The gravitational two-body problem in the vicinity of the light ring: Insights from the black-hole-ring toy model

Shahar Hod

*The Ruppin Academic Center, Emeq Hefer 40250, Israel*

*and*

*The Hadassah Institute, Jerusalem 91010, Israel*

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Abstract

The physical properties of an axisymmetric black-hole-ring system are studied analytically within the framework of general relativity to second order in the dimensionless mass ratio $\mu \equiv m/M$. In particular, we analyze the asymptotic behaviors of the binding-energy and the total angular-momentum of the two-body system in the vicinity of the light ring at $R = 3M$, where the circular orbit becomes null. We find that both quantities diverge quadratically in $\mu(1 - 3M/R)^{-1}$ at the light ring. The reported divergent behavior of the physical quantities stems from the second order spin-orbit interaction between the black hole and the orbiting object (the dragging of inertial frames by the orbiting ring). It is shown that this composed black-hole-ring toy model captures some of the essential features of the conservative dynamics of the (astrophysically more interesting) black-hole-particle system. In particular, we show that both systems share the same quadratic divergent behavior of the physical quantities near the light ring. Moreover, we prove that both systems are characterized by the same ratio $\frac{E^{(2)}(R \to 3M)}{J^{(2)}(R \to 3M)} = \frac{1}{3\sqrt{3}}$, where $E^{(2)}$ and $J^{(2)}$ are the divergent second order (self-interaction) expansion coefficients of the binding-energies and the angular-momenta of the systems, respectively.
I. INTRODUCTION.

The gravitational two-body problem has attracted much attention over the years from both physicists and mathematicians, see [1–20] and references therein. As is well known, the two-body problem in Newtonian gravity is characterized by a particularly elegant and simple analytic solution [1]. In fact, Newtonian gravity provides a very accurate mathematical description for the dynamics of a wide variety of astrophysical binaries in the weak-field limit. However, the dynamics of close binaries composed of black holes and superdense neutron stars are characterized by strong-gravity effects which cannot be treated properly within the limited framework of Newtonian mechanics. Instead, the dynamics of compact binaries in the strong-gravity regime should be described by the (mathematically more complicated) equations of the general theory of relativity.

The Einstein equations which describe the dynamics of close binaries in the strong-gravity regime are very complex and have no compact analytic solution. Despite this fact, the two-body problem in general relativity can be tackled using an approximation method which is based on a perturbative treatment in powers of the symmetric mass ratio

$$\mu \equiv \frac{Mm}{(M + m)^2},$$

where $M$ and $m$ are the masses of the two compact objects. (We use gravitational units in which $G = c = 1$).

Two important physical quantities which characterize the circular two-body dynamics are the binding-energy and the total angular-momentum of the system. The binding energy $E_{\text{binding}}$ of the two-body system is given by the difference between the total gravitational energy of the system at infinity and the total gravitational energy of the system at a finite separation:

$$E_{\text{binding}}(x) = E(x = 0) - E(x).$$

Here

$$x \equiv \left[(M + m)\Omega\right]^{2/3}$$

is a convenient invariant (and dimensionless) coordinate constructed from the characteristic frequency $\Omega$ of the circular orbit [20, 21]. The binding-energy can be expanded in powers of
the dimensionless mass ratio $\mu$:

$$E_{\text{binding}}(x)/M = \sum_{k=1}^{\infty} E^{(k)}(x) \cdot \mu^k .$$  \hspace{1cm} (4)

Likewise, the angular-momentum $J$ of the two-body system can be expressed as a power series of the dimensionless mass ratio $\mu$:

$$J(x)/M^2 = \sum_{k=1}^{\infty} J^{(k)}(x) \cdot \mu^k .$$  \hspace{1cm} (5)

In the zeroth-order ($\mu \to 0$) approximation the spacetime metric is described by the physical parameters of the larger object (the central black hole) while the smaller object (the ‘test-particle’) follows a geodesic of the black-hole spacetime. In this test-particle limit the system is characterized by the well-known relations \[2\]

$$E^{(1)}(x) = \frac{1}{\sqrt{(1-3x)}} - 1 \quad \text{and} \quad J^{(1)}(x) = \frac{1}{\sqrt{x(1-3x)}} .$$  \hspace{1cm} (6)

Realistic astrophysical binaries are often composed of an orbiting object whose mass $m$ is smaller but non-negligible as compared to the mass $M$ of the central black hole. In these situations the zeroth-order (test-particle) approximation is no longer valid and one should take into account the gravitational self-force corrections to the orbit \[7–20\]. These corrections (which are second-order in the symmetric mass ratio $\mu$) take into account the finite mass of the orbiting object.

The gravitational self-force has two distinct contributions: (1) It is responsible for dissipative effects that cause the orbiting particle to emit gravitational waves \[6, 7\]. (2) The self-force due to the finite mass of the orbiting object is also responsible for conservative corrections [of order $O(\mu^2)$] to the binding-energy and to the total angular-momentum of the two-body system. Following Refs. \[9–20\], in the present study we shall focus on these conservative corrections to the orbit.

II. THE BLACK-HOLE-PARTICLE SYSTEM NEAR THE LIGHT RING

The conservative second-order (self-interaction) contributions to the binding-energy and to the total angular-momentum of the two-body (black-hole-particle) system, $E^{(2)}_{\text{particle}}(x)$ and $J^{(2)}_{\text{particle}}(x)$, were computed most recently in \[20, 22\]. The mathematical expressions of these
physical quantities are quite cumbersome \cite{20}, but a remarkably simple quadratic divergence
of both these quantities was observed in the vicinity of the unperturbed light ring \cite{23} at
\[ x = \frac{1}{3} \, 20: \]
\[
E_{\text{particle}}^{(2)}(z \to 0) = \left( \frac{1}{27} - \frac{1}{12} \zeta \right) \cdot z^{-2} \quad \text{and} \quad J_{\text{particle}}^{(2)}(z \to 0) = \left( \frac{1}{3\sqrt{3}} - \frac{\sqrt{3}}{4} \zeta \right) \cdot z^{-2}, \tag{7}
\]
where
\[
z \equiv 1 - 3x . \tag{8}
\]
Here \( \zeta \) is a “fudge” factor which was introduced in \cite{20}. Using numerical computations, the
value of this fudge factor was estimated in \cite{20} to be
\[
\zeta \approx 1 . \tag{9}
\]

We would like to point out that the physical quantities \( E^{(2)}(x) \) and \( J^{(2)}(x) \) satisfy the
simple relation [see Eq. (7)]
\[
\frac{E_{\text{particle}}^{(2)}(z \to 0)}{J_{\text{particle}}^{(2)}(z \to 0)} = \frac{1}{3\sqrt{3}} \tag{10}
\]
in the vicinity of the light ring. It is worth emphasizing that this relation holds true regardless
of the value of the (numerically computed) fudge factor \( \zeta \).

It is worth noting that the quadratic divergence of the physical quantities which charac-
terize the dynamics of the black-hole-particle system near the light ring [see Eq. (7)] could
only be inferred using numerical computations, see Fig. 2 of \cite{20}. The main goal of the
present Letter is to analyze a simple toy model which captures, at least qualitatively, some
important features of the fundamental two-body problem in general relativity. In particu-
lar, we would like to provide an analytical explanation for the quadratic divergence of the
self-interaction quantities \( E^{(2)}(x) \) and \( J^{(2)}(x) \) in the vicinity of the light ring.

III. THE BLACK-HOLE-RING SYSTEM NEAR THE LIGHT RING

In a recent paper \cite{24} we proposed to model the conservative behavior of the two-body
system using the analytically solvable model of a stationary axisymmetric ring of particles
in orbit around a central black hole. This composed system was analyzed in detail by
Will \cite{19}. We have shown \cite{24} that this toy model captures, at least qualitatively, some
important features of the conservative dynamics of the (astrophysically more relevant) black-hole-particle system \cite{25}. In particular, like the orbiting particle, the rotating ring can drag the generators of the black-hole horizon \cite{19}.

It is well-known \cite{19} that local inertial frames are dragged by an orbiting object. In fact, because of the dragging of inertial frames by the orbiting object, one can have a Schwarzschild-like black hole with zero angular-momentum \((J_H = 0)\) but with a non-zero angular-velocity \((\omega_H \neq 0)\) [see Eq. (12) below]. Specifically, it was found in \cite{19} that the angular-momentum of a black hole which is perturbed by an orbiting ring is given by \cite{26}

\[
J_H = 4M^3\omega_H - 8Mmx^3J^{(1)}(x) ,
\]  

(11)

where the leading-order dimensionless angular momentum of the ring \(J^{(1)}(x)\), is given by the expression \cite{6}. Thus, to first-order in the symmetric mass ratio \(\mu\) of the system [see Eq. (11)], a zero angular momentum black hole \((J_H = 0)\) in the composed black-hole-ring system is characterized by a non-zero angular velocity \(\omega_H\):

\[
M\omega_H = 2x^3J^{(1)}(x) \cdot \mu .
\]  

(12)

The relation (12) [and, in particular, the fact that \(\omega_H(J_H = 0) \neq 0\)] is a direct consequence of the dragging of inertial frames by the orbiting ring \cite{19,27}.

The binding-energy of the black-hole-ring system, \(E_{\text{binding}}(x)\), can be expanded in the form (4). From the results presented in \cite{19} for the composed black-hole-ring system one finds after some algebra that the leading-order expansion coefficient \(E^{(1)}_{\text{ring}}(x)\) [the coefficient of the \(O(\mu)\) term in the expansion (11)] is given by the expression \cite{6}. In addition, one finds \cite{19} that the \(O(\mu^2)\) contribution to the energy budget of the black-hole-ring system in the vicinity of the light ring \((z \to 0)\) is dominated by the divergent term

\[
\mu E^{(2)}_{\text{ring}}(z \to 0) = -\frac{4}{27z}M\omega_HJ^{(1)} .
\]  

(13)

The expression (13) represents a spin-orbit interaction between the spinning black hole (which is characterized by the horizon angular velocity \(\omega_H\)) and the rotating ring (which is characterized by the angular momentum \(mJ^{(1)}\)). It is worth emphasizing that \(\omega_H\) is linear in the mass \(m\) of the orbiting ring [see Eq. (12)]. Thus, this spin-orbit interaction term represents a second-order self-interaction term of order \(O(\mu^2)\). Taking cognizance of Eqs. (6), (12), and (13), we find \cite{28}

\[
E^{(2)}_{\text{ring}}(z \to 0) = -\frac{8}{243} \cdot z^{-2}
\]  

(14)
in the vicinity of the light ring.

Likewise, the total angular-momentum of the black-hole-ring system, \( J(x) \), can be expanded in the form \( [5] \). From the results presented in \( [19] \) for the composed black-hole-ring system one finds after some algebra that the leading-order expansion coefficient \( J_{\text{ring}}^{(1)}(x) \) [the coefficient of the \( O(\mu) \) term in the expansion \( [5] \)] is given by \( [6] \). In addition, one finds \( [19] \) that the \( O(\mu^2) \) contribution to the angular-momentum of the black-hole-ring system in the vicinity of the light ring \((z \to 0)\) is dominated by the divergent term

\[
\mu J_{\text{ring}}^{(2)}(z \to 0) = -\frac{4}{3}M\omega Hz^{-3/2} .
\]  

Taking cognizance of Eqs. \( [6] \), \( [12] \), and \( [15] \), we find \( [28] \)

\[
J_{\text{ring}}^{(2)}(z \to 0) = -\frac{8}{27\sqrt{3}} \cdot z^{-2}
\]

in the vicinity of the light ring.

It is worth emphasizing that the perturbation expansions \( [4] \) and \( [5] \) are valid in the regime \( E^{(2)}(x) \cdot \mu^2 \ll E^{(1)}(x) \cdot \mu \ll 1 \) [and likewise \( J^{(2)}(x) \cdot \mu^2 \ll J^{(1)}(x) \cdot \mu \ll 1 \)]. Thus, the divergent behaviors \( [14] \) and \( [16] \) are valid in the regime

\[
\mu^{2/3} \ll z \ll 1 .
\]  

Inspection of Eqs. \( [14] \) and \( [16] \) reveals that the binding-energy and the total angular-momentum of the black-hole-ring system diverge quadratically in \( \mu z^{-1} \) at the light ring. Remarkably, the physical quantities of the original black-hole-particle system share this same divergent behavior (that is, a quadratic divergence in \( \mu z^{-1} \)) in the vicinity of the light ring, see Eq. \( [7] \).

Moreover, the physical quantities \( E_{\text{ring}}^{(2)} \) and \( J_{\text{ring}}^{(2)} \) which characterize the black-hole-ring system satisfy the simple ratio

\[
\frac{E_{\text{ring}}^{(2)}(z \to 0)}{J_{\text{ring}}^{(2)}(z \to 0)} = \frac{1}{3\sqrt{3}},
\]

which is identical to the corresponding ratio \( \frac{E_{\text{particle}}^{(2)}(z \to 0)}{J_{\text{particle}}^{(2)}(z \to 0)} = \frac{1}{3\sqrt{3}} \) [see Eq. \( [10] \)] satisfied by the physical quantities of the original black-hole-particle system!
IV. SUMMARY AND DISCUSSION

The physical properties of a black-hole-ring system were analyzed in the vicinity of the photon orbit at $R = 3M$, where the circular orbit of the ring becomes null. We have shown that this simple toy model may capture some important features of the conservative dynamics of the (physically more interesting) black-hole-particle system. In particular, this model provides a simple analytic explanation for the quadratic divergence of the self-interaction quantities $E^{(2)}(x)$ and $J^{(2)}(x)$ in the vicinity of the light ring, see Eqs. (14) and (16).

Moreover, we have shown that the black-hole-particle system and the black-hole-ring system share the same relation

$$\frac{E^{(2)}_{\text{particle}}(R \to 3M)}{J^{(2)}_{\text{particle}}(R \to 3M)} = \frac{E^{(2)}_{\text{ring}}(R \to 3M)}{J^{(2)}_{\text{ring}}(R \to 3M)} = \frac{1}{3\sqrt{3}}$$

between the second-order expansion coefficients.

The present toy model suggests that the second-order spin-orbit interaction between the black hole and the orbiting object [the dragging of inertial frames by the orbiting object, see Eq. (12)] is the main element determining the observed (quadratic) divergent behavior of the self-interaction quantities in the vicinity of the light ring.

Finally, it is worth pointing out the simple relations [see Eqs. (7), (14), and (16)]

$$\frac{E^{(2)}_{\text{particle}}(R \to 3M)}{E^{(2)}_{\text{ring}}(R \to 3M)} = \frac{J^{(2)}_{\text{particle}}(R \to 3M)}{J^{(2)}_{\text{ring}}(R \to 3M)} = \frac{81}{32} \left( \zeta - \frac{4}{9} \right)$$

between the self-interaction quantities of the black-hole-particle system and the corresponding physical quantities of the black-hole-ring system. We note that these ratios would be equal to 1 if the fudge factor $\zeta$ [see Eq. (9)] equals $68/81$.

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[1] H. Goldstein, *Classical Mechanics*, Addison-Wesley, Reading (1980); P. A. Sundararajan, Ph.D. Thesis, Massachusetts Institute of Technology (2009).
[2] J. M. Bardeen, W. H. Press and S. A. Teukolsky, Astrophys. J. 178, 347 (1972).
[3] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, (Oxford University Press, New York, 1983).
[4] S. L. Shapiro and S. A. Teukolsky, *Black holes, white dwarfs, and neutron stars: The physics of compact objects* (Wiley, New York, 1983).
[5] V. Cardoso, A. S. Miranda, E. Berti, H. Witek and V. T. Zanchin, Phys. Rev. D 79, 064016 (2009).
[6] M. Favata, Phys. Rev. D 83, 024028 (2011).
[7] A. Ori and K. S. Thorne, Phys. Rev. D. 62, 124022 (2000); A. Buonanno and T. Damour, Phys. Rev. D 62, 064015 (2000).
[8] E. Poisson, Living Rev. Relativity 7, 6 (2004).
[9] C. O. Lousto, Class. and Quant. Grav. 22, S369 (2005).
[10] S. Detweiler, in *Mass and Motion in General Relativity*, edited by L. Blanchet, A. Spallicci, and B. Whiting (Springer, 2011).
[11] L. Barack, Class. and Quant. Grav. 26, 213001 (2009).
[12] S. Detweiler, Phys. Rev. D 77, 124026 (2008).
[13] N. Sago, L. Barack, and S. Detweiler, Phys. Rev. D 78, 124024 (2008).
[14] T. S. Keidl, A. G. Shah, J. L. Friedman, D. Kim, and L. R. Price, Phys. Rev. D 82, 124012 (2010).
[15] A. Shah, T. Keidl, J. Friedman, D. Kim, and L. Price, Phys. Rev. D 83, 064018 (2011).
[16] T. Damour, Phys. Rev. D 81, 024017 (2010).
[17] L. Barack and N. Sago, Phys. Rev. Lett. 102, 191101 (2009); L. Barack and N. Sago, Phys. Rev. D 81, 084021 (2010).
[18] M. Favata, Phys. Rev. D 83, 024027 (2011).
[19] C. M. Will, The astrophysical Journal 191, 521 (1974); C. M. Will, The astrophysical Journal 196, 41 (1975).
[20] S. Akcay, L. Barack, T. Damour, and N. Sago, Phys. Rev. D 86, 104041 (2012).
[21] Note that $x$ reduces to $M/R$ in the test-particle ($m \to 0$) limit, where $R$ is the Schwarzschild radial coordinate associated with the central black hole (the larger object).
[22] It should be emphasized that the authors of Ref. [20] focused on the *conservative* circular dynamics of the two-body system. It is only in this non dissipative regime that the energy...
and angular momentum of the system are conserved quantities.

[23] The light ring is often referred to as the null circular geodesic of the black-hole spacetime.

[24] S. Hod, Phys. Rev. D 87, 024036 (2013).

[25] It should be emphasized that this toy model, being axially-symmetric, can not describe the most important characteristic of the gravitational two-body problem: the emission of gravitational radiation. Nevertheless, following Refs. [9–20], in the present study we shall focus on the conservative dynamics of the two-body system. That is, following Refs. [9–20] we shall ignore the emission of gravitational waves.

[26] Note that the results presented in [19] are expressed in terms of the irreducible mass $M_{ir}$ of the black hole. For a ‘bare’ Schwarzschild black hole the irreducible mass coincides with the mass $M$ of the black hole. For the black-hole-ring system considered in [19] one finds $M_{ir} = M[1+O(\mu^2)]$. Taking cognizance of Eq. (6), one finds that the $O(\mu^2)$ difference between $M_{ir}$ and $M$ does not affect the leading-order divergent behaviors [see Eqs. (14) and (16) below] of the $O(\mu^2)$ correction terms. In addition, the results presented in [19] are expressed in terms of the dimensionless ratio $M_{ir}/R$, where $R$ is the proper circumferential radius of the ring. The invariant coordinate $x$ defined in (3) reduces to $M/R$ in the test-particle ($m \rightarrow 0$) limit. For finite $m$ values one has $x = \frac{M}{R}[1+O(\mu)]$. Taking cognizance of Eq. (6), one finds that the $O(\mu)$ difference between $x$ and $M/R$ does not affect the leading-order divergent behaviors [see Eqs. (14) and (16) below] of the $O(\mu^2)$ correction terms. Finally, we note that the symmetric mass ratio $\mu$ [see Eq. (1)] is closely related to the dimensionless mass ratio $q \equiv m/M$: $\mu = q[1+O(q)]$. Taking cognizance of Eq. (6), one finds that the leading-order divergent coefficients [see Eqs. (14) and (16) below] of the $O(\mu^2)$ correction terms would also describe the leading-order divergent coefficients of the corresponding $O(q^2)$ corrections terms.

[27] In particular, note that a ‘bare’ (unperturbed) Schwarzschild black hole is characterized by the relation $J_H = 4M^3\omega_H$, which implies the simple relation $\omega_H(J_H = 0) = 0$.

[28] Here we have used the fact that $x \rightarrow \frac{1}{3}$ in the vicinity of the light ring.