Abstract

In this paper, the concept of tangent space for finite product of C-n,m-selective Banach manifolds is introduced. Using this notion, the concept of differentiation of the mappings f: M_0 → N_0 is extended to the differentiation of the mappings g: M_1 × M_2 → N_1 × N_2, where M_i and N_i are m_i-selective and g_i-selective Banach manifolds for i ∈ {0, 1, 2}, respectively. Moreover, the notions of vector field and tensor field over m-selective Banach manifolds are established.

Keywords: Observer, Selective Banach Manifold, Tangent Space, Tensor Field

1. Introduction

The problem of finding a mathematical model that can change the signature of a metric on a manifold was an interesting topic for both physicists and mathematicians\(^1\). In 2004, it was highly expected to be solved due to introducing the mathematical model of one dimensional observer. Then, considering the concept of the observer over an n-dimensional manifold, the notion of selective manifold was introduced. In fact, it was a realistic approach to the problem of unity in physics\(^1\). In 2009, the concept of a multi-dimensional observer was presented, and by using this notion, a new version of Tychonoff Theorem was proved, and an index for topological entropy was given\(^4\). In 2011, the notion of synchronization for continuous time dynamical systems from the observer viewpoint was established\(^5\). This notion is a generalization of synchronization, and it is proved that the future of the points of the sets in which two dynamical systems are relative probability synchronized are the same up to homomorphism determined by a relative probability synchronization\(^6\). Then, the idea of relative metric spaces as a mathematical model consistent with physical phenomena was considered.

Recently, the concept of the selective manifold over a Banach space was introduced, and its properties were studied. Also, the concept of the α-level differentiation of the mappings between selective Banach manifolds was presented, and a new version of chain rule theorem was proved for the mappings between selective Banach manifolds. In addition, the notion of the tangent space of a selective Banach manifold at a given point was presented.

In this paper, the concept of tangent space for a finite product of selective Banach manifolds is defined, and using it, the notion of the α-level differentiation of mappings between such selective Banach manifolds is presented. In the end, concepts of vector field and tensor field are introduced. In the rest of this section, we provide some preliminaries on observers and selective Banach manifolds.

Let \(M\) be a set, and \(J\) be an index set composed of finite elements. A n-dimensional observer on \(M\) is a mapping \(m: M \rightarrow \prod_{j \in J} I_j\), where \(I_j = [0,1]\) for every \(j \in J\). Let \(\mu: M \rightarrow \prod_{j \in J} I_j\) and \(\eta: M \rightarrow \prod_{j \in J} I_j\) be two n-dimensional observers on \(M\). We write \(\eta \subseteq \mu\) if \(\eta_j(x) \leq \mu_j(x)\) for all \(x \in M\) and \(j \in J\). Moreover, we write \(\eta \subset \mu\) if \(\eta_j \subseteq \mu_j\) and for every \(x \in M\), there exists \(j \in J\) such that \(\eta_j(x) < \mu_j(x)\). The n-dimensional observers \(\mu \cap \eta\) over \(M\) are defined as

\[
(\mu \cap \eta)_j(x) = \inf\{\mu_j(x), \eta_j(x)\};
\]

\[
(\mu \cup \eta)_j(x) = \sup\{\mu_j(x), \eta_j(x)\};
\]
Let, $\tau_\mu$ be a collection of subsets of $\mu$ for which the following conditions are satisfied:

a1) $\mu \in \tau_\mu$ and $\chi^\mu_j \in \tau_\mu$, where $\chi^\mu_j(x) = \prod_{j \neq 0} 0$;

a2) $\chi \cap \eta \in \tau_\mu$, whenever $\chi, \eta \in \tau_\mu$;

a3) if $\{\tau_\mu\}_{\mu \in A}$ is any collection of $\tau_\mu$, then $\bigcup_{\mu \in A} \tau_\mu \in \tau_\mu$, where $A$ is an index set.

$\tau_\mu$ and $(M, n, \tau_\mu)$ are called the $\mu$-topology on $M$ and the relative topological space, respectively.

The observer $a$ is called the constant observer on $M$ if at least one of its elements is constant, i.e. for some $j \in J$, $a(x)$ is constant for all $x \in M$. The collection of all constant observers on $M$ is denoted by $C$. Suppose that $a$ and $b$ be two constant observers on $M$ for which $a < b$. If $\lambda \in \tau_\mu$, then $\lambda^{-1}(a, b)$ is defined as

$$\{x \in M \mid a(x) < \lambda(x) < b(x) \quad \forall j \in J\}.$$

Moreover, the topology that is generated by all $\lambda^{-1}(a, b)$ is called the $\mu$-topology over $(M, \tau_\mu)$.

Let $S_i \subseteq \bigcup_{j \in C} (\tau_\mu)^j$, and $S = \{U | U = U_0 \cap \mu^{-1}(a, b) \in C & U \subseteq S_i\}.

A pair $(U_\phi, \phi)$ is called a chart for $M$ if $U_\phi \subseteq S$ and $\phi$ is a one to one mapping over $U_\phi$ so that $\phi(U_\phi)$ is a $\mu$-Banach manifold. The set $D = \{(U_\phi, \phi) | (U_\phi, \phi) is a chart for M\}$ is called a $\mu$-structure if

b1) $\mu(M) = \mu(U_{\phi_1} \cup U_{\phi_2})$;

b2) if $(U_\phi, \phi) \in D$ for $i \in \{1, 2\}$, then $\mu(U_\phi) = \mu(U_{\phi_2})$, then there exist a one to one and onto map $h_0: U_\phi \rightarrow U_{\phi_2}$ that is a $\mu$-Banach diffeomorphism $h: \phi(U_\phi) \rightarrow \phi(U_{\phi_2})$ such that $h_{\phi} = \phi_{h_0}$.

Let $E$ be a Banach space. $(M, D)$ is called a $\mu$-selective Banach manifold modeled over $E$ if $D$ is a $\mu$-structure.

1.2 Definition

Let $(M, D)$ be a $C^\mu$-structure. There exists a $C^\mu$-structure, $A$, on $M$ such that every $C^\mu$-structure, $D_\mu$, on $M$ with $\mu \leq D_\mu$, is equivalent to $A$.

Any $C^\mu$-structure, $A$, that satisfies the conditions of Theorem 1.2 is called a $C^\mu$ maximal $\mu$-structure.

1.3 Definition

$(M, A, \mu)$ is called a $C^\mu$-selective Banach manifold if $A$ is a $C^\mu$ maximal $\mu$-structure on $M$ and $A = \{U_{\phi_1} \times V_{\phi_2}, \phi_{\sigma_1} \times \phi_{\sigma_2} | (U_{\phi_1}, \phi_{\sigma_1}) \in D_{M_1} \& (V_{\phi_2}, \phi_{\sigma_2}) \in D_{M_2}\}$.

1.4 Theorem

Let $(M_1, D_{M_1})$ be a $C^\mu$-selective Banach manifold for $i \in \{1, 2\}$. Then, $(M_1 \times M_2, \Omega)$ is a $C^\mu$-selective Banach manifold, where

$$\Omega = \{(U_{\phi_1} \times V_{\phi_2}, \phi_{\sigma_1} \times \phi_{\sigma_2}) \mid (U_{\phi_1}, \phi_{\sigma_1}) \in D_{M_1} \& (V_{\phi_2}, \phi_{\sigma_2}) \in D_{M_2}\}.$$

1.5 Definition

Let the mapping $f: M_1 \rightarrow M_2$ be as above, $p \in M$ and $r \leq 0$. We say that $f$ is $(r, a)$-differentiable at $p$ if the following conditions are satisfied:

a) $K_{f, a} \neq \Phi$;

b) $\mu_{p} f = \mu_{a}$;

c) $f: (V_{\phi}, \psi) \in D_{M_2}$ and $\mu_{2}(V_{\phi}) = a$, then $\mu_{2}(f(V_{\phi})) = \mu_{2}(f^{-1}(V_{\phi}))$. 

\[ \mu_{2}(f(V_{\phi})) = \mu_{2}(f^{-1}(V_{\phi})). \]
c4) if \((U_\phi, \Phi, V_\psi, \omega) \in K_{f,a}\), then the mapping 
\(\psi f o f^{-1} : \Phi(U_\phi) \to \psi(V_\psi)\) is a \(C^2\) map in a neighborhood of \(\phi(p)\).

Condition (c4) of Definition 1.5 implies that the definition of the \((r, a)\) - differentiable mapping is independent of the choice of the charts. We say that the mapping \(f\) is continuous at \(p\) if \(\psi f o f^{-1}\) is a continuous map at \(\phi(p)\). We say that the mapping \(f\) is \((r, a)\) - differentiable on \(M\) if \(f\) is \((r, a)\) - differentiable at every \(p \in M\). If \(r = \infty\), then we say that \(f\) is \(a\) - smooth.

### 1.6 Theorem

Let \((M_i, D_{M_i})\) be a \(C^r\mu_i\) - selective Banach manifold for \(i \in \{1, 2\}\), and \(f : M_1 \to M_2\) is a map. If \((V_\psi, \omega) \in D_{M_2}\), then there exists \((U_\phi, \Phi) \in D_{M_1}\) such that \(f^{-1}(V_\psi) = U_\phi\).

### 1.7 Theorem

Let \((M_i, D_{M_i})\) be a \(C^r\mu_i\) - selective Banach manifold over a Banach space \(E\), for \(i \in \{1, 2, 3\}\), and \(f : M_1 \to M_2\) and \(g : M_2 \to M_3\) be two \((r, a)\) - differentiable maps. Then, the mapping \(gof : M_2 \to M_3\) is \((r, a)\) - differentiable.

### 2. Tangent Space for \(C^n\mu\) - Selective Banach Manifolds

In this section, we discuss the concept of the \(a\) - level tangent space to a \(C^r\mu\) - selective Banach manifold at a given point, and study its properties. Throughout this section, we assume that every vector space is defined over the field \(F\), where \(F = \mathbb{R}\) or \(F = \mathbb{C}\).

Let \((M, A, \mu)\) be a \(C^r\) selective Banach manifold over a Banach space \(E\), \(n \geq 1\), and the mapping 

\[ \mu : M \to \prod_{j \in J} I_j; \]

is an observer, where \(I_j = [0, 1]\), and \(J = \{1, \ldots, m\}\). For every \(p \in M\) and \(\alpha \in \text{Im} \mu\), we define an \((r, a)\) - differentiable multi-path through \(p\) as a multi-function \(\gamma : (-1, 1) \to M\) that satisfies the following conditions:

\[ \gamma(p) = 0; \]

\[ \gamma \circ f : (-1, 1) \to E \] is a \(C^r\) map for every \((U_\phi, \Phi) \in A; \]

where \(U_\phi \cap \gamma(-1, 1) \neq \Phi, \mu(U_\phi) = a\), and \(r \geq 1\). We denote the set of all \((r, a)\) - differentiable multi-paths through \(p\) by \(V^{(r,a)}\).

Let \((M, D_{M})\) be a \(C^n\mu\) - selective Banach manifold over a Banach space \(E\), and \(p \in M\). Given two \((r, a)\) - differentiable multi-paths \(\gamma\) and \(\eta\) passing through \(p\), we write \(\gamma \sim \eta\) if

\[ (\phi \circ \eta)'(0) = (\phi \circ \eta)'(0); \]

for every chart \((U_\phi, \Phi) \in A\) around \(p\), where \(U_\phi \cap \gamma(-1,1) \neq \Phi, U_\phi \cap \eta((-1,1)) \neq \Phi\), and \(\mu(U_\phi) = a\). This relation is an equivalent relation.

By an \(a\) - level geometric tangent vector of \(M\) at \(p\), we mean the equivalence class \([\gamma]\) of an \((r, a)\) - differentiable multi-path \(\gamma\) passing through \(p\). We define the \(a\) - level geometric tangent space of \(M\) at \(p\) as the set \(\frac{V^{(r,a)}(M)}{\sim}\) of all \(a\) - level geometric tangent vector of \(M\) at \(p\), we denote it by \(T^{(r,a)}_p(M)\).

Let \(\gamma\) be an \((r, a)\) - differentiable multi-path passing through \(p\), \((U_\phi, \Phi) \in A, U_\phi \cap \gamma((-1,1)) = \Phi, \mu(U_\phi) = a\). We define the \(j\)th component of \(\gamma\) by

\[ \gamma^U_j : (-1,1) \to U_\phi \] with \(\gamma^U_j(t) = \gamma(t) \cap U_\phi; \]

where \(t \in \gamma^{-1}(U_\phi), j \in J, \text{Card} I < \infty\).

Let \(r \geq 1\). The mapping \(\gamma^U_j\) is a \(C^r\) map for every \(j \in J\).

Restriction \(\sim\) to \(U_\phi\), we have \([\gamma_j^U]\in T_p(U_\phi)\). We define

\[ A^{(r,a)} = \{(U_\phi, \Phi) \in A | \mu(U_\phi) = a, p \in U_\phi\}. \]

Now, we define \(\sim\) as follows: \((U_\phi, \Phi) \sim(U_\phi, \Phi)\) if the following conditions are satisfied:

\[ \phi(U_\phi \cap U_\phi)\text{ is a Banach embedded submanifold of } \Phi(U_\phi); \]

\[ \text{there exists a Banach diffeomorphism } f_i : \Phi(U_\phi \cap U_\phi) \to \Phi(U_\phi) \text{ for } i \in \{1, 2\}. \]

It is straightforward to show that there exists a bijection \(h : T^{(r,a)}_p(M) \to \prod_{(U_\phi, \Phi)} T_p(U_\phi)\) that gives a vector space structure to \(T^{(r,a)}_p(M)\) by transferring the structure of \(T^\alpha_{(U_\phi, \Phi)}(U_\phi)\) to \(T^{(r,a)}_p(M)\).

### 2.1 Theorem

Let \((M_i, D_{M_i})\) be a \(C^n\mu_i\) - selective Banach manifold and \(a \in \text{Im} \mu_i\) for \(i \in \{1, 2\}\), and \(f : M_1 \to M_2\) be an \((r, a)\) - differentiable mapping at \(p \in M_1\). We denote the \(a\) - level
differentiation of \( f \) at \( p \) by \( d^a_p(f) \), and define it as the linear mapping

\[
d^a_p(f): T^a_p(M_1) \rightarrow T^a_p(M_2)
\]

\[
[f] \mapsto [f \circ \gamma];
\]

where \( \gamma: (-1,1) \rightarrow M_1 \), and \( \gamma(0) = p \).

Let \((M_1, D_{M_1})\) be a \( C^r \) - selective Banach manifold modeled over a Banach space \( E_i \) and \( a \in \text{Im } \mu_i \) for \( i \in \{1, 2\} \). Then, \( M_1 \times M_2 \) is a \( C^r \) - selective Banach manifold that is modeled over the Banach space \( E_i \times E_i \) where

\[
\mu: M_1 \times M_2 \rightarrow [0,1] \text{ by } \mu(x, y) = (\mu_1(x), \mu_2(y)).
\]

Let \((m_i, m_j) \in M_1 \times M_2 \) be given. We consider multi-path \( \gamma \) passing through \( p \) by:

\[
\gamma: (-1,1) \rightarrow M_1 \times M_2
\]

\[
\gamma(t) = (\gamma_1(t), \gamma_2(t)): \gamma(0) = (m_1, m_2);
\]

where \( \gamma_1: (-1,1) \rightarrow M_1 \) and \( \gamma_2(0) = m_i \) for \( i \in \{1, 2\} \).

### 2.2 Definition

A multi-path \( \gamma: (-1,1) \rightarrow M_1 \times M_2 \) passing through \( p = (m_1, m_2) \in M_1 \times M_2 \) is an \( (r, a) \) - differentiable map if \( (\phi \circ \nu) \circ \gamma: (-1,1) \rightarrow E_1 \times E_2 \) is a \( C^r \) map for every chart \((U \times V, \phi \circ \nu) \in D_{M_1 \times M_2}\) where \( (U \times V, \phi \circ \nu) \cap \gamma((-1,1)) \neq \emptyset \), and \( \mu(U \times V) = (a, a) \).

\( (\phi \circ \nu) \circ \gamma: (-1,1) \rightarrow E_1 \times E_2 \) is differentiable iff \( (\phi \circ \nu) \circ \gamma_i: (-1,1) \rightarrow E_i \) is differentiable for \( i \in \{1, 2\} \). Thus

\[
d^a\gamma = (d^a\gamma_1, d^a\gamma_2).
\]

We denote the set of all \( (r, a) \) - differentiable multi-paths passing through \( p = (m_1, m_2) \) by \( W^{p,a} \).

We define the relation \( \sim \) as follows:

\[
\gamma \sim \eta \iff \frac{d(\gamma \circ \nu)}{dt}(0) = \frac{d(\eta \circ \nu)}{dt}(0);
\]

where \( (U_\phi, \phi) \in D_{M_1 \times M_2} \), \( U_\phi \cap \gamma((-1,1)) \neq \emptyset \) and \( \mu(U_\phi) = (a, a) \). It is easy to check that \( \sim \) is an equivalence relation on \( W^{p,a} \). We denote \( W^{p,a}_p \sim \) by \( T^p_{\phi}(M_1 \times M_2) \), and we call it the \( a \)-level geometric tangent space to \( M_1 \times M_2 \) at \( p = (m_1, m_2) \).

### 2.3 Theorem

Let \((M, D_M)\) be a \( C^r \) - selective Banach manifold over the Banach space \( E_i \) and \( a \in \text{Im } \mu_i \) for \( i \in \{1, 2\} \). Then, there exists a one to one correspondence between \( T^1_{(m_1, m_2)}(M_1 \times M_2) \) and \( T^{2, a}_{m_1}(M_1) \times T^{2, a}_{m_2}(M_2) \) that gives a vector space structure to \( T^1_{(m_1, m_2)}(M_1 \times M_2) \) by transferring the structure of \( T^{2, a}_{m_1}(M_1) \times T^{2, a}_{m_2}(M_2) \) to \( T^1_{(m_1, m_2)}(M_1 \times M_2) \).

#### Proof.

Let

\[
f: T^1_{(m_1, m_2)}(M_1 \times M_2) \rightarrow T^{2, a}_{m_1}(M_1) \times T^{2, a}_{m_2}(M_2)
\]

\[
f([\gamma]) = ([\gamma_1], [\gamma_2]);
\]

where \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \), and \( t \in (-1,1) \). Clearly, \( f \) gives to \( T^1_{(m_1, m_2)}(M_1 \times M_2) \) a vector space structure by transferring the structure of \( T^{2, a}_{m_1}(M_1) \times T^{2, a}_{m_2}(M_2) \) to \( T^1_{(m_1, m_2)}(M_1 \times M_2) \).

### 2.4 Definition

Let \((M, D_M)\) be a \( C^r \) - selective Banach manifold modeled over \( E_i \) \((N_i, D_{N_i})\) be a \( C^r \) - selective Banach manifold modeled over \( E_i \) \((N_i, D_{N_i})\) where \( a \in \text{Im } \gamma_i \), \( a \in \text{Im } \gamma_i \) for \( i \in \{1, 2\} \). We say that \( f: M_1 \times M_2 \rightarrow N_1 \times N_2 \) is \( (r, a) \) - differentiable if the following conditions are satisfied:

\begin{enumerate}
  \item \((\gamma_1 \times \gamma_2) \circ f = \mu_1 \times \mu_2;
  \item \text{there exists } (U_\phi, \phi) \in D_{M_1} \text{ and } (V_\nu, \nu) \in D_{N_1} \text{ so that } \\
    \mu_1(U_\phi) = \gamma_1(V_\nu) = a \quad \text{and } f(U_\phi \times U_\phi) \subseteq V_\nu \times V_\nu;
  \item \text{where } i \in \{1, 2\};
  \item \text{if } (W_\eta, \eta) \in D_{N_1 \times N_2}, \text{ and } a = \gamma_1(W_\eta) = \gamma_2(Z_\delta),
    \text{ then } \gamma_1 \times \gamma_2 = \mu_1 \times \mu_2(f^{-1}(W_\eta \times Z_\delta));
  \item \text{is a } C^r \text{ map between Banach spaces.}
\end{enumerate}

Let \((M, A, \mu)\) be a \( C^r \) selective Banach manifold. For given \( p \in M \), we define the relation \( \sim \) on

\[
A^{p,a} = \{(U_\phi, \phi) \in A \mid p \in U_\phi \text{ and } \mu(U_\phi) = a\};
\]

as follows: \((U_\phi, \phi) \sim (U_\phi, \phi)\) if
g1) $\phi(U_\phi \cap U_{\phi'})$ is a Banach embedded submanifold of $\phi(U_\phi)$;

g2) there exists a Banach diffeomorphism $f_i : \phi(U_\phi \cap U_{\phi'}) \rightarrow \phi(U_\phi)$;

for $i \in \{1, 2\}$.

Clearly, $\sim'$ is an equivalence relation on $A^{r,\alpha}$. By the use of a choice function, we can choose an element of each class. We denote the set of these elements by $B^{r,\alpha}$.

2.5 Definition

Let $f : M \rightarrow \bigcup_{p \in M} \left( \prod_{(U_\phi, \phi) \in B^{r,\alpha}} R \right)$ be a mapping. We say that $f$ is $(r, \alpha)$ - level differentiable if $f(p) \in \prod_{(U_\phi, \phi) \in B^{r,\alpha}} R$ and $\pi_\phi \circ f \circ \phi^{-1}$ is a $C^r$ map, where $(U_\phi, \phi) \in B^{r,\alpha}$, and $\pi_\phi$ is the projection on the component $(U_\phi, \phi)$. We denote the set of such mappings by $G^{r,\alpha}(M)$. $G^{r,\alpha}(M)$ endowed with the multiplication and the addition of functions in components is a vector space over $R$.

3. Selective $(r, \alpha)$ - Vector Fields

In this section, we put forward then notion of selective $(r, \alpha)$ - vector fields. We suppose that $(M, A, \mu)$ is a $C^{r,1}$ selective Banach manifold for $r \geq 1$.

3.1 Definition

A mapping $X : G^{r+1,\alpha}(M) \rightarrow G^{r,\alpha}(M)$ is called a selective $(r, \alpha)$ - vector field with observer $\mu$ if:

h1) $X(fg) = X(f)g + fX(g)$ for $f, g \in G^{r+1,\alpha}$,

h2) $X(af + g) = aX(f) + X(g)$, where $a \in R$, and $f, g \in G^{r,\alpha}(M)$.

3.2 Theorem

Let $\zeta^{r,1}(M)$ be the set of $C^{r,1}$ functions on $M$, and $X$ be an $(r, \alpha)$ - vector field on $M$. Define the mapping

$$X_p : \zeta^{r,1}(M) \rightarrow \prod_{(U_\phi, \phi) \in B^{r,\alpha}} R$$

by

$$X_p(f) = X(\prod_{(U_\phi, \phi) \in B^{r,\alpha}} f(p))$$

where $p \in M$. Then $X_p \in T_p^{r,\alpha}(M)$.

**Proof.**

Let $f, g \in \zeta^{r,1}(M)$ be given. Then

$$X_p(fg) = X(\prod_{(U_\phi, \phi) \in B^{r,\alpha}} f(p)) \cdot (\prod_{(U_\phi, \phi) \in B^{r,\alpha}} g(p))$$

$$= X(\prod_{(U_\phi, \phi) \in B^{r,\alpha}} f(p)) \cdot (\prod_{(U_\phi, \phi) \in B^{r,\alpha}} g(p))$$

$$= X(\prod_{(U_\phi, \phi) \in B^{r,\alpha}} f(p)) + fX(g)(p)$$

It is easy to check that $X_p$ is linear. Therefore, $X_p \in T_p^{r,\alpha}(M)$.

The next theorem gives a description of $X$ over $U_{\phi}p = \prod_{(U_\phi, \phi) \in B^{r,\alpha}} U_{\phi}p$.

3.3 Theorem

The mapping $X$ is an $(r, \alpha)$ - vector field on $U_{\phi}p$ if there exists an $(r, \alpha)$ - vector field $X_{U}^p$ defined on $U_{\phi}$ such that $X = \prod X_{U}^p$.

**Proof.**

Suppose that the $(r, \alpha)$ - vector field $X_{U}^p$ be given, and

$$X = \prod (X_{U}^p)$$

Then for given $f, g \in G^{r,\alpha}(U_{\phi}p)$, we have

$$X(fg) = X(\prod_{(U_\phi, \phi) \in B^{r,\alpha}} f^U_p) \cdot (\prod_{(U_\phi, \phi) \in B^{r,\alpha}} g^U_p)$$

The second property is straightforward from calculus. Therefore, $X$ is an $(r, \alpha)$ - vector field.
Conversely, let $X$ be an $(r, a)$ - vector field on $U_{\phi, p}$ and $(U_{\phi, \theta}) \in B^{p, a}$. We define the mapping $X_{p}^{U_{\phi, \theta}}$ on $U_{\phi, p}$ by

$$X_{p}^{U_{\phi, \theta}}(f) = \prod_{(U_{\phi, \theta})}(X(f))$$

It is to check that $X_{p}^{U_{\phi, \theta}}$ is an $(r, a)$ - vector field on $U_{\phi, p}$ and $X = \prod_{(U_{\phi, \theta})}X_{p}^{U_{\phi, \theta}}$.

### 3.4 Definition

The $a$ - level tensor product for $T_{p, a}^{\mu} (M)$ and $(T_{p, a}^{\mu} (M))^{*}$ is a vector space “$T_{p, a}^{\mu} (M) \otimes (T_{p, a}^{\mu} (M))^{*}$” endowed with a mapping

$$\tau: T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*} \rightarrow T_{p, a}^{\mu} (M) \otimes (T_{p, a}^{\mu} (M))^{*}$$

such that for every bilinear mapping

$$\phi: T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*} \rightarrow X$$

where $X$ is a vector space, there exists a linear mapping

$$\phi: zT_{p, a}^{\mu} (M) \otimes (T_{p, a}^{\mu} (M))^{*} \rightarrow X$$

So that the diagram of Figure 1 commutes: i.e. $\phi = \phi \circ \tau$.

### 3.5 Theorem

The $a$ - level tensor products $T_{p, a}^{\mu} (M) \otimes (T_{p, a}^{\mu} (M))^{*}$ are unique up to isomorphism approximation. That is for two $a$ - level tensor products $\tau_{1}: T_{p, a}^{\mu} (M) \otimes (T_{p, a}^{\mu} (M))^{*} \rightarrow T_{1}$ and $\tau_{2}: T_{p, a}^{\mu} (M) \otimes (T_{p, a}^{\mu} (M))^{*} \rightarrow T_{2}$, there exists a unique isomorphism $i : T_{1} \rightarrow T_{2}$ such that the diagram of Figure 2 commutes: i.e. $\tau_{2} = i \circ \tau_{1}$.

**Proof.**

First, we show that for a given $a$ - level tensor product $\tau: T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*} \rightarrow T$, the only mapping $f$:

$$T_{p, a}^{\mu} (M) \otimes (T_{p, a}^{\mu} (M))^{*} \rightarrow T$$

is the identity mapping $id_{T}$. That is the diagram of Figure 3 commutes, where $f = id_{T}$. In fact, for the given bilinear mapping $\tau: T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*} \rightarrow T$, there exists a linear mapping $\phi : T \rightarrow T$ such that the diagram of Figure 4 commutes. Since the identity mapping satisfies this condition, then by uniqueness of $\phi : T \rightarrow T$ we have $\phi = id_{T}$. Now, using Definition 3.4 for the case that $\phi: T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*} \rightarrow X$ is replaced by $\tau_{2}: T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*} \rightarrow T_{2}$, results that the diagram of Figure 5 commutes for a unique linear mapping $\phi_{2} : T_{2} \rightarrow T_{1}$, there exists a unique linear mapping $\phi_{2} : T_{2} \rightarrow T_{1}$ so that the diagram of Figure 6 commutes. Therefore, the

**Figure 1.** The diagram of the existence of $T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*}$

**Figure 2.** The diagram of the uniqueness of $T_{p, a}^{\mu} (M) \times (T_{p, a}^{\mu} (M))^{*}$

**Figure 3.** The diagram of $f = id_{T}$

**Figure 4.** The diagram of $\phi = id_{T}$
The identity mapping since it is compatible with the map $\tau_2$. Thus, $(\phi_2)^{-1} = \phi_1$, i.e. $\phi_1$ and $\phi_2$ are invertible. Therefore, both $\phi_1$ and $\phi_2$ are isomorphisms.

4. Conclusion

In this paper, the concept of tangent space for finite product of $C^\infty\mu$-selective Banach manifolds is put forward. Next, the concept of differentiation of the mappings $f : M \to N$, where $M$ and $N$ are $\gamma$-selective and $\mu$-selective Banach manifolds respectively, is extended to the differentiation of the mappings $g : M_i \times M_2 \to N_i \times N_p$, where $M_i$ and $N_i$ are $\gamma_i$-selective and $\mu_i$-selective Banach manifolds for $i \in \{1, 2\}$, respectively. Moreover, the notions of vector field and tensor field over $\mu$-selective Banach manifolds are introduced.

5. References

1. Khadekar GS, Kondwar G. Spherically symmetric static fluids in Rosen’s bimetric theory of gravitation. J Dyn Syst Geom Theor. 2006; 4(1): 95–102.
2. Molaei MR. Relative semi-dynamical systems. International Journal of Uncertainty, Fuzziness and Knowledge-based Systems. 2004; 12(2):237–43.
3. Molaei MR. Relative vector fields. J Interdiscip Math. 2006; 9(3):499–506.
4. Molaei MR. Observational modeling of topological spaces. Chaos Solitons & Fractals. 2009; 42(1):615–9.
5. Molaei MR. The concept of synchronization from the observer’s viewpoint. Chankaya University Journal of Science and Engineering. 2011; (8)2:255–62.
6. Santilli MR. Iso-, Geno-, Hyper-mechanics for matter, their isoduals for antimatter and their novel applications in physics, chemistry and biology. J Dyn Syst Geom Theor. 2003; 1(2):121–93.