Tuning Rules for Control of Nonlinear Mechanical Systems

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September 8, 2020

Abstract

In this paper, we propose several rules to tune the gains of Passivity-Based Controllers for a class of nonlinear mechanical systems. Such tuning rules aim to prescribe a desired behavior to the closed-loop system, where we are particularly interested in attenuating oscillations and improving the rise time of the transient response. Hence, the resulting controllers stabilize the plant and simultaneously address the performance in terms of oscillations, damping ratio, and rise time of the transient response of the closed-loop system. Moreover, the closed-loop system analysis provides a clear insight into how the kinetic energy, the potential energy, and the damping of the mechanical system are related to its transient response, endowing in this way the tuning rules with a physical interpretation. Additionally, we corroborate the analytical results through the practical implementation of a controller that stabilizes a two-degrees-of-freedom (2DoF) planar manipulator, where the control gains are tuned following the proposed rules.

1 Introduction

New technological trends have created new control challenges in which current linear techniques are not adequate as the nonlinearities phenomena are no longer negligible. Therefore, new approaches must be developed. In contrast to the linear methods, the development of a general framework for control of nonlinear systems is hindered by the complexity of the nonlinear dynamics. Thus, the current nonlinear techniques are available only for special classes of systems [1], and in general, these are subjected to the need to solve partial differential equations (PDEs). Furthermore, the vast majority of the nonlinear control methods only focus in the stability of the closed-loop system, without providing any insight into how to tune the control gains, and consequently, disregarding the performance of the closed-loop system. Nevertheless, in several cases, it is essential to ensure a prescribed performance to solve a task at hand, e.g., applications involving physical systems that require high precision such as those found in aerospace, medical, semiconductor manufacturing, among other industries.

While there exist several papers that achieve performance in terms of $L_2$ stability (see [2], [3], [4]), the literature to find gains to achieve performance in terms of oscillations and (or) rise time is scarce. In this line of research, we find [5] where the authors propose a methodology for tuning the damping gain in switched-mode power converters, and [6] where the authors investigate how by modifying the initial conditions of a dynamic controller, the transient response of a class of mechanical system can be improved.

On the linear counterpart, there exist several tuning methodologies mostly oriented for PID controllers, which is not surprising since PIDs dominate overwhelmingly in industrial applications due to its simple strategy, where the main implementation problem is reduced to select the suitable gains: Proportional, Integral, and Derivative gains [7]. The most used rules to tune such gains are the methods proposed by Ziegler and Nichols (ZN) [8] for Single-Input Single-Output (SISO) systems. Other examples of tuning methodologies for PID controllers may be found in [9], [10], and [11]. However, some disadvantages of the aforementioned methods include the use of heuristic approaches to derive the rules, the use of a first or second-order time-delay model to approximate the real plant or the necessity of solving complex optimization problems that involve linear matrix inequalities.

Due to the simple structure of PID controllers and the suitability of Passivity-Based Control (PBC) techniques to stabilize physical systems, recently, several authors have paid particular attention to the so-called PID-PBC approach, see for instance [3], [4], [12], [13]. Some advantages of this approach are that the $L_2$ stability for the closed-loop system is guaranteed and, the methodology does not require the solution of PDEs.

*The work of Carmen Chan-Zheng is supported by the University of Costa Rica.
However, to the best of the authors’ knowledge, there is no available literature that provides guidelines on how to tune the gain of PID-PBCs.

Inspired by the seminal work of Brayton and Moser [14] and saddle point matrix theory, the main contribution of this work is the development of a method to select the gains of PID-PBCs such that these controllers are suitable to stabilize Multiple-Input Multiple-Output (MIMO) systems while ensuring that the transient response of the closed-loop system has no overshoot or has a damping ratio with prescribed bounds, and has a prescribed upper bound for the rise time. The remainder of this paper is structured as follows: in Section 2 we provide the preliminaries and the problem formulation. In Section 3 we present the main results of this paper. In Section 4 we apply our tuning rules to control a 2DoF planar manipulator. We finalize the paper with some concluding remarks and future work in Section 5.

Notation: We denote the $n \times n$ identity matrix as $I_n$ and the $n \times m$ matrix of zeros as $0_{n \times m}$. For a given smooth function $f : \mathbb{R}^m \to \mathbb{R}$, we define the differential operator $\nabla_x f := (\frac{\partial f}{\partial x})^T$ and $\nabla^2_x f := \frac{\partial^2 f}{\partial x^2}$. For a smooth mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, we define the $ij$–element of its $n \times m$ Jacobian matrix as $(\nabla_x F)_{ij} := \frac{\partial F_i}{\partial x_j}$. When clear from the context the subindex in $\nabla$ is omitted. Given a distinguished element $x_* \in \mathbb{R}^n$ we define the constant matrix $F_* := F(x_*) \in \mathbb{R}^{n \times m}$. For a given matrix $A \in \mathbb{R}^{n \times n}$, we denote by $\lambda_{\text{min}}(A)$ and $\lambda_{\text{max}}(A)$ the minimum eigenvalue and the maximum eigenvalue of $A$, respectively. We say that $A$ is positive semi-definite, denoted as $A \succeq 0$, if $A = A^T$ and $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$, and positive-definite, denoted as $A > 0$, if its symmetric and $x^T A x > 0$ for all $x \neq 0$. For a positive (semi-)definite matrix $A$ and a vector $x \in \mathbb{R}^n$, we define the weighted Euclidean norm as $\|x\|_A^2 := x^T A x$.

2 Preliminaries and problem setting

In this section, we summarize some properties of a class of saddle point matrices, which are the cornerstone in the development of the tuning rules presented in Section 3. Although not mentioned explicitly, in the pioneering work of Brayton and Moser [14], the authors use these properties to verify the behavior of the transient response. Then, we provide the port-Hamiltonian (pH) representation of the class of mechanical systems for which our tuning rules are suitable. Finally, we describe some details of the PID-Passivity-Based controller implemented in this work.

2.1 Some properties of a class of saddle point matrices

Consider the following linear system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\Phi \begin{bmatrix} x \\ y \end{bmatrix}, \quad \Phi := \begin{bmatrix} X & Z^T \\ -Z & Y \end{bmatrix}$$

(1)

where $X \in \mathbb{R}^{n \times n}$ is positive-definite, $Z \in \mathbb{R}^{m \times n}$ has full row rank, $Y \in \mathbb{R}^{m \times m}$ is positive semi-definite, $\dot{x} \in \mathbb{R}^n$, and $\dot{y} \in \mathbb{R}^m$, where $m \leq n$. The structure of $\Phi$ corresponds to a class of saddle point matrices, which reveals interesting properties of the system (1) through the analysis of the spectrum of $X$, $Y$, and $Z$. In particular, we are interested in the results below in Theorem 1 and Corollary 1 the proof of which may be found in [15] and [16], respectively.

Theorem 1 Assume $Y = 0_{m \times m}$. Let $(\lambda_\Phi, [v; w])$ be an eigenpair of $\Phi$. Then, $\lambda_\Phi$ is real if and only if:

$$\left( \frac{v^* X v}{v^* v} \right)^2 \geq 4 \frac{v^* (Z^T Z) v}{v^* v}.$$  

(2)

Furthermore, under some mild conditions, the entire spectrum of $\Phi$ is real, see Corollary 2.6 of [15].

Corollary 1 Assume $Y = 0_{m \times m}$. Let $X$ be positive-definite and let $\lambda_\Phi \in \mathbb{C}$ be any eigenvalue of $\Phi$. Then, the following statements are true:

i) If $\Re(\lambda_\Phi) \neq 0$, then

$$\frac{1}{2} \lambda_{\text{min}}(X) \leq \Re(\lambda_\Phi) \leq \frac{1}{2} \lambda_{\text{max}}(X)$$

(3)

ii) If $\Re(\lambda_\Phi) = 0$, then

$$\min\{\lambda_{\text{min}}(X), \lambda_{\text{min}}(Z X^{-1} Z^\top)\} \leq \lambda_\Phi \leq \lambda_{\text{max}}(X).$$

(4)
Throughout this work, we consider mechanical systems that admit a pH representation of the form

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix}
= \begin{bmatrix}
0_{n \times n} & I_n \\
-I_n & -D(q,p)
\end{bmatrix} \nabla H(q,p) + \begin{bmatrix}
0_{n \times m} \\
G
\end{bmatrix} u,
\]

\[
H(q,p) = \frac{1}{2} p^T M^{-1}(q)p + V(q)
\]

\[
y = G^T M^{-1}(q)p = G^T \dot{q}.
\]

where \(q, p \in \mathbb{R}^n\) are the generalized positions and momenta vectors, respectively, \(u \in \mathbb{R}^m\) is the control vector with \(m \leq n\), \(y \in \mathbb{R}^n\) is the passive output, \(D(q,p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}\) is the damping matrix, which positive semi-definite, \(H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) is the Hamiltonian, \(M(q) : \mathbb{R}^n \to \mathbb{R}^{n \times n}\) is the mass-inertia matrix satisfying \(M(q) > 0\), \(V(q) : \mathbb{R}^n \to \mathbb{R}\) is the potential energy of the system, and \(G \in \mathbb{R}^{n \times m}\) is the input matrix, where \(\text{rank}(G) = m\).

To formulate the problem under study we, first, identify the set of assignable equilibria for (5), which is given by

\[
\mathcal{E} = \{(q,p) \in \mathbb{R}^n \times \mathbb{R}^n \mid G^T \nabla V(q) = 0, p = 0\}.
\]

Then, we consider PID-like passivity-based controllers of the form (e.g. [3, 4, 12])

\[
u = -K_p y - K_I (\gamma(q) + \kappa) - K_D \dot{y},
\]

where the gains \(K_P, K_I, K_D \in \mathbb{R}^{m \times m}\) are positive semi-definite, \(\gamma(q) := G^T q\), and \(\kappa \in \mathbb{R}^m\) is a constant vector that is used to assign the equilibrium for the closed-loop system, see [4] for further details. Hence, the closed-loop system (5)-(6) takes the form

\[
\begin{bmatrix}
\dot{q} \\
\dot{p}
\end{bmatrix} = \Upsilon^{-1}(q) F(q,p) \Upsilon^{-\top}(q) \nabla H_d(q,p)
\]

where

\[
H_d(q,p) := H(q,p) + \frac{1}{2} \|\gamma(q) + \kappa\|^2_{K_I} + \frac{1}{2} \|y\|^2_{K_P},
\]

\[
\Upsilon(q) := \begin{bmatrix}
I_n \\
G K_D \nabla q y & I_n + G K_D G^T M^{-1}(q)
\end{bmatrix},
\]

\[
F(q,p) := \begin{bmatrix}
0_{n \times n} \\
I_n \\
0_{n \times n} \\
-I_n & -D(q,p) - G K_D G^T
\end{bmatrix}.
\]

Now, we formulate the problem under study as follows. **Problem setting:** given \((q_*, 0_n) \in \mathcal{E}\), propose a method to choose the gains \(K_P, K_I, K_D\) in (6) such that the closed-loop system (7) has an asymptotically stable equilibrium at \((q_*, 0_n)\) and its transient response does not exhibit oscillations, has a prescribed damping ratio or rise time.

**Remark 1** The passivity properties of the control law (5) ensure that the closed-loop system (7) is \(\mathcal{L}_2\) stable, see [3] for further details.

**Remark 2** As indicated in [4], the implementation of the term \(K_D \dot{y}\) is subject to two conditions, namely, \(i\) \(y\) must be of relative degree one, and \(ii\) \(u\) can be expressed as a function of the state vector without singularities. However, for systems of the form (5), the mentioned conditions are always verified, where \(i\) is directly satisfied and some simple computations show that the controller has no singularities since the matrix

\[
\Psi(q) := I_m + K_D G^T M^{-1}(q) G
\]

has full rank for every \(K_D \geq 0\).

## 3 Tuning rules

In this section, first, we thoroughly describe our approach to obtain the tuning rules. To facilitate the analysis, we linearize the system and convert the drift vector field into a class of saddle point matrices by similarity transformation. Bear The main benefit of this particular form is that this reveals a clear relationship between the damping, the potential energy, and the kinetic energy, which is used later to propose the tuning rules.
3.1 Linearizing and obtaining the saddle point form

To obtain the linearized dynamics of (7), we introduce the following vectors:

\[ \tilde{q} := q - q^*, \quad \tilde{p} := p. \]  

(9)

Then, the linearized system around the equilibrium point \((q^*, 0)\) corresponds to:

\[
\begin{bmatrix}
\dot{\tilde{q}} \\
\dot{\tilde{p}}
\end{bmatrix} = \Upsilon^{-1} F_* \Upsilon^{-\top} \nabla^2 H_{ds} \begin{bmatrix}
\tilde{q} \\
\tilde{p}
\end{bmatrix}
\]  

(10)

where \(\Upsilon\) and \(F_*\) are defined as in (8). Next, in order to obtain the saddle point, we define

\[
\begin{align*}
\mathcal{R} &:= GK_P G^\top + D_*, \\
\mathcal{P} &:= GK_i G^\top + \nabla^2 V_*, \\
\mathcal{W} &:= GK_D G^\top + M_*
\end{align*}
\]

and \(\phi_P, \phi_W \in \mathbb{R}^{n \times n}\) are full rank matrices satisfying

\[
\begin{align*}
\mathcal{W}^{-1} &= \phi_W^\top \phi_W, \\
\mathcal{P} &= \phi_P^\top \phi_P.
\end{align*}
\]

(11)

Subsequently, we define the similarity transformation matrix \(T \in \mathbb{R}^{2n \times 2n}\) and new coordinates \(z \in \mathbb{R}^{2n}\) such that

\[
\begin{bmatrix}
\tilde{q} \\
\tilde{p}
\end{bmatrix} = T \begin{bmatrix}
\tilde{q} \\
\tilde{p}
\end{bmatrix}.
\]

(13)

Therefore, the linearized system in \(z\) coordinates corresponds to

\[
\dot{z} = -\mathcal{N} z,
\]

(14)

where \(\mathcal{N}\) belongs to a class of saddle point matrices as (1) with \(X := \phi_W \mathcal{R} \phi_W^\top, Z := \phi_P \phi_P^\top,\) and \(Y := 0_{n \times n}\).

Finally, to characterize the eigenvalues of \(\mathcal{N}\), denote with \((\lambda_N, v)\) an eigenpair of (14) with \(\lambda_N \in \mathbb{C}\) and \(v \in \mathbb{R}^n\). Then, \(\lambda_N\) is given by the following expression (see [15]):

\[
\lambda_N := \frac{1}{2} \left( \frac{v^* \phi_W \mathcal{R} \phi_W^\top v}{v^* v} \pm \sqrt{\left( \frac{v^* \phi_W \mathcal{R} \phi_W^\top v}{v^* v} \right)^2 - 4 \frac{v^* \phi_W \mathcal{P} \phi_W^\top v}{v^* v}} \right).
\]

(15)

Note that the terms \(\mathcal{R}, \mathcal{P}\) and \(\mathcal{W}\) are related to the damping injection, the potential energy and the kinetic energy, respectively. In the sequel, we demonstrate that if a specific relation holds, then the system presents a particular transient response.

Remark 3: The spectrum is invariant to similarity transformation.

3.2 Removing the overshoot

The oscillations of a transient response is characterized by the dominant pair of complex-conjugated poles of the system. The peak of such oscillations corresponds to the maximum overshoot of the system [17]. Here, we provide a condition such that system (10) presents a no-overshoot response. In other words, the matrix \(\mathcal{N}\) from system (10) must contain only real spectrum.

From Theorem [1], an eigenvalue of \(\mathcal{N}\) is real if and only if condition [2] holds, that is, the discriminant of [15] is nonnegative. Then, to extend condition [2] to all the eigenvalues of \(\mathcal{N}\), we propose the following:

Proposition 1: The spectrum of the system (10) is real and nonnegative if the following is satisfied:

\[
4\lambda_{\max}(\mathcal{P})\lambda_{\max}(\mathcal{W}) \leq \lambda_{\min}(\mathcal{R})^2
\]

(16)

Proof: Let \(\eta := \phi_W v\), then, expression [2] can be rewritten as:

\[
4 \frac{\eta^* \mathcal{P} \eta \eta^* \mathcal{W} \eta}{\eta^* \eta} \leq \left( \frac{v^* \mathcal{R} v}{v^* v} \right)^2.
\]

(17)
Therefore, if condition (16) holds, then inequality (2) is satisfied for any \( \lambda \) since

\[
4 \eta^* P \eta \eta^* W \eta \leq 4 \lambda_{\text{max}}(P) \lambda_{\text{max}}(W)
\]

\[
\leq (\lambda_{\text{min}}(R))^2
\]

\[
\leq \left( \frac{\eta^* R \eta}{\eta^* \eta} \right)^2.
\]

Any eigenvalue of \( \mathcal{N} \) is characterized by expression (15), then it follows that

\[
0 \leq \frac{\lambda_{\text{min}}(R)}{\lambda_{\text{max}}(W)} \leq \frac{\nu^* \phi^*_W \phi^*_W v}{\nu^* v} \implies \Re(\lambda) \leq 0.
\]

(19)

**Remark 4** Condition (16) is less conservative than Corollary 2.6 of [15].

**Remark 5** The equality case in (16) corresponds to a critical damped response.

### 3.3 Prescribing the bounds for the damping ratio

The tuning rule described with Proposition 1 might be restrictive for some applications that need a faster rise time. However, this is usually achieved at the expense of a transient response with overshoot and oscillations. If this performance is acceptable, we propose a rule to improve the rise time by tuning the bounds of the damping ratio of the spectrum of (10).

Denote with \( \lambda_N \in \mathbb{C} \) any eigenvalue of \( \mathcal{N} \), then, the standard definition of the damping ratio of \( \lambda_N \) is given by [18]:

\[
\zeta_N := \frac{|\Re(\lambda_N)|}{\sqrt{\Re(\lambda_N)^2 + \Im(\lambda_N)^2}}
\]

where \( 0 \leq \zeta_N \leq 1 \).

From (20), note that the damping ratio of the spectrum of \( \mathcal{N} \) belongs to the interval \([0, 1]\), which is conservative. We can rewrite the definition of (20) in terms of \( R, P \) and \( W \) and provide less conservative bounds, i.e.,

**Proposition 2** Denote with \( (\lambda_N, v) \) any eigenpair of (14) with \( \lambda_N \in \mathbb{C} \) and \( v \in \mathbb{R}^n \), then, the damping ratio of \( \lambda_N \) is given by

\[
\zeta_N := \frac{1}{2} \frac{v^* X v}{v^* v} \left( \sqrt{\frac{v^* Z^T Z v}{v^* v}} \right)^{-1}
\]

where this is bounded by

\[
\zeta_{\text{min}} \leq \zeta_N \leq \zeta_{\text{max}},
\]

where

\[
\zeta_{\text{min}} := \max \left\{ 0, \frac{\lambda_{\text{min}}(R)^2}{4 \lambda_{\text{max}}(W) \lambda_{\text{max}}(P)} \right\}
\]

\[
\zeta_{\text{max}} := \min \left\{ 1, \frac{\lambda_{\text{max}}(R)^2}{4 \lambda_{\text{min}}(W) \lambda_{\text{min}}(P)} \right\}.
\]

**Proof:** From the proof of Corollary 1 (see [16]), we have that:

\[
\Re(\lambda_N) = \frac{1}{2} \frac{v^* X v}{v^* v}.
\]

(24)

Next, note that expression (15) follows from solving the quadratic equation (see [15])

\[
\lambda_N^2 - \frac{v^* X v}{v^* v} \lambda_N + \frac{v^* Z^T Z v}{v^* v} = 0.
\]

(25)
Substituting \((24)\) in \((25)\) yields
\[
\frac{\nu^*Z^T Z \nu}{\nu^* \nu} = \Re(\lambda_N)^2 + \Im(\lambda_N)^2, \tag{26}
\]
Then, expression \((21)\) follows from substituting \((24)\) and \((26)\) in \((20)\). By rewriting \((21)\) we have that
\[
\zeta^2 = \frac{1}{4} \frac{(\eta^* R \eta)^2}{(\eta^* W \eta)(\eta^* P \eta)}, \tag{27}
\]
therefore, \((22)\) follows from:
\[
\frac{\lambda_{\min}(R)^2}{\lambda_{\max}(W) \lambda_{\max}(P)} \leq \frac{(\eta^* R \eta)^2}{(\eta^* W \eta)(\eta^* P \eta)} \leq \frac{\lambda_{\max}(R)^2}{\lambda_{\min}(W) \lambda_{\min}(P)}. \tag{28}
\]

### 3.4 Prescribing the upper bound for the rise time

In this section, we proceed to characterize the upper bound of the rise time for system \((10)\) based on the work of [16]. We define the rise time \(t_r \in \mathbb{R}^+\) as the time taken by the system to reach 98% of its steady state value. The rise time is influenced directly by the real part of the pole closest to the imaginary axis. Consider the following three scenarios:

**S1:** The spectrum of \(N\) is purely real.

**S2:** The elements of the spectrum of \(N\) have an imaginary part different from zero.

**S3:** Otherwise.

Based on this premise, we define \(t_{ru} \in \mathbb{R}^+\) as the nominal rise time of the system, then, we propose the following:

**Proposition 3** Denote with \(\Re(\lambda_u)\) the lower bound for the real part of the spectrum of \(N\). Then, the rise time of the response of \((14)\) is bounded from above by \(t_{ru} \in \mathbb{R}^+\) where this is defined as
\[
t_{ru} := \frac{4}{\Re(\lambda_u)} \tag{29}
\]
where \(\Re(\lambda_u)\) is given by
\[
\Re(\lambda_u) = \begin{cases} 
\min\{\lambda_{\min}(W^{-1} R), \lambda_{\min}(R^{-1} P)\} & \text{if } S1 \\
\frac{1}{2} \lambda_{\min}(W^{-1} R) & \text{if } S2 \\
\min\{\frac{1}{2} \lambda_{\min}(W^{-1} R), \lambda_{\min}(R^{-1} P)\} & \text{if } S3.
\end{cases} \tag{30}
\]

**Proof:** The decay ratio of System \((14)\) is bounded by \(\Re(\lambda_u)\), therefore, expression
\[
\exp^{-\Re(\lambda_u) t_{ru}} = 0.0183 \tag{31}
\]
calculates the upper bound of the time where all the outputs of the systems have reached 98% of the desired equilibrium point. Expression \((29)\) follows immediately by rearranging \((31)\). Finally, \((30)\) follows immediately from substituting \((11)\) in Corollary 1.

\(\square\)

**Remark 6** \(S1\) can be ensured by using Proposition 1 while \(S2\) can be ensured with the condition \(\frac{1}{4} \frac{\lambda_{\max}(R)^2}{\lambda_{\min}(W) \lambda_{\max}(P)} < 1\) from Proposition 3.

**Remark 7** Proposition 3 can be used as a tuning rule in combination with Propositions 1 and 2. For example, note that the pair \(\{\lambda_{\min}(R), \lambda_{\max}(P)\}\) is used in Proposition 1 to ensure a “no-overshoot” behavior, therefore, the pair \(\{\lambda_{\max}(R), \lambda_{\min}(P)\}\) can be used to prescribe the upper bound of the rise time.

**Remark 8** For implementation purposes, the expression \(\lambda_{\min}(X^{-1} Y)\) can be approximated with \(\frac{\lambda_{\min}(Y)}{\lambda_{\max}(X)}\), however, this might be conservative since \(\frac{\lambda_{\min}(Y)}{\lambda_{\max}(X)} \leq \lambda_{\min}(X^{-1} Y)\).
3.5 Discussion

Some additional observations from this section are discussed below:

i) **About the natural damping**: note that the tuning rules require some knowledge of the natural damping $D(q,p)$ of the system, which can be challenging in practice. Nevertheless, we stress the fact that the tuning rules will work even with a rough estimate as the closed-loop system will remain stable. Some caveats of working with the rough estimate include changes in the oscillatory behavior and deviation of the bounds. Such variations may provide some intuition about the real bounds of the natural damping. For example, when applying Proposition 1 to achieve a critically damped response, if the system presents an over-damped (resp. under-damped) response, then the real damping value is larger (resp. smaller) than the nominal.

ii) **Improving the performance of a stable system**: if the open-loop system \( \text{(5)} \) is stable, then, the controller \( \text{(6)} \) can be used to improve its performance.

iii) **The underactuated case**: if the lower bound of $D(q,p)$ is different from zero for the underactuated case, i.e. $m < n$, then, Proposition 1 can be applied to reduce the oscillations to the minimum of such system.

iv) **Region of attraction**: the proposed tuning rules are based on the linearized system, i.e., these rules are valid in a neighborhood of the nonlinear system around the equilibrium point \((q^*,0)\). This region corresponds to the domain of attraction (see [19] and [20] for methods to estimate this region).

4 Experimental Results

In this section, we use the energy-shaping plus damping injection controller described in [3] to stabilize a Quanser 2 DoF Rigid Planar Manipulator (see [21] for reference manual) at the desired equilibrium \((q^*,0_2)\) with $q^* = \text{col}(0.6, 0.8)$ and to prescribe a desired behavior to the transient response. The manipulator model is described as in \( \text{(5)} \) with $n = 2$, $V(q) = 0$, $G = I_2$ and

$$M(q) = \begin{bmatrix} a_1 + a_2 + 2b \cos(q_2) & a_2 + b \cos(q_2) \\ a_2 + b \cos(q_2) & a_2 \end{bmatrix}.$$  \(32\)

The system setup and parameters values\(^1\) are shown in Fig. 1 and Table 1 respectively.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{setup.png}
\caption{Experimental Setup: 2 DoF Planar Manipulator}
\end{figure}

\begin{table}[h]
\centering
\begin{tabular}{|l|c|}
\hline
Parameter & Value \\
\hline
$a_1$ & 0.1476 \\
$a_2$ & 0.0725 \\
b & 0.0858 \\
$D$ & diag(0.07, 0.03) \\
\hline
\end{tabular}
\caption{System parameters}
\end{table}

\(^1\)Nominal values obtained from the reference manual.
To illustrate the applicability of our tuning rules, we first obtain the results for a “No Rule” scenario for comparison purposes. Then, we perform the following experiments:

**E1:** system without oscillations (Proposition 1).

**E2:** system with oscillations (Proposition 2 with $0.4 \leq \zeta \leq 0.8$).

Table 2 contains the gains calculated by using Propositions 1 and 2 for each experiment. Furthermore, Table 3 presents the upper bound estimation for the rise time using Proposition 3 for each experiment. The results of the angular position trajectories for Link 1 (L1) and Link 2 (L2) are shown in Fig. 2 and Fig. 3, respectively.

For the reader convenience, Table 3 shows the experimental results for the rise time from both experiments. Furthermore, a video with the experimental results can be found in this link: https://youtu.be/aHPv-mKK_eI

| Table 2: Proportional, Integral and Derivative Gains |
|-----------------------------------------------|
| **No Rule** | **E1** | **E2** |
| $K_P$ | diag(1,0.5) | diag(7.3972,9.2) | diag(3.9136, 4.1710) |
| $K_I$ | diag(50,30) | diag(35,20) | diag(50,45) |
| $K_D$ | diag(0,0) | diag(0,0) | diag(0.08,0.15) |

| Table 3: Experimental rise time vs Nominal rise time |
|-----------------------------------------------|
| **No Rule** | **E1** | **E2** |
| L1 | L2 | L1 | L2 | L1 | L2 |
| Experimental (sec) | 0.662 | 0.2740 | 1.0160 | 2.1940 | 0.5680 | 0.3300 |
| Nominal (sec) | 3.397 | 1.846 | 0.966 |

Figure 2: Trajectories for angular position of Link 1. Initial conditions $(q, p) = (0, 0)$

Figure 3: Trajectories for angular position of Link 2. Initial conditions $(q, p) = (0, 0)$
Comparing E2 with E1 and “No Rule”, it can be seen clearly that there is a trade-off between oscillations and the rise time, i.e., the faster the rise time, the more overshoot/oscillations the transient response will exhibit. Additionally, note that by tuning the kinetic term in E2, this improved the settling time with respect to the “No Rule” scenario.

Finally, although the nominal values in Table 3 are conservative, the rise time of each output is upper bounded; therefore, we can ensure that every trajectory has reached the 98% of its final value by the nominal value. However, note that there is a particular case in E1 where the time taken for L2 is larger than the nominal. As mentioned in Section 3.5, as a consequence of working with a rough estimate of the natural damping, a deviation from the real value will occur. In this particular case, the nominal rise time is given by the expression

\[ t_{ru} = 4/\lambda_{\text{min}}(R^{-1}P) = 4\lambda_{\text{max}}(RP^{-1}), \]

where it can be seen that the rise time is proportional to the upper bound of the damping matrix \( R \). Consequently, this rule suggests that the real damping is actually larger than the nominal provided by the manufacturer.

5 Concluding remarks and future work

Our results have shown that transforming the pH structure into other coordinates reveals interesting spectral properties that can be used to improve the transient response for the nonlinear mechanical systems. Furthermore, it is clear from the tuning rules that there is an underlying relationship between the potential energy \( (P) \), the kinetic energy \( (W) \), and the damping \( (R) \), which combinations result in a specific transient response. As seen in the experiments, the proposed tuning rules can prescribe the desired performance in terms of the oscillation, the damping ratio, and the rise time to a nonlinear MIMO mechanical systems.

Possible future research includes providing a less conservative estimation of the region where our rules are valid. A strategy to achieve this is to provide tuning rules that do not rely on the linearization of the system, i.e., to find tuning rules that can be applied directly to the nonlinear system. Additionally, we expect to extend this methodology to other domains, such as electrical circuits or electro-mechanical systems.

References

[1] R. Ortega, J. A. L. Perez, P. J. Nicklasson, and H. J. Sira-Ramirez, *Passivity-based control of Euler-Lagrange systems: mechanical, electrical and electromechanical applications*. Springer Science & Business Media, 2013.

[2] A. J. Van der Schaft and A. Van Der Schaft, *L2-gain and passivity techniques in nonlinear control*, vol. 2. Springer, 2000.

[3] M. Zhang, P. Borja, R. Ortega, Z. Liu, and H. Su, “PID passivity-based control of port-Hamiltonian systems,” *IEEE Transactions on Automatic Control*, vol. 63, no. 4, pp. 1032–1044, 2017.

[4] P. Borja, R. Ortega, and J. M. A. Scherpen, “New Results on Stabilization of port-Hamiltonian Systems via PID Passivity-based Control,” *IEEE Transactions on Automatic Control*, pp. 1–1, 2020.

[5] D. Jeltsema and J. M. A. Scherpen, “Tuning of passivity-preserving controllers for switched-mode power converters,” *IEEE Transactions on Automatic Control*, vol. 49, no. 8, pp. 1333–1344, 2004.

[6] D. A. Dirksz and J. M. A. Scherpen, “Tuning of dynamic feedback control for nonlinear mechanical systems,” in *2013 European Control Conference (ECC)*, pp. 173–178, IEEE, 2013.

[7] K. J. Åström and T. Hägglund, “The future of PID control,” *Control engineering practice*, vol. 9, no. 11, pp. 1163–1175, 2001.

[8] J. Ziegler and N. Nichols, “Optimum settings for automatic controllers,” 1993.

[9] S. Skogestad, “Simple analytic rules for model reduction and PID controller tuning,” *Journal of process control*, vol. 13, no. 4, pp. 291–309, 2003.

[10] S. Boyd, M. Hast, and K. J. Åström, “MIMO PID tuning via iterated LMI restriction,” *International Journal of Robust and Nonlinear Control*, vol. 26, no. 8, pp. 1718–1731, 2016.

[11] I. Ruiz-López, G. Rodríguez-Jiménez, and M. García-Alvarado, “Robust MIMO PID controllers tuning based on complex/real ratio of the characteristic matrix eigenvalues,” *Chemical Engineering Science*, vol. 61, no. 13, pp. 4332–4340, 2006.
[12] J. G. Romero, A. Domaire, R. Ortega, and P. Borja, “Global stabilisation of underactuated mechanical systems via pid passivity-based control,” IFAC-PapersOnLine, vol. 50, no. 1, pp. 9577–9582, 2017.

[13] B. Jayawardhana, R. Ortega, E. Garcia-Canseco, and F. Castanos, “Passivity of nonlinear incremental systems: Application to pi stabilization of nonlinear rlc circuits,” Systems & control letters, vol. 56, no. 9-10, pp. 618–622, 2007.

[14] R. Brayton and J. Moser, “A theory of nonlinear networks. I,” Quarterly of Applied Mathematics, vol. 22, no. 1, pp. 1–33, 1964.

[15] M. Benzi and V. Simoncini, “On the eigenvalues of a class of saddle point matrices,” Numerische Mathematik, vol. 103, no. 2, pp. 173–196, 2006.

[16] S.-Q. Shen, T.-Z. Huang, and J. Yu, “Eigenvalue estimates for preconditioned nonsymmetric saddle point matrices,” SIAM journal on matrix analysis and applications, vol. 31, no. 5, pp. 2453–2476, 2010.

[17] B. T. Kulakowski, J. F. Gardner, and J. L. Shearer, Dynamic modeling and control of engineering systems. Cambridge University Press, 2007.

[18] K. J. Åström and R. M. Murray, Feedback systems: an introduction for scientists and engineers. Princeton university press, 2010.

[19] H. Khalil, Nonlinear systems, vol. 3. Prentice hall Upper Saddle River, NJ, 2002.

[20] T. Kloiber and P. Kotyczka, “Estimating and enlarging the domain of attraction in IDA-PBC,” in 2012 IEEE 51st IEEE Conference on Decision and Control (CDC), pp. 1852–1858, IEEE, 2012.

[21] Quanser, “2 DOF Serial Flexible Joint, Reference Manual,” vol. Doc. No. 800, Rev 1, 2013.