Characterizations of umbilic hypersurfaces in warped product manifolds

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Abstract We consider the closed orientable hypersurfaces in a wide class of warped product manifolds, which include space forms, deSitter-Schwarzschild and Reissner-Nordström manifolds. By using an integral formula or Brendle’s Heintze-Karcher type inequality, we present some new characterizations of umbilic hypersurfaces. These results can be viewed as generalizations of the classical Jellet-Liebmann theorem and the Alexandrov theorem in Euclidean space.

Keywords Umbilic, k-th mean curvature, warped products

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1 Introduction

The characterization of hypersurfaces with constant mean curvature in warped product manifolds has attracted much attention recently. There are at least three types of results: Jellet-Liebmann type, Alexandrov type, and stability type.

Classical Jellet-Liebmann theorem, also referred to as the Liebmann-Süss theorem, asserts that any closed star-shaped (or convex) immersed hypersurface in Euclidean space with constant mean curvature is a round sphere. This has been generalized to a class of warped products by Montiel [9]. Similar results are also obtained for hypersurfaces with constant higher order mean curvature or Weingarten hypersurfaces in warped products (see [1,5,12]).

The classical Alexandrov theorem states that any closed embedded hypersurface of constant mean curvature in Euclidean space is a round sphere. This was generalized to a class of warped product manifolds by Brendle [4]. The key step in his proof is the Minkowski type formula and a Heintze-Karcher type inequality, which also works for Weingarten hypersurfaces (cf. [5,12]). Kwong

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et al. [6] proved Alexandrov type results for closed embedded hypersurfaces with radially symmetric higher order mean curvature in a class of warped products.

From a variational point of view, hypersurfaces of constant mean curvature in a Riemannian manifold are critical points of the area functional under variations preserving a certain enclosed volume (see [2,3]). Under the assumption of stability, constant mean curvature hypersurfaces in warped products are studied in [8,11], etc.

In this paper, we prove the Jellet-Liebmann type theorems and an Alexandrov type theorem for certain closed hypersurfaces including constant mean curvature hypersurfaces in some class of warped product manifolds.

Throughout this paper, we assume that $\overline{M}^{n+1} = [0, r) \times_{\lambda} P^n$ ($0 < r \leq \infty$) is a warped product manifold endowed with a metric

$$\overline{g} = dr^2 + \lambda^2(r)g^P,$$

where $(P, g^P)$ is an $n$-dimensional closed Riemannian manifold ($n \geq 2$) and $\lambda: [0, r) \rightarrow [0, +\infty)$ is a smooth positive function, called the warping function.

We first consider a hypersurface $x: M^n \rightarrow \overline{M}^{n+1}$ immersed in a warped product $\overline{M}^{n+1}$ whose mean curvature $H$ satisfies

$$H = \phi(r),$$

where $\phi(r) = x^*(\Phi(r))$ and $\Phi(r)$ is a radially symmetric positive function on $\overline{M}$; or

$$H^{-\alpha} = \langle \lambda \partial_r, \nu \rangle,$$

where $\nu$ is a normal vector of $M$ and $\alpha > 0$ is a constant.

Notice that hypersurfaces satisfying (1) are critical points of the area functional under variations preserving a weighted volume (see Appendix). And hypersurfaces satisfying (2) can be seen as self-similar solutions to the curvature flows expanding by $H^{-\alpha}$. Furthermore, both of them can be regarded as generalizations of constant mean curvature hypersurfaces.

We now introduce the following definition of locally star-shapedness for the convenience.

**Definition 1** Suppose that $(\overline{M}^{n+1}, \overline{g})$ is a warped product manifold and $x: M \rightarrow \overline{M}$ is an immersion of an orientable hypersurface $M^n$ in $\overline{M}$. The hypersurface $x(M)$ is **locally star-shaped** if $\langle \partial_r, \nu \rangle > 0$ holds everywhere on $M$.

**Remark 1** When $\overline{M}$ is a Euclidean space, locally star-shapedness is a weaker condition compared with star-shapedness. In fact, it implies that $x(M)$ can be locally represented as a graph on a sphere centered at the origin. To the best of our knowledge, it is not clear whether a locally star-shaped hypersurface in a warped product manifold must be embedded. Actually, we can construct a locally star-shaped curve in $\mathbb{R}^2$ (see Fig. 1) which has a self-intersection point.
Our first main result is the Jellet-Liebmann type theorems of these hypersurfaces.

**Theorem 1** Suppose that \((\overline{M}^{n+1}, \overline{g})\) is a warped product manifold satisfying
\[
\text{Ric}^P \geq (n-1)(\lambda'^2 - \lambda'\lambda'')g^P,
\]
and \(x: M \to \overline{M}\) is an immersion of a closed orientable hypersurface \(M^n\) in \(\overline{M}\). If \(x(M)\) is locally star-shaped and satisfies
\[
\langle \nabla H, \partial_r \rangle \leq 0,
\]
then \(x(M)\) must be totally umbilic.

**Remark 2** If \(M\) has constant mean curvature, Theorem 1 reduces to the Jellett-Liebmann type theorem proved by Montiel [9].

Applying Theorem 1 to hypersurfaces satisfying (1), we obtain the following result.

**Corollary 1** Under the same assumption of Theorem 1, if \(x(M)\) is locally star-shaped and satisfies
\[
H = \phi(r),
\]
where \(\phi(r) = x^*(\Phi(r))\) and \(\Phi(r)\) is a positive non-increasing function of \(r\), then \(x(M)\) must be totally umbilic.

**Remark 3** An Alexandrov type theorem for the above hypersurfaces under the embeddedness assumption was obtained by Kwong et al. [6].

The following example shows that the non-increasing assumption on \(\Phi(r)\) is necessary.

**Example 1** Let \(\Sigma\) be an ellipsoid given by
\[
\Sigma = \left\{ y \in \mathbb{R}^{n+1} \mid y_1^2 + \cdots + y_n^2 + \frac{y_{n+1}^2}{\alpha^2} = 1 \right\}.
\]
The mean curvature of $\Sigma$ is

$$H = \frac{a}{n\sqrt{a^2 + 1 - r^2}} \left( n - 1 + \frac{1}{a^2 + 1 - r^2} \right) =: \Phi(r).$$

It is easy to check that $\Phi$ is increasing for $r$.

Another application of Theorem 1 is about hypersurfaces satisfying (2).

**Corollary 2** Under the same assumption of Theorem 1, if $x(M)$ is strictly convex and satisfies

$$H^{-\alpha} = \langle \lambda \partial_r, \nu \rangle,$$

where $\alpha > 0$ is a constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Similar results hold for higher order mean curvature under stronger assumptions. Let $\sigma_k(\kappa)$ denote the $k$-th elementary symmetric polynomial of principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ of $x(M)$, i.e.,

$$\sigma_k(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}.$$

Then the $k$-th mean curvature is given by $H_k = \sigma_k(\kappa) / \binom{n}{k}$. A hypersurface $x(M)$ is $k$-convex, if, at any point of $M$, principal curvatures $\kappa \in \Gamma_k := \{ \mu \in \mathbb{R}^n \mid \sigma_i(\mu) > 0, 1 \leq i \leq k \}$.

**Theorem 2** Suppose that $M^{n+1} = [0, \bar{r}) \times \lambda P^n$ is a warped product manifold, where $(P, g^P)$ is a closed Riemannian manifold with constant sectional curvature $\varepsilon$ and

$$\frac{\lambda''(r)}{\lambda(r)} + \frac{\varepsilon - (\lambda'(r))^2}{(\lambda(r))^2} \geq 0. \tag{4}$$

Let $x: M \to \bar{M}$ be an immersion of a closed orientable hypersurface $M^n$ in $\bar{M}$. For any fixed $k$ with $2 \leq k \leq n - 1$, if $x(M)$ is $k$-convex, locally star-shaped, and satisfies

$$\langle \nabla H_k, \partial_r \rangle \leq 0,$$

then $x(M)$ must be totally umbilic.

If we require that the inequality in (4) is strict as in [4,5] and $H_k = \text{constant}$, then we obtain the following result.

**Corollary 3** Suppose that $M^{n+1} = [0, \bar{r}) \times \lambda P^n$ is a warped product manifold, where $(P, g^P)$ is a closed Riemannian manifold with constant sectional curvature $\varepsilon$ and

$$\frac{\lambda''(r)}{\lambda(r)} + \frac{\varepsilon - (\lambda'(r))^2}{(\lambda(r))^2} > 0. \tag{5}$$

Let $x: M \to \bar{M}$ be an immersion of a closed orientable hypersurface $M^n$ in $\bar{M}$. For any fixed $k$ with $2 \leq k \leq n - 1$, if $x(M)$ is $k$-convex, locally star-shaped, and $H_k = \text{constant}$, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$. 
Remark 4 Corollary 3 implies that the embeddedness condition in [5, Theorem 2] can be replaced by the locally star-shapedness.

Corollary 4 Under the same assumption of Theorem 2, if for any fixed $k$ with $2 \leq k \leq n - 1$, $x(M)$ is $k$-convex, locally star-shaped, and satisfies

$$H_k = \phi(r),$$

where $\phi(r) = x^*(\Phi(r))$ and $\Phi(r)$ is a positive non-increasing function of $r$, then $x(M)$ must be totally umbilic.

Corollary 5 Under the same assumption of Theorem 2, if for any fixed $k$ with $2 \leq k \leq n - 1$, $x(M)$ is strictly convex and satisfies

$$H_k^{\alpha} = \langle \lambda \partial_r, \nu \rangle,$$

where $\alpha > 0$ is a constant, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Now, we turn to the warped product manifold $\overline{M}^{n+1} = [0, \bar{r}) \times_\lambda P^n$, where $(P, g^P)$ is a closed Riemannian manifold with constant sectional curvature $\varepsilon$. As in [4,12], we list four conditions of the warping function $\lambda: [0, \bar{r}) \rightarrow [0, +\infty)$:

(C1) $\lambda'(0) = 0$ and $\lambda''(0) > 0$;
(C2) $\lambda'(r) > 0$ for all $r \in (0, \bar{r})$;
(C3) the function

$$2\frac{\lambda''(r)}{\lambda(r)} - (n - 1)\varepsilon - (\lambda'(r))^2\frac{1}{(\lambda(r))^2}$$

is non-decreasing for $r \in (0, \bar{r})$;
(C4) (5) holds for all $r \in (0, \bar{r})$.

Instead of locally star-shapedness or convexity, under the embeddedness assumption, we study hypersurfaces satisfying

$$H_k^{\alpha} = \langle \lambda \partial_r, \nu \rangle,$$  \hspace{1cm} (6)

and prove the following Alexandrov type theorem.

Theorem 3 Suppose that $(\overline{M}, \overline{g})$ is a warped product manifold satisfying conditions (C1)–(C4), or the space forms $\mathbb{R}^{n+1}$, $\mathbb{S}^{n+1}_+$, and $\mathbb{H}^{n+1}$. Let $x: M \rightarrow \overline{M}$ be an embedding of a connected closed orientable hypersurface $M^n$ in $\overline{M}$. If $H_k > 0$ and $x(M)$ satisfies

$$H_k^{\alpha} = \langle \lambda \partial_r, \nu \rangle$$  \hspace{1cm} (7)

for some fixed $k$ with $1 \leq k \leq n$ and $\alpha \geq 1/k$, then $x(M)$ is a slice $\{r_0\} \times P$ for some $r_0 \in (0, \bar{r})$.

Remark 5 Theorem 3 is a new application of the Minkowski type formula and Brendle’s Heintze-Karcher type inequality. Additionally, one needs Hölder’s inequality to force the equality holds in Heintze-Karcher type inequality.
Remark 6  It is interesting to compare Theorem 3 with Corollary 5 when $\overline{M} = \mathbb{R}^{n+1}$. In this case, (7) is actually the equation of self-similar solution of $H_{k}^{-\alpha}$ flow since $\lambda' = 1$. With embeddedness and less convexity requirement (only $H_{k} > 0$), Theorem 3 leads to the conclusion including the case of $k = n$. In general warped product manifolds, the geometric meaning of (7) is unclear.

Remark 7  Throughout this paper, the assumptions for the ambient spaces $\overline{M}$ are satisfied by space forms, the deSitter-Schwarzschild manifolds, the Reissner-Nordström manifolds, and many other manifolds (cf. [4]).

This paper is organized as follows. In Section 2, we list some useful properties of warped products. In Section 3, we derive an integral formula which is the key to the proof of our main theorems. In Section 4, we present the proofs of Theorem 1, 2, and the corollaries. In Section 5, we prove Theorem 3. In Appendix, we show that a hypersurface with a given positive mean curvature function is the critical point of the area functional under variations preserving weighted volume. Throughout the paper, the summation convention is used unless otherwise stated.

2 Preliminaries of warped products

In this section, we list some basic properties of warped products $(\overline{M} = [0, \overline{r}) \times_{\lambda} P^{n}, \overline{g})$ given above (see [10]).

Proposition 1  Suppose $U, V \in \Gamma(TP)$. The Levi-Civita connection $\nabla$ of a warped product $(\overline{M} = [0, \overline{r}) \times_{\lambda} P, \overline{g})$ satisfies

i) $\nabla_{\partial_{r}} \partial_{r} = 0$,

ii) $\nabla_{\partial_{r}} V = \nabla_{V} \partial_{r} = \frac{\lambda'}{\lambda} V$,

iii) $\nabla_{V} U = \nabla^{P}_{V} U - \frac{\lambda'}{\lambda} \overline{g}(V, U) \partial_{r}$,

where $\nabla^{P}$ is the Levi-Civita connection of $(P, g^{P})$.

Remark 8  From Proposition 1, we know that any slice $\{r\} \times P$ in a warped product $\overline{M} = [0, \overline{r}) \times_{\lambda} P$ is totally umbilic.

Proposition 2  Suppose $Y_{1}, Y_{2}, Y_{3}, Y_{4} \in \Gamma(TP)$. The $(0, 4)$-Riemannian curvature tensor $\overline{\text{Rm}}$ of a warped product $(\overline{M} = [0, \overline{r}) \times_{\lambda} P, \overline{g})$ satisfies

i) $\overline{\text{Rm}}(\partial_{r}, Y_{1}, \partial_{r}, Y_{2}) = -\frac{\lambda''}{\lambda} \overline{g}(Y_{1}, Y_{2})$,

ii) $\overline{\text{Rm}}(\partial_{r}, Y_{1}, Y_{2}, Y_{3}) = 0$,

iii)

$$\overline{\text{Rm}}(Y_{1}, Y_{2}, Y_{3}, Y_{4}) = \lambda^{2} \text{Rm}^{P}(Y_{1}, Y_{2}, Y_{3}, Y_{4}) - \frac{\lambda^{2}}{\lambda^{2}} (\overline{g}(Y_{1}, Y_{3}) \overline{g}(Y_{2}, Y_{4}) - \overline{g}(Y_{2}, Y_{3}) \overline{g}(Y_{1}, Y_{4})),$$

where $\text{Rm}^{P}$ is the $(0, 4)$-Riemannian curvature tensor of $(P, g^{P})$. 

Suppose \( U, V \in \Gamma(T \bar{M}) \). The Ricci curvature tensor \( \overline{\text{Ric}} \) of a warped product manifold \( (\bar{M} = [0, \bar{r}] \times \lambda P, \bar{g}) \) satisfies

i) \( \overline{\text{Ric}}(\partial_r, \partial_r) = -n \frac{\lambda''}{\lambda} \),

ii) \( \overline{\text{Ric}}(\partial_r, V) = 0 \),

iii) \( \overline{\text{Ric}}(V, U) = \text{Ric}^P(V, U) - (\frac{\lambda''}{\lambda} + (n - 1)\frac{\lambda'^2}{\lambda^2})\bar{g}(V, U) \),

where \( \text{Ric}^P \) is the Ricci curvature tensor of \( (P, g^P) \).

For the convenience, we introduce the Kulkarni-Nomizu product \( \overline{\otimes} \). For any two \((0, 2)\)-type symmetric tensors \( h \) and \( w \), \( h \overline{\otimes} w \) is the 4-tensor given by

\[
(h \overline{\otimes} w)(X_1, X_2, X_3, X_4) = h(X_1, X_3)w(X_2, X_4) + h(X_2, X_4)w(X_1, X_3) - h(X_1, X_4)w(X_2, X_3) - h(X_2, X_3)w(X_1, X_4).
\]

The following result is a corollary of Proposition 2 through a straightforward calculation and the proof is given for the completeness.

Proposition 4 Suppose that \( (P, g^P) \) is a Riemannian manifold with constant sectional curvature \( \varepsilon \). The Riemannian curvature tensor \( \overline{\text{Rm}} \) of a warped product manifold \( \bar{M} = [0, \bar{r}] \times \lambda P \) can be expressed as follows:

\[
\overline{\text{Rm}} = \frac{\varepsilon - \lambda'^2}{2\lambda^2} \bar{g} \overline{\otimes} \bar{g} - \left( \frac{\lambda''}{\lambda} + \frac{\varepsilon - \lambda'^2}{\lambda^2} \right) \bar{g} \overline{\otimes} dr^2. \tag{8}
\]

Proof Let

\[
e_A, e_B, e_C, e_D \in \Gamma(T \bar{M}), \quad r_A = \bar{g}(e_A, \partial_r), \quad e_A^* = e_A - r_A \partial_r.
\]

Using Proposition 2, we have

\[
\overline{\text{Rm}}(e_A, e_B, e_C, e_D)
= \overline{\text{Rm}}(e_A^*, e_B^*, e_C^*, e_D^*)
= \overline{\text{Rm}}(e_A^*, r_B \partial_r, e_C^*, r_D \partial_r)
= \overline{\text{Rm}}(r_A \partial_r, e_B^*, r_C \partial_r, e_C^*)
= \lambda'^2 \overline{\text{Rm}}^P(e_A^*, e_B^*, e_C^*, e_D^*)
- \frac{\lambda'^2}{\lambda^2} (\bar{g}(e_A^*, e_C^*)\bar{g}(e_B^*, e_D^*) - \bar{g}(e_B^*, e_C^*)\bar{g}(e_A^*, e_D^*))
- \frac{\lambda''}{\lambda} (r_{BD} \bar{g}(e_A^*, e_C^*) - r_{BC} \bar{g}(e_A^*, e_D^*) - r_{AC} \bar{g}(e_B^*, e_C^*) + r_{AC} \bar{g}(e_B^*, e_D^*)).
\]

Since

\[
\bar{g}(e_A^*, e_C^*) = \bar{g}(e_A, e_C) - r_{AC} = (\bar{g} - dr^2)(e_A, e_C),
\]

we know

\[
\bar{g}(e_A^*, e_C^*)\bar{g}(e_B^*, e_D^*) - \bar{g}(e_B^*, e_C^*)\bar{g}(e_A^*, e_D^*)
= \frac{1}{2} (\bar{g} - dr^2) \overline{\otimes} (\bar{g} - dr^2)(e_A, e_B, e_C, e_D)
\]

and

\[
r_{BD} \bar{g}(e_A^*, e_C^*) - r_{BC} \bar{g}(e_A^*, e_D^*) - r_{AC} \bar{g}(e_B^*, e_C^*) + r_{AC} \bar{g}(e_B^*, e_D^*)
= (\bar{g} - dr^2) \overline{\otimes} dr^2(e_A, e_B, e_C, e_D).
\]
Using the fact that the sectional curvatures of $(P, g^P)$ is a constant $\varepsilon$, i.e.,

$$\text{Rm}^P = \frac{\varepsilon}{2} g^P \otimes g^P,$$

we have

$$\lambda^2 \text{Rm}^P(e_A^*, e_B^*, e_C^*, e_D^*) = \frac{\varepsilon}{2\lambda^2} (\bar{g} - dr^2) \otimes (\bar{g} - dr^2)(e_A, e_B, e_C, e_D).$$

Combining these together, we obtain (8). \[\square\]

3 An integral formula

In this section, we obtain an integral formula by the divergence theorem, which is the key to the proof of our main results.

Let $x : M \to \overline{M}$ be an immersion of a closed orientable hypersurface $M^n$ into a warped product $\overline{M}^{n+1} = [0, \overline{r}) \times_{\lambda} P^n$ endowed with a metric $\bar{g} = dr^2 + \lambda^2(r)g^P$. Let $\nu$ be a unit normal vector field of $M$ and $h = (h_{ij})$ denote the second fundamental form with respect to an orthonormal frame $\{e_1, \ldots, e_n\}$ on $M$ defined by

$$h_{ij} = \langle \nabla_i x, \nabla_j \nu \rangle.$$

The principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ are the eigenvalues of $h$. Thus, the $k$-th elementary symmetric polynomials of principal curvatures can be expressed as

$$\sigma_k(\kappa(h)) = \frac{1}{k!} \delta^{i_1 \cdots i_k}_{j_1 \cdots j_k} h_{i_1 j_1} \cdots h_{i_k j_k},$$

where $\delta^{i_1 \cdots i_k}_{j_1 \cdots j_k}$ is the generalized Kronecker symbol. Let $\sigma_{k;i}(\kappa)$ denote $\sigma_k(\kappa)$ with $\kappa_i = 0$ and $\sigma_{k;ij}(\kappa)$, with $i \neq j$, denote the symmetric function $\sigma_k(\kappa)$ with $\kappa_i = \kappa_j = 0$. For convenience, we let

$$\sigma_0(\kappa) = \sigma_{0;i}(\kappa) = \sigma_{0;ij}(\kappa) = 1$$

for some fixed $i$ and $j$.

The following proposition is from a standard calculation (see also [5,6]) and the proof is given for the completeness.

**Proposition 5** Under an orthonormal frame such that $h_{ij} = \kappa_i \delta_{ij}$, we have the equality

$$\sum_i \nabla_i (\frac{\partial \sigma_k(h)}{\partial h_{ij}}) = -\sum_{p \neq j} R_{vpjp} \sigma_{k-2;jp}(\kappa)$$

for any fixed $j$ and $2 \leq k \leq n$.

**Proof** Let $\tilde{h} = I + th$. Then

$$\sigma_n(\tilde{h}) = \sigma_n(I + th) = \sum_{k=0}^{n} t^k \sigma_k(h).$$

(9)
Using
\[
\frac{\partial \sigma_n(\tilde{h})}{\partial h_{ij}} = t(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h}),
\]
\[
\sum_{i=1}^{n} \nabla_i (\tilde{h}^{-1})_{ij} = -t(\tilde{h}^{-1})_{ip}(\tilde{h}^{-1})_{qj} \nabla_i h_{pq},
\]
for arbitrary \( t \) small and the Codazzi equation
\[
\nabla_i h_{pq} = \nabla_q h_{pi} + R_{\nu p q i},
\]
we have
\[
\sum_{k=1}^{n} t^k \nabla_i \frac{\partial \sigma_k(h)}{\partial h_{ij}} = \nabla_i \frac{\partial \sigma_n(\tilde{h})}{\partial h_{ij}} = t^2 \sigma_n(\tilde{h})(-\tilde{h}^{-1})_{ip}(\tilde{h}^{-1})_{qj} \nabla_i h_{pq} + (\tilde{h}^{-1})_{ij}(\tilde{h}^{-1})_{pq} \nabla_i h_{pq} = t^2(\tilde{h}^{-1})_{pq}(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h})(-\nabla_q h_{pi} + \nabla_i h_{pq}) = t^2(\tilde{h}^{-1})_{pq}(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h})R_{\nu p q i}.
\]
(10)

Now, we choose a local orthonormal frame \( \{e_1, \ldots, e_n\} \) such that \( h_{ij} = \kappa_i \delta_{ij} \).

Then
\[
(\tilde{h}^{-1})_{ij} = \frac{\delta_{ij}}{1 + t\kappa_i}.
\]

Thus,
\[
(\tilde{h}^{-1})_{pq}(\tilde{h}^{-1})_{ij} \sigma_n(\tilde{h})R_{\nu p q i} = -\frac{\overline{R}_{\nu p j p}}{(1 + t\kappa_j)(1 + t\kappa_p)} \prod_{l=1}^{n} (1 + t\kappa_l)
\]
\[
= -\overline{R}_{\nu p j p} \prod_{l \in \{1, \ldots, n\} \setminus \{j, p\}} (1 + t\kappa_l)
\]
\[
= -\overline{R}_{\nu p j p} \sum_{k=2}^{n} t^{k-2} \sigma_{k-2;jp}(\kappa).
\]

Combining with (10), we have
\[
\sum_{k=1}^{n} t^k \nabla_i \frac{\partial \sigma_k(h)}{\partial h_{ij}} = -\overline{R}_{\nu p j p} \sum_{k=2}^{n} t^k \sigma_{k-2;jp}(\kappa).
\]

Comparing the coefficients of \( t^k \), we have
\[
\sum_{i} \nabla_i \left. \frac{\partial \sigma_k(h)}{\partial h_{ij}} \right|_{\kappa} = - \sum_{p \neq j} \overline{R}_{\nu p j p} \sigma_{k-2;jp}(\kappa) \quad (11)
\]
for each \( k \in \{2, \ldots, n\} \). \( \square \)
Denote
\[ \eta = x^* \left( \int_0^r \lambda(s) \, ds \right), \quad u = \langle \lambda \partial_r, \nu \rangle. \]

Then we have the following integral formula.

**Lemma 1** Suppose that \( x(M) \) is a closed hypersurface of \( \overline{M} \). Then the following equality holds:
\[
\int_M \{ -(n-k) \langle \nabla \sigma_k, \lambda \partial_r \rangle + (n-k) \sigma_1 \sigma_k - n(k+1) \sigma_{k+1} \} u - n R_{\nu p j p} \langle \lambda \partial_r, e_j \rangle \sigma_{k-1; j p} \} d\mu = 0.
\]

**Proof** From a straightforward calculation, we have
\[
\nabla_i \left( k \sigma_k \nabla_i \eta - n \frac{\partial \sigma_k}{\partial h_{i j}} \nabla_j u \right)
= k \langle \nabla \sigma_k, \nabla \eta \rangle + k \sigma_k \Delta \eta - n \nabla_i \left( \frac{\partial \sigma_k}{\partial h_{i j}} \right) \nabla_j u - n \frac{\partial \sigma_k}{\partial h_{i j}} \nabla_i \nabla_j u
= k \langle \nabla \sigma_k, \nabla \eta \rangle + k \sigma_k (n \lambda' - \sigma_1 u) - n \nabla_i \left( \frac{\partial \sigma_k}{\partial h_{i j}} \right) \langle \lambda \partial_r, h_{i j} e_l \rangle
+ n \frac{\partial \sigma_k}{\partial h_{i j}} (- \lambda' h_{i j} - \langle \lambda \partial_r, h_{i j} e_l \rangle + h_{i j} h_{i j} u)
= k \langle \nabla \sigma_k, \lambda \partial_r \rangle + (n-k) \sigma_k \sigma_1 u - n(k+1) \sigma_{k+1} u
- n \nabla_i \left( \frac{\partial \sigma_k}{\partial h_{i j}} \right) \langle \lambda \partial_r, h_{i j} e_l \rangle - n \langle \lambda \partial_r, \nabla \sigma_k \rangle - n \frac{\partial \sigma_k}{\partial h_{i j}} \langle \lambda \partial_r, e_l \rangle \overline{R}_{\nu j l i}.
\]

When \( 2 \leq k \leq n \), using Proposition 5, we know that
\[
- n \nabla_i \left( \frac{\partial \sigma_k}{\partial h_{i j}} \right) \langle \lambda \partial_r, h_{i j} e_l \rangle - n \frac{\partial \sigma_k}{\partial h_{i j}} \langle \lambda \partial_r, e_l \rangle \overline{R}_{\nu j l i}
= n \overline{R}_{\nu p j p} \sigma_{k-2; j p} \langle \lambda \partial_r, e_j \rangle - n \sigma_{k-1; j p} \langle \lambda \partial_r, e_l \rangle \overline{R}_{\nu j i l}
= n \overline{R}_{\nu p j p} \langle \lambda \partial_r, e_j \rangle \langle \sigma_{k-2; j p} \kappa_j - \sigma_{k-1; j p} \rangle
= - n \overline{R}_{\nu p j p} \langle \lambda \partial_r, e_j \rangle \sigma_{k-1; j p}.
\]

For \( k = 1 \), the above equality is established by a direct computation.
Combining these equalities and using the divergence theorem, we finish the proof. \( \square \)

**4 Proofs of main theorems**

**Proof of Theorem 1** Using Lemma 1 for \( k = 1 \), we know that
\[
\int_M \{ -n(n-1) \langle \nabla H, \lambda \partial_r \rangle + ((n-1) \sigma_1^2 - 2n \sigma_2) u - n \text{Ric}(\nu, \lambda \partial_r^\top) \} \, d\mu = 0, \quad (12)
\]
where $\partial^\top_r$ denotes the tangent part of $\partial_r$.

From the Newton inequality and the assumption that $x(M)$ is locally star-shaped, we have

\[
((n - 1)\sigma^2_1 - 2n\sigma_2)u \geq 0.
\]

And the equality of (13) occurs if and only if $\kappa_1 = \cdots = \kappa_n$.

Let

\[
\nu^P = \nu - \langle \nu, \partial_r \rangle \partial_r.
\]

Since

\[
\text{Ric}(\nu, \lambda \partial^\top_r) = \text{Ric}(\nu, \lambda \partial_r) - \text{Ric}(\nu, \nu)u
\]

\[
= u\left(-n \frac{\lambda''}{\lambda} + n \frac{\lambda''}{\lambda} \langle \partial_r, \nu \rangle^2 - \text{Ric}^P(\nu^P, \nu^P) + (\frac{\lambda''}{\lambda} + (n - 1)\frac{\lambda'^2}{\lambda^2})|\nu^P|^2 \right)
\]

\[
= -u\left(\text{Ric}^P(\nu^P, \nu^P) + (n - 1)\left(\frac{\lambda''}{\lambda} - \frac{\lambda'^2}{\lambda^2}\right)|\nu^P|^2 \right)
\]

\[
= -u(\text{Ric}^P(\nu^P, \nu^P) + (n - 1)(\lambda\lambda'' - \lambda'^2)g^P(\nu^P, \nu^P)),
\]

we know $\text{Ric}(\nu, \lambda \partial^\top_r) \leq 0$ by assumption.

Combining these estimates with $\langle \nabla H, \partial_r \rangle \leq 0$, we obtain that the left-hand side of (12) is nonnegative. This implies that the inequalities are actually equalities at any point of $M$. Thus, $x(M)$ is totally umbilic. \(\square\)

**Remark 9** In the previous proof, when

\[
\text{Ric}^P > (n - 1)(\lambda'^2 - \lambda\lambda'')g^P,
\]

we also obtain $\partial^\top_r = 0$. This means that $\partial_r$ is the normal vector of $x(M)$, which implies that $x(M)$ is a slice.

**Proof of Theorem 2** Under the assumption, it follows from (8) that

\[
\overline{R}_{vpjp} = -\left(\frac{\lambda''}{\lambda} + \frac{\varepsilon - \lambda'^2}{\lambda^2}\right)\langle \partial_r, \nu \rangle \langle \partial_r, e_j \rangle
\]

for any fixed $p$ and $j \neq p$. Using

\[
\sum_{p \neq j} \sigma_{k-1;jp} = (n - k)\sigma_{k-1;j},
\]

we know

\[
-\sum_p \overline{R}_{vpjp} \langle \lambda \partial_r, e_j \rangle \sigma_{k-1;jp} = \sum_p u\left(\frac{\lambda''}{\lambda} + \frac{\varepsilon - \lambda'^2}{\lambda^2}\right)\langle \partial_r, e_j \rangle^2 \sigma_{k-1;jp}
\]

\[
= (n - k)u\left(\frac{\lambda''}{\lambda} + \frac{\varepsilon - \lambda'^2}{\lambda^2}\right)\langle \partial_r, e_j \rangle^2 \sigma_{k-1;j}
\]

\[
\geq 0,
\]
where the inequality follows from that \( \frac{\lambda'}{\lambda} + \frac{\varepsilon \lambda^2}{\lambda^2} \geq 0 \) and \( x(M) \) is locally star-shaped and \( k \)-convex.

Similar to the previous proof, we know that
\[
-(n-k)\langle \nabla \sigma_k, \lambda \partial_r \rangle \geq 0
\]
and
\[
((n-k)\sigma_1 \sigma_k - n(k+1)\sigma_{k+1})u \geq 0.
\]
But Lemma 1 shows that the integral of these terms are zero. Thus, we know that all these inequalities are actually equalities, which implies \( x(M) \) is totally umbilic.

\[\square\]

Proof of Corollary 3 Since \( H_k = \text{constant} \), we know \( \nabla \sigma_k = 0 \). From the proof of Theorem 2,
\[
-R_{\nu p j p} (\lambda \partial_r, e_j) \sigma_{k-1;jp} = 0.
\]
Combining with condition (5), we obtain \( |\partial_r^\top| = 0 \), which implies that \( x(M) \) is a slice.

\[\square\]

Proofs of Corollaries 1 and 4 From \( H_k = \phi(r) \), by a direct calculation, we have
\[
\langle \nabla H_k, \partial_r \rangle = \phi' |\partial_r^\top|^2.
\]
Since \( \Phi(r) \) is non-increasing, we know \( \phi'(r) \leq 0 \). Thus,
\[
\langle \nabla H_k, \partial_r \rangle \leq 0. \tag{14}
\]

By Theorem 1 or 2, we finish the proof.

\[\square\]

Proofs of Corollaries 2 and 5 From \( H_k^{-\alpha} = u \), we have
\[
\langle \nabla H_k, \partial_r \rangle = -\frac{1}{\alpha} u^{-\frac{1}{\alpha}-1} \langle \nabla u, \partial_r \rangle = -\frac{1}{\alpha} u^{-\frac{1}{\alpha}-1} \lambda \kappa_i(e_i, \partial_r)^2.
\]
Since \( x(M) \) is strictly convex, from \( u = H_k^{-\alpha} \), we know \( u > 0 \). By \( \alpha > 0 \), we have (14).

From the proof of Theorem 1 or 2, we know
\[
\langle \nabla H_k, \partial_r \rangle = 0.
\]
This implies that \( \partial_r \) is the normal vector of \( x(M) \), which means that \( x(M) \) is a slice.

\[\square\]

5 Proof of Theorem 3

In this section, we give the proof of Theorem 3. By [7, Lemma 2.3], we know that \( x(M) \) is \( k \)-convex from \( H_k > 0 \). Thus, Maclaurin’s inequality
\[
H_k^{1/k} \leq H_k^{1/(k-1)} \tag{15}
\]
holds. From Brendle’s Heintze-Karcher type inequality established in [4],
\[ \int_M ud\mu \leq \int_M \frac{\lambda'}{H_1} d\mu \]
and the Minkowski type formula (see [5])
\[ \int_M H_k u d\mu \geq \int_M H_{k-1}^\alpha \lambda' d\mu, \]
combining with Maclaurin’s inequality and \( H^{-\alpha}_k = u \), we obtain
\[ \int_M H^{-\alpha}_k \lambda' d\mu \leq \int_M \frac{\lambda'}{H_1} d\mu \leq \int_M H^{-1/k}_k \lambda' d\mu \tag{16} \]
and
\[ \int_M H^{1-\alpha}_k \lambda' d\mu \geq \int_M H_{k-1} \lambda' d\mu. \tag{17} \]

By Hölder’s inequality, Maclaurin’s inequality (15), and (17), we have
\[
\int_M H^{-1/k}_k \lambda' d\mu \leq \left( \int_M H^{-p/p}_k H^{-p/k}_k \lambda' d\mu \right)^{1/p} \left( \int_M H_{k-1} \lambda' d\mu \right)^{(p-1)/p} \\
\leq \left( \int_M H^{-1+p/k-1+k/p-1}_{k-1} \lambda' d\mu \right)^{1/p} \left( \int_M H^{1-\alpha}_k \lambda' d\mu \right)^{(p-1)/p}. \tag{18}
\]

Choose \( p \) such that
\[ p - 1 + \frac{1}{k} = \alpha. \]
Then \( p = (k\alpha + k - 1)/k \). This implies \( p > 1 \) when \( \alpha > 1/k \). Inequality (18) becomes
\[
\int_M H^{-1/k}_k \lambda' d\mu \leq \left( \int_M H^{-\alpha}_k \lambda' d\mu \right)^{k/(k\alpha + k - 1)} \left( \int_M H^{1-\alpha}_k \lambda' d\mu \right)^{(k\alpha - 1)/(k\alpha + k - 1)}. \tag{19}
\]

Using Hölder’s inequality, (15), and (17) again as before, we obtain
\[
\int_M H^{1-\alpha}_k \lambda' d\mu \leq \left( \int_M H^{-1+k+p-pk\alpha}/k \lambda' d\mu \right)^{1/p} \left( \int_M H^{1-\alpha}_k \lambda' d\mu \right)^{(p-1)/p}.
\]
Equivalently,
\[ \int_M H^{1-\alpha}_k \lambda' d\mu \leq \int_M H^{-1+k+p-pk\alpha}/k \lambda' d\mu. \tag{20} \]

Now, we choose \( p \) such that
\[ \frac{-1 + k + p - pk\alpha}{k} = -\alpha. \]
Then $p = (k\alpha - 1 + k)/(k\alpha - 1)$. Thus, inequality (20) is

$$\int_M H_k^{1-\alpha} \lambda' d\mu \leq \int_M H_k^{-\alpha} \lambda' d\mu.$$ 

Substituting the above inequality into (19), we obtain

$$\int_M H_k^{-1/k} \lambda' d\mu \leq \int_M H_k^{-\alpha} \lambda' d\mu.$$ 

Combining the above inequality with (16) when $\alpha > 1/k$ or directly from (16) if $\alpha = 1/k$, we know that the equality of the Heintze-Karcher type inequality occurs for $\alpha \geq 1/k$. As in [4], we finish the proof.

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**Appendix**

Let $\overline{M}^{n+1}$ be an oriented Riemannian manifold, and let $x: M^n \to \overline{M}^{n+1}$ be an immersion of a closed smooth $n$-dimensional manifold $M$ into $\overline{M}^{n+1}$. Suppose that a smooth map $X: (-\varepsilon, \varepsilon) \times M \to \overline{M}$ is a normal variation satisfying

$$\frac{\partial}{\partial t} X = -f\nu,$$

where $f$ is a smooth function on $M$ and $\nu$ is the unit normal of $X(t, M)$.

We introduce the weighted volume $V: (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$V(t) = \int_{[0,t] \times M} X^*(e^\Psi d\overline{\mu}), \quad (A1)$$

where $d\overline{\mu}$ is a standard volume element of $\overline{M}$ and $\Psi$ is a smooth function on $\overline{M}$. Thus, $V(t)$ represents the (oriented) weighted volume sweeping by $M$ on the time interval $[0, t)$. By the same calculations as in [3], we have

$$V'(t) = \int_M fX^*(e^\Psi) d\mu, \quad (A2)$$

where $d\mu$ is the volume element of $M$ with respect to the induced metric.

**Definition A1** A variation of $x: M \to \overline{M}$ is called a weighted volume-preserving variation if $V(t) \equiv 0$.

Denote

$$J(t) = A(t) + nH_0 V(t), \quad (A3)$$
where
\[ A(t) = \int_M d\mu, \quad H_0 = \frac{\int_M H d\mu}{\int_M X^*(e^\psi) d\mu}. \]

Then
\[ J'(0) = \int_M n f (-H + H_0 e^\psi) d\mu, \]

where \( \psi \) denotes \( x^*(\Psi) \).

**Proposition A1** The following three statements are equivalent:

(i) the mean curvature of \( x(M) \) satisfies \( H = C e^\psi \) for a constant \( C \);

(ii) for all weighted volume-preserving variations, \( A'(0) = 0 \);

(iii) for arbitrary variations, \( J'(0) = 0 \).

**Proof** It is easy to check (i) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (ii). Now, we show (ii) \( \Rightarrow \) (i).

We can choose \( f = -H e^{-\psi} + H_0 \) since \( \int_M f e^\psi d\mu = 0 \). From
\[ 0 = J'(0) = \int_M n (-H e^{-\psi} + H_0)^2 e^\psi d\mu, \]

we know \(-H e^{-\psi} + H_0 = 0\). Thus, we finish the proof. \( \square \)

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