Where do perturbative and non-perturbative QCD meet?

G.S. Bali\textsuperscript{a}, P. Boyle\textsuperscript{b}\textsuperscript{*} and C.T.H. Davies\textsuperscript{a}

\textsuperscript{a}Department of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, UK
\textsuperscript{b}Department of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3JZ, UK

We computed the static potential and Wilson loops to $O(\alpha^2)$ in perturbation theory for different lattice quark and gluon actions. In general, we find short distance lattice data to be well described by “boosted perturbation theory”. For Wilson-type fermions at present-day quark masses and lattice spacings agreement within 10\% between measured “$\beta$-shifts” and those predicted by perturbation theory is found. We comment on prospects for a determination of the real world QCD running coupling.

1. INTRODUCTION

Perturbative results on physical quantities are useful in lattice simulations in several ways:

\begin{itemize}
  \item to qualitatively understand “improvement” and “lattice artefacts”,
  \item to numerically validate that the continuum limit exists and is reached as $\beta \to \infty$,
  \item for predicting the “$\beta$-shift”, resulting from massive sea quarks,
  \item to estimate quantities that are hard to obtain otherwise, e.g. the static quark self energy,
  \item to calculate $\alpha_s$ from low energy QCD phenomenology.
\end{itemize}

In view of this we calculated Wilson loops and the static potential with massive Wilson, Sheikholeslami-Wohlert (SW) and Kogut-Susskind (KS) fermions to $O(\alpha^2)$ \cite{1}. We define the potential, $aV_{\text{int}}(Ra) = v_{L,1}(R)\alpha_L + v_{L,2}(R)\alpha_L^2 + \cdots$, where $V_{\text{int}}(Ra) = V(Ra) - V(\infty)$. The static quark self energy, $V(\infty) = 2\delta m_{\text{stat}} \propto \alpha a^{-1} + \cdots$, vanishes in dimensional regularisation but diverges on the lattice as the continuum limit is approached. For SU(3) we find, $a\delta m_{\text{stat}} = 2.1172743 \alpha_L + [11.143(3) + n_f Y_f/\alpha_L^2]$, with $Y_f = -0.36846(6)$ for massless KS quarks and $Y_f = -0.42333(6) + 0.05162(2)c_{SW} - 0.58702(2)c_{SW}^2$ for Wilson-SW quarks. While our value of the Wilson-SW $Y_f$ agrees with Ref. \cite{2} we find a 4 \sigma discrepancy in the gluonic contribution.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Wilson quark contribution to $v_{L,2}$.}
\end{figure}

2. “$\Delta K_1$” AND THE “$\beta$-SHIFT”

The $\overline{MS}$ scheme is related to the lattice scheme via $\alpha_{\overline{MS}}(a^{-1}) = \alpha_L + b_1 a^2 + \cdots$, with the conversion factor $b_1 = -\pi/(2N) + k_1 N + K_1(ma)n_f$, where the numerical constant $k_1$ is known for a variety of gluonic actions and $K_1(0)$ is known for Wilson, SW and KS quarks with Wilson glue. Further couplings can be defined, e.g. from the potential in position space, $\alpha_R(\mu) = -r V_{\text{int}}(r)/C_F$, $\mu = r^{-1}$. While the $\overline{MS}$ scheme is “mass-independent”, the $\beta$-function coefficients $\beta_i$ of the above “$R$” scheme depend on $m/\mu$ and hence on the quark mass; universality is lost and the $\beta_i$ are specific for each dimensionful observ-
would also expect rotational symmetry to be restored at finite $a$ for distances $r \gg a$. These two cases become distinguishable in perturbation theory once an external scale $m$ is introduced and hence the situation $r < m$ differs from $r > m$.

At finite $a$ and $m > 0$ the lattice scheme becomes mass-dependent too, as indicated by the function $K_1(ma) = K_1(0) + \Delta K_1(ma)$ within $b_1$ above. We denote the fermionic contribution to $v_{L,2}$ as $n_f v_{L,2}^f$ and $V_{L,2}^f = v_{L,2}^f/(4\pi)^2$. The latter, multiplied by $-R$, is displayed in Fig. 1 for the example of Wilson fermions. In the massless case we find the expected logarithmic running, proportional to the fermionic contribution to $\beta_0$ (straight line). The offset at small $R$ is related to $K_1(ma)$, which we determine by matching the lattice potential at $R \gg 1$ to the known $\overline{MS}$ result. $\Delta K_1$ is only unique in the limit $m = 0$ (where it vanishes). For massive quarks universality is lost and $\Delta K_1$ will depend on the observable that is matched. In the case of the position space potential (and force) we obtain the asymptotic behaviour, $\Delta K_1(ma) = 0.0011(3) - K_1(0) - \ln(ma)/(3\pi)$ for $ma \rightarrow \infty$. The logarithmic term is universal and guarantees the massive fermions to decouple from the $\beta$ function.

We display the size of this correction, relative to $K_1(0) < 0$ in Fig. 2. Note that in the limit $ma \rightarrow \infty$ the ratio $-\Delta K_1/K_1$ diverges towards negative values. For the two $O(a)$ improved fermionic actions the mass dependence is minimal while this effect can be very significant for Wilson fermions and has to be taken into account in any calculation that uses the potential as an intermediate scheme.

As can be seen from Fig. 3 at $r \ll m^{-1}$, i.e. $R \ll (ma)^{-1}$, the effective Coulomb coupling is screened with the same logarithmic slope as in the massless case. At $r \gg m^{-1}$ the heavy quarks decouple and do not contribute to the running anymore but just to the overall normalisation. It is here that the un-quenched potential can be matched to the quenched one by adjusting the coupling constant $\beta = 3/(2\pi \alpha_{\overline{MS}})$: $\beta^{(n_f)} = \beta^{(0)} + \Delta \beta$, $\Delta \beta = n_f [\ln(Dma) + 3\pi \Delta K_1(ma)]/(2\pi^2) + O(\beta^{-1})$. We find the numerical values, $D = 0.448(2), 0.0238(1)$ and $0.726(2)$, for Wilson, SW and KS fermions, respectively. As $ma \rightarrow \infty$ the

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**Figure 2.** Matching with the $\overline{MS}$ scheme.

**Figure 3.** Comparison with the measured $\beta$-shift.
\( \Delta K_1 \) term guarantees that \( \Delta \beta \to 0 \) while \( \Delta \beta \) diverges as \( ma \to 0 \): in the light quark limit the quenched and un-quenched theories decouple.

In Fig. 3 we compare our perturbative prediction to numerical data with \( n_f = 2 \) Wilson [4], SW [5] and KS [6] fermions, obtained at lattice spacings \( 4 < r_0/a < 6.5 \) and quark masses \( 0.1 \geq ma \geq 0.01 \). The Wilson and SW results fall onto a universal curve that differs by less than 10 % from the prediction while the KS results deviate much more and some \( \beta \) dependence is evident. Whether this is due to a slower convergence of the perturbative series or due to \( n_f \) not being a multiple of four is an open question. The qualitative agreement between prediction and simulation for Wilson-type quarks indicates that, at least at present masses, physics at hadronic scales is not strongly affected by quark loops, which is consistent with the phenomenological success of the quenched approximation.

3. THE QCD RUNNING COUPLING

We determine the running coupling \( \alpha_{\overline{MS}}(3.401\, a^{-1}) \) from the average plaquette, following the method detailed in Ref. [3]. The result is then converted into units of \( r_0 [4–6] \) and numerically evolved to the scale \( \mu = 10r_0 \) (Fig. 4) via the four-loop \( \beta \) function. While the mass dependence of the plaquette is rather weak, the term \( \Delta K_1(ma) \) has a statistically significant impact on the Wilson results at masses \( ma > 0.02 \).

The bottom-left \( n_f = 0 \) pentagon corresponds to the ALPHA collaboration result, \( \alpha_{\overline{MS}}(\mu) = 0.157(4) \), the other ones have been obtained from the average plaquette within a \( \beta \)-window that covers the range of lattice spacings, spanned by the dynamical simulations. We note that the \( n_f = 0 \) values obtained by use of the two different methods agree. All dynamical results can be fitted to polynomials in \( mr_0 \) (solid curves). The SW results [5] have been obtained at fixed \( r_0 \) while the two Wilson [4] and three KS [6] data sets have been produced at fixed \( \beta \). We observe a significant lattice spacing dependence, in contrast to the \( n_f = 0 \) case. Extrapolating to the limits \( mr_0 \approx 0.01 \) and \( a/r_0 \to 0 \), the latter linearly for Wilson and quadratically for KS fermions, we obtain the consistent estimate, \( \alpha_{\overline{MS}}^{(2)}(m_Z) - \alpha_{\overline{MS}}^{(0)}(\mu) = 0.030(6) \). Simulations at additional lattice spacings are mandatory to control the extrapolation to the continuum limit. Furthermore, numerical data at \( n_f \neq 0,2 \) are essential, before \( \alpha_{\overline{MS}}^{(5)}(m_Z) \) can be predicted with confidence. Once such results are available an error of 2 % appears to be realistic.

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