Some mixed Hodge structures on $l^2$-cohomology groups of coverings of Kähler manifolds

Pascal Dingoyan

Abstract We give methods to compute $l^2$-cohomology groups of a covering manifold obtained by removing the pullback of a normal crossing divisor to a covering of a compact Kähler manifold. We prove that in suitable quotient categories, these groups admit natural mixed Hodge structure whose graded pieces are given by the expected Gysin maps.

Mathematics Subject Classification 32L10 (32C35, 14C30)

1 Introduction

The Hodge decomposition of the cohomology ring of a compact Kähler manifold $X$ defines a Hodge structure. This gives strong relations between the topology of the manifold and the holomorphic structure. This decomposition still holds for an infinite covering $p : \tilde{X} \to X$ once one restricts to the space of square integrable harmonic forms. Let us recall two achievements in this setting.

The $l^2$-cohomology groups of the covering $p : \tilde{X} \to X$ are the De Rham cohomology groups $H^k_{d(2)}(\tilde{X}) = \text{Ker}(d)/\text{Im}(d)$ of the square integrable forms on $\tilde{X}$. The reduced $l^2$-cohomology groups $\text{Ker}(d)/\text{Im}(d)$ are isomorphic to the harmonic spaces $H^k_{d(2)}(\tilde{X})$ [2, 9, 10, 16, 18, 36, 54]. Assume that $p : \tilde{X} \to X$ is a Galois covering with Galois group $G$.

Then a fundamental result of Atiyah [2] is that the Euler characteristic of $X$ is equal to the $l^2$-Euler characteristic of $p : \tilde{X} \to X$. Note that the $l^2$-harmonic spaces are modules over the Von Neumann algebra $N(G)$ generated by the left action of $G$ on $l^2(G)$. This allows to define a Von Neumann dimension.
When \( X \) is a Kähler manifold, the \( d \)-harmonic spaces on \( \tilde{X} \) admit a Hodge decomposition according to bi-type: \( H^d_{d(2)}(\tilde{X}) = \oplus_{p+q=r} H^{(p,q)}_{d(2)}(\tilde{X}) \). If moreover \( \tilde{X} \) is the universal covering space of \( X \), Gromov [30,31] proves that the non vanishing of \( H^1_{d(2)}(\tilde{X}) \) implies that there exists a proper equivariant holomorphic map from \( \tilde{X} \) to the unit disc.

Deligne [11,12] discovered that the cohomology ring of any algebraic variety, open or singular, carries a mixed Hodge structure (a real filtration whose quotients have a Hodge structure subject to natural compatibilities). In this article, one will put such mixed Hodge structure on the cohomology groups of the quasi-projective manifolds \( X, D_i, \tilde{X}, \tilde{D}_i \). An example is provided by one irreducible smooth divisor \( D \) in \( X \): the Gysin–Leray exact sequence \( \ldots \rightarrow H^{p-2}(D, \mathbb{C}) \overset{i_*}{\rightarrow} H^p(X, \mathbb{C}) \rightarrow H^p(X \setminus D, \mathbb{C}) \rightarrow \ldots \) enables one to filter \( H^p(X \setminus D, \mathbb{C}) \) by subspaces whose quotients carry Hodge structures. In general, a spectral sequence considers further relations between the divisors.

Now, the \( l^2 \)-cohomology groups of \( H^{-1}(X \setminus D) \rightarrow X \setminus D \) are defined as the cohomology over \( X \setminus D \) of the locally constant sheaf \( p_*\mathcal{C}|_{X \setminus D} \), the sheaf of locally constant square integrable functions in the fiber of \( p \). This sheaf is isomorphic to \( l^2(G) \otimes_{\mathbb{Z}[G]} p\mathbb{Z}_{\tilde{X} \setminus p^{-1}(D)} \) and the cohomology groups are isomorphic to the groups of equivariant cohomology of \( p^{-1}(X \setminus D) \) with values in \( l^2(G) \) (see Sect. 2.6.6).

The building blocks of the mixed Hodge structure on \( H^*(X \setminus D, p_*\mathcal{C}) \) will be the Hodge structures on the harmonic spaces \( H^d_{d(2)}(\tilde{X}), H^d_{d(2)}(p^{-1}(D_i)), H^d_{d(2)}(p^{-1}(D_i \cap D_j)), \ldots \) and Gysin’s morphisms between them. Note that \( G \) acts co-compactly on the manifolds \( \tilde{X}, p^{-1}(D_i), \ldots \).

We first transpose the sheaf theoretic development of mixed Hodge theory as in Deligne [11] to the \( l^2 \)-setting. To this aim, we use the definition of Campana–Demailly [8] for the \( l^2 \)-cohomology groups of \( G \)-equivariant coherent analytic sheaves. This gives enough functoriality to obtain the Gysin’s morphisms between the non reduced \( l^2 \)-cohomology groups.

In order to use the Hodge structure on the harmonic spaces, we work in a quotient category or localized category [25,32,60]. In the quotient category, the morphisms from the \( \mathbb{Z} \)-harmonic spaces \( H^{(p,q)}_{d(2)}(p^{-1}(\cap_{0\leq i \leq q} D_i)) \) to the un reduced Dolbeault cohomology groups \( H^{p,q}_{d(2)}(p^{-1}(\cap_{0\leq i \leq q} D_i)) \) are isomorphisms. Hence the non reduced parts of the Dolbeault cohomology groups become isomorphic to zero.

Then, the weight spectral sequence, which abuts to the \( l^2 \)-cohomology groups of \( H^{-1}(X \setminus D) \rightarrow X \setminus D \), becomes a spectral sequence of Hodge structures. As in Deligne [11], it degenerates at \( E_2 \) and is therefore computable (see Theorem 1.3).

The initial motivation of this work was the study of non compact divisors in the universal covering of \( X \), whose irreducible components are compact. We refer to the article of Nori [42] for results on this question. Here we focus on the technical
Some mixed Hodge structures on $l^2$-cohomology groups

part of mixed Hodge structure modulo some torsion theory. In a subsequent paper, functoriality, geometrical and analytical applications will be given.

Before giving detailed statements, we comment on the use of torsion theory. The precise definition of a torsion theory will be given in Sect. 2.4. It is a quite standard tool in $l^2$-cohomology (see Farber [23], Eyssidieux [21], Lück [36], Sauer-Thom [49] and it appears implicitly in Cheeger-Gromov [9], Shubin [54]). We combine the torsion theory with a theory of $l^2$-sheaves (Eyssidieux [21], Campana–Demailly [8]): we can then link the simplicial, the topological and the analytical $l^2$-cohomology in the framework of the mixed Hodge structures.

In the quotient category, the torsion modules are by definition isomorphic to zero and isomorphisms are defined up to torsion modules. Hence it may be useful to interpret in the original category an isomorphism in the localised category. In the case of a Galois covering, the Von Neumann algebra $N(G)$ of the groups $G$ acts faithfully on the $l^2$-cohomology groups. We can work modulo the modules with vanishing $V$ on Neumann dimension and results may be interpreted in terms of the orbit $N(G)x$ of elements. This leads to a $\partial\bar{\partial}$-lemma (Corollary 3.13) up to a weak isomorphism.

The following theorem gives a functorial description of the $l^2$-Hodge decomposition. It will be used later to relate the $l^2$-cohomology groups over $X$, $D$, $D_i \cap D_j$, . . .

Let $(p_*(2)\Omega, d) \to X$ be the complex of $l^2$-direct image of holomorphic forms: a germ at $x \in X$ is given by a square integrable holomorphic form in a neighborhood $p^{-1}(V)$ of $p^{-1}(x)$ [8] and Sect. 2.6.

**Theorem 1.1** Let $p : \tilde{X} \to (X, \omega)$ be a Galois covering of a compact hermitian manifold with covering group $G$. Let $N(G)$ be the von Neumann algebra of $G$.

1. There exists a Hodge to De Rham spectral sequence of $N(G)$-modules

   $$H_{\bar{\partial}(2)}^{p,q}(\tilde{X}) \simeq H^q(X, p_*(2)\Omega^p) \Rightarrow \bigoplus_{r=p+q} H^{p+q}_d(\tilde{X}).$$

2. Assume that $\omega$ is a Kähler metric. Let $\tau$ be a real torsion theory such that $\text{Im}\bar{\partial}/\text{Im}\partial$ is a torsion module. Then the Hodge to De Rham spectral sequence

   $$H_{\bar{\partial}(2)}^{p,q}(\tilde{X}) \Rightarrow H^{p+q}_d(\tilde{X})$$

   degenerates at $E_1$ in the quotient category $\text{Mod}(N(G))/\tau$ and the isomorphism

   $$\bigoplus_{p+q=r} H^{(p,q)}_{\bar{\partial}(2)}(\tilde{X}) \simeq H^r_d(\tilde{X})$$

   in $\text{Mod}(N(G))/\tau$ defines a $\tau$-Hodge structure on $H^r_d(\tilde{X})$.

Standard torsion theories are $\tau_{\text{dim}}$ and $\tau_{U(G)}$: the torsion theory $\tau_{\text{dim}}$ is such that modules of zero $G$-dimension are torsions (we use the generalised Von-Neumann dimension of [36, Chap. 6]).

Let $U(G)$ be the ring of operators affiliated to $N(G)$: It is the ring of unbounded operators on $l^2(G)$ that commutes with the right action of $G$ on $l^2(G)$. It is isomorphic to the quotient ring of $N(G)$ by the multiplicative set of weak isomorphism (see [36, Chap. 8]). Then $\tau_{U(G)}$ is the torsion theory such that a module $M$ is a torsion module.
if $\mathcal{U}(G) \otimes_{N(G)} M$ is isomorphic to zero. When the covering $p : \tilde{X} \to X$ is Galois and $X$ is compact, then $\text{Im} \tilde{\alpha} / \text{Im} \bar{\alpha}$ is a $\tau_{\mathcal{U}(G)}$-torsion module.

The following theorem enables one to define a mixed Hodge structure on the $l^2$-cohomology groups of $p^{-1}(X \setminus D) \to X \setminus D$. It gives two filtrations for computing these groups. The weight filtration $W$ measures the singularities along $D$ of some representative of a cohomology class in $H^2(X \setminus D, p_*(\mathcal{O}_\tilde{X}))$. The Hodge filtration is related to the decomposition of forms into bi-type.

Let $p_*(\mathcal{O}_\tilde{X}) = l^2(G, \mathbb{R}) \otimes_{\mathbb{R}[G]} p_!(\mathbb{R}_\tilde{X})$ and $p_*(\mathbb{C}) = l^2(G, \mathbb{C}) \otimes_{\mathbb{C}[G]} p_!(\mathbb{C}_\tilde{X})$ be the sheaves on $X$ of locally constant functions which are square integrable in the fibers of $p$. Let $(p_*(\mathcal{O}_{\tilde{X}}(\log D), d))$ be the complex of sheaves on $X$ of $l^2$-direct image ([8] and Sect. 2.6) of the logarithmic forms with pole on $D$. This complex is bi-filtered by the weight filtration $W$ (see Sect. 4.1) and the Hodge filtration $F$.

**Theorem 1.2** Let $D = D_1 \cup \cdots \cup D_r$ be a normal crossing divisor in $X$. Set $D_I = \bigcup_{I \subseteq \{1, \ldots, n\}} \bigcap_{i \in I} D_i$, $(1 \leq l \leq n)$ and $D_0 = X$.

1. The group $H(X \setminus D, p_*(\mathbb{C}))$ is isomorphic as a $N(G)$-module to $H^2(X, (p_*(\mathcal{O}_{\tilde{X}}(\log D)), d))$.
2. Let $\tau$ be a real torsion theory on $\text{Mod}(N(G))$ and let $\text{Mod}(N(G))/\tau$ be the quotient category. Assume the $l^2$-Hodge to De Rham spectral sequences of each $p^{-1}(D_l)$, $l \in \{0, \ldots, r\}$, degenerates in $\text{Mod}(N(G))/\tau$ as in Theorem 1.1. Then:
   (i) The weight spectral sequence for $H^2(X, (p_*(\mathcal{O}_{\tilde{X}}(\log D)), W, d))$ whose $E_1^{p, q}$-term is isomorphic to $H^q_{d(2)}(p^{-1}(D_p))$ degenerates at $E_2$ in $\text{Mod}(N(G))/\tau$.
   (ii) The Hodge spectral sequence for $H^2(X, (p_*(\mathcal{O}_{\tilde{X}}(\log D)), F, d))$ which has term $E_1^{p, q}$ isomorphic to $H^q(X, p_*(\mathcal{O}_{\tilde{X}}(\log D)))$ degenerates at $E_1$ in $\text{Mod}(N(G))/\tau$.
   (iii) Define the weight filtration $W$ on $H^k(X \setminus Y, p_*(\mathbb{R}))$ to be the shifted filtration $W[k] := W_{-k}$ of the filtration induced by the weight spectral sequence.
      Define the Hodge filtration $F$ on $H^k(X \setminus Y, p_*(\mathbb{C}))$ to be the filtration induced by the Hodge spectral sequence.
      Then $(H^k(X \setminus D, p_*(\mathbb{R})), W, F)$ is a mixed Hodge structure in $\text{Mod}(N(G))/\tau$.

We note further that for a torsion theory $\tau$ defined in terms of the group $G$ only, such as $\tau_{\text{dim}}$ or $\tau_{\mathcal{U}(G)}$, the above mixed Hodge structure in $\text{Mod}(N(G))/\tau$ is functorial in the category of $G$-covers. Moreover two compactifications of $X' \setminus D$ which are dominated by a third one will define isomorphic mixed Hodge structures.

The reduction of the above spectral sequences to $\text{Mod}(N(G))/\tau_{\text{dim}}$ or $\text{Mod}(N(G))/\tau_{\mathcal{U}(G)}$ always defines mixed Hodge structures. The isomorphism $E_2(W) \simeq E_\infty(W)$ gives the expected realisation in term of harmonic spaces and Gysin maps (notations of Corollary 5.1):

**Theorem 1.3** Let $p : \tilde{X} \to X$ be a Galois covering of a compact Kähler manifold. Let $G$ be its group of deck transformations and let $N(G)$ be its von Neumann algebra. Then in the quotient category $\text{Mod}(N(G))/\tau_{\text{dim}}$ or $\text{Mod}(N(G))/\tau_{\mathcal{U}(G)}$, the
space $Gr^W_l H^k(X \setminus \mathcal{D}, p_*(\mathbb{C}))$ is isomorphic to the middle homology of the Gysin sequence

$$\mathcal{H}^k_{d(2)}(p^{-1}(\mathcal{D}_{l+1})) \to \mathcal{H}^k_{d(2)}(p^{-1}(\mathcal{D}_l)) \to \mathcal{H}^k_{d(2)}(p^{-1}(\mathcal{D}_{l-1})).$$

Moreover

$$Gr^p_F H^{p+q}(X \setminus \mathcal{D}, p_*(\mathbb{C})) \cong \mathcal{H}^q_{d(2)}(\tilde{X}, \Omega^p_X \log(p^{-1}(\mathcal{D}))).$$

Part of the graded module associated to the weight filtration on $H^\cdot(X \setminus \mathcal{D}, p_*(\mathbb{C}))$ has a combinatorial description:

A dual CW complex $\tilde{K}$ is attached to $(\tilde{X}, p^{-1}(\mathcal{D})) \to (X, \mathcal{D})$ (see Sect. 5.2): A connected component of $p^{-1}(\mathcal{D}_{k+1})$ defines a $k$-cell. It is attached along $(k - 1)$-cells corresponding to connected components of $p^{-1}(\mathcal{D}_k)$ which contain it.

**Corollary 1.4** Let $n = \dim \mathbb{C} X$. In $\text{Mod}(N(G))/\tau_{\text{dim}}$ or $\text{Mod}(N(G))/\tau_{\text{dim}(G)}$, the space $Gr^W_{2n} H^{2n-(k+1)}(X \setminus \mathcal{D}, p_*(\mathbb{C}))$ is isomorphic to the $k$-th (reduced) relative $l^2$-homology group $\tilde{H}_{k,2}(||\tilde{K}|, ||\tilde{K}(\infty)||)$ of the dual CW-complex associated to $(\tilde{X}, p^{-1}(\mathcal{D})) \to (X, \mathcal{D})$.

Here $\tilde{K}(\infty)$ is the set of cells whose isotropy subgroups under the action of $G$ are infinite.

## 2 Preliminaries

### 2.1 Real structures

#### 2.1.1 The Godement resolution

(see Godement [26] or Bredon [4]). Let $X$ be a topological space. Let $R$ be a ring and $\mathcal{R}$ be the sheaf of rings it defines. Let $\mathcal{A}$ be a sheaf of (left) $\mathcal{R}$-modules. Let $\mathcal{C}^\cdot (X, \mathcal{A}) := (\mathcal{C}^\cdot_{\text{God}}(X, \mathcal{A}), d)$ be the Godement resolution [26, p. 168] by sheaves of $\mathcal{R}$-modules with $\mathcal{R}$-linear differential $d$. Let $\Gamma(X, \cdot)$ be the functor of global sections.

Then

$$\mathcal{A} \to \mathcal{C}^\cdot (X, \mathcal{A}) \quad \mathcal{A} \to \mathcal{C}^\cdot (X, \mathcal{A}) = (\Gamma(X, \mathcal{C}^\cdot_{\text{God}}(X, \mathcal{A})), d)$$

are covariant additive exact functors with values in the category of differential $\mathcal{R}$-sheaves, resp. cochain $R$-complexes. If $X$ is clear from the context, we write $\mathcal{C}(\mathcal{A}), \ldots$

If $(\mathcal{F}, d)$ is a differential sheaf, let $\mathcal{C}^\cdot (\mathcal{F}^\cdot)$ be the total complex associated to the double complex $\mathcal{C}^d_{\text{God}}(X, \mathcal{F}^p)$. Let $j : Y \to X$ be a continuous map and $(\mathcal{F}^\cdot, d)$ be a differential sheaf of $\mathcal{R}$-modules on $Y$. One sets $Rj_*\mathcal{F}^\cdot := j_*\mathcal{C}^\cdot (\mathcal{F}^\cdot)$

#### 2.1.2 Real structures

Assume that $R$ is a $\mathbb{R}$-algebra, then any sheaf of $\mathcal{R}$-modules is also a sheaf of $\mathbb{R}$-vector spaces so that $\mathcal{A} \otimes R \mathbb{C}$ is a sheaf of $\mathcal{R} \otimes R \mathbb{C}$-left modules. Let $i : \mathcal{A} \to \mathcal{A} \otimes R \mathbb{C}$.
Then

\[ j_*(\mathcal{A}) \otimes_{\mathbb{R}} \mathbb{C} \to j_*(\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}) \quad (1) \]

\[ C^*(X, \mathcal{A}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{C^*(i) \otimes 1_C} C^*(X, \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}) \quad (2) \]

\[ C^*(X, \mathcal{A}) \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{C^*(i) \otimes 1_C} C^*(X, \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}) \quad (3) \]

are $\mathcal{R} \otimes_{\mathbb{R}} \mathbb{C}$-isomorphisms (resp. $R \otimes_{\mathbb{R}} \mathbb{C}$-isomorphisms).

A real structure on a $\mathcal{R} \otimes_{\mathbb{R}} \mathbb{C}$-sheaf $\mathcal{B}$ is a $\mathcal{R}$-subsheaf $\mathcal{A} \to \mathcal{B}$ such that $\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C} \overset{i \otimes 1_C}{\to} \mathcal{B}$ is an isomorphism. It induces a real structure on the Godement resolution.

Then $H^*(X, \mathcal{A}) \otimes_{\mathbb{R}} \mathbb{C} \to H^*(X, \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C})$ is a $R \otimes_{\mathbb{R}} \mathbb{C}$-isomorphism.

**Definition 2.1** Let $G$ be a discrete group. Let $N_l(G)$ (resp. $N_r(G)$) be the left (resp. right) von Neumann algebra of $G$ generated by operators on $L^2(G):=L^2(G, \mathbb{C})$ of left (resp. right) convolution by elements in $\mathbb{C}[G]$. Then one sets $N(G):=N_l(G)$.

The $\mathbb{R}[G]$-isomorphism $L^2(G, \mathbb{C}) \ni a \mapsto Re(a)+iIm(a) \in L^2(G, \mathbb{R}) \otimes \mathbb{C}$ induces decompositions

\[ \text{End}_\mathbb{C}(L^2(G, \mathbb{R}) \otimes \mathbb{C}) \simeq \text{End}_\mathbb{R}(L^2(G, \mathbb{R})) \otimes \mathbb{C} \quad (4) \]

\[ \text{End}_{\mathbb{C}[G]}(L^2(G, \mathbb{R}) \otimes \mathbb{C}) \simeq \text{End}_{\mathbb{R}[G]}(L^2(G, \mathbb{R})) \otimes \mathbb{C} \quad (5) \]

\[ N_r(G, \mathbb{C}) \simeq N_r(G, \mathbb{R}) \otimes \mathbb{C} \quad (6) \]

The left action of $G$ on $L^2(G, \mathbb{R})$ defines a left action of $G$ on $\text{End}_\mathbb{R}(L^2(G, \mathbb{R}))$ and $\text{End}_\mathbb{C}L^2(G, \mathbb{C})$ so that $g(m(a))=(gm)(ga)$. The set of continuous invariant morphisms for this action, $N_r(G, \mathbb{R})$ and $N_r(G, \mathbb{C})$, are algebras. An element $\rho(f)$ in $N_r(G, \mathbb{C})$ is represented by right convolution with a function $f \in L^2(G, \mathbb{C})$ which is moderate $[15, 13.8.3]: \exists C \geq 0 : \forall g \in \mathbb{C}[G] . \| g \ast f \|_2 \leq C \| g \|_2$. Then $\rho(f)$ belongs to $N_r(G, \mathbb{R})$ is equivalent to $f$ real valued.

The symmetric statements hold if we work with the right Von Neumann algebra generated by the right regular representation. Hence $N(G, \mathbb{C}) \simeq N(G, \mathbb{R}) \otimes \mathbb{C}$.

**2.2 A lemma on von Neumann algebras** In this section we recall basic definitions about a (finite) von Neumann algebra $M$, its representations and the Murray–von Neumann dimension.

Our goal is the Lemma 2.15 which enables to lift, up to a weak isomorphism, to the source a vector in the closure of the range of a $M$-Fredholm operator. This lemma, of analytical independent interest, will give a non trivial example of torsion theory.

For this section we refer to Dixmier [15], Pedersen [43], Takesaki [57]. A survey on von Neumann algebras is given in [36, chap. 9].

**Definition 2.2** Let $H_0$ be some (separable) Hilbert space. A $C^*$-subalgebra $M$ of $\mathcal{B}(H_0)$ is a von Neumann algebra if $M$ is weakly closed $\iff M$ is strongly closed $\iff M$ is equal to its bi-commutant in $\mathcal{B}(H)$. 

\[ \otimes \] Springer
One let $M'$ be the commutant of $M$ in $B(H_0)$. Let $H$ be some Hilbert space. A representation $\pi : M \to B(H)$ is normal if it is continuous on bounded increasing net of $M$. Normality of $\pi$ implies that Ker $(\pi)$ and $\pi(M)$ are Von Neumann algebras. Then $(H, \pi)$ (or simply $H$) is called a M-Hilbert module. We recall that weakly isomorphic M-Hilbert modules are M-isometric:

**Lemma 2.3** Let $f : (H_1, \pi_1) \to (H_2, \pi_2)$ be a bounded M-linear weak isomorphism (injective with dense range). Let $f = u\pi$ be its polar decomposition. Then $u : H_1 \to H_2$ is a M-isometry.

**Definition 2.4** Let $M$ be a Von Neumann algebra on a separable Hilbert space. Let $M_+$ be the cone of positive operators in $M$.

(1) A trace is a function $t : M_+ \to [0, +\infty]$ such that if $\lambda > 0$ and $x, y \in M_+$ then $t(\lambda x) = \lambda t(x)$, $t(x+y) = t(x)+t(y)$ and for all unitary $u \in A$, $t(uxu^*) = t(x)$.

(2) A state is a positive linear functional of norm one: $\phi \in M'$ such that $\phi(M_+) \subset \mathbb{R}^+$ and $\phi(1) = 1$.

(3) A trace or a state is normal if it is continuous on limit of increasing nets.

(4) A faithful trace is called finite if $t(1) < +\infty$ (and $M$ is then called finite von Neumann algebra). It is called semi finite if $M_+^t = \{y \in M_+ : t(y) < +\infty\}$ is weakly dense in $M_+$. Then for all $x \in M_+$, $t(x) = \sup_{y\leq x, y\in M_+^t} t(y)$.

**Example 2.5** (1) Let $\delta_e \in l^2(G)$ be the dirac function at the unit element $e$ of $G$. The trace of $n \in N_l(G)$ or $N_r(G)$ is tr$_{N_l(G)}n := \langle n(\delta_e), \delta_e \rangle$.

(2) Let $(H, \pi)$ be a M-Hilbert module. Let $\xi \in H$ be a vector of unit norm. Then $x \mapsto \langle \pi(x)\xi, \xi \rangle$ is a normal positive linear functional of norm 1, called the vector state associated to $(\pi, H, \xi)$.

Traces and states satisfy Cauchy–Schwartz inequality:

$$|\varphi(y^*x)|^2 \leq \varphi(x^*x)\varphi(y^*y) \quad \text{on} \quad M_2^\varphi = \{x \in M : \varphi(x^*x) < +\infty\}. \quad (7)$$

Therefore, the left kernel $L_\varphi = \{x \in M : \varphi(x^*x) = 0\}$ of a trace or a state is a linear space. One says that $\varphi$ is faithful if $L_\varphi = 0$. Define a scalar product on $M_2^\varphi$ (which is equal to $M/\text{Ker} (\varphi)$ if $\varphi$ is a state) by $(\xi_x, \xi_y) := \varphi(y^*x)$ with $x \mapsto \xi_x$ be the quotient map.

The GNS (Gelfand–Naimark–Segal) construction is obtained by completion of this pre-Hilbert space (see [43, 3.3]). Recall that a representation $(\pi, H, \xi)$ of $M$ is said to be cyclic if $\pi(M)\xi$ is dense in $H$.

**Lemma 2.6** (1) Let $\varphi$ be a positive linear functional on $M$. There exists a cyclic representation $(\pi_\varphi, H_\varphi, \zeta_\varphi)$ such that $(\pi_\varphi(x)\zeta_\varphi, \zeta_\varphi) = \varphi(x)$.

(2) Let $\varphi'$ be a positive linear functional such that $\varphi' \leq \varphi$, then there exists a unique $a \in \pi_\varphi(M)'$, $0 \leq a \leq 1$, such that $\varphi'(x) = (\pi_\varphi(x)a\zeta_\varphi, \zeta_\varphi)$.

(3) Hence $(\pi_\varphi', H_\varphi', \zeta_\varphi')$ is a subrepresentation of $(\pi_\varphi, H_\varphi, \zeta_\varphi)$.

(4) Let $\varphi$ be the vector state associated to $(\pi, H, \xi)$, then $M/\text{Ker} (\varphi) \ni x \mapsto x\xi \in H$ defines a M-linear isometry between $(\pi_\varphi, H_\varphi, \zeta_\varphi)$ and $(\pi, \pi(M)^\varphi, \zeta^H, \xi)$.
(5) Assume that $\varphi$ is a faithful state, then for any state $\varphi'$ on $M$, $(\pi_{\varphi'}, H_{\varphi'}, \zeta_{\varphi'})$ is a sub-representation of $(\pi_{\varphi}, H_{\varphi}, \zeta_{\varphi})$.

2.2.1 The standard form. The unitary invariance of a trace $t$ implies that $t(xy) = t(yx)$ on $M'_2$. This gives further properties to the associated GNS space: Let $(M, t)$ be a von Neumann algebra with a normal faithful tracial state (a continuous positive linear form $\varphi$ such that $\varphi(1) = 1$ and $\varphi$ is a normal trace). Then $(x, y) \mapsto t(y^*x) = (x, y)_t$ is a scalar product.

Definition 2.7 (1) Let $(\pi_t, l^2(M, t), \zeta_t)$ be the Hilbertian representation obtained through the GNS construction. It is called the standard form associated to $(M, t)$.

(2) The isometric densely defined operator $M \ni x \mapsto x^* \in M$ extends to $J : L^2(M, t) \to L^2(M, t)$ which is conjugate linear, isometric and involutive.

(3) If $x \in M$, the maps $\lambda(x) : y \mapsto xy$ and $\rho(x) : y \mapsto yx$, defined on $M$, extend uniquely as $\rho(x), \lambda(x) \in B(l^2(M, t))$.

A vector $\zeta \in l^2(M, t)$ defines two closed, densely defined, unbounded operators:

$$\lambda(\zeta) : D(\lambda(\zeta)) = M\zeta \ni x\zeta \mapsto \rho(x)\zeta \in l^2(M, t)$$

$$\rho(\zeta) : D(\rho(\zeta)) = M\zeta \ni x\zeta \mapsto \lambda(x)\zeta \in l^2(M, t)$$

Lemma 2.8 (1) $\lambda(M)$ and $\rho(M)$ are von Neumann subalgebras on $l^2(M, t)$ such that $J\lambda(M)J = \rho(M)$ and $\lambda(M)' = \rho(M)$.

(2) Let $\zeta \in l^2(M, t)$. Then $\rho(\zeta)$ is bounded iff $\lambda(\zeta)$ is bounded iff $\zeta \in M\zeta_t$.

Proof see [57, Chap. 5 th.2.22 (p. 324) and Lemma 2.21].

Example 2.9 Let $G$ be a discrete group, let $N_l(G), N_r(G)$ left (resp. right) von Neumann algebra generated by the left (resp. right) translations. Then the standard form of $(N_l(G), tr)$ is $(\lambda, l^2(G), e)$ and $N_l(G)' = N_r(G)$ and $N_r(G)' = N_l(G)$ [15, I.5.2].

2.2.2 The Murray–von Neumann dimension. Let $M'$ be the commutant of $M$ acting on $l^2(M)$. Let $M$ act on $l^2(M) \otimes l^2(\mathbb{N})$ through the faithful representation $x \mapsto x \otimes 1$. It is called amplification [57, p. 184]. The structure theorem of normal morphisms between von Neumann algebras [15] implies:

Lemma 2.10 Let $(H, \pi)$ be a separable $M$-module. Then there exists a $M$-isometry $u : H \to l^2(M) \otimes l^2(\mathbb{N})$ such that $ux = (x \otimes 1)u$.

Let $p$ be the orthogonal projection on $u(H)$. Then $p$ belongs to the commutant of $M$ acting on $l^2(M) \otimes l^2(\mathbb{N})$ which is equal to $M' \otimes B(l^2(\mathbb{N})) : p = (p_{ij})$ decomposes as a matrix such that $p_{ij} \in M'$. 

Definition 2.11 The trace of $p$ is defined by $\text{Tr}(p) = \sum_{i \in \mathbb{N}} \tau(p_{ii})$. It does not depend on the embedding $u$. It is called the von Neumann dimension of $H$, denoted $\dim_{(M, \tau)} H$ or $\dim_M H$ if $\tau$ is understood.

Definition 2.12 [36, 54] A morphism $h : H_1 \to H_2$ between $M$-Hilbert modules is called $M$-Fredholm if $\dim_M \ker(h) < +\infty$ and there exists $L \subset \text{Ran}(h)$ such that $\dim_M H_2 \ominus L < +\infty$. 

Springer
Then \((\text{loc. cit.})\) \(h\) is a M-Fredholm morphism iff \(\dim_M \ker (h^*) < +\infty\) and if \(h^* h = \int \lambda dE_\lambda\) is the spectral decomposition of \(h^* h\) then there exists \(\lambda > 0\) such that \(\text{Tr}_ME_\lambda < +\infty\).

**Definition 2.13** (Murray–von Neumann [40, Chap. XVI])

1. Let \(M\) be a finite von Neumann algebra on \(H\). A closed densely defined operator \(h : \text{Dom}(h) \to H\) is said to be affiliated to \(M\) if it commutes with \(M\): for all unitary \(u \in M\), \(u \text{Dom}(h) = \text{Dom}(h)\) and \(uh = hu\).
2. \([40,54]\) A linear subspace \(L\) in \(H\) is said to be essentially dense if for any \(\epsilon > 0\), there exists a closed \(M\)-Hilbert submodule \(L_\epsilon\) contained in \(L\) such that \(\dim_M(H \ominus L_\epsilon) \leq \epsilon\).

We have the following important properties:

**Lemma 2.14** \([40,54]\) Let \(M\) be a finite von Neumann algebra.

1. Let \(h : H_1 \to H_2\) be a morphism of \(M\)-Hilbert modules. Let \(L\) be a linear subspace of \(H_2\) which is essentially dense. Then \(h^{-1}(L)\) is essentially dense in \(H_1\).
2. Assume that \(h\) is \(M\)-Fredholm. Then \(\text{Ran}(h)\) is essentially dense in \(\overline{\text{Ran}(h)}\).
3. Assume that \(h : H_1 \to H_2\) is a \(M\)-Fredholm weak isomorphism. Then for any closed \(M\)-subspace \(F\) in \(H_2\), \(h_{|h^{-1}(F)} : h^{-1}(F) \to F\) is a weak isomorphism.

The closed densely defined unbounded operators \(\rho(x)\) with \(x \in \ell^2(M,t)\) are examples of operators affiliated to \(\rho(M)\). The bi-commutant theorem implies that \(h\) is affiliated to \(M\) iff \(f(h) \in M\) for every bounded Borel function on \(\text{Spec}(h)\) (see [43, 5.3.10]). In particular, if \(h \geq 0\) then \(h\) is affiliated to \(M\) iff \((1 + \epsilon h)^{-1} h \in M\) for some \(\epsilon > 0\).

An essential property of affiliated operators to \((M, t)\), a finite von Neumann algebra, is that they form an algebra (a property valid for more general semi-finite von Neumann algebra, see [36, Chap. 8], [40, Chap. XVI], [54]): The spectral theorem implies that the domain of an operator affiliated to \(M\) is essentially dense and the above lemma proves that intersection of two essentially dense subspaces is an essentially dense subspace.

The following lemma is a variation on the description of \(\overline{\pi(M)x}\) in term of affiliated operators as in [40, 9.2], developed as Radon–Nykodim theorems in [17,43, 5.3]:

**Lemma 2.15** Let \((M, t)\) be a finite von Neumann algebra. Let \(\pi : M \to \mathcal{B}(H)\) be a faithful representation of \(M\) as a von Neumann subalgebra of \(\mathcal{B}(H)\). Let \(h\) be an operator in the commutant \(\pi(M)'\) of \(\pi(M)\) which is \(M\)-Fredholm. For any \(\xi \in \overline{\text{Ran}(h)}\), there exists \(r \in M\) such that \(\pi_t(r)\) (Sect. 2.2.1) is injective with dense range and \(\pi(r)\xi \in \text{Ran}(h)\).

**Proof** (1) First, assume that \(H = \ell^2(M, t)\) and that \(\rho(\xi), h \in M'\) have dense ranges. Then \(h = \rho(r)\) with \(r \in M\) (Lemma 2.8). Let \(h^{-1} \circ \rho(\xi) = up\) be the polar decomposition of the operator \(h^{-1} \circ \rho(\xi)\) affiliated to \(M'\). Then \(u\) is an isometry, \(p\) is positive, \(u, p\) are affiliated to \(M'\). But \(up = [up(1 + p)^{-1}](1 + p)\). Note that \(1 + p \geq 1\) hence \((1 + p)^{-1}, p(1 + p)^{-1}\) belong to \(M'\). From Lemma 2.8, there
exists \(a, y \in M\), such that \((1 + p)^{-1} = \rho(a)\) and \(\rho(y) = up(1 + p)^{-1}\). Then \(\rho(\xi) \circ \rho(a) = h \circ \rho(y)\). But \(\rho(\xi) \circ \rho(a) = \rho(a\xi)\) and \(h \circ \rho(y) = \rho(h(y))\) (identity is valid on the dense subset \(M\)). Then \(\operatorname{Ran}(\rho(a)) = \operatorname{Dom}(1 + p)\) is dense.

(2) In general, let \(\xi \in \operatorname{Ran}(h)\). Then \(\pi(M)\xi \subset \operatorname{Ran}(\tilde{h})\). Let \(h_1 : \operatorname{Ker}(h)^{\perp} \to \operatorname{Ran}(\tilde{h})\) and \(h_1 = h^{\perp} + id_{l^2(M)} : \operatorname{Ker}(h)^{\perp} \oplus l^2(M, t) \to \operatorname{Ran}(\tilde{h}) \oplus l^2(M, t)\). Set \(\pi_1 = \pi \oplus \lambda\). Lemma 2.14 implies that

\[
h_1 : h_1^{-1} \left(\pi_1(M)(\xi \oplus 1)\right) \to \pi_1(M)(\xi \oplus 1)
\]

is a \(M\)-linear weak isomorphism, for \(h_1\) is a \(M\)-Fredholm weak isomorphism.

Moreover the vector state \(x \mapsto \langle \pi(x)\xi, \xi \rangle + t(x)\) associated to \((\xi, 1)\) dominates \(t\). Using the GNS construction Lemma 2.6 (5), we deduce that there exists \(M\)-isomorphisms \(U_1 : h_1^{-1} \left(\pi_1(M)(\xi \oplus 1)\right) \to l^2(M, t)\) and \(U_2 : \pi_1(M)(\xi \oplus 1) \to l^2(M, t)\). Setting \(\tilde{\xi} = U_2(\xi, 1)\) and \(\tilde{h} = U_2 \circ h_1 \circ U_1^{-1}\), we are reduced to the first case of the proof.

From the proof, one sees that if \(x \in l^2(G)\), then the conductor of the affiliated operator \(\rho(x)\) to \(N_r(G)\) is non trivial.

For further reference, we give the following example of weak isomorphism:

**Lemma 2.16** Let \(G\) be an infinite discrete group and let \(\mu \in l^1(G, \mathbb{R})\) be a probability measure on \(G\) such that the subgroup generated by the support of \(\mu\) is equal to \(G\). Then convolution with \(1 - \mu\) defines a weak isomorphism on \(l^2(G)\).

**Proof** The hypothesis implies that the only \(\mu\)-harmonic function in \(l^2(G)\) is the null function (see Woess [64, p. 159]).

### 2.3 \(G\)-Hilbert modules and von Neumann dimension

**Let** \(G\) be a discrete group. A (left) \(G\)-Hilbert module is a Hilbert space \(V\) with a unitary action \(U(\cdot)\) of \(G\) such that \(V\) is \(G\)-isometric to a \(G\)-invariant subspace of the free Hilbert \(G\)-module \(H \otimes l^2(G)\). Then the von Neumann algebra generated by \(\{U(\cdot)\}\) is isomorphic to \(N_l(G)\). If \(\dim H < +\infty\), then \(V\) is said to be finitely generated. Let \(V\) be embedded as a closed \(G\)-invariant subset of \(H \otimes l^2(G)\). Let \(P \in N_r(G)\otimes \mathcal{B}(H)\) be the orthogonal projection onto \(V\). Then \(\dim_{N_l(G)} V = Tr P\).

**Example 2.17** (see [54, Section 3]) If \(\tilde{E} \to \tilde{X}\) is the pullback of a smooth vector bundle \(E \to X\) under a \(G\)-covering map \(\tilde{X} \to X\), then the space of sections of \(\tilde{E}\) with coefficients in a Sobolev space is a \(G\)-Hilbert module (see Sect. 3.1.2).

Let \(\text{Mod}(N(G))\) be the category of \(N(G)\)-modules where \(N(G)\) is viewed as an abstract ring. A finitely generated projective \(N(G)\)-module is represented by an idempotent matrix \(A \in M_n(N(G))\). Define \(\dim_{N(G)} P := \text{tr}_{N(G)} A\). Following Lück [36] Chap. 6, we define the dimension function of a \(N(G)\)-module \(M\) by

\[
\dim_{N(G)} M := \sup\{\dim_{N(G)} P : P \subset M \text{ a finitely generated projective submodule}\}.
\]

\(\copyright\) Springer
If $V$ is a Hilbert $N(G)$-module, the two dimension functions agree: Their dimension is the supremum over the finite dimensional subspaces [36, p. 21 th.1.12 and th. 6.24].

2.4 Localisation at a Torsion theory In this section we introduce the main categorical tool: quotient of an abelian category $A$ by a subcategory $B$. This gives the category for computing “$\text{Mod}B$”. Serre [52] used it in algebraic topology to obtain isomorphisms modulo finite groups. The motivating example of isomorphism between the $l^2$-cohomology and the reduced $l^2$-cohomology up to torsion was discussed in the introduction.

An equivalent well known construction is Verdier’s localisation of the category $A$ by a class of morphisms $I$, so that morphisms in $I$ become isomorphisms in $I^{-1}A$ (see [60]).

We work with the category of modules over a ring $R$. Then we speak of torsion theory and localisation at a torsion theory.

Definition 2.18 (1) A Serre subcategory $T$ of an abelian category $A$ is a full subcategory $T$ of $A$ such that for any exact sequence $0 \to A \to B \to C \to 0$ in $A$, $B$ belongs to $T$ iff $A$ and $C$ belongs to $T$.

(2) In the category $\text{Mod}(R)$ of $R$-modules over a ring $R$, a Serre class $T$ defines a hereditary torsion theory $\tau = (T, F)$ on $R$ (Vas [59], Golan [27]): Modules in $T$ are torsion modules. Define the class of free modules by $F = \{ F \in \text{Mod}(R) : \forall T \in T, \text{Hom}_R(T, F) = 0 \}$.

Lemma 2.19 Let $\tau = (T, F)$ be a hereditary torsion theory, then

(1) $F$ is closed under submodules, direct products and extension.

(2) Any $M \in \text{Mod}(R)$ has a unique maximal $\tau$-torsion submodule, denoted $T_\tau(M)$.

(3) A hereditary torsion theory is cogenerated by an injective module $E$ : $\tau = (T, F)$ with $T \ni S \iff \text{Hom}_R(C, E) = 0$.

(4) The functor $T_\tau(\_): \text{Mod}(R) \to T$ is a left exact functor ($N \subset M$ implies that $T_\tau(M) \cap N = T_\tau(N)$).

Proof (2) Let $\{N_i, i \in \Lambda\}$ be the set of all torsion submodules of $M$. The trivial module $\{0\}$ belongs to this set. Then $T_\tau(M) := \sum_{i \in \Lambda} N_i$ satisfies required properties for it is a homomorphic image of $\oplus_{i \in \Lambda} N_i$ which is torsion.

(3), (4) See [27, p. 5 and p. 24].

Definition 2.20 (1) Let $\tau_1$ and $\tau_2$ be torsion theories, then $\tau_1$ is smaller than $\tau_2$ (\(\tau_1 \leq \tau_2\)) if $T_1 \subset T_2$ iff $F_1 \supset F_2$.

(2) Then if $C$ is a class in $\text{Mod}(R)$, the hereditary torsion theory generated by $C$ is the smallest hereditary torsion theory $\tau$ such that $C \subset T$.

2.4.1 Quotient category. Let $T$ be a Serre subcategory of an Abelian category $A$. There exists a quotient category $A/T$ and an exact functor $A \to A/T$ with the universal property of factorisation of functor mapping objects in $T$ to the zero object (Grothendieck [32, (1.11)], Gabriel [25, Chap. 3], Verdier [60, §2]).

Definition 2.21 Let $\tau = (T, F)$ be a hereditary torsion theory on $\text{Mod}(R)$. Let $\text{Mod}(R)/\tau$ be the quotient category of $\text{Mod}(R)$ by the Serre class $T$. 

\[ \square \]
(i) The objects of $\text{Mod}(R)/\tau$ are identical with the objects of $\text{Mod}(R)$;
(ii) The morphisms of $\text{Mod}(R)/\tau$ are elements of the following inductive limit
\[
\lim\limits_{\longrightarrow} \{\text{Hom}_R(M', N/N') : M' \subset M, N' \subset N \text{ and } M/M', N' \in T\}.
\]

Let $\alpha \in \text{Hom}_{N(G)}(M, N)$ and let $[\alpha]$ be its image in the quotient category. Then
$[\alpha]$ is a monomorphism (resp. epimorphism, resp. isomorphism) iff $\text{Ker}(\alpha)$ (resp.
$\text{Coker}(\alpha)$, resp. $\text{Ker}(\alpha)$ and $\text{Coker}(\alpha)$) is a torsion module.

2.4.2 Remark Another description of $\text{Mod}(R)/\tau$ is through fractions: Let $\mathcal{I}$ be the
set of morphisms in $\text{Mod}(R)$ such that $\text{Ker}(f)$ and $\text{Coker}(f)$ are in $\mathcal{T}$. Then $\mathcal{I}$
is a multiplicative system and $\mathcal{I}^{-1}\text{Mod}(R)$ and $\text{Mod}(R)/\tau$ are equivalent categories
([60,62, 10.3.4 and ex. 10.3.2]). Hence a morphism $[f_1] \in \text{Hom}_{\text{Mod}(R)/\tau}(M, N)$ is
represented by a left fraction $M \xrightarrow{i} M' \xrightarrow{f_1} N$ with $i \in \mathcal{I}$.

2.4.3 Torsion theory and complexification. Let $R$ be a ring which is also an $\mathbb{R}$-algebra
then $\mathbb{R} \otimes \mathbb{C}$ is a $\mathbb{C}$-algebra. If $A \in \text{Mod}(R)$ then the $(R, \mathbb{C})$-isomorphism $(\mathbb{R} \otimes \mathbb{C}) \otimes_R A \simeq A \otimes \mathbb{C}$ gives a structure of $R \otimes \mathbb{C}$-module to $A \otimes \mathbb{C}$. The forgetful functor
$\phi : \text{Mod}(R \otimes \mathbb{C}) \ni B \mapsto B_R \in \text{Mod}(R)$ is faithful and exact and has a left adjoint
\[
(R \otimes \mathbb{C}) \otimes : \text{Hom}_R(A, E_R) \simeq \text{Hom}_{\mathbb{R} \otimes \mathbb{C}}((R \otimes \mathbb{C}) \otimes_R A, E) \quad (8)
\]

Let $J = \phi(i1d_B)$ be the automorphism in $B_R$ defined by multiplication by $i$ in $B$
and let $J_C = J \otimes 1_C : B_R \otimes \mathbb{C} \to B_R \otimes \mathbb{C}$ be its complex linear extension. Let $\tilde{B}$ be
the complex conjugate $R \otimes \mathbb{C}$-module associated to $B$: Its underlying abelian group is $B$, and the module structure is given by the following representation $\rho$ of $R \otimes \mathbb{C}$ in $\text{End}(B)$ : $\rho(\rho \otimes c)(b) = (\rho \otimes \overline{c}).b$. The natural antilinear $R$-isomorphism $B \to \tilde{B}$
defines a functor on $\text{Mod}(R \otimes \mathbb{C})$. Then antilinear $R$-linear maps from $B$ to $B'$ are
in bijection with $R \otimes \mathbb{C}$-linear maps from $B$ to $\overline{B'}$ (or from $\overline{B}$ to $B'$). An $R \otimes \mathbb{C}$-
isomorphism from $B$ to $\tilde{B}$ is equivalent with a real structure $\alpha : B \simeq_{\mathbb{R} \otimes \mathbb{C}} A \otimes \mathbb{C}$
on $B$ ($A \in \text{Mod}(R)$), which in turn is equivalent to a conjugation $C$ on $B$ (an $R$-linear
involution $C$ on $B_R$ such that $CJ = -JC$). This defines an antilinear $\mathbb{R}$-isomorphism
$\text{Hom}_{\mathbb{R} \otimes \mathbb{C}}(A \otimes \mathbb{C}, \tilde{E}) \simeq \text{Hom}_{\mathbb{R} \otimes \mathbb{C}}(A \otimes \mathbb{C}, E)$.

Lemma 2.22 (1) Let $B \in \text{Mod}(R \otimes \mathbb{C})$, then $B_R \otimes \mathbb{C} \simeq_{\mathbb{R} \otimes \mathbb{C}} B \otimes \tilde{B}$.

(2) Let $C$ be a conjugation on $B \in \text{Mod}(R \otimes \mathbb{C})$. Then $C_C := C \otimes 1 d_C : B_R \otimes \mathbb{C} \to$
$B_R \otimes \mathbb{C}$ maps $B$ to $\tilde{B}$.

Proof (1) Let $P_{\pm} = \frac{1d \pm iJ_C}{2} : B_R \otimes \mathbb{C} \to B_R \otimes \mathbb{C}$ be the projection onto the
eigenspaces $\{J_C = \pm i1d\}$. One checks that $B \ni b \mapsto P_+(b \otimes 1) \in \{J_C = i1d\}$ and $\tilde{B} \ni b \mapsto P_-(b \otimes 1) \in \{J_C = -i1d\}$ are $R \otimes \mathbb{C}$-isomorphisms.

(2) The complex extension $C_C$ of $C$ is an involution which anticommutes with $J$. \text{□}

Note that $B \ni b \mapsto P_+(b \otimes 1) \in B_R \otimes \mathbb{C}$ is a splitting of the natural surjection
$B_R \otimes \mathbb{C} \to B$. 

\text{Springer}
Definition 2.23 (see [27]) Let $\gamma : R \to R \otimes_R C$ be the above ring extension.

(i) Let $\tau = (T, F)$ be a torsion theory on $\text{Mod}(R)$. Then $\gamma_*\tau$ is the torsion theory on $\text{Mod}(R \otimes_R C)$ such that $B \in \text{Mod}(R \otimes_R C)$ is $\gamma_*\tau$-torsion iff $B_R$ is $\tau$-torsion.

(ii) Let $\tau_C = (T_C, F_C)$ be a torsion theory on $\text{Mod}(R \otimes_R C)$. Then $\gamma^*\tau_C$ is the torsion theory on $\text{Mod}(R)$ such that $A \in \text{Mod}(R)$ is $\gamma^*\tau_C$-torsion iff $A \otimes_R C$ is $\tau_C$-torsion.

One checks that if $\tau$ is cogenerated by an injective $I$ (Lemma 2.19), then $\gamma_*\tau$ is cogenerated by the injective $I \otimes_R C$. Hence $\gamma^*\gamma_*\tau = \tau$ for $I \otimes_R C \simeq_R I^2$. However if $\tau_C$ is a torsion theory on $\text{Mod}(R \otimes_R C)$, $\gamma_*\gamma^*\tau_C$ is in general strictly smaller than $\tau_C : B$ torsion does not imply $B \oplus \bar{B}$ torsion.

Definition 2.24 Let $\tau_C = (T_{\tau_C}, F_{\tau_C})$ be a torsion theory on $\text{Mod}(R \otimes_R C)$ with torsion functor $T_{\tau_C}$ (Lemma 2.19). The following properties are equivalent:

(i) $\gamma_*\gamma^*\tau_C = \tau_C$.

(ii) There exists a torsion theory $\tau$ on $\text{Mod}(R)$ such that $\tau_C = \gamma_*\tau$.

(iii) $T_{\tau_C}$ is stable by conjugation.

(iv) $\forall B \in \text{Mod}(R \otimes_R C), T_{\tau_C}(B) = \overline{T_{\tau_C}(B)}$ (Sect. 2.4.3). A (hereditary) torsion theory on $\text{Mod}(R \otimes_R C)$ is real if it satisfies one of the above properties.

Example 2.25 Let $\tau_C$ be any torsion theory then $\gamma_*\gamma^*\tau_C$ is a real torsion theory.

Corollary 2.26 (1) Let $(\tau, \tau_C, \tau')$ be torsion theories on $(R, R \otimes_R C, R)$ such that $\tau \leq \gamma^*\tau_C$ and $\tau_C \leq \gamma_*\tau'$ (iff $T_{\tau} \otimes C \subset T_{\tau_C}$ and $[T_{\tau_C}] R \subset T_{\tau'}$). There exists exact functors

$$
(\cdot \otimes C) : \text{Mod}(R)/\tau \to \text{Mod}(R \otimes_R C)/\tau_C \\
(\cdot)_R : \text{Mod}(R \otimes_R C)/\tau_C \to \text{Mod}(R)/\tau'
$$

such that the following diagrams commute:

$$
\begin{array}{ccc}
\text{Mod}(R) & (\cdot \otimes C) & \text{Mod}(R \otimes_R C) \\
\downarrow & & \downarrow \\
\text{Mod}(R)/\tau & (\cdot \otimes C) & \text{Mod}(R \otimes_R C)/\tau_C \\
\end{array}
\quad
\begin{array}{ccc}
\text{Mod}(R \otimes_R C) & (\cdot)_R & \text{Mod}(R) \\
\downarrow & & \downarrow \\
\text{Mod}(R \otimes_R C)/\tau_C & (\cdot)_R & \text{Mod}(R)/\tau'
\end{array}
$$

(2) In particular if $\tau = \tau'$ and $\tau_C = \gamma_*\tau$, then $(\cdot \otimes C, (\cdot)_R)$ is a pair of adjoint functors.

(3) Let $(\tau', \gamma_*\tau')$ be torsion theories on $(R, R \otimes_R C)$ such that $\tau' \leq \tau$. Then there exists an exact functor from $(\text{Mod}(R)/\tau', \text{Mod}(R \otimes_R C)/\gamma_*\tau')$ to $(\text{Mod}(R)/\tau, \text{Mod}(R \otimes_R C)/\gamma_*\tau)$ which commutes with tensor product up to an equivalence.

Proof The kernel of $\text{Mod}(R) \xrightarrow{(\cdot \otimes C)} \text{Mod}(R \otimes_R C) \to \text{Mod}(R \otimes_R C)/\tau_C$ is spanned by the modules $M$ such that $M \otimes_R C$ is $\tau_C$-torsion which follows from $M$ being $\tau$-torsion. One concludes from [25, p. 368 Cor. 2 and Cor. 3] or [22, 15.9]. The other assertions are proved in the same way.

\( \square \)

Springer
Notation 2.27 Let $\tau$ be a (hereditary) torsion theory on Mod$(R)$. Then $\tau \otimes \mathbb{C} := \gamma_\tau \tau$ is the real torsion theory on Mod$(R \otimes_{\mathbb{R}} \mathbb{C})$ associated to $\tau$.

Definition 2.28 Define a category $\mathcal{R}$ of modules with real structure modulo $\tau$:

An object is a pair $(B, \alpha)$, $B \in$ Mod$(R \otimes_{\mathbb{R}} \mathbb{C})/\tau \otimes \mathbb{C}$ with $\alpha : B \to A \otimes_{\mathbb{R}} \mathbb{C}$ an isomorphism in Mod$(R \otimes_{\mathbb{R}} \mathbb{C})/\tau \otimes \mathbb{C}$.

A morphism $f : (B, \alpha) \to (B', \alpha')$ in $\mathcal{R}$ is a map $f : B \to B'$ such that there exists $g : A \to A'$ with $(g \otimes 1_{\mathbb{C}}) \circ \alpha = \alpha' \circ f$.

Lemma 2.29 (1) The category $\mathcal{R}$ is abelian.

(2) A conjugation $C$ on $B \in$ Mod$(R \otimes_{\mathbb{R}} \mathbb{C})/\tau \otimes \mathbb{C}$ (an involution on $B_R \in$ Mod$(R)/\tau$ which anticommutes with $(1d_B)_R$) defines a real structure modulo $\tau : B \cong Ker(C - 1d_{B_R}) \otimes \mathbb{C}$. It induces an isomorphism $C_{\mathbb{C}, \bar{B}} : \bar{B} \to B$ in Mod$(R \otimes_{\mathbb{R}} \mathbb{C})/\tau \otimes \mathbb{C}$ (see Lemma 2.22).

Notation 2.30 Let $B' \to B$ be a subobject of a module with real structure modulo $\tau$.

Then $\bar{B'}$ will be identified with the subobject $\bar{B} \to \bar{B}$ with dense range $\mathcal{R}$ in particular, if $F'$ is a filtration on $B$, then $\bar{F'} := (\bar{F'})$ will be called the conjugate filtration on $B$.

2.5 Examples of torsion theories

2.5.1 A result of Dickson [13]. If $\mathcal{C}$ is a class of modules defines

$$L(\mathcal{C}) := \{B \in$ Mod$(R) : \forall C \in \mathcal{C}, \hom_R(B, C) = 0\}, \quad (9)$$

$$R(\mathcal{C}) := \{B \in$ Mod$(R) : \forall C \in \mathcal{C}, \hom_R(C, B) = 0\}. \quad (10)$$

The torsion theory generated by $\mathcal{C}$ is given by $\mathcal{T}_\mathcal{C} = LR(\mathcal{C})$.

2.5.2 Multiplicative systems. Let $S$ be a subset in $R$ stable by multiplication. Then $\mathcal{T}_S = \{M \in$ Mod$(R) : \forall m \in M, \exists s \in S \text{ s.t. } sm = 0\}$ is a Serre class. We will use the torsion theory generated by $\operatorname{Ran} f^{H_2}/\operatorname{Ran} f$ with $f : H_1 \to H_2$ a bounded morphism between $N(G)$-Hilbert modules. If $A \in$ Mod$(N(G))$, let

$$S = \{r \in N(G) \text{ with dense range : } \exists a \in A \text{ with } ra = 0\} \subset \cup_{a \in A}(0 : a).$$

The properties of essentially $G$-dense subsets (see [54] or [40, Chap. XVI]) imply that $S$ is a multiplicative system. A $N(G)$-module $M$ is $\tau_S = (\mathcal{T}_S, \mathcal{F}_S)$-torsion iff $\forall m \in M, \exists s \in S \text{ s.t. } sm = 0$. Note that intersection of multiplicative systems is a multiplicative system.

2.5.3 The torsion $\mathcal{T}_{\dim}$. In [36, Chap. 6], Lück defines a dimension function $\dim_{N(G)} : \operatorname{Mod}(N(G)) \to [0, +\infty]$ which is additive on short exact sequences and which coincides with von Neumann dimension for $N(G)$-Hilbert modules. Therefore

$$\mathcal{T}_{\dim} = \{M \in \operatorname{Mod}(N(G)) : \dim_{N(G)} M = 0\}$$

Springer
is a Serre class and defines a torsion theory $\tau_{\text{dim}}$ on $N(G)$. This torsion theory is real for a $N(G)$-module $P$ is projective finitely generated iff $\tilde{P}$ is (see Sect. 2.4.3).

Standard examples of $\tau_{\text{dim}}$-torsion modules are given in Corollary 3.11: Modules of the shape $\frac{A}{A}$, with $A$ a $G$-invariant subspace of a finite $G$-dimensional $G$-Hilbert modules $H$, are zero dimensional. This follows from the normality of the dimension function (see a proof in Corollary 3.11).

2.5.4 The torsion $T_{U(G)}$. The algebra $U(G)$ of affiliated operators to $N(G)$ is studied in [36,40, Chap. 8] and Reich [45]. An affiliated operator (to $N(G)$) is a $G$-equivariant unbounded operator $f : \text{dom}(f) \subset l^2(G) \to l^2(G)$. Let $M \in \text{Mod}(N(G))$. Define $T_{U}(M) := \text{Ker}(M \to U(G) \otimes_{N(G)} M)$. This defines a torsion class

$$T_{U(G)} = \{ M \in \text{Mod}(N(G)) : T_{U}(M) = M \}$$

and a torsion theory $\tau_{U(G)}$. An element $m \in M$ belongs to $T_{U(G)}(M)$ iff there exists an $r \in N(G)$ which is a not a divisor of zero (iff $r$ is a weak isomorphism) such that $rm = 0$. Indeed $r$ becomes invertible in $U(G)$. According to [59, p. 673], a module $F \in \text{Mod}(N(G))$ is $\tau_{U(G)}$-torsion free iff $F$ is flat. This torsion theory is real for $F$ is flat iff $\tilde{F}$ is.

Examples of $\tau_{U(G)}$ torsion modules are given in Corollary 3.11. We present the case of a module generated by elements with infinite isotropy.

Lemma 2.31 Let $H$ be an infinite finitely generated subgroup of the discrete group $G$. Then $l^2(G) \otimes_{\mathbb{C}[G]} \mathbb{C}[G/H]$ is a $\tau_{U(G)}$-torsion module.

Proof Let $\mu \in l^1(H, \mathbb{R}) \subset l^1(G, \mathbb{R})$ be a probability measure with finite support generating $H$. Lemma 2.16 implies $\lambda(1-\mu)$ is a weak isomorphism on $l^2(G, \mathbb{C}) \cong \mathbb{C}[H] \otimes_{\mathbb{C}[H]} \mathbb{C}$. But $U(G) \otimes_{N(G)} l^2(G) \otimes_{\mathbb{C}[G]} \mathbb{C}[G/H]$ is spanned by elements $r \otimes f \otimes g H$ with $r \in U(G)$, $f \in l^2(G, \mathbb{C})$ and $g \in G$. Then

$$r \otimes f \otimes g H = r \lambda(g(1-\mu))^{-1} \otimes \lambda(g(1-\mu)) f \otimes g H = r \lambda(g(1-\mu))^{-1}$$

$$\otimes f \rho((1-\bar{\mu})g^{-1}) \otimes g H = r \lambda(g(1-\mu))^{-1} \otimes f \otimes (1-\bar{\mu}).e H = 0$$

for $1-\bar{\mu} = 0$ in $\mathbb{C}[G/H]$. \hfill $\Box$

Remark 2.32 Let $p : \tilde{X} \to X$ be a covering of a compact manifolds. We will see that if the covering group $G$ is Galois, then the torsion theories $\tau_{\text{dim}}$ and $\tau_{U(G)}$ on $\text{Mod}(N(G))$ give valuable informations on the $l^2$-cohomology groups of $p : \tilde{X} \setminus p^{-1}(D) \to X \setminus D$.

On the opposite, if one considers a covering map $p : \tilde{X} \to X$ with trivial automorphism, then $N(G)$ is isomorphic to $\mathbb{C}$ and a torsion theory is either trivial or any module is torsion.

2.6 The sheaves of $l^2$-direct images Let $p : \tilde{X} \to X$ be a covering map between complex manifolds. Let $G$ be the group of deck transformations. Let $N(G) := N(G, \mathbb{C})$ be the its left von Neumann algebra (the bi-commutant of the set of left translations.
acting on $l^2(G, \mathbb{C})$ and let $\mathcal{N}(G)$ (or $\mathcal{N}(G, \mathbb{C})$) be the sheaf of rings it defines. Let $\mathcal{N}(G, \mathbb{R})$ be the sheaves of rings defined by $\mathcal{N}(G, \mathbb{R})$ (Sect. 2.1).

According to Campana–Demailly [8], if $\mathcal{F}$ is a coherent analytic sheaf on $X$, there exists a sheaf $p_*(\mathcal{F})$ called $l^2$-direct image such that $p_*(.)$ is an exact functor from the category of coherent analytic sheaves on $X$ to the category of sheaves on $X$ ([8, prop2.6]). The sheaf $p_*(\mathcal{O}_X)$ is the sheaf associated to the presheaf $V \mapsto \mathcal{O}(p^{-1}(V)) \cap L^2(p^{-1}(V), \mathbb{C})$. We change from notations of [8], our $p_*(\mathcal{F})$ is written there $p_*(\tilde{\mathcal{F}}$. Campana–Demailly [8, Cor 2.7] prove:

**Lemma 2.33** For any analytic coherent sheaf $\mathcal{F}$, the morphism $p_*\mathcal{O} \otimes \mathcal{F} \to p_*(\mathcal{F})$ is an isomorphism.

This isomorphism defines on $p_*(\mathcal{F})$ a structure of $\mathcal{N}(G)$-sheaf compatible with the natural structure of $\mathbb{Z}[G]$-sheaf.

**Definition 2.34** Let $K$ be a subring of $\mathbb{C}$. Let $p_*(K)$ be the locally constant sheaf defined by the presheaf

$$U \to \{ f \in L^2(p^{-1}(U), \mathbb{C}), f \text{ is } K \text{-valued and locally constant} \}.$$ 

Then $p_*(\mathbb{R})$ is a $\mathcal{N}(G, \mathbb{R})$-module, $p_*(\mathbb{C})$ is a $\mathcal{N}(G, \mathbb{C})$-module. When $p : \tilde{X} \to X$ is Galois, $p_*(\mathbb{C})$ is isomorphic to $l^2(G, \mathbb{C}) \otimes \mathbb{Z}[G] p!(\mathbb{Z}_{\tilde{X}})$ as sheaves of left $\mathcal{N}(G, \mathbb{C})$-modules.

**Definition 2.35** Let $\mathcal{F} \to X$ be a coherent analytic sheaf (or a constant sheaf).

The $l^2$-cohomology groups $H^*(X, p_*(\mathcal{F}))$ of $p : \tilde{X} \to X$ with values in $p^*\mathcal{F}$ are the cohomology groups of the sheaf $p_*(\mathcal{F})$ over $X$.

**Lemma 2.36** Let $D : E \to F$ be a differential operator with holomorphic coefficients between holomorphic vector bundles. Then there exists an operator $p_*(D) : p_*(E) \to p_*(F)$.

**Proof** In local trivialisation $\mathcal{O}^n \simeq E$, $\mathcal{O}^m \simeq F$, $D$ is given as $\sum_{|I| \leq r} a_I \partial_I$ with $a_I$ an holomorphic function. But $p_*(E)$, and $p_*(F)$ are $\mathcal{O}$ modules, hence it is enough to study $D = \partial_I$. The claim is then a consequence of the Cauchy inequalities.

**Lemma 2.37**

$$0 \to p_*(\mathbb{C}) \to p_*(\mathcal{O}) \xrightarrow{d} p_*(\mathcal{O})^{\Omega^1} \to \ldots \to p_*(\mathcal{O})^{\Omega^n} \to 0$$

is well-defined and exact.

**Remark** It seems natural to extend the functor $l^2$-direct images to an exact functor from the category of $D_X$-coherent modules to the category of $\mathcal{N}(G)$-sheaves.

**2.6.1 Link to singular cohomology.** In this section we recall that the cohomology of a locally constant sheaf is isomorphic to the singular cohomology with local coefficients.
and to the equivariant cohomology of the universal cover \( \tilde{X} \) with values in a \( \pi_1(X) \)-module (Eilenberg [19], Steenrod [55], Whitehead [63]). We refer to Dimca [14, section 2.5] for a more detailed presentation.

A bundle of groups \( L \) (or local system of groups) on \( X \) is a covariant functor from the fundamental groupoid of \( X \) to the category of abelian groups ([63] p. 257).

A locally constant sheaf \( L \) defines a bundle of groups \( L \) for a locally constant sheaf over a simply connected space is constant ([38] I.2). Hence a path \( x \) be the constant path at \( x \). Then \( L(x) := L(y(x)) \) is a \( \pi_1(X, x_0) \)-module through the representation \( \rho_L : \pi_1(X, x_0) \ni \gamma \mapsto L(\gamma) \in Hom(L(y(x)), L(y(x0))). \)

A \( \pi_1(X) \)-module \( M \) defines a locally constant sheaf: the sheaf of cross sections of the fiber bundle \( \tilde{X} \times_{\pi_1(X)} M \to X. \)

Let \( K \) be the kernel of the monodromy representation \( \rho_L \). Then the pullback of the bundle of groups \( L \) and of the locally constant sheaf \( L \) are constant on \( p_K : X_K \to X \), the covering with fundamental group \( K \), \( L(x_0) \) is a \( \pi_1(X, x_0)/K \)-module.

Let \( C^p(X, \mathbb{L}) \) be the group of singular cochains with values in \( \mathbb{L} \): This is the set of functions \( c \) which assigns to a singular simplex \( \sigma : \Delta_p \to X \) an element \( c(\sigma) \in L(\sigma(e_0)) \). This is a group under addition of functional values. This lead to a complex \( (C^p(X, \mathbb{L}), \delta) \) whose cohomology \( H^k_{\text{sing}}(X, \mathbb{L}) := H^k(C^p(X, \mathbb{L}), \delta) \) is by definition the singular cohomology group of \( X \) with values in the local system \( L \) (see [63] p. 270).

Let \( \mathcal{L} \) be a locally constant sheaf over \( X \) with associated local system \( L \). Let \( \delta \) be the differential sheaf associated to the presheaf \( U \to (C^p(U, L(U)), \delta) \) (equivalent to the definition given in [4, I.7]). It defines a \( \Gamma(X, \mathbb{L}) \)-acyclic resolution of \( \mathcal{L} \) and provides an isomorphism between the sheaf cohomology groups of \( \mathcal{L} \) and the singular cohomology groups of \( L \) (see [4, Chap. III]).

Let \( X_K \to X \) be the cover of \( X \) with group of deck transformations \( \pi_1(X, x_0)/K \). Let \( (C(X_K), \delta) \) be its singular chain complex. Then \( (\text{Hom}_{\mathbb{L}}[\pi_1(X, x_0)/K](C(X_K), L(x_0)), \delta) \) is a complex whose cohomology \( H^k(X_K, L(x_0)) \) is by definition the group of equivariant cohomology of \( X_K \) with values in \( L(x_0) \). From [19, §. 24], we have an isomorphism \( H^k_e(X_K, L(x_0)) = H^k_{\text{sing}}(X, L). \)

### 2.7 Mixed Hodge structures

We recall or adapt definitions and fundamental results from Deligne’s mixed Hodge structures [11, 12]. The main points are contained in two theorems: the theorem on strictness of morphisms and the theorem on degenerescence of the two spectral sequences associated to a mixed Hodge complex (see [20,41,61] for introductions to mixed Hodge structures).

#### 2.7.1 Preliminaries

We recall (see Peters-Steenbrink [44, p. 49]) that a pseudomorphism \( f : K \to L \) between two complexes \( K \) and \( L \) in an abelian category is a chain of morphisms

\[
K \to K_1 \to K_2 \to \cdots \to K_n = L
\]

such that \( f_1, \ldots, f_n \) are quasi-isomorphisms. It induces a morphism in the derived category. One says that \( f : K \to L \) is a pseudo-isomorphism when \( f \) is a
quasi-isomorphism. When $K^\cdot$ and $L^\cdot$ are filtered complexes it is understood that morphisms are filtered and that a quasi-isomorphism is a filtered quasi-isomorphism ($Gr_F^\cdot(f_\cdot)$ is a quasi-isomorphism).

For further reference, we recall [11, 1.4.6] that if $(K^\cdot, d^\cdot)$ is a complex in an abelian category then:
- The canonical filtration $\tau_{\leq}$ on $K$ is defined by: $\tau_{\leq p}K^n$ is equal to $K^n$ if $n < p$, to $\text{Ker}(d^p)$ if $n = p$ and is the null object if $p < n$.
- The trivial filtration is defined by: $(\sigma_{\geq p}K)^n$ is equal to the null object if $n < p$ and $K^n$ if $p \leq n$.

A quasi-isomorphism $f : K^\cdot \to L^\cdot$ defines a filtered quasi-isomorphism $f : (K^\cdot, \tau_{\leq}) \to (L^\cdot, \tau_{\leq})$.

**Definition 2.38** Let $R$ be a ring which is also an $\mathbb{R}$-algebra. Let $\tau$ be a (hereditary) torsion theory on $\text{Mod}(R)$.

1. A mixed Hodge structure $H = (H_R, W, F)$ in $\text{Mod}(R)/\tau$ is given by
   (i) a left $R$-module $H_R$,
   (ii) a filtration $W$ on $H_R$ in $\text{Mod}(R)/\tau$,
   (iii) a filtration $F$ on $H_R \otimes_\mathbb{R} \mathbb{C}$ in $\text{Mod}(R \otimes_\mathbb{R} \mathbb{C})/\tau \otimes \mathbb{C}$ such that $W_\mathbb{C}, F, \bar{F}$ (Notation 2.30) are opposed (see [11] 1.2) in $\text{Mod}(R \otimes_\mathbb{R} \mathbb{C})/\tau \otimes \mathbb{C}$.

2. A morphism of mixed Hodge structure $f : H \to H'$ in $\text{Mod}(R)/\tau$ is a morphism $f \in H\text{om}_{\text{Mod}(R)/\tau}(H_R, H'_R)$ such that $f$ is compatible with $W$ and $f_\mathbb{C}$ is compatible with $F$.

Theorem 1.2.10 of Deligne [11] implies:

**Theorem 2.39** The category of mixed Hodge structure in $\text{Mod}(R)/\tau$ is abelian. A morphism $f : H \to H'$ between MHS in $\text{Mod}(R)/\tau$ is strict for the filtrations.

In this article, we use only mixed Hodge complexes over $\mathbb{R}$-algebras. This reflects the use of $I^2(G, \mathbb{R})$.

**Definition 2.40** Let $R$ be a ring which is also an $\mathbb{R}$-algebra. Let $\tau'$ and $\tau$ be torsion theories on $\text{Mod}(R)$ such that $\tau'$ is smaller than $\tau$.

A $(\tau', \tau)$-Hodge complex $(K_R, (K_R \otimes \mathbb{C}, F))$ of weight $m$ is given by

1. A bounded below complex of modules $K_R$ in $\text{Mod}(R)/\tau'$
2. A bounded below filtered complex of modules $(K_R \otimes \mathbb{C}, F)$ in $\text{Mod}(R \otimes_\mathbb{R} \mathbb{C})/\tau' \otimes \mathbb{C}$.
3. A pseudo-morphism of bounded below complexes $\alpha : K_R \to K_R \otimes_\mathbb{C}$ (comparison morphism) in $\text{Mod}(R)/\tau'$ such that $\alpha \otimes Id : K_R \otimes \mathbb{C} \to K_R \otimes \mathbb{C}$ is a pseudo-isomorphism.

The isomorphism $H(K_R) \otimes \mathbb{C} \to H(K_R \otimes \mathbb{C})$ (in $\text{Mod}(R \otimes \mathbb{C})/\tau' \otimes \mathbb{C}$') defines a real structure on $H(K_R \otimes \mathbb{C})$.

One requires that

1. $d$ is strictly compatible with $F$ in $\text{Mod}(R \otimes \mathbb{C})/\tau \otimes \mathbb{C}$.
2. $F$ and $\bar{F}$ (Notation 2.30) are $m + k$-opposed on $H^k(K_R \otimes \mathbb{C}) \simeq H^k(K_R) \otimes \mathbb{C}$ in $\text{Mod}(R \otimes \mathbb{C})/\tau \otimes \mathbb{C}$.
Definition 2.41 (Following [44]) Let $R$ be a ring which is also an $\mathbb{R}$-algebra. Let $\tau'$, $\tau$ be torsion theories on $R$ such that $\tau'$ is smaller than $\tau$.

A $(\tau', \tau)$-mixed Hodge complex $((K_R, W), (K_R \otimes \mathbb{C}, F), \beta)$ consists in

1. A bounded below filtered complex $(K_R, W)$ in $\text{Mod}(R)/\tau'$,
2. A bounded below bi-filtered complex $(K_R \otimes \mathbb{C}, W, F)$ in $\text{Mod}(R \otimes \mathbb{C})/\tau' \otimes \mathbb{C}$ and a pseudo-morphism $\beta : (K_R, W) \to (K_R \otimes \mathbb{C}, W)$ (comparison morphism) in the category of bounded below filtered complexes in $\text{Mod}(R)_{/\tau'}$ inducing a pseudo-isomorphism $\beta \otimes \text{Id}_\mathbb{C} : (K_R \otimes \mathbb{C}, W) \to (K_R \otimes \mathbb{C}, W)$.
3. One requires that for each $n$, $(Gr^W_n(K_R), (Gr^W_n(K_R \otimes \mathbb{C}, F)))$ with pseudo-morphism $Gr^W_n(\beta) : Gr^W_n(K_R) \to Gr^W_n(K_R \otimes \mathbb{C})$ is a $(\tau', \tau)$-Hodge complex of weight $n$.

A $(\tau', \tau)$-mixed Hodge complex will also be called a mixed Hodge complex $\text{mod}(\tau', \tau)$.

Example 2.42 (1) If $\tau' = ([0], \text{Mod}(R))$ is the trivial torsion theory (no non zero torsion submodule), then a $(\tau', \tau)$-mixed Hodge complex will be referred as a $\tau$-mixed Hodge complex or a mixed Hodge complex modulo $\tau$. Therefore complexes and pseudo-morphisms are data in $\text{Mod}(R)$ and properties of degenerescence are assumed in $\text{Mod}(R)_{/\tau}$.

(2) When $\tau' = \tau$, we will speak of mixed Hodge complex in $\text{Mod}(R)_{/\tau}$. Hence each mixed Hodge complex modulo $\tau$ defines a mixed Hodge complex in $\text{Mod}(R)_{/\tau}$.

The following theorem contains the “lemme des deux filtrations” ([11, 1.3] and [12, 7.2]):

Theorem 2.43 Let $((K_R, W), (K_R \otimes \mathbb{C}, W, F))$ be a mixed Hodge complex $\text{mod}(\tau', \tau)$. Then the recurrent filtration and the direct filtration induced by $F$ on $E^p,q_r(K_R \otimes \mathbb{C}, W)$ are equal in $\text{Mod}(R \otimes \mathbb{C})_{/\tau' \otimes \mathbb{C}}$. The sequence $0 \to E_r(F^p K, W) \to E_r(K, W) \to E_r(K/F^p K) \to 0$ is exact for any $1 \leq r \leq +\infty$ and for any $p$. The weight spectral sequence $(E_r(K_R, W))_r$ degenerates at $E_2$ in $\text{Mod}(R)_{/\tau}$. The Hodge spectral sequence $(E_r(K, F))_r$ degenerates at $E_1$ in $\text{Mod}(R \otimes \mathbb{C})_{/\tau' \otimes \mathbb{C}}$.

Proof We have the following data within a torsion theory $\tau'$ smaller than $\tau$: There exists a left fraction

$$
\begin{array}{ccc}
(K_R, W) & \xrightarrow{\beta_1} & (K'_R \otimes \mathbb{C}, W) \\
& \uparrow \beta_2 & \\
(K_R \otimes \mathbb{C}, W, F) & \xrightarrow{\hat{q}_i} & \end{array}
$$

Springer
with $\beta_1$ a morphism in $\text{Mod}(R)_{/\tau}$ and $\beta_2$ a quasi-isomorphism in $\text{Mod}(R \otimes R \mathbb{C})_{/\tau \otimes \mathbb{C}}$ such that $\beta_1 \otimes 1_\mathbb{C}$ is a filtered quasi-isomorphism. If, $r \geq 1$, the morphisms

$$E_r(\beta_1) : E_r(K_R, W) \to E_r(K'_R \otimes \mathbb{C}, W)$$

(11)

$$E_r(\beta_1 \otimes 1_\mathbb{C}) : E_r(K_R \otimes \mathbb{C}, W) \to E_r(K'_R \otimes \mathbb{C}, W)$$

(12)

$$E_r(\beta_2) : E_r(K_R \otimes \mathbb{C}, W) \to E_r(K'_R \otimes \mathbb{C}, W)$$

(13)

define morphisms of spectral sequences

$$E_r(\beta) = E_r(\beta_2)^{-1} \circ E_r(\beta_1)$$

(14)

$$E_r(\beta \otimes 1_\mathbb{C}) = E_r(\beta_2)^{-1} \circ E_r(\beta_1 \otimes 1_\mathbb{C})$$

(15)

Then $E_r(\beta_1) \otimes 1_\mathbb{C} \simeq E_r(\beta_1 \otimes 1_\mathbb{C})$ is an isomorphism. Moreover the morphism of spectral sequences $(E_r(\beta))_r$ defines a real structure $\alpha_r$ on $E_r(K_R \otimes \mathbb{C}, W)$ such that $d_r$ is real, for

$$d_r \simeq (E_r(\beta) \otimes 1_\mathbb{C})(d_r \otimes 1_\mathbb{C}).$$

Note that the real structure induced on $E_{r+1} \simeq H(E_r)$ by $E_r$ is the same than its real structure. The differential $d_r$ is compatible with the direct filtrations and their conjugates.

One reduces modulo $\tau$: One obtains a real structure modulo $\tau$ on each term of the spectral sequence, with $d_r$ real and compatible with the direct filtrations.

(1) By hypothesis, $F_d = F_{d^*} = F_{\text{rec}}$ and their conjugates define a Hodge structure modulo $\tau$ of weight $-p + (p + q) = q$ on $E^{p,q}_1(W) = H^{p+q}(Gr_W^p)$. But $d_1$ is real and compatible with $F$ so that it is a morphism of Hodge structures in $\text{Mod}(R)_{/\tau}$. It is therefore strict for $F$ in $\text{Mod}(R \otimes \mathbb{C})_{/\tau \otimes \mathbb{C}}$.

(2) Proposition (7.2.5) in [12] implies $F_d = F_{d^*} = F_{\text{rec}}$ modulo $\tau \otimes \mathbb{C}$ on $E_2(W)$. From [11] (1.2.10), generalised in Theorem 2.39, the category of mixed Hodge structures in $\text{Mod}(R)_{/\tau}$ is abelian. Hence $(E^{p,q}_2, \alpha^{p,q}_2, F_{\text{rec}})$ is a Hodge structure of weight $q$ in $\text{Mod}(R)_{/\tau}$. The differential $d_2$ is a morphism of Hodge structures in $\text{Mod}(R)_{/\tau}$ for it is real and compatible with $F_{\text{rec}}$ and its conjugate. This implies that $d_2$ is strict modulo $\tau \otimes \mathbb{C}$. But $d_2 : E^{p,q}_2 \to E^{p+2,q-1}_2$ is a morphism of Hodge structures of different weights. It must vanish. An induction argument implies that $d_r = 0$ if $r \geq 2$.

(3) One concludes from section (7.2) of [12]: The following sequence is exact for any $1 \leq r \leq +\infty$ and any $p \in \mathbb{Z}$:

$$0 \to E_r(F^p K, W) \to E_r(K, W) \to E_r(K/F^p K, W) \to 0,$$

and the spectral sequence $E(K, F)$ degenerates at $E_1$. \hfill $\Box$

Note that the abelian category $\text{Mod}(R)_{/\tau}$ admits inductive limits and that the localisation functor $\text{Mod}(R) \to \text{Mod}(R)_{/\tau}$ commutes with inductive limits [25, prop. 9, p. 378]. Hence, we consider the category $\mathcal{M}(R)_{/\tau}$ of sheaves with values in $\text{Mod}(R)_{/\tau}$.
**Definition 2.44** On a topological space $X$, a $(\tau', \tau)$-cohomological Hodge complex of weight $m$ $(\mathcal{K}_R, (\mathcal{K}_R \otimes \mathbb{C}, F))$ consists in

1. A bounded below complex $\mathcal{K}_R$ in $\mathcal{M}(R)/\tau'$,
2. A bounded below filtered complex $(\mathcal{K}_R \otimes \mathbb{C}, F)$ in $\mathcal{M}(R \otimes \mathbb{C})/\tau' \otimes \mathbb{C}$,
3. A pseudo-morphism of bounded below complexes $\alpha : \mathcal{K}_R \rightarrow (\mathcal{K}_R \otimes \mathbb{C})$ (first comparison morphism) in $\mathcal{M}(R)/\tau'$, such that $\alpha \otimes \text{id} : \mathcal{K}_R \otimes \mathbb{C} \rightarrow \mathcal{K}_R \otimes \mathbb{C}$ is a pseudo-isomorphism in $\mathcal{M}(R \otimes \mathbb{C})/\tau' \otimes \mathbb{C}$ and

$R\Gamma(\mathcal{K})$ is a $(\tau', \tau)$-Hodge complex of weight $m$.

**Definition 2.45** A $(\tau', \tau)$-cohomological mixed Hodge complex of sheaves $\mathcal{K} = ((\mathcal{K}_R, W); (\mathcal{K}_C, F, W), \beta)$ is given by complexes of sheaves in $(\mathcal{M}(R)/\tau', \mathcal{M}(R \otimes \mathbb{C})/\tau' \otimes \mathbb{C})$ and pseudomorphism $\beta : (\mathcal{K}_R, W) \rightarrow (\mathcal{K}_C, W)$ in that category, such that $\beta \otimes 1_\mathbb{C}$ is a quasi-isomorphism and such that for all $m \in \mathbb{Z}$, $Gr^m_{\mathbb{W}} \mathcal{K}$ is a $(\tau', \tau)$-cohomological Hodge complex of sheaves.

**Definition 2.46** When $\tau' = (0, \text{Mod}(R))$ is the trivial torsion theory, $(\mathcal{K}_R, (\mathcal{K}_R \otimes \mathbb{C}, F))$ (resp. $(\mathcal{K}_R, W); (\mathcal{K}_C, F, W), \beta)$) is called a $\tau$-cohomological Hodge complex of sheaves of weight $m$ (resp. a $\tau$-cohomological mixed Hodge complex of sheaves).

In this article, we will mostly deal with $\tau' = (0, \text{Mod}(R))$ the trivial torsion theory. Hence we will use complexes of sheaves of $R$-modules.

### 3 The Hodge to De Rham spectral sequence

#### 3.1 The local Hodge to De Rham spectral sequence

From now on, $p : \tilde{X} \rightarrow X$ will be a covering of a complex connected manifold $X$ by a complex manifold $\tilde{X}$, not necessarily connected. Let $G$ be the group of covering transformations.

**Definition 3.1** When $G$ acts transitively, one says that $p : \tilde{X} \rightarrow X$ is a $G$-covering.

**3.1.1 Sobolev spaces (see [54, p. 511])** Fix a hermitian (later Kähler) metric on $X$ and take its pullback on $\tilde{X}$.

Let $\Lambda^k_{\mathbb{R}} := \Lambda^k_{\mathbb{R}}(X)$, $\Lambda^k := \Lambda^k_{\mathbb{R}} \otimes \mathbb{C}$ and $\Lambda^p,q := \Lambda^p T^*_{\mathbb{C}}(1,0)(X) \otimes \Lambda^q T^*_{\mathbb{C}}(0,1)(X)$ be respectively the bundles of real $k$-forms, of complex $k$-forms, and of bi-degree $(p, q)$-forms on $X$. Let $\Lambda^k_{\mathbb{R}}, \Lambda^k$ and $\Lambda^p,q$ be the associated sheaves of differential forms. Let $(E, h_E)$ be a hermitian complex vector bundle on $X$ (or a riemannian real vector bundle). Let $(\tilde{E}, h_{\tilde{E}})$ be its pullback by $p$.

**Definition 3.2** We define the following classical spaces of sections of $\tilde{E}$, following [54, §. 3]:

1. The Hilbert space of square integrable sections $(L^2(\tilde{X}, \tilde{E}), \|\cdot\|_{L^2(\tilde{X}, \tilde{E})})$.
2. The Frechet space of smooth sections with compact support $C^\infty_c(\tilde{X}, \tilde{E})$.
3. The space of distributional sections $\mathcal{D}'(\tilde{X}, \tilde{E})$. 

\[ \text{Springer} \]
(4) The Sobolev space \( S^j(\tilde{X}, \tilde{E}) \), \( j \in \mathbb{N} \), is the space of \( u \in \mathcal{D}'(\tilde{X}, \tilde{E}) \) such that 
\( \forall 0 \leq i \leq j, \ \nabla^i u \in L^2(\tilde{X}, \tilde{E}) \) (with \( \nabla \) a connection on \( E \)). Then \( ||u||^2_{S^j(\tilde{X}, \tilde{E})} := \sum_{0 \leq i \leq j} ||\nabla^i u||^2_{L^2(\tilde{X}, \tilde{E})} \).

(5) Denote also \( S^{+\infty}(\tilde{X}, \tilde{E}) = \cap_{j \in \mathbb{N}} S^j(\tilde{X}, \tilde{E}) \).

### 3.1.2 Local Sobolev spaces uniform with respect to \( p \).

Let \( U \) be an open subset in \( X \). Let \( j \in \mathbb{N} \cup \{+\infty\} \). Define the local Sobolev space uniform with respect to \( p_{*(2)} S^j_{loc}(U, E) \) as the set \( \{ \alpha \in \mathcal{D}'(p^{-1}(U), \tilde{E}), \ n \theta \in \mathcal{C}_c^{\infty}(U), (\theta \circ p)\alpha \in S^j(\tilde{X}, \tilde{E}) \} \).

### 3.1.3 Sheaf of uniform Sobolev spaces.

Let \( j \in \mathbb{N} \cup \{+\infty\} \), let \( p_{*(2)} S^j \mathcal{A}^{p.q}(E) \) be the sheaf on \( X \) associated to the presheaf \( U \mapsto p_{*(2)} S^j_{loc}(U, \Lambda^{p.q} \otimes \tilde{E}) \) and set \( p_{*(2)} S^j \mathcal{A}^{p.q}(E) = \bigoplus_{p+q=n} p_{*(2)} S^j \mathcal{A}^{p.q}(E) \).

Let \( D : C^\infty(X, E) \to C^\infty(X, E') \) be a differential operator on \( X \) acting on hermitian vector bundles. Let \( L^2(p^{-1}(U), \tilde{E}) \cap \text{Dom}(D) \) be the space of square summable sections \( \alpha \) such that \( D\alpha \) is square summable.

Let \( p_{*(2)} (\mathcal{E} \cap \text{Dom}(D))) \) be the sheaf generated by the presheaf \( U \mapsto [L^2(p^{-1}(U), \tilde{E}) \cap \text{Dom}(D)] \).

Note that the above sheaves do not depend on the smooth metrics on \( X \) and \( E \) and that these are sheaves of \( N(G) \)-modules.

Let \( p_{*(2)} S^j \mathcal{A}^{p.q}_{\mathbb{R}} \) be the sheaf of \( N(G, \mathbb{R}) \)-modules on \( X \) associated to the presheaf of real forms \( U \mapsto p_{*(2)} S^j_{loc}(U, \Lambda^{p.q}_{\mathbb{R}}) \).

**Lemma 3.3** (1) The complex \( (p_{*(2)} \Omega^\cdot, d) \) is a \( N(G) \)-resolution of \( p_{*(2)} \mathbb{C} \).

(2) Let \( k \in \mathbb{N} \cup \{+\infty\} \). Then \( (p_{*(2)} S^k \cdot \mathcal{A}^{p.q}_{\mathbb{R}}, d) \) is a \( N(G, \mathbb{R}) \)-resolution of \( p_{*(2)} \mathbb{R} \) and \( (p_{*(2)} S^k \cdot \mathcal{A}^{p.q}, d) \) is a \( N(G) \)-resolution of \( p_{*(2)} \mathbb{C} \).

(3) The sheaf \( p_{*(2)} S^j \mathcal{A}^{p.q} \simeq p_{*(2)} S^j \mathcal{A}^0 \otimes \mathcal{A} \mathcal{A}^{p.q} \) is a fine \( N(G) \)-sheaf.

**Lemma 3.4** Let \( k \in \mathbb{N}, k \geq n \). Let \( F \) be the Hodge filtration.

(1) The morphism \( (p_{*(2)} \Omega^\cdot, d, F) \to (p_{*(2)} S^k \cdot \mathcal{A}^{p.q}, d, F) \) is a filtered quasi-isomorphism.

(2) The following complex of \( N(G) \)-sheaves is exact:

\[
\begin{array}{c}
0 \to p_{*(2)} \Omega^p \xrightarrow{i} p_{*(2)} S^k \mathcal{A}^{p.q} \xrightarrow{j} p_{*(2)} S^{k-1} \mathcal{A}^{p.q} \xrightarrow{k} \ldots \\
\to p_{*(2)} S^{k-n} \mathcal{A}^{p.q} \to 0.
\end{array}
\]

**Proof** One notes that (2) is equivalent to (1) for \( \text{Gr}_F^p (p_{*(2)} S^k \cdot \mathcal{A}^{p.q}, d, F) \) is the Dolbeault complex \( (p_{*(2)} S^k \cdot \mathcal{A}^{p.q}, d, F) \). One will proves the more general fact that

\[
(p_{*(2)} \Omega^\cdot, d, F) \to (p_{*(2)} S^k \cdot \mathcal{A}^{p.q}, d, F) \to (p_{*(2)} \mathcal{A}^{p.q} \cap \text{Dom}(d), F, d)
\]

are filtered quasi-isomorphism. Indeed the \( E_1 \)-term of the spectral sequence of the third complex is \( p_{*(2)} \mathcal{A}^{p.q} \cap \text{Dom}(d) \cap \ker(\overline{\partial})/\ker(\overline{\partial}) \). The assertions are local over the base manifold \( X \).

One may work over an open chart \( U \) of \( X \) which is biholomorphic to some ball \( B(0, 2) \) in \( \mathbb{C}^n \). Let \( x \) be the center of the chart. Then the covering \( p^{-1}(U) \to U \) is isomorphic.
to \( U \times p^{-1}(x) \to U \). The standard estimates for the resolution of the \( \overline{\partial} \)-operator on a strictly pseudoconvex domains will implies the lemma. Let \( f \) be a \((p, q)\)-form, \( q \geq 1 \), on the strictly pseudoconvex domain \( U \subset \subset \mathbb{C}^n \) which belongs to \( \text{Dom}(d) \cap \text{Ker}(\overline{\partial}) \subset \text{Ker}(\overline{\partial}) \). Then \( f \in \text{Dom}(\overline{\partial}) \). Let \( N : L^2(\U, \Lambda^{p,q}) \to L^2(\U, \Lambda^{p,q}) \) be the (bounded) \( \overline{\partial} \)-Neumann operator (see [56, p. 280–282]) so that \( \text{Ran}(N) \subset \text{Dom}(\overline{\partial}^* + \overline{\partial}^*) \) and \( f = \overline{\partial}^*Nf + \overline{\partial}^*\overline{\partial}f \). The last term is vanishing for \( \overline{\partial}N = N\overline{\partial} \) on \( \text{Dom}(\overline{\partial}) \). Then \( f = Nf \) and \( f \in H^2_{\text{loc}}(\U) \) implies \( (\overline{\partial}^* Nf) \in L^2(\U, \Lambda^{p,q}), Nf \in H^{s+2}_{\text{loc}}(\U) \) (usual Sobolev space in euclidian space). Hence if \( U_1 \subset \subset U \), the map \( H^s(U) \cap \text{Ker}(\overline{\partial}) \in \mathbb{L}^2(U, \Lambda^{p,q}) \) \( \overset{\sim}{\rightarrow} \) \( f \mapsto Nf \in H^{s+1}(U_1) \) is continuous.

Let \( B(0, 1) \simeq U_1 \subset \subset U \). Let \( \alpha \in (p_{\ast}(2)A^{p,q} \cap \text{Dom}(d) \cap \text{ker}(\overline{\partial})) \), \( q \geq 1 \), which belongs to \( p_{\ast}(2)S^sA^{p,q}_2(U) \) for some \( s \geq 0 \). Then \( \beta = (\overline{\partial}^* N\alpha|_{U_1 \times \{y\}})_{y \in p^{-1}(x)} \in [L^2(p^{-1}(U_1) \cap \text{Dom}(\overline{\partial})] \cap p_{\ast}(2)S^{s+1}A^{p,q-1}(U_1) \) is such that \( \overline{\partial}\beta = \alpha|_{U_1} \). This proves that

\[
0 \rightarrow p_{\ast}(2)\Omega^p \overset{i}{\rightarrow} p_{\ast}(2)(A^{p,0} \cap \text{Dom}(d)) \overset{\overline{\partial}}{\rightarrow} p_{\ast}(2)(A^{p,1} \cap \text{Dom}(d)) \overset{\overline{\partial}}{\rightarrow} \ldots \rightarrow p_{\ast}(2)(A^{p,n} \cap \text{Dom}(d)) \rightarrow 0
\]

and

\[
0 \rightarrow p_{\ast}(2)\Omega^p \overset{i}{\rightarrow} p_{\ast}(2)S^kA^{p,0} \overset{\overline{\partial}}{\rightarrow} p_{\ast}(2)S^{k-1}A^{p,1} \overset{\overline{\partial}}{\rightarrow} \ldots \rightarrow p_{\ast}(2)S^{k-n}A^{p,n} \rightarrow 0
\]

are exacts. Hence \((p_{\ast}(2)S^k . \cdot, d, F) \rightarrow (p_{\ast}(2)A^\cdot \cap \text{Dom}(d), F, d)\) is a filtered quasi-isomorphism.

### 3.2 The global Hodge to De Rham spectral sequence

#### 3.2.1 Global Sobolev spaces

We assume now that \( X \) is a hermitian manifold of bounded geometry [53, Appendix 1]. An example is a covering of a compact hermitian manifold.

Let \( E \) be a uniformly bounded hermitian vector bundle on \( X \) (loc. cit.), then the smooth sections with compact support are dense in the Sobolev space \( S^m(\tilde{X}, \tilde{E}) \) \((m \in \mathbb{N})\). Moreover any \( C^\infty\)-bounded uniformly elliptic differential operator is essentially self-adjoint [53, prop. 4.1]. Hence \( A := (1 + \Delta)^{\frac{1}{2}} : L^2(\tilde{X}, \oplus_p \Lambda^p) \to L^2(\tilde{X}, \oplus_p \Lambda^p) \) (defined through the functional calculus) defines the isomorphisms \( A^m : S^k(\tilde{X}, \oplus_p \Lambda^p) \to S^{k-m}(\tilde{X}, \oplus_p \Lambda^p) \) (see e.g. Roe [46, th.5.5]). From the (non unitary) isomorphisms

\[
(S^j(\tilde{X}, \oplus_p \Lambda^p), \| \cdot \|_{S^j(\tilde{X}, \oplus_p \Lambda^p)}) \overset{\sim}{\rightarrow} (D(A^j), \| A^j \cdot \|_{L^2(\tilde{X}, \oplus_p \Lambda^p)})
\]

define a new Sobolev norm by \( \| \alpha \|_j = \| A^j \alpha \|_{L^2(\tilde{X}, \oplus_p \Lambda^p)} \).

If \( j \geq 1 \), let

\[
d : (S^j(\tilde{X}, \Lambda^p), \| \cdot \|_j) \to (S^{j-1}(\tilde{X}, \Lambda^{p+1}), \| \cdot \|_{j-1})
\]

\[
\overline{\partial} := \overline{\partial}_q : (S^j(\tilde{X}, \Lambda^{p,q}), \| \cdot \|_j) \to (S^{j-1}(\tilde{X}, \Lambda^{p,q+1}), \| \cdot \|_{j-1})
\]
be the bounded operators induced by the differentials $d$ and $\bar{\partial}$. Then $A^k : (S^j(X, \Lambda^\cdot), ||.|.|) \to (S^{j-k}(X, \Lambda^\cdot), ||.|.|_{j-k})$ is an isometric isomorphism and operators $\partial, \bar{\partial}$ and $A$ commute if the metric is Kähler.

### 3.2.2 The Hodge to De Rham spectral sequence

The Hodge to De Rham spectral sequence is the spectral sequence for the filtered complex $(S, F) = (S^{k-}(X, \Lambda^\cdot), d, F)$. Then $F^p S^{k-(p+q)}(X, \Lambda^p.q) = \oplus_{p'+q=r, p' \geq p} S^{k-(p'+q)}(\bar{X}, \Lambda^{p'.q})$. Hence, $(E_0^{p,q}(S, F), d_0) \simeq_{N(G)} (S^{k-p-q}(\bar{X}, \Lambda^{p.q}), \partial)$ and the Kodaira’s decomposition [54, p. 499] reads

$$
(E_1^{p,q}(S, F), d_1) \simeq_{N(G, \mathbb{C})} \frac{\text{Ker}(\partial_q)}{\text{Im}(\partial_{q-1})} \oplus \frac{\mathcal{H}^{p,q}(\bar{X}) \oplus \text{Im}(\partial_{q-1})}{\text{Im}(\partial_{q-1})}.
$$

### 3.3 The degenerescence of the Hodge to De Rham spectral sequence

Let $\mathcal{H}^{p,q}_{\partial(2)}(\bar{X})$ be the space of square integrable $\Delta_d$-harmonic forms and $\mathcal{H}^{p,q}_{\bar{\partial}(2)}(\bar{X})$ be the space of square integrable $\Delta_{\bar{\partial}}$-harmonic $(p, q)$-forms. Let $(\mathcal{H}^{p,q}_{\partial(2)}(\bar{X}), F)$ be the complex with trivial differential, and Hodge filtration. Assume the metric is Kähler. Then for all $r \geq 0$,

$$E_0^{p,q}(\mathcal{H}^{p,q}_{\partial(2)}(\bar{X}), F) = \mathcal{H}^{p,q}_{\partial(2)}(\bar{X}) = E_r^{p,q}(\mathcal{H}^{p,q}_{\partial(2)}(\bar{X}), F).$$

**Lemma 3.5** Assume the metric is Kähler and of bounded geometry. Let $\mathcal{H}^{p,q}_{\partial(2)}(\bar{X})$ be the space of square integrable harmonic $(p, q)$-forms. Fix an integer $k$ greater than $\dim_{\mathbb{R}} X$. Let $H^{p,q}_{\partial(2)}(\bar{X}) := H^q((S^{k-}(\bar{X}, \Lambda^{p.\cdot}), \bar{\partial}))$. Let $\tau$ be a torsion theory such that

(i) The spectral sequence for $(S, F) = (S^{k-}(\bar{X}, \Lambda^\cdot), d, F)$ degenerates at $E_1$ in $\text{Mod}(N(G, \mathbb{C}))/\tau$. The differential $d$ is strictly compatible with $F$ in $\text{Mod}(N(G, \mathbb{C}))/\tau$.

(ii) $E_1^{p,q}(S, F) \simeq E_\infty^{p,q}(S, F) \simeq \mathcal{H}^{p,q}_{\partial(2)}(\bar{X})$ in $\text{Mod}(N(G, \mathbb{C}))/\tau$.

**Proof** (i) Note that $i : (\mathcal{H}^{p,q}_{\partial(2)}(\bar{X}), F) \to (S^{k-}(\bar{X}, \Lambda^\cdot), F)$ is a morphism of filtered $N(G)$-modules. Let $\tau$ be a torsion theory on $N(G)$ such that $E_1(i)$ is an isomorphism in $\text{Mod}(N(G))/\tau$. Then $E_r(i)$ is an isomorphism for any $r \geq 1$. But the spectral sequence of $(\mathcal{H}^{p,q}_{\partial(2)}(\bar{X}), F)$ degenerates so that the spectral sequence for $(S^{k-}(\bar{X}, \Lambda^\cdot), d, F)$ degenerates at $E_1$ in $\text{Mod}(N(G))/\tau$.

(ii) The assertion follows for $(E_1^{p,q}(S, F), d_1) \simeq_{N(G, \mathbb{C})} \mathcal{H}^{p,q}_{\partial(2)}(\bar{X}) \oplus \frac{\text{Im}(\partial_{q-1})}{\text{Im}(\partial_{q-1})}$. □

**Definition 3.6** The torsion theory generated by $C = \{ \text{Coker}(\mathcal{H}^{p,q}_{\partial(2)} \to H^{p,q}_{\partial(2)}, p, q \geq 0) \}$ is the smallest torsion theory on $\text{Mod}(N(G))$ which satisfies the above lemma. Let $\tau_\partial$ (or $\tau_{\partial, \bar{X}}$) be the torsion theory it defines on $\text{Mod}(N(G))$. 

© Springer
Remark 3.7 The use of the unitary isometry $A^m$ between various Sobolev spaces proves that the torsion theory $\tau_\partial$ does not depend on the order $k$ in the Sobolev complex $(S^k\cdot (\tilde{X}, \Lambda^\cdot), d, F)$. Moreover, following Bruning Lesch [7, Th. 2.12] (smoothing of cohomology), it can be shown that the torsion theory $\tau_\partial$ is indeed an invariant of the elliptic complex $(C^\infty(\tilde{X}, \oplus_p \Lambda^p), d)$ and the complete Kähler metric.

3.4 The case of a compact Kähler manifold $X$

Lemma 3.8 Let $X$ be a compact complex hermitian manifold and let $E$ be a hermitian vector bundle. Then $\Gamma(X, p_{n(2)}S^jA^{p,q}(E)) = S^j(\tilde{X}, \Lambda^{p,q} \otimes \tilde{E})$.

Definition 3.9 ([54, prop 1.13], [36, chap. 1]) A bounded complex $(L^\cdot, d^\cdot)$ of Hilbert $G$-modules is $G$-Fredholm if $\bigoplus d_i : \bigoplus L_i \to \bigoplus L_i$ is $G$-Fredholm (see Definition 2.12).

In the following lemma, the manifold $\tilde{X}$ is not necessarily connected.

Lemma 3.10 (Atiyah [2], see also [36,54]) Let $\tilde{X} \to \tilde{X}/G = X$ be a $G$-covering of a compact complex manifold (see Definition 3.1). Then the complexes of Hilbert modules

$(S^{k-\cdot}(\tilde{X}, \Lambda^{p-\cdot}), \partial) \text{ and } (S^{k-\cdot}(\tilde{X}, \Lambda^\cdot), d)$

are $G$-Fredholm.

Corollary 3.11 With the same hypothesis,

1. The $N(G)$-module $\overline{\text{Im}(\partial)} / \overline{\text{Im}(\partial)}$ is a $\dim N(G)$-torsion module.

2. $\forall x \in \overline{\text{Im}(\partial)}$, there exists $r \in N(G)$ such that $\ker(r) = 0$ and $rx \in \overline{\text{Im}(\partial)}$.

Therefore

\[
\mathcal{U}(G) \otimes N(G) \frac{\overline{\text{Im}(\partial)}}{\overline{\text{Im}(\partial)}} = 0.
\]

Proof (1) Lemma 2.12 of [54] implies that $h = \overline{\partial} : H_1 = \overline{\text{Im}(\partial)} \to H_2 = \overline{\text{Im}(\partial)}$ is $G$-Fredholm, hence lemma 1.15 of [54] implies that $\text{Im}(h)$ is $G$-dense in its closure: $\forall \epsilon > 0$, there exists $L_\epsilon \subset \text{Im}(h)$, a closed $G$-invariant subspace, such that $\dim N(G) \overline{\text{Im}(h)} \oplus L_\epsilon \leq \epsilon$. In this example, we may take $L_\epsilon := \text{Im}(\overline{\partial} \circ l_{[\eta, +\infty]}(\partial_\Gamma))$ (functional calculus) with $\eta > 0$ small enough. This is equivalent to $\dim N(G) \overline{\text{Im}(h)} = \dim N(G) \overline{\text{Im}(h)}$ hence $\dim N(G) \overline{\text{Im}(h)} = 0$.

(2) Apply the Lemma 2.15. \hfill \Box

Analogue proof holds for an elliptic complex of vector bundles:

Theorem 3.12 Let $\tilde{X} \to \tilde{X}/G = X$ be a $G$-cover of a compact complex manifold. Let $(E^\cdot, d^\cdot)$ be an elliptic complex ($d^\cdot$ is a differential operator of order one) between vector bundles on $X$. Let $(S^{j-\cdot}(\tilde{X}, \tilde{E}^\cdot), d^\cdot)$ be the associated Sobolev complex on $p : \tilde{X} \to X$. 

\textcopyright Springer
(1) The complex $(S^1 - (\tilde{X}, \tilde{E}^*), d)$ is $G$-Fredholm.

(2) The module $\frac{\text{Im}(d)}{\text{Im}(d)}$ is of $G$-dimension zero and $\mathcal{U}(G) \otimes_{\mathcal{N}(G)} \frac{\text{Im}(d)}{\text{Im}(d)} = 0$.

This implies that if $d : L^2(\tilde{X}, E) \to L^2(\tilde{X}, F)$ acts as an unbounded elliptic operator, then $\mathcal{U}(G) \otimes \frac{\text{Im}(d)}{\text{Im}(d)} = 0$: One uses conjugation by $(1 + \partial^* d)^{-1}$ and $(1 + dd^*)^{-1}$. One may also considers currents in negative Sobolev scale (e.g. Dirac measure on a point).

**Corollary 3.13** (A $\partial \bar{\partial}$-lemma) Let $p : \tilde{X} \to (X, \omega)$ be a $G$-covering of a compact Kähler manifold. Let $\alpha$ be a $d$-closed square integrable $(p, q)$-form on $\tilde{X}$ which is orthogonal to the harmonic forms. Then there exists a weak isomorphism $r \in N(G)$, there exists a square integrable form $\gamma$ on $\tilde{X}$ such that $r \alpha = \partial \bar{\partial} \gamma$.

**Proof** Above corollary implies that there exists a weak isomorphism $r' \in N(G)$ and $\gamma_1 \in \text{Im} \partial, \gamma_2 \in \text{Im} \bar{\partial}$ such that $r' \alpha = \partial \gamma_1 + \bar{\partial} \gamma_2$. Then there exists a weak isomorphism $r''$ such that $r'' \gamma_1 = \bar{\partial} \gamma_3$ and $r'' \gamma_2 = \partial^* \gamma_4$. Hence $r'' r' \gamma = \partial \bar{\partial} \gamma_3 = \partial \bar{\partial}^* \gamma_4$ is $\partial \bar{\partial}$-closed. But the metric is Kähler, hence $\partial^* \bar{\partial} = \partial \bar{\partial}^*$. Therefore $\partial \bar{\partial} \gamma_4 \in \text{Ker} (\bar{\partial}) \cap \text{Ker} (\bar{\partial})^*$ is vanishing. □

The combination of sheaf theory and torsion theory enables to recover a result of Dodziuk [16] in the case of Hermitian manifold. Standard sheaf theory proves that unreduced De Rham $l^2$-cohomology groups and unreduced simplicial $l^2$-cohomology groups are isomorphic. The use of torsion theory is needed to provide an isomorphism between the reduced cohomology groups.

**Corollary 3.14** Let $p : \tilde{X} \to \tilde{X}/G = X$ be a $G$-covering. Then the combinatorial reduced $l^2$-cohomology and the analytical reduced $l^2$-cohomology are isomorphic in $N(G)/\tau_{\text{dim}}$.

**Proof** Let $f : K \to X$ be a $C^1$-triangulation of $X$ [63]: $K$ is a rectilinear complex in some euclidian space and $f$ is a $C^1$-map which is a homeomorphism. In what follows, we identify $K$ and $M$.

Let $K'$ be the first barycentric subdivision of $K$ [35,51]. Let $v \in K_0$ be a vertex in $K$. Let $F_v = St(v, K')$ be the closed star of $v$ in $K'$. The set $F_{v_0} \cap \cdots \cap F_{v_k}$ is non empty if and only if $[v_0, \ldots, v_k]$ is a simplex of $K$. Hence the simplicial complex defined by the nerve of the closed covering $\mathcal{M} = \{ F_v, v \text{ a vertex of } K \}$ is identified with $K$, the simplicial complex of $K$. Let $S = [v_0, \ldots, v_k] \in K_k$ be a $k$-dimensional simplex of $K$. Define $F_S := F_{v_0} \cap \cdots \cap F_{v_k}$. Then $F_S = S^*$ is equal to the dual cell of $[v_0, \ldots, v_k]$ (which is homeomorphic to a $(n-k)$-closed ball).

If $\mathcal{F}$ is a sheaf on $X$, let $(\mathcal{C}(\mathcal{M}, \mathcal{F}), \delta)$ be the differential sheaf $U \to \prod_{S \in \mathcal{F}} \mathcal{F}(\mathcal{F}_S \cap U)$ ([26] II.5.2). Then $(\mathcal{C}(\mathcal{M}, p_{*(2)} \mathbb{C}), \delta)$ is a resolution of $p_{*(2)} \mathbb{C}$, $(\mathcal{C}(\mathcal{M}, p_{*(2)} \mathbb{C}), \delta) \to (\mathcal{C}(\mathcal{M}, p_{*(2)} S^{k-\cdot -A^*}), \delta + d)$ and $(p_{*(2)} S^{k-\cdot -A^*}, d) \to (\mathcal{C}(\mathcal{M}, p_{*(2)} S^{k-\cdot -A^*}), \delta + d)$ are quasi-isomorphisms of $\mathcal{N}(G)$-sheaves. Each term in theses complexes are $\Gamma(X,\cdot)$-acyclic for $\mathcal{F}_S$ is contractible and $p_{*(2)} \mathbb{C}$ is locally constant; and the sheaves $p_{*(2)} S^{k-\cdot} - A^*$ are fine. Hence [26, II.5.2]:

$$H^k(X, p_{*(2)} \mathbb{C}) \simeq H^k_\delta(\Gamma(X, \mathcal{C}(\mathcal{M}, p_{*(2)} \mathbb{C}))) \simeq H^k_{d(2)}(\tilde{X}).$$
Let \((\tilde{K}, \delta)\), \(\tilde{K}'\) be the pullback simplicial structures, and define \(\tilde{F}_S := \tilde{F}_{v_0} \cap \cdots \cap \tilde{F}_{v_k}\). Then \(\tilde{F}_S = \tilde{S}^*\) is equal to the dual cell of \(\tilde{S}\). Hence

\[
H^k_S(\Gamma(X, \mathcal{C}(\mathcal{M}, p_{*(2)}\mathbb{C}))) \simeq_{N(G)} H^k(\text{Hom}_{\mathbb{C}[G]}(\mathcal{C}(\tilde{K}), l^2(G)))
\]

(we use the right \(\mathbb{C}[G]\)-module structure in \(\tilde{K}\), and the \(N(G) - \mathbb{C}[G]\)-bi-module structure in \(l^2(G)\)). The reduction with respect to the torsion dimension \(\tau_{dim}\) gives the result, for the modules \(\text{Im}(d)/\text{Im}(d)\) defined by the combinatorial differential (see Sect. 2.5.3) or analytical differential (see above) have \(N(G)\)-dimension zero. \(\Box\)

Note that Dodziuk proves that the isomorphism is given by integration of harmonic forms on the cells of a pull back of a triangulation.

### 3.5 Pure Hodge structures

**Theorem 3.15** Let \(X\) be a connected Hermitian manifold and \(p : \tilde{X} \to X\) be a covering, not necessarily connected, with covering transformations group \(G\). Let \(k \geq n = \dim_{\mathbb{C}} X\).

1. (i) The morphism \(((p_{*(2)}\Omega^-, d), F) \to ((p_{*(2)}\mathcal{S}^{k-\cdot}, \mathcal{A}^-, d), F)\) is a filtered quasi-isomorphism of \(\mathcal{N}(G)\)-sheaves such that \(\text{Gr}_F p_{*(2)}\mathcal{S}^{k-\cdot}, \mathcal{A}^-\) is \(\Gamma\)-acyclic.
   (ii) The morphism \(((p_{*(2)}\Omega^-, d) \to (p_{*(2)}\mathcal{S}^{k-\cdot}, \mathcal{A}^-, d)\) is a quasi-isomorphism of \(\mathcal{N}(G)\)-sheaves. It defines a real structure on \(\mathbb{H}(X, (p_{*(2)}\Omega^-, d))\) compatible with the real structure given by the Godement resolution and the pseudo-isomorphism (Sect. 2.7.1) \(\mathbb{C}(p_{*(2)}\mathbb{C}) \to \mathcal{A}(p_{*(2)}\mathbb{C}, d)\).

2. Assume the manifold \(X\) is compact. Then the Frölicher spectral sequence

\[
H^q(X, p_{*(2)}\Omega^p) \Rightarrow \mathbb{H}^{p+q}(X, (p_{*(2)}\Omega^-, d)) \simeq_{N(G, \mathbb{C})} H^{p+q}(X, p_{*(2)}\mathbb{C}),
\]

is isomorphic to the Hodge to De Rham spectral sequence \(H^{p+q}_{d(2)}(\tilde{X}) \Rightarrow H^{p+q}_{d(2)}(\tilde{X})\) (see Sect. 3.2.2).

3. Assume the manifold \(X\) is a compact Kähler manifold. Let \(\tau\) be a torsion theory on \(\text{Mod}(N(G))\) greater than \(\tau_{\mathbb{P}}\). Then the Frölicher spectral sequence degenerates in \(\text{Mod}(N(G))_{/\tau}\). Hence \(d\) is strict for \(F\) in \(\text{Mod}(N(G))_{/\tau}\):

\[
\text{Gr}_F^q H^{p+q}(X, p_{*(2)}\mathbb{C}) \simeq_{\text{Mod}(N(G))_{/\tau}} H^q(X, p_{*(2)}\Omega^p).
\]

4. Assume moreover that \(\tau\) is real. Then the Hodge filtration on the hypercohomology \(F^p \mathbb{H}(X, p_{*(2)}\Omega^-) := \text{Im}(\mathbb{H}(X, F_{p_{*(2)}\Omega^-}) \to \mathbb{H}(X, (p_{*(2)}\Omega^-, d)))\) and its complex conjugate \(\tilde{F}\) are k-opposed on \(\mathbb{H}^k(X, p_{*(2)}\Omega^-)\) in \(\text{Mod}(N(G))_{/\tau}\). It defines a pure Hodge structure of weight \(k\) on \(H^k(X, p_{*(2)}\mathbb{R})\) in \(\text{Mod}(N(G))_{/\tau}\) (see Definition 2.38).

**Proof** (1) (i) was proved in Lemma 3.4.

(ii) \((p_{*(2)}\Omega^-, d) \to (p_{*(2)}\mathcal{S}^{k-\cdot}, \mathcal{A}^-, d)\) is a quasi-isomorphism for \(p_{*(2)}\mathbb{C} \to (p_{*(2)}\Omega^-, d)\) and \(p_{*(2)}\mathbb{C} \to (p_{*(2)}\mathcal{S}^{k-\cdot}, \mathcal{A}^-, d)\) are resolutions. But \((\mathcal{A}_{\mathbb{C}}, d) = \mathbb{C}\) Springer
\((A_\mathbb{R}, d) \otimes_{\mathbb{Z}} \mathbb{C}\) and the following diagram is commutative:

\[
\begin{array}{ccc}
(p_{*}(2)S^k_{-}, A^\mathbb{R}_{\mathbb{C}}, d) & \longrightarrow & (p_{*}(2)\Omega^i, d) \\
\downarrow & & \downarrow \\
(p_{*}(2)S^k_{-}, A^\mathbb{R}_{\mathbb{C}}, d) & \longrightarrow & (p_{*}(2)S^k_{-}, A^\mathbb{R}_{\mathbb{C}}, d)
\end{array}
\]

(2) Sheaves \(p_{*}(2)S^k_{-}, A^\mathbb{R}_{\mathbb{C}}\) and \(p_{*}(2)S^k_{-}, A^\mathbb{R}_{\mathbb{C}}\) are \(\Gamma(X, -)-acyclic\) and \(\Gamma(X, p_{*}(2)S^j A^\mathbb{R}_{\mathbb{C}}) = S^j(\tilde{X}, \Lambda^\mathbb{R}_{\mathbb{C}})\) for \(X\) is compact. Hence \(\mathbb{H}^r(X, p_{*}(2)\Omega^i) \cong N(G)H^r(S^k_{-}, (\tilde{X}, \Lambda^\mathbb{R}_{\mathbb{C}}), d)\).

Note that \(p_{*}(2)S^j A^p,q \rightarrow Gr^p_F(p_{*}(2)S^j A^{p,q})\) is \(\Gamma(X, -)-acyclic\). Hence the spectral sequence for \((\mathbb{H}(X, p_{*}(2)\Omega^i), F)\) is (isomorphic) given by the spectral sequence of \(N(G)\)-modules

\[
H^{p+q}(S^k_{-}, (\tilde{X}, \Lambda^\mathbb{R}_{\mathbb{C}}), d) \iff E_1^{pq} = H^q(S^k_{-}, (\tilde{X}, \Lambda^{\mathbb{R}_{\mathbb{C}}}))
\]

(3) From Lemma 3.5, the above spectral sequence degenerates in \(\text{Mod}(N(G))/\tau\) and \(H_{d_{\mathfrak{d}}}(i, n-i)(\tilde{X}) \rightarrow E_1^{(i,n-i)} \cong E_1^{(i,n-i)}\) is an isomorphism. Hence \(\oplus_{i \geq p} H_{d_{\mathfrak{d}}}(i, n-i)(\tilde{X}) \rightarrow F^p H^n(X, p_{*}(2)\mathbb{C})\) is an isomorphism.

(4) If moreover \(\tau\) is real then \(\oplus_{i \geq q} H_{d_{\mathfrak{d}}}(i, n-i)(\tilde{X}) = \oplus_{i \geq q} H_{d_{\mathfrak{d}}}(i, n-i)(\tilde{X}) \rightarrow F^q H^n(X, p_{*}(2)\mathbb{C})\) (Notation 2.30) is also an isomorphism. From [11] (1.2.4), we conclude that in \(\text{Mod}(N(G))/\tau\), the filtrations \(F\) and \(\tilde{F}\) on \(H^n(X, p_{*}(2)\mathbb{C})\) are \(n\)-opposed. \(\square\)

**Example 3.16** (1) Let \(\alpha : p_{*}(2)\mathbb{R} \rightarrow (p_{*}(2)\Omega^i, d^-)\) be the natural map. We have seen that \((p_{*}(2)\mathbb{R}, (p_{*}(2)\Omega^i, F), \alpha)\) is a CHC of sheaves of weight 0 (Definition 2.44) modulo \(\tau_{\dim}\) when \(G\) is Galois, or \(\tau_{\dim}^-\) (Definition 2.23) in general. In the following diagram, the maps \(R\Gamma(i)\) (Hodge to De Rham) and \(m\) (\(\Gamma\)-acyclic sheaves) are filtered quasi-isomorphisms.

\[
\begin{array}{ccc}
R\Gamma(p_{*}(2)\mathbb{R}) & \xrightarrow{R\Gamma(\alpha)} & R\Gamma(p_{*}(2)\Omega^i, F) \\
\downarrow & & \downarrow \\
R\Gamma(p_{*}(2)S^k_{-}, A^\mathbb{R}_{\mathbb{C}}, d, F) & \xrightarrow{R\Gamma(i)} & R\Gamma(p_{*}(2)S^k_{-}, A^\mathbb{R}_{\mathbb{C}}, d, F)
\end{array}
\]

Therefore if \(r \geq 1\),

\[
E_r(m, F)^{-1} \circ E_r(R\Gamma(i), F) : E_r(R\Gamma(p_{*}(2)\Omega^i, F)) \rightarrow E_r(\Gamma(S^k_{-}, A^\mathbb{R}_{\mathbb{C}}, d, F))
\]

defines an isomorphism of spectral sequences. The second degenerates modulo \(\tau_{\dim}\) when \(G\) is Galois (\(E_1 \simeq E_\infty\) in \(\text{Mod}(N(G))/\tau_{\dim}\)).

\(\square\) Springer
(1bis) Let $\tau$ be a torsion theory such that $(p_{*}(2)\mathbb{R}, (p_{*}(2)\Omega^{*}, F), \alpha)$ is a CHC of sheaves of weight 0 modulo $\tau$.

Let $k, m \in \mathbb{Z}$. One checks that $((2i\pi)^{k}\mathbb{Z} \otimes p_{*(2)\mathbb{R}}[m], (p_{*(2)\Omega}[m], F[k + m]), \alpha[m](2i\pi)^{k})$, where $F[k]^{+} := F^{p+k}$ is the filtration shifted $k$-steps to the left, is a CHC of sheaves of weight $m - 2k$ modulo $\tau$. Note the new conjugation induced by $\alpha(2i\pi)^{k}$ is $(-1)^{k}$ times the original one.

(2) Assume that $\tilde{X} \to X$ is such that each connected component $\tilde{X}_{i}$ of $\tilde{X}$ is compact and $G$ is transitive on fibers. Let $G_{i}$ be the stabiliser of $\tilde{X}_{i}$ and let $p_{i}$ be the restriction of $p$ to $X_{i}$.

Then $d$ and $\overline{d}$ have closed ranges. Indeed

$$S^{k}(\tilde{X}, \Lambda^{p}) \ni \alpha \mapsto (g \mapsto g^{*}\alpha|_{\tilde{X}_{i}}) \in l^{2}(G, S^{k}(\tilde{X}_{i}, \Lambda^{p}))^{G_{i}}$$

is a $G$-equivariant isometric isomorphism which commutes with $d$ and $\overline{d}$. But the functor of invariant with respect to $G_{i}$ is exact on $\mathbb{Q}[G_{i}]$-modules, hence $H^{k}(\tilde{X}) \cong l^{2}(G, H^{k}(\tilde{X}_{i}))^{G_{i}}$. The cohomology is reduced, and the Hodge structure is isomorphic to that of $\tilde{X}_{i}$ twisted by that of $l^{2}(G, \mathbb{C})$.

If $\mathcal{F}$ is a sheaf for which $p_{*(2)}\mathcal{F}$ is defined then the $l^{2}$-cohomology is separated and

$$H^{k}(\tilde{X}, p_{*(2)}\mathcal{F}) \cong_{N(G)} l^{2}(G, H^{k}(X_{i}, p^{*}_{i}\mathcal{F}))^{G_{i}}$$

and

$$\dim_{N(G)}l^{2}(G, H^{k}(X_{i}, p^{*}_{i}\mathcal{F}))^{G_{i}} = \frac{\dim_{\mathbb{C}} H^{k}(X_{i}, p^{*}_{i}\mathcal{F})}{|G_{i}|}.$$ 

### 3.6 Addendum: smoothing of cohomology

It is well known that the cohomology of currents is isomorphic to the cohomology of smooth forms. Here we impose moreover that forms are square integrable in the fiber of $p$.

**Lemma 3.17** (1) Let $U$ be some open set in $X$. Then $H^{\cdot}(\Gamma(U, p_{*(2)}S^{\infty}\mathcal{A}^{i}), d) \to H^{\cdot}(\Gamma(U, p_{*(2)}S^{\infty^{\infty}}\mathcal{A}^{i}), d)$ is an isomorphism.

(2) Assume that $X$ is a Kähler manifold of bounded geometry. Then for any closed $l$-form $\alpha$ in $\text{Dom}[(1 + \Delta)^{2}]$ on $\tilde{X}$, there exists $(\beta, \gamma) \in S^{l-1}(\tilde{X}, \Lambda^{l-1}) \times S^{\infty}(\tilde{X}, \Lambda)_{a}$ such that $\alpha = d\beta + \gamma$.

**Proof** (1) Note that $(p_{*(2)}S^{\infty}\mathcal{A}^{i}, d) \to (p_{*(2)}S^{\infty^{\infty}}\mathcal{A}^{i}, d)$ is a quasi-isomorphism.

The result follows for theses sheaves are flabby.

(2) This follows from [7, th.2.12 ] and Sect. 3.2.1 above. \(\square\)

We refer to [7, th.2.12 or th.3.5] for more general results on smoothing of cohomology in Hilbert complexes.

### 3.7 An application: mixed Hodge structure on the $l^{2}$-cohomology of a covering of a normal crossing divisor

Let $Y = Y_{1} \cup \cdots \cup Y_{p}$ be a compact normal crossing
divisor in a smooth Kähler manifold. Let \( p : \tilde{Y} \to Y \) be a covering of \( Y \). It induces covering maps \( p := p_q : \tilde{Y}_q \to Y_q \) with \( Y_q = \sqcup_{i_0 < \cdots < i_q} Y_{i_0} \cap \cdots \cap Y_{i_q} \).

A classical generalisation of the Mayer–Vietoris argument relates the cohomology of \( Y \) in any coefficient systems to the cohomology of the subspace \( Y_q \). This defines a mixed Hodge structure on \( H^*(Y, p_*(\mathbb{C})) \) in \( \text{Mod}(N(G))/\tau \) for each \( H^*(Y_q, p_*(\mathbb{C})) \) carries a Hodge structure in \( \text{Mod}(N(G))/\tau \), if \( \tau \) is big enough:

**Theorem 3.18** Let \( \tau \) be a real torsion theory on \( \text{Mod}(N(G)) \) such that for any \( q \), \( \tau \) is greater than \( \tau_{Y_q} \).

1. Then \( (R\Gamma(Y, p_*(\mathbb{R})), (R\Gamma(Y, \Pi_*(p_*(\mathbb{R}))), W), (R\Gamma(Y, s(p_*(\Omega_Y)), W, F), R\Gamma(i)) \) is a \( \tau \)-mixed Hodge complex.

2. The weight spectral sequence \( H^p(Y_q, p_{q*}(\mathbb{C})) \Rightarrow H^{p+q}(Y, p_*(\mathbb{C})) \) with differential \( d^{p,q}_1 = s^{p,q}_q \) is a spectral sequence of Hodge structures in \( \text{Mod}(N(G))/\tau \) and degenerates at \( E_2 \).

3. The filtration \( W[n] \), \( \mathbb{H}^n(Y, p_*(\mathbb{C})) = \text{Im}(\mathbb{H}^n(Y, (\sigma_{\geq n}, p_*(\mathbb{C})), \delta)) \Rightarrow H^n(Y, p_*(\mathbb{C})) \) and \( F \cdot H^n(Y, p_*(\mathbb{C})) \simeq \mathbb{H}^n(Y, s(\sigma_{\geq n} p_*(\mathbb{C}))) \) induces a mixed Hodge structure on \( H^n(Y, p_*(\mathbb{C})) \) in \( \text{Mod}(N(G))/\tau \).

**Proof** This is a consequence of the Theorem 2.43 and general theorems on cohomological mixed Hodge complexes (see [12, 44, Th.3.18 (I) and (II))].

Let \( K_Y := ((p_*(\mathbb{R})), (\Pi_*(p_*(\mathbb{R}))), W), (s(p_*(\Omega_Y)), W, F), i) \). The complex of sheaves \( (\Pi_*(p_*(\mathbb{C})), \delta^{p,q}_q) \) is a resolution of \( p_*(\mathbb{C}) \) for it is isomorphic to the usual Mayer–Vietoris resolution of \( Y \) [20, 3.5.4] tensorised by the locally constant sheaf \( p_*(\mathbb{C}) \). Theorem 3.15 implies that \( (R\Gamma(Y, Gr^W_{\tau q} K_Y), F) \) is a \( \tau \)-Hodge complex of

\( \copyright \) Springer
weight $-q$ such that $H^n R \Gamma (Y, Gr^W_{-q} K_Y) \simeq H^n_{\mathbb{R}}(Y_q, p_{\ast}(2) \mathbb{C})$. But $i \otimes 1_{\mathbb{C}}$ is a quasi-isomorphism. Hence $K_Y$ is a $\tau$-cohomological Hodge complex of sheaves and $(R \Gamma (Y, p_{\ast}(2) \mathbb{R}), (R \Gamma (Y, \Pi \ast p_{\ast}(2) \mathbb{R})_Y), W), (R \Gamma (Y, s(p_{\ast}(2) \omega_Y), W, F), R \Gamma (i))$ is a $\tau$-mixed Hodge complex.

$\square$

Example 3.19 Assume the irreducible components of $p^{-1}(Y)$ are compact. Then $\tau$ may be the trivial torsion theory.

4 Mixed Hodge structures on the complement of a normal crossing divisor

We follow now the strategy of Deligne [11] to put a mixed Hodge structure on the $l^2$-cohomology groups associated to $p : \tilde{X} \setminus p^{-1}(D) \to X \setminus D$.

In Sect. 4.1, we prove that the Leray spectral sequence which abuts to $H^\ast (X \setminus D, p_{\ast}(2) \mathbb{C})$ is isomorphic to the weight spectral sequence associated to the complex of square integrable forms with logarithmic singularity along the pullback of $D$.

In Sect. 4.2, we use the existence of the Hodge structure mod $\tau$ on the $l^2$-cohomology of the compact manifolds $X, D_i, D_i \cap D_j, \ldots$ as it was developed in the previous section. We prove that the weight spectral sequence is a spectral sequence of Hodge structures mod $\tau$ which degenerates at $E_2$.

In Sect. 4.4, we give a description of the first page in the weight spectral sequence in term of the homology associated to the $l^2$-Gysin morphisms of the inclusions $\ldots \to D_i \cap D_j \to D_i \to X$. Section 5.1 will provide additional informations.

4.1 Local setting We refer to [11] and [44, Chap. 4]. for this section. Let $X$ be a complex manifold and $D \subset X$ be a normal crossing divisor. Let $j : U = X \setminus D \to X$ be the injection. Let $\Omega^p_X (\log D)$ be the $\mathcal{O}_X$-subsheaf of $j_{\ast} \Omega^p_U$ of meromorphic forms with logarithmic poles on $D : \alpha \in \Omega^p_X (\log D)$ if $\alpha$ and $d\alpha := j_{\ast} d^j_{\ast} \alpha$ have pole of order at most one on $D$. Let $(z) : V \to D(0, 1)^n$ be a holomorphic chart centered at $x \in X$ such that $D \cap V = \{ z_1, \ldots, z_k = 0 \}$. Then $\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}, dz_{k+1}, \ldots, dz_n$ is a free basis of $\Omega^1_X (\log D)_x$ and $\Omega^p_X (\log D) = \wedge^p \Omega^1_X (\log D)$ is a locally free $\mathcal{O}_X$-sheaf.

Let $(\wedge \{ \frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k} \})$ be the free antisymmetric $\mathbb{C}$-algebra built on $\frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k}$. The weight filtration $\mathcal{W}$ is defined by

$$W_m \Omega^p_X (\log D) = \begin{cases} 0 & \text{for } m < 0 \\ \Omega^\ast_X \wedge \Omega^m_X (\log D) & \text{for } 0 \leq m \leq p \\ \Omega^p_X (\log D) & \text{for } p \leq m \end{cases}$$

4.1.1 Residues. Let $D = \bigcup_{t \in T} D_t$ be the decomposition of $D$ into smooth irreducible components. If $I$ is a subset of $T$, let $a_I : D_I = \bigcap_{t \in I} D_t \to X$ be the natural injection. Let $m \in \mathbb{N} \setminus \{ 0 \}$. Then set $D_m = \bigcup_{|I|=m} D_I$ and $a_m = \bigcup_{|I|=m} a_I$. Define $D_\emptyset = D_0 = X, a_\emptyset = a_0 = 1d$. Let $t \in T$. The residu map

$$Res_t = Res_{D_t} : \Omega_X (\log D) \to (a_t)_{\ast} \Omega_{D_t}^{-1} \left( \log D_t \cap \left( \bigcup_{j \neq t} D_j \right) \right)$$
is defined in a coordinate neighborhood \((U, (z))\) : \(Res_{D_t}(\eta \wedge \frac{dz_i}{z_i} + \eta') = \eta|_{D_t}\) if \(D_t \cap U = \{z_i = 0\}\) and \(\eta\) and \(\eta'\) do not contain \(\frac{dz_i}{z_i}\). Then \(Res_{D_t}\) is \(\mathcal{O}_X\)-linear and commutes with \(d\).

Let \(I = (t_1, \ldots, t_k)\) be an ordered \(k\)-tuple. Define

\[
Res_I = Res_{t_1} \circ \cdots \circ Res_{t_k} : \Omega_X^*(\log D) \to (a_I)_*\Omega_{D_I}^{-m} \left( \log D_I \cap \left( \sum_{j \notin I} D_j \right) \right)
\]

Then \(Res_I\) is \(\mathcal{O}_X\)-linear and commutes with \(d\). Let \(\Lambda \cdot T\) be the free antisymmetric \(\mathbb{C}\)-algebra on \(T\). We still denote by \(\Lambda \cdot T\) the constant sheaf on \(X\) it defines. Let \(I \in T^k\), set \(\Lambda^I t := t_1 \wedge \cdots \wedge t_k\). Let \(I\) be a \(k\)-tuples of distincts elements and \(J = \sigma(I)\) be a permutation of \(I\). Then

\[
\Lambda^I t \otimes Res_I = \Lambda^J t \otimes Res_J
\]

for \(Res_{\sigma(I)} = \epsilon(\sigma)Res_I\) with \(\epsilon(\sigma)\) the signature of \(\sigma\). If \(I = (t_1, \ldots, t_t) \in T^t\), set \(\{I\} = \{t_1, \ldots, t_t\} \subset T\). Then if \(\sharp\{I\} = l\), there is a well-defined map

\[
Res_{\{I\}} := \Lambda^I t \otimes Res_I : \Omega_X^*(\log D) \to (a_I)_*\Omega_{D_I}^{-l} \left( \log D_I \cap \left( \sum_{j \notin I} D_j \right) \right) \otimes \Lambda^t T.
\]

One sets \(Res_m = \bigoplus_{|\{I\}|=m} Res_{\{I\}}\).

Choice of an ordering on \(T\) gives a trivialisation \(Res_m \simeq \bigoplus_{1 \leq i < \cdots < m} Res_{(t_1, \ldots, t_m)}\).

From [11] 3.1.5.2, the map

\[
Res_m : (Gr^W_m \Omega_X^*(\log D), d) \to (a_m)_*(\Omega_{D_m}^{-m}, d) \otimes \Lambda^m T
\]

(16)

is an isomorphism of complexes.

Let \(p : \tilde{X} \to X\) be a covering. If \(f : Y \to X\) is a continuous map, let \(f^*p : \tilde{Y} \to Y\) be the induced covering.

**Lemma 4.1** The residu morphism \(Res_m\) induces an isomorphism

\[
p_{*(2)}Res_m : (Gr^W_mp_{*(2)}\Omega_X^*(\log D), d) \to (a_m)_*(a_m^*p_{*(2)}\Omega_{D_m}^{-m}, d) \otimes \Lambda^m T.
\]

**Proof** From [8], the functor \(p_{*(2)}\) is exact from the category of coherent sheaves of \(\mathcal{O}\)-modules to the category of sheaves on \(X\). Moreover the natural map \(p_{*(2)}(a_m)_*\Omega_{D_m}^{-m} \to (a_m)_*(a_m^*p_{*(2)}\Omega_{D_m}^{-m})\) is an isomorphism [8, prop. 2.9]. Therefore (16) implies that

\[
0 \to (p_{*(2)}W_{m-1}\Omega_X^*(\log D), d) \to (p_{*(2)}W_m\Omega_X^*(\log D), d)
\]

\[
\to (a_m)_*((a_m^*p_{*(2)}\Omega_{D_m}^{-m}, d) \otimes \Lambda^m T \to 0
\]

is exact. A computation in a local chart prove that the last maps define a map of differential complexes. \(\square\)
From Lemma 2.37, the following complexes are well defined (see 2.7.1 for the \( \tau_{\leq} \)-filtration).

**Proposition 4.2** (1) *The maps of filtered complexes*

\[
(p_{\ast}(2)\Omega_X \log D, W, d) \xleftarrow{\alpha} (p_{\ast}(2)\Omega_X \log D, \tau_{\leq}, d) \xrightarrow{\beta} (j_{\ast}(j^\ast p)_{\ast}(2)\Omega_X \log D, \tau_{\leq}, d) \tag{17}
\]

are filtered quasi-isomorphisms.

(2) *This defines an isomorphism between the Leray spectral sequence for \( j_{\ast}(j^\ast p)_{\ast}(2) \)
\( \mathbb{C} \) and the spectral sequence for the hypercohomology of the filtered complex \( (p_{\ast}(2)\Omega_X \log D, W, d) \).

(3) *One deduces the \( N(G) \)-isomorphisms:

\[
\mathbb{H}^r (X, (p_{\ast}(2)\Omega_X \log D, d)) \cong \mathbb{H}^r (X, j_{\ast}(j^\ast p)_{\ast}(2)\Omega_X \log D, d)) \\
\cong \mathbb{H}^r (X \setminus \Omega_1, j_{\ast}(j^\ast p)_{\ast}(2)\Omega_X \log D, d)) \cong H^r (X \setminus \Omega_1, p_{\ast}(2)\mathbb{C}).
\]

**Proof** The corresponding statement for a trivial covering map is proposition 3.1.8 of [11]. Let \((z) : V \to D(0, 1)^n\) be a chart in \(X\) such that \(D \cap V = \{z_1 = \cdots = z_k = 0\}\). Let \(x\) be the center of the chart.

Define a residue map \( R_m : \Gamma(V \setminus D, \Omega^m) \cap \Ker(d) \to \mathbb{C}^{c(m,k)} \) through integration on \(m\)-cycles \(\{|z_1| = \epsilon_1, \ldots, |z_m| = \epsilon_m\}, 1 \leq i_1 < \cdots < i_m \leq k\).

It is known (see Griffiths–Harris [28]) that \(d\Gamma(V \setminus D, \Omega^{m-1}) = \Ker(R_m)\). An explicit (continuous) left inverse \(S_m\) is constructed in [28, p 451–452]. The construction given there works for any closed holomorphic form, and is not restricted to polar singularities. It maps logarithmic forms to logarithmic forms.

Fix a trivialisation of \(G\)-coverings \(p^{-1}(V) \cong V \times p^{-1}(x)\). This defines

\[
\mathbb{H}^r (V, (p_{\ast}(2)\Omega_X \log D, d)) \cap \Ker(d) \to l^2(p^{-1}(x))^c(m,k)
\]

and \(S_m : \Ker(R_m) \to \Gamma(p^{-1}(V \setminus D), p_{\ast}(2)\Omega^{m-1})\) a left inverse of \(d\) which maps logarithmic forms to logarithmic forms. Then

\[
l^2(p^{-1}(x), \mathbb{C}) \otimes \left( \Lambda^m \left[ \frac{dz_1}{z_1}, \ldots, \frac{dz_k}{z_k} \right] \right) \to H^m (\Gamma(V, p_{\ast}(2)\Omega_X \log D), d) \\
\to H^m (\Gamma(V \setminus D, p_{\ast}(2)\Omega^\prime), d)
\]

are isomorphisms.

(1) Maps \(\alpha\) and \(\beta\) are filtered morphisms. We proved that \(\beta : (p_{\ast}(2)\Omega_X \log D, d) \to (j_{\ast}(j^\ast p)_{\ast}(2)\Omega_D^\prime, d)\) is a quasi-isomorphism hence a filtered quasi-isomorphism for the canonical filtration.

Now \(\hat{D}_m \to D_m\) is a covering of a manifold. Hence, using (Lemma 4.1), the sheaf

\[
\mathcal{H}^r (\mathcal{G}_m^W (p_{\ast}(2)\Omega_X \log D, d)) \cong \mathcal{H}^r ((a_{m,p})_{\ast}(2)\Omega_{D_{\hat{m}}}^{-m}, d)
\]
is vanishing if \( i \neq m \) and is isomorphic (as \( \mathcal{N}(G) \)-sheaf) to \( (a_{m}^{*})_{\tau(2)}C_{D_{m}} \) if \( i = m \) (Lemma 2.37).

This implies that \( H^{m}((p_{*}(2)\Omega_{X}(log D), d)) \cong a_{*}(a_{m}^{*})_{\tau(2)}C_{D_{m}} \) as \( \mathcal{N}(G) \)-sheaves. Hence \( \alpha \) is a filtered quasi-isomorphism.

(2) Note that \( ((j^{*})p)_{\tau(2)}O_{U}, d \) is a \( \mathcal{N}(G) \)-resolution of \( (j^{*})p_{(2)}C \) by \( j_{\tau} \)-acyclic sheaves: This follows from [8, th. 3.6] and \( j_{\tau} \) is a Stein morphism. Then \( R^{m}j_{\tau}((j^{*})p_{\tau(2)}C) \) is isomorphic to \( H^{m}((j^{*})p_{\tau(2)}O_{U}, \theta)) \) as \( \mathcal{N}(G) \)-sheaves. This defines a pseudo-isomorphism \( (Rj_{\tau}(j^{*})p_{\tau(2)}C, \tau_{\leq}) = (j_{\tau}C((j^{*})p_{\tau(2)}C, \tau_{\leq})) \rightarrow (j_{\tau}(j^{*})p_{(2)}\Omega_{X\backslash D}, d) \) which composed with \( E_{r}(\alpha) \circ E_{r}(\beta)^{-1} \) induces an isomorphism of spectral sequences.

(3) The quasi-isomorphism \( (p_{*}(2)\Omega_{X}(log D), d) \rightarrow (j_{\tau}(j^{*})p_{\tau(2)}\Omega_{X\backslash D}, d) \) implies
\[
H^{i}(X, p_{*}(2)\Omega_{X}(log D)) \cong H^{i}(X, j_{\tau}(j^{*})p_{\tau(2)}\Omega_{X\backslash D}) \cong H^{i}(X\backslash D, (j^{*})p_{\tau(2)}\Omega_{X\backslash D}) \cong H^{i}(X\backslash D, (j^{*})p_{\tau(2)}C) \] as \( N(G) \)-modules.

\[\square\]

**Definition 4.3** Let \( \tau_{\bar{D}_{\bar{j}}} \) be the Serre category generated by \( \cup_{1 \leq u \leq} \tau_{j_{\bar{i}}, \bar{D}_{\bar{f}}} \) (see Definition 3.6).

**Lemma 4.4** (1) In the following diagram, the maps \( f_{1} \) and \( f_{2} \) are filtered quasi-isomorphisms:
\[
(Rj_{\tau}(j^{*})p_{\tau(2)}R, \tau_{\leq}) \xrightarrow{i} (Rj_{\tau}(j^{*})p_{\tau(2)}C, \tau_{\leq}) \xrightarrow{f_{1}} (Rj_{\tau}(j^{*})p_{\tau(2)}\Omega_{X\backslash D}, \tau_{\leq}, d) \]
\[
\xrightarrow{f_{2}} (p_{\tau(2)}\Omega_{X}(log D), W, d) \xrightarrow{\alpha} (p_{\tau(2)}\Omega_{X}(log D), \tau_{\leq}, d) \xrightarrow{\beta} (j_{\tau}(j^{*})p_{\tau(2)}\Omega_{X\backslash D}, \tau_{\leq}, d) \]

(2) This defines the second comparison morphism \( \tilde{\beta} : (Rj_{\tau}(j^{*})p_{\tau(2)}R, \tau_{\leq}) \rightarrow (p_{\tau(2)}\Omega_{X}(log D), W, d) \) such that \( \tilde{\beta} \otimes 1_{C} : (Rj_{\tau}(j^{*})p_{\tau(2)}C, \tau_{\leq}) \rightarrow (p_{\tau(2)}\Omega_{X}(log D), W, d) \) is a pseudo-isomorphism.

(3) Let \( \tau \) be a real torsion theory greater than \( \gamma^{*}\tau_{\bar{D}_{\bar{j}}} \) (Definition 2.23). Assume that \( X \) is a compact Kähler manifold. Then \( K := [(Rj_{\tau}(j^{*})p_{\tau(2)}R, \tau_{\leq} = W); (p_{\tau(2)}\Omega_{X}(log D), W, F, \beta) \] is a \( \tau \)-mixed Hodge complex of \( \mathcal{N}(G, R) \)-sheaves. Hence \( R\Gamma(Gr^{W}_{m}K) \) is a Hodge complex of weight \( m \) in \( \text{Mod}(N(G, C))_{/\tau} \) (see Definition 2.40).

**Proof** (1) According to Sect. 2.1.1,
\[
Rj_{\tau}((j^{*})p)_{\tau(2)}R \otimes C \xrightarrow{i^{\otimes}1_{C}} Rj_{\tau}((j^{*})p)_{\tau(2)}C \]
is a \( \mathcal{N}(G) \)-isomorphism. This defines a filtered isomorphism for the canonical filtration. From Proposition 4.2 and exactness of the Godement resolution, \( f_{1} \) is a quasi-isomorphism. Also \( f_{2} \) is quasi-isomorphism. Hence \( f_{1}, f_{2} \) are filtered quasi-isomorphisms for the canonical filtration.
Lemma 4.5 The residue morphism

\[ p_{\ast(2)} \text{Res}_m : Gr^W_m \tau_p \Omega_X(\log D) \to a_m(\tau_p) \Sigma_{D_m} \otimes \Lambda^m T \]

maps \( R^m j_{\ast} p_{\ast(2)} \mathbb{R} [-m] \xrightarrow{q_i} Gr^\tau_m \tau_p \mathbb{R} \) to \((2i\pi)^{-m} a_m(\tau_p) \otimes \Lambda^m T [-m]. \)

(3,a) The following lemma is proved as in [11, prop. 3.1.9].

Lemma 4.5 The residue morphism

\[ p_{\ast(2)} \text{Res}_m : Gr^W_m \tau_p \Omega_X(\log D) \to a_m(\tau_p) \Sigma_{D_m} \otimes \Lambda^m T \]

maps \( R^m j_{\ast} p_{\ast(2)} \mathbb{R} [-m] \xrightarrow{q_i} Gr^\tau_m \tau_p \mathbb{R} \) to \((2i\pi)^{-m} a_m(\tau_p) \otimes \Lambda^m T [-m]. \)

(3,b) According to Example 3.16, \( \mathcal{K}_{D_m} \) defined by

\[
(2i\pi)^{-m} \mathbb{Z} \otimes (a_m p)_{\ast(2)} \mathbb{R} [-m], \\
((a_m p)_{\ast(2)} \Sigma_{D_m} [-m], F_{D_m} [-m], d), (2i\pi)^{-m} \alpha_{D_m}) \tag{18}
\]

is a \( \mathcal{N}(G, \mathbb{R}) \)-Hodge complex of sheaves of weight \( m \) modulo \( \tau \). From Lemma 4.1, we deduce that \( (R^m j_{\ast} p_{\ast(2)} \mathbb{R} [-m], Gr^W_m \tau_p \Omega_X(\log D), Gr^\tau_m(\tilde{\beta})) \) is a cohomological Hodge complex of sheaves of weight \( m \) modulo \( \tau : R\Gamma(Gr^W_{\tau p} \mathcal{K}) \simeq R\Gamma(\mathcal{K}_{D_m}) \) is a Hodge complex of weight \( m \).

4.2 Global setting

One gets our first theorem on mixed Hodge structure on \( L^2 \)-cohomology (compare with [12, §8] and [44, th. 3.18]):

Theorem 4.6 Let \( p : \tilde{X} \to X \) be a covering of a compact Kähler manifold with covering transformations group \( G \). Let \( \Gamma \) be the global section functor over \( X \), from the category of \( \mathcal{N}(G, \mathbb{R}) \)-sheaves of modules (resp. \( \mathcal{N}(G) \)-sheaves of modules) to the category of \( N(G, \mathbb{R}) \)-modules (resp. \( N(G) \)-modules). Let \( R\Gamma \) be its derived functor realized through the Godement resolution.

Assume that a torsion theory \( \tau \) on \( N(G, \mathbb{R}) \) is chosen so that for each \( p \in \mathbb{Z} \),

\[ R\Gamma(Gr^W_{\tau p} \mathcal{K}) := [(R\Gamma(R^p j_{\ast} p_{\ast(2)} \mathbb{R})), (R\Gamma(Gr^W_{\tau p} \Omega_X(\log D), F))], R\Gamma(Gr^W_{\tau p} \tilde{\beta}) \]

is a Hodge complex in \( N(G, \mathbb{R})/\tau \), then

\[ R\Gamma(\mathcal{K}) := [(R\Gamma(j_{\ast} p_{\ast(2)} \mathbb{R}, \tau_\leq), (R\Gamma(p_{\ast(2)} \Omega_X(\log D), W, F), R\Gamma(\tilde{\beta})) \]

is a mixed Hodge complex in \( N(G, \mathbb{R})/\tau \). Therefore:

(i) The spectral sequence for \( (R\Gamma(p_{\ast(2)} \Omega_X(\log D)), W, d), \) which \( E_1^{p,q} \)-term is \( H^{q-p}(Gr^W_{\tau p} \Omega_X(\log D), d) \) degenerates at \( E_2 \) in \( N(G, \mathbb{C})/\tau \otimes \mathbb{C}. \)
(ii) The differential $d_1$ on $E_1^{-p,q} (R\Gamma (p \ast (2) \Omega_X (\log D)), W) \simeq H^{-2p+q} (D_p, (a_\ast^p p) \ast (2) \mathbb{C})$ is real, and is a morphism of Hodge structures in $\text{Mod}(N(G)) / \tau$.

(iii) Through the isomorphism

$$Gr_p^{\tau'} H^{-p+q} (X \setminus D, p \ast (2) \mathbb{R}) \otimes \mathbb{C} \simeq Gr_p^W H^{-p+q} (X, (p \ast (2) \Omega_X (\log D), d)),$$

the Hodge filtration induces a Hodge structure (modulo $\tau$). It is the same that the Hodge structure induced by the isomorphism $E_2 (R\Gamma (p \ast (2) \Omega_X (\log D)), W) \simeq E_\infty (R\Gamma (p \ast (2) \Omega_X (\log D)), W)$.

(iv) The spectral sequence in $N(G, \mathbb{C}) / \tau \otimes \mathbb{C}$

$$E_1^{-p,q} (R\Gamma (p \ast (2) \Omega_X (\log D)), F) \simeq H^q (X, p \ast (2) \Omega_X^p (\log D))$$

$$\Rightarrow \mathbb{H}^{p+q} (X, p \ast (2) \Omega_X (\log D))$$

degenerates at $E_1$.

(v) Define the weight filtration $W$ on $H^k (X \setminus D, p \ast (2) \mathbb{C})$, to be the shifted filtration $k$ step to the left of the filtration induced by the Leray spectral sequence:

$$W. H^k (X \setminus D, p \ast (2) \mathbb{R}) = \text{Im} (\mathbb{H}^k (X, \tau_\leq, -k Rj_* p \ast (2) \mathbb{R}) \to H^k (X \setminus D, p \ast (2) \mathbb{R})).$$

It is equal to $\text{Im} (\mathbb{H}^k (X, (W_\leq -k p \ast (2) \Omega_X^1 (\log D), d)) \to \mathbb{H}^k (X \setminus D, p \ast (2) \mathbb{C})).$

Let $F$ be the Hodge filtration on $H^k (X \setminus D, p \ast (2) \mathbb{C})$, then $(H^k (X \setminus D, p \ast (2) \mathbb{R}, W, F)$ is a mixed Hodge structure in $\text{Mod}(N(G)) / \tau$.

**Proof** Recall that $\mathcal{C} (\cdot)$ is an exact functor to the category of flabby sheaves and that $R\Gamma (\cdot) := \Gamma (X, \mathcal{C} (\cdot))$ is an exact functor from the category of complex of $\mathcal{N}(G)$-sheaves on $X$ to the category of $N(G)$-complexes. Consider the following diagram:

$$\begin{array}{cccc}
R\Gamma (\tilde{\beta}) : & (R\Gamma p \ast (2) \Omega_X (\log D), W, d) & \xrightarrow{g_3} & (R\Gamma (p \ast (2) \Omega_X (\log D)), \tau_\leq, d) \\
(R\Gamma R j_\ast (j^* p) \ast (2) \mathbb{R})_{\ast \leq} & (R\Gamma R j_\ast (j^* p) \ast (2) \mathbb{C}, \tau_\leq) & \xrightarrow{R\Gamma (f_1)} & (R\Gamma R j_\ast (j^* p) \ast (2) \Omega^2, \tau_\leq, d) \\
R\Gamma (\beta) : & (R\Gamma R j_\ast (j^* p) \ast (2) \mathbb{R}, \tau_\leq) & \xrightarrow{R\Gamma (f_2)} & (R\Gamma (p \ast (2) \Omega_X (\log D)), \tau_\leq, d) \\
\end{array}$$

Set $g_1 = R\Gamma (f_1) \circ R\Gamma (i)$. Then $g_{1,C} = g_1 \otimes 1_C$. Referring to the proof of Theorem 2.43, defines $g_2 = g_2 \circ g_3$ and $\beta_1 = g_1$. This defines pseudo-morphisms

$$\beta = R\Gamma (\tilde{\beta}) : (R\Gamma R j_\ast (j^* p) \ast (2) \mathbb{R}, \tau_\leq) \dashrightarrow (R\Gamma p \ast (2) \Omega_X (\log D), W, d) \hspace{1cm} (19)$$

$$\beta_C : (R\Gamma R j_\ast (j^* p) \ast (2) \mathbb{C}, \tau_\leq) \dashrightarrow (R\Gamma p \ast (2) \Omega_X (\log D), W, d). \hspace{1cm} (20)$$

Using the exactness of $R\Gamma$ and the previous lemma, one sees that $\beta_C$ is a pseudo-isomorphism.
Moreover
\[ E_1^{p,q}(R\Gamma K) \simeq H^{p+q}(R\Gamma (G_i^W K)) \simeq H^{p+q}(R\Gamma (K_{D_p})) \]
has a Hodge structure of weight \( q \) modulo \( \tau \) [see (18)]. One concludes from Theorem 2.43 and Section 7.2 of [12].

Example 4.7 (1) Let \( \tau_{\tilde{\partial},\tilde{D}} \) be the Serre category generated by
\[
\begin{align*}
\Im \tilde{\partial}, \tilde{\partial} : \bigoplus_{p,q} \iota S^{k-q,(p,q)}(p^{-1}(D)) & \to \bigoplus_{p,q} \iota S^{k-q-1,(p,q+1)}(p^{-1}(D))
\end{align*}
\]
where \( k \) is an integer big enough and \( D_{\tilde{\partial}} = \tilde{X} \). Then \( \gamma^*\tau_{\tilde{\partial},\tilde{D}} \) (Definition 2.23) is the smallest torsion theory which fulfills the above assumptions.

(2) Assume that \( p : \tilde{X} \to X \) is a Galois covering, then the torsion theory associated to the dimension function or the \( U(G) \)-torsion modules fulfills the above assumptions.

4.3 Interpretation in equivariant cohomology We give three isomorphisms for the cohomology groups of the local system \( p_{s(2)}\mathbb{C} \) over \( X \setminus D \) (see Sect. 2.6.1). For simplicity, we assume that \( p : \tilde{X} \to X \) is a Galois covering.

One knows that \( H^k(X\setminus D, p_{s(2)}\mathbb{C}) \) is isomorphic to \( H^k(\text{Hom}_{\mathbb{C}[G]}(C,(p^{-1}(X\setminus D)), l^2(G))) \), the singular equivariant cohomology of \( p^{-1}(X\setminus D) \) with coefficient in \( l^2(G) \) [19].

\( D \) admits a basis of neighborhood \( V \) such that \( V \) retracts onto \( D \) and \( X \setminus D \) retracts onto \( X \setminus V \). One may assume that \( \tilde{V} \) and \( X \setminus V \) are manifolds with boundary. Using the invariance of cohomology of a locally constant sheaf through homotopy [38, cor.I.3.5], we get that \( H^k(X\setminus D, p_{s(2)}\mathbb{C}) \to H^k(X\setminus V, p_{s(2)}\mathbb{C}) \) is an isomorphism. Using a triangulation \( T \) of the manifold with boundary \( X \setminus V \), we get an isomorphism \( H^k(X\setminus V, p_{s(2)}\mathbb{C}) \simeq H^k(\text{Hom}_{\mathbb{C}[G]}(\tilde{T}, l^2(G))) \) as in Corollary 3.14.

To obtain a direct combinatorial description of the group \( H^k(X\setminus D, p_{s(2)}\mathbb{C}) \), we use the Leray cover defined by the open stars of a triangulation of \( X \): Let \( f : K \to X \) be a \( C^1 \)-triangulation of \( X \) [63] such that \( D \) is a subcomplex. Let \( v \in K_0 \) be a vertex in \( K \).

Let \( U_v = St(v, K) \) be the open star of \( v \) in \( K \) (Munkres [39, §.2]). The open covering \( \mathcal{M} = \{ U_v, v \in K_0 \setminus D \} \) of \( X \setminus D \) is such that any finite intersection of its elements is acyclic for any locally constant sheaf. Let \( K(\neg) \) be the subcomplex of simplexes \( \sigma \) of \( K \) such that \( \sigma \) does not intersect \( D \). Then the nerve of \( \mathcal{M} \) is isomorphic as a simplicial set to \( K(\neg D) \). Let \( \tilde{K}(\neg D) \) and \( \tilde{K} \) be the pullback simplicial complexes. Using Leray’s theorem ([4, III.4.13] or [26, II.5.2.4]), we get as in the proof of Corollary 3.14,
\[ H^k(X\setminus D, p_{s(2)}\mathbb{C}) \simeq_{N(G)} \tilde{H}^k(\mathcal{M}, p_{s(2)}\mathbb{C}) \simeq_{N(G)} H^k(\text{Hom}_{\mathbb{C}[G]}(\tilde{K}(\neg D), l^2(G))). \]
This description as the homology of a complex of $G$-Hilbert modules allows one to define a reduced cohomology which is isomorphic to the corresponding harmonic space ([36, 1.1.4], [54]). The two last complexes have finite von Neumann dimensions. Hence their reduced cohomology are isomorphic to their cohomology in $\text{Mod}(N(G))/\tau_{dG}$ or $\text{Mod}(N(G))/\tau_{\text{dim}}$.

**Corollary 4.8** There exists a mixed Hodge structure on $H^k(\text{Hom}_{\mathbb{C}[G]}(C,(p^{-1}(X \setminus D)), i^2(G)))$ in $\text{Mod}(N(G))/\gamma^{*}\tau_{G,D}$. In particular in $\text{Mod}(N(G))/\tau_{\text{dim}}$, there exists a mixed Hodge structure on $\tilde{H}^k_{(2)}(K(\neg D))$ (reduced cohomology).

### 4.4 Interpretation of $(E_1, d_1)$

One aim of this section is formula (23) below which interprets $d_1$ through the Gysin morphisms. For this, we use a smooth logarithmic complex as in Griffiths–Schmid [29, p. 73].

**Definition 4.9** Let $p_{\ast}(2)S^{\infty \cdot}(\log D)$ be the subsheaf of $j_{\ast}(j^{*}p_{\ast}(2)S^{\infty \cdot}A)$ such that a germ $\alpha$ belongs to $[p_{\ast}(2)S^{\infty \cdot}(\log D)]_{x}$ if

$$h\alpha \in [p_{\ast}(2)S^{\infty \cdot}A]_{x} \text{ and } h\partial \alpha \in [p_{\ast}(2)S^{\infty \cdot}A]_{x}$$

(21)

where $h$ is a defining equation for $D$ at $x$.

From its definition, $(p_{\ast}(2)S^{\infty \cdot}(\log D), d)$ is a complex. Moreover

**Lemma 4.10** $p_{\ast}(2)S^{\infty \cdot}(\log D) = p_{\ast}(2)S^{\infty \cdot}A \otimes_{\mathcal{O}} \Omega_{X}(\log D)$.

**Proof** One first proves the corresponding result in a coordinate chart $(V, (z))$ for a trivial one-sheeted covering. This will give estimates that carries over to the local situation of a trivialised covering $p^{-1}(V) \simeq V \times p^{-1}(x)$.

(1) Let us first recall a proof in a coordinate chart $(V, (z_1, \ldots , z_n))$: The hypothesis implies that $h\alpha$ and $(h\alpha) \wedge \frac{\partial h}{\partial \overline{z}}$ extends as smooth forms. Assume that $h = z_1 \ldots z_k$ in $(V, (z))$ and that $h\alpha = \sum I \beta_I d^I$ where $I$ is a subset of $K = \{1, \ldots , k\}$ and the $\beta_I$'s are smooth forms on $V$ which belong to the exterior algebra generated by $dz_{i+1}, \ldots , dz_n$. Then $\gamma = (h\alpha) \wedge \frac{\partial h}{\partial \overline{z}} = \sum_{i \in I} \frac{\beta_I}{z_i} d^I \wedge dz_i$ is a smooth form on $V \setminus \{z_1 = 0\}$ such that any of its partial derivative $\frac{\partial |A|}{\partial z^i} \beta_I$ admits a limit on $V \cap D$. Let $p \in (z_1 = 0) \setminus (z_2, \ldots , z_k = 0)$. Then

$$0 = \lim_{V \setminus D \ni p \to p} z_1 \frac{\partial |B|}{\partial \overline{z}^B} \gamma = \lim_{V \setminus D \ni p \to p} \sum_{i \in I} \frac{z_1}{z_i} \frac{\partial |B|}{\partial \overline{z}^B} \beta_I d^I \wedge dz_i$$

$$= \sum_{I \neq 1} \frac{\partial |B|}{\partial \overline{z}^B} \beta_I (p) dz^I \wedge dz_i$$

Hence if $I \neq 1$ then $\frac{\partial |B|}{\partial \overline{z}^B} \beta_I (p) = 0$. By continuity, this holds for any $p \in \{z_1 = 0\}$. This implies that $\beta_I \in z_1 A(V)$ (Malgrange [37], Schwartz [50, th.2]). In the same way we see that if $i \neq 1$ then $\beta_I \in z_i A(V)$. Hence $\beta_I \in z^{K-I} A(V)$ by [50, th.3] and $\alpha = \sum_{I} \frac{\beta_I}{z_k-I} dz_i := \sum_{I} \alpha_I \frac{dz_i}{z_i}$ belongs to $A \otimes_{\mathcal{O}} \Omega_{X}(\log D)(V)$.® Springer
(2) But \( z^{K-l} A(V) \) is a closed ideal, therefore the surjective maps \( \alpha_I \mapsto z^{K-l} \alpha_I \) are open: for all compact subset \( K_1 \subset V, \) all \( m_1 \in \mathbb{N}, \) all \( I \subset \{1, \ldots, k\}, \) there exists compact subsets \( K_2 \subset K_3 \subset V, \) integers \( m_2 \leq m_3 \) and positive numbers \( C_1, C_2 \) such that \( \| \alpha_I \|_{C^m(K_1)} \leq C_1 \| z^{K-l} \alpha_I \|_{C^m(K_2)} \leq C_2 \| h \alpha \|_{H^{m_3}(K_3)}. \) The last norm is the Sobolev norm of order \( m_3. \) This implies that \( p_{*}(2)S^\infty \Omega^\ast_X(\log D) = p_{*}(2)S^\infty A^\ast \otimes \Omega^\ast_X(\log D). \)

**Lemma 4.11** The sheaf \( p_{*}(2)S^\infty A^0 \) is a flat \( \mathcal{O}_X \)-module.

**Proof** The statement for a trivial one sheeted covering is the flatness theorem of Malgrange [37]: let \( \mathcal{I}_x \subset \mathcal{O}_x \) be a finitely generated ideal. Let \( V \) be a neighborhood of \( x \) such that \( \mathcal{I}(V) \) admits a finite presentation

\[
\mathcal{O}^k(V) \xrightarrow{r} \mathcal{O}^p(V) \xrightarrow{g} \mathcal{I}(V) \rightarrow 0
\]

(the map \( r \) gives the module of relations and \( g : h \rightarrow \sum_{i=1}^p h_i g_i \) is defined using generators of \( \mathcal{I}(V) \)).

The flatness of \( C^\infty(V) \) on \( \mathcal{O}(V) \) implies that \( (C^\infty(V))^k \xrightarrow{r} (C^\infty(V))^p \xrightarrow{g} \mathcal{I}(V).C^\infty(V) \) is exact. Hence \( \text{Im}(r) \) is closed and the above exact sequence splits. If \( V \) is small enough, the pullback of this splitting provides a splitting of \( p_{*}(2)S^\infty A^0(V)^k \rightarrow p_{*}(2)S^\infty A^0(V)^p \rightarrow \mathcal{I}(V).p_{*}(2)S^\infty A^0(V). \) By Tougeron [58] I.4, we conclude that \( Tor^\mathcal{O}_x^1(\mathcal{O}_x/\mathcal{I}_x, p_{*}(2)S^\infty A^0_x) = 0. \)

**Corollary 4.12** (1) The morphism \( (p_{*}(2)\Omega^\ast_X(\log D), d) \rightarrow (p_{*}(2)S^\infty_{-p}(\log D), d) \) is a quasi-isomorphism. Moreover the morphism

\[
(p_{*}(2)\Omega^\ast_X(\log D), d, W, F) \rightarrow (p_{*}(2)S^\infty_{-p}(\log D), d, W, F)
\]

defines a bi-filtered \( \Gamma \)-acyclic resolution.

(2) Let \( I \) be a \( p \)-tuple of distinct elements in \( T. \) Then there is a map

\[
\text{Res}_I : p_{*}(2)S^\infty_{-p}(\log D) \rightarrow p_{*}S^\infty_{-p} \left( \log D_I \cap \left( \sum_{j \notin I} D_j \right) \right).
\]

(3) The maps in the following diagram are filtered quasi-isomorphisms:

\[
\text{Res}_I : (\text{Gr}^W_I p_{*}(2)\Omega^\ast_X(\log D), d, F) \rightarrow (\text{Gr}^W_I p_{*}(2)S^\infty_{-p}(\log D), d, F)[-l]
\]

\[
\text{Res}_I : (\text{Gr}^W_I p_{*}(2)S^\infty_{-p}(\log D), d, F) \rightarrow (\text{Gr}^W_I p_{*}(2)S^\infty_{-p}(\log D), d, F)[-l]
\]
4.4.1 Gysin morphisms. Let $I \subset T$ be such that $\# I = p$. Fix a (hermitian) metric on $D_I$ and choose an orthogonal splitting $T(X)_{|D_I} \simeq T_{D_I} \oplus N_{D_{I}/X}$. For any $\epsilon > 0$ small enough, the exponential map induces a diffeomorphism $N_{D_{I}/X}(\epsilon) \xrightarrow{\epsilon_I} U_I(\epsilon)$ from a tubular neighborhood of $D_I$ in $N_{D_{I}/X}$ into a neighborhood of $D_I$ in $X$. Define a retraction $r_I : U_I(\epsilon) \to D_I$ by composing $e_I^{-1}$ with the bundle projection $N_{D_{I}/X} \to D_I$. Let $n = \dim X$. One choose $\epsilon_1, \ldots, \epsilon_n$ such that the domains of the maps $r_I : U_I(\epsilon_\rho) \to D_I$, $\# I = p$, satisfy the conditions $\cap_{I \in I, \# I = q} U_I(\epsilon_q) \subset U_I(\epsilon_p)$. Hence the intersection of the tubular neighborhoods of $D_1$ and $D_2$ is contained in the tubular neighborhood of $D_1 \cap D_2$ (for simplicity assume that $X$ is of bounded geometry). Then $r_I$ is a smooth submersion and $r_I^*$ maps local Sobolev space of order $s$ to local Sobolev space of order $s$ (for tensor product of separated variables commutes with the Fourier transform).

Let $s_I$ be a section of $[D_I]$ vanishing on $D_I$, $t \in T$. Let $h_I$ be a hermitian metric on $[D_I]$, such that $|s_I(x)| = 1$ if $x \notin U_I(\epsilon_I')$ with $\epsilon_I' < \epsilon_I$. Let $\eta_I = \frac{1}{2\pi} \partial \log |s_I|^2$. The Poincaré–Lelong formula reads $d[\eta_I] = [D_I] - \omega_t$ where $\omega_t$ is the first Chern form of $([D_I], h_I)$. Assume that $X$ is compact. Then these currents are supported in a compact neighborhood of $D_I$. For $t \in T^p$, define $\eta_I = \eta_{t_1} \wedge \cdots \wedge \eta_{t_p}$. If $t \notin [I]$, we have

$$supp(\omega_t \wedge \eta_I) \subset \subset U_{[I] \cup [I]}(\epsilon_{[I] + 1}).$$ (22)

We lift this construction by the covering map. Note that a connected component of $p^{-1}(U_I(\epsilon_I))$ contains only one connected component of $p^{-1}(D_I)$.

Let us introduce the following notations: for $t \in T^p$ such that $\# [I] = p$, denote $r_{I, t}, \eta_I, \ldots$ the lifts $p^*(r_{I, t}), p^*\eta_I, \ldots$ to $p^{-1}(U_{[I]})$.

Let $[\alpha] \in H^{q-2p}(D_I, (a_I^\gamma)^* S^{k-\cdot , \cdot})$. From Lemma 3.17, we may assume that $\alpha$ belongs to $\Gamma(p^{-1}(D_I), \mathcal{A}^{q-2p}) \cap_{\nu \in [I]} \text{Dom}[(1 + \Delta_{p^{-1}(D_I)})^\nu]$ so that $r_{I, t}^*([\alpha]) \eta_I \in \Gamma(\cap_{\nu \in [I]} U_I(\epsilon_I), (p_{\#(2)^*} S^{\infty, q-2p}(\log D_I)))$. Denote by the same symbol the extension of $r_{I, t}^*([\alpha]) \eta_I$ by zero to $\tilde{X}$. Then $r_{I, t}^*([\alpha]) \eta_I$ is an element of $\Gamma(\tilde{X}, S^{\infty, q-2p}(\log D_I))$ such that $\text{Res}_t(r_{I, t}^*([\alpha]) \eta_I) = \alpha$. From Corollary 4.12, one represents a class in $H^{q-2p-\nu}(X, p_{\#(2)^*}(G_W^{-1} \Omega_{\tilde{X}}(\log D_I), \nu))$ by a form $\frac{1}{p!} \sum_{t} r_{I, t}^*([\alpha]) \eta_I \in \Gamma(\tilde{X}, S^{\infty, q-2p}(\log D_I))$, with the convention that $I \to [I]$ is antisymmetrical in $I$. Hence $\alpha_I = \epsilon(t)\alpha$ where $\epsilon(t)$ is the signature of the permutation $I \to ([t], \hat{I})$. Then,

$$d(r_{I, t}^*([\alpha]) \eta_I) = 0 \quad (-1)^{q-2p} r_{I, t}^*([\alpha]) \sum_{I \in I} \epsilon(t) \omega_t \eta_I = (-1)^{q-2p} \sum_{I \in I} r_{I, t}^*([\alpha], \eta_I)$$

$$d \sum_I r_{I, t}^*([\alpha]) \eta_I = (-1)^{q-2p} \sum_I \sum_J r_{I, J}^*([\alpha, \eta_I]) \omega_t \eta_J$$

$$\text{Res}_J d \sum_I r_{I, t}^*([\alpha]) \eta_I = (-1)^{q-2p} a_J a_J^* \left( \sum_I \omega_t r_{I, J}^*([\alpha, \eta_I]) \right) \otimes \Lambda^J t$$

Therefore

$$d_{I \in T^p} \sum_{I \in T^p} \alpha_I \otimes \Lambda^J t = \sum_{J \in T^{p-1}} (-1)^{q-2p} a_J^* \left( \sum_I \omega_t r_{I, J}^*([\alpha, \eta_I]) \right) \otimes \Lambda^J t$$ (23)

\(\odot\) Springer
4.5 $l^2$-characteristics In this section, we state Atiyah’s equalities between $l^2$-Euler characteristics and Euler characteristics of $X \setminus D$ under the hypothesis that the covering $p : \tilde{X} \to X$ is Galois. Let $\tau_{dim}$ be the torsion theory such that $\mathcal{T}_{dim} = \{ M \in \text{Mod}(N(G)) : \dim_{N(G)} M = 0 \}$ (see Sect. 2.5.3). Theorem 3.15 shows that $\tau_{dim}$ satisfies hypothesis of Theorem 4.6. Hence the weight spectral sequence degenerates at $E_2$ and the Hodge spectral sequence degenerates at $E_1$ in $\text{Mod}(N(G)) / \tau_{dim}$.

**Proposition 4.13** Let $\tilde{X} \to X$ be a $G$-cover of a compact Kähler manifold of dimension $n$ and let $D \to X$ be a normal crossing divisor. Then $H^n(X \setminus D, (j^*p)_{*}(2) \mathbb{C})$ has finite von Neumann dimension. Moreover if $F$ is the Hodge filtration given by the Theorem 4.6 then

$$Gr_F^p H^i(X \setminus D, (j^*p)_{*}(2) \mathbb{C}) = H^i_{\mathcal{J}}(\tilde{X}, \Omega_X^p (\log D)) \text{ in Mod}(N(G)) / \tau_{dim} \quad (24)$$

$$\sum_{i=0}^{n} (-1)^i \dim_{N(G)} Gr_F^p H^i(X \setminus D, (j^*p)_{*}(2) \mathbb{C})$$

$$= \sum_{i=0}^{n} (-1)^i \dim \mathbb{C} Gr_F^p H^i(X \setminus D, \mathbb{C}) \quad (25)$$

$$\chi(2)(p^{-1}(X \setminus D)) = \sum_{i} (-1)^i \chi(2)(p^{-1}(D_i))$$

$$= \sum_{i} (-1)^i \chi(p^{-1}(D_i)) = \chi(X \setminus D). \quad (26)$$

*Proof* Let $\tilde{H}^k_{\mathcal{J}}(\tilde{X}, p^*(\Omega_X^p (\log D)))$ be the Reduced Dolbeaut cohomology group. Then in $\text{Mod}(N(G)) / \tau_{dim}$, we have $H^k(X, p_{*}(2)\Omega_X^p (\log D)) \simeq \tilde{H}^k_{\mathcal{J}}(\tilde{X}, p^*(\Omega_X^p (\log D)))$ (see [8, 3.4 and 3.14]) and

$$Gr_F^p H^i(X \setminus D, (j^*p)_{*}(2) \mathbb{C}) \simeq H^{i-p}(X, p_{*}(2)\Omega_X^p (\log D))$$

$$\simeq H^{i-p}_{\mathcal{J}}(\tilde{X}, p^*(\Omega_X^p (\log D)))$$

is finite $N(G)$-dimensional. Using Atiyah’s theorem (see [2,8]), we get $\chi(2)(\tilde{X}, p^*(\Omega_X^p (\log D))) = \chi(X, \Omega_X^p (\log D))$. But Deligne’s theorem [11] gives $Gr_F^p H^i(X \setminus D, \mathbb{C}) \simeq H^{i-p}(X, \Omega_X^p (\log D))$. Hence

$$\sum_{i=0}^{n} (-1)^i \dim_{N(G)} Gr_F^p H^i(X \setminus D, (j^*p)_{*}(2) \mathbb{C}) = \sum_{i=0}^{n} (-1)^i \dim \mathbb{C} Gr_F^p H^i(X \setminus D, \mathbb{C})$$

$$\sum_{i=0}^{n} (-1)^i \dim_{N(G)} H^i(X \setminus D, (j^*p)_{*}(2) \mathbb{C}) = \sum_{i=0}^{n} (-1)^i \dim \mathbb{C} H^i(X \setminus D, \mathbb{C})$$

Using the weight spectral sequence, we get $\sum_{i,j} (-1)^{i+j} E_1^{i,j} = \sum_{i,j} (-1)^{i+j} E_2^{i,j}$ in the Grothendieck group of the category $\text{Mod}(N(G))$. From Lemma 4.1 and
Theorem 3.15, one gets

$$E^{i,j}_1 \simeq H^{j-2i}(D_1, ((a^*_i p)_{s(2)} \Omega_{D_1}, d)) \simeq H^{j-2i}(D_1, (a^*_i p)_{s(2)} \mathbb{C})$$

This gives the equalities in the last line using additivity of dimension functions. \(\square\)

Note in particular that \(F^n H^k(X \setminus D, (j^* p)_{s(2)} \mathbb{C}) = \text{Gr}_p^n H^k \simeq H^{k-n}(X, p_{s(2)}(K \otimes [D])) = \tilde{H}^{k-n}_{\delta(2)}(\tilde{X}, K_{\tilde{X}} \otimes [p^{-1}(D)])\) in \(\text{Mod}(N(G))/\tau_{\text{dim}}\).

### 4.6 Examples of \((\tau', \tau)\)-cohomological mixed Hodge complex (CMHC)

We have seen that \(\mathcal{K}:=[(R_j^*) (j^* p)_{s(2)} \mathbb{R}, (R_j^*) (j^* p)_{s(2)} \mathbb{R}, \tau), (p_{s(2)} \Omega_X (\log D), W, F), \tilde{\beta}]\) is a \((0, \gamma^* \tau_{\tilde{\beta}}, \tilde{\beta})\)-CMHC (see Lemma 4.4 for a definition of \(\tilde{\beta}\)).

The following complex is functorial under morphism and its behavior under change of group may be studied through homological algebra.

Recall that if \(p : \tilde{X} \to X\) is a \(G\)-covering map between locally compact spaces, then the functor \(p_! p^*\) from the category of sheaves on \(X\) to the category of \(\mathbb{Z}[G]\)-sheaves on \(X\) is exact (Iversen [34, p. 99 and p. 315]). Moreover, we have an isomorphism \(p_! p^*(F) \simeq (p_! \mathbb{Z}) \otimes F\) as \(\mathbb{Z}[G]\)-sheaves.

Then tensoring over \(\mathbb{C}[G]\) by \(N(G)\) gives a functor from the category of \(\mathbb{R}\)-mixed Hodge complexes of sheaves on \(X\) to the category of \(N(G)\)-mixed Hodge complexes of sheaves modulo the dimension function: the functor \(N: [\text{Sheaves over } X] \to [\mathcal{N}(G) - \text{Sheaves over } X]\) given by \(\mathcal{F} \to \mathcal{N}(G) \otimes_{\mathbb{C}[G]} p_! p^* \mathcal{F}\) is exact for a local model is an induced trivial module tensorised with \(\mathbb{C}[G]\). There are natural maps \(\mathcal{N}(G) \otimes_{\mathbb{C}[G]} p_! p^* \mathcal{F} \to i^2(\mathcal{N}(G) \otimes_{\mathbb{C}[G]} p_! p^* \mathcal{F})\) defined by \(n \otimes f \mapsto n(\delta_c) \otimes f\). In the case where \(\mathcal{F}\) is holomorphic coherent or locally constant sheaf, this map takes values into \(p_{s(2)} \mathcal{F}\). But is far from being surjective. However, if we reduce to the category of sheaves in \(\text{Mod}(N(G))/\tau_{\text{dim}}\), then we get from the usual CMHC \(\mathcal{K}\) on \(X\) a CMHC \(\text{mod}(\tau_{\text{dim}}, \tau_{\text{dim}}) \mathcal{N}(G) \otimes p_! p^* \mathcal{K}\).

### 5 Examples

#### 5.1 The \(E_1\) page of the spectral sequences

We give below the first terms of the unreduced \((E_1, d_1)\) complexes of the weight spectral sequence (after a choice of ordering of the divisors) and of the Hodge spectral sequence.

#### 5.1.1 The weight filtration

The opposite of the weight index is the column index. We recall that the shift in the weight filtration on the \(i^2\)-cohomology groups of \(p^{-1}(X \setminus D) \to X \setminus D\) is given by \(\text{Gr}^W_i H^i(X \setminus D, p_{s(2)} \mathbb{C}) \simeq \text{Gr}^W_{i-j} H^{j+i}(X, (p_{s(2)} \Omega_X (\log D), d))\). For example, in suitable quotient category, homology at column \(-i\) of the top line will compute \(\text{Gr}^W_i H^i(X \setminus D, p_{s(2)} \mathbb{C})\).
Some mixed Hodge structures on $l^2$-cohomology groups

\begin{equation}
\begin{array}{c}
\cdots \\
H^0_{(2)}(p^{-1}(D_3)) \xrightarrow{d_1} H^2_{(2)}(p^{-1}(D_2)) \xrightarrow{d_1} H^4_{(2)}(p^{-1}(D_1)) \xrightarrow{d_1} H^6_{(2)}(\tilde{X}) \\
0 \\
0 \\
H^1_{(2)}(p^{-1}(D_2)) \xrightarrow{d_1} H^3_{(2)}(p^{-1}(D_1)) \xrightarrow{d_1} H^5_{(2)}(\tilde{X}) \\
0 \\
0 \\
H^0_{(2)}(p^{-1}(D_2)) \xrightarrow{d_1} H^2_{(2)}(p^{-1}(D_1)) \xrightarrow{d_1} H^4_{(2)}(\tilde{X}) \\
0 \\
0 \\
0 \\
0 \\
H^1_{(2)}(\tilde{X}) \\
0 \\
0 \\
0 \\
Gr_{-3} W \\
Gr_{-2} W \\
Gr_{-1} W \\
Gr_0 W
\end{array}
\end{equation}

We refer to the formula (23) for an interpretation of $d_1$ in term of the Gysin morphisms.

The homology of the $i$-th line at the column $Gr_{W_-}^i$ is $E_2^{0,i}(W_-)$. But Proposition 4.2 implies

\[ Gr_{W_-}^i H^i(X \setminus D, \ p^*_*(C)) = \text{Im}(H^i(X, \ p^*_*(C)) \rightarrow H^i(X \setminus D, \ p^*_*(C))). \]

Hence under the hypothesis of Theorem 4.6, the kernel of the restriction mapping is isomorphic in $\text{Mod}(N(G))/\tau$ to the image of the Gysin homomorphism.

5.1.2 The Hodge filtration. The $E_1$-page of the Hodge filtration has differential $d_1$ induced by the operator $\partial$. We recall that the $l^2$-cohomology groups $H^i(X, \ p^*_*(C))$ of a holomorphic vector bundle are isomorphic to the unreduced $l^2$-Dolbeault cohomology groups $H^i_{\partial}(X, \ p^*_*(E))$ (see [8]). In the case of a surface this gives:

\[ H^i_{\partial}(\tilde{X}, \ p^*_*(E)) \]
\[ H^2(X, p_*(2)\mathcal{O}) \xrightarrow{d_1} H^2(X, p_*(2)\Omega_X^1(\log D)) \xrightarrow{d_1} H^2(X, p_*(2)K \otimes [D]) \]

\[ H^1(X, p_*(2)\mathcal{O}) \xrightarrow{d_1} H^1(X, p_*(2)\Omega_X^1(\log D)) \xrightarrow{d_1} H^1(X, p_*(2)K \otimes [D]) \]

\[ H^0(X, p_*(2)\mathcal{O}) \xrightarrow{d_1} H^0(X, p_*(2)\Omega_X^1(\log D)) \xrightarrow{d_1} H^0(X, p_*(2)K \otimes [D]) \]

\[ \text{Gr}_F^0 \quad \text{Gr}_F^1 \quad \text{Gr}_F^2 \]

5.1.3 Reduction with respect to the dimension torsion theory. Let \( p : \tilde{X} \to \tilde{X}/G = X \) be a \( G \)-covering of the complex compact Kähler manifold. Let us choose a Kähler metric on each divisor \( D_l \) of \( X \), and a hermitian metric on the vector bundle \( \Omega_X^1(\log D) \to X \). And consider the pullback structures on \( X \) and \( p^{-1}(D_l) \).

Let \( q \) be one of the projections from the \( l^2 \)-cohomology group onto the respective harmonic spaces (which are induced by the orthogonal projections from the square integrable cocycle to the harmonic spaces). Then the reduction of \( q \) in \( \text{Mod}(N(G))/\tau_{\dim} \) or \( \text{Mod}(N(G))/\tau_{\ell(G)} \) is isomorphic to the identity (see Corollary 3.11) and the torsion theory \( \tau_{\dim} \) satisfies hypothesis of Theorem 4.6. Therefore:

**Corollary 5.1** In \( \text{Mod}(N(G))/\tau_{\dim} \) or \( \text{Mod}(N(G))/\tau_{\ell(G)} \), we have the following isomorphisms.

(i) For the weight spectral sequence:

\[ E_{1}^{l+k+l}(W_-) = \mathbb{H}^{l+k}(X, \text{Gr}_F^W p_*(2)\Omega_X^1(\log D)) \cong \mathcal{H}_{d(2)}^{l+k-1}(p^{-1}(D_l)) \]

\[ [d_1] = [q \circ d_1 \circ q] \]

\[ E_{2}^{l+k+l}(W_-) \cong \text{Homology of} \left( \mathcal{H}_{d(2)}^{l+k-2}(p^{-1}(D_{l+1})) \xrightarrow{q \circ d_1} \mathcal{H}_{d(2)}^{l+k-1}(p^{-1}(D_l)) \right) \]

(ii) For the Hodge spectral sequence:

\[ E_{1}^{l,k+l}(F) \cong \mathcal{H}_{d(2)}^{l}(\tilde{X}, p^*\Omega_X^l(\log D)) \text{ and } [d_1] = 0. \]

5.2 The dual \( CW \)-complex associated to \( p : (\tilde{X}, p^{-1}(D)) \to (X, D) \) and a combinatorial interpretation of the highest weight cohomology The top line of \( (E_1(W), d_1) \) is interpreted through the dual complex associated to the divisors. A \( k \)-cell of this complex is a connected component of \( p^{-1}(D_{k+1}) \). The \( l^2 \)-homology of this dual complex is interesting: if the action of the infinite group \( G \) is cocompact, the reduction in \( \tau_{\dim} \) or \( \tau_{\ell(G)} \) only keeps track of the complex associated to the compact
connected components, for the other components have infinite isotropy subgroups, see Lemma 5.4.

On the other hand, the $l^2$-harmonic forms of maximal degree of a connected covering manifold is vanishing when the covering is infinite. This means that, when one neglects torsion, the only non trivial top dimensional $l^2$-cohomology spaces which enters in the top line of $(E_1(W), d_1)$ are associated to compact connected component. In this case they are multiple of the dual of the fundamental class.

This correspondence between cells with finite isotropy of the dual complex and dual of the fundamental class of compact connected components is formalised in Lemma 5.5. In the case of a Galois cover, one then describes, modulo $\tau_{d}(G)$ or $\tau_{dim}$, the groups $Gr^W_{2n} H^+(X\setminus D)$ as the relative $l^2$-holonomy of the dual complex relative to the complex of cells with infinite isotropy (Theorem 5.6).

Consideration of non Galois coverings leads us to distinguish between non compact connected component and component with infinite isotropy. When working in general non Galois coverings, we may use the edge homomorphisms (Sect. 5.4) in unreduced cohomology in order to derive informations from the simplicial $l^2$-holonomy of the dual simplicial complex.

5.2.1 The $\Delta$-complex $\tilde{K}$. Let $T_k \ni \beta \mapsto D_\beta$ be a parameterisation of the set of connected components $D_\beta$ of $\sqcup_{l=1}^k D_l$. We assume that $T_1 = T$. Let $\mathcal{P}(T)$ be the set of subset of $T$. Let $n : T_k \rightarrow \mathcal{P}(T)$ be the map such that $D_\beta$ is a connected component of $\cap_{l \in n(\beta)} D_l$ and $\forall n(\beta) = k$. Let $\tilde{T}_k \ni \beta \mapsto \tilde{D}_\beta$ be a parameterisation of the set of connected components of $p^{-1}(\sqcup_{l=1}^k D_l)$. Then $G$ acts on $\tilde{T}_k$ and $p$ induces a projection $p : \tilde{T}_k \rightarrow T_k$. We recall that $D_{\emptyset} = D_0 = X$ so that $T_0$ is a set with a single element, and $\tilde{T}_0 \ni \gamma \mapsto \tilde{X}_\gamma$ parameterizes the set of connected components of $\tilde{X}$. In order to have uniform notations, the component $\tilde{X}_\gamma$ will also be denoted by $D_\gamma$.

We now define the $\Delta$-set $\tilde{K}$ (Rourke Sanderson [47]): Let $(T, \leq)$ be an order on $T$. Let $\mathcal{F}_{inc}([k], T)$ be the set of injective increasing maps from $([k], \leq) = ([0, \ldots , k], \leq)$ to $(T, \leq)$. A $k$-cell is an element $(\beta, o) \in \tilde{T}_{k+1} \times \mathcal{F}_{inc}([k], T)$ such that $\tilde{D}_\beta$ is a connected component of $p^{-1}(D_{\emptyset} \cap \cdots \cap D_{\emptyset(k)})$. In the following the map $o \in \mathcal{F}_{inc}([k], T)$ will be identified with the $(k + 1)$-uple $(o(0), \ldots , o(k)) \in T^{k+1}$.

Let $f : [s] \rightarrow [k]$ be an injective increasing map. It induces the $f$-th face map $\tilde{K}(f) : \tilde{K}_k \rightarrow \tilde{K}_s$. Let $(\beta, o) \in \tilde{K}_k$. Then $\tilde{K}(f)(\beta, o) = (\gamma, o \circ f)$ where $\gamma \in \tilde{T}_s$ is the connected component of $p^{-1}(\cap_{l \in f(o(s))} D_l)$ which contains $\tilde{D}_\beta$.

Let $\Delta_k$ be the standard simplex in $\mathbb{R}^k$. Then $f : [s] \rightarrow [k]$ induces the usual linear injection $f : \Delta_s \rightarrow \Delta_k$. A geometric realization $|\tilde{K}|$ of $\tilde{K}$ is formed from the disjoint union $\sqcup_{l \in \mathcal{L}} \tilde{K}_l \times \Delta_l$ by identifying pairs $((\beta, o), f(x))$ and $(\tilde{K}(f)(\beta, o), x)$ ([47, §2]).

Let $C_k(\tilde{K})$ be the free $\mathbb{Z}$-module with basis the $k$-simplices $(\beta, o)$ in $\tilde{K}_k$. Let $f_m : [k] \rightarrow [k]$ be the usual $m$-th face map and let $\partial_m(\beta, o) := \tilde{K}(f_m)(\beta, o)$. The boundary map $\partial : C_k(\tilde{K}) \rightarrow C_{k-1}(\tilde{K})$ is defined by $\partial(\beta, o) = \sum_{m=0}^k (-1)^m \partial_m(\beta, o) = \sum_{i \in o} \epsilon(i)(\beta, o)$ where if $i = o(m)$, we set $(\beta, o) = (\tilde{\beta}, \tilde{o}) := \partial_m(\beta, o)$ and $\epsilon(i) = (-1)^m$ is the signature of the permutation $o \mapsto (i, \tilde{o})$.

$G$ acts naturally on $\tilde{K}$, $|\tilde{K}|$ and $|\tilde{K}|$ is a $G$-CW complex (see [36, 1.2.1]). Moreover any other choice of ordering $(T, \leq')$ on $T$ is given by an increasing permutation.
\( \sigma : (T, \leq) \to (T, \leq') \) which defines \( G \)-isomorphic \( \Delta \)-complexes, \( G \)-homeomorphic geometric realizations and \( G \)-homotopic chain complexes.

Let \( \epsilon : \tilde{K}_0 \to \tilde{K}_{-1} = \tilde{T}_0 \) be the augmentation map which assigns to \((\beta, t)\) the element \( \gamma \in \tilde{T}_0 \) such that \( \tilde{D}_\beta \subset \tilde{X}_\gamma \). Let \((\tilde{K}_\epsilon, \partial)\) be the augmented complex with respect to \( \epsilon \).

The group \( G \) acts naturally on \((\tilde{K}_\epsilon, \partial)\).

One notes that \( \epsilon \) sends a non compact connected divisor (resp. with infinite isotropy) to a non compact connected component of \( p^{-1}(X) \) (resp. with infinite isotropy).

**Definition 5.2** Recall that \( G \) acts on \( \tilde{K} \).

1. Let \( \tilde{K}(\infty) \) be the \( \Delta \)-complex consisting of simplexes \((\beta, o)\) of \( \tilde{K} \) whose isotropy subgroup \( G(\beta, o) \) under the action of \( G \) is infinite.
2. Let \( \tilde{K}(n.c.) \) be the \( \Delta \)-complex consisting of simplexes \((\beta, o)\) such that \( \tilde{D}_\beta \) is non compact.
3. Let \( \tilde{K}_\epsilon(\infty) \) and \( \tilde{K}_\epsilon(n.c.) \) be the corresponding augmented complexes.

Notations: Let \( A \) be an abelian group, and \( K' \) be a \( \Delta \)-subcomplex of a \( \Delta \)-complex \( K \) with augmentation \( \epsilon : K_0 \to K_{-1} \). Then the induced augmented complex \( K'_\epsilon \) is defined by \( \epsilon : K'_0 \to \epsilon(K'_0) = K'_{-1} \). Then \((C(\tilde{K}_\epsilon, K'_\epsilon, A), \partial) = ([C(\tilde{K}_\epsilon)/C(K'_\epsilon)] \otimes A, \partial \otimes 1_A)\) is the relative augmented chain complex with coefficients in \( A \).

**Remark** One can compute the relative homology groups \( H_k(\tilde{K}_\epsilon, K'_\epsilon) \) using a modified boundary map which annihilates boundaries lying in \( K'_\epsilon \) (see [24, remark II.4.8]). Such boundary maps will appear in computation of the relative homology of square integrable chains:

Let \( C_k(K'_\epsilon) \), respectively \( C_k(\tilde{K}_\epsilon\setminus K'_\epsilon) \), be the free abelian groups generated by the simplexes contained in \( K'_\epsilon \), respectively not contained in \( K'_\epsilon \). Let \( \alpha' : C(K'_\epsilon) \to C(\tilde{K}_\epsilon) \), \( a : C(\tilde{K}_\epsilon\setminus K'_\epsilon) \to C(K'_\epsilon) \), \( b' : C(\tilde{K}_\epsilon) \to C(K'_\epsilon) \), \( b : C(K'_\epsilon) \to C(K'_\epsilon) \otimes C(\tilde{K}_\epsilon\setminus K'_\epsilon) \) be the natural maps defining the splitting \( C(\tilde{K}_\epsilon) = C(\tilde{K}_\epsilon\setminus K'_\epsilon) \oplus C(K'_\epsilon) \). The maps \( \partial = b_{k-1} \partial a_k : C(\tilde{K}_\epsilon\setminus K'_\epsilon) \to C_{k-1}(\tilde{K}_\epsilon\setminus K'_\epsilon) \) define a complex \((C(\tilde{K}_\epsilon\setminus K'_\epsilon), \partial)\). The maps \( i_k : C(\tilde{K}_\epsilon\setminus K'_\epsilon) \to C_K(\tilde{K}_\epsilon) \to C_K(\tilde{K}_\epsilon)/C_K(K'_\epsilon) \) define an isomorphism of complexes \( i : (C(\tilde{K}_\epsilon\setminus K'_\epsilon), \partial) \to (C_K(\tilde{K}_\epsilon), \partial) \).

Hence the relative homology groups \( H_k(K, K') \) are isomorphic to the homology of the complex \((C(\tilde{K}_\epsilon\setminus K'_\epsilon), \partial)\). If \( K' \) is a \( G - \Delta \)-subcomplex then \( i \) is an isomorphism of \( \mathbb{Z}[G] \)-complexes.

**Example 5.3**

1. Assume \( \tilde{X} \) is connected non compact and \( G \) is infinite then \( \tilde{K}_{-1} = \tilde{K}_{-1}(\infty) = \tilde{K}_{-1}(n.c.) = \{\tilde{X}\} \). Hence the relative chain complexes \((C(\tilde{K}_\epsilon, \tilde{K}_\epsilon(\infty)), \partial)\) and \((C(\tilde{K}, \tilde{K}(\infty)), \partial)\) are equal.
2. If \( G \) is finite then \( \tilde{K}_\epsilon(\infty) = \emptyset \).
3. If the action of \( G \) on \( p : \tilde{X} \to X \) is proper then \( \tilde{K}(\infty) = \tilde{K}(n.c.) \). Moreover, \( \tilde{K}_\epsilon \setminus \tilde{K}_\epsilon(n.c.) \) is the set of (closed) simplexes which are faces of finitely many simplexes.

Let \((C^\text{sing}(|\tilde{K}|), \partial)\) be the complex of singular chains.
Lemma 5.4  (1) Let $\tilde{K}(\infty)$ be the $\Delta$-complex consisting of simplexes $(\beta, o)$ whose isotropy subgroup $G_{(\beta, o)}$ under the action of $G$ is infinite. Then the map of relative chain complexes $(C, (\tilde{K}, \tilde{K}(\infty), \mathbb{Q}), \partial) \to (C^{\text{sing}}(|\tilde{K}|, |\tilde{K}(\infty)|, \mathbb{Q}), \partial)$ is a $\mathbb{Q}[G]$-homotopy equivalence (see [36, p. 264]).

(2) Assume that $G$ is infinite. Then the morphism

$$(l^2(G) \otimes_{\mathbb{C}[G]} C, (\tilde{K}_e, \mathbb{C}), \partial) \to (l^2(G) \otimes_{\mathbb{C}[G]} C, (\tilde{K}_e, \tilde{K}(\infty), \mathbb{C}), \partial)$$

is an isomorphism in $\text{Mod}(N(G))/\tau_{\mathbb{U}(G)}$ (see [36, th. 6.54]).

(3) Define a complex of square integrable chains by

$$(\mathcal{C}^{(2)}(\tilde{K}_e \setminus \tilde{K}(\infty)))$$

$$= \left\{ \sum a_{(\beta, o)}(\beta, o), \text{ with } (\beta, o) \in \tilde{K}_e \setminus \tilde{K}(\infty) \right\} / \sum |a_{(\beta, o)}|^2 |G_{\beta}|^{-1} < +\infty $$

with boundary

$$\partial(\beta, o) = \left\{ \begin{array}{ll}
\sum \partial_m(\beta, o) \epsilon(\beta, o) (-1)^m \partial_m(\beta, o) & \text{if } (\beta, o) \notin \tilde{K}_0, \\
\epsilon(\beta, o) = \gamma & \text{if } (\beta, o) \in \tilde{K}_0 \setminus \tilde{K}(\infty), \gamma \notin \tilde{K}_1(\infty), \\
0 & \text{otherwise.}
\end{array} \right.$$  

(27)

Assume the number of $G$-orbits of $\tilde{K}_e \setminus \tilde{K}(\infty)$ is finite then

$$(l^2(G) \otimes_{\mathbb{C}[G]} C, (\tilde{K}_e, \tilde{K}(\infty), \mathbb{C}), \partial)$$

is $\mathbb{C}[G]$-isomorphic to $(\mathcal{C}^{(2)}(\tilde{K}_e \setminus \tilde{K}(\infty)), \partial)$.

Proof. Indeed $C_k(\tilde{K}) \simeq \bigoplus_{\sigma \in \Sigma_k} \mathbb{Z}[G/G_{\sigma}]$ where $\Sigma_k$ is a set of representatives for the $G$-orbits of $k$-simplexes (compare with Brown [6, p. 68 example 5.5b]). Let $\tilde{K}'$ be a sub-$\Delta$-complex of $\tilde{K}$. Let $C(\tilde{K} \setminus \tilde{K}') \simeq \bigoplus_{\sigma \in \Sigma_{\tilde{K}}} \mathbb{Z}[G/G_{\sigma}]$ be the free group on cells in $\tilde{K} \setminus \tilde{K}'$. Then $C(\tilde{K} \setminus \tilde{K}') \to C(\tilde{K}, \tilde{K}')$ is a $\mathbb{Z}[G]$-isomorphism whose inverse defines a $\mathbb{Z}[G]$-splitting of $0 \to C(\tilde{K} \setminus \tilde{K}') \to C(\tilde{K}) \to C(\tilde{K}, \tilde{K}')$. Assume moreover that $\tilde{K}(\infty) \subset \tilde{K}'$. Then $C(\tilde{K}, \tilde{K}') \otimes \mathbb{Q}$ and $C^{\text{sing}}(|\tilde{K}|, |\tilde{K}'|) \otimes \mathbb{Q}$ are projective $\mathbb{Q}[G]$-modules ([6, ex.4 p. 30]). If $\tilde{K}'$ is a sub-$\Delta$-complex then the map of relative chain complexes $(C(\tilde{K}, \tilde{K}'), \partial) \to (C^{\text{sing}}(|\tilde{K}|, |\tilde{K}'|), \partial)$ is a quasi-isomorphism (Hatcher [33, th.2.27, p. 137]). We conclude that the $\mathbb{Q}[G]$-quasi-isomorphism of projective complexes $(C(\tilde{K}, \tilde{K}(\infty), \mathbb{Q}), \partial) \to (C^{\text{sing}}(|\tilde{K}|, |\tilde{K}(\infty)|, \mathbb{Q}), \partial)$ is a homotopy equivalence ([6, p. 29, th.8.4]). (This prove 1).

Let us prove (2). The proof of (1) shows that $0 \to C(\tilde{K}(\infty)) \to C(\tilde{K}_e) \to C(\tilde{K}(\infty)) \to 0$ is a split exact $\mathbb{Z}[G]$-sequence. Hence the following sequence is exact:

$$0 \to l^2(G) \otimes_{\mathbb{C}[G]} C(\tilde{K}(\infty), \mathbb{C}) \to l^2(G) \otimes_{\mathbb{C}[G]} C(\tilde{K}_e, \mathbb{C}) \to$$

$$l^2(G) \otimes_{\mathbb{C}[G]} C(\tilde{K}_e, \tilde{K}(\infty), \mathbb{C}) \to 0.$$  

From Lemma 2.31, the first term is isomorphic to zero in $\text{Mod}(N(G))/\tau_{\mathbb{U}(G)}$. 

\(\square\) Springer
Lemma 5.5 Let $\beta \in \tilde{T}_k$ and let $n_\beta$ be the degree of the covering map $\tilde{D}_\beta \to D_{p(\beta)} = p(\tilde{D}_\beta)$ (see Sect. 5.2.1).

Let $(C_\beta^{(2)}(\tilde{K}_e \setminus \tilde{K}_e(\infty)), \partial)$ be the complex of square integrable chains

$$\left\{ \sum a_{(\beta,o)}(\beta,o), \text{ with } (\beta,o) \in \tilde{K}_e, \tilde{K}_e(n,c) \text{ s.t. } \sum |a_{(\beta,o)}|^2(n_\beta)^{-1} < +\infty \right\}$$

with bounded boundary map $\partial$ given in (27) with $\tilde{K}_e(\infty)$ replaced by $\tilde{K}_e(n,c)$.

There exists an isomorphism of complexes

$$(C_\beta^{(2)}(\tilde{K}_e \setminus \tilde{K}_e(n,c)), \partial) \to (H_{d(2)}^{2(n-1)}(p^{-1}(D_{+1})), \varphi \circ d_1),$$

which is compatible with the action of $G$.

Proof If $\beta \in \tilde{T}_k$, then $H_{d(2)}^{2(n-k)}(\tilde{D}_\beta)$ is vanishing if $\tilde{D}_\beta$ is non compact. If $\tilde{D}_\beta$ is compact, we denote by $C_\beta$ the unique harmonic form on $\tilde{D}_\beta$ such that $\int_{\tilde{D}_\beta} C_\beta = 1$. Then $|C_\beta|^2 = V_\beta^{-1} = (n_\beta V_{p(\beta)})^{-1}$ with $V_\beta$ (resp. $V_{p(\beta)}$) is the volume of $\tilde{D}_\beta$ with respect to a pullback metric on $D_{p(\beta)}$ (resp. the volume of $D_{p(\beta)}$).

One extends $C_\beta$ by zero to $p^{-1}(D_k)$. This extension, still denoted by $C_\beta$, satisfies the relations $C_\beta \rho(g) := g^* C_\beta = C_{g^{-1} \beta}$. Let $\Sigma_k$ be a set of representatives for the $G$-orbits of $k$-simplices. There is a hilbertian orthogonal decomposition $H_{d(2)}^{2(n-k)}(p^{-1}(D_k)) = \bigoplus_{\beta \in \tilde{T}_k \setminus \tilde{T}(n,c)} H_{d(2)}^{2(n-k)}(\tilde{D}_\beta)$ and an isometric isomorphism

$$H_{d(2)}^{2(n-k)}(p^{-1}(D_k)) \cong \bigoplus_{\beta \in \Sigma_k \setminus \tilde{T}(n,c)} |l^2(G/G_\beta) \text{ given by}$$

$$f \mapsto \bigoplus_{\beta \in \Sigma_k} \langle gKf, C_\beta \rangle \varphi^{-1} >.$$}

Then, we define

$$r : C_\beta^{(2)}(\tilde{K}_e \setminus \tilde{K}_e(n,c)) \to H_{d(2)}^{2(n-1)}(p^{-1}(D_{+1})) \otimes \Lambda^\sigma T$$

$$\sum_{(\beta,o) \in \tilde{K}_e, \tilde{K}_e(n,c)} a_{(\beta,o)}(\beta,o) \mapsto \sum_{(\beta,o)} a_{(\beta,o)} C_\beta \otimes \Lambda^\sigma T$$

It converts the left action on chains to the right action on cohomology classes.

We prove now that it is a chain map. The use of exterior differential forms in computation of $d_1$ lead us to use antisymmetric chains: if $\sigma$ is a permutation of $[k]$
then \((\beta, o \circ \sigma)\) is by definition equal to \(\varepsilon(\sigma)(\beta, o)\) where \(\varepsilon(\sigma)\) is the signature of \(\sigma\).

Therefore a chain in \(L^2(G) \otimes C[\mathbb{G}] C(\bar{K}_e, \bar{K}_e(\infty))\) is defined by \(\sum (\beta, [o]) a(\beta, o)(\beta, o)\) where \([o] = \{o(0), \ldots, o(k)\}\) is identified with the orbit class under the permutation group of the injective map \(o: [k] \to T^{k+1}\) and \(o \to a(\beta, o)\) is antisymmetric in \(o\). If \(\beta \in \tilde{T}^k\) or \(T_k\), let \(r_{\beta, \ldots}\) denotes the restrictions of \(p^*(r_{np(\beta)})\) \(\ldots\) to \(U_\beta\), that connected component of \(p^{-1}(U_{np(\beta)}(\varepsilon_1))\) which contains \(\tilde{D}_\beta\) (notations of Sect. 4.4.1).

With these conventions, we associate to the square integrable singular chain \(c = \sum (\beta, [o]) a(\beta, o)(\beta, o) \in C^2_k(\bar{K}_e \setminus \bar{K}_e(\text{n.c.}))\) the logarithmic form (see Sect. 4.4.1)

\[
L(c) = \sum_{(\beta, [o])} a(\beta, o)r^*_\beta(C_\beta) \eta(\beta, o) = \sum_{[o]} r^*_o \left( \sum_{\beta} a(\beta, o)C_\beta \right) \eta_o
\]

whose residu is

\[
r(c) = \sum_{[o]} \left( \sum_{(\beta, o)} a(\beta, o)C_\beta \right) \otimes \Lambda^o t \in \bigoplus_{[o]} \mathcal{H}^{2(n-k+1)}_{d(2)}(p^{-1}(D_o) \otimes \Lambda^{k+1}T).
\]

Here \(\eta(\beta, o)\) is the extension by 0 to \(\tilde{X}\) of \(p^*(\eta(0(0), \ldots, o(k)))|_{U_\beta}\). Let \((\gamma, o') \in \tilde{K}_{k-1}\). Then, as in (Sect. 4.4.1),

\[
\text{Res}_\gamma dL(c) = a^*_\gamma \left( \sum_{\tilde{D}_\beta \subset \tilde{D}_\gamma} (-1)^{2(n-k-1)} \sum_{i \in T \setminus o'([k-1])} a(\beta, (i, o')) \omega_i r^*_\beta(C_\beta) \right) \otimes \Lambda^{o'} t.
\]

Let \(q_\gamma\) be the orthogonal projection \(L^2(\tilde{D}_\gamma, \Lambda^{2(n-k)}) \to \mathcal{H}^{2(n-k)}_{d(2)}(\tilde{D}_\gamma)\). It is non vanishing only if \(\tilde{D}_\gamma\) is compact. In this case,

\[
\int_{\tilde{D}_\gamma} \omega_i r^*_\beta C_\beta = \int_{p^{-1}(D_i) \cap \tilde{D}_\gamma} r^*_\beta C_\beta = \int_{\tilde{D}_\beta} r^*_\beta C_\beta = 1
\]

for \(\text{supp}(r^*_\beta C_\beta) \cap p^{-1}(D_i) \cap \tilde{D}_\gamma = \tilde{D}_\gamma\) (see Sect. 4.4.1). Hence \(q_\gamma(\omega_i r^*(C_\beta)) = C_\gamma\) and

\[
q_\gamma \circ \text{Res}_\gamma dL(c) = (-1)^{2(n-k-1)} \sum_{\tilde{D}_\beta \subset \tilde{D}_\gamma} \left( \sum_{i \in T \setminus o'([k-1])} a(\beta, (i, o')) \right) C_\gamma \otimes \Lambda^{o'} t
\]

\[
= \left( \sum_{(\beta, [o])} a(\beta, o) \epsilon((\beta, o) : (\gamma, o')) \right) C_\gamma \otimes \Lambda^{o'} t
\]

where \(\epsilon((\beta, o) : (\gamma, o'))\) is vanishing if \(\tilde{D}_\beta \not\subset \tilde{D}_\gamma\) and is equal to the signature of the permutation mapping \(o\) to \((i, o')\) if \(\tilde{D}_\beta \subset \tilde{D}_\gamma\) and \(o([k]) \setminus o'([k-1]) = \{i\}\).
Let \( q = \bigoplus_{\gamma} q_{\gamma} \) be the orthogonal projection \( L^{2}(p^{-1}(D_{k}), \Lambda^{2(n-k)}) \to \mathcal{H}_{d(2)}^{2(n-k)}(p^{-1}(D_{k})) \). It will be identified with the induced projection \( H_{d(2)}^{2(n-k)}(p^{-1}(D_{k})) \to \mathcal{H}_{d(2)}^{2(n-k)}(p^{-1}(D_{k})) \). Then

\[
q \circ d_{1}(r(c)) = \sum_{\gamma, \{o'\}} q_{\gamma} \circ \text{Res}_{\gamma} dL(c) = \sum_{(\beta, \{o\})} a(\beta, o) \mathcal{C}_{\gamma} \otimes \Lambda^{o' t}
\]

\[
\sum_{(\beta, \{o\})} \sum_{i \in \{o\}} a(\beta, o) \mathcal{C}_{i} \otimes \Lambda^{i t} = r(\partial c).
\]

One notes that the boundary maps are bounded for

\[
\sum_{\gamma \in \tilde{T}_{k}} \left( \sum_{\tilde{D}_{\beta} \subset \tilde{D}_{\gamma}} \left| a(\tilde{\beta}, o) \right| \right)^{2} \frac{1}{n_{\gamma}} \leq \sum_{\gamma} \left( \sum_{\tilde{D}_{\beta} \subset \tilde{D}_{\gamma}} \left| a(\tilde{\beta}, o) \right|^{2} n_{\beta} \right) \left( \sum_{\tilde{D}_{\beta} \subset \tilde{D}_{\gamma}} \frac{n_{\beta}}{n_{\gamma}} \right) \leq (k + 1) \left( \sum_{\beta} \left| a(\beta, o) \right|^{2} \frac{1}{n_{\beta}} \right) \max_{\gamma \in \tilde{T}_{k}} \#(\beta \in T_{k+1} : D_{\beta} \subset D_{\gamma}).
\]

The following theorem holds for any real torsion theory greater or equal to \( \tau_{U}(G) \) (e.g. \( \tau_{\text{dim}} \)).

**Theorem 5.6** Let \( \tilde{X} \to X \) be a G-covering of a compact n-dimensional Kähler manifold, let \( D \) be a normal crossing divisor in \( X \) and let \( \tilde{K} \) be the associated dual complex. Let \( \tilde{K}_{\epsilon}(\infty) \) be the augmented subcomplex of cells of infinite isotropy subgroups. Theses cells are in one-to-one correspondence with the set of the non compact connected components of \( \bigsqcup_{I \in \mathcal{P}(T)} p^{-1}(D_{I}) \).

There exists isomorphisms in \( \text{Mod}(N(G)) \)

\[
r : (l^{2}(G) \otimes \mathbb{C}[G] C.(\tilde{K}_{\epsilon}, \tilde{K}_{\epsilon}(\infty), \partial)) \to (\mathcal{H}_{d(2)}^{2(n-(-1))}(p^{-1}(D_{-1})), q \circ d_{1})
\]

which induce the isomorphisms in \( \text{Mod}(N(G))/\tau_{U}(G) \)

\[
H_{k,(2)}(\tilde{K}_{\epsilon}, \tilde{K}_{\epsilon}(\infty)) \to \text{Gr}_{W}^{2n} H^{2n-k}(X \setminus D, p_{*}(\mathbb{C})) \text{ for } k \geq -1.
\]

Hence \( \text{Gr}_{W}^{2n} H^{2n-k}(X \setminus D, p_{*}(\mathbb{C})) \) is non vanishing in \( \text{Mod}(N(G))/\tau_{U}(G) \) implies that there exists compact connected components in \( p^{-1}(D_{k}) (k \geq 0) \).

\( \odot \) Springer


Proof The hypothesis implies that \( \tilde{K}_e(\infty) = \tilde{K}_e(n.c.) \), \( |G_\beta| = n_\beta \) and the set of orbits of \( \tilde{K}_e \setminus \tilde{K}_e(\infty) \) is finite. Then \( (l^2(G) \otimes \mathbb{C}[G] C.(\tilde{K}_e, \tilde{K}_e(\infty)), \partial) \) is identified with \( (C^{(2)}_{-}(\tilde{K}_e \setminus \tilde{K}_e(n.c.)), \partial) \) and we conclude from Lemmas 5.4 and 5.5.

Corollary 5.1 implies that \( H_{k-1}(2)(\tilde{K}_e, \tilde{K}_e(\infty)) \rightarrow Gr_{2n}^{W} H^{2n-k}(X \setminus D, p_{*(2)}\mathbb{C}) \) is an isomorphism in \( \text{Mod}(N(G))/\tau_{\text{u}(G)} \) if \( G \) is infinite. \( \square \)

Remark 5.7 (1) Let \( \dim_{\mathbb{C}} \tilde{D}_\beta = n - k \) and let \( G_\beta \) be the stabiliser of the irreducible component \( \tilde{D}_\beta \). Then the operator \( \text{Im}(\tilde{\partial}) : L^2(\tilde{D}_\beta, \lambda^{2n-2k-1}) \rightarrow L^2(\tilde{D}_\beta, \lambda^{2n-2k}) \) has close range iff \( G_\beta \) is non-amenable. (see Brooks [5], Saloff–Coste Woess [48]).

(2) In \( \text{Mod}(N(G))/\tau_{\text{u}(G)} \), we have \( H_{k,(2)}(\tilde{K}, \tilde{K}(\infty)) \simeq H_{k,(2)}(\tilde{K}, \tilde{K}(\infty)) \) where \( H_{k,(2)}(\tilde{K}, \tilde{K}(\infty)) \) is the (simplicial) harmonic space associated to the finitely generated hilbertian complex \( (l^2(G) \otimes \mathbb{C}[G] C.(\tilde{K}, \tilde{K}(\infty)), \partial) \).

5.3 Normal crossing divisor such that \( X \setminus D \) is Stein Assume that \( X \setminus D \) is Stein. Then the sheaves \( (j^*p)^{\ast}(\Omega_{X \setminus D}) \) are \( \Gamma(X \setminus D, ,)-\text{acyclic} (([8]). From Proposition 4.2, we deduce:

Lemma 5.8 Let \( p : \tilde{X} \rightarrow X \) be a covering of a complex manifold with transformation group \( G \). Assume that \( X \setminus D \) is Stein. Then \( H(X \setminus D, p_{*}(2)\mathbb{C}) \) is \( N(G) \)-isomorphic to \( H(\Gamma(X \setminus D, (j^*p)^{\ast}(\Omega_{2})), d) \). In particular, \( k > \dim X \implies H^k(X \setminus D, p_{*}(2)\mathbb{C}) = 0 \).

This last statement may be proved by other topological methods.

One deduces that the dual complex associated to \( p^{-1}(D) \) is acyclic for the reduced \( l^2 \)-cohomology up to degree \( \dim_{\mathbb{C}} X - 1 \):.

Proposition 5.9 Let \( p : \tilde{X} \rightarrow X \) be a \( G \)-covering of a compact Kähler manifold. Let \( D \) be a normal crossing divisor in \( X \). Assume that \( X \setminus D \) is Stein. Then the relative homology group \( H_{i}(\tilde{K}(\tilde{K}(\infty))) \) of the dual complex is vanishing in \( \text{Mod}(N(G))/\tau_{\text{u}(G)} \) when \( i + 1 < n = \dim_{\mathbb{C}} X \).

Proof This is a consequence of Theorem 5.6 and the above lemma. \( \square \)

Example 5.10 Assume \( X \) is projective and \( \dim_{\mathbb{C}} X = n \geq 2 \). Let \( D \) be a divisor in \( X \) and let \( D_v \) be a generic hyperplane section such that \( D \cup D_v \) is a normal crossing divisor. We compare the dual CW-complex \( \tilde{K} \) associated to \( p^{-1}(D) \) with \( \tilde{K}' \) the one associated to \( p^{-1}(D \cup D_v) \). We recover directly the vanishing result of \( H_{i}(l^2(G) \otimes \mathbb{C}[G] C.(\tilde{K}', \tilde{K}'(\infty), \mathbb{C})) \) for \( i \leq n - 2 \).

Roughly speaking, \( \tilde{K}' \) is obtained from the formal cone \( w \ast \tilde{K} \) by removing \( n \)-cells and replacing a formal \((n - 1)\)-dimensional cell \( w \ast (\beta, o) \), with \( \tilde{D}_\beta \) a curve, by the collection of \((n - 1)\)-cells which parameterizes points in \( \tilde{D}_\beta \cap p^{-1}(D_v) \) and attached along there common boundary \( w \ast \partial(\beta, o) \cup (\beta, o) \): Let \( \tilde{T}_k \ni y \rightarrow \tilde{D}_y \) be a set which parameterizes the connected components of codimension \( k \) of \( p^{-1}(\cup_{i=1}^k \cup T \cup \{v\} \cup D) \).

Let us assume that \( T_k \subset T' \). One sets \( \tilde{D}_w = p^{-1}(D_v) \). We choose an ordering on \( T' = T \cup \{v\} \) such that \( v \) is the biggest element of \( T' \). Let \( (y, o') \in \tilde{K}_{k-1} \) be such

\( \square \) Springer
that $\tilde{D}_Y \subset p^{-1}(D_\nu)$ and let $k \geq 2$. There exists a unique $(\beta, o)$ in $\tilde{K}_{n-2}$ such that $\tilde{D}_Y \subset \tilde{D}_\beta \cap p^{-1}(D_\nu)$. Hence $(o'(0), \ldots, o'(k-1)) = (o(0), \ldots, o(k-2), w)$ and $\partial_{k-1}(\gamma, o') = (\beta, o)$. The Lefschetz’s theorem implies that $\tilde{D}_Y = \tilde{D}_\beta \cap p^{-1}(D_\nu)$ if dim$_C \tilde{D}_\beta \geq 2$. When $\tilde{D}_\beta$ is a curve, $p^{-1}(D_\nu) \cap \tilde{D}_\beta$ is a discrete set of cardinality $n_\beta^\nu(p(\tilde{D}_\beta) \cap D_\nu)$ with $n_\beta$ the degree of the covering $\tilde{D}_\beta \to p(\tilde{D}_\beta)$. Let $n-1 \geq k \geq 2$, we can define a map $\star w : \tilde{K}_{k-2} \to \tilde{K}_{k-1}$ by $\star w((\beta, o)) := (\gamma, o')$. Then $\tilde{K}_{k-1} = \star w(\tilde{K}_{k-2}) \cup \tilde{K}_{k-1}$. Choose arbitrarily one connected component $\tilde{D}_Y$ in $\tilde{D}_\beta \cap p^{-1}(D_\nu)$. We define $\star w : \tilde{K}_{n-2} \to \tilde{K}_{n-1}$ by $\star w((\beta, o)) = (\gamma, (o, w))$.

Therefore $\tilde{K}_{\leq n-2}$ is embedded in the subcomplex $\star w(\tilde{K}_{\leq n-2}) \cup \tilde{K}_{\leq n-1} \cup \{w\}$ of $\tilde{K}'$. This subcomplex is isomorphic to a cone with vertex $w$ over $\tilde{K}_{\leq n-2}$ (see Munkres [39, p. 43]). This implies that $H_i(I^2(G) \otimes \mathbb{C}[G] \tilde{K}') = 0$ when $i \leq n - 2$. Hence $H_i(I^2(G) \otimes \mathbb{C}[G] C, (\tilde{K}', \tilde{K}'(\infty), \mathbb{C}))$ is vanishing for $i \leq n - 2$.

### 5.4 Edge homomorphisms

One states here the Edge homomorphisms of the non reduced weight spectral sequences: From $E^{l,k}_1(W_-) \simeq H_k^{l+2l}(p^{-1}(D_\nu))$, we get that $E^{l,k}_r(W_-), (r \geq 1)$, is non vanishing implies $-n \leq l \leq 0$ and $-2l \leq k \leq 2n$.

**Lemma 5.11** (1) There exists an epimorphism $E^{0,k}_2(W_-) \to \text{Im}(H^k(\tilde{X}, p_*(\mathbb{C})) \to H^k(\tilde{X} \setminus D, p_*(\mathbb{C})).$

(2) There exists a monomorphism $Gr^W_2 H^{2n-l}(X \setminus D, p_*(\mathbb{C})) \to E^{2,2n}_2(W_-).

(3) The following sequence is exact:

$$
H^{2n-2}(X \setminus D, p_*(\mathbb{C})) \to E^{2,2n}_2(W_-) \xrightarrow{d_2} E^{0,2n-1}_2(W_-) \to H^{2n-1}(X \setminus D, p_*(\mathbb{C})) \to E^{2,2n}_2(W_-) \to 0
$$

**Proof** We refer to ([26, II.4.5, p. 81]) for these well known assertions. 

### 5.5 Kähler hyperbolic manifolds

Let $p : \tilde{X} \to (X, \omega)$ be a $G$-cover. Following Gromov [31], the form $\tilde{\omega} = p^* \omega$ is said to be $d$-bounded if $\tilde{\omega} = d\eta$ with $\eta$ a 1-form which is bounded with respect to $\tilde{\omega}$. We recall the following theorems on Kähler hyperbolic manifolds [31, 1.4.A and 2.5]:

**Theorem 5.12** (Gromov)

(i) Let $(\tilde{X}, \tilde{\omega})$ be a complete Kähler manifold of real dimension $n = 2m$ and assume that $\tilde{\omega} = d\eta$ where $\eta$ is a bounded 1-form on $\tilde{X}$. Then there exists a strictly positive constant $\lambda_0 \geq \text{const}_n ||\eta||_{\infty}^{-1}$ such that every $l^2$-form $\psi$ of degree $p \neq m$ satisfies the inequality

$$
\langle \psi, \Delta \psi \rangle \geq \lambda_0^2 \langle \psi, \psi \rangle.
$$

Furthermore, the above inequality is satisfied by the $l^2$-forms of degree $m$ which are orthogonal to the harmonic $m$-forms.
(ii) Let $(\tilde{X}, \tilde{\omega}) \to (X, \omega)$ be the universal cover of a Kähler manifold. Assume that $\tilde{\omega}$ is $d$-(bounded). Then the space $H^{p,q}_{\bar{\partial}(2)}(\tilde{X})$ of harmonic $l^2$-forms on $X$ of bi-degree $(p, q)$ is non vanishing if $p + q = m$.

A spectral gap for $\Delta$ acting on forms on any degree implies that $d$, $\delta$, $\bar{\partial}$ and $\bar{\partial}^*$ have closed ranges for the metric is complete Kähler and $\Delta = \Delta_{\bar{\partial}}$.

Moreover, for any complex submanifold $i : Y \to X$, the pullback induced metric $i^*\omega$ of the pullback $G$-cover $i^*p : (\tilde{Y}, i^*\omega) \to (Y, i^*\omega)$ is also $d$-(bounded). Hence the property of being $d$-(bounded), and then of spectral gap in any degree, in the $G$-cover is hereditary.

From the Lefschetz theorem [3], if $X$ is a projective manifold and $Y$ is a generic intersection of at most $n - 2$ hyperplane sections then $\Pi_1(Y) \simeq \Pi_1(X)$. Hence the above theorem implies that $H^{p,q}_{\bar{\partial}(2)}(\tilde{Y}) \neq 0$ if $p + q = \dim Y$.

**Theorem 5.13** Let $p : (\tilde{X}, \tilde{\omega}) \to (X, \omega)$ be a $G$-cover such that $\tilde{\omega}$ is $d$-(bounded). Let $D$ be a normal crossing divisor in $X$. Then any torsion theory $\tau$ fulfills assumption of Theorem 4.6. Moreover the weight spectral sequence degenerates at $E_1$:

$$Gr_{k+l}^W H^k(X \setminus D, p_{*(2)}\mathbb{C}) \simeq \begin{cases} H^{k-l}_{d(2)}(p^{-1}(D)) & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

Therefore we may choose the trivial torsion theory $\tau = ([0], \text{Mod}(N(G)))$ (no non zero torsion modules) in the statement of Theorem 4.6.

**Proof** Recall that $D_l := \cap_{t \in I} D_t \to X$. Indeed the $l^2$-cohomology groups $H^{p,q}_{\bar{\partial}(2)}(p^{-1}(D_l))$ are reduced for $\bar{\partial}_D$ has a closed range. Hence $\tau_{\bar{\partial}, p^{-1}(D_l)} = (0, \text{Mod}(N(G)))$ (Definition 3.6) is the trivial torsion theory: the $l^2$-Hodge to De Rham spectral sequence of $p^{-1}(D_l)$ degenerates. Therefore $\tau_{\bar{\partial}, p^{-1}(D_l)}$ (Definition 4.3) is also trivial. According to Lemma 4.4 (3), this implies that for each $l \in \mathbb{Z}$, $R\Gamma(Gr^W_l K) := \{(\Gamma^R l^p_{*} p_{*(2)} \mathbb{R}), (R\Gamma Gr^W_l p_{*(2)} \Omega^*_X(\log D), F), R\Gamma(Gr^W_l \tilde{\beta})\}$ is a Hodge complex in $N(G, \mathbb{R})$ as required in Theorem 4.6.

In the $E_1$-stage of the weight spectral sequence, the only non vanishing terms are $H^{n-l}_{d(2)}(p^{-1}(D_l))$ which are isomorphic to the corresponding harmonic spaces. Therefore the weight spectral sequence degenerates at $E_1$, $Gr_{n+l}^W H^n(X \setminus D, p_{*(2)}\mathbb{C}) \simeq H^{n-l}_{d(2)}(p^{-1}(D_l))$ and $H^k(U, p_{*(2)}\mathbb{C})$ is vanishing if $k \neq n$. \hfill $\square$

**Acknowledgments** My heartly thanks go to S. Vassout and G. Skandalis for illuminating discussions on Von Neumann algebras. Thanks to B. Klingler and P. Eyssidieux for stimulating discussions on the subject. I thank the referee for its valuable work and helpful comments. I thank G. Courtois, S. Diverio, E. Falbel, V. Minerbe and M. Wolff for their remarks and corrections on the initial version of this article.

**References**

1. Ancona, V., Gaveau, B.: Differential forms on singular varieties, volume 273 of Pure and Applied Mathematics (Boca Raton), De Rham and Hodge theory simplified. Chapman & Hall/CRC, Boca Raton (2006)
2. Atiyah, M.F.: Elliptic operators, discrete groups and von Neumann algebras. In: Colloque "Analyse et Topologie" en l’Honneur de Henri Cartan (Orsay, 1974), pp. 43–72. Astérisque, No. 32–33. Soc. Math. Paris (1976)
3. Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A.: Compact complex surfaces, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 2nd edn. Springer, Berlin (2004)
4. Bredon, G.E.: Sheaf theory, volume 170 of Graduate Texts in Mathematics, 2nd edn. Springer, New York (1997)
5. Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. 56(4), 581–598 (1981)
6. Brown, K.S.: Cohomology of groups, volume 87 of Graduate Texts in Mathematics. Springer, New York 1994 Corrected reprint of the 1982 original (1994)
7. Brüning, J., Lesch, M.: Hilbert complexes. J. Funct. Anal. 108(1), 88–132 (1992)
8. Campana, F., Delattre, J.-P.: Cohomologie $L^2$ sur les revêtements d’une variété complexe compacte. Ark. Mat. 39(2), 263–282 (2001)
9. Cheeger, J., Gromov, M.: Bounds on the von Neumann dimension of $L^2$-cohomology and the Gauss–Bonnet theorem for open manifolds. J. Differ. Geom. 22(1), 1–34 (1985)
10. Cheeger, J., Gromov, M.: $L^2$-cohomology and group cohomology. Topology 25(2), 189–215 (1986)
11. Deligne, P.: Théorie de Hodge. II. Inst. Hautes Études Sci. Publ. Math. 40, 5–57 (1971)
12. Deligne, P.: Théorie de Hodge. III. Inst. Hautes Études Sci. Publ. Math. 44, 5–77 (1974)
13. Dickson, S.E.: A torsion theory for Abelian categories. Trans. Am. Math. Soc. 121, 223–235 (1966)
14. Dimca, A.: Sheaves in topology. Universitext, Springer, Berlin (2004)
15. Dixmier, J.: Les algèbres d’opérateurs dans l’espace hilbertien (algèbres de von Neumann). Gauthier-Villars Éditeur, Paris, 1969. Deuxième édition, revue et augmentée, Cahiers Scientifiques, Fasc. XXV (1969)
16. Dodziuk, J.: de Rham-Hodge theory for $L^2$-cohomology of infinite coverings. Topology 16(2), 157–165 (1977)
17. Dye, H.A.: The Radon–Nikodým theorem for finite rings of operators. Trans. Am. Math. Soc. 72, 243–280 (1952)
18. Eckmann, B.: Introduction to $l_2$-methods in topology: reduced $l_2$-homology, harmonic chains, $l_2$-Betti numbers. Isr. J. Math. 117, 183–219 (2000) (notes prepared by Guido Mislin)
19. Eilenberg, S.: Homology of spaces with operators. I. Trans. Am. Math. Soc. 61:378–417 [errata, 62, 548 (1947).
20. El Zein, F.: Introduction à la théorie de Hodge mixte. Actualités Mathématiques. Hermann, Paris (1991)
21. Eyssidieux, P.: Invariants de von Neumann des faisceaux analytiques cohérents. Math. Ann. 317(3), 527–566 (2000)
22. Faith, C.: Algebra: rings, modules and categories. I. Springer, New York (1973) (Die Grundlehren der mathematischen Wissenschaften, Band 190)
23. Farber, M.S.: Homological algebra of Novikov–Shubin invariants and Morse inequalities. Geom. Funct. Anal. 6(4), 628–665 (1996)
24. Ferrario, D.L., Piccinini, R.A.: Simplicial structures in topology. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York (2011) (translated from the 2009 Italian original by Maria Nair Piccinini)
25. Gabriel, P.: Des catégories abéliennes. Bull. Soc. Math. France 90, 323–448 (1962)
26. Godement, R.: Topologie algébrique et théorie des faisceaux. Hermann, Paris, : Troisième édition revue et corrigée. Publications de l’Institut de Mathématique de l’Université de Strasbourg, XIII, Actualités Scientifiques et Industrielles, No (1973). 1252
27. Golan, J.S.: Torsion theories, volume 29 of Pitman Monographs and Surveys in Pure and Applied Mathematics. Longman Scientific & Technical, Harlow (1986)
28. Griffiths, P., Harris, J.: Principles of algebraic geometry. Wiley Classics Library. Wiley, New York (1994) (reprint of the 1978 original)
29. Griffiths, P., Schmid, W.: Recent developments in Hodge theory: a discussion of techniques and results. In: Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), pp. 31–127. Oxford University Press, Bombay (1975)
30. Gromov, M.: Sur le groupe fondamental d’une variété kählerienne. C. R. Acad. Sci. Paris Sér. I Math. 308(3), 67–70 (1989)
31. Gromov, M.: Kähler hyperbolicity and $L^2$-Hodge theory. J. Differ. Geom. 33(1), 263–292 (1991)
Some mixed Hodge structures on $l^2$-cohomology groups

32. Grothendieck, A.: Sur quelques points d’algèbre homologique. Tôhoku Math. J. (2) 9, 119–221 (1957)
33. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
34. Iversen, B.: Cohomology of Sheaves. Universitext. Springer, Berlin (1986)
35. Lefschetz, S.: Introduction to Topology. Princeton Mathematical Series, vol. 11. Princeton University Press, Princeton (1949)
36. Lück, W.: $L^2$-invariants: theory and applications to geometry and $K$-theory, volume 44 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer, Berlin (2002)
37. Malgrange, B.: Ideals of differentiable functions. Tata Institute of Fundamental Research Studies in Mathematics, No. 3. Tata Institute of Fundamental Research, Bombay (1967)
38. Mebkhout, Z., Narváez-Macarro, L.: Le théorème de constructibilité de Kashiwara. In Éléments de la théorie des systèmes différentiels. Images directes et constructibilité (Nice, 1990), vol. 46 of Travaux en Cours, pp. 47–98. Hermann, Paris (1993)
39. Munkres, J.R.: Elements of Algebraic Topology. Addison-Wesley Publishing Company, Menlo Park (1984)
40. Murray, F.J., Von Neumann, J.: On rings of operators. Ann. Math. (2) 37(1), 116–229 (1936)
41. Nicolaescu, L., et al.: Mixed hodge structures. http://www.nd.edu/~lnicolae/Hodge.htm
42. Nori, M.V.: Zariski’s conjecture and related problems. Ann. Sci. École Norm. Sup. (4) 16(2), 305–344 (1983)
43. Pedersen, G.K.: $C^*$-algebras and their automorphism groups, volume 14 of London Mathematical Society Monographs. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London (1979)
44. Peters, C.A.M., Steenbrink, J.H.M.: Mixed Hodge structures, volume 52 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer, Berlin (2008)
45. Reich, H.: Group von Neumann algebras and related algebras. PhD thesis, Universität Göttingen (1999)
46. Roe, J.: An index theorem on open manifolds. I, II. J. Differ. Geom. 27(1):87–113; 115–136 (1988)
47. Rourke, C.P., Sanderson, B.J.: ∆-sets. I. Homotopy theory. Quart. J. Math. Oxford Ser. (2) 22, 321–338 (1971)
48. Saloff-Coste, L., Woess, W.: Transition operators on co-compact $G$-spaces. Rev. Mat. Iberoam. 22(3), 747–799 (2006)
49. Sauer, R., Thom, A.: A spectral sequence to compute $L^2$-Betti numbers of groups and groupoids. J. Lond. Math. Soc. (2) 81(3), 747–773 (2010)
50. Schwartz, L.: Division par une fonction holomorphe sur une variété analytique complexe. Summa Brasil. Math. 3, 181–209 (1955)
51. Seifert, H., Threlfall, W.: Seifert and Threlfall: a textbook of topology, volume 89 of Pure and Applied Mathematics. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980 (translated from the German edition of 1934 by Michael A. Goldman, With a preface by Joan S. Birman, With “Topology of 3-dimensional fibered spaces” by Seifert, translated from the German by Wolfgang Heil)
52. Serre, J.-P.: Groupes d’homotopie et classes de groupes abéliens. Ann. Math. (2) 58, 258–294 (1953)
53. Shubin, M.A.: Spectral theory of elliptic operators on noncompact manifolds. Astérisque 207(5), 35–108 (1992) [Méthodes semi-classiques, Vol. 1 (Nantes, 1991)]
54. Shubin, M.A.: $L^2$ Riemann-Roch theorem for elliptic operators. Geom. Funct. Anal. 5(2), 482–527 (1995)
55. Steenrod, N.E.: Homology with local coefficients. Ann. Math. (2) 44, 610–627 (1943)
56. Takesaki, M.: Theory of operator algebras. I. volume 124 of Encyclopaedia of Mathematical Sciences. Springer, Berlin (2002). Reprint of the first (1979) edition. Operator Algebras and Non-commutative Geometry, 5
57. Togneto, J.-C.: Idéaux de fonctions différentiables. Springer, Berlin (1972) (Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 71)
58. Vas, L.: Torsion theories for finite von Neumann algebras. Commun. Algebra 33(3), 663–688 (2005)
59. Verdier, J.-L.: Des catégories dérivées des catégories abéliennes. Astérisque (239):xii+253 pp (1996) [with a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis (1997)]
60. Voisin, C.: Théorie de Hodge et géométrie algébrique complexe, volume 10 of Cours Spécialisés [Specialized Courses]. Société Mathématique de France, Paris (2002)
62. Weibel, C.A.: An introduction to homological algebra, volume 38 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (1994)
63. Whitehead, J.H.C.: On $C^1$-complexes. Ann. Math. (2) 41, 809–824 (1940)
64. Woess, W.: Denumerable Markov chains. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2009. Generating functions, boundary theory, random walks on trees (2009)