Regret Bounds for Discounted MDPs

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Abstract

Reinforcement learning (RL) has traditionally been understood from an episodic perspective; the concept of non-episodic RL, where there is no restart and therefore no reliable recovery, remains elusive. A fundamental question in non-episodic RL is how to measure the performance of a learner and derive algorithms to maximize such performance. Conventional wisdom is to maximize the difference between the average reward received by the learner and the maximal long-term average reward. In this paper, we argue that if the total time budget is relatively limited compared to the complexity of the environment, such comparison may fail to reflect the finite-time optimality of the learner. We propose a family of measures, called $\gamma$-regret, which we believe to better capture the finite-time optimality. We give motivations and derive lower and upper bounds for such measures. A follow-up work [He et al., 2020] has improved both our lower and upper bound, the gap is now closed at $\tilde{\Theta}\left(\frac{\sqrt{SAT}}{(1-\gamma)^2}\right)$.

1 Introduction

Reinforcement learning (RL) is concerned with how an algorithm should interact with a (partially) unknown Markov decision process (MDP) to maximize the cumulative reward. We distinguish between two types of RL: episodic RL, where the learner is reset to a starting distribution periodically; and non-episodic RL, where the learner strictly operates on the MDP without interruption. Theoretical analysis has shown that in the tabular setting, an episodic RL algorithm can be expected to perform almost as well as the optimal episodic non-stationary policy in the long run (Azar et al., 2017; Zanette and Brunskill, 2019; Dann, 2019).

In this work, we are interested in the metric of evaluating non-episodic RL algorithms. In literature, the average reward received by an RL algorithm has been compared with the optimal gain (Mahadevan, 1996) a stationary policy can achieve from the starting state (Jaksch et al., 2010; Bartlett and Tewari, 2012; Fruit et al., 2018; Ortner, 2020). Intuitively, the optimal gain is the maximal average-reward a policy can achieve when it operates on the MDP for infinitely many steps.

However, there exist certain nuisances in comparing a learning algorithm that interacts with the MDP for only a finite amount of steps (say, $T$ steps) with a policy that runs on the MDP for an infinite amount of steps. Previous work has rationalized this comparison by assuming the MDP has relatively short “mixing time”. While the concept of mixing time is only formally defined for Markov chains, different authors have different interpretations of it in the case of MDPs. For example, Jaksch et al. (2010) assumes the MDP has diameter $D \ll T$, and Bartlett and Tewari (2012); Fruit et al. (2018) assume the MDP weakly communicates and the optimal bias vector $h^*$ has bias-span $\text{sp}(h^*) \ll T$; Ortner (2020) chooses to consider the maximal mixing time of the Markov chains induced by all the policies. To give a concrete example, consider the MDP in Figure [1]. The MDP is formed by two sub-MDPs connected by $N$ middle states. One sub-MDP is less

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We can define the maximum $\gamma$-discounted total rewards generated by starting from a state $s$ and following policy $\pi$ to choose the next action repeatedly, i.e.

$$V^\pi_s = \mathbb{E}_{a_h^0 = s} \left[ \sum_{h=0}^{\infty} \gamma^h r_h' \right].$$

We can define the maximum $\gamma$-discounted total rewards from state $s$ by

$$V^*_{s, \gamma} = \sup_{\pi} V^\pi_s.$$

The $\gamma$-regret is defined by

$$\text{Regret}_s(T) = \sum_{h=0}^{T-1} (1 - \gamma) V^*_{s, \gamma} - \sum_{h=0}^{T-1} r_h.$$
3 Related Work

Theoretical analysis on non-episodic RL are typically done through the notion of average-reward regret, which is defined by

$$\text{Regret}_\gamma(T) = \sum_{h=0}^{T-1} \rho^*_h - \sum_{h=0}^{T-1} r_h,$$

where $\rho^*_h$ is the maximal gain that can be achieved by a (stationary) policy starting from state $s_h$. For a more detailed introduction, see e.g. Mahadevan (1996). It can be shown that the average-reward regret can be related to $\gamma$-regret by

$$\lim_{\gamma \to 1} \text{Regret}_\gamma(T) = \text{Regret}_\gamma^*(T)$$

Current analysis of $\text{Regret}_\gamma^*(T)$ all assume that the MDP is at least weakly communicating, and therefore $\rho^*_h$ does not depend on $s_h$. The analysis was pioneered by Jaksch et al. (2010), who identified the diameter of the MDP, $D$, or any related measure, to be necessary when bounding $\text{Regret}_\gamma^*(T)$, and provided a lower bound of $\Omega(\sqrt{DSAT})$, which is still the best lower bound to date. They also derived an upper bound of $\tilde{O}(DS\sqrt{AT})$.

The definition of $\gamma$-regret is also quite related to a notion called sample complexity of exploration, introduced in Kakade et al. (2003). Specifically, this complexity, which we denote by $N_{\gamma}(\epsilon,\delta)$, is the smallest integer such that with probability at least $1-\delta$, there are at most $N_{\gamma}(\epsilon,\delta)$ different $h$ such that $E[\Delta_h] > \epsilon$, where

$$\Delta_h = V^{\pi^*_h,\gamma} - \sum_{t=0}^{\infty} \gamma^t r_{h+t}.$$

The quantity $\Delta_h$ can be related to $\gamma$-regret by noting that

$$(1-\gamma) \sum_{h=0}^{T-1} \Delta_h - \frac{1}{1-\gamma} \leq \text{Regret}_\gamma(T) \leq (1-\gamma) \sum_{h=0}^{T-1} \Delta_h + \frac{1}{1-\gamma}.$$  

However, $N_{\gamma}(\epsilon,\delta)$ itself does not measure directly the performance of a learner in the first $T$ steps. For example, a $T$-step optimal learner could have very large $N_{\gamma}(\epsilon,\delta)$ simply because it is not optimal after $T$ steps. The best upper bound on $N_{\gamma}(\epsilon,\delta)$ to date is $O \left( \frac{SA}{\epsilon^2 (1-\gamma)^2} \right)$ (Szita and Szepesvári, 2010), while the best lower bound is $\tilde{\Omega} \left( \frac{SA}{\epsilon^2 (1-\gamma)^2} \right)$ (Lattimore and Hutter, 2012). More discussion on the connection between $N_{\gamma}(\epsilon,\delta)$ and $\text{Regret}_\gamma(T)$ can be found in Section 6.1.

The algorithm we use to prove the upper bounds on $\gamma$-regret is an adaption of Jin et al. (2018, Algorithm 1), with modifications to handle the cyclic dependency in the non-episodic setting. For a detailed comparison see Section 6.3. It also looks visually similar to Dong et al. (2019, Algorithm 1); however, the goal of Dong et al. (2019) is to propose a model-free algorithm that has low sample complexity of exploration, and the proof technique therein is very different from ours and Jin et al. (2018).

4 Lower Bounds

In this section, we give two lower bounds on $\gamma$-regret. They both have $\sqrt{T}$ dependency on $T$ for sufficiently large $T$. The first one scales with $\sqrt{S}$ but has has a worse dependency on $\gamma$; the second one does not scale with $S$ but has a better dependency on $\gamma$. The proofs can be found in the appendix.
Algorithm 1 Tabular Double Q-Learning with Upper Confidence

Input: $\gamma$, $S$, $A$, $p$

Parameters: $b_t = \frac{2}{1-\gamma} \sqrt{\frac{\ln\left(\frac{2SA^2}{p}\right)}{(1-\gamma)t}}$ for $t \geq 1$, $\alpha_t = \frac{2-\gamma}{1+t-\gamma}$ for $t \geq 1$

for $(s,a) \in S \times A$  \hspace{1cm} $\triangleright$ start initialization

\begin{align*}
Q^0(s,a), Q^1(s,a) & \leftarrow \frac{1}{1-\gamma} \\
N^0(s,a), N^1(s,a) & \leftarrow 0
\end{align*}

$s_0 \leftarrow$ reset

for $h \leftarrow 0, 1, \cdots, \infty$  \hspace{1cm} $\triangleright$ main loop

\begin{align*}
\iota & \leftarrow h \bmod 2 \\
\alpha_t & \leftarrow \max_{a' \in A} Q^\iota(s_h, a') \\
s_{h+1}, r_h & \leftarrow \text{NEXT}(a_h) \\
N^\iota(s_h, a_h) & \leftarrow N^\iota(s_h, a_h) + 1 \\
t & \leftarrow N^\iota(s_h, a_h) \\
Q^\iota(s_h, a_h) & \leftarrow (1 - \alpha_t)Q^\iota(s_h, a_h) \\
& \quad + \alpha_t (r + b_t + \gamma \max_{a' \in A} Q^1 - (s_{h+1}, a'))
\end{align*}

Theorem 1. For any positive integers $S, A, T$, $\gamma \in [0, 1)$, and any (possibly randomized) learning algorithm, there exists an MDP such that

$$
\mathbb{E}[\text{Regret}_{\gamma}(T)] \geq \begin{cases} 
\frac{T}{20}, & \text{if } T \leq SA, \\
\frac{\sqrt{SAT}}{20\sqrt{2}}, & \text{otherwise}.
\end{cases}
$$

Theorem 2. For any $\gamma \in \left(\frac{2}{3}, 1\right)$, positive integers $A \geq 30, T \geq \frac{A}{1-\gamma}$, and any (possibly randomized) learning algorithm, there exists a two-state MDP such that

$$
\mathbb{E}[\text{Regret}_{\gamma}(T)] \geq \frac{\sqrt{AT}}{2304(1-\gamma)^{\frac{3}{2}}} - \frac{1}{1-\gamma}.
$$

5 Upper Bounds

In this section, we will introduce a tabular version of the double Q-learning algorithm proposed in Hasselt (2010), and then use it to prove an upper bound on the $\gamma$-Regret.

The tabular version is described in Algorithm 1. Compared to the original neural-network-oriented version, the tabular version has certain specifications that are crucial for theoretical analysis:

- The two Q-value functions have to be initialized to $\frac{1}{1-\gamma}$, or for that matter, the maximal possible discounted cumulative return if it is known.
- The behavioral policy (i.e., the strategy used to choose which action to take) cannot be arbitrary as in the original version. The action has to be taken greedily based on the Q-value function that immediately gets updated afterward.
- In the original version, in each round, one of the two Q-value functions gets chosen, for example, randomly, and gets updated from the other one. In the tabular version, the updates have to happen in a strictly alternating fashion — the Q-value function gets updated in a round is used for updating the other Q-value function in the next round.
- Soft update and upper confidence bounds are used when updating the Q-value functions.
Our upper bounds are presented in the following theorem.

**Theorem 3.** For any $\gamma \in [0, 1)$ and $p \in (0, 1]$, with probability at least $1 - p$, for any positive integer $T$, Algorithm 1 has

$$
\text{Regret}_\gamma(T) \leq \frac{14\sqrt{SAT \ln \left( \frac{\pi^2 SAT^2}{p} \right)}}{(1 - \gamma)^{\frac{3}{2}}} + 2SA + 4 \frac{1}{1 - \gamma},
$$

and consequently,

$$
\mathbb{E} \left[ \text{Regret}_\gamma(T) \right] \leq \frac{14\sqrt{SAT \ln \left( \frac{\pi^2 SAT^3}{p} \right)}}{(1 - \gamma)^{\frac{3}{2}}} + 2SA + 5 \frac{1}{1 - \gamma}.
$$

### 5.1 Proof of Theorem 3

The proof will be in the same style as in [Jin et al. (2018)](https://www.jin-rl.com/), with technical modifications to handle the cyclic dependencies in the non-episodic setting. Recall that in Algorithm 1, for any $t \geq 1$, $\alpha_t = \frac{2 - \gamma}{1 - \gamma}$. We furthermore define $\alpha_0 = 1$. Let $\alpha^i_t = \alpha_0 \prod_{j=i+1}^t (1 - \alpha_t)$; it is easy to verify that $\sum_{i=0}^t \alpha^i_t = 1$. Define by $Q_h$ and $N_h$ the $Q^0$ and $N^0$ function at the beginning of iteration $h$ if $h$ is even, or the $Q^1$ and the $N^1$ function at the beginning of iteration $h$ if $h$ is odd. Let $n_h = N_h(s_h, a_h)$. For $i = 1, 2, \cdots, n_h$, let $p_{\text{prev}_i}(h)$ be the $i$th smallest $h' < h$ such that $h'$ and $h$ have the same parity, $s_{h'} = s_h$, and $a_{h'} = a_h$. Define

$$
V_h(s) = \max_{a} Q_h(s, a), \quad \Delta_h = V_{s_h}^{*, \gamma} - \sum_{t=0}^{\infty} \gamma^t r_{h+t},
$$

$$
\bar{R}(s, a) = \mathbb{E}_{r \sim R(s, a)}[r], \quad \bar{r}_h = \bar{R}(s_h, a_h),
$$

$$
V_{M(s, a)}^{*, \gamma} = \mathbb{E}_{s' \sim M(s, a)}\left[ V_{s'}^{*, \gamma} \right], \quad Q_{s, a}^{*, \gamma} = \bar{R}(s, a) + V_{M(s, a)}^{*, \gamma},
$$

$$
\phi_h = V_h(s_h) - V_{s_h}^{*, \gamma}, \quad \delta_h = \phi_h + \Delta_h.
$$

The following lemmas are adapted from [Jin et al. (2018)](https://www.jin-rl.com/); the proofs can be found in the appendix.

**Lemma 4.** The following statements are true: (i). $\frac{\ln(C \cdot t)}{\sqrt{T}} \leq \sum_{i=1}^t \alpha^i_t \sqrt{\frac{\ln(C \cdot t)}{i}} \leq 2 \cdot \frac{\ln(C \cdot t)}{\sqrt{T}}$ for any $t \geq 1$ and $C \geq e$; (ii). $\sum_{i=1}^t (\alpha^i_t)^2 \leq \frac{2}{(1 - \gamma)^t}$ for any $t \geq 1$; (iii). $\sum_{i=1}^\infty \alpha^i_t = 2 - \gamma$ for any $i \geq 1$.

**Lemma 5.** For any $h$,

$$
Q_h(s_h, a_h) - Q_{s_h, a_h}^{*, \gamma} = \alpha_0^h \left( \frac{1}{1 - \gamma} - Q_{s, a}^{*, \gamma} \right) + \sum_{i=1}^{n_h} \alpha^i_{n_h} b_i + \gamma \sum_{i=1}^{n_h} \alpha^i_{n_h} \phi_{p_{\text{prev}_i}+1}
$$

$$
+ \sum_{i=1}^{n_h} \alpha^i_{n_h} \left( \bar{r}_{p_{\text{prev}_i}} - \bar{r}_{p_{\text{prev}_i}} + \gamma \left( V_{s_{p_{\text{prev}_i}+1}}^{*, \gamma} - V_{s_{p_{\text{prev}_i}+1}}^{*, \gamma} \right) \right).
$$

**Lemma 6.** Define random variables $r_{s, a, i}$ to be the reward received after taking action $a$ on state $s$ the $i$th time, and $s'_{s, a, i}$ to be the next state when receiving reward $r_{s, a, i}$, then for any $T$, with probability at least $1 - p$, the following hold simultaneously

(i). For any $h$, $0 \leq Q_h(s_h, a_h) - Q_{s_h, a_h}^{*, \gamma} \leq \alpha_0^h + \gamma \sum_{i=1}^{n_h} \alpha^i_{n_h} \phi_{p_{\text{prev}_i}+1} + 3\beta_{n_h}$, where $\beta_t = \frac{2}{t^{(1 - \gamma)/t}} \sqrt{\frac{\ln \left( \frac{\pi^2 SAT^2}{p} \right)}{(1 - \gamma)t}}$

if $t \geq 1$ and $\beta_0 = 0$.

(ii). $\sum_{h=0}^{T-1} \bar{r}_h - r_h + \gamma \left( V_{M(s_h, a_h)}^{*, \gamma} - V_{s_h+1}^{*, \gamma} \right) \leq \xi_T$, where $\xi_T = \frac{\sqrt{T} \ln \left( \frac{1}{2p} \right)}{1 - \gamma}$. 


We are now ready to begin our proof. From now on all the calculation will condition on the events where the statements in Lemma \[6\] are true. It is important to notice that in this case we have that for any \( h, \phi_h \geq 0 \) and \( \Delta_h \leq \delta_h \). First note that

\[
\delta_h = Q_h(s_h, a_h) - \sum_{t=0}^{\infty} \gamma^t r_{h+t}
\]

\[
= (Q_h(s_h, a_h) - Q_{s_h, a_h}^\gamma) + \left( Q_{s_h, a_h}^\gamma - \sum_{t=0}^{\infty} \gamma^t r_{h+t} \right)
\]

\[
\leq \alpha_{n_h}^0 \cdot \frac{1}{1-\gamma} + \gamma \sum_{i=1}^{n_h} \alpha_i^0 \phi_{prev_i^h+1} + 3\beta_{n_h} + \bar{r}_h - r_h + \gamma \left( V_{M(s_h, a_h)}^\gamma - \sum_{t=0}^{\infty} \gamma^t r_{h+1+t} \right)
\]

where (a) is due to Lemma \[6\](i). Therefore, according to Lemma \[4\](ii) we have

\[
\sum_{h=0}^{T-1} \delta_h \leq \frac{1}{1-\gamma} \sum_{h=0}^{T-1} \alpha_{n_h}^0 + \gamma \sum_{h=0}^{T-1} \alpha_i^0 \phi_{prev_i^h+1} + \gamma \sum_{h=0}^{T-1} \phi_{h+1} + \xi_T + 3 \sum_{h=0}^{T-1} \beta_{n_h}.
\]

Using the fact that \(|\delta_h| \leq \frac{1}{1-\gamma}\) for any \( h \) and rearranging the terms, we get

\[
(1-\gamma) \sum_{h=0}^{T-1} \delta_h \leq \frac{1}{1-\gamma} \sum_{h=0}^{T-1} \alpha_{n_h}^0 + \gamma \left( \sum_{h=0}^{T-1} \alpha_i^0 \phi_{prev_i^h+1} - \sum_{h=0}^{T-1} \phi_{h+1} \right) + \xi_T + 3 \sum_{h=0}^{T-1} \beta_{n_h} + \frac{2\gamma}{1-\gamma}. \tag{1}
\]

To continue the calculation, first note that \( \alpha_{n_h}^0 \) is 1 if \( n_h = 0 \) and 0 otherwise, therefore

\[
\sum_{h=0}^{T-1} \alpha_{n_h}^0 = \sum_{h=0}^{T-1} 1_{n_h = 0} \leq 2SA.
\]

Next note that

\[
\sum_{h=0}^{T-1} \sum_{i=1}^{n_h} \alpha_{n_h}^i \phi_{prev_i^h+1} - \sum_{h=0}^{T-1} \phi_{h+1} \leq \sum_{h=0}^{T-1} \phi_h \sum_{t=h+1}^{\infty} \alpha_t^{n_h} - \sum_{h=0}^{T-1} \phi_{h+1}
\]

\[
\leq \frac{2 - \gamma}{2} \sum_{h=0}^{T-1} \phi_h - \sum_{h=0}^{T-1} \phi_h + \phi_0
\]

\[
= (1-\gamma) \sum_{h=0}^{T-1} \phi_h + \frac{1}{1-\gamma}
\]

\[
= (1-\gamma) \sum_{h=0}^{T-1} (\delta_h - \Delta_h) + \frac{1}{1-\gamma},
\]

where (a) is because of Lemma \[4\](iii). Now going back to (1) we get

\[
(1-\gamma) \sum_{h=0}^{T-1} \delta_h \leq \frac{2SA + 3}{1-\gamma} + (1-\gamma) \sum_{h=0}^{T-1} (\delta_h - \Delta_h) + \xi_T + 3 \sum_{h=0}^{T-1} \beta_{n_h}
\]

\[
\iff (1-\gamma) \sum_{h=0}^{T-1} \Delta_h \leq \frac{2SA + 3}{1-\gamma} + \xi_T + 3 \sum_{h=0}^{T-1} \beta_{n_h}
\]
Finally, note that

\[
\sum_{h=0}^{T-1} \beta_{n_h} \leq 2 \sqrt{\ln \left( \frac{\pi^2 \text{SAT}^2}{p} \right)} \sum_{h=0}^{T-1} \frac{1_{n_h \geq 1}}{n_h} \cdot \sqrt{\frac{1}{n_h}}
\]

\[
= 2 \sqrt{\ln \left( \frac{\pi^2 \text{SAT}^2}{p} \right)} \sum_{s,a} N_T(s,a) \sum_{t=1}^{\frac{T}{2}} \sqrt{\frac{1}{t}}
\]

\[
\leq 2 \sqrt{\ln \left( \frac{\pi^2 \text{SAT}^2}{p} \right)} \sum_{s,a} 2 \sqrt{N_T(s,a)}
\]

\[
\leq 4 \sqrt{\text{SAT} \ln \left( \frac{\pi^2 \text{SAT}^2}{p} \right)} (1 - \gamma)^{1.5},
\]

where (a) is because \( \sum_{t=1}^{T/2} \frac{1}{t} \leq 2 \sqrt{T} \) and (b) is by Cauchy-Schwarz inequality and the fact that \( \sum_{s,a} N_T(s,a) \leq T \). Therefore,

\[
(1 - \gamma) \sum_{h=0}^{T-1} \Delta_h \leq \frac{2 \text{SA} + 3}{1 - \gamma} + \frac{\sqrt{2}}{1 - \gamma} \sqrt{T \ln \left( \frac{3}{2p} \right)} + \frac{12 \text{SAT} \ln \left( \frac{\pi^2 \text{SAT}^2}{p} \right)}{(1 - \gamma)^{1.5}}
\]

\[
\leq \frac{2 \text{SA} + 3}{1 - \gamma} + \frac{14 \text{SAT} \ln \left( \frac{\pi^2 \text{SAT}^2}{p} \right)}{(1 - \gamma)^{1.5}}
\]

(2)

On the other hand, we have

\[
(1 - \gamma) \sum_{h=0}^{T-1} \Delta_h = \sum_{h=0}^{T-1} (1 - \gamma) V_{s_h}^\star - (1 - \gamma) \sum_{h=0}^{T-1} \sum_{t=0}^{\infty} \gamma^t r_{h+t}
\]

\[
\geq \sum_{h=0}^{T-1} (1 - \gamma) V_{s_h}^\star - \sum_{u=0}^{T-1} r_u - (1 - \gamma) \sum_{u=T}^{\infty} r_u \sum_{v=u-T+1}^{\infty} \gamma^v
\]

\[
= \text{Regret}_\gamma(T) - (1 - \gamma) \sum_{u=T}^{\infty} r_u \sum_{v=u-T+1}^{\infty} \gamma^v
\]

\[
\geq \text{Regret}_\gamma(T) - \frac{1}{1 - \gamma}.
\]

(3)

Combining together (2) and (3), we arrive at

\[
\text{Regret}_\gamma(T) \leq \frac{14 \text{SAT} \ln \left( \frac{\pi^2 \text{SAT}^2}{p} \right)}{(1 - \gamma)^{1.5}} + \frac{2 \text{SA} + 4}{1 - \gamma}.
\]

This concludes the proof.
6 Technical Discussions

6.1 Converting Sample Complexity of Exploration to \(\gamma\)-Regret

Recall that the sample complexity of exploration \(N_\gamma(\epsilon, \delta)\) is the smallest integer such that with probability at least \(1 - \delta\), there are at most \(N(\epsilon, \delta)\) different \(h\) such that

\[
V_{sh}^* > \epsilon - \mathbb{E} \left[ \sum_{t=0}^{\infty} \gamma^t r_{h+t} \right]
\]

It is easy to see that

\[
\mathbb{E} \left[ \text{Regret}_{\gamma}(T) \right] \in O \left( \inf_{\epsilon} N_\gamma \left( \epsilon, \frac{1}{T} \right) + \epsilon T (1 - \gamma) + \frac{1}{1 - \gamma} \right).
\]

Plugging in the best existing bound for \(N_\gamma(\epsilon, \delta)\), which is \(\tilde{O} \left( \frac{SA}{\epsilon^2 (1 - \gamma)^6} \right)\) from Szita and Szepesvári (2010), we arrive at an upper bound of \(\tilde{O} \left( T \frac{SA}{(1 - \gamma)^2} \right)\) on \(\mathbb{E} \left[ \text{Regret}_{\gamma}(T) \right]\). It may seem that this bound has better dependencies on \(S, A\), and \(\gamma\), but this is not the case. In fact, we have the following inequalities:

\[
\tilde{O} \left( T \frac{SA}{(1 - \gamma)^2} \right) \geq \begin{cases} \tilde{O}(T), & \text{if } T < \frac{SA}{(1 - \gamma)}, \\ \tilde{O} \left( \frac{\sqrt{SAT}}{(1 - \gamma)^2} \right), & \text{otherwise}. \end{cases}
\]

Note that in the above inequalities \(\tilde{O}(T)\) is a trivial upper bound on \(\text{Regret}_{\gamma}(T)\) for any \(T\), while \(\tilde{O} \left( \frac{\sqrt{SAT}}{(1 - \gamma)^2} \right)\) has a worse dependency on \(\gamma\) than our upper bound.

If the upper bounds on \(N_\gamma(\epsilon, \delta)\) were to hold uniformly over all possible \(\epsilon\), then we could translate the (uniform) upper bound on \(N_\gamma(\epsilon, \delta)\) into upper bounds on \(\gamma\)-regret in a better way. In fact, if the best existing upper bound on \(N_\gamma(\epsilon, \delta)\), \(\tilde{O} \left( \frac{SA}{\epsilon^2 (1 - \gamma)^6} \right)\), was to hold uniformly over all possible \(\epsilon\), then

\[
\mathbb{E} \left[ \text{Regret}_{\gamma}(T) \right] \in O \left( (1 - \gamma) \left( \int_{\epsilon_0}^{\frac{1}{\epsilon_0}} \frac{SA}{\epsilon^2 (1 - \gamma)^6} + T \epsilon_0 \right) + \frac{1}{1 - \gamma} \right),
\]

we could get an upper bound on \(\gamma\)-regret as good as \(\tilde{O} \left( \frac{\sqrt{SAT}}{(1 - \gamma)^2} \right)\). We can see that even in this imagined ideal scenario the translated upper bound still has a worse dependency on \(\gamma\) than ours.

6.2 Lower Bound Proof Techniques

Our proof of the second lower bound (Theorem 2) on \(\gamma\)-regret is an adaptation of the proof for the average-reward setting in Jaksch et al. (2010, Theorem 5). The major challenge the \(\gamma\)-regret formulation brings is that the value function, now being the discounted total return instead of the long-term average, can vary from state to state. While in the two-state MDP case this is still manageable by explicitly writing out the exact formula of the value functions for each state, as we have done in the proof of Theorem 2, it becomes unclear how we should generalize to MDPs with \(S\) states.

6.3 Upper Bounds Proof Techniques

Our derivation of the upper bounds on \(\gamma\)-regret is inspired by Jin et al. (2018), who showed that a specific tabular version of Q-Learning (Watkins and Dayan, 1992) has near-optimal regret in the episodic setting. Their analysis, however, is not directly applicable to the non-episodic setting.
First, in the episodic setting, there are $H$ value functions $Q_0, Q_1, \cdots, Q_{H-1}$ to be learned, each $Q_i$ depends only on $Q_{j>i}$ — there is no cyclic dependency; on the other hand, in the non-episodic setting, there is only one single value function, so a hierarchical induction in the analysis is not possible. To deal with self-dependency, we find it very useful to replace the regular Q-learning with double Q-learning [Hasselt, 2010], which has been widely used in deep reinforcement learning since it was introduced [Hasselt et al., 2016; Hessel et al., 2018].

Second, a key ingredient in the proof of Jin et al. (2018) is the choice of learning rate $\alpha_t = \frac{H+1}{H+1}$ — a nice consequence of this choice is that the total per-episode-step regret blow-up is $(1 + 1/H)^H$; since there are at most $H$ steps in each episode, the total blow-up is $(1 + 1/H)^H$, which is upper bounded by the constant $e$ regardless how large $H$ is. The same quantity $(1 + 1/H)^H$ also appeared in Azar et al. (2017) for the same reason. However, in the non-episodic setting, the blow-up could become arbitrarily large because the learner is not reset every $H$ steps; therefore, different techniques are required to control the blow-up of the regret.

References

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A Proof of Theorem 1

The proof will essentially be a reduction to the multi-armed bandit lower bounds. Fix $S, A, T, \gamma,$ and a learning algorithm, we construct an MDP as follows: if we label all the states by $0, 1, \cdots, S - 1$, then the learner goes from state $i$ to state $((i + 1) \mod S)$ deterministically regardless the action taken, and the reward function is specified by

$$R(s, a) = \begin{cases} 1, & \text{w.p. } \frac{1}{2} + 1_{a = a_s} \cdot \epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

where $\epsilon$ and $a_s$ will be specified later. It is easy to see that

$$\mathbb{E} \left[ \text{Regret}_\gamma(T) \right] = \left( T \cdot \left( \frac{1}{2} + \epsilon \right) - \mathbb{E} \left[ \sum_{u=0}^{T-1} r_u \right] \right).$$

(5)

Let $T_s$ be the number of actions the learner takes at state $s$ in the first $T$ steps, we have that

$$T_s \geq \left\lceil \frac{T}{S} \right\rceil \geq \frac{T}{2S} \text{ if } T \geq S. \tag{6}$$

Also denote $r_{s,i}$ the $i$th reward received when taking action at state $s$, then we have that

$$\mathbb{E} \left[ \text{Regret}_\gamma(T) \right] = \sum_{s=1}^{S} \left( T_s \cdot \left( \frac{1}{2} + \epsilon \right) - \mathbb{E} \left[ \sum_{v=1}^{T_s} r_{s,v} \right] \right). \tag{7}$$

According to the lower bounds for multi-armed bandits, e.g. Theorem 7.1 and its construction in Auer et al. (1995), there exists $\epsilon = \frac{1}{4} \min \left( \sqrt{A \gamma T}, 1 \right)$ and $a_s$ (recall that $R(s, a)$ is defined from $\epsilon$ and $a_s$) such that for any $s$ we have

$$T_s \cdot \left( \frac{1}{2} + \epsilon \right) - \mathbb{E} \left[ \sum_{v=1}^{T_s} r_{s,v} \right] \geq \frac{1}{20} \min \left( \sqrt{AT_s}, T_s \right).$$

Finally, note that if $T \leq SA$, then $T_s \leq A$ for any $s$ and consequently $\min \left( \sqrt{AT_s}, T_s \right) = T_s$; on the other hand, if $T > SA$, then $T_s \geq A$ for any $s$ and consequently for any $s$, $\min \left( \sqrt{AT_s}, T_s \right) = \sqrt{AT_s} \geq \sqrt{\frac{AT}{2S}}$, where the last inequality follows from (6). Going back to (7) gives us the desired lower bounds.

B Proof of Theorem 2

We will construct an MDP similar to the one in the proof of Jaksch et al. (2010) Theorem 5) for our proof. Specifically, the MDP has two states 0 and 1; the learner receives reward 0 in state 0 and reward 1 in state 1, regardless the action taken; the learner goes from state 1 to state 0 with probability $1 - \gamma$ regardless the action taken; the learner goes from state 0 to state 1 with probability $1 - \gamma + 1_{a = a^*} \cdot \epsilon$ when action $a$ is taken, where $\epsilon = \frac{1}{24} \sqrt{\frac{A(1 - \gamma)}{T}}$ and $a^*$ will be chosen later. It is easy to see that $\epsilon \leq 1 - \gamma$ since we assumed that $T \geq \frac{A}{1 - \gamma}$. By definition, we have that

$$V^*_0 = \gamma(1 - \gamma + \epsilon)V^*_{1,\gamma} + \gamma(\gamma - \epsilon)V^*_0,$$

$$V^*_1 = 1 + (1 - \gamma)V^*_0 + \gamma^2 V^*_{1,\gamma}.$$
We can solve the above equations to get
\[ V_0^{*} = \gamma - \gamma^2 + \gamma \epsilon, \quad V_1^{*} = 1 - \gamma^2 + \gamma \epsilon. \] (8)\[ V_0^{*} = \frac{\gamma - \gamma^2 + \gamma \epsilon}{(1 - \gamma)(1 - 2\gamma^2 + \gamma + \gamma \epsilon)}, \quad V_1^{*} = \frac{1 - \gamma^2 + \gamma \epsilon}{(1 - \gamma)(1 - 2\gamma^2 + \gamma + \gamma \epsilon)}. \] (9)

Note that because \( \epsilon \leq 1 - \gamma \), we have in the denominators of (8) and (9) that \((1 - \gamma)(1 - 2\gamma^2 + \gamma + \gamma \epsilon) \in [(1 - \gamma)^2, 4(1 - \gamma)^2]\)

Let \( N_0 \) and \( N_1 \) be the number of steps (in the first \( T \) steps) that the learner is in state 0 and 1 respectively, and let \( N_0^{*} \) be the number of steps (in the first \( T \) steps) the learner is in state 0 and takes action \( a^* \), using the same argument as in the proof of Jaksch et al. (2010, Theorem 5), we have that
\[
E[N_1] \leq \frac{T}{2} + E[N_0^*] \cdot \frac{\epsilon}{1 - \gamma} + \frac{1}{2(1 - \gamma)}, \quad (11)
\]
\[
E[N_0^*] \leq \frac{T}{2A} + \frac{1}{2A(1 - \gamma)} + \frac{\epsilon T}{2} \sqrt{\frac{T}{A(1 - \gamma)}} + \frac{\epsilon T}{2(1 - \gamma)\sqrt{A}}. \quad (12)
\]

Therefore,
\[
(1 - \gamma)^{-1} \cdot E[\text{Regret}(T)] \geq E[N_0] \cdot V_0^{*} + E[N_1] \cdot \left( V_1^{*} - \frac{1}{1 - \gamma} \right)
\]
\[
= \frac{E[N_0] \cdot \gamma(1 - \gamma + \epsilon) - E[N_1] \cdot \gamma(1 - \gamma)}{(1 - \gamma)(1 - 2\gamma^2 + \gamma + \gamma \epsilon)} \quad (c)
\]
\[
\geq \frac{T\gamma^2 - \gamma - \frac{\gamma^2}{2(1 - \gamma)} - E[N_0^*] \cdot \gamma(2 - 2\gamma + \epsilon)}{(1 - \gamma)(1 - 2\gamma^2 + \gamma + \epsilon)} \quad (b)
\]
\[
\geq \frac{T\gamma - 1 - 3\epsilon \cdot \left( \frac{\epsilon T}{2A} + \frac{\epsilon T}{2} \sqrt{\frac{T}{A(1 - \gamma)}} + \frac{\epsilon T}{2(1 - \gamma)\sqrt{A}} \right)}{(1 - \gamma)(1 - 2\gamma^2 + \gamma + \epsilon)} \quad (c)
\]
\[
\geq \frac{\sqrt{AT}(1 - \gamma) - 1 - \left( \frac{\sqrt{AT}(1 - \gamma)}{576} + \frac{\sqrt{AT}(1 - \gamma)}{192} \right)}{(1 - \gamma)(1 - 2\gamma^2 + \gamma + \epsilon)} \quad (d)
\]
\[
= \frac{\sqrt{AT}(1 - \gamma) - 1}{(1 - \gamma)(1 - 2\gamma^2 + \gamma + \epsilon)} \quad (e)
\]
\[
\geq \frac{\sqrt{AT}}{2304(1 - \gamma)\sqrt{(1 - \gamma)^2 - 1}} \quad (f)
\]

where (a) is due to (11) and the fact that \( N_0 + N_1 = T \), (b) is due to (12), (c) is because by assumption \( T \geq \frac{\gamma}{1 - \gamma} \geq \frac{\gamma}{\gamma} \) and \( \epsilon \leq 1 - \gamma \), (d) is by substituting \( \epsilon \) with the chosen value and recall that by assumption \( A \geq 30 \) and \( T(1 - \gamma) \geq 1 \), (e) is because by our assumption \( \gamma \in \left( \frac{2}{3}, 1 \right) \), (f) is due to (10). Rearranging the terms concludes the proof.

C Proof of Lemma 4

For (ii) and (iii), the same proof as in Jin et al. (2018), Lemma 4.1.(b)-(c) can be applied, with \( H \) replaced by \( \frac{\gamma}{1 - \gamma} \), and note that in proving (iii) the requirement for \( n \) and \( k \) to be positive integers in their proof can
be relaxed to \( n \) and \( k \) being real numbers that are at least 1. We will prove (i) by induction on \( t \). The base case \( t = 1 \) holds because \( \alpha_{t}^{1} = 1 \). Assuming the statement is true for \( t \), then on one hand,

\[
\sum_{i=1}^{t+1} \frac{\alpha_{t+1}^{i} \sqrt{\ln (C \cdot i)}}{i} = (a) \frac{\sqrt{\ln (C \cdot (t+1))}}{t+1} + (1 - \alpha_{t+1}) \sum_{i=1}^{t} \frac{\alpha_{t}^{i} \sqrt{\ln (C \cdot i)}}{i}
\]

\[
\geq \alpha_{t+1} \frac{\sqrt{\ln (C \cdot (t+1))}}{t+1} + (1 - \alpha_{t+1}) \sum_{i=1}^{t} \frac{\sqrt{\ln (C \cdot t)}}{t}
\]

where in (a) we used the definition of \( \alpha_{t}^{i} \), in (b) we used the induction assumption, and in (c) is because \( x \mapsto \frac{\ln (C \cdot x)}{x} \) is a non-increasing function when \( C \geq e \) and \( x \geq 1 \). On the other hand, we have

\[
\sum_{i=1}^{t+1} \frac{\alpha_{t+1}^{i} \sqrt{\ln (C \cdot i)}}{i} \leq (b) \frac{\sqrt{\ln (C \cdot (t+1))}}{t+1} + 2(1 - \alpha_{t+1}) \frac{\sqrt{\ln (C \cdot t)}}{t}
\]

\[
\leq \frac{2 - \gamma}{2 + t - (t+1)\gamma} \frac{\sqrt{\ln (C \cdot (t+1))}}{t+1} + \frac{2t(1 - \gamma)}{2 + t - (t+1)\gamma} \frac{\sqrt{\ln (C \cdot t)}}{t}
\]

\[
\leq \frac{2 - \gamma}{2 + t - (t+1)\gamma} \frac{\sqrt{\ln (C \cdot (t+1))}}{t+1} + \frac{2\sqrt{t(1 - \gamma)}\sqrt{t+1}}{2 + t - (t+1)\gamma} \frac{\sqrt{\ln (C \cdot t)}}{t+1}
\]

\[
= 2 \sqrt{\frac{\ln (C \cdot (t+1))}{t+1}}
\]

where in (a) we used the definition of \( \alpha_{t}^{i} \), in (b) we used the induction assumption, and in (c) we used the definition of \( \alpha_{t} \). Therefore, the statement in (i) is true for any \( t \geq 1 \). First note that

D Proof of Lemma \( 5 \)

We have that

\[
Q_{h}(s_{h}, a_{h}) = \alpha_{n_{h}}^{0} \frac{1}{1 - \gamma} + \sum_{i=1}^{n_{h}} \alpha_{n_{h}}^{i} \left( r_{\text{prev}_{h}}^{i} + b_{i} + \gamma V_{\text{prev}_{h}+1}^{i} \left( s_{\text{prev}_{h}+1} \right) \right)
\]

\[
Q_{s_{h}a_{h}}^{*\gamma} = \sum_{i=0}^{n_{h}} \alpha_{n_{h}}^{i} \left( r_{\text{prev}_{h}}^{i} + \gamma V_{M(s_{h}, a_{h})}^{\ast\gamma} \right) = \alpha_{n_{h}}^{0} Q_{s_{h}a_{h}}^{\ast\gamma} + \sum_{i=1}^{n_{h}} \alpha_{n_{h}}^{i} \left( r_{\text{prev}_{h}}^{i} + \gamma V_{M(s_{h}, a_{h})}^{\ast\gamma} \right),
\]

where in (a) we used the fact that \( \sum_{i=0}^{t} \alpha_{t}^{i} = 1 \) for any \( t \) and the definition of \( Q_{s_{h}a_{h}}^{*\gamma} \). Therefore we have

\[
Q_{h}(s_{h}, a_{h}) - Q_{s_{h}a_{h}}^{*\gamma} = \alpha_{n_{h}}^{0} \left( \frac{1}{1 - \gamma} - Q_{s_{h}a_{h}}^{*\gamma} \right) + \sum_{i=1}^{n_{h}} \alpha_{n_{h}}^{i} \left( r_{\text{prev}_{h}}^{i} - r_{\text{prev}_{h}}^{i} + b_{i} + \gamma \left( V_{\text{prev}_{h}+1}^{i} \left( s_{\text{prev}_{h}+1} \right) - V_{M(s_{h}, a_{h})}^{\ast\gamma} \right) \right)
\]

\[
= \alpha_{n_{h}}^{0} \left( \frac{1}{1 - \gamma} - Q_{s_{h}a_{h}}^{*\gamma} \right) + \sum_{i=1}^{n_{h}} \alpha_{n_{h}}^{i} b_{i} + \gamma \sum_{i=1}^{n_{h}} \alpha_{n_{h}}^{i} \phi_{\text{prev}_{h}+1}^{i}
\]

\[
+ \sum_{i=1}^{n_{h}} \alpha_{n_{h}}^{i} \left( r_{\text{prev}_{h}}^{i} - r_{\text{prev}_{h}}^{i} + \gamma \left( V_{\text{prev}_{h}+1}^{i} \left( s_{\text{prev}_{h}+1} \right) - V_{M(s_{h}, a_{h})}^{\ast\gamma} \right) \right).
\]
E Proof of Lemma 6

It suffices to show that (i) happens with probability at least $1 - \frac{p}{3}$ and (ii) happens with probability at least $1 - \frac{2p}{3}$.

We focus on (i) first. The case where $n_h = 0$ is trivial, so we assume $n_h \geq 1$. Fix any $s, a, t$, let $x_i = \alpha_i \left( r_{s,a,i} - \bar{R}(s,a) + \gamma \left( V_{s,a}^{*} - V_{M(s,a)}^{*} \right) \right)$. We can see that $\{x_i\}_{i=1}^t$ is a Martingale difference sequence and $|x_i| \leq \frac{\alpha_i}{1 - \gamma}$, therefore by Azuma-Hoeffding inequality we have that with probability at least $1 - \frac{2p}{\pi^2 S A t^2}$,

$$\left| \sum_{i=1}^{t} x_i \right| \leq \frac{1}{1 - \gamma} \sqrt{2 \ln \left( \frac{\pi^2 S A t^2}{p} \right) \sum_{i=1}^{t} (\alpha_i)^2} \leq \frac{1}{1 - \gamma} \sqrt{ \frac{4 \ln \left( \frac{\pi^2 S A t^2}{p} \right)}{(1 - \gamma)t} } = \beta_t,$$

where in (a) we used Lemma 4(ii). Using a union bound, the above inequalities hold for all $s, a, t \geq 1$ with probability at least

$$1 - SA \sum_{t=1}^{\infty} \frac{2p}{\pi^2 S A t^2} = 1 - \frac{p}{3}.$$ 

According to Lemma 6, it suffices to show that with probability at least $1 - \frac{p}{3}$, we have that for any $s, a, t$,

$$0 \leq \sum_{i=1}^{t} \alpha_i b_i + \sum_{i=1}^{t} \alpha_i \left( r_{s,a,i} - \bar{R}(s,a) + \gamma \left( V_{s,a}^{*} - V_{M(s,a)}^{*} \right) \right) \leq 3 \beta_t, \quad (13)$$

and then the first inequality in (i) follows by induction and the second inequality in (i) follows naturally. In fact, to see (13), first note that by Lemma 4(i) we have that

$$\beta_t \leq \sum_{i=1}^{t} \alpha_i b_i \leq 2 \beta_t$$

and the previous arguments showed that with probability at least $1 - \frac{p}{3}$ we have that for any $s, a, t$,

$$\left| \sum_{i=1}^{t} \alpha_i \left( r_{s,a,i} - \bar{R}(s,a) + \gamma \left( V_{s,a,i}^{*} - V_{M(s,a,i)}^{*} \right) \right) \right| \leq \beta_t.$$

This concludes the proof that (i) is true with probability at least $1 - \frac{p}{3}$.

Next we focus on (ii). Let $y_h = \bar{r}_h - r_h + \gamma \left( V_{M(s_h,a_h)}^{*} - V_{s_h,a_h}^{*} \right)$. We can see that $\{y_h\}_{t=0}^{T-1}$ is a Martingale difference sequence and $|y_h| \leq \frac{1}{1 - \gamma}$, therefore by Azuma-Hoeffding inequality we have that with probability at least $1 - \frac{2p}{3}$,

$$\sum_{h=1}^{T-1} y_h \leq \frac{1}{1 - \gamma} \sqrt{ \frac{2T \ln \left( \frac{3}{2p} \right) }{2p} } = \xi_T.$$

This concludes the proof that (ii) is true with probability at least $1 - \frac{2p}{3}$.