Blow-up solutions concentrated along minimal submanifolds for some supercritical elliptic problems on Riemannian manifolds

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Abstract. Let $\text{(}M, g\text{)}$ and $\text{(}K, \kappa\text{)}$ be two Riemannian manifolds of dimensions $m$ and $k$, respectively. Let $\omega \in C^2(N), \omega > 0$. The warped product $M \times_\omega K$ is the $(m + k)$-dimensional product manifold $M \times K$ furnished with metric $g + \omega^2 \kappa$. We prove that the supercritical problem

$$-\Delta_{g+\omega^2\kappa} u + h u = u^{p+2\pm \varepsilon}, \quad u > 0,$$

in $(M \times_\omega K, g + \omega^2 \kappa)$

has a solution concentrated along a $k$-dimensional minimal submanifold $\Gamma$ of $M \times_\omega N$ as the real parameter $\varepsilon$ goes to zero, provided the function $h$ and the sectional curvatures along $\Gamma$ satisfy a suitable condition.

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1. Introduction and statement of main results

We deal with the semilinear elliptic equation

$$-\Delta_{g} u + h u = u^{p-1}, \quad u > 0,$$

in $(\mathcal{M}, g),$ (1.1)

where $(\mathcal{M}, g)$ is an $n$-dimensional compact Riemannian manifold without boundary, $h$ is a $C^1$-real function on $\mathcal{M}$ such that $-\Delta_{g} + h$ is coercive and $p > 2$.

The compactness of the embedding $H^1_{g} (\mathcal{M}) \hookrightarrow L^p_{g} (\mathcal{M})$ for any $p \in (2, 2^*_n)$, where

$$2^*_n := \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3, \\ +\infty & \text{if } n = 2, \end{cases}$$
ensures that
\[ \inf_{u \in \mathcal{H}_0^1(M), u \neq 0} \frac{\int_M \left( |\nabla g u|^2 + hu^2 \right) \, d\mu_g}{\left( \int_M |u|^p \, d\mu_g \right)^{2/p}} \]
is achieved, and so problem (1.1) always has a solution for any \( p \in (2, 2^*_n) \).

In the critical case \( p = 2^*_n \), the situation turns out to be more complicated. In particular, the existence of solutions is related to the position of the potential \( h \) with respect to the geometric potential
\[ h_g := \frac{n - 2}{4(n - 1)} S_g, \]
where \( S_g \) is the scalar curvature of the manifold. If \( h \equiv h_g \), then problem (1.1) is referred to as the Yamabe problem and it always has a solution (see Aubin [1, 2], Schoen [10], Trudinger [11] and Yamabe [12] for early references on the subject). When \( h < h_g \) somewhere in \( M \), existence of a solution is guaranteed by a minimization argument (see, for example, Aubin [1, 2]). The situation is extremely delicate when \( h \geq h_g \) because blow-up phenomena can occur as pointed out by Druet in [6, 7].

The supercritical case \( p > 2^*_n \) is even more difficult to deal with. A first result in this direction is a perturbative result due to Micheletti, Pistoia and Vétois [9]. They considered the almost critical problem (1.1) when \( p = 2^*_n \pm \varepsilon \); i.e., if \( p = 2^*_n - \varepsilon \), problem (1.1) is slightly subcritical and if \( p = 2^*_n + \varepsilon \), problem (1.1) is slightly supercritical. They proved the following result.

**Theorem 1.1.** Assume that \( n \geq 6 \) and \( \xi_0 \in \mathcal{M} \) is a nondegenerate critical point of \( h - \frac{n - 2}{4(n - 1)} S_g \). Then

(i) if \( h(\xi_0) > \frac{n - 2}{4(n - 1)} S_g(\xi_0) \), then the slightly subcritical problem (1.1) with \( p = 2^*_n - 1 - \varepsilon \) has a solution \( u_\varepsilon \) which concentrates at \( \xi_0 \),

(ii) if \( h(\xi_0) < \frac{n - 2}{4(n - 1)} S_g(\xi_0) \), then the slightly supercritical problem (1.1) with \( p = 2^*_n - 1 - \varepsilon \) has a solution \( u_\varepsilon \) which concentrates at \( \xi_0 \) as \( \varepsilon \to 0 \).

Now, for any integer \( 0 \leq k \leq n - 3 \), let
\[ 2^*_{n,k} = \frac{2(n - k)}{n - k - 2} \]
be the \((k + 1)st\) critical exponent. We remark that \( 2^*_{n,k} = 2^*_{n-k,0} \) is nothing but the critical exponent for the Sobolev embedding \( H^1_g(M) \to L^q_g(M) \) when \((M, g)\) is an \((n - k)\)-dimensional Riemannian manifold. In particular, \( 2^*_{n,0} = \frac{2n}{n - 2} \) is the usual Sobolev critical exponent.

We can summarize the results proved by Micheletti, Pistoia and Vétois just saying that problem (1.1), when \( p \to 2^*_n \) (i.e., \( k = 0 \)), has positive solutions blowing up at points. Note that a point is a 0-dimensional manifold! Therefore, a natural question arises, **does problem (1.1) have solutions blowing up at k-dimensional submanifolds when \( p \to 2^*_{n,k} \)?**

In the present paper, we give a positive answer when \((\mathcal{M}, g)\) is a warped product manifold.
We recall the notion of warped product introduced by Bishop and O’Neill in [3]. Let \((M, g)\) and \((K, \kappa)\) be two Riemannian manifolds of dimensions \(m\) and \(k\), respectively. Let \(\omega \in C^2(M)\), \(\omega > 0\), be a differentiable function. The warped product \(\mathcal{M} = M \times_\omega K\) is the product (differentiable) \(n\)-dimensional \((n := m + k)\) manifold \(M \times K\) furnished with the Riemannian metric \(g = g + \omega^2 \kappa\). The function \(\omega\) is called a warping function. For example, every surface of revolution (not crossing the axis of revolution) is isometric to a warped product, with \(M\) the generating curve, \(K = S^1\) and \(\omega(x)\) the distance from \(x \in M\) to the axis of revolution.

It is not difficult to check that if \(u \in C^2(M \times_\omega K)\), then
\[
\Delta_g u = \Delta_g u + \frac{m}{\omega} g(\nabla_g f, \nabla_g u) + \frac{1}{\omega^2} \Delta_\kappa u. \tag{1.2}
\]
Assume that \(h\) is invariant with respect to \(K\); i.e., \(h(x, y) = h(x)\) for any \((x, y) \in M \times K\). If we look for solutions to (1.1) which are invariant with respect to \(K\), i.e., \(u(x, y) = v(x)\), then by (1.2) we immediately deduce that \(u\) solves (1.1) if and only if \(v\) solves
\[
-\Delta_g v - \frac{m}{\omega} g(\nabla_g f, \nabla_g v) + hv = v^{p-1} \quad \text{in} \ (M, g), \tag{1.3}
\]
or equivalently
\[
- \text{div}_g (\omega^m \nabla_g v) + \omega^m hv = \omega^m v^{p-1}, \quad v > 0, \quad \text{in} \ (M, g). \tag{1.4}
\]
Here we are interested in studying problem (1.4) when the exponent \(p\) approaches the higher critical exponent \(2^*_{n,k} = 2^*_m\), i.e., \(p = 2^*_m - \varepsilon\) for some small real parameter \(\varepsilon\). It is clear that if \(v\) is a solution to problem (1.3) which concentrates at a point \(\xi_0 \in M\), then \(u(x, y) = v(x)\) is a solution to problems (1.1) which concentrates along the fiber \(\{\xi_0\} \times K\), which is a \(k\)-dimensional submanifold of \(M\). It is important to notice that the fiber \(\{\xi_0\} \times K\) is totally geodesic in \(M \times_\omega K\) (and in particular a minimal submanifold of \(M \times_\omega K\)) if \(\xi_0\) is a critical point of the warping function \(\omega\).

Therefore, we are led to study the more general anisotropic almost critical problem
\[
- \text{div}_g (a(x) \nabla_g u) + a(x) hu = a(x) u^{m+2 \varepsilon}, \quad u > 0, \quad \text{in} \ (M, g), \tag{1.5}
\]
where \((M, g)\) is an \(m\)-dimensional compact Riemannian manifolds, \(a \in C^2(M)\) with \(\min_M a > 0\), \(h \in C^2(M)\) such that the anisotropic operator
\[
- \text{div}_g (a(x) \nabla_g u) + a(x) hu
\]
is coercive and \(\varepsilon \in \mathbb{R}\).

Our main result reads as follows.

**Theorem 1.2.** Assume that \(m \geq 9\) and \(\xi_0 \in M\) is a nondegenerate critical point of \(a\). Then

(i) if
\[
h(\xi_0) > \frac{m - 2}{4(m - 1)} S_g(\xi_0) - \frac{3(m - 2)}{2(m - 1)} \frac{\Delta_g a(\xi_0)}{a(\xi_0)},
\]
then, if $\varepsilon > 0$ is small enough, the slightly subcritical problem (1.5) has a solution $u_\varepsilon$ which concentrates at $\xi_0$ as $\varepsilon \to 0$,

(ii) if

$$h(\xi_0) < \frac{n-2}{4(n-1)}S_g(\xi_0) - \frac{3(m-2)}{2(m-1)} \frac{\Delta_g a(\xi_0)}{a(\xi_0)},$$

then, if $\varepsilon < 0$ is small enough, the slightly supercritical problem (1.5) has a solution $u_\varepsilon$ which concentrates at $\xi_0$ as $\varepsilon \to 0$.

In particular, Theorem 1.2 applies to the case $a = \omega^m$ where $\omega$ is the warping function. We recall that if $\xi_0$ is a critical point of the warping function $\omega$, then the fiber $\Gamma := \{\xi_0\} \times K$ is a minimal $k$-dimensional submanifold of the warped product manifold $M \times \omega K$ equipped with the metric $g = g + \omega^2 \kappa$. Let

$$\Sigma_g(\Gamma) := \frac{m-2}{4(m-1)}S_g(\xi_0) - \frac{3m(m-2)}{2(m-1)} \frac{\Delta_g \omega(\xi_0)}{\omega(\xi_0)},$$

which turns out to be a weighted mean of sectional curvatures of $\Gamma$. From the above discussion and Theorem 1.2 we immediately deduce the following result concerning the supercritical problem (1.1).

**Theorem 1.3.** Assume that $m \geq 9$, $h$ is invariant with respect to $K$ and $p = 2^*_m - \varepsilon$. Then

(i) if $h(\Gamma) > \Sigma_g(\Gamma)$, then, if $\varepsilon > 0$ is small enough, the supercritical problem (1.1) has a solution $u_\varepsilon$, invariant with respect to $K$, which concentrates along $\Gamma$ as $\varepsilon \to 0$,

(ii) if $h(\Gamma) < \Sigma_g(\Gamma)$, then, if $\varepsilon < 0$ is small enough, the supercritical problem (1.1) has a solution $u_\varepsilon$, invariant with respect to $K$, which concentrates along $\Gamma$ as $\varepsilon \to 0$.

Let us state some open problems about the anisotropic problem (1.5).

(a) Theorem 1.2 holds true when $m \geq 9$. The interesting question concerns the low dimensions $m = 3, \ldots, 8$. For example, one could ask if the results obtained by Druet [6, 7] for the Yamabe problem in low dimensions are true anymore.

(b) Theorem 1.2 holds true when the potential $h$ is different from the geometric potential $\Sigma(\Gamma)$. It is interesting to see what happens when $h$ coincides somewhere with $\Sigma(\Gamma)$. In particular, it could be interesting to obtain similar results to those obtained by Esposito, Pistoia and Vetois [8] for the Yamabe problem.

(c) Theorem 1.2 concerns the case when $p$ is close to $2^*_m$ but different from it. Is it possible to establish an existence result for the pure critical case $p = 2^*_m$ similar to the results obtained for the Yamabe problem?

Finally, we ask if similar results hold true in the nonsymmetric case. More precisely, if $p \to 2^*_{m,k}$ for some integer $k \geq 1$, does there exist a solution to problem (1.1) which blows up along a minimal $k$-dimensional submanifold $\Gamma$ provided the potential $h$ is different from a suitable weighted mean of
sectional curvatures of $\Gamma$? Recently, Davila, Pistoia and Vaira [4] gave a positive answer when $k = 1$.

This paper is organized as follows. In Section 2 we introduce the necessary framework. In Section 3 we reduce the problem to a finite-dimensional one via a Lyapunov–Schmidt reduction and in Section 4 we study the finite-dimensional problem and we prove Theorem 1.2.

2. Setting of the problem

Let $H$ be the Hilbert space $H^1_g(M)$ endowed with the scalar product

$$\langle u, v \rangle_H = \int_M a(x)\nabla_g u \nabla_g v \, d\mu_g + \int_M a(x)h(x)uv \, d\mu_g.$$ 

Let $\| \cdot \|_H$ be the norm induced by $\langle \cdot, \cdot \rangle_H$, which is equivalent to the usual one. We also denote the usual $L^q$-norm of a function $u \in L^q_g(M)$ by

$$|u|_g = \left( \int_M |u|^q d\mu_g \right)^{1/q}.$$ 

Let $i^*_H : L^{2m^{*}}_g(M) \to H$ be the adjoint operator of the embedding $i : H \to L^{2m^{*}}_g(M)$; that is,

$$u = i^*_H(v) \iff \langle i^*_H(v), \varphi \rangle_H = \int_M v \varphi \, d\mu_g \quad \forall \varphi \in H$$

$$\iff \int_M a(x)\nabla_g u \nabla_g \varphi \, d\mu_g + \int_M a(x)h(x)u \varphi \, d\mu_g = \int_M v \varphi \, d\mu_g \quad \forall \varphi \in H.$$ 

We recall the following inequality (see [9]):

$$|i^*_H(w)|_s \leq C |w|_s^{\frac{m}{m+2s}} \quad \text{for} \quad w \in L^{\frac{ms}{m+2s}}, \quad (2.1)$$

where $s > \frac{2m^{*}}{m-2} = 2^{*}_m$ so that $\frac{m s}{m+2s} > \frac{2m}{m+2}$.

We consider the Banach space $H_\varepsilon = H^1_g(M) \cap L^{s_\varepsilon}_g(M)$ with the norm

$$\|u\|_{H_\varepsilon} = \|u\|_H + |u|_{s_\varepsilon},$$

where we set

$$s_\varepsilon = \begin{cases} 2^{*}_m - \frac{m}{2}\varepsilon & \text{if } \varepsilon < 0, \\ 2^{*}_m & \text{if } \varepsilon > 0. \end{cases}$$

We remark that in the subcritical case $\varepsilon > 0$ the space $H_\varepsilon$ is nothing but the space $H^1_g(M)$ with norm $\| \cdot \|_H$. By (2.1), we easily deduce that

$$\|i^*_H(w)\|_{H_\varepsilon} \leq c |w|^{\frac{m s_\varepsilon}{m+2s_\varepsilon}} \quad \text{if} \quad w \in L^{\frac{ms_\varepsilon}{m+2s_\varepsilon}}. \quad (2.2)$$

Finally, we can rewrite (1.5) as

$$u = i^*_H(a(x)f_\varepsilon(u)), \quad u \in H_\varepsilon, \quad (2.3)$$

where

$$f_\varepsilon(u) = (u^+)^{2^{*}-1-\varepsilon} \quad \text{and} \quad u^+ = \max\{u, 0\}.$$
Now, let us introduce the main ingredient to build a solution to problem (2.3), namely the standard bubble

\[ U(z) := \frac{\alpha_m}{(1 + |z|^2)^{\frac{m-2}{2}}}, \quad z \in \mathbb{R}^m, \]

where \( \alpha_m := (m(m-2))^{\frac{m-2}{4}}. \) It is well known that the functions

\[ U_{\delta,y}(z) := \delta^{-\frac{m-2}{2}} U\left(\frac{z-y}{\delta}\right), \quad z, y \in \mathbb{R}^m, \delta > 0, \]

are all the positive solutions to the critical problem \(-\Delta U = U^{2^*_m - 1}\) on \( \mathbb{R}^m. \)

We are going to read the euclidean bubble \( U_{\delta,y} \) on the manifold \( M \) via geodesic coordinates. Let \( \chi \) be a smooth cutoff function such that \( 0 \leq \chi \leq 1, \chi(z) = 1 \) if \( z \in B(0, r/2) \subset \mathbb{R}^m, \chi(z) = 0 \) if \( z \in \mathbb{R}^m \setminus B(0, r), |\nabla \chi| \leq 2/r, \) where \( r \) is the injectivity radius of \( M. \) Let us define on \( M \) the function

\[ W_{\delta,\eta}(x) = \begin{cases} \chi(\exp^{-1}\xi_0(x))\delta^{-\frac{m-2}{2}} U(\delta^{-1}\exp^{-1}\xi_0(x) - \eta) & \text{if } x \in B_g(\xi_0, r), \\ 0 & \text{if } x \in M \setminus B_g(\xi, r). \end{cases} \]

We will look for a solution of (2.3) or, equivalently of (1.5), as \( u = W_{\delta,\eta} + \Phi, \) where

\[ \delta = \delta_\varepsilon(t) = \sqrt{|\varepsilon|t} \quad \text{for some } t > 0 \text{ and } \eta \in \mathbb{R}^m. \]

We remark that as \( \varepsilon \) goes to 0, the function \( W_{\delta,\eta} \) blows up at the point \( \xi_0. \) The remainder term \( \Phi \) belongs to the space \( K_{\delta,\eta}^\perp, \) which is introduced as follows. It is well known that any solution to the linearized equation

\[ -\Delta \psi = (2^*_m - 1)U^{2^*_m - 2} \psi \]

is a linear combination of the functions

\[ V_0(z) = \frac{\partial}{\partial \delta} \left[ \delta^{-\frac{m-2}{2}} U(\delta^{-1}z) \right] \bigg|_{\delta = 1}, \quad V_i(z) = \frac{\partial U}{\partial z_i}(z), \quad i = 1, \ldots, m. \]

Let us define on \( M \) the functions

\[ Z_{\delta,\eta}^i(x) = \begin{cases} \chi(\exp^{-1}\xi_0(x))\delta^{-\frac{m-2}{2}} V_i(\delta^{-1}\exp^{-1}\xi_0(x) - \eta) & \text{if } x \in B_g(\xi_0, r), \\ 0 & \text{if } x \in M \setminus B_g(\xi, r). \end{cases} \]

Let us introduce the spaces

\[ K_{\delta,\eta} = \text{span} \langle Z_{\delta,\eta}^0, \ldots, Z_{\delta,\eta}^m \rangle, \]

\[ K_{\delta,\eta}^\perp = \left\{ \Phi \in H_\varepsilon : \langle \Phi, Z_{\delta,\eta}^i \rangle_H = 0 \text{ for } i = 1, \ldots, m \right\}. \]
In order to solve problem (2.3) we will solve the following couple of equations:

\[ \pi_{\delta,\eta}(W_{\delta,\xi} + \Phi - i_H^*(a f\varepsilon(W_{\delta,\eta} + \Phi))) = 0, \tag{2.6} \]
\[ \pi_{\perp,\delta,\eta}(W_{\delta,\eta} + \Phi - i_H^*(a f\varepsilon(W_{\delta,\eta} + \Phi))) = 0, \tag{2.7} \]
where \( \pi_{\delta,\eta} : H_{\varepsilon} \to K_{\delta,\eta} \) and \( \pi_{\perp,\delta,\eta} : H_{\varepsilon} \to K_{\delta,\eta}^\perp \) are the orthogonal projection and \( \Phi \in H_{\varepsilon} \cap K_{\delta,\eta}^\perp. \)

### 3. The finite-dimensional reduction

First of all, we solve equation (2.7). We set

\[ L_{\varepsilon,\delta,\eta}(\Phi) := \pi_{\perp,\delta,\eta}\{ \Phi - i_H^*(a(x)f'_{\varepsilon}(W_{\delta,\eta} + \Phi)) \}, \]
\[ N_{\varepsilon,\delta,\eta}(\Phi) := \pi_{\perp,\delta,\eta}\{ i_H^*(a(x)(f_{\varepsilon}(W_{\delta,\eta} + \Phi) - f_{\varepsilon}(W_{\delta,\eta}) - f'_{\varepsilon}(W_{\delta,\eta})[\Phi])) \}, \]
\[ R_{\varepsilon,\delta,\eta} := \pi_{\delta,\eta}\{ i_H^*(a(x)f_{\varepsilon}(W_{\delta,\eta})) - W_{\delta,\eta} \}, \]
so equation (2.7) reads as

\[ L_{\varepsilon,\delta,\eta}(\Phi) = N_{\varepsilon,\delta,\eta}(\Phi) + R_{\varepsilon,\delta,\eta}. \tag{3.1} \]

**Lemma 3.1.** For any real numbers \( \alpha \) and \( \beta \) with \( 0 < \alpha < \beta \), there exists a positive constant \( C_{\alpha,\beta} \) such that, for \( \varepsilon \) small, for any \( \eta \in \mathbb{R}^m \), for any real number \( t \in [\alpha, \beta] \) and for any \( \Phi \in H_{\varepsilon} \cap K_{\delta,\eta}^\perp(t) \), it holds that

\[ \|L_{\varepsilon,\delta,\eta}(\Phi)\|_{H,s_{\varepsilon}} \geq C_{\alpha,\beta}\|\Phi\|_{H,s_{\varepsilon}}. \tag{3.2} \]

**Proof.** The proof is the same of [9, Lemma 3.1] which we refer the reader to. \( \square \)

**Lemma 3.2.** If \( m \geq 9 \), for any real numbers \( \alpha \) and \( \beta \) with \( 0 < \alpha < \beta \), there exists a positive constant \( C_{\alpha,\beta} \) such that, for \( \varepsilon \) small enough, for any \( \eta \in \mathbb{R}^m \) and for any real number \( t \in [\alpha, \beta] \),

\[ \|R_{\varepsilon,\delta,\eta}(t)\|_{H,s_{\varepsilon}} \leq C_{\alpha,\beta}\varepsilon \|\log|\varepsilon|\|. \]

**Proof.** By definition of \( i_H^* \) and (2.2),

\[ \|R_{\varepsilon,\delta,\eta}(t)\|_{H,s_{\varepsilon}} \leq c \left( \|a(x)f_{\varepsilon}(W_{\delta,\eta}(t),\eta) + a(x)\Delta_g W_{\delta,\eta}(t),\eta\|_{m+2s_{\varepsilon}} \right. \]
\[ \left. + \nabla_g a(x)\nabla_g W_{\delta,\eta}(t),\eta - a(x)hW_{\delta,\eta}(t),\eta\|_{m+2s_{\varepsilon}} \right) \]
\[ \leq C_{\alpha,\beta}\varepsilon \|\log|\varepsilon|\|. \]

Using [9, Lemma 3.2], by direct computation it is easy to prove that

\[ \|f_{\varepsilon}(W_{\delta,\eta}(t),\eta) + \Delta_g W_{\delta,\eta}(t),\eta - hW_{\delta,\eta}(t),\eta\|_{m+2s_{\varepsilon}} \leq C_{\alpha,\beta}\varepsilon \|\log|\varepsilon|\|. \]

It remains to estimate the term

\[ \|\nabla_g a\nabla_g W_{\delta,\eta}(t),\eta\|_{m+2s_{\varepsilon}}. \]
Since \( \xi_0 \) is a critical point for the function \( a \), we have

\[
\left\| \nabla g a \nabla_g W_{\delta \varepsilon(t), \eta} \right\|_q^q \leq C \max_{1 \leq j \leq m} \int_{|y| < r} \left| y \right|^q \left| \frac{\partial}{\partial y_j} \left( \chi(y) \delta^{-\frac{m-2}{2}} U \left( \frac{y}{\delta} - \eta \right) \right) \right| \left| g(y) \right|^\frac{1}{2} dy
\]

\[
\leq C \max_{1 \leq j \leq m} \int_{|y| < r} \left| y \right|^q \left| \delta^{-\frac{m-2}{2}} \frac{\partial}{\partial y_j} \chi(y) U \left( \frac{y}{\delta} - \eta \right) \right|^q dy
\]

\[
+ C \max_{1 \leq j \leq m} \int_{|y| < r} \left| y \right|^q \left| \delta^{-\frac{m-2}{2}} \chi(y) \frac{\partial}{\partial y_j} U \left( \frac{y}{\delta} - \eta \right) \right|^q dy
\]

\[
\leq C \max_{1 \leq j \leq m} \int_{\frac{r}{\delta} \leq |z| \leq \frac{r}{\delta}} \left| \delta z \right|^q \left| \delta^{-\frac{m-2}{2}} U (z - \eta) \right|^q \delta^m dz
\]

\[
+ C \max_{1 \leq j \leq m} \int_{|z| \leq \frac{r}{\delta}} \left| \delta z \right|^q \left| \delta^{-\frac{m-2}{2}} \frac{\partial U}{\partial z_j} (z - \eta) \right|^q \delta^m dz
\]

\[
\leq C \delta^{-\frac{m-2}{2}} q + C \delta^{m+q-\frac{m}{q}}.
\]

Therefore,

\[
\left\| \nabla g a \nabla_g W_{\delta \varepsilon(t), \eta} \right\|_q \leq \delta^2,
\]

(3.3)

because \( m \geq 9 \) and if \( q = \frac{ms}{m+2s} \) then \( 1 - \frac{m}{2} + \frac{m}{q} = 2 \) if \( \varepsilon > 0 \) or \( 1 - \frac{m}{2} + \frac{m}{q} \) close to 2 if \( \varepsilon < 0 \) and sufficiently small. This concludes the proof. \( \square \)

**Proposition 3.3.** If \( m \geq 9 \), for any real numbers \( \alpha \) and \( \beta \) with \( 0 < \alpha < \beta \), there exists a positive constant \( C_{\alpha, \beta} \) such that, for \( \varepsilon \) small enough, for any \( \eta \in \mathbb{R}^m \) and for any real number \( t \in [\alpha, \beta] \), there exists a unique solution \( \Phi_{\delta \varepsilon(t), \eta} \in H_{\varepsilon} \cap K_{\delta \varepsilon(t), \eta}^1 \) of equation (2.7) such that

\[
\left\| \Phi_{\delta \varepsilon(t), \eta} \right\|_{H, s, \varepsilon} \leq C_{\alpha, \beta} \left| \varepsilon \right| \left| \log \left| \varepsilon \right| \right|.
\]

Moreover, \( \Phi_{\varepsilon, \delta \varepsilon(t), \eta} \) is continuously differentiable with respect to \( t \) and \( \eta \).

**Proof.** In order to solve equation (2.7) we look for a fixed point for the operator

\[
T_{\varepsilon, \delta \varepsilon(t), \eta} = T : H_{\varepsilon} \cap K_{\delta \varepsilon(t), \eta}^1 \to H_{\varepsilon} \cap K_{\delta \varepsilon(t), \eta}^1
\]

defined by

\[
T_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi) = L_{\varepsilon, \delta \varepsilon(t), \eta}^{-1} \left\{ N_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi) + R_{\varepsilon, \delta \varepsilon(t), \eta} \right\}.
\]

We have

\[
\left\| T_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi) \right\|_{H, s, \varepsilon} \leq \left\| N_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi) \right\|_{H, s, \varepsilon} + \left\| R_{\varepsilon, \delta \varepsilon(t), \eta} \right\|_{H, s, \varepsilon},
\]

\[
\left\| T_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi_1) - T_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi_2) \right\|_{H, s, \varepsilon} \leq \left\| N_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi_1) - N_{\varepsilon, \delta \varepsilon(t), \eta} (\Phi_2) \right\|_{H, s, \varepsilon}.
\]

Since \( m \geq 9 \), a simple application of the mean value theorem gives

\[
|f_{\varepsilon}(x + y) - f_{\varepsilon}(x + z) - f'_{\varepsilon}(x)(y - z)| \leq C |y - z| (|y| + |z|) |x + |y| + |z| |^{2m - 3 - \varepsilon}
\]
for all $x > 0, x, y, z \in \mathbb{R}$. Thus

$$\left\| N_{\varepsilon, \delta_{\varepsilon}(t), \eta}(\Phi_1) - N_{\varepsilon, \delta_{\varepsilon}(t), \eta}(\Phi_2) \right\|_{H, s \varepsilon}$$

$$\leq \left\| f_{\varepsilon}(W_{\delta_{\varepsilon}(t), \eta} + \Phi_1) - f_{\varepsilon}(W_{\delta_{\varepsilon}(t), \eta} + \Phi_2) - f'_{\varepsilon}(W_{\delta_{\varepsilon}(t), \eta})(\Phi_1 - \Phi_2) \right\| \frac{m_{s \varepsilon}}{m + 2s \varepsilon}$$

$$\leq C \left\| \Phi_1 - \Phi_2 \right\|_{\beta_{\varepsilon}} \left( \left\| \Phi_1 \right\|_{\beta_{\varepsilon}} + \left\| \Phi_2 \right\|_{\beta_{\varepsilon}} \right)$$

$$\times \left( \left\| W_{\delta_{\varepsilon}(t), \eta} \right\|_{\beta_{\varepsilon}} + \left\| \Phi_1 \right\|_{\beta_{\varepsilon}} + \left\| \Phi_2 \right\|_{\beta_{\varepsilon}} \right)^{2^*_m - 3 - \varepsilon},$$

(3.4)

where $\beta_{\varepsilon} = \frac{m_{s \varepsilon}}{m + 2s \varepsilon} (2^* - 1 - \varepsilon)$, so

$$\beta_{\varepsilon} = \begin{cases} s \varepsilon & \text{if } \varepsilon < 0, \\ 2^*_m - \varepsilon \frac{2m}{m + 2} & \text{if } \varepsilon > 0. \end{cases}$$

In particular,

$$\left\| N_{\varepsilon, \delta_{\varepsilon}(t), \eta}(\Phi_1) \right\|_{H, s \varepsilon}$$

$$\leq \left\| f_{\varepsilon}(W_{\delta_{\varepsilon}(t), \eta} + \Phi_1) - f_{\varepsilon}(W_{\delta_{\varepsilon}(t), \eta} + \Phi_2) - f'_{\varepsilon}(W_{\delta_{\varepsilon}(t), \eta})(\Phi_1 - \Phi_2) \right\| \frac{m_{s \varepsilon}}{m + 2s \varepsilon} \(3.5)$$

Thus, by (3.5) and by Lemma 3.2 we have

$$\left\| T_{\varepsilon, \delta_{\varepsilon}(t), \eta}(\Phi) \right\|_{H, s \varepsilon} \leq \left\| \Phi \right\|_{H, s \varepsilon}^2 + C_{\alpha, \beta} \left\| \varepsilon \right\| \left\| \log \left\| \varepsilon \right\| \right\|,$$

so if $\left\| \Phi \right\|_{H, s \varepsilon} \leq 2C_{\alpha, \beta} \left\| \varepsilon \right\| \left\| \log \left\| \varepsilon \right\| \right\|$ and for $\varepsilon$ small,

$$\left\| T_{\varepsilon, \delta_{\varepsilon}(t), \eta}(\Phi) \right\|_{H, s \varepsilon} \leq 2C_{\alpha, \beta} \left\| \varepsilon \right\| \left\| \log \left\| \varepsilon \right\| \right\|.$$

Moreover, by (3.4) we have

$$\left\| T_{\varepsilon, \delta_{\varepsilon}(t), \eta}(\Phi_1) - T_{\varepsilon, \delta_{\varepsilon}(t), \eta}(\Phi_2) \right\|_{H, s \varepsilon} \leq K \left\| \Phi_1 - \Phi_2 \right\|_{H, s \varepsilon}$$

for some $K < 1$ if $\left\| \Phi_i \right\|_{H, s \varepsilon} \leq 2C_{\alpha, \beta} \left\| \varepsilon \right\| \left\| \log \left\| \varepsilon \right\| \right\|$ and $\varepsilon$ small enough.

Hence, a contraction mapping argument proves that the map $T_{\varepsilon, \delta_{\varepsilon}(t), \eta}$ admits a fixed point $\Phi_{\delta_{\varepsilon}(t), \eta}$. The regularity of $\Phi_{\delta_{\varepsilon}(t), \eta}$ with respect to $\eta$ and $t$ follows by standard arguments using the implicit function theorem. □

4. The reduced problem and proof of Theorem 1.2

Let $J_{\varepsilon} : H_{\varepsilon} \to \mathbb{R}$ be the energy associated with problem (1.5) defined by

$$J_{\varepsilon}(u) = \frac{1}{2} \int_M a(x) \left( |\nabla g u|^2 + h(x) u^2 \right) d\mu_g - \frac{1}{2m-\varepsilon} \int_M a(x) u^2 t d\mu_g.$$

(4.1)

It is well known that any critical point of $J_{\varepsilon}$ is a solution to problem (1.5).

Let us introduce the reduced energy

$$\tilde{J}_{\varepsilon}(t, \eta) := J_{\varepsilon} \left( W_{\delta_{\varepsilon}(t), \eta} + \Phi_{\delta_{\varepsilon}(t), \eta} \right),$$

where $W_{\delta_{\varepsilon}(t), \eta}$ is defined in (2.4), $\delta_{\varepsilon}(t) = \sqrt{|\varepsilon|} t$ (see (2.5)) and $\Phi_{\delta_{\varepsilon}(t), \eta}$ is given in Proposition 3.3.
Proposition 4.1.

(i) If \((t, \eta)\) is a critical point of \(\tilde{J}_\varepsilon\), then \(W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}\) is a solution of (2.6) and then is a solution of problem (1.5).

(ii) We have

\[
\tilde{J}_\varepsilon (t, \eta) = a(\xi_0) \left[ a_m - b_m \varepsilon \log |\varepsilon| + c_m \varepsilon + d_m |\varepsilon| \Phi (t, \eta) \right] + o(|\varepsilon|) \tag{4.2}
\]

\(C^1\)-uniformly with respect to \(\eta \in \mathbb{R}^m\) and \(t \in [\alpha, \beta]\). Here

\[
\Phi (t, \eta) = \begin{cases} 
\frac{2(m-1)}{(m-2)(m-4)} \left[ h(\xi_0) - \frac{m-2}{4(m-1)} S_g(\xi_0) + \frac{3(m-2)}{2(m-1)} \Delta g a(\xi_0) \right] \\
+ \frac{1}{2} \frac{D_g^2 a(\xi_0)[\eta, \eta]}{a(\xi_0)} \end{cases} 
\]

\[
= \frac{1}{2} \frac{D_g^2 a(\xi_0)[\eta, \eta]}{a(\xi_0)} \left( t - \frac{\varepsilon (m-2)^2}{8 \log t} \right) \tag{4.3}
\]

and \(a_m, \ldots, d_m\) are constants depending only on \(m\).

Proof. The proof of (i) is quite standard and can be obtained by arguing exactly as in the proof of [9, Proposition 2.2].

The proof of (ii) follows in two steps.

Step 1. We prove that

\[
\tilde{J}_\varepsilon (t, \eta) = J_\varepsilon \left( W_{\delta_\varepsilon(t), \eta} \right) + o(|\varepsilon|)
\]

\(C^1\)-uniformly with respect to \(\eta \in \mathbb{R}^m\) and \(t \in [\alpha, \beta]\).

First, let us prove the \(C^0\)-estimate. We have

\[
\tilde{J}_\varepsilon (t, \eta) - J_\varepsilon \left( W_{\delta_\varepsilon(t), \eta} \right) = \int_M a(x) \left( \nabla_g W_{\delta_\varepsilon(t), \eta} \nabla \Phi_{\delta_\varepsilon(t), \eta} + h W_{\delta_\varepsilon(t), \eta} \Phi_{\delta_\varepsilon(t), \eta} \right) d\mu_g
\]

\[
- f_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \Phi_{\delta_\varepsilon(t), \eta} d\mu_g
\]

\[
- \int_M a(x) \left( (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta})^{2^*-\varepsilon} - (W_{\delta_\varepsilon(t), \eta})^{2^*-\varepsilon} \right) \Phi_{\delta_\varepsilon(t), \eta} d\mu_g + \frac{1}{2} \| \Phi_{\delta_\varepsilon(t), \eta} \|^2_{H^1}
\]

Since

\[
\int_M a(x) \nabla_g W_{\delta_\varepsilon(t), \eta} \nabla \Phi_{\delta_\varepsilon(t), \eta} d\mu_g = - \int_M \nabla_g a(x) \nabla_g W_{\delta_\varepsilon(t), \eta} \Phi_{\delta_\varepsilon(t), \eta} d\mu_g
\]

\[
- \int_M a(x) \Delta_g W_{\delta_\varepsilon(t), \eta} \Phi_{\delta_\varepsilon(t), \eta} d\mu_g,
\]

we need an estimate of the term

\[
\int_M \nabla_g a(x) \nabla_g W_{\delta_\varepsilon(t), \eta} \Phi_{\delta_\varepsilon(t), \eta} d\mu_g.
\]

By (3.3) we get

\[
\left\| \nabla_g a(x) \nabla_g W_{\delta_\varepsilon(t), \eta} \right\|_{\frac{2m}{m+2}} = O(|\varepsilon|)
\]
and, by Hölder’s inequality and Proposition 3.3, we obtain
\[ \left| \int_M \nabla_g a(x) \nabla_g W_{\delta_\varepsilon(t), \eta} \Phi_{\delta_\varepsilon(t), \eta} d\mu_g \right| = o(|\varepsilon|). \]

The following estimate is analogous to (4.11) in [9, Lemma 4.2]: for any \( \theta \in (0, 1) \),
\[ \| - \Delta_g W_{\delta_\varepsilon(t), \eta} + h W_{\delta_\varepsilon(t), \eta} - f_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \|_{\frac{2m}{m+2}} = o(|\varepsilon|^{\theta}). \]
Thus
\[ \left| \int_M a(x) \left( - \Delta_g W_{\delta_\varepsilon(t), \eta} + h W_{\delta_\varepsilon(t), \eta} - f_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \right) \Phi_{\delta_\varepsilon(t), \eta} d\mu_g \right| = o(|\varepsilon|). \]

Similarly, following again [9, Lemma 4.2] it is easy to prove that
\[ \left| \int_M a(x) \left( (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta})^{2^* - \varepsilon} - (W_{\delta_\varepsilon(t), \eta})^{2^* - \varepsilon} - f_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \Phi_{\delta_\varepsilon(t), \eta} \right) d\mu_g \right| = o(|\varepsilon|), \]
which concludes the proof of the \( C^0 \)-estimate.

Now, let us prove the \( C^1 \)-estimate. We point out that
\[ \frac{1}{2 \varepsilon} Z_{\delta_\varepsilon(t), \eta}^0 \to a(\xi_0) \| V_k \|_{D^{1,2}(\mathbb{R}^m)} \] for \( 0 \leq k \leq m \). (4.6)

We have the terms
\[ \frac{\partial}{\partial t} J_\varepsilon (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}) - \frac{\partial}{\partial t} J_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \]

and
\[ \frac{\partial}{\partial \eta_k} J_\varepsilon (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}) - \frac{\partial}{\partial \eta_k} J_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \] for \( 1 \leq k \leq m \).

By (4.4) and (4.5) we have
\[ \frac{\partial}{\partial t} J_\varepsilon (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}) - \frac{\partial}{\partial t} J_\varepsilon (W_{\delta_\varepsilon(t), \eta}) = \left[ J_\varepsilon (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}) - J_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \right] \left[ \frac{\partial}{\partial t} W_{\delta_\varepsilon(t), \eta} \right] \\
+ \left[ J_\varepsilon (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}) - J_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \right] \left[ \frac{\partial}{\partial t} \Phi_{\delta_\varepsilon(t), \eta} \right] \\
= \left[ J_\varepsilon (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}) - J_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \right] \left[ \frac{1}{2 \varepsilon} Z_{\delta_\varepsilon(t), \eta}^0 \right] \\
+ \left[ J_\varepsilon (W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}) - J_\varepsilon (W_{\delta_\varepsilon(t), \eta}) \right] \left[ \frac{\partial}{\partial t} \Phi_{\delta_\varepsilon(t), \eta} \right]. \]
Analogously,
\[
\frac{\partial}{\partial \eta^k} J_\varepsilon \left( W_{\delta_\varepsilon}(t), \eta + \Phi_{\delta_\varepsilon}(t), \eta \right) - \frac{\partial}{\partial \eta^k} J_\varepsilon \left( W_{\delta_\varepsilon}(t), \eta \right) = \left( J_\varepsilon' \left( W_{\delta_\varepsilon}(t), \eta + \Phi_{\delta_\varepsilon}(t), \eta \right) - J_\varepsilon' \left( W_{\delta_\varepsilon}(t), \eta \right) \right) \left[ Z_{\delta_\varepsilon}(t), \eta \right] \\
+ J_\varepsilon' \left( W_{\delta_\varepsilon}(t), \eta + \Phi_{\delta_\varepsilon}(t), \eta \right) \left[ \frac{\partial}{\partial \eta^k} \Phi_{\delta_\varepsilon}(t), \eta \right].
\]

For \(0 \leq k \leq m\), we have
\[
\left( J_\varepsilon' \left( W_{\delta_\varepsilon}(t), \eta + \Phi_{\delta_\varepsilon}(t), \eta \right) - J_\varepsilon' \left( W_{\delta_\varepsilon}(t), \eta \right) \right) \left[ Z_{\delta_\varepsilon}(t), \eta \right] = - \int_M \nabla_g a(x) \nabla_g Z_{\delta_\varepsilon}(t), \eta \Phi_{\delta_\varepsilon}(t), \eta \, d\mu_g \\
+ \int_M a(x) \left( -\Delta_g Z_{\delta_\varepsilon}(t), \eta + h Z_{\delta_\varepsilon}(t), \eta - f_\varepsilon'(W_{\delta_\varepsilon}(t), \eta) Z_{\delta_\varepsilon}(t), \eta \right) \Phi_{\delta_\varepsilon}(t), \eta \, d\mu_g \\
+ \int_M a(x) \left( f_\varepsilon(W_{\delta_\varepsilon}(t), \eta + \Phi_{\delta_\varepsilon}(t), \eta) - f_\varepsilon(W_{\delta_\varepsilon}(t), \eta) \right) \\
- f_\varepsilon'(W_{\delta_\varepsilon}(t), \eta) \Phi_{\delta_\varepsilon}(t), \eta \right) Z_{\delta_\varepsilon}(t), \eta \, d\mu_g.
\]

At this point, by (4.6), we have
\[
\left| \int_M \nabla_g a(x) \nabla_g Z_{\delta_\varepsilon}(t), \eta \Phi_{\delta_\varepsilon}(t), \eta \, d\mu_g \right| \\
\leq \| \Phi_{\delta_\varepsilon}(t), \eta \| L^2 \| \nabla_g a(x) \nabla_g Z_{\delta_\varepsilon}(t), \eta \| L^2 = o(|\varepsilon|).
\]

Arguing as in (426) of [9, Lemma 4.2], we have that, for \(0 \leq k \leq m\) and for \(\theta \in (0, 1)\),
\[
\| -\Delta_g Z_{\delta_\varepsilon}(t), \eta + h Z_{\delta_\varepsilon}(t), \eta - f_\varepsilon'(W_{\delta_\varepsilon}(t), \eta) Z_{\delta_\varepsilon}(t), \eta \|_{\frac{2m}{m+2}} = O(|\varepsilon|^{\theta}).
\]
This implies that
\[
\left| \int_M a(x) \left( -\Delta_g Z_{\delta_\varepsilon}(t), \eta + h Z_{\delta_\varepsilon}(t), \eta - f_\varepsilon'(W_{\delta_\varepsilon}(t), \eta) Z_{\delta_\varepsilon}(t), \eta \right) \Phi_{\delta_\varepsilon}(t), \eta \, d\mu_g \right| = o(|\varepsilon|).
\]
Again, arguing as in [9, Lemma 4.2], we have
\[
\left| \int_M a(x) \left( f_\varepsilon(W_{\delta_\varepsilon}(t), \eta + \Phi_{\delta_\varepsilon}(t), \eta) - f_\varepsilon(W_{\delta_\varepsilon}(t), \eta) \right) \\
- f_\varepsilon'(W_{\delta_\varepsilon}(t), \eta) \Phi_{\delta_\varepsilon}(t), \eta \right) Z_{\delta_\varepsilon}(t), \eta \, d\mu_g \right| = o(|\varepsilon|).
\]
It remains to estimate the term
\[
J_\varepsilon' \left( W_{\delta_\varepsilon}(t), \eta + \Phi_{\delta_\varepsilon}(t), \eta \right) \left[ \frac{\partial}{\partial t} \Phi_{\delta_\varepsilon}(t), \eta \right].
\]
By (2.7), since $\Phi_{\delta, t, \eta} \in K_{\delta, t, \eta}^\perp$, we have
\[
J'_\varepsilon (W_{\delta, t, \eta} + \Phi_{\delta, t, \eta}) \left[ \frac{\partial}{\partial t} \Phi_{\delta, t, \eta} \right] = \sum_{j=0}^{m} \lambda^j_{\delta, t, \eta} \left\langle Z^j_{\delta, t, \eta}, \frac{\partial}{\partial t} \Phi_{\delta, t, \eta} \right\rangle_H = - \sum_{j=0}^{m} \lambda^j_{\delta, t, \eta} \left\langle \frac{\partial}{\partial t} Z^j_{\delta, t, \eta}, \Phi_{\delta, t, \eta} \right\rangle_H.
\]

By easy computation we have
\[
\left\| \frac{\partial}{\partial t} Z^j_{\delta, t, \eta} \right\|_H = O(1),
\]
\[
\left\| \frac{\partial}{\partial \eta_k} Z^j_{\delta, t, \eta} \right\|_H = O(1) \text{ for all } 1 \leq k \leq m.
\]

By [9, Lemma 4.2] we have that
\[
\sum_{j=0}^{m} \left| \lambda^j_{\delta, t, \eta} \right| = O(|\varepsilon|^{1/2}).
\]

This, in light of Proposition 3.3, ensures that
\[
J'_\varepsilon (W_{\delta, t, \eta} + \Phi_{\delta, t, \eta}) \left[ \frac{\partial}{\partial t} \Phi_{\delta, t, \eta} \right] = o(|\varepsilon|).
\]

The estimate for
\[
J'_\varepsilon (W_{\delta, t, \eta} + \Phi_{\delta, t, \eta}) \left[ \frac{\partial}{\partial \eta_k} \Phi_{\delta, t, \eta} \right], \quad k \in [1, m],
\]
can be obtained in a similar way. This concludes Step 1 of the proof.

**Step 2.** We prove that $J_\varepsilon (W_{\delta, t, \eta})$ satisfies expansion (4.2) $C^1$-uniformly with respect to $\eta \in \mathbb{R}^m$ and $t \in [\alpha, \beta]$.

Let us prove the $C^0$-estimate. It holds that
\[
J_{\varepsilon} (W_{\delta, t, \eta}) = a(\xi_0) \left( \frac{1}{2} \int_M |\nabla g W_{\delta, \eta}|^2 d\mu_g + \frac{1}{2} \int_M h(x) W^2_{\delta, \eta} d\mu_g \right)
\]
\[
- \frac{1}{2^{*} m - \varepsilon} \int_M W^2_{\delta, \eta} d\mu_g )
\]
\[
+ \frac{1}{2} \int_M (a(x) - a(\xi_0)) |\nabla g W_{\delta, \eta}|^2 d\mu_g + \frac{1}{2} \int_M (a(x) - a(\xi_0)) h(x) W^2_{\delta, \eta} d\mu_g
\]
\[
- \frac{1}{2^{*} m - \varepsilon} \int_M (a(x) - a(\xi_0)) W^2_{\delta, \eta} d\mu_g.
\]
First of all, let us estimate the integrals $I_1$, $I_2$ and $I_3$:

\[
I_1 = \frac{\delta^{m+2}}{2} \int_{\mathbb{R}^m} g^{ij}(y) \frac{\partial}{\partial y_i} \left( \chi(y - \delta \eta) U \left( \frac{y}{\delta} - \eta \right) \right) \\
\times \frac{\partial}{\partial z_j} \left( \chi(y - \delta \eta) U \left( \frac{y}{\delta} - \eta \right) \right) |g(y)|^{1/2} \, dy
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^m} g^{ij}(\delta(z + \eta)) \frac{\partial}{\partial z_i} (\chi(\delta z) U(z)) \frac{\partial}{\partial z_j} (\chi(\delta z) U(z)) |g(\delta(z + \eta))|^{1/2} \, dz
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^m} g^{ij}(\delta(z + \eta)) \frac{\partial U}{\partial z_i}(z) \frac{\partial U}{\partial z_j}(z) |g(\delta(z + \eta))|^{1/2} \, dz + o(\delta^2)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^m} \left( \delta_{ij} + \frac{\delta^2}{2} \sum_{i,j,r,k=1}^n \frac{\partial^2 g^{ij}}{\partial y_r \partial y_k}(0)(z_r + \eta_r)(z_k + \eta_k) \right) \frac{\partial U}{\partial z_i}(z) \frac{\partial U}{\partial z_j}(z) \\
\times \left( 1 - \frac{\delta^2}{4} \sum_{s,r,k=1}^n \frac{\partial^2 g^{ss}}{\partial y_r \partial y_k}(0)(z_r + \eta_r)(z_k + \eta_k) \right) \, dz + o(\delta^2)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^m} |\nabla U|^2 \, dz + \frac{\varepsilon |t|}{4} \sum_{i,j,r,k=1}^m \frac{\partial^2 g^{ij}}{\partial y_r \partial y_k}(0) \eta_r \eta_k \int_{\mathbb{R}^m} \frac{\partial U}{\partial z_i}(z) \frac{\partial U}{\partial z_j}(z) \, dz
\]

\[
- \frac{\varepsilon |t|}{8} \sum_{s,r,k=1}^m \frac{\partial^2 g^{ss}}{\partial y_r \partial y_k}(0) \eta_r \eta_k \int_{\mathbb{R}^m} |\nabla U|^2 \, dz + o(\varepsilon)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^m} |\nabla U|^2 \, dz + \frac{\varepsilon |t|}{4} \sum_{i,j,r,k=1}^m \frac{\partial^2 g^{ij}}{\partial y_r \partial y_k}(0) \eta_r \eta_k \int_{\mathbb{R}^m} \left( \frac{U'(z)}{|z|} \right)^2 \, dz
\]

\[
+ \frac{\varepsilon |t|}{4} \sum_{i,r,k=1}^m \frac{\partial^2 g^{ii}}{\partial y_r \partial y_k}(0) \eta_r \eta_k \int_{\mathbb{R}^m} \left( \frac{U'(z)}{|z|} \right)^2 \, dz
\]

\[
- \frac{\varepsilon |t|}{8} \sum_{i,k=1}^m \frac{\partial^2 g^{ii}}{\partial y_k^2}(0) \int_{\mathbb{R}^m} (U'(z))^2 \, dz
\]

\[
- \frac{\varepsilon |t|}{8} \sum_{i,r,k=1}^m \frac{\partial^2 g^{ii}}{\partial y_r \partial y_k}(0) \eta_r \eta_k \int_{\mathbb{R}^m} (U'(z))^2 \, dz + o(\varepsilon).
\]

We set $\tilde{h}(y) = h(\exp_{\xi_0}(y))$. Then we get

\[
I_2 = \frac{\delta^{m+2}}{2} \int_{\mathbb{R}^m} \tilde{h}(y) \chi^2(y - \delta \eta) U^2 \left( \frac{y}{\delta} - \eta \right) |g(y)|^{1/2} \, dy
\]
\[
\frac{\delta^2}{2} \int_{\mathbb{R}^m} \tilde{h}(\delta(z + \eta))U^2(z) |g(\delta(z + \eta))|^{1/2} \, dz + o(\delta^2)
\]
\[
= \frac{\delta^2}{2} \int_{\mathbb{R}^m} \left( \tilde{h}(0) + O(\delta) \right) U^2(z)(1 + O(\delta^2)) \, dz + o(\delta^2)
\]
\[
= \frac{|\varepsilon| t}{2} \tilde{h}(0) \int_{\mathbb{R}^m} U^2(z) \, dz + o(\varepsilon).
\]

We notice that, by direct computation and considering that \( \delta = \sqrt{|\varepsilon| t} \),

\[
\frac{1}{2^* m - \varepsilon} = \frac{1}{2^* m} + \frac{\varepsilon}{(2^* m)^2} + o(\varepsilon);
\]

\[
\delta^e \frac{m - 2}{2^* m - \varepsilon} = 1 + \frac{m - 2}{4} \varepsilon \log (|\varepsilon| t) + o(\varepsilon);
\]

\[
U^{2^* m - \varepsilon} = U^{2^* m - \varepsilon} U^{2^* m \log U} + o(\varepsilon).
\]

Therefore,

\[
I_3 = \frac{\delta^{e \frac{m - 2}{2^* m - \varepsilon}}}{2^* m - \varepsilon} \int_{\mathbb{R}^m} \chi^{2^* m - \varepsilon}(y - \delta \eta) U^{2^* m - \varepsilon} \left( \frac{y}{\delta} - \eta \right) |g(y)|^{1/2} \, dy
\]

\[
= \frac{\delta^e \frac{m - 2}{2^* m - \varepsilon}}{2^* m - \varepsilon} \int_{\mathbb{R}^m} U^{2^* m - \varepsilon}(z) |g(\delta(z + \eta))|^{1/2} \, dz + o(\delta^2)
\]

\[
= \frac{\delta^e \frac{m - 2}{2^* m - \varepsilon}}{2^* m - \varepsilon} \int_{\mathbb{R}^m} U^{2^* m - \varepsilon}(z)
\]

\[
\times \left( 1 - \frac{\delta^2}{4} \sum_{i,r,k=1}^n \frac{\partial^2 g^{ii}}{\partial y_i \partial y_k}(0)(z_r + \eta_r)(z_k + \eta_k) \right) \, dz + o(\delta^2)
\]

\[
= \frac{\delta^e \frac{m - 2}{2^* m - \varepsilon}}{2^* m - \varepsilon} \int_{\mathbb{R}^m} U^{2^* m - \varepsilon}(z)
\]

\[
\times \left( 1 - \frac{\delta^2}{4} \sum_{i,r,k=1}^n \frac{\partial^2 g^{ii}}{\partial y_i \partial y_k}(0)(z_r z_k + \eta_r \eta_k) \right) \, dz + o(\delta^2)
\]

\[
= \left( \frac{1}{2^* m} + \frac{\varepsilon}{(2^* m)^2} \right) \left( 1 + \frac{m - 2}{4} \varepsilon \log (|\varepsilon| t) \right)
\]

\[
\times \int_{\mathbb{R}^m} \left( U^{2^* m}(z) - \varepsilon U^{2^* m}(z) \log U(z) \right)
\]

\[
\times \left( 1 - \frac{|\varepsilon| t}{4} \sum_{i,r,k=1}^n \frac{\partial^2 g^{ii}}{\partial y_i \partial y_k}(0)(z_r z_k + \eta_r \eta_k) \right) \, dz + o(\varepsilon)
\]

\[
= \frac{1}{2^* m} \int_{\mathbb{R}^m} U^{2^* m} + \frac{\varepsilon}{2^* m} \left( \frac{1}{2^* m} \int_{\mathbb{R}^m} U^{2^* m}(z) \, dz - \int_{\mathbb{R}^m} U^{2^* m}(z) \log U(z) \, dz \right)
\]
\[
+ \frac{1}{2^*} \frac{m - 2}{4} \varepsilon \log (|\varepsilon| t) \int_{\mathbb{R}^m} U^2_m(z) dz \\
- \frac{1}{2^*} \frac{m - 2}{4} \varepsilon \log (|\varepsilon| t) \int_{\mathbb{R}^m} U^2_m(z) dz \\
- \frac{1}{2^*} \frac{m - 2}{4} \varepsilon \log (|\varepsilon| t) \int_{\mathbb{R}^m} U^2_m(z) dz \\
- \sum_{i,k=1}^m \frac{\partial^2 g^{ii}}{\partial y_r \partial y_k} (0) \int_{\mathbb{R}^m} U^2_m(z) \, dz \\
- \frac{1}{2^*} \frac{m - 2}{4} \varepsilon \log (|\varepsilon| t) \int_{\mathbb{R}^m} U^2_m(z) dz + o(\varepsilon).
\]

Therefore, we get

\[
I_1 + I_2 - I_3 = 1 - \frac{m - 2}{8} \varepsilon \log (|\varepsilon| t) - c_m \varepsilon \\
+ \sum_{i,k=1}^m \frac{\partial^2 g^{ii}}{\partial y_r \partial y_k} (0) \eta_r \eta_k \int_{\mathbb{R}^m} U^2_m(z) \, dz \\
+ \sum_{i,r,k=1}^m \frac{\partial^2 g^{ii}}{\partial y_r \partial y_k} (0) \eta_r \eta_k \int_{\mathbb{R}^m} U^2_m(z) \, dz \\
+ 1 \int_{\mathbb{R}^m} \left( \frac{U'(z)}{|z|} \right)^2 z_i^2 dz - \frac{1}{8} \int_{\mathbb{R}^m} \left( U'(z) \right)^2 dz + o(\varepsilon)
\]

\[
= \frac{K_m^{-1}}{m} \left[ 1 - \frac{(m - 2)^2}{8} \varepsilon \log (|\varepsilon| t) - c_m \varepsilon \\
+ \frac{2(m - 1)}{(m - 2)(m - 4)} \left( h(\xi) - \frac{(m - 2)}{4(m - 1)} S_g(\xi) \right) |\varepsilon| t + o(\varepsilon) \right],
\]

where \( c_m \) is a constant depending only on \( m \),

\[ K_m := \sqrt{\frac{4}{m(m - 2) \omega_m^{2/m}}} \]
and $\omega_m$ is the volume of the unit $m$-sphere (see also [9, Lemma 4.1]). Here we used the following facts:

$$\sum_{i,j=1}^m \frac{\partial^2 g_{ii}}{\partial z_j^2}(0) - \sum_{i,j=1}^m \frac{\partial^2 g_{ij}}{\partial z_i \partial z_j}(0) = S_g(\xi_0),$$

$$\frac{1}{2} \int_{\mathbb{R}^m} |\nabla U|^2 z_i^2 dz - \frac{1}{2^*} \int_{\mathbb{R}^m} U^{2^*} z_i^2 dz = \int_{\mathbb{R}^m} \left(\frac{\partial U}{\partial z_i}\right)^2 z_i^2 dz$$

$$\int_{\mathbb{R}^m} \left(\frac{U'(z)}{|z|}\right)^2 z_i^4 dz = 3 \int_{\mathbb{R}^m} \left(\frac{U'(z)}{|z|}\right)^2 z_i^2 z_j^2 dz$$

$$\int_{\mathbb{R}^m} \left(\frac{U'(z)}{|z|}\right)^2 z_i z_j z_k z_h dz = \frac{1}{2} \int_{\mathbb{R}^m} \left(\frac{U'(z)}{|z|}\right)^2 z_i^4 dz (\delta_{ij} \delta_{hk} + \delta_{ik} \delta_{jh} + \delta_{ih} \delta_{jk}).$$

Now, let us estimate the integrals $I_4, I_5$ and $I_6$. We set

$$\tilde{a}(y) = a(\exp_{\xi_0}(y)),$$

and we denote by $\frac{\partial \tilde{a}}{\partial y_s}$ the derivative of $\tilde{a}$ with respect to its $s$th variable. Therefore, we have

$$I_4 = \frac{1}{2} \int_{\mathbb{R}^m} \left[ \sum_{i,j} \tilde{a}(\delta(z + \eta)) - \tilde{a}(0) \right] g^{ij}(\delta(z + \eta))$$

$$\times \frac{\partial}{\partial z_i} (\chi(\delta z) U(z)) \frac{\partial}{\partial z_j} (\chi(\delta z) U(z)) |g(\delta(z + \eta))|^{1/2} dz$$

$$= \frac{\delta^2}{4} \int_{\mathbb{R}^m} \sum_{s,k} \frac{\partial^2 \tilde{a}}{\partial y_s \partial y_k}(0)(z_s + \eta_s)(z_k + \eta_k) |\nabla U|^2 dz + o(\delta^2)$$

$$= \sum_{s,k} \frac{\delta^2}{4} \frac{\partial^2 \tilde{a}}{\partial y_{s} \partial y_{k}}(0) \int_{\mathbb{R}^m} z_s z_k |\nabla U|^2 dz$$

$$+ \sum_{s,k} \frac{\delta^2}{4} \frac{\partial^2 \tilde{a}}{\partial y_{s} \partial y_{k}}(0) \eta_s \eta_k \int_{\mathbb{R}^m} |\nabla U|^2 dz + o(\delta^2)$$

$$= |\varepsilon| \left\{ \frac{1}{4} \sum_k \frac{\partial^2 \tilde{a}}{\partial y_k^2}(0) \int_{\mathbb{R}^m} z_k^2 |\nabla U|^2 dz \right.$$
and by the mean value theorem we get for some $\theta \in (0,1)$,
\[
I_5 = \frac{\delta^2}{2} \int_{\mathbb{R}^m} \left| \tilde{a}(\delta(z + \eta)) - \tilde{a}(0) \right| \tilde{h}(\delta(z + \eta)) \chi^2(\delta z) U^2(z) g(\delta(z + \eta)) \frac{1}{|z|} dz \\
= \frac{\delta^4}{2} \int_{\mathbb{R}^m} \sum_{s,k} \frac{\partial^2 \tilde{a}}{\partial y_s \partial y_k} (\delta \theta z) \cdot (z_s + \eta_s)(z_k + \eta_k) \tilde{h}(\delta(z + \eta)) \chi^2(\delta z) U^2(z) \frac{1}{|z|} dz \\
= O(\delta^3) = o(\varepsilon).
\]

Finally, using
\[
\left| \int_{\mathbb{R}^m} |z|^2 \left[ U^{2^*}_{m-\varepsilon}(z) - U^{2^*}(z) \right] dz \right| \leq |\varepsilon| \int_{\mathbb{R}^m} |z|^2 U^{2^*}_{m-\varepsilon}(z) \ln U(z) dz = o(1)
\]
and also
\[
\frac{\delta^{m-2}}{2^*_{m-\varepsilon}} = \frac{1}{2^*_{m-\varepsilon}} + o(1) = \frac{1}{2^*_{m}} + o(1),
\]
we get
\[
I_6 = \frac{\delta^2}{2} \int_{\mathbb{R}^m} \left[ \tilde{a}(\delta(z + \eta)) - \tilde{a}(0) \right] \chi^{2^*_{m-\varepsilon}}(\delta z) U^{2^*_{m-\varepsilon}}(z) g(\delta(z + \eta)) \frac{1}{|z|} dz \\
= \frac{\delta^2}{2} \int_{\mathbb{R}^m} \sum_{s,k} \frac{\partial^2 \tilde{a}}{\partial y_s \partial y_k} (0)(z_s + \eta_s)(z_k + \eta_k) U^{2^*_{m-\varepsilon}}(z) dz + o(\delta^2) \\
= \frac{\delta^2}{2} \int_{\mathbb{R}^m} \sum_{s,k} \frac{\partial^2 \tilde{a}}{\partial y_s \partial y_k} (0) \eta_s \eta_k \int_{\mathbb{R}^m} U^{2^*_{m-\varepsilon}}(z) dz \\
+ \frac{\delta^2}{2} \int_{\mathbb{R}^m} \sum_{k} \frac{\partial^2 \tilde{a}}{\partial y_k^2} (0) z_k^2 U^{2^*_{m}}(z) dz + o(\delta^2) \\
= \frac{|\varepsilon|}{2} \int_{\mathbb{R}^m} \sum_{k} \frac{\partial^2 \tilde{a}}{\partial y_k^2} (0) z_k^2 U^{2^*_{m}}(z) dz \\
+ \sum_{s,k} \frac{\partial^2 \tilde{a}}{\partial y_s \partial y_k} (0) \eta_s \eta_k \int_{\mathbb{R}^m} U^{2^*_{m}}(z) dz \right) + o(|\varepsilon|) \\
= |\varepsilon| \left( \frac{m^2 - 2}{4m^2} \int_{\mathbb{R}^m} |z|^2 U^{2^*_{m}}(z) dz \\
+ \frac{m^2 - 2}{4m^2} \sum_{s,k} \int_{\mathbb{R}^m} U^{2^*_{m}}(z) dz \right) + o(|\varepsilon|).
\]

Therefore,
\[
I_4 + I_5 - I_6 \\
= |\varepsilon| \left\{ \sum_{k=1}^{m} \frac{\partial^2 \tilde{a}}{\partial y_k^2} (0) \left[ \frac{1}{4m} \int_{\mathbb{R}^m} |z|^2 \nabla U^2 \right] - \frac{m^2 - 2}{4m^2} \int_{\mathbb{R}^m} |z|^2 U^{2^*_{m}}(z) dz \right\}
\]
\[ + \sum_{k,s=1}^{m} \frac{\partial^2 \tilde{a}}{\partial y_s \partial y_k}(0) \eta_s \eta_k \left[ \frac{1}{4} \int_{\mathbb{R}^m} |\nabla U|^2 \, dz ight. \\
\left. - \frac{m - 2}{4m} \int_{\mathbb{R}^m} U^{2m}(z) \, dz \right] \right\} + o(|\varepsilon|). \]

A straightforward computation shows that

\[ \frac{1}{4m} \int_{\mathbb{R}^m} |\nabla U|^2 \, dz - \frac{m - 2}{4m^2} \int_{\mathbb{R}^m} |z|^2 U^{2m}(z) \, dz = \frac{3K_{m}^{-m}}{m(m - 4)} \]

and

\[ \frac{1}{4} \int_{\mathbb{R}^m} |\nabla U|^2 \, dz - \frac{m - 2}{4m} \int_{\mathbb{R}^m} U^{2m}(z) \, dz = \frac{K_{m}^{-m}}{2m}. \]

It is enough to use the following ingredients. For any positive real numbers \( p \) and \( q \) such that \( p - q > 1 \), we let

\[ I_{q}^{p} = \int_{0}^{\infty} \frac{r^q}{(1 + r)^p} \, dr. \]

In particular, there hold that

\[ I_{q+1}^{p+1} = \frac{p - q - 1}{p} I_{q}^{p} \quad \text{and} \quad I_{q+1}^{p+1} = \frac{q + 1}{p - q - 1} I_{q}^{p+1}. \]

Moreover, we have

\[ I_{m}^{m} = \frac{2K_{m}^{-m}}{\alpha_{m}^{2}(m - 2)^{2}\omega_{m-1}}. \]

Finally, we collect all the previous estimates and we get the \( C^0 \)-estimate, taking into account that

\[ \Delta_{g} a(\xi_0) = \sum_{k=1}^{m} \frac{\partial^2 \tilde{a}}{\partial y_k^2}(0) \quad \text{and} \quad D_{g}^2 a(\xi_0)[\eta, \eta] = \sum_{k,s=1}^{m} \frac{\partial^2 \tilde{a}}{\partial y_k \partial y_s}(0) \eta_k \eta_s. \]

Now, let us prove the \( C^1 \)-estimate. We first consider the derivative of the term

\[ A(\delta_{\varepsilon}(t), \eta) = \frac{1}{2} \int_{M} a(x)|\nabla_{g} W_{\delta_{\varepsilon}(t), \eta}|^2 \, d\mu_{g} \]

with respect to \( t \). Since \( \delta'_{\varepsilon}(t) \delta_{\varepsilon}(t) = \frac{\varepsilon}{2} \), we get

\[ \frac{\partial}{\partial t} A(\delta_{\varepsilon}(t), \eta) = \delta'_{\varepsilon}(t) \frac{\partial}{\partial \delta} A(\delta_{\varepsilon}(t), \eta) \]

\[ = \frac{\delta'_{\varepsilon}(t)}{2} \frac{\partial}{\partial \delta} \int_{\mathbb{R}^m} \tilde{a}(\delta(z + \eta))g^{ij}(\delta(z + \eta)) \]

\[ \times \frac{\partial}{\partial z_i} (\chi(\delta z)U(z)) \frac{\partial}{\partial z_j} (\chi(\delta z)U(z)) |g(\delta(z + \eta))|^{1/2} \, dz \]
\[
\begin{align*}
&= \frac{\delta'(t)}{2} \int_{\mathbb{R}^m} \frac{\partial^2 a}{\partial y_k \partial y_r}(0)(z_r + \eta_r)(z_k + \eta_k)|\nabla U(z)|^2 dz \\
&+ \frac{\delta'(t)}{2} \int_{\mathbb{R}^m} \frac{\partial^2 g^{ij}}{\partial y_k \partial y_r}(0)(z_r + \eta_r)(z_k + \eta_k) \\
&\times \frac{\partial U(z)}{\partial z_i} \frac{\partial U(z)}{\partial z_j} dz - \frac{\delta'(t)\delta}{4} \int_{\mathbb{R}^m} a(0)|\nabla U(z)|^2 \\
&\times \frac{\partial^2 g^{ss}}{\partial y_k \partial y_r}(0)(z_r + \eta_r)(z_k + \eta_k) dz + o(\varepsilon) \\
&= \frac{\varepsilon}{4} \frac{\partial^2 a}{\partial y_k \partial y_r}(0)\eta_k\eta_r \int_{\mathbb{R}^m} |\nabla U(z)|^2 dz \\
&+ \frac{\varepsilon}{4} \frac{\partial^2 g^{ij}}{\partial y_k \partial y_r}(0) \int_{\mathbb{R}^m} |\nabla U(z)|^2 z^2 dz \\
&+ \frac{\varepsilon}{4} \frac{\partial^2 g^{ij}}{\partial y_k \partial y_r}(0) \int_{\mathbb{R}^m} \left( \frac{U'(z)}{|z|} \right)^2 z^2 dz \\
&+ \frac{\varepsilon}{4} \frac{\partial^2 g^{ss}}{\partial y_k \partial y_r}(0) \int_{\mathbb{R}^m} |\nabla U(z)|^2 dz \\
&- \frac{\varepsilon}{8} \frac{\partial^2 g^{ss}}{\partial y_k \partial y_r}(0) \int_{\mathbb{R}^m} |\nabla U(z)|^2 z^2 dz + o(\varepsilon).
\end{align*}
\]

Here we recognize the derivative of \( I_1 \) with respect to \( t \).

In a similar way we consider \( \frac{\partial}{\partial \eta_k} A(\delta_{\varepsilon}(t), \eta) \) for \( 1 \leq k \leq m \). We have

\[
\frac{\partial}{\partial \eta_k} A(\delta_{\varepsilon}(t), \eta) = \frac{1}{2} \int_{\mathbb{R}^m} \tilde{a}(\delta(z + \eta))g^{ij}(\delta(z + \eta)) \frac{\partial}{\partial z_i} (\chi(\delta z)U(z))
\]
\[ \begin{align*}
&\times \frac{\partial}{\partial z_j} \left( \chi(\delta z) U(z) \right) |g(\delta(z + \eta))|^{1/2} dz \\
&= \frac{\delta}{2} \int_{\mathbb{R}^m} \frac{\partial \tilde{a}}{\partial y_k} (\delta(z + \eta)) g^{ij}(\delta(z + \eta)) \\
&\times \frac{\partial U(z)}{\partial z_i} \frac{\partial U(z)}{\partial z_j} |g(\delta(z + \eta))|^{1/2} dz \\
&+ \frac{\delta}{2} \int_{\mathbb{R}^m} \tilde{a}(\delta(z + \eta)) \frac{\partial^2 g^{ij}}{\partial y_k \partial y_r} (\delta(z + \eta)) \frac{\partial U(z)}{\partial z_i} \frac{\partial U(z)}{\partial z_j} dz \\
&+ \frac{\delta}{2} \int_{\mathbb{R}^m} \tilde{a}(\delta(z + \eta)) \frac{\partial U(z)}{\partial z_i} \frac{\partial |g|^{1/2}}{\partial y_k} (\delta(z + \eta)) dz + o(\varepsilon) \\
&= \frac{\delta^2}{2} \int_{\mathbb{R}^m} \frac{\partial^2 \tilde{a}}{\partial y_k \partial y_r} (0)(z_r + \eta_r) |\nabla U(z)|^2 dz \\
&+ \frac{\delta^2}{2} \int_{\mathbb{R}^m} \tilde{a}(0) \frac{\partial^2 g^{ij}}{\partial y_k \partial y_r} (0)(z_r + \eta_r) \frac{\partial U(z)}{\partial z_i} \frac{\partial U(z)}{\partial z_j} dz \\
&- \frac{\delta^2}{4} \int_{\mathbb{R}^m} \tilde{a}(0) |\nabla U(z)|^2 \frac{\partial^2 g^{ss}}{\partial y_k \partial y_r} (0)(z_r + \eta_r) dz + o(\varepsilon) \\
&= \frac{\varepsilon}{2} \frac{\partial^2 \tilde{a}}{\partial y_k \partial y_r} (0) \eta_r \int_{\mathbb{R}^m} |\nabla U(z)|^2 dz \\
&+ \frac{\varepsilon}{2} \tilde{a}(0) \eta_r \frac{\partial^2 g^{ii}}{\partial y_k \partial y_r} (0) \int_{\mathbb{R}^m} \left( \frac{U'(z)}{|z|} \right)^2 z_i^2 dz \\
&- \frac{\varepsilon}{4} \tilde{a}(0) \eta_r \frac{\partial^2 g^{ss}}{\partial y_k \partial y_r} \int_{\mathbb{R}^m} |\nabla U(z)|^2 dz + o(\varepsilon).
\end{align*} \]

Here we recognize the derivative of \( I_1 \) with respect to \( \eta_k \).

We argue in a similar way for all the other addenda in \( J_\varepsilon(W_{\delta_\varepsilon(t), \eta}) \). The claim follows easily. \( \square \)

**Proof of Theorem 1.2.** Set

\[ \Theta(\xi_0) := h(\xi_0) - \frac{m - 2}{4(m - 1)} S_g(\xi_0) + \frac{3m(m - 2)}{2(m - 1)} \Delta g^\omega(\xi_0). \]

Let

\[ t_0 := \begin{cases} 
\frac{b_m}{\Theta(\xi_0)} & \text{if } \varepsilon > 0, \\
-\frac{b_m}{\Theta(\xi_0)} & \text{if } \varepsilon < 0.
\end{cases} \]
Since $\xi_0$ is a nondegenerate critical point of $\omega$, a straightforward computation shows that the point $(t_0, 0)$ is a nondegenerate critical point of the function $\Phi$. Therefore, by Proposition 4.1(ii), if $\varepsilon$ is small enough, there exists $(t_\varepsilon, \eta_\varepsilon) \in (0, +\infty) \times \mathbb{R}^m$ which is a critical point of $\tilde{J}_\varepsilon$ such that $(t_\varepsilon, \eta_\varepsilon) \to (t_0, 0)$ as $\varepsilon \to 0$. Finally, by Proposition 4.1(i), we deduce that $W_{\delta_\varepsilon(t), \eta} + \Phi_{\delta_\varepsilon(t), \eta}$ is a solution of problem (1.5) which blows up at the point $\xi_0$ as $\varepsilon \to 0$. □

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