CONGRUENCES BETWEEN HILBERT MODULAR FORMS OF WEIGHT 2, AND SPECIAL VALUES OF THEIR $L$-FUNCTIONS

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Abstract. The purpose of this paper is to show how a congruence between (the Fourier coefficients of) a Hilbert cusp form and a Hilbert Eisenstein series of parallel weight 2 gives rise to congruences between algebraic parts of critical values of their $L$-functions. This is a generalization of a result of V. Vatsal.

0. Introduction

0.1. Introduction. In this paper, we study a way to obtain congruences between special values of $L$-functions from a congruence between a Hilbert cusp form and a Hilbert Eisenstein series of parallel weight 2. Our result is a generalization of the work of V. Vatsal [Vat] for elliptic modular forms.

Let $F$ be a totally real number field of degree $n$ with narrow class number $h_F^n = 1$. Let $\Delta_F$ denote the discriminant of $F$. Let $n$ be an integer such that $n$ is prime in $6\Delta_F$. Let $p$ be a prime number such that $p > n + 2$ and $p$ is prime to $6n\Delta_F$. We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Let $O$ be the ring of integers of a finite extension $K$ of $\mathbb{Q}_p$ and $\varpi$ a uniformizer of $O$. Let $M_2(n, O)$ (resp. $S_2(n, O)$) denote the space of Hilbert modular (resp. cusp) forms of parallel weight 2 and level $n$ with coefficients in $O$. Let $\Gamma_0$ denote the set of all cusps of $H^n$ and $\Gamma_0$ be an integral ideal of $O$ and embeddings $\mathbb{Q} \hookrightarrow \mathbb{Q}_p \hookrightarrow \mathbb{C}$. Let $O$ be the ring of integers of a finite extension $K$ of $\mathbb{Q}_p$ and $\varpi$ a uniformizer of $O$. Let $M_2(n, O)$ (resp. $S_2(n, O)$) denote the space of Hilbert modular (resp. cusp) forms of parallel weight 2 and level $n$ with coefficients in $O$ (see (1.2)). Let $Y(n)$ be the Shimura variety $\Gamma_1(\mathfrak{d}_F, [t_1], n) \backslash \Gamma_0 \backslash \text{Hom}(F, \mathbb{R})$ defined by (1.1), and let $Y(n)^{\text{BS}}$ be the Borel–Serre compactification of $Y(n)$ (cf. (2.1)). Let $C$ denote the set of all cusps of $Y(n)$, and let $D_s$ denote the boundary of $Y(n)^{\text{BS}}$ at $s \in C$. Let $C_\infty$ be the subset of $C$ consisting of cusps $\Gamma_0(\mathfrak{d}_F, [t_1], n)$-equivalent to the cusp $\infty$ (where $\Gamma_0(\mathfrak{d}_F, [t_1], n)$ is the congruence subgroup defined in (1.1). Let $D_{C_\infty}(n)$ denote the union of $D_s$ for all $s \in C_\infty$.

Theorem 0.1 (=Theorem 6.1). Let $\varphi$ and $\psi$ be totally even (resp. totally odd) $\mathcal{O}$-valued narrow ray class characters of $F$ with conductor $m_\varphi$ and $m_\psi$ such that $m_\varphi m_\psi = n$ and $e$ the character $\text{sgn} \text{Hom}(F, \mathbb{R})$ (resp. 1) of the Weyl group $W_G$ (for the definition, see before Proposition 2.3). Put $\chi = \varphi \psi$. Let $E$ denote the Hilbert Eisenstein series $E_2(\varphi, \psi) = E_2(n, \mathcal{O})$ associated to the pair $(\varphi, \psi)$ with character $\chi$ (see Proposition 2.3). Assume that $\varphi$ is non-trivial and the algebraic Iwasawa $\mu$-invariants of $\mathcal{U}_m^{\text{ferm}}$ and $\mathcal{U}_m^{\text{ferm}}(\psi)$ (for the definition, see [Was] §13.3) are 0. Let $f \in S_2(n, \mathcal{O})$ be a normalized Hecke eigenform for all Hecke operators with character $\chi$. We assume the following four conditions:

(a) $f \equiv E \pmod{\varpi}$ (for the definition, see before Theorem 6.1);
(b) the local components $H^n(\vartheta(Y(n)^{\text{BS}}), \mathcal{O})_m$ and $H^{n+1}(Y(n), \mathcal{O})_m$ are torsion-free, where $m$ is the Eisenstein maximal ideal of $\mathcal{H}_2(n, \mathcal{O})$ defined before Theorem 6.9;
(c) the local component $H^n(D_{C_\infty}(n), \mathcal{O})_{m_E}$ is torsion-free, where $m_E$ is the maximal ideal of $\mathcal{H}_2(n, \mathcal{O})$ defined before Proposition 5.3.

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(d) there exists a prime ideal \( q \) of \( \mathfrak{O}_F \) dividing \( n \) such that \( C(q, E) \neq N(q) \) (mod \( \mathfrak{O} \)), where \( C(q, E) \) denotes the \( U(q) \)-eigenvalue \( \varphi(q) + \psi(q)N(q) \) of \( E \).

Then there exist \( \Omega'_t \in \mathbb{C}^1 \) and \( u \in \mathbb{C}^1 \) such that, for every narrow ray class character \( \eta \) of \( F \) of conductor \( m_\eta \) such that \( n|m_\eta \) and \( \eta = \epsilon \) on \( W_G = \mathbb{A}^\times_{E,\infty}/\mathbb{A}^\times_{F,\infty,+} \), the both values \( \tau(\eta^{-1})D(1, f, \eta)/(2\pi\sqrt{-1})^n\Omega'_t \) and \( \tau(\eta^{-1})D(1, E, \eta)/(2\pi\sqrt{-1})^n \) belong to \( \mathcal{O}(\eta) \) and the following congruence holds:

\[
\tau(\eta^{-1}) \frac{D(1, f, \eta)}{(2\pi\sqrt{-1})^n\Omega'_t} = u\tau(\eta^{-1}) \frac{D(1, E, \eta)}{(2\pi\sqrt{-1})^n} \text{ in } \mathcal{O}(\eta)/\mathfrak{O}.
\]

Here \( \tau(\eta^{-1}) \) denotes the Gauss sum attached to \( \eta^{-1} \) (see (1.14)), \( D(1, *, \eta) \) is given by the Dirichlet series in the sense of G. Shimura (see (1.14)), and \( \mathcal{O}(\eta) \) denotes the ring of integers of the field generated by \( \im(\eta) \) over \( K \).

This result is a generalization of the result of Vatsal [Vat] in the case where \( F = \mathbb{Q} \) and weight \( k = 2 \). However, the methods to prove the main theorem have some limitations, such as the need for the torsion-freeness of the compact support cohomology and the boundary cohomology. In the case where \( F \) is a real quadratic field, the torsion-freeness is satisfied under some conditions (see Proposition 5.8 and 5.9). We also give an example of a congruence between a Hilbert cusp form and a Hilbert Eisenstein series satisfying all the assumptions of the main theorem (Example 5.10).

Remark 0.2. (1) The assumption on the algebraic Iwasawa \( \mu \)-invariants of \( \mathcal{Q}^{\text{ker}(\varphi)} \) and \( \mathcal{Q}^{\text{ker}(\psi)} \) is satisfied if \( \mathcal{Q}^{\text{ker}(\varphi)} \) and \( \mathcal{Q}^{\text{ker}(\psi)} \) are abelian extensions over \( \mathbb{Q} \) by Ferrero-Washington theorem (see, for example, [Was, §7.5, Theorem 7.15]).

(2) The assumption \( n|m_\eta \) is used in a cohomological description of the special value \( \tau(\eta^{-1})D(1, E, \eta)/(2\pi\sqrt{-1})^n \) as follows. Since \( \varphi \) is non-trivial, \( E \) vanishes at cusps \( \Gamma_0(\mathcal{O}_F[t_1], n) \) equivalent to the cusp \( \infty \). The assumption \( n|m_\eta \) allows us to describe the special value in terms of Mellin transforms relevant to cusps \( \Gamma_0(\mathcal{O}_F[t_1], n) \)-equivalent to \( \infty \) (Proposition 2.5 and 2.6).

We give an outline of the proof of the main theorem (Theorem 0.1=Theorem 6.1) below in order to clarify its complicated structure, the methods used, and the places where the assumptions are necessary. The proof consists of five steps.

**Step 1.** To prove Mellin transforms for the relative cohomology classes of \( E \) and \( f \).

Since \( \varphi \) is non-trivial, \( E = E_2(\varphi, \psi) \) vanishes at every \( s \in C_\infty \). Therefore we can define the relative cohomology class \( [\omega_E]_{\text{rel}} \) (resp. \( [\omega_f]_{\text{rel}} \)) associated to \( E \) (resp. \( f \)) in \( H^n(Y(n)_{BS}, D_{C_\infty}(n); \mathbb{C}) \), whose image in \( H^n(Y(n), \mathbb{C}) \) is the cohomology class \( [\omega_E] \) (resp. \( [\omega_f] \)) associated to \( E \) (resp. \( f \)). Then we prove that the value \( \tau(\eta^{-1})D(1, E, \eta)/(2\pi\sqrt{-1})^n \) (resp. \( \tau(\eta^{-1})D(1, f, \eta)/(2\pi\sqrt{-1})^n \)) is expressed as a linear combination of the images of \( [\omega_E]_{\text{rel}} \) (resp. \( [\omega_f]_{\text{rel}} \)) under the evaluation maps with integral coefficients (Proposition 2.6 and 2.7), where we use the assumption that weight \( k = 2 \). The proof is based on the method of T. Oda [Oda], H. Hida [Hida94], and T. Ochiai [Ochi] for a Hilbert cusp form. By the assumption \( n|m_\eta \), we can generalize the Mellin transform to a Hilbert modular form vanishing at every \( s \in C_\infty \) as mentioned in Remark 0.2 (2).

**Step 2.** To prove the integrality of the restriction of the cohomology class associated to a Hilbert modular form to the boundary.

For \( h \in M_2(n, \mathcal{O}) \) and \( s \in C \), we prove that the image of the cohomology class \( [\omega_h] \) of \( h \) belonging to \( H^n(Y(n), \mathbb{C}) = H^n(Y(n)_{BS}, \mathbb{C}) \) under the restriction map to \( H^n(D_s, \mathbb{C}) \) is integral, that is, \( \text{res}([\omega_h]) \in H^n(D_s, \mathcal{O})/(\mathcal{O}-\text{torsion}) \) (see Proposition 3.4). For the proof, we
express the image of $[\omega_h]$ in the group cohomology $H^n(\Gamma, O_F, [t], n, \mathbb{C})$ in terms of multiple integrals (see Proposition-Definition 5.2) (following the method of H. Yoshida [Yo] in the case where $F$ is a real quadratic field), and we explicitly compute its restriction to the boundary (generalizing the method of G. Stevens [St]) in the case where $F = \mathbb{Q}$. The author does not know any other means to prove the integrality, for instance, using de Rham cohomology.

**Step 3.** To prove the integrality of $[\omega_E]_{\text{rel}}$ and $[\omega_f]_{\text{rel}}/\Omega_{\text{f}}$.

For the Eisenstein series $E$, we first prove the rationality of $[\omega_E]$ and $[\omega_E]_{\text{rel}}$, that is, $[\omega_E] \in H^n(Y(n), K)$ (Proposition 5.2) and $[\omega_E]_{\text{rel}} \in H^n(Y(n)_{\text{BS}}, D_{C_{\infty}}(n); K)$ (Proposition 6.1); the latter follows from the former and a vanishing result on $H^{n-1}(D_{C_{\infty}}(n), \mathbb{C})$ (Proposition 5.3). The proof of the vanishing result is based on an explicit computation of the action of Hecke operators at places dividing the level $n$ on $H_{n-1}(D_{C_{\infty}}(n), \mathbb{C})$, where we use the assumptions that $h_F^+ = 1$ and weight $k = 2$. Moreover, under the assumption on the algebraic Iwasawa $\mu$-invariants and the assumptions (b), (c), and (d), we prove the integrality of $[\omega_E]$ and $[\omega_E]_{\text{rel}}$, that is, $[\omega_E] \in H^n(Y(n), O)/(O\text{-torsion})$ and $[\omega_E]_{\text{rel}} \in H^n(Y(n)_{\text{BS}}, D_{C_{\infty}}(n); O)/(O\text{-torsion})$, and further the mod $\varpi$ non-vanishing of $[\omega_E]$ and $[\omega_E]_{\text{rel}}$ (Corollary 5.6). The proof is based on the method of C. Skinner (in the case where $F = \mathbb{Q}$) and T. Berger [Be] (in the case where $F$ is an imaginary quadratic field); our result follows from the Mellin transform for the relative cohomology class $[\omega_E]_{\text{rel}}$ (mentioned in Step 1), the integrality res$([\omega_E]) \in H^n(\partial (Y(n)_{\text{BS}}), O)/(O\text{-torsion})$ (mentioned in Step 2), and the Iwasawa main conjecture for totally real number fields (proved by A. Wiles [Wil]).

For the cusp form $f$, we prove that there exists $\Omega_{\text{f}} \in \mathbb{C}^n$ such that the class $[\omega_f]/\Omega_{\text{f}}$ belongs to $H^n_{\text{par}}(Y(n), O)/(O\text{-torsion})$ and its reduction modulo $\varpi$ does not vanish. The key ingredients of the proof are the Hecke-equivariance of the canonical homomorphism $H^n_{\text{par}}(Y(n), K) \to H^n_{\text{par}}(Y(n), \mathbb{C})$ induced by the fixed embedding $K \to \mathbb{C}$ and the Eichler-Shimura-Harder isomorphism $H^n_{\text{par}}(Y(n), \mathbb{C}[\epsilon]) \simeq S_2(n, \mathbb{C})$ as Hecke modules (see (1.7)), where the left-hand side is the $\epsilon$-isotypic part. The proof of the Eichler-Shimura-Harder isomorphism is based on an explicit computation of the action of $W_G$ on the space of invariant differential forms, where we use the assumption $h_F^+ = 1$.

**Step 4.** To prove the main theorem.

We prove the congruence between the special values $\tau(\eta^{-1})D(1, E, \eta)/(2\pi\sqrt{-1})^n$ and $\tau(\eta^{-1})D(1, f, \eta)/(2\pi\sqrt{-1})^n$. It follows from the Mellin transforms for the relative cohomology classes $[\omega_E]_{\text{rel}}$ and $[\omega_f]_{\text{rel}}/\Omega_{\text{f}}$ (mentioned in Step 1) and the congruence between $[\omega_E]_{\text{rel}}$ and $[\omega_f]_{\text{rel}}/\Omega_{\text{f}}$ (mentioned in Step 5).

**Step 5.** To prove the congruence between $[\omega_E]_{\text{rel}}$ and $[\omega_f]_{\text{rel}}/\Omega_{\text{f}}$.

We prove the congruence between the integral cohomology classes $[\omega_E]$ and $[\omega_f]/\Omega_{\text{f}}$ in the parabolic cohomology $H^n_{\text{par}}(Y(n), O)/\varpi$ (Theorem 7.1). Moreover, we lift the congruence to the relative cohomology classes $[\omega_E]_{\text{rel}}$ and $[\omega_f]_{\text{rel}}/\Omega_{\text{f}}$ in $H^n(Y(n)_{\text{BS}}, D_{C_{\infty}}(n); O)/\varpi$ by using Proposition 5.3 (mentioned in Step 3). The proof of the former is based on multiplicity one results for the $E$-part $\overline{M}_E$ and $\text{Fil}^n(\overline{M}_E)$ of the $f$-part $\overline{M}_f$, and integral $p$-adic Hodge theory.

Here $\overline{M}_E$ (resp. $\overline{M}_f$) is the quotient $\overline{M}/\varpi$ (resp. $\overline{M}/\varpi$) of the torsion-free $E$-part (resp. $f$-part) $\overline{M}_E$ (resp. $\overline{M}_f$) of the integral log-crystalline cohomology by $\varpi$ (for the definition, see [7.6, 7.2]). By the theory of Eisenstein cohomology and the $q$-expansion principle over $\mathbb{C}$, we prove that the dimension of $\overline{M}_E$ over $O/\varpi$ is 1 and $\overline{M}_E = \text{Fil}^n(\overline{M}_E)$ (Proposition 7.3), where we use the assumption (d) on Hecke eigenvalues at places dividing the level. Since the dimension of $\text{Fil}^n(\overline{M}_f)$ over $O/\varpi$ is also 1, we obtain $\overline{M}_E = \text{Fil}^n(\overline{M}_E) \simeq \text{Fil}^n(\overline{M}_f)$ (Proposition 7.3 and 7.4), where the second isomorphism follows from the assumptions (b).
and (c), and the congruence (a) between Hecke eigenvalues including at places dividing the level. Now the assertion follows from integral $p$-adic Hodge theory (§7.4). The result of this step may be regarded as an analogue of the multiplicity one theorem for modulo $p$ parabolic cohomology in the case where the residual Galois representation $\tilde{\rho}_f (= \rho_f \pmod{\mathfrak{p}})$ associated to $f$ is reducible. When $\tilde{\rho}_f$ is irreducible, under some assumptions, the multiplicity one theorem has been proved by M. Dimitrov [Dim2] for a general totally real number field.

The organization of this paper is as follows.

In §1, we summarize results on Hilbert modular varieties and Hilbert modular forms in the analytic and algebraic settings. Moreover, we state basic properties of Hilbert Eisenstein series (Proposition 1.2 and 1.3), which are of great utility in the following sections.

In §2, we give a cohomological description of special values of $L$-functions associated to a Hilbert modular form vanishing at cusps $\Gamma_0(\mathcal{O}_F[t_1], n)$-equivalent to $\infty$ (Proposition 2.3 and 2.6). The evaluation of the associated cohomology class on Hilbert modular cycles produces special values of $L$-functions.

In §3, we prove the integrality of the restriction of the cohomology class associated to a Hilbert Eisenstein series to the boundary of the Borel–Serre compactification of the Hilbert modular variety (Proposition 3.4).

In §4, we recall the theory of Eisenstein cohomology and the Eichler–Shimura–Harder homomorphism. We prove the Eichler–Shimura–Harder isomorphism (4.7) for the $\varepsilon$-part.

In §5, we generalize Stevens’s result [Ste2] on the integrality of the cohomology class associated to an elliptic modular form. We prove the integrality of the cohomology class $[\omega_E]$ associated to $E$ (Corollary 5.6).

In §6, we prove the main theorem (Theorem 0.1=Theorem 6.1). The key ingredient of our proof is the congruence between the integral cohomology classes of $f$ and $E$ in the parabolic cohomology, whose proof is postponed to §7 (Theorem 7.1). Combining with Proposition 2.3 and 2.6, we obtain the main theorem.

In §7, we prove the congruence between the integral cohomology classes of $f$ and $E$ in the parabolic cohomology (Theorem 7.1) by combining the theory of Eisenstein cohomology and integral $p$-adic Hodge theory.

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0.2. **Notation.** Let $\hat{\mathbb{Z}}$ denote $\prod_l \mathbb{Z}_l$, where $l$ runs over all rational primes. We abbreviate $\mathbb{A}_Q$, the ring of adeles of $\mathbb{Q}$, to $\mathbb{A}$. We fix a rational prime number $p > 3$. We fix algebraic closures $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ and $\overline{\mathbb{Q}}_p$ of the field of $p$-adic numbers $\mathbb{Q}_p$, and embeddings $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}}_p \to \mathbb{C}$, where $\mathbb{C}$ denotes the field of complex numbers.
Let \( F \) be a totally real number field unramified at \( p, n \) the degree \([F; \mathbb{Q}]\) of the extension \( F/\mathbb{Q} \), and \( \sigma_F \) the ring of integers of \( F \). For a place \( v \) of \( F \) (resp. a non-zero prime ideal \( q \) of \( \sigma_F \)), let \( F_v \) (resp. \( F_q \)) denote the completion at \( v \) (resp. the \( q \)-adic completion) of \( F \). Let \( \sigma_{F_v} \) denote the ring of integers of \( F_v \), and \( \sigma_F \) the product of \( \sigma_{F_v} \) over all non-zero prime ideals \( q \) of \( \sigma_F \). Let \( J_F \) denote the set of embeddings of \( F \) into \( \mathbb{R} \). For \( a \in F \) and \( \nu \in J_F \), let \( a' \) denote \( \nu(a) \). We have \( F \otimes \mathbb{Q} \mathbb{R} \simeq \mathbb{R}^{J_F} \), and write \( (F \otimes \mathbb{Q} \mathbb{R})^+ \) for the subgroup of \((F \otimes \mathbb{Q} \mathbb{R})^x\) corresponding to \((\mathbb{R}^+_x)^{J_F}\), where \( \mathbb{R}_x^+ \) denotes the multiplicative group of positive real numbers. As usual, \( \mathbb{A}_F \) denotes the ring of adeles of \( F \), which is the product of the finite part \( \mathbb{A}_{F,f}(\simeq \widehat{\mathbb{D}}_F \otimes \sigma_F F) \) and the infinite part \( \mathbb{A}_{F,\infty}(\simeq F \otimes \mathbb{Q} \mathbb{R}) \). For \( x \in \mathbb{A}_F \) and a place \( v \) of \( F \), \( x_0, x_\infty \), and \( x_v \) denote the finite component \( \in \mathbb{A}_{F,f} \), the infinite component \( \in \mathbb{A}_{F,\infty} \), and the \( v \)-component \( \in F_v \) of \( x \), respectively. For \( x \in \mathbb{A}_F \), a subset \( X \) of \( \mathbb{A}_F \), and a non-zero ideal \( n \) of \( \sigma_F \), we write \( x_n \) and \( X_n \) for the images of \( x \) and \( X \) in \( \prod q | n F_q \), where \( q \) denotes a non-zero prime ideal of \( \sigma_F \). Let \( N \) denote the norm map \( N_{F/Q} \) of the extension \( F/Q \), \( \mathfrak{d}_F \subset \sigma_F \) the different of \( F \), and \( \Delta_F \) the discriminant \( N(\mathfrak{d}_F) \) of \( F \), which is prime to \( p \) by assumption. Let \( \mathbb{C}_F^+ \) denote the narrow ideal class group of \( F \). We have an isomorphism \( F^\times \mathbb{A}_F/\widehat{\mathbb{D}}_F(F \otimes \mathbb{Q} \mathbb{R})^+_\mathfrak{d} \simeq \mathbb{C}_F^+ \) sending the class of \( x \in \mathbb{A}_F^\times \) to the class of the fractional ideal \( [x] := \prod q^{\mathcal{v}_q(\mathfrak{d}x)} \), where \( q \) runs over the set of all non-zero prime ideals of \( \sigma_F \).

For a non-zero ideal \( \mathfrak{b} \) of \( \sigma_F \), let \( \mathbb{C}_F(b) \) denote the narrow ray class group of \( F \) modulo \( \mathfrak{b} \). By a narrow ray class character of \( F \) modulo \( \mathfrak{b} \), we mean a homomorphism \( \chi : \mathbb{C}_F^+(\mathfrak{b}) \to \mathbb{C}^\times \). The conductor of \( \chi \) is the smallest divisor \( \mathfrak{m}_\chi \) of \( \mathfrak{b} \) such that \( \chi \) factors through \( \mathbb{C}_F^+(\mathfrak{m}_\chi) \). For a narrow ray class character \( \chi \) of \( F \) modulo \( \mathfrak{b} \), there exists \( r = (r_\nu)_\nu \in (\mathbb{Z}/2\mathbb{Z})^{J_F} \) such that

$$\chi((\alpha)) = \text{sgn}((\alpha)^r) \quad \text{for all} \quad \alpha \in F^\times \quad \text{satisfying} \quad \alpha \equiv 1 \pmod{\mathfrak{b}}.$$  

Here \( \text{sgn}((x)) \) for \( x \in \mathbb{R}^\times \) denotes the sign of \( x \) and \( \text{sgn}((\alpha)^r) = \prod \nu \in J_F \text{sgn}(\alpha')^{r_\nu} \), where we identify \( J_F \) with the set of infinite places of \( F \). We call \( r \) the sign of \( \chi \). We say that \( \chi \) is totally even (resp. totally odd) if \( r_\nu = 0 \) (resp. \( r_\nu = 1 \)) for all \( \nu \in J_F \).

For an algebraic group \( H \) defined over \( \mathbb{Q} \), \( H(\mathbb{R}) \) is abbreviated to \( H_\infty \) and \( H_{\infty,+} \) denotes the connected component of \( H_\infty \) containing the unit. Let \( G \) be the reductive algebraic group \( \text{Res}_{F/Q}(\text{GL}_2/F) \) over \( \mathbb{Q} \), where \( \text{Res}_{F/Q} \) denotes the Weil restriction of scalars. We have \( G_\infty = \text{GL}_2(\mathbb{R})^{J_F} \), \( G_{\infty,+} = \text{GL}_2(\mathbb{R})^{J_F^+} \), and \( G(A) = \text{GL}_2(\mathbb{A}_F) \). Let \( B \) denote the Borel subgroup of \( G \) consisting of upper triangular matrices, and let \( U \) denote its unipotent radical.

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### 1. Hilbert modular varieties and Hilbert modular forms

#### 1.1. Analytic Hilbert modular varieties

In this subsection, we recall the definition of analytic Hilbert modular varieties. For more detail, refer to [Dim2] §1.1.

Let \( \mathfrak{H} \) be the upper half plane \( \{z \in \mathbb{C} \mid \text{Im}(z) > 0\} \). The group \( \text{GL}_2(\mathbb{R}) \) acts on \( \mathfrak{H} \) by linear fractional transformations. We can extend the action to \( \text{GL}_2(\mathbb{R}) \) by defining the action of \( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \) on \( \mathfrak{H} \) by \( z \mapsto -\bar{z} \). We define the action of \( G_\infty = \text{GL}_2(\mathbb{R})^{J_F} \) on \( \mathfrak{H}^{J_F} \) by \( (g_\nu)_{\nu \in J_F} \cdot (z_\nu)_{\nu \in J_F} = (g_\nu z_\nu)_{\nu \in J_F} \). Let \( i = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathfrak{H}^{J_F} \). Let \( K_\infty \) and \( K_{\infty,+} \) be the stabilizers of \( i \) in \( G_\infty \) and \( G_{\infty,+} \), respectively.
For a non-zero ideal \( n \) of \( \mathfrak{O}_F \), we define the open compact subgroup \( K_1(n) \) of \( G(\mathbb{A}_F) \) by

\[
K_1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\hat{\mathbb{Z}}) \middle| c \in n, d - 1 \in n \right\}.
\]

The adelic Hilbert modular variety of level \( K_1(n) \) is defined by

\[
Y(n) = G(\mathbb{Q}) \backslash G(\mathbb{A}_F) / K_1(n)K_{\infty,+} = G(\mathbb{Q})_+ \backslash G(\mathbb{A}_F)_+ / K_1(n)K_{\infty,+},
\]

where \( G(\mathbb{A}_F)_+ = G(\mathbb{A}_F)G_{\infty,+} \) and \( G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G_{\infty,+} \).

Then \( Y(n) \) is a disjoint union of finitely many arithmetic quotients of \( \mathfrak{g}_F \) as follows. Let \( a \) be a fractional ideal of \( F \). We consider the following congruence subgroups of \( G(\mathbb{Q})_+ \):

\[
\Gamma_0(a, n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{O}_F^{\times} \begin{pmatrix} a^{-1} & \ast \\ \ast & \ast \end{pmatrix} \middle| ad - bc \in \mathfrak{O}_F^{\times} \right\},
\]

\[
\Gamma_1(a, n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(a, n) \middle| d \equiv 1 \mod n \right\},
\]

\[
\Gamma_1^n(a, n) = \Gamma_1(a, n) \cap \text{SL}_2(F),
\]

where \( \mathfrak{O}_F^{\times} \) denotes the subgroup of \( \mathfrak{O}_F^{\times} \) consisting of totally positive units.

Let \( \text{Cl}_F^\dagger \) be the narrow ideal class group of \( F \) and \( h_1^+ \) the narrow class number \( \#\text{Cl}_F^\dagger \) of \( F \). Choose and fix \( t_1, \ldots, t_{h_1^+} \in \mathbb{A}_F^\times \) such that \( t_{i,\infty} = 1 \) and the corresponding fractional ideals \( [t_1], \ldots, [t_{h_1^+}] \) form a complete set of representatives of \( \text{Cl}_F^\dagger \). We put

\[
x_i = \begin{pmatrix} D^{-1}t_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} \in G(\mathbb{A})_+.
\]

We define the analytic Hilbert modular varieties \( Y_i \) by

\[
Y_i = \Gamma_1^\natural(\mathfrak{O}_F[t_i], n) \backslash \mathfrak{g}_F^{\dagger},
\]

where \( \mathfrak{g}^{\dagger} \) denotes \( \Gamma / (\Gamma \cap F^{\times}) \) for a congruence subgroup \( \Gamma \) of \( G(\mathbb{Q})_+ \). Then, by the strong approximation theorem, we have the following description of \( Y(n) \):

\[
Y(n) \simeq \prod_{1 \leq i \leq h_1^+} Y_i,
\]

given by sending the class of \( x_ig \in Y(n) \) to the class of \( g[t_i] \in Y_i \) for \( g \in G_{\infty,+} \).

We also need the following varieties:

\[
Y_i^\dagger(n) = \prod_{1 \leq i \leq h_1^+} Y_i^{-1}, \quad Y_i^{-1} = \Gamma_1^\natural(\mathfrak{O}_F[t_i], n) \backslash \mathfrak{g}_F^{\dagger}.
\]

1.2. Analytic Hilbert modular forms. In this subsection, we fix notation concerning the spaces of Hilbert modular forms, following [Dim2] §1.2.2.

Let \( k \) be an integer \( \geq 2 \) and \( \mathfrak{n} \) a non-zero ideal of \( \mathfrak{O}_F \). Let \( t = \sum_{i \in J_F} t \in \mathbb{Z}[J_F] \).

Let \( M_k(n, \mathfrak{C}) \) (resp. \( S_k(n, \mathfrak{C}) \)) denote the \( \mathbb{C} \)-vector space \( G_{kt,J_F}(K_1(n)) \) (resp. \( S_{kt,J_F}(K_1(n)) \)) of holomorphic Hilbert modular (resp. cusp) forms of weight \( kt \) and of level \( K_1(n) \) defined in [Dim2] Definition 1.2. Let \( \chi \) be a Hecke character of \( F \) of type \( -(k - 2)t \) whose conductor divides \( n \). Let \( M_k(n, \chi, \mathfrak{C}) \) (resp. \( S_k(n, \chi, \mathfrak{C}) \)) denote the subspace \( G_{kt,J_F}(K_1(n), \chi) \) (resp. \( S_{kt,J_F}(K_1(n), \chi) \)) of \( G_{kt,J_F}(K_1(n)) \) (resp. \( S_{kt,J_F}(K_1(n)) \)) of elements with character \( \chi \) defined in [Dim2] Definition 1.3.
For a fractional ideal \( \mathfrak{a} \) of \( F \), let \( M_k(\Gamma_1(\mathfrak{a}, n), \mathbb{C}) \) (resp. \( S_k(\Gamma_1(\mathfrak{a}, n), \mathbb{C}) \)) denote the space \( \text{G}_{kt, J_F}(\Gamma_1(\mathfrak{a}, n); \mathbb{C}) \) (resp. \( \text{S}_{kt, J_F}(\Gamma_1(\mathfrak{a}, n); \mathbb{C}) \)) of holomorphic Hilbert modular (resp. cusp) forms of weight \( kt \) and of level \( \Gamma_1(\mathfrak{a}, n) \) defined in [Dim2] Definition 1.4.

Then we have canonical isomorphisms (cf. [Hida91] p.323 and [Hida88] (2.6a)):

\[
(1.6) \quad M_k(n, \mathbb{C}) \simeq \bigoplus_{1 \leq i \leq h_F} M_k(\Gamma_1(\mathcal{O}_F[t_i], n), \mathbb{C}), \quad S_k(n, \mathbb{C}) \simeq \bigoplus_{1 \leq i \leq h_F} S_k(\Gamma_1(\mathcal{O}_F[t_i], n), \mathbb{C}).
\]

1.3. Hecke operators on analytic Hilbert modular forms. Let \( n \) be a non-zero ideal of \( \mathfrak{o}_F \). In this subsection, we recall the definition of the Hecke operators acting on \( M_k(n, \mathbb{C}) \) and \( S_k(n, \mathbb{C}) \), following [Dim2] §1.10.

Let \( \Delta(n) \) be the following semigroup:

\[
\Delta(n) = G(\mathbb{A}_F) \cap \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{O}_F) \bigg| c \in \mathfrak{n}\mathfrak{o}_F, \ d \in \mathfrak{o}_F^\times \text{ whenever } q|n \right\},
\]

where \( q \) is a non-zero prime ideal of \( \mathfrak{o}_F \). For \( y \in \Delta(n) \), we define the action of the double coset \( K_1(n)yK_1(n) \) on \( M_k(n, \mathbb{C}) \) (resp. \( S_k(n, \mathbb{C}) \)) by

\[
(1.7) \quad f|K_1(n)yK_1(n)|(x) = \sum_i f(xy_i^{-1}),
\]

where \( K_1(n)yK_1(n) = \prod_i K_1(n)y_i \). By the definition of \( M_k(n, \mathbb{C}) \) and \( S_k(n, \mathbb{C}) \), the right-hand side is independent of the choice of the representative set \( \{y_i\}_i \).

We define the Hecke operator \( T(q^e) \) (resp. \( S(q^e) \)) for a non-negative integer \( e \), a non-zero prime ideal \( q \) of \( \mathfrak{o}_F \) (resp. prime ideal \( q \) of \( \mathfrak{o}_F \) prime to \( n \)), and a uniformizer \( \varpi_q \) of \( \mathfrak{o}_F \) by the action of the double coset \( K_1(n)\left( \begin{pmatrix} \varpi_q^e & 0 \\ 0 & 1 \end{pmatrix} \right) K_1(n) \) (resp. \( K_1(n)\left( \begin{pmatrix} 0 & \varpi_q^e \\ \varpi_q^e & 0 \end{pmatrix} \right) K_1(n) \)). We note that these operators are independent of the choice of \( \varpi_q \). We put \( T(q^e) = T(\varpi_q^e) \) and \( S(q^e) = S(\varpi_q^e) \) (resp. \( U(q^e) = T(\varpi_q^e) \)) for a non-negative integer \( e \) and a non-zero prime ideal \( q \) prime to \( n \) (resp. prime ideal \( q \) dividing \( n \)). We define \( T(m) = \prod_{q|n} T(q^e(q)) \) and \( S(m) = \prod_{q|n} S(q^e(q)) \) for all non-zero ideal \( m = \prod_{q|n} q^e(q) \) of \( \mathfrak{o}_F \) prime to \( n \) and \( U(m) = \prod_{q|n} U(q^e(q)) \) for all non-zero ideal \( m = \prod_{q|n} q^e(q) \) of \( \mathfrak{o}_F \), where \( q \) is a non-zero prime ideal.

The definition of the Hecke operators acting on \( M_k(\Gamma_1(a, n), \mathbb{C}) \) and \( S_k(\Gamma_1(a, n), \mathbb{C}) \) and their relation (via (1.6)) to the adelic ones recalled above are explicitly given in [ShL] §2.

For a subalgebra \( \mathcal{A} \) of \( \mathbb{C} \), let \( \mathcal{H}_k(n, \mathcal{A}) \) (resp. \( \mathcal{S}_k(n, \mathcal{A}) \)) be the commutative \( \mathcal{A} \)-subalgebra of \( \text{End}_{\mathbb{C}}(M_k(n, \mathbb{C})) \) (resp. \( \text{End}_{\mathbb{C}}(S_k(n, \mathbb{C})) \)) generated by \( T(m) \), \( S(m) \) for all non-zero ideal \( m = \prod_{q|n} q^e(q) \) of \( \mathfrak{o}_F \) prime to \( n \) and \( U(m) \) for all non-zero ideal \( m = \prod_{q|n} q^e(q) \) of \( \mathfrak{o}_F \).

1.4. Dirichlet series associated to Hilbert modular forms. In this subsection, we recall the definition and properties of the Dirichlet series associated to a Hilbert modular form, following [ShL] §2.

Let \( h \in M_k(n, \mathbb{C}) \) and \( h_i \in M_k(\Gamma_1(\mathcal{O}_F[t_i], n), \mathbb{C}) \) such that \( h = (h_i)_{1 \leq i \leq h_F} \) under the isomorphism (1.6). Then \( h_i \) has the Fourier expansion of the form

\[
(1.8) \quad h_i(z) = c_{\infty}(t_i)^{-1} h \)N(t_i)^{k/2} + \sum_{0 \leq c \leq t_i} c(t_i)^{-1} h \)N(x)^{k/2} \epsilon_F(x)
\]

given by [ShL] (2.18) and [Hida88] Proposition 4.1. Here the notion \( \gg 0 \) means totally positive, \( m \mapsto c(m, h) \) is a function on the set of all fractional ideals of \( F \) vanishing outside
the set of integral ideals, and \( e_F \) denotes the additive character of \( F \backslash \mathbb{A}_F \) characterized by \( e_F(x) = \exp(2\pi i x) \) for \( x \in \mathbb{A}_F \). We put
\[
a_\infty(0, h_i) = c_\infty([t_i]^{-1}, h)N([t_i])^{k/2} \text{ and } a_\infty(\xi, h_i) = c([\xi]^{-1}, h)N(\xi)^{k/2}
\]
for any \( 0 \ll \xi \in [t_i] \). We also put
\[
C_{\infty,i}(0, h) = N([t_i])^{-k/2}a_\infty(0, h_i),
\]
\[
C(m, h) = N(m)^{k/2}c(m, h)
\]
for all non-zero ideals \( m \) of \( \mathfrak{o}_F \). Let \( \eta \) be a finite Hecke character of \( F \). The Dirichlet series in the sense of Shimura \([\text{Shi}, (2.25)]\) is defined by
\[
\sum_m C(m, h)\eta(m)N(m)^{-s} \text{ for } s \in \mathbb{C},
\]
where \( m \) runs over the set of all non-zero ideals of \( \mathfrak{o}_F \). It converges absolutely if Re\( (s) \) is sufficiently large and extends to a meromorphic function on the complex plane (see, for example, \([2.3]\) in this paper). For each \( h \in M_k(n, \mathbb{C}) \), let \( D(s, h, \eta) \) denote this meromorphic function. If \( \eta \) is the trivial character, we simply write \( D(s, h) \) for \( D(s, h, \eta) \).

1.5. Duality theorem between Hecke algebras and Hilbert modular forms. Recall that, for \( h = (h_l)_{1 \leq l \leq h_F^+} \in M_k(n, \mathbb{C}) \), \( h_i \in M_k(I_1(\mathfrak{d}_F[t_i], n), \mathbb{C}) \) has the Fourier expansion of the form \([12]\). For a subring \( A \) of \( \mathbb{C} \), we put
\[
M_k(\Gamma_1(\mathfrak{d}_F[t_i], n), A) = M_k(\Gamma_1(\mathfrak{d}_F[t_i], n), \mathbb{C}) \cap A[[e_F(\xi z) : \xi = 0 \text{ or } 0 \ll \xi \in F]],
\]
\[
S_k(\Gamma_1(\mathfrak{d}_F[t_i], n), A) = S_k(\Gamma_1(\mathfrak{d}_F[t_i], n), \mathbb{C}) \cap A[[e_F(\xi z) : \xi = 0 \text{ or } 0 \ll \xi \in F]],
\]
\[
M_k(n, A) = \bigoplus_{1 \leq i \leq h_F^+} M_k(\Gamma_1(\mathfrak{d}_F[t_i], n), A), \quad S_k(n, A) = \bigoplus_{1 \leq i \leq h_F^+} S_k(\Gamma_1(\mathfrak{d}_F[t_i], n), A).
\]

By \([\text{Hida88}, \text{Theorem 4.11}]\) and \([\text{Hida91}, \text{Theorem 2.2 (ii)}]\), the space \( S_k(n, A) \) (resp. \( M_k(n, A) \)) is stable under the action of \( \mathcal{H}_k(n, A) \) (resp. \( \mathbb{H}_k(n, A) \)).

**Theorem 1.1** (Duality theorem). Assume that \( p \) is prime to the discriminant \( \Delta_F \) of \( F \). Let \( K \) be a finite extension of the field \( \Phi_p \) defined in Proposition \([3.4]\). \( \mathbb{O} \) its ring of integers, and \( \kappa \) the residue field. Assume that \( \kappa \) contains the residue fields for all primes \( \mathfrak{p} \) of \( \mathfrak{o}_F \) over \( p \). Then, for \( A = K = \mathbb{O} \), the following \( \mathbb{A} \)-bilinear map is a perfect pairing:
\[
\langle \cdot, \cdot \rangle : \mathbb{H}_2(n, A) \times M_2(n, A) \rightarrow A; (t, f) \mapsto C(\mathfrak{o}_F, f|t).
\]

**Proof.** The duality theorem between \( \mathcal{H}_2(n, A) \) and \( S_2(n, A) \) is well-known \([\text{Hida88}, \text{Theorem 5.1}]\). We follow the arguments in the proof of \([\text{Hida88}, \text{Theorem 5.1}]\) and \([\text{Hida91}, \text{Theorem 2.2 (iii)}]\). In the case \( A = K \), the proof is the same as that of \([\text{Hida91}, \text{Theorem 2.2 (ii)}]\).

Suppose that \( A = \mathbb{O} \). It suffices to prove that the \( \mathbb{O} \)-linear homomorphism \( M_2(n, \mathbb{O}) \rightarrow \text{Hom}_\mathbb{O}(\mathbb{H}_2(n, \mathbb{O}), \mathbb{O}) \) induced by the pairing is an isomorphism. If \( \phi : \mathbb{H}_2(n, \mathbb{O}) \rightarrow \mathbb{O} \) is an \( \mathbb{O} \)-linear map, then we can extend it to a \( K \)-linear map \( \phi : \mathbb{H}_2(n, K) \rightarrow K \). Thus, by the duality theorem for a field \( K \), we get \( f \in M_2(n, K) \) such that \( \langle t, f \rangle = \phi(t) \) for all \( t \in \mathbb{H}_2(n, \mathbb{O}) \). For a non-zero ideal \( m = \prod_{q \mid \mathfrak{p}} q^{e(q)} \prod_{q \ni \mathfrak{p}} q^{r(q)} \) of \( \mathfrak{o}_F \), we put \( V(m) = \prod_{q \mid \mathfrak{p}} T(q)^{c(q)} \prod_{q \ni \mathfrak{p}} U(q)^{t(q)} \). Then we have \( C(m, f) = C(\mathfrak{o}_F, fV(m)) = (V(m), f) = \phi(V(m)) \in \mathbb{O} \). Here the first equality follows from \([\text{Shi}, (2.20)]\). Suppose that the constant term of \( f \) does not belong to \( \mathbb{O} \), that is, \( a_\infty(0, f_i) \notin \mathbb{O} \) for some \( i \). Let \( r \in \mathbb{Z} \) be the positive integer such that \( \varpi^ra_\infty(0, f_i) \in \mathbb{O}^{\times} \). Then the \( q \)-expansion of \( \varpi^ra_\infty(0, f_i) \) modulo \( \varpi \). By \([\text{An–Go}]\), the kernel of the \( q \)-expansion map on the sum of the spaces of Hilbert modular forms with coefficients in
\(\mathbb{F}_p\) of parallel weight \(k\) \((k \in \mathbb{Z}, k \geq 0)\) defined in Definition 1.3 is generated by \(H_{p-1}\), where \(H_{p-1}\) is the Hasse invariant, which is a Hilbert modular form of level 1 and of parallel weight \(p-1\). Therefore we have \(c\tau f_{\varphi} = c\tau a_{\infty}(0, f_{\varphi}) \pmod{c}\tau\) = \(\alpha(H_{p-1} - 1)\) for some \(\alpha \in \mathcal{O}/c\tau\). This is a contradiction because the weight of \(H_{p-1}\) is \((p-1)t > 2t\). \(\square\)

1.6. Hilbert Eisenstein series. In this subsection, we recall the definition and properties of the Hilbert Eisenstein series, following [Shi §3].

Let \(\varphi\) (resp. \(\psi\)) be a narrow ray class character of \(F\) (cf. Notation), whose conductor is denoted by \(m_\varphi\) (resp. \(m_\psi\)). Let \(q\) (resp. \(r\)) \(\in (\mathbb{Z}/2\mathbb{Z})^{\infty}\) be the sign of \(\varphi\) (resp. \(\psi\)). We may regard \(\varphi\) (resp. \(\psi\)) as a function on the set of all non-zero ideals of \(\mathfrak{O}_F\) by defining \(\varphi(m) = 0\) (resp. \(\psi(m) = 0\)) if \(m\) is not prime to \(m_\varphi\) (resp. \(m_\psi\)). Then a function \(\text{sgn}(x)^r\psi(x\mathfrak{h}^{-1})\) of \(x \in \mathfrak{h}\) depends only on \(x\) modulo \(m_\varphi\mathfrak{h}\) for a fractional ideal \(\mathfrak{h}\) of \(F\).

Let \(\tau(\psi)\) be the Gauss sum attached to \(\psi\) defined by

\[
\tau(\psi) = \sum_{x \in m_\varphi^{-1}\mathfrak{d}_F^{-1}/\mathfrak{d}_F^{-1}} \text{sgn}(x)^r\psi(xm_\varphi\mathfrak{d}_F)e_F(x).
\]

The following is obtained by [Shi Proposition 3.4] and [Da–Da–Po Proposition 2.1]:

**Proposition 1.2.** Let \(k\) be an integer \(\geq 2\) such that \((k, \cdots, k) \equiv q + r \pmod{(2\mathbb{Z})^{\infty}}\). Then there exists \(E_k(\varphi, \psi) = (E_k(\varphi, \psi)_1)_{1 \leq i \leq h_\mathfrak{F}} \in M_k(m_\varphi, m_\psi, \varphi, \psi, \mathbb{C})\), called a Hilbert Eisenstein series, satisfying the following properties.

1. \(D(s, E_k(\varphi, \psi)) = L(s, \varphi)L(s - k + 1, \psi)\).
2. \(C(m, E_k(\varphi, \psi)) = \sum_{c|m} \varphi\left(\frac{m}{c}\right)\psi(c)N(c)^{k-1}\) for each integral ideal \(m\) of \(F\).
3. If \(m_\varphi \neq \mathfrak{o}_F\), then \(C_{\infty}(0, E_k(\varphi, \psi)) = 0\). If \(m_\varphi = \mathfrak{o}_F\), then

\[
C_{\infty}(0, E_k(\varphi, \psi)) = 2^{-m}\varphi^{-1}(t_i)L(1 - k, \varphi^{-1}\psi).
\]

**Proposition 1.3.** Assume that \([F : \mathbb{Q}] > 1\), \(h_\mathfrak{F}^+ = 1\), and \(\mathfrak{d}_F[t_1] = \mathfrak{o}_F\). Under the same notation and assumptions as Proposition 1.2, the constant term \(a_c(0, E_k(\varphi, \psi)_1)\) of \(E_k(\varphi, \psi)_1\) at a cusp \(c \in \mathbb{P}^1(F)\) is given by the following: fix \(\alpha = \left(\begin{array}{cc} x & \beta \\ y & \delta \end{array}\right) \in \text{SL}_2(\mathfrak{o}_F)\) such that \(c = \alpha(\infty)\). If \(y \notin m_\varphi\) and \(\psi \neq 1\), then \(a_c(0, E_k(\varphi, \psi)_1) = 0\). If \(y \in m_\varphi\) or \(\psi = 1\), then

\[
a_c(0, E_k(\varphi, \psi)_1) = \frac{N(\mathfrak{d}_F)^{-k/2}}{2^m} \frac{\tau(\varphi\psi^{-1})}{\tau(\psi^{-1})}\left(\frac{N(m_\varphi)}{N(m_\psi)}\right)^k \text{sgn}(y)^r\psi(-ym_\psi^{-1})\text{sgn}(x)^r\psi^{-1}(x)
\]

\[
\times \left(\prod_{q|m_\varphi, m_\psi, q|m_\psi^{-1}} (1 - \varphi\psi^{-1}(q)N(q)^{-k})\right) L(1 - k, \varphi^{-1}\psi).
\]

**Remark 1.4.** The assumption \(h_\mathfrak{F}^+ = 1\) allows us to simplify some computations. In the general case, the constant terms at all cusps are computed by T. Ozawa [Oza].

**Proof.** We follow the arguments in the proof of [Da–Da–Po Proposition 2.1] and [Fre Chapter III, Theorem 4.9]. We put \(a = m_\varphi\) and \(b = m_\psi\). First, we recall the construction of the Eisenstein series \(E_k(\varphi, \psi)\) given in [Shi §3] and [Da–Da–Po Proposition 2.1]. Let \(U\) be the subgroup \(\{u \in \mathfrak{o}_F^* \mid N(u)^k = 1, u \equiv 1 \pmod{ab}\}\) of \(\mathfrak{o}_F^*\), which is of finite index. For \(z \in \mathfrak{S}_F^+\)
and $s \in \mathbb{C}$ with $\text{Re}(2s + k) > 2$, we define
\[
E_k(\varphi, \psi)_1(z, s) = N([t_1])^{-1/2}[\alpha_{[U]}^s : U]^{-1} \Gamma(k) N(b)^{-1} \tau(\psi) \sum_{h \in \text{Cl}_F} \sum_{a \in \mathcal{H}} \sum_{t \in \mathbb{H}} \sum_{d \in \mathcal{D}^1_{F}[t_1]^{-1}h} \times \text{sgn}(a)^q \varphi(ah^{-1}) \text{sgn}(-t)^r \psi(-tb \delta_{F}[t_1]h^{-1}) N(h)^{k-1} \times E_{k, U}(z, s; a, t; ah, \delta_{F}^{-1}[t_1]^{-1}h),
\]
where $\text{Cl}_F$ is the ideal class group of $F$ and
\[
E_{k, U}(z, s; a, t; ah, \delta_{F}^{-1}[t_1]^{-1}h) = \Delta_{F}^{-1/2} N(h)^{-1} (-1)^{kn} (2\pi \sqrt{-1})^{-kn} \sum_{(a', b') \in R/(a', b') \neq (0,0)} (a'z + b')^{-k} |a'z + b'|^{-2s}.
\]
Here the last sum runs over a complete set $R$ of representatives for the quotient of $(a + ah) \times (t + \delta_{F}^{-1}[t_1]^{-1}h)$ by the diagonal multiplication of $U$. This series converges if $\text{Re}(2s + k) > 2$ and is analytically continued to a holomorphic function on the whole complex plane ([Shi, p.656]). The Eisenstein series $E_k(\varphi, \psi)_1(z)$ is defined to be $E_k(\varphi, \psi)_1(z, 0)$, which is a holomorphic function of $z \in \mathfrak{f}^{\text{fr}}$ ([Shi, p.656]). For $z \in \mathfrak{f}^{\text{fr}}$, we have
\[
E_{k, U}(z, s; a, t; ah, \delta_{F}^{-1}[t_1]^{-1}h) = \frac{\Delta_{F}^{-1/2} N(h) (-2\pi \sqrt{-1})^{-kn}}{R/(a', b') \neq (0,0)} \sum_{(a'z + b') \in R} (a'z + b')^{-k} |a'z + b'|^{-2s}
\]
Note that only the terms for $(a', b')$ with $a'z + b'y = 0$ in the series contribute to the constant term of $E_k(\varphi, \psi)_1|_{\alpha = 0}$. We put $C = \Delta_{F}^{-1/2} \Gamma(k)^{n}[\alpha_{[U]}^s : U]^{-1} N(h)^{-1} (2\pi \sqrt{-1})^{-kn}$.

(1) First suppose that $y \notin \mathfrak{b}$. Since $\delta_{F}[t_1] = ah$ and $b'y = -a't \in (y)b^{-1}h \cap \mathfrak{h}$, we see that $b'y^{-1}$ is not prime to $b$ and hence $\text{sgn}(-b')^{-1} \psi^{-1}(-b'h^{-1}) = 0$ if $b \neq 1$. Thus the constant term $a_{c}(0, E_k(\varphi, \psi)_1)$ is equal to 0 if $b \neq 1$.

Consider the case $b = 1$. The constant term of $E_k(\varphi, \psi)_1|_{\alpha = 0}$ is equal to the value of
\[
(1.14) \quad C \cdot N([t_1])^{-k/2} \sum_{h \in \text{Cl}_F} \sum_{(a', b') \in R/(a', b') \neq (0,0)} \sum_{a', b'} \text{sgn}(a'z + b') \varphi(a'h^{-1}) N(h)^{k} (a'z + b')^{-k-2s}
\]
at $s = 0$. We note that the map $(a', b') \mapsto a'z + b'\delta$ from the set of pairs $(a', b')$ in (1.14) to $\mathfrak{h} - \{0\}$ is bijective. Indeed, the inverse map is given by $d \mapsto (-dy, dx)$. Thus the value of (1.14) at $s = 0$ is equal to the value of
\[
(1.15) \quad C \cdot N([t_1])^{-k/2} \sum_{h \in \text{Cl}_F} \sum_{d \in \mathfrak{d}, d \neq 0} \sum_{a', b'} \text{sgn}(-dy) \varphi(-dy'h^{-1}) N(h)^{k} d^{-k-2s}
\]
at $s = 0$. Here the last sum runs over a complete set $\mathfrak{d}'$ of representatives for the quotient of $\mathfrak{h}$ by the multiplication of $U$. Since the map $(h, d) \mapsto dh^{-1}$ from the set of pairs $(h, d)$ in (1.15) to the set of all non-zero ideals of $\mathfrak{o}_F$ is a surjective $[\alpha_{[U]}^\infty : U]$-to-1 map, the value of (1.15) at $s = 0$ is equal to
\[
C \cdot N([t_1])^{-k/2} \text{sgn}(-y) \varphi(-y)[\alpha_{[U]}^\infty : U] L(k, \varphi).
\]
Therefore, the functional equation for the Hecke $L$-functions (see, for example, [MI Theorem 3.3.1]) implies that the constant term $a_{c/0}(E_k(\varphi), 1_2)$ is equal to
\[
a_{c/0}(E_k(\varphi), 1_2) = \frac{N(\mathcal{O}_F)^{-k/2}}{2^n} \tau(\varphi) N(m\varphi)^{-k} \text{sgn}(-y)^q \varphi(-y)L(1-k, \varphi^{-1}).
\]

(2) Next suppose that $y \in \mathfrak{b}$. The constant term of $E_k(\varphi, 1_2)|_{\alpha}$ is equal to the value of
\[
C \cdot N([t_1])^{-k/2} N(\mathfrak{b})^{-1} \tau(\psi) \sum_{\mathfrak{b} \in \mathcal{C}_F} \sum_{(a', b') \in \mathcal{B}(\mathfrak{d}, \mathfrak{h}) \neq (0, 0)} \mathfrak{N}(\mathfrak{b})^k \times \text{sgn}(a'^q \varphi(a'h^{-1}) \text{sgn}(-b') \psi^{-1}(-b'b\mathfrak{h}^{-1})(a'b + b'd)^{-k-2s}
\]
at $s = 0$. We note that the map $(a', b') \mapsto a'b + b'd$ from the set of pairs $(a', b')$ in (1.16) to $\mathfrak{b}^{-1} \mathfrak{h} \setminus \{0\}$ is bijective. Indeed, the inverse map is given by $d \mapsto (-dy, dx)$. Thus the value of (1.16) at $s = 0$ is equal to the value of
\[
C \cdot N([t_1])^{-k/2} N(\mathfrak{b})^{-1} \tau(\psi) \sum_{\mathfrak{b} \in \mathcal{C}_F} \sum_{d \in \mathcal{B}(\mathfrak{d}, \mathfrak{h}) \neq (0, 0)} \mathfrak{N}(\mathfrak{b})^k \times \text{sgn}(-dy)^q \varphi(-dy\mathfrak{h}^{-1}) \text{sgn}(-dx)^q \psi^{-1}(-dx\mathfrak{h}^{-1}) N(\mathfrak{b})^k N(d)^{-k-2s}
\]
at $s = 0$. Since the map $(\mathfrak{h}, d) \mapsto d\mathfrak{h}^{-1}$ from the set of pairs $(\mathfrak{h}, d)$ in (1.17) to the set of all non-zero ideals of $\mathfrak{o}_F$ is a surjective $[\mathfrak{o}_F^\times : U]$-to-1 map, the value of (1.17) at $s = 0$ is equal to
\[
C \cdot N([t_1])^{-k/2} N(\mathfrak{b})^{-1} \tau(\psi) \text{sgn}(-y)^q \varphi(-y) \text{sgn}(-x)^q \psi^{-1}(-x)
\times \varphi(\mathfrak{b}^{-1}) N(\mathfrak{b})^k [\mathfrak{o}_F^\times : U] L(k, \varphi^{-1}) \prod_{q|m, q|m_F^{-1}} (1 - \varphi^{-1}(q) N(q)^{-k}).
\]

Therefore, in the same way as above, our assertion follows from the functional equation for the Hecke $L$-functions. \hfill \square

1.7. **Geometric Hilbert modular varieties.** In this subsection, we fix notation concerning the integral models of Hilbert modular varieties and their compactifications, following [Dim2] and [Dim–Ti].

We fix a non-zero ideal $\mathfrak{n}$ of $\mathfrak{o}_F$. We put $\Delta = N(\mathfrak{n}\mathfrak{o}_F)$. Let $\mathfrak{m}_\alpha$ be the closed subscheme of $\mathbb{G}_m \otimes \mathfrak{o}_F^{-1}$ given by the $\mathfrak{n}$-torsion points of $\mathbb{G}_m \otimes \mathfrak{o}_F^{-1}$. Let $\mathfrak{c}$ be a fractional ideal of $F$. We consider the contravariant functor $\mathcal{F}_c^1$ (resp. $\mathcal{F}_c^1$) from the category of $\mathbb{Z}[1/\Delta]$-schemes to the category of sets sending a scheme $S$ to the set of isomorphism classes of triples $((A, \iota), \lambda, \alpha)$ (resp. $(A, \iota), [\lambda], \alpha)$), where $(A, \iota)$ is a Hilbert-Blumenthal abelian variety (HBAV for short) over $S$ endowed with a $\mathfrak{c}$-polarization $\lambda$ (resp. the $\mathfrak{o}_F^{\times}$-orbit $[\lambda]$ of a $\mathfrak{c}$-polarization $\lambda$) and a $\mu_\alpha$-level structure $\alpha$ (for the definitions, see, for example, [Dim2] §1.3]).

Throughout the paper, we assume that
\[
(1.18) \quad (\mathfrak{n}, 6\Delta_F) = 1 \quad \text{and the quotient of the group } \Gamma_1(\mathfrak{c}, \mathfrak{n}) \text{ by its torsion is torsion-free.}
\]

Then the functor $\mathcal{F}_c^1$ is representable by a quasi-projective, smooth, geometrically connected $\mathbb{Z}[1/\Delta]$-scheme $M_c^1 = M(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$ of relative dimension $n = [F : \mathbb{Q}]$ ([Dim–Ti] Theorem 4.1) and [Dim3] Lemma 2.1).

The functor $\mathcal{F}_c^1$ has a coarse moduli scheme $M_c = M(\Gamma_1(\mathfrak{c}, \mathfrak{n}))$, which is the quotient of $M_c^1$ by $\mathfrak{o}_F^{\times}/\mathfrak{o}_F^{\times}$ ([Dim–Ti] Corollary 4.2)]. Here $\mathfrak{o}_F^{\times}$ denotes the subgroup of $\mathfrak{o}_F^{\times}$ consisting of totally positive units, $\mathfrak{o}_F^{\times}$ denotes the subgroup of $\mathfrak{o}_F^{\times}$ consisting of elements congruent to
1 modulo \(n\), and the finite group \(\sigma_{F, +}/\sigma_{F, n}^2\) acts on \(M_{1, \epsilon}\) by \([\epsilon] \cdot ((A, \iota), \lambda, \alpha) = ((A, \iota), \iota(\epsilon) \circ \lambda, \alpha)\) for \([\epsilon] \in \sigma_{F, +}/\sigma_{F, n}^2\). Also, \(M_{\epsilon}\) is a quasi-projective, smooth, geometrically connected \(\mathbb{Z}[1/\Delta]\)-scheme of relative dimension \(n = [F : \mathbb{Q}]\). We put

\[
M^1 = \coprod_{1 \leq i \leq h_F^+} M^1_{[t_i]}, \quad M = \coprod_{1 \leq i \leq h_F^+} M_{[t_i]},
\]

where \(\{[t_i]\}_{1 \leq i \leq h_F^+}\) is the complete set of representatives of \(\text{Cl}_F^+\) fixed in \([1.1]\).

Let \(M^1_{t_{\text{tor}}}\) (resp. \(M^1_{\text{tor}}\)) denote the toroidal compactification of \(M^1_{t}\) (resp. \(M_{t}\)) constructed in \([\text{Dim}].\) The scheme \(M^1_{t_{\text{tor}}}\) (resp. \(M^1_{\text{tor}}\)) is smooth and proper over \(\mathbb{Z}[1/\Delta]\) and the boundary \(M_{t_{\text{tor}}} - M^1_{t}\) (resp. \(M^1_{\text{tor}} - M_{t}\)) is a relative simple normal crossing divisor of \(M_{t_{\text{tor}}}\) (resp. \(M^1_{\text{tor}}\)) \((\text{Dim} 7.2)\). We put

\[
M^1_{t_{\text{tor}}} = \coprod_{1 \leq i \leq h_F^+} M^1_{[t_i]}, \quad M_{t_{\text{tor}}} = \coprod_{1 \leq i \leq h_F^+} M_{[t_i]}.
\]

### 1.8. Geometric Hilbert modular forms

In this subsection, we recall the definition of the geometric Hilbert modular form, following \([\text{Dim}2], \S 1\) and \([\text{Ti–Xi}], \S 2\).

We keep the notation in \([1.7]\). Throughout the paper, we assume that

\[(1.19) \quad \text{for each } i, \ [t_i] \text{ is prime to } p.\]

Let \(\pi : \mathcal{A} \to M^1_{[t_i]}\) denote the universal HBAV. The morphism \(\pi\) extends to a morphism of semi-abelian schemes \(\pi : \mathcal{G} \to M^1_{t_{\text{tor}}}\) \((\text{Dim–Ti} \text{ Theorem } 6.4)\). Let \(\mathcal{A}_{t_{\text{tor}}}\) denote the toroidal compactification of \(\mathcal{A}\) constructed in \([\text{Dim–Ti}]\). Let \(\mathcal{W}_{1, [t_i]}\) (resp. \(\mathcal{H}^1_{1, [t_i]}\)) denote the sheaf \(\mathcal{O}_{\mathcal{G}/M^1_{[t_i]}}\) (resp. \(\mathcal{H}^1_{t_{\text{tor}}/M^1_{t_{\text{tor}}}}\)) defined in \([\text{Dim}2], \S 1.9\), which is a locally free \(\mathcal{O}_{M^1_{[t_i]} \otimes \mathcal{O}_{F'}}\)-module of rank 1 (resp. rank 2). We put \(\mathcal{W}_{1, [t_i]} = \mathcal{W}_{1, [t_i]}^2 \mathcal{O}_{M^1_{[t_i]} \otimes \mathcal{O}_{F'}} \mathcal{H}^1_{1, [t_i]}\).

Let \(\widetilde{F}\) be the Galois closure of \(F\) in \(\overline{\mathbb{Q}}\), \(F'\) the field generated by elements \(\epsilon^{1/2}\) for all \(\epsilon \in \sigma_{F, +}\) over \(\widetilde{F}\), and \(\sigma_{F'}\) the ring of integers of \(F'\). For a \(\mathbb{Z}[1/\Delta]\)-scheme \(S\), we write \(S_{\sigma_{F'}}\) for the base change of \(S\) to \(\text{Spec}(\sigma_{F'}[1/\Delta])\). As explained in \([\text{Dim}2], \S 1.6, \S 1.9]\), \(\mathcal{W}_{1, [t_i]}\) (resp. \(\mathcal{H}^1_{1, [t_i]}\)) descends to a locally free \(\mathcal{O}_{M^1_{[t_i]} \otimes \sigma_{F'}}\) \(\mathcal{O}_{\mathcal{G}/M^1_{[t_i]} \otimes \mathcal{O}_{F'}}\) \(\mathcal{W}_{1, [t_i]}\) (resp. \(\mathcal{H}^1_{1, [t_i]}\)). We put \(\mathcal{U}_{[t_i]} = \mathcal{W}_{1, [t_i]}^2 \mathcal{O}_{M^1_{[t_i]} \otimes \sigma_{F'}} \mathcal{H}^1_{1, [t_i]}\).

Let \(D^1_{[t_i]}\) (resp. \(D_{[t_i]}\)) denote the boundary \(M^1_{[t_i]} - M_{[t_i]}\) (resp. \(M^1_{[t_i]} - M_{[t_i]}\)) of \(M^1_{t_{\text{tor}}}\) (resp. \(M^1_{\text{tor}}\)). For \(X = M^1_{[t_i]}\) (resp. \(M^1_{[t_i]}\) and \(D = D^1_{[t_i]}\) (resp. \(D_{[t_i]}\)), let \((X, L)\) denote the log scheme in the sense of \([\text{Kato}], (1.5) (1)\). For a \(\mathbb{Z}[1/\Delta]\)-algebra \(R\) and a \(\mathbb{Z}[1/\Delta]\)-algebra \((S, L)\) to \((\text{Spec}(R), \text{triv})\) denote the scheme \(\text{Spec}(R)\) endowed with the trivial log structure and let \((S, L)_R\) denote the base change of \((S, L)\) to \((\text{Spec}(R), \text{triv})\). Let \(\Omega^1_{X/R} (\log\ldots)\) denote the logarithmic differential module obtained by the log smooth morphism \((X, L)_R \to (\text{Spec}(\mathbb{Z}[1/\Delta]), \text{triv})_R\) in the sense of \([\text{Kato}], (1.7)\).

Let \(k\) be an integer \(\geq 2\). For \(\omega = \omega_{[t_i]}\) (resp. \(\omega_{[t_i]}\)) and \(\nu = \nu_{[t_i]}\) (resp. \(\nu_{[t_i]}\)), we put \(\omega^{(k-2)} = \omega^{(k-2)} \otimes \nu^{-k2/}\).
\[ M_{M_{[t],[n]}}^{1,\text{tor}} \] (resp. \( M_{M_{[t],[n]}}^{1,\text{tor}} \)). We define the spaces of Hilbert modular forms of weight \( kt \) and of level \( \Gamma_{1}(O_{F}[t],[n]) \) and \( \Gamma_{1}(O_{F}[t],[n]) \) with coefficients in \( R \) to be

\[ M_{k}(\Gamma_{1}(O_{F}[t],[n]), R) = H^{0}(M_{M_{[t],[n]}}^{1,\text{tor}}(\log(D_{[t]})), \omega_{M_{[t],[n]}}^{(k-2)}) \otimes \Omega_{M_{[t],[n]}}^{n}, R) \]

\[ M_{k}(\Gamma_{1}(O_{F}[t],[n]), R) = H^{0}(M_{M_{[t],[n]}}^{1,\text{tor}}(\log(D_{[t]})), \omega_{M_{[t],[n]}}^{(k-2)}) \otimes \Omega_{M_{[t],[n]}}^{n}, R) \]

respectively. If \( F \neq \mathbb{Q} \), then, by the Koecher’s principle, we have

\[ M_{k}(\Gamma_{1}(O_{F}[t],[n]), R) = H^{0}(M_{M_{[t],[n]}}^{1,\text{tor}}(\log(D_{[t]})), \omega_{M_{[t],[n]}}^{(k-2)}) \otimes \Omega_{M_{[t],[n]}}^{n}, R) \]

\[ M_{k}(\Gamma_{1}(O_{F}[t],[n]), R) = H^{0}(M_{M_{[t],[n]}}^{1,\text{tor}}(\log(D_{[t]})), \omega_{M_{[t],[n]}}^{(k-2)}) \otimes \Omega_{M_{[t],[n]}}^{n}, R) \]

We define the space of Hilbert cusp forms of weight \( kt \) and of level \( \Gamma_{1}(O_{F}[t],[n]) \) and \( \Gamma_{1}(O_{F}[t],[n]) \) with coefficients in \( R \) to be

\[ S_{k}(\Gamma_{1}(O_{F}[t],[n]), R) = \text{im} \left( H^{0}(M_{M_{[t],[n]}}^{1,\text{tor}}(\log(D_{[t]})), \omega_{M_{[t],[n]}}^{(k-2)}) \otimes \Omega_{M_{[t],[n]}}^{n}, R) \right) \]

\[ S_{k}(\Gamma_{1}(O_{F}[t],[n]), R) = \text{im} \left( H^{0}(M_{M_{[t],[n]}}^{1,\text{tor}}(\log(D_{[t]})), \omega_{M_{[t],[n]}}^{(k-2)}) \otimes \Omega_{M_{[t],[n]}}^{n}, R) \right) \]

We put

\[ M_{k}(M_{1}, R) = \bigoplus_{1 \leq i \leq h_{F}^{+}} M_{k}(\Gamma_{1}(O_{F}[t],[n]), R) \]

\[ S_{k}(M_{1}, R) = \bigoplus_{1 \leq i \leq h_{F}^{+}} S_{k}(\Gamma_{1}(O_{F}[t],[n]), R) \]

1.9. Hecke correspondences. In this subsection, we fix notation concerning the Hecke correspondence \( T(a) \) and \( U(a) \), following \[ \text{Dim}2 \] §2.4 and \[ \text{Ki–La} \] §1.11.

Let \( a \) be a non-zero ideal of \( O_{F} \). We fix a pair \((i, j)\) such that \([t] a = [t] j \) in \( C_{F}^{1} \). We consider the functor \( F_{a,i,j} \) from the category of \( Z[1/\Delta]\)-schemes to the category of sets sending a scheme \( S \) to the set of isomorphism classes of quintuples \((A, i), \lambda, \alpha, C, \beta\). Here \((A, i)\) is a \( HBAV \) over \( S \), endowed with a \([t] \)-polarization \( \lambda \) and a \( \mu_{a} \)-level structure \( \alpha \), \( C \) is an \( \phi_{F} \)-stable closed subshift of \( A[a] \), which is disjoint from \( \alpha(\mu_{a}) \) etale locally isomorphic to the constant group scheme \( \phi_{F}/a \) over \( \phi_{F}, \) and \( \beta \) is the \( \phi_{F} \)-strictly orbit-isomorphisms \((t[a], (t[a], j) +) \approx ([t] j, [t] j +) \), where \( c_{+} = c \cap (F \otimes \mathbb{R})_{C}^{+} \) is the totally positive cone for a fractional ideal \( c \) of \( F \).

The functor \( F_{a,i,j} \) is representable by \( M_{1}^{1} \) constructed in \[ \text{Ki–La} \] §1.9. Put \( M_{1}^{1} = \prod_{1 \leq i \leq h_{F}^{+}} M_{1}^{1,\text{tor}}(A, i). \) Then the two projections \( \pi_{1}, \pi_{2} : M_{1}^{1} \rightarrow M_{1}^{1} \) given in \[ \text{Ki–La} \] §1.9 induce algebraic correspondences \( T(a) \) and \( U(a) \) on \( M_{1}^{1} \). We define the Hecke correspondence \( T(a) \) and \( U(a) \) on \( M_{1}^{1,\text{tor}}(A, i) \) by taking a toroidal compactification of \( M_{1}^{1} \) (see the proof of \[ \text{Dim}2 \] Corollary 2.7).

Thus, we get an action of \( T(a) \) and \( U(a) \) on the spaces \( M_{k}(M_{1}, R) \) and \( S_{k}(M_{1}, R) \) (\[ \text{Dim}2 \] §2.4 and \[ \text{Ki–La} \] §1.11) and hence we obtain an action of \( T(a) \) and \( U(a) \) on \( M_{k}(M, R) \) and \( S_{k}(M, R) \) by using the projection \( \sum_{[t] a = [t] j} \frac{1}{[t] a, \phi F_{a,n}^{1} [t] a, \phi F_{a,n}^{2} [t] a} \) \( M_{k}(M_{1}, R) \rightarrow M_{k}(M, R) \).

According to \[ \text{Dim}2 \] §2.4 and \[ \text{Ki–La} \] §1.11, this action over \( \mathbb{C} \) coincides with the usual Hecke operator as \[ \text{Le}. \]

2. Mellin transform

The purpose of this section is to give a cohomological description of special values of the \( L \)-functions defined in \[ \text{Le} \] associated to a Hilbert modular form vanishing at cusps
\[ \Gamma_0(\mathfrak{d}_F[t_1], n) \)-equivalent to the cusp \( \infty \) (Proposition 2.15 and 2.16), where we use the assumption \( h_F^n = 1 \).

2.1. Borel–Serre compactification. In this subsection, we recall the Borel–Serre compactification of \( Y_i \) defined by (1.3). For more detail, refer to [Ha §2.1] and [Hida93 §1.8].

We fix an integer \( i \) with \( 1 \leq i \leq h_F^n \) and abbreviate \( \Gamma_1(\mathfrak{d}_F[t_1], n) \) to \( \Gamma \).

Let \((\mathcal{S}^I_F)^{\text{BS}}\) denote the Borel–Serre compactification of \( \mathcal{S}^I_F \), which is a locally compact manifold on which \( \text{GL}_2(F) \) acts. We can describe the boundary of \((\mathcal{S}^I_F)^{\text{BS}}\) at the cusp \( \infty \) as follows. Put \( X = \{ (y, x) \in (F \otimes \mathbb{R})_+^i \times (F \otimes \mathbb{R}) \mid \prod_{i \in J_F} y_i = 1 \} \). Then we have

\[
(2.1) \quad \mathcal{S}^I_F \xrightarrow{\sim} X \times \mathbb{R}^n_{+i}; (x_i + \sqrt{-1}y_i)_{i \in J_F} \mapsto \left( \left( \prod_{i \in J_F} y_i \right)^{-1} y_i, x_i \right)_{i \in J_F}, \prod_{i \in J_F} y_i,
\]

which is compatible with the action of \( \Gamma_{\infty} \). Here \( \Gamma_{\infty} \) denotes the stabilizer of \( \infty \) in \( \Gamma \), which acts trivially on the second factor of the right-hand side. The compactification of \( \mathcal{S}^I_F \) at the cusp \( \infty \) is given by \( X \times (\mathbb{R}^n_{+i} \cup \{ \infty \}) \).

Let \( Y_i^{\text{BS}} \) denote the Borel–Serre compactification \( \overline{\mathcal{S}^I_F}^{\text{BS}} \) of \( Y_i \). Then \( Y_i^{\text{BS}} \) is a compact manifold and its boundary at a cusp \( s \), which is denoted by \( D_s \), is given by \( \Gamma_s \setminus \alpha(X \times \{ \infty \}) \), where \( \Gamma_s \) denotes the stabilizer of \( s \) in \( \Gamma \) and \( \alpha \in \text{SL}_2(F) \) such that \( s = \alpha(\infty) \).

2.2. Fundamental domain. In this subsection, we construct a relative homology class, which is related to special values of the \( L \)-functions attached to a Hilbert modular form.

We keep the notation in §2.1. Let \( E \) be a subgroup of \( \mathcal{S}_F^{\infty} \) of finite index and \( \varepsilon_1, \cdots, \varepsilon_{n-1} \) a \( \mathbb{Z} \)-basis of \( E \). We note that a fundamental domain of \((\mathbb{R}^n_{+i})^{\infty} / E \) is given by

\[
\Omega_E = \prod_{1 \leq j \leq n-1} \{ \varepsilon_j^{r_j} \mid r_j \in [0, 1) \} \times \mathbb{R}^n_{+i} \xrightarrow{\sim} X \times \mathbb{R}^n_{+i} \xrightarrow{\sim} \mathcal{S}^I_F;
\]

\[
(\varepsilon_1^{r_1}, \cdots, \varepsilon_{n-1}^{r_{n-1}}, -\log(r_{n})) \mapsto ((\varepsilon^r, 0), -\log(r_{n})) \mapsto \sqrt{-1}y^n \varepsilon^r,
\]

where \( r = (r_1, \cdots, r_{n-1}) \in [0, 1)^{n-1} \), \( (\varepsilon^r)^{\prime} = \prod_{1 \leq j \leq n-1} (\varepsilon_j^{r_j})^{\prime} \) for \( i \in J_F \), \( \varepsilon^r = ((\varepsilon^r)^{\prime})_{i \in J_F} \), and \( y = -\log(r_{n}) \). We put

\[
\overline{\Omega}_E = \prod_{1 \leq j \leq n-1} \{ \varepsilon_j^{r_j} \mid r_j \in [0, 1) \} \times (\mathbb{R}_{\geq 0} \cup \{ \infty \}).
\]

We define a singular \( n \)-cube \( \ell_i : [0, 1]^n \rightarrow \overline{\Omega}_E \rightarrow (\mathcal{S}^I_F)^{\text{BS}} \) by

\[
(r_1, \cdots, r_n) \mapsto (\varepsilon_1^{r_1}, \cdots, \varepsilon_{n-1}^{r_{n-1}}, -\log(r_{n})) \mapsto \sqrt{-1}y^n \varepsilon^r.
\]

Let \( c_{E,i} \) denote the composition of \( \ell_i \) and the canonical map \( (\mathcal{S}^I_F)^{\text{BS}} \rightarrow Y_i^{\text{BS}} \).

2.3. Mellin transform. In this subsection, we give a Mellin transform for a Hilbert modular form \( \in \text{M}_k(\Gamma_1(\mathfrak{d}_F[t_1], n), \mathbb{C}) \) vanishing at cusps \( \Gamma_0(\mathfrak{d}_F[t_1], n) \)-equivalent to the cusp \( \infty \). For the proof, we must need to prove analytic properties of the \( L \)-functions, which are obtained by P-B. Garrett for a Hilbert cusp form \( \in \text{S}_k(\Gamma(\mathfrak{n}), \mathbb{C}) \) of level \( \Gamma(n) \) ([Ga §1.9, p.37, Theorem]).

Here \( \Gamma(n) \) is the principal congruence subgroup of level \( n \). In order to do it, we strictly follow the argument in the method of Garrett. We keep the notation in §1.1, §2.1 and §2.2.

**Proposition 2.1.** Let \( h \in \text{M}_k(\Gamma_1(\mathfrak{d}_F[t_1], n), \mathbb{C}) \). For \( s \in \mathbb{C} \) such that \( \text{Re}(s) \gg 0 \), the integral

\[
\int_{\text{image of } c_{E,i}} y^{(s-1)n} w(\tilde{h})
\]
converges absolutely and extends to a meromorphic function on the complex plane, which is holomorphic at $s = 1$. Here $w(h)$ is defined by (2.2) and $\tilde{h}(z) = h(z) - a_\infty(0, h)$.

**Proof.** For $s \in \mathbb{C}$ such that $\text{Re}(s) \gg 0$, the integral above is just equal to the following:

$$(2.2) \quad \int_{[0,1]^n-1} \int_{\sqrt{m} \mathbb{R}_+^n} y^{(s-1)} w(\tilde{h}) = \int_{[0,1]^n-1} \left( \int_{\sqrt{-1}\mathbb{R}^n} + \int_{0}^{\sqrt{-1}} \right) y^{(s-1)} w(\tilde{h}).$$

We calculate the second term. Put $s = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \in G_{\mathbb{R},+}$. By the pull-back formula, we have

$$(2.3) \quad \int_{[0,1]^n-1} \int_{0}^{\sqrt{-1}} y^{(s-1)} w(\tilde{h}) = - \int_{[0,1]^n-1} \int_{\sqrt{-1}} \int_{0}^{\sqrt{-1}} y^{(1-s)} \sigma \cdot w(\tilde{h}|\sigma)$$

$$- \int_{[1,0]^n-1} \int_{\sqrt{-1}} \int_{0}^{\sqrt{-1}} y^{(1-s)} \sigma \cdot w(a_\infty(0, h|\sigma)) - \int_{[0,1]^n-1} \int_{0}^{\sqrt{-1}} y^{(s-1)} w(a_\infty(0, h)).$$

An elementary calculation shows that the second (resp. third) term of (2.3) is absolutely convergent for $\text{Re}(s) > k$ (resp. $\text{Re}(s) \geq 1$) and holomorphic at $s = 1$. Thus, for the proof, it suffices to show that the first terms of (2.2) and (2.3) are absolutely convergent and holomorphic at $s = 1$. Hence we reduce it to showing that the integral

$$(2.4) \quad \int_{[a,b]^n-1} \int_{1}^{\infty} y^{(s-1)} \tilde{h}(\sqrt{-1}y^{\frac{1}{2}} \varepsilon^{\tau}) y^m dr dy$$

is absolutely convergent and holomorphic at $s = 1$ for any $a, b \in \mathbb{R}$ with $a \leq b$ and non-negative integer $m$. There is a constant $M > 0$ such that $\xi(N, \xi) > M$ for any $0 < \xi \in [\xi]$. Then there is a constant $\varepsilon > 0$ such that $\xi(N, \xi) > M + \varepsilon$ for any such $\xi$. Thus, by the same argument as in [Ga, p.29], we have an estimate

$$\exp \left( \pi n M \frac{1}{2} y^{\frac{1}{2}} \right) \left| \tilde{h}(\sqrt{-1}y^{\frac{1}{2}} \varepsilon^{\tau}) \right| \leq \sum_{0 < \xi \in [\xi]} |a_\infty(\xi, h)| \exp \left( -\pi \left( 2 - \frac{M}{M + \varepsilon} \right) \frac{1}{2} \right) \text{Tr}(\xi y^{\frac{1}{2}} \varepsilon^{\tau}).$$

Since $\tilde{h}(z)$ is absolutely convergent, so is the latter series. Hence there are constants $C, C' > 0$ such that

$$\left| \tilde{h}(\sqrt{-1}y^{\frac{1}{2}} \varepsilon^{\tau}) \right| \leq C \exp \left( -C' y^{\frac{1}{2}} \right)$$

for any $y \geq 1$ and $r \in [a, b]^{n-1}$. Therefore the integral (2.4) is dominated by

$$\int_{[a,b]^{n-1}} \int_{1}^{\infty} \exp(-C'y^{\frac{1}{2}}) y^{|\text{Re}(s)|-1+m} dr dy$$

and hence is absolutely convergent and a holomorphic function of $s \in \mathbb{C}$. \hfill $\square$

We assume that $h_{\bar{F}}^\perp = 1$. We fix a Hilbert cusp form $f$ and the Hilbert Eisenstein series $E_2(\varphi, \psi)$ given in Proposition 1.2 satisfying the following conditions:

$$f \in S_2(n, \chi, \mathbb{C})$$

and $E_2(\varphi, \psi) \in M_2(n, \chi, \mathbb{C})$ vanishes at cusps $\Gamma_0(\partial_F[t_1], n)$-equivalent to the cusp $\infty$.

We simply write $h = f$ or $E_2(\varphi, \psi)$. We express the special values of the Dirichlet series $D(1, h, \eta)$ as a Mellin transform for $h$ (cf. [Oda §16], [Hida94 §7, §8], and [Ochi §3]).
Let $\eta$ be a $\mathbb{Q}$-valued narrow ray class character of $F$ whose conductor is denoted by $m_\eta$ such that $m_\eta$ is prime to $\mathcal{O}_F[t_1]$ and $n|m_\eta$. Let $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times$ (resp. $(m_\eta^{-1}/\mathcal{O}_F)^\times$) be the subset of $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times$ (resp. $m_\eta^{-1}/\mathcal{O}_F$) consisting of elements whose annihilator is $m_\eta$. We fix a non-canonical isomorphism of $\mathcal{O}_F$-modules $m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1} \cong m^{-1}/\mathcal{O}_F \cong \mathcal{O}_F/m_\eta$ and a non-canonical bijection induced from it $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times \cong (m_\eta^{-1}/\mathcal{O}_F)^\times$. Hence we may canonically identify $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times/\mathcal{O}_{F,+}^\times$ with a subgroup of $\text{Cl}_F^+(m_\eta)$ under the canonical extension
$$1 \to (\mathcal{O}_F/m_\eta)^\times/\mathcal{O}_{F,+}^\times \to \text{Cl}_F^+(m_\eta) \to \text{Cl}_F^+ \to 1.$$  

Let $\eta_1$ be the function on $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times/\mathcal{O}_{F,+}^\times$ defined by $\eta_1(\bar{b}) = \eta(bm_\eta\mathcal{O}_F[t_1])$. We note that $\eta_1(\xi\bar{b}) = \eta(\xi)\eta_1(\bar{b})$ for any $\bar{b} \in (m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times/\mathcal{O}_{F,+}^\times$ and $0 \leqslant \xi \in [t_1]$ prime to $m_\eta$. Let $E$ denote $\eta_{F,m_\eta,+} := \{ e \in \mathcal{O}_{F,+}^\times \mid e \equiv 1 \pmod{m_\eta}\}$. We fix a complete set $S$ (resp. $T$) of representatives of $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times/\mathcal{O}_{F,+}^\times$ in $m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}$ (resp. $\mathcal{O}_{F,+}^\times/E$ in $\mathcal{O}_{F,+}^\times$) satisfying the condition that
$$\text{(2.6)} \quad \text{every cusp } b \in S \text{ is } \Gamma_0(\mathcal{O}_F[t_1],1_n)\text{-equivalent to the cusp } \infty.$$  

Here we note that the existence of such set follows from the assumption $n|m_\eta$. Indeed, fix a generator $m$ (resp. $c$) of $m_\eta$ (resp. $\mathcal{O}_F[t_1]$) and a set $S'$ of representatives of $(\mathcal{O}_F/m_\eta)^\times$ satisfying the condition that each $x \in S'$ is prime to $mc$. Then $\{x/mc \mid x \in S'\}$ is a complete set of representatives of $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times/\mathcal{O}_{F,+}^\times$. The assumption $n|m_\eta$ implies $mc \in \mathcal{O}_F[t_1]$ and hence there is $$\left(\frac{x}{mc}, \ast, \ast\right) \in \Gamma_0(\mathcal{O}_F[t_1],1_n).$$  

Let $\bar{b}$ denote the image of $b \in S$ in $(m_\eta^{-1}\mathcal{O}_F^{-1}[t_1]^{-1}/\mathcal{O}_F^{-1}[t_1]^{-1})^\times/\mathcal{O}_{F,+}^\times$ under the canonical map. We have
$$\text{(2.7)} \quad N([t_1])^{s-k/2} \sum_{b \in S} \sum_{u \in T} \eta_1(\bar{b})^{-1}h_1(z + bu)$$
$$= N([t_1])^{s-k/2} \sum_{0 \leqslant \xi \in [t_1]} a_{\infty}(\xi, h_1) \sum_{b \in S} \sum_{u \in T} \eta_1(\bar{b})^{-1}e_F(\xi bu)e_F(\xi z)$$
$$= \tau(\eta^{-1})N([t_1])^{s-k/2} \sum_{0 \leqslant \xi \in [t_1]} a_{\infty}(\xi, h_1)\eta(\xi[t_1]^{-1})e_F(\xi z).$$  

Here the last equality follows from Shih (3.11). By taking $\Omega_E = \prod_{u \in T} u^{-1}\Omega_{\mathcal{O}_{F,+}^\times}$, we have
$$N([t_1])^{s-k/2} \sum_{b \in S} \eta_1(\bar{b})^{-1} \int_{\Omega_E} h_1(z + b)\varphi^{(s-1)} dz_{J_F}$$
$$= N([t_1])^{s-k/2} \sum_{b \in S} \eta_1(\bar{b})^{-1} \sum_{u \in T} \int_{u^{-1}\Omega_{\mathcal{O}_{F,+}^\times}} h_1(z + bu)\varphi^{(s-1)} dz_{J_F}$$
$$= N([t_1])^{s-k/2} \sum_{b \in S} \eta_1(\bar{b})^{-1} \sum_{u \in T} \int_{\Omega_{\mathcal{O}_{F,+}^\times}} h_1(z + bu)\varphi^{(s-1)} dz_{J_F}$$
$$= \int_{\Omega_{\mathcal{O}_{F,+}^\times}} N([t_1])^{s-k/2} \sum_{b \in S} \sum_{u \in T} \eta_1(\bar{b})^{-1}h_1(z + bu)\varphi^{(s-1)} dz_{J_F}.$$
Here we note that each integral is well-defined by using Proposition 2.1 and the condition (2.6). By using the expansion of (2.7), for $\Re(s) \gg 0$, we have
\[
N([t_1])^{s-k/2} \sum_{b \in S} \eta_1(\overline{b})^{-1} \int_{\sqrt{t_1(F \otimes \mathbb{R})}_+} h_1(z + b) y^{(s-1)t} dz_{JF} = \tau(\eta^{-1}) N([t_1])^{s-k/2} \sum_{0 \leq \xi \leq [t_1]} a_{\infty}(\xi, h_1) \eta(\xi [t_1]^{-1}) \int_{\Omega_{\infty, +}} e_F(\xi z) y^{(s-1)t} dz_{JF}.
\]
\[
= \tau(\eta^{-1}) \sum_{0 \leq \xi \leq [t_1]} a_{\infty}(\xi, h_1) \eta(\xi [t_1]^{-1}) N([t_1])^{-k/2} N(\xi [t_1]^{-1})^{s} \int_{\Omega_{\infty, +}} e_F(\xi z)(\xi y)^{(s-1)t} \prod_{\nu \in J_F} d\xi^e z_i.
\]
\[
= \tau(\eta^{-1}) \sum_{\xi \in F_{+}} a_{\infty}(\xi, h_1) \eta(\xi [t_1]^{-1}) N([t_1])^{-k/2} N(\xi [t_1]^{-1})^{s} \int_{\sqrt{t_1(F \otimes \mathbb{R})}_+} e_F(\xi z)(\xi y)^{(s-1)t} \prod_{\nu \in J_F} d\xi^e z_i.
\]
\[
= \tau(\eta^{-1}) D(s, h, \eta)(2\pi)^{-sn} \sqrt{-1} \Gamma(s)^n.
\]
Here we note that each integral is well-defined by using Proposition 2.1 and we may regard $h_1(z + b)$ as a function on $\sqrt{t_1(F \otimes \mathbb{R})}_+ / E$ since $h_1(uz + b) = h_1(z + b)$ for any $u \in E$. Furthermore, the integrals in the first line of this equation are independent of the choice of a lift $b$ of $\overline{b}$. Hence the integral depends only on the image $b$ of $b$ in $(m_{\eta}^{-1} \mathfrak{d}_F^{-1}[t_1]^{-1} / \mathfrak{d}_F^{-1}[t_1]^{-1})^\times / \mathfrak{o}_{F,+}$ and it shall be denoted by
\[
\int_{\sqrt{t_1(F \otimes \mathbb{R})}_+} h_1(z + \overline{b}) y^{(s-1)t} dz_{JF}.
\]
Therefore we obtain the following Mellin transform:

**Proposition 2.2.** Assume that $h_1^c = 1$. Let $h = f$ or $E_2(\varphi, \psi)$ as (2.5) and $\eta$ a $\mathbb{Q}$-valued narrow ray class character of $F$ whose conductor is denoted by $m_{\eta}$ such that $m_{\eta}$ is prime to $\mathfrak{o}_F[t_1]$ and $n|m_{\eta}$. Then we have
\[
\sum_{b \in S} \eta_1(\overline{b})^{-1} \int_{\sqrt{t_1(F \otimes \mathbb{R})}_+ / \mathfrak{o}_{F,m_{\eta}}^\times} h_1(z + \overline{b}) dz_{JF} = \tau(\eta^{-1}) D(1, h, \eta)(-2\pi \sqrt{-1})^{-n}.
\]

**Remark 2.3.** As mentioned above, the assumption $n|m_{\eta}$ and the conditions (2.5) and (2.6) imply that the integrals are well-defined. If $h$ is a Hilbert cusp form, then the Mellin transform as Proposition 2.2 is satisfied without the assumption $n|m_{\eta}$.

We consider a Mellin transform in the anti-holomorphic case. Let $W_G$ denote the Weyl group $K_{\infty}/K_{\infty,+}$, which is identified with $\{ w_J \mid J \subset J_F \}$, where $w_J \in K_{\infty}$ such that $w_{J, t} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if $t \in J$ and $w_{J, t} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ if $t \in J_F \setminus J$. Then $W_G$ acts on the space of Hilbert modular forms via $h \mapsto h_J := h[K_{\infty} w_J K_{\infty}]$ for each subset $J$ of $J_F$.

**Proposition 2.4.** Under the same notation and assumptions as Proposition 2.2, we have
\[
\sum_{b \in S} \eta_1(\overline{b})^{-1} \int_{\sqrt{t_1(F \otimes \mathbb{R})}_+ / \mathfrak{o}_{F,m_{\eta}}^\times} h_{J, t}(z + \overline{b}) dz_{J} = \tau(\eta^{-1}) D(1, h, \eta) n_{\infty}(\nu_J)(-2\pi \sqrt{-1})^{-n},
\]
where $dz_J$ is defined by (3.1) and $\nu_J \in \mathfrak{h}_{F, \infty}$ such that $\nu_{J,t} = 1$ if $t \in J$ and $\nu_{J,t} = -1$ if $t \in J_F \setminus J$. 
Proof. Since $h^+_F = 1$, we can take $a \in \mathfrak{o}_K^+$ such that $\iota(a) > 0$ if $a \in J$ and $\iota(a) < 0$ if $a \in J_F \backslash J$.

By putting $\gamma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, the action of $[K_\infty, \gamma, K_\infty]$ on $Y(n) = Y_1$ is given by $z \mapsto \gamma^{-1} z$.

Then, by definition, we have $h_{J,1}(z) = h_1(\gamma^{-1} z)(-1)^{2(|J_F \backslash J|)}$. Hence we obtain

$$h_1(\gamma^{-1} z) = \sum_{\mu \in [t_1], \mu = J} c(\mu|t_1|^{-1}, \mu)|N(\mu)|e_F(\sqrt{-1}\mu y_{\infty}^{(n)} \mu x_\infty) .$$

Here $\{\mu\} = \{\mu \in J_F | \mu^\ell > 0\}$ and $y_{\infty, \ell} = y_{\infty, \ell}$ (resp. $-y_{\infty, \ell}$) if $\ell \in J$ (resp. $\ell \in J_F \backslash J$). Now our assertion follows from the same argument as in the proof of Proposition 2.2.

2.4 Relation between cohomology class and Dirichlet series. In this subsection, we give a cohomological description of special values of the $L$-functions.

We keep the notation in (2.3). As the previous subsection, we assume that $h^+_F = 1$. We fix a lift $b \in S$ of $\tilde{b} \in (m_\infty^{-1} \mathfrak{o}_K^{-1}[t_1]^{-1}/\mathfrak{o}_K^{-1}[t_1]^{-1})^\times / \mathfrak{o}_K^+$. We consider the Hilbert modular variety $Y(n)$ defined by (1.1). Let $C(1) \Phi F[t_1], n)$ denote the set of all cusps of $Y(n)$. Let $C_\infty$ be the subset of $C(1)(\Phi F[t_1], n)$ consisting of cusps $\Gamma_0(\Phi F[t_1], n)$-equivalent to the cusp $\infty$.

We consider the following subset $H_b$ of $\Phi Y_F$:

$$H_b := b + \sqrt{-1}(F \otimes \mathbb{R})^\times \rightarrow \Phi Y_F .$$

We define an action of $\Phi F_{m, n}, +$ on $H_b$ by $\varepsilon \cdot (z_i)_{\in J_F} = (\varepsilon z_i - (\varepsilon - 1)b)_{\in J_F}$. Since $(\varepsilon - 1)b \in \Phi F^{-1}[t_1]|^{-1}$ for any $\varepsilon \in \Phi F_{m, n}, +$ we see that $\varepsilon \cdot (z_i)_{\in J_F}$ is $\Gamma_1(\Phi F[t_1], n)$-equivalent to $(z_i)_{\in J_F}$.

Therefore we have $H_b / \Phi F_{m, n}, + \rightarrow Y(n)$ and it induces $H_n^p(Y(n), A) \rightarrow H_n^p(H_b / \Phi F_{m, n}, +, A)$ for $A = \mathcal{O}, K$, or $\mathbb{C}$.

We define subsets $H_b^{BS}, \partial_\infty, \text{ and } \partial_b$ of $X \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$ as follows. Let $X_b$ denote the image of $H_b$ in $X$ under the composition of the isomorphism (2.1) and the projection to $X$. We have $H_b \simeq X_b \times \mathbb{R}_+^\times$. We define $H_b^{BS}, \partial_\infty, \text{ and } \partial_b$ by

$$H_b^{BS} = X_b \times (\mathbb{R}_{\geq 0} \cup \{\infty\}), \quad \partial_\infty = X_b \times \{\infty\}, \quad \partial_b = X_b \times \{0\} .$$

The action of $\Phi F_{m, n}, +$ on $H_b$ extends canonically to an action on $H_b^{BS}$. We put $\alpha_b(\mathcal{O}) = \begin{pmatrix} -b & 1 + b^2 \\ -b & 0 \end{pmatrix}$. Note that $\alpha_b(\infty) = b$. The embedding $H_b \hookrightarrow \Phi Y_F$ (resp. the composition $H_b \xrightarrow{\alpha_b} H_b \hookrightarrow \Phi Y_F \xrightarrow{\alpha_b} \Phi Y_F$) induces an $\Phi F_{m, n}, +$-equivariant map

$$(2.8) \quad H_b \cup \partial_\infty \hookrightarrow \Phi Y_F \cup (X \times \{\infty\}) \quad \text{ (resp. } H_b \cup \partial_b \hookrightarrow \Phi Y_F \cup \alpha_b(X \times \{\infty\}))$$

because $\alpha_b(b + \sqrt{-1}y) = b + \sqrt{-1}y$. Therefore we have $H_b^{BS} / \Phi F_{m, n}, + \rightarrow Y(n)^{BS}$ and it induces

$$(2.9) \quad H_n^n(Y(n)^{BS}, D_{C_n}(n); A) \rightarrow H_n^n(Y(n)^{BS}, D_{b, \infty}(n); A)$$

$$\rightarrow H_n^n(H_b^{BS} / \Phi F_{m, n}, +, \partial_b / \Phi F_{m, n}, + \cup \partial_\infty / \Phi F_{m, n}, +; A) \simeq H_n^n(H_b / \Phi F_{m, n}, +, A)$$

for $A = \mathcal{O}, K$, or $\mathbb{C}$. Here $D_s$ is the boundary of $Y(n)^{BS}$ at a cusp $s$ as explained in (2.1) $D_{C_n}(n) = \prod_{s \in C_n} D_s$, and $D_{b, \infty}(n) = D_b \cup D_\infty$.

We define the evaluation map

$$(2.10) \quad \text{ev}_{b, 1, A} : H_n^n(Y(n)^{BS}, D_{C_n}(n); A) \rightarrow A$$

by the composition of (2.9) and the trace map $H_n^n(H_b / \Phi F_{m, n}, +, A) \rightarrow A$, where

$$H_n^n(Y(n)^{BS}, D_{C_n}(n); A) := H_n^n(Y(n)^{BS}, D_{C_n}(n); A) / A$$

$A$-torsion.
Note that the definition of $ev_{h,1,A}$ depends only on $\bar{b}$ (because if $\bar{b} = \bar{b}'$, then there is $\gamma \in \Gamma_1(\mathcal{O}_F[t_1], n)$ such that $b = \gamma(b')$ and $\gamma(\infty) = \infty$ and hence it shall be denoted by $ev_{h,1,A}$.

In order to give a cohomological description of the $L$-functions, we recall the relative de Rham theory, which is proved by A. Borel [Bo, Theorem 5.2] for general locally symmetric spaces. Let $\Omega^\bullet(Y(n), \mathbb{C})$ denote the complex of $\mathbb{C}$-valued $C^\infty$-differential $\Gamma_1(\mathcal{O}_F[t_1], n)$-invariant forms in $\mathcal{F}^{\mathcal{F}}$. Let $\Omega^\bullet_{\mathrm{rel}}(Y(n), D_{C_\infty}(n); \mathbb{C})$ denote the complex of forms in $\Omega^\bullet(Y(n), \mathbb{C})$ which, together with their exterior differentials, are fast decreasing at every $s \in C_\infty$. By the proof of [Bo Theorem 5.2] on the stalks at the boundary, we have

$$H^n_{\mathrm{dR}}(Y(n), \Omega^\bullet_{\mathrm{rel}}(Y(n), D_{C_\infty}(n); \mathbb{C})) \simeq H^n(Y(n)^{\mathbb{B}_p}, D_{C_\infty}(n); \mathbb{C}).$$

Let $h = f$ or $\mathcal{E}_2(\varphi, \psi)$ as (2.5). Let $[\omega_h]$ denote the Betti cohomology class in $H^n(Y(n), \mathbb{C})$ attached to the de Rham cohomology class of $\omega(h)$. By the isomorphism (2.11) and the condition (2.5), we can define the relative cohomology class $[\omega_h]_{\mathrm{rel}}$ in $H^n(Y(n)^{\mathbb{B}_p}, D_{C_\infty}(n); \mathbb{C})$ attached to the relative de Rham cohomology class of $\omega(h)$ whose image in $H^n(Y(n), \mathbb{C})$ is $[\omega_h]$. Now, by combining these observations and Proposition 2.2, we obtain the following:

**Proposition 2.5.** Let $h = f$ or $\mathcal{E}_2(\varphi, \psi)$ as (2.5). Let $A = \mathcal{O}$, $\mathcal{K}$, or $\mathbb{C}$. Assume that $h_F^* = 1$ and $a[\omega_h]_{\mathrm{rel}} \in H^n(Y(n)^{\mathbb{B}_p}, D_{C_\infty}(n); A)$ for some $a \in A$. Let $\eta$ be a $Q$-valued narrow ray class character of $F$ whose conductor is denoted by $m_\eta$ such that $m_\eta$ is prime to $\mathcal{O}_F[t_1]$ and $n|m_\eta$. Then we have

$$A(\eta) \ni \sum_{b \in S} \eta_1(\bar{b})^{-1} ev_{h,1,A}(a[\omega_h]_{\mathrm{rel}}) = a\tau(\eta^{-1})D(1, h, \eta)(-2\pi \sqrt{-1})^{-n}.$$

We treat the anti-holomorphic case. By the description of the action of $[K_\infty w_J K_\infty]$ on $Y(n)$ mentioned in the proof of Proposition 2.4, we see that the action of $[K_\infty w_J K_\infty]$ preserves the component $D_{C_\infty}(n)$ and hence $[K_\infty w_J K_\infty]$ acts on $\hat{H}^{\infty}(Y(n)^{\mathbb{B}_p}, D_{b,\infty}(n); A)$. We obtain the following proposition by the same argument as in the proof of Proposition 2.6 using Proposition 2.4 instead of Proposition 2.2.

**Proposition 2.6.** Under the same notation and assumptions as Proposition 2.4 and Proposition 2.2, we have

$$A(\eta) \ni \sum_{b \in S} \eta_1(\bar{b})^{-1} ev_{h,1,A}(a[\omega_h]_{\mathrm{rel}}[K_\infty w_J K_\infty]) = a\tau(\eta^{-1})D(1, h, \eta)\eta_\infty(\nu_J)(-2\pi \sqrt{-1})^{-n}.$$

3. **Integrality of cohomology classes at boundary**

The purpose of this section is to prove the integrality of the restriction of the Betti cohomology class associated to a Hilbert Eisenstein series to the boundary of the Borel–Serre compactification of $Y(n)$ (Proposition 3.4). The proof is based on the comparison theorem between Betti cohomology and group cohomology, and an explicit computation of the restriction of a group cocycle associated to a Hilbert Eisenstein series to the boundary.

### 3.1. Cocycles associated to Hilbert modular forms

In this subsection, we construct a group cocycle associated to a Hilbert modular form, which is a generalization of the Eichler–Shimura cocycle in the case where $F = \mathbb{Q}$. We strictly follow the argument in the method of H. Yoshida [Yo]. We fix $i$ with $1 \leq i \leq h_F^*$ and abbreviate $\Gamma_1(\mathcal{O}_F[t_1], n)$ to $\Gamma$.

First we recall the definition of group cohomology. Let $R$ be a commutative ring and $M$ a left $R[\Gamma]$-module. For a non-negative integer $q$, let $C_q$ denote the space of functions on $\Gamma^q$
with values in $M$. The differential map $d^q: C^q \to C^{q+1}$ is given by
\[
d^q u(\gamma_1, \ldots, \gamma_{q+1}) = \frac{1}{q!} \sum_{1 \leq j \leq q} (-1)^j u(\gamma_1, \ldots, \gamma_{j-1}, \gamma_j, \gamma_{j+1}, \ldots, \gamma_q) + (-1)^{q+1} u(\gamma_1, \ldots, \gamma_q).
\]
The associated $q$-th cohomology group of $\Gamma$ with coefficients in $M$ is given by
\[
H^q(\Gamma, M) = \ker(d^q: C^q \to C^{q+1})/\operatorname{im}(d^{q-1}: C^{q-1} \to C^q).
\]

For a subset $J$ of $J_F$, we put $z^J_i = z_i$ (resp. $\overline{z}_i$) if $i \in J$ (resp. $i \in J_F \setminus J$) and
\[(3.1)\]
\[
dz_j = \bigwedge\vphantom{\int_1^2}{\cap}_{i \in J_F} dz^J_i.
\]

For a $\mathbb{Z}$-algebra $A$, $u, v \in A$, and a non-negative integer $\ell$, we put
\[
\left[\begin{array}{c}
u \\
v\end{array}\right]^{\ell} = (u^\ell, u^{\ell-1}v, \ldots, uv^{\ell-1}, v^\ell).
\]

Let $L_2(A)$ denote the space of column vectors $A^{\ell+1} \simeq \operatorname{Sym}^\ell(A^2)$. We define the $\ell$-th symmetric tensor representation $\rho_\ell$ of $GL_2(A)$ on $L_2(A)$ by
\[
\rho_\ell(g) \left[\begin{array}{c}u \\
v\end{array}\right]^{\ell} = \left[\begin{array}{c}g(u) \\
v\end{array}\right]^{\ell}.
\]

Let $k$ be an integer $\geq 2$. Recall that $\bar{F}$ is the Galois closure of $F$ in $\mathbb{Q}$ and $F'$ is the field generated by elements $\varepsilon_{\ell/2}$ for all $\varepsilon \in A_2^\infty$. For an $\mathfrak{o}_{F'}$-algebra $A$, we define an $A[[\mathfrak{m}_2(\mathfrak{o}_{F'}) \cap GL_2(F')^{\mathfrak{o}_{F'}}]]$-module $L_{kt}(A)$ as follows: let $L_{kt}(A)$ be the $A$-module $\otimes_{i \in J_F} L_{k-2}(A)$ with a left action by
\[
g \bullet P = \det(\rho(\gamma)^{(2-k)\ell}) \rho(\gamma)P,
\]
where $\rho = \otimes_{i \in J_F} \rho_{k-2}$. Note that $G(A_f)$ naturally acts on $L_{kt}(A \otimes_{\mathfrak{o}_{F'}} \mathfrak{A}_{F'}(\gamma))$. We consider the $i$-part $L_{kt,i}(A)$ of $L_{kt}(A)$ defined by
\[
L_{kt,i}(A) = L_{kt}(A \otimes_{\mathfrak{o}_{F'}} F') \cap x_i \bullet L_{kt}(A \otimes_{\mathfrak{o}_{F'}} \mathfrak{A}_{F'}).
\]

Hereafter, in this subsection, we abbreviate $L_{kt,i}(A)$ to $L_{kt}(A)$. For a holomorphic function $h$ on $\mathcal{H}^{J_F}$, we define an $L_{kt}(\mathbb{C})$-valued holomorphic differential $n$-form $\omega(h)$ on $\mathcal{H}^{J_F}$ by
\[(3.2)\]
\[
\omega(h) = h(z) \otimes \bigotimes_{i \in J_F} \left[\begin{array}{c}z_i \\
1\end{array}\right]^{k-2} dz_{J_F}.
\]
If $h \in M_k(\Gamma, \mathbb{C})$, then we have
\[
g^* \omega(h) = \det(\rho(g)^{(2-k)\ell}) \rho(g) h(z) \otimes \bigotimes_{i \in J_F} \left[\begin{array}{c}z_i \\
1\end{array}\right]^{k-2} dz_{J_F}
\]
for $g \in \text{GL}_2(\mathbb{C})^{J_F}$. Here $(h\gamma)(z) := \det(\rho(\gamma)^{(k-1)\ell}) j_i(gz) = k\gamma h(gz)$. Since $(h\gamma)(z) = h(z)$ for $\gamma \in \Gamma$ and the center $\Gamma \cap F'$ of $\Gamma$ acts trivially on $L_{kt}(\mathbb{C})$, we obtain the pull-back formula
\[(3.3)\]
\[
\gamma^* \omega(h) = \gamma \bullet \omega(h)
\]
for any $\gamma \in \Gamma$ of $\gamma$. 


Hereafter, in this subsection, we put \( J_F = \{ \iota_1, \cdots, \iota_n \} \) and \( z_i = z_{i,0} \) for \( z \in J_F \). We fix a base point \( w = (w_i)_{1 \leq i \leq n} \in J_F \). We define an \( L_{ht}(\mathbb{C}) \)-valued holomorphic function by

\[
F(z) = \int_{w_1}^{z_1} \cdots \int_{w_n}^{z_n} \omega(h).
\]

For \( \gamma \in \Gamma \), we define a function \( \gamma \ast F \) on \( J_F \) by \( \gamma \ast F(z) = \gamma \cdot F(\gamma^{-1}z) \). We note that

\[
\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} (\gamma \ast F - F)(z) = 0.
\]

**Lemma 3.1.** ([Yq Chapter V, Lemma 5.1]). Let \( D \) be an open contractible domain in \( \mathbb{C}^n \). Let \( f \) be a holomorphic function on \( D \).

1. Assume that

\[
\frac{\partial}{\partial z_1} \cdots \frac{\partial}{\partial z_n} f(z) = 0 \quad \text{for} \quad z = (z_1, \cdots, z_n) \in D.
\]

Then there exist holomorphic functions \( g_i \) on \( D \) such that \( g_i(z) \) is independent of \( z_i \) and \( f \) is decomposed into \( f(z) = \sum_{1 \leq i \leq n} g_i(z) \).

2. Moreover, assume that \( f \) is decomposed into \( f(z) = \sum_{1 \leq i \leq n} g_i(z) \) as (1) and \( n \geq 2 \). If \( f(z) \) is independent of \( z_1 \), then there exist holomorphic functions \( h_i \) on \( D \) such that \( h_i(z) \) is independent of \( z_1 \) and \( z_i \) and \( f \) is decomposed into \( f(z) = \sum_{2 \leq i \leq n} h_i(z) \).

By applying Lemma 3.1 (1) to \( -(\gamma \ast F - F) \), we obtain a decomposition

\[
-(\gamma \ast F - F)(z) = \sum_{1 \leq i \leq n} g_i^{(1)}(\gamma)(z),
\]

where \( g_i^{(1)}(\gamma) \) is a holomorphic function on \( J_F \) and \( g_i^{(1)}(\gamma)(z) \) is independent of \( z_i \).

We explicitly describe \( g_n^{(1)}(\gamma)(z) \) as follows. We have

\[
(\gamma \ast F - F)(z) = \int_{\gamma w_1}^{\gamma z_1} \cdots \int_{\gamma w_n}^{\gamma z_n} \left( \int_{w_1}^{z_1} + \int_{w_n}^{z_n} \right) \omega(h) - \int_{w_1}^{z_1} \cdots \int_{w_n}^{z_n} \omega(h)
\]

\[
= \int_{\gamma w_1}^{\gamma z_1} \cdots \int_{\gamma w_{n-1}}^{\gamma z_{n-1}} \left( \int_{w_1}^{z_1} \cdots \int_{w_{n-1}}^{z_{n-1}} - \int_{w_n}^{z_n} \right) \omega(h).\]

By applying Lemma 3.1 (1) to the second term in the second line of (3.3), we can choose \( -g_n^{(1)}(\gamma)(z) \) as the first term in the second line of (3.3):

\[
g_n^{(1)}(\gamma)(z) = \int_{\gamma w_1}^{\gamma z_1} \cdots \int_{\gamma w_{n-1}}^{\gamma z_{n-1}} \int_{w_n}^{z_n} \omega(h).
\]

By regarding (3.5) as a 1-cocycle, we obtain

\[
dg_n^{(1)}(\gamma_1, \gamma_2)(z) = - \sum_{1 \leq i \leq n-1} d g_i^{(1)}(\gamma_1, \gamma_2)(z)
\]

for \( \gamma_1, \gamma_2 \in \Gamma \), where \( d \) is the boundary map in group cohomology. Since the left-hand side is independent of \( z_n \) and \( d g_i^{(1)}(\gamma_1, \gamma_2)(z) \) is independent of \( z_i \), by Lemma 3.1 (2), we obtain a decomposition

\[
dg_n^{(1)}(\gamma_1, \gamma_2)(z) = \sum_{1 \leq i \leq n-1} g_i^{(2)}(\gamma_1, \gamma_2)(z),
\]

where \( g_i^{(2)}(\gamma_1, \gamma_2) \) is a holomorphic function and \( g_i^{(2)}(\gamma_1, \gamma_2)(z) \) is independent of \( z_i \).
We explicitly describe \( g^{(2)}_{n-1}(\gamma_1, \gamma_2)(z) \) as follows. A direct calculation shows that

\[
(3.9) \quad g^{(2)}_{n-1}(\gamma_1, \gamma_2)(z) = \int_{\gamma_1}^{z_1} \cdots \int_{\gamma_1}^{z_m} \int_{\gamma_1}^{w_{n-1}} \int_{\gamma_1}^{w_n} \omega(h) \]

By repeating this argument, we obtain a decomposition

\[
(3.10) \quad g^{(2)}_{n-1}(\gamma_1, \gamma_2)(z) = \int_{\gamma_1}^{z_1} \cdots \int_{\gamma_1}^{z_m} \int_{\gamma_1}^{w_{n-1}} \int_{\gamma_1}^{w_n} \omega(h).
\]

By the same argument as above, we can choose \( g^{(2)}_{n-1}(\gamma_1, \gamma_2)(z) \) as the first term in the last equation of (3.9):

\[
(3.11) \quad (-1)^m d g^{(m-1)}_{n-m+2}(\gamma_1, \cdots, \gamma_m)(z) = \sum_{1 \leq i \leq n-m+1} g_i^{(m)}(\gamma_1, \cdots, \gamma_m)(z)
\]

for each \( m \) with \( 2 \leq m \leq n-1 \) and \( \gamma_1, \cdots, \gamma_m \in \Gamma \), where

\[
g_{n-m+1}(\gamma_1, \cdots, \gamma_m)(z) = \int_{\gamma_1}^{z_1} \cdots \int_{\gamma_1}^{z_m} \int_{\gamma_1}^{w_{n-1}} \int_{\gamma_1}^{w_n} \omega(h).
\]

Hence we have an explicit formula

\[
d g^{(n-1)}_{2}(\gamma_1, \cdots, \gamma_n)(z) = \int_{\gamma_1}^{\gamma_n} \cdots \int_{\gamma_1}^{w_{n-1}} \int_{\gamma_1}^{w_n} \omega(h).
\]

Therefore we obtain the following theorem (1) because it is a constant function:

**Proposition-Definition 3.2.** Let \( h \in M_\delta(\Gamma, \mathbb{C}) \) and \( w = (w_i)_{1 \leq i \leq n} \in \delta^{IF} \) a base point.

1. For \( \gamma_i \in \Gamma \) and a lift \( \gamma_i \in \Gamma \) of \( \gamma_i \), the following map \( \pi_{h,w} : \Gamma \rightarrow L_{kt}(\mathbb{C}) \) is an \( n \)-cocycle:

\[
\pi_{h,w}(\gamma_1, \cdots, \gamma_n) = \int_{\gamma_1}^{\gamma_n} \cdots \int_{\gamma_1}^{w_{n-1}} \int_{\gamma_1}^{w_n} \omega(h).
\]

2. The cohomology class \( [\pi_h] \in H^n(\Gamma, L_{kt}(\mathbb{C})) \) defined by \( \pi_{h,w} \) does not depend on the choice of the base point \( w \in \delta^{IF} \).

**Proof.** The assertion (2) follows from [Yo] Theorem 5.2. □
3.2. Integrality of $n$-cocycles at boundary. We keep the notation in \([3.1]\). In this subsection, for $E \in M_k(\Gamma, \mathcal{O})$, we prove the integrality of the image of $[\pi E]$ under the restriction map.

We define two maps $b_1$ and $b_2$ from $\overline{G(\mathbb{Q})}_{\gamma}^n$ to $L_{kt}(\mathbb{C})$ by

\[
b_1(\overline{\gamma}, \ldots, \overline{\gamma}) = \gamma_1 \int \gamma_2 \cdots \gamma_n w_1 \cdot \cdots \int w_{n-1} \int_0^{\sqrt{-1} \infty} \omega(E) = \gamma_1 \int \gamma_2 \cdots \gamma_n w_1 \cdot \int w_{n-1} \int_0^{w_n} \omega(a_\infty(0, E)),
\]

\[
b_2(\overline{\gamma}, \ldots, \overline{\gamma}) = \int \gamma_1 \cdots \gamma_n w_1 \cdot \int \gamma_2 \cdots \gamma_n w_1 \cdot \int w_{n-1} \int_0^{\sqrt{-1} \infty} \omega(E) = \int \gamma_1 \cdots \gamma_n w_1 \cdot \int w_{n-1} \int_0^{w_n} \omega(a_\infty(0, E)),
\]

where $\tilde{E}(z) = E(z) - a_\infty(0, E)$. We note that the same argument as in the proof of Proposition \([2.1]\) shows that $b_1(\overline{\gamma}, \ldots, \overline{\gamma})$ and $b_2(\overline{\gamma}, \ldots, \overline{\gamma})$ converge absolutely.

We define a new map $\pi_{E,w}^b : G(\mathbb{Q})_{\gamma}^n \to L_{kt}(\mathbb{C})$ by

\[
(3.12) \quad \pi_{E,w}^b(\overline{\gamma}, \ldots, \overline{\gamma}) = \pi_{E,w}(\overline{\gamma}, \ldots, \overline{\gamma}) + b_1(\overline{\gamma}, \ldots, \overline{\gamma}) - b_2(\overline{\gamma}, \ldots, \overline{\gamma}).
\]

**Proposition 3.3.** For $E \in M_k(\Gamma, \mathbb{C})$, the map $\pi_{E,w}^b$ satisfies the following properties.

1. The value $\pi_{E,w}^b(\overline{\gamma}, \ldots, \overline{\gamma})$ is independent on $w_n$.
2. $\pi_{E,w}^b$ is cohomologous to $\pi_{E,w}$.

**Proof.** The assertion (1) follows from a direct calculation. For the proof of (2), we put

\[
v(\overline{\gamma}, \ldots, \overline{\gamma}) = \int \gamma_1 \cdots \gamma_{n-1} w_1 \cdot \int \gamma_{n-2} \cdots \gamma_n w_1 \cdot \int w_{n-1} \int_0^{\sqrt{-1} \infty} \omega(E) = \int \gamma_1 \cdots \gamma_{n-2} w_1 \cdot \int \gamma_{n-1} \cdots \gamma_n w_1 \cdot \int w_{n-1} \int_0^{w_n} \omega(a_\infty(0, E)).
\]

Now the assertion follows from the following claim:

\[
(3.13) \quad dv(\overline{\gamma}, \ldots, \overline{\gamma}) = \pi_{E,w}^b(\overline{\gamma}, \ldots, \overline{\gamma}) - \pi_{E,w}(\overline{\gamma}, \ldots, \overline{\gamma}).
\]

For the proof of the claim \((3.13)\), it suffices to prove the following:

(i) $\overline{\gamma} \bullet v(\overline{\gamma}, \ldots, \overline{\gamma}) = b_1(\overline{\gamma}, \ldots, \overline{\gamma})$;

(ii) $\sum_{1 \leq n-j \leq n-1} (-1)^{n-j} v(\overline{\gamma}_1, \ldots, \overline{\gamma}_{n-j} \overline{\gamma}_{n-j+1}, \ldots, \overline{\gamma_n}) + (-1)^n v(\overline{\gamma}, \ldots, \overline{\gamma}) = -b_2(\overline{\gamma}, \ldots, \overline{\gamma})$.

The assertion (i) follows from a direct calculation. For the proof of (ii), it suffices to show the following \((*)_k\) by induction on $1 \leq n - k \leq n - 1$:

\[
(*)_k \quad \sum_{n-k \leq n-j \leq n-1} (-1)^{n-j} v(\overline{\gamma}_1, \ldots, \overline{\gamma}_{n-j} \overline{\gamma}_{n-j+1}, \ldots, \overline{\gamma_n}) + (-1)^n v(\overline{\gamma}, \ldots, \overline{\gamma}) = (-1)^{n-k} \left\{ \int \gamma_1 \cdots \gamma_n w_1 \cdot \int \gamma_{n-k} \cdots \gamma_{n-k+1} w_{k+1} \cdot \int w_n \int_0^{\sqrt{-1} \infty} \omega(E) \right. \]

\[
- \int \gamma_1 \cdots \gamma_{n-k+1} w_{k+1} \cdot \int \gamma_{n-k} \cdots \gamma_{n-k+1} w_{k+1} \cdot \int w_n \int_0^{w_n} \omega(a_\infty(0, E)) \right\}.
\]
Suppose that $k = 1$. A direct calculation shows $(\ast)_1$:

$$
(-1)^{n-1}v(\overline{\gamma}_1, \ldots, \overline{\gamma}_{n-1}, \overline{\gamma}_n) + (-1)^nv(\overline{\gamma}_1, \ldots, \overline{\gamma}_{n-1})
$$

$$
= (-1)^{n-1}\left\{ \sum_{n-k-1 \leq j \leq n-1} (-1)^{n-j}v(\overline{\gamma}_1, \ldots, \overline{\gamma}_{n-j-1}, \overline{\gamma}_{n-k-j}, \overline{\gamma}_{n-k-j+1}, \overline{\gamma}_{n-k-j+2}, \ldots, \overline{\gamma}_n) + (-1)^nv(\overline{\gamma}_1, \ldots, \overline{\gamma}_{n-1}) \right\}
$$

By using Proposition 3.3, we prove the main result of this section:

**Proposition 3.4.** Assume that $h_F^2 = 1$ and $\mathcal{O}_F[t_1] = \mathcal{O}_F$. Let $\Phi_p$ be the composite field of $t_F((F')'(\sqrt{-1}))$ in $\overline{\mathbb{Q}}_p$ for all $i \in J_F$ and $\mathcal{O}$ the ring of integers of a finite extension $K$ of $\Phi_p$.

Here $\iota_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ is the fixed embedding and $F'$ is the field defined in (9.2). Let $k$ be an even integer such that $2 \leq k \leq p$. Let $E \in M_k(\Gamma, \mathcal{O})$. Then we have the following properties:

1. The cohomology class $\text{res}([\pi_E])$ is integral, that is, $\text{res}([\pi_E]) \in \bigoplus_{s \in C(\Gamma)} \hat{H}^n(\Gamma_s, L_{k,1}(\mathcal{O})).$

Here $\hat{H}^n(\Gamma_s, L_{k,1}(\mathcal{O}))$ denotes the image of $H^n(\Gamma_s, L_{k,1}(\mathcal{O})) \to H^n(\Gamma_s, L_{k,1}(K))$.

2. Suppose that $E$ vanishes at a cusp $s \in C(\Gamma)$. Then we have $\text{res}([\pi_E]) = 0$ in $\hat{H}^n(\Gamma_s, L_{k,1}(\mathcal{O})).$

**Proof.** We treat the case $s = \infty$. Let $\overline{\gamma}_1, \ldots, \overline{\gamma}_n \in \overline{\Gamma}_\infty$. By Proposition 3.3 (1), the value $\pi_{E, w}(\overline{\gamma}_1, \ldots, \overline{\gamma}_n)$ defined by (9.12) is independent on $w_n$. The first terms of $b_1(\overline{\gamma}_1, \ldots, \overline{\gamma}_n)$
and \( b_2(\gamma_1, \ldots, \gamma_n) \) converge to 0 when \( w_n \) tends to \( \sqrt{-1} \infty \). Hence we obtain

\[
\pi_{E, (\sqrt{-1}, \ldots, \sqrt{-1}, w_n)}(\gamma_1, \ldots, \gamma_n) = \lim_{w_n \to -\sqrt{-1} \infty} \int_{\gamma_1 \cdots \gamma_n \cdot \sqrt{-1}} \cdots \int_{\gamma_1 \sqrt{-1}} \left( \int_{\gamma_1 w_n} - \int_{\gamma_1 0} + \int_{0 w_n} \right) \omega(a_\infty(0, E)) = \int_{\gamma_1 \cdots \gamma_n \cdot \sqrt{-1}} \cdots \int_{\gamma_1 \sqrt{-1}} \int_{0} \omega(a_\infty(0, E)).
\]

Here the first equality follows from that \( \gamma_1 \cdot \omega(a_\infty(0, E)) = \gamma_1^* \omega(a_\infty(0, E)) \), where we use the assumption that \( k \) is even. Since \( \gamma_i \in \text{GL}_2(\mathfrak{o}_F) \cap B_{\infty, +} \) for each \( i \), \( \gamma_1 \cdots \gamma_j \epsilon \in \mathcal{O} \) for each \( j \) and \( \epsilon \in \{ -\sqrt{-1}, 0 \} \). Thus \( \pi_{E, (\sqrt{-1}, \ldots, \sqrt{-1}, w_n)}(\gamma_1, \ldots, \gamma_n) \) belongs to \( L_{k, 1}(\mathcal{O}) \), where we use the assumption that \( k \leq p \). Hence the image of \( \text{res}(\pi_E) \) in the \( \infty \)-part is integral.

We treat the general case \( s \in C(\Gamma) \). By the assumption \( h_F^+ = 1 \), we can take \( \alpha \in \text{GL}_2(\mathfrak{o}_F) \) such that \( s = \alpha(\infty) \). Let \( \gamma_1, \ldots, \gamma_n \in \Gamma \). Let \( B_1, \ldots, B_n \in \text{GL}_2(\mathfrak{o}_F) \cap B_{\infty, +} \) such that \( \gamma_i = \alpha B_i \alpha^{-1} \). By the pull-back formula \( \alpha^* \omega(E) = \alpha \cdot \omega(E) \), we have

\[
\pi_{E, (\sqrt{-1}, \ldots, \sqrt{-1}, \omega w_n)}(\gamma_1, \ldots, \gamma_n) = \alpha \cdot \pi_{E, (\sqrt{-1}, \ldots, \sqrt{-1}, w_n)}(B_1, \ldots, B_n).
\]

Now the same argument as in the case \( s = \infty \) replacing \( E \) by \( E | \alpha \) shows that the image of \( \text{res}(\pi_E) \) in the \( s \)-part is integral.

\[
\square
\]

4. Eisenstein cohomology and Eichler–Shimura–Harder isomorphism

The purpose of this section is to recall theory of Eisenstein cohomology and the Eichler–Shimura–Harder isomorphism. We use the assumption \( h_F^+ = 1 \) to prove the Eichler–Shimura–Harder isomorphism (1.7) by using an explicit computation of the action of the Weyl group.

4.1. Eisenstein cohomology. The purpose of this subsection is to prove the following proposition, where \( Y = Y(n) \) and \( H^n_{\text{Eis}}(Y, \mathbb{C}) \) is the Eisenstein cohomology defined by (1.1).

**Proposition 4.1.** (1) The Hodge number of \( H^n_{\text{Eis}}(Y, \mathbb{C}) \) is equal to \( n \), that is, \( H^n_{\text{Eis}}(Y, \mathbb{C}) = F^n H^n_{\text{Eis}}(Y, \mathbb{C}) \).

(2) \( H^n_{\text{Eis}}(Y, \mathbb{C}) \) is stable under the Hecke correspondences.

**Proof.** (1) The assertion for \( H^n_{\text{Eis}}(Y^1(n), \mathbb{C}) \) is obtained by E. Freitag ([Fre], Chapter III, Proposition 3.5 and Theorem 4.9)]. We follow the arguments in the method of Freitag. We fix \( j \) with \( 1 \leq j \leq h_F^+ \) and abbreviate \( \Gamma_j(\mathfrak{d}_F, [n]) \) to \( \Gamma_j \). We put \( J_F = \{ t_1, \ldots, t_n \} \) and \( z_i = z_{t_i} \) for \( z \in \mathcal{H}_F \). For \( z \in \mathcal{H}_F \), we put \( N(z) = \prod \mathfrak{p} \leq \mathfrak{n} z_i \).

The proof consists of three steps.

**Step 1:** To give a basis of \( H^{n-1}(\Gamma_1, \mathbb{C}) \) and \( H^n(\Gamma_1, \mathbb{C}) \) over \( \mathbb{C} \) for each cusp \( t \in C(\Gamma) \). (Here we use a basis of \( H^{n-1}(\Gamma_t, \mathbb{C}) \) to prove Proposition 5.2).

We treat the case \( t = \infty \). We prove that a basis of \( H^{n-1}(\Gamma_\infty, \mathbb{C}) \) (resp. \( H^n(\Gamma_\infty, \mathbb{C}) \)) over \( \mathbb{C} \) is given by

\[
\omega^{n-1}_\infty = \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{n-1}}{y_{n-1}} \quad \text{(resp. } \omega^n_\infty = dx_1 \wedge \cdots \wedge dx_n)\).
\]

Put \( D = \{ z \in \mathcal{H}_F \mid N(y) = 1 \} \). The group \( \Gamma_\infty \) which consists of transformations of the form \( z \mapsto \varepsilon z + b \) with \( N(\varepsilon) = 1 \) acts on \( D \). We identify \( D \) with \( \mathbb{R}^{2n-1} \) by \( z \mapsto (x_1, \ldots, x_n, u_1, \ldots, u_{n-1}) \) with coordinates \( \{ x_i \}_{1 \leq i \leq n} \) and \( \{ u_i := \log(y_i) \}_{1 \leq i \leq n-1} \). Since \( \Gamma_\infty \backslash \mathcal{H}_F \) is homeomorphic to \( \mathbb{R} \times (\Gamma_\infty \backslash D) \) by \( z \mapsto (\log(N(y)), N(y)^{-1/n} z) \), the canonical
embedding $\Gamma_\infty \setminus D \hookrightarrow \Gamma_\infty \setminus \mathcal{B}^P$ is a homotopy equivalence. Hence it induces $H^* (\Gamma_\infty, \mathbb{C}) \simeq H^* (\Gamma_\infty \setminus \mathcal{B}^P, \mathbb{C}) \simeq H^* (\Gamma_\infty \setminus D, \mathbb{C})$.

We consider a $\Gamma_\infty$-invariant harmonic differential $m$-form $\omega = \sum_{b,c} f_{b,c} (x,u) dx_b \wedge du_c$ on $D$. By the same argument as in [Fre, p.145, 146], the functions $f_{b,c} (x,u)$ are independent of $x$, and if $f_{b,c} (x,u) \neq 0$, then $b = \phi$ or $\{1, \cdots, n\}$. We treat the case $b = \phi$ (the case $b = \{1, \cdots, n\}$ is similar). Since $H^{n-1} (\Gamma_\infty \setminus D, \mathbb{C})$ is isomorphic to the de Rham cohomology of a lattice $\log (\mathfrak{m}_{\mathfrak{p}})$ of $\mathbb{R}^{n-1}$, the same argument as in [Fre, p.146] shows that $\omega_{\infty}^{-1}$ is a basis of $H^{n-1} (\Gamma_\infty, \mathbb{C})$.

We treat the general case $t \in C(\Gamma)$. Let $\alpha \in G(\mathbb{Q})$ be such that $t = \alpha (\infty)$. The canonical map $\alpha : D_\infty \overset{\cong}{\rightarrow} D_t$ induces $\alpha^{-1} (\Gamma \alpha)_{\infty} \setminus D_\infty \overset{\cong}{\rightarrow} \Gamma_{\infty} \setminus D_t$. Now our assertion follows from the same argument as in the case $t = \infty$ by replacing $\Gamma$ by $\alpha^{-1} \Gamma \alpha$ (cf. [Fre, p.154]).

**Step 2**: To construct the Eisenstein operator $E : \bigoplus_{t \in C(\Gamma)} H^n (\Gamma_t, \mathbb{C}) \to H^n (\Gamma, \mathbb{C})$.

We may assume $t = \infty$ by the same argument as in Step 1. As mentioned in the proof of [Fre, Chapter III, Remark 3.1], $\omega_{\infty}^n$ is cohomologous to $dz_1 \wedge \cdots \wedge dz_n$ up to a constant factor. We put $\omega_{\infty} : = dz_1 \wedge \cdots \wedge dz_n$. The Eisenstein operator $E$ is defined by symmetrization

$$E (\omega_{\infty}) = \sum_{M \in \Gamma \setminus \Gamma} M^* \omega_{\infty} := \lim_{s \to 0} \sum_{M \in \Gamma \setminus \Gamma} |N(j(M,z))|^{-2s} M^* \omega_{\infty},$$

(cf. [Fre, Chapter III, Proposition 3.3]). Here, for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $j(M,z) = (c_i z_i + d_i)_{1 \leq i \leq n}$ and $M^* \omega_{\infty} = N(j(M,z))^{-2} \omega_{\infty}$. We note that, by using analytic continuation ([Shl, Proposition 3.2]) of Eisenstein series of the type

$$E_{2,0} (z,s) = \sum_{M \in \Gamma \setminus \Gamma} N(j(M,z))^{-2} |N(j(M,z))|^{-2s},$$

$E (\omega_{\infty})$ is expressed by $E (\omega_{\infty}) = E_{2,0} (z,0) \omega_{\infty}$. Hence $E$ is well-defined.

**Step 3**: To show that $E$ is a section of the restriction map $H^n (\Gamma, \mathbb{C}) \to H^n (\Gamma_t, \mathbb{C})$ for every $t \in C(\Gamma)$.

As discussed in the proof of [Fre, Chapter III, Proposition 3.3], our assertion follows from that the constant term of $E_{2,0} (z,0)$ at $t$ is equal to 1 (resp. 0) if $t$ is $\Gamma$-equivalent to $\infty$ (resp. $t$ is not $\Gamma$-equivalent to $\infty$).

We define the Eisenstein cohomology $H^n_{\text{Eis}} (Y, \mathbb{C})$ to be the image of $E$:

$$H^n_{\text{Eis}} (Y, \mathbb{C}) = \text{im}(E).$$

Hence the Hodge number of $H^n_{\text{Eis}} (Y, \mathbb{C})$ is equal to $n$ because $E_{2,0} (z,0)$ is holomorphic.

(2) Let us fix $\Gamma' = \Gamma (t_j \cap \mathcal{D})$ and $\alpha \in G(\mathbb{Q})$ such that $\Gamma \alpha \Gamma'$ is expressed as a finite disjoint union $\Gamma \alpha \Gamma' = \coprod_{i \in I} \Gamma_{\alpha_j}$. For the proof, it suffices to show that

$$E (\omega_{\gamma}) (\Gamma \alpha \Gamma') = E (\omega_{\gamma}) (\Gamma \alpha \Gamma').$$

We may assume $t = \infty$ by the same argument as in Step 1 and 2. By the definition of $E$, the left-hand side of (4.2) is equal to

$$E (\omega_{\infty}) (\Gamma \alpha \Gamma') = \lim_{s \to 0} \sum_{\gamma \in \Gamma \setminus \Gamma \alpha \Gamma'} |N(j(\gamma,z))|^{-2s} \gamma^* \omega_{\infty}.$$

We consider the right-hand side of (4.2). For each $s \in \mathbb{P}^1 (F)$, we put $\mathcal{S} = \{ \gamma \in \Gamma \setminus \Gamma \alpha \Gamma' \mid \gamma(s) = \infty \}$. Note that $\Gamma \setminus \Gamma \alpha \Gamma' = \bigcup_{s \in \mathbb{P}^1 (F)} \mathcal{S}_s$. Since, for each $s \in \mathbb{P}^1 (F)$, there exist a
unique \( t \in C(\Gamma') \) and a unique \( M \in \Gamma'_{\mathfrak{p}} \backslash \Gamma' \) such that \( M(s) = t \), we have
\[
\Gamma_{\infty} \backslash \Gamma \Gamma' = \coprod_{t \in C(\Gamma')} \coprod_{M \in \Gamma'_{\mathfrak{p}} \backslash \Gamma'} \mathcal{S}_{M^{-1}(t)}.
\]
We put \((\omega'_t)_{t \in C(\Gamma')} = \omega_{\infty} |[\Gamma \Gamma']\). We claim that
\[
\omega'_t = \sum_{\gamma \in \mathcal{S}_t} \gamma^* \omega_{\infty}.
\]
For the moment, we admit the claim \((1.1)\). Hence we obtain
\[
E(\omega_{\infty} |[\Gamma \Gamma']\)) = \sum_{t \in C(\Gamma')} \sum_{\gamma \in \mathcal{S}_t} E(\gamma^* \omega_{\infty})
\]
\[
= \lim_{s \to 1} \sum_{t \in C(\Gamma')} \sum_{\gamma \in \mathcal{S}_t} \sum_{M \in \Gamma'_{\mathfrak{p}} \backslash \Gamma'} |N(j(\gamma M, z))|^{-2s} (\gamma M)^* \omega_{\infty}.
\]
Here the first equality follows from \((1.3)\) and the second equality follows from the definition of \( E \). Therefore our assertion \((1.2)\) follows from \( \mathcal{S}_t \cdot M = \mathcal{S}_{M^{-1}(t)} \), \((1.1)\) and \((1.3)\).

Thus it remains to prove the claim \((1.5)\). We decompose \( \Gamma \Gamma' \) into a disjoint union: \( \Gamma \Gamma' = \coprod_{i \in I^t} \Gamma \beta_i \Gamma' \) and \( \Gamma \beta_i \Gamma' = \coprod_{j \in J^t_i} \Gamma \beta \delta_i \beta_j \) with \( \delta_i \beta_j \in \Gamma' \). By the definition of the Hecke operator acting on the boundary cohomology \( \text{Hida93} (3.1c) \), we have \( \omega'_t = \sum_{i \in I^t_o} \sum_{j \in J^t_i} (\beta \delta_i \beta_j)^* \omega_{\beta_i}(t) \).

Here \( I^t_o = \{ i \in I^t \mid \beta_i(t) = \Gamma\text{-equivalent to } \infty \} \). For each \( i \in I^t_o \), we may assume that \( \beta_i(t) = \infty \). Now, for the proof of claim, it suffices to show that \( \mathcal{S}_t = \coprod_{i \in I^t_o} \coprod_{j \in J^t_i} \mathcal{S}_{\beta \delta_i \beta_j} \).

The inclusion \( \supseteq \) (resp. \( \subseteq \)) follows from \( \beta \delta_i \beta_j(t) = \infty \) (resp. the decomposition \( \Gamma_{\infty} \backslash \Gamma \Gamma' = \coprod_{i \in I^t_o} \coprod_{j \in J^t_i} \mathcal{S}_{\beta \delta_i \beta_j} \)).

4.2. Partial Eichler–Shimura–Harder isomorphism. In this subsection, we prove the Eichler–Shimura–Harder isomorphism \((1.7)\), where we use the assumption \( h^+_F = 1 \).

Let \( K \) be a finite extension of \( \mathbb{Q}_p \) and \( \mathcal{O} \) the ring of integers of \( K \). For \( A = \mathcal{O}, K, \) or \( \mathbb{C} \), let \( H^*_\text{par}(Y(n), A) \) denote the compact support cohomology of \( Y(n) \) with coefficients in \( A \), and let \( H^*_\text{cusp}(Y(n), A) \) denote the parabolic cohomology of \( Y(n) \) with coefficients in \( A \), that is, \( H^*_\text{cusp}(Y(n), A) = \text{im} (H^*_\text{par}(Y(n), A) \to H^*(Y(n), A)) \). We have the decomposition
\[
H^*(Y(n), \mathbb{C}) \simeq H^*_\text{par}(Y(n), \mathbb{C}) \oplus H^*_\text{cusp}(Y(n), \mathbb{C})
\]
(see Step 3 and \((1.1)\) in the proof of Proposition \((1.1)\). Let \( \hat{H}^*_\text{par}(Y(n), \mathcal{O}) \) denote the image of \( H^*_\text{par}(Y(n), \mathcal{O}) \to H^*_\text{par}(Y(n), K) \).

By \( \text{Hida93} \) Theorem 1.1, if the degree \( n = [F : \mathbb{Q}] \) is even, then the \( \mathbb{C} \)-vector space \( H^*_\text{par}(Y(n), \mathbb{C}) / H^*_\text{cusp}(Y(n), \mathbb{C}) \) is spanned by the cohomology classes of the invariant forms \( \omega_{J'} = \sum_{j \in J'} y_{j}^{-2} dx_{j} \wedge dy_{j} \) for all subsets \( J' \) of \( J_F \) such that \( |J'| = n/2 \), where \( H^*_\text{cusp}(Y(n), \mathbb{C}) \) denotes the cuspidal cohomology of \( Y(n) \). Both \( H^*_\text{par}(Y(n), \mathbb{C}) \) and \( H^*_\text{cusp}(Y(n), \mathbb{C}) \) are \( W_G \)-modules \( \text{Hida88} \) \((7)\). We assume that \( h^+_F = 1 \). As mentioned after Proposition \((2.5)\) for each subset \( J \) of \( J_F \), the action of \((1_+, \iota)_{\iota \in J}, (-1_+, \iota)_{\iota \in J_F \setminus J} \in W_G \) on \( Y(n) \) is given by
\[
((x_1 + \sqrt{-1} y_1)_{\iota \in J}, (x_1 + \sqrt{-1} y_1)_{\iota \in J_F \setminus J}) \mapsto (\xi^* (x_1 + \sqrt{-1} y_1)_{\iota \in J}, (-\xi^* (x_1 + \sqrt{-1} y_1)_{\iota \in J_F \setminus J})
\]
for some \( \xi \in \mathfrak{p}_F^* \). Thus, in the case \( n \) is even, if a character \( \epsilon \) of \( W_G \) satisfies \( \not\exists \{ t \in J_F \mid \epsilon(t) = -1 \} \neq \emptyset, \) then \( H^*_\text{par}(Y(n), \mathbb{C})[\epsilon] = H^*_\text{cusp}(Y(n), \mathbb{C})[\epsilon] \). Here, for a \( W_G \)-module \( V \), \( \epsilon[V] \) denotes the \( \epsilon \)-isotypic part \( \{ v \in V \mid w \cdot v = \epsilon(w)v \text{ for all } w \in W_G \} \). Hence we obtain
\[
H^*_\text{par}(Y(n), \mathbb{C})[\epsilon] \simeq H^*_\text{cusp}(Y_1(n), \mathbb{C})[\epsilon] \simeq S_2(n, \mathbb{C})
\]
as Hecke modules (cf. [Hida94, §2, §3]). Thus the Hecke algebra \( \mathcal{H}_2(n, O) \) is isomorphic to the \( O \)-subalgebra of \( \text{End}_O \left( \tilde{H}^n_{\text{par}}(Y(n), O)[\epsilon] \right) \). We have the decomposition

\[
H^n(Y(n), \mathbb{C})[\epsilon] \cong \tilde{H}^n_{\text{par}}(Y(n), \mathbb{C})[\epsilon] \oplus \tilde{H}^n_{\text{Eis}}(Y(n), \mathbb{C})[\epsilon].
\]

By Proposition 4.1 we have a homomorphism \( \mathbb{H}_2(n, O) \to \text{End}_O \left( \tilde{H}^n(Y(n), O)[\epsilon] \right) \). For every ideal \( I \) of \( \mathbb{H}_2(n, O) \), let \( I[\epsilon] \) denote the image of \( I \) under this homomorphism.

## 5. Rationality and Integrality of Cohomology Classes

The purpose of this section is to prove the rationality (Proposition 5.4) and integrality (Corollary 5.6) of the cohomology class associated to the Hilbert Eisenstein series \( E \) attached to a pair of Hecke characters of \( F \) satisfying the following (Eis condition). We use the assumption \( h_F^+ = 1 \) to prove a vanishing result on the cohomology of \( D_{C, \infty}(n) \) (Proposition 5.3).

Let \( \Phi_p \) be the field introduced in Proposition 3.4. We fix a finite extension \( K \) of \( \Phi_p \). Let \( O \) be the ring of integers of \( K \), \( \varpi \) a uniformizer, and \( \kappa \) the residue field.

We assume that \( h_F^+ = 1 \). Let \( n \) be a non-zero ideal of \( \sigma_F \) such that \( n \) is prime to \( 6p \Delta_F \) and \( \mathfrak{d}_F[t] \). Let us fix narrow ray class characters \( \varphi \) and \( \psi \) of \( F \) satisfying \( \mathfrak{m}_{\varphi, \psi} = \mathfrak{m}_\varphi \mathfrak{m}_\psi = n \) and

(Eis condition) \( \varphi \) and \( \psi \) are \( O \)-valued and totally even (resp. totally odd), \( \varphi \) is non-trivial, and the algebraic Iwasawa \( \mu \)-invariants of

the splitting fields \( \mathbb{Q}^{\text{ker}(\varphi)} \) and \( \mathbb{Q}^{\text{ker}(\psi)} \) are equal to 0 (see Remark 0.2).

Let \( E \) denote the Hilbert Eisenstein series \( E_2(\varphi, \psi) \in M_2(n, \mathbb{C}) \) attached to \( \varphi \) and \( \psi \) as Proposition 1.2. Note that \( E \) satisfies (2.5) by Proposition 1.3. We define the character \( \epsilon_E \) of \( W_G \) by \( \epsilon_E = \text{sgn}^{1/F} \) (resp. \( \epsilon_E = 1 \)) if both \( \varphi \) and \( \psi \) are totally even (resp. totally odd).

Here we identify \( W_G = K_{\infty}/K_{\infty, +} \) with \( \{ \pm 1 \}^{1/F} \) by the determinant map. Put \( \chi = \varphi \psi \).

### Remark 5.1.

We note that \( [\omega_E]^F = 0 \) in \( H^n(Y(n), \mathbb{C}) \), where \( [\omega_E]^F \) stands for the projection of \( [\omega_E] \) to the \( \epsilon_E \)-part. Indeed, for a narrow ray class character \( \theta \) of \( F \) such that \( \mathfrak{m}_\theta = \mathfrak{m}_\varphi \mathfrak{m}_\psi = n \) and \( \theta = \epsilon_E \) on \( W_G \cong \mathbb{A}_F^{\times}/\mathbb{A}_F^{\times, +} \), under the same notation as (2.4) we have

\[
(5.1) \quad \sum_{b \in S} \eta(b)^{-1} \psi_{b, 1, c}([\omega_E]_{10}^c) = \tau(\eta^{-1}) \frac{1}{(2\pi)^n} D(1, E, \eta)
\]

\[
= \frac{(-1)^n}{2^n \Delta_F^{1/2}} \frac{\tau(\varphi \psi) \varphi \psi(m_\varphi) \theta(m_\psi)}{\tau(\psi) \psi(m_\psi) \theta(m_\varphi)} \cdot L(0, \theta^{-1} \psi)L(0, \theta \varphi^{-1}) \neq 0,
\]

where \( \eta \) denotes \( \theta \varphi^{-1} \psi^{-1} \). Here the first equality follows from Proposition 2.6 and Proposition 2.6; the second equality follows from Proposition 1.2 (1), the functional equation for Hecke \( L \)-functions (see, for example, [Miyake, Theorem 3.3.1]), and the fact that \( \eta \varphi = \theta \psi^{-1} \) is totally odd and \( [Miyake, (3.3.1)] \), and \( L(0, \theta^{-1} \psi)L(0, \theta \varphi^{-1}) \neq 0 \) follows from the fact that both \( \theta \psi^{-1} \) and \( \theta \varphi^{-1} \) are totally odd and the functional equation for Hecke \( L \)-functions (see, for example, [Da–Da–Po, Lemma 1.1]). Hence \( [\omega_E]^F \neq 0 \). Now our assertion follows from Proposition 4.1 and the \( q \)-expansion principle over \( \mathbb{C} \).
5.1. **Rationality of cohomology classes.** In this subsection, we prove the rationality of the cohomology classes of $E$ in $H^n(Y(n), ζ)$ and $H^n(Y(n) | BS, D_{C∞}(n); ζ)$.

**Proposition 5.2.** The cohomology class $[ω_E]$ is rational, that is, $[ω_E] ∈ H^n(Y(n), K)$.

**Proof.** Let $p_E$ denote the maximal ideal of $H_2(n, ζ) ⊗ K$ generated by $T(q) − C(q, E), S(q) − χ^{-1}(q)$ for all non-zero prime ideals $q$ of $ζ$ prime to $n$ and $U(q) − C(q, E)$ for all non-zero prime ideals $q$ of $ζ$ dividing $n$. By Proposition 5.1 $[ω_E] = [ω_E]_K$. Hence there is $c ∈ H^n(Y(n), K)_{p_E}[ζ]$ mapping to $[ω_E]$. We have $[ω_E] = c ∈ H^n(Y(n), ζ)_{p_E}[ζ]$. The isomorphism $[17]$ and the $q$-expansion principle over $ζ$ imply $H^n(Y(n), ζ)_{p_E}[ζ] = 0$. Hence $[ω_E] = c ∈ H^n(Y(n), K)$.

In order to prove the rationality of the relative cohomology class, we need to show a vanishing result on the cohomology of $D_{C∞}(n)$. We abbreviate $Γ_1(ζ | [t], n)$ to $Γ$ and $Γ_0(ζ | [t], n)$ to $Γ_0$. Let $q$ be a non-zero prime ideal of $ζ$ dividing $n$. Since $h^2_F = 1$, we can choose and fix a totally positive generator $q_4$ (resp. $c$) of $q$ (resp. $ζ | [t]$). For the proof, we need the following:

\[
(5.2) \quad Γ \begin{pmatrix} 1 & 0 \\ 0 & g_q \end{pmatrix} Γ = \prod_{b ∈ ζ | [t]^{-1}/ζ | [t]^{-1} q} Γ \begin{pmatrix} 1 & b \\ 0 & g_q \end{pmatrix};
\]

\[
(5.3) \quad γΓ \begin{pmatrix} 1 & 0 \\ 0 & g_q \end{pmatrix} Γγ^{-1} = Γγ \begin{pmatrix} 1 & 0 \\ 0 & g_q \end{pmatrix} γ^{-1} Γ = Γ \begin{pmatrix} 1 & 0 \\ 0 & g_q \end{pmatrix} Γ \text{ for } γ ∈ Γ_0;
\]

where $b$ runs over a complete set of representatives of $ζ | [t]^{-1}/ζ | [t]^{-1} q$. In order to show (5.2) and (5.3), we may assume $ζ | [t] = ζ$ because $\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix} Γ_1(ζ | [t], n) \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} = Γ_1(ζ, n)$ and $\begin{pmatrix} 1 & 0 \\ 0 & e^{-1} \end{pmatrix} Γ_0(ζ | [t], n) \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} = Γ_0(ζ, n).

First we show (5.2). By taking the inverse and multiplying $g_q$, it suffices to show that

\[
Γ \begin{pmatrix} g_q & 0 \\ 0 & 1 \end{pmatrix} Γ = \prod_{b ∈ ζ | q} \begin{pmatrix} g_q & b \\ 0 & 1 \end{pmatrix} Γ.
\]

For any $β = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ∈ Γ \begin{pmatrix} g_q & 0 \\ 0 & 1 \end{pmatrix} Γ$, we have $a, b, c, d ∈ ζ, c ≡ 0 (mod n), d ≡ 1 (mod n)$, and $det(β) = g_q u$ for some $u ∈ ζ^×$. Since $q$ divides $n$, we have $(c, d) = 1$. Hence there is $γ_1 = \begin{pmatrix} d & * \\ -c & * \end{pmatrix} ∈ Γ$ with $det(γ_1) = 1$ such that

\[
βγ_1 \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} det(β) & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g_q & b' \\ 0 & 1 \end{pmatrix}.
\]

We show (5.3). The first equality of (5.3) follows from the fact that $Γ$ is a normal subgroup of $Γ_0$ and the second equality of (5.3) follows from the same argument as in the proof of (5.2). Indeed, for $β = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = γ \begin{pmatrix} g_q & 0 \\ 0 & 1 \end{pmatrix} γ^{-1}$, we have $a, b, c, d ∈ ζ, c ≡ 0 (mod n), d ≡ 1 (mod n), det(β) = g_q$, and $q$ divides $n$.

We put $α_q = \begin{pmatrix} 1 & b \\ 0 & g_q \end{pmatrix}$. Note that, for $s ∈ F$, the condition $s ∈ C_∞$ is equivalent to the condition $εs = a/c$ for some $a, c ∈ ζ$ such that $(a, c) = 1$ and $c ≡ 0 (mod n)$. Hence, if $s ∈ C_∞$, then $α_q(s) ∈ C_∞$. Therefore $U(q)$ preserves the component $D_{C_∞}(n)$. Let $H_2(n, ζ)'$ be the
commutative $\mathcal{O}$-subalgebra of $\text{End}_{\mathcal{O}}(H^{n-1}(D_{C_\infty}(n), \mathcal{O})) \oplus \text{End}_{\mathcal{O}}(H^n(Y(n)_{\text{BS}}, D_{C_\infty}(n); \mathcal{O})) \oplus \text{End}_{\mathcal{O}}(H^n(Y(n), \mathcal{O})) \oplus \text{End}_{\mathcal{O}}(H^n(D_{C_\infty}(n), \mathcal{O}))$ generated by $U(q)$ for all non-zero prime ideals $q$ of $\mathfrak{o}_F$ dividing $n$, and $m'_\mathfrak{q}$ the maximal ideal of $\mathbb{H}_2(n, \mathcal{O})'$ generated by $\varpi$ and $U(q) - C(q, \mathcal{E})$ for all non-zero prime ideals $q$ of $\mathfrak{o}_F$ dividing $n$.

**Proposition 5.3.** Assume that $C(q, \mathcal{E}) \not\equiv N(q) \pmod{\varpi}$ for some prime ideal $q$ dividing $n$. Then $H^{n-1}(D_{C_\infty}(n), \mathcal{O})_{m'_\mathfrak{q}} = 0$.

*Proof.* By Step 1 in the proof of Proposition 5.1 for each $t \in C_\infty$ such that $t = \gamma(\infty)$ with $\gamma \in \Gamma_0$, a basis of $H^{n-1}(D_t, \mathbb{C})$ is given by $\omega_t := (\gamma^{-1})^* (\omega_{n-1})$. We claim that

\begin{equation}
\omega_t U(q) = N(q) \omega_t
\end{equation}

for any $t \in C_\infty$ and any prime ideal $q$ of $\mathfrak{o}_F$ dividing $n$.

For the moment, we admit the claim (5.4). We have

\begin{equation}
H^{n-1}(D_{C_\infty}(n), \mathcal{O})_{m'_\mathfrak{q}} \simeq \prod_{p \in \mathbb{H}_2(n, \mathcal{O})'} H^{n-1}(D_{C_\infty}(n), \mathcal{O}) \otimes_{\mathbb{H}_2(n, \mathcal{O})'} K(p),
\end{equation}

where $p$ runs over the set of maximal ideals of $\mathbb{H}_2(n, \mathcal{O})' \otimes K$ such that $p \cap \mathbb{H}_2(n, \mathcal{O})' \subset m'_\mathfrak{q}$, and $K(p)$ denotes the residue field of $p$. Let $\varphi_p$ denote the mod $p$ map $\mathbb{H}_2(n, \mathcal{O})' \otimes K \to K(p)$. The condition $\ker(\varphi_p) \cap \mathbb{H}_2(n, \mathcal{O})' \subset m'_\mathfrak{q}$ is equivalent to the condition $\varphi_p(U(q)) \equiv C(q, \mathcal{E}) \pmod{m_{K(p)}}$ for all non-zero prime ideals $q$ of $\mathfrak{o}_F$ dividing $n$. By (5.1), $\varphi_p(U(q)) = N(q)$. Now our assumption implies $H^{n-1}(D_{C_\infty}(n), \mathcal{O})_{m'_\mathfrak{q}} = 0$ as desired.

Thus it remains to prove the claim (5.4). In order to do it, under the canonical isomorphism $H^{n-1}(\partial(Y(n)_{\text{BS}}), \mathbb{C}) \simeq \{(c_s)_s \in \bigoplus_{s \in \mathbb{P}^1(F)} H^{n-1}(D_s, \mathbb{C}) \mid \gamma^*(\omega_{\gamma(s)}) = \omega_s \text{ for } \gamma \in \Gamma, s \in \mathbb{P}^1(F)\}$, we explicitly describe the action of $U(q)$ on the right-hand side.

We first treat the case $t = \infty$. By using the decomposition (5.2) and the definition of the Hecke operator acting on the boundary cohomology [Hida93 (3.1c)], we have

\begin{equation}
\left(\omega_\infty I | \Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_b \end{pmatrix} \Gamma \right)_s = \sum_{\alpha_b(s) \sim \Gamma} (\alpha_b)^*(\omega_{\alpha_b(s)}),
\end{equation}

where $b$ runs over a complete set of representative of $\mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1}q$ such that $\alpha_b(s)$ is $\Gamma$-equivalent to $s$. By (5.2) and (5.3), for $s \in C_\infty$, $\alpha_b(s)$ is $\Gamma$-equivalent to $s$. Indeed, for $s = \gamma_0(\infty)$ with $\gamma_0 \in \Gamma_0$, we have $\Gamma = \prod_{b} \Gamma_0^{-1} = \prod_{b} \Gamma_0^{-1} \Gamma_0$. Then $\gamma_0^{-1} \gamma_0 \Gamma_0 = \alpha_b$ for some $\gamma \in \Gamma$ and $b' \in \mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1}q$ and hence $\alpha_b(s) = \gamma^{-1}(s)$. Therefore, if $s$ is not $\Gamma$-equivalent to $\infty$, then (5.5) is 0 and

\begin{equation}
\left(\omega_\infty I | \Gamma \begin{pmatrix} 1 & 0 \\ 0 & g_b \end{pmatrix} \Gamma \right)_\infty = \sum_{b \in \mathfrak{d}_F^{-1}[t_1]^{-1}/\mathfrak{d}_F^{-1}[t_1]^{-1}q} (\alpha_b)^*(\omega_\infty) = N(q) \omega_\infty.
\end{equation}

Here the last equality follows from that $\omega_\infty$ is invariant under the action of the standard Borel subgroup $B_{C_\infty}$. We treat the general case $t \in C_\infty$. Let $\gamma \in \Gamma_0$ such that $t = \gamma(\infty)$. The canonical map $\gamma : D_{C_\infty}(n) \to D_{C_\infty}(n)$ induces $\gamma^* : H^{n-1}(D_{C_\infty}(n), \mathbb{C}) \to H^{n-1}(D_{C_\infty}(n), \mathbb{C})$. We have $\gamma^*(\omega_t) \in H^{n-1}(D_{C_\infty}, \mathbb{C})$. Hence $(\gamma^{-1})^*(\gamma^*(\omega_t))U(q)) = N(q) \omega_t$. Now the assertion follows from (5.3).
Proposition 5.4. Under the same notation and assumptions as Proposition 5.3, \([\omega_E]_{rel}\) is rational, that is, \([\omega_E]_{rel} \in H^n(Y(n))^{BS}, D_{C_\infty}(n): K)\).

Proof. By Proposition 5.2, there is \(c \in H^n(Y(n))^{BS}, D_{C_\infty}(n): K)_{m_E}^O\) mapping to \([\omega_E] \in H^n(Y(n), K)_{m_E}^O\). The difference \(c - [\omega_E]_{rel}\) is in the image of \(H^{n-1}(D_{C_\infty}(n), \mathbb{C})_{m_E}^O\) and Proposition 5.3 implies \([\omega_E]_{rel} = c\).

5.2. Denominator ideal. In this subsection, we recall the definition of the denominator ideal in the sense of T. Berger ([Be, §4.1]).

Let \(H^n(Y(n), \mathcal{O})\) denote the image of \(H^n(Y(n)) \to H^n(Y(n), K)\). For \(c \in H^n(Y(n), K)\), let \(\delta(c)\) denote the denominator ideal of \(c\), that is,

\[\delta(c) = \{a \in \mathcal{O} \mid ac \in H^n(Y(n), \mathcal{O})\}.\]

5.3. Congruence modules and integrality of cohomology classes. In this subsection, we determine the structure of the congruence module associated to \(E\) by using the denominator ideal of \([\omega_E]\). As an application, we prove the integrality of \([\omega_E]\). The proof is based on the method of T. Berger [Be, §4] and M. Emerton [Eme, Proposition 4, Theorem 5].

We abbreviate \(T_1(\mathfrak{d}_F[t_1], n)\) to \(\Gamma\). Let \(\mathfrak{p}_F\) be the prime ideal of \(\mathbb{H}_2(n, \mathcal{O})\) generated by \(T(q) - C(q, E), S(q) - \chi^{-1}(q)\) for all non-zero prime ideals \(q\) of \(\mathfrak{o}_F\) prime to \(n\) and \(U(q) - C(q, E)\) for all non-zero prime ideals \(q\) of \(\mathfrak{o}_F\) dividing \(n\). Let \(\mathcal{P}_E\) denote the image of \(\mathfrak{p}_F\) under the canonical surjection \(\mathbb{H}_2(n, \mathcal{O}) \to \mathbb{H}_2(n, \mathcal{O})\). The module \(\mathbb{H}_2(n, \mathcal{O})/\mathcal{P}_E\) is the congruence module associated to \(E\).

In order to determine the structure of \(\mathbb{H}_2(n, \mathcal{O})/\mathcal{P}_E\), we use an element \(A_{1,s} \in \mathbb{H}_2(n, \mathcal{O})\) for a cusp \(s \in C(\Gamma)\) defined as follows. The space \(M_2(n, \mathcal{O})\) of modular forms introduced in §4.5 can be identified with the space \(M_2(M, \mathcal{O})\) of geometric modular forms defined in §4.8 (see, for example, [Hida88, p.329–333] and Definition 4.5). Hence, if \(f = f_1 \in M_2(n, \mathcal{O})\), then the constant term of \(f_1\) at \(s\) belongs to \(\mathcal{O}\) by the \(q\)-expansion principle. Now, by using the duality theorem (Theorem 4.1), we can define \(A_{1,s} \in \mathbb{H}_2(n, \mathcal{O})\) as the element corresponding to an \(\mathcal{O}\)-linear map \(f \mapsto a_s(0, f_1)\) from \(M_2(n, \mathcal{O})\) to \(\mathcal{O}\), where \(a_s(0, f_1)\) denotes the constant term of \(f_1\) at \(s\).

Let \(s_0 \in C(\Gamma)\) such that \(v_p(a_{s_0}(0, E_1)) \leq v_p(a_s(0, E_1))\) for every \(s \in C(\Gamma)\), where \(v_p\) denotes the \(p\)-adic valuation. We put

\[C = a_{s_0}(0, E_1).\]

Let \(\mathbb{H}_2(n, \mathcal{O})\) be the commutative \(\mathcal{O}\)-subalgebra of \(\text{End}_{\mathcal{O}}(H^n_c(Y(n), \mathcal{O})) \oplus \text{End}_{\mathcal{O}}(H^n(Y(n), \mathcal{O}))\)

\[\oplus \text{End}_{\mathcal{O}}(H^n(\partial(Y(n))^{BS}, \mathcal{O})) \oplus \text{End}_{\mathcal{O}}(H^n(Y(n))^{BS}, \mathcal{O}))\]

generated by \(T(q), S(q)\) for all non-zero prime ideals \(q\) of \(\mathfrak{o}_F\) prime to \(n\) and \(U(q)\) for all non-zero prime ideals \(q\) of \(\mathfrak{o}_F\) dividing \(n\), and \(m\) the maximal ideal of \(\mathbb{H}_2(n, \mathcal{O})\) generated by \(\varpi\) and \(T(q) - C(q, E), S(q) - \chi^{-1}(q)\) for all non-zero prime ideals \(q\) of \(\mathfrak{o}_F\) prime to \(n\) and \(U(q) - C(q, E)\) for all non-zero prime ideals \(q\) of \(\mathfrak{o}_F\) dividing \(n\).

Theorem 5.5. Let \(p\) be a prime number \(> 3\) such that \(p\) is prime to \(n\) and \(\Delta_F\). We assume the following two conditions (a) and (b):

(a) \(H^n(\partial(Y(n))^{BS}, \mathcal{O})_m, H^{n+1}(Y(n), \mathcal{O})_m, H^n(D_{C_\infty}(n), \mathcal{O})_m\) are torsion-free, where \(m_\mathbb{E}\) is the maximal ideal of \(\mathbb{H}_2(n, \mathcal{O})\) defined before Proposition 5.3.

(b) \(C(q, E) - N(q) (\mod \varpi)\) for some prime ideal \(q\) dividing \(n\).

Then there are isomorphisms of \(\mathcal{O}\)-modules

\[\mathbb{H}_2(n, \mathcal{O})[\epsilon_\mathbb{E}] ((\mathcal{P}_E + \sum_{s \in C(\Gamma)} \mathcal{O}A_{1,s})[\epsilon_\mathbb{E}] \simeq \mathbb{H}_2(n, \mathcal{O})/\mathcal{P}_E \simeq \mathcal{O}/C.\]
Here the notion $[\epsilon]_\kappa$ is defined at the end of §1.4.

Proof. We prove the assertion by constructing the following surjective $\mathcal{O}$-linear morphisms (5.6), (5.7), (5.8), and (5.9) whose composition is the identity:

$$
\mathcal{O}/\mathcal{C} \xrightarrow{(5.6)} \mathcal{H}_2(n, \mathcal{O})[\epsilon]/(pE + \sum_{s \in C(\Gamma)} \mathcal{O}A_{1,s})[\epsilon] \xrightarrow{(5.7)} \mathcal{H}_2(n, \mathcal{O})/\mathcal{P}_\mathcal{E} \xrightarrow{(5.8)} \mathcal{O}/\delta_G \xrightarrow{(5.9)} \mathcal{O}/\mathcal{C}.
$$

First we construct the surjective morphisms (5.6) and (5.7). By the definition of $A_{1,s}$, we have $A_{1,s} = a_s(0, E_1)$ in $\mathcal{H}_2(n, \mathcal{O})/pE \simeq \mathcal{O}$. Hence we obtain a surjective $\mathcal{O}$-linear map by $1 \mapsto 1$:

$$
(5.6) \quad \mathcal{O}/\mathcal{C} \rightarrow \mathcal{H}_2(n, \mathcal{O})[\epsilon]/(pE + \sum_{s \in C(\Gamma)} \mathcal{O}A_{1,s})[\epsilon].
$$

The canonical surjection $\mathcal{H}_2(n, \mathcal{O}) \rightarrow \mathcal{H}_2(n, \mathcal{O})$ induces

$$
(5.7) \quad \mathcal{H}_2(n, \mathcal{O})[\epsilon]/(pE + \sum_{s \in C(\Gamma)} \mathcal{O}A_{1,s})[\epsilon] \rightarrow \mathcal{H}_2(n, \mathcal{O})/\mathcal{P}_\mathcal{E}.
$$

Put $G = E/C \in M_2(n, \mathcal{O})$. Consider the cohomology class $[\omega_G]^e \in H^n(Y(n), \mathbb{C})[\epsilon]_m$. By Remark 4.1 we have $[\omega_G]^e = [\omega_G] \neq 0$. By Proposition 5.2 $[\omega_G] \in H^n(Y(n), K)$. Let $\delta_G$ denote the denominator ideal $\delta([\omega_G]^e)$ of $[\omega_G]^e$ defined in (5.2). Next we construct the surjective morphism

$$
(5.8) \quad \mathcal{H}_2(n, \mathcal{O})/\mathcal{P}_\mathcal{E} \rightarrow \mathcal{O}/\delta_G.
$$

By Proposition 3.4 (1), $\text{res}([\omega_G]^e) \in H^n(\partial Y(n)^{\text{BS}}, \mathcal{O})[\epsilon]_m$. The image of $\text{res}([\omega_G]^e)$ under the connecting homomorphism $H^n(\partial Y(n)^{\text{BS}}, K)[\epsilon]_m \rightarrow H^{n+1}(Y(n), K)[\epsilon]_m$ is 0. Hence, by the assumption (a) on $H^{n+1}_c(Y(n), \mathcal{O})_m$, there is $c \in H^n(Y(n), \mathcal{O})[\epsilon]_m$ such that $\text{res}(c) = \text{res}([\omega_G]^e)$. We have $c - [\omega_G]^e \in H^n_{\text{par}}(Y(n), K)[\epsilon]_m$. Fix a generator $d$ of $\delta_G$. Put $e_0 = d(c - [\omega_G]^e) \in H^n(Y(n), \mathcal{O})[\epsilon]_m$. The assumption (a) on $H^n(\partial (Y(n)^{\text{BS}}), \mathcal{O})_m$ implies $e_0 \in H^n_{\text{par}}(Y(n), \mathcal{O})[\epsilon]_m$. We may assume $e_0 \neq 0$. Indeed, if $e_0 = 0$, then $c = [\omega_G]^e$ and hence $\delta_G = \mathcal{O}$. Let $e_0, \ldots, e_v$ be an $\mathcal{O}$-basis of $H^n_{\text{par}}(Y(n), \mathcal{O})[\epsilon]_m$. For $t \in \mathcal{H}_2(n, \mathcal{O})$, we write

$$
t(e_0) = \sum_{0 \leq i \leq v} \lambda_i(t)e_i
$$

with $\lambda_i(t) \in \mathcal{O}$. Thus the $\mathcal{O}$-linear surjective morphism defined by

$$
\mathcal{H}_2(n, \mathcal{O}) \rightarrow \mathcal{O}/\delta_G; t \mapsto \lambda_0(t)
$$

induces the required morphism (5.8).

Finally we construct the surjective morphism

$$
(5.9) \quad \mathcal{O}/\delta_G \rightarrow \mathcal{O}/\mathcal{C}.
$$

In order to do it, it suffices to show that $\delta_G \subset (C)$. We fix a generator $d$ of $\delta_G$. Then we have $d[\omega_G] \in \widetilde{H}^n(Y(n), \mathcal{O})$. Moreover, under the assumption (b), Proposition 5.4 implies $d[\omega_G]_{\text{rel}} \in H^n(Y(n)^{\text{BS}}, D_{C_{\text{rel}}}(n); K)$. We claim that $d[\omega_G]_{\text{rel}}$ is integral, that is,

$$
(5.10) \quad d[\omega_G]_{\text{rel}} \in \widetilde{H}^n(Y(n)^{\text{BS}}, D_{C_{\text{rel}}}(n); \mathcal{O}).
$$

For the moment, we admit the claim (5.10). Let $\eta_p$ be a non-trivial primitive narrow ray class character of $F$ corresponding to a finite order character of $\text{Gal}(F(\zeta_p^{\infty})/F)$ such that
\( \eta_p = \epsilon_p \) on \( W_G \cong \mathbb{A}_{F,\infty}^\times / \mathbb{A}_{F,\infty,+}^\times \). Put \( \eta = \eta_p \varphi^{-1} \psi^{-1} \). Note that \( n|\eta \)\( m_\eta \) implies the following:

\[
\mathcal{O}(\eta) \ni \sum_{b \in S} \eta(b)^{-1} \text{ev}_{b,1,\mathcal{O}}(d[\omega_G]_{rel}) = \frac{d}{C} \frac{2^n \Delta_{E}^{1/2}}{\tau(\psi \varphi \psi)(m_\eta) \eta_p(m_{\varphi \psi})} \cdot L(0, \eta_p^{-1} \psi)L(0, \eta_p \varphi^{-1}).
\]

Here the equality follows from \( C[\omega_G] = [\omega_E] \) and the same argument as in the proof of \( (5.1) \), and the integrality of the value follows from \( (5.10) \), Proposition \( 2.5 \) and Proposition \( 2.6 \).

Note that the second and third terms in the second line of \( (5.11) \) are prime to \( p \). Moreover, by the condition on the \( \mu \)-invariants in (Eis condition) with the help of the Iwasawa main conjecture for totally real number fields proved by A. Wiles \([\text{Wil}]\), the \( p \)-adic valuation of \( L(0, \eta_p^{-1} \psi) \) and \( L(0, \eta_p \varphi^{-1}) \) are smaller than that of \( \varpi \) for all but finitely many narrow ray class character \( \eta_p \) of \( F \) such that \( \eta_p = \epsilon_p \) on \( W_G \). Therefore we obtain \( C \mid d \) as required.

Thus it remains to prove the claim \( (5.10) \). We have an exact sequence

\[
H^{n-1}(D_{C_{\infty}}(n), \mathcal{O})_{m_{\varphi \psi}} \to H^n(Y(n)_{BS}, D_{C_{\infty}}(n), \mathcal{O}) \to H^n(D_{C_{\infty}}(n), \mathcal{O})_{m_{\varphi \psi}}.
\]

By Proposition \( 5.3 \), \( H^{n-1}(D_{C_{\infty}}(n), \mathcal{O})_{m_{\varphi \psi}} \) is torsion. By the assumption (a), \( H^n(D_{C_{\infty}}(n), \mathcal{O})_{m_{\varphi \psi}} \) is torsion-free. Therefore we obtain an exact sequence

\[
0 \to \hat{H}^n(Y(n)_{BS}, D_{C_{\infty}}(n), \mathcal{O}) \to \hat{H}^n(Y(n), \mathcal{O})_{m_{\varphi \psi}} \to H^n(D_{C_{\infty}}(n), \mathcal{O})_{m_{\varphi \psi}}.
\]

Now the claim \( (5.10) \) follows from this exact sequence. \( \square \)

By the proof of Theorem \( 5.5 \) we get \( \delta_G = (C) \) and hence we obtain the following:

**Corollary 5.6.** Under the same assumptions as Theorem \( 5.5 \) we have

\[
\omega_E \in \hat{H}^n(Y(n), \mathcal{O}) \setminus \varpi \hat{H}^n(Y(n), \mathcal{O}),
\]

\[
|\omega_E|_{rel} \in \hat{H}^n(Y(n)_{BS}, D_{C_{\infty}}(n), \mathcal{O}) \setminus \varpi \hat{H}^n(Y(n)_{BS}, D_{C_{\infty}}(n), \mathcal{O}).
\]

### 5.4. Real quadratic field case

In this subsection, we give an example of a congruence between a Hilbert cusp form and a Hilbert Eisenstein series.

We use the same notation as in the proof of Theorem \( 5.5 \). We abbreviate \( \Gamma_1(\mathfrak{n}_F[t_1], \mathfrak{n}) \) to \( \Gamma \) and \( \Gamma \cap SL_2(F) \) to \( \Gamma_1 \). Hereafter, in this subsection, we assume that \( F \) is a real quadratic field with \( h^+_1 = 1 \). First we show the following lemma.

**Lemma 5.7.** Assume the following four conditions (1), (2), (3), and (4):

1. \( H^2_\text{par}(Y(n), \mathcal{O}) \) is torsion-free;
2. \( H^2(\varphi (Y(n))_{BS}, \mathcal{O}) \) is torsion-free;
3. \( C(\mathfrak{q}, \mathcal{E}) \neq N(\mathfrak{q}) \pmod{\varpi} \) for some prime ideal \( \mathfrak{q} \) dividing \( \mathfrak{n} \);
4. the ideal \( (C) \neq 0, \mathcal{O} \).

Then there exist a finite extension \( K \) of \( K \) with the ring of integer \( \mathcal{O} \hookrightarrow \mathcal{O}' \) and a uniformizer \( \varpi' \) such that \( (\varpi') \cap \mathcal{O} = (\varpi) \) and a Hecke eigenform \( f \in \mathcal{S}_2(\mathfrak{n}, \mathcal{O}') \) for all \( T(\mathfrak{q}) \) and \( U(\mathfrak{q}) \) with character \( \chi \) such that \( f \equiv \mathcal{E} \pmod{\varpi'} \).

**Proof.** As discussed in the proof of \( (5.5) \) of Theorem \( 5.5 \) the assumption (4) implies \( e_0 \neq 0 \in H^2_{\text{par}}(Y(n), \mathcal{O})_{\mathcal{E}} \). Hence \( e_0 \) is cohomologous to \( -[\omega_E] \pmod{\varpi} \) and the Hecke eigenvalues of \( e_0 \) are the same as those of \( -[\omega_E] \pmod{\varpi} \) for all \( t \in \mathcal{H}_2(\mathcal{O}) \). Now the Deligne-Serre lifting lemma \( ([\text{Del–Sc}] \text{ Lemma 6.11}) \) in the case \( R = \mathcal{O}, M = H^2_{\text{par}}(Y(n), \mathcal{O})_{\mathcal{E}}, \) and \( \mathfrak{T} = \mathcal{H}_2(\mathcal{O}) \) says that there exist a finite extension \( K' \) of \( K \) with the ring of integer \( \mathcal{O} \hookrightarrow \mathcal{O}' \) and...
a uniformizer \( \varpi' \) such that \( (\varpi') \cap \mathcal{O} = (\varpi) \) and a non-zero eigenvector \( e \in \widetilde{H}^2_{\text{par}}(Y(n), \mathcal{O})[\epsilon_{f}] \otimes \mathcal{O}' \) for all \( t \in \mathcal{H}_2(n, \mathcal{O}) \) with eigenvalues \( \lambda(t) \) such that \( \lambda(V(q)) \equiv C(q, \mathbf{E}) \pmod{\varpi'} \) for all non-zero prime ideals \( q \) of \( \mathcal{O} \) prime to \( n \) (resp. dividing \( n \)) and \( V(q) = T(q) \) (resp. \( U(q) \)).

By the isomorphism \( (1.7) \), we obtain a Hecke eigenform \( f \in S_2(n, \mathbb{C}) \) for all \( T(q) \) and \( U(q) \) such that \( e = [\omega_f] \). By using the relation between Hecke eigenvalues and Fourier coefficients, we may assume that \( f \in S_2(n, \mathcal{O}') \) with character \( \chi \). Therefore we obtain the congruence \( f \equiv E \pmod{\varpi'} \).

In order to give an example of a congruence between a Hilbert cusp form and a Hilbert Eisenstein series, we prove (1) and (2) of Lemma 5.7 in certain cases (Proposition 5.8 and 5.9) and give a Hilbert Eisenstein series satisfying (3) and (4) of Lemma 5.7 based on a numerical table in \([\text{Oka}] \) (Example 5.10).

**Proposition 5.8.** Assume that \( n \) is prime to \( 6 \Delta_F \). If \( p \) is prime to \( 6n \) and \( \mathbb{Z}(\mathcal{O}_{F, \ell}^\times / \mathcal{O}_{F, \ell}^{\times 2}) \), then the assumption (1) of Lemma 5.7 is satisfied.

**Proof.** The Poincaré–Lefschetz duality theorem says that \( H^2(Y(n), \mathcal{O}) \cong H_1(Y(n), \mathcal{O}) \). Hence it suffices to show that the maximal abelian quotient \( \Gamma_{\text{ab}} \) of \( \Gamma \) is \( p \)-torsion-free. Since \( n \) is prime to 2, we have \( \Gamma_{\text{ab}} = \Gamma^1 \) and \( \Gamma^1/\Gamma^1 \cong \mathcal{O}_{F, \ell}^\times / \mathcal{O}_{F, \ell}^{\times 2} \). Thus, by our assumption, it suffices to show that \( (\Gamma^1)_{\text{ab}} \) is \( p \)-torsion-free. By taking conjugation, we may assume \( \Gamma^1 = \Gamma_1(\mathcal{O}, n) \cap SL_2(\mathcal{O}) \). Then \( (\Gamma^1)_{\text{ab}} \) is torsion (\([\text{Sc}] \) Theorem 3)) and there is a non-zero ideal \( m \) of \( \mathcal{O} \) such that the principal congruence subgroup \( \Gamma(m) \) satisfies \( \Gamma(m) \subset [\Gamma^1 : \Gamma^1] \subset \Gamma^1 \) (\([\text{Sel}] \) Corollary 3 of Theorem 2)). We have \( (\Gamma^1)_{\text{ab}} \cong (\Gamma^1/\Gamma(m))_{\text{ab}} \). We estimate the order of the right-hand side. Put \( H = \Gamma^1/\Gamma(m) \). We have decompositions \( SL_2(\mathcal{O})/\Gamma(m) = \prod_i SL_2(\mathcal{O}/q_i) \) and \( H = \prod_i H_{q_i} \). For each \( i \), we define \( \tilde{H}_{q_i} \) by the cartesian diagram

\[
\begin{array}{ccc}
H_{q_i} & \longrightarrow & SL_2(\mathcal{O}/q_i) \\
\downarrow & & \downarrow \\
\tilde{H}_{q_i} & \longrightarrow & SL_2(\mathcal{O}_{F, q_i})
\end{array}
\]

We fix a prime ideal \( q = q_i \) of \( \mathcal{O} \) and a positive integer \( r = r_i \). Let \( l \) denote the prime number such that \( (l) = q \cap \mathbb{Z} \). The assertion follows from the following:

**Claim** (a) \( \tilde{H}_{q}^{\text{ab}} = 1 \) in the case \( \tilde{H}_q = SL_2(\mathcal{O}_{F, q}) \) and \( (q, 6) = 1 \);

(b) \( \tilde{H}_{q}^{\text{ab}} \) is an \( l \)-group in the case \( \tilde{H}_q = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{q^r} \right\} \).

The assertion (a) is obtained by \([\text{Fe–Si}] \) Proposition 2.6. The assertion (b) follows from the following facts (i), (ii), and (iii): (i) \( \tilde{H}_q \) is generated by all elementary unipotents in \( \tilde{H}_q \); (ii) \( \tilde{\Gamma}(q^{4r}) \subset EL_2(q^{2r}) \); (iii) \( EL_2(q^{2r}) \subset [\tilde{H}_q : \tilde{H}_q] \). Here \( EL_2(\mathcal{O}) \) denotes the subgroup of \( SL_2(\mathcal{O}) \) generated by all elementary unipotents, and for a non-negative integer \( m \), \( \tilde{\Gamma}(q^m) = \ker(SL_2(\mathcal{O}) \to SL_2(\mathcal{O}/q^m)) \) and \( EL_2(q^m) = EL_2(\mathcal{O}) \cap \tilde{\Gamma}(q^m) \). Indeed, (i) implies that the image of \( \tilde{H}_q/(\tilde{H}_q \cap \tilde{\Gamma}(q)) \) in \( SL_2(\mathcal{O}/q) \) is generated by \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and hence it is an \( l \)-group. Since \( (\tilde{H}_q \cap \tilde{\Gamma}(q^m))/(\tilde{H}_q \cap \tilde{\Gamma}(q^{m+1})) \) is an \( l \)-group for a non-negative integer \( m \), (ii) and (iii)
implies \( \hat{H}_q/\hat{\Gamma}(q^r) \) is an \( l \)-group. Hence \( \hat{H}_q^{ab} \) is an \( l \)-group. The facts (i) and (ii) follow from

\[
\begin{pmatrix}
\alpha & b \\
c & d
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
-a^{-1}c & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-a^{-1} & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
a^{-1} & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-a^{-1}(1-a^{-2}) & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
a^{-1}b & 1
\end{pmatrix}.
\]

The fact (iii) follows from the same argument as in the proof of [Gha, Proposition 2.6] in the case \( n = 2 \) and general \( \mathfrak{n} \mathfrak{F}_q/q \) and the commutator relation [Fe–Si (1)].

Let \( \varepsilon_0 \) denote the fundamental unit of \( F \). We put \( \varepsilon_+ = \varepsilon_0 \) (resp. \( \varepsilon_0^\circ \)) if \( N(\varepsilon_0) = 1 \) (resp. \( N(\varepsilon_0) = -1 \).

**Proposition 5.9.** If \( p \nmid N(\varepsilon_+ - 1) \) and \( n \) is a prime ideal \( q \) of \( \mathfrak{o}_F \) such that \( q \) is prime to \( 6\Delta_F \), then the assumption (2) of Lemma 5.7 is satisfied.

**Proof.** We may assume \( \Gamma = \Gamma_1(\mathfrak{o}_F, n) \) by taking conjugation. Fix a cusp \( s \in \mathcal{C}(\Gamma) \). As mentioned in [Gha, p.260], \( H^2(\Gamma_s, O) \) is torsion-free if and only if \( H^1(\hat{\Gamma}_s, K/O) \) is divisible.

Main tools for the proof of the divisibility are the description \( H^1(\hat{\Gamma}_s, K/O) = H^1(\alpha^{-1}\hat{\Gamma} \cap B_\infty, K/O) \) and the Hochschild–Serre spectral sequence

\[
E^{i,j}_2 = H^i(\alpha^{-1}\hat{\Gamma} \cap B_\infty, H^j(\alpha^{-1}\hat{\Gamma} \cap U_\infty, K/O)) \Rightarrow H^{i+j}(\alpha^{-1}\hat{\Gamma} \cap B_\infty, K/O),
\]

where \( \alpha \in \text{SL}_2(\mathfrak{o}_F) \) such that \( \alpha(\infty) = s \), \( B_\infty \) denotes the standard Borel subgroup of upper triangular matrices, \( U_\infty \) denotes the unipotent radical of \( B_\infty \), and the bar means image in \( \text{GL}_2(F)/(\text{GL}_2(F) \cap F^\times) \). Let \( T_\infty \) denote the standard torus of \( B_\infty \). By the same argument as in [Gha, §3.4.2], our assertion follows from the following (5.12), (5.13), and (5.14):

\[
\alpha^{-1}\hat{\Gamma} \cap U_\infty \simeq q^{1-e} \quad \text{if } (y,q) = q^e;
\]

\[
\alpha^{-1}\hat{\Gamma} \cap T_\infty \simeq \mathfrak{o}^\times_{F,+};
\]

\[
1 \to \alpha^{-1}\hat{\Gamma} \cap U_\infty \to \alpha^{-1}\hat{\Gamma} \cap B_\infty \to \alpha^{-1}\hat{\Gamma} \cap T_\infty \to 1.
\]

Fix \( \alpha = \begin{pmatrix}
x & \beta \\ y & \delta
\end{pmatrix} \in \text{SL}_2(\mathfrak{o}_F) \) such that \( \alpha(\infty) = s \). We may assume that if \( (y,q) = 1 \), then \( (\delta,q) = q \). Indeed, since \( (xq,y) = 1 \), there is \( \begin{pmatrix}
x & \beta \\ y & \delta
\end{pmatrix} \in \text{SL}_2(\mathfrak{o}_F) \) with \( (\delta,q) = q \).

First we prove (5.12). Suppose that \( \begin{pmatrix}
1 & b \\ 0 & 1
\end{pmatrix} \in \alpha^{-1}\hat{\Gamma} \cap U_\infty \). The direct calculation shows that the condition \( \alpha \begin{pmatrix}
1 & b \\ 0 & 1
\end{pmatrix} \alpha^{-1} \in \Gamma \) is equivalent to the condition \( bx^2 \in \mathfrak{o}_F \), \( by^2 \in \mathfrak{q} \), and \( bxy \in \mathfrak{q} \). Since \( (x,y) = 1 \), we have \( b \in \mathfrak{o}_F \). If \( (y,q) = q^e \), then \( b \in q^{1-e} \) as desired.

Next we prove (5.13). Suppose that \( \begin{pmatrix}
a & 0 \\ 0 & d
\end{pmatrix} \in \alpha^{-1}\hat{\Gamma} \cap T_\infty \). As in the proof of (5.12), the direct calculation shows that if \( (y,q) = 1 \) (resp. \( (y,q) = q \)), then \( a \equiv 1 \pmod{q} \) (resp. \( d \equiv 1 \pmod{q} \)) and hence \( \begin{pmatrix}
a & 0 \\ 0 & d
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\ 0 & a^{-1}d
\end{pmatrix} \begin{pmatrix}
a & 0 \\ 0 & d
\end{pmatrix} = \begin{pmatrix}
ad^{-1} & 0 \\ 0 & 1
\end{pmatrix} \).

Finally we prove (5.14). For \( \begin{pmatrix}
a & b \\ 0 & d
\end{pmatrix} \in \alpha^{-1}\hat{\Gamma} \cap B_\infty \), it suffices to show that \( \begin{pmatrix}
1 & -a^{-1}b \\ 0 & 1
\end{pmatrix} \in \alpha^{-1}\hat{\Gamma} \cap U_\infty \). As in the proof of (5.12), it follows from the condition \( \alpha \begin{pmatrix}
a & b \\ 0 & d
\end{pmatrix} \alpha^{-1} \in \Gamma \). \( \square \)
Example 5.10. We give an example satisfying the assumptions of Lemma 5.7 in the case $F = \mathbb{Q}(\sqrt{2})$ with $h_F^+ = 1$, $\Delta_F = 8$, and $\varepsilon_0 = 1 + \sqrt{2}$. By [Oka] §4, p.1137, for the non-trivial character $\chi$ of $\text{Gal}(F(\sqrt{2})/F)$ whose conductor is a prime ideal (5) of $\mathfrak{O}_F$, we have

$$L(-1, \chi) = \frac{28}{9}.$$

A pair of characters $\varphi = \chi^{-1}$ and the trivial character $\psi = 1$ satisfies (Eis condition). One can see that $p = 7$ with $(p, 6\Delta_F) = 1$ and the Eisenstein series $E_2(\varphi, \psi)$ of level $\Gamma_1(\mathfrak{O}_F, (5))$ satisfy all the assumptions of Lemma 5.7.

6. CONGRUENCES BETWEEN $L$-VALUES

The purpose of this section is to prove the main theorem (Theorem 6.1 = Theorem 6.1) of this paper. We use the assumption $h_F^+ = 1$ in the proof of the isomorphism of Theorem 6.1 between a relative cohomology and the corresponding partial parabolic cohomology. We keep the notation in §5.

6.1. Canonical periods. Let $f \in S_2(n, \mathfrak{O})$ be a normalized Hecke eigenform for all $T(q)$ and $U(q)$ with character $\chi$. Let $\epsilon$ denote $\epsilon_E$ defined at the beginning of §5.

We define the canonical period $\Omega_f^\epsilon$ of $f$. Let $p_\mathfrak{m}$ denote the prime ideal of $\mathcal{H}_3(n, \mathfrak{O})$ generated by $T(q) - C(q, f)$ and $S(q) - \chi^{-1}(q)$ for all non-zero prime ideals $\mathfrak{q}$ of $\mathfrak{O}_F$ prime to $n$ and $U(q) - C(q, f)$ for all non-zero prime ideals $\mathfrak{q}$ of $\mathfrak{O}_F$ dividing $n$. The isomorphism (4.7) and the $q$-expansion principle over $\mathbb{C}$ imply that $\dim_C(H_{\text{par}}^n(Y(n), \mathcal{O})[p_\mathfrak{m}, \epsilon]) = 1$ and $\text{rank}_\mathcal{O} \left( H_{\text{par}}^n(Y(n), \mathcal{O})[p_\mathfrak{m}, \epsilon] \right) = 1$. Choose a generator $[\delta_\mathfrak{p}]^\epsilon$ of $H_{\text{par}}^n(Y(n), \mathcal{O})[p_\mathfrak{m}, \epsilon]$. Let $[\omega_f]^\epsilon$ denote the projection of $[\omega_f]$ to the $\epsilon$-part. We define the canonical period $\Omega_f^\epsilon \in \mathbb{C}^\times$ of $f$ by

$$[\omega_f]^\epsilon = \Omega_f^\epsilon [\delta_\mathfrak{p}]^\epsilon.$$

6.2. Congruences between special values. For modular forms $f, g \in M_2(n, \mathfrak{O})$, we define the congruence $f \equiv g \pmod{\mathfrak{m}}$ by $C(m, f) \equiv C(m, g) \pmod{\mathfrak{m}}$ for all non-zero ideals $\mathfrak{m}$ of $\mathfrak{O}_F$.

Theorem 6.1. Let $p$ be a prime number such that $p \geq n + 2$ and $p$ is prime to $n$ and $6\Delta_F$. Assume that $h_F^+ = 1$. Let $\varphi$ and $\psi$ be primitive narrow ray class characters of $F$ satisfying (Eis condition) at the beginning of §5 and $\epsilon$ the character $\epsilon_E$ of the Weyl group $W_2$ defined after (Eis condition). Put $\chi = \varphi \psi$. Let $f \in S_2(n, \mathfrak{O})$ be a normalized Hecke eigenform for all $T(q)$ and $U(q)$ with character $\chi$. We assume the following conditions (a), (b), (c):

(a) $f \equiv E \pmod{\mathfrak{m}}$;
(b) $H^n(\vartheta(Y(n)_{\text{BS}}), \mathcal{O}), H^{n+1}(Y(n), \mathcal{O})_m$, and $H^n(D_{\mathfrak{m}}(n), \mathcal{O})_m$ are torsion-free, where $m$ (resp. $m_E$) is the maximal ideal of $\mathcal{H}_2(n, \mathcal{O})$ (resp. $\mathbb{H}_2(n, \mathcal{O})^E$) defined before Theorem 5.3 (resp. Proposition 5.3);
(c) $C(q, E) \not\equiv N(q) \pmod{\mathfrak{m}}$ for some prime ideal $\mathfrak{q}$ dividing $n$.

Then there exists $u \in \mathfrak{O}^\times$ such that, for every narrow ray class character $\eta$ of $F$, whose conductor is denoted by $\mathfrak{n}_\eta$, such that $\mathfrak{n}_\eta \mathfrak{m}$ and $\eta = \epsilon$ on $W_2 \cong \mathbb{A}_F^\times/\mathbb{A}_F^\times+$, the both values $\tau(\eta^{-1}) D(1, f, \eta)/(2\pi \sqrt{-1})^n \Omega_f^\epsilon$ and $\tau(\eta^{-1}) D(1, E, \eta)/(2\pi \sqrt{-1})^n$ belong to $\mathcal{O}(\eta)$ and the following congruence holds:

$$\tau(\eta^{-1}) D(1, f, \eta)/(2\pi \sqrt{-1})^n \Omega_f^\epsilon = u \tau(\eta^{-1}) D(1, E, \eta)/(2\pi \sqrt{-1})^n \in \mathcal{O}(\eta)/\mathfrak{m}.$$
Here $\tau(\eta^{-1})$ denotes the Gauss sum attached to $\eta^{-1}$ (cf. (1.13)), $D(1, \ast, \eta)$ is the Dirichlet series recalled in (1.4). $O(\eta)$ denotes the ring of integers of the field $K(\eta)$ generated by $\text{im}(\eta)$ over $K$.

**Remark 6.2.** In general, the value $\tau(\eta^{-1})D(1, E, \eta)/(2\pi\sqrt{-1})^n$ is non-zero in $O(\eta)/\varpi$ by the proof of Theorem 5.5 (see after (5.11)).

**Proof.** For the moment, we admit Theorem 7.1 which shall be proved in §6.6. We abbreviate $Y(n)$ to $Y$ and $D_{C_\infty}(n)$ to $D_{C_\infty}$. For $A = O$ or $K$, we define the partial parabolic cohomology $H^n_{\text{par}}(Y, D_{C_\infty}; A)$ to be the image of

$$H^n_{\text{par}}(Y, D_{C_\infty}; A) = \text{im}\left( H^n_{\text{par}}(Y, D_{C_\infty}; A) \to H^n(Y, D_{C_\infty}; A) \right)$$

and put

$$\tilde{H}^n_{\text{par}}(Y, D_{C_\infty}; O) = \text{im}\left( H^n_{\text{par}}(Y, D_{C_\infty}; O) \to H^n(Y, D_{C_\infty}; O) \right)$$

$$\tilde{H}^n_{\text{par}}(Y, D_{C_\infty}; O) = \text{im}\left( H^n_{\text{par}}(Y, D_{C_\infty}; O) \to H^n_{\text{par}}(Y, D_{C_\infty}; K) \right).$$

By Proposition 5.5, $H^n_{\text{par}}(Y, D_{C_\infty}; K)_{m'_E} \to H^n_{\text{par}}(Y, D_{C_\infty}; K)_{m'_E}$ is an isomorphism and induces an isomorphism

$$\tilde{H}^n_{\text{par}}(Y, D_{C_\infty}; O)_{m'_E} \simeq \tilde{H}^n_{\text{par}}(Y, D_{C_\infty}; O)_{m'_E}.$$ 

Hence, by Corollary 5.6 and the definition of $[\delta^r_{\text{rel}}]$ in (6.1) $[\omega_{E_{\text{rel}}}^r]$ and $[\delta^r_{\text{rel}}] = [\omega_{\text{rel}}^r]/\Omega_f$ belong to $\tilde{H}^n_{\text{par}}(Y, D_{C_\infty}; O)$. Furthermore Theorem 7.1 and the isomorphism (6.1) imply

$$[\delta^r_{\text{rel}}] = u[\omega_{E}^r_{\text{rel}}]$$

for some $u \in O^\times$. Now our assertion follows from Proposition 2.5 and Proposition 2.6. 

7. Congruences between cohomology classes

The purpose of this section is to prove Theorem 7.1 that is, a congruence between a Hilbert eigenform and a Hilbert Eisenstein series gives rise to corresponding congruence between the associated cohomology classes under certain assumptions.

In this section, we assume that $2 \leq n \leq p - 2$ and $K$ is a finite extension of the composite field of $\iota_p(F')$ and $\Phi_p$. Here $\iota_p: \overline{Q}/\mathfrak{p} \to \overline{Q}_p$ is the fixed embedding and $F'$ (resp. $\Phi_p$) is the field defined in §1.3 (resp. Proposition 4.1). Let $O$ be the ring of integers of $K$, $\varpi$ a uniformizer of $O$, and $\kappa$ the residue field of $O$.

**7.1. Comparison theorem for torsion cohomology.** In this subsection, we briefly review the fully faithful functor from the category of finitely generated filtered $\varphi$-modules to the category of representations of $\mathcal{G}_{\mathcal{O}_p} = \text{Gal}((\overline{Q}_p)/Q_p)$ on $O$-modules of finite length, and state the comparison theorem between the parabolic étale cohomology and the parabolic log-cristalline cohomology for Hilbert modular varieties, which we shall use in the following subsections.

For a non-negative integer $r$, let $\mathbf{MF}^r_\mathcal{O}$ denote the category whose objects are the following triples $(M, (\text{Fil}^i M)_{i \in \mathbb{Z}}, (\varphi_M^i)_{i \in \mathbb{Z}})$:

1. $(M, (\text{Fil}^i M)_{i \in \mathbb{Z}}, (\varphi_M^i)_{i \in \mathbb{Z}})$ is a finitely generated $O$-module;
2. $(\text{Fil}^i M)_{i \in \mathbb{Z}}$ is a decreasing filtration on $M$ by $O$-submodules such that $\text{Fil}^0 M = M$ and $\text{Fil}^{r+1} M = 0$;
3. $\varphi_M^r: \text{Fil}^r M \to M$ is an $O$-linear homomorphism such that $\varphi_M^r |_{\text{Fil}^{r+1} M} = p \varphi_M^{r+1}$ and $\sum_{i=0} \varphi_M^i (\text{Fil}^i M) = M$. 


A morphism in $\text{MF}_G^r$ is a homomorphism of filtered $\mathcal{O}$-modules compatible with $\phi^*$. It is known that any morphism $\eta: M \to M'$ in $\text{MF}_G^r$ is strict with respect to the filtrations, that is, $\eta(\text{Fil}_i^r M) = \text{Fil}_i^r M' \cap \eta(M)$ for each $i \in \mathbb{Z}$ ([Fal, Theorem 5.3]). This implies that $\text{MF}_G^r$ is an abelian category as follows. Let $\eta: M \to M'$ be a morphism in $\text{MF}_G^r$, and let $\eta$ denote $\eta$ regarded as a homomorphism of underlying $\mathcal{O}$-modules. Then the $\mathcal{O}$-module $N := \ker(\eta)$ with $\phi^*_N$ defined by $\text{Fil}_i^r N = N \cap \text{Fil}_i^r M$ and $\phi^*_N = \phi^*_M|_N$, respectively, belongs to $\text{MF}_G^r$ and gives the kernel of $\eta$ in $\text{MF}_G^r$. Let $N'$ denote $\text{coker}(\eta)$. We define a filtration $\text{Fil}_i^r N'$ and an $\mathcal{O}$-linear homomorphism $\phi^*_N$ by $\text{Fil}_i^r N' = \text{Fil}_i^r M'/\eta(\text{Fil}_i^r M)$ and the homomorphism induced by $\phi^*_M$ and $\phi^*_M'$, respectively. Note that $\text{Fil}_i^r N' \to N'$ is injective because $\eta$ is strict, and hence $\text{Fil}_i^r N'$ may be regarded as an $\mathcal{O}$-submodule of $N'$. The triple $(N', (\text{Fil}_i^r N')_{i \in \mathbb{Z}}, (\phi^*_N)_{i \in \mathbb{Z}})$ belongs to $\text{MF}_G^r$ and gives the cokernel of $\eta$ in $\text{MF}_G^r$. The strictness of $\eta$ further shows that we have $\text{Fil}_i(\text{im}(\eta)) = \eta(M) \cap \text{Fil}_i M' = \eta(\text{Fil}_i^r M) \simeq \text{Fil}_i^r(\text{im}(\eta))$ and hence $\text{im}(\eta) = \text{coker}(\eta)$ in $\text{MF}_G^r$.

Let $\text{MF}_G^r$ denote the full subcategory of $\text{MF}_G^r$ consisting of objects $M$ satisfying $\omega M = 0$. Let $\text{Rep}_\mathcal{O}(G_{\mathbb{Q}_p})$ denote the category of representations of $G_{\mathbb{Q}_p}$ on $\mathcal{O}$-modules of finite length. For an integer $r$ such that $0 \leq r \leq p - 2$, there exists a fully faithful functor

$$T_{\text{cris}}: \text{MF}_G^r \to \text{Rep}_\mathcal{O}(G_{\mathbb{Q}_p})$$

given by J-M. Fontaine and G. Laffaille ([Fal, Theorem 5.3] [see also [Br, Theorem 3.2.4.6] = [Tsu, Theorem 5.1] and [Br, Theorem 3.2.4.7] for an alternative proof with an extension to the log-smooth reduction case, but without compact support], for $(X_{\text{tor}}, X) = (M^{1,\text{tor}}, M)$ or $(M^{1,\text{tor}}, M)$ defined in [L, 7] there are canonical $G_{\mathbb{Q}_p}$-equivariant $\mathcal{O}$-linear isomorphisms

$$H^n_{\text{et}}(X_{\mathbb{Q}_p}, \mathcal{O}) \simeq T_{\text{cris}}\left( H^n_{\text{log-cris},c}(X_{\mathbb{Q}_p}^\text{tor}) \otimes_{\mathbb{Z}_p} \mathcal{O} \right),$$

$$H^n_{\text{et},c}(X_{\mathbb{Q}_p}, \mathcal{O}) \simeq T_{\text{cris}}\left( H^n_{\text{log-cris},c}(X_{\mathbb{Q}_p}^\text{tor}) \otimes_{\mathbb{Z}_p} \mathcal{O} \right).$$

For $? = \phi$ or $c$ and $A = \mathcal{O}$ or $K$, we simply write $H^n_{\text{log-cris},?}(X_{\mathbb{Q}_p}^\text{tor}) \otimes_{\mathbb{Z}_p} A$ for $H^n_{\text{log-cris},?}(X_{\mathbb{Q}_p}^\text{tor})A$. For $A = \mathcal{O}$ or $K$, we define $H^n_{\text{et},\text{par}}(X_{\mathbb{Q}_p}, A)$ and $H^n_{\text{log-cris,par}}(X_{\mathbb{Q}_p}^\text{tor})A$ by

$$H^n_{\text{et,par}}(X_{\mathbb{Q}_p}, A) = \text{im} \left( H^n_{\text{et},c}(X_{\mathbb{Q}_p}, A) \to H^n_{\text{et}}(X_{\mathbb{Q}_p}, A) \right),$$

$$H^n_{\text{log-cris,par}}(X_{\mathbb{Q}_p}^\text{tor})A = \text{im} \left( H^n_{\text{log-cris},c}(X_{\mathbb{Q}_p}^\text{tor})A \to H^n_{\text{log-cris}}(X_{\mathbb{Q}_p}^\text{tor})A \right).$$

We obtain the following $G_{\mathbb{Q}_p}$-equivariant $\mathcal{O}$-linear isomorphisms from (7.1) and (7.2):

$$H^n_{\text{et,par}}(X_{\mathbb{Q}_p}, \mathcal{O}) \simeq T_{\text{cris}}\left( H^n_{\text{log-cris,par}}(X_{\mathbb{Q}_p}^\text{tor}) \mathcal{O} \right).$$

For $? = \phi$ or par, we put

$$\tilde{H}^n_{\text{et},?}(M_{\mathbb{Q}_p}, \mathcal{O}) = \text{im} \left( H^n_{\text{et},?}(M_{\mathbb{Q}_p}, \mathcal{O}) \to H^n_{\text{et},?}(M_{\mathbb{Q}_p}, K) \right),$$

$$\tilde{H}^n_{\text{log-cris},?}(M_{\mathbb{Q}_p}^\text{tor})\mathcal{O} = \text{im} \left( H^n_{\text{log-cris},?}(M_{\mathbb{Q}_p}^\text{tor})\mathcal{O} \to H^n_{\text{log-cris},?}(M_{\mathbb{Q}_p}^\text{tor})K \right).$$
We define objects $\tilde{H}_{\text{et},\text{par}}^n(M_\Q, \kappa)$ of $\text{Rep}_{\log, \text{cris}}^p(G_{\Q_p})$ and $\tilde{H}_{\text{log,cris},\text{par}}^n(M_{\text{tor}})_\kappa$ of $\text{MF}^p_{\O}$ by

$$
\tilde{H}_{\text{et},\text{par}}^n(M_\Q, \kappa) = \tilde{H}_{\text{et},\text{par}}^n(M_\Q, \O)/\varpi, \quad \tilde{H}_{\text{log,cris},\text{par}}^n(M_{\text{tor}})_\kappa = \tilde{H}_{\text{log,cris},\text{par}}^n(M_{\text{tor}})_\O/\varpi.
$$

### 7.2. Analogue of a multiplicity one theorem

In this subsection, we state the main theorem of [7] which shall be proved in [7].

**Theorem 7.1.** Under the same notation and assumptions as Theorem 6.7, $[\delta_\ell]^t \pmod{\varpi}$ and $[\rho_\E]^t \pmod{\varpi}$ belong to $\tilde{H}_{\text{et},\text{par}}^n(M_\Q, \kappa)$ and there exists $u \in \O^*$ such that

$$
[\delta_\ell]^t = u[\rho_\E]^t \text{ in } \tilde{H}_{\text{et},\text{par}}^n(M_\Q, \kappa).
$$

**Remark 7.2.** M. Dimitrov [Dim2] Theorem 6.7] proved that a multiplicity one theorem holds for the $\ell$-parts of $\tilde{H}_{\text{et},\text{par}}^n(M_\Q, \kappa)$ and $\tilde{H}_{\text{et},\text{par}}^n(M_\Q, \O)$ if the residual Galois representation $\rho_\ell$ is irreducible under some assumptions.

In the rest of this subsection, we introduce some objects of $\text{Rep}_{\log, \text{cris}}^p(G_{\Q_p})$ and $\text{MF}^p_{\O}$ associated to $\E$ and $\ell$, which will be used in the proof of Theorem 7.1.

For $? = \phi$ or $c$, let $T(\alpha)_{\text{et}}$ and $U(\alpha)_{\text{et}}$ (resp. $T(\alpha)_{\text{dr}}$ and $U(\alpha)_{\text{dr}}$) be the Hecke operators on $H^n_{\text{et},?(M_\Q, \Q_p)}$ (resp. $H^n_{\text{log,dr},?(M_{\text{tor}})}$) induced by the Hecke correspondences $T(a)$ and $U(a)$ on $M_{\text{tor}}(\Q_p)$ (resp. $M_{\text{tor}}(\Q_p)$) (see §1.9), respectively. We define the Hecke operators $T(\alpha)_{\text{cris}}$ and $U(\alpha)_{\text{cris}}$ on $H^n_{\text{et},?(M_{\Q_p})}$ via the isomorphism $H^n_{\log,?(M_{\text{tor}})} \simeq H^n_{\text{log,cris},?(M_{\text{tor}})}$. By the de Rham conjecture ([Fa–Jo, §VIII]), the comparison isomorphism $c_{M,\text{dr}}$ between $H^n_{\text{et},?(M_{\Q_p})}$ and $H^n_{\text{log,cris},?(M_{\text{tor}})}$ is Hecke-equivariant (cf. [Fa–Jo Theorem 1.2]). Since $c_{X,\text{dR}}$ is compatible with the Hecke isomorphism $c_{X,\text{cris}}$, $H^n_{\text{et},?(X_{\Q_p})}$ and $H^n_{\text{log,cris},?(X_{\text{tor}})}$, $X = M^1$ or $M$, the isomorphism $c_{X,\text{cris}}$ is Hecke-equivariant and $H^n_{\log,?(M_{\text{tor}})}$ is stable under $T(\alpha)_{\text{cris}}$ and $U(\alpha)_{\text{cris}}$.

Set $D = M_{\text{tor}} - M$. Let $H_2(n, \O)$ be the commutative $\O$-subalgebra of $\text{End}_\O(\tilde{H}_{\text{et},c}(M_\Q, \O)) \oplus \text{End}_\O(\tilde{H}_{\text{et},?}(D_\Q, \O)) \oplus \text{End}_\O(\tilde{H}_{\text{et},?}^{n+1}(M_\Q, \O))$ generated by $T(\alpha)_{\text{et}}$ for all non-zero prime ideals $q$ of $\O_F$ dividing $n$, $U(\alpha)_{\text{et}}$ for all non-zero prime ideals $q$ of $\O_F$ dividing $n$. Let $p_\E$ (resp. $p_\ell$) be the prime ideal of $H_2(n, \O)$ generated by $T(\alpha)_{\text{et}} - C(q, \E)$ (resp. $T(\alpha)_{\text{et}} - C(q, \ell)$) for all non-zero prime ideals $q$ of $\O_F$ prime to $n$ and $U(\alpha)_{\text{et}} - C(q, \E)$ (resp. $U(\alpha)_{\text{et}} - C(q, \ell)$) for all non-zero prime ideals $q$ of $\O_F$ dividing $n$. Let $m$ denote the maximal ideal $(\varpi, p_\E, p_\ell)$. We may regard $H_2(n, \O)$ as the $\O$-subalgebra of $\text{End}_{\O}^p(\tilde{H}_{\text{et},?}(M_{\text{tor}}) \oplus \text{End}_{\O}(\tilde{H}_{\text{et},?}(M_{\text{tor}}) \oplus \text{End}_{\O}(\tilde{H}_{\text{et},?}^{n+1}(M_{\text{tor}})))$.

We shall consider the $\ell$-parts of $\tilde{H}_{\text{et},\text{par}}^n(M_\Q, \O)$ and $\tilde{H}_{\text{log,cris},\text{par}}^n(M_{\text{tor}})_\O$ etc. defined by

$$
\mathcal{T} = \tilde{H}_{\text{et},\text{par}}^n(M_\Q, \kappa)[m], \quad \mathcal{T}_\ell = \tilde{H}_{\text{et},\text{par}}^n(M_\Q, \O)[p_\ell], \quad \mathcal{T}_f = \mathcal{T}_f/\varpi,
$$

$$
\mathcal{M} = \tilde{H}_{\text{log,cris},\text{par}}^n(M_{\text{tor}})_\kappa[m], \quad \mathcal{M}_\ell = \tilde{H}_{\text{log,cris},\text{par}}^n(M_{\text{tor}})_\O[p_\ell], \quad \mathcal{M}_f = \mathcal{M}_f/\varpi.
$$

A main tool for our proof is the $\E$-parts of $\tilde{H}_{\text{et}}^n(M_\Q, \O)$ and $\tilde{H}_{\text{log,cris}}^n(M_{\text{tor}})_\O$ defined by

$$
\mathcal{T}_\E = \tilde{H}_{\text{et}}^n(M_\Q, \O)[p_\E], \quad \mathcal{M}_\E = \tilde{H}_{\text{log,cris}}^n(M_{\text{tor}})_\O[p_\E].
$$

We obtain the following isomorphisms from (7.3) and (7.1):

$$
\mathcal{T} \simeq T_{\text{cris}}(\mathcal{M}), \quad \mathcal{T}_\ell \simeq T_{\text{cris}}(\mathcal{M}_\ell), \quad \mathcal{T}_f \simeq T_{\text{cris}}(\mathcal{M}_f), \quad \mathcal{T}_\E \simeq T_{\text{cris}}(\mathcal{M}_\E).
$$
Lemma 7.3. The canonical morphisms \( \tilde{\mathcal{T}}_f \to \mathcal{T} \) and \( \tilde{\mathcal{M}}_f \to \mathcal{M} \) are injective.

Proof. The assertion follows from the fact that \( \tilde{H}^n_{et,par}(\mathcal{M}_{\mathcal{Q}}, \mathcal{O})/(\tilde{H}^n_{et,par}(\mathcal{M}_{\mathcal{Q}}, \mathcal{O})[\mathfrak{p}_f]) \) and \( \tilde{H}^n_{log-cris,par}(M_{tor})\mathcal{O}/(\tilde{H}^n_{log-cris,par}(M_{tor})\mathcal{O}[\mathfrak{p}_f]) \) are torsion-free. \( \square \)

7.3. Multiplicity one for \( \text{Fil}^n(\tilde{M}) \).

Theorem 7.4. There exists a canonical isomorphism \( \text{Fil}^n(\tilde{H}^n_{log-cris,par}(M_{tor})\mathcal{O}) \simeq S_2(\mathfrak{n}, \mathcal{O}) \).

Proof. By the degeneration of the Hodge spectral sequences for \( H^*(\mathcal{M}_{\mathcal{Q}}\mathcal{O}, \Omega^n_{M_{tor}}(\log(D))) \) and \( H^*(\mathcal{M}_{\mathcal{Q}}\mathcal{O}, \Omega^n_{M_{tor}}(-\log(D))) \) [Fre, Theorem 4.1 and 4.1*], we have canonical isomorphisms

\[
\text{Fil}^n(\tilde{H}^n_{log-cris}(M_{tor})\mathcal{Z}_p) \simeq \text{Fil}^n(\tilde{H}^n_{log-cris}(M_{tor})\mathcal{Z}_p) \simeq H^0(\mathcal{M}_{\mathcal{Q}}\mathcal{O}, \Omega^n_{M_{tor}}(\log(D))),
\]

\[
\text{Fil}^n(\tilde{H}^n_{log-cris,c}(M_{tor})\mathcal{Z}_p) \simeq \text{Fil}^n(\tilde{H}^n_{log-cris,c}(M_{tor})\mathcal{Z}_p) \simeq H^0(\mathcal{M}_{\mathcal{Q}}\mathcal{O}, \Omega^n_{M_{tor}}(\log(D))).
\]

Note \( \Omega^n_{M_{tor}}(-\log(D)) = \Omega^n_{M_{tor}} \) for the last isomorphism. Therefore \( \text{Fil}^n(\tilde{H}^n_{log-cris,par}(M_{tor})\mathcal{O}) \) is canonically isomorphic to the image of the homomorphism

\[
H^0(\mathcal{M}_{\mathcal{Q}}\mathcal{O}, \Omega^n_{M_{tor}}/\mathcal{O}(\log(D))),
\]

which is identified with \( S_2(\mathfrak{n}, \mathcal{O}) \) by the Koecher’s principle and the \( q \)-expansion principle because \( S_2(\mathfrak{n}, \mathcal{C}) \) is identified with \( \text{im}(H^0(\mathcal{M}_{\mathcal{C}}\mathcal{O}, \Omega^n_{M_{tor}}/\mathcal{C}) \to H^0(\mathcal{M}_{\mathcal{C}}\mathcal{O}, \Omega^n_{M_{tor}}/\mathcal{C}(\log(D)))) \) [Fre, Chapter II, §4]. \( \square \)

Proposition 7.5. (1) The dimension of \( \text{Fil}^n(\tilde{M}) \) over \( \kappa \) is equal to 1.

(2) The homomorphism \( \text{Fil}^n(\tilde{M}_f) \to \text{Fil}^n(\tilde{M}) \) is an isomorphism.

Proof. (1) We have \( \text{Fil}^n(\tilde{M}) = (S_2(\mathfrak{n}, \mathcal{O})/\mathcal{Z}[\mathfrak{m}] \) by Theorem [7.4]. By the duality theorem \( S_2(\mathfrak{n}, \mathcal{O}) \simeq \text{Hom}_\mathcal{O}(\mathfrak{H}(\mathfrak{n}, \mathcal{O}), \mathcal{O}) \) [Hida88, Theorem 5.1], we obtain \( (S_2(\mathfrak{n}, \mathcal{O})/\mathcal{Z})[\mathfrak{m}] \simeq \text{Hom}_\mathcal{O}(\mathfrak{H}(\mathfrak{n}, \mathcal{O})/\mathfrak{m}, \kappa) \) and the dimension of the right-hand side over \( \kappa \) is equal to 1.

(2) By Theorem [7.4] we have \( \text{Fil}^n(\tilde{M}_f) \simeq S_2(\mathfrak{n}, \mathcal{O})[\mathfrak{p}_f] \) and the right-hand side is a free \( \mathcal{O} \)-module of rank 1. Hence the claim follows from (1), \( \text{Fil}^n(\tilde{M}_f) = \text{Fil}^n(\tilde{M}_f)/\mathcal{Z}\text{Fil}^n(\tilde{M}_f) \), and Lemma [7.3]. \( \square \)

7.4. Multiplicity one for \( \tilde{M}_E \).

Proposition 7.6. Assume that \( h^+_p = 1 \) and \( C(q, \mathcal{E}) \neq N(q) \) for some prime ideal \( q \) dividing \( \mathfrak{n} \). Then \( \tilde{M}_E \) is free of rank 1 over \( \mathcal{O} \) and \( \text{Fil}^n(\tilde{M}_E) = \tilde{M}_E \).

Proof. The U(q)-eigenvalue of each invariant form \( \omega_{pJ} \) defined in [5.2] is equal to \( N(q) \) because

\[
\omega_{pJ}|U(q) = \sum_{b \in \mathfrak{g}_F/\mathfrak{q}} \begin{pmatrix} 1 & b \\ 0 & \mathfrak{q}_b \end{pmatrix} \omega_{pJ} = N(q)\omega_{pJ}.
\]

Here the first equality follows from [5.2] and the second equality follows from that \( \omega_{pJ} \) is invariant under the action of the standard Borel subgroup \( B_{\mathcal{Q}} \). Then, by the same arguments as in the proof of [4.7], we see that \( H^0_{par}(Y(\mathfrak{n}, \mathcal{C})[\mathcal{E}] \simeq H^0_{cusp}(Y(\mathfrak{n}, \mathcal{C})[\mathcal{E}] \) and the right-hand side is 0 because \( H^0_{cusp}(Y(\mathfrak{n}, \mathcal{C}) \simeq \bigoplus_{w \in W_{\mathfrak{g}_2}} S_2(\mathfrak{n}, \mathcal{C}) \) as Hecke modules (cf. [Hida94, Corollary 2.2]) and the \( q \)-expansion principle. Hence, by [1.0], we obtain \( H^n(Y(\mathfrak{n}, \mathcal{C})[\mathcal{E}] = H^n_{par}(Y(\mathfrak{n}, \mathcal{C})[\mathcal{E}] \).

By combining with Proposition [1.1] (1), we have \( H^n(Y(\mathfrak{n}, \mathcal{C})[\mathcal{E}] = \)
Fil^n(H^n(Y(n), \mathbb{C})|\mathfrak{p}_E)). Since H^n(Y(n), \mathbb{C}) \cong H^n(M^\flat_{\text{G}})^{\omega}, \Omega^n_{M^\flat_{\text{G}}/\mathcal{O}(\log(D))}) as filtered modules, we see that \(M_E = \text{Fil}^n(M_E) \cong H^n(M^\flat_{\text{G}})^{\omega}, \Omega^n_{M^\flat_{\text{G}}/\mathcal{O}(\log(D))}|\mathfrak{p}_E). Here the last isomorphism follows from \(7.4\). The last term is free of rank 1 over \(\mathcal{O}\) by the \(q\)-expansion principle.

Combining with Corollary 5.6 we obtain the following:

**Corollary 7.7.** Under the same assumptions as Theorem 5.3, \(\widetilde{T}_E\) is free \(\mathcal{O}\)-module of rank 1 generated by \([\omega_E]\).

7.5. The morphism \(\widetilde{T}_E \to \widetilde{T}\).

**Lemma 7.8.** Assume that both \(H^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})_m\) and \(H^n(Y(n), \mathcal{O})_m\) are torsion-free, where \(m\) is the maximal ideal of \(\mathcal{H}_2(n, \mathcal{O})\) defined before Theorem 5.5. Then the exact sequence \(0 \to H^n_{\text{par}}(Y(n), \mathcal{O}) \to H^n(\partial(Y(n)^{\text{BS}}), \mathcal{O}) \to H^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})\) induces an exact sequence \(0 \to \widetilde{H}^n_{\text{par}}(Y(n), \mathcal{O})_m/\omega \to \widetilde{H}^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})_m/\omega \to \widetilde{H}^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})_m/\omega.

**Proof.** We omit the coefficient \(O\) of the cohomology groups to simplify the notation. Let \(N\) be the image of \(H^n(Y(n))\) in \(H^n(\partial(Y(n)^{\text{BS}}))\). Then we have an exact sequence \(0 \to N_m \to H^n(\partial(Y(n)^{\text{BS}}))_m \to H^n(Y(n))_m\), whose last term is torsion-free by assumption. Therefore \(N_m/\omega \to H^n(\partial(Y(n)^{\text{BS}}))_m/\omega\) is injective. Since \(N_m\) is torsion-free by assumption, we have \(H^n(\partial(Y(n))_m/\omega \to H^n(\partial(Y(n)^{\text{BS}}))_m/\omega\) and obtain an exact sequence \(0 \to \widetilde{H}^n_{\text{par}}(Y(n))_m \to \widetilde{H}^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})_m \to 0\). By taking the reduction modulo \(\omega\), we obtain an exact sequence \(0 \to \widetilde{H}^n_{\text{par}}(Y(n))_m/\omega \to \widetilde{H}^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})_m/\omega \to \widetilde{H}^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})_m/\omega \to 0\) because \(N_m\) is torsion-free. This completes the proof.

Note that \(M_m = M_m\) for a \(\mathcal{H}_2(n, \mathcal{O})\)-module \(M\). Hence, by Lemma 7.8 we may regard \(\widetilde{T}\) as a submodule of \((\widetilde{H}^n_{\text{par}}(M^n_{\mathfrak{Q}}), \mathcal{O})/\omega)\), under the assumption of the lemma.

**Proposition 7.9.** Under the same assumptions as Theorem 7.1, the natural homomorphism \(\widetilde{T}_E/\omega \to (\widetilde{H}^n_{\text{par}}(M^n_{\mathfrak{Q}}), \mathcal{O})/\omega)\) is injective and its image is contained in \(\mathfrak{t}\).

**Proof.** One can prove the first claim in the same way as in Lemma 7.3. The assumption \(E \equiv f(\text{mod } \omega)\) is equivalent to \(E - f \in \mathfrak{Z}M_2(n, \mathcal{O})\) by the \(q\)-expansion principle (\([\text{Dim}^2\text{Proposition 1.10 (i)}]\) Applying Proposition 7.4 (1) (resp. (2)) to the cohomology class of \(\omega^{-1}(E - f) \in M_2(n, \mathcal{O})\) (resp. \(f \in S_2(n, \mathcal{O})\)), we obtain \(\omega^{-1}\text{res}(\omega_E) = \text{res}(\omega^{-1}(E - f)) \in \widetilde{H}^n(\partial(Y(n)^{\text{BS}}), \mathcal{O})\). Now the second claim follows from Corollary 7.7 and Lemma 7.8.

7.6. **Proof of Theorem 7.1** In this subsection, we prove Theorem 7.1. Let \(\widetilde{T}_E\) (resp. \(\widetilde{M}_E\)) denote the quotient \(\widetilde{T}_E/\omega\), resp. \(\widetilde{M}_E/\omega\) in \(\text{Rep}^p_{\text{cris}}(G_{\mathbb{Q}_p})\) (resp. \(\text{MF}^p_{\mathcal{O}}\)). By Lemma 7.3 and Proposition 7.9 we have the following monomorphisms in \(\text{Rep}^p_{\text{cris}}(G_{\mathbb{Q}_p})\) and \(\text{MF}^p_{\mathcal{O}}\):

\[
\begin{array}{ccc}
\widetilde{T}_E \xrightarrow{\alpha_T} \mathcal{T} & \xrightarrow{\beta_T} & \mathcal{T}_f,
\end{array}
\]

\[
\begin{array}{ccc}
\widetilde{M}_E \xrightarrow{\alpha_M} \mathcal{M} & \xrightarrow{\beta_M} & \mathcal{M}_f.
\end{array}
\]

We define the action of \(W_\mathcal{G}\) on the underlying \(\mathcal{O}\)-modules of \(\widetilde{T}_E, \mathcal{T}, \text{ and } \mathcal{T}_f\) via the comparison isomorphism between étale and Betti cohomologies induced by the fixed embedding \(\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}\). Then the morphisms \(\alpha_T\) and \(\beta_T\) are \(W_\mathcal{G}\)-equivariant. By Proposition 7.5 (1) and Proposition 7.6 we have \(\alpha_M(M_E) = \text{Fil}^n(M_E)\). Hence, by Proposition 7.5 (2), we see that there exists a subobject \(L\) of \(\mathcal{T}_f\) in \(\text{Rep}^p_{\text{cris}}(G_{\mathbb{Q}_p})\) such that \(\beta_T(L) = \alpha_T(\widetilde{T}_E)\). By Remark 5.1 and
Corollary 7.7, we have $\widetilde{T}_E = \widetilde{T}_E[\epsilon]$, which implies that $L$ is $W_G$-stable and $L = L[\epsilon]$. Since the isomorphism (1.7) over $\mathbb{C}$ says that the dimension of $\widetilde{T}_E[\epsilon]$ over $\kappa$ is equal to 1, we obtain $L = \widetilde{T}_E[\epsilon]$. This completes the proof because $[\delta T]^*(mod \varpi)$ and $[\omega E]^*(mod \varpi)$ are bases of $\widetilde{T}_E[\epsilon]$ and $\widetilde{T}_E$, respectively (cf. (6.1) Corollary 4.1).

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