ON BALAZARD, SAIAS, AND YOR’S EQUIVALENCE TO THE RIEMANN HYPOTHESIS

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Abstract. Balazard, Saias, and Yor proved that the Riemann Hypothesis is equivalent to a certain weighted integral of the logarithm of the Riemann zeta-function along the critical line equaling zero. Assuming the Riemann Hypothesis, we investigate the rate at which a truncated version of this integral tends to zero, answering a question of Borwein, Bradley, and Crandall and disproving a conjecture of the same authors. A simple modification of our techniques gives a new proof of a classical Omega theorem for the function $S(t)$ in the theory of the Riemann zeta-function.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. In [1], Balazard, Saias, and Yor gave an elegant proof of the formula

$$\int_{\Re(s)=1/2} \frac{\log |\zeta(s)|}{|s|^2} |ds| = 2\pi \sum_{\beta>1/2} \log \left| \frac{\rho}{1-\rho} \right|,$$

where the sum runs over the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with real part strictly greater than $1/2$. Since the Riemann Hypothesis (RH) states that $\beta = 1/2$ for all the nontrivial zeros of $\zeta(s)$, it follows that RH is equivalent to the expression

$$\int_{\Re(s)=1/2} \frac{\log |\zeta(s)|}{|s|^2} |ds| = 0.$$  

(1.2)

This equivalence led Borwein, Bradley, and Crandall [2] to study the function

$$I(T) = \int_{-T}^{T} \frac{\log \left| \zeta\left(\frac{1}{2}+it\right) \right|}{\frac{1}{4} + t^2} \, dt.$$  

Since by (1.2), RH is equivalent to the assertion that $I(T) \to 0$ as $T \to \infty$, they asked the following question: What are the admissible positive values of $\alpha$ such that $I(T) = O(T^{-\alpha})$ as $T \to \infty$ on RH? Based upon numerical evidence, they conjectured that $I(T) = O(T^{-2})$.

In this note, we answer their question and disprove their conjecture by showing that $I(T) = O(T^{-\alpha})$ for any fixed positive $\alpha < 2$ as $T \to \infty$, but that $I(T) \neq O(T^{-2})$. Precisely, we prove the following theorem.

Theorem 1.1. Assume RH. Then we have

$$I(T) = O\left(\frac{1}{T^2 \left( \log \log T \right)^2} \right)$$  

(1.3)

and

$$I(T) = \Omega\left(\frac{1}{T^2 \left( \log \log T \right)^{3/2}} \right)$$  

(1.4)

as $T \to \infty$.

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Additionally, by estimating the tails of the integral in (1.1) we obtain an unconditional formula for $I(T)$ in terms of the nontrivial zeros of the Riemann zeta-function.

**Theorem 1.2.** For $T \geq 3$, we have

$$I(T) = 2\pi \sum_{-T \leq \gamma \leq T, \beta > 1/2} \log \left| \frac{\rho}{1 - \rho} \right| + O\left(\frac{1}{T^2 \log T}\right).$$

(1.5)

Through a straightforward modification of our argument it can be shown that the Lindelöf Hypothesis implies that the error term in (1.5) is $o(T^{-2} \log T)$ as $T \to \infty$. We remark that the proof of Theorem 1.2 does not give a new proof of (1.1) since we merely truncate the integral. However, we will show how to adapt the method used to prove Theorem 1.2 to give a simple, new proof of (1.1) that relies only on standard techniques in complex analysis.

In the final section, we give a new proof of a classical Omega theorem of Montgomery for the function $S(t)$.

2. Various lemmas

Our first two lemmas concern integrals of the logarithm of the Riemann zeta-function (one unconditional and the other conditional upon RH).

**Lemma 2.1.** Uniformly for $1 \leq c \leq 2$ and $t \geq 3$ we have

$$\int_c^{1/2} \left| \log \zeta(\sigma + it) \right| d\sigma \ll \log t.$$

**Proof.** See Lemma $\beta$ of Titchmarsh [11].

**Lemma 2.2.** Assume RH. Then for $t \geq T \geq 3$ we have

$$\int_T^t \log \left| \zeta(1/2 + iu) \right| du \ll \frac{\log t}{(\log \log t)^2}.$$

**Proof.** Under the assumption of RH, Cauchy’s theorem implies that

$$\int_T^t \log \left| \zeta(1/2 + iu) \right| du = -\int_{1/2}^{3/2} \arg \zeta(\sigma + it) d\sigma + \int_{1/2}^{3/2} \arg \zeta(\sigma + iT) d\sigma + O(1).$$

We will bound the first integral on the right-hand side of this equation. The second integral can be handled similarly.

Let $\sigma_t = 1/2 + (\log \log t)^{-1}$ and write

$$\int_{1/2}^{3/2} \arg \zeta(\sigma + it) d\sigma = I_1 + I_2 + I_3,$$

(2.1)

where $I_1$ is the portion of the integral over $[1/2, \sigma_t]$, $I_2$ is the portion over $[\sigma_t, 3/4]$, and $I_3$ is the portion over $[3/4, 3/2]$. By Theorem 13.21 of [9], we have $\arg \zeta(\sigma + it) \ll \log t / \log \log t$ for $\sigma \geq 1/2$. Thus,

$$I_1 \ll \frac{\log t}{(\log \log t)^2}.$$

For $\sigma_t \leq \sigma < 3/4$ it follows from Corollary 13.16 of [9] that $\arg \zeta(\sigma + it) \ll (\log t)^{(2-2\sigma)} / \log \log t$. Hence

$$I_2 \ll \frac{\log t}{(\log \log t)^2}.$$

Finally, Corollary 13.16 of [9] also implies that $\arg \zeta(\sigma + it) \ll (\log t)^{1/2}$ uniformly for $3/4 \leq \sigma \leq 3/2$, and we have

$$I_3 \ll (\log t)^{1/2}.$$

The lemma now follows by inserting the estimates for $I_1, I_2$ and $I_3$ into (2.1).
Next we prove two key lemmas which are used to prove the estimate \([\text{1.4}]\) in Theorem \([\text{1.1}]\).

**Lemma 2.3.** Assume RH. For any sequence of complex numbers \(\{r(n)\}\) let

\[
R(t) = \sum_{n \leq N} \frac{r(n)}{n^t}.
\]

Then uniformly for \(1/2 \leq \alpha \leq 2\), \(h \in \mathbb{R}\), \(N > 1\), \(T \geq 3\), and \(\varepsilon > 0\) we have

\[
\int_T^{2T} \log (\alpha + it + ih) |R(t)|^2 dt = T \sum_{m n \leq N} \frac{\Lambda(n)r(m)r(mn)}{n^{\alpha + ih \log n}} + O\left( N \log T N^{3/2 + \varepsilon} \sum_{n \leq N} |r(n)|^2 \right).
\]

**Proof.** Let \(c = 1 + (\log N)^{-1}\), \(R(s) = \sum_{n \leq N} r(n) n^{-s}\), and \(\overline{R}(s) = \sum_{n \leq N} \overline{r(n)} n^{-s}\). We shall consider the case \(1/2 \leq \alpha \leq c\). The remaining case \(c \leq \alpha \leq 2\) is treated similarly to \(I_3\) below.

By the elementary inequality \(2|ab| \leq |a|^2 + |b|^2\) it follows that

\[
|r(m)r(n)| \left( \frac{m}{n} \right)^{\sigma - \alpha} \leq \frac{1}{2} \left( |r(m)|^2 \Delta + |r(n)|^2 m^2(\sigma - \alpha) \right)
\]

for any \(\Delta > 0\). Thus,

\[
|\mathcal{R}(s - \alpha - ih)\overline{\mathcal{R}}(\alpha + ih - s)| \leq \sum_{m n \leq N} |r(m)r(n)| \left( \frac{m}{n} \right)^{\sigma - \alpha}
\]

\[
\ll \left( \Delta \sum_{m \leq N} \frac{1}{m^{2(\sigma - \alpha)}} + \frac{1}{\Delta} \sum_{m \leq N} m^{2(\sigma - \alpha)} \right) \sum_{n \leq N} |r(n)|^2
\]

\[
\ll \left( \Delta N^{1-2(\sigma - \alpha)} \log N + \frac{N^{1+2(\sigma - \alpha)}}{\Delta} \right) \sum_{n \leq N} |r(n)|^2
\]

uniformly for \(\alpha \leq \sigma \leq c\). Choosing \(\Delta = N^{2(\sigma - \alpha)}(\log N)^{-1/2}\), we conclude that

\[
|\mathcal{R}(s - \alpha - ih)\overline{\mathcal{R}}(\alpha + ih - s)| \ll N(\log N)^{1/2} \sum_{n \leq N} |r(n)|^2 \tag{2.2}
\]

uniformly for \(\alpha \leq \sigma \leq c\).

Let \(\mathcal{C}\) be the positively oriented rectangle with vertices at \(\alpha + i(T + h)\), \(c + i(T + h)\), \(c + i(2T + h)\), and \(\alpha + i(2T + h)\). We write

\[
i \int_{\mathcal{C}} \log \zeta(s) \mathcal{R}(s - \alpha - ih)\overline{\mathcal{R}}(\alpha + ih - s) \, ds = I_1 + I_2 + I_3 + I_4,
\]

where \(I_1, I_2, I_3, I_4\) are the parts of the integral over the left, bottom, right, and top edges of \(\mathcal{C}\), respectively. Cauchy’s theorem implies that

\[
I_1 + I_2 + I_3 + I_4 = 0.
\]

Thus, after an obvious variable change, we have

\[
\int_T^{2T} \log \zeta(\alpha + it + ih) |R(t)|^2 \, dt = -I_3 + O\left( |I_2| + |I_4| \right) \tag{2.3}
\]

By \((2.2)\) and Lemma \([\text{2.1}]\) we have

\[
|I_2| + |I_4| \ll N(\log NT)^{3/2} \sum_{n \leq N} |r(n)|^2. \tag{2.4}
\]

It remains to estimate \(I_3\).
In $I_3$, we express $\log \zeta(s)$ as an absolutely convergent Dirichlet, interchange summation and integration, and then integrate term-by-term to obtain

$$- I_3 = T \sum_{mn \leq N} \frac{\Lambda(n) r(m) r(mn)}{n^\alpha \log n} + O \left( \sum_{k=2}^{\infty} \sum_{m,n \leq N \atop n \neq km} \frac{\Lambda(k)}{k^\epsilon \log k} \frac{|r(m) r(n)|}{|n|} \left( \frac{n}{m} \right)^{c-\alpha} \right). \tag{2.5}$$

To bound the error term, we first note that

$$\sum_{m,n \leq N \atop n \neq km} \frac{|r(m) r(n)|}{|n|} \left( \frac{n}{m} \right)^{c-\alpha} \ll \Delta \sum_{n \leq N} \frac{|r(n)|^2}{m^{2(c-\alpha)} |n|} + \frac{1}{\Delta} \sum_{m \leq N} \frac{|r(m)|^2}{n^{2(c-\alpha)} |n|}$$

for any $\Delta > 0$. Next, using standard techniques, we have

$$\sum_{m \leq N \atop n \neq km} \frac{1}{m^{2(c-\alpha)} |n|} \ll N^{1-2(c-\alpha)(\log N)^2} \quad \text{and} \quad \sum_{n \leq N \atop n \neq km} \frac{n^{2(c-\alpha)}}{|n|} \ll N^{1+2(c-\alpha) \log N}$$

uniformly in $k$. Hence

$$\sum_{m,n \leq N \atop n \neq km} \frac{|r(m) r(n)|}{|n|} \left( \frac{n}{m} \right)^{c-\alpha} \ll \left( \Delta N^{1-2(c-\alpha)(\log N)^2} + \frac{N^{1+2(c-\alpha) \log N}}{\Delta} \right) \sum_{n \leq N} |r(n)|^2.$$

Choosing $\Delta = N^{2(c-\alpha) (\log N)^{-1/2}}$, it follows that the big-O term in (2.5) is

$$\ll N^{(\log N)^{3/2}} \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^\epsilon \log k} \sum_{n \leq N} |r(n)|^2 \ll N^{(\log N)^{3/2} \log \log N} \sum_{n \leq N} |r(n)|^2.$$

The lemma now follows from this estimate and (2.5)–(2.6). \qed

**Lemma 2.4.** Let $\mu$ and $\nu$ be fixed non-negative integers, $N > 1$, and $h \in [0, (\log \log N)^{-1}]$. Then there exist two real-valued arithmetic functions $r^+(n)$ and a positive constant $C$ (depending on $\mu$ and $\nu$) such that

$$\sum_{mn \leq N} \frac{\Lambda(n) \mu^+(h \log n) r^+(m) r^+(mn)}{\sqrt{n (\log n)^\nu}} \sum_{n \leq N} |r^+(n)|^2 \geq C h^\mu (\log N)^{1/2} (\log \log N)^{\mu - \nu + 1/2}$$

and

$$\sum_{mn \leq N} \frac{\Lambda(n) \mu^-(h \log n) r^-(m) r^-(mn)}{\sqrt{n (\log n)^\nu}} \sum_{n \leq N} |r^-(n)|^2 \leq -C h^\mu (\log N)^{1/2} (\log \log N)^{\mu - \nu + 1/2}.$$

**Proof.** Our proof of this lemma is based upon the ideas in the proof of Theorem 2.1 of Soundararajan [10]. We shall prove the first inequality. The second inequality can be proved similarly by choosing $r^-(n) = \mu(n) r(n)$, where $\mu(n)$ is the M"obius function and $r(n)$ is defined below. Throughout the proof, the letter $p$ denotes a prime number.

We choose $r^+(n)$ to be the multiplicative function $r(n)$ supported on square-free integers and defined on primes $p$ by

$$r(p) = \begin{cases} \frac{L((\log p)^\nu}{\sqrt{p}}, & \text{if } A < p < B, \\ 0, & \text{otherwise.} \end{cases}$$
Therefore, the parameters \(A, B\) and \(L\) are chosen so that

\[
A = L^2(\log L)^{2\nu+1}, \quad B = L^2, \quad \text{and} \quad L^2(\log B)^{2\nu+1} = (2\nu + 1) \log N.
\]

We note that with our choice we have \(r(p) \ll 1, L \asymp (\log N)^{1/2}(\log \log N)^{-\nu-1/2}\), and \(\log B < (3/2) \log \log N\), so that \(\sin(h \log p) \gg (h \log p)/2\) for \(h \in [0, (\log \log N)^{-1}]\) and \(p < B\).

With \(r^+(n) = r(n)\), the denominator on the left-hand side of the first inequality is

\[
\sum_{n \leq N} |r(n)|^2 \leq \sum_{n=1}^{\infty} r(n)^2 = \prod_p (1 + r(p)^2). 
\]

To estimate the numerator, we use Rankin’s trick which asserts that for any sequence of non-negative real numbers \(\{a_n\}\), and any \(\alpha > 0\) we have

\[
\sum_{n > x} a_n \leq x^{-\alpha} \sum_{n > x} a_n n^{\alpha} \leq x^{-\alpha} \sum_{n=1}^{\infty} a_n n^{\alpha}.
\]

Therefore,

\[
\sum_{mn \leq N} \frac{\Lambda(n) \sin^\mu(h \log n)r(m)r(mn)}{\sqrt{n} (\log n)^\nu} = \sum_{n \leq N} \frac{\Lambda(n) \sin^\mu(h \log n)r(n)}{\sqrt{n} (\log n)^\nu} \sum_{m \leq N/n} r(m)^2 \\
= \sum_{n \leq N} \frac{\Lambda(n) \sin^\mu(h \log n)r(n)}{\sqrt{n} (\log n)^\nu} \prod_{p|n} (1 + r(p)^2) + O \left( h^\mu \sum_{n \leq N} \frac{\Lambda(n)r(n)}{\sqrt{n} (\log n)^{\nu-\mu}} \left( \frac{n}{N} \right)^\alpha \prod_{p|n} (1 + p^\alpha r(p)^2) \right). 
\]

Here we have used the inequality \(|\sin x| \leq x\) for \(x \geq 0\) in the big-\(O\) term. Note that \(r(n)\) is supported on square-free integers, and the inequalities \(\sin(h \log p) \gg h \log p\) and \(r(p) \ll 1\) hold for all \(p < B\). Using these observations we see that the ratio of the main term in (2.6) to \(\sum_{n \leq N} |r(n)|^2\) is

\[
\gg \sum_{p \leq N} \frac{\sin^\mu(h \log p) \cdot r(p)}{\sqrt{p} (\log p)^{\nu-1}(1+r(p)^2)} = L \sum_{A < p < B} \frac{\sin^\mu(h \log p) \cdot \log p}{p (1+r(p)^2)} \\
= Lh^\mu \left( \frac{(\log B)^{\mu+1}}{\mu+1} - \frac{(\log A)^{\mu+1}}{\mu+1} \right) + O((\log B)^\mu) \\
\gg Lh^\mu (\log \log N)^{\mu+1} \gg h^\mu (\log N)^{1/2}(\log \log N)^{\mu-\nu+1/2}.
\]

On the other hand, the error term in (2.6) is

\[
\ll h^\mu N^{-\alpha} \left( \sum_{A < p < B} \frac{(\log p)^{\mu+1}}{p^{1-\alpha}(1+p^\alpha r(p)^2)} \right) \prod_p (1 + p^\alpha r(p)^2) \\
\ll (1 + \alpha \log B) h^\mu N^{-\alpha} \left( \sum_{A < p < B} \frac{(\log p)^{\mu+1}}{p} \right) \prod_p (1 + p^\alpha r(p)^2). 
\]
Note that $B = L^3$ and $L \ll (\log N)^{1/2}$. So by Rankin’s trick (with exponent taken to be 1/2) we have
\[
\sum_{n \leq N} |r(n)|^2 = \sum_{n=1}^{\infty} |r(n)|^2 + O \left( \frac{L^2}{N^{1/2}} \sum_{A < p < B} \frac{\log p)^{2\nu}}{\sqrt{p}} \right)
\]
\[
= \prod_p (1 + r(p)^2 + O \left( \frac{L^2}{N^{1/2}} B^{3/2} (\log B)^{2\nu} \right) \gg \prod_p (1 + r(p)^2).
\]
Choosing $\alpha = (\log L)^{-2}$, we see that the ratio of (2.7) to $\sum_{n \leq N} |r(n)|^2 \gg \prod_p (1 + r(p)^2)$ is
\[
\ll h^\mu L N^{-\alpha} (\log B)^{\mu+1} \prod_p \left( \frac{1 + p^\alpha r(p)^2}{1 + r(p)^2} \right)
\]
\[
\ll h^\mu L (\log B)^{\mu+1} \exp \left\{ \frac{-\alpha}{\log N} + \sum_{A < p < B} (p^\alpha - 1) \frac{L^2 (\log p)^{2\nu}}{p} \right\}
\]
\[
\ll h^\mu L (\log B)^{\mu+1} \exp \left\{ \frac{-\alpha}{\log N} + \frac{\alpha L^2}{2\nu + 1} \left( (\log B)^{2\nu+1} - (\log A)^{2\nu+1} \right) + O(\alpha^2 L^2 (\log B)^{2\nu+2}) \right\}
\]
\[
\ll h^\mu L (\log B)^{\mu+1} \exp \left\{ \frac{-\alpha L^2 (\log A)^{2\nu+1}}{2\nu + 1} \right\} = o(h^\mu (\log N)^{1/2} (\log \log N)^{\mu-\nu+1/2})
\]
since $L (\log B)^{\mu+1} \ll (\log N)^{1/2} (\log \log N)^{\mu-\nu+1/2}$ by our choices of $A, B,$ and $L$. Combining the estimates, the lemma follows. \qed

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Our proof of (1.3) follows from Lemma 2.2 while our proof of (1.4) is a consequence of the following Omega theorem.

**Theorem 3.1.** Assume RH. Then as $t \to \infty$, we have
\[
\int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + it)| \, du = \Omega_{\pm} \left( h \sqrt{\frac{\log t}{\log \log t}} \right)
\]
uniformly for $h \in [0, (\log \log t)^{-1}]$.

**Proof.** We prove this theorem using Soundararajan’s resonance method.

Let $R(t) = \sum_{n \leq N} r(n)n^{-it}$ and observe that
\[
\max_{T \leq t \leq 2T} \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + it)| \, du \geq \frac{\int_{T}^{2T} \left\{ \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + it)| \, du \right\} |R(t)|^2 dt}{\int_{T}^{2T} |R(t)|^2 dt} \quad (3.1)
\]
and
\[
\min_{T \leq t \leq 2T} \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + it)| \, du \leq \frac{\int_{T}^{2T} \left\{ \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + it)| \, du \right\} |R(t)|^2 dt}{\int_{T}^{2T} |R(t)|^2 dt}. \quad (3.2)
\]
Making the substitution $u = t + h_1$, using Lemma 2.3 with $\alpha = 1/2$, and integrating with respect to $h_1$, the double integral in the numerators in (3.1) and (3.2) is
\[
= \Re \int_{-h}^{h} \int_{T}^{2T} \log \zeta(\frac{1}{2} + it + ih_1)|R(t)|^2 \, dt \, dh_1
\]
\[
= 2T \sum_{mn \leq N} \frac{\Lambda(n) r(m) \overline{r(mn)} \sin(h \log n)}{\sqrt{n(\log n)^2}} + O \left( h N (\log T N)^{3/2+\varepsilon} \sum_{n \leq N} |r(n)|^2 \right). \quad (3.3)
\]
Furthermore, Montgomery and Vaughan’s mean-value theorem for Dirichlet polynomials (Corollary 3 of [8]) implies that

$$\int_T^{2T} |R(t)|^2 \, dt = (T + O(N)) \sum_{n \leq N} |r(n)|^2. \quad (3.4)$$

Choosing $N = T(\log T)^{-2}$, Lemma 2.4 and equations (3.1)–(3.4) imply that

$$\max_{T \leq t \leq 2T} \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + iu)| \, du \geq c_1 h \sqrt{\frac{\log T}{\log \log T}}$$

and

$$\min_{T \leq t \leq 2T} \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + iu)| \, du \leq -c_2 h \sqrt{\frac{\log T}{\log \log T}}$$

uniformly for $h \in [0, (\log \log N)^{-1}]$, where $c_1$ and $c_2$ are (computable) positive constants. The theorem follows. \hfill \Box

We now prove Theorem 1.1.

**Proof of Theorem 1.1.** We first prove (1.3). Assuming RH, (1.2) implies that

$$\int_{-\infty}^{\infty} \log |\zeta(\frac{1}{2} + it)| \frac{1}{\frac{1}{4} + t^2} \, dt = 0.$$

Since the integrand is even, it follows that

$$I(T) = -2 \int_T^{\infty} \log |\zeta(\frac{1}{2} + it)| \frac{1}{\frac{1}{4} + t^2} \, dt.$$

Integrating by parts and applying Lemma 2.2 we have

$$I(T) = -2 \int_T^{\infty} \frac{1}{\frac{1}{4} + t^2} \left( \int_T^{t} \log |\zeta(\frac{1}{2} + iu)| \, du \right) \, dt$$

$$= -4 \int_T^{\infty} \frac{t}{(\frac{1}{4} + t^2)^2} \left( \int_T^{t} \log |\zeta(\frac{1}{2} + iu)| \, du \right) \, dt$$

$$\ll \int_T^{\infty} \frac{1}{t^3 (\log \log t)^2} \, dt \ll \frac{1}{T^2 (\log \log T)^2}.$$

This completes the proof of (1.3).

We now prove (1.4). Let $h \in [0, (\log \log t)^{-1}]$ and suppose, for sake of contradiction, that

$$I(t) = o\left( \frac{1}{t^2} \sqrt[3]{\frac{\log t}{\log \log t^3}} \right).$$

Then for $t - h \leq u \leq t + h$ we have

$$I(u) - I(t - h) = o\left( \frac{1}{t^2} \sqrt[3]{\frac{\log t}{\log \log t^3}} \right), \quad (3.5)$$
as well. Integrating by parts yields
\[ \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + iu)| \, du = \int_{t-h}^{t+h} (\frac{1}{4} + u^2) \, d\left( \int_{\frac{1}{2} + iu}^{u} \frac{\log |\zeta(\frac{1}{2} + iv)|}{\frac{1}{4} + v^2} \, dv \right) \]
\[ = \left( \frac{1}{4} + (t+h)^2 \right) \int_{t-h}^{t+h} \frac{\log |\zeta(\frac{1}{2} + iv)|}{\frac{1}{4} + v^2} \, dv \]
\[ - \int_{t-h}^{t+h} 2u \left( \int_{\frac{1}{2} + iu}^{u} \frac{\log |\zeta(\frac{1}{2} + iv)|}{\frac{1}{4} + v^2} \, dv \right) \, du. \]

Using the assumption (3.5) twice, it follows that
\[ \int_{t-h}^{t+h} \log |\zeta(\frac{1}{2} + iu)| \, du = o\left( \sqrt{\frac{\log t}{(\log \log t)^3}} \right). \]

If \( h = (\log \log t)^{-1} \), this contradicts Theorem 3.1, and thus proves (1.4).

\[ \square \]

4. Proof of Theorem 1.2

In this section, we use contour integration to prove Theorem 1.2. We also show how this method can be modified to give a new proof of (1.1) that relies solely on standard techniques from complex analysis.

**Proof of Theorem 1.2** First, suppose that \( T \) is not an ordinate of a zero of \( \zeta(s) \) and consider
\[ \frac{1}{i} \int_{\frac{1}{2} + iT}^{\frac{1}{2} + i\infty} \frac{\log \zeta(s)}{s(1-s)} \, ds. \]

Let \( S \) be subset of the region \( \sigma > 1/2 \) and \( t > T \), that excludes all the horizontal segments \( 1/2 + i\gamma \) to \( \beta + i\gamma \). It follows that \( \log \zeta(s) \) is a single-valued analytic function in \( S \). Moreover, along each branch cut from \( 1/2 + i\gamma \) to \( \beta + i\gamma \) the values of \( \log \zeta(s) \) on the upper and lower cuts differ by \( 2\pi i \). Therefore, moving the contour in the above integral from \( \Re(s) = 1/2 \) to \( \Re(s) = \infty \) yields
\[ \frac{1}{i} \int_{\frac{1}{2} + iT}^{\frac{1}{2} + i\infty} \frac{\log \zeta(s)}{s(1-s)} \, ds = 2\pi i \sum_{\gamma > T} \int_{\beta + i\gamma}^{\beta + i\gamma} \frac{1}{s(1-s)} \, ds + \frac{1}{i} \int_{\frac{1}{2} + iT}^{\infty} \log \zeta(s) \, ds. \] (4.1)

Also, we have
\[ \int_{\beta + i\gamma}^{\beta + i\gamma} \frac{1}{s(1-s)} \, ds = \log(\rho) - \log\left( \frac{1}{2} + i\gamma \right) - \log(1 - \rho) + \log\left( \frac{1}{2} - i\gamma \right). \] (4.2)

For \( \sigma \geq 2 \) we have \( \log \zeta(s) \ll 2^{-\sigma} \) uniformly in \( t \). From this and Lemma 2.1 it follows that
\[ \int_{\frac{1}{2} + iT}^{\infty} \frac{\log \zeta(s)}{s(1-s)} \, ds \ll \frac{1}{T^2} \left( \int_{\frac{1}{2}}^{2} + \int_{2}^{\infty} \right) |\log \zeta(\sigma + iT)| \, d\sigma \ll \frac{1}{T^2} (\log T + 1). \]

Taking the real parts in (4.1), and using the above estimate and (4.2), we deduce that
\[ \int_{T}^{\infty} \frac{\log |\zeta(\frac{1}{2} + it)|}{\frac{1}{4} + t^2} \, dt = 2\pi i \sum_{\gamma > T} \log \left| \frac{\rho}{1 - \rho} \right| + O\left( \frac{1}{T^2} \log T \right). \]

Similarly, it can be shown that
\[ \int_{-\infty}^{-T} \frac{\log |\zeta(\frac{1}{2} + it)|}{\frac{1}{4} + t^2} \, dt = 2\pi i \sum_{\gamma < -T} \log \left| \frac{\rho}{1 - \rho} \right| + O\left( \frac{1}{T^2} \log T \right). \]
Combining these two estimates and then differencing the resulting formula with (1.1) completes the proof of the theorem in the case when \( T \neq \gamma \). If \( T = \gamma \), we note that for all sufficiently small \( \varepsilon > 0 \) the estimate in (1.5) holds for \( T = \gamma + \varepsilon \). The theorem now follows in this case by letting \( \varepsilon \to 0^+ \).

\[ \square \]

**Proof of (1.1).** Consider the integral

\[
\frac{1}{i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{\log((s-1)\zeta(s))}{s(1-s)} \, ds.
\]

Arguing as in the previous proof, we move the contour from \( \Re(s) = 1/2 \) to \( \Re(s) = \infty \) and deduce that

\[
\frac{1}{i} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \frac{\log((s-1)\zeta(s))}{s(1-s)} \, ds = 2\pi \sum_{\beta > 1/2} \int_{\beta + i\gamma}^{\beta + i\gamma} \frac{1}{s(1-s)} \, ds + \frac{1}{i} \int_{C} \frac{\log((s-1)\zeta(s))}{s(1-s)} \, ds,
\]

where \( C \) is the positively oriented circle centered at \( s = 1 \) with radius \( 1/4 \). By the calculus of residues and the fact that \( \lim_{s \to 1} ((s-1)\zeta(s)) = 1 \) the last integral equals zero. Thus, by this and (4.2), taking the real parts in (4.3) gives

\[
\int_{-\infty}^{\infty} \log \left| \frac{\frac{1}{2} + it}{\frac{1}{4} + t^2} \right| \, dt = 2\pi \sum_{\beta > 1/2} \log \left| \frac{\rho}{1 - \rho} \right|.
\]

Note that by residue calculus (or otherwise) we have

\[
\int_{-\infty}^{\infty} \log \left| \frac{\frac{1}{2} + it}{\frac{1}{4} + t^2} \right| \, dt = \frac{1}{2} \int_{-\infty}^{\infty} \log \left( \frac{1}{4} + t^2 \right) \, dt = 0.
\]

This completes the proof. \[ \square \]

5. **Montgomery’s Omega Theorem for \( S(t) \)**

Let \( N(t) \) denote the number of non-trivial zeros \( \rho = \beta + i\gamma \) of the Riemann zeta-function with \( 0 < \gamma \leq t \). It is well-known that

\[
N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O \left( \frac{1}{t} \right)
\]

for \( t \geq 10 \). Here, if \( t \) is not equal to an ordinate of a zero of \( \zeta(s) \), the function \( S(t) \) is defined by

\[
S(t) = \frac{1}{\pi} \Im \log \zeta \left( \frac{1}{2} + it \right),
\]

where the branch of logarithm is obtained by continuous variation along the line segments joining the points \( 2, 2 + it \), and \( \frac{1}{2} + it \), starting with \( \arg \zeta(2) = 0 \). If \( t \) corresponds to an ordinate of a zero of \( \zeta(s) \) we set

\[
S(t) = \frac{1}{2} \lim_{\varepsilon \to 0} \left\{ S(t+\varepsilon) + S(t-\varepsilon) \right\}.
\]

Assuming RH, it is known that

\[
|S(t)| \leq \left( \frac{1}{4} + o(1) \right) \frac{\log t}{\log \log t}
\]

as \( t \to \infty \) \[4\]. In this section, we illustrate how Lemmas \( 2.3 \) and \( 2.4 \) in \( \S 3 \) can be used to give a new proof of Montgomery’s result \( 7 \) that

\[
S(t) = \Omega_\pm \left( \sqrt{\frac{\log t}{\log \log t}} \right)
\]

(5.1)

...
assuming RH. Tsang [13] gave an alternate proof of (5.1). In contrast to the proofs of Montgomery and Tsang, our proof uses the resonance method.

Proof of (5.1). Define the auxiliary function

\[ S_1(t) = \int_0^t S(u) \, du \]

and note that

\[ \max_{t \leq u \leq t+h} \pm S(u) \geq \frac{1}{h} \int_t^{t+h} \pm S(u) \, du = \frac{\pm(S_1(t+h) - S_1(t))}{h}. \tag{5.2} \]

We use a result of Littlewood (see Theorem 3 of [6] or Theorem 9.9 of [12]) that

\[ S_1(t) = \frac{1}{\pi} \int_{1/2}^{2} \log |\zeta(\sigma + it)| \, d\sigma + O(1). \]

Now taking the real part of the integral in Lemma 2.3 and integrating with respect to \( \alpha \) from 1/2 to 2 yields

\[
\int_T^{2T} S_1(t+h) |R(t)|^2 \, dt = \frac{T}{\pi} \sum_{mn \leq N} \frac{\Lambda(n) r(m) \overline{r(mn)}}{\sqrt{n (\log n)^2}} \cos(h \log n) + O\left( T \sum_{n \leq N} |r(n)|^2 \right) \\
+ O\left( N(\log TN)^{1/2} \sum_{n \leq N} |r(n)|^2 \right) + O\left( \int_T^{2T} |R(t)|^2 \, dt \right).
\]

Choosing \( N = T(\log T)^{-2} \) and noting that

\[ \int_T^{2T} |R(t)|^2 \, dt = \left( T + O(N) \right) \sum_{n \leq N} |r(n)|^2, \]

we obtain

\[
\frac{\pm \int_T^{2T} (S_1(t+h) - S_1(t)) |R(t)|^2 \, dt}{\int_T^{2T} |R(t)|^2 \, dt} = \frac{2}{\pi} \sum_{mn \leq N} \frac{\Lambda(n) r(m) \overline{r(mn)}}{\sqrt{n (\log n)^2}} \sin^2 \left( \frac{h}{2} \log n \right) + O(1).
\]

Using Lemma 2.4 with \( \mu = \nu = 2 \) to estimate the ratio of sums on the right-hand side of the above expression, we deduce that

\[
\max_{T \leq t \leq 2T} \pm (S_1(t+h) - S_1(t)) \gg h^2 \sqrt{\log T \log \log T}
\]

uniformly for \( h \in [0, (\log \log N)^{-1}] \). Combining this inequality with the observation in (5.2) and choosing \( h = (\log \log N)^{-1} \), the estimate (5.1) follows. \( \square \)

We remark that using the resonance method in a different way, the estimate in (5.1) can be refined. In [3], assuming RH, it is shown that

\[
\max_{T \leq t \leq 2T} S(t) \geq \frac{1}{\pi} \sqrt{\frac{\log t}{\log \log t}} + O\left( \frac{\sqrt{\log t}}{\log \log t} \right)
\]

and

\[
\min_{T \leq t \leq 2T} S(t) \leq -\frac{1}{\pi} \sqrt{\frac{\log t}{\log \log t}} + O\left( \frac{\sqrt{\log t}}{\log \log t} \right).
\]

These are conditional analogues of Soundararajan’s unconditional Omega theorem for \( |\zeta(\frac{1}{2} + it)| \) in [10]. It does not seem, however, that the method in [3] can be modified to prove Theorem 3.1.
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