Abstract

We provide a simple and generic adaptive restart scheme for convex optimization that is able to achieve worst-case bounds matching (up to constant multiplicative factors) optimal restart schemes that require knowledge of problem specific constants. The scheme triggers restarts whenever there is sufficient reduction of a distance-based potential function. This potential function is always computable. We apply the scheme to obtain the first adaptive restart algorithm for saddle-point algorithms including primal-dual hybrid gradient (PDHG) and extragradient. The method improves the worst-case bounds of PDHG on bilinear games, and numerical experiments on quadratic assignment problems and matrix games demonstrate dramatic improvements for obtaining high-accuracy solutions. Additionally, for accelerated gradient descent (AGD), this scheme obtains a worst-case bound within 60% of the bound achieved by the (unknown) optimal restart period when high accuracy is desired. In practice, the scheme is competitive with the heuristic of O’Donoghue and Candes [1].

1 Introduction

Periodically re-initializing an algorithm from its output, known as restarting, is a popular method for generically speeding up optimization algorithms. More precisely, given an algorithm with a sublinear convergence rate and an additional regularity assumption (e.g., strong convexity) periodically restarting the algorithm can improve its convergence from sublinear to linear [2]. The difficulty is in choosing the length of the restart period. One can resolve this issue by running a grid search on a log-scale to find the best restart period [3], but this can be many times more computationally intensive than running with a single restart period. Furthermore, a restart interval length that changes as the algorithm runs, adapting to the local curvature of a function, may outperform a fixed restart period. Addressing these two issues motivates the study of adaptive restart schemes.

Adaptive restarting was pioneered by O’Donoghue and Candes [1] for Nesterov’s accelerated gradient descent (AGD) [4]. Their paper suggests restarting AGD whenever the function value increases. They observe that this heuristic is competitive with selecting the best restart period via hyperparameter search. They also provide a theoretical analysis showing that for quadratics, as the number of iterations tends to infinity, the worst-case performance of their method is almost optimal. Despite its success in practice, one drawback of O’Donoghue and Candes’s [1] restart scheme is that its theory only applies to quadratics. A large number of papers have followed this work by trying to develop restart schemes that avoid hyperparameter searches while obtaining stronger guarantees. These schemes either miss the optimal bound by log factors [5–7], or are based on maintaining a monotonely increasing minimum restart interval length which may limit adaptivity [8, 9].

Furthermore, there is little study of restart schemes in the context of saddle point algorithms. Cham-bolle and Pock [10] show that a restart scheme can be used to improve the sublinear converge rate of primal-dual hybrid gradient (PDHG) under a uniform convexity assumption. Zhao [11] explores
restart schemes for stochastic PDHG. However, these papers neither offer adaptive schemes nor provide linear convergence guarantees. There are many papers showing the linear convergence of saddle point algorithms without restarts [12–16]. For example, in the bilinear setup it is known that the last iterate of PDHG [17] and extragradient [18, 19] converge linearly to the optimal solution. However, all these results obtain suboptimal convergence guarantees, even in the simple bilinear setup.

Our contributions

1. We provide a simple adaptive restart scheme where restarts are triggered using a distance based potential function. In a general setup we precisely characterize the linear convergence rate of this scheme.

2. We apply this scheme to specific algorithms. For PDHG it obtains the optimal convergence rate for bilinear games. To the best of our knowledge this provides the first example of this rate being achieved by a saddle point algorithm. For AGD on smooth strongly convex functions it yields the optimal convergence rate up to small constant factors.

3. We run experiments with this restart scheme. For PDHG, on matrix games and linear programming relaxations of quadratic assignment problems, our scheme dramatically improves on the ability of “vanilla” PDHG to find high accuracy solutions. For AGD, on regularized logistic regression and LASSO problems, our scheme is competitive with the scheme of O’Donoghue and Candes [1].

Notation

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+$ the positive real numbers, and $\mathbb{N}$ the set of natural numbers (starting from one). Let $\ln(\cdot)$ and $e$ refer to the natural log and exponential respectively. Let $\lambda_{\min}(\cdot)$ denote the minimum eigenvalue of a matrix. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ for $n, m \in \mathbb{N}$ be closed convex sets and denote $W = X \times Y$ where for all $w \in W$ we denote $w_x \in X$ as the primal component and $w_y \in Y$ as the dual component. Let $\| \cdot \|$ be an arbitrary norm on elements of $W$ and define $B_r(w) := \{ \hat{w} \in W : \| w - \hat{w} \| \leq r \}$.

2 Our adaptive restart scheme

Function \texttt{GenericAdaptiveRestartScheme}(\omega_0):

\begin{verbatim}
for $i = 1, \ldots, \infty$ do
    $u_i^0, s_i^0 \leftarrow \text{InitializeAlgorithm}(\omega_{i-1})$;
    $t \leftarrow 0$;
    repeat
        $t \leftarrow t + 1$;
        $u_i^t, s_i^t \leftarrow \text{OneStepOfAlgorithm}(u_i^{t-1}, s_i^{t-1})$;
    until restart condition (1) holds;
    $\tau_i \leftarrow t, \omega_i \leftarrow u_i^t$;
end
\end{verbatim}

\texttt{GenericAdaptiveRestartScheme} presents our restart scheme. We use $i$ to index the restart epochs, $\tau_i$ to represent the length of the $i$th restart epoch, and $\omega_i$ to be the point produced by the algorithm at the end of the $i$th restart epoch. Within each restart epoch $i$, $t$ keeps track of the number of iterations. The object $s_i^t$ is the internal state of the algorithm. For example, for AGD it would include momentum information; for PDHG it contains the current iterate and $t$. The vector $u_i^t$ is what the algorithm outputs at iteration $t$ to obtain its sublinear convergence guarantees. For example, for primal dual hybrid gradient this would be the running average of the iterates since the last restart.

Our restart condition. Let $\beta \in (0, 1)$ be an instance-independent parameter (e.g., $\beta = 1/2$) and $\phi : \mathbb{R} \to \mathbb{R}$ a instance-independent function. For $i > 1$, restart if

$$\frac{\| u_i^t - \omega_{i-1} \|}{\phi(t)} \leq \beta \frac{\| \omega_{i-1} - \omega_{i-2} \|}{\phi(\tau_{i-1})};$$

(1)
for $i = 1$, for simplicity we suggest restarting when $t = 1$, but for the analysis we allow the initial restart length $\tau_1$ to be arbitrary. As detailed in Section 5, the function $\phi$ is selected to correspond to the sublinear convergence bound of the algorithm. For example, AGD converges proportional to $1/(t + 1)^2$, and therefore $\phi(t) := (t + 1)^2$. Similarly, PDHG converges proportional to $1/t$, and therefore $\phi(t) := t$.

There are many other possible restart conditions found in literature. For example, a popular restart scheme for AGD is to restart when $f(w_i^t) - \inf_{w \in W} f(w) \leq \beta (f(\omega_{i-1}) - \inf_{w \in W} f(w))$. In other words, when the potential function $f(w_i^t) - \inf_{w \in W} f(w)$ is reduced by a constant factor. Unfortunately, this assumes knowledge of the minimum of $f$, which is rarely available. Our restart scheme can also be viewed as decreasing the potential function

$$\phi(t) := \left\| w_i^t - \omega_{i-1} \right\|$$

but unlike, $f(w_i^t) - \inf_{w \in W} f(w)$ this quantity is always computable. The idea behind choosing this as a potential function is that for many algorithms as (2) decreases the distance to optimality decreases. This is made explicit later in Lemma 3.

3 Assumptions required for adaptive restarts

This section defines a broad set of conditions under which we will be able to adaptively restart an algorithm to obtain linear convergence. The main property we require is that without restarts the algorithm reduces the localized duality gap which is defined as

$$\Delta_r(w) := \sup_{(x,y) \in B_r(w)} f(w_x, y) - f(x, w_y)$$

where $r \in \mathbb{R}^+$, $w_x$ denotes the primal iterate and $w_y$ the dual iterate. We prepend the word ‘localized’ to ‘duality gap’ because we radius $r$ may be significantly smaller than the distance to optimality. If the radius is larger than the distance to optimality then it represents a valid duality gap. Let $W^* := \{ w \in W : \Delta_r(w) = 0, \forall r \in \mathbb{R}^+ \}$ be the set of optimal solutions. Note if we are interested in only minimizing a function then we can let $W = X$ and (3) simplifies to $\Delta_r(w) = f(w) - \inf_{\tilde{w} \in B_r(w)} f(\tilde{w})$.

**Assumption 1.** Suppose that for all $w \in W$ and $r_a \in (0, \infty)$, $r_b \in [0, \infty)$ the inequality $\Delta_{r_a}(w) \leq \max\{1, r_b/r_a\} \Delta_{r_b}(w)$ holds.

Assumption 1 is extremely mild, and as Lemma 1 demonstrates it suffices for $f$ to be convex-concave for it to hold. The proof of Lemma 1 appears in Appendix A.1.

**Lemma 1.** Let $X$ and $Y$ be closed convex sets. Also, suppose $f(x, y)$ is continuous, convex in $x$ for all $y \in Y$, and concave in $y$ for all $x \in X$ then Assumption 1 holds.

**Function** GenericIterativeAlgorithm($\omega$):

1. $w^0, s^0 \leftarrow$ InitializeAlgorithm($\omega$);
2. for $t = 1, \ldots, \infty$ do
   1. $w^t, s^t \leftarrow$ OneStepOfAlgorithm($w^{t-1}, s^{t-1}$);
3. end

Consider a generic iterative algorithm described in GenericIterativeAlgorithm. To prove our results we need $\Delta_r(w)$ to be reduced in a predictable manner as per Assumption 2.

**Assumption 2.** Consider the sequence $\{w^t\}_{t=0}^\infty$ generated by GenericIterativeAlgorithm. We assume that for all $w \in W$ and $t \in \mathbb{N}$, $\Delta_{r}\Delta_{r_b}(w^t) \leq \frac{C(1+\beta)^2\epsilon^2}{\phi(t)}$ where $r = \|w^t - \omega\|$.

Assumption 2 is strongly related to the standard sublinear convergence bound for an algorithm. For example, as we show in Section 5, for AGD $\beta$ takes any value in $(0, 1)$, $C = 2L$ and $\phi(t) = (t + 1)^2$ where $L$ is the smoothness constant of $f$. We note that if $\phi$ takes the form $\phi(k) = (k + c_1)c_2$ for constants $c_1 \geq 0$ and $c_2 > 0$ then $\phi$ is a strictly increasing log concave function.
Assumption 3. Consider the sequence \( \{w^t\}_{t=0}^\infty \) generated by GenericIterativeAlgorithm. Suppose that there exists some function \( Q : \mathbb{R} \rightarrow \mathbb{R}^+ \) that is bounded from above and satisfies \( \|w^t - w^*\| \leq Q(t)\|w - w^*\| \) for all \( w^* \in W^* \) and \( t \in \mathbb{N} \).

The choice of the function \( Q \) in Assumption 3 affects the final linear convergence bound. To achieve the tightest linear convergence bound one should aim to select the smallest (valid) \( Q(t) \) possible. Usually \( Q(t) > 1 \) and it may even be a constant as is the case for PDHG. We further need the localized duality gap to satisfy an error bound property, this is the purpose of Assumption 4.

Assumption 4. There exists some constant \( \theta \in (0, \infty) \) such that for all \( w \in W \) that if \( r^* = \|W^* - w\| \) then \( \theta r^* \leq \Delta_r(w) \).

Theorem 1. Consider the sequence \( \{w^t\}_{t=0}^\infty \) generated by GenericIterativeAlgorithm. Suppose \( t \in \mathbb{N} \), and that Assumptions 1, 2, and 4 hold. Then, \( \|W^* - w^t\| \geq \beta \|w^t - \omega\| \).

Proof. We obtain,

\[
\theta \|W^* - w^t\|^2 \leq \Delta_{W^* - w^t}\|w^t\|^2
\]

by Assumption 4,

\[
\leq \max \left\{ \frac{\|W^* - w^t\|}{\beta \|w^t - \omega\|} \right\} \Delta_{\beta \|w^t - \omega\|}(w^t)
\]

by Assumption 1,

\[
\leq \max \left\{ \frac{\|W^* - w^t\|}{\beta \|w^t - \omega\|} \right\} C(1 + \beta)^2 \|w^t - \omega\|^2
\]

by Assumption 2.

Rearranging to bound \( \|W^* - w^t\| \) gives the result. \( \square \)

Define \( t^* \in [1, \infty) \) as any solution to

\[
\frac{\phi(t^* - 2)}{(1 + Q(t^* - 2))^2} \geq \frac{(1 + \beta)^2}{\beta^2} \hat{\kappa}.
\]

Note a solution to (4) must exist because \( \phi(t) \) is unbounded from above and \( Q(t) \) is bounded from above. The quantity \( t^* \) is useful for summarizing our results. In particular, as detailed in the proof of Theorem 1, \( t^* \) represents an upper bound on the restart interval length given \( \tau_1 \leq t^* \). But first we prove Lemma 4 which makes a statement on when restarts are triggered. The proof of Lemma 4 appears in Section B.1.

Lemma 4. Consider GenericAdaptiveRestartScheme. Suppose that Assumptions 1, 2, 3, and 4 hold. If \( \phi(t) \geq \max\{\sqrt{\phi(t^* - 2)} / \phi(\tau_{i-1}), \phi(t^* - 2)\} \) and \( t \in \mathbb{N} \) then \( \frac{\|w^t - \omega_{t-1}\|}{\phi(t)} \leq \frac{\|w_{t-1} - \omega_{t-2}\|}{\phi(\tau_{i-1})} \), i.e., a restart is triggered.

4
Theorem 1. Let $\epsilon \in (0, 1)$ and suppose that Assumptions 1, 2, 3, and 4 hold. Consider the sequence \( \{\omega_i\}_{i=0}^\infty \), \( \{\tau_i\}_{i=1}^\infty \) generated by GenericAdaptiveRestartScheme. Then for
\[
n = \left\lceil \log_{1/\beta} \left( \frac{1 + Q(\tau_1)}{\epsilon} \max \left\{ 1, \frac{\phi(t^*)}{\tilde{\phi}(\tau_1)} \right\} \right) \right\rceil
\]
the inequality \( \frac{\|W^* - \omega_n\|}{\|W^* - \omega_0\|} \leq \epsilon \) holds and \( \sum_{i=1}^n \tau_i \leq t^* n + 2(\tau_1 - t^*)^+ \).

The proof of Theorem 1 appears in Appendix B.2. Proof sketch. To show that \( \frac{\|W^* - \omega_n\|}{\|W^* - \omega_0\|} \leq \epsilon \) we use that the potential function is decreasing by $\beta$ at each iteration and that it upper bounds the distance to optimality (Lemma 3). We use carefully use Lemma 4 to bound \( \sum_{i=1}^n \tau_i \).

5 Theory applied to specific algorithms

Definition 1. A function $f : X \to \mathbb{R}$ is strongly convex if for all $x \in X$, $\lambda_{\min}(\nabla^2 f(x)) \geq \alpha$.

Definition 2. A differentiable function $f : W \to \mathbb{R}$ is $L$-smooth if \( \|\nabla f(w) - \nabla f(w')\|_2 \leq L \|w - w'\|_2 \) for all $w, w' \in W$.

Fact 1 will be useful for showing standard sublinear bounds imply Assumption 2. In particular, it allows us to change the center of the ball in a bound from $\bar{w}$ to $w$.

Fact 1. For any $\bar{w}, w \in W$ with $r = \|w - \bar{w}\|$ we have $\sup_{(x, y) \in B_{\beta r}(\bar{w})} (\tilde{w}, y) \leq \sup_{(x, y) \in B_{(1 + \beta r)}(\bar{w})} f(w_x, y) - f(x, y)$.

Proof. Holds because by the triangle inequality: $B_{\beta r}(\bar{w}) \subseteq B_{(1 + \beta r)}(\bar{w})$. \( \square \)

5.1 Primal dual hybrid gradient

Assumption 5 (Standard setup for PDHG adapted from [10]). Let $X$ be a closed convex set, $f(x, y) = y^T A x + G(x) - F^*(y)$ where $F : X \to \mathbb{R}$ and $G : X \to \mathbb{R}$ are continuous convex functions.

By Assumption 5, the convex conjugate $F^* : Y \to \mathbb{R}$ is also a convex and continuous, with $Y$ closed and convex. We will use the Euclidean norm to measure distances and $\gamma \in (0, \infty)$ as the step size for the algorithm\(^1\).

Lemma 5 (Theorem 1 of [10]). Suppose Assumption 5 holds. Let $L = \|A\|_2$, then AdaptiveRestartPDHG with $L_\gamma < 1$ satisfies, $\sup_{(\bar{x}, \tilde{y}) \in B_{\bar{w}}(\bar{y})} (x^T_1, \tilde{y}) - f(x, y) \leq \frac{\|\gamma \|}{\gamma}$ where $x_1 = x_1, \tilde{y} = \omega_{i-1}$, for all $R \in R^+$. Furthermore, $\|w_{i+1} - w^*\|_2 \leq (1 - \gamma^2 L^2)^{-1/2} \|\omega_{i-1} - w^*\|_2$ for all $w^* \in W^*$.

Theorem 2. Suppose Assumptions 4 and 5 hold. Let $L = \|A\|_2$, $\theta \in (0, L)$, $\gamma \in (0, 1 / L)$, $\beta \in (0, 1)$, and $\epsilon \in (0, 1)$. Define $q := (1 - \gamma^2 L^2)^{-1/2}$ and $t^* := \frac{(1 + \vartheta)^2(1 + \beta)^2}{\gamma^2 \gamma} + 2$. Consider the sequence \( \{\omega_i\}_{i=0}^\infty \), \( \{\tau_i\}_{i=1}^\infty \) generated by AdaptiveRestartPDHG. Then for
\[
n = \left\lceil \log_{1/\beta} \left( \frac{1 + q}{\epsilon \max \left\{ 1, \frac{t^*}{\tau_1} \right\} \right) \right\rceil
\]
the inequality \( \frac{\|W^* - \omega_n\|}{\|W^* - \omega_0\|} \leq \epsilon \) holds and \( \sum_{i=1}^n \tau_i \leq t^* n + 2(\tau_1 - t^*)^+ \).

Proof. By Lemma 1 and Assumption 5, Assumption 1 holds. By combining Fact 1 and 5 with $R = (1 + \beta) r$, we observe that Assumption 2 holds with $\phi(t) = t$ and $C = \frac{\beta}{\gamma}$. Lemma 5 implies Assumption 3 holds with $Q(t) := q$. Therefore we have established the premise of Theorem 1 which implies the desired result. \( \square \)

\(^1\)For simplicity we assume that the primal and dual step sizes are equal. However, all these results go through when the primal and dual step sizes are different. One can obtain such results by diagonal scaling of the primal and dual variables, or by changing the norm to $\|w\| = \sqrt{\|w_x\|_2^2 / \gamma_x + \|w_y\|_2^2 / \gamma_y}$ where $\gamma_x, \gamma_y \in (0, \infty)$ are the primal and dual step sizes, respectively.
Corollary 1 simply instantiates Theorem 2 with $\beta = 1/2$ and $\gamma = 0.7L$. These particular values are chosen to (approximately) minimize the constant factors.

**Corollary 1.** Suppose Assumption 4 and 5 holds. Let $L = \|A\|_2$, $\theta \in (0, L]$, $\gamma = 0.7L$, $\beta = 0.5$, and $\epsilon \in (0, 1)$. Then AdaptiveRestartPDHG requires at most $57 \frac{L}{\theta} \ln \left( \frac{2}{\theta \epsilon} \right) + \max \left\{ 57 \frac{L}{\theta} \ln \left( \max \left\{ 1, \frac{77L}{\theta \epsilon} \right\} \right), 2\tau_1 \right\}$ calls to OneStepOfPDHG until some $w_n$ satisfies $\|w_n - w^*\|_2 \leq \epsilon$.

**Proof.** Follows from the bound in Theorem 3. In particular, observing that for $\gamma = 0.7L$, $\beta = 1/2$ we have $q = 0.51^{-1/2}, 1 + 0.51^{-1/2} \leq 3$, $t^* = \frac{(1+q)^2(1+\beta)^2}{2\gamma^2} + 2 \leq \frac{(1+0.51^{-1/2})^2 \eta}{0.7 \times 2} \leq 39 \frac{L}{\theta}$, $t^*/\ln(2) \leq 57 \frac{L}{\theta}$. 

Using almost the same argument, one can also show that our restart scheme applied to extragradient obtains the same worst-case complexity as Corollary 1 (up to constant factors). We give the details for this in Appendix C.2.

Let us discuss our result in the bilinear setup. Recall that by Lemma 2 in the bilinear setup $\theta$ is the minimum nonzero singular value of $A$ which we denote by $\sigma_{\min}$. Using standard lower bound arguments one can show that our bound $O(L^2/\sigma_{\min}^2 \ln(1/\epsilon))$ is optimal for bilinear games (see Appendix C.3). In contrast, the best known guarantees for saddle point algorithms in the bilinear setup [18, 19] study the last iterate and give bounds of the form $O(L^2/\sigma_{\min}^2 \ln(1/\epsilon))$.

### 5.2 Accelerated gradient descent

For this subsection, let $f(x) = a(x) + b(x)$ where $a : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function and $b : \mathbb{R}^n \to \mathbb{R}$ is a continuous convex function which is possibly nonsmooth. We assume that the quantity $p_t(y) := \argmin_{y \in \mathbb{R}^n} \nabla a(y)^T(x - y) + b(x) + \frac{1}{2\tau} \|x - y\|^2_2$ is easily computable. We use FISTA with backtracking line search [24] as the basic algorithm that we then integrate with GenericAdaptiveRestartScheme. We label this integrated algorithm AdaptiveRestartAGD and include its full description in Appendix C.1. For conciseness, define: $\tilde{\kappa} := \frac{L \eta}{\theta}$ where $\eta \in (1, \infty)$ is the backtracking parameter for AdaptiveRestartAGD, $f$ is $L$-smooth and $\alpha$-strongly convex. Theorem 3 is an application of Theorem 1 to AGD. The proof involves establishing Assumptions 1, 2, 3, and 4 which is follows from standard results [24]. Corollary 2 simply instantiates Theorem 3 with $\beta = 1/4$. The value $\beta = 1/4$ is designed to (approximately) minimize the constant factors. The proof of Theorem 3 and Corollary 2 appears in Appendix C.1.1.

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For comparing with [18] note that in their setup $L = \sqrt{\lambda_{\max}(A^T A)}$ and $\sigma_{\min} = \sqrt{\lambda_{\min}(A^T A)}$. 

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Theorem 3. Let $\beta, \epsilon \in (0, 1)$ and suppose that $f$ is $\alpha$-strongly convex and $L$-smooth with minimizer $w^*$. Define $t^* := 1 + \sqrt{k}(\rho + \sqrt{\rho^2 + 4\rho})$ where $\rho := \frac{1+\beta}{\beta}$. Consider the sequence $\{\omega_i\}_{i=0}^\infty, \{\tau_i\}_{i=1}^\infty$ generated by AdaptiveRestartAGD. Then for

$$n = \left\lfloor \log_{1/\beta}(2/\epsilon) + 3 \log_{1/\beta}(\max\{1, t^*/\tau_1\}) \right\rfloor$$

the inequality $\frac{\|w^* - \omega_{i+1}\|}{\|w^* - w_i\|} \leq \epsilon$ holds and $\sum_{i=1}^n \tau_i \leq t^* n + 2(t^* - t_1)^{+}$. 

Corollary 2. Suppose $f$ is $\alpha$-strongly convex and $L$-smooth with minimizer $w^*$. Let $\beta = 1/4$ and $\epsilon \in (0, 1)$. Then AdaptiveRestartAGD requires at most $8.5(\sqrt{n} + 1) \ln (\frac{2}{\epsilon}) + \max\left\{26\sqrt{n} \ln \left(\frac{12\sqrt{n}}{\tau_i}\right), 2\tau_1\right\}$ calls to OneStepOfAGD until some $\omega_{i_0}$ satisfies $\frac{\|w^* - \omega_{i_0}\|}{\|w^* - w_{i_0}\|} \leq \epsilon$.

Compared with other restart schemes in literature the coefficient of $8.5$ on the $\sqrt{n} \ln(1/\epsilon)$ term in Corollary 2 is small. For [8] (with $\mu_0 = 1$) the coefficient is $\approx 32$, for [9] it is $\approx 45$, and an exact hyperparameter search on the restart period yields a coefficient of $\approx 5.4$ (Remark 2).

6 Numerical results

We test the proposed adaptive restart scheme applied to the PDHG and AGD algorithms, comparing both with no restarts and with the best fixed-period restart chosen by grid search. For AGD we additionally compare with [1]. For PDHG, we modified the implementation in the Python library ODL [25], while we implemented AGD in Julia. All experiments use $\tau_1 = 1$. We note that in these experiments, the adaptive restart scheme is applied heuristically as we expect, but have not established, that Assumption 4 holds globally. In all the examples involving PDHG (Figure 1 and 2) the average performs much worse than the current so we only plot the current iterate for the no-restart scheme. For the restart schemes, as our theory suggests, we plot the running average since the last restart. All source code is available at https://github.com/google-research/generic-adaptive-restarts. More details on the experiments are found in Appendix D.

6.1 Matrix games

Given a matrix $A \in \mathbb{R}^{m \times n}$, a matrix game is the simple saddle-point problem defined by $f(x, y) = y^T A x$, $X = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x \geq 0\}$, and $Y = \{y \in \mathbb{R}^m : \sum_{j=1}^m y_j = 1, y \geq 0\}$. Following [26], we use synthetic instances from two families with $m = n = 100$: first where the coefficients of $A$ are sampled independently uniformly over $[-1, -\frac{1}{n}]$, and second from a standard Normal distribution. We measure the saddle-point residual defined as $\Delta_{\infty}(w)$ at each iteration. We solve 50 instances from each family and plot the median residual with error bands for the 10th to 90th percentile. Results are in Figure 1. The adaptive scheme is consistent with the best fixed restart scheme, and both are dramatically better than PDHG without restarts.

![Figure 1: Matrix games (normal on left, uniform on right). The best fixed restart period is found via grid search on {8, 32, 128, 512, 2048}.](image-url)
6.2 Quadratic assignment problem relaxations

We select two linear programming (LP) instances, qap15 and nug08-3rd, from the Mittelmann collection set [27], a standard benchmark set for LP. These two problems are relaxations of quadratic assignment problems [28], a classical NP-hard combinatorial optimization problem. These instances are known to be challenging for traditional solvers and amenable to first-order methods [29].

We encode the LP \( \min_x c^T x \) subject to \( Ax = b \), \( l \leq x \leq u \) in saddle-point form as \( f(x, y) = c^T x - y^T Ax - b^T y \), \( X = \{ x \in \mathbb{R}^n : l \leq x \leq u \} \), and \( Y = \mathbb{R}^m \). Given a point \((\hat{x}, \hat{y})\), the residual is measured as the \( \ell_2 \) norm of the vector concatenating the primal infeasibilities, dual infeasibilities, and the primal-dual objective gap. We additionally calculated the last iteration at which there is a change in the set of variables at either their upper or lower bounds as the last active set change, a point that appears associated with the beginning of faster convergence. Results are in Figure 2.

![Figure 2: PDHG performance on LP instances nug08-3rd (left), qap15 (right). The best fixed restart period is found via grid search on \{64, 256, 1024, 4096, 16384, 65536\}. Dots indicate restarts.](image)

6.3 Logistic regression and LASSO

To test our restart scheme for AGD we run on L1-regularized logistic regression and LASSO problems downloaded from the LIBSVM dataset [30, https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/]. We solve problems of the form \( f(x) = \sum_i l_i(a_i^T x) + \lambda \|x\|_1 \) where \( \lambda \) is the regularization parameter, \( a_i \) is the ith row of the data matrix, and \( l_i \) is the loss function. Let \( b_i \) denote the data label. For LASSO we use \( l_i(c) = \frac{1}{2}(c - b_i)^2 \), for L1-regularized logistic regression we use the log logistic loss \( l_i(c) = \log(1 + \exp(c \text{sign}(b_i))) \). The data matrix is preprocessed by (i) removing empty columns, (ii) adding an intercept, and (iii) normalizing the columns. Statistics for the problems are given in Table 2 in the Appendix. We run AdaptiveRestartAGD with \( l_0 = 1 \), \( \eta = 5/4 \) and \( \beta = 1/4 \). Figure 3 shows that our scheme is competitive with the function scheme of O’Donoghue and Candes [1] and in some instances (Duke breast cancer) does much better.

![Figure 3: Tests on AGD. From left to right: E2006-tfidf (LASSO), rcv1.binary (logistic), Duke breast cancer (logistic). The best fixed restart period is found via grid search on \{128, 256, 512, 1024, 2048\}. Dots indicate restarts. “O’D and C” is the scheme of [1].](image)

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A Proofs from Section 3

A.1 Proof of Lemma 1

Proof. Suppose that \( r_a \geq r_b \), then it immediately follows that
\[
\sup_{(x,y) \in B_r(x)} f(w, y) - f(x, w) \leq \sup_{(x,y) \in B_r(x)} f(w, y) - f(x, w).
\]
Therefore consider the case that \( r_a \leq r_b \). Let \((x_a, y_a) \in \arg\max_{(x,y) \in B_{r_a}(x)} f(w, y) - f(x, w) \) and \((x_b, y_b) \in \arg\max_{(x,y) \in B_{r_b}(x)} f(w, y) - f(x, w) \). By convexity \( f(w + \lambda (x_b - x_w), w_y) \leq (1 - \lambda)f(w) + \lambda f(x_b, w_y) \) with \( \lambda = r_a/r_b \). Rearranging this inequality yields:
\[
f(w) - f(x_b, w_y) \leq \frac{f(w) - f(x_a, w_y)}{\lambda} = \frac{r_b}{r_a} (f(w) - f(x_a, w_y))
\]
where the second inequality uses that \((x_b, w_y) \in B_{r_a}(x) \). By the same argument (using concavity of \( f \) in \( y \) instead of convexity of \( f \) in \( x \)),
\[
f(w, y_b) - f(w) \leq \frac{r_b}{r_a} (f(w, y_a) - f(w))
\]
Adding these inequalities together yields,
\[
f(w, y_b) - f(x_b, w_y) \leq \frac{r_b}{r_a} (f(w, y_a) - f(x_a, w_y))
\]
as required. \( \square \)

A.2 Proof of Lemma 2

Proof. Start by considering the special case that \( c = 0, b = 0 \), and \( A \) is diagonal (possibly nonsquare). Later we will show the general case reduces to this. Let \( D \) be a diagonal matrix with entries
\[
D_{ii} = \begin{cases} 1 & A_{ii} \neq 0 \\ 0 & A_{ii} = 0 \end{cases}
\]
Let \( X^* \times Y^* \) denote the set of saddle points for this problem. Then \( X^* = \{ x : Ax = 0 \} = \{ x : Dx = 0 \}, Y^* = \{ y : y^T A = 0 \} = \{ y : y^T D = 0 \}. \) In which case
\[
\| W^* - w \|_2 = \left\| \begin{pmatrix} Dx \\ y^T D \end{pmatrix} \right\|_2,
\]
which implies that
\[
\Delta_r(w) = \sup_{(x,y) \in B_{r^*}(w)} f(w, y) - f(x, w)
\]
\[
\geq r^* \left\| \begin{pmatrix} Dx \\ y^T A \end{pmatrix} \right\|_2
\]
\[
\geq \sigma r^* \left\| \begin{pmatrix} Dx \\ y^T D \end{pmatrix} \right\|_2
\]
\[
\geq \sigma (r^*)^2.
\]
Next, consider the general case. We will reduce it to the case we just analyzed by shifting and rotating the space. In particular, consider the singular value decomposition \( A = U \Sigma V^T \) where \( \Sigma \) is an \( n \times m \) diagonal matrix, and the matrices \( U \) and \( V \) are orthogonal. Define \( \bar{x} = V^T(x - x^*) \) and \( \bar{y} = U^T(y - y^*) \). Therefore,
\[
c^T x + y^T A x + b^T y
\]
\[
= c^T \bar{x} + (y - y^*)^T A (x - x^*) + b^T y + (y^*)^T A x + y^T A x^* - (y^*)^T A x^*
\]
\[
= c^T \bar{x} + (y - y^*)^T A (x - x^*) + b^T y - c^T x - b^T y - (y^*)^T A x^*
\]
\[
= \bar{y}^T \Sigma \bar{x} - (y^*)^T A x^*.
\]
where the third inequality uses that $Ax^* = -b$ and $(y^*)^T A = -c$. Note that because $U$ and $V$ are orthogonal: (i) $(\bar{y}, \bar{x})$ is a saddle point for $\gamma^T \Sigma \bar{X}$ if and only if $(x, y)$ is a saddle point for $f$, (ii) $\|\bar{x} - \bar{x}'\|_2 = \|V(x - x')\|_2 = \|x - x'\|_2$ and $\|\bar{y} - \bar{y}'\|_2 = \|U^T(y - y')\|_2 = \|y - y'\|_2$ for $\bar{x}' = V^T(x - x^*)$ and $\bar{y}' = U^T(y - y^*)$. It follows that the result holds in the general case. □

B Proof of results from Section 4

B.1 Proof of Lemma 4

**Lemma 6.** Consider the sequence $\{w^t\}_{t=0}^\infty$ generated by GenericIterativeAlgorithm. Suppose that $t \in N$ and Assumption 3 holds then $\|w^t - \omega\| \leq (1 + Q(t))\|W^* - \omega\|$.

**Proof.** Let $w^* := \arg\min_{w \in W^*} \|w - \omega\|$. By the triangle inequality and definition of $Q(t)$ (as given in Assumption 3), $\|w^t - \omega\| \leq \|w^t - w^*\| + \|w^* - \omega\| \leq (1 + Q(t))\|w^* - \omega\|$. □

**Lemma 7.** Consider GenericAdaptiveRestartScheme. Suppose that Assumptions 1, 2, 3, and 4 hold. If $\phi(t) \geq (1 + Q(t))\frac{1 + \beta}{\beta} \max\{\sqrt{k\phi(t_1 - 1)}, \frac{k}{\beta}\}$ then $t \in N$ then $\frac{\|w^*_t - \omega\|_1}{\phi(t)} \leq \frac{1}{\beta\phi(t_1 - 1)}$.

**Proof.** If $\|W^* - \omega_1\| \geq \beta\|\omega_1 - \omega_2\|$ then

$$\frac{\|W^* - \omega_1\|}{\phi(t)} \leq \frac{1}{\phi(t)} \left(1 + \beta\right)\frac{\tilde{\kappa}}{\phi(t_1 - 1)} \|\omega_1 - \omega_2\|$$

(Lemma 3)

By the assumed lower bound on $\phi(t)$.

If $\|W^* - \omega_1\| < \beta\|\omega_1 - \omega_2\|$ then

$$\frac{\|W^* - \omega_1\|}{\phi(t)} \leq \frac{1}{\phi(t)} \left(1 + \beta\right)\sqrt{\frac{\tilde{\kappa}}{\phi(t_1 - 1)}} \|\omega_1 - \omega_2\|$$

(Lemma 3)

By the assumed lower bound on $\phi(t)$.

Therefore,

$$\frac{\|w^*_t - \omega_1\|}{\phi(t)} \leq \frac{\|W^* - \omega_1\|}{\phi(t)} \leq \frac{\|W^* - \omega_1\|}{\phi(t_1 - 1)} \leq \beta\frac{\|\omega_1 - \omega_2\|}{\phi(t_1 - 1)}$$

where the first inequality uses Lemma 6, and the second inequality uses the established bound. □

**Proof of Lemma 4.** The inequality $\phi(t) \geq \max\{\sqrt{\phi(t^* - 2)}\phi(t_1 - 1), \phi(t^* - 2)\}$ implies that $\phi(t) \geq \phi(t^* - 2)$. Therefore $t \geq t^* - 2 \Rightarrow Q(t) \leq Q(t^* - 2)$. It follows by (4) and $1 \leq Q(t) \leq Q(t^* - 2)$ that

$$(1 + Q(t))\frac{1 + \beta}{\beta} \max\{\sqrt{\phi(t_1 - 1)}\phi(t^* - 2), \frac{1 + \beta}{\beta}\} \leq \max\{\sqrt{\phi(t^* - 2)}\phi(t_1 - 1), \phi(t^* - 2)\}.$$ 

The result follows by Lemma 7. □

B.2 Proof of Theorem 1

**Lemma 8.** Suppose that Assumptions 1, 2, 3, and 4 hold. Then, GenericAdaptiveRestartScheme generates a sequence $\{\tau_i\}_{i=1}^\infty$ satisfying $\tau_i \leq \max\{\tau_i, t^* - 1\}$ and $\sum_{j=1}^\infty (\tau_j - t^*)^+ \leq 2(\tau_1 - t^*)^+$.

**Proof.** By Lemma 4 and (1) we have

$$\phi(\tau_i - 2) < \max\left\{\phi(t^* - 2), \sqrt{\phi(t^* - 2)}\phi(t_1 - 1)\right\} \leq \max\left\{\phi(t^* - 2), \phi\left(\frac{\tau_1 - t^*}{2}ight)\right\}.$$ 

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If \( t_{i-1} \leq t^* - 2 \) then \( \phi(t_i - 1) < \phi(t^* - 2) \Rightarrow t_i - 1 < t^* - 2 \Rightarrow t_i < t^* - 1 \). If \( t_{i-1} > t^* - 2 \) then by log concavity of \( \phi \) we have \( \phi(t_i - 1) < \phi\left(\frac{t_{i-1} + t^*}{2}\right) \) which by monotonicity of \( \phi \) implies \( t_i \leq \frac{t_{i-1} + t^*}{2} \). Subtracting \( t^* \) from the latter inequality and using induction it follows that if \( t_j > t^* \) for all \( j \in \{1, \ldots, i\} \) then \( t_j - t^* \leq \frac{t_1 - t^*}{2^{i-j+1}} \leq \frac{t_1 - t^*}{2^{i-1}} \). This implies for all \( n \in \mathbb{N} \) that \( \sum_{i=1}^n (t_i - t^*)^+ \leq (t_1 - t^*)^+ \sum_{i=1}^n 2^{i-1} \leq 2(t_1 - t^*)^+ \), where the last inequality uses the standard bound on the sum of a geometric series.

**Lemma 9.** Consider the sequence \( \{\omega_i\}_{i=0}^{\infty} \) generated by \texttt{GenericAdaptiveRestartScheme}. Suppose that Assumptions 1, 2, 3, and 4 hold. Then for all \( n \in \mathbb{N} \), \( \|W^* - \omega_n\| \leq \|\omega_1 - \omega_0\| \max \left\{1, \frac{\phi(t^*)}{\phi(t_1)}\right\} \beta^n \).

**Proof.** By definition of \texttt{GenericAdaptiveRestartScheme}, for all \( i \in \mathbb{N} \) with \( i > 1 \), \( \frac{\omega_i - \omega_{i-1}}{\phi(t_{i-1})} \leq \frac{\omega_i - \omega_{i-1}}{\phi(t_{i-1})} \) which by induction implies

\[
\|\omega_n - \omega_{n-1}\| \leq \|\omega_1 - \omega_0\| \frac{\phi(t_n)}{\phi(t_1)} \beta^{n-1}.
\]

Moreover,

\[
\|W^* - \omega_n\| \leq \max \left\{1, \frac{\phi(t^*)}{\phi(t_n)}\right\} \|\omega_n - \omega_{n-1}\|
\]

by Lemma 3, or

\[
\|W^* - \omega_n\| \leq \beta \|\omega_n - \omega_{n-1}\| \|\omega_1 - \omega_0\| \frac{\phi(t^*)}{\phi(t_n)} \beta^{n-1}.
\]

By the previous inequality and (5) we deduce

\[
\|W^* - \omega_n\| \leq \|\omega_1 - \omega_0\| \beta \max \left\{1, \frac{\phi(t^*)}{\phi(t_n)}\right\} \frac{\phi(t_n)}{\phi(t_1)} \beta^n = \|\omega_1 - \omega_0\| \max \left\{\frac{\phi(t_n)}{\phi(t_1)} \frac{\phi(t^*)}{\phi(t_n)}\right\} \beta^n.
\]

If \( \frac{\phi(t_n)}{\phi(t_1)} > 1 \) \( \Rightarrow t_n > t_1 \) then by Lemma 8 \( t_1 \leq t^* \Rightarrow \frac{\phi(t_n)}{\phi(t_1)} < \frac{\phi(t^*)}{\phi(t_1)} \Rightarrow \max \left\{\frac{\phi(t_n)}{\phi(t_1)} \frac{\phi(t^*)}{\phi(t_n)}\right\} \leq \max \left\{1, \frac{\phi(t^*)}{\phi(t_1)}\right\} \).

**Proof of Theorem 1.** We have

\[
\|W^* - \omega_n\| \leq \|\omega_1 - \omega_0\| \max \left\{1, \frac{\phi(t^*)}{\phi(t_1)}\right\} \frac{\phi(t_n)}{\phi(t_1)} \beta^n
\]

by Lemma 9,

\[
\leq \|\omega_1 - \omega_0\| \frac{\epsilon}{1 + Q(t_1)}
\]

by definition of \( n \),

\[
\leq \epsilon \|W^* - \omega_0\|
\]

by Lemma 6.

If \( t_1 \leq t^* \) then by Lemma 9 we deduce that \( t_i \leq t^* \) for all \( i \). If \( t_1 > t^* \) then using Lemma 8 we deduce that \( \sum_{i=1}^n t_i \leq t^* n + \sum_{i=1}^n (t_i - t^*)^+ \leq t^* n + 2(t_1 - t^*) \).

**C Supplementary material for Section 5**

**C.1 Accelerated gradient descent**

**Lemma 10.** Let \( f \) be convex and \( L \)-smooth with \( L \geq \ell_0^0 \). At any inner iteration of \texttt{AdaptiveRestartAGD} and for all \( x \in X \) the following inequality holds,

\[
f(w_i^t) - f(x) \leq 2\eta L \|\omega_i - x\|^2 \left(\frac{1}{t+1}\right)^2.
\]

**Proof.** See Theorem 4.4 of [24] and note that the proof does not use that \( x^* \) is a minimizer, so we can replace it with any \( x \in X \).
Function InitializeAGD(w):
    return w, w, 1;

Function OneStepOfAGD(w, v, λ, η):
    Find the smallest nonnegative integer k such that with ℓ = η^k ℓ
    \[ f(p_ℓ(v)) \leq a(v) + (p_ℓ(v) - v)^T a(v) + \frac{ℓ}{2} \|p_ℓ(v) - v\|^2 + b(p_ℓ(v)). \]
    Set \( ℓ_+ = η^k ℓ \) and compute \( w_+ \leftarrow p_ℓ_+(v) \);
    \( λ_+ \leftarrow \frac{1 + \sqrt{1 + 4\epsilon^2}}{2\epsilon} \);
    \( v_+ = w_+ + \frac{λ_+}{λ_+} (w_+ - v) \);
    return \( w_+, v_+, λ_+, ℓ_+ \)

Function AdaptiveRestartAGD(ω_0, ℓ_0, η):
    for \( i = 1, \ldots, ∞ \) do
        \( w_0^i, v_0^i, λ_0^i \leftarrow \) InitializeAGD(ω_{i-1}, ℓ_{i-1});
        \( t \leftarrow 0 \);
        repeat
            \( t \leftarrow t + 1; \)
            \( w_i^t, v_i^t, λ_i^t, ℓ_i^t \leftarrow \) OneStepOfAGD(w_{i-1}^t, v_{i-1}^t, λ_{i-1}^t, ℓ_{i-1}^t, η) ;
            until restart condition (1) holds;
        \( τ_i \leftarrow t; \)
        \( \omega_i \leftarrow w_i^τ \);
    end

C.1.1 Proof of Theorem 3 and Corollary 2

Proof of Theorem 3. Assumption 1 holds by Lemma 1. By Fact 1 and 10, Assumption 2 holds with \( C = 2Lη \) and \( φ(k) = (k + 1)^2 \). By Lemma 10 with \( x = w^* \) and strong convexity,

\[
\frac{2ηL\|ω_{i-1} - w^*\|^2}{(t+1)^2} \geq f(w_i^t) - f(w^*) \geq \frac{α}{2}\|w_i^t - w^*\|^2 \geq 2\sqrt{\frac{ηL}{α(t+1)^2}}\|ω_{i-1} - w^*\|_2 \geq \|w_i^t - w^*\|_2.
\]

Therefore, Assumption 3 holds with \( Q(t) = \frac{2}{t+1} \sqrt{λ} \). Assumption 4 with \( θ = α/2 \) holds by \( α \)-strong convexity.

With the premise of Theorem 1 established it only remains to make the value of \( t^* \) explicit and simplify the bound. From the definition of \( C \) and \( θ, \hat{κ} = 4κ \), we claim that

\[ t^* = 1 + \sqrt{κ}(ρ + \sqrt{ρ^2 + 4ρ}) \]

is a solution to (4). To establish this claim note that this value of \( t^* \) implies that

\[ (t^* - 1)^2 - 2ρ\sqrt{κ}(t^* - 1) - 4ρ\hat{κ} = 0 \]

by the quadratic formula

\[ \Rightarrow t^* - 1 = 2ρ\sqrt{κ} \left( 1 + \frac{2\sqrt{κ}}{t^* - 1} \right) \]

moving terms to the RHS and dividing by \( t^* - 1 \)

\[ \Rightarrow \frac{(t^* - 1)^2}{1 + \frac{2\sqrt{κ}}{t^* - 1}} = 4ρ^2\hat{κ} \]

squaring both sides and rearranging.

which establishes (4).

As \( β ∈ (0, 1) \Rightarrow ρ ≥ 2 \) the previous equation implies \( t^* ≥ 2\sqrt{κ} \). Therefore

\[ \left( 1 + \frac{2\sqrt{κ}}{τ_i + 1} \right) \max \left\{ 1, \frac{(t^* + 1)^2}{(τ_i + 1)^2} \right\} \leq 2 \max \left\{ 1, \frac{(t^* + 1)^3}{(τ_i + 1)^3} \right\}. \]

\[ \Box \]

Proof of Corollary 2. Follows by substituting \( β = 1/4 \) into the bound in Theorem 3, which yields

\[ t^* = \frac{1}{1 + (5 + 3√5)\sqrt{κ}} \]

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\[ \frac{f^*}{\ln(4)} \leq 1 + 8.5\sqrt{\kappa}. \]

**Remark 2.** It is worth contrasting the bound in Corollary 2 with the bound that one achieves if \( \kappa \) is known and one picks the fixed restart period \( T \) that minimizes the worst-case bound. By Lemma 10 and strong convexity, \( \|w_{i+1} - w^*\|_2 \leq \frac{2\sqrt{\kappa}}{T} \|w_i - w^*\|_2 \Rightarrow n \leq \ln \left( \frac{\|w_{i+1} - w^*\|_2}{\|w_i - w^*\|_2} \right) / \ln \left( \frac{2\sqrt{\kappa}}{T} \right), \)

which implies that \( \|w_{n+1} - w^*\|_2 \leq \epsilon \) after at most \( T \ln \left( \frac{\|w_{n+1} - w^*\|_2}{\epsilon} \right) / \ln \left( \frac{2\sqrt{\kappa}}{T} \right) \) iterations. We can (approximately) minimize this upper bound with respect to \( T \) by exactly minimizing \( T / \ln \left( \frac{2\sqrt{\kappa}}{T} \right) \) yielding \( T = e\sqrt{2\kappa} \) and an overall bound of \( (e\sqrt{2\kappa} + 1) \ln \left( \frac{\|w_{n+1} - w^*\|_2}{\epsilon} \right) / \ln \left( \frac{2\sqrt{\kappa}}{T} \right) \).

### C.2 Extragradient

This subsection is based on Bubeck [31]. For general Bregman divergences this algorithm is called mirror prox; when the Euclidean norm is used (as is the case here) it is known as extragradient.

Define,

\[ g(u) := \left( \nabla_x f(x, y), -\nabla_y f(x, y) \right). \]

**Function InitializeExtragradient(\( u \)):**

return 0, \( u \);

**Function OneStepOfExtragradient(\( \bar{u}, u, t, \gamma \)):**

\[ v \leftarrow \arg\min_{w \in W} g(u)^T w + \frac{1}{2} \| w - u \|_2^2; \]

\[ u^+ \leftarrow \arg\min_{w \in W} g(v)^T w + \frac{1}{2} \| w - u \|_2^2; \]

\[ \bar{u}^+ \leftarrow \frac{\bar{u} + u^+}{2}; \]

return \( \bar{u}^+, u^+ \);

**Function AdaptiveRestartExtragradient(\( \omega_0, \gamma \)):**

for \( i = 1, \ldots, \infty \) do

\[ u_i^0, u_i^0 \leftarrow \text{InitializeExtragradient}(\omega_{i-1}); \]

\( t \leftarrow 0; \)

repeat

\[ t \leftarrow t + 1; \]

\[ u_i^t, u_i^t \leftarrow \text{OneStepOfExtragradient}(u_i^{t-1}, u_i^{t-1}, t, \gamma); \]

until restart condition (1) holds;

\( \tau_i \leftarrow t, \omega_i \leftarrow u_i^t; \)

end

Lemma 11 follows the proof of Bubeck [31, Theorem 4.4]. The original proof is given by Nemirovski [32, Proposition 2.2].

**Lemma 11.** Suppose \( f(x, y) \) is \( L \)-smooth, convex in \( x \) for all \( y \in Y \), and concave in \( y \) for all \( x \in X \). Then \( \text{AdaptiveRestartExtragradient} \) with \( \gamma \in (0, 1/L] \) satisfies, \( \sup_{(\bar{x}, \bar{y}) \in B_R(\bar{w})} f(x_i^t, \bar{y}) - f(\bar{x}, \bar{y}) \leq \frac{R^2}{\gamma} \) where \( \omega_i := u_i^t, (\bar{x}, \bar{y}) := \omega_{i-1}, \) for all \( R \in \mathbb{R}^+ \). Furthermore, \( \|w_i^t - w^*\|_2 \leq \|\omega_i - w^*\|_2 \) for all \( w^* \in W^* \).

**Proof.** Repeating the proof of Theorem 4.4 of [31] with \( \nabla f(y_{t+1}) \) replaced with \( g(u_i^t) \) yields

\[ g(u_i^t)^T (u_i^t - w) \leq \frac{\|w - u_i^{t-1}\|_2^2 - \|w - u_i^t\|_2^2}{\gamma}. \]

From the previous inequality, using that \( f \) is convex-concave we deduce

\[ f(\bar{x}_i, y) - f(x, \bar{y}_i) \leq \frac{\|w - u_i^{t-1}\|_2^2 - \|w - u_i^t\|_2^2}{\gamma} \]

(6)
with \( w = (x, y) \) and \( u_t^i = (\hat{x}_t^i, \hat{y}_t^i) \). By Jensen’s inequality and telescoping, (6) implies

\[
f(x_t^i, y) - f(x, y_t^i) \leq \frac{1}{t} \sum_{k=1}^{t} (f(\hat{x}_k, y) - f(x, \hat{y}_k)) \leq \frac{\|w_t^i - \omega_{t-1}\|^2}{\gamma t}.
\]

Therefore the bound on the duality gap holds. Also, since for any saddle point \((x^*, y^*) = w^* \in W^*\) we have \( f(x_t^i, y) - f(x^*, y^*) \geq 0 \) by (6) we deduce \( \|w_* - u_t^i\|_2 \leq \|w_* - u_{t-1}^i\|_2 \). This implies \( \|w_* - u_t^i\|_2 \leq \|w_* - \omega_{t-1}\|_2 \) and therefore by the triangle inequality,

\[
\|w_* - u_t^i\|_2 \leq \frac{1}{t} \sum_{k=1}^{t} \|w_* - u_k^i\|_2 \leq \|w_* - \omega_{t-1}\|_2.
\]

**Theorem 4.** Suppose \( f(x, y) \) is \( L \)-smooth, convex in \( x \) for all \( y \in Y \), and concave in \( y \) for all \( x \in X \). Further suppose Assumptions 4 holds. Let \( \theta \in (0, L], \gamma \in (0, 1/L], \beta \in (0, 1), \) and \( \epsilon \in (0, 1) \). Define \( t^* := \frac{4(1 + \beta)^2}{\beta^2} - \frac{1}{\gamma \beta} + 2 \). Consider the sequence \( \{\omega_i\}_{i=0}^{\infty} \) \( \{\tau_i\}_{i=1}^\infty \) generated by AdaptiveRestartExtragradient. Then for

\[
n = \left\lfloor \log_{1/\beta} \left( \frac{2}{\epsilon} \max \left\{ 1, \frac{t^*}{\tau_1} \right\} \right) \right\rfloor
\]

the inequality \( \|W^* - \omega_n\| \leq \epsilon \) holds and \( \sum_{i=1}^{n} \tau_i \leq t^* n + 2(\tau_1 - t^*)^+ \).

**Proof.** By Lemma 1, Assumption 1 holds. By combining Fact 1 and 11 with \( R = (1 + \beta)r \), we observe that Assumption 2 holds with \( \phi(t) = t \) and \( C = 2/\gamma \). Lemma 11 implies Assumption 3 holds with \( Q(t) := 1 \). Therefore we have established the premise of Theorem 1 which implies the desired result.

**C.3 Lower bounds for saddle point algorithms**

With appropriate indexing of their iterates, saddle point algorithms such as primal-dual hybrid gradient, extragradient and their restarted variants satisfy

\[
\begin{align*}
x_t &\in x_0 + \text{span}(\nabla_x f(x_0, y_0), \ldots, \nabla_x f(x_{t-1}, y_{t-1})) \quad (7a) \\
y_t &\in y_0 + \text{span}(\nabla_y f(x_0, y_0), \ldots, \nabla_y f(x_t, y_t)) \quad (7b)
\end{align*}
\]

On bilinear games with \( c = y_0 = x_0 = 0 \), (7) simplifies to

\[
\begin{align*}
y_t &\in \text{span}(Ax_0 - b, \ldots, Ax_{t-1} - b) \\
x_t &\in \text{span}(A^T y_0, \ldots, A^T y_t),
\end{align*}
\]

which implies

\[
x_t \in \text{span}(A^T (Ax_0 - b), \ldots, A^T (Ax_{t-1} - b)). \quad (8)
\]

Therefore to form a lower bound for saddle point algorithms it will suffice to consider algorithms that satisfy (8). We provide a lower bound for algorithms satisfying (8) in Theorem 5. The proof of Theorem 5 is essentially identical to the proof of Bubeck [31, Theorem 3.15], the proof is just reframed in terms of a bilinear game instead of minimizing a quadratic.

**Theorem 5.** For any \( 0 < \gamma_{\min} < \gamma_{\max} \), there exists a matrix \( A \) and vector \( b \) with \( \gamma_{\max} \) and minimum singular value greater than \( \gamma_{\min} \) such that for any black-box procedure satisfying (8) with \( x_0 = 0 \), one has

\[
\|x_t - x_*\| \geq \left( \frac{\gamma_{\max} - 1}{\gamma_{\min} + 1} \right)^{t-1} \|x_0 - x_*\|
\]

where \( x^* \) is the unique primal solution to the saddle point problem \( f(x, y) = y^T Ax + b^T y \).
Proof. Let $T \in \mathbb{R}^{k \times k}$ be a tridiagonal matrix with 2 on the diagonal and $-1$ on the upper and lower diagonals. Note that $x^T T x = 2 \sum_{i=1}^k x(i)^2 - 2 \sum_{i=1}^{k-1} x(i) x(i+1) = x(1)^2 + x(k)^2 + \sum_{i=1}^{k-1} (x(i) - x(i+1))^2$, which implies $0 \leq T \leq 4I$. Define, $A := \sqrt{\gamma_{\max}^2 - \gamma_{\min}^2} T + \gamma_{\min}^2 I$, $b := 2(A^T)^{-1} e_1$. Since $A^T A = A^2 = \gamma_{\max}^2 T + \gamma_{\min}^2 I$, we deduce that the minimum singular value of $A$ is greater than $\sqrt{\lambda_{\min}(A^T A)} \geq \gamma_{\min}$, and the maximum singular value of $A$ is less than $\sqrt{\lambda_{\max}(A^T A)} \leq \gamma_{\max}$. The remainder of the proof continues exactly as the proof of Bubeck [31, Theorem 3.15] except with $\kappa = \gamma_{\max}^2 / \gamma_{\min}^2$ and $\alpha = \gamma_{\min}^2$.

Note, for $\gamma_{\max} / \gamma_{\min} \gg 1$ we have

$$
\frac{\gamma_{\max} - 1}{\gamma_{\min} + 1} ^ {t-1} \approx \exp \left( -2(t-1) \frac{\gamma_{\max}}{\gamma_{\min}} \right).
$$

D More experimental details

D.1 Matrix games

The step size used for the matrix games is $\gamma = \frac{\sqrt{0.9}}{||A||_2}$ where the operator norm $||A||_2$ is computed using numpy.linalg.norm. The constant $\sqrt{0.9}$ was taken from ODL and was not tuned. $\beta = \frac{1}{2}$ is used in the restart scheme.

D.2 Quadratic assignment problem relaxations

For each instance we first fix the ratio of primal and dual step sizes by a hyperparameter sweep over $\{10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^0, 10^1, 10^2, 10^3, 10^4, 10^5\}$. We run PDHG without restarts for 1000 iterations and select the ratio $r$ such that the final iterate has the smallest residual. Given $r$, the dual step size is chosen as $\gamma_i = \frac{\sqrt{0.9}}{r ||A||_2}$ and the primal step size chosen as $\gamma_x = r \gamma_y$. The operator norm $||A||_2$ is estimated using ODL’s implementation of power iteration. $\beta = \frac{1}{2}$ is used in the restart scheme. Table 1 lists the dimensions of the two instances.

| Instance   | # variables | # constraints | # nonzeros in $A$ |
|------------|-------------|---------------|-------------------|
| nug08-3rd  | 20,448      | 19,728        | 139,008           |
| qap15      | 22,275      | 6,331         | 110,700           |

Table 1: Dimensions of the quadratic assignment problem relaxations.

D.3 Logistic regression and LASSO

Given runs with different fixed periods, we pick the algorithm that gets the function value below $10^{-8}$ earliest (in terms of number of iterations) as the best. Table 2 lists statistics of the instances.

| Dataset                  | Problem                | # data points | # features | regularizer |
|--------------------------|------------------------|---------------|------------|-------------|
| E2006-tfidf [33]         | LASSO                  | 16,087        | 150,360    | 1.0         |
| rcv1.binary [34]         | regularized logistic loss | 20,242    | 47,236     | $10^{-1}$   |
| Duke breast cancer [35, 36] | regularized logistic loss | 44         | 7,129      | $10^{-2}$   |

Table 2: Statistics of test problems for AGD.

D.4 A hard example for the function scheme of O’Donoghue and Candes [1]

Consider the following 1-smooth function:

$$
h_\delta(\eta) = \begin{cases} 
\eta^2 / 2 & \eta \geq -\delta \\
-\delta \eta - \delta^2 / 2 & \eta < -\delta.
\end{cases}
$$
which is plotted in Figure 4 for $\delta = 0.1$. Using this function we will construct a hard example for AGD,

$$f(x) = \sum_{i=1}^{n} i h_\delta(x_i) + \frac{\alpha}{2} \|x\|_2^2$$

(9)

with $n = 500$, $\delta = 10^{-4}$, $\alpha = 10^{-4}$ starting from $x = -1$. Note this problem is $n$-smooth and $\alpha$-strongly convex. The unique minimizer is $x = 0$. In this situation the restart interval that minimizes the worst-case bound is $\approx e \sqrt{n/\alpha} \approx 6000$. From Figure 5, we can see that our method and hyper-parameter searching on the restart period produces restarts of roughly this magnitude. The function restart scheme of O’Donoghue and Candès [1] in contrast restarts too frequently, causing the algorithm to run slower than vanilla AGD. Intuitively, the sharp transition in the smoothness at $-\delta$ causes the function value to regularly increase and stopping the scheme of O’Donoghue and Candès [1] from building up momentum. However, once it enters the neighborhood of the minimizer where the function is quadratic it quickly converges.

Figure 4: Plot of the function $h_\delta$ for $\delta = 0.1$

Figure 5: Results for minimizing the function (9). The best fixed restart period is found via grid search on $\{128, 256, 512, 1024, 2048, 4096, 8192\}$. 