Coherent Sheaves, Chern Classes, and Superconnections on Compact Complex-Analytic Manifolds

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Dedicated to the memory of Igor Shafarevich on the occasion of his 100th Anniversary

Abstract

We construct a twist-closed enhancement of the category $\mathcal{D}_{\text{coh}}^b(X)$, the bounded derived category of complexes of $\mathcal{O}_X$-modules with coherent cohomology, by means of the DG-category of $\bar{\partial}$-superconnections. Then we apply the techniques of $\bar{\partial}$-superconnections to define Chern classes and Bott-Chern classes of objects in the category, in particular, of coherent sheaves.
1 Introduction

The derived category of coherent sheaves is known to be a meaningful homological invariant of an algebraic variety. The basic motivation for the authors of the present paper was the wish to understand to which extent the derived category is a good invariant for complex analytic manifolds.

Let $X$ be a smooth compact complex-analytic manifold and $\mathcal{D}^{b}_{\text{coh}}(X)$ the derived category of $\mathcal{O}_{X}$-modules with bounded coherent cohomology. There are some indications that this category is not as good as it is in the algebraic case. First, a result in [V] implies that $\mathcal{D}^{b}_{\text{coh}}(X)$ are equivalent for all K3 surfaces $X$ with no curves. Hence, the derived category does not ‘feel’ the complex geometry of the generic K3 surface. This situation is very different for algebraic varieties. If the canonical or anticanonical class of $X$ is ample, then the variety can be uniquely reconstructed from its derived category [BO]. The number of abelian varieties derived equivalent to a given one is finite [O]. There is at most countably many algebraic varieties in a given class of derived equivalence ([AncTo]). An example of a projective variety with infinite number of derived partners is a 3-dimensional projective space with 8 suitably chosen points blown up [L].

Second, a wonderful property of the derived category of coherent sheaves on a smooth proper algebraic variety is that it satisfies a property similar to Brown representability. Namely, the category is saturated, i.e. all bounded cohomological functors with values in vector spaces are representable (see [BK1], [BVdB]). It was shown in [BVdB] that for the derived category of a smooth compact complex surface with no curves (say, a generic K3) this property does not hold. It was also conjectured in loc.cit. that if the derived category is saturated then the variety is (an analytification of) an algebraic space. The conjecture was proven by B. Toën and M. Vaquié [TV], though they used an a priori stronger (DG) version of saturatedness.

It was probably these facts that lead to the common opinion that the derived category was not a meaningful homological invariant in the complex-analytic case. Literally, this was correct. However, there was some ambiguity in what sort of natural structure is reasonable to fix when considering derived categories. In particular, it was mentioned quite a time ago that it made sense to consider triangulated categories together with enhancements, a sort of enrichments of the categories with a DG-structures [BK2]. Results of this paper suggest that some good (that is, twist-closed) enhancements are very relevant to complex-analytic geometry of manifolds.

We construct a DG-enhancement, $\mathcal{C}_{X}$, of $\mathcal{D}^{b}_{\text{coh}}(X)$ by $\bar{\partial}$-superconnections. Its objects are DG-modules over the DG-algebra of Dolbeault forms, with suitable properties. Our first result is that the homotopy category of this DG-category is equivalent to $\mathcal{D}^{b}_{\text{coh}}(X)$. A similar enhancement was independently considered by J. Block [Block]. In particular, we discuss the fully faithfulness of the functor that gives the equivalence. Also we do not assume that every object in $\mathcal{D}^{b}_{\text{coh}}(X)$ has a presentation by a finite complex of locally free sheaves of $\mathcal{O}_{X}$-modules (as in the proof of Lemma 4.6 in loc.cit). A counter-example of C. Voisin [Vo] shows that this does not hold for all complex-analytic manifolds. The work by N. Pali [P1], [P2] where he described coherent sheaves in terms of $\bar{\partial}$-connections on special sheaves of modules over smooth functions was inspiring for us at the early
stage of this project.

An important property of the category $C_X$ is that it is twist-closed, which means that the functors constructed via twisted complexes in the category are representable. Twisted complexes are solutions of the Maurer-Cartan equation with values in the (degree one) endomorphisms of objects in the DG-category. Note that twist-closed categories are pre-triangulated, i.e. so-called one-sided twisted complexes are representable, but it is crucial for constructing moduli of objects in the complex-analytic set-up to have representability of all twisted complexes.

Using the dg-enhancement via superconnections we discuss Chern character for objects in $D^b_{\text{coh}}(X)$. To this end, we extend any $\bar{\partial}$-superconnection to a (non-flat) De Rham superconnection in the sense of Quillen [Q] via a suitable choice of Hermitian metrics. As a result, we see that the $p$-th coefficient of the Chern character for an object in $D^b_{\text{coh}}(X)$ is represented by a closed $(p,p)$-form.

This suggests the idea to define characteristic classes of objects in $D^b_{\text{coh}}(X)$ which lie in Bott-Chern cohomology. The developed techniques for Chern classes of superconnections allows us to do this in a relatively easy way.

We leave for another publication the extension of the theory of $\bar{\partial}$-superconnections to non-compact varieties $X$ and its application to constructing moduli spaces of objects in $D^b_{\text{coh}}(X)$.

When proving the theorem on the DG-enhancement via $\bar{\partial}$-superconnections we follow our Kavli IPMU preprint in 2011 [BR]. The applications to Chern and Bott-Chern characteristic classes was presented by the second-named author in his talk on the Russian-Japanese conference "Categorical and analytic invariants in Algebraic Geometry II" at Kavli IPMU, Tokyo University, in 2015 [R]. We are pleased to dedicate this work to the memory of a famous mathematician and strong personality which was Igor Rostislavovich Shafarevich.

Having prepared this paper for publication, we learned about the sizeable preprint by J.-M. Bismut, Shu Shen and Zhaoting Wei [BSW], where the authors addressed similar questions, but also discuss Grothendieck-Riemann-Roch theorem. It would be interesting to compare the approaches of the two papers.

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2 DG-Enhancements

In this section we recall some facts on DG-categories and introduce the notion of twist-closed DG-categories.

DG-categories can be considered as a particular class of $A_\infty$-categories. The theory has a direct generalization to the $A_\infty$-case. We avoid this more general context here, because natural enhancements which one meets in complex geometry have a DG-structure.

Let $\mathcal{C}$ be an additive DG-category over a field. This means that we have finite direct sums of objects, morphisms between any two objects constitute a $\mathbb{Z}$-graded complex of vector spaces over the field, and Leibniz rule for the composition of morphisms is satisfied. We will denote the space of degree $i$ morphisms in $\mathcal{C}$ by $\mathbb{R}\text{Hom}^i$.

Two objects $A, B \in \mathcal{C}$ are said to be DG-isomorphic if there exists an invertible degree zero closed morphism $f \in \mathbb{R}\text{Hom}^0(A, B)$. Accordingly, one defines DG-isomorphism of functors.

We assume that the category is equipped with an equivalence $T: \mathcal{C} \to \mathcal{C}$, called translation functor, together with a DG-isomorphism of functors:

$$\mu: \text{id} \to T,$$

where $\mu$ has degree $-1$.

If $\mathcal{C}$ is a DG-category, then its homotopy category $\mathcal{H}o\mathcal{C}$ is defined as a category with the same objects as in $\mathcal{C}$, but morphisms are the degree zero homology of complexes of morphisms in $\mathcal{C}$.

Now we explain the ideology (based on [BK2]) of twisted complexes and functors, which they represent.

A twisted complex in a DG-category $\mathcal{C}$ is a pair $W = (E, \alpha)$, where $E$ is an object in $\mathcal{C}$ and $\alpha$, a twisting cochain, is in $\mathbb{R}\text{Hom}^1(E, E)$ and satisfies Maurer-Cartan equation:

$$d\alpha + \alpha^2 = 0.$$

Any twisted complex $T = (E, \alpha)$ defines a contravariant functor

$$h^W: \mathcal{C} \to \mathcal{C}'(\text{Vect}),$$

where $\mathcal{C}'(\text{Vect})$ stands for the category of complexes of vector spaces. The functor is defined on $A \in \mathcal{C}$ by:

$$A \mapsto \mathbb{R}\text{Hom}^\bullet(A, E),$$

but the differential in $\mathbb{R}\text{Hom}^\bullet(A, E)$ is twisted by $\alpha$:

$$d\alpha = d_{\mathbb{R}\text{Hom}^\bullet(A, E)} + \alpha.$$

The Maurer-Cartan equation for $\alpha$ implies that $d^2\alpha = 0$.

We will also consider twisted complexes of a particular form. A twisted complex $W = (E, \alpha)$ is called one-sided, if the object $E$ is decomposed into a finite direct sum $E = \oplus E_i$ and $\alpha$ is given by a strictly upper triangular matrix with respect to this decomposition.
The basic example of a one-sided twisted complex is obtained by taking \( E \) to be of the form

\[
E = E_1 \oplus E_2
\]

and \( \alpha \) to belong to \( \mathbb{R}\text{Hom}^1(E_1, E_2) \), a subspace in \( \mathbb{R}\text{Hom}^1(E, E) \). Note that \( \alpha^2 \) is automatically zero for such \( \alpha \). Therefore, the Maurer-Cartan equation reduces to \( d\alpha = 0 \). Thus \( \alpha \) can be interpreted as a closed degree zero morphism \( E_1 \to E_2[1] \).

In the following definition, DG-categories are assumed to be additive and equipped with translation functors.

**Definition 2.1**

(i) A DG-category is called **twist-closed** if \( h^W \) is representable for any twisted complex \( W \).

(ii) A DG-category is called **pre-triangulated** if \( h^W \) is representable for any one-sided twisted complex \( W \).

The following theorem was proved in \([BK2]\).

**Theorem 2.2**

If \( \mathcal{C} \) is pre-triangulated, then \( \mathcal{H}\text{o}\mathcal{C} \) is naturally triangulated.

The idea behind this theorem is that the twisting cochain \( \alpha \) in a one-sided twisted complex of the form (1) defines a morphism \( \tilde{\alpha} : E_1 \to E_2[1] \) in \( \mathcal{H}\text{o}\mathcal{C} \) and the object that represents the functor \( h^W \) gives a cone of \( \tilde{\alpha} \) in the homotopy category. Thus the basic hereditary problem of the axiomatics of triangulated categories that the cones of morphisms are not canonical is resolved by lifting morphisms in a triangulated category to closed morphisms in an appropriate DG-category.

Another reason why the DG-context looks to be more suitable is that the pre-triangulatedness is a property of a DG-category, while to make a category triangulated one has to put an extra structure on the category (to fix a class of exact triangles). The price to pay for transferring into the DG-world is that one has to consider DG-categories up to an appropriate equivalence relation, i.e. there is always a variety of equivalent choices for DG-categories ”representing” a given triangulated category.

**Definition 2.3**

If \( \mathcal{D} \) is a triangulated category, then a pre-triangulated category \( \mathcal{C} \) together with an equivalence of triangulated categories \( \mathcal{H}\text{o}\mathcal{C} \to \mathcal{D} \) is said to be an enhancement of \( \mathcal{D} \). The category \( \mathcal{D} \) is then said to be enhanced. A functor between two DG-categories is said to be a quasi-equivalence if it induces an equivalence of the corresponding homotopy categories.

It is clear from definitions that a twist-closed DG-category is pre-triangulated, hence its homotopy category is naturally triangulated. It will be crucial for our further constructions to have enhancements which are twist-closed.

**NB!** The twist-closedness is not preserved under quasi-equivalences.

The following example shows that a standard enhancement of the derived category of coherent sheaves on an algebraic variety is not twist-closed.
2.4 Example. Let $X$ be an algebraic variety with the structure sheaf $\mathcal{O}_X$ and $\mathcal{D} = \mathcal{D}_{\text{coh}}(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with bounded coherent cohomology. Consider the DG-category $\mathcal{C} = I(X)$ of bounded below complexes of injective $\mathcal{O}_X$-modules with bounded coherent cohomology. By definition, this is a full subcategory in the DG-category $\mathcal{C}^*(\mathcal{O}_X\text{-mod})$ of complexes of $\mathcal{O}_X$-modules. It is known that $I(X)$ (not the $\mathcal{C}^*(\mathcal{O}_X\text{-mod})$) is an enhancement of $\mathcal{D}_{\text{coh}}^b(X)$ \cite{BK2}, \cite{BLL}.

Let $E$ be a complex in $I(X)$ with the differential $d$, such that some terms of $E$ are not coherent $\mathcal{O}_X$-modules (typically, neither of them is). Consider the twisted complex $W = (E, \alpha)$ with $\alpha = -d$. One can easily see that the functor $h^W$ is not representable.

Indeed, it is represented by the object in $\mathcal{C}^b(\mathcal{O}_X\text{-mod})$ of complexes of $\mathcal{O}_X$-modules. Its cohomology is not coherent.

Here is an example of a twist-closed enhancement.

2.5 Example. Let $\mathcal{D} = \mathcal{D}^b(\text{mod-}A)$ be the bounded derived category of (right) modules over an associative algebra $A$ of finite global dimension. Consider the DG-category $\mathcal{C} = P(A)$ consisting of perfect complexes, i.e. finite complexes of finitely generated projective $A$-modules. This is a full subcategory in the DG-category $\mathcal{C}^b(\text{mod-}A)$ of all complexes of $A$-modules. Again, $P(A)$ (and not $\mathcal{C}^b(\text{mod-}A)$) is an enhancement of $\mathcal{D}^b(\text{mod-}A)$. Every twisted complex $W = (E, \alpha)$ over this category produces a functor $h^W$ which is representable by the same graded module $E$ but with the new differential

$$d_T = d_E + \alpha.$$ 

This is a perfect complex. Hence the enhancement is twist-closed.

3 An Enhancement via $\bar{\partial}$-superconnections

We are going to construct a twist-closed enhancement of $\mathcal{D}_{\text{coh}}^b(X)$. It will be a DG-category $\mathcal{C} = \mathcal{C}_X$ whose homotopy category is equivalent to $\mathcal{D}_{\text{coh}}^b(X)$. The idea of the construction can be explained via Koszul duality applied to the algebra of differential operators (cf. \cite{Kap}). This goes along the following lines.

3.1 A Viewpoint via Koszul Duality

Denote $\mathcal{A}^{i,j} = \mathcal{A}^{i,j}_X$ the sheaf of smooth complex-valued $(i,j)$-forms on $X$. For the sake of simplicity, we will also use the notation $\mathcal{A}^i = \mathcal{A}^i_X = \mathcal{A}^{0,i}$. In particular, $\mathcal{A}^0_X$ is the sheaf of smooth functions on $X$. We regard $\mathcal{A}^*$ as a (sheaf of) dg-rings endowed with Dolbeault’s differential $\bar{\partial}$, and use the notation $\mathcal{A}^\natural$ for the same ring considered as a graded ring with no differential. We use notation $\mathcal{A}^+ = \mathcal{A}^+_X = \oplus_{i>0} \mathcal{A}^i_X$ for the positive part of Dolbeault complex. If $\mathcal{E}$ is a locally free $\mathcal{A}^0$-module, then $\mathcal{A}^{i,j}(\mathcal{E}) = \mathcal{A}^{i,j}_X(\mathcal{E}) = \mathcal{A}^{i,j}_X \otimes_{\mathcal{A}^0_X} \mathcal{E}$ denotes the sheaf of smooth $(i,j)$-forms on $X$ with values in $\mathcal{E}$. Similarly, we put $\mathcal{A}(\mathcal{E}) = \mathcal{A}_X(\mathcal{E}) = \mathcal{A}_X \otimes \mathcal{E}$.

A locally free coherent sheaf $E$ on $X$ is given by a smooth vector bundle $\mathcal{E}$ on $X$ with a flat $\bar{\partial}$-connection $\nabla$, the sheaf $E$ being the sheaf of $\nabla$-horizontal sections. One can interpret such a connection as a module over the sheaf of algebras $\mathcal{D}^b_X$ of $\bar{\partial}$-differential
operators on \( X \). Algebra \( D_X^\partial \) is filtered by degree of differential operators. The associated graded algebra is the symmetric algebra \( S^*(T_X^{1,1}) \) over \( A^\partial_X \) of the sheaf \( T_X^{0,1} \) of vector fields of type \((0,1)\) on \( X \). The quadratic dual to it over \( A^\partial_X \) is the sheaf of graded algebras of smooth \((0,i)\)-forms on \( X \):
\[
\mathcal{A}^\cdot_X = \oplus \mathcal{A}^i_X.
\]
A version of the non-homogeneous quadratic Koszul duality (cf. [Kap]) implies that the Koszul dual to \( D_X^\partial \) itself is \( A^\cdot_X = (A^\cdot_X, \bar{\partial}) \), i.e. \( A^\cdot_X \) regarded as a sheaf of DG-algebras equipped with Dolbeault differential \( \bar{\partial} \).

Note that \( A^0_X \) has a natural structure of a \( D_X^\partial \)-module. Besides, \( A^0_X \) is the zero component of the filtration on the algebra \( D_X^\partial \). Then, the Koszul duality at the level of algebras amounts to the following statement.

**Lemma 3.2** If \( X \) is a complex-analytic manifold of dimension \( n \), then there is quasi-isomorphism of sheaves of DG-algebras:
\[
\mathbb{R}Hom_{D_X^\partial}(A^0_X, A^0_X) \simeq A^\cdot_X.
\]

**Proof.** Consider the \( \bar{\partial} \)-version of the Spencer complex (see [Sab]), i.e. a locally free resolution of \( A^0_X \) by left \( D_X^\partial \)-modules:
\[
0 \to D_X^\partial \otimes A_X^\partial \Lambda^n T_X^{0,1} \to \ldots \to D_X^\partial \otimes A_X^\partial \Lambda^2 T_X^{0,1} \to D_X^\partial \otimes A_X^\partial T_X^{0,1} \to D_X^\partial \to A_X^0 \to 0.
\]
Applying this resolution to calculating the required \( \mathbb{R}Hom_{D_X^\partial}(A^0_X, A^0_X) \) gives the result. \( \square \)

A result of N. Pali (when slightly reformulated) identifies the abelian category of coherent \( \mathcal{O}_X \)-modules with a suitable subcategory of \( D_X^\partial \)-modules.

**Theorem 3.3** [P1] For a complex-analytic manifold \( X \), the category of coherent \( \mathcal{O}_X \)-modules is equivalent to the category of \( D_X^\partial \)-modules which, as \( A^\partial_X \)-modules, locally have a finite resolution by finite rank free \( A^0_X \)-modules.

Koszul duality typically states that appropriate derived categories of modules over Koszul dual algebras are equivalent. Thus, Lemma 3.2 together with Theorem 3.3 give a reason to search for an enhancement of \( D^\partial_{coh}(X) \) among DG-categories of suitable \( A_X \)-DG-modules. Note that Pali’s proof uses a choice of norms and Leray-Koppelman operators to prove the existence of solutions of a complicated system of differential equations. The results of the present paper imply Pali’s theorem with no use of such involved analytic techniques.

For the case of a locally free \( \mathcal{O}_X \)-module \( E \), take \( \mathcal{E} = A^0 \otimes \mathcal{O}_X \) \( E \). Then, the \( \bar{\partial} \)-connection \( \nabla : \mathcal{E} \to A^0.1(\mathcal{E}) \) can be extended to a differential in \( A(\mathcal{E}) \). As a result, \( A(\mathcal{E}) \) acquires a natural structure of a DG-module over \( A \). This is the module which corresponds to the sheaf \( E \) of \( \nabla \)-horizontal sections, suggested by Koszul duality.
We extend this correspondence to arbitrary objects in $D^b_{coh}(X)$. It is natural to call $\mathcal{A}$-DG-modules of suitable type by $\bar{\partial}$-superconnections (see below). It is easy to construct a superconnection corresponding to an arbitrary object in $D^b_{coh}(X)$ if every coherent sheaf on $X$ has a resolution by locally free $\mathcal{O}_X$-modules, which is true if $X$ is the analytification of an algebraic variety (the resolution can be constructed via an ample system of line bundles [Illu]) or if $X$ is a compact analytic surface as proved by Schuster [S].

In general, locally free resolutions do not exist. A counterexample with a complex torus is due to C. Voisin [Vo]. This makes the issue more subtle.

3.4 The category of $\bar{\partial}$-superconnections

Before we start, let us agree on some notation concerning graded modules over supercommutative graded rings. Let $\mathcal{R}$ be such a ring, and $\mathcal{M}$ and $\mathcal{N}$ two graded left modules over $\mathcal{R}$. We denote by $\text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ a left graded module of homomorphisms from $\mathcal{M}$ to $\mathcal{N}$ which satisfy $\forall \phi \in \text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N}), \forall \mu \in \mathcal{M}, \forall \omega \in \mathcal{R},$

$$\phi(\omega \cdot \mu) = (-1)^{\deg \phi \cdot \deg \omega} \omega \cdot \phi(\mu).$$

Note that every left graded $\mathcal{R}$-module can be made a right module by defining

$$\mu \cdot \omega := (-1)^{\deg \mu \cdot \deg \omega} \omega \cdot \mu.$$

Then, what we defined as $\text{Hom}_{\mathcal{R}}(\mathcal{M}, \mathcal{N})$ becomes simply homomorphisms which are $\mathcal{R}$-linear with respect to the right-multiplication.

Let $M$ be a bounded (left) DG-module over $\mathcal{A}$ (more precisely, a sheaf of modules). If $\bar{D}$ is a differential in $M$, $s$ a section of $M$, and $\omega$ an element in $\mathcal{A}$, then the Leibniz rule has to be satisfied:

$$\bar{D}(\omega \cdot s) = \bar{\partial} \omega \cdot s + (-1)^{\deg \omega} \bar{\partial} s$$

(2)

Forgetting the differential, but not the grading in $M$, we obtain a graded module, $M^g$, over $\mathcal{A}^g$. Define the objects of the category $\mathcal{C} = \mathcal{C}_X$ to be DG-modules $M$ over $\mathcal{A}$ for which $M^g$ are locally free graded modules of finite rank over $\mathcal{A}^g$. If $M$ is in $\mathcal{C}$, then it is, in particular, a bounded graded complex of locally free $\mathcal{A}^0$-modules. We say that objects of $\mathcal{C}$ are flat $1 \bar{\partial}$-superconnections, which is inspired by Quillen’s terminology in [Q].

Let $M$ and $N$ be in $\mathcal{C}$. Then the graded sheaf $\mathcal{H}om_{\mathcal{A}^g}(M^g, N^g)$ is endowed with the structure of a left $\mathcal{A}^g$-module and a differential $\bar{D}_{\mathcal{H}om}(\phi) = \bar{D}_N \circ \phi - (\omega \cdot \phi \cdot \bar{\partial} M)$, which are related by the Leibniz rule. That is to say, sections $\phi \in \mathcal{H}om^*_\mathcal{A}(M, N)$, $\omega \in \mathcal{A}$, and $s \in M$ satisfy the sign rule:

$$\phi(\omega \cdot s) = (-1)^{\deg \phi \cdot \deg \omega} \omega \cdot \phi(s),$$

(3)

$$\bar{D}_{\mathcal{H}om}(\omega \cdot \phi) = \bar{\partial} \omega \cdot \phi + (-1)^{\deg \omega} \bar{\partial} \phi,$$

(4)

$1$Until Section 4, we deal mainly with flat superconnections. Therefore, we will frequently omit the word 'flat'.

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Thus, the complex $\mathcal{H}om^\bullet_{\mathcal{A}}(M, N) = (\mathcal{H}om_{\mathcal{A}^\bullet}(M^\bullet, N^\bullet), \mathcal{D}_{\mathcal{H}om})$ is also an object in $\mathcal{C}$. We define morphisms in the dg-category $\mathcal{C}$ as a complex of global sections of $\mathcal{H}om^\bullet_{\mathcal{A}}(M, N)$:

$$\mathbb{R}\mathcal{H}om^\bullet_{\mathcal{C}}(M, N) := \Gamma(X, \mathcal{H}om^\bullet_{\mathcal{A}}(M, N)),$$

equipped with the differential $\mathcal{D}_{\mathbb{R}\mathcal{H}om}$ inherited from $\mathcal{D}_{\mathcal{H}om}$.

**Proposition 3.5** $\mathcal{C}$ is a twist-closed DG-category.

**Proof.** The translation functor is defined by the shift of grading of DG-$\mathcal{A}$-modules. Hence, we need to prove that every twisted complex in $\mathcal{C}$ is representable.

Let $M$ be in $\mathcal{C}$, $\mathcal{D}$ the differential in $M$, and $\alpha \in \mathbb{R}\mathcal{H}om^1_{\mathcal{C}}(M, M)$ a solution of the Maurer-Cartan equation, $\mathcal{D}_{\mathcal{H}om} \alpha + \alpha^2 = 0$. Since $\alpha$ satisfies the sign rule (3) with $\deg \alpha = 1$, then $\mathcal{D} + \alpha$ satisfies the Leibniz rule (2) and $(\mathcal{D} + \alpha)^2 = 0$. Hence $M(\alpha)$, the same $\mathcal{A}^\bullet$-module $M^\bullet$, but with a new differential, $\mathcal{D} + \alpha$, is again a DG-$\mathcal{A}$-module. Clearly, it belongs to $\mathcal{C}$ and is a representing object for the twisted complex defined by the pair $(M, \alpha)$. $\square$

**Corollary 3.6** Category $\mathcal{C}$ is pre-triangulated. The homotopy category $\mathcal{H}o(\mathcal{C})$ is triangulated.

Any object in $\mathcal{C}_X$ is by restriction along the embedding $\mathcal{O}_X \to \mathcal{A}_X^0$ a complex of $\mathcal{O}_X$-modules, because $\mathcal{O}_X$ is $\bar{\partial}$-horizontal. If $M$ and $N$ are in $\mathcal{C}_X$, then a closed morphism in $\mathbb{R}\mathcal{H}om^0_{\mathcal{C}}(M, N)$ obviously defines a morphism of complexes of the corresponding $\mathcal{O}_X$-modules. Homotopy-equivalent closed morphisms define isomorphic morphisms in $\mathcal{D}^b(\mathcal{O}_X\text{-mod})$, because the derived category factors through the homotopy category of complexes of $\mathcal{O}_X$-modules. Hence we obtain a functor:

$$\Phi : \mathcal{H}o(\mathcal{C}_X) \to \mathcal{D}^b(\mathcal{O}_X\text{-mod}). \quad (5)$$

The rest of this section is devoted to proving that $\mathcal{C}_X$ defines an enhancement of $\mathcal{D}^b_{\text{coh}}(X)$ via the functor $\Phi$.

**Theorem 3.7** Let $X$ be a compact smooth complex-analytic manifold. Then $\Phi$ is a triangulated equivalence between the homotopy category $\mathcal{H}o(\mathcal{C}_X)$ and $\mathcal{D}^b_{\text{coh}}(X)$.

The strategy of the proof is as follows. First, we will show that any $\bar{\partial}$-superconnection is a complex of $\mathcal{O}_X$-modules with bounded coherent cohomology. Second, we will show that functor $\Phi$ is fully faithful, i.e it retains homomorphisms between any two objects in $\mathcal{H}o(\mathcal{C}_X)$. Third, we will prove that any object in $\mathcal{D}^b_{\text{coh}}(X)$ is quasi-isomorphic to a $\bar{\partial}$-superconnection. The proof relies on the description of the local structure of $\bar{\partial}$-superconnections.
3.8 The Local Structure of $\bar{\partial}$-Superconnections

Every finite complex of locally free $\mathcal{O}_X$-modules $E^\bullet$ yields a $\bar{\partial}$-superconnection by taking a tensor product of complexes

$$\mathcal{A}_X \otimes_{\mathcal{O}_X} E^\bullet$$

with $\bar{\partial} \otimes \text{id}_E$ as a differential. Here we will prove the technical statement which claims that any flat $\bar{\partial}$-superconnection is locally isomorphic (gauge-equivalent) to a $\bar{\partial}$-superconnection of this kind. We believe that this fact is of independent interest.

Let $M$ be a $\bar{\partial}$-superconnection. Since $M^\natural$ is locally free over $\mathcal{A}^\natural$, it can be non-canonically presented in the form:

$$M^\natural = \mathcal{A}^\natural \otimes_{\mathcal{A}_0^X} E^\bullet \quad (6)$$

where $E^\bullet = M/A^+M$ is a finitely generated graded locally free $\mathcal{A}_0^X$-module ($A^+ = \oplus_{i>0} \mathcal{A}_i^X$). To have this presentation, choose an $\mathcal{A}_0^X$-linear splitting of the projection $M \to E^\bullet$ and use the $\mathcal{A}^\natural$-module structure of $M^\natural$. Every $E^i$ can be understood as a smooth complex vector bundle on $X$. Consider the (non-canonical) bigrading of $M^\natural$ where $A^i \otimes E^j$ has bidegree $(i, j)$. The initial (canonical) grading of $\mathcal{A}^i \otimes E^j$ in $M$ has the total degree $(i + j)$.

The differential $\bar{D}$ in this module has the following decomposition with respect to this bigrading:

$$\bar{D} = \bar{\gamma} + \bar{\nabla} + \sum_{i \geq 2} \bar{\beta}_i. \quad (7)$$

Here $\bar{\gamma}$ is the component of bidegree $(0, 1)$, $\bar{\nabla}$ the component of bidegree $(1, 0)$, and $\bar{\beta}_i$ the component of degree $(i, 1-i)$. The Leibniz rule implies that $\bar{\gamma}$ is an endomorphism of $M^\natural$ of total degree 1, hence it satisfies the sign rule (3). Therefore, $\bar{\gamma}$ is fully defined by $\mathcal{A}_0^X$-module homomorphisms $\bar{\gamma}_j : E^j \to E^{j+1}$. $\bar{\nabla}$ can be understood as a set of not necessarily flat $\bar{\partial}$-connections $\bar{\nabla}_j$ on $E^j$. An important point is that these connections are not necessarily flat (one can say, these are only homotopically flat, cf., the 3-d line of eq. (9) below), which does not allow us at this stage to endow $E^j$ with the structure of a holomorphic vector bundle. By the Leibniz rule again, the components $\bar{\beta}_i$'s correspond to $\mathcal{A}^0_X$-morphisms $E^j \to \mathcal{A}^i \otimes E^{j-i+1}$. In this notation, the (bi)graded components of the condition

$$\bar{D}^2 = 0 \quad (8)$$

read as a sequence of equations:

$$\bar{\gamma}^2 = 0, \quad [\bar{\gamma}, \bar{\nabla}] = 0, \quad \bar{\nabla}^2 + [\bar{\gamma}, \bar{\beta}_2] = 0, \quad [\bar{\nabla}, \bar{\beta}_2] + [\bar{\gamma}, \bar{\beta}_3] = 0, \quad [\bar{\nabla}, \bar{\beta}_3] + \bar{\beta}_2^2 + [\bar{\gamma}, \bar{\beta}_4] = 0, \quad (9)$$

---

2The (bi)degree of a homogeneous operator acting in a (bi)graded space is that how much it changes the (bi)degree of any element it acts upon.
and so on. Note that here and from now on, the bracket $[\cdot, \cdot]$ denotes a supercommutator, so, in eq. (9), it is actually an anticommutator, because all the participants are of odd degree in this case, for example, $[\bar{\gamma}, \nabla] = \bar{\gamma}\nabla + \nabla\bar{\gamma}$.

Note that if all $\bar{\beta}_i$’s were zero these equations would be equivalent to saying that $\nabla$ give a holomorphic structure (that is a flat $\bar{\partial}$-connection) in all vector bundles $E^j$’s, and, then, $\bar{\gamma}$ would be a holomorphic differential in a complex of holomorphic vector bundles.

Now, we choose a point $x \in X$ and an open neighborhood of $x$ in analytic topology on $X$. We consider local gauge transformations of the form:

$$\bar{D}' = e^{-\phi \bar{D}} e^{\phi}$$

with $\phi$ an $\mathcal{A}^i$-module endomorphism of $M^2$, which has degree 0 with respect to the canonical grading. Clearly, $\phi$ is defined by its values on $\mathcal{E}^\ast$. Thus, we interpret $\phi$ as an element of $\text{Hom}^0_{\mathcal{A}^0}(\mathcal{E}^\ast, \mathcal{A}^i \otimes \mathcal{E}^\ast)$. We say that the gauge parameter $\phi$ is strict if it has a decomposition

$$\phi = \phi_1 + \phi_2 + \ldots$$

where $\phi_i \in \otimes_j \text{Hom}^0_{\mathcal{A}^j}(\mathcal{E}^j, \mathcal{A}^i \otimes \mathcal{E}^{j-i})$ over the neighborhood of $x$. In other words, for a strict gauge parameter, the component $\phi_0 \in \text{Hom}^0_{\mathcal{A}^0}(\mathcal{E}^\ast, \mathcal{E}^\ast)$ is zero. Note, that this condition does not depend on the non-canonical splitting (6).

The corresponding gauge transformation, $\exp\phi$, will be also called strict. Such gauge transformations can be regarded as a change of the non-canonical bigrading of $M$, which is the same as a choice of an isomorphism $\mathcal{A}^2 \otimes \mathcal{E}^\ast \xrightarrow{\sim} M^2$, which in turn is the same as an automorphism $\mathcal{A}^i \otimes \mathcal{E}^\ast \xrightarrow{\sim} \mathcal{A}^i \otimes \mathcal{E}^\ast$ trivial on $\mathcal{E}^0 = \mathcal{A}^0 \otimes \mathcal{E}^0$.

Transformation (10) for components of $\bar{D}'$ reads:

$$\bar{\gamma}' = \bar{\gamma},$$

i.e. $\gamma$ does not change,

$$\bar{\nabla}' = \bar{\nabla} + [\bar{\gamma}, \phi_1],$$

$$\bar{\beta}'_2 = \bar{\beta}_2 + [\bar{\gamma}, \phi_2] + \frac{1}{2} \bar{\gamma} \phi_1^2 + \frac{1}{2} \phi_1^2 \bar{\gamma} - \phi_1 \bar{\gamma} \phi_1 + [\bar{\nabla}, \phi_1], \text{ etc.}$$

The following lemma confirms that every superconnection is locally isomorphic to a complex of holomorphic vector bundles (cf., [Block, Lemma 4.5]).

**Local Lemma 3.9** Any flat $\bar{\partial}$-superconnection over a polydisc can be transformed by a strict gauge transformation of the form (10) to the form (7) with all $\bar{\beta}_i$’s being zero.

The proof will easily follow from a technical lemma 3.12 below. An immediate consequence of the above Local Lemma is the following

**Corollary 3.10** (i) Every $\bar{\partial}$-superconnection on a polydisc is isomorphic to $\mathcal{A}_X \otimes_{\mathcal{O}_X} \mathcal{E}^\ast$, where $\mathcal{E}^\ast$ is a finite complex of free $\mathcal{O}_X$-modules,

(ii) Cohomology sheaves of any $\bar{\partial}$-superconnection are coherent sheaves of $\mathcal{O}_X$-modules.
Proof. By Lemma 3.9, for any $\bar{\partial}$-superconnection $M$ we can find a strict gauge transformation that gives an operator $\bar{D}$ of the form (7) with all $\hat{\beta}_i$’s being zero. This implies that $M$ is isomorphic to $\mathcal{A}_X \otimes_{\mathcal{O}_X} E^*$, where $\hat{\gamma}$ plays the role of the differential in $E^*$ and $\bar{\nabla}$ consists of the $\bar{\partial}$-connections on all $\mathcal{A}_X \otimes_{\mathcal{O}_X} E^*$’s.

The claim (ii) is local, thus we can assume $M = \mathcal{A}_X \otimes_{\mathcal{O}_X} E^*$ as in (i). As a complex of $\mathcal{O}_X$-modules it is quasiisomorphic to $E^*$, hence it has coherent cohomology. $\square$

3.11 RELATIVE SUPERCONNECTIONS ON THE PRODUCT OF POLYDISCS

Let $U$, $W$ be complex manifolds and $\pi: U \to W$ a projection with smooth fibres. Denote by $\mathcal{A}_\pi^* = \oplus \mathcal{A}_m^i$ the sheaf of rings on $U$ of relative $(0,i)$-forms. $\mathcal{A}_\pi^*$ is a dg-algebra with the differential $\bar{\partial}_\pi$, which is given by the relative Dolbeault operator, $\bar{\partial}_\pi: \mathcal{A}_\pi^i \to \mathcal{A}_\pi^{i+1}$.

Let now $M^*$ be a dg-module over $\mathcal{A}_\pi^*$. We call $M^*$ a relative $\bar{\partial}$-superconnection if it is locally free over $\mathcal{A}_\pi^2$.

Given $U, W, \pi$ be as above, consider a graded smooth vector bundle $\mathcal{E}^*$ on $U$ endowed with a relative $\bar{\partial}$-connection $\bar{\nabla}: \mathcal{E}^i \to \mathcal{A}_\pi^{1i} \otimes \mathcal{E}^i$, where $\bar{\nabla}$ obeys the Leibniz $\bar{\partial}_\pi$-rule. $\bar{\nabla}$ can be extended to $\mathcal{A}_\pi^2 \otimes \mathcal{E}^*$ by the graded Leibniz rule with respect to $\mathcal{A}_\pi^*$. If $\bar{\nabla}$ is flat, i.e. $\bar{\nabla}^2 = 0$, we get a relative $\bar{\partial}$-superconnection of the form $(\mathcal{A}_\pi^2 \otimes \mathcal{E}^*, \bar{\nabla})$. Let us call a superconnection of this particular shape an ordinary relative $\bar{\partial}$-connection.

In Lemma 3.12 below, we shall need the following construction. If $U = V \times W$ and $\pi_V, \pi_W$ are two projections on the factors, then we have obvious isomorphisms of relative and absolute forms: $\mathcal{A}_{\pi_V}^* = \pi_W^* \mathcal{A}_W$, $\mathcal{A}_{\pi_W}^* = \pi_V^* \mathcal{A}_V$, and an isomorphism $\mathcal{A}_U^* = \mathcal{A}_V^* \otimes_{\mathcal{O}_V^*} \mathcal{A}_W$. This agrees with differentials, because the Dolbeault operator on $U$, $\bar{\partial}_U$, can be split into the sum $\bar{\partial}_U = \bar{\partial}_V + \bar{\partial}_W$ in an obvious sense. ($\bar{\partial}_V$ acts “along $V$”, that is increases by 1 only the grading in $\mathcal{A}_{\pi_V}^*$; similarly, $\bar{\partial}_W$ acts “along $W$”.)

Let $\mathcal{E}^*$ be a graded smooth vector bundle over $U$. Suppose we have two relative $\bar{\partial}$-superconnections of the form $(\mathcal{A}_{\pi_W}^2 \otimes \mathcal{E}^*, \bar{D}_V)$ and $(\mathcal{A}_{\pi_W}^2 \otimes \mathcal{E}^*, \bar{D}_W)$, where $\bar{D}_V$ and $\bar{D}_W$ obey the Leibniz rules with respect to the corresponding projections. Given this, we can extend $\bar{D}_V$ from $\mathcal{A}_{\pi_W}^2 \otimes \mathcal{E}^*$ to $\mathcal{A}_U^2 \otimes \mathcal{E}^*$ by the Leibniz $\bar{\partial}_V$-rule, and similarly for $\bar{D}_W$, and construct the operator $\bar{D} = \bar{D}_V + \bar{D}_W$ on $\mathcal{A}_U^2 \otimes \mathcal{E}^*$ which satisfies the Leibnitz $\bar{\partial}_U$-rule. If additionally $[\bar{D}_V, \bar{D}_W] = 0$, then we get a $\bar{\partial}$-superconnection $(\mathcal{A}_U^2 \otimes \mathcal{E}^*, \bar{D})$ on $U$.

Now consider an $n$-dimensional polydisc $U$ which is presented as the product $U = V \times W$, where $V$ has coordinates $z^1, \ldots, z^m$, $m > 1$, and $W$ coordinates $z^{m+1}, \ldots, z^n$. Let $V'$ be the polydisc with $m - 1$ coordinates, $z^1, \ldots, z^{m-1}$. Thus, $V$ is a product $V = V' \times V_1$, where $V_1$ is 1-dimensional polydisc with coordinate $z^m$ as a coordinate. Let $W'$ be the polydisc $W' = V_1 \times W$. We have a new decomposition into the product: $U = V' \times W'$.

Lemma 3.12 Let $M$ be a $\bar{\partial}$-superconnection over the polydisc $U = V \times W$ as above, $\dim V > 1$. Suppose the differential in $M$ is of the form

$$\bar{D} = \bar{D}_V + \bar{\nabla}_W,$$
where $\nabla_W$ is an ordinary relative $\bar{\partial}$-connection along $W$, and $\bar{D}_V$ a relative $\bar{\partial}$-superconnection along $V$. Then there exists a strict gauge transformation $\bar{D}' = e^{-\phi} \bar{D} e^\phi$ such that

$$\bar{D}' = \bar{D}_{V'} + \nabla_{W'},$$

where $\nabla_{W'}$ is an ordinary relative $\bar{\partial}$-connection along $W'$, and $\bar{D}_{V'}$ a relative $\bar{\partial}$-superconnection along $V'$.

Proof. Since $\bar{D} = \bar{D}_V + \nabla_W$, it can be written (cf., eq. (7)) as

$$\bar{D} = \bar{\gamma} + \nabla + \sum_{i=2}^m \bar{\beta}_i,$$

where $\bar{\gamma}$, $\nabla$, $\bar{\beta}_i$'s are operators of degree 1 on sections of $M^i$ defined by morphisms

$$\bar{\gamma} : \mathcal{E}^j \to \mathcal{E}^{j+1},$$

$$\bar{\beta}_i : \mathcal{E}^j \to \pi_{V'}^* A_i^1 \otimes \mathcal{E}^{j-i+1},$$

and $\nabla$ is a first order differential operator mapping $\mathcal{E}^j$ to $\pi_{V'}^* A_V^1 \otimes \mathcal{E}^j$.

All of these, $\bar{\gamma}$, $\nabla$, $\bar{\beta}_i$'s, (anti)commute with $\nabla_W$, because $[\bar{D}_V, \nabla_W] = 0$. We shall say, they depend holomorphically on $W$.

Let us now consider $V$ as a product $V = V' \times V_1$. Any $(0,i)$-form on $V$ can be decomposed as

$$\omega = \omega_{i,0} + \omega_{i-1,1},$$

where

$$\omega_{i-1,1} = \eta_{i-1,0} dz_m,$$

and $\omega_{i,0}$ and $\eta_{i-1,0}$ are Dolbeault forms on $V'$ of degree $i$ and $i - 1$ respectively.

Analogously, we have the decomposition of $\bar{D}_V = \bar{\gamma} + \nabla + \sum_{i=2}^m \bar{\beta}_i$. In particular,

$$\nabla = \nabla_{1,0} + \nabla_{0,1},$$

$$\bar{\beta}_k = (\bar{\beta}_k)_{k,0} + (\bar{\beta}_k)_{k-1,1}.$$
As we have mentioned, $\nabla_{0,1}$ is acyclic in positive degree, hence, the above equation has a solution $\phi_1 = (\phi_1)_{1,0}$. Note, that the conditions $\phi_0 = 0$ and $\phi_i = (\phi_i)_{i,0}$ guaranty that $\nabla_{0,1}' = \nabla_{0,1}$.

Let us use this result as the base of induction in $k$ and suppose $(\beta_s)_{s-1,1} = 0$ for $s \leq k$. Then, the equation of vanishing of $(\beta_{k+1})_{k,1}$ reads as

$$(\beta_{k+1})_{k,1} + [\nabla_{0,1}, (\phi_k)_{k,0}] = 0. \quad (13)$$

This again has a solution. Moreover, since $\beta_k$’s depend holomorphically on $W$ and so is $\nabla_{0,1}$, solutions $\phi_k$ to eq. (13) can also be chosen to commute with $\nabla_W$.

Thus, we can kill the $(\beta_i)_{i-1,1}$ for all $i = 2, \ldots, m$. At the end, we get

$$\bar{D}' = e^{-\phi} \bar{D}e^{\phi} = \bar{D}'_{1,0} + \nabla_{0,1},$$

where $(\bar{D}'_{1,0})^2 = 0$, $(\nabla_{0,1})^2 = 0$, and $[\bar{D}'_{1,0}, \nabla_{0,1}] = 0$. Since $\phi$ commutes with $\nabla_W$, we also have that

$$D' = e^{-\phi}(\bar{D} + \nabla_W)e^{\phi} = \bar{D}'_{1,0} + \nabla_{0,1} + \nabla_W.$$

Let us rewrite this as

$$\bar{D}' = \bar{D}' + \nabla_{W'},$$

where $\bar{D}' = \bar{D}'_{1,0}$ and $\nabla_{W'} = \nabla_{0,1} + \nabla_W$, which yields the result. □

**Proof of Lemma 3.9** It follows by use of Lemma 3.12 and induction in decreasing $m$. □

### 3.13 Full Faithfulness

Let $M$ and $N$ be $\bar{\partial}$-superconnections. We regard them as complexes of $\mathcal{O}_X$-modules and consider the complex of derived local homomorphisms $\mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N)$, which is the object in $D(\mathcal{O}_X\text{-mod})$ too.

Fix a complex $I(N)$ of injective $\mathcal{O}_X$-modules together with a quasi-isomorphism $N \to I(N)$. It induces a morphism of complexes

$$\mu : \text{Hom}_{\mathcal{O}_X}(M, N) \to \text{Hom}_{\mathcal{O}_X}(M, I(N)).$$

Consider the composition $\phi$ of the natural map $\text{Hom}_{\mathcal{A}}(M, N) \to \text{Hom}_{\mathcal{O}_X}(M, N)$ with $\mu$. Since injective sheaves are acyclic with respect to the functor $\text{Hom}(U, -)$, for every $\mathcal{O}_X$-module $U$ (cf. [KS, Corollary 2.4.2]) we have a functorial isomorphism in $D(\mathcal{O}_X\text{-mod})$:

$$\mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N) \simeq \text{Hom}_{\mathcal{O}_X}(M, I(N)). \quad (14)$$

Hence we have a commutative diagram:

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{A}}(M, N) & \xrightarrow{\phi} & \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N) \\
\downarrow{\mu} & & \\
\text{Hom}_{\mathcal{O}_X}(M, N) & \xrightarrow{\mu} & \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N)
\end{array}$$
Lemma 3.14  \( \phi \) induces a quasi-isomorphism \( \text{Hom}^*_A(M, N) \simeq \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N) \).

Proof. We need to show that \( \phi \) induces an isomorphism of cohomology sheaves. This is a local statement. Hence, we can assume that \( X \) is a polydisc. According to Corollary 3.10 we can replace \( M \) and \( N \) by Dolbeault bicomplexes of finite complexes \( E_1^* \) and \( E_2^* \) of free \( \mathcal{O}_X \)-modules:

\[
M = A \otimes_{\mathcal{O}_X} E_1^*, \quad N = A \otimes_{\mathcal{O}_X} E_2^*.
\]

Then the complex \( \text{Hom}^*_A(M, N) \) is isomorphic to Dolbeault complex of the complex of sheaves of local homomorphisms \( A \otimes_{\mathcal{O}_X} \text{Hom}_{\mathcal{O}_X}(E_1^*, E_2^*) \).

If \( E_1^* \) and \( E_2^* \) both consist of single locally free sheaves \( E_1 \) and \( E_2 \), then this complex is obviously quasi-isomorphic to \( \text{Hom}_{\mathcal{O}_X}(E_1, E_2) \). On the other hand, \( M \) and \( N \) are quasi-isomorphic to, respectively, \( E_1 \) and \( E_2 \), hence \( \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N) = \text{Hom}_{\mathcal{O}_X}(E_1, E_2) \), i.e. the statement of the lemma is clear for this case.

We shall use now the induction on the length of the complexes \( E_1^* \) and \( E_2^* \). If one of them, say \( E_1^* \), has length greater than one, then we decompose it into a short exact sequence of complexes

\[
0 \to E_1'' \to E_1^* \to E_1''' \to 0
\]

where \( E_1'' \) and \( E_1''' \) are complexes of free sheaves of shorter length. Since \( A \) is flat over \( \mathcal{O}_X \) by Theorem A.12 in the Appendix, functor \( A \otimes_{\mathcal{O}_X} (-) \) is exact on the derived categories. Let

\[
M' \to M \to M'' \tag{15}
\]

be the decomposition of the object \( M \) induced by the above exact sequence.

By induction, we know that \( \phi \) induces quasi-isomorphisms:

\[
\text{Hom}^*_A(M', N) \simeq \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M', N),
\]

\[
\text{Hom}^*_A(M'', N) \simeq \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M'', N)
\]

Cohomology sheaves of \( \text{Hom}^*_A(M, N) \) and \( \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N) \) fit into two long exact sequences, obtained by applying the functors \( \text{Hom}^*_A(-, N) \) and \( \mathbb{R}\text{Hom}_{\mathcal{O}_X}(-, N) \) to the triangle (15), and \( \phi \) gives a morphism of these long sequences. The quasi-isomorphism for \( M \) follows from the lemma on 5 homomorphisms applied to this diagram. \( \square \)

Proposition 3.15 The functor \( \Phi : \mathcal{H}o(\mathcal{C}_X) \to \mathcal{D}^b(\mathcal{O}_X\text{-mod}) \) is fully faithful.

Proof. By the standard property of local \( \mathbb{R}\text{Hom} \) applied to two \( \bar{\partial} \)-superconnections \( M \) and \( N \), one can recover the global homomorphisms by the formula:

\[
\text{Hom}_{\mathcal{O}_X\text{-mod}}(M, N) = \mathbb{H}^0(X, \mathbb{R}\text{Hom}_{\mathcal{O}_X}(M, N)),
\]

where \( \mathbb{H}^n \) stands for the hypercohomology of a complex of \( \mathcal{O}_X \)-modules. In view of Lemma 3.14, we can replace the right hand side in this equality and get:

\[
\text{Hom}_{\mathcal{O}_X\text{-mod}}(M, N) = \mathbb{H}^0(X, \text{Hom}^*_A(M, N)).
\]
There is a standard spectral sequence converging to the hypercohomology:

\[ H^i(X, \mathcal{H}om^j_A(M, N)) \rightarrow H^{i+j}(\mathcal{H}om^*_A(M, N)) \]

Since sheaves \( \mathcal{H}om^j_A(M, N) \) are fine, cohomology \( H^i(X, \mathcal{H}om^j_A(M, N)) \) are trivial for \( i \geq 0 \). The spectral sequence degenerates, and yields the equality of \( \text{Hom}_{D(O_X-\text{mod})}(M, N) \) with the 0-th homology of the complex of global sections \( \Gamma(X, \mathcal{H}om^*_A(M, N)) \), which is exactly \( \text{Hom}_{\mathcal{H}oc}(M, N) \). □

3.16 EVERY COHERENT SHEAF IS QUASISOMORPHIC TO A \( \bar{\partial} \)-SUPERCONNECTION

In this Subsection we construct a \( \bar{\partial} \)-superconnection that is quasi-isomorphic to a given coherent sheaf.

Let \( \mathcal{A}_\omega \) be the sheaf of complex-valued real-analytic functions on \( X \) regarded as a real-analytic manifold. Since the sheaf \( \mathcal{A}_\omega \) is identified with the restriction to the diagonal \( X \subset X \times \bar{X} \) of the sheaf of holomorphic functions \( \mathcal{O}_{X \times \bar{X}} \), which is coherent, \( \mathcal{A}_\omega \) is also coherent.

Let \( F \) be a coherent sheaf of \( \mathcal{O}_X \)-modules, i.e. it is finitely presented. Then, \( \mathcal{F}_\omega = \mathcal{A}_\omega \otimes_{\mathcal{O}_X} F \) is also finitely presented (\( \otimes \) is right-exact). Hence, \( \mathcal{F}_\omega \) is coherent as an \( \mathcal{A}_\omega \)-module, because \( \mathcal{A}_\omega \) is coherent.

**Lemma 3.17** For a coherent sheaf of \( \mathcal{O}_X \)-modules \( F \), the sheaf of \( \mathcal{A}^0 \)-modules \( \mathcal{F} = \mathcal{A}^0 \otimes_{\mathcal{O}_X} F \) has a locally free resolution \( \mathcal{E}^* \) over \( \mathcal{A}^0 \):

\[
0 \rightarrow \mathcal{E}^{-n} \rightarrow \cdots \rightarrow \mathcal{E}^0 \rightarrow \mathcal{F} \rightarrow 0 \tag{16}
\]

**Proof.** As it is explained by Atiyah and Hirzebruch in [AH1], the famous Grauert’s result [G] on the existence of a fundamental system of Stein neighborhoods of any real-analytic manifold \( X \) in its complexification, together with theorems A and B of Cartan, easily implies, on a compact \( X \), the existence of a finite locally free resolution for an arbitrary coherent \( \mathcal{A}_\omega \)-module. Fix such a resolution for \( \mathcal{F}_\omega \):

\[
0 \rightarrow \mathcal{E}^{-n}_{\omega} \rightarrow \cdots \rightarrow \mathcal{E}^0_{\omega} \rightarrow \mathcal{F}_\omega \rightarrow 0 \tag{17}
\]

By a result of Malgrange [M] \( \mathcal{A}^0 \) is a flat ring over \( \mathcal{A}_\omega \). Hence, by taking tensor product with \( \mathcal{A}^0 \) over \( \mathcal{A}_\omega \), we obtain an \( \mathcal{A}^0 \)-resolution \( \mathcal{E}^* = \mathcal{A}^0 \otimes_{\mathcal{A}_\omega} \mathcal{E}^*_{\omega} \) for \( \mathcal{F} \) of the form (16).

For reader’s convenience, we will give another, somewhat more explicit, construction for the resolution of the form (16), which, however, still relies on [M]. In a sufficiently small neighborhood \( U \) of any point \( x \in X \), the sheaf \( \mathcal{F}|_U \) has a finite resolution by finitely generated free \( \mathcal{A}^0_U \)-modules. Indeed, the sheaf \( F|_U \) has a finite resolution by free \( \mathcal{O}_U \)-modules. The sheaf \( \mathcal{A}^0_U \) is flat over \( \mathcal{O}_U \) by proposition A.13 (see Appendix). Thus, we can take the tensor product of the resolution with \( \mathcal{A}^0_U \) and get the required local resolution.
On a compact manifold, every \( \mathcal{A}^0 \)-module which is locally generated by a finite number of sections is also globally generated by a finite number of sections, because \( \mathcal{A}^0 \) is a fine sheaf. This is clearly applicable to sheaves which locally have free resolutions. Now if we have an epimorphism between two sheaves which have local resolutions, then the kernel is a sheaf which also has such a resolution. Apply this to the epimorphism \( (\mathcal{A}^0)^s \to \mathcal{F} \), which exists because \( \mathcal{F} \) is generated by a finite number of global sections. We get that the kernel has also free resolutions locally on \( X \) and is generated over \( \mathcal{A}^0 \) by finite number of sections. Hence we can iterate the process and construct the resolution until the kernel becomes locally free. This must happen after a finite number of iterations, because the manifold, being compact, is covered by a finite number of open sets on which a finite free resolution exists. Just take the resolution of the length equal to the maximum of lengths of these free resolutions on this finite number of open sets.

\[ \square \]

Now denote by \( \nabla_F \) the differential \( \bar{\partial} \) acting on \( \mathcal{A} \otimes \mathcal{A}^0 \mathcal{F} = \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \) along the first tensor factor. It makes \( \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \) into a DG-module over \( \mathcal{A} \). It is not a \( \bar{\partial} \)-superconnection, because in general it is not a locally free as an \( \mathcal{A}^2 \)-module, but we will construct a \( \bar{\partial} \)-superconnection quasi-isomorphic to \( \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \).

Choose a resolution \( \mathcal{E}^* \) as in (16) and put \( M^2 = (\mathcal{A} \otimes \mathcal{E})^2 \) as a graded module over \( \mathcal{A}^2 \) with the grading given by the sum of gradings on \( \mathcal{A} \) and \( \mathcal{E}^* \). Let \( \bar{\gamma} \) be the differential in the resolution \( \mathcal{E}^* \). We denote also \( \bar{\gamma} \) its \( \mathcal{A}^2 \)-linear (as of operator of degree 1) extension to \( M^2 \). We denote \( \bar{\gamma}_0 \) the map \( \mathcal{E}^0 \to \mathcal{F} \), its \( \mathcal{A}^2 \)-linear extension to the map \( (\mathcal{A} \otimes \mathcal{E})^2 \to (\mathcal{A} \otimes \mathcal{F})^2 \), and the extension to the map \( M^2 \to (\mathcal{A} \otimes \mathcal{F})^2 \) which is zero on other components, \( (\mathcal{A} \otimes \mathcal{E})^2 \to (\mathcal{A} \otimes \mathcal{F})^2 \) for \( i \neq 0 \).

**Theorem 3.18** \( M^2 \) can be endowed with a structure of a \( \bar{\partial} \)-superconnection such that \( \bar{\gamma}_0 \) furnishes a quasi-isomorphism \( \bar{\gamma}_0 : M \to \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{F} \).

**Proof.** We claim that there exists a system of (non-flat, in general) \( \bar{\partial} \)-connections \( \nabla_i \) on \( \mathcal{E}^i \)'s which commute with \( \bar{\gamma} \) and such that

\[ \bar{\gamma}_0 \nabla_0 = \nabla_F \bar{\gamma}_0. \]  

(18)

First, we construct a \( \bar{\partial} \)-connection on \( \mathcal{E}^0 \) compatible with \( \bar{\gamma}_0 \), \( \nabla_F \) in this sense.

Applying functor \( \Gamma(X, \mathcal{A}^1 \otimes_{\mathcal{A}^0} (-)) \) to the sequence (16) gives an exact sequence, because the sheaf \( \mathcal{A}^1 \) is flat over \( \mathcal{A}^0 \) and all the sheaves in the sequence are fine, hence acyclic. In particular, \( \Gamma(X, \mathcal{A}^1 \otimes_{\mathcal{A}^0} \mathcal{E}^0) \to \Gamma(X, \mathcal{A}^1 \otimes_{\mathcal{A}^0} \mathcal{F}) \) is an epimorphism.

If \( \mathcal{E}^0 \) is a free \( \mathcal{A}^0 \)-module, i.e. a trivial smooth vector bundle, then take a basis \( \{s_i\} \) of its sections and define \( \nabla(s_i) \) to be any element in \( \mathcal{A}^1 \otimes \mathcal{E}^0 \) such that \( \bar{\gamma}_0 \nabla(s_i) = \nabla_F \bar{\gamma}_0(s_i) \). Such an element exists in view of the above epimorphism on global sections. If \( \mathcal{E}^0 \) is not a trivial bundle, then consider the direct sum \( S = \mathcal{E}^0 \oplus \mathcal{G} \) which is a trivial vector bundle. Take, as above, a connection \( \nabla_S : S \to \mathcal{A}^1 \otimes S \) on \( S \) compatible with the composite map \( S \overset{p}{\to} \mathcal{E}^0 \overset{\bar{\gamma}_0}{\to} \mathcal{F} \), where \( p : S \to \mathcal{E}^0 \) is the projection. Then we have:

\[ \bar{\gamma}_0 p_0 \nabla_S = \nabla_F \bar{\gamma}_0 p_0. \]
The restriction $\nabla_0 = p_0 \nabla_S|_{\mathcal{E}^0}$ of this connection to $\mathcal{E}^0$ will define a connection on $\mathcal{E}^0$ compatible with $\gamma_0$, that is $\gamma_0 \nabla_0 = \nabla_F \gamma_0$.

The connections $\nabla_i$’s on other $\mathcal{E}^i$’s, commuting with $\bar{\gamma}$ are constructed consecutively by decreasing $i$ and using the same argument with replacing the sequence (16) by its truncation

$$0 \to \mathcal{E}^{-n} \to \cdots \to \mathcal{E}^{-i} \to \mathcal{F}_i \to 0,$$

where $\mathcal{F}_i$ is the kernel of $\gamma_0$ and $\mathcal{F}_i$, for $i \geq 2$, is the kernel of $\bar{\gamma} : \mathcal{E}^{-i+1} \to \mathcal{E}^{-i+2}$.

Using the Leibniz rule, we extend $\nabla$ to a differential operator (also denoted by $\nabla$) on $\mathcal{M}^2 = \mathcal{A} \otimes \mathcal{E}^*$ which has bidegree $(1, 0)$ and (anti)commutes with $\bar{\gamma}$.

We want to find a differential in $\mathcal{M}^2$ of the form $\mathcal{D} = \bar{\gamma} + \nabla + \sum_{i \geq 2} \bar{\beta}_i$, where $\bar{\beta}_i$’s are $\mathcal{A}^0$-module endomorphisms of $\mathcal{M}$ of bidegree $(i, 1 - i)$. Recall that the equation $\mathcal{D}^2 = 0$ implies a series of equations (9) on components of $\mathcal{D}$.

We proceed by induction. Suppose we have already found $\bar{\beta}_i$’s, for $i = 2, \ldots, k - 1$, which satisfy the first $k - 1$ equations of (9). Then the $k$-th equation looks like:

$$[\bar{\gamma}, \bar{\beta}_k] + u_k = 0,$$

where $u_k$ depends on $\nabla, \bar{\beta}_2, \ldots, \bar{\beta}_{k-1}$.

Note that $u_k \in \Gamma(X, \mathcal{A}^k \otimes \mathcal{A}^0 \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{E}^*)$ and Bianchi identity implies $[\bar{\gamma}, u_k] = 0$.

The exact sequence (16) can be interpreted as a quasi-isomorphism of $\mathcal{E}^*$ with $\mathcal{F}$, which implies the isomorphism in $\mathcal{D}^b(\mathcal{A}^0-\text{mod})$:

$$\mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{E}^* = \mathcal{R} \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{F}.$$

Therefore, the $[\bar{\gamma}, \cdot]$-complex $\mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{E}^*$ has trivial cohomology in all negative degrees. Since $\mathcal{A}^k$ is a flat sheaf over $\mathcal{A}^0$ and all the sheaves in the complex are fine, hence acyclic, the complex $\Gamma(X, \mathcal{A}^k \otimes \mathcal{A}^0 \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{E}^*)$ has trivial cohomology in all negative degrees for each $k$. The differential in this complex is exactly $[\bar{\gamma}, \cdot]$. This implies that the equation (19) has solution for any $k \geq 3$, because, then, $u_k$ is a closed element of negative degree $2 - k$ in this complex.

We are left only with the 3rd equation in (9), where $u_2 = \nabla^2 \in \Gamma(X, \mathcal{A}^2 \otimes \mathcal{A}^0 \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{E})$. The obstruction to the solution of the equation $[\bar{\gamma}, \bar{\beta}_2] + u_2 = 0$ lies in the zeroth cohomology of the complex $\Gamma(X, \mathcal{A}^2 \otimes \mathcal{A}^0 \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{E}^*)$, which, by above, equals $\Gamma(X, \mathcal{A}^2 \otimes \mathcal{A}^0 \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{F})$. Note that $\nabla$ is a chain presentation for $\nabla_F$, because $\nabla$ commutes with $\bar{\gamma}$ and in view of eq. (18). This implies that, under the identification of the zeroth cohomology of the complex $\Gamma(X, \mathcal{A}^1 \otimes \mathcal{A}^0 \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{E}^*)$ with $\Gamma(X, \mathcal{A}^1 \otimes \mathcal{A}^0 \mathcal{E}^{nd}_{\mathcal{A}^0} \mathcal{F})$, the connection $\nabla$ corresponds to $\nabla_F$. Then, the cohomology class of $u_2 = \nabla^2$ is identified with $\nabla_F^2 = 0$. Therefore, the obstruction to solving the equation $[\bar{\gamma}, \bar{\beta}_2] + u_2 = 0$ with respect to $u_2$ also vanishes.

Now let us check that the $\tilde{\mathcal{D}}$-superconnection $M$ constructed in this way is quasi-isomorphic to $\mathcal{F}$. We have the chain map $\gamma_0 : M \to \mathcal{A} \otimes \mathcal{F}$. The filtration of $M$ induced by the standard filtration of $\mathcal{A}$ gives a spectral sequence which implies that $\gamma_0$ is a quasi-isomorphism. The fact that the cohomology of $\mathcal{A} \otimes \mathcal{F}$ is $\mathcal{F}$ can be checked locally. Take locally a resolution $\mathcal{E}^*$ of $\mathcal{F}$ by free $\mathcal{O}_X$-modules. Consider the bicomplex $\mathcal{A} \otimes \mathcal{E}^*$. The comparison of the two spectral sequences of the bicomplex completes the proof of Theorem 3.18. □
Since the functor $\Phi$ is fully faithful by Proposition 3.15, it follows that its image is a full triangulated subcategory in $D^b_{\text{coh}}(X)$ that contains all coherent sheaves. Therefore, the essential image coincides with $D^b_{\text{coh}}(X)$. This concludes the proof of Theorem 3.7.

4 Chern Classes of Superconnections

One way to define Chern classes with values in de Rham cohomology for any coherent sheaf $F$ on a smooth complex manifold is to choose an appropriate real-analytic or smooth resolution for the sheaf $\mathcal{F} = \mathcal{A}^0 \otimes_{\mathcal{O}_X} F$ as in Lemma 3.17, apply the standard construction of Chern-Weil to the members of the resolution and to alternate them. According to the example of C. Voisin [Vo], not every coherent sheaf has a resolution by holomorphic vector bundles. Hence, Chern classes of the members of the resolution might not have $(p,p)$-type.

We refine this approach via superconnections in order to establish the fact that the classes of $F$ does have $(p,p)$-type, and then apply this technique to defining Bott-Chern classes of coherent sheaves and, more generally, of objects in $D^b_{\text{coh}}(X)$. We define Chern classes for objects in the derived category $D^b_{\text{coh}}(X)$ by using the presentation of them by $\bar{\partial}$-superconnections. We will see that this allows us to establish the property of the Chern classes to be of the diagonal type in a relatively easy way.

An alternative approach to define Chern classes of coherent sheaves is, first, to get rid of the torsions by induction in the dimension of their supports and, then, to choose a suitable birational transformation of the manifold, which reduces the problem to the case when the coherent sheaf is locally free (cf. [Griv]).

4.1 Chern Classes of de Rham Superconnections

We follow here the ideas of Quillen [Q] and Freed [F]. We acknowledge also expositions in [BSW, Qi].

In this subsection we consider a smooth real manifold $X$ and the sheaf $\Lambda^*$ of graded algebras of smooth complex-valued forms on $X$. We consider $\Lambda^*$ as a $\mathbb{Z}/2$ graded sheaf of algebras. Let $\mathcal{M}$ be a $\mathbb{Z}/2$ graded module locally free over this algebra and $\mathcal{D} : \mathcal{M} \to \mathcal{M}$ an odd differential operator (not necessarily squared to 0), satisfying the ordinary Leibniz rule for local sections $s$ of $\mathcal{M}$ and $\alpha$ of $\Lambda^*$:

$$\mathcal{D}(\alpha s) = d\alpha \cdot s + (-1)^{\lvert \alpha \rvert} \alpha \mathcal{D}(s),$$

where $d$ is de Rham’s differential.

We say that $(\mathcal{M}, \mathcal{D})$ is a de Rham superconnection. Note the essential difference in our terminology: for de Rham superconnections we don’t require $\mathcal{D}^2 = 0$, while for $\bar{\partial}$-superconnections $\bar{\partial}^2 = 0$.

Define the curvature of $(\mathcal{M}, \mathcal{D})$ by the formula:

$$\mathcal{F} = \mathcal{D}^2.$$
This is an even $\Lambda^\bullet$-endomorphism of $\mathcal{M}$. There is a supertrace functional, $\text{tr}$, which is defined on endomorphisms of a locally free module over a (super)commutative algebra $\Lambda^\bullet$ and takes values in $\Lambda^\bullet$.

Consider the form $\omega_k \in \Lambda^{\text{even}}$ defined by the (super)trace of the $k$-th power of $F$:

$$\omega_k := \text{tr} F^k.$$  

Note that $\omega_k$ is a sum of forms of various even degrees. By the same argument as for the standard Chern classes, we see that $\omega_k$ is a closed form. Indeed:

$$d(\text{tr} F^k) = \text{tr}[\mathcal{D}, F^k] = k \text{tr}([\mathcal{D}, F] F^{k-1}) = 0,$$

because of the Bianchi identity:

$$[\mathcal{D}, F] = [\mathcal{D}, \mathcal{D}^2] = 0.$$

Here and henceforth we denote $[\cdot, \cdot]$ the supercommutator with respect to the grading on forms. Define the $k$-th coefficient of the Chern character of $\mathcal{M}$ via the cohomology class of $\omega_k$:

$$\text{ch}_k \mathcal{M} = \frac{1}{k!} \left( \frac{i}{2\pi} \right)^k [\omega_k].$$

(21)

Any replacement of $\mathcal{D}$ with another odd operator $\mathcal{D}'$ in $\mathcal{M}$ satisfying the same Leibniz rule does not change the cohomology class of $\omega_k$. Indeed, the set of such operators is an affine space, hence we can consider a smooth path $\mathcal{D}(t)$ connecting $\mathcal{D}$ with $\mathcal{D}'$. Let $a(t) = \frac{d}{dt} \mathcal{D}(t)$. Then again in view of the Bianchi identity, we have:

$$\frac{d}{dt} \omega_k(t) = k \text{tr}([\mathcal{D}(t), a(t)] F^{k-1}(t)) = k d(\text{tr} a(t) F^{k-1}(t)),$$

i.e. the derivative is an exact form, which implies that the cohomology class of $\omega_k(t)$ is constant along the path.

Consider $\mathcal{E} = \mathcal{M}/\Lambda^+ \mathcal{M}$, where $\Lambda^+$ is the positive degree part of $\Lambda^\bullet$. $\mathcal{E}$ is a $\mathbb{Z}/2$ graded vector bundle: $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. By choosing a splitting $\mathcal{E} \to \mathcal{M}$ of the quotient map $\mathcal{M} \to \mathcal{E}$, we get a non-canonical isomorphism of $\Lambda^\bullet$-modules $\mathcal{M} = \Lambda^\bullet \otimes \mathcal{E}$. We can choose $\mathcal{D}$ to be a direct sum of ordinary connections in $\mathcal{E}^+$ and $\mathcal{E}^-$ extended in the standard way to $\Lambda^\bullet \otimes \mathcal{E}^+$ and $\Lambda^\bullet \otimes \mathcal{E}^-$. It follows that the curvature of $\mathcal{D}$ is a direct sum of the curvatures of the connections on $\mathcal{E}^+$ and $\mathcal{E}^-$. Then, according to the Chern-Weil definition of Chern classes for vector bundles, the cohomology class $\frac{1}{k!} \left( \frac{i}{2\pi} \right)^k [\omega_k]$ lies in $H^2_{dR}(X, \mathbb{C})$ and equals to the $k$-th component of the Chern character for $\mathcal{E}$:

$$\text{ch}_k \mathcal{M} = \text{ch}_k \mathcal{E}^+ - \text{ch}_k \mathcal{E}^-.$$  

(22)

Note that the right hand side is nothing but the $\text{ch}_k \mathcal{E}$ for a $\mathbb{Z}/2$-graded vector bundle $\mathcal{E}$. It is well known that $\text{ch}_k \mathcal{E}$ and, hence, $\text{ch}_k \mathcal{M}$ belong in fact to $H^2_{dR}(X, \mathbb{Q})$.  

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4.2 Chern Classes of $\bar{\partial}$-Superconnections

Now let $X$ be a complex manifold and $M$ a flat $\bar{\partial}$-superconnection with differential $\bar{\partial}_M : M \to M$ on $X$. Consider the quotient $\mathcal{E} = M/\mathcal{A}^0+M$. It has a $\mathbb{Z}$ grading: $\mathcal{E} = \mathcal{E}^\ast$.

Let us consider $X$ as a real manifold and replace $M$ by a module $\mathcal{M}$ over the algebra of smooth complex-valued forms bigraded in a standard way by their $(p,q)$-types $\Lambda^\ast = \Lambda^{p,q}$:

$$\mathcal{M} = \Lambda^{p,q} \otimes \Lambda^0 \otimes M$$

Define the operator $\bar{\partial} : \mathcal{M} \to \mathcal{M}$ as an extension of $\bar{\partial}_M$ from $M$ to $\mathcal{M}$ by the following Leibniz rule for local forms $\alpha \in \Lambda^{p,q}$ and local sections $s$ of $M$:

$$\bar{\partial}(\alpha s) = \bar{\partial} \alpha \cdot s + (-1)^{|\alpha|} \alpha \bar{\partial}_M(s).$$

We define the Chern character of a $\bar{\partial}$-superconnection $\mathcal{M}$ as the Chern character of an arbitrary de Rham superconnection $\mathcal{M}$, that is $\text{ch}_k(M) = \text{ch}_k(\mathcal{M})$. We will prove that the $\text{ch}_k$-component of this character has a $(k,k)$-type, what means (on a not necessarily Kähler manifold) a de Rham class which has a representative by a $(k,k)$-form.

A choice of splitting $\mathcal{E}^\ast \to M$ for the factorization map $M \to \mathcal{E}^\ast$ defines a non-canonical $\Lambda^\ast$ module isomorphism $M = \Lambda^\ast \otimes \mathcal{E}^\ast$ and an $\Lambda^\ast$ module isomorphism:

$$\mathcal{M} = \Lambda^{p,q} \otimes \mathcal{E}^\ast.$$  \hspace{1cm} (23)

We shall consider $\mathcal{M}$ as a $\mathbb{Z}/2\mathbb{Z}$-graded module $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$, where

$$\mathcal{M}_+ = \bigoplus_{p+q+i \in 2\mathbb{Z}} \Lambda^{p,q} \otimes \mathcal{E}^i, \quad \mathcal{M}_- = \bigoplus_{p+q+i \in 2\mathbb{Z}+1} \Lambda^{p,q} \otimes \mathcal{E}^i.$$  \hspace{1cm} (24)

Let us choose arbitrary Hermitian forms $h_i$’s, on each vector bundle $\mathcal{E}^i$. We denote $(\cdot,\cdot)$ the sesquilinear pairing defined by $h_i$’s on $\mathcal{E}^\ast = \bigoplus \mathcal{E}^i$. We use the same notation for its extension to a graded $\Lambda^{p,q}$-valued pairing on $\mathcal{M}$, which is $\Lambda^{p,q}$-sesquilinear with respect to the complex conjugation on forms in $\Lambda^{p,q}$. In other words, for any $s_1, s_2 \in \mathcal{M}$, we get a pairing $(s_1, s_2) \in \Lambda^{p,q}$, and for any $\alpha \in \Lambda^{p,q}$ this pairing satisfies:

$$(s_1, \alpha s_2) = (-1)^{|\alpha||s_1|} \alpha(s_1, s_2), \quad (\alpha s_1, s_2) = \bar{\alpha}(s_1, s_2).$$

Let $D$ be an operator on $\mathcal{M}$ uniquely defined by the equation on arbitrary local sections $s_1$ and $s_2$ of $\mathcal{M}$:

$$\bar{\partial}(s_1, s_2) = (Ds_1, s_2) + (-1)^{|s_1|}(s_1, Ds_2),$$  \hspace{1cm} (25)

or, equivalently:

$$\partial(s_1, s_2) = (\bar{D}s_1, s_2) + (-1)^{|s_1|}(s_1, \bar{D}s_2).$$  \hspace{1cm} (26)

The existence and uniqueness of such $D$ is similar to the existence and uniqueness of a Hermitian connection in a holomorphic vector bundle compatible with the $\bar{\partial}$-operator on local sections of the bundle.
Note that $D$ satisfies the Leibniz rule:

$$D(\alpha s) = \partial \alpha \cdot s + (-1)^{|\alpha|} \alpha D(s).$$

Make $\mathcal{M}$ into a de Rham superconnection by introducing an odd (in $\mathbb{Z}/2$ grading) differential operator $\mathcal{D}$ on $\mathcal{M}$:

$$\mathcal{D} = D + \bar{D}.$$ 

Then $\mathcal{D}$ satisfies:

$$d(s_1, s_2) = (\mathcal{D}s_1, s_2) + (-1)^{|s_1|}(s_1, \mathcal{D}s_2).$$

Since $\bar{D}^2 = 0$, we have that:

$$0 = \partial^2(s_1, s_2) = (D^2s_1, s_2) + (-1)^{|s_1|+1}(Ds_1, \bar{D}s_2) + (-1)^{|s_1|}(Ds_1, \bar{D}s_2) + (s_1, \bar{D}^2s_2) = (D^2s_1, s_2)$$

As this holds for any $s_1, s_2 \in \mathcal{E}^*$, it follows that $D^2 = 0$. We can say that $D$ defines a flat $\partial$-superconnection on the module $\tilde{M} = \mathcal{A}^{*,0} \otimes \mathcal{A}^{0,*} \mathcal{E}^*$. Then the curvature $\mathcal{F}$ of the de Rham superconnection $\mathcal{D}$ has the expression:

$$\mathcal{F} = \mathcal{D}^2 = [D, \bar{D}].$$ (27)

Now, we fix a (non-canonical) isomorphism $\mathcal{M} = \mathcal{A}^{*,*} \otimes \mathcal{A}^{0,0} \mathcal{E}^*$ as in eq. (23), and decompose (not necessarily $\mathcal{A}^{*,*}$-linear) the operators acting on sections of $\mathcal{M}$ into their homogeneous components with respect to the tridegree, where an operator with degree $(p, q, r)$ acts as $\mathcal{A}^{a,b} \otimes \mathcal{A}^{0,0} \mathcal{E}^c \rightarrow \mathcal{A}^{a+p,b+q} \otimes \mathcal{A}^{0,0} \mathcal{E}^{c+r}$, that is, it shifts the grading as:

$$(a, b, c) \rightarrow (a + p, b + q, c + r).$$

We denote $\text{deg}(A) = (d_1(A), d_2(A), d_3(A))$ the triple degree of a homogeneous operator $A$. Similarly, the tridegree of a section $s$ of $\mathcal{A}^{a,b} \otimes \mathcal{A}^{0,0} \mathcal{E}^c$ is denoted as

$$\text{deg}(s) = (d_1(s), d_2(s), d_3(s)) := (a, b, c).$$

Also, for a form $\omega$ in $\mathcal{A}^{a,b}$, we will use the notation for its bidegree:

$$\text{deg}(\omega) = (d_1(\omega), d_2(\omega)) := (a, b).$$

Since the pairing $(\cdot, \cdot)$ on $\mathcal{M}$ is sesquilinear, for any two homogeneous sections $s_1$ and $s_2$ of $\mathcal{M}$ we have that

$$\text{deg}((s_1, s_2)) = (d_2(s_1) + d_1(s_2), d_1(s_1) + d_2(s_2))$$ (28)

To simplify notation, we denote $\bar{\beta}_0 := \bar{\gamma}$ and $\bar{\beta}_1 := \bar{\nabla}$ the components in the expression (7) for superconnection $\bar{D}$. Then, the degree decomposition for $\bar{D}$ reads:

$$\bar{D} = \sum_{i \geq 0} \bar{\beta}_i,$$ (29)

where $\bar{\beta}_i$ is homogeneous of degree

$$\text{deg}(\bar{\beta}_i) = (0, i, 1 - i).$$ (30)
Lemma 4.3  The operator $D$ has the following degree decomposition:

$$D = \sum_{i \geq 0} \beta_i,$$

(31)

where $\beta_i$ has degree $(i, 0, i - 1)$.

Proof. Let $\omega_{(P,Q)}$ denote the bidegree $(P, Q)$ component of an inhomogeneous form $\omega$ in $\mathcal{A}^*$. Consider a decomposition of $D$ in its tridegree components,

$$D = \sum D^{p,q,j}.$$ 

Let us choose homogeneous local sections $s_1, s_2$ of $\mathcal{M}$, and denote $\text{deg}((s_1, s_2)) = (A, B)$. Formula (25) implies that

$$\sum_{p,q,i} (D^{p,q,j} s_1, s_2)_{(P,Q)} = \bar{\partial}(s_1, s_2)_{(P,Q)} - \sum_i (-1)^{|s_1|} (s_1, \bar{\beta}_i s_2)_{(P,Q)}.$$  (32)

Then, we notice that all terms in the right hand side vanish unless $P = A$ and $Q = B + d_3(s_2) - d_3(s_1) + 1$. Indeed, note that $(s_1, s_2) = 0$ unless $d_3(s_1) = d_3(s_2)$, hence $\text{deg}(\bar{\partial}(s_1, s_2)) = (A, B + 1) = (A, B + d_3(s_2) - d_3(s_1) + 1)$. Also, $(s_1, \bar{\beta}_i s_2) = 0$ if $d_3(s_1) \neq d_3(\bar{\beta}_i s_2)$, while $d_3(s_1, \bar{\beta}_i s_2) = B + i$ and $d_3(\bar{\beta}_i s_2) = d_3(s_2) + 1 - i$, which implies the same formula the degree of the last term in (32).

Since $\text{deg}((D^{p,q,j} s_1, s_2)) = (A + q, B + p)$ and $d_3(D^{p,q,j} s_1) = d_3(s_1) + j$, we have that $(D^{p,q,j} s_1, s_2)_{(P,Q)} = 0$ unless $q = 0, p = d_3(s_2) - d_3(s_1) + 1$, and $d_3(s_1) + j = d_3(s_2)$. Since $s_1, s_2$ are arbitrary, we conclude that $D^{p,q,j} = 0$ unless $q = 0$ and $p = j + 1$, which proves the lemma. $\square$

In view of the formulas (27), (29) and (31), the curvature has the form:

$$\mathcal{F} = \sum_{i,j \geq 0} [\beta_i, \bar{\beta}_j].$$  (33)

Decompose the Chern form $\omega_k = \text{tr} \mathcal{F}^k$ into $(p, q)$-components: $\omega_k = \sum \omega_k^{p,q}$. We shall show that the form has trivial all the off-diagonal entries.

Lemma 4.4  We have that $\omega_k^{p,q} = 0$, if $p \neq q$.

Proof. According to (29) and (31), the possible degrees for $D$ are $(p, 0, p - 1)$, where $p \geq 0$, and the degrees for $\mathcal{F}$ are $(0, q, -q + 1)$, where $q \geq 0$. Then possible degrees for $\mathcal{F} = [D, \bar{D}]$ and for its powers $\mathcal{F}^k$ are $(p, q, p - q)$, for some $p, q \geq 0$.

Note that the trace of an endomorphism is zero on components with degrees $(p, q, r)$ if $r \neq 0$, hence the result. $\square$

Now, we vary the Hermitian forms in vector bundles $\mathcal{E}^j$'s with a parameter $t$ according to the rule:

$$h_j(t) = t^i h_j.$$  (34)

This defines a variation of the de Rham superconnection, $\mathcal{D}(t)$, and its curvature, $\mathcal{F}(t)$. Let us see how the Chern form $\omega_k(t) = \text{tr} \mathcal{F}(t)^k$ varies. Decompose the form $\omega_k(t)$ into its $(p, p)$-components with account of the result of Lemma 4.4: $\omega_k(t) = \sum_{p \geq 0} \omega_k^{p,p}(t)$. In particular, $\omega_k = \omega_k(1) = \sum \omega_k^{p,p}$.
Lemma 4.5 The following equality holds: \( \bar{\omega}_k^{p,p}(t) = t^{p-k} \omega_k^{p,p} \).

Proof. Note that \( \bar{D} \) does not change under the variation of metrics. The operator \( D(t) \) is defined by eq. (25) via \( \bar{D} \) and \( h_i(t) \) and we are going to determine how it depends on \( t \). To this end we extract various homogeneous components of eq. (25) as we did in proving Lemma 4.3, cf., eq. (32).

First, we take \( s_1, s_2 \), such that \( d_3(s_1) = d_3(s_2) = j \). Then, the hermitian form \((s_1, s_2)\) is determined by the metric \( h_j(t) \) in only one component, \( \mathcal{E}^j \), of \( \mathcal{E}^* \) and we write it as \( (s_1, s_2)(t) = t^j(s_1, s_2)_j \) in agreement with eq. (34). We consider now the following component of eq. (25),

\[
\bar{\partial}(s_1, s_2)(t) = (\beta_1(t)s_1, s_2)(t) + (-1)^{|s_1|}(s_1, \bar{\beta}_1 s_2)(t) \implies t^i \bar{\partial}(s_1, s_2)_j = t^i(\beta_1(t)s_1, s_2)_j + (-1)^{|s_1|}t^i(s_1, \bar{\beta}_1 s_2)_j.
\]

We conclude from this relation, that \( \beta_1 \) does not depend of \( t \), which is part of the assertion of the lemma.

Second, to find \( \beta_i(t) \) for \( i \neq 1 \), we take \( d_3(s_2) = j, d_3(s_1) = j+1-i \). In this case, the term in the left hand side of eq. (25) vanishes and we are left with the following relation

\[
(\beta_i(t)s_1, s_2)(t) + (-1)^{|s_1|}(s_1, \bar{\beta}_i s_2)(t) = t^j(\beta_i(t)s_1, s_2)_j + (-1)^{|s_1|}t^{j-1}(s_1, \bar{\beta}_i s_2)_{j+1-i} = 0.
\]

Thus, we conclude that, for all \( i \),

\[
\beta_i(t) = t^{i-1} \beta_i.
\]

We turn now to \( \omega_k^{p,p} \) and notice that, by definition,

\[
\omega_k^{p,p}(t) = \sum \prod_{s=1}^k [\beta_{p_s}(t), \bar{\beta}_{q_s}],
\]

and the sum runs over \( k \)-tuples of pairs \((p_1, q_1), \ldots, (p_k, q_k)\), such that \( \sum p_s = p \) and \( \sum q_s = q \).

Then, we observe, that \( \omega_k^{p,p}(t) = t^N \omega_k^{p,p}, \) where \( N = \sum_{s=1}^k (p_s - 1) = p - k \), whence the result. \( \square \)

Let \( H^p_{dR}(X) \) denote the de Rham cohomology classes which can be represented by a closed \( 2p \)-form of type \((p, p)\). Since the cohomology class of \( \omega_k \) does not depend on connection, it cannot in particular vary with \( t \) when the Hermitian structure varies according to eq. (34). Then, Lemmas 4.4 and 4.5 imply the following

Proposition 4.6 The \( k \)-th Chern character of a \( \bar{D} \)-superconnection is of type \((k, k)\), that is, \( \text{ch}_k(M) \in H^k_{dR}(X) \).

4.7 Remark. As a matter of fact, the above Proposition follows immediately from Lemma 4.4 alone and the general properties of the Chern character of a superconnection described in Subsection 4.1, namely that the \( k \)-th character \( \text{ch}_k M \) belongs to \( H^{2k}_{dR}(X, \mathbb{Q}) \), cf., eq. (22). Thus, in principle, we do not need Lemma 4.5 to prove the last Proposition 4.6. However, the technical result of that Lemma will be necessary below in the next Subsection.
4.8 The Chern Character of a Coherent Analytic Sheaf

We are going to show that the definition of a Chern character given above in Subsection 4.2 descends to $\mathcal{D}^b_{\text{coh}}(X)$ via the equivalence of categories $\mathcal{D}^b_{\text{coh}}(X) \cong \mathcal{H}o(\mathcal{C}_X)$. First, recall

**Lemma 4.9** Let $\mathcal{E}_1^i \xrightarrow{\varphi} \mathcal{E}_2^i$ be a morphism of complexes of smooth vector bundles (locally free $\mathcal{A}^0$-modules in terminology of the present paper). If $\varphi$ is a quasi-isomorphism then $\text{ch}(\mathcal{E}_1^i) = \text{ch}(\mathcal{E}_2^i)$, where $\text{ch}(E^i) = \sum (-1)^i \text{ch}(E^i)$.

*Proof.* Denote $\gamma_1$, $\gamma_2$ the differentials in $\mathcal{E}_1^i$ and $\mathcal{E}_2^i$, respectively, and consider the cone of $\varphi$ realised as a complex $C^i = \mathcal{E}_1^{i+1} \oplus \mathcal{E}_2^i$ with a differential, which maps $(a, b) \in C^i$ to $(-\gamma_1(a), \varphi(a) + \gamma_2(b)) \in C^{i+1}$. Obviously, $\text{ch}(C^i) = \text{ch}(E_2^i) - \text{ch}(E_1^i)$. On the other hand, if $\varphi$ is a quasi-isomorphism, the complex $C^i$ is acyclic and, hence, $\text{ch}(C^i) = 0$, whence the statement of the lemma. □

Let $M_1$ and $M_2$ be two objects in the category $\mathcal{C}_X$, that is two $\bar{\partial}$-superconnections on $X$. Let $\mathcal{E}_\alpha := M_\alpha/\mathcal{A}^+ M_\alpha$, $\alpha = 1, 2$. These are complexes of vector bundles. Suppose now that $M_1$ and $M_2$ are isomorphic in the category $\mathcal{H}o(\mathcal{C}_X)$. This implies (but not equals to) that there is a quasi-isomorphism $M_1 \xrightarrow{\varphi} M_2$. This quasi-isomorphism descends to a quasi-isomorphism $\mathcal{E}_1^i \xrightarrow{\varphi} \mathcal{E}_2^i$. On the other hand, $\text{ch}(M_\alpha) = \text{ch}(\mathcal{E}_\alpha)$ as it has been defined in eq. (22). Hence, by Lemma 4.9, $\text{ch}(M_1) = \text{ch}(M_2)$ and we conclude that the Chern character depends only on the isomorphism class in $\mathcal{H}o(\mathcal{C}_X)$. By equivalence of categories it means that the Chern character is also defined on the isomorphism classes of objects in $\mathcal{D}^b_{\text{coh}}(X)$. This gives, in particular, a definition of the Chern character for coherent sheaves on $X$.

4.10 Bott-Chern Cohomology

We have seen above that, given a $\mathbb{Z}/2$-graded module $\mathcal{M}$ over $\Lambda^* = \mathcal{A}^{\bullet\bullet}$ with a superconnection $\mathcal{D}$ on it, one constructs de Rham’s cohomology classes $\text{ch}_k(\mathcal{M}) = \left[ \frac{1}{2\pi i} \text{tr} \left( \frac{1}{2\pi i} \mathcal{K} \right)^k \right] \in H^{2k}_{\text{dR}}(X, \mathbb{Q})$, where $\mathcal{K} = \mathcal{D}^2$ is the curvature and $\text{ch}_k(\mathcal{M})$ does not depend on the superconnection, but only on $\mathcal{M}$ itself. Let us now show that if $(\mathcal{M}, \mathcal{D})$ comes from a $\bar{\partial}$-superconnection $(\mathcal{M}, \bar{\partial})$ as in Subsection 4.2, one can define $\text{ch}_k(\mathcal{M})$ as an element in more refined cohomology.

Let us consider the following cohomology groups for a complex-analytic manifold $X$, \n
\[ H^{p,p}(X) := \frac{\{\text{complex } d\text{-closed } (p, p)\text{-forms on } X\}}{\{d\text{-exact forms}\}} \]  \hspace{1cm} (35) \n
and \n
\[ H^{p,p}_{\text{BC}}(X) := \frac{\{\text{complex } d\text{-closed } (p, p)\text{-forms on } X\}}{\{\bar{\partial}\bar{\partial}\text{-exact forms}\}}. \]  \hspace{1cm} (36) \n
The latter are known as Bott-Chern cohomology, and we have obviously a surjection $H^{p,p}_{\text{BC}}(X) \twoheadrightarrow H^{p,p}(X)$. On a Kähler manifold, by Hodge theory and $\partial\bar{\partial}$-lemma, we have that $H^{p,p}_{\text{BC}}(X) = H^{p,p}(X) = H^p(X, \Omega^p_X)$, which is not true in general, cf., [AngTo].
In Subsection 4.2 we saw that for a $\overline{\partial}$-superconnection $M$, $\text{ch}_k(M) \in H^{k,k}(X)$. It is known that, for a holomorphic bundle $E$, its Chern classes, or character can in fact be defined as elements of Bott-Chern cohomology (36). Let us show that the same is true for $\overline{\partial}$-superconnections and, hence, also for coherent sheaves on complex-analytic manifolds.

Let $(M, D)$ be a $\overline{\partial}$-superconnection on $X$. Recall (Subsection 4.2) that in order to define $\text{ch}(M)$ we chose a splitting

$$\mathcal{M} := \mathcal{A}^{\bullet \bullet} \otimes \mathcal{A}^0 \cdot M \xrightarrow{\sim} \mathcal{A}^{\bullet \bullet} \otimes \mathcal{A}^0 \mathcal{E}^*,$$

and hermitian metrics on $\mathcal{E}^*$'s. Then we get in $\mathcal{M}$ a de Rham superconnection $D = D + \overline{D}$, where $D$ depends on the metric chosen and satisfies (cf., eq. (25))

$$\overline{\partial} (\phi, \psi) = (D\phi, \psi) + (-1)^{\phi}(\phi, D\psi).$$

(38)

Here $\phi$ and $\psi$ are sections of $\mathcal{M}$, and $(\ , \ )$ is the $\mathcal{A}^{\bullet \bullet}$-valued pairing defined by the metric in $\mathcal{E}^*$.

Let $\mathcal{F} = \mathcal{D}^2$, consider the Chern forms $\omega_k = \text{tr}\mathcal{F}^k$ and discuss how everything depends on the choices made, i.e. the splitting (37) and the metric in $\mathcal{E}^*$. A change of the splitting is the same as an $\mathcal{A}^{\bullet \bullet}$-automorphism of $\mathcal{A}^{\bullet \bullet} \otimes \mathcal{A}^0 \mathcal{E}^*$, which is the same as a strict gauge transformation. The forms $\omega_k$ are obviously invariant upon this. Let us now consider an infinitesimal variation of the metric. It can be described by a hermitian section $\delta h$ of $\mathcal{E}nd \mathcal{E}^*$ as $\delta(\phi, \psi) = (\phi, \delta h \psi) = (\delta h \phi, \psi)$. The infinitesimal variation of (38) reads (note, that $\overline{D}$ does not change with $\delta h$):

$$\overline{\partial} (\delta h \phi, \psi) = (\delta h D\phi, \psi) + (-1)^{\phi}(\delta h \phi, D\psi) + (\delta D \phi, \psi).$$

(39)

On the other hand, eq. (38) with $\phi$ replaced by $\delta h \phi$ gives

$$\overline{\partial} (\delta h \phi, \psi) = (D\delta h \phi, \psi) + (-1)^{\phi}(\delta h \phi, D\psi).$$

(40)

By comparing (39) and (40), we obtain that $\delta D = D\delta h - \delta h D$.

Since $D^2 = \overline{D}^2 = 0$, the curvature of $D + \overline{D}$ equals $\mathcal{F} = [\overline{D}, D]$ and we get $\delta \mathcal{F} = [\overline{D}, [D, \delta h]]$. Then, recalling that $[D, \mathcal{F}] = [\overline{D}, \mathcal{F}] = 0$, we obtain that

$$\delta \text{tr}\mathcal{F}^k = k \text{tr} \delta \mathcal{F} \mathcal{F}^{k-1} = k \text{tr} [\overline{D}, [D, \delta h]] \mathcal{F}^{k-1} = k \overline{\partial} \text{tr} \delta h \mathcal{F}^{k-1}.$$

This shows that $\delta \omega_k$ is $\overline{\partial} \partial$-exact and, thus, the closed form $\omega_k$ gives a well defined class in $H^{k,k}_{\text{BC}}(X)$. Note, that $a \text{ priori}$ the form $\omega_k$ has components $\omega_k = \sum_{p=0}^{k} \omega_k^{(p,p)}$, where $\omega_k^{(p,p)}$ is a $(p, p)$-form. However, the metric-rescaling argument in Subsection 4.2 (cf., Lemma 4.5) together with the present argument show that $\omega_k^{(p,p)}$ are $\overline{\partial} \partial$-exact for $p < k$.

As a result, we get the following theorem.

**Theorem 4.11** The Chern character of a $\overline{\partial}$-superconnection $M$ is well-defined as an element in Bott-Chern cohomology

$$\text{ch}_k(M) \in H^{k,k}_{\text{BC}}(X) \cap H^{2k}_{\text{dR}}(X, \mathbb{Q}).$$
Appendix

In this Appendix we will prove, for the reader’s convenience, that, on a smooth complex manifold \( X \) of dimension \( n \), the sheaf of rings \( \mathcal{A}_X^0 \) is flat over the sheaf of rings \( \mathcal{O}_X \) of holomorphic functions on \( X \), that is, the functor \( F \mapsto F \otimes_{\mathcal{O}_X} \mathcal{A}_X^0 \) is exact on the category of sheaves of \( \mathcal{O}_X \)-modules. For sheaves, flatness is only the property of local rings at each point of \( X \). For that reason, we pass to local rings. Let us choose an arbitrary point \( x \in X \) and denote \( \mathcal{O} = \mathcal{O}_{X,x} \) the local ring of holomorphic functions, \( \mathcal{C}^\infty = \mathcal{A}_X^0 \) the local ring of complex valued smooth functions, and \( \mathcal{C}^\omega \) the local ring of complex-valued real-analytic functions at \( x \). We prove below that \( \mathcal{C}^\infty \) is flat over its subring \( \mathcal{O} \subset \mathcal{C}^\infty \). By definition, this means that for any short exact sequence of \( \mathcal{O} \)-modules,

\[
0 \to M_1 \to M_2 \to M_3 \to 0,
\]

the corresponding sequence

\[
0 \to M_1 \otimes_{\mathcal{O}} \mathcal{C}^\infty \to M_2 \otimes_{\mathcal{O}} \mathcal{C}^\infty \to M_3 \otimes_{\mathcal{O}} \mathcal{C}^\infty \to 0
\]
is exact as well. Our proof will result from a recollection of results in literature, the most important and strongest part of which are due to Malgrange ([M], and in more detail below).

Actually we shall prove a stronger property of the pair \( \mathcal{O} \subset \mathcal{C}^\infty \). All the rings and ring homomorphisms below are assumed to be unital.

**Definition A.1** Let \( E \) be a module over a commutative ring \( A \). Then \( E \) is called faithfully flat if the following two conditions hold:

(i) \( E \) is \( A \)-flat;
(ii) for any \( A \)-module \( M \), the equality \( E \otimes_A M = 0 \) implies \( M = 0 \).

In other words, the functor \( M \mapsto E \otimes_A M \) on \( A \)-modules is (i) exact and (ii) faithful. The following useful properties of flat algebras can be found, for example, in [AM, Exercise 16 in Ch.3].

**Proposition A.2** Let \( \rho : A \to B \) be a homomorphism of rings. Suppose \( B \) is flat as an \( A \)-module (one says \( B \) is a flat \( A \)-algebra). Then, the following propositions are equivalent:

(i) \( B \) is faithfully flat over \( A \);
(ii) for any ideal \( a \subset A \), one has that \( \rho^{-1}(\rho(a)B) = a \);
(iii) the induced map \( \text{Spec} B \to \text{Spec} A \) is surjective;
(iv) for any maximal ideal \( m \subset A \), one has that \( \rho(m)B \neq B \);
(v) for any \( A \)-module \( M \), the map \( x \mapsto x \otimes 1 : M \to M \otimes_A B \) is injective.

Another convenient necessary and sufficient condition of faithful flatness is given by the following proposition taken from [M, §4 of Ch.III].

**Proposition A.3** Given a ring and a subring, \( A \subset B \), the ring \( B \) is faithfully flat over \( A \) if and only if the \( A \)-module \( B/A \) is flat.
Proof. Let us assume first that the \( A \)-module \( B/A \) is flat. It follows then from the short exact sequence

\[
0 \to A \to B \to B/A \to 0 \quad (41)
\]

that \( B \) is \( A \)-flat as well. Let us tensor the sequence (41) with an arbitrary \( A \)-module \( M \):

\[
0 \to M \to M \otimes_A B \to M \otimes_A B/A \to 0.
\]

Since \( B/A \) is flat, the sequence remains exact, which implies condition (ii) of Definition A.1 (cf. also (v) of Proposition A.2).

Let us now assume that \( B \) is a faithfully flat \( A \)-algebra. Choosing an arbitrary short exact sequence of \( A \)-modules and tensoring it with the sequence (41) we obtain the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
M_1 & \to & M_2 \\
\downarrow & & \downarrow \\
M_1 \otimes_A B & \to & M_2 \otimes_A B \\
\downarrow & & \downarrow \\
M_1 \otimes_A B/A & \to & M_2 \otimes_A B/A \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

The first line, \( 0 \to M_1 \to M_2 \to M_3 \to 0 \), is exact by the assumption made. The second line is exact, because we assume that \( B \) is \( A \)-flat. The columns are exact, which results from the faithful flatness of \( B \) and Proposition A.2(v). Altogether, this implies exactness of the third line, whence the \( A \)-flatness of \( B/A \). \( \square \)

The flatness obeys the transitivity property. Thus, for three nested rings \( A \subset B \subset C \), if \( B \) is \( A \)-flat and \( C \) is \( B \)-flat, then \( C \) is \( A \)-flat. Similar result holds for faithful flatness:

**Lemma A.4** If, for three rings \( A \subset B \subset C \), \( B \) is faithfully flat over \( A \), and \( C \) is faithfully flat over \( B \), then \( C \) is faithfully flat over \( A \).

**Proof.** Since the ordinary flatness is inherited by transitivity, the statement follows from A.2(iii). Alternatively, consider the short exact sequence of \( A \)-modules

\[
0 \to B/A \to C/A \to C/B \to 0.
\]

We know that \( B/A \) and \( C/B \) are \( A \)-flat. This implies that \( C/A \) is \( A \)-flat. \( \square \)

The following lemma is a slightly weakened version of \([M\text{, Proposition } 4.7]\).

**Lemma A.5** If, for three rings \( A \subset B \subset C \), \( C \) is faithfully flat over \( A \), and \( C \) is faithfully flat over \( B \), then \( B \) is faithfully flat over \( A \).
Proof. In the short exact sequence \((*)\), we know that \(C/A\) and \(C/B\) are \(A\)-flat. This implies the \(A\)-flatness of \(B/A\). \(\square\)

Our further argument will be based on the results of Malgrange [M] and, in the first place, on the following strong result [M, §4 of Ch.III, Theorem 1.1 and Corollary 1.12 of Ch.VI].

**Theorem A.6** The local ring of smooth functions \(C^\infty\) is faithfully flat over its subring \(C^\omega\), the local ring of real-analytic functions.

This theorem is actually formulated in [M] for real-valued functions. It can however be readily extended to complex-valued functions (see [AH2]). Let \(C^\infty_{\mathbb{R}}\) and \(C^\omega_{\mathbb{R}}\) denote the local rings of real-valued smooth and real-analytic functions respectively. The theorem of Malgrange asserts that \(C^\infty_{\mathbb{R}}\) is faithfully flat over \(C^\omega_{\mathbb{R}}\) and, thus, \(C^\infty_{\mathbb{R}}/C^\omega_{\mathbb{R}}\) is \(C^\omega_{\mathbb{R}}\)-flat. For any homomorphism of rings \(A \to B\) and any flat \(A\)-module \(M\), the \(B\)-module \(B \otimes_A M\) is \(B\)-flat. Therefore, the module \((C^\infty_{\mathbb{R}}/C^\omega_{\mathbb{R}}) \otimes_{C^\omega_{\mathbb{R}}} C^\omega\) is \(C^\omega\)-flat. On the other hand, we find that

\[
(C^\infty_{\mathbb{R}}/C^\omega_{\mathbb{R}}) \otimes_{C^\omega_{\mathbb{R}}} C^\omega = (C^\infty_{\mathbb{R}}/C^\omega_{\mathbb{R}}) \otimes_{C^\omega_{\mathbb{R}}} (C^\omega_{\mathbb{R}} \otimes_{C^\omega_{\mathbb{R}}} C) = (C^\infty_{\mathbb{R}}/C^\omega_{\mathbb{R}}) \otimes_{C^\omega_{\mathbb{R}}} C = C^\infty/\omega,
\]

whence \(C^\infty/\omega\) is \(C^\omega\)-flat, which shows that \(C^\infty\) is faithfully flat over \(C^\omega\).

In view of transitivity, it remains to prove that the local ring of real-analytic functions, \(C^\omega\), is faithfully flat over its subring of holomorphic functions, \(O\). Flatness and faithful flatness agree with completions of Noetherian rings. Namely (see [AM, Proposition 10.14], [B, Theorem 3(iii) in Ch.III, §3, n°4]):

**Proposition A.7** Let \(A\) be a Noetherian ring, choose any ideal \(a \subset A\), and denote \(\hat{A}\) the \(a\)-adic completion of \(A\). Then, \(\hat{A}\) is a flat \(A\)-algebra.

In the case of local Noetherian rings we have more (see [M, Theorem 4.9 in Ch.III] and [B, Ch.III, §3, n°5]).

**Proposition A.8** Let \(A\) be a local Noetherian ring with maximal ideal \(m \subset A\) and denote \(\hat{A}\) its \(m\)-adic completion. Then, \(\hat{A}\) is a faithfully flat \(A\)-algebra.

**Proof.** This follows from Propositions A.7 and A.2(iv). \(\square\)

The following proposition can be found in [AM, Exercise 12 in Ch.10].

**Proposition A.9** Let \(A\) be a Noetherian ring and \(B = A[[x_1, \ldots, x_n]]\) the \(A\)-algebra of formal power series in \(n\) variables. Then \(B\) is faithfully flat over \(A\).

**Proof.** The ring \(B\) can be viewed as an \(a\)-adic completion of the ring of polynomials \(A[x_1, \ldots, x_n]\), where \(a = (x_1, \ldots, x_n)\) is the ideal of polynomials with zero a constant term. The flatness of \(B\) over \(A\) is then implied by Proposition A.7. The faithful flatness follows now by Proposition A.2(iv). \(\square\)

We need some properties of the rings \(O\) and \(C^\omega\). Both are local Noetherian rings [M, Theorem 3.8 of Ch.III]. Let now \(\hat{O} = \mathbb{C}[[z_1, \ldots, z_n]]\) be the ring of formal power series in \(n\) variables (the completion of the local ring \(O\)) and \(\hat{C}^\omega = \mathbb{C}[[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]]\) the ring of formal power series in \(2n\) variables (the completion of the local ring \(C^\omega\)). Both rings, \(\hat{O}\) and \(\hat{C}^\omega\), are local Noetherian rings as well. Moreover,
Proposition A.10  The following properties hold:
(i) $\hat{O}$ is faithfully flat over $\mathcal{O}$;
(ii) $\hat{C}^\omega$ is faithfully flat over $\mathcal{C}\omega$;
(iii) $\hat{C}^\omega$ is faithfully flat over $\hat{O}$.

Proof. (i) and (ii) follow by A.8, while (iii) follows from the isomorphism $\hat{C}^\omega \simeq \hat{O}[\bar{z}_1, \ldots, \bar{z}_n]$ and A.9. □

Proposition A.11  The ring $\mathcal{C}\omega$ is faithfully flat over its subring $\mathcal{O}$.

Proof. The embeddings of rings $\mathcal{O} \subset \hat{O} \subset \hat{C}^\omega$ show that $\hat{C}^\omega$ is faithfully flat over $\mathcal{O}$ by Proposition A.10(i) and (iii), and by transitivity A.4. The result follows then from the embeddings $\mathcal{O} \subset \mathcal{C}^\omega \subset \mathcal{C}\omega$ by Proposition A.10(ii) and Lemma A.5. □

The last proposition together with the theorem of Malgrange (Proposition A.6) implies now by the transitivity argument A.4:

Theorem A.12  The ring $\mathcal{C}\infty$ is faithfully flat over its subring $\mathcal{O}$.

Corollary A.13  On a complex analytic manifold $X$, the sheaf $\mathcal{A}_X^0$ of rings of smooth functions is faithfully flat over the sheaf $\mathcal{O}_X$ of rings of holomorphic functions.

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