Cocycles for Boson and Fermion Bogoliubov Transformations

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Abstract

We discuss unitarily implementable Bogoliubov transformations for charged, relativistic bosons and fermions, and we derive explicit formulas for the 2-cocycles appearing in the group product of their implementers. In the fermion case this provides a simple field theoretic derivation of the well-known cocycle of the group of unitary operators on a Hilbert space modeled on the Hilbert Schmidt class and closely related to the loop groups. In the boson case the cocycle is obtained for a similar group of pseudo-unitary (symplectic) operators. We also give formulas for the phases of one-parameter groups of implementers and, more generally, families of implementers which are unitary propagators with parameter dependent generators.

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1 Introduction

The formalism for quantizing relativistic fermions in external fields is not only essential for quantum field theory, but it plays also a crucial role in the representation theory of the affine Kac-Moody algebras and the Virasoro algebra [1]. Indeed, this connection has led to a most fruitful interplay between physics and mathematics (see e.g. [2]).

The geometric approach to this subject by means of determinant bundles over infinite dimensional Grassmannians [3, 4] seems to be preferred by mathematicians but is quite abstract and different from the physicists’ tradition. There is, however, another rigorous approach in the spirit and close to quantum field theory, namely the theory of quasi-free second quantization (QFSQ) of fermions. Lundberg [5] was probably the first who formulated its abstract framework in an elegant and concise way, and he used it to construct in general abstract current algebras providing (by restriction) representations of the affine Kac-Moody algebras and the Virasoro algebra [1]. Later on this formalism was worked out in all mathematical detail by Carey and Ruijsenaars [6].

Besides its conceptual simplicity, QFSQ of fermions has another advantage, namely it has a natural boson counterpart (which to our knowledge is not the case for the Grassmannian approach [3, 4]). Ruijsenaars [6] in his comprehensive work on Bogoliubov transformations of charged, relativistic particles made clear and exploited the formal analogy of bosons and fermions, and he was able to derive most of the corresponding formulas for the two cases in a parallel way. Though very transparent and simple, Ruijsenaar’s QFSQ of bosons did not become very popular probably due to the fact that it deviates substantially from the traditional approach to boson quantum field theory (which is usually formulated in terms of Weyl operators (= exponentiated fields) [8] rather than the fields itself).

Though in their extensive work on QFSQ Carey and Ruijsenaars [6] restricted themselves to the fermion case, it is rather straightforward to derive most of their results for the boson case as well [9], and, moreover, to develop a $\mathbb{Z}_2$-graded formalism — a super-version of QFSQ — [9, 10, 11] comprising the boson and the fermion case and extending these in a non-trivial way. Especially, the resulting current super algebras naturally provide representations of the $\mathbb{Z}_2$-graded extensions of the affine Kac-Moody algebras and the Virasoro algebra [10, 11].
The implementers of Bogoliubov transformations [7] are an "integrated version" of the currents referred to above, i.e., the abstract current algebra provides the Lie algebra of the Lie group generated by these implementers. It is well-known that the essential, non-trivial aspect of the current algebras in the occurrence of a Schwinger term [3, 10] which — from a mathematical point of view — is a non-trivial Lie algebra 2-cocycle. On the group level, this corresponds to a non-trivial Lie group 2-cocycle arising in the product relations of the implementers [3, 4].

In this paper we prove the explicit formulas for these group 2-cocycles in the standard phase convention [7] by a simple, direct calculation. Moreover, we derive a general formula for the phase relating the one-parameter group of implementers obtained by exponentiating a current (via Stone’s theorem [12]) to the implementer given by the general formula in [7], and we generalize this to unitary propagators [13] generated by "time" dependent currents.

Our results can be regarded as a supplement to [7]. However (in contrast to [7]), our proofs are completely parallel in the boson and the fermion case: we introduce a symbol $X \in \{B, F\}$ which is equal to $B$ in the boson and to $F$ in the fermion case, and all our formulas and arguments are given in terms of this variable $X$. To this aim we introduce the symbols

$$\text{deg}(B) \equiv 1, \quad \text{deg}(F) \equiv 0$$
$$\varepsilon_B \equiv +1, \quad \varepsilon_F \equiv -1,$$

i.e. $\varepsilon_X = (-)^{\text{deg}(X)}$, and

$$[a, b]_X \equiv ab - \varepsilon_X ba,$$

i.e. $[\cdot, \cdot]_B$ is the commutator and $[\cdot, \cdot]_F$ is the anticommutator.

In the fermion case we were not able to obtain the result in the same generality as Ruijsenaars [7], but our formulas are restricted to some neighborhood of the identity containing the topologically trivial Bogoliubov transformations leaving the particle number and the charge unchanged [7, 8]. Moreover, our formula for the group 2-cocycle is well-known in the fermion case [3]. As it plays an essential role in the theory of the loop groups [3] with all its applications [4], we hope that our alternative proof is nevertheless useful. To our knowledge, the formula in the boson case is new. Moreover, these 2-cocycles play a crucial role for the construction of current algebras for bosons and fermions in (3 + 1)-dimensions (arising from Bogoliubov transformations which are not unitarily implementable but require some additional "wave
function renormalization”) given recently by the author [14] (the formulas were used in [14] without proof).

The plan of this paper is as follows. In the next Sect. we introduce our notation and summarize the facts we need about Bogoliubov transformations and currents algebras within the framework of QFSQ of bosons and fermions. Our results are presented in Sect. 3 and their proofs are given in Sect. 4. We conclude with a few remarks in Sect. 5.

2 Preliminaries

(a) Second Quantization: Let be a separabel Hilbert space and the direct sum of two subspaces \( h_+ \) and \( h_- \): \( h = h_+ \oplus h_- \), and \( T_\pm \) the orthogonal projections in \( h \) onto \( h_\pm \): \( h_\pm = T_\pm h \). \( h_+ \) can be thought of as the one-particle space and \( h_- \) as the one-antiparticle space. We write \( \mathcal{B}(h) \), \( \mathcal{B}_2(h) \), and \( \mathcal{B}_1(h) \) for the bounded, the Hilbert-Schmidt, and the trace-class operators on \( h \) [12], respectively, and for any linear operator \( A \) on \( h \),

\[
A_{\varepsilon\varepsilon'} \equiv P_\varepsilon AP_{\varepsilon'} \quad \forall \varepsilon, \varepsilon' \in \{+, -\}.
\]

(3)

We denote as \( \mathcal{F}_B(h) \) (\( \mathcal{F}_F(h) \)) the boson (fermion) Fock space over \( h \) with the vacuum \( \Omega \) and creation and annihilation operators \( a^*(f) \) and \( a(f) \), \( f \in h \) obeying canonical commutator (anticommutator) relations

\[
[a(f), a^*(g)]_X = (f, g) \\
[a(f), a(g)]_X = 0 \quad \forall f, g \in h
\]

(4)

((\cdot, \cdot) is the inner product in \( h \)), and

\[
a(f)\Omega = 0, \quad a^*(f) = a(f)^* \quad \forall f \in h
\]

(5)

as usual [8] (\( * \) denotes the Hilbert space adjoint). Moreover, we introduce the particle number operator \( N \) inducing a natural \( \mathbb{N}_0 \)-gradation in \( \mathcal{F}_X(h) \)

\[
\mathcal{F}_X(h) = \bigoplus_{\ell=0}^{\infty} h_X^{(\ell)}, \quad h_X^{(\ell)} = (P_\ell - P_{\ell-1})\mathcal{F}_X(h)
\]

with \( h_X^{(\ell)} \) the \( \ell \)-particle subspace and \( P_\ell \) the orthogonal projection onto the vectors with particle number less or equal to \( \ell \) [8], and

\[
\mathcal{D}_X^{(\ell)}(h) \equiv \{ \eta \in \mathcal{F}_X(h) | \exists \ell < \infty : P_\ell \eta = \eta \}
\]

(6)
is the set of finite particle vectors; note that $\mathcal{D}_X^f(h)$ is dense in $\mathcal{F}_X(h)$ \cite{15}. Similar as Ruijsenaars \cite{7}, we introduce the field operators

$$
\Phi^+(f) \equiv a^*(T_+f) + a(JT_-f)
$$
$$
\Phi(f) \equiv a(T_+f) - \varepsilon_X a^*(JT_-f) \quad \forall f \in h
$$

with $J$ a conjugation in $h$ commuting with $T_\pm$. Then the (anti-) commutator relations (3) result in

$$
[\Phi(f), \Phi^+(g)]_X = (f, g)
$$
$$
[\Phi(f), \Phi(g)]_X = 0 \quad \forall f, g \in h,
$$

and

$$
\Phi(f) \equiv \Phi^+(q_X f)^* \quad \forall f \in h
$$

with $q_X = T_+ - \varepsilon_X T_-$, i.e.

$$
q_B \equiv P_+ - P_-, \quad q_F \equiv 1.
$$

Remark 2.1: Note that the operators $a^*(f)$ and $\Phi^+(f) (f \in h)$ are bounded in the fermion- but unbounded in the boson case \cite{8}. However, in the later case, $\mathcal{D}_X^f(h)$ (11) provides a common, dense, invariant domain for all these operators due to the estimate \cite{7, 11} (which trivially hold for $X = F$ as well)

$$
\|a^*(f)P_\ell\| \leq (\ell + 1)\|f\| \quad \forall f \in h, \ell \in \mathbb{N}
$$

(with $\|\cdot\|$ we denote the operator norm and the Hilbert space norm), and they are closed operators on $\mathcal{F}_B(h)$. Hence (8) and similar eqs. below have to be understood as relations on $\mathcal{D}_X^f(h)$.

Remark 2.2: Note that Ruijsenaars \cite{7} uses the field operators $\tilde{\Phi}^*(f) = \Phi^+(f)$ and $\tilde{\Phi}(f) = \tilde{\Phi}^+(f)^* = \Phi(q_X f) (f \in h)$ deviating from ours in the boson case $X = B$. Our definition is more convenient for discussing the current algebras (see below).

(b) Bogoliubov Transformations: Let $U$ be a closed, invertible operator on $h$. Then the transformation

$$
\alpha_U : \Phi^+(f) \mapsto \alpha_U(\Phi^+(f)) \equiv \Phi^+(U f) \quad \forall f \in h
$$

leaves the relations (8), (9) invariant if and only if

$$
U q_X U^* = U^* q_X U = q_X.
$$
We denote such an \( U \) as \( X \)-unitary\(^1\), and \( \alpha_U \) (12) is a Bogoliubov transformation (BT). It is called \textit{unitarily implementable} if there is an unitary operator \( \tilde{\Gamma}(U) \) on \( \mathcal{F}_X(h) \) such that

\[
\tilde{\Gamma}(U)\Phi^+(f) = \Phi^+(Uf)\tilde{\Gamma}(U) \quad \forall f \in h, \tag{14}
\]

and the well-known necessary and sufficient condition for this to be the case is the Hilbert-Schmidt criterium \cite{7}

\[
U_+\!, U_- \in B_2(h). \tag{15}
\]

(Though not completely obvious, by a little thought one can convince oneself that our \( \tilde{\Gamma}(U) \) is identical with the implementer \( \tilde{U} \) defined in [7], eqs. (2.10) and (2.18)). We denote the group of all \( X \)-unitary operators on \( h \) obeying this condition as \( G_X(h) \). Furthermore, we introduce the set \( \mathcal{U}^{(0)}(h) \) of all closed, invertible operators \( U \) on \( h \) such that \( U_- \) has a bounded inverse \((U_-)^{-1}\) on \( h_- \), and

\[
G^{(0)}_X(h) \equiv G_X(h) \cap \mathcal{U}^{(0)}(h). \tag{16}
\]

A crucial difference between the boson and the fermion case is that (13) and (13) for \( X = B \) (but not for \( X = F \)) imply that \( U \in \mathcal{U}^{(0)}(h) \), hence

\[
G^{(0)}_B(h) = G_B(h), \tag{17}
\]

whereas there are plenty of \( U \in G_F(h) \) not contained in \( G^{(0)}_F(h) \) \cite{5}, and there is a one-to-one correspondence between \( G^{(0)}_B(h) \) and \( G^{(0)}_F(h) \) showing that there are many more fermion than boson BTs: Indeed, for \( U \in \mathcal{U}^{(0)}(h) \) the eq.

\[
T_- = UT_+ - T_+Z + UT_-Z = ZT_+ - T_+U + ZT_-U \tag{18}
\]

has a unique solution \( Z \in \mathcal{U}^{(0)}(h) \),

\[
\begin{align*}
Z_{++} &= U_{++} - U_{+-}(U_{--})^{-1}U_{-+} \\
Z_{+-} &= U_{+-}(U_{--})^{-1} \\
Z_{-+} &= -(U_{--})^{-1}U_{-+} \\
Z_{--} &= (U_{--})^{-1},
\end{align*} \tag{19}
\]

\footnote{note that \( F \)-unitary=unitary, and what we call \( B \)-unitary was denoted as \textit{pseudo-unitary} in [5]}
and this defines a bijective mapping

\[ \sigma : \mathcal{U}^{(0)}(h) \to \mathcal{U}^{(0)}(h); U \mapsto \sigma(U) \equiv Z \]  

(20)

with the following properties

\[ \sigma(\sigma(U)) = U \quad \forall U \in \mathcal{U}^{(0)}(h) \]
\[ \sigma(U)^{-1} = \sigma(U^{-1}) \quad \forall U \in \mathcal{U}^{(0)}(h) \]
\[ U \in G_B(h) \iff \sigma(U) \in G_B^{(0)}(h) \]
\[ U \in G_F^{(0)}(h) \iff \sigma(U) \in G_F(h) \]  

(21)

following from \[ (18) \] (to prove the last two relations, take the adjoint of \[ (18) \] and use \[ (13) \] and \[ T_B T_B = q_B T_B = \pm T_B \]).

The theory of boson and fermion BTs can be developed parallelly and on equal footing only if one restricts oneself in the fermion case to \( G_F^{(0)}(h) \) (see \[ 7 \]). This will be done in the following.

**Remark 2.3:** Note that \( G_F^{(0)}(h) \) is not a subgroup of \( G_F(h) \). Fermion BTs with \( U \in G_F(h) \) not contained in \( G_F^{(0)}(h) \) play an important role, e.g. for anomalies and the boson fermion correspondence (see e.g. \[ 4 \] and \[ 6 \]).

**c) Implementers:** The explicit formulas for the implementers \( \tilde{\Gamma}(U) \), \( U \in G_X^{(0)}(h), X \in \{B, F\} \), can be found in Ref. \[ 7 \]. We shall only need

\[ \tilde{\Gamma}(U) \Omega = N(U)e^{Z(U)} + a^* a^* \Omega \quad \forall U \in G_X^{(0)}(h) \]  

(22)

with \( Z(U) \equiv \sigma(U) \) \[ (19) \],

\[ N(U) = \det(1 - \varepsilon_X(Z(U)_{+-})^* Z(U)_{+-})^{\varepsilon_X/2} \]  

(23)

a normalization constant (\( \det(\cdot) \) is the Fredholm determinant \[ 10 \]), where we use the notation

\[ Aa^* a^* = \sum_{n=1}^{\infty} \lambda_n a^* (f_n)a^* (J g_n) \]  

(24)

for any operator \( A \in B_2(h) \) represented in the standard form \[ 12 \]

\[ A = \sum_{n=1}^{\infty} \lambda_n f_n(g_n, \cdot) \]  

(25)
with \( \{f_n\}_{n=1}^{\infty} \) and \( \{g_n\}_{n=1}^{\infty} \) orthonormal systems of vectors in \( h \) and \( \lambda_n \) complex numbers.

**Remark 2.4:** Note that the determinant in (23) exists if and only if \( Z(U)_{+} \in B_2(h) \) \( \{1\} \), and this is the case for all \( U \in G_X^{(0)}(h) \).

**Remark 2.5:** From the estimate \( \{1\} \) it follows that (23) is well-defined for \( A \in B_1(h) \), and the estimate \( \{1\} \)

\[
\|A^{*}a^{*}P_{\ell}\| \leq (\ell + 2)\|A\|_2 \quad \forall \ell \in \mathbb{N}
\]

shows that this definition extends naturally to all \( A \in B_2(h) \).

**(c) Current Algebras:** For \( A \in B(h) \), \( e^{itA} \) is in \( G_X(h) \) for all \( t \in \mathbb{R} \) if and only if

\[
q_X A^{*}q_X = A
\]

and

\[
A_{+}, A_{-} \in B_2(h). \tag{27}
\]

We denote a \( A \in B(h) \) obeying \( \{23\} \) as \( X \)-self-adjoint\(^\text{*} \), and as \( g_X(h) \) the set of all \( X \)-self-adjoint operators obeying \( \{27\} \). \( g_X(h) \) is the Lie algebra of the Lie group \( G_X(h) \) (with \( i^{-1} \times \)commutator as Lie bracket), and it is a Banach algebra with the norm \( \|\cdot\|_2 \),

\[
\|A\|_2 = \|A_{++}\| + \|A_{--}\| + \|A_{+-}\|_2 + \|A_{-+}\|_2 \tag{28}
\]

(\( \|\cdot\|_2 \) is the Hilbert-Schmidt norm \( \{12\} \)).

For \( A = q_X A^{*}q_X \) in the form \( \{23\} \), eq. \( \{11\} \) allows us to define

\[
Q(A) \equiv \sum_{n=1}^{\infty} \lambda_n \Phi^{*}(f_n)\Phi(g_n) \tag{29}
\]

on \( \mathcal{D}_X^{2}(h) \), and from \( \{8\} \) and \( \{3\} \) we have

\[
\begin{align*}
\{Q(A), \Phi^{*}(f)\} &= \Phi^{*}(Af) \quad \forall f \in h \\
\{Q(A), Q(B)\} &= Q([A, B]) \\
Q(A)^{*} &= Q(A)
\end{align*}
\]

([\cdot, \cdot] is the commutator as usual) for all \( A, B \in B_2(h) \). Thus by defining

\[
\hat{d} \Gamma(A) \equiv Q(A) - \langle \Omega, Q(A)\Omega \rangle = Q(A) + \varepsilon_X tr(T A) \tag{30}
\]

\(^2\)note that \( F \)-self-adjoint=\( \text{self-adjoint} \)
\(< \cdot, \cdot >\) is the scalar product in \(\mathcal{F}_X(h)\) and \(\text{tr}(\cdot)\) the trace in \(h\); the last equality follows from (29), (7), and (5)) we obtain
\[
[d\hat{\Gamma}(A), d\hat{\Gamma}(B)] = d\hat{\Gamma}(\{A, B\}) + S(A, B) \tag{31}
\]
with \(S(A, B) = -\varepsilon_X \text{tr}(T_- [A, B])\) (note that the r.h.s of this is equal to \(Q([A, B])\) for \(A, B \in B_1(h)\), or equivalently (by using the cyclicity of the trace)
\[
S(A, B) = -\varepsilon_X \text{tr}(A_- B_{+-} - B_{-+} A_{+-}) \tag{32}
\]
which obviously is purely imaginary for all \(A, B\) obeying (26). Moreover,
\[
[d\hat{\Gamma}(A), \Phi^+(f)] = \Phi^+(Af) \quad \forall f \in h \tag{33}
\]
and \(d\hat{\Gamma}(A)\) is essentially self-adjoint implying (we denote the self-adjoint extension of \(d\hat{\Gamma}(A)\) by the same symbol)
\[
e^{itd\hat{\Gamma}(A)}\Phi^+(f) = \Phi^+(e^{itA}f)e^{itd\hat{\Gamma}(A)} \quad \forall t \in \mathbb{R}, f \in h. \tag{34}
\]
One can prove the estimate \[11\]
\[
\|d\hat{\Gamma}(A)P_\ell\| \leq 8\ell \|A\|_2 \quad \forall \ell \in \mathbb{N} \tag{35}
\]
showing that the definition of \(d\hat{\Gamma}(A)\) naturally extends from \(A = q_X A^* q_X \in B_1(h)\) to \(A \in g_X(h)\), and all the relations \([11]\)--\([35]\) are valid for all \(A, B \in g_X(h)\). The relations \([11]\) provide the (abstract) boson \((X = B)\) and fermion \((X = F)\) current algebras, and \(S(\cdot, \cdot)\) is the Schwinger term. Due to the anti-symmetry and the Jacobi identity obeyed by the commutator, \(S(\cdot, \cdot)\) obeys 2-cocycle relations and it is therefore a (non-trivial) 2-cocycle of the Lie algebra \(g_X(h)\), and \(d\hat{\Gamma}(\cdot)\) provides a representation of a central extension of \(g_X(h)\).

Note that by construction
\[
< \Omega, d\hat{\Gamma}(A)\Omega > = 0 \quad \forall A \in g_X(h). \tag{36}
\]

**Remark 2.6:** The operators \(d\hat{\Gamma}(A), A \in g_X(h)\), are unbounded in general, and (due to \([35]\) and \(d\hat{\Gamma}(A)P_\ell = P_{\ell+2}d\hat{\Gamma}(A)P_\ell\) for all \(\ell \in \mathbb{N}\) \(\mathcal{D}_X(h)\) \[8\] provides a common, dense, invariant domain of essential self-adjointness for all these operators \([11]\).
3 Results

(a) First Result: For $U \in G_X(h)$ the defining relation (14) determine the implementer $\hat{\Gamma}(U)$ only up to a phase factor $\in U(1) \equiv \{e^{i\varphi}|0 \leq \varphi < 2\pi\}$, and its unique definition requires an additional fixing of this phase ambiguity. A convenient and natural choice for this is $\langle \Omega, \hat{\Gamma}(U)\Omega \rangle >$ real and positiv $\forall U \in G_X^{(0)}(h)$, and in fact this is the convention used in eqs. (22) and (23). From the explicit formulas in [7] one can easily see that then

$$\hat{\Gamma}(U)^* = \hat{\Gamma}(U^{-1}) = \hat{\Gamma}(q_XU^*q_X) \quad \forall U \in G_X^{(0)}(h).$$

(38)

From (14) it follows that for $U, V \in G_X^{(0)}(h)$, the unitary operators $\hat{\Gamma}(U)\hat{\Gamma}(V)$ and $\hat{\Gamma}(UV)$ both implement the same BT $\alpha_{UV}$, hence they must be equal up to a phase,

$$\hat{\Gamma}(U)\hat{\Gamma}(V) = \chi(U, V)\hat{\Gamma}(UV)$$

(39)

with $\chi$ a function $G_X(h) \times G_X(h) \to U(1)$ determined by (39) and the phase convention used for the implementers. From the assoziativity of the operator product we conclude that $\chi$ satisfies the relation

$$\chi(U, V)\chi(UV, W) = \chi(V, W)\chi(U, VW) \quad \forall U, V, W \in G_X(h),$$

(40)

and changing the phase convention for the implementers,

$$\hat{\Gamma}(U) \longrightarrow \beta(U)\hat{\Gamma}(U) \quad \forall U \in G_X^{(0)}(h)$$

(41)

with $\beta : G_X^{(0)}(h) \to U(1)$ some smooth function, amounts to changing

$$\chi(U, V) \longrightarrow \chi(U, V)\delta\beta(U, V)$$

(42)

with

$$\delta\beta(U, V) \equiv \frac{\beta(U)\beta(V)}{\beta(UV)} \quad \forall U, V \in G_X(h)$$

(43)

satisfying (40) trivially. Eq. (40) is a 2-cocycle relation, a function $\chi : G_X(h) \times G_X(h) \to U(1)$ satisfying it is a 2-cocycle, and a 2-cocycle of the form $\delta\beta$ eq. (43) is a 2-coboundary of the group $G_X(h)$ [17].
The first result of this paper is the explicit formula for the 2-cocycle $\chi$ defined by (37) and (39):

$$\chi(U, V) = \left( \frac{\det(1 + (V^* - -)^{-1} (V^* + - (U^*) + -)) \varepsilon^{X/2}}{\det(1 + (U^* - -)^{-1} U^* + - V^* - -)^{-1})} \right)$$

(44)

for all $U, V \in G_X^{(0)}(h)$.

**Remark 3.1:** Obviously, $V^* + - , U^* + - \in B_2(h)$ is sufficient for the existence of the determinants in (44) [16].

**Remark 3.2:** Let $G_X, 0(h)$ be the group of all $X$-unitary operators $U$ on $h$ obeying

$$(U - 1) \in B_1(h).$$

(45)

Then it is easy to see that for $U, V \in G_{X,0}(h) \cap U^{(0)}(h)$,

$$\chi(U, V) = \delta \beta_0(U, V), \quad \beta_0(U) = \left( \frac{\det(T_+ + U^* - -)}{\det(T_+ + U^* + -)} \right)^{\varepsilon X/2},$$

(46)

(cf. eq. (35)), i.e. $\chi$ (44) is a trivial 2-cocycle for for $G_{X,0}(h)$. However, for $G_X^{(0)}(h) \ni U \notin G_{X,0}(h)$, $\beta_0(U)$ (46) does not exist in general showing that $\chi$ (44) is a non-trivial 2-cocycle for the group $G_X(h)$.

**Remark 3.3:** Note that in the fermion case $X = F$, eq. (37) can be used to determine the phase of the implementers $\hat{\Gamma}(A)$ only for $U \in G_F^{(0)}(h)$ (as the l.h.s. of (37) is zero for all other $U \in G_F(h)$ [4]), and that eq. (14) gives the 2-cocycle $\chi$ of the group $G_F(h)$ only locally, i.e. in the neighborhood $G_F^{(0)}(h) \subset G_F(h)$ of the identity.

(b) Second Result: From (14) and (34) it follows that for $A \in g_X(h)$ and $t \in \mathbb{R}$, the unitary operators $e^{it\hat{\Gamma}(A)}$ and $\hat{\Gamma}(e^{itA})$ both implement the same BT $\alpha_{\text{eitA}}$, hence they must be equal up to a phase $\eta(A; t)$,

$$e^{it\hat{\Gamma}(A)} = \eta(A; t)\hat{\Gamma}(e^{itA}) \quad \forall A \in g_X(h), t \in \mathbb{R}.$$

(47)

The second result of this paper is the explicit formula for $\eta(A, t)$ defined by (17), (37) and (29), (30):

$$\eta(A; t) = \exp \left( i \int_0^t dr \varphi(e^{irA}, A) \right) \quad \forall A \in g_X(h)$$

(48)
for all $t \in \mathbb{R}$ in the boson case $X = B$, and for all $t \in \mathbb{R}$ such that $e^{isA} \in G^{(0)}_F(h)$ for all $|s| < |t|$ in the fermion case $X = F$, with $\varphi$ given by

$$\varphi(U, A) \equiv \frac{\varepsilon_X}{2} \text{tr}((U_-)^{-1}U_+A_+ + (A^*)_- + (U^*)_+ - (A^*)_-)$$

$\forall U \in G_X(h), A \in g_X(h)$. \hfill (49)

**Third Result:** The formula (48) can be easily generalized to unitary propagators $u(t, s)$ generated by a family of $t$-dependent $X$-self-adjoint operators $A(t)$: Let

$$A(\cdot) : \mathbb{R} \to g_X(h), t \mapsto A(t)$$

be continuous in the $\| \cdot \|_2$-norm. Then the eq.

$$\frac{\partial}{\partial s} u(s, t) = iA(s)u(s, t)$$

$$u(t, t) = 1 \quad \forall s, t \in \mathbb{R}$$

(51)
can be solved by $X$-unitary operators $u(s, t)$ obeying

$$u(r, s)u(s, t) = u(r, t) \quad \forall r, s, t \in \mathbb{R}$$

(52)
(see e.g. [13], Section X.12). As a third result of this paper we show that

$$u(s, t) \in G_X(h) \quad \forall s, t \in \mathbb{R},$$

(53)
and that the second quantized version of (51)

$$\frac{\partial}{\partial s} U(s, t) = id\hat{\Gamma}(A(s))U(s, t)$$

$$U(t, t) = 1 \quad \forall s, t \in \mathbb{R}$$

(54)
can be solved by unitary operators $U(s, t)$ on $F_X(h)$; moreover

$$U(s, t) = \eta(A(\cdot), s, t)\hat{\Gamma}(u(s, t)) \quad \forall (s, t) \in I^2$$

(55)
with

$$\eta(A(\cdot), s, t) = \exp \left( i \int_t^s dr \varphi(u(s, r), A(r)) \right)$$

(56)
and $\varphi$ eq. (49); the domain of validity for this is $I^2 = \mathbb{R}^2$ in the boson case $X = B$, and $I^2$ the set of all $(s, t) \in \mathbb{R}^2$ such that $u(s, t) \in G^{(0)}_X(h)$ in the fermion case $X = F$. 

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Remark 3.4: Though the second result is a special case of the third one, we prefer to state and prove it independently: the non-trivial part of the later is the existence of the unitary propagator $U(s, t)$ (which is trivial for $t$-independent generators $A$), whereas the proof of eqs. (55)–(56) can be given by a straightforward extension of the one for (17)–(18).

4 Proofs

(a) Proof of the First Result: We first proof the following Lemma:

Lemma:

Let $A_{+-}, B_{+-} \in B_{2}(h)$. Then

$$< e^{A_{+-}a^*a} \Omega, e^{B_{+-}a^*a} \Omega > = \det(1 - \varepsilon \chi(A_{+-})*B_{+-})^{-\varepsilon \chi}.$$  \hfill (57)

Proof of the Lemma: By assumption,

$$T \equiv (A_{+-})*B_{+-} \in B_{1}(h),$$ \hfill (58)

and we assume at first that $\|A_{+-}\|, \|B_{+-}\| < 1$. Similarly as in [7] one can easily show then that

l.h.s. of (57) = $\sum_{n=0}^{\infty} a^{(x)}_{n}$ \hfill (59)

with

$$a^{(x)}_{n} = \frac{1}{n!} \sum_{m_{1},m_{2},\cdots,m_{n}=1}^{\infty} \sum_{\pi \in P_{n}} \text{sign}(\pi)^{\text{deg}(\chi)} \prod_{i=1}^{n} (e_{m_{i}}, T e_{m_{x(i)}})$$ \hfill (60)

$\{e_{m}\}_{m=0}^{\infty}$ a complete, orthonormal basis in $T_{-}h$, $P_{n}$ the set of all permutation of $\{1,2,\cdots,n\}$, and $\text{sign}(\pi) = 1(-1)$ for even (odd) permutations $\pi \in P_{n}$.

Now obviously

$$\prod_{i=1}^{n} (e_{m_{i}}, T e_{m_{x(i)}}) = (e_{\tilde{m}_{1}}, T e_{\tilde{m}_{1}})(e_{\tilde{m}_{2}}, T e_{\tilde{m}_{2}})$$

$$\times \cdots (e_{\tilde{m}_{N_{1}}}, T e_{\tilde{m}_{N_{1}}})(e_{\tilde{m}_{N_{1}+1}}, T e_{\tilde{m}_{N_{1}+1}})$$

$$\times (e_{\tilde{m}_{N_{1}+2}}, T e_{\tilde{m}_{N_{1}+2}})(e_{\tilde{m}_{N_{1}+3}}, T e_{\tilde{m}_{N_{1}+3}})(e_{\tilde{m}_{N_{1}+4}}, T e_{\tilde{m}_{N_{1}+4}})$$

$$\times (e_{\tilde{m}_{N_{1}+2N_{2}+1}}, T e_{\tilde{m}_{N_{1}+2N_{2}+1}})(e_{\tilde{m}_{N_{1}+2N_{2}+2}}, T e_{\tilde{m}_{N_{1}+2N_{2}+2}})$$

$$\times (e_{\tilde{m}_{N_{1}+2N_{2}+3}}, T e_{\tilde{m}_{N_{1}+2N_{2}+3}})(e_{\tilde{m}_{N_{1}+2N_{2}+4}}, T e_{\tilde{m}_{N_{1}+2N_{2}+4}})$$

$$\times \cdots$$ \hfill (61)
with \((\tilde{m}_1, \tilde{m}_2, \cdots, \tilde{m}_n) = (m_{\sigma(1)}, m_{\sigma(2)}, \cdots, m_{\sigma(n)})\) for some \(\sigma \in \mathcal{P}_n\) and non-negative integers \(N_1, N_2, \cdots N_n\) obeying
\[
\sum_{\alpha=1}^{n} \alpha N_\alpha = n
\]
and determined by \(\pi \in \mathcal{P}_n\). Hence we have
\[
\sum_{m_1, m_2, \cdots, m_n=1}^{\infty} \prod_{i=1}^{n} (e_{m_i} T e_{m_{\sigma(i)}}) = \text{tr}(T^{N_1}) \text{tr}(T^{2 N_2}) \cdots \text{tr}(T^n)^{N_n} = \prod_{\alpha=1}^{n} \text{tr}(T^\alpha)^{N_\alpha}.
\]
From (61) we can deduce that
\[
\text{sign}(\pi) = \prod_{\alpha=1}^{n} (-)^{(\alpha-1)N_\alpha},
\]
and we can write
\[
\alpha_n^{(X)} = \frac{1}{n!} \sum_{N_1, N_2, \cdots, N_n}^{(n)} K_n(N_1, N_2, \cdots N_n) \prod_{\alpha=1}^{n} (-)^{\deg(X)(\alpha-1)N_\alpha} \text{tr}(T^\alpha)^{N_\alpha}.
\]
with \(\sum_{N_1, N_2, \cdots, N_n}^{(n)}\) the sum over all non-negative integers \(N_1, N_2, \cdots N_n\) obeying (62), and \(K_n(N_1, N_2, \cdots N_n)\) denoting the number of permutations \(\pi \in \mathcal{P}_n\) leading to a term containing \(\prod_{\alpha=1}^{n} \text{tr}(T^\alpha)^{N_\alpha}\). In the following we determine these numbers by simple combinatorics.

There are \(n\) numbers \((\tilde{m}_1, \tilde{m}_2, \cdots, \tilde{m}_n)\), hence we have \(n\) possibilities to choose \(\tilde{m}_1\), \((n-1)\) possibilities for \(\tilde{m}_2\), \(\cdots\), \((n-N_1+1)\) possibilities to choose \(\tilde{m}_{N_1}\); however \(N_1! = N_1(N_1-1) \cdots 1\) of these \(n(n-1) \cdots (n-N_1+1)\) choices are equal as permutations of \((\tilde{m}_1, \tilde{m}_2, \cdots, \tilde{m}_{N_1})\) have to be identified. Hence we have
\[
\frac{n!}{(n-N_1)! N_1!}
\]
different choices for \((\tilde{m}_1, \tilde{m}_2, \cdots, \tilde{m}_{N_1})\) for producing a term containing \(\text{tr}(T)^{N_1}\).

In order to produce a term containing in addition \(\text{tr}(T^2)^{N_2}\), we have \((n-N_1)\) possibilities left to choose \(\tilde{m}_{N_1+1}\), \(\cdots\), \((n-N_1-2N_2+1)\) possibilities to choose \(\tilde{m}_{N_1+2N_2}\). Again we have to identify the choices obtained by permuting \((\tilde{m}_{N_1+1}, \cdots, \tilde{m}_{N_1+2N_2})\); moreover (due to the cyclicity of the trace), we
must identify the choices which differ only by changing $m_{N_1+1}$ and $m_{N_1+2}$, \ldots, or $m_{N_1+2N_2-1}$ and $m_{N_1+2N_2}$; all together we have

$$\frac{n!}{(n - N_1)! N_1! (n - N_1 - 2N_2)! 2^{N_2}}$$

different choices for $(\tilde{m}_1, \tilde{m}_2, \ldots, \tilde{m}_{N_1+2N_2})$ leading to a term containing $\text{tr}(T)^{N_1} \text{tr}(T^2)^{N_2}$. Continuing these considerations, we arrive at the conclusion that there are

$$\frac{n!}{(n - N_1)! N_1! (n - N_1 - 2N_2)! 2^{N_2} (n - N_1 - 2N_2 - 3N_3)! 3^{N_3} \ldots} = n! \prod_{\alpha=1}^{N_1} \frac{1}{N_\alpha!} = K_n(N_1, N_2, \ldots, N_n)$$

(64)

different permutations $\pi \in \mathcal{P}_n$ leading to a $\prod_{\alpha=1}^{N_1} \text{tr}(T^\alpha)^{N_\alpha}$-term. (As a simple check, one can easily convince oneself that

$$\sum_{N_1, N_2, \ldots} K_n(N_1, N_2, \ldots, N_n) = n!.$$  

With that we obtain

$$a_{n}^{(X)} = \sum_{N_1, N_2, \ldots} \prod_{\alpha=1}^{N_1} \varepsilon_X^{\alpha-1} N_\alpha \frac{\text{tr}(T^\alpha)^{N_\alpha}}{N_\alpha! \alpha^{N_\alpha}}$$

leading to

$$\sum_{n=0}^\infty a_{n}^{(X)} = \sum_{N_1, N_2, \ldots=0}^\infty \prod_{\alpha=1}^{N_1} \frac{1}{N_\alpha!} \left( \varepsilon_X^{-1} \frac{\text{tr}(T^\alpha)}{\alpha} \right)^{N_\alpha} = \prod_{\alpha=1}^\infty \exp \left( \varepsilon_X^{-1} \frac{\text{tr}(T^\alpha)}{\alpha} \right)$$

$$= \exp \left( -\varepsilon_X \text{tr}(\log(1 - \varepsilon_X T)) \right) = \text{det}(1 - \varepsilon_X T)^{-\varepsilon_X}$$

where we freely interchanged infinite sums and products (which is allowed as everything converges absolutely for all $T \in B_1(h), \|T\| < 1$).

The validity of (57) for general $T$ follows from the observation that due to the considerations above,

$$< e^{\varepsilon_1 A - a^* a^* \Omega} e^{\varepsilon_2 B a^* a^* \Omega} > = \text{det}(1 - \varepsilon_X (\tilde{z}_1 A_{+ -})^* z_2 B_{+ -})^{-\varepsilon_X}, \quad z \in \mathbb{C}$$

is valid for $|z_1| < 1/\|A_{+ -}\|, |z_2| < 1/\|B_{+ -}\|$, and both sides of this eq. are analytic have a unique analytic extension to $z_1 = z_2 = 1$.  

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Remark 4.1: In the fermion case $X = F$, eq. (57) was given by Ruijsenaars [18].

Proof of (44): With (39) we have (cf. (38))
\[
< \hat{\Gamma}(U^{-1})\Omega, \hat{\Gamma}(V)\Omega > = \chi(U, V) < \Omega, \hat{\Gamma}(UV)\Omega >
\]
and with (22)
\[
\chi(U, V) = \frac{N(U^{-1})N(V)}{N(UV)} E(U, V)
\]
where
\[
E(U, V) = < e^{Z(U^{-1})}_{+} a^* a \Omega, e^{Z(V)}_{+} a^* a \Omega >= \det(1 - \varepsilon_X (Z(U^{-1})_{+})*Z(V)_{+})^{-\varepsilon_X}
\]
(we used the Lemma above). Now obviously for all $U \in \mathbf{G}^{(0)}_X(h)$ (cf. (19))
\[
(Z(U^{-1})_{+})^* = \varepsilon_X Z(U)_{-}
\]
hence with (19)
\[
1 - \varepsilon_X (Z(U^{-1})_{+})*Z(V)_{+} = 1 - (U_{-})^{-1}U_{-}V_{-}V_{-}^{-1} = T_+ + (U_{-})^{-1}(UV)_{-}V_{-}^{-1},
\]
and we can write
\[
E(U, V) = \det'((U_{-})^{-1}(UV)_{-}V_{-}^{-1})^{-\varepsilon_X}
\]
with
\[
\det'( \cdots ) \equiv \det(T_+ + \cdots)
\]
the determinant on $T_- h$. Similarly one obtains from (23)
\[
N(U) = \det'((U_{-})^{-1}(U_{-})^{-1})^{\varepsilon_X/2}
\]
(which can also be deduced from $N(U)^{-2} = E(U^{-1}, U)$ and $(U^{-1})_{-} = U_{-}^*$),
and by simple properties of determinants this results in
\[
\chi(U, V) = \left( \frac{\det'(V_{-})^{-1}(V^* U^*)_{-}U_{-}^*}{\det'(U_{-})^{-1}(UV)_{-}V_{-}^{-1}} \right)^{\varepsilon_X/2}
\]
. (65)
Using
\((UV)_{--} = U_{--}V_{--} + U_{--}V_{+-}\)
and its adjoint we obtain (44).

**Remark 4.2:** Note that we can write (65) also as
\[
\chi(U, V) = \left(\frac{\det'(V_{--} - (V^*U^*)_{--}U_{--})}{\det'(U_{--} - (UV)_{--}V^*)}\right)^{\varepsilon x/2}.
\] (66)

(b) **Proof of the Second Result:** Let \(A \in g_X(h)\) and \(s, t, \varepsilon \in \mathbb{R}\). We write for simplicity
\[
\chi(e^{isA}, e^{itA}) \equiv \chi'(s, t).
\] (67)

Due to the cocycle relation (39) we have
\[
\chi'(s, t) = \chi'(t, \varepsilon) \chi'(s + t, \varepsilon),
\]
and by iteration
\[
\chi'(s, t) = \left(\prod_{\nu=0}^{N-1} \frac{\chi'(t + \nu\varepsilon, \varepsilon)}{\chi'(s + t + \nu\varepsilon, \varepsilon)}\right) \chi'(s, t + N\varepsilon)
\]
valid for arbitrary \(N \in \mathbb{N}\). Choosing \(t + N\varepsilon = 0\) and noting that \(\chi'(s, 0) = 1\), we obtain the formula
\[
\chi'(s, t) = \left(\prod_{\nu=0}^{N-1} \frac{\chi'(t - \nu t/N, -t/N)}{\chi'(s + t - \nu t/N, -t/N)}\right) \forall N \in \mathbb{N}.
\] (68)

For \(s, t\) such that \(e^{isA}, e^{itA}, e^{i(s+t)A} \in G_X^{(0)}(h)\) we have
\[
\log (\chi'(s, -t/N)) = -c(s)t/N + O((t/N)^2), \quad c(s) = \frac{d}{dt} \log (\chi'(s, t)) \big|_{t=0}
\] (note that \(\chi'(s, t)\) is continuously differentiable), hence
\[
\log (\chi'(s, t)) = \sum_{\nu=0}^{N-1} (c(t + s - \nu t/N) - c(t - \nu t/N)) t/N + O(t/N)
\]
\[
= \int_0^t dr (c(s + r) - c(r)) + O(t/N)
\]
\[
= \log (\eta'(t + s)) - \log (\eta'(s)) - \log (\eta'(t)) + O(t/N)
\]
with
\[
\log (\eta'(s)) = \int_0^s drc(r).
\] (70)

In the limit \( N \to \infty \) we obtain
\[
\chi'(s, t) = \left( \frac{\eta'(s)\eta'(t)}{\eta'(s + t)} \right)^{-1},
\] (71)
equivalent to
\[
\hat{\Gamma}'(t) \equiv \eta'(t)\hat{\Gamma}(e^{itA})
\]
obeying
\[
\hat{\Gamma}'(s)\hat{\Gamma}'(t) = \hat{\Gamma}'(s + t).
\]

Moreover, a simple calculation yields
\[
\frac{d}{dt} \omega < \Omega, \hat{\Gamma}'(t)\Omega > \big|_{t=0} = 0
\]
(cf. (69)). From the uniqueness of an implementer of a Bogoliubov transformation with a fixed phase we can conclude thus that
\[
\hat{\Gamma}'(t) = e^{itd\hat{\Gamma}(A)}
\]
yielding (67) with
\[
\eta(A; t) = \eta'(t).
\] (72)

Now
\[
c(s) = i\varphi(e^{isA}, A)
\] (73)
with
\[
\varphi(U, A) \equiv \frac{d}{dt} \log (\chi(U, e^{itA})) \big|_{t=0},
\] (74)
and with (66) we obtain (69).

(c) **Proof of the Third Result:** Let \( s, t \in \mathbb{R} \). As \( A(\cdot) \) (50) is continuous in the \( \| \cdot \|_2 \)-norm, we have from the principle of uniform boundedness \( [3] \) that
\[
\| A(r) \|_2 < \alpha(s, t) \quad \forall r \in [s, t]
\] (75)
([s, t] \subset \mathbb{R} denotes the closed interval inbetween \( s \) and \( t \)) for some finite \( \alpha(s, t) \), and obviously \( u(r, t) \) (51) is uniformly bounded in the operator norm for all \( r \in [s, t] \) as well. From (51) we obtain
\[
u(s, t) = 1 + \int_t^s driA(r)u(r, t),
\]
hence
\[ \|u(s, t)\|_2 \leq 1 + |s - t| \alpha(s, t) \sup_{r \in [s, t]} \|u(r, t)\| \]
proving (53).

From (35) and (29), (30) we have
\[ \|d\hat{\Gamma}(A(r))P_\ell\| \leq 8\ell \alpha(s, t) \]
\[ d\hat{\Gamma}(A(r))P_\ell = P_{\ell+2}d\hat{\Gamma}(A(r))P_\ell \quad \forall \ell \in \mathbb{N}, r \in [s, t]. \quad \text{(76)} \]

Eq. (54) is formally equivalent to
\[ U(s, t) = 1 + \int_t^s dr \, id\hat{\Gamma}(A(r))U(r, t), \]
suggesting to define \( U(s, t) \) as power series on \( D^f(h) \)
\[ U(s, t) = 1 + \sum_{n=1}^{\infty} R_n(s, t) \quad \text{(77)} \]
with
\[ R_n(s, t) = \int_t^s dr_1 id\hat{\Gamma}(A(r_1)) \int_t^{r_1} dr_2 id\hat{\Gamma}(A(r_2)) \cdots \int_t^{r_{n-1}} dr_{n-1} id\hat{\Gamma}(A(r_{n-1})). \quad \text{(78)} \]

Indeed, with (76) we can estimate
\[ \|R_n(s, t)P_\ell\| \leq |\int_t^s dr_1 \int_t^{r_1} dr_2 \cdots \int_t^{r_{n-1}} dr_{n-1} |d\hat{\Gamma}(A(r_1))P_{\ell+2n-2}\|d\hat{\Gamma}(A(r_2))P_{\ell+2n-4}\|d\hat{\Gamma}(A(r_{n-1}))P_\ell| \]
\[ \leq (8\alpha(s, t))^n \ell(\ell + 2) \cdots (\ell + 2n - 2) \frac{|s - t|^n}{n!}, \]
and with the ratio test for power series we conclude that (77), (78) is well-defined on \( D^f(h) \) for \( 16\alpha(s, t)|s - t| < 1 \). Then for sufficiently small \( |s - t|, |r - t|, |r - s| \), (74) and (78) imply that
\[ U(r, s)U(s, t) = U(r, t) \quad \text{(79)} \]
and
\[ U(s, t)^* = U(t, s) \quad \text{(80)} \]
we recall that all $d\hat{\Gamma}(A(r))$, $r \in \mathbb{R}$, are essentially self-adjoint on $\mathcal{D}(\hat{h})$ showing that $U(s,t)$ can be uniquely extended to a unitary operator on $\mathcal{F}_X(h)$ such that (79) remains true. Then we can use (79) to extend the definition of $U(s,t)$ to all $s,t \in \mathbb{R}$, and it is easy to see that these satisfy (54).

Having established the existence of $U(s,t)$ (54), the validity of (55) and (56) can be proved similarly as (47)–(49): Defining

\[
\chi'_s(r,t) \equiv \chi(u(r,s), u(s,t))
\] (81)

we deduce from the cocycle relation (44) that

\[
\chi'_s(r,t) = \frac{\chi'_t(s,t + \epsilon)}{\chi'_t(r,t + \epsilon)} \chi'_s(r,t + \epsilon)
\]

\[
= \left( \prod_{\nu=1}^{N} \frac{\chi'_{t+(\nu-1)\epsilon}(s,t + \nu\epsilon)}{\chi'_{t+(\nu-1)\epsilon}(r,t + \nu\epsilon)} \right) \chi'_s(r,t + N\epsilon),
\]

and with

\[
t + N\epsilon = s, \quad \chi'_s(r,s) = 1
\]

we obtain similarly as above

\[
\chi'_s(r,t) = \left( \frac{\eta'_s(t)\eta'_r(s)}{\eta'_r(t)} \right)^{-1}
\] (82)

with

\[
\log (\eta'_s(t)) = -\int_t^s dqc(s,q),
\] (83)

\[
c(s,q) = \frac{\partial}{\partial r} \log \left( \chi(u(s,q), u(q,r)) \right)_{r=q} = -i\varphi(u(s,q), A(q))
\] (84)

and $\varphi$ (49), proving (55), (56) by the same argument as above (note that

\[
\frac{d}{dis} < \Omega, \eta'_s(t)\hat{\Gamma}(u(s,t))\Omega > |_{s=t} = 0
\]

(cf. (64))).
5 Final Remarks

Remark 5.1: It is easy to see that the current algebra \((31)\) is in fact the Lie algebra version of \((39)\). Indeed, it follows from \((47)\) and \(d\eta(A, t)/dt|_{t=0} = 0\) that
\[
\frac{d}{dt} \hat{\Gamma}(e^{iA})|_{t=0} = \forall A \in g_X(h),
\]

hence
\[
[d\hat{\Gamma}(A), d\hat{\Gamma}(B)] = \frac{d}{ds dt} \hat{\Gamma}(e^{isA})\hat{\Gamma}(e^{itB})\hat{\Gamma}(e^{-isA})\hat{\Gamma}(e^{-itB})|_{s=t=0} = d\hat{\Gamma}([A, B]) + S(A, B) \forall A, B \in g_X(h)
\]

with
\[
S(A, B) = \frac{d}{ds dt} \chi(e^{isA}, e^{itB})\chi(e^{-isA}, e^{-itB})\chi(e^{isA}e^{itB}, e^{-isA}e^{-itB})|_{s=t=0};
\]

with \(\chi[14]\) this can be shown to coincide with \((32)\).

Remark 5.2: For simplicity, we restricted our discussion in this paper to currents \(d\hat{\Gamma}(A)\) with bounded operators \(A\) on \(h\). However, it is straightforward to extend all our results from the Lie algebra \(g_X(h)\) to certain Lie algebras \(g_{T_X,2}(h; H)\) of unbounded operators \(A\) on \(h\) with a common, dense, invariant domain of definition which are naturally associated with some given self-adjoint operator \(H\) on \(h\) \([11]\), and with the results of \([11]\) it is straightforward to show that \((47), (48)\) and \((55), (56)\) hold for all \(A \in g_{T_X,2}(h; H)\) and mappings
\[
A(\cdot) : \mathbb{R} \rightarrow g_{T_X,2}(h; H), t \mapsto A(t)
\]
continuous in the natural topology of \(g_{T_X,2}(h; H)\) \([11]\), respectively. Hence in general one can use these formulas for unbounded operators as well.

Remark 5.3: As an application of \((55)\), we mention that it can be used to construct the time evolution \(U(s, t)\) on the second quantized level if the time evolution \(u(s, t)\) on the 1-particle level is given and \(u(s, t) \in G_X(0)\).
For example, the later condition is fulfilled for bosons or fermions in external Yang-Mills fields in \((1 + 1)\)- (but not in higher) dimensions (in the fermion case, \((55)\) is valid only as long as \(u(s, t)\) does not produce level crossing).
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