Hall viscosity induced transverse voltage in two-dimensional Fermi liquids

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The absence of parity and time-reversal symmetry in two-dimensional Fermi liquids gives rise to non-dissipative transport features characterized by the Hall viscosity. For non-vanishing magnetic fields, the Hall viscous force directly competes with the Lorentz force, since both mechanisms contribute to the Hall voltage. In this work, we present a channel geometry that allows us to uniquely distinguish these two contributions and derive, for the first time, their functional dependence on all external parameters. In particular, the ratio of Hall viscous to Lorentz force contributions decreases with the width and slip-length of our channel, while it increases with its carrier density and electron-electron mean free path. Therefore, for typical materials such as GaAs, the Hall viscous contribution dominates the Lorentz signal up to a few tens of millitesla. This paves the way to uniquely measure and identify Hall viscous signals in experiment.

Introduction. The idea of describing electrons in solids via hydrodynamics goes back to the discovery of the Gurzhi effect in (Al)GaAs quantum wires [1–3]. Recently, hydrodynamic transport has received renewed attention due to the accessibility of the hydrodynamic regime in modern materials [4–7], even beyond the Fermi liquid regime [8–10]. In particular, two-dimensional systems that violate parity invariance are of special interest, since they exhibit novel non-dissipative hydrodynamic transport coefficients, such as the Hall viscosity [11–13]. Recent experiments in graphene have shown that \( \eta_H \) may be of the same order of magnitude as the shear viscosity \( \eta \) and therefore is expected to significantly affect the fluid transport [17]. However, current theoretical and experimental works [15–20] do not provide a quantitative answer to the functional dependence of Hall viscous effects on external parameters describing the system.

To address this open issue, we predict in this work a Hall viscous contribution \( \Delta V_{\text{m}} \) to the total Hall voltage \( \Delta V_{\text{tot}} = \Delta V_{\text{m}} + \Delta V_B \) measured across a two-dimensional channel in the Fermi liquid regime, with \( \Delta V_B \) the Lorentz contribution. We derive the functional dependence of \( \Delta V_{\text{m}} \) on all external parameters. In particular, we evaluate its dependence on the wire width \( W \), on the impurity \( l_{\text{imp}} \) and slip lengths \( l_s \), as well as on the equilibrium carrier density \( \rho_0 \) and the magnetic field \( B \). This allows to explicitly distinguish \( \Delta V_{\text{m}} \) from \( \Delta V_B \) in two-dimensional Hall setups, such as the structure shown in Fig. 1. In this way, we provide a direct answer to the question of how to distinguish Hall viscous and Lorentz force contributions to the total Hall voltage. For both weak and strong magnetic fields as well as for clean systems our approach is analytical, whereas for intermediate field regimes we numerically solve the Navier-Stokes equations. In particular, we demonstrate that \( \Delta V_B \) is proportional to the velocity profile integrated over the channel width, and therefore proportional to the total fluid momentum. In contrast, \( \Delta V_{\text{m}} \) exclusively depends on the gradient of the velocity profile at the boundaries of the system. Based on this, we analytically prove that for small fields and clean systems with Poiseuille-like (parabolic) velocity profile the ratio \( \Delta V_{\text{m}} / \Delta V_B \) is determined completely by the interplay of length scales defining the system. While this ratio increases with the transverse electron-electron mean free path \( l_{\text{ee}} \), it decreases with \( W, l_s \) and \( l_{\text{imp}} \). Since \( l_{\text{ee}} \) and \( l_{\text{imp}} \) are density dependent, this ratio acquires a density dependence as well. In particular, in the absence of impurities \( \Delta V_{\text{m}} / \Delta V_B = O(\rho_0^2) \), whereas for weak impurity strength there exists an additional \( O(\rho_0^2) \) contribution. Hence, it is possible to achieve \( \Delta V_{\text{m}} / \Delta V_B \gg 1 \) by tuning the width and density of the sample. For \( \rho_0 = 9.1 \times 10^{11} \text{cm}^{-2} \) and \( W = 3 \mu \text{m} \), such a regime is in many realized in GaAs for \( B \lesssim 20 \mu \text{T} \). Beyond the weak field limit, we numerically show that increasing \( B \) strongly reduces \( \Delta V_{\text{m}} / \Delta V_B \), due to the suppression of \( \eta \) and \( \eta_H \) [13]. For large magnetic fields, for example in GaAs for \( B \gtrsim 0.5 \mu \text{T} \), impurities dominate the transport, which causes an Ohmic (flat) fluid profile with vanishing \( \Delta V_{\text{m}} / \Delta V_B \). The critical field \( B_c \) at which the

![FIG. 1. Setup to distinguish Hall viscous from Lorentz force contributions to the total Hall voltage. The red curve shows the hydrodynamic velocity profile \( v_y(x) \) in a channel of length \( L \) and width \( W \ll L \) in presence of an electric field \( \vec{E} \) and an out-of-plane magnetic field \( \vec{B} \). Momentum transfer to the boundaries, captured by the slip length \( l_s \), is shown by the red dashed curve. The total Hall force \( \vec{F}_{\text{tot}} \) induces a transverse pressure gradient, giving rise to a gradient in the chemical potential \( \Delta \mu \). Illustrated (initially) by the (dashed) black line.](cond-mat.mes-hall)
transition \( \Delta V_{\text{nh}} / \Delta V_B = 1 \) occurs, increases with decreasing \( W,  l_s, l_{\text{imp}}, \) and \( l_{\text{ce}}^{-1} \), even beyond the weak field limit. For GaAs we find \( B_s \sim \mathcal{O}(10 \text{mT}) \), making our predictions experimentally measurable.

**Model.** Our setup is shown in Fig. 1. We analyze the hydrodynamic flow of non-relativistic electrons along a two-dimensional channel under the combined influence of a DC electric field \( \mathbf{E} = -E_0 \mathbf{e}_x \) and an out-of-plane magnetic field \( \mathbf{B} = -B \mathbf{e}_z \). To justify our hydrodynamic approach, we assume that \( l_{\text{ce}} \) is the shortest length scale in our system. In particular, it is smaller than the cyclotron radius \( r_c = m_{\text{eff}} v_F / |eB| \), which is defined in terms of the effective electron mass \( m_{\text{eff}} \), the electron charge \( e < 0 \) and the Fermi velocity \( v_F \). Additionally, we assume that our system is incompressible and has translational invariance in longitudinal, as well as vanishing current in transversal direction. These assumptions lead to the hydrodynamic variables [21]

\[
v = v_x(y) \mathbf{e}_x, \quad \mu = \mu_0 + \Delta \mu, \quad T = T_0 = \text{constant}, \quad (1)
\]

where \( \Delta \mu \) is the local chemical potential fluctuation about the global equilibrium value \( \mu_0 \) [22]. The dynamical equations for this ansatz are obtained by applying the non-relativistic Navier-Stokes equations for incompressible fluids [23–26],

\[
\begin{align*}
\eta \partial^2_y v_x &= e \left( \rho_0 + \frac{\partial \rho}{\partial \mu} \Delta \mu \right) E_x + \frac{\rho_0 v_F m_{\text{eff}}}{l_{\text{imp}}} v_x, \quad (2) \\
\partial_y \rho &= \mu_0 \partial_x \Delta \mu = (eB \rho_0 - \eta_l \partial^2_x) v_x. \quad (3)
\end{align*}
\]

Here, \( \rho = \rho_0 + \rho_0 \Delta \mu \) defines the pressure in terms of \( \Delta \mu \). In two dimensions, the dynamics of incompressible, non-relativistic fluids is entirely captured by the shear and Hall viscosities [13, 27]

\[
\begin{align*}
\eta &= \frac{\eta_0}{1 + (2 l_{\text{ce}} / r_c)^2}, \quad \eta_l = \frac{2 \eta_0 l_{\text{ce}} / r_c}{1 + (2 l_{\text{ce}} / r_c)^2},
\end{align*}
\]

(4)

where \( \eta_0 = m_{\text{eff}} \rho_0 v_F l_{\text{ce}} / 4 \) is the shear viscosity for vanishing magnetic field \( B = 0 \). The main goal of this work is to determine the total transverse voltage drop

\[
\Delta V_{\text{tot}} = \Delta \mu(W) - \Delta \mu(0) / e
\]

as a response to \( E_x \), and to separate the Hall viscous and Lorentz force contributions therein. Therefore, we solve Eqs. (2) and (3) under the general boundary conditions

\[
\begin{align*}
v_x(-l_s) = v_x(W + l_s) &= 0, \\
\Delta \mu|_{y=0} + \Delta \mu|_{y=W} &= 0.
\end{align*}
\]

The slip length \( l_s \) characterizes the velocity profile at the boundaries, as shown in Fig. 1. For \( l_s = 0 \), the fluid flow is Poiseuille-like (parabolic), while \( l_s \rightarrow \infty \) defines the diffusive regime with a flat \( v_x(y) \) [28].

**Transverse Response.** The total Hall voltage consists of two different terms, namely the Hall viscous and the Lorentz force contributions, satisfying \( \Delta V_{\text{tot}} = \Delta V_{\text{nh}} + \Delta V_B \). To analytically derive the functional dependence of these building blocks, we first restrict ourselves to weak magnetic fields, defined by \( l_G / r_c \ll 1 \). The Gurzhi length \( l_G = \sqrt{\eta_l / \rho_0 m_{\text{eff}} v_F} \) quantifies the relative strength of impurity to shear effects. While the flow is Poiseuille-like for \( l_G / W \gg 1 \), it becomes Ohmic for \( l_G / W \ll 1 \). In general, the assumption of weak magnetic fields allows us to expand the velocity profile and hence the Navier-Stokes equations in powers of \( B \). With this ansatz, the linearized Navier-Stokes Eqs. (2, 3) predict the total Hall voltage of Eq. (5) to be given by [26]

\[
\Delta V_{\text{tot}} = \frac{e \eta_l m_{\text{eff}} v_F}{m_{\text{eff}} v_F} \left[ l_G \left( \frac{m_{\text{eff}} v_F \eta_l}{e \eta_l m_{\text{eff}} v_F} - B \right) \right] \\
\times \left[ A_1 \sinh(W l_G^{-1}) + A_2 \cosh(W l_G^{-1}) - 1 \right] - BW.
\]

(7)

where \( A_1, A_2 \) are functions of \( l_G \) and \( W \), explicitly defined in Ref. [26]. In the limit \( W / l_G \ll 1 \), where shear effects dominate, Eq. (7) implies up to first order in \( l_{\text{imp}}^{-1} \)

\[
\begin{align*}
\Delta V_{\text{nh}} &= \frac{\eta_l}{\eta} E_x \left[ W + \rho_0 m_{\text{eff}} v_F W^3 + 6 l_s W^2 + 6 l_s^2 W \right], \\
\Delta V_B &= \frac{e \rho_0 B E_x}{12 \eta} \left( W^3 + 6 l_s W^2 + 6 l_s^2 W \right).
\end{align*}
\]

(8)

For weak magnetic fields, Eq. (4) predicts that \( \eta_l \) linearly increases while \( \eta \) decreases as a function of \( B \). As a consequence, Eq. (5) implies

\[
\Delta V_{\text{nh}} / \Delta V_B = \frac{l_{\text{ce}}^2}{6 W^2 + l_s W + l_s^2} + 2 l_{\text{ce}}^2 / l_{\text{imp}}^2.
\]

(9)

Thus, for small \( B \) and \( W / l_G \ll 1 \), this ratio is com-
Summarizing, the distinct dependence of \( \Delta V_{\eta} \) on \( B \) in comparison to Lorentz force signal as the the carrier Hall viscous contribution becomes strongly enhanced in magnetic fields. Hence, in the weak field limit and for \( l_s \ll l_{\text{imp}} \), Eq. (10) predicts \( \Delta V = \frac{e l_{\text{imp}}}{m_{\text{eff}}} B W \).

The dependence of \( \Delta V_{\eta} \) on the magnetic field is more subtle. Figure 3 shows that for weak magnetic fields with Poiseuille-like velocity profile, the gradient \( \partial_y v_x(y)|_{y=0,W} \) increases as a function of \( B \). According to Eq. (11), this corresponds to an enhancement of \( \Delta V_{\eta} \). As discussed above, impurities dominate the bulk transport for large \( B \), implying an Ohmic (flat) velocity profile. This again decreases \( \partial_y v_x(y)|_{y=0,W} \) and therefore reduces \( \Delta V_{\eta} \). Altogether, this implies that systems in which Hall viscous effects dominate the weak magnetic field regime are eventually always driven to \( \Delta V_{\eta}/\Delta V_B < 1 \). In particular, the transition \( \Delta V_{\eta}/\Delta V_B = 1 \) occurs at a critical magnetic field \( B_c \), which strongly depends on the width, slip-length and electron-electron mean free path of the system.

This is explicitly shown in Fig. 3 which illustrates \( \Delta V_{\eta}/\Delta V_B \) as a function of \( W, l_s, l_{\text{imp}} \) and \( l_{\text{imp}} \). According to Eq. (11), the dependence of this ratio on these parameters can be explained by analyzing the corresponding velocity profiles, depicted in Fig. 2. We observe that as \( l_s \) increases, the gradient \( \partial_y v_x(y)|_{y=0,W} \) decreases, lead-
ing to a reduction of $\Delta V_{\text{tot}}$. In contrast, since the overall fluid momentum is enhanced, $\Delta V_B$ increases as a function of the slip length. Therefore, in total, $\Delta V_{\text{tot}}$ monotonically decreases as a function of $l_s$, as illustrated in Fig. 3. Increasing the channel width leads to the same result, since it also increases the overall fluid momentum and at the same time decreases $\partial_y v_x(y)|_{y=0.1W}$. In contrast, $\Delta V_{\text{ee}}/\Delta V_B$ scales oppositely with the electron-electron mean free path. According to Eq. (4), the viscosities $\eta, \eta_{\text{ee}}$ as well as the ratio $\eta_{\text{ee}}/\eta$ increase as a function of $l_{\text{ee}}$ as long as $l_{\text{ee}} \ll r_c$. Effectively, this reduces the overall fluid momentum but increases the Hall viscous voltage contribution. This dependence changes for $l_{\text{ee}}/r_c \gtrsim 1$. However, in this limit the applicability of hydrodynamics breaks down. Last but not least, Fig. 3 illustrates that as a consequence of the hydrodynamic assumption $l_{\text{ee}} \ll l_{\text{imp}}, l_{\text{imp}}$ does not significantly change $\Delta V_{\text{ee}}/\Delta V_B$. All our numerical results agree with Eq. (9), which analytically describes the functional dependence of $\Delta V_{\text{ee}}/\Delta V_B$ in weak magnetic fields.

To demonstrate the experimental relevance of our predictions, Fig. 4 finally shows $\Delta V_{\text{tot}}, \Delta V_B$ and $\Delta V_{\text{tot}}$ in GaAs as a function of the magnetic field [8, 9]. For $B \ll B_c \approx 20\text{mT}$, the Hall viscous voltage contribution exceeds the Lorentz signal and gives rise to a Hall viscosity dominated transport regime. Since with the exception of $l_{\text{ee}}$ it is possible to evaluate all parameters in Eqs. (2)-(4) precisely, measuring $\Delta V_{\text{ee}}$ and $\Delta V_B$ with increasing $B$, enables us to determine the electron-electron mean free path in this sample by fitting theoretical and experimental data. In general, this procedure works for any Fermi liquid and is therefore a powerful tool to evaluate the interaction strength in those systems. For $B \gg B_c$, Fig. 4 illustrates that transport is dominated by the Lorentz signal. Eventually, this leads to the Ohmic, impurity driven velocity profile, implying Eq. (12). Our present approach does not incorporate the formation of Landau levels which in GaAs occurs beyond the order of a few tens of millitesla. Hence, our predictions pave the way for an experimental identification of Hall viscous effects and explicitly illustrate how this response can be used to determine the electron-electron interaction strength in hydrodynamic systems.

Possible future directions of our approach include two-dimensional massive Dirac systems, where parity and time reversal symmetry are already broken for vanishing magnetic fields [34] and a torsional Hall viscous term is expected to be present [12, 35].

Acknowledgments. We thank Carlos Hoyos, Andrew Lucas, Hartmut Buhmann and Laurens Molenkamp for useful discussions. We acknowledge financial support through the DFG via SFB 1170 ‘ToCoTronics’, the ENB Graduate School on Topological Insulators and through the Würzburg-Dresden Cluster of Excellence on Complexity and Topology in Quantum Matter - ct.qmat (EXC 2147, project-id 39085490).

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SUPPLEMENTAL MATERIAL

Equations of motion. We derive the equations of motion for a charged two-dimensional Fermi liquid confined to a channel geometry and under the influence of a DC electric field $E$ and an out-of-plane magnetic field $B$ (cf. Eqs. (1)-(2) of the main text). In particular, this set of equations is given by the charge continuity as well as by the momentum conservation (Navier-Stokes) equations,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,$$

$$\rho \left( \partial_t + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla p + \eta \nabla^2 \mathbf{v} + \eta \mathbf{B} \times \mathbf{e}_z + \rho \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} \right) - \frac{\rho_0 v_0 T_0}{l_{\text{imp}}} \mathbf{v}.$$

In our analysis, we consider a steady, hydrodynamic flow of electrons, which is translationally invariant along the $\mathbf{e}_z$-direction. Moreover, we assume a vanishing current in the $\mathbf{e}_y$-direction, implying that the velocity profile takes the form $\mathbf{v} = v_x(y) \mathbf{e}_x$. To obtain an inhomogeneous, non-trivial velocity profile, our system needs to deviate from global thermal equilibrium. However, in order to be able to linearize Eqs. (S1) and (S2), we assume that variations of the chemical potential and temperature are small compared to their global equilibrium values

$$\mu(y) = \mu_0 + \Delta \mu(y) \quad \text{with} \quad \Delta \mu(y) \ll \mu_0,$$

$$T(y) = T_0 + \Delta T(y) \quad \text{with} \quad \Delta T(y) \ll T_0.$$

For typical Fermi liquids, such as GaAs, $\mu_0 = \mathcal{O}(50 \text{ meV})$ whereas $T_0 = \mathcal{O}(1 \text{ K})$. Note, that due to time and translational invariance in the $\mathbf{e}_z$-direction, $\Delta \mu$ and $\Delta T$ vary only in the $\mathbf{e}_y$-direction. In terms of $\mu(y)$ and $T(y)$, pressure and density fluctuations in our system are given by

$$\rho(y) = \rho_0 + \frac{\partial \rho_0}{\partial \mu_0} \Delta \mu + \frac{\partial \rho_0}{\partial T_0} \Delta T,$$

$$p(y) = p_0 + \frac{\partial p_0}{\partial \mu_0} \Delta \mu + \frac{\partial p_0}{\partial T_0} \Delta T = p_0 + \rho_0 \Delta \mu + s_0 \Delta T,$$

where $s_0$ is the equilibrium entropy of the fluid. With these assumptions in place, Eq. (S1) is trivially satisfied. Therefore, the dynamics of $v_x(y)$, $\Delta \mu(y)$ and $\Delta T(y)$ are completely described by the set of Navier-Stokes equations (S2). In particular, the gradients of $\Delta \mu(y)$ and $\Delta T(y)$ appear only through the pressure gradient in Eq. (S2). For our system, the gradient of temperature is negligible compared to the gradient of the chemical potential. To see this, we write

$$\partial_y p = \rho_0 \mu_0 \partial_y \left( \frac{\Delta \mu}{\mu_0} \right) + s_0 T_0 \partial_y \left( \frac{\Delta T}{T_0} \right).$$

Due to the assumptions in Eq. (S3), the dimensionless gradients $\partial_y(\Delta \mu/\mu_0)$ and $\partial_y(\Delta T/T_0)$ are of the same order of magnitude. As a consequence, the relative strength of the chemical potential contribution to the temperature contribution in Eq. (S5) is given by $U = \rho_0 \mu_0 / s_0 T_0$. 

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If $\mathcal{U} \gg 1$, the chemical potential gradient dominates over temperature fluctuations and vice versa. For typical Fermi liquids, such as GaAs, we find $\mathcal{U} \approx 10^{16}$. Hence, in those samples the gradient of temperature is clearly negligible. Thus, we assume $T(y) = T_0$ and define

$$\rho(y) = \rho_0 + \frac{\partial \rho_0}{\partial \mu} \Delta \mu(y) \land p(y) = p_0 + \rho_0 \Delta \mu(y) . \quad (S6)$$

Substituting this into Eq. (S2), leads to

$$\eta \frac{\partial^2}{\partial y^2} \psi_{\text{x}}(y) = \left( \rho_0 + \frac{\partial \rho_0}{\partial \mu} \Delta \mu \right) \left( eE_x + \frac{\nu \mu_{\text{eff}}}{l_{\text{imp}}} \psi_{\text{x}}(y) \right) , \quad (S7)$$

$$\partial_y p = -eB \left( \rho_0 + \frac{\partial \rho_0}{\partial \mu} \Delta \mu \right) - \eta \theta_y \frac{\partial^2}{\partial y^2} \psi_{\text{x}}(y) . \quad (S8)$$

So far, Eqs. (S7)-(S8) still deviate from Eqs. (1)-(2) of the main text, due to additional terms $\propto \Delta \mu(y) \psi_{\text{x}}(y)$. However, these terms induce non-linear corrections to our physical observables in terms of the electromagnetic fields. Therefore, in the context of linear response theory, we are allowed to drop these additional terms without loss of generality. This leads to Eqs. (1)-(2) of the main text

$$\eta \frac{\partial^2}{\partial y^2} \psi_{\text{x}}(y) = \left( \rho_0 + \frac{\partial \rho_0}{\partial \mu} \Delta \mu \right) eE_x + \rho_0 \frac{\nu \mu_{\text{eff}}}{l_{\text{imp}}} \psi_{\text{x}}(y) , \quad (S9)$$

$$\partial_y p = \rho_0 \partial_y \Delta \mu = \left( eB \rho_0 - \eta \theta_y \frac{\partial^2}{\partial y^2} \psi_{\text{x}}(y) \right) . \quad (S10)$$

For our setup, we supplement these equations with no-slip boundary conditions of the form

$$\psi_{\text{x}}(-l_s) = \psi_{\text{x}}(W + l_s) = 0 , \quad (S11)$$

$$\Delta \mu\big|_{y=0} + \Delta \mu\big|_{y=W} = 0 . \quad (S12)$$

In the following, we provide an explicit solution of the Navier-Stokes Eqs. (S9)-(S10) under these boundary conditions for weak and strong magnetic fields, since in that regime the physical properties of the analytic solutions become most apparent.

**Weak magnetic field regime.** The weak magnetic field limit $r_s \gg \lambda_G$ allows for the expansion of the velocity profile and hence of the Navier-Stokes equations in powers of $B$. Technically, this expansion is achieved by introducing the dimensionless parameter $\epsilon \ll 1$, satisfying

$$B \to \epsilon B , \quad \eta\lambda_G \to \epsilon \eta \lambda_G , \quad \Delta \mu \to \epsilon \Delta \mu , \quad \psi_{\text{x}} = v_0^\text{x} + \epsilon v_1^\text{x} . \quad (S13)$$

In particular, this assumes that to first order the system responds linearly in terms of the magnetic field. With this ansatz the Navier-Stokes Eqs. (S9)-(S10) reduce to

$$\eta \frac{\partial^2}{\partial y^2} v_0^\text{x}(y) - \frac{m_{\text{eff}} \nu \rho_0}{l_{\text{imp}}} v_0^\text{x}(y) = e \rho_0 \nu E_x , \quad (S14)$$

$$\left( eB \rho_0 - \eta \theta_y \frac{\partial^2}{\partial y^2} \right) v_0^\text{x}(y) = \rho_0 \partial_y \Delta \mu(y) , \quad (S15)$$

$$\eta \frac{\partial^2}{\partial y^2} v_1^\text{x}(y) - \frac{m_{\text{eff}} \nu \rho_0}{l_{\text{imp}}} v_1^\text{x}(y) = e \nu \frac{\partial \rho_0}{\partial \mu} \Delta \mu(y) . \quad (S16)$$

To find an explicit solution of this set of equations we first determine $v_0^\text{x}(y)$ by solving Eq. (S14), which is the zero-field Poiseuille flow equation in the presence of impurities. Once we have obtained $v_0^\text{x}(y)$, we calculate $\Delta \mu(y)$ by integrating Eq. (S15). Substituting this quantity in Eq. (S16) finally allows us to derive the first order velocity correction $v_1^\text{x}(y)$. Explicitly, we find

$$v_0^\text{x}(y) = -\frac{eE_x \eta l_{\text{imp}}}{\nu \mu_{\text{eff}} l_{\text{imp}}} \left( A_1 \cosh [y\lambda_G^{-1}] + A_2 \sinh [y\lambda_G^{-1}] + 1 \right) \quad (S17)$$

$$\Delta \mu(y) = \frac{e \nu \mu_{\text{eff}} E_x}{\eta l_{\text{imp}}} \left[ l_G^{\nu} \frac{m_{\text{eff}} \nu \theta}{\eta l_{\text{imp}}} - eB \right]$$

$$\times \left( A_1 \sinh [y\lambda_G^{-1}] + A_2 \cosh [y\lambda_G^{-1}] - eB y \right) + \Gamma , \quad (S18)$$

where for clarity we defined

$$A_1 = -\cosh \left[ \frac{W}{2\lambda_G} \right] \sech \left[ \frac{2l_s + W}{2\lambda_G} \right] , \quad (S19)$$

$$A_2 = \sinh \left[ \frac{W}{2\lambda_G} \right] \sech \left[ \frac{2l_s + W}{2\lambda_G} \right] , \quad (S20)$$

$$\Gamma = -\frac{e \nu \mu_{\text{eff}} E_x}{2 \eta l_{\text{imp}}} \left[ l_G^{\nu} \frac{m_{\text{eff}} \nu \theta}{\eta l_{\text{imp}}} - eB \right]$$

$$\times \left( A_1 \sinh \left[ W\lambda_G^{-1} \right] + A_2 \cosh \left[ W\lambda_G^{-1} \right] + 1 \right) - eB W . \quad (S21)$$

Before we explicitly present our result for $v_1^\text{x}(y)$, we want to emphasize that Eq. (S18) predicts the total Hall voltage $\Delta V_{\text{tot}} = [\Delta \mu(W) - \Delta \mu(0)]/e$, measured across the sample. To derive Eq. (6) of the main text, we need to expand the hyperbolic functions in Eqs. (S19)-(S21) up to third order in $W/l_G$. As a result, we find

$$\Delta V_{\text{mi}} = \frac{\eta l_G^{\nu} E_x}{\eta} \left[ W + \frac{\rho_0 \nu \mu_{\text{eff}} \nu E_x}{12 \eta l_{\text{imp}}} (W^3 + 6l_s W^2 + 6l_s^2 W) \right] \quad (S22)$$

$$\Delta V_{\text{B}} = \frac{\epsilon \rho_0 B E_x}{12 \eta} \left( W^3 + 6l_s W^2 + 6l_s^2 W \right) . \quad (S23)$$

In Eq. (8) of the main text, we have illustrated the density dependence of these voltage contributions,

$$\Delta V_{\text{mi}} = f_1 \rho_0 + f_2 (n_{\text{imp}}) \land \Delta V_{\text{B}} = f_3 \rho_0^{-2} . \quad (S24)$$

However, so far we did not specify the functions $f_{1,2,3}$. This will be the purpose of the following paragraph. The linear, impurity independent scaling of $\Delta V_{\text{mi}}$ is defined as the $l_{\text{imp}} \to \infty$ limit of Eq. (S22). The density dependence of this term is given by (cf. Eq. (3) of the main text)

$$\Delta V_{\text{mi}}^0 = \frac{\eta l_G^{\nu} E_x W}{r_c} = \frac{2 \rho_0 \nu \mu_{\text{eff}} \nu E_x W|eB|}{m_{\text{eff}}}$$

$$= \frac{12 \eta \lambda_G^3 E_x W|eB|}{F_{\pi} m_{\text{eff}} k_B T^2} \ln^2 \left( \frac{\hbar^2 \pi \rho_0}{m_{\text{eff}} k_B T} \right) \quad (S25)$$

Here, according to Refs. 13 31 38 39, we introduced the second momentum relaxation time

$$\tau_{ee} = \frac{l_c}{\nu E_x} = \frac{6 \lambda_G^3}{F_{\pi} m_{\text{eff}} k_B T^2} \ln^2 \left( \frac{\hbar^2 \pi \rho_0}{m_{\text{eff}} k_B T} \right) . \quad (S26)$$
where $k_B$ is the Boltzmann constant and $F_x$ is a geometric factor, characterizing electron-electron scattering amplitudes. Since density variations do not significantly change the $\ln^2(\mu/k_B T)$ terms in the Fermi liquid regime $\mu \gg k_B T$, we treat $f_1$ as density independent function. Moreover, the impurity contribution to $\Delta V_{\text{imp}}$ is given by

$$\Delta V_{\text{imp}} = \frac{\eta H E_x}{\eta} \rho_0 m_{\text{eff}} \frac{\rho_{\text{imp}}}{12 \eta \tau_{\text{imp}}} (W^3 + 6l_s W^2 + 6l_s^2 W) \quad (S27)$$

$$= \frac{|eB|m_{\text{eff}}^2 \nu^3 n_{\text{imp}} E_x}{3h^3 \pi} (W^3 + 6l_s W^2 + 6l_s^2 W) = f_2.$$  

Here, we considered point like impurities with concentration $n_{\text{imp}}$, scattering strength $\nu_0$ and inverse momentum relaxation time $\tau_{\text{imp}}^{-1} = \nu_0 l_{\text{imp}}^{-1} = m_{\text{eff}} \nu_0^3 \hbar^{-3} n_{\text{imp}}$. In the same manner, Eq. (S23) evolves to

$$\Delta V_B = \frac{e\rho_0 B E_x}{12 \eta} (W^3 + 6l_s W^2 + 6l_s^2 W) \quad (S28)$$

$$= \frac{e \rho_0 B E_x}{3m_{\text{eff}} v_F^2 \tau_{ee}} (W^3 + 6l_s W^2 + 6l_s^2 W)$$

$$= \frac{e m_{\text{eff}} B E_x}{6\pi \hbar^2 \rho_0^2 \tau_{ee}} (W^3 + 6l_s W^2 + 6l_s^2 W) = f_3 \rho_0^{-2}.$$  

Having derived the explicit density dependence of $\Delta V_{\text{imp}}$ and $\Delta V_B$, we proceed in presenting our solution for $v_1^x(y)$. Therefore, we substitute Eq. (S18) into Eq. (S16), which leads to the first order velocity correction

$$v_1^x(y) = (C_1 + \lambda_1 y) \cosh[y \eta^{-1}_{\text{G}}] + (C_2 + \lambda_2 y) \sinh[y \eta^{-1}_{\text{G}}] - A_3 + A_4 y.$$  

Here, for clarity, we defined the following functions

$$\lambda_1 = \frac{e^2 l_{\text{imp}}^2 E_x^2 \frac{\nu_0}{\eta H}}{2m_{\text{eff}} v_F} \left( eB - \frac{m_{\text{eff}} v_F \eta H}{l_{\text{imp}}} \right) A_2,$$  

$$\lambda_2 = \frac{e^2 l_{\text{imp}}^2 E_x^2 \frac{\nu_0}{\eta H}}{2m_{\text{eff}} v_F} \left( eB - \frac{m_{\text{eff}} v_F \eta H}{l_{\text{imp}}} \right) A_1,$$  

$$A_3 = e l_{\text{imp}} E_x \frac{\nu_0}{m_{\text{eff}} v_F \rho_0}, \quad A_4 = \frac{e^2 l_{\text{imp}}^2 E_x B}{m_{\text{eff}} v_F^2 \rho_0}.$$  

$$C_1 = \cosh[(2l_s + W)l_{\text{G}}^{-1}] \left( \sinh[(l_s + W)l_{\text{G}}^{-1}] \left[ A_3 - l_s \left( \lambda_2 \sinh[l_s l_{\text{G}}^{-1}] - \lambda_1 \cosh[l_s l_{\text{G}}^{-1}] - A_4 \right) \right] \right.$$  

$$- \sinh[l_s l_{\text{G}}^{-1}] \left[ (\lambda_2 l_s - A_3 + W) \sinh[(l_s + W)l_{\text{G}}^{-1}] + l_s l_s + W) \cosh[(l_s + W)l_{\text{G}}^{-1} + A_3 l_s + W] \right)$$.  

$$C_2 = - \cosh[(2l_s + W)l_{\text{G}}^{-1}] \left( 2(A_4 W - A_3) \cosh[l_s l_{\text{G}}^{-1}] + 2A_3 \cosh[(l_s + W)l_{\text{G}}^{-1}] \right.$$  

$$+ \lambda_3 W \sinh[(2l_s + W)l_{\text{G}}^{-1}] + \sinh[W l_{\text{G}}^{-1}] + \lambda_1 W \left[ \cosh[(2l_s + W)l_{\text{G}}^{-1}] + \cosh[W l_{\text{G}}^{-1}] \right]$$

$$+ 2l_s \left[ \lambda_1 \cosh[(2l_s + W)l_{\text{G}}^{-1}] + \cosh[W l_{\text{G}}^{-1}] + A_4 \cosh[(l_s + W)l_{\text{G}}^{-1}] + \cosh[l_s l_{\text{G}}^{-1}] \right] + \lambda_2 \sinh[W l_{\text{G}}^{-1}] \right)/2.$$  

Notice, that for $\eta_H \neq 0$, the first order correction $v_1^x(y)$ breaks the reflection symmetry of the entire velocity profile with respect to $y = W/2$. Explicitly this is caused by the linear terms in powers of $y$ within Eq. (S29). Moreover, since $C_{1,2}, \lambda_{1,2}$ and $A_{1,2,3,4}$ are non-linear in $E_x$ and $l_{\text{imp}}$. Eq. (S29) implies that the first order correction $v_1^x \ll v_0^x$ in the linear response regime.

**Strong magnetic field regime.** We now move on to the discussion of strong magnetic fields, implicitly defined by $r_c \ll l_{\text{G}}$. In this limit, $\eta$ and $\eta_H$ tend to zero (cf. Eq. (3) of the main text), which strongly simplifies Eqs. (S9)–(S10). In particular, these equations yield

$$v_x = -\frac{e l_{\text{imp}} E_x}{m_{\text{eff}} v_F} \wedge \partial_y \Delta \mu = e B v_x.$$  

Here, we again dropped a term $\propto \Delta \mu E_x$ due to its negligibility in the linear response regime. Consequently, Eq. (S33) predicts the transverse voltage drop $\Delta V_B$ in the strong magnetic field regime, explicitly given by Eq. (10) of the main text.