Knotted handle decomposing spheres for handlebody-knots

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Abstract. We show that a handlebody-knot whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves. As an application, we show that the handlebody-knots $6_{14}$ and $6_{15}$ are not equivalent. We also show that certain genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors. As an application, we show that the exteriors of $6_{14}$ and $6_{15}$ are not homeomorphic.

1. Introduction.

A genus $g$ handlebody-knot is a genus $g$ handlebody embedded in the 3-sphere $S^3$. Two handlebody-knots are equivalent if one can be transformed into the other by an isotopy of $S^3$. A handlebody-knot is trivial if it is equivalent to a handlebody standardly embedded in $S^3$, whose exterior is a handlebody. We denote by $E(H) = S^3 - \text{int} H$ the exterior of a handlebody-knot $H$.

Definition 1.1. A 2-sphere $S$ in $S^3$ is an $n$-decomposing sphere for a handlebody-knot $H$ if

1. $S \cap H$ consists of $n$ essential disks in $H$, and
2. $S \cap E(H)$ is an incompressible and not boundary-parallel surface in $E(H)$.

In some cases it might be suitable to replace the condition (2) in Definition 1.1 with the condition

2'. $S \cap E(H)$ is an incompressible, boundary-incompressible, and not boundary-parallel surface in $E(H)$,

although we adopt the condition (2) in this paper. The two definitions are equivalent if $n = 1$, or $n = 2$ and $E(H)$ is boundary-irreducible.

For two $n$-decomposing spheres $S$ and $S'$ for a handlebody-knot $H$, $S$ is isotopic to $S'$ if there is an isotopy of $S^3$ from $S$ to $S'$ such that $S$ remains being an $n$-decomposing sphere throughout the isotopy.

A handlebody-knot $H$ is reducible if there exists a 1-decomposing sphere for $H$, where we remark that (2) follows from (1) when $n = 1$. A handlebody-knot is irreducible if it is not reducible. A handlebody-knot $H$ is irreducible if $E(H)$ is boundary-irreducible.

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The converse is true for a genus two handlebody-knot $H$. In particular, for a genus two handlebody-knot $H$, the following are equivalent:

1. $H$ is irreducible.
2. $\pi_1(E(H))$ is indecomposable with respect to free products.
3. $E(H)$ is boundary-prime (cf. [16, 2.10 Definition]).
4. $E(H)$ is boundary-irreducible.

By [18], we have the equivalence between (1) and (2). By [7], we have the equivalence between (2) and (3) for a handlebody-knot $H$ of arbitrary genus. The conditions (3) and (4) are equivalent if $E(H)$ is not a solid torus (cf. [16, Proposition 2.15]). We remark that there is an irreducible genus $g \not= 2$ handlebody-knot whose exterior is not boundary-irreducible (cf. [16, Theorem 5.4]).

A genus two handlebody-knot [17] and a trivial handlebody-knot can be uniquely decomposed by 1-decomposing spheres into handlebody-knots each of which has no 1-decomposing spheres. The uniqueness is not known for genus $g \geq 3$ handlebody-knots.

**Definition 1.2.** A 2-sphere $S$ in $S^3$ is a knotted handle decomposing sphere for a handlebody-knot $H$ if

1. $S \cap H$ consists of two parallel essential disks in $H$, and
2. $S \cap E(H)$ is an incompressible and not boundary-parallel surface in $E(H)$.

We say that a 2-sphere $S$ bounds $(B,K;H)$ if $S$ bounds a 3-ball $B$ so that $S \cap H$ consists of two parallel essential disks in $H$, and that $H \cup E(B)$ is equivalent to a regular neighborhood of a nontrivial knot $K$. A knotted handle decomposing sphere for $H$ bounds $(B,K;H)$. A 2-sphere $S$ which bounds $(B,K;H)$ is not always a knotted handle decomposing sphere for $H$ (see the left picture of Figure 1). In this paper, we represent a handlebody-knot by a spatial trivalent graph whose regular neighborhood is the handlebody-knot as shown in Figure 1. Then the intersection of the spatial trivalent graph and the 2-sphere indicates two disks.

If $H$ is a genus $g \geq 2$ handlebody-knot whose exterior is boundary-irreducible, then a 2-sphere $S$ which bounds $(B,K;H)$ is a knotted handle decomposing sphere for $H$, where we note that $g \geq 2$ implies that $S \cap E(H)$ is not boundary-parallel in $E(H)$, and that the boundary-irreducibility implies the incompressibility of $S \cap E(H)$. A trivial handlebody-knot has no knotted handle decomposing sphere by the following lemma.

**Lemma 1.3 ([14, Lemma 2.2]).** An incompressible surface properly embedded in a handlebody cuts it into handlebodies.
The pair $6_{14}$, $6_{15}$ is the remaining pair of handlebody-knots whose fundamental groups are isomorphic. In Section 2, we show that a handlebody-knot whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves (Theorem 2.2), where we note that Koda and the third author [10] have successfully removed the assumption that the exterior is boundary-irreducible. As an application, we show that the handlebody-knots $6_{14}$ and $6_{15}$ are not equivalent (Example 2.6). In Section 3, we show that certain genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors (Theorem 3.1). As an application, we show that the exteriors of the handlebody-knots $6_{14}$ and $6_{15}$ are not homeomorphic (Example 3.5).

### 2. A unique decomposition for a handlebody-knot.

Let $H$ be a handlebody-knot in $S^3$, and $S$ a knotted handle decomposing sphere for $H$ which bounds $(B, K; H)$. Let $A$ be an annulus properly embedded in $E(H) - \text{int } B$ so that $A \cap S = l$ is an essential loop in the annulus $S \cap E(H)$, and that $A \cap \partial H = l'$ bounds an essential disk $D$ in $H$, where $\partial A = l \cup l'$ (see Figure 2). Put $T = (S \cap E(H)) \cup (B \cap \partial H)$. Let $A'$ be an annulus obtained from $T$ by cutting along $l$ and pasting two parallel copies of $A$, where $T$ is slightly isotoped so that $T \cap H = \emptyset$. Then we have a new knotted handle decomposing sphere $S'$ obtained from $A'$ by attaching two parallel copies of $D$ to $\partial A'$. We say that $S'$ is obtained from $S$ by an *annulus-move* along $A$. For example, in Figure 3, $S'$ is obtained from $S$ by an annulus-move along $A$.

A set $S = \{S_1, \ldots, S_n\}$ of knotted handle decomposing spheres for a handlebody-knot $H$ is *unnested* if each sphere $S_i$ bounds $(B_i, K_i; H)$ so that $B_i \cap B_j = \emptyset$ for $i \neq j$. 

![Figure 1.](image1.png)

![Figure 2. An annulus-move along $A$.](image2.png)
An unnested set $S$ is maximal if $n \geq m$ for any unnested set $\{S'_1, \ldots, S'_m\}$ of knotted handle decomposing spheres for $H$. By the Haken–Kneser finiteness theorem [4], [8], there exists a maximal unnested set of knotted handle decomposing spheres for $H$. By Schubert’s theorem [15], $K_i$ is prime for any $i$ if $S$ is maximal.

**Lemma 2.1.** Let $H$ be a handlebody-knot whose exterior is boundary-irreducible. Let $S = \{S_1, \ldots, S_n\}$ be an unnested set of knotted handle decomposing spheres for $H$ such that $S_i$ bounds $(B_i, K_i; H)$ and that $K_i$ is prime for any $i$. Let $S' = \{S'_1, \ldots, S'_m\}$ be a set of 2-decomposing spheres for $H$. Then $S$ can be deformed so that $S_i \cap S'_j = \emptyset$ for any $i, j$ by isotopies and annulus-moves.

**Proof.** Let $A_i = S_i \cap E(H)$ for $i = 1, \ldots, n$ and $A'_j = S'_j \cap E(H)$ for $j = 1, \ldots, m$. We may assume that $A_i \cap A'_j$ consists of essential arcs or loops in both $A_i$ and $A'_j$, and that $|A_i \cap A'_j|$ is minimal by isotopies and annulus-moves for each pair $(i, j)$.

Suppose that $A_i \cap A'_j$ consists of essential arcs for some $i$ and $j$. Let $\Delta$ be a component of $A'_j \cap B_i$ which is cobounded by two adjacent arcs of $A_i \cap A'_j$ in $A'_j$. Since the arcs $\partial \Delta \cap \partial H$ are essential in the annulus $\partial H \cap B_i$ by the minimality of $|A_i \cap A'_j|$, $\partial \Delta$ winds around $B_i - \text{int} H$ longitudinally twice. By attaching a 2-handle $N(\Delta)$ to the solid torus $E(B_i - \text{int} H)$, we have a once punctured lens space $L(2, q)$, which contradicts Alexander’s theorem [1]. Hence $A_i \cap A'_j$ consists of essential loops for any pair $i$ and $j$.

Let $F$ be an outermost subannulus of $A'_j$ which is cut by $(\bigcup_{k=1}^n A_k) \cap A'_j$ for some $j$. Let $A_i$ be the annulus such that $F \cap A_i \neq \emptyset$. If $F$ is contained in $B_i$, then by the primeness of $K_i$, we can isotope off $F$ from $B_i$. Hence $F$ is in the outside of $B_i$. Then by an annulus move for $S_i$ along the annulus $F$, we can reduce $|A_i \cap A'_j|$. This contradicts to the minimality of $|A_i \cap A'_j|$. \hfill $\square$

**Theorem 2.2.** A handlebody-knot $H$ whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves.

**Proof.** Let $S = \{S_1, \ldots, S_n\}$, $S' = \{S'_1, \ldots, S'_m\}$ be maximal unnested sets of knotted handle decomposing spheres for $H$ such that $S_i$ and $S'_j$ bound $(B_i, K_i; H)$ and $(B'_j, K'_j; H)$, respectively. By Lemma 2.1, we can deform $S'$ so that $S_i \cap S'_j = \emptyset$ for any $i, j$ by isotopies and annulus-moves. We also deform $S'$ so that $B_i \cap B'_j = \emptyset$ by isotopies if $B_i \cap B'_j$ is homeomorphic to $S^2 \times I$, where $I$ is an interval. Then we have $B_i \subset B'_j$, $B'_j \subset B_i$, or $B_i \cap B'_j = \emptyset$ for any $i, j$. Since $S'$ is maximal, for any $B_i$, there exists a 3-ball...
$B'_j$ such that $B_i \subset B'_i$ or $B'_j \subset B_i$. Since $K_i$ and $K'_j$ are prime, $S_i$ is parallel to $S'_j$. This gives a one-to-one correspondence between $S$ and $S'$. Hence a maximal unnested set of knotted handle decomposing spheres for $H$ is unique up to isotopies and annulus-moves.

**Proposition 2.3.** Let $H$ be a genus $g$ handlebody-knot whose exterior is boundary-irreducible. Let $\{S_1, \ldots, S_n\}$ be an unnested set of knotted handle decomposing spheres for $H$ such that $S_i$ bounds $(B_i, K_i; H)$ for any $i$. Put $H' := H \cup B_{m+1} \cup \cdots \cup B_n$. Then $\{S_1, \ldots, S_m\}$ is an unnested set of knotted handle decomposing spheres for $H'$, or $g = 1$ and $m = 1$.

**Proof.** Suppose that $S_i \in \{S_1, \ldots, S_m\}$ is not a knotted handle decomposing sphere for $H'$. If $S_i \cap E(H')$ is compressible in $E(H')$, then $S_i \cap E(H)$ is also compressible in $E(H)$, a contradiction. If $S_i \cap E(H')$ is parallel to an annulus $A \subset \partial E(H')$ in $E(H')$, then $A$ contains some annuli of $(B_{m+1} \cup \cdots \cup B_n) \cap \partial H'$. This shows that $g = 1$ and $m = 1$.

**Proposition 2.4.** Let $H$ be a genus $g \geq 2$ handlebody-knot, $S$ a 2-sphere which bounds $(B, K; H)$. If $E(H \cup B)$ is boundary-irreducible, then so is $E(H)$.

**Proof.** Suppose that $E(H)$ is boundary-reducible and let $D$ be a compressing disk in $E(H)$. Since $E(H \cup B)$ is boundary-irreducible, $D$ intersects with the annulus $A = S \cap E(H)$. Since $E(H)$ is irreducible, we may assume that $D \cap A$ consists of essential arcs in $A$. Since the knot $K$ is nontrivial, an outermost disk of $D$ gives a compressing disk in $E(H \cup B)$. This is a contradiction.

An $(n$-component$)$ handlebody-link is a disjoint union of $n$ handlebodies embedded in the 3-sphere $S^3$. A non-split handlebody-link is a handlebody-link whose exterior is irreducible.

**Proposition 2.5.** Let $H$ be a handlebody-knot, $S$ a 2-sphere which bounds $(B, K; H)$. Suppose that $H - \text{int} B$ is a non-split handlebody-link whose exterior is boundary-irreducible. If $H - \text{int} B$ is 2-component handlebody-link or $E(H \cup B)$ is a handlebody, then $E(H)$ is boundary-irreducible.

**Proof.** Suppose that $E(H)$ is boundary-reducible. Let $D$ be a compressing disk in $E(H)$. Put $A = S \cap E(H)$. If $D \cap A \neq \emptyset$, then we may assume that $D \cap A$ consists of essential arcs in $A$, since $E(H)$ is irreducible. Since the knot $K$ is nontrivial, an outermost disk $\delta$ of $D$ is contained in $E(H \cup B)$. If $H - \text{int} B$ is not a handlebody-knot, then the arc $\delta \cap (H - \text{int} B)$ connects the different components of $H - \text{int} B$ on $\partial(H - \text{int} B)$, a contradiction. If $E(H \cup B)$ is a handlebody, then $\delta$ cuts $E(H \cup B)$ into a 3-manifold homeomorphic to $E(H - \text{int} B)$, which is a handlebody by Lemma 1.3. This implies that $H - \text{int} B$ is trivial, which contradicts that $E(H - \text{int} B)$ is boundary-irreducible. Then $D \cap A = \emptyset$, and so $D$ is in $E(H - \text{int} B)$. Since $E(H - \text{int} B)$ is boundary-irreducible, $D$ is inessential in $E(H - \text{int} B)$. Let $D'$ be a disk in $\partial E(H - \text{int} B)$ such that $\partial D' = \partial D$.

Let $D_1, D_2$ be the disks such that $S \cap H = D_1 \cup D_2$. If $D' \cap (D_1 \cup D_2) = \emptyset$, then $\partial D'$ is inessential in $\partial E(H)$, which contradicts that $D$ is essential in $E(H)$. If
$D' \cap (D_1 \cup D_2) = D_1$ or $D' \cap (D_1 \cup D_2) = D_2$, then the 2-sphere $S' = D' \cup D$ can be slightly isotoped so that $S' \cap (H - \text{int } B) = \emptyset$, which contradicts that $H - \text{int } B$ is non-split, since $S'$ separates $D_1$ and $D_2$. Thus $D_1, D_2 \subseteq D'$. If $H - \text{int } B$ is not a handlebody-knot, then $D'$ connects the different components of $H - \text{int } B$ on $\partial (H - \text{int } B)$, a contradiction. If $E(H \cup B)$ is a handlebody, then the 2-sphere $S' = D' \cup D$ can be slightly isotoped so that $D'$ is properly embedded in $H - \text{int } B$. Then $S'$ separates a handlebody $E(H \cup B)$ into a solid torus and a handlebody which is homeomorphic to the exterior of $H - \text{int } B$. This contradicts that $H - \text{int } B$ is nontrivial. \hfill \Box

**Example 2.6.** We show that any two of the handlebody-knots $5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15}$ and $6_{15}^*$ are not equivalent, where $5_4, 6_{14}$ and $6_{15}$ are the handlebody-knots depicted in Figure 4, and $5_4^*, 6_{14}^*$ and $6_{15}^*$ are their mirror images, respectively.

Let $H$ be one of the handlebody-knots $5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15}$ and $6_{15}^*$. Let $S$ be the knotted handle decomposing sphere for $H$ depicted in Figure 4, where $S$ bounds $(B, K; H)$ and $K$ is a trefoil knot. By Proposition 2.5, $E(H)$ is boundary-irreducible. By Proposition 2.3, $\{S\}$ is a maximal unnested set of knotted handle decomposing spheres for $H$, since the trivial handlebody-knot $H \cup B$ has no knotted handle decomposing sphere. Then $S$ is unique by Theorem 2.2, which implies that the pair $(K, H - \text{int } B)$ is an invariant of $H$. Hence any two of the handlebody-knots $5_4, 5_4^*, 6_{14}, 6_{14}^*, 6_{15}$ and $6_{15}^*$ are not equivalent.

![Figure 4](image)

**Proposition 2.7.** There exists a sequence of handlebody-knots $H_i$ ($i \in \mathbb{N} \cup \{0\}$) satisfying the following conditions.

- $H_0$ is the trivial genus two handlebody-knot, which has no knotted handle decomposing sphere.
- For $i \geq 1$, $H_i$ has a unique knotted handle decomposing sphere $S_i$ which bounds $(B_i, K_i; H_i)$.
- For $i \geq 1$, $H_i \cup B_i$ is equivalent to $H_{i-1}$ as a handlebody-knot.

**Proof.** Let $H_0$ be the trivial genus two handlebody-knot. For $i \geq 1$, let $H_i$ be the genus two handlebody-knot with $i - 1$ tangles $T$ and a 2-sphere $S_i$ bounding $(B_i, K_i; H_i)$ as depicted in Figure 5. Then $H_i \cup B_i$ is equivalent to $H_{i-1}$. We remark that $H_1$ is the irreducible handlebody-knot $6_{14}$, whose exterior is boundary-irreducible. It follows by Proposition 2.4 that $H_i$ is boundary-irreducible for $i \geq 1$. Then $S_i$ is a knotted handle decomposing sphere for $H_i$.

We prove by induction on $i$ that $S_i$ is a unique knotted handle decomposing sphere for $H_i$. We already showed that $S_1$ is a unique knotted handle decomposing sphere for
Figure 5.

$H_1$ in Example 2.6. Assume that $S_{i-1}$ is a unique knotted handle decomposing sphere for $H_{i-1}$. Suppose that $S_i$ is not a unique knotted handle decomposing sphere for $H_i$. Then, by Lemma 2.1 and Theorem 2.2, there is a knotted handle decomposing sphere $S'_i$ for $H_i$ which bounds $(B'_i, K'_i; H_i)$ such that the set $\{S_i, S'_i\}$ is a maximal unnested set of knotted handle decomposing spheres for $H_i$.

Let $K_{i-1}$ be the core of $H_i - \text{int} B_i$, which is a satellite knot. Let $T'$ be the tangle obtained from $T$ and 3 half twists as the leftmost tangle of $K_{i-1}$ in Figure 5. Then $T$ and $T'$ are prime tangles (cf. [5]). Since $K_{i-1}^-$ is obtained from $T'$ and $i-2$ copies of $T$ by tangle sum, $K_{i-1}^-$ is a prime knot [12]. It follows by Proposition 2.3 that $S'_i$ corresponds to $S_{i-1}$. Hence $K_{i-1}^-$ is the positive trefoil knot, and $(H_i \cup B_i) - \text{int} B_i'$ is a regular neighborhood of $K_{i-1}^-$. A loop $l$ of $S'_i \cap \partial H_i$ is in $\partial (H_i - \text{int} B_i)$, since the set $\{S_i, S'_i\}$ is unnested.

If $l$ is essential in $\partial (H_i - \text{int} B_i)$, then $l$ is a meridian loop of a solid torus $H_i - \text{int} B_i$. By the primeness of $K_{i-1}^-$, the positive trefoil knot $K_i^-$ is equivalent to the satellite knot $K_{i-1}^-$ for $i > 1$, a contradiction.

If $l$ is inessential in $\partial (H_i - \text{int} B_i)$, then $l$ bounds a disk $D$ in $\partial (H_i - \text{int} B_i)$. Let $D_1, D_2$ be the disks such that $S_i \cap H_i = D_1 \cup D_2$. Since $l$ is essential in $\partial H_i$, $D \cap (D_1 \cup D_2) \neq \emptyset$. If $D$ contains both $D_1$ and $D_2$, then $l$ is a separating loop in $\partial H_i$ and $\partial H_{i-1}$, which contradicts that $S_{i-1} \cap \partial H_{i-1}$ consists of non-separating disks. Thus $D$ contains either $D_1$ or $D_2$, which implies that $l$ is parallel to the loops of $S_i \cap \partial H_i$. Then $H_i - \text{int} B_i$ and $(H_i \cup B_i) - \text{int} B'_i$ are equivalent as handlebody-knots. It follows that $K_{i-1}^-$ and $K_{i-1}^-$ are equivalent, which contradicts that $K_{j-1}^-$ has a non-trivial Fox 3-coloring if and only if $j$ is odd, since the replacement of the tangle $T$ with the trivial tangle does not change the number of Fox 3-colorings.

Therefore $S_i$ is a unique knotted handle decomposing sphere for $H_i$. This completes the proof.

Proposition 2.7 suggests that the following theorem holds. Actually, the theorem is true by the recent work of Koda and the third author [10]. Then Proposition 2.7 gives a concrete example which has a hierarchy of any depth.
Theorem 2.8. For any handlebody-knot \( H \), there exists a unique sequence of handlebody-knots \( H_0, \ldots, H_m = H \) satisfying the following conditions.

- \( H_0 \) has no knotted handle decomposing sphere.
- For \( 1 \leq i \leq m \), \( H_i \) has a unique maximal unnested set of knotted handle decomposing spheres \( \{ S_{i,1}, \ldots, S_{i,n_i} \} \), where each \( S_{i,j} \) bounds \( (B_{i,j}, K_{i,j}; H_i) \).
- For \( 1 \leq i \leq m \), \( H_i \cup B_{i,1} \cup \cdots \cup B_{i,n_i} \) is equivalent to \( H_{i-1} \) as a handlebody-knot.

3. Handlebody-knots and their exteriors.

In this section, we show that certain genus two handlebody-knots with a knotted handle decomposing sphere can be determined by their exteriors. As an application, we show that the exteriors of the handlebody-knots \( 6_{14} \) and \( 6_{15} \) are not homeomorphic.

Theorem 3.1. For \( i = 1, 2 \), let \( H_i \) be an irreducible genus two handlebody-knot with a knotted handle decomposing sphere \( S_i \) bounding \( (B_i, K_i; H_i) \) such that \( B_i \) contains all spheres in a maximal unnested set of knotted handle decomposing spheres for \( H_i \). Suppose that \( E(H_i \cup B_i) \) is a handlebody and that \( H_i - \text{int} B_i \) is a nontrivial handlebody-knot for \( i = 1, 2 \). Then \( H_1 \) and \( H_2 \) are equivalent if and only if there is an orientation preserving homeomorphism from \( E(H_1) \) to \( E(H_2) \).

An annulus \( A \) properly embedded in a 3-manifold is essential if \( A \) is incompressible and not boundary-parallel. To prove Theorem 3.1, we give some lemmas.

Lemma 3.2 ([2, 15.26 Lemma]). Let \( K \) be a knot in \( S^3 \). If \( E(K) \) contains an essential annulus \( A \), then either

1. \( K \) is a composite knot and \( A \) can be extended to a decomposing sphere for \( K \),
2. \( K \) is a torus knot and \( A \) can be extended to an unknotted torus or
3. \( K \) is a cable knot and \( A \) is the cabling annulus.

Lemma 3.3 ([9, Lemma 3.2]). If \( A \) is an essential annulus in a genus two handlebody \( W \), then either

1. \( A \) cuts \( W \) into a solid torus \( W_1 \) and a genus two handlebody \( W_2 \) and there is a complete system of meridian disks \( \{ D_1, D_2 \} \) of \( W_2 \) such that \( D_1 \cap A = \emptyset \) and \( D_2 \cap A \) is an essential arc in \( A \), or
2. \( A \) cuts \( W \) into a genus two handlebody \( W' \) and there is a complete system of meridian disks \( \{ D_1, D_2 \} \) of \( W' \) such that \( D_1 \cap A \) is an essential arc in \( A \).

We say that an annulus \( A \) is obtained from a knotted handle decomposing sphere \( S \) for a handlebody-knot \( H \) when \( A = S \cap E(H) \).

Lemma 3.4. Let \( H \) be an irreducible genus two handlebody-knot with a knotted handle decomposing sphere \( S \) bounding \( (B, K; H) \) such that \( B \) contains all spheres in a maximal unnested set of knotted handle decomposing spheres for \( H \). Suppose that \( E(H \cup B) \) is a handlebody and that \( H - \text{int} B \) is a nontrivial handlebody-knot. Then any essential separating annulus in \( E(H) \) is isotopic to either a cabling annulus for \( H - \text{int} B \) or an annulus obtained from a knotted handle decomposing sphere for \( H \).
Proof. Let \( A' \) be an essential separating annulus in \( E(H) \). Assuming that \( A' \) cannot be obtained from a knotted handle decomposing sphere for \( H \), we show that \( A' \) is a cabling annulus for \( H \setminus \text{int} B \). Put \( A = S \cap E(H) \) and \( W = E(H \cup B) \). We may assume that \( A \cap A' \) consists of essential arcs or loops in both \( A \) and \( A' \), and that \( |A \cap A'| \) is minimal by isotopies. As the proof of Lemma 2.1, we may assume that \( A \cap A' \) consists of essential loops.

If \( \partial A' \) is contained in \( B \), then \( A' \) is an annulus obtained from a knotted handle decomposing sphere for \( H \), since each loop of \( \partial A' \) is parallel to \( \partial (S \cap H) \). Hence there is a loop \( C \) of \( \partial A' \) contained in \( W \).

Suppose \( A \cap A' \neq \emptyset \). Let \( F \) be the outermost subannulus on \( A' \) containing \( C \), which is an annulus properly embedded in \( W \). Since \( A' \) is incompressible in \( E(H) \), \( F \) is incompressible in \( W \). By the minimality of \( |A \cap A'| \), \( F \) is not boundary-parallel in \( W \). Let \( D \) be a disk in \( E(H \setminus \text{int} B) \) such that \( D \cap W = F \) and \( D \cap B \) is a disk \( D_0 \) in \( B \). If \( C \) is essential in \( \partial (H \setminus \text{int} B) \), then \( E(H \setminus \text{int} B) \) is boundary-reducible, which implies that \( H \setminus \text{int} B \) is trivial, a contradiction. Hence \( C \) is inessential in \( \partial (H \setminus \text{int} B) \). Let \( D' \) be the disk in \( \partial (H \setminus \text{int} B) \) such that \( \partial D' = C \). Let \( D_1, D_2 \) be the disks such that \( S \cap H = D_1 \cup D_2 \). If \( C \) is parallel to \( \partial D_0 \) on \( \partial (H \cup B) \), then \( F \) is an annulus obtained from a knotted handle decomposing sphere for the trivial genus two handlebody-knot \( H \cup B \), a contradiction. Thus \( D_1, D_2 \subset D' \) or \( (D_1 \cup D_2) \cap D' = \emptyset \), which contradicts that the 2-sphere \( S' = D' \cup D \) separates \( D_1 \) and \( D_2 \), where \( S' \) is slightly isotoped so that \( D' \) is properly embedded in \( H \setminus \text{int} B \). Hence \( A \cap A' = \emptyset \), which implies that \( A' \subset W \).

The annulus \( A' \) is incompressible in \( W \), since it is incompressible in \( E(H) \). If \( A' \) is boundary-parallel in \( W \), then \( A' \) is parallel to \( A \) and is obtained from a knotted handle decomposing sphere for \( H \), since \( A' \) is not boundary-parallel in \( E(H) \). Hence \( A' \) is essential in the genus two handlebody \( W \).

By Lemma 3.3, the separating annulus \( A' \) cuts \( W \) into a solid torus \( W_1 \) and a genus two handlebody \( W_2 \) so that \( A' \) winds around \( W_1 \) at least twice. If \( A \) is contained in \( \partial W \cap W_1 \), then by attaching a 2-handle \( N(D) \) to the solid torus \( W_1 \), we have a once punctured lens space \( L(p, q) \) \( (p \geq 2) \), where \( D \) is a component of \( S \cap H \). This contradicts Alexander’s theorem [1]. Thus \( A \) is contained in \( \partial W \cap W_2 \) and \( A' \) cuts \( W \cup B \) into \( W_1 \) and \( W_2 \cup B \).

Suppose that \( A' \) is compressible in \( W \cup B \). Let \( D \) be a compressing disk for \( A' \) in \( W \cup B \). Then \( D \) is contained in \( W_2 \cup B \), since \( A' \) is incompressible in \( W \). By attaching a 2-handle \( N(D) \) to the solid torus \( W_1 \), we have a once punctured lens space \( L(p, q) \) \( (p \geq 2) \), a contradiction. Thus \( A' \) is incompressible in \( W \cup B \). Suppose that \( A' \) is boundary-parallel in \( W \cup B \). Since \( A' \) is not boundary-parallel in \( W \), \( W_2 \cup B \) is a solid torus \( A' \times I \). Then the solid torus \( W_1 \) is isotopic to \( W \cup B = E(H \setminus \text{int} B) \), which implies that \( H \setminus \text{int} B \) is trivial, a contradiction. Thus \( A' \) is not boundary-parallel in \( W \cup B \). Therefore \( A' \) is essential in \( W \cup B = E(H \setminus \text{int} B) \), which is the exterior of the tunnel number one knot represented by the core curve of \( H \) \( \setminus \text{int} B \). By Lemma 3.2, \( A' \) is a cabling annulus for \( H \) \( \setminus \text{int} B \), where we note that a tunnel number one knot is prime. □

Proof of Theorem 3.1. If \( H_1 \) and \( H_2 \) are equivalent, then there is an orientation preserving self-homeomorphism of \( S^3 \) which sends \( H_1 \) to \( H_2 \), which gives an orientation preserving homeomorphism from \( E(H_1) \) to \( E(H_2) \).
Suppose that there is an orientation preserving homeomorphism \( f \) from \( E(H_1) \) to \( E(H_2) \). Since any cabling annulus cuts off a solid torus from \( E(H_2) \), it follows from Lemma 3.4 that \( f(S_1 \cap E(H_1)) = S_2 \cap E(H_2) \). Since \( E(H_i - \text{int} B_i) \) and \( B_i - \text{int} H_i \) are exteriors of knots, by the Gordon-Luecke theorem [3], both of the restrictions of \( f \) to \( E(H_1 - \text{int} B_1) \) and \( B_1 - \text{int} H_1 \) are extended to homeomorphisms of \( S^3 \). Hence \( f \) can be extended to a homeomorphism \( \hat{f} \) of \( S^3 \) such that \( \hat{f}(S_1) = S_2 \) and \( \hat{f}(H_1) = H_2 \). □

Example 3.5. By Example 2.6, neither 6_15 nor 6_1^* is equivalent to 6_14. We recall that each of them has a unique knotted handle decomposing sphere. By Theorem 3.1, there is no orientation preserving/reversing homeomorphism from \( E(6_14) \) to \( E(6_15) \). Hence \( E(6_{14}) \) and \( E(6_{15}) \) are not homeomorphic.

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