Entangled random pure states with orthogonal symmetry: exact results

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Abstract

We compute analytically the density \( \varrho_{N,M}(\lambda) \) of Schmidt eigenvalues, distributed according to a fixed-trace Wishart–Laguerre measure, and the average Rényi entropy \( \langle S_q \rangle \) for reduced density matrices of entangled random pure states with orthogonal symmetry (\( \beta = 1 \)). The results are valid for arbitrary dimensions \( N = 2k, M \) of the corresponding Hilbert space partitions, and are in excellent agreement with numerical simulations.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement is one of the most distinctive features of quantum systems. Recently it has attracted much attention in view of possible applications to quantum information and quantum computation problems [1, 2]. In these domains, one is often interested in creating states with large entanglement, thus raising the question how to give a quantitative measure of entanglement. Pure bipartite systems (defined below) constitute a typical example where well-behaved entanglement quantifiers can be defined, such as the von-Neumann or Rényi entropies of either subsystem [2], the so-called concurrence for two-qubit systems [3] or other entanglement monotones [4, 5].

Introducing a source of randomness in quantum entanglement problems is the key to address typical properties of such states. In this paper we focus on random pure quantum states in bipartite systems, where many analytical results have been obtained in recent times (see e.g. [6] for an excellent review).

More precisely, we consider a bi-partition of an \( NM \)-dimensional Hilbert space \( \mathcal{H}^{(NM)} \), as \( \mathcal{H}^{(NM)} = \mathcal{H}_A^{(N)} \otimes \mathcal{H}_B^{(M)} \), where we assume without loss of generality that \( N \leq M \). For example, \( A \) may be taken as a given subsystem (say a set of spins) and \( B \) may represent the environment (e.g. a heat bath). Take a quantum state \( |\psi\rangle \) of the composite system and let \( |i^A\rangle \)
and $|α^B⟩$ be two complete bases of $\mathcal{H}_A^{(N)}$ and $\mathcal{H}_B^{(M)}$ respectively. The state $|ψ⟩$ can then be expanded as a linear combination 

$$|ψ⟩ = \sum_{i=1}^{N} \sum_{α=1}^{M} x_{i,α} |i^A⟩ ⊗ |α^B⟩$$  

(1)

whose coefficients $x_{i,α}$’s form the entries of a rectangular $(N \times M)$ matrix $X$.

The possible features of $|ψ⟩$ we are considering here are as follows.

- **Entanglement:** we say that $|ψ⟩$ is an entangled state if it cannot be expressed as a direct product of two states belonging to the two subsystems $A$ and $B$. In other words, in order for $|ψ⟩$ to be fully unentangled (completely separable), the coefficients $x_{i,α}$ must have the product form $x_{i,α} = a_i b_α$ for all $i$ and $α$ in a certain basis. In this case, the state $|ψ⟩ = |θ^A⟩ ⊗ |θ^B⟩$ can be written as a Kronecker product of two states $|θ^A⟩ = \sum_{i=1}^{N} a_i |i^A⟩$ and $|θ^B⟩ = \sum_{α=1}^{M} b_α |α^B⟩$ belonging respectively to the two subsystems $A$ and $B$.

- **Randomness:** suppose that the expansion coefficients $x_{i,α}$ are random variables drawn from a certain probability distribution. In this case, we say that $|ψ⟩$ is a random state, and here we focus on the simplest and most common case where $x_{i,α}$’s are independent and identically distributed (real or complex) Gaussian variables.

- **Purity:** the density matrix of the composite system is simply given by $ρ = |ψ⟩⟨ψ|$ with the constraint $\text{Tr}[ρ] = 1$, or equivalently $⟨ψ|ψ⟩ = 1$. Note that the composite system may instead be in a mixed state, with a density matrix of the form

$$ρ = \sum_k p_k |ψ_k⟩⟨ψ_k|,$$

(2)

where $|ψ_k⟩$’s are the pure states of the composite system and $0 \leq p_k \leq 1$ are the probabilities that the composite system is in the $k$th pure state, with $\sum_k p_k = 1$. We will not consider this case here, and we refer to [7] and references therein for recent results on mixed states.

Let $|ψ⟩$ be an entangled pure state of a bipartite quantum system. Its density matrix can then be straightforwardly expressed as

$$ρ = \sum_{i,α, j,β} x_{i,α} x^*_{j,β} |i^A⟩⟨j^A| ⊗ |α^B⟩⟨β^B|,$$

(3)

where the Roman indices $i$ and $j$ run from 1 to $N$ and the Greek indices $α$ and $β$ run from 1 to $M$. We normalize the pure state $|ψ⟩$ to unity so that $\text{Tr}[ρ] = 1$.

Tracing out the environmental degrees of freedom (i.e. those of the subsystem $B$) leads to the definition of the reduced density matrix $ρ_A = \text{Tr}_B[ρ]$:

$$ρ_A = \text{Tr}_B[ρ] = \sum_{α=1}^{M} |α^B⟩⟨α^B|$$

(4)

Using the expansion in equation (3) one gets

$$ρ_A = \sum_{i,j=1}^{N} \sum_{α=1}^{M} x_{i,α} x^*_{j,α} |i^A⟩⟨j^A| = \sum_{i,j=1}^{N} W_{ij} |i^A⟩⟨j^A|,$$

(5)

where $W_{ij}$’s are the entries of the $N \times N$ matrix $W = X X^\dagger$. In an analogous way, one could obtain the reduced density matrix $ρ_B = \text{Tr}_A[ρ]$ of the subsystem $B$ in terms of the $M \times M$
matrix $W' = \mathcal{X}' \mathcal{X}'$ and find that $W$ and $W'$ share the same set of nonzero (positive) real eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. In the diagonal basis, one can express $\rho_A$ as

$$\rho_A = \sum_{i=1}^{N} \lambda_i |\lambda^A_i\rangle \langle \lambda^A_i|,$$

where $|\lambda^A_i\rangle$'s are the normalized eigenvectors of $W = \mathcal{X} \mathcal{X}^\dagger$ and similarly for $\rho_B$. The original composite state $|\psi\rangle$ in this diagonal basis reads

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} |\lambda^A_i\rangle \otimes |\lambda^B_i\rangle.$$

Equation (7) is known as the Schmidt decomposition, and the normalization condition $\langle \psi | \psi \rangle = 1$, or equivalently $\text{Tr}[\rho] = 1$, imposes a constraint on the eigenvalues, $\sum_{i=1}^{N} \lambda_i = 1$.

It is useful to remark that while each individual state $|\lambda^A_i\rangle \otimes |\lambda^B_i\rangle$ in the Schmidt decomposition in equation (7) is unentangled, their linear combination $|\psi\rangle$, in general, is entangled, and therefore the state $|\psi\rangle$ cannot, in general, be written as a direct product $|\psi\rangle = |\phi^A\rangle \otimes |\phi^B\rangle$ of two states of the respective subsystems. Knowledge of the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of the matrix $W$ is essential in providing information about how entangled a pure state is. Typical entanglement quantifiers include the Rényi entropy of order $q \geq 1$

$$S_q := \frac{1}{1-q} \log \left[ \sum_{i=1}^{N} \lambda_i^q \right],$$

which converges to the von Neumann entropy $S_{VN} = -\sum_{i=1}^{N} \lambda_i \ln \lambda_i$ for $q \rightarrow 1$. The Rényi and von Neumann entropies attain their minimum value 0 when one of the eigenvalues reaches its maximum value 1 and all the others are zero, which corresponds to completely unentangled states, while they attain their maximum value $\ln N$ in the situation where all eigenvalues are equal ($\lambda_i = 1/N$ for all $i$). In this case, all the states in the Schmidt decomposition (7) are equally present and the state $|\psi\rangle$ is maximally entangled.

So far, we have considered an arbitrary pure state in equation (1) with a fixed coefficient matrix $\mathcal{X} = [x_{i,a}]$. This state is called random if the coefficients are drawn from an underlying Gaussian distribution (real or complex) $\text{Prob}[\mathcal{X}] \propto \exp[-\beta \frac{1}{2} \text{Tr}(\mathcal{X}' \mathcal{X})]$ where the Dyson index $\beta = 1, 2$ corresponds respectively to real and complex $\mathcal{X}$ matrices. While generally $\{x_{i,a}\}$ are complex, real coefficients are important for systems enjoying a time-reversal (or any anti-unitary) symmetry. In these cases, it is known that the eigenfunctions can be chosen to be real, and the corresponding ensembles are the ’orthogonal’ ones ($\beta = 1$). Exact results for the statistics of random orthogonal states are very scarce [8, 9]. It is the goal of this paper to fill this gap and to present exact results for the average density of Schmidt eigenvalues (one-point function) and the average Rényi entropy, valid for arbitrary dimensions $N = 2k, M$ of the corresponding Hilbert space partitions.

Conversely, analytical results for spectral statistics of random pure states with broken time-reversal symmetry ($\beta = 2$) abound. The joint probability density (JPD) of Schmidt eigenvalues was derived by Lloyd and Pagels [10] (see equation (14) below), and using this result Page [11] computed the average von Neumann entropy for large $N, M$ and found

$$\langle S_{VN} \rangle \approx \ln(N) - \frac{N}{2M}.$$

Since $\ln(N)$ is the maximal possible value of von Neumann entropy for the subsystem $A$, in the limit when $M \gg N$, the average entanglement entropy of a random pure state is close to
maximal\(^1\). Later, the same result was shown to hold for the \(\beta = 1\) case \([12]\). In the same paper, Page also conjectured from numerical experiments that the average von Neumann entropy for finite \(N, M\), and \(\beta = 2\) should read
\[
\langle S_{VN} \rangle = \frac{1}{2N} \sum_{k=N+1}^{MN} \frac{1}{k} - \frac{M - 1}{2N},
\]
a result that was independently proven by many researchers soon after \([13]\) also in a non-extensive setting \([14]\). Recently, many efforts have been directed towards the study of other statistical quantities for finite \((N, M)\), and full distributions of interesting observables. We mention here the

- density of Schmidt eigenvalues (one-point function) for \(\beta = 2\) and finite \((N, M)\), derived independently in \([15, 16]\);
- universality of eigenvalue correlations for \(\beta = 2\) \([17]\);
- distribution of the minimum eigenvalue for \(\beta = 1, 2\) and finite \((N, M)\), derived in \([8]\) where a conjecture by Znidaric \([18]\) was proven (see also \([9]\) for a related result);
- average fidelity between quantum states \([19]\) and distribution of so-called \(G\)-concurrence \([4]\) for \(\beta = 2\);
- distribution of so-called purity (i.e. \(S_2\)) for small \(N\) \([20]\), and phase transitions in its Laplace transform for large \(N\) \([21]\);
- full distribution of Rényi entropies (including large deviation tails), computed in \([22]\) for large \(N = M\) and all \(\beta\)s using a Coulomb gas method. As a byproduct, the authors also obtain in \([22]\) the average and variance of Rényi entropy valid for large \(N = M\) as \(^2\)
\[
\langle S_q \rangle \approx \ln N - \ln \bar{s}(q) - \frac{1}{2},
\]
\[
\text{Var}(S_q) \approx \frac{q}{2\beta N^2},
\]
where
\[
\bar{s}(q) = \frac{4q\Gamma(q + 1/2)}{\sqrt{\pi}\Gamma(q + 2)}.
\]

We will compare in section 3 the asymptotic result (11) with our exact formula for the average \(\langle S_q \rangle\) for \(\beta = 1\) (see equation (42)) and find that (42) converges to (11) very quickly for low \(q\), thus including the most relevant cases \(q = 1, 2\). Conversely, the rate of convergence progressively deteriorates as \(q\) increases (see section 3).

In order to proceed, we now summarize the basic ingredients of the calculation. The joint density of Schmidt eigenvalues \(\lambda_i \in [0, 1]\) \([10]\) reads
\[
\mathcal{P}(\lambda_1, \ldots, \lambda_N) = C^{(\beta)}_{N, M} \left( \sum_{i=1}^{N} \lambda_i - 1 \right) \prod_{i=1}^{N} \lambda_i^{(M-N+i)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta,
\]
where \(C^{(\beta)}_{N, M}\) is a normalization constant known exactly for any \(\beta\), and the Dyson index \(\beta = 1, 2\) identifies respectively systems with preserved (orthogonal) or broken (unitary) time-reversal.

\(^1\) Note, however, that the probability of the maximally entangled microscopic state (where all Schmidt eigenvalues are close to each other) decays very quickly as \(N\) increases, a result that is based on the exact evaluation of the full large deviation tails \([22]\).

\(^2\) Note that the limit \(q \to 1\) of equation (11) is consistent with equation (9) already derived by Page.
symmetry. The delta function guarantees that Tr[ρ] = 1 and implies that the typical eigenvalue scales as λ ∼ 1/N.

Another JPD of eigenvalues which is closely related to (14) is from the Wishart–Laguerre (WL) ensemble of random matrices [23, 24] of the form \( \mathcal{W} = X^\dagger X \), where \( X \) is a Gaussian \( M \times N \) matrix with real or complex entries. The joint distribution of the \( N \) nonnegative eigenvalues of \( \mathcal{W} \) is known as \( P(\mathcal{W}) (\lambda_1, \ldots, \lambda_N) \):

\[
P^{(W)}(\lambda_1, \ldots, \lambda_N) = \left[ K^{(\beta)}_{N,M} \right]^{-1} e^{-2 \sum_{i=1}^N \lambda_i} \prod_{i=1}^N \lambda_i^{(2(M-N)-1)} \prod_{j<k} |\lambda_j - \lambda_k|^\beta,
\]

where \( K^{(\beta)}_{N,M} \) is a known normalization constant. Therefore, the JPD (14) can be seen as a fixed-trace version of the WL ensemble. The presence of a fixed-trace constraint has crucial consequences on the spectral properties of random matrix ensembles [26, 27]. The goal of this paper is to compute exactly the one-point marginal (average density) \( \varrho_{N,M}(x) \) for \( \beta = 1 \), defined as

\[
\varrho_{N,M}(x) = \left\langle \frac{1}{N} \sum_{i=1}^N \delta(x - \lambda_i) \right\rangle.
\]

Writing down this average explicitly, one is led to

\[
\varrho_{N,M}(\lambda_1) := C^{(1)}_{N,M} \int_{[0,1]^{N-1}} d\lambda_2 \cdots d\lambda_N \delta \left( \sum_{i=1}^N \lambda_i - 1 \right) \prod_{i=1}^N \lambda_i^{\nu-1} \prod_{j<k} |\lambda_j - \lambda_k|,
\]

where \( \nu = M - N \). Computing this \((N-1)\)-fold integral is the main technical challenge. Note that in the large \( N, M \) limit with \( c = N/M \) fixed, the average density can be computed for all \( \beta \)s using a Coulomb gas technique [22] and has the scaling form:

\[
\varrho_\nu(x) = N \varrho_\nu^\star (Nx),
\]

where

\[
\varrho_\nu^\star(x) := \frac{1}{2\pi c x} \sqrt{(L_+(c) - x)(x - L_-(c))},
\]

where the edge points \( L_{(\pm)} = c(\sqrt{1/c} \pm 1)^2 \).

The average density is important in order to obtain averages of so-called linear statistics with a simple one-dimensional integration as

\[
\left\langle \sum_{i=1}^N f(\lambda_i) \right\rangle = N \int_0^1 d\lambda \varrho_{N,M}(\lambda) f(\lambda).
\]

In particular, we consider \( f(x) = x^q \) for the Rényi entropy in section 3.

The paper is organized as follows. In section 2, we compute the one-point density for \( \beta = 1 \) and any \( N = 2k, M \) using a Laplace transform method, which is technically transparent and avoids the unnecessarily heavy formalism used for earlier derivations of the \( \beta = 2 \) case [16]. In section 3 we use the obtained result to compute the average Rényi entropies for \( \beta = 1 \) and compare them with the asymptotic formula (11) for large \( N \). Then we provide some conclusions in section 4.

3 A linear statistics is a quantity of the form \( \mathcal{O} = \sum_{i=1}^N f(\lambda_i) \), where \( f(x) \) is any smooth function.
2. Density of Schmidt eigenvalues for the orthogonal case $\beta = 1$ ($N$ even)

In this case the JPD (14) of Schmidt eigenvalues reads

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) = C_{N,M} \delta \left( \sum_{i=1}^{N} \lambda_i - 1 \right) \prod_{i=1}^{N} \lambda_i^{i-\frac{1}{2}} \prod_{j<k} |\lambda_j - \lambda_k|, \quad (21)$$

where we put $C_{N,M} \equiv C_{N,M}^{(1)}$ and $\nu = M - N$.

The goal is to compute the density of eigenvalues for finite $(N, M)$, i.e. the marginal

$$\varrho_{N,M}(\lambda_1) := C_{N,M} \mathcal{N} \left[ \int_{[0,1)^{N-1}} d\lambda_2 \cdots d\lambda_N \delta \left( \sum_{i=1}^{N} \lambda_i - 1 \right) \prod_{i=1}^{N} \lambda_i^{i-\frac{1}{2}} \prod_{j<k} |\lambda_j - \lambda_k| \right], \quad (22)$$

which is normalized to 1, i.e. $\int_{[0,1)^{N-1}} d\lambda N \varrho_{N,M}(\lambda_1, 1) dt = 1$. In the orthogonal case the marginalization constant reads $C_{N,M} = \frac{\Gamma(1/(\sqrt{2} \nu))}{\Gamma(1/(M-N)/2) \Gamma(1/(N-N)/2)}$.

We first define $\varrho_{N,M}(\lambda_1, t) := \varrho_{N,M}(\lambda_1, 1)$, where

$$\hat{\varrho}_{N,M}(\lambda_1, t) := C_{N,M} \int_{[0,1)^{N-1}} d\lambda_2 \cdots d\lambda_N \delta \left( \sum_{i=1}^{N} \lambda_i - t \right) \prod_{i=1}^{N} \lambda_i^{i-\frac{1}{2}} \prod_{j<k} |\lambda_j - \lambda_k|, \quad (23)$$

is an auxiliary function that we are going to compute exactly.

We next take the Laplace transform of (23):

$$\int_{0}^{\infty} dt \hat{\varrho}_{N,M}(\lambda_1, t) e^{-\nu t} = C_{N,M} \int_{[0,\infty)^{N-1}} d\lambda_2 \cdots d\lambda_N e^{-\nu \sum_{i=1}^{N} \lambda_i} \left( \prod_{i=1}^{N} \lambda_i^{i-\frac{1}{2}} \prod_{j<k} |\lambda_j - \lambda_k| \right), \quad (24)$$

where on the rhs we have extended the range of integration to the full positive semi-axis. This is harmless in view of the unit norm constraint. The integral on the rhs can be written in the form

$$\int_{[0,\infty)^{N-1}} d\lambda_2 \cdots d\lambda_N e^{-\nu \sum_{i=1}^{N} \lambda_i} \left( \prod_{i=1}^{N} \lambda_i^{i-\frac{1}{2}} \prod_{j<k} |\lambda_j - \lambda_k| \right) = \frac{K_{N,M}}{N(2\nu)^{-1/2}M/N/2} \varrho^{(WL)}_{N,M}(2\lambda_1), \quad (25)$$

where

$$K_{N,M} \equiv K_{N,M}^{(1)} \int_{[0,\infty)^{N-1}} d\lambda_1 \cdots d\lambda_N e^{-\nu \sum_{i=1}^{N} \lambda_i} \left( \prod_{i=1}^{N} \lambda_i^{i-\frac{1}{2}} \prod_{j<k} |\lambda_j - \lambda_k| \right) = 2^{N/2}(\sqrt{\nu}/2)^{-N} \prod_{j=0}^{N-1} \Gamma \left( \frac{3+\nu}{2} \right) \Gamma \left( \frac{M-N+j+1}{2} \right) \quad (26)$$

is the normalization constant of the JPD of eigenvalues of a WL ensemble with $\beta = 1$, which can be derived from the Laguerre–Selberg integral (see e.g. [28]), and

$$\varrho^{(WL)}_{N,M}(\lambda_1) = N(K_{N,M})^{-1} \int_{[0,\infty)^{N-1}} d\lambda_2 \cdots d\lambda_N e^{-\nu \sum_{i=1}^{N} \lambda_i} \left( \prod_{i=1}^{N} \lambda_i^{i-\frac{1}{2}} \prod_{j<k} |\lambda_j - \lambda_k| \right) \quad (27)$$

is the one-point density of the WL ensemble, normalized to $N$.

The spectral density (one-point function) for the WL ensemble $\varrho^{(WL)}_{N,M}(\lambda_1)$ at even $N$ is known as

$$\varrho^{(WL)}_{N,M}(x) = \frac{\varrho^{(WL)}_{N,M}(\lambda_1)}{4} x^{(\nu-1)/2} e^{-x/2} \int_{0}^{\infty} dx' \text{sgn}(x-x')(x')^{(\nu-1)/2} e^{-x'/2} S(x, x'; \nu, N), \quad (28)$$
where
\[ S(x, x'; v, N) := \sum_{j=0}^{N-2} \frac{(j + 1)!}{(j + v)!} \left[ L^v_{j+1}(x')L^v_j(x) - L^v_j(x)\hat{L}^v_{j+1}(x') \right], \tag{29} \]

where \( \hat{L}^v_j(x) \) are the Laguerre functions defined by the sum
\[ L^v_N(z) = \frac{\Gamma(v + N + 1)}{N!} \sum_{k=0}^{N} \frac{(-N)_k}{k! \Gamma(v + k + 1)} z^k, \tag{30} \]

where \((x)_n = \Gamma(x + n)/\Gamma(x)\) and \(\text{sgn}(z) = z/|z|\). The explicit formula (28) can be most conveniently derived by taking the \(\mu \to 0\) limit of equation (4.14) in [29]. Equivalent but less handy expressions can be found in [30], while the general formalism based on skew-orthogonal polynomials is in [24, 31].

In order to take the inverse Laplace transform of (25), some work is needed. First, we make a change of variable \(x' = xz\) in (28), obtaining
\[ \theta_N,M(x) = \frac{\theta(x)}{4} e^{-x/2} \int_0^\infty dz \text{sgn}(1 - z)z^{(v-1)/2} e^{-xz/2} S(x, xz; v, N). \tag{31} \]

For later convenience, we now define and compute the following inverse Laplace transform:
\[ \Psi_k(t, x, z; N_1, N_2; v) := \mathcal{L}^{-1} \left[s^k e^{-x(t+z)} L^v_{N_1}(2sx)L^v_{N_2}(2sxz)\right](t). \tag{32} \]

Using the general definition of Laguerre functions (30) and the following elementary Laplace inverse:
\[ \mathcal{L}^{-1}[s^\alpha e^{-b]}](t) = \frac{(t-b)^{-\alpha-1} \theta(t-b)}{\Gamma(-\alpha)}, \tag{33} \]

where \(\theta(x)\) is the Heaviside step function, it is straightforward to get
\[ \Psi_k(t, x, z; N_1, N_2; v) = \frac{\Gamma(v + N_1 + 1)\Gamma(v + N_2 + 1)}{N_1!N_2!} \]
\[ \times \sum_{m=0}^{N_1} \sum_{\ell=0}^{N_2} \frac{(-N_1)_m(-N_2)_\ell}{m!\Gamma(v + m + 1)\Gamma(v + \ell + 1)\Gamma(-k - m - \ell)} \]
\[ \times (2x)^{m+\ell} z^k (t - x(1+z))^{-k-m-\ell-1} \theta(t - x(1+z)). \tag{34} \]

Combining everything together, we obtain for \(\hat{\rho}_{N,M}(\lambda_1, t)\)
\[ \hat{\rho}_{N,M}(\lambda_1, t) = \frac{C_{N,M}K_{N,M}}{2\pi i M N/2} \Gamma(2\lambda_1) \int_0^\infty dz \text{sgn}(1 - z)z^{(v-1)/2} \]
\[ \times [\Psi_k(t, \lambda_1, z; j, j + 1; v) - \Psi_k(t, \lambda_1, z; j + 1, j; v)], \tag{35} \]

where
\[ \kappa = M - N - MN/2 + 1. \tag{36} \]

The sought density \((t = 1)\) can then be written in the compact form:
\[ \rho_{N,M}(x) = N_{N,M} \sum_{j=0}^{N-2} \sum_{m=0}^j \sum_{\ell=0}^{j+1} \hat{\rho}^{(j)}_{lm} x^{v+m+\ell} \]
\[ \times \int_0^\infty dz \text{sgn}(1 - z)z^{(v-1)/2} (1 - x(1+z))^{-k-m-\ell-1} \phi_{lm}(z) \theta(1 - x(1+z)), \]

\[ \int_0^\infty dz \text{sgn}(1 - z)z^{(v-1)/2} (1 - x(1+z))^{-k-m-\ell-1} \phi_{lm}(z) \theta(1 - x(1+z)). \]
where

\[ \mathcal{N}_{N,M} = \frac{C_{N,M} K_{N,M}}{2^{1+MN/2-N}} \]

(37)

\[
\phi_{\ell m}(z) = z^\ell - z^m. 
\]

(39)

The integral in \( z \) can now be performed exactly. Let

\[ \Xi(a, b; x) = \int_0^{(1-x)/x} dz \, \text{sgn}(1-z)z^a(1-x(1+z))^b, \quad \text{Re}[a, b] > -1. \]

(40)

One has

\[ \Xi(a, b; x) = x^{-1-a}(1-x)^{1+a+b} \times \begin{cases} B(a+1, b+1), & \text{for } 1/2 \leq x \leq 1 \\ -B(a+1, b+1) + 2B(x/(1-x), a+1, b+1), & \text{for } 0 \leq x \leq 1/2 \end{cases} \]

where \( B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b) \) is Euler’s beta function and \( B(z, a, b) = \int_0^z du u^{a-1}(1-u)^{b-1} \) is the incomplete beta function.

Eventually one gets for the density of Schmidt eigenvalues for \( \beta = 1 \)

\[
\varrho_{N,M}(x) = N_{N,M} \sum_{j=0}^{N-2} \sum_{m=0}^{j+1} \phi_{\ell m}(x) \times \begin{cases} \Xi \left( \frac{1}{2} + \ell, -\kappa - m - \ell - 1; x \right), & 0 \leq x \leq 1. \end{cases} 
\]

(41)

Equation (41) is the main result of this section\(^4\). The obtained exact formula is the starting point to compute averages of linear statistics using formula (20). In the next section, we are going to compute the average Rényi entropy at finite \( N, M \) and compare it with the exact asymptotic result for large \( N = M \) obtained in [22]. Note that \( \langle \lambda \rangle = \int_0^1 d\lambda \varrho_{N,M}(\lambda) = 1/N \) in agreement with the general scaling argument that typically \( \lambda \sim 1/N \) due to the trace constraint \( \sum_{j=1}^N \lambda_j = 1 \).

In figure 1 we plot the density (41) for \( \beta = 1, 2 \) for \( N = M = 6 \), and in figure 2 the density (41) for \( N = 6, M = 12 \) together with the large \( N \) density (18). In figure 3 we compare the theoretical density with numerical results, obtained as follows [32, 33].

(i) We generate \( n = 5 \times 10^4 \) real Gaussian \( M \times N \) matrices \( X \) (where \( N = 6, M = 8 \)).

(ii) For each instance we construct the Wishart matrix \( W = X^T X \).

(iii) We diagonalize \( W \) and collect its \( N \) real and non-negative eigenvalues \( \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N\} \).

(iv) We define a new set of variables \( 0 \leq \lambda_i \leq 1 \) as \( \tilde{\lambda}_i = \frac{\lambda_i}{\sum_{j=1}^N \lambda_j} \), for \( i = 1, \ldots, N \). The set of variables \( \lambda_i \) is guaranteed to be sampled according to the measure (14).

(v) We construct a normalized histogram of \( \lambda_i \).

The agreement between theory and simulations is excellent.

\(^4\) Following analogous but much quicker steps, one can also derive the already known one-point density for \( \beta = 2 \) in a much simpler way. For example, for \( N = M \) we obtain

\[ \varrho_{N,N}^{(2\beta)}(x) = \frac{\Gamma(N^2)}{N} \sum_{k=0}^{N-1} \sum_{\ell,m=0}^{\ell} \frac{(-k)_\ell (-k)_m}{(\ell)_\ell (m)_m} x^{\ell+m}(1-x)^{N^2-2\ell-m} \frac{\Gamma(N^2-1-\ell-m)}{\Gamma(N^2-1) \Gamma(N^2-1-\ell-m)}, \quad 0 \leq x \leq 1. \]
Figure 1. Density of Schmidt eigenvalues for $\beta = 1$ (solid blue line, equation (41)) and $\beta = 2$ (dashed red line), both for $N = M = 6$.

Figure 2. Density of Schmidt eigenvalues for $\beta = 1$ (equation (41)) for $N = 6, M = 12$ (solid line). The dashed red line shows the corresponding large $N$ density (18) for $\epsilon = N/M = 1/2$.

3. Average Rényi entropy

The average Rényi entropy $\langle S_q \rangle$ is computed from density (41) as

$$\langle S_q \rangle = \frac{1}{1 - q} \log \left[ N \int_0^1 dx \chi^q \rho_{N,M}(x) \right]. \quad (42)$$

The integral $\mathcal{J}(q)$ can be computed easily. First, define the function

$$\varphi(a, b, c, d) := \int_0^1 dz z^a (1 + z)^b B(z, c, d). \quad (43)$$
Then the sought formula for $J(q)$ reads

$$J(q) = N_{N,M} \sum_{j=0}^{N-2} \sum_{m=0}^{j+1} \sum_{\ell=0}^{j+1} c_{jm}^{(l)} \left[ \theta_j(m+\ell, q) - \theta_m(m+\ell, q) \right],$$

where

$$\theta_j(\alpha, q) := B \left( \frac{\nu - 1}{2} + r + 1, -\kappa - \alpha \right) G_j(\alpha, q) + 2\phi \left( q + \frac{\nu + 1}{2} + \alpha - 1 - r, \frac{\nu - 1}{2} + r - \kappa - \alpha, \frac{\nu - 1}{2} + r + 1, -\kappa - \alpha \right).$$

$$G_j(\alpha, q) := B \left( q + \alpha - r + \frac{\nu + 1}{2}, \frac{\nu - 1}{2} + r - \kappa - \alpha + 1 \right) - \frac{2}{q + \alpha - r + \frac{\nu + 1}{2}}\times_2 F_1 \left( q + \alpha - r + \frac{\nu + 1}{2}, q + \nu - \kappa + 1; q + \alpha - r + \frac{\nu + 1}{2} + 1; -1 \right).$$

Here, $\,_2F_1(a, b; c; x)$ is a hypergeometric function defined by the series

$$\,_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k.$$

In figures 4 and 5 we compare respectively $\langle S_2 \rangle$ (average purity) and $\langle S_{60} \rangle$ as a function of $N(= M)$ for $\beta = 1, 2$ with the large $N$ asymptotic formula (11) from [22].

We find that:

(i) The average Rényi entropy for systems with time-reversal symmetry ($\beta = 1$) is always lower than systems of the same size where this symmetry is broken ($\beta = 2$). This fact is in agreement with recent findings [34] about the so-called single-particle or one-magnon states, where real states have lower entanglement measured in terms of two-spin entanglement content than the case of complex states. We have checked that this feature persists for $N \neq M$, where a large $N$ formula for general $q$ is not yet available.

(ii) For low $q$, the finite and large $N$ results are in excellent agreement already for $N \sim 6$, for both $\beta = 1, 2$. This means that for the most relevant cases of average von Neumann entropy ($q \to 1$) and purity ($q = 2$), one can safely use equation (11) from [22] as an excellent approximation for any practical purposes. Conversely, the quality of the
approximation decays as $q$ increases, up to the limit $S_\infty \to -\ln \lambda_{\text{max}}$ (where $\lambda_{\text{max}}$ is the largest Schmidt eigenvalue), and one has to consider larger and larger subsystems in order to reach a satisfactory agreement (see figure 5). The discrepancy between $\beta = 1$ and $\beta = 2$ is also more pronounced in the case of high $q$, and the convergence to the asymptotic limit is much slower for the $\beta = 1$ case.

4. Conclusions

In summary, we have computed exactly the density $\varrho_{N,M}(\lambda)$ of Schmidt eigenvalues for bipartite entanglement of random pure states with orthogonal (time-reversal) symmetry ($\beta = 1$). The result is valid for any finite dimensions $N \leq M$ (with $N$ even) of the corresponding Hilbert space partitions. Using the exact formula we derived and simple
linear statistics, we computed the average Rényi entropy $\langle S_q \rangle$ for the $\beta = 1$ case, which was previously unavailable. We find that the exact values for the averages at $N = M$ converge very quickly to the asymptotic $N \to \infty$ formula derived in [22] for low values of the parameter $q$, thus including the most relevant cases of the von Neumann entropy ($q \to 1$) and the so-called purity ($q = 2$). As $q$ is increased, the speed of convergence deteriorates for both $\beta = 2$ and $\beta = 1$, and the latter value for the average Rényi entropy is consistently lower than the former for the same values of parameters $q, N, M$, even at $N \neq M$.

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References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Peres A 1993 Quantum Theory: Concepts and Methods (Dordrecht: Kluwer)
[3] Hill S and Wooters W K 1997 Entanglement of a pair of quantum bits Phys. Rev. Lett. 78 5022
Wooters W K 1998 Entanglement of formation of an arbitrary state of two qubits Phys. Rev. Lett. 80 2245
[4] Cappellini V, Sommers H-J and Życzkowski K 2006 Distribution of $G$ concurrence of random pure states Phys. Rev. A 74 062322
[5] Gour G 2005 Family of concurrence monotones and its applications Phys. Rev. A 71 012318
[6] Majumdar S N 2010 Extreme eigenvalues of Wishart matrices: application to entangled bipartite system arXiv:1005.4515
[7] Osipov V A, Sommers H-J and Życzkowski K 2010 Random Bures mixed states and the distribution of their purity J. Phys. A: Math. Theor. 43 055302
[8] Majumdar S N, Bohigas O and Lakshminarayan A 2008 Exact minimum eigenvalue distribution of an entangled random pure state J. Stat. Phys. 131 33
[9] Chen Y, Liu D-Z and Zhou D-S 2010 Smallest eigenvalue distribution of the fixed-trace Laguerre beta-ensemble J. Phys. A: Math. Theor. 43 315303
[10] Lloyd S and Pagels H 1988 Complexity as thermodynamic depth Ann. Phys., NY 188 186
[11] Lubkin E 1978 J. Math. Phys. 19 1028
[12] Page D N 1995 Average entropy of a subsystem Phys. Rev. Lett. 71 1291
[13] Bandyopadhyay J N and Lakshminarayan A 2002 Testing statistical bounds on entanglement using quantum chaos Phys. Rev. Lett. 89 060402
[14] Foong S K and Kanno S 1994 Proof of Page’s conjecture on the average entropy of a subsystem Phys. Rev. Lett. 72 1148
Sánchez-Ruiz J 1995 Simple proof of Page’s conjecture on the average entropy of a subsystem Phys. Rev. E 52 5653
Sen S 1996 Average entropy of a quantum subsystem Phys. Rev. Lett. 77 1
[15] Malacarne L C, Mendes R S and Lenzi E K 2002 Average entropy of a subsystem from its average Tsallis entropy Phys. Rev. E 65 046131
[16] Sommers H-J and Życzkowski K 2004 Statistical properties of random density matrices J. Phys. A: Math. Gen. 37 S457
[17] Kubotani H, Adachi S and Toda M 2008 Exact formula of the distribution of Schmidt eigenvalues for dynamical formation of entanglement in quantum chaos Phys. Rev. Lett. 100 240501
Adachi S, Toda M and Kubotani H 2009 Random matrix theory of singular values of rectangular complex matrices. I. Exact formula of one-body distribution function in fixed-trace ensemble Ann. Phys. 324 2278
[18] Liu D-Z and Zhou D-S 2010 Local statistical properties of Schmidt eigenvalues of bipartite entanglement for a random pure state Int. Math. Res. Not. doi:10.1093/imrn/rnq091
[19] Znidaric M 2007 Entanglement of random vectors J. Phys. A: Math. Theor. 40 F105
[19]Życzkowski K and Sommers H-J 2005 Average fidelity between random quantum states Phys. Rev. A 71 032313
[20]Giraud O 2007 J. Phys. A: Math. Theor. 40 F1053
[21]Facchi P, Marzolino U, Parisi G, Pascazio S and Scardicchio A 2008 Phase transitions of bipartite entanglement Phys. Rev. Lett. 101 050502
De Pasquale A, Facchi P, Parisi G, Pascazio S and Scardicchio A 2010 Phase transitions and metastability in the distribution of the bipartite entanglement of a large quantum system Phys. Rev. A 81 052324
[22]Nadal C, Majumdar S N and Vergassola M 2010 Phase transitions in the distribution of bipartite entanglement of a random pure state Phys. Rev. Lett. 104 110501
Nadal C, Majumdar S N and Vergassola M 2010 Statistical distribution of quantum entanglement for a random bipartite state arXiv:1006.4091
[23]Wishart J 1928 The generalised product moment distribution in samples from a normal multivariate population Biometrika 20A 32
[24]Mehta M L 2004 Random Matrices 3rd edn (New York: Academic)
[25]James A T 1964 Distribution of matrix variates and latent roots derived from normal samples Ann. Math. Stat. 35 475
[26]Akemann G, Cicuta G M, Molinari L and Vernizzi G 1999 Compact support probability distributions in random matrix theory Phys. Rev. E 59 1489
Akemann G, Cicuta G M, Molinari L and Vernizzi G 1999 Non-universality of compact support probability distributions in random matrix theory Phys. Rev. E 60 5287
[27]Lakshminarayan A, Tomsovic S, Bohigas O and Majumdar S N 2008 Extreme statistics of complex random and quantum chaotic states Phys. Rev. Lett. 100 044103
[28]Luque J-G and Thibon J-Y 2003 Hankel hyperdeterminants and Selberg integrals J. Phys. A: Math. Gen. 36 5267
[29]Akemann G, Phillips M J and Sommers H-J 2010 The chiral Gaussian two-matrix ensemble of real asymmetric matrices J. Phys. A: Math. Theor. 43 085211
[30]Verbaarschot J 1994 The spectrum of the Dirac operator near zero virtuality for $N_c = 2$ and chiral random matrix theory Nucl. Phys. B 426 559
[31]Nagao T and Wadati M 1991 Correlation functions of random matrix ensembles related to classical orthogonal polynomials J. Phys. Soc. Japan 60 3298
[32]Bengtsson I and Życzkowski K 2006 Geometry of Quantum States (New York: Cambridge University Press)
[33]Życzkowski K. and Sommers H-J 2001 Induced measures in the space of mixed quantum states J. Phys. A: Math. Gen. 34 7111
[34]Lakshminarayan A and Subrahmanyam V 2003 Entanglement sharing in one-particle states Phys. Rev. A 67 052304