Multistage Problems on a Global Budget

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Abstract

Time-evolving or temporal graphs gain more and more popularity when studying the behavior of complex networks. In this context, the multistage view on computational problems is among the most natural frameworks. Roughly speaking, herein one studies the different (time) layers of a temporal graph (effectively meaning that the edge set may change over time, but the vertex set remains unchanged), and one searches for a solution of a given graph problem for each layer. The twist in the multistage setting is that the found solutions may not differ too much between subsequent layers. We relax on this notion by introducing a global instead of the so far local budget view. More specifically, we allow for few disruptive changes between subsequent layers but request that overall, that is, summing over all layers, the degree of change is upper-bounded. Studying several classic graph problems (both NP-hard and polynomial-time solvable ones) from a parameterized angle, we encounter both fixed-parameter tractability and parameterized hardness results. Somewhat surprisingly, we find that sometimes the global multistage versions of NP-hard problems such as \textsc{Vertex Cover} turn out to be computationally easier than the ones of polynomial-time solvable problems such as \textsc{Matching}.

1 Introduction

Recognizing the need to address the continuous evolution of networks and the steady demand for maintenance due to instances changing over time, Eisenstat et al. \cite{eisenstat2016multistage} and Gupta et al. \cite{gupta2018multistage} introduced what is now known as the “multistage view” on combinatorial optimization problems. Roughly speaking, focusing on graphs\textsuperscript{1}, the idea is to study a series of graphs over fixed vertex set and changing edge set (indeed, this is nowadays known as a standard model of temporal graphs), and the goal is to find for each graph of the series (known as the layers of the temporal graph) a solution of the studied computational problem where solutions to subsequent layers are not “too different”

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\textsuperscript{1}In general, the multistage view applies to all kinds of combinatorial optimization problems.
from each other. For instance, consider the well-known Vertex Cover problem. Here, the goal is to find for each layer of the temporal graph a small vertex cover and to guarantee that the vertex cover sets between two subsequent layers differ not too much (the degree of change is upper-bounded by a given parameter). Thus, this can be interpreted as a conservative (no dramatic changes allowed for the solution sets) view on solving problem instances that evolve over time. Clearly, the static case (no evolution takes place, that is, there is only one layer) is a special case, leading to many computational hardness results in this setting based already on the hardness of the static version.

Since the works of Eisenstat et al. [7] and Gupta et al. [9], who mainly focused on approximation algorithms, meanwhile there has been quite some further development in studying multistage versions of computational problems. For instance, there have been recent studies on Multistage Matching [1], Multistage Knapsack [3], and Online Multistage Subset Maximization [2]. In particular, due to natural parameterizations in this problem setting such as ‘number of layers’ or ‘maximum degree of change’ between the solutions for subsequent instances, also parameterized complexity studies have recently been started [8].

We modify the meanwhile standard multistage model from a local to global view in terms of how many changes between solution sets for subsequent instances are allowed. Whereas in the original model there is a parameter upper-bounding the maximum degree of change between every pair of subsequent layers of a temporal graph, we now introduce a more global view by only upper-bounding the sum of changes. Intuitively, one may say while we still keep the evolutionary view on dynamically changing instances and corresponding solutions, our new model allows for occasional disruptive changes between subsequent layers while overall the degree of change is still clearly limited.²

While, after providing the formal definitions, our results are reviewed in Table 1, next we discuss the main findings from our work. We provide results both for classic NP-hard graph problems (including a global multistage version of Vertex Cover) and classic problems solvable in polynomial time (including a global multistage version of Matching). We consider three central parameters: $k$, an upper bound on the size of the solution for a layer; $\ell$, the global budget upper-bounding the sum of changes between all layer solutions; $\tau$, the number of layers (equivalently, the number of time steps). Our three key messages are as follows:

1. There is a general systematic algorithmic approach that may lead to fixed-parameter tractability results with respect to combined parameter $k + \ell$. Indeed, along these lines we also obtain polynomial kernels with respect to the combined parameter $k + \tau$. We exemplify this approach by providing corresponding results for global multistage versions of the NP-hard problems Vertex Cover and Path Contraction.

2. One encounters (parameterized) computational hardness results for the single parameters $k$, $\ell$, and $\tau$ and even some combinations thereof; the main technical contribution here is to show W-hardness results for the parameter $k$.

3. Global multistage versions of polynomial-time solvable problems such as Matching, $s$-$t$-Path, and $s$-$t$-Cut turn out to be computationally harder than versions of the above NP-hard problems. More specifically, we spot W-hardness results for the combined parameter $k + \ell$.

²We are only aware of a Bachelor Thesis [11] supervised in the Berlin group where such a global multistage view has been adopted for the specific case of Vertex Cover.
In summary, our first systematic study of global multistage problems contributes a rich and promising new scenario in the fast growing field of studying temporal graph problems.

Our work is structured as follows. In Section 2 we provide basic notation and definitions. In Section 3 we present our general approach for gaining fixed-parameter tractability results for global multistage versions of NP-hard problems. In Section 4 we show corresponding parameterized hardness results. These complement our framework by proving tightness in the sense that an adaption to the single parameter $k$ or to a wider class of graph problems are impossible under standard assumptions. In Section 5 we then study global multistage versions of polynomial-time solvable problems and encounter simple but surprising parameterized hardness results, contrasting our more positive results for NP-hard problems in Section 3. We conclude in Section 6.

2 Preliminaries

For a graph $G = (V,E)$, we define $V(G) = V$ and $E(G) = E$. Take $v \in V$. We denote by $E(G(v)) = \{v,u\} \in E(G)$ the edges incident to $v$ in $G$ and by $N(v)$ the neighborhood of $v$, i.e., $N(v) = \{u : \{v,u\} \in E(G)\}$. Let $[x]$ be the set $\{1, \ldots x\}$.

**Definition 1** (Temporal Graph). We represent a temporal graph $\mathcal{G}$ using an ordered sequence of static graphs (called layers): $\mathcal{G} = \langle G_1, G_2, \ldots, G_\tau \rangle$. The subscripts $i \in [\tau]$ indexing the graphs in the sequence are the discrete time steps $1$ to $\tau$, where $\tau$ is known as the lifetime of $\mathcal{G}$. The underlying graph $G$ of $\mathcal{G}$ is defined as $V(G) = V(G_1) = \ldots = V(G_\tau)$ and $E(G) = E(G_1) \cup \ldots \cup E(G_\tau)$.

Let $\mathcal{X}$ be a parameterized graph problem where we are given a graph $G$ and a nonnegative integer $k$ (called the parameter) and we are searching for a set of size at most $k$ vertices and/or edges that satisfies some graph property $\mathcal{P}_X$ in $G$. For instance, if $\mathcal{X}$ is the problem Vertex Cover, then we search for a set of at most $k$ vertices that satisfies the property being a vertex cover in $G$.

We now generalize parameterized problems from graphs to temporal graphs as follows.

**Definition 2** (Global Multistage Problem for graph property $\mathcal{P}_X$). We are given a triple $(\mathcal{G}, k, \ell)$ where $\mathcal{G} = \langle G_1, \ldots, G_\tau \rangle$ is a temporal graph and the goal is to find sets $S_1, \ldots, S_\tau$, each of size at most $k$, such that $S_i$ satisfies $\mathcal{P}_X$ in $G_i$ ($i \in [\tau]$) and the total number of new elements from $S_i$ to $S_{i+1}$ ($i \in [\tau - 1]$) is bounded by $\ell$, that is, $\sum_{i \in [\tau-1]} |S_{i+1} \setminus S_i| \leq \ell$. Note that $k$ bounds the solution size in each step whereas $\ell$ bounds the number of so-called relocations.

| Problem                  | $k$   | $\ell$ | $k + \ell$ | $k + \tau$ |
|--------------------------|-------|--------|------------|------------|
| Vertex Cover             | $W[1]$ para-NP $k^{O(k+\ell)}\text{poly}(n)$ kernel |
| Path Contraction         | ? para-NP $k^{O(k+\ell)}\text{poly}(n)$ kernel |
| Cluster Edge Deletion    | $W[1]$ para-NP ? |
| Planar (Edge) Dominating Set | $W[2]$ para-NP $W[2]^\dagger$ |
| $s$-$t$-Path             | $W[2]$ para-NP $W[2]^\dagger$ |
| $s$-$t$-Cut              | $W[2]$ para-NP $W[2]^\dagger$ |
| Matching                 | $W[1]$ para-NP $W[1]^\dagger$ |

Table 1: Summary of our results. "?" denotes open cases. † hardness prevails even for $\ell = 0.$
Note that, while we only bound the number of additions explicitly, the number of deletions is implicitly bounded by $\ell + |S_1| - |S_\tau|$. Furthermore, one may usually take $|S_1| = |S_2| = \cdots = |S_\tau|$, in which case the number of additions and deletions must be equal.

3 An FPT-Framework for NP-hard Problems Gone Globally Multistage

In this section, we introduce a framework to show fixed-parameter tractability for the combined parameter solution size plus number of relocations $k + \ell$ for the multistage version of NP-hard problems that admit a certain type of kernel. We exemplify our framework by apply it to the multistage version of Vertex Cover and Path Contraction.

3.1 The Framework

Many parameterized graph problems $\mathcal{X}$ have a kernel, i.e., for every instance $I$ of $\mathcal{X}$ with parameter $k \in \mathbb{N}$, there is an instance $I'$ of the same problem type whose size is bounded by $f(k)$ for some function $f$, independent from the instance size, such that every solution for $I'$ can be transformed into a solution for $I$ and vice versa. We now want to use this fact to describe a general framework for solving the global multistage problem. However, to apply our framework on a parameterized graph problem, the problem needs a kernel with an extra property: Each vertex and/or edge of a minimal solution of size at most $k$ in $I$ must be contained in the instance $I'$. Such a kernel is known as full kernel and was first introduced by Damaschke [5]. In the following, we call the reductions that are used to get the instance $I'$ full reductions. Note that our framework will also work for a slightly more general version where each vertex and/or edge of a minimal solution of size at most $k$ in $I$ only needs to be uniquely represented by a vertex and/or edge of instance $I'$. As we see in subsequent sections, many kernels for parameterized graph problems can be turned into full kernels. Furthermore we restrict ourselves to problems that are monotonous in the sense that every superset of a solution is still a solution.

For the following, assume that $\mathcal{X}$ is a parameterized graph problem with parameter $k$ such that, on $n$-vertex graphs, poly($n$) time suffice to compute a full kernel of size $f(k)$ for $\mathcal{X}$ and to verify a solution. We next describe our framework for solving the global multistage problem $(\mathcal{G}, k, \ell)$ for graph property $\mathcal{P}_X$ on a temporal graph $\mathcal{G} = \langle G_1, \ldots, G_\tau \rangle$ with life time $\tau$.

Step 1 Compute a full kernel $G'_i$ for all $G_i$ ($i = |\tau|$). We so obtain a sequence of kernels $\mathcal{G}' = \langle G'_1, \ldots, G'_\tau \rangle$. (There are several problems where we can add vertices without changing the solution. For such problems, we can turn $\mathcal{G}'$ easily into a kernel (i.e., a temporal graph) by adding vertices such that each time step has the same vertex set.)

Step 2 Guess a sequence $L$ of at most $\ell + k$ integer pairs $(x, y)$ where $0 \leq x \leq f(k)$ and $1 \leq y \leq f(k)$. Intuitively, a pair $(x, y) \in L$ with $x \neq 0$ instructs us to replace the $x$th vertex/edge of the current solution set $S$ with the $y$th vertex/edge of $G'_i$. If $x = 0$, then only the $y$th vertex is added.

Step 3 Start with $S = \emptyset$ and $i = 1$. Repeat this step as long as $i \leq \tau$. If $S$ does not satisfy $\mathcal{P}_X$ on $G'_i$, take (and remove) the first pair of $L$ and modify $S$ accordingly as described above. Otherwise, if $S$ does satisfy $\mathcal{P}_X$ on $G'_i$, set $S_i := S$ and increment $i$ by 1.
Step 4 If \( L \) runs out before the iteration finishes, or if at any point \(|S| > k\), restart from Step 2. Otherwise, a solution has been found.

**Theorem 1.** Let \( \mathcal{X} \) be a monotonous graph problem such that polynomial time suffices to compute a full kernel of size \( f(k) \) and to verify a solution. Given an \( n \)-vertex temporal graph \( G = \langle G_1, \ldots, G_\tau \rangle \) as well as parameters \( k \) and \( \ell \), the following holds for the global multistage problem \( (G, k, \ell) \) for graph property \( \mathcal{P}_\mathcal{X} \):

1. it can be solved in \((f(k)+1)^{2(\ell+k)}\text{poly}(n)\) time and
2. it has a polynomial-time-computable kernel of at most \((f(k)\tau)^2\) vertices and at most \((f(k)\tau)^2\) edges.

**Proof.** Let \( f(k) \) be the a bound on the size of the kernel in each time step. Then the runtime of the algorithm above can be bounded by \((f(k)+1)^{2(\ell+k)}\text{poly}(n)\). We next show that the algorithm above solves the global multistage problem.

Begin by observing that, due to the assumed monotonicity, the requirement \( \sum_{i \in [\tau-1]} |S_{i+1}\setminus S_i| \leq \ell \), may be replaced by \(|S_1| + \sum_{i \in [\tau-1]} |S_{i+1}\setminus S_i| \leq k + \ell \). Clearly the first bound implies the second. Conversely, given any solution that obeys the second bound, we may do the following. As long as \( \sum_{i \in [\tau-1]} |S_{i+1}\setminus S_i| > \ell \), take \( i \) as the first time step where \( N := S_{i+1}\setminus S_i \) is not empty, pick an arbitrary element from \( N \) and add it to all of \( S_1, \ldots, S_i \). It is easy to check that this will yield a solution with \( \sum_{i \in [\tau-1]} |S_{i+1}\setminus S_i| \leq \ell \). Thus, if the algorithm finds a solution, then it is a valid solution for \( G' \) and this solution translates into a valid solution for \( G \).

It remains to show that the algorithm finds a solution for \( G \) if it exists. Let us consider a solution for \( G \), which we can turn into a solution for \( G' \), on which we focus in the following. W.l.o.g., we can assume that the solution for \( G' \) changes the vertices/edges over time as late as possible. By our guess, we determine the vertices of the solution \( S_1 \) in \( G_1 \). Furthermore, whenever the vertices in the solution set change over the time, we also can guess that change in our sequence of pairs. Knowing \( S_1 \) and the sequence, our algorithm finds the solution for \( G' \) and thus for \( G \). Finally note that, if some instance \( I = (G_i, k) \) is a no-instance, then the instance \( I' = (G'_i, k) \) is also a no-instance and the algorithm above finds no solution.

We now show how to turn \( G' \) into a temporal graph with the same vertex set in every time step. Assume we can add vertices to an instance of \( \mathcal{P}_\mathcal{X} \) such that the change maintains solvability. Our sequence of graphs has at most \( f(k)\tau \) different vertices. If \( \mathcal{X} \) allows us to add these vertices into every time step without further edges, then our kernel has \( f(k)\tau \) vertices and \( f(k)^2\tau \) edges. Otherwise, we can bound the number of vertices only by \((f(k)\tau)^2\) edges.

The running time of our framework above can be easily improved if the solution set of a monotonous graph problem \( \mathcal{X} \) only consists of either vertices or edges—as it is usually the case. Then we can replace \( f(k) \)-term in the run time by the number of vertices and edges, respectively, in the kernel.

### 3.2 Application to Vertex Cover

As a first application of our framework we show **Global Multistage Vertex Cover.** In a graph \( H, C \subseteq V(H) \) is a vertex cover if for every \( \{u, v\} \in E(H) : v \in C \) or \( u \in C \). In the problem **Vertex Cover**, we are given \( (H, k) \) where \( H \) is a graph and \( k \in \mathbb{N} \). The goal is to find a vertex cover of size at most \( k \) in \( H \).
**Problem (Global Multistage Vertex Cover).** We are given a triple \((\mathcal{G}, k, \ell)\) where \(\mathcal{G} = \langle G_1, \ldots, G_\tau \rangle\) is a temporal graph and the goal is to find sets \(S_1, \ldots, S_\tau\), each of size at most \(k\), such that \(S_i \subseteq V(G_i)\) is a vertex cover for \(G_i\) (\(i \in [\tau]\)) and the total number of new elements from \(S_i\) to \(S_{i+1}\) (\(i \in [\tau-1]\)) is bounded by \(\ell\), that is, \(\sum_{i \in [\tau-1]} |S_{i+1} \setminus S_i| \leq \ell\).

For a static \(n\)-vertex graph \(H\), **Vertex Cover** with parameter \(k\) admits a full kernel of at most \(k^2 + 2k\) vertices and at most \(k^2 + k\) edges by applying the following three reduction rules exhaustively as shown by Damaschke [5].

**Rule 1.** An isolated vertex \(v\) can be deleted.

**Rule 2.** If a vertex \(v\) has more than \(k + 1\) incident edges, delete all except \(k + 1\) of its incident edges.

**Rule 3.** If Rule 1 and 2 can not be applied any more and \(H\) has more than \(k^2 + 2k\) vertices conclude that the given graph is a no-instance.

Clearly, Rule 1 is full since an isolated vertex can never be part of any minimal solution for \(G_i\). Rule 2 is also full since a vertex \(v\) with \(k + 1\) edges is part of every solution of size \(k\), i.e., all deleted edges are covered by \(v\). If we apply Rule 3, then one can easily see that we turn a no-instance to a no-instance. We so get a sequence of full kernels and we can apply our framework as described in the previous section.

To obtain a temporal graph as kernel, we can simply add to each \(G_i\) the set \(\bigcup_{j \in [\tau]} V(G_j) \setminus V(G_i)\) as isolated vertices. We so obtain a kernel consisting of at most \((k^2 + 2k)\tau\) vertices and at most \((k^2 + k)\tau\) edges.

By our framework, we can conclude the following.

**Theorem 2.** Assume that an instance \((\mathcal{G}, k, \ell)\) of **Global Multistage Vertex Cover** with \(n\) and \(\tau\) being the number of vertices and time steps, respectively, of \(\mathcal{G}\) is given. The instance can be solved in \((k^2 + 2k + 1)^{2(\ell+k)}\tau\text{poly}(n)\) time. Moreover, the problem has a kernel of at most \((k^2 + 2k)\tau\) vertices and at most \((k^2 + k)\tau\) edges.

### 3.3 Application to Path Contraction

For a graph \(H\) and a subset of its edges \(C \subseteq E(H)\), we write \(H/C\) for the graph obtained from \(H\) by contracting each edge in \(C\). (Contracting an edge is done by identifying its endpoints and removing any loops or parallel edges afterwards.) In the **Path Contraction** problem, we are given a graph \(H\) and an integer \(k \in \mathbb{N}\), and the goal is to find \(C \subseteq E(H)\) such that every connected component of \(H/C\) is a path. Again, the multistage adaption is straightforward.

**Problem (Global Multistage Path Contraction).** We are given a triple \((\mathcal{G}, k, \ell)\) where \(\mathcal{G} = \langle G_1, \ldots, G_\tau \rangle\) is a temporal graph and the goal is to decide whether there exist \(S_1, \ldots, S_\tau\) with \(S_i \subseteq E(G)\) (\(G\) being the underlying graph of \(\mathcal{G}\)), each of size at most \(k\), such that \(G_i/S_i\) is a disjoint union of paths for every \(i \in [\tau]\), and the total number of new elements from \(S_i\) to \(S_{i+1}\) is bounded by \(\ell\), that is, \(\sum_{i \in [\tau-1]} |S_{i+1} \setminus S_i| \leq \ell\).

**Path Contraction** has a problem kernel with at most \(5k + 3\) vertices and at most \((5k + 3)^2\) edges with respect to solution size \(k\) [10]. To obtain the kernel on a given instance \((H, k)\), apply the following rules.
Rule 1. If any connected component of $H$ contains an edge $e = \{u, v\} \in E(H)$ whose removal disconnects it into two connected components that contain at least $k + 2$ vertices each, then contract the edge $e$.

Rule 2. If Rule 1 is not applicable and any connected component has more than $5k + 3$ vertices, conclude that $H$ is a no-instance.

Rule 1 is full since every path obtained from $H$ after the contraction of $k$ edges has some vertices connected to $u$ and some connected to $v$ so that it does not make sense to contract edge $e$. In other words, every minimal solution of size at most $k$ must not contain edge $e$. Rule 2 is surely full. To obtain a kernel for the multistage problem, observe as before that adding isolated vertices does not affect the solution.

**Theorem 3.** Assume that an instance $(G, k, \ell)$ of Global Multistage Path Contraction with $n$ and $\tau$ being the number of vertices and time steps, respectively, of $G$ is given. The instance can be solved in $(5k+3)^{k+\ell}2^\tau\text{poly}(n)$ time. Moreover, the problem has a kernel of at most $(5k+3)^{\ell+2}\tau$ vertices and at most $(5k+3)^{\ell+2}\tau$ edges.

### 4 Parameterized Hardness for NP-hard Problems Gone Globally Multistage

In this section, we point out the limits of parameterization of the global multistage version of NP-hard problems. It is clear that the global multistage version of an NP-hard problem is NP-hard for $\tau = 1$ and the number of relocations $\ell = 0$. We can further show that the global multistage version of Vertex Cover and Cluster Edge Deletion are $W[1]$-hard with respect to solution size $k$. For Planar (Edge) Dominating Set, we can even show $W[2]$-hardness with respect to the combined parameter solution size plus number of relocations $k + \ell$ for their global multistage version.

#### 4.1 Hardness for the Parameter $k$

In this subsection, we show that Vertex Cover and Cluster Edge Deletion become $W[1]$-hard with respect to solution size $k$ in their global multistage version. This is done by a reduction from Clique, which is well-known to be $W[1]$-hard with respect to solution size [6]. The problem Clique asks for a given graph $H$ and a number $\tilde{k}$ whether $H$ contains a complete subgraph of size $\tilde{k}$. We derive the following theorem.

**Theorem 4.** Global Multistage Vertex Cover and Global Multistage Cluster Edge Deletion are $W[1]$-hard parameterized by the solution size $k$.

For this, we describe our reduction from Clique using Global Multistage Vertex Cover as our examplary problem. We highlight the necessary properties of the vertex and edge gadget and prove the correctness of the reduction based on these properties. Afterwards, we show how to build the vertex and edge gadget for the remaining problem.
Figure 1: An example of the edge gadget for the edge \( e = \{v_1, v_3\} \) in a graph with three vertices \( v_1, v_2, \) and \( v_3 \) for \( \tilde{k} = 3 \). The \( 8\tilde{k}^2 \) repetitions of layers 1 to 3 (with different center vertices!) are not drawn to keep the figure readable. Furthermore, each copy sets contains only three instead of twelve vertices. The edges are only contained in the layer in which the right end vertex (\( w_i \) or \( v_e \)) lies. The copy sets are marked by dashed circles.

### 4.1.1 Vertex Cover

We introduce our reduction from \textsc{Clique} using \textsc{Global Multistage Vertex Cover} as our exemplary problem.

Let \( (H, \tilde{k}) \) be an instance of \textsc{Clique}.\(^3\) Let \( V(H) = \{v_1, v_2, \ldots, v_n\} \) and \( E(H) = \{e_1, e_2, \ldots, e_m\} \). We construct a temporal graph \( G \) as follows.

**Vertex Gadget.** For each vertex \( v_i \in V(H) \), the graph \( G \) contains a set \( V_i \) of \( 4\tilde{k} \) vertices \( v_i^1, \ldots, v_i^{4\tilde{k}} \). We call this set \( V_i \) the \textit{copy set} of \( v_i \).

**Edge gadget.** An edge gadget for an edge \( e = \{v_p, v_q\} \) consists of \( 8\tilde{k}^2 n+1 \) layers. It contains \( 8\tilde{k}^2 n \) vertices \( w_{e}^{1}, w_{e}^{2}, \ldots, w_{e}^{8\tilde{k}^{2}n} \) and a vertex \( v_e \), which are all isolated in all but one layer. In the \((i+jn)\)-th \((j \in [8\tilde{k}^2])\) layer of the gadget, the vertex \( w_{e}^{i+jn} \) is connected to all vertices in \( V_i \). In the \((8\tilde{k}^2 n+1)\)-th layer, the vertex \( v_e \) is connected to all vertices in \( V_q \) and \( V_p \). See Fig. 1 for an example.

**Instance.** Before all edge gadgets, in a first time step, we have a clique consisting of \( 4\tilde{k}^2 + 2 \) new vertices. The vertices of this clique are isolated in all other layers. Afterwards, there are \( m \) edge gadgets, one for each edge of \( H \). We set \( k := 4\tilde{k}^2 + 1 \) and \( \ell = 8mk^2(n-k)+\tilde{k}(\tilde{k}+1)+\left(m-\left(\frac{5}{2}\right)\right) \).

\(^3\)We assume that \( H \) is simple, as parallel edges can be replaced by a single edge. Furthermore, we assume that \( \tilde{k} \geq 3 \), as otherwise, \textsc{Clique} can be solved in polynomial time.
We can observe that \( k = O(\tilde{k}^2) \). Note that all layers but the first are stars. We call the vertex forming the center of the star in the \( i \)-th layer the center vertex \( c_i \).

**Properties.** We can observe the following properties of any solution \( S_1, \ldots, S_\tau \).

1. In the first layer, \( k \) vertices of the clique in this layer have to be contained in \( S_1 \).
2. In the \( (i + jn) \)-th layer of an edge gadget for an edge \( e \), the whole copy set \( V_i \) is contained in \( S_{i+jn} \), or \( u_{e}^{i+jn} \) is contained in \( S_{i+jn} \). There are \( 8\tilde{k}^2 \) of these layers for each vertex \( v_i \in V(H) \) for each edge gadget.
3. In the last layer of an edge gadget for an edge \( e = \{v_p, v_q\} \), the copy sets \( V_p \) and \( V_q \) are contained in \( S_{8\tilde{k}^2n+1} \), or \( v_e \) is contained in \( S_{8\tilde{k}^2n+1} \).
4. Every copy set has size \( 4\tilde{k} > \tilde{k} + 2 \).
5. All vertices not contained in a copy set are isolated in all but one layers. In each layer but the first, there is at most one non-isolated vertex that is not contained in a copy set.

It is easy to see that the reduction is computable in polynomial time. We now show that the \( \mathcal{W}[1] \)-hardness reduction from \textsc{Clique} is correct. Therefore, we show that \( H \) has a clique of size \( \tilde{k} \) if and only if the \textsc{Global Multistage Vertex Cover} instance \( I = (G, k, \ell) \) admits a solution.

**Lemma 1.** If \( H \) contains a clique of size \( \tilde{k} \), then the \textsc{Global Multistage Vertex Cover} instance \( (G, k, \ell) \) admits a solution.

**Proof.** Let \( C = \{v_{i_1}, v_{i_2}, \ldots, v_{i_\tilde{k}}\} \subseteq V(H) \) be a clique in \( H \). We construct a solution \( S \) for \( I \) as follows. In the first layer, \( k \) arbitrary vertices of the clique in this layer are contained in the vertex cover \( S_1 \). In all other layers, the vertex cover contains the copy sets \( V_j \) for all \( j \in [\tilde{k}] \). We denote the set of these vertices by \( S := \bigcup_{j=\tilde{k}}^\tilde{k} V_j \). The remaining vertex of the vertex cover is the vertex \( c_i \) in all layers \( G_i \) in which \( S \) is not a vertex cover. Otherwise, this vertex does not relocate.

This is indeed a vertex cover of size \( k \) in each layer. The \( 4\tilde{k}^2 \) vertices contained in \( S \) are only relocated from the first layer to the second layer, resulting in \( 4\tilde{k}^2 \) relocations. By Properties 2 and 3, the last vertex from the vertex cover relocates in each edge gadget of an edge of a clique \( 8\tilde{k}^2(n - \tilde{k}) \) times, while it relocates \( 8\tilde{k}^2(n - \tilde{k}) + 1 \) in all other edge gadgets. Thus, the total number of relocations is \( 4\tilde{k}^2 + 8m\tilde{k}^2(n - \tilde{k}) + m - (\tilde{k}^2) = \ell \).

It remains to show that any solution to the \textsc{Global Multistage Vertex Cover} instance \( (G, k, \ell) \) induces a clique of size \( \tilde{k} \) in \( H \). To do so, we first show that we can assume that the solution has some structural properties.

**Lemma 2.** If there exists a solution \( S \), then there exists a solution in which each \( S_i \) contains at most one center vertex, and no \( S_i \) contains the center vertex of any layer \( j > i \).

**Proof.** Given any solution, note that every vertex of \( S_1 \) has to be contained in the clique in the first layer by Property 1. Thus, in \( S_1 \) no vertex is the center vertex of any layer of any edge gadget. If a relocation to a center vertex happens, we can delay this relocation until the layer in which the center vertex is not isolated (as this layer is unique by Property 5).
Now suppose that \( S_i \) is the first layer containing two center vertices. By our delay modification, \( S_i \) then contains exactly two center vertices, one being \( c_i \) and the other being \( c_j \) with \( j < i \). Let \( v \) be the vertex which was relocated to \( c_i \) in layer \( i \). Then we may instead relocate \( c_j \) to \( c_i \) and still have a valid solution, because \( c_j \) will be isolated in all future layers by Property 5.

We now prove a lower bound on the number of relocations needed in any edge gadget.

**Lemma 3.** Let \( S = (S_1, \ldots, S_r) \) be a solution for the Global Multistage Vertex Cover instance \((G, k, \ell)\). Consider an edge gadget for an edge \( e = \{v_p, v_q\} \), and let \( G_e \) be the first and \( G_t \) be the last layer of the edge gadget. Assume that \( S_{s-1} \) contains no center vertex from a layer of the edge gadget, and that every layer contains at most one center vertex. Define \( \alpha := \min_{S'} |S_i \Delta S'| + |S_t \Delta S'| \) where the minimum is taken over all sets \( S' \subseteq V \) of size \( k \) that contain exactly \( k \) cover sets and whose remaining vertex is a center vertex for one of the layers of the edge gadget. Furthermore, let

\[
\beta := \begin{cases} 
0 & \text{if } V_p \cup V_q \subseteq S_t \\
1 & \text{otherwise.}
\end{cases}
\]

During the edge gadget at least \( 8\tilde{k}^2(n - \tilde{k}) + \frac{k-2}{k} \alpha - 1 + \beta \) relocations are needed.

**Proof.** Consider any block of \( n \) consecutive layers within the edge gadget for \( e \) and let \( F \) be the set of all copy sets which are completely covered in at least one of the \( n \) layers. Set \( \gamma := |F| - \tilde{k} \).

If \( \gamma < 0 \), then we have to cover \( n - |F| = n - \tilde{k} - \gamma \) center vertices, resulting in at least \( n - \tilde{k} - \gamma \) relocations within this block. If \( \gamma \geq 0 \), then at least \( \gamma \tilde{k} - 1 \) relocations are then required within this block by Property 4 as \( |V_i| = 4k \). Furthermore, \( n - \tilde{k} - \gamma \) additional relocations are needed to cover the vertices \( y_i \cup x_{j} \) for all \( V_i \not\in F \) by Property 2. Together, these are at least \( n - \tilde{k} - \gamma + \gamma \tilde{k} - 1 \) relocations (as there are that many layers) by the second part of Property 2. Thus, the lower bounds on the number of relocations are clearly minimized for \( \gamma = 0 \). Since there are \( 8\tilde{k}^2 \) disjoint sets of \( n \) consecutive layers, at least \( 8\tilde{k}^2(n - \tilde{k}) \) relocations are needed.

If \( \gamma < 0 \) holds throughout, then at least \( 8\tilde{k}^2 \geq \alpha - 1 + \beta \) additional relocations are needed by the above lower bound. Otherwise, at least \( \frac{k-2}{k} \alpha \) further relocations happen, as we only counted relocations towards the center vertex of a layer, or (for \( \gamma > 0 \)) two out of at least \( \tilde{k} \) relocations of non-center vertices. If \( S_s \) and \( S_t \) differ in more than one vertex, then these vertices are non-center vertices, as there is at most one center vertex in each \( S_i \) by Lemma 2. Thus, at least \( \frac{k-2}{k} \alpha - 1 \) additional relocations are required.

If \( \beta = 1 \), then the center vertex of the last layer \( c_t \) has to be contained in \( S_t \) by Property 3, which requires an additional relocation. Thus, we get in total at least \( 8\tilde{k}^2(n - \tilde{k}) + \alpha - 1 + \beta \) relocations.

With Lemmas 2 and 3 we can derive the following lemma which completes the correctness proof of our reduction.

**Lemma 4.** Any solution needs at least \( \ell \) relocations and a solution with \( \ell \) relocation exists if and only if \( G \) contains a clique of size \( \tilde{k} \).

**Proof.** Fix a solution obeying Lemma 2. By Lemma 3, we know that at least \( m \cdot 8\tilde{k}^2(n - \tilde{k}) + 4\tilde{k}^2 \) relocations happen in edge gadgets (the \(+4\tilde{k}^2 \) comes from the first layer), plus one edge for each edge gadget for an edge \( \{v_p, v_q\} \) such that \( V_p \) or \( V_q \) is not completely contained in the vertex cover.
in the last layer of an edge gadget. Define $\mathcal{V}^i := \{v_j : V_j \subseteq S_i\}$ to be the set of vertices whose copy set is completely contained in $S_i$.

If $\mathcal{V}^i$ and $\mathcal{V}^{i+1}$ differ for $i \geq 2$, let $v_p \in \mathcal{V}^{i+1} \setminus \mathcal{V}^i$. At least $\frac{k-2}{k}24k = 4(\tilde{k} - 2) > \tilde{k}$ relocations where used to add $V_p$ to $S_{i+1}$, while $\beta$ can reduce only for $\tilde{k} - 1$ edge gadgets (the gadgets for the edges $\{v_p, v_q\}$ with $v_q \in \mathcal{V}^{i+1}$). Thus, if $\gamma$ vertices of $\mathcal{V}^i$ are changed, then this needs at least $m - \binom{\tilde{k}}{2} + \gamma \tilde{k} - \gamma(\tilde{k} - 1) > m - \binom{\tilde{k}}{2}$ additional relocations, and $\gamma = 0$ has to hold by the definition of $\ell$.

If $\mathcal{V}^i := \{v_j : V_j \subseteq S_i\}$ does not change after the second layer (i.e., $\mathcal{V}^i = \mathcal{V}^{i+1}$ for $i \geq 2$), there are by Lemma 3 at least $m - |E(H[\mathcal{V}^i])|$ additional relocations (one for each edge gadget not corresponding to an edge from $E(H[\mathcal{V}^i])$), implying $|E(H[\mathcal{V}^i])| \geq \binom{\tilde{k}}{2}$ by the bound $\ell$ on relocations. Thus, $\mathcal{V}^i$ is a clique of size $\tilde{k}$.

We remark that, by inserting additional layers in between the existing ones, $\mathcal{G}$ can be modified to allow for solutions with $\max_i(S_{i+1} \setminus S_i) \leq 1$, thus showing hardness even if the local budget, that is the number of allowed modifications between adjacent layers, is also restricted.

4.1.2 Cluster Edge Deletion

We now show $\mathsf{W}[1]$-hardness with respect to solutions size $k$ for GLOBAL MULTISTAGE CLUSTER EDGE DELETION. Recall that, for a graph $H$, a set $D \subseteq E(H)$ is called a cluster edge deletion set if $H - D$ is a cluster graph (i.e., all connected components are cliques).

**Problem (Global Multistage Cluster Edge Deletion).** We are given a triple $(\mathcal{G}, k, \ell)$ where $\mathcal{G} = (G_1, \ldots, G_\tau)$ is a temporal graph and the goal is to find sets $S_1, \ldots, S_\tau$, each of size at most $k$, such that such that $S_i \subseteq E(G_i)$ is a cluster edge deletion set for $G_i$ $(i \in \llbracket \tau \rrbracket)$ and the total number of new elements from $S_i$ to $S_{i+1}$ $(i \in \llbracket \tau - 1 \rrbracket)$ is bounded by $\ell$, that is, $\sum_{i \in \llbracket \tau - 1 \rrbracket} |S_{i+1} \setminus S_i| \leq \ell$.

We again reduce from CLIQUE. Let $(H, \tilde{k})$ be an instance of CLIQUE. We construct a temporal graph $\mathcal{G}$ as follows.

**Vertex gadget.** For each vertex $v_i \in V(H)$, we add a clique of size $4\tilde{k} + 1$ containing a special vertex $u_i$. The edges adjacent to $u_i$ in this clique are the copy set of $v_i$, and denoted by $E_i$.

**Edge gadget.** An edge gadget for an edge $e = \{v_p, v_q\}$ consists $8\tilde{k}^2 n + 1$ layers. Again, it contains $8\tilde{k}^2 n$ vertices $w_i^1, w_i^2, \ldots, w_i^{8\tilde{k}^2 n}$, which are all isolated in all but one layer. In the $(i + jn)$-th $(j \in \llbracket \tilde{k}^2 \rrbracket)$ layer of the gadget, the vertex $w_i^{i+jn}$ is connected to $u_i$. In the $(8\tilde{k}^2 n + 1)$-th layer, the edge $\{u_p, u_q\}$ exists.

**Instance.** Before all edge gadgets, in a first layer, we have $8\tilde{k}^2 + 1$ vertex-disjoint paths of length two, ensuring that the solution in the first layer consists of $8\tilde{k}^2 + 1$ edges of these paths. The vertices of the paths are isolated in all other layers. Afterwards, there are $m$ edge gadgets, one for each edge of Graph $H$. This time, we set $k = 4\tilde{k}^2 + 1$ and $\ell = 8m\tilde{k}^2(n - \tilde{k}) + \tilde{k}^2 + (m - \binom{\tilde{k}}{2}) + \tau$. 

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Properties. We can observe the following properties of any solution $S_1, \ldots, S_\tau$.

1. In the first layer, $k$ edges of the disjoint paths of length three in this layer have to be contained in $S_1$.

2. In the $(i+jn)$-th layer of an edge gadget, the whole copy set $E_i$ is contained in $S_{i+jn}$, or the edge $\{u_i, w_{i+jn}^i\}$ is contained in $S_{i+jn}$. There are $8\tilde{k}^2$ of these layers for each vertex $v_i \in V(H)$ for each edge gadget.

3. In the last layer of an edge gadget for an edge $e = \{v_p, v_q\}$, the copy sets $E_p$ and $E_q$ are contained in $S_{8k^2n+1}$, or the edge $\{u_p, v_p\}$ is contained in $S_{8k^2n+1}$.

4. Every copy set has size $4\tilde{k} > \tilde{k} + 2$.

5. All edges not contained in a copy set are contained in only one layer. In each layer but the first, there is at most one edge which is not contained in a copy set.

With these properties, we can show that $H$ has a clique of size $\tilde{k}$ if and only if $G$ has a temporal cluster edge deletion set of size $k$ in a very similar way than in Section 4.1.1.

4.2 Hardness for the Combined Parameter $k + \ell$

Some classical graph problems do not admit a full kernel. Two of them are the Planar Dominating Set and the Edge Dominating Set problem. We will show here that their inability to fit into our framework of Section 3.1 grounds in the fact that these problems are actually $W[2]$-hard for the combined parameter $k + \ell$.

4.2.1 Planar Dominating Set

We begin with the former, which is defined as follows. Recall that, for a graph $H$, a set $D \subseteq V(H)$ is called a dominating set if $D \cup \bigcup_{d \in D} N(d) = V(H)$.

Problem (Global Multistage Planar Dominating Set). Given a temporal graph $\mathcal{G} = \langle G_1, \ldots, G_\tau \rangle$ that is planar in every time step and parameters $k, \ell$, find sets $S_1, \ldots, S_\tau$, each of size at most $k$, such that $S_i \subseteq V(G_i)$ is a dominating set for $G_i$ ($i \in \{1, \ldots, \tau\}$) and the total number of new elements from $S_i$ to $S_{i+1}$ ($i \in \{1, \ldots, \tau - 1\}$) is bounded by $\ell$. 

![Figure 2: For each vertex $v_i$ in the graph $H$, a graph $G_i$ is constructed where vertex $w$ is connected to all vertices except $v_i$ and $v_i$ is connected only to its neighbors in $H$.](image)
Theorem 5. Global Multistage Planar Dominating Set is \(W[2]\)-hard when parameterized by the solutions size \(k\) and no relocations are allowed \((\ell = 0)\).

Proof. The theorem can be shown by a reduction from the \(W[2]\)-hard Dominating Set problem \([4]\). Given an instance \(I = (H, k')\) of that problem, we construct an instance \(I' = (G, k, \ell = 0)\) of Global Multistage Planar Dominating Set as follows.

Let \(w\) and \(w'\) be two new vertices. For each vertex \(v_i \in V(H)\) \((i = 1, \ldots, n)\) create a time layer \(G_i\) with \(V(G_i) = V(H) \cup \{w, w'\}\) and \(E(G_i) = \{(v_i, u)\mid \forall u \in N(v_i)\} \cup \{\{w, u\} \mid \forall u \in V(H) \setminus \{v_i\}\} \cup \{\{w, w'\}\}\). For an example see Fig. 2. Observe that in every time step \(i = 1, \ldots, n\), \(G_i\) without \(v_i\) forms a star and connecting \(v_i\) with some vertices of the star leads to a planar graph.

We claim that \(H\) contains a dominating set \(D\) of size \(k\) if and only if \(G\) contains a multistage dominating set \(D'\) of size \(k + 1\). For the forward direction, choose \(D' = D \cup \{w\}\). For every time step \(i\), vertex \(v_i\) is dominated in \(G_i\) by a vertex in \(D\). All other vertices are dominated in \(G_i\) by vertex \(w\). For the reverse direction, we have to dominate \(w'\) in each time step. W.l.o.g., we can therefore assume that \(w \in D'\). Then \(D \setminus \{w'\}\) must be a dominating set in \(H\) since every vertex \(v_i\) is, at time step \(i\), only adjacent to its neighbors from \(H\). \(\square\)

4.2.2 Planar Edge Dominating Set

The problem Global Multistage Planar Edge Dominating Set is quite similar to the previous one, but with edges instead of vertices. However, in this setting we may even restrict our temporal graph to have planar underlying graph. Recall that, for a graph \(H, D \subseteq E(H)\) is called an edge dominating set if \(\bigcup_{(v, w) \in D} E(H(v)) \cup E(H(w)) = E(H)\).

Problem (Global Multistage Planar Edge Dominating Set). We are given a triple \((G, k, \ell)\) where \(G = (G_1, \ldots, G_\tau)\) is a temporal graph with planar underlying graph \(G\). The goal is to find sets \(S_1, \ldots, S_\tau\), each of size at most \(k\), such that \(S_i \subseteq E(G_i)\) is an edge dominating set for \(G_i\) \((i \in [\tau])\) and the total number of new elements from \(S_i\) to \(S_{i+1}\) \((i \in [\tau - 1])\) is bounded by \(\ell\), that is, \(\sum_{i \in [\tau - 1]} |S_{i+1} \setminus S_i| \leq \ell\).

Theorem 6. Global Multistage Planar Edge Dominating Set is \(W[2]\)-hard when parameterized by the solutions size \(k\) and no relocations are allowed \((\ell = 0)\).

Proof. We will use a reduction from Set Cover here, which is well known to be \(W[2]\)-hard \([4]\). A Set Cover instance consists of an integer \(n\) and \(F \subseteq 2^{[n]}\) a family of subsets of \([n]\). The task is to find a subset \(S \subseteq F\) of size at most \(k\) such that \(\bigcup S = [n]\).

From this, we construct a temporal graph as follows. Let \(V := \{\ast\} \cup [n] \cup F\) be the set of vertices. At time step \(i \in [n]\), we define the edge set as \(E(G_i) := \{\ast, i\}, \{\ast, F\} : F \in F \land i \in F\}.\) For an example see Fig. 3. Note that the underlying graph \(G\) is then a star with center vertex \(\ast\). We claim that \(G\) contains an edge dominating set of size \(k\) if and only if that Set Cover instance has a solution of size \(k\).

Begin by observing that, w.l.o.g., \(\bigcup F = [n]\) and \(\emptyset \notin F\). We may further assume that any solution to Global Multistage Planar Edge Dominating Set does only contain edges of the form \(\{\ast, F\}\) with \(F \in F\) since any other edge may be exchanged for one of these.

Now the claimed equivalence follows by including the edge \(\{\ast, F\}\) in the edge dominating set for \(G\) if and only if \(F\) is contained in the Set Cover solution. \(\square\)
Figure 3: An example of a temporal graph \( G = \langle G_1, \ldots, G_5 \rangle \) for the reduction from Set Cover using the family \( F = \{ F_1 = \{1,2,5\}, F_2 = \{3,4,5\}, F_3 = \{1,3\}, F_4 = \{1,5\} \} \). Isolated vertices are not shown.

In summary, we have showed that the temporal multistage version of dominating set is W[2]-hard for the parameter \( k + \ell \) even when each layer is a planar graph or if the underlying graph is the line graph of a planar graph.

5 Hardness of Polynomial-Time Solvable Problems Gone Globally Multistage

We now show hardness for the global multistage version of several polynomial-time solvable problems.

**Theorem 7.** Global Multistage \( s\)-\( t\)-Path, Global Multistage \( s\)-\( t\)-Cut are W[2]-hard and Global Multistage Matching is W[1]-hard parameterized by solution size \( k \), even if the number of relocations is zero (i.e. \( \ell = 0 \)).

5.1 \( s\)-\( t\)-Path

We show that the global multistage version of \( s\)-\( t\)-Path with no relocations is W[2]-hard parameterized by solution size \( k \). The problem is formally defined as follows.

**Problem (Global Multistage \( s\)-\( t\)-Path).** Given a temporal graph \( G = \langle G_1, \ldots, G_\tau \rangle \), two vertices \( s, t \in V(G) \), and an integer \( k \), decide whether there exists a set \( F \subseteq E(G) \) of at most \( k \) edges such that for every \( i \in [\tau] \), there exists an \( s\)-\( t\)-path in \( G'_i := (V(G_i), E(G_i) \cap F) \).

This is done by a reduction from the W[2]-hard Hitting Set problem [4]. Given a finite set \( U = \{u_1, \ldots, u_n\} \), a set \( F = \{F_1, F_2, \ldots, F_m\} \subseteq 2^U \) of subsets of \( U \), and an integer \( k' \), the problem Hitting Set asks whether \( U \) contains a subset \( X \) of size \( k' \) such that \( X \) intersects with every set in \( F \). This problem is well-known to be W[2]-hard parameterized by solution size \( k' \).

**Construction.** Given an instance \( \mathcal{I} = (U, F, k') \) of Hitting Set, we design an instance of Global Multistage \( s\)-\( t\)-Path as follows. The graph \( G \) contains two vertices \( s \) and \( t \). Furthermore, for each \( u_i \in U \), there are two vertices \( v_i \) and \( w_i \). For each set \( F_j \in \mathcal{S} \), the \( j \)-th layer contains
Figure 4: An example for the $j$-th layer of $G$ for the reduction from Hitting Set for a subset $S_j = \{u_1,u_3,u_5\} \subseteq U$ with $U = \{u_1,u_2,u_3,u_4,u_5\}$.

for each element $u_i \in F_j$ the edges $\{s,v_i\}$, $\{v_i,w_i\}$, and $\{w_i,t\}$, as shown Fig. 4a. Finally, we set $k := 3k'$.

**Lemma 5.** The instance $I$ admits a hitting set of size $k'$ if and only if $(G,k)$ admits a solution of size $k$.

**Proof.** We now show both directions separately.

$(\Rightarrow)$ Let $X := \{u_{i_1}, u_{i_2}, \ldots, u_{i_{k'}}\}$ be a hitting set.

We set $Y := \{\{s,v_{i_p}\}, \{v_{i_p}, w_{i_p}\}, \{w_{i_p}, t\} : p \in [k']\}$. Note that $|Y| = 3|X| = 3k' = k$, so it remains to show that there exists an $s$-$t$-path in each layer. Consider the $j$-th layer. Since $X$ is a hitting set, there exists some $p \in [k']$ such that $u_{i_p} \in F_j$. Therefore, the edges $\{s,v_{i_p}\}, \{v_{i_p}, w_{i_p}\}, \{w_{i_p}, t\}$ are contained in both $Y$ and $E(G_j)$. Thus, there exists an $s$-$t$-path in the $j$-th layer.

$(\Leftarrow)$ Let $Y$ be a solution for the $s$-$t$-Path instance $(G,k)$.

We set $X := \{u_i : \{\{s,v_i\}, \{v_i,w_i\}, \{w_i,t\}\} \subseteq F\}$. We have $|X| \leq \frac{1}{3}|Y| \leq \frac{1}{3}k = k'$. It remains to show that $X$ is a hitting set. Consider the set $F_j$. Since $Y$ is a solution of $(G,k)$, there exists an $s$-$t$-path in the $j$-th layer. All these paths are of the form $s - v_i - w_i - t$ for an $u_i \in F_j$. Therefore, there is an $u_i \in F_j \cap X$. Thus, $X$ is a hitting set.

The $W[2]$-hardness for Global Multistage $s$-$t$-Path follows directly from Lemma 5 since $k = O(k')$.

**5.2 $s$-$t$-Cut**

We now consider the problem of separating $s$ and $t$ in each layer. We show that the global multistage version of $s$-$t$-Cut with no relocations problem is $W[2]$-hard parameterized by solutions size $k$.

**Problem (Global Multistage $s$-$t$-Cut).** Given a temporal graph $G = (G_1,\ldots,G_r)$, two vertices $s$ and $t$, and an integer $k$, decide whether there exists a set $F \subseteq E(G)$ of at most $k$ edges such that for every $i \in [r]$, there exists no $s$-$t$-path in $G_i := (V(G_i),E(G_i) \setminus F)$. The set $F$ is called a temporal $s$-$t$-cut.

We show that this problem is $W[2]$-hard by a reduction similar to the reduction for Global Multistage $s$-$t$-Path. We again reduce from the $W[2]$-hard Hitting Set problem.
**Construction.** Given an instance $I = (U, S, k')$ of Hitting Set, we design an instance of Global Multistage $s$-$t$-Cut as follows. The graph $G$ contains two vertices $s$ and $t$. Furthermore, for each $u_i \in U$, there are two vertices $v_i$ and $w_i$. For each set $S_j = \{u_{i_1}, u_{i_2}, \ldots, u_{i_r}\} \in S$, the $j$-th layer contains the $s$-$t$-path $s - v_{i_1} - v_{i_2} - v_{i_3} - \cdots - v_{i_r} - w_{i_r} - t$, see Fig. 4b for an example. We set $k := k'$.

**Lemma 6.** The instance $I$ admits a hitting set of size $k'$ if and only if $G$ has a temporal $s$-$t$-cut of size $k$.

**Proof.** We show both directions separately.

$(\Rightarrow)$ Let $X = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ be a hitting set. We define $F := \{v_{i_j}, w_{i_j} : j \in [k]\}$, and claim that $F$ separates $s$ and $t$ in each layer.

Consider the $j$-th layer of $G$. This layer contains a unique $s$-$t$-path, which contains the edge $\{v_i, w_i\}$ for each $u_i \in S_j$. As $X$ is a hitting set, there is an edge of this path contained in $F$ and, thus, $F$ is a temporal $s$-$t$-cut.

$(\Leftarrow)$ Let $F$ separate $s$ and $t$ in each layer and let $|F| \leq k$. If $F$ contains an edge of the form $\{s, v_i\}$, $\{v_i, w_j\}$ for $j \neq i$, or $\{w_i, t\}$, then we can replace this edge by $\{v_i, w_i\}$, as any $s$-$t$-path containing such an edge contains also the edge $\{v_i, w_i\}$. Thus, we assume that $F = \{v_i, w_i : i \in I\}$ for some $I \subseteq [n]$ with $|I| = k$.

Define $X := \{u_i : i \in I\}$. Clearly, $|X| = |I| = k$, so it remains to show that $X$ is a hitting set. Consider a set $S_j$. Since $F$ is an $s$-$t$-cut in $G_j$, it contains an edge $e = \{v_i, w_i\}$ of the unique $s$-$t$-path in this layer. Thus, we have $u_i \in X \cap S_j$ and, therefore, $X$ hits $S_j$. \hfill $\square$

The W[2]-hardness for Global Multistage $s$-$t$-Cut follows directly from Lemma 6 since $k' = k$.

### 5.3 Matching

Last but not least, we show that the global multistage version of Matching with no relocations is W[1]-hard parameterized by solutions size $k$. The problem is formally defined as follows.

**Problem (Global Multistage Matching).** Given a temporal graph $G = (G_1, \ldots, G_T)$, and an integer $k$, decide whether there exists a set $F \subseteq E(G)$ of at least $k$ edges such that, for every $i \in [T]$, the edge set $E(G_i) \cap F$ is a matching. The set $F$ is called a temporal matching.

We will show that this problem is W[1]-hard by a reduction from Independent Set. Given a graph $H$ and an integer $k'$, the problem Independent Set asks whether $H$ contains a set of pairwise non-adjacent vertices of size at least $k'$. It is well-known to be W[1]-hard parameterized by solutions size $[4]$.

**Construction.** Given an instance $I = (H, k')$ of Independent Set, we construct an instance of Global Multistage Matching as follows. Let $E(H) = \{e_1, e_2, \ldots, e_m\}$ with $e_i = \{v_i, w_i\}$. The vertices of $G$ are $V(G) = V(H) \cup \{c\}$, where $c$ is not contained in $V(H)$. The graph $G$ has $m$ layers. The $i$-th layer of $G$ contains the edges $\{v_i, c\}$ and $\{w_i, c\}$, see Fig. 5 for an example. Finally, we set $k := k'$.

**Lemma 7.** The graph $H$ contains an independent set of size $k'$ if and only if $G$ contains a temporal matching of size $k$.  

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Figure 5: An example for the $i$-th layer of $G$ for the Global Multistage Matching reduction from Independent Set for an edge $e_i = \{v_i, w_i\}$ and $V(H) = \{v_i, w_i, v, v', v''\}$.

Proof. We show both directions separately.

($\Rightarrow$) Let $X$ be an independent set in $H$. Define $M := \{\{v, c\} : v \in X\}$. Clearly, $|M| = k$, so it remains to show that $M \cap E(G_i)$ is a matching for all $i \in \tau$. Every layer containing an edge $\{v_i, c\}$ contains only one other edge, namely $\{w_i, c\}$. If $v_i \in X$ (w.i. $\in X$ is symmetric), then $w_i \notin X$, as $X$ is an independent set. Thus, $F \cap E(G_i) = \{\{v, c\}\}$ is a matching. If neither $v_i \in X$ nor $w_i \in X$, then $F \cap E(G_i) = \emptyset$ is a matching.

($\Leftarrow$) Let $F \subseteq E(G)$ be a temporal matching of size $k$. Define $X := \{v \in V(H) : \{v, c\} \in F\}$. Clearly, $|X| = k$. It remains to show that $X$ is an independent set. So assume that $e_i \in E(H[X])$. Then $\{v_i, c\} \in F$ and $\{w_i, c\} \in F$, contradicting the assumption that $F$ is a temporal matching. $\square$

The $W[1]$-hardness for Global Multistage Matching follows directly from Lemma 7 since $k' = k$.

6 Conclusion

We presented a fairly general framework to derive fixed-parameter tractability results (including polynomial kernelization) for global multistage versions of classic NP-hard problems. A particular technical feature herein is to show how to derive “temporal kernels” from known static ones—a method that to the best of our knowledge has not yet been used in the design of algorithms for temporal (graph) problems. Our results are complemented by several parameterized hardness results.

As to challenges for future work, we also refer to Table 1. In particular, we left open the existence of (polynomial) kernels for the combined parameter $\tau + k$ for problems that are solvable in polynomial time in the static setting (this includes Matching). Clearly, another natural challenge is to further explore the applications and limitations of our presented framework. We firmly believe that studying temporal problems in the global multistage view may be of practical as well as theoretical interest in the world of complex network analysis.

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