Research on a family of n-Scroll Chaos Generators

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Abstract. This paper studies a family of n-scroll chaos generators using a modified Chua’s circuit. A mathematic model of the generators is established, the relationship between equilibrium points and scrolls is also analyzed, and a general theorem for generation of n-scroll chaos attractors is given. Numerical simulation is illustrated, showing excellent agreement with our theoretical predictions.

1. Introduction
The general Chua’s Circuit is a general circuit which generates chaotic double scroll attractors. See the detailed basic characteristics of Chua’s circuit in [1-7].

Based on the Chua’s circuit, this paper makes a deep study on a family of n-Scroll chaotic generators. First, the features of 3 piece-wise linear implementation of the nonlinearity are analyzed, then they are extended to multi piecewise linear sections. Then the equilibrium points are divided into 2 kinds and are deeply analyzed for them. Finally, based on previous deduction and analysis, a general theorem to generate n-scroll chaos attractors is proposed and discussed. The simulation results are consistent with the theorem very well, which also proves the proposed theorem.

2. The General Chua’s Circuit
The general Chua’s Circuit is shown in Fig 1:

Figure 1. The General Chua’s Circuit Diagram
where \(N_R\) is the Chua’s diode which can be expressed as \(h(V_{C1}) = G_b V_{C1} + 0.5(G_a-G_b)(|V_{C1}+B_p| - |V_{C1}-B_p|)\), and \(V_{C1}, V_{C2}\) stands for the voltage of \(C_1, C_2\), \(i_L\) is the current of \(L\), \(B_p\) is the threshold which drives the diode.
A general uniform formula of the Chua’s Circuit is described in [3-4]:

\[
\begin{align*}
\dot{x} &= \alpha [y - h(x)] \\
\dot{y} &= x - y + z \\
\dot{z} &= -\beta y
\end{align*}
\] (1)

Where \( h(x) \) is the characteristics of the Chua’s diode, uniformly expressed as \( h(x) = bx + 0.5(a-b)|(x+1)|+0.5(b-a)|(x-1) | \) in which \( a, b \) is positive integer stands for the slopes of the piecewise linearity of \( h(x) \). This is a typical circuit of double scroll attractor generator. \[5\]

3. n-Scroll Chaos Generators

3.1. generation of n-Scroll attractors

In order to generate n-Scroll attractors, [8-10] proposed a method to further extend the characteristics of the piecewise linearity in nonlinear part to \( 2^M - 1 \) piece, which described in following equation:

\[
h(x) = m_{M-1}x + \frac{1}{2} \sum_{i=1}^{M-1} (m_{i-1} - m_i)(|x + c_i| - |x - c_i|)
\] (2)

Where \( m \) and \( c \) are series which can be expressed as \( m = [m_0, m_1, m_2, \ldots, m_{M-1}] \) \( c = [c_1, c_2, \ldots, c_{M-1}] \). The lengths of the series are respectively \( M, M-1 \). The n-Scroll attractors could be realized only if a set of proper values of \( \alpha, \beta, m, c \) are specified.

To generate n-scroll attractors, [8][9] gave a series of values of \( \alpha = 9, \beta = 14.286 \),

- 2 scroll case: \( m = [-1/7 2/7], c = [0 1] \)
- 4 scroll case: \( m = [-1/7 2/7 -4/7 2/7], c = [0 1 2.15 3.6] \)
- 6 scroll case: \( m = [-1/7 2/7 -4/7 2/7 -4/7 2/7], c = [0 1 2.15 3.6 8.2 13] \)
- 3 scroll case: \( m = [0.9/7 -3/7 3.5/7 -2.4/7], c = [0 1 2.15 4] \)
- 5 scroll case: \( m = [0.9/7 -3/7 3.5/7 -2.7/7 4/7 -2.4/7], c = [0 1 2.15 3.6 6.2 9] \)
- 7 scroll case: \( m = [0.9/7 -3/7 3.5/7 -2.4/7 2.52/7 -1.68/7 2.52/7 -1.68/7], c = [1 2.15 3.6 6.2 9 14 25] \)

The simulation results are shown in Fig 2:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Phase diagram of x and y when n=2,3,4,5,6,7}
\end{figure}
3.2. Equilibrium Points Analysis

Figure 3 shows the composure of the previous 7-scroll attractors and the characteristics of \( h(x) \) for the convenience of analysis where red line stands for 7-scroll attractors while blue one is \( h(x) \).

**Figure 3.** 7-scroll attractors and \( h(x) \)

The equilibrium points can be derived from:

\[
\begin{align*}
\dot{x} &= \alpha[y - h(x)] = 0 \\
\dot{y} &= x - y + z = 0 \\
\dot{z} &= -\beta y = 0
\end{align*}
\]

Then Eq. 3 can be deduced as:

\[
\begin{align*}
h(x) &= 0 \\
x &= -z \\
y &= 0
\end{align*}
\]

The equilibrium point can be expressed as \((x_{eq0} 0 -x_{eq})\). From above it can be calculated through definition of \( h(x) = 0 \), which means the intersection points of \( h(x) \) with X axis.

\[
h(x) = m_{M-1}x + \frac{1}{2} \left( \sum_{j=1}^{M-1} (m_{r1} - m_j) \left| x + c_j \right| - \left| x - c_j \right| \right) = 0
\]

Apparently \((0 0 0)\) is one of the equilibrium point. \( x_{eq0} \) stands for the first equilibrium point, while the second equilibrium point \( x_{eq1} \) can be known from

\[
m_{u}c_1 + m_r (x_{eq1} - c_1) = 0 \\
x_{eq1} = c_1 - m_r c_1 / m_r
\]

for those \( r \geq 2 \) situations, \( x_{eqr} \) can be calculated from the following formulas:

\[
\begin{align*}
\therefore m_{u}c_1 + \sum_{j=1}^{r-1} m_j (c_{j+1} - c_j) + m_r (x_{eqr} - c_r) &= 0 \\
\therefore x_{eqr} &= c_r - \left( m_{u}c_1 + \sum_{j=1}^{r-1} m_j (c_{j+1} - c_j) \right) / m_r
\end{align*}
\]
Where \( r = 2, 3, \ldots, M-1 \) and \( x_{eqr} \) stands for the \( r \)th equilibrium point, because \( h(x) \) is symmetric about Y axis, then \( -x_{eqr} \) must also be another equilibrium point. So there is totally \( 2M-1 \) equilibrium points in the system.

For each equilibrium point, the Jacobi matrix can be derived from:
\[
J = \begin{pmatrix}
-\alpha h(x) & \alpha & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{pmatrix}
\] (8)

For \( (0 0 0) \) and \( (x_{eqr} 0 -x_{eqr}) \), their Jacobi matrixes are respectively:
\[
J_0 = \begin{pmatrix}
-\alpha m_0 & \alpha & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{pmatrix}, \quad J_i = \begin{pmatrix}
-\alpha m_i & \alpha & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{pmatrix}, \quad \text{where } i=1, 2, \ldots, M-1.
\]

So the equilibrium points can be divided into 2 kinds according their trace of Jacobi matrixes:

- The slope of \( h(x) \) which passes through every equilibrium point is negative. Through computation, the eigenvalues of Jacobi matrix for that kind of equilibrium points have such relations: one real eigenvalue is negative while the other two are conjugate whose real part is positive as following expressed.
  \( \lambda_1 < 0, \quad \lambda_2 = \lambda_3^* \quad \text{and} \quad \text{Re}(\lambda_2)>0 \)

The trace of the Jacobi matrix is described as the following:
\[
\text{trace}(J) = \frac{1}{(dV)} \frac{1}{dt} (dV) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} dx_i = \sum_{i=1}^{N} \frac{\partial F_i(x)}{\partial x_i} = \lambda_1 + \lambda_2 + \lambda_3
\]

It is easy to confirm that \( \text{trace}(J) < 0 \) which means this is a dissipation system. The track in the phase diagram must be convergent to the attractors.

- In another situation, the slope of \( h(x) \) which passes through every equilibrium point is positive, while the eigenvalues of Jacobi matrix for the equilibrium points have such relations: one real eigenvalue is positive while other two are conjugate whose real part is negative as following expressed.
  \( \lambda_1 > 0, \quad \lambda_2 = \lambda_3^* \quad \text{and} \quad \text{Re}(\lambda_2)<0 \)

Also It is easy to know that \( \text{trace}(J) > 0 \) which means this is a non-dissipation system. The track in the phase diagram must be emanative to attractors\(^{[11-16]}\). Simulations also proves this conclusion.

4. Results

To design n-Scroll generators, the set of values of series \( m \) and \( c \) must be properly specified. And the slopes of the piecewise linear section of \( h(x) \) is alternatively positive and negative which is summarized as following: (‘+’ stands for the slope of that piece is positive, ‘-’ means that the slope is negative)

- In the case when \( n \) is odd
  - \( n=3 \)
    - + - + - +
  - \( n=5 \)
    - + - + - + - + -
While in the case when \( n \) is even

- \( n=2 \)
- \( n=4 \)
- \( n=6 \)

Then after the previous analysis and deduction, it can be summarized as the theorem:

**Theorem:**

To generate \( n \)-Scroll attractors, the system must meet such criterion:

1. The system must be chaotic. One of criterion is that the maximum Lyapunov exponent of the system must be positive.
2. When \( n \) is odd, the series \( m \) has \( n+1 \) elements, and the number of piecewise section is \( 2n+1 \); While on the other hand, when \( n \) is even, \( m \) has \( n \) elements, and the number of piecewise section is \( 2n-1 \).
3. The slope of each piecewise linear section is alternatively positive and negative.
4. For this system, the intersection points of \( h(x) \) with \( X \) axis are equilibrium points. For one equilibrium point, if the slope of the piecewise part of \( h(x) \) on the point is positive, it’s easy to confirm that the trace of its Jacobi matrix is negative, so that equilibrium point is an attractor. While on the other hand, when the slope is negative, the trace of its Jacobi matrix is positive, so it is not an attractor.
5. The values of \( m \) and \( c \) must be properly specified.

5. **Conclusion**

Based on the Chua’s circuit, this paper makes a deep study on \( n \)-Scroll chaotic attractor generators. First, it analyzes the features of piecewise linear implementation of the nonlinearity, extended it to multi piecewise linear sections. Then the equilibrium points are divided into 2 kinds and are deeply analyzed. Finally, the theorem to generate \( n \)-scroll is also proposed and discussed.

The work still to come next is that: find the general rule to generate \( n \)-Scroll chaos attractors and apply it into chaotic secure communication, and then realize it on hardware.

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