Nash equilibrium for risk-averse investors in a market impact game with transient price impact

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Abstract

We consider a market impact game for n risk-averse agents that are competing in a market model with linear transient price impact and additional transaction costs. For both finite and infinite time horizons, the agents aim to minimize a mean-variance functional of their costs or to maximize the expected exponential utility of their revenues. We give explicit representations for corresponding Nash equilibria and prove uniqueness in the case of mean-variance optimization. A qualitative analysis of these Nash equilibria is conducted by means of numerical analysis.

Keywords: Market impact game, high-frequency trading, Nash equilibrium, transient price impact, market impact, predatory trading

1 Introduction

In a market impact game, financial agents compete with each other in a market framework where each trade creates price impact. Early papers on this subject, such as Brunnermeier and Pedersen [6], Carlin et al. [7], and Schöneborn and Schied [20], consider risk-neutral agents that are active in a linear Almgren–Chriss market impact model. Already in this relatively simple setup, interesting effects appear, such as transitions in the predatory or cooperative behavior of agents. Extensions to risk-averse agents in the Almgren–Chriss framework were given, e.g., in [8, 17].

In this paper, we consider risk-averse agents that are active in a discrete-time model with linear transient price impact. For single-agent optimization problems, such price-impact models were introduced by Obizhaeva and Wang [14] and later further developed, e.g., in [2, 10]. A market impact game with two risk-neutral agents was first considered by Schöneborn [19], who observed that equilibrium strategies may exhibit strong oscillations between buy and sell trades if trading speed is sufficiently high. This situation was further investigated in [16, 18], where the model was also enhanced by introducing additional quadratic transaction costs, whose strength is parameterized by a number θ ≥ 0. It was shown in particular that there exists an explicitly given critical value θ∗ > 0 such that the equilibrium strategies show at least some oscillations for θ < θ∗, whereas all oscillations disappear for θ ≥ θ∗.

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In [18], only two competing risk-neutral agents are considered. The main goal of the present paper is to extend the results and observations from [18] to a more flexible setting, in which an arbitrary (but finite) number of agents optimize their strategies under risk aversion. More precisely, the agents either minimize a mean-variance functional of the trading costs over deterministic strategies or they maximize the expected CARA utility of their revenues over adaptive strategies. We show that both problems admit an identical Nash equilibrium, which is given in explicit form and which is unique in the case of mean-variance optimization. Then we use numerical analysis of the equilibrium strategies to determine numerically the critical threshold for the transaction costs above which all oscillations cease.

If agents exhibit strictly positive risk aversion, it is possible to study the market impact game with an infinite time horizon. This question is interesting when one does not want to impose an externally given time horizon and instead aims at an intrinsic derivation of a trading horizon. Show that such an infinite-horizon market impact game admits a Nash equilibrium in case $\theta$ is equal to the critical value $\theta^*$, which was determined numerically for finite time horizons. If $\theta \neq \theta^*$, a Nash equilibrium may not exist.

This paper is organized as follows. In Section 2.1, we present the setup and state all results for a finite time horizon. Section 2.2 contains our discussion of the market impact game with infinite time horizon. All proofs are given in Section 3.

## 2 Main results

### 2.1 Finite time horizon

We consider an $n$-agent extension of the discrete-time market impact model with linear transient price impact that was studied, e.g., in [11, 14, 18, 19]. This model is sometimes also called the discrete-time linear propagator model, and we refer to [11] for a discussion and further background.

Suppose that $n$ financial agents are active in a market impact model for one risky asset. As commonly assumed in the market impact literature, the unaffected price process $S^0 = (S^0_t)_{t \geq 0}$ will be a square-integrable and right-continuous martingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. An important special case will be the Bachelier model of the form

$$S^0_t = S_0 + \sigma B_t, \quad t \geq 0,$$

for constants $S_0, \sigma > 0$ and a standard Brownian motion $B$. All agents trade at a finite number of times $0 \leq t_0 < t_1 < \cdots < t_N$. The trading strategy of agent $i$ will be a vector $\xi_i = (\xi_{i,0}, \ldots, \xi_{i,N})^\top$ where $\xi_{i,k}$ represents the number of shares sold at time $t_k$. That is, $\xi_{i,k} > 0$ represents a sell order and $\xi_{i,k} < 0$ means a buy order. The matrix of all strategies is denoted by $\Xi = [\xi_1, \ldots, \xi_n]$.

When all the agents apply their strategies, the asset price is given by

$$S^\Xi_t = S^0_t - \sum_{t_k < t} \left[ G(t - t_k) \sum_{i=1}^n \xi_{i,k} \right],$$

where $G : \mathbb{R}_+ \to \mathbb{R}_+$ is called the decay kernel. The quantity $G(t - t_k)$ describes the time-$t$ price impact of a unit transaction made at time $t_k \leq t$. When agent $j$ first places an order $\xi_{j,k} > 0$ at time $t_k$, the asset price is moved linearly from $S^\Xi_{t_k}$ to $S^\Xi_{t_k^+} := S^\Xi_{t_k} - G(0)\xi_{j,k}$. The liquidation cost for agent $j$ is thus:

$$-\frac{1}{2} (S^\Xi_{t_k^+} + S^\Xi_{t_k})\xi_{j,k} = \frac{G(0)}{2} \xi_{j,k}^2 - S^\Xi_{t_k} \xi_{j,k}.$$

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Suppose that immediately after agent $j$, another agent $i$ places an order $\xi_{i,k} > 0$. The liquidation cost for agent $i$ is the following:

$$-rac{1}{2}(S^\xi_{t_k+} + S^\xi_{t_k-} - G(0)\xi_{i,k})\xi_{i,k} = \frac{G(0)}{2}\xi^2_{i,k} - S^\xi_{t_k}\xi_{i,k} + G(0)\xi_{j,k}\xi_{i,k},$$

where $G(0)\xi_{j,k}\xi_{i,k}$ is an additional cost term due to the latency in execution time. On average, fifty percent of times, the order of agent $j$ will be executed before the order of agent $i$. The latency costs for agent $i$ at time $t_k$ will thus be of the form

$$\frac{1}{2}G(0)\sum_{j \neq i} \xi_{i,k}\xi_{j,k}.$$

In addition to the execution costs described above, we follow [16, 18] in assuming quadratic transaction costs $\theta\xi^2_{i,k}$ with $\theta \geq 0$. One of our main goals will be to analyze the qualitative effects these transaction costs will have on optimal strategies. Such quadratic transaction costs are often used to model "slippage" arising from temporary price impact; see [3, 5] and [10, Section 2.2]. Moreover, one can argue as in Proposition 2.6 of [18] to see that our quadratic transaction cost function can be replaced by proportional transaction costs in a neighborhood of the origin without affecting the Nash equilibrium we are going to derive. Since the main difference of quadratic and proportional transaction costs is their behavior at the origin, one may therefore guess that similar results as obtained in the following sections for quadratic transaction costs might also hold for proportional transaction costs. We have thus motivated the following definition.

**Definition 2.1.** Given a time grid $T = \{t_0, t_1, \ldots, t_N\}$, the execution costs of the strategy $\xi_i$ given all other strategies $\xi_j$ with $j \neq i$ are defined as

$$\mathcal{C}_T(\xi_i | \xi_{-i}) = \sum_{k=0}^{N} \left[ \frac{G(0)}{2}\xi^2_{i,k} - S^\xi_{t_k}\xi_{i,k} + \frac{G(0)}{2} \sum_{j \neq i} \xi_{i,k}\xi_{j,k} + \theta\xi^2_{i,k} \right],$$

(3)

where $\xi_{-i} = [\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n]$. In the sequel, we will suppose that agent $i$ has an initial position of $X_i \in \mathbb{R}$ shares and is constraint to hold a zero terminal position by the end of the trading day. It is often assumed [14, 18] that agents aim to minimize the expected costs over the following class of strategies,

$$\mathcal{X}(X_i, T) = \left\{ \xi = (\xi_0, \ldots, \xi_N) \mid \xi_i \text{ is } \mathcal{F}_{t_i}\text{-measurable and bounded and } \sum_{i=0}^{N} \xi_i = X_i \right\}.$$

In practice, however, it is also popular to incorporate the agents’ risk aversion and to optimize the following mean-variance functional of the trading costs,

$$\text{MV}_{\gamma}(\xi_i | \xi_{-i}) = \mathbb{E}[\mathcal{C}_T(\xi_i | \xi_{-i})] + \frac{\gamma}{2} \text{Var}[\mathcal{C}_T(\xi_i | \xi_{-i})].$$

(4)

Here, $\gamma \geq 0$ is a risk-aversion parameter. For $\gamma > 0$, the mean-variance functional (4) is typically only time-consistent if strategies are deterministic; see, e.g., [3, 13]. Therefore, its minimization is typically restricted to the class of deterministic strategies in $\mathcal{X}(X_i, T)$, which we denote by

$$\mathcal{X}_{\text{det}}(X_i, T) = \left\{ \xi \in \mathcal{X}(X_i, T) \mid \xi \text{ is deterministic} \right\} = \left\{ \xi \in \mathbb{R}^{|T|} \mid 1^\top \xi = X_i \right\}.$$
for \( \mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^{N+1} \). It can also make sense to maximize the expected utility of the revenues, which are nothing else than the negative costs. Here, we will use the following utility functional,

\[
U_\gamma(\xi_i|\xi_{-i}) := \mathbb{E}[u_\gamma(-\mathcal{C}_T(\xi_i|\xi_{-i}))],
\]

where \( u_\gamma(x) \) is the following exponential – or CARA – utility function,

\[
u_\gamma(x) = \begin{cases} 
\frac{1}{\gamma}(1 - e^{-\gamma x}) & \text{if } \gamma > 0, \\
-x & \text{if } \gamma = 0.
\end{cases}
\]

Due to the time consistency of the expected utility functional, we can consider its maximization over all adapted strategies from the class \( \mathcal{X}(X_i, T) \). Moreover, as, e.g., in \([7, 16, 17, 18, 21]\), we assume henceforth that each agent has full information about the strategies used by the other agents.

**Definition 2.2.** Suppose there are \( n \) agents with initial inventories \( X_1, \ldots, X_n \in \mathbb{R} \) and risk aversion parameter \( \gamma \geq 0 \) and that \( T := \{t_0, t_1, \ldots, t_N\} \) is a fixed time grid.

(a) A *Nash equilibrium for mean-variance optimization* is a collection of strategies \( (\xi_i^*, \ldots, \xi_n^*) \in \mathcal{X}_{\text{det}}(X_1, T) \times \cdots \times \mathcal{X}_{\text{det}}(X_n, T) \) such that each \( \xi_i^* \) minimizes the mean-variance functional \( \text{MV}_\gamma(\xi|\xi_{-i}) \) over \( \xi \in \mathcal{X}_{\text{det}}(X_i, T) \).

(b) A *Nash equilibrium for CARA utility maximization* is a collection of strategies \( (\xi_i^*, \ldots, \xi_n^*) \in \mathcal{X}(X_1, T) \times \cdots \times \mathcal{X}(X_n, T) \) such that each \( \xi_i^* \) maximizes the CARA utility functional \( U_\gamma(\xi|\xi_{-i}) \) over \( \xi \in \mathcal{X}(X_i, T) \).

In the preceding definition, we have assumed that all agents share the same risk aversion parameter \( \gamma \geq 0 \). The case in which the agents have different risk aversion parameters is a straightforward but tedious extension of the current model. It will significantly complicate the notation while not providing significant additional insights. For this reason, we will only consider the case of identical risk aversion parameters. Now let

\[
\varphi(t) := \text{Var}(S_t^0) \quad \text{for } t \geq 0.
\]

We define for \( \theta, \gamma \geq 0 \),

\[
\Gamma_{ij}^{\gamma, \theta} = G(|t_i - t_j|) + \gamma \varphi(t_{i \wedge t_j}) + 2\theta \delta_{ij}, \quad \text{for } i, j = 0, 1, \ldots, N,
\]

where \( \delta_{ij} \) is the Kronecker delta. Then we define

\[
\tilde{\Gamma}_{ij} = \begin{cases} 
0 & \text{if } i < j, \\
\frac{1}{2} \Gamma_{ij}^{0.0} & \text{if } i = j, \\
\Gamma_{ij}^{0.0} & \text{if } i > j.
\end{cases}
\]

Note that \( \Gamma_{ij}^{0.0} = \Gamma + \Gamma^T \). We further define

\[
v = \frac{1}{1^T[\Gamma_{ij}^{\gamma, \theta} + (n - 1)\tilde{\Gamma}]^{-1}1} \left[\Gamma_{ij}^{\gamma, \theta} + (n - 1)\tilde{\Gamma}\right]^{-1}1, \quad \text{(7)}
\]

\[
w = \frac{1}{1^T[\Gamma_{ij}^{\gamma, \theta} - \tilde{\Gamma}]^{-1}1} \left[\Gamma_{ij}^{\gamma, \theta} - \tilde{\Gamma}\right]^{-1}1.
\]

Recall that a function \( g : \mathbb{R} \to \mathbb{R} \) is called strictly positive definite (in the sense of Bochner) if for all \( n \in \mathbb{N} \) and \( s_1, \ldots, s_n \in \mathbb{R} \), the matrix \( (g(s_i - s_j))_{i,j=1,\ldots,n} \) is positive definite.

**Assumption 2.3.** We henceforth assume that the function \( \mathbb{R} \ni x \mapsto G(|x|) \) is strictly positive definite.
Assumption 2.3 is satisfied as soon as $G$ is convex, nonincreasing, and nonconstant; see, e.g., [2, Proposition 2] for a short proof. It implies that the matrix $\Gamma^{0,0}$ is positive definite for all time grids $T$. As observed in [2], Assumption 2.3 also rules out the existence of price manipulation strategies in the sense of Huberman and Stanzl [12]. Now we can state our first result on the existence and uniqueness of a Nash equilibrium. It extends Theorem 2.5 from [18], where the case $n = 2$ and $\gamma = 0$ was treated.

**Theorem 2.4.** Suppose Assumption 2.3 holds. Then, for any time grid $T$, parameters $\theta, \gamma \geq 0$, initial inventories $X_1, \ldots, X_n \in \mathbb{R}$, and $\bar{X} = \frac{1}{n} \sum_{j=1}^{n} X_j$, the strategies

$$
\xi^*_i = \bar{X} v + (X_i - \bar{X}) w, \quad i = 1, \ldots, n.
$$

(9)

form the unique Nash equilibrium for mean-variance optimization. If, moreover, $S^0$ is a Bachelier model of the form (1), then the strategies (9) also form a Nash equilibrium for CARA utility maximization.

**Remark 2.5.** Note that the Nash equilibrium for mean-variance optimization is unique, but that we do not know whether the Nash equilibrium for CARA utility maximization is also unique. This has to do with the larger class of adapted strategies that is admitted for CARA utility maximization. However, it follows easily from the first part of Theorem 2.4 that the strategies (9) form a unique Nash equilibrium for CARA utility maximization when $S^0$ is a Bachelier model and all agents are restricted to use deterministic strategies.

It follows from Theorem 2.4 that in the following two special cases the Nash equilibrium has a particularly simple structure:

- if $X_1 = \cdots = X_n$, then $\xi^*_i = X_1 v$ for $i = 1, \ldots, n$;
- if $X_1 + \cdots + X_n = 0$, then $\xi^*_i = X_i w$ for $i = 1, \ldots, n$.

It was shown in Corollary 1 of [2] that, for convex and nonincreasing $G$ and convex $\phi$, single-agent strategies ($n = 1$) are always buy-only or sell-only. On the other hand, Schöneborn [19] observed that for $G(t) = e^{-t}$, $n = 2$, $\gamma = 0$, and $\theta = 0$ the equilibrium strategies oscillate between buy and sell orders. These oscillations are thus a genuine effect of the interaction between the two agents. This effect was explained in [16, 18, 19] as a result of the need for protection against predatory trading by the competitor. These oscillations also have a similarity to the “hot-potato” game between high-frequency traders during the flash crash of May 10, 2010 (see [9, p. 2]). In [18], the influence of $\theta$ on the oscillations of the equilibrium strategies was analyzed for $n = 2$ and $\gamma = 0$. It was found that there exists a critical level $\theta^*$ such that the equilibrium strategies show at least some oscillations for $\theta < \theta^*$, whereas all oscillations disappear for $\theta \geq \theta^*$. In [18], the critical level $\theta^*$ was identified as $\frac{1}{4} G(0)$. Here, our goal is to analyze numerically the influence of the number $n$ of agents and the level $\gamma$ of their risk aversion on the value of $\theta^*$. To this end, we make the following assumptions for the remainder of this section.

**Assumption 2.6.** Throughout the remainder of Section 2.1, we make the following assumptions.

(i) We have $T = 1$ and the time grid is equidistant: $T_N := \{ \frac{k}{N} | k = 0, 1, \ldots, N \}$ for $N \in \mathbb{N}$.

(ii) $G$ is of the form $G(t) = e^{-t}$.

(iii) $S^0$ is a Bachelier model of the form (9) with $\sigma = 1$.

In Figure 1 we observe that increasing the risk aversion $\gamma$ does not stop the oscillations in the vector $v$. On the contrary, increasing $\gamma$ actually magnifies the oscillations during the early trading periods. Increasing the level $\theta$ of transaction costs, however, will clearly diminish the size of oscillations. For
fixed $n$, $N$, and $\gamma$, we can therefore look for that level $\theta_v = \theta_v(n, N, \gamma)$ at which $\min_i v_i$ becomes nonnegative. In Figure 2, we provide numerical surface plots for the function $(N, \gamma) \mapsto \theta_v(n, N, \gamma)$ with $n = 2$ and $n = 5$. Together with additional simulations carried out by the authors, Figures 2 and 3 suggest that for each $n$ there is a critical level at which all oscillations of $v$ cease and that it is given by

$$\theta_v^*(n) = \sup_{N, \gamma} \theta_v(n, N, \gamma) = \frac{n - 1}{4}.$$ 

This conjecture is consistent with the theoretical results obtained in [2] and [18] for $n = 1$ and $n = 2$, respectively.

Figure 1: Vector $v$ with $\gamma = 0$ (left), $\gamma = 1$ (middle) and $\gamma = 3$ (right) for $n = 2$, $N = 100$, and $\theta = 0$.

Figure 2: Surface plots of $\theta_v(n, N, \gamma)$ with $n = 2$ (left) and $n = 5$ (right) with respect to $N$ and the risk aversion parameter $\gamma$. 
Figure 3: Surface plot (left) and level curves (right) of $\theta_v(n, N, \gamma)$ with respect to the number of players $n$ and the risk aversion parameter $\gamma$ for $N = 500$.

Now we turn to vector $w$. The first observation is that $w$ is independent of the number $n$ of agents. Thus, the critical level $\theta_w$ at which oscillations cease is a function of $N$ and $\gamma$ only. For $\gamma = 0$, $w$ must be identical to the one studied in [16, 18], and it follows from Theorem 2.7 of [18] that the critical transaction cost level in this case is $\frac{1}{4}$. Moreover, it can be seen from Figures 4 and 5 that increasing the risk aversion $\gamma$ does have a diminishing effect on the oscillations of $w$. Therefore, we conjecture that

$$\theta_w^* = \sup_{N, \gamma} \theta_w(N, \gamma) = \frac{1}{4}.$$

This conjecture is also supported by Figure 6 and consistent with Theorem 2.7 of [18]. It is interesting to note that for $\gamma = 0$ and $\theta = \frac{1}{4}$, the vector $w$ has a particularly simple structure. This is stated in the following theoretical result.

**Proposition 2.7.** Under Assumption 2.6, for $\theta = \frac{1}{4}$ and $\gamma = 0$,

$$w_0 = \cdots = w_{N-1} = \frac{1 - e^{-1/N}}{N(1 - e^{-1/N}) + 1}$$

and

$$w_N = \frac{1}{N(1 - e^{-1/N}) + 1}.$$
2.2 Infinite time horizon

For non-vanishing risk aversion $\gamma > 0$, it is possible to study our problem also for an infinite time horizon. The intuitive reason is that any risk-averse investor will automatically try to liquidate any position held in an asset whose price process is a martingale.

**Assumption 2.8.** Throughout Section 2.2, we make the following assumptions.

(i) The time grid is $\mathbb{N}_0 = \{0, 1, \ldots\}$.

(ii) $G$ is of the form $G(t) = e^{-\rho t}$ for some $\rho > 0$.

(iii) $S^0$ is a Bachelier model of the form $[9]$. 
Under Assumption 2.8 (i), the strategy of an agent \( i \) with initial position \( X_i \in \mathbb{R} \) will be represented by a sequence \( \xi = (\xi_0, \xi_1, \ldots) \) of random variables such that each \( \xi_i \) is \( \mathcal{F}_i \)-measurable, the random variable \( \xi \) takes values in \( \ell^1 \) and is bounded in \( \ell^\infty \), and \( \sum_{k=0}^{\infty} \xi_k(\omega) = X_i \) for each \( \omega \in \Omega \). The set of all these strategies will be denoted by \( \mathcal{X}(X_i, N_0) \). Again, the class of all deterministic strategies in \( \mathcal{X}(X_i, N_0) \) will be denoted by \( \mathcal{X}_{\text{det}}(X_i, N_0) \). Since \( \ell^1 \subset \ell^2 \), it is clear that (3) can be extended as follows to strategies \( \xi_i \in \mathcal{X}(X_i, N_0), i = 1, \ldots, n \),

\[
\mathcal{C}_{N_0}(\xi_i | \xi_{-i}) = \sum_{k=0}^{\infty} \left[ \frac{G(0)}{2} \xi_{i,k}^2 - S_{i,k}^\xi \xi_{i,k} + \frac{G'(0)}{2} \sum_{j \neq i} \xi_{j,k} + \theta \xi_{i,k}^2 \right].
\]

Again, each agent will aim to minimize the following mean-variance functional,

\[
\text{MV}_\gamma(\xi_i | \xi_{-i}) = \mathbb{E}[\mathcal{C}_{N_0}(\xi_i | \xi_{-i})] + \frac{\gamma}{2} \text{Var}[\mathcal{C}_{N_0}(\xi_i | \xi_{-i})], \quad \xi_i \in \mathcal{X}_{\text{det}}(X_i, N_0),
\]
or to maximize the CARA utility functional

\[
U_\gamma(\xi_i | \xi_{-i}) = \mathbb{E}[u_\gamma(-\mathcal{C}_{N_0}(\xi_i | \xi_{-i}))], \quad \xi_i \in \mathcal{X}(X_i, N_0).
\]

The notion of Nash equilibria for mean-variance optimization and CARA utility maximization can be defined in exactly the same way as in Definition 2.2. However, it is not clear \textit{a priori} whether the formulas (7) and (8) for \( v \) and \( w \) can also be extended to an infinite time horizon, because it is no longer clear whether the vector \( \mathbf{1} \) belongs to the range of the linear operators \( \Gamma^\gamma,\theta + (n-1)\tilde{\Gamma} \) and \( \Gamma^\gamma,\theta - \tilde{\Gamma} \). The following result states the existence of an infinite-horizon Nash equilibrium in a specific situation.

**Theorem 2.9.** In addition to Assumption 2.8, suppose that \( \gamma > 0 \) and

\[
\theta = \theta^* := \frac{n-1}{4}.
\]

Then there exist unique positive solution \( \alpha \) and \( \beta \) of the two equations

\[
0 = \frac{1}{e^{(\alpha+\rho)} - 1} - \frac{n}{e^{(\alpha-\rho)} - 1} - \frac{\gamma \sigma^2 e^{-\alpha}}{(1-e^{-\alpha})^2}, \quad (11)
\]

\[
0 = 2\theta + \frac{1}{2} + \frac{1}{e^{(\beta+\rho)} - 1} - \frac{\gamma \sigma^2 e^{-\beta}}{(1-e^{-\beta})^2}, \quad (12)
\]

Moreover, \( \alpha \in (0, \rho) \). For these, we define \( v \in \ell^1 \) through

\[
v_0 = \frac{e^\alpha - 1}{e^\alpha - e^{\alpha-\rho}} \quad \text{and, for } i = 1, 2, \ldots, \quad v_i = \frac{e^{-\alpha i}}{1 - e^{-\alpha i}} + \frac{1}{1 - e^{\alpha i - \rho}}
\]

and \( w \in \ell^1 \) through

\[
w_i = (1 - e^{-\beta}) e^{-\beta i}.
\]

Then, for any initial positions \( X_1, \ldots, X_n \), the strategies

\[
\xi_i^* = \bar{X} v + (X_i - \bar{X}) w, \quad i = 1, \ldots, n.
\]

form a Nash equilibrium for mean-variance optimization in \( \mathcal{X}_{\text{det}}(X_1, N_0) \times \cdots \times \mathcal{X}_{\text{det}}(X_n, N_0) \) and a Nash equilibrium for CARA utility maximization in \( \mathcal{X}(X_1, N_0) \times \cdots \times \mathcal{X}(X_n, N_0) \).
Remark 2.10. Consider first the case in which $X_1 + \cdots + X_n = 0$. Then it will follow from the proof of Theorem 2.9 that the formula $\xi^*_i = X_i w$ provides a Nash equilibrium for every choice of $\theta \geq 0$. In the case $X_1 = \cdots = X_n$, however, our proof yields that there will not exist a Nash equilibrium whose strategies decay exponentially in time unless the condition $\theta = \theta^* := \frac{n-1}{4}$ is satisfied. As a matter of fact, we conjecture that no Nash equilibrium exists unless (10) or $X_1 + \cdots + X_n = 0$ holds. The situation is very similar to the one of Theorem 4.5 in [16], where a continuous-time version of the game for $n = 2$ and $\gamma = 0$ was analyzed. It was shown there that a continuous-time Nash equilibrium can exist only if $\theta = \theta^*$ or $X_1 = X_2 = 0$.

The qualitative behavior of the respective solutions $\alpha$ and $\beta$ of (11) and (12) is plotted in Figures 7 and 8. In case $n = 1$, we have the following explicit result.

**Proposition 2.11.** If $n = 1$, then the solution $\alpha$ of (11) is given by

$$
\alpha = \cosh^{-1}\left[\frac{\gamma\sigma^2 \cosh(\rho) + 2 \sinh(\rho)}{\gamma\sigma^2 + 2 \sinh(\rho)}\right].
$$

(14)
3 Proofs

Lemma 3.1. An admissible strategy $\xi_i \in \mathcal{X}_{\text{det}}(X_i, T)$ given all the competitors’ strategies $\xi_j \in \mathcal{X}_{\text{det}}(X_j, T)$ with $j \neq i$ has the following mean-variance functional:

$$\text{MV}_\gamma(\xi_i|\xi_{-i}) = -X_iS_0^0 + \frac{1}{2} \xi_i^\top \Gamma^0.0 \xi_i + \xi_i^\top \Gamma \sum_{j \neq i} \xi_j. \quad (15)$$

Proof. Since all strategies are deterministic,

$$\mathbb{E}[\mathcal{C}_T(\xi_i|\xi_{-i})] = \sum_{k=0}^{N} \left[ \frac{G(0)}{2} \xi_{i,k}^2 - \mathbb{E}[S_{t_k}^\xi \xi_{i,k}] + \frac{G(0)}{2} \sum_{j \neq i} \xi_{i,k} \xi_{j,k} + \theta \xi_{i,k}^2 \right].$$

Since $S^0$ is a martingale,

$$\sum_{k=0}^{N} \mathbb{E}[S_{t_k}^\xi \xi_{i,k}] = \sum_{k=0}^{N} \xi_{i,k} \mathbb{E}[S_{t_k}^0] - \sum_{k=0}^{N} \xi_{i,k} \sum_{m=0}^{k-1} \left( G(t_k - t_m) \sum_{j=1}^{n} \xi_{j,m} \right)$$

$$= X_iS_0^0 - \sum_{k=0}^{N} \xi_{i,k} \sum_{m=0}^{k-1} \xi_{i,m} G(t_k - t_m)$$

$$- \sum_{k=0}^{N} \xi_{i,k} \sum_{m=0}^{k-1} G(t_k - t_m) \sum_{j \neq i} \xi_{j,m}.$$ 

Moreover, using matrix notation,

$$\frac{G(0) + 2\theta}{2} \sum_{k=0}^{N} \xi_{i,k}^2 + \sum_{k=0}^{N} \xi_{i,k} \sum_{m=0}^{k-1} \xi_{i,m} G(t_k - t_m) = \frac{1}{2} \left[ 2\theta \sum_{k=0}^{N} \xi_{i,k}^2 + \sum_{k=0}^{N} \xi_{i,k} \xi_{i,m} G(|t_k - t_m|) \right]$$

$$= \frac{1}{2} \xi_i^\top \Gamma^0.0 \xi_i.$$ 

and

$$\sum_{k=0}^{N} \xi_{i,k} \left[ \frac{G(0)}{2} \sum_{j \neq i} \xi_{j,k} + \sum_{m=0}^{k-1} G(t_k - t_m) \sum_{j \neq i} \xi_{j,m} \xi_{j,m} \right] = \xi_i^\top \Gamma \sum_{j \neq i} \xi_j.$$ 

Using again that $\xi_i$ are deterministic and the martingale property of $S^0$,

$$\text{Var}[\mathcal{C}_T(\xi_i|\xi_{-i})] = \text{Var}\left[ \sum_{k=0}^{N} S_{t_k}^\xi \xi_{i,k} \right] = \text{Var}\left[ \sum_{k=0}^{N} S_{t_k}^0 \xi_{i,k} \right] = \sum_{p,q=0}^{N} \xi_{p,q} \text{Cov}(S_{t_p}^0, S_{t_q}^0) = \sum_{p,q=0}^{N} \xi_{p,q} \varphi(t_p \land t_q).$$ 

By substituting the preceding results into (4), we obtain the desired formula:

$$\text{MV}_\gamma(\xi_i|\xi_{-i}) = \mathbb{E}[\mathcal{C}_T(\xi_i|\xi_{-i})] + \frac{\gamma}{2} \text{Var}[\mathcal{C}_T(\xi_i|\xi_{-i})]$$

$$= -X_iS_0^0 + \frac{1}{2} \xi_i^\top \Gamma^0.0 \xi_i + \xi_i^\top \Gamma \sum_{j \neq i} \xi_j + \frac{\gamma}{2} \sum_{p,q=0}^{N} \xi_{p,q} \varphi(t_p \land t_q)$$

$$= -X_iS_0^0 + \frac{1}{2} \xi_i^\top \Gamma^0.0 \xi_i + \xi_i^\top \Gamma \sum_{j \neq i} \xi_j.$$ 

$\square$
We will use the convention of saying that an \( n \times n \)-matrix \( A \) is *positive* if \( x^\top Ax > 0 \) for all nonzero \( x \in \mathbb{R}^n \), which makes sense also if \( A \) is not necessarily symmetric. Clearly, for a positive matrix \( A \) there is no nonzero \( x \in \mathbb{R}^n \) for which \( Ax = 0 \), and so \( A \) is invertible.

**Lemma 3.2.** For all \( \gamma, \theta \geq 0 \) and \( n \geq 1 \), the matrices \( \Gamma^{\gamma, \theta} \), \( \widetilde{\Gamma} \), \( \Gamma^{\gamma, \theta} - \widetilde{\Gamma} \) and \( \Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma} \) are positive.

*Proof.* By Lemma 3.2 in [18], the matrices \( \Gamma \), \( \Gamma^{0, \theta} \), \( \widetilde{\Gamma} \), and \( \Gamma^{0, \theta} - \widetilde{\Gamma} \) are positive. Since \( C := (\varphi(t_i \wedge t_j))_{i,j=1,\ldots,N} \) is the covariance matrix of the random variables, \( S^0_{t_i}, \ldots, S^0_{t_N} \), it is nonnegative definite. It follows that \( \Gamma^{\gamma, \theta} = \Gamma^{0, \theta} + \gamma C \) and \( \Gamma^{\gamma, \theta} - \widetilde{\Gamma} = (\Gamma^{0, \theta} - \widetilde{\Gamma}) + \gamma C \) are positive as well. Hence, \( \Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma} \) is also positive. \( \Box \)

**Lemma 3.3.** For a given time grid \( \mathbb{T} \) and initial values \( X_1, \ldots, X_n \in \mathbb{R} \), there exists at most one Nash equilibrium for mean-variance optimization.

*Proof.* The proof is similar to the one of Lemma 3.3 in [18] or Lemma 4 in [17] and hence omitted. \( \Box \)

Now we are ready to prove Theorem 2.4.

**Proof of Theorem 2.4.** By Definition 2.2 and Lemma 3.1, we have the following linear-quadratic optimization problem: for all \( i = 1, \ldots, n \),

\[
\begin{align*}
\text{MV}_\gamma(\xi^*_i | \xi^*_i) &= -X_i S^0_0 + \min_{\xi_i \in \mathcal{P}_{\det}(X_i, \mathbb{T})} \left( \frac{1}{2} \xi^\top_i \Gamma^{\gamma, \theta} \xi_i + \xi^\top_i \widetilde{\Gamma} \sum_{j \neq i} \xi_j \right).
\end{align*}
\]

The constraint \( \xi_i \in \mathcal{P}_{\det}(X_i, \mathbb{T}) \) can be re-written as the linear equality constraint \( 1^\top \xi_i = X_i \).

To solve this problem, we use the Lagrange multiplier theorem [4, pp. 276-283] to obtain \( \alpha_i \in \mathbb{R} \) for \( i = 1, \ldots, n \) such that the optimal strategies satisfy the following necessary conditions:

\[
\begin{align*}
\Gamma^{\gamma, \theta} \xi^*_i + \widetilde{\Gamma} \sum_{j \neq i} \xi^*_j &= \alpha_i 1, \\
1^\top \xi^*_i &= X_i.
\end{align*}
\] (16)

We will show below that these equations are also sufficient for our optimization problem. Summing over \( i \) in the first line of (16) yields

\[
[\Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma}] \sum_{j=1}^n \xi^*_j = \sum_{j=1}^n \alpha_j 1.
\]

By Lemma 3.2 \( \Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma} \) is an invertible matrix. Thus,

\[
\sum_{j=1}^n \xi^*_j = \sum_{j=1}^n \alpha_j [\Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma}]^{-1} 1
\]

\[
= \frac{1^\top \sum_{j=1}^n \alpha_j [\Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma}]^{-1} 1}{1^\top [\Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma}]^{-1} 1} [\Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma}]^{-1} 1
\]

\[
= \sum_{j=1}^n \frac{1^\top \xi^*_j}{1^\top [\Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma}]^{-1} 1} [\Gamma^{\gamma, \theta} + (n-1)\widetilde{\Gamma}]^{-1} 1
\]

\[
= \sum_{j=1}^n X_j v,
\] (17)
where we have used the second condition from (16) in the final step.

Now consider the first conditions in (16). Pick the \( i \)th equation, multiply by \((n - 1)\), and then subtract the other \( (n - 1) \) equations from it. We get

\[
\Gamma^{\gamma, \theta}[(n - 1)\xi_i^* - \sum_{j \neq i} \xi_j^*] - \tilde{\Gamma}[(n - 1)\xi_i^* - \sum_{j \neq i} \xi_j^*] = [(n - 1)\alpha_i - \sum_{j \neq i} \alpha_j]1.
\]

Further simplifications show that

\[
(\Gamma^{\gamma, \theta} - \tilde{\Gamma})[n\xi_i^* - \sum_{j = 1}^n \xi_j^*] = [n\alpha_i - \sum_{j = 1}^n \alpha_j]1.
\]

The matrix \( \Gamma^{\gamma, \theta} - \tilde{\Gamma} \) is invertible by Lemma 3.2. If follows that

\[
n\xi_i^* - \sum_{j = 1}^n \xi_j^* = [nX_i - \sum_{j = 1}^n X_j]w.
\]

Using (17) and (18) now gives

\[
\xi_i^* = \bar{X}v + (X_i - \bar{X})w
\]

where \( \bar{X} = \frac{1}{n} \sum_{j = 1}^n X_j \).

Now we show that the equations (16) are sufficient for the minimization of our mean-variance functional. To this end, we rewrite the objective mean-variance functional as follows:

\[
\frac{1}{2} \xi_i^\top \Gamma^{\gamma, \theta} \xi_i + \xi_i^\top \tilde{\Gamma} \sum_{j \neq i} \xi_j^* = \frac{1}{2} \xi_i^\top \Gamma^{\gamma, \theta} \xi_i + g_i^\top \xi_i,
\]

where \( g_i = \tilde{\Gamma}\sum_{j \neq i} \xi_j^* \). Next, for \( i = 1, \ldots, n \), we consider arbitrary \( \eta_i \in \mathcal{X}_{\text{det}}(X_i, T) \). Then, by (16),

\[
\frac{1}{2} \eta_i^\top \Gamma^{\gamma, \theta} \eta_i + g_i^\top \eta_i - \left[ \frac{1}{2} \xi_i^\top \Gamma^{\gamma, \theta} \xi_i^* + g_i^\top \xi_i^* \right] = \frac{1}{2} (\eta_i + \xi_i^*)^\top \Gamma^{\gamma, \theta} (\eta_i - \xi_i^*) + g_i^\top (\eta_i - \xi_i^*)
\]

\[
= \frac{1}{2} (\Gamma^{\gamma, \theta})^\top (\eta_i + \xi_i^*) + g_i^\top (\eta_i - \xi_i^*)
\]

\[
= \left[ (\Gamma^{\gamma, \theta} \xi_i^* + g_i) + \frac{1}{2} (\Gamma^{\gamma, \theta})^\top (\eta_i - \xi_i^*) \right]^\top (\eta_i - \xi_i^*)
\]

\[
= \alpha_i 1^\top (\eta_i - \xi_i^*) + \frac{1}{2} (\eta_i - \xi_i^*)^\top \alpha_i 1 + \frac{1}{2} (\eta_i - \xi_i^*)^\top (\eta_i - \xi_i^*) \geq 0,
\]

with equality if and only if \( \eta_i = \xi_i^* \). Altogether, we obtain that (9) defines the unique Nash equilibrium in \( \mathcal{X}_{\text{det}}(X_1, T) \times \cdots \times \mathcal{X}_{\text{det}}(X_n, T) \).

Now we turn to CARA utility maximization. We first note that the cost functional \( C_T(\xi_i|\xi_{-i}) \) is a Gaussian random variable if \( S^0 \) is a Bachelier model and the strategies \( \xi_i, \ldots, \xi_n \) are deterministic. Therefore, for \( \gamma > 0 \),

\[
U_{\gamma}(\xi_i|\xi_{-i}) = \frac{1}{\gamma} \left( 1 - \exp \left( \gamma \mathbb{E}[C_T(\xi_i|\xi_{-i})] + \frac{\gamma^2}{2} \text{Var}[C_T(\xi_i|\xi_{-i})] \right) \right) = u_{\gamma}( - \text{MV}_{\gamma}(\xi_i|\xi_{-i}) )
\]

For \( \gamma = 0 \), we clearly have

\[
U_0(\xi_i|\xi_{-i}) = - \mathbb{E}[C_T(\xi_i|\xi_{-i})] = - \text{MV}_0(\xi_i|\xi_{-i}).
\]
Therefore, mean-variance optimization and CARA utility maximization are equivalent when performed over the class of deterministic strategies. Next, suppose that the strategies $\xi_{-i}$ are deterministic. Then it follows as in Theorem 2.1 of [15] that the maximizer of the functional $U_\gamma(\xi|\alpha)$ over the class of all adapted strategies $\xi \in \mathcal{X}(X_i, T)$ is deterministic. That is, the maximization of $U_\gamma(\xi|\alpha)$ over $\mathcal{X}(X_i, T)$ is equivalent to the maximization over $\mathcal{X}_{det}(X_i, T)$. Therefore, it now follows as in the proof of Corollary 2.1 of [17] that the strategies (9) form a Nash equilibrium for CARA utility maximization.

Proof of Proposition 2.7. Let us define a vector $\omega$ by $\omega_i = 1$ for $i = 0, \ldots, N-1$ and $\omega_N = 1/(1-e^{-1/N})$. Then the assertion will follow if

$$(\Gamma^{0,\theta} - \tilde{\Gamma})\omega = c\mathbf{1}$$

with $c = \frac{1}{1-e^{-\frac{1}{N}}}$. To this end, we note that, with $\delta_{ij}$ denoting the Kronecker delta,

$$
\left( (\Gamma^{0,\theta} - \tilde{\Gamma})\omega \right)_i = \left( (\tilde{\Gamma}^\top + 2\theta \mathbf{1})\omega \right)_i = (\tilde{\Gamma}^\top \omega)_i + (2\theta \mathbf{1}\omega)_i = G(0)\omega_i + \sum_{j=i+1}^{N-1} G\left( \frac{j}{N} - \frac{i}{N} \right) \omega_j + G\left( 1 - \frac{i}{N} \right) \omega_N + 2\theta \sum_{j=0}^{N-1} \delta_{ij}\omega_j
$$

and

$$
= \left( \frac{1}{2} + 2\theta \right) \omega_i + \sum_{j=i+1}^{N-1} G\left( \frac{j}{N} - \frac{i}{N} \right) \omega_j + G\left( 1 - \frac{i}{N} \right) \omega_N.
$$

Since $\theta = \frac{1}{4}$, we have

$$
\left( (\Gamma^{0,\theta} - \tilde{\Gamma})\omega \right)_i = \sum_{j=i}^{N-1} G\left( \frac{j}{N} - \frac{i}{N} \right) \omega_j + G\left( 1 - \frac{i}{N} \right) \omega_N = e^{\frac{1}{N}} \sum_{j=i}^{N-1} e^{-\frac{i}{N}} + \frac{e^{-(1-\frac{i}{N})}}{1-e^{-1/N}} = \frac{1}{1-e^{-1/N}}.
$$

This proves (20) and hence the assertion.

Now we prepare for the proof of Theorem 2.9.

Lemma 3.4. For $\gamma, \sigma, \rho > 0$, the following equation has a unique positive solution $\alpha$,

$$(1) \quad \frac{1}{e^{(\alpha + \rho)}} - \frac{n}{e^{(\alpha - \rho)}} - \frac{\gamma e^{-\alpha}}{(1 - e^{-\alpha})^2} = 0.$$

Moreover, $\alpha \in (0, \rho)$.

Proof. By rearranging equation (21) we get

$$
0 = \frac{-(e^\rho - e^{-\rho}) + (n-1)(e^{-\alpha} - e^\rho)}{(e^\alpha + e^{-\alpha}) - (e^\rho + e^{-\rho})} + \frac{\gamma e^{-\alpha}}{2 - (e^\alpha + e^{-\alpha})}
$$

$$
= \frac{\gamma e^{-\alpha}}{2 - 2 \cosh(\alpha)} + \frac{-2 \sinh(\rho) + (n-1) \left( \cosh(\alpha) - \sinh(\alpha) - (\cosh(\rho) + \sinh(\rho)) \right)}{2 \cosh(\alpha) - 2 \cosh(\rho)}
$$

$$
= \frac{\gamma e^{-\alpha}}{2 - 2 \cosh(\alpha)} - \frac{\sinh(\rho)}{\cosh(\alpha) - \cosh(\rho)} + \left( \frac{n-1}{2} \right) \left[ 1 - \frac{\sinh(\alpha) + \sinh(\rho)}{\cosh(\alpha) - \cosh(\rho)} \right] =: f(\alpha)
$$

(22)
Clearly, when \( \alpha > \rho > 0 \), then \( f(\alpha) < 0 \). Therefore, if a zero of \( f \) exists, it must be within \((0, \rho)\). One easily sees that
\[
\lim_{\alpha \downarrow 0} f(\alpha) = -\infty \quad \text{and} \quad \lim_{\alpha \uparrow \rho} f(\alpha) = +\infty.
\]

Hence, \( f \) admits at least one zero in \((0, \rho)\). Moreover,
\[
\frac{df(\alpha)}{d\alpha} = 2\gamma^2 \sinh(\alpha) \frac{\sinh(\alpha) \sinh(\rho)}{(2 - 2 \cosh(\alpha))^2} + \frac{\sinh(\alpha) \sinh(\rho)}{(\cosh(\alpha) - \cosh(\rho))^2}
\]
\[
+ \left( \frac{n - 1}{2} \right) \left[ \frac{\sinh(\alpha) (\sinh(\alpha) + \sinh(\rho))}{(\cosh(\alpha) - \cosh(\rho))^2} - \frac{\cosh(\alpha)}{\cosh(\alpha) - \cosh(\rho)} \right] > 0,
\]
and so the zero must be unique. \( \square \)

**Lemma 3.5.** Suppose that \( \gamma, \sigma, \rho > 0 \) and \( \theta \geq 0 \). Then the following equation has a unique positive solution \( \beta \),
\[
2\theta + 1 + \frac{1}{e^{(\beta + \rho)} - 1} - \frac{\gamma^2 e^{-\beta}}{(1 - e^{-\beta})^2} = 0.
\]

**Proof.** Let
\[
g(\beta) = 2\theta + 1 + \frac{1}{e^{(\beta + \rho)} - 1} - \frac{\gamma^2 e^{-\beta}}{(1 - e^{-\beta})^2}.
\]
Clearly,
\[
\lim_{\beta \downarrow 0} g(\beta) = -\infty \quad \text{and} \quad \lim_{\beta \uparrow \infty} g(\beta) = 2\theta + \frac{1}{2} > 0.
\]
Hence, there exists at least one zero in \((0, \infty)\). Next, we look at
\[
\frac{dg(\beta)}{d\beta} = - \frac{e^{(\beta + \rho)}}{(e^{(\beta + \rho)} - 1)^2} + \frac{\gamma^2 e^{-\beta}}{(1 - e^{-\beta})^2} + \frac{2\gamma^2 e^{-2\beta}}{(1 - e^{-\beta})^3}
\]
\[
= \gamma^2 e^{(\beta + \rho)} (e^{(\beta + \rho)} - 1)^{-2} \left[ \frac{(e^{(\beta + \rho)} - 1)(e^{-\beta} + 1)}{e^{(2\beta + \rho)}(1 - e^{-\beta})^3} - \frac{1}{\gamma^2} \right].
\]
Note that \( \frac{dg(\beta)}{d\beta} > 0 \) if
\[
\frac{(e^{(\beta + \rho)} - 1)(e^{-\beta} + 1)}{e^{(2\beta + \rho)}(1 - e^{-\beta})^3} > \frac{1}{\gamma^2} > 0.
\]
Here, \( \frac{(e^{(\beta + \rho)} - 1)(e^{-\beta} + 1)}{e^{(2\beta + \rho)}(1 - e^{-\beta})^3} \) is strictly decreasing and bounded below by \( e^\rho \) because for all \( \beta > 0 \),
\[
\frac{d}{d\beta} \left( \frac{(e^{(\beta + \rho)} - 1)(e^{-\beta} + 1)}{e^{(2\beta + \rho)}(1 - e^{-\beta})^3} \right) = - \frac{2e^{(\beta + \rho)}(e^{(\beta + \rho)} - 1)(2e^{(\beta + \rho)} - e^\beta + e^\rho - 2)}{(e^\beta - 1)^4} < 0,
\]
and because by L'Hôpital's Rule,
\[
\lim_{\beta \uparrow \infty} \left( \frac{(e^{(\beta + \rho)} - 1)(e^{-\beta} + 1)}{e^{(2\beta + \rho)}(1 - e^{-\beta})^3} \right) = \lim_{\beta \uparrow \infty} \frac{(e^{(\beta + \rho)} - 1)(e^{-\beta} + 1)}{e^{(2\beta + \rho)}} = \lim_{\beta \uparrow \infty} \frac{2e^{(\beta + \rho)}(e^{(\beta + \rho)} - 1)}{2e^{(2\beta + \rho)}} = \lim_{\beta \uparrow \infty} e^\rho - e^{-\beta} = e^\rho.
\]
Therefore, we have two cases to consider:

1. when \( \gamma^2 \geq e^{-\rho} \), we have that \( \frac{dg(\beta)}{d\beta} > 0 \) for all \( \beta > 0 \). It follows that there exists a unique \( \beta > 0 \) such that \( g(\beta) = 0; \)
2. when $0 < \gamma \sigma^2 < e^{-\rho}$, we can always find a unique $\beta^*$ such that $\frac{dg(\beta)}{d\beta} > 0$ for $\beta < \beta^*$ and $\frac{dg(\beta)}{d\beta} < 0$ for $\beta > \beta^*$. In other words, $(\beta^*, g(\beta^*))$ is the global maximum of $g(\beta)$ on $(0, \infty)$. Now suppose $g(\beta^*) \leq 0$. We know that $g(\beta)$ is strictly decreasing on $(\beta^*, \infty)$, then for all $\beta \in (\beta^*, \infty)$, $g(\beta) < g(\beta^*) \leq 0$, which contradicts (24). Therefore, $g(\beta^*) > 0$, and $0 < 2\theta + \frac{1}{2} < g(\beta) < g(\beta^*)$ for all $\beta \in (\beta^*, \infty)$, which implies a zero cannot exist in $(\beta^*, \infty)$. Since $g(\beta)$ is strictly increasing on $(0, \beta^*)$, it follows that there exists a unique $\beta \in (0, \beta^*)$ such that $g(\beta) = 0$.

In either case, there exists a unique positive solution $\beta$ that solves (23).

**Proof of Theorem 2.9** Let us first extend the matrices (5) and (6) to our infinite time horizon by letting

$$\Gamma_{ij}^{\gamma, \theta} = e^{-\rho(i-j)} + \gamma \sigma^2(i \land j) + 2\theta \delta_{ij}, \quad i, j = 0, 1, \ldots,$$

and

$$\tilde{\Gamma}_{ij} = \begin{cases} 0 & \text{if } i < j, \\ \frac{1}{2} \Gamma_{ij}^{0,0} & \text{if } i = j, \\ \Gamma_{ij}^{0,0} & \text{if } i > j. \end{cases}$$

Next, let $\alpha$ be as provided by Lemma 3.4 and define a vector $\nu \in \ell^1$ by

$$\nu_0 = \frac{1}{1 - e^{(\alpha - \rho)}}, \quad \text{and, for } i = 1, 2, \ldots, \quad \nu_i = e^{-\alpha i}.$$

We will now show that

$$[\Gamma^{\gamma, \theta} + (n - 1)\tilde{\Gamma}]\nu = c\mathbf{1} \quad \text{for } c = \frac{\gamma \sigma^2 e^{-\alpha}}{(1 - e^{-\alpha})^2} > 0,$$

where $1 \in \ell^\infty$ is the sequence $(1, 1, \ldots)$. Using our assumption $\theta = \theta^* = \frac{n-1}{4}$, we get that

$$\left( [\Gamma^{\gamma, \theta} + (n - 1)\tilde{\Gamma}]\nu \right)_i = \Gamma_{i0}^{\gamma, 0} \nu_0 + \sum_{j=1}^{\infty} \Gamma_{ij}^{\gamma, 0} \nu_j + [2\theta \delta_{i0} + (n - 1)\tilde{\Gamma}_{i0}] \nu_0 + \sum_{j=1}^{\infty} [2\theta \delta_{ij} + (n - 1)\tilde{\Gamma}_{ij}] \nu_j$$

$$= e^{-\rho} \nu_0 + \sum_{j=1}^{\infty} [e^{-\rho(i-j)} + \gamma \sigma^2(i \land j)] e^{-\alpha j} + (n - 1) e^{-\rho} \nu_0 + (n - 1) \sum_{j=1}^{i} e^{-\rho(i-j)} e^{-\alpha j}$$

$$= n e^{-\rho} \nu_0 + \sum_{j=1}^{\infty} [e^{-\rho(i-j)} + \gamma \sigma^2(i \land j)] e^{-\alpha j} + (n - 1) e^{-\rho} \sum_{j=1}^{i} e^{-(\alpha - \rho)j}.$$

Expanding the center term gives,

$$\sum_{j=1}^{\infty} [e^{-\rho(i-j)} + \gamma \sigma^2(i \land j)] e^{-\alpha j} = \sum_{j=1}^{i} [e^{-\rho(i-j)} + \gamma \sigma^2 j] e^{-\alpha j} + \sum_{j=i+1}^{\infty} [e^{-\rho(j-i)} + \gamma \sigma^2 j] e^{-\alpha j}$$

$$= e^{-\rho} \sum_{j=1}^{i} e^{-(\alpha - \rho)j} + \gamma \sigma^2 \sum_{j=1}^{i} e^{-\alpha j} + e^{\rho} \sum_{j=i+1}^{\infty} e^{-(\alpha - \rho)j} + \gamma \sigma^2 \sum_{j=i+1}^{\infty} e^{-\alpha j}.$$
Thus,
\[
\left([\Gamma^{\gamma, \theta} + (n-1)\overline{\Gamma}]\nu \right)_i
= ne^{-\rho_i} \nu_0 + ne^{-\rho_i} \sum_{j=1}^i \frac{1}{e^{-(\alpha+\rho)j} + e^{\rho_i}} e^{-(\alpha+\rho)j} + e^{\rho_i} \sum_{j=i+1}^\infty e^{-(\alpha+\rho)j} + \gamma \sigma^2 \sum_{j=1}^i j e^{-\alpha j} + \gamma \sigma^2 \sum_{j=i+1}^\infty e^{-\alpha j}
\]

\[
= ne^{-\rho_i} \nu_0 + n \frac{e^{-\rho_i}}{e^{(\alpha+\rho)} - 1} + \frac{e^{-\alpha i}}{e^{(\alpha+\rho)} - 1} + \frac{1}{e^{(\alpha+\rho)} - 1} \frac{\gamma \sigma^2 e^{-\alpha}}{1 - e^{-\alpha}}
\]

\[
= ne^{-\rho_i} \left[ \frac{1}{e^{(\alpha+\rho)} - 1} + \frac{1}{e^{(\alpha+\rho)} - 1} \right] + e^{-\alpha i} \left[ \frac{1}{e^{(\alpha+\rho)} - 1} \right] + \frac{n}{e^{(\alpha+\rho)} - 1} \frac{\gamma \sigma^2 e^{-\alpha}}{1 - e^{-\alpha}} + \frac{\gamma \sigma^2 e^{-\alpha}}{1 - e^{-\alpha}}
\]

\[
= \frac{\gamma \sigma^2 e^{-\alpha}}{(1 - e^{-\alpha})^2},
\]

where we have used $\nu_0 = \frac{1}{1-e^{(\alpha+\rho)}}$ and our equation (21) in the final step. This establishes (25). Now we can define
\[
v = \frac{1}{1^\top \nu} \nu = \frac{1}{e^{\alpha} - 1} + \frac{1}{e^{\alpha} - \rho_i},
\]

which satisfies the equivalent of (17) in our setting with infinite time horizon.

Let us now deal with the vector $w$. To this end, we take $\beta$ as provided by Lemma 3.5 and define $w \in \ell^1$ by $w_i = e^{-\beta_i}$. Then
\[
[(\Gamma^{\gamma, \theta} - \overline{\Gamma})w]_i = (2\theta + \frac{1}{2})e^{-\beta_i} + e^{\beta_i} \sum_{j=i+1}^\infty e^{-(\beta_i+\rho)j} + \gamma \sigma^2 \sum_{j=1}^i j e^{-\beta j} + \gamma \sigma^2 \sum_{j=i+1}^\infty e^{-\beta j}
\]

\[
= e^{-\beta_i} \left[ 2\theta + \frac{1}{2} + \frac{1}{e^{(\beta+\rho)} - 1} - \frac{\gamma \sigma^2 e^{-\beta}}{1 - e^{-\beta}} \right] + \frac{\gamma \sigma^2 e^{-\beta}}{1 - e^{-\beta}}
\]

\[
= \frac{\gamma \sigma^2 e^{-\beta}}{(1 - e^{-\beta})^2}.
\]

It follows that we can define
\[
w = \frac{1}{1^\top \omega} \omega = \frac{e^{-\beta} - 1}{e^{-\beta} - \omega},
\]

which satisfies the equivalent of (8) in our setting with infinite time horizon.

Finally, if initial positions $X_1, \ldots, X_n \in \mathbb{R}$ are given, and we define $\xi_1, \ldots, \xi_n$ via (13), then it is straightforward to verify that of the first-order conditions (16) are verified with our current choices for $v$, $w$, $\Gamma^{\gamma, \theta}$, and $\overline{\Gamma}$. As in (19), these yield that $\xi_1, \ldots, \xi_n$ form a Nash equilibrium for mean-variance optimization. As in the proof of Theorem 2.4, one then concludes that this is also a Nash equilibrium for CARA utility maximization.}

**Proof of Proposition 2.11.** If $n = 1$ and $\gamma > 0$, we want to find an $\alpha$ that satisfies
\[
\frac{1}{e^{(\alpha+\rho)} - 1} - \frac{1}{e^{(\alpha-\rho)} - 1} - \frac{\gamma \sigma^2 e^{-\alpha}}{(1 - e^{-\alpha})^2} = 0.
\]

If follows from (22) that
\[
\frac{\gamma \sigma^2}{2 - 2 \cosh(\alpha)} - \frac{\sinh(\rho)}{\cosh(\alpha) - \cosh(\rho)} = 0.
\]

By rearranging the equation we have
\[
\cosh(\alpha) = \frac{\gamma \sigma^2 \cosh(\rho) + 2 \sinh(\rho)}{\gamma \sigma^2 + 2 \sinh(\rho)}.
\]
Since $\rho > 0$, we have $\sinh(\rho) > 0$ and $\cosh(\rho) > 1$, which implies,

$$\frac{\gamma \sigma^2 \cosh(\rho) + 2 \sinh(\rho)}{\gamma \sigma^2 + 2 \sinh(\rho)} > 1.$$  

Therefore, we can solve for $\alpha$ and obtain

$$\alpha = \cosh^{-1}\left(\frac{\gamma \sigma^2 \cosh(\rho) + 2 \sinh(\rho)}{\gamma \sigma^2 + 2 \sinh(\rho)}\right).$$

Furthermore, since $\rho > 0$ implies that

$$\frac{\gamma \sigma^2 \cosh(\rho) + 2 \sinh(\rho)}{\gamma \sigma^2 + 2 \sinh(\rho)} < \cosh(\rho)$$

and $\cosh^{-1}(\cdot)$ is an increasing function, we have that $0 < \alpha < \rho$.

References

[1] A. Alfonsi, A. Fruth, and A. Schied. Constrained portfolio liquidation in a limit order book model. *Banach Center Publications*, 83:9–25, 2008.

[2] A. Alfonsi, A. Schied, and A. Slynko. Order book resilience, price manipulation, and the positive portfolio problem. *SIAM J. Financial Math.*, 3:511–533, 2012.

[3] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–39, 2000.

[4] D. P. Bertsekas. *Nonlinear programming*. Athena Scientific Optimization and Computation Series. Athena Scientific, Belmont, MA, second edition, 1999.

[5] D. Bertsimas and A. Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1:1–50, 1998.

[6] M. K. Brunnermeier and L. H. Pedersen. Predatory trading. *Journal of Finance*, 60(4):1825–1863, August 2005.

[7] B. I. Carlin, M. S. Lobo, and S. Viswanathan. Episodic liquidity crises: cooperative and predatory trading. *Journal of Finance*, 65:2235–2274, 2007.

[8] R. A. Carmona and J. Yang. Predatory trading: a game on volatility and liquidity. *Preprint*, 2011.

[9] CFTC-SEC. Findings regarding the market events of May 6, 2010. Report, 2010.

[10] J. Gatheral. No-dynamic-arbitrage and market impact. *Quant. Finance*, 10:749–759, 2010.

[11] J. Gatheral and A. Schied. Dynamical models of market impact and algorithms for order execution. In J.-P. Fouque and J. Langsam, editors, *Handbook on Systemic Risk*, pages 579–602. Cambridge University Press, 2013.

[12] G. Huberman and W. Stanzl. Price manipulation and quasi-arbitrage. *Econometrica*, 72(4):1247–1275, 07 2004.
[13] J. Lorenz and R. Almgren. Mean-variance optimal adaptive execution. *Appl. Math. Finance*, 18(5):395–422, 2011.

[14] A. Obizhaeva and J. Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16:1–32, 2013.

[15] A. Schied, T. Schöneborn, and M. Tehranchi. Optimal basket liquidation for CARA investors is deterministic. *Applied Mathematical Finance*, 17:471–489, 2010.

[16] A. Schied, E. Strehle, and T. Zhang. High-frequency limit of Nash equilibria in a market impact game with transient price impact. *SIAM J. Financial Math.*, 8(1):589–634, 2017.

[17] A. Schied and T. Zhang. A state-constrained differential game arising in optimal portfolio liquidation. *Math. Finance*, 27(3):779–802, 2017.

[18] A. Schied and T. Zhang. A market impact game under transient price impact. *To appear in Mathematics of Operations Research*, 2018.

[19] T. Schöneborn. Trade execution in illiquid markets. Optimal stochastic control and multi-agent equilibria. Doctoral dissertation, TU Berlin, 2008.

[20] T. Schöneborn and A. Schied. Liquidation in the face of adversity: stealth vs. sunshine trading. SSRN Preprint 1007014, 2009.

[21] E. Strehle. Optimal execution in a multiplayer model of transient price impact. *To appear in Market Microstructure and Liquidity*, 2018.