Gravitoelectromagnetism and other decompositions of the Riemann tensor

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Abstract

We present gravitoelectromagnetism and other decompositions of the Riemann tensor from the differential-geometrical point of view.

The word gravitomagnetism is now in common use, even if some few people (see e.g. page 1306 of Ref. [1]) proposed to write gravimagnetism instead. Three of the many recent papers on this topic are Refs. [2], [3] and [4]. What is the idea behind this topic?

The Maxwell field can be represented by the tensor $F_{ij}$, possessing $4^2 = 16$ real components. By use of antisymmetry, this figure reduces to 6, but even a six-dimensional space is hard to imagine. However, by decomposing it into two 3-vectors – the electric and the magnetic 3-vector – the geometric meaning of all the components can be made clear, and this is the form where the contact with experimentalists becomes possible. Mathematically, these 3-vectors can be obtained by the additional introduction of a unit time-like vector field $u^i$ and well-known transvections with it and with the pseudo-tensor $\epsilon_{ijkl}$.

Now, the analogous procedure shall be made with the gravitational field. We have the Riemann tensor $R_{ijkl}$, possessing $4^4 = 256$ real components. By use of the known symmetries, this figure reduces to 20, but this twenty-dimensional space is even harder to imagine. In the next pages, I will present five different possibilities how to arrange this set of components to get a better understandable system (for details see e.g. Ref. [5]).

The Riemann tensor $R_{ijkl}$ of a space-time of dimension $n \geq 3$ can be decomposed according to several different criteria:
1. The usual one into the Weyl tensor $C_{ijkl}$ plus a term containing the Ricci tensor $R_{ij}$ plus a term containing the Riemann curvature scalar $R$.
2. Two trace-less parts plus the trace.
3. The Weyl tensor plus only one additional term.
4. Two divergence-free parts plus the trace.
5. The gravitoelectromagnetic point of view.

One should know, that the mathematical construction of the gravitoelectric field is much elder than the word “gravitoelectric”: Einstein calculated the gravitoelectric correction to the Newtonian potential of the sun - even if Einstein himself never had used that word. Concerning the invariant characterization of gravitomagnetism one should be careful, because non-flat space-times exist, whose curvature invariants are all vanishing, see e.g. Ref. [6].

We use the following two properties of the Riemann tensor

\[ R_{ijkl} = -R_{ijlk}, \quad (1) \]
\[ R_{ijkl} = R_{klij}. \quad (2) \]

The Ricci tensor is the trace of the Riemann tensor:

\[ R_{ij} = g^{kl} R_{ijkl}, \quad (3) \]

where $g_{ij}$ denotes the metric of the space-time, and the Riemann curvature scalar is the trace of the Ricci tensor

\[ R = g^{kl} R_{kl}. \quad (4) \]

The sign conventions are defined such that in Euclidean signature, the curvature scalar of the standard sphere is positive.

For any symmetric tensor $H_{ij}$ we define another tensor $H^*_{ijkl}$ via

\[ H^*_{ijkl} = H_{ik} g_{jl} + H_{jl} g_{ik} - H_{il} g_{jk} - H_{jk} g_{il}. \quad (5) \]

Then the tensor $H^*_{ijkl}$ automatically fulfils the identities eqs. (1) and (2). For the special case $H_{ij} = g_{ij}$ we get the simplified form

\[ g^*_{ijkl} = 2g_{ik} g_{jl} - 2g_{il} g_{jk}. \quad (6) \]
1 The usual decomposition

The Weyl tensor $C_{ijkl}$ is the trace-less part of the Riemann tensor, i.e.

$$g^{ik}C_{ijkl} = 0; \quad (7)$$

it vanishes identically for $n = 3$. Using the notation of eqs. (5) and (6) we make the ansatz

$$R_{ijkl} = C_{ijkl} + \alpha R_{ijkl}^* + \beta R g_{ijkl}^*. \quad (8)$$

Then the coefficients $\alpha$ and $\beta$ have to be specified such that the condition eq. (7) becomes an identity. This condition determines the coefficients $\alpha$ and $\beta$ uniquely, and the result is the following:

$$\alpha = \frac{1}{n-2} \quad \text{and} \quad \beta = \frac{-1}{2(n-1)(n-2)}. \quad (9)$$

Thus, we get the usual formula

$$R_{ijkl} = C_{ijkl} + \frac{1}{n-2} \left(R_{ikjl} + R_{jlgi} - R_{ilgk} - R_{jkgi}\right)$$

$$- \frac{1}{(n-1)(n-2)} R (g_{klij} - g_{lji}g_{jk}). \quad (10)$$

2 The decomposition using trace-less parts

In distinction to the previous subsection, we now perform a more consequent decomposition into trace and trace-less parts. To this end we define $S_{ij}$ as the trace-less part of the Ricci tensor, i.e. $g^{ij}S_{ij} = 0$ with $S_{ij} = R_{ij} + \kappa R g_{ij}$ possessing the unique solution $\kappa = -1/n$, i.e.,

$$S_{ij} = R_{ij} - \frac{1}{n} R g_{ij}. \quad (11)$$

Then the analogous equation to eq. (8) is

$$R_{ijkl} = C_{ijkl} + \gamma S_{ijkl}^* + \delta R g_{ijkl}^*. \quad (12)$$

Again, eq. (5) has been applied. Eq. (12) becomes a correct identity if and only if

$$\gamma = \frac{1}{n-2} \quad \text{and} \quad \delta = \frac{1}{2n(n-1)}. \quad (13)$$
So we get

\[
R_{ijkl} = C_{ijkl} + \frac{1}{n-2} \left( S_{ik}g_{jl} + S_{jl}g_{ik} - S_{il}g_{jk} - S_{jk}g_{il} \right)
+ \frac{1}{n(n-1)} R (g_{ik}g_{jl} - g_{il}g_{jk}) .
\]  

(14)

3  Decomposition into two parts

Let us define a tensor

\[
L_{ij} = R_{ij} + \zeta R g_{ij}
\]

(15)
such that a parameter \( \varepsilon \) exists which makes

\[
R_{ijkl} = C_{ijkl} + \varepsilon L_{ijkl}^*
\]

(16)

becoming a true identity. It turns out that this is possible iff

\[
\zeta = \frac{-1}{2(n-1)} \quad \text{and} \quad \varepsilon = \frac{1}{n-2} .
\]

(17)

Thus, we can write eq. (15) as

\[
L_{ij} = R_{ij} - \frac{1}{2(n-1)} R g_{ij}
\]

(18)

and eq. (16) as

\[
R_{ijkl} = C_{ijkl} + \frac{1}{n-2} (L_{ik}g_{jl} + L_{jl}g_{ik} - L_{il}g_{jk} - L_{jk}g_{il}) .
\]

(19)

4  Decomposition into divergence-free parts

Now, besides the identities eqs. (1) and (2), we also use identities involving the covariant derivatives, denoted by a semicolon, of the Riemann tensor. The Bianchi identity reads

\[
R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0 .
\]

(20)

Its trace can be obtained by transvection with \( g^{ik} \) and reads

\[
R_{jl;m} + R^{i}_{jlm;i} - R_{jm;l} = 0 .
\]

(21)
It should be mentioned, that the transvection with respect to other pairs of indices does not lead to further identities. The Einstein $E_{ij}$ tensor is defined as

$$E_{ij} = R_{ij} + \lambda R g_{ij},$$

where $\lambda$ has to be chosen such that the Einstein tensor is divergence-free, i.e.,

$$E_{j;i} = 0.$$

(23)

Using the trace of eq. (21) (again, there is essentially only one such trace),

$$2R_{i;i} - R_{ii} = 0,$$

(24)

we uniquely get $\lambda = -1/2$, i.e. the Einstein tensor is

$$E_{ij} = R_{ij} - \frac{1}{2} R g_{ij}.$$  

(25)

With the ansatz

$$R_{ijkl} = W_{ijkl} + \eta E_{ijkl}^* + \vartheta R g_{ijkl},$$

(26)

it holds: The coefficients $\eta$ and $\vartheta$ are uniquely determined by the requirements that eq. (26) is an identity, and the divergence of the tensor $W_{ijkl}$ vanishes:

$$W_{ijkl}^{i;i} = 0.$$  

(27)

We get the following values of the constants:

$$\eta = 1 \quad \text{and} \quad \vartheta = \frac{1}{4},$$

(28)

and then:

$$R_{ijkl} = W_{ijkl} + E_{ik}g_{jl} + E_{jl}g_{ik} - E_{il}g_{jk} - E_{jk}g_{il}$$

$$+ \frac{1}{2} R (g_{ik}g_{jl} - g_{il}g_{jk})$$

(29)

defines a decomposition of the Riemann curvature tensor into the divergence-free tensors $W_{ijkl}$, $E_{ij}$, $g_{ij}$ and the scalar $R$.

It should be mentioned, that for every $n > 2$, the four tensors $R_{ij}$ eq. (3), $S_{ij}$ eq. (11), $L_{ij}$ eq. (18) and $E_{ij}$ eq. (25) represent four different tensors.

It is a remarkable fact, that the coefficients in eqs. (25) and (29) do not depend on the dimension $n$. 

5
5 The gravitoelectromagnetic point of view

The gravitoelectromagnetic point of view can be obtained by the additional introduction of a unit time-like vector field $u^i$ and well-known transvections with it and with the pseudo-tensor $\epsilon_{ijkl}$.

References

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