Symmetries in the $g_{9/2}$ shell

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We consider symmetries which arise when two-body interaction matrix elements with isospin $T = 0$ are set equal to a constant in a single-$j$-shell calculation. The nucleus $^{96}$Cd is used as an example.

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I. INTRODUCTION

The recent discovery of a $J = 16^+$ isomer in $^{96}$Cd by Nara Singh et al. [1] was noted in a work on symmetries in the $g_{9/2}$ shell by Zamick and Robinson [2]. In a previous work by these authors [3], the main emphasis was on the $f_{7/2}$ shell, although the equations were written in a general way so as to apply to any shell. In this work we elaborate on the work in Ref. [2] by giving detailed wave functions and energies.

First, however, we would like to point out that the $g_{9/2}$ shell has, in recent years, been a beehive of activity both experimentally and theoretically. For example, in contrast to the $f_{7/2}$ shell, where the spectrum of three identical particles would be identical to that of five particles (or three holes) in the single-$j$-shell approximation, one can have different spectra in the $g_{9/2}$ shell. It was noted by Escuderos and Zamick [4] that, with a seniority-conserving interaction, the $(J = 21/2^+)$–$(J = 3/2^+)$ splitting (maximum and minimum spins for three identical particles) was the same for five identical particles as for three. However, with a $Q \cdot Q$ interaction (which does not conserve seniority), the splittings were equal in magnitude but opposite in sign.

Even more interesting, it was noted in Ref. [4] that for four identical particles (or holes, e.g. $^{96}$Pd) there are three $J = 4^+$ states—two with seniority 4 and one with seniority 2. In general, seniority is not conserved in the $g_{9/2}$ shell. Despite this fact, it was found that no matter what interaction was used, one eigenstate emerged that was always the same for all interactions—this was a seniority-4 state. The other two states were a mix of seniorities 2 and 4. A unique state also emerged for $J = 6^+$. This observation led to considerable activity with an intent to explain this behaviour as seen in Refs. [5,6].

Another recently emerging topic has to do with $T = 0$ pairing in which the pairs are coupled to the maximum angular momentum, which for the $g_{9/2}$ shell is $J = 9^+$ (see Refs. [12,13]). In the Nature article [12], an experiment is presented in which an almost equal-spaced spectrum is found in $N = Z$ $^{92}$Pd for $J = 0^+$, $2^+$, $4^+$, and $6^+$. This is well reproduced by the shell model but also by $T = 0$ pairing with maximum alignment of pairs. The $2^+$ state is lower in $^{92}$Pd than in $^{96}$Pd, which is not surprising because the latter is semi-magic. An interesting observation in Ref. [12] is that, although the spectrum looks vibrational, the $B(E2)$'s might obey a rotational formula rather than vibrational. It would be of interest to have experimental verification.

Recently the current authors addressed a different problem [16], but one that has some implications for the above topic. We studied the question of isomerism for systems of three protons and one neutron, e.g. $^{96}$Ag. We found that for the upper half of a $j$ shell, be it $f_{7/2}$ or $g_{9/2}$ shell, the $J = J_{\text{max}}$ two-body matrix element is much more attractive than in the lower half. The states in question have $J = 7^+$ and $9^+$ respectively. This was necessary to explain the isomeric behaviours of states in various nuclei, e.g. why the lifetime of the $J = 12^+$ state in $^{52}$Fe is much longer than in $^{44}$Ti, or why the $J = 16^+$ states in $^{96}$Cd and the $J = 15^+$ states in $^{96}$Ag are isomeric.

In that work, we also used the previously determined two-body matrix elements of Coraggio et al. [17] as well as our own. With regards to the works of Refs. [12–15], this suggests the $J = J_{\text{max}}$ pairing is a better approximation in the upper half of a $j$ shell rather than the lower half.

It should be added that one of us had previously studied the problem of the effects of varying two-body matrix elements in the $f_{7/2}$ region [18] and that one can learn a lot by studying the explicit $f_{7/2}$ wave functions in Ref. [6], which are based on previous works in Refs. [19,20].

But let us not lose sight of what this paper is about—partial dynamical symmetries (PDS) in $^{96}$Cd. One obvious distinction of our work relative to that of Refs. [12–15] is that we are dealing with higher spins than the ones that they consider. In the Nature article [12], they measure $J = 0, 2, 4,$ and $6$ and also discuss $J = 8$, whereas the PDS that we consider occur for $J = 11^+$ and beyond. Thus, our work may be regarded as different but complimentary to theirs.

At the time of this writing, the $16^+$ isomer is the only known excited state of $^{96}$Cd [1]. It decays to a $15^+$ isomer in $^{96}$Ag. In previous work by Escuderos and Zamick [3], it was noted that the single-$j$ approximation for a few holes relative to $Z = 50$, $N = 50$ works fairly well. However, they cautioned that this approximation does not work at all for a few particles relative to $Z = 40$, $N = 40$. The relevant shell is, of course, $g_{9/2}$.

II. THEORETICAL FRAMEWORK

The symmetry in question mentioned above comes from setting two-body interaction matrix elements with isospin $T = 0$ to be constant, whilst keeping the $T = 1$ matrix elements unchanged [2]. It does not matter what
the $T = 0$ constant is as far as symmetries are concerned, but it does affect the relative energies of states of different isospins. Briefly stated, for the four-hole system $^{96}$Cd, we then have a PDS, one which involves $T = 0$ states with angular momenta that do not exist for $T = 2$ states in a pure $g_{9/2}$ configuration. That is to say, the PDS will not occur for states with angular momenta which can occur in the $(g_{9/2})^4$ configuration of four identical particles ($^{96}$Pd). Although many 6-$j$ and 9-$j$ relations were used in Refs. 2, 3, to describe why these symmetries are partial, one simple argument is illuminating. If a given angular momentum can occur for, say, four $g_{9/2}$ neutron holes ($T = 2$), then there is a constraint on the $T = 0$ states with the same angular momentum—namely their wave functions have to be orthogonal to the $T = 2$ states. This constraint prevents the occurrence of a PDS.

For qualifying $T = 0$ states, the PDS consists of $J_p$ and $J_n$ being good dual quantum numbers. That is to say, the wave function of a state will have only one $(J_p, J_n)$ and $(J_n, J_p)$. Another way of saying this is that $J_p - J_n$ is a good quantum number. Furthermore, states with different total angular momentum $J$ but with the same $(J_p, J_n)$ will be degenerate.

We have given a physical argument for the PDS. We can also explain it mathematically. There are both off-diagonal and diagonal conditions. The former is needed to explain why $(J_p, J_n)$ are good dual quantum numbers. The reason is the vanishing of the $9j$-symbol

$$\left\{ \begin{array}{ccc} j & j & (2j - 1) \\ j & j & (2j - 1) \\ (2j - 1) & (2j - 3) & (4j - 4) \end{array} \right\} = 0 \quad (1)$$

Next we need diagonal conditions to explain why states with the same $(J_p, J_n)$ are degenerate. These are given by

$$\left\{ \begin{array}{ccc} j & j & (2j - 3) \\ j & j & (2j - 1) \\ (2j - 3) & (2j - 1) & I \end{array} \right\} = \frac{1}{4(4j - 5)(4j - 1)}$$

for $I = (4j - 4), (4j - 5), (4j - 7)$ and

$$\left\{ \begin{array}{ccc} j & j & (2j - 1) \\ j & j & (2j - 1) \\ (2j - 1) & (2j - 1) & I \end{array} \right\} = \frac{1}{2(4j - 1)^2} \quad (2)$$

for $I = (4j - 4), (4j - 2), (4j - 2)$.

### III. RESULTS

The partial dynamical symmetry manifests itself is best illustrated by examining Tables II and III. Here we use the two-body INTd matrix elements from Zamick and Escuderos to perform single-j-shell calculations of the energies and wave functions of $^{96}$Cd. Actually, it does not matter what charge-independent interaction is used to illustrate the symmetry that will emerge. Let us first focus on the $J = 11^+$ and $J = 12^+$ states. Relative to Table II we see certain simplicities for the $J = 11^+$ states in Table III (where the $T = 0$ two-body interaction matrix elements are set to a constant). For the lowest state, the only non-zero components are $(J_p, J_n) = (4, 8)$ and $(8, 4)$; for the second state, they are $(6, 8)$ and $(8, 6)$. This confirms what we said above: $(J_p, J_n)$ are good dual quantum numbers. Nothing special happens to $J = 11^+$, $T = 1$ states.

We show results for $J = 12^+$ as a counterpoint. We see that nothing special happens as we go from Table II to Table III no PDS. The reason for this is, as discussed above, that four identical $g_{9/2}$ nucleons can have $J = 12^+$, but, because of the Pauli Principle, they cannot couple to $J = 11^+$.

The other states with $J = 13, 14, 15, 16$ cannot occur for four identical nucleons and are therefore subject to the PDS. Note certain degeneracies, e.g. $J = 11, 13,$ and 14 states, all with $(J_p, J_n) = (6, 8)$ and $(8, 6)$, have the same energy $E = 5.3798$ MeV. The proof of all these properties are contained in Refs. 2, 3.

The $J = 16^+$, which was experimentally discovered by Nara Singh et al. [1] is correctly predicted to be isomeric in Table II. It lies below the lowest $15^+$ or $14^+$ states. In Table III however, the $J = 16^+$ state lies above the lowest $J = 14^+$ state and is degenerate with the second $J = 14^+$ state ($E = 5.6007$ MeV). Clearly, fluctuations in the $T = 0$ matrix elements are responsible for making the $J = 16^+$ isomeric.

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1. B.S. Nara Singh et al., Phys. Rev. Lett. 107, 172502 (2011).
2. L. Zamick and S.J.Q. Robinson, Phys. Rev. C 84, 044325 (2011).
3. S.J.Q. Robinson and L. Zamick, Phys. Rev. C 63, 064316 (2001).
4. A. Escuderos and L. Zamick, Phys. Rev. C 73, 044302 (2006).
5. L. Zamick, Phys. Rev. C 75, 064305 (2007).
6. A. Escuderos, L. Zamick, and B.F. Bayman, Wave functions in the $f_{7/2}$ shell, for educational purposes and ideas, http://arxiv.org/abs/nucl-th/0506050 (2005).
7. P. Van Isacker and S. Heinze, Phys. Rev. Lett. 100, 052501 (2008).
8. L. Zamick and P. Van Isacker, Phys. Rev. C 78, 044327 (2008).
9. Chong Qi, Phys. Rev. C 83, 014307 (2011).
10. Chong Qi, Z.X. Xu, and R.J. Liotta, Nucl. Phys. A 884–885, 21–35 (2012).
11. I. Talmi, Nucl. Phys. A 846, 31 (2010).
Table I: Wave functions and energies (in MeV, at the top) of selected states of $^{96}$Cd calculated with the INTd interaction (see text).

| $J$ = 11 |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $J_p$ | $J_n$ | $T=1$ | $T=1$ | $T=1$ | $T=1$ |
| 4 | 8 | 0.4709 | -0.6359 | -0.2463 | 0.3092 | -0.4544 | 0.1051 |
| 6 | 6 | 0.2229 | 0.0000 | 0.8712 | 0.0000 | -0.3121 | -0.3065 |
| 6 | 8 | 0.4607 | -0.3092 | -0.0631 | -0.6359 | 0.4432 | -0.2956 |
| 8 | 4 | 0.4709 | 0.6359 | -0.2463 | -0.3092 | -0.4544 | 0.1051 |
| 8 | 6 | 0.4607 | 0.3092 | -0.0631 | 0.6359 | 0.4432 | -0.2956 |
| 8 | 8 | 0.2869 | 0.0000 | 0.3343 | 0.0000 | 0.3110 | 0.8421 |

| $J$ = 12 |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $J_p$ | $J_n$ | $T=1$ | $T=1$ | $T=1$ | $T=2$ |
| 4 | 8 | 0.4364 | 0.3052 | -0.3894 | -0.3592 | -0.5903 | 0.2957 |
| 6 | 6 | 0.7797 | -0.4079 | 0.0000 | 0.2927 | 0.0000 | -0.3742 |
| 6 | 8 | 0.0344 | 0.5602 | -0.5903 | 0.2078 | 0.3894 | -0.3766 |
| 8 | 4 | 0.4364 | 0.3052 | 0.3894 | -0.3592 | 0.5903 | 0.2957 |
| 8 | 6 | 0.0344 | 0.5602 | 0.5903 | 0.2078 | -0.3894 | -0.3766 |
| 8 | 8 | 0.0940 | 0.1402 | 0.0000 | 0.7550 | 0.0000 | 0.6337 |

| $J$ = 13 |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $J_p$ | $J_n$ | $T=1$ | $T=1$ |  |
| 6 | 8 | 0.7071 | 0.6097 | -0.3581 |  |
| 8 | 6 | -0.7071 | 0.6097 | -0.3581 |  |
| 8 | 8 | 0.0000 | 0.5065 | 0.8623 |  |

| $J$ = 14 |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $J_p$ | $J_n$ | $T=1$ |  |
| 6 | 8 | 0.6943 | -0.1339 | -0.7071 |  |
| 8 | 6 | 0.6943 | -0.1339 | 0.7071 |  |
| 8 | 8 | 0.1894 | 0.9819 | 0.0000 |  |

| $J$ = 15 |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $J_p$ | $J_n$ | $T=1$ |  |
| 6.2789 |  |

| $J$ = 16 |  |  |  |  |  |  |
|---|---|---|---|---|---|---|
| $J_p$ | $J_n$ | $T=1$ |  |
| 4.9371 |  |

[12] B. Cederwall et al., Nature (London) 469, 68 (2011).
[13] S. Zerguine and P. Van Isacker, Phys. Rev. C 83, 064314 (2011).
[14] C. Qi, J. Blomqvist, T. Bäck, B. Cederwall, A. Johnson, R.J. Liotta, and R. Wyss, Phys. Rev. C 84, 021301(R) (2011)
[15] Z.X. Xu, C. Qi, J. Blomqvist, R.J. Liotta, and R. Wyss, Nucl. Phys. A 877, 51 (2012).
[16] L. Zamick and A. Escuderos, Nucl. Phys. A 889, 8 (2012).
[17] L. Coraggio, A. Covello, A. Gargano, and N. Itaco, Phys. Rev. C 85, 034335 (2012).
[18] L. Zamick and S.J.Q. Robinson, Phys. Atom. Nucl. 65, 740 (2002).
[19] B.F. Bayman, J.D. McCullen, and L. Zamick, Phys. Rev. Lett. 11, 215 (1963).
[20] J.N. Ginocchio and J.B. French, Phys. Lett. 7, 137 (1963).
Table II: Wave functions and energies (in MeV, at the top) of selected states of $^{96}$Cd calculated with the INTd interaction (see text) with $T = 0$ matrix elements set to zero.

| $J_p$ | $J_n$ | $T = 1$ | $T = 1$ | $T = 1$ | $T = 1$ |
|-------|-------|---------|---------|---------|---------|
| 4     | 8     | 0.7071  | 0.0000  | 0.2933  | -0.5491 | 0.3351  | -0.0121 |
| 6     | 8     | 0.0000  | 0.0000  | 0.2913  | 0.5605  | 0.6482  | -0.4253 |
| 8     | 4     | -0.7071 | 0.0000  | 0.2933  | -0.5491 | 0.3351  | -0.0121 |
| 8     | 6     | 0.0000  | -0.7071 | 0.5350  | 0.0396  | -0.4111 | -0.2079 |
| 8     | 8     | 0.0000  | 0.0000  | 0.4130  | 0.2822  | 0.1319  | 0.8558  |

| $J_p$ | $J_n$ | $T = 1$ | $T = 1$ | $T = 2$ |
|-------|-------|---------|---------|---------|
| 4     | 8     | 0.5699  | 0.2803  | -0.0961 | -0.4783 | 0.5208  | 0.2957  |
| 6     | 6     | 0.5712  | -0.7151 | 0.1498  | 0.0000  | 0.0000  | -0.3742 |
| 6     | 8     | 0.0925  | 0.3679  | 0.4629  | -0.5208 | -0.4738 | -0.3766 |
| 8     | 4     | 0.5699  | 0.2803  | -0.0961 | 0.4783  | -0.5208 | 0.2957  |
| 8     | 6     | 0.0925  | 0.3679  | 0.4629  | 0.5208  | 0.4783  | -0.3766 |
| 8     | 8     | -0.0846 | -0.2465 | 0.7284  | 0.0000  | 0.0000  | 0.6337  |

| $J_p$ | $J_n$ | $T = 1$ | $T = 1$ |
|-------|-------|---------|---------|
| 6     | 8     | 0.7071  | 0.5265  | -0.4721 |
| 8     | 6     | -0.7071 | 0.5265  | -0.4721 |
| 8     | 8     | 0.0000  | 0.6676  | 0.7445  |

| $J_p$ | $J_n$ | $T = 1$ | $T = 1$ |
|-------|-------|---------|---------|
| 6     | 8     | 0.7071  | 0.0000  | -0.7071 |
| 8     | 6     | 0.7071  | 0.0000  | 0.7071  |
| 8     | 8     | 0.0000  | 1.0000  | 0.0000  |

| $J_p$ | $J_n$ | $T = 1$ |
|-------|-------|---------|
| 8     | 8     | 1.0000  |

| $J_p$ | $J_n$ | $T = 1$ |
|-------|-------|---------|
| 8     | 8     | 5.6007  |