The Hodge filtrations of monodromic mixed Hodge modules and the irregular Hodge filtrations

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Abstract

For an algebraic vector bundle $E$ over a smooth algebraic variety $X$, a monodromic $D$-module on $E$ is decomposed into a direct sum of some $O$-modules on $X$. We show that the Hodge filtration of a monodromic mixed Hodge module is decomposed with respect to the decomposition of the underlying $D$-module. By using this result, we endow the Fourier-Laplace transform $M^\wedge$ of the underlying $D$-module $M$ of a monodromic mixed Hodge module with a mixed Hodge module structure. Moreover, we describe the irregular Hodge filtration on $M^\wedge$ concretely and show that it coincides with the Hodge filtration at all integer indices.

Contents

0 Introduction 2

1 Monodromic mixed Hodge modules 5
   1.1 Monodromic mixed Hodge modules on vector bundles . . . . . . . . . . . . . 5
   1.2 The case where $E$ is a trivial bundle of rank one . . . . . . . . . . . . . 7
   1.3 Example: normal crossing type . . . . . . . . . . . . . . . . . . . . . . . . . . 10

2 The Hodge filtration of monodromic mixed Hodge modules 13

3 The Fourier-Laplace transform of a monodromic mixed Hodge module 21
   3.1 The Fourier-Laplace transform of a $D$-module . . . . . . . . . . . . . . . . . 21
   3.2 The Fourier-Laplace transform of a monodromic mixed Hodge module . . . . 23

4 Irregular Hodge filtrations 28
   4.1 Irregular Hodge filtrations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 28
   4.2 The Fourier-Laplace transforms of $R$-modules and the irregular Hodge filtrations 33

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0 Introduction

The present paper is a continuation of our previous paper [20].

In this paper, we only deal with algebraic objects. Let $E$ be an algebraic vector bundle over a smooth algebraic variety $X$ of finite type over $\mathbb{C}$ and $\pi: E \to X$ be the projection.

Since $\pi$ is affine morphism, we will identify an algebraic $D$-module $M$ on $E$ with an algebraic $\pi^* D$-module $\pi^* M$ (see Lemma 1.1). We denote by $E_E$ the Euler vector field on $E$.

Then, if an algebraic $D$-module $M$ on $E$ is monodromic (see Definition 1.3), we have a decomposition

$$ M = \bigoplus_{\beta \in \mathbb{R}} M^\beta, $$

where $M_\beta = \bigcup_{l \geq 0} \ker((E_E - \beta)^l: \pi_* M \to \pi_* M)$ (see Proposition 1.6). A monodromic mixed Hodge module is a mixed Hodge module on $E$ whose underlying $D$-module is monodromic.

Then, our first main result is the following.

**Theorem 0.1 (Theorem 2.1).** Let $M$ be a monodromic mixed Hodge module on $E$ and $M$ the underlying $D$-module. Then, the Hodge filtration $F^*_M \subset M$ is decomposed with respect to the decomposition (1):

$$ F^*_M = \bigoplus_{\beta \in \mathbb{R}} F^*_M \beta, $$

where $F^*_M \beta := F^*_M \cap M^\beta$.

**Remark 0.2.** Theorem 0.1 was shown in a different way in a recent preprint [3] by Chen-Dirks.

When the rank of $E$ is one, this result was already shown in [20]. We will prove it by using the fact that the pull-back of a monodromic $D$-module on $\mathbb{C}^n$ by the blowing up morphism $\hat{\mathbb{C}}^n \to \mathbb{C}^n$ is a monodromic $D$-module on a line bundle (Lemma 1.11) and some results for monodromic mixed Hodge modules on line bundles in [20].

We consider the Fourier-Laplace transform $M^\wedge$ of a $D$-module $M$ on $E$, which is a $D$-module on the dual vector bundle $\pi^\vee: E^\vee \to X$. When $E$ is trivial: $E \simeq X \times \mathbb{C}^n$ (then, $E^\vee$ is also trivial: $E \simeq X \times \hat{\mathbb{C}}^n$) and $X$ is affine, we can identify $D$-modules on $E$ (resp. $E^\vee$) with the $\Gamma(E; D)$-modules (resp. $\Gamma(E^\vee; D)$) of their global sections. In this case, $M^\wedge$ is $M$ as a set
and its $\Gamma(E^\vee; D)$-module structure is defined so that for $P \in \Gamma(X; D)$ and $1 \leq i \leq n$ we have

$$P \cdot m^\wedge = (Pm)^\wedge,$$

$$\zeta_i \cdot m^\wedge = (\partial_{\zeta_i} m)^\wedge,$$

and

$$\partial \zeta_i \cdot m^\wedge = -(z_i m)^\wedge,$$

where $(z_1, \ldots, z_n)$ (resp. $(\zeta_1, \ldots, \zeta_n)$) is the coordinate system of $\mathbb{C}^n$ (resp. the dual $\mathbb{C}^n$) and $m^\wedge$ is a global section in $M^\wedge$ corresponding to a global section $m \in M$ (see Lemma 3.6). Furthermore, in this case, for a $\Gamma(X; O)$-submodule $F \subset M$, we define a $\Gamma(X; O)$-submodule $F^\wedge \subset M^\wedge$ as

$$F^\wedge := \{m^\wedge \in M^\wedge \mid m \in F\}.$$

Even in a general case (not necessary $E$ is trivial), we can define an $O_X$-submodule $F^\wedge$ of $\pi^*_M M^\wedge$ for an $O_X$-module $F$ of $\pi_* M$ (Definition 3.7). Recall that the underlying $D$-module of a mixed Hodge module is regular holonomic. In general, even if $M$ is regular, $M^\wedge$ may not be regular. Therefore, even if $M$ is the underlying $D$-module of a mixed Hodge module, $M^\wedge$ may not be so. Nevertheless, it is known that when a $D$-module $M$ is monodromic and regular, so is $M^\wedge$ (see Lemma 3.9). Therefore, for the underlying $D$-module $M$ of a monodromic mixed Hodge module, $M^\wedge$ may be equipped with a mixed Hodge module structure. Reichelt [11] gave definitions of mixed Hodge module structures on the homogeneous $A$-hypergeometric $D$-modules, which are expressed as the Fourier-Laplace transform of certain monodromic $D$-modules. Moreover, Reichelt-Walther [12] defined a mixed Hodge module whose underlying $D$-module is the Fourier-Laplace transform of the underlying $D$-module of a monodromic mixed Hodge module. By a different method from theirs, using Theorem 0.1 (for a line bundle), we show the following.

**Proposition 0.3** (Definition 3.20). For a monodromic mixed Hodge module $M$ on $E$, we can naturally define a mixed Hodge module $M^\wedge$ on $E^\vee$ whose underlying $D$-module is $M^\wedge$.

When $E$ is a line bundle, this result was already proved in [20]. To show it in the general case, we will describe $M^\wedge$ in terms of the Fourier-Laplace transform on a line bundle (Lemma 3.18) and use the results in [20]. We can describe the Hodge filtration of $M^\wedge$ concretely below (Corollary 0.6).

Next, we consider the irregular Hodge filtrations. For a holomorphic function $f$ on a smooth manifold, the exponential $D$-module $e^f$ is not regular in general. Since the underlying $D$-module of a mixed Hodge module is regular and holonomic, we can not apply the theory of mixed Hodge module to endow it with a natural Hodge filtration. Nevertheless, for the underlying $D$-module $M$ of a mixed Hodge module $M$ on $E$, Esnault-Sabbah-Yu [5] and Sabbah-Yu [17] defined a natural filtration $F^\text{irr} \gamma (M \otimes e^f)$ of the exponentially twisted $D$-module $M \otimes e^f$ for $\alpha \in [0, 1]$, called the irregular Hodge filtration. Note that combining $F^\text{irr} \gamma (M \otimes e^f)$ for all $\alpha \in [0, 1]$, we can consider the filtration $\{F^\gamma (M \otimes e^f)\}_{\gamma \in \mathbb{R}}$ indexed by $\mathbb{R}$, not only $\mathbb{Z}$. We thus obtain natural filtrations (also called the irregular Hodge filtrations) on the Fourier-Laplace transforms and their stalks: the twisted de Rham cohomologies, so that they are generalizations of the ones defined in Deligne [4], Yu [23] and Sabbah [14]. Moreover, Sabbah [15] established the category of irregular Hodge modules, which contains the category of mixed Hodge modules, as a full subcategory of the category of integrable mixed twistor $D$-modules introduced by Mochizuki [7]. The $D$-module $M^\wedge$ with the irregular
Hodge filtration defines an object in the category of irregular Hodge modules, which is not a mixed Hodge module in general. In general, it is difficult to compute the irregular Hodge filtrations concretely. However, in our situation, due to Theorem 0.1 we can describe the irregular Hodge filtration \( F^\text{irr}_{\alpha+p} M^\wedge \) of \( M^\wedge \) in terms of the original Hodge filtration \( F_M \) as follows.

**Theorem 0.4** (Theorem 4.23). For \( \alpha \in [0, 1) \) and \( p \in \mathbb{Z} \), we have

\[
F^\text{irr}_{\alpha+p} M^\wedge = \bigoplus_{\beta \in \mathbb{R}} (F_{p+\lfloor \alpha - \beta \rfloor} M^\beta)^\wedge,
\]

where \( \lfloor \alpha - \beta \rfloor \) is the largest integer less than or equal to \( \alpha - \beta \).

We will prove it by describing the rescaled module (Subsection 4.1) and its Kashiwara-Malgrange filtration explicitly (Lemma 4.22).

Eventually, on \( M^\wedge \), we have two Hodge filtrations: the first one is the Hodge filtration \( \{ F_p M^\wedge \}_{p \in \mathbb{Z}} \) defined in Proposition 0.3 and the second one is the irregular Hodge filtration \( \{ F^\text{irr}_{\alpha+p} M^\wedge \}_{p \in \mathbb{Z}} \) (for \( \alpha \in [0, 1) \)). In fact, these filtrations are equal at all integer indices.

**Theorem 0.5** (Theorem 4.39). For \( p \in \mathbb{Z} \), we have

\[
F^\text{irr}_p M^\wedge = F_p M^\wedge.
\]

In particular, we can say that “the irregular Hodge filtration of the Fourier-Laplace transform of the monodromic mixed Hodge module is in fact the Hodge filtration”. Remark that the filtration \( \{ F^\text{irr}_\gamma M^\wedge \}_{\gamma \in \mathbb{R}} \) jumps at also non-integer numbers in general.

Combining this result with Theorem 0.4 the Hodge filtration (defined by Proposition 0.3) can be described as follows.

**Corollary 0.6** (Corollary 4.40). For \( p \in \mathbb{Z} \), we have

\[
F_p M^\wedge = \bigoplus_{\beta \in \mathbb{R}} (F_{p+\lceil -\beta \rceil} M^\beta)^\wedge.
\]

Finally, we consider the irregular Hodge filtration “at infinity”. Let \( \widetilde{E} \) (resp. \( \widetilde{E}^\vee \)) be the projective compactification of \( E \) (resp. \( E^\vee \)). We denote by \( j: E \hookrightarrow \widetilde{E} \) and \( j^\vee: E^\vee \hookrightarrow \widetilde{E}^\vee \) the inclusions and \( D_\infty \) (resp. \( D_\infty^\vee \)) the divisor \( \widetilde{E} \setminus E \) (resp. \( \widetilde{E}^\vee \setminus E^\vee \)). For a \( D \)-module \( M \) on \( E \) let \( N \) be the pushforward \( j_* M \) of \( M \) by \( j \), which is a \( D \)-module on \( \widetilde{E} \) (recall that our \( D \)-modules are algebraic). Then, we can consider the Fourier-Laplace transform \( N^\wedge \) of \( N \), which is a \( D \)-module on \( \widetilde{E}^\vee \). We will see that this is equal to \( j^\vee_* M^\wedge \) (Lemma 3.5). For a mixed Hodge module \( M \) on \( E \) and its extension \( N \) to \( \widetilde{E} \) whose underlying \( D \)-module is \( N \), we can also consider the irregular Hodge filtration on \( N^\wedge = (j^\vee_* M^\wedge) \). Then, we can compute explicitly the irregular Hodge filtration on \( N^\wedge \) (Theorem 4.30) and get the following result.

**Corollary 0.7** (Corollary 4.32). The irregular Hodge filtration \( \{ F_{\alpha+p} N^\wedge \}_{p \in \mathbb{Z}} \) for \( \alpha \in [0, 1) \) has the strict specializability property along \( D_\infty^\vee \).
For the definition of the strict specializability, see Definition 1.12. Remark that this result is shown in a more general setting in a recent preprint by Mochizuki [8].

By the definition of \(N\), we have \(N = N[*D_\infty]\) (see Lemma 4.13). For the definition of \([*D_\infty]\), see Fact 2.3. In particular, the Hodge filtration of \(N\) is described in terms of \(F_\bullet V^D_\infty N\), where \(V^D_\infty N\) is the Kashiwara-Malgrange filtration of \(N\) along \(D_\infty\). For the \(D\)-module with a filtration \((N^\wedge, F^\bullet \text{irr} N^\wedge)\), we denote by \((N^\wedge, F^\bullet \text{irr} N^\wedge)[*D_\infty^\vee]\) the \(D\)-module \(N^\wedge[*D_\infty^\vee] := N^\wedge(*D_\infty^\vee)(= N^\wedge)\) with the filtration \(F_\bullet (N^\wedge[*D_\infty^\vee])\) defined by the same formula as usual “\([*D_\infty]\)“ for the localization of a mixed Hodge module. Then, we have the following.

**Corollary 0.8** (Corollary 4.35). We have \((N^\wedge, F^\bullet \text{irr} N^\wedge) = (N^\wedge, F^\bullet \text{irr} N^\wedge)[*D_\infty^\vee]\).

By Corollaries 0.7 and 0.8, we can say that “the irregular Hodge filtration of the Fourier-Laplace transform of a monodromic mixed Hodge module has the same properties as the usual Hodge filtrations”. To be more precise, we can conclude as follows. Let \(\widetilde{M}^\wedge\) be the extension of the mixed Hodge module \(M^\wedge\) on \(E^\vee\) to \(\mathbf{E}^\vee\) such that \(\widetilde{M}^\wedge = M^\wedge[*D_\infty^\vee]\), whose underlying \(D\)-module is denoted by \(\widetilde{M}^\wedge\) (in fact, there exists such an extension by the definition of “algebraic” mixed Hodge module in [19]). Note that we have \(\widetilde{M}^\wedge = N^\wedge\). Then, by Theorem 0.5 and Corollary 0.8 we have the following.

**Corollary 0.9** (Corollary 4.50). The irregular Hodge filtration \(\{F^p_{\text{irr}} N^\wedge\}_{p \in \mathbb{Z}}\) indexed by integers is the Hodge filtration of the mixed Hodge module \(\widetilde{M}^\wedge\).

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## 1 Monodromic mixed Hodge modules

### 1.1 Monodromic mixed Hodge modules on vector bundles

In this subsection, we recall the notion of monodromic \(D\)-module on a vector bundle and some basic facts. We refer to [2] and [20]. Let \(X\) be a smooth algebraic variety of finite type over \(\mathbb{C}\), \(O_X\) the structure sheaf on \(X\) and \(D_X\) the sheaf of differential operators on \(X\). Basically, we consider only algebraic left \(D\)-modules in this paper. Moreover, all the \(D\)-modules and \(O\)-modules are quasi-coherent.
Let \( \pi: E \to X \) be an algebraic vector bundle and \( M \) a (quasi-coherent) \( D \)-module (resp. \( O \)-module) on \( E \). Then, \( \pi_*M \) is a \( \pi_*D \)-module (resp. \( \pi_*O \)-module). Conversely, for a \( \pi_*D \)-module \( N \) (resp. \( \pi_*O \)-modules) on \( X \), we define a \( D_E \)-module (resp. \( O_E \)-modules) \( \pi^*N \) as

\[
\pi^*N := D_E \otimes_{\pi^{-1}\pi_*D_X} \pi^{-1}N, \\
(\text{resp. } \pi^*N := O_E \otimes_{\pi^{-1}\pi_*O_X} \pi^{-1}N)
\]

Because \( \pi \) is an affine morphism (i.e. the inverse image of an affine open subset of \( X \) is affine), we have the following equivalence.

**Lemma 1.1** (see Proposition 7.10 of Brylinski [2]). The functors \( \pi_* \) and \( \pi^* \) define an equivalence of categories between the category of quasi-coherent \( D_E \)-modules (resp. \( O_E \)-modules) and that of quasi-coherent \( \pi_*D_E \)-modules (resp. \( \pi_*O_E \)-modules). The same assertion holds for the categories of coherent \( D \) or \( O \)-modules.

In the following, we identify \( D_E \)-modules (resp. \( O_E \)-modules) with \( \pi_*D_E \)-modules (resp. \( \pi_*D_E \)-modules).

**Remark 1.2.** When \( X \) is the one point set, \( E \) is just a vector space \( \mathbb{C}^n \). We sometimes consider a \( D \)-module on an affine subset of \( \mathbb{C}^n \), such as \( (\mathbb{C}^*)^n \), where \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \). In this case, we have the equivalence similar to Lemma 1.1. For example, we identify \( D((\mathbb{C}^*)^n) \)-modules with \( \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}][\partial_{z_1}, \ldots, \partial_{z_n}] \)-modules, where \( \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}][\partial_{z_1}, \ldots, \partial_{z_n}] \) is a ring of differential operators on \( (\mathbb{C}^*)^n \).

Let \( V \) be a vector space of rank \( n \) and \( e_1, \ldots, e_n \) a basis of \( V \). The coordinates of \( V \) associated to the basis \( e_1, \ldots, e_n \) is the isomorphism \( \mathbb{C}^n \cong V \) which sends \( (z_1, \ldots, z_n) \in \mathbb{C}^n \) to \( z_1e_1 + \cdots + z_ne_n \in V \). It is easy to see that the vector field \( \sum_{i=1}^n z_i \partial_{z_i} \) on \( V \) does not depend on the choice of the basis \( e_1, \ldots, e_n \). It is called the Euler vector field on \( V \). More generally, for any local trivialization \( \pi^{-1}(U) \cong U \times \mathbb{C}^n (U \subset X) \) of the vector bundle \( \pi: E \to X \) and bundle coordinates \( (x_1, \ldots, x_m, z_1, \ldots, z_n) \in U \times \mathbb{C}^n \), the vector field \( \sum_{i=1}^n z_i \partial_{z_i} \) on \( \pi^{-1}(U) \) does not depend on the choice of the local trivialization and we thus obtain a vector field on \( E \). It is called the Euler vector field on \( E \) and we denote it by \( \mathcal{E}_E \). We can regard \( \mathcal{E}_E \) as a section of \( \pi_*D_E \).

**Definition 1.3.** Let \( M \) be a \( D \)-module on \( E \). We say that \( M \) is monodromic if for any (local algebraic) section \( m \in \pi_*M \) there is a polynomial \( b(u) \in \mathbb{C}[u] \) such that \( b(\mathcal{E}_E)m = 0 \). Moreover, if all the roots of the minimal polynomial of such \( b(u) \) is in \( \mathbb{Q} \) (resp. \( \mathbb{R} \)) for any \( m \), we say that \( M \) is \( \mathbb{Q} \)-monodromic (resp. \( \mathbb{R} \)-monodromic).

Since we only consider \( \mathbb{Q} \)-monodromic \( D \)-module in this paper, we will say “monodromic” as “\( \mathbb{Q} \)-monodromic”.

**Remark 1.4.** For a subset \( V \) of \( \mathbb{C}^n \) (e.g. \( (\mathbb{C}^*)^n \)) and a \( D \)-module \( M \) on \( X \times V \), we also define “\( M \) is monodromic” in the same way as Definition 1.3.

**Remark 1.5.** Remark that \( E \) is equipped with a natural \( \mathbb{C}^* \)-action. Assume that \( M \) is regular holonomic. Let \( K \) be the perverse sheaf corresponding to \( M \) via the Riemann-Hilbert
correspondence. Then, \(M\) is monodromic if and only if \(K\) is cohomologically locally constant on each \(\mathbb{C}^*\)-orbit in \(E\), i.e. each cohomology \(H^j(K)\) is locally constant on each \(\mathbb{C}^*\)-orbit (Proposition 7.12 of Brylisnki [2]).

Note that we can also endow \(\pi_* M\) with an \(\mathcal{O}_X\)-modules structure by the adjunction \(\mathcal{O}_X \to \pi_* \mathcal{O}_E \to \pi_* M\). For a \(\mathcal{O}_E\)-module \(M\) and \(\beta \in \mathbb{Q}\), we define a \(\mathcal{O}_X\)-submodule \(M^\beta\) of \(\pi_* M\) by

\[
M^\beta := \bigcup_{l \geq 0} \text{Ker}((\mathcal{E}_E - \beta)^l : \pi_* M \to \pi_* M).
\]

(3)

**Proposition 1.6.** A \(\mathcal{O}_E\)-module \(M\) is monodromic if and only if \(M\) (recall that we identify it with \(\pi_* M\)) is a direct sum of the family of \(\mathcal{O}_X\)-modules \(\{M^\beta\}_{\beta \in \mathbb{R}}\) as

\[
M = \bigoplus_{\beta \in \mathbb{R}} M^\beta.
\]

(4)

**Proof.** The proof is similar to the proof of Proposition 1.7 of [20].

**Remark 1.7.** If \(E\) is trivial and we fix a trivialization \(E \simeq X \times \mathbb{C}^n\), we can endow \(M^\beta\) with a natural \(\mathcal{O}_X\)-module structure. Because a lift of a section of \(\mathcal{D}_X\) to \(E\) is not unique, we can not define a natural \(\mathcal{D}_X\)-module structure on \(M^\beta\) in general.

**Remark 1.8.** Submodules, quotient modules and extensions (in the category of \(\mathcal{D}_E\)-modules) of monodromic \(\mathcal{D}_E\)-modules are monodromic again.

Let us consider a mixed Hodge module \(M = (M, F_\bullet M, K, W_\bullet K)\) on \(E\), where \(M\) is the underlying \(\mathcal{D}\)-module, \(F_\bullet M\) is the Hodge filtration, \(K\) is the underlying \(\mathbb{Q}\)-perverse sheaf and \(W_\bullet K\) is the weight filtration.

**Definition 1.9.** We say that \(M\) is monodromic if \(M\) is monodromic.

### 1.2 The case where \(E\) is a trivial bundle of rank one

Let us recall some results for a monodromic mixed Hodge module \(M = (M, F_\bullet M, K, W_\bullet K)\) in the case where \(E\) is a trivial bundle of rank one i.e. \(E \simeq X \times \mathbb{C}_z\). We fix this trivialization in this subsection. See [20] for details.

Note that in this case \(M^\beta(= \bigcup_{l \geq 0} \text{Ker}((z\partial_z - \beta)^l : M \to M))\) is not only an \(\mathcal{O}_X\)-module, it is a \(\mathcal{D}_X\)-module (see Remark 1.7). Moreover, we have a decomposition

\[
M = \bigoplus_{\beta \in \mathbb{R}} M^\beta,
\]

(5)

where the \(\mathcal{D}_X[z](\partial_z)\)-module structure is defined by using the morphisms \(z: M^\beta \to M^{\beta+1}\) and \(\partial_z: M^\beta \to M^{\beta-1}\) (in fact, \(M^\beta = 0\) for \(\beta \notin \mathbb{Q}\)).

We say that \(M\) is \(\mathbb{K}(= \mathbb{Q}\) or \(\mathbb{R}\))-specializable if there exists the Kashiwara-Malgrange filtration of \(M\) indexed by \(\mathbb{K}\). We denote by \(\{V_\beta^M\}_{\beta \in \mathbb{R}}\) the Kashiwara-Malgrange filtration.
of $M$ along $z$, where the index is defined to be $\text{gr}^\beta_M = V^\beta_M/V^\gamma_M$ is killed by $(z\partial_z - \beta)^l$ for some $l \geq 0$. Sometimes $\text{gr}^\beta_M$ is abbreviated to $\text{gr}^\beta_M$. The decomposition (5) leads to the following.

**Proposition 1.10** (Proposition 1.15 of [20]). Let $M$ be a monodromic coherent $D$-module on $X \times \mathbb{C}_z$. Then, $M$ is specializable, i.e. the Kashiwara-Malgrange filtration of $M$ along $t = 0$ exists and we have

$$V_z^\gamma M = \bigoplus_{\beta \geq \gamma} M^\beta.$$

Therefore, we have

$$\text{gr}^\gamma_M = M^\gamma.$$

In particular, the $\alpha$-nearby cycle and the vanishing cycle of $M$ along $z$ are described as follows:

$$\psi_{z, \alpha} M (= \text{gr}^\alpha_M) = M^\alpha, \quad \text{and}$$

$$\phi_{z, 1} M (= \text{gr}^{-1}_M) = M^{-1}.$$

Moreover, the morphism $\psi_{z, 0} M \to \phi_{z, 1} M$ (resp. the morphism $\var : \phi_{z, 0} M \to \phi_{z, 1} M$) is $-\partial_z : M^0 \to M^{-1}$ (resp. $z : M^{-1} \to M^0$).

In this situation, the Hodge filtration is decomposed with respect to the decomposition (5).

**Proposition 1.11** (Theorem 2.2 of [20]). For $p \in \mathbb{Z}$, the Hodge filtration $F_p M \subset M$ is decomposed as

$$F_p M = \bigoplus_{\beta \in \mathbb{R}} F_p M^\beta,$$

where $F_p M^\beta := F_p M \cap M^\beta$ and the $\mathcal{O}_X[z]$-module structure of the right hand side is defined by the morphisms $z : F_p M^\beta \to F_p M^{\beta+1}$.

Let us recall the strict specializability for a filtered $D$-module. See [19] and [16] for details. Let $(M, F_p M)$ be a holonomic $D$-module $M$ with a good filtration on $X \times \mathbb{C}_z$. We set $F_p \text{gr}^\beta_M := F_p M \cap V^\beta_M/F_p M \cap V^{\gamma_B} M$.

**Definition 1.12.** We say that $(M, F_p M)$ is strictly $\mathbb{K}(= \mathbb{Q}$ or $\mathbb{R}$)-specializable along $t$ if $M$ is $\mathbb{K}$-specializable and for any $p \in \mathbb{Z}$

(i) for any $\beta > -1$, $z : F_p \text{gr}^\beta_M \to F_p \text{gr}^\beta_M$ is surjective, and

(ii) for any $\beta < 0$, $\partial_z : F_p \text{gr}^\beta_M \to F_{p+1} \text{gr}^{\beta-1}_M$ is surjective.

This property is one of the important constraints on mixed Hodge modules. Proposition 1.11 and the strict specializability lead the following.

**Lemma 1.13** (Lemma 2.4 of [20]). For the underlying filtered $D$-module $(M, F_p M)$ of a monodromic mixed Hodge module on $X \times \mathbb{C}_z$, $l \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}$ we have

$$F_p M^{\alpha+l} = z^l F_p M^{\alpha} \quad (\alpha \in (-1, 0]), \quad \text{and}$$

$$F_p M^{\alpha-l} = \partial_z z^{l} F_p M^{\alpha} \quad (\alpha \in [-1, 0]).$$
In particular, we have

\[ F_p M = \left( \bigoplus_{l \geq 1} \bigoplus_{[-1,0]} \partial \bar{z} F_{p-l} M^\alpha \right) \oplus F_p M^{-1} \oplus \left( \bigoplus_{l \geq 0} \bigoplus_{\alpha \in (-1,0]} z^l F_p M^\alpha \right). \]  

(8)

We can describe the category \( \text{MHM}^p_{\text{mon}}(X \times \mathbb{C}_z) \) of monodromic graded polarizable mixed Hodge modules on \( X \times \mathbb{C}_z \) as follows. We consider a tuple \((\mathcal{M}_{(-1,0)}, T_s, N, \mathcal{M}_{-1}, c, v)\), where \( \mathcal{M}_{(-1,0)} \) and \( \mathcal{M}_{-1} \) are graded polarizable mixed Hodge modules on \( X \) and \( T_s : \mathcal{M}_{(-1,0)} \simeq \mathcal{M}_{(-1,0)}, N : \mathcal{M}_{(-1,0)} \rightarrow \mathcal{M}_{(-1,0)}(-1), c : \mathcal{M}_0 := \text{Ker}(T_s - 1) \subset \mathcal{M}_{(-1,0)} \rightarrow \mathcal{M}_{-1} \) and \( v : \mathcal{M}_{-1} \rightarrow \mathcal{M}_0(-1) \) are morphisms in the category of mixed Hodge modules with the following properties:

(i) \( T_s \) commutes with \( N \).

(ii) The underlying \( D \)-module \( \mathcal{M}_{(-1,0)} \) of \( \mathcal{M}_{(-1,0)} \) is decomposed as

\[ \mathcal{M}_{(-1,0)} = \bigoplus_{\alpha \in (-1,0] \cap \mathbb{Q}} M_\alpha, \]

where \( M_\alpha := \text{Ker}(T_s - \exp(-2\pi \sqrt{-1} \alpha)) \subset \mathcal{M}_{(-1,0)}. \)

(iii) \( vc : \mathcal{M}_0 \rightarrow \mathcal{M}_0(-1) \) is \(-N\).

We denote by \( \mathcal{G}(X) \) the category of such tuples \((\mathcal{M}_{(-1,0)}, T_s, N, \mathcal{M}_{-1}, c, v)\).

Let \( \mathcal{M} = (M, F_\bullet M, K, W_\bullet K) \) be a monodromic mixed Hodge module on \( X \times \mathbb{C}_z \). We define \( \mathcal{M}_{(-1,0)} \) as the nearby cycle \( \psi_z M \) of \( M \) along \( z \), \( \mathcal{M}_{-1} \) the unipotent vanishing cycle \( \phi_{t,1} M, T_s \) (resp. \( N \)) the semi-simple part (resp. \( \frac{1}{2\pi \sqrt{-1}} \) times the logarithm of the unipotent part) of the monodromy automorphism of \( \psi_z M \) and \( c \) (resp. \( v \)) the morphism \( \psi_z M \rightarrow \phi_{t,1} M \) (resp. \( \phi_{t,1} M \rightarrow \psi_z M(-1) \)). Then, the tuple \((\mathcal{M}_{(-1,0)}, T_s, N, \mathcal{M}_{-1}, c, v)\) is an object in \( \mathcal{G}(X) \).

In this way, we get a functor

\[ F : \text{MHM}^p_{\text{mon}}(X \times \mathbb{C}_z) \rightarrow \mathcal{G}(X). \]

Proposition 1.14 (Theorem 3.5 of [20]). The functor \( F \) induces an equivalence of categories.

In particular, we can reconstruct \( M \) from the tuple \((\psi_z M, T_s, N, \phi_{t,1} M, \text{can}, \text{var})\).

Remark 1.15. As stated in Remark 1.14, we can also consider monodromic mixed Hodge modules on \( X \times \mathbb{C}_z^* \). Then, we have a similar statement to Proposition 1.14 and Proposition 1.13 for a monodromic mixed Hodge modules \( M \) on \( X \times \mathbb{C}_z^* \) (see [20]). In this case, \( M \) is decomposed as

\[ M = \bigoplus_{\beta \in \mathbb{R}} M^\beta, \]

where \( M^\beta \) is defined as in the case of that on \( X \times \mathbb{C}_z \), and for \( \alpha \in (-1,0] \) and \( k \in \mathbb{Z} \) we have

\[ M^{\alpha+k} = z^k M^\alpha. \]

Moreover, as a corresponding assertion to Lemma 1.13, we have

\[ F_* M = \bigoplus_{\alpha \in (-1,0]} \bigoplus_{k \in \mathbb{Z}} z^k F_* M^\alpha. \]
1.3 Example: normal crossing type

Let us consider monodromic $D$-modules on $E = X \times \mathbb{C}^n$ with a stronger condition. Let $(z_1, \ldots, z_n)$ be the standard coordinates of $\mathbb{C}^n$ and $\pi$ the projection $X \times \mathbb{C}^n \to X$.

**Definition 1.16.** A $D$-module $M$ on $X \times \mathbb{C}^n$ is of normal crossing type if for any section $m \in \pi_*M$ and $1 \leq i \leq n$ there exists a polynomial $b(u) \in \mathbb{C}[u]$ such that $b(z_i \partial_{z_i})m = 0$. In other words, for any $1 \leq i \leq n$, $M$ is monodromic on $(X \times \mathbb{C}_{z_1} \times \ldots \mathbb{C}_{z_{i-1}} \times \mathbb{C}_{\beta_{i+1}} \times \ldots \mathbb{C}_{z_n}) \times \mathbb{C}_{\partial_{z_i}}$, where we regard $(X \times \mathbb{C}_{z_1} \times \ldots \mathbb{C}_{z_{i-1}} \times \mathbb{C}_{z_{i+1}} \times \ldots \mathbb{C}_{z_n}) \times \mathbb{C}_{\partial_{z_i}}$ as a rank one vector bundle over $X \times \mathbb{C}_{z_1} \times \ldots \mathbb{C}_{z_{i-1}} \times \mathbb{C}_{z_{i+1}} \times \ldots \mathbb{C}_{z_n}$.

**Remark 1.17.** Let $M$ be a regular holonomic $D$-module $M$ of normal crossing type on $\mathbb{C}^n$ and $K$ be the perverse sheaf corresponding to $M$. For $1 \leq k \leq n$ and $1 \leq i_1, \ldots, i_k \leq n$, we set

$$V_{i_1,\ldots,i_k} := (\bigcap_{1 \leq s \leq k} \{z_s = 0\}) \setminus (\bigcup_{s \notin \{i_1,\ldots,i_k\}} \{z_s = 0\}) \subset \mathbb{C}^n.$$  

Then, the restriction of each cohomology of $K$ to $V_{i_1,\ldots,i_k}$ or $\mathbb{C}^n \setminus \bigcup_{1 \leq s \leq n} \{z_s = 0\}$ is locally constant by Remark 1.5. Conversely, if $K$ has this property for a regular holonomic $D$-module on $\mathbb{C}^n$, $M$ is of normal crossing type.

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ we set

$$M^\beta := \bigcap_{i=1}^n \bigcup_{l \geq 0} (\text{Ker}((z_i \partial_{z_i} - \beta_i)^l : \pi_*M \to \pi_*M)).$$

Then, we can regard $M^\beta$ as a $D_X$-module.

As mentioned, a $D$-module $M$ of normal crossing type is also a monodromic $D$-module with respect to any $z_i$-direction. Therefore, we can apply the results in Subsection 1.2 inductively. For example, we have the following.

**Lemma 1.18.** $M$ is of normal crossing type if and only if $M$ is decomposed as

$$M = \bigoplus_{\beta \in \mathbb{R}^n} M^\beta.$$  

For $\gamma \in \mathbb{R}$, $M^\gamma = \bigcup_{l \geq 0} \text{Ker}(\mathcal{E}_E - \beta \mathbb{I}^l) \subset \pi_*M$ is a $D_X$-module (see Remark 1.7), where $\mathcal{E}_E = \sum_{i=1}^n z_i \partial_{z_i}$. Moreover, it is decomposed as follows.

**Lemma 1.19.** We have

$$M^\gamma = \bigoplus_{\beta \in \mathbb{R}^n} M^\beta$$  

as a $D_X$-module.

**Proof.** Let $m$ be a section of $M^\beta$ and $l_0$ an integer large enough so that $(z_i \partial_{z_i} - \beta_i)^{l_0}m = 0$ for any $1 \leq i \leq n$. We set $\gamma := \beta_1 + \cdots + \beta_n$. Then we have

$$(\mathcal{E}_E - \gamma)^{l_0}m = 0,$$
and thus obtain
\[ M^\beta \subset M^\gamma. \quad (9) \]

Conversely, by Lemma 1.18, \( m \in M^\gamma \) is decomposed as \( m = m_1 + \cdots + m_k \), where \( m_s(\neq 0) \in M^\beta_s \) and \( \beta_s \neq \beta_t \) for \( s \neq t \). By (9) and the decomposition (1), we have the converse inclusion \( M^\gamma \subset \bigoplus_{\beta_1 + \cdots + \beta_n = \gamma} M^\beta \). This implies the desired assertion.

\[ \square \]

Remark 1.20. For \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \) and \( 1 \leq i \leq n \), it is easy to see that
\[
\begin{align*}
z_i^*M^\beta &\subset M^{\beta + e_i} \quad \text{and} \\
\partial z_i^*M^\beta &\subset M^{\beta - e_i},
\end{align*}
\]
where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \). Moreover, in a similar way to the proof of Proposition 1.10 of [20], we can see that the morphism
\[
\begin{align*}
z_i : M^\beta &\to M^{\beta + e_i} \\
(\text{resp. } \partial z_i : M^\beta &\to M^{\beta - e_i})
\end{align*}
\]
is an isomorphism if \( \beta_i \neq -1 \) (resp. \( \beta_i \neq 0 \)). Therefore, the \( D \)-module \( M \) is determined by the following data:

(i) The family of \( D_X \)-modules \( \{M^\alpha\}_{\alpha \in [-1,0]^n} \).

(ii) The nilpotent endomorphisms \( z_i \partial z_i - \alpha_i : M^\alpha \to M^\alpha \) for \( \alpha \in [-1,0]^n \) and \( 1 \leq i \leq n \).

(iii) For \( 1 \leq i \leq n \) and \( \alpha \in [-1,0]^n \) with \( \alpha_i = -1 \) (resp. \( \alpha_i = 0 \)), the morphism \( z_i : M^\alpha \to M^{\alpha + e_i} \) (resp. \( \partial z_i : M^\alpha \to M^{\alpha - e_i} \)) such that the composition \( \partial z_i \circ z_i : M^\alpha \to M^\alpha \) (resp. \( z_i \circ \partial z_i : M^\alpha \to M^\alpha \)) is equal to \( z_i \partial z_i + 1 \) (resp. \( z_i \partial z_i \)) defined in (ii).

We assume that \( M \) is coherent. Let \( V_{z_i}^*M \) be the Kashiwara-Malgrange filtration of \( M \) along \( z_i \). Then, by Proposition 1.10 we have the following.

Lemma 1.21. For a coherent \( D \)-module \( M \) of normal crossing type, the Kashiwara-Malgrange filtration of \( M \) along \( z_i \) is exists and we have
\[
V_{z_i}^*M = \bigoplus_{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \atop \beta_i \geq \gamma} M^\beta.
\]
In particular, for \( \alpha \in (-1,0] \) the \( \alpha \)-nearby cycle \( \psi_{z_i,\alpha}M := \text{gr}_{V_{z_i}^*}^\alpha M \) (resp. the unipotent vanishing cycle \( \phi_{z_i,1}M := \text{gr}_{V_{z_i}^*}^{-1} M \)) of \( M \) can be described as
\[
\begin{align*}
\psi_{z_i,\alpha}M &= \bigoplus_{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \atop \beta_i = \alpha} M^\beta \\
(\text{resp. } \phi_{z_i,1}M &= \bigoplus_{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \atop \beta_i = -1} M^\beta).
\end{align*}
\]
The previous lemma implies that the nearby cycle $\psi_{z_i,\alpha}M$ and the vanishing cycle $\phi_{z_i,1}M$ is again a $D$-module of normal crossing type on $X \times \mathbb{C}_{z_1} \times \cdots \mathbb{C}_{z_{i-1}} \times \mathcal{O} \times \mathbb{C}_{z_{i+1}} \times \cdots \mathbb{C}_{z_n}$. This allows us to prove the following proposition.

**Proposition 1.22.** Let $\mathcal{M}$ be a mixed Hodge module on $X \times \mathbb{C}^n$ whose underlying $D$-module $M$ is of normal crossing type. Then, the Hodge filtration $F_pM$ is decomposed as

$$F_pM = \bigoplus_{\beta \in \mathbb{R}^n} F_pM^\beta,$$

where $F_pM^\beta := F_pM \cap M^\beta$ and the $\mathcal{O}_X[z_1, \ldots, z_n]$-module structure of the right hand side is defined by the morphisms $z_i : F_pM^\beta \to F_pM^{\beta + e_i}$ for $\beta \in \mathbb{R}^n$.

**Proof.** The proof is by induction on $n$. The assertion for $n = 1$ is Proposition 1.11. Suppose that the statement is proved for $n = n_0 - 1$ ($n_0 \geq 2$) and consider the case where $n = n_0$. We set $M_{z_n}^\beta := \bigcup_{l \geq 1} \text{Ker}(z_n\partial_{z_n} - \beta)^l \subset \pi_*M$. Since $M$ is monodromic with respect to the $z_n$-direction on $(X \times \mathbb{C}^{n-1}) \times \mathbb{C}_{z_n}$, we have

$$M = \bigoplus_{\beta \in \mathbb{R}} M_{z_n}^\beta \quad \text{and} \quad (10)\ F_pM = \bigoplus_{\beta \in \mathbb{R}} F_pM_{z_n}^\beta, \quad (11)$$

where $F_pM_{z_n}^\beta = F_pM \cap M_{z_n}^\beta$. Note that for if $\beta$ is in $(-1,0]$ (resp. $\beta$ is $-1$), we have $M_{z_n}^\beta = \psi_{z_n,\beta}M$ (resp. $M_{z_n}^\beta = \phi_{z_n,1}M$). As mentioned above, the $\alpha$-nearby cycle ($\alpha \in (-1,0]$) and the unipotent vanishing cycle of $M$ are of normal crossing type. Moreover, they are a direct summand of a mixed Hodge module and their Hodge filtrations are $F_\bullet M_\alpha$ and $F_\bullet M_\alpha$ up to shift. Therefore, by the induction hypothesis, we have

$$F_pM_{z_n}^\alpha = \bigoplus_{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_{\beta_n = \alpha}} F_pM^\beta \quad \text{and,}$$

$$F_pM_{z_n}^{-1} = \bigoplus_{\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_{\beta_n = -1}} F_pM^\beta.$$

By $(11)$ and the strict specializability along $z_n = 0$, we obtain the desired assertion. \qed

In particular, combining it with Lemma 1.19, we have the following.

**Corollary 1.23.** In the situation of Proposition 1.22, we have

$$F_pM = \bigoplus_{\gamma \in \mathbb{R}} F_pM^\gamma, \quad \text{and}$$

$$F_pM^\gamma = \bigoplus_{\beta \in \mathbb{R}^n_{\beta_1 + \cdots + \beta_n = \gamma}} F_pM^\beta.$$
In the next section, we will generalize this assertion to the case for general monodromic mixed Hodge modules, which is not necessarily of normal crossing type.

**Remark 1.24.** For a mixed Hodge module of normal crossing type $M$ and $\beta = (\beta_1, \ldots, \beta_n) \in (-1,0]^n$, it is easy to see

$$M^\beta = \psi_{z_1,\beta_1} \cdots \psi_{z_n,\beta_n} M.$$  

Moreover, for example, for $\beta = (-1,\beta_2, \ldots, \beta_n) \in \{-1\} \times (-1,0]^{n-1}$, we have

$$M^\beta = \phi_{z_1,1} \psi_{z_2,\beta_2} \cdots \psi_{z_n,\beta_n} M.$$  

A similar statement holds for any $\beta \in [-1,0]^{n-1}$. Then, we can generalize the gluing: Proposition 1.14 to the normal crossing case. In particular, the mixed Hodge module of normal crossing type $M$ can be reconstructed from the family of mixed Hodge modules $\{\Psi_1 \cdots \Psi_n M\}(\psi_{1,\alpha}, \ldots, \psi_{n,\alpha})$ with some morphisms between them, where $\Psi_i$ is $\psi_{z_i,\alpha} (\alpha \in (-1,0))$ or $\phi_{z_i,1}$.

## 2 The Hodge filtration of monodromic mixed Hodge modules

Let $M = (M,F\bullet M,K,W\bullet K)$ be a monodromic mixed Hodge module on a vector bundle $\pi: E \to X$ on a smooth algebraic variety $X$. By Proposition 1.6 we have the decomposition

$$M = \bigoplus_{\beta \in \mathbb{R}} M^\beta.$$  

For $p \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, we define an $O_X$-submodule of $M^\beta$ as

$$F_p M^\beta := \pi_* (F_p M) \cap M^\beta (\subset \pi_* M).$$

Then, the direct sum $\bigoplus_{\beta \in \mathbb{R}} F_p M^\beta$ is a $\pi_* O_E$-submodule of $\bigoplus_{\beta \in \mathbb{R}} M^\beta$. Therefore, by Lemma 1.1 we can also regard $\bigoplus_{\beta \in \mathbb{R}} F_p M^\beta$ as an $O_E$-submodule of $M$. The purpose of this section is to show the following theorem.

**Theorem 2.1.** For $p \in \mathbb{Z}$, the Hodge filtration $F_p M$ is decomposed as

$$F_p M = \bigoplus_{\beta \in \mathbb{R}} F_p M^\beta.$$  

**Remark 2.2.** This result was shown in a different way in a recent preprint [2] by Chen-Dirks.

Since it is enough to show this theorem locally on $X$, we may assume that $E$ is a trivial bundle $X \times \mathbb{C}^n$ and $X$ is affine (therefore, we identify $M$ with the module of its global sections). Let $(z_1, \ldots, z_n)$ be the coordinates of $\mathbb{C}^n$. We set $D_1 := \{z_1 = 0\} \subset X \times \mathbb{C}^n$ and $V_1 := E \setminus D_1$.

Let us recall some basic properties of the localization $M[*D_1]$ and the dual localization $M[[D_1]]$ of a mixed Hodge module $M$. For details, see [10] and [7]. We denote by $M[*D_1]$ (resp. $M[[D_1]]$) the underlying $D$-module of $M[*D_1]$ (resp. $M[[D_1]]$). The stupid localization $M(*D_1)$ (resp. $(M,F\bullet M)(*D_1)$) along $D_1$ of a $D$-module $M$ (resp. a filtered $D$-module $(M,F\bullet M)$) is the $D_E(*D_1)(= D_E \otimes_{\mathbb{C}[z_1]} \mathbb{C}[z_1^{\pm 1}])$-module (resp. the filtered $D_E(*D_1)$-module) defined as

$$M \otimes_{\mathbb{C}[z_1]} \mathbb{C}[z_1^{\pm 1}]$$

13
(resp. $(M \otimes_{\mathbb{C}[z_1]} \mathbb{C}[z_1^{\pm 1}], F_\bullet (M \otimes_{\mathbb{C}[z_1]} \mathbb{C}[z_1^{\pm 1}]))$).

$(M, F_\bullet M)(\ast D_1)$ along $z_1$ of a filtered $D$-module $(M, F_\bullet M)$ is the $D_E(\ast D_1)(= D_E \otimes_{\mathbb{C}[z_1]} C[z_1^{\pm 1}])$-module defined as

$$(M \otimes_{\mathbb{C}[z_1]} C[z_1^{\pm 1}], F_\bullet (M \otimes_{\mathbb{C}[z_1]} C[z_1^{\pm 1}])).$$

Let $V_{z_1}\ast M$ be a Kashiwara-Malgrange filtration of a $D$-module $M$ along $z_1$.

**Fact 2.3** (see [1], [16] and [7]). Let $M = (M, F_\bullet M, K, W_\bullet M)$ be a mixed Hodge module on $E = X \times \mathbb{C}^\circ$. Then, we have the following.

(i) The underlying $D$-modules are as follows:

$$M[\ast D_1] = M(\ast D_1) = M \otimes_{\mathbb{C}[z_1]} C[z_1^{\pm 1}],$$
and

$$M[\mid D_1] = D(D(M)(\ast D_1)) = D_E \otimes_{V_{z_1}D_E} V_{z_1}^{> -1}M,$$

where $V_{z_1}D_E$ is the $V$-filtration of $D_E$ along $z_1$ and $D$ is the duality functor between the category of mixed Hodge modules.

(ii) We have an (canonical) isomorphism $D(M[\ast D_1]) \simeq (DM)[\mid D_1]$.

(iii) There is natural morphisms $M \rightarrow M[\ast D_1]$ and $M[\mid D_1] \rightarrow M$ whose restriction to $V_1$ are isomorphisms. In particular, the stupid localizations of the underlying filtered $D$-modules of $M[\ast D_1]$ and $M[\mid D_1]$ are the stupid localization $(M, F_\bullet M)(\ast D_1)$ of the underlying filtered $D$-module of $M$, and we have

$$V_{z_1}^{> -1}M(\ast D_1) = V_{z_1}^{> -1}M[\mid D_1] = V_{z_1}^{> -1}M.$$

(iv) We have

$$V_{z_1}^{> -1}M[\ast D_1] = z_1^{-1}V_{z_1}^{> 0}M,$$
and

$$F_pV_{z_1}^{> -1}M[\ast D_1] = z_1^{-1}F_pV_{z_1}^{> 0}M \quad (p \in \mathbb{Z}).$$

(v) The Hodge filtrations are described as follows:

$$F_p(M[\ast D_1]) = \sum_{k \geq 0} \partial_{z_1}^k F_{p-k}V_{z_1}^{> -1}M,$$
and

$$F_p(M[\mid D_1]) = \sum_{k \geq 0} \partial_{z_1}^k \otimes F_{p-k}V_{z_1}^{> -1}M,$$

where $F_{p-k}V_{z_1}^{> -1}M = F_{p-k}M \cap V_{z_1}^{> -1}M$ and $F_{p-k}V_{z_1}^{> -1}M = F_{p-k}M \cap V_{z_1}^{> -1}M$. With (iv), in particular, the filtered $D$-modules of $M[\ast D_1]$ and $M[\mid D_1]$ are determined only by the stupid localization $(M, F_\bullet M)(\ast D_1)$.

The following is a simple consequence of (i) of Fact 2.3

**Lemma 2.4.** If $M$ is monodromic, then $M(\ast D_1)$ and $M(\mid D_1)$ are also monodromic.
Let \( \rho : \mathbb{C}^n \to \mathbb{C}^n \) be the blowing up of \( \mathbb{C}^n \) at the origin. Remark that \( \mathbb{C}^n \) is a subvariety of \( \mathbb{C}^n \times \mathbb{P}^{n-1} \) and \( \rho \) is the projection to \( \mathbb{C}^n \). We write \( q : \mathbb{C}^n \to \mathbb{P}^{n-1} \) the projection to \( \mathbb{P}^{n-1} \). Let \( [y_1 : \ldots : y_n] \) be the homogeneous coordinates of \( \mathbb{P}^{n-1} \). Define \( U_1 \) as a local chart \( \{ y_1 \neq 0 \} \subset \mathbb{P}^{n-1} \) of \( \mathbb{P}^{n-1} \). We use the same symbol \( (y_2, \ldots, y_n) \) for the coordinates of \( U_1 \), i.e. \((y_2, \ldots, y_n) \in U_1 \) is the point \([1: y_2: \ldots: y_n] \in \mathbb{P}^{n-1} \). Then, we have

\[
\mathcal{C}_s^* \times U_1 \overset{\simeq}{\to} q^{-1}(U_1) \overset{\simeq}{\to} V_1
\]

(12)

Then, (12) sends the vector field \( s \partial_s \) to the pullback \( \rho^* \mathcal{E} \). By this corollary, we have a decomposition

\[
M_1 = \bigoplus_{\beta \in \mathbb{R}} M_1^\beta,
\]

(14)

The following simple lemma reduces a problem for a monodromic \( D \)-module on a vector bundle to that for a monodromic \( D \)-module on a line bundle.

**Lemma 2.5.** The isomorphism (12) sends the vector field \( s \partial_s \) on \( \mathcal{C}_s^* \times U_1 \) to the vector field \( \mathcal{E} = \sum_{i=1}^n z_i \partial_{z_i} \) on \( V_1 \).

**Proof.** Let \( G := (g_1, \ldots, g_n) \) be the morphism (12). Then, (12) sends the vector field \( s \partial_s \) to

\[
z_1 \sum_{k=1}^n (\partial g_k / \partial s \circ G^{-1}) \partial_{z_k}.
\]

(13)

Since

\[
\partial g_k / \partial s = \begin{cases} 1 & (k = 1) \\ y_k & (k \neq 1) \end{cases}
\]

we have

\[
(13) = z_1 \partial_{z_1} + z_1 \sum_{k=2}^n (z_k / z_1) \partial_{z_k}
\]

\[
= \mathcal{E}.
\]

\[\square\]

We write \( \rho_1 \) for the induced isomorphism \( q^{-1}(U_1) \overset{\simeq}{\to} V_1 \) by \( \rho \) and \( L \rho_1^* \) the pullback functor of the category of \( D \)-modules. Since \( \rho_1 : q^{-1}(U_1) \overset{\simeq}{\to} V_1 \) is an isomorphism, \( L \rho_1^* \mathcal{M}_1 \) is just the pullback \( \rho_1^* \mathcal{M}_1 = O_{q^{-1}(U_1)} \otimes_{\rho_1^{-1}O_{U_1}} \mathcal{M}_1 \) of the pullback functor \( \rho_1^* \) for \( \mathcal{M}_1 \) on \( V_1 \). We just write \( \rho_1^* \mathcal{M}_1 \) for \( L \rho_1^* \mathcal{M}_1 \). Note that any section \( m \in \rho_1^* \mathcal{M}_1 \) can be expressed as \( m = m' \otimes m'' \) for some \( m' \in M_1 \). The morphisms \( \rho \) and \( \rho_1 \) induces morphisms \( X \times \mathbb{C}^n \to X \times \mathbb{C}^n \) and \( X \times q^{-1}(U_1) \to X \times V_1 \), denoted by the same symbols \( \rho \) and \( \rho_1 \). For a monodromic \( D \)-module \( \mathcal{M}_1 \) on \( X \times V_1 \), we set \( \tilde{M}_1 := \rho^* \mathcal{M}_1 \). Lemma 2.5 immediately deduces the following.

**Corollary 2.6.** A \( D \)-module \( \mathcal{M}_1 \) on \( X \times V_1 \) is monodromic (in the sense of Remark 1.4) if and only if the \( D \)-module \( \tilde{M}_1 \) on \( X \times q^{-1}(U_1) \) is monodromic with respect to the \( s \)-direction.

By this corollary, we have a decomposition

\[
\tilde{M}_1 = \bigoplus_{\beta \in \mathbb{R}} \tilde{M}_1^\beta,
\]

(14)
where \( \widetilde{M}_1^\beta = \bigcup_{i \geq 0} (\text{Ker}((s \partial_a - \beta)^i) : \widetilde{M}_1 \to \widetilde{M}_1) \). We can regard \( \widetilde{M}_1^\beta \) as a \( DX \times U_1 \)-module (see Remark \[1.7\]). Recall that we also have

\[
M_1 = \bigoplus_{\beta \in \mathbb{R}} M_1^\beta, \tag{15}
\]

where \( M_i^\beta = \bigcup_{i \geq 0} (\text{Ker}((\mathcal{E} - \beta)^i) : M_1 \to M_1) \). Let us see the relationship between \[14\] and \[15\].

**Lemma 2.7.** For \( \beta \in \mathbb{R} \) we have

\[
\widetilde{M}_1^\beta = 1 \otimes \rho_1^{-1}(M_1^\beta) (= O_{X \times U_1} \otimes \rho_1^{-1}(M_1^\beta)),
\]
as \( DX \times U_1 \)-modules.

**Proof.** By Lemma \[2.5\] for a section \( m \in M_1 \) and \( 1 \otimes m \in \widetilde{M}_1 \), we have

\[
s \partial_a (1 \otimes m) = 1 \otimes (\mathcal{E}m).
\]

Therefore, if the section \( m \) is in \( M_1^\beta \), the section \( 1 \otimes m \in \widetilde{M}_1 \) is in \( \widetilde{M}_1^\beta \). Hence, we have \( 1 \otimes \rho_1^{-1}(M_1^\beta) \subseteq \widetilde{M}_1^\beta \). Let us show the reverse inclusion. Any section \( m \in \widetilde{M}_1 \) can be expressed as \( m = \sum_{i=0}^{\infty} (1 \otimes m_i) \) with \( m_i \in M_i^\beta \) for some \( \beta_i \in \mathbb{R} \) by \[15\]. Assume that \( m \) is in \( \widetilde{M}_1^\beta \). Since \( 1 \otimes m_i \) is in \( \widetilde{M}_1^\beta \) as already shown, the decomposition \[14\] implies that \( \sum_{\beta_i \neq \beta} (1 \otimes m_i) = 0 \). Therefore, we have \( m = \sum_{\beta_i = \beta} (1 \otimes m_i) \) and we thus obtain \( \widetilde{M}_1^\beta \subseteq 1 \otimes \rho_1^{-1}(M_1^\beta) \). \[ \square \]

For a monodromic mixed Hodge module \( M_1 \) on \( X \times V_1 \), we consider the pullback \( H^0 \rho_1^\dagger M_1 \) of \( M_1 \) by \( \rho_1 \) as a mixed Hodge module, whose underlying \( D \)-module is \( \rho_1^\dagger M_1 \). We set \( \widetilde{M}_1 := H^0 \rho_1^\dagger M_1 \) and \((\widetilde{M}_1, F_\bullet \widetilde{M}_1)\) is the underlying filtered \( D \)-module on \( X \times q^{-1}(U_1) \). Since \( \rho_1 \) is an isomorphism, the Hodge filtration \( F_\bullet \widetilde{M}_1 \) is just the pullback of the Hodge filtration \( F_\bullet M_1 \) as that of \( O_{X \times V_1} \)-modules:

\[
F_p \widetilde{M}_1 = O_{X \times q^{-1}(U_1)} \otimes \rho_1^{-1}O_{X \times V_1} \rho_1^{-1}F_p M_1.
\tag{16}
\]

In order to prove Proposition \[2.9\] below, we need the following

**Lemma 2.8.** We have

\[
F_p \widetilde{M}_1 \cap \widetilde{M}_1^\beta = 1 \otimes \rho_1^{-1}(F_p M_1 \cap M_1^\beta). \tag{17}
\]

**Proof.** By Lemma \[2.7\] and \[16\], the right hand side of \[17\] is contained in the left hand side of \[17\]. Let \( m \) be a section in the left hand side. By \[16\], we can write \( m = 1 \otimes m' \) for some \( m' \in F_p M_1 \). Let \( m' = \sum_{i=0}^{\infty} m_i' \) be the decomposition, where \( m_i' \in M_i^\beta \) for some \( \beta_i \in \mathbb{R} \) with \( \beta_i \neq \beta_j \) (\( i \neq j \)). By Lemma \[2.7\] \( 1 \otimes m_i' \) is in \( \widetilde{M}_1^\beta \). Since \( m \in \widetilde{M}_1^\beta \), we have \( 1 \otimes m_i' = 0 \), i.e. \( m_i' = 0 \) if \( \beta_i \neq \beta \). Hence, \( m' \in M_i^\beta \) and we thus conclude that \( m \) is in the right hand side of \[17\]. \[ \square \]
Combining Corollary 2.6 and Proposition 1.11, we have the following.

**Proposition 2.9.** For a monodromic mixed Hodge module $M_1$ on $X \times V_1$, we have a decomposition of the Hodge filtration as

$$F_p M_1 = \bigoplus_{\beta \in \mathbb{R}} F_p M_1^\beta, \quad (18)$$

where $(M_1, F_{\bullet} M_1)$ is the underlying filtered $D$-module of $M_1$ and $F_p M_1^\beta = F_p M_1 \cap M_1^\beta$.

**Proof.** By Corollary 2.6, $\tilde{M}_1$ is monodromic on $X \times q^{-1}(U_1) \simeq X \times \mathbb{C}^*_s \times U_1$ with respect to the $s$-direction. Therefore, by Proposition 1.11, we have

$$F_p \tilde{M}_1 = \bigoplus_{\beta \in \mathbb{R}} F_p \tilde{M}_1^\beta, \quad (19)$$

where $F_p \tilde{M}_1^\beta = F_p \tilde{M}_1 \cap \tilde{M}_1^\beta$. Moreover, by Lemma 2.8 we have

$$F_p \tilde{M}_1 = \bigoplus_{\beta \in \mathbb{R}} 1 \otimes \rho_1^{-1}(F_p M_1 \cap M_1^\beta)$$

$$= 1 \otimes \rho_1^{-1}(\bigoplus_{\beta \in \mathbb{R}} F_p M_1 \cap M_1^\beta). \quad (20)$$

Note that $\bigoplus_{\beta \in \mathbb{R}} F_p M_1 \cap M_1^\beta$ is an $O_{X \times V_1}$-submodule of $M_1$. Then, since $\rho_1$ is an isomorphism, from the equalities (16) and (20) we get the desired equality. \qed

Let $M = (M, F_{\bullet} M, K, W_{\bullet} M)$ be a monodromic mixed Hodge module on $X \times \mathbb{C}^n$. We set $M_1 := M|_{X \times V_1}$. Its underlying filtered $D$-module is $(M_1, F_{\bullet} M_1) := (M, F_{\bullet} M)|_{X \times V_1}$.

**Corollary 2.10.** For $p \in \mathbb{Z}$, $F_p V_{z_1}^{-1} M$ is decomposed with respect to the decomposition $M = \bigoplus_{\beta \in \mathbb{R}} M_1^\beta$, i.e. we have

$$F_p V_{z_1}^{-1} M = \bigoplus_{\beta \in \mathbb{R}} F_p V_{z_1}^{-1} M \cap M_1^\beta.$$

**Proof.** By the strict specializability along $z_1$ of the filtered $D$-module $(M, F_{\bullet} M)$ (Definition 1.12), we have (see Proposition 3.2.2 of [18] or Exercise 11.1 of [21])

$$F_p V_{z_1}^{-1} M = j_*(F_p M_1) \cap V_{z_1}^{-1} M, \quad (21)$$

where $j$ is the inclusion $X \times V_1 \hookrightarrow X \times \mathbb{C}^n$ and the intersection in the right hand side is taken in $j_* M_1 = M[z_1^{\pm 1}]$. By Proposition 2.9 we have

$$F_p M_1 = \bigoplus_{\beta \in \mathbb{R}} F_p M_1^\beta. \quad (22)$$

By Lemma 2.11 below, we have

$$V_{z_1}^{-1} j_* M_1 \cap (j_* M_1)^\beta = V_{z_1}^{-1} j_* M_1 \cap j_* (M_1^\beta),$$

17
where \((j_*, M_1)^\beta\) is defined similarly to \([3]\) and \(j_*(M_1^\beta)\) is a \(C\)-submodule of \(j_*M_1\) generated by \(\{j_*m \in j_*M_1 \mid m \in M_1^\beta\}\). Therefore, since \(V_{z_1} >-1 j_*M_1 = V_{z_1} >-1 M\), we have
\[
V_{z_1} >-1 M \cap M^\beta = V_{z_1} >-1 j_*M_1 \cap j_*(M_1^\beta).
\]
Hence, we have
\[
V_{z_1} >-1 M \cap j_*(F_p M^\beta) = V_{z_1} >-1 M \cap F_p M^\beta.
\] (23)
Combining \([21], [22]\) and \([23]\), we obtain
\[
F_p V_{z_1} >-1 M = \bigoplus_{\beta \in \mathbb{R}} F_p M \cap V_{z_1} >-1 M \cap M^\beta.
\]

The following was used in the proof of Corollary \([2.10]\)

**Lemma 2.11.** We have
\[
V_{z_1} >-1 j_*M_1 \cap (j_* M_1)^\beta = V_{z_1} >-1 j_*M_1 \cap j_*(M_1^\beta).
\] (24)

**Proof.** It is obvious that the left hand side contained in the right hand side. For a section \(m\) in the right hand side of \([24]\), since \(V_{z_1} >-1 j_*M_1 = V_{z_1} >-1 M, m\) is a section of \(M\) with \(((E - \beta)^l m)|_{X \times V_1} = 0\) for some \(l \geq 0\). Therefore, \((E - \beta)^l m\) is a section of \(V_{z_1} >-1 j_*M_1\) whose support is contained in \(z_1 = 0\). Since the multiplication by \(z_1\) on \(V_{z_1} >-1 j_*M_1\) is injective, we have \((E - \beta)^l m = 0\), i.e. \(m\) is in \((j_* M_1)^\beta\). \(\square\)

As mentioned, if \(M\) is monodromic, the localizations \(M[*D_1]\) and \(M[!)D_1]\) are also monodromic. Corollary \([2.10]\) deduces the following.

**Corollary 2.12.** If \(M \simeq M[*D_1]\) or \(M \simeq M[!)D_1]\), the assertion stated in Theorem \([2.1]\) is true.

**Proof.** Suppose that \(M = M[*D_1]\). Then, by Corollary \([2.10]\), \(F_p V_{z_1} >0 M\) is decomposed as \(F_p V_{z_1} >0 M = \bigoplus_{\beta \in \mathbb{R}} F_p V_{z_1} >0 M \cap M^\beta\). Hence, by (iv) of Fact \([2.3]\), \(F_p V_{z_1} >-1 M[*D_1]\) is also decomposed. Since \(F_p M = \sum_{k \geq 0} \partial_{z_1}^k F_p M V_{z_1} >-1 M\) by (v) of Fact \([2.3]\) we thus obtain the decomposition of \(F_p M\) in this case. The case of \(M = M[!)D_1]\) can be proved in the same way. \(\square\)

Let us recall the Beilinson’s maximal extension. See \([1], [16], [7], [10]\) and \([9]\) for details. We consider the vector space \(I^{\varepsilon, k} := C e_\varepsilon \oplus \cdots \oplus C e_k\) with the basis \(e_\varepsilon, \ldots, e_k\) for \(\varepsilon = 0, 1\) and \(k \in \mathbb{Z}_{\geq 1}\) Moreover, we consider the nilpotent endomorphism \(N\) of \(I^{\varepsilon, k}\) so that \(N e_j = e_{j-1}\) (with \(e_{-1} := 0\)). For a \(D\)-module \(M\) on \(X \times \mathbb{C}^n\) with the assumption \(M = M[*D_1]\), we consider the new \(D\)-module \(I^{\varepsilon, k} : M \otimes C I^{\varepsilon, k}\) on \(X \times \mathbb{C}^n\) so that \(z_1 \partial_{z_1} (m \otimes e_j) = z_1 \partial_{z_1} m \otimes e_j + m \otimes e_{j-1}\) and \(\partial_{z_2} (m \otimes e_j) = \partial_{z_2} m \otimes e_j (s \neq 1)\). The morphism \(I^{0, k} \rightarrow I^{1, k}\) induces the morphism \(M^{0,k} \rightarrow M^{1,k}\). Therefore, we can consider the morphism \(M^{0,k}[!)D_1] \rightarrow M^{1,k}[*)D_1\) as the
composition of the morphisms $M^{0,k}[!D_1] \to M^{0,k}[*D_1]$ and $M^{0,k}[*D_1] \to M^{1,k}[*D_1]$. Then, the kernel of the natural morphism

$$M^{0,k}[!D_1] \to M^{1,k}[*D_1]$$

does not depend on sufficiently large $k \geq 1$, i.e. the inductive limit $\varprojlim_k \text{Ker}(M^{0,k}[!D_1] \to M^{1,k}[*D_1])$ exists. So, we define

$$\Xi_{z_1}M := \varprojlim_k \text{Ker}(M^{0,k}[!D_1] \to M^{1,k}[*D_1]).$$

We can generalize this construction to mixed Hodge modules; for a mixed Hodge module $M$ on $X \times \mathbb{C}^n$, we can define a mixed Hodge module $M^{\varepsilon,k}$ on $X \times \mathbb{C}^n$, a morphism $M^{0,k}[!D_1] \to M^{1,k}[*D_1](-1)$ and $\Xi_{z_1}M = \varprojlim_k \text{Ker}(M^{0,k}[!D_1] \to M^{1,k}[*D_1])$, which are compatible with the corresponding objects for the underlying $D$-module of $M$. Note that the filtered $D$-module $(\Xi_{z_1}M, F^\bullet \Xi_{z_1}M)$ depends only on the stupid localization $(M, F^\bullet M)(*D_1)$.

**Fact 2.13** (see loc. cit.). Let $M = (M, F^\bullet M, K, W^\bullet M)$ is a mixed Hodge module on $E = X \times \mathbb{C}^n$. Then, we have the following.

(i) There are natural morphisms between mixed Hodge modules

$$a: \psi_{z_1,1}M \to \Xi_{z_1}M, \text{ and } b: \Xi_{z_1}M \to \psi_{z_1,1}M(-1).$$

(ii) Consider the complex:

$$\psi_{z_1,1}M \to \Xi_{z_1}M \oplus \phi_{z_1,1}M \to \psi_{z_1,1}M(-1), \quad (25)$$

where the first morphisms is $a \oplus \text{can}$ and the second morphism is $b + \text{var}$. Then, the cohomology in the middle degree of this complex at $\Xi_{z_1}M \oplus \phi_{z_1,1}M$ is isomorphic to $M$.

(iii) Let $\text{Glue}(X \times \mathbb{C}^n, D_1)$ be the category of tuples $(M', M''_c, v)$, where $M'$ is a mixed Hodge module on $X \times V_1$ which is the restriction of a mixed Hodge module on $X \times \mathbb{C}^n$, $M''_c$ is a mixed Hodge module on $X \times D_1$ and $c$ (resp. $v$) is a morphism $\psi_{z_1,1}M' \to M''_c$ (resp. $M''_c \to \psi_{z_1,1}M'(-1)$) such that the endomorphism $v \circ c$ of $\psi_{z_1,1}M'$ is the nilpotent endomorphism of $\psi_{z_1,1}M'$. Then, the functor $M \mapsto (M|_{X \times V_1}, \phi_{z_1,1}M, \text{can}, \text{var})$ induces an equivalence of categories between the category of mixed Hodge modules and $\text{Glue}(X \times \mathbb{C}^n, D_1)$.

If $M$ is monodromic, it is easy to see that if $M^{\varepsilon,k}$ is also monodromic. Hence, $M^{\varepsilon,k}[*D_1]$ and $M^{\varepsilon,k}[!*D_1]$ are also monodromic. Therefore, $\Xi_{z_1}M$ is also monodromic. Then, we have the following.

**Corollary 2.14.** For a monodromic mixed Hodge module $M$ on $X \times \mathbb{C}^n$, the Hodge filtration $F^\bullet \Xi_{z_1}M$ is decomposed as $F^\bullet \Xi_{z_1}M = \bigoplus_{\beta \in \mathbb{R}} F^\beta \Xi_{z_1}M \cap (\Xi_{z_1}M)^\beta$.

**Proof.** By Corollary 2.12 $F^pM^{\varepsilon,k}[!*D_1]$ and $F^pM^{\varepsilon,k}[*D_1]$ are decomposed. Therefore, $F^p \Xi_{z_1}M$, i.e. the kernel of the morphism $F^pM^{\varepsilon,k}[!*D_1] \to F^pM^{\varepsilon,k}[*D_1]$ for sufficiently large $k \geq 1$, is also decomposed. \qed
For the proof of Theorem 2.1, another lemma is needed.

**Lemma 2.15.** Let $M$ be a monodromic $D$-module on $X \times \mathbb{C}^n$ and $V^\bullet M$ the Kashiwara-Malgrange filtration along $z_1 = 0$. For $\gamma \in \mathbb{R}$ and a section $m \in V^\gamma M$, let $m$ be a decomposition $m = \sum_{k=1}^{k_0} m_k$, where $m_k$ is in $M^{\beta_k}$ for some $\beta_k \in \mathbb{R}$ with the condition $\beta_{k_1} \neq \beta_{k_2}$ for $k_1 \neq k_2$. Then, we have $m_k \in V^\gamma M$ for any $1 \leq k \leq k_0$.

**Proof.** For each $1 \leq k \leq k_0$, let $\delta_k \in \mathbb{R}$ be the biggest number such that $m_k \in V^{\delta_k} M$. We may assume that $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_{k_0}$. In particular, we have $m_k \in V^{\delta_k}_1 M$ for any $k$. If $\delta_1 \geq \gamma$, the claim is obvious, so we suppose $\delta_1 < \gamma$. For sufficiently large $l_1 \geq 0$, $(z_1 \partial_{z_1} - \delta_1)^{l_1} m_k$ is in $V^{\delta_1}_{z_1} M$ for any $k$. On the other hand, since $m_k$ is in $M^{\delta_k}$, we can take sufficiently large $l_2 \geq 0$ such that $(\mathcal{E} - \beta_k)^{l_2} m_k = 0$ for any $k$. Set $\mathcal{E}' := \sum_{i \geq 2} z_i \partial_{z_i}$. Then, we have

$$\sum_{j=0}^{l_2} C_{l_2,j}(z_1 \partial_{z_1} - \delta_1)^j (\mathcal{E}' - (\beta_k - \delta_1))^{l_2-k} m_k = 0,$$

where $C_{l_2,j}$ are the binomial coefficients. Therefore, for any $k$, there exists a polynomial $H_k$ in $z_1 \partial_1, \ldots, z_n \partial_n$ such that

$$(\mathcal{E}' - (\beta_k - \delta_1))^{l_2} m_k = H_k(z_1 \partial_{z_1} - \delta_1) m_k.$$

Hence, there exists a polynomial $H_k'$ in $z_1 \partial_1, \ldots, z_n \partial_n$ such that

$$(\mathcal{E}' - (\beta_k - \delta_1))^{l_1} m_k = H_k'(z_1 \partial_{z_1} - \delta_1)^{l_1} m_k.$$

Therefore, the section $[m_k] \in \text{gr}^{\delta_1}_{V^\gamma} M$ is in $(\text{gr}^{\delta_1}_{V^\gamma} M)^{\beta_k - \delta_1}$, where we set

$$(\text{gr}^{\delta_1}_{V^\gamma} M)^{\beta_k - \delta_1} = \bigcup_{l \geq 1} \ker(\mathcal{E}' - (\beta_k - \delta_1))^l (\subset \text{gr}^{\delta_1}_{V^\gamma} M).$$

Moreover, since $\delta_1 < \gamma$, we have

$$\sum_{i=1}^{k_0} [m_k] = [m] = 0 \quad (\text{in } \text{gr}^{\delta_1}_{V^\gamma} M). \quad (26)$$

However, it is easy to check (in the same way as in the proof of Proposition 1.6) that $(\text{gr}^{\delta_1}_{V^\gamma} M)^{\beta} \cap (\text{gr}^{\delta_1}_{V^\gamma} M)^{\beta'} = 0$ for $\beta \neq \beta'$ (this is true even if we do not yet know that $\text{gr}^{\delta_1}_{V^\gamma} M$ is not monodromic). Combining this fact with (26), we have $[m_k] = 0$ in $\text{gr}^{\delta_1}_{V^\gamma} M$. However, this contradicts with $[m_1] \neq 0$. This completes the proof.

**Corollary 2.16.** For a monodromic $D$-module on $X \times \mathbb{C}^n$ and $\alpha \in (-1, 0]$, the $\alpha$-nearby cycle $\psi_{z_1, \alpha} M = \text{gr}^\alpha_{V^\gamma} M$ and the unipotent vanishing cycle $\phi_{z_1, 1} M = \text{gr}^{-1}_{V^\gamma} M$ are also monodromic on $X \times \mathbb{C}^{n-1}(= X \times \{z_1 = 0\} \times \mathbb{C}_{z_2} \times \cdots \times \mathbb{C}_{z_n})$.

**Proof.** For $\alpha \in [-1, 0]$, let $[m] \in \text{gr}^\alpha_{V^\gamma} M$ be a section represented by a section $m \in V^\alpha M$. By the previous lemma, we can decompose $m$ as

$$m = \sum_{k=1}^{k_0} m_k,$$
where $m_k$ is in $M^{\beta_k}$ for some $\beta_k \in \mathbb{R}$ and $V^\alpha_{z_1} M$. Therefore, it is enough to see $[m_k]$ is killed by some power of $(\ell' - \delta)$ for some $\delta \in \mathbb{R}$. In the same way as in the proof of Lemma 2.15, there is an integer $l \geq 0$ and a polynomial $H_k$ in $z_1 \partial_1, \ldots, z_n \partial_n$ such that
\[(\ell' - (\beta_k - \alpha))^l m_k = H_k(z_1 \partial_1 - \alpha)m_k.\]
Since $[m_k]$ is killed by some power of $z_1 \partial_1 - \alpha$, this implies that $[m_k]$ is in $(\text{gr}^\alpha_{z_1} M)^{\beta_k-\alpha}$, and the proof is complete.

**Proof of Theorem 2.1.** If $n = 1$, the assertion is true by Proposition 1.11. We use the induction on $n$. Consider the case $n \geq 2$. By (ii) of Fact 2.13, $M$ is isomorphic to the cohomology in the middle degree of the complex (25). By Corollary 2.14 and Corollary 2.16 with the inductive assumption, all the terms of (25) are monodromic and the Hodge filtrations are decomposed with respect to the decomposition of the underlying $D$-modules. Hence, so is its cohomology in the middle degree. This completes the proof.

3 The Fourier-Laplace transform of a monodromic mixed Hodge module

In this section, we consider the Fourier-Laplace transform of a monodromic mixed Hodge module.

3.1 The Fourier-Laplace transform of a $D$-module

First, let us recall the notion of the Fourier-Laplace transform of a $D$-module. We refer to [2]. Let $X$ be a smooth algebraic variety, $\pi: E \to X$ an algebraic vector bundle on $X$, $\pi^\vee: E^\vee \to X$ the dual vector bundle of $E$ and $\varphi: E \times_X E^\vee \to \mathbb{C}$ the pairing between $E$ and $E^\vee$. Moreover, let $\mathcal{E}^-\varphi$ be the integrable connection $(\Omega_{E \times_X E^\vee}, d - d\varphi)$; we regard it as a $D$-module. We denote by $p$, $q$ the projections $E \times_X E^\vee \to E$ and $E \times_X E^\vee \to E^\vee$. For a morphism $f: Y \to Z$ between the manifolds $Y$ and $Z$ and a complex of $D$-modules $N_1$ (resp. $N_2$) on $Y$ (resp. $Z$), let $f^!N_1$ be the pushforward of $N_1$ (which is denoted by $\int f^! N_1$ in [6]), and $f^!N_2$ the pullback of $N_2$, which is also expressed as $Lf^!N_2[\dim Y - \dim Z]$. Recall that $f^!$ (resp. $f^!$) corresponds to $Rf_!$ (resp. $f^!$) under the Riemann-Hilbert correspondence.

**Definition 3.1.** For a $D$-module $M$ on $E$, we define the Fourier-Laplace transform $M^\wedge$ as
\[M^\wedge = q!(p^* M \otimes_{\Omega_{E \times_X E^\vee}} \mathcal{E}^-\varphi).\]

**Remark 3.2.** Since $p$ is a projection, we have $H^j p^!M = 0$ ($j \neq -n$) and
\[H^{-n} p^! M = H^0 Lp^* M = p^* M (= \Omega_{E \times_X E^\vee} \otimes_{p^{-1} \Omega_E} p^{-1} M).\]

**Remark 3.3.** It is not difficult to see that 0-th cohomology is the only non-trivial cohomology of $q!(p^* M \otimes_{\Omega_{E \times_X E^\vee}} \mathcal{E}^-\varphi)$. Therefore, we will sometimes write $M^\wedge$ for $H^0 q!(p^* M \otimes_{\Omega_{E \times_X E^\vee}} \mathcal{E}^-\varphi)$. It is known that $(\cdot)^\wedge$ defines an exact functor.
Let us consider the projective version of the above definition. Define \( \widetilde{E} \) (resp. \( \widetilde{E}^\vee \)) as the projective compactification of \( E \) (resp. \( E^\vee \)) i.e. the projective bundle of the direct sum of \( E \) (resp. \( E^\vee \)) and the trivial bundle over \( X \). We use the same symbol \( \pi \) and \( \pi^\vee \) for their projection to \( X \). Moreover, we denote by \( j: E \hookrightarrow \widetilde{E} \) (resp. \( j^\vee: E^\vee \hookrightarrow \widetilde{E}^\vee \)) the inclusion of \( E \) (resp. \( E^\vee \)) to \( \widetilde{E} \) (resp. \( \widetilde{E}^\vee \)) and \( D_\infty \) (resp. \( D_\infty^\vee \)) the divisor \( \widetilde{E} \setminus E \) (resp. \( \widetilde{E}^\vee \setminus E^\vee \)). We use the same symbol \( D_\infty \) (resp. \( D_\infty^\vee \)) for the divisor \( D_\infty \times_X E^\vee \) (resp. \( E \times_X D_\infty^\vee \)) of \( E \times_X E^\vee \). Let \( \tilde{p} \) (resp. \( \tilde{q} \)) be the projection \( \widetilde{E} \times_X \widetilde{E}^\vee \to \widetilde{E} \) (resp. \( \widetilde{E} \times_X \widetilde{E}^\vee \to \widetilde{E}^\vee \)) and \( \varphi \) the rational function on \( \widetilde{E} \times_X \widetilde{E}^\vee \) defined as the pairing of \( \widetilde{E} \) and \( \widetilde{E}^\vee \), whose pole divisor is \( D_\infty \cup D_\infty^\vee \) (we use the same symbol as \( \varphi: E \times_X E^\vee \to \text{C} \)). Let \( \mathfrak{e}^{-\varphi} \) be the meromorphic connection \( (O_{\widetilde{E} \times_X \widetilde{E}^\vee}(*D_\infty \cup D_\infty^\vee), d - d_\varphi) \).

**Definition 3.4.** For a \( D \)-module \( N \) on \( \widetilde{E} \), we define the Fourier-Laplace transform \( N^\wedge \) as

\[
N^\wedge = H^0 \tilde{q}_! (\tilde{p}^! N \otimes \mathfrak{e}^{-\varphi}).
\]

Since our \( D \)-modules are algebraic, \( N^\wedge \) is expressed as follows.

**Lemma 3.5.** For a \( D \)-module \( N \) on \( \widetilde{E} \), we have

\[
N^\wedge \simeq j^\vee_* (N|_E)^\wedge.
\]

**Proof.** By the definition of \( \mathfrak{e}^{-\varphi} \), we have

\[
\tilde{p}^! N \otimes \mathfrak{e}^{-\varphi} = (j \times j^\vee)_* (p^! (N|_E) \otimes \mathfrak{e}^{-\varphi}).
\]

Therefore, we obtain

\[
N^\wedge \simeq H^0 \tilde{q}_! ((j \times j^\vee)_* (p^! (N|_E) \otimes \mathfrak{e}^{-\varphi})).
\]

\[
\simeq j^\vee_* H^0 q_! (p^! (N|_E) \otimes \mathfrak{e}^{-\varphi})
\]

\[
= j^\vee_* (N|_E)^\wedge.
\]

\( \square \)

Let us consider the case when \( X \) is affine and \( E \) is trivial, i.e. \( E \simeq X \times \text{C}^n \). Let \( (z_1, \ldots, z_n) \) be the standard coordinates of \( \text{C}^n \) (we sometimes write \( \text{C}^n_\mathbb{C} \) to emphasize the coordinates), \( \text{C}^n_\mathbb{C} \) the dual vector space of \( \text{C}^n_\mathbb{C} \), \( (\zeta_1, \ldots, \zeta_n) \) the dual coordinates of \( \text{C}^n_\mathbb{C} \). Then, we have \( E^\vee \simeq X \times \text{C}^n_\mathbb{C} \). Remark that we can identify a \( D \)-module \( M \) with the \( \Gamma(E; D_E) \)-module \( \Gamma(E; M) \). Recall that since \( q \) is a projection, the pushforward \( q_! \) is described in terms of the relative de Rham complex (see Proposition 1.5.28 of \[6\]). The following is well-known.

**Lemma 3.6.** (i) There is a ring isomorphism \( \Gamma(E^\vee, D_{E^\vee}) \simeq \Gamma(E; D_E) \) which sends \( P \in D_X \) to the same element \( P \) and \( \zeta_i \) (resp. \( \partial_{\zeta_i} \)) to \( \partial_{z_i} \) (resp. \( -z_i \)).

(ii) For a \( D \)-module \( M \) on \( E \), the Fourier-Laplace transform \( M^\wedge \) is \( M \) as a \( \text{C} \)-module and its \( \Gamma(E^\vee, D_{E^\vee}) \)-module structure is induced from the original \( \Gamma(E; D_E) \)-module structure via the isomorphism \( \Gamma(E^\vee, D_{E^\vee}) \simeq \Gamma(E; D_E) \).
We will introduce a similar statement for $R$-modules in the next section (Lemma 4.19). We can prove this lemma in the same way as the proof written there. We write $m^\wedge$ for the section of $m^\wedge$ corresponding to $m \in M$. By this lemma, we have
\[
\begin{align*}
\zeta_i \cdot m^\wedge &= (\partial_{z_i} m)^\wedge \\
\partial z_i \cdot m^\wedge &= -(z_i m)^\wedge.
\end{align*}
\] (27)

If we take two trivializations $\varphi_i : E \simeq X \times \mathbb{C}^n$ ($i = 1, 2$) and a section $m \in M$, the section $m^\wedge$ for the trivialization $\varphi_1$ (we write $(m^\wedge)_1$ for it) does not coincide with $m^\wedge$ for $\varphi_2$ (we write $(m^\wedge)_2$ for it), i.e. "$m^\wedge$" depends on the choice of the trivialization. However, they are equal up to a multiplicative factor, i.e. there is a holomorphic function $A(x) \in \Gamma(X; \mathcal{O}_X) \subset \Gamma(E; \mathcal{O}_E)$ such that we have
\[ (m^\wedge)_2 = A(x)(m^\wedge)_1. \]

Therefore, for an $\mathcal{O}_X$-submodule $F$ of $\pi_* M$, the $\mathcal{O}_X$-submodule
\[ F^\wedge := \{ m^\wedge \in \pi_* M^\wedge \mid m \in F \} \quad (28) \]
of $\pi_* M^\wedge$ does not depend on the choice of the trivialization $E \simeq X \times \mathbb{C}^n$. Hence, the following definition is well-defined.

**Definition 3.7.** For a $D$-module $M$ on $E$ ($E$ is not necessary trivial) and an $\mathcal{O}_X$-submodule $F$ of $\pi_* M$, we define an $\mathcal{O}_X$-submodule
\[ F^\vee := \{ m^\vee \in \pi_* M^\vee \mid m \in F \} \]
of $\pi_* M^\vee$ so that for any local trivialization $\pi^{-1}(U) \simeq U \times \mathbb{C}^n$ ($U \subset X$ is affine) we have
\[ (F^\vee)|_U = (F|_U)^\wedge, \]
where the RHS is the one defined by (28).

### 3.2 The Fourier-Laplace transform of a monodromic mixed Hodge module

Next, we consider a monodromic $D$-module $M$ on a (not necessary trivial) vector bundle $E$. We use the notation defined in the previous subsection. Recall that $M^\beta$ is defined as
\[ M^\beta = \bigcup_{l \geq 0} \text{Ker}((\mathcal{E} - \beta)^l) \subset \pi_* M, \]
where $\mathcal{E}$ is the Euler vector field on $E$.

**Proposition 3.8.** If $M$ is monodromic, then so is $M^\wedge$. Moreover, we have
\[ (M^\wedge)^\beta = (M^{-\beta-n})^\wedge, \]
as $\mathcal{O}_X$-modules for any $\beta \in \mathbb{R}$, where the RHS is defined by Definition 3.7.

**Proof.** We may assume $X$ is affine and $E$ is trivial, i.e. $E \simeq X \times \mathbb{C}^n$. Then, we use the description by Lemma 3.6. Consider a section $m^\wedge \in M^\wedge$ for $m \in M^\beta$. We denote by $\mathcal{E}^\vee$ the Euler vector field $\sum_{i=1}^n \zeta_i \partial_{z_i}$ on $E^\vee$. By (27), we have
\[ \mathcal{E}^\vee m^\wedge = ((-\mathcal{E} - n) m)^\wedge. \]
Therefore, $(\mathcal{E}^\vee + n + \beta)^l m^\wedge$ is zero for some $l \geq 0$. This implies the assertion. \qed
If \( M \) is a holonomic \( D \)-module, so is \( M^\wedge \). On the other hand, \( M^\wedge \) may not be regular in general even if \( M \) is regular since \( \mathcal{E}^\phi \) is not regular at infinity. Hence, \( M^\wedge \) may not be the underlying \( D \)-module of a mixed Hodge module. In general, it is an underlying \( D \)-module of a mixed twistor \( D \)-module (see Subsection 3.2). Nevertheless, we have the following.

**Lemma 3.9** (Théorème 7.24 of Brylinski [2]). *If \( M \) is monodromic regular holonomic \( D \)-module, so is \( M^\wedge \).*

See also Proposition 1.24 of [20]. From this lemma, it may be possible to endow \( M^\wedge \) with a mixed Hodge module structure. In fact, when \( E \) is of rank 1, we constructed a mixed Hodge module whose underlying \( D \)-module is \( M^\wedge \) in Subsection 3.7 of [20].

**Lemma 3.10** (Subsection 3.7 of [20]). *Let \( M = (M,F_\bullet M,K,W_\bullet K) \) be a monodromic mixed Hodge module. Assume that \( E \) is of rank 1 and \( E \) is trivialized by an isomorphism \( E \simeq X \times \mathbb{C}_t \). Then, we can endow \( M^\wedge \) with a natural mixed Hodge module structure, i.e. we can define a good filtration \( F_p^\wedge M \) of \( M^\wedge \) and \((M^\wedge,F_\bullet^\wedge M)\) is the underlying filtered \( D \)-module of a mixed Hodge module \( M^\wedge \), with the following property: for \( \beta \in \mathbb{R} \) we have*

\[
F_p^\wedge (M^\wedge)^\beta := F_p^\wedge M^\wedge \cap (M^\wedge)^\beta = (F_{p+1+|\beta|} M^{-\beta-1})^\wedge, \tag{29}
\]

*under the isomorphism \((M^\wedge)^\beta = (M^{-\beta-1})^\wedge \) (Proposition 3.3).*

Let us recall the idea of this result and describe the Hodge filtration explicitly. Assume that \( E \) is of rank 1 and \( E \) is trivialized by an isomorphism \( E \simeq X \times \mathbb{C}_t \). Let us consider the object in \( \mathcal{G}(X) \) (defined in Subsection 1.2):

\[
((\phi_{t,1} M \oplus \psi_{t,1} \neq 1 M, 1 \oplus T_s^{-1}, \text{can o var } \oplus N), \psi_{t,1} M(-1), -\text{var}, \text{can}), \tag{30}
\]

where \( T_s \) (resp. \( N \)) is the semisimple part (resp. \( \frac{1}{2\pi i} \) times the logarithm of the unipotent part) of the monodromy automorphism. By Proposition 1.11, we get a mixed Hodge module which will be denoted by \( M^\wedge \) on \( X \times \mathbb{C}_t \). One can see that the underlying \( D \)-module of \( M^\wedge \) is \( M^\wedge \). We set \( M^\wedge = (M^\wedge,F_\bullet^\wedge M^\wedge,K^\wedge,W_\bullet^\wedge K^\wedge) \). The perverse sheaf \( K^\wedge \) is the Fourier-Sato transform of \( K \). In the setting of Lemma 3.10 let \( M = \bigoplus_{\beta \in \mathbb{R}} M^\beta \) and \( M^\wedge = \bigoplus_{\beta \in \mathbb{R}} (M^\wedge)^\beta \) be the decompositions. By Proposition 1.11 we have \( F_\bullet^\wedge M = \bigoplus_{\beta \in \mathbb{R}} F_\bullet M^\beta \) and \( F_\bullet^\wedge M^\wedge = \bigoplus_{\beta \in \mathbb{R}} F_\bullet (M^\wedge)^\beta \). By Proposition 3.25 of [20], we have (29) for \( \beta \in \mathbb{R} \).

**Remark 3.11.** In Proposition 3.25 of [20], only (29) for \( \beta \in [-1,0] \) is stated. However, it is easy to verify that (29) holds for any \( \beta \in \mathbb{R} \) by the strict specializability.

**Remark 3.12.** There are other possible mixed Hodge modules whose underlying \( D \)-modules are \( M^\wedge \). In this paper, we always take the one which corresponds to (30) so that it coincides with the irregular Hodge filtration (see Theorem 1.39).

**Remark 3.13.** If we take two trivializations \( \varphi_i : E \simeq X \times \mathbb{C}_t \) \((i = 1, 2)\), we obtain two mixed Hodge modules \((\varphi_i)_! M^\wedge \) by Lemma 3.10 and get two Hodge module structures on \( M^\wedge = (\varphi_i)_! ((\varphi_i)_! M^\wedge) \) \((i = 1, 2)\). However, one can see that these coincide by using Remark 3.16 below. As a consequence, we can generalize Lemma 3.10 to the case of a (not necessary trivial) line bundle.
Remark 3.14. The correspondence $M \mapsto M^\wedge$ defines an exact functor between the categories of mixed Hodge modules on the line bundles $E$ and $E^\vee$. Moreover, this induces a functor between their derived categories.

We will generalize Lemma 3.10 to the case of a (not necessary trivial) line bundle. To that end, we will express the Fourier-Laplace transformation on a vector bundle of any rank in terms of the Fourier-Laplace transform on some vector bundle of rank 1 with some functors between the categories of $D$-modules. Note that we want to take those that fit the theory of mixed Hodge modules as those functors. Therefore, for example, as a pullback functor for a morphism $f$, we use $f^\dagger$ not $f^*$ in the following.

We will use the following lemmas.

Lemma 3.15 (Corollaire 6.7 of Brylinski [2]). Let $E$ and $F$ be a vector bundle over $X$ and $f : E \to F$ a morphism of vector bundles. We denote by $t^f$ its transpose morphism $t^f : F^\vee \to E^\vee$ between the dual vector bundles. Then, for a $D$-module $M$ on $E$ there is a natural isomorphism between the complexes of $D_F^\vee$-modules

$$ (f^\dagger M)^\wedge \simeq (t^f)^\dagger M^\wedge [n_F - n_E], $$

where $n_E$ (resp. $n_F$) is the rank of $E$ (resp. $F$).

Remark 3.16. Let $\varphi_i : E \simeq X \times \mathbb{C}^n (i = 1, 2)$ be two trivializations of a trivial vector bundle $E$. Then, $\varphi_1 \circ \varphi_2^{-1}$ is an isomorphism between vector bundles. The $D$-module structure of $((\varphi_1)^\dagger M)^\wedge$ is described by Lemma 3.15. In this case, the $D$-module $((\varphi_1)^\dagger M)^\wedge$ is isomorphic to $(t^f(\varphi_1 \circ \varphi_2^{-1}))^\dagger((\varphi_2)^\dagger M)^\wedge$ through the natural morphism in Lemma 3.15.

Lemma 3.17 (Corollaire 6.7 of Brylinski [2]). Let $X$ and $Y$ be smooth algebraic varieties, $f : Y \to X$ a morphism and $E$ a vector bundle over $X$. We denote by $u$ (resp. $u^\vee$) the natural morphism from the pullback vector bundle $f^* E$ (resp. $f^* E^\vee (= (f^* E)^\vee)$) of $E$ (resp. $E^\vee$) by $f$ to $E$ (resp. $E^\vee$). Then, for a $D$-module $M$ on $X$ we have a natural isomorphism

$$(u^\dagger M)^\wedge \simeq (u^\vee)^\dagger M^\wedge.$$
Similarly, we can regard the second projection $E^\vee \times_X E^\vee \to E^\vee$ as the base change of $E^\vee$ by $E^\vee \to X$:

\[
\begin{array}{ccc}
E^\vee \times_X E^\vee & \longrightarrow & E^\vee \\
\downarrow & & \downarrow \\
E^\vee & \longrightarrow & X.
\end{array}
\]

This morphism $E^\vee \times_X E^\vee \to E^\vee$ is $p^\vee$. Let $\iota$ be the inclusion

\[
\iota: E^\vee \simeq \{1\} \times E^\vee \hookrightarrow C^\vee \times E^\vee.
\]

Then, we have the following.

**Lemma 3.18** (Proposition 6.11 of Brylinski [2]). For a $D$-module $M$ on $E$ we have a natural isomorphism

\[
M^\wedge \simeq H^1 \iota^\dagger((H^0 \omega_1 H^{-n} p^\dagger M)^\wedge).
\]

**Proof.** In the derived category of $D$-modules, we have

\[
\iota^\dagger(\omega_1 p^\dagger M)^\wedge[1 - n] \simeq \iota^\dagger(\iota(\omega)(p^\dagger M)^\wedge \quad \text{(by Lemma 3.15)}
\]

\[
\simeq \iota^\dagger(\iota(\omega)(p^\dagger))^\wedge \quad \text{(by Lemma 3.17)}
\]

\[
\simeq M^\wedge. \quad (33)
\]

Note that we have $M^\wedge = H^0 M^\wedge$ and $p^\dagger M = H^{-n} p^\dagger M[n]$. Therefore, the $j$-th cohomology of the complex $(\omega_1 p^\dagger M)^\wedge \simeq (\iota(\omega)(p^\dagger M)^\wedge[n - 1]$ is 0 for $j > -n$, and hence by (33) we have

\[
M^\wedge = H^0 M^\wedge
\]

\[
\simeq H^{1-n} \iota^\dagger(\omega_1 p^\dagger M)^\wedge
\]

\[
\simeq H^1 \iota^\dagger H^{-n} (\omega_1 p^\dagger M)^\wedge
\]

\[
\simeq H^1 \iota^\dagger (H^0 \omega_1 H^{-n} p^\dagger M)^\wedge.
\]

By Proposition 3.18, the Fourier-Laplace transformation on a vector bundle of any rank can always be expressed in terms of the Fourier-Laplace transform on some vector bundle of rank 1. The following lemma was essentially shown in [2]. For convenience, we present a proof.

**Lemma 3.19** (Brylinski [2]). If $M$ is monodromic, so is $H^0 \omega_1 H^{-n} p^\dagger M$ on $C \times E^\vee$. In particular, $(H^0 \omega_1 H^{-n} p^\dagger M)^\wedge$ is monodromic on $C^\vee \times E^\vee$.

**Proof.** It is enough to show the assertion under the assumption that $E$ is trivial and $X$ is one point: $E \simeq \mathbb{C}^n$. We express any object in an algebraic way. Let $z = (z_1, \ldots, z_n)$ be the coordinates of $\mathbb{C}^n$ and $\zeta = (\zeta_1, \ldots, \zeta_n)$ its dual coordinates. Note that we have

\[
H^{-n} p^\dagger M \simeq \mathbb{C}[z, \zeta] \otimes_{\mathbb{C}[z]} M.
\]

26
We decompose $\omega$ into

$$i_\omega: E \times E \rightarrow \mathbb{C}_s \times E \times E^\vee$$

$$(z, \zeta) \mapsto (\langle z, \zeta \rangle, z, \zeta)$$

and

$$p_\omega: \mathbb{C}_s \times E \times E \rightarrow \mathbb{C}_s \times E^\vee$$

$$(s, z, \zeta) \mapsto (s, \zeta).$$

Then, we have

$$H^0(i_\omega)_! H^{-n}p_! M \simeq H^0(p_\omega)_! H^0(i_\omega)_! H^{-n}p_! M.$$ 

Set $N := H^0(i_\omega)_! H^{-n}p_! M$. Then, we can express $N$ as

$$N \simeq (\mathbb{C}[z, \zeta] \otimes_{\mathbb{C}[z]} M) \otimes_{\mathbb{C}} \mathbb{C}[\partial_z].$$

Let $\text{DR}_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee}(N)$ be the relative de Rham complex:

$$N \rightarrow \Omega^1_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee} \otimes N \rightarrow \cdots \rightarrow \Omega^n_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee} \otimes N,$$

where $\Omega^i_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee} \otimes N$ is in degree 0. Then, we can express $H^0(p_\omega)_! N$ as

$$H^0(p_\omega)_! N = H^0(p_\omega)_! (\text{DR}_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee}(N)),$$

i.e. the cokernel of the morphism

$$\Omega^{n-1}_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee} \otimes N \rightarrow \Omega^n_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee} \otimes N.$$

Set $dz := dz_1 \wedge \cdots \wedge dz_n$. We fix the isomorphism $\Omega^n_{\mathbb{C}_s \times E \times E^\vee / \mathbb{C}_s \times E^\vee} \simeq O_{\mathbb{C}_s \times E \times E^\vee} dz$. Then, a section of $H^0(p_\omega)_! N$ can be represented by a sum of some sections in the form:

$$dz \otimes f(\zeta) \otimes m \otimes \partial^l_s,$$

where $f(\zeta) \in \mathbb{C}[\zeta]$, $m \in M$ and $l \in \mathbb{Z}_{\geq 0}$. Note that a section of $H^0(p_\omega)_! N$ in the form:

$$dz \otimes \partial_{z_k}(f(\zeta) \otimes m \otimes \partial^l_s) = dz \otimes f(\zeta) \otimes \partial_{z_k} m \otimes \partial^l_s + dz \otimes (-\zeta_k f(\zeta) \otimes m \otimes \partial^l_{s+1})$$

is zero. Therefore, we have

$$[dz \otimes f(\zeta) \otimes \mathcal{E}_m \otimes \partial^l_s] = [dz \otimes (z, \zeta)(f(\zeta) \otimes m \otimes \partial^l_s)],$$

in $H^0(p_\omega)_! N$, where $\mathcal{E} = \sum_{i=1}^n z_i \partial_{z_i}$. Moreover, since

$$(s \partial_s + l + 1)\partial^l_s = \partial^{l+1}_s,$$

we have

$$(s \partial_s + l + 1)[dz \otimes f(\zeta) \otimes m \otimes \partial^l_s] = dz \otimes (z, \zeta)(f(\zeta) \otimes m \otimes \partial^{l+1}_s)$$

$$= dz \otimes f(\zeta) \otimes \mathcal{E}_m \otimes \partial^l_s \quad \text{(by (34))}.$$
By the assumption that $M$ is monodromic, there exists a polynomial $b(u) \in \mathbb{C}[u]$ such that $b(\mathcal{E})m = 0$. Hence, we obtain

$$b(s \partial_s + l + 1)[dz \otimes f(\zeta) \otimes m \otimes \partial \delta_s] = [dz \otimes f(\zeta) \otimes b(\mathcal{E})m \otimes \partial \delta_s] = 0,$$

in $H^0(p_\omega) \cap N$. We thus conclude that $H^0\omega \mathcal{E}^{-n}p^\perp M$ is monodromic on $\mathbb{C}_s \times E^\vee$ and this completes the proof. 

$H^0\omega \mathcal{E}^{-n}p^\perp M$ is an object in the derived category of mixed Hodge modules on a line bundle $\mathbb{C} \times E^\vee$ over $E^\vee$, where we use the same symbols $H^0\omega$ and $H^0\mathcal{E}$ as the functors between the categories of mixed Hodge modules. Moreover, by Lemma 3.19 $H^0\omega \mathcal{E}^{-n}p^\perp M$ is a monodromic mixed Hodge module. Therefore, by Lemma 3.18 we have $M^\wedge \simeq H^1\iota^!(\mathcal{H}^0\omega \mathcal{E}^{-n}p^\perp M)^\wedge$.

**Definition 3.20.** Let $p: E \to X$ be a vector bundle whose rank is greater than 2 and $M$ a monodromic mixed Hodge module on $E$. Then, we define a mixed Hodge module $M^\wedge$ whose underlying $D$-module is $M^\wedge$ as

$$M^\wedge := H^1\iota^!(\mathcal{H}^0\omega \mathcal{E}^{-n}p^\perp M)^\wedge(1).$$

(35)

**Remark 3.21.** The Tate twist “(1)” is needed so that Theorem 4.39 below holds.

It is not easy to compute the Hodge filtration of $M^\wedge$ directly from the definition because the pushforward of the mixed Hodge module is complicated object in general. However, in the next section, we will compare the Hodge filtration $F_* M^\wedge$ with the irregular Hodge filtration (see Theorem 4.39), and by virtue of it, we will get a concrete description of the Hodge filtration of $M^\wedge$ (Corollary 4.40).

## 4 Irregular Hodge filtrations

### 4.1 Irregular Hodge filtrations

As mentioned in the previous section, the exponentially twisted module $\mathcal{E}^{-\varphi}$, in particular the Fourier-Laplace transform of a regular holonomic $D$-module, is not always regular, and the Fourier-Laplace transform is not always equipped with any mixed Hodge module structure since the underlying $D$-module of a mixed Hodge module is regular. Nevertheless, we endowed a natural mixed Hodge module structure on the Fourier-Laplace transform of the underlying $D$-module of a monodromic mixed Hodge module (Lemma 3.10 and Definition 3.20). On the other hand, Esnault-Sabbah-Yu [5], Sabbah-Yu [17] defined a natural filtration called the irregular Hodge filtration on the exponentially twisted module, in particular the Fourier-Laplace transform of the underlying $D$-module of a mixed Hodge module, which are a generalization of the filtration on the twisted de Rham cohomologies defined by Deligne [4], Yu [23] and
Example 4.1. Let $(\mathcal{D})$ be the same symbols as in the theory of $\mathcal{D}$-functors, localizations and Beilinson’s gluing, even for we can define the 6-operations, the Kashiwara-Malgrange filtrations, the nearby-vanishing $R_{\lambda}$, for a $\mathcal{D}$-module of the filtered $\lambda$-modules. Moreover, we denote by $\mathcal{R}_{\lambda}$ the sheaf of subalgebras in $\mathcal{D}$ generated by $\mathcal{R}_{\lambda}$ and a divisor $\Theta_{\lambda}$, which is a section of $\mathcal{D}$-modules to the theory of $\mathcal{D}$-modules (for example, like $f^{-1}$ and $f^{!}$). In particular, for an $\mathcal{R}_{\lambda}$-module $\mathcal{M}$ and a divisor $D \subset X$, we can define a localization (resp. dual localization) $\mathcal{M}[\ast D]$ (resp. $\mathcal{M}[\ast !D]$) of $\mathcal{M}$ along $D$, which has the same properties as described in Fact 2.3 (see loc. cit.). Remark that $\mathcal{M}[\ast D]$ is not equal to the naive localization $\mathcal{M}(\ast D) = \mathcal{M} \otimes_{\mathcal{O}_{X \times \mathbb{C}}{\mathcal{D}}^{\ast}} \mathcal{O}_{X \times \mathbb{C}}{\mathcal{D}}^{\ast D}$ in general. If $\mathcal{M}$ is the Rees module $\mathcal{R}_{\lambda}M$ of a filtered $\mathcal{D}$-module $(M, F_{\lambda}M)$ (see example 1.1), $\mathcal{R}_{\lambda}M[\ast D]$ (resp. $\mathcal{R}_{\lambda}M[\ast !D]$) coincides with the Rees module of the filtered $\mathcal{D}$-module $(M[\ast D], F_{\lambda}M[\ast D])$ (resp. $(M[\ast !D], F_{\lambda}M[\ast !D])$). Moreover, the strict ($\mathbb{Q}$ or $\mathbb{R}$)-specializability explained in Definition 1.2 can also be generalized for $\mathcal{M}$-modules. Moreover, we have the notion of holonomicity for $\mathcal{M}$-modules.

The category $\text{MTM}^\text{int}_{\text{good}}(X; \mathbb{Q})$ of integrable mixed twistor $\mathcal{D}$-modules with good $\mathbb{Q}$-structures, introduced by Mochizuki [7], contains the category $\text{MHM}(X)$ of mixed Hodge modules as a full subcategory, which is a generalization of the category of pure twistor $\mathcal{D}$-modules introduced by Sabbah [13] and the category of mixed twistor structures introduced by Simpson [22]. Moreover, Sabbah [15] defined an abelian full subcategory $\text{IrrMHM}(X; \mathbb{Q})$, called the category of irregular mixed Hodge modules (with $\mathbb{Q}$-structures), of $\text{MTM}^\text{int}_{\text{good}}(X; \mathbb{Q})$ which contains $\text{MHM}(X)$. We can write

$$\text{MHM}(X) \subset \text{IrrMHM}(X; \mathbb{Q}) \subset \text{MTM}^\text{int}_{\text{good}}(X; \mathbb{Q}).$$
We will explain them in a little more detail.

A mixed twistor $D$-module $\mathcal{F} \in MTM_{\text{good}}(X; \mathbb{Q})$ is a pair of $R_X^{\text{int}}$-modules $\mathcal{M}_1$, $\mathcal{M}_2$, a sesqui-linear pairing $C$ of $\mathcal{M}_1$ and $\mathcal{M}_2$ and a weight filtration with a $\mathbb{Q}$-structure satisfying some conditions. For a mixed Hodge module $M = (\mathcal{M}, F, M, K, W, K)$, we can construct a natural mixed twistor $D$-module $\mathcal{F} = (\mathcal{M}_1, \mathcal{M}_2, C) \in MTM_{\text{good}}(X; \mathbb{Q})$ such that $\mathcal{M}_2 = R_F M$ (see Proposition 13.5.4 of [7]), and this construction defines the inclusion $\text{MHM}(X) \subset MTM_{\text{good}}(X; \mathbb{Q})$ above (i.e. a fully faithful exact functor $\text{MHM}(X) \hookrightarrow MTM_{\text{good}}(X; \mathbb{Q})$).

**Remark 4.2.** In [7], the “underlying $R$-module” of an algebraic mixed twistor $D$-module $\mathcal{F}$ on $X$ is an $R(\ast H)$-module $\mathcal{M}$ on a compactification $\overline{X}$ of $X$, where we set $H := \overline{X} \setminus X$ (see Definition 14.1.1). However, since we believe the difference in the terminology will not cause any confusion, we call $\mathcal{M} := \mathcal{M}|_X$ the underlying $R$-module in this paper.

**Remark 4.3.** In the following, we do not consider the weight filtrations and the $\mathbb{Q}$-structures of mixed twistor $D$-modules. So, we forget them and treat an object in $MTM_{\text{good}}(X; \mathbb{Q})$ as a $R$-triple $(\mathcal{M}_1, \mathcal{M}_2, C)$ with some conditions.

**Notation 4.4.** Let $Y$ be another smooth algebraic variety and $f : X \rightarrow Y$ be a morphism. In Section 14 of [7], the functors

$$T_{f_!}, T_{f!} : D^bMTM_{\text{good}}(X; \mathbb{Q}) \rightarrow D^bMTM_{\text{good}}(Y; \mathbb{Q})$$

$$T_{f*}, T_{f!} : D^bMTM_{\text{good}}(Y; \mathbb{Q}) \rightarrow D^bMTM_{\text{good}}(X; \mathbb{Q})$$

are defined, each of which is compatible with the corresponding functor in the theory of mixed Hodge modules. For an underlying integrable $R$-module $\mathcal{M}$ of a mixed twistor $D$-module $\mathcal{F}$ on $X$, we denote by $T_{f_!} \mathcal{M}$ the underlying complex of integrable $R$-modules of $T_{f_!} \mathcal{F}$. $T_{f!} \mathcal{M}$, $T_{g*} \mathcal{N}$ and $T_{g!} \mathcal{N}$ are defined in the same way for an underlying $R$-module of a mixed twistor $D$-module $\mathcal{N}$ on $Y$.

**Remark 4.5.** For a morphism $f : X \rightarrow Y$ and an underlying $R$-module of a mixed twistor $D$-module on $X$, the object $T_{f_!} \mathcal{M}$ is not the same “the $D$-module theoretical pushforward” $f_! \mathcal{M}$ in general, even though the underlying (complex of) $D$-modules are equal. If $f$ is projective, we have $T_{f_!} \mathcal{M} = f_! \mathcal{M}$. In the general case, we first take a smooth variety $\overline{X}$ (resp. $\overline{Y}$) containing $X$ (resp. $Y$) such that $H_X := \overline{X} \setminus X$ (resp. $H_Y := \overline{Y} \setminus Y$) is a divisor, and a proper morphism $\overline{f} : \overline{X} \rightarrow \overline{Y}$ which induces $f : X \rightarrow Y$. Moreover, let $\mathcal{M}$ be the underlying $R$-module of a mixed twistor $D$-module on $\overline{X}$ whose restriction $\mathcal{M}|_X$ is $\mathcal{M}$. Then, $T_{f_!} \mathcal{M}$ is expressed as

$$T_{f_!} \mathcal{M} = T_{\overline{f}_!} (\mathcal{M}|_{\ast H_X})|_Y.$$ 

The same result holds for $T_{f!} \mathcal{M}$. Similarly, $T_{f*}$ is neither “the scheme theoretical pullback” $f^*$ or “the $D$-module theoretical pullback” $f^!$ in general.

Next, we review about the rescaling of $R$-modules. We consider a complex plane $\mathbb{C}_\tau$ with the coordinates $\tau$ and set $\mathcal{X} := X \times \mathbb{C}_\tau$ and $\mathcal{X}_0 := X \times \{\tau = 0\}$. Let $j : X \times \mathbb{C}_\tau \times \mathbb{C}_\lambda \hookrightarrow \mathcal{X} \times \mathbb{C}_\lambda$ be the inclusion, $q : X \times \mathbb{C}_\tau^* \times \mathbb{C}_\lambda \rightarrow \mathcal{X} \times \mathbb{C}_\lambda$ the projection, and $\mu$ a morphism

$$\mu : X \times \mathbb{C}_\tau^* \times \mathbb{C}_\lambda \rightarrow X \times \mathbb{C}_\lambda \quad ((x, \tau, \lambda) \mapsto (x, \lambda/\tau)).$$

30
Then, for an (algebraic) $R^\text{int}_X$-module $\mathcal{M}$ (an object on $X \times \mathbb{C}_\lambda$) we consider the pullback $\mu^* \mathcal{M} = O_{X \times \mathbb{C}_\lambda} \otimes \mu^{-1} \mathcal{M}$ as $O$-module and its pushforward (as an algebraic object) $j_* \mu^* \mathcal{M}$ by $j$. The object $j_* \mu^* \mathcal{M}$ is an $O_{X \times \mathbb{C}_\lambda}(\{\tau = 0\})$-module and denoted by $\mathcal{M}$. Remark that in the analytic setting we have to modify the definition a bit not to use the pushforward by the open embedding $j$ (2.2.a of [15]), but in the algebraic setting our definition is enough. The $O_{X \times \mathbb{C}_\lambda}(\{\tau = 0\})$-module $\mathcal{M}$ can be endowed with a natural $R^\text{int}_X(\{\tau = 0\})$-module structure so that for a section $m \in \mathcal{M}$ and a vector field $\theta$ on $X$, we have

\begin{align*}
\lambda(1 \otimes m) &= \tau \otimes \lambda m, \\
\lambda \theta(1 \otimes m) &= \tau \otimes \lambda \theta m, \\
\partial_\tau(1 \otimes m) &= -1 \otimes \lambda^2 \partial_\lambda m, \quad \text{and} \\
\lambda^2 \partial_\lambda(1 \otimes m) &= \tau \otimes \lambda^2 \partial_\lambda m.
\end{align*}

(36)

This $R^\text{int}_X(\{\tau = 0\})$-module $\mathcal{M}$ is called the rescaling of $\mathcal{M}$. We say that $\mathcal{M}$ is well-rescalable if the $R^\text{int}_X(\{\tau = 0\})$-module $\mathcal{M}$ is strictly $\mathbb{R}$-specializable and regular along $\tau = 0$ (Definition 2.19 of [15]).

The notions of rescaling and well-rescalability generalize to $R$-triples and filtered $R$-triples (2.3.d of [15]). Then, the category of irregular mixed Hodge modules $\text{IrrMHM}(X; \mathbb{Q})$ (with $\mathbb{Q}$-structure) is defined as the full subcategory of $\text{MTM}^\text{int}_\text{good}(X; \mathbb{Q})$, which consists of graded well-rescalable filtered $R$-triples whose rescaling are also in $\text{MTM}^\text{int}_\text{good}(X; \mathbb{Q})$ (see Definitions 2.50 and 2.52 of [15]). By Proposition 2.68 of [15], the subcategory $\text{MHM}(X) \subset \text{MTM}^\text{int}_\text{good}(X; \mathbb{Q})$ is contained in $\text{IrrMHM}(X; \mathbb{Q})$.

The important fact is that “the exponential twist” is contained in $\text{IrrMHM}(X; \mathbb{Q})$ although not in $\text{MHM}(X)$, as explained below. Let $\varphi$ be a rational function on $X$ and $P(\subset X)$ its pole divisor. Then, we can define a $R^\text{int}_X(\ast P \times \mathbb{C}_\lambda)$-module $O_{X \times \mathbb{C}_\lambda}(\ast P \times \mathbb{C}_\lambda) \cdot e^{\varphi / \lambda}$ so that $O_{X \times \mathbb{C}_\lambda}(\ast P \times \mathbb{C}_\lambda) \cdot e^{\varphi / \lambda}$ is an $O_{X \times \mathbb{C}_\lambda}$-module and

\begin{align*}
\lambda \theta \cdot (m e^{\varphi / \lambda}) &= (\lambda \theta m) e^{\varphi / \lambda} + \theta(\varphi) e^{\varphi / \lambda}, \quad \text{and} \\
\lambda^2 \partial_\lambda \cdot m e^{\varphi / \lambda} &= - (\varphi m) e^{\varphi / \lambda}
\end{align*}

for $m \in O_{X \times \mathbb{C}_\lambda}(\ast P \times \mathbb{C}_\lambda)$ and a vector field $\theta$ on $X$. This module is twistor-specializable along $P \times \mathbb{C}_\lambda$ and we define $e^{\varphi / \lambda}_X := (O_{X \times \mathbb{C}_\lambda}(\ast P \times \mathbb{C}_\lambda) e^{\varphi / \lambda}[\ast (P \times \mathbb{C}_\lambda)]$ (see Proposition 3.3 of [17]).

**Lemma 4.6** (Proposition 3.3 of [17], Theorem 0.2 and 2.4.g of [15]). The object $e^{\varphi / \lambda}_X$ underlies an object of $\text{MTM}^\text{int}_\text{good}(X; \mathbb{Q})$. More strongly, this belongs to $\text{IrrMHM}(X; \mathbb{Q})$, i.e. the Rees module of an underlying filtered $D$-module of a mixed Hodge module on $X$, the $R^\text{int}_X$-module $\mathcal{M} \otimes_{O_{X \times \mathbb{C}_\lambda}} e^{\varphi / \lambda}_X$ underlies an object of $\text{IrrMHM}(X; \mathbb{Q})$.

For an $R_X$-module $\mathcal{M}$, we set

$$
\Xi_{\text{DR}}(\mathcal{M}) := \mathcal{M} / (\lambda - 1) \mathcal{M} \in \text{Mod}(D_X),
$$

and call it the underlying $D$-module of $\mathcal{M}$. An important feature of well-rescalable good $R^\text{int}_X$-module is that we can define a natural good filtration on $\Xi_{\text{DR}}(\mathcal{M})$ called the irregular Hodge filtration. Let us recall the definition.
Through the identification \( X \times \mathbb{C}_\lambda^* \) with the image of the diagonal embedding \( X \times \mathbb{C}_\lambda^* \hookrightarrow X \times \mathbb{C}_\tau \times \mathbb{C}_\lambda \) (\( \tau \) is the parameter for the rescaling), we have
\[
i^*_\tau=\lambda R_{X \times \mathbb{C}_\tau/C_\tau}(\ast\{\tau = 0\}) = R_X \otimes_{\mathcal{C}[\lambda]} \mathbb{C}[\lambda^{\pm 1}], \tag{37}
\]
where \( i^*_\tau \) is the inclusion \( \{\tau = \lambda\} \hookrightarrow X \times \mathbb{C}_\tau \times \mathbb{C}_\lambda \) and \( R_{X \times \mathbb{C}_\tau/C_\tau} \) is the subalgebra of \( R_{X \times \mathbb{C}_\tau} \) generated by \( R_X \) and \( O_{X \times \mathbb{C}_\tau \times \mathbb{C}_\lambda} \) (which does not contain "\( \partial\) ").

Let \( \pi^0 : X \times \mathbb{C}_\lambda^* \to X \) be the projection, \( \mathcal{M} \) a well-rescalable good \( R^\text{int}_X \)-module and \( M \) its underlying \( D \)-module \( \Xi_{\text{DR}}(\mathcal{M}) \).

**Lemma 4.7** (Remark 2.20 of [15]). *We have an isomorphism between \( R_X \otimes_{\mathcal{C}[\lambda]} \mathbb{C}[\lambda^{\pm 1}] \)-modules (not \( R^\text{int}_X \)-modules)*
\[
i^*_\tau=\lambda \mathcal{M} \cong \pi^0 M \tag{38}
\]
which sends a section \( 1 \otimes (1 \otimes m) \in O_{X \times \mathbb{C}_\lambda^*} \otimes i^*_\tau=\lambda \mathcal{M} \) to \( 1 \otimes [m] \in O_{X \times \mathbb{C}_\lambda^*} \otimes \pi^0 M \) where \( m \) is a section of \( \mathcal{M} \), \( 1 \otimes m \) is the one in \( \mathcal{M} \), and \([m]\) is the image of \( m \) under \( \mathcal{M} \to \Xi_{\text{DR}}(\mathcal{M}) \). Moreover, under the isomorphism \( \Xi_{\text{DR}}(\mathcal{M}) \), the natural action \( \lambda^2 \partial_\lambda \) on \( \pi^0 M \) corresponds to the action \( \lambda^2 \partial_\lambda + \tau \partial_\tau \) on \( i^*_\tau=\lambda \mathcal{M} \). More precisely, for \( k \in \mathbb{Z} \) and a section \( m \in \mathcal{M} \), the section \( 1 \otimes (\lambda^2 \partial_\lambda + \tau \partial_\tau)(\lambda^k \otimes m) \) of \( i^*_\tau=\lambda \mathcal{M} \) corresponds to \( (\lambda^2 \partial_\lambda)(\lambda^k \otimes m)(= k\lambda^{k+1} \otimes m) \) under the isomorphism \( \Xi(\mathcal{M}) \).

*Proof.* Since \( \mu \circ i^*_\tau=\lambda : X \times \mathbb{C}_\lambda \simeq X \times \{\tau = \lambda\} \to X \times \mathbb{C}_\lambda \) is the morphism \((x, \lambda) \to (x, 1)\), by [33] we have [33]. We remark that \( 1 \otimes (\lambda^k \otimes m) \) corresponds to \( \lambda^k \otimes m \) under [33]. Then, the second statement follows from the definition of the actions [33] of \( \mathcal{M} \). \( \square \)

\( \pi^0 M \) has a natural grading induced by the \( \lambda \)-adic filtration
\[
\lambda^k O_{X \times \mathbb{C}_\lambda^*} \otimes M \subset \pi^0 M.
\]
Here, for brevity we write \( \lambda^k O_{X \times \mathbb{C}_\lambda^*} \otimes M \) for \( \lambda^k O_{X \times \mathbb{C}_\lambda^*} \otimes \pi^0 M \). Then, the \( k \)-th graded piece is \( \text{gr}^k \pi^0 M \) \( \lambda^k \otimes M \). The corresponding graded module is denoted by \( \text{gr}(\pi^0 M)(= \bigoplus_{k \in \mathbb{Z}} \text{gr}^k \pi^0 M) \), which is \( O_X[\lambda^{\pm 1}] \otimes M \). We can regard it as a \( R_F D_X \)-module. Remark that by the definition of well-rescalability we can consider the Kashiwara-Malgrange filtration \( V^*_\tau(\mathcal{M}) \) along \( \tau = 0 \) of \( \mathcal{M} \).

**Lemma 4.8** (Lemma 2.21 of [15]). For \( \beta \in \mathbb{R} \), we have \((\tau - \lambda)V^\beta_\tau(\mathcal{M}) = (\tau - \lambda)\mathcal{M} \cap V^\beta_\tau(\mathcal{M}) \). Therefore, we obtain an inclusion
\[
i^*_\tau=\lambda V^\beta_\tau(\mathcal{M}) \hookrightarrow i^*_\tau=\lambda \mathcal{M}.
\]
In particular, we can regard \( i^*_\tau=\lambda V^\beta_\tau(\mathcal{M}) \) as a submodule of \( \pi^0 M \) by Lemma [4.7].

For \( \alpha \in [0, 1) \), the \( \lambda \)-adic filtration \( \lambda^k \otimes M \subset \pi^0 M \) induces a filtration on \( i^*_\tau=\lambda V^{-\alpha}_\tau(\mathcal{M}) \). The corresponding graded module \( \text{gr}(i^*_\tau=\lambda V^{-\alpha}_\tau(\mathcal{M})) \) is a graded \( R_F D_X \)-submodule of \( \text{gr}(\pi^0 M) = O_X[\lambda^{\pm 1}] \otimes M \). Since \( \text{gr}(\pi^0 M) \) is a strict graded \( R_F D_X \)-module, so is \( \text{gr}(i^*_\tau=\lambda V^{-\alpha}_\tau(\mathcal{M})) \). Therefore, \( \text{gr}(i^*_\tau=\lambda V^{-\alpha}_\tau(\mathcal{M})) \) comes from the Rees module associated to a filtration of \( M \).
Definition 4.11. For $\alpha \in [0,1)$, the irregular Hodge filtration $F^\text{irr}_{\alpha+p}M$ is the unique good filtration of the $D$-module $M$ indexed by $\mathbb{Z}$ such that the corresponding Rees module $R_{F^\text{irr}_{\alpha+p}}M = \bigoplus_{p \in \mathbb{Z}} F^\text{irr}_{\alpha+p}M^p(\subset M[\lambda^{\pm 1}])$ is equal to $\text{gr}(i^*_\tau=\lambda V_\tau^{-\alpha}(\mathcal{M}))$.

We can regard the family $\{F^\text{irr}_{\alpha+p}M\}_{\alpha \in [0,1), p \in \mathbb{Z}}$ as a filtration of $M$ indexed by $\mathbb{R}$. If $\mathcal{M}$ comes from a filtered $D$-module, the irregular Hodge filtration is equal to the original one as follows.

Proposition 4.10 (Proposition 2.40 of [15]). For a filtered $D$-module $(M,F^\text{irr}_pM)$, the corresponding Rees module $\mathcal{M} = R_F M$ is a well-rescalable good $R^\text{int}_X$-module. Moreover, the irregular Hodge filtration $F^\text{irr}_pM$ is equal to the original filtration $F_pM$. In particular, $F^\text{irr}_0M$ jumps only at the integers.

For an irregular Hodge module $\mathcal{T}$, the underlying $R^\text{int}_X$-module $\mathcal{M}$ is well-rescalable and good. So, we can consider the irregular Hodge filtration on $\Xi_{\text{DR}}(\mathcal{M})$. As already mentioned, for a mixed Hodge module $M = (M,F^\text{irr}_pM,K,W,M)$, we can regard it as an irregular Hodge module whose underlying $R^\text{int}_X$-module is the Rees module $R_F M$. Therefore, by the proposition above, the irregular Hodge filtration on $\Xi_{\text{DR}}(\mathcal{M}) = M$ is the original Hodge filtration $F_pM$.

4.2 The Fourier-Laplace transforms of $R$-modules and the irregular Hodge filtrations

In Section 3 we introduced the Fourier-Laplace transform of a $D$-module on a vector bundle (or the projective compactification of a vector bundle). For a monodromic mixed Hodge module, we endowed the Fourier-Laplace transform of its underlying $D$-module with a structure of mixed Hodge module (Definition 3.20). As explained there, in general, we can not define “the Fourier-Laplace transform of a (non-monodromic) mixed Hodge module” in the category of mixed Hodge modules. However, for a (not necessary monodromic) mixed Hodge module, if we regard it as an integrable mixed twistor $D$-module as explained in Subsection 4.1 we can naturally define “the Fourier-Laplace transform” of it in the category of irregular mixed Hodge module. To explain it, we first recall the definition of the Fourier-Laplace transform of an $R$-module and its basic properties.

Let $\pi: E \to X$ be a vector bundle on a smooth algebraic variety $X$ and $\pi^\vee: E^\vee \to X$ its dual bundle. We use the notations defined in Section 3 for a vector bundle $E$. For example, $\tilde{E}$ (resp. $\tilde{E}^\vee$) is the projective compactification of $E$ (resp. $E^\vee$). Moreover, $\varphi$ the rational function on $\tilde{E} \times_X \tilde{E}^\vee$ defined as the pairing of $\tilde{E}$ and $\tilde{E}^\vee$, whose pole divisor is $D_\infty \cup D_\infty^\vee$. Recall that the $R^\text{int}$-modules $\mathcal{E}^{E^\vee/\lambda}_{\tilde{E} \times_X \tilde{E}^\vee}$ and $\mathcal{E}^{E/\lambda}_{\tilde{E} \times_X \tilde{E}^\vee}$ are the underlying $R$-modules of mixed twistor $D$-modules (Lemma 4.3).

Definition 4.11. For the underlying $R^\text{int}$-module $\mathcal{M}$ (resp. $\mathcal{N}$) of a mixed twistor $D$-module on $E$ (resp. $\tilde{E}$), we define the Fourier-Laplace transform $\mathcal{M}^\wedge$ (resp. $\mathcal{N}^\wedge$) as

\[
\mathcal{M}^\wedge = H^0\mathcal{q}_*(p^*\mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda}_{E \times_X E^\vee})
\]

(resp. $\mathcal{N}^\wedge = H^0\mathcal{q}_*(\tilde{p}^*\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda}_{\tilde{E} \times_X \tilde{E}^\vee}[*{(D_\infty \cup D_\infty^\vee)}]$). (39)
Lemma 4.12. Let $\mathcal{N}$ be the underlying $R^{\text{int}}$-module of a mixed twistor $D$-module on $\tilde{E}$. Then, we have
\[(\mathcal{N}^\wedge)|_{E^\vee} = (\mathcal{N}|_{E})^\wedge.\]

Proof. By the definition (see Remark 4.5), we have
\[(\mathcal{N}|_{E})^\wedge = H^0\hat{q}_*(\hat{p}^*(\mathcal{N}|_{E}) \otimes \mathcal{O}^{-\varphi/\lambda})
= H^0\tilde{q}_!(\tilde{p}^*\mathcal{N} \otimes \mathcal{O}^{-\varphi/\lambda}[\ast D_{\infty} \cup D_{\infty}^\vee])|_{E^\vee}
= (\mathcal{N}^\wedge)|_{E^\vee}.\]

Corollary 4.13. In the setting of Lemma 4.12, we have
\[\mathcal{N}^\wedge \simeq (\mathcal{J}_!^\vee (\mathcal{N}|_{E})^\wedge)[\ast D_{\infty}].\]

Proof. Since $\mathcal{N}$ is the underlying $R$-module of a mixed twistor $D$-module, we have
\[(\tilde{p}^*\mathcal{N} \otimes \mathcal{O}^{-\varphi/\lambda})[\ast (D_{\infty} \cup D_{\infty}^\vee)] = (\tilde{p}^*\mathcal{N} \otimes \mathcal{O}^{-\varphi/\lambda})[\ast D_{\infty}][\ast D_{\infty}^\vee]. \tag{40}\]
By Lemma 3.2.12 of [7] or Corollary 9.7.1 of [16], we have
\[\tilde{\mathcal{N}} = \tilde{q}_!(\tilde{p}^*\mathcal{N} \otimes \mathcal{O}^{-\varphi/\lambda}[\ast D_{\infty} \cup \ast D_{\infty}^\vee])
= \tilde{q}_!(\tilde{p}^*\mathcal{N} \otimes \mathcal{O}^{-\varphi/\lambda}[\ast D_{\infty} \cup D_{\infty}^\vee][\ast D_{\infty}^\vee]).\]
Since $\tilde{q}_!(\tilde{p}^*\mathcal{N} \otimes \mathcal{O}^{-\varphi/\lambda}[\ast D_{\infty} \cup D_{\infty}^\vee])|_{E^\vee}$ is $(\mathcal{N}|_{E})^\wedge$, we have the desired assertion. \qed

Remark 4.14. If $\mathcal{M}$ (resp. $\mathcal{N}$) is the Rees module of a filtered $D$-module $(M, F_\bullet M)$ (resp. $(N, F_\bullet N)$), the underlying $D$-module of $\mathcal{M}^\wedge$ (resp. $\mathcal{N}^\wedge$) is the Fourier-Laplace transform $M^\wedge$ (resp. $N^\wedge$) defined in Section 3.

By Lemma 4.6, we obtain the following.

Proposition 4.15. If $\mathcal{M}$ (resp. $\mathcal{N}$) is the Rees module of the underlying filtered $D$-module of a mixed Hodge module, $\mathcal{M}^\wedge$ (resp. $\mathcal{N}^\wedge$) is the underlying $R^{\text{int}}$-module of an irregular mixed Hodge module.

Remark 4.16. For a monodromic mixed Hodge module, we defined a mixed Hodge module whose underlying $D$-module is the Fourier-Laplace transform of its underlying $D$-module (Definition 3.20). As explained in Lemma 4.6, we can regard it as an irregular mixed Hodge module. On the other hand, we have another “Fourier-Laplace transform” made from a monodromic mixed Hodge module, which appeared in Proposition 4.15. So we have two definitions of “the Fourier-Laplace transform of a monodromic mixed Hodge module” in the category of irregular mixed Hodge modules. In general, the two are different, but they are related to each other. We will observe it in Subsection 4.4.

Let us see $\mathcal{M}^\wedge$ and $\mathcal{N}^\wedge$ have better descriptions. We need the following lemma.
Lemma 4.17. For the underlying $R^\text{int}$-module $\mathcal{N}$ on $\tilde{E}$ of a mixed twistor $D$-module, we have

$$(\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})[\ast D_\infty \cup D'_\infty] = (\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})(\ast D_\infty)[\ast D'_\infty].$$

Proof. This proof is inspired by the proof of Proposition A.2.7 of [13] and the one of Lemma 3.1 of [17]. We assume that $X$ is one point variety. We can prove in the general case in the same way. In this case, $E$ and $E'$ are vector spaces of rank $n$. Let $(z_1, \ldots, z_n)$ be the coordinates of $E$ and $(\zeta_1, \ldots, \zeta_n)$ its dual coordinates of $E'$. We write $\mathbb{C}^n$ (resp. $\mathbb{C}^n$) for $E$ (resp. $E'$) with the coordinates $(z_1, \ldots, z_n)$ (resp. $(\zeta_1, \ldots, \zeta_n)$). Moreover, $\mathbb{P}^n$ (resp. $\mathbb{P}^n$) is the projective compactification of $E = \mathbb{C}^n$ (resp. $E = \mathbb{C}^n$). Remark that $D_\infty$ (resp. $D'_\infty$) is the divisor $\mathbb{P}^n \setminus \mathbb{C}^n$ (resp. $\mathbb{P}^n \setminus \mathbb{C}^n$).

By the equality (40), it is enough to show

$$(\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})[\ast D_\infty] = (\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})(\ast D_\infty).$$

Let $[z_0 : z_1 : \ldots : z_n]$ (resp. $[\zeta_0 : \zeta_1 : \ldots : \zeta_n]$) be the homogeneous coordinates of $\mathbb{P}^n$ (resp. $\mathbb{P}^n$) and $U_i := \{z_i = 1\} (\approx \mathbb{C}^n)$ (resp. $U_j := \{\zeta_j = 1\}$) an open subset of $\mathbb{P}^n$ (resp. $\mathbb{P}^n$) with the coordinates $(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n)$ (resp. $(\zeta_0, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_n)$) for $i, j = 0, \ldots, n$. Then, $\{U_i\}_i$ (resp. $\{U_j\}_j$) is a covering of $\mathbb{P}^n$ (resp. $\mathbb{P}^n$). Note that $D_\infty$ (resp. $D'_\infty$) is defined by $z_0$ (resp. $\zeta_0$). Therefore, the assertion is clear on $U_0 \times \mathbb{P}^n$. So, it is enough to prove the equality (41) on $U_1 \times U'_1$, $U_1 \times U'_2$ and $U_1 \times U'_3$. We may assume $n = 2$.

On $U_1 \times U'_1$. Let $V^\bullet(\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})$ be the Kashiwara-Malgrange filtration along $z_0$. For a section $m \in \mathcal{N}|_{U_1}$ and $m \otimes 1 \in \overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda}|_{U_1 \times U'_1}$, we have $z_0^k (m \times 1) \in V^{r_0-1}(\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})$ for some $k \geq 0$ by an standard property of the $V$-filtration. Let $k_0 \geq 0$ be the smallest $k$ and assume $k_0 \geq 1$. Remark that on $U_1 \times U'_1$ (with the coordinates $(z_0, z_2, \zeta_1, \zeta_2)$), we have $\varphi = (1/z_0)(\zeta_1 + z_2 \zeta_2)$. Then, we have

$$(\partial_{\zeta_1}) z_0^{k_0} (m \otimes 1) = - (1/z_0) \cdot z_0^{k_0} (m \otimes 1)$$

$$= - z_0^{k_0-1} (m \otimes 1).$$

Since the operators $\partial_{\zeta_1}$ preserves the filtration $V^{\bullet}_{z_0}$, the section $z_0^{k_0-1}(m \otimes 1)$ is also in $V^{r_0-1}_{z_0}$. This contradicts the definition of $k_0$. Therefore, we have $k_0 = 0$, i.e. $m \otimes 1$ is in $V^{r_0-1}_{z_0}(\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})$. This implies that $V^\bullet_{z_0}(\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})$ is constant on $U_1 \times U'_1$ and we thus obtain the equality (41).

On $U_1 \times U'_2$. Similarly to the previous case, for a section $m \in \mathcal{N}|_{U_1}$, we take the minimum $k_0$ such that $z_0^{k_0} (m \otimes 1) \in V^{r_0-1}_{z_0}(\overline{p^*}\mathcal{N} \otimes \mathcal{E}^{-\varphi/\lambda})$ and assume $k_0 \geq 1$. We have $\varphi = (1/(z_0 \zeta_0))(1 + z_2 \zeta_2)$ on $U_1 \times U'_2$. Then, we have

$$(\zeta_0^2 \partial_{\zeta_0} + \zeta_2 \zeta_0 \partial_{\zeta_2}) z_0^{k_0} (m \otimes 1) = z_0^{k_0-1} (m \otimes 1).$$

Therefore, $k_0$ is 0 and thus by the same argument in the previous case, we obtain the equality (41).

On $U_1 \times U'_3$. We can prove it in the same way.
As mentioned in Remark 4.5, $T_q$ is not $q \uparrow$ in general. Nevertheless, the following holds.

**Corollary 4.18.** For the underlying $R^{\text{int}}$-module $\mathcal{M}$ of a mixed twistor $D$-module on $E$, we have

$$\mathcal{M}^\wedge = H^0 q_!(p^* \mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda}).$$

**Proof.** Let $\mathcal{N}$ be the underlying $R^{\text{int}}$-module of a mixed twistor $D$-module on $\widetilde{E}$ whose restriction $\mathcal{N}|_E$ is $\mathcal{M}$. Then, by Lemma 4.17 we have

$$\mathcal{M}^\wedge = H^0 q_!(\mathcal{N}^\wedge \otimes \mathcal{E}^{-\varphi/\lambda}[\ast D_{\infty} \cup D_{\infty}^\vee])|_{E^\vee},$$

$$= H^0 q_!(\mathcal{N}^\wedge \otimes \mathcal{E}^{-\varphi/\lambda}(\ast D_{\infty})[\ast D_{\infty}^\vee])|_{E^\vee}.$$  

By Lemma 3.2.12 of [7] or Corollary 9.7.1 of [16] again, the last term is equal to

$$H^0 q_!(\mathcal{N}^\wedge \otimes \mathcal{E}^{-\varphi/\lambda}(\ast D_{\infty}))[\ast D_{\infty}^\vee],$$

i.e.

$$H^0 q_!(\mathcal{N}^\wedge \otimes \mathcal{E}^{-\varphi/\lambda}(\ast D_{\infty})).$$

This is equal to $H^0 q_!(\mathcal{N}^\wedge \otimes \mathcal{E}^{-\varphi/\lambda})$, which completes the proof. \hfill \qed

Finally, let us describe the Fourier-Laplace transform when $X$ is affine and $E$ is trivial. Fix the trivialization $E \simeq X \times \mathbb{C}^n$. As above, we write $\mathbb{C}^n_2$ (resp. $\mathbb{C}^n_1$) for $\mathbb{C}^n$ with the coordinates $(z_1, \ldots, z_n)$ (resp. the dual coordinates $(\zeta_1, \ldots, \zeta_n)$). Due to Corollary 4.13, in order to know $(\mathcal{N}|_E)^\wedge$, we need to know $(\mathcal{N}|_E)$, i.e. $(\mathcal{M}|_E)$.

**Lemma 4.19.** Let $\mathcal{M}$ be the underlying $R^{\text{int}}$-module of a mixed twistor $D$-module on $E = X \times \mathbb{C}^n_2$ for a smooth affine variety $X$. Then, as a $\mathbb{C}[\lambda]$-module, $\Gamma(X \times \mathbb{C}^n_2; \mathcal{M})$ is isomorphic to $\Gamma(X \times \mathbb{C}^n_2; \mathcal{M}^\wedge)$ and under this identification, the vector field $\theta$ on $X$ acts as the same theta, $\theta \otimes \lambda^{-1}$ acts as $\delta_{z_i}$ and $\delta_{\zeta_i}$ acts as $-z_i$. Moreover, $\lambda^2 \partial_{\lambda}$ acts as $\lambda^2 \partial_{\lambda} + \lambda \mathcal{E}\mathcal{C}_2$, where $\mathcal{E}\mathcal{C}_2 = \sum_{i=1}^n \zeta_i \delta_{z_i}$.

**Proof.** We may assume $X$ is one point variety, i.e. $E = \mathbb{C}^n_2$. We set $\mathcal{M}_1 := p^* \mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda}$ and $\mathcal{A}^k := \Omega_{\mathbb{C}^n_2 \times \mathbb{C}^n_1/\mathbb{C}^n_2}^k \otimes \mathbb{C}[\lambda] \lambda^{-k}$, where $\Omega^k_{\mathbb{C}^n_2 \times \mathbb{C}^n_1/\mathbb{C}^n_2}$ is the sheaf of relative holomorphic $k$-forms. Then, the $R$-module structure of $\mathcal{M}_1$ defines the connection $\mathcal{M}_1 \to \mathcal{A}^1 \otimes \mathcal{M}_1$ $(m \mapsto \sum_{i=1}^n d_z^i/\lambda \otimes \delta_{z_i} m)$. This morphism is naturally extended to the relative de Rham complex

$$\mathcal{M}_1 \to \mathcal{A}^1 \otimes \mathcal{M}_1 \to \cdots \to \mathcal{A}^n \otimes \mathcal{M}_1,$$

where the rightmost term is of degree 0. Fixing the isomorphisms $\mathcal{A}^1 \simeq \oplus_{i=1}^n \mathcal{O}_{\mathbb{C}^n_2 \times \mathbb{C}^n_2}dz_i/\lambda$ and $\mathcal{A}^k \simeq \bigwedge^k (\oplus_{i=1}^n \mathcal{O}_{\mathbb{C}^n_2 \times \mathbb{C}^n_2}dz_i/\lambda)$, one can see that this complex is the Koszul complex of the $R$-module $\mathcal{M}_1$ with respect to the regular sequence $\delta_{z_1}, \ldots, \delta_{z_n}$. Therefore, the only non-trivial cohomology is the 0-th one and that is

$$\mathcal{M}_1/ \sum_{i=1}^n \delta_{z_i} \mathcal{M}_1$$

36
by the identification \( \mathcal{A}^n \simeq O_{\mathbb{C}^n} dz_1 \wedge \cdots \wedge dz_n / \lambda^n \). The pushforward of an \( R \)-module by a projection can be expressed as a relative de Rham cohomology. Therefore, by Corollary 4.18 we have

\[
\mathcal{M}^\wedge \simeq H^0 q_* (\mathcal{M}_1 / \sum_{i=1}^n \partial_i \mathcal{M}_1)
\]

in the category of \( \mathcal{R}^{\text{int}} \)-modules. Taking the global sections of \( q_*(\mathcal{M}_1 / \sum_{i=1}^n \partial_i \mathcal{M}_1) \), since we have \( \Gamma(\mathbb{C}_z \times \mathbb{C}_\lambda^\wedge; M \wedge) \simeq \Gamma(\mathbb{C}_z; M) \tau / \sum_{i=1}^n \partial_i \Gamma(\mathbb{C}_z; M) \tau \).

By looking at the actions on \( \Gamma(\mathbb{C}_z; M) \tau \), we thus get the second assertion.

Combining Lemma 4.19 and Corollary 4.13, we now understand the \( \mathcal{R}^{\text{int}} \)-module structure of \( \mathcal{N}^\wedge \).

**Definition 4.20.** In the setting of Lemma 4.19 for a section \( m \in \Gamma(X \times \mathbb{C}_z^\wedge; \mathcal{M}) \) we denote by \( m^\wedge \) the corresponding section of \( \mathcal{M}^\wedge \) under the isomorphism \( \Gamma(X \times \mathbb{C}_z^\wedge; \mathcal{M}) \simeq \Gamma(X \times \mathbb{C}_z^\wedge; \mathcal{N}^\wedge) \).

In terms of the terminology in the proof of Lemma 4.19 \( m^\wedge \) is the class represented by a section \( (dz_1 \wedge \cdots \wedge dz_n / \lambda^n) \otimes m \in \mathcal{A}^n \otimes \mathcal{M}_1 \). Then, by Lemma 4.19 we have

\[
\begin{align*}
\lambda \cdot m^\wedge &= (\lambda m)^\wedge, \\
\zeta_i \cdot m^\wedge &= (\partial_{z_i} m)^\wedge, \\
\partial_{\zeta_i} \cdot m^\wedge &= -(z_i m)^\wedge, \quad \text{and} \\
\lambda^2 \partial_{\lambda} \cdot m^\wedge &= ((\lambda^2 \partial_{\lambda} + \lambda E_{\mathbb{C}_z}) m)^\wedge.
\end{align*}
\]

(42)

**Remark 4.21.** The section \( m^\wedge \) depends on the choice of the trivialization of \( E \). However, similarly to Definition 3.7 we can define an \( O_{X \times \mathbb{C}_\lambda} \)-module \( \mathcal{F}^\wedge \) of \( \pi_\lambda^{-1} \mathcal{M}^\wedge \) for an \( O_{X \times \mathbb{C}_\lambda} \)-submodule \( \mathcal{F} \) of \( \pi_\lambda \mathcal{M} \).

**4.3 Fourier-Laplace transforms of a monodromic mixed Hodge modules**

In this subsection, we will compute the irregular Hodge filtration of the Fourier-Laplace transform of a monodromic mixed Hodge module. To simplify the description, we will consider the Fourier-Laplace transforms of mixed Hodge modules on \( \mathbb{C}_\lambda^\wedge \) (with the coordinates \( (z_1, \ldots, z_n) \)). However, we can generalize the results to the Fourier-Laplace transform on a general vector bundle (see Remark 4.38).

We will use the notation defined in the previous subsection. Let \( \mathcal{M} = (M, F \cdot M, K, W \cdot K) \) be a mixed Hodge module on \( \mathbb{C}_z^\wedge \) and \( \mathcal{N} = (N, F \cdot N, K', W \cdot K') \) the pushforward of \( \mathcal{M} \) by the inclusion \( j: \mathbb{C}_z^\wedge \hookrightarrow \mathbb{P}_z^n \) (since our module is always algebraic, we can consider such an object). We denote by \( \mathcal{M} \) (resp. \( \mathcal{N} \)) the corresponding \( \mathcal{R}^{\text{int}}_{\mathbb{C}_z^\wedge} \)-module (resp. \( \mathcal{R}^{\text{int}}_{\mathbb{P}_z^n} \)) of \( (M, F \cdot M) \) (resp. \( (N, F \cdot N) \)).
Proof. We remark that since \( \Lambda^\cdot|_{\mathbb{C}_z^\cdot} = \mathcal{M}^\wedge \) and \( N^\cdot|_{\mathbb{C}_z^\cdot} = M^\wedge \) (see Corollary 4.13). By Proposition 4.15 \( N^\cdot \) is also the underlying \( R^\text{int} \)-module of an irregular mixed Hodge module. Then, as explained in subsection 4.1, \( \Xi_{\text{DR}}(\mathcal{N}^\wedge) = N^\wedge \) is equipped with the irregular Hodge filtration \( F^\cdot \text{irr} N^\wedge \) (Definition 4.9).

In the following, we assume that \( M \) is monodromic. Then, by Proposition 1.6 and Theorem 2.1 we have the decompositions

\[
M = \bigoplus_{\beta \in \mathbb{R}} M^\beta \quad \text{and} \quad F^\cdot M = \bigoplus_{\beta \in \mathbb{R}} F^\cdot M^\beta,
\]

where \( M^\beta = \bigcup_{i \geq 0} \ker(\mathcal{E}_{\mathbb{C}^\cdot} - \beta)^i \). Therefore, we have

\[
\mathcal{M} = \bigoplus_{\beta \in \mathbb{R}} F_p M^\beta \lambda^p.
\]

\( \mathbb{C}^n \) is the dual space of \( \mathbb{C}_z^n \) with the dual coordinates \((\zeta_1, \ldots, \zeta_n)\). Let \([\zeta_0: \cdots: \zeta_n]\) be the homogeneous coordinates of \( \mathbb{P}^n_\zeta = \mathbb{P}(\mathbb{C}^n) \) and \( \{U^\cdot \}_{i=0}^n (U^\cdot_0 = \{\zeta_i \neq 0\}(\approx \mathbb{C}^n) \subset \mathbb{P}^n_\zeta) \) the affine open covering of \( \mathbb{P}^n_\zeta \). Note that \( U^\cdot_0 = \mathbb{C}^n_\zeta \). To understand the irregular Hodge filtration \( F^\cdot \text{irr} N^\wedge \), we will compute the restriction of \( F^\cdot \text{irr} N^\wedge \) to each affine open subset \( U^\cdot_i \) respectively.

### 4.3.1 The irregular Hodge filtration on \( M^\wedge \)

First, we compute \( F^\cdot \text{irr} N^\wedge|_{\mathbb{C}^\cdot} (= F^\cdot \text{irr} M^\wedge) \). In order to do that, we need to compute \( V^\cdot_{\tau} (\mathcal{M}^\wedge) \), where \( \mathcal{M}^\wedge \) is the rescaled module of \( M^\wedge \) and \( \tau \) the rescaling parameter (see Subsection 4.1). Since \( \mathbb{C}^n_\zeta \times \mathbb{C}_\tau \times \mathbb{C}_\lambda \) is affine, we identify the sheaves on it with the modules of global sections of them and they are represented by the same symbol by abuse of notation. Then, \( \mathcal{M}^\wedge \) is \( O_{\mathbb{C}^\cdot} [\tau^\pm 1, \lambda] \otimes O_{\mathbb{C}_z}[\lambda] \cdot \mathcal{M}^\wedge \) as an \( O_{\mathbb{C}^\cdot} \times \mathbb{C}_\tau \times \mathbb{C}_\lambda (\ast \{\tau = 0\}) \)-module with an \( R^\text{int} \)-module action defined as \( \mathbf{12} \). Moreover, recall that (a global section of) \( \mathcal{M}^\wedge \) can be expressed as \( m^\wedge \) for \( m \in \mathcal{M} \) (see Definition 4.20).

#### Lemma 4.22. In this setting, we have

\[
V^\cdot_{\tau} (\mathcal{M}^\wedge) = \bigoplus_{i, p \in \mathbb{Z}, \beta \in \mathbb{R}} \tau^i \otimes (F_p M^\beta \lambda^p)^\wedge. \quad (43)
\]

Proof. We remark that since \( \mathcal{M}^\wedge \) is strictly \( \mathbb{R} \)-specializable along \( \tau \) (by the definitions of the well-rescalability and the irregular Hodge module), Kashiwara-Malgrange filtration \( V^\cdot_{\tau} (\mathcal{M}^\wedge) \) exists and each graded piece \( \operatorname{gr}^\cdot_{\tau} (\mathcal{M}^\wedge) \) is strict. Therefore, if a non-zero section \( s \in \mathcal{M}^\wedge \) is killed by \( (\tau \partial_{\tau} - \gamma \lambda)^l \) for some \( l \geq 0 \), the section \( s \) is in \( V^\cdot_{\tau} (\mathcal{M}^\wedge) \) and not in \( V^\cdot_{\tau} (\mathcal{M}^\wedge) \).

Let \( m \) be a section of \( F_p M^\beta \). Then, \( m \lambda^p \) is in \( \mathcal{M} \). By \( \mathbf{12} \), we have

\[
\lambda^2 \partial_{\lambda} (m \lambda^p)^\wedge = ((\mathcal{E}_{\mathbb{C}^\cdot} + p) m \lambda^{p+1})^\wedge.
\]
Therefore, by (36), for \(i \geq 0\) we have
\[
\tau \partial_{\tau}(\tau^i \otimes (m\lambda^p)^{\wedge}) = (i\lambda\tau^i + \tau^{i+1}\partial_{\tau})(1 \otimes (m\lambda^p)^{\wedge}) = i\tau^{i+1} \otimes (m\lambda^{p+1})^{\wedge} - \tau^{i+1} \otimes (\lambda^2\partial_{\lambda}(m\lambda^p)^{\wedge}) = i\tau^{i+1} \otimes (m\lambda^{p+1})^{\wedge} - \tau^{i+1} \otimes ((p + \xi_{\nu_2})m\lambda^{p+1})^{\wedge}.
\] (44)

Hence, we have
\[
(\tau \partial_{\tau} - (i - p - \beta)\lambda)(\tau^i \otimes (m\lambda^p)^{\wedge}) = -\tau^{i+1} \otimes ((\xi_{\nu_2} - \beta)m\lambda^{p+1})^{\wedge}.
\]

By induction, for \(l \geq 1\) we obtain
\[
(\tau \partial_{\tau} - (i - p - \beta)\lambda)^l(1 \otimes (m\lambda^p)^{\wedge}) = (-1)^l\tau^{i+l} \otimes ((\xi_{\nu_2} - \beta)^l m\lambda^{p+1})^{\wedge}.
\]

Therefore, \(\tau^i \otimes (m\lambda^p)^{\wedge} \in \mathcal{T}(\mathcal{M}^{\wedge})\) is killed by \((\tau \partial_{\tau} - (i - p - \beta)\lambda)^l\) for sufficiently large \(l \geq 0\) and hence \(\tau^i \otimes (m\lambda^p)^{\wedge}\) is in \(V^\wedge_i V_{p-\beta}(\mathcal{T}(\mathcal{M}^{\wedge}))\) for the reason stated at the beginning of this proof. We thus conclude that the RHS of (43) is contained in the LHS.

Any section \(s \in V^\wedge_i V_{p-\beta}(\mathcal{T}(\mathcal{M}^{\wedge}))\) is a sum of some sections \(\tau^i \otimes (m\lambda^p)^{\wedge}\) for some \(i \in \mathbb{N}, p \in \mathbb{Z}\) and \(m \in F_p M^\beta\). Let \(s = s_{\gamma_1} + \cdots + s_{\gamma_k}\) be the decomposition of \(s\) such that \(s_{\gamma_j}(\neq 0) \in \sum_{i-p-\beta=\gamma_j} \tau^i \otimes (F_p M^\beta \lambda^p)^{\wedge}\) and \(\gamma_1 \leq \cdots \leq \gamma_k\). As we proved, \(s_{\gamma_1}\) is in \(V^\wedge \sum_{i-p-\gamma_1}(\mathcal{T}(\mathcal{M}^{\wedge}))\) and not in \(V^\wedge_i V_{p-\gamma_1}(\mathcal{T}(\mathcal{M}^{\wedge}))\). Hence, \(\gamma_1\) is greater than \(\gamma\). This implies that the LHS of (43) is contained in the RHS.

This completes the proof.

Recall that for \(\alpha \in [0, 1)\) we can regard \(\text{gr}(i^\wedge_{\tau=\lambda} V^\wedge_{\alpha} (\mathcal{T}(\mathcal{M}^{\wedge})))\) as a submodule of \(\text{gr}(i^\wedge_{\tau=\lambda}(\mathcal{T}(\mathcal{M}^{\wedge}))) = \text{gr}(\pi^\wedge_{\alpha} M^{\wedge}) \simeq O_{\nu_2}^{[\lambda^\wedge]} \otimes O_{\nu_2} M^{\wedge}\) (see Subsection 4.1). Note that the isomorphism
\[
\text{gr}(i^\wedge_{\tau=\lambda}(\mathcal{T}(\mathcal{M}^{\wedge}))) \simeq O_{\nu_2}^{[\lambda^\wedge]} \otimes O_{\nu_2} M^{\wedge}
\] (45)
is defined so that (a section of the LHS represented by) the section \(1 \otimes (m\lambda^p)^{\wedge} \in \mathcal{T}(\mathcal{M}^{\wedge})\) corresponds to \(1 \otimes m^{\wedge}\). Moreover, the Rees module of the irregular Hodge filtration \(F^\wedge_{\alpha+p} M^{\wedge}\) is equal to \(\text{gr}(i^\wedge_{\tau=\lambda} V^\wedge_{\alpha} (\mathcal{T}(\mathcal{M}^{\wedge})))\) (see Definition 4.9). Therefore, in order to know \(F^\wedge_{\alpha+p} M^{\wedge}\) it remains to see the image of \(\text{gr}(i^\wedge_{\tau=\lambda} V^\wedge_{\alpha} (\mathcal{T}(\mathcal{M}^{\wedge})))\) under the isomorphism (45).

**Theorem 4.23.** For \(\alpha \in [0, 1)\) and \(\beta \in \mathbb{R}\), we define a filtration \(F^\wedge_{\bullet}(M^\wedge)^{\beta}\) of \(M^\wedge\) as
\[
F^\wedge_{\alpha+p}(M^\wedge)^{\beta} := (F_{p+[\alpha-\beta]} M^\wedge)^{\wedge}.
\]

Then, we have
\[
F^\wedge_{\alpha+p} M^{\wedge} = \bigoplus_{\beta \in \mathbb{R}} F^\wedge_{\alpha+p}(M^\wedge)^{\beta}.
\]

**Proof.** By Lemma 4.22 we have
\[
i^\wedge_{\tau=\lambda} V^\wedge_{\alpha}(\mathcal{T}(\mathcal{M}^{\wedge})) = i^\wedge_{\tau=\lambda} \bigoplus_{i,p \in \mathbb{Z}, \beta \in \mathbb{R} \, i-p-\beta \geq -\alpha} \tau^i \otimes (F_p M^\beta \lambda^p)^{\wedge}.
\]
Under the identification (45), the RHS is (as a subset of $O_{\mathbb{C}^n}[\lambda^{\pm 1}] \otimes M^\wedge$)

$$
\bigoplus_{i \in \mathbb{Z}} \sum_{p \in \mathbb{Z}, \beta \in \mathbb{R}} \lambda^i \otimes (F_p M^\beta)^\wedge.
$$

(46)

Note that the condition $i - p - \beta \geq -\alpha$ is equivalent to

$$i + [\alpha - \beta] \geq p,$$

where $[\alpha - \beta]$ is the largest integer less than or equal to $\alpha - \beta$. Therefore, (46) is equal to

$$
\bigoplus_{i \in \mathbb{Z}, \beta \in \mathbb{R}} \lambda^i \otimes (F_{i+[\alpha-\beta]} M^\beta)^\wedge.
$$

This implies the desired result. \hfill \Box

4.3.2 The irregular Hodge filtration of $M^\wedge$ at infinity

Next, let us consider the irregular Hodge filtration on $N^\wedge|_{U_i^n}$ for $i = 1, \ldots, n$. Since they can all be computed in the same way, we will consider the case where $i = n$. In this subsection, we assume that $n \geq 2$. However this assumption is not essential; the argument proceeds in exactly the same way also for the case $n = 1$. Actually, all the results hold also in that case (after changing the notations appropriately). Let $(\zeta'_0, \zeta'_1, \ldots, \zeta'_{n-1})$ be the coordinates of $U'_n(\cong \mathbb{C}^n)$ so that the point of $\mathbb{P}^n_\mathbb{R}$ corresponding to $(\zeta'_0, \zeta'_1, \ldots, \zeta'_{n-1})$ is $[\zeta'_0 : \zeta'_1 : \cdots : \zeta'_{n-1} : 1]$. Then, we have $U'_n \cap U'_n = \{ \zeta'_0 \neq 0 \} = \{ \zeta_n \neq 0 \}$, $D'_n \cap U'_n = \{ \zeta'_0 = 0 \}$, $\zeta'_0 = 1/\zeta_n$ and $\zeta'_i = \zeta_i/\zeta_n$ for $i = 1, \ldots, n-1$ on $U'_n \cap U'_n$. Moreover, we have

$$
\mathcal{E}_{U'_n}(= \sum_{i=1}^n \zeta_i \partial_{\zeta_i}) = -\zeta_n \partial_{\zeta'_n}.
$$

(47)

Let $j'_n$ be the inclusion $U'_n \cap U'_n \to U'_n$. By Lemma 4.13, we have

$$(\mathcal{N}^\wedge)|_{U'_n} \simeq (j'_n)^!(\mathcal{M}^\wedge)|_{U'_n \cap U'_n}[*\{\zeta'_0 = 0\}].
$$

(48)

Since $U'_n \cap U'_n$ (resp. $U'_n$) is affine, we may identify $(\mathcal{M}^\wedge)|_{U'_n \cap U'_n}^\wedge$ (resp. $(j'_n)^!(\mathcal{M}^\wedge)|_{U'_n \cap U'_n}^\wedge$) with the module of its global sections and regard it as $\Gamma(U'_n \cap U'_n; R_{U'_n \cap U'_n})$-module (resp. $\Gamma(U'_n; R_{U'_n})$-module). Then, we can write $(j'_n)^!(\mathcal{M}^\wedge)|_{U'_n \cap U'_n}^\wedge$ in an algebraic way as

$$
\mathcal{M}^\wedge \otimes_{\mathbb{C}[\zeta_n]} \mathbb{C}[\zeta_n^{\pm 1}] |_{U'_n \cap U'_n}^\wedge.
$$

Moreover, we can write the RHS of (48) as

$$
\mathcal{M}^\wedge \otimes_{\mathbb{C}[\zeta_n]} \mathbb{C}[\zeta_n^{\pm 1}] |_{U'_n \cap U'_n}^\wedge.
$$

Its underlying $D$-module is $M^\wedge \otimes_{\mathbb{C}[\zeta_n]} \mathbb{C}[\zeta_n^{\pm 1}]$. Remark that the underlying $D$-module of $(\mathcal{M}^\wedge)|_{U'_n \cap U'_n}^\wedge$ is also expressed as the same $M^\wedge \otimes_{\mathbb{C}[\zeta_n]} \mathbb{C}[\zeta_n^{\pm 1}]$ (under the identification of the sheaf of module and the module of its global sections). However, in the following, we always regard $M^\wedge \otimes_{\mathbb{C}[\zeta_n]} \mathbb{C}[\zeta_n^{\pm 1}]$ as the underlying $D$-module of $(j'_n)^!(\mathcal{M}^\wedge)|_{U'_n \cap U'_n}^\wedge$, i.e. $M^\wedge \otimes_{\mathbb{C}[\zeta_n]} \mathbb{C}[\zeta_n^{\pm 1}]$ is a $D$-module on $U'_n$ (not $U'_n \cap U'_n$). Moreover, for a section $m \in M$, the section $m^\wedge \otimes 1 \in M^\wedge \otimes_{\mathbb{C}[\zeta_n]} \mathbb{C}[\zeta_n^{\pm 1}]$ is simply denoted by $m^\wedge$ if there is no confusion.
Lemma 4.24. If we regard $U'_n$ as a trivial line bundle $C_{\zeta'_n} \times (C_{\zeta'_1} \times \cdots \times C_{\zeta'_{n-1}})$ over $(C_{\zeta'_1} \times \cdots \times C_{\zeta'_{n-1}})$, the $D$-module $M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp]$ is monodromic on this line bundle, i.e. we have

$$M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp] = \bigoplus_{\beta \in \mathbb{R}} (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp])_{\zeta'_0}^\beta,$$

where we set

$$(M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp])_{\zeta'_0}^\beta := \bigcup_{l \geq 0} \mathrm{Ker} (\zeta'_0 \partial_{\zeta'_0} - \beta)^l (\subset M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp]).$$

In particular, for $\beta \in \mathbb{R}$ we have

$$\mathrm{gr}_V^\beta (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp]) \simeq (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp])_{\zeta'_0}^\beta,$$

as $\Gamma(C_{\zeta'_1} \times \cdots \times C_{\zeta'_{n-1}}; O)$-modules, where $\mathrm{gr}_V^\beta$ is the graded piece of the Kashiwara-Malgrange filtration $V_{\zeta'_0}^\bullet$ along $\zeta'_0 = 0$. Moreover, for $\beta \in \mathbb{R}$ we have

$$(M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp])_{\zeta'_0}^\beta = \sum_{j \in \mathbb{Z}, \gamma \in \mathbb{R}} \zeta'^j_0 (M^\wedge)^{\gamma}, \quad (49)$$

where $\zeta'^j_0 (M^\wedge)$ is the subset of $M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp]$ generated by $\{ \zeta'^j_0 m^\wedge (= m^\wedge \otimes \zeta'^j_0) \in M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp] \mid m \in M^\gamma \}$ and the RHS of (49) is the subset $\{ \sum_{i=1}^k s_i \in \zeta'^j_0 (M^\wedge) \wedge (j_i \in \mathbb{Z}, \gamma_i \in \mathbb{R}, j_i + \gamma_i + n = \beta) \}$ of $M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp]$ (in other words, it is not the $\Gamma(U'_n; O)$-module generated by $\{ \zeta'^j_0 (M^\wedge) \}$, but the $\Gamma(C_{\zeta'_1} \times \cdots \times C_{\zeta'_{n-1}}; O)$-module generated by them).

Proof. As we already remarked, for a section $m \in M$, we write $m^\wedge$ for the section $m^\wedge \otimes 1 \in M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp]$. For $m \in M^\gamma$ and $j \in \mathbb{Z}$, consider a section $\zeta'^j_0 m^\wedge$. Then, we have

$$\zeta'_0 \partial_{\zeta'_0} (\zeta'^j_0 m^\wedge) = \zeta'^j_0 (j + \zeta'_0 \partial_{\zeta'_0}) m^\wedge = \zeta'^j_0 ((j + \partial_{\zeta'_0}) m^\wedge).$$

This implies the desired assertions.

Remark 4.25. Similar to $\sum_{j \in \mathbb{Z}, \gamma \in \mathbb{R}} \zeta'^j_0 (M^\wedge)$ above, for a family $\{ A_i \}$ of subsets of $(M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp])_{\zeta'_0}^\beta$, we denote by $\bigcap_i A_i$ the $\Gamma(C_{\zeta'_1} \times \cdots \times C_{\zeta'_{n-1}}; O)$-submodule of $(M^\wedge \otimes_{C[\zeta_n]} C[\zeta_{n-1}^\perp])_{\zeta'_0}^\beta$ generated by $\{ A_i \}$, not the $\Gamma(U'_n; O)$-module generated by them, when no confusion arises.

For $\beta \in \mathbb{R}$, we define a positive integer $j_\beta \in \mathbb{Z}_{\geq 0}$ by

$$j_\beta := \max\{-\beta - n - 1, 0\}.$$

We will use the following elementary lemma.
Lemma 4.26. (i) For any $\beta \in \mathbb{R}$ and $j \in \mathbb{Z}_{\geq 0}$, the inequality $j + \beta + n \geq -1$ holds if and only if the inequality $j \geq j_\beta$ holds.

(ii) For any $\beta \in \mathbb{R}$, we have
\[ j_\beta + \beta + n \geq -1. \]

(iii) For $\beta \in \mathbb{R}$ and $r \geq 0$, if $j_\beta + \beta + n \geq r$, we have $j_\beta = j_{\beta-1} = \ldots = j_{\beta-r-1} = 0$.

Corollary 4.27. We have
\[ V_{\zeta_0}^{-1}(\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}][\{\zeta'_0 = 0\}]) = \sum_{j \geq 0, p \in \mathbb{Z}, \beta \in \mathbb{R}, j \geq j_\beta} \zeta_{0,j}^\beta(F_p M^\beta \lambda^p)^\wedge. \]  

Proof. By Fact 2.3 for $\gamma > -1$ we have
\[ V_{\zeta_0}^{-1}(\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}][\{\zeta'_0 = 0\}]) \supseteq \mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}], \]

By Lemma 4.24 the RHS is equal to
\[ \sum_{j, p \in \mathbb{Z}, \beta \in \mathbb{R}, j + \beta + n \geq \gamma} \zeta_{0,j}^\beta(F_p M^\beta \lambda^p)^\wedge. \]

Since $V_{\zeta_0}^{-1}(\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}][\{\zeta'_0 = 0\}])$ is $\zeta_{0,j}^{-1}V_{\zeta_0}^{-1}(\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}][\{\zeta'_0 = 0\}])$ (see Fact 2.3), we have
\[ V_{\zeta_0}^{-1}(\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}][\{\zeta'_0 = 0\}]) = \sum_{j, p \in \mathbb{Z}, \beta \in \mathbb{R}, j + \beta + n \geq -1} \zeta_{0,j}^\beta(F_p M^\beta \lambda^p)^\wedge. \]

Moreover, for $j < 0$, $\beta \in \mathbb{R}$ with $j + \beta + n \geq -1$ and $p \in \mathbb{Z}$, we have
\[
\zeta_{0,j}^\beta(F_p M^\beta \lambda^p)^\wedge = \zeta_{n,j}^{-1}(F_p M^\beta \lambda^p)^\wedge = (\zeta_{n,j}^{-1} F_p M^\beta \lambda^p)^\wedge \\
\supseteq (F_\beta M^\beta + j \lambda^{p-j})^\wedge \\
= \zeta_{0,j}^\beta(F_{p-j} M^\beta + j \lambda^{p-j})^\wedge.
\]

The last term is contained in the RHS of (50). Therefore, together with (i) of Lemma 4.26, the RHS of (51) is equal to the RHS of (50). This completes the proof.

Recall again that $\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}][\{\zeta'_0 = 0\}]$ is a submodule of $\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}]$ generated by $V_{\zeta_0}^{-1}(\mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}])$ (Fact 2.3). Therefore, by Corollary 4.27 we have the following.

Corollary 4.28. We have
\[ \mathcal{M}^\wedge \otimes_{\mathbb{C}[[n]]} \mathbb{C}[z_n^{\pm 1}][\{\zeta'_0 = 0\}] = \sum_{k, j \geq 0, p \in \mathbb{Z}, \beta \in \mathbb{R}, j \geq j_\beta} \zeta_{0,j}^{k_j} F_{p+k_0} M^\beta \lambda^p)^\wedge. \]
Note that the rescaled module $\tau(\mathcal{M}^\wedge \otimes_{\mathbb{C}[n]} \mathbb{C}[\zeta_\pm^\pm][*\{\zeta_0' = 0\}])$ is
\[ O_{U_n'}[\tau^{-1}, \lambda] \otimes O_{U_n'}[\lambda] (\mathcal{M}^\wedge \otimes_{\mathbb{C}[n]} \mathbb{C}[\zeta_\pm^\pm][*\{\zeta_0' = 0\}]), \]
as an $O_{U_n' \times C_r \times C_A}(*\{\tau = 0\})$-module with an $R^{\text{in}}$-module action \[36\].

**Lemma 4.29.** We have
\[ V_2^\gamma(\tau(\mathcal{M}^\wedge \otimes_{\mathbb{C}[n]} \mathbb{C}[\zeta_\pm^\pm][*\{\zeta_0' = 0\}]))) = \bigoplus_{i \in \mathbb{Z}} \sum_{k, j \geq 0, p \in \mathbb{Z}, \beta \in \mathbb{R}} \tau^i \otimes \partial_{C_0'}^k \zeta_0^j (F_p M^\beta \lambda^p)^\wedge. \quad (52) \]

**Proof.** As in the proof of Lemma 4.22 if a section $s \in \tau(\mathcal{M}^\wedge \otimes_{\mathbb{C}[n]} \mathbb{C}[\zeta_\pm^\pm][*\{\zeta_0' = 0\}])$ is killed by $(\tau \partial_{\tau} - \gamma \lambda)^l$ for some $l \geq 0$, $s$ is in $V_2^\gamma(\tau(\mathcal{M}^\wedge \otimes_{\mathbb{C}[n]} \mathbb{C}[\zeta_\pm^\pm][*\{\zeta_0' = 0\}])).$

Let $\beta$ be a real number and $m$ a section of $F_p M^\beta$. For $j, k, \in \mathbb{Z}_{\geq 0}$ with $j + \beta + n \geq 1$ and $i \in \mathbb{Z}$, we consider a section $\tau^i \otimes \partial_{C_0'}^k \zeta_0^j (m \lambda^p)^\wedge \in \tau(\mathcal{M}^\wedge \otimes_{\mathbb{C}[n]} \mathbb{C}[\zeta_\pm^\pm][*\{\zeta_0' = 0\}])$. Recall that we have
\[ \partial_{C_0} = -\lambda \zeta \epsilon \mathcal{C}_n^\wedge \quad \text{and} \quad \zeta_0' = \zeta_0^{-1}, \]
on $U_n' \cap U_n'$ and
\[ \lambda^2 \partial_{\lambda} (m \lambda^p)^\wedge = ((\lambda^2 \partial_{\lambda} + \lambda \epsilon \mathcal{C}_n^\wedge) m \lambda^p)^\wedge = ((p + \epsilon \mathcal{C}_n^\wedge) m \lambda^p + 1)^\wedge. \]

Therefore, we have
\[ \lambda^2 \partial_{\lambda}(\partial_{C_0'}^k \zeta_0^j (m \lambda^p)^\wedge) = \lambda^2 \partial_{\lambda}(((-\lambda \zeta \epsilon \mathcal{C}_n^\wedge)^k \zeta_0^j (m \lambda^p)^\wedge)
= (k \lambda(-\lambda \zeta \epsilon \mathcal{C}_n^\wedge)^k \zeta_0^j + (-\lambda \zeta \epsilon \mathcal{C}_n^\wedge)^k (-\lambda \zeta \epsilon \mathcal{C}_n^\wedge)^k \zeta_0^j \lambda^2 \partial_{\lambda}(m \lambda^p)^\wedge
= k^{(\lambda \zeta \epsilon \mathcal{C}_n^\wedge)^k \zeta_0^j (m \lambda^{p+1})} + (\lambda \zeta \epsilon \mathcal{C}_n^\wedge)^k \zeta_0^j (p + \epsilon \mathcal{C}_n^\wedge) m \lambda^{p+1})^\wedge
= \partial_{C_0'}^k \zeta_0^j ((k + p + \epsilon \mathcal{C}_n^\wedge) m \lambda^{p+1})^\wedge. \]

By using this, we obtain
\[ \tau \partial_{\tau}(\tau^i \otimes \partial_{C_0'}^k \zeta_0^j (m \lambda^p)^\wedge) = i \tau^{i+1} \otimes \partial_{C_0'}^k \zeta_0^j (m \lambda^{p+1})^\wedge - \tau^{i+1} \otimes \lambda^2 \partial_{\lambda}(\partial_{C_0'}^k \zeta_0^j (m \lambda^p)^\wedge)
= i \tau^{i+1} \otimes \partial_{C_0'}^k \zeta_0^j (m \lambda^{p+1})^\wedge - \tau^{i+1} \otimes \partial_{C_0'}^k \zeta_0^j ((k + p + \epsilon \mathcal{C}_n^\wedge) m \lambda^{p+1})^\wedge
= \tau^{i+1} \otimes \partial_{C_0'}^k \zeta_0^j ((i - k - p - \epsilon \mathcal{C}_n^\wedge) m \lambda^{p+1})^\wedge. \]

Therefore, we have
\[ (\tau \partial_{\tau} - (i - k - p - \beta) \lambda)(\tau^i \otimes \partial_{C_0'}^k \zeta_0^j (m \lambda^p)^\wedge) = -\tau^{i+1} \otimes \partial_{C_0'}^k \zeta_0^j ((\epsilon \mathcal{C}_n^\wedge - \beta) m \lambda^{p+1})^\wedge, \]
and hence
\[ (\tau \partial_{\tau} - (i - k - p - \beta) \lambda)^l(\tau^i \otimes \partial_{C_0'}^k \zeta_0^j (m \lambda^p)^\wedge) = (-1)^l \tau^{i+l} \otimes \partial_{C_0'}^k \zeta_0^j ((\epsilon \mathcal{C}_n^\wedge - \beta)^l m \lambda^{p+l})^\wedge, \]

43
for $l \geq 0$. Since the RHS is zero for sufficiently large $l \geq 0$, we conclude that $\tau^i \otimes \partial^k_{\zeta_0} \zeta^{j}_{\zeta_0} (m \lambda^p)^{\wedge}$ is killed by $(\tau \partial_{x} -(i - k - p - \beta)\lambda)^{l}$ for some $l \geq 0$ and hence $\tau^i \otimes \partial^k_{\zeta_0} \zeta^{j}_{\zeta_0} (m \lambda^p)^{\wedge}$ is in $V_{\tau}^{i-k-p-\beta} (\mathcal{M}^{\wedge} \otimes \mathbb{C}[z_n] \mathbb{C}[\zeta_n^{\pm 1}][\zeta_0 = 0])$. We thus conclude that the RHS of (52) is contained in the LHS.

In the same way as the proof of Lemma 4.22 we can show the LHS of (52) is contained in the RHS.

Now, we can compute the irregular Hodge filtration $F^{\text{irr}}_{\alpha \ast} (M^{\wedge} \otimes \mathbb{C}[z_n] \mathbb{C}[\zeta_n^{\pm 1}]) (= F^{\text{irr}}_{\alpha + p \ast} N^{\wedge} |_{U_{n}^{\gamma}})$ in the same way as for $F^{\text{irr}}_{\ast} M^{\wedge}$.

**Theorem 4.30.** For $\alpha \in [0, 1]$ and $p \in \mathbb{Z}$ we have

$$F^{\text{irr}}_{\alpha \ast} (M^{\wedge} \otimes \mathbb{C}[z_n] \mathbb{C}[\zeta_n^{\pm 1}]) = \left( \sum_{j \geq 0, \beta \in \mathbb{R}} \zeta^{j}_{\zeta_0} + (F_{p + 1 - \alpha + \beta} M^{\beta})^{\wedge} \right)$$

$$+ \left( \sum_{k \geq 0, \beta \in \mathbb{R}} \partial^k_{\zeta_0} \zeta^{j}_{\zeta_0} (F_{p - k + 1 - \alpha + \beta} M^{\beta})^{\wedge} \right).$$

(53)

**Proof.** The proof is similar to that of Theorem 4.23. By Lemma 4.29 we have

$$i_{\tau = \lambda}^{\ast} V_{\tau}^{\alpha} (\mathcal{M}^{\wedge} \otimes \mathbb{C}[z_n] \mathbb{C}[\zeta_n^{\pm 1}][\zeta_0 = 0]) = i_{\tau = \lambda}^{\ast} \left( \sum_{\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}} \lambda^i \otimes \partial^k_{\zeta_0} \zeta^{j}_{\zeta_0} (F_{p} M^{\beta})^{\wedge} \right).$$

Therefore, the Rees module of the irregular Hodge filtration (a submodule of $O_{U^{\gamma}_{n}} [\lambda^{\pm 1}] \otimes (M^{\wedge} \otimes \mathbb{C}[z_n] \mathbb{C}[\zeta_n^{\pm 1}])$) is

$$\bigoplus_{i \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{R}} \sum_{\mathbb{Z} \times \mathbb{Z} \times \mathbb{R}} \lambda^i \otimes \partial^k_{\zeta_0} \zeta^{j}_{\zeta_0} (F_{p} M^{\beta})^{\wedge}.$$

Hence, after replacing $i$ by $p$ we have

$$F^{\text{irr}}_{\alpha \ast} (M^{\wedge} \otimes \mathbb{C}[z_n] \mathbb{C}[\zeta_n^{\pm 1}]) = \sum_{k \geq 0, \beta \in \mathbb{R}} \partial^k_{\zeta_0} \zeta^{j}_{\zeta_0} (F_{p - k - 1 - \alpha} M^{\beta})^{\wedge}. \quad \text{(54)}$$

Therefore, by (i) and (ii) of Lemma 4.26 the RHS of (53) is contained in the RHS of (54).

We show the converse inclusion. For $k_0, j_0 \in \mathbb{Z}_{\geq 0}, \beta_0 \in \mathbb{R}$ with $j_0 \geq j_0$ ( $\iff j_0 + \beta_0 + n \geq -1$) and $m \in F_{p - k_0 - \beta_0 + \alpha} M^{\beta_0}$, we consider a section $\partial^k_{\zeta_0} \zeta^{j}_{\zeta_0} M^{\wedge}$, which is in the RHS of (54). By (i) of Lemma 4.26 we have $j_0 \geq j_0$. We set $j'_0 := j_0 - j_0 \in \mathbb{Z}_{\geq 0}$ so that $j_0 + j'_0 = j_0$. 

44
The case: \( j_0 \geq k_0 \)  
In this case, we have
\[
\partial^{k_0} \zeta_0^{j_0} m^\wedge = \sum_{l=0}^{k_0} \partial_{\zeta_0^{j_0}}^{l} \zeta_0^{j_0} m^\wedge \quad \text{(for some } a_s \in \mathbb{Z})
\]
\[
= \sum_{l=0}^{k_0} a_s \zeta_0^{j_0} \zeta_0^{j_0} m^\wedge \quad \text{(by Lemma 4.24)}
\]
\[
\in \sum_{l=0}^{k_0} \zeta_0^{j_0} \zeta_0^{j_0} m^\wedge (\partial_{\zeta_0^{j_0}}(E_{S_g^n} + n))^{k_0-l} m^\wedge
\]
\[
\subset \sum_{l=0}^{k_0} \zeta_0^{j_0} \zeta_0^{j_0} m^\wedge (F_{[p-(k_0-l)+\alpha_j]} M^{\alpha_j-(k_0-l)})^\wedge.
\]

The last term is contained in the first part of RHS of (53) since \( j_0' - l \geq j_0' - k_0 \geq 0 \).

The case: \( j_0 < k_0 \)  
If \( j_0' = 0 \), \( \partial^{k_0} \zeta_0^{j_0} m^\wedge \) is in the RHS of (53). So, we assume \( j_0' \geq 1 \). In this case, we “divide” \( \partial^{k_0} \zeta_0^{j_0} m^\wedge \) as
\[
\partial^{k_0} \zeta_0^{j_0} m^\wedge = \partial^{k_0-j_0'} \zeta_0^{j_0} \zeta_0^{j_0} m^\wedge.
\]

Then, by (53) for \( \partial^{k_0-j_0'} \zeta_0^{j_0} m^\wedge \), the section \( \partial^{k_0-j_0'} \partial^{j_0} \zeta_0^{j_0} m^\wedge \) is contained in
\[
\sum_{l=0}^{j_0'} \partial^{k_0-j_0'} \zeta_0^{j_0} \zeta_0^{j_0} m^\wedge (F_{[p-(k_0-l)+\alpha_j]} M^{\alpha_j-(k_0-l)})^\wedge.
\]

Then, by induction on the exponent of \( \partial^{j_0} \) (remark that \( k_0 - j_0' < k_0 \) since \( j_0' \geq 1 \)), we conclude that the RHS of (56) is contained in the RHS of (53).

Moreover, we have the following.

**Corollary 4.31.**  
(i) For \( \alpha \in [0,1), p \in \mathbb{Z} \) and \( \gamma \in \mathbb{R}_{\geq-1} \), we have
\[
F^{\text{irr}}_{\alpha+p} V^\gamma_{\zeta_0^0} (M^\wedge \otimes \mathbb{C}_{[\zeta_0]} \mathbb{C}_{[\zeta_0^{\pm 1}]}) = \sum_{j \geq 0, \beta \in \mathbb{R}} \zeta_0^{j_0+j} (F_{p+\alpha_j} M^{\beta_j})^\wedge.
\]

(ii) For \( \alpha \in [0,1), p \in \mathbb{Z} \) and \( \gamma \in \mathbb{R}_{<0} \), we have
\[
F^{\text{irr}}_{\alpha+p} V^\gamma_{\zeta_0^0} (M^\wedge \otimes \mathbb{C}_{[\zeta_0]} \mathbb{C}_{[\zeta_0^{\pm 1}]}) = \sum_{k \geq 0, \beta \in \mathbb{R}} \zeta_0^{j_0+j} (F_{p-k+\alpha_j} M^{\beta_j})^\wedge,
\]
where the sum \( \sum \) in the RHS is the one defined in Remark 4.25.

**Proof.** Recall that \( M^\wedge \otimes \mathbb{C}_{[\zeta_0]} \mathbb{C}_{[\zeta_0^{\pm 1}]} \) is monodromic with respect to the \( \zeta_0' \)-direction by Lemma 4.24 and we have
\[
(M^\wedge \otimes \mathbb{C}_{[\zeta_0]} \mathbb{C}_{[\zeta_0^{\pm 1}]})_{\zeta_0}^{j} = \sum_{j \in \mathbb{Z}, \gamma \in \mathbb{R}} \zeta_0^{j_0+j} (M^\gamma)^\wedge,
\]

45
Therefore, we have \( \partial_{s} \) is contained in \( V_{\gamma} \) and only if \( j_{\beta} + \beta + n + j \geq \gamma \) (resp. \( j_{\beta} + \beta + n - k \geq \gamma \)). Hence, by Theorem 4.30, we have

\[
F_{\alpha+p}^{\gamma} V_{\gamma}^{\gamma} (M^\wedge \otimes \mathbb{C}[\zeta_n]) \subset \mathbb{C}[\zeta_n^{\pm 1}] = 0. \tag{60}
\]

To prove (ii), we assume that \( \gamma \in \mathbb{R}_{>0} \). The first term of (59) is contained in \( V_{\gamma}^{\gamma} (M^\wedge \otimes \mathbb{C}[\zeta_n]) \) unless \( s = 0 \). Hence, for \( \gamma < 0 \), we have

\[
F_{\alpha+p}^{\gamma} V_{\gamma}^{\gamma} (M^\wedge \otimes \mathbb{C}[\zeta_n]) \simeq \sum_{k \geq 0, \beta \in \mathbb{R}} \partial_{s_{\gamma}}^{k} \zeta_{0}^{j_{\beta}} (F_{p+k+\alpha-\beta} M^{\beta})^\wedge. \tag{61}
\]

\[\square\]

**Corollary 4.32.** For \( \alpha \in [0, 1] \), the irregular Hodge filtration \( F_{\alpha+p}^{\gamma} (M^\wedge \otimes \mathbb{C}[\zeta_n]) \subset \mathbb{C}[\zeta_n^{\pm 1}] \) satisfies the strict specializability property along \( \zeta_{0} = 0 \).

**Proof.** First, let us see the condition (i) in Definition 4.12. For \( \gamma > -1 \), by (i) of Corollary 4.31, we have

\[
F_{\alpha+p}^{\gamma} V_{\gamma}^{\gamma} (M^\wedge \otimes \mathbb{C}[\zeta_n]) \simeq \sum_{j \geq 0, \beta \in \mathbb{R}} \partial_{s_{\gamma}}^{j} \zeta_{0}^{j_{\beta}+j} (F_{p+j+\alpha-\beta} M^{\beta})^\wedge. \tag{61}
\]

It is enough to see that for \( \gamma = \gamma_0 > 0 \) and a section \( \sigma \) in the RHS of (61), \( \zeta_{0}^{-1} \sigma \) is in the RHS of (61) for \( \gamma = \gamma_0 - 1 \). Consider a section \( \zeta_{0}^{j_{\beta}+j} m^\wedge \) for \( j_{\beta} + \beta + n + j = \gamma_0 \) and \( m \in F_{p+j+\alpha-\beta} M^{\beta} \). If \( j \geq 1 \), it is clear that \( \zeta_{0}^{-1} (\zeta_{0}^{j_{\beta}+j} m^\wedge) \) is in the RHS of (61) for \( \gamma = \gamma_0 - 1 \).
So, let us assume that \( j = 0 \). By (iii) of Lemma 4.26, we have \( j_\beta = j_{\beta-1} = 0 \). Therefore, we have
\[
ζ_0^{j-1} \cdot ζ_0^{j_\beta} m^\wedge = ζ_0^{j_\beta}(\partial_{\zeta_0} m)^\wedge
\]
\[
\subset ζ_0^{j_\beta}(F_{p\cdot[\alpha-\beta]+1}M^{\beta-1})^\wedge
\]
\[
= ζ_0^{j_\beta-1}(F_{p\cdot(\alpha-\beta-1)}M^{\beta-1})^\wedge \quad (\text{by } j_\beta = j_{\beta-1} = 0).
\]
The last term is contained in the RHS of (61) for \( γ = γ_0 - 1 \). This completes the proof of the condition (i) in Definition 1.12.

Let us check the condition (ii) in Definition 1.12. By (ii) of Corollary 4.31, for \( γ < 0 \) we have
\[
F_{\alpha+p}^{\text{irr}}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]) = \sum_{k \geq 0, \beta \in \mathbb{R}} \partial_{\zeta_0}^k ζ_0^{j_\beta}(F_{p-k+\beta}M^\beta)^\wedge.
\] (62)

For \( γ_0 < -1 \) (not \( < 0 \)), consider a section \( \partial_{\zeta_0}^k ζ_0^{j_\beta} m^\wedge \) with \( j_\beta + \beta + n - k = γ_0 \) and \( m \in F_{p-k+\beta}M^\beta \). Since \( j_\beta + \beta + n \geq -1 \) by (ii) of Lemma 4.26, we have \( k \geq 1 \). Moreover, \( \partial_{\zeta_0}^{k-1} ζ_0^{j_\beta} m^\wedge \) is in the RHS of (62) for \( γ = γ_0 - 1 \). Therefore, we have
\[
F_{\alpha+p}^{\text{irr}}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]) \subset ζ_0^{j_\beta} \cdot F_{\alpha+p-1}^{\text{irr}}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]),
\]
which is the condition (ii) in Definition 1.12.

This completes the proof. \( \square \)

Remark 4.33. Corollary 4.32 is derived by Theorem 1.6 of Mochizuki [8], which is an assertion about the strict specializability in a more general setting. The above proof is a concrete verification of this fact.

Finally, we check that “the irregular Hodge filtration at infinity is localized”.

Corollary 4.34. (i) For \( α \in [0, 1) \), we have
\[
F_{\alpha+p}^{\text{irr}} V_{\zeta_0}^{-1}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]) = ζ_0^{j_\beta} F_{\alpha+p}^{\text{irr}} V_{\zeta_0}^{0}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]).
\]

(ii) We have
\[
F_{\alpha+p}^{\text{irr}}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]) = \sum_{k \geq 0} \partial_{\zeta_0}^k F_{\alpha+p-k}^{\text{irr}} V_{\zeta_0}^{-1}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]).
\]

Proof. The assertion (i) of Corollary 4.31 implies that
\[
F_{\alpha+p}^{\text{irr}} V_{\zeta_0}^{-1}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]) \subset ζ_0^{j_\beta} F_{\alpha+p}^{\text{irr}} V_{\zeta_0}^{0}(M^\wedge \otimes \mathbb{C}[\zeta_n] \mathbb{C}[\zeta_n^{\pm 1}]).
\]
Conversely, we consider \( \zeta_0^{j_0} \cdot \zeta_0^{j_\beta} (F_{p+\alpha-\beta}^+ M^\beta)^\wedge \), where \( \zeta_0^{j_0} \cdot \zeta_0^{j_\beta} (F_{p+\alpha-\beta}^+ M^\beta)^\wedge \) is the term for \( s = 0 \) and \( l = 0 \) in \( \beta \). Since \( j_\beta + \beta + n \geq 0 \), by (iii) of Lemma 4.26 we have \( j_\beta = j_{\beta-1} = 0 \). Therefore, we have

\[
\zeta_0^{j_0} \cdot \zeta_0^{j_\beta} (F_{p+\alpha-\beta}^+ M^\beta)^\wedge = \zeta_0^{j_0} (F_{p+\alpha-\beta}^+ M^\beta)^\wedge \\
=(\partial_{zn} F_{p+\alpha-\beta}^+ M^\beta)^\wedge \\
\subset (F_{p+\alpha-\beta+1}^+ M^{\beta-1})^\wedge \\
= \zeta_0^{j_\beta} (F_{p+\alpha-(\beta-1)}^+ M^{\beta-1})^\wedge \quad \text{(by } j_\beta = j_{\beta-1}).
\]

The last term is contained in \( F_{\alpha+p}^{\text{irr}} V_0^{-1} (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}]) \). This completes the proof of (i).

(ii) follows from the strict specializability Corollary 4.32.

For a filtered \( D \)-module \( (M, F M) \) on \( U_{i}^\wedge \), we define \( (M, F M)[\{\zeta_0' = 0\}] \) as the filtered \( D \)-module \( M(\{\zeta_0' = 0\}) \) with the filtration defined by the same formula as (v) of Fact 2.3. Then, we can write

\[
(M^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}], F_{\alpha+i}^{\text{irr}} (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}]) = (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}], F_{\alpha+i}^{\text{irr}} (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}]))[\{\zeta_0' = 0\}]
\]

This corollary means that the irregular Hodge filtration has the same properties as the Hodge filtration of the localization of an usual mixed Hodge module.

Obviously, we have the same statement also for the irregular Hodge filtration of \( N^\wedge \bigcap U_{i}^\wedge \) for \( i = 1, \ldots, n - 1 \). Therefore, we can restate Corollaries 4.32 and 4.34 as follows.

**Corollary 4.35.** For \( \alpha \in [0, 1) \), the irregular Hodge filtration \( F_{\alpha+i}^{\text{irr}} N^\wedge \) has the strict specializability property along \( D_{i}^\wedge \). Moreover, we have

\[
(N^\wedge, F_{\alpha+i}^{\text{irr}} N^\wedge) = (N^\wedge, F_{\alpha+i}^{\text{irr}} N^\wedge)[\{D_{i}^\wedge\}].
\]

**Remark 4.36.** We will later prove that the filtration \( \{F_{p}^{\text{irr}} N^\wedge\}_{p \in \mathbb{Z}} \) is the Hodge filtration of a mixed Hodge module (Corollary 4.50). Since the Hodge filtration of a mixed Hodge module is strictly specializable along any divisor, Corollary 4.35 for \( \alpha = 0 \) follows also from this fact. Corollary 4.35 is an improvement on it since it says that the strict specializability properties hold also for other \( \alpha \in [0, 1) \).

**Remark 4.37.** By (53), the restriction of \( F_{\alpha+i}^{\text{irr}} (M^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}]) \) to \( U_{i}^\wedge \cap U_{n}^\wedge \) is

\[
\sum_{\beta \in \mathbb{R}} (F_{p+\alpha-\beta}^+ M^\beta)^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}] = (F_{p+\alpha}^+ M^\beta)^\wedge \otimes_{C[\zeta_n]} C[\zeta_n^{\pm 1}].
\]

Therefore, the computation (53) is consistent with Theorem 4.28.

**Remark 4.38.** We remark that we can generalize all the results in this subsection, especially Theorems 4.23 and 4.30 to mixed Hodge modules on a vector bundle on a smooth algebraic variety. For this purpose, it is enough to prove them in the case of trivial vector bundles. We omit the details.
4.4 The irregular Hodge filtration and the mixed Hodge module structure of $M^\wedge$

We continue to consider the setting of the previous subsection. In Section 3, we defined a mixed Hodge module structure on $M^\wedge$ and thus $M^\wedge$ is equipped with the Hodge filtration $F_pM^\wedge$. On the other hand, in the previous subsection we computed the irregular Hodge filtration $F^\text{irr}_\alpha M^\wedge$ on $M^\wedge$ for $\alpha \in [0, 1)$. In this subsection, we will prove the following.

**Theorem 4.39.** We have the equality

$$F^\text{irr}_p M^\wedge = F_p M^\wedge$$

for any $p \in \mathbb{Z}$.

By Theorem 4.23, we have the following.

**Corollary 4.40.** For $p \in \mathbb{Z}$, we have

$$F_p M^\wedge = \bigoplus_{\beta \in \mathbb{R}} (F_{p+\lfloor -\beta \rfloor} M^\beta)^\wedge.$$

Recall that the mixed Hodge module structure on $M^\wedge$ is defined by the formula (35). Since it is difficult to compute the Hodge filtration of the push forward of a mixed Hodge module in general, it is also difficult to compute $F_p M^\wedge$ just following the definition. So, we take a different approach, which takes the advantage of the strength of the theory of mixed twistor $D$-modules and the irregular Hodge theory.

**Notation 4.41.** For an $R^\text{int}$-module $\mathcal{M}$ and an integer $l \in \mathbb{Z}$, we set

$$\mathcal{M}(l) := \lambda^l \mathcal{M}.$$

Note that if $\mathcal{M}$ is the Rees module $R_F M$ corresponding to a filtered $D$-module $(M, F^\bullet M)$, we have

$$\mathcal{M}(l) = R_F(M(l)),$$

where $M(l)$ is the Tate twist of the filtered $D$-module $(M, F^\bullet M)$.

We need to generalize Lemmas 3.15 and 3.17 to $R$-modules.

**Lemma 4.42.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the underlying $R^\text{int}$-modules of mixed twistor $D$-modules on a smooth algebraic variety $X$. Assume that the intersection of the characteristic varieties $\text{Ch}(\mathcal{M}_1)$ and $\text{Ch}(\mathcal{M}_2)$ is contained in a zero section $\mathbb{C}_\lambda \times T^*X$. Then, for an algebraic variety $Y$ with a morphism $f: Y \to X$, we have

$$T^f \mathcal{M}_1 \otimes T^f \mathcal{M}_2 = T^f(\mathcal{M}_1 \otimes \mathcal{M}_2)(d_f)[d_f],$$

where we put $d_f := \dim Y - \dim X$.

**Remark 4.43.** In [8], we say “$\mathcal{M}_1$ and $\mathcal{M}_2$ are non-characteristic” if the assumption in Lemma 4.42 holds.
Proof. Let $\Delta_X : X \to X \times X$ and $\Delta_Y : Y \to Y \times Y$ be the diagonal embeddings. By the assumption, $\Delta_X$ is non-characteristic with respect to $\mathcal{M}_1 \boxtimes \mathcal{M}_2$. Therefore, by Corollary 4.56 of [8], $\dim X(= 2\dim X - \dim X)$-th one is the only non-trivial cohomology of $T\Delta_X^!(\mathcal{M}_1 \boxtimes \mathcal{M}_2)$, and we have

$$H^{\dim X}(T\Delta_X^!(\mathcal{M}_1 \boxtimes \mathcal{M}_2)) \simeq \Delta_X^!(\mathcal{M}_1 \boxtimes \mathcal{M}_2)(- \dim X). \tag{63}$$

The RHS is $\mathcal{M}_1 \otimes \mathcal{M}_2(- \dim X)$. By using this fact, we have

$$Tf! \mathcal{M}_1 \otimes Tf! \mathcal{M}_2 \simeq Tf! (\mathcal{M}_1 \boxtimes \mathcal{M}_2)(\dim Y)[\dim Y] \simeq Tf! T\Delta_Y^!(\mathcal{M}_1 \boxtimes \mathcal{M}_2)(\dim Y)[\dim Y] \simeq Tf! (\mathcal{M}_1 \boxtimes \mathcal{M}_2)(d_f)[d_f].$$

Let $E$ be an algebraic vector bundle over a smooth algebraic variety $X$.

Lemma 4.44. Let $Y$ be a smooth algebraic variety and $f : Y \to X$ a morphism. We denote by $u$ (resp. $u^\vee$) the natural morphism $f^*E \to E$ (resp. $f^*E^\vee \to E^\vee$). For the underlying $R^{\text{int}}$-module $\mathcal{M}$ of an integrable mixed twistor $D$-module on $E$, we have a natural isomorphism in the category of $R^{\text{int}}$-modules

$$(H^j(Tu! \mathcal{M}))^\wedge \simeq H^j(T(u^\vee)! \mathcal{M}^\wedge) \quad (j \in \mathbb{Z}).$$

Proof. We consider the following diagram

$$
\begin{array}{ccc}
E & \xrightarrow{p} & E \times X \\
\downarrow{u} & & \downarrow{u \times u^\vee} \\
E & \xrightarrow{q} & E^\vee
\end{array}
\quad f^*E & \xrightarrow{f^*E \times_Y f^*E^\vee} & f^*E^\vee
\preceq
\quad f^*E \times_Y f^*E^\vee & \xrightarrow{u} & f^*E^\vee.

Note that since any projection is non-characteristic with respect to any $R$-module, we have

$$T(p^! \varphi)^! \simeq (p^! \varphi)^!(n_E)[n_E],$$

where $n_E$ is the rank of $E$. Moreover, we have

$$\xi_{f^*E \times_Y f^*E^\vee} \simeq (u \times u^\vee)^! \xi_{E \times_X E} \simeq f!(u \times u^\vee)^! \xi_{E \times_X E}(-d_f)[-d_f].$$

50
Therefore, we have
\[
(H^j(T^1 u^*)^\lambda) \simeq T^j q_* ((p)^*(H^j T^1 u^*) \otimes E^{-\varphi/\lambda})
\]
\[
\simeq H^j T^1 q_* ((p)^*(T^1 u^*) \otimes E^{-\varphi/\lambda})
\]
\[
\simeq H^j T^1 q_* ((T^1 (p)^* (T^{1} u^*) \otimes T(u \times u^\dagger) E^{-\varphi/\lambda})(-d_f - n_E)[-d_f - n_E]
\]
\[
\simeq H^j T^1 q_* ((T^1 (u \times u^\dagger) (T^1 (p)^* M) \otimes T(u \times u^\dagger) E^{-\varphi/\lambda})(-d_f - n_E)[-d_f - n_E]
\]
\[
\simeq H^j T^1 q_* ((T^1 (u \times u^\dagger) (T^1 (p)^* M) \otimes E^{-\varphi/\lambda}))(-n_E)[-n_E]
\]
\[
\simeq H^j T^1 (u^\dagger) ((T^1 (p)^* M) \otimes E^{-\varphi/\lambda}))(-n_E)[-n_E]
\]
\[
\simeq H^j T^1 (u^\dagger) ((T^1 (p)^* M) \otimes E^{-\varphi/\lambda}))
\]
\[
= H^j T^1 (u^\dagger)^* \mathcal{M}^\lambda,
\]
where the second isomorphism follows from the exactness of Fourier-Laplace transform, the
4-th isomorphism follows from Lemma 4.42 and the 6-th isomorphism follows from the base
change: Proposition 14.3.27 of [7].

Lemma 4.45. Let \( f : X \to Y \) be a morphism between smooth algebraic variety \( X \) and \( Y \) and \( \mathcal{M} \) (resp. \( \mathcal{L} \)) be the underlying \( \mathcal{R}^{\mathcal{M}} \)-module of an integrable mixed twistor \( D \)-module (resp. a smooth integrable mixed twistor \( D \)-module, i.e. an admissible variation of mixed twistor structure) on \( X \) (resp. \( Y \)). Then, we have the following isomorphism in the derived category of \( \mathcal{R}^{\mathcal{M}} \)-modules:
\[
T^j f_* (\mathcal{M} \otimes f^* \mathcal{L}) \simeq T^j f_* \mathcal{M} \otimes \mathcal{L}.
\]

Proof. Take a smooth variety \( \overline{X} \) containing \( X \) such that \( H_X := \overline{X} \backslash X \) is a divisor in \( \overline{X} \),
and a proper morphism \( \overline{f} : \overline{X} \to Y \) which induces \( f : X \to Y \). Moreover, take the underlying \( \mathcal{R}^{\mathcal{M}} \)-module \( \mathcal{M} \) of a mixed twistor \( D \)-module on \( \overline{X} \) whose restriction \( \mathcal{M}|_X \) is \( \mathcal{M} \). Then, we have
\[
T^j f_* (\mathcal{M} \otimes f^* \mathcal{L}) = T^j f_* ((\mathcal{M} \otimes \overline{f}^* \mathcal{L})[*H_X]).
\]
Let \( \Delta_X : \overline{X} \hookrightarrow \overline{X} \times \overline{X} \) and \( \Delta_Y : Y \hookrightarrow Y \times Y \) be the diagonal embedding. Then, by Proposition 4.58 of [8], we have
\[
(\mathcal{M} \otimes \overline{f}^* \mathcal{L})[*H_X] \simeq T^1 \Delta_X (\mathcal{M}[*H_X] \otimes \overline{f}^* \mathcal{L})(\dim X)[\dim X].
\]
Therefore, we have
\[
T^j f_* (\mathcal{M} \otimes f^* \mathcal{L}) \simeq T^j f_* T^1 \Delta_X (\mathcal{M}[*H_X] \otimes \overline{f}^* \mathcal{L})(\dim X)[\dim X]
\]
\[
\simeq T^j \Delta_Y (\mathcal{M} \otimes \overline{f}^* \mathcal{L})(\dim X)[\dim X]
\]
\[
\simeq T^j \Delta_Y (\mathcal{M} \otimes \overline{f}^* \mathcal{L})(\dim X)[\dim X].
\]
where we used the base change for the second isomorphism. Let \( p_{X,i} \) (resp. \( p_{Y,i} \)) be the \( i \)-th projection \((i = 1, 2)\) of \( X \times X \) (resp. \( Y \times Y \)). Then, we have

\[
((f \times f)^\dagger (\overline{\mathcal{M}} \otimes f^* L)) = (f \times f)^\dagger (p_{X,1}^* \overline{\mathcal{M}}[\ast H_X] \otimes p_{Y,2}^* L)
\]

where the final isomorphism follows from the projection formula. Moreover, since \( T p_{X,1}^! \simeq p_{X,1}^!(\dim X)[\dim X] \), we have

\[
((f \times f)^\dagger (p_{X,1}^* \overline{\mathcal{M}}[\ast H_X])) \simeq T (f \times f)^\dagger (p_{X,1}^* \overline{\mathcal{M}}[\ast H_X])(-\dim X)[-\dim X]
\]

where we used the base change formula for the third isomorphism. Combining it with (64) and (65), we obtain

\[
T f_* (\overline{\mathcal{M}} \otimes f^* L) \simeq T (T f_* (\overline{\mathcal{M}} \otimes L))(\dim Y)[\dim Y]
\]

where the last isomorphism follows from (63) (or Proposition 4.58 of [8]).

**Lemma 4.46.** Let \( F \) be another vector bundle over \( X \), \( f: E \to F \) a morphism and \( n_E \) (resp. \( n_F \)) the rank of the vector bundle \( E \) (resp. \( F \)). We denote by \( ^t f: F^\vee \to E^\vee \) its transpose morphism. For the underlying \( R^{\text{int}} \)-module \( \mathcal{M} \) of an integrable mixed twistor \( D \)-module on \( E \), we have a natural isomorphism in the category of \( R^{\text{int}} \)-modules

\[
(H^j T f_* \mathcal{M})^\wedge \simeq H^{j+n_f}(T (^t f)^! \overline{\mathcal{M}} \otimes L)(n_f),
\]

where we put \( n_f := n_E - n_F \).

**Proof.** We consider the following diagram

\[
\begin{array}{c}
E \xrightarrow{p} E \times X \xrightarrow{q} E^\vee \\
\downarrow p' \quad \downarrow 1 \times f \quad \downarrow q' \\
F \xrightarrow{f'} F \times X \xrightarrow{q'} F^\vee
\end{array}
\]

52
Then, in the same way as the argument in the proof of Lemma 4.44, we have

\[(H^3T\mathcal{M})^\wedge \simeq H^3Tt_*(p^*T\mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda})\]
\[
\simeq H^3Tq_*(T(p^*T\mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda})(-n_F)[-n_F]
\]
\[
\simeq H^3Tq_*(T(f \times 1)_*(p^n)^\wedge \mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda})(-n_F)[-n_F]
\]
\[
\simeq H^3Tq_*(T(p^n)^\wedge \mathcal{M} \otimes (f \times 1)^*\mathcal{E}^{-\varphi/\lambda})(-n_F)[-n_F]
\]  

(66)

where we used the base change formula for the third isomorphism and Lemma 4.45 for the 4-th isomorphism. Since $1 \times t\mathfrak{f}$ is non-characteristic with respect to $\mathcal{E}^{-\varphi/\lambda}_{E \times X E^v}$, we have

\[(f \times 1)^*\mathcal{E}^{-\varphi/\lambda}_{E \times X E^v} \simeq (1 \times t\mathfrak{f})^*\mathcal{E}^{-\varphi/\lambda}_{E \times X E^v}
\]
\[
\simeq T(1 \times t\mathfrak{f})^\wedge \mathcal{E}^{-\varphi/\lambda}_{E \times X E^v}(n_f)[n_f].
\]

Therefore, we have

\[
T(p^n)^\wedge \mathcal{M} \otimes (f \times 1)^*\mathcal{E}^{-\varphi/\lambda}(n_f)[n_f]
\]
\[
\simeq T(1 \times t\mathfrak{f})^\wedge \mathcal{E}^{-\varphi/\lambda}(n_f)[n_f]
\]
\[
\simeq (1 \times t\mathfrak{f})^\wedge \mathcal{E}^{-\varphi/\lambda}(n_f)[n_f]
\]

where we used Lemma 4.42 for the second isomorphism. Combining it with (66), we have

\[(H^3T\mathcal{M})^\wedge \simeq H^3Tt_*(1 \times t\mathfrak{f})^\wedge (T(p^n)^\wedge \mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda})(-n_F)[-n_F]
\]
\[
\simeq H^3T(1 \times t\mathfrak{f})^\wedge Tq_*(p^n)^\wedge \mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda}(n_f)[n_f]
\]
\[
\simeq H^3T(1 \times t\mathfrak{f})^\wedge Tq_*(\mathcal{M} \otimes \mathcal{E}^{-\varphi/\lambda})(n_f)
\]

where we used the base change formula for the second isomorphism. This completes the proof. 

For a vector bundle $E$ over $X$, we use the morphisms $\omega: E \times X E \to \mathbb{C} \times E$, $p: E \times X E^v \to E$ and $\iota: E^v \hookrightarrow \mathbb{C}^v \times E^v$ defined in Section 3 just before Lemma 3.18.

**Lemma 4.47.** For the underlying $R^{\text{int}}$-module $\mathcal{M}$ of an integrable mixed twistor $D$-module on $E$, we have the isomorphism:

\[
\mathcal{M}^\wedge \simeq H^1T\iota^\wedge (H^0\omega^*p^*\mathcal{M})^\wedge (1).
\]

**Proof.** Since $p$ is a projection, we have

\[
T(p^n)^\wedge \mathcal{M} \simeq p^*\mathcal{M}(\dim X)[\dim X].
\]

Moreover, since $(H^2\omega^*\mathcal{M})^\wedge$ is monodromic by Lemma 3.19, $\iota$ is non-characteristic with respect to $(H^3T\omega^*\mathcal{M})^\wedge$. Therefore, we have

\[
T(\iota)^\wedge (H^0\omega^*p^*\mathcal{M})^\wedge \simeq \iota^*\mathcal{M}^\wedge (1) \simeq (H^0\omega^*\mathcal{M})^\wedge (-1)[-1].
\]



53
Therefore, we have
\[ H^{1T_1}(H^{0T_1} \omega \ast p^\ast \mathcal{M})^\wedge \simeq H^{1T_1}(H^{nE-1T_1} \omega)^1 (p^\ast \mathcal{M})^\wedge (n_E - 1) \quad \text{(by Lemma 4.46)} \]
\[ \simeq H^{1T_1}(H^{nE-1T_1} \omega)^1 (H^{nE T_1} p^\ast \mathcal{M})^\wedge (-n_E) (n_E - 1) \]
\[ \simeq H^{1T_1}(H^{nE-1T_1} \omega)^1 (H^{nE T_1} p^\ast \mathcal{M})^\wedge (-1) \quad \text{(by Lemma 4.44)} \]
\[ \simeq H^{0T}(p^\ast \circ \iota \circ \omega \circ \iota)^1 \mathcal{M}^\wedge (-1) \]
\[ \simeq \mathcal{M}^\wedge (-1). \]
\[ \square \]

Let us consider the irregular Hodge filtration of the right hand side of (67). We define the terminology: “monodromic $R$-modules” in a similar way to Definition 1.3. Remark that if an $R$-module $\mathcal{M}_1$ on $\mathbb{C}^\vee \times \mathbb{E}^\vee$ is monodromic with respect to $\mathbb{C}^\vee$-direction, $\iota$ is non-characteristic with respect to $\mathcal{M}_1$. In particular, we have $H^{1T_1} \mathcal{M}_1 \simeq \iota^\ast \mathcal{M}_1(1)$.

**Lemma 4.48.** Let $\mathcal{M}_1$ be the underlying $R$-module of an irregular Hodge module on $\mathbb{C}^\vee \times \mathbb{E}^\vee$ and its underlying $D$-module is denoted by $M_1$. Assume that $\mathcal{M}_1$ (resp. $\tau \mathcal{M}_1$) is monodromic with respect to the $\mathbb{C}^\vee$-direction (resp. $\mathbb{C}_\tau$-direction). Moreover, assume that $H^{1T_1} \mathcal{M}_1(\tau \mathcal{M}_1(1))$ is the underlying $R$-module of an irregular Hodge module on $\mathbb{E}^\vee$. Then, for $\alpha \in [0, 1)$, we have

\[ F_{\alpha + \bullet}^\ast M_1 = \iota^\ast F_{\alpha + \bullet + 1}^\ast M_1, \quad (68) \]

where we regard $\iota^\ast M_1$ as the underlying $D$-module of $H^{1T_1} \mathcal{M}_1$.

**Proof.** Consider the rescaling $\tau(\iota^\ast \mathcal{M}_1)$ (resp. $\tau \mathcal{M}_1$), which is an object on $\mathbb{C}_\tau \times \mathbb{E}^\vee$ (resp. $\mathbb{C}_\tau \times \mathbb{C}^\vee \times \mathbb{E}^\vee$). We denote by the same symbol $\iota$ the inclusion $\mathbb{C}_\tau \times \mathbb{E}^\vee \hookrightarrow \mathbb{C}_\tau \times \mathbb{C}^\vee \times \mathbb{E}^\vee$. Then, by the definition, it is obvious that we have

\[ \tau(\iota^\ast \mathcal{M}_1) \simeq \iota^\ast \tau \mathcal{M}_1. \]

Note that for $\gamma \in \mathbb{R}$ we have

\[ V_\gamma(\tau \mathcal{M}_1) = \bigoplus_{\beta \geq \gamma} (\tau \mathcal{M}_1)^\gamma, \quad (69) \]

where $(\tau \mathcal{M}_1)^\gamma = \bigoplus_{k \geq 0} \ker(\tau \partial_\gamma - \gamma)^k \subset \tau \mathcal{M}_1$. Moreover, it is clear that for $\beta \in \mathbb{R}$ we have

\[ (\iota^\ast \tau M_1)^\beta = \iota^\ast (\tau M_1)^\beta. \]

Therefore, we have

\[ V_\gamma \iota^\ast \tau \mathcal{M}_1 = \bigoplus_{\beta \geq \gamma} \iota^\ast (\tau \mathcal{M}_1)^\beta \]
\[ = \iota^\ast V_\gamma(\tau \mathcal{M}_1) \quad \text{(by (69))}. \]

Hence, we obtain

\[ \iota_{\tau = \lambda}^\ast V_\beta(\iota^\ast \tau \mathcal{M}_1(1)) = \lambda^{-1} \iota^\ast (\iota_{\tau = \lambda}^\ast V_\beta(\tau \mathcal{M}_1)), \]

in $\iota^\ast M_1[\lambda^{\pm 1}]$. This equality means the equality (68).\[ \square \]
Remark 4.49. In [8] (see Theorem 1.5 in loc. cit.), Lemma 4.48 and some stronger results are shown in a more general situation. For example, $H^1_t \N$ is always an irregular Hodge module. But, we do not need it here.

Proof of Theorem 4.39. By Lemma 4.47, we have
\[ F_{p}^{\text{irr}} M^\wedge \simeq F_{p-1}^\text{irr} H^1 t^\dagger (H^0 \omega_t H^{-n} p^\dagger M)^\wedge. \]

By Lemma 4.48, the RHS is equal to
\[ t^\ast F_{p}^{\text{irr}} (H^0 \omega_t H^{-n} p^\dagger M)^\wedge. \] (70)

By (29) and Theorem 4.23, the Hodge filtration (defined by Lemma 3.10) of the Fourier-Laplace transform of a monodromic mixed Hodge module on a line bundle coincides with the irregular Hodge filtration (for $\alpha = 0$). Therefore, for $p \in \mathbb{Z}$ we have
\[ F_{p}^{\text{irr}} (H^0 \omega_t H^{-n} p^\dagger M)^\wedge = F_{p} (H^0 \omega_t H^{-n} p^\dagger M)^\wedge, \] (71)

where the RHS is the Hodge filtration defined by Lemma 3.10. Hence, (70) is equal to
\[ t^\ast F_{p} (H^0 \omega_t H^{-n} p^\dagger M)^\wedge. \]

On the other hand, by definition 3.20, we have
\[ F_{p} M^\wedge = F_{p-1} H^1 t^\dagger (H^0 \omega_t H^{-n} p^\dagger M)^\wedge. \]

By the definition of the pullback functor $H^1 t^\dagger$ (between the category of mixed Hodge modules), we have
\[ F_{p-1} H^1 t^\dagger (H^0 \omega_t H^{-n} p^\dagger M)^\wedge \simeq t^\ast F_{p} (H^0 \omega_t H^{-n} p^\dagger M)^\wedge. \]

Combining these equality together, we obtain
\[ F_{p} M^\wedge = F_{p}^{\text{irr}} M^\wedge. \]

\[ \square \]

Finally, we discuss the relationship between the irregular Hodge filtration and the Hodge filtration of $M^\wedge$ “at infinity”. Let $M^\wedge$ be the mixed Hodge module defined in Definition 3.20. Moreover, let $\widetilde{M}^\wedge$ be the mixed Hodge module which is the unique extension of $M^\wedge$ to $D^\wedge_{\xi}$ such that $\widetilde{M}^\wedge = \widetilde{M}^\wedge [\ast D^\wedge_{\infty}]$. We denote by $\widetilde{M}^\wedge$ the underlying $D$-module. Then, we have (by Lemma 3.5)
\[ \widetilde{M}^\wedge = N^\wedge, \]

where $N$ is the one defined in the first part of Subsection 4.3. By Theorem 4.39 and Corollary 4.35 we have the following.

Corollary 4.50. We have
\[ F_{p}^{\text{irr}} N^\wedge = F_{p} \widetilde{M}^\wedge, \]

for any $p \in \mathbb{Z}$. In particular, the irregular Hodge filtration $\{ F_{p}^{\text{irr}} N^\wedge \}_{p \in \mathbb{Z}}$ (for $\alpha = 0$) is the Hodge filtration of a mixed Hodge module.
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