HESSIAN ESTIMATES FOR DIRICHLET AND NEUMANN EIGENFUNCTIONS OF LAPLACIAN

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ABSTRACT. By methods of stochastic analysis on Riemannian manifolds, we develop two approaches to determine an explicit constant $c(D)$ for an $n$-dimensional compact manifold $D$ with boundary such that

$$\frac{1}{n} \|\phi\|_\infty \leq \|\text{Hess}\phi\|_\infty \leq c(D)\lambda \|\phi\|_\infty$$

holds for any Dirichlet eigenfunction $\phi$ of $-\Delta$ with eigenvalue $\lambda$. Our results provide the sharp Hessian estimate $\|\text{Hess}\phi\|_\infty \lesssim \lambda^{n+\frac{3}{4}}$. Corresponding Hessian estimates for Neumann eigenfunctions are derived in the second part of the paper.

1. INTRODUCTION

Let $D$ be an $n$-dimensional compact Riemannian manifold with smooth boundary $\partial D$. We write $(\phi,\lambda) \in \text{Eig}(\Delta)$ if $\phi$ is a Dirichlet eigenfunction of $-\Delta$ on $D$ with eigenvalue $\lambda > 0$, i.e. $-\Delta \phi = \lambda \phi$. We always assume eigenfunctions $\phi$ to be normalized in $L^2(D)$ such that $\|\phi\|_{L^2} = 1$. According to [16], there exist two positive constants $c_1(D)$ and $c_2(D)$ such that

$$c_1(D) \sqrt{\lambda} \|\phi\|_\infty \leq \|\nabla \phi\|_\infty \leq c_2(D) \sqrt{\lambda} \|\phi\|_\infty, \quad (\phi,\lambda) \in \text{Eig}(\Delta),$$

(1.1)

where we write $\|\nabla \phi\|_\infty := \|\nabla \phi\|_{L^\infty}$ for simplicity. An analogous statement for Neumann eigenfunctions has been derived by Hu, Shi and Xue [9]. Subsequently, by methods of stochastic analysis on Riemannian manifolds, Arnaudon, Thalmaier and Wang [2] determined explicit constants $c_1(D)$ and $c_2(D)$ in (1.1) for Dirichlet and Neumann eigenfunctions. From this, together with the uniform estimate of $\phi$ (see [8][7][12]),

$$\|\phi\|_\infty \leq c_D \lambda^{\frac{n+1}{4}},$$

for some positive constant $c_D$, the optimal uniform bound of the gradient writes as

$$\|\nabla \phi\|_\infty \lesssim \lambda^{\frac{n+1}{4}}.$$

Results of this type have been used to study gradient estimates for unit spectral projection operators and to give a new proof of Hörmander’s multiplier theorem, see [24][25][26].

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Concerning higher order estimates of eigenfunctions, not much is known. Very recently, Steinerberger [17] studied Laplacian eigenfunctions of $-\Delta$ with Dirichlet boundary conditions on bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary and proved a sharp Hessian estimate for the eigenfunctions which reads as

$$
\|\text{Hess} \phi\|_\infty \leq \lambda \frac{n+1}{n}
$$

where

$$
\|\text{Hess} \phi\|_\infty := \sup \{ \|\text{Hess} \phi(v, v)(x) : x \in \mathbb{R}^n, v \in \mathbb{R}^n, |v| = 1 \}. 
$$

To the best of our knowledge, higher order estimates of eigenfunctions for Euclidean domains first appeared in [6] (see Lemma C.1 in the Appendix there which is easily adapted to cover our situation).

It is a natural question under which geometric assumptions such estimates extend to compact manifolds (with boundary). Following the lines of [2], one may ask the question how for the Hessian to derive explicit numerical constants $C_1(D)$ and $C_2(D)$ such that

$$
C_1(D)\lambda \|\phi\|_\infty \leq \|\text{Hess} \phi\|_\infty \leq C_2(D)\lambda \|\phi\|_\infty, \quad (\phi, \lambda) \in \text{Eig}(\Delta). \quad (1.2)
$$

Note that for eigenfunctions of the Laplacian, one trivially has

$$
|\text{Hess} \phi| \geq \frac{1}{n} |\Delta \phi| = \frac{\lambda}{n} |\phi|,
$$

and thus there is always the obvious lower bound

$$
\frac{\|\text{Hess} \phi\|_\infty}{\|\phi\|_\infty} \geq \frac{\lambda}{n}.
$$

For this reason, we concentrate in the sequel on upper bounds for $\|\text{Hess} \phi\|_\infty/\|\phi\|_\infty$.

In [2] a derivative formula for Dirichlet eigenfunctions has been given from where an upper bound

$$
|\Delta + \text{Hess}^\# \phi| \leq C_1 \lambda \|\phi\|_\infty,
$$

on the boundary. For $x \in \partial D$ we use the convention that $x$ is defined up to the first hitting time $\tau_D = \inf\{t > 0 : X_t(x) \in \partial D\}$ of the boundary. For $x \in \partial D$ we use the convention that $X_t(x)$ is defined with lifetime $\tau_D = 0$; in this case the subsequent statements usually hold automatically.

Assume that $X_t$ is a Brownian motion on $D \setminus \partial D$ with generator $\frac{1}{2} \Delta$, and write $X_t(x)$ to indicate the starting point $X_0 = x$. Then $X_t(x)$ is defined up to the first hitting time $\tau_D = \inf\{t > 0 : X_t(x) \in \partial D\}$ of the boundary. For $x \in \partial D$ we use the convention that $X_t(x)$ is defined with lifetime $\tau_D = 0$; in this case the subsequent statements usually hold automatically.

Suppose that $Q_t : T_xD \to T_{X_t(x)}D$ is defined by

$$
DQ_t = -\frac{1}{2} \text{Ric}^\#(Q_t) dt, \quad Q_0 = \text{id},
$$

where $D := //d//^{-1}$ with $// := ///\partial D : T_xD \to T_{X_t(x)}D$ parallel transport along $X_t(x)$ and $\text{Ric}^\#(v)(w) = \text{Ric}(v, w)$ for $v, w \in TD$. Suppose that $(\phi, \lambda) \in \text{Eig}(\Delta)$. Then, for $v \in T_xM$ and any $k \in C^1([0, \infty); \mathbb{R})$, i.e., $k$ bounded with bounded derivative, the process

$$
k(t) e^{-\lambda t/2} \langle \nabla \phi(X_t), Q_t(v) \rangle - e^{-\lambda t/2} \phi(X_t) \int_0^t \langle k(s) Q_s v, //dB_s \rangle, \quad t \leq \tau_D
$$

is a martingale. From this, by taking expectation, a formula involving $\nabla \phi$ can be obtained which allows to derive an upper bound for $|\nabla \phi|$ on $D$ by estimating $|\nabla \phi|$ on the boundary $\partial D$ and carefully choosing the function $k$. Along this circle of ideas, our aim is to establish a similar strategy for the Hessian of an eigenfunction $\phi$.

In view of the fact that $P_t \phi = e^{-\lambda t/2} \phi$ where $P_t$ is the semigroup generated by $\frac{1}{2} \Delta$, we focus first on martingales which are appropriate for attaining uniform Hessian estimates of eigenfunctions. Let us start with some background on Bismut type formulas for second-order derivatives of heat semigroups. A second-order differential formula for the heat semigroup $P_t$ was first obtained by Elworthy and Li [5, 13] for a non-compact manifold, however with restrictions on the curvature of the manifold. An intrinsic formula for $\text{Hess} P_t \phi$ has been given by Stroock [18] for a compact Riemannian manifold, and a localized version of such a formula was obtained in [11, 13] adopting martingale arguments. For the Hessian of the Feynman-Kac semigroup of an operator $\Delta + V$ with a potential function $V$ on manifolds, we refer the reader to [14, 15, 19].
For a complete Riemannian manifold \( M \) without boundary, an appropriate version of a Bismut-type Hessian formula gives the following estimate (see [3, Corollary 4.3] and Lemma 2.2 or Corollary 3.2 with \( \sigma_1 = \sigma_2 = 0 \):

\[
\|\text{Hess} P_t f\|_{\infty} \leq \left( K_1 \sqrt{t} + \frac{K_2 t}{2} \right) e^{K_0 t} \|f\|_{\infty} + \frac{2}{t} e^{K_0 t} \|f\|_{\infty}
\]

where

\[
K_0 := \sup \left\{ -\text{Ric}(v, v) : y \in M, v \in T_y M, |v| = 1 \right\};
\]

\[
K_1 := \sup \| R(y) : y \in M \};
\]

\[
K_2 := \sup \{ |(d^* R + \nabla \text{Ric})^\phi(v, w)(y) : y \in M, v, w \in T_y M, |v| = |w| = 1 \}
\]

and

\[
|R(y) := \sup \left\{ \sum_{i,j=1}^n R(e_i, v, w, e_j)^2(y) : |v| \leq 1, |w| \leq 1 \right\}
\]

for an orthonormal base \( \{e_i\}_{i=1}^n \) of \( T_y M \).

Thus if \( f = \phi \) and \( (\phi, \lambda) \in \text{Eig}(\lambda) \), then

\[
\|\text{Hess} \phi\|_{\infty} \leq \left( K_1 \sqrt{t} + \frac{K_2 t}{2} \right) e^{(K_0 + \lambda/2) t} \|\phi\|_{\infty} + \frac{2 e^{(K_0 + \lambda/2) t}}{t} \|\phi\|_{\infty}
\]

for any \( t > 0 \). Letting \( t = \frac{1}{K_0 + \lambda/2} \) then yields the estimate

\[
\frac{\|\text{Hess} \phi\|_{\infty}}{\|\phi\|_{\infty}} \leq \left( K_1 \sqrt{\frac{2}{2K_0 + \lambda}} + \frac{K_2}{2K_0 + \lambda} \right) e^{(\lambda + 2K_0) t}.
\]

To carry over such results to (compact) manifolds \( D \) with boundary, the influence of the boundary has to be studied. In this paper, we shall adopt a martingale approach to the Hessian of Dirichlet eigenfunctions. This approach is based on the construction of a suitable martingale which builds a relation between \( \text{Hess} \phi \) and \( d^* \phi \) and then to estimate \( C_2(D) \) in (1.2) by searching for explicit constants \( C_1, C_2 \) and \( C_3 \) such that

\[
\|\text{Hess} \phi\|_{\infty} \leq C_1 \|\text{Hess} \phi\|_{\partial D, \infty} + C_2 \|\nabla \phi\|_{\partial D, \infty} + C_3 \|\nabla \phi\|_{\infty}
\]

where \( \|\text{Hess} \phi\|_{\partial D, \infty} := \sup_{x \in \partial D} |\text{Hess} \phi(x) \) and \( \|\nabla \phi\|_{\partial D, \infty} := \sup_{x \in \partial D} |\nabla \phi(x) \). The final estimate for \( |\text{Hess} \phi| \) is then received by combining the last inequality with estimate (1.1) in [2].

Let us start with the general principle behind the construction of the relevant martingale. Let \( k \in C_b^1((0, \infty); \mathbb{R}) \) and define an operator-valued process \( W_t^k : T_x D \otimes T_x D \to T_{X_t(x)} D \) as solution to the following covariant Itô equation

\[
\text{D}W_t^k(v, w) = R(\text{d}B_t, Q_t(k(t)v)) Q_t(w) - \frac{1}{2} (d^* R + \nabla \text{Ric})^\phi(Q_t(k(t)v), Q_t(w)) dt - \frac{1}{2} \text{Ric}^\phi(W_t^k(v, w)) dt,
\]

with initial condition \( W_0^k(v, w) = 0 \). Here the operator \( d^* R \) is defined by \( d^* R(v_1, v_2) := -\text{tr} \nabla R(v_1, v_2) v_2 \) and thus satisfies

\[
\langle d^* R(v_1, v_2), v_3 \rangle = \langle (\nabla_{v_3} \text{Ric}^\phi)(v_1), v_2 \rangle - \langle (\nabla_{v_2} \text{Ric}^\phi)(v_3), v_1 \rangle
\]

for all \( v_1, v_2, v_3 \in T_x D \) and \( x \in D \). Then the process

\[
M_t := e^{\frac{\lambda}{2} t/2} \text{Hess} \phi(Q_t(k(t)v), Q_t(v)) + e^{\frac{\lambda}{2} t/2} d\phi(W_t^k(v, v)) - e^{\frac{\lambda}{2} t/2} d\phi(Q_t(v)) \int_0^t \langle Q_s(k(s)v), \text{d}B_s \rangle
\]

is a martingale on \([0, \tau_D]\) in the sense that \( (M_{t \wedge \tau_D})_{t \geq 0} \) is a globally defined martingale where \( \tau_D = \inf\{t > 0 : X_t(x) \in \partial D \} \) denotes the first hitting time of \( X(x) \) of the boundary \( \partial D \). The martingale property of (1.5) then allows to establish an inequality of the type (1.4) by equating the expectations.
at time 0 and at time $t \wedge \tau_D$. This approach then requires to estimate the boundary values of $|d\phi|$ and $|\text{Hess}\phi|$, in order to obtain the wanted upper bound for $\|\text{Hess}\phi\|_{\infty}$. To this end, we establish the required estimates in Lemmas 2.4-2.5 by using the information on the second fundamental form $II$ and the second derivative of $N$, where for $X, Y \in T_x\partial D$ and $x \in \partial D$, the second fundamental form is defined by

$$II(X, Y) = -\langle \nabla X N, Y \rangle.$$ 

Finally, let

$$\ell(t) := \ell_{k,\sigma}(t) := \begin{cases} 
\cos \sqrt{kt} - \frac{\sigma}{\sqrt{k}} \sin \sqrt{kt}, & k > 0, \\
1 - \sigma t, & k = 0, \\
\cosh \sqrt{-kt} - \frac{\sigma}{\sqrt{-k}} \sinh \sqrt{-kt}, & k < 0,
\end{cases} \quad (1.6)$$

We state now the first main result of this paper.

**Theorem 1.1.** Let $D$ be a compact Riemannian manifold with smooth boundary $\partial D$. Let $K_0, K_1, K_2, \sigma$ be non negative constants such that $\text{Ric} \geq -K_0$, $|R| \leq K_1$ and $|d^*\text{Ric} + \nabla R\text{Cot}| \leq K_2$ on $D$, and that $|II| \leq \sigma$. Assume that the distance function $\rho_0$ is smooth on the tubular neighborhood $\partial_{r_1} D := \{ x \in D : \rho_0(x) \leq r_1 \}$ of $\partial D$. Let $k, \beta, \gamma$ be constants such that $|\text{Sect}| \leq k$ on $\partial_{r_1} D$, and that

$$|\nabla (\Delta \rho_0)| \leq \beta, \quad \|\Delta^2 \rho_0\| \leq \gamma \quad \text{on } \partial_{r_0} D,$$

where $r_0 = r_1 \wedge t^{-1}(1/2)$. Then for any non-trivial $(\phi, \lambda) \in \text{Eig}(\Delta)$,

$$\frac{\|\text{Hess}\phi\|_{\infty}}{\|\phi\|_{\infty}} \leq C_{\lambda}(D) \lambda$$

where

$$C_{\lambda}(D) \leq 2(n-1)e^{\sigma} \left( \frac{\alpha}{\lambda} + \sqrt{\frac{2}{\pi \lambda}} \right) + 2\alpha e^{\frac{1}{2}n\pi r_0 + \frac{1}{2}} \max \left\{ \frac{1 + 2K_0}{\lambda^2} + \frac{\sigma}{n r_0 + 2\sigma}, \frac{2e^{\frac{1}{2}n\pi r_0}}{\sqrt{\lambda}} \right\}$$

$$+ \frac{3}{r_0} \left( \frac{\alpha^2 + 2\beta}{\alpha^2 + 2\sigma} + \frac{6 \sigma}{\alpha + 2\sigma} \right) + \left( \frac{6}{\alpha + 2\sigma} \right) \lambda + \frac{\alpha}{\alpha + 2\sigma}$$

$$+ \sqrt{\frac{2}{\pi \lambda}} \left( \frac{2}{\alpha + 2\sigma} + 3\beta + \frac{K_1}{16 \sqrt{\pi}} \right) \left( \frac{1}{\alpha + 2\sigma} \right) + \left( \frac{1 + K_0}{\lambda^2} \right)$$

$$+ 4e^{\frac{1}{2}n\pi r_0 + \frac{1}{2}} \max \left\{ \sqrt{1 + \frac{2K_0}{\lambda^2} + \frac{\sigma}{n r_0 + 2\sigma}}, \frac{2e^{\frac{1}{2}n\pi r_0}}{\sqrt{\lambda}} \right\}$$

$$\times \left( \frac{2}{\alpha + 2\sigma} + 3\beta + \frac{K_1}{16 \sqrt{\pi}} \right) \left( \frac{1}{\alpha + 2\sigma} \right) + \left( \frac{1 + K_0}{\lambda^2} \right) \right\} \quad (1.8)$$

for $\alpha = 2(n-1)\max(\sigma, k)$.

**Remark 1.2.** 1) Adopting the estimate above, obviously $C_{\lambda}(D)$ is decreasing in $\lambda$, and hence $C_{\lambda}(D) \leq C_{\lambda_1}(D)$ where $\lambda_1$ is the first Dirichlet eigenvalue of $-\Delta$ which gives

$$\frac{\|\text{Hess}\phi\|_{\infty}}{\|\phi\|_{\infty}} \leq C_{\lambda_1}(D) \lambda.$$ 

2) If the manifold has constant sectional curvature and mean curvature on $\partial_{r_0} D$, i.e. $H = \theta$, $\text{Sect} = k$ on $\partial_{r_0} D$, then for $\rho_0(x) \leq t^{-1}(0) \wedge r_0$,

$$\Delta \rho_0 = \frac{t_{\theta, (n-1)k}}{t_{\theta, (n-1)k}^{\rho_0}}(\rho_0).$$
As a consequence, the upper bound of $|\nabla (\Delta \rho)|$ and $|\Delta^2 \rho|$ can be calculated explicitly, as

$$
|\nabla (\Delta \rho)|(x) \leq 4((n-1)k + \sigma^2), \quad |\Delta^2 \rho|(x) \leq 8 \max \{\sigma, \sqrt{(n-1)k}((n-1)k + \sigma^2),
$$

for $\rho_\partial(x) \leq \ell_0 \wedge \ell^{-1}(1/2)$.

For the general case, from the second variation formula of $\rho_\partial$ (see (2.9) below) we see that further information about $|\nabla|_2, |\nabla^2|_2, |R|, |\nabla R|$ and $|\nabla^2 R|$ on $\partial D$ is needed to derive an upper bound of $|\nabla (\Delta \rho)|$ and $|\Delta^2 \rho|$.

Turning now to Hessian estimates for Neumann eigenfunctions, let us denote by $\text{Eig}_N(\Delta)$ the set of non-trivial $(\phi, \lambda)$ for the Neumann eigenproblem, i.e., $\phi$ is non-constant, $\Delta \phi = -\lambda \phi$ and $N\phi|_{\partial D} = 0$ for the unit inward normal vector field $N$ of $\partial D$. Proceeding along the previous ideas, the main difference is that we can no longer consider the process only up to the first hitting the boundary $\partial D$. When constructing the suitable martingales, the boundary behaviour of the process must be included a priori. We will use the reflecting Brownian motion as base process to deal with this question. Due to recent work on Bismut-type Hessian formula for the Neumann semigroup [4], we have the following formula linking $\text{Hess} Pf$ and $df$ intrinsically:

$$
\text{Hess} Pf(v, v) = \mathbb{E}\left[ -df(\tilde{Q}(v)) \int_0^t \langle \tilde{Q}(s), df(B_s) + df(\tilde{W}_s(v, v)) \rangle \right],
$$

where $\tilde{Q}$ and $\tilde{W}$ are defined in (3.1) and (3.2) in Section 3. By observing the fact that $P_t \phi = e^{-\frac{t}{\lambda} \Delta} \phi$ and estimating $\tilde{Q}$ and $\tilde{W}$ carefully under suitable curvature conditions, we obtain the following theorem which gives an upper estimate for $\text{Hess} \phi$ of the type (1.2) with an explicit constant $C_2(D)$.

**Theorem 1.3.** Let $D$ be an $n$-dimensional compact Riemannian manifold with boundary $\partial D$. Let $K_0, K_1, K_2$ be non-negative constants such that $\text{Ric} \geq -K_0, |R| \leq K_1$ and $|\text{d}^* R + \nabla \text{Ric}| \leq K_2$ on $D$, and let $\sigma_1, \sigma_2, \sigma$ be non-negative constants such that $-\sigma_1 \leq \Pi \leq \sigma$ and $|\nabla^2 N - R(N)| \leq \sigma_2$ on the boundary $\partial D$. Assume the distance function $\rho_\partial$ to the boundary $\partial D$ is smooth on $\partial D := \{x \in D : \rho_\partial(x) \leq r_1 \}$ and let $k$ be constant such that $\text{Sect} \leq k$ on $\partial r_1 D$. Then for any non-trivial $(\phi, \lambda) \in \text{Eig}_N(\Delta)$,

$$
\frac{||\text{Hess} \phi||_\infty}{||\phi||_\infty} \leq C_{N, \lambda}(D) \lambda
$$

where

$$
C_{N, \lambda}(D) = \begin{cases} 
K_1 + 2K_0 + 2\sigma_1 \left( \frac{K_2}{\lambda} + 2\sigma_1 \right) 
+ \frac{K_2 + 2\sigma_2 \left( \frac{K_2}{\lambda} + 2\sigma_1 \right)}{\lambda \sqrt{2\lambda + 4K_0 + 4\sigma_1 \left( \frac{K_2}{\lambda} + 2\sigma_1 \right)}} e^{\frac{2nr_1}{\lambda}} & 
\text{if } r_0 = r_1 \\
+ \frac{\sigma_2 nr_0}{2\lambda} \sqrt{2\lambda + 4K_0 + 4\sigma_1 \left( \frac{n}{r_0} + 2\sigma_1 \right)} e^{\frac{2nr_1}{\lambda}} & 
\text{if } r_0 < r_1
\end{cases}
$$

for $r_0 = r_1 \wedge \ell^{-1}(0)$. Denoting by $\lambda_1$ the first Neumann eigenvalue of $-\Delta$, then

$$
\frac{||\text{Hess} \phi||_\infty}{||\phi||_\infty} \leq C_{N, \lambda_1}(D) \lambda.
$$

The remainder of the paper is organized as follows. In Section 2 we first show for Dirichlet eigenfunctions

$$
||\text{Hess} \phi||_\infty / ||\phi||_\infty \leq C_{\lambda}(D) \lambda
$$

(1.9)

by verifying that the process (1.5) is a martingale, in combination with boundary estimates for $|\text{Hess} \phi|$. Section 3 deals with Neumann eigenfunctions where we give a proof of Theorem 1.3 by using Bismut type Hessian formulae for the Neumann semigroup and an estimate of the local time.
2. Hessian estimates for Dirichlet Eigenfunctions

This section is dedicated to the approach described in the Introduction. In fact, the proof of Theorem 1.1 is also divided into two steps by first showing Theorem 2.1 with some auxiliary function \( h \), which will be constructed in Section 2.3.

2.1. Preliminary. We start by defining the fundamental martingale which will serve as basis for our method.

**Theorem 2.1.** On a compact Riemannian manifold \( D \) with boundary \( \partial D \), let \( X_t(x) \) be a Brownian motion starting from \( x \in D \) and denote by \( \tau_D = \inf\{ t \geq 0 : X_t(x) \in \partial D \} \) its first hitting time of \( \partial D \). Define \( Q_t \) and \( W^k_t \) as above where \( k \in C^1_b([0,\infty);\mathbb{R}) \). Then, for \((\phi, \lambda) \in \text{Eig}(\Delta) \) and \( v \in T_xD \), the process

\[
e^{\lambda t/2} \text{Hess} \phi(Q_t(k(t)v), Q_t(v)) + e^{\lambda t/2} d\phi(W^k_t(v, v))
- e^{\lambda t/2} d\phi(Q_t(v)) \int_{0}^{t} \langle Q_s(k(s)v), \parallel dB_s \rangle
\]

is a martingale on \([0, \tau_D]\).

**Proof.** Due to the compactness of \( D \) it is sufficient to check that (2.1) is a local martingale on \([0, \tau_D]\). Fixing a time \( T > 0 \), for \( v \in T_xD \), we let

\[
N_t(v, v) = \text{Hess} P_{T-t} \phi(Q_t(v), Q_t(v)) + (dP_{T-t} \phi)(W_t(v,v)), \quad t \leq T \wedge \tau_D,
\]

where

\[
W_t(v, v) = Q_t \int_{0}^{t} Q_r^{-1} R(\parallel dB_r, Q_r(v))Q_r(v) - \frac{1}{2} Q_t \int_{0}^{t} Q_r^{-1} (d^* R + \nabla \text{Ric}) \phi(Q_r(v), Q_r(v)) dr.
\]

Then \( N_t(v, v) \) is a local martingale, see for instance the proof of [20] Lemma 2.7] with potential \( V \equiv 0 \). Since \((\phi, \lambda) \in \text{Eig}(\Delta) \), we know that \( P_{T-t} \phi(x) = e^{-\lambda(T-t)/2} \phi(x) \) and thus

\[
e^{\lambda t/2} \text{Hess} \phi(Q_t(v), Q_t(v)) + e^{\lambda t/2} (d\phi)(W_t(v, v))
\]

is also a local martingale. Furthermore, consider

\[
N^k_t(v, v) := e^{\lambda t/2} \text{Hess} \phi(Q_t(k(t)v), Q_t(v)) + (e^{\lambda t/2} d\phi)(W^k_t(v, v)).
\]

According to the definition of \( W^k_t(v, v) \), resp. \( W_t(v, v) \), and in view of the fact that \( N_t(v, v) \) is a local martingale, it is easy to see that

\[
e^{\lambda t/2} \text{Hess} \phi(Q_t(k(t)v), Q_t(v)) + (e^{\lambda t/2} d\phi)(W^k_t(v, v))
- \int_{0}^{t} e^{\lambda s/2} \text{Hess} \phi(Q_s(k(s)v), Q_s(v)) ds
\]

is a local martingale as well. From the formula

\[
e^{\lambda t/2} d\phi(Q_t(v)) = d\phi(v) + \int_{0}^{t} e^{\lambda s/2} (\text{Hess} \phi)(\parallel dB_s, Q_s(v))
\]

it follows that

\[
\int_{0}^{t} e^{\lambda s/2} (\text{Hess} \phi)(Q_s(k(s)v), Q_s(v)) ds - e^{\lambda t/2} d\phi(Q_t(v)) \int_{0}^{t} \langle Q_s(k(s)v), \parallel dB_s \rangle
\]

is a local martingale. We conclude that

\[
(e^{\lambda t/2} \text{Hess} \phi)(Q_t(k(t)v), Q_t(v)) + (e^{\lambda t/2} d\phi)(W^k_t(v, v))
- e^{\lambda t/2} d\phi(Q_t(v)) \int_{0}^{t} \langle Q_s(k(s)v), \parallel dB_s \rangle
\]

is a local martingale. \( \square \)

We shall use the following estimate to proceed with the Hessian formula for \( \phi \).
Lemma 2.2. Assume that \( \text{Ric} \geq -K_0, |R| \leq K_1 \) and \( |d^* R + \nabla \text{Ric}| \leq K_2 \) on \( D \) for non-negative constants \( K_0, K_1 \) and \( K_2 \). Let \( k \in C^1_p([0, \infty); \mathbb{R}) \). For \( t \geq 0 \) and \( \delta > 0 \), it holds

\[
|Q_t| \leq e^{K_0t/2} \quad \text{and} \quad \mathbb{E}
\left[
W^k_t(v, k(t)v) \mathbb{I}_{[t \leq \tau_D]} \right] \leq \left( K_1 \left( \int_0^t k(s)\, ds \right)^{1/2} + \frac{K_2}{2} \int_0^t |k(s)|\, ds \right) e^{K_0t} |k(t)|,
\]

where \( K_0, K_1 \) and \( K_2 \) are defined as in (2.3).

**Proof.** The first inequality follows from the lower Ricci curvature bound condition and the definition of \( Q_t \). According to the definition of \( W^k_t \), it is easy to see that

\[
W^k_t(v, v) = Q_t \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v) - \frac{1}{2} Q_t \int_0^t Q^{-1}_s (d^* R + \nabla \text{Ric})^2(Q_s(k(s)v), Q_s(v))\, ds.
\]

Note that for \( 0 \leq s \leq t \), the damped parallel transport \( Q_{s,t} = Q_t Q^{-1}_s : T_x D \to T_x D \) satisfying

\[
DQ_{s,t} = -\frac{1}{2} \nabla \text{Ric}^s(Q_{s,t}) dt, \quad Q_{s,s} = \text{id},
\]

Thus the lower bound of Ricci curvature \( -K_0 \) yields

\[
|Q_{s,t}| \leq e^{K_0(t-s)/2}.
\]

Then we have

\[
\mathbb{E}\left[ W^k_t(v, v) \mathbb{I}_{[t \leq \tau_D]} \right] \leq \mathbb{E}\left[ \mathbb{I}_{[t \leq \tau_D]} |Q_t \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v)\} \right]
\]

\[
+ \frac{1}{2} \mathbb{E}\left[ \mathbb{I}_{[t \leq \tau_D]} |Q_t \int_0^t Q^{-1}_s (d^* R + \nabla \text{Ric})^2(Q_s(k(s)v), Q_s(v))\, ds \right]
\]

\[
\leq e^{K_0t} \mathbb{E}\left[ \mathbb{I}_{[t \leq \tau_D]} e^{-K_0t} Q_t \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v)\} \right]^{1/2}
\]

\[
+ \frac{K_2}{2} \mathbb{E}\left[ \mathbb{I}_{[t \leq \tau_D]} e^{K_0t} \int_0^t e^{K_0s} |k(s)|\, ds \right].
\]

Moreover,

\[
d|e^{-K_0t} Q_t \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v)\}^2\]

\[
= 2 e^{-K_0t} \left\{ R(/_{s} dB_s, Q_t(k(t)v))Q_t(v), Q_t \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v)\} \right\}
\]

\[
+ e^{-K_0t} \left| R^\#(Q_t(k(t)v), Q_t(v)) \right|_{\text{HS}}^2 dt
\]

\[
- e^{-K_0t} \text{Ric} \left( Q_t, \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v), Q_t \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v)\} \right) dt
\]

\[
- K_0 e^{-K_0t} |Q_t \int_0^t Q^{-1}_s \{R(/_{s} dB_s, Q_s(k(s)v))Q_s(v)\}^2 dt
\]

\[
\leq e^{-K_0t} \left| R^\#(Q_t(k(t)v), Q_t(v)) \right|_{\text{HS}}^2 dt \leq K_1^2 e^{-K_0t} |Q_t|^4 k(t)^2 dt \leq K_1^2 e^{K_0t} k(t)^2 dt, \quad t \leq \tau_D.
\]

Combining this with (2.5), we have

\[
\mathbb{E}\left[ W^k_t(v, v) \mathbb{I}_{[t \leq \tau_D]} \right] \leq K_1 e^{K_0t} \left( \int_0^t e^{K_0s} k(s)^2\, ds \right)^{1/2} + \frac{K_2}{2} e^{-K_0t} \int_0^t |k(s)|\, ds.
\]

We then complete the proof. \( \square \)
By the results above, the following Hessian formula for eigenfunctions $\phi$ is obtained.

**Theorem 2.3.** Let $D$ be a compact Riemannian manifold with boundary $\partial D$. Let $X(x)$ be a Brownian motion starting from $x \in D$ and $\tau_D$ be its first hitting time of $\partial D$. Suppose that $k$ is a non-negative function in $C^1_b((0,\infty);\mathbb{R})$ such that $k(0) = 1$. Then for $(\phi, \lambda) \in \text{Eig}(\Delta)$, $t \geq 0$ and $v \in T_xD$,

$$(\text{Hess} \phi)(v, v) = \mathbb{E}^x \left[ e^{\frac{(t \wedge \tau_D)k}{2}} (\text{Hess} \phi)(Q_{t \wedge \tau_D}(k(t \wedge \tau_D)v), Q_{t \wedge \tau_D}(v)) + e^{\frac{(t \wedge \tau_D)k}{2}} (\text{div}(\phi))(W^k_{t \wedge \tau_D}(v, v)) \right] - \mathbb{E}^x \left[ e^{\frac{(t \wedge \tau_D)k}{2}} (\text{div}(\phi))(Q_{t \wedge \tau_D}(v)) \int_0^{t \wedge \tau_D} \langle Q_s(k(s)v), \sqrt{s}dB_s \rangle \right].$$

(2.6)

**Proof.** The claim follows by taking expectation of the martingale (2.1) at time 0 and $t \wedge \tau_D$. Recall that $|Q_s| \leq e^{k_{l}t/2}$. For $x \in \partial D$ formula (2.6) is obviously tautological since $\tau_D \equiv 0$. \hfill $\square$

To derive Hessian estimates of $\phi$ from Theorem 2.3 requires estimates of Hess $\phi$ on the boundary $\partial D$. To this end, we first note the following observation. Since $\phi = 0$ on the boundary $\partial D$, we have $\nabla \phi = N(\phi)N$.

**Lemma 2.4.** For $x \in \partial D$ let $H(x)$ be the mean curvature of the boundary. Then

$$N^2(\phi)(x) = -H(x)N(\phi)(x), \quad x \in \partial D.$$ 

**Proof.** For $x \in \partial D$, we have

$$0 = \lambda \phi(x) = \Delta \phi(x) = \text{div}(\nabla \phi)(x) = \text{div}(N(\phi))N(x) = \langle \nabla N(\phi), N(x) + N(\phi) \text{div}(N)(x) \rangle.$$

Taking into account that $\text{div}(N)(x) = H(x)$, the proof is completed. \hfill $\square$

The following lemma is taken from [2] Proposition 2.5 and allows to estimate the values of $|\nabla \phi|$ on the boundary.

**Lemma 2.5.** Let $\alpha_0 \in \mathbb{R}$ such that

$$\Delta \rho_0 \leq \alpha_0$$

outside $\text{Cut}(\partial D)$. Then for any $t > 0$,

$$||\nabla \phi||_{\partial D, \infty} = ||N(\phi)||_{\partial D, \infty} \leq ||\phi||_{\infty} e^{\frac{t}{2}} \left( \alpha_0^+ + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} \right).$$

In particular,

$$||\nabla \phi||_{\partial D, \infty} \leq ||\phi||_{\infty} e^{\frac{1}{2}} \left( \alpha_0^+ + \frac{\sqrt{2\lambda}}{\sqrt{\pi}} \right).$$

(2.8)

**Remark 2.6.** With constants $K_0, \theta > 0$ such that $\text{Ric} \geq -K_0$ on $D$ and $H \geq -\theta$ on the boundary $\partial D$, where $H(x)$ is the mean curvature of $D$ at $x \in D$, let

$$\alpha_0 = \max \left\{ \theta, \sqrt{(n - 1)K_0} \right\}.$$ 

Then estimate (2.7) holds true for this $\alpha_0$. \hfill $\square$

Next, we introduce some results on local time estimate of reflecting Brownian motion, which is also a tool in the boundary estimate of $|\text{Hess} \phi|$. Let us recall some basic notations on it. The reflecting Brownian motion on $D$ with generator $\frac{1}{2} A$ satisfies the SDE

$$dX_t = \eta_t \circ dB_t^x + \frac{1}{2} N(X_t) d\lambda_t, \quad X_0 = x,$$

where $B_t^x$ is a standard Brownian motion on the Euclidean space $T_x \mathbb{R}^n$ and $\lambda_t$ is the local time supported on $\partial D$ (see [23] for details). Now we turn to the problem of estimating $\mathbb{E}[e^{\alpha t/2}]$ for $\alpha > 0$ by exploiting a specific class of functions $h$. 
Lemma 2.7. Suppose that $h \in C^\infty(D)$ such that $h \geq 1$ and $N \log h \geq 1$. For $\alpha > 0$ let

$$K_{h,\alpha} = \sup \{-\Delta \log h + \alpha |\nabla \log h|^2\}.$$

Then

$$\mathbb{E}[e^{\alpha t/2}] \leq ||h||_{\infty}^\alpha \exp \left(\frac{\alpha}{2} K_{h,\alpha t}\right).$$

Proof. By the Itô formula we have

$$dh^{-\alpha}(X_t) = \langle \nabla h^{-\alpha}(X_t), \|dW_t\rangle + \frac{1}{2} \Delta h^{-\alpha}(X_t) dt + \frac{1}{2} Nh^{-\alpha}(X_t) dt.$$

Hence,

$$M_t := h^{-\alpha}(X_t) \exp \left( -\frac{\alpha}{2} K_{h,\alpha t} + \frac{\alpha}{2} \int_0^t N \log h(X_s) ds \right)$$

is a local submartingale. Therefore, by Fatou’s lemma and taking into account that $h \geq 1$, we get

$$\mathbb{E} \left[ h^{-\alpha}(X_t) \exp \left( -\frac{\alpha}{2} K_{h,\alpha t} + \frac{\alpha}{2} \int_0^t N \log h(X_s) ds \right) \right] \leq \mathbb{E} \left[ h^{-\alpha}(X_t) \exp \left( -\frac{\alpha}{2} K_{h,\alpha t} (t \land \tau_D) + \frac{\alpha}{2} \int_0^{\tau \wedge \tau_D} N \log h(X_s) ds \right) \right] \leq h^{-\alpha}(x) \leq 1.$$

Since $N \log h(x) \geq 1$ we conclude that

$$\mathbb{E} \left[ \exp \left( \frac{\alpha}{2} t \right) \right] \leq \mathbb{E} \left[ \exp \left( \frac{\alpha}{2} \int_0^t N \log h(X_s) ds \right) \right] \leq ||h||_{\infty}^\alpha \exp \left( \frac{\alpha}{2} K_{h,\alpha t} \right).$$

At the end of this subsection, we introduce some results on Hessian comparison of $\rho_\theta$. Let $p$ be the orthogonal projection of $x$ on $\partial D$, and let $\gamma(s) = \exp_p(sN), s \in [0, \rho_\theta(x)]$ be the geodesic from $p$ to $x$. Let $\{J(s)\}_{s \in [0, \rho_\theta(x)]}$ be the Jacobi field along $\gamma$ such that $J(\rho_\theta(x)) = v$ for $v \in T_xD$, and $J(0) = -\Pi^p(J(0)) \in T_\rho \partial D$, where $\langle \Pi^p(J(0)), w \rangle = \Pi(J(0), w)$ for $w \in T_p \partial D$. From the variation formula of $\rho_\theta$, we know that

$$\text{Hess} \rho_\theta(v, v) = -\Pi(J(0), J(0)) + \int_0^{\rho_\theta(x)} \left( |J(s)|^2 - \langle R(\dot{\gamma}(s), J(s))\dot{\gamma}(s), J(s) \rangle \right) ds.$$

The following result is essentially due to Kasue [10,11] (see also Theorem A.1 in [21]).

Lemma 2.8 (Hessian Comparison). Let $\sigma$ and $k$ be non-negative constants such that $|\Pi| \leq \sigma$ and $|\text{Sect}| \leq k$ on $\partial \rho_\theta D$, where $\rho_\theta$ is smooth $\partial \rho_\theta D$. Then

$$\frac{\ell_{k,\sigma}^\ell(\rho_\theta(x))}{\ell_{k,\sigma}^\ell(\rho_\theta(x))} \leq \text{Hess} \rho_\theta(v, v) \leq \frac{\ell_{k,\sigma}^{\ell-k}(\rho_\theta(x))}{\ell_{k,\sigma}^{\ell-k}(\rho_\theta(x))}, \quad \rho_\theta \leq r_0 \land \ell_{k,\sigma}^{1-\ell}(\frac{1}{2}).$$

Moreover, for $\rho_\theta(x) \leq r_0 \land \ell_{k,\sigma}^{-1}(\frac{1}{2})$,

$$|\text{Hess} \rho_\theta| \leq 2 \max \{\sigma, \sqrt{k} \}.$$

Proof. The proof of first inequality can be found in [23 Theorem 1.2.2]. Based on this, it is easy to have for $k, \sigma \geq 0$,

$$\text{Hess} \rho_\theta(v, v) \leq \max \{\sigma, \sqrt{k} \}.$$

For $\rho_\theta(x) \leq r_0 \land \ell_{k,\sigma}^{-1}(\frac{1}{2})$,

$$\text{Hess} \rho_\theta(v, v) \geq \frac{\ell_{k,\sigma}^\ell(\rho_\theta(x))}{\ell_{k,\sigma}^\ell(\rho_\theta(x))} \geq 2 \ell_{k,\sigma}(\rho_\theta(x)) \geq -2 \max \{\sigma, \sqrt{k} \}.$$
We then complete proof of the second inequality. □

2.2. Hessian estimate of Dirichlet eigenfunctions. Lemmas 2.4, 2.5 and 2.7 allow to derive an estimate of \(|\text{Hess} \phi|\) on the boundary \(\partial D\).

**Lemma 2.9.** Let \(K_0, \sigma\) be non-negative constants such that \(\text{Ric} \geq -K_0, \ |\!|\!| \leq \sigma\). Suppose that the distance function \(\rho_0\) is smooth on \(\partial_{r_0} D := \{ x : \rho_0(x) \leq r_0 \}\) for some constant \(r_0 > 0\). Then for \(x \in \partial D\),

\[
\|\text{Hess}(\phi)\|_{\partial D, \infty} \leq (n - 1)\sigma \|N(\phi)\|_{\partial D, \infty} + \|h\|_{\infty}^\sigma e^{\frac{1}{2}(K_0 + \sigma K_{0, r})t} \left( C_1 \|D\|_{\infty} + C_2 \sqrt{r} \right) \|\phi\|_{\infty} + \|h\|_{\infty}^\sigma e^{\frac{1}{2}(K_0 + \sigma K_{0, r})t} \left( C_3 \|\phi\|_{\infty} + \|\nabla \phi\|_{\infty} \right) + \|h\|_{\infty}^\sigma e^{\frac{1}{2}(K_0 + \sigma K_{0, r})t} \sqrt{C_4} \|\text{Hess} \phi\|_{\infty}
\]

where \(h \in C^\infty(D)\) such that \(h \geq 1\) and \(N \log h \geq 1\) and

\[K_{h, \sigma} = \sup[-\Delta \log h + \sigma|\nabla \log h|^2],\]

and the constant \(C_1, C_2, C_3, C_4\) are defined as

\[
\begin{align*}
C_1 &= \|\Delta \rho_0\|_{\partial_0 D}, \\
C_2 &= \|\Delta(\phi(\rho_0))\Delta \rho_0 + 2\phi'(\rho_0)\nabla(\Delta \rho_0) + \psi(\rho_0)(\lambda \Delta \rho_0 + \Delta^2 \rho_0)\|_{\partial_0 D}, \\
C_3 &= \|\Delta(\phi(\rho_0)) + 2\phi'(\rho_0)\Delta \rho_0 + \psi(\rho_0)(3\nabla(\Delta \rho_0) + \lambda)\|_{\partial_0 D}, \\
C_4 &= \|2\phi'(\rho_0) + 2(n - 1)\psi(\rho_0)\|_{\partial_0 D}.
\end{align*}
\]

where \(\psi \in C^2(\mathbb{R}_+, [0, 1])\) satisfies \(\psi(0) = 1, \psi'(0) = 0\) and \(\psi(r) = 0\) for \(r > r_0\).

**Proof.** Given \(x \in \partial D\), let \(\{X_i\}_{1 \leq i \leq n}\) be an orthonormal basis of \(T_x D\) with \(X_i = N\). Then

\[
|\text{Hess}(\phi)(X_i, X_j)| = |\nabla \phi(X_i, X_j)| = |\langle \nabla X_i \nabla \phi, X_j \rangle|
\]

By assumption we have \(|\!|\!| \leq \sigma\). If \(X_i, X_j \in T_x \partial D\), i.e. \(i, j \neq 1\), then \(\langle \nabla \phi, X_j \rangle_{\partial D} = 0\) and

\[
|\text{Hess}(\phi)(X_i, X_j)| = |-N(\phi)\nabla X_i \nabla X_j| \leq \sigma |N(\phi)|. \quad (2.10)
\]

If \(X_i = X_j = N\), i.e. \(i = j = 1\), then \(\nabla_N N|_{\partial D} = 0\) and

\[
|\text{Hess}(\phi)(N, N)| = |N^2(\phi)| \leq |H N(\phi)| \leq (n - 1)\sigma |N(\phi)|. \quad (2.11)
\]

If \(X_j \in T_x \partial D\) and \(X_i = N\) (i.e. \(j \neq 1\) and \(i = 1\)), then

\[
|\text{Hess}(\phi)(X_j, N)|_{\partial D} = |N X_j(\phi)|. \quad (2.12)
\]

In order to get control on (2.12), we shall use a probabilistic argument based on the Brownian motion on \(D\) reflected at the boundary. Before going into the details, we make a general remark on the extension of vector fields from \(\partial D\) to a tubular neighborhood of the boundary.

**Remark 2.10.** Assuming that the boundary \(\partial D\) is smooth, let \(N\) be the unit inward normal vector field \(N\) on \(\partial D\). Furthermore, let

\[
\Phi : [0, r_0] \times \partial D \rightarrow D, \quad (r, x) \mapsto \exp_x(r N),
\]

be the geodesic from \(x \in \partial D\) orthogonal to \(\partial D\) and parametrized by its arc length \(r\). As the differential of \(\Phi\) at any point \((0, x)\) has full rank, we find \(e_0 > 0\) such that \(\Phi\) is a diffeomorphism from \([0, e_0] \times \partial D\) onto the open neighborhood \(\{ x \in D : \rho_\beta(x) < e_0 \}\) of \(\partial D\) in \(D\). This allows to extend \(N\) to a tubular (collar) neighborhood of \(\partial D\) as \(\Phi_* N\). By construction then \(\nabla \Phi N = 0\). If \(X\) is a vector field on \(\partial D\) tangential to \(\partial D\), we extend it to the neighborhood of \(\partial D\) as being independent of the real variable.
in the product $[0, e_0] \times \partial D$. By construction, close to the boundary, the distance function $\rho_\beta(x) = \text{dist}(x, \partial D)$ is smooth and satisfies $N = \nabla \rho_\beta$.

Let $N$ be the extension of the normal vector field to a tubular neighborhood $\partial_r D := \{x : \rho_\beta(x) \leq r_0\}$ of $\partial D$ and define

$$\varphi(x) = \psi(\rho_\beta(x)) \text{div}(\phi N), \quad x \in \partial_r D,$$

(2.14)

where $\psi \in C^2(\mathbb{R}^+, [0, 1])$ satisfies $\psi(0) = 1$, $\psi'(0) = 0$ and $\psi(r) = 0$ for $r > r_0$. Using the formula $\text{div}(\phi N) = N(\phi) + \phi \text{div}(N)$, along with Lemma 2.4, we observe for $x \in \partial D$,

$$N(\varphi)(x) = \psi'(0) \text{div}(\phi N) + N(\text{div}(\phi N)) = 0.$$

Thus $\varphi$ satisfies the Neumann boundary conditions on $D$.

Let now $X_i$ be the reflecting Brownian motion on $D$ and $P_t^f f(x) = \mathbb{E}^x[f(X_t)]$ for $f \in \mathcal{B}(D)$ the corresponding Neumann semigroup. According to the Kolmogorov equation,

$$\varphi(x) = P_t^f(\varphi)(x) - \frac{1}{2} \int_0^t P_s^f(\Delta \varphi)(x) \, ds.$$

Taking derivative on both sides of the above equation yields

$$X_i(\varphi)(x) = X_i(P_t^f(\varphi))(x) - \frac{1}{2} \int_0^t X_i(P_s^f(\Delta \varphi))(x) \, ds$$

where $X_i$ is tangential to $\partial D$. We first observe that for $x \in \partial D$,

$$X_i(\varphi)(x) = X_i(\psi(\rho_\beta)) \text{div}(\phi N)(x) + \psi(\rho_\beta) X_i(\text{div}(\phi N))(x) = X_i(\text{div}(\phi N))(x)$$

$$= X_i N(\phi)(x) + X_i(\phi)(x) \text{div}(N)(x) + \phi(x) X_i(\text{div}(N))(x)$$

$$= X_i N(\phi)(x).$$

To deal with the upper bound, we use the Bismut formula established in [23, Theorem 3.2.1] for the compact manifold $D$, which gives

$$|\nabla P_t^f f| \leq \frac{1}{\sqrt{t}} e^{\frac{1}{4} K_0 t} \mathbb{E}^t[|e^{\sigma l}|]^\frac{1}{2} ||f||_\infty,$$

where $l_t$ is the local time supported on $\partial D$. By Lemma 2.7, derived in the previous subsection, we have

$$\mathbb{E}^x[|e^{\sigma l}|] \leq ||h||^{2\sigma} \exp(\sigma K_{h,2\sigma} t),$$

where $h \in C^\infty(D)$ such that $h \geq 1$ and $N \log h \geq 1$ and

$$K_{h,2\sigma} = \sup[-\Delta \log h + 2\sigma |\nabla \log h|^2].$$

We then conclude that

$$|X_i N(\phi)(x) \leq ||h||^{2\sigma} e^{\frac{1}{4} (K_0 + \sigma K_{h,2\sigma}) t} \left[ \frac{1}{\sqrt{t}} ||\nabla \phi||_{B(x, r_0)} + \sqrt{t} \||\Delta \phi||_{B(x, r_0)} \right].$$

(2.15)

According to the definition of $\varphi$ in (2.14), we have

$$||\varphi||_\infty \leq ||\nabla \phi||_\infty + ||\text{div}(N)||_{\partial_\beta D} ||\phi||_\infty$$

By commutation rules, we calculate

$$\Delta(\psi(\rho_\beta)) \text{div}(\phi N)) = \Delta(\psi(\rho_\beta)) \text{div}(\phi N)) + 2\psi'(\rho_\beta) N(\phi N) + \psi(\rho_\beta) \Delta(\text{div}(\phi N))$$

$$= \Delta(\psi(\rho_\beta)) (\phi \text{div}(N) + N(\phi)) + 2\psi'(\rho_\beta) (\phi N(\text{div}(N)) + N(\phi) \text{div}(N) + N^2(\phi))$$

$$+ \psi(\rho_\beta) \Delta(\text{div}(\phi N))$$

(2.16)

and

$$\Delta(\text{div}(\phi N)) = \text{div}((\Box - \text{Ric}^b)(\phi N))$$
\[ \text{and as a consequence} \]

\[ \text{div}(\nabla_{\sigma} \phi) = \sum_{i=1}^{n} (\langle \nabla e_i(\phi), \nabla e_i \rangle + e_i(\phi) \text{div}(\nabla e_i)) \]

\[ = \langle \text{Hess}_{\phi} \phi, \nabla \rangle + \langle \nabla \phi, \sum_{i=1}^{n} \text{div}(\nabla e_i) e_i \rangle \]

\[ = \langle \text{Hess}_{\phi} \phi, \nabla \rangle + \langle \nabla \phi, \nabla (\text{div}(\nabla)) \rangle. \]

Combining this with (2.17) yields

\[ \Delta (\text{div}(\phi N)) = \phi \Delta (\text{div}(N)) - \Delta (\phi N) - 2 \langle \text{Hess}(\phi), \nabla N \rangle + 2 \langle \nabla \phi, \nabla (\text{div}(N)) \rangle \]

\[ + \langle \Box N, \nabla \phi \rangle - \text{Ric}(N, \nabla \phi). \]

From the fact that \( N = \nabla \rho_0 \) and the Weitzenböck formula, we observe that

\[ \text{div}(N) = \Delta \rho_0, \quad \nabla N = \text{Hess} \rho_0, \quad \text{and} \quad \langle \Box N, \nabla \phi \rangle - \text{Ric}(N, \nabla \phi) = \langle \nabla \Delta \rho_0, \nabla \phi \rangle. \]  

(2.18)

Combining the equations (2.16), (2.17) and (2.18) with (2.15), we finally conclude that

\[ |X_i N(\phi)(x)| \leq ||h||_\infty^{\sigma} e^{\frac{1}{2}(K_0 + 2t K_{2,\sigma})^t} \left( C_1 \frac{1}{\sqrt{t}} + C_2 \sqrt{t} \right) ||\phi||_\infty + ||h||_\infty^{\sigma} e^{\frac{1}{2}(K_0 + 2t K_{2,\sigma})^t} \left( \frac{1}{\sqrt{t}} + C_3 \sqrt{t} \right) ||\nabla \phi||_\infty \]

\[ + ||h||_\infty^{\sigma} e^{\frac{1}{2}(K_0 + 2t K_{2,\sigma})^t} \sqrt{t} C_4 ||\text{Hess} \phi||_\infty \]

where

\[ C_1 = ||\Delta \rho_0||_{\delta_{D}}, \]

\[ C_2 = ||\Delta (\rho_0) \nabla \rho_0 + 2 \rho' (\rho_0) |\nabla (\Delta \rho_0) + \psi (\rho_0) (\lambda \Delta \rho_0 + \Delta^2 \rho_0)||_{\delta_{D}}, \]

\[ C_3 = ||\Delta (\rho_0) + 2 \rho' (\rho_0) \Delta \rho_0 + \psi (\rho_0) (3 |\nabla (\Delta \rho_0) + \lambda)||_{\delta_{D}}, \]

\[ C_4 = ||2 \rho' (\rho_0) + 2 (n-1) \rho_0 ||\text{Hess} \rho_0||_{\delta_{D}}. \]

The proof is completed by combining the above estimate with (2.10) and (2.11). □

Combining the estimates in Lemmas 2.5 and 2.9 with Theorem 2.3, we are now in a position to prove our main result.

**Theorem 2.11.** Let \( D \) be a compact Riemannian manifold with boundary \( \partial D \). Let \( K_0, K_1, K_2 \) and \( \sigma \) be non-negative constants such that \( \text{Ric} \geq -K_0, |R| \leq K_1 \) and \( |\text{Ric}| \leq K_2 \) on \( D \), and that \( |\nabla| \leq \sigma \) on the boundary \( \partial D \). Assume the distance function \( \rho_0 \) is smooth on the tubular neighborhood \( \partial_0 D := \{ x : \rho_0(x) \leq r_0 \} \) of \( \partial D \) for some constant \( r_0 > 0 \), and let \( \alpha, \beta, \gamma \in \mathbb{R} \) be such that

\[ |\text{Hess} \rho_0| \leq \frac{\alpha}{n-1}, \quad |\nabla (\Delta \rho_0)| \leq \beta, \quad |\Delta^2 \rho_0| \leq \gamma \quad \text{on} \ \partial_0 D. \]  

(2.19)

For \( h \in C^\infty(D) \) with \( \min_D h = 1 \) and \( N \log h|_{\partial D} \geq 1 \), then

\[ \frac{||\text{Hess} \phi||}{||\phi||_\infty} \leq 2(n-1) e^{\sigma \left( \alpha + \sqrt{\frac{2 \lambda}{\pi}} \right)} + 2 \alpha ||h||_\infty^{\sigma} \sqrt{\max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{2,\sigma}}, 2 \sqrt{\sigma} ||h||_\infty^{\sigma} \left( \frac{6}{r_0} + 2 \sigma \right) \right\}} \]
By Lemmas 2.5 and 2.9, we have
\[
\frac{3}{r_0} (\alpha^2 + 2\beta) + \frac{4}{r_0} \alpha + 2\lambda \alpha + \gamma
\]
\[
+ \left( \frac{\alpha}{r_0} + \frac{3}{r_0} + \lambda + K_1 \right) \frac{K_2}{\Delta \lambda}
\]
\[
+ 4e \|h\|_\infty^2 \max \left\{ \sqrt{\lambda + 2K_0 + \alpha K_{h,2\sigma}}, 2 \sqrt{\epsilon} \|h\|_\infty^2 \left( \frac{6}{r_0} + 2\alpha \right) \right\}
\]
\[
\times \left( \alpha + \frac{1}{4} \sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}} \sqrt{\lambda + K_0} \right).
\]

**Proof.** According to formula (2.6) we have
\[
|\text{Hess} \phi(v,v)| = E \left[ e^{\lambda(t\wedge \tau_D)/2} \text{Hess} \phi(Q_{t\wedge \tau_D}(k(t \wedge \tau_D)v), Q_{t\wedge \tau_D}(v)) \right]
\]
\[
+ E \left[ e^{\lambda(t\wedge \tau_D)/2} d\phi(W_{t\wedge \tau_D}^p(v,v)) \right]
\]
\[
- E \left[ e^{\lambda(t\wedge \tau_D)/2} d\phi(Q_{t\wedge \tau_D}(v)) \int_0^{t\wedge \tau_D} \langle Q_s(k(s)v), /s dB_s \rangle \right].
\]

Taking \(k(s) = (t-s)/t\) for \(s \in [0,t]\) in the equation yields
\[
|\text{Hess} \phi(v,v)| \leq E \left[ 1_{\{\tau_D \leq t\}} e^{\lambda(t\wedge \tau_D)/2} t^{-\tau_D} \|\text{Hess}(\phi)\|_{0,D,\infty} \right]
\]
\[
+ |d\phi|_{\infty} \left( K_1 \frac{\sqrt{t} + K_2}{2} \right) e^{(\frac{1}{2} + K_0)t} \frac{\epsilon(\frac{1}{2} + K_0)t}{\sqrt{t}}
\]
\[
+ |d\phi|_{\infty} e^{(\frac{1}{2} + K_0)t}. \tag{2.20}
\]

By Lemmas 2.5 and 2.9 we have
\[
|\text{Hess} \phi(v,v)| \leq E \left[ 1_{\{\tau_D \leq t\}} e^{\lambda(t\wedge \tau_D)/2} t^{-\tau_D} \left[ \max \left\{ \|H\|_{0,D,\infty}, \epsilon \right\} \|N\phi\|_{0,D,\infty} \right] \right]
\]
\[
+ |d\phi|_{\infty} \left( K_1 \frac{\sqrt{t} + K_2}{2} \right) e^{(\frac{1}{2} + K_0)t} \frac{\epsilon(\frac{1}{2} + K_0)t}{\sqrt{t}}
\]
\[
+ |d\phi|_{\infty} e^{(\frac{1}{2} + K_0)t}. \tag{2.21}
\]

where \(C_1, C_2, C_3\) and \(C_4\) are defined as in Lemma 2.9. Combining this with the fact that
\[
\frac{t-\tau_D}{t} \frac{1}{\sqrt{t-\tau_D}} = \frac{\sqrt{t-\tau_D}}{t} \leq \frac{1}{\sqrt{t}}
\]
and then substituting back into (2.21) and using (2.8), we obtain
\[
|\text{Hess} \phi(v,v)| \leq (n-1) \epsilon e^{(\frac{1}{2} + K_0)t} \frac{\epsilon(\frac{1}{2} + K_0)t}{\sqrt{t}} \|\phi\|_{\infty}
\]
Now let

\[ t = t_0 := \frac{1}{\max \{ \lambda + 2K_0 + \sigma K_{h,2\sigma}, 4e \| h \|_\infty^2 C_4^2 \}}. \]

Then

\[ \| h \|_\infty^\sigma e^{\frac{1}{2} + K_0 + \frac{\sigma K_{h,2\sigma}}{2}} \sqrt{t_0} C_4 \| \text{Hess } \phi \|_\infty \leq \frac{1}{2} \| \text{Hess } \phi \|_\infty \]

and then inequality (2.22) becomes

\[
|\text{Hess } \phi(v, v)| \leq 2(n - 1)e\left(\alpha + \sqrt{\frac{2\lambda}{\pi}}\right) \| \phi \|_\infty^\sigma
+ 2C_1 \| h \|_\infty^\sigma \sqrt{e} \max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{h,2\sigma}}, 2e \| h \|_\infty^\sigma C_4 \right\} \| \phi \|_\infty^\sigma
+ \frac{C_2}{C_4} \| \phi \|_\infty + \frac{C_3}{C_4} \| d\phi \|_\infty^\sigma
\]

\[
+ 2 \sqrt{e} (\| h \|_\infty^\sigma + 1) \max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{h,2\sigma}}, 2e \| h \|_\infty^\sigma C_4 \right\} \| d\phi \|_\infty^\sigma
+ \frac{2K_1}{\sqrt{e}} \sqrt{\max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{h,2\sigma}}, 2e \| h \|_\infty^\sigma C_4 \right\}} \| d\phi \|_\infty^\sigma
+ \frac{K_2}{\sqrt{e}} \sqrt{\max \left\{ \lambda + 2K_0 + \sigma K_{h,2\sigma}, 4e \| h \|_\infty^2 C_4^2 \right\}} \| d\phi \|_\infty^\sigma
\]

\[
\leq 2\alpha e\left(\alpha + \sqrt{\frac{2\lambda}{\pi}}\right) \| \phi \|_\infty^\sigma
+ 2C_1 \| h \|_\infty^\sigma \sqrt{e} \max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{h,2\sigma}}, 2e \| h \|_\infty^\sigma C_4 \right\} \| \phi \|_\infty + \frac{C_2}{C_4} \| \phi \|_\infty
\]

\[
+ 4 \sqrt{e} \| h \|_\infty^\sigma \max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{h,2\sigma}}, 2e \| h \|_\infty^\sigma C_4 \right\} \| d\phi \|_\infty^\sigma
+ \left( \frac{C_3}{C_4} + \frac{K_1}{C_4} + \frac{K_2}{4 \sqrt{e} C_4} \right) \| d\phi \|_\infty^\sigma
\]

(2.23)

It is known from Arnaudon, Thalmaier and Wang [2] Eq. (2.8) that

\[
\frac{\| d\phi \|_\infty}{\| \phi \|_\infty} \leq \sqrt{e} \left( \alpha + \sqrt{\frac{2}{\pi}} (\lambda + K_0) + \frac{\lambda + K_0}{4(\alpha + \sqrt{\frac{2}{\pi}} (\lambda + K_0))} \right)
\]

\[
\leq \sqrt{e} \left( \alpha + \left( \sqrt{\frac{2}{\pi}} + \frac{1}{4} \sqrt{\frac{2}{\pi}} \right) \sqrt{\lambda + K_0} \right).
\]
Note here we use the upper bound $\alpha + \sqrt{\frac{2}{\pi}}$ of $f(t, \alpha)$ defined in [2] to simplify the upper bound in [2] Eq. (2.8)]. Next, combining this with (2.23) implies that

$$\frac{\|\text{Hess} \phi\|}{\|\phi\|_\infty} \leq 2e(n-1)e \left( \alpha + \sqrt{\frac{2}{\pi}} \right)$$

$$+ 2C_1\|h\|_\infty \sqrt{\max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{h,2r}}, 2\sqrt{e}\|h\|_{\infty}^r C_4 \right\}} + \frac{C_2}{C_4}$$

$$+ \left( \frac{C_3}{C_4} + \frac{K_1}{4\sqrt{eC_4^2}} \right) \sqrt{e} \left( \alpha + \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{4} \frac{\pi}{2}} \sqrt{\lambda + K_0} \right) \right)$$

$$+ 4e\|h\|_\infty^r \max \left\{ \sqrt{\lambda + 2K_0 + \sigma K_{h,2r}}, 2\sqrt{e}\|h\|_{\infty}^r C_4 \right\} \left( \alpha + \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{2}{4} \frac{\pi}{2}} \sqrt{\lambda + K_0} \right) \right).$$

Using condition 2.19 the constants $C_1, C_2, C_3$ and $C_4$ now become

$$C_1 = \alpha,$$

$$C_2 = \|\phi\|_\infty^r (\alpha^2 + 2\beta) + \|\phi''\|_\infty^r \alpha + \left( \lambda \alpha + \gamma \right),$$

$$C_3 = 3\|\phi''\|_\infty^r \alpha + \|\phi''\|_\infty^r + 3\beta + \lambda,$$

$$C_4 = 2\|\phi''\|_\infty^r.$$  (2.24)

Let

$$\psi(r) = \begin{cases} \left( \frac{\sigma}{r_0} \right)^3, & 0 \leq r \leq r_0; \\ 0, & r > r_0. \end{cases}$$

Then $\psi' \leq \frac{3}{r_0}$ and $\psi'' \leq \frac{6}{r_0^2}$. Form these estimates, the constants $C_1, C_2, C_3$ and $C_4$ are further explicit.

\[ \square \]

2.3. **Proof of Theorem 1.1.** In this subsection we describe F.-Y. Wang’s construction of functions $h$ satisfying the requirements of Lemma 2.7 (see [22, p. 1436] or [23, Theorem 3.2.9] for the details). His construction is performed under the following condition.

**Condition (A)** There exist a non-negative constant $\sigma$ such that $1 \leq \sigma$ and a positive constant $r_1$ such that the distance function $\rho_0$ to the boundary $\partial D$ is smooth on $\partial r_1 D := \{ x \in D : \rho_0(x) \leq r_1 \}$. Moreover, $\text{Sect} \leq k$ on $\partial r_1 D$ for some positive constant $k$.

Under Condition (A), based on the Hessian comparison theorem, F.-Y. Wang then constructs a function $h$ satisfying the necessary properties of Lemma 2.7 (see [22, p. 1436] or [23, Theorem 3.2.9] for the notation and the precise result), along with explicit upper bounds for $\|h\|_{\infty}$ and the constant $K_{h,\alpha}$. Modifying his construction one may take

$$\log h(x) = \frac{1}{\Lambda_0} \int_0^{\rho_0(x)} \left( \ell(s) - \ell(r_0) \right)^{1-n} ds \int_{s \wedge r_0}^r \|(u) - \ell(r_0)\|^{n-1} du$$  (2.26)

where $\ell = \ell_{\sigma, \ell}$ is defined in (1.6), $r_0 := r_1 \wedge \ell^{-1}(0)$ and

$$\Lambda_0 := (1 - \ell(r_0))^{1-n} \int_0^{r_0} (\ell(s) - \ell(r_0))^{n-1} ds.$$  (2.27)

Then from the proof of [21, Theorem 1.1], we get:

$$K_{h,\alpha} \leq K_\sigma := \frac{n}{r_0} + \alpha \quad \text{and} \quad \|h\|_\infty \leq e^{\frac{1}{r_0} \sqrt{\lambda K_0}}.$$  (2.28)

Using the $h$ constructed above, we are now able to complete the proof of Theorem 1.1.
Proof of Theorem 1.1 Using $h$ defined in (2.26) and substituting the estimates (2.27), we replace

$$K_{h,2\sigma}, \|h\|_\infty,$$

by

$$\frac{n}{r_0} + 2\sigma, \ e^{\frac{n}{r_0}/2},$$

respectively. By Lemma 2.8, we see that the upper bound $\alpha$ in Theorem 2.19 can be chosen as $2(n - 1)\max\{\sigma, \sqrt{k}\}$. This completes the proof of inequality (1.9). $\Box$

3. HESSIAN ESTIMATES FOR NEUMANN EIGENFUNCTIONS

We also use a stochastic approach to prove Theorem 1.3. Let us first recall the Hessian formulas for the Neumann semigroups, established recently in [4]. The reflecting Brownian motion on $D$ with generator $\frac{1}{2}\Delta$ satisfies the SDE

$$dX_t = f_t \circ dB_t + \frac{1}{2}N(X_t)dl_t, \quad X_0 = x,$$

where $B_t$ is a standard Brownian motion on the Euclidean space $T_xD \equiv \mathbb{R}^n$. We write again $X_t = X_t(x)$ to indicate the starting point $x \in D$ (which may be on the boundary $\partial D$). Here $f_t : T_xD \rightarrow T_{X_t(x)}D$ denotes the $\nabla$-parallel transport along $X_t(x)$ and $l_t$ the local time of $X_t(x)$ supported on $\partial D$. Note that the reflecting Brownian motion $X_t(x)$ is defined for all $t \geq 0$.

Suppose that $\tilde{Q}_t : T_xD \rightarrow T_{X_t(x)}D$ satisfies

$$D\tilde{Q}_t = -\frac{1}{2}Ric\tilde{Q}_t dt + \frac{1}{2}(\nabla N)\tilde{Q}_t dl_t, \quad \tilde{Q}_0 = id. \quad (3.1)$$

For $k \in C^1_b((0,\infty);\mathbb{R})$ define an operator-valued process $\tilde{W}^k_1 : T_xD \otimes T_xD \rightarrow T_{X_t(x)}D$ as solution to the following covariant Itô equation

$$D\tilde{W}^k_1(v,w) = R((/\tilde{Q}_t(\tilde{Q}_t(k(t)v))),\tilde{Q}_t(w)) dt$$

$$-\frac{1}{2}(d^\ast R + \nabla Ric)\tilde{Q}_t(k(t)v),\tilde{Q}_t(w) dt$$

$$-\frac{1}{2}(\nabla^2 N - R(N))\tilde{Q}_t(k(t)v),\tilde{Q}_t(w) dl_t$$

$$-\frac{1}{2}Ric\tilde{Q}_t(\tilde{W}^k_1(v,w)) dt + \frac{1}{2}(\nabla N)\tilde{W}^k_1(v,w) dl_t, \quad (3.2)$$

with initial condition $\tilde{W}^k_0(v,w) = 0$.

**Theorem 3.1 ([4]).** Let $D$ be a compact Riemannian manifold with boundary $\partial D$. Let $X(x)$ be the reflecting Brownian motion on $D$ with starting point $x$ (possibly on the boundary) and denote by $P_t f(x) = \mathbb{E}[f(X_t(x))]$ the corresponding Neumann semigroup acting on $f \in B_b(D)$. Then, for $v \in T_xD$, $t \geq 0$ and $k \in C^1_b((0,\infty);\mathbb{R})$,$$

$$Hess P_t f(v,v) = \mathbb{E}

\left\{-df(\tilde{Q}_t(v)) \int_0^t \langle \tilde{Q}_t(k(s)v), f_t \circ dB_to \rangle + df(\tilde{W}^k_1(v,v)) \right\}.$$

Estimating $\tilde{W}^k$ and $\tilde{Q}$ explicitly, we can get pointwise bounds for the Hessian of Neumann eigenfunctions.

**Corollary 3.2.** We keep the assumptions of Theorem 3.1. Let $K_0, K_1, K_2$ and $\sigma_1, \sigma_2$ be non-negative constants such that $\text{Ric} \geq -K_0$, $|R| \leq K_1$ and $|d^\ast R + \nabla \text{Ric}| \leq K_2$ on $D$, and $\Pi \geq -\sigma_1$, $|\nabla^2 N + R(N)| < \sigma_2$ on the boundary $\partial D$. Then, for $(\phi, \lambda) \in \text{Eig}_N(D)$,

$$|\text{Hess} \phi(x) | \leq e^{\frac{1}{2}d_t + K_0} \mathbb{E}[e^{\frac{1}{2}d_t}] \left( \frac{1}{\sqrt{t}} + K_1 \sqrt{t} + \frac{K_2}{2} t \right) \|d\phi\|_\infty.$$
\[+ \frac{\sigma_2}{2} e^{(K_0 + \frac{1}{2}) t} \mathbb{E}\left( e^{\frac{1}{2}\sigma_1 l} \int_0^t e^{\frac{1}{2}\sigma_1 l_s} \, dl_s \right) \|d\phi\|_\infty.\]

**Proof.** By [4] Theorem 4.1] the Hessian of the semigroup can be estimated as

\[|\text{Hess } P_t f| \leq \left( K_1 \sqrt{t} + \frac{K_2}{2} t + \frac{1}{\sqrt{t}} \right) \mathbb{E}\left[ e^{\frac{1}{2}\sigma_1 l} \right] e^{K_0 t} \|\nabla f\|_\infty + \frac{\sigma_2}{2} \mathbb{E}\left( e^{\frac{1}{2}\sigma_1 l} \int_0^t e^{\frac{1}{2}\sigma_1 l_s} \, dl_s \right) e^{K_0 t} \|\nabla f\|_\infty.\]

We complete the proof by observing that \( P_t \phi = e^{-\lambda t/2} \phi \).

Combining Theorem 3.2 and Lemma 2.7 we are now in a position to prove Theorem 1.3.

**Theorem 3.3.** Let \( D \) be an \( n \)-dimensional compact Riemannian manifold with boundary \( \partial D \). Let \( K_0, K_1, K_2, \sigma_1, \sigma_2 \) be non-negative constants such that \( \text{Ric} \geq -K_0, |R| \leq K_1 \) and \( |d^* R + \nabla \text{Ric}| \leq K_2 \) on \( D \), and that \( \Pi \geq -\sigma_1 \) and \( |\nabla^2 N - R(N)| \leq \sigma_2 \) on the boundary \( \partial D \). For \( h \in C^\infty(D) \) with \( \min_D h = 1 \) and \( N \log h|_{\partial D} \geq 1 \), let \( K_{h,\alpha} := \sup_D (-\Delta \log h + \alpha |\nabla \log h|^2) \) with \( \alpha \) a non-negative constant. Then for any non-trivial \( (\phi, t) \in \text{Eig}_N(D) \),

\[ \frac{\|\text{Hess } \phi\|_\infty}{\|\phi\|_\infty} \leq C_{N,t}(D)_t \]

where

\[ C_{N,t}(D) = e\left( 1 + \frac{K_1 + 2K_0 + 2\sigma_1 K_{h,2r_1} + K_2 + 2\sigma_2 K_{h,2r_1}}{\lambda} \right) \left( \frac{\|h\|^{3r_1}_{\infty}}{\|h\|^{2r_1}_{\infty}} \right)^{\lambda} \left( \sqrt{2\lambda + 4K_0 + 4\sigma_1 K_{h,2r_1}} \right)^{\alpha} + \frac{\sigma_2 e}{\lambda} \right) \] \[ \left( \sqrt{2\lambda + 4K_0 + 4\sigma_1 K_{h,2r_1}} \right)^{\alpha} \ln \left( \frac{\|h\|^{3r_1}_{\infty}}{\|h\|^{2r_1}_{\infty}} \right). \]

**Proof.** By Lemma 2.7 we have

\[ \mathbb{E}[e^{\sigma_1 l_t}] \leq \mathbb{E}[e^{\sigma_1 l}] \leq \|h\|_{\infty}^{2\sigma_1} \exp((\sigma_1 K_{h,2r_1})t), \]

and

\[ \mathbb{E}[e^{\sigma_1 l_t}] \leq \|h\|_{\infty}^{2\sigma_1} \exp((\sigma_1 K_{h,2r_1})t). \]

Moreover, we observe that

\[ \mathbb{E}\left[ e^{\frac{1}{2}\sigma_1 l} \int_0^t e^{\frac{1}{2}\sigma_1 l_s} \, dl_s \right] \leq \frac{2(\mathbb{E}[e^{(\sigma_1 + \epsilon) l}] - 1)}{\sigma_1 + \epsilon} \]

\[ \leq \frac{2}{\sigma_1 + \epsilon} (\|h\|_{\infty}^{2\sigma_1} \exp((\sigma_1 + \epsilon) K_{h,2r_1}) - 1) \]

\[ \leq \frac{2}{\sigma_1 + \epsilon} (\|h\|_{\infty}^{2\sigma_1} \exp((\sigma_1 + \epsilon) K_{h,2r_1}) - 1) \]

\[ \leq 4\|h\|_{\infty}^{2\sigma_1} \ln \|h\|_{\infty} + 2\|h\|_{\infty}^{2\sigma_1} \exp((\sigma_1 + \epsilon) K_{h,2r_1}) K_{h,2r_1}. \]

Letting \( \epsilon \) tend to 0, we arrive at

\[ \mathbb{E}\left[ e^{\frac{1}{2}\sigma_1 l} \int_0^t e^{\frac{1}{2}\sigma_1 l_s} \, dl_s \right] \leq 4\|h\|_{\infty}^{2\sigma_1} \ln \|h\|_{\infty} + 2\|h\|_{\infty}^{2\sigma_1} \exp((\sigma_1 K_{h,2r_1})t) K_{h,2r_1}. \]

Therefore, combining this with Theorem 3.2 we obtain

\[ \frac{\|\text{Hess } \phi\|_\infty}{\|d\phi\|_\infty} \leq e^{\frac{1}{2}\lambda + K_0 r} \left( \frac{1}{\sqrt{t}} + K_1 \sqrt{t} + \frac{K_2}{2} \right)\|h\|^{2r_1}_{\infty} \exp((\sigma_1 K_{h,2r_1})t) \]

\[ + \frac{\sigma_2 e^{\frac{1}{2}\lambda + K_0 r}}{2 \ln \|h\|_{\infty} + K_{h,2r_1} t}\|h\|^{2r_1}_{\infty} \exp((\sigma_1 K_{h,2r_1})t) \]
\[
\leq e^{(\lambda+K_0)t}\left(\frac{1}{\sqrt{t}} + K_1 \sqrt{t} + \frac{K_2}{2}\right)||h||_{\infty}^{2\sigma_1} \exp(\sigma_1 K_{h,2\sigma_1} t) + \sigma_2 e^{(\lambda+K_0)t} [2\ln ||h||_{\infty} + K_{h,\sigma_1}] ||h||_{\infty}^{2\sigma_1} \exp(\sigma_1 K_{h,2\sigma_1} t).
\]

Let \( t = (\lambda + 2K_0 + 2\sigma_1 K_{h,2\sigma_1})^{-1} \). Then we get

\[
\frac{||\text{Hess } \phi||_{\infty}}{||\phi||_{\infty}} \leq \left( \frac{K_1}{\sqrt{\lambda + 2K_0 + 2\sigma_1 K_{h,2\sigma_1}}} + \sqrt{\lambda + 2K_0 + 2\sigma_1 K_{h,2\sigma_1}} \right) + \frac{K_2 + 2\sigma_2 K_{h,\sigma_1}}{2(\lambda + 2K_0 + 2\sigma_1 K_{h,2\sigma_1})} + 2\sigma_2 \ln ||h||_{\infty} ||h||_{\infty}^{2\sigma_1} \sqrt{e}.
\]

On the other hand, from [2], it has already been shown that

\[
\frac{||d\phi||_{\infty}}{||\phi||_{\infty}} \leq \frac{1}{\sqrt{t}} e^{(\sigma_1 K_0 + \lambda)t} \leq \frac{1}{\sqrt{t}} ||h||_{\infty}^{\sigma_1} \exp\left(\frac{1}{2}(\lambda + \sigma_1 K_{h,2\sigma_1} + K_0)t\right).
\]

Let \( t = (\lambda + K_0 + \sigma_1 K_{h,2\sigma_1})^{-1} \). Then we get

\[
\frac{||d\phi||_{\infty}}{||\phi||_{\infty}} \leq \sqrt{\lambda + K_0 + \sigma_1 K_{h,2\sigma_1}} ||h||_{\infty}^{\sigma_1} \sqrt{e}.
\]

We then conclude that

\[
\frac{||\text{Hess } \phi||_{\infty}}{||\phi||_{\infty}} \leq \left( \lambda + K_1 + 2K_0 + 2\sigma_1 K_{h,2\sigma_1} + \frac{K_2 + 2\sigma_2 K_{h,\sigma_1}}{2(\lambda + 2K_0 + 2\sigma_1 K_{h,2\sigma_1})} + 2\sigma_2 \ln ||h||_{\infty} \sqrt{\lambda + K_0 + \sigma_1 K_{h,2\sigma_1}} \right) ||h||_{\infty}^{2\sigma_1} \sqrt{e}.
\]

\( \square \)

**Proof of Theorem 3.3** From the conditions we see that Condition (A) is satisfied. Then, the Hessian estimate of Neumann eigenfunctions in Theorem 3.3 remain valid by substituting the \( h \) defined in (2.26). Then under replacing

\[ K_{h,\alpha} \quad \text{and} \quad ||h||_{\infty} \]

by

\[ K_{\alpha} := \frac{n}{r_0} + \alpha \quad \text{and} \quad e^{\alpha r_0/2} \]

respectively, the conclusion is just listed in Theorem 1.3. \( \square \)

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