A connection between HH3 and KdV with one source

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Abstract

In the system made of Korteweg-de Vries with one source, we first show by applying the Painlevé test that the two components of the source must have the same potential. We then explain the natural introduction of an additional term in the potential of the source equations while preserving the existence of a Lax pair. This allows us to prove the identity between the travelling wave reduction and one of the three integrable cases of the cubic Hénon-Heiles Hamiltonian system.

Keywords: KdV with source, cubic Hénon-Heiles, Lax pair, Painlevé test.

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1 Introduction

In several soliton equations, it is possible to add a so-called “source term” without destroying the soliton property. For instance, the Korteweg-de Vries (KdV) equation

$$b u_t + \left( u_{xx} - \frac{3}{a} u^2 \right)_x = 0, \ a, b = \text{constants},$$

(1)
retains its solitonic property when extended to [21, 26]

\[
\begin{align*}
E_1 &\equiv bu_t + \left( u_{xx} - \frac{3}{a} u^2 - dv \right)_x = 0, \quad a, b, d = \text{constants}, \\
E_2 &\equiv v_{xx} + \left( -\frac{u}{a} + \mu_1 \right) v = 0, \\
E_3 &\equiv w_{xx} + \left( -\frac{u}{a} + \mu_2 \right) w = 0,
\end{align*}
\]

provided \( \mu_1 = \mu_2 = \mu \). The parameter \( \mu \) is then inessential and it could be removed by the Galilean transformation

\[
(u, x, t) \rightarrow (u + a\mu, x + 6\frac{\mu}{b}t, t).
\]

If one denotes \( W = vw \), \( Z = v/w \), the transformed system for \((u, W, Z)\),

\[
\begin{align*}
\left( \partial_x^3 - 4 \left( \frac{a}{u} - \mu \right) \partial_x - 2 \frac{u_x}{a} \right) W &= 0, \\
(W \log Z)_x &= 0,
\end{align*}
\]

allows the easy elimination of \( W \) and \( Z \), resulting in the KdV6 equation [17, 18],

\[
\left( \partial_x^3 - \frac{4}{a} U_x \partial_x - \frac{2}{a} U_{xx} \right) \left( bU_t + U_{xxx} - \frac{3}{a} U_x^2 \right) = 0, \quad U_x = u - \alpha\mu.
\]

Solutions of system (2) with \( \mu_1 = \mu_2 \) have been investigated by means of the inverse scattering method [22] and the Darboux transformation [27, 19]. In the more general situation \( \mu_1 \neq \mu_2 \), it has been noticed [20] that one can also build a variety of solutions.

The purpose of this paper is to present some new results concerning the KdV with one source system (2). In section 2, we first examine the system (2) and prove that a necessary condition to pass the Painlevé test is \( \mu_1 = \mu_2 \). In section 3, we introduce an extension of this system admitting a Lax pair, and we show that this generalized KdV with one source admits a reduction which can be identified to one of the three integrable sets of equations of motion of the cubic Hénon-Heiles Hamiltonian system.

## 2 The Painlevé test

Among the possible leading behaviours

\[
u \sim v_0 \chi^{p_1}, \quad v \sim v_0 \chi^{p_2}, \quad w \sim w_0 \chi^{p_3}, \quad v_0 v_0 v_0 w_0 \neq 0,
\]

in which \( \chi \) denotes the expansion variable near a movable singularity \( \varphi = \varphi(x, t) \) [6]

\[
\chi = \left( \frac{\varphi_x}{\varphi} - \frac{\varphi_{xx}}{2\varphi_x} \right)^{-1},
\]

there exist at least two families in which all \( p_i \) are negative integers, these are

\[
\begin{align*}
\text{F1} : \quad &p = (-2, -1, -1), \quad u_0 = 2a, \quad (v_0, w_0) = \text{arbitrary}, \\
\text{F2} : \quad &p = (-2, -2, -2), \quad u_0 = 6a, \quad v_0 w_0 = -\frac{72a}{d}.
\end{align*}
\]
with \( p = (p_1, p_2, p_3) \), and their respective Fuchs indices are

\[
\begin{align*}
F1 &: -1, 0, 0, 3, 3, 4, 6, \\
F2 &: -3, -1, 0, 4, 5, 6, 8.
\end{align*}
\]

(9)

When one checks the existence of the Laurent series, one finds the following conditions for the absence of movable logarithms,

\[
F1 : Q_6 \equiv (\mu_1 - \mu_2)^2 v_0 w_0 = 0,
\]

(10)

and

\[
F2 : \begin{cases}
Q_6 \equiv (\mu_1 - \mu_2)(bC + 3(\mu_1 + \mu_2)) = 0, \\
Q_8 \equiv 2S Q_6 + (\mu_1 - \mu_2)(20S_{xx} - 20S^2 - 4bC_{xx} + 200u_4 + 5(\mu_1 - \mu_2)^2) = 0,
\end{cases}
\]

(11)

where \( C = C(x, t), S = S(x, t) \) are functions given by the singular manifold [6]

\[
S = \left( \frac{\varphi_{xx}}{\varphi_x} \right)_x - \frac{1}{2} \left( \frac{\varphi_{xx}}{\varphi_x} \right)^2, \quad C = -\frac{\varphi_t}{\varphi_x},
\]

(12)

and \( u_4 \) is the arbitrary coefficient introduced at Fuchs index \( i = 4 \). Therefore a necessary condition for the system (2) to pass the Painlevé test is that \( \mu_1 = \mu_2 \).

3 Link with the cubic Hénon-Heiles

The last two equations of system (4) each admit a first integral related to the Wronskian of \( v \) and \( w \),

\[
\begin{align*}
2G(t) &= WW_{xx} - \frac{1}{2} W_x^2 - 2 \left( \frac{u}{a} - \mu \right) W^2, \\
W(\log Z)_x &= g(t), \\
g(t) &= v_x w - vw_x, \quad G(t) = -\frac{1}{2} g^2(t).
\end{align*}
\]

(13-15)

The conservation of the Wronskian is of course intrinsic to the original system (2) (with \( \mu_1 = \mu_2 \)), and it manifests itself by the natural introduction in the transformed system for \( (u, W, Z) \) of one arbitrary function of time, thus ensuring the conservation of the total differential order (seven) between the original and the transformed systems.

One deduces that the field

\[
Q = W^{1/2}
\]

obeys a nonlinear ODE of the Ermakov-Pinney type [13, 25]

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{u}{a} + \mu - \frac{G(t)}{Q^4} \right) Q = 0,
\]

(17)

i.e. an equation which only differs from the linear ODE for \( v \) or \( w \) in system (2) by the contribution of \( G(t) \).

The main point is that the KdV with one source system (2) is incomplete, in the sense that the term \( G(t)/W^2 \) can be added to the potential of the linear equation for \( v \) and
while retaining the Painlevé property. Indeed, the extrapolation of (2) to an arbitrary value of $G(t)$

\[
\begin{aligned}
F_1 &\equiv bu_t + \left( u_{xx} - \frac{3}{a} u^2 - dW \right)_x = 0, \\
F_2 &\equiv v_{xx} + \left( -\frac{u}{a} + \mu - \frac{G(t)}{v^2 w^2} \right) v = 0, \\
F_3 &\equiv w_{xx} + \left( -\frac{u}{a} + \mu - \frac{G(t)}{v^2 w^2} \right) w = 0,
\end{aligned}
\]  

(18)

admits the second order matrix Lax pair

\[
\Psi_x = L \Psi, \quad \Psi_t = M \Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},
\]

(19)

\[
L = \begin{pmatrix} u/a - \mu + \lambda & 1 \\ 0 & \frac{d}{4a \lambda} \end{pmatrix}, \quad bM = \begin{pmatrix} -u_x/a - \frac{d(vw)_x}{4a \lambda} & 1 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix},
\]

\[
a_{12} = \frac{2u}{a} - 4(\lambda - \mu) + \frac{dvw}{2a \lambda},
\]

\[
a_{21} = \left( \frac{u}{a} + \lambda - \mu \right) \left( \frac{u}{a} - 4(\lambda - \mu) \right) - \frac{u_{xx}}{a} - \frac{dG(t)}{2a \lambda vw} + \frac{dvw}{2a} - \frac{dvw_x}{2a \lambda} = 0,
\]

in which $\lambda$ is the spectral parameter. The zero-curvature vanishing condition

\[
b(L_t - M_x + [L, M]) \equiv \frac{d(wF_2 + vF_3)}{4a \lambda} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} F_1/a & d(wF_2 + vF_3)/2a \lambda \\ 0 & 0 \end{pmatrix},
\]

(20)

is indeed equivalent to the condition that the lhs $F_1, F_2, F_3$ of the system (18) simultaneously vanish.

Let us now prove that the system (18) admits for $G(t)$ arbitrary a noncharacteristic reduction to an ODE system which is complete in the Painlevé sense. The transformed system of (18) for $(u, W = vw, Z = v/w)$ reads

\[
\begin{aligned}
bv_t + \left( u_{xx} - \frac{3}{a} u^2 - dW \right)_x &= 0, \\
\left( \partial_x^3 - 4 \left( \frac{u}{a} - \mu - \frac{G(t)}{W^2} \right) \partial_x - 2 \left( \frac{u}{a} - \frac{G(t)}{W^2} \right)_x \right) W &= 0, \\
W(\log Z)_x &= g(t),
\end{aligned}
\]

(21)

the second equation is independent of $G(t)$ and therefore admits the first integral $G(t)$ defined in (13), and the first two equations define a closed subsystem for $u, W$. The resulting system for $(u, Q = W^{1/2})$ is

\[
\begin{aligned}
bv_t + \left( u_{xx} - \frac{3}{a} u^2 - dQ^2 \right)_x &= 0, \\
\left( \partial_x^2 - \frac{u}{a} + \mu - \frac{G(t)}{Q^4} \right) Q &= 0.
\end{aligned}
\]

(22)

As to the cubic Hénon-Heiles Hamiltonian, it is defined as [16, 14]

\[
H = \frac{1}{2} (p_1^2 + p_2^2 + c_1 q_1^2 + c_2 q_2^2) + \alpha q_1 q_2^2 - \frac{1}{3} \beta q_1^3 + \frac{c_3}{2q_2^2}, \quad \alpha \neq 0,
\]

(23)
in which $p_i = p_i(\xi)$, $q_i = q_i(\xi)$ ($i = 1, 2$) and $\alpha, \beta, c_1, c_2, c_3$ are constants. The corresponding equations of motion,
\[
\begin{align*}
\frac{d^2 q_1}{d\xi^2} + c_1 q_1 - \beta q_1^2 + \alpha q_2^2 &= 0, \\
\frac{d^2 q_2}{d\xi^2} + c_2 q_2 + 2\alpha q_1 q_2 - \frac{c_3}{q_2} &= 0,
\end{align*}
\tag{24}
\]
pass the Painlevé test (in the dependent variables $q_1, q_2^2$) for only three sets of values of $(\beta/\alpha, c_1, c_2)$ \cite{4, 5, 15}: $(\beta/\alpha = -1, c_1 = c_2)$, $(\beta/\alpha = -6)$, $(\beta/\alpha = -16, c_1 = 16c_2)$. These three cases are equivalent to the stationary reduction of three fifth order soliton equations, respectively known as the Sawada-Kotera (SK), fifth order Korteweg-de-Vries (KdV$_5$) and Kaup-Kupershmidt (KK) equations, which belong respectively to the BKP, KP and CKP hierarchies.

The link between the extended KdV with one source system (18) now becomes obvious. Under the traveling wave reduction
\[
G(t) = \text{constant, } \xi = x - ct,
\tag{25}
\]
the two-component system is readily identified to the Hamilton equations (24), with
\[
\begin{align*}
 u &= q_1, \quad Q = W^{1/2} = (vw)^{1/2} = q_2, \\
c_1 &= -bc, \quad \beta = \frac{3}{a}, \quad \alpha = -d, \quad c_2 = \mu, \quad \alpha = -\frac{1}{2a}, \quad c_3 = G(t),
\end{align*}
\tag{26}
\]
which imply
\[
\frac{\beta}{\alpha} = -6.
\tag{27}
\]

Therefore the traveling-wave reduction of the extended KdV with one source system (18) is the cubic HH system corresponding to KdV$_5$.

Since one cannot include additional terms in the system (24) without destroying its Painlevé property \cite{11, 14, 10} (completeness property), this proves that the initial system (2) (with $\mu_1 = \mu_2$) was incomplete. This is why the extended KdV with one source system (18) deserves to be qualified as “complete”.

Remark. As proven in Ref. \cite{12}, the number of degrees of freedom in (23) can be extended arbitrarily.

4 Conclusion

It is known that the cubic and quartic Hénon-Heiles Hamiltonians pass the Painlevé test only for seven sets of coefficients. The seven Hénon-Heiles Hamiltonians all have the Painlevé property and have been extensively studied \cite{3, 8, 9, 2}. However, the explicit integration of three of the quartic cases, namely 1:6:1, 1:6:8, 1:12:16, is not yet optimal, in the sense that the expressions are quite intricate. The reason is that one could not yet associate each of these three cases to an optimal PDE system \cite{1}. The link established in this paper between a PDE with source system and one of the cubic Hénon-Heiles Hamiltonians strongly suggests that there could exist three privileged systems of the type PDE+source, whose reduction $x - ct$ would be identical to the Hamilton equations of the quartic cases 1:6:1, 1:6:8, 1:12:16. A preliminary step in this direction has been made in \cite[Eq. (A.24)]{7} for the 1:6:1 and 1:6:8 cases, whose integration with the autonomous F-VI equation \cite{10} needs to be improved.
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References

[1] Sarah Baker, Squared eigenfunction representations of integrable hierarchies, PhD Thesis, University of Leeds (1995).
[2] S. Baker, V.Z. Enol’skii and A.P. Fordy, Integrable quartic potentials and coupled KdV equations, Phys. Lett. A 201 (1995) 167–174. arXiv:hep-th/9504087v1.
[3] M. Błaszak, A generalized Hénon-Heiles system and related integrable Newton equations, J. Math. Phys 35 (1994) 1693–1709.
[4] T. Bountis, H. Segur and F. Vivaldi, Integrable Hamiltonian systems and the Painlevé property, Phys. Rev. A 25, 1257–1264 (1982).
[5] Chang Y.F., M. Tabor and J. Weiss, Analytic structure of the Hénon-Heiles Hamiltonian in integrable and nonintegrable regimes, J. Math. Phys. 23 (1982) 531–538.
[6] R. Conte, Invariant Painlevé analysis of partial differential equations, Phys. Lett. A 140 (1989) 383–390.
[7] R. Conte and M. Musette, The Painlevé handbook (Springer, Berlin, 2008).
[8] R. Conte, M. Musette and C. Verhoeven, Explicit integration of the Hénon-Heiles Hamiltonians, J. Nonlinear Mathematical Physics 12 Supp. 1 (2005) 212–227. arXiv:nlin/0412057v1.
[9] R. Conte, M. Musette and C. Verhoeven, Completeness of the cubic and quartic Hénon-Heiles Hamiltonians, Theor. Math. Phys. 144 (2005) 888–898. arXiv:nlin/0507011v1.
[10] C.M. Cosgrove, Higher order Painlevé equations in the polynomial class, I. Bureau symbol $P^2$, Stud. Appl. Math. 104 (2000) 1–65.
[11] J. Drach, Sur l’intégration par quadratures de l’équation $\frac{d^2y}{dx^2} = [\varphi(x) + h]y$, C. R. Acad. Sc. Paris 168 (1919) 337–340.
[12] J.C. Eilbeck, V.Z. Enol’skii, V.B. Kuznetsov and D.V. Leykin, Linear $r$–matrix algebra for systems separable in parabolic coordinates, Phys. Lett. A 180 (1993) 208–214.
[13] V. P. Ermakov, Équations différentielles du deuxième ordre. Conditions d’intégrabilité sous forme finale. Univ. Izv. Kiev (1880) Ser. 3, No. 9, 1–25. [English translation by A. O. Harin, 29 pages].
[14] A.P. Fordy, The Hénon-Heiles system revisited, Physica D 52 (1991) 204–210.
[15] B. Grammaticos, B. Dorizzi and R. Padjen, Painlevé property and integrals of motion for the Hénon-Heiles system, Phys. Lett. A 89 (1982) 111–113.
[16] M. Hénon and C. Heiles, The applicability of the third integral of motion: some numerical experiments, Astron. J. 69 (1964) 73–79.

[17] A. Karasu-Kalkani, A. Karasu, A. Sakovich, S. Sakovich and R. Turhan, A new integrable generalization of the Korteweg-de Vries equation, J. Math. Phys. 49 (2008) 073516.

[18] B.A. Kupershmidt, KdV6: an integrable system, Phys. Lett. A 372 (2008) 2634–2639.

[19] R. Lin, Y. Zeng and Wen-Xiu Ma, Solving the KdV hierarchy with self-consistent sources by inverse scattering method Physica A 291 (2001) 287–298.

[20] Ma Wen-Xiu, Complexiton solutions of the Korteweg-de Vries equations with self-consistent sources, Chaos, solitons and fractals 26 (2005) 1452–1458.

[21] V.K. Mel'nikov, On equations for interactions, Lett. Math. Phys. 7 (1983) 129–136.

[22] V.K. Mel’nikov, Integration method of the Korteweg-de Vries equation with a self-consistent source, Phys. Lett. A 133 (1988) 493–496.

[23] V.K. Mel’nikov, Exact solutions of the Korteweg-de Vries equation with a self-consistent source, Phys. Lett. A 128 (1988) 488–492.

[24] P. Painlevé, Mémoire sur les équations différentielles dont l’intégrale générale est uniforme, Bull. Soc. Math. France 28 (1900) 201–261.

[25] E. Pinney, The nonlinear differential equation $y''(x) + p(x)y(x) + c/y^3(x) = 0$, Proc. Amer. Math. Soc. 1 (1950) 681–681.

[26] V.E. Zakharov and E.A. Kuznetsov, Multi-scale expansions in the theory of systems integrable by the inverse scattering transform, Physica D 18 (1986) 455–463.

[27] Y. Zeng, W. Ma and Y. Shao, Two binary Darboux transformations for the KdV hierarchy with self-consistent sources, J. Math. Phys. 42 (2001) 2113–2128.