Lattice QCD with domain wall quarks and applications to weak matrix elements

T. Blum∗ and A. Soni†

∗Department of Physics, Brookhaven National Laboratory, Upton, NY 11973

Using domain wall fermions, we estimate $B_K(\mu \approx 2 \text{GeV}) = 0.628(47)$ in quenched QCD which is consistent with previous calculations. At $6/g^2 = 6.0$ and 5.85 we find the ratio $f_K/m_\rho$ in agreement with the experimental value, within errors. These results support expectations that $O(a)$ errors are exponentially suppressed in low energy ($E \ll a^{-1}$) observables, and indicate that domain wall fermions have good scaling behavior at relatively strong couplings. We also demonstrate that the axial current numerically satisfies the lattice analog of the usual continuum axial Ward identity.

A basic feature of the strong interactions has been missing in lattice calculations, the $SU(N_f)_L \times SU(N_f)_R$ chiral flavor symmetry of the light quarks. We recently reported\cite{1} on calculations using a new discretization for simulations of QCD, domain wall fermions (DWF)\cite{2,3}, which preserve chiral symmetry on the lattice in the limit of an infinite extra 5th dimension. There it was demonstrated that DWF exhibit remarkable chiral behavior\cite{1} even at relatively large lattice spacing and modest extent of the fifth dimension. Here we give further results using DWF which are of direct phenomenological interest\cite{4}.

In addition to retaining chiral symmetry, DWF are also “improved” in another important way. In the limit that the number of sites in the extra dimension, $N_s$, goes to infinity, the leading discretization error in the effective four dimensional action for the light degrees of freedom goes like $O(a^2)$. This theoretical dependence is deduced from the fact that the only operators available to cancel $O(a)$ errors in the effective action are not chirally symmetric. For finite $N_s$, $O(a)$ corrections are expected to be exponentially suppressed with the size of the extra fifth dimension. Our calculations for $B_K$ show a weak dependence on $a$ that is easily fit to an $a^2$ ansatz.

We use the boundary fermion variant of DWF developed by Shamir. For details, consult Ka-plan\cite{2} and Shamir\cite{3}. See Ref.\cite{6} for a discussion of the 4d chiral Ward identities (CWI) satisfied by DWF. Our simulation parameters are summarized in Table 1.

Table 1
Summary of simulation parameters. $M$ is the five dimensional Dirac fermion mass, and $m$ is the coupling between layers $s = 0$ and $N_s - 1$.

| 6/g^2 | size          | M   | m   | # conf |
|-------|---------------|-----|-----|--------|
| 5.85  | $16^3 \times 32 \times 14$ | 1.7 | 0.075 | 34     |
| 5.85  | $16^3 \times 32 \times 10$ | 1.7 | 0.05  | 36     |
| 6.0   | $16^3 \times 32 \times 10$ | 1.7 | 0.075 | 39     |
| 6.3   | $24^3 \times 60 \times 10$ | 1.5 | 0.075 | 25     |

We begin with the numerical investigation of the lattice PCAC relation. The CWI are satisfied exactly on any configuration since they are derived from the corresponding operator identity. We checked this explicitly in our simulations. In the asymptotic large time limit, we find for the usual PCAC relation

$$2 \sinh \left( \frac{a m_\pi}{2} \right) \frac{\langle A_\mu | \pi \rangle}{\langle J_5 | \pi \rangle} = 2m + 2 \frac{\langle J_{5a} | \pi \rangle}{\langle J_5 | \pi \rangle}, \quad (1)$$

which goes over to the continuum relation for $a m_\pi \ll 1$ and $N_s \to \infty$ (see Ref.\cite{6} for operator definitions). The second term on the r.h.s. is
anomalously and vanishes as \( N_s \to \infty \). It is a measure of explicit chiral symmetry breaking induced by the finite 5th dimension. At \( 6/g^2 = 6.0 \) and \( N_s = 10 \) we find the l.h.s. of Eq. 1 to be 0.1578(2) and 0.1083(3) for \( m \) = 0.075 and 0.05, respectively. The anomalous contributions for these two masses are 2x(0.00385(5) and 0.00408(12)), which appears to be roughly constant with \( m \). Increasing \( N_s \) to 14 at \( m = 0.05 \), the anomalous contribution falls to (2x) 0.00152(8) while the l.h.s. is 0.1026(6), which shows that increasing \( N_s \) really does take us towards the chiral limit.

Next we investigate the matrix element of \( O_{LL} \). At \( 6/g^2 = 5.85 \) the two data points extrapolate linearly to -0.0005(100) at \( m = 0 \). At \( 6/g^2 = 6.0 \) the three data points extrapolate to -0.004(9). At \( 6/g^2 = 6.3 \), the two points extrapolate to 0.05(3). This slight overshoot is not unexpected since the values of \( m \) used here correspond to rather heavy quarks. In our initial study we found a similar behavior [1], and as the quark mass was lowered, the required linear behavior set in. All of the above results are for \( N_s = 10 \) except at \( 6/g^2 = 5.85 \) where \( N_s = 14 \) was used for reasons explained below.

![Figure 1. The kaon B parameter. The cross (not used in the fit) denotes the partially unquenched result using KS sea quarks[8].](image)

In Fig. 1 we show the kaon B parameter. The results for \( B_K \) depend weakly on \( 6/g^2 \), and are well fit to a pure quadratic in \( a \). We find \( B_K (\mu = a^{-1}) = 0.628(47) \) in the continuum limit. This value is already consistent with previous results [8] though it does not include the perturbative running of \( B_K \) to a common energy scale. This requires a perturbative calculation to determine the renormalization of \( O_{LL} \), which has not yet been done. The energy scale at \( 6/g^2 = 6.0 \) is roughly 2 GeV. Also, we find \( B_K (\mu = a^{-1}) = 0.67(4) \) on a set of 20 Kogut-Susskind lattices with \( m_{KS} = 0.01 \) and \( 6/g^2 = 5.7 \) and the same five dimensional lattice volume as the point at \( 6/g^2 = 6.0 \).

At \( 6/g^2 = 6.0 \), we have also calculated \( B_K \) using the partially conserved axial current \( A_{\mu}^a (x) \) (and the analogous vector current). This point split conserved current requires explicit factors of the gauge links to be gauge invariant. Alternatively a gauge non-invariant operator may be defined by omitting the links; the two definitions become equivalent in the continuum limit. Results for the gauge non-invariant operators agree within small statistical errors with those obtained with naive currents, Fig. 1(see Ref. [6,1] for operator definitions). The results for the gauge invariant operators receive appreciable perturbative corrections which bring the two results into agreement [8].

Assuming PCAC, the pseudoscalar decay constant is determined from the measurement of \( \langle 0 | J^0_\pi | P \rangle \). We find \( f_K = 159(14) \) MeV and 164(12) MeV for \( 6/g^2 = 6.0 \) and 5.85, respectively. The errors are statistical and do not include the error in the lattice spacing determination from \( am_{\rho} \). The central values agree with experiment, \( f_{K+} = 160 \) MeV. The lattice spacing determinations from \( am_{\rho} \) are \( a^{-1} = 1.53(27) \), 2.09(21), and 3.20(81) GeV at \( 6/g^2 = 5.85 \), 6.0, and 6.3, respectively. Alternatively, we may form the dimensionless ratio \( f_K/m_{\rho} \). We find for \( 6/g^2 = 5.85 \) and 6.0, \( f_K/m_{\rho} = 0.213(42) \) and 0.206(27), where we have added the statistical errors naively in quadrature. The experimental result is 0.208. The decay constant calculated di-
rectly from the matrix element of the partially conserved axial current agrees with results using the matrix element of the pseudoscalar density, up to the anomalous contribution which is slightly less than the statistical error. While the above indicate good scaling behavior, they must be checked further with improved statistics and a fully covariant fitting procedure. More importantly, the continuum limit still has to be taken: a recent precise calculation using quenched Wilson quarks by the CP-PACS collaboration gives a value for $f_K/m_\rho$ in the continuum limit that is inconsistent with experiment \cite{10}, so the above agreement with experiment may be fortuitous.

![Figure 2. The pion mass squared.](image)

In Fig. 2 we show the pion mass squared as a function of $m$. The data at $6/g^2 = 6.0$ and 6.3 are consistent with chiral perturbation theory. At $6/g^2 = 5.85$ the two masses extrapolate to 0.045(10) for $N_s = 10$ and 0.031(13) for $N_s = 14$. This discrepancy is probably not due to higher order terms in the chiral expansion since the physical quark masses are light compared to the masses at the other couplings, and the curvature would have the wrong sign. We see a large downward shift in $m^2_\pi$ as $N_s$ goes from 10 to 14. However, increasing $N_s$ to 18 at $m = 0.075$ has a negligible effect (the point at $m = 0.05$ should also be checked since the slope may increase). We note that the anomalous contribution to Eq. 1 is more than double the value at $6/g^2 = 6.0$. In the case of the vector Schwinger model, it was found that topology changing gauge configurations can induce significant explicit chiral symmetry breaking effects \cite{5}. Further investigation is required.

Our study shows that DWF are an attractive alternative for lattice QCD calculations where chiral symmetry is crucial. DWF yield good agreement with expectations from chiral perturbation theory without the complicated mixing of operators required with Wilson quarks, or the entanglement of flavor and space-time degrees of freedom as with Kogut-Susskind quarks. Up to exponentially small corrections, DWF maintain the full chiral symmetry of QCD at relatively strong couplings, and thus should have more continuum-like behavior. The data presented here seem to indicate just that, though future studies with improved statistics are needed to confirm this. This improved scaling may compensate for the added cost of the extra dimension.

We thank S. Aoki and Y. Shamir for useful discussions. This research was supported by US DOE grant DE-AC0276CH0016. The numerical computations were carried out on the NERSC T3E.

REFERENCES

1. T. Blum and A. Soni, PR D56, 174 (1997).
2. D. Kaplan, PLB 288, 342 (1992).
3. Y. Shamir, NPB 406, 90 (1993).
4. Much of this work was reported in T. Blum and A. Soni, [hep-lat/9706023] sub. to PRL.
5. P. M. Vranas, [hep-lat/9705023].
6. Y. Shamir and V. Furman, NPB 439, 54 (1995).
7. S. Aoki, et al., NPB (Proc.Suppl. 53), 341 (1997); S. Sharpe, NPB (Proc.Suppl. 34), 403 (1994); S. Aoki, et al., [hep-lat/9705032].
8. F. Brown, et al., PRL 71, 1677 (1993); W. Lee and M. Klomfass, [hep-lat/9608089].
9. N. Ishizuka, et al., PRL 71, 24 (1993).
10. Plenary talk by T. Yoshić, this volume.