NON-UNIQUENESS OF CONVEX BODIES WITH PRESCRIBED VOLUMES OF SECTIONS AND PROJECTIONS

FEDOR NAZAROV, DMITRY RYABOGIN AND ARTEM ZVAVITCH

AbSTRACT. We show that if \( d \geq 4 \) is even, then one can find two essentially different convex bodies such that the volumes of their maximal sections, central sections, and projections coincide for all directions.

1. Introduction

As usual, a convex body \( K \subset \mathbb{R}^d \) is a compact convex subset of \( \mathbb{R}^d \) with non-empty interior. We assume that \( 0 \in K \). We consider the central section function \( A_K \):

\[
A_K(u) = \operatorname{vol}_{d-1}(K \cap u^\perp), \quad u \in S^{d-1},
\]

the maximal section function \( M_K \):

\[
M_K(u) = \max_{t \in \mathbb{R}} \operatorname{vol}_{d-1}(K \cap (u^\perp + tu)), \quad u \in S^{d-1},
\]

and the projection function \( P_K \):

\[
P_K(u) = \operatorname{vol}_{d-1}(K|u^\perp), \quad u \in S^{d-1}.
\]

Here \( u^\perp \) stands for the hyperplane passing through the origin and orthogonal to the unit vector \( u \), \( K \cap (u^\perp + tu) \) is the section of \( K \) by the affine hyperplane \( u^\perp + tu \), and \( K|u^\perp \) is the projection of \( K \) to \( u^\perp \). Observe that \( A_K \leq M_K \leq P_K \). It is well known, \([Ga]\), that for origin-symmetric bodies each of the functions \( M_K = A_K \) and \( P_K \) determines the convex body \( K \subset \mathbb{R}^d \) uniquely. More precisely, either of the conditions

\[
M_{K_1}(u) = M_{K_2}(u) \quad \forall u \in S^{d-1},
\]

and

\[
P_{K_1}(u) = P_{K_2}(u) \quad \forall u \in S^{d-1},
\]

implies \( K_1 = K_2 \), provided \( K_1, K_2 \) are origin-symmetric and convex.

In this paper, we address the (im)possibility of analogous results for not necessarily symmetric convex bodies.

It is well known, \([BF]\), that on the plane there are convex bodies \( K \) that are not Euclidean discs, but nevertheless satisfy \( M_K(u) = P_K(u) = 1 \) for all \( u \in S^1 \). These are the bodies of constant width 1.

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In 1929 T. Bonnesen asked whether every convex body $K \subset \mathbb{R}^3$ is uniquely defined by $P_K$ and $M_K$, (see [BF], page 51). We note that in any dimension $d \geq 3$, it is not even known whether the conditions $M_K \equiv c_1$, $P_K \equiv c_2$ are incompatible for $c_1 < c_2$.

In 1969 V. Klee asked whether the condition $M_{K_1} \equiv M_{K_2}$ implies $K_1 = K_2$, or, at least, whether the condition $M_K \equiv c$ implies that $K$ is a Euclidean ball, see [Kl1].

Recently, R. Gardner and V. Yaskin, together with the second and the third named authors constructed two bodies of revolution $K_1$, $K_2$ such that $K_1$ is origin-symmetric, $K_2$ is not origin-symmetric, but $M_{K_1} \equiv M_{K_2}$, thus answering the first version of Klee’s question but not the second one (see [GRYZ]).

The main results we will present in this paper are the following.

**Theorem 1.** If $d = 4$, there exists a convex body of revolution $K \subset \mathbb{R}^d$ satisfying $M_K \equiv \text{const}$ that is not a Euclidean ball.

**Theorem 2.** If $d \geq 4$ is even, there exist two essentially different convex bodies of revolution $K_1$, $K_2 \subset \mathbb{R}^d$ such that $A_{K_1} \equiv A_{K_2}$, $M_{K_1} \equiv M_{K_2}$, and $P_{K_1} \equiv P_{K_2}$.

Theorem 1 answers the question of Klee in $\mathbb{R}^4$, and Theorem 2 answers the analogue of the question of Bonnesen in even dimensions.

**Remark 1.** Theorem 1 is actually true in all dimensions, but the construction for $d \neq 4$ is long and rather technical, so we will present it in a separate paper.

We borrowed the general idea of the construction of the bodies $K_1$ and $K_2$ in Theorem 2 from [RY], which attributes it to [GV] and [GSW]. It can be easily understood from the following illustration.

![Figure 1. Two small-eared round faces in a cap](image)

Here the ”ears” and the ”cap” will be made very small in order not to destroy the convexity of the bodies.

The paper is organized as follows. In Section 2 we reduce the problem to finding a non-trivial solution to two integral equations. In Section 3 we prove Theorem 1. In Section 4 we prove Theorem 2.
2. REDUCTION TO A SYSTEM OF INTEGRAL EQUATIONS

From now on, we assume that $d \geq 3$. We will be dealing with the bodies of revolution

$$K_f = \{ x \in \mathbb{R}^d : x_2^2 + x_3^2 + \ldots + x_d^2 \leq f^2(x_1) \},$$

obtained by the rotation of a smooth concave function $f$ supported on $[-1, 1]$ about the $x_1$-axis.

Note that $K$ is rotation invariant, thus every its hyperplane section is equivalent to a section by a hyperplane with normal vector in the second quadrant of the $(x_1, x_2)$-plane.

Lemma 1. Let $L(\xi) = L(s, h, \xi) = s\xi + h$ be a linear function with slope $s$, and let $H(L) = \{ x \in \mathbb{R}^d : x_2 = L(x_1) \}$ be the corresponding hyperplane. Then the section $K \cap H(L)$ is of maximal volume iff

$$\int_{-x}^{y} (f^2 - L^2)^{(d-4)/2} L = 0,$$

where $-x$ and $y$ are the first coordinates of the points at which $L$ intersects the graphs of $-f$ and $f$ respectively (see Figure 2).

**Figure 2.** View of $K$ and $H(L)$ in $(x_1, x_2)$-plane.

*Proof.* Fix $s > 0$. Observe that the slice $K \cap H(L) \cap H_\xi$ of $K \cap H(L)$ by the hyperplane $H_\xi = \{ x \in \mathbb{R}^d : x_1 = \xi \}$, $-1 < \xi < 1$, is the $(d - 2)$-dimensional Euclidean ball
\{(x_3, x_4, \ldots, x_d) : x_3^2 + \ldots + x_d^2 \leq r^2\} \text{ of radius } r = \sqrt{f^2(\xi) - L^2(\xi)}. \text{ Hence,}

(5) \quad \text{vol}_{d-1}(K \cap H(L)) = v_{d-2}\sqrt{1 + s^2} \int_{-y(s)}^{-x(s)} (f^2(\xi) - L^2(\xi))^{(d-2)/2} d\xi,

where \(v_{d-2}\) is the volume of the unit ball in \(\mathbb{R}^{d-2}\).

The section \(K \cap H(L)\) is of maximal volume iff

\[ \frac{d}{dh}\text{vol}_{d-1}(K \cap H(L)) = 0, \]

where in the only if part we use the Theorem of Brunn, [Ga]. Computing the derivative, we conclude that for a given \(s \in \mathbb{R}\), the section \(K \cap H(L)\) is of maximal volume iff (4) holds. \(\square\)

Lemma 2. Let \(L(s, \xi) = s\xi + h(s)\) be a family of linear functions parameterized by the slope \(s\). For each \(L\) in our family, define the hyperplane \(H(L)\) by \(H(L) = \{x \in \mathbb{R}^d : x_2 = L(x_1)\}\), (see Figure 2). The corresponding family of sections is of constant \(d - 1\)-dimensional volume iff

(6) \quad \int_{-x}^{y} (f^2 - L^2)^{(d-2)/2} = \text{const} \frac{\sqrt{1 + s^2}}{1 + s^2}, \quad \text{for all} \quad s > 0.

Proof. The right hand side in (5) is constant iff (6) holds. \(\square\)

3. The case \(d = 4\)

Observe that when \(d = 4\), the system of equations (4), (6) simplifies to

\[ \int_{-x}^{y} L = 0, \quad \text{and} \quad \int_{-x}^{y} (f^2 - L^2) = \text{const} \frac{\sqrt{1 + s^2}}{1 + s^2}, \quad \text{for all} \quad s > 0. \]

In this case we will show that the maximal sections correspond to level intervals, see Proposition 1 below. We will also prove that the values of the maximal section function \(M_K\) depend on the distribution function \(t \to |\{f > t\}|\) only. More precisely, we have

Theorem 3. Let \(d = 4\), \(K = \{x \in \mathbb{R}^4 : x_3^2 + x_4^2 + x_1^2 \leq f^2(x_1)\}\), and let

\[ u = u(s) = \left(-\frac{s}{\sqrt{1 + s^2}}, \frac{1}{\sqrt{1 + s^2}}, 0, 0\right) \in \mathbb{S}^3, \quad s > 0. \]

Then,

(7) \quad M_{K_f}(u) = \pi \sqrt{1 + s^2} \left(\frac{2}{3} t^2 |\{f > t\}| + \int_{t}^{\infty} 2\tau |\{f > \tau\}| d\tau\right),

where \(t\) is the unique solution of the equation \(s = 2t/|\{f > t\}|\).

In particular, if \(f_1\) and \(f_2\) are equimeasurable (i.e., for every \(\tau > 0\), we have \(|\{f_1 > \tau\}| = |\{f_2 > \tau\}|\)), then \(M_{K_{f_1}} \equiv M_{K_{f_2}}\).
Theorem 3 is a simple consequence of the following two propositions.

**Proposition 1.** Let \( f, s, u(s) \) be as in Theorem 3. Then the section of maximal volume in the direction \( u(s) \) is the one that corresponds to the line joining \((-x, -t)\) and \((y, t)\), where \( t \) is such that \( s = 2t/\{|f > t|\} \), \( 0 < t < \max_{\xi \in [-1,1]} f(\xi) \), (see Figure 3).

**Proof.** Fix \( s > 0 \). Since the distribution function is decreasing to 0, there exists a unique \( t \) satisfying \( s = 2t/\{|f > t|\} \). To prove that

\[
\int_{-x}^{y} L(\xi) d\xi = 0
\]

observe that two shaded triangles on Figure 3 are congruent. \( \square \)

**Proposition 2.** Let \( K, f, t, x, y \) be as in the previous proposition, and let the line \( L \) be passing through the points \((-x, -t), (y, t)\). Then (7) holds.

**Proof.** Note that

\[
\int_{-x}^{y} L^2 = (x + y)\frac{t^2}{3} = \frac{t^2}{3} \{|f > t|\},
\]

and

\[
\int_{-x}^{y} f^2 = \int_{\{f > t\}} f^2 = t^2 \{|f > t|\} + \int_{t}^{\infty} 2\tau \{|f > \tau|\} d\tau.
\]

\( \square \)
Proof of Theorem 1. Let $f_o(\xi) = \sqrt{1 - \xi^2}$, $\xi \in [-1, 1]$. Take a concave function $f$ on $[-1, 1]$ such that $f \neq f_o$ and $f$ is equimeasurable with $f_o$.

4. Proof of Theorem 2

Let $\varphi$ and $\psi$ be two smooth functions supported on the intervals $D = [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ and $E = [1 - \delta, 1]$ respectively, where $0 < \delta < \frac{1}{8}$. Define

$$f_+(\xi) = f_o(\xi) + \varepsilon \varphi(\xi) - \varepsilon \varphi(-\xi) + \varepsilon \psi(\xi),$$

and

$$f_-(\xi) = f_o(\xi) - \varepsilon \varphi(\xi) + \varepsilon \varphi(-\xi) + \varepsilon \psi(\xi),$$

where $\varepsilon > 0$ is so small that $f_\pm$ are concave on $[-1, 1]$ (see Figure 4).

![Figure 4. Graph of functions $f_\pm$](image)

Define $K_1 = K_{f_+}$ and $K_2 = K_{f_-}$.

Observe that

(8) $f_+(\xi) = f_-(\xi)$ \(\forall \xi \in [-1, 1] \setminus (D \cup (-D))\), and $f_+(\xi) = f_-(\xi)$ \(\forall \xi \in D \cup (-D)\).

We can choose $\varepsilon$ so small that $K_1$ and $K_2$ are very close to the Euclidean ball, and the sections of the maximal volume of $K_1$ and $K_2$ are very close to the central sections of the ball.

In particular, the intersection points $x = x(s)$ and $y = y(s)$ satisfy

(9) $x(s), y(s) > \frac{5}{8}$ if $s \leq \frac{\sqrt{7}}{3}$, and $x(s), y(s) < \frac{7}{8}$ if $s \geq \frac{\sqrt{7}}{3}$,

for both bodies.

First, we show that $P_{K_1} \equiv P_{K_2}$ and $A_{K_1} \equiv A_{K_2}$.

Observe that we have $h_{K_1}(u) = h_{K_2}(u)$ and $\rho_{K_1}(u) = \rho_{K_2}(u)$ for all directions $u = (\xi, \sqrt{1 - \xi^2}, 0, 0, ..., 0) \in S^{d-1}$, $\xi \in [0, 1] \setminus D$. Observe also that $h_{K_1}(u) = h_{K_2}(-u)$
and \( \rho_{K_1}(u) = \rho_{K_2}(-u) \) for all directions \( u = (\xi, \sqrt{1 - \xi^2}, 0, 0, ..., 0) \in S^{d-1}, \xi \in D \). Hence, the non-ordered pairs

\[
\{h_{K_1}(u), h_{K_1}(-u)\}, \quad \text{and} \quad \{h_{K_2}(u), h_{K_2}(-u)\}
\]

coincide for all \( u \in S^{d-1} \), and so do the pairs

\[
\{\rho_{K_1}(u), \rho_{K_1}(-u)\}, \quad \text{and} \quad \{\rho_{K_2}(u), \rho_{K_2}(-u)\}.
\]

By the result of Goodey, Schneider and Weil, \[GSW\], we have \( P_{K_1} \equiv P_{K_2} \). Also,

\[
A_{K_1}(\theta) = \frac{1}{d-1} \int_{S^{d-1} \cap \theta^\perp} \frac{\rho_{K_1}^{d-1}(-u) + \rho_{K_1}^{d-1}(u)}{2} d\sigma(u) \]

\[
= \frac{1}{d-1} \int_{S^{d-1} \cap \theta^\perp} \frac{\rho_{K_2}^{d-1}(-u) + \rho_{K_2}^{d-1}(u)}{2} d\sigma(u) = A_{K_2}(\theta)
\]

for all \( \theta \in S^{d-1} \).

It remains to show that

\[
(10) \quad M_{K_1} \equiv M_{K_2}.
\]

Assume first that \( d = 4 \). The functions \( f_+ \) and \( f_- \) are equimeasurable, so \[10\] follows from Theorem 3.

Let now \( d \geq 6 \) be even. Note that in this case, \( p = \frac{d-2}{2} \in \mathbb{N} \).

We claim that we can choose \( \varphi \) such that \[10\] holds. This result will be a consequence of the following two propositions.

**Proposition 3.** If \( \varepsilon \) is small enough, then for

\[
u = u(s) = (-\frac{s}{\sqrt{1 + s^2}}, \frac{1}{\sqrt{1 + s^2}}, 0, 0, ..., 0) \in S^{d-1}\n\]

we have

\[
(11) \quad M_{K_1}(u) = M_{K_2}(u),
\]

provided \( s \geq \frac{\sqrt{7}}{3} \).

**Proof.** By \[3\], if the sections \( K_1 \cap H(L_1), K_2 \cap H(L_2) \) are the sections of the maximal volume, corresponding to the same slope \( s \geq \frac{\sqrt{7}}{3} \), then, they are the sections of two symmetric (to each other) bodies corresponding to \( \psi = 0 \). Hence, \[11\] holds by symmetry. \( \square \)

To formulate the second proposition we will need the following result, which is a consequence of the Borsuk-Ulam Theorem.

**Lemma 3.** There exists \( \varphi \neq 0 \) such that

\[
(12) \quad \int_{-\frac{5}{8}}^{\frac{5}{8}} f_+^{2j}(\xi)\xi^l d\xi = \int_{-\frac{5}{8}}^{\frac{5}{8}} f_-^{2j}(\xi)\xi^l d\xi
\]

for all \( j = 0, ..., p \) and \( l = 0, ..., 2(p - j) \).
Proof. We will choose \( \varphi \) using the Borsuk-Ulam Theorem.

For \( j \) and \( l \) as above, consider the vector \( a = a(\varphi) \) with coordinates

\[
a_{j,l}(\varphi) = \frac{2}{5} \int_{-\frac{5}{8}}^{\frac{2}{5}} f^2_j(\xi)\xi^l d\xi - \frac{2}{5} \int_{-\frac{5}{8}}^{\frac{2}{5}} f^2_j(\xi)\xi^l d\xi.
\]

We will view this vector as an element of \( \mathbb{R}^{n(p)} \) with appropriately chosen \( n(p) \).

For \( x = (x_0, x_1, ..., x_n(p)) \in S^{n(p)} \) define \( \varphi_x = \sum_{j=0}^{n(p)} x_j \varphi_j \), where \( \varphi_j \) are smooth not identically zero functions with pairwise disjoint supports contained in \( D \), and let

\[
B : (x_0, x_1, ..., x_n(p)) \rightarrow \varphi_x \rightarrow a(\varphi_x)
\]

be our map from \( S^{n(p)} \) to \( \mathbb{R}^{n(p)} \). By the definition of \( f_\pm \) the map \( B \) is odd. Hence, by the Borsuk-Ulam Theorem, one can choose \( x \) (and hence \( \varphi_x \)) in such a way that \( a(\varphi_x) = 0 \).

Proposition 4. Let \( L = L(s, h, \xi) \) be as above and let \( 0 \leq s \leq \frac{\sqrt{7}}{3} \). Then,

\[
\int_{-x}^{y} (f^2_+ - L^2)^p = \int_{-x}^{y} (f^2_- - L^2)^p.
\]

Moreover,

\[
\int_{-x}^{y} (f^2_+ - L^2)^{p-1} L = 0 \quad \text{if and only if} \quad \int_{-x}^{y} (f^2_- - L^2)^{p-1} L = 0.
\]

In particular, (11) holds for \( 0 \leq s \leq \frac{\sqrt{7}}{3} \).

Proof. We start with the proof of (13). We open parentheses and observe that all we need to prove is

\[
\int_{-x(s)}^{y(s)} f^{2j}_+(\xi)\xi^l d\xi = \int_{-x(s)}^{y(s)} f^{2j}_-(\xi)\xi^l d\xi
\]

for all \( 0 \leq s \leq \frac{\sqrt{7}}{3} \), and for all \( j = 0, ..., p \) and \( l = 0, ..., 2(p-j) \). By (9), this follows from (12) since \( f_+(\xi) = f_-(\xi) \) for \( |\xi| \geq \frac{5}{8} \).

Similarly, (12) implies (14) for \( j = 0, ..., p-1 \) and \( l = 0, ..., 2(p-1-j) + 1 \).

Thus, (10) follows from Propositions 3, 4. This finishes the proof of the Theorem.

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Department of Mathematics, Kent State University, Kent, OH 44242, USA
E-mail address: nazarov@math.kent.edu
E-mail address: ryabogin@math.kent.edu
E-mail address: zvavitch@math.kent.edu