Quenched Computation of the Complexity of the Sherrington-Kirkpatrick Model.

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Introduction

The frozen phase of mean field spin glass models displays, in the thermodynamic limit, a very high number of stable and metastable states. Such a feature is the consequence of the disorder and the frustration characterizing spin glasses and causing the onset of many different configurations of spins minimizing the thermodynamic potential, organized in the phase space in a rather complicated way. In order to comprehend the structure of the landscape of the thermodynamic potential below a given critical temperature a very important theoretical tool is the complexity, else called, in the framework of structural glasses, configurational entropy, i.e. the logarithm of the number of metastable states the system can evolve to. The organization of thermodynamic states in complex systems is fundamental, e.g., in the understanding of the dynamic properties.

A certain class of disordered mean-field spin-glass models, including, e.g., the p-spin interaction models, provides a proper description for the main features of structural glasses. These are models displaying a stable one-step Replica Symmetry Breaking (1RSB) frozen phase, very well representing the glassy phase of amorphous magnets (i.e. actual spin glasses), no dynamical transition precedes the static one and the frozen phase is a Full Replica Symmetry Breaking (FRSB) spin-glass phase. Even though the dynamic behavior of systems whose static phase is properly represented by a 1RSB solution turns out to be extensive, the existence of an extensive complexity for the SK model has been put forward since the very early stages of theoretical modeling of such materials.

In the last twenty years basically two different approaches have been presented for the behavior of the complexity in the SK model. The first one was originally introduced by Bray and Moore (BM); there the complexity was analyzed both in the annealed approximation, i.e. as the logarithm of the disordered average of the number of states, and as quenched average, i.e. the average of the logarithm. The second one was initially proposed by Parisi and Potters who showed that the complexity could be obtained by calculating the partition function of m distinct real replicas of the system and provided the connection with the previous BM formalism by means of a generalization of the two-group Ansatz. In that framework they computed an annealed solution using the so-called ‘unbroken’ two-group Ansatz. As we will see, such an Ansatz turns out to be equivalent to impose a certain supersymmetry on the complexity. Indeed, in Refs. [13,14], explicit computation has led to a supersymmetric complexity different from the BM one.
Over the years it has become more and more evident that an important role in the study of the complexity of disordered systems is played by the so-called Becchi-Rouet-Stora-Tyutin supersymmetry (BRST-SUSY), first discovered in the framework of the quantization of gauge theories. In the context of stochastic field equations it can be shown that the generating functional of correlation functions averaged over disorder leads to an action displaying BRST SUSY. This formally coincides with the average over the quenched random couplings of the number of solutions of the mean-field equations for the SK model. In such a model, the property of BRST-SUSY has been analyzed in Refs. [14,17] and a possible explanation of their physical meaning has been presented by the authors in Ref. [18].

In particular, in Ref. [14] evidence for a BRST-SUSY complexity functional was brought about. The difference with the BM complexity turns out, actually, not to be at the functional level, but simply the new solution is a second saddle point of the same functional. This means that there is just one BRST-SUSY functional (the BM complexity) displaying (at least) two saddle points, one satisfying the BRST-SUSY and the other one not. Both solutions identify an extensive complexity, computing the number of solutions of the Thouless-Anderson-Palmer (TAP) equations in the annealed approximation. In Ref. [18] the authors analyzed the two solutions and their problems reaching the conclusion that both of them had to be discarded as candidate quenched complexities of the SK model. So far for the annealed level.

In the present paper we go further analyzing the quenched complexity of the SK model in the FRSB scheme, modified in order to generalize the approach of Monasson (applicable to systems with 1RSB stable static phase). The aim is to see whether or not it is possible to build a stable (or marginally stable) solution with an extensive complexity, such as in 1RSB-stable systems like p-spin models.

The program we apply amounts to compute the quenched complexity in the FRSB scheme with proper assumptions. Our computation scheme is formally a particular case of the one starting from the three-dimensional antiparabolic equation proposed by Bray and Moore in Ref. [22]. The solution is, however, more generic than the one obtained, under important constraints, by Bray, Moore and Young. Indeed, our assumptions are less strict than those of Ref. [23], thus allowing for the construction of a complexity as a function of the free energy. Both that solution and the present one are BRST-SUSY invariant. The main results of the present paper were anticipated in Ref. 24.

In Secs. I-II we recall the replica computation for the SK model making the Ansatz of infinite replica symmetry breaking in the variational form of Sommers and Dupont. The SK model is stable only in the limit of infinite breaking of the replica symmetry. Even then the stability with respect to overlap fluctuations is only marginal. In Sec. III we formally generalize the Legendre transform approach to the case of FRSB stable systems and we modify the replica computation scheme accordingly, adopting a modified FRSB scheme in the spirit of Monasson investigation of hidden states for glass-like models. In Sec. IV we show the results for the complexity and the thermodynamic observables obtained both by direct numerical solution and by analytical expansion around the critical point up to very high order, within the modified FRSB scheme.

The stability of the solution is discussed in Sec. V, where we show that, in the SK model, for any value of the complexity different from zero, even marginal stability is always violated. This means that, requiring BRST-SUSY, no exponential growth of the number of states with the size of the system at any value of the free energy can be obtained.

What about a BRST-SUSY-breaking FRSB quenched complexity? Since any supersymmetric solution would suffer of the same instability (see Sec. V), in order to give account for an extensive complexity this remains the only alternative. In Sec. VI we give the most general expressions of the FRSB complexity and of the differential FRSB equations, without imposing any symmetry. Such equations are, basically, those already presented in Refs. [22,23]. We rederive everything using the two-group Ansatz formalism, where the breaking of the BRST-SUSY is more easily carried out and the BRST-SUSY invariant complexity is described by the unbroken two-group. From a purely formal point of view, the equations are equal to those treated in the study of temperature chaos by means of coupled real replicas, with different boundary conditions. Their resolution is currently under investigation.

I. THE FRSB SK COMPLEXITY

The Hamiltonian of the SK model is

$$\mathcal{H} = -\sum_{i<j}^{1,N} J_{ij} \sigma_i \sigma_j$$

(1)

where $N$ is the number of Ising spins $\sigma_i \pm 1$ and $J_{ij}$ are quenched Gaussian random variables with variance $\overline{J_{ij}^2} = 1/N$ and mean $\overline{J_{ij}} = 0$. The over-bar denotes the average over disorder.

Using the replica trick the average free energy can be expressed in terms of the spin-overlap replica matrix $(a, b = 1, \ldots, n; n$ being the number of replicas):$\beta f^{\text{rep}} = -\frac{\beta^4}{4} + \frac{\beta^2}{4} \lim_{n \to 0} \frac{1}{n} \sum_{a \neq b} q_{ab}^2$

$$- \lim_{n \to 0} \frac{1}{n} \log \text{Tr}_\sigma \exp \left( \frac{\beta^2}{2} \sum_{a \neq b} q_{ab} \sigma^a \sigma^b \right)$$

(2)

where $\sigma^a$ is the spin variable relative to the replica $a$ and the trace is taken in the replica space.
In the limit of FRSB the matrix \( q_{ab} \) is parametrized by a continuous non decreasing function \( q(x) \), \( 0 \leq x \leq 1 \), and the free energy functional (2) becomes

\[
\beta f^{\text{rep}}(\beta) = -\frac{\beta^2}{3} \left[ 1 - 2q(1) + \int_0^1 dx \, q^2(x) \right] + \beta \int_{-\infty}^{+\infty} dy \, P_0(y) \, \phi(0, y)
\]

where

\[
P_0(y) = \frac{1}{\sqrt{2\pi q(0)}} \exp \left( -\frac{y^2}{2q(0)} \right)
\]

and \( \phi(0, y) \) is the solution evaluated at \( x = 0 \) of the Parisi antiparabolic differential equation:

\[
\dot{\phi}(x, y) = -\frac{\dot{q}(x)}{2} \left[ \phi''(x, y) + x \beta \phi'(x, y)^2 \right] \tag{5}
\]

with the boundary condition

\[
\phi(1, y) = T \log (2 \cosh \beta y) \tag{6}
\]

We have used the standard notation, denoting derivatives with respect to \( x \) by a dot and derivatives with respect to \( y \) by a prime.

II. VARIATIONAL METHOD

Functional (3) has to be extremized with respect to \( q(x) \). In order to implement the extremization one can include the Parisi equation (5) and the boundary condition at \( x = 1 \), Eq. (6) into the free energy functional via Lagrange multipliers \( P(x, y) \):

\[
\frac{\delta}{\delta P(x, y)} (\beta f^{\text{rep}}(\beta) + \beta \int_{-\infty}^{+\infty} dy \, P(1, y) \left[ \phi(1, y) - T \log (2 \cosh \beta y) \right]) = \int_{-\infty}^{+\infty} dy \, P(x, y) \left[ \phi''(x, y) \right.
\]

\[
\left. + x \beta \phi'(x, y)^2 \right] \tag{7}
\]

The functional \( f^{\text{rep}}_\beta \) is stationary with respect to variations of \( P(x, y) \), \( P(1, y) \), \( \phi(x, y) \), \( \phi(0, y) \) and of the overlap function \( q(x) \). Variations with respect to \( P(x, y) \) and \( P(1, y) \) give back equation (5) and its boundary condition (6). Stationarity with respect to variations of \( \phi(x, y) \) and \( \phi(0, y) \) leads to a partial differential equation for \( P(x, y) \) and to its boundary condition at \( x = 0 \):

\[
\upsilon(x, y) = \frac{\dot{q}(x)}{2} P''(x, y) - x \beta \dot{q}(x) P(x, y) \phi'(x, y)^2 \tag{8}
\]

\[
P(0, y) = P_0(y)
\]

Eventually, variation with respect to \( q(x) \) leads to

\[
q(x) = \int_{-\infty}^{+\infty} dy \, P(x, y) \phi'(x, y)^2 \tag{10}
\]

The Lagrange multiplier \( P(x, y) \) can be interpreted as a probability distribution.\(^{25}\) The value \( q(x) \) of the overlap function can be associated with the time scale \( \tau_x \) such that for times of order \( \tau_x \) states with an overlap equal to \( q(x) \) or greater can be reached by the system. In this picture, \( P(x, y) \) becomes the probability distribution of frozen local fields \( y \) at the time scale labeled by \( x \).\(^{25}\)

Starting from Eq. (10) and from the interpretation of \( P(x, y) \) as distribution of local fields we notice that the function \( m(x, y) = \phi'(x, y) \) represents the local magnetization over the time-scale \( x \). It obeys the equation

\[
m(x, y) = -\frac{\dot{q}(x)}{2} m''(x, y) - x \beta \dot{q}(x) m(x, y) m'(x, y) \tag{11}
\]

with initial condition

\[
m(1, y) = \tanh(\beta y) \tag{12}
\]

In terms of the local magnetization one can express both the equilibrium magnetization and the field-cooled susceptibility of the system:

\[
m_{\text{eq}} = \int_{-\infty}^{+\infty} dy \, P(0, y) \, m(0, y) \tag{13}
\]

\[
\chi_{\text{FC}} = \int_{-\infty}^{+\infty} dy \, P(0, y) \, m'(0, y) \tag{14}
\]

A. Marginal Stability of the FRSB Solution

In the SK model, with FRSB stable low temperature phase, the stability condition can be written as

\[
1 - 2q(1) + \int_{-\infty}^{+\infty} dy \, P(1, y) \, m^4(1, y)
\]

\[
= T^2 \int_{-\infty}^{+\infty} dy \, P(1, y) \left[ m'(1, y)^2 \right] \tag{15}
\]

Deriving equation (10) with respect to \( x \) we obtain the fundamental relation:\(^{25,29}\)

\[
\int_{-\infty}^{+\infty} dy \, P(x, y) \left[ m'(x, y) \right]^2 = 1 \tag{16}
\]

and inserting Eq. (16) in Eq. (15) the latter turns out to be marginally satisfied.

Notice that Eq. (16) can be used as well to simplify the expression of some observables. If the external magnetic field is zero, the minimum value of \( q, q(0) \), is zero \( [m(0, 0) = 0] \). That causes Eq. (9) to become \( P(0, y) = \delta(y) \). As a consequence, Eq. (16) yields \( m'(0, 0)^2 = 1 \), \( m'(0, 0) = \pm 1 \), thus implying \( m_{\text{eq}} = 0 \) and \( \chi_{\text{FC}} = 1 \) in the frozen spin-glass phase.
III. THE QUENCHED COMPUTATION

A. Varying the breaking point of the order parameter \(q(x)\)

The solution of the self-consistency equation (10) of the FRSB solution in the SK model leads to a monotonic continuous order parameter function \(q(x)\) with \(x \in [0,1]\) representing a continuous “replica symmetry breaking index” or, in the dynamical interpretation of the replica computation, a “time-scale label”. In the absence of external magnetic field \(q(x)\) grows from zero at \(x = 0\) to its maximum value for \(x = x_c\). From there on it remains constant: \(q(x) = q(x_c) = q(1), \forall x \in [x_c, 1]\).

The Edwards-Anderson parameter \(q_{EA} = q(1)\), also called self-overlap, is a physical quantity representing the overlap of two configurations belonging to the same spin-glass state. This scenario naturally emerges solving the Parisi equation (5) with boundary condition (6) imposed at \(x = 1\). The value of \(x_c\) is a decreasing function of the temperature, as shown in e.g. Ref. [34].

If, instead than at \(x = 1\), we set the boundary condition of the Parisi equation at a certain \(x_b < x_c\), the overlap function extremizing the free energy functional develops a discontinuity \(\Delta q = q(1|x_b) - q(x_b)\) at \(x_b\), where \(q(x_b)\) is the value of the overlap at the left side of \(x_b\), while \(q(1|x_b)\) is the value at the right side (see e.g. Fig. 1). In general, the value of \(q(1|x_b)\) will be different from (larger than) \(q_{EA}\). If the boundary condition (6) or, equivalently (12), is imposed at any \(x_b\) less than one, but larger than \(x_c\), no change in behavior occurs: the value \(q_{EA}\) of the Edwards-Anderson self-overlap is reached anyway and the point at which this happens does not shift. The above construction is a step forward with respect to what was done in Refs. [10,23] where the formal construction was performed keeping such a difference but the solutions were computed setting it to zero.

Such a generalization can be directly implemented in the FRSB scheme by imposing condition (18) at \(x = x_b\) and, at the same time by allowing for a discontinuity \(\Delta q\).

This leads to the following free energy functional:

\[
\beta f_{\text{var}}^{\text{rep}} = -\frac{\beta^2}{4} \left[ 1 - 2q(1|x_b) \right] - \frac{\beta^2}{4} \int_0^1 dx \, q(x)^2 \quad (17)
\]

\[
- \beta \int_{-\infty}^{\infty} dy \, P_0(y) \, \phi(0, y) + \beta \int_{-\infty}^{\infty} dy \, P(x, y) \left[ \phi(x, y) - \phi_1(x, y) \right] - \beta \int_0^{x_b} dx \int_{-\infty}^{\infty} dy \, P(x, y) \left\{ \phi'(x, y) + \frac{\phi''(x, y)}{2} \right\}
\]

\[
\phi_1(x_b, y) \equiv \frac{1}{\beta} \log \int Dz \, p(x_b, y, z) \quad (18)
\]

\[
Dz \equiv dz \, e^{-z^2/2} \quad (19)
\]

\[
p(x_b, y, z) \equiv \left[ 2 \cosh \left( \beta y + \beta z \sqrt{\Delta q} \right) \right]^{x_b} \quad (20)
\]

\[
\Delta q \equiv q(1|x_b) - q(x_b) \quad (21)
\]

If \(x_b > x_c\), \(\Delta q\) is zero and the boundary condition (18) becomes

\[
\phi_1(x_b, y) = \frac{1}{\beta} \log 2 \cosh \beta y = \phi(1, y) \quad (22)
\]

This way we recover the usual unconstrained FRSB solution of the SK model.

It is worth noticing that solving Eq. (5) simply with boundary condition (6) imposed at \(x_b < x_c\) just leads to same result, i.e. to the same discontinuous \(q(x)\). The choice we make here of introducing \(\Delta q\) already in the boundary condition just makes practical computation easier.

For numerical purposes the study of the equation (11) for the local magnetization \(m(x, y) = \phi'(x, y)\) is better suited. For \(x \leq x_b\), the stationarity Eqs. (11), (12) are replaced by

\[
m(x_b, y) = m_1(x_b, y) = \left\langle \tanh \left( \beta y + \beta z \sqrt{\Delta q} \right) \right\rangle \quad (23)
\]

where the average is defined as

\[
\left\langle o(y, z) \right\rangle = \frac{\int Dz \, o(y, z) \, p(x_b, y, z)}{\int Dz \, p(x_b, y, z)} \quad (24)
\]

![FIG. 1: Overlap functions q(x) at T = 0.6 evaluated resuming the series expansion in Tc - T (Tc = 1). The dashed line represents the usual FRSB q(x) whereas the continuous line is q(x) computed with an imposed breaking point x_b = 0.3 < x_c = 0.530229(1). In the latter case q(x) reaches a plateau at a given x_0. Both functions verify the relationship \int_0^1 q(x) dx = 1 - T.](image-url)
for overlap functions do satisfy the relation $1$ for the $1\text{RSB}$ solution. All overlap functions satisfy $1\leq b<\frac{m}{T}$ instead than at $x=x_b$. For $x_b<x_c=0.54173(1)$ a discontinuity occurs. Notice that for $x_b=0.0841$ ($x_{\text{1RSB}}=0.0845(2)$) the continuous part of $q(x)$ goes to zero in the whole interval $[0,x_{\text{1RSB}}]$. At $x_b=x_{\text{1RSB}}$ the free energy of the 1RSB solution becomes lower then the relative one in the whole interval $[0, x]$. We notice that Eq. (23) for $x_m=0.0841(b)$ the continuous part of $q(x)$ becomes a step function. This happens because at $x=x_b$, i.e. $\Delta q=0$, becomes Eq. (12).

In Fig. 1 we plot $q(x)$ at $T=0.6$ and $x_b<x_c$ by means of series expansions around the critical point and subsequent Padé resummation. In Fig. 2 we show $q(x)$ at $T=0.2$ for different values of $x_b<x_c$, obtained by numerical resolution of the Parisi equation. Notice that for $x_b$ less than a particular value (we can call it $x_{\text{1RSB}}$) $q(x)$ becomes a step function. This happens because at $x_b \leq x_{\text{1RSB}}$ the free energy of the 1RSB solution becomes lower then the relative one for the FRSB solution. All overlap functions do satisfy the relation $1 - \int dx \ q(x) = T$.

B. The Coupled Real Replicas Approach\textsuperscript{10,11} Generalized to FRSB-stable Spin-Glasses

The equations discussed in the previous section can be obtained by considering $m$ equivalent copies (real replicas) of the system interacting through an infinitesimal coupling.

If $Z_J$ is the partition function for a given realization of the couplings $J_{ij}$, the starting point for such a computation is the replicated partition function $Z_{\text{Rep}}$, where $n$ is the number of replicas of a single copy of the system and $m$ are the real replicas of the system interacting between each other through an infinitesimal coupling. Afterward we send $n$ to zero, keeping $m$ finite\textsuperscript{10,11} (see also Sec. VI).

Because of the interaction between the $m$ copies the resulting solution in the space of $n \times m$ replicas is not just the one we would get for $n$ replicas repeated $m$ times. In other words, the values of the elements of the overlap-matrix of each copy are not the same one would get in absence of the coupling.

Using the permutation symmetry of the $m$ replicas it is possible to perform proper permutations of lines and rows of $q_{ab}$ in order to get the highest overlap values grouped together in a diagonal block of dimension $m$. Sending $n \to 0$ leads to a finite $m \in [0,1]$, representing a sort of density of the number of copies.

Setting a value $x_b<x_c$ at which we impose the boundary condition of the FRSB equations, is equivalent to cut the refinement of replica symmetry breaking inside the diagonal blocks of the Parisi matrix at $q(x_b)$. Beyond this point no further breaking occurs and all the elements of the matrix inside the diagonal blocks take the same value $q(1|x_b)=q(x_b)$. Such a quantity is always something less than the Edwards-Anderson self-overlap $[q(1|x_b)<q(1|x_c)]$. This can be easily understood since it is a sort of average of the monotonously increasing overlap function over the $q$ values labeled by $x>x_b$.

Identifying $x_b$ with $m$, when the 'copies density' $m=x_b$ is chosen higher than the critical value $x_c$ the refinement is complete and the solution coincides with the one without discontinuity.

C. Complexity as Legendre Transform of the FRSB Replica Thermodynamic Potential

Making the identification $m=x_b$ and looking at the analogy with systems displaying a stable 1RSB frozen phase we can generalize the approach or Ref. \textsuperscript{11} to FRSB systems, defining also in this case the complexity $\Sigma$ as the Legendre Transform of replicated free energy $f_{\text{Rep}}$ as

$$\Sigma(f) = \max_{x_b} [\beta x_b f - \beta x_b f_{\text{Rep}}]$$

with the free energy $f$ and $\beta x_b$ as Legendre conjugated variables:

$$\beta x_b = \frac{\partial \Sigma}{\partial f}$$

$$f = \frac{\partial x_b f_{\text{Rep}}}{\partial x_b}$$
Exploiting Eq. (29), the complexity can be equivalently expressed as
\[ \Sigma(f) = \beta x_b \frac{\partial f_{\text{rep}}}{\partial x_b} \bigg|_{x_b(f)} \]  
(30)
where the relation \( x_b(f) \), or vice-versa \( f(x_b) \), is yielded, e.g., by Eq. (29). If \( x_b < x_c \), \( f_{\text{rep}} \) is not stationary with respect to the breaking parameter \( x_b \) and, therefore, \( \Sigma(f) \neq 0 \) in some \( f \)-interval \([f_0, f_{\text{th}}]\). As \( x_b \to x_c \) the unconstrained FRSB solution is recovered, thus \( f(x_c) = f_{\text{eq}} \) and it also coincides with the lower band-edge \( f_0 \) at which the complexity goes to zero.

**IV. THE FRSB BRST-SUSY SOLUTION AND ITS PROPERTIES.**

The explicit expressions for the free energy and the complexity as functions of \( x_b \) are obtained from Eqs. (29) and (30) and respectively:
\[
\begin{align*}
f(x_b) &= \frac{\beta x_b}{4} \left[ q^2(1|x_b) - q^2(1|x_b) \right] \\
&+ \int_{-\infty}^{\infty} dy \, P(x_b, y) \left[ \phi_1(x_b, y) - \frac{1}{\beta x_b} \langle \log p(x_b, y, z) \rangle \right] - \frac{\beta}{4} \left[ 1 - 2q(1|x_b) + \int_0^1 dx \, q^2(x) \right] - \phi(0,0) \\
\Sigma(x_b) &= \frac{(\beta x_b)^2}{4} \left[ q(1|x_b)^2 - q(x_b^-)^2 \right] \\
&+ \int_{-\infty}^{\infty} dy \, P(x_b, y) \left[ \beta x_b \phi_1(x_b, y) - \langle \log p(x_b, y, z) \rangle \right]
\end{align*}
\]
(31)
(32)

In particular, the free energy can be written as
\[ f = e - Ts \]  
(33)
where \( e \) is the internal energy density and \( s \) the entropy contribution inside a single spin-glass state.

Thermodynamic relations are valid for any \( x_b \) and, therefore, the expressions for energy and entropy are
\[
\begin{align*}
u(\beta) &= \frac{\partial \beta f_{\text{rep}}}{\partial \beta} = -\frac{\beta}{2} \left[ 1 - \int_0^1 dx \, q^2(x) \right] \\
s(\beta, x_b) &= \beta \left[ \beta \frac{\partial f_{\text{rep}}}{\partial \beta} \bigg|_{x_b} - x_b \frac{\partial f_{\text{rep}}}{\partial x_b} \bigg|_{x_b} \right] \\
&= -\frac{\beta^2}{4} \left[ 1 - q(1|x_b)^2 \right] - \beta^2 x_b \Delta q \left[ 1 - (1 - x_b)q(1|x_b) \right] \\
&+ \int_{-\infty}^{\infty} dy \, P(x_b, y) \left[ \frac{1}{x_b} \langle \log p(x_b, y, z) \rangle - y \right] m_1(x_b, y)
\end{align*}
\]
(34)
(35)
where we used the fact that the replica entropy \( \beta^2 \frac{\partial f_{\text{rep}}}{\partial \beta} \) is equal to \( s + \Sigma/x_b \). Notice that, as \( \Delta q \to 0 \), the expression of the entropy tends to the usual FRSB one.

The solution we are considering satisfies the BRST SUSY.\textsuperscript{14,17} We will show later on (Sec. VI) how this can be simply proved, starting from a general formulation involving more than one order parameter and reducing to the present one when certain relations among them, stating the supersymmetry, are imposed.

In the following we present our results for the complexity in the FRSB\textsubscript{3q} Ansatz obtained both by numerical resolution of Parisi equation and by analytical expansion around \( T_s \).

**A. Numerical Resolution of FRSB Equations with Pseudo-Spectral Integration**

We performed an accurate numerical study of Eqs. (8), (11) with imposed breaking at \( x_b < x_c \), exploiting the pseudo-spectral numerical integration method used, e.g., in Refs. [34–36]. As a comparison we also performed the study of the 1RSB solution and its complexity (analyzed and discussed in Ref. [14]) for the same values of the temperature.

In figure 3 we show the behavior of the FRSB\textsubscript{3q} complexity as a function of the free energy. We observe that in the SK model the complexity (30) is non-zero in a given interval \( f \in [f_0, f_{\text{th}}] \) As shown in the inset of Fig. 3, the interval of free energies \( f \) in which the complexity is non zero increases as \( T \) decreases. This is also the dominion of variation of \( f_{\text{rep}}(f) \).

Above the static transition between paramagnet and FRSB spin-glass, at temperature \( T_s \), the complexity is zero. Below \( T_s \), if we look at the behavior of the replicated thermodynamic potential \( f_{\text{rep}} \) both as a function of the state free energy \( f \) and as a function of the breaking parameter \( x_b \) (see Figs. 4 and 5) we can see that the extremal value of the replicated free energy is always at the lowest value of the free energy \( f \); \( f_0 = f(x_c) = f_{\text{rep}}(x_c) \). This is the value of vanishing complexity and coincides with the equilibrium thermodynamic free energy \( f_{\text{eq}} \).

We define \( x_{\text{max}} \) the value of \( x_b \) corresponding to \( f_{\text{th}} \) and to the maximum complexity (see Figs. 5-6): In figure 5 according to the interpretation mutated from 1RSB-stable systems,\textsuperscript{14} we can see that for any \( x_{\text{max}} < x < x_c \) the values of \( f \) are increasing as \( x_b \) decreases: they should be the free energy values of the most significant states. Their number is exponentially large and equal to \( \exp[\lambda \Sigma(f)] \). Beyond \( x_{\text{max}} \) the solution is said 'unphysical' \( f(x) \) decreases with \( x \).

On the contrary, as \( x \) decreases from \( x_c \) to \( x_{\text{max}} \), the total free energy \( f_{\text{rep}} \) decreases. Indeed \( f_{\text{rep}} \) is the free energy computed by the replica trick and, thus, it has to be maximized with respect to variations of the order parameter values. Adopting the usual, unconstrained, scheme of computation the maximum value of the SK free energy, \( f_{\text{rep}} = f_{\text{eq}} \) occurs for \( x_b \gtrless x_c \). If we compute it at any other value of \( x_b \), \( f_{\text{rep}} \) is less than this. In figure 5 the behaviors of both \( f_{\text{rep}} \) and \( f \) as functions of the imposed breaking parameter are shown at \( T = 0.2 \).
Having analyzed in detail the behaviors of both the complexity and the replicated free energy we summarize that (a) the maximum of $f^\text{rep}(f)$ or, equivalently, the minimum of $\Phi(f) = f - T\Sigma(f)$, is given by the extremum $f_0$ of the $f$-support (the equilibrium free energy in the FRSB stable SK model), for any temperature, whereas in models where a $T_b \neq T_s$ exists (1RSB stable systems) such a stationary point is somewhere in the middle of the support, going to $f_0$ as temperature decreases\cite{7,18} (b) the support of the curve $\Sigma(f)$ increases from zero as temperature is decreased from the critical point (Fig. 3), whereas in 1RSB stable systems, below $T_s$, the $f$-support decreases as $T$ is lowered.

Such apparent contradictions can be solved studying the Ising $p$-spin model, which has both a 1RSB and a FRSB phase.\cite{20} Such an analysis will be presented elsewhere.

A closer analysis of the complexity of the SK model at a given temperature as a function of the free energy rises yet another, more serious, problem. In figure 7 the BRST-SUSY quenched complexity is plotted as a function of the free energy together with the BRST annealed complexity obtained with a 1RSB $q(x)_{14,18}$ at $T = 0.2$. Convexity implies that the annealed average must be larger than the quenched one and this must hold at any value of $f$. As we can see from the figures 7-8, the threshold value $f_{th}$ in the quenched case (vertical line on the right side in Fig. 8) is greater than in the annealed one (left side vertical line). This leads to a violation of the convexity because in the interval of $f$ values between the annealed and the quenched threshold free energies the annealed complexity is zero while the quenched one is finite. Also their maximum values violate the convexity requirement as it is shown in Fig. 8. In Fig. 9 the maximum value of $\Sigma$ is shown versus temperature. In the limit $T \to 0$ we get, for the maximum value of the complexity, the limit

$$\lim_{T \to 0} \Sigma^{\text{max}}(T) = 0.00778(5)$$

larger than the value obtained in the annealed approximation for the SK model ($\Sigma^{\text{max}}(0) = 0.0073$),\cite{14} corresponding to a 1RSB statics in absence of external fields.

### B. Analytical Resolution with Padé Resummation of Series Expansion

As we have seen (figure 1), $q(x)$ is a continuous monotonous function for $x < x_b$. At some $x_0 < x_b$ it develops a plateau such that $q(x) = q(x_0)$ for $x_0 \leq x \leq x_b$,
in particular \( q(x_b^-) = q(x_0) \). Setting \( x_b \) near the break point of the Parisi solution, \( x_c \), and defining their difference as \( \epsilon = x_c - x_b \), we derive the following relationships valid at any temperature:

\[
\begin{align*}
\Delta q & = O(\epsilon) \\
x_0 & = x_c - O(\epsilon) \\
q(1) & = q_c(1) - O(\epsilon^2) \\
q(x_0) & = q_c(1) - O(\epsilon) \\
q(x) & = q_c(x) + O(\epsilon^4) \quad x < x_0 \\
f_{\text{rep}}(\epsilon) & = f_{\text{rep}}^c + O(\epsilon^5)
\end{align*}
\]

where the subscript \( c \) stays for the case \( x \geq x_c \), i.e. the unconstrained FRSB solution of the SK model. In particular, \( q_c(1) = q_{\text{EA}} \) and \( f_{\text{rep}}^c = f_{\text{eq}} \).

From the derivative of Eq. (29) we see that

\[
\left[ 2 \frac{\partial f_{\text{rep}}}{\partial x_b} + x_b \frac{\partial^2 f_{\text{rep}}}{\partial x_b^2} \right]_{x_b = x_{\text{max}}} = 0
\]

the state free energy \( f \) takes its maximal value as a function of \( x_b \); as a consequence, smaller values of \( x_b \) have no physical meaning. Notice that at \( x_b = x_{\text{max}} \) the complexity \( \Sigma(x_b) \) is also maximal and \( \Sigma_{\text{max}} = \Sigma(x_{\text{max}}) \) represents the total complexity at that temperature.

Decreasing \( x_b \), the value \( q(x_0) \) of the lowest plateau decreases; in particular, at \( x_b = x_{1\text{RSB}} \) we have \( \Delta q = q(1|x_b) \), that is the FRSB solution free energy becomes higher than the one of the 1RSB function with \( q_0 = 0 \) and \( q_1 = q(1|x_b) \). However, \( x_{1\text{RSB}} < x_{\text{max}} \) and, therefore, it is always outside the "physical" range of \( x_b \)’s.

We have computed the function \( q(x) \) and the various quantities of interest in a series expansion in powers of \( \epsilon \).
and of the reduced temperature \( T = T_s - T \) \((T_s = 1)\) at very high orders through the methods described in Ref. [34]. The various series turn out to be asymptotics but they can be resummed e.g. through Padé approximants yielding very precise results in the whole low temperature phase. For the total complexity \( \Sigma_{\text{max}}(T) \) at temperature \( T \) we have the following series expansion:

\[
\Sigma_{\text{max}}(T) = \frac{\tau^6}{81} - \frac{2 \tau^7}{81} + \frac{187 \tau^8}{1215} - \frac{9938 \tau^9}{10935} + O(\tau^{10})
\]

Higher orders show that this series is strongly asymptotics (see Appendix), but Padé approximants of it give good results, in particular we have

\[
\lim_{T \to 0} \Sigma_{\text{max}}(T) = 0.0077(2)
\]

in agreement with the result obtained by numerical evaluation of the differential equations of the preceding section. If we compare the quenched complexity with the annealed one,\(^9,14,18\) we can observe that, at any order in \( \tau \), the convexity property is conserved up to the threshold energy value of the annealed solution, \( f_{\text{th}}^a \) (compare with figure 8), while it is violated soon after. At the threshold \( f_{\text{th}}^a \) the difference between the quenched and the annealed complexity is

\[
\Sigma(f_{\text{th}}^a) - \Sigma_a(f_{\text{th}}^a) = -\frac{3 + 2 \sqrt{2}}{2187} \tau^9 + O(\tau^{10}) < 0
\]

On the contrary, the difference between the maximum value of the two complexities is,

\[
\Sigma_{\text{max}}^a(\tau) - \Sigma_{\text{max}}(\tau) = \frac{\tau^8}{243} - \frac{2 \tau^9}{135} + \frac{3394 \tau^{10}}{5467} + O(\tau^{11})
\]

that is positive to the leading order, thus violating convexity. The above expansions up to very high order is reported in Appendix.

The self-overlap of the states with highest free energy is always smaller than the Edwards-Anderson parameter \( q_{EA} \), see figure 10, in particular near \( T = 0 \), the self-overlap of the states with maximal complexity is

\[
q(1|x_{\text{max}}) = 1 - 0.4T + O(T^2)
\]

while the Edwards-Anderson parameter has a \( T^2 \) dependence.

V. VIOLATION OF MARGINAL STABILITY

To solve the above mentioned problems the analysis must, then, be deepened. The behaviour of the complexity in itself not being enough to understand the properties of the system. We go, then, further analyzing its stability in the FRSB\(\Delta q\) scheme of computation.

A. The Plefka Criterion

A necessary condition for the physical relevance of the solution is the Plefka criterion,\(^39,40\) stating that the quantity

\[
x_P = 1 - \beta^2(1 - m^2),
\]

where the overbar denotes the average over quenched disorder, must be positive or zero. Such quantity coincides with the replicon eigenvalue \( \Lambda_R \) in the replica computation.

In order to compute \( x_P \) we need the quantity \( P(x^+_b, y) \). To this aim one can consider \( q \) as the independent variable instead of \( x \). The discontinuities in \( q(x) \) become plateaus of \( x(q) \) and vice versa (see figure 11). The functions \( P(q, y) \) and \( m(q, y) \) satisfy the differential equations

\[
\partial_q m(q, y) = -\frac{1}{2} m''(q, y) - \beta x(q) m(q, y) m'(q, y)
\]

where \( q_1 = q(1|x_{b}) \). Notice that the previous equations do not depend on the derivatives of \( x(q) \); therefore we can safely use them also where \( x(q) \) is discontinuous. This would not be true if we had used \( q(x) \) instead of \( x(q) \).

By solving these equations at fixed \( x = x_b \) we obtain:

\[
\int dy \ P(q_1, y) \ tanh^k \beta y = \int dy \ \tanh^k \beta y
\]

\[
\times \int dy' G_{\Delta q}(y - y') P(q_b, y') \cosh^{2k}(\beta y) \]

\[
\int dy'' G_{\Delta q}(y'' - y') \cosh^{2k}(\beta y'' - \beta y')
\]

FIG. 10: higher curve: \( q_{EA} \) as a function of \( \tau = 1 - T \); lower curve: self-overlap \( q(1|x_{\text{max}}) \) of the states with maximal complexity as a function of \( \tau = 1 - T \), note the finite slope at \( T = 0 \).
where \( G_\Delta(z) = \exp[-z^2/(2\Delta)] \) and \( q_b \equiv q(x_b) \). By a simple change of variables the last expression can be rewritten as:

\[
\int dy \, P(q_1, y) \tanh^k \beta y = \int dy \, P(q_b, y) \left( \tanh \beta (z \sqrt{\Delta q} + y) \right)
\]

where the average \( \langle \ldots \rangle \) is defined in Eq. (24), so that the Plefka parameter finally reads:

\[
x_p = 1 - \beta^2 \left[ 1 - 2q_1 + \int dy \, P(q_b, y) \left( \tanh \beta (\beta y + \beta z \sqrt{\Delta q}) \right) \right]
\]

In order to derive the Plefka parameter \( x_p \) we have to compute disorder averages of the single-site magnetization \( m^k \). They can be expressed in terms of the cavity field \( y \) which is the field induced by \( N \) spins on the \( N+1 \)-th spin \( s_0 \), averaged in the absence of the spin \( s_0 \). Introducing the distribution function \( P(y) \) of the cavity field, averaged with respect to the states and to the disorder, we have:

\[
\bar{m}^k = \int dy \, P(y) \tanh^k \beta y
\]

It can be shown that the distribution of the cavity field at equilibrium is simply related to the function \( P(x, y) \) introduced to compute the Parisi solution \( q(x) \). We now briefly discuss how the same results can be obtained using the cavity method. According to the Parisi picture the average number of equilibrium states with given free energy \( F \) inside a cluster \( \Gamma \) is given by

\[
dN(F) = \exp[-x(q)\beta F_{\Gamma} + x\beta F]dF
\]

Where \( F_{\Gamma} \) is the free energy of the cluster. The cavity method derives the expression of the free energy and the function \( q(x) \) by simply using the condition that the previous distribution (58) must reproduce itself when a new spin \( s_0 \) is added to a system of \( N \) spins. If we assume that there is a complexity curve \( \Sigma(f) \) we can apply the cavity method to reproduce the recipe of Ref. [11]. The only difference with the standard equilibrium treatment is the starting point, i.e. the average number of metastable states with free energies near a given free energy \( F^* = Nf^* \), which now reads:

\[
dN(F) = \exp \left[ N\Sigma(f^*) - \frac{\partial \Sigma}{\partial f} \bigg|_{f=f^*} F^* + \frac{\partial \Sigma}{\partial f} \bigg|_{f=f^*} F \right] dF
\]

If we make the formal identification

\[
\beta x = \left. \frac{\partial \Sigma}{\partial f} \right|_{f=f^*}
\]

the derivation of chapter V of Ref. [41] can be applied straightforwardly. There is a crucial difference, however, between the two expressions: the first quantity is \( O(1) \) for the states at the equilibrium free energy while the second quantity is \( O(e^{CN}) \) for the metastable states. Correspondingly, in the equilibrium case the final result coincides with the true equilibrium free energy, while in the latter case the result is the free energy of a system in which the states \( \alpha \) are weighted with a non-Boltzmann weight proportional to \( e^{-\beta x F_\alpha} \), where \( \beta x = \partial \Sigma(f^*)/\partial f \) is the Legendre conjugate of \( f \). A Legendre transform must therefore be applied to obtain the complexity, just as in Ref. [11].

In the context of the cavity method the distribution \( P(y) \) of the cavity field \( y \) inside a state, required to compute objects like \( m^k \), can be identified with the function \( P(x, y) \) at the value of \( x \) corresponding to the self-overlap of the states in the framework of replica computation (see Sec. II). The physical meaning of the functions \( P(x, y) \) and \( m(x, y) \) at any \( x \) can be understood and their evolution operators rederived. The function \( P(x, y) \) describes the distribution with respect to the disorder of the average cavity field \( y \) of a cluster labelled by \( q(x) \) and is computed taking into account the presence of the spin \( s_0 \). The function \( m(x, y) \) yields the magnetization, averaged over the cluster, of the spin \( s_0 \) when the average cavity field is \( y \). However, the weights assigned to the subclusters to compute the averages are not the Boltzmann weights. Indeed, the averages inside a cluster \( \Gamma \) are defined as:

\[
\langle O \rangle_\Gamma = \frac{\sum_\alpha e^{-\beta x F_\alpha} \langle O \rangle_\alpha}{\sum_\alpha e^{-\beta x F_\alpha}}
\]

Note that the previous average coincides with the Boltzmann averages only at the level of the configurations corresponding to \( x = 1 \). Thus the magnetization inside a state is actually the thermodynamic magnetization in presence of an average cavity field \( y \), i.e. \( m(1, y) = \tanh(\beta y) \). The peculiarity of these averages is that the subclusters with lower free energy no longer dominate the average, and all the subclusters have the same (infinitesimal) weight. As a consequence, the quantity \( \int dy \, P(x, y) \, m(x, y)^2 \), the average of the overlap inside a cluster, is simply given by \( q(x) \), yielding the variational equation (10). Had we weighted the subclusters with their Boltzmann weight this would not be true since few states would have dominated forcing us to consider diagonal contributions proportional to the self-overlap. The cavity method thus yields the evolution operators for \( P(x, y) \) and \( m(x, y) \) which coincide with the equations (8), (11).

**B. Overlap Discontinuity and Replica Eigenvalue**

In this section we present a simple argument in order to show that the FRSB quenched solution must be dis-
continuous at $x_b$ in order to implement the recipe of Ref. [11] but, at the same time, this discontinuity causes the replicon eigenvalue $\Lambda_R = x_P$ to be negative.

We start considering the expression of the replicated variational free energy as a function of $q(x)$ reported in Eq. (18). This expression must be extremized with respect to all parameters but $x_b$. Therefore, as noted in Ref. [25], at the extremum the total derivative with respect to $x_b$ of the function $f^{rep}$ is zero when $\Delta q = 0$, instead, if $\Delta q$ is small and that the corresponding function $\Sigma(f)$ is similar to the one of the equilibrium Parisi solution ($\epsilon = 0$). In particular, we assume that the function $x(q)$ displays a region with $\partial_q x(q) \neq 0$ on the left of the point $q_b$. This is precisely the behaviour we have encountered in the analyse of the SK model in the preceding sections. Our analysis will be valid only at small $\epsilon$, which corresponds to the region of free energy near the equilibrium free energy. This region however is quite important since it is natural to assume that the function $\Sigma(f)$, if it exists, is a continuous function of $f$ going to zero at the equilibrium free energy. Furthermore, the outcome of the present analysis is confirmed by numerical evaluation for which it is verified in the whole range of $x_b$ values, as we will show in the following.

It is useful to introduce the function $E(q)$:

\[
E(q) = q - \int dy \, P(q, y) \, m(q, y)^2
\]

The variational equations impose that $E(q) = 0$ where $\partial_q x(q) \neq 0$. Instead, if $x(q)$ is constant, e.g. for $q_1 \leq q \leq q_f$, we must have $E(q) = 0 = E(q_f)$ at the end points but not in the rest of the interval. In other words, if there is a plateau in $x(q)$ [which corresponds to a discontinuity in $q(x)$] the equation $q = \int dy \, P(q, y) \, m(q, y)^2$ must be verified only at the starting point and at the ending point. We mention that this is sufficient to show that the FRSB solution with a discontinuity at $x_b$ coincides with the partial derivative:

\[
\frac{\partial f^{rep}}{\partial x_b} = \frac{\beta}{4} \left[ q^2(1|x_b) - q^2(x_b^-) \right] + \frac{1}{\beta x_b} \int dy \, P(x_b^-, y)
\]

\[
\times \left[ \log \int Dz \, p(x_b, y, z) - \langle \log p(x_b, y, z) \rangle \right]
\]

where $p(x_b, y, z)$ is defined in Eq. (20). The above quantity is zero when $\Delta q = 0$, therefore we need a discontinuity of $q(x)$ at $x = x_b$ in order to compute the Legendre transform, Eq. (27).

In the following we show that the presence of the discontinuity leads to a negative replicon. The discontinuity corresponds to a plateau in the function $x(q)$ starting at $q_b$ and ending at $q_1$. The function $E(q)$ defined earlier must be equal to zero at $q_b$ and $q_1$ but not between these points, see figure 11. We work under the hypothesis that $\epsilon = x_e - x_0$ is small and that the corresponding function $x(q)$ is similar to the one of the equilibrium Parisi solution ($\epsilon = 0$). In particular, we assume that the function $x(q)$ displays a region with $\partial_q x(q) \neq 0$ on the left of the point $q_b$. This is precisely the behaviour we have encountered in the analysis of the SK model in the preceding sections. Our analysis will be valid only at small $\epsilon$, which corresponds to the region of free energy near the equilibrium free energy. This region however is quite important since it is natural to assume that the function $\Sigma(f)$, if it exists, is a continuous function of $f$ going to zero at the equilibrium free energy. Furthermore, the outcome of the present analysis is confirmed by numerical evaluation for which it is verified in the whole range of $x_b$ values, as we will show in the following.

It is useful to introduce the function $E(q)$:

\[
E(q) = q - \int dy \, P(q, y) \, m(q, y)^2
\]

The derivatives of the function $E(q)$ with respect to $q$ can be evaluated using the equations for $P(q, y)$ and $m(q, y)$ written above and expressed only in terms of derivatives with respect to $y$.

The central point of the argument is the observation that the replicon eigenvalue can be expressed as the derivative $\partial_q E(q)$ evaluated at $q_1$. Indeed at $q_1$ we have $m = \tanh \beta y$ which satisfies $m' = \beta(1 - m^2)$, thus $\partial_q E(q_1) = x_P$. This property leads immediately to the result that each model with a continuous FRSB $q(x)$, such as the SK model, is marginally stable, see e.g. [25,29] and Sec. II A.
The previous expressions allow to study the behaviour of the derivatives of $E(q)$ at the point $q_b$, where $x(q)$ is discontinuous (see fig. 11). Indeed, since the functions $P(q, y)$ and $m(q, y)$ are continuous at $q_b$, we can relate the derivatives on the right of $q_b$ to the derivatives on the left of $q_b$, which are identically zero. We have, then,

$$
\partial_q E(q_b^+) = \partial_q E(q_b^-) \rightarrow \partial_q E(q_b^+) = 0 \quad (68)
$$

$$
\partial_q^2 E(q_b^+) = 2\beta(x_0 + \Delta x) \int dy \, P(q, y)[m'(q, y)]^3 - \int dy \, P(q, y)[m''(q, y)]^2
$$

$$
= \partial_q^2 E(q_b^-) + 2\beta \Delta x \int dy \, P(q, y)[m'(q, y)]^3
$$

$$
= c_1 \Delta x > 0 \quad (70)
$$

with $\Delta x = x_b - x_0$. Analogously we have:

$$
\partial_q^3 E(q_b^+) = -2 \beta \partial_q x(q_b^-) \int dy \, P(q, y)[m'(q, y)]^3 + O(\Delta x)
$$

$$
= -c_2 + O(\Delta x) < 0 \quad (71)
$$

The last inequalities in (70) and (71) follow from the fact that the function $m(q, y)$ entering $\int dy \, P(q, y)[m'(q, y)]^3$ is equal to $\tanh \beta y$ at first order in $\Delta x$ and therefore at this order the integral is strictly positive. Now the relation

$$
0 = E(q_b^1) = \frac{c_1}{2} \Delta x \Delta q^2 - \frac{c_2}{6} \Delta q^3 + O(\Delta q^4) \quad (72)
$$

implies

$$
\Delta q = (3c_1/c_2) \Delta x + O(\Delta x^2) \quad (73)
$$

and

$$
\partial_q E(q_b^1) = -\frac{3}{2} (\Delta x^2) \frac{c_2^2}{c_1} + O(\Delta x^3) < 0 . \quad (74)
$$

The previous argument can be generalized to any mean field spin-glass model with a FRSB phase under the same hypothesis for the function $g(x)$ at $x_b$ near $x_c$, i.e. a discontinuity at $x = x_b$ and a region with $\partial_q x(q) \neq 0$ on the left side of $q_b = q(x_b)$. Indeed, in presence of multiple $p$-spin interaction the only change in $E(q)$ is that we must substitute the first term $q$ with a function $\partial_q f(q)$ which depends on the interaction strength. In general, the Plefka parameter $x_P$, or equivalently the replicon $\Lambda_R$, can be expressed as $\partial_q^2 f(q) - \int dy \, P(q, y) (1 - m^2)^2$, thus it can be again identified with $\partial_q E(q_b^1)$. The function $\partial_q f(q)$ does not depend on $x(q)$ and therefore it does not affect the difference between derivatives on the left and on the right side of $q_b$ and the inequalities in Eqs. (70) and (71) still hold, leading again to a negative replicon.

The argument can also be generalized to soft-spin systems (with soft-spin distribution normalized to $\langle S^2 \rangle = 1$): in general at $q_1$ we have

$$
m(q_1, y) = \frac{\int P(S) e^{\beta y S} dS}{\int P(S) e^{\beta y S} dS} \rightarrow m' = \beta (1 - m^2) \quad (75)
$$

thus the identification of the replicon with the derivative $\partial_q E(q)$, computed at $q = q_1 = q(1|x_b)$, holds once again.

C. BRST-SUSY Subextensive Complexity

Our most important result is that the quenched FRSB BRST-SUSY solution here described must be rejected on physical ground at any value of free energy apart from the equilibrium one.

We have checked that the BRST-SUSY solution studied in Sec. IV does not satisfy the Plefka criterion [equation (49)], i.e. the replicon eigenvalue is negative as soon as $x_b < x_c$. In figures 12 and 13 we plot the replicon, Eq. (56) versus the breaking parameter $x_b$ and the free energy $f$: it turns out to be always negative for $x_b < x_c$, going to zero from below as $x_b \rightarrow x_c$ and remaining zero for any value of $x_b$ above the self-consistent point $x_c$. Even going beyond the annealed approximation for the SK model,14,18 where the relative 1RSB static phase was not satisfying Plefka’s criterion, the condition $\Lambda_R \geq 0$ remains violated. The FRSB BRST-SUSY solution with a non zero $\Delta q$ is not even marginally stable. In order to have marginal stability it must be $\Delta q = 0$ that means $x_b \geq x_c$, i.e. the breaking point cannot be imposed but it has to be determined self-consistently.

For such a marginally stable solution, however, $\Sigma$ is zero, thus there is no way to implement the Legendre transform approach to compute any physically well defined complexity. If BRST-SUSY holds, there is no exponentially large number of states at any energy level above $f_{eq}$ in the SK model.

In terms of free energies, $f_{eq}$ is equivalent to the lower band-edge $f_0 = f(x_c)$, since we are computing the statistics using the right FRSB scheme. This means that, provided the BRST-SUSY complexity actually counts some extensive number of TAP solutions, they have no physical meaning. As discussed in Ref. [18] the violation of the Plefka criterion leads also to a mathematical inconsistency with a given assumption implicit also in the present solution (i.e. the condition $B = 0$, see Refs. [9,14,18,22,23]). As we have previously shown in the present section, a direct computation indicates that this result can be extended to generic mean-field models with a continuous FRSB $q(x)$, e.g. the Ising $p$-spin model below the Gardner transition (see Ref. [5]).

Since this solution is unphysical we are led to the conclusion that, holding SUSY, the number of states of the system is subextensive and the states have all the same free energy per spin equal to the equilibrium value. Furthermore all the states verify $x_P = 0$. 


the quenched complexity

$$\Sigma_q = \frac{1}{N} \log \rho_s(f) \quad (76)$$

in a more general formulation, inspired by the Boltzmann microscopic definition of entropy. It reduces to the one we just employed if particular supersymmetric relations are satisfied. The FRSB expression for the quenched complexity has been presented for the first time, as far as we know, in Ref. [22], as

$$\Sigma_q = -\lambda q - u T - \Delta (1 - q) - \frac{\Delta^2}{2\beta^2} + \frac{1}{2\beta^2} \int_0^1 dx \left[ \rho^2(x) + \eta^*(x) \eta(x) \right] + \phi(0, 0, 0) \quad (77)$$

where $\phi(0, 0, 0)$ is the “equilibrium”, zero local fields value of the function $\phi(x, h_1, h_2)$ solution of the antiparabolic three-dimensional differential equation

$$\dot{\phi} = -\frac{1}{2} \frac{\partial^2 \phi}{\partial h_1^2} + x \left( \frac{\partial \phi}{\partial h_1} \right)^2 - \rho \left[ \frac{\partial^2 \phi}{\partial h_1 \partial h_2} + x \frac{\partial \phi}{\partial h_1} \frac{\partial \phi}{\partial h_2} \right] \quad (78)$$

where the derivative with respect to $x$ is indicated by the dot. The boundary condition at $x = 1$ is

$$\phi(1, h_1, h_2) = \log \int_{-1}^1 \text{d}m \ e^{\mathcal{L}_{\text{full}}(m; u, q, \Delta, \lambda, \eta(1), \rho(1), \eta^*(1))} \quad (79)$$

$$\mathcal{L}_{\text{full}}(m; u, q, \Delta, \lambda, \eta(1), \rho(1), \eta^*(1)) = -\frac{1}{2} \log \left[ 2 \pi (\beta^2 q - \eta(1)) \right] + \log \left( \frac{1}{1 - m^2} + B \right)$$

$$+ \left[ \tanh^{-1} m - (\Delta + \rho(1)) m + h_1 \right]^2$$

$$+ \left( \lambda - \eta^*(1) \right) m^2 + uf(m; q) + h_2 m \quad (80)$$

$$f(m; q) = -\log 2 + \frac{1}{2} \log (1 - m^2)$$

$$+ \frac{1}{2} \rho \left[ \frac{1}{1 - m^2} + B \right] \quad (81)$$

The order parameters $q, \lambda, \Delta, \eta(x), \eta^*(x), \rho(x)$ have to be determined self-consistently by extremising (77). Since we will reformulate everything in the generalized two-group Ansatz notation we will present the self-consistency equations later on, once the linear transformations to that formalism will be applied. We only mention that $\eta(x)$ is proportional to what was called $q(x)$ up to now and that the BRST-SUSY implies a dependency of $\Delta$ and $\lambda$ on $q$ and of $\eta^*$ and $\rho$ on $\eta$. The number $q$ is the value of all the elements of the diagonal submatrices of the two-group Ansatz, and it is usually set equal to $q(1)$ in the unbroken two group scheme.
A. BRST relations

Before going on describing of the two group Ansatz we very shortly recall the relations among order parameters satisfied by systems that are invariant under BRST-SUSY.

In the BM notation, the macroscopic BRST relations, at the level of order parameters, are:

\[ \Delta = - \frac{\beta^2}{2} u^q \]  
\[ \lambda = \frac{\beta^2}{8} u^q \]

for the annealed approximation and

\[ \rho_{ab} = \frac{\beta^2}{2} u^q \rho_{ab} \]

\[ \eta_{ab}^* = \frac{\beta^2}{4} u^q \]

in the quenched RSB case (see Ref. [44], where a detailed derivation is presented in the notation of Ref. [14]).

B. Formulation in the two-group Ansatz

In this section we will first recall the approach of Ref. [10] in order to reformulate Eqs. (77 - 80) in the two-group formalism. The motivation to perform such a transformation is that, from the point of view of the BRST-SUSY, imposing the BRST relations is equivalent to set equal to zero all order parameters involved but \( q(x) \).

The two-group Ansatz was introduced by Bray and Moore in Ref. [12] in order to solve the instability problem of the replica symmetric solution of the SK model. In Ref. [11] Monasson showed how the formalism of Legendre transforms can be applied to mean-field disordered models through the \textit{pinning of real replicas} in a configuration space extended to \( m \) copies of the system.

Parisi and Potters explained how the BM action can be obtained through the method of Monasson provided that the symmetry between real replicas is broken according to a generalized two-group Ansatz. Being \( n \) the number of replicas introduced to compute the quenched average and \( m \) the number of real copies, they analyzed

\[
\lim_{n \to 0} \frac{1}{n} \log \mathbb{Z}^{mn} = \text{ext}_f \left[ \log \rho_\lambda(f) - m \beta g N f \right]
\]

where \( \mathbb{Z}^{mn} \) is the partition function of \( n \times m \) copies of the system. In terms of the replicated matrix-parameter \( \mathbb{Q} \), its average is [cfr. Eq. (2)]

\[
\mathbb{Z}^{mn} = \text{ext} \exp \left\{ N \left[ \frac{\beta^2}{4} \left( mn - \text{Tr} \mathbb{Q}^2 \right) \right] \right\}
\]

where the indexes \( a, b = 1, \ldots, n \) while \( c, d = 1, \ldots, m \). The four index matrix \( \mathbb{Q}_{ab}^{cd} \) can be expressed as the composition of \( n^2 \) sub-matrices \( \mathbb{Q}_{ab} \) of dimension \( m \times m \) of the form:

\[
\mathbb{Q}_{ab} = \begin{pmatrix}
\mathbb{Q}_{ab}^+ & \mathbb{Q}_{ab}^{-m} \\
\mathbb{Q}_{ab}^{-m} & \mathbb{Q}_{ab}^-
\end{pmatrix}
\]

The matrices \( \mathbb{Q}_{ab}^\pm \) are further parameterized as

\[
\mathbb{Q}_{ab}^\pm = \mathbb{Q}_{ab} \pm \frac{A_{ab}}{y} + \frac{C_{ab}}{2y^2}
\]

Furthermore, \( \mathbb{Q}_{ab}^{cc} \equiv 0 \). In Ref. [10] the last term was denoted by \( B_{ab}/y^2 \). We write \( C_{ab} = 2B_{ab} \) in order to obtain more symmetric expressions in the following and to avoid confusion with the parameter \( B \) in Eq. (80).

Evaluating Eq. (86) with this Ansatz, one obtains a complexity that can be formally connected to the one of Bray, Moore and Young through a given change of variables.

For what concerns the diagonal sub-matrices - corresponding to the annealed case - the transformations bringing from the notation of Ref. [22] (PP) to the one of the generalized two-group [10] (PP) are:

BM PP (two-group)
\[ q = \mathbb{Q} \]
\[ \Delta = \beta^2 \left( A + \frac{m}{2} \mathbb{Q} \right) \]
\[ \lambda = \frac{\beta^2}{2} \left( C + m A + \frac{m^2}{4} \mathbb{Q} \right) \]
\[ u = -m \]

Writing equations (77)-(81) with the substitutions (90)-(93), allows for an immediate connection between the breaking of the matrices structure into two groups and breaking the BRST-SUSY. Indeed, Eqs. (91) and (92) transform into the BRST relations (82)-(83) if we set \( A = C = 0 \), i.e. if we do not break the matrix structure at all (\textit{unbroken two-group}). On the contrary, setting values of \( A \) and \( C \) different from zero amounts to break the BRST-SUSY and leads to independent values of \( q, \Delta \) and \( \lambda \).

The changes of variables for the off diagonal terms \( \eta_{ab}, \eta_{ab}^* \) and \( \rho_{ab} \) are:

BM PP (two-group)
\[ \eta_{ab} = \beta^2 \mathbb{Q}_{ab} \]
\[ \rho_{ab} = -\beta^2 \left( A_{ab} + \frac{m}{2} \mathbb{Q}_{ab} \right) \]
\[ \eta_{ab}^* = \beta^2 \left( C_{ab} + m A_{ab} + \frac{m^2}{4} \mathbb{Q}_{ab} \right) \]

Setting \( A_{ab} = C_{ab} = 0 \) in the two-group construction reproduces the generalized BRST relations for the replica symmetry breaking case, Eqs. (84)-(85).
Is there any FRSB solution breaking the BRST-SUSY? If a complexity indeed exists the lower band-edge of its
dominion in free energy must coincide with the SK equi-
librium free energy $f_{eq}$, computed making use of the
FRSB scheme. Such a value has been found also by
Bray, Moore and Young, starting from the point $u = u_0$
such that $\Sigma(u_0) = 0$ a BRST-SUSY solution develops.
Indeed, their solution was based on an assumption equiv-
lent to the BRST-SUSY and, therefore, it was not the
quenched analogue of the BM solution (see Sec. VI C).

As we just underlined, the BRST relations for the or-
der parameters involved in the computation can be cast
in a very straightforward way if the notation of general-
ized two-group Ansatz of Ref. [10] is used.

In the Full RSB limit the matrices $Q_{ab}$, $A_{ab}$ and $C_{ab}$
tend respectively to the functions $q(x)$, $a(x)$ and $c(x)$ as
$n \to 0$. In such limit the expression for the quenched
complexity (77) (with $B = 0$) becomes

$$
\Sigma_q = -\beta u f - \beta^2 (1 - Q) \left[ A - \frac{u}{2} Q \right] + \phi(0,0,0)
+ \beta^2 \frac{1}{4} \int_0^1 dx \left[ 2 \left( A^2 - a^2(x) \right) - 4u \left( A Q - a(x)q(x) \right)
+ 2 \left( Q C - q(x)c(x) \right) + u^2 \left( Q^2 - q^2(x) \right) \right]
$$

with boundary condition

$$
\phi(1, y_1, y_2) = \log \int_{-1}^1 dm \ e^{L_{full}}
$$

$$
L_{full} = \beta u f(m; q) - \beta m \left( \frac{u}{2} y_1 - y_2 \right)
+ \log \left( \frac{1}{1 - m^2} \right) - \frac{1}{2} \log \left[ 2\pi \beta^2 (Q - q(1)) \right]
+ \beta^2 \frac{m^2}{2} \left[ C - c(1) - u (A - a(1)) + \frac{u^2}{4} (Q - q(1)) \right]
- \frac{1}{2\beta^2 (Q - q(1))} \left( \tanh^{-1} m - \beta y_1 \right)
- \beta^2 m \left[ (A - a(1)) - \frac{u}{2} (Q - q(1)) \right] \right] \right]
$$

$$
\beta f(m; q) = -\log 2 + \frac{1}{2} m \tanh^{-1} m - \frac{\beta^2}{4} (1 - Q^2)
$$

and where the local fields $h_1$, $h_2$ in Eq. (78) have been changed in $y_1$, $y_2$ following the linear transformation:

$$
y_1 = -\frac{h_1}{\beta}
$$

$$
y_2 = -\frac{u h_1}{2 \beta} + \frac{h_2}{\beta}
$$

Setting

$$
m \equiv \tanh(\beta y_1 + \beta z \sqrt{\Delta q})
$$

the above formula (99) becomes

$$
\phi(1, y_1, y_2) = -\frac{\beta^2}{4} u (1 - Q^2)
+ \log \int_{-\infty}^{\infty} Dz p(-u, y_1, z) \ e^{L_2}
$$

$$
L_2(z, y_1, y_2) \equiv \beta \tanh(\beta y_1 + \beta z \sqrt{\Delta q})
\times \left[ \frac{\Delta a}{\sqrt{\Delta q}} + y_2
+ \frac{\beta}{2\Delta q} \tanh(\beta y_1 + \beta z \sqrt{\Delta q}) \left( \Delta q c - (\Delta a)^2 \right) \right]
$$

where $p(u, y, z)$ is defined in Eq. 20.

The self-consistency equations for the order parameters are:

$$
q(x) = \int dy_1 dy_2 P(x, y_1, y_2) [\partial_y q(x, y_1, y_2)]^2
$$

$$
\frac{\partial a(x)}{\partial x} - u q(x) = \int dy_1 dy_2 P(x, y_1, y_2)
$$

$$
\partial_y q(x, y_1, y_2) \partial_y \phi(x, y_1, y_2)
$$

$$
c(x) - 2u a(x) + u^2 q(x)
$$

$$
= \int dy_1 dy_2 P(x, y_1, y_2) [\partial_y a(x, y_1, y_2)]^2
$$

$$
Q = \langle \langle m^2 \rangle \rangle
$$

$$
A = \frac{1}{2Q - q(1)} \left\{ Q a(1) - (Q - q(1))(1 - Q(1 + u)) \right\}
\frac{1}{2\beta^2 (2Q - q(1))} \langle \langle m (\tanh^{-1} m - \beta y_1) \rangle \rangle
$$

$$
C = Q + 2A + u (1 - 2Q + 2A) - u^2 Q
$$

$$
-2 \frac{\Delta a}{\Delta q} \left[ A - Q + 1 - \frac{u}{2} Q \right] - Q \left( \frac{\Delta a}{\Delta q} \right)^2
$$

$$
- \frac{1}{2\Delta q} \left[ 1 - \frac{1}{2\beta^2 \Delta q} \langle \langle (\tanh^{-1} m - \beta y_1)^2 \rangle \rangle \right]
$$

where we have made use of Eq. (110) in Eq. (111) and Eqs. (110), (111) to yield Eq. (112). The average $\langle \langle \langle \ldots \rangle \rangle \rangle$ is defined as

$$
\langle \langle O(m, y_1, y_2) \rangle \rangle = \int dy_1 dy_2 P(1, y_1, y_2)
$$

$$
\times \int_{-1}^{1} dm \ O(m, y_1, y_2) e^{L_{full}}
$$

or else, using Eq. (104), as
\[
\langle O(z, y_1, y_2) \rangle = \int_0^1 dy_1 \, dy_2 \, P(1, y_1, y_2) \quad (114)
\]
\[\times \frac{\int_0^\infty Dz O(z, y_1, y_2) \, p(-u, u, y_1, z) \, e^{cz}}{\int_0^\infty Dz \, p(-u, y_1, z) \, e^{cz}} \]

The saddle point equations (110)-(112) for the elements of the diagonal matrices can, then, be written as

\[
Q = \left\{ \tanh(\beta \, z \, \sqrt{\Delta q + \beta \, y_1}) \right\} \quad (115)
\]
\[
Q \, \Delta a + A \, \Delta q = -\Delta q[1 - Q(1 + u)]
\]
\[
+ \frac{\sqrt{\beta} \, \Delta q}{\epsilon} \left\{ \int \left( \tanh(\beta \, z \, \sqrt{\Delta q + \beta \, y_1}) \right) \right\}
\]
\[
C = Q + 2A + u(1 - 2Q + 2A) - u^2Q
\]
\[
- \frac{\Delta a}{\Delta q} \left( A - Q + 1 - \frac{u}{2} \right) - Q \left( \frac{\Delta a}{\Delta q} \right)^2
\]
\[
- \frac{1}{2\Delta q} \left( 1 - \Delta q \langle z^2 \rangle \right)
\]

The Parisi Eq. (5) and the boundary condition Eq. (18) can be obtained as a BRST-SUSY reduction of Eqs. (98)-(99), setting \( A = a(x) = C = c(x) = 0 \), but leaving \( Q \neq q(1) \). Furthermore, in our notation, \( q(1) \) becomes \( q(x_b) \) whereas \( Q \) is \( q(1|x_b) \) as represented in the self-consistency Eq. (26).

If a BRST-SUSY-breaking solution exists it must be a solution of the above equations with, as necessary (but not sufficient) features: \( a(x) \neq 0 \) and/or \( c(x) \neq 0 \), a lower band-edge equal to \( f_{eq} \) and a positive \( x_P \) decreasing to zero as \( f \to f_{eq} \).

C. An instance of BRST-SUSY solution

In Ref. [23] the initial condition of the very same equation (5) is given by Eq. (14), that we report here for clarity:

\[
Z(h_1, h_2) = \int_{-1}^1 \frac{dm}{\sqrt{2\pi(Q_{bm} - \eta(1))}} \left( \frac{1}{1 - m^2} \right)
\]
\[\exp\left\{-\frac{\left( \tanh^{-1} m - \hat{\Delta} m + h_1 \right)^2}{2(Q_{bm} - \eta(1))}\right\}
\[+ \left( \lambda - \frac{1}{2} \eta'(1) \right) m^2 + u \, f(m; q) + m \, h_2 \}
\]
\[f(m; q) = -\log 2 + \frac{1}{2} \log(1 - m^2)\]
\[+ \frac{1}{2} m \, \tanh^{-1} m - \frac{\beta^2}{4} \left( 1 - q_{EA}^2 \right)\]

The above expression coincides with Eq. (80) if we set \( B = 0, Q_{bm} = \beta^2 q \) and \( \hat{\Delta} = \Delta + \nu(1) \). The symbol \( f \) is what we call \( \beta f \) in the present paper.

Such a function is also identical to our boundary condition Eq. (18), provided a suitable change of notation is performed. First of all the \( x \)-range has to be shifted as

\[x \in [0, 1] \rightarrow x \in [0, x_b]\]

so that any parameter computed in \( x = 1 \) will result as computed in \( x_b \) in our notation. Then we identify \( u = -x_b \) and on the line of transformations (90)-(92), (94)-(96) we make the following change of variables:

\[Q_{bm} = \beta^2 q(1|x_b) \quad \eta(x) = \beta^2 q(x) \quad (119)\]
\[\lambda = \frac{x_b^2}{16} \beta^2 q(1|x_b) \quad \eta^*(x) = \frac{x_b^2}{4} \beta^2 q(x) \quad (120)\]
\[\Delta = \frac{x_b}{2} \beta^2 q(1|x_b) \quad \rho(x) = -\frac{x_b}{2} \beta^2 q(x) \quad (121)\]

If \( x_b \geq x_c \), \( q(1|x_b) = q_{EA} \). Moreover the local fields are trasformed as in Eqs. (102)-(98), with \( y_1 = y \) and setting \( y_2 = 0 \) without loss of generality:

\[h_1 = -\beta y \quad h_2 = \frac{x_b}{2} \beta y \quad (122)\]

and the integration variable \( m \) (TAP-site-magnetization) is changed in \( z \) using Eq. (104).

The function \( \phi(x, y_1, y_2) \) involved is linearly connected to the solution of Eq. (5) with boundary condition (18) by the relation

\[\phi_{BMY}(x, y, 0) = \beta x_b \phi_{Legendre}(x, y) + \frac{\beta^2 x_b}{4} (1 - q_1)^2 \quad (123)\]

Exploiting Eqs. (118)-(122), (104) and (123), the complexity becomes equal to the Legendre transform of \( f_{rep} \), given by Eq. (27).

The actual computation performed by Bray, Moore and Young set, furthermore, the difference \( \Delta q = (Q_{bm} - \eta(1))/\beta^2 = q(1|x_b) - q(x_b) \) equal to zero.

D. Discussion

The resolution of Eq. (98) keeping \( A, C, a(x) \) and \( c(x) \) different from zero from the beginning should lead to a BRST-SUSY-breaking complexity. Such a solution would display some self-consistently determined order parameters, given by Eqs. (107)-(112), breaking the supersymmetry and allowing for the construction of a quenched complexity, function of some \( x_b \) break parameter and its conjugated \( f \).

From what is already known about the annealed complexity and the property of marginal stability of the SK model at the lowest free energy values, we expect this solution to fulfill precise requirements. First of all there cannot exist states of any kind below \( f_{eq} \), that is the minimal free energy value achievable by the system at a given temperature. Therefore the lower band edge \( f_0 \),
that is less than $f_{eq}$ in the annealed approximation, must coincide with it in the quenched computation. Moreover the system is known to be marginally stable, i.e. the replicon, or Plefka parameter, is zero at $f_{eq}$.

The BRST-SUSY-breaking annealed solution displays a strictly positive $x_P$ for any value of the $f$-conjugated $u$ variable for which the complexity is positive. Numerical evaluations of this parameter, directly counting the number of solutions of the TAP equations for a $N$ spins system yield different, apparently contradictory, results. In Ref. [46] Bray and Moore performed the direct evaluation of the number of solutions of TAP equations and in Ref. [47] Plefka has carried out an analogous probe considering modified TAP equations such that only solutions of the original equations with $x_P \geq 0$ are selected. In both cases a finite size scaling analysis reveals that the replicon eigenvalue/Plefka parameter goes to zero as $N^{-1/3}$, thus indicating an exclusively marginal kind of stability. On the contrary, a very recent numerical study by Cagnaga et. al[50] displays an $x_P$ behavior consistent with the one obtained for the annealed BRST-SUSY-breaking solution, hinting a strictly positive $x_P$ in the whole free energy support for which $\Sigma > 0$ (see Ref. [18] for the anneal lead complexity $x_P$). Even about the free energy density support numerical findings disagree, Plefka’s results hinting for a vanishing support[47] as $N$ increases, whereas the results of Ref. [48] show a finite support (even consistent with the annealed support, at least for $N \leq 80$).

Further analysis would be worthwhile to better understand the whole picture. In Ref. [9] the solution of the annealed saddle point equations was shown also to satisfy the quenched ( replica symmetric) saddle point equations for all $u$ above a certain, negative, value $u$. This includes $u = 0$, to which the largest value of the complexity corresponds. Over the $f$-support this equivalent to say that the annealed solution holds for $f \in [f(\Sigma), f_u]$ but is unstable in the range $[f_0, f(\Sigma)]$. Therefore, going to the exact quenched computation $x_P$ should decrease from $x_P(f(\Sigma)) > 0$ to $x_P(f_{eq}) = 0$ in a well defined solution.

The ensemble of TAP-solutions over which the saddle points are computed is of unknown nature, by this meaning that we cannot know a priori if the TAP stationarity points counted by our complexity are minima or saddles of any order and whether they lead to the correct linear susceptibility ($x_P \geq 0$) or not ($x_P < 0$). As a consequence, we are not able to say whether we are counting all possible solutions (in which case a complexity breaking the supersymmetric invariance would be mathematical inconsistent) or only some solutions belonging to a particular subset (in which case the invariance would not be a necessary requirement anymore). Recently Apelmeier et. al[52] have shown that the BRST-SUSY-breaking, else said BM, 9 complexity is counting stationary points of the spin-glass free energy landscape, that are minima with one flat direction and that lead to $\chi_l = \beta^x (1 - q)$. The flat direction is bounded to an isolated zero eigenvalue that can only be found considering subextensive corrections to the complexity saddle point. These very same corrections amount to a prefactor of $\exp N \Sigma^{SP}$ that turns out to be zero at all orders in a $1/N$ expansion when computed on the BRST-SUSY-breaking saddle point. This was shown by Kurchan[49] for the case $u = 0$, corresponding to the maximum value of such complexity, (and to zero complexity in BRST-SUSY case). That argument can be extended to any value of $u$ (or $f$), indicating that the prefactor can be either exactly zero or exponentially small in $N$. Since the Morse theorem must hold even for the BRST-SUSY-breaking solution the vanishing prefactor should go like $\exp( - N \Sigma^{SP})$ at $u = 0$.

The zero prefactor and the isolated eigenvalue are different ways of expressing the fact that metastable states in the SK model are of a different nature with respect to those of the $p$-spin interacting model, with $p > 3$. An isolated eigenvalue has, indeed, been found also in the spherical $p$-spin model[5] but there it is always positive. This difference is crucial to distinguish between the dynamical behaviour of spin-glasses and structural glasses relaxation. In the case of $p > 2$, representing a good mean-field model for structural glasses, metastable states are proper local minima (see e.g. [3-5] for the spherical $p$-spin model or [20] for Ising $p$-spin) whereas in the SK model they turn out to be stationary points of different kinds in the thermodynamic limit.

VII. CONCLUSIONS

The main motivation for our work has been the need for a clearer understanding of the two different approaches that have been presented for the behavior of the complexity in the SK model up to now: the first one was originally introduced by Bray and Moore[9] and the second one initially proposed by Parisi and Potters[10] and explicitly carried out (in a somehow different conceptual framework) quite recently in Ref. [14].

In Ref. [10] it was shown that the complexity could be obtained by calculating the partition function of $m$ distinct real replicas of the system[11] and provided the connection with the previous BM formalism by means of a generalization of the two-group Ansatz.12 In that framework they assumed an ‘unbroken’ two-group Ansatz in order to perform explicit computation. This has been finally performed exploiting the knowledge of BRST SUSY[15] (formally equivalent to the unbroken two-group)[18] and leading to a supersymmetric complexity different the BM one.[14]

The difference with the BM complexity has turned out not to be at the functional level, but simply the new solution is a second saddle point of the same functional.[18] This means that there is just one BRST-SUSY functional, the BM complexity functional[9,10] displaying (at least) two saddle points, one satisfying the BRST-SUSY[14] and the other one breaking it[9]. Both solutions identify an extensive complexity, computing the number of solutions of the Thouless-Anderson-Palmer
(TAP) equations in the annealed approximation. In Ref. [18] the authors analyzed the two annealed solutions and their problems reaching the conclusion that both of them had to be discarded as candidate complexities of the SK model.

In this paper we have faced the quenched calculation and the analysis of the properties of the complexity of the Sherrington-Kirkpatrick model making use of the Full Replica Symmetry Breaking Ansatz. Such complexity can be seen as the Legendre transform of the replicated free energy potential. A detailed description of the behavior of the complexity is presented in Sec. IV. Results are obtained both by numerical extremization of the replica free energy in the FRSB limit at a given temperature below the critical one (T_s) and by analytical expansion around the critical temperature. The complexity we find is the quenched analogue of the annealed complexity studied in Ref. [14]. As such, it satisfies BRST-SUSY invariance. Even though going to the exact quenched computation cures some deficiency intrinsic in the annealed approximation (e.g. the lower band-edge now coincides with the equilibrium free energy) other problems arise or old inconsistencies are not overcome. The most important outcome is that any attempt to build a BRST-SUSY complexity using the Legendre transform approach at finite temperature leads to a negative replicon everywhere in the complexity support apart from the lower band-edge/equilibrium value f_eq, where the replicon is zero and replica stability is marginal. This means that the solutions of the mean field TAP equations that our observable is counting are not physically relevant. More specifically they do not express a local linear susceptibility equal to β(1 − f_eq). This is related, in the replica language, to an instability against overlap fluctuations. Moreover, this leads to an even more serious problem, since a negative τ_p is mathematically inconsistent with the B = 0 solution (see Sec. VI) of the saddle points equations for complexity, as shown in the Appendix of Ref. [18].

We are, then, brought to the conclusion that a complexity satisfying BRST-SUSY, such as, e.g. the one of p-spin models (p > 2), cannot yield any extensive contribution in the SK model. A subextensive complexity was yield by the analysis performed by van Mourik and Coolen in a dynamical reformulation of Parisi solution in terms of a multiple (infinite) hierarchy of decoupled time-scales, in which a small fraction of slower spins plays the role of an effective quenched disorder for the faster spins. Since in their construction the quantity of ‘slow’ spins turns out to be subextensive with respect to the quantity of ‘fast’ spins, the entropy of the slowest dynamic variable, i.e. the complexity, is zero. This means, though, that a “supersymmetric assumption” is hidden somewhere in their procedure.

We eventually discussed the possibility of computing a BRST-SUSY-breaking quenched complexity, by means of the tools presented in Sec. VI. Even though the nature of the metastable states counted by such complexity would be different from the one of structural glass metastable states, and conceptually less intuitive, an extensive, thermodynamically stable, complexity can, in principle, still be computed.

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Appendix

In this appendix we show the expansion of some observables in the ’reduced’ temperature τ ≡ T_s − T, where T_s = 1 is the critical temperature of the SK model. The following expansion are strongly asymptotics at higher orders, but, as already anticipated in Sec. IV B, Padé resummation allows for converging results.

The maximum complexity as a function of the reduced temperature is:

\[ \Sigma_{\text{max}}^\tau = \Sigma_{0}^\tau + \frac{\tau^6}{81} - \frac{2\tau^7}{81} + \frac{187\tau^8}{1215} - \frac{9938\tau^9}{10935} + \frac{2313541\tau^{10}}{382725} - \frac{10350722\tau^{11}}{22963} + \frac{1912213832\tau^{12}}{51667875} \]

\[ - \frac{5717233507\tau^{13}}{17222625} + \frac{12852517881914\tau^{14}}{3978426375} - \frac{18216238077263593\tau^{15}}{537087560625} + \frac{133461584458391896554\tau^{16}}{34910691440625} + O(\tau^{19}) \]

The difference between the maximum value of quenched and annealed complexities is given by

\[ \Sigma_{\text{max}}^\tau - \Sigma_{0}^\tau = \frac{\tau^8}{243} - \frac{2\tau^9}{135} + \frac{3394\tau^{10}}{54675} - \frac{1381\tau^{11}}{4725} + \frac{9052283\tau^{12}}{7381125} - \frac{788437\tau^{13}}{273375} \]

\[ - \frac{6757091354\tau^{14}}{217005075} + \frac{84666457147717\tau^{15}}{107417512125} - \frac{136466739459370883\tau^{16}}{10741751212500} \]
The difference between the threshold free energies yield by quenched and annealed computation is

$$f_{th}(\tau) - f_{th}^{\infty}(\tau) = \frac{\tau^7}{162} - \frac{13 \tau^8}{810} + \frac{2711 \tau^9}{54675} + \frac{70474 \tau^{10}}{38275} - \frac{1701148 \tau^{11}}{5740875} + \frac{453457547 \tau^{12}}{8268600} - \frac{269738513263 \tau^{13}}{21158395021340794679} + \frac{2411167500 \tau^{14}}{477411165000} - \frac{21158395021340794679 \tau^{15}}{3258331201125000} + O(\tau^{16})$$

The above equation is expanded to order $\tau^{16}$, which is a significant contribution in the study of spin glasses and related systems.