Research article

On the constant coefficients of a certain recurrence relation: A simple proof

Milica Andelić a, Carlos M. da Fonseca b,c,⁎

a Department of Mathematics, Kuwait University, Safat 13060, Kuwait
b Kuwait College of Science and Technology, Doha District, Safat 13133, Kuwait
c Chair of Computational Mathematics, University of Deusto, 48007 Bilbao, Basque Country, Spain

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A B S T R A C T

In this note, we provide a short proof for the explicit formulas of the coefficients of a particular 3-term recurrence relation derived from a k-periodic recurrence. Any of the recurrences can be naturally interpreted in terms of determinants of Hessenberg matrices families.

1. Introduction

The sequence

0, 1, 1, 3, 5, 13, 23, 59, 105, 269, 479, 1227, 2185, 5597,…

is coined in the On-Line Encyclopedia of Integer Sequences as A005824. It was considered in [1] by Shallit in the context of the worst-case behavior of three iterative algorithms for computing the Jacobi symbol. It can be recursively defined as

\[ f_n = \begin{cases} 
2f_{n-1} + f_{n-2} & \text{if } n \text{ is odd}, \\
 f_{n-1} + 2f_{n-2} & \text{if } n \text{ is even}, 
\end{cases} \]

for \( n ≥ 2 \), with initial conditions \( f_0 = 0 \) and \( f_1 = 1 \). However, in the On-Line Encyclopedia of Integer Sequences, it is stated originally as

\[ f_n = 5f_{n-2} - 2f_{n-4}. \]

Over the past few decades, it has been independently claimed in different areas of mathematics that, for a given \( k \)-periodic term recurrence relation

\[ f_n = \begin{cases} 
a_1f_{n-1} - b_1f_{n-2} & \text{if } n \equiv 1 \pmod{k}, \\
a_2f_{n-1} - b_2f_{n-2} & \text{if } n \equiv 2 \pmod{k}, \\
a_3f_{n-1} - b_3f_{n-2} & \text{if } n \equiv 3 \pmod{k}, \\
\vdots & \\
a_{k-1}f_{n-1} - b_{k-1}f_{n-2} & \text{if } n \equiv k-1 \pmod{k}, \\
a_kf_{n-1} - b_kf_{n-2} & \text{if } n \equiv 0 \pmod{k}, 
\end{cases} \]

(1.1)

with \( n ≥ 2 \) and assuming that \( f_0 = 1 \) and \( f_1 = a_1 \), there exist constants \( a \) and \( b \) such that

\[ f_n = af_{n-k} - bf_{n-2k}. \]

(1.2)

Cooper in [2] finds explicitly these constants for \( k = 2, \ldots, 6 \). In each proof, \( n \pmod{k} \) is considered separately. Motivated by [2], Carson in [3] provides the formula for \( a \) and \( b \), using a recursive process of row elimination. Nevertheless, the particular case when \( a_i = 1 \), for all \( i \), was considered earlier by Lehmer in [4]. This particular case has been rediscovered in [5] using a rather intricate technique.

It is interesting to notice that previously this problem had been independently investigated in orthogonal polynomial theory. Indeed, in [6] this case was analyzed. However, earlier we find this problem in the seminal work by Geronimo and Van Assche [7] or, for \( k = 2 \) and \( a_i \)’s constant, as an exercise in the Chihara’s masterful book [8, p.91]. The papers [9, 10] by Geronimus are also worth mentioning.

The aim of this note is to provide a short proof to (1.2) as an immediate consequence of a less-known result on determinant of certain tridiagonal matrices due to Rózsa [11]. A determinantal formula for the recurrence relation is stated.

2. The main result

The sequences satisfying the recurrence relation (1.1) were considered by Perron in [12]. However, they can be traced back to the foundational work by Stieltjes [13, 14]. On the other hand, the connection between (1.1) and the determinant of tridiagonal k-Toeplitz matrices

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\[
A_n = \begin{pmatrix}
a_1 & 1 & & & \\
b_1 & a_2 & 1 & & \\
& b_2 & a_3 & 1 & \\
& & \ddots & \ddots & \ddots \\
& & & b_{k-1} & a_k & 1
\end{pmatrix}
\]  \quad (2.1)

is quite useful, but not always conveniently explored (see, e.g., [15, 16, 17, 18]).

The determinant of \( A_n \) has been independently rediscovered over the decades in many instances (cf., e.g., [19, 20, 21]). Notwithstanding, the first general formula is due to Rózsza [11] in 1969 and it stands relatively ignored. Namely, setting

\[
\Delta_{i,j} = \det \begin{pmatrix} a_i & 1 & \cdots & \cdots & b_{j-1} \\
& a_i & 1 & \cdots & b_{j-1} \\
& & \ddots & \ddots & \ddots \\
& & & a_i & 1 \\
\end{pmatrix},
\]

for \( 1 \leq i \leq j \leq k \), and

\[
\Delta_{i,j} = \begin{cases} 0 & \text{if } j < i - 1 \\ 1 & \text{if } j = i - 1. \end{cases}
\]

for \( j < i \), then the determinant of \( A_n \) is given by

\[
\det A_n = (\sqrt{b_1 - b_2})^n \Delta_{1,n} U_q(x) + \sqrt{b_1 - b_2} \Delta_{1,n+2} U_{q+2}(x)
\]  \quad (2.2)

with \( n = qk + r \),

\[
x = \frac{\Delta_{1,k} - b_1 \Delta_{2,k-1}}{2 \sqrt{b_1 - b_2}}.
\]

and \( \{U_q(x)\}_{x \geq 0} \) being the Chebyshev polynomials of the second kind. Recall that these polynomials satisfy the three-term recurrence relation

\[
U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad \text{for all } n = 1, 2, \ldots.
\]  \quad (2.3)

with initial conditions \( U_0(x) = 1 \) and \( U_1(x) = 2x \). Now, taking into account that \( n - k = (q - 1)k + r \), \( n - 2k = (q - 2)k + r \), and using (2.2) and (2.3), we get

\[
\det A_n = (\sqrt{b_1 - b_2})^n \Delta_{1,n} U_q(x) + \sqrt{b_1 - b_2} \Delta_{1,n+2} U_{q+2}(x)
\]

while for \( k = 2 \) one obtains (2.4)

\[
f_n = (a_1a_2 - b_1 - b_2)f_{n-2} - b_1b_2 f_{n-4},
\]

for \( k = 3 \), one gets

\[
f_n = (a_1a_2a_3 - a_1b_1 - a_2b_2 - a_3b_1)f_{n-3} - b_1b_2b_3 f_{n-6},
\]

which coincide with the particular formulas one can find in the recent literature.

Moreover, according for instance to [22, 23], one can represent (1.2) in terms of the determinant of a Hessenberg matrix, namely

\[
f_n = \det \begin{pmatrix} f_{1} & \cdots & f_{k} & \cdots & f_{2k} \\
-1 & 0 & \cdots & \cdots & -1 \\
& -1 & 0 & \cdots & -1 \\
& & \ddots & \ddots & \ddots \\
& & & -1 & 0
\end{pmatrix}_{2k},
\]

where all non-mentioned entries should be read as zero. Furthermore, \( a \) and \(-b\) lie in the \( k \)th and \( 2k \)th superdiagonals, respectively.

\section{Conclusion}

Any \( k \)-periodic recurrence relation can be defined as a single recurrence relation. In this note, we provide a short proof for the formulas of the coefficients of such relation, recalling less-known results in the literature.

\section{Declarations}

\section{Author contribution statement}

M. Andelíć: Analyzed and interpreted the data.

C.M. da Fonseca: Analyzed and interpreted the data; wrote the paper.

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