Comments on an article by Fomin, Fulton, Li, and Poon

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August 30, 2022

Abstract

In [5, 1], the authors studied a cone connecting the spectrum of an Hermitian matrix with the singular spectrum of its off-diagonal block. The purpose of this note is to explain why their description is incomplete and how to remedy it.

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1 Introduction

Let \( p \geq q \geq 1 \) and \( n = p + q \). We consider the map \( \pi : \text{Herm}(n) \to M_{p,q}(\mathbb{C}) \) that associates to an Hermitian matrix \( X \in \text{Herm}(n) \) its off-diagonal block \( \pi(X) \in M_{p,q}(\mathbb{C}) \). The spectrum of an Hermitian \( n \times n \) matrix \( X \) is denoted by \( \lambda(X) = (\lambda_1 \geq \cdots \geq \lambda_n) \) and the singular spectrum of \( Y \in M_{p,q}(\mathbb{C}) \) is denoted by \( s(Y) = (s_1 \geq \cdots \geq s_q \geq 0) \).

Consider the cone

\[
A(p, q) = \{(\lambda(X), s(\pi(X))), \ X \in \text{Herm}(n)\}.
\]
In [5][1], the authors state that an element \((\lambda, s)\) belongs to \(A(p, q)\) if and only if

\[
\sum_{i \in I} \lambda_i - \sum_{j \in J^o} \lambda_j \geq 2 \sum_{k \in K} s_k
\]

holds for all triple \((I, J, K)\) of subsets of \([q] = \{1, \cdots, q\}\) that belongs to \(\bigcup_{r \leq q} LR^q_r\). Here \(LR^q_r\) denotes the list of triples of cardinal \(r\) defined inductively by Horn [2], and we have denoted \(J^o = \{n + 1 - \ell, \ell \in J\}\).

The purpose of this note is to explain why the inequalities of the type (1) are not sufficient to describe \(A(p, q)\).

In the next section, we will see that a direct application of the O’Shea-Sjamaar Theorem [6] shows that \(A(p, q)\) is described by the following inequalities:

\[
\sum_{i \in I} \lambda_i - \sum_{j \in J^o} \lambda_j \geq 2 \sum_{k \in K \cap [q]} s_k - 2 \sum_{k \in K^o \cap [q]} s_k
\]

where the triplets \((I, J, K)\) belongs to \(\bigcup_{r<n} LR^n_r\). Using the main result of [7], we can show (see [8]) that we can restrict this system by considering uniquely triplets \((I, J, K)\) satisfying the following conditions :

- \(I, J, K\) are of cardinal \(r \leq q\),
- \(I \cap J^o = \emptyset\),
- \(K = K_+ \cup (K_-)^o\) where \(K_+, K_-\) are disjoint subsets of \([q]\).

In the example \(A(3,3)\) that we detail in Section 3 we find two inequalities of the type (2) which are independent of those of the type (1):

\[
\begin{align*}
\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2 - \lambda_5 - \lambda_6 & \geq 2(s_1 - s_2 - s_3) \\
\lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4 - \lambda_6 & \geq 2(s_1 - s_2 - s_3).
\end{align*}
\]

The independence of the last two inequalities is ensured by the example \(\lambda_0 = (1,1,1,-1,-1), s_0 = (1,0,0)\). The element \((\lambda_0, s_0)\) does not verify the inequality (3), and then \((\lambda_0, s_0) \notin A(3,3)\), whereas it verifies all the inequalities of the type (1).

Acknowledgement

I would like to thank Michèle Vergne for giving me the \(LR^3_3\) list, and my colleague Bijan Mohammadi for providing me with the example \((\lambda_0, s_0)\).
2 An application of the O’Shea-Sjamaar Theorem

We work with the reductive real Lie groups $G := U(p, q)$ and $\tilde{G} := GL_n(\mathbb{C})$. Let us denote by $\iota : G \to \tilde{G}$ the canonical embedding.

The subgroup $K := U(n)$ is a maximal compact subgroup of $\tilde{G}$. Let $\tilde{p} := \text{Herm}(n) \subset \mathfrak{gl}_n(\mathbb{C})$ be the subspace of Hermitian matrices.

The subgroup $\tilde{K} := \tilde{K} \cap U(p, q) \simeq U(p) \times U(q)$ is a maximal compact subgroup of $\tilde{G}$, and the subspace $\mathfrak{p} := \tilde{p} \cap \mathfrak{g}$ admits a natural identification with $M_{p,q}(\mathbb{C})$:

$$X \in M_{p,q}(\mathbb{C}) \mapsto \begin{pmatrix} 0 & X \cr X^* & 0 \end{pmatrix} \in \mathfrak{p}.$$

2.1 Complexification and antiholomorphic involution

The complexification of the group $G$ is $G_{\mathbb{C}} := GL_n(\mathbb{C})$. We consider the antiholomorphic involution $\sigma$ on $G_{\mathbb{C}}$ defined by $\sigma(g) = I_{p,q}(g^*)^{-1}I_{p,q}$, where $I_{p,q} = \text{Diag}(I_p, -I_q)$. The subgroup $G$ is the fixed point set of $\sigma$.

The complexification of the group $\tilde{G}$ is $\tilde{G}_{\mathbb{C}} := GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$. The inclusion $\tilde{G} \hookrightarrow \tilde{G}_{\mathbb{C}}$ is given by the map $g \mapsto (\tilde{g}, \tilde{g})$. We consider the antiholomorphic involution $\tilde{\sigma}$ on $\tilde{G}_{\mathbb{C}}$ defined by $\tilde{\sigma}(g_1, g_2) = (\overline{g_2}, \overline{g_1})$. The subgroup $\tilde{G}$ corresponds to the fixed point set of $\tilde{\sigma}$.

The embedding $\iota : G \hookrightarrow \tilde{G}$ admits a complexification $\iota_{\mathbb{C}} : G_{\mathbb{C}} \hookrightarrow \tilde{G}_{\mathbb{C}}$ defined by $\iota_{\mathbb{C}}(g) = (g, \overline{\sigma}(g))$: notice that $\iota_{\mathbb{C}} \circ \sigma = \tilde{\sigma} \circ \iota_{\mathbb{C}}$.

The groups $U = U(n)$ and $\tilde{U} = U(n) \times U(n)$ are respectively maximal compact subgroups of $G_{\mathbb{C}}$ and $\tilde{G}_{\mathbb{C}}$. The embedding $\iota_{\mathbb{C}} : U \hookrightarrow \tilde{U}$ is defined by $\iota_{\mathbb{C}}(k) = (k, I_{p,q}^*I_{p,q})$. The fixed point subgroups of the involutions are $U^\sigma = K$ and $\tilde{U}^\tilde{\sigma} = \tilde{K}$.

At the level of Lie algebra, we have a morphism $\iota_{\mathbb{C}} : \mathfrak{g}_n(\mathbb{C}) \hookrightarrow \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ defined by $\iota_{\mathbb{C}}(X) = (X, \overline{\sigma}(X))$ where $\sigma(X) = -I_{p,q}^*X^*I_{p,q}$.

2.2 Orthogonal projection

We use on $\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})$ the euclidean norm $\|(X, Y)\|^2 = \text{Tr}(XX^*) + \text{Tr}(YY^*)$. The subspace orthogonal to the image of $\iota_{\mathbb{C}}$ is $\{(X, -\sigma(X)), X \in \mathfrak{gl}_n(\mathbb{C})\}$. Hence the orthogonal projection

$$\pi : \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \longrightarrow \mathfrak{gl}_n(\mathbb{C}),$$

is defined by the relations $\pi(X, Y) = \frac{1}{2}(X + \overline{\sigma(Y)})$. Note that $\pi$ commutes with the involutions $: \pi \circ \tilde{\sigma} = \sigma \circ \pi$.

We restrict the projection $\pi$ to different subspaces:

- the projection $\pi : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$ is defined by $\pi(X) = \frac{1}{2}(X + \sigma(X))$.

- the projection $\pi : \tilde{\mathfrak{p}} = \text{Herm}(n) \to \mathfrak{p} \simeq M_{p,q}(\mathbb{C})$ is defined so that $X = \begin{pmatrix} * & \pi(X) \\ \pi(X)^* & * \end{pmatrix}$. 

3
the projection $\pi : \tilde{u} \to u$ is defined by $\pi(X, Y) = \frac{1}{2}(X + IP_{p,q}YP_{p,q})$.

The involution $\hat{\sigma}$ defines on $u = u(n) \times u(n)$ an orthogonal decomposition $u = u^\hat{\sigma} \oplus u^{-\hat{\sigma}}$ where $V \in \tilde{u}^\pm$ if $\hat{\sigma}(V) = \pm V$. In the same way, we have an orthogonal decomposition $u = u^\sigma \oplus u^{-\sigma}$.

The map $X \mapsto iX$ defines two isomorphisms $\tilde{p} \simeq \tilde{u}^{-\hat{\sigma}}$ and $p \simeq u^{-\sigma}$ that fit into the following commutative diagram

$$
\begin{array}{ccc}
\tilde{p} & \xrightarrow{\pi} & p \\
\downarrow & & \downarrow \\
\tilde{u}^{-\hat{\sigma}} & \xrightarrow{\pi} & u^{-\sigma}.
\end{array}
$$

2.3 O'Shea-Sjamaar's Theorem

If $A \in u(n)$, the corresponding adjoint orbit $O_A = \{gAg^{-1}, g \in U(n)\}$ is entirely determined by the spectrum $\lambda(iA)$ of the Hermitian matrix $iA$.

Recall that $K \simeq U(p) \times U(q)$ acts canonically $p \simeq M_{p,q}(C)$. For any $Y \in p$, the orbit $\mathcal{V}_Y := \{kYk^{-1}, k \in K\}$ is entirely determined by the singular spectrum $s(Y)$.

We start with some basic facts.

**Lemma 2.1** Let $X, X' \in u(n)$.

1. $O_X \times O_{X'} \cap \tilde{u}^{-\hat{\sigma}} \neq \emptyset$ if and only if $O_X = O_{X'}$.

2. Let $(Z, -\bar{Z}) \in O_X \times O_X \cap \tilde{u}^{-\hat{\sigma}}$. Then $O_X \times O_X \cap \tilde{u}^{-\hat{\sigma}}$ is equal to the orbit $\tilde{K} \cdot (Z, -\bar{Z}) := \{(gZg^{-1}, -\overline{gZg^{-1}}), g \in U(n)\}$.

3. When $Y \in u(n)^{-\sigma}$, the intersection $O_Y \cap u(n)^{-\sigma}$ is equal to the orbit $\mathcal{V}_Y$.

**Proof** : $(A, B) \in \tilde{u}^{-\hat{\sigma}}$ if and only if $B = -\overline{A}$. Suppose that $O_X \times O_{X'}$ contains an element $(A, -\overline{A}) \in \tilde{u}^{-\hat{\sigma}}$. Then $\lambda(iX) = \lambda(iA)$ and $\lambda(iX') = \lambda(i\overline{A})$. Since $\lambda(iA) = \lambda(i\overline{A})$ we obtain $\lambda(iX) = \lambda(iX')$, and then $O_X = O_{X'}$.

Let $\lambda(iX) = (\lambda_1 \geq \cdots \geq \lambda_n)$. The orbit $O_X$ contains the diagonal matrix $\Delta = \frac{1}{i}\text{Diag}(\lambda_1, \cdots, \lambda_n)$, and the product $O_X \times O_X$ contains $\Delta, \Delta = (\Delta, -\overline{\Delta}) \in \tilde{u}^{-\hat{\sigma}}$. The first point is proved and the two other points are classical (see [9], Example 2.9).

We can now state the application of the O'Shea-Sjamaar Theorem that interest us.

**Theorem 2.2** Let $X \in u(n)$ and $Y \in u(n)^{-\sigma}$. The following statements are equivalent:

1. $O_Y \subset \pi(O_X \times O_X)$.

2. $O_Y \cap u^{-\sigma} \subset \pi(O_X \times O_X \cap \tilde{u}^{-\hat{\sigma}})$.
3. \((\lambda(iX), s(iY)) \in A(p, q)\).

4. \(2\mathcal{O}_Y \subset \mathcal{O}_X + \mathcal{O}_\overline{X}\).

Proof : The equivalence 1. \(\iff\) 2. is the consequence of the O’Shea-Sjamaar Theorem (see [6], Section 3).

The equivalence 1. \(\iff\) 4. is a direct consequence of the definition of the projection \(\pi : \tilde{u} \to u\). Since \(\pi(A, B) = \frac{1}{2} (A + I_{p,q} \overline{B}I_{p,q})\), we see that \(\pi(\mathcal{O}_X \times \mathcal{O}_X) = \frac{1}{2} (\mathcal{O}_X + \mathcal{O}_\overline{X})\).

The equivalence 2. \(\iff\) 3. follows from the commutative diagram (4): the inclusion \(\mathcal{O}_Y \cap u^{-\sigma} \subset \pi(\mathcal{O}_X \times \mathcal{O}_X \cap \tilde{u}^{-\sigma})\) is equivalent to

\[
U(p) \times U(q) \cdot (iY) \subset \pi\left(U(n) \cdot (iX)\right).
\]

and by definition the last inclusion is equivalent to 3.. \(\square\)

2.4 Horn inequalities

Let us denote by \(\mathbb{R}^n_+\) the set of weakly decreasing \(n\)-tuples of real numbers. To each \(a \in \mathbb{R}^n_+\), we associate the orbit \(\mathcal{O}_a := \{X \in \text{Herm}(n), \lambda(X) = a\}\). We consider the Horn cone

\[
\text{Horn}(n) := \{(x, y, z) \in (\mathbb{R}^n_+)^3, \ O_z \subset O_x + O_y\}.
\]

Denote the set of cardinality \(r\)-subsets \(I = \{i_1 < i_2 < \cdots < i_r\}\) of \([n] := \{1, \ldots, n\}\) by \(\mathcal{P}_r^n\). To each \(I \in \mathcal{P}_r^n\) we associate a weakly decreasing sequence of non-negative integers \(\mu(I) = (\mu_1 \geq \cdots \geq \mu_r)\) where \(\lambda_a = i_a - a\) for \(a \in [r]\).

Definition 2.3 Let \(1 \leq r < n\). \(LR^n_r\) refers to the set of triplet \((I, J, K) \in (\mathcal{P}_r^n)^3\) such that \((\mu(I), \mu(J), \mu(K)) \in \text{Horn}(r)\).

The following theorem was conjectured by Horn [2] and proved by a combination of the works of Klyachko [3] and Knutson-Tao [4]. If \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(I \subset [n]\), we define \(|x|_I = \sum_{i \in I} x_i\) and \(|x| = \sum_{i=1}^n x_i\).

Theorem 2.4 The triplet \((x, y, z) \in (\mathbb{R}^n_+)^3\) belongs to \(\text{Horn}(n)\) if and only if the following conditions holds:

- \(|x| + |y| = |z|\),
- \(|x|_I + |y|_J \geq |z|_K\), for any \(r < n\) and any \((I, J, K) \in LR^n_r\).
2.5 Inequalities determining \(A(p,q)\)

Let us denote by \(\mathbb{R}^q_{++}\) the set of weakly decreasing \(q\)-tuples of non-negative real numbers. Let \(Y \in p\) and let \(s(Y) = (s_1, \cdots, s_q) \in \mathbb{R}^q_{++}\) be its singular spectrum. For \(s \in \mathbb{R}^q_{++}\), we define the \(K\)-orbit \(\mathcal{V}_s := \{Y \in p, s(Y) = s\}\). A standard result asserts that \(\mathcal{V}_s\) contains the matrix

\[
Y(s) := \begin{pmatrix}
0 & 0 & M(s) \\
0 & 0 & 0 \\
(M(s))^* & 0 & 0
\end{pmatrix}
\]

with \(M(s) = \begin{pmatrix}
0 & \cdots & s_1 \\
\vdots & \ddots & \vdots \\
s_q & \cdots & 0
\end{pmatrix}\).

The spectrum of \(Y(s)\) is equal to \(\nu(s) = (s_1, \cdots, s_q, 0, \cdots, 0, -s_q, \cdots, -s_1) \in \mathbb{R}^n_+\). Hence we see that the \(K\)-orbit \(\mathcal{V}_s\) is contained in \(O_{\nu(s)} := \{X \in \text{Herm}(n), \lambda(X) = \nu(s)\}\).

If \(\lambda = (\lambda_1, \cdots, \lambda_n)\), we denote by \(\lambda^*\) the vector \((-\lambda_n, \cdots, -\lambda_1)\); we see that \(\lambda(-X) = \lambda(X)^*\) for any \(X \in \text{Herm}(n)\).

Using the equivalence 3. \(\Leftrightarrow\) 4. of Theorem 2.2 we obtain the following equivalent statements:

- \((\lambda, s) \in A(p,q)\)
- \(\exists (X, Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s\) such that \(Y = \pi(X)\)
- \(\exists (X, Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s, 2\mathcal{O}_{Y/i} \subset \mathcal{O}_{X/i} + \mathcal{O}_{X/i}^\perp\)
- \(\exists (X, Y) \in \mathcal{O}_\lambda \times \mathcal{V}_s, 2\mathcal{O}_Y \subset \mathcal{O}_X + \mathcal{O}_{X^\perp}\)
- \(2\mathcal{O}_{\nu(s)} \subset \mathcal{O}_\lambda + \mathcal{O}_{\lambda^*}\)
- \((\lambda, \lambda^*, 2\nu(s)) \in \text{Horn}(n)\).

Thanks to Theorem 2.4 we can conclude with the following description of \(A(p,q)\).

**Theorem 2.5** An element \((\lambda, s) \in \mathbb{R}^n_+ \times \mathbb{R}^q_{++}\) belongs to \(A(p,q)\) if and only if

\[
(\star)_{I,J,K} \quad |\lambda|_I - |\lambda|_J, K \geq 2|s|_{K\cap\lbrack q\rbrack} - 2|s|_{K^*\cap\lbrack q\rbrack}
\]

for any \(r < n\) and any \((I, J, K) \in LR^a_r\).

**Remark 2.6** In the formulation of the previous theorem we have used that \(|\lambda^*|_J = -|\lambda|_{J,^\circ}\) and \(|\nu(s)|_K = |s|_{K\cap\lbrack q\rbrack} - |s|_{K^*\cap\lbrack q\rbrack}\).

**Remark 2.7** As we have said in the introduction, we can restrict the system of inequalities in Theorem 2.5 by considering uniquely triplets \((I, J, K) \in LR^a_r\) with \(r \leq q\) (see [8]).
3 Examples

3.1 Computation of $A(2, 2)$

The set $LR^4_1$ corresponds to the set of triplets $(i, j, k)$ of elements of $[4]$ such that $i + j = k + 1$: the corresponding (non-trivial) inequalities are

$$\lambda_1 - \lambda_4 \geq 2s_1, \quad \lambda_2 - \lambda_4 \geq 2s_2, \quad \lambda_1 - \lambda_3 \geq 2s_2.$$

The set $LR^2_2$ corresponds to the set of triplets $(I = \{i_1 < i_2\}, J = \{j_1 < j_2\}, K = \{k_1 < k_2\})$ of subsets of $[4]$ satisfying Horn’s conditions:

1. $i_1 + i_2 + j_1 + j_2 = k_1 + k_2 + 3$,
2. $i_1 + j_1 \leq k_1 + 1$, $i_1 + j_2 \leq k_2 + 1$, $i_2 + j_1 \leq k_2 + 1$.

Here the inequality $(*_{I,J,K})$ is non-trivial only in one case: when $I = J = K = \{1, 2\}$ we obtain $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(s_1 + s_2)$.

We summarize our computations as follows.

**Proposition 3.1** An element $(\lambda, s) \in \mathbb{R}^4_+ \times \mathbb{R}^2_+$ belongs to $A(2, 2)$ if and only if the following conditions holds

- $\lambda_1 - \lambda_4 \geq 2s_1, \quad \lambda_2 - \lambda_4 \geq 2s_2, \quad \lambda_1 - \lambda_3 \geq 2s_2$.
- $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \geq 2(s_1 + s_2)$.

3.2 Computation of $A(3, 3)$

The non-trivial inequalities associated to $LR^6_1$ are

$$\begin{align*}
\lambda_1 - \lambda_6 & \geq 2s_1, \\
\lambda_2 - \lambda_6 & \geq 2s_2, \\
\lambda_1 - \lambda_5 & \geq 2s_3 \quad (6)
\end{align*}$$

The non-trivial inequalities associated to $LR^6_2$ are

$$\begin{align*}
\lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 & \geq 2(s_1 + s_2), \\
\lambda_1 + \lambda_2 - \lambda_4 - \lambda_6 & \geq 2(s_1 + s_3), \\
\lambda_1 + \lambda_3 - \lambda_5 - \lambda_6 & \geq 2(s_1 + s_3) \quad (7)
\end{align*}$$
Note that the inequality \( \lambda_1 + \lambda_4 - \lambda_5 - \lambda_6 \geq 2(s_2 + s_3) \) is not valid, even if it looks like the previous ones, since the triplet \( \{1, 4\}, \{1, 2\}, \{2, 3\} \) does not belong to \( LR_2^6 \).

The non-trivial inequalities associated to \( LR_3^6 \) are

\[
\begin{align*}
\lambda_1 + \lambda_3 + \lambda_4 - \lambda_2 - \lambda_5 - \lambda_6 & \geq 2(s_1 - s_2 - s_3) \\
\lambda_1 + \lambda_2 + \lambda_5 - \lambda_3 - \lambda_4 - \lambda_6 & \geq 2(s_1 - s_2 - s_3) \\
\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 & \geq 2(s_1 + s_2 + s_3).
\end{align*}
\]

(8)

In the case of \( LR_3^6 \), the trivial inequalities are those induced by inequalities obtained with \( LR_1^6 \) and \( LR_2^6 \). For example, the inequalities corresponding to the triplets \( \{(1, 2, 5)\}, \{(2, 3, 4)\}, \{(2, 3, 6)\} \) and \( \{(1, 2, 4)\}, \{(1, 2, 3)\}, \{(1, 2, 4)\} \) of \( LR_2^6 \) are respectively

\[
\begin{align*}
\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 & \geq 2(-s_1 + s_2 + s_3) \quad \text{and} \quad \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2 - s_3).
\end{align*}
\]

The former is induced by \( \lambda_1 - \lambda_4 \geq 2s_3 \) obtained with \( LR_1^6 \) and \( \lambda_2 - \lambda_3 \geq 0 \geq s_2 - s_1 \) while the latter is induced by \( \lambda_1 + \lambda_2 - \lambda_5 - \lambda_6 \geq 2(s_1 + s_2) \) obtained with \( LR_2^6 \) and \( s_3 \geq 0 \).

**Proposition 3.2** An element \( (\lambda, s) \in \mathbb{R}_+^6 \times \mathbb{R}_+^3 \) belongs to \( \mathcal{A}(3, 3) \) if and only if the inequalities listed in (6), (7) and (8) are satisfied.

**Remark 3.3** The cone \( \mathcal{A}(3, 3) \subset \mathbb{R}^5 \) corresponds to the intersection of the Horn cone \( \text{Horn}(6) \subset \mathbb{R}^{18} \) with the subspace \( \{(\lambda, \lambda^*, 2\nu(s)), (\lambda, s) \in \mathbb{R}^3 \times \mathbb{R}^2\} \). It is striking that \( \mathcal{A}(3, 3) \) is determined by 23 inequalities while \( \text{Horn}(6) \) is described with a minimal list of 536 inequalities.

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