Some results on the subadditivity condition of syzygies

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Abstract
Let $S = K[x_1, \ldots, x_n]$, where $K$ is a field, and $t_i$ denotes the maximal shift in the minimal graded free $S$-resolution of the graded algebra $S/I$ at degree $i$. In this paper, we prove:

- If $I$ is a monomial ideal of $S$ and $a \geq b - 1 \geq 0$ are integers such that $a + b \leq \text{proj dim}(S/I)$, then
  $$ t_{a+b} \leq t_a + t_1 + t_2 + \cdots + t_b - \frac{b(b-1)}{2}. $$

- If $I = I_\Delta$ where $\Delta$ is a simplicial complex such that $\dim(\Delta) < t_a - a$ or $\dim(\Delta) < t_b - b$, then
  $$ t_{a+b} \leq t_a + t_b. $$

- If $I$ is a monomial ideal that minimally generated by $m_1, \ldots, m_r$ such that $\frac{lcm(m_1, \ldots, m_r)}{lcm(m_1, \ldots, \widehat{m_i}, \ldots, m_r)} \notin K$ for all $i$, where $\widehat{m_i}$ means that $m_i$ is omitted, then $t_{a+b} \leq t_a + t_b$ for all $a, b \geq 0$ with $a + b \leq \text{proj dim}(S/I)$.

Keywords Betti numbers · Simplicial complex · Monomial ideal · Subadditivity condition

1 Introduction
Let $S = K[x_1, \ldots, x_n]$, where $K$ is a field and let $I$ be a graded ideal of $S$ and suppose $S/I$ has minimal graded free $S$-resolution
$$ 0 \to F_p = \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{pj}} \to \cdots \to F_1 = \bigoplus_{j \in \mathbb{N}} S(-j)^{\beta_{1j}} \to F_0 = S \to S/I \to 0. $$

The numbers $\beta_{ij} = \beta_{ij}(S/I)$, where $i, j \geq 0$, are called the graded Betti numbers of $I$, which count the elements of degree $j$ in a minimal generator of $(i + 1)$-th syzygy:
\[ \text{Syz}_{i+1}(S/I) = \ker(F_i \to F_{i-1}). \] Let \( t_i \) denote the maximal shifts in the minimal graded free \( S \)-resolution of \( S/I \), namely

\[ t_i = t_i(S/I) := \max(j : \beta_{ij}^S(S/I) \neq 0). \]

We say that \( I \) satisfies the \textit{subadditivity condition} if \( t_{a+b} \leq t_a + t_b \) for all \( a, b \geq 0 \) and \( a + b \leq \) \( p \), where \( p \) is the projective dimension of \( I \).

It is known that graded ideals may not satisfy the subadditivity condition as shown by the counter example in [1, Section 6]. However, no counter examples are known for monomial ideals. For edge ideals of graphs the inequality \( t_{a+b} \leq t_a + t_b \) was shown by Fernández-Ramos and Gimenez [2, Theorem 4.1]. The same inequality has been shown later for any monomial ideal by Srinivasan and Herzog [3, Corollary 4]. Yazdan Pour [4, Corollary 3.5] independently proved the same result. Bigdeli and Herzog [5, Theorem 1] proved the subadditivity condition, when \( I \) is the edge ideal of a chordal graph or a whisker graph. Some more results regarding subadditivity have been obtained by Khoury and Srinivasan [6, Theorem 2.3], the Abedelfatah and Nevo [7, Theorem 1.3] and Faridi [8, Theorem 3.7].

In this paper we prove in Theorem 3.5 that if \( I \) is a monomial ideal of \( S \) and \( a \geq b - 1 \geq 0 \) are integers such that \( a + b \leq \) \( \text{proj dim}(S/I) \), then

\[ t_{a+b} \leq t_a + t_1 + t_2 + \cdots + t_b - \frac{b(b-1)}{2}, \]

which generalizes the well-known inequality \( t_{a+1} \leq t_a + t_1 \).

In Theorem 3.3, we prove that if \( I = I_\Delta \) where \( \Delta \) is a simplicial complex such that \( \dim(\Delta) < t_a - a \) or \( \dim(\Delta) < t_b - b \), then \( t_{a+b} \leq t_a + t_b \). The proof of both Theorems 3.3 and 3.5 uses a combinatorial topological argument.

In Theorem 4.1, we prove algebraically, using Taylor resolution, that the subadditivity condition holds when \( I \) is a monomial ideal that minimally generated by \( m_1, \ldots, m_r \) such that \( \frac{\text{lcm}(m_1, \ldots, m_r)}{\text{lcm}(m_1, \ldots, \hat{m}_i, \ldots, m_r)} \notin K \) for all \( i \), where \( \hat{m}_i \) means that \( m_i \) is omitted.

\section{2 Preliminaries}

Fix a field \( K \). Let \( S = K[x_1, \ldots, x_n] \) be the graded polynomial ring with \( \text{deg}(x_i) = 1 \) for all \( i \), and \( M \) be a graded \( S \)-module. The integer \( \beta_{ij}^S(M) = \dim_K \text{Tor}_i^S(M, K) \) is called the \( (i,j) \)th \textit{graded Betti number} of \( M \). Note that if \( I \) is a graded ideal of \( S \), then \( \beta_{i+1,j}^S(S/I) = \beta_{i,j}^S(I) \) for all \( i, j \geq 0 \).

For a simplicial complex \( \Delta \) on the vertex set \( \Delta_0 = [n] = \{1, \ldots, n\} \), its \textit{Stanley–Reisner ideal} \( I_\Delta \subset S \) is the ideal generated by the squarefree monomials \( x_F = \prod_{i \in F} x_i \) with \( F \not\in \Delta \). \( F \subset [n] \). The dimension of the face \( F \) is \( |F| - 1 \) and the \textit{dimension of \( \Delta \) is} \( \text{max}\{\dim F : F \in \Delta\} \).

For \( W \subset V \), we write

\[ \Delta[W] = \{ F \in \Delta : F \subset W \} \]

for the induced subcomplex of \( \Delta \) on \( W \). We denote by \( \beta_i(\Delta) = \dim_K \widetilde{H}_i(\Delta; K) \) the dimension of the \( i \)-th reduced homology group of \( \Delta \) with coefficients in \( K \). The following result is known as Hochster’s formula for graded Betti numbers.

\begin{theorem}[Hochster] Let \( \Delta \) be a simplicial complex on \([n]\). Then
\end{theorem}
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\[ \beta_{i,j}(S/I_\Delta) = \sum_{W \subseteq [n], |W| = i+j} \beta_{j-1}(\Delta[W];K) \]

for all \( i, j \geq 0 \).

If \( \Delta_1 \) and \( \Delta_2 \) are two subcomplexes of \( \Delta \) such that \( \Delta = \Delta_1 \cup \Delta_2 \), then there is a long exact sequence of reduced homologies, called the Mayer–Vietoris sequence

\[
\cdots \to \tilde{H}_i(\Delta_1 \cap \Delta_2;K) \to \tilde{H}_i(\Delta_1;K) \oplus \tilde{H}_i(\Delta_2;K) \to \tilde{H}_i(\Delta;K) \\
\to \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2;K) \to \cdots
\]

3 The main theorems

Lemma 3.1 Let \( \Delta = \Delta_1 \cup \cdots \cup \Delta_t \) be a union of subcomplexes. If

\[ \tilde{H}_{j-r+1}\left( \bigcap_{m=1}^r \Delta_{i_m};K \right) = 0 \]

for all \( 1 \leq i_1 < \cdots < i_r \leq t \), then \( \tilde{H}_j(\Delta;K) = 0 \).

Proof We prove the assertion by induction on \( t \). It is trivial for \( t = 1 \). Let \( t = 2 \). Using the Mayer–Vietoris sequence

\[
\cdots \to \tilde{H}_j(\Delta_1;K) \oplus \tilde{H}_j(\Delta_2;K) \to \tilde{H}_j(\Delta;K) \to \tilde{H}_{j-1}(\Delta_1 \cap \Delta_2;K) \to \cdots
\]

and the assumptions \( \tilde{H}_j(\Delta_1;K) = \tilde{H}_j(\Delta_2;K) = \tilde{H}_{j-1}(\Delta_1 \cap \Delta_2;K) = 0 \), we get \( \tilde{H}_j(\Delta;K) = 0 \).

Let \( t > 2 \). Now, we consider the Mayer–Vietoris sequence

\[
\cdots \to \tilde{H}_j(\Delta_1;K) \oplus \tilde{H}_j(\Delta_2 \cup \cdots \cup \Delta_t;K) \to \tilde{H}_j(\Delta;K) \\
\to \tilde{H}_{j-1}((\Delta_1 \cap \Delta_2) \cup \cdots \cup (\Delta_1 \cap \Delta_t);K) \to \cdots
\]

by induction assumption, we have

\[ \tilde{H}_j(\Delta_2 \cup \cdots \cup \Delta_t;K) = \tilde{H}_{j-1}((\Delta_1 \cap \Delta_2) \cup \cdots \cup (\Delta_1 \cap \Delta_t);K) = 0. \]

Hence \( \tilde{H}_j(\Delta;K) = 0 \) as desired. \( \square \)

Proposition 3.2 Let \( \Delta \) be a simplicial complex on the set \([n]\) and \( W \subseteq [n] \) such that \( |W| = t_a + s + l + 1 \), where \( a \leq \text{proj dim}(S/I_\Delta) \), \( s \geq 0 \) and \( l \geq 1 \). Then for all \( A \subseteq W \) with \( |A| = s' + l \) where \( 0 \leq s' \leq s \), we have

\[ \tilde{H}_{l-a+s}\left( \bigcup_{B \subseteq A : |B| = l} (\Delta[W \setminus B];K) \right) = 0. \]

Proof We prove the proposition by induction on \( l \). Let \( l = 1 \). Note that for all \( 1 \leq r \leq |A| \), we have \( \beta_{a,l+a+2-r}(S/I_\Delta) = 0 \). So

\[ \square \]
\[
\widehat{H}_{t_a-a+\delta-r+1}(\Delta[W\backslash B_1] \cap \cdots \cap \Delta[W\backslash B_r];K) = 0.
\]

for all distinct singleton subsets \(B_1, \ldots, B_r\) of \(A\). By Lemma 3.1, we obtain that
\[
\widehat{H}_{t_a-a+\delta}(\bigcup_{B \subseteq A: |B|=l} \Delta[W\backslash B];K) = 0.
\]

Let \(l > 1\) and \(A = \{b_1,\ldots, b_s, b_{s+1}, \ldots, b_{s'+1}\}\). For \(1 \leq j \leq s' + 1\), denote by \(B_j\) the set of all \(B \subseteq A\) such that \(|B| = l\), \(b_j \in B\) and \(\{b_1, \ldots, b_{j-1}\} \cap B = \emptyset\). For \(1 \leq j \leq s' + 1\), let \(\Delta_j = \bigcup_{B \in B_j} \Delta[W\backslash B]\). We have that
\[
\bigcup_{B \subseteq A: |B|=l} \Delta[W\backslash B]) = \Delta_{i_1} \cup \cdots \cup \Delta_{i_r}.
\]

Let \(1 \leq i_1 < \cdots < i_r \leq s' + 1\). Denote by \(W'\) the set \(W\backslash \{b_{i_1}, \ldots, b_{i_r}\}\) and by \(A'\) the set \(A\backslash \{b_1, \ldots, b_{i_r}\}\). Let \(\tilde{s} = s - r + 1\) and \(\tilde{l} = l - 1\). We have that \(|W'| = t_a + \tilde{s} + \tilde{l} + 1\) and
\[
\Delta_{i_1} \cap \cdots \cap \Delta_{i_r} = \bigcup_{B \subseteq A': |B|=\tilde{l}} \Delta[W'\backslash B].
\]

By induction hypothesis, we have that
\[
\widehat{H}_{t_a-a+\delta}(\Delta_{i_1} \cap \cdots \cap \Delta_{i_r};K) = 0.
\]

By Lemma 3.1 we get
\[
\widehat{H}_{t_a-a+\delta}(\bigcup_{B \subseteq A': |B|=\tilde{l}} \Delta[W'\backslash B];K) = 0.
\]

Now, we prove the main results.

**Theorem 3.3** Let \(\Delta\) be a simplicial complex on the set \([n]\) and \(a, b\) are non negative integers such that \(a, b \leq \text{proj dim}(S/I_{\Delta})\). If \(\text{dim}(\Delta) < t_a - a\) or \(\text{dim}(\Delta) < t_b - b\), then \(t_{a+b} \leq t_a + t_b\).

**Proof** Without loss of generality, assume that \(\text{dim}(\Delta) < t_b - b\). We have to show that \(\beta_{a+b, t_a-t_b+1}(S/I_{\Delta}) = 0\) for all integer \(r \geq 0\). Let \(W \subseteq [n]\) of size \(t_a + t_b + r + 1\) and \(A\) be any subset of \(W\) with \(|A| = t_a + r\), where \(r \geq 0\). Since \(\Delta\) does not contains a face of dimension \(t_b - b\), it follows that
\[
\Delta[W] = \bigcup_{B \subseteq A: |B|=b} \Delta[W\backslash B].
\]

By 3.2 (taking \(s = s' = t_b - b + r\) and \(l = b\)), it follows that
\[
\tilde{H}_{t_a-a+t_b-b+r}(\Delta[W];K) = 0.
\]
Example 3.4 Let $\Delta$ be a graph on the vertices $[n]$ and assume that there is two disjoint non-edges in $G$. So $t_2 \geq 4$ and $\dim(\Delta) = 1 < t_2 - 2$. By Theorem 3.3, it follows that $t_{a+2} \leq t_a + t_2$ for all $a$ such that $a + 2 \leq \proj \dim(S/I_\Delta)$.

Theorem 3.5 If $I$ is a monomial ideal of $S$, $b \geq 1$ and $a \geq b - 1$ are integers such that $a + b \leq \proj \dim(S/I)$, then

$$t_{a+b} \leq t_a + t_1 + t_2 + \cdots + t_b - \frac{b(b - 1)}{2}.$$ 

Proof By polarization, we may assume that $I$ is a squarefree ideal, and so $I = I_\Delta$, where $\Delta$ is a simplicial complex on $[n']$. First, we prove that

$$t_{c+1} \leq t_c + t_d - d + 1, \quad (1)$$

for all $d \geq 1$ and $d - 1 \leq c \leq \proj \dim(S/I)$. Assume on contrary, that $\beta_{c+1,\Delta+d+r+2}(S/I) \neq 0$ for some $r \geq 0$. It follows that there exists a subset $W$ of $[n']$ so that $|W| = t_c + t_d - d + r + 2$ and $H_{t_{c+1},\Delta+d+r}(\Delta[W]; K) \neq 0$. In particular, $\proj \dim(S/I_{\Delta[W]}) \geq c + 1$.

If $\deg(m) \geq t_d - d + r + 2$ for all minimal generator $m$ of $I_{\Delta[W]}$, then $\beta_{1,s}(S/I_{\Delta[W]}) = 0$ for all $x \leq t_d - d + r + 1$. So $\beta_{d+1}(S/I_{\Delta[W]}) = 0$ for all $x \leq t_d - d + r + d = t_d + r$. It follows that $\proj \dim(S/I_{\Delta[W]}) < d \leq c + 1$. This is a contradiction.

We obtain that there is a subset $A$ of $W$ with $|A| = t_d - d + r + 1$ and

$$\Delta[W] = \bigcup_{B \subseteq A: |B| = 1} \Delta[W \setminus B].$$

By 3.2 (taking $s = s' = t_d - d + r$ and $l = 1$), it follows that

$$\tilde{H}_{t_{c+1},t_d+d+r}(\Delta[W]; K) = 0,$$

a contradiction.

Now, we prove by induction on $1 \leq b' \leq b$ that

$$t_{a+b'} \leq t_a + t_1 + t_2 + \cdots + t_{b'} - \frac{b'(b' - 1)}{2}.$$ 

The case $b' = 1$ follows by (1). Let $b' > 1$. By the induction hypothesis, we have

$$t_{a+b'} = t_{(a+b'-1)+1} \leq t_{a+b'-1} + t_{b'} - b' + 1 \leq t_a + t_1 + \cdots + t_{b'-1} - \frac{(b'-1)(b'-2)}{2} + t_{b'} - b' + 1 = t_a + t_1 + t_2 + \cdots + t_{b'} - \frac{b'(b' - 1)}{2}.$$
4 The Taylor resolution

Let $I$ be a monomial ideal with $G(I) = \{m_1, \ldots, m_r\}$. For a subset $F$ of $G(I)$, set $\text{lcm}(F) = \text{lcm}\{m_i : m_i \in F\}$ and define a formal symbol $[F]$ with multidegree equal to $\text{lcm}(F)$. Let $T_0 = S$ and for each $i \geq 1$, let $T_i$ be the free $S$-module with basis $\{[F] : [F] = i\}$. Note that $T_i$ is a multigraded $S$-module.

Let $\phi_0 : T_0 \to S/I$ be the canonical homomorphism and define the multigraded differential $\phi_i : T_i \to T_{i-1}$ by

$$[F] \longmapsto \sum_{k=1}^{i} (-1)^{k-1} \cdot \frac{\text{lcm}(F)}{\text{lcm}(F \setminus \{m_j\})} \cdot [F \setminus \{m_j\}]$$

where $F = \{m_j, \ldots, m_i\}$, written with the indices in increasing order.

The free resolution

$$\mathbb{T} : 0 \to T_r \xrightarrow{\phi_r} \cdots \to T_1 \xrightarrow{\phi_1} T_0 \xrightarrow{\phi_0} S/I \to 0$$

is called the Taylor resolution of $I$.

We denote by $H_a(\mathbb{T})$ the $a$-th homology group, of the chain complex

$$\mathbb{T} : 0 \to T_r \otimes_S K \xrightarrow{\phi_r \otimes_K K} \cdots \to T_1 \otimes_S K \xrightarrow{\phi_1 \otimes_K K} T_0 \otimes_S K \to 0.$$

Note that $\beta_{a,b}(S/I) = \dim_K(H_a(\mathbb{T}))_b$ and for all $0 \leq j \leq r$

$$T_j \otimes_S K \cong T_j \otimes_S S/M \cong T_j/MT_j = \overline{T}_j$$

where $M$ is the maximal ideal $(x_1, \ldots, x_n)$ of $S$.

Proposition 4.1 Let $I$ be a monomial ideal with $G(I) = \{m_1, \ldots, m_r\}$. If there exists $0 \neq [F] \in H_{a+b}(\mathbb{T})$ such that $\text{deg}(\text{lcm}(F)) = t_{a+b}$, then $t_{a+b} \leq t_a + t_b$.

Proof Without loss of generality, assume that $F = \{w_1, \ldots, w_{a+b}\}$ and

$$G(I) = \{m_1, \ldots, m_{r-(a+b)}, w_1, \ldots, w_{a+b}\}.$$  

Set $F_1 = \{w_1, \ldots, w_a\}$ and $F_2 = \{w_{a+1}, \ldots, w_{a+b}\}$. Since $[F] \in \ker \overline{\phi}_{a+b}$, it follows that $\frac{\text{lcm}(F)}{\text{lcm}(G)} \notin K$, for all $G \subseteq F$. So we have that $[F_1] \in \ker \overline{\phi}_{a}$ and $[F_2] \in \ker \overline{\phi}_{b}$. If $[F_1] \in \text{Im} \overline{\phi}_{a+1}$, then there exist $[F_{1,1}], \ldots, [F_{1,s}] \in \overline{T}_{a+1}$ and $a_1, \ldots, a_s \in K$ such that

$$[F_1] = a_1 \overline{\phi}_{a+1}[F_{1,1}] + \cdots + a_s \overline{\phi}_{a+1}[F_{1,s}].$$

Note that $\text{lcm}(F_j) = \text{lcm}(F_{1,j})$ for all $1 \leq j \leq s$, so $F_{1,j} \cap F_2 = \emptyset$ for all $1 \leq j \leq s$. For all $1 \leq j \leq s$ let $\hat{F}_{1,j} = F_{1,j} \cup F_2$ and assume that the elements of $\hat{F}_{1,j}$ are ordered as the order of $G(I)$. We obtain that $[F] = a_1 \overline{\phi}_{a+b+1}[\hat{F}_{1,1}] + \cdots + a_s \overline{\phi}_{a+b+1}[\hat{F}_{1,s}]$, which contradicts to the fact that $[F] \notin \text{Im} \overline{\phi}_{a+b+1}$. Similarly, $[F_2] \notin \text{Im} \overline{\phi}_{b+1}$.

It follows that

$$t_{a+b} = \text{deg}(\text{lcm}(F)) \leq \text{deg}(\text{lcm}(F_1)) + \text{deg}(\text{lcm}(F_2)) \leq t_a + t_b.$$ 

$\square$
Corollary 4.2 Let $I$ be a monomial ideal with $G(I) = \{m_1, \ldots, m_r\}$. If $\frac{\lcm(m_1, \ldots, m_i)}{\lcm(m_1, \ldots, \hat{m}_i, \ldots, m_r)} \notin K$ for all $i$, where $\hat{m}_i$ means that $m_i$ is omitted, then $t_{a+b} \leq t_a + t_b$.

Proof Let $[F]$ be a generator of $T_{a+b}$ with $\deg(\lcm(F)) = t_{a+b}$. By the assumption, it follows that $[F] \in \ker \phi_{a+b}$ and $[F] \notin \text{Im} \phi_{a+b+1}$. Hence the assertion follows from Proposition 4.1.

Example 4.3 Let $S = K[a, b, c]$ and $I$ be the ideal of $S$ which generated by $G(I) = \{a^4bc, b^3c^2, c^5a^3\}$. Note that
\[
\frac{\lcm(G(I))}{\lcm(b^3c^2, c^5a^3)} = a, \quad \frac{\lcm(G(I))}{\lcm(a^4bc, c^5a^3)} = b^2, \quad \frac{\lcm(G(I))}{\lcm(a^4bc, b^3c^2)} = c^3.
\]
So by Corollary 4.2, $I$ satisfies the subadditivity condition.

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