On the rate of convergence of image classifiers based on convolutional neural networks

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February 20, 2020

Abstract
Image classifiers based on convolutional neural networks are defined, and the rate of convergence of the misclassification risk of the estimates towards the optimal misclassification risk is analyzed. Under suitable assumptions on the smoothness and structure of the a posteriori probability a rate of convergence is shown which is independent of the dimension of the image. This proves that in image classification it is possible to circumvent the curse of dimensionality by convolutional neural networks.

AMS classification: Primary 62G05; secondary 62G20.

Key words and phrases: Curse of dimensionality, convolutional neural networks, image classification, rate of convergence.

1 Introduction

1.1 Scope of this article
Deep neural networks are nowadays among the most successful and most widely used methods in machine learning, see, e.g., Schmidhuber (2015), Rawat and Wang (2017), and the literature cited therein. In many applications the most successful networks are deep convolutional networks, see, e.g., Krizhevsky, Sutskever and Hinton (2012) and Kim (2014) concerning applications in image classification or language recognition, resp. These networks can be considered as a special case of deep feedforward neural networks, where symmetry constraints are imposed on the weights of the networks. For general deep feedforward neural networks it was recently shown that under suitable compository assumptions on the structure of the regression function these networks are able to achieve dimension reduction in estimation of high-dimensional regression functions (cf., Kohler

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and Krzyżak (2017), Bauer and Kohler (2019), Schmidt-Hieber (2019), Kohler and Langer (2019) and Suzuki and Nitanda (2019)). The purpose of this article is to characterize situations in image classification, where deep convolutional neural networks can achieve a similar dimension reduction.

1.2 Image classification

Let $d_1, d_2 \in \mathbb{N}$ and let $(X,Y), (X_1,Y_1), \ldots, (X_n,Y_n)$ be independent and identically distributed random variables with values in $[0,1]^{(1,\ldots,d_1) \times (1,\ldots,d_2)} \times \{0,1\}$.

Here we use the notation $[0,1]^J = \{(a_j)_{j \in J} : a_j \in [0,1] \ (j \in J)\}$ for a nonempty and finite index set $J$, and we describe a (random) image from (random) class $Y \in \{0,1\}$ by a (random) matrix $X$ with $d_1$ columns and $d_2$ rows, which contains at position $(i,j)$ the grey scale value of the pixel of the image at the corresponding position.

Let $\eta(x) = \mathbf{P}\{Y = 1|X = x\} \ (x \in [0,1]^{(1,\ldots,d_1) \times (1,\ldots,d_2)})$ be the so-called a posteriori probability of class 1. Then we have

$$\min_{f:[0,1]^{(1,\ldots,d_1) \times (1,\ldots,d_2)} \to \{0,1\}} \mathbf{P}\{f(X) \neq Y\} = \mathbf{P}\{f^*(X) \neq Y\},$$

where $f^*(x) = \begin{cases} 1, & \text{if } \eta(x) > \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$

is the so-called Bayes classifier (cf., e.g., Theorem 2.1 in Devroye, Györfi and Lugosi (1996)). Set

$D_n = \{(X_1,Y_1), \ldots, (X_n,Y_n)\}$.

In the sequel we consider the problem of constructing a classifier

$$f_n = f_n(\cdot, D_n) : [0,1]^{(1,\ldots,d_1) \times (1,\ldots,d_2)} \to \{0,1\}$$

such that the misclassification risk

$$\mathbf{P}\{f_n(X) \neq Y|D_n\}$$

of this classifier is as small as possible. Our aim is to derive a bound on the expected difference of the misclassification risk of $f_n$ and the optimal misclassification risk, i.e., we want to derive an upper bound on

$$\mathbf{E}\left\{\mathbf{P}\{f_n(X) \neq Y|D_n\} - \min_{f:[0,1]^{(1,\ldots,d_1) \times (1,\ldots,d_2)} \to \{0,1\}} \mathbf{P}\{f(X) \neq Y\}\right\}$$

$$= \mathbf{P}\{f_n(X) \neq Y\} - \mathbf{P}\{f^*(X) \neq Y\}.$$
1.3 Plug-in classifiers

We will use plug-in classifiers of the form

\[ f_n(x) = \begin{cases} 1, & \text{if } \eta_n(x) \geq \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases} \]

where

\[ \eta_n(\cdot) = \eta_n(\cdot, \mathcal{D}_n) : [0,1]^{\{1,\ldots,d_1\} \times \{1,\ldots,d_2\}} \to \mathbb{R} \]

is an estimate of the aposteriori probability \( \eta \). It is well-known that such plug-in classifiers satisfy

\[ P\{f_n(X) \neq Y|\mathcal{D}_n\} - P\{f^*(X) \neq Y\} \leq 2 \cdot \int |\eta_n(x) - \eta(x)| P_X(dx) \]

(cf., e.g., Theorem 1.1 in Györfi et al. (2002)), which implies (via the Cauchy-Schwartz inequality)

\[ P\{f_n(X) \neq Y\} - P\{f^*(X) \neq Y\} \leq 2 \cdot \sqrt{E \left\{ \int |\eta_n(x) - \eta(x)|^2 P_X(dx) \right\}}. \quad (2) \]

Hence we can derive an upper bound on the difference between the expected misclassification risk of our estimate and the minimal possible value from a bound on the expected \( L_2 \) error of the estimate \( \eta_n \) of the aposteriori probability.

It is well-known that the bound in (2) is not tight, therefore classification is easier than regression estimation (cf., Devroye, Gőrő and Lugosi (1996)). In the sequel we will nevertheless solve an image classification problem via regression estimation, because this will enable us to impose conditions on the underlying distribution by restricting the structure of the aposteriori probability. And, as we will see in the next subsection, it is easy to formulate such restrictions such that they seem to be natural assumptions in image classification applications.

1.4 A hierarchical max-pooling model for the aposteriori probability

In order to derive nontrivial rate of convergence results on the difference between the misclassification risk of any estimate and the minimal possible value it is necessary to restrict the class of distributions (cf., Cover (1968) and Devroye (1982)). In the sequel we will use assumptions on the structure and the smoothness of the aposteriori probability.

The basic idea behind the formulation of our condition is the following: Consider an application where a human has to decide about a class of an image, e.g., the human has to decide whether a given image contains a specific traffic sign or not. Then the human will survey the whole image and look at each subpart of the image whether it contains the traffic sign or not. By looking at a subpart, the human can estimate a probability that this subpart contains the traffic sign. It is then natural to assume that the probability that the whole image contains a traffic sign is simply the maximum of the probabilities
for each subpart of the image. This idea leads to the definition of a max-pooling model for the aposteriori probability introduced below.

Furthermore, we take decision whether a given subpart of the image contains a traffic sign or not by taking several decisions whether the image contains parts of a traffic sign or not, and by combining these decisions about the different parts hierarchically. This idea leads to the hierarchical model introduced below.

By combining both ideas we get our main model, the hierarchical max-pooling model for the aposteriori probability, which we introduce next. In order to formulate this we need the following notation: For $M \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$ we define

$$x + M = \{x + z : z \in M\}.$$ 

For $I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}$ and $x = (x_i)_{i \in \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} \in [0, 1]^{\{1, \ldots, d_1\} \times \{1, \ldots, d_2\}}$ we set

$$x_I = (x_i)_{i \in I}.$$

**Definition 1** a) We say that $m : [0, 1]^{\{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} \to \mathbb{R}$ satisfies a max-pooling model with index set

$$I \subseteq \{0, \ldots, d_1 - 1\} \times \{0, \ldots, d_2 - 1\},$$

if there exist a function $f : [0, 1]^{\{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} \to \mathbb{R}$ such that

$$m(x) = \max_{(i,j) \in \mathbb{Z}^2 : (i,j) + I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} f(x_{(i,j)} + I) \quad (x \in [0, 1]^{\{1,\ldots, d_1\} \times \{1,\ldots, d_2\}}).$$

b) Let $I = \{0, \ldots, 2^l - 1\} \times \{0, \ldots, 2^l - 1\}$ for some $l \in \mathbb{N}_0$. We say that

$$f : [0, 1]^{\{1, \ldots, 2^l\} \times \{1, \ldots, 2^l\}} \to \mathbb{R}$$

satisfies a hierarchical model of level $l$, if there exists functions

$$g_{k,s} : \mathbb{R}^4 \to [0, 1] \quad (k = 1, \ldots, l, s = 1, \ldots, 4^{l-k})$$

such that we have

$$f = f_{l,1}$$

for some $f_{k,s} : [0, 1]^{\{1, \ldots, 2^k\} \times \{1, \ldots, 2^k\}} \to \mathbb{R}$ recursively defined by

$$f_{k,s}(x) = g_{k,s}(f_{k-1,4(s-1)+1}(x_{\{1, \ldots, 2^{k-1}\} \times \{1, \ldots, 2^{k-1}\}}),$$

$$f_{k-1,4(s-1)+2}(x_{\{2^{k-1}+1, \ldots, 2^k\} \times \{1, \ldots, 2^{k-1}\}}),$$

$$f_{k-1,4(s-1)+3}(x_{\{1, \ldots, 2^{k-1}\} \times \{2^{k-1}+1, \ldots, 2^k\}}),$$

$$f_{k-1,4(s-1)}(x_{\{2^{k-1}+1, \ldots, 2^k\} \times \{2^{k-1}+1, \ldots, 2^k\}}))$$

$x \in [0, 1]^{\{1, \ldots, 2^k\} \times \{1, \ldots, 2^k\}}$)

for $k = 2, \ldots, l, s = 1, \ldots, 4^{k-l},$ and

$$f_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = g_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) \quad (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} \in [0, 1])$$. 

4
for \( s = 1, \ldots, 4^{l-1} \).

c) We say that \( m : [0,1]^{\{1,\ldots,d_1\} \times \{1,\ldots,d_2\}} \rightarrow \mathbb{R} \) satisfies a **hierarchical max-pooling model of level** \( l \) (where \( 2^l \leq \max\{d_1,d_2\} \)), if \( m \) satisfies a max-pooling model with index set
\[
I = \{0, \ldots, 2^l - 1\} \times \{0, \ldots, 2^l - 1\}
\]
and the function \( f : [0,1]^{\{1,\ldots,2^l\} \times \{1,\ldots,2^l\}} \rightarrow \mathbb{R} \) in the definition of this max-pooling model satisfies a hierarchical model with level \( l \).

d) Let \( p = q + s \) for some \( q \in \mathbb{N}_0 \) and \( s \in (0,1] \), and let \( C > 0 \). We say that a hierarchical max-pooling model is \((p,C)\)-smooth if all functions \( g_{k,s} \) in its definition are \((p,C)\)-smooth (see Subsection 1.7 for the definition of \((p,C)\)-smoothness).

### 1.5 Main result

The main contributions in this paper are as follows: First, we introduce the above setting for the mathematical analysis of an image classification problem. Here our main idea is to use plug-in classification estimates, which allows us to restrict the underlying class of distributions by imposing constraints on the structure and the smoothness of the a posteriori probability. The main advantage of this approach is that we can introduce in this setting with the above \((p,C)\)-smooth hierarchical max-pooling model a natural condition for applications. Second, we analyze the rate of convergence of deep convolutional neural network classifiers (with ReLU activation function) in this context. Here we show in Theorem 1 below that in case that the a posteriori probability satisfies a \((p,C)\)-smooth hierarchical max-pooling model, the expected misclassification risk of the estimate converges toward the minimal possible value with rate
\[
n^{-\frac{p}{p+4}}
\]
(up to some logarithmic factor). Since this rate of convergence does not depend on the dimension \( d_1 \cdot d_2 \) of the image, this shows that under suitable assumptions on the structure of the a posteriori probability it is possible to circumvent the curse of dimensionality in image classification by using convolutional neural networks.

### 1.6 Discussion of related results

Convolutional neural networks introduced by Le Cun et al. (1989) have become the leading techniques in pattern recognition applications, cf., e.g., Le Cun et al. (1998), LeCun, Bengio and Hinton (2015), Goodfellow, Bengio and Courville (2016), Rawat and Wang (2017), and the literature cited therein.

As mentioned by Rawat and Wang (2017), despite the empirical success of these methods the theoretical proof of why they succeed is lacking. In fact there are only a few papers addressing theoretical properties of these networks. Several papers used the idea that properly defined convolutional neural networks are able to mimic deep feedforward neural networks and obtained rate of convergence results for estimates based on convolutional neural networks similar to feedforward neural networks estimates (cf., e.g., Oono
and Suzuki (2019) and the literature cited therein). The drawback of this approach is that in this way it is not possible to identify situations in which convolutional neural networks are superior to standard feedforward neural networks. Generalization bounds for convolutional neural networks have been analyzed in Lin and Zhang (2019). In several papers it was shown that gradient descent is able to find the global minimum of the empirical loss function in case of overparametrized convolutional neural networks, cf., e.g., Du et al. (2019). But, as was shown by a counterexample in Kohler and Krzyżak (2019), overparametrized deep neural networks do not, in general, generalize well. In an abstract setting, very interesting approximation properties of deep convolutional neural networks have been obtained by Yarotsky (2018). However, it is unclear how one can apply these results in statistical estimation problem.

Much more is known about standard deep feedforward neural networks. Here it was recently shown that under suitable compository assumptions on the structure of the regression function these networks are able to achieve dimension reduction in estimation of high-dimensional regression functions (cf., Kohler and Krzyżak (2017), Bauer and Kohler (2019), Schmidt-Hieber (2019), Kohler and Langer (2019) and Suzuki and Nitanda (2019)). Imaiizumi and Fukamizu (2019) derived results concerning estimation by neural networks of piecewise polynomial regression functions with partitions having rather general smooth boundaries. Eckle and Schmidt-Hieber (2019) and Kohler, Krzyżak and Langer (2019) showed that the least squares neural network regression estimates based on deep neural networks can achieve the rate of convergence results similar to piecewise polynomial partition estimates where partition is chosen in an optimal way.

Classification is a problem theory of which has been intensively studied in statistics, see e.g., the book Devroye, Györfi and Lugosi (1996) which discusses probabilistic theory of pattern recognition in depth. This theory can of course be applied to image classification, but due to the high dimension of the input in image classification this will not lead to useful results. To the best of our knowledge there do not exist until now papers which analyze the rate of convergence of image classifiers and are able to achieve sufficient, and for applications satisfying, dimension reduction. The classification problem with standard deep feedforward neural networks has been analyzed in Kim, Ohn and Kim (2019).

A Bayesian approach towards the analysis of images, which can be used e.g. for feature extraction, can be found in Chang et al. (2017).

A related problem to image classification is image reconstruction or image denoising. Here quite a few theoretical results exist, see, e.g., Korostelev and Tsybakov (1993) and the literature cited therein.

1.7 Notation

Throughout the paper, the following notation is used: The sets of natural numbers, natural numbers including 0, integers and real numbers are denoted by \( \mathbb{N} \), \( \mathbb{N}_0 \), \( \mathbb{Z} \) and \( \mathbb{R} \), respectively. For \( z \in \mathbb{R} \), we denote the smallest integer greater than or equal to \( z \) by \( \lceil z \rceil \). Let \( D \subseteq \mathbb{R}^d \) and let \( f : \mathbb{R}^d \to \mathbb{R} \) be a real-valued function defined on \( \mathbb{R}^d \). We write \( x = \arg \min_{z \in D} f(z) \) if \( \min_{z \in D} f(z) \) exists and if \( x \) satisfies \( x \in D \) and \( f(x) = \min_{z \in D} f(z) \).
For $f : \mathbb{R}^d \to \mathbb{R}$

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|$$

is its supremum norm, and the supremum norm of $f$ on a set $A \subseteq \mathbb{R}^d$ is denoted by

$$\|f\|_{\infty, A} = \sup_{x \in A} |f(x)|.$$

Let $p = q + s$ for some $q \in \mathbb{N}_0$ and $0 < s \leq 1$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called $(p, C)$-smooth, if for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ with $\sum_{j=1}^d \alpha_j = q$ the partial derivative

$$\frac{\partial^q f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(x)$$

exists and satisfies

$$\left| \frac{\partial^q f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(x) - \frac{\partial^q f}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(z) \right| \leq C \cdot \|x - z\|^s$$

for all $x, z \in \mathbb{R}^d$.

Let $\mathcal{F}$ be a set of functions $f : \mathbb{R}^d \to \mathbb{R}$, let $x_1, \ldots, x_n \in \mathbb{R}^d$ and set $x_i^n = (x_1, \ldots, x_n)$. A finite collection $f_1, \ldots, f_N : \mathbb{R}^d \to \mathbb{R}$ is called an $\varepsilon$-cover of $\mathcal{F}$ on $x_1^n$ if for any $f \in \mathcal{F}$ there exists $i \in \{1, \ldots, N\}$ such that

$$\frac{1}{n} \sum_{k=1}^n |f(x_k) - f_i(x_k)| < \varepsilon.$$

The $\varepsilon$-covering number of $\mathcal{F}$ on $x_1^n$ is the size $N$ of the smallest $\varepsilon$-cover of $\mathcal{F}$ on $x_1^n$ and is denoted by $N_1(\varepsilon, \mathcal{F}, x_1^n)$.

For $z \in \mathbb{R}$ and $\beta > 0$ we define $T_\beta z = \max\{-\beta, \min\{\beta, z\}\}$. If $f : \mathbb{R}^d \to \mathbb{R}$ is a function and $\mathcal{F}$ is a set of such functions, then we set

$$(T_\beta f)(x) = T_\beta (f(x)) \quad \text{and} \quad T_\beta \mathcal{F} = \{T_\beta f : f \in \mathcal{F}\}.$$  

1.8 Outline of the paper

In Section 2 the convolutional neural network image classifiers used in this paper are defined. The main result is presented in Section 3 and proven in Section 4.

2 Convolutional neural network image classifiers

In the sequel we define a convolutional neural network with $L$ convolutional layers, one linear layer and one max-pooling layer. The network has $k_r$ channels (also called feature maps) in the convolutional layer $r$ and the convolution in layer $r$ will be performed by using a window of values of layer $r - 1$ of size $M_r$, where $r \in \{1, \ldots, L\}$. We will denote the input layer as the convolutional layer 0 with $k_0 = 1$ channels, and the network will use the ReLU activation function

$$\sigma(x) = \max\{x, 0\}.$$
Then we define recursively the weights
\[ w = \left( w_{i,j,s_1,s_2}^{(r)} \right)_{1 \leq i,j \leq M_r, s_1 \in \{1,\ldots,k_r-1 \}, s_2 \in \{1,\ldots,k_r \}, r \in \{1,\ldots,L \} }, \]
for the bias in each channel and each convolutional layer, and the output weights
\[ w_{\text{bias}} = \left( w_{s_2}^{(r)} \right)_{s_2 \in \{1,\ldots,k_r \}, r \in \{1,\ldots,L \} }, \]
\[ w_{\text{out}} = \left( w_{(i_2,j_2),s_2}^{(r)} \right)_{i_2 \in \{1,\ldots,d_1-\sum_{k=1}^{L} M_k+L \}, j_2 \in \{1,\ldots,d_2-\sum_{k=1}^{L} M_k+L \}, s_2 \in \{1,\ldots,k_l \} }. \]

The output of the network is
\[ f_{w,w_{\text{bias}},w_{\text{out}}} (x) = \max \left\{ \sum_{s_2=1}^{k_1} w_{(i_2,j_2),s_2}^{(L)} o_{(i_2,j_2),s_2}^{(L)} : i_2 \in \{1,\ldots,d_1 - \sum_{k=1}^{L} M_k + L \}, j_2 \in \{1,\ldots,d_2 - \sum_{k=1}^{L} M_k + L \} \right\}, \]
where \( o_{(i_2,j_2),s_2}^{(L)} \) is the output of the last convolutional layer, which is recursively defined as follows:

We start with
\[ o_{(i,j),1}^{(0)} = x_{i,j} \text{ for } i \in \{1,\ldots,d_1 \} \text{ and } j \in \{1,\ldots,d_2 \}. \]

Then we define recursively
\[ o_{(i_2,j_2),s_2}^{(r)} = \sigma \left( \sum_{s_1=1}^{k_{r-1}} \sum_{t_1,t_2 \in \{1,\ldots,M_r \}} w_{t_1,t_2,s_1,s_2}^{(r)} o_{(t_1,t_2-1,j_2-t_2-1),s_1}^{(r-1)} + w_{s_2}^{(r)} \right) \]
for \( i_2 \in \{1,\ldots,d_1 - \sum_{k=1}^{r} M_k + r \}, j_2 \in \{1,\ldots,d_2 - \sum_{k=1}^{r} M_k + r \} \) and \( r \in \{1,\ldots,L \} \).

Here we assume that \( d_1, d_2, M_1, \ldots, M_L \) and \( L \) are chosen such that the following holds.

Let \( \mathcal{F}(L, k, M) \) be the set of all functions of the above form with parameters \( L, k = (k_1, \ldots, k_L) \) and \( M = (M_1, \ldots, M_L) \). Let
\[ \eta_n = \arg \min_{f \in \mathcal{F}(L,k,M)} \frac{1}{n} \sum_{i=1}^{n} |Y_i - f(X_i)|^2 \]
be the least squares estimate of \( \eta(x) = \mathbb{E}[Y|X = x] \). Then our estimate \( f_n \) is defined by
\[ f_n(x) = \begin{cases} 1, & \text{if } \eta_n(x) \geq \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases} \]
3 Main result

Our main result is the following theorem, which present an upper bound on the distance between the expected misclassification risk of our plug-in classifier and the optimal misclassification risk.

**Theorem 1** Let \(d_1, d_2 \in \mathbb{N}, p \geq 1\) and \(C > 0\). Let \((X, Y), (X_1, Y_1), \ldots, (X_n, Y_n)\) be independent and identically distributed \([0, 1]^{d_1} \times [0, 1]^{d_2} \times \{0, 1\}\)-valued random variables. Assume that the aposteriori probability \(\eta(x) = P\{Y|X = x\}\) satisfies a \((p, C)\)-smooth hierarchical max-pooling model of finite level \(l\). Choose

\[
L_n = \left\lfloor c_1 \cdot n^\frac{4}{2(2p+4)} \right\rfloor
\]

for \(c_1 > 0\) sufficiently large, choose \(r \in \mathbb{N}\) sufficiently large, and set

\[
L = \frac{4^l - 1}{3} \cdot L_n,
\]

\[
k_s = \frac{2 \cdot 4^l + 4}{3} + r \quad (s = 1, \ldots, L),
\]

and

\[
M_s = 2^t \quad \text{for} \quad 4^{t-1} \cdot L_n + \cdots + 4^{l-1} \cdot L_n < s \leq 4^{t-1} \cdot L_n + \cdots + 4^{l-t} \cdot L_n, \quad t \in \{1, \ldots, l\},
\]

and define the estimate \(f_n\) as in Section 2. Then

\[
P\{f_n(X) \neq Y\} - \min_{f:\{0,1\}^{d_1} \times \{0,1\}} P\{f(X) \neq Y\} \leq c_2 \cdot (\log n)^2 \cdot n^{-\frac{2p}{2p+4}}.
\]

**Remark 1.** The rate of convergence in Theorem 1 does not depend on the dimension \(d_1 \cdot d_2\) of \(X\), hence the estimate is able to circumvent the curse of dimensionality under the above structural assumption on \(\eta\).

**Remark 2.** In the proof of Theorem 1 we show that the expected \(L_2\) error of our estimate of the aposteriori probability tends to zero with the rate of convergence

\[
n^{-\frac{2p}{2p+4}}
\]

(up to some logarithmic factor). According to Stone (1982) this is the optimal minimax rate of convergence for estimation of \((p, C)\) smooth functions defined on \(\mathbb{R}^4\). Since our \((p, C)\)-smooth hierarchical max-pooling model includes functions defined by a max-pooling of a \((p, C)\)-smooth function applied to only four of its component, we conjecture that (3) is in our setting the optimal rate of convergence for estimation of the aposteriori probability.

**Remark 3.** It follows from the proof of Theorem 1 that the result also holds in case that we require in the definition of \(F(L, k, M)\) that the filter satisfies

\[
w_{i,j;1,s}^{(r)} = 0 \quad \text{whenever} \quad i \notin \{1, M_r\} \text{ or } j \notin \{1, M_r\}
\]
for all \( r \in \{1, \ldots, L\}, s_1 \in \{1, \ldots, k_r-1\} \) and \( s_2 \in \{1, \ldots, k_r\} \). In this case the convolutional neural network in Theorem 1 combines at position \((i, j)\) in layer \( r \) only the outputs corresponding to the four positions

\[(i, j), (i + M_r - 1, j), (i, j + M_r - 1) \text{ and } (i + M_r - 1, j + M_r - 1).\]

## 4 Proofs

### 4.1 Auxiliary results

In this subsection we present several auxiliary results from the literature which we will use in the proof of Theorem 1. Our first result is a well-known bound on the misclassification risk of the plug-in classifiers.

**Lemma 1** Define \((X, Y), (X_1, Y_1), \ldots, (X_n, Y_n), \) and \( D_n, \eta, f^* \) and \( f_n \) as in Section 1. Then

\[
P\{f_n(X) \neq Y|D_n\} \leq P\{f^*(X) \neq Y\} \leq 2 \cdot \int |\eta_n(x) - \eta(x)|P_X(dx)
\]

holds.

**Proof.** See Theorem 1.1 in Györfi et al. (2002).

Our next result is a bound on the expected \(L_2\) error of the (truncated) least squares regression estimate.

**Lemma 2** Let \((X, Y), (X_1, Y_1), \ldots, (X_n, Y_n)\) be independent and identically distributed \(\mathbb{R}^d \times \mathbb{R}\)-valued random variables. Assume that the distribution of \((X, Y)\) satisfies

\[
E\{\exp(c_3 \cdot Y^2)\} < \infty
\]

for some constant \(c_3 > 0\) and that the regression function \(m(\cdot) = E\{Y|X = \cdot\}\) is bounded in absolute value. Let \(\tilde{m}_n\) be the least squares estimate

\[
\tilde{m}_n(\cdot) = \arg\min_{f \in \mathcal{F}_n} \frac{1}{n} \sum_{i=1}^{n} |Y_i - f(X_i)|^2
\]

based on some function space \(\mathcal{F}_n\) consisting of functions \(f : \mathbb{R}^d \to \mathbb{R}\) and set \(m_n = T_{c_4 \log(n)} \tilde{m}_n\) for some constant \(c_4 > 0\). Then \(m_n\) satisfies

\[
E \int |m_n(x) - m(x)|^2 P_X(dx) \leq c_5 \cdot (\log(n))^2 \cdot \sup_{x_1 \in (\mathbb{R}^d)^n} \left( \log \left( N_1 \left( \frac{1}{n^{c_3 \log(n)}}, T_{c_4 \log(n)} \mathcal{F}_n, x_1^n \right) \right) + 1 \right)
\]

holds.
\[\inf_{f \in F_n} \int |f(x) - m(x)|^2 P_X(dx)\]

for \(n > 1\) and some constant \(c_5 > 0\), which does not depend on \(n\) or the parameters of the estimate.

**Proof.** This result follows in a straightforward way from the proof of Theorem 1 in Bagirov, Clausen and Kohler (2009). A complete proof can be found in the supplement of Bauer and Kohler (2019).

Our third auxiliary result is an approximation result for \((p, C)\)-smooth functions by very deep feedforward neural networks.

**Lemma 3** Let \(d \in \mathbb{N}\), let \(f : \mathbb{R}^d \to \mathbb{R}\) be \((p, C)\)-smooth for some \(p = q + s, q \in \mathbb{N}\) and \(s \in (0, 1]\), and \(C > 0\). Let \(M \in \mathbb{N}\) be sufficiently large and let \(\sigma : \mathbb{R} \to \mathbb{R}\) be the ReLU activation function

\[\sigma(x) = \max\{x, 0\}\]

Then there exists a feedforward neural network \(f_{\text{net}}\) with

\[L = 4M^d - 1 + (2p + 4(q + 1)d) \cdot \log_4(M) \cdot \left[\log_2(\max\{d, q\} + 1)\right]\]

hidden layers and at most

\[r = 2^d \cdot \left(4d^2 + 18d + 2 \left(\frac{d + q}{d}\right) \cdot (4\lceil e^d \rceil + \max\{2d, 9q\} + 13)\right)\]

neurons per layer, such that

\[\sup_{x \in [-1, 1]^d} |f(x) - f_{\text{net}}(x)| \leq c_6 \cdot M^{-2p}.

**Proof.** See Theorem 2 in Kohler and Langer (2019). An alternative proof of a closely related result can be found in Yarotsky and Zhevnerchuk (2019), see Theorem 4.1 therein.

\(\square\)

**4.2 An approximation result for convolutional neural networks**

In this subsection we describe in Lemma 5 below a connection between fully connected neural networks and convolutional neural networks, which will enable us to derive in the proof of Theorem 4 an approximation result for the hierarchical max-pooling models by the convolutional neural networks. Before we do this we present a bound on the error we make in case that we replace the functions \(g_{k,s}\) in a hierarchical model by some approximations of them.

**Lemma 4** Let \(d_1, d_2, l \in \mathbb{N}\) with \(2^l \leq \min\{d_1, d_2\} + l\) and set \(I = \{0, 1, \ldots, 2^l - 1\} \times \{0, 1, \ldots, 2^l - 1\}\). Define \(m\) and \(\bar{m}\) by

\[m(x) = \max_{(i,j) \in \mathbb{Z}^2 : (i,j) + I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} f(x(i,j) + I)\]
and
\[
m(x) = \max_{(i,j) \in \mathbb{Z}^2 : (i,j)+I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} \tilde{f}(x(i,j)+I),
\]
where \( f \) and \( \tilde{f} \) satisfy
\[
f = f_{l,1} \quad \text{and} \quad \tilde{f} = \tilde{f}_{l,1}
\]
for some \( f_{k,s}, \tilde{f}_{l,1} : \mathbb{R}^{\{1, \ldots, 2^k\} \times \{1, \ldots, 2^k\}} \to \mathbb{R} \) recursively defined by
\[
f_{k,s}(x) = g_{k,s}(f_{k-1, (s-1)+1}(x_{\{1, \ldots, 2^k-1\} \times \{1, \ldots, 2^k-1\}}),
\]
\[
f_{k-1, (s-1)+1}(x_{\{2^k-1+1, \ldots, 2^k\} \times \{1, \ldots, 2^k-1\}}),
\]
\[
f_{k-1, (s-1)+3}(x_{\{1, \ldots, 2^k-1\} \times \{2^k-1+1, \ldots, 2^k\}}),
\]
\[
f_{k-1, (s-1)+4}(x_{\{2^k-1+1, \ldots, 2^k\} \times \{2^k-1+1, \ldots, 2^k\}})
\]
and
\[
\tilde{f}_{k,s}(x) = \tilde{g}_{k,s}(\tilde{f}_{k-1, (s-1)+1}(x_{\{1, \ldots, 2^k-1\} \times \{1, \ldots, 2^k-1\}}),
\]
\[
\tilde{f}_{k-1, (s-1)+1}(x_{\{2^k-1+1, \ldots, 2^k\} \times \{1, \ldots, 2^k-1\}}),
\]
\[
\tilde{f}_{k-1, (s-1)+3}(x_{\{1, \ldots, 2^k-1\} \times \{2^k-1+1, \ldots, 2^k\}}),
\]
\[
\tilde{f}_{k-1, (s-1)+4}(x_{\{2^k-1+1, \ldots, 2^k\} \times \{2^k-1+1, \ldots, 2^k\}})
\]
for \( k = 2, \ldots, l, s = 1, \ldots, 4^{l-k} \), and
\[
f_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = g_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})
\]
and
\[
\tilde{f}_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = \tilde{g}_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})
\]
for \( s = 1, \ldots, 4^{l-1} \). Assume that all functions \( g_{k,s} : \mathbb{R}^4 \to [0, 1] \) are Lipschitz continuous regarding the Euclidean distance with Lipschitz constant \( C > \frac{1}{2} \). Then for any \( x \in [0, 1]^{\{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} \) it holds:
\[
|m(x) - \tilde{m}(x)| \leq \frac{(2 \cdot C)^l - 1}{2 \cdot C - 1} \cdot \max_{i \in \{1, \ldots, l\}, s \in \{1, \ldots, 4^{l-i}\}} \|g_{i,s} - \tilde{g}_{i,s}\|_{[0,1]^4, \infty}.
\]

**Proof.** If \( a_1, b_1, \ldots, a_n, b_n \in \mathbb{R} \), then
\[
|\max_{i=1, \ldots, n} a_i - \max_{i=1, \ldots, n} b_i| \leq \max_{i=1, \ldots, n} |a_i - b_i|.
\]
Indeed, in case \( a_1 = \max_{i=1, \ldots, n} a_i \geq \max_{i=1, \ldots, n} b_i \) (which we can assume w.l.o.g.) we have
\[
|\max_{i=1, \ldots, n} a_i - \max_{i=1, \ldots, n} b_i| = a_1 - \max_{i=1, \ldots, n} b_i \leq a_1 - b_1 \leq \max_{i=1, \ldots, n} |a_i - b_i|.
\]
Consequently it suffices to show
\[
\max_{(i,j) \in \mathbb{Z}^2 : (i,j)+I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} |f(x(i,j)+I) - \tilde{f}(x(i,j)+I)|
\]
\[ \leq \frac{(2 \cdot C)^l - 1}{2 \cdot C - 1} \cdot \max_{i \in \{1, \ldots, l\}, s \in \{1, \ldots, 4^l - 1\}} \| g_{i,s} - \bar{g}_{i,s}\|_{[0,1)^4,\infty}. \]

This in turn follows from
\[ |f_{k,s}(x) - \bar{f}_{k,s}(x)| \leq \frac{(2 \cdot C)^k - 1}{2 \cdot C - 1} \cdot \max_{i \in \{1, \ldots, k\}, s \in \{1, \ldots, 4^k - 1\}} \| g_{i,s} - \bar{g}_{i,s}\|_{[0,1)^4,\infty} \quad (5) \]

for all \( k \in \{1, \ldots, l\} \), all \( s \in \{1, \ldots, 4^{l-k}\} \) and all \( x \in [0,1)^{(1, \ldots, 2^l)} \times (1, \ldots, 2^l) \), which we show in the sequel by induction on \( k \).

For \( k = 1 \) and \( s \in \{1, \ldots, 4^{l-1}\} \) we have
\[
|f_{1,s}(x) - \bar{f}_{1,s}(x)| = |g_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) - \bar{g}_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})| \leq \| g_{1,s} - \bar{g}_{1,s}\|_{[0,1)^4,\infty}.
\]

Assume now that (5) holds for some \( k \in \{1, \ldots, l-1\} \). Then the triangle inequality and the Lipschitz assumption on \( g \) imply
\[
|f_{k+1,s}(x) - \bar{f}_{k+1,s}(x)| \leq |g_{k+1,s} (f_{k,4}(s-1)+1(x_{1,1}, \ldots, 2^{k-1}) \times (1, \ldots, 2^{k-1})) - f_{k,4}(s-1)+1(x_{1,1}, \ldots, 2^{k-1}) \times (1, \ldots, 2^{k-1})| + \| g_{k+1,s} - f_{k+1,s}\|_{[0,1)^4,\infty} \leq C \cdot \left( |f_{k,4}(s-1)+1(x_{1,1}, \ldots, 2^{k-1}) \times (1, \ldots, 2^{k-1})| - f_{k,4}(s-1)+1(x_{1,1}, \ldots, 2^{k-1}) \times (1, \ldots, 2^{k-1})| \right)^2.
\]

\[ \leq (2 \cdot C) \cdot \left( \frac{2 \cdot C^k - 1}{2 \cdot C - 1} \right) \cdot \max_{i \in \{1, \ldots, k\}, s \in \{1, \ldots, 4^{k-1}\}} \| g_{i,s} - \bar{g}_{i,s}\|_{[0,1)^4,\infty} \]

\[ \leq \frac{(2 \cdot C)^{k+1} - 1}{2 \cdot C - 1} \cdot \max_{i \in \{1, \ldots, k+1\}, s \in \{1, \ldots, 4^{k-1}\}} \| g_{i,s} - \bar{g}_{i,s}\|_{[0,1)^4,\infty} \]

\[ \square \]
Lemma 5 Let \( d_1, d_2, l, L_{\text{net}}, r_{\text{net}} \in \mathbb{N} \) and for \( k \in \{1, \ldots, l\} \) and \( s \in \{1, \ldots, 4^{l-1}\} \) let

\[ g_{\text{net}, k, s} : \mathbb{R}^4 \to \mathbb{R} \]

be defined by a feedforward neural network with \( L_{\text{net}} \) hidden layers and \( r_{\text{net}} \) neurons per hidden layer and ReLU activation function. Set \( I = \{0, \ldots, 2^l - 1\} \times \{0, \ldots, 2^l - 1\} \) and define \( \bar{m} : [0, 1]^{\{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} \to \mathbb{R} \) by

\[ \bar{m}(x) = \max_{(i,j) \in \mathbb{Z}^2 : (i,j) + I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}} \bar{f} \left( x_{(i,j)+I} \right), \]

where \( \bar{f} \) satisfies

\[ \bar{f} = \bar{f}_{1,1} \]

for some \( \bar{f}_{k,s} : [0, 1]^{\{1, \ldots, 2^k\} \times \{1, \ldots, 2^k\}} \to \mathbb{R} \) recursively defined by

\[ \bar{f}_{k,s}(x) = \bar{g}_{\text{net}, k, s} \left( \bar{f}_{k-1,4\cdot(s-1)+1}(x_{\{1, \ldots, 2^k-1\} \times \{1, \ldots, 2^k-1\}}) \right), \]

\[ \bar{f}_{k-1,4\cdot(s-1)+2}(x_{\{2^k-1,1, \ldots, 2^k\} \times \{1, \ldots, 2^k-1\}}) \]

\[ \bar{f}_{k-1,4\cdot(s-1)+3}(x_{\{1, \ldots, 2^k-1\} \times \{2^k-1,1, \ldots, 2^k\}}) \]

\[ \bar{f}_{k-1,4\cdot(s-1)+4}(x_{\{2^k-1,1, \ldots, 2^k\} \times \{2^k-1,1, \ldots, 2^k\}}) \]

for \( k = 2, \ldots, l, s = 1, \ldots, 4^{l-k}, \) and

\[ \bar{f}_{1,s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = \bar{g}_{\text{net}, 1, s}(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) \]

for \( s = 1, \ldots, 4^{l-1}. \) Set

\[ l_{\text{net}} = \frac{4^l - 1}{3} \cdot L_{\text{net}}, \]

\[ k_s = \frac{2 \cdot 4^l + 4}{3} + r_{\text{net}} \quad (s = 1, \ldots, l_{\text{net}}) \]

and

\[ M_s = 2^t \quad \text{for} \ 4^{t-1} \cdot L_{\text{net}} + \cdots + 4^{t-1} \cdot L_{\text{net}} < s \leq 4^{t-1} \cdot L_{\text{net}} + \cdots + 4^{l-t} \cdot L_{\text{net}} \]

\((t \in \{1, \ldots, l\}). \) Then there exists some \( m_{\text{net}} \in \mathcal{F}(l_{\text{net}}, k, M) \) such that

\[ \bar{m}(x) = m_{\text{net}}(x) \]

holds for all \( x \in [0, 1]^{\{1, \ldots, d_1\} \times \{1, \ldots, d_2\}}. \)

In order to prove Lemma 5 we will use the following auxiliary result.

Lemma 6 Let \( d_1, d_2, r_0, t, L_{\text{net}}, r_{\text{net}} \in \mathbb{N} \) and let \( \sigma(x) = \max\{x, 0\} \) be the ReLU activation function. Define

\[ o^{(0)}_{(i,j),1} = x_{i,j} \quad \text{for} \ i \in \{1, \ldots, d_1\} \text{ and} \ j \in \{1, \ldots, d_2\} \]
We define the weights (10) by using the weights of such that
\[
\begin{align*}
\sigma \left( \sum_{s_t=1}^{k_r-1} \sum_{s_1=1}^{t_1,t_2 \in \{1, \ldots, M_t\}} w_{t_1,t_2,s_1,s_2}^{(r)} \cdot o_{(i_2+t_1-1,j_2+t_2-1),s_1}^{(r-1)} + w_{s_2}^{(r)} \right)
\end{align*}
\]
for \( r = 1, 2, \ldots, r_0 + \text{L}_{\text{net}} + 1 \), \( i_2 \in \{ 1, \ldots, d_1 - \sum_{k=1}^{r} M_k + r \} \), \( j_2 \in \{ 1, \ldots, d_2 - \sum_{k=1}^{r} M_k + r \} \), \( s_t \in \{ 1, \ldots, t + \text{r}_{\text{net}} \} \) and \( s_2 \in \{ 1, \ldots, t + \text{r}_{\text{net}} \} \). Let \( g_{\text{net}} : \mathbb{R}^4 \to \mathbb{R} \) be a standard feedforward neural network with \( \text{L}_{\text{net}} \) hidden layers and \( \text{r}_{\text{net}} \) neurons per hidden layer. Set \( I = \{ 0, \ldots, 2^{M-1} \} \times \{ 0, \ldots, 2^{M-1} \} \), let \( f_1, \ldots, f_4 : [0, 1]^{\{1, \ldots, 2^M\} \times \{1, \ldots, 2^M\}} \to \mathbb{R} \) be functions and let \( s_{2,t} \in \{ 1, \ldots, t + \text{r}_{\text{net}} \} \) \( (i = 1, \ldots, 10) \). Assume that for all \( (i_2, j_2) \in \mathbb{Z}^2 \) with \( (i_2, j_2) + I \subseteq \{1, \ldots, d_1 \} \times \{1, \ldots, d_2 \} \) the following four assumptions hold:
\[
\begin{align*}
o_{(i_2,j_2),s_2,1}^{(r_0)} + o_{(i_2,j_2),s_2,2}^{(r_0)} &= f_1(x_{(i_2,j_2)} + I), \\
o_{(i_2+2^{M-1},j_2),s_2,3}^{(r_0)} + o_{(i_2+2^{M-1},j_2),s_2,4}^{(r_0)} &= f_2(x_{(i_2,j_2)} + I), \\
o_{(i_2,j_2+2^{M-1}),s_2,5}^{(r_0)} + o_{(i_2,j_2+2^{M-1}),s_2,6}^{(r_0)} &= f_3(x_{(i_2,j_2)} + I), \\
o_{(i_2+2^{M-1},j_2+2^{M-1}),s_2,7}^{(r_0)} + o_{(i_2+2^{M-1},j_2+2^{M-1}),s_2,8}^{(r_0)} &= f_4(x_{(i_2,j_2)} + I).
\end{align*}
\]
Then there exist weights
\[
\begin{align*}
w_{t_1,t_2,s_1,s_2}^{(r)} \quad (r \in \{ r_0 + 1, \ldots, r_0 + \text{L}_{\text{net}} + 1 \})
\end{align*}
\]
satisfying
\[
\begin{align*}
w_{t_1,t_2,s_1,s_2}^{(r)} = 0 \quad \text{if} \ t_1 \notin \{1, 2^{M} \} \text{ or } t_2 \notin \{1, 2^{M} \},
\end{align*}
\]
such that
\[
\begin{align*}
o_{(i_2,j_2),s_2,9}^{(r_0 + \text{L}_{\text{net}} + 1)} + o_{(i_2,j_2),s_2,10}^{(r_0 + \text{L}_{\text{net}} + 1)}
= g_{\text{net}}(f_1(x_{(i_2,j_2)} + I), f_2(x_{(i_2,j_2)} + I), f_3(x_{(i_2,j_2)} + I), f_4(x_{(i_2,j_2)} + I))
\end{align*}
\]
holds for all \( (i_2, j_2) \in \mathbb{Z}^2 \) with \( (i_2, j_2) + I \subseteq \{1, \ldots, d_1 \} \times \{1, \ldots, d_2 \} \).

**Proof.** We define the weights (10) by using the weights of \( g_{\text{net}} \). Here we assume that \( g_{\text{net}} \) is given by
\[
\begin{align*}
g_{\text{net}}(x) = \sum_{i=1}^{\text{L}_{\text{net}}} w_{1,i}^{(1,3)} f_i^{(1,3)}(x) + w_{1,0}^{(1,3)}
\end{align*}
\]
for \( f_i^{(L)} \)'s recursively defined by
\[
\begin{align*}
f_i^{(s)}(x) = \sigma \left( \sum_{j=1}^{\text{L}_{\text{net}}} w_{i,j}^{(s-1)} f_j^{(s-1)}(x) + w_{i,0}^{(s-1)} \right)
\end{align*}
\]
for \( i \in \{1, \ldots, r_{\text{net}}\} \), \( s \in \{2, \ldots, L_{\text{net}}\} \), and
\[
f_i^{(1)}(x) = \sigma \left( \sum_{j=1}^{4} w_{i,j}^{(0)} x(j) + w_{i,0}^{(0)} \right) \quad (i \in \{1, \ldots, r_{\text{net}}\}).
\]

We modify the above weights \( w_{i,j}^{(s)} \) of \( g_{\text{net}} \) such that the four inputs of \( g_{\text{net}} \) are replaced by \( (6) \)–\( (9) \). This means that at layer \( r_0 + 1 \) the only nonzero weights in the filter are

\[
\begin{align*}
& w_{1,1,s_1,s_2}^{(r_0+1)}, w_{1,1,s_2,2,s_2}^{(r_0+1)}, w_{1,2^M,2s_3,s_2}^{(r_0+1)}, w_{1,2^M,2s_4,s_2}^{(r_0+1)}, \\
& w_{2^M,1,s_2,5,s_2}^{(r_0+1)}, w_{2^M,1,2s_6,s_2}^{(r_0+1)}, w_{2^M,2^M,2s_7,s_2}^{(r_0+1)}, w_{2^M,2^M,2s_7,s_8}^{(r_0+1)} \quad (s_2 \in \{t+1, \ldots, t + r_{\text{net}}\}).
\end{align*}
\]

And for \( r \in \{r_0 + 2, \ldots, r_0 + L_{\text{net}} + 1\} \) the only nonzero weights in the filter are

\[
w_{(1,1),s_1,s_2}^{(r)} \quad (s_1, s_2 \in \{t + 1, \ldots, t + r_{\text{net}}\}).
\]

Furthermore we define the weights of the filter in layer \( r_0 + L_{\text{net}} + 1 \) such that the output of \( g_{\text{net}} \) multiplied by 1 and by \(-1\) is used as input for \( o_{(i_2,j_2),s_2,9}^{(r_0+L_{\text{net}}+1)} \) and \( o_{(i_2,j_2),s_2,10}^{(r_0+L_{\text{net}}+1)} \) : resp.

The construction of this convolutional neural network together with the assumptions \( (6) \)–\( (9) \) implies that the corresponding convolutional neural networks produce simultaneously for any \((i, j) \in \mathbb{Z}^2 \) with \((i, j) + I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\} \) the following outputs:

\[
\begin{align*}
o_{(i_2,j_2),s_2,9}^{(r_0+L_{\text{net}}+1)} &= \sigma \left( \sum_{i=1}^{r_{\text{net}}} w_{(1,1),i}^{(L_{\text{net}})} \cdot o_{(i_2,j_2),s_2,11}^{(r_0+L_{\text{net}})} + w_{1,0}^{(L_{\text{net}})} \right), \\
o_{(i_2,j_2),s_2,10}^{(r_0+L_{\text{net}}+1)} &= \sigma \left( - \sum_{i=1}^{r_{\text{net}}} w_{(1,1),i}^{(L_{\text{net}})} \cdot o_{(i_2,j_2),s_2,11}^{(r_0+L_{\text{net}})} - w_{1,0}^{(L_{\text{net}})} \right), \\
o_{(i_2,j_2),r_0+1+i}^{(r)} &= \sigma \left( \sum_{j=1}^{r_{\text{net}}} w_{i,j}^{(r-r_0-1)} \cdot o_{(i_2,j_2),t+j}^{(r-1)} + w_{i,0}^{(r-r_0-1)} \right)
\end{align*}
\]

for \( r \in \{r_0 + 2, \ldots, r_0 + L_{\text{net}} + 1\} \), and

\[
o_{(i_2,j_2),r_0+1+i}^{(r_0+1)} = \sigma \left( \sum_{j=1}^{4} w_{i,j}^{(0)} f_1(x_{(i_2,j_2)+I}) + w_{i,0}^{(0)} \right).
\]

The definition of \( g_{\text{net}} \) implies

\[
\sum_{i=1}^{r_{\text{net}}} w_{(1,1),i}^{(L_{\text{net}})} \cdot o_{(i_2,j_2),s_2,11}^{(r_0+L_{\text{net}})} + w_{1,0}^{(L_{\text{net}})}
\]

\[
= g_{\text{net}} \left( f_1(x_{(i_2,j_2)+I}), f_2(x_{(i_2,j_2)+I}), f_3(x_{(i_2,j_2)+I}), f_4(x_{(i_2,j_2)+I}) \right).
\]
for all \((i_2, j_2) \in \mathbb{Z}^2\) with \((i_2, j_2) + I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}\), from which we can conclude that

\[
\begin{align*}
& o^{(r_0 + L_{\text{net}} + 1)}(i_2, j_2) + o^{(r_0 + L_{\text{net}} + 1)}(i_2, j_2) \\
& = \max \{ g_{\text{net}}(f_1(x_{(i_2,j_2)+1}), f_2(x_{(i_2,j_2)+1}), f_3(x_{(i_2,j_2)+1}), f_4(x_{(i_2,j_2)+1})), 0 \} \\
& + \max \{-g_{\text{net}}(f_1(x_{(i_2,j_2)+1}), f_2(x_{(i_2,j_2)+1}), f_3(x_{(i_2,j_2)+1}), f_4(x_{(i_2,j_2)+1})), 0 \} \\
& = g_{\text{net}}(f_1(x_{(i_2,j_2)+1}), f_2(x_{(i_2,j_2)+1}), f_3(x_{(i_2,j_2)+1}), f_4(x_{(i_2,j_2)+1}))
\end{align*}
\]

holds for all \((i_2, j_2) \in \mathbb{Z}^2\) with \((i_2, j_2) + I \subseteq \{1, \ldots, d_1\} \times \{1, \ldots, d_2\}\). \hfill \Box

**Proof of Lemma 5.** In the proof we will use the network \(f_{id}: \mathbb{R} \rightarrow \mathbb{R}\) defined by

\[
f_{id}(x) = \sigma(x) + \sigma(-x) = \max\{x, 0\} + \max\{-x, 0\} = x,
\]

which enables us to save a value computed in layer \(k\) by a sum of the outputs of two neurons in layer \(k + 1\).

The idea of our proof is to choose the filter (and the bias weights) such that our convolutional neural network saves in channels corresponding to position \((i, j)\) the values of \(x_{i,j}, f_{1,s}(x_{(i,j)+1})\) \((s = 1, \ldots, 4^{l-1})\), \(f_{2,s}(x_{(i,j)+1})\) \((s = 1, \ldots, 4^{l-2})\), \ldots, \(f_{4,s}(x_{(i,j)+1})\) \((s = 1)\). The values will be saved by using the above idea of saving a value in two neurons by using the input weights of \(f_{id}\), and once they have been computed they will be propagated to the next level using the output weights of \(f_{id}\). Here we will save the value again in two neurons using the input weights of \(f_{id}\), and in order to do this within one layer we merge the input weights of \(f_{id}\) of the next layer with the output weights of the previous layer. To do this we need 2 neurons for each of the above values, so altogether

\[
2 \cdot (1 + 4^{l-1} + 4^{l-2} + \cdots + 4^0) = 2 \cdot \left(1 + \frac{4^l - 1}{4 - 1}\right) = 2 \cdot \frac{4^l + 4}{3}
\]

neurons. Furthermore, we will need \(r_{\text{net}}\) additional neurons to compute the networks \(g_{\text{net},k,s}\). So altogether we need

\[
k_s = \frac{2 \cdot 4^l + 4}{3} + r_{\text{net}}
\]

many channels in each convolutional layer \(s\).

In the first convolutional layer we copy \(x_{i,j}\) in the first two channels, and we propagate these values in the successive layers using the weights of \(f_{id}\). So after the first layer we have available the input in the first two channels in all convolutional layers.

Set

\[
M_s = 2^{s \in \{1, 2, \ldots, 4^{l-1} \cdot L_{\text{net}}\}}.
\]

Starting already in parallel in the first layer, we compute successively in layers

\[
1, 2, \ldots, 4^{l-1} \cdot L_{\text{net}}
\]

17
in the channels
\[ \frac{2 \cdot 4^l + 4}{3} + 1, \frac{2 \cdot 4^l + 4}{3} + 2, \ldots, \frac{2 \cdot 4^l + 4}{3} + r_{net} \]
the networks \( g_{net,1,1}, g_{net,1,2}, \ldots \) by applying Lemma \( \Box \). Here the computation of \( g_{net,1,s} \) takes place in layers \( (s - 1) \cdot L_{net} + 1, \ldots, s \cdot L_{net} \). As input it uses the first two channels for \( s > 1 \), and in case \( s = 1 \) it selects its input from the input of the convolutional network. The computed function value is then saved in the two channels which we use to save the value of \( g_{net,1,s} \), i.e., the computed function value is saved in the channels \( 2 + 2s - 1 \) and \( 2 + 2s \). Here we propagate again the value of these neurons successively to the next layer. So after layer \( 4^{l-1} \cdot L_{net} \) we have available the values of all \( f_{1,s} \) in the channels \( 2 + 1, \ldots, 2 + 2 \cdot 4^{l-1} \).

We next use these values to compute analogously all values of \( f_{2,s} \) \( (s = 1, \ldots, 4^{l-2}) \) in the layers
\[ 4^{l-1} \cdot L_{net} + 1, 4^{l-1} \cdot L_{net} + 2, \ldots, 4^{l-1} \cdot L_{net} + 4^{l-2} \cdot L_{net} \]
using
\[ M_s = 2^2 \quad (s \in \{ 4^{l-1} \cdot L_{net} + 1, 4^{l-1} \cdot L_{net} + 2, \ldots, 4^{l-1} \cdot L_{net} + 4^{l-2} \cdot L_{net} \}). \]
So after layer \( 4^{l-1} \cdot L_{net} + 4^{l-2} \cdot L_{net} \) we have available the values of all \( f_{2,s} \) in the channels \( 2 + 2 \cdot 4^{l-1} + 1, \ldots, 2 + 2 \cdot 4^{l-1} + 2 \cdot 4^{l-2} \).

Now we proceed in the same way, where we double the value of \( M_s \) each time we have finished computing the values of \( g_{net,k,s} \) for some value of \( k \). At layer
\[ l_{net} = 4^{l-1} \cdot L_{net} + 4^{l-2} \cdot L_{net} + \cdots + 4^0 \cdot L_{net} = \frac{4^l - 1}{3} \cdot L_{net} \]
we have computed the value of all \( f_{1,1}(x_{(i,j)+I}) \). We use the output weights to compute all these values (by summing up each time the outputs of the two corresponding neurons) and carry out max-pooling operation. This finishes the computation of the network.

By using an induction on \( k \) it is easy to see that Lemma \( \Box \) implies that the above network satisfies for any \( k \in \{ 1, \ldots, l \} \) and any \( (i, j) \in \mathbb{Z}_2 \) with \( (i, j) + I \subseteq \{ 1, \ldots, d_1 \} \times \{ 1, \ldots, d_2 \} \)
\[ \bar{f}_{k,s}(x_{(i,j)+I}) = o^{(4^{l-1})-L_{net}+\cdots+4^{(l-k)}-L_{net}}_{(i,j),2+2^{l-1}+\cdots+2^{l-k}+1+2s-1} + o^{(4^{l-1})-L_{net}+\cdots+4^{(l-k)}-L_{net}}_{(i,j),2+2^{l-1}+\cdots+2^{l-k}+1+2s} \]
This implies that the output of our network is given by
\[
\max \left\{ \begin{array}{c}
o^{(l_{net})}_{(i,j),2+2^{l-1}+\cdots+2^{l-1}+1+2} + o^{(l_{net})}_{(i,j),2+2^{l-1}+\cdots+2^{l-1}+2} : \\
(i,j) \in \mathbb{Z}_2, (i,j) + I \subseteq \{ 1, \ldots, d_1 \} \times \{ 1, \ldots, d_2 \}
\end{array} \right\} 
\]
\[
= \max \left\{ \bar{f}(x_{(i,j)+I}) : (i,j) \in \mathbb{Z}_2, (i,j) + I \subseteq \{ 1, \ldots, d_1 \} \times \{ 1, \ldots, d_2 \} \right\} 
= \bar{m}(x).
\]
\[ \square \]
4.3 A bound on the covering number

The purpose of the subsection is to show the following bound on the covering number of $\mathcal{F}(L, k, M)$.

**Lemma 7** Let $\sigma(x) = \max\{x, 0\}$ be the ReLU activation function, define $\mathcal{F}(L, k, M)$ as in Section 2 and set

$$k_{\text{max}} = \max\{k_1, \ldots, k_L\} \quad \text{and} \quad M_{\text{max}} = \max\{M_1, \ldots, M_L\}.$$ 

Assume $c_4 \cdot \log n \geq 2$. Then we have for any $\epsilon \in (0, 1]$:

$$\sup_{x \in [\mathbb{R}^{(1, \ldots, d_1) \times (1, \ldots, d_2)}]^n} \log \left( \mathcal{N}_1 \left( \epsilon, \mathcal{F}(l, k, M), x_1^n \right) \right) \leq c_7 \cdot L^2 \cdot (\log L) \cdot \log \left( \frac{\log n}{\epsilon} \right)$$

for some constant $c_7 > 0$ which depends on $d_1, d_2, k_{\text{max}}$ and $M_{\text{max}}$.

**Proof.** Let $\mathcal{F}$ be the set of all neural networks $f : [0, 1]^{(1, \ldots, d_1) \times (1, \ldots, d_2)} \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{s_2=1}^{k_1} w_{(1,1), s_2} \cdot o^{(L)}_{(1,2), s_2}$$

where $o^{(L)}_{(1,1), s_2}$ is recursively defined by

$$o^{(r)}_{(1,2), s_2} = \sigma \left( \sum_{s_1=1}^{k_{r-1}} \sum_{t_1, t_2 \in \{1, \ldots, M_r\}} w^{(r)}_{t_1, t_2, s_1, s_2} \cdot o^{(r-1)}_{(t_2, j_2), s_2} + w^{(r)}_{s_2} \right)$$

for $i_2 \in \{1, \ldots, d_1 - \sum_{k=1}^{r} M_k + r\}$, $j_2 \in \{1, \ldots, d_2 - \sum_{k=1}^{r} M_k + r\}$ and $r \in \{1, \ldots, L\}$, and by

$$o^{(0)}_{(1, j), 1} = x_{i, j} \quad \text{for} \quad i \in \{1, \ldots, d_1\} \quad \text{and} \quad j \in \{1, \ldots, d_2\}.$$ 

Then for any $f \in T_{c_4 \cdot \log n} \mathcal{F}(k, M)$ there exist $f_1, \ldots, f_{d_1 \cdot d_2} \in T_{c_4 \cdot \log n} \mathcal{F}$ such that

$$f(x) = \max\{f_1(x), f_2(x), \ldots, f_{d_1 \cdot d_2}(x)\}$$

holds for all $x \in [0, 1]^{(1, \ldots, d_1) \times (1, \ldots, d_2)}$. Using (10) this implies

$$\mathcal{N}_1 \left( \epsilon, T_{c_4 \cdot \log n} \mathcal{F}(l, k, M), x_1^n \right) \leq (\mathcal{N}_1 \left( \epsilon, T_{c_4 \cdot \log n} \mathcal{F}, x_1^n \right))^{d_1 \cdot d_2},$$

hence it suffices to show

$$\log \left( \mathcal{N}_1 \left( \epsilon, T_{c_4 \cdot \log n} \mathcal{F}, x_1^n \right) \right) \leq c_8 \cdot L^2 \cdot (\log L) \cdot \log \left( \frac{\log n}{\epsilon} \right),$$

(11)
The set $\mathcal{F}$ is a set of feedforward neural networks with $L$ hidden layers, $k_{\text{max}} \cdot d_1 \cdot d_2$ neurons per hidden layer, which is parameterized by the following parameters

$$w = \left( w^{(r)}_{i,j,s_1,s_2} \right)_{1 \leq i,j \leq M_r, s_1 \in \{1,\ldots,k_{r-1}\}, s_2 \in \{1,\ldots,k_r\}, r \in \{1,\ldots,L\},}$$

and

$$w_{\text{bias}} = \left( w^{(r)}_{s_2} \right)_{s_2 \in \{1,\ldots,k_r\}, r \in \{1,\ldots,L\}}.$$

So the total number of free parameters in this set of neural networks is bounded from above by

$$3 \cdot M_{\text{max}}^2 \cdot k_{\text{max}}^2 \cdot L.$$

Here each value of these parameters is used as a value of several weights in the neural networks in $\mathcal{F}$.

Let $T_{c_4 \cdot \log n} \mathcal{F}^+ = \{ (x, y) \in [0,1]^{\{1,\ldots,d_1\} \times \{1,\ldots,d_2\}} \times \mathbb{R} : y \leq f(x) \} : f \in T_{c_4 \cdot \log n} \mathcal{F}$

be the set of all subgraphs of $T_{c_4 \cdot \log n} \mathcal{F}$, and denote its VC dimension by $V_{T_{c_4 \cdot \log n} \mathcal{F}^+}$ (cf., e.g., Definition 9.6 in Györfi et al. (2002)). Similarly define $\mathcal{F}^+$ and $V_{\mathcal{F}^+}$.

Then Theorem 6 in Bartlett et al. (2019) implies

$$V_{\mathcal{F}^+} \leq c_9 \cdot 3 \cdot M_{\text{max}}^2 \cdot k_{\text{max}}^2 \cdot L \cdot L \cdot \log L \leq c_10 \cdot L^2 \cdot \log L.$$

Using

$$V_{T_{c_4 \cdot \log n} \mathcal{F}^+} \leq V_{\mathcal{F}^+},$$

we can conclude from this together with Lemma 9.2 and Theorem 9.4 in Györfi et al. (2002)

$$N_1(\epsilon, T_{c_4 \cdot \log n} \mathcal{F}, x_1^n) \leq 3 \cdot \left( \frac{4e \cdot c_4 \cdot \log n}{\epsilon} \cdot \log \frac{6e \cdot c_4 \cdot \log n}{\epsilon} \right)^{V_{T_{c_4 \cdot \log n} \mathcal{F}^+}} \leq 3 \cdot \left( \frac{6e \cdot c_4 \cdot \log n}{\epsilon} \right)^{2 \cdot c_{10} \cdot L^2 \cdot \log L}.$$

This completes the proof of (11). \qed

### 4.4 Proof of Theorem 1

W.l.o.g. we assume that $n$ is so large that $c_4 \cdot \log n \geq 2$ holds. Then $z > 1/2$ holds if and only if $T_{c_4 \cdot \log n} \eta_n(x) \geq 1/2$ holds, and consequently we have

$$f_n(x) = \begin{cases} 1, & \text{if } T_{c_4 \cdot \log n} \eta_n(x) \geq 1/2 \\ 0, & \text{elsewhere} \end{cases}$$
Hence Lemma 1 implies that it suffices to show
\[
\mathbb{E} \int |T_{c_4 \log n} \eta_n(x) - \eta(x)|^2 \mathbb{P}_X(dx) \leq c_{11} \cdot (\log n)^4 \cdot n^{-\frac{2p}{p+1}}.
\]
By Lemma 2 we know
\[
\mathbb{E} \int |T_{c_4 \log n} \eta_n(x) - \eta(x)|^2 \mathbb{P}_X(dx) \\
\leq c_{12} \cdot (\log(n))^2 \cdot \sup_{x \in [0,1]} \left( \log \left( N_1 \left( \frac{1}{n \cdot c_4 \log(n)}, T_{c_4 \log(n)} \mathcal{F}(l,k,M) \right) \right) + 1 \right) \\
+ 2 \cdot \inf_{f \in \mathcal{F}(l,k,M)} \int |f(x) - m(x)|^2 \mathbb{P}_X(dx).
\]
Application of Lemma 7 yields
\[
c_{12} \cdot (\log(n))^2 \cdot \sup_{x \in [0,1]} \left( \log \left( N_1 \left( \frac{1}{n \cdot c_4 \log(n)}, T_{c_4 \log(n)} \mathcal{F}(l,k,M) \right) \right) + 1 \right) \\
\leq c_{13} \cdot (\log n)^3 \cdot L^2 \cdot \log L.
\]
Next we derive a bound on the approximation error
\[
\inf_{f \in \mathcal{F}(l,k,M)} \int |f(x) - m(x)|^2 \mathbb{P}_X(dx).
\]
For \( k \in \{1, \ldots, l\} \) and \( s \in \{1, \ldots, 4^l - k\} \) let \( g_{net,k,s} : \mathbb{R}^4 \to \mathbb{R} \) be the neural network from Lemma 3 which satisfies
\[
\|g_{k,s} - g_{net,k,s}\|_{[0,1]^4,\infty} \leq c_{14} \cdot L_n^{-\frac{2p}{p+1}},
\]
and define \( m \in \mathcal{F}(l,k,M) \) as in Lemma 5. Then Lemmata 4 and 5 imply
\[
\inf_{f \in \mathcal{F}(l,k,M)} \int |f(x) - m(x)|^2 \mathbb{P}_X(dx) \leq c_{15} \cdot \max_{k \in \{1, \ldots, l\}, s \in \{1, \ldots, 4^l - k\}} \|g_{k,s} - g_{net,k,s}\|_{[0,1]^4}^2 \\
\leq c_{16} \cdot L_n^{-p}.
\]
Putting in the values of \( L \) and \( L_n \) and summarizing the above results, the proof is complete. \( \square \)

5 Acknowledgment

The authors would like to thank Luc Devroye for a fruitful discussion of the topic of this paper.
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