ROBUST OBSERVER-BASED CONTROL FOR DISCRETE-TIME SEMI-MARKOV JUMP SYSTEMS WITH ACTUATOR SATURATION

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Abstract. This paper investigates the control synthesis for discrete-time semi-Markov jump systems with nonlinear input. Observer-based controllers are designed in this paper to achieve a better performance and robustness. The nonlinear input caused by actuator saturation is considered as a group of linear controllers in the convex hull. Moreover, the elapse time and mode dependent Lyapunov functions are investigated and sufficient conditions are derived to guarantee the $H_{\infty}$ performance index. The largest domain of attraction is estimated as the existing saturation in the system. Finally, a numerical example is utilized to verify the effectiveness and feasibility of the developed strategy.

1. Introduction. It is well known that, Markov jump system is widely studied in recent decades as its outstanding performance in modeling the practical system, such as sudden changes of the component and abrupt variations in structures [4, 9]. However, the biggest limitation of Markov jump systems applying in practical systems is that the sojourn time of one mode only follows exponential or geometric distribution in a continuous-time system or a discrete-time system [14]. Owing to the restriction of memoryless hypothesis, semi-Markov jump systems are proposed to describe more extensive systems with the sojourn time subject to any kinds of distributions. Markov hypothesis is only satisfied at the jump instant, however, within the time interval, the memoryless property is not guaranteed as the non-exponential distribution of the sojourn time. That is the reason we call this kind of system as semi-Markov jump system.

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As this advantage, semi-Markov jump system becomes more important in practical modeling other than Markov jump system. Such as reliability analysis [2], DNA analysis [3], insurance and finance [10], earthquake modeling [1], etc. By comparing with Markov models and semi-Markov models for crack deterioration on flexible pavements, the semi-Markov models presents a better performance obviously [24]. Furthermore, the failure rate of a component always follows Weibull distribution instead of the exponential distribution we assumed before. However, as the complexity of the calculation and concepts, the theoretical results for semi-Markov jump systems are very limited. Moreover, previous studies are mainly concentrate on the continuous-time cases [23, 18], which motivates us to study more control problems subject to discrete-time settings. In the view of concept, the transition rate in semi-Markov jump systems can be seen as time varying which is sojourn time dependent. Different from the nonhomogeneous Markov jump system with time varying transition probability [27, 19], transition probability in semi-Markov jump system is not only dependent on the transition probability from current state to the next state but also dependent on the distribution of the sojourn time. The semi-Markov kernel approach is proved to be the most effective method to solve the problem of discrete-time semi-Markov systems [29]. Moreover, elapse time is developed to analysis the stability of the system which is distinguish from the whole time variable in Markov jump systems [25, 26]. That is the strategy we used in this paper.

On the other side, actuator saturation is inevitable problem in practical systems. Take an example, in a wind turbine system, the amplitude and rate of the blade rotation both have their limited working regions as the design characteristic [12]. Another example is the wheeled mobile robot, in which the drive torques supplied by motors at the rear wheels is said to be saturated [7]. In a similar way, there exists restrictions in almost all the actuators. To be more specific, the overstep and shortage of the limitation of the capacity may leads to the nonlinearity of the actuator, which is regarded as the saturation of actuators. It is obviously that this nonlinearity may cause the unstable and bad performance of the system. Therefore, the stability and stabilization considering actuator saturation has achieved much attention in recent years [8, 6, 31]. As the widely application in practical systems, many approaches have been proposed to handle the actuator saturation problems, such as [32, 13, 17]. From another point of view, the wind turbine system is more appropriate modeling as semi-Markov jump systems according to [11]. Therefore, it is essential to consider actuator saturation into the semi-Markov jump systems.

To our knowledge, control problems for semi-Markov jump system in discrete-time domain subject to nonlinear input have not been studied before, which result to our research in this paper.

Robust observer based controller has been widely investigated in the past decades. For the reason that the traditional feedback control method use the system state directly, which overlooked unavailable of the state. The observer utilized with the same structure of the system is to measure the state such that the designed controller could stabilize the system and achieve the disturbance rejection. Robust observer-based control problem has been proposed in [20]. Furthermore, the observer-based controller is designed for linear parameter varying (LPV) systems by linear matrix inequalities (LMIs) [15]. For Markov jump systems, observer-based controllers are given in [22]. Moreover, robust controllers are designed for nonhomogeneous Markov
jump systems with actuator saturation in [28]. Recently, [21] gives sufficient conditions of nonhomogeneous semi-Markov jump systems based on observer-based controllers. However, observer-based controller design for discrete-time semi-Markov jump systems are limited which motivates this paper.

This paper concentrates on robust control scheme for semi-Markov jump systems with external disturbances subject to nonlinear input. Three contributions in our work are summarized as: (i) It is the first attempt to consider actuator saturation for discrete-time semi-Markov jump systems with designing observer-based controllers. (ii) The bounded sojourn time is applied to reduce the conservatism of the proposed semi-Markov kernel approach. (iii) the largest attraction domain is estimated to make sure the system is σ-MSS for the designed controllers.

**Notation:** Throughout this paper, an $n$-dimension, $m$-dimension, $p$-dimension and $q$-dimension Euclidean spaces are given as $R^n$, $R^m$, $R^p$ and $R^q$. $\mathbb{N}_+$ and $\mathbb{N}$ indicate for the positive and non-negative integers respectively. In addition, $\mathbb{N}_{[a,b]}$ represents the value taken from the constraints set between $a$ and $b$. Moreover, the superscript ”$T$” denotes the transposition while ”$-1$” denotes the matrix inverse. $\| \star \|_2$ represents the 2-norm of a function, which can be understood as the energy of the function. $| \star |$ means the row value of a vector and $E\{\star \}$ represents the expectation of a function. $\bigcap_{i=1}^{N} P_i$ is the intersection of multiple matrices $P_i$.

## 2. Preliminaries and problem formulation

Considering a discrete-time semi-Markov linear jump system with nonlinear input as:

\[
\begin{align*}
    x(k+1) &= A(r_k)x(k) + B(r_k)\alpha(u(k)) + D(r_k)\omega(k) \\
    y(k) &= C_1(r_k)x(k) \\
    z(k) &= C_2(r_k)x(k)
\end{align*}
\]

where $x_k \in R^n$ is the vector of state variables, $\alpha(u(k))$ represents the nonlinear input, $\omega_k \in R^m$ is the disturbance satisfied $\|\omega_k\|_2 = \sqrt{\sum_{k=0}^{\infty} \omega_k^T \omega_k} \leq \varpi$, $\varpi > 0$, which is supposed to be energy-bounded. $z(k) \in R^p$ is the control output and $y(k) \in R^q$ is the measured output. Parameters with with appropriate dimensions are $A(r_k)$, $B(r_k)$, $B_{\omega}(r_k)$, $C(r_k)$ and $D(r_k)$. Stochastic variable $\{r_k\}$ represents the system mode taking values in a given finite set $\Gamma = \{1, 2, \cdots, N\}$, which governed by semi-Markov jump process.

For better understanding of discrete-time semi-Markov jump systems, the following concepts are first given as below.

In this paper, we define the random process $\{k_n\} \in \mathbb{N}_+$ to represent the instant time at $n$th jump. Then, the random process $\{R_n\} \in \Gamma$ represents the mode at the instant time $k_n$, which is called embedded Markov chain (EMC). Hence, $\{R_n\}$ can be regarded as the index of $\{r_k\}$ at the time instant. According to the above, the sojourn time $\{T_n\} \in \mathbb{N}_+$ is defined as $T_{n+1} = k_{n+1} - k_n$, which means the time interval between the $n$th jump and the $(n + 1)$th jump. According to the description above, the following definitions of Markov renewal chain and semi-Markov chain are given as follow.

**Definition 2.1.** [29] Define a stochastic process $\{R_n, k_n\}$ is a Markov renewal chain (MRC) if we have the following description:
\[\pi_{ij}(\theta) = \Pr(R_{n+1} = j, T_{n+1} = \theta | R_0, \ldots, R_n = i; k_0, \ldots, k_n)\]
\[= \Pr(R_{n+1} = j, T_{n+1} = \theta | R_n = i)\]
where \(\Pi(\theta) = [\pi_{ij}(\theta)]_{i,j \in \Gamma}\) is regarded as discrete-time semi-Markov kernel (SMK). The property of \(\sum_{i,j} \pi_{ij}(\theta) = 1\) with \(\pi_{ij}(0) = 0\) should be satisfied according to [29]. The EMC \(\{R_n\}\) is associated to the MRC \(\{R_n, k_n\}\) from [5]. Additionally, the transition probability of MRC is described as \(\Lambda = [\lambda_{ij}]_{i,j \in \Gamma}\) and the property of embedded Markov chain is given as \(\lambda_{ij} = \Pr(R_{n+1} = j | R_n = i)\) with \(\lambda_{ii} = 0\) and \(\sum_{j=1}^{N} \lambda_{ij} = 1\).

**Definition 2.2.** [29] Consider a MRC \(\{R_n, k_n\}\), the stochastic process \(\{r_k\}\) is a semi-Markov chain (SMC) associated to the MRC \(\{R_n, k_n\}\), if
\[r_k = R_{N(k)}, k \in \mathbb{N}_+\]
where \(N(k) = \max\{n \in \mathbb{N} | k_n \leq k\}\) is the last jump in interval \([0, k]\). Therefore, \(r_k\) gives the system states at time \(k\).

**Remark 1.** As defined in Definition 1, the memoryless property of the embedded Markov chain is still satisfied. In other words, the future state and the sojourn time of current state are dependent on the current state instead of all past states. Moreover, the mode change only happens at the jump time instant \(k_n\) instead of the sampling time \(k\). We define \(\tau = k - k_n\) as the elapsed time from the current state for analyzing the stability of the system (1) according to Definition 2.

**Remark 2.** The semi-Markov kernel \(\pi_{ij}(\theta)\) is sojourn time dependent, which is composed by both transition probability of EMC \(\lambda_{ij}\) and the distribution of the sojourn time \(\rho_{ij}(\theta)\). Therefore, the probability density function (PDF) of sojourn time is dependent on both the current state and the future state which is defined as \(\rho_{ij}(\theta) = \Pr(T_{n+1} = \theta | R_{n+1} = j, R_n = i)\) with \(\rho_{ii}(\theta) = 0\).

As the system state is difficult to be measured, we have the following system form with the observer-based controller:
\[
\begin{align*}
\dot{x}(k+1) &= A(r_k)x(k) + B(r_k)\alpha(u(k)) + H(r_k)y(k) - \hat{y}(k) \\
\dot{\hat{y}}(k) &= C_1(r_k)\hat{x}(k) \\
\alpha(u(k)) &= \alpha(K(r_k, \tau))\hat{x}(k)
\end{align*}
\]
where \(x(k) \in \mathbb{R}^n\) is the estimated state of the observer, \(\hat{y}(k)\) is the estimated output of the observer, \(H(r_k)\) and \(K(r_k, \tau)\) are the observer and the controller gains to be obtained. The Input nonlinear in this paper is mainly considered as the standard actuator saturation defined as \(\alpha(u(k)) = [\alpha(u(1)), \alpha(u(2)), \ldots, \alpha(u(k))]\), where \(\alpha(u(i)) = sign(u_i)\min\{1, |u_i|\}\). Here \(\alpha\) denotes both the vector value and scale value of the saturation function. \(sign(x)\) is the sign function.

**Definition 2.3.** [16] For the given positive symmetric matrices \(P, F\), an ellipsoid set and a symmetric polyhedron set are defined as follows, respectively:
\[
\Omega(P, 1) = \{x(k) \in \mathbb{R}^n : x^T(k)Px(k) \leq 1\}.
\]
\[
\mathcal{L}(F) = \{x(k) \in \mathbb{R}^n : |f_a x(k)| \leq 1, a = 1, 2, \ldots, m\}.
\]
Remark 3. The symbol $\mathfrak{S}(F)$ means the linear region of the saturated control $u = \alpha(Fx)$. If there exists $|x| \leq 1$ satisfied for all $x \in \mathfrak{S}(P, 1)$, $\mathfrak{S}(P, 1)$ is said to be invariant under controller $u = \alpha(Fx)$.

Lemma 2.4. For the given vectors $v(k)$ and $\nu(k)$, if $v(k) < 1$, then, $\alpha(u(k)) = \sum_{l=1}^{2m} \xi_l(M_lu(k) + M_l^-v(k))$, where $0 \leq \xi_l \leq 1$ and $\sum_{l=1}^{2m} \xi_l = 1$. Suppose that each element in $m \times m$ matrices $M_l$ is either 1 or 0 and $M_l^- = I - M_l$.

Remark 4. Let $v(k) = F(r_k, \tau)x(k)$ for system (1), we have $|v| < 1$ satisfied when $x(k) \in \mathfrak{S}(F)$. Then, nonlinear input can be designed as $\alpha(u(k)) = \sum_{l=1}^{2m} \xi_l(M_lK(r_k, \tau) + M_l^-F(r_k, \tau))x(k)$ according to Lemma 1, which is composed by a set of matrices with several rows from $K(r_k, \tau)$ and the others from $F(r_k, \tau)$.

Following gives the block of the system we considered. By defining the estimated error $e(k) = x(k) - \hat{x}(k)$, the close-loop semi-Markov jump system from (1) and (2) is obtained as:

\[
\begin{align*}
\hat{x}(k + 1) &= \hat{A}(r_k)\hat{x}(k) + \hat{D}(r_k)\omega(k) \\
\hat{z}(k) &= \hat{C}(r_k)\hat{x}(k)
\end{align*}
\]

where $\hat{x}(k) = [e^T(k) x^T(k)]^T$, $\hat{D}(r_k) = [D^T(r_k) D^T(r_k)]^T$, $\hat{C}(r_k) = [0 C_2^T(r_k)]^T$ and

\[
\hat{A}(r_k) = \begin{bmatrix}
A(r_k) - H(r_k)C_1(r_k) & 0 \\
-B(r_k) \cdot \hat{M}_1(r_k, \tau) & A(r_k) + B(r_k) \cdot \hat{M}_1(r_k, \tau)
\end{bmatrix},
\]

\[
\hat{M}_1(r_k, \tau) = \sum_{l=1}^{2m} \xi_l(M_lK(r_k, \tau) + M_l^-F(r_k, \tau)).
\]

Definition 2.5. [30] Consider any given initial mode $r_0 \in \Gamma$ and state $\hat{x}(0) \in R^n$, system (3) is said to be mean square stable (MSS) with $\omega(k) = 0$ if

\[
\lim_{k \to \infty} E\{\hat{x}^T(k)\hat{x}(k)\} = 0
\]

However, as the infinity of the sojourn time is non-existent in practical systems, the upper bound of the sojourn time is considered in this paper for finite feasible sufficient conditions. For the reason that the sojourn time follows any kind of distribution rather than geometric distribution in Markov jump system, based on Definition 4, the approximation error $\sigma$ is necessary in giving the new stability definition. Thus, $\sigma$-mean square stability ($\sigma$-MSS) is presented as follow.

Definition 2.6. [29] Consider any given initial mode $r_0 \in \Gamma$, state $\hat{x}(0) \in R^n$ and the maximum of sojourn time $T_{\max}$, system (3) is said to be $\sigma$-error mean square stable with $\omega(k) = 0$ if

\[
\lim_{k \to \infty} E\{\hat{x}^T(k)\hat{x}(k)\} | \hat{x}(0), r_0, T_{n+1} \leq T_{\max} | R_n = i = 0
\]

with

\[
\sigma = \sum_{i=1}^{n} |\ln(\Psi_i(T_{\max}^i))|,
\]
where $$\Psi_3(\theta) = \Pr(T_{n+1} \leq \theta | R_n = i) = \sum_{l=0}^{\theta} \sum_{j \in \Gamma} \pi_{ij}(l)$$. $$\Psi_3(\theta)$$ is the cumulative density function (CDF) of the sojourn-time in system mode $$i$$.

**Remark 5.** The symbol $$\sigma$$ means the approximation error compared with the classical mean square stability. When $$T_{\text{max}}^i$$ increases, error $$\sigma$$ will decrease. It means that when $$T_{\text{max}} \rightarrow \infty$$ for any $$i \in \Gamma$$, accordingly we have $$\sigma \rightarrow 0$$. Consequently, $$\sigma$$-MSS can be transformed to the classical MSS. More details are given in Table 1 in simulation part.

**Definition 2.7.** [30] If for given $$\gamma > 0$$, the following inequality holds, then, discrete-time semi-Markov jump system (3) is satisfactory with a $$H_{\infty}$$ performance index $$\gamma$$:

$$
\|T_{\omega}\|_{\infty}^2 = \sum_{k=0}^{\infty} z^T(k)z(k) / \sum_{k=0}^{\infty} \omega^T(k)\omega(k) \leq \gamma^2
$$

(6)

With the consideration of nonlinear input, the objective of this paper is to design controllers based on observers, such that the close-loop system (3) is $$\sigma$$-MSS and $$H_{\infty}$$ performance index is guaranteed. Moreover, the largest domain of the attraction is obtained.

3. **Main results.** In this part, an observer-based controller formed as in Remark 4 is designed to stabilize system (3) and then $$H_{\infty}$$ performance index is utilized to obtain the robustness of the system (3). The sufficient conditions are given according to Definition 5 in the form of LMIs. Finally, estimation of attraction domain is presented to enlarge the attraction domain.

3.1. **Controller design analysis.**

**Theorem 3.1.** For given initial state $$x_0$$ and matrices $$M_I, M^{-}_I$$, if for all $$i,j \in \Gamma$$, $$T_{\text{max}}^i \in \mathbb{N}_+$$, there exists a set of $$P_1(i, \tau) > 0$$, $$P_2(i, \tau) > 0$$ and $$\tau \in \mathbb{N}_0, T_{\text{max}}^i$$, such that the following inequalities are satisfied. Then, system (3) is said to be $$\sigma$$-MSS in the region $$\bigcap_{i=1}^{N} \Omega(\hat{P}(i, \tau))$$:

$$
\begin{bmatrix}
-P_1(i, \tau) & 0 & A^T(i) - C^T(i)H^T(i) & -\hat{M}_I^T(i, \tau)B^T(i) \\
* & -P_2(i, \tau) & 0 & A^T(i) + \hat{M}_I^T(i, \tau)B^T(i) \\
* & * & -P^{-1}_1(i, \tau + 1) & 0 \\
* & * & * & -P^{-1}_2(i, \tau + 1)
\end{bmatrix} < 0
$$

(7)

$$
\sum_{\theta=1}^{T_{\text{max}}^i} \sum_{i \neq j, \theta \in \Gamma} \eta_{ij}(\theta)\{P_1(j, 0) - P_1(i, \theta)\} < 0
$$

(8)

$$
\sum_{\theta=1}^{T_{\text{max}}^i} \sum_{i \neq j, \theta \in \Gamma} \eta_{ij}(\theta)\{P_2(j, 0) - P_2(i, \theta)\} < 0
$$

(9)

$$
\Omega(\hat{P}(i, \tau), 1) \in \mathcal{L}(\hat{F}(i))
$$

(10)

where $$\eta_{ij}(\theta) = \pi_{ij}(\theta)/\rho_i(\theta)$$ with $$\rho_i(\theta) = \sum_{\theta=1}^{T_{\text{max}}^i} \sum_{i \neq j, \theta \in \Gamma} \pi_{ij}(\theta)$$, $$\hat{M}_I(i, \tau) = M_I K(i, \tau) + M^{-}_I F(i, \tau)$$. Then, $$K(i, \tau)$$ is the controller gain.
Proof. To reduce the conservatism, we set the Lyapunov function as:

$$V(x(k), r_k, \delta(k)) = x^T(k)\bar{P}(r_k, \delta(k))x(k)$$

(11)

The stability analysis is divided into two parts. First is mode stay in $r_k = i$ as sampling time $k \in [k_n, k_{n+1} - 2]$, we have:

$$\Delta V(x(k), r_k, \delta(k)) = V(x(k + 1), r_{k+1}, \delta(k + 1)) - V(x(k), r_k, \delta(k))$$

$$= x^T(k + 1)\bar{P}(i, \delta(k + 1))x(k + 1) - x^T(k)\bar{P}(i, \delta(k))x(k)$$

$$= x^T(k)[\bar{A}^T(i)\bar{P}(i, \delta(k + 1))\bar{A}(i) - \bar{P}(i, \delta(k))]x(k) < 0$$

we set $\delta(k) = \tau$, $\delta(k + 1) = \tau + 1$, $\bar{P}(i, \delta(k + 1)) = \begin{bmatrix} P_i(i, \delta(k + 1)) & 0 \\ 0 & P_i(i, \delta(k)) \end{bmatrix}$ and

$$\bar{P}(i, \delta(k)) = \begin{bmatrix} P_i(i, \delta(k)) & 0 \\ 0 & P_i(i, \delta(k)) \end{bmatrix},$$

then inequality (7) comes from (12) by using Schur complement. According to (12), we have

$$V(x(k_n), r_{k_n}, \delta(k_n)) - V(x(k_n - 1), r_{k_n-1}, \delta(k_n - 1)) < 0$$

(13)

We also have

$$V(x(k_{n+1}), r_{k_{n+1} - 1}, \delta(k_{n+1})) - V(x(k_{n+1} - 1), r_{k_{n+1} - 1}, \delta(k_{n+1} - 1)) < 0$$

(14)

In the same way, we have

$$V(x(k_{n+1} - 1), r_{k_{n+1} - 1}, \delta(k_{n+1} - 1)) - V(x(k_{n} - 1), r_{k_{n} - 1}, \delta(k_{n} - 1)) < 0$$

(15)

Thus, we have

$$V(x(k_{n+1}), r_{k_{n+1} - 1}, T_{n+1}) - V(x(k_n), r_{k_n}, 0) < 0$$

(16)

Then, when the system changes from $i$ to $j$, which means $k \in [k_n+1, k_{n+1}]$, we have

$$\mathbb{E}[V(x(k_{n+1}), r_{k_{n+1} + 1}, \delta(k_{n+1}))|x(k_n), r_{k_n}] - V(x(k_n), r_{k_n}, 0)$$

$$\leq \mathbb{E}[V(x(k_{n+1}), j, 0)] - V(x(k_n), i, 0)$$

$$= x^T(k_{n+1})[\sum_{\theta = 1}^{T_{n+1}} \sum_{i \neq j, j \in \Gamma} \eta_{ij}(\theta)(P(j, 0) - P(i, T_{n+1}))]x(k_{n+1}) < 0$$

(17)

where $\eta_{ij}(\theta) = \pi_{ij}(\theta)/\rho_i(\theta)$ with $\rho_i(\theta) = \sum_{\theta = 1}^{T_{n+1}} \sum_{i \neq j, j \in \Gamma} \pi_{ij}(\theta)$.

When (17) is satisfied, which infers to (8) and (9) are sufficient conditions for stabilization. That is to say system (3) is $\sigma$-MSS with the controller constructed as below:

$$u(k) = \sum_{i=1}^{2^n} \xi_i(M_iK(r_k, \tau) + M_i^-F(r_k, \tau))x(k)$$

The proof is completed. \[\square\]

3.2. $H_\infty$ performance analysis. In this subsection, with the consideration of the energy bounded external disturbances, $H_\infty$ performance index is utilized to analyze the robustness and feasibility of system (3).

Theorem 3.2. For given initial state $x_0$ and matrices $M_i$, $M_i^-$, If for all $i, j \in \Gamma$, $T_{\max}^i \in \mathbb{N}_+$, there exists a set of $Q_1(i, \tau) > 0$, $Q_2(i, \tau) > 0$, $G_1(i) > 0$, $G_2(i) > 0$, $Z(i, \tau)$, $Y(i, \tau)$ and $\tau \in \mathbb{N}[0, T_{\max}^i]$, such that the following inequalities are satisfied.
Then, system (3) is said to be $\sigma$-MSS in the region $\bigcap_{i=1}^{N} \tilde{P}(i, \tau)$ with the controller $K(i, \tau) = Z(i, \tau)G^{-1}(i)$:

$$
\begin{bmatrix}
    a_{11} & 0 & 0 & 0 & G_{1T}^T(i)B^T & -\Xi(i, \tau) \\
    a_{22} & 0 & G_{1T}^T(i)C_{T}^T(i) & 0 & G_{3T}^T(i)A^T(i) + \Xi(i, \tau) \\
    * & * & -\gamma^2 I & 0 & D^T(i) & D^T(i) \\
    * & * & * & -I & 0 & 0 \\
    * & * & * & * & -Q_1(i, \tau + 1) & 0 \\
    * & * & * & * & * & -Q_2(i, \tau + 1)
\end{bmatrix} < 0
$$

(18)

$$
\sum_{\theta=1}^{T_{\text{max}}} \sum_{i \neq j, j \in \Gamma} \eta_{ij}(\theta)\{Q_1(j, 0) - Q_1(i, \theta)\} < 0
$$

(19)

$$
\sum_{\theta=1}^{T_{\text{max}}} \sum_{i \neq j, j \in \Gamma} \eta_{ij}(\theta)\{Q_2(j, 0) - Q_2(i, \theta)\} < 0
$$

(20)

$$
\Omega(\tilde{P}(i, \tau), 1) \in \mathcal{L}(\mathcal{F}(i))
$$

(21)

where $a_{11} = -G_{1T}^T(i) - G_1(i) + Q_1(i, \tau)$, $a_{22} = -G_{2T}^T(i) - G_2(i) + Q_2(i, \tau)$, $\Xi(i, \tau) = Z^T(i, \tau)M_1^TB^T(i) + Y^T(i, \tau)M_1^{T}B^T(i)$, $\tilde{A} = A(i) - H(i)C_1(i)$, $F(i, \tau) = Y(i, \tau)$, $G^{-1}(i)$ and $\eta_{ij}(\theta) = \pi_{ij}(\theta)/\rho_i(\theta)$ with $\rho_i(\theta) = \sum_{\theta=1}^{T_{\text{max}}} \sum_{i \neq j, j \in \Gamma} \pi_{ij}(\theta)$.

Proof. As we know that according to Definition 6, we have

$$
J(k) = \mathbb{E}\left\{\sum_{k=0}^{\infty} [z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k)]\right\}
$$

$$
\leq \mathbb{E}\left\{\sum_{k=0}^{\infty} [z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k)] + \Delta V(x(k), r_k, \delta(k))\right\}
$$

(22)

Then, we consider the sum of any time interval between $[k_n, k_{n+1} - 1]$ as following:

$$
\sum_{k=k_n}^{k_{n+1}-1} J(k) \leq \sum_{k=k_n}^{k_{n+1}-1} \mathbb{E}\{z^T(k)z(k) - \gamma^2 \omega^T(k)\omega(k) + \mathbb{E}[x^T(k_{n+1})P(i, \theta)x(k_{n+1})] - x^T(k_n)P(i, 0)x(k_n)\}
$$

$$
\leq \sum_{\theta=1}^{T_{\text{max}}} \sum_{i \neq j, j \in \Gamma} \eta_{ij}(\theta)\{\sum_{k=k_n}^{k_{n+1}-1} |z^T_kz_k - \gamma^2 \omega^T_k\omega_k| + \mathbb{E}[x^T(k_{n+1})P(i, \theta)x(k_{n+1})] - x^T(k_n)P(i, 0)x(k_n)\}
$$

(23)

As $k_{n+1} = k_n + T_{n+1}$, and then the right side of equation (23) can be expressed as:

$$
\sum_{\tau=0}^{T_{n+1}} [z^T(k_n + \tau)z(k_n + \tau) - \gamma^2 \omega^T(k_n + \tau)\omega(k_n + \tau) + x^T(k_n + \tau + 1)P(i, \tau + 1)x(k_n + \tau + 1) - x^T(k_n + \tau)P(i, \tau)x(k_n + \tau)] < 0
$$

(24)
Then, substituting \( z(k_n + \tau) = C_2(i)x(k_n + \tau) \) and if (24) is satisfied, we have \( J(k) < 0 \) is guaranteed, which means,

\[
x^T(k_n + \tau)C_2^T(i)C_2(i)x(k_n + \tau) - \gamma^2 \omega^T(k_n + \tau)\omega(k_n + \tau) + [\hat{A}(i)x(k_n + \tau) + \hat{D}(i)\omega(k_n + \tau)]^TP(i, \tau + 1)[\hat{A}(i)x(k_n + \tau) + \hat{D}(i)\omega(k_n + \tau)] - x^T(k_n + \tau)P(i, \tau)x(k_n + \tau)
\]

\[
= \begin{bmatrix} x(k_n + \tau) \\ \omega(k_n + \tau) \end{bmatrix}^T \phi \begin{bmatrix} x(k_n + \tau) \\ \omega(k_n + \tau) \end{bmatrix} < 0
\]

By Schur complement, formula (26) is equivalent to (27).

\[
\begin{bmatrix}
-P_1(i, \tau) & 0 & 0 & 0 & \tilde{A}^T(i) & -[B(i)\tilde{M}_l(i, \tau)]^T \\
* & -P_2(i, \tau) & 0 & C_2^T(i) & 0 & R^T(i, \tau) \\
* & * & -\gamma^2I & 0 & D^T(i) & D^T(i) \\
* & * & * & -I & 0 & 0 \\
* & * & * & * & -P_1^{-1}(i, \tau + 1) & 0 \\
* & * & * & * & * & -P_2^{-1}(i, \tau + 1)
\end{bmatrix} < 0
\]

Let \( Q_1(i, \tau) = P_1^{-1}(i, \tau) \), \( Q_2(i, \tau) = P_2^{-1}(i, \tau) \), \( Q_3(i, \tau + 1) = P_1^{-1}(i, \tau + 1) \), \( Q_2(i, \tau + 1) = P_2^{-1}(i, \tau + 1) \), \( R(i, \tau) = A(i) + B(i)\tilde{M}_l(i, \tau) \).

Then, (27) can be transformed to (28) as

\[
\begin{bmatrix}
-Q_1^{-1}(i, \tau) & 0 & 0 & 0 & \tilde{A}^T(i) & -[B(i)\tilde{M}_l(i, \tau)]^T \\
* & -Q_2^{-1}(i, \tau) & 0 & C_2^T(i) & 0 & R^T(i, \tau) \\
* & * & -\gamma^2I & 0 & D^T(i) & D^T(i) \\
* & * & * & -I & 0 & 0 \\
* & * & * & * & -Q_1(i, \tau + 1) & 0 \\
* & * & * & * & * & -Q_2(i, \tau + 1)
\end{bmatrix} < 0
\]

Multiply diag\{\( G_1^T(i, \tau), G_2^T(i, \tau), I, I, I \)\} and diag\{\( G_1(i, \tau), G_2(i, \tau), I, I, I, I \)\} on the left and right side of (28), respectively. As \( G(i) + G^T(i) - Q(i, \tau) \leq G(i)Q^{-1}(i, \tau)G^T(i) \), inequality (18) is sufficient for the satisfaction of (28).

Without loss of generality, (9) and (10) are also guaranteed when system mode changes from \( i \) to \( j \), thus, we have (19) and (20) satisfied. We also have controllers under condition (21) are constructed as \( u(k) = \sum_{i=1}^{2^n} \xi_i(M_iK(r_k, \tau) + M_iF(r_k, \tau)x(k)) \). This completes the proof.

3.3. **Attraction domain estimation.** In this subsection, the largest attraction domain estimation of system (3) is given in the form of the optimization problem as following.

**Theorem 3.3.** For given initial state \( x_0 \) and matrices \( M_l, M_l^- \), If for all \( i, j \in \Gamma, T_{\text{max}}^i \in \mathbb{N}_+ \), there exists a set of \( Q_1(i, \tau) > 0, Q_2(i, \tau) > 0, G_1(i) > 0, G_2(i) > 0, K(i, \tau), F(i, \tau) \) and \( \tau \in \mathbb{N}_{[0, T_{\text{max}}]} \), such that the following optimization problem is solved. Then, the largest domain of attraction for system (3) is obtained

\[
\min_{X_n, Y_n} \lambda
\]
s.t.

\[
\begin{bmatrix}
  a_{11} & 0 & 0 & G_{1}^T(i)C_{2}^T(i) & 0 & -G_{1}^T(i)A^T(i) & -\Xi(i, \tau) \\
  a_{22} & 0 & 0 & G_{2}^T(i)C_{2}^T(i) & 0 & 0 & O(i, \tau) \\
  * & * & -\gamma^2 I & 0 & 0 & 0 & 0 \\
  * & * & * & -I & D^T(i) & D^T(i) & <0 \\
  * & * & * & * & -I & 0 & 0 \\
  * & * & * & * & * & -Q_1(i, \tau + 1) & 0 \\
  * & * & * & * & * & * & -Q_2(i, \tau + 1)
\end{bmatrix}
\]

(30)

\[
\sum_{\theta=1}^{T_{\max}} \sum_{i \neq j \in \Gamma} \eta_{ij}(\theta)\{Q_1(j, 0) - Q_1(i, \theta)\} < 0
\]  

(31)

\[
\sum_{\theta=1}^{T_{\max}} \sum_{i \neq j \in \Gamma} \eta_{ij}(\theta)\{Q_2(j, 0) - Q_2(i, \theta)\} < 0
\]  

(32)

\[
\begin{bmatrix}
  -\lambda R & I \\
  * & -\hat{Q}(i, \tau)
\end{bmatrix} < 0
\]  

(33)

\[
\begin{bmatrix}
  -1 & \hat{f}_i \\
  * & -\hat{Q}(i, \tau)
\end{bmatrix} < 0
\]  

(34)

where \(O(i, \tau) = G_{2}^T(i)A^T(i) + \Xi(i, \tau)\), \(K(i, \tau) = Z(i, \tau)G^{-1}(i)\), \(F(i, \tau) = Y(i, \tau)G^{-1}(i)\), \(\Xi(i, \tau) = Z^T(i, \tau)M^T_{\rho}B^T(i) + Y^T(i, \tau)(M^T_{\rho})^TB^T(i)\).

Proof. The proof procedure has been shown in Theorem 2, here we give the proof of (33). As we know that

\[
\alpha \Omega(R) \in \Omega(\hat{P}(i, \tau), 1)
\]

(35)

Then, we can get

\[
\alpha^2 x^T(0)\hat{P}(i, \tau)x(0) \leq 1
\]

which can be written as:

\[
\begin{bmatrix}
  -1/\alpha^2 & x^T(0) \\
  * & \hat{P}^{-1}(i, \tau)
\end{bmatrix} < 0
\]

(36)

Then we have the following inequality hold

\[
\begin{bmatrix}
  -1/\alpha^2 & x^T(0) \\
  * & \hat{Q}(i, \tau)
\end{bmatrix} < 0
\]

(37)

We choose an ellipsoid reference set, then we have \(\{x \in R : x^T Rx \leq 1\}\) and (37) is equivalent to (33) when \(\lambda = -1/\alpha^2\).

Furthermore, according to

\[
\Omega(\hat{P}(i, \tau), 1) \in \mathcal{L}([\hat{F}(i))
\]

(38)

The following inequality should be satisfied if we want (38) guaranteed, which is equal to (34).

\[
\begin{bmatrix}
  -1 & \hat{f}_i \\
  * & -\hat{Q}(i, \tau)
\end{bmatrix} < 0
\]

(39)
Remark 6. In this part, a reference ellipsoid set $\Omega(R,1)$ is applied to estimate the largest attraction domain of the system (1) with actuator saturation. Then, the estimation of the largest attraction domain problem becomes the optimization problem solving by getting the minimum value of $\lambda$.

4. Example and simulation. A numerical example is presented in this section to verify the effectiveness and feasibility of the mentioned design method. Furthermore, the largest attraction domain is estimated according to the optimization problem solving.

Consider discrete-time semi-Markov jump system (1) with three modes and the relevant parameters are given as following:

$$A(1) = \begin{bmatrix} -0.36 & 0.69 \\ -1.81 & 1.97 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 0.34 & 0.62 \\ -0.37 & 1.36 \end{bmatrix}, \quad A(3) = \begin{bmatrix} 0.34 & 0.62 \\ -0.37 & 1.36 \end{bmatrix}$$

$$B(1) = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, \quad B(2) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad B(3) = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

$$B_\omega(1) = B_\omega(2) = B_\omega(3) = \begin{bmatrix} -0.3 \\ 0.1 \end{bmatrix}, \quad C(1) = C(2) = C(3) = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$D(1) = D(2) = D(3) = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}^T$$

In this part, we set the external disturbances as $\omega(k) = 0.2e^{-0.02k}\sin(k)$. According to the description above, the transition probability of the MRC is assumed as:

$$\lambda_{ij} = \begin{bmatrix} 0 & 0.5072 & 0.4928 \\ 0.5357 & 0 & 0.4643 \\ 0.1507 & 0.8493 & 0 \end{bmatrix}$$

Then, the probability of sojourn time is assumed to follow different kinds of distribution, such as Bernoulli, Geometric and Weibull distributions, etc. Thus, we have 

$$\rho_{12}(t) = 0.6^t * 0.4^{10-t} * 10!/10-t!*t!, \quad \rho_{13}(t) = 0.7^t * 0.3^{10-t} * 10!/10-t!*t!, \quad \rho_{21}(t) = 0.4(t-1)^{1.3} - 0.4^t, \quad \rho_{23}(t) = 0.6(t-1)^{1.5} - 0.6^t, \quad \rho_{31}(t) = 0.5(t-1)^{1.3} - 0.5^t,$$

$$\rho_{32}(t) = 0.4(t-1)^{1.5} - 0.4^t,$$ where $a!$ represents the factorial of $a$.

As mentioned in Definition 5, the changes of error $\sigma$ for different maximum value of the sojourn time is presented in Table 1 to demonstrate the difference between MSS and $\sigma$-MSS, which is the main reason why the MSS is inapplicable in analysing the discrete-time semi-Markov process. From the values in Table 1, we could find that as the sojourn time is large enough, the error reference $\sigma$ is almost 0. Then, the transition probability becomes constant one and $\sigma$-MSS becomes MSS.

The controllers can be obtained by solving Theorem 2 and the domain attraction can be obtained by Theorem 3.

Fig.1 and Fig.2 show the mode changes and state responses of the system for a given initial state $x_0 = [-1 \quad 1]^T$ and Fig.3 gives the feasible region from Theorem 2 (in dashed line) and the largest domain attraction from Theorem 3 (in solid line). Obviously, the system is $\sigma$-MSS under the observer-based controllers. The optimal solution of parameter $\lambda$ in Theorem 3 is given as $\lambda = 0.0067$, which implies the close-loop system with actuator saturation keeps $\sigma$-MSS under the largest domain attraction by designed controllers.

5. Conclusion. The observer-based controllers are designed for the discrete-time semi-Markov jump system with consideration of nonlinear input in this paper. With energy bounded disturbances, $H_\infty$ performance is analyzed. To handle the nonlinear
Table 1. The error $\sigma$ for different maximum sojourn time in different mode

| Maximum sojourn time $T_{\text{max}}^i$ | error $\sigma$ |
|---------------------------------------|----------------|
| $T_{\text{max}}^1$ | $T_{\text{max}}^2$ | $T_{\text{max}}^3$ |
| 3 | 4 | 3 | $\ln(0.032)=-3.442$ |
| 4 | 4 | 3 | $\ln(0.1048)=-2.2557$ |
| 4 | 4 | 4 | $\ln(0.1061)=-2.2434$ |
| 6 | 6 | 6 | $\ln(0.4859)=-0.7218$ |
| 8 | 8 | 8 | $\ln(0.9028)=-0.1023$ |
| 10 | 10 | 10 | $\ln(1)=0$ |

Figure 1. The response of the jump mode

Figure 2. The response of the system state

input caused by actuator saturation, a group of linear feedbacks are placed into the convex hull to replace the nonlinear input and ellipsoids are applied to estimate the domain of attraction. Finally, the largest attraction domain is given by solving the optimization problem. A numerical example is used to illustrate the stability and stabilization of the proposed problem and the largest domain is given for this example.
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