Bounds on Black Hole Entropy in Unitary
Theories of Gravity

Ram Brustein (1), A.J.M. Medved (2)
(1) Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel,
(2) Physics Department, University of Seoul, Seoul 130-743 Korea
E-mail: ramyb@bgu.ac.il, allan@physics.uos.ac.kr

ABSTRACT: We consider unitary and weakly coupled theories of gravity that extend
Einstein gravity and reduce to it asymptotically at large distances. Our discussion is
restricted to such theories that, similarly to Einstein gravity, contain black holes as
semiclassical states in a range of scales. We show that, at a given scale, the entropy
of these black holes has to be larger than the number of elementary light species in
the theory. Our bound follows from the observation that the black hole entropy has
to be larger than the product of its mass and horizon radius (in units of Planck’s
constant divided by the speed of light) and the fact that, for any semiclassical black
hole, this product has to be larger than the number of light species. For theories
that obey our assumptions, the bound resolves the “species problem”: the tension
between the geometric, species-independent nature of black hole entropy and the
proportionality of ordinary thermodynamic entropy to the number of species. We
then show that, when black holes in Einstein’s theory are compared to those in the
extended theories at a fixed value of mass, the entropy of the Einstein black holes will
always be minimal. Similar considerations are also applied to the entropy density of
black branes in anti-de Sitter space.

KEYWORDS: Black Holes, Models of Quantum Gravity.
1. Introduction

The entropy of a black hole (BH) depends on a purely geometric property, its area in units of Newton’s constant $G$. The geometric nature of the BH entropy persists for extensions of Einstein gravity, for which it is given by Wald’s Noether-charge formula. Then, as shown in [6], one can always re-express Wald’s geometric entropy in the standard area-law form but in theory-dependent units that are determined by the gravitational coupling. For Einstein’s theory, the coupling is simply Newton’s constant but, for a general theory, this is not necessarily true. On the other hand, ordinary thermodynamic entropy is proportional to the number of light species $N$. So, apparently, if a sufficiently large number of light species existed at a given length scale, then the thermodynamic entropy of the matter that formed a BH larger than this scale would exceed any value of entropy that does not depend on $N$. Consequently, there appears to be a severe tension between the purely geometric entropy of BHs and the second law of thermodynamics. This argument seems to
suggest that BHs limit the number of light species and has been called “the species problem” [7].

In this paper, we consider unitary and weakly coupled theories of gravity that extend Einstein gravity. We further assume that these theories reduce to Einstein gravity asymptotically at large distances. The main differences between Einstein gravity and the extensions under consideration are then expected to occur at small distances. By unitary, we mean that the theory is free of ghosts below its ultraviolet (UV) momentum cutoff scale. Since the theory is weakly coupled, we can define the number of light species $N$ more precisely [8]. We wish to consider theories that, at an energy scale $\Lambda = 1/l$, have a finite number $N(l)$ of light species with a mass $m$ less than this scale and with a decay width $\Gamma$ less than their mass; $\Gamma < m < \Lambda$. The decay width $\Gamma$ is defined as the inverse of the lifetime of the state, which is the time that it takes for the first transition to a lower energy state via the emission of a light quantum. The number $N(l)$ includes the graviton and possibly other gravitational degrees of freedom. Of particular interest is the case that the energy scale $\Lambda$ is at the UV cutoff; $\Lambda_{UV} = 1/l_{UV}$.

We assume that the coupling of all the light species $N(l)$ is such that they can be at thermal equilibrium at (inverse) temperature $\beta = l$. We consider only metric theories of gravity and define the scale $l_{UV}$ for such theories as the scale above which exchanges of metric perturbations in elementary particle processes become strong. Obviously, it follows that, for curvatures less than $1/l_{UV}^2$, gravity is weak and semiclassical. It may well be that Einstein’s gravity is modified for scales well above $l_{UV}$; for example, if large extra dimensions of size $R > l_{UV}$ exist.

Let us recall the properties of any semiclassical BH, as described in [8], since these are central to the following discussion. We consider, as in [8], neutral, static and non-rotating BHs and recall that these can be described in terms of three parameters: the mass $M$, the Schwarzschild radius $R_S$ and the inverse temperature $\beta = 1/T$. By the no-hair theorem [9], these parameters must be related in a simple way for any theory of gravity; however, the precise relation is theory dependent. The following conventions are used throughout: $\hbar, k_B, c = 1$, the number of dimensions of the
spacetime is $D = d + 1 = n + 2 \geq 4$ but we mostly limit ourselves to $D = 4$. As our emphasis is on the parametrical behavior of quantities, numerical factors of order unity are consistently neglected. The subscripts $E$ and $X$ are always used to denote quantities in Einstein’s theory and a generic extension, respectively. If no subscript appears on a certain quantity then it is relevant to all theories; for example, $R_{S,E}$ is the Schwarzschild radius of a BH in Einstein gravity, $R_{S,X}$ is the Schwarzschild radius of a BH in an extended gravity, and $R_S$ denotes a Schwarzschild radius of a BH in either. For simplicity, we mainly consider static “Schwarzschild-like” BHs in an asymptotically flat spacetime. Analogous bounds can typically be formulated for more complicated scenarios.

As in [8], we assume that semiclassical BHs are

i) Black bodies,

ii) Classically black.

Assumption $i)$ can be stated as

$$- \frac{dM}{dt} = N(\beta) \beta^{-4} R_S^2.$$ 

Here, $N(\beta)$ is the number of light species into which the BH can decay. Assumption $ii)$ implies that the quantum wavelength of particles emitted by the BH is at least as large as $R_S$. Since, according to assumption $i)$, the BHs are black bodies, this implies that the energy of emitted quanta $R_S^{-1}$ also bounds the BH temperature, so

$$R_S/\beta \leq 1 . \quad (1.1)$$

Following [10]-[14] and [8], we further assume that, for semiclassical BHs, the size and inverse temperature decrease slower than the speed of light:

$$(a) \quad - \frac{dR_S}{dt} < 1 ,$$

$$(b) \quad - \frac{d\beta}{dt} < 1 .$$
Additionally, the fraction of the mass loss of the BH has to be small during both the thermal and the light crossing time scales:

\[
(c) \quad - \frac{R_S}{M} \frac{dM}{dt} < 1 ,
\]

\[
(d) \quad - \frac{\beta}{M} \frac{dM}{dt} < 1 .
\]

Finally, we require that the BH be metastable or

\[
(e) \quad \frac{\Gamma}{M} < 1 .
\]

The width \( \Gamma \) is defined the same as for the elementary species: the inverse of the time it takes for the first transition from the BH to a lower energy state by the emission of a light quantum. In the preceding discussion, we have ignored grey factors and numerical factors related to the statistics of the species. None of the decay channels of the BH are expected to be parametrically suppressed at energies \( \sim 1/\beta \), which is the main energy range of the BH emission. Since the BHs are assumed to be black bodies, it follows that \( N(\beta) \) is equal to the number of species that can be in thermal equilibrium at (inverse) temperature \( \beta \). In [8], it was shown that BHs which satisfy all the above assumptions obey the important inequality

\[
R_S M > N(\beta) .
\]  (1.2)

As argued by Dvali and others [10]-[14], through the implementation of non-perturbative consistency arguments, the radial size of a black hole for Einstein’s theory is limited by the minimal scale \( l_{UV;E} \) such that \( l_{UV;E}^2 = G_E N_E (l_{UV;E}) \). Physically, \( l_{UV;E} \) is the effective UV cutoff for the theory, below which semiclassical gravity can no longer be trusted. With this scale as a lower bound on \( R_E \), along with the standard area–entropy law \( S = R_{S;E}^2 / G_E \), the inequality

\[
S_E > N_E (l_{UV;E})
\]  (1.3)

then follows.
2. The number of species bounds black hole entropy

We wish to generalize bound (1.3) and find out whether it is also valid for extensions of Einstein gravity.

Let us relate the BH entropy \( S_X \) to its mass \( M_X \). Using our assumption that semiclassical BH’s exist for a range of masses, we can then consider the situation of a BH changing its mass, size and temperature such that the resulting state is another semiclassical BH. This allows us to invoke the first law of BH mechanics in its integral form.

Recalling the differential form of the first law of BH mechanics, we have

\[
\frac{dS_X}{dM_X} = \beta_X(M_X) \; ;
\]

from which the BH entropy is given by

\[
S_X = \int \beta_X(M_X) dM_X .
\]

Since bound (1.4) is valid for all values of the mass, \( \beta(M_X) \geq R_{S;X}(M_X) \) and thus

\[
\int \beta_X(M_X) dM_X \geq \int R_X(M_X) dM_X .
\]

For a Schwarzschild BH in Einstein’s theory, bound (2.3) is saturated, \( \beta_E(M_E) = R_{S;E}(M_E) \), leading to the integrated first law \( S_E = R_{S;E} M_E \). Meanwhile, for any weakly coupled extension, \( R_{S;X}(M_X) = M_X [1 + I_X(M_X)] \) in Planck units, where the “correction” \( M_X I_X \) is known to be positive definite and of order unity (see Eq. (13) of \cite{8} and the surrounding discussion). It follows that

\[
\int R_{S;X}(M_X) dM_X = R_{S;X} M_X ,
\]

up to a parametrically small correction (the difference between \( \int I_X M_X dM_X \) and \( I_X M_X^2 \), which is generically of order 1) and a constant of integration. We deal with the latter by extrapolating the semiclassical regime down to \( M_X = 0 \) and then making the natural assumption that, just like for Einstein gravity, the integration constant is zero. That is, a vanishing mass means a non-existent BH horizon with the corresponding limit \( R_{S;X}(M_X \to 0) \to 0 \) being well defined.
Combining Eqs. (2.2), (2.4) and bound (2.3), we obtain the entropy bound

\[ S_X \geq R_{S,X} M_X . \]  

(2.5)

Then, combining inequality (1.2) and bound (2.5), we arrive at the desired inequality for any semi-classically allowed value of \( \beta_X \):

\[ S_X > N_X(\beta_X) . \]  

(2.6)

Since inequality (2.6) is valid for all scales, it is also valid for \( \beta_X \sim l_{UV;X} \), and so

\[ S_X > N_X(l_{UV;X}) , \]  

(2.7)

which is the obvious analogue of Eq. (1.3).

We expect that \( N_X(l) \) will monotonically increase with increasing energy scale or decreasing \( l \). Hence, it is expected that the strongest possible bound on the entropy is when \( \beta_X \) reaches the UV cutoff \( l_{UV;X} \) for the generic theory; that is, the inequality (2.7) is the most restrictive one.

3. The entropy of black holes in Einstein gravity is minimal

We can make further use of bound (2.5) to show that \( S_X(M) \geq S_E(M) \) for a fixed mass \( M \).

Our other key input is that, for a unitary extension of Einstein gravity, any gravitational coupling in the vicinity of the horizon can only increase from its Einstein value. To understand this point, let us consider small perturbations of the background (Schwarzschild-like) metric near the horizon: \( g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \) with \( h_{\mu\nu} \ll 1 \).

Since the perturbations are small, the one-particle exchange approximation can be invoked to determine the coupling of the gravitons. For those near the BH horizon, it is appropriate to use a flat-space form of the 1PI propagator. Indeed, in spite of the curvature of the BH background, a particle near the horizon will “observe” an effective geometry that is two dimensional and conformally flat. This becomes evident, as we next discuss, from the near-horizon form of the Klein–Gordon equation.
For a Schwarzschild-like BH solution the metric can be expressed in the form
\[ ds^2 = -f(r)dt^2 + \frac{1}{g(r)}dr^2 + r^2d\Omega^2_n, \]
where \( f(r) \) and \( g(r) \) have simple zeroes at the horizon \( r = R_S \), while \( \frac{\partial_r f(r)|_{r=R_S}}{\partial_r g(r)|_{r=R_S}} \) are both finite and non-vanishing. It is convenient to introduce the usual radial “tortoise” coordinate \( r^* \) such that \( dr^* \equiv dr/\sqrt{fg} \); then \( ds^2 = f(r^*) \left[ -dt^2 + dr^2 \right] + r^2d\Omega^2_n \). Let us, for simplicity, consider the Klein–Gordon equation for a scalar (the results also apply to gravitons): \( (\Box \phi - m^2)\phi = 0 \) or
\[ \left[ -\frac{1}{f}\partial_t^2 + \frac{1}{r^2f}\partial_{r^*} \left( r^2\partial_{r^*} \right) + \frac{1}{r^2f}\partial^2_{\Omega} - m^2 \right] \phi = 0, \] (3.1)
where \( \partial^2_{\Omega} \) represents the sum over angular derivatives. Multiplying through by \( f \) and using the defining relation for \( r^* \), we obtain
\[ \left[ -\partial_t^2 + \frac{\sqrt{fg}}{r}\partial_{r^*} + \partial^2_{r^*} + \frac{1}{r^2f}\partial^2_{\Omega} - fm^2 \right] \phi = 0. \] (3.2)
Taking the near-horizon limit, we find that Eq. (3.2) reduces to
\[ \left[ -\partial_t^2 + \partial^2_{r^*} \right] \phi = 0. \] (3.3)
That is, given a scattering process in the vicinity of the horizon, any of the involved particles are essentially massless and effectively “perceive” a two-dimensional, flat spacetime.

Let us now discuss a two-graviton scattering process in a flat spacetime. For Einstein gravity, only massless spin-2 gravitons are exchanged, but gravitons can, for a general theory, be either massless or massive and either spin-0 or spin-2. Particles of any other spin, in particular vectors, can not couple linearly to a conserved source and may therefore be ignored. Consequently, the 1PI graviton propagator \( [D(q^2)]_{\nu,\rho,\alpha,\beta} \equiv \langle h_{\mu,\nu}(q)h_{\lambda,\rho}(-q) \rangle \) has the following irreducibly decomposed form [15]-[18]:
\[ [D(q^2)]_{\nu,\rho,\alpha,\beta} = (\rho_E(q^2) + \rho_{NE}(q^2)) \left[ \delta_{\mu,\nu}\delta_{\alpha,\beta} - \frac{1}{2}\delta_{\mu,\rho}\delta_{\alpha,\beta} \right] \frac{G_E}{q^2} \]
\[ + \sum_i \tilde{\beta}_{\mu,\rho,\alpha,\beta}^{\lambda,\alpha,\beta} \left( \delta_{\mu,\nu}\delta_{\alpha,\beta} - \frac{1}{3}\delta_{\mu,\rho}\delta_{\alpha,\beta} \right) \frac{G_E}{q^2 + m_i^2} \]
\[ + \sum_j \tilde{\beta}_{\mu,\rho,\alpha,\beta}^{\lambda,\alpha,\beta} \delta_{\mu,\rho}\delta_{\alpha,\beta} \frac{G_E}{q^2 + m_j^2}. \] (3.4)
Here, the positive number $q^2 = -q^\mu q_\mu$ is the flat-spacetime (spacelike) momentum, the propagator is evaluated in the vacuum state and $G_E$ is the $D$-dimensional Newton’s constant. Also, the Einstein and “Non-Einstein” parts of the gravitational couplings $\rho$ are indicated by the subscripts $E$ and $NE$. We have distinguished between the massless spin-2 particles, massive spin-2 particles with mass $m_i$ and scalar particles with mass $\tilde{m}_j$. The $\rho$’s are dimensionless couplings that generally depend on the energy scale $q$; in particular, $\rho_E(0) = 1$ and $\rho_{NE}(0) = 0$.

Although valid for a flat spacetime, Eq. (3.4) is also applicable to a scattering near a BH horizon where the spacetime is, as shown above, effectively two-dimensional and flat. The internal states are limited to massless particles propagating strictly in the radial direction, so that all the $m$’s vanish and $q^2 \to q^2_{2D} = q_t^2 - q_r^2$. Hence,

\[
[D_{r=R}(q^2 = q^2_{2D})]_{\mu}^{\nu} = \left(\rho_E(q_{2D}^2) + \rho_{NE}(q_{2D}^2)\right) \left[\delta_{\mu}^{\beta} \delta_{\alpha}^{\nu} - \frac{1}{2} \delta_{\mu}^{\nu} \delta_{\alpha}^{\beta}\right] \frac{G_E}{q_{2D}^2} + \sum_i \rho_{NE}(q_{2D}^2) \left(\delta_{\mu}^{\beta} \delta_{\alpha}^{\nu} - \frac{1}{3} \delta_{\mu}^{\nu} \delta_{\alpha}^{\beta}\right) \frac{G_E}{q_{2D}^2} + \sum_j \tilde{\rho}_{NE}(q_{2D}^2) \delta_{\mu}^{\nu} \delta_{\alpha}^{\beta} \frac{G_E}{q_{2D}^2},
\]

(3.5)

whereas $[D_{r=R}(q^2 \neq q^2_{2D})]_{\mu}^{\nu} = 0$.

Eq. (3.3) describes the annihilation of a graviton at one point in the near-horizon region followed by its creation at another point in the same region, with virtual processes taking place in between. It is important to emphasize that the $D$-dimensional origin of the propagator is still encoded in Eq. (3.5). The external indices $\mu, \nu, \alpha, \beta$ can take on any of the original $D$-dimensional values. Additionally, each $D$-dimensional field leads to many two-dimensional fields having the same two-dimensional momentum but with a slightly different angular profile. So that, even though the internally scattered particles are massless and restricted to radial motion, the $\rho$’s still keep track of the original $D$-dimensional fields.

For a unitary theory, all of the couplings must remain positive at all energy scales \cite{17}; meaning that the propagator can only increase relative to its Einstein value. Let us recall the Kallen–Lehmann representation of the 1PI propagator (see,
e.g., Ch. 12 of [13]). We observe that the \( \rho \)'s are spectral densities that can be microscopically evaluated with the insertion of a complete set of single- and multi-particle states; schematically, \( \rho = \sum_n <0|h|n><n|h|0> \). If all such inserted states have a positive norm, there can only be positive contributions to a given \( \rho \). By the same token, a generalized theory that extends Einstein's can only make a positive contribution to any Einstein spectral density or gravitational coupling. To rephrase, any additional degree of freedom will open up additional intermediating attractive channels. The extra channels can only act to increase the overlap between two given graviton states at two different points, thus leading to an increase in the associated gravitational coupling.

Let us recall that, for a generalized gravity theory, the BH entropy is given by Wald’s Noether-charge formula [3, 4, 5], which can then be re-expressed [6] as an area law with a theory-dependent gravitational coupling \( G_X \). For Einstein’s theory, \( G_X = G_E \) is simply Newton’s constant. More generally, \( G_X \) determines the strength of the r,t-polarized gravitons in the vicinity of the horizon and, thus, can also be extracted from the near-horizon propagator for the \( h_{rt} \) gravitons [6, 20]. Then, from Eq. (3.5), it follows that \( G_X \geq G_E \) must hold true at any given energy scale. We cannot, however, directly apply Eq. (3.5) to determine how the entropy \( S_X \) compares to the Einstein value (as was done previously for the shear viscosity of a brane theory via the \( h_{xy} \) gravitons [21]). This is because, for a BH (and unlike for a black brane), the horizon radius also depends on the strength of the coupling.

We would like to compare the generalized entropy \( S_X \) to its Einstein counterpart \( S_E \) at a fixed value of the BH mass, as a microcanonical calculation is the most appropriate one for a static BH. However, the mass is defined at infinity, whereas we require a quantity that is determined at the horizon. For \( D = 4 \), a natural choice is to use \( \tilde{M}_X \equiv R_X/G_X \). For Einstein gravity, \( \tilde{M}_E = M_E \).

The Wald prescription tells us that the corresponding BH entropy is \( S_X = R_{S;X} \tilde{M}_X = R_{S;X}^2/G_X \) or \( S_X = G_X \tilde{M}_X^2 \). Let us first choose \( \tilde{M}_X = \tilde{M}_E \) and take
the appropriate ratio to obtain
\[
\frac{S_X}{S_E|\bar{M}_X=\bar{M}_E} = \frac{G_X}{G_E} \geq 1. \tag{3.6}
\]

Additionally, using bound (2.3) and the fact that \(S_X = R_{S,X}\bar{M}_X\), we obtain \(\bar{M}_X \geq M_X\). Then, recalling that \(\bar{M}_E = M_E\), we have
\[
\frac{S_X}{S_E} = \frac{G_X\bar{M}_X^2}{G_E\bar{M}_E^2} = \frac{\bar{M}_X^2}{\bar{M}_E^2} \frac{G_X}{G_E} \geq 1. \tag{3.7}
\]
Finally, fixing the mass \(M_X = M_E\), we arrive at
\[
\frac{S_X}{S_E|M_X=M_E} = \frac{\bar{M}_X^2}{\bar{M}_E^2} \frac{G_X}{G_E} \geq 1. \tag{3.8}
\]

Previously we have found that \(S_X > N_X\) and \(S_E > N_E\). Can we now conclude that \(N_X(l_{UV,X}) \geq N_E(l_{UV,E})\)? It is already known that \(l_{UV,X} \geq l_{UV,E}\) [8]. Moreover, at any given scale, a weakly coupled unitary extension of Einstein’s theory has at least as many degrees of freedom as that of Einstein gravity. That is, \(N_X(l) \geq N_E(l)\) can be expected to hold for any \(l\) within the semi-classical regime of both theories. It thus follows that \(N_X(l_{UV,X}) \geq N_E(l_{UV,E})\) is in fact valid, where we have used the UV scale \(l_{UV,X}\) in the arguments for both \(N_X\) and \(N_E\).

4. The entropy density of black branes in AdS

Some of the preceding analysis can be extended to the case of an asymptotically anti-de Sitter (AdS) brane theory, where the quantity of interest is the entropy density of the black brane. The near-horizon form of an AdS brane metric is essentially the same as that of a BH in an asymptotically flat spacetime. Hence, we can apply the same reasoning as before and again use the flat-space form of the 1PI propagator or Eq. (3.5). In this context, the stable tachyons of AdS space do not pose a special problem. Since masses come attached with a factor of \(f\) (cf. Eq. (3.2)), any finite-mass tachyon will also behave as a massless species. As it is well known, the condition of stability restricts the magnitude of any tachyon mass to be finite and small, \(|m^2| \lesssim d/L^2\) [23], where \(L\) is the AdS radius of curvature.
We wish to compare the entropy density $s_X$ for a generalized theory to the entropy density $s_E$ of Einstein’s theory, and do so at a common value for the Hawking temperature, $T_X = T_E \equiv T$, with all other charges fixed. This canonical framework is the obvious analogue of comparing BHs at fixed mass, given that the energy of a brane is infinite. Then, since $s_E \sim T^n/G_E$ and $s_X \sim T^n/G_X$, any difference in the densities must be due solely to the gravitational couplings. Using our previous arguments, we expect that $G_X \geq G_E$, and so an “inverted bound” should follow:

$$s_X \leq s_E.$$  \hfill \text{(4.1)}

However, this is not the story in its entirety. A precursor is the fact that bound (4.1) is not invariant under field redefinitions.

To compute $G_X$, we can again make use of the Wald formalism \cite{3, 4, 5} as prescribed in \cite{6}. For this coupling to deviate physically from $G_E$, it is necessary that the Lagrangian depends explicitly and non-linearly on the Riemann tensor in a polarization-dependent form. Any deviation in $G_X$ from $G_E$ that is independent of the graviton polarization is not physically meaningful, as $G_X$ could always be recalibrated to be equal to $G_E$, with all other couplings then being rescaled by the same amount.

However, theories that depend on higher powers of the Riemann tensor lead to, generically, equations of motion with more than two time derivatives and, therefore, to propagating ghosts. Well-known non-generic exceptions are the Kaluza–Klein scenarios, whereby $G_X$ may change from $G_E$ but in such a way that $s_X = s_E$ is always maintained \cite{21}. In the Kaluza–Klein scenarios one needs to keep an infinite number of higher derivative terms.

The only consistent unitary extensions of Einstein gravity with a finite number of additional higher-curvature terms are those of the Lovelock class \cite{24, 25, 26}. But, for any Lovelock brane theory, one finds that $G_X = G_E$ must be true! To demonstrate this last point, let us first express the brane metric in a generic form, assuming only translational invariance on the brane and a static spacetime:

$$ds^2 = -a(r)f(r)dt^2 + \frac{1}{b(r)f(r)}dr^2 + \sum_{i=1}^{n} c(r)dx_i^2.$$  \hfill \text{(4.2)}
Here, $f(r)$ has a simple zero at the horizon $R_S$; otherwise, the radial functions $a(r)$, $b(r)$, $c(r)$ and their derivatives (including that of $f(r)$) are positive and regular in the exterior region of the spacetime.

For a Lovelock theory, the non-Einstein contributions to the Wald entropy are strictly constructed out of contractions of 4-index Riemann tensorial components that do not carry any $r$ and/or $t$ indices [27, 28]. That is, these contributions will be made up only of contractions of $\mathcal{R}_{x_i x_j x_i x_j}$ (with $\{i \neq j\} = \{1, 2, \ldots, n\}$) and permutations thereof. Now consider that $\mathcal{R}_{x_i x_j x_i x_j} = \Gamma_{x_i x_{11}}^{x_{11}} \Gamma_{x_j x_j}^{r} \sim fb[\partial c]^2/c$. At the horizon, $f$ is vanishing, while $b$, $c$ and $\partial c$ are all regular — meaning that any of these tensorial components and, hence, the non-Einstein contributions to the Wald entropy must be vanishing for this Lovelock brane theory.

As explained above, $s_X/s_E = G_E/G_X$ when evaluated at the same value of temperature. For Lovelock theories, the implication is clear:

$$\frac{s_X}{s_E} = 1. \quad (4.3)$$

Note that this equality of entropy densities is applicable to both the gravity theories and their field-theory duals [30, 31].

Equation (4.3) can be used to directly verify the so-called KSS bound [29]. This bound asserts that $\eta/s \geq 1/4\pi$ — where $\eta$ is the shear viscosity of the brane and its field-theory dual alike — and can be elegantly restated as $\eta_X/s_X \geq \eta_E/s_E$. The key point here is that the factors of temperature now cancel out of this ratio. And so this is really a direct comparison of gravitational couplings, which are generally different for the different polarizations involved (with $s$ and $\eta$ implicating the $r,t$- and $x_i,x_j$-polarized gravitons respectively [20]). In a previous work [21], we have used Eqs. (3.4, 3.5) to argue that $\eta_X \geq \eta_E$. (Note that the horizon is the relevant surface for calculations of $\eta$, even for the dual theory [32].) Taken together with either Eq. (4.1) or (4.3), the KSS bound immediately follows.

5. Summary and Conclusion

To summarize, we have demonstrated two novel bounds for the entropy of a semi-
classical BH in a unitary and weakly coupled extension of Einstein gravity. We have established that, at any applicable energy scale up to the UV momentum cutoff scale, this entropy must be larger than the number of elementary light species in the theory at the given scale. Then, we have verified that, for different theories compared at a fixed value of mass, the entropy will always be minimal for Einstein gravity. Some of the same basic principles were also applied to the entropy density of a generalized AdS brane theory.

The bound \( S > N \) provides a resolution to the so-called species problem. It places a fundamental and universal limit on the entropy of a semiclassical BH in terms of the number of light particle species. The bound implies that the BH entropy apparently does “know” about the various particle species but, at the same time, is able to maintain its status as a purely geometric quantity. Our considerations suggest the likely route for resolving the tension between the two different facets of the BH entropy: In a theory with an excessively large number of elementary fields, BHs that have less than the required entropy are not stable enough to be a part of the semiclassical spectrum. They either decay too quickly to some other states, either elementary particles or BHs after they form, or do not have enough time to form at all. It would be interesting to understand in detail the dynamical physical mechanisms that act to enforce the bound \( S > N \) when it is near saturation.

The fact that the entropy of BHs in Einstein gravity is minimal compared to any BH of the same mass in an extended theory implies that, when one considers bounds on the entropy of BHs and the significance of such bounds, then it is enough to establish them for Einstein’s theory. Since Einstein gravity is much simpler to analyze than most of its extensions, this bound is technically very useful. It would be interesting to determine explicitly whether this bound applies to string theoretical BHs. Our results suggest that it does. In [33] the entropy bound is in fact used to constrain the effective number of species in string theory.
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