Abstract

In this paper, we present Lyapunov-based robust and adaptive controllers for the finite time stabilization of a perturbed chain of integrators with bounded uncertainties. The proposed controllers can be designed for integrator chains of any arbitrary length. The uncertainty bounds are known in the robust control problem whereas they are unknown in the adaptive control problem. Both controllers are developed from a class of finite time stabilization controllers for pure integrator chains. Lyapunov-based design permits to calculate upper bound on convergence time.

Key words: Finite time Stabilization; Perturbed integrator chain; Robust Control; Adaptive Control; Lyapunov Analysis.

1 Introduction

The problem of finite time stabilization of perturbed integrator chains arises in many practical cases of robust nonlinear control. Usually this uncertainty is bounded by the physical limits of the system, however the bounds may be known or unknown.

For the case of known uncertainty bounds, Levant used homogeneity approach to demonstrate finite time stabilization of integrator systems [1,2]. Laghrouche et al. [3] presented a two part integral sliding mode based control to deal with the finite time stabilization and uncertainty rejection separately. Dinuzzo et al. proposed another method in [4], where finite time stabilization is treated as Robust Fuller’s problem using Higher Order Sliding Mode. Defoort et al. [5] developed a robust MIMO controller, using a constructive algorithm with geometric homogeneity based finite time stabilization.

The problem is more challenging if the uncertainty bounds are unknown. For this, the control design should (a) not require the uncertainty bounds and (b) avoid gain overestimation [6]. Huang et al. [7] used dynamic gain adaptation for first order systems. Their method does not solve the gain overestimation problem because the gains cannot decrease. Plestan et al. [6,8] proposed a sliding mode approach, in which the gains are decreased slowly, after sliding mode is reached. Shtessel et al. [9] also presented a Second Order adaptive gain super-twisting SMC for non-overestimation of the control gains. Glumineau et al. [10] used impulsive sliding mode for adaptive control of a double integrator system.

In this paper, we present Lyapunov-based controllers for the finite time stabilization of arbitrary order perturbed integrator chains with bounded uncertainties. There are two main contributions in this paper. First, a robust controller is developed that stabilizes an integrator chain of arbitrary length, if the bounds of the uncertainty are known. The advantage is that it can be developed from a class of finite time controllers for pure integrator chains. Then the controller is extended to an adaptive controller for the case where the bounds on the uncertainty are unknown. This controller aims to converge the states to a neighborhood of the origin. However, the states may leave the neighborhood within a region around it, which depends upon the unknown bounds. Therefore, we do not solve the problem of reaching in finite time an arbitrary neighborhood of the origin.

The paper is organized as follows: problem formulation is discussed in Section 2, robust and adaptive controllers...
are presented in Sections 3 and 4 respectively and conclusion is given in Section 5.

2 Problem Formulation

Let us consider an uncertain nonlinear system:

\[ \begin{align*}
\dot{z}_i &= z_{i+1}, \quad i = 1, \ldots, r - 1, \\
\dot{z}_r &= \varphi(t) + \gamma(t)u.
\end{align*} \tag{1} \]

where \( z \in \mathbb{R}^r \) is the state vector and \( u \in \mathbb{R} \) is the control input. The functions \( \varphi \) and \( \gamma \) are arbitrary measurable functions that represent bounded uncertainty:

\[ (H1) \quad \varphi(t) \in I_\varphi := [-\bar{\varphi}, \bar{\varphi}], \quad \gamma(t) \in I_\gamma := [\gamma_m, \bar{\gamma}] \],

where \( \bar{\varphi}, \gamma_m, \bar{\gamma} \) are positive constants. In consequence, we are in fact dealing with the differential inclusion

\[ \dot{z}_r \in I_\varphi + uI_\gamma. \tag{2} \]

The control objective is to stabilize System (2) to the origin in finite time. Since these controllers are discontinuous feedback laws \( u = U(z) \), solutions of (2) will fall under differential inclusions and need to be understood in Filippov sense, i.e. the right hand vector set is enlarged at the discontinuity points of (2) to the convex hull of the set of velocity vectors obtained by approaching \( z \) from all the directions in \( \mathbb{R}^r \), while avoiding zero-measure sets [11].

3 Design of robust controller

In this section, we develop a controller for stabilizing System (2), assuming that the bounds on \( \varphi \) and \( \gamma \) are known. This controller is derived from a class of Lyapunov-based controllers that guarantee finite time stabilization of pure integrator chains, and satisfy some additional geometric conditions. This chain is given as

\[ \begin{align*}
\dot{z}_i &= z_{i+1}, \quad i = 1, \ldots, r - 1, \\
\dot{z}_r &= \gamma u(z) + \bar{\varphi}sign(u_0(z)) + \varphi.
\end{align*} \tag{3} \]

Let us recall the theorem:

**Theorem 1** [12] Consider System (3). Suppose there exist a continuous state-feedback control law \( u = u_0(z) \), a positive definite \( C^1 \) function \( V_1 \) defined on a neighborhood \( \bar{U} \subset \mathbb{R}^r \) of the origin and real numbers \( c > 0 \) and \( 0 < \alpha < 1 \), such that the condition \( \dot{V}_1 + cV_1^\alpha(z(t)) \leq 0 \), if \( z(t) \in \bar{U} \) is true for every trajectory \( z \) of System (3). Then all trajectories of System (3) with the feedback \( u_0(z) \) which stay in \( \bar{U} \) converge to zero in finite time. If \( \bar{U} = \mathbb{R}^r \) and \( V_1 \) is radially unbounded, then System (3) with the feedback \( u_0(z) \) is globally finite time stable with respect to the origin.

Based on this theorem, we develop a robust controller for System (2).

**Theorem 2** Consider System (2) subject to Hypothesis (H1). Then the following control law stabilizes System (2) to the origin in finite time:

\[ u = (u_0 + \varphi sign(u_0))/\gamma_m, \tag{4} \]

where \( u_0(z) \) is any state-feedback control law that satisfies the hypotheses of Theorem 1 and obeys the following additional conditions: for every \( z \in \bar{U}, \)

\[ \frac{\partial V_1}{\partial z_r}(z)u_0(z) \leq 0, \quad \text{and} \quad u_0(z) = 0 \Rightarrow \frac{\partial V_1}{\partial z_r}(z) = 0. \tag{5} \]

If \( \bar{U} = \mathbb{R}^r \) and \( V_1 \) is radially unbounded, then System (2) with the feedback \( u(z) \) is globally finite time stable with respect to the origin.

**Proof of Theorem 2.** Under the control law \( u \) defined in (4), System (2) can be rewritten as:

\[ \begin{align*}
\dot{z}_i &= z_{i+1}, \quad i = 1, \ldots, r - 1, \\
\dot{z}_r &= \gamma u_0(z) + \frac{\bar{\varphi}}{\gamma_m}sign(u_0(z)) + \varphi.
\end{align*} \tag{6} \]

The Conditions in (5) mean that \( \partial V_1/\partial z_r sign(u_0) \) defines a continuous and non positive function of the time along trajectories of System (6). The time derivative of the Lyapunov function \( V_1 \) verifying the hypotheses of Theorem 1 along a non trivial trajectory of System (6) inside \( \bar{U} \) is given as

\[ V_1 = \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} (\varphi + \gamma u), \]

\[ \leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} u_0 + \frac{\partial V_1}{\partial z_r} sign(u_0) (\bar{\varphi} - |\varphi|) \]

\[ \leq \sum_{i=1}^{r-1} \frac{\partial V_1}{\partial z_i} z_{i+1} + \frac{\partial V_1}{\partial z_r} u_0 \leq -cV_1^\alpha. \]

This implies that any a non-trivial trajectory \( z \) reaches zero and stays there in finite time. \( \blacksquare \)

It can be verified that the controllers proposed by Hong [13] and Huang [14] satisfy the hypotheses of Theorem 1 and Condition (5). Considering Hong’s controller, we denote \( |a|^\theta := |a|^\theta sign(a) \forall a \in \mathbb{R} \) and \( \theta > 0 \). Let \( k < 0 \) and \( l_1, \ldots, l_r \) positive real numbers. For \( z = (z_1, \ldots, z_r) \), we define for \( i = 0, \ldots, r - 1 \):

\[ p_i = 1 + (i - 1)k, \]

\[ v_0 = 0, \quad v_{i+1} = -l_{i+1}[|z_{i+1}|^{\beta_i} - |v_i|^{\beta_i}](\alpha_i+1/\beta_i), \tag{7} \]

where \( \alpha_i = p_{i+1}/p_i, \) for \( i = 1, \ldots, r, \) and, for \( k < 0 \) sufficiently small,

\[ \beta_0 = p_2, \quad (\beta_i + 1)p_{i+1} = \beta_0 + 1 > 0, \quad i = 1, \ldots, r - 1. \]

Consider the positive definite radially unbounded function \( V_1 : \mathbb{R}^r \rightarrow \mathbb{R}^+ \) given by
Let us now consider that uncertainty bounds \( \gamma_m, \gamma_M \) and \( \bar{\varphi} \) of System (2) are unknown. For any \( a \in \mathbb{R} \), let \( \sigma(a) \) be the standard saturation function defined by \( \sigma(a) = a/\max(1, |a|) \). For \( \varepsilon > 0 \), \( \alpha \in \mathbb{R} \), we define \( \nu_\varepsilon(a) = 0.5 + 0.5\varepsilon \left( (|a| - 0.75\varepsilon) / (0.25\varepsilon) \right) \). The following controller is proposed:

\[
u(t) = \gamma u_0(z) + \bar{\varphi} \text{sign}(u_0(z)),
\]

where \( u_0 \) is a homogeneous controller that satisfies the hypotheses of Theorem 1 and fulfills Condition (5). The adaptive function \( \tilde{\varphi} = \alpha + \delta \nu_\varepsilon(z) \) and \( \bar{\varphi}(t) \) is defined by the ODE \( \dot{\bar{\varphi}}(t) = k_\nu \nu_\varepsilon(V(z)) - (1 - \nu_\varepsilon(V(z))) [\bar{\varphi}]^\alpha \), with the initial condition \( \bar{\varphi}(0) = 0 \). The new terms are defined as \( \kappa, \delta > 0, \eta \in (0, 1) \), \( k > 0 \) and \( V_1 \) is a homogeneous Lyapunov function which also satisfies Theorem 1 and Condition (5). Then the following theorem provides the main result for the adaptive case.

**Theorem 3** Consider System (2) under the feedback control law (11). Then, \( \forall \varepsilon, \exists \Delta, c' > 0 \) and \( 0 < c' < 1 \) such that the following conditions are satisfied for any initial condition \( z_0 \in \bar{U} \).

(i) \( \lim_{t \to \infty} V_1(z(t)) \leq \varepsilon \), \( \limsup_{t \to \infty} V_1(z(t)) \leq \varepsilon \);

(ii) \( \limsup_{t \to \infty} |\bar{\varphi}| \leq 2 \Phi + k (\Delta^{1-\alpha} / (c(1-\alpha))) \),

where \( \Phi := (\tilde{\varphi} + (\kappa_{\gamma_m} - 1)^2 / (4\gamma_m\delta)) / \gamma_m \), \( \Delta := (c' - \alpha' + c'(1 - \alpha')\gamma_m\bar{\Phi}^2 / (2k))^{1-\alpha'} \).

**Proof of Theorem 3:** We first demonstrate that when the system states are in the domain \( V_1 > \varepsilon \), the controller brings them to the domain \( V_1 \leq \varepsilon \) in finite time. Then, it is proved that once \( \varepsilon \) reaches the domain \( V_1 \leq \varepsilon \), it stays in the domain \( V_1 \leq \Delta \) for all consecutive time instances and \( \bar{\varphi} \) is upper-bounded after a sufficiently large time. It can be noted that \( \bar{\varphi} \) is a non-negative function.

We argue by contradiction in order to prove that \( \lim_{t \to \infty} V_1(z(t)) \leq \varepsilon \). Supposing there exists \( t \) such that \( V_1(t) > \varepsilon \) for every \( t \geq \bar{t} \), then according to the dynamics of \( \bar{\varphi} \), we get \( \bar{\varphi} = \kappa \) for \( t \geq \bar{t} \). This implies that for \( t \geq \bar{t} \), \( \varphi \) is increasing and \( \bar{\varphi} > \Phi \). Since we have

\[
V_1 = \frac{\partial V_1}{\partial z_2} + ... + \frac{\partial V_1}{\partial z_r} (\gamma [\gamma u_0 + \bar{\varphi} \text{sign}(u_0)] + \varphi) \\
= \frac{\partial V_1}{\partial z_2} + ... + \frac{\partial V_1}{\partial z_r} u_0 + \frac{\partial V_1}{\partial z_r} \left( -u_0 + \kappa_\gamma u_0 + \gamma \delta |u_0|^2 + \gamma \varphi \text{sign}(u_0) + \varphi \right),
\]

\[
\leq -cV_1^\alpha \left( \frac{\partial V_1}{\partial z_r} (\kappa_{\gamma_m} - 1) |u_0| + \gamma \delta |u_0|^2 + \gamma \varphi \text{sign}(u_0) + \varphi \right),
\]

\[
= -cV_1^\alpha \left( \frac{\partial V_1}{\partial z_r} \left( \gamma \delta |u_0| + \frac{\kappa_{\gamma_m} - 1}{2\gamma_m \delta} \right) + \gamma \varphi \right),
\]

\[
\leq -cV_1^\alpha - \frac{\partial V_1}{\partial z_r} \left( \frac{\gamma \delta |u_0| + \frac{\kappa_{\gamma_m} - 1}{2\gamma_m \delta}}{4\gamma_m \delta} + \gamma \varphi \right),
\]

\[
\leq -cV_1^\alpha - \gamma_m \frac{\partial V_1}{\partial z_r} (\varphi - \Phi) \leq -cV_1^\alpha.
\]
Then $V_1(z)$ converges to zero in finite time, which contradicts the hypothesis. The functions $u_0$ and $V_1$ are homogeneous, which according to [15], means that
\[
\exists \epsilon', \alpha' > 0 : |\partial V_1 / \partial z_r| \leq \epsilon' V_1^{\alpha'},
\]
where $\epsilon' = \max_{\{z:V_1(z) = 1\}} |\partial V_1 / \partial z_r|$, $\alpha' = \kappa_2 / \kappa_1$. The terms $\kappa_2$ and $\kappa_1$ are the respective degrees of homogeneity of $\partial V_1 / \partial z_r$ and $V_1$. We suppose now that $V_1 < \epsilon$. Considering (13), let us estimate the overshoot in the worst case condition with respect to uncertainty. For $V_1(z(0)) = \epsilon$ and $\dot{\varphi}(0) = 0$, we get
\[
\dot{V}_1 \leq -c V_1^{\alpha} - \gamma_0 c V_1^{\alpha'} (\dot{\varphi} - \bar{\Phi}), \quad \dot{\varphi} = k.
\]
The overshoot $\Delta$ of $V_1$ holds for $\dot{V}_1 = 0$ at $t = T_M$. We get $\dot{\varphi}(T_M) = \bar{\Phi} - \left(\left(c \Delta^{(1-\alpha')}/(c \alpha')\right) \leq \bar{\Phi}, \text{ and then}
T_M \leq \bar{\Phi}/k.\right)$ An upper bound of $\Delta$ can be estimated as
\[
\Delta = \left(\epsilon^{1-\alpha'} + (c' (1 - \alpha') \gamma_m \bar{\Phi}^2/(2k)) \right)^{1/\alpha}.\]
We now estimate an upper bound of $\lim_{t \to \infty} \bar{\Phi}$. Consider the case $V_1(z(0)) = \epsilon$ with $V_1(z(0)) \geq 0$, in this case we have $\dot{\varphi}(0) < \Phi$. For $t = T_M$, i.e., $\dot{V}_1 = 0$, we get $\dot{\varphi}(T_M) \leq \bar{\Phi} + \dot{\varphi}(0) < \bar{\Phi}. \dot{\varphi}$ will increase until time $T_f$ where $\dot{\bar{\Phi}}(T_f) = 0$ and $V_1(z(T_f)) \geq 0$. The worst case is calculated with respect to the boundary of $\bar{\Phi}$, using $\dot{V}_1 \leq -c V_1^{\alpha}$ and $\dot{\varphi} = k$. Here $T_f$ corresponds to $V_1(z(T_f)) = 0$, i.e., $T_f - T_m = \left(\Delta^{(1-\alpha)}\right) / (c(1 - \alpha))$, which implies that
\[
\dot{\varphi}(T_f) \leq \dot{\varphi}(T_M) + k(T_f - T_M)
= 2 \bar{\Phi} + \left(k \Delta^{(1-\alpha)}\right) / (c(1 - \alpha)).
\]

Discussion: The adaptive functions $\hat{\gamma}$ and $\hat{\varphi}$ are chosen non-negative and $\hat{\gamma}$ is strictly positive with a term proportional to $|u_0|$. It satisfies the condition $\partial \hat{\gamma} / \partial |u_0| > 0$, which is sufficient to ensure that the states will not diverge, irrespectively of $\hat{\varphi}$. The second adaptive function $\hat{\varphi}$ ensures the convergence of the state to a neighborhood of zero. Its dynamics can be defined explicitly by:
\[
\hat{\varphi} = \begin{cases} 
k & \text{if } V_1 \geq \epsilon \\
\frac{2k}{\epsilon} (\epsilon - V_1) \frac{1}{\epsilon} (\epsilon) \leq V_1 \leq \epsilon, \\
\frac{2k}{\epsilon} (\epsilon) \leq V_1 \leq \frac{\epsilon}{2}, \end{cases}
\]

5 CONCLUSIONS

In this paper, we have presented a robust and an adaptive controller for the finite time stabilization of perturbed integrated chains with bounded uncertainty. The robust controller is designed using the knowledge of the uncertainty bounds, and it converges exactly to zero. In the case of unknown uncertainty bounds, the adaptive controller make the state enter in finite time an arbitrary neighborhood of zero an infinite number of time while staying in a bounded neighborhood of zero of depending of the uncertainty bounds. The proof of convergence of both controllers has been demonstrated through Lyapunov analysis, and the calculation of upper bound of the convergence time has also been presented.

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