SYMMETRIC \((q, \alpha)\)-STABLE DISTRIBUTIONS.
PART I: FIRST REPRESENTATION

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Abstract

The classic central limit theorem and \(\alpha\)-stable distributions play a key role in probability theory, and also in Boltzmann-Gibbs (BG) statistical mechanics. They both concern the paradigmatic case of probabilistic independence of the random variables that are being summed. A generalization of the BG theory, usually referred to as nonextensive statistical mechanics and characterized by the index \(q\) (\(q = 1\) recovers the BG theory), introduces special (long range) correlations between the random variables, and recovers independence for \(q = 1\). Recently, a \(q\)-central limit theorem consistent with nonextensive statistical mechanics was established\(^{[1]}\) which generalizes the classic Central Limit Theorem. In the present paper we introduce and study symmetric \((q, \alpha)\)-stable distributions. The case \(q = 1\) recovers the Lévy \(\alpha\)-stable distributions.

1 Introduction

In paper\(^{[1]}\) a generalization of the classic central limit theorem applicable to nonextensive statistical mechanics\(^{[2]}\)\(^{[3]}\) (which recovers the usual, Boltzmann-Gibbs statistical mechanics as the \(q = 1\) particular instance), was presented; for reviews of nonextensive statistical mechanics see\(^{[4]}\)\(^{[5]}\)\(^{[6]}\). We follow here along the lines of that paper. One of the important aspects of this generalization is that it concerns the case of random variables correlated in a special manner. On the basis of the \(q\)-Fourier transform \(F_q\) introduced there (\(F_1\) being the classical Fourier transform), and the function

\[ z(s) = \frac{1 + s}{3 - s}, \]

we described attractors of conveniently scaled limits of sums of \(q\)-independent random variables\(^1\) with a finite \((2q - 1)\)-variance\(^2\). This description was essentially based on the mapping

\[ F_q : \mathcal{G}_q[2] \rightarrow \mathcal{G}_{z(q)}[2], \tag{1} \]

\(^1\) \(q\)-independence corresponds to standard probabilistic independence if \(q = 1\), and to specific long-range (global in physics terminology) correlations if \(q \neq 1\).
\(^2\) We required there \(1 \leq q < 2\). Denoting \(Q = 2q - 1\), it is easy to see that this condition is equivalent to the finiteness of the \(Q\)-variance with \(1 \leq Q < 3\); see also\(^{[7]}\).
where $G_q[2]$ is the set of $q$-Gaussians (the number 2 in the notation will soon become transparent).

In the current work, which consists of two parts, we will introduce and study a $q$-analog of the $\alpha$-stable Lévy distributions. In this sense, the present paper is a conceptual continuation of paper [1]. For simplicity we will analyze only symmetric densities in the one-dimensional case. The classic theory of $\alpha$-stable distributions $\mathcal{L}(\alpha)$ was originated by Paul Lévy and developed by Lévy, Gnedenko, Feller and others (see, for instance, [8, 9, 10, 11, 12] and references therein for details and history). The $\alpha$-stable distributions found a huge number of applications in various practical studies [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23], confirming the frequent nature of these distributions.

We introduce a class of random variables $\mathcal{L}_q(\alpha)$, which we call $(q, \alpha)$-stable distributions. Namely, we consider the symmetric densities $f(x)$ with asymptotics $f \sim C|x|^{-1+\frac{1}{1+\alpha(q-1)}}$, $|x| \to \infty$, where $1 \leq q < 2$, $0 < \alpha < 2$, and $C$ is a positive constant. We establish that linear combinations and scaling limits of sequences of $q$-independent random variables with $(q, \alpha)$-stable distributions are again random variables with $(q, \alpha)$-stable distributions. These facts justify that $\mathcal{L}_q(\alpha)$ form a class of stable distributions. To this end, we note that $\mathcal{L}_q(\alpha)$ for fixed $q \in [1, 2)$ and $\alpha \in (0, 2)$ coincides with the set of symmetric Lévy distributions $\mathcal{L}_{sym}(\gamma)$, where

$$\gamma = \gamma(q, \alpha) = \frac{\alpha(2-q)}{1+\alpha(q-1)}.$$  

However, the $(q, \alpha)$-stability holds for specifically correlated random variables ($q$-independence exhibits a special correlation). Thus for each fixed $q \in [1, 2)$ we have the set of $q$-independent $(q, \alpha)$-stable distributions. If $q = 1$ then $q$-independence becomes usual independence and $\gamma(1, \alpha) = \alpha$, implying $\mathcal{L}_1(\alpha) = \mathcal{L}_{sym}(\alpha)$. The main purpose of the current paper is to classify $(q, \alpha)$-stable distributions in terms of their densities depending on the parameters $1 \leq q < 2$ (or equivalently $1 \leq Q < 3$, $Q = 2q - 1$) and $0 < \alpha \leq 2$. We establish the mapping

$$F_q : G_q[2] \to G_q[\alpha],$$  

where $G_q[\alpha]$ is the set of functions $\{be^{-\beta|x|^\alpha}, \ b > 0, \beta > 0\}$, and

$$q^L = \frac{3 + Q\alpha}{1 + \alpha}, \quad Q = 2q - 1,$$

i.e.,

$$\frac{2}{q^L - 1} = \frac{1 + \alpha}{1 + \alpha(q-1)}.$$  

The particular case $q = Q = 1$ recovers $q^L = \frac{3 + \alpha}{1+\alpha}$, already known in the literature. Denote $Q_1 = \{(Q, \alpha) : 1 \leq Q < 3, \alpha = \frac{3}{2}\}$, $Q_2 = \{(Q, \alpha) : 1 \leq Q < 3, 0 < \alpha < 2\}$ and $Q = Q_1 \cup Q_2$. Note that the case $(Q, \alpha) \in Q_1$ for $q$-independent random variables with a finite $Q$-variance was studied in [1]. For $(Q, \alpha) \in Q_2$ the $Q$-variance is infinite. We will focus our analysis namely on the latter case. Note that the case $\alpha = 2$, in the framework of the present description like in that of the classic $\alpha$-stable distributions, becomes peculiar.

In Part II we study the attractors of scaled sums, and expand the results of paper [1] to the region $Q$ generalizing the mapping (1) into the form

$$F_{q,\alpha(q)} : G_q[\alpha] \to G_{xo(q)}[\alpha], \ 1 \leq q < 2, \ 0 < \alpha \leq 2,$$

\(^3\)Hereafter $g(x) \sim h(x), x \to a$, means that $\lim_{x \to a} \frac{g(x)}{h(x)} = 1.$
where
\[ \zeta_\alpha(s) = \frac{\alpha - 2(1 - q)}{\alpha} \] and \[ z_\alpha(s) = \frac{\alpha q + 1 - q}{\alpha + 1 - q}. \]

Note that, if \( \alpha = 2 \), then \( \zeta_2(q) = q \) and \( z_2(q) = (1 + q)/(3 - q) \), thus recovering the mapping (1), and consequently, the result of [1].

These two descriptions of \((q, \alpha)\)-stable distributions, based on mappings (2) and (3), respectively, can be unified to the scheme

\[
\mathcal{L}(q, \alpha) \xrightarrow{F_q} \mathcal{G}_q(\alpha) \xleftarrow{F_{\zeta(q)}} \mathcal{G}_{\zeta(q)}(2)
\]

(4)

\[
\downarrow F_q
\]

\[
\mathcal{G}_q(2)
\]

which gives the full picture of interrelations between the values of parameters \( q \in [1, 2) \) and \( \alpha \in (0, 2) \).

2 Basic operations of \( q \)-algebra

We recall briefly the basic operations of \( q \)-algebra. Indeed, the analysis we will conduct is entirely based on the \( q \)-structure of nonextensive statistical mechanics (for more details see [24, 4] and references therein). To this end, we recall the well known fact that the classical Boltzmann-Gibbs entropy \( S_{BG} = -\sum p_i \ln p_i \) satisfies the additivity property. Namely, if \( A \) and \( B \) are two independent subsystems, then \( S_{BG}(A + B) = S_{BG}(A) + S_{BG}(B) \). However, the \( q \)-generalization of the classic entropy introduced in [2] and given by \( S_q = \frac{1 - \sum p_i^q}{q - 1} \) with \( q \in \mathbb{R} \) and \( S_1 = S_{BG} \), does not possess this property if \( q \neq 1 \). Instead, it satisfies the pseudo-additivity (or \( q \)-additivity) \[ S_q(A + B) = S_q(A) + S_q(B) + (1 - q) S_q(A) S_q(B). \]

Inherited from the right hand side of this equality, the \( q \)-sum of two given real numbers, \( x \) and \( y \), is defined as \( x \oplus_q y = x + y + (1 - q) xy \). The \( q \)-sum is commutative, associative, recovers the usual summing operation if \( q = 1 \) (i.e. \( x \oplus y = x + y \)), and preserves 0 as the neutral element (i.e. \( x \oplus 0 = x \)). By inversion, we can define the \( q \)-subtraction as \( x \ominus_q y = \frac{x - y}{1 + (1 - q)y} \). The \( q \)-product for \( x, y \) is defined by the binary relation \( x \otimes_q y = [x^{1-q} + y^{1-q} - 1]_+^{1-q} \). Here the symbol \([x]_+\) means that \([x]_+ = x \) if \( x \geq 0 \), and \([x]_+ = 0 \) if \( x < 0 \). This operation also commutative, associative, recovers the usual product when \( q = 1 \), and preserves 1 as the unity. The \( q \)-product is defined if \( x^{1-q} + y^{1-q} \geq 1 \). Again by inversion, it can be defined the \( q \)-division: \( x \oslash_q y = (x^{1-q} - y^{1-q} + 1)^{1-q} \).

3 \( q \)-generalization of the exponential and cyclic functions

Now we introduce the \( q \)-exponential and \( q \)-logarithm [24], which play an important role in the nonextensive theory. These functions are denoted by \( e^q_x \) and \( \ln_q x \) and respectively defined as \( e^q_q = [1 + (1 - q)x]_+^{1-q} \) and \( \ln_q x = \frac{x^{1-q} - 1}{1-q} \), \((x > 0)\). The entropy \( S_q \) can be conveniently rewritten in the form

\[
S_q = \sum p_i \ln_q \frac{1}{p_i}.
\]

Let us mention now the main properties of these functions, which we will use essentially in this paper. For the \( q \)-exponential the relations \( e_q[x] = e_q x e_q y \) and \( e_q x + y = e_q x \ominus_q e_q y \) hold true.
These relations can be written equivalently as follows: $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$ and $\ln_q(xy) = (\ln_q x) \oplus_q (\ln_q y)$. The $q$-exponential and $q$-logarithm have the asymptotics

$$e^x_q = 1 + x + \frac{q}{2} x^2 + o(x^2), \; x \to 0,$$

and

$$\ln_q(1 + x) = x - \frac{q}{2} x^2 + o(x^2), \; x \to 0,$$

respectively. The $q$-product and $q$-exponential can be extended to complex numbers $z = x + iy$ (see [11, 25, 26]).

In addition, for $q \neq 1$ the function $e^x_q$ can be analytically extended to the complex plain except the point $z_0 = -1/(1 - q)$ and defined as the principal value along the cut $(-\infty, z_0)$. If $q < 1$, then, for real $y$, $|e^{iy}_q| \geq 1$ and $|e^{iy}_q| \sim K_q(1 + y^2)^{-1/(1-q)}$, $y \to \infty$, with $K_q = (1 - q)^{1/(1-q)}$. Similarly, if $q > 1$, then $0 < |e^{iy}_q| \leq 1$ and $|e^{iy}_q| \to 0$ if $|y| \to \infty$.

**Lemma 3.1** Let $A_n(q) = \prod_{k=0}^n a_k(q)$, where $a_k(q) = q - k(1-q)$. Then the following power series expansion holds

$$e^x_q = 1 + z + z^2 \sum_{n=0}^\infty \frac{A_n(q)}{(n+2)!} z^n, \; |z| < \frac{1}{|1-q|}.$$

**Corollary 3.2** Let $I_q = (-1/|1-q|, 1/|1-q|)$. For arbitrary real number $x \in I_q$ the equation

$$e^{ix}_q = \{1 - x^2 \sum_{n=0}^\infty \frac{(-1)^n A_{2n}(q)}{(2n+2)!} x^{2n}\} + i\{x - x^2 \sum_{n=0}^\infty \frac{(-1)^n A_{2n+1}(q)}{(2n+3)!} x^{2n+1}\}$$

holds.

Define for $x \in I_q$ the functions $q$-cos and $q$-sin by formulas

$$\cos_q(x) = 1 - x^2 \sum_{n=0}^\infty \frac{(-1)^n A_{2n}(q)}{(2n+2)!} x^{2n},$$

and

$$\sin_q(x) = x - x^2 \sum_{n=0}^\infty \frac{(-1)^n A_{2n+1}(q)}{(2n+3)!} x^{2n+1}.$$

In fact, $\cos_q(x)$ and $\sin_q(x)$ is defined for all real $x$ by using appropriate power series expansions. Properties of $q$-sin, $q$-cos, and corresponding $q$-hyperbolic functions, were studied in [27]. Here we note that the $q$-analogs of the well known Euler’s formulas read

**Corollary 3.3** (i) $e^{ix}_q = \cos_q(x) + i \sin_q(x)$;

(ii) $\cos_q(x) = \frac{e^{ix}_q + e^{-ix}_q}{2}$;

(iii) $\sin_q(x) = \frac{e^{ix}_q - e^{-ix}_q}{2i}$.

**Lemma 3.4** The following equality holds:

$$\cos_q(2x) = e^{2(1-q)x^2}_{2q-1} - 2 \sin^2_{2q-1}(x).$$

This property reflects the possible extensivity of $S_q$ in the presence of special correlations [30, 31, 32, 33].
Proof. The proof follows from the definitions of \( \cos_q(x) \) and \( \sin_q(x) \), and from the fact that \( (e^x_q)^2 = e^{2x_q/(1+q)} \) (see Lemma 2.1 in [1]).

Denote \( \Psi_q(x) = \cos_q 2x - 1 \). It follows from Equation (10) that
\[
\Psi_q(x) = (e^{2(1-q)x^2} - 1) - 2 \sin_{2q-1}^2(x).
\]

**Lemma 3.5** Let \( q \geq 1 \). Then we have

1. \(-2 \leq \Psi_q(x) \leq 0\);
2. \( \Psi_q(x) = -2q x^2 + o(x^3), \ x \to 0 \).

Proof. It follows from (10) that \( \Psi_q(x) \leq 0 \). Further, \( \sin_q(x) \) can be written in the form (see [27]) \( \sin_q(x) = \rho_q(x) \sin[\varphi_q(x)] \), where \( \rho_q(x) = (e^{(1-q)x^2})^{1/2} \) and \( \varphi_q(x) = \frac{\arctan(1-q)x}{1-q} \). This yields \( \Psi_q(x) \geq -2 \) if \( q \geq 1 \). Using the asymptotic relation (5), we get
\[
e^{2(1-q)x^2} - 1 = 2(1 - q)x^2 + o(x^3), \ x \to 0.
\]

It follows from (5) that
\[
-2 \sin_{2q-1}^2(x) = -2x^2 + o(x^3), \ x \to 0.
\]

The relations (10), (11) and (12) imply the second part of the statement.

**Remark 3.6** It is not hard to verify that in the case \( q > 1 \) for \( x > (q - 1)^{-1} \) the representation
\[
e^{-x_q} = [(q - 1)x]^{-\frac{1}{q-1}} \left( 1 - \frac{1}{(1-q)^2x^2} + \frac{1}{(1-q)^4x^2} \sum_{n=0}^{\infty} \frac{(-1)^n A_n(q)}{(n+2)!(q-1)^{2n}} \frac{1}{x^n} \right)
\]
follows from Lemma 3.1.

### 4 q-Fourier transform for symmetric functions

The q-Fourier transform for \( q \geq 1 \), based on the q-product, was introduced in [1] and played a central role in establishing the q-analog of the standard central limit theorem. Formally the q-Fourier transform for a given function \( f(x) \) is defined by the formula
\[
F_q[f](\xi) = \int_{-\infty}^{\infty} e^{ix\xi} \otimes_q f(x)dx.
\]

For discrete functions \( f_k, k = 0, \pm 1, \ldots \), this definition takes the form
\[
F_q[f](\xi) = \sum_{k=-\infty}^{\infty} e^{ik\xi} \otimes_q f(k).
\]

In the future we use the same notation in both cases. We also call (13) or (14) the q-characteristic function of a given random variable \( X \) with an associated density \( f(x) \), using the notations \( F_q(X) \) or \( F_q(f) \) equivalently.
It should be noted that, if in the formal definition (13), $f$ is compactly supported, then integration has to be taken over this support, although, in contrast with the usual analysis, the function $e^{i2\xi_q} \otimes_q f(x)$ under the integral does not vanish outside the support of $f$. This is an effect of the $q$-product.

The following lemma establishes the relation of the $q$-Fourier transform without using the $q$-product.

**Lemma 4.1** The $q$-Fourier transform can be written in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x)e^{i2\xi_q(f(x))^{q-1}}dx.$$  \hspace{1cm} (15)

**Remark 4.2** Note that, if the $q$-Fourier transform of a given function $f(x)$ defined by the formal definition in (13) exists, then it coincides with the expression in (15). The $q$-Fourier transform determined by the formula (15) has an advantage when compared to the formal definition: it does not use the $q$-product, which is, as we noticed above, restrictive in use.

Further to the properties of the $q$-Fourier transform established in [1], we note that, for symmetric densities, the assertion analogous to Lemma 4.1 is true with the $q$-cos.

**Lemma 4.3** Let $f(x)$ be an even function. Then its $q$-Fourier transform can be written in the form

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x)\cos_q(x\xi[f(x)]^{q-1})dx.$$  \hspace{1cm} (16)

**Proof.** Notice that, because of the symmetry of $f$,\[
\int_{-\infty}^{\infty} e^{i2\xi_q} \otimes_q f(x)dx = \int_{-\infty}^{\infty} e^{-i2\xi_q} \otimes_q f(x)dx.
\]

Taking this into account, we have

$$F_q[f](\xi) = \frac{1}{2} \int_{-\infty}^{\infty} \left( e^{i2\xi_q} \otimes_q f(x) + e^{-i2\xi_q} \otimes_q f(x) \right) dx.$$  

Applying Lemma 4.1 we obtain

$$F_q[f](\xi) = \int_{-\infty}^{\infty} f(x) \frac{e^{i2\xi_q[f(x)]^{q-1}} + e^{-i2\xi_q[f(x)]^{q-1}}}{2} dx,$$

which coincides with (16).  \hspace{1cm} $\blacksquare$

Further, denote $H_{q,\alpha} = \{ f \in L_1 : f(x) \sim C|x|^{-1+\alpha(q-1)}, \ |x| \to \infty \}$. It is readily seen that $\phi(q, \alpha) = \frac{\alpha+1}{1+\alpha(q-1)} > 1$ for all $\alpha \in (0,2)$ and $q \in [1,2)$. Moreover, $\phi(q, \alpha)(2q-1) < 3$ for all $\alpha \in (0,2)$ and $q \in [1,2)$, which implies $\sigma_2^{2q-1}(f) = \infty$.\footnote{In [1] we did not require the condition for $f(x)$ to be symmetric if $\sigma_2^{2q-1}(f) < \infty$.}

**Lemma 4.4** Let $f(x), x \in R$, be a symmetric probability density function of a given random variable. Further, let either

(i) the $(2q-1)$-variance $\sigma_{2q-1}^2(f) < \infty$ (associated with $(2q-1, \alpha) \in Q_1$), or

(ii) $f(x) \in H_{q,\alpha}$, where $(2q-1, \alpha) \in Q_2$.  

Further to the properties of the $q$-Fourier transform established in [1], we note that, for symmetric densities, the assertion analogous to Lemma 4.1 is true with the $q$-cos.
Then, for the $q$-Fourier transform of $f(x)$, the following asymptotic relation holds true:

$$F_q[f](\xi) = 1 - \mu_{q,\alpha}|\xi|^\alpha + o(|\xi|^\alpha), \xi \to 0,$$

where

$$\mu_{q,\alpha} = \begin{cases} \frac{\sigma_{2q-1}^2}{2^{2q-1}} \nu_{2q-1}, & \text{if } \alpha = 2; \\
\frac{2^{1-\alpha(1+\alpha(1-q))}}{2-q} \int_0^\infty \frac{\Psi_q(y)}{y^{1+\alpha(1-q)}} dy, & \text{if } (2q-1, \alpha) \in \mathbb{Q}_2.
\end{cases}$$

with $\nu_{2q-1}(f) = \int_{-\infty}^\infty|f(x)|^{2q-1} dx$.

**Remark 4.5** Stable distributions require $\mu_{q,\alpha}$ to be positive. We have seen (Lemma 3.5) that if $q \geq 1$, then $\Psi_q(x) \leq 0$ (not being identically zero), which yields $\mu_{q,\alpha} > 0$.

**Proof.** First, we assume that $\alpha = 2$. Using Lemma 4.1, we have

$$F_q[f](\xi) = \int_{-\infty}^\infty (\circ_q e^{ix\xi}) \cos_q(x\xi[f(x)]^{q-1}) dx = \int_{-\infty}^\infty f(x) \cos_q(x\xi[f(x)]^{q-1}) dx.$$  \hspace{1cm} (19)

Making use of the asymptotic expansion (5) we can rewrite the right hand side of (19) in the form

$$F_q[f](\xi) = \int_{-\infty}^\infty f(x) \left(1 + \xi f(x)[f(x)]^{q-1} - q/2\xi^2 [f(x)]^{2(q-1)}\right) dx + o(\xi^3) =$$

$$1 - (q/2)\xi^2 \sigma_{2q-1}^2 \nu_{2q-1} + o(\xi^3), \xi \to 0,$$

from which the first part of Lemma follows.

Now, assume $(2q-1, \alpha) \in \mathbb{Q}_2$. We apply Lemma 4.3 to obtain

$$F_q[f](\xi) - 1 = \int_{-\infty}^\infty f(x)[\cos_q(x\xi[f(x)]^{q-1}) - 1] dx =$$

$$2 \int_0^N f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx + 2 \int_0^\infty f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx,$$

where $N$ is a sufficiently large finite number. In the first integral we use the asymptotic relation

$$\Psi_q(x) = \frac{q}{2} x^2 + o(x^3),$$

which follows from Lemma 3.5, and get

$$2 \int_0^N f(x) \Psi_q\left(\frac{2\xi[f(x)]^{q-1}}{2}\right) dx =$$

$$- q\xi^2 \int_0^N x^2 f^{2q-1}(x) dx + o(\xi^3), \xi \to 0. \hspace{1cm} (20)$$

In the second integral taking into account the hypothesis of the lemma with respect to $f(x)$, we have

$$2 \int_0^\infty f(x) \Psi_q\left(\frac{x\xi[f(x)]^{q-1}}{2}\right) dx = 2C \int_0^\infty \frac{1}{x^{1+\alpha(1-q)}} \Psi_q\left(\frac{x^{1-\alpha(1+q-1)}}{2}\right) dx.$$  \hspace{1cm} (18)

We use the substitution

$$x^{2-q}/x^{1+\alpha(1-q)} = 2y/\xi.$$
in the last integral, and obtain

\[
2 \int_{N}^{\infty} f(x) \Psi_q \left( \frac{x \xi [f(x)]^{q-1}}{2} \right) dx = \\
- \frac{2^{2-\alpha}(1 + \alpha(q - 1))C}{2 - q} |\xi|^\alpha \int_{0}^{\infty} \Psi_q(y) \frac{dy}{y^{\alpha+1}} + o(|\xi|^\alpha), \xi \to 0.
\] 

Hence, the obtained asymptotic relations (20) and (21) complete the proof. ■

5 Weak convergence of correlated random variables

This section we start with introduction of the notion of \(q\)-independence in particular case. More general definition and some examples are given in [1]. We will also introduce two types of convergence, namely, \(q\)-convergence and weak \(q\)-convergence and establish their equivalence.

**Definition 5.1** Two random variables \(X\) and \(Y\) are said to be \(q\)-independent if

\[
F_q[X + Y](\xi) = F_q[X](\xi) \otimes_q F_q[Y](\xi).
\] 

In terms of densities, the relation (22) can be rewritten as follows. Let \(f_X\) and \(f_Y\) be densities of \(X\) and \(Y\) respectively, and let \(f_{X+Y}\) be the density of \(X+Y\). Then

\[
\int_{-\infty}^{\infty} e^{ix\xi} \otimes_q f_{X+Y}(x) dx = F_q[f_X](\xi) \otimes_q F_q[f_Y](\xi).
\] 

For \(q = 1\) the conditions (22) turns into the well known relation

\[
F[f_X * f_Y] = F[f_X] \cdot F[f_Y]
\]

between the convolution (noted \(*\)) of two densities and the multiplication of their (classical) Fourier images, and holds for independent \(X\) and \(Y\). If \(q \neq 1\), then \(q\)-independence describes a specific correlation.

**Definition 5.2** Let \(X_1, X_2, ..., X_N, ...\) be a sequence of identically distributed random variables. Denote \(Y_N = X_1 + ... + X_N\). By definition, the sequence \(X_N, N = 1, 2, ...\) is said to be \(q\)-independent (or \(q\)-i.i.d.) if for all \(N = 2, 3, ...\), the relations

\[
F_q[Y_N][\xi] = F_q[X_1](\xi) \otimes_q ... \otimes_q F_q[X_N](\xi)
\] 

hold.

**Definition 5.3** A sequence of random variables \(X_N, N = 1, 2, ...,\) is said to be \(q\)-convergent to a random variable \(X_\infty\) if \(\lim_{N \to \infty} F_q[X_N](\xi) = F_q[X_\infty](\xi)\) locally uniformly in \(\xi\).

Evidently, this definition is equivalent to the weak convergence (denoted by “\(\Rightarrow\)”) of random variables if \(q = 1\). For \(q \neq 1\) denote by \(W_q\) the set of continuous functions \(\phi\) satisfying the condition

\[
|\phi(x)| \leq C(1 + |x|)^{-\frac{\alpha}{q-1}}, x \in \mathbb{R}.
\]

\[6\] We assume \(X\) and \(Y\) to have the zero \(q\)-means. For the definition of \(q\)-independence for random variables with non-zero \(q\)-means see [1].
Definition 5.4 A sequence of random variables $X_N$ with the density $f_N$ is called weakly q-convergent to a random variable $X_\infty$ with the density $f$ if $\int_\mathbb{R} f_N(x) dm_q \rightarrow \int_\mathbb{R} f(x) dm_q$ for arbitrary measure $m_q$ defined as $dm_q(x) = \phi_q(x) dx$, where $\phi_q \in W_q$. We denote the q-convergence by the symbol $\xrightarrow{q}$. 

Lemma 5.5 Let $q > 1$. Then $X_N \Rightarrow X_0$ yields $X_N \xrightarrow{q} X_0$. 

The proof of this lemma immediately follows from the obvious fact that $W_q$ is a subset of the set of bounded continuous functions.

Recall that a sequence of probability measures $\mu_N$ is called tight if, for an arbitrary $\epsilon > 0$, there is a compact $K_\epsilon$ and an integer $N_\epsilon^*$ such that $\mu_N(R^d \setminus K_\epsilon) < \epsilon$ for all $N \geq N_\epsilon^*$.

Lemma 5.6 Let $1 < q < 2$. Assume a sequence of random variables $X_N$, defined on a probability space with a probability measure $P$, and associated densities $f_N$, is q-convergent to a random variable $X$ with an associated density $f$. Then the sequence of associated probability measures $\mu_N = P(X_N^{-1})$ is tight.

Proof. Assume that $1 < q < 2$ and $X_N$ is a q-convergent sequence of random variables with associated densities $f_N$ and associated probability measures $\mu_N$. We have

$$\frac{1}{R} \int_{-R}^R (1 - F_q[f_N](\xi)) d\xi = \frac{1}{R} \int_{-R}^R (1 - \int_R f_N e^{ix\xi f_N^{-1}} dx) d\xi =$$

$$\int_R \left( \frac{1}{R} \int_{-R}^R (1 - e^{ix\xi f_N^{-1}}) d\xi \right) d\mu_N(x).$$

(25)

It is not hard to verify that

$$\frac{1}{R} \int_{-R}^R e^{ix\xi t} d\xi = \frac{2\sin \frac{1}{R-q}(Rx(2-q)t)}{Rx(2-q)t}.$$ 

(26)

It follows from (25) and (26) that

$$\frac{1}{R} \int_{-R}^R (1 - F_q[f_N](\xi)) d\xi = 2 \int_{-\infty}^\infty \left( 1 - \frac{\sin \frac{1}{R-q}(x(2-q)R f_N^{-1})}{Rx(2-q)f_N^{-1}} \right) d\mu_N(x).$$

(27)

Since $1 < q < 2$ by assumption, $\frac{1}{R-q} > 1$ as well. It is known [27, 28, 29] that for any $q' > 1$ the properties $\sin_q'(x) \leq 1$ and $(\sin_q'(x))/x \rightarrow 1$, $x \rightarrow 0$ hold. Moreover, $(\sin_q'(x))/x \leq 1, \forall x \in R$. Suppose, $\lim_{|x| \rightarrow \infty} |x| f_N^{-1} = L_N$, $N \geq 1$. Divide the set $\{N \geq N_0\}$ into two subsets $A = \{N_j \geq N_0 : L_{N_j} > 1\}$ and $B = \{N_k \geq N_0 : L_{N_k} \leq 1\}$. If $N \in A$, since $\frac{1}{R-q} \leq 1$, there is a number $a > 0$ such that

$$\frac{1}{R} \int_{-R}^R (1 - F_q[f_N](\xi)) d\xi \geq 2 \int_{|x| \geq a} \left( 1 - \frac{1}{R|x|(2-q)f_N^{-1}} \right) d\mu_N(x)$$

$$\geq C \mu_N (|x| \geq a), \ C > 0 \ \forall N \in A,$$

for $R$ small enough. Now taking into account the q-convergence of $X_N$ to $X$ and, if necessary, taking $R$ smaller, for any $\epsilon > 0$, we obtain

$$\mu_N (|x| \geq a) \leq \frac{1}{CR} \int_{-R}^R (1 - F_q[f_0](\xi)) d\xi < \epsilon, \ \forall N \in A.$$
If \( N \in B \) then there exist constants \( b > 0, \delta > 0 \), such that
\[
f_N(x) \leq \frac{L_N + \delta}{|x|^{q-1}} \leq \frac{1 + \delta}{|x|^{q-1}}, \quad |x| \geq b, \forall N \in B.
\]
Hence, we have
\[
\mu_N(|x| > b) = \int_{|x| > b} f_N(x) \, dx \leq (1 + \delta) \int_{|x| > b} \frac{dx}{|x|^{q-1}}, \quad N \in B.
\]

Since, \( 1/(q-1) > 1 \), for any \( \epsilon > 0 \) we can select a number \( b_\epsilon \geq b \) such that \( \mu_N(|x| > b_\epsilon) < \epsilon, \ N \in B \). As far as \( A \cup B = \{N \geq N_0 \} \) the proof of the statement is complete. ■

Further, we introduce the function
\[
D_q(t) = D_q(t; a) = te^{iat^{q-1}} = t(1 + i(1 - q)at^{q-1})^{-\frac{1}{q-1}},
\]
defined on \([0, 1]\), where \( 1 < q < 2 \) and \( a \) is a fixed real number. Obviously \( D_q(t) \) is continuous on \([0, 1]\) and differentiable in the interval \((0, 1)\). In accordance with the classical Lagrange average theorem for any \( t_1, t_2, 0 \leq t_1 < t_2 \leq 1 \) there exists a number \( t_* \), \( t_1 < t_* < t_2 \) such that
\[
D_q(t_1) - D_q(t_2) = D_q'(t_*)(t_1 - t_2),
\]
where \( D_q' \) means the derivative of \( D_q(t) \) with respect to \( t \).

Consider the following Cauchy problems for the Bernoulli equation
\[
y' - \frac{1}{t}y = \frac{ia(1-q)}{t}y^q, \quad y(0) = 0,
\]
It is not hard to verify that \( D_q(t) \) is a solution to the problems \([30]\).

**Lemma 5.7** For \( D_q'(t) \) the estimate
\[
|D_q'(t; a)| \leq C(1 + |a|)^{-\frac{q}{q-1}}, \quad t \in (0, 1), \ a \in R^1,
\]
holds, where the constant \( C \) does not depend on \( t \).

**Proof.** It follows from \([28]\) and \([30]\) that
\[
|y'(t)| \leq t^{-1}|y + ia(1-q)y^q| = |e^{iat^{q-1}} + ia(1-q)(e^{iat^{q-1}})^q| = |1 + ia(1-q)t^{q-1}|^{-\frac{q}{q-1}} \leq C(1 + |a|)^{-\frac{q}{q-1}}, \ t \in (0, 1).
\]

■

**Theorem 5.8** Let \( 1 < q < 2 \) and a sequence of random vectors \( X_N \) be weakly \( q \)-convergent to a random vector \( X \). Then \( X_N \) is \( q \)-convergent to \( X \).

**Proof.** Assume \( X_N \), with associated densities \( f_N \), is weakly \( q \)-convergent to a \( X \), with an associated density \( f \). The difference \( \mathcal{F}_q[f_N](\xi) - \mathcal{F}_q[f_N](\xi) \) can be written in the form
\[
\mathcal{F}_q[f_N](\xi) - \mathcal{F}_q[f_N](\xi) = \int_{R^d} (D_q(f_N(x)) - D_q(f(x))) \, dx,
\]
where \( D_q(t) = D_q(t; a) \) is defined in \([28]\) with \( a = x\xi \). It follows from \([29]\) and \([31]\) that
\[
|\mathcal{F}_q[f_N](\xi) - \mathcal{F}_q[f_N](\xi)| \leq C \int_{R^d} |(1 + |x|)^{-\frac{q}{q-1}} (f_N(x) - f(x))| \, dx,
\]
which yields \( \mathcal{F}_q[f_N](\xi) \to \mathcal{F}_q[f_N](\xi) \) for all \( \xi \in R^d \). ■
Theorem 5.9 Let $1 < q < 2$ and a sequence of random vectors $X_N$ with the associated densities $f_N$ is $q$-convergent to a random vector $X$ with the associated density $f$ and $F_q[f](\xi)$ is continuous at $\xi = 0$. Then $X_N$ weakly $q$-converges to $X$.

Proof. Now assume that $f_N$ converges to $f$ in the sense of $q$-convergence. It follows from Lemma 5.6 that the corresponding sequence of induced probability measures $\mu_N = P(X_N^{-1})$ is tight. This yields relatively weak compactness of the sequence $\mu_N$. Theorem 5.8 implies that each weakly convergent subsequence $\{\mu_{N_j}\}$ of $\mu_N$ converges to $\mu = P(X^{-1})$. Hence, $\mu_N \Rightarrow \mu$, or the same, $X_N \Rightarrow X$. Now applying Lemma 5.5 we complete the proof. ■

6 Symmetric $(q, \alpha)$-stable distributions. First representation

In this section we introduce symmetric $(q, \alpha)$-stable distributions and give the description based on the mapping (2). In accordance with this description $q$ takes any value in $[1, 2)$, however we distinguish the cases $\alpha = 2$ and $\alpha 

Definition 6.1 A random variable $X$ is said to have a $(q, \alpha)$-stable distribution if its $q$-Fourier transform is represented in the form $e_q^{-\beta |\xi|^{\alpha}}$, with $\beta > 0$. We denote by $L_q(\alpha)$ the set of all $(q, \alpha)$-stable distributions.

Denote $G_q(\alpha) = \{be_q^{-\beta |\xi|^{\alpha}}, b > 0, \beta > 0\}$. In other words $X \in L_q(\alpha)$ if $F_q[f] \in G_q(\alpha)$ with $b = 1$. Note that if $\alpha = 2$, then $G_q(2)$ represents the set of $q$-Gaussians and $L_q(2)$ - the set of random variables whose densities are $q_s$-Gaussians, where $q_s = (3q - 1)/(1 + q)$.

Proposition 6.2 Let $q$-independent random variables $X_j \in L_q(\alpha), j = 1, ..., m$. Then for any constants $a_1, ..., a_m$,

$$\sum_{j=1}^{m} a_jX_j \in L_q(\alpha).$$

Proof. Let

$$F_q[X_j](\xi) = e_q^{-\beta |\xi|^{\alpha}}, j = 1, ..., m.$$ 

Using the properties $e_q^{x} \otimes e_q^{y} = e_q^{x+y}$ and $F_q[aX](\xi) = F_q[X](a^{-q} |\xi|)$, it follows from the definition of the $q$-independence that

$$F_q[\sum_{j=1}^{m} a_jX_j] = e_q^{-\beta |\xi|^{\alpha}}, \beta = \sum_{j=1}^{m} \beta_ja_j^{\alpha(2-q)} > 0.$$

Remark 6.3 Proposition 6.2 justifies the stability of distributions in $L_q(\alpha)$. Recall that if $q = 1$ then $q$-independent random variables are independent in the usual sense. Thus, if $q = 1$, $0 < \alpha < 2$, then $L_1(\alpha)$ coincides with symmetric $\alpha$-stable Lévy distributions $L_{sym}(\alpha)$.

Further we show that the $q$-weak limits of sums

$$Z_N = \frac{1}{s_N(q, \alpha)} (X_1 + ... + X_N), N = 1, 2, ...$$

as $N \rightarrow \infty$, where $s_N(q, \alpha), N = 1, 2, ...$, are some reals (scaling parameter), also belong to $L_q(\alpha)$.  

1
Definition 6.4 A sequence of random variables $Z_N$ is said to be $q$-convergent to a $(q, \alpha)$-stable distribution, if
\[
\lim_{N \to \infty} F_q[Z_N](\xi) \in G_q(\alpha) \text{ locally uniformly by } \xi.
\]

Theorem 1. Assume $(2q-1, \alpha) \in \mathbb{Q}_2$. Let $X_1, X_2, \ldots, X_N, \ldots$ be symmetric $q$-independent random variables and all having the same probability density function $f(x) \in H_{q, \alpha}$. Then $Z_N$, with
\[
s_N(q, \alpha) = \left(\mu_{q, \alpha} N\right)^{-\frac{1}{(\alpha-1)q}},
\]
is $q$-convergent to a $(q, \alpha)$-stable distribution, as $N \to \infty$.

Remark 6.5 By definition $\mathbb{Q}_2$ excludes the value $\alpha = 2$. The case $\alpha = 2$, in accordance with the first part of Lemma 4.4, coincides with Theorem 2 of [1]. Note in this case $L_q(2) = G_q(2)$, where $q^* = \frac{3q-1}{q+1}$.

Proof. Assume $(Q, \alpha) \in \mathbb{Q}_2$. Let $f$ be the density associated with $X_1$. First we evaluate $F_q(X_1) = F_q(f(x))$. Using Lemma 4.4 we have,
\[
F_q[f](\xi) = 1 - \mu_{q, \alpha} |\xi|^\alpha + o(|\xi|^\alpha), \xi \to 0.
\]
Denote $Y_j = N^{-\frac{1}{\alpha}} X_j$, $j = 1, 2, \ldots$. Then $Z_N = Y_1 + \ldots + Y_N$. Further, it is readily seen that, for a given random variable $X$ and real $a > 0$, the equality $F_q[aX](\xi) = F_q[X](a^{2-q}\xi)$ holds. It follows from this relation that $F_q(Y_j) = F_q[f]\left(\frac{\xi}{(\mu_{q, \alpha} N)^{1/\alpha}}\right)$, $j = 1, 2, \ldots$. Moreover, it follows from the $q$-independence of $X_1, X_2, \ldots,$ and the associativity of the $q$-product that
\[
F_q[Z_N](\xi) = F_q[f]\left((\mu_{q, \alpha} N)^{-\frac{1}{\alpha}} \xi\right) \in G_q(\alpha) \text{ (N factors)}.
\]
Hence, making use of the expansion (4) for the $q$-logarithm, Eq. (34) implies
\[
\ln_q F_q[Z_N](\xi) = N \ln_q F_q[f]\left((\mu_{q, \alpha} N)^{-\frac{1}{\alpha}} \xi\right) = N \ln_q (1 - \frac{|\xi|^\alpha}{N} + o((|\xi|^\alpha)/N)) = - |\xi|^\alpha + o(1), \quad N \to \infty,
\]
locally uniformly by $\xi$.

Hence, locally uniformly by $\xi$,
\[
\lim_{N \to \infty} F_q(Z_N) = e_q^{-|\xi|^\alpha} \in G_q(\alpha).
\]
Thus, $Z_N$ is $q$-convergent to a $(q, \alpha)$-stable distribution, as $N \to \infty$. 

This theorem links the classic Lévy distributions with their $q_{\alpha}^L$-Gaussian counterparts. Indeed, in accordance with this theorem, a function $f$, for which
\[
f \sim C/x^{(\alpha+1)/(1+\alpha(q-1))}, \quad |x| \to \infty,
\]
is in $L_q(\alpha)$, i.e. $F_q[f](\xi) \in G_q(\alpha)$. It is not hard to verify that there exists a $q_{\alpha}^L$-Gaussian, which is asymptotically equivalent to $f$. Let us now find $q_{\alpha}^L$. Any $q_{\alpha}^L$-Gaussian behaves asymptotically $C_1/|x|^\eta = C_2/|x|^{2/(q_{\alpha}^L - 1)}$, $C_j = \text{const}$, $j = 1, 2$, i.e. $\eta = 2/(q_{\alpha}^L - 1)$. Hence, we obtain the relation
\[
\frac{\alpha + 1}{1 + \alpha(q - 1)} = \frac{2}{q_{\alpha}^L - 1}.
\]
Solving this equation with respect to $q_{\alpha}^L$, we have
\[
q_{\alpha}^L = \frac{3}{\alpha + 1} + \frac{Q\alpha}{\alpha + 1}, \quad Q = 2q - 1,
\]
linking three parameters: $\alpha$, the parameter of the $\alpha$-stable Lévy distributions, $q$, the parameter of correlation, and $q_{\alpha}^L$, the parameter of attractors in terms of $q_{\alpha}^L$-Gaussians (see Fig. 2 (left)). Equation (38) identifies all $(Q, \alpha)$-stable distributions with the same index of attractor $G_{q_{\alpha}^L}$ (See Fig. 1), proving the following proposition.
Proposition 6.6 Let $1 \leq Q < 3$ (or $Q = 2q - 1$) and $0 < \alpha < 2$. Then all distributions $X \in \mathcal{L}_q(\alpha)$, where the pairs $(Q, \alpha)$ satisfy the equation

$$\frac{3 + Q\alpha}{\alpha + 1} = q^L_{\alpha},$$

have the same attractor asymptotically equivalent to $q^L_{\alpha}$-Gaussian.

Figure 1: All pairs of $(Q, \alpha)$ on the indicated curves are associated with the same $q^L_{\alpha}$-Gaussian. Two curves corresponding to two different values of $q^L_{\alpha}$ do not intersect. In this sense these curves represent the constant levels of $q^L_{\alpha}$ or $\eta = 2/(q^L_{\alpha} - 1)$. The line $\eta = 1$ joins the points $(Q, \alpha) = (1, 0.0 - 0)$ and $(3 - 0, 2)$; the line $\eta = 2$ joins the Cauchy distribution (noted $\mathcal{C}$) with itself at $(Q, \alpha) = (1, 1)$ and at $(2, 2)$; the $\eta = 3$ line joins the points $(Q, \alpha) = (1, 2.0 - 0)$ and $(5/3, 2)$ (by $\epsilon$ we simply mean to give an indication, and not that both infinitesimals coincide). The entire line at $Q = 1$ and $0 < \alpha < 2$ is mapped into the line at $\alpha = 2$ and $5/3 \leq q^L_{\alpha} < 3$.

In the particular case $Q = 1$, we recover the known connection between the classical Lévy distributions ($q = Q = 1$) and corresponding $q^L_{\alpha}$-Gaussians. Put $Q = 1$ in Eq. (38) to obtain

$$q^L_{\alpha} = \frac{3 + \alpha}{1 + \alpha}, \quad 0 < \alpha < 2.$$  \hfill (39)

When $\alpha$ increases between 0 and 2 (i.e. $0 < \alpha < 2$), $q^L_{\alpha}$ decreases between 3 and $5/3$ (i.e. $5/3 < q^L_{\alpha} < 3$): See Figs. 2 (left) and 3 (left).

It is useful to find the relationship between $\eta = \frac{2}{q^L_{\alpha} - 1}$, which corresponds to the asymptotic behaviour of the attractor depending on $(\alpha, Q)$. Using formula (38), we obtain (Fig. 2 right)

$$\eta = \frac{2(\alpha + 1)}{2 + \alpha(Q - 1)}.$$  \hfill (40)

Proposition 6.7 Let $X \in \mathcal{L}_Q(\alpha)$, $1 \leq Q < 3$, $0 < \alpha < 2$. Then the associated density function $f_X$ has asymptotics $f_X(x) \sim |x|^\eta$, $|x| \to \infty$, where $\eta = \eta(Q, \alpha)$ is defined in (40).

Remark 6.8 If $Q = 1$ (classic Lévy distributions), then $\eta = \alpha + 1$, as is well known.
Analogous relationships can be obtained for other values of $Q$. We call, for convenience, a $(Q, \alpha)$-stable distribution a $Q$-Cauchy distribution, if its parameter $\alpha = 1$. We obtain the classic Cauchy-Poisson distribution if $Q = 1$. The corresponding line can be obtained cutting the surface in Fig. 2 (right) along the line $\alpha = 1$. For $Q$-Cauchy distributions we have

$$q^L_1(Q) = \frac{3 + Q}{2} \quad \text{and} \quad \eta = \frac{4}{Q + 1},$$

(41)

respectively (see Figs. 2).

The relationship between $\alpha$ and $q^L_1$ for typical fixed values of $Q$ are given in Fig. 3 (left). In this figure we can also see, that $\alpha = 1$ (Cauchy) corresponds to $q^L_1 = 2$ (in the $Q = 1$ curve). In Fig. 3 (right) the relationships between $Q$ ($Q = 2q - 1$) and $q^L_\alpha$ are represented for typical fixed values of $\alpha$.

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