CAUCHY MATRIX SOLUTIONS OF SOME LOCAL AND NONLOCAL COMPLEX EQUATIONS

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We develop a Cauchy matrix reduction technique that enables us to obtain solutions for the reduced local and nonlocal complex equations from the Cauchy matrix solutions of the original nonreduced systems. Specifically, by imposing local and nonlocal complex reductions on some Ablowitz–Kaup–Newell–Segur-type equations, we study some local and nonlocal complex equations involving the local and nonlocal complex modified Korteweg–de Vries equation, the local and nonlocal complex sine-Gordon equation, the local and nonlocal potential nonlinear Schrödinger equation, and the local and nonlocal potential complex modified Korteweg–de Vries equation. Cauchy matrix-type soliton solutions and Jordan block solutions for the aforesaid local and nonlocal complex equations are presented. The dynamical behavior of some of the obtained solutions is analyzed with graphical illustrations.

Keywords: local and nonlocal complex reductions, AKNS-type equations, Cauchy matrix solutions, dynamics

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1. Introduction

The complex integrable equations have been of considerable interest recently in many areas of mathematical physics. Some famous examples include the nonlinear Schrödinger (NLS) equation, the complex modified Korteweg–de Vries (cmKdV) equation, the complex sine-Gordon (csG) equation, and so on. The NLS equation

\[ 2iut + u_{xx} + 8|u|^2u = 0 \]  \hspace{1cm} (1.1)

usually appears in the description of deep water waves [1], vortex filaments [2], and the collapse of Langmuir waves in plasma physics [3]. The cmKdV equation

\[ 4ut + u_{xxx} + 24|u|^2ux = 0 \]  \hspace{1cm} (1.2)

has been used as a model for the nonlinear evolution of plasma waves [4] and a molecular chain model [5], as well as for short pulses in optical fibers [6]. The csG equation

\[ u_{xt} + 4u + 8u \partial^{-1}(|u|^2)_t = 0 \]  \hspace{1cm} (1.3)

appeared in general relativity [7] and in describing propagation of optical pulses in nonlinear media [8]. In Eqs. (1.1)–(1.3), the dependent variable \( u \) is a function of continuous coordinates \((x, t) \in \mathbb{R}^2; i \) is the

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imaginary unit, $|\cdot|$ denotes the modulus, and $\partial^{-1}$ stands for an inverse operator of $\partial = \partial/\partial x$, such that $\partial \partial^{-1} = \partial^{-1} \partial = 1$. It is well known that all the three above equations can be derived from the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy by complex reductions.

In recent years, the study of parity–time symmetric integrable models has become a focus of attention in the theory of integrable systems. The parity–time symmetric integrable models play important roles in quantum physics and other areas of physics, such as quantum chromodynamics, electric circuits, optics, Bose–Einstein condensates, and so on. One of the famous examples is the reverse-space NLS equation

$$iu_{t} + u_{xx} + u^{2}u^{*}(-x) = 0,$$  (1.4)

which was first proposed by Ablowitz and Musslimani [9]. Here and hereafter, the asterisk denotes complex conjugation. This equation is parity–time symmetric because it is invariant under the action of the parity–time operator, i.e., the joint transformations $x \rightarrow -x$, $t \rightarrow -t$ and complex conjugation. Equation (1.4) is Lax-integrable and admits infinitely many conservation laws. Besides, it can be solved by the inverse scattering transform [10]. Because Eq. (1.4) involves two different locations $\{x,t\}$ and $\{-x,t\}$, this model can be used to describe two-place physics phenomena [11]. Up to now, several methods have been used to search for solutions of the reverse space NLS equation (1.4), such as the Riemann–Hilbert approach, Hirota’s bilinear method, the Darboux transformation, and others [12]–[17].

Besides the study of the nonlocal NLS equation, various results for the nonlocal cmKdV equation have also been obtained. The inverse scattering transform for the reverse time–space nonlocal cmKdV equation with nonzero boundary conditions at infinity was presented in [18]. In [19], the nonlocal cmKdV equation was shown to be gauge equivalent to a spin-like model. Consequently, dark solitons, W-type solitons, M-type solitons, and periodic solutions for the nonlocal cmKdV equation were obtained with the help of Darboux transformations. In [20], a simple variable transformation was introduced to convert the nonlocal cmKdV equation into the local cmKdV equation. As a result, multisoliton and quasiperiodic solutions for the nonlocal cmKdV equation were constructed. By using the Ablowitz–Musslimani reduction formulas, the 1-, 2-, and 3-soliton solutions of the nonlocal cmKdV equations were found in [21]. Based on the double Wronskian solutions for the AKNS hierarchy, a reduction technique was developed by imposing a constraint on the two basic vectors in double Wronskians such that two potential functions in the AKNS hierarchy obey some nonlocal relations, which allowed obtaining solutions of the nonlocal cmKdV equation [22]. This method proves its efficiency in general, and it has been applied to many nonlocal systems, such as nonlocal discrete soliton equations [23], [24] and nonlocal nonisospectral soliton equations [25]–[27].

As regards the nonlocal csG equation, its solutions were also studied in [22]. Furthermore, the covariant hodograph transformations between nonlocal short pulse models and the nonlocal multicomponent csG equation were discussed [28].

The Cauchy matrix approach, as a systematic method for constructing integrable equations together with their solutions, was first proposed by Nijhoff and his collaborators [29], [30] in investigating the soliton solutions of the Adler–Bobenko–Suris lattice list. This method is actually a by-product of the linearization approach that was first proposed by Fokas and Ablowitz [31] and was generalized to discrete integrable systems by Nijhoff, Quispel et al. (see, e.g., [32]–[35]). In [36], a generalized Cauchy matrix scheme was proposed to construct more types of exact solutions for the Adler–Bobenko–Suris lattice list beyond the soliton solutions. The (generalized) Cauchy matrix approach actually arose from the well-known Sylvester equation in matrix theory [37]. In [38], this method was used to discuss the relations between the Sylvester equation and some continuous integrable equations, including the KdV equation, the mKdV equation, the Schwarzian KdV equation, and the sG equation. All of these were shown to arise from the same Sylvester equation and to be expressible as some discrete equations for elements $S^{(i,j)}$ of some matrix-valued master function.
Recently, motivated by the Cauchy matrix approach and the understanding of dispersion relations, the first author of this paper studied the connections between the Sylvester equation and some AKNS equations [39], involving the second-order AKNS equation, the third-order AKNS equation, the negative-order AKNS equation, the second-order potential AKNS equation, and the third-order potential AKNS equation. Consequently, Cauchy matrix-type soliton solutions, Jordan block solutions, and generic soliton–Jordan-block mixed solutions for these AKNS-type equations were derived. Within that approach, Cauchy matrix-type soliton solutions and Cauchy matrix-type Jordan block solutions for the nonlocal NLS equation were also investigated [40].

In this paper, we discuss Cauchy matrix solutions of some local and nonlocal complex integrable equations that can be obtained by imposing reductions on the above AKNS-type equations except for the second-order AKNS equation. The models include local and nonlocal cmKdV equations, local and nonlocal csG equations, local and nonlocal potential NLS equations, and local and nonlocal potential cmKdV equations. For brevity in what follows, we say that the AKNS-type equations mentioned above, except the second-order AKNS equation, are respectively the AKNS(3) equation, the AKNS(−1) equation, the pAKNS(2) equation, and the pAKNS(3) equation.

This paper is organized as follows. In Sec. 2, we briefly review some AKNS-type equations and their local and nonlocal complex reductions. The corresponding Cauchy matrix solutions are also listed. In Sec. 3, Cauchy matrix solutions of the local and nonlocal cmKdV equation and the local and nonlocal csG equation are given. Dynamics of some obtained solutions are analyzed and illustrated by asymptotic analysis. In Sec. 4, we present the Cauchy matrix solutions and the dynamics of the local and nonlocal potential NLS equation and the local and nonlocal potential cmKdV equation. Section 5 contains conclusions and some remarks.

2. AKNS-type equations and Cauchy matrix solutions

As a preliminary part, we present an account of some AKNS-type equations and their local and nonlocal complex reductions. The equations involve the AKNS(3), AKNS(−1), pAKNS(2), and pAKNS(3) equations. Moreover, we recall the Cauchy matrix solutions of these equations, including Cauchy matrix-type soliton solutions and Cauchy matrix-type Jordan block solutions. For more details, we refer the reader to [39].

2.1. AKNS-type equations. In [41], Ablowitz, Kaup, Newell, and Segur proposed a spectral problem, named the AKNS spectral problem,

\[ \Phi_x = P\Phi, \quad P = \begin{pmatrix} -\lambda & 2u \\ -2v & \lambda \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}, \quad (2.1) \]

with the spectral parameter \( \lambda \) and potentials \( u = u(x, t) \) and \( v = v(x, t) \), which provides integrable backgrounds for the NLS equation, the mKdV equation, and the sG equation. With the time evolution defined as

\[ \Phi_t = Q\Phi, \quad Q = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \quad (2.2) \]

the compatibility condition \( \Phi_{xt} = \Phi_{tx} \) or, moreover, the zero-curvature equation \( P_t - Q_x + [P, Q] = 0 \), leads to the equations

\[ A = 2\partial^{-1}(-v, u) \begin{pmatrix} -B \\ C \end{pmatrix} - \lambda_x x + A_0, \quad (2.3a) \]

\[ 2 \begin{pmatrix} u \\ -v \end{pmatrix}_t = L \begin{pmatrix} -B \\ C \end{pmatrix} - 2\lambda \begin{pmatrix} -B \\ C \end{pmatrix} + 4A_0 \begin{pmatrix} u \\ v \end{pmatrix} - 4\lambda_x \begin{pmatrix} xu \\ xv \end{pmatrix}, \quad (2.3b) \]
where $A_0$ is a constant and
\[
L = \begin{pmatrix} -\partial & 0 \\ 0 & \partial \end{pmatrix} + 8 \begin{pmatrix} u \\ v \end{pmatrix} \partial^{-1}(-v,u). \tag{2.4}
\]

Setting $\lambda_t = 0$ and $A_0 = \lambda^n$ and expanding $(B,C)^T$ as
\[
\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} B_j \\ C_j \end{pmatrix} \lambda^{n-j}, \tag{2.5}
\]
we use (2.3b) to derive the positive-order AKNS hierarchy
\[
2^{n-1} \begin{pmatrix} u \\ -v \end{pmatrix}_t = L^n \begin{pmatrix} u \\ v \end{pmatrix}, \quad n = 1, 2, \ldots, \tag{2.6}
\]
where $L$ given by (2.4) is the recursion operator. Next, we take $\lambda_t = 0$ as previously, and let $A_0 = \lambda^{-n}$. Substituting the expansion
\[
\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{n} \begin{pmatrix} B_j \\ C_j \end{pmatrix} \lambda^{-n-1} \tag{2.7}
\]
in (2.3b) and taking $(B_1,C_1)^T$ such that $L(-B_1,C_1)^T = -4(u,v)^T$, we then arrive at the negative-order AKNS hierarchy
\[
L^n \begin{pmatrix} u \\ -v \end{pmatrix}_t = 2^{n+1} \begin{pmatrix} u \\ v \end{pmatrix}, \quad n = 1, 2, \ldots. \tag{2.8}
\]

In the positive-order AKNS hierarchy (2.6), one of the most salient or characteristic equations is the AKNS(3) equation, which has the form
\[
4u_t + u_{xxx} + 24uuv_x = 0, \tag{2.9a}
\]
\[
4v_t + v_{xxx} + 24uvv_x = 0. \tag{2.9b}
\]
Under the complex reduction
\[
v(x,t) = \delta u^*(\sigma x,\sigma t), \quad \delta, \sigma = \pm 1, \quad x, t \in \mathbb{R}, \tag{2.10}
\]
we obtain the local and nonlocal cmKdV equations
\[
4u_t + u_{xxx} + 24\delta uu^*(\sigma x,\sigma t)u_x = 0. \tag{2.11}
\]
It is obvious that Eq. (2.11) is preserved under the transformations $u \to -u$ and $u \to \pm iu$. When $\sigma = -1$, Eq. (2.11) is the reverse time–space cmKdV equation.

The first AKNS-type equation in the negative-order AKNS hierarchy (2.8) is the AKNS(-1) equation (see [42])
\[
u_{xt} + 4u + 8u \partial^{-1}(uv)_t = 0, \tag{2.12a}
\]
\[
v_{xt} + 4v + 8v \partial^{-1}(uv)_t = 0. \tag{2.12b}
\]
Imposing the reduction
\[
v(x,t) = \delta u^*(\sigma x,\sigma t), \quad \delta, \sigma = \pm 1, \quad x, t \in \mathbb{R}, \tag{2.13}
\]
on Eq. (2.12) gives rise to the local and nonlocal (nonpotential) csG equations

\[ u_{xt} + 4u + 8\delta u \partial^{-1}(uu^*(\sigma x, \sigma t))_t = 0. \]  

(2.14)

Equation (2.14) is also preserved under the transformations \( u \to -u \) and \( u \to \pm iu \). When \( \sigma = -1 \), Eq. (2.14) is the reverse time–space csG equation.

Because both \( u \) and \( v \) tend to zero as \( |x| \to \infty \) and \( uv \) is a conserved density, we can alternatively write (2.12) as

\[ u_{xt} + 8uw = 0, \quad v_{xt} + 8vw = 0, \quad w_x = (uv)_t, \]  

(2.15)

where \( w = \partial^{-1}(uv)_t + 1/2 \) is an auxiliary function. By complex reduction (2.13) together with the relation \( w(x, t) = \sigma w^*(\sigma x, \sigma t) \), one obtain another form of the local and nonlocal csG equations

\[ u_{xt} + 8uw = 0, \quad w_x = \delta(uu^*(\sigma x, \sigma t))_t. \]  

(2.16)

Besides the AKNS hierarchy, there exists another hierarchy that can also be viewed as the AKNS type. We call it the potential AKNS hierarchy. The first nontrivial member is the pAKNS(2) equation, namely,

\[ 2q_t - q_{xx} + \frac{q_x q_x}{r} = 0, \]  

(2.17a)

\[ 2r_t - r_{xx} + \frac{q_x r_x}{q} = 0. \]

(2.17b)

Under the complex reduction

\[ r(x, t) = q^*(\sigma x, t), \quad \sigma = \pm 1, \quad t \to -it, \quad x, t \in \mathbb{R}, \]  

(2.18)

Eq. (2.17) leads to the local and nonlocal potential NLS equations

\[ 2iq_t - q_{xx} + \frac{2q_x q_x^*(\sigma x)}{q^*(\sigma x)} = 0, \]  

(2.19)

which are preserved under the transformations \( q \to -q \) and \( q \to \pm iq \). When \( \sigma = 1 \), (2.19) is the local potential NLS equation, which was first introduced in [32], [43] and is related to the equation of motion for the Heisenberg ferromagnet with uniaxial anisotropy [44]. When \( \sigma = -1 \), Eq. (2.19) is the reverse-space potential NLS equation.

The second nontrivial member of the potential AKNS hierarchy is the pAKNS(3) equation

\[ 4q_t + q_{xxx} - \frac{q_x q_x r_x}{r} - \frac{q_x r_x}{q q^2} (q_x r - q r_x) = 0, \]  

(2.20a)

\[ 4r_t + r_{xxx} - \frac{q_x r_x}{q^2} (q_x r - q r_x) = 0. \]  

(2.20b)

Under the complex reduction

\[ r(x, t) = q^*(\sigma x, \sigma t), \quad \sigma = \pm 1, \quad x, t \in \mathbb{R}, \]  

(2.21)

Eq. (2.20) yields the local and nonlocal potential cmKdV equations

\[ 4q_t + q_{xxx} - \frac{3q_x r_x}{q^*(\sigma x, \sigma t)} - \frac{q_x q_x^*(\sigma x, \sigma t)}{qq^2(\sigma x, \sigma t)} (q_x q_x^*(\sigma x, \sigma t) - q q_x^*(\sigma x, \sigma t)) = 0, \]  

(2.22)

which are also preserved under the transformations \( q \to -q \) and \( q \to \pm iq \). When \( \sigma = -1 \), Eq. (2.22) corresponds to the reverse time–space potential cmKdV equation.
It is truly noteworthy that the pAKNS(2) equation (2.17) and the pAKNS(3) equation (2.20) are respectively related to the AKNS(2) equation

\[
2u_t - u_{xx} - 8u^2v = 0, \quad (2.23a) \\
2v_t + v_{xx} + 8uv^2 = 0 \quad (2.23b)
\]

and the AKNS(3) equation (2.9) by the transformations

\[
2u = \frac{q_x}{r}, \quad 2v = -\frac{r_x}{q}. \quad (2.24)
\]

### 2.2. Cauchy matrix solutions.

The section is devoted to Cauchy matrix solutions of the AKNS-type equations mentioned above, including Cauchy matrix-type soliton solutions and Cauchy matrix-type Jordan block solutions.

Cauchy matrix solutions of the AKNS(3) equation (2.9) and the AKNS(−1) equation (2.12) can be summarized in the following theorem.

**Theorem 1.** The functions

\[
\begin{align*}
    u & = s_2^T(I - M_2M_1)^{-1}r_2, \\
    v & = s_1^T(I - M_1M_2)^{-1}r_1
\end{align*} \quad (2.25a, b)
\]

solve the AKNS (3) equation (2.9) and AKNS (−1) equation (2.12) if the components \( K_j \in \mathbb{C}_{N_j \times N_j} \), \( M_1 \in \mathbb{C}_{N_1 \times N_2} \), \( M_2 \in \mathbb{C}_{N_2 \times N_1} \), and \( r_j, s_j \in \mathbb{C}_{N_j \times 1} \) (\( j = 1, 2 \)) satisfy the set of defining equations

\[
\begin{align*}
    K_1M_1 - M_1K_2 & = r_1s_2^T, \\
    K_2M_2 - M_2K_1 & = r_2s_1^T, \\
    r_{j,x} & = (-1)^{j-1}K_jr_j, \\
    s_{j,x} & = (-1)^{j-1}K_j^Ts_j, \\
    r_{j,t} & = (-1)^{j}K_j^Tr_j, \\
    s_{j,t} & = (-1)^{j}(K_j^T)^ns_j
\end{align*} \quad (2.26a-c)
\]

respectively with \( n = 3 \) and \( n = −1 \), where \( N_1 + N_2 = 2N \).

For notational brevity here and in what follows, we omit the index indicating the size of each unit matrix \( I \).

Cauchy matrix solutions of the pAKNS(2) equation (2.17) and the pAKNS(3) equation (2.20) are presented in the following theorem.

**Theorem 2.** The functions

\[
\begin{align*}
    q & = s_2^T(I - M_2M_1)^{-1}K_2^{-1}r_2 - s_2^TJ_2M_2(I - M_2J_2)^{-1}K_2^{-1}r_1 - 1, \\
    r & = s_1^T(I - M_1M_2)^{-1}K_1^{-1}r_1 - s_1^TJ_1M_1(I - M_1J_1)^{-1}K_1^{-1}r_2 - 1
\end{align*} \quad (2.27a, b)
\]

solve the pAKNS (2) equation (2.17) and pAKNS (3) equation (2.20) if the components \( K_j \in \mathbb{C}_{N_j \times N_j} \), \( M_1 \in \mathbb{C}_{N_1 \times N_2} \), \( M_2 \in \mathbb{C}_{N_2 \times N_1} \), and \( r_j, s_j \in \mathbb{C}_{N_j \times 1} \) (\( j = 1, 2 \)) satisfy the set of defining equations (2.26) respectively with \( n = 2 \) and \( n = 3 \), where \( N_1 + N_2 = 2N \).

According to the analysis above, we know that Cauchy matrix solutions of the AKNS-type equations (2.9), (2.12), (2.17), and (2.20) are determined by the defining equation set (2.26). Equations (2.26b) and (2.26c) are used to determine the dispersion relations \( r_j, s_j \) (\( j = 1, 2 \)) and the two equations in (2.26a)
are used to determine \( M_1 \) and \( M_2 \). Both equations in (2.26a) are the Sylvester equations and have a unique solution for \( M_1 \) and \( M_2 \) if \( \mathcal{E}(K_1) \cap \mathcal{E}(K_2) = \emptyset \), where \( \mathcal{E}(K_1) \) and \( \mathcal{E}(K_2) \) respectively denote the eigenvalue sets of \( K_1 \) and \( K_2 \) (see [37]). We assume that \( K_1 \) and \( K_2 \) satisfy such a condition. Additionally, we suppose that \( 0 \notin \mathcal{E}(K_1) \cup \mathcal{E}(K_2) \) and \( 1 \notin \mathcal{E}(M_1M_2) \) in order to guarantee the invertibility of the matrices \( K_1 \), \( K_2 \) and \( I - M_1M_2 \). Because \( M_1M_2 \) and \( M_2M_1 \) have the same nonzero eigenvalues, the matrix \( I - M_2M_1 \) is also invertible. It is worth noting that the variables \( u, v, q, \) and \( r \) are invariant and system (2.26) is covariant under the similarity transformations

\[
K_j = \gamma_j^{-1}K_j\gamma_j, \quad r_j = \gamma_j^{-1}r_j, \quad s_j = \gamma_j^T s_j \quad (j = 1, 2),
\]

\[
M_1 = \gamma_1^{-1}M_1\gamma_2, \quad M_2 = \gamma_2^{-1}M_2\gamma_1,
\]

where \( \gamma_1 \) and \( \gamma_2 \) are transformation matrices.

To derive the explicit solutions, we only need to solve the canonical equations

\[
\Omega_1M_1 - M_1\Omega_2 = r_1s_2^T, \quad \Omega_2M_2 - M_2\Omega_1 = r_2s_1^T, \tag{2.29a}
\]

\[
r_{j,x} = (-1)^{j-1}\Omega_j r_j, \quad s_{j,x} = (-1)^{j-1}\Omega_j^T s_j, \tag{2.29b}
\]

\[
r_{j,t} = (-1)^{j}\Omega_j^t r_j, \quad s_{j,t} = (-1)^{j}\Omega_j^{-t} s_j \tag{2.29c}
\]

where \( j = 1, 2 \), and \( \Omega_1 \) and \( \Omega_2 \) are the respective canonical forms of the matrices \( K_1 \) and \( K_2 \). From the constraints on \( K_1 \) and \( K_2 \), we know that

\[
\mathcal{E}(\Omega_1) \cap \mathcal{E}(\Omega_2) = \emptyset, \quad 0 \notin \mathcal{E}(\Omega_1) \cup \mathcal{E}(\Omega_2). \tag{2.30}
\]

Equations (2.29b) and (2.29c) are linear, which implies expressions for \( r_j \) and \( s_j \) in the form

\[
r_j = \exp((-1)^{j-1}(\Omega_j x - \Omega_j^t t))C_j^+, \quad s_j = \exp((-1)^{j-1}(\Omega_j^T x - (\Omega_j^T)^{-t})D_j^+), \tag{2.31}
\]

where \( C_j^+ \), \( D_j^+ \) are constant column vectors, \( j = 1, 2 \). The key point of the procedure to solve Eq. (2.29a) is to factor \( M_1 \) and \( M_2 \) into triplets as \( M_1 = F_1G_1H_2 \) and \( M_2 = F_2G_2H_1 \), where \( \{F_j, H_j\} \subseteq \mathbb{C}_{N_j \times N_j} \), \( G_1 \subseteq \mathbb{C}_{N_1 \times N_2} \), and \( G_2 \subseteq \mathbb{C}_{N_2 \times N_1} \). For the detailed calculations, we refer the refer to [38].

In what follows, we just list two kinds of solutions involving Cauchy matrix-type soliton solutions and Cauchy matrix-type Jordan block solutions, where the subscripts \( D \) and \( J \) usually correspond to the respective diagonal and Jordan block \( \{\Omega_j\} \). If

\[
\Omega_j = \Omega_{D,j} = \text{Diag}(k_{j,1}, k_{j,2}, \ldots, k_{j,N_j}), \tag{2.32}
\]

for \( j = 1, 2 \), then we have

\[
r_j = \exp((-1)^{j-1}(\Omega_{D,j} x - \Omega_{D,j}^t t))C_{D,j}^+, \quad s_j = \exp((-1)^{j-1}(\Omega_{D,j}^T x - (\Omega_{D,j}^T)^{-t})D_{D,j}^+),
\]

\[
M_1 = \exp(\Omega_{D,1} x - \Omega_{D,1}^t t)C_{D,1}G_1D_{D,2}\exp(-\Omega_{D,2} x + \Omega_{D,2}^t t), \quad M_2 = \exp(-\Omega_{D,2} x + \Omega_{D,2}^t t)C_{D,2}(-G_1^T)D_{D,1}\exp(\Omega_{D,1} x - \Omega_{D,1}^t t), \tag{2.33}
\]

where

\[
C_{D,j}^+ = (c_{j,1}, c_{j,2}, \ldots, c_{j,N_j})^T, \quad D_{D,j}^+ = (d_{j,1}, d_{j,2}, \ldots, d_{j,N_j})^T,
\]

\[
C_{D,j} = \text{Diag}(c_{j,1}, c_{j,2}, \ldots, c_{j,N_j}), \quad D_{D,j} = \text{Diag}(d_{j,1}, d_{j,2}, \ldots, d_{j,N_j}), \tag{2.34}
\]

\[
G_D = (g_{i,m})_{N_1 \times N_2}, \quad g_{i,m} = \frac{1}{k_{1,i} - k_{2,m}}.
\]

In this case, we can derive Cauchy matrix-type soliton solutions.
If

$$\Omega_j = \Omega_{j,j} = \begin{pmatrix} k_{j,1} & 0 & 0 & \ldots & 0 \\ 1 & k_{j,1} & 0 & \ddots & \vdots \\ 0 & 1 & k_{j,1} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 1 & k_{j,1} \end{pmatrix}_{N_j \times N_j}$$

(2.35)

for \( j = 1, 2 \), then we have

$$r_j = \exp((-1)^j(\Omega_{j,j}x - \Omega^n_{j,j}t))C^+_j,$$

$$s_j = \exp((-1)^j(\Omega^T_{j,j}x - (\Omega^T_{j,j})^n))D^+_j,$$

$$M_1 = \exp(\Omega_{j,1}x - \Omega^n_{j,1}t)C^{-1}_{j,1}G_jD_{j,2}\exp(-\Omega^T_{j,2}x + (\Omega^T_{j,1})^n)t),$$

$$M_2 = \exp(-\Omega_{j,2}x + \Omega^n_{j,2}t)C^{-1}_{j,2}(-G^T_j)D^{-1}_{j,1}\exp(\Omega^T_{j,1}x - (\Omega^T_{j,1})^n)t),$$

(2.36)

where

$$C^+_{j,j} = e_{j,1}I, \quad D^+_{j,j} = d_{j,1}I, \quad C^-_{j,j} = e_{j,1}I, \quad D^-_{j,j} = d_{j,1}I,$$

$$G_j = (g_{i,m})_{N_1 \times N_2}, \quad g_{i,m} = C^{i-1}_{i+m-2}(-1)^{i+1}(k_{1,1} - k_{2,1})^{i+m-1}.$$  

(2.37)

with \( I = (1, 0, \ldots, 0)^T \) and \( C^i_m = \frac{m!}{i!(m-i)!} \) for \( m \geq i \). In this case, we can derive Cauchy matrix-type Jordan block solutions.

In the next two sections, we present the Cauchy matrix reduction technique to consider the Cauchy matrix solutions for the local and nonlocal equations (2.11), (2.14), (2.19), and (2.22). For this, we take \( N_1 = N_2 = N \). In addition, for convenience, we set \( k_i = k_{2,i}, c_i = c_{2,i}, d_i = d_{2,i}, \) and \( \vartheta_i = c_i d_i \) with \( i = 1, 2, \ldots, N \).

### 3. Cauchy matrix solutions for equations (2.11) and (2.14)

In this section, we discuss Cauchy matrix solutions of local and nonlocal cmKdV equations (2.11) and local and nonlocal csG equations (2.14). We construct suitable constraints for the pairs \((\Omega_1, \Omega_2)\), \((r_1, r_2)\), \((s_1, s_2)\), and \((M_1, M_2)\) in the defining equation set (2.29) such that solutions (2.25) satisfy reductions (2.10) and (2.13).

#### 3.1. Local and nonlocal cmKdV equations

To derive solutions of the local and nonlocal cmKdV equations (2.11), we restrict ourselves to \( n = 3 \) in Theorem 1. For the Cauchy matrix solution of Eq. (2.11), we show the results in the following theorem.

**Theorem 3.** The function

$$u(x, t) = s_T^r(I - M_2M_1)^{-1}r_2,$$

(3.1)

solves the local and nonlocal cmKdV equations (2.11), where the matrices and vectors in the right-hand side satisfy Eqs. (2.29) \((n = 3)\) and simultaneously obey the constraints

$$r_1 = \varepsilon T \cdot r_2^*(\sigma x, \sigma t), \quad s_1 = \varepsilon(T^r)^{-1}s_2^*(\sigma x, \sigma t), \quad M_1 = -\delta T M_2^*(\sigma x, \sigma t)T^*,$$

(3.2)

where \( T \in \mathbb{C}_{N \times N} \) is a constant matrix such that

$$\Omega_1T + \sigma T \Omega_2^* = 0, \quad C_1^+ = \varepsilon T(C_2^*)^*, \quad D_1^+ = \varepsilon(T^r)^{-1}D_2^r, \quad \varepsilon^2 = \varepsilon^* = \delta.$$  

(3.3)
**Proof.** This is shown by direct computation. Under assumption (3.3), starting from (2.31) we have

$$r_1(x, t) = \exp(\Omega_1 x - \Omega_1^t t)C_1^+ = \exp(T(-\Omega_2^3 x + \Omega_2^3 \sigma t)T^{-1})C_1^+ = T \exp(-\Omega_2^3 x + \Omega_2^3 \sigma t)T^{-1}C_1^+ = \epsilon T \cdot r_2^*(\sigma x, \sigma t).$$

(3.4)

A similar computation yields $s_1(x, t) = \epsilon(T^T)^{-1}s_1^*(\sigma x, \sigma t)$. Substituting $\Omega_1 = -\sigma T \Omega_2^2 T^{-1}$ in the first equation in (2.29a), using the second equation in (2.29a), and noticing the solvability of the Sylvester equation, we arrive at the relation $M_1(x, t) = -\delta \sigma T M_2^*(\sigma x, \sigma t)T^*$. 

Using relations (3.2) and (3.3), we immediately obtain

$$v(x, t) = s_2^T(I - M_1 M_2)^{-1}r_1 = \epsilon^2 s_2^T(\sigma x, \sigma t)(I - M_2^*(\sigma x, \sigma t)M_1^*(\sigma x, \sigma t))^{-1}r_2^*(\sigma x, \sigma t) = \delta \epsilon^2 r_2^*(\sigma x, \sigma t),$$

which is nothing but the reduction in (2.10). The theorem is proved.

**3.1.1. Exact solutions.** According to Theorem 3, solutions of local and nonlocal cmKdV equations (2.11) can be expressed as

$$u(x, t) = s_2^T(I + \delta \sigma M_2^2 T^*(\sigma x, \sigma t)T^*)^{-1}r_2,$$

(3.5a)

where the components $r_2$ and $s_2$ are

$$r_2 = \exp(-\Omega_2 x + \Omega_2^3 t)C_2^+, \quad s_2 = \exp(-\Omega_2^2 T x + (\Omega_2^2)^3 t)D_2^+,$$

(3.5b)

and the matrices $M_2$ and $T$ are defined by the relation

$$\Omega_2 M_2^2 T + \sigma M_2^2 T \Omega_2^* = \epsilon r_2 s_2^T(\sigma x, \sigma t).$$

(3.5c)

**Remark 1.** Redefining $M_2^2 T \rightarrow M_2^2$, we can simplify solution (3.5a) together with (3.5c) to

$$u(x, t) = s_2^T(I + \delta \sigma M_2^2 (\sigma x, \sigma t))^{-1}r_2,$$

(3.6a)

$$\Omega_2^2 M_2^2 + \sigma M_2^2 \Omega_2^* = \epsilon r_2 s_2^T(\sigma x, \sigma t).$$

(3.6b)

| $(\epsilon, \sigma)$ | Soliton solution | Sylvester equation |
|---------------------|------------------|-------------------|
| (1, 1)              | $u(x, t) = s_2^T(I + M_2^{(1)}(x, t))^{-1}r_2$ | $\Omega_{D,2}^2 M_2^{(1)} + \Omega_{D,2}^{(1)} \Omega_{D,2}^* = r_2 s_2^T$ |
| (1, 1)              | $u(x, t) = s_2^T(I - M_2^{(2)}(x, t)^{-1}r_2$ | $\Omega_{D,2}^2 M_2^{(2)} + \Omega_{D,2}^{(2)} \Omega_{D,2}^* = \epsilon r_2 s_2^T$ |
| (1, -1)             | $u(x, t) = s_2^T(I - M_2^{(3)}(x, t)^{-1}r_2$ | $\Omega_{D,2}^2 M_2^{(3)} - M_2^{(3)} \Omega_{D,2}^* = r_2 s_2^T(\sigma x, \sigma t)$ |
| (1, -1)             | $u(x, t) = s_2^T(I + M_2^{(4)}(x, t)^{-1}r_2$ | $\Omega_{D,2}^2 M_2^{(4)} - M_2^{(4)} \Omega_{D,2}^* = \epsilon r_2 s_2^T(\sigma x, \sigma t)$ |

**Table 1.** Soliton solutions of local and nonlocal cmKdV equations (2.11)

We now pay attention to Eqs. (3.5b) and (3.6) to give the expressions for solutions of local and nonlocal cmKdV equations (2.11) with different $(\epsilon, \sigma)$. To obtain soliton solutions, we take $\Omega_2 = \Omega_{D,2}$. We list the soliton solution formulas with different $(\epsilon, \sigma)$ in Table 1. In the table, the vectors $r_2$, $s_2$ and matrices $M_2^{(j)} = M_2^{(j)}(x, t)$ ($j = 1, 2, 3, 4$) are given by

$$r_2 = \exp(-\Omega_{D,2}^2 x + \Omega_{D,2}^3 t)C_{D,2}^+, \quad s_2 = \exp(-\Omega_{D,2}^2 x + (\Omega_{D,2}^3)^3 t)D_{D,2}^+,$$

(3.7a)

$$M_2^{(1)} = -i M_2^{(2)} = \exp(-\Omega_{D,2}^2 x + \Omega_{D,2}^3 t)C_{D,2}^- G_{D}^{(12)} D_{D,2}^+ \exp(-\Omega_{D,2}^2 x + \Omega_{D,2}^3 t),$$

(3.7b)

$$M_2^{(3)} = -i M_2^{(4)} = \exp(-\Omega_{D,2}^2 x + \Omega_{D,2}^3 t)C_{D,2}^- G_{D}^{(34)} D_{D,2}^+ \exp(-\Omega_{D,2}^2 x + \Omega_{D,2}^3 t),$$

(3.7c)
where $C_{D,2}^\pm$ and $D_{D,2}^\pm$ are defined in (2.34) and

$$G_D^{(12)} = (g_{i,m}^{(12)})_{N \times N}, \quad g_{i,m}^{(12)} = \frac{1}{k_i + k_m^*}, \quad (3.8a)$$

$$G_D^{(34)} = (g_{i,m}^{(34)})_{N \times N}, \quad g_{i,m}^{(34)} = \frac{1}{k_i - k_m^*}. \quad (3.8b)$$

To derive the Cauchy matrix-type Jordan block solutions, we take $\Omega_2 = \Omega_{1,2}$ and summarize the Jordan block solutions in Table 2, where we use the notation

$$r_2 = \exp(-\Omega_{1,2} x + \Omega_{1,2}^3 t) C_{1,2}^+, \quad s_2 = \exp(-\Omega_{1,2}^T x + (\Omega_{1,2}^T)^3 t) D_{1,2}^+, \quad (3.9a)$$

$$\tilde{M}_2^{(1)} = -t \tilde{M}_2^{(2)} = \exp(-\Omega_{1,2} x + \Omega_{1,2}^3 t) C_{1,2}^+ \tilde{G}_{1,2}^+ D_{1,2}^+, \exp(-\Omega_{1,2}^T x + (\Omega_{1,2}^T)^3 t), \quad (3.9b)$$

$$\tilde{M}_2^{(1)} = -t \tilde{M}_2^{(2)} = \exp(-\Omega_{1,2} x + \Omega_{1,2}^3 t) C_{1,2}^- \tilde{G}_{1,2}^- D_{1,2}^+, \exp(\Omega_{1,2}^T x - (\Omega_{1,2}^T)^3 t), \quad (3.9c)$$

with $C_{1,2}^+$ and $D_{1,2}^+$ defined in (2.37) and

$$\tilde{G}_{1} = (\tilde{g}_{i,m})_{N \times N}, \quad \tilde{g}_{i,m} = C_{i+m+2} (-1)^{m+1} \frac{1}{(k_1 + k_1^*)^{i+m+1}}, \quad (3.10a)$$

$$\tilde{G}_{1} = (\tilde{g}_{i,m})_{N \times N}, \quad \tilde{g}_{i,m} = C_{i+m-2} (-1)^{m+1} \frac{1}{(k_1 - k_1^*)^{i+m+1}}. \quad (3.10b)$$

| $(\varepsilon, \sigma)$ | Jordan block solution | Sylvester equation |
|------------------------|-----------------------|-------------------|
| $(1, 1)$ | $u(x, t) = s_2^T (I + \tilde{M}_2^{(1)} \tilde{M}_2^{(1)*})^{-1} r_2$ | $\Omega_{1,2} \tilde{M}_2^{(1)} + \tilde{M}_2^{(1)} \Omega_{1,2} = r_2 s_2^T$ |
| $(i, 1)$ | $u(x, t) = s_2^T (I - \tilde{M}_2^{(2)} \tilde{M}_2^{(2)*})^{-1} r_2$ | $\Omega_{1,2} \tilde{M}_2^{(2)} + \tilde{M}_2^{(2)} \Omega_{1,2} = i r_2 s_2^T$ |
| $(1, -1)$ | $u(x, t) = s_2^T (I - \tilde{M}_2^{(1)} \tilde{M}_2^{(1)*})(-x,t)^{-1} r_2$ | $\Omega_{1,2} \tilde{M}_2^{(1)} - \tilde{M}_2^{(1)} \Omega_{1,2} = r_2 s_2^T(-x,-t)$ |
| $(i, -1)$ | $u(x, t) = s_2^T (I + \tilde{M}_2^{(2)} \tilde{M}_2^{(2)*})(-x,t)^{-1} r_2$ | $\Omega_{1,2} \tilde{M}_2^{(2)} - \tilde{M}_2^{(2)} \Omega_{1,2} = i r_2 s_2^T(-x,-t)$ |

### 3.1.2. Dynamics.

We now present some explicit solutions of the local and nonlocal cmKdV equations (2.11). Moreover, we identify their dynamical properties. For brevity, we introduce the notation

$$\xi_i = 2(k_i^3 t - k_i x), \quad e^{\theta_{im}} = \frac{1}{(k_i^* - k_m^*)^2}, \quad e^{\varphi_{im}} = \frac{1}{(k_i^* + k_m^*)^2}, \quad (3.11)$$

where $i, m = 1, 2, \ldots, N$.

When $N = 1$, 1-soliton solutions of Eq. (2.11) can be described as

$$\varepsilon = 1, \quad \sigma = 1 : \quad u_{11, \varepsilon, \sigma} = e^{-\xi_1} \frac{\partial_1}{|\partial_1|^2 e^{\xi_1} + \theta_{11}}, \quad (3.12a)$$

$$\varepsilon = i, \quad \sigma = 1 : \quad u_{11, \varepsilon, \sigma} = e^{-\xi_1} \frac{\partial_1}{|\partial_1|^2 e^{\xi_1} + \theta_{11}}, \quad (3.12b)$$

$$\varepsilon = 1, \quad \sigma = -1 : \quad u_{11, \varepsilon, \sigma} = e^{-\xi_1} \frac{\partial_1}{|\partial_1|^2 e^{-\xi_1} + \theta_{11}}, \quad (3.12c)$$

$$\varepsilon = i, \quad \sigma = -1 : \quad u_{11, \varepsilon, \sigma} = e^{-\xi_1} \frac{\partial_1}{|\partial_1|^2 e^{-\xi_1} + \theta_{11}}, \quad (3.12d)$$

These solutions map into the ones obtained in [22] by simple transformations.
Because these solutions have a complex form, we analyze the dynamics of $|u_{11}|^2$. With no loss of generality, we discuss solutions (3.12a) and (3.12c).

We start with the 1-soliton solution in (3.12a). Setting

$$k_1 = \alpha + i \beta$$

(3.13)

there, we show by direct calculation that in the case $\varepsilon = 1$ and $\sigma = 1$, we have

$$|u_{11}|^2_{(\varepsilon, \sigma)} = \alpha^2 \sech^2[2\alpha(x - (\alpha^2 - 3\beta^2)t + h)], \quad h = \frac{1}{2\alpha} \ln \frac{2\alpha}{|\partial_1|}.$$  (3.14)

This solution describes a stable unidirectional traveling wave, which travels with a fixed amplitude $\alpha^2$, a constant velocity $\alpha^2 - 3\beta^2$, and the crest trajectory given by $x = (\alpha^2 - 3\beta^2)t - h$. When $\alpha^2 = 3\beta^2$, Eq. (3.14) describes a stationary soliton wave. Figure 1 shows a stationary soliton wave and two moving soliton waves.

We next consider solution (3.12c). Substituting (3.13) in solution (3.12c) in the case $\varepsilon = 1$ and $\sigma = -1$, we obtain

$$|u_{11}|^2_{(\varepsilon, \sigma)} = \frac{16|\beta^2 \partial_1|^2 e^{4\alpha(\alpha^2 t - 3\beta^2 t - x)}}{|\partial_1|^4 + 16|\beta \partial_1|^2 \cos[4\beta(x - 3\alpha^2 t + \beta^2 t)]}.$$  (3.15)

Because of the cosine function in the denominator, there is a quasiperiodic phenomenon. When $|\partial_1|^2 = 4\beta^2$, solution (3.15) has singularities on the straight lines

$$x(t) = (3\alpha^2 - \beta^2)t + \frac{\kappa\pi}{2\beta}, \quad \kappa \in \mathbb{Z},$$  (3.16)

and when $|\partial_1|^2 \neq 4\beta^2$, solution (3.15) is nonsingular and reaches its extrema on the straight lines

$$x(t) = (3\alpha^2 - \beta^2)t + \frac{1}{4\beta} \left( \gamma + 2\kappa \pi - \arcsin \frac{\alpha(|\partial_1|^4 + 16|\beta \partial_1|^2)}{8|\beta \partial_1|^2 \sqrt{\alpha^2 + \beta^2}} \right), \quad \kappa \in \mathbb{Z},$$  (3.17)

where $\sin \gamma = \alpha/\sqrt{\alpha^2 + \beta^2}$. The velocity is $3\alpha^2 - \beta^2$. When $3\alpha^2 = \beta^2$, (3.15) is a stationary solution. For a given $t$, $|u_{11}|^2_{(\varepsilon, \sigma)}$ with $\varepsilon = 1$ and $\sigma = -1$ tends to zero as $x \rightarrow \infty$ for $\alpha > 0$ and as $x \rightarrow -\infty$ for $\alpha < 0$. We show this solution in Fig. 2.

When $N = 2$, with no loss of generality we just list 2-soliton solutions of (2.11) in the cases $\varepsilon = 1$, $\sigma = 1$ and $\varepsilon = 1$, $\sigma = -1$. They are given by

$$\varepsilon = 1, \quad \sigma = 1 : \quad u_{12,(\varepsilon, \sigma)} = \frac{u_{D,2}}{u_{D,1}},$$  (3.18a)

where

$$u_{D,1} = 1 + \sum_{i=1}^{2} \sum_{j=1}^{2} \partial_i \partial_j e^{\xi_i + \xi_j + \theta_{ij}} + |\partial_1 \partial_2|^2 |k_1 - k_2|^4 e^{\xi_1 + \xi_2 + \xi_3 + \xi_4 + \theta_{12} + \theta_{21} + \theta_{34} + \theta_{43}},$$

(3.18b)

and

$$\varepsilon = 1, \quad \sigma = -1 : \quad u_{12,(\varepsilon, \sigma)} = \frac{u_{D,4}}{u_{D,3}},$$  (3.19a)
Fig. 1. 1-soliton solution $|u_{11}|^2_{(\varepsilon, \sigma)}$ with $\varepsilon = 1$ and $\sigma = 1$, Eq. (3.14), with $c_1 = d_1 = 1 + i$ at $t = 1$ (solid line) and $t = 0$ (dashed line): (a) a stationary wave for $k_1 = \sqrt{3} + i$, and a traveling wave for (b) $k_1 = 1 + i$ and (c) $k_1 = 1.5 + 0.5i$.

Fig. 2. Contour plot of a 1-soliton solution $|u_{11}|^2_{(\varepsilon, \sigma)}$ with $\varepsilon = i$ and $\sigma = 1$, Eq. (3.15): (a) level plot for $c_1 = d_1 = 1$ and $k_1 = 0.01 + i$ in the range $x \in [-3, 3]$, $t \in [-3, 3]$; (b) waves at $t = 0.1$ with $k_1 = 0.06 + i$ (solid line) and $k_1 = -0.06 + i$ (dashed line).
where

\[ u_{D,3} = 1 + \sum_{i=1}^{2} \sum_{j=1}^{2} \partial_{i}^{n} \partial_{j} \epsilon \alpha_{i} - \xi_{i} + \epsilon_{ij} + \sum_{i=1}^{2} \partial_{i}^{n} \partial_{j} \epsilon \alpha_{i} - \xi_{i} + \epsilon_{ij}, \]

\[ + |\partial_{1} \partial_{2}| k_{2} - k_{2}^{4} e^{\xi_{1} - \xi_{2} - \xi_{11} + \epsilon_{12} + \epsilon_{21} + \epsilon_{22}}, \] (3.19b)

\[ u_{D,4} = \sum_{i=1}^{2} \partial_{i} \epsilon \xi_{i} + \partial_{1} \partial_{2} (k_{1} - k_{2})^{2} e^{\xi_{1} + \xi_{2}} \left( \sum_{i=1}^{2} \partial_{i}^{n} \epsilon \alpha_{i} - \xi_{i} + \epsilon_{11} + \epsilon_{12} \right). \]

We show these two solitary waves in Figs. 3 and 4.

We finally discuss the simplest Jordan block solutions (\( N = 2 \)). We still take \( \epsilon = 1, \sigma = 1 \) and \( \epsilon = 1, \sigma = -1 \) as two examples. The Jordan block solutions of Eq. (2.11) are as follows:

\[ \epsilon = 1, \sigma = 1 : \quad u_{13,13, \epsilon, \sigma} = \frac{u_{14, \epsilon}}{u_{1,1}} \] (3.20a)

where

\[ u_{1,1} = (|\partial_{1}|^{2} e^{\xi_{1} + \xi_{1}} + e^{2\theta_{11}})^{2} + \]

\[ + |\partial_{1}|^{2} e^{\xi_{1} + \xi_{11} - 3\theta_{11}} (1 - 2 e^{\theta_{11}})^{2} + (1 - e^{\theta_{11}})(\tau^{2} + \tau^{*2}) - 2 e^{\theta_{11}} |\tau|^{2} + |\tau|^{4}, \]

\[ u_{1,2} = \partial_{1} e^{\xi_{1} - 4\theta_{11}} \times \]

\[ \times \left( 1 - |\partial_{1}|^{2} e^{\xi_{1} + \xi_{11} + 2 \theta_{11}} \left( 1 - 4 e^{\theta_{11}} + \tau^{*}(\tau + \tau^{*}) + |\tau|^{2} e^{\theta_{11}/2} - 2 \sqrt{2} \tau \sinh \frac{\theta_{11} + \ln 2}{2} \right) \right), \] (3.20b)

with \( \tau = 3k_{1}^{2}t - x, \) and

\[ \epsilon = 1, \sigma = -1 : \quad u_{13,13, \epsilon, \sigma} = \frac{u_{4, \epsilon}}{u_{3,3}} \] (3.21a)

where

\[ u_{3,3} = (|\partial_{1}|^{2} e^{\xi_{1} - \xi_{1}} + e^{-2\epsilon_{11}})^{2} + \]

\[ + |\partial_{1}|^{2} e^{\xi_{1} + \xi_{11} - 3\epsilon_{11}} (4 e^{\epsilon_{11}} + e^{\epsilon_{11}}(\tau + \tau^{*})^{2} + 4|\tau|^{2}) + \]

\[ + |\tau|^{2} + 2 e^{\epsilon_{11}/2}(1 + 2 e^{\epsilon_{11}} + |\tau|^{2})(\tau + \tau^{*}) \right), \] (3.21b)

\[ u_{4,4} = \partial_{1} e^{\xi_{1} - 4\epsilon_{11}} \left( 1 + |\partial_{1}|^{2} e^{\xi_{1} + \xi_{11} + 2\epsilon_{11}} (1 + 4 e^{\epsilon_{11}} + 2 e^{\epsilon_{11}/2}(\tau + 2\tau^{*}) + 3|\tau|^{2} + \right) \]

\[ + \tau^{2} + \tau e^{-\epsilon_{11}/2}(1 + \tau^{*2}) \right). \]

We show these two solitary waves in Figs. 5 and 6.

### 3.2. Local and nonlocal csG equations.

We now consider the Cauchy matrix solutions of local and nonlocal csG equations (2.14). We set \( n = -1 \) in (2.26). Similarly to the foregoing, the Cauchy matrix solutions of Eq. (2.14) can be summarized by the following theorem. We skip the proof because it is similar to the one for Theorem 3.

**Theorem 4.** The function

\[ u(x, t) = s_{4}^{T}(I - M_{2} M_{1})^{-1} r_{2}, \] (3.22)

solves local and nonlocal csG equations (2.14), where the matrices and vectors in the right-hand side satisfy Eqs. (2.29) \( (n = -1) \) and constraints (3.2) where \( T \in \mathbb{C}^{N \times N} \) is a constant matrix satisfying (3.3).
Fig. 3. Shape and motion of 2-soliton solutions $|u_{12}|_{(\epsilon, \sigma)}^2$ with $\epsilon = 1$ and $\sigma = 1$, given by (3.18), for $k_1 = 0.2 + 0.8i$, $k_2 = 0.25 + 0.1i$ and $c_1 = c_2 = d_1 = d_2 = 1$: 2D plot at (a) $t = -8$, (b) $t = 0$, and (c) $t = 8$.

Fig. 4. Shape and motion of 2-soliton solutions $|u_{12}|_{(\epsilon, \sigma)}^2$ with $\epsilon = 1$ and $\sigma = -1$, given by (3.19), for $k_1 = 0.3 + i$, $k_2 = 0.01 + 0.5i$, $c_1 = 2$ and $c_2 = d_1 = d_2 = 1$: (a) contour plot in the range $x \in [-10, 10]$, $t \in [-20, -11]$; (b) waves at $t = -5$ (solid line) and $t = -11$ (dashed line).
Fig. 5. Shape and motion of Jordan block solutions $|u_{13}|^2(\varepsilon, \sigma)$ with $\varepsilon = 1$ and $\sigma = 1$, given by (3.20), for $k_1 = 0.28 - 0.28i$ and $c_1 = d_1 = 1$: 2D plot at (a) $t = -10$, (b) $t = 2$, and (c) $t = 7$.

Fig. 6. Shape and motion of Jordan block solutions $|u_{13}|^2(\varepsilon, \sigma)$ with $\varepsilon = 1$ and $\sigma = -1$, given by (3.21), for $k_1 = 0.02 + i$ and $c_1 = d_1 = 1$: 2D plot at (a) $t = -0.5$, (b) $t = 0$, and (c) $t = 0.5$. 

1527
3.2.1. Exact solutions. According to Theorem 4, solutions of local and nonlocal csG equations (2.14) are still given by (3.5a), where the components $r_2(x,t)$ and $s_2(x,t)$ are of form

$$r_2 = \exp(-\Omega_2 x + \Omega_2^{-1} t)C_2^+,$$  
$$s_2 = \exp(-\Omega_2^T x + (\Omega_2^T)^{-1} t)D_2^+,$$  
(3.23)

and $M_2$ and $T$ are determined by (3.5c).

Tables 1 and 2 also provide the respective soliton solutions and Jordan block solutions of the local and nonlocal csG equations (2.14), where in Table 1 we set $r = (\Omega_2x - (\Omega_2)^{-1}t)$.

We now discuss the dynamics of 1-soliton solutions (3.24). We consider two solutions (3.24c) and (3.24d) as the examples. Substituting (3.13) in (3.24a) leads to

$$2\Omega_2 x + 2r C_2^+ D_2^+,$$  
(3.24a)

$$2\Omega_2 x + 2r C_2^+ D_2^+,$$  
(3.24b)

$$2\Omega_2 x + 2r C_2^+ D_2^+,$$  
(3.24c)

$$2\Omega_2 x + 2r C_2^+ D_2^+,$$  
(3.24d)

where $\zeta_j = 2(k_j^{-1}t - k_j x)$, $j = 1,2$.

3.2.2. Dynamics. We now discuss the dynamics of 1-soliton solutions (3.24). We consider two solutions (3.24a) and (3.24c) as the examples. Substituting (3.13) in (3.24a) leads to

$$2\Omega_2 x + 2r C_2^+ D_2^+,$$  
(3.24a)

This wave travels with a fixed amplitude $\alpha^2$ and a constant velocity $1/(\alpha^2 + \beta^2)$. The crest trajectory is $x = t/(\alpha^2 + \beta^2) - h$. There is no stationary wave because of the positive velocity. Figure 7 shows a traveling soliton wave.

For solution (3.24c), we have

$$2\Omega_2 x + 2r C_2^+ D_2^+,$$  
(3.24c)

In quite a similar fashion, there is a quasiperiodic phenomenon because of the appearance of the cosine function. When $|\theta_1|^2 = 4\beta^2$, solution (3.26) has singularities on the straight lines

$$x(t) = \frac{t}{\alpha^2 + \beta^2} + \frac{\kappa \pi}{2\beta}, \quad \kappa \in \mathbb{Z},$$  
(3.27)

1528
Fig. 7. Waves of 1-soliton solution (3.25) with $k_1 = c_1 = d_1 = 1 + i$ at $t = 5$ (solid line) and $t = -5$ (dashed line).

Fig. 8. Shape and motion of 1-soliton solutions $|u_{21}|^2_{(\varepsilon, \sigma)}$ with $\varepsilon = 1$ and $\sigma = -1$, given by (3.26), for $c_1 = d_1 = 1$: (a) contour plot for $k_1 = 0.01 + i$ in the range $x \in [-3, 3], t \in [-3, 3]$; (b) waves at $t = 0.1$ with $k_1 = 0.06 + i$ (solid line) and $k_1 = -0.06 + i$ (dashed line).

and when $|\vartheta_1|^2 \neq 4\beta^2$, solution (3.26) is nonsingular and reaches its extrema on the straight lines

$$
x(t) = -\frac{t}{\alpha^2 + \beta^2} + \frac{1}{4\beta} \left( \gamma + 2\kappa \pi - \arcsin \frac{\alpha(|\vartheta_1|^4 + 16\beta^4)}{8|\vartheta_1|^2 \sqrt{\alpha^2 + \beta^2}} \right), \quad \kappa \in \mathbb{Z},
$$

(3.28)

where $\gamma$ is same as in (3.17). The velocity is $-1/(\alpha^2 + \beta^2)$. For a given $t$, $|u_{21}|^2_{(\varepsilon, \sigma)}$ with $\varepsilon = 1$ and $\sigma = -1$ tends to zero as $x \to \infty$ for $\alpha > 0$ and as $x \to -\infty$ for $\alpha < 0$. We show this solution in Fig. 8.

The 2-soliton solutions of (2.14) in the cases $\varepsilon = 1, \sigma = 1$ and $\varepsilon = 1, \sigma = -1$ are as follows:

$$
\varepsilon = 1, \sigma = 1: \quad u_{22, (\varepsilon, \sigma)} = \frac{u'_{D, 2}}{u'_{D, 1}},
$$

(3.29a)

where

$$
u'_{D, 1} = 1 + \sum_{i=1}^{2} \sum_{j=1}^{2} \vartheta_i^* \vartheta_j e^{\vartheta_i + \theta_{ij}} +
\vartheta_1 \vartheta_2 (k_1 - k_2) e^{\vartheta_1 + \vartheta_2} e^{\vartheta_{11} + \theta_{11} + \theta_{21} + \theta_{22}},
$$

(3.29b)

$$
u'_{D, 2} = \sum_{i=1}^{2} \vartheta_i e^{\vartheta_i} + \vartheta_1 \vartheta_2 (k_1 - k_2) e^{\vartheta_1 + \vartheta_2} e^{\vartheta_{11} + \theta_{11} + \theta_{21} + \theta_{22}}.
$$
We use a similar strategy, in other words, we impose suitable constraints on the pairs (\(2.19\)) and the local and nonlocal potential cmKdV equations (\(2.22\)).

\[ u'_{D,4} = \frac{2}{u'_{D,0}}, \]

where

\[ u'_{D,0} = \frac{2}{u'_{D,1}} \]

and

\[ \varepsilon = 1, \sigma = -1: \quad u_{12,\varepsilon,\sigma} = \frac{u'_{\Omega,4}}{u_{D,0}}, \tag{3.30a} \]

We show \(|u_{22,\varepsilon,\sigma}|^2\) for \(\varepsilon = 1, \sigma = 1\) and \(\varepsilon = 1, \sigma = -1\) in Figs. 9 and 10.

For the simplest Jordan block solutions, we start with the case \(\varepsilon = 1, \sigma = 1\). Then the solution is

\[ \varepsilon = 1, \sigma = 1: \quad u_{23,\varepsilon,\sigma} = \frac{u'_{1,2}}{u'_{1,1}}, \tag{3.31a} \]

where

\[ u'_{1,1} = |\vartheta|^2 e^{\xi_i + \zeta_i} + e^{-2\xi,12} + \vartheta^2 e^{\xi_i + \zeta_i - 2\xi_{11}} (1 + (t - x)^2 - 2e^{\xi_{11}}), \]

\[ u'_{1,2} = \vartheta e^{\xi_i - 2\xi_{11}} \bigg( 1 - |\vartheta|^2 e^{\xi_i + 2\xi_{11}} \bigg( 1 - 4e^{\xi_{11}} + (e^{-\xi_{11}/2}(t - x) + 2)(t - x)^2 - 2\sqrt{2}(t - x) \sinh \frac{\vartheta_{11} + \ln 2}{2} \bigg) \bigg). \tag{3.31b} \]

In the case \(\varepsilon = 1, \sigma = -1\), the Jordan block solution is given by

\[ \varepsilon = 1, \sigma = -1: \quad u_{23,\varepsilon,\sigma} = \frac{u'_{1,4}}{u'_{1,3}}, \tag{3.32a} \]

where

\[ u'_{1,3} = |\vartheta|^2 e^{\xi_i - \zeta_i} + e^{-2\xi_{11}} + |\vartheta|^2 e^{\xi_i - 2\xi_{11}} (t - x)^2 (4e^{2\xi_{11}} + 8e^{\xi_{11}} + (t - x)^2)^2 + 4e^{\xi_{11}/2}(1 + (t - x)^2 + 2e^{\xi_{11}/2}) \bigg) \bigg), \tag{3.32b} \]

\[ u'_{1,4} = \vartheta e^{\xi_i - 4\xi_{11}} \bigg( 1 + |\vartheta|^2 e^{\xi_i + 2\xi_{11}} (1 + 4(t - x)^2) + 4e^{\xi_{11}} + 6(t - x)e^{\xi_{11}/2} + e^{-\xi_{11}/2}(t - x)(1 + (t - x)^2) \bigg) \bigg). \]

We show \(|u_{23,\varepsilon,\sigma}|^2\) for \(\varepsilon = 1, \sigma = 1\) and \(\varepsilon = 1, \sigma = -1\) in Figs. 11 and 12.

4. Cauchy matrix solutions of equations (2.19) and (2.22)

In this section, we turn our attention to Theorem 2 and construct Cauchy matrix solutions of the local and nonlocal potential NLS equations (2.19) and the local and nonlocal potential cmKdV equations (2.22). We use a similar strategy, in other words, we impose suitable constraints on the pairs \((\Omega_1, \Omega_2), (r_1, r_2), (s_1, s_2), (M_1, M_2)\) in the set of defining equations (2.29) such that (2.27) coincide with the reductions in (2.18) and (2.21).
Fig. 9. Shape and motion of 2-soliton solutions $|u_{22}|(\varepsilon, \sigma)$ with $\varepsilon = 1$ and $\sigma = 1$, given by (3.29), for $k_1 = 0.15 + 0.5i$, $k_2 = 0.05 + 0.01i$, and $c_1 = c_2 = d_1 = d_2 = 1$: 2D plot at (a) $t = -0.05$, (b) $t = 0$, and (c) $t = 0.03$.

Fig. 10. Shape and motion of 2-soliton solutions $|u_{22}|(\varepsilon, \sigma)$ with $\varepsilon = 1$ and $\sigma = -1$ given by (3.30), for $k_1 = 0.03 + 2.5i$, $k_2 = 0.01 + 0.5i$, $c_1 = 2$, and $c_2 = d_1 = d_2 = 1$: (a) contour plot in the range $x \in [-10, 0]$, $t \in [-8, -4]$; (b) waves at $t = -6$ (solid line) and $t = -5$ (dashed line).
Fig. 11. Shape and motion of Jordan block solutions $|u_{23}|_{(\varepsilon, \sigma)}$ with $\varepsilon = 1$ and $\sigma = 1$, given by (3.31), for $k_1 = 0.25 + 0.3i$ and $c_1 = 2i$: 2D plot at (a) $t = -0.6$, (b) $t = -0.5$, and (c) $t = 0.3$.

Fig. 12. Shape and motion of Jordan block solutions $|u_{23}|_{(\varepsilon, \sigma)}$ with $\varepsilon = 1$ and $\sigma = -1$, given by (3.32), for $k_1 = 0.055 + i$ and $c_1 = d_1 = 1$: 2D plot at (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$. 
4.1. Local and nonlocal potential NLS equations. To find the solutions for local and nonlocal potential NLS equations (2.19) we reduce from the pAKNS(2) equation solution (2.27) in the case $n = 2$. The following theorem reveals the Cauchy matrix solutions for local and nonlocal potential NLS equations (2.19).

**Theorem 5.** The function

$$q = s_2^T(I - M_2M_1)^{-1}\Omega_2^{-1}r_2 - s_2^T(I - M_1M_2)^{-1}\Omega_1^{-1}r_1 - 1 \quad (4.1)$$

solves the local and nonlocal potential NLS equations (2.19), where the matrices and vectors in the right-hand side satisfy Eqs. (2.29) ($n = 2$) and the constraints

$$r_1 = \varepsilon T \cdot r_2^*(\sigma x), \quad s_1 = \varepsilon(T^T)^{-1}s_2^*(\sigma x), \quad M_1 = TM_2^*(\sigma x)T^* \quad (4.2)$$

with a constant matrix $T \in \mathbb{C}_{N \times N}$ such that

$$\Omega_1 T + \sigma T \Omega_2^* = 0, \quad C_1^+ = \varepsilon TC_2^+, \quad D_1^+ = \varepsilon(T^T)^{-1}D_2^+, \quad \varepsilon^2 = \varepsilon^* = -\sigma. \quad (4.3)$$

**Proof.** This is again proved by direct computation. Under assumptions (4.3), we have

$$r_1(x) = \exp(\Omega_1x + i\Omega_1^2t)C_1^+ = \exp(-(T\Omega_2^*\sigma xT^{-1} - T(i\Omega_2^*)^tT^{-1})C_1^+ =
= T\exp(-\Omega_2^*\sigma x - (i\Omega_2^*)^tT^{-1})C_1^+ = \varepsilon T \cdot r_2^*(\sigma x),$$

and the other relations in (4.2) follow similarly. Using relations (4.2) and (4.3), we then show by straightforward calculation that

$$r(x) = s_2^T(I - M_2M_1)^{-1}\Omega_1^{-1}r_1 - s_1^T(I - M_2M_1)^{-1}\Omega_2^{-1}r_1 - 1 =
= \varepsilon r_2^*(\sigma x)(I - M_2^*(\sigma x))^{-1}\Omega_2^{-1}r_1^*(\sigma x) -
- s_2^T(I - M_2^*(\sigma x))M_2^*(\sigma x)\Omega_1^{-1}r_1^*(\sigma x) - 1 = q^*(\sigma x),$$

which coincides with the reduction in (2.18).

**4.1.1. Exact solutions.** Under conditions (4.2) and (4.3), we can write solution (4.1) in an equivalent form as

$$q = s_2^T(I - M_2M_2^*(\sigma x))^{-1}\Omega_2^{-1}r_2 - \varepsilon^*s_2^TM_2(I - M_2^*(\sigma x)M_2^*)^{-1}\Omega_2^{-1}r_2^*(\sigma x) - 1, \quad (4.4a)$$

where

$$r_2 = \exp(-\Omega_2x + i\Omega_2^2t)C_2^+, \quad s_2 = \exp(-\Omega_2^T x - i(\Omega_2^T)^t)D_2^+, \quad (4.4b)$$

and $M_2$ is determined by

$$\Omega_2 M_2 + \sigma M_2 \Omega_2^* = \varepsilon r_2 s_2^T(\sigma x). \quad (4.4c)$$

Here, we followed the treatment discussed in Remark 1 by absorbing the matrix $T$ into $M_2$.

In what follows, we construct soliton solutions and Jordan block solutions of local and nonlocal potential NLS equations (2.19). We seek the soliton solutions by taking $\Omega_2 = \Omega_{D,2}$. In this case, solution (4.4a) can be expressed as

$$q_1 = s_2^T(I - M_2M_2^*(\sigma x))^{-1}\Omega_{D,2}^{-1}r_2 - \varepsilon^*s_2^TM_2(I - M_2^*(\sigma x)M_2^*)^{-1}\Omega_{D,2}^{-1}r_2^*(\sigma x) - 1. \quad (4.5)$$
We let $M_2^{(1)}(x)$, $M_2^{(2)}(x)$, $M_2^{(3)}(x)$, and $M_2^{(4)}(x)$ denote the respective solutions of (4.4c) with $\varepsilon = i$, $\varepsilon = -i$, $\varepsilon = 1$, and $\varepsilon = -1$. It is worth noting that the solutions $q_{1,\varepsilon}$ with $\varepsilon = \pm i$ coincide because $M_2^{(2)}(x) = -M_2^{(3)}(x)$. Besides, the solutions $q_{1,\varepsilon}$ with $\varepsilon = \pm 1$ also coincide because $M_2^{(4)}(x) = -M_2^{(3)}(x)$. In what follows, we therefore present the solutions for $\varepsilon = i$ and $\varepsilon = 1$.

When $\varepsilon = i$, the soliton solution of Eq. (2.19) is given by

$$q_{1,i} = s_2^T (I - M_2^{(1)}(x)) - 1 \Omega_2^{-1} r_2 + is_2^T M_2^{(1)}(x) (I - M_2^{(1)}(-x)) - 1 \Omega_2^{-1} r_2^* - 1,$$  

where

$$r_2 = \exp(-\Omega_{D,2} x - i\Omega_{D,2}^2 t) C_{D,2}^+,$$  

$s_2 = \exp(-\Omega_{D,2} x - i\Omega_{D,2}^2 t) D_{D,2}^+$,  

$$M_2^{(1)} = i \exp(-\Omega_{D,2} x - i\Omega_{D,2}^2 t) C_{D,2}^- D_{D,2}^-$ \exp(-\Omega_{D,2} x + i\Omega_{D,2}^2 t),$$  

with $C_{D,2}^+$ and $D_{D,2}^+$ given in (2.34) and $G_{D}^{(12)}$ defined by (3.8a). Specially, the 1-soliton solution with $\varepsilon = i$ is

$$q_{1,i} = \frac{k_1^i (\varphi_1 e^{-\varphi} - k_1) - |\varphi_1|^2 k_1^i e^{-(\varphi + \varphi^*) + \theta_{11}}}{k_1^i (1 - |\varphi_1|^2 e^{-(\varphi + \varphi^*) + \theta_{11}})},$$  

where $\varphi = 2k_1(x + ik_1 t)$.

When $\varepsilon = 1$, the soliton solution is

$$q_{1,1} = s_2^T (I - M_2^{(3)}(x)) - 1 \Omega_2^{-1} r_2 - s_2^T M_2^{(3)}(x) (I - M_2^{(3)}(-x)) - 1 \Omega_2^{-1} r_2^* - 1,$$  

where $r_2$ and $s_2$ are the same as in (4.7a), and

$$M_2^{(3)} = \exp(-\Omega_{D,2} x - i\Omega_{D,2}^2 t) C_{D,2}^- G_{D}^{(34)} D_{D,2}^+ \exp(\Omega_{D,2} x + i\Omega_{D,2}^2 t),$$  

with $C_{D,2}^+$ and $D_{D,2}^+$ given in (2.34) and $G_{D}^{(34)}$ defined by (3.8b). The 1-soliton solution for $\varepsilon = 1$ is

$$q_{1,1} = \frac{k_1^i (\varphi_1 e^{-\varphi} - k_1) - |\varphi_1|^2 k_1^i e^{-(\varphi + \varphi^*) + \theta_{11}}}{k_1^i (1 + |\varphi_1|^2 e^{-(\varphi + \varphi^*) + \theta_{11}})},$$  

where $\ell = 2k_1(x + ik_1 t)$.

To derive the Jordan block solutions, we take $\Omega_2 = \Omega_{j,2}$. Some solutions are listed as follows:

$$\varepsilon = i: \quad q_{2,i} = s_2^T (I - \tilde{M}_2 \tilde{M}_2) - 1 \Omega_{j,2}^{-1} r_2 + is_2^T \tilde{M}_2 (I - \tilde{M}_2 \tilde{M}_2) - 1 \Omega_{j,2}^{-1} r_2^* - 1,$$

$$\varepsilon = 1: \quad q_{2,1} = s_2^T (I - \tilde{M}_2 \tilde{M}_2(-x)) - 1 \Omega_{j,2}^{-1} r_2 - s_2^T \tilde{M}_2 (I - \tilde{M}_2(-x) \tilde{M}_2) - 1 \Omega_{j,2}^{-1} r_2^* - 1,$$

where

$$r_2 = \exp(-\Omega_{j,2} x - i\Omega_{j,2}^2 t) C_{j,2}^+,$$  

$s_2 = \exp(-\Omega_{j,2} x - i\Omega_{j,2}^2 t) D_{j,2}^+$,

$$\tilde{M}_2 = i \exp(-\Omega_{j,2} x - i\Omega_{j,2}^2 t) C_{j,2}^- D_{j,2}^- \exp(-\Omega_{j,2} x + i\Omega_{j,2}^2 t),$$  

$$\tilde{M}_2 = \exp(-\Omega_{j,2} x - i\Omega_{j,2}^2 t) C_{j,2}^- G_{j,2} T D_{j,2}^- \exp(\Omega_{j,2} x + i\Omega_{j,2}^2 t).$$
4.1.2. Dynamics. We now examine the dynamics of $|q_1|^2$ given by (4.8) and (4.11). Using expansion (3.13) and setting

\[ c_1 = c^{(1)} + ic^{(2)}, \quad d_1 = d^{(1)} + id^{(2)} \]

in solution (4.8), we deduce the solution

\[ \varepsilon = i : \quad |q_1|^2 = \frac{q_{D,2}}{q_{D,1}}, \]

where

\[ q_{D,1} = (\alpha^2 + \beta^2)^2(4\alpha^2e^{2\varsigma_1} - |\theta_1|^2e^{2\varsigma_2})^2, \]

\[ q_{D,2} = (\alpha^2 + \beta^2)^2(16\alpha^4e^{4\varsigma_1} + |\theta_1|^4e^{4\varsigma_2}) + 8\alpha^2|\theta_1|^2(\Theta_1 + \Theta_2) + |\theta_1|^2(\alpha^4 - \beta^4)e^{2(\varsigma_1 + \varsigma_2)} - 32\alpha^4(\alpha^2 + \beta^2)(\Theta_1 \sin \iota_1 + \Theta_2 \cos \iota_1)e^{3\varsigma_1 + \varsigma_2} - 8\alpha^2|\theta_1|^2(2\alpha \beta \Theta_1 + \Theta_2(\alpha^2 - \beta^2)) \cos \iota_1 - (2\alpha \beta \Theta_2 - \Theta_1(\alpha^2 - \beta^2)) \sin \iota_1)e^{\varsigma_1 + 3\varsigma_2}. \]

Here and hereafter, $\varsigma_1 = 2\alpha x$, $\varsigma_2 = 4\alpha \beta t$, $\iota_1 = 2(\beta x + (\alpha^2 - \beta^2)t)$, and

\[ \Theta_1 = (c^{(1)}d^{(2)} + c^{(2)}d^{(1)})\alpha - (c^{(1)}d^{(1)} - c^{(2)}d^{(2)})\beta, \]

\[ \Theta_2 = (c^{(1)}d^{(1)} - c^{(2)}d^{(2)})\alpha + (c^{(1)}d^{(2)} + c^{(2)}d^{(1)})\beta. \]

There is a quasiperiodic phenomenon because of the appearance of sine and cosine functions in the numerator. It is obvious that $|q_1|^2$ with $\varepsilon = i$ has singularities on the straight line $x = 2\beta t - h$. The wave speed is $2\beta$. It is worth noting that if $k_1 = c_1 = d_1$, then

\[ |q_1|^2(x, t)|_{t = \frac{2\pi - \ln \beta}{4\pi}} \rightarrow 0 \quad \text{as} \quad x \rightarrow \frac{\kappa \pi}{\beta} \quad (\kappa \in \mathbb{Z}). \]

In particular, $|q_1|^2(x, t)|_{t = -\ln \beta/4\beta^2}$ is an even function of $x$ and tends to 0 as $x \rightarrow 0$ (see Fig. 13(d)). Besides, we see that $|q_1|^2 \rightarrow 1$ as $x \rightarrow \infty$ for a fixed $t$, which means that the background level here is 1. We show this solution in Fig. 13.

To proceed, we next consider (4.11). Under expansions (3.13) and (4.12), we have

\[ \varepsilon = 1 : \quad |q_1|^2 = \frac{q_{D,4}}{q_{D,3}}, \]

where

\[ q_{D,4} = (\alpha^2 + \beta^2)^2|\theta_1|^4e^{4\varsigma_2} + 16\beta^4[\Theta_1 + \Theta_2]e^{2(\varsigma_2 - \varsigma_1)} + (\alpha^2 + \beta^2)^2 - 2(\alpha^2 + \beta^2)(\Theta_1 \sin \iota_1 + \Theta_2 \cos \iota_1)e^{3\varsigma_1 + \varsigma_2} + 8|\theta_1|^2(\beta^4 - \alpha^4) \cos \iota_2 - 2\alpha \beta(\alpha^2 + \beta^2) \sin \iota_2 + (2\alpha \beta \Theta_1 + \Theta_2(\alpha^2 - \beta^2)) \cos(\iota_2 - \iota_1) + (2\alpha \beta \Theta_2 - \Theta_1(\alpha^2 - \beta^2)) \sin(\iota_2 - \iota_1)e^{3\varsigma_1 + \varsigma_2} \]

\[ q_{D,3} = (\alpha^2 + \beta^2)^2(16\beta^4 + |\theta_1|^4e^{4\varsigma_2} - 8|\theta_1|^2e^{2\varsigma_2} \cos \iota_2); \]

where $\iota_2 = 4\beta x$. The solution $|q_1|^2$ with $\varepsilon = 1$ has singularities at the points

\[ (x, t) = \left( \frac{\kappa \pi}{2\beta}, \frac{1}{4\alpha \beta} \ln \frac{2\beta}{|\theta_1|} \right), \quad \kappa \in \mathbb{Z}, \]

and behaves quasiperiodically. We show this solution in Fig. 14.
Fig. 13. Shape and motion of $|q_1|^2$ for $\varepsilon = i$, Eq. (4.13), for $c_1 = d_1 = 1 + i$: (a) contour plot for $k_1 = 1 + i$ in the range $x \in [-4, 4], t \in [-7, 7]$; 2D plot for $k_1 = 1.1 + i$ at (b) $t = 0$ and (c) $t = 0.6$; 2D plot for $k_1 = 1 + i$ at (d) $t = 0$ and (e) $t = 0.6$.

Fig. 14. Shape and motion of $|q_1|^2$ for $\varepsilon = 1$, given by (4.15), for $k_1 = c_1 = d_1 = 1 + i$: (a) contour plot in the range $x \in [1, 10]$ and $t \in [-0.3, 0.3]$; (b) 2D plot at $t = 0$. 

1536
4.2. Local and nonlocal potential cmKdV equations. We finally consider the construction of Cauchy matrix solutions of local and nonlocal potential cmKdV equations (2.22). For this, we take \( n = 3 \) in (2.29) and summarize the result in the following theorem.

**Theorem 6.** The function

\[
q = s_2^T(I - M_2 M_1)^{-1}\Omega_2^{-1}r_2 - s_2^T M_2 (I - M_1 M_2)^{-1}\Omega_1^{-1}r_1 - 1, \tag{4.16}
\]

solves the local and nonlocal potential cmKdV equations (2.22), where the matrices and vectors in the right-hand side satisfy Eqs. (2.29) \( (n = 3) \) and the constraints

\[
r_1 = \varepsilon T \cdot r_2^*(\sigma x, \sigma t), \quad s_1 = \varepsilon (T^T)^{-1}s_2^*(\sigma x, \sigma t), \quad M_1 = TM_2^*(\sigma x, \sigma t)T^*, \tag{4.17}
\]

where \( T \in \mathbb{C}_{N \times N} \) is a constant matrix such that

\[
\Omega_1 T + \sigma T \Omega_2^* = 0, \quad C_1^* = \varepsilon TC_2^*, \quad D_1^* = \varepsilon (T^T)^{-1}D_2^*, \quad \varepsilon^2 = \varepsilon^* = -\sigma. \tag{4.18}
\]

We omit the proof and proceed with deriving Cauchy matrix-type soliton solutions and Jordan block solutions for local and nonlocal potential cmKdV equations (2.22).

4.2.1. Exact solutions. Using (4.17) and (4.18) in (4.16), we can rewrite solution (4.16) as

\[
q = s_2^T(I - M_2 M_2^*(\sigma x, \sigma t))^{-1}\Omega_2^{-1}r_2 - \varepsilon^* s_2^T M_2 (I - M_2^*(\sigma x, \sigma t)M_2^*)^{-1}\Omega_2^{-1}r_2^* - 1, \tag{4.19}
\]

where \( T \) is absorbed into \( M_2 \) and the components are given by (3.5b) and (3.6b). Similarly to the foregoing, we present the solutions corresponding to \( \varepsilon = i \) and \( \varepsilon = 1 \). The soliton solutions in these two cases are given by

\[
\varepsilon = i : \quad q_{3,\varepsilon} = s_2^T(I - M_2^{(1)}M_2^{(1)*})^{-1}\Omega_2^{-1}r_2^* + \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} + \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} - 1, \tag{4.20a}
\]

\[
\varepsilon = 1 : \quad q_{3,\varepsilon} = s_2^T(I - M_2^{(3)}M_2^{(3)*}(-x, -t))^{-1}\Omega_2^{-1}r_2^* - \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} - 1, \tag{4.20b}
\]

where \( r_2, s_2, M_2^{(1)}, \) and \( M_2^{(3)} \) are given by (3.7). The corresponding 1-soliton solutions are expressed by

\[
\varepsilon = i : \quad q_{3,\varepsilon} = \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} + \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} - 1, \tag{4.21a}
\]

\[
\varepsilon = 1 : \quad q_{3,\varepsilon} = \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} + \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} - 1. \tag{4.21b}
\]

The Jordan block solutions are

\[
\varepsilon = i : \quad q_{4,\varepsilon} = s_2^T(I - \hat{M}_2 M_2^*)^{-1}\Omega_2^{-1}r_2^* + \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} + \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} - 1, \tag{4.22a}
\]

\[
\varepsilon = 1 : \quad q_{4,\varepsilon} = s_2^T(I - \hat{M}_2 M_2^{*(\sigma x, \sigma t)})^{-1}\Omega_2^{-1}r_2^* - \frac{k_1^* (\partial_1 e^{\xi_1} - k_1)}{|k_1|^2(1 + |\partial_1|^2 e^{\xi_1} + 1)} - 1, \tag{4.22b}
\]

where \( r_2, s_2, \hat{M}_2, \) and \( \bar{M}_2 \) are given by (3.9).
4.2.2. Dynamics. We now study the dynamics of soliton solutions (4.21). Substituting (3.13) and (4.12) in (4.21a) we obtain

\[ \varepsilon = i : \quad |q_3|^2 = \frac{q_{D,2}}{q_{D,1}}, \]  

(4.23a)

where

\[
q_{D,1} = ((\alpha^2 + \beta^2)(4\alpha^2e^{2\mu_1} - |\theta_1|^2e^{2\mu_2}))^2, \\
q_{D,2} = 16(\alpha^4(\Theta_1^2 + \Theta_2^2) - |\alpha\beta\theta_1|^2(\alpha^2 + \beta^2))e^{2(\mu_1+\mu_2)} + \\
+ (\alpha^2 + \beta^2)^2(4\alpha^2e^{2\mu_1} + |\theta_1|^2e^{2\mu_2})^2 - \\
- 32\alpha^4(\alpha^2 + \beta^2)(\Theta_1\sin \nu_1 + \Theta_2\cos \nu_1)e^{3\mu_1+\mu_2} - \\
- 8|\alpha\theta_1|^2((\Theta_1(\alpha^2 - \beta^2) - 2\alpha\beta\Theta_2)\sin \nu_1 + (2\alpha\beta\Theta_1 + (\alpha^2 - \beta^2)\Theta_2)\cos \nu_1)e^{\mu_1+3\mu_2};
\]

with \( \mu_1 = 2\alpha(x + 3\beta^2t), \mu_2 = 2\alpha^3t \) and \( \nu_1 = 2\beta(x - (3\alpha^2 - \beta^2)t). \) There is a quasiperiodic phenomenon and \( |q_3|^2 \) with \( \varepsilon = i \) has singularities on the straight line \( x = (\alpha^2 - 3\beta^2)t - h. \) The speed is \( \alpha^2 - 3\beta^2. \) We depict this solution in Fig. 15. When \( k_1 = c_1 = d_1 = 1 + i, \)

\[ |q_3|^2(x, t)|_{t = \kappa\pi/2} \to 0 \quad as \quad x \to -\kappa\pi. \]

We note that \( |q_3|^2(x, 0) \) is an even function of \( x \) (see Fig. 15(d)). The background level here is 1.

Solution (4.21b) is as follows:

\[ \varepsilon = 1 : \quad |q_3|^2 = \frac{q_{D,4}}{q_{D,3}}, \]

(4.24a)

where

\[
q_{D,4} = 16\beta^4(\Theta_1^2 + \Theta_2^2)e^{2\mu_3} + \\
+ 8\beta^2[(|\theta_1|^2((\alpha^2 - \beta^2)\Theta_2 + 2\alpha\beta\Theta_1) - 4\beta^2(\alpha^2 + \beta^2)\Theta_2)\cos \nu_1 + \\
+ (|\theta_1|^2(2\alpha\beta\Theta_2 + (\beta^2 - \alpha^2)\Theta_1) - \\
- 4\beta^2(\alpha^2 + \beta^2)\Theta_1)\sin \nu_1]e^{2\mu_2+\mu_3+\mu_4} + \\
+ (\alpha^2 + \beta^2)[(\alpha^2 + \beta^2)(|\theta_1|^4 + 16\beta^4) + \\
+ 8|\beta\theta_1|^2((\beta^2 - \alpha^2)\cos 2\nu_1 - 2\alpha\beta\sin 2\nu_1)]e^{2(\mu_2+\mu_4)},
\]

(4.24b)

\[
q_{D,3} = (\alpha^2 + \beta^2)^2(|\theta_1|^4 + 16\beta^4 - 8|\beta\theta_1|^2\cos 2\nu_1)e^{2(\mu_2+\mu_4)};
\]

where \( \mu_3 = 4\alpha(\alpha^2 - 3\beta^2)t \) and \( \mu_4 = 2\alpha(x - 3\beta^2t). \) The solution \( |q_3|^2_{(\varepsilon = 1)} \) has singularities at the points

\[ x = (3\alpha^2 - \beta^2)t + \frac{\kappa\pi}{2\beta}, \quad \kappa \in \mathbb{Z} \]

and behaves quasiperiodically. We show this solution in Fig. 16.

5. Conclusions and some remarks

We have exhibited a Cauchy matrix reduction technique that enables us to obtain solutions of the reduced local and nonlocal complex equations from the Cauchy matrix solutions of the original nonreduced systems. We investigated the Cauchy matrix solutions of four local and nonlocal complex equations, including the local and nonlocal cmKdV equations, local and nonlocal csG equations, local and nonlocal potential NLS equations, and local and nonlocal potential cmKdV equations. For each reduced equation, we gave its soliton solutions and Jordan block solutions. In particular, we presented explicit expressions of 1-soliton solutions, 2-soliton solutions, and the simplest Jordan block solutions. The dynamical behavior of these solutions was analyzed with graphical illustrations.
Fig. 15. Shape and motion of $|q_3|^2$ for $\varepsilon = i$, given by (4.23), for $c_1 = d_1 = 1 + i$: (a) contour plot for $k_1 = \sqrt{3} + i$ in the range $x \in [-0.5, 0.2]$ and $t \in [-40, 40]$; (b) 2D plot at $t = 0$; (c) contour plot for $k_1 = 1 + i$ in the range $x \in [-5, 5]$ and $t \in [-2, 2]$; (d) 2D plot at $t = 0$.

Fig. 16. Shape and motion of $|q_3|^2$ for $\varepsilon = 1$, given by (4.24), for $d_1 = 1 + i$: (a) contour plot for $k_1 = 0.9 + i$ and $c_1 = 2 + i$ in the range $x \in [4, 15]$ and $t \in [-2, 2]$; (b) 2D plot at $t = 0$; (c) contour plot for $k_1 = c_1 = 1 + i$ in the range $x \in [4, 15]$ and $t \in [-2, 2]$; (d) 2D plot at $t = 0$. 1539
We found that local and nonlocal cmKdV equations (2.11) and local and nonlocal csG equations (2.14) formally share the same solutions, but have different dispersion relations; the same also applies to local and nonlocal potential NLS equations (2.19) and local and nonlocal potential cmKdV equations (2.22). The matrix $T$ was needed to relate $(\Omega_1, C_1^+, D_1^+, M_1, r_1, s_1)$ to $(\Omega_2, C_2^+, D_2^+, M_2, r_2, s_2)$, but it was subsequently absorbed into $M_2$ in the expressions for the solutions. Besides, we observed that all 1-soliton solutions, 2-soliton solutions, and the simplest Jordan block solutions of the nonlocal complex equations exhibit quasiperiodic behavior.

We finish the paper by the following remarks.

First, the reduction technique given in this paper can be used to study the local and nonlocal complex reductions of the AKNS hierarchies (2.6) and (2.8). The most direct way is to extend dispersion relations

$$2u_{t_2} - u_{xx} - 8u^2v = 0, \quad 2v_{t_2} + v_{xx} + 8uv^2 = 0,$$

$$2u_{t_n} + u_{xt_{n-1}} + 8u\partial^{-1}(uv)t_{n-1} = 0, \quad n = 3, 4, \ldots, (5.1)$$

its Cauchy matrix solutions are still expressed by (2.25), with Eqs. (2.29a) and the general dispersion relations

$$r_j \rightarrow \exp\left((-1)^j\left(\Omega_j x - \sum_{n=2}^{\infty} \Omega_j^n t_n\right)\right)C_j^+, \quad s_j \rightarrow \exp\left((-1)^j\left(\Omega_j^T x - \sum_{n=2}^{\infty} \Omega_j^n t_n\right)\right)D_j^+.$$  

Here, we neglect the trivial member of the hierarchy $u_{t_1} - u_x = 0, v_{t_1} - v_x = 0$. For the even members in hierarchy (5.1), we consider the complex reduction $v(x, t) = \delta u^*(\sigma x, \sigma t)$, $\delta, \sigma = \pm 1$. Tables 1 and 2 respectively show soliton solutions and Jordan block solutions of the higher-order local and nonlocal cmKdV equations. For the odd members, we introduce the complex reduction $v(x, t) = \delta u^*(\sigma x, t)$, $\delta, \sigma = \pm 1$, $t \rightarrow -it$. The solutions of the higher-order local and nonlocal NLS equations can be obtained without any problem.

We note that compared with the double Wronskian reduction technique developed in [22], the Cauchy matrix reduction technique has a deficiency: we cannot obtain the Cauchy matrix solutions of the nonlocal real equations reduced from the AKNS-type equations. For instance, to derive Cauchy matrix solutions for the nonlocal real mKdV equation

$$4u_t + u_{xxx} + 24\delta uu(-x, -t)u_x = 0, \quad \delta = \pm 1,$$

we should replace the first equation in (3.3) with $\Omega_1 T - T\Omega_2 = 0$, where $T$ is an invertible matrix. This implies that the matrices $\Omega_1$ and $\Omega_2$ share the same eigenvalues, which contradicts condition (2.30).

Finally, we return to pAKNS equations (2.17) and (2.20). These two equations were first derived from [39] by using the Cauchy matrix approach. The pAKNS equation (2.20) with $r = q$ reduces to the equation

$$4q_t + q_{xxx} - \delta \frac{q_r q_{xx}}{q} = 0, \quad (5.2)$$

which is called the potential mKdV equation [30]. In [45], a noncommutative version of (5.2) was introduced. Moreover, its mirror counterpart, recursion operators, hierarchies, and a class of explicit solutions were also derived. Therefore, constructing the potential AKNS hierarchy and discussing its integrability is an interesting problem worthy of consideration.
**Conflicts of interest.** The authors declare no conflicts of interest.

**REFERENCES**

1. V. E. Zakharov, “Stability of periodic waves of finite amplitude on the surface of a deep fluid,” *J. Appl. Mech. Tech. Phys.*, 9, 190–194 (1968).
2. H. Hasimoto, “A soliton on a vortex filament,” *J. Fluid Mech.*, 51, 477–485 (1972).
3. V. E. Zakharov, “Collapse of Langmuir waves,” *Sov. Phys. JETP*, 35, 908–912.
4. C. F. F. Karney, A. Sen, and F. Y. F. Chu, “Nonlinear evolution of lower hybrid waves,” *Phys. Fluids*, 22, 940–952 (1979).
5. O. B. Gorbacheva and L. A. Ostrovsky, “Nonlinear vector waves in a mechanical model of a molecular chain,” *Phys. D*, 8, 223–228 (1983).
6. S. C. Anco, M. Mohiuddin, and T. Wolf, “Traveling waves and conservation laws for complex mKdV-type equations,” *Appl. Math. Comput.*, 219, 679–698 (2012).
7. H. J. de Vega, J. Ramírez Mittelbrunn, M. Ramón Medrano, and N. G. Sánchez, “The general solution of the 2D sigma model stringy black hole and the massless complex sine-Gordon model,” *Phys. Lett. B*, 323, 133–138 (1994), arXiv: hep-th/9312085.
8. Q.-Han Park and H. J. Shin, “Field theory for coherent optical pulse propagation,” *Phys. Rev. A*, 57, 4621–4642 (1998).
9. M. J. Ablowitz and Z. H. Musslimani, “Integrable nonlocal nonlinear Schrödinger equation,” *Phys. Rev. Lett.*, 110, 064105, 5 pp. (2013).
10. M. J. Ablowitz and Z. H. Musslimani, “Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation,” *Nonlinearity*, 29, 915–946 (2016).
11. S. Y. Lou, “Alice–Bob systems, $\hat{P}$–$\hat{T}$–$\hat{C}$ symmetry invariant and symmetry breaking soliton solutions,” *J. Math. Phys.*, 59, 083507, 20 pp. (2018).
12. D. Sinha and P. K. Ghosh, “Symmetries and exact solutions of a class of nonlocal nonlinear Schrödinger equations with self-induced parity-time-symmetric potential,” *Phys. Rev. E*, 91, 042908, 14 pp. (2015).
13. T. I. Valchev, “On Mikhailov’s reduction group,” *Phys. Lett. A*, 379, 1877–1880 (2015).
14. M. J. Ablowitz, X.-D. Luo, and Z. H. Musslimani, “Inverse scattering transform for the nonlocal nonlinear Schrödinger equation with nonzero boundary conditions,” *J. Math. Phys.*, 59, 011501, 42 pp. (2018).
15. Z. Yan, “Integrable $\hat{P}\hat{T}$-symmetric local and nonlocal vector nonlinear Schrödinger equations: A unified two-parameter model,” *Appl. Math. Lett.*, 47, 61–68 (2015).
16. K. Chen and D.-J. Zhang, “Solutions of the nonlocal nonlinear Schrödinger hierarchy via reduction,” *Appl. Math. Lett.*, 75, 82–88 (2018).
17. M. Li and T. Xu, “Dark and antidark soliton interactions in the nonlocal nonlinear Schrödinger equation with the self-induced parity–time-symmetric potential,” *Phys. Rev. E*, 91, 033202, 8 pp. (2015).
18. X.-D. Luo, “Inverse scattering transform for the complex reverse space-time nonlocal modified Korteweg–de Vries equation with nonzero boundary conditions and constant phase shift,” *Chaos*, 29, 073118, 13 pp. (2019).
19. L.-Y. Ma, S.-F. Shen, and Z.-N. Zhu, “Soliton solution and gauge equivalence for an integrable nonlocal complex modified Korteweg–de Vries equation,” *J. Math. Phys.*, 58, 103501, 12 pp. (2017).
20. B. Yang and J. Yang, “Transformations between nonlocal and local integrable equations,” *Stud. Appl. Math.*, 40, 178–201 (2017).
21. M. Gürses and A. Pekcan, “Nonlocal modified KdV equations and their soliton solutions by Hirota method,” *Commun. Nonlinear Sci. Numer. Simul.*, 67, 427–448 (2019).
22. K. Chen, X. Deng, S. Lou, and D.-J. Zhang, “Solutions of nonlocal equations reduced from the AKNS hierarchy,” *Stud. Appl. Math.*, 141, 113–141 (2018).
23. X. Deng, S.-Y. Lou, and D.-J. Zhang, “Bilinearisation-reduction approach to the nonlocal discrete nonlinear Schrödinger equations,” *Appl. Math. Comput.*, 332, 477–483 (2018).
24. W. Feng, S.-L. Zhao, and Y.-Y. Sun, “Double Casoratian solutions to the nonlocal semi-discrete modified Korteweg–de Vries equation,” *Internat. J. Modern Phys. B*, 34, 2050021, 14 pp. (2020).
25. S.-M. Liu, H. Wu, and D.-J. Zhang, “New dynamics of the classical and nonlocal Gross–Pitaevskii equation with a parabolic potential,” Rep. Math. Phys., 86, 271–292 (2020), arXiv:2003.01865.
26. W. Feng and S.-L. Zhao, “Soliton solutions to the nonlocal non-isospectral nonlinear Schrödinger equation,” Internat. J. Modern Phys. B, 34, 2050219, 14 pp. (2020).
27. H. J. Xu and S. L. Zhao, “Local and nonlocal reductions of two nonisospectral Ablowitz–Kaup–Newell–Segur equations and solutions,” Symmetry, 13, 23, 23 pp. (2021).
28. K. Chen, S.-M. Liu, and D.-J. Zhang, “Covariant hodograph transformations between nonlocal short pulse models and the AKNS(−1) system,” Appl. Math. Lett., 88, 230–236 (2019).
29. F. Nijhoff, J. Atkinson, and J. Hietarinta, “Soliton solutions for ABS lattice equations: I. Cauchy matrix approach,” J. Phys. A: Math. Theor., 42, 404005, 34 pp. (2009).
30. J. Hietarinta, N. Joshi, and F. W. Nijhoff, Discrete Systems and Integrability, Cambridge Texts in Applied Mathematics, Vol. 54, Cambridge Univ. Press, Cambridge (2016).
31. A. S. Fokas and M. J. Ablowitz, “Linearization of the Korteweg–de Vries and Painlevé II equations,” Phys. Rev. Lett., 47, 1096–1100 (1981).
32. F. W. Nijhoff, G. R. W. Quispel, J. van der Linden, and H. W. Capel, “On some linear integral equations generating solutions of nonlinear partial differential equations,” Phys. A, 119, 101–142 (1983).
33. F. W. Nijhoff, H. W. Capel, and G. L. Wiersma, “Integrable lattice systems in two and three dimensions,” in: Geometric Aspects of the Einstein Equations and Integrable Systems (Scheveningen, The Netherlands, August 26–31, 1984), Lecture Notes in Physics, Vol. 239 (R. Martini, ed.), Springer, Berlin–New York (1985), pp. 263–302.
34. D.-J. Zhang, S.-L. Zhao, and F. W. Nijhoff, “Direct linearization of extended lattice BSQ systems,” Stud. Appl. Math., 129, 220–248 (2012).
35. W. Fu and F. W. Nijhoff, “Direct linearizing transform for three-dimensional discrete integrable systems: the lattice AKP, BKP and CKP equations,” Proc. R. Soc. London Ser. A, 473, 20160195, 22 pp. (2017).
36. D.-J. Zhang and S.-L. Zhao, “Solutions to ABS lattice equations via generalized Cauchy matrix approach,” Stud. Appl. Math., 131, 72–103 (2013).
37. J. Sylvester, “Sur l’équation en matrices $px = qx$,” C. R. Acad. Sci. Paris, 99, 67–71, 115–116 (1884).
38. D.-D. Xu, D.-J. Zhang, and S.-L. Zhao, “The Sylvester equation and integrable equations: I. The Korteweg–de Vries system and sine-Gordon equation,” J. Nonlinear Math. Phys., 21, 382–406 (2014).
39. S.-L. Zhao, “The Sylvester equation and integrable equations: The Ablowitz–Kaup–Newell–Segur system,” Rep. Math. Phys., 82, 241–263 (2018).
40. W. Feng and S.-L. Zhao, “Cauchy matrix type solutions for the nonlocal nonlinear Schrödinger equation,” Rep. Math. Phys., 84, 75–83 (2019).
41. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, “Nonlinear-evolution equations of physical significance,” Phys. Rev. Lett., 31, 125–127 (1973).
42. D.-J. Zhang, J. Ji, and S.-L. Zhao, “Soliton scattering with amplitude changes of a negative order AKNS equation,” Phys. D, 238, 2361–2367 (2009).
43. F. W. Nijhoff, “On some “Schwarzian equations” and their discrete analogues,” in: Algebraic Aspects of Integrable Systems: In memory of Irene Dorfman, Progress in Nonlinear Differential Equations and Their Applications, Vol. 26 (A. S. Fokas, I. M. Gelfand, eds.), Birkhäuser, Boston, MA (1996), pp. 237–260.
44. G. R. W. Quispel and H. W. Capel, “The nonlinear Schrödinger equation and the anisotropic Heisenberg spin chain,” Phys. Lett. A, 88, 371–374 (1982).
45. S. Carillo, M. Lo Schiavo, E. Porten, and C. Schiebold, “A novel noncommutative KdV-type equation, its recursion operator, and solitons,” J. Math. Phys., 59, 043501, 14 pp. (2018), arXiv:1704.03208.