Research Article

A Restriction for Singularities on Collapsing Orbifolds

Yu Ding

Department of Mathematics and Statistics, California State University, Long Beach, CA 90840, USA

Correspondence should be addressed to Yu Ding, yding@csulb.edu

Received 8 August 2011; Accepted 5 September 2011

Academic Editors: S. Kar, U. Lindström, and E. H. Saidi

Copyright © 2011 Yu Ding. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Every point \( p \) in an orbifold \( X \) has a neighborhood that is homeomorphic to \( G_p \backslash B_r(0) \), where \( G_p \) is a finite group acting on \( B_r(0) \subset \mathbb{R}^n \), so that \( G_p \backslash B_r(0) \) is compact. Assume \( X \) is a Riemannian orbifold with isolated singularities that is collapsing, that is, \( X \) admits a sequence of metrics \( g_i \) with uniformly bounded curvature, so that, for any \( x \in X \), the volume of \( B_1(x) \), with respect to the metric \( g_i \), goes to 0 as \( i \to \infty \). For such \( X \), we prove that \( |G_p| \leq (2\pi/0.47)^{n(n-1)} \) for all singularities \( p \in X \).

1. Introduction

An \( n \)-dimensional Riemannian orbifold, \( X \), is a metric space so that the following is true: for any \( x \in X \), there exists \( r = r(x) > 0 \) and a Riemannian metric \( \tilde{g}_x \) on \( B_{2r}(0) \subset \mathbb{R}^n \), a finite group \( G_x \) (the isotropy group) acting on \( (B_r(0), \tilde{g}_x) \) by isometries, so that \( G_x(0) = 0 \), and there is an isometry \( \iota_x : B_r(x) \to G_x \backslash B_r(0) \) with \( \iota_x(x) = 0 \) (see [1]). We call \( x \in X \) a regular point if \( |G_x| = 1 \); otherwise, \( x \) is a singular point. We say the curvature of \( X \) satisfies

\[
|K_X| \leq \kappa^2, \tag{1.1}
\]

if the sectional curvature \( K \) of every \( (B_r(0), \tilde{g}_x) \) above satisfies \( |K| \leq \kappa^2 \). We say \( X \) is collapsing, if \( X \) admits a sequence of metrics, \( g_i \), with uniformly bounded curvature, so that, for any \( x \in X \),

\[
\lim_{i \to \infty} \operatorname{Vol}_{g_i}(B_1(x)) = 0. \tag{1.2}
\]

As an example, consider the standard \( \mathbb{Z}_m = \mathbb{Z} / m\mathbb{Z} \) action on the sphere \( S^2 \).

The quotient orbifold \( X_m = \mathbb{Z}_m \backslash S^2 \) will be arbitrarily collapsed when \( m \to \infty \) (see Figure 1). However, for any fixed \( m \), \( X_m \) can be collapsed only to a certain degree; it does not support a sequence of collapsing metrics. In fact, for each one of the two singularities
on $X_m$, there is a neighborhood that is isometric to $Z_m \setminus \mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with some $Z_m$ invariant metric. Therefore if $g_i$ is a collapsing sequence of metrics on $X_m$, we get a corresponding sequence $\tilde{g}_i$ of pullback metrics on $S^2$; every $\tilde{g}_i$ is smooth. Observe $\text{Vol}(S^2, \tilde{g}_i) = m\text{Vol}(X_m, g_i)$, where $m$ is fixed and $\lim_{i \to \infty} \text{Vol}(X_m, g_i) = 0$, thus $\lim_{i \to \infty} \text{Vol}(S^2, \tilde{g}_i) = 0$. If the diameter of $(X_m, g_i)$ stays bounded, we immediately get a contradiction to the Gauss-Bonnet theorem; in general, we can use the result in [2] to conclude that $S^2$ admits an $F$-structure, in particular the Euler characteristic $\chi(S^2)$ vanishes—this is a contradiction since clearly $\chi(S^2) = 2$.

On the other hand, consider the double of a 2-dimensional rectangle. Clearly it admits a flat metric, thus we obtain a sequence of collapsing metrics by rescale. Notice, in this example, for each of the four singularities, the isotropy group $G_\epsilon$ has order 2, a quite small number.

Intuitively, these examples suggest that when an orbifold is collapsing, a conelike singularity cannot be too “sharp,” that is, there should be some bound in $|G_\epsilon|$. The main result of this paper is as follows.

**Theorem 1.1.** Assume $X$ is a compact, collapsing orbifold, $p \in X$ is an isolated singularity. Then $|G_p| \leq (2\pi/0.47)^{n(n-1)}$.

If $X$ has an isolated singularity $p$, then the dimension of $X$ must be even, and $G_p \subset SO(n)$. Theorem 1.1 fails if we drop the requirement that $x$ is an isolated singularity; for example, we can take any orbifold $X'$ and let $X = X \times S^1$; by shrinking the $S^1$ factor, we see $X$ is collapsing while there is no restriction on singularities of $X'$. The bound $|G_p| \leq (2\pi/0.47)^{n(n-1)}$ has its root in the Bieberbach theorem of crystallographic groups and Gromov’s almost flat manifold theorem.

Clearly, Theorem 1.1 is a corollary of the following.

**Theorem 1.2.** For any $L > 0$, there is $\epsilon = \epsilon(n, L)$ so that if $X$ is an orbifold with all singularities $q \in X$ satisfying $|G_q| < L$, Vol$(B_1(q)) < \epsilon$, then $|G_p| \leq (2\pi/0.47)^{n(n-1)}$ for any isolated singularity $p \in X$.

**Remark 1.3.** The bound $|G_p| \leq (2\pi/0.47)^{n(n-1)}$ in Theorem 1.1 is not sharp. When $n = 2$, it is not hard to see that either $|G_p| = 2$ or $X$ is a flat orbifold. Therefore by Polya and Niggli’s classification of crystallographic groups on $\mathbb{R}^2$ [3, page 105] or [4, page 228], we actually have $|G_p| \leq 6$ for collapsing 2 orbifolds.

A *nilmanifold*, $\Gamma \setminus N$, is the quotient of the (left) action of a discrete, uniform subgroup $\Gamma \subset N$, on a simply connected nilpotent Lie group $N$. Left invariant vector fields (LIVFs) can be defined on $\Gamma \setminus N$. An affine diffeomorphism of $\Gamma \setminus N$ is a diffeomorphism that maps any local LIVF to some local LIVF. In general, a right invariant vector field (RIVF) cannot be defined globally in $\Gamma \setminus N$, unless this vector field is in the center of the Lie algebra of $N$. However, the right invariant vector fields, not the left invariant ones, are Killing fields of left invariant metrics on $\Gamma \setminus N$. An *infranal orbifold* is the quotient of a nilmanifold by the action of a finite group $H$ of affine diffeomorphisms. If the action $H$ is free, we get an infranal manifold.
In our previous work, [5], we generalized the Cheeger-Fukaya-Gromov nilpotent Killing structure [6] and the Cheeger-Gromov F-structure, [2, 7], to collapsing orbifolds. In particular, sufficiently collapsed X can be decomposed into a union of orbits. Each orbit $O_p$ is the orbit of the action of a sheaf $\mathfrak{n}$ of nilpotent Lie algebras, which comes from local RIVFs on a nilmanifold fibration in the frame bundle $FX$. Therefore every $O_p$ is an infranil orbifold. The proof of Theorem 1.2 is based on the relation between singularities on $X$ and singularities within an orbit $O_p$ in $X$, as well as the nilmanifold fibration on $FX$.

X is called almost flat, if

$$\sup |K_X|^{1/2} \cdot \text{Diam } X \leq \delta_n,$$

where Diam $X$ is the diameter of $X$, $\delta_n$ is a small constant that depends only on $n$. In [8], Gromov proved that an almost flat manifold $M$ has a finite, normal covering space $\tilde{M} = \Gamma \setminus N$ that is a nilmanifold. Subsequently, Ruh [9] proved that $M$ is diffeomorphic to $\Lambda \setminus N$, where $\Lambda \supset \Gamma$ is a discrete subgroup in the affine transformation group of $N$. In [10], Ghanaat generalized this to an almost flat orbifold $X$, under the assumption that $X$ is good in the sense of Thurston [1], that is, $X$ is the global quotient of a simply connected manifold $M$. There are examples of orbifolds that are not good, see [1]. In fact, without much effort, one can remove the assumption that $X$ is good.

**Proposition 1.4.** If $X$ is an almost flat orbifold, then $X$ is an infranil orbifold.

Precisely, there is a nilmanifold $\tilde{X} = \Gamma \setminus \tilde{N}$, a finite group $H$ acting on $\tilde{X}$ by affine diffeomorphism, so that $X$ is diffeomorphic to $H \setminus \tilde{X}$. The order of $H$ is bounded by $c_n \leq (2\pi / 0.47)^{n(n-1)/2}$.

Moreover, there is a sequence of metrics $g_j$ so that $\text{Diam}(X, g_j) \to 0$.

The proof is almost the same as [11, 12]; the only difference is one must replace the exponential map by the develop map (see [5, 13]) and modify the definition of Gromov product in [11] accordingly.

The proof of Theorem 1.2 does not depend on Proposition 1.4. On the other hand, Proposition 1.4 implies Theorem 1.2 for almost flat orbifolds immediately, even without the assumption that the singularities are isolated.

**Remark 1.5.** If $p \in X$ is an isolated singularity, then, near $p$, $X$ is homeomorphic to (and in the metric sense, close to) a metric cone over a space form of dimension $n - 1$. When $n = 4$, the $4 - 1 = 3$-dimensional space forms were first classified by Threlfall and Seifert, they used the fact that $SO(4)$ is locally isomorphic to $SO(3) \times SO(3)$; [3, chapter 7] or [4] for details.

**Remark 1.6.** By the work of Anderson, Gao, Nakajima, Tian, Yang, and others, orbifolds with discrete singularities appear naturally as Gromov-Hausdorff limits of noncollapsing Einstein metrics with a uniform $L^{n/2}$ curvature bound; see [14] for a recent survey. In particular, for Kahler-Einstein metrics, there is a complex structure on the limit $X$.

**2. Proof of Theorem 1.2**

If $X$ is an infranil orbifold, then it is easy to obtain the bound in Theorem 1.1. Since the proof contains some ideas for the general case, we give full details.

**Lemma 2.1.** Assume $X$ is an infranil orbifold. Then $|G_x| \leq (2\pi / 0.47)^{n(n-1)/2}$.

**Proof.** Assume $X = \Lambda \setminus N$, where $N$ is a simply connected nilpotent Lie group, $\Lambda$ is a discrete group of affine diffeomorphisms on $N$ so that $X = \Lambda \setminus N$ is compact. If $N$ is
abelian, then $X$ is a flat orbifold, $\Lambda$ is a discrete group of isometries on $N = \mathbb{R}^n$ that acts properly discontinuously. So the conclusion follows from (the proof of) Bieberbach’s theorem on crystallographic groups. In fact, it is well known that the maximal rotational angle of any $\lambda \in \Lambda$ is either 0 or at least $1/2$. Thus the bound comes from a standard packing argument; notice $n(n - 1)/2 = \dim SO(n)$ and the bi-invariant metric on $SO(n)$ has positive curvature.

We prove the general case by induction on dimension of $X$. Remember that $\Lambda$ contains a normal subgroup $\Gamma$ of infinite index, so that $\Gamma$ is a uniform, discrete subgroup of $N$ and $X$ is the quotient of the $\Lambda/\Gamma$ action on the nilmanifold $\tilde{X} = \Lambda \setminus N$. Clearly $G_x$ embeds in $\Lambda/\Gamma$, that is, $G_x = \{ \bar{\lambda} \in \Lambda/\Gamma \mid \bar{\lambda}x = \bar{x} \}$; here we choose a point $\bar{x}$ in $\tilde{X} = \Lambda \setminus N$ that projects to $x \in \Lambda \setminus N$.

Let $C$ be the center of $N$, then $C$ is connected, of positive dimension. Since any $\lambda \in \Lambda$ is affine diffeomorphism, $\lambda$ moves a $C$-coset in $N$ to a $C$-coset. Therefore $\Lambda/\Gamma$ acts on the nilmanifold $\tilde{X}^* = (\Gamma/(\Gamma \cap C)) \setminus (N/C)$, the quotient $\tilde{X}^*$ is an infranil orbifold of lower dimension. Let $\pi : \tilde{X} \to \tilde{X}^*$ be the projection, and assume $\pi(\bar{x}) = \bar{x}^*$. Thus we have a homomorphism

$$h : G_x \longrightarrow G_x^*.$$  

(2.1)

$\tilde{X}$ is a torus bundle over $\tilde{X}^*$, the fiber is $T = (\Gamma \cap C) \setminus C$. Assume $\bar{\lambda} \in \Lambda/\Gamma$ is in $\ker h$, the kernel of $h$, then $\bar{\lambda}$ fixes every $T$ fiber in $\tilde{X}$. If, in addition, $\bar{\lambda}$ fixes every point in the $T$ fiber passing through $\bar{x}$, we claim $\bar{\lambda}$ must be identity. In fact, on $N$ we have $\lambda(z) = a \cdot A(z)$, where $a \in N$ and $A$ is a Lie group automorphism of $N$; if $\bar{\lambda}$ fixes every point in one $T$ fiber, then $A$ is identity on the center $C \subset N$. This implies that $\bar{\lambda}$ is a translation on every $T$ fiber. Since $\bar{\lambda}$ is of finite order and fixes every point in one $T$-fiber, $\bar{\lambda}$ must be identity. Therefore any element $\bar{\lambda} \in \ker h$ is decided by its restriction on the $T$ fiber passing through $x$; so $\ker h$ is isomorphic to a finite group of affine diffeomorphisms on $T$ that fixes $\bar{x} \in T$, thus $|\ker h|$ can be bounded by Bieberbach’s theorem. Since

$$|G_x| \leq |G_x^*| \cdot |\ker h|,$$  

(2.2)

the conclusion follows by induction.

In [5], the existence of nilpotent Killing structure of Cheeger-Fukaya-Gromov [6] is generalized to sufficiently collapsed orbifolds. We briefly review this construction.

As in the manifold case, one can define the frame bundle $FX$ of an orbifold $X$. If $B_r(x) \subset X$ is isometric to $G_x \setminus B_r(0)$, where $G_x$ is a finite group acting on $B_r(0) \subset \mathbb{R}^n$, then locally $FX$ is $G_x \setminus FB(0, r)$, where $FB(0, r)$ is the orthonormal frame bundle over $B_r(0)$, and $G_x$ acts on $FB(0, r)$ by differential, that is, $\tau \in G_x$ moves a frame $\mathbf{u}$ to $\tau \mathbf{u}$. Therefore $FX$ is a manifold; strictly speaking, $FX$ is not a fiber bundle. Let $\pi : FX \to X$ be the projection.

Moreover, there is a natural $SO(n)$ action on $FX$; on the frames over regular points, this $SO(n)$ action is the same one as in the manifold case; however, at the frames over singular points, this action is not free. As in the work of Fukaya [15], see also [5], any Gromov-Hausdorff limit $Y$ of a collapsing sequence $FX$, is a manifold. Following [6], for sufficiently collapsing orbifolds, locally we have an $SO(n)$-equivariant fibration

$$Z \to FX \overset{f}{\to} Y,$$  

(2.3)

where the fiber $Z$ is a nilmanifold, $Y$ is a smooth manifold with controlled geometry.
As in [6], we can put a canonical affine structure on the $Z$ fibers, that is, a canonical way to construct a diffeomorphism from a fiber $Z$ to the nilmanifold $\Gamma \backslash N$. In particular, there is a sheaf $n$, of a nilpotent Lie algebra of vector fields on $FX$. Sections of $n$ are local right invariant vector fields on the nilmanifold fibers $Z$. By integrating $n$, we get a local action of a simply connected nilpotent Lie group, $\mathfrak{g}$, on $FX$. Therefore we also call a $Z$ fiber an orbit, and we can write $Z = \tilde{\mathcal{O}}$.

The fibration $f : FX \to Y$ is $SO(n)$-equivariant, so any $Q \in SO(n)$ moves a $Z$ fiber to a (perhaps another) $Z$ fiber by affine diffeomorphism. Moreover, the $SO(n)$ action on $n$ is locally trivial, that is, if $A \in so(n)$ is sufficiently small, then $e^{\lambda} \in SO(n)$ moves a section, $n(U)$, of $n$ on any open set $U \subset FX$, to itself (over $U \cap Ue^{\lambda}$) ([6, Proposition 4.3]). In particular, the sheaf $n$ induces a sheaf, which we also denote by $n$, on the orbifold $X$ away from the singular points. An orbit $\tilde{\mathcal{O}} = Z$ on $X$ projects down to an orbit $\mathcal{O}$ on $X$.

Assume $\tilde{q} \in FX$ is any frame over $q \in X$. Let
\[
I(q) = \{ Q \in SO(n) \mid Z_qQ = Z_{\tilde{q}} \}
\] (2.4)
be the the isotropy group of an orbit $Z_q = \mathcal{O}_q \subset FX$. We will simply write $I(q)$ by $I$. Let $I_0$ be the identity component of $I$. It can be shown that, restricted on $Z = \tilde{\mathcal{O}}_{\tilde{q}}$, the action of $I_0$ is identical to the action of a torus, and the Lie algebra of this torus, $I_0$, is in the center of $n$ (see [5, 6] for more details). Consider the nilmanifold
\[
\tilde{\mathcal{O}} = \frac{\tilde{\mathcal{O}}_{\tilde{q}}}{I_0}.
\] (2.5)
Therefore $Z_{\tilde{q}} = \tilde{\mathcal{O}}_{\tilde{q}}$ is a torus bundle over $\tilde{\mathcal{O}}_{\tilde{q}}$. Notice, on $Z = \tilde{\mathcal{O}}_{\tilde{q}}$, $I$ moves $I_0$ fibers to $I_0$ fibers, thus the orbit $\mathcal{O}_q$ is the quotient of $\tilde{\mathcal{O}}_{\tilde{q}}$ by the action of the finite group $I/I_0$. Therefore $\mathcal{O}_q$ is an infranil orbifold. In particular, the singularities within $\mathcal{O}_q$ satisfy the bound in Lemma 2.1.

It is important to remark that the above structure is not trivial.

**Lemma 2.2.** Let $L$ be any integer. Then there is $e \in e(n, L)$, so that if $X$ is an orbifold with $|G_x| \leq L$, $\text{Vol}(B_t(x)) \leq e$ for all $x \in X$, then every $n$-orbit $\mathcal{O}$ on $X$ is of positive dimension.

**Proof (sketch).** For any unit vector $A \in so(n)$, the bound in $|G_x|$ implies that $e^{\lambda}A$ does not have fixed point in $\tilde{\mathcal{O}}_{\tilde{q}}$ unless $t = 0$ or $|t| \leq cL^{-1}$. However, for sufficiently collapsed orbifolds, there is a vector $B$ in the center of $n$ so that $B$ generates a closed loop in $\tilde{\mathcal{O}}_{\tilde{q}}$ that is shorter than $cL^{-1}$, therefore $B$ cannot be in the Lie algebra of $I_0$, which is in both $so(n)$ and the center of $n$. Thus the orbit $\tilde{\mathcal{O}}_{\tilde{q}}$ is not contained in a single $SO(n)$ orbit in $FX$, so $\mathcal{O}_q$ is of positive dimension in $X = FX/\text{SO}(n)$ (see [5] for more details). □

**Proof of Theorem 1.2.** Assume $p \in X$ is an isolated singular point, $\tilde{p} \in \pi^{-1}(p)$ is in $FX$. $Z_{\tilde{p}} = \tilde{\mathcal{O}}_{\tilde{p}}$ is the fiber that projects to $\mathcal{O}_p$. Let $I(p)$, $I_0$, $\tilde{\mathcal{O}}_{\tilde{p}}$ be as above. Let
\[
K_p = \{ Q \in SO(n) \mid \tilde{p}Q = \tilde{p} \}.
\] (2.6)
Thus $K_p$ is a subgroup of $I$, and $|K_p| = |G_p|$. Let
\[
K_{\tilde{p}} = \{ Q \in K_p \mid (\tilde{p}I_0)Q = (\tilde{p}I_0) \forall \tilde{p}' \in Z_{\tilde{p}} \}.
\] (2.7)
Thus $K_{\tilde{p}}$ is a normal subgroup of $K_p$. 
Lemma 2.3. If $Q \in K_p$ fixes every point in $Z_p$, then $Q$ is the identity in $SO(n)$.

Proof. Potentially $Q$ may fix every point in $Z_p$ while moving some points of $FX$ that are outside $Z_p$. We will rule out this possibility.

By assumption, $p$ is an isolated singularity. For any $Q$ that is not identity, the connected component of the fixed point set of $Q$ that passes through $\tilde{p}$ must project to $p$ under $\pi : FX \to X$, because away from $\pi^{-1}(p)$ the $SO(n)$ action is free. Therefore $\pi(Z_p) = p$ is a single point in $X$, and this contradicts the fact that the $n$-orbits on $X$ are of positive dimension; see Lemma 2.2.

In particular, we have a faithful representation of $K_p$ in the affine group of $Z_p$, that is, we can identify $K_p$ with the restricted action of the group $K_p$ on $Z_p$. Take any $Q \in K_p$ that is not identity in $SO(n)$. By definition $Q$ fixes $\tilde{p}$. If $Q$ fixes every point in $\tilde{p}I_0$, as in Lemma 2.1, $Q$ is a translation on every $I_0$ fiber; because $Q$ is of finite order and $Q$ moves every $I_0$ fiber to itself, $Q$ necessarily fixes every point in $Z$; thus $Q$ is identity. So $Q$ rotates the tangent plane of $\tilde{p}I_0$ at $\tilde{p}$. Therefore $K_p^{-}$ is isomorphic to a finite group of affine diffeomorphisms on the torus $\tilde{p}I_0$.

By the Bieberbach theorem,

$$|K_p^{-}| \leq \left(\frac{2\pi}{0.47}\right)^{k(k-1)/2}, \quad k = \dim I_0. \quad (2.8)$$

Recall that Bieberbach’s theorem implies that all finite subgroups of $SL(n, \mathbb{Z})$ have a uniform upper bound in order. We have

$$O_p = \frac{H}{(Z_p / I_0)}, \quad (2.9)$$

where $H = I / I_0$ is a finite group. Let $H_p$ be the subgroup of $H$ that fixes $p$. Now we get an embedding

$$\frac{K_p}{K_p^-} \subset H_p. \quad (2.10)$$

By Lemma 2.1,

$$|H_p| \leq \left(\frac{2\pi}{0.47}\right)^{i(i-1)/2}, \quad i = \dim O_p. \quad (2.11)$$

Thus

$$|G_p| = |K_p| = \left|\frac{K_p}{K_p^-}\right| \cdot \left|K_p^{-}\right| \leq \left(\frac{2\pi}{0.47}\right)^{n(n-1)}. \quad (2.12)$$

Acknowledgment

The author is grateful to Professor Tian for very helpful suggestions.
References

[1] W. Thurston, *The Geometry and Topology of 3-Manifolds*, Princeton University, 1979.
[2] J. Cheeger and M. Gromov, “Collapsing Riemannian manifolds while keeping their curvature bounded. II,” *Journal of Differential Geometry*, vol. 32, no. 1, pp. 269–298, 1990.
[3] J. A. Wolf, *Spaces of Constant Curvature*, AMS Chelsea Publishing, Providence, RI, USA, 6th edition, 2011.
[4] W. P. Thurston, *Three-Dimensional Geometry and Topology. Vol. 1*, vol. 35, Princeton University Press, Princeton, NJ, USA, 1997.
[5] Y. Ding, “F-structure on collapsed orbifolds,” http://www.csulb.edu/~yding/orbifold.pdf.
[6] J. Cheeger, K. Fukaya, and M. Gromov, “Nilpotent structures and invariant metrics on collapsed manifolds,” *Journal of the American Mathematical Society*, vol. 5, no. 2, pp. 327–372, 1992.
[7] J. Cheeger and M. Gromov, “Collapsing Riemannian manifolds while keeping their curvature bounded—I,” *Journal of Differential Geometry*, vol. 23, no. 3, pp. 309–346, 1986.
[8] M. Gromov, “Almost flat manifolds,” *Journal of Differential Geometry*, vol. 13, no. 2, pp. 231–241, 1978.
[9] E. A. Ruh, “Almost flat manifolds,” *Journal of Differential Geometry*, vol. 17, no. 1, pp. 1–14, 1982.
[10] P. Ghanaat, “Diskrete Gruppen und die Geometrie der Repèrebündel,” *Journal für die Reine und Angewandte Mathematik*, vol. 492, pp. 135–178, 1997.
[11] P. Buser and Karcher, “Gromov’s Almost Flat Manifolds,” *Astérisque*, vol. 81, 1981.
[12] P. Ghanaat, “Almost Lie groups of type $\mathbb{R}^n$,” *Journal für die Reine und Angewandte Mathematik*, vol. 401, pp. 60–81, 1989.
[13] X. X. Chen and G. Tian, “Ricci flow on Kähler-Einstein manifolds,” *Duke Mathematical Journal*, vol. 131, no. 1, pp. 17–73, 2006.
[14] M. T. Anderson, “A survey of Einstein metrics on 4-manifolds,” in *Handbook of Geometric Analysis, No. 3*, vol. 14, pp. 1–39, International Press, Somerville, Mass, USA, 2010.
[15] K. Fukaya, “A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters,” *Journal of Differential Geometry*, vol. 28, no. 1, pp. 1–21, 1988.
Submit your manuscripts at http://www.hindawi.com