Non-uniform Continuity of the Generalized Camassa–Holm Equation in Besov Spaces

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Abstract
In this paper, we consider the Cauchy problem for the generalized Camassa–Holm equation proposed by Hakkaev and Kirchev (Commun Partial Differ Equ 30:761–781, 2005). We prove that the solution map of the generalized Camassa–Holm equation is not uniformly continuous on the initial data in Besov spaces. Our result includes the present work Li et al. (Differ Equ 269:8686–8700, 2020) on Camassa–Holm equation with \( Q = 1 \) and extends the previous non-uniform continuity in Sobolev spaces Mi and Mu (Monatsh Math 176:423–457, 2015) to Besov spaces.

Keywords Generalized Camassa–Holm equation · Non-uniform continuous dependence · Besov spaces

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1 Introduction

In this paper, we are concerned with the Cauchy problem for the generalized Camassa–Holm equation introduced by Hakkaev and Kirchev (2005)

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + \frac{(Q+2)(Q+1)}{2} u^2 \frac{\partial u}{\partial x} = \left( \frac{Q}{2} u^{Q-1} \frac{\partial u}{\partial x} + u^Q \frac{\partial u}{\partial x} \right)_{xx}, \\
 u(0, x) = u_0, \quad x \in \mathbb{R},
\end{array}
\right.
\end{align*}
\]

where \( Q \geq 1 \) is a positive integer and \( u(t, x) \) stands for the fluid velocity at time \( t \geq 0 \) in the spatial direction.

Hakkaev and Kirchev (2005) proved that the Cauchy problem for (1.1) is locally well-posed in Sobolev space \( H^s(\mathbb{R}) \) with \( s > \frac{3}{2} \). The local well-posed in the sense of Hadamard in Besov space \( B^s_{p,r}(\mathbb{R}) \), \( s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \), \( 1 \leq p, r \leq \infty \) or \( B^{\frac{3}{2}}_{2,1}(\mathbb{R}) \) were established in Mi and Mu (2013), Yan et al. (2014a, b), respectively. Mi and Mu (2015) showed that the Cauchy problem for (1.1) is locally well-posed in Sobolev space \( H^s \) with \( s > \frac{3}{2} \) for both the periodic and the non-periodic case. In addition, they proved that the solution map is not uniformly continuous.

When \( Q = 1 \), Eq. (1.1) is reduced to the famous Camassa–Holm (CH) equation

\[
\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} + 3u \frac{\partial u}{\partial x} = 2u_\chi u_{xx} + uu_{xxx},
\]

which was first found by Fokas and Fuchssteiner (1981) and later derived as a water wave model by Camassa and Holm (1993). The CH equation is completely integrable (Camassa and Holm 1993; Constantin 2001), has bi-Hamiltonian structure (Fokas and Fuchssteiner 1981; Constantin 1997), possesses an infinity of conservation laws and admits exact peakon solutions of the form \( ce^{-|x-ct|} \), describing an essential feature of the travelling waves of largest amplitude (Camassa and Holm 1993; Constantin and Escher 2007, 2011).

The local well-posedness and ill-posedness for the Cauchy problem of the CH equation (1.2) in Sobolev spaces and Besov spaces were studied in the series of papers (Constantin and Escher 1998a, b; Danchin 2001; Guo et al. 2019, 2022; Li and Yin 2016; Li et al. 2022). After the non-uniform continuity for some dispersive equations was studied by Kenig et al. (2001), non-uniform dependence of the CH equation has been studied by several authors. The first non-uniform dependence result for CH equation was established by Himonas and Misiołek (2005) in \( H^s(\mathbb{R}) \) with \( s \geq 2 \) on the circle using explicitly constructed travelling wave solutions and this was then sharpened to \( s > \frac{3}{2} \) in Himonas and Kenig (2009) on the line and Himonas et al. (2010) on the circle. The above-mentioned works utilize the method of approximate solutions in conjunction with delicate commutator and multiplier estimates. Danchin (2001, 2003) showed the local existence and uniqueness of strong solutions to (1.2) with the initial data in \( B^s_{p,r} \) for \( s > \max\{1 + \frac{1}{p}, \frac{3}{2}\} \), \( 1 \leq p \leq \infty \), \( 1 \leq r < \infty \). The continuous properties of the solutions on the initial data have been supplemented by Li and Yin (2016). Very recently, Li et al. (2020) have sharpened the results in Li and Yin (2016) by showing that the solution map is not uniformly continuous, which depends

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deeply on the estimates of the transfer equation in Besov space and the constructed high–low-frequency smooth initial data.

For studying the non-uniform continuity of the generalized Camassa–Holm equation, it is more convenient to express (1.1) in the following equivalent nonlocal form

\[
\begin{aligned}
&u_t + u \partial_x u = -\partial_x \left(1 - \partial_x^2 \right)^{-1} \left[ Q^2 + 3Q + \frac{Q}{2} u \partial_x u \right]^2 - 1 \left[ Q^2 + 3Q + \frac{Q}{2} u \partial_x u \right], \\
&u(0, x) = u_0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad x \in \mathbb{R}.
\end{aligned}
\]

(1.3)

Up to the present, there is no result for the non-uniform continuous dependence of Eq. (1.3) in Besov space and it seems more difficult due to the fact that the structure of (1.3) when \( Q \geq 2 \) is much more complicated than that of the CH equation. On the one hand, when bounding the core part \( u - u_0 + t(u_0) \partial_x u_0 \) by \( t^2 \) in the small time near \( t = 0 \), because of the high nonlinearity in convection term, the method developed for the CH equation in Li et al. (2020) will only aggravate this difficulty. To overcome this difficulty, we add a new term \( t \mathbf{P}(u_0) \), and by using the differential mean value theorem, it is transformed into the estimates of nonlocal term \( \mathbf{P}(u) - \mathbf{P}(u_0) \) and convective term \( u \partial_x u - (u_0) \partial_x u_0 \), which is valid for any bounded set in the above working Besov space. Thus, for any bounded set \( u_0 \) in working space, the corresponding solution \( \mathbf{S}_t(u_0) \) can be approximated by a function of first degree of time \( t \) with respect to initial data \( u_0 \). That is,

\[
\mathbf{S}_t(u_0) = u_0 - t(u_0) \partial_x u_0 + t \mathbf{P}(u_0) + t^2 \mathcal{O}(u_0),
\]

where \( \mathcal{O}(u_0) \) is a bounded quantity of some Besov norm of \( u_0 \). Therefore, the non-uniform continuity of \( \mathbf{S}_t(u_0) \) is transformed into that of \( u_0 - t(u_0) \partial_x u_0 + t \mathbf{P}(u_0) \), which is the essence in our paper and is different from that in Li et al. (2020). On the other hand, more precise product and multiplier estimates are necessary.

Our main result is stated as follows.

**Theorem 1.1** Let \( s > \max \left\{ 1 + \frac{1}{p}, \frac{3}{2} \right\} \), \( 1 \leq p, r \leq \infty \). The solution map \( u_0 \rightarrow \mathbf{S}_t(u_0) \) of the initial value problem (1.3) is not uniformly continuous from any bounded subset of \( B^s_{p,r}(\mathbb{R}) \) into \( \mathcal{C}([0, T]; B^s_{p,r}(\mathbb{R})) \). More precisely, there exist two sequences \( g_n \) and \( f_n \) such that

\[
\| f_n \|_{B^s_{p,r}} \lesssim 1, \quad \lim_{n \rightarrow \infty} \| g_n \|_{B^s_{p,r}} = 0,
\]

but

\[
\liminf_{n \rightarrow \infty} \| \mathbf{S}_t(f_n + g_n) - \mathbf{S}_t(f_n) \|_{B^s_{p,r}} \gtrsim t, \quad t \in [0, T_0],
\]

with small positive time \( T_0 \) for \( T_0 \leq T \).

**Remark 1.1** Our result includes the present work (Li et al. 2020) on Camassa–Holm equation with \( Q = 1 \).
Remark 1.2 Since $B_{2,2}^s = H^s$, our result covers and extends the previous non-uniform continuity in Sobolev space (Mi and Mu 2015) to Besov space.

Remark 1.3 Compared with the method developed for the CH equation in Li et al. (2020), we use the new term $u_0 - tu_0^Q \partial_x u_0 + t \tilde{P}(u_0)$ to approximate the solution $S_t(u_0)$ by using the differential mean value theorem. The proof is more concise and general which can work for the generalized CH-type equation. In fact, following the idea of our paper, our result can hold for the following CH-type equation:

$$u_t + u^Q \partial_x u = \tilde{P}(u), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R},$$

if $\tilde{P}$ satisfying $\tilde{P}(0) = 0$ and

$$\|\tilde{P}(u) - \tilde{P}(v)\|_{B_{r,s}^{-1}(\mathbb{R})} \lesssim \|u - v\|_{B_{p,s}^{-1}(\mathbb{R})} \|u\|_{B_{p,r}(\mathbb{R})}^{O-1},$$

$$\|\tilde{P}(u) - \tilde{P}(v)\|_{B_{p,r}(\mathbb{R})} \lesssim \|u - v\|_{B_{p,r}(\mathbb{R})} \|u\|_{B_{p,r}(\mathbb{R})}^{O_q},$$

$$\|\tilde{P}(u)\|_{B_{p,r}(\mathbb{R})} \lesssim \|u\|_{W^{1,\infty}(\mathbb{R})} \|u\|_{B_{p,r}(\mathbb{R})}^{O_q},$$

$$\|\tilde{P}(u) - \tilde{P}(v)\|_{B_{r,s}^{1}(\mathbb{R})} \lesssim \|u - v\|_{B_{r,s}^{1}(\mathbb{R})} \|u\|_{B_{r,s}(\mathbb{R})}^{O_q}.$$

Notations: Given a Banach space $X$, we denote the norm of a function on $X$ by $\| \cdot \|_X$, and

$$\| \cdot \|_{L^\infty_T(X)} = \sup_{0 \leq t \leq T} \| \cdot \|_X.$$

The symbol $A \lesssim B$ means that there is a uniform positive constant $C$ independent of $A$ and $B$ such that $A \leq CB$.

2 Littlewood–Paley Analysis

In this section, we will review the definition of Littlewood–Paley decomposition and nonhomogeneous Besov space, and then list some useful properties. For more details, the readers can refer to Bahouri et al. (2011).

There exists a couple of smooth functions $(\chi, \varphi)$ valued in $[0, 1]$, such that $\chi$ is supported in the ball $B \triangleq \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \}$, $\varphi$ is supported in the ring $C \triangleq \{ \xi \in \mathbb{R}^d : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \}$ and $\varphi \equiv 1$ for $\frac{4}{3} \leq |\xi| \leq \frac{3}{2}$. Moreover,

$$\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,$$

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,$$

$$|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j} \cdot) \cap \text{Supp } \varphi(2^{-j'} \cdot) = \emptyset,$$
Then, we can define the nonhomogeneous dyadic blocks $\Delta_j$ and nonhomogeneous low-frequency cut-off operator $S_j$ as follows:

$$
\begin{align*}
\Delta_j u &= 0, \text{ if } j \leq -2, \quad \Delta_{-1} u = \chi(D) u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \\
\Delta_j u &= \varphi(2^{-j} D) u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u), \text{ if } j \geq 0, \\
S_j u &= \sum_{j'=-\infty}^{j-1} \Delta_j' u.
\end{align*}
$$

**Definition 2.1** (Bahouri et al. (2011)) Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R}^d)$ consists of all tempered distributions $u$ such that

$$
||u||_{B^s_{p,r}(\mathbb{R}^d)} \triangleq \left\| (2^{js} ||\Delta_j u||_{L^p(\mathbb{R}^d)})_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.
$$

Then, we have the following product and multiplier estimates, which will play an important role in the estimates of nonlocal term $P(u)$ and convective term $u^Q \partial_3 u$.

**Lemma 2.1** (Bahouri et al. 2011) (1) Algebraic properties: \( \forall s > 0, B^s_{p,r}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \) is a Banach algebra. \( B^s_{p,r}(\mathbb{R}^d) \) is a Banach algebra \( \iff B^s_{p,\infty}(\mathbb{R}^d) \iff L^\infty(\mathbb{R}^d) \iff s > \frac{d}{p} \) or \( s = \frac{d}{p}, r = 1 \).

(2) For any \( s > 0 \) and \( 1 \leq p, r \leq \infty \), there exists a positive constant \( C = C(d, s, p, r) \) such that

$$
||uv||_{B^s_{p,r}(\mathbb{R}^d)} \leq C \left( ||u||_{L^\infty(\mathbb{R}^d)} ||v||_{B^s_{p,r}(\mathbb{R}^d)} + ||v||_{L^\infty(\mathbb{R}^d)} ||u||_{B^s_{p,r}(\mathbb{R}^d)} \right).
$$

(3) Let \( m \in \mathbb{R} \) and \( f \) be an \( S^m \) multiplier (i.e. \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is smooth and satisfies that \( \forall \alpha \in \mathbb{N}_d^d \), there exists a constant \( C_\alpha \) such that \( |\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^m - |\alpha| \) for all \( \xi \in \mathbb{R}^d \). Then, the operator \( f(D) \) is continuous from \( B^s_{p,r} \) to \( B^s_{p,r} \).

(4) Let \( 1 \leq p, r \leq \infty \) and \( s > \text{max}\{1 + \frac{1}{p}, \frac{3}{2}\} \). Then, we have

$$
||uv||_{B^{s-2}_{p,r}(\mathbb{R}^d)} \leq C ||u||_{B^{s-2}_{p,r}(\mathbb{R}^d)} ||v||_{B^{s-1}_{p,r}(\mathbb{R}^d)}.
$$

Hence, for the terms \( P(u) \) and \( P(v) \), we have

$$
||P(u) - P(v)||_{B^{s-1}_{p,r}(\mathbb{R}^d)} \lesssim ||u - v||_{B^{s-1}_{p,r}(\mathbb{R}^d)} ||u, v||_{B^{s-1}_{p,r}(\mathbb{R}^d)},
$$

$$
||P(u) - P(v)||_{B^s_{p,r}(\mathbb{R}^d)} \lesssim ||u - v||_{B^s_{p,r}(\mathbb{R}^d)} ||u, v||_{B^s_{p,r}(\mathbb{R}^d)},
$$

$$
||P(u)||_{B^{s+1}_{p,r}(\mathbb{R}^d)} \lesssim ||u||_{B^{s+1}_{p,r}(\mathbb{R}^d)} ||u||_{B^s_{p,r}(\mathbb{R}^d)}.
$$
Lemma 2.2 (Bahouri et al. 2011; Li and Yin 2017) Let $1 \leq p, r \leq \infty$ and $\sigma > - \min\{\frac{1}{p}, 1 - \frac{1}{p}\}$. There exists a constant $C = C(p, r, \sigma)$ such that for any smooth solution to the following linear transport equation:

$$\partial_t f + v \partial_x f = g, \quad f|_{t=0} = f_0.$$ 

We have

$$\sup_{s \in [0,t]} \| f(s) \|_{B^\sigma_{p,r}(\mathbb{R})} \leq C e^{C V_p(v, t)} \left( \| f_0 \|_{B^\sigma_{p,r}(\mathbb{R})} + \int_0^t \| g(\tau) \|_{B^\sigma_{p,r}(\mathbb{R})} d\tau \right), \quad (2.4)$$

with

$$V_p(v, t) = \begin{cases} \int_0^t \| \partial_x v(s) \|_{L^\infty(\mathbb{R})} ds, & \text{if } \sigma < 1 + \frac{1}{p}, \\
\int_0^t \| \partial_x v(s) \|_{B^\sigma_{p,r}(\mathbb{R})} ds, & \text{if } \sigma = 1 + \frac{1}{p} \text{ and } r > 1, \\
\int_0^t \| \partial_x v(s) \|_{B^{\sigma-1}_{p,r}(\mathbb{R})} ds, & \text{if } \sigma > 1 + \frac{1}{p} \text{ or } \{ \sigma = 1 + \frac{1}{p} \text{ and } r = 1 \}. \end{cases}$$

3 Non-uniform Continuous Dependence

In this section, we will give the proof of Theorem 1.1. Before proceeding further, we need to introduce several important propositions to show that the solution $S_t(u_0)$ can be approximated by $u_0 - t(u_0) Q \partial_x u_0 + tP(u_0)$ in a small time near $t = 0$.

Firstly, we establish the estimates of the difference between the solution $S_t(u_0)$ and initial data $u_0$ in different Besov norms. That is

Proposition 3.1 Assume that $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ with $1 \leq p, r \leq \infty$ and $\| u_0 \|_{B^s_{p,r}} \lesssim 1$. Then, under the assumptions of Theorem 1.1, we have

$$\| S_t(u_0) - u_0 \|_{B^{s-1}_{p,r}} \lesssim t \| u_0 \|_{B^{s-1}_{p,r}} \lesssim t \| u_0 \|_{B^s_{p,r}},$$

$$\| S_t(u_0) - u_0 \|_{B^s_{p,r}} \lesssim t (\| u_0 \|_{B^{s+1}_{p,r}} + \| u_0 \|_{B^{s-1}_{p,r}}),$$

$$\| S_t(u_0) - u_0 \|_{B^{s+1}_{p,r}} \lesssim t (\| u_0 \|_{B^{s+1}_{p,r}} + \| u_0 \|_{B^{s-1}_{p,r}}),$$

$$\| S_t(u_0) - u_0 - t \nu_0 \|_{B^s_{p,r}} \lesssim t^2 (\| u_0 \|_{B^{s+1}_{p,r}} + \| u_0 \|_{B^{s-1}_{p,r}}),$$

where $\nu_0 = -u_0^Q \partial_x u_0 + P(u_0)$.

Proof For simplicity, denote $u(t) = S_t(u_0)$. Firstly, according to the local well-posedness result (Mi and Mu 2013; Yan et al. 2014a,b), there exists a positive time $T = T(\| u_0 \|_{B^s_{p,r}}, s, p, r)$ such that the solution $u(t)$ belongs to $C([0, T]; B^s_{p,r})$. Moreover, by Lemmas 2.1–2.2, for all $t \in [0, T]$ and $\gamma \geq s - 1$, there holds

$$\| u(t) \|_{B^\gamma_{p,r}} \leq C \| u_0 \|_{B^\gamma_{p,r}}. \quad (3.1)$$
Now we shall estimate the different Besov norms of the term $u(t) - u_0$, which can be bounded by $t$ multiplying the corresponding Besov norms of initial data $u_0$.

For $t \in [0, T]$, using (4) with $v = 0$ and product estimates (2) in Lemma 2.1, we have from (3.1) that

$$
||u(t) - u_0||_{B^s_{p,r}} \leq \int_0^t ||\partial_\tau u||_{B^s_{p,r}} d\tau
\leq \int_0^t ||P(u)||_{B^s_{p,r}} d\tau + \int_0^t ||uQ\partial_\tau u||_{B^s_{p,r}} d\tau
\leq t(||u||_{L^\infty(B^s_{p,r})} + ||u||_{L^\infty(B^s_{p,r})} ||u_x||_{L^\infty(B^s_{p,r})})
\leq t(||u||_{L^\infty(B^s_{p,r})} + ||u||_{L^\infty(B^s_{p,r})} ||u_x||_{L^\infty(B^s_{p,r})})
\leq t(||u||_{B^s_{p,r}}^s + ||u||_{B^s_{p,r}} ||u_0||_{B^s_{p,r}}),
$$

where we have used that $B^{s-1}_{p,r}$ is a Banach algebra with $s - 1 > \max\{\frac{1}{p}, \frac{1}{2}\}$ in the fourth inequality.

Following the same procedure of estimates as above, we have

$$
||u(t) - u_0||_{B^s_{p,r}} \leq \int_0^t ||\partial_\tau u||_{B^s_{p,r}} d\tau
\leq \int_0^t ||P(u)||_{B^s_{p,r}} d\tau + \int_0^t ||uQ\partial_\tau u||_{B^s_{p,r}} d\tau
\leq t(||u||_{L^\infty(B^s_{p,r})} ||u||_{L^\infty(B^s_{p,r})})
\leq t(||u||_{B^s_{p,r}}^s + ||u||_{B^s_{p,r}} ||u_0||_{B^s_{p,r}}),
$$

and

$$
||u(t) - u_0||_{B^{s+1}_{p,r}} \leq \int_0^t ||\partial_\tau u||_{B^{s+1}_{p,r}} d\tau
\leq \int_0^t ||P(u)||_{B^{s+1}_{p,r}} d\tau + \int_0^t ||uQ\partial_\tau u||_{B^{s+1}_{p,r}} d\tau
\leq t(||u||_{L^\infty(B^{s+1}_{p,r})} ||u||_{L^\infty(B^{s+1}_{p,r})} + ||u||_{L^\infty(B^{s+1}_{p,r})} ||u||_{L^\infty(B^{s+1}_{p,r})})
\leq t(||u||_{B^{s+1}_{p,r}}^s + ||u||_{B^{s+1}_{p,r}} ||u_0||_{B^{s+1}_{p,r}}),
$$

where we have used $||u||_{B^{s+1}_{p,r}} \leq ||u||_{B^{s+1}_{p,r}} ||u||_{B^{s-1}_{p,r}}$, which can be proved by recurrence method in the third inequality. In fact, using product law (2) of Lemma 2.1, we have

$$
||uQ||_{B^{s+1}_{p,r}} \leq ||u||_{L^\infty} ||uQ^{-1}||_{B^{s+1}_{p,r}} + ||u||_{B^{s+1}_{p,r}} ||uQ^{-1}||_{L^\infty}
\leq ||u||_{B^{s-1}_{p,r}} ||uQ^{-1}||_{B^{s+1}_{p,r}} + ||u||_{B^{s+1}_{p,r}} ||u||_{B^{s+1}_{p,r}} ||u||_{B^{s+1}_{p,r}}.
$$
Next, we estimate the Besov norm for the term \( u(t) - u_0 - tv_0 \) which can be bounded by \( t^2 \) multiplying the Besov norms of initial data \( u_0 \). Using (4) and product estimates (2) in Lemma 2.1, we obtain from the first three results of Proposition 3.1 that

\[
\| u(t) - u_0 - tv_0 \|_{B^s_{p,r}} \leq \int_0^t \| \partial_t u - v_0 \|_{B^s_{p,r}} \, dt \\
\leq \int_0^t \| P(u) - P(u_0) \|_{B^s_{p,r}} \, dt + \int_0^t \| u^Q \partial_t u - u_0^Q \partial_t u_0 \|_{B^s_{p,r}} \, dt \\
\leq \int_0^t \| u(t) - u_0 \|_{B^s_{p,r}} \, dt + \int_0^t \| u(t) - u_0 \|_{B^{s-1}_{p,r}} \| u(\tau) \|_{B^s_{p,r}} \, d\tau \\
+ \int_0^t \| u(\tau) - u_0 \|_{B^{s+1}_{p,r}} \| u_0 \|_{B^s_{p,r}} \, d\tau \\
\leq C t^2 (\| u_0 \|_{B^{s+1}_{p,r}} + \| u_0 \|_{B^{s-1}_{p,r}} + \| u_0 \|_{B^{s+1}_{p,r}} + \| u_0 \|_{B^{s+2}_{p,r}} ).
\]

Thus, we complete the proof of Proposition 3.1. □

**Proof of Theorem 1.1** Let \( \hat{\phi} \in C^\infty_0(\mathbb{R}) \) be an even, real-valued and non-negative function on \( \mathbb{R} \) and satisfy

\[
\hat{\phi}(x) = \begin{cases} 
1, & \text{if } |x| \leq \frac{1}{4}, \\
0, & \text{if } |x| \geq \frac{1}{2}.
\end{cases}
\]

Define the high-frequency function \( f_n \) and the low-frequency function \( g_n \) by

\[
f_n = 2^{-ns} \phi(x) \sin \left( \frac{17}{12} 2^n x \right), \quad g_n = \frac{12}{17} 2^{-s} \hat{\phi}(x), \quad n \gg 1.
\]

It has been showed in Li et al. (2020) that \( \| f_n \|_{B^{s}_{p,r}} \lesssim 2^{n(\sigma-s)} \).

Set \( u^n_0 = f_n + g_n \), consider Eq. (1.3) with initial data \( u^n_0 \) and \( f_n \), respectively. Obviously, we have

\[\| u^n_0 - f_n \|_{B^s_{p,r}} = \| g_n \|_{B^s_{p,r}} \leq C 2^{-\frac{n}{\sigma}} .\]

which means that

\[
\lim_{n \to \infty} \| u^n_0 - f_n \|_{B^s_{p,r}} = 0.
\]
It is easy to show that \(|u_0^n, f_n| |B_{p,r}^{\sigma-1} \lesssim 2^{-\frac{n}{\sigma}}\) and

\[|u_0^n, f_n| |B_{p,r}^\sigma \leq C 2^{(\sigma-s)n} \quad \text{for} \quad \sigma \geq s,\]

which imply

\[
\left( |u_0^n| |B_{p,r}^{\sigma+1} + |u_0^n| |B_{p,r}^{\sigma+1} |u_0^n| |B_{p,r}^{\sigma+1} + |u_0^n| |B_{p,r}^{\sigma+1} |u_0^n| |B_{p,r}^{\sigma+1} \right) \lesssim 1,
\]

\[
\left( |f_n| |B_{p,r}^{\sigma+1} + |f_n| |B_{p,r}^{\sigma+1} |f_n| |B_{p,r}^{\sigma+1} + |f_n| |B_{p,r}^{\sigma+1} |f_n| |B_{p,r}^{\sigma+1} \right) \lesssim 1,
\]

Furthermore, since \(u_0^n\) and \(f_n\) are both bounded in \(B_{p,r}^s\), according to Proposition 3.1, we deduce that

\[
||S_t(u_0^n) - S_t(f_n)||_{B_{p,r}^s} \geq t \left( |u_0^n| Q \partial_x u_0^n - (f_n) Q \partial_x f_n - P(u_0^n) + P(f_n) \right)_{B_{p,r}^s} - ||g_n||_{B_{p,r}^s} - Ct^2
\]

\[
\geq t \left( |u_0^n| Q \partial_x u_0^n - (f_n) Q \partial_x f_n \right)_{B_{p,r}^s} - C 2^{-s} - Ct^2.
\]

(3.2)

Notice that

\[
(u_0^n) Q \partial_x u_0^n - (f_n) Q \partial_x f_n = g_n Q \partial_x f_n + (u_0^n) Q \partial_x g_n + ((u_0^n) Q - f_n Q - g_n Q) \partial_x f_n,
\]

using Lemma 2.1, after simple calculation, we obtain

\[
||(u_0^n) Q - f_n Q - g_n Q) \partial_x f_n| |_{B_{p,r}^s} \leq C \left( |(u_0^n) Q - f_n Q - g_n Q) ||L^\infty|| f_n||_{B_{p,r}^{s+1}} + C \left| \partial_x f_n \right|_{L^\infty} \right) \left( |(u_0^n) Q - f_n Q - g_n Q) ||B_{p,r}^s\right) \leq C 2^{-n(s-1)},
\]

which together with fact

\[
\lim_{n \to \infty} ||g_n Q \partial_x f_n| |_{B_{p,r}^s} \geq M
\]

for some positive \(M\) that has been showed in Li et al. (2020) yield

\[
\lim_{n \to \infty} ||S_t(f_n + g_n) - S_t(f_n)||_{B_{p,r}^s} \geq t \quad \text{for} \quad t \text{ small enough}.
\]

This completes the proof of Theorem 1.1.
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