metric $g$. Assume next that there exists an element $a_{ij} \in H$ for any pair of $h_i, h_j \in H_x$ such that $g|_{L(t)}(h_i(t)p(t), h_j(t)p(t)) = a_{ij}$, here $a_{ij}$ does not depend on $t$. This means that we have such a translation of the set $L \cap P$ along transversal $P$ that is an isometry on $L$. Suppose this translation is not a turn. This again can always be done locally. Note that then $h_i(t) = h_i(t_x) + b_i(t)$, $h_i \in H_x$.

**Definition 1.** Let us call transversal $P$ invariant one if $hP \cap P \neq \emptyset$ implies $hP = P$.

**Statement 1.** Let $(M, F)$ be a manifold with foliation of codimension 1 given by locally free action of the commutative group $H$. Let there exist a connected closed transversal $P$, which meets the condition (*).

Then we can deform the action of the group $H$ so that the orbits of the action of $H$ (leaves of the foliation $F$) preserve but the transversal $P$ becomes invariant with respect to the action of $H$.

- Fix a point $x \in P$.
  1) Condition (*) (existence of the set $H_x$) implies the existence of $h_1 = \inf_{\|h\|} H_x = \{ h \in \mathcal{H} | hx \in P \}$, where $\| \cdot \|$ is a standard Euclidean norm in $\mathbb{R}^n$.
  2) Let us consider next a map $p : \mathbb{R} \to M$, $p(\mathbb{R}) = P$, $p(0) = x$ and set for any $y \in P \gamma(sh_1)(y) = h_1(b_1(t))^{-1}y$. To clarify the construction one must consider two coordinate charts adapted to the foliations in the neighbourhoods of points $x$ and $h_1x$. Since $h_1$ is a local diffeomorphism there exists $\varepsilon > 0$ such that for $(x - \varepsilon, x + \varepsilon) \subset P$ we have $h_1(x - \varepsilon, x + \varepsilon) \cap P =$
\{h_1x\}. Also since codimension of \(F\) equals 1, there exists locally — in a neighbourhood of \((x-\varepsilon, x+\varepsilon)\) — a continuous map \(h_t : (x-\varepsilon, x+\varepsilon) \to H\).

Condition \((*)\) (existence of \(h_i(t)\)) implies that the deformation can be defined for all \(P\). Note that \(\forall s \in \mathbb{R} \setminus \mathbb{Z} \quad sh_1 P \cap P = \emptyset\) since otherwise \(h_1\) is not an inf \(H_x\).

3) This deformed action is correctly defined. If there exists a point \(x \in M\) \(h_1x = x\) then by condition \((*)\) (absence of turns) for any \(y \in L_x\) \(h_1y = y\).

This action is naturally continuous on \(P \times [0, \|h_1\|] \times \mathbb{R}^{n-1}\) with respect to the coordinate \(sh_1 \in H\). It is continuous in a neighbourhood of 0 and since it can be continuously spread up to any point by condition \((*)\) (existence of \(h_i(t), h_i \in H_x\)) it is continuous for any other \(t \in \mathbb{R}\).

Let us apply this construction for all \(h \in H_x\).

Note that each consequent application of the preceding construction the dimension of the nondeformed group action lessens by 1. So we must show that this algorithm can not be applied more than \(n\) times. Condition \((*)\) (no turns) implies that we get a subgroup of the group of isometries \(\mathbb{R}^n \to \mathbb{R}^n\) which does not contain rotations so it consists only of translations. Since \(P\) is a closed submanifold of \(M\) the generators set of the considered subgroup is not more than \(n\) \([13]\).

One can prove the last fact otherwise: This subgroup of isometries consists of the maps \(Ax + b, A \in O(n), b \in \mathbb{R}^n\). Let there exist \((A - I)^{-1}\), then \(\|(A^n + \ldots + I)b\| \leq \|(A - I)^{-1}\|\|A^{n-1} - I\||\|b\| \leq 2\|(A - I)^{-1}\||\|b\|\) for any \(n \in \mathbb{Z}\). This violates the condition \((*)\) (the set of translations consists of isometries). Thus there exists at least one coordinate for each transformation \(\alpha_i(x) = A_ix + b\), with respect to which \(\alpha_i\) is operator of the type \(\begin{pmatrix} 1 & 0 \\ 0 & B_i \end{pmatrix}x + \begin{pmatrix} b \\ b_1 \end{pmatrix}\) \((b \neq 0)\). Then there can be no more than one translation in this direction since otherwise in the neighbourhood of the point \(0 = \lim_{k \to \infty} bm_k - b'n_k\) either condition \((*)\) — there are turns if the set \(Bib_2\) is closed, or otherwise \(P\) is not closed. The construction of this group gives us
its commutativity. Let us prove now that it is generated by no more than \( n \) elements. Commutativity implies that for all \( i, j \) \((A_i - I)b_j = (A_j - I)b_i\). Let \( A_i = \begin{pmatrix} 1 & 0 \\ 0 & B_i \end{pmatrix} \), then for any \( j \) one has \( A_{j,1} - \delta_{11} b_{11} = 0 \), so there are no more than \( n \) different \( A_j \), consequently there are no more than \( n b_j \). Commutativity implies that \( A_i \) are diagonal matrices which together with the fact that the action of these isometries conserves orientation on any subset of \( \mathbb{R}^n \) (since the foliation is generated by the action of the orientable group \( \mathbb{R}^n \)) gives us only the identity matrix for each \( A_i \). Thus our group is a commutative finitely generated subgroup of \( Iso(\mathbb{R}^n) \) without cyclic subgroup or the translation group. \( \triangleright \)

Note that the condition \((*)\) (no turns) of the previous statement is necessary for the existence of the invariant Ehresmann connection if the set \( L \cap P \) is finite and \( L \cong H \). Otherwise we get the translation \( a \) along transversal that is not a shift so the action of the \( h \) which translates a point \( x \) of \( L \cap P \) into the \( a(x) \) will produce infinitely many points \( L \cap P \subset L \) which contradicts the assumption.

**Example 1.** Let us consider the foliation on \( M = \mathbb{R}^2 \times \mathbb{S}^1 \) as the natural fibre bundle with the standard leaf \( \mathbb{R}^2 \). Let us take the transversal \( P \) which overlap torus \( \mathbb{S}^1 \times \mathbb{S}^1 \) naturally inserted into \( M \), finite number of times but more than once (Seifert foliation). Since the condition \((*)\) of the previous statement is not fulfilled (the desired pair is the pair of points lying on the intersection of the transversal \( P \) and \( \mathbb{R}^2 \)), \( P \) does not define Ehresmann connection invariant with respect to the action of the additive group \( \mathbb{R}^2 \).

**Note 1.** Suppose that we do not want to deform the action of the group. Here we can define a sequence of transversals which may converge pointwise to the transversal which will define the Ehresmann connection. Let us take as in the beginning of the proof of the previous statement \( x \in P \), \( h_1 \in H_x \). Set \( P_1 = h_1 P \) then consider \( P_2 = (h_1 + 1/2h(t))P_1 \) where \( h(0) = h_{12}(0) \), for \( x \in P \) one has \( h_{12}^{-1}(x)x \in h_1 P \) and \( h(t_0) = e \) for \( P(t_0) = h_1 x \), at the same
time \([0,1]h_{1/2} \cap P_2 = \emptyset\). And so on. The sequence of transversals \((P_n)_{n \in \mathbb{N}}\) converges pointwise to an invariant transversal if and only if the infinite sum \(\sum_{i \in \mathbb{N}} a_i\) where \(a_i = h_{i,i+1}(x)\) converges on any leaf of the foliation \(F\).

**Example 2.** Consider the foliation on \(S^1 \times \mathbb{R}\) generated by the images of the lines, \(l_b: y = ax + b\), (the constant \(a \neq 0\), \(b \in \mathbb{R}\) under the natural map \(\mathbb{R}^2 \rightarrow S^1 \times \mathbb{R}, (x,y) \mapsto ([x],y)\). Let \(P\) be the transversal which is defined in the coordinate chart \(U \cong \mathbb{R} \times (-1/2,1/2)\), \(x \in \mathbb{R}, y \in (-1/2,1/2)\) as follows:

\[
y = \begin{cases} 
0 & x \leq 0; \\
x & x \in (0,1/4]; \\
-x + 1/2 & x \in (1/4,1/2]; \\
0 & x > 1/2
\end{cases}
\]

If we take this transversal as the first member of the sequence described in the previous note then there exists the limit transversal of this sequence which is given by the constant equality \(y = 0\).

1.2. **Properties of invariant transversals.** Let us consider the connected transversal \(P \subset M\) of the foliation \(F\) on \(M\) generated by the locally free action of the commutative group \(H\). Let \(u\) denote the isotropy group of the set \(P\) by \(H_P = \{h \in H|\forall x \in P, hx \in P\}\).

**Lemma 1.** There is no path that lies in \(H_P\) and connects \(x\) and \(y\) for any pair \(x,y \in H_P, x \neq y\).

- Assume the contrary. Let there exists a path \(\gamma : [0,1] \rightarrow H, \gamma[0,1] \in H\). Then for arbitrary point \(x \in P, (\gamma[0,1])x \subset P\). Thus \(P\) is not a transversal. This contradiction implies the result. \(\triangleright\)

**Statement 2.** If \(P\) is an invariant transversal then for any point \(x \in M\) the isotropy group \(H_x \subset H_P\).

- Again assume the contrary. Then in a neighbourhood \(U(x)\) of the point \(x \in P\) which does not meet the condition of the statement for any \(h_x \in I_x\)
\[ h_x U(x) \not\in P, \] but at the same time \[ h_x U(x) \cap P \neq \emptyset. \] So \( P \) does not define a connection. ▽

Let \( P \) be an invariant transversal on the foliated manifold \((M, F)\). Consider the distribution of \( TM \) tangent to the manifold \( P \). Let us transfer this distribution with the help of the action of \( H \) to the whole manifold \( M \). The final distribution is smooth one since \( P \) is a smooth submanifold of \( M \) and the action of \( H \) is also smooth. Let us define the Ehresmann connection on \((M, F)\) as follows: we set \( \Pi(t, t') = s(t)v(t') \) for the vertical curve \( v: [0, 1] \to L \), here \( v(t) = s(t)v(0) \), \( s: [0, 1] \to H \), \( s(0) = e \), \( s(1)v(0) = v(1) \), and the horizontal curve \( w: [0, 1] \to M \), \( w = hw_0 \), here \( w_0 \in P \). Thus we obtain the rectangle \( \Pi \), unique due to the commutativity of the group \( H \).

Consider then the set \( \Gamma \) of the invariant transversals on \( M \) with isotropy groups \( H_\gamma \), \( \gamma \in \Gamma \). Each transversal of \( \Gamma \) gives rise to an Ehresmann connection. Let \( \gamma, \gamma' \in \Gamma \). We shall now define the map \( h : \mathbb{R} \to H \), \( h(t)\gamma(t) = \gamma'(t) \). Consider a leaf \( L \subset M \) and a pair of points \( x \in \gamma, x' \in \gamma' \), \( x, x' \in L \). There exists \( h' \in H \) such that \( xh' = x' \). Set \( h : \mathbb{R} \to \mathbb{R}^n \) as follows: Put \( h(0) = h' \). Note then that in some neighbourhood of the curve \( l: [0, 1] \to H \), \( l(0) = e \), \( l(1) = h' \) one can define a horizontal translation of this curve along transversals \( \gamma \preceq \gamma' \). It suffices now to show that this map can be extended to the complete transversals. There are several possibilities:

1) The ends of the translated curve have limit points \( x_1, x_2 \) on the both transversals. Let \( x_1, x_2 \) lie on the same leaf \( L \). Then there either exists \( \lim_{t \to x_1} h(t) \) or not. In the first case set \( h(x_1) := \lim_{t \to x_1} h(t) \) and extend the translation by the rule described above. Let now \( h(t) \) be unbounded as \( t \to x_1 \). Since the transversal \( \gamma \) is complete there exists \( x'_1 \in \gamma \cap L \). Let us consider \( \operatorname{Sat}(U(x'_1)) \). There exists \( t \in U'(x_1) \) such that \( \gamma(t) = h(t)x'' \) for some \( x'' \in U(x'_1) \). Hence \( x_1 = h(t)x'_1 \) which contradicts the assumption since in the neighbourhood of the singular leaf one can translate one transversal into another with the help of the continuous bounded map. Now let \( x_1, x_2 \) lie on the different leaves. Then by completeness of the both transversals
there exists an interval intersecting both the leaf passing through the limit point which lies on the other transversal as well as all the leaves near it. So the transversals are not invariant.

2) There are no limit points on both transversals. The construction is finished and the map \( h \) is correctly defined.

3) The last possibility — there exists a limit point \( x \) only on one transversal. Consider \( \text{Sat}( (x - \varepsilon, x + \varepsilon)) \). The point \( y \in \gamma' \cap L_x \) since the second transversal is complete by assumption. Consider then \( \text{Sat}(y - \varepsilon, y + \varepsilon) \). Since there are no limit points on \( \gamma' \) this curve is infinitely close to the leaf \( L_x \) thus intersecting any adjacent leaf on the one or the other side of the limit leaf \( L_x \). Now by considering a leaf from \( \text{Sat}(y - \varepsilon, y + \varepsilon) \), we arrive to the contradiction with the invariance of \( \gamma' \).

Note that for a fixed \( s \in [0, 1] \), \( sh(t) \gamma(t) \in \Gamma \). Fix a transversal \( \gamma_0 \in \Gamma \), then for each pair \( \gamma, \gamma' \in \Gamma \) one can define \( \gamma + \gamma' \in \Gamma \). For a fixed point \( x \in \gamma_0 \) there exist uniquely defined maps \( h, h' : \mathbb{R} \rightarrow H, \gamma(t) = h(t)\gamma_0(t), \gamma'(t) = h'(t)\gamma_0(t) \). Consider now \( (\gamma + \gamma')(t) = (h(t) + h'(t))\gamma_0(t) \). Isotropy group of the last transversal naturally coincides with \( H, H_{\gamma'} \). Thus one can define addition of the two transversals along the path connecting it.

The same operation can be defined for two Ehresmann connections generated by two invariant transversals. So let \( \nabla, \nabla' \) be two Ehresmann connections invariant with respect to the action of the group \( H \). Fix a point \( x_0 \in M \). Now let us consider transversals \( P \) and \( P' \) horizontal with respect to \( \nabla \) and \( \nabla' \) respectively. Let \( P \) and \( P' \) pass through the fixed point \( x_0 \). Let us construct \( P'' = P + P' \) along zero path \( \gamma([0, 1]) = x_0 \). Thus there is a structure of the additive group on the set of invariant Ehresmann connections.

**Example 3.** Let us consider different flows on the torus \( \mathbb{T}^2 \) as the set of transversals. Let the leaves of the foliation be meridians. Consider the set of transversals generated by flows \( \Gamma \supset \Gamma' \cong \mathbb{R} \), here the map \( \Gamma' \rightarrow \mathbb{R} \) is given by formula \( \gamma \mapsto a, (a, 1) \) being a vector of the flow \( \gamma \).
Example 4. Consider the foliation on $M = \mathbb{R}^2 \setminus \{0\}$, generated by circles with common center 0. Let us define an action of the group $H = \mathbb{R}$ on $M$ as follows: $\mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$: $R_h x = e^{ih/|x|}x$. Then there is no connection invariant with respect to the action of $H$, but one can deform given action of $\mathbb{R}$ as follows: $R'_h x = e^{ih}x$. Let us show by reductio ad absurdum that there is no invariant Ehresmann connection on $M$ with the foliation generated by undeformed action of $H$. Assume the existence of such connection, then the tangent vector at a point $x \in \mathbb{R}^2$ to the horizontal transversal (which exists since the considered foliation is the foliation of codimension equal to 1) under the action of the element from isotropy group $H_x \cong \mathbb{Z}$ of the leaf $L_x \ni x$ transforms to the collinear one. This is impossible since $\frac{d}{dx} e^{ih/x} = -\frac{1}{x^2} e^{ih/x} \neq 0$.

Now let $\text{codim} F > 1$ and consider global connected transversal $N \subset M$. Let us then consider a set of curves $\Phi$ of $N$. Note that $\forall \phi \in \Phi$ the existence of $h' \phi \cap \phi \neq \emptyset$ implies as in the proof of the first statement the existence of the map $h : \mathbb{R} \rightarrow H$, $h(t) \phi \subset N$. Let $h(t) \phi \in \Phi$. Let also the condition (*) be true for any curve of $\Phi$. Thus either $h' h(t) \phi = \phi$ or $(h' \phi) \cap \phi$ is discreet. Consider the subset $H^c_{\phi_0} \subset H_{\phi_0}$ defined for a curve $\phi_0 \in \Phi$. We deform the action of $H$ as in the proof of the first statement to get a connection on $\text{Sat}(\phi_0)$, invariant under the action of $h \in H^c_{\phi_0}$. Let us then consider the set of curves $\Phi_0 = \bigcup h_n^* \phi$, where $h_1 = \inf \lim\|h^d_{\phi_0}\}$ and deform the action of $H$ on $\text{Sat}(\Phi_0)$.

1) Assume first that $h_1 d\gamma(x) \in T_{h_1 x} N$ for $x = \gamma(0) \in N$ (this can always be done). Note that if there exists a neighbourhood of identity $U(e) \in H$ such that $U(e) \setminus \{e\} N \cap N = \emptyset$ then $\dim M \setminus \overline{\Phi_0} = \dim M$.

2) Now we find ourselves in the conditions of the third statement.

3) If $\overline{\text{Sat}(\Phi_0)} = M$ then the algorithm of construction is finished, otherwise we must consider another curve $\phi' \in \Phi$ and continue the construction for it.

1) implies that the dimension of the final distribution of $TM$ equals $p = \dim N$. 
The construction algorithm implies that one can construct the invariant connection on $M$ if there exists at least one connection on $(M, F)$ in which any two leaves can be connected by horizontal line. Note that the invariant connection may not be integrable.

1.3. **Analysis of conditions of Statement 1.** Now let transversal $P$ be not a closed subset of $M$. Then if $\gamma \cap L$ is dense in $L' \subset L$, $L' = \{x_0\}H_1$, for $H_1$ being a subgroup of $H$ it is sometimes possible to get an invariant connection on $M$. Consider for $\gamma(0) = x_0$, $t_1 = \inf\{t \in [0, \infty) | \gamma(t) \in L(0)\}$, then apply the algorithm to $h = \gamma(t_1)\gamma(0)^{-1}$. Note that in this case $\gamma$ generates the distribution on $TM$. We can apply Statement 1 to $H \cong \mathbb{R}$, $P \cong L$. Note that the translation $f : L \to L$ of the leaf $L$ along $\gamma$ defined with the help of the first intersection point of $\gamma \cap L$ with respect to a fixed point $x_0 \in \gamma \cap L$ can be made an isometry since $\forall n \in \mathbb{Z}$, $f^n(x_0) \neq x_0$. The same can be done in case the leaf intersecting $\gamma$ is dense or $\gamma \cap L = L$ and $\overline{\gamma \cap L \gamma} = \gamma$. This means that we pass to the isometry group on $L$. Note that the results above hold true only under conditions $(\ast)$ for each of considered maps.

Now let $\gamma$ be not a closed subset of $M$ and not everywhere dense. Consider $x_0 \in \mathbb{T} \setminus \gamma$. Let us try to deform action of the group according to the algorithm of the first statement for $x_n \in \gamma \cap B_{\frac{1}{n}}(x_0)$. Note that $h_1(n) \to 0$ as $n \to \infty$. Then a point $x \in \omega$ after the deformation of the group action becomes unaccessible from any other point of the leaf $L$. Thus $\gamma$ can not be an invariant transversal.

**Note 2.** *Let the transversal $\gamma$ be either not closed submanifold of $M$ or violate condition $(\ast)$ (not an isometry). Then the method described in the paper [12] gives us a compact connected transversal $S$ on the foliated manifold $M$. Note that later on we shall try to avoid this method in order either to deform existing non-compact transversal or to get the compact one with some special properties.*
If $\omega(\gamma(0))$ being the $\omega$-limit set of $\gamma$ is closed one-dimensional submanifold of $M$ then we can consider it as new $\gamma$.

**Statement 3.** Let there exist a riemannian metric on $M$. Suppose for $u \in F \subset TM$, $\|u\| = 1$, $(v, u) \leq \alpha < 1$ for any $v \in TM$, $\|v\| = 1$, $v$ being a tangent vector to $\gamma$. Then if for a fixed leaf $L$ there exists an isolated limit point $x \in \gamma \backslash \gamma|_{L}$ then

1) There exists a limit set $\omega$ of the transversal $\gamma$ which is homeomorphic to $S^1$.

2) $\omega$ is a total transversal of the foliation $F$ if there exists at least one compact leaf intersecting $\omega$.

- 1) Let us first introduce a parametrization on $\gamma(t)$ such that $x_k = \gamma(t+k)$. The set of functions $f^k$ with graphs $\gamma|_{\gamma \cap x_k \neq \emptyset}$ is uniform continuous in the charts adapted to the foliation in a neighbourhood of $x$. Hence there locally exists a limit curve $\omega(x)$ transversal to $F$. The curve $\omega$ is closed since otherwise $x$ is not an isolated limit point. Sliding along $\gamma(t)$, $t \in [0,1]$ by compactness of this interval we get a limit set $\omega$ (maybe not unique) which is naturally homeomorphic to $S^1$. The set $\omega$ is transversal to the leaves of the foliation $F$ by construction.

2) Compactness of $\omega$ implies the existence of leaves which lie in a neighbourhood of $\omega$. Let us show that there are no other leaves. First consider $T = \inf\{t \in \mathbb{R} | \forall s > t \in L, L_s \cap \omega \neq \emptyset\}$. Since Sat($\omega$) is open $\gamma(T) \in L$, $L \cap \omega = \emptyset$. Then $(T, +\infty)$ is covered by the infinite set of intervals of the type $[a_n, a_{n+1})$, $n \in \mathbb{Z}$, for which we have $a_n \to T$ as $n \to -\infty$ and $a_n \to +\infty$ as $n \to +\infty$. Thus the topological structure of the arbitrary neighbourhood of the limit point $\gamma(T)$ is non-trivial which contradicts with the definition of the foliation.$\triangleright$

**Note 3.** One can show compactness of the $\omega$-limit set of the transversal $P$ in case $x$ is not an isolated limit point using the same methods as in the
proof of the closeness of \( \omega(x) \). At the same time little can be said on the dimension of this limit set.

**Example 5.** Assume that the foliation on torus \( T^2 = \mathbb{R}^2/\mathbb{Z}^2 \) is given by parallels. Consider the integral curve of an irrational flow as the transversal \( P \). Then for any point \( x \in T^2 \), \( \omega(x) = T^2 \).

Let us now clarify the condition of the second part of the previous statement by the following examples:

**Example 6.** Let a foliation on \((-1, 1) \times S^1\) be given by gluing two Reeb foliations on \((0, 1) \times S^1\). Then there exists a complete transversal diffeomorphic to \( \mathbb{R} \) which converges to \( \{-1/2\} \times S^1 \) and \( \{1/2\} \times S^1 \). But neither of the limit sets is total transversal since they do not intersect the leaf \( \{0\} \times S^1 \).

**Example 7.** Consider Klein bottle as foliated manifold \( M \). Define the foliation on it by meridians between two of which we put the Reeb foliation on \((0, 1) \times S^1\). Again any limit set of any complete transversal does not intersect two boundary leaves.

**Note 4.** Note that in case all the leaves passing through \( \omega \) are homeomorphic to \( \mathbb{R} \) each leaf intersects \( \omega \) only at one point. Assume that each leaf of the foliation is noncompact. Then there exists a subgroup \( H_1 \) of \( \mathbb{R}^n \), homeomorphic to \( \mathbb{R} \) such that for any \( h \in H_1 \), \( h\omega \cap \omega = \emptyset \).

- The following statements are true due to the facts proved above together with the construction from the next statement.

1) There exists a connection on \( \text{Sat}(\omega) \) invariant with respect to the action of the group \( H \). \( \text{Sat}(\omega) \) is an open set since each point \( x \) of it possess an open neighbourhood of the type \( U(h)(\alpha_1, \alpha_2) \), where \( x = h\alpha \) for \( \alpha \in (\alpha_1, \alpha_2) \subset \omega \), \( h \in U(h) \subset H \) and \( U(h) \) being an open set.

2) There exists a sequence \( (h^ny) \to L_x \), \( n \to \infty \) for any \( y \in \omega \), here \( L_x \) is a leaf passing through the point \( x \in P \), which is a limit point of the previous
statement and $h^n \in \{h \in H|hy \in \omega\}$. Thus $\forall h \in H \exists n \in \mathbb{N}, \forall m > n \ h^m p \cap \omega = \emptyset$.

Hence for all $h \in H \setminus \{e\} \ h\omega \cap \omega = \emptyset$. >

Note that the set $\text{Sat}(P_1) \cap P_2$ is always an open subset of $P_2$ for any two transversals $P_1$ and $P_2$ on the manifold with foliation of codimension 1.

**Statement 4.** Assume that condition \((\ast)\) (no turns part of it) is violated but the set $P$ is closed hence homeomorphic to the circle $\mathbb{S}^1$. Then the following statements hold true:

1) The number of intersection points of the leaf $L'$ with $P$ between any fixed pair of the intersection points of the fixed leaf $L$ with $P$ with respect to the translation of the pair of points along transversal is constant.

2) There exists a number $N \in \mathbb{N}$ such that for any $L \in P$ the cardinality of the set $L \cap P$ is not more than $N$ if the translation along transversal does not contain shifts.

3) Again if the translation along transversal does not contain shifts then there is no leaf intersecting $P$ infinitely many times.

- 1) It suffices to consider the construction of the first statement with the only possible difficulty, i.e. there may not exist $h \in H, hP = P$, since otherwise there again may exist $n \in \mathbb{N}, h^n = e$. Since the translation along transversal $P$ is a continuous map the proposition holds true.

2) The map generated by the map $x_i \to x_i + 1$ generated by translation along transversal can be lifted to the isometry $U : H \simeq \mathbb{R}^n \leftrightarrow \mathbb{R}^n$; besides since $UH = H$ and $U^n = Id$ we have $U \in O_n(\mathbb{R}^n)$ (the set of orthogonal transformations of $\mathbb{R}^n$). Let us consider the leaf $L$ intersecting $P$ minimal possible number of times (let us denote this number by $n$) then each other leaf by 1) must intersect $P$ by the set of cardinality multiple of $n$. Since the transformation $U$ preserves while sliding along leaves the cyclic group $<U(L)>$ is a subgroup of $<U(L)>$. Thus on any leaf we get the action of the subgroup of the orthogonal translations and if we do not assume the
existence of the leaf that intersects $P$ maximal finite number of times then $P$ is not closed. This contradiction completes the proof.

3) Let there exist a leaf $L$ such that $L \cap P$ is infinite. Let us consider the set of orthogonal transformations $U(L, P)$ on $\mathbb{R}^n$. Let this set be generated by the set of points $L \cap P$. Since $U(n)$ is a compact set there exists a limit point for $U(L, P)$, so again $P$ is not closed. ▷

**Note 5.** If $P$ is compact then the condition (* ) holds true.

- The proposition is obvious if all the leaves are compact. Otherwise let us prove this fact for foliations of the dimension equal to 1 (the proof is similar in general case). Let condition (* ) (no turns part) be satisfied. The same holds true in case there exists a curvature since the operators $U(L)$ are unitary. Let the leaf $L$ intersect $P$ infinite number of times then the only obstruction while translating $[x_i, x_j] \subset L$ along $P$ is as follows: $x_{n+i} \to C$, $x_{j+n} \to \infty$ as $n \to \infty$. But this contradicts the definition of the foliation since then the set $(x_{n+i})_{n \in \mathbb{N}}$ has a limit point on $P$ and this limit point is infinitely close to its image under the translation by elements from the neighbourhood of $C$. Or we can point out the existence of $N \in \mathbb{Z}$ such that $x_{N+i} = x_j$. Note that there exists a correctly defined set $h_1(t)$, $t \in S^1$ since the transversal is compact and this set is defined for a neighbourhood of any point of it. ▷

**Example 8.** Let us describe the set $L \cap P \subset L$ and the set of its transformations in some partial cases.

1. If $H = \mathbb{R}^1$, then the transformations preserving discreet set of points and an orientation on $\mathbb{R}$ can be only shifts.

2. If $H = \mathbb{R}^2$ then the following cases of the set $L \cap P$ are possible:

   a) $L \cap P$ is finite then the deformations which conserve it are the turns on the angle $2\pi/k$, $k \in \mathbb{N}$.

   b) $L \cap P$ is countable then the deformations can be only turns by the angles: $\pi/3$ ($\mathbb{R}^2$ by isosceles hexagons), $2\pi/3$ (triangles), $\pi/2$ (squares),
\[ \pi \text{ (stripes), along with the shifts in subsequent directions. The invariant transversal then passes through the centers of the given figures. Note that by construction of the translation along } P \text{ the centers can not be points of } L \cap P. \]

**Corollary 1.** Let \( M \) be a manifold with the foliation \( F \) of codimension 1, generated by a locally free action of the commutative Lie group \( H \). Assume that there exists a compact transversal on \( (M, F) \). Then there exists an invariant Ehresmann connection on \( (M, F) \).

- It suffices to show the existence of the one-dimensional full transversal which meets the conditions of the previous statement. First note that the map \( U(L_M) : \mathbb{R}^n \rightarrow \mathbb{R}^n \), for \( L_M \) being a leaf with the minimal number of intersection points with \( P \), naturally has a fixed point which slides along \( P \) to the neighbouring leaf \( L \). Since \( U(L_M) \) can be uniquely extended to the neighbourhood of the leaf \( L_M \) we get the result. \( \triangleright \)

Condition \((\ast)\) (isometry part) can be analyzed in the following way: if \([x(t), x'(t)] \rightarrow \infty\), then the unique non-trivial situation which can not be improved with the help of the last method described below is \( x(t) \rightarrow x_0, \) \( x'(t) \rightarrow x'_0\), where \( x_0, x'_0 \) belong to different leaves. Then there naturally exists an element \( h \in H \) such that the distance between \( shx_0 \) and \( shx'_0 \) infinitely increases as \( s \in \mathbb{R} \). Let us consider now the two-dimensional foliation on the set \( sh\gamma, s \in \mathbb{R} \). The only foliation such that the transversal \( \gamma \) can not be improved with the help of the method described below is a Reeb foliation. Thus the transversal \( P \) intersecting leaves between \( L \) and \( L' \) is not compact or more precisely is the union \(( -\infty, a] \cup [b, +\infty)\). Then there also is no closed transversal connecting \( x_0 \) with \( x'_0 \) (otherwise on this transversal exists a point which is invariant under the action of \( sh \) for all \( s \in \mathbb{R} \)). Hence the manifold \( M \) splits into two saturated sets as follows: \( \text{Sat}([a, b]) \cup \text{Sat}(( -\infty, a] \cup [b, +\infty)) \). Note that in the latter case we can consider one of the ends \(( -\infty, a] \) or \(( b, +\infty)\) as new transversal. Note also
that there can be no more than $|\pi_1(M)|$ of such obstacles. This is true since there exists a loop which is a concatenation of the set $[a + \varepsilon, b - \varepsilon]$ and an interval connecting point $x_1 = x(t_1) = a + \varepsilon$ with $x'_1 = x'(t_2) = b - \varepsilon$. If this loop is contractible then there exists $h : [a + \varepsilon, b - \varepsilon] \to H$ $h[a + \varepsilon, b - \varepsilon][a + \varepsilon, b - \varepsilon] = pt$, which is impossible.

Now let us consider two constructions that help us to deform a transversal $\gamma$ so that it satisfy condition (\*) (isometry part). Assume that $\gamma(t) \to \omega \subset L$, $t \to \infty$ and that there exists $t' \in \mathbb{R}$ such that $\forall t > t' \omega(t) \not\in \gamma$. Note that in this case $\gamma([t', +\infty))$ lies in the saturated neighbourhood of the leaf $L$. Let us construct $\gamma'$ as follows: since there exists $t \in \mathbb{R}$ such that $\gamma(t) \in \omega$, the set $H\gamma(t - \varepsilon, t + \varepsilon) \cap \omega \neq \emptyset$ and there exist $\delta > 0$, $h \in H$ and $t \in \mathbb{R}$ such that $\gamma(t) \in U_\delta(\omega)$. Note that for some $t_0 > t'$ $\gamma([t_0, +\infty) \in \text{Sat}(\gamma(t - \varepsilon, t + \varepsilon))$. Let us construct $\gamma'$ by gluing of the set $\gamma(-\infty, t)$ to the part of $\gamma$ converging to $L$ from the proper side, i.e. with $t'' \in \mathbb{R}$, $\gamma(t'') \in \text{Sat}(\gamma(t - \varepsilon, t + \varepsilon))$, $t'' > t'$. Thus we get a curve connecting $L$ with itself. The only difficulty here is the possibility of self-intersection of the deformed transversal $P'$. But by Note 2 there then exists a transversal homeomorphic to $S^1$.

The second case. Let $x \in M$ be a limit point of the transversal $\gamma$. Also let there exist a coordinate chart such that for $t_n \in \mathbb{R}$, $t_n \to \infty$, $|d\gamma(t_n)/dt| \to \infty$, $(n \to \infty)$. Consider a number $C \in \mathbb{R}^+$. Let us deform $\gamma$ as follows: consider a function $\phi_n : \mathbb{R} \to H$, $\phi|_{(-\infty, t_n-\varepsilon)} = e_i |d\phi(t_n)\gamma(t_n)| < C$, $\phi|_{(t_n, +\infty)} = \phi(t_n)$ in a neighbourhood of $t_n$. The curve $\gamma' = \gamma \prod_{n \in \mathbb{N}} \phi_n$ then does not have the same property as the curve $\gamma$ at the point $x$. Again there may appear new points on $\gamma'$ with this property.

2. Structure of $C_0(M)|_L$.

Let $(M, F)$ be a foliated manifold with the foliation generated by the locally free action of a commutative group $H$. Assume that $H$ is a one-dimensional group. Assume also that $(M, F)$ possess an invariant integrable Ehresmann connection. Let $P$ be an invariant transversal. Let us consider
a continuous map $F : P \to P$ generated by the action of $a \in H_P$. Since $C_0(M) \in C_0 \times ([0, a])$ it suffices to describe structure of $C_0(P)$. Let us consider a sequence $(d(f^n(x), f^n(y)))_{n \in \mathbb{N}}$ for each pair of points $x, y \in P$. In case 0 is an accumulation point of this sequence then there are two possibilities: 1) there exists a point $z \in P$ and subsequence $(d(f_{n_i}(x), z)) \to 0$ as $n_i \to \infty$; 2) there is no accumulation point. The second case is not interesting since then it is possible to deform metric on $P$ so that $d(x_{n_i}, x_{n_j}) \geq \varepsilon$ for some $\varepsilon > 0$. In the first case then the restriction of any function $g \in C_0(M)$ on the leaf $L_x$ possess a subsequence $|g(a_{n_i}x) - g(z)| \to 0$. Define a function $g_z$ on $\mathbb{R}^+ = \bigcup_{i \in \mathbb{N}} [a_{n_i}, a_{n_i+1}]$ s a restriction of $g$ on the set of intervals correspondent to the set of points $a_{n_i}x$. The function $g_z$ behaves as in the third case of [6] where the component $f_1$ is bounded but does not belong to $C_0(\mathbb{R})$.

Now let us describe the structure of $C_0(M)|_L$ for arbitrary leaf $L \in F$. Fix $\varepsilon > 0$. We shall consider a finite number of limit points $z_k, k = 1, \ldots, n(\varepsilon, K)$ for any leaf $L$ on the compact subset of $P$. Then let us put into consideration the set of functions $g_{z_k}$ approximating the given function $g \in C_0(M)|_L$ with precision $\varepsilon$ on the compact subset $K$ of $P$ with respect to the supremum norm. The function $g_{z_k}$ is defined on the set of intervals $[a_{n_k}, a_{n_k+1}]$, here $a_{n_k}x \in U_b(z_k), \forall t \in [0, 1] g_{z_k}(a_{n_k}tax) = g(a_{n_k}tax_k)$. Note that $g_{z_k}$ is a bounded function with discontinuities in the set of points $\bigcup_{k \in \mathbb{N}} \{a_{n_k}x\} \bigcup \{a_{n_k+1}x\}$. Now as in the third case of [6] we deform $g_{z_k}$ to $g'_{z_k} \in C_b(L_x)$, so that $\| (g - g'_{z_k}) | \bigcup_{k \in \mathbb{N}} [a_{n_k}, a_{n_k+1}] \| < \varepsilon$. Consider now $g_1 = g - \sum_{i=1}^{n(\varepsilon)} g'_{z_i}$, as new function and approximate it with precision $\varepsilon/2$ on a larger compact $K'$ of $P$ and so on. As the result we get the function $g' \in C_0(\mathbb{R}^+)$. Thus we get a sequence of the finite sets of functions the behavior of which was described in [6] and which approximate the target function $g$ with any precision on any compact subset of $P$. Hence we get an inclusion $C_0(M)|_L \subset C_0(L) + \times_{\mathbb{Z}} C([0, 1])$. The last space is a space with
the norm defined by the algorithm of approximation. That is this norm is as follows: \[ \|f\| = \|f_0\|_0 + \sum_{l \in \mathbb{N}, k_1 < i < k_2} \max \left\{ \|f_i\|_0 \right\}. \]

The only question that needs clarification is the question of independence of the approximating point set \((z_i) \subset P\) on the target function. To avoid this difficulty one must consider two functions \(f_1, f_2 : M \to \mathbb{R}\) such that \(\|f'_1\|^2 + \|f'_2\|^2 \neq 0\). Let us then unite two sets of approximating points for these functions into one. Let us show now that it turns possible to find a function \(g_i\) from the space constructed in the previous paragraph such that \(\|g - g_i\|_0 < \varepsilon\) for any function \(g \in C_0(M)\) and \(\varepsilon > 0\). (\(\Rightarrow\)). Let there exist at least one isolated point \(x_0 \in L \cap P\). Then for \(f, f(x_0) \neq 0\) one has \((f - f_i)(x_0) = f(x_0)\), thus the sequence \(f_i\) does not converge to \(f\) with respect to the sup norm.

(\(\Leftarrow\)). By assumption and construction of the approximation process \(\forall \varepsilon > 0 \ \forall f \in C_0(M)|_L \ \exists f_i \ \forall x \in \text{supp} f, \ ((f_i - f)(x) < \varepsilon).\)

2) \(\Rightarrow\). The algorithm finishes on the \(i\)-th step, hence \(d(x_j, x_k) \geq \frac{\varepsilon}{2i}\). Since by construction \((x_j)\) is compact it is finite.

(\(\Leftarrow\)). Evident (cf. the third case of [6]). \(\triangleright\)

Note here that the first case of [6] can be described as the set consisting only of the first component. The second case then shows a possibility to
find set which does not contain the first component of the decomposition. The third case is then the case in which both components are present.

Let us consider a set $P' \subset P$ such that for some atlas there exists $C > 0 \|f^n\| < C$, $n \in \mathbb{N}$. Then we can slightly deform a tail of the decomposition given above. It seems natural that for some problems we shall need only those parts of the decomposition whose Chesaro mean $n'/n$ is the maximal. In certain cases we can prove the

**Note 6.** Consider two sets $S_1, S_2$ of points which define the second part of the decomposition. Define for each $i \in (n_1)$ ($n_j$, $j = 1, 2$ is the set consisting of the intersection points of $\mathbb{N}$ with $S_i$) $a_i$ which equals number of points from $(n_2)$ that lie between the $i$-th and the $i+1$-st element from $(n_1)$. If $a_i \leq K \in \mathbb{N}$ the Hausdorff dimension $d_1$ of the set $S_1P$, defined as the limit $f^{(n_1)}P$ is not greater than that defined as the limit $f^{(n_2)}P$.

- We must consider a covering $(A_i)$ of the set $S_1$. In case there exists a set with the power non-equal to 0 in the mean such that for any $x_1 \in S_1$ there exists a number $i(x_1) > 0$ $i_2 \in [-i_1, i_1]$ ($f^{i_2}(x_1) \in S_2$ we can construct a covering of the second set by the first one.

This statement holds true also in general case since for any isolated point $x \in L \cap P \dim \{x\} = 0$.

**Example 9.** Consider a foliation on $\mathbb{R}^2 \times [0,1]/ \simeq$, here the relation $\simeq$ is given as follows: $(x,0) \simeq (e^{2i \frac{x}{|x|}}, 1)$. Assume that Ehresmann connection is given by distribution tangent to $\mathbb{R}^2 \times \{pt\}$, $pt \in [0,1]$. Then the maximal set for any leaf not passing through $S_1 = \{x \in \mathbb{R}^2||x| = 1\}$ is $S_1$ of dimension equal to 1, at the same time any other point of the intersection naturally is isolated so of dimension 0. Note that any leaf passing through points inside $S_1$ has another limit set — point $\{0\}$ of dimension not less than that of any other point in the intersection of this leaf and $P$.

Note also that the Chesaro convergence of $n_1/n_2$ does not imply the previous statement. It suffices to consider the first sequence consisting of
Assume that the members of the second sequence equal to the members of the first one but their number is such that \( a_n = \sqrt{n} \).

Let us define a characteristic function for any point \( x \) which is one of the points generating the second part of the decomposition. This function is built as follows: \( \forall \varepsilon > 0 \) consider the frequency of the intersection of the given leaf \( L \) and \( B_\varepsilon(x) \) \( \nu_\varepsilon(x) = \lim_{N \to \infty} N_1/N \), here \( N_1 \) i number of the points on the leaf from \([0, N]\) which lie in \( B_\varepsilon(x) \). Consider \( \lim_{\varepsilon \to 0} \nu_\varepsilon(x) \). Now the most important function for any \( \varepsilon > 0 \) is \( f_\varepsilon : N \to N \), \( f_\varepsilon = N_1(n) \). By construction \( f_\varepsilon(n) \leq kn, k \in [0, 1] \) and \( f_\varepsilon > f_\delta \) for \( \delta < \varepsilon \). Set \( f_0(x, k) = \sup\{n(k)|a^{n(k)}x_0 \to x\}, (x_0 \in L \cap P \) is some fixed point\). This mapping is similar to the return map of [10], but is greater than equal to the inverse of it.

Thus there either exists a function \( g(t), g \neq \text{const} g(0) = 0 \) such that \( \lim_{\varepsilon \to 0} \lim_{t \to 0} g(t)f_\varepsilon(1/t) = 1 \) or \( \lim_{\varepsilon \to 0} f_\varepsilon(1/t) = 0 \). Let us show that in the last case \( x \) is an isolated point. Let us consider \( f(x, n) \) generated by the sequence \( x_n \in B_{1/n}(x) \cap L \). This function on infinity is less or equal than any \( f_\varepsilon(n) \) for any \( \varepsilon > 0 \), at the same time by assumption \( x \) is not isolated, thus \( \lim_{n \to \infty} f(x, n) = \infty \). So the behavior of the singular point is uniquely described by the behavior of the function \( g \) at zero and we can introduce an order on the set of these points \( x \geq y \Leftrightarrow \lim_{t \to 0} g(x, t)/g(y, t) < \infty \). Consider now \( X_f = \{x \in P | f(x, t) = f(t)\} \). Naturally the closure of this set contains only points greater with respect to the relation given above. First mention that on the one hand \( \forall U_\varepsilon(x') \exists \varepsilon > 0, x \in U_\varepsilon(x'), (f_{\varepsilon'}(x') \geq f_\varepsilon(x)) \). On the other hand let \( x_k \to y a^{n_i+k}x_0 \to x_k \), then the sequence \( (x_j), x_{2k}, \ldots x_{2k+1}, x_j \to y \) as \( j \to \infty \) and the growth at \( \infty \) of \( (n_i + k)_j \) coincides with that of \( n_i \).

The same function can be defined for the set \( S_0 = \{p \in L \cap P \} \) there exists an open neighbourhood \( U(p) \) of the point \( p \) such that \( U(p) \cap (L \cap P) = \{p\} \). Fist not that the set \( S_0 \subset \mathbb{R}^n \) is at most countable one. Then fix an order on the set \( \phi : \mathbb{N} \leftrightarrow L \cap P \). Note that \( \phi^{-1}(S_0) \subset \mathbb{N} \). Now for a point \( s \in S_0 \),
$s = \phi(0)$ we put $f_0(s) = 0$. Finally for $\phi^{-1}(s) \in \mathbb{Z}$ we get

$$f(\phi^{-1}(s)) = \begin{cases} f(\phi^{-1}(s-1)) + 1, & \text{if } s \in S_0; \\ f(\phi^{-1}(s-1)), & \text{otherwise}. \end{cases}$$

Now we point out that $X_M = \bigcap_{f \mid X_f \neq \emptyset} X_{g \leq f}$ is closed and non-empty in case $P$ or one of the considered sets $X_{g \geq f}$ is compact. Thus we have proved the

**Statement 6.** If $P$ or one of the sets $X_{g \leq f}$ is compact then there exists a maximal element on the set of points with the order $\geq$.

The statements of the following note are too evident to give the proofs but rather important to ignore.

**Note 7.** 1) For any function $f$ the set $X_f$ is invariant with respect to $H_P$.

2) $X_{\min} = L \cap P$.

3) $L \cap P \cap X_{\max} \neq \emptyset$ implies $X_{\min} = X_{\max} = L \cap P$.

**Corollary 2.** The function $f_0$ can either vanish or be equal to the trivial function $f : \mathbb{N} \to \mathbb{N}, n \mapsto n$.

- To prove this consider a sequence of points $(x_n)$ of $L : \bigcap P$ such that $x_n \to x \in L \cap P$. Now, $x \notin S_0$. Assume now that there also exists $x' \in L \cap P, x' \in S_0$. Then there must exist $h \in H$ such that $x' = hx$, but then since the action of the group $H$ is continuous $hx_n \to x'$. This contradiction with the first statement of the previous note proves the statement. $\triangleright$

**Example 10.** Kronecker irrational flow on the torus $T^2$. Consider the parallel $P$ diffeomorphic to $S^1$ of the torus transversal to the leaves of the foliation. The set $X_{\min} = X_{\max} = P$ since the only nonempty closed subset of $S^1$ invariant with respect to the rotation rationally independent with $2\pi$ is $S^1$ itself. Note that in case the set of points that return to the fixed one is the sequence defined by Fibonacci numbers (appendix C of [10]), then for any $x \in P f_0(x,t) \geq \log_{\sqrt{\frac{\sqrt{5}+1}{2}}} \sqrt{5}t$. 

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Note that in case \( L \cap P \subset P \) does not contain nontrivial sets invariant with respect to the action of \( H_P \) then \( X_{\text{max}} = L \cap P \). The following example shows that the latter is possible also in other case.

**Example 11.** Denjoy \( C^1 \)-vector field on the torus \( \mathbb{T}^2 \) \[16\]. Again consider the irrational flow on torus \( \mathbb{T}^2 \) as a foliation. Consider next the countable set of intervals \( \{ I_m = [0, l_m] ; m \in \mathbb{Z}, l_m > 0 \} \) such that

1) \( \sum_{i \in \mathbb{Z}} l_i = l < +\infty \),

2) \( \lim_{m \to \infty} l_m/l_{m+1} = 1 \).

Substitute then the point \( a^m x \in P \) by the interval \( I_m, m \in \mathbb{Z} \), where \( P \cong \mathbb{S}^1 \) is the parallel transversal to the leaves of the foliation on the torus \( \mathbb{T}^2 \), \( a \) is the rotation of the transversal \( P \) on the angle rationally independent with \( 2\pi, x \in P \) being a fixed point.

Let us introduce now the set of mappings \( f_m : I_m \to I_{m+1} \) with the following properties:

1) \( \frac{df_m}{dt} > 0 \);

2) there exists a number \( \delta_m > 0 \) such that the derivative \( \frac{df_m}{dt} \) equals 1 on intervals \( [0, \delta_m) \) and \([l_m - \delta_m, l_m] \);

3) \( \min(1, l_m/l_{m+1}) - (1 - l_{m+1}/l_m)^2 \leq \frac{df_m}{dt} \leq \max(1, l_{m+1}/l_m) + (1 - l_{m+1}/l_m)^2 \).

Let us add this map to the shift \( a : \mathbb{S}^1 \to \mathbb{S}^1 \). Thus we get a mapping \( f : \mathbb{S}^1 \to \mathbb{S}^1 \).

Then the leaf passing through any point of \( I_m, m \in \mathbb{Z} \) intersects \( P \) in closed nowhere dense set invariant with respect to the map \( f \). Thus the set of intersection points of this leaf coincides with the \( X_{\text{max}} \) constructed above. Also any leaf which does not pass through \( I_m, m \in \mathbb{Z} \) is dense on the set of similar leaves. Moreover the closure of the set \( L \cap P \) contains also points from \( \partial(I_m) \). In this case the maximal subset is a closure of \( L \cap P \subset P \) as in the preceding example. Note that this set contains the invariant with respect to the action of \( f \) closed subset consisting of \( \bigcup_{m \in \mathbb{Z}} \partial I_m \).
Since for any \( g \) the set \( X_g \) is invariant with respect to the action of \( H_P \) we can consider the following condition:

(*) The characteristic function \( f(x, t) \) does not depend on \( x \in P \) in some neighbourhood \( U(x_0) \) of the fixed point \( x_0 \in L \cap P \). The following statement is then evident.

**Statement 7.** Condition (*) implies \( X_{\text{max}} \subset \partial P \cap L \).

3. **Family of Shrödinger operators.**

Let us again consider a manifold \( M \) with the foliation \( F \) generated by the action of the abelian group \( H \) and integrable Ehresmann connection invariant with respect to the action of the group \( H \). Let us put into consideration the operator \( \mathcal{H} = -\nabla + V \) on \( L^2(H \times P) \), defined for any leaf of the foliation \( F \), here \( V \in C_0(M) \) and \( \nabla = \sum_{i=1}^{\dim H} \frac{\partial}{\partial x_i} \), \( x_i = 1, \dim H \) being coordinates on \( H \). Thus \( \mathcal{H} \) defines a family of Shrödinger operators \( (\mathcal{H}_p)_{p \in H} \), \( \mathcal{H}_p : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) for any leaf \( L \), here \( n = \dim H \). So each \( \mathcal{H}_p = -\nabla + V_p \) for \( V_p \in C_0(M)|_{L_p} \). At the rest of the paper we assume that \( V_p \) depends upon \( p \in P \) analytically. The good illustration of the method gives the paper [9].

**Statement 8.** Let \( H = \mathbb{S}^1 \). Then the spectrum of the Shrödinger operator \( \mathcal{H} = -\frac{d^2}{dx^2} + V \) depends on the transversal coordinate \( p \in P \) continuously.

- Let \( \mu \) be an \( H \)-invariant measure on the transversal \( P \). Consider the decomposition of the set of the target spaces \( L_2 \)

\[
L^2(H \times P) = \int_P L^2(H) d\mu = \int_P \int_{[0,2\pi)^n} H' \frac{d^n \theta}{(2\pi)^n} d\mu,'
\]

here \( H' \simeq l_2 \) and the isomorphism \( L^2(\mathbb{R}^n = H, d^n x) \leftrightarrow \int_{[0,2\pi)^n} H' \frac{d^n \theta}{(2\pi)^n} \) is the Floquet decomposition \( U' : L^2(\mathbb{R}^n = H, d^n x) \to \int_{[0,2\pi)^n} H' \frac{d^n \theta}{(2\pi)^n} \) given as follows:

\[
(U' f)_{\theta}(x) = \sum_{m \in \mathbb{Z}^n} e^{-i \theta m} f(x + \sum m_i a_i),
\]
$a_i, i = 1, n$ being the basis of the group $H$ (not such of the group $H_P$). Then theorems XIII.64 and XII.11 from \cite{13} and Foubini theorem imply the statement. $	riangleright$

Assume now that $H$ is not a compact group and for any $x \in P$ the set $H_P\{x\}$ is dense in $P$. Assume also that there exists a $H_P$-invariant metric on $P$. Thus we deal with specific ergodic transformations on $P$. Then it can be shown that $C_0(M)|_L$ consists of limit almost-periodic functions with periods rational dependent with $a \in H_P$.

**Lemma 2.** The spectrum of the Shrödinger operator $\mathcal{H} = -\frac{d}{dx} + V$ with limit almost periodic potential $V$ is a pointwise limit of a set of spectra of operators $\mathcal{H}_n = -\frac{d}{dx} + V_n$, here $V_n$ are limit almost-periodic functions such that $V_n \to \|\cdot\|_{\text{sup}} V$, ($n \to \infty$).

- Consider $\varepsilon > 0$ and $\|\mathcal{H}_n - \mathcal{H}\| \leq \varepsilon$. Let us also take into consideration spaces $L_{n,m} = \int (-1/2, 1/2]l' dq$, where $l' = l_2(\{1, \ldots, m\})$ and the operator $U$ is again a Floquet decomposition. Then one have for the set of operators $\mathcal{H}'_n = \mathcal{H}_n|_{L_{n,m}}$. At the same time $\mathcal{H}_n$ is a bounded operator for any $n \in \mathbb{N}$. Then Corollary 2 of \cite{15} implies the statement for $\mathcal{H}'_n$, thus any point of the spectrum of $\mathcal{H}$ is close to some such point of $\mathcal{H}_n$ for sufficiently large number $n \in \mathbb{N}$. $	riangleright$

**Corollary 3.** If there exists an $H_P$-invariant riemannian metric on $P$ then spectrum of the family of Schrödinger operators continuously depend on $p \in P$.

**Note 8.** Note that similar statements can be translated to the case of the complex-valued potential $V : M \to \mathbb{C}$. This can be done using results of \cite{14} \cite{15}.

**Statement 9.** Let transversal $P \simeq \mathbb{R}^n$ and $\forall x \neq 0 H_x = n\mathbb{Z}$ ($n \in \mathbb{N}$), $H_0 = \mathbb{Z}$. Then the following statements hold true:
1) Spectrum of the family of Hill operators with periodic real-valued potentials is homeomorphic to $L \times S^{n-1}$, here $L$ is the union of graphs of functions $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \mathbb{N}$, $f_{ni+k}(x) = 1/k \arctan(x)$, $f_{ni} = 0$, $k = 1, \ldots, n-1$, $i \in \mathbb{N}$.

2) Assume that for any $x \in P$ and $E \in \Lambda_x$ ($\Lambda_x$ is the spectrum of the operator $H_p$ at the point $x \in P$) $\Delta_x(E) \neq 0$. If $V$ is a complex-valued function then arcs of the spectrum with the greatest period can interchange rotating over $0 \in P$ only in case operator $H_0$ at the point $0$ possess closed analytic arc in the spectrum.

- $V$ is the real-valued function. There exists only one non-trivial case — $P \simeq \mathbb{R}^2$, since the only alternative to the given case is the spiral line over \{0\} × $\mathbb{R}$ ∈ $P$ × $\mathbb{R}$. Then the spectrum of each operator $H_0$ is not purely discreet one which contradicts paragraph (a) of the Theorem XIII.89 from [13].

Assume that the condition of the paragraph 2) holds true then passing to the limit $p \to 0$ we obtain the arc in the spectrum of $H_0$ analytic in each internal point [15]. Then by periodicity of the potential $V$ the arc of the operator $H_0$, which by the last note splits into $n$ arcs of the operators $H_x$, $x \in P \setminus \{0\}$ is the connected set i.e. $n-1$ arcs of the spectrum of the operator $H_x$ for $x \neq 0$ glue with each other. Thus invariance of these arcs under the rotation over $0$ — which at the same time gives rise to the transformation of the spaces $L^2(\mathbb{R})$ — the other arcs also glue. >

Using the density functions of the second part of the paper one can find out which eigenvalues of the family of operators are more important.

References

[1] R.A. Blumenthal and J.J. Hebda, An analogue of the holonomy bundle for a foliated manifold. C. R. Acad. Sci., Paris, Sér. I 303 (1986), 931 – 934.

[2] R.A. Blumenthal and J.J. Hebda, Complementary distributions which preserve the leaf geometry and applications to totally geodesic foliations. Quart. J. Math. Oxford 35 (1984), 383 – 392.
[3] R.A. Blumenthal and J.J. Hebda, Ehresmann connections for foliations. Indiana Univ. Math. J. 33 (1984), 597 – 611.

[4] F. Cadet, Deformation quantization using groupoids. Case of toric manifolds. arXiv:math.OA/0305261 v2 20 May 2003.

[5] N. Dunford and J.T. Schwartz, Linear Operators. I. General theory. (Pure and Applied Mathematics. Vol. 6). New York and London: Interscience Publishers. 1958.

[6] P. N. Ivanshin, Structure of function algebras on foliated manifolds. Lobachevskii J. Math. 14 electronic only (2004), 39 – 54.

[7] P. N. Ivanshin, Algebras of functions on groupoid of some special foliations. Southwest J. Pure Appl. Math. 2003 No.1, electronic only (2003), 96 – 108.

[8] Sh. Kobayashi and K. Nomizu, Foundations of differential geometry. I. New York-London: Interscience Publishers, a division of John Wiley & Sons. 1963.

[9] F. Lledo and O. Post, Generating spectral gaps by geometry. arXiv:math-ph/0406032 v1 15 Jun 2004.

[10] J. Milnor, Dynamics in one complex variable. Introductory lectures. Wiesbaden: Vieweg. 1999.

[11] P. Molino, Riemannian foliations. Progress in Mathematics, Vol. 73. Boston-Basel: Birkhäuser. 1988.

[12] S.P. Novikov, Topology of foliations. (Russian, English) Trans. Mosc. Math. Soc. 14 (1965), 268 – 304; translation from Tr. Mosk. Mat. Obshch. 14 (1965), 248 – 278.

[13] M. Reed and B. Simon, Methods of modern mathematical physics. IV: Analysis of operators. New York - San Francisco - London: Academic Press. 1978.

[14] F.S. Rofe-Beketov, The spectrum of non-selfadjoint differential operators with periodic coefficients (Russian, English) Sov. Math., Dokl. 4 (1963), 1312 – 1315; translation from Dokl. Akad. Nauk SSSR 152 (1963), 1312-1315.

[15] K.C. Shin, On the shape of spectra for non-self-adjoint periodic Schrödinger operators. arXiv:math-ph/0404015 v1 6 Apr 2004.

[16] I. Tamura, Topology of foliations: an introduction. Translations of Mathematical Monographs. 97. Providence, RI: American Mathematical Society (AMS). 1992.

[17] N. I. Zhukova and G.V. Chubarov, Aspects of the qualitative theory of suspended foliations. J. Difference Equ. Appl. 9, No. 3-4 (2003), 393 – 405.

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