Bulk topological proximity effect in multilayer systems

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We investigate the bulk topological proximity effect in multilayer lattice systems. We show that one can introduce topological properties into a system composed of multiple trivial layers by coupling to a single nontrivial layer described by the Haldane model. This phenomenon depends not only on the number of layers but also on their arrangement, which can lead to the emergence of dark states in multilayer systems. The response of a trivial system to the proximity of a topological insulator appears to be highly nonlocal, in contrast to the proximity effect observed in context of superconductivity. We also find a range of parameters where our system is semimetallic with features similar to the ones observed in three-dimensional topological states. This is promising from the perspective of bridging two- and three-dimensional topologically protected states of matter.

The proximity effect of superconductivity is a well studied phenomenon. By bringing a superconducting material with a finite, local, $U(1)$ order parameter into contact with a nonsuperconducting one, the order parameter is inherited into the bulk of the nonsuperconducting material. This effect has an analogue for topological materials [1], even though topological materials do not possess a local order parameter. Hsieh et al. [2] describe the effect of a Chern insulator with Chern number $C = 1$ coupled to a topologically trivial two-dimensional material. At finite coupling the trivial material also becomes topologically nontrivial with an opposite Chern number $C = -1$. It is important to emphasize that this is pure bulk physics and must be differentiated from topological edge states. The effect is studied for bilayer systems composed of a graphene layer coupled to a Haldane layer [3] in Refs. [4, 5]. Here, Ref. [4] introduces a topological invariant for open systems which makes it possible to compute Chern number of a subsystem, e.g., for single layer. This technique gives evidence of the emergence of the $C = -1$ Chern number in the trivial layer. Another study investigates bilayers of two Haldane insulators with opposite Chern numbers and found various topological many-body phases, especially if two-body interactions in one layer are applied [6]. A spinful bilayer system of stacked Kane-Mele layers [7] has been investigated in Ref. [8], and various types of bulk proximity effects involving topologically nontrivial systems coupled to few topologically trivial layers have been recently studied in real materials, both theoretically [9, 10] as well as experimentally [11].

The current understanding of the problem suggests that the main difference between the superconducting and the bulk topological proximity effects is that the latter does not possess a local order parameter. On top of that, a Chern insulator is chiral which is manifested in the sign of its Chern number. The proximity of the nontrivial layer induces topological properties with opposite chirality [2, 5]. The previous understanding was that the trivial layer compensates the Chern number of the nontrivial layer when brought in its proximity. Following this it is a priori not evident how, e.g., two equal trivial layers would compensate the Chern number of a third, nontrivial one. As the problem of the topological proximity effect has been studied so far mostly in two-dimensional bilayer systems, there remains much to be understood about the problem for multilayer systems. We aim to fill this gap in order to further understand the bulk topological proximity effect and how it differs from the proximity effect observed in superconducting systems. Studying multilayer systems will also help understanding differences between two- and three-dimensional topologically protected states of matter which host vastly different physics [12, 13].

Three-layer system – We begin our investigation with a detailed study of a three-layer system, which is described by a tight-binding model with spinless, noninteracting fermions. In such a system, there are two graphene layers (GL) which have a nearest-neighbor (NN) hopping amplitude $t_1$ and a staggered potential $m$. The remaining third layer (HL) is described by the topological Haldane model [3] with NN hopping $t_1$ and with next-nearest-neighbor (NNN) hopping $t_2$. The NNN hopping occurs with an associated phase shift of $\Phi = \pi/2$, which keeps the system particle-hole symmetric at half filling. The layers are coupled through a tunneling process with amplitude $r$. The Hamiltonian in momentum space has the form

$$
\mathcal{H}(k) = \begin{pmatrix}
\hat{d}_1(k) \cdot \vec{\sigma} & r \cdot 1 & 0 \\
r \cdot 1 & \hat{d}_2(k) \cdot \vec{\sigma} & r \cdot 1 \\
0 & r \cdot 1 & \hat{d}_3(k) \cdot \vec{\sigma}
\end{pmatrix},
$$

with $\mathbb{1}$ ($\mathbb{0}$) being the $2 \times 2$ unit (zero) matrix, $\vec{\sigma}$ is a vector of Pauli matrices. The $2 \times 2$ structure appears due to two sublattices of the honeycomb lattice. The

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the associated topological indices, or to two GLs together (without the HL), with topological index \( I_{12} \). The results of numerical calculations are presented in Fig. 1 a) and e) for configuration I and in Fig. 1 b) and f) for configuration II. For vanishing interlayer tunneling \( r = 0 \) in both configurations I and II, the gap is open due to the staggered potential \( m \) and the NNN hopping \( t_2 \), and the system is topologically nontrivial thanks to the HL. Correspondingly, the Chern number of the full system is \( C = 1 \), which is a sum of Chern numbers of each layer. As we increase the coupling strength, in both configurations I and II, the gap decreases and eventually closes at the \( K = \frac{2\pi}{3} \left( 1, \frac{1}{\sqrt{3}} \right) \) point in the BZ at some critical value \( r_1 \), which marks a phase transition of the system to a topologically trivial state with \( C = 0 \). Note that the values of \( r_1 \) are different for the two configurations. We also note that the behavior for \( r > r_1 \) is significantly different in the two configurations.

In configuration I, the system remains in the topologically trivial state for small range of coupling strengths \( r \). At some critical value \( r_2 \), the gap closes at the \( K' = \frac{2\pi}{3} \left( 1, -\frac{1}{\sqrt{3}} \right) \) point in the BZ leading to another phase transition. We also note that the two GLs change the values of their respective topological indices \( I_\alpha \) sequentially. For weak coupling, the indices have values \( I_1 = I_2 = 0 \). Then at \( r_1 \), the \( I_1 \) index of the layer neighboring HL changes to \( I_1 = -1 \) and after the second phase transition at \( r_2 \) the \( I_2 \) index of the last GL changes to 1. The topological index \( I_{12} \) of the two GLs follows \( I_{12} = I_1 + I_2 \).

The HL topological index \( I_{HL} = 1 \) remains unchanged for all \( r \) in both cases unless the gap is closed. Therefore, we have \( C = I_{HL} + I_{12} = I_{HL} + I_1 + I_2 \). In configuration II, the system remains in a topologically trivial state for all \( r > r_1 \). The individual behavior of the indices \( I_\alpha \) is also different. The initial values \( I_1 = I_2 = 0 \) and \( I_{HL} = 1 \) remain the same for all \( r \) where they are properly defined. However, the index of two GLs together \( I_{12} \) does change to \( I_{12} = -1 \) at \( r_1 \). Therefore, we have \( C = I_{HL} + I_{12} \neq I_{HL} + I_1 + I_2 \). The difference between the two configurations I and II can be explained by the emergence of a dark state in the sandwiched configuration II. We choose this naming in correspondence with dark states in open quantum systems [16–18]. Dark states in our system can be engineered knowing the eigenstates \( v_\pm(k) \) of the graphene Hamiltonian \( \tilde{g}(k) \cdot \sigma \). We notice, that the six-component statevector \([v_\pm(k), 0, 0, -v_\mp(k)]^T\) is an eigenstate of the Hamiltonian (1) in the configuration II with the same energy as the eigenstate \( v_\mp(k) \) of graphene. We therefore obtain a state that is completely decoupled from the HL, due to the vanishing amplitude at the central layer, and insensitive to the coupling strength \( r \). On the other hand, states \([v_\pm(k), 0, 0, v_\pm(k)]^T\) are coupled stronger to the HL, compared to a single-layer state, with an effective coupling strength \( r^{\text{eff}} = \sqrt{2r} \). As a result, the configura-

We first examine the bulk properties of the system in the two spatial configurations I and II. To determine topological properties of the system we calculate its Chern number \( C \) as a function of varying coupling strength \( r \). We use Fukui’s method [15] on the discretized BZ to numerically obtain \( C \). To determine topological properties of each of the layers separately we employ the method developed in Ref. [4], in which a topological invariant \( I_\alpha \) is calculated based on the single-particle density matrix of a subsystem \( \alpha \). In our case the subsystem either corresponds to one of the GLs, with \( I_1 \) and \( I_2 \) being

Figure 1. Band gap (blue) and Chern number (orange) of the full three-layer system for a) configuration I and b) configuration II, which are schematically represented in c) and d), respectively. In e) and f) the topological indices \( I_1 \) and \( I_2 \) for the individual GLs as well as \( I_{12} \) for subsystem consisting of both GLs are plotted for smaller range of coupling strengths. For both configurations at intermediate couplings \( 2.9 < r < 6.6 \) the gap is closed at three pairs of Dirac points that lie on a closed contour in the BZ, which is represented in g). With increasing \( r \) the Dirac points move along the contour as depicted by arrows. Used parameters are \( t_1 = 4 \) in all layers, \( m = 1 \) in GLs, and \( t_2 = 1 \) with \( \Phi = \pi/2 \) in the HL.

three-dimensional vector \( \tilde{d}_i(k) \) represents the Hamiltonian of the decoupled layer \( i \) in the Bloch sphere representation, and \( \mathbf{k} = (k_x, k_y) \) is a vector in the twodimensional Brillouin zone (BZ) of the honeycomb lattice. The three layers can be arranged in two different ways: Configuration I, with the HL being on top of the two GLs, as depicted in Fig. 1 c), where \( \tilde{d}_1 = \tilde{d}_h \) and \( \tilde{d}_2 = \tilde{d}_3 = \tilde{d}_g \), or configuration II, with the HL sandwiched between two GLs as shown in Fig. 1 d), where \( \tilde{d}_2 = \tilde{d}_h \) and \( \tilde{d}_1 = \tilde{d}_3 = \tilde{d}_g \). The components of \( \tilde{d}_g/h \) are given by \( d_{g/h}(k) = d_{h/g}(k) = -t_1 \sum_{i=1}^{3} \cos(k \cdot \mathbf{a}_i) \), \( d_{g/y}(k) = d_{h/y}(k) = -t_1 \sum_{i=1}^{3} \sin(k \cdot \mathbf{a}_i) \), \( d_{g,z} = m \), and \( d_{h,z} = -2t_2 \sum_{i=1}^{3} \sin(k \cdot \mathbf{b}_i) \) [5]. The vectors \( \mathbf{a}_i \) and \( \mathbf{b}_i \) (\( |\mathbf{a}_i| = 1 \)) link the NNs and NNNs within the honeycomb lattice, respectively. We assume that the layers are parallel to the xy-plane, AA-stacked [14] in the z direction, and that the system is half-filled.

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tion II can be mapped onto a bilayer problem, which was investigated in Ref. [4, 5], and a decoupled, effective GL. The above arguments no longer hold for the configuration I where such an eigenstate with vanishing amplitude at the HL does not exist.

In both considered configurations of the three-layer system, at intermediate values of \( r \) the gap closes again at \( r < \approx 2.9 \), remains closed for a certain range of coupling strengths, and finally reopen at \( r > \approx 6.6 \) and remains open for \( r \to \infty \). In contrast to previous gap closing instances, this one does not occur at the \( K \) or \( K' \) points but rather at points from a subset \( D \) of the BZ given by

\[
D = \left\{ k : \tilde{a}_h(k) = \tilde{a}_g(k) \right\}. \tag{2}
\]

This set forms a closed contour in the BZ depicted in Fig. 1 g). At \( r < \) the gap closes at points \( k \in D \) that lie on the lines connecting the \( K \) point and its three neighboring \( K' \) points. The system becomes semimetallic with three pairs of Dirac points, one for each \( K' \) neighbor of \( K \). The semimetallic properties emerge even though our system, which has honeycomb structure, is exposed to curvature. As \( r \) is increased the monopole and the antimonopole that were initially created at the same \( k \in D \) point move away from one another along the \( D \) contour, as depicted in Fig. 1 g) with arrows. Eventually, at \( r > \), the monopoles and antimonopoles annihilate in new pairs, and the gap opens again. This phenomenon is reminiscent of what occurs in three-dimensional topological systems [19, 20], where one also observes Dirac points created as monopole-antimonopole pairs, evolving through the BZ and eventually annihilating in the same or different pairs. This similarity suggests that studying multilayer systems could bridge our understanding of two- and three-dimensional topological systems. We have also numerically checked that the semimetallic feature is robust against introducing slightly different properties for each layer, i.e., varying NN hopping amplitude \( t_1 \), staggered potential \( m \) and interlayer coupling \( r \) for each layer. An explanation of the mechanism responsible for the gap closing and formation of Dirac-point pairs can be found in the Supplemental Material [21] and also in Ref. [6].

To better understand the observations for different configurations of layers we investigate the system in the strong coupling limit. We perform a perturbation expansion with respect to terms \( m, t_2 \ll r \), details of which can be found in the Supplemental Material [21]. When the coupling is sufficiently strong, the spectrum splits into pairs of bands separated by an energy offset of order \( r \). The dispersion of different bands reads

\[
E_{\pm}(k, \kappa_z) = -2r \cos(\kappa_z) + \epsilon_{\pm}(k), \tag{3}
\]

where \( \kappa_z \in \{ \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \ldots \} \) and \( \epsilon_{\pm}(k) \) are the eigenenergies of the Haldane model with renormalized parameters \( t_1^{\text{eff}}, t_2^{\text{eff}}, m^{\text{eff}}, \Phi^{\text{eff}} = \Phi = \pi/2 \). Different pairs of bands vary with respect to the \( z \) dependence of their wavefunction. Because of this effective Haldane model description, each band can have a finite Chern number. As the different pairs of bands are independent from one another within the strong coupling approximation and because the sum of the Chern numbers of bands from a given pair is zero, the topological properties of the entire half-filled system will be determined by the pair that lies closest to the Fermi energy \( E_F \). If \( E_F \) lies between the two bands of such a pair, the system as a whole will acquire a finite Chern number \( C \) of the lower band. In the half-filled strong coupling limit only the states with \( \kappa_z = \pi/2 \) (with energies \( E_{\pm}(k, \kappa_z) = \epsilon_{\pm}(k) \)) are relevant for the Chern number of the full system.

We note that none of the bands is associated with any of the layers in particular. The wavefunctions are completely delocalized with respect to the layer index. This poses an issue for the interpretation of the topological index \( I_\alpha \). Trivial or nontrivial topology is a property of a band rather than a layer. Therefore, unless bands are approximately localized on specific layers, using index \( I_\alpha \) might lead to inconsistencies. While in Ref. [4] bands could be approximately associated with layers thanks to the weak coupling and different energy scales in the HL and the GL, this is no longer the case in our three-layer system. In the sandwiched configuration II, even in the weak coupling limit, the bands are always delocalized between the two GLs, due to their degeneracy at \( r = 0 \) and the symmetry of the layer arrangement. As a result, layer specific topological indices \( I_1 \) and \( I_2 \) remain zero for all coupling strengths even though the topological indices \( C \) and \( I_2 \) change at the critical coupling strength, c.f. Fig. 1. We note that at \( r = 0 \) the GLs are degenerate with each other but not with the HL. We also note that in the configuration II the presence of the dark state can be connected to the state in the strong coupling limit which has vanishing wavefunction amplitude at the middle HL.

Multilayer system – We next consider how our study of a three-layer system can be generalized to a case with \( L \) layers. We begin with the investigation of the strong coupling limit. For detailed derivations see the Supplemental Material [21]. Similarly, as in the three-layer case, the spectrum splits into \( L \) pairs of bands if \( r/L \gg t_1, t_2, m \) is the dominant energy scale in the system. Each pair is described by a renormalized Haldane model and is approximately decoupled from other bands. The Chern number of the system, shown in Fig. 2 for \( r = 50 \), is determined by the pair closest to the Fermi energy. In the half-filled case, we immediately notice that all systems with an even number of layers \( L \) will be topologically trivial at strong coupling. That is because the energy dispersion for a pair \( (\pm) \) of bands is given by Eq. (3) with \( \kappa_z \in \{ \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \ldots \} \). For even \( L \) we have al-
ways $\kappa_z \neq \pi/2$ and for each $\kappa_z$ the two bands, $E_\pm(k, \kappa_z)$ and $E_\mp(k, \kappa_z)$, are either both greater than or smaller than zero. For an odd number of layers $L$ there exists a state with $\kappa_z = \pi/2$ such that $E_\pm(k, \kappa_z) = \epsilon_\pm(k)$ and the two corresponding bands will determine the topological properties of the entire system. We then need to compare the renormalized Haldane parameters to determine whether the system is topologically trivial or not. These parameters are $m_{\text{eff}} = (1 - |N|^2 \sin^2(\pi h/2)) m$ and $t_{\text{eff}} = |N|^2 \sin^2(\pi h/2) t_2$ with $h$ being the layer index of the HL. $|N|^2$ is a normalization factor. All cases with even $h$ will be trivial due to vanishing $\sin^2(\pi h/2)$ term, which leads to a diverging ratio of $m_{\text{eff}}/t_2$ term. The remaining cases can be topologically nontrivial if $m_{\text{eff}}/t_{\text{eff}} = (m/t)(1 - |N|^2)/|N|^2 < 3\sqrt{3}$ [3]. The normalization factor for $\kappa_z = \pi/2$ and for odd $L$ reads $|N|^2 = 2/(L+1)$. Thus for $m = t_2 = 1$ the system can be topologically nontrivial for $L$ being smaller than a critical value $L_c = 6\sqrt{3} + 1 \approx 11.4$. The numerically determined phase boundary agrees well with the perturbationally obtained critical number of layers $L_c$.

The predictions of the strong coupling limit in the multilayer case should be robust for a wide range of coupling strengths, even when the bands start overlapping, because one needs to close the gap in order to change topological properties of the system. However, as already observed for the case of three layers, at intermediate $r$ the gap closes. In multilayer systems this gap closing might be different than in three-layer systems, where we observed semimetallic properties. In particular, emergence of multiple dark states with different energies [21] might lead to closing the gap and making the system metallic. However, since the dark states do not couple to the HL, the remainder of the system should still feature semimetallic properties with pairs of Dirac points propagating along a specific contour in the BZ. More generally, splitting of the Hilbert space into a subspace of dark states and a subspace of states coupled to the HL results in effectively two separate systems with different properties. While the subsystem of the dark states can be metallic, the other can feature chiral edge states or semimetallic properties. Similar instances of many-body phases are, e.g., Refs. [22–24].

**Conclusion** – In conclusion, we have presented a detailed study of AA-stacked honeycomb lattice multilayer systems with a single topological layer described by the Haldane model. As in the previously discussed bilayer systems [2, 4, 5], a single HL can open a gap and induce topologically nontrivial properties in the neighboring, previously topologically trivial layers. However, we also observe multiple new phenomena, such as emergence of dark states which are effectively decoupled from HL, or the onset of semimetallic properties. We are convinced that studying such multilayer systems might help in finding connections between two- and three-dimensional topological states of matter. One example is the emergence of a semimetallic phase in our system. Also a mapping to a weak topological insulator in the weak coupling regime might be thinkable. We furthermore show that one has to be careful when using layer-specific topological indices, as topological properties are a feature of bands rather than layers. This is an example of nonlocality of topological properties and marks a further distinction between the bulk topological and the superconducting proximity effect. We expect that our results can be easily generalized to other models, e.g., generalized Hofstadter and Kane-Mele models. Studying interactions in these systems will certainly be of interest in the near future. On a mean-field level, Hubbard interactions will renormalize the staggered potential and thus stabilize the topological states found, e.g., in Fig. 2. We also expect that the systems used in our letter could be realized experimentally using a combination of shaken optical lattices [25, 26], synthetic dimensions [5] and spin-dependent optical potentials [27].
We consider a multilayer generalization of the three-layer system in tight-binding approximation considered in the main text. There are $L$ honeycomb lattice layers AA-stacked along the $z$ axis each being parallel to the $xy$-plane and infinite. The first $L_1$ and the last $L_2 = L - L_1 - 1$ are GLs, the $(h = L_1 + 1)$th is a HL. The full model in $k$-space is captured by the following Hamiltonian

$$\mathcal{H}(k) = \begin{pmatrix} \ddot{g}_y(k) \cdot \tilde{\sigma} & r \cdot 1 & 0 \\ r \cdot 1 & \ddot{h}_y(k) \cdot \tilde{\sigma} & r \cdot 1 \\ 0 & r \cdot 1 & \ddot{g}_y(k) \cdot \tilde{\sigma} \end{pmatrix} ,$$

with terms defined in the main text. As a sidenote we point to the fact that this Hamiltonian, while producing correct dispersions and Berry curvatures, is not periodic with respect to shift to a neighboring copy of the BZ which would require additional unitary transformation [28]. We use this version of the Hamiltonian, as it has more concise notation.

We take the strong coupling limit of our system $r/L \gg t_1, t_2, m$, in which we can consider special properties of the Haldane layer as a small perturbation. The unperturbed Hamiltonian $H_0(k)$ is then of form (4) with block terms $\ddot{g}_y(k) \cdot \tilde{\sigma}$ on the diagonal, including the $h$ position. The perturbation Hamiltonian $H_\ell(k)$ vanishes everywhere except for the $h$ diagonal block and is given by $(\ddot{h}_y(k) - \ddot{g}_y(k)) \cdot \tilde{\sigma}$. We first diagonalize the unperturbed Hamiltonian, which corresponds to solving a one-dimensional tight-binding chain with open boundary conditions (OBC) combined with diagonalizing $2 \times 2$ matrix of graphene in $k$-space. One can show that the eigenstates are product states of standing waves in $z$ direction with eigenstates of graphene in $xy$-plane

$$v_{\pm}(k, \kappa_z) = \mathcal{N}[\alpha_{\pm}(k) \sin(\kappa_z), \beta_{\pm}(k) \sin(\kappa_z), \alpha_{\pm}(k) \sin(2\kappa_z), \beta_{\pm}(k) \sin(2\kappa_z), \ldots \alpha_{\pm}(k) \sin(L\kappa_z), \beta_{\pm}(k) \sin(L\kappa_z)]^T ,$$

where $\mathcal{N}$ is the normalization factor and $\kappa_z \in \{ \frac{\pi}{L+1}, \frac{2\pi}{L+1}, \ldots, \frac{L\pi}{L+1} \}$. The eigenvalues read

$$E_{\pm}^{(0)}(k, \kappa_z) = -2r \cos(\kappa_z) + \epsilon_{g,\pm}(k) ,$$

with $\epsilon_{g,\pm}(k)$ being the energy bands of graphene. The factors $\alpha_{\pm}(k)$ and $\beta_{\pm}(k)$ are components of the normalized eigenvectors of graphene. Now we consider the correction due to $H_\ell(k)$. We focus on the first order correction to the eigenstates [29]

$$v_{\pm}^{(1)}(k, \kappa_z) = \sum_{\kappa'_z \neq \kappa_z} \sum_{\gamma \in \{+1\}} \left[ (v_\gamma(k, \kappa'_z))^\dagger H_\ell(k) v_{\pm}(k, \kappa_z) \right] \frac{E_\pm(k, \kappa_z) - E_\gamma(k, \kappa'_z)}{E_\pm(k, \kappa_z) - E_\pm(k, \kappa_z)} v_\gamma(k, \kappa'_z) .$$

We notice that for sufficiently large $r/L \gg t_1, t_2, m, \ldots$ we obtain $|E_{\pm}(k, \kappa_z) - E_\gamma(k, \kappa'_z)| \gg |E_{\pm}(k, \kappa_z) - E_\pm(k, \kappa_z)|$ and only the terms with the same $\kappa_z$ will contribute significantly to $v_{\pm}^{(1)}(k, \kappa_z)$. This is true also for higher order terms in perturbation expansion [29]. Therefore, we can neglect mixing between levels with different $\kappa_z$. This simplifies the problem to a set of independent pairs of states with the same $\kappa_z$. The effective Hamiltonian for such a pair reads

$$H_{\text{eff}}(k, \kappa_z) = -2r \cos(\kappa_z) 1 + \ddot{g}_y(k) \cdot \tilde{\sigma} + |\mathcal{N}|^2 \sin^2(\kappa_z) h \left[ \ddot{h}_y(k) - \ddot{g}_y(k) \right] \cdot \tilde{\sigma} ,$$

with $|\mathcal{N}|^2 \sin^2(\kappa_z) h$ representing the amplitude of the $\kappa_z$ state at the HL which has index $h$. If we set the same value of NN hopping $t_1$ in the GLs and the HL, the $x$ and $y$ components of $\ddot{g}_y$ and $\ddot{h}_y$ are identical. Then the effective Hamiltonian for given $\kappa_z$ has the form of the standard Haldane model with energy shift $-2r \cos(\kappa_z)$, NN hopping amplitude $t_1$, and an effective staggered potential and NNN hopping given by $m_{\text{eff}} = (1 - |\mathcal{N}|^2 \sin^2(\kappa_z) h) m$ and $t_{\text{eff}}^3 = |\mathcal{N}|^2 \sin^2(\kappa_z) h t_2$, respectively. The ratio of these effective parameters $m_{\text{eff}}$ and $t_{\text{eff}}^3$ determines the topological properties of a given pair of energy bands. In Fig. 3 we present exemplarily the spectrum for $L = 5$, $L_1 = L_2 = 2$, $t_1 = t_2 = m = 1$, and $r = 12$ in the cylinder geometry, i.e., periodic boundary conditions in $x$ direction and open ones in $y$ direction. We observe a clear separation of bands into pairs. Depending on effective parameters $m_{\text{eff}}$ and $t_{\text{eff}}^3$, different for each pair, the bands have Chern numbers being either $C = 0$ or $\pm 1$. This is reflected by the edge states shown in blue/orange. The half-filled system as a whole has a Chern number $C = 1$ and a chiral edge state crosses the gap, due to the properties of a middle bands pair, which is separated by the Fermi energy $E_F = 0$. 

Strong coupling limit
Figure 3. Band structure of a system with $L = 5$, $L_1 = L_2 = 2$, $t_1 = t_2 = m = 1$, and $r = 12$ with open boundary conditions in $y$ direction. The edges are of zig-zag type. Bands can be split into 5 pairs, each of which is described by slightly different effective Haldane model. The second and fourth pair of bands have no edge state crossing the gap due to $m_{\text{eff}}/t_{\text{eff}}^2 > 3\sqrt{3}$. The remaining pairs have chiral edge states crossing the gap between them. The Fermi energy $E_F = 0$ separates two bands of the middle pair resulting in the nontrivial topological properties of a system as a whole.

Dark states for arbitrary coupling and number of layers

The emergence of dark states in multilayer systems described by the Hamiltonian (4) can be shown also in the regime of arbitrary coupling strength $r$. In order to find the condition for their existence we apply a unitary transformation to (4) of the form $U^\dagger(k)H(k)U(k)$, where

$$U(k) = \begin{pmatrix} \ddots & U_g(k) & 0 & 0 \\ 0 & U_g(k) & 0 & 0 \\ 0 & 0 & U_g(k) & \ddots \end{pmatrix},$$

and where $U_g(k)$ diagonalizes the graphene Hamiltonian $\tilde{d}_g(k) \cdot \vec{\sigma}$. This transformation leaves coupling blocks with $r$ terms unchanged as they are proportional to the unit matrix, diagonalizes $2 \times 2$ blocks which correspond to GLs, and modifies the $2 \times 2$ block of the HL in some way, which is irrelevant for the existence of dark states. Now we consider the GLs above and below the HL. Here, the first $L_1$ GLs are coupled by $r$ but not coupled to the HL and the same applies for the $L_2$ GLs below the HL. If these two separate systems share an eigenvalue, then a dark state of the following form exists:

$$v_{\text{DS}}(k) = [\alpha \psi_1, \alpha \psi_2, \ldots, \alpha \psi_{2L_1}, 0, 0, \beta \phi_1, \beta \phi_2, \ldots, \beta \phi_{2L_2}]^T,$$

where $[\psi_1, \psi_2, \ldots, \psi_{2L_1}]^T$ is an eigenstate of $L_1$ layers above HL with eigenvalue $\lambda(k)$ and $[\phi_1, \phi_2, \ldots, \phi_{2L_2}]^T$ is an eigenstate of $L_2$ layers below HL, which has the same eigenvalue $\lambda$. The factors $\alpha$ and $\beta$ are chosen such that the interference is destructive on the HL when applying the Hamiltonian $U^\dagger(k)H(k)U(k)$ to the state in Eq. (10). In the above case, $\alpha$ and $\beta$ can always be found, since (i) components $\psi_i$ and $\phi_i$ do not depend on $k$ and (ii) in the eigenbasis of the Hamiltonian $U^\dagger(k)H(k)U(k)$ the states with nonvanishing odd elements $\psi_{2n+1} \neq 0 \neq \phi_{2m+1}$ necessarily have vanishing even elements $\psi_{2n} = 0 = \phi_{2m}$, and vice versa. Thus it is sufficient to set $\alpha/\beta = -\phi_1/\psi_{2L_1-1}$ or $\alpha/\beta = -\phi_2/\psi_{2L_1}$.

Semimetal

In this section we show why for the intermediate coupling strengths the gap closes and remains closed for a finite range of $r$ values. We focus on the three-layer system and HL on top of the two GLs which corresponds to the
configuration I. Generalization to configuration II and multilayered systems is discussed at the end of this section. The Hamiltonian is given by Eq. (1) of the main text. For \( k_d \in \mathcal{D} \), defined in the main text we apply the unitary matrix defined by Eq. (9) in the following way:

\[
\begin{pmatrix}
\mathbb{U}(k_d) & 0 & 0 \\
0 & \mathbb{U}(k_d) & 0 \\
0 & 0 & \mathbb{U}(k_d)
\end{pmatrix}
\begin{pmatrix}
d_h(k_d) \cdot \hat{\sigma} & r \cdot \mathbb{1} & 0 \\
r \cdot \mathbb{1} & d_g(k_d) \cdot \hat{\sigma} & r \cdot \mathbb{1} \\
0 & r \cdot \mathbb{1} & d_g(k_d) \cdot \hat{\sigma}
\end{pmatrix}
\begin{pmatrix}
\mathbb{U}(k_d) & 0 & 0 \\
0 & \mathbb{U}(k_d) & 0 \\
0 & 0 & \mathbb{U}(k_d)
\end{pmatrix}
\begin{pmatrix}
\epsilon_+(k_d) & 0 & r & 0 & 0 & 0 \\
0 & \epsilon_-(k_d) & 0 & r & 0 & 0 \\
r & 0 & \epsilon_+(k_d) & 0 & 0 & r \\
0 & r & 0 & \epsilon_-(k_d) & 0 & r \\
0 & 0 & r & 0 & \epsilon_+(k_d) & 0 \\
0 & 0 & 0 & r & 0 & \epsilon_-(k_d)
\end{pmatrix}, \tag{11}
\]

Here, \( \epsilon_{\pm}(k_d) \) are the eigenvalues of both Haldane and graphene tight-binding model as \( k_d \in \mathcal{D} \). The eigenspace of Eq. (11) splits into two orthogonal subspaces related to eigenvalues \( \epsilon_+ \) and \( \epsilon_- \), which satisfy \( \epsilon_+ = -\epsilon_- \) for all \( k_d \.

The Hamiltonian thus has eigenvalues \( E_{\pm,i}(k_d) \in \{ \epsilon_+(k_d) + \sqrt{2r}, \epsilon_+(k_d) - \sqrt{2r} \} \). As the subspaces for \( \epsilon_+ \) and \( \epsilon_- \) are independent, we conclude that at \( \sqrt{2r} = \epsilon_+(k_d) \) a degeneracy occurs between states with \( \epsilon_+(k_d) - \sqrt{2r} = \epsilon_-(k_d) + \sqrt{2r} = 0 \). We make three essential observations for this effect:

1. along the contour \( \mathcal{D} \) of \( k_d \) points, the values of \( \epsilon_{\pm}(k_d) \) change smoothly in a periodic manner, which leads to gap closing at single points rather the entire line of \( k_d \)'s at the same time,

2. there are three pairs of such points which have a low energy dispersion forming Dirac cones,

3. at \( r_\text{c} \) – the left boundary of a gapless region – the pairs of these Dirac points emerge at the line connecting \( \mathbf{K} \) and \( \mathbf{K}' \) points in BZ. For increasing \( r_\text{c} < r < r_\text{g} \) these pairs of Dirac points evolve through the BZ along the \( \mathcal{D} \) contour in opposite directions, and for \( r_\text{g} \) – the right boundary of the gapless region – the Dirac points annihilate in different pairs, as discussed in the main text.

Because the Dirac points at intermediate \( r \) are created and annihilated in pairs, it is intuitive to assume that they form a monopole-antimonopole pairs. Below we provide more rigorous arguments to back up this statement. First, we consider a fact that the \( \mathcal{D} \) contour is symmetric with respect to the axis connecting \( \mathbf{K} \) and \( \mathbf{K}' \) points. We define \( \mathbf{K} = \frac{2\pi}{3} (1, \frac{1}{\sqrt{3}}) \) and \( \mathbf{K}' = \frac{2\pi}{3} (1, -\frac{1}{\sqrt{3}}) \). Then if we represent one Dirac point as \( k_d = (\frac{2\pi}{3} - \bar{k}_x, k_y) \) we immediately get that \( k_d' = (\frac{2\pi}{3} + \bar{k}_x, k_y) \) will also be a Dirac point. This is due to the following relation

\[
\vec{d}_{g/h}(k_d) \cdot \hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{3}} \end{pmatrix} \cdot \left( \vec{d}_{g/h}(k_d') \cdot \hat{\sigma} \right)^* \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{2\pi}{3}} \end{pmatrix}. \tag{12}
\]

In other words, the \( \mathbf{k} \)-resolved Hamiltonians of the system at \( k_d \) and \( k_d' \) are related by a complex conjugation combined with a unitary gauge transformation. Because of the complex conjugation, the two Dirac points have opposite topological properties, hence we refer to them as a monopole-antimonopole pair. As a result of Dirac points occurring in pairs the gap closing should not affect topological properties of the full system.

We close this section with a generalizing remark for multilayer systems. We notice that the definition of \( \mathcal{D} \) is independent of the number of layers and the layer index of the HL. Therefore, our reasoning seems to also apply to multilayered systems. However, there are certain differences. For systems with more than three layers one can expect more than three pairs of Dirac points forming. On top of that, one can also expect dark states as discussed above. These can render the full system metallic consisting of two decoupled subsystems, one of which is topological or semi-metallic system as discussed in the main text. Finally, one could consider a system in which different layers differ also with respect to their staggered potential \( m \) and NN hopping amplitude \( t_1 \). While analytic calculations become much more involved in such case, the results can be obtained numerically. We found that the presence of the Dirac points in the gapless region is robust to small such perturbations (results obtained for a three-layer system are not shown in here).

**Edge state spectra**

We now investigate the bulk-boundary correspondence in the three-layer system. We choose a system which is infinite in the \( x \) direction and has zig-zag edges in the \( y \) direction. Because the edge states of the GLs can couple
Figure 4. Band structures of the systems with zig-zag edges in the $y$ direction for a) sandwiched HL, b) HL on top of GLs. The boundary at lower values of $y$ is different for HL and GLs, but for higher values of $y$ all layers share the same boundary, as depicted in the insets. Color represents average position $\langle y \rangle$ of a state along the $y$ axis. We set $r = 2$ in both configurations. Other parameters of the system are set as in Fig. 1.

to edge states of the HL, we employ a similar approach as in Ref. [2, 5] in which we take the GLs and the HL of different lengths in $y$ direction which we call terrace configuration. However, in contrast to Ref. [5], we take the HL to be shorter in order to determine the position of edge state induced in proximity effect on the GLs. The number of lattice sites in $y$ direction is $N_{GL}^y = 100$ for the GL and $N_{HL}^y = 50$ for the HL and we set $r = 2$ ($t_1$, $t_2$ and $m$ are as in Fig. 1. in the main text). The setups are schematically depicted in Fig. 4, where we also plot the band structure of the system and in color the average position in $y$ direction $\langle y \rangle$ of each state. We choose $r = 2$ where configuration I is topologically nontrivial and configuration II is trivial.

In configuration II, Fig. 4 a), we observe graphene dark state bands which are gapped and decoupled from the rest of the system. They can be recognized by $\langle y \rangle \approx 50$ in grey due to their complete delocalization with respect to the $y$ position. They also have two flat edge states, characteristic for graphene with zig-zag edges [30, 31]. The dispersion of the dark states is identical to the one of a single GL. The two bands with distribution similar to the one of the dark states and with $\langle y \rangle \approx 25$ (orange), correspond to the terrace of the GLs, sites with $y < 50$ which are not coupled to the HL. Most importantly, we observe two edge states crossing the gap in the middle of the system $\langle y \rangle \approx 50$. Contrary to the dark states these states are localized at the edge of the HL. The edge states reside on all three of the layers, and therefore the bulk topological proximity effect induces an edge state in graphene that, while being localized close to the edge of the intrinsically nontrivial HL, resides in the bulk of graphene. On the other edge, shared by GL and HL, the gap is not crossed by edge states due to their hybridization. We conclude that at $r = 2$ the system in configuration II can either have two counter-propagating edge states or no edge state. This is in agreement with our bulk investigations, Fig. 1., which predicts the system to be topologically trivial at $r = 2$.

In configuration I, Fig. 4 b), we observe that the system becomes metallic. This is solely due to the geometry of the GLs being longer than the HL. The metallic properties originate from the terrace parts of two coupled GLs that extend beyond HL. This can be identified by the orange color corresponding to $\langle y \rangle \approx 25$. The terrace of the GL has edge states at the $y = 0$ edge, characteristic to a zig-zag edge of graphene. However, they have only one edge state at the onset of the HL, which crosses the gap (of the trilayer part of the system). The system has also another edge state localized around $y = 100$. In conclusion, the system in configuration I has one chiral edge state, as expected from bulk considerations Fig. 1., identified here by the two edge states at the opposite boundaries of the three-layer part of the system, while the metallic terrace can be approximately treated as separate system. This shows that a topological insulator can exhibit edge states not only at the boundary with a trivial insulator, but also a metallic system (here bilayer graphene).

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