Generalization of the Geiger-Nuttall law and alpha clustering in heavy nuclei

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Abstract. A generalization of the Geiger-Nuttall law is deduced, which is valid for the radioactivity of all clusters (including α particles), by considering the clusterization and subsequent decay of nucleons within the nucleus. This universal decay law (UDL) is a linear relation between the half-lives of the decaying clusters and the corresponding Q-values. In this universal decay law (UDL) the penetrability is still a dominant quantity. By using three free parameters only, one finds that all known ground state to ground state radioactive decays are explained rather well. This allows us to search for new cluster decay modes and to carry out a simple and model-independent study of the decay properties of nuclei over the whole nuclear chart. It also helps in distinguishing the role played by pairing collectivity in the clustering process in heavy nuclei.

It is exactly a century ago that the first striking correlation between the half-lives of radioactive decay processes and the Q-values of the emitted particle was found in α-decay systematics by Geiger and Nuttall [1] as,

$$\log T_{1/2} = a Q_\alpha^{-1/2} + b,$$

where $a$ and $b$ are constants. Geiger and collaborators published three papers in that issue of Philosophical Magazine. All these papers were communicated by Rutherford. Indeed, its revolutionary explanation by Gamow [2] and also by Gurney and Condon [3] required to accept the probabilistic interpretation of Quantum Mechanics. The Gamow theory reproduced the Geiger-Nuttall law nicely. One can assert that this is an effective theory where concepts like “frequency of escape attempts” have to be introduced. In fact, a proper calculation of the decay process needs to address first the clustering of the nucleons at a certain distance outside the nuclear surface and, in a second step, the evaluation of the penetrability through the Coulomb and centrifugal barriers. The first step is a challenging undertaking because a proper description of the cluster in terms of its components requires a microscopic many-body framework that is very complicated. This is the reason why usually effective approaches are used when dealing with clusterization. That is, one evaluates the penetrability, which is an easy task especially if semiclassical approaches are applied. The cluster is assumed to be like a little ball moving in a sort of vacuum within the mean field potential, bouncing on and reflected off the internal wall of the potential.

The Geiger-Nuttall law in the form of Eq. (1) has limited prediction power since the coefficients $a$ and $b$ change for the decays of each isotopic series [4, 5], as can be seen from the left panel of Fig. 1. Intensive works have been done trying to generalize the Geiger-Nuttall
law for a universal description of all detected $\alpha$ decay events [6, 7]. Still one may then wonder why effective approaches have been so successful. The reason is that the $\alpha$-particle formation probability usually varies from nucleus to nucleus much less than the penetrability. In the logarithm scale of the Geiger-Nuttall law the differences in the formation probabilities are usually small fluctuations along the straight lines predicted by that law [5] for different isotopic chains. Although successful, this semiclassical picture collides with basic quantum mechanics, since even if a cluster existed on the mother nucleus the Pauli principle would hinder any free motion of the cluster within the potential that traps the cluster inside the nucleus. What is missing in this picture is the possibility that the cluster is not “pre-formed” in the mother nucleus. In other words, one has to evaluate the probability that the cluster indeed is present on the nuclear surface. This can be done within the framework of the shell model. The importance of a proper treatment of $\alpha$ decay was attested by a recent calculation which shows that the different lines can be merged in a single line. One thus obtained a generalization of the Geiger-Nuttall law, which holds for all isotopic chains and all cluster radioactivities [8, 9]. In this universal decay law (UDL) the penetrability is still a dominant quantity. By using three free parameters only, one finds that all known ground-state to ground-state radioactive decays are explained rather well. This good agreement is a consequence of the smooth transition in the nuclear structure that is often found when going from a nucleus to its neighboring nuclei. This is also the reason why, e.g., the BCS approximation works so well in many nuclear regions. Recently the UDL is also generalized to describe the proton decay process [10], which allows us to distinguish the role played by nuclear deformation on the radioactive decay properties.

In this contribution I briefly review the idea behind the UDL. The relation between cluster decay and clusterization was first noticed by analyzing the decay processes in terms of the R-matrix theory. One thus obtains for the half-life of the decaying nucleus the expression [11],

$$T_{1/2} = \frac{\hbar \ln 2}{\Gamma_c} \approx \frac{\ln 2}{\nu} \left| H_i^+ (\chi, \rho) \right|^2 \frac{RF_c(R)}{2},$$

(2)

where $\nu$ is the outgoing velocity of the emitted particle. $H_i^+$ is the Coulomb-Hankel function.
and its arguments are standard [8]. The penetrability is proportional to $|H_0^+(\chi, \rho)|^{-2}$. Eq. (2) is valid for all clusters and for spherical as well as deformed cases [12].

The quantity $F_c(R)$ is the formation amplitude of the decaying cluster at distance $R$. The important feature that we used in the derivation of the UDL is that the decay width is independent upon the matching radius $R$. Our aim is to find few quantities that determine the half-life. Expanding in these quantities in terms of few parameters we hope to be able to find, at the lowest order of perturbation, an expression of the half-life, which is as simple as the Geiger-Nuttall law but valid in general, i.e., for all isotopic series as well as all type of clusters. With this in mind we notice that the Coulomb-Hankel function can be well approximated by an analytic formula, which for ground state to ground state transitions in even-even nuclei, i.e., for $l = 0$, reads [13],

$$H_0^+(\chi, \rho) \approx (\cot \beta)^{1/2} \exp \left[ \chi(\beta - \sin \beta \cos \beta) \right],$$

where the cluster $Q_c$-value is $Q_c = \mu \nu^2/2$. One sees that $\cos^2 \beta$ would be a small quantity if $Z_c Z_q$ is large, i.e., for heavy and superheavy systems [8, 9]. In this case one can expand the last term in a power series of $\cos \beta$. By defining the quantities $\chi' = Z_c Z_d \sqrt{A/Q_c}$ and $\rho' = \sqrt{A Z_c Z_d (A_d^{1/3} + A_c^{1/3})}$ where $A = A_d A_c / (A_d + A_c)$, one gets, after some simple algebra,

$$\log T_{1/2} = a \chi' + b \rho' + \log \left( \frac{\cot \beta \ln 2}{\nu R^2 |F_c(R)|^2} \right) + o(3),$$

where $a = e^2 \pi \sqrt{2m}/(\hbar \ln 10)$ and $b = -4 e \sqrt{2m R_0}/(\hbar \ln 10)$ are constants ($m$ is the nucleon mass). The first two terms dominate the Coulomb penetration and $o(3)$ corresponds to the remaining small terms. But still the strong dependence of the half-life upon the formation probability in the third term of Eq. (4) has to be taken into account.

The formation amplitude $F_c(R)$ can be extracted from the experimental half-lives data by,

$$\log |RF_c(R)| = \frac{1}{2} \log \left[ \frac{\ln 2}{\nu} |H_0^+(\chi, \rho)|^2 \right] - \frac{1}{2} \log T_{1/2}^{\text{Expt.}}.$$  

We found that Eq. (4) can be written as a simple linear formula, which properly takes into account the strong dependence of the formation amplitude upon the cluster as well as the mother nuclear structure to a first order of approximation. This we have archived by exploiting the property that for a given cluster $N_0 \equiv RF_c(R)/H_0^+(\chi, \rho)$ does not depend upon $R$. Proceeding as above one readily obtains the relation,

$$\log |RF_c(R)| \approx \log |R'F_c(R')| + \frac{2e \sqrt{2m}}{\hbar \ln 10} \left( \sqrt{R_0'} - \sqrt{R_0} \right) \rho',$$

where $R' = R_0'(A_d^{1/3} + A_c^{1/3})$ is a value of the radius that differs from $R$. Since for a given cluster any nuclear structure would be carried by the terms $RF_c(R)$ and $R'F_c(R')$ in exactly the same fashion, Eq. (6) implies that the formation amplitude is indeed linearly dependent upon $\rho'$. Therefore one can write,

$$\log T_{1/2} = a \chi' + b \rho' + c.$$  

This equation holds for all cluster radioactivities. A straightforward conclusion from the UDL is that $\log T_{1/2}$ depends linearly upon $\chi'$ and $\rho'$. This to be valid should include the Geiger-Nuttall law as a special case. One sees that this is indeed the case since $\rho'$ remains constant for a given $\alpha$-decay chain and $\chi' \propto Q_c^{-1/2}$. We have chosen the parameters $a$, $b$ and $c$ by fitting experimental data of ground state to ground state cluster decay transitions in even-even nuclei. For details of these procedure and of the formalism itself see Refs. [8, 9]. We analyzed g.s. to
g.s. radioactive decays of even-even nuclei starting with alpha-decay, for which there is a large amount of experimental data. The UDL reproduces the available experimental half-lives within a factor of about 2.2. We found that although the UDL reproduces nicely most available experimental data, as expected, there is a case where it fails by a large factor. This corresponds to the α decays of nuclei with neutron numbers equal to or just below \( N = 126 \) [14, 15]. The reason for this large discrepancy is that in \( N \leq 126 \) nuclei the α formation amplitudes are much smaller than the average quantity predicted by the UDL. The case that shows the most significant hindrance corresponds to the α decay of the nucleus \(^{210}\)Po. We found that the formation amplitude in \(^{210}\)Po is hindered with respect to the one in \(^{212}\)Po due to the hole character of the neutron states in the first case. This is a manifestation of the mechanism that induces clusterization, which is favored by the presence of high-lying configurations. Such configurations are more accessible in the neutron-particle case of \(^{212}\)Po than in the neutron-hole case of \(^{210}\)Po.

The quantity \( F_c(R) \) is the formation amplitude of the decaying cluster at distance \( R \). The amplitude of the wave function in the internal region is the formation amplitude, i.e.,

\[
F_c(R) = \int dR d\xi_d d\xi_c |\Psi(\xi_d)\phi(\xi_c)Y_l(R)]_{jm,MM}^* |\Psi_m(\xi_d,\xi_c, R),
\]

where \( d, c \) and \( m \) label the daughter, emitting cluster and mother nuclei, respectively. \( \Psi \) are the intrinsic wave functions and \( \xi \) the corresponding intrinsic coordinates. \( \phi(\xi_c) \) is a Gaussian function of the relative coordinates of the nucleons that constitute the cluster. The important feature that we used in the derivation of the UDL is that the decay width is independent upon the matching radius \( R \). The problem of evaluating the α decay width properly played a fundamental role in the evolution of the shell model itself. The shell model is more than a model featuring single-particle motion; in principle, with a large enough basis, it should be capable of taking into account any correlations, including correlations of inducing clusterization. This was clear to the pioneers who started to use the shell model for the description of α decay. The results were discouraging since the theoretical decay rates were smaller than the corresponding experimental values by 4–5 orders of magnitude [16], depending on the "reasonable" value to be chosen for the nuclear radius. Due to this failure, doubt arose about the validity of the shell model itself [17]. Since the matching radius \( R \) in Eq. (2) has to be chosen at a distance beyond the point where the cluster was formed, i.e., beyond the range of the nuclear force and Pauli exchanges, the formation amplitude had to be or should have been evaluated at rather large distances. However, that would have required shell models for the mother and daughter nuclei with large bases. With the very limited shell-model spaces used at that time, the region of prominent four-particle correlation was not reached at all and as a compromise, the radius \( R \) was chosen to be well within the interaction region. One of the problems with those early microscopic calculations was that the residual nucleon-nucleon interaction was not known well enough. Soon after the pairing interaction had been adapted to nuclei [18, 19]. It was also applied to α decay [20]. It was then found with great relief that the pairing interaction, which links many shell-model configurations together, highly enhances the calculated α-decay width. Although the calculated widths were still too small by orders of magnitude, it was clear that this discrepancy had to be attributed to the small shell-model spaces allowed by the computing facilities at that time. To their credit, the pioneers had deep insight into the role of configuration mixing. The fundamental role of configuration mixing was only confirmed by actual large-scale calculations [21, 22]. The physics behind the enhancement induced by configuration mixing is that, with the participation of high-lying configurations, the pairing interaction clusters the two neutrons and the two protons on the nuclear surface [23]. The two-neutron and two-proton wave-function terms add up constructively in the surface region. It was found that the mechanism that induces clustering is the same that produces the pairing collective, which is manifested in
an strong increase in the form factor corresponding to the corresponding transfer cross section. This property gives rise to a giant pairing resonances, which corresponds to the most collective of the pairing states lying, as the standard (particle-hole) giant resonances, high in the spectrum.

The kink observed experimentally should be related to the difference in clusterization induced by the pairing force in these two cases. To analyze the clustering features we will consider only the spin-singlet component, i.e., \((\chi_1\chi_2)_0\), of the two-body wave function, since that is the only part entering the intrinsic \(\alpha\)-particle wave function. This component has the form,

\[
\Psi_2(r_1, r_2; \theta_{12}) = \frac{1}{4\pi} \sum_{p \leq q} \sqrt{\frac{2j_p + 1}{2}} X(pq; gs)\varphi_p(r_1)\varphi_q(r_2) P_l(c \cos \theta_{12}),
\]

where \(\varphi\) is the single-particle wave function and \(P_l\) is the Legendre polynomial of order \(l\) satisfying \(P_l(\cos 0) = 1\) (notice that for the ground states studied here it is \(l_p = l_q\)). As mentioned above, the pairing vibrations show strong clustering features as the number of single-particle states is increased [23]. But another manifestation of the pairing collectivity is an enhancement of the wave function on the nuclear surface. The reason of this enhancement is that all configurations contribute with the same phase in the building up of the two-particle wave function on the nuclear surface. The same mechanism increases the \(\alpha\) formation amplitude and, therefore, the relative values of the wave functions of \(^{210}\text{Pb}(gs)\), \(^{210}\text{Po}(gs)\) and \(^{206}\text{Pb}(gs)\) on the nuclear surface give a measure of the importance of the corresponding formation amplitudes.

![Figure 2](image-url)

**Figure 2.** The square of the two-body wave function \(|\Psi_2(r_1, r_2, \theta)|^2\) with \(r_1 = 9\) fm for the two neutrons in \(^{210}\text{Pb}\) (left) and two neutron holes in \(^{206}\text{Pb}\) (right).

To study the behavior of the two-particle wave functions we will apply Eq. (9) with \(r_1 = 9\) m. We have plotted in Fig. 2 \(\Psi_2(r_1, r_2, \theta)\) as a function of \(r_2\) and \(\theta\). One sees that the wave functions are indeed strongly enhanced at the nuclear surface, as expected. But the important feature for us is that the enhancement is strongest in \(^{210}\text{Pb}(gs)\) and weakest in \(^{206}\text{Pb}(gs)\). This is because there is a relatively small number of configurations in the hole-hole case. In addition, the radial wave functions corresponding to the high-lying particle states extend farther out in space with respect to the hole configurations. With these two-body wave functions we proceeded to evaluate the \(\alpha\) formation amplitudes in \(^{212}\text{Po}(gs)\) and \(^{210}\text{Po}(gs)\) [14]. One thus finds that with \(R = 9\) fm the observed ratio between the formation amplitudes in \(^{212}\text{Po}\) and \(^{210}\text{Po}\) can be reproduced nicely.

In conclusion, in this paper we have first showed the mechanisms that induce clusterization of nucleons inside the nucleus. Using the conditions that are required to produce clusters, we have
then obtained a simple formula that provides with great precision the half-lives corresponding to cluster decay. The formula is valid for all kind of clusters and for all isotopic series, as expected since we derived it from the general description of the decay half-life.

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