THE CONVEX HULL OF A CONVEX SPACE CURVE WITH FOUR VERTICES

JAKOB BOHR, STEEN MARKVORSEN, AND MATTEO RAFFAELLI

Abstract. We obtain an upper bound for the volume of the convex hull of a simple closed Frenet curve having exactly four vertices, i.e., four points of vanishing torsion, and lying on the boundary of its convex hull. Moreover, we show that the upper bound is attained when the curve intersects every plane in at most four points, a condition studied by Scherk and Segre in the 1930s. The proof hinges on the fact that, under the four-vertex assumption, the convex hull is a union of line segments, and so it admits an elementary parametrization. We also comment on a question posed by Newson in 1899.

1. Introduction

In 1934, Bonnesen and Fenchel [2, p. 117] proposed the following isoperimetric problem; see also [3, section A28].

Problem 1.1. Among all closed curves in $\mathbb{R}^3$ of a given length, find the one whose convex hull has maximal volume.

Remark 1.2. It is clear that Problem 1.1 makes sense in arbitrary dimension and also for open curves.

As of today, only some special cases of Problem 1.1 are solved: the case of open convex space curves [6], where “convex” means that no plane intersects the curve in more than three points; the case of closed curves in $\mathbb{R}^{2d}$ [19], again assuming that no hyperplane intersects the curve in more that $2d$ points; and that of closed space curves satisfying certain symmetry assumptions [13]. For other related results, see [5, 15, 23, 4].

While the interest in Problem 1.1 has generated many important results about convex hulls and their volumes, most of them are valid under the restrictive convexity assumption mentioned above. On the other hand, closed convex curves do not exist in odd dimensions [1, Corollary 4.6]. Hence, not much is known about the volume of the convex hull of an arbitrary closed space curve, the only relevant estimate being the one given by Tilli in [22, Corollary 1.4].

The purpose of this short note is to present a volume estimate for the convex hull of a simple closed convex curve in $\mathbb{R}^3$ having (positive curvature and) precisely four vertices, i.e., four points of zero torsion; here, and in the rest of the paper, by convex we mean that the curve lies on the boundary of its convex hull. While these conditions are restrictive, it is known from the work of Scherk and Segre during the
1930s that if a simple closed convex curve intersects every plane in at most four points, then it has exactly four vertices \([15, 21, 16, \text{p. 357}]\); moreover, the converse does not hold \([8]\). Recall that four is the minimum number of vertices any such curve can have, and that the torsion changes sign at each of them \([20, 10]\).

To state our main results, let \(\text{Imm}^{k \geq 1}([a, b], \mathbb{R}^3)\) be the space of \(C^k\) curves \(\beta : [a, b] \to \mathbb{R}^3\) with positive speed. We say that \(\beta\) is closed if \(\beta^{(h)}(a) = \beta^{(h)}(b)\) for \(h \leq k\), and that \(\beta\) is simple if it is injective on \([a, b]\). Once and for all, let \(\gamma \in \text{Imm}^3([a, b], \mathbb{R}^3)\) be a simple closed convex curve with unit speed and nowhere vanishing curvature, and let \(\text{conv}(\gamma)\) be its convex hull.

**Theorem 1.3.** If \(\gamma\) has exactly four vertices, then

\[
\text{vol}(\text{conv}(\gamma)) \leq \frac{1}{24} \int_a^b \int_a^b |\det(\gamma'(t_1), \gamma'(t_2), \gamma(t_1) - \gamma(t_2))| \, dt_1 \, dt_2,
\]

where \(\text{vol}\) denotes volume.

Clearly, the integrand in (1) is bounded from above by \(\text{diam}(\gamma) := \max_{[a, b]^2} |\gamma(t_1) - \gamma(t_2)|\).

**Corollary 1.4.** If \(\gamma\) has exactly four vertices, then

\[
\text{vol}(\text{conv}(\gamma)) < \frac{1}{24} \text{diam}(\gamma) \text{length}(\gamma)^2.
\]

**Remark 1.5.** Since \(\gamma\) is closed, we have \(\text{length}(\gamma) > 2 \text{diam}(\gamma)\), which implies \(\text{vol}(\text{conv}(\gamma)) < 1/48 \text{length}(\gamma)^3\) when the number of vertices is four. This should be compared with the general estimate in \([22, \text{Corollary 1.3}]\), which predicts \(\text{vol}(\text{conv}(\gamma)) < 1/36 \text{length}(\gamma)^3\).

**Remark 1.6.** If \(\gamma\) lies on a sphere of radius \(R\), then \(\text{diam}(\gamma) \leq 2R\). So \(\text{vol}(\text{conv}(\gamma)) < R/12 \text{length}(\gamma)^2\) when it has exactly four vertices.

The proof of Theorem 1.3 whose details are worked out in the next two sections, may be summarized as follows. When \(\gamma\) has exactly four vertices, its convex hull is a union of chords, i.e., line segments with endpoints on \(\gamma([a, b])\), and so it admits a parametrization of the form \((t_1, t_2, u) \mapsto \gamma(t_1) + u(\gamma(t_2) - \gamma(t_1)).\) Besides, the convexity assumption guarantees that \(\gamma\) is star-shaped with respect to any point \(p\) in the interior of its convex hull. Projecting \(\gamma\) to a sphere centered at \(p\), we obtain a spherical curve \(\gamma_p\) having as many pairs of antipodal points as there are line segments passing from \(p\). In particular, the projected curve must have at least two pairs of antipodal points, for otherwise \(p\) would be a boundary point. This means that \(p\) is covered at least four times by the parametrization, and the inequality follows.

In \([21]\), Segre proves that if \(\gamma\) meets every plane in at most four points, then not only is the convex hull a union of chords, but any point in its interior is contained in exactly two chords. Hence our proof of Theorem 1.3 leads to the following corollary.

**Corollary 1.7.** If no plane intersects \(\gamma\) in more than four points, then

\[
\text{vol}(\text{conv}(\gamma)) = \frac{1}{24} \int_a^b \int_a^b |\det(\gamma'(t_1), \gamma'(t_2), \gamma(t_1) - \gamma(t_2))| \, dt_1 \, dt_2.
\]

As we explain in section 4, Corollary 1.7 gives a partial answer to an old question of Newson \([14]\).
2. The radial projection

Let \( o \) be the origin of \( \mathbb{R}^3 \), and suppose that \( o \in \text{int} \, \text{conv}(\gamma) \). Note that, since \( \gamma \) is convex, no ray emanating from \( p \in \text{int} \, \text{conv}(\gamma) \) intersects \( \gamma \) more than one point. Hence the radial projection (of \( \gamma \) with respect to \( p \))

\[
\gamma_p := (\gamma - p)/\|\gamma - p\|
\]

is a well-defined simple closed curve on \( S^2 \). The purpose of this section is to record the following basic observation, which will play a key role in the proof of Theorem 1.3,

\[\text{Lemma 2.1.} \quad \text{Suppose that } \gamma_p := \gamma/\|\gamma\| \text{ intersects its antipodal reflection at a single pair of points } \pm q \in S^2, \text{ and let } C_q \text{ be the great circle tangent to } \gamma_p \text{ at } q. \text{ Then, for any sufficiently small } p \in \mathbb{R}^3 \text{ not parallel to any element of } C_q, \text{ the curve } \gamma_p := (\gamma - p)/\|\gamma - p\| \text{ either intersects its antipodal reflection at two pairs of points or does not intersect it at all; if it does, then } \gamma_{-p} \text{ does not.} \]

\[\text{Proof.} \quad \text{To begin with, note that when } \gamma_p \text{ and its antipodal reflection intersect at a single pair of points, they must do so tangentially, as otherwise they would cross each other [9]. Note 8.2]; hence the great circle } C_q \text{ is tangent to both } \gamma_p \text{ and } -\gamma_p \text{ at } \pm q. \text{ Further note that if } p \text{ is transversal to the plane containing } C_q, \text{ then it points to the same side of } \gamma_p \text{ in a neighborhood of } \pm q \text{ in } \gamma_p, \text{i.e., to the same region bounded by } \gamma_p. \text{ Clearly, as long as } p \text{ is sufficiently small, if } p \text{ points to the side of } \gamma_p \text{ having area less than } 2\pi, \text{ then the curve } \gamma_p \text{ will intersect its antipodal reflection four times, while } \gamma_{-p} \text{ will be disjoint from it.} \]

\[\square\]

3. Proof of Theorem 1.3

Here we prove Theorem 1.3. The first step amounts to showing that under the four-vertex assumption, any point in the convex hull of \( \gamma \) can be written as a convex combination of just two points of \( \gamma([a, b]) \).

\[\text{Lemma 3.1 ([12, 24]).} \quad \text{If } \gamma \text{ has exactly four vertices, then } \text{conv}(\gamma) \text{ is a union of chords, i.e., line segments with endpoints in } \gamma([a, b]). \]

\[\text{Proof.} \quad \text{Since } \gamma \text{ is connected, a theorem of Fenchel [7, 11] tells us that any point in } \text{conv}(\gamma) \text{ can be written as the convex combination of at most three points of } \gamma. \text{ Hence, assuming the hypothesis of the lemma and applying [17, Corollary 2], we deduce that the boundary of } \text{conv}(\gamma) \text{ is a union of chords. As a consequence, as explained in [12], also the interior has the same property; the argument is reproduced below for the sake of completeness.} \]

Let \( p \) be a point in the interior, which we may assume to coincide with the origin of \( \mathbb{R}^3 \), and let \( p_1 \) and \( p_2 \) be the intersections of the \( z \)-axis with the boundary of \( \text{conv}(\gamma) \). Then we know by assumption that \( p_i \) (\( i = 1, 2 \)) belongs to a line segment with endpoints \( q_i, r_i \in \gamma([a, b]) \). Next, extending \( q_1, q_2 \) to a continuous map \( q: [1, 2] \to \gamma([a, b]) \), define \( r_t \) as the intersection of the image of \( \gamma \) with the plane spanned by \( q_t \) and the \( z \)-axis, and \( p_t \) as the intersection of the chord \( (q_t, r_t) \) with the \( z \)-axis. It is clear that \( p_t \) depends continuously on \( t \) and agrees with \( p_1, p_2 \) for \( t = 1, 2 \). Hence there exists \( t \in [1, 2] \) such that the chord \( (q_t, r_t) \) intersects the \( z \)-axis in the point \( p \), which is the desired conclusion.

\[\square\]

Next we show that any point in the interior of the convex hull is hit by at least two chords.
Lemma 3.2. Suppose that $\text{conv}(\gamma)$ is a union of chords. If $p \in \text{conv}(\gamma)$ belongs to only one chord, then $p \in \partial \text{conv}(\gamma)$.

Proof. As usual, suppose that $o \in \text{int conv}(\gamma)$. It is clear that, for any $p \in \text{int conv}(\gamma)$, the radial projection

$$\gamma_p := (\gamma - p)/\|\gamma - p\|$$

has as many pairs of antipodal points as there are chords passing from $p$. Proceeding by contradiction, we suppose that $\gamma_o$ meets its antipodal reflection at a single pair of points. Then we know by Lemma 2.1 that some nearby projections will intersect their antipodal reflections four times, i.e., will have two pairs of antipodal points, while some will not intersect them at all. But since $\text{conv}(\gamma)$ is a union of chords, this implies that $p \in \partial \text{conv}(\gamma)$, contradicting our assumption that $p \in \text{int conv}(\gamma)$. \(\square\)

Now we are ready to finalize our proof.

Proof of Theorem 1.3. By Lemma 3.1, the map $\sigma : [a, b] \times [0, 1] \rightarrow \mathbb{R}^3$ defined by

$$\sigma(t_1, t_2, u) = \gamma(t_1) + u(\gamma(t_2) - \gamma(t_1))$$

is a parametrization of $\text{conv}(\gamma)$. In particular, note that $\text{conv}(\gamma)$ is covered at least twice by $\sigma$, because $\sigma(t_1, t_2, u) = \sigma(t_2, t_1, 1 - u)$. Furthermore, Lemma 3.2 tells us that any point in the interior of $\text{conv}(\gamma)$ is contained in at least two chords of $\gamma$. Hence the volume of $\text{conv}(\gamma)$ satisfies

$$\text{vol}(\text{conv}(\gamma)) \leq \frac{1}{4} \int \int [\det(J_\sigma)] \, dV,$$

where $J_\sigma$ denotes the Jacobian matrix of $\sigma$. Since

$$\det(J_\sigma) = (u - u^2) \det(\gamma'(t_1), \gamma'(t_2), \gamma(t_2) - \gamma(t_1)),$$

integration with respect to $u$ gives

$$\text{vol}(\text{conv}(\gamma)) \leq \frac{1}{24} \int_a^b \int_a^b [\det(\gamma'(t_1), \gamma'(t_2), \gamma(t_2) - \gamma(t_1))] \, dt_1 \, dt_2,$$

as desired. \(\square\)

4. Newson’s Challenge

Here we briefly comment on a question posed by Newson in 1899.

Let $\beta \in \text{Imm}^1([a, b], \mathbb{R}^3)$ be a simple closed unit-speed curve lying on a plane containing the origin. Then the area enclosed by $\beta$ is given by

$$\frac{1}{2} \left| \int_a^b \beta'(t) \times \beta(t) \, ds \right|.$$  

In [14], Newson derived the area formula (or, rather, an equivalent version of it) by interpreting $\beta([a, b])$ as a polygon with an infinite number of sides, i.e., as the limiting case of the expression

$$\frac{1}{2} \left| \sum_{i=1}^n \beta_i \times \beta_{i+1} \right|,$$

which measures the area of an inscribed polygon with $n$ vertices. He then challenged the reader of the Annals to come up with a three-dimensional version of [13] by interpreting a closed surface as a polyhedron with infinitely many faces. While we
do not provide a complete answer to this problem, in the following we explain how Newson’s idea may be used to derive formula \(2\).

As before, let \(\gamma \in \text{Imm}^3([a, b], \mathbb{R}^3)\) be a simple closed unit-speed curve with nowhere vanishing curvature, lying on the boundary of its convex hull, and suppose that no plane intersects \(\gamma\) in more than four points. Partitioning \([a, b]\) into \(n-1\) sufficiently small subintervals \([t_i, t_{i+1}]\), we may approximate \(\gamma\) by the polygonal line with vertices \(\gamma_i := \gamma(t_i)\). Then, as \(n\) grows large,

\[
\bigcup_{i,j} T_{i,j} \to \text{conv}(\gamma),
\]

where \(T_{i,j}\) denotes the tetrahedron \((\gamma_i, \gamma_{i+1}, \gamma_j, \gamma_{j+1})\), and each point in the interior of \(T_{i,j}\) is contained in the interior of exactly two tetrahedra. Since \(T_{i,j}\) has signed volume

\[
\frac{1}{6} \det(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_i, \gamma_{j+1} - \gamma_j),
\]

it follows that

\[
\text{vol}(\text{conv}(\gamma)) \approx \frac{1}{24} \sum_{i,j} |\det(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_i, \gamma_{j+1} - \gamma_j)|,
\]

where the extra factor of \(1/2\) comes from the equality \(T_{i,j} = T_{ji}\). Thus, letting \(n \to \infty\), we obtain \(2\).

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**References**

1. Carlos Améndola, Darrick Lee, and Chiara Meroni, *Convex hulls of curves: Volumes and signatures*, arXiv:2301.09405, 2023.
2. T. Bonnesen and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow, ID, 1987. MR 920366
3. Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy, *Unsolved problems in geometry*, Problem Books in Mathematics, Springer, New York, 1991. MR 1107516
4. J. de Dios Pont, P. Ivanisvili, and J. Madrid, *A new proof of the description of the convex hull of space curves with totally positive torsion*, arXiv:2201.12932v2, 2023.
5. Douglas Derry, *Convex hulls of simple space curves*, Canadian J. Math. 8 (1956), 383–388. MR 82140
6. E. Egerváry, *On the smallest convex cover of a simple arc of space-curve*, Publ. Math. Debrecen 1 (1949), 65–70. MR 36021
7. W. Fenchel, *Über Krümmung und Windung geschlossener Raumkurven*, Math. Ann. 101 (1929), 238–252.
8. Mohammad Ghomi, *Converse of Scherk–Segre theorem on the number of vertices of a convex space curve*, MathOverflow, https://mathoverflow.net/a/484132 (version: 2024-12-19).
9. , *Tangent lines, inflections, and vertices of closed curves*, Duke Math. J. 162 (2013), no. 14, 2691–2730. MR 3127811
10. , *Torsion of locally convex curves*, Proc. Amer. Math. Soc. 147 (2019), no. 4, 1699–1707. MR 3910434
11. Olof Hanner and Hans Rådström, *A generalization of a theorem of Fenchel*, Proc. Amer. Math. Soc. 2 (1951), 589–593. MR 44142
12. Paata Ivanishvili, *When is the convex hull of two space curves the union of lines?*, MathOverflow, https://mathoverflow.net/a/209240 (version: 2015-06-15).
13. Z. A. Melzak, *The isoperimetric problem of the convex hull of a closed space curve*, Proc. Amer. Math. Soc. 11 (1960), 265–274. MR 116263
14. H. B. Newson, *On the volume of a polyhedron*, Ann. of Math. (2) 1 (1899/00), no. 1-4, 108–110. MR 1502261
15. A. A. Nudel’man, *Isoperimetric problems for the convex hulls of polygonal lines and curves in multidimensional spaces*, Math. USSR-Sb. 25 (1975), no. 2, 276–294.
16. William F. Pohl, *On a theorem related to the four-vertex theorem*, Ann. of Math. (2) 84 (1966), 356–367. MR 198363
17. M. C. Romero-Fuster and V. D. Sedykh, *A lower estimate for the number of zero-torsion points of a space curve*, Beiträge Algebra Geom. 38 (1997), no. 1, 183–192. MR 1447996
18. Peter Scherk, *Über reelle geschlossene Raumkurven vierter Ordnung*, Math. Ann. 112 (1936), no. 1, 743–766. MR 1513072
19. I. J. Schoenberg, *An isoperimetric inequality for closed curves convex in even-dimensional Euclidean spaces*, Acta Math. 91 (1954), 143–164. MR 65944
20. V. D. Sedykh, *Four vertices of a convex space curve*, Bull. London Math. Soc. 26 (1994), no. 2, 177–180. MR 1272305
21. Beniamino Segre, *Intorno alle ovali sghembe, e su di un’estensione del teorema di Cavalieri–Lagrange alle funzioni di due variabili*, Accad. Ital. Mem. Cl. Sci. Fis. Mat. Nat. 7 (1936), 365–397.
22. Paolo Tilli, *Isoperimetric inequalities for convex hulls and related questions*, Trans. Amer. Math. Soc. 362 (2010), no. 9, 4497–4509. MR 2645038
23. V. A. Zalgaller, *Extremal problems on the convex hull of a space curve*, St. Petersburg Math. J. 8 (1997), no. 3, 369–379. MR 1402286
24. P. B. Zatitskiy, P. Ivanisvili, and D. M. Stolyarov, *Bellman vs Beurling: Sharp estimates of uniform convexity for Lp spaces*, St. Petersburg Math. J. 27 (2016), no. 2, 333–343. MR 3444467

Department of Applied Mathematics and Computer Science, Technical University of Denmark, Matematiktorvet, Building 303B, 2800 Kongens Lyngby, Denmark

Department of Applied Mathematics and Computer Science, Technical University of Denmark, Matematiktorvet, Building 303B, 2800 Kongens Lyngby, Denmark

Email address: stema@dtu.dk

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332
Email address: raffaelli@math.gatech.edu