A novel mechanism for dynamical generation of elementary fermion mass: lattice evidence

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Abstract

We establish from first principles the existence of a non-perturbative mechanism leading to elementary particle mass generation in models with gauge fields, fermions and scalars if an exact invariance forbids power divergent fermion masses and fermionic chiral symmetries broken at UV scale are maximally restored. We use lattice simulations to show that in the Nambu–Goldstone phase a fermion mass term, unrelated to the Yukawa operator, is dynamically generated. In models with electro-weak interactions weak boson masses are also generated opening new scenarios for beyond the Standard Model physics.
1 Introduction

The Standard Model (SM) of particle physics, in spite of its impressive success in describing a plethora of processes mediated by electro-weak (EW) and strong interactions, is widely believed to represent only an effective low energy theory as it neither accounts for dark matter and the quantum aspects of gravity nor provides enough CP-violation for baryogenesis. Within the SM the masses of fermions and EW gauge bosons are described in terms of a well established symmetry breaking pattern \cite{1}. But since the Higgs mass and self-interaction as well as the fermion-to-Higgs Yukawa couplings are unrelated free parameters the SM is by construction unable to shed light on the problems of EW scale naturalness \cite{2} and fermion mass hierarchy \cite{3}.

There have been numerous attempts of building phenomenologically viable models where the EW scale is stable under quantum corrections, either because the basic theory enjoys an approximate symmetry larger than in the SM or because the EW and Higgs mass scales are related in a fixed way to a fundamental dynamical scale. Examples of the first kind of approaches are given by the many models based on SuperSymmetry (SUSY) \cite{4,5}, which, besides having the problem of explaining SUSY breaking, are presently disfavoured owing to the experimental exclusion of SUSY particles with mass up to few TeV \cite{6}. Approaches of the second kind typically assume the existence of some new interaction that gets strong at energies at or above the EW scale, as well as of heavy fermionic particles subjected to it. The original TechniColor idea \cite{7,8} could nicely explain the EW boson masses, but attempts of reproducing heavy fermion masses in Extended TechniColor (ETC) models \cite{9,10} face severe problems to comply with experimental constraints on flavour changing neutral currents, even in subsequently developed Walking ETC model variants \cite{11,12,13,14}. In such frameworks the Higgs boson is a composite state formed owing to the beyond SM strong force and gets a mass quite smaller than the new physics scale. This scale, in view of recent EW precision analyses \cite{15}, should be lying above 5-10 TeV. Other ways to comply with experimental constraints and possibly address the flavour hierarchy problem are represented by the Partially Composite Higgs models \cite{16,17} or by models with extra dimensions \cite{18,19}.

At variance with previous attempts to solve the open problems of the SM, a novel, intrinsically non-perturbative mechanism for \textit{elementary fermion mass generation}, was conjectured in Ref. \cite{20}. This mechanism is expected to be at work in non-Abelian gauge models where (as usual) 1) chiral transformations acting on fermions and scalars are exact symmetries, but (deviating from common assumptions) 2) purely fermionic chiral symmetries undergo an explicit breaking at the UV cutoff scale. When bare parameters are “naturally” tuned so as to minimize fermion chiral breaking, in the effective Lagrangian (EL) \cite{21} no Yukawa term occurs, but operators of non-perturbative origin that violate fermion chiral symmetries, among which a fermion mass term, are \textit{conjectured} to appear, if the scalar potential is such that the theory lives in its Nambu–Goldstone (NG) phase. Once EW interactions are introduced, the same mechanism also yields massive $W^\pm$, $Z^0$ bosons and a composite Higgs boson in the $W^+W^-$, $Z^0Z^0$, or $tt$ channel \cite{22}. A similar result was claimed long ago \cite{23} by advocating an admittedly non-natural mechanism for fermion mass generation with a somewhat problematic formulation of the composite Higgs framework \cite{24}.

In this paper by means of pioneering lattice simulations (lacking analytical methods) the actual occurrence of the non-perturbative mass generation mechanism of Ref. \cite{20}, within the simplest (“toy”, yet highly non-trivial) $d = 4$ gauge model where it could take place, is for the first time established from first principles. In section \ref{sec:2} we summarize the theoretical background of the mass generation mechanism within the simple model we investigate. In section \ref{sec:3} we present our lattice simulation setup and the numerical evidence for the dynamical generation of elementary fermion mass. In section \ref{sec:4} we draw our conclusions and discuss how the established mechanism of elementary particle mass generation can be exploited to build realistic beyond the SM models.
2 Mass generation in a toy model

Besides non-Abelian gauge bosons and Dirac fermions, scalar fields are included in the model to have an explicit breaking of fermion chiral transformations together with an extended chiral symmetry forbidding power divergent fermion mass corrections. The field content is thus

- an SU(3) gauge field, $A_μ^c$ ($c = 1, 2, ..., 8$), with bare (renormalized) coupling $g_0$ ($g_S$),
- one Dirac fermion doublet, $Q = (u, d)^T$, transforming as a colour triplet under SU(3),
- one complex scalar doublet, $\varphi = (\varphi_0 + i\varphi_3, -\varphi_2 + i\varphi_1)^T$, invariant under SU(3).

Adopting the $2 \times 2$ matrix notation $\Phi = [\varphi | -i\tau^2\varphi^*]$, the toy model Lagrangian, $\mathcal{L}_{\text{toy}}(Q, A, \Phi)$, takes the form

$$\mathcal{L}_{\text{toy}} = \mathcal{L}_k(Q, A, \Phi) + \mathcal{V}(\Phi) + \mathcal{L}_W(Q, A, \Phi) + \mathcal{L}_Y(Q, \Phi),$$

with $\mathcal{L}_k$ and $\mathcal{V}$ representing standard kinetic terms and scalar potential. The model has a hard UV cutoff $\Lambda_{\text{UV}} \sim b^{-1}$. In $\mathcal{L}_{\text{toy}}$ there are also a Yukawa term, $\mathcal{L}_Y(Q, \Phi) = \eta(\bar{Q}_L\Phi Q_R + \bar{Q}_R\Phi^\dagger Q_L)$, and a non-standard term

$$\mathcal{L}_W(Q, A, \Phi) = \frac{\mu^2}{2}\bar{\rho}(\bar{Q}_L\gamma^\mu\Phi D_\mu Q_R + \bar{Q}_R\gamma^\mu\Phi^\dagger D_\mu Q_L).$$

The term $\mathcal{L}_W$, being a $\Lambda_{\text{UV}}^{-2} \times d = 6$ operator, leaves the model power-counting renormalizable [20], exactly like it happens for the Wilson term in lattice QCD [25, 26], but induces a breaking of the purely fermionic chiral symmetries. The combined effect of $\mathcal{L}_W$ and strong interactions in the NG phase will turn out to have a key impact on the form of the EL (the generating functional of proper vertices [21]) describing the renormalized model at the quantum level. After renormalization, $\Lambda_{\text{UV}}$ can be taken much larger than any physical scale, either external momenta or $\Lambda_S$, the latter denoting the renormalization group invariant (RGI) scale of the theory.

Among other symmetries, the Lagrangian [1] is invariant under the global transformations ($\Omega_{L/R} \in \text{SU}(2)$)

$$\chi_L \times \chi_R = [\tilde{\chi}_L \times (\Phi \rightarrow \Omega_L \Phi)] \times [\tilde{\chi}_R \times (\Phi \rightarrow \Omega_R^\dagger \Phi)],$$

$$\tilde{\chi}_L/R : Q_{L/R} \rightarrow \Omega_{L/R} Q_{L/R}, \quad \bar{Q}_{L/R} \rightarrow \bar{Q}_{L/R} \Omega_{L/R}^\dagger.$$  

No power divergent fermion mass term $\sim \Lambda_{\text{UV}}(\bar{Q}_L Q_R + \bar{Q}_R Q_L)$ can be generated as it is not $\chi_L \times \chi_R$ invariant.

2.1 Wigner phase and fermion chirality restoration

The Lagrangian [1] is not invariant under the purely fermionic chiral transformations $\tilde{\chi}_L \times \tilde{\chi}_R$. However, it was shown in [20] that in the phase with positive renormalized squared scalar mass ($\mu^2 > 0$), where the $\chi_L \times \chi_R$ symmetry is realized à la Wigner, a critical value of the Yukawa coupling, $\eta_{\text{cr}}$, exists at which up to $O(b^2)$ corrections the effective Yukawa term vanishes. The renormalized Schwinger–Dyson equations (SDE), say for the $\tilde{\chi}_L$ transformations (no sum over $i = 1, 2, 3$), read

$$\partial_\mu (Z_{ji} \tilde{D}_{\mu}^L(x) \hat{O}^j(0)) \equiv (\bar{\eta} - \eta) \langle (\tilde{D}_{\mu}^L(x) \hat{O}^j(0)) + O(b^2) \rangle,$$

$$\tilde{D}_{\mu}^L = \bar{Q}_L \frac{\tau^i_\mu}{2} \Phi Q_R - \bar{Q}_R \Phi^\dagger \frac{\tau^i_\mu}{2} Q_L.$$  

where $|x| \gg b$ and a $\tilde{\chi}_L$-covariant operator $\hat{O}^j$ has been inserted. The current $\tilde{J}_{\mu}^L$ is detailed in [20]. Owing to parity, similar SDE hold for the right-handed transformations $\tilde{\chi}_R$. At the value
\[ \eta = \eta_{\text{cr}}(g_0^2, \rho, \lambda_0) \] that solves the equation \( \eta - \tilde{\eta}(\eta; g_0^2, \rho, \lambda_0) = 0 \) the SDEs take the form of Ward–Takahashi identities (WTI) and the fermionic chiral transformations \( \tilde{\chi}_L \times \tilde{\chi}_R \) become approximate symmetries \[ \tilde{\mathcal{L}}_{\text{toy}} \] of the model \[ (1) \]. In Eq. \( (5) \) the dimensionless coefficient \( \tilde{\eta} = \tilde{\eta}(\eta; g_0^2, \rho, \lambda_0) \) stems from the mixing of the \( \tilde{\chi}_L \)-variation of \( \tilde{\mathcal{L}}_W \) with the variation of \( \tilde{\mathcal{L}}_Y \), while \( Z_j \) is a multiplicative renormalization factor which at \( \eta = \eta_{\text{cr}} \) is free from logarithmic UV divergencies. In the Wigner phase the elementary particle content (gluons, fermions, four mass degenerate real scalars) and the symmetries of the model constrain the form of the EL so that its \( d \leq 4 \) piece is closely analogous to the expression of \( \tilde{\mathcal{L}}_{\text{toy}} \) and reads

\[
\Gamma_{4W}^W \equiv \Gamma_{\mu_2^0 > 0} = \Gamma_k(A, Q, \Phi) +
+ \frac{1}{4} (FF) + \hat{Q}_L \mathcal{P} Q_L + \hat{Q}_R \mathcal{P} Q_R + \frac{1}{2} \text{tr} [\partial_\mu \Phi^i \partial_\mu \Phi],
\]

where \( \hat{\mathcal{V}}(\Phi) \) is the effective scalar potential.

The Wigner phase is hence well suited to determine the peculiar \( \eta \) value, \( \eta_{\text{cr}} = \eta_{\text{cr}}(g_0^2, \rho, \lambda_0) \), at which the effective Yukawa term disappears from Eq. \( (7) \). We stress that, neglecting \( O(b^2) \) artifacts, \( \tilde{\eta} \) and \( \eta_{\text{cr}} \) are independent of the subtracted scalar mass \( \mu_2^0 \) (see appendix A) and thus equal in the Wigner and NG phase. From Eq. \( (5) \) it is clear that \( \eta_{\text{cr}} \) can be determined, e.g. as the value of \( \eta \) where, for \( i = 1, 2, 3 \) and \( x \neq 0 \), one has

\[
\partial_\mu \langle \tilde{A}_\mu^i (x) \tilde{D}_P^i (0) \rangle = 0,
\]

\[
\tilde{A}_\mu^i = \tilde{J}_L^i - \tilde{J}_R^i,
\]

\[
\tilde{D}_P^i = \hat{Q}_L \left\{ \Phi, \frac{\tau^i}{2} \right\} Q_R - \hat{Q}_R \left\{ \frac{\tau^i}{2}, \Phi^i \right\} Q_L.
\]

An equivalent but statistically less noisy method is discussed below (see section 3). It should also be noted that the existence of an \( \eta_{\text{cr}}, \) where the \( \tilde{\chi}_L \times \tilde{\chi}_R \) transformations become a symmetry of the theory up to \( O(b^2) \) corrections, is a general property of the Wigner phase independently of the specific form of \( \tilde{\mathcal{L}}_W \) chosen in Eq. \( (2) \). Changing \( \tilde{\mathcal{L}}_W \), owing to renormalizability and universality, would merely modify the values of \( \eta_{\text{cr}} \) and \( Z_j \) as well as the \( O(b^2) \) artifacts.

### 2.2 Nambu–Goldstone phase and “NP anomaly”

Most interesting is the situation where \( \mu_2^0 < 0 \) because \( \mathcal{V}(\Phi) \) has a double-well shape with \( \langle \Phi^i \Phi \rangle = v^2 \mathbb{1}, v \neq 0 \), and the \( \chi_L \times \chi_R \) symmetry is realized \( \tilde{\text{a}} \ la \) NG. In large volume, under an infinitesimal symmetry breaking perturbation, the \( \chi_L \times \chi_R \) symmetry will be spontaneously broken to \( \text{SU}(2) \). Moreover, owing to \( v \neq 0 \), at \( \eta = \eta_{\text{cr}} \) the residual \( O(b^2) \) and \( \tilde{\chi}_L \times \tilde{\chi}_R \) violating action terms polarize the vacuum that is degenerate as a result of the dynamical \( \tilde{\chi}_L \times \tilde{\chi}_R \) spontaneous breaking ensuing from strong interactions.

The realization of the \( \chi_L \times \chi_R \) invariance \( \tilde{\text{a}} \ la \) NG has a key impact on the low energy physics. Three elementary massless Goldstone bosons appear in the spectrum and the corresponding fields must be included in the EL, where they also occur in interaction terms. In Ref. \[ 20 \] it was argued that at \( \eta = \eta_{\text{cr}} \) in the NG phase the EL describing the model \[ (1) \] should include \( \tilde{\chi}_L \times \tilde{\chi}_R \) violating terms of NP origin, among which a fermion mass term.

The conjecture of Ref. \[ 20 \] can be checked by studying the SDE associated with the \( \tilde{\chi}_L \times \tilde{\chi}_R \) transformations (see e.g. Eq. \( (5) \)) taken between low energy states in the NG phase. Since \( \tilde{\mathcal{L}}_{\text{toy}} \) is
parity invariant, it is convenient to consider an axial SDE and evaluate (at $x_0 \gg b$) the effective PCAC mass, i.e. the matrix element ratio
\[ \frac{Z_A}{Z_P} m_{AWI} = \frac{Z_A \sum_x \delta_0(A^\dagger_0(x)P^i(0))}{2Z_P \sum_x (P^i(x)P^i(0))} \bigg|_{\eta_{cr}}, \quad P^i = \tilde{Q} \frac{\tau^i}{2} Q, \] (11)
where $Z_A = Z_f|_{\eta_{cr}}$ and $Z_P$ are renormalization factors. If $\tilde{\chi}_L \times \tilde{\chi}_R$ violating terms of NP origin were absent in the EL, at $\eta = \eta_{cr}$ one should find $\frac{Z_A}{Z_P} m_{AWI} \to 0$ as $b \sim \Lambda_{UV}^{-1} \to 0$. Actually our lattice simulations (see section 3) give strong numerical evidence that $\frac{Z_A}{Z_P} m_{AWI}$ does not vanish in the continuum limit. We outline below our theoretical interpretation in the EL framework of this striking result.

In the NG phase the EL describing the model (1) can be expressed in terms of effective fermion, gauge and scalar fields. The latter, in view of $v \neq 0$, are conveniently rewritten [25] in terms of Goldstone ($\zeta_{1,2,3}$) and massive ($\zeta_0$) scalar fields
\[ \Phi = RU, \quad R = (v + \zeta_0), \quad U = \exp[i\eta^{-1}r^k\zeta_k], \] (12)
where $U$ is a dimensionless effective field transforming as $U \to \Omega_L U \Omega_R^\dagger$ under $\chi_L \times \chi_R$. It represents the exponential Goldstone boson map and makes sense only if $v \neq 0$. At $\eta = \eta_{cr}$ the $d \leq 4$ piece of the full EL, $\Gamma^{NG}$, is expected [20] to read (see also Eq. (7))
\[ \Gamma^{NG}_4 = c_2 \Lambda^2 S \text{tr} (\partial_\mu U^\dagger \partial_\mu U) + c_1 \Lambda_S [Q_L U Q_R + h.c.] + \tilde{c} \Lambda_S R \text{tr} (\partial_\mu U^\dagger \partial_\mu U) + \Gamma_{\rho < 0} + O(\Lambda^2_S/\nu^2), \] (13)
where the term $\propto c_1$ describes NP breaking of $\tilde{\chi}_L \times \tilde{\chi}_R$ and provides an effective mass for the fermion fields. In fact, when $U$ is expanded around the identity, one gets
\[ c_1 \Lambda_S [Q_L U Q_R + \bar{Q}_R U^\dagger Q_L] = c_1 \Lambda_S \bar{Q}Q[1 + O(\zeta/v)], \] (14)
i.e. a fermion mass term plus a host of more complicated, non-polynomial $\bar{Q} - \zeta_{1,2,3}$ particles $- Q$ interactions.

The NP term (14) was conjectured in [20] to arise dynamically from the interplay of strong interactions and $\tilde{\chi}_L \times \tilde{\chi}_R$ breaking at the UV cutoff scale in the basic $\mathcal{L}_{toy}$ theory [1]. The occurrence of such a $\tilde{\chi}_L \times \tilde{\chi}_R$ violating term in the EL (13) is essential to reproduce the non-zero value that we find for the correlator $Z_A \sum_x \delta_0(A^\dagger_0(x)P^i(0))|_{\eta = \eta_{cr}}$ in Eq. (11), despite the fact that the local operators $\tilde{A}^\dagger_0$ and $P^i$ transform differently under $\tilde{\chi}_L \times \tilde{\chi}_R$. The coefficient $c_1$ in Eq. (14) has been argued in Ref. [20] to be an $O(g_3^2)$, odd function of $\rho$, with a $\lambda_0$ dependence arising only at high loop orders. As for its dependence on the scalar squared mass, $c_1$ is expected to stay finite in the limit $-\bar{\mu}^2 > \Lambda^2_S$ (which is of phenomenological interest, see Refs. [20, 22]) and to be non-zero only for $\bar{\mu}^2 < 0$. Indeed in our interpretation the NP term (14) arises from dynamical $\tilde{\chi}_L \times \tilde{\chi}_R$ spontaneous breaking, which is in turn effective only in the NG phase where the vacuum gets polarized owing to $\tilde{\chi}_L \times \tilde{\chi}_R$ breaking effects $\propto b^2 v$ in the fermionic action.

A proper understanding of the NP terms in the EL expression (13) requires considering what is the natural extension of $\tilde{\chi}_L \times \tilde{\chi}_R$ symmetry in the presence of weak interactions [22]. In that context one finds that maximal restoration of the (extended) $\tilde{\chi}_L \times \tilde{\chi}_R$ symmetry, besides constraining $\eta$ and $\rho$ (see Eq. (2)) to their critical values, entails the vanishing of the coefficient $\tilde{c}$. Thus in $\Gamma^{NG}_4$ (see Eq. (13)) only the NP terms with coefficients $c_1$ and $c_2$ remain, which are responsible for the dynamical mass of fermions and weak bosons, respectively [20, 22].
In general each NP term in $\Gamma^{NG}$ corresponds to an operator violating $\tilde{\chi}_L \times \tilde{\chi}_R$ in the renormalized SDE. At $\eta = \eta_{cr}$ all such NP terms occurring in the r.h.s. of the $\tilde{\chi}_L \times \tilde{\chi}_R$ SDE, if non-zero, must be RGI, as the l.h.s. of the SDE is. Indeed conservation up to $O(b^2)$ of the $Z_{\mu}^{J^{L(R)}}$ currents makes the l.h.s. of the $\tilde{\chi}_L \times \tilde{\chi}_R$ SDE scale invariant independently of $\mu_0^2$. The full NG phase EL, $\Gamma^{NG} \supset \Gamma_4^{NG}$ contains of course an infinite number of local terms of arbitrarily high dimension, among which those of NP origin that break the approximate $\tilde{\chi}_L \times \tilde{\chi}_R$ symmetry. The occurrence of these NP and RGI terms in the EL will be referred to as a “NP anomaly” in the restoration of $\tilde{\chi}_L \times \tilde{\chi}_R$ symmetry.

3 Lattice study and results

We describe the main steps and results of our numerical study of the model (1) in a lattice regularization consistent with the exact $\chi_L \times \chi_R$ symmetry. The technical details of our lattice setup are discussed in the appendix B. Lattice $\chi_L \times \chi_R$ invariance implies, besides certain relations between renormalization constants (e.g. $Z_{\chi} = Z_{\psi}$), that discretization errors are of $O(b^2)$ only ($n$ integer), as we prove in the appendix C. Since the arguments of Ref. [20] imply that the “NP anomaly” leading to elementary fermion mass generation (if any) occurs even if virtual fermion effects are neglected, in this first investigation we decided to work in the quenched fermion approximation [29]. Quenching entails key numerical simplifications, as in this case scalar and gauge fields can be generated and renormalized independently from each other. In the appendix D further details on our data analysis are given compared to what is discussed in the present section.

For a given choice of bare gauge coupling ($\beta = 6/g_0^2$), scalar potential parameters ($\lambda_0$, $m_0^2 - m_{cr}^2 = \mu_0^2/Z_{m_0}^2$) and strength ($\rho$) of the $L_W$ term, we tune the bare coupling, $\eta$, so as to restore the fermionic chiral symmetry $\tilde{\chi}_L \times \tilde{\chi}_R$. This task is conveniently carried out in the Wigner phase by looking for the value of $\eta$ at which

$$r_{AWI}(\eta_{cr}, \rho, \lambda_0, g_0^2) = \mathcal{N}(x_0, y_0; \eta, \rho, \lambda_0, g_0^2) = 0, \quad (15)$$

$$\mathcal{N}(x_0, y_0; \eta, \rho, \lambda_0, g_0^2) = b^6 \sum_{x, y} \langle P^1(0) | \partial_0^{FW} \tilde{A}_0^{1,BW} | x \rangle \varphi_0(y) \rangle, \quad (16)$$

$$\mathcal{D}(x_0, y_0; \eta, \rho, \lambda_0, g_0^2) = b^6 \sum_{x, y} \langle P^1(0) | \tilde{D}_P(x) \varphi_0(y) \rangle, \quad \varphi_0 = \frac{1}{2} \text{Tr}[\Phi]. \quad (17)$$

Here $\partial_0^{FW}$ is a forward lattice derivative and $\tilde{A}_0^{1,BW}$ the backward one-point-split lattice version of $\tilde{Q}\bar{\gamma}_0\gamma_5\tilde{Q}$. $\tilde{D}_P$ and $P^1$ are given in Eqs. (10) and (11), respectively. In the correlators $\mathcal{N}(x_0, y_0; \eta, \rho, \lambda_0, g_0^2)$ and $\mathcal{D}(x_0, y_0; \eta, \rho, \lambda_0, g_0^2)$ the time distances $x_0$ and $y_0 - x_0$ can be separately optimized, e.g. in order to isolate the lowest-lying pseudoscalar (PS) meson and one-Φ-particle states. In Fig. 1 we show the behaviour of $r_{AWI}$ as a function of $\eta$ at the three chosen values of the bare gauge coupling, $\beta = 5.75$, 5.85 and 5.95. In the quenched approximation these $\beta$-values correspond to lattice spacings of about 0.15, 0.12 and 0.10 fm, respectively, if we conventionally assume for the Sommer scale $r_0 = 0.5$ fm (as in the case of QCD). The resulting values of $\eta_{cr}(6/\beta, \rho, \lambda_0)|_{\rho = 1.96}$ are denoted by red squares.

In the NG phase we always work at $\eta = \eta_{cr}$ and propagate onto the observables the uncertainty on $\eta_{cr}$, which gives the dominant contribution to their error. First we compute the effective PCAC mass of Eq. (11). For practical reasons $Z_P^{-1}$ is evaluated in a hadronic scheme defined in the Wigner phase by taking $Z_P^{-1} r_0^{-2} = G_{PS}^W = \langle 0 | P^1 | \text{PS meson} \rangle^W$ at the subtraction point $M_{PS}$. As for $Z_{\tilde{A}}$, we
have exploited the equality $Z_A = Z_V$, entailed by the $\chi_L \times \chi_R$ symmetry, and actually evaluated $Z_V$, which in our setup can be obtained from an exact WTI.

Figure 1: $r_{AWI}$ as a function of $\eta$ at $\beta = 5.75$, 5.85 and 5.95. Red squares denote the values of $\eta_{cr}$ at which $r_{AWI} = 0$.

Figure 2: $2m_{AWI}^R r_0 \equiv 2r_0 m_{AWI} Z_V Z_P^{-1}$ against $(b/r_0)^2$ in the NG phase and its linear extrapolation to the continuum limit.

In Fig. 2 we plot the renormalized quantity $2r_0 m_{AWI} Z_V Z_P^{-1}$ as a function of $(b/r_0)^2$. A linear extrapolation shows that its continuum limit lies about three standard deviations away from zero. Secondly, as another check that the effective PCAC mass $Z_A m_{AWI}$ is non-zero and indicates the presence in the EL of a fermion mass term, we have computed from the $\sum_x \langle P^1(0) P^1(x) \rangle$ correlator the mass of the lowest lying pseudoscalar meson state. Our data for $r_0 M_{PS}$ are plotted in Fig. 3 as a function of $(b/r_0)^2$ together with the best fit linear extrapolation to vanishing lattice spacing. The figure shows that the continuum limit of $r_0 M_{PS}$ lies above zero by more than five standard deviations. On the other hand, if $r_0 M_{PS}$ had a vanishing continuum limit, one should see $r_0^2 M_{PS}^2$ approaching zero as $b^2 \to 0$ with a $b^4$-rate. On a $(b/r_0)^4$ scale our $r_0^2 M_{PS}^2$ data lie very close to the
continuum limit (see appendix D.2) and are incompatible with the “no mechanism” hypothesis.

Finally we checked that at $\eta = \eta_{cr}(\rho)$ the magnitude of $Z_{AWI} m_{AWI}$ (and of $M_{PS}^2$) is actually controlled by the strength of fermionic chiral breaking at the $b^{-1} \sim \Lambda_{UV}$ scale as measured by the coefficient $\rho$ of the $L_W$ term [2]. An increase of $\rho$ by a factor 1.5 yields an increase of $Z_{AWI} m_{AWI}$ by a factor of about 2.4 (see appendix D.3) in good agreement with expectations from [20] and the section 2. For a correct interpretation of this finding one should notice that in more realistic models supporting the mass generation mechanism under discussion and accounting for weak interactions, parameters (like $\rho$) that control the strength of fermionic chiral breaking and hence the NP mass terms are not really free, rather they are all constrained by the conditions of maximal restoration of $\tilde{\chi}_L \times \tilde{\chi}_R$ symmetry that are appropriate in that context (see Ref. [22] and appendix E).

4 Conclusions and outlook

In this paper by means of lattice simulations we have established from first principles the dynamical generation of an elementary fermion mass term as a “non-perturbative anomaly” in the effective Lagrangian of a non-Abelian SU(3) gauge model (see Eq. (1)). The model contains a doublet of strongly interacting fermions coupled to a colourless complex scalar doublet via renormalizable Yukawa and higher dimensional ($L_W$ in Eq. (2)) operators. Once the bare parameters are chosen so as to ensure maximal restoration of fermion chiral symmetries and be in the spontaneously broken phase of the exact symmetry acting on fermions and scalars, a renormalization group invariant fermion mass of the order of the $\Lambda$ parameter ($\Lambda_S$) is generated.

This result represents a ground-breaking progress in quantum field theory as it establishes the occurrence of a non-perturbative obstruction (“anomaly”) to the recovery of broken fermionic chiral symmetries, which is responsible for the dynamically generated fermion mass term. The latter, being unrelated to the Yukawa operator, is expected to become independent of the scalar field expectation value ($v$) in the limit $\Lambda_S/v \to 0$. A numerical check of this point is left for a future study.

From a phenomenological viewpoint, EW interactions can be included in the model by promoting the exact $\chi_L \times U(1)_Y \subset \chi_L \times \chi_R$ invariance to a gauge symmetry without running into the
problem of tree-level flavour changing neutral currents. As in this case the global $\tilde{\chi}_L \times \tilde{\chi}_R$ transformations act both on fermions and weak gauge bosons, one can show that i) the requirement of maximal $\tilde{\chi}_L$ restoration leads to stringent constraints on (the sum of the squares of) all the $\tilde{\chi}$ breaking action parameters and ii) a unique mechanism generates NP mass terms for fermions and weak gauge bosons \[22\].

These results provide a very interesting starting point to build beyond the SM models based on our novel mechanism for generating the mass of elementary particles. Since all masses will parametrically be of the order of the theory RGI scale, the latter must be much larger than $\Lambda_{QCD} \sim 300$ MeV, if the mass of the top quark or of the EW gauge bosons have to be reproduced. This observation suggests the existence of a new non-Abelian gauge interaction that gets strong at the scale $\Lambda_T \gg \Lambda_{QCD}$, as well as of new fermionic elementary particles with an $O(\Lambda_T)$ NP mass. First very rough estimates \[20, 22\] seem to hint at $\Lambda_T = O(a$ few TeV). As the condition of $\tilde{\chi}_L$ restoration entails the decoupling of the isosinglet component of the basic scalar field $\Phi$, one ends up with models of the Composite Higgs \[16, 17\] type, where the Higgs boson is a bound state \[20, 22\] in the $WW + ZZ$ channel formed owing to the new strong interaction.

In the theoretical framework sketched above we see also a chance of understanding the observed hierarchy of fermion masses. Denoting by $c_{1,f}/\Lambda_T$ the dynamical mass of each SM fermion $f$ and by $g$ the gauge coupling of the strongest gauge interaction which the right-handed component of the $f$–fermion field is subjected to, it turns out \[20\] that $c_{1,f}$ is $O(g^4)$ for the heaviest fermion generation and possibly of higher order for the other generations. In this way one can understand e.g. the top to $\tau$ mass ratio \[22\]. Further remarks about extending the model \[1\] to a phenomenologically sensible theory can be found in appendix \[E\].

In summary, combining the condition of maximal restoration of the fermionic chiral symmetries broken explicitly by the UV regularization (a weak form of ’t Hooft naturalness) with the assumption (common to all Composite Higgs approaches) of a new non-Abelian gauge force with $\Lambda_T = O(a$ few TeV) and owing to the non-perturbative anomaly we have numerically demonstrated here, a novel and conceptually simple way of giving mass to elementary fermions and EW gauge bosons has been found. Our result can open new promising horizons for the long standing efforts to relate the EW scale to the scale of new physics, at which new resonances should be seen in accelerator experiments, and to understand the couplings and the nature of the observed Higgs boson.

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Appendices

A  $m^2_\phi$ independence of $\eta_{cr}$

The critical value of the bare Yukawa coupling, $\eta_{cr}$, is defined as the value of $\eta$ that solves the equation $\eta - \tilde{\eta}(\eta; g^2_0, \rho, \lambda_0) = 0$. The dimensionless coefficient function $\tilde{\eta}(\eta; g^2_0, \rho, \lambda_0)$ characterizes the UV power divergent mixing of the $d = 6$ operator $O^d_6$ with the $d = 4$ density $\bar{D}^4_L$. Here $\bar{D}^4_L$ and $O^d_6$ stand for the operators that are obtained from the variation of the $L_Y$ and $L_W$ terms in Eq. (18), respectively, under an isotriplet (index $i$) $\bar{\chi}_L$ transformation. The expression of $\bar{D}^4_L$ is given in Eq. (6), while the detailed form of $O^d_6$ can be found in Sect. III D of Ref. [20].

In what follows we explain why the dimensionless mixing coefficient $\tilde{\eta}$, and consequently also $\eta_{cr}$, is independent of the scalar squared mass parameter $m^2_\phi$, or more precisely of its subtracted counterpart $\mu^2_\phi \equiv m^2_\phi - m^2_{cr}$ (we neglect here a dimensionless renormalization factor, as it is irrelevant for our arguments). This property is a consequence of renormalizability, locality and straightforward dimensional considerations. We start by observing that in a correlator of renormalized fields with one insertion of $O^d_6$ the leading UV divergence is given by quadratically divergent Feynman diagrams and is insensitive to the shift $\mu^2_\phi \rightarrow \mu^2_\phi + \delta m^2$. Indeed, if we imagine expanding in $\delta m^2$ these diagrams, we see that for dimensional reasons the terms in the expansion beyond the first one will be necessarily suppressed by at least one relative power of $b^2\delta m^2$. Hence they do not contribute to $\tilde{\eta}$.

At the NP level one might worry about the possibility that $\mu^2_\phi$ enters $\tilde{\eta}$ scaled by $\Lambda^2_s$. We now prove that the presence of either positive or negative powers of the ratio $\mu^2_\phi/\Lambda^2_s$ in $\tilde{\eta}$ would violate the general principles of renormalizability in Quantum Field Theories, which are assumed to hold in our $L_{toy}$ model.

1) **Positive mass powers** - Terms of the kind $\mu^2_\phi/\Lambda^2_s$, i.e. terms displaying inverse powers of the RGI scale of the theory, cannot appear in a renormalizable field theory. Apart from the fact that this kind of dependence would completely ruin the Feynman diagrams power counting, from the argument given above and renormalizability to the NP level in $g^2_0$ it follows that corrections $\sim \mu^2_\phi/\Lambda^2_s$ to $\tilde{\eta}$ can never occur.

In fact, one explicitly checks that any correlator of renormalized fields with one insertion of $O^d_6$ can be expanded in powers of $\eta$, $\rho$ and $\lambda_0$ yielding a series where each term corresponds to a diagram consisting only of renormalized effective vertices with external legs and/or scalar lines sewn together with a number of scalar field propagators. In all terms of this series the $\mu^2_\phi$ dependence comes only from the scalar propagators, since the renormalized effective vertices, which by construction here resum diagrams with gluon and fermion (but no scalar) internal lines, are independent of $\mu^2_\phi$, up to immaterial $O(b^2)$ artifacts. Hence, under the shift $\mu^2_\phi \rightarrow \mu^2_\phi + \delta m^2$, a correction to the correlator of the kind $\delta m^2/\Lambda^2_s$ does not occur. Indeed, were such a correction present, it could only arise as the product of a relative $O(b^2\delta m^2)$ correction from the scalar propagators times an $O(b^{-2}\Lambda^2_s)$ contribution from the $\mu^2_\phi$-independent renormalized effective vertices. But, as the renormalized effective vertices are by definition UV finite, this is manifestly impossible, thereby ruling out terms of the kind $\mu^2_\phi/\Lambda^2_s$ in the correlator.

2) **Negative mass powers** - Terms of the kind $\Lambda^2_s/\mu^2_\phi$, i.e. terms displaying inverse powers of mass parameters cannot show up in mixing coefficients. In fact, IR divergencies, revealed in the limit of vanishingly small mass, can only appear in matrix elements and not in the expression of the renormalized operators in terms of the bare ones. This is the consequence of the fact that one can determine the mixing coefficients by only using correlators at non-exceptional momenta that
do not develop IR divergencies in the massless limit.

| $\eta_{cr}^A$ | $\eta_{cr}^B$ | $b^2 \mu_\phi^2$ | $r_0^2 \mu_\phi^2$ |
|----------------|----------------|-----------------|-----------------|
| -1.145(6)      | -1.136(6)      | 0.0504(10)      | 1.22(3)         |
| -1.138(2)      | -1.132(2)      | 0.0171(10)      | 0.41(3)         |

Table 1: The values of $\eta_{cr}^A$ and $\eta_{cr}^B$ at fixed bare couplings $g_\sigma^2 = 6/5.95$, $\rho = 1.96$, $\lambda_0 = 0.6022$ for the values of $\mu_\phi^2 \equiv (m_\phi^2 - m_\sigma^2)$ given in the 3rd (4th) column in lattice ($r_0$) units.

Finally \(^1\) as a check of consistency, we repeated the numerical computation of $\eta_{cr}$ at $\beta = 5.95$ reducing the subtracted squared scalar mass, $r_0^2 \mu_\phi^2 \equiv r_0^2 (m_\phi^2 - m_\sigma^2)$, from 1.22 to 0.41, see Table 1. For this smaller value of the subtracted scalar mass we worked at the same bare parameters and lattices as employed for the larger one, see the block of Tab. 3 corresponding to $\beta = 5.95$. Adopting the same analysis procedure as discussed in appendix D.1, we find that the change in the estimated value of $\eta_{cr}$ observed upon reducing $\mu_\phi^2$ by a factor of three is not significant within our statistical errors. Moreover the change in the central $\eta_{cr}$ value obtained with the two different values of $\mu_\phi^2$ in Table 1 but the same choice of time plateau for $r_{AWH}$ is at a given $\mu_\phi^2$ as small as the central value of $\eta_{cr}^A - \eta_{cr}^B$, which is certainly an $O(b^2)$ lattice artifact in the computation of $\eta_{cr}$. These findings altogether are nicely consistent with $\eta_{cr} = \eta_{cr}(g_\sigma^2, \rho, \lambda_0)|_{g_\sigma^2 = 6/5.95, \rho = 1.96, \lambda_0 = 0.6022}$ being independent of the value of $\mu_\phi^2$ up to $O(b^2)$ lattice artifacts.

B Lattice formulation

Lattice simulations of models with gauge, fermion and scalar fields are rather uncommon in the literature. See, however, Ref. \([30]\) for a quenched study where, unlike here, the scalar field belongs to the fundamental representation of the SU(2) gauge group but no $\bar{\chi}_L \times \bar{\chi}_R$ symmetry violating terms are included.

For our purposes it is crucial to adopt a lattice formulation preserving the exact $\chi_L \times \chi_R$ symmetry, while including terms breaking the purely fermionic $\bar{\chi}_L \times \bar{\chi}_R$ invariance. We have thus chosen the lattice regularized action of the model \([1]\) to be of the form

$$S_L = b^4 \sum_x \left\{ \mathcal{L}_k^g[U] + \mathcal{L}_k^\chi(\Phi) + \mathcal{V}(\Phi) + QD_L[U, \Phi] \right\},$$

$$\mathcal{L}_k^g[U] : \text{SU}(3) \text{ plaquette action},$$

$$\mathcal{L}_k^\chi(\Phi) + \mathcal{V}(\Phi) = \frac{1}{2} \text{tr} [\Phi^\dagger (-\partial_\mu \partial_\nu) \Phi] + \frac{m_\phi^2}{2} \text{tr} [\Phi^\dagger \Phi] + \frac{\lambda_0}{4} (\text{tr} [\Phi^\dagger \Phi])^2,$$

where $\Phi = \varphi_0 \mathbb{1} + i \varphi_j \tau^j$ and the symbol $U$ denotes the set of gauge links. Setting $F(x) \equiv [\varphi_0 \mathbb{1} + i \gamma_5 \tau^j \varphi_j](x)$ the lattice Dirac operator reads

$$(D_L[U, \Phi]Q)(x) = \gamma_\mu \tilde{\nabla}_\mu Q(x) + \eta F(x) Q(x) +$$

$$- b^2 \rho \frac{1}{2} F(x) \tilde{\nabla}_\mu \tilde{\nabla}_\mu Q(x) +$$

$$- b^2 \rho \frac{1}{4} \left[ (\partial_\mu F(x) U_\mu(x) \tilde{\nabla}_\mu Q(x + \hat{\mu}) +$$

$$+ (\partial_\mu F(x) U_\mu^\dagger(x - \hat{\mu}) \tilde{\nabla}_\mu Q(x - \hat{\mu}) \right].$$

\(^1\)A full understanding of this paragraph requires reading of section 3 of the main text, plus possibly some details of the appendices \([3]\) and \([3]\).
Gauge covariant lattice derivatives are defined via
\begin{align}
\tilde{\nabla}_\mu f(x) & \equiv \frac{1}{2}(\nabla_\mu + \nabla^*_\mu)f(x), \\
b\nabla_\mu f(x) & \equiv U_\mu(x)f(x + \hat{\mu}) - f(x), \\
b\nabla^*_\mu f(x) & \equiv f(x) - U^*_\mu(x - \hat{\mu})f(x - \hat{\mu}).
\end{align}
(20)

The partial derivatives \( \partial_\mu \), \( \partial^*_\mu \) are the forward, backward derivatives \( \nabla_\mu \), \( \nabla^*_\mu \), respectively, with \( U_\mu = I \).

The lattice action \([18]\) describes 2 (isospin) \( \times \) 16 (doublers) fermionic flavours, even in the \( b \to 0 \) limit. In fact, the fermion Lagrangian term proportional to \( \rho \), which is a lattice regularization of the \( \mathcal{L}_W \) term of Eq. \([2]\), being the product of \( b^2 \) times a (dimension six) fermion bilinear operator with two derivatives, does not remove the doublers in the continuum limit. However, for the goals of the present quenched study the presence of doublers in the valence sector, which does not alter the \( \beta \)-function and hence preserves the asymptotic freedom of the model, is harmless. With respect to the unquenched case, the quenched approximation is expected to entail only some systematic error (typically of order 10–20\% based on Lattice QCD experience) in the value of the coefficient \( c_1 \) of the NP fermion mass term (see Eq. \([13]\)) that is possibly generated in the NG phase EL. For analogous unquenched studies a lattice Dirac operator \( D_L[U, \Phi] \) with no doublers would of course be needed, which could e.g. be built by using the domain-wall \([31]\) or overlap \([32]\) fermion kernel for two flavours with the addition of suitable Yukawa and \( \mathcal{L}_W \)-like terms.

### B.1 Fermionic regularization

Following closely Refs. \([33], [34], \) and \([35]\) for staggered fermions, one can analyze the fermion flavour content of the action \([18]\). We first rewrite its fermionic sector in terms of the field \( \chi(x) = A_x^{-1}Q(x) \) with \( A_x = \gamma_1^{x_1} \gamma_2^{x_2} \gamma_3^{x_3} \gamma_4^{x_4} \), then we pass to the so-called flavour basis
\begin{align}
q_{\alpha,a,B}(y) = \sum_\xi \bar{U}(2y, 2y + \xi)[A_\xi]_{\alpha,a}(1 - b\xi_\mu \tilde{\nabla}_\mu)\chi^B(2y + \xi),
\end{align}
(22)

where \( \alpha, a \) and \( B \), all taking values in \( \{1, 2, 3, 4\} \), denote Dirac, taste and replica indices. While the \( \chi^B(x) \) fields are defined on the fine lattice \( x_\mu = 2y_\mu + \xi_\mu, \xi_\mu = 0, 1 \) with \( \bar{U}(2y, 2y + \xi) \) denoting the average of link products along the shortest paths from \( 2y \) to \( 2y + \xi \), the flavour basis fields \( q_{\alpha,a,B}(y) \) live on the coarse lattice with coordinate \( y_\mu \) (here we have set \( b = 1 \)). In terms of these fields close to the continuum limit the lattice fermionic action reads
\begin{align}
S^F_L = \sum_{y,B} \bar{q}^B(y) \left\{ \sum_\mu (\gamma_\mu \otimes \mathbb{1})D_\mu + \eta \mathcal{F}^B(y) \right\} q^B(y) + O(b^2),
\end{align}
(23)

where for each \( B \) value the matrix \( \mathcal{F}^B(y) = \phi_0(2y)(\mathbb{1} \otimes \mathbb{1}) + s_B i \tau^i \phi_i(2y)(\gamma_5 \otimes t_5) \), \( s_{1,2} = 1 = -s_{3,4} \), has a “Dirac \( \otimes \) taste” structure with \( t_\mu = \gamma_\mu^T \) taste matrices. Unlike in the original \( Q(x) \) basis, in the flavour basis \( q(y) \) defined via Eq. \([22]\) space-time and flavour degrees of freedom are completely disentangled from each other. As the action \([23]\) is diagonal in taste and replica indices up to \( O(b^2) \), it obviously describes 32 fermion species, namely 4 replicas of the 4 tastes of the isospin doublet \( Q^T = (u, d) \).

The quark bilinears expressed in the \( Q \) basis, once summed over one \( 2^4 \) hypercube of the fine lattice, have well defined quantum numbers in the classical continuum limit, as one checks by rewriting them in the flavour \( q(y) \) basis. For example the isotriplet pseudoscalar density
\begin{align}
P^i(x) = \bar{Q}(x)\gamma_5 \tau^i Q(x), \quad x_\mu = 2y_\mu + \xi_\mu
\end{align}
(24)
summed over the hypercube \((\xi_\mu = 0, 1)\) yields in flavour basis the corresponding isotriplet, taste diagonal density
\[
\frac{1}{16} \sum_\xi P^i(2y + \xi) = \sum_{B=1}^4 s_B \bar{q}^B(y)(\gamma_5 \otimes t_5) \frac{\tau^i}{2} q^B(y) + O(b^2). \tag{25}
\]

Similarly for the lattice backward isotriplet axial current
\[
\tilde{A}^{i,BW}_\mu(x) = \frac{1}{2} \bar{Q}(x - \hat{\mu})\gamma_\mu \gamma_5 \frac{\tau^i}{2} U_\mu(x - \hat{\mu}) Q(x) + \frac{1}{2} \bar{Q}(x)\gamma_\mu \gamma_5 \frac{\tau^i}{2} U_\mu(x - \hat{\mu}) Q(x - \hat{\mu}), \quad x = 2y + \xi
\]

summing over the hypercube one finds in the flavour basis the corresponding isotriplet, taste diagonal current,
\[
\frac{1}{16} \sum_\xi \tilde{A}^{i,BW}_\mu(2y + \xi) = \sum_{B=1}^4 s_B \bar{q}^B(y)(\gamma_\mu \gamma_5 \otimes t_5) \frac{\tau^i}{2} q^B(y) + O(b^2). \tag{27}
\]

### B.2 Renormalizability and other technicalities

Beyond tree level, loop effects do not generate \(d \leq 4\) terms besides the operators that are already present in the lattice action \(18\). This follows from the symmetries (gauge, \(H(4), \chi_L \times \chi_R, C, P, T\)) of the model \(18\) which were discussed in Ref. \(20\) and are preserved by the lattice regularization. The latter is chosen so as to respect the “spectrum doubling symmetry” \(36\) of naive fermions, implying that the action \(18\) is invariant under the transformations
\[
Q(x) \rightarrow Q'(x) = e^{-ix_\mu H} M_H Q(x),
\]
\[
\bar{Q}(x) \rightarrow \bar{Q}'(x) = \bar{Q}(x) M_H^\dagger e^{ix_\mu H}, \tag{28}
\]

where \(H\) is an ordered set of four-vector indices \(H \equiv \{\mu_1, ..., \mu_h\}\), \((\mu_1 < \mu_2 < ... < \mu_h)\). For \(0 \leq h \leq 4\) there are 16 four-vectors \(\pi_H\), with \(\pi_H,\mu = \pi\) if \(\mu \in H\) or \(\pi_H,\mu = 0\) otherwise and 16 matrices \(M_H \equiv (\gamma_5 \gamma_1, ..., \gamma_5 \gamma_{16})\). The use of symmetric (gauge covariant) lattice derivatives \(\bar{\nabla}_\mu\) acting on \(Q\) in the \(L_W\) term is dictated by the need of preserving the spectrum doubling symmetry. The latter, being an exact symmetry of \(S_L\), is an invariance of the lattice effective action \(\Gamma_L[U, \Phi, Q]\). Hence all fermion species, including doublers, enter in a symmetric way in the effective action, in particular in the non-irrelevant local terms \(\sum_y \sum_B \bar{q}^B(y)(\gamma_\mu \otimes 1) D_\mu q^B(y)\) and \((\eta - \bar{\eta}) \sum_y \sum_B \bar{q}^B(y) F^B(y) q^B(y)\) (in the flavour basis).

These nice renormalization properties imply in particular that \(\eta_{cr}\), as determined from the vanishing of \(r_{AW1}\) (Eq. \(15\)), is unique for all fermion species.

As the model \(18\) is studied in the quenched approximation, problems with exceptional configurations of gauge and scalar fields supporting very small eigenvalues of \(D_L\) plague, as expected, the Monte Carlo sampling of fermionic observables. In order to avoid such problems and have a robust IR cutoff in the fermionic propagators we include in the lattice action \(18\) an additional “twisted mass” term \(37, 38\)
\[
S_L^{tm} = b^4 \mu \sum_x \bar{Q}(x) i \gamma_5 \tau^3 Q(x). \tag{29}
\]

This term does not spoil power counting renormalizability, it only breaks in a soft way the \(\chi_L \times \chi_R\) symmetry and can thus be safely removed at the end of the calculations by taking (at fixed lattice spacing) the limit \(\mu \rightarrow 0\).
B.3 Ignoring fermionic disconnected diagrams

In our quenched study of the lattice model (18) the calculation of the correlators appearing in Eq. (11) and Eqs. (16)-(17) was carried out by consistently neglecting all the fermionic disconnected Wick contractions. The latter arise in correlators with isospin non-singlet operators because of the occurrence of isospin changing fermion-antifermion-Φ vertices and are statistically quite noisy. Neglecting fermionic disconnected diagrams is, however, justified in our case as it affects neither the results for the mixing coefficient \( \bar{\eta} \) (which determines \( \eta_{cr} \), see Eq. (5)) nor the pseudoscalar isotriplet meson mass, \( M_{PS} \). It entails only an immaterial (and numerically small) modification of the renormalized matrix element ratio \( \frac{Z_{AWI}}{Z_{P}} \), see Eq. (11).

The arguments behind the statements above are given in this section. Following [39], we call “one-boundary” diagrams the fermionic connected diagrams in which valence fermions localized at \( y_{source} = 0 \) are contracted with the valence quarks sitting at \( x \neq y_{source} \), and “two-boundaries” diagrams those in which valence quarks and antiquarks are contracted with each other separately at \( y_{source} = 0 \) and \( x \neq y_{source} \). The valence fermion structure of the two-boundaries Wick contractions that were neglected in evaluating all the correlators considered in this paper is shown in the diagrams of Fig. 4.

![Figure 4: The typical two-boundaries diagram neglected in our work, with fermionic bilinear operators inserted at times \( y_{source} = 0 \) and \( x_0 > 0 \). The meaning of the cuts through one of the valence fermion loops (left panel) or just between them (right panel) is discussed below Eq. (30).](image)

As for the evaluation of \( \bar{\eta} - \eta \) the key observation is that the bare SDE of \( \tilde{\chi}_L \) and/or \( \tilde{\chi}_R \) transformations (see Eqs. (3.15)-(3.16) of Ref. [20]), as well as their renormalized counterparts, see e.g. Eq. (5) for the \( \tilde{\chi}_L \) SDE, can be split in two independent identities according to the number of diagram boundaries ensuing from the way valence quarks are contracted. The reason is that the set of intermediate states of the two types of Wick contractions are not identical, so consequently the \( x_0 \)-functional dependence of one-boundary and two-boundaries diagrams must be separately matched between the l.h.s. and the r.h.s. of the SDE. The quantity \( \bar{\eta} - \eta \), being an operator mixing coefficient, is a factor common to both one-boundary and two-boundaries diagrams. In other words, \( \bar{\eta} \), and thus \( \eta_{cr} \), can be in principle computed equally well by restricting attention to either one-boundary or two-boundaries Wick contractions.

The mass \( M_{PS} \) of the isotriplet pseudoscalar meson, as obtained, say, from the correlator (here \( x = (x, x_0) \))

\[
C^{11}(x_0) = b^3 \sum_x \langle P^1(y_{source})P^1(x) \rangle_{y_{source} = 0},
\]

(30)
can also be evaluated by just restricting attention to the one-boundary Wick contractions. To see that two-boundaries diagrams do not affect the value of \( M_{PS} \), let us examine them in terms of their possible cuts, see Fig. 4. One can either cut through one of the two fermion loops stemming from the Wick contractions at \( y_{source} = 0 \) and at \( x \neq y_{source} \), or one can cut in the middle of the two-boundaries diagram (no fermionic lines are cut in this case). In the first two cases one exposes terms decreasing exponentially in Euclidean time with \( M_{PS} \), but one also recognizes from the cut diagram topology that these terms merely contribute to the renormalization of either \( P^1(y_{source}) \) or
\( P^1(x) \), but not to \( M_{PS} \). In the other case, owing to the quenched approximation, the intermediate states have nothing to do with a pseudoscalar meson, rather they are made up of an odd number of scalar \( \Phi \)-particles plus gluons.

Unlike \( \eta \) and \( M_{PS} \), the value of bare correlator ratio \( m_{AWI} \) (Eq. (11)) gets instead modified if two-boundaries diagrams are neglected. The reason is that two-boundaries diagrams contribute, as we just saw discussing the case of \( M_{PS} \), to the renormalization of the isotriplet pseudoscalar quark density and the axial current four-divergence, as well as to mixing of these quark bilinears with other operators involving scalar fields but no fermions. In the \textit{quenched} lattice \( L_{toy} \) model of interest to us, however, the properly renormalized ratio, \( m_{AWI}^R \equiv \frac{Z_A}{Z_P} m_{AWI} \), is obtained (at \( \eta = \eta_{cr} \)) if one consistently ignores all two-boundaries diagrams in the evaluation of the renormalization constants \( Z_A \) and \( Z_P \), which is what we have done in our numerical study.

A proof of the above statement can be given as follows. Let us consider in the quenched approximation the model \( L_{toy} \) (see Eq. (18)) and its extension to two generations of fermion doublets, i.e. an analogous model with fermion sector action \( S^F_L = \bar{Q}D_LQ + \bar{Q}'D_LQ' \), with \( Q = (u, d) \) and \( Q' = (u', d') \) entering with identical \( \tilde{\chi} \) violating parameters, \( \rho \) and \( \eta \). Owing to fermion quenching, the latter model will admit an EL description analogous to the one of \( L_{toy} \) (see Eq. (13)), with the fermion sector simply replicated for the \( Q \)- and the \( Q' \)- generations. In particular the \( \tilde{\chi} \) violating NP terms in the EL will occur multiplied by coefficients \( c_1, c_2, \ldots \) that are generation-independent and have the same value as for \( L_{toy} \). In the two generations lattice model introduced above the non-vanishing of \( c_1 \) is equivalent to the non-vanishing of the renormalized correlator ratio \( \frac{Z_A}{Z_P} m_{AWI} \)\text{gen. off–diag} \), where

\[
m^\text{gen. off–diag}_{AWI} = \left. \frac{\sum_x \partial_0 \langle \bar{Q}\gamma_0\gamma_5 \tilde{\gamma}_1^I Q'(x)\bar{Q}'\gamma_5 \tilde{\gamma}_1^J Q(0) \rangle}{2 \sum_x \langle \bar{Q}\gamma_0\gamma_5 \tilde{\gamma}_1^I Q'(x)\bar{Q}'\gamma_5 \tilde{\gamma}_1^J Q(0) \rangle} \right|_{\eta_{cr}}
\]

and \( \bar{Q}\gamma_0\gamma_5 \tilde{\gamma}_1^I Q' \) is the current of a \( \tilde{\chi} \)-axial \textit{generation off-diagonal} transformation. But this renormalized correlator ratio of the two generations model, to which no two-boundaries diagrams contribute owing to the generation off-diagonal nature of the axial current, is just equal to the ratio \( \frac{Z_A}{Z_P} m_{AWI} \) that is computed in the one generation model \( L_{toy} \) by neglecting two-boundaries diagrams both in \( m_{AWI} \) and in \( \frac{Z_A}{Z_P} \). The latter ratio (at \( \eta = \eta_{cr} \)) is hence a renormalized and \( \tilde{\chi} \) covariant quantity that is non-vanishing if and only if the coefficient \( c_1 \) in the EL expression (13) is non-zero.

### B.4 Gauge and scalar sector renormalization

Owing to the quenched approximation, gauge and scalar configurations can be independently generated. Similarly, action parameter renormalization can be carried out independently for gauge and scalar fields. The gauge coupling is renormalized by keeping the Sommer length scale \( r_0 \) \[10\] \[41\] fixed in physical units. In the following for mere orientation we take its value from calculations of the static quark potential in QCD and set \( r_0 = 0.5 \, \text{fm} \sim 1/(394 \, \text{MeV}) \).

The choice of the phase in which the theory lives is made by taking \( m_\sigma^2 \) either larger (Wigner) or smaller (NG phase) than the critical value \( m_{cr}^2 \), at which the scalar susceptibility diverges (in practice it has a sharp peak on a large enough space-time volume). The values of the critical mass are listed in Tab. \[18\].

In our study of \( m_{AWI} \) and \( M_{PS} \) in the NG phase the bare scalar parameters in \[18\] at each value of the lattice spacing are determined by the renormalization conditions

\[
M_\sigma^2 r_0^2 = 1.284 \quad \text{and} \quad \lambda_R \equiv \frac{M_\sigma^2}{2v_R^2} = 0.441.
\]

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Moreover the renormalized scalar vev, \( v_R = Z_R^{1/2} v \), where \( Z_\phi \) denotes the scalar field \( \phi \) renormalization constant, is fixed in physical units by the condition \( v_R^2 v_0^2 = 1.458 \). The \( \phi \) field renormalization constant is computed from

\[
Z_\phi^{1/2} = \left[ M_\sigma^2 \left( b^4 \sum_y \langle \varphi_0(y) \varphi_0(0) \rangle - V_4 v^2 \right) \right]^{-1},
\]

where \( V_4 = L^3 \times T \) is the lattice four-volume and the mass \( M_\sigma \) of the one-\( \varphi_0 \) state is extracted from the time decay of the correlator \( b^3 \sum_x \langle \varphi_0(x_0, x) \varphi_0(0, 0) \rangle \). Once \( M_\sigma \), \( Z_\phi \) and the bare scalar vev \( v \) are known, \( \lambda_R \) is easily evaluated.

In a numerical simulation on a finite lattice the scalar vev is always zero, even if \( m^2_\phi < m^2_{cr} \). Hence following Refs. [42, 43] the technique of “fixing” the axial part of the global \( \chi_L \times \chi_R \) symmetry was exploited in order to get \( \langle \Phi \rangle = v \mathbb{1} \neq 0 \) in the NG phase. As this technique gives the correct NG phase results up to finite size effects, for the observables discussed in the section 3 we checked, by simulating at three different volumes (with linear size \( L \) covering the range \( 2.0 < L < 3.4 \) fm), that finite-\( L \) systematic errors on our results at \( L \approx 2.4 \) fm are well below the uncertainties shown in Figs. 2 and 3 or those quoted in appendix D. More in detail, the finite-\( L \) systematic errors on \( M_{PS} \) and \( m_{AWI} \) were found to be marginally significant compared to statistical errors of 1-2\% and 3-5\%, respectively.

| \( b/\text{fm} \) | \( b^2 m_{cr}^2 \) | \( r_0^2 M_\sigma^2 \) | \( \lambda_R \) | \( \beta \) | \( b^2 m_\phi^2 \) | \( \lambda_0 \) |
|-------------|-------------|-------------|----------|----------|-------------|----------|
| 0.152       | -0.5269(12) | 1.278(6)    | 0.437(4) | 5.75     | -0.5941     | 0.5807   |
| 0.123       | -0.5357(11) | 1.286(6)    | 0.441(4) | 5.85     | -0.5805     | 0.5917   |
| 0.102       | -0.5460(10) | 1.290(7)    | 0.444(5) | 5.95     | -0.5756     | 0.6022   |

Table 2: Gauge and scalar bare parameters \( \beta, m^2_\phi, \lambda_0 \) and the corresponding measured value of the renormalized quantities \( r_0^2 M_\sigma^2 \) and \( \lambda_R \) for the three lattice spacings we have considered. The value of the critical mass \( b^2 m_{cr}^2 \) which defines the phase transition point is also given.

The bare parameters used in the NG-phase simulations are listed in Tab. 2. Concerning the simulations in the Wigner phase aimed at determining \( \eta_{cr} \), we have worked at the same lattice resolution (i.e. \( \beta \)) and bare quartic scalar coupling (\( \lambda_0 \)) as employed in the NG phase, \( (m^2_\phi - m^2_{cr}) r_0^2 = 1.22 \) within errors. The latter condition entails a small and uniform \( O(b^2(m^2_\phi - m^2_{cr})) \) contribution to the lattice artifacts in the resulting \( \eta_{cr} \) values at the three lattice spacings. These UV cutoff artifacts are then propagated in the analysis of NG observables (see appendix D) and safely removed by extrapolating to the continuum limit.

A comment about the triviality of the scalar quartic coupling is in order here. Triviality implies that in order to really take the limit \( \Lambda_{UV} \sim b^{-1} \rightarrow \infty \) while keeping fixed the aforementioned low energy renormalization conditions, among which the one on \( \lambda_R \) (see Eq. 32), the bare quartic coupling \( \lambda_0 \) should be made to diverge logarithmically with increasing \( \Lambda_{UV} \). Of course in practice this is not needed at all, rather it is sufficient to vary the lattice spacing \( b \sim \Lambda_{UV}^{-1} \) by a factor of two or three in order to be able, within the numerical errors, to estimate by an extrapolation (which is typically linear in \( b^2 \)) the continuum limit value of the renormalized observables.

### C Lattice artifacts \( O(b^{2n}) \)

We show in this section that in the lattice model [18] only discretization errors of order \( O(b^{2n}) \) (\( n \) positive integer) can show up in the continuum limit of renormalized quantities. This property is a
consequence of the exact invariance of the lattice action under the global $\chi_L \times \chi_R$ transformations, the form of which is given by Eqs. (3)-(4), with $\Omega_L$ and $\Omega_R$ independent SU(2) matrices.

The proof is based on the observation that for symmetry reasons the Symanzik local effective Lagrangian $\mathcal{L}_{\text{toy}}$ describing the lattice artifacts of the model, with the subtracted scalar squared mass $\mu_0^2 \equiv m_0^2 - m_r^2$ set to an $O(b^0)$ value, cannot involve operator terms with odd engineering mass dimension, thereby taking the form

$$L_{\text{toy}}^{\text{Sym}} = L_4 + \sum_{n=1}^{\infty} b^{2n} L_{4+2n}.$$  

(34)

In fact, by properly combining parity ($P$), time reversal ($T$), a finite $\chi_L \times \chi_R$ transformation and a finite vector U(1) transformation, which all correspond to exact lattice symmetries, one can define a lattice transformation which, in analogy with the Wilson Lattice QCD case discussed in Ref. [38], we call $\mathcal{D}_d$. The action of $\mathcal{D}_d$ on the lattice fields reads

$$Q(x) \to iQ(x^{PT}), \quad \bar{Q}(x) \to i\bar{Q}(x^{PT}), \quad \Phi(x) \to -\Phi(x^{PT}), \quad U_\mu(x) \to U^\dagger_\mu(x^{PT}),$$

(35)

where the SU(3) gauge link transformation implies for the gauge potentials $A_\mu(x) \to -A_\mu(x^{PT})$, $\mu = 0, 1, 2, 3$. One can also check that the gauge covariant derivatives transform so that e.g. $D_\mu Q(x) \to -i\tilde{D}_\mu Q(x^{PT})$, while for the plaquette $U_{\mu \neq \nu}(x)$ we have $U_{\mu \neq \nu}(x) \to U^\dagger_{\mu \neq \nu}(x^{PT})$.

We thus see that all the fields entering the basic lattice action $\mathcal{L}_{\text{toy}}$ under the operation $\mathcal{D}_d$ get transformed in their counterparts localized at the $PT$-reflected coordinate $x^{PT} = -x$ times a phase $(-1)^{d_F}$, where $d_F$ is the engineering mass dimension of the field $F$, namely $d_A = d_\Phi = -1$ for gluons and scalars, $d_Q = d_{\bar{Q}} = (-1)^{3/2} = i$ for fermion fields. By the arguments of Ref. [46], the invariance of the lattice action $\mathcal{L}_{\text{toy}}$ under $\mathcal{D}_d$ implies an analogous invariance of the formal Symanzik effective Lagrangian $\mathcal{L}_{\text{toy}}^{\text{Sym}}$ under the continuum counterpart of $\mathcal{D}_d$, which in turn rules out from it all the local terms with odd engineering mass dimension, leading to Eq. (34).

By using again the exact $\mathcal{D}_d$ invariance and the standard arguments of Ref. [46] about the lattice artifact corrections to local operators, it is also clear that the discretization errors on the matrix elements of a local operator $\mathcal{O}(x)$ taken between states at momenta parametrically much smaller than $b^{-1}$ can be fully described in terms of the Symanzik effective Lagrangian $\mathcal{L}_{\text{toy}}^{\text{Sym}}$ and the effective local operator $\mathcal{O}_0(x) + \sum_{n=1}^{\infty} b^{2n} \mathcal{O}_{2n}$.

The same kind of arguments can be straightforwardly extended to prove that the power divergent mixings of a local operator can only involve even powers of $b^{-1}$.

### D. Data analysis details

#### D.1 Analysis of section 3 "Lattice study and results"

In the Wigner phase we compute $r_{\text{AWI}}$ from the ratio (15) for the values of the bare parameters $\eta$ and $\mu$ listed in Tab. 3 and for a value of $\tau \equiv y_0 - x_0 = 0.6$ fm chosen conveniently (see below) and kept fixed for the three lattice spacings. We fit the dependence of $r_{\text{AWI}}$ on $\eta$ and $\mu$ through the linear ansatz $r_{\text{AWI}} = K_0 + K_1 \eta + K_2 \mu$, which turns out to describe very well our data. We thus identify $\eta_{cr}$ with $-K_0/K_1$. The resulting values of $\eta_{cr}$ for the three lattice spacings corresponding to $\beta = 5.75, 5.85, 5.95$ are given in Tab. 4.

The typical profile of $r_{\text{AWI}}$ in the euclidean time (Fig. 5) exhibits two plateaus. The $r_{\text{AWI}}$ values shown in Fig. 1 and the $\eta_{cr}$ results of Tab. 4 all refer to the time plateau region at $x_0$ smaller
Table 3: Simulation parameters in the Wigner phase. We used 480 gauge and scalar paired configurations at each simulation point. Each of these bosonic configurations is different from any other by either the scalar field or both the scalar and the gauge ones.

| b/fm  | \((L^4 \times T)/b^4\) | \(\eta\)  | \(b\mu\) |
|-------|-----------------|--------|--------|
| 0.152 | \(16^3 \times 32\) | -1.1505 | 0.0180 0.0280 0.0480 |
| \((\beta = 5.75)\) |                      | -1.1898 | 0.0180 0.0280 0.0480 |
|       |                  | -1.3668 | 0.0180 0.0280 0.0480 |
| 0.123 | \(16^3 \times 40\) | -1.0983 | 0.0224 0.0316 0.0387 |
| \((\beta = 5.85)\) |                      | -1.1375 | 0.0120 0.0172 0.0224 0.0387 |
|       |                  | -1.2944 | 0.0224 0.0387 |
| 0.102 | \(20^3 \times 48\) | -0.9761 | 0.0186 0.0321 |
| \((\beta = 5.95)\) |                      | -1.0354 | 0.0186 0.0321 |
|       |                  | -1.0771 | 0.0186 0.0321 |

Table 4: The values of \(\eta_{cr}\) at our three lattice spacings for \(r_0^2(m_0^2 - m_{cr}^2) \approx 1.22\) and the listed values of \(\rho\) and \(\lambda_0\).

| \(\beta\) | \(\rho\) | \(\lambda_0\) | \(\eta_{cr}\) |
|---------|--------|-------------|-------------|
| 5.75    | 1.96   | 0.5807      | -1.271(10)  |
| 5.85    | 1.96   | 0.5917      | -1.207(8)   |
| 5.95    | 1.96   | 0.6022      | -1.145(6)   |

than \(T/2\) that we call region A. In this case, as \(\tau = 0.6\) fm, the \(x_0\) range is chosen in such a way that the three-point correlators in Eqs. (16)-(17) are dominated by the PS-meson state at times between 0 and \(x_0\), by the one-\(\Phi\)-particle state at times between \(x_0\) and \(y_0 = x_0 + \tau\) and by the vacuum at times larger than \(y_0\). The time plateau region of \(r_{AWI}\) called region B corresponds instead to \(x_0\) larger than \(T/2\). In this situation the same three-point correlators are dominated by the vacuum at times between 0 and \(x_0\), by a PS-meson plus one-\(\Phi\)-particle state at times between \(x_0\) and \(y_0\) and by one-PS-meson state at times larger than \(y_0\). The \(r_{AWI}\) values one can extract from the time region B are expected to differ from those coming from the time region A only due to \(O(b^2)\) lattice artifacts. In fact, we check in Fig. 6 that the difference in the values of \(\eta_{cr}\) obtained by evaluating \(r_{AWI}\) from the two plateaus of regions A and B vanishes as an \(O(b^2)\) quantity in the continuum limit.

In the NG phase we compute \(M_{PS}\) and \(m_{AWI}\) for the values of \(\eta\) and \(\mu\) listed in Tab. 5. In terms of the lattice fermion operators (24)-(26) \(m_{AWI}\) reads (see Eq. (11))

\[
m_{AWI} = \frac{\partial_{0}^{FW} \sum_{x} \langle \tilde{A}_{0}^{1,BW}(x) P_{1}(0) \rangle}{2 \sum_{x} \langle P_{1}(x) P_{1}(0) \rangle} |_{\eta_{cr}}. \tag{36}
\]

We extrapolate to \(\eta_{cr}\) and \(\mu = 0\) with the ansatz

\[
M_{PS}^2 = Y_0 + Y_1 \eta + Y_2 \mu + Y_3 \eta^2 + Y_4 \mu^2 + Y_5 \eta \mu \quad \text{and} \quad m_{AWI}^2 = Y_0 + Y_1 \eta + Y_2 \mu + Y_3 \mu^2.
\]

The definition of a renormalized counterpart of the matrix element ratio that we have suggestively denoted by \(m_{AWI}\) (see Eq. (11)) requires computing both \(Z_P^{-1}\), which, as discussed in the section, is provided by an hadronic matrix element evaluated for \((\eta, \mu) \to (\eta_{cr}, 0)\) in the Wigner phase, and \(Z_{\tilde{A}} = Z_{\tilde{J}}|_{\eta_{cr}} = Z_{\tilde{V}}\). Since this last renormalization factor is independent of the twisted mass parameter \(\mu\) appearing in the action term (29), it can be accurately computed by imposing, at
Figure 5: A typical plot of $r_{AWI}$ against $x_0/b$, here for the case of $\eta = -1.1374$, $b\mu = 0.0224$ and $\beta = 5.85$. The two time regions where $r_{AWI}$ shows a plateau are $A : x_0 \sim [0.9, 1.8]$ fm and $B : x_0 \sim [2.7, 3.3]$ fm.

Figure 6: The difference $\eta^A_{cr} - \eta^B_{cr}$ vs. $b^2/r_0^2$, where $\eta^A_{cr}(B)$ is estimated by using the plateau value of $r_{AWI}$ in the time region A(B). The value of $\eta^A_{cr} - \eta^B_{cr}$ extrapolated to the continuum limit is well compatible with zero.
\[ \eta = \eta_{cr}, \text{ the validity of the } \tilde{\chi} \text{ SDE involving the bare lattice vector current } \tilde{V}^{2,BW} = J_{L}^{2,BW} + J_{R}^{2,BW}, \]

namely

\[ Z_{\tilde{V}} \partial_{0}^{FW} \sum_{x} \langle \tilde{V}_{0}^{2,BW}(x)P^{1}(0) \rangle_{\eta_{cr}} = 2 \mu \sum_{x} \langle P^{1}(x)P^{1}(0) \rangle_{\eta_{cr}} + O(b^2). \]

Indeed from the exact WTI of \( \chi_{L} \times \chi_{R} \) symmetry that is satisfied by the lattice \( \chi \)-vector current \( V_{\nu}^{2} \),

\[ \partial_{\nu}V_{\nu}^{2}(x) = 2 \mu P^{1}(x), \quad (37) \]

we learn that the combination \( 2 \mu P^{1} \) needs neither to be renormalized nor rescaled by any factor. In practice the estimates of \( Z_{\tilde{V}} \) that we obtained in the NG phase by imposing Eq. (37) at the simulation points in Tab. 5 exhibit only a smooth dependence on \( \eta \) and \( \mu \), which allowed us to extrapolate them to \( \eta \to \eta_{cr} \) and \( \mu \to 0 \) by assuming the polynomial ansatz \( a + b \eta + c \mu + d \mu^2 \).

| \( b/\text{fm} \) | \( \eta \) | \( \mu \) |
|----------------|---------|---------|
| 0.152          | -1.2714 | 0.0050  |
| 0.123          | -1.2105 | 0.0040  |
| (\( \beta = 5.75 \)) |         |         |
| 0.123          | -1.1474 | 0.0066  |
| (\( \beta = 5.95 \)) |         |         |

Table 5: Simulation parameters in the NG phase. We used 60 gauge and scalar paired configurations at each simulation point. The spatial extension of the lattice is fixed to \( L \sim 2.4 \) fm, while the time extension is \( T = 6.0 \) fm for the coarsest lattice spacing and \( T = 4.9 \) fm for the two other ones.

### D.2 Check of analysis stability and systematics

In order to check the stability of the results presented in section 3 and the underlying analysis of appendix D.1, we have studied the impact of modifying in turn

1. the plateau choice for \( r_{AWI} \), which was taken in the time region either A or B of Fig. 5;

2. the fit function for \( r_{AWI} \) in the Wigner phase, which was chosen to be either of the form \( K_{0} + K_{1} \eta + K_{2} \mu \) or extended by adding the non-linear term \( K_{4} \mu^2 \) (data are very well linear in \( \eta \)). In addition the mean value over the results from the two fits above was considered;

3. the choice of the time plateau for \( M_{PS} \) and \( m_{AWI} \), taken to be either \([1.4,2.2]\) fm or \([1.1,2.4]\) fm;
the fit functions for \( m_{AWI} \) and \( M_{PS}^2 \) in the NG phase, which were both either truncated to \( Y_0 + Y_1\eta + Y_2\mu \) or extended by adding in turn terms non-linear in \( \eta \) and/or \( \mu \), such as \( Y_4\mu^2 \), \( Y_3\eta^2 + Y_4\mu^2 \) or \( Y_3\eta^2 + Y_4\mu^2 + Y_5\eta\mu \).

For each combination of the above choices the properly renormalized results of the fits to \((\eta, \mu) = (\eta_{cr}, 0)\), i.e. \( r_0M_{PS} \) and \( 2r_0m_{AWI}Z\sqrt{Z'}Z^{-1} \), were extrapolated linearly in \( b^2/r_0^2 \) to the continuum limit. To estimate the impact of possible O(\( b^4 \)) and higher order lattice artifacts, we have also considered a continuum extrapolation excluding the coarsest lattice spacing, but imposing that the two choices of taking either \( \eta_{cr}^A \) or \( \eta_{cr}^B \) yield the same continuum value. For a typical choice of the time plateau and the fit function in \( \eta \) and \( \mu \) for \( M_{PS}r_0 \), in Fig. 7 we show the continuum limit extrapolations obtained by working at \( \eta_{cr}^A \) or at \( \eta_{cr}^B \) and using data at three lattice spacings, as well as the combined continuum extrapolation that results from using data on the two finer lattices at both \( \eta_{cr}^A \) and \( \eta_{cr}^B \). The analogous comparison for the renormalized ratio \( 2r_0m_{AWI}Z\sqrt{Z'}Z^{-1} \) is presented in Fig. 8.

Figure 7: \( M_{PS}r_0 \) data and continuum limit extrapolations as obtained by working at \( \eta_{cr}^A \) or at \( \eta_{cr}^B \) and using data at three lattice spacings as well as combining data on the two finer lattices at both \( \eta_{cr}^A \) and \( \eta_{cr}^B \) under the constraint of a common continuum limit. A conservative choice of the time plateau for \( M_{PS} \) and the fit ansatz \( M_{PS}^2 = Y_0 + Y_1\eta + Y_2\mu + Y_3\eta^2 + Y_4\mu^2 + Y_5\eta\mu \) were adopted.

The combination of the variants discussed above produced 72 different analyses, none of which gave a result for \( m_{AWI} \) or \( M_{PS} \) that is compatible with zero (i.e. with the no-mechanism scenario). The median over all the analyses, excluding only a few ones for \( M_{PS}^2 \) that yield a \( \chi^2 \) value per degree of freedom larger than 2, gives

\[
\begin{align*}
 r_0M_{PS} &= 0.93 \pm 0.09 \pm 0.10, \\
 2r_0m_{AWI}Z\sqrt{Z'}Z^{-1} &= 1.20 \pm 0.39 \pm 0.19,
\end{align*}
\]

where the first error is statistical and the second systematic. These two error terms were estimated by attributing equal weight to all analyses, following the method of Ref. [47] (see there the text around Eq. (28)).

Further strong evidence for a non-zero fermion mass in the NG phase EL comes from the following “reductio ad absurdum” argument. Let us assume that no fermion mass term is generated. If this were the case, at \( \eta = \eta_{cr} \) one should have \( M_{PS} = O(b^2) \), and hence \( M_{PS}^2 \) would approach

\[0.0\]
Figure 8: Data and continuum limit extrapolations for $2m_{AWI r0}^R \equiv 2r0m_{AWI}Z_VZ_P^{-1}$ in the same cases as for Fig. 7. The fit ansatz $m_{AWI} = Y_0 + Y_1 \eta + Y_2 \mu + Y_3 \mu^2$ was adopted.

Figure 9: $r_0^2 M^2_{PS}$ against $(b/r_0)^4$ in the NG phase: full (dotted) straight lines show the continuum extrapolation linear in $b^4$ using data at all three (only the two finest) lattice spacings.
a vanishing continuum limit with only $O(b^4)$ artifacts. In Fig. 9 we plot $r_0^2 M_{PS}^2$ as a function of $(b/r_0)^4$ (the original data are those at $\eta = \eta_c^4$ already shown in Fig. 3, and a subset of those in Fig. 7). On a $(b/r_0)^4$ scale the lattice results for $r_0^2 M_{PS}^2$ lie very close to the continuum limit, which appears to be non-zero by more than five standard deviations, whatever ansatz is taken for the continuum extrapolation. Thus the hypothesis of “no mechanism” is not supported by lattice data.

D.3 Studies at larger values of $\rho$ and $v$

Once a fermion mass term breaking $\tilde{\chi}_L \times \tilde{\chi}_R$ symmetry is found in the continuum limit of the critical theory in its NG phase, as conjectured in Ref. [20], it is important to check the behaviour of this mass as a function of $\rho$ and the scalar field expectation value $v$.

Since as $\rho \to 0$ also $\eta_{cr}$ vanishes [20], at $\rho = 0$ the $\tilde{\chi}_L \times \tilde{\chi}_R$ transformations become exact symmetries of the lattice model (15) and $m_{AWI}$ vanishes, as well as the infinite volume value of $M_{PS}$. One thus expects that $m_{AWI}$ and $M_{PS}^2$ are increasing functions of $\rho$, and arguments were given in Ref. [20] according to which for small $\rho$ they should increase at least as $O(\rho^2)$. The main results of this work were produced at $\rho = 1.96$.

In order to check the $\rho$ dependence of $m_{AWI}$ and $M_{PS}^2$ we have thus studied the case of a 1.5 times larger $\rho$ value, namely $\rho = 2.94$, for the intermediate lattice spacing ($\beta = 5.85, a \sim 0.123 \text{ fm}$) and for the same scalar sector parameters as specified in appendix B.4.

The list of simulations we performed at $\rho = 2.94$ is given in Tab. 6. We limited ourselves to a few simulations since due to the larger $\rho$ value the CPU time effort for inverting the Dirac matrix for similar volumes and twisted mass values increased by a factor 5 to 10 with respect to the case of $\rho = 1.96$.

| phase | $(L_3^3 \times T)/b^4$ | $\eta$ | $b\mu$ |
|-------|--------------------------|--------|--------|
| Wigner | 16$^3 \times 32$ | -1.765 | 0.0140, 0.0210, 0.0280 |
|       |               | -1.7160 | 0.0210 |
|       |               | -1.6670 | 0.0210 |
| NG    | 20$^3 \times 40$ | -1.8197 | 0.0050, 0.0100, 0.0150 |
|       |               | -1.7808 | 0.0100 |

Table 6: Simulation parameters at $\rho = 2.94$, $\beta = 5.85$. We use 480 gauge and scalar paired configurations in the Wigner phase and 60 in the NG phase at each simulation point.

In the Wigner phase, using the data analysis procedure of appendix D.1, we found that changing $\rho$ from 1.96 to 2.94 leads to a substantial change of $\eta_{cr}$, from $-1.207(8)$ to $-1.838(13)$. Notice that the increase of $\eta_{cr}$ is almost perfectly linear in $\rho$, in line with the theoretical remark [20] that $\eta_{cr}$ is an $O(\rho)$ quantity, and an odd function of $\rho$.

In the NG phase we observed a significant increase of $2 r_0 m_{AWI} Z_A Z_{P}^{-1}$ from 1.15(0.21) to 2.80(0.36) (see Fig. 10) as well as a similar increase of $M_{PS}^2 r_0^2$. These findings are in nice agreement with theoretical expectations from Ref. [20]. For their interpretation in the context of a revised concept of universality in more realistic models based on the NP mass mechanism under discussion, see the remarks at the end of the section 3 and in appendix E.

Due to CPU time limitations we leave for a future study the investigation of the behaviour of $m_{AWI}$ and $M_{PS}^2$ in the critical theory as a function of the scalar field expectation value $v$, which is
motivated by the remarks in section 2.2 on the dependence of \( c_1 \) on \( \hat{\mu}_\phi^2 \).

### E Towards a realistic model

We extend here the discussion started in the section 2 about how one can possibly build particle physics models based on the NP mass generation mechanism discussed and identified in this paper. In order to get a phenomenologically sensible theory one has to extend the model (1) by introducing, besides EW interactions, also a “Tera-strong” force and a set of “Tera-fermions” (we take this name from Ref. \[48\]) that communicate with ordinary matter (quarks or leptons) via strong and/or EW interactions \[22\]. The “Tera-strong” force should be given by a new non-Abelian gauge interaction that becomes strong at the scale \( \Lambda_T \gg \Lambda_{\text{QCD}} \), where \( \Lambda_T \) can be roughly estimated to be in the few TeV range \[20, 22\].

Models of this type without tree-level flavour changing neutral currents can be easily formulated, if SU(2)_L \times U(1)_Y invariant EW interaction terms are introduced in the same way as in the SM, e.g. by coupling SU(2)_L gauge bosons only to left-handed Tera-fermions arranged in doublets of definite hypercharge. By then taking all right-handed Tera-fermions to be weak isospin singlets with given hypercharge the usual relation \( Q_{em} = T_3 + Y/2 \) between electric charge \( Q_{em} \), third weak isospin component \( T_3 \) and hypercharge \( Y \) is always fulfilled. Owing to the sizeable NP mass of the “Tera-fermions”, the lightest Tera-mesons are expected to be rather heavy, i.e. with a mass around 2-3 times \( \Lambda_T \), implying that these extended models have a good chance of passing the EW precision tests, in particular those concerning the \( S \)-parameter \[49, 50\]. By treating the chiral SU(2)_L \times U(1)_Y interactions in perturbation theory and carrying out unquenched lattice simulation of models including QCD and “Tera-strong” gauge forces, it is in principle possible to relate with controllable errors the NP mass of the top quark to the mass of the lightest Tera-mesons, which should be seen as resonances in accelerator experiments at sufficiently high energy.

Lattice methods can also be useful to check and possibly establish the existence in the \( WW + ZZ \) channel of one composite Higgs state with mass \( M_h \sim 125 \text{ GeV} \), which is expected to form owing to the interactions of weak gauge bosons with “Tera-fermions” that are subjected to EW and “Tera-strong” interactions. For a first step in this direction see Ref. \[51\]. If the existence of such a
composite Higgs state is confirmed, then perturbative unitarity of the (longitudinal) $WW$ scattering amplitude at momentum scales well below $\Lambda_T$, for which the Higgs state plays a crucial role, would make very plausible that the Higgs couplings are close to the present experimental findings and deviate from their SM value only by small $O(p^2/\Lambda_T^2)$ effects, with $p^2$ standing for $M_h^2$ or the typical momentum scale of the process where the Higgs couplings are measured.

A striking and interesting feature of our mass generation mechanism is the observed dependence of low energy physical quantities on the parameters of the $d \geq 6$ breaking Lagrangian terms. To all orders in the perturbative expansion in the gauge coupling these terms would represent irrelevant details of the UV completion of the critical model. But at the NP level, in the NG phase of the critical theory, the $d \geq 6$ breaking operators appearing in the UV regulated classical Lagrangian produce relevant $O(\Lambda_{UV}^0)$ effects, because the symmetries of our model allow for the occurrence in the EL of RGI finite NP terms, such as those with coefficients $c_1$ and $c_2$ in Eq. (13). As first pointed out in Ref. [20] and discussed in the section 2.2, this dynamical phenomenon is due to a subtle interplay between the residual $O(b^2)$ terms violating $\tilde{\chi}_L \times \tilde{\chi}_R$ and the strong interactions, which are responsible for spontaneous breaking of the approximate $\tilde{\chi}_L \times \tilde{\chi}_R$ symmetries. Of course no such a phenomenon would be observed if the critical model were defined in a UV regularization preserving exactly $\tilde{\chi}_L \times \tilde{\chi}_R$ invariance, e.g. at $\rho = 0$ and $\eta = \eta_{cr}|\rho=0 = 0$. This remark, which is indeed consistent with our numerical finding that $c_1$ increases upon increasing $\rho$, see appendix D.3, also implies that non-Abelian gauge models including fermions and scalars in general fall into different universality classes depending on whether the UV regularization breaks or possibly preserves the purely fermionic chiral symmetries.

In other words, the “NP anomaly” to the recovery of broken fermionic chiral symmetries that we demonstrated in our work appears to require a non-trivial revision of the concept of universality, as compared to its standard formulation which can be proven to all orders of perturbation theory (only). This important point certainly deserves a more thorough investigation. At the moment it seems to us conceivable that the kind of universality that is realized in the context of maximal $\tilde{\chi}$ restoration carries over also to the NP mass terms in the EL [22]. In this new NP framework universality should thus mean that a set of UV complete models endowed with different $\tilde{\chi}$ breaking Lagrangian terms lead to the same low energy observables, once the conditions of maximal $\tilde{\chi}$ restoration are fulfilled at correspondingly different critical values of the relevant bare parameters. The low energy physics should thus be controlled only by the gauge couplings, the number of fermions and their transformation properties under the gauge groups, plus possibly a finite number of dimensionless coefficients parametrizing the relative strength of the $\tilde{\chi}$ breaking terms for the different fermion generations and having a magnitude bounded by the conditions of maximal $\tilde{\chi}_L$ restoration.

We finally note that a beyond the SM model supporting the mass generation mechanism we have discussed and endowed with a reasonable content of (standard and Tera) matter fields, if a suitable, non-standard choice of hypercharges for Tera-fermions is made, can yield unification of EW and strong (and perhaps even Tera-strong) gauge couplings at an energy scale around $10^{18}$ GeV and to a precision level that is as good as for the unification achieved in the Minimal SUSY Model.

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