Large gravitons and near-horizon
diffeomorphisms

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ABSTRACT: Usual gauge fixing procedures in classical general relativity rely on the existence of solutions of a second order wave equation. We propose to use these equations to relate asymptotic symmetries at infinity to asymptotic symmetries of a black hole horizon, in tune with recent proposals. We illustrate the construction for the BTZ and four-dimensional Kerr black holes. We find in both cases a realization of the group of diffeomorphisms of the real line.

KEYWORDS: Large gauge transformations. Asymptotic symmetries. Black holes.
1 Introduction

Black holes are seemingly simple objects: most of known examples involve exact solutions for Einstein’s equations, which is something rare. In four dimensions and lower, uniqueness theorems guarantee that black hole metrics are the natural endpoint for the evolution of gravitational systems, whereas entropic arguments posit the same fate even in higher dimensions. No-hair theorems corroborate this point of view by stating that the classical black hole is completely characterized by its observable charges.

There is, however, a major difference, between a black hole and this “structureless particle” point of view: the existence of an event horizon. This furnishes the black hole with a structure which can be seen both as geometric and thermodynamic, with profound implications for its physical description. The event horizon is a surface of infinite redshift, and as such its description defies the “effective field theory” point of view pervasive from other branches of high energy physics.

The black-hole horizon is usually defined classically as a global property of the space-time: the boundary of the causal future of the “asymptotic past” and the causal past of the “asymptotic future”. Although we should point out at this point that
alternative definitions do exist: particularly the notion of trapped surface, which has the advantage of being local, and in fact is much more manageable, specially in numerical simulations. The terms “asymptotic past” and “asymptotic future” deserve further qualification: they fall into a more generic notion of “asymptotic infinity”, which tries to define the analogue of an “isolated system” when gravity is present [1].

The notion of asymptotic infinity accomplishes this analogue, at the expense of giving up some of the diffeomorphism invariance. The analogue, as well as the fixing of the diffeomorphism invariance, seem necessary for a proper definition of space-time observable charges, like energy or angular momentum, which depends explicitly on a notion of Poincaré invariance. In turn, Poincaré invariance is expected to hold only infinitely far from the sources, so, in dealing with non-trivial cases, one arrives at the notion of “asymptotic isometry” [2]. The most famous examples are the Bondi-Metzner-Sachs (BMS) group in four dimensional general relativity with zero cosmological constant [3, 4] and the Brown-Henneaux (BH) group in three-dimensional gravity with negative cosmological constant [5]. In both cases, one cannot extricate the “global isometries” – the isometries of the vacuum solutions – from the group of asymptotic symmetries. In the Brown-Henneaux case, the charges associated to the asymptotic isometries generate a non-trivial Virasoro algebra, which gives a consistent geometric interpretation for the physical degrees of freedom associated with the black hole background.

In terms of local physics, this attribution shares many parallels with the situation in non-abelian gauge theories, where the local degrees of freedom are often complemented by “large gauge transformations”. Would-be gauge transformations – in the gravity case, coordinates transformations – which are no longer duplicate descriptions of the same physical configuration since they change the value of observables of the system. Unlike instantons in non-abelian gauge fields, however, there is no gauge-invariant way to think about those transformations as localized, nor any known topological invariant associated to it. Their association with bona-fide, physical local degrees of freedom of the black hole is then problematic, even when the numerical checks seem to match, as in the Brown-Henneaux case.

Proposals for local, geometric diffeomorphisms that count the black hole degrees of freedom (“black hole hair”) along the directions above have been put forward, by a number of authors over the years. See [6, 7] for examples relevant to our discussion. They have been, however, always plagued by the unclear message of gauge invariance.
Recently Hawking and collaborators proposed to tackle the problem by transposing the concept of asymptotic isometry to the horizon [8, 9] – see also Donnay et al. [10, 11] – which could in principle solve not only the problem of counting the Bekenstein-Hawking entropy formula but also the information paradox problem.

In this article we propose that the problem of gauge invariance can be consistently solved by linking the approximate isometries at infinity to the ones at the horizon, via the gauge fixing conditions. This recipe, though dependent on the particular dynamical model whose solution is the black hole under consideration, has the advantage of explicating the gauge choices involved in ascribing the diffeomorphisms, while keeping the local aspect of the proposal. We will revise some of the important notions in Sections 2 and 3, and work out explicitly the BTZ and Kerr cases in Section 4 and 5, respectively. We close with a summary and some prospects.

2 Asymptotics and symmetries

Many papers and textbooks cover the issues of asymptotic simplicity, asymptotic isometries and supertranslations, see, for instance, [1] and chapter 11 in [12]. We will focus on the anti-de Sitter and flat case. In both of them, there is a non-vanishing Ω function which serves to link the physical metric $g_{ab}$ to a non-physical metric $\hat{g}_{ab}$ by means of a conformal transformation:

$$g_{ab} = \Omega^{-2} \hat{g}_{ab},$$

(2.1)

in such a way that the asymptotic region – far from sources – can be mapped to the pre-image of $0 < \Omega < \epsilon$. The “conformal boundary” $\Omega \to 0$ is a region added to the unphysical manifold in such a way that it is topologically closed. The causal structure of the conformal boundary depends on the model considered: if the cosmological constant is negative, the boundary is space-like – save for two points – with the topology of a cylinder. If the cosmological constant is zero, the boundary is divided in five pieces, two topologically given by $\mathbb{R} \times S^2$, called past and future null infinity, and three points, past and future timelike infinity and spacelike infinity. Obviously many issues about the asymptotic behavior of fields are “swept under the rug”, particularly the behavior at the “boundaries of the boundaries”: both timelike infinites, and spacelike infinity in the case of asymptotically flat spacetimes; time-like infinites in the case of negative cosmological constant. With all its shortcomings, the procedure is rich enough to tackle
the issues raised in the introduction.

From the definition of the conformal boundary as the $\Omega = 0$ level surface a lot of structure arises. Consider vacuum Einstein’s equation written in terms of the unphysical metric $\hat{g}_{ab}$:

$$R_{ac} = \frac{k(D-1)}{\ell^2} g_{ac}, \quad \text{or}$$

$$\hat{R}_{ac} + (D - 2) \frac{\hat{\nabla}_a \hat{\nabla}_c \Omega}{\Omega} - \frac{(D - 1)\hat{g}_{ac}\hat{g}^{bd} \frac{\hat{\nabla}_b \Omega \hat{\nabla}_d \Omega}{\Omega} + \hat{g}_{ac}\hat{g}^{bd} \frac{\hat{\nabla}_b \hat{\nabla}_d \Omega}{\Omega}}{\Omega^2} = \frac{k(D-1)\hat{g}_{ac}}{\ell^2 \Omega^2},$$

(2.3)

where in the second line we wrote the Ricci tensor associated with $g_{ac}$ in terms of hatted quantities, which are associated with $\hat{g}_{ac}$. In the first equation $k = 0, +1, -1$ corresponding to the flat [12], de Sitter and anti-de Sitter [13] cases respectively. Regularity of the unphysical metric at $\Omega = 0$ requires that, at the boundary:

$$\hat{g}^{bd} \hat{\nabla}_b \hat{\nabla}_d \Omega = -\frac{k}{\ell^2} \quad \text{at} \quad \Omega = 0,$$

(2.4)

so the normal vector $n^a = \hat{g}^{ab} \hat{\nabla}_b \Omega$ to the $\Omega = 0$ surface will be spacelike for negative cosmological constant, timelike for positive cosmological constant and null for the flat case. By multiplying $\Omega$ by a non-vanishing function $\omega$ at the boundary one can further ensure that, near $\Omega = 0$,

$$\hat{g}^{bd} \hat{\nabla}_b \Omega \hat{\nabla}_d \Omega = -\frac{k}{\ell^2} + \mathcal{O}(\Omega^2),$$

(2.5)

and therefore the vanishing of the $\mathcal{O}(\Omega^{-1})$ term in (2.3) requires:

$$\hat{\nabla}_a \hat{\nabla}_c \Omega = 0 \quad \text{at} \quad \Omega = 0,$$

(2.6)

which in turn implies that the vector $n^a$ is covariantly constant at $\Omega = 0$. This construction still allows for a gauge symmetry: one may still multiply the $\Omega$ function by a nowhere vanishing function $\omega$: $\Omega \to \omega \Omega$, which is constant along $n^a$: $n^a \hat{\nabla}_a \omega = 0$. This remaining conformal symmetry allows us to fix the conformal structure of the boundary, and, by choosing a particular set of coordinates ("Bondi coordinates" [14]), write a metric for the asymptotic boundary as induced by the interior non-physical metric $\hat{g}_{ab}$.
This induced metric can be fixed to be that of a flat cylinder for asymptotic anti-de Sitter spaces \( k = -1 \) and a null line times a standard, constant curvature sphere for the asymptotically flat case \( k = 0 \). More importantly, the remaining conformal symmetry of the unphysical metric gives rise to the conformal symmetries. The usual isometry condition \( \mathcal{L}_\xi g_{ab} = 0 \) can only be expected to hold at \( \Omega = 0 \) ("infinitely far away"), so we define an asymptotic symmetry as a vector field \( \xi^a \) which has a smooth limit to the surface \( \Omega = 0 \) such that

\[
\Omega^2 \mathcal{L}_\xi g_{ab} = 0, \quad \text{at} \quad \Omega = 0. \tag{2.7}
\]

Expanding the Lie derivative in terms of the unphysical metric we find [15]:

\[
\tilde{\nabla}^a \xi^b + \tilde{\nabla}^b \xi^a - 2\tilde{g}^{ab} \xi^c \tilde{\nabla}_c \Omega = 0 \quad \text{at} \quad \Omega = 0, \tag{2.8}
\]

which can be used to recover the asymptotic symmetries of vacuum space – the (anti)-de Sitter group for \( k \neq 0 \) and the Poincaré group for \( k = 0 \). Because the condition is only enforced at \( \Omega = 0 \), we have an equivalence class of solutions: two solutions of (2.7) \( \xi^a \) and \( \xi'^a \) generate the same asymptotic symmetry if the vector fields coincide at \( \Omega = 0 \).

Now we can specialize to the two cases we are going to address here:

### 2.1 Three-dimensional anti-de Sitter space-times

In this case one can find coordinates \( z, u, v \) such that the metric has the asymptotic form:

\[
ds^2 = \frac{dz^2 + du \, dv}{z^2}, \tag{2.9}
\]

where one takes \( \Omega = z \) and then the unphysical metric \( \hat{g}_{ab} \) is locally three-dimensional Minkowski. The generic solution of (2.7) is given by the Brown-Henneaux generators [5]:

\[
\ell_n = \frac{1}{2} e^{2nu} \left( -n \, z \, \frac{\partial}{\partial z} + \frac{\partial}{\partial u} \right), \quad \bar{\ell}_n = \frac{1}{2} e^{2nv} \left( n \, z \, \frac{\partial}{\partial z} + \frac{\partial}{\partial v} \right), \quad n \in \mathbb{Z}, \tag{2.10}
\]

with each set satisfying the Witt algebra:

\[
[\ell_n, \ell_m] = -(n - m)\ell_{n+m}. \tag{2.11}
\]
The generators $\ell_{-1}, \ell_0, \ell_1$ and their barred counterparts generate the three-dimensional anti-de Sitter algebra $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. The remaining operators induce generic, local conformal transformations at spatial infinity $i^0$. Heuristically, scale transformations of $i^0$ induced by $\ell_n, \bar{\ell}_n$ can be “undone” by a translation of $z$ [13].

2.2 Four-dimensional asymptotically flat space-times

Now $n^a$, the normal vector to the surface $\Omega = 0$ is null, and one can construct coordinates $u, \Omega, \theta, \phi$ such that the unphysical metric at future null infinity is given by:

$$d\hat{s}^2 = 2 du d\Omega + d\theta^2 + \sin^2 \theta d\phi^2.$$  \hspace{1cm} (2.12)

In these coordinates $n^a = \partial / \partial u$. Along with the usual Poincaré symmetries, one can check that $\xi^a = \alpha n^a$ satisfies that $\Omega^2 \hat{\nabla}_a g_{ab} = 0$ at $\Omega = 0$ for generic $\alpha$ satisfying $n^a \hat{\nabla}_a \alpha = \partial_a \alpha = 0$. Since $\alpha$ is now a function of $\theta$ and $\phi$ only it can be expanded in spherical harmonics. The $l = 0$ and $l = 1$ pieces completes the Poincaré group whereas the higher harmonics form an abelian algebra called “supertranslations”. The whole space of solutions is called the BMS group.

One important fact about supertranslations for the following analysis is that the solutions $\xi^a = \alpha n^a$ can be obtained from a potential. We note that the gradient of $\Phi = \alpha \Omega$ with $\alpha$ as above induces a vector field:

$$\xi^a = \hat{g}^{ab} \hat{\nabla}_b \Phi = \alpha n^a + \Omega \hat{g}^{ab} \hat{\nabla}_b \alpha,$$  \hspace{1cm} (2.13)

which is equal to the supertranslation $\alpha n^a$ at the boundary $\Omega = 0$. By the considerations after (2.7), the gradient of $\Phi$ is then equivalent to a supertranslation. One notes that solutions for infinitesimal isometries of flat space $\partial_a \zeta_b + \partial_b \zeta_a = 0$ – the flat space Killing equation have a similar decomposition, where $\zeta_a^\mu$ are associated to translations whereas $\zeta_{\mu
u}^a = x_\mu (\partial_\nu)^a - x_\nu (\partial_\mu)^a$ are the Lorentz generators.

3 Fluctuations and the general gist

Before we turn to the specific cases, let us digress over metric perturbations. Let us suppose that we start with a solution of Einstein’s equation, with possibly a cosmological
constant:

\[ R_{ab} = \frac{k(D-1)}{\ell^2} g_{ab}. \]  

(3.1)

One can check [12] that, if one changes the metric by a “small amount” \( \delta g_{ab} = h_{ab} \), the change in the Ricci tensor is, to first order:

\[ \delta R_{ac} = -\frac{1}{2} \nabla_a \nabla_c h^b_b - \frac{1}{2} \nabla_b \nabla^b h_{ac} + \nabla_c (\nabla^b h_{ab})_b + R_{bca} h^{bd} + R_{(c}^d h_{a)d}, \]  

(3.2)

where indices are raised and contracted with the unperturbed metric \( g_{ab} \). The Ricci and Riemann tensor are also computed with respect to \( g_{ab} \).

The equation for the fluctuations:

\[ \delta R_{ab} = \frac{k(D-1)}{\ell^2} \delta g_{ab} \]  

(3.3)

sets constraints on the \( D(D+1) \) dynamical components of the metric perturbation. However, due to diffeomorphism invariance, \( h_{ab} \) and \( h'_{ab} = h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a \) are physically indistinguishable for well-defined vector fields \( \xi_b \). We can use this gauge freedom to make \( h_{ab} \) traceless \( g^{ab} h_{ab} = 0 \) and transverse \( \nabla^c h_{cb} = 0 \). Generically, upon a gauge transformation:

\[ g'^{ab} h'_{ab} = g^{ab} h_{ab} + \nabla_c \xi^c, \quad \nabla^c h'_{cb} = \nabla^c h_{cb} + \nabla^2 \xi_b + \nabla^c \nabla_b \xi_c, \]  

(3.4)

which we will equal to zero to write differential equations for \( \xi_c \). The traceless transverse gauge is achieved by solving:

\[ \nabla^b \xi_b = -\frac{1}{2} g^{ab} h_{ab}, \quad \nabla^2 \xi_b + \frac{k(D-1)}{\ell^2} \xi_b = -\nabla^c h_{cb} - \frac{1}{2} \nabla_b h, \]  

(3.5)

which define the vector field \( \xi_c \) up to a solution of the homogeneous equations:

\[ \nabla^b \xi_b = 0, \quad \nabla^2 \xi_b + \frac{k(D-1)}{\ell^2} \xi_b = 0. \]  

(3.6)

Both gauge transformations are necessary to reduce the number of graviton degrees of freedom to their true value – zero in three dimensions and two in four. However both solutions assume fast enough fall-offs at infinity: after all, an acton of the Poincaré group can change your stationary black hole solution to a moving black hole.
Which brings us to the main point of this paper. The asymptotic symmetries at
the conformal boundary described in the last section induce, via (3.6), an “active”
transformation in the interior. By the discussion above, these are physical: they do
change the physical properties of the background. Specifically, one can follow [16] and
associate with an infinitesimal coordinate transformation $\xi^a$ a charge, for instance, in
flat pure Einstein-Hilbert theory:

$$Q[\xi] = \int_\Sigma \epsilon_{aba_1...a_{D-2}} \nabla^a \xi^b,$$

one can recover the total mass $M$ associated to time translations and total angular
momentum $J$ associated to azimuthal rotations. The status of a charge associated
to the generic solution of (2.7) is less clear. In the Brown-Henneaux case, the Weyl
subgroup of the asymptotic algebra consists of exactly two charges, associated with $\ell_0$
and $\bar{\ell}_0$, from which one can extract the mass and angular momentum – see (4.1) below.
In the BMS case all supertranslations commute, so they may be associated to the space-
time “hair”. Regardless of their interpretation, the action of the Brown-Henneaux and
BMS generators which do not commute with either $M$ and $J$ cannot be pure gauge
transformations because they change the value for those observables. In the BMS case,
we take a supertranslation $\xi^a = \alpha(\theta, \phi)n^a$ and find that, asymptotically:

$$[\alpha(\theta, \phi) \partial_u, \partial_\phi] = - (\partial_\phi \alpha) \partial_u,$$

which does not vanish, even at the conformal boundary. By placing $\xi^a$ as a boundary
condition of the residual transformation (3.6), one can then induce a coordinate trans-
formation at the interior which is physical, and may be associated to a true degree of
freedom. In the remaining of the paper we will illustrate this idea with the BTZ and the
Kerr black hole to construct coordinate transformations at the horizon, as in [10, 17].
We use BTZ as a “proof of concept” because the whole structure is readily integrable.
The particular problem of relating asymptotic to near horizon symmetries in BTZ was
tackled by Compère et al. [18], albeit using a dynamic-dependent symplectic form, and
proposals for BTZ hair have been presented by Afshar et al. in [19]. The program
has also been carried on for the particular Schwarzschild case by Compère and Long
[20, 21].
4 The three-dimensional BTZ black hole

The BTZ metric for a asymptotically AdS$_3$ ($k = -1$) black hole with mass $M = r_+^2 + r_-^2$ and angular momentum $J = 2r_+ r_- :$ [22, 23],

$$ds^2 = - \left( \frac{r^2 - r_+^2}{r^2} \right) dt^2 + \frac{r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left( d\phi - \frac{r_+ r_-}{r^2} dt \right)^2$$  (4.1)

is an asymptotically AdS$_3$ space-time with $\ell = 1$, which can be seen heuristically from the large $r$ expansion of (4.1). The difference between the BTZ metric and usual AdS$_3$ is a global one: upon the change of variables:

$$u = \frac{r_+ - r_-}{2}(\phi + t), \quad \cosh^2 \varrho = \frac{r^2 - r_+^2}{r_+^2 - r_-^2}, \quad v = \frac{r_+ + r_-}{2}(\phi - t)$$  (4.2)

one recovers the left-right invariant metric of the SL(2, $\mathbb{R}$) group manifold:

$$ds^2 = d\varrho^2 + du^2 + dv^2 + 2 \cosh 2\varrho du dv.$$  (4.3)

Let us exploit the SL(2, $\mathbb{R}$) symmetry of the BTZ geometry. Define the Killing-Cartan form:

$$\eta_{ij} = \frac{1}{2} \text{Tr}(\sigma^i\sigma^j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$  (4.4)

and its inverse $\eta^{ij}$. From the Euler decomposition of the SL(2, $\mathbb{R}$) group manifold:

$$g = e^{u\sigma^3} e^{\varrho\sigma^1} e^{v\sigma^3} = \begin{pmatrix} e^u e^v \cosh \varrho & e^{u-v} \sinh \varrho \\ e^{-u+v} \sinh \varrho & e^{-u-v} \cosh \varrho \end{pmatrix},$$  (4.5)

we define the covariant current components:

$$J_i = \frac{1}{2} \eta_{ij} \text{Tr}(dgg^{-1}\sigma^j), \quad \bar{J}_i = \frac{1}{2} \eta_{ij} \text{Tr}(g^{-1} dg\sigma^j).$$  (4.6)

Using the inverse metric $g^{ab}$ to (4.3), we associate to each component a vector field $J^a_i$
and $\bar{J}^a_i$. Explicitly:

$$J_3 = \frac{\partial}{\partial u}, \quad J_+ = e^{2u} \left[ \frac{\partial}{\partial \varrho} - \frac{\cosh 2\varrho}{\sinh 2\varrho} \frac{\partial}{\partial u} + \frac{1}{\sinh 2\varrho} \frac{\partial}{\partial v} \right],$$

$$J_- = e^{-2u} \left[ \frac{\partial}{\partial \varrho} + \frac{\cosh 2\varrho}{\sinh 2\varrho} \frac{\partial}{\partial u} - \frac{1}{\sinh 2\varrho} \frac{\partial}{\partial v} \right];$$

$$\bar{J}_3 = \frac{\partial}{\partial v}, \quad \bar{J}_+ = e^{-2v} \left[ \frac{\partial}{\partial \varrho} - \frac{1}{\sinh 2\varrho} \frac{\partial}{\partial u} + \frac{\cosh 2\varrho}{\sinh 2\varrho} \frac{\partial}{\partial v} \right],$$

$$\bar{J}_- = e^{2v} \left[ \frac{\partial}{\partial \varrho} + \frac{1}{\sinh 2\varrho} \frac{\partial}{\partial u} - \frac{\cosh 2\varrho}{\sinh 2\varrho} \frac{\partial}{\partial v} \right].$$

(4.7)

Which satisfies the algebra:

$$[J_3, J_\pm] = \pm 2J_\pm, \quad [J_+, J_-] = 4J_3, \quad [J_i, J_j] = 0$$

(4.8)

$$[\bar{J}_3, \bar{J}_\pm] = \mp 2\bar{J}_\pm, \quad [\bar{J}_+, \bar{J}_-] = -4\bar{J}_3.$$

Where commutators are represented by usual vector field brackets (Lie derivative). From these we define the structure constants $C_{ij}^k$, via $[J_i, J_j] = C_{ij}^k J_k$ and the antisymmetric tensor $\epsilon_{ijk} = C_{ij}^l \eta_{lk}$. One can show that $\epsilon_{3+} = 4$, $\epsilon_{3-} = -1$ and that:

$$\epsilon^{ijk} \epsilon_{ijk} = 3! \det(\eta_{ij}) = -24.$$

(4.11)

A little algebra shows that:

$$\nabla_a (J_i)_b = \epsilon_{abc} J^c_i, \quad \nabla_a (\bar{J}_i)_b = -\epsilon_{abc} \bar{J}^c_i,$$

(4.12)

where $\epsilon_{abc}$ is the volume form extracted from (4.3).

Following the discussion from last section, we will define “large gravitons” as the solutions of the homogeneous equations (3.6) which asymptote to the Brown-Henneaux generators at the boundary (2.10). Defining the scalars $\xi_i = J_i^a \xi_a$, we have, using (4.12), that the vector Laplacian can be written as:

$$J_i^a \nabla^2 \xi_a = \nabla^2 \xi_i + \epsilon_i^{jk} \nabla_j \xi_k + 2\xi_i, \quad \bar{J}_i^a \nabla^2 \bar{\xi}_a = \nabla^2 \bar{\xi}_i - \epsilon_i^{jk} \nabla_j \bar{\xi}_k + 2\bar{\xi}_i,$$

(4.13)

where $\nabla_i = J_i^a \nabla_a$ – and analogously for $\nabla_i$, or simply the directional derivative on the direction $J_i$. The second order equation for $\xi_a$ can now be cast as a system of first
order equations for $\xi_i$:

$$\epsilon_i^{jk}\nabla_j\xi_k = \sigma_i, \quad \epsilon_i^{jk}\nabla_j\sigma_k = 4\nabla^2\xi_i - 4\sigma_i - 4\nabla_i(\nabla_j\xi^j) = -8\sigma_i. \quad (4.14)$$

For the case considered here, we can solve both equations, zero divergence and zero Laplacian, by setting $\sigma_i = 0$. Incidentally, for the case of “massive perturbations”, the equation is similar to a normal mode:

$$\epsilon_i^{jk}\nabla_j\xi_k = \mu\xi_i, \quad (4.15)$$

where $\mu$ is related to the mass of the perturbation, and $\xi_i$ is now a non-trivial linear combination of the vector field components and its rotational derivative $\sigma_i$.

The equations for $\bar{\xi}_i = \bar{J}_i^a\bar{\xi}_a$ are obtained similarly:

$$\epsilon_i^{jk}\nabla_j\bar{\xi}_k = \bar{\sigma}_i, \quad \epsilon_i^{jk}\nabla_j\bar{\sigma}_k = 4\nabla^2\bar{\xi}_i + 4\bar{\sigma}_i - 4\nabla_i(\nabla_j\bar{\xi}^j) = 8\bar{\sigma}_i. \quad (4.16)$$

For massless modes, one then has:

$$\epsilon_i^{jk}\nabla_j\xi_k = \epsilon_i^{jk}\bar{\nabla}_j\bar{\xi}_k = 0. \quad (4.17)$$

One can see that these equations are equivalent by introducing the matrix:

$$L_{ij} = g_{ab}\bar{J}_i^a\bar{J}_j^b, \quad (4.18)$$

satisfying $L_i^j L^k_j = \delta_i^k$, with indices lowered and raised by $\eta$. Vectors – and tensors – can be decomposed on either basis, and $L_i^j$ implements the change:

$$\xi_j = L_i^j\bar{\xi}_j, \quad \bar{\xi}_i = \xi_j L^j_i. \quad (4.19)$$

The strategy for solving (4.17) for $\xi_a$ which asymptotes to the Brown-Henneaux generators is simplified due to the fact that $\ell_n$ only depends on $u$ and $\bar{\ell}_n$ only on $v$. Since the “squared” operator

$$\epsilon_i^{jk}\nabla_j(\epsilon_k^{lm}\nabla_l\xi_m) = 4\nabla^2\xi_i \quad (4.20)$$
is proportional to the scalar Laplacian, each component of $\xi_i$ satisfies:

$$\nabla^2 \xi_i = \left[ \frac{\partial}{\partial z} \left( z(z - 1) \frac{\partial}{\partial z} \right) + \frac{1}{16z(z - 1)} \left( (2z - 1) \frac{\partial^2}{\partial u \partial v} - \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} \right) \right] \xi_i = 0,$$

(4.21)

where $z = \cosh^2 \varrho$. Assuming the solution for $\xi_3$ is only a function of $z$ and $u$ the solution is given readily:

$$\xi_3 = -\frac{p^2 - 1}{2} \left[ c_+ \left( \frac{z - 1}{z} \right)^{p/2} + c_- \left( \frac{z}{z - 1} \right)^{p/2} \right] e^{2pu},$$

(4.22)

and the other components are obtained from $\epsilon_{ijk} \nabla_j \xi_k = 0$:

$$\xi_+ = \frac{p(p - 1)}{2} \left[ c_+ \left( \frac{z - 1}{z} \right)^{(p+1)/2} + c_- \left( \frac{z}{z - 1} \right)^{(p+1)/2} \right] e^{2(p+1)u},$$

(4.23)

$$\xi_- = -\frac{p(p + 1)}{2} \left[ c_+ \left( \frac{z - 1}{z} \right)^{(p-1)/2} + c_- \left( \frac{z}{z - 1} \right)^{(p-1)/2} \right] e^{2(p-1)u},$$

(4.24)

where $2p$ is the eigenvalue for $J_3$. By the same token, we have for $\bar{\xi}_i$ satisfying $\epsilon_{ijk} \nabla_j \bar{\xi}_k = 0$ only depending on $z$ and $v$:

$$\bar{\xi}_3 = \frac{q^2 - 1}{2} \left[ \bar{c}_+ \left( \frac{z - 1}{z} \right)^{q/2} + \bar{c}_- \left( \frac{z}{z - 1} \right)^{q/2} \right] e^{2qv},$$

(4.26)

$$\bar{\xi}_+ = \frac{q(q + 1)}{2} \left[ \bar{c}_+ \left( \frac{z - 1}{z} \right)^{(q-1)/2} + \bar{c}_- \left( \frac{z}{z - 1} \right)^{(q-1)/2} \right] e^{2(q-1)v},$$

(4.27)

$$\bar{\xi}_- = -\frac{q(q - 1)}{2} \left[ \bar{c}_+ \left( \frac{z - 1}{z} \right)^{(q+1)/2} + \bar{c}_- \left( \frac{z}{z - 1} \right)^{(q+1)/2} \right] e^{2(q+1)v},$$

(4.28)

The constants sitting in front of the expressions for $\xi_3$ and $\bar{\xi}_3$ were chosen so that $\xi^a$ and $\bar{\xi}^a$ asymptote to the Brown-Henneaux generators in a simpler expression.

As one can see, no mention to the actual BTZ metric (4.1) was made, the whole construction being a priori defined on a sort of covering space where $u$ and $v$ cover the whole plane. Coming back to the $t$ and $\phi$ coordinates, we find that $p$ and $q$ are related
to the frequency and angular momentum – eigenvalues of $\partial_t$ and $\partial_\phi$ by:

$$p = -\frac{i}{2} \frac{\omega - m}{r_+ - r_-}, \quad q = \frac{i}{2} \frac{\omega + m}{r_+ + r_-}. \quad (4.29)$$

Using (4.12) one may calculate the metric perturbation due to $\xi^a$: $h_{ab} = \nabla_a \xi_b + \nabla_b \xi_a$. Defining null coordinates on the $\varrho - u$ plane, $u_\pm = u \pm \frac{1}{2} \log(\tanh \varrho)$, we have:

$$h_{ab} = -2p(p^2 - 1) \left[ c_+ e^{2pu_+} (du_+)_a (du_+)_b + c_- e^{2pu_-} (du_-)_a (du_-)_b \right]. \quad (4.30)$$

Thus, for purely ingoing solution near the black hole horizon ($\varrho = 0$), we must set $c_- = 0$. Note that for a single valued solution,

$$p = i \frac{m}{r_+ - r_-}, \quad (4.31)$$

with $m$ integer, in order that $2pu = i m (\phi + t)$. So (4.30) corresponds to ingoing and outgoing waves of frequency equal to minus the angular momentum. The $(\phi + t)$ dependence of the phase indicate the “chiral”, left-moving character of these modes. The right-moving modes are calculated from $\bar{\xi}_a$. In fact, we need only change $(u, p, c_\pm) \to (v, q, \bar{c}_\pm)$ in (4.30) to obtain the solution:

$$\bar{h}_{ab} = -2q(q^2 - 1) \left[ \bar{c}_+ e^{2qv_+} (dv_+)_a (dv_+)_b + \bar{c}_- e^{2qv_-} (dv_-)_a (dv_-)_b \right], \quad (4.32)$$

where $v_\pm = v \pm \frac{1}{2} \log(\tanh \varrho)$. The interpretation of the null coordinates $v_\pm$ are different now: one can check that they correspond to outgoing and ingoing coordinates, respectively, in BTZ. Thus our physical solution requires $\bar{c}_+ = 0$. According to (4.2),

$$q = i \frac{\bar{m}}{r_+ + r_-}, \quad (4.33)$$

with $\bar{m}$ integer, in such a way that $2qv = i \bar{m} (\phi - t)$. 

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In terms of the $\varrho, u, v$ coordinates, the diffeomorphism $\xi^a$ can be written as:

$$
\xi^a = c_+ e^{2pu_+} \left[ -\frac{p(p + \cosh 2\varrho)}{2 \sinh 2\varrho} \partial_v + \frac{p^2 + p \cosh 2\varrho + \sinh^2 2\varrho}{2 \sinh^2 2\varrho} \partial_u - \frac{p(p \cosh 2\varrho + 1)}{2 \sinh^2 2\varrho} \partial_v \right] 
+ c_- e^{2pu_-} \left[ \frac{p(p - \cosh 2\varrho)}{2 \sinh 2\varrho} \partial_v + \frac{p^2 - p \cosh 2\varrho + \sinh^2 2\varrho}{2 \sinh^2 2\varrho} \partial_u - \frac{p(p \cosh 2\varrho - 1)}{2 \sinh^2 2\varrho} \partial_v \right].
$$

(4.34)

At spatial infinity ($\varrho = \infty$), where $u_+ = u_- \to u$,

$$
\xi^a \to (c_+ + c_-) e^{2pu} \left( -\frac{p \partial}{2 \partial \varrho} + \frac{1}{2} \frac{\partial}{\partial u} \right),
$$

(4.35)

as expected, since we constructed the diffeomorphism to be the Brown-Henneaux generators (2.10) there. Let us define two new basis of vectors in which the diffeomorphisms can be better represented. First, define the coordinate

$$
\chi = u + v = r_+ \left( \phi - \frac{r_+}{r_+} t \right) = r_+ \phi_H
$$

(4.36)

related to the co-rotating angular coordinate $\phi_H$ at the event horizon. Now, we introduce the basis of vectors $\{\partial_+, \bar{\partial}_+, \partial_\chi^+\}$ dual to $du_+, dv_+$ and $d\chi$:

$$
\partial_+ = \frac{\partial_u - \partial_v}{2} + \frac{\sinh 2\varrho}{2} \partial_e, \quad \bar{\partial}_+ = \frac{\partial_v - \partial_u}{2} + \frac{\sinh 2\varrho}{2} \partial_e, \quad \partial_\chi^+ = \frac{\partial_u + \partial_v}{2} - \frac{\sinh 2\varrho}{2} \partial_e,
$$

(4.37)

and the corresponding dual basis to $du_-, dv_-$ and $d\chi$:

$$
\partial_- = \frac{\partial_u - \partial_v}{2} - \frac{\sinh 2\varrho}{2} \partial_e, \quad \bar{\partial}_- = \frac{\partial_v - \partial_u}{2} - \frac{\sinh 2\varrho}{2} \partial_e, \quad \partial_\chi^- = \frac{\partial_u + \partial_v}{2} + \frac{\sinh 2\varrho}{2} \partial_e.
$$

(4.38)

In these two sets of basis, the diffeomorphisms can be written as

$$
\xi^a = c_+ e^{2pu_+} \left[ \frac{1}{2} \partial_+ - \frac{p(p + 1)}{4 \sinh^2 \varrho} \bar{\partial}_+ + \left( \frac{1}{2} - \frac{p(p - 1)}{4 \cosh^2 \varrho} \right) \partial_\chi^+ \right] 
+ c_- e^{2pu_-} \left[ \frac{1}{2} \partial_- - \frac{p(p - 1)}{4 \sinh^2 \varrho} \bar{\partial}_- + \left( \frac{1}{2} - \frac{p(p + 1)}{4 \cosh^2 \varrho} \right) \partial_\chi^- \right],
$$

(4.39)
\[ \tilde{c}_a = c_+ e^{2qv_+} \left[ \frac{1}{2} \partial_+ - \frac{q(q+1)}{4 \sinh^2 \omega} \partial_+ + \left( \frac{1}{2} - \frac{q(q-1)}{4 \cosh^2 \omega} \right) \partial_{\chi} \right] + c_- e^{2qv_-} \left[ \frac{1}{2} \partial_- - \frac{q(q-1)}{4 \sinh^2 \omega} \partial_- + \left( \frac{1}{2} - \frac{q(q+1)}{4 \cosh^2 \omega} \right) \partial_{\chi} \right] . \] (4.40)

Written in terms of the coordinates \( u_\pm \) and \( v_\pm \), we can better understand the role of the set of constants \( c_\pm \) and \( \tilde{c}_\pm \). We begin by related to the usual time and radial BTZ coordinates \( t \) and \( r \) in (4.1), can be seen to satisfy the asymptotic values

\[
\begin{align*}
t = \infty : \quad & \begin{cases} u_+ + u_- = \infty, \\
v_+ + v_- = -\infty, \end{cases} \quad t = -\infty : \begin{cases} u_+ + u_- = -\infty, \\
v_+ + v_- = \infty, \end{cases} \\
r = r_+ : \quad & \begin{cases} u_+ - u_- = -\infty, \\
v_+ - v_- = -\infty. \end{cases}
\end{align*}
\] (4.41)

Hence one can identify the future event horizon \( H^+ \) as located at the outgoing coordinates \( u_-=\infty \) and \( v_+=-\infty \), while the past horizon \( H^- \) at the ingoing coordinates \( u_+=-\infty \) and \( v_-=\infty \). Now, the left diffeomorphism (4.39) act at the horizons as

\[
\begin{align*}
H^+ : \quad & \begin{cases} u_+ \to u_+ + \frac{c_+}{2} e^{2pu_+}, \\
\chi \to \chi + \frac{c_+}{2} \left( 1 - \frac{p(p-1)}{2} \right) e^{2pu_+}, \end{cases} \\
H^- : \quad & \begin{cases} u_- \to u_- + \frac{c_-}{2} e^{2pu_-}, \\
\chi \to \chi + \frac{c_-}{2} \left( 1 - \frac{p(p+1)}{2} \right) e^{2pu_-}. \end{cases}
\end{align*}
\] (4.42)
and the right diffeomorphism (4.40) as
\[
\mathcal{H}^+ : \begin{cases} 
v_- \to v_- + \frac{\bar{c}_-}{2} e^{2qv_-} \\
\chi \to \chi + \frac{\bar{c}_-}{2} \left( 1 - \frac{q(q + 1)}{2} \right) e^{2qv_-}, \end{cases} \tag{4.44}
\]
\[
\mathcal{H}^- : \begin{cases} 
v_+ \to v_+ + \frac{\bar{c}_+}{2} e^{2qv_+} \\
\chi \to \chi + \frac{\bar{c}_+}{2} \left( 1 - \frac{q(q - 1)}{2} \right) e^{2qv_+}. \end{cases} \tag{4.45}
\]

This rather simple form can be cast in a mode-decomposition independent form, for instance, if \( c_- = 0 \), the transformation of \( \mathcal{H}^+ \) is given by:
\[
u_+ \to u_+ + \epsilon f(u_+), \quad \chi \to \chi + \epsilon \left( f(u_+) + \frac{1}{4} f'(u_+) - \frac{1}{8} f''(u_+) \right), \tag{4.46}
\]
which can be compared with the literature as a different gauge choice for the large diffeomorphisms. One can also expect this behavior from the generic solution presented in [24]. We also note that the condition \( c_- = 0 \) leaves the past horizon invariant. One can then, relate the solutions obtained for \( c_+ = 0 \) and \( c_- = 0 \) in (4.22) by a time reversal. A similar condition will arise in the next section.

5 The Kerr black hole

The Kerr metric for mass \( M \) and angular momentum \( J = aM \) is suitably described in the Kinnersley null tetrad basis, see [25]:
\[
\ell = e_1 = \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} + \frac{a}{\Delta} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial r}, \quad n = e_2 = \frac{\Delta}{2\Sigma} \left( \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} + \frac{a}{\Delta} \frac{\partial}{\partial \phi} - \frac{\partial}{\partial r} \right),
\]
\[
m = e_3 = \frac{1}{\sqrt{2} \sigma} \left( ia \sin \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \phi} \right), \quad \bar{m} = e_4 = (m)^*.
\]
where \( t, r, \theta, \phi \) are the Boyer-Lindquist coordinates and
\[
\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-), \quad \sigma = r + ia \sin \theta, \quad \Sigma = \sigma \sigma^* = |\sigma|^2, \tag{5.2}
\]
and the metric is defined by saying that the only non-vanishing inner products are:

\[ \ell \cdot n = -1, \quad m \cdot \bar{m} = 1. \] (5.3)

This defines the local Minkowski metric to be:

\[ \eta^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \] (5.4)

The Kinnersley basis is particularly useful for dealing with perturbations of the Kerr metric. Separability of the equations is achieved for scalar, spinorial, vector and gravitational perturbations. Our problem is to find non-trivial solutions to the equations

\[ \nabla^2 \xi^a = 0, \quad \nabla_a \xi^a = 0, \] (5.5)

which approach supertranslations at null infinity \( I^\pm \). In general, the problem can be cast into the solution for the potential of a spin-1 perturbation of the Kerr black hole. These are defined by the solution to the vacuum Maxwell equations

\[ \nabla^a F_{ab} = \nabla_{[a} F_{bc]} = 0 \] (5.6)

in terms of the vector potential \( A_a \) such that \( F_{ab} = \nabla_a A_b - \nabla_b A_a \). We have that the vacuum Maxwell equations generate a solution to (5.5) if we choose the Lorenz gauge \( \nabla_a A^a = 0 \), given that the Kerr metric is Ricci-flat. Expressions for the general potential in [26] use the “ingoing and outgoing radiation gauges” \( \ell^a A_a = 0 \) or \( n^a A_a = 0 \), which do not suit our purposes because the gauge transformation is as hard as the problem we want to solve. As in (2.13), we will suppose that \( \xi^a \) is a gradient \( \xi^a = \nabla_a \Phi \) to begin with. Then one can obtain a solution to (5.5) by just considering \( \Phi \) to be a solution of the wave equation \( \nabla^2 \Phi = 0 \), with the asymptotic boundary condition from (2.13):

\[ \Phi \approx \Omega \alpha(\theta, \phi), \quad \text{at} \quad \Omega = 0. \] (5.7)
Again, the divergence-free condition is satisfied by assumption and:

$$\nabla^2 \nabla_b \Phi = \nabla_b \nabla^2 \Phi + R_b{}^d \nabla_d \Phi \quad (5.8)$$

also vanishes in Ricci-flat backgrounds like the the Kerr metric. The scalar Laplacian can be written in terms of the Ricci rotation coefficients $\gamma_{\rho \mu \nu} = (e^a_\rho)(e^b_\mu)(\nabla_a e^b_\nu)$:

$$\nabla^2 \Phi = \eta^{\mu \nu} (e^a_\mu)(e^b_\nu) \nabla_a \nabla_b \Phi = \eta^{\mu \nu} e^a_\mu (e^b_\nu(\Phi)) - \eta^{\mu \nu} \gamma_{\rho \mu \nu} \eta^{\rho \sigma} e^\sigma_\sigma(\Phi), \quad (5.9)$$

where we used the definition of vectors as directional derivatives to write $(e^a_\mu) \nabla_a \Phi \equiv e^a_\mu(\Phi)$. In terms of the Newman-Penrose symbols, which for the Kerr metric can be found in [25], the Laplacian becomes:

$$\nabla^2 \Phi = -\ell (n(\Phi)) - n(\ell(\Phi)) + m(\bar{m}(\Phi)) + \bar{m}(m(\Phi))$$

$$+ (\bar{\rho} + \bar{\rho}^* - \bar{\epsilon} - \bar{\epsilon}^*)n(\Phi) + (\bar{\gamma}^* + \bar{\gamma} - \bar{\mu}^* - \bar{\mu})\ell(\Phi)$$

$$- (\bar{\tau}^* - \bar{\pi} + \bar{\alpha} - \bar{\beta}^*)m(\Phi) - (\bar{\tau} - \bar{\pi}^* + \bar{\alpha}^* - \bar{\beta})\bar{m}(\Phi). \quad (5.10)$$

We continue by introducing the differential operators:

$$\ell = D = \frac{r^2 + a^2}{\Delta} \frac{\partial}{\partial t} + \frac{a}{\Delta} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial r}, \quad n = -\frac{\Delta}{2\Sigma} \bar{D};$$

$$m = \frac{1}{\sqrt{2} \sigma} Q = \frac{1}{\sqrt{2} \sigma} \left( i a \sin \theta \frac{\partial}{\partial t} + i \frac{\partial}{\sin \theta \partial \phi} + \frac{\partial}{\partial \theta} \right), \quad \bar{m} = \frac{1}{\sqrt{2} \sigma^*} \bar{Q}, \quad (5.11)$$

in terms of which we can write the Laplacian operator as a sum of two anticommutators:

$$\nabla^2 \Phi = \frac{1}{\Sigma} \left( \frac{1}{2\Delta} \{ \Delta D, \Delta \bar{D} \} + \frac{1}{2 \sin^2 \theta} \{ \sin \theta Q, \sin \theta \bar{Q} \} \right). \quad (5.12)$$

It can be checked that both terms of the sum in the brackets commute. Writing each explicitly in terms of the Boyer-Lindquist coordinate operators:

$$\frac{1}{2\Delta} \{ \Delta D, \Delta \bar{D} \} = \frac{\partial}{\partial r} \Delta \frac{\partial}{\partial r} - \frac{1}{\Delta} \left( (r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \phi} \right)^2, \quad (5.13)$$

$$\frac{1}{2 \sin^2 \theta} \{ \sin \theta Q, \sin \theta \bar{Q} \} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( a \sin^2 \theta \frac{\partial}{\partial t} + \frac{\partial}{\partial \phi} \right)^2. \quad (5.14)$$
The wave operator is now separable and the solutions can be written in terms of confluent Heun equations. We will write the generic solution for frequency $\omega$, angular momentum $m$ and angular quantum number $l$ by:

$$\Phi_{\omega,l,m}(r,\theta,\phi) = e^{-i\omega t}(C^\infty h^+_{\omega,l,m}(r) + C^- h^-_{\omega,l,m}(r))_0 S_{lm}(\cos \theta)e^{im\phi},$$

(5.15)

where $h^\pm$ are confluent Heun functions [27] and $0 S_{lm}$ are the scalar spheroidal harmonics [28]. In the zero frequency limit, $l$ reduces to the usual spherical quantum number, $0 S_{lm}$ to the usual spherical harmonic $Y^m_l$ and the separation constant between the angular and radial equation is $l(l+1)$. The radial functions have the asymptotic behavior – see [29]:

$$h^\pm_{\omega,l,m} = \frac{e^{\pm i\omega r}}{r^{1\mp i\omega}(r^+ + r^-)}(1 + O(r^{-1})),$$

(5.16)

which poses us a problem for the boundary conditions at $I^\pm$, given by (5.7), because (5.16) has an essential singularity at $\Omega \simeq (r \pm t)^{-1} = 0$, unless $\omega = 0$.

One can actually show that there is a solution: if one goes to the unphysical metric $\hat{g}_{ab}$ a little algebra shows that if $\Phi$ satisfies the wave equation with respect to the metric $g_{ab}$, then $\hat{\Phi} = \Omega^{-1/6} \Phi$ satisfies:

$$\hat{g}^{ab}\hat{\nabla}_a \hat{\nabla}_b \hat{\Phi} - \frac{1}{6} \hat{R} \hat{\Phi} = 0,$$

(5.17)

where $\hat{R}$ is the Ricci scalar associated with $\hat{g}_{ab}$. As $\Omega = 0$, the asymptotic flat space-time approaches the asymptotic structure of Minkowski space, and there is a gauge choice where $\hat{g}_{ab}$ approaches the metric of the standard Einstein static universe:

$$ds^2_{EE} = -dx^2 + dy^2 + \sin^2 y(d\theta^2 + \sin^2 \theta \, d\phi^2),$$

(5.18)

which has constant Ricci scalar: $\hat{R}_{EE} = 24$. In this case, one can verify [30] that (5.17) has a Green’s function:

$$G_{EE} = \frac{1}{2\pi^2 \cos(\Delta x) - \cos(\Delta s)},$$

(5.19)

where

$$\Delta s = \cos y \cos y' + \sin y \sin y' (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'))$$

(5.20)
is the invariant distance on the 3-sphere. One sees that $I^\pm$ is mapped to $x \pm y = \pi$, where
the static universe metric can be continued without any problems. By propagating
using the left or right-moving part of $G_{\text{EE}}$ one can obtain a solution at the interior with
the prescribed boundary behavior. Note that $\hat{\Phi} = \alpha$ is finite at $\Omega = 0$.

As in the case with the BTZ black hole, there is an ambiguity with the choice of
constants $c^\infty_\pm$. Like there, we may fix this by requiring that the induced diffeomorphism
is “outward moving” near the outer horizon $r_+$. As it happens, for $\omega = 0$ the radial
equation simplifies to Riemann’s differential equation form, and the generic solution
can be written in terms of Gauss’ hypergeometric functions. Near the outer horizon,
one can find the asymptotic behavior:

$$\Phi_{l,m}(r) \approx (c^+_+(r - r_+)\theta^+ + c^+_-(r - r_-)^{-i\theta^+})P^m_l(\cos \theta)e^{im\phi}, \quad \text{near } r = r_+. \quad (5.21)$$

However, from the discussion at (3.8), if one takes the viewpoint that the “only true
observables” are the mass and the angular momentum, the dependence on $\theta$ above is
spurious. One then is led to consideration of the asymptotic potentials of the form:

$$\Phi_{l,m}(r) \approx c^+_+(r - r_+)\theta^+ + c^+_-(r - r_-)^{-i\theta^+}e^{im\phi}, \quad \text{at } r = r_+. \quad (5.22)$$

The coefficients at the horizon $c^\pm_\pm$ are linear combinations of $c^\infty_\pm$, the linear transformation
matrix entries are called the connection coefficients. The radial exponent has the
nice interpretation:

$$\theta^+ = \frac{r_-m(r_+ + r_-)}{r_+ - r_-} = \frac{1}{2\pi} \frac{m\Omega_+}{T_+}, \quad (5.23)$$

where $\Omega_+$ and $T_+$ are the angular velocity and the temperature of the horizon at
$r = r_+$. Therefore, $\theta^+$ is, up to a factor of $2\pi$, the increase in entropy of the black hole
by absorption of a wave with zero energy and angular momentum $m$.

In terms of the tortoise radial coordinate:

$$r_* = r + \frac{r_+(r_+ + r_-)}{r_+ - r_-}\log \left(\frac{r}{r_+} - 1\right) - \frac{r_-(r_+ + r_-)}{r_+ - r_-}\log \left(\frac{r}{r_-} - 1\right), \quad (5.24)$$

the coordinate transformation can be interpreted as a chiral shift of the co-moving
coordinates and a shift of $r$, which places a conformal transformation. The transformation is best seen in Teukolsky-Eddington-Finkelstein coordinates [31] – see also [32].
Defining $\tilde{\phi}_\pm$ as
\[ d\tilde{\phi}_\pm = d\phi \pm \frac{a}{r^2 + a^2}dr_* = d\phi \pm \frac{a}{\Delta}dr, \tag{5.25} \]
we can put the covariant form of the transformation in the rather simple form:
\[ (\xi_m)_a = im c^+_m e^{im\tilde{\phi}_+} (d\tilde{\phi}_+)_a + im c^-_m e^{im\tilde{\phi}_-} (d\tilde{\phi}_-)_a \tag{5.26} \]
The structure is then quite similar to (4.30) and (4.32). In terms of vector fields (supposing either $c^+_\pm = 0$, and $r \simeq r_+$):
\[ (\xi_m)^a \propto im e^{im\tilde{\phi}_\pm} \frac{1}{\Sigma\Delta} \left( \frac{\partial}{\partial t} + \Omega_+ \frac{\partial}{\partial \phi} \pm \frac{\partial}{\partial r_*} \right), \tag{5.27} \]
approaching the “radial-temporal” $\ell, n$ elements of Kinnersley null tetrad near the outer horizon, depending on whether one chooses purely ingoing or outgoing transformations.

However, in taking purely ingoing modes at the horizon, one has to fix $c^+_\pm$ and $c^-\pm$ appropriately. This fixing seems strange from the boundary conditions placed at $I^+$, since this condition is actually placed at $H^+$. The situation resembles to the “in-mode” vs. “up-mode” decomposition of black hole scattering, see Fig. 1. In terms of the actual choice of solution of the wave equation both of them use the time reversal symmetry to identify the “outgoing” function at the future null infinity $I^+$ or future horizon $H^+$ with the “ingoing” function at past null infinity $I^-$ or the past horizon $H^-$. The use of time reversal is necessary because, generically, there is no canonical way to associate the space of physical states at $I^+$ to the one at $I^-$, and hence no way to compute scattering coefficients. By the time reversal recipe, the coefficients of normalized waves have the interpretation of scattering coefficients.

For our purposes, the in-mode choice selects the unique solution by setting $c^+_\pm = 0$. One then faces the problem of interpreting the $u^-\pm$ component in (5.15). We will interpret this just as the scattering problem: as a time-reversal induced supertranslation at $I^-$. By this recipe, the outgoing supertranslations associated to the horizon are very specific combinations of supertranslations at $I^+$ and $I^-$. From a physical point of view, this interpretation seems in league with the experiments performed by an asymptotic observer in order to detect the black hole. We note that this particular feature distinguishes the Kerr case from the BTZ case, with the former believed to be closer to the generic case.
Figure 1. Respectively, the in- and up-mode of wave scattering at the outer horizon of a black hole. The in-mode is characterized by the solution of the wave equation in the Kerr background with $c^+ = 0$ whereas in the up-mode $c^\infty = 0$ is chosen.

6 Discussion

In this paper we proposed to use the gauge fixing procedure for perturbations in classical general relativity to associate asymptotic symmetries to bulk symmetries. In the case where the space-time has a black hole, these will induce transformations at the horizon, in a way reminiscent of recent proposals by [10, 11, 17, 18, 33] – among others. One finds that these have always “zero-frequency”, corresponding to “soft-gravitons” in the IR-limit, which have been discussed by numerous authors – see [34, 35] for a recent overview. The induced transformations at the horizon seem to have the generic behavior of coordinate transformations involving the co-rotating angular variable $\phi_R$, along with suitable scalings on the radial direction. One also sees that both in three and four dimensions the condition that these transformations are outgoing at the horizon $\mathcal{H}^+$ selects the solution uniquely. Because of the non-commuting nature of (some of) the supertranslations at infinity to the mass and angular momentum, we know that these constitute “large gauge transformations” and should be treated as true degrees of freedom of the theory.

Given the generality of the elements, one cannot help but wonder whether the construction generalizes to higher dimensions. In a series of articles, Barnich and
collaborators [36–39] helped with the issue of supertranslations in different dimensions and settings, so it seems an easy target. The connections of the diffeomorphisms outlined here and conformal field theory [7, 40] cannot be overlooked. The existence of constraints posed by the gauge choice (3.6) are the key piece behind the appearance of central terms in the representation of the conserved quantities in classical mechanics [41]. A positive result for the appearance of central terms in the representation algebra would be most interesting in the long standing problem of holographic description of asymptotically flat space-times [42, 43]. We also hope to address the problem of quantization in future work.

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