Exponential inequalities under the sub-linear expectations with applications to laws of the iterated logarithm

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Abstract

Kolmogorov’s exponential inequalities are basic tools for studying the strong limit theorems such as the classical laws of the iterated logarithm for both independent and dependent random variables. This paper establishes the Kolmogorov type exponential inequalities of the partial sums of independent random variables as well as negatively dependent random variables under the sub-linear expectations. As applications of the exponential inequalities, the laws of the iterated logarithm in the sense of non-additive capacities are proved for independent or negatively dependent identically distributed random variables with finite second order moments. For deriving a lower bound of an exponential inequality, a central limit theorem is also proved under the sub-linear expectation for random variables with only finite variances.

Keywords: sub-linear expectation; capacity; Kolmogorov’s exponential inequality; negative dependence; laws of the iterated logarithm; central limit theorem.

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1 Introduction and notations.

Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus, c.f. Denis and Martini (2006), Gilboa (1987), Marinacci (1999), Peng (1997, 1999, 2006, 2008a) etc. This paper considers the general sub-linear expectations and related non-additive probabilities generated by them. Though some specified sub-linear expectations...
such as $g$-expectation, $G$-expectation, are introduced from stochastic calculus (c.f., Peng (1997, 1999, 2006, 2008a)), the general sub-linear expectation is introduced by Peng (2008b) in a general function space and is a natural extension of the classical linear expectation with the linear property being replaced by the sub-additivity and positive homogeneity (c.f. Definition 2.1 below). However, this simple generalization provides a very flexible framework to model non-additive probability problems and produces many interesting properties different from those of the linear expectations. For example, one constant is not enough to characterize the mean or variance of a random variable in a sub-linear expectation space, the limit in the law of large numbers is no longer a contact, and, comparing to the classical one-dimensional normal distribution which is characterized by the Stein equation, an ordinary differential equation (ODE), a normal distribution under the sub-linear expectation is characterized by a time-space parabolic partial different equation (PDE). Roughly speaking, a sub-linear expectation is related to a group of unknown linear expectations and the distribution under a sub-linear expectation is related to a group of probabilities (c.f. Lemma 2.4 of Peng (2008b)). For more properties of the sub-linear expectations, one can refer to Peng (2008b), where the notion of independent and identically distributed random variables under the sub-linear expectations is introduced and the weak convergence such as central limit theorems and weak laws of large numbers are studied.

The motivation of this paper is to study the laws of the iterated logarithm under reasonable conditions. Basically, the classical law of the iterated logarithm is established through the Kolmogorov type exponential inequalities for both independent and negatively dependent random variables (c.f. Petrov (1995), Shao and Su (1999)). The main purpose of this paper is to establish the Kolmogorov type exponential inequalities for independent random variables as well as negatively dependent random variables in the general sub-linear expectation spaces. By applying these inequalities, we prove that the laws of the iterated logarithm holds for independent random variables as well as negatively dependent random variables under the condition that only the second order moments are finite. It is shown that for a sequence $\{X_n; n \geq 1\}$ of independent and identically distributed random variables with finite variances, the law of the iterated logarithm holds if and only if the sub-linear means are zeros and the Choquet integral of $X_1^2/\log \log |X_1|$ is finite. Also, for deriving a lower bound of an exponential inequality for independent and identically distributed ran-
dom variables, we prove a central limit theorem under only the condition that the second order moments are finite, which improves the central limit theorem of Peng (2008b) (c.f. Remark 3.2 below). Because the sub-linear expectation is not additive, many powerful tools for linear expectations and probabilities such as the martingale method, the stopping time, the symmetrization method are not valid, so that the study of the limit theorems becomes much more technical even after the exponential inequalities are established. In the next section, we give some notations under the sub-linear expectations including independence and negative dependence. In Section 3 we give the main results. The proof is given in the last section.

2 Basic Settings

We use the framework and notations of Peng (2008b). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\text{Lip}}(\mathbb{R}_n)$, where $C_{l,\text{Lip}}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions $\varphi$ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}_n,$$

for some $C > 0, m \in \mathbb{N}$ depending on $\varphi$.

$\mathcal{H}$ is considered as a space of “random variables”. In this case we denote $X \in \mathcal{H}$.

Remark 2.1 It is easily seen that if $\varphi_1, \varphi_2 \in C_{l,\text{Lip}}(\mathbb{R}_n)$, then $\varphi_1 \vee \varphi_2, \varphi_1 \wedge \varphi_2 \in C_{l,\text{Lip}}(\mathbb{R}_n)$ because $\varphi_1 \vee \varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2 + |\varphi_1 - \varphi_2|), \varphi_1 \wedge \varphi_2 = \frac{1}{2}(\varphi_1 + \varphi_2 - |\varphi_1 - \varphi_2|)$.

Definition 2.1 A sub-linear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a functional $\hat{\mathbb{E}}: \mathcal{H} \to \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;

(b) Constant preserving : $\hat{\mathbb{E}}[c] = c$;

(c) Sub-additivity: $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ whenever $\hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$ is not of the form $+\infty - \infty$ or $-\infty + \infty$;

(d) Positive homogeneity: $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \lambda \geq 0$. 


Here $\mathbb{R} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{F}, \mathbb{E})$ is called a sub-linear expectation space. Give a sub-linear expectation $\mathbb{E}$, let us denote the conjugate expectation $\mathbb{E}$ of $\mathbb{E}$ by

$$\mathbb{E}[X] := -\mathbb{E}[-X], \quad \forall X \in \mathcal{F}.$$  

From the definition, it is easily shown that $\mathbb{E}[X] \leq \mathbb{E}[X]$, $\mathbb{E}[X + c] = \mathbb{E}[X] + c$ and $\mathbb{E}[X - Y] \geq \mathbb{E}[X] - \mathbb{E}[Y]$ for all $X, Y \in \mathcal{F}$ with $\mathbb{E}[Y]$ being finite. Further, if $\mathbb{E} |X|$ is finite, then $\mathbb{E}[X]$ and $\mathbb{E}[X]$ are both finite. Denote

$$\mathcal{L} = \{X \in \mathcal{F} : \mathbb{E}|X| < \infty\}.$$  

**Definition 2.2** (Peng (2006, 2008b))

(i) **(Identical distribution)** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{F}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$ if

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{1,Lip}(\mathbb{R}_n).$$  

A sequence $\{X_n ; n \geq 1\}$ of random variables is said to be identically distributed if $X_i \overset{d}{=} X_1$ for each $i \geq 1$.

(ii) **(Independence)** In a sub-linear expectation space $(\Omega, \mathcal{F}, \mathbb{E})$, a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{F}$ is said to be independent to another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{F}$ under $\mathbb{E}$ if for each test function $\varphi \in C_{1,Lip}(\mathbb{R}_m \times \mathbb{R}_n)$ we have

$$\mathbb{E} [\varphi(X, Y)] = \mathbb{E} [\mathbb{E} [\varphi(x, Y)] |_{x=X}],$$  

whenever $\varphi(x) := \mathbb{E} [||\varphi(x, Y)||] < \infty$ for all $x$ and $\mathbb{E} [||\varphi(X)||] < \infty$.

(iii) **(IID random variables)** A sequence of random variables $\{X_n ; n \geq 1\}$ is said to be independent and identically distributed (IID), if $X_i \overset{d}{=} X_1$ and $X_{i+1}$ is independent to $(X_{i+1}, \ldots, X_n)$ for each $i \geq 1$.

From the definition of independence, it is easily seen that, if $Y$ is independent to $X$, $X, Y \in \mathcal{L}$, and $X \geq 0, \mathbb{E}[Y] \geq 0$, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]. \quad (2.1)$$
Further, if $Y$ is independent to $X$ and $0 \leq X, Y \in \mathcal{L}$, then

$$
\hat{E}[XY] = \hat{E}[X]\hat{E}[Y], \quad \hat{E}[XY] = \hat{E}[X]\hat{E}[Y].
$$

(2.2)

It is important to note that, in general, the independence of $Y$ and $X$ under the sub-linear expectation does not imply $\hat{E}[YX] = \hat{E}[Y]\hat{E}[X]$. So, even the simple moments as $\hat{E}[X_1 + \cdots + X_n]^2$ and $\hat{E}[(X_1 + \cdots + X_n)^4]$ of the partial sums of independent random variables under the sub-linear expectations cannot be estimated by the classical method. Without good inequalities for the partial sums, a lot of limit theorems under the sub-linear expectations remain unknown.

Motivated by the above properties (2.1) and (2.2), we give the concept of negative dependence under the sub-linear expectation.

**Definition 2.3**

(i) **(Negative dependence)** In a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}$ is said to be negatively dependent (ND) to another random vector $X = (X_1, \ldots, X_m), X_i \in \mathcal{H}$ under $\hat{E}$ if for each pair of test functions $\varphi_1 \in C_{l, Lip}(\mathbb{R}_m)$ and $\varphi_2 \in C_{l, Lip}(\mathbb{R}_n)$ we have

$$
\hat{E}[\varphi_1(X)\varphi_2(Y)] \leq \hat{E}[\varphi_1(X)]\hat{E}[\varphi_2(Y)]
$$

whenever either $\varphi_1, \varphi_2$ are coordinatewise nondecreasing or $\varphi_1, \varphi_2$ are coordinatewise non-increasing with $\varphi_1(X) \geq 0, \hat{E}[\varphi_2(Y)] \geq 0, \hat{E}[|\varphi_1(X)\varphi_2(Y)|] < \infty, \hat{E}[|\varphi_1(X)|] < \infty, \hat{E}[|\varphi_2(Y)|] < \infty$.

(ii) **(ND random variables)** Let $\{X_n; n \geq 1\}$ be a sequence of random variables in the sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$. $X_1, X_2, \ldots$ are said to be negatively dependent if $X_{i+1}$ is negatively dependent to $(X_1, \ldots, X_i)$ for each $i \geq 1$.

It is obvious that, if $\{X_n; n \geq 1\}$ is a sequence of independent random variables and $f_1(x), f_2(x), \ldots \in C_{l, Lip}(\mathbb{R})$, then $\{f_n(X_n); n \geq 1\}$ is also a sequence of independent random variables; if $\{X_n; n \geq 1\}$ is a sequence of negatively dependent random variables and $f_1(x), f_2(x), \ldots \in C_{l, Lip}(\mathbb{R})$ are non-decreasing (resp. non-increasing) functions, then $\{f_n(X_n); n \geq 1\}$ is also a sequence of negatively dependent random variables.
Next, we introduce the capacities corresponding to the sub-linear expectations. Let $G \subset F$. A function $V : G \rightarrow [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \quad \text{and} \quad V(A) \leq V(B) \quad \forall \ A \subset B, \ A, B \in G.$$ 

It is called to be sub-additive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in G$ with $A \cup B \in G$.

Let $(\Omega, \mathcal{H}, \hat{E})$ be a sub-linear space, and $\hat{E}$ be the conjugate expectation of $\hat{E}$. It is natural to define the capacity of a set $A$ to be the sub-linear expectation of the indicator function $I_A$ of $A$. However, $I_A$ may be not in $\mathcal{H}$. So, we denote a pair $(\mathcal{V}, \mathcal{V})$ of capacities by

$$\mathcal{V}(A) := \inf \{ \hat{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H} \}, \quad \mathcal{V}(A) := 1 - \mathcal{V}(A^c), \quad \forall A \in F,$$

where $A^c$ is the complement set of $A$. Then

$$\mathcal{V}(A) := \hat{E}[I_A], \quad \mathcal{V}(A) := \hat{E}[I_A], \quad \text{if} \ I_A \in \mathcal{H}$$

$$\hat{E}[f] \leq \mathcal{V}(A) \leq \hat{E}[g], \quad \hat{E}[f] \leq \mathcal{V}(A) \leq \hat{E}[g], \quad \text{if} \ f \leq I_A \leq g, \ f, g \in \mathcal{H}. \quad (2.3)$$

It is obvious that $\mathcal{V}$ is sub-additive. But $\mathcal{V}$ and $\hat{E}$ are not. However, we have

$$\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathcal{V}(B) \quad \text{and} \quad \hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y] \quad (2.4)$$

due to the fact that $\mathcal{V}(A^c \cap B^c) = \mathcal{V}(A^c \setminus B) \geq \mathcal{V}(A^c) - \mathcal{V}(B)$ and $\hat{E}[-X -Y] \geq \hat{E}[-X] - \hat{E}[Y]$.

Also, we define the Choquet integrals/expectations $(C_{\mathcal{V}}, C_{\mathcal{V}})$ by

$$C_{\mathcal{V}}[X] = \int_0^\infty V(X \geq t)dt + \int_0^0 [V(X \geq t) - 1]dt$$

with $V$ being replaced by $\mathcal{V}$ and $\mathcal{V}$ respectively.

It can be verified that (c.f., Lemma 4.3 (iii)), if $\lim_{c \rightarrow \infty} \hat{E}[\lfloor |X| - c \rfloor^+] = 0$, then

$$\hat{E}[|X|] \leq C_{\mathcal{V}}(|X|). \quad (2.5)$$

3 Main results

In this section, we give the mains results. We first give the upper bound of the exponential inequalities for negatively dependent random variables, then a lower bound of an exponential inequality for independent and identically distributed random variables. For deriving this lower bound, we give a new central limit theorem. At last, we give the laws of the iterated logarithm.
3.1 Exponential inequalities

Let \{X_1, \ldots, X_n\} be a sequence of random variables in \((\Omega, \mathcal{A}, \widehat{E})\). Set \(S_n = \sum_{k=1}^{n} X_k\), \(B_n = \sum_{k=1}^{n} \widehat{E}[X_k^2]\) and \(M_{n,p} = \sum_{k=1}^{n} \widehat{E}[|X_k|^p], p \geq 2\). The following is our main result on the exponential inequalities. For the exponential inequalities for classical negatively dependent random variables, one can refer to Su, Zhao and Wang (1997), Shao (2000) etc. The comparison method of Shao (2000) is not valid under the sub-linear expectation \(\widehat{E}\) because of the non-additivity.

**Theorem 3.1** Let \(\{X_1, \ldots, X_n\}\) be a sequence of negatively dependent random variables in \((\Omega, \mathcal{A}, \widehat{E})\) with \(\widehat{E}[X_k] \leq 0\). Then

(a) For all \(x, y > 0\),
\[
\mathbb{V}(S_n \geq x) \leq \mathbb{V}\left(\max_{k \leq n} X_k \geq y\right) + \exp\left\{ -\frac{x^2}{2(xy + B_n)}\left(1 + \frac{2}{3} \ln\left(1 + \frac{xy}{B_n}\right)\right)\right\}; \quad (3.1)
\]

(b) For all \(x > 0\) and \(0 < \delta \leq 1\),
\[
\mathbb{V}(S_n \geq x) \leq 2\left(\frac{6(1 + \delta)}{\delta}\right)^p \frac{M_{n,p}}{x^p} + \exp\left\{ -\frac{x^2}{2B_n(1 + \delta)}\right\}, \quad \forall p \geq 2; \quad (3.2)
\]

(c) We have
\[
C_V[(S_n^+)^p] \leq p^p C_V\left[(\max_{k \leq n} X_k^+)^p\right] + pB_n^{p/2}
\]
\[
\leq p^p \sum_{k=1}^{n} C_V[(X_k^+)^p] + pB_n^{p/2}, \quad \forall p \geq 2. \quad (3.3)
\]

The following corollary gives the estimates of \(\mathbb{V}(S_n \geq x)\).

**Corollary 3.1** Let \(\{X_1, \ldots, X_n\}\) be a sequence of independent random variables in \((\Omega, \mathcal{A}, \widehat{E})\) with \(\widehat{E}[X_k] \leq 0\). Then

(a) For all \(x, y > 0\),
\[
\mathbb{V}(S_n \geq x) \leq \mathbb{V}\left(\max_{k \leq n} X_k \geq y\right) + \exp\left\{ -\frac{x^2}{2(xy + B_n)}\left(1 + \frac{2}{3} \ln\left(1 + \frac{xy}{B_n}\right)\right)\right\}; \quad (3.4)
\]

(b) For all \(x > 0\) and \(0 < \delta \leq 1\),
\[
\mathbb{V}(S_n \geq x) \leq 2\left(\frac{6(1 + \delta)}{\delta}\right)^p \frac{M_{n,p}}{x^p} + \exp\left\{ -\frac{x^2}{2B_n(1 + \delta)}\right\}, \quad \forall p \geq 2; \quad (3.5)
\]
By choosing \( p = 2 \) in (3.5) we obtain
\[
V(S_n \geq x) \leq C \frac{\sum_{k=1}^{n} \hat{E}[X_k^2]}{x^2}, \quad \forall x > 0.
\] (3.6)

The next theorem gives a lower bound of an exponential inequality for independent and identically distributed random variables.

**Theorem 3.2** Suppose that \( \{X_n; n \geq 1\} \) is a sequence of independent and identically distributed random variables with \( \hat{E}[X_1] = \hat{E}[-X_1] = 0 \) and \( \lim_{c \to \infty} \hat{E}[(X_1^2 - c)^+] = 0 \). Write \( \sigma^2 = \hat{E}[X_1^2] \). Let \( \{y_n\} \) be a sequence of positive numbers such that \( y_n \to \infty, y_n/\sqrt{n} \to 0 \). Then for any \( |b| < \sigma, \epsilon > 0 \) and \( \delta > 0 \) with \( (b/\sigma)^2 + \delta < 1 \), there exists \( n_0 \) such that
\[
V\left(\left|\frac{S_n}{y_n\sqrt{n}} - b\right| \leq \epsilon\right) \geq \exp\left(-\left|\frac{b}{\sigma}\right| + \delta \right) \frac{y_n^2}{2}, \quad \forall n \geq n_0.
\] (3.7)

**Remark 3.1** (3.5) is the Rosenthal type inequality under the Choquet expectation. (3.2) is the Fuk and Nagaev type inequality. (3.1) is the upper bound of the Kolmogorov type exponential inequality and (3.7) is the lower bound (c.f. Lemmas 7.1 and 7.2 of Petorv (1995)). However, the lower bound cannot be established in the same as that of Lemma 7.2 of Petorv (1995) because the sub-linear expectation is not additive over unjoint events.

### 3.2 A central limit theorem

For proving Theorem 3.2, we need the following central limit theorem for independent and identically distributed random variables with only finite variances, which is of independent interest.

**Theorem 3.3** (CLT) Suppose that \( \{X_n; n \geq 1\} \) is a sequence of independent and identically distributed random variables with \( \hat{E}[X_1] = \hat{E}[-X_1] = 0 \) and \( \lim_{c \to \infty} \hat{E}[(X_1^2 - c)^+] = 0 \). Then for any continuous function \( \varphi \) satisfying \( |\varphi(x)| \leq C(1 + x^2) \),
\[
\lim_{n \to \infty} \hat{E}\left[\varphi\left(\frac{S_n}{\sqrt{n}}\right)\right] = \tilde{E}[\varphi(\xi)],
\] (3.8)
where \( \xi \sim N(0, [\sigma, \sigma]) \) under \( \hat{E} \). Further, if \( p \geq 2 \) and \( \lim_{c \to \infty} \hat{E}[|X_1|^p - c]^+] = 0 \), then (3.8) holds for any continuous function \( \varphi \) satisfying \( |\varphi(x)| \leq C(1 + |x|^p) \) .
Remark 3.2 Peng (2008b) pointed that (3.8) holds for all continuous function \( \varphi \) satisfying a polynomial growth condition: 

\[ |\varphi(x)| \leq C(1 + |x|^k) \text{ for some } k \text{ (c.f. his Theorem 5.1).} \]

However, in his proof for a bounded and Lipschitz continuous function \( \varphi \), the \((2 + \alpha)\)-th moment \( \hat{\mathbb{E}}[|X_i|^{2+\alpha}] \) needs to be assumed bounded (c.f. the proof of his Lemma 5.4). Also, when a continuous function is extended to a continuous function \( \varphi \) satisfying 

\[ |\varphi(x)| \leq C(1 + |x|^{p-1}) \text{ for } p > 0, \]

the following condition is needed (c.f. his Lemma 5.5, where \( Y_n = S_n/\sqrt{n} \)):

\[ \sup_n \hat{\mathbb{E}} \left[ \frac{|S_n|}{\sqrt{n}} \right]^p < \infty, \]

which is not verified in Peng (2008b). As explained in Section 1, such moment inequalities are not obvious under the sub-linear expectations. Now, note Theorem 3.3 and

\[ \hat{\mathbb{E}}[|X_1|^p] = \hat{\mathbb{E}}[|X_n|^p] \leq 2^{p-1}(\hat{\mathbb{E}}[|S_n|^p] + \hat{\mathbb{E}}[|S_{n-1}|^p]). \]

A sufficient and necessary condition for (3.8) to hold for any continuous function satisfying a polynomial growth condition is that \( \hat{\mathbb{E}}[|X_1|^p] < \infty \) for all \( p > 0 \). It is also important to note that \( |x|^k \) is a continuous function satisfying a polynomial growth condition, but \( e^x \) and \( e^{x^2} \) are not.

3.3 The law of the iterated logarithm

Before we give the laws of iterated logarithm, we need some more notations about the sub-linear expectations and capacities.

**Definition 3.1** (I) A sub-linear expectation \( \hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R} \) is called to be countably sub-additive if it satisfies

(\text{e}) **Countable sub-additivity:** \( \hat{\mathbb{E}}[X] \leq \sum_{n=1}^{\infty} \hat{\mathbb{E}}[X_n] \), whenever \( X \leq \sum_{n=1}^{\infty} X_n \), \( X, X_n \in \mathcal{H} \) and \( X \geq 0, X_n \geq 0, n = 1, 2, \ldots \);

It is called to continuous if it satisfies

(\text{f}) **Continuity from below:** \( \hat{\mathbb{E}}[X_n] \uparrow \hat{\mathbb{E}}[X] \) if \( 0 \leq X_n \uparrow X \), where \( X_n, X \in \mathcal{H} \);

(\text{g}) **Continuity from above:** \( \hat{\mathbb{E}}[X_n] \downarrow \hat{\mathbb{E}}[X] \) if \( 0 \leq X_n \downarrow X \), where \( X_n, X \in \mathcal{H} \).
(II) A function \( V : \mathcal{F} \to [0, 1] \) is called to be countably sub-additive if
\[
V \left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} V(A_n) \quad \forall A_n \in \mathcal{F}.
\]

(III) A capacity \( V : \mathcal{F} \to [0, 1] \) is called a continuous capacity if it satisfies

(III1) **Continuity from below:** \( V(A_n) \uparrow V(A) \) if \( A_n \uparrow A \), where \( A_n, A \in \mathcal{F} \);

(III2) **Continuity from above:** \( V(A_n) \downarrow V(A) \) if \( A_n \downarrow A \), where \( A_n, A \in \mathcal{F} \).

It is obvious that a continuous sub-additive capacity \( V \) (resp. a sub-linear expectation \( \hat{E} \)) is countably sub-additive. The “the convergence part” of the Borel-Cantelli Lemma is still true for a countably sub-additive capacity.

**Lemma 3.1** (Borel-Cantelli’s Lemma) Let \( \{A_n, n \geq 1\} \) be a sequence of events in \( \mathcal{F} \). Suppose \( V \) is a countably sub-additive capacity. If \( V(A_n) < \infty \), then
\[
V(A_n \ i.o.) = 0, \quad \text{where } \{A_n \ i.o.\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.
\]

**Proof.** By the monotonicity and countable sub-additivity, it follows that
\[
0 \leq V\left( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \right) \leq V \left( \bigcup_{i=n}^{\infty} A_i \right) \leq \sum_{i=n}^{\infty} V(A_i) \to 0 \quad \text{as } n \to \infty.
\]

**Remark 3.3** It is important to note that the condition that “\( X \) is independent to \( Y \) under \( \hat{E} \)” does not implies that “\( X \) is independent to \( Y \) under \( V \)” because the indicator functions \( I\{X \in A\} \) and \( I\{X \in A\} \) are not in \( C_{t, \text{Lip}}(\mathbb{R}) \), and also, “\( X \) is independent to \( Y \) under \( V \)” does not implies that “\( X \) is independent to \( Y \) under \( \hat{E} \)” because \( \hat{E} \) is not an integral with respect to \( V \). So, we have not “the divergence part” of the Borel-Cantelli Lemma.

Because \( V \) may be not countably sub-additive in general, we define an outer capacity \( V^* \) by
\[
V^*(A) = \inf \left\{ \sum_{n=1}^{\infty} V(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n \right\}, \quad V^*(A) = 1 - V^*(A^c), \quad A \in \mathcal{F}.
\]
Then it can be shown that \( V^*(A) \) is a countably sub-additive capacity with \( V^*(A) \leq V(A) \) and the following properties:

(a*) If \( V \) is countably sub-additive, then \( V^* \equiv V \).
(b*) If $I_A \leq g, g \in \mathcal{H}$, then $V^*(A) \leq \widehat{E}[g]$. Further, if $\widehat{E}$ is countably sub-additive, then

$$\widehat{E}[f] \leq V^*(A) \leq V(A) \leq \widehat{E}[g], \quad \forall f \leq I_A \leq g, f, g \in \mathcal{H}. \quad (3.1)$$

(c*) $V^*$ is the largest countably sub-additive capacity satisfying the property that $V^*(A) \leq \widehat{E}[g]$ whenever $I_A \leq g \in \mathcal{H}$, i.e., if $V$ is also a countably sub-additive capacity satisfying $V(A) \leq \widehat{E}[g]$ whenever $I_A \leq g \in \mathcal{H}$, then $V(A) \leq V^*(A)$.

In this subsection, we let $\{X_n; n \geq 1\}$ be a sequence of identically distributed random variables in $(\Omega, \mathcal{H}, \widehat{E})$. Denote $\sigma^2 = \widehat{E}[X_1^2]$, $\bar{\sigma}^2 = \widehat{E}[\sigma_1^2]$, $a_n = \sqrt{2n \log \log n}$, where $\log x = \ln(x \vee e)$. The following is the law of the iterated logarithm for independent random variables and negatively dependent random variables. For the law of the iterated logarithm for classical negatively dependent random variables, one can refer to Shao and Su (1999), Zhang (2001a) etc.

**Theorem 3.4** (a) Suppose that $X_1, X_2, \ldots$ are negatively dependent with $\widehat{E}[X_1] = \widehat{E}[-X_1] = 0$, $\lim_{c \to \infty} \widehat{E}[(X_1^2 - c)^+] = 0$ and

$$C_V\left[\frac{X_1^2}{\log \log |X_1|}\right] < \infty. \quad (3.2)$$

Then

$$V^*\left(\left\{\liminf_{n \to \infty} \frac{S_n}{a_n} < -\bar{\sigma}\right\} \cap \left\{\limsup_{n \to \infty} \frac{S_n}{a_n} > \bar{\sigma}\right\}\right) = 0. \quad (3.3)$$

(b) Suppose that $X_1, X_2, \ldots$ are independent, $V^*$ is continuous and $\widehat{E}$ is countably sub-additive. If

$$V^*(\left\{\limsup_{n \to \infty} \frac{|S_n|}{a_n} = +\infty\right\}) < 1, \quad (3.4)$$

then $\widehat{E}[X_1] = \widehat{E}[-X_1] = 0$, and (3.2) holds.

(c) Suppose that $X_1, X_2, \ldots$ are independent, $V$ is continuous. Assume that (3.4) holds.

Then we have (3.3). Further, if $\lim_{c \to \infty} \widehat{E}[|X_1| - c]^+] = 0$, then $\widehat{E}[X_1] = \widehat{E}[-X_1] = 0$.

**Remark 3.4** Theorem 3.4 (a) can be regarded as the direct part which gives sufficient conditions for the law of the iterated logarithm to hold, and Theorem 3.4 (b) (c) can be regarded the inverse part which gives the necessary conditions. According this Theorem, we conjecture that the necessary and sufficient conditions for (3.3) are $\widehat{E}[X_1^2] < \infty$, $\widehat{E}[X_1] = \widehat{E}[-X_1] = 0$ and (3.2).
Corollary 3.2 Suppose $X_1, X_2, \ldots$ are independent with $\lim_{c \to \infty} \hat{E} \left[ (X_1^2 - c)^+ \right] = 0$, $\hat{E}[X_1] = \hat{E}[-X_1] = 0$ and (3.2). If $\mathcal{V}$ is continuous, then we have

(I) \[ \mathcal{V} \left( \sigma \leq \limsup_{n \to \infty} \frac{S_n}{a_n} \leq \sigma \right) = 1 \]

and

\[ \mathcal{V} \left( -\sigma \leq \liminf_{n \to \infty} \frac{S_n}{a_n} \leq -\sigma \right) = 1. \]

(II) Suppose that $C(\{x_n\})$ is the cluster set of a sequence of $\{x_n\}$ in $\mathbb{R}$, then

\[ \mathcal{V} \left( C \left\{ \frac{S_n}{a_n} \right\} = \left[ \liminf_{n \to \infty} \frac{S_n}{a_n}, \limsup_{n \to \infty} \frac{S_n}{a_n} \right] \supset (-\sigma, \sigma) \right) = 1. \]

If $\mathcal{V}^*$ is continuous and $\hat{E}$ is countably sub-additive, then we also have the conclusions (I) and (II) with $\mathcal{V}$ being replaced by $\mathcal{V}^*$.

Remark 3.5 In the proof of Corollary 3.2, the central limit theorems under the Peng’s framework are used. Though a lot of results on the central limit theorems for related classical negatively dependent random variables can be found in literature (c.f. Newman (1984), Newman and Wright (1981), Su, Zhao and Wang (1997), Zhang (2001b) etc), it is difficult to establish a central limit theorem for non-independent random variables under the sub-linear expectations. So, we have no version of Corollary 3.2 for negatively dependent random variables.

Remark 3.6 Chen and Hu (2013) tried to establish the law of the iterated logarithm for bounded independent and identically distributed random variables. They have introduced a clever method to obtain the lower bound $\sigma$ of $\limsup_{n \to \infty} \frac{S_n}{a_n}$. To establish the upper bound $\overline{\sigma}$, they have to assume the boundness of the random variables. The central limit theorem of Peng (2008b) is not utilized correctly to get the moment inequalities and exponential inequalities. Chen and Hu (2013) obtained the moment inequality $\hat{E}[|S_n|^r] \leq c_r n^{r/2}$ by taking $\varphi(x) = |x|^r$ in (3.8) (c.f. their Lemma 6) and obtained the exponential inequalities by taking $\varphi(x) = e^{\lambda x^2}$. As we have pointed in Remark 3.2 in Peng’s proof, to show that (3.8) is true for all continuous functions satisfying a polynomial growth condition, the moment inequality $\hat{E}[|S_n|^r] \leq c_r n^{r/2}$ needs to be verified at first for all $r \geq 2$. This is a circle. Also,
similarly as in the study of the moment convergence under the classical expectation/integral, when taking \( \varphi(x) = e^{\lambda x^2} \), the uniform integrability of \( \exp\{\lambda (S_n/\sqrt{n})^2\} \) needs to be verified at first, which is not an easy work even in the case of the classical linear expectations.

4 Proofs

We first show the exponential inequities, then the central limit theorem, and at last the law of the iterated logarithm. The Hölder’s inequality under the sub-linear expectation will be used frequently in our proofs, which can be proved by the same may under the linear expectation due to the properties of the monotonicity and sub-additivity, and the elementary inequality \(|xy| \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q\), where \( p, q > 1 \) are two real numbers satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Lemma 4.1** (Hölder’s inequality) Let \( p, q > 1 \) be two real numbers satisfying \( \frac{1}{p} + \frac{1}{q} = 1 \). Then for two random variables \( X, Y \) in \( (\Omega, \mathcal{H}, \hat{E}) \) we have

\[
\hat{E}[|XY|] \leq \left( \hat{E}[|X|^p] \right)^{\frac{1}{p}} \left( \hat{E}[|Y|^q] \right)^{\frac{1}{q}}
\]

whenever \( \hat{E}[|X|^p] < \infty, \hat{E}[|Y|^q] < \infty \).

We also need the Rosenthal type inequalities under \( \hat{E} \) which have been obtained by Zhang (2014).

**Lemma 4.2** (Rosenthal’s inequality) (a) Let \( \{X_1, \ldots, X_n\} \) be a sequence of independent random variables in \( (\Omega, \mathcal{H}, \hat{E}) \) with \( \hat{E}[X_k] = 0, k = 1, \ldots, n \). Then

\[
\hat{E}\left[\max_{k \leq n} (S_n - S_k)^p \right] \leq C_p \left\{ \sum_{k=1}^{n} \hat{E}[|X_k|^p] + \left( \sum_{k=1}^{n} \hat{E}[|X_k|^2] \right)^{p/2} \right\}, \quad \text{for } p \geq 2. \tag{4.1}
\]

In particular,

\[
\hat{E}\left[(S_n^+)^p \right] \leq C_p \left\{ \sum_{k=1}^{n} \hat{E}[|X_k|^p] + \left( \sum_{k=1}^{n} \hat{E}[|X_k|^2] \right)^{p/2} \right\}, \quad \text{for } p \geq 2. \tag{4.2}
\]

(b) Let \( \{X_1, \ldots, X_n\} \) be a sequence of negatively dependent random variables in \( (\Omega, \mathcal{H}, \hat{E}) \). Then

\[
\hat{E}\left[\max_{k \leq n} |S_k|^p \right] \leq C_p \left\{ \sum_{k=1}^{n} \hat{E}[|X_k|^p] + \left( \sum_{k=1}^{n} \hat{E}[|X_k|^2] \right)^{p/2} \right. \left. + \left( \sum_{k=1}^{n} \left( (\hat{E}[X_k])^- + (\hat{E}[X_k])^+ \right) \right)^{p} \right\}, \quad \text{for } p \geq 2. \tag{4.3}
\]
4.1 Proofs of the exponential inequalities

Proof of Theorem 3.1 Let $Y_k = X_k \wedge y$, $T_n = \sum_{k=1}^{n} Y_k$. Then $X_k - Y_k = (X_k - y)^+ \geq 0$ and $\mathbb{E}[Y_k] \leq \mathbb{E}[X_k] \leq 0$. Note that $\varphi(x) := e^{t(x/y)}$ is a bounded non-decreasing function and belongs to $C_{l, Lip}(\mathbb{R})$ since $0 \leq \varphi'(x) \leq te^{ty}$ if $t > 0$. It follows that for any $t > 0$,

$$V(S_n \geq x) \leq V(\max_{k \leq n} X_k \geq y) + V(T_n \geq x).$$

and

$$V(T_n \geq x) \leq e^{-tx} \mathbb{E}[e^{tT_n}] \leq e^{-tx} \prod_{k=1}^{n} \mathbb{E}[e^{tY_k}],$$

be the definition of the negative dependence. Note

$$e^{tY_k} = 1 + ty + \frac{e^{ty} - 1 - ty}{y^2} Y_k^2 \leq 1 + ty + \frac{e^{ty} - 1 - ty}{y^2} Y_k^2.$$

We have

$$\mathbb{E}[e^{tY_k}] \leq 1 + \frac{e^{ty} - 1 - ty}{y^2} \mathbb{E}[Y_k^2] \leq \exp\left\{ \frac{e^{ty} - 1 - ty}{y^2} \mathbb{E}[X_k^2] \right\}.$$

Choosing $t = \frac{1}{y} \ln \left( 1 + \frac{xy}{B_n} \right)$ yields

$$V(T_n \geq x) \leq e^{-tx} \exp\left\{ \frac{e^{ty} - 1 - ty}{y^2} B_n \right\} = \exp\left\{ \frac{x}{y} \frac{x(B_n + xy + 1)}{y(1 + \frac{xy}{B_n})} \ln \left( 1 + \frac{xy}{B_n} \right) \right\}. \quad (4.4)$$

Applying the elementary inequality

$$\ln(1 + t) \geq \frac{t}{2(1 + t)} + \frac{t^2}{2(1 + t)^2} \left( 1 + \frac{2}{3} \ln(1 + t) \right)$$

yields

$$\left( \frac{B_n}{xy} + 1 \right) \ln \left( 1 + \frac{xy}{B_n} \right) \geq 1 + \frac{xy}{2(xy + B_n)} \left( 1 + \frac{2}{3} \ln \left( 1 + \frac{xy}{B_n} \right) \right).$$

(3.1) is proved.

For (3.2), if $xy \leq \delta B_n$, then

$$\frac{x^2}{2(xy + B_n)} \left( 1 + \frac{2}{3} \left( 1 + \frac{xy}{B_n} \right) \right) \geq \frac{x^2}{2B_n(1 + \delta)}.$$

Suppose $xy \geq \delta B_n$. Note

$$M_{n,p}^+ := \sum_{k=1}^{n} \mathbb{E}[(Y_k^+)^p] \leq B_n y^{p-2} \leq \delta^{-1} xy^{p-1},$$
and that the function \((1 + \frac{1}{t}) \ln(1 + t)\) is increasing in \(t \geq 0\). It follows that
\[
\left( \frac{B_n}{xy} + 1 \right) \ln \left( 1 + \frac{xy}{B_n} \right) \geq \left( \frac{M_{n,p}}{xy^{p-1}} + 1 \right) \ln \left( 1 + \frac{xy^{p-1}}{M_{n,p}} \right)
\]
\[
\geq 1 + \frac{xy^{p-1}}{2(xy^{p-1} + M_{n,p})} \left( 1 + \frac{2}{3} \ln \left( 1 + \frac{xy^{p-1}}{M_{n,p}} \right) \right)
\]
\[
\geq 1 + \frac{\delta}{3(1 + \delta)} \ln \left( 1 + \frac{xy^{p-1}}{M_{n,p}} \right) \geq 1 + \frac{\delta}{3(1 + \delta)} \ln \left( 1 + \frac{xy^{p-1}}{M_{n,p}} \right).
\]

Then by (4.4) we have
\[
\mathbb{V}(T_n \geq x) \leq \left( \frac{M_{n,p}}{M_{n,p} + xy^{p-1}} \right) \frac{\delta x}{3(1 + \delta) x}.
\]

Let \(y = \frac{\delta x}{3(1 + \delta)}\). We conclude that
\[
\mathbb{V}(S_n \geq x) \leq \mathbb{V}\left( \max_{k \leq n} X_k \geq y \right) + \exp \left\{ -\frac{x^2}{2B_n(1 + \delta)} \right\} + \left( \frac{M_{n,p}}{M_{n,p} + xy^{p-1}} \right) \frac{\delta x}{3(1 + \delta) x}
\]
\[
\leq \frac{M_{n,p}}{y^p} + \exp \left\{ -\frac{x^2}{2B_n(1 + \delta)} \right\} + \frac{M_{n,p}}{M_{n,p} + xy^{p-1}}
\]
\[
\leq 2 \left( \frac{6(1 + \delta)}{\delta} \right)^p \frac{M_{n,p}}{x^p} + \exp \left\{ -\frac{x^2}{2B_n(1 + \delta)} \right\}.
\]

(3.2) is proved.

Note that
\[
C_{\mathbb{V}}[(X^+)^p] = \int_0^\infty \mathbb{V}(X^p > x) dx = \int_0^\infty px^{p-1} \mathbb{V}(X > x) dx.
\]

We put \(y = x/r\), where \(r = p > p/2\), in (4.4), then multiply both sides of this inequality by \(px^{p-1}\). We find that
\[
px^{p-1} \mathbb{V}(S_n^+ \geq x) \leq px^{p-1} \mathbb{V}\left( \max_{k \leq n} X_k^+ \geq \frac{x}{r} \right) + px^{p-1} \left( 1 + \frac{x^2}{r B_n} \right)^{-r}.
\]

By integrating on the positive half-line, we conclude (3.3). □

**Proof of Corollary 3.1** Note, by (2.4),
\[
\mathbb{V}(S_n \geq x) \leq \mathbb{V}(\max_{k \leq n} X_k \geq x) + \mathbb{V}(T_n \geq x),
\]
\[
\mathbb{E}[e^{tY_k}] \leq 1 + t \mathbb{E}[Y_k] + e^{ty} - 1 - ty \mathbb{E}[Y_k^2] \leq 1 + e^{ty} - 1 - ty \mathbb{E}[Y_k^2],
\]
and \(\mathbb{V}(T_n \geq x) \leq e^{-tx} \mathbb{E}[e^{tT_n}] \leq e^{-tx} \prod_{k=1}^n \mathbb{E}[e^{tY_k}]\). The proof is similar to that of Theorem 3.1 □
Proof of Theorem 3.2. It is important to note the independence under $\widehat{E}$ is defined through continuous functions in $C_{l,Lip}$ and the indicator function of an event is not continuous. We need to modify the indicator function by functions in $C_{l,Lip}$. For $t > 0$, let

$$N =: [nt^2/y_n^2], \ m = [y_n^2/t^2]; \ r = \sqrt{n}\sqrt{y_n}/(tm)$$

and let $g_\epsilon$ be a function satisfying that its derivatives of each order are bounded, $g_\epsilon(x) = 1$ if $x \geq 1$, $g_\epsilon(x) = 0$ if $x \leq 1 - \epsilon$, and $0 \leq g_\epsilon(x) \leq 1$ for all $x$, where $0 < \epsilon < 1$. Then

$$g_\epsilon(\cdot) \in C_{c}^\infty(\mathbb{R}) \subset C_{l,Lip}(\mathbb{R}) \text{ and } I\{x \geq 1\} \leq g_\epsilon(x) \leq I\{x > 1 - \epsilon\}. \quad (4.5)$$

Define a function $\phi(x) = 1 - g_{1/2}\left(\frac{2|x|}{et}\right)$. Then

$$\{\frac{S_n}{y_n\sqrt{n}} - b \leq \epsilon\} \supset \left\{b - \epsilon/2 \leq \frac{S_{Nm}}{y_n\sqrt{n}} \leq b + \epsilon/2\right\} \cap \left\{\frac{S_n - S_{Nm}}{y_n\sqrt{n}} \leq \epsilon/2\right\}
\supset \left\{tm(b - \epsilon/2) \leq \frac{S_{Nm}}{r} \leq tm(b + \epsilon/2)\right\} \cap \left\{\frac{S_n - S_{Nm}}{y_n\sqrt{n}} \leq \epsilon/2\right\}
\supset \bigcap_{i=1}^{m} \left\{\left|\frac{S_{Ni} - S_{N(i-1)}}{rt} - b\right| \leq \epsilon/2\right\} \cap \left\{\frac{S_n - S_{Nm}}{y_n\sqrt{n}} \leq \epsilon/2\right\}
\supset \bigcap_{i=1}^{m} \left\{\phi\left(\frac{S_{Ni} - S_{N(i-1)}}{rt} - b\right) \leq 1 - g_{1/2}\left(\frac{2}{\epsilon} \left|\frac{S_n - S_{Nm}}{y_n\sqrt{n}}\right|\right)\right\} \cdot$$

It follows that

$$I\left\{\frac{S_n}{y_n\sqrt{n}} - b \leq \epsilon\right\} \geq \prod_{i=1}^{m} \phi\left(\frac{S_{Ni} - S_{N(i-1)}}{rt} - b\right) \left(1 - g_{1/2}\left(\frac{2}{\epsilon} \left|\frac{S_n - S_{Nm}}{y_n\sqrt{n}}\right|\right)\right).$$

Note that $\{S_{Ni} - S_{N(i-1)}, i = 1, \ldots, m, S_n - S_{Nm}\}$ are independent under $\widehat{E}$ (and $\widehat{E}$). By (2.3), we have

$$\mathbb{V}\left(\left|\frac{S_n}{y_n\sqrt{n}} - b \leq \epsilon\right|\right) \geq \left(\widehat{E}\left[\phi(S_N/(rt) - b)\right]\right)^m \left(1 - \overline{E}\left[g_{1/2}\left(\frac{2}{\epsilon} \left|\frac{S_n - S_{Nm}}{y_n\sqrt{n}}\right|\right)\right]\right).$$

Note

$$0 \leq \overline{E}\left[g_{1/2}\left(\frac{2}{\epsilon} \left|\frac{S_n - S_{Nm}}{y_n\sqrt{n}}\right|\right)\right] \leq \mathbb{V}\left(\left|\frac{S_n - S_{Nm}}{y_n\sqrt{n}} \geq \epsilon/4\right|\right) \leq \frac{16\sigma^2 n - Nm}{\epsilon^2 n^2 y_n} \rightarrow 0.$$

By applying Theorem 3.3 it follows that

$$\liminf_{n \rightarrow \infty} y_n^{-2} \ln \mathbb{V}\left(\left|\frac{S_n}{y_n\sqrt{n}} - b \leq \epsilon\right|\right) \geq \liminf_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} t^{-2} m^{-1} \ln \mathbb{V}\left(\left|\frac{S_n}{y_n\sqrt{n}} - b \leq \epsilon\right|\right) \geq \liminf_{t \rightarrow \infty} \liminf_{n \rightarrow \infty} t^{-2} \ln \widehat{E}\left[\phi(S_N/(rt) - b)\right] \leq \liminf_{t \rightarrow \infty} t^{-2} \ln \widehat{E}\left[\phi(\xi/t - b)\right].$$

16
Note $\xi/t \sim \mathcal{N}(0, [\sigma^2/t^2, \pi^2/t^2])$. By Lemma 5 of Chen and Hu (2013)

$$
\mathcal{E} [\phi(\xi/t - b)] \geq \exp \left\{ -\frac{1}{2} \left( \frac{bt}{\sigma} \right)^2 \right\} \mathcal{E} [\phi(\xi/t)] \geq \exp \left\{ -\frac{1}{2} \left( \frac{bt}{\sigma} \right)^2 \right\} \mathcal{V}(|\xi| \leq ct/4).
$$

It follows that

$$
\liminf_{t \to \infty} t^{-2} \ln \mathcal{E} [\phi(\xi/t - b)] \geq -\frac{1}{2} \left( \frac{b}{\sigma} \right)^2.
$$

The proof of Theorem 3.2 is completed. □.

4.2 Proofs of the central limit theorem

For showing Theorem 3.3, we let $Y_j = (-\sqrt{J}) \vee (X_j \wedge \sqrt{J})$, $T_n = \sum_{j=1}^n Y_j$. Suppose $\varphi$ is a bounded and (global) Lipschitz continuous function. We first show that

$$
\lim_{n \to \infty} \mathbb{E} [\varphi \left( \frac{T_n}{\sqrt{n}} \right)] = \mathbb{E} [\varphi(\xi)].
$$

We use the Peng (2008b)'s argument for a bounded and Lipschitz continuous function, which is a version of the Stein method under the sub-linear expectations. Here we only give the difference. The main difference is that $Y_j$’s are not identically distributed and $\mathbb{E}[Y_j], \mathbb{E}[-Y_j]$ are no zeros.

First, we have the following facts.

(F1) $\mathbb{E}[(X_1^2 - j)^+] \to 0$ as $j \to \infty$, and

$$
\frac{\sum_{j=1}^n \mathbb{E}[|X_j - Y_j|]}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.
$$

(F2)

$$
\frac{\sum_{j=1}^n \mathbb{E}[|Y_j|^{2+\alpha}]}{n^{1+\alpha/2}} \to 0 \quad \text{as} \quad n \to \infty, \quad \forall \alpha > 0.
$$

(F3) For any $p \geq 2$,

$$
\mathbb{E}[|T_n|^p] \leq C_p n^{p/2}.
$$

In fact, for (F1), note

$$
\mathbb{E}|X_j - Y_j| \leq \mathbb{E} \left[ (|X_1| - \sqrt{J})^+ \right] \leq j^{-1/2} \mathbb{E} \left[ (X_1^2 - j)^+ \right].
$$

(F1) is obvious.

For (F2), note that $\mathbb{E}[|Y_j|^{2+\alpha}] \leq c^{2+\alpha} + j^{\alpha/2} \mathbb{E}[(X_1^2 - c)^+]$ for any $c > 1$. So, (4.8) is true.
For (F3), by the Rosenthal inequality (4.3) and the fact (4.7) we have

\[
\hat{E}[|T_n|^p] \leq C_p \sum_{j=1}^{n} \hat{E}[|Y_j|^p] + C_p \left(\sum_{j=1}^{n} \hat{E}[|Y_j|^2]\right)^{p/2} + C_p \left(\sum_{j=1}^{n} \left[\hat{E}[Y_j^+] + (\hat{E}[Y_j^-])^p\right]\right.
\]

\[
\leq C_p n^{p/2-1} \sum_{j=1}^{n} \hat{E}[X_j^2] + C_p \left(\sum_{j=1}^{n} \left[\hat{E}[|X_j - Y_j|]\right]^p\right) \leq C_p n^{p/2}.
\]

And so, (4.9) is true.

Now, for a small but fixed \( h > 0 \), let \( V(t, x) \) be the unique viscosity solution of the following equation,

\[\partial_t V + G(\partial_x^2 V) = 0, \quad (t, x) \in [0, 1 + h] \times \mathbb{R}, \quad V|_{t=1+h} = \varphi(x),\]

where \( G(\alpha) = 1 \left( \sigma^2 \alpha^+ - \sigma^2 \alpha^- \right) \). Then by the interior regularity of \( V \),

\[\|V\|_{C^{1+\alpha/2,2+\alpha}(0,1] \times \mathbb{R}) < \infty, \text{ for some } \alpha \in (0, 1).\]

According to the definition of \( G \)-normal distribution, we have \( V(t, x) = \hat{E}[\varphi(x+\sqrt{t+h-t} \xi)]. \)

In particular,

\[V(h, 0) = \hat{E}[\varphi(\xi)], \quad V(1 + h, x) = \varphi(x).\]

It is obvious that, if \( \varphi(\cdot) \) is a global Lipschitz function, i.e., \( |\varphi(x) - \varphi(y)| \leq C|x - y| \), then \( |V(t, x) - V(t, y)| \leq C|x - y| \) and \( |V(t, x) - V(s, x)| \leq C\hat{E}[|\xi|]|t - s|^{1/2} \). So, \( |V(1 + h, x) - V(1, x)| \leq C\hat{E}[|\xi|]\sqrt{h} \) and \( |V(h, 0) - V(0, 0)| \leq C\hat{E}[|\xi|]\sqrt{h} \). Let \( \delta = \frac{1}{n^2} \), \( T_0 = 0 \). Following the proof of Lemma 5.4 of Peng (2008b), it is sufficient to show that

\[\lim_{n \to \infty} \hat{E}[V(1, \sqrt{\delta} T_n)] = V(0, 0). \quad (4.10)\]

Applying the Taylor’s expansion yields

\[V(1, \sqrt{\delta} T_n) - V(0, 0) = \sum_{i=0}^{n-1} \left\{ [V((i + 1)\delta, \sqrt{\delta} T_{i+1}) - V(i\delta, \sqrt{\delta} T_{i+1})] + [V(i\delta, \sqrt{\delta} T_{i+1}) - V(i\delta, \sqrt{\delta} T_i)] \right\}\]

\[= \sum_{i=0}^{n-1} \{ I_i^\delta + J_i^\delta \},\]
with \( |I_{i}^{j} | \leq C \delta^{1+\alpha/2}(1 + |Y_{i+1}|^\alpha + |Y_{i+1}|^{2+\alpha}) \),

\[
J_{i}^{j} = \partial_{i} V(i\delta, \sqrt{\delta} T_{i}) \delta + \frac{1}{2} \partial_{xx}^{2} V(i\delta, \sqrt{\delta} T_{i}) Y_{i+1}^{2} \delta + \partial_{x} V(i\delta, \sqrt{\delta} T_{i}) Y_{i+1} \sqrt{\delta} \\
= \left( \partial_{i} V(i\delta, \sqrt{\delta} T_{i}) \delta + \frac{1}{2} \partial_{xx}^{2} V(i\delta, \sqrt{\delta} T_{i}) X_{i+1}^{2} \delta + \partial_{x} V(i\delta, \sqrt{\delta} T_{i}) X_{i+1} \sqrt{\delta} \right) \\
+ \left( \frac{1}{2} \partial_{xx}^{2} V(i\delta, \sqrt{\delta} T_{i})(Y_{i+1}^{2} - X_{i+1}^{2}) \delta + \partial_{x} V(i\delta, \sqrt{\delta} T_{i})(Y_{i+1} - X_{i+1}) \sqrt{\delta} \right) \\
=: J_{i}^{j}_{\delta, 1} + J_{i}^{j}_{\delta, 2},
\]

where \( C \) is a constant. And so

\[
\hat{E} \left\{ \sum_{i=0}^{n-1} J_{i}^{j}_{\delta, 1} \right\} - \sum_{i=0}^{n-1} \left\{ \hat{E}[|J_{i}^{j}_{\delta, 2}|] + \hat{E}[|I_{i}^{j}|] \right\} \\
\leq \hat{E}[V(1, \sqrt{\delta} T_{n}) - V(0, 0)] \leq \hat{E} \left\{ \sum_{i=0}^{n-1} J_{i}^{j}_{\delta, 1} \right\} \sum_{i=0}^{n-1} \left\{ \hat{E}[|J_{i}^{j}_{\delta, 2}|] + \hat{E}[|I_{i}^{j}|] \right\}.
\]

By noting the fact (F2), we have

\[
\sum_{i=0}^{n-1} \hat{E}[|I_{i}^{j}|] \leq C(\frac{1}{n})^{1+\alpha/2} \sum_{i=1}^{n-1} \left( 1 + \hat{E}[|Y_{i}|^\alpha] + \hat{E}[|Y_{i}|^{2+\alpha}] \right) \to 0.
\]

For \( J_{i}^{j}_{\delta, 1} \), note \( \hat{E}[X_{i+1}^{2}] = \sigma^{2} \), \( \hat{E}[X_{i+1}^{2}] = \sigma^{2} \), \( \hat{E}[X_{i+1}] = \hat{E}[X_{i+1}] = 0 \). It follows that

\[
\hat{E} \left[ J_{i}^{j}_{\delta, 1} | X_{1}, \ldots, X_{n} \right] = \left[ \partial_{i} V(i\delta, \sqrt{\delta} T_{i}) + G(\partial_{xx}^{2} V(i\delta, \sqrt{\delta} T_{i})) \right] \delta = 0.
\]

It follows that \( \hat{E} \left[ \sum_{i=0}^{n-1} J_{i}^{j}_{\delta, 1} \right] = \hat{E} \left[ \sum_{i=0}^{n-1} J_{i}^{j}_{\delta, 1} \right] = \ldots = 0 \). For \( J_{i}^{j}_{\delta, 2} \), note

\[
\hat{E} \left[ |\partial_{xx}^{2} V(i\delta, \sqrt{\delta} T_{i})| \right] \leq C \left[ 1 + (i\delta)^{\alpha/2} + \hat{E}[|\sqrt{\delta} T_{i}|^\alpha] \right] \leq C
\]

by the Hölder inequality and Fact 3. Similarly, \( \hat{E} \left[ |\partial_{x} V(i\delta, \sqrt{\delta} T_{i})| \right] \leq C \). It follows that

\[
\sum_{i=0}^{n-1} \hat{E}[|J_{i}^{j}_{\delta, 2}|] \leq \sum_{i=0}^{n-1} \left\{ \hat{E}[|\partial_{xx}^{2} V(i\delta, \sqrt{\delta} T_{i})|] \hat{E}[|X_{i+1}^{2} - Y_{i+1}^{2}|] \delta \\
+ \hat{E}[|\partial_{x} V(i\delta, \sqrt{\delta} T_{i})|] \hat{E}[|X_{i+1} - Y_{i+1}|] \sqrt{\delta} \right\} \\
\leq C \frac{1}{n} \sum_{i=1}^{n} \hat{E}(X_{i} - j)^{+} + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \hat{E}[|X_{i} - Y_{i}|] \to 0,
\]

by the independence and Fact 1. (4.10) is now proved and hence (4.6) follows. Finally, by the Lipschitz continuity of \( \varphi \), we have

\[
\left| \hat{E} \left[ \varphi \left( \frac{S_{n}}{\sqrt{n}} \right) \right] - \hat{E} \left[ \varphi \left( \frac{T_{n}}{\sqrt{n}} \right) \right] \right| \leq C \frac{\sum_{j=1}^{n} \hat{E}[|X_{j} - Y_{j}|]}{\sqrt{n}} \to 0,
\]
by Fact (F1). So, for a bounded and Lipschitz continuous function \( \varphi \), (3.8) is verified.

If \( \varphi \) is a bounded and uniformly continuous function, we define a function \( \varphi_\delta \) as a convolution of \( \varphi \) and the density of a normal distribution \( N(0, \delta) \), i.e.,

\[
\varphi_\delta = \varphi \ast \psi_\delta, \quad \text{with } \psi_\delta(x) = \frac{1}{\sqrt{2\pi\delta}} \exp \left\{ -\frac{x^2}{2\delta} \right\}.
\]

Then \( |\varphi'_\delta(x)| \leq \sup_x |\varphi(x)|\delta^{-1/2} \) and \( \sup_x |\varphi_\delta(x) - \varphi(x)| \to 0 \) as \( \delta \to 0 \). As proved, (3.8) holds for each \( \varphi_\delta \). So, it holds for \( \varphi \).

Finally, suppose \( p \geq 2 \), \( \lim_{c \to \infty} \mathbb{E}[(|X_1|^p - c)^+] = 0 \), and \( \varphi \) is a continuous function satisfying \( \varphi(x) \leq C(1 + |x|^p) \). Give a number \( N > 1 \). Define \( \varphi_1(x) = \varphi((-N) \vee (x \wedge N)) \) and \( \varphi_2(x) = \varphi(x) - \varphi_1(x) \). Then \( \varphi_1 \) is a bounded and uniformly continuous function and

\[
|\varphi_2(x)| \leq 4C|x|^p \mathbb{I}\{|x| > N\} \leq 4C(2|x|^p - N)^+ = 8C(|x|^p - N/2)^+.
\]

So

\[
\mathbb{E} \left[ \varphi \left( \frac{S_n}{\sqrt{n}} \right) \right] - \mathbb{E}[\varphi(\xi)]
\leq \mathbb{E} \left[ \varphi_1 \left( \frac{S_n}{\sqrt{n}} \right) \right] - \mathbb{E}[\varphi_1(\xi)] + 8C\mathbb{E} \left[ (|S_n/\sqrt{n}|^p - N/2)^+ \right] + 8C\mathbb{E} \left[ (|\xi|^p - N/2)^+ \right].
\]

Hence, it is sufficient to show that

\[
\lim_{N \to \infty} \limsup_{n \to \infty} \mathbb{E} \left[ (|S_n/\sqrt{n}|^p - N)^+ \right] = 0. \quad (4.11)
\]

Let \( \hat{Y}_j = X_j - Y_j \), , \( \hat{S}_n = \sum_{j=1}^n (\hat{Y}_j - \mathbb{E}[\hat{Y}_j]) \). Then

\[
S_n^+ \leq T_n^+ + \hat{S}_n^+ + \sum_{j=1}^n \mathbb{E}[|\hat{Y}_j|],
\]

\[
(|S_n^+ / \sqrt{n}|^p - N)^+ \leq \left( 2^{p-1}|T_n^+ / \sqrt{n}|^p - N \right)^+ + 2^{p-1}|\hat{S}_n^+ / \sqrt{n}|^p + 2^{p-1}\left( \sum_{j=1}^n \mathbb{E}[|\hat{Y}_j|] / \sqrt{n} \right)^p.
\]

We have shown that

\[
\sum_{j=1}^n \mathbb{E}[|\hat{Y}_j|] / \sqrt{n} = \sum_{j=1}^n \mathbb{E}[|X_j - Y_j|] / \sqrt{n} \to 0
\]

by Fact (F1), and

\[
\mathbb{E} \left[ (2^{p-1}|T_n^+ / \sqrt{n}|^p - N)^+ \right] \leq N^{-1}2^{2p-2}\mathbb{E} \left[ |T_n / \sqrt{n}|^{2p} \right] \leq N^{-1}C_p
\]
by Fact (F3). Applying (4.2) yields
\[\hat{E}\left[\left|\frac{S^n_+}{\sqrt{n}}\right|^p\right] \leq C_p n^{-p/2} \sum_{j=1}^{n} \hat{E}[|\hat{Y}_j|^p] + C_p \left(n^{-1} \sum_{j=1}^{n} \hat{E}[|\hat{Y}_j|^2]\right)^{p/2}\]
\[\leq C_p n^{-1} \sum_{j=1}^{n} \hat{E}\left[\left(|X_1| - j^{1/2}\right)^p\right] \to 0.\]

It follows that
\[\lim_{N \to \infty} \limsup_{n \to \infty} \hat{E}\left[\left|\frac{S^n_+}{\sqrt{n}} - N\right|^+\right] = 0.\]

Similarly,
\[\lim_{N \to \infty} \limsup_{n \to \infty} \hat{E}\left[\left|\frac{S^n_-}{\sqrt{n}} - N\right|^+\right] = 0.\]

Hence (4.11) is proved and the proof is now completed. □.

If consider \(Y_j = (-j) \lor (X_j \land j)\) and the following equation, instead,
\[\partial_t V + \overline{G}(\partial_x V) = 0,\]
where \(\overline{G}(\alpha) = \overline{\mu} \alpha^+ - \overline{\mu} \alpha^-\), one can prove the following weak law of large numbers. The proof is similar to that of Theorem 3.3 and so omitted.

**Corollary 4.1 (WLLN)** Suppose \(\{X_n; n \geq 1\}\) is a sequence of independent and identically distributed random variables with \(\hat{E}[X_1] = \overline{\mu}, \hat{E}[X_1] = \overline{\mu}\) and \(\lim_{c \to \infty} \hat{E}[|X_1| - c]^+ = 0\). Then for any continuous function \(\varphi\) satisfying \(|\varphi(x)| \leq C(1 + |x|)\),
\[\lim_{n \to \infty} \hat{E}\left[\varphi\left(\frac{S^n}{n}\right)\right] = \sup_{\mu \leq x \leq \overline{\mu}} \varphi(x).\] (4.12)

Further, if \(p \geq 1\) and \(\lim_{c \to \infty} \hat{E}[|X_1|^p - c]^+ = 0\), then (4.12) holds for any continuous function \(\varphi\) satisfying \(|\varphi(x)| \leq C(1 + |x|^p)\).

### 4.3 Proofs of the laws of the iterated logarithm

For proving the law of the iterated logarithm, we need more properties of the sub-linear expectations and capacities. We define an extension of \(\hat{E}\) on the space of all random variables by
\[\mathbb{E}[X] = \inf\{\hat{E}[Y] : X \leq Y, Y \in \mathcal{H}\}.\]

Then \(\mathbb{E}\) is a sub-linear expectation on the space of all random variables, and
\[\mathbb{E}[X] = \hat{E}[X] \ \forall X \in \mathcal{H}, \ \forall (A) = \mathbb{E}[I_A] \ \forall A \in \mathcal{F}.\]
Lemma 4.3 Suppose $X \in \mathcal{H}$.

(i) Then for any $\delta > 0$,
\[ \sum_{n=1}^{\infty} V(|X| \geq \delta a_n) < \infty \iff C_V \left( \frac{X^2}{\log \log |X|} \right) < \infty. \]

(ii) If $C_V \left( \frac{X^2}{\log \log |X|} \right) < \infty$, then for any $\delta > 0$ and $p > 2$,
\[ \sum_{n=1}^{\infty} \mathbb{E} \left[ \left( |X| \wedge (\delta a_n) \right)^p \right] < \infty. \]

(iii) For any $0 \leq b < c < \infty$,
\[ \mathbb{E}[|X| \wedge c] \leq \int_{0}^{c} V(|X| > x)dx, \quad \mathbb{E}[|X| I\{b \leq |X| \leq c\}] \leq \int_{b}^{c} V(|X| > x)dx. \quad (4.13) \]

If \( \lim_{c \to +\infty} \mathbb{E}[(|X| - c)^+] = 0 \) or \( \mathbb{E} \) is countably sub-additive, then \( \mathbb{E}[|X|] \leq C_V(|X|) \).

Proof. (i) It is sufficient to note that
\[ \left\{ c_1 \frac{|X|^2}{\log \log |X|} > n \right\} \subset \{ |X| \geq \delta a_n \} \subset \left\{ c_2 \frac{|X|^2}{\log \log |X|} > n \right\} \]
for some $0 < c_1 < c_2$, and
\[ \sum_{n=1}^{\infty} V(|X| > n) < \infty \iff \int_{0}^{\infty} V(|X| > x)dx. \]

(iii) Suppose that $Y$ is a bounded random variable. Choose $n > 2$ such that $|Y| < n$. Then
\[ |Y| = \sum_{i=1}^{n} |Y| I\{i-1 < |Y| \leq i\} \leq \sum_{i=1}^{n} i(I\{|Y| > i-1\} - I\{|Y| > i\}) \]
\[ \leq 1 + \sum_{i=1}^{n} I\{|Y| > i\}. \]

It follows that
\[ \mathbb{E}[|Y|] \leq 1 + \sum_{i=1}^{n} V(|Y| > i) \leq 1 + \int_{0}^{\infty} V(|Y| > x)dx \leq 1 + C_V[|Y|] \quad (4.14) \]
by the (finite) sub-additivity of $\mathbb{E}$. By considering $|Y|/\epsilon$ instead of $|Y|$, we have
\[ \mathbb{E} \left[ \frac{|Y|}{\epsilon} \right] \leq 1 + C_V \left[ \frac{|Y|}{\epsilon} \right] = 1 + \frac{1}{\epsilon} C_V[|Y|]. \]

That is \( \mathbb{E}[|Y|] \leq \epsilon + C_V[|Y|] \). Taking $\epsilon \to 0$ yields \( \mathbb{E}[|Y|] \leq C_V[|Y|] \). Now, taking $Y = |X| \wedge c$ and $Y = |X| I\{b \leq |X| \leq c\}$ completes the proof of (4.13).
If \( \lim_{c \to +\infty} \hat{E}[(|X| - c)^+] = 0 \), taking \( c \to \infty \) yields \( \hat{E}[|X|] = \lim_{c \to \infty} \hat{E}[|X| \wedge c] \leq C_V(|X|) \). If \( \hat{E} \) is countably sub-additive, then so is \( E \). Taking \( n = \infty \) in (4.14) and using the countable sub-additivity yields the same conclusion for any \( Y \).

(ii) Let \( f(x) \) be the inverse function of \( \sqrt{x \log \log x} \). By (c),

\[
\hat{E}[(|X| \wedge (\delta a_n))^p] \leq \int_0^{(\delta a_n)^p} \mathbb{V}(|X|^p > x) \, dx = p \int_0^{\delta a_n} x^{p-1} \mathbb{V}(|X| > x) \, dx 
\leq 2 \int_0^{\delta n} (2 \log \log y)^{\frac{p}{2}} y^{\frac{p}{2}-1} \mathbb{V}(f(|X|) > y) \, dy.
\]

It follows that

\[
\sum_{n=16}^{\infty} \frac{\hat{E}[(|X| \wedge (\delta a_n))^p]}{a_n^p} \leq 2 \sum_{n=16}^{\infty} a_n^{-p} \int_0^{\delta n} (2 \log \log y)^{\frac{p}{2}} y^{\frac{p}{2}-1} \mathbb{V}(f(|X|) > y) \, dy 
\leq 4 \int_0^{\infty} (2 \log \log x)^{-p/2} \int_0^{\delta x} (2 \log \log y)^{\frac{p}{2}} y^{\frac{p}{2}-1} \mathbb{V}(f(|X|) > y) \, dy \, dx 
\leq 4 \int_0^{\infty} (2 \log \log y)^{\frac{p}{2}} y^{\frac{p}{2}-1} \mathbb{V}(f(|X|) > y) \, dy \int_{y/\delta}^{\infty} (2 \log \log x)^{-p/2} \, dx 
\leq c_3 \int_0^{\infty} \mathbb{V}(f(|X|) > y) \, dy = c_3 C_V[f(|X|) \leq c_3 C_V \left( \frac{|X|^2}{\log \log |X|} \right) < \infty. \quad \square
\]

**Proof of Theorem 3.4.** The proof will be completed via three steps. The first step is to show (a). The second step is show that (3.4) implies (3.2). In the last step, we show that the means are zeros.

**Step 1.** We show (a). Without loss of generality, we assume \( \hat{E}[X_1^2] = 1 \). We will show that

\[
\limsup_{n \to \infty} \frac{S_n}{a_n} \leq 1 \quad \text{a.s.} \quad \mathbb{V}^*, \quad i.e. \quad \mathbb{V}^* \left( \left\{ \frac{S_n}{a_n} > 1 \right\} \text{ i.o.} \right) = 0. \quad (4.15)
\]

For given \( 0 < \epsilon < 1 \), let \( b_n = \frac{\epsilon}{30} \sqrt{n / \log \log n}, \quad a_n = \sqrt{2n \log \log n} \). Let \( Y_k = (-b_k) \vee (X_k \wedge b_k), \quad \tilde{Y}_k = X_k - Y_k \). By the countable sub-additivity of \( \mathbb{V}^* \), (4.15) will follow if we have shown that

\[
\sum_{k=1}^{n} \frac{\tilde{Y}_k^+}{a_n} \to 0 \quad \text{a.s.} \quad \mathbb{V}^* \quad \text{and} \quad \sum_{k=1}^{n} \frac{\tilde{Y}_k^-}{a_n} \to 0 \quad \text{a.s.} \quad \mathbb{V}^* \quad \text{and} \quad \limsup_{n \to \infty} \frac{\sum_{k=1}^{n} Y_k}{a_n} \leq (1 + \epsilon)^2 \quad \text{a.s.} \quad \mathbb{V}^*. \quad (4.16)
\]

and

\[
\limsup_{n \to \infty} \frac{\sum_{k=1}^{n} y_k}{a_n} \leq (1 + \epsilon)^2 \quad \text{a.s.} \quad \mathbb{V}^*. \quad (4.17)
\]
We first show (4.16). We only consider $\hat{Y}_k^+$ since the proof for $\hat{Y}_k^-$ is similar. Let $g_k(\cdot)$ be a smooth function satisfying (4.5). By (2.3),
\[
\sum_{k=1}^{\infty} \mathbb{V}(\hat{Y}_k^+ > a_k) \leq \sum_{k=1}^{\infty} \mathbb{V}(X_k > a_k + b_k) \leq \sum_{k=1}^{\infty} \mathbb{V}(X_k > a_k)
\]
\[
\leq \sum_{k=1}^{\infty} \mathbb{E}\left[g_1/2(X_k/(2a_k))\right] = \sum_{k=1}^{\infty} \mathbb{E}\left[g_1/2(X_1/(2a_k))\right]
\]
\[
\leq \sum_{k=1}^{\infty} \mathbb{V}(X_1 > a_k/2) \leq cC_{\mathcal{V}} \left[\frac{|X_1|^2}{\log \log |X_1|}\right] < \infty.
\]
Hence $\mathbb{V}^*(\{\hat{Y}_k^+ > a_k\} \ i.o.) = 0$ by the Borel-Cantelli lemma. So, it is sufficient to show that
\[
\frac{\sum_{k=1}^{n}(\hat{Y}_k^+ \wedge a_k)}{a_n} \rightarrow 0 \ \text{a.s.} \mathbb{V}^*.
\] (4.18)

Note that the random variables $\hat{Y}_k^+ \wedge a_k$'s are non-negative. It is sufficient to show that
\[
\frac{\sum_{k=2^{n-1}}^{2^n+1}(\hat{Y}_k^+ \wedge a_k)}{a_{2^{n+1}}} \rightarrow 0 \ \text{a.s.} \mathbb{V}^*.
\]

Let $Z_k = (\hat{Y}_k^+ \wedge a_k - \mathbb{E}[\hat{Y}_k^+ \wedge a_k])$. Then for any $\delta > 0$,
\[
\mathbb{V}\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} Z_k}{a_{2^{n+1}}} \geq \delta \right) \leq \frac{C}{2^n+1} \sum_{k=2^n+1}^{2^{n+1}} \mathbb{E}[|Z_k|^p] + \exp\left\{ - \frac{\delta^2(a_{2^{n+1}})^2}{4 \sum_{k=2^n+1}^{2^{n+1}} \mathbb{E}[Z_k^2]} \right\}
\]
by (3.2). Note $\mathbb{E}Z_k^2 \leq 4\mathbb{E}\left[\left((X_1 - b_k)^+\right)^2\right] \rightarrow 0$. By Lemma 4.3 (ii), we have
\[
\sum_{n=1}^{\infty} \mathbb{V}\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} Z_k}{a_{2^{n+1}}} \geq \delta \right) \leq C \sum_{k=1}^{\infty} \mathbb{E}[|X_k \wedge a_k|^p] + \sum_{n=1}^{\infty} \exp\left\{ - 2 \log \log 2^{n+1} \right\} < \infty.
\]
Hence by the sub-additivity of $\mathbb{V}^*$ and the Borel-Cantelli Lemma, we have
\[
\mathbb{V}^* \left( \left\{ \frac{\sum_{k=2^n+1}^{2^{n+1}} Z_k}{a_{2^{n+1}}} > \delta \right\} \ i.o. \right) = 0, \ \forall \delta > 0.
\]

On the other hand,
\[
\frac{\sum_{k=2^n+1}^{2^{n+1}} \mathbb{E}[(\hat{Y}_k^+) \wedge a_k]}{a_{2^{n+1}}} \leq C \frac{\sum_{k=2^n+1}^{2^{n+1}} (\log \log k)^{1/2}}{a_{2^{n+1}}} \mathbb{E}[X_1^2 I\{|X_1| \geq b_k\}] \rightarrow 0.
\]
Hence, (4.18) is proved. So (4.16) holds.

Next, we show (4.17). Let $n_k = \lceil e^{k-\alpha} \rceil$, where $0 < \alpha < \frac{\epsilon}{1+\epsilon}$. Then $n_{k+1}/n_k \rightarrow 1$ and $\frac{n_{k+1} - n_k}{n_k} \approx C \cdot \frac{n_k}{k}$. For $n_k < n \leq n_{k+1}$, we have
\[
\sum_{i=1}^{n} Y_k \leq \sum_{i=1}^{n_k} Y_i + \max_{n_k < n \leq n_{k+1}} \left( \sum_{i=n_k+1}^{n} (-Y_i) \right)
\]
\[
\leq \sum_{i=1}^{n_k} (Y_i - \mathbb{E}[Y_i]) + \max_{n_k < n \leq n_{k+1}} \left( \sum_{i=n_k+1}^{n} (-Y_i) \right) + \sum_{i=1}^{n_k} \left| \mathbb{E}[Y_i] \right|
\]
\[
=: I_k + II_k + III_k.
\]
For the third term, from the fact that $\hat{\EE}[X_i] = \EE[-X_i] = 0$ it follows that $|\hat{\EE}[Y_i]| = |\EE[Y_i] - \hat{\EE}[X_i]| \leq \hat{\EE}(|X_i| - b_i) \leq b_i^{-1} \hat{\EE}[X_i^2 - b_i^2]$ and $|\EE[-Y_i]| = |\EE[-X_i] - \hat{\EE}[(|X_i| - b_i)] \leq b_i^{-1} \hat{\EE}[(|X_i| - b_i)]^2$. Hence

\[
\frac{III_k}{a_{n_k}} = o(1) \frac{\sum_{i=1}^{n_k} (\log \log n_k)^{1/2}}{(n_k \log \log n_k)^{1/2}} \to 0. \tag{4.19}
\]

For the second term, by applying the Rosenthal inequality we have

\[
\forall (II_k \geq \delta a_{n_k}) \leq c \frac{\sum_{i=n_k+1}^{n_k+1} \hat{\EE}[|Y_i|^p]}{a_{n_k}^p} + c \left( \frac{\sum_{i=n_k+1}^{n_k+1} \hat{\EE}[|Y_i|^2]}{a_{n_k}^2} \right)^{p/2} + c \left( \frac{\sum_{i=n_k+1}^{n_k+1} \hat{\EE}[X_i^2]}{a_{n_k}^2} \right)^{p/2} + c \left( \frac{n_{k+1} - n_k}{n_k \log \log n_k} \right)^p.
\]

By Lemma (4.3) (ii), it follows that

\[
\sum_{k=1}^{\infty} \forall (II_k \geq \delta a_{n_k}) \leq c \sum_{i=1}^{\infty} \frac{\hat{\EE}[|X_i| \land a_i]^p}{a_i^p} + c \sum_{k=1}^{\infty} \left( \frac{1}{k^{\alpha}} \right)^{p/2} < \infty, \forall \delta > 0,
\]

whenever we choose $p > 2$ such that $\alpha p/2 > 1$. It follows that

\[
\forall^* \left( \frac{II_k}{a_{n_k}} > \delta \right) i.o. = 0, \forall \delta.
\] \tag{4.20}

Finally, we consider the first term $I_k$. Let $y = 2b_{n_k+1}$ and $x = (1 + \epsilon)a_{n_k+1}$. Then $|Y_i - \hat{\EE}[Y_i]| \leq y$ and $xy \leq \delta_{n_k+1}$. By (3.1), we have

\[
\forall (I_k \geq (1 + \epsilon)^2 a_{n_k}) \leq \exp \left\{ -\frac{(1 + \epsilon)^2 a_{n_k}^2}{2(en_k/10 + \sum_{i=1}^{n_k} \hat{\EE}[Y_i^2])} \right\}.
\]

Since

\[
\left| \hat{\EE}[X_i^2] - \hat{\EE}[Y_i^2] \right| \leq \hat{\EE}[X_i^2 - Y_i^2] = \hat{\EE}[(X_i^2 - b_i^2)] \to 0, \text{ as } i \to \infty,
\]

we have $\sum_{i=1}^{n_k} \hat{\EE}[Y_i^2] \leq (1 + \epsilon/2)n_k \hat{\EE}X_i^2 = (1 + \epsilon/2)n_k$ for $k$ large enough. It follows that

\[
\sum_{k=k_0}^{\infty} \forall (I_k \geq (1 + \epsilon)^2 a_{n_k}) \leq \sum_{k=k_0}^{\infty} \exp \left\{ -(1 + \epsilon)^2 \log n_k \right\} \leq \sum_{k=k_0}^{\infty} \frac{c}{k^{(1+\epsilon)(1-\alpha)}} < \infty
\]

if $\alpha$ is chosen such that $(1 + \epsilon)(1 - \alpha) > 1$. It follows that by the countably sub-additivity and the Borel-Cantelli Lemma again,

\[
\forall^* \left( \left\{ \frac{I_k}{a_{n_k}} > (1 + \epsilon)^2 \right\} \text{ i.o.} \right) = 0. \tag{4.21}
\]

25
Combining (4.19)-(4.21) yields (4.17). The proof of (4.15) is proved.

From (4.15), it follows that

\[
\liminf_{n \to \infty} S_n/a_n = -\limsup_{n \to \infty} -S_n/a_n \geq -1 \quad \text{a.s.} V^*.
\]

The part (a) is proved.

**Step 2.** We show (3.4) =⇒ (3.2). Suppose \( Cl \left[ X^2 \right] \leq \infty \). Then, by (2.3) and Lemma 4.3 (i),

\[
\sum_{j=1}^{\infty} \mathbb{V}(|X_j| > M(1-\epsilon)a_j) \geq \sum_{j=1}^{\infty} \mathbb{E} \left[ g_e \left( \frac{|X_j|}{Ma_j} \right) \right] = \sum_{j=1}^{\infty} \mathbb{E} \left[ g_e \left( \frac{|X_1|}{Ma_j} \right) \right]
\]

\[
\geq \sum_{j=1}^{\infty} \mathbb{V}(|X_1| > Ma_j) = \infty, \quad \forall M > 0.
\]

If \( \{X_j; j \geq 1\} \) were independent under \( V \) or \( V^* \), then one can use the standard argument of the Borel-Cantelli lemma to show that

\[
V^*\left( \{ |X_j| > M(1-\epsilon)a_j \} \right) \quad \text{i.o.} = 1.
\]

Now, since the indicator functions are not in \( Cl, \text{Lip} \), we introduce the smoothing method. Let \( Z_j = g_{1/2} \left( \frac{|X_j|}{Ma_j} \right) \). Then \( 0 \leq Z_j \leq 1 \) and \( \mathbb{E}[Z_j - \mathbb{E}[Z_j]]^2 \leq 2\mathbb{E}[Z_j] \). Note \( \mathbb{E}[-Z_j + \mathbb{E}[Z_j]] = 0 \).

Then by (3.6),

\[
\mathbb{V}\left( \sum_{j=1}^{n} Z_j \leq \frac{1}{2} \sum_{j=1}^{n} \mathbb{E}[Z_j] \right) = \mathbb{V}\left( \sum_{j=1}^{n} (-Z_j + \mathbb{E}[Z_j]) \geq \frac{1}{2} \sum_{j=1}^{n} \mathbb{E}[Z_j] \right) 
\]

\[
\leq C \frac{8 \sum_{j=1}^{n} \mathbb{E}[Z_j]}{(\sum_{j=1}^{n} \mathbb{E}[Z_j])^2} = C \frac{8 \sum_{j=1}^{n} \mathbb{E}[Z_j]}{\sum_{j=1}^{n} \mathbb{E}[Z_j]} \to 0.
\]

So

\[
\mathbb{V}\left( \sum_{j=1}^{n} Z_j > \frac{1}{2} \sum_{j=1}^{n} \mathbb{E}[Z_j] \right) \to 1 \quad \text{as } n \to \infty.
\]

If \( V \) is continuous as assumed in Theorem 3.4 (b), then \( V \equiv V^* \). If \( \mathbb{E} \) is sub-additive as assumed in Theorem 3.4 (c), then

\[
V^*\left( |X| \geq c \right) \leq V\left( |X| \geq c \right) \leq \mathbb{E} \left[ g_{1/2}(\delta/|c|) \right] \leq V^*\left( |X| \geq c(1-\delta) \right), \quad \forall \delta > 0 \tag{4.23}
\]

by (2.3) and (3.1). Write \( A_n = \{ \sum_{j=1}^{n} Z_j > \frac{1}{4} \sum_{j=1}^{n} \mathbb{E}[Z_j] \} \). In either case, we have

\[
V^*\left( A_n \right) \geq \mathbb{V}\left( \sum_{j=1}^{n} Z_j > \frac{1}{2} \sum_{j=1}^{n} \mathbb{E}[Z_j] \right) \to 1 \quad \text{as } n \to \infty.
\]
Now, by the continuity of $\mathbb{V}$,*

$$\mathbb{V}^* \left( \limsup_{n \to \infty} \frac{|X_n|}{a_n} > M(1 - \epsilon) \right) = \mathbb{V}^* \left( \left\{ \frac{|X_j|}{M a_j} > (1 - \epsilon) \right\} \ i.o. \right) \quad (4.24)$$

$$\geq \mathbb{V}^* \left( \sum_{j=1}^{\infty} g_c \left( \frac{|X_j|}{M a_j} \right) = \infty \right) = \mathbb{V}^* \left( A_n \ i.o. \right) \geq \lim_{n \to \infty} \mathbb{V}^* \left( A_n \right) = 1.$$ 

On the other hand,

$$\limsup_{n \to \infty} \frac{|X_n|}{a_n} \leq \limsup_{n \to \infty} \left( \frac{|S_n|}{a_n} + \frac{|S_{n-1}|}{a_n} \right) \leq 2 \limsup_{n \to \infty} \frac{|S_n|}{a_n}.$$ 

It follows that

$$\mathbb{V}^* \left( \limsup_{n \to \infty} \frac{|S_n|}{a_n} > m \right) = 1, \ \forall m > 0.$$ 

Hence

$$\mathbb{V}^* \left( \limsup_{n \to \infty} \frac{|S_n|}{a_n} = +\infty \right) = \lim_{m \to \infty} \mathbb{V}^* \left( \limsup_{n \to \infty} \frac{|S_n|}{a_n} > m \right) = 1,$$

which contradict (3.4). So, (3.2) holds.

**Step 3.** Finally, we show $\widehat{\mathbb{E}}[X_1] = \widehat{\mathbb{E}}[-X_1] = 0$. If $\widehat{\mathbb{E}}$ is countably additive as assumed in (b). Then (3.2) implies $\widehat{\mathbb{E}} \left[ \frac{X^2}{\log \log |X|} \right] \leq C \mathbb{V} \left[ \frac{X^2}{\log \log |X|} \right] < \infty$ by Lemma 4.3 (iii). And so $\lim_{c \to \infty} \widehat{\mathbb{E}}(\{X_1 \leq c\}) = 0$. Note $\lim_{c \to \infty} \widehat{\mathbb{E}}(\{X_1 \geq c\} \cup \{X_1 \leq c\}) = E[X_1]$. Write $Y_j = (c) \vee (X_j \wedge c)$. Then $\widehat{\mathbb{E}}[Y_j] = \widehat{\mathbb{E}}[Y_1] \to \widehat{\mathbb{E}}[X_1]$ as $c \to +\infty$. So, for $c$ large enough, by (3.6) we have

$$\mathbb{V} \left( \frac{S_n}{n} < \widehat{\mathbb{E}}[X_1] - 2\epsilon \right) \leq \mathbb{V} \left( - \sum_{k=1}^{n} (Y_j - \widehat{\mathbb{E}}[Y_j]) > ne \right) + \mathbb{V} \left( \sum_{k=1}^{n} |X_j - Y_j| > ne/2 \right)$$

$$\leq C \sum_{k=1}^{n} \widehat{\mathbb{E}} \left( \frac{|X_j - Y_j|^2}{n^2 \epsilon^2} \right) + \sum_{k=1}^{n} \frac{\widehat{\mathbb{E}}[|X_j - Y_j|]}{n \epsilon}$$

$$\leq \frac{C c^2}{n \epsilon^2} + \frac{\widehat{\mathbb{E}}(\{|X_1| \leq c\})}{\epsilon} \to 0 \ \text{as} \ n \to \infty \ \text{and then} \ c \to \infty.$$

By (4.23), it follows that

$$\lim_{n \to \infty} \mathbb{V}^* \left( \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] - 3\epsilon \right) \geq \lim_{n \to \infty} \mathbb{V} \left( \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] - 2\epsilon \right) = 1.$$ 

By the continuity of $\mathbb{V}^*$,

$$\mathbb{V}^* \left( \limsup_{n \to \infty} \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] - 2\epsilon \right) \geq \limsup_{n \to \infty} \mathbb{V}^* \left( \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] - 3\epsilon \right) = 1.$$ 

It follows that

$$\mathbb{V}^* \left( \limsup_{n \to \infty} \frac{S_n}{n} \geq \widehat{\mathbb{E}}[X_1] \right) = \lim_{\epsilon \downarrow 0} \mathbb{V}^* \left( \limsup_{n \to \infty} \frac{S_n}{n} > \widehat{\mathbb{E}}[X_1] - 3\epsilon \right) = 1, \quad (4.25)$$

27
by the continuity of \( V^* \) again. Write

\[
A = \left\{ \limsup_{n \to \infty} \frac{S_n}{n} \geq \hat{\mathbb{E}}[X_1] \right\}, \quad B = \left\{ \limsup_{n \to \infty} \frac{|S_n|}{a_n} = +\infty \right\}.
\]

By (3.4), (4.25) and the sub-additivity of \( V^* \), it follows that

\[
V^*(AB^c) = V^*(A \setminus AB) \geq V^*(A) - V^*(AB) \geq V^*(A) - V^*(B) > 0.
\]

However, on \( B^c \) we have \( \limsup_{n \to \infty} \frac{|S_n|}{a_n} = 0 \). It follows that \( \hat{\mathbb{E}}[X_1] \leq 0 \). Similarly, \( \hat{\mathbb{E}}[-X_1] \leq 0 \). Hence \( \hat{\mathbb{E}}[X_1] = \hat{\mathbb{E}}[-X_1] = 0 \). (b) and (c) are now proved.

\[\square\]

**Proof of Corollary 3.2.** By Theorem 3.4 we have

\[
\limsup_{n \to \infty} \frac{|S_n|}{a_n} \leq \sigma \ a.s. V^*.
\]

Let \( n_k = k^k \). Then \( n_{k-1}/n_k \to 0 \) and \( a_{n_{k-1}}/a_{n_k} \to 0 \). Note

\[
\frac{S_{n_k}}{a_{n_k}} = \frac{S_{n_k} - S_{n_k-1}}{2(n_k - n_{k-1}) \log \log n_k} \sqrt{1 - \frac{n_{k-1}}{n_k}} + \frac{S_{n_{k-1}}}{a_{n_{k-1}}} a_{n_{k-1}} - \frac{a_{n_{k-1}}}{a_{n_k}}.
\]

So, it is sufficient to show that for any \( b \) with \( |b| < \sigma \),

\[
\liminf_{k \to \infty} \left| \frac{S_{n_k} - S_{n_k-1}}{2(n_k - n_{k-1}) \log \log n_k} - b \right| < \epsilon \ a.s. V^*, \ \forall \epsilon > 0.
\]

By the independence of the sequence \( \{S_{n_k} - S_{n_{k-1}}; k \geq 2\} \) and the smoothing argument as showing (4.24) from (4.22), it is sufficient to prove

\[
\sum_{k=1}^{\infty} V \left( \left| \frac{S_{n_k} - S_{n_{k-1}}}{2(n_k - n_{k-1}) \log \log n_k} - b \right| < \epsilon \right) = \infty, \ \forall \epsilon > 0. \tag{4.26}
\]

Applying Theorem 3.2 with \( y_n = \sqrt{2 \log \log n_k} \) yields

\[
V \left( \left| \frac{S_{n_k} - S_{n_{k-1}}}{2(n_k - n_{k-1}) \log \log n_k} - b \right| < \epsilon \right) \geq \exp \left\{ -((b/\sigma)^2 + \delta) \log \log n_k \right\} \geq c_k^{-((b/\sigma)^2 + \delta)},
\]

if \( k \) is large enough, where \( (b/\sigma)^2 + \delta < 1 \). (4.26) follows and the proof is completed. \[\square\]
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