Quantum information in loop quantum gravity

Daniel R. Terno
Perimeter Institute for Theoretical Physics, 31 Caroline St. N., Waterloo ON, Canada
N2L 2Y5
E-mail: dterno@perimeterinstitute.ca

Abstract. A coarse-graining of spin networks is expressed in terms of partial tracing, thus allowing to use tools of quantum information theory. This is illustrated by the analysis of a simple black hole model, where the logarithmic correction of the Bekenstein-Hawking entropy is shown to be equal to the total amount of correlations on the horizon. Finally other applications of entanglement to quantum gravity are briefly discussed.

1. Introduction
Loop Quantum Gravity (LQG) is a canonical quantization of General Relativity, which relies on a 3+1 decomposition of space-time [1, 2]. It describes the states of 3d geometry and their evolution in time through the implementation of a Hamiltonian constraint. The states of the canonical hypersurface are the spin networks, which represent polymeric excitations of the gravitational field. Spin networks are also used to describe quantum states in the path integral approach to the quantization of gravity — spin foams [3, 4].

The main prediction in the LQG framework is a discrete spectrum of geometric operators. In particular, a surface can be regarded as made of elementary patches of a finite quantized area. Based on this structure one can not only study the entropy associated to a surface [1, 2, 5, 6], but also analyze the information-theoretical aspects of the corresponding states [7, 8, 9, 10].

In this work I describe a precise relationship between a coarse-graining of a spin network and a partial tracing. This allows to discuss entanglement in spin networks and to relate it to the black hole entropy.

2. Spin networks, coarse graining and partial tracing
A spin network is a graph $\Gamma$ with vertices $v$ and oriented edges $e$. The spin network state is the assignment of a SU(2) representation $V^e$ to each edge $e$ and a SU(2)-invariant linear map (an intertwiner) $I_v : \bigotimes_{\text{ingoing}} V^e \rightarrow \bigotimes_{\text{outgoing}} V^e$ to each vertex $v$. Denote the Hilbert space of intertwiners at the vertex $v$ as $\mathcal{H}^0_v \equiv \text{Int}(\bigotimes_{\text{ingoing}} V^e \rightarrow \bigotimes_{\text{outgoing}} V^e)$.

A spin network state $|\Gamma, \vec{j}, \vec{i}\rangle$ defines a function of the holonomies along the graph edges $T_{\{\Gamma, j, i\}}[\gamma]$. For a fixed graph and a fixed assignment of the representations we omit their labels and denote a basis state in $\mathcal{H}^0 \equiv \bigotimes_v \mathcal{H}^0_v$ as $|\iota_1 \ldots \iota_V\rangle$, where each $\iota_v$ enumerates the intertwiners at the vertex $v$. The corresponding function $T_\iota[\gamma]$ equals to the tensor contraction...
of the matrix representations of the group elements $\bigotimes_e D^j_e(g_e)$ with the intertwiner $\bigotimes_v \mathcal{I}^v$, \[
T^v_c[g] \equiv \langle g|t_1 \ldots t_V \rangle \equiv \text{tr} \bigotimes_{v=1}^{V} S_v \bigotimes_{e(v)=1} D^{j_e}(g_e) \cdot \mathcal{I}^v, \tag{1}
\]
where the matrix elements of the holonomies $g_e$ along the edges outgoing from the same vertex are grouped together. For each vertex the total number of the adjacent edges $E_v$ equals to the number of incoming and outgoing edges, $E_v = T_v + S_v$. It is a gauge invariant function, its value is preserved under the (residual) action of the SU(2) gauge group at the graph’s vertices. Such gauge invariant functions, called gauge invariant cylindrical functions, are the wave functions of quantum geometry [1, 2, 3].

A note on normalization: in the following it is convenient to assume that the intertwiners are normalized as $\|\mathcal{I}\| = 1$, and the factors $\sqrt{\bar{d}_j} = \sqrt{2j+1}$ are absorbed into representation matrices. As a result, the spin network states are normalized to one, $\langle \bar{I}|\bar{I}' \rangle = \delta_{\bar{I}\bar{I}'}$. An arbitrary pure state is given by $|\Psi\rangle = \sum'_c c'_v|\bar{I}'\rangle$, with $\sum'_c |c'_v|^2 = 1$.

Consider a closed connected spin network based on the oriented graph $\Gamma$ and a bounded connected region $B$ of this spin network. The interior $\text{int}(B)$ of $B$ consists of the vertices $v \in B$ and the edges between them. The exterior $\text{ext}(B)$ of $B$ consists of all other vertices. Its boundary $\partial B$ consists in edges $e$ such that one of its end vertices is inside $B$ and the other outside. The state of $B$ is the tensor product of all the intertwiners attached to the vertices $v \in \text{int}(B)$: $\mathcal{H}_B = \mathcal{H}_{\text{int}(B)} \equiv \bigotimes_{v \in B} \mathcal{H}_v$. The state of $\text{ext}(B)$ is the tensor product of all the intertwiners attached to the vertices $v \in \text{ext}(B)$: $\mathcal{H}_{\text{ext}(B)} \equiv \bigotimes_{e \in \partial B} \mathcal{H}_e$. The Hilbert space of boundary states $\mathcal{H}_{\partial B}$ is the space of intertwiners between the representations $j_e$ attached to the edges crossing the boundary $\partial B$. It is the space of states of $A$ if we coarse-grain it to a single vertex.

In the simplest coarse-graining procedure one contracts the intertwiners attached to each internal vertex and thus obtains an intertwiner between the edges crossing the boundary $\partial B$. One can glue these same intertwiners using a non-trivial parallel transport between the internal vertices, i.e., using non-trivial group elements $g_e$ on each internal edge $e \in \text{int}(B)$. To maintain gauge invariance one proceed as follows [14].

For an arbitrary set of group elements $\{g_e, e \in E_B\}$, the parallel-transport dependent boundary state is
\[
\int_{\text{SU}(2)} dg \text{tr} \bigotimes_{e \in \partial B} D^{j_e}(g_e) / \sqrt{\bar{d}_j} \bigotimes_{e \in \text{int}(B)} D^{j_e}(g_e) \bigotimes_{v \in \text{int}(B)} \mathcal{I}_v \in \mathcal{H}_{\partial B}, \tag{2}
\]
where $\epsilon_e, e \in E_B$ is a sign $\pm$ depending on whether the edge $e$ is ingoing ($s(e) \notin B$) or outgoing ($s(e) \in B$). The trace is taken over all the SU(2) representations $V_{\epsilon_e}$ around each vertex $v \in A$. The integration over SU(2) insures that the resulting tensor is SU(2) invariant, thus an intertwiner, and properly normalized [14].

Due to the global SU(2) invariance and to the SU(2) invariance of the intertwiners $\mathcal{I}_v, v \in A$, not every distinct set of group elements $\{g_e, e \in E_B\} \in \text{SU}(2)^{E_B}$ leads to a distinct boundary state. To get distinct states one has to quotient by the gauge invariance. The simplest orbit is the one-point orbit defined by $g_e = 1, \forall e \in E_B$, which corresponds to the contraction of the internal intertwiners.

Recall the standard definition of a reduced density operator [11, 13]. Consider two subsystems $A$ and $B$, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with the direct product basis $|mn\rangle \equiv |m\rangle_A \otimes |n\rangle_B$. The corresponding wave functions are $\langle x|m \rangle = \psi_m(x)$ for $A$ and $\langle y|n \rangle = \phi_n(y)$ for $B$. A generic state $|\Psi\rangle \in \mathcal{H}$ is given by a linear combination $|\Psi\rangle = \sum_{m,n} c_{mn}|m\rangle|n\rangle$. If the matrix elements of the operator $O$ are $\langle mn|O|m'n\rangle = o_{m'mn}c_{mn}e_{m'n} = \text{tr}(\rho^A_Q)$, then
\[
\langle \Psi|O|\Psi\rangle = \sum_{m'n} o_{m'mn}c_{mn}e_{m'n} = \text{tr}(\rho^A_Q), \tag{3}
\]
where the reduced density operator $\rho^A_{\Phi} \equiv \text{tr}_B \rho_{\Psi}$ is obtained by tracing out the subsystem $B$, $\rho^A_{mnv} = \rho_{mn,m'n}$. In the coordinate basis the operator $O$ is given by

$$O(x, y; x', y') = o(x, x')\delta(y - y'),$$  \hspace{1cm} (4)$$

so thanks to the orthonormality of the functions $\phi_n$ the reduced density operator is

$$\rho^A_{\Phi}(x, x') = \int dyd'\Psi(x, y)\overline{\Psi}(x', y')\delta(y - y') = c_{mn}c_{m'n}\psi_m(x)\overline{\psi_{m'}}(x'),$$  \hspace{1cm} (5)$$

with $\langle \Psi|O|\Psi \rangle = \int dx dx' o(x, x')\rho^A_{\Phi}(x, x')$.

It is easy to see that Eq. (3) gives a natural definition of the partial trace on the abstract space $\mathcal{H}^0$. Consider an operator $O_1$ that pertains to a single vertex (we label this vertex as $v = 1$). Its matrix elements are given by

$$\langle i' | O_1 | i'' \rangle = o(\iota_1, \iota'_1) \prod_{v=2}^V \delta_{\iota_v \iota'_v}. \hspace{1cm} (6)$$

For example, a volume operator is a sum of vertex operators $[1, 2]$. Then the reduced density matrix of $|\Psi\rangle$ is given by $\rho^A_{\Phi, i_1' i'_1} = \sum_{i_2, \ldots, i_V} c_{i_1 i_2 \ldots i_V} c_{i'_1 i'_2 \ldots i'_V}$. In this expression a single vertex $v = 1$ comprises the system $A$, while the rest of the vertices belong to $B = \text{ext}(A)$.

Cylindrical functions admit no natural separation of variables in $O_1[g, g']$ that is analogous to Eq. (4). However, the coarse-graining procedures that were described above allow to introduce the reduced subsystems in the language of cylindrical functions. A total coarse-graining over the edges that belong to $\text{ext}(A)$ leads to

$$T^1_i [g] = \bigotimes_{e_1 = 1}^E D^{i_1} (g_{e_1}) \cdot T^1 i \otimes \mathcal{M}^{i_2 \ldots i_V}, \hspace{1cm} (7)$$

where $\mathcal{M}^{i_2 \ldots i_V} = \text{tr} \bigotimes_{e \in \text{ext}(A)} D^{i_e} (g_e) \otimes \bigotimes_{v \in \text{ext}(A)} \mathcal{T}_v \in \mathcal{H}_{\mathcal{D}A}$. It plays a role of $\psi_m(x)$ which are used to expand the reduced operators on $A$. The reduced operator $O^1_i [g, g']$ and the reduced density matrix $\rho^A_{\Psi, [g, g']}$ follow from their functional matrix elements,

$$O^1_i [g, g'] = \sum_{i'_{1}, \ldots, i'_{V}} T^1_{i'_{1}} [g] o(\iota_1, \iota'_1) T^1_{i'_{2}} [g'], \hspace{0.5cm} \rho^A_{\Psi, [g, g']} = \sum_{i'_{2}, \ldots, i'_{V}} c_{i'_{1}} e^*_{i'_{1}} T^1_{i'_{2}} [g] T^1_{i'_{2}} [g']. \hspace{1cm} (8)$$

where $|\vec{a}\rangle \equiv |i'_{1}, i'_2, \ldots, i'_V\rangle$. The expression for $\rho^A_{\Psi, [g, g']}$ also results as a generalization of Eq. (5). The generalization to other types of “local” operators is straightforward [14].

3. Black hole entropy

A generic surface on a spin network background is thus described as a set of patches, each punctured by a unique link of the spin network. The spin network defines how the patches, and therefore the whole surface, are embedded in the surrounding 3d space and describes how the surface folds. For a closed surface the region of the spin network which is inside the surface defines an intertwining between its patches.

An important remark is that any spin $j$ representation $V^j$ can be decomposed as a symmetrized tensor product of $2j$ spin-$\frac{1}{2}$ representations $V^{j/2}$. Therefore, one can interpret that a fundamental patch or elementary surface is a spin-$\frac{1}{2}$ representation. All higher spin patches can be constructed from such elementary patches. For example, considering two spin-$\frac{1}{2}$
patches, they can form a spin 0 representation or a spin 1 representation: in one case, the two patches are folded on one another and cancel each other, while in the later case they add coherently to form a bigger patch of spin 1. Considering an arbitrary surface, one can then look at it at the fundamental level decomposing it into spin-1/2 patches, or one can look at it at a coarse-grained level decomposing the same surface into bigger patches of spin $s > 1/2$. From this point of view, the size of the patches used to study a surface is like the choice of a ruler of fixed size used by the observer to analyze the properties of the object. Thus one can study the coarse-graining or renormalisation of these quantities when one observes the surface at a bigger scale, using bigger patches to characterize the surface [10].

Considering the horizon as a closed surface the interior of a black hole is described by (a superposition of) spin networks whose boundary puncturing the horizon define the patches of the horizon surface. For an external observer only the horizon information is relevant. For him the bulk spin network is fully coarse-grained and the state of the $n$ patches on the boundary belongs to the tensor product $V^{j_1} \otimes \ldots \otimes V^{j_n}$. The only constraint on physical states is that they should be globally gauge invariant, i.e. SU(2) invariant, such that the possible horizon states are the intertwiners (invariant tensors) between the representations $V^{j_i}$ and the area is fixed at some value $A$.

This model is different from the standard loop quantum gravity approach, which studies the classically induced boundary theory on the black hole horizon (or generically any isolated horizon) [5, 6]. Since the problem of identifying horizons within the quantum states of geometry in full theory is yet unsolved, we model the black hole as a region of the quantum space with a boundary (the horizon) such that the only information about the geometry of the internal region accessible to the external observer is information which can be measured on the boundary. Our results can therefore be applied to any closed surface.

In the simplest scenario, the black hole entropy calculation is reduced to counting the number of distinct SU(2) invariant states on the space of $2n$ qubits (spin-1/2 states) with the horizon area $A = a_{1/2}2n$. The ignorance of a particular microstate makes the statistical state under consideration to be a maximally mixed state $\rho$ on the space of intertwiners $\mathcal{H}^0 = \text{Int}(\mathbb{C}^2)^{\otimes 2n}$. Its orthogonal decomposition is simply

$$\rho = \frac{1}{N} \sum_i |\mathcal{I}^i\rangle\langle\mathcal{I}^i|,$$

where $|\mathcal{I}^i\rangle$ form a basis of $\mathcal{H}^0$ and $N \equiv \dim \mathcal{H}^0$. A straightforward calculation gives the Bekenstein-Hawking entropy and its logarithmic correction:

$$S \equiv -\text{tr} \rho \log \rho = \log N \sim 2n \log 2 - \frac{3}{2} \log n. \quad (10)$$

In a coarse-grained model of a black hole one considers horizon states to be given by intertwiners between $m$ representations of a fixed spin $s$, so the horizon area is $A = a_s m$. The space $\mathcal{H} = (V^s)^{\otimes m}$ decomposes as

$$\bigotimes_{j=0}^{m} \mathbb{C}^s \cong \bigoplus_{j} \mathcal{H}^j \equiv \bigoplus_{j} V^j \otimes \sigma^s_{m,j}, \quad (11)$$

where $V^j$ is the irreducible spin-$j$ representation of SU(2), and $\sigma^s_{m,j}$ is the degeneracy subspace. From the asymptotic form of the multiplicities $c^s_{m,j} = \dim \sigma^s_{m,j}$ it follows that

$$S \sim m \log s - \frac{3}{2} \log m. \quad (12)$$
4. Entanglement and correlations

Entanglement can be loosely defined as an exhibition of stronger-than-classical correlations between the subsystems. Recently it became one of the main resources of quantum information theory [11, 12]. We are interested to find how much entanglement is contained in an arbitrary bipartite splitting of this state. For a pure state $|\Psi\rangle$ there is a unique measure of entanglement—the degree of entanglement, which is the von Neumann entropy of either of its reduced density matrices $\rho_{A,B}^{\Psi}$. I use here only one of the measures of the mixed state entanglement, namely the entanglements of formation. It is possible to show that for the states (9) all measures of entanglement coincide [9]. The entanglement of formation is defined as follows. A state $\rho$ can be decomposed as a convex combination of pure states, $\rho = \sum_{\alpha} w_{\alpha} |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}|$. The entanglement of formation is the averaged degree of entanglement of the pure states $|\Psi_{\alpha}\rangle$ (the von Neumann entropy of their reduced density matrices) minimized over all possible decompositions

$$E_F(\rho) = \inf_{\{\Psi_{\alpha}\}} \sum_{\alpha} w_{\alpha} S(\rho_{\alpha}).$$

(13)

To simplify the notation consider again the qubit model of a black hole. Let the $2n$ qubits be divided into the groups of $2k \leq n$ and $2n - 2k$ qubits. The corresponding Hilbert spaces are $\mathcal{H}_A \equiv (\mathbb{C}^2)^{\otimes 2k}$ and $\mathcal{H}_B \equiv (\mathbb{C}^2)^{\otimes 2n-2k}$, respectively. Using the decomposition of Eq. (11) twice, the intertwiner space can be decomposed as follows:

$$\mathcal{H}^0 = V^0 \otimes \sigma_{2n,0} = \bigoplus_{j=0}^{k} V^0_{(j)} \otimes (\sigma_{2k,j} \otimes \sigma_{2n-2k,j}),$$

(14)

where $V^0_{(j)}$ is the singlet state in $V^j \otimes V^j$. Hence the dimensionality of $\mathcal{H}^0$ is related to the multiplicities of the degeneracy subspaces through $N = c_{2n,0} = \sum_{j=0}^{k} c_{2k,j} c_{2n-2k,j}$.

The basis states of Alice and Bob are respectively labeled as $|j, m, a_j\rangle$ and $|j, m, b_j\rangle$. Here $0 \leq j \leq k (\leq n - k)$ and $-j \leq m \leq j$ have their usual meaning and the degeneracy labels, $a_j$ and $b_j$, enumerate the different subspaces $V^j$

$$|I_{a_j,b_j}\rangle \equiv \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{j-m} |j, -m, a_j\rangle \otimes |j, m, b_j\rangle$$

(15)

It is possible to show that the entanglement of formation of the state $\rho$ is:

$$E_F(\rho) = \frac{1}{N} \sum_{j=0}^{k} c_{2k,j} c_{2n-2k,j} \log(2j+1),$$

(16)

with the analogous expressions for any spin $s \geq \frac{1}{2}$. In the large $n$ limit the case of equal splitting is especially interesting. Then

$$E_F(\rho|n : n) \sim \frac{1}{2} \log n,$$

(17)

which becomes $E_F \sim \log m/2$ for any $s$ [10].

More generally, for all bipartite partitions of the horizon spin network with sufficiently large number of edges comprising the smaller space, and for any $s \geq \frac{1}{2}$, it is possible to show that the quantum mutual information between the black hole horizon and its parts is is three times the entanglement between the halves,

$$I_\rho(A : B) = S(\rho_A) + S(\rho_B) - S(\rho) \simeq 3SE(\rho|A : B).$$

(18)
In particular, if the ratio between the number of qubits is kept fixed while \( n \) is arbitrary, the logarithmic correction \( \frac{3}{2} \log n \) asymptotically equals to \( I_\rho(A : B) \), so the deviation of the black hole entropy from its classical value equals to the total amount of correlations between the halves of spin networks that describe it [10]. Hence in a model where the black hole horizon would be constructed out of independent uncorrelated qubits, the entropy would scale linearly in the number of qubits \( 2n \). However, the requirement of invariance under SU(2) creates correlations between the horizon qubits, which are revealed through the logarithmic correction \( \frac{3}{2} \log n \) to the entropy law formula.

Returning to the qubit black hole it is interesting to note that a fraction of unentangled states in Eq. (15) when a pair of qubits is segregated from the rest is \( s_0^{(2)} \sim \frac{1}{4} + \frac{3}{8n} \). It leads to an interesting coincidence with the evaporation model [15] and allows to speculate about corrections to it.

Moreover, using the relations between coarse-graining and partial tracing it is possible to investigate the entanglement in spin-networks [14] and its possible role in the emergence of classical geometry.

Finally, let me mention the “information loss paradox” [17]. While it is not obvious that the unitarity must persist in the process of creation and evaporation of black holes, consideration of the matter alone is not sufficient to convincingly preserve it [16]. Entanglement between gravitational and matter degrees of freedom may the way to restore it. In the simplest scenario initially the spacetime is approximately and the matter is in some state \( \rho \). Ascribe a state \( \Phi \) to the geometry that corresponds to a classical nearly Minkowski metric. The evolution that ends in the black hole evaporation is unitary and is schematically described as \( \Xi = U(\Phi \otimes \rho)U^\dagger \), where \( \Xi \) is the final entangled state of matter and gravity. Reduced density operators give predictions for the gravitational background and the matter distribution on it. The evolution of matter is obtained by tracing out the gravitational degrees of freedom and is a completely positive non-unitary map [12]. If we assume that the initial states are pure, then the entropy of a reduced density operator is exactly the degree of entanglement between matter and gravity, \( E(\Xi) \). Hence, the increase in the entropy of matter is not an expression of information loss, but a measure of the created entanglement, i.e. redistribution of information.

Acknowledgments
I thank Etera Livine for helping to introduce me to quantum gravity and many enjoyable collaborations that provide the bulk of this contribution.

[1] Thiemann T 2003 *Lectures on Loop Quantum Gravity*, Lect. Notes Phys. 631 41.
[2] Ashtekar A and Lewandowski J 2004 *Class. Quant. Grav.* 21 R53.
[3] Perez A 2004 *Preprint* gr-qc/0409061.
[4] Oriti D 2001 *Rept. Prog. Phys.* 64 1489.
[5] Ashtekar A, Baez J, Krasnov K 2000 *Adv. Theor. Math. Phys.* 4, 1.
[6] Domagala M and Lewandowski J 2004 *Class. Quant. Grav.* 21 5233.
[7] Girelli F and Livine E R 2005 *Class. Quant. Grav.* 22 3295.
[8] Dreyer O, Markopoulou F and Smolin L 2004 *Preprint* hep-th/0409056.
[9] Livine E R and Terno D R 2005 *Phys. Rev. A* 72 022307.
[10] Livine E R and Terno D R 2005 *Preprint* gr-qc/0508085.
[11] Peres A 1993 *Quantum Theory: Concepts and Methods* (Dordrecht: Kluwer).
[12] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (New York: Cambridge University Press).
[13] Landau L D and Lifshitz E M 1981 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon).
[14] Livine E R and Terno D R 2005 *Preprint* gr-qc/0505068.
[15] Bekenstein J D and Mukhanov V F 1995 *Phys. Lett.* B360 7.
[16] Peres A and Terno D R 2004 *Rev. Mod. Phys.* 76 93.
[17] Terno D R 2005 *Preprint* gr-qc/0505068.