Regularizing properties of the double layer potential of second order elliptic differential operators

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Abstract: We prove the validity of regularizing properties of a double layer potential associated to the fundamental solution of a nonhomogeneous second order elliptic differential operator with constant coefficients in Schauder spaces by exploiting an explicit formula for the tangential derivatives of the double layer potential itself. We also introduce ad hoc norms for kernels of integral operators in order to prove continuity results of integral operators upon variation of the kernel, which we apply to layer potentials.

Keywords: Double layer potential, second order differential operators with constant coefficients.

2000 Mathematics Subject Classification: 31B10.

1 Introduction.

In this paper, we consider the double layer potential associated to the fundamental solution of a second order differential operator with constant coefficients. Throughout the paper, we assume that

\[ n \in \mathbb{N} \setminus \{0, 1\}, \]

where \( \mathbb{N} \) denotes the set of natural numbers including 0. Let \( \alpha \in [0, 1] \), \( m \in \mathbb{N} \setminus \{0\} \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{m, \alpha} \). Let \( \nu \equiv (\nu_l)_{l=1, \ldots, n} \) denote the external unit normal to \( \partial \Omega \). Let \( N_2 \) denote the number of multi-indexes \( \gamma \in \mathbb{N}^n \) with \( |\gamma| \leq 2 \). For each

\[ a \equiv (a_\gamma)_{|\gamma| \leq 2} \in C^{N_2}, \]  

we set

\[ a^{(2)} \equiv (a_{lj})_{l,j=1, \ldots, n}, \quad a^{(1)} \equiv (a_j)_{j=1, \ldots, n}, \quad a \equiv a_0. \]
with \( a_{ij} \equiv 2^{-1} a_{e_j + e_j} \) for \( j \neq l \), \( a_{jj} \equiv a_{e_j + e_j} \), and \( a_j \equiv a_{e_j} \), where \( \{ e_j : j = 1, \ldots, n \} \) is the canonical basis of \( \mathbb{R}^n \). We note that the matrix \( a^{(2)} \) is symmetric. Then we assume that \( a \in \mathbb{C}^{N_2} \) satisfies the following ellipticity assumption

\[
\inf_{\xi \in \mathbb{R}^n, |\xi| = 1} \text{Re} \left\{ \sum_{|\gamma| = 2} a_{\gamma} \xi^\gamma \right\} > 0, \tag{1.2}
\]

and we consider the case in which

\[
a_{lj} \in \mathbb{R} \quad \forall \, l, j = 1, \ldots, n. \tag{1.3}
\]

Then we introduce the operators

\[
P[a, D]u \equiv \sum_{l,j=1}^n \partial_{x_l} (a_{lj} \partial_{x_j} u) + \sum_{l=1}^n a_l \partial_{x_l} u + au,
\]

\[
B_\Omega^* v \equiv \sum_{l,j=1}^n \nu_l \partial_{x_j} v - \sum_{l=1}^n \nu_l a_l v,
\]

for all \( u, v \in C^2(\Omega) \), and a fundamental solution \( S_a \) of \( P[a, D] \), and the double layer potential

\[
w[\partial \Omega, a, S_a, \mu](x) \equiv \int_{\partial \Omega} \mu(y) B_{\Omega,y}^* (S_a(x - y)) \, d\sigma_y \tag{1.4}
\]

\[
= -\int_{\partial \Omega} \mu(y) \sum_{l,j=1}^n a_{lj} \nu_l(y) \frac{\partial S_a}{\partial x_j}(x - y) \, d\sigma_y
\]

\[
- \int_{\partial \Omega} \mu(y) \sum_{l=1}^n \nu_l(y) a_l S_a(x - y) \, d\sigma_y 
\]

\( \forall x \in \mathbb{R}^n \),

where the density or moment \( \mu \) is a function from \( \partial \Omega \) to \( \mathbb{C} \). Here the subscript \( y \) of \( B_{\Omega,y}^* \) means that we are taking \( y \) as variable of the differential operator \( B_{\Omega,y}^* \). The role of the double layer potential in the solution of boundary value problems for the operator \( P[a, D] \) is well known (cf. e.g., Günter [14], Kupradze, Gegelia, Basheleishvili and Burchuladze [20], Mikhlin [23]).

The analysis of the continuity and compactness properties of the integral operator associated to the double layer potential is a classical topic. In particular, it has long been known that if \( \mu \) is of class \( C^{m, \alpha} \), then the restriction of the double layer potential to the sets

\[
\Omega^+ \equiv \Omega, \quad \Omega^- \equiv \mathbb{R}^n \setminus \text{cl}\Omega,
\]

can be extended to a function of \( C^{m, \alpha}(\text{cl}\Omega^+) \) and to a function of \( C^{m, \alpha}_{\text{loc}}(\text{cl}\Omega^-) \),
respectively (cf. e.g., Miranda [24], Wiegrer [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [3].)

In case $n = 3$ and $\Omega$ is of class $C^{1,\alpha}$ and $S_a$ is the fundamental solution of the Laplace operator, it has long been known that $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is a linear and compact operator in $C^{1,\alpha}(\partial \Omega)$ and is linear and continuous from $C^0(\partial \Omega)$ to $C^{0,\alpha}(\partial \Omega)$ (cf. Schauder [30], [31], Miranda [24]).

In case $n = 3$, $m \geq 1$ and $\Omega$ is of class $C^{m+1}$ and if $P[a, D]$ is the Laplace operator, Günther [14, Ch. II, § linear and compact operator in $C^{1,\alpha}(\partial \Omega)$ and is linear and continuous from $C^0(\partial \Omega)$ to $C^{0,\alpha}(\partial \Omega)$ (cf. Schauder [30], [31], Miranda [24].)

In case $n = 3$, $m \geq 1$ and $\Omega$ is of class $C^{m+1}$ and if $P[a, D]$ is the Laplace operator, Günther [14, Ch. II, § 21, Thm. 3] has proved that $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is bounded from $C^{m-1,\alpha'}(\partial \Omega)$ to $C^{m,\alpha}(\partial \Omega)$ for $\alpha' \in [\alpha, 1[$ and that accordingly it is compact in $C^{m,\alpha}(\partial \Omega)$.

Faber, Jodeit and Rivière [12] have proved that if $\Omega$ is of class $C^1$ and if $P[a, D]$ is the Laplace operator, then $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is compact in $L^p(\partial \Omega)$ for $p \in ]1, +\infty[$. Later Hofmann, M. Mitrea and Taylor [10] have proved the same compactness result under more general conditions on $\partial \Omega$.

In case $n = 2$ and $\Omega$ is of class $C^{2,\alpha}$ and if $P[a, D]$ is the Laplace operator, Schippers [32] has proved that $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is continuous from $C^0(\partial \Omega)$ to $C^{1,\alpha}(\partial \Omega)$.

In case $n = 3$ and $\Omega$ is of class $C^2$ and if $P[a, D]$ is the Helmholtz operator, Colton and Kress [2] have developed previous work of Günther [14] and Mikhlin [23] and proved that the operator $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is bounded from $C^{0,\alpha}(\partial \Omega)$ to $C^{1,\alpha}(\partial \Omega)$ and that accordingly it is compact in $C^{1,\alpha}(\partial \Omega)$.

Wiegner [36] has proved that if $\gamma \in \mathbb{N}^n$ has odd length and $\Omega$ is of class $C^{m,\alpha}$, then the operator with kernel $(x - y)^{\gamma}|x - y|^{-(m-1-|\gamma|)}$ is continuous from $C^{m-1,\alpha}(\partial \Omega)$ to $C^{m,\alpha}(\partial \Omega)$ (and a corresponding result for the exterior of $\Omega$).

In case $n = 3$, $m \geq 2$ and $\Omega$ is of class $C^{m,\alpha}$ and if $P[a, D]$ is the Helmholtz operator, Kirsch [18] has proved that the operator $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is bounded from $C^{m-1,\alpha}(\partial \Omega)$ to $C^{m,\alpha}(\partial \Omega)$ and that accordingly it is compact in $C^{m,\alpha}(\partial \Omega)$.

von Wahl [35] has considered the case of Sobolev spaces and has proved that if $\Omega$ is of class $C^\infty$ and if $S_a$ is the fundamental solution of the Laplace operator, then the double layer improves the regularity of one unit on the boundary.

Then Heinemann [15] has developed the ideas of von Wahl in the frame of Schauder spaces and has proved that if $\Omega$ is of class $C^{m+5}$ and if $S_a$ is the fundamental solution of the Laplace operator, then the double layer improves the regularity of one unit on the boundary, i.e., $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is linear and continuous from $C^{m,\alpha}(\partial \Omega)$ to $C^{m+1,\alpha}(\partial \Omega)$.

Mazëya and Shaposhnikova [22] have proved that $w[\partial \Omega, a, S_a, \cdot]_{|\partial \Omega}$ is continuous in fractional Sobolev spaces under sharp regularity assumptions on the boundary and if $P[a, D]$ is the Laplace operator.

Mitrea [29] has proved that the double layer of second order equations and systems is compact in $C^{0,\beta}(\partial \Omega)$ for $\beta \in [0, \alpha[$ and bounded in $C^{0,\alpha}(\partial \Omega)$ under the assumption that $\Omega$ is of class $C^{1,\alpha}$. Then by exploiting a formula
for the tangential derivatives such results have been extended to compactness and boundedness results in $C^{1,\beta}(\partial\Omega)$ and $C^{1,\alpha}(\partial\Omega)$, respectively.

Mitrea, Mitrea, and Verdera [28] have proved that if $q$ is a homogeneous polynomial of odd order, then the operator with kernel $q(x-y)|x-y|^{-(n-1)-\deg(q)}$ maps $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(c\partial\Omega)$.

In this paper we are interested in the regularizing properties of the operator $w[\partial\Omega, a, S_{a},\cdot]|_{\partial\Omega}$ in Schauder spaces under the assumption that $\Omega$ is of class $C^{m,\alpha}$. We prove our statements by exploiting tangential derivatives and an inductive argument to reduce the problem to the case of the action of $w[\partial\Omega, a, S_{a},\cdot]|_{\partial\Omega}$ on $C^{0,\alpha}(\partial\Omega)$ instead of flattening the boundary with parametrization functions as done by other authors. We mention that the idea of exploiting an inductive argument together with a formula for the tangential gradient in order to prove continuity and compactness properties of the double layer potential has been exploited by Kirsch [18, Thm. 3.2] in case $n=3$ and $P[a,D]$ equals the Helmholtz operator and $S_{a}$ is the fundamental solution satisfying the radiation condition. The tangential derivatives of $f \in C^{1}(\partial\Omega)$ are defined by the equality

$$M_{lr}[f] \equiv \nu_{l}\frac{\partial \tilde{f}}{\partial x_{r}} - \nu_{r}\frac{\partial \tilde{f}}{\partial x_{l}}$$

on $\partial\Omega$, for all $l, r \in \{1, \ldots, n\}$. Here $\tilde{f}$ denotes an extension of $f$ to an open neighbourhood of $\partial\Omega$, and one can easily verify that $M_{lr}[f]$ is independent of the specific choice of the extension $\tilde{f}$ of $f$. Then we prove an explicit formula for

$$M_{lr}[w[\partial\Omega, a, S_{a},\mu]](x) - w[\partial\Omega, a, S_{a}, M_{lr}[\mu]](x) \quad \forall x \in \partial\Omega, \quad (1.5)$$

for all $\mu \in C^{1}(\partial\Omega)$ and $l, r \in \{1, \ldots, n\}$ (see formula (9.2).)

We note that Günter [14, Ch. II, §10, (42)] contains a formula for the partial derivatives of the double layer with respect to the variables in $\mathbb{R}^{n}$ in case $n=3$ and $P[a,D]$ equals the Laplace operator (see (7.11) in case of the Laplace operator.) A similar formula can be found in Kupradze, Gegelia, Basheleishvili and Burchuladze [20, Ch. V, §6, (6.11)] for the elastic double layer potential in case $n=3$. Schwab and Wendland [33] have proved that the difference in (1.5) can be written in terms of pseudodifferential operators of order $-1$. Dindoš and Mitrea have proved a number of properties of the double layer potential. In particular, [7, Prop. 3.2] proves the existence of integral operators such that the gradient of the double layer potential corresponding to the Stokes system can be written as a sum of such integral operators applied to the gradient of the moment of the double layer. Duduchava, Mitrea, and Mitrea [11] analyze various properties of the tangential derivatives. Duduchava [10] investigates partial differential equations on hypersurfaces and Bessel potential operators. In particular [10, point B of the proof of Lem. 2.1] analyzes the properties of a commutator of a
Bessel potential operator and of a tangential derivative. Hofmann, Mitrea and Taylor [16, (6.2.6)] prove a general formula for the tangential derivatives of the double layer potential corresponding to second order elliptic homogeneous equations and systems in explicit terms.

The formula (9.2) we compute here extends a formula of [21] for the Laplace operator, which has been computed with arguments akin to those of Günter [14, Ch. II, § 10, (42)], and a formula of [8] for the Helmholtz operator, and can be considered as a variant of the formula of Hofmann, Mitrea and Taylor [16, (6.2.6)] for the second order nonhomogeneous elliptic differential operator $P^a[a, D]$.

Formula (9.2) involves auxiliary operators, which we analyze in section 8. We have based our analysis of the auxiliary operators involved in formula (9.2) on the introduction of boundary norms for weakly singular kernels and on a result of joint continuity of weakly singular integrals both on the kernel of the integral and on the functional variable of the corresponding integral operator (see section 6.) For fixed choices of the kernel and for some choices of the parameters, such lemmas are known (cf. e.g., Kirsch and Hettlich [19, Thm. 3.17, p. 121].) The authors believe that the methods of section 6 may be applied to simplify also the exposition of other classical proofs of properties of layer potentials.

By exploiting formula (9.2), we can prove that $w[\partial \Omega, a, S_n, \cdot, ]|_{\partial \Omega}$ induces a linear and continuous operator from $C^m(\partial \Omega)$ to the generalized Schauder space $C^{m, \alpha}(\partial \Omega)$ of functions with $m$-th order derivatives which satisfy a generalized $\omega_\alpha$-Hölder condition with

$$\omega_\alpha(r) \sim r^\alpha|\ln r| \quad \text{as } r \to 0,$$

and that $w[\partial \Omega, a, S_n, \cdot, ]|_{\partial \Omega}$ induces a linear and continuous operator from $C^{m, \beta}(\partial \Omega)$ to $C^{m, \alpha}(\partial \Omega)$ for all $\beta \in [0, \alpha]$. In particular, the double layer potential has a regularizing effect on the boundary if $\Omega$ is of class $C^{m, \alpha}$. As a consequence of our result, $w[\partial \Omega, a, S_n, \cdot, ]|_{\partial \Omega}$ induces a compact operator from $C^m(\partial \Omega)$ to itself, and from $C^{m, \omega_\alpha(\cdot)}(\partial \Omega)$ to itself, and from $C^{m, \alpha}(\partial \Omega)$ to itself when $\Omega$ is of class $C^{m, \alpha}$.

2 Notation

We denote the norm on a normed space $X$ by $\| \cdot \|_X$. Let $X$ and $Y$ be normed spaces. We endow the space $X \times Y$ with the norm defined by $\| (x, y) \|_{X \times Y} \equiv \|x\|_X + \|y\|_Y$ for all $(x, y) \in X \times Y$, while we use the Euclidean norm for $\mathbb{R}^n$. For standard definitions of Calculus in normed spaces, we refer to Deimling [4]. If $A$ is a matrix with real or complex entries, then $A^T$ denotes the transpose matrix of $A$. The set $M_n(\mathbb{R})$ denotes the set of $n \times n$ matrices with real entries. Let $D \subseteq \mathbb{R}^n$. Then $\overline{D}$ denotes the closure of $D$, and $\partial D$ denotes the boundary of $D$, and $\text{diam}(D)$ denotes the diameter of $D$. The
symbol $|\cdot|$ denotes the Euclidean modulus in $\mathbb{R}^n$ or in $\mathbb{C}$. For all $R \in ]0, + \infty[$, $x \in \mathbb{R}^n$, $x_j$ denotes the $j$-th coordinate of $x$, and $B_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let $\Omega$ be an open subset of $\mathbb{R}^n$. The space of $m$ times continuously differentiable complex-valued functions on $\Omega$ is denoted by $C^m(\Omega, \mathbb{C})$, or more simply by $C^m(\Omega)$. Let $s \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^s$. Then $Df$ denotes the Jacobian matrix of $f$. Let $\eta \equiv (\eta_1, \ldots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \ldots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{\eta_1} f \ldots \partial^{\eta_n} f}{\partial x_1^{\eta_1} \ldots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions $f$ whose derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\partial \Omega$ is denoted $C^m_b(\Omega)$. The subspace of $C^m_b(\partial \Omega)$ whose derivatives up to order $m$ are bounded is denoted $C^m_b(\partial \Omega)$. Then $C^m_b(\partial \Omega)$ endowed with the norm $\|f\|_{C^m_b(\partial \Omega)} \equiv \sum_{|\eta| \leq m} \sup_{\partial \Omega} |D^\eta f|$ is a Banach space.

Now let $\omega$ be a function of $[0, + \infty]$ to itself such that

$$\omega \text{ is increasing and } \lim_{r \to 0^+} \omega(r) = 0, \quad (2.1)$$

and that

$$\sup_{r \in [0, 1]} \omega^{-1}(r)r < \infty. \quad (2.2)$$

If $f$ is a function from a subset $\mathcal{D}$ of $\mathbb{R}^n$ to $\mathbb{C}$, we set

$$|f : \mathcal{D}|_{\omega} \equiv \sup \left\{ \frac{|f(x) - f(y)|}{\omega(|x - y|)} : x, y \in \mathcal{D}, x \neq y \right\}.$$ 

If $|f : \mathcal{D}|_{\omega} < \infty$, we say that the function $f$ is $\omega(\cdot)$-Hölder continuous. Sometimes, we simply write $|f|_{\omega}$ instead of $|f : \mathcal{D}|_{\omega}$. If $\omega(r) = r$, and if $|f : \mathcal{D}|_{\omega} < \infty$, then we say that $f$ is Lipschitz continuous and we set $\text{Lip}(f) \equiv |f : \mathcal{D}|_{\omega}$. The subspace of $C^0(\mathcal{D})$ whose functions are $\omega(\cdot)$-Hölder continuous is denoted $C^{0, \omega(\cdot)}(\mathcal{D})$, and the subspace of $C^0(\mathcal{D})$ whose functions are Lipschitz continuous is denoted $\text{Lip}(\mathcal{D})$.

Let $\Omega$ be an open subset of $\mathbb{R}^n$. The subspace of $C^m(\partial \Omega)$ whose functions have $m$-th order derivatives that are $\omega(\cdot)$-Hölder continuous is denoted $C^{m, \omega(\cdot)}(\partial \Omega)$. Then we set

$$C^{m, \omega(\cdot)}_b(\partial \Omega) \equiv C^{m, \omega(\cdot)}(\partial \Omega) \cap C^m_b(\partial \Omega).$$

The space $C^{m, \omega(\cdot)}_b(\partial \Omega)$, equipped with its usual norm

$$\|f\|_{C^{m, \omega(\cdot)}_b(\partial \Omega)} = \|f\|_{C^m(\partial \Omega)} + \sum_{|\eta| = m} |D^\eta f : \Omega|_{\omega},$$

is well-known to be a Banach space.

Obviously, $C^{m, \omega(\cdot)}_b(\partial \Omega) = C^{m, \omega(\cdot)}(\partial \Omega)$ if $\Omega$ is bounded (and in this case, we shall always drop the subscript $b$.) The subspace of $C^m(\partial \Omega)$ of
those functions $f$ such that $f|_{\partial \Omega \cap \mathbb{B}_n(0,R)} \in C^m,\omega(\mathbb{B}(\text{cl}\Omega \cap \mathbb{B}_n(0,R)))$ for all $R \in [0, +\infty]$ is denoted $C^m,\omega_{\text{loc}}(\text{cl}\Omega)$. Clearly, $C^m,\omega(\text{cl}\Omega) = C^m,\omega_{\text{loc}}(\text{cl}\Omega)$ if $\Omega$ is bounded.

Particularly important is the case in which $\omega(\cdot)$ is the function $r^\alpha$ for some fixed $\alpha \in [0,1]$. In this case, we simply write $\cdot : \partial\Omega|_{\alpha}$ instead of $\cdot : \partial\Omega|_{r^\alpha}$, and $C^m,\omega(\partial\Omega)$ instead of $C^m,r^\alpha(\partial\Omega)$, and $C^m,\omega_{\text{loc}}(\partial\Omega)$ instead of $C^m,r^\alpha_{\text{loc}}(\partial\Omega)$. We observe that property (2.2) implies that

$$C^m,1_{\partial\Omega} \subseteq C^m,\omega(\partial\Omega).$$

For the definition of a bounded open Lipschitz subset of $\mathbb{R}^n$, we refer for example to Nečas [29, §1.3]. Let $m \in \mathbb{N} \setminus \{0\}$. We say that a bounded open subset $\Omega$ of $\mathbb{R}^n$ is of class $C^m,\omega$, if for every $P \in \partial\Omega$, there exist an open neighborhood $W$ of $P$ in $\mathbb{R}^n$, and a diffeomorphism $\psi \in C^m,\omega(\mathbb{B}_n, \mathbb{R}^n)$ of $\mathbb{B}_n \equiv \{x \in \mathbb{R}^n : |x| < 1\}$ onto $W$ such that $\psi(0) = P$, $\psi(\{x \in \mathbb{B}_n : x_n = 0\}) = W \cap \partial\Omega$, $\psi(\{x \in \mathbb{B}_n : x_n < 0\}) = W \cap \Omega$ (is said to be a parametrization of $\partial\Omega$ around $P$). Now let $\Omega$ be bounded and of class $C^m,\omega$. By compactness of $\partial\Omega$ and by definition of set of class $C^m,\omega$, there exist $P_1, \ldots, P_r \in \partial\Omega$, and parametrizations $\{\psi_i\}_{i=1, \ldots, r}$, with $\psi_i \in C^m,\omega(\mathbb{B}_n, \mathbb{R}^n)$ such that $\cup_{i=1}^r \psi_i(\{x \in \mathbb{B}_n : x_n = 0\}) = \partial\Omega$. Let $h \in \{1, \ldots, m\}$. Let $\omega$ be as in (2.1), (2.2). Let

$$\sup_{r \in [0,1]} \omega^{-1}(r)r^\alpha < \infty. \quad (2.3)$$

We denote by $C^{h,\omega}(\partial\Omega)$ the linear space of functions $f$ of $\partial\Omega$ to $\mathbb{R}$ such that $f \circ \psi_i(\cdot, 0) \in C^{h,\omega}(\mathbb{B}_{n-1})$ for all $i = 1, \ldots, r$, and we set

$$\|f\|_{C^{h,\omega}(\partial\Omega)} \equiv \sup_{i=1, \ldots, r} \|f \circ \psi_i(\cdot, 0)\|_{C^{h,\omega}(\mathbb{B}_{n-1})}. \quad \forall f \in C^{h,\omega}(\partial\Omega).$$

It is well known that by choosing a different finite family of parametrizations as $\{\psi_i\}_{i=1, \ldots, r}$, we would obtain an equivalent norm. In case $\omega(\cdot)$ is the function $r^\alpha$, we have the spaces $C^{h,\alpha}(\partial\Omega)$.

It is known that $(C^{h,\omega}(\partial\Omega), \|\cdot\|_{C^{h,\omega}(\partial\Omega)})$ is complete. Moreover condition (2.3) implies that the restriction operator is linear and continuous from $C^{h,\omega}(\partial\Omega)$ to $C^{h,\omega}(\text{cl}\Omega)$.

We denote by $d\sigma$ the area element of a manifold imbedded in $\mathbb{R}^n$. We retain the standard notation for the Lebesgue spaces.

**Remark 2.4** Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0,1]$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^m,\omega$.

Let $\omega$ be as in (2.1), (2.2). If $h \in \{1, \ldots, m\}$, $h < m$, then $m-1 \geq 1$ and $\Omega$ is of class $C^{m-1,\omega}$ and condition (2.2) implies the validity of condition (2.4) with $\alpha$ replaced by $1$, and thus we can consider the space $C^{h,\omega}(\partial\Omega)$ even if we do not assume condition (2.3). If instead $h = m$, the definition
we gave requires (2.3).

**Remark 2.5** Let \( \omega \) be as in (2.1). Let \( D \) be a subset of \( \mathbb{R}^n \). Let \( f \) be a bounded function from \( D \) to \( \mathbb{C} \), \( a \in ]0, +\infty[ \). Then,

\[
\sup_{x, y \in D, \ |x - y| \geq a} \frac{|f(x) - f(y)|}{\omega(|x - y|)} \leq \frac{2}{\omega(a)} \sup_D \ |f|.
\]

Thus the difficulty of estimating the Hölder quotient \( \frac{|f(x) - f(y)|}{\omega(|x - y|)} \) of a bounded function \( f \) lies entirely in case \( 0 < |x - y| < a \). Then we have the following well known extension result. For a proof, we refer to Troianiello [34, Thm. 1.3, Lem. 1.5].

**Lemma 2.6** Let \( m \in \mathbb{N} \setminus \{0\}, \; \alpha \in ]0, 1[ \), \( j \in \{0, \ldots, m\} \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{m, \alpha} \). Let \( R \in ]0, +\infty[ \) be such that \( \text{cl} \Omega \subseteq B_n(0, R) \). Then there exists a linear and continuous extension operator \( \tilde{f} \) of \( C_j, \alpha(\partial \Omega) \) to \( C_j, \alpha(\text{cl} B_n(0, R)) \), which takes \( \mu \in C_j, \alpha(\partial \Omega) \) to a map \( \tilde{\mu} \in C_j, \alpha(\text{cl} B_n(0, R)) \) such that \( \tilde{\mu}|_{\partial \Omega} = \mu \) and such that the support of \( \mu \) is compact and contained in \( B_n(0, R) \). The same statement holds by replacing \( C^{m, \alpha} \) by \( C^m \) and \( C_j, \alpha \) by \( C^j \).

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^1 \). The tangential gradient \( D_{\partial \Omega} f \) of \( f \in C^1(\partial \Omega) \) is defined as

\[
D_{\partial \Omega} f \equiv D\tilde{f} - (\nu \cdot D\tilde{f})\nu \quad \text{on } \partial \Omega,
\]

where \( \tilde{f} \) is an extension of \( f \) of class \( C^1 \) in an open neighborhood of \( \partial \Omega \), and we have

\[
\frac{\partial \tilde{f}}{\partial x_r} - (\nu \cdot D\tilde{f})\nu_r = \sum_{l=1}^n M_{l \nu}[f] \nu_l \quad \text{on } \partial \Omega,
\]

for all \( r \in \{1, \ldots, n\} \). If \( a \) is as in (1.1), (1.2), then we also set

\[
D_a f \equiv (D_{a, r} f)_{r=1,\ldots,n} \equiv D\tilde{f} - \frac{D\tilde{f} a^{(2)}_\nu}{\nu^r a^{(2)}_\nu} \nu_r \quad \text{on } \partial \Omega.
\]

Since

\[
D_{a, r} f = \frac{\partial \tilde{f}}{\partial x_r} - \frac{D\tilde{f} a^{(2)}_\nu}{\nu^r a^{(2)}_\nu} \nu_r = \sum_{l=1}^r M_{l \nu}[f] \left( \frac{\sum_{h=1}^n a_{l h} \nu_h}{\nu^r a^{(2)}_\nu} \right) \quad \text{on } \partial \Omega, \; (2.7)
\]

for all \( r \in \{1, \ldots, n\} \), \( D_a f \) is independent of the specific choice of the extension \( \tilde{f} \) of \( f \). We also need the following well known consequence of the Divergence Theorem.

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Lemma 2.8 Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^1$. If $\varphi, \psi \in C^1(\partial\Omega)$, then

$$
\int_{\partial\Omega} M_{lj}[\varphi] \psi \, d\sigma = - \int_{\partial\Omega} \varphi M_{lj}[\psi] \, d\sigma
$$

for all $l, j \in \{1, \ldots, n\}$.

Next we introduce the following auxiliary Lemmas, whose proof is based on the definition of norm in a Schauder space.

Lemma 2.9 Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.

(i) A function $f \in C^1(\partial\Omega)$ belongs to $C^{m,\omega}(\cdot)(\partial\Omega)$ if and only if $M_{lr}[f] \in C^{m-1,\omega}(\cdot)(\partial\Omega)$ for all $l, r \in \{1, \ldots, n\}$.

(ii) The norm $\| \cdot \|_{C^{m,\omega}(\partial\Omega)}$ is equivalent to the norm on $C^{m,\omega}(\cdot)(\partial\Omega)$ defined by

$$
\|f\|_{C^{m,\omega}(\partial\Omega)} + \sum_{l,r=1}^n \|M_{lr}[f]\|_{C^{m-1,\omega}(\cdot)(\partial\Omega)} \quad \forall f \in C^{m,\omega}(\partial\Omega).
$$

Then we have the following (see also Remark 2.4.)

Lemma 2.10 Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $h \in \{1, \ldots, m\}$. Then the following statements hold.

(i) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Then $M_{lj}$ is linear and continuous from $C^h(\partial\Omega)$ to $C^{h-1,\omega}(\partial\Omega)$ for all $l, j \in \{1, \ldots, n\}$. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(ii) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Let $a$ be as in (1.1), (1.2). Then the function from $C^h(\partial\Omega)$ to $C^{h-1,\omega}(\partial\Omega, \mathbb{R}^n)$, which takes $f$ to $D_a f$ is linear and continuous. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(iii) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Then the space $C^h(\partial\Omega)$ is continuously imbedded into $C^{h-1,\omega}(\partial\Omega)$. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(iv) Let $h < m$. Let $\psi_1, \psi_2$ be as in (2.1), (2.2). Let condition

$$
\sup_{r \in [0,1]} \psi_2^{-1}(r) \psi_1(r) < \infty
$$

Then we have the following (see also Remark 2.4.)

Lemma 2.10 Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $h \in \{1, \ldots, m\}$. Then the following statements hold.

(i) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Then $M_{lj}$ is linear and continuous from $C^h(\partial\Omega)$ to $C^{h-1,\omega}(\partial\Omega)$ for all $l, j \in \{1, \ldots, n\}$. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(ii) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Let $a$ be as in (1.1), (1.2). Then the function from $C^h(\partial\Omega)$ to $C^{h-1,\omega}(\partial\Omega, \mathbb{R}^n)$, which takes $f$ to $D_a f$ is linear and continuous. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(iii) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Then the space $C^h(\partial\Omega)$ is continuously imbedded into $C^{h-1,\omega}(\partial\Omega)$. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(iv) Let $h < m$. Let $\psi_1, \psi_2$ be as in (2.1), (2.2). Let condition

$$
\sup_{r \in [0,1]} \psi_2^{-1}(r) \psi_1(r) < \infty
$$

Then we have the following (see also Remark 2.4.)

Lemma 2.10 Let $m \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $h \in \{1, \ldots, m\}$. Then the following statements hold.

(i) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Then $M_{lj}$ is linear and continuous from $C^h(\partial\Omega)$ to $C^{h-1,\omega}(\partial\Omega)$ for all $l, j \in \{1, \ldots, n\}$. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(ii) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Let $a$ be as in (1.1), (1.2). Then the function from $C^h(\partial\Omega)$ to $C^{h-1,\omega}(\partial\Omega, \mathbb{R}^n)$, which takes $f$ to $D_a f$ is linear and continuous. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(iii) Let $h < m$. Let $\omega$ be as in (2.1), (2.2). Then the space $C^h(\partial\Omega)$ is continuously imbedded into $C^{h-1,\omega}(\partial\Omega)$. If we further assume that $\omega$ satisfies condition (2.3), then the same statement holds also for $h = m$.

(iv) Let $h < m$. Let $\psi_1, \psi_2$ be as in (2.1), (2.2). Let condition

$$
\sup_{r \in [0,1]} \psi_2^{-1}(r) \psi_1(r) < \infty
$$

Then we have the following (see also Remark 2.4.)
hold. Then $C^{h,\psi_1(\cdot)}(\partial \Omega)$ is continuously imbedded into $C^{h,\psi_2(\cdot)}(\partial \Omega)$.
If we further assume that $\psi_j$ satisfies condition (2.3) for $j \in \{1, 2\}$, then the same statement holds also for $h = m$.

(v) Let $h < m$. Let $\psi_1, \psi_2, \psi_3$ be as in (2.1), (2.2). Let conditions
\[ \sup_{j=1,2} \sup_{r \in [0,1]} |\psi_j'(r)| < \infty \]
hold. Then the pointwise product is bilinear and continuous from $C^{h,\psi_1(\cdot)}(\partial \Omega) \times C^{h,\psi_2(\cdot)}(\partial \Omega)$ to $C^{h,\psi_3(\cdot)}(\partial \Omega)$. If we further assume that $\psi_j$ satisfies condition (2.3) for $j \in \{1, 2, 3\}$, then the same statement holds also for $h = m$.

**Lemma 2.11** Let $\Omega$ be a bounded open Lipschitz subset of $\mathbb{R}^n$. Let $\psi_1, \psi_2, \psi_3$ be as in (2.1), (2.2). Let conditions
\[ \sup_{j=1,2} \sup_{r \in [0,1]} |\psi_j'(r)| < \infty \]
hold. Then the pointwise product is bilinear and continuous from the space
$C^{0,\psi_1(\cdot)}(\partial \Omega) \times C^{0,\psi_2(\cdot)}(\partial \Omega)$ to $C^{0,\psi_3(\cdot)}(\partial \Omega)$.

## 3 Preliminary inequalities

We first introduce the following elementary lemma on matrices.

**Lemma 3.1** Let $\Lambda \in M_n(\mathbb{R})$ be invertible. Let $|\Lambda| \equiv \sup_{|x|=1} |\Lambda x|$. Then

(i) Let $\tau_\Lambda \equiv \max\{|\Lambda|, |\Lambda^{-1}|\}$. Then
\[ |\Lambda^{-1}| |x| \leq |\Lambda x| \leq \tau_\Lambda |x| \quad \forall x \in \mathbb{R}^n. \]

(ii) Let $r \in [0, +\infty[$. Then
\[ |\Lambda^{-1}x|^{-r} \leq |\Lambda|^{-r} |x|^{-r} \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \]

**Proof.** Statement (i) is well known. We now consider statement (ii). Let $x \in \mathbb{R}^n \setminus \{0\}$. Then we have
\[ |x| = |\Lambda(\Lambda^{-1}x)| \leq |\Lambda| |\Lambda^{-1}x|. \]
Hence, $|\Lambda^{-1}x| \geq |\Lambda|^{-1} |x|$ and the statement follows. \(\square\)

Then we introduce the following elementary lemma, which collects either known inequalities or variants of known inequalities, which we need in the sequel.

**Lemma 3.2** Let $\gamma \in \mathbb{R}$. Let $\Lambda \in M_n(\mathbb{R})$ be invertible. The following statements hold.

(i)
\[ \frac{1}{2} |x' - y| \leq |x'' - y| \leq 2|x' - y|, \]

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\[
\frac{1}{2r^2_\Lambda} |\Lambda x' - \Lambda y| \leq |\Lambda x'' - \Lambda y| \leq 2r^2_\Lambda |\Lambda x' - \Lambda y|,
\]
for all \(x', x'' \in \mathbb{R}^n, x' \neq x'', y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|).\]

(ii)

\[
|x' - y| \gamma \leq 2 |x'' - y| \gamma, \quad |x'' - y| \gamma \leq 2 |x' - y| \gamma, \quad |\Lambda x' - \Lambda y| \gamma \leq (2r^2_\Lambda) |\Lambda x'' - \Lambda y| \gamma, \quad |\Lambda x'' - \Lambda y| \gamma \leq (2r^2_\Lambda) |\Lambda x' - \Lambda y| \gamma,
\]
for all \(x', x'' \in \mathbb{R}^n, x' \neq x'', \ y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)\).

(iii)

\[
||x' - y|\gamma - |x'' - y|\gamma| \leq (2^\gamma - 1)|x' - y|^\gamma \quad \forall y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|),
\]
for all \(x', x'' \in \mathbb{R}^n, x' \neq x''\).

(iv) There exist \(m_\gamma, m_\gamma(\Lambda) \in [0, +\infty]\) such that

\[
||x' - y|\gamma - |x'' - y|\gamma| \leq m_\gamma|x' - x''||x' - y|^\gamma, 
|\|\Lambda x' - \Lambda y\|\gamma - |\|\Lambda x'' - \Lambda y\|\gamma| \leq m_\gamma(\Lambda)|\|\Lambda x' - \Lambda y\|\gamma - |\|\Lambda x'' - \Lambda y\|\gamma| \leq m_\gamma(\Lambda)|\|\Lambda x' - \Lambda y\|\gamma - |\|\Lambda x'' - \Lambda y\|\gamma|^{-1}
\]
for all \(x', x'' \in \mathbb{R}^n, x' \neq x'', \ y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|)\).

(v)

\[
|\ln |x' - y| - \ln |x'' - y|| \leq 2|x' - x''|^{-1} |x' - y|^{-1} \quad \forall y \in \mathbb{R}^n \setminus \mathbb{B}_n(x', 2|x' - x''|),
\]
for all \(x', x'' \in \mathbb{R}^n, x' \neq x''\).

**Proof.** The first two inequalities of statement (i) follows by the triangular inequality. Then we have

\[
|\Lambda x' - \Lambda y| \leq \tau_\Lambda |x' - y| \leq \tau_\Lambda 2|x'' - y| \leq 2r^2_\Lambda |\Lambda x'' - \Lambda y|,
\]
and thus the first of the second two inequalities of statement (i) holds true. The second of the second two inequalities of statement (i) can be proved by interchanging the roles of \(x'\) and \(x''\).

We now prove only the second inequalities in statements (ii), (iv). Indeed the first inequalities follow by the second ones and by the equality \(\tau_\Lambda = 1\) when \(\Lambda\) is the identity matrix. The first of the second inequalities in (ii) for \(\gamma \geq 0\) follows by raising the inequality \(|\Lambda x' - \Lambda y| \leq (2r^2_\Lambda)|\Lambda x'' - \Lambda y|\) of statement (i) to the power \(\gamma\). Instead for \(\gamma < 0\) the same inequality follows by raising the inequality \(|\Lambda x'' - \Lambda y| \leq (2r^2_\Lambda)|\Lambda x' - \Lambda y|\) of statement (i) to the power \(\gamma\). The second of the second inequalities of (ii) can be proved by interchanging the roles of \(x'\) and \(x''\).
Statement (iii) follows by a direct application of (ii). To prove (iv) and (v), we follow Cialdea [12 §8]. We first consider (iv), and we assume that $|\Delta x' - \Delta y| \leq |\Delta x'' - \Delta y|$. By the Lagrange Theorem, there exists $\zeta \in [\Delta x' - \Delta y, [\Delta x'' - \Delta y]|$ such that

$$ ||\Delta x' - \Delta y|| \gamma - ||\Delta x'' - \Delta y|| \gamma | \leq |\gamma| \zeta^{\gamma - 1} ||\Delta x' - \Delta y| - ||\Delta x'' - \Delta y|| |. $$

If $\gamma \geq 1$, the inequality $\zeta \leq |\Delta x'' - \Delta y|$ and (i) imply that

$$ \zeta^{\gamma - 1} \leq |\Delta x'' - \Delta y|^{\gamma - 1} \leq (2\tau_\Lambda^{\gamma - 1}) |\Delta x' - \Delta y|^{\gamma - 1}. $$

If $\gamma < 1$, then inequalities $\zeta \geq |\Delta x' - \Delta y|$ and $\tau_\Lambda \geq 1$ imply that

$$ \zeta^{\gamma - 1} \leq |\Delta x' - \Delta y|^{\gamma - 1} \leq (2\tau_\Lambda^{\gamma - 1}) |\Delta x' - \Delta y|^{\gamma - 1}. $$

Then we have

$$ ||\Delta x' - \Delta y|| \gamma - ||\Delta x'' - \Delta y|| \gamma | \leq |\gamma| (2\tau_\Lambda^{\gamma - 1}) |\Delta x' - \Delta y| - ||\Delta x'' - \Delta y|| | \leq |\Delta x' - \Delta y|, $$

which implies the validity of (iv). Similarly, in case $|\Delta x' - \Delta y| > |\Delta x'' - \Delta y|$, we can prove that (3.3) holds with $x'$ and $x''$ interchanged. Then (ii) implies the validity of (iv).

We now consider statement (v) and we assume that $|x' - y| \leq |x'' - y|$. By the Lagrange Theorem, there exists $\zeta \in [|x' - y|, |x'' - y|]$ such that

$$ |\ln |x' - y| - \ln |x'' - y|| \leq \zeta^{-1} ||x' - y| - |x'' - y|| \leq \zeta^{-1} |x' - x''|. $$

By the above assumption, $\zeta^{-1} \leq |x' - y|^{-1}$, and thus statement (v) follows. Similarly, if $|x' - y| > |x'' - y|$, we can prove that (3.3) holds with $x'$ and $x''$ interchanged and (i) implies that $\zeta^{-1} \leq |x'' - y|^{-1} \leq 2|x' - y|^{-1}$, which implies the validity of (v).

\[ \square \]

**Lemma 3.5** Let $G$ be a nonempty bounded subset of $\mathbb{R}^n$. Then the following statements hold.

(i) Let $F \in \text{Lip}(\partial B_n \times [0, \text{diam}(G)])$ with

$$ \text{Lip}(F) \equiv \left\{ \frac{|F(\theta', r') - F(\theta'', r'')|}{|\theta' - \theta''| + |r' - r''|} : (\theta', r'), (\theta'', r'') \in \partial B_n \times [0, \text{diam}(G)], (\theta', r') \neq (\theta'', r'') \right\}. $$

Then

$$ \left| F \left( \frac{x' - y}{|x' - y|}, |x' - y| \right) - F \left( \frac{x'' - y}{|x'' - y|}, |x'' - y| \right) \right| \leq \text{Lip}(F) |x' - x''|, $$

(3.6)
implies that the left hand side of (3.6) is less or equal to 
\[ \forall y \in G \setminus B_n(x', 2|x' - x''|), \]
for all \( x', x'' \in G, x' \neq x'' \). In particular, if \( f \in C^1(\partial \mathbb{B}_n \times \mathbb{R}, \mathbb{C}), \) then
\[
M_{f,G} \equiv \sup \left\{ \left| f \left( \frac{x' - y}{|x' - y|} \right) - f \left( \frac{x'' - y}{|x'' - y|} \right) \right| \left| \frac{x' - y}{|x' - y|} \right| \left| \frac{x'' - y}{|x'' - y|} \right| : x', x'' \in G, x' \neq x'', y \in G \setminus B_n(x', 2|x' - x''|) \right\} < \infty .
\]
(ii) Let \( W \) be an open neighbourhood of \( \text{cl}(G - G) \). Let \( f \in C^1(W, \mathbb{C}) \).
Then
\[
\tilde{M}_{f,G} \equiv \sup \left\{ \left| f(x' - y) - f(x'' - y) \right| \left| x' - x'' \right|^{-1} : x', x'' \in G, x' \neq x'', y \in G \right\} < \infty .
\]
Here \( G - G \equiv \{ y_1 - y_2 : y_1, y_2 \in G \} \).

Proof. We first consider statement (i). The Lipschitz continuity of \( F \) implies that the left hand side of (3.6) is less or equal to
\[
\text{Lip}(F) \left\{ \left| \frac{x' - y}{|x' - y|} - \frac{x'' - y}{|x'' - y|} \right| + \left| \frac{1}{|x' - y|} - \frac{1}{|x'' - y|} \right| \right\}
\leq \text{Lip}(F) \left\{ \left| \frac{x'' - y}{|x'' - y|} \right| - \left| \frac{x' - y}{|x' - y|} \right| + \frac{1}{|x' - y|} \left| x'' - y \right| - \left| x' - y \right| + \left| x' - x'' \right| \right\}
\leq \text{Lip}(F) \left\{ \left| \frac{x'' - y}{|x'' - y|} \right| - \left| \frac{x' - y}{|x' - y|} \right| + \frac{1}{|x' - y|} \left| x'' - x'' \right| + \left| x' - x'' \right| \right\}
\leq \text{Lip}(F) \left\{ \left| \frac{x' - x''}{|x' - x''|} \right| \right\} ,
\]
and thus inequality (3.6) holds true.

Since \( \partial \mathbb{B}_n \times \mathbb{R} \) is a manifold of class \( C^\infty \) imbedded into \( \mathbb{R}^{n+1} \), there exists \( F \in C^1(\mathbb{R}^{n+1}) \) which extends \( f \). Since \( \partial \mathbb{B}_n \times [0, \text{diam}(G)] \) is a compact subset of \( \mathbb{R}^{n+1} \), \( F \) is Lipschitz continuous on \( \partial \mathbb{B}_n \times [0, \text{diam}(G)] \) and the second part of statement (i) follows by inequality (3.6).

We now consider statement (ii). Since \( f \in C^1(W, \mathbb{C}) \), \( f \) is Lipschitz continuous on the compact set \( \text{cl}(G - G) \), and statement (ii) follows. \( \square \)

Then we have the following well known statement.
Lemma 3.7 Let $\alpha \in [0,1]$. Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then there exists $c_{\Omega,\alpha} > 0$ such that

$$|\nu(y) \cdot (x - y)| \leq c_{\Omega,\alpha} |x - y|^{1+\alpha} \quad \forall x, y \in \partial \Omega.$$ 

Next we introduce a list of classical inequalities which can be verified by exploiting the local parametrizations of $\partial \Omega$.

Lemma 3.8 Let $\Omega$ be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the following statements hold.

(i) Let $\gamma \in ]-\infty,n-1[.$ Then

$$c_{\Omega,\gamma}' \equiv \sup_{x \in \partial \Omega} \int_{\partial \Omega} \frac{d\sigma_y}{|x - y|^{\gamma}} < +\infty.$$ 

(ii) Let $\gamma \in ]-\infty,n-1[.$ Then

$$c_{\Omega,\gamma}'' \equiv \sup_{x',x'' \in \partial \Omega, x' \neq x''} \left| x' - x'' \right|^{-(n-1)+\gamma} \int_{\partial \Omega \setminus B_n(x',2|x' - x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma}} < +\infty.$$ 

(iii) Let $\gamma \in ]n-1, +\infty[.$ Then

$$c_{\Omega,\gamma}''' \equiv \sup_{x',x'' \in \partial \Omega, x' \neq x''} \left| x' - x'' \right|^{-(n-1)+\gamma} \int_{\partial \Omega \setminus B_n(x',2|x' - x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma}}$$

is finite.

(iv) Let $\gamma \in ]0,1[.$ Then

$$c_{\Omega,\gamma}^iv \equiv \sup_{x',x'' \in \partial \Omega, 0 < |x' - x''| < 1/e} \left| \ln |x' - x''| \right|^{-1} \int_{\partial \Omega \setminus B_n(x',2|x' - x''|)} \frac{d\sigma_y}{|x' - y|^{n-1}} < +\infty.$$ 

4 Preliminaries on the fundamental solution

We first introduce a formula for the fundamental solution of $P[a, D]$. To do so, we follow a formulation of Dalla Riva [3 Thm. 5.2, 5.3] and Dalla Riva, Morais and Musolino [5 Thm. 3.1, 3.2] (see also John [17], and Miranda [24] for homogeneous operators, and Mitrea and Mitrea [27, p. 203].)

Theorem 4.1 Let $a$ be as in (1.1), (1.2). Let $S_a$ be a fundamental solution of $P[a, D]$. Then there exist an analytic function $A_0$ from $\partial \mathbb{B}_n$ to $\mathbb{C}$, and an analytic function $A_1$ from $\partial \mathbb{B}_n \times \mathbb{R}$ to $\mathbb{C}$, and $b_0 \in \mathbb{C}$, and an analytic
function $B_1$ from $\mathbb{R}^n$ to $\mathbb{C}$ such that $B_1(0) = 0$, and an analytic function $C$ from $\mathbb{R}^n$ to $\mathbb{C}$ such that

$$S_a(x) = |x|^{2-n}A_0\left(\frac{x}{|x|}\right) + |x|^{3-n}A_1\left(\frac{x}{|x|}, |x|\right) + b_0 \ln |x| + B_1(x) \ln |x| + C(x),$$

(4.2)

for all $x \in \mathbb{R}^n \setminus \{0\}$, and such that both $b_0$ and $B_1$ equal zero if $n$ is odd. Moreover,

$$|x|^{2-n}A_0\left(\frac{x}{|x|}\right) + \delta_{2,n}b_0 \ln |x|$$

is a fundamental solution for the principal part $\sum_{i,j=1}^n \partial_{x_i}(a_{ij}\partial_{x_j})$ of $P[a,D]$. Here $\delta_{2,n}$ denotes the Kronecker symbol. Namely,

$$\delta_{2,n} = 1 \text{ if } n = 2, \quad \delta_{2,n} = 0 \text{ if } n > 2.$$

Then we have the following.

**Corollary 4.3** Let $a$ be as in (1.1), (1.2). Let $S_a$ be a fundamental solution of $P[a,D]$. Then the following statements hold.

(i) If $n \geq 3$, then there exists one and only one fundamental solution of the principal part $\sum_{i,j=1}^n \partial_{x_i}(a_{ij}\partial_{x_j})$ of $P[a,D]$ which is positively homogeneous of degree $2 - n$ in $\mathbb{R}^n \setminus \{0\}$.

(ii) If $n = 2$, then there exists one and only one fundamental solution $S(x)$ of the principal part $\sum_{i,j=1}^n \partial_{x_i}(a_{ij}\partial_{x_j})$ of $P[a,D]$ such that

$$\beta_0 \equiv \lim_{x \to 0} \frac{S(x)}{\ln |x|} \in \mathbb{C}, \quad \int_{\partial B_n} S \, d\sigma = 0,$$

and such that $S(x) - \beta_0 \ln |x|$ is positively homogeneous of degree 0 in $\mathbb{R}^n \setminus \{0\}$.

**Proof.** We retain the notation of Theorem 4.1. We first consider statement (i). By Theorem 4.1 the function $|x|^{2-n}A_0\left(\frac{x}{|x|}\right)$ is a fundamental solution of the principal part of $P[a,D]$ and is clearly positively homogeneous of degree $2 - n$. Now assume that $u$ is a fundamental solution of the principal part of $P[a,D]$ and that $u$ is positively homogeneous of degree $2 - n$ in $\mathbb{R}^n \setminus \{0\}$. Then the difference

$$w(x) \equiv |x|^{2-n}A_0\left(\frac{x}{|x|}\right) - u(x)$$

defines an entire real analytic function in $\mathbb{R}^n$ and is positively homogeneous of degree $2 - n$ in $\mathbb{R}^n \setminus \{0\}$. In particular,

$$\lambda^{n-2}w(\lambda x) = w(x) \quad \forall (\lambda, x) \in [0, +\infty[ \times (\mathbb{R}^n \setminus \{0\}),$$

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\[ \lambda^{(n-2)+|\beta|}D^\beta w(\lambda x) = D^\beta w(x) \quad \forall (\lambda, x) \in [0, +\infty[ \times (\mathbb{R}^n \setminus \{0\}) , \]

and accordingly

\[ \lambda^{(n-2)+|\beta|}D^\beta w(\lambda x) = D^\beta w(x) \quad \forall (\lambda, x) \in [0, +\infty[ \times (\mathbb{R}^n \setminus \{0\}) , \]

for all \( \beta \in \mathbb{N}^n \). Then by letting \( \lambda \) tend to 0, we obtain \( D^\beta w(0) = 0 \) for all \( \beta \in \mathbb{N}^n \). Since \( w \) is real analytic, we deduce that \( w \) is equal to 0 in \( \mathbb{R}^n \) and thus statement (i) holds.

We now assume that \( n = 2 \). By Theorem 4.1, the function \( S(x) \equiv A_0(\frac{x}{|x|}) = \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma + b_0 \ln |x| \) is a fundamental solution of the principal part of \( P[a, D] \) and satisfies the conditions of statement (ii). We now assume that \( u \) is another fundamental solution of the principal part as in (ii). Then the difference

\[ w(x) \equiv A_0(\frac{x}{|x|}) - \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma + b_0 \ln |x| - u(x) \]

defines an entire real analytic function in \( \mathbb{R}^n \) and we have

\[ 0 = \lim_{x \to 0} \ln |x| = \lim_{x \to 0} \frac{A_0(\frac{x}{|x|}) - \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma}{\ln |x|} + b_0 - \lim_{x \to 0} \frac{u(x)}{\ln |x|}, \]

and accordingly

\[ b_0 = \lim_{x \to 0} \frac{u(x)}{\ln |x|} \equiv \beta_0 \in \mathbb{C}. \]

Then our assumption implies that the analytic function

\[ u(x) - \beta_0 \ln |x| = u(x) - b_0 \ln |x|, \]

is positively homogeneous of degree 0 in \( \mathbb{R}^n \setminus \{0\} \). Hence, there exists a function \( g_0 \) from \( \partial B_n \) to \( \mathbb{C} \) such that

\[ u(x) - b_0 \ln |x| = g_0(\frac{x}{|x|}) \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \]

In particular, \( g_0 \) is real analytic and

\[ w(x) = A_0(\frac{x}{|x|}) - \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma + b_0 \ln |x| - (g_0(\frac{x}{|x|}) + b_0 \ln |x|) \]

\[ = A_0(\frac{x}{|x|}) - \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma - g_0(\frac{x}{|x|}). \]

Moreover, \( w \) must be positively homogeneous of degree 0 in \( \mathbb{R}^n \setminus \{0\} \). Since \( w \) is continuous at 0, \( w \) must be constant in the whole of \( \mathbb{R}^n \). Since \( \int_{\partial B_n} w \, d\sigma = \int_{\partial B_n} S \, d\sigma - \int_{\partial B_n} u \, d\sigma = 0 \), such a constant must equal 0 and thus \( A_0(\frac{x}{|x|}) - \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma = g_0(\frac{x}{|x|}) \) for all \( x \in \mathbb{R}^n \setminus \{0\} \). Hence, \( u(x) = A_0(\frac{x}{|x|}) - \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma + b_0 \ln |x| \) and statement (ii) follows. \( \square \)
Then we can introduce the following.

Definition 4.4 Let \( a \) be as in (1.1), (1.2). We define as normalized fundamental solution of the principal part of \( P[a,D] \), to be the only fundamental solution of Corollary 4.3

By Theorem 4.1 and by Corollary 4.3, the normalized fundamental solution of the principal part of \( P[a,D] \) equals

\[
|x|^{2-n} A_0 \left( \frac{x}{|x|} \right)
\]

if \( n \geq 3 \) and

\[
A_0 \left( \frac{x}{|x|} \right) = \frac{1}{2\pi} \int_{\partial B_n} A_0 \, d\sigma + b_0 \ln |x|
\]

if \( n = 2 \), where \( A_0 \) and \( b_0 \) are as in Theorem 4.1. We now see that if the principal coefficients of \( P[a,D] \) are real, then the normalized fundamental solution of the principal part of \( P[a,D] \) has a very specific form. To do so, we introduce the fundamental solution \( S_n \) of the Laplace operator. Namely, we set

\[
S_n(x) \equiv \left\{ \begin{array}{ll}
\frac{1}{s_n} \ln |x| & \forall x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n = 2, \\
\frac{1}{|x|^{2-n} s_n} |x|^{2-n} & \forall x \in \mathbb{R}^n \setminus \{0\}, \text{ if } n > 2,
\end{array} \right.
\]

where \( s_n \) denotes the \((n-1)\) dimensional measure of \( \partial B_n \). Then we have the following elementary statement, which can be verified by the chain rule and by Corollary 4.3 (cf. e.g., Dalla Riva [4].)

Lemma 4.5 Let \( a \) be as in (1.1), (1.2), (1.3). Then there exists an invertible matrix \( T \in M_n(\mathbb{R}) \) such that

\[
a^{(2)} = TT^t
\]  (4.6)

and the function

\[
S_{a^{(2)}}(x) \equiv \frac{1}{\sqrt{\det a^{(2)}}} S_n(T^{-1} x) \quad \forall x \in \mathbb{R}^n \setminus \{0\},
\]

coincides with the normalized fundamental solution of the principal part of \( P[a,D] \) if \( n \geq 3 \), and coincides with the normalized fundamental solution of the principal part of \( P[a,D] \) up to an additive constant if \( n = 2 \).

Then Theorem 4.1, and Corollary 4.3, and Lemma 4.5 imply the validity of the following.

Corollary 4.7 Let \( a \) be as in (1.1), (1.2), (1.3). Let \( T \in M_n(\mathbb{R}) \) be as in (4.6). Let \( S_n \) be a fundamental solution of \( P[a,D] \).

Then there exist an analytic function \( A_1 \) from \( \partial B_n \times \mathbb{R} \) to \( \mathbb{C} \), and an analytic function \( B_1 \) from \( \mathbb{R}^n \) to \( \mathbb{C} \) such that \( B_1(0) = 0 \), and an analytic
function $C$ from $\mathbb{R}^n$ to $\mathbb{C}$ such that
\[
S_a(x) = \frac{1}{\sqrt{\det a^{(2)}}}S_n(T^{-1}x) + |x|^{3-n}A_1\left(\frac{x}{|x|}, |x|\right) + (B_1(x) + b_0(1 - \delta_{2,n})) \ln |x| + C(x),
\]
for all $x \in \mathbb{R}^n \setminus \{0\}$, and such that both $b_0$ and $B_1$ equal zero if $n$ is odd. Moreover, \[
\frac{1}{\sqrt{\det a^{(2)}}}S_n(T^{-1}x)
\]
is a fundamental solution for the principal part of $P[a, D]$.

Next we prove the following technical statement.

**Lemma 4.9** Let $a$ be as in (1.1), (1.2). Let $S_a$ be a fundamental solution of $P[a, D]$. Let $G$ be a nonempty bounded subset of $\mathbb{R}^n$.

(i) Let $\gamma \in [0, 1]$. Then
\[
C_{0, S_a, G, n-1-\gamma} \equiv \sup_{0 < |x| \leq \text{diam}(G)} |x|^{n-1-\gamma}|S_a(x)| < +\infty. \tag{4.10}
\]
If $n > 2$, then (4.10) holds also for $\gamma = 1$.

(ii)
\[
\tilde{C}_{0, S_a, G} \equiv \sup \left\{ \frac{|x' - y'|^{n-1}}{|x' - x''|} |S_a(x' - y) - S_a(x'' - y)| : x', x'' \in G, x' \neq x'', y \in G \setminus \mathbb{B}_n(x', 2|x' - x''|) \right\} < \infty.
\]

**Proof.** Statement (i) is an immediate consequence of formula (4.2). We now prove statement (ii). To do so, we resort to formula (4.2) and we set
\[
A(\theta, r) \equiv A_0(\theta) + rA_1(\theta, r) \quad \forall (\theta, r) \in \partial \mathbb{B}_n \times \mathbb{R},
\]
\[
B(x) \equiv b_0 + B_1(x) \quad \forall x \in \mathbb{R}^n.
\]
Then Lemmas 3.2 and 3.5 imply that
\[
|S_a(x' - y) - S_a(x'' - y)| \leq |x' - y|^{2-n} \left| A \left( \frac{x' - y}{|x' - y|}, |x' - y| \right) - A \left( \frac{x'' - y}{|x'' - y|}, |x'' - y| \right) \right| + \left| A \left( \frac{x'' - y}{|x'' - y|}, |x'' - y| \right) \right| |x' - y|^{2-n} - |x'' - y|^{2-n} |
\]
\[+ |\ln |x' - y|| |B(x' - y) - B(x'' - y)|
\]
Lemma 4.11 Let $a$ be as in (1.1), (1.2), (1.3). Let $T \in M_n(\mathbb{R})$ be as in (4.8). Let $S_a$ be a fundamental solution of $P[a,D]$. Let $B_1, C$ be as in Corollary 4.7. Let $G$ be a nonempty bounded subset of $\mathbb{R}^n$. Then the following statements hold.

(i) There exists a real analytic function $A_2$ from $\partial \mathbb{B}_n \times \mathbb{R}$ to $\mathbb{C}^n$ such that

\[
DS_a(x) = \frac{1}{s_n \sqrt{\det a^{(2)}}} [T^{-1} x]^{-n} x^t (a^{(2)})^{-1} + |x|^{2-n} A_2 \left( \frac{x}{|x|} \right) + DB_1(x) \ln |x| + DC(x) \quad \forall x \in \mathbb{R}^n \setminus \{0\} .
\]

(ii) $C_{1,a,G} \equiv \sup_{0 < |x| \leq \text{diam}(G)} |x|^{n-1} |DS_a(x)| < +\infty .

(iii) $\tilde{C}_{1,a,G} \equiv \sup \left\{ \frac{|x'-y|^n}{|x'-x''|^n} |DS_a(x') - DS_a(x'')| : x', x'' \in G, x' \neq x'', y \in G \setminus \mathbb{B}_n(x', 2|x'-x''|) \right\} < \infty .

Since $A$ is continuous on the compact set $\partial \mathbb{B}_n \times [0, \text{diam}(G)]$, and $B$ and $C$ are continuous on the compact set $\text{cl}(G - G)$, there exists $c > 0$ such that

\[
|S_a(x' - y) - S_a(x'' - y)| \\
\leq c|x' - x''| \left\{ |x' - y|^{1-n} + \frac{1}{|x'-y|} + \ln |x'-y| + 1 \right\} \\
\leq c|x' - x''| |x' - y|^{1-n} \\
\times \{ 1 + |x' - y|^{n-2} + |x' - y|^{n-1} \ln |x'-y| + |x'-y|^{n-1} \},
\]

and thus statement (ii) holds. \qed
Thus if we set
\[ \text{for all } x \]
\[ \text{we can set } \beta \text{ the function} \]
\[ \text{By the analyticity of } A \text{ of the function} \]
\[ \text{Hence, formula (4.12) implies that} \]
\[ \text{Next we turn to the proof of statement (ii). By Lemma 3.1 (ii) and by} \]
\[ \text{the Schwartz inequality, we have} \]
\[ \text{imply the validity of statement (i).} \]

Next we consider
\[ \text{By the Fundamental Theorem of Calculus, we have} \]
\[ \text{Thus if we set} \]
\[ \text{the function } \beta \text{ is analytic and satisfies the equality} \]
\[ \text{Then we can set} \]
\[ \text{By the analyticity of } A_1 \text{ and } \beta, \text{and by the equality} r^{n-3} \theta^b_0 (1- \delta_{2,n}) = 0 \]
\[ \text{if } n = 2, \text{the function } A_2 \text{ is analytic. Hence, equalities (4.14) and (4.15) imply the validity of statement (i).} \]

Next we turn to the proof of statement (ii). By Lemma 3.1 (ii) and by the Schwartz inequality, we have
\[ |T^{-1} x|^{-n} |x^t (a^{(2)})^{-1}| \leq |x|^{-n} |T|^n |(a^{(2)})^{-1}|. \]

Hence, formula (4.13) implies that
\[ |x|^{n-1} |DS_a(x)| \leq \frac{1}{s_n \sqrt{\det a^{(2)}}} |T|^n |(a^{(2)})^{-1}| \]
\[ + \left\{ |x| A_2 \left( \frac{x}{|x|}, |x| \right) + (|x|^{n-1} \ln |x|) DB_1(x) + |x|^{n-1} DC(x) \right\}, \]

Proof. By formula (4.8) and by the chain rule, we have
\[ DS_a(x) = \frac{1}{s_n \sqrt{\det a^{(2)}}} |T^{-1} x|^{-n} x^t (a^{(2)})^{-1} + (3 - n) |x|^2 - n x^t |x| A_1 \left( \frac{x}{|x|}, |x| \right) \]
\[ + |x|^{-3-n} \left\{ DA_1 \left( \frac{x}{|x|}, |x| \right) [\ln |x| I - x \otimes x |x|^{-1}] x^{-2} + \frac{\partial A_1}{\partial r} \left( \frac{x}{|x|}, |x| \right) x^t |x| \right\} \]
\[ + DB_1(x) \ln |x| + (B_1(x) + b_0 (1 - \delta_{2,n})) \frac{x^t |x|}{|x|^2} + DC(x), \]

for all \( x \in \mathbb{R}^n \setminus \{0\} \), where we have still denoted by \( A_1 \) any analytic extension of the function \( A_1 \) of Corollary 4.7 to an open neighbourhood of \( \partial B_n \times \mathbb{R} \) in \( \mathbb{R}^{n+1} \) and where \( x \otimes x \) denotes the matrix \( (x_i x_j)_{i,j=1,...,n} \).
for all $x \in \mathbb{R}^n \setminus \{0\}$. Then the continuity of $A_2$ on the compact set $\partial \mathbb{B}_n \times [0, \text{diam}(G)]$ and the continuity of $DB_1$ and $DC$ on the compact set $\text{cl} \mathbb{B}_n(0, \text{diam}(G))$ imply the validity of statement (ii).

We now turn to statement (iii). Let $x', x'' \in G$, $x' \neq x''$, $y \in G \setminus \mathbb{B}_n(x', 2|x' - x'|)$. By statement (i), we have

$$|DS_n(x' - y) - DS_n(x'' - y)| \leq \frac{1}{s_n \sqrt{\det a^{(2)}}} \left|T^{-1}(x' - y)|^{-n}(x' - y)^t a^{(2)} |^{-1} - |T^{-1}(x'' - y)|^{-n}(x'' - y)^t a^{(2)} |^{-1}\right|$$

$$+ |x' - y|^{2-n} A_2 \left(\frac{x' - y}{x' - y}, |x' - y|\right) - |x'' - y|^{2-n} A_2 \left(\frac{x'' - y}{x'' - y}, |x'' - y|\right)$$

$$+ |x' - x''| |a^{(2)}|^{-1} |T^{-1}(x' - y)|^{-n} - |T^{-1}(x'' - y)|^{-n}$$

$$+ DC(x' - y) - DC(x'' - y).$$

We first estimate the first summand in the right hand side of inequality (4.16). By the triangular inequality, we have

$$\left|T^{-1}(x' - y)|^{-n}(x' - y)^t a^{(2)} |^{-1} - |T^{-1}(x'' - y)|^{-n}(x'' - y)^t a^{(2)} |^{-1}\right|$$

$$\leq |x' - y| |a^{(2)}|^{-1} \left|T^{-1}(x' - y)|^{-n} - |T^{-1}(x'' - y)|^{-n}\right|$$

$$+ |x' - x''| |a^{(2)}|^{-1} |T^{-1}(x' - y)|^{-n}.$$ (4.17)

Then Lemmas 3.1 (ii), 3.2 (ii), (iv) with $\gamma = -n$, $\Lambda = T^{-1}$ imply that

$$\left|T^{-1}(x' - y)|^{-n} - |T^{-1}(x'' - y)|^{-n}\right| \leq m_{-n}(T^{-1}) |T^{-1} x' - T^{-1} x''| |T^{-1} x' - T^{-1} y|^{-n-1}$$

$$\leq m_{-n}(T^{-1}) |T^{-1} x' - x''| |x'' - y|^{-n-1},$$

$$|T^{-1}(x'' - y)|^{-n} \leq |T^n|x'' - y|^{-n},$$

$$|x'' - y|^{-n} \leq 2^n |x' - y|^{-n}.$$ (4.18)

Next we estimate the second summand in the right hand side of inequality (4.16). By Lemmas 3.2 (iv) and 3.3 (i), the second summand is less or equal to

$$|x' - y|^{2-n} - |x'' - y|^{2-n} \left|A_2 \left(\frac{x'' - y}{x'' - y}, |x'' - y|\right)\right|$$

$$\leq 2^n |x' - y|^{-n}.$$ (4.19)
\[ +|x' - y|^{2-n} A_2(\frac{x' - y}{|x' - y|}, |x' - y|) - A_2(\frac{x'' - y}{|x'' - y|}, |x'' - y|) \]
\[ \leq m_{2-n}|x' - x''||x' - y|^{2-n-1} \sup_{\partial B_n \times [0, \text{diam}(G)]} |A_2| \]
\[ + |x' - y|^{2-n}\left( \sum_{j=1}^{n} M_{A_2,j,G}\right)|x' - x''||x' - y|^{-1}. \]

Next we estimate the third summand in the right hand side of inequality (4.16). By Lemmas 3.2 (v) and 3.5 (ii), the third summand is less or equal to
\[ \ln |x' - y| - \ln |x' - y| |DB_1(x'' - y)| \]
\[ + |x' - y||DB_1(x'' - y) - DB_1(x' - y)| \]
\[ \leq 2|x' - x''||x' - y|^{-1} \sup_{G \subset G} |DB_1| + \left( \sum_{j=1}^{n} \tilde{M}_{\frac{\alpha_n}{\alpha_j},G}\right)|x' - x''||\ln |x' - y|| \]
\[ \leq |x' - x''||x' - y|^{-n}\left\{ 2|x' - y|^{n-1} \sup_{G \subset G} |DB_1| \right. \]
\[ \left. + \left( \sum_{j=1}^{n} \tilde{M}_{\frac{\alpha_n}{\alpha_j},G}\right)|x' - y|^n \ln |x' - y|| \right\}. \]

Finally, Lemma 3.5 (ii) implies that
\[ |DC(x' - y) - DC(x'' - y)| \leq \left( \sum_{j=1}^{n} \tilde{M}_{\frac{\alpha_n}{\alpha_j},G}\right)|x' - x''| \]
\[ \leq |x' - x''||x' - y|^{-n}\left( \sum_{j=1}^{n} \tilde{M}_{\frac{\alpha_n}{\alpha_j},G}\right) \sup_{(x',y) \in G \times G} |x' - y|^n. \]

Then inequalities (4.16)–(4.21) imply the validity of statement (iii). \(\square\)

5 Preliminary inequalities on the boundary operator

We now turn to estimate the kernel \(B_{\Omega, y}(S_\alpha(x - y))\) of the double layer potential of (1.4). We do so under assumption (1.3). To do so, we introduce some basic inequalities for \(B_{\Omega, y}(S_\alpha(x - y))\) by means of the following.

**Lemma 5.1** Let \(\alpha\) be as in (1.1), (1.2), (1.3). Let \(T \in M_n(\mathbb{R})\) be as in (4.6). Let \(S_\alpha\) be a fundamental solution of \(P[\alpha, D]\).
Let $\alpha \in ]0, 1]$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^1, \alpha$. Then the following statements hold.

(i) If $\alpha \in ]0, 1]$, then

$$b_{\Omega, \alpha} \equiv \sup_{x,y} \left\{ |x-y|^{n-1-\alpha} |B_{\Omega, y}^2 (S_n(x-y))| : x, y \in \partial \Omega, x \neq y \right\} < +\infty.$$  

If $n > 2$, then (5.2) holds also for $\alpha = 1$.

(ii)

$$\tilde{b}_{\Omega, \alpha} \equiv \sup_{x', y'} \left\{ \frac{|x'-y'|^{n-\alpha}}{|x'-x''|} |B_{\Omega, y'}^2 (S_n(x'-y')) - B_{\Omega, y}^2 (S_n(x''-y'))| : x', y' \in \partial \Omega, x' \neq x'', y \in \partial \Omega \setminus B_n(x', 2|x'-x''|) \right\} < +\infty.$$  

Proof. By Lemma 4.11 (i), we have

$$B_{\Omega, y}^2 (S_n(x-y)) = -DS_n(x-y)a^{(2)} \nu(y) - \nu'(y)a^{(1)} S_n(x-y)$$  

(5.3)

$$= -\frac{1}{s_n \sqrt{\det a^{(2)}}} |T^{-1}(x-y)|^{-n} (x-y)^\nu(y)$$

$$-|x-y|^2 A_2 \frac{x-y}{|x-y|} a^{(2)} \nu(y)$$

$$-DB_1(x-y)a^{(2)} \nu(y) \ln |x-y| - DC(x-y)a^{(2)} \nu(y)$$

$$-\nu'(y)a^{(1)} S_n(x-y) \quad \forall x, y \in \partial \Omega, x \neq y.$$  

Then by Lemmas 3.1 (ii), 3.7, 4.9 (i), and by the equality in (5.3), we have

$$|x-y|^{n-1-\alpha} |B_{\Omega, y}^2 (S_n(x-y))| \leq \frac{1}{s_n \sqrt{\det a^{(2)}}} \alpha^{(2)} |T|^n |x-y|^{-n+1+\alpha+n-1-\alpha}$$

$$+|x-y|^{2-1-\alpha} |a^{(2)}| A_2 \frac{x-y}{|x-y|} |x-y|$$

$$+|x-y|^{n-1-\alpha} \ln |x-y| |a^{(2)}| DB_1(x-y)|$$

$$+|x-y|^{n-1-\alpha} |a^{(2)}| DC(x-y)| + |a^{(1)}| C_{0, S_n, \partial \Omega, n-1-\alpha},$$

for all $x, y \in \partial \Omega$, $x \neq y$. If either $\alpha \in ]0, 1]$ or if $\alpha \in [0, 1]$ and $n > 2$, then the right hand side is bounded for $x, y \in \partial \Omega$, $x \neq y$. Hence, we conclude that statement (i) holds true.

Next we consider statement (ii).

$$|B_{\Omega, y}^2 (S_n(x'-y)) - B_{\Omega, y}^2 (S_n(x''-y))|$$  

(5.4)
for all \( \mathbf{x}, \mathbf{x}'' \in \partial \Omega, \mathbf{x} \neq \mathbf{x}'', \mathbf{y} \in \partial \Omega \setminus \mathcal{B}_n(\mathbf{x}', 2|\mathbf{x}' - \mathbf{x}'|) \). Next we denote by \( J_1 \) the first term in the right hand side of (5.4). By Lemmas 3.1 (ii), 3.2 (ii) and (iv) with \( \gamma = -n, \Lambda = T^{-1} \), and by Lemma 3.7, we have

\[
J_1 \leq \frac{1}{s_n \sqrt{\det a^{(2)}}} \times \left\{ \frac{|T^{-1}(\mathbf{x} - \mathbf{y})|^{-n} - |T^{-1}(\mathbf{x}' - \mathbf{y})|^{-n}}{s_n \sqrt{\det a^{(2)}}} \right\} \]

\[
\leq \frac{1}{s_n \sqrt{\det a^{(2)}}} \times \left\{ \frac{m_n(T^{-1})}{|T^{-1}|^{n+1} |\mathbf{x}' - \mathbf{x}'|^{n+1}} \right\},
\]

for all \( \mathbf{x}', \mathbf{x}'' \in \partial \Omega, \mathbf{x}' \neq \mathbf{x}'', \mathbf{y} \in \partial \Omega \setminus \mathcal{B}_n(\mathbf{x}', 2|\mathbf{x}' - \mathbf{x}'|) \). Next we note that

\[
|(\mathbf{x}' - \mathbf{x}'')^T \nu(y)| \leq |(\mathbf{x}' - \mathbf{x}'')^T (\nu(y) - \nu(x'))| + |(\mathbf{x}' - \mathbf{x}'')^T \nu(x')| \]

\[
\leq |\mathbf{x}' - \mathbf{x}'||\nu|_\alpha |\mathbf{x}' - \mathbf{x}'|^\alpha + c_{\Omega, \alpha} |\mathbf{x}' - \mathbf{x}'|^{1+\alpha} \]

\[
\leq |\mathbf{x}' - \mathbf{x}'||\mathbf{x}' - \mathbf{y}|^\alpha (\nu|_\alpha + c_{\Omega, \alpha}),
\]

and that accordingly

\[
J_1 \leq \frac{|\mathbf{x}' - \mathbf{x}'|}{s_n \sqrt{\det a^{(2)}}} \times \left\{ \frac{m_n(T^{-1})}{|T^{-1}|^{n+1} |\mathbf{x}' - \mathbf{x}'|^{n+1}} \right\},
\]

for all \( \mathbf{x}', \mathbf{x}'' \in \partial \Omega, \mathbf{x}' \neq \mathbf{x}'', \mathbf{y} \in \partial \Omega \setminus \mathcal{B}_n(\mathbf{x}', 2|\mathbf{x}' - \mathbf{x}'|) \). Next we denote
by $J_2$ the sum of the terms different from $J_1$ in the right hand side of (5.4). Then Lemma 3.2 (iv), (v) and Lemmas 3.5, 4.9 (ii) imply that

$$J_2 \leq |a^{(2)}| \left( \sum_{j=1}^{n} M_{A_2,j,\partial \Omega} \right) \frac{|x' - x''|}{|x' - y|} |x' - y|^{2-n}$$

(5.7)

$$+ |a^{(2)}| \sup_{\partial B_n \times \{0, \text{diam}(\partial \Omega)\}} |A_2| m_{2-n} |x' - x''| |x' - y|^{1-n}$$

$$+ |a^{(2)}| \left( \sum_{j=1}^{n} \bar{M}_{\partial B_n, \partial \Omega} \right) |x' - x''| | \ln |x' - y||$$

$$+ |a^{(2)}| \sup_{\partial \Omega - \partial \Omega} |DB_1| 2 \frac{|x' - x''|}{|x' - y|}$$

$$+ M_{C} |x' - x''| + \tilde{C}_{0,S_n,\partial \Omega} |a^{(1)}| \frac{|x' - x''|}{|x' - y|^{n-1}}$$

for all $x', x'' \in \partial \Omega$, $x' \neq x''$, $y \in \partial \Omega \setminus B_n(x', 2|x' - x''|)$. By inequalities (5.3), (5.6), (5.7), we conclude that statement (ii) holds.

6 Boundary norms for kernels

For each subset $A$ of $\mathbb{R}^n$, we find convenient to set

$$\Delta_A \equiv \{(x, y) \in A \times A : x = y\}.$$

We now introduce a class of functions on $(\partial \Omega)^2 \setminus \Delta_{\partial \Omega}$ which may carry a singularity as the variable tends to a point of the diagonal, just as in the case of the kernels of integral operators corresponding to layer potentials defined on the boundary of an open subset $\Omega$ of $\mathbb{R}^n$.

**Definition 6.1** Let $G$ be a nonempty bounded subset of $\mathbb{R}^n$. Let $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$. We denote by $\mathcal{K}_{\gamma_1, \gamma_2, \gamma_3}(G)$ the set of continuous functions $K$ from $(G \times G) \setminus \Delta_G$ to $\mathbb{C}$ such that

$$\|K\|_{\mathcal{K}_{\gamma_1, \gamma_2, \gamma_3}(G)} \equiv \sup \left\{ |x - y|^{\gamma_1} |K(x, y)| : x, y \in G, x \neq y \right\}$$

$$+ \sup \left\{ \frac{|x' - y|^{\gamma_2}}{|x' - x''|^{\gamma_3}} |K(x', y) - K(x'', y)| :$$

$$x', x'' \in G, x' \neq x'', y \in G \setminus B_n(x', 2|x' - x''|) \right\} < +\infty.$$

One can easily verify that $(\mathcal{K}_{\gamma_1, \gamma_2, \gamma_3}(G), \| \cdot \|_{\mathcal{K}_{\gamma_1, \gamma_2, \gamma_3}(G)})$ is a Banach space.

**Remark 6.2** Let $a$ be as in (1.7), (1.8). Let $S_a$ be a fundamental solution of $P[a, D]$. 25
Obviously, \( \omega \) takes integrable in \( C \) to \( C \) if \( \partial \) is continuous. Moreover, the bilinear map from \( K, \mu \) to \( \Omega \) is continuous.

**Proposition 6.3** Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Let \( \gamma_1 \in (-\infty, 1] \), \( \gamma_2, \gamma_3 \in \mathbb{R} \). Then the following statements hold.

(i) If \( (K, \mu) \in K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega) \times L^\infty(\partial \Omega) \), then the function \( K(x, \cdot)\mu(\cdot) \) is integrable in \( \partial \Omega \) for all \( x \in \partial \Omega \), and the function \( u[\partial \Omega, K, \mu] \) from \( \partial \Omega \) to \( C \) defined by

\[
u[\partial \Omega, K, \mu](x) \equiv \int_{\partial \Omega} K(x, y)\mu(y) d\sigma_y \quad \forall x \in \partial \Omega,
\]

is continuous. Moreover, the bilinear map from \( K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega) \times L^\infty(\partial \Omega) \) to \( C^0(\partial \Omega) \), which takes \( (K, \mu) \) to \( u[\partial \Omega, K, \mu] \) is continuous.

(ii) If \( \gamma_1 \in [n-2, n-1], \gamma_2 \in [n-1, +\infty[, \gamma_3 \in [0, 1], (n-1) - \gamma_2 + \gamma_3 \in [0, 1], \) then the bilinear map from \( K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega) \times L^\infty(\partial \Omega) \) to the space \( C^0, min\{n-1-\gamma_1, n-1-\gamma_2+\gamma_3\}(\partial \Omega) \), which takes \( (K, \mu) \) to \( u[\partial \Omega, K, \mu] \) is continuous.

(iii) If \( \gamma_1 \in [n-2, n-1], \gamma_2 = n-1, \gamma_3 \in [0, 1], \) then the bilinear map from \( K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega) \times L^\infty(\partial \Omega) \) to the space \( C^0, max\{r^{n-1-\gamma_1}, \omega_{\gamma_3}(r)\}(\partial \Omega) \), which takes \( (K, \mu) \) to \( u[\partial \Omega, K, \mu] \) is continuous.
Proof. By definition of norm in $K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega)$, we have

$$|K(x, y)\mu(y)| \leq \|K\|_{K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega)} \|\mu\|_{L^\infty(\partial \Omega)} \frac{1}{|x - y|^{\gamma_1}} \quad \forall (x, y) \in (\partial \Omega)^2 \setminus D_{\partial \Omega}.$$ 

Then the function $K(x, \cdot)\mu(\cdot)$ is integrable in $\partial \Omega$ for all $x \in \partial \Omega$, and the Vitali Convergence Theorem implies that $u[\partial \Omega, K, \mu]$ is continuous on $\partial \Omega$ (cf. e.g., Folland [13] 2.33 p. 60, p. 180). By Lemma 3.8 (i), we also have

$$\left| \int_{\partial \Omega} K(x, y)\mu(y) \, d\sigma_y \right| \leq \|K\|_{K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega)} \|\mu\|_{L^\infty(\partial \Omega)} C_{\gamma_1, \gamma_3}. \quad (6.5)$$

Hence, statement (i) follows. Next we turn to estimate the Hölder coefficient of $u[\partial \Omega, K, \mu]$ under the assumptions of statements (ii) and (iii). Let $x', x'' \in \partial \Omega$, $x' \neq x''$. By Remark 2.3 there is no loss of generality in assuming that $0 < |x' - x''| \leq r_{\gamma_3}$. Then the inclusion $B_n(x', 2|x' - x''|) \subseteq B_n(x'', 3|x' - x''|)$ and the triangular inequality imply that

$$|u[\partial \Omega, K, \mu](x') - u[\partial \Omega, K, \mu](x'')| \quad (6.6)$$

$$\leq \|\mu\|_{L^\infty(\partial \Omega)} \left\{ \int_{B_n(x', 2|x' - x''|) \cap \partial \Omega} |K(x', y)| \, d\sigma_y + \int_{B_n(x'', 3|x' - x''|) \cap \partial \Omega} |K(x'', y)| \, d\sigma_y + \int_{\partial \Omega \setminus B_n(x', 2|x' - x''|)} |K(x', y) - K(x'', y)| \, d\sigma_y \right\}. \quad (6.7)$$

Then Lemma 3.3 (ii) implies that

$$\int_{B_n(x', 2|x' - x''|) \cap \partial \Omega} |K(x', y)| \, d\sigma_y + \int_{B_n(x'', 3|x' - x''|) \cap \partial \Omega} |K(x'', y)| \, d\sigma_y \quad (6.7)$$

$$\leq \|K\|_{K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega)} \left\{ \int_{B_n(x', 2|x' - x''|) \cap \partial \Omega} \frac{d\sigma_y}{|x' - y|^{\gamma_1}} + \int_{B_n(x'', 3|x' - x''|) \cap \partial \Omega} \frac{d\sigma_y}{|x'' - y|^{\gamma_1}} \right\}$$

$$\leq \|K\|_{K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega)} 2c_{\Omega, \gamma_1} |x' - x''|^{(n-1)-\gamma_1}.$$ 

Moreover, we have

$$\int_{\partial \Omega \setminus B_n(x', 2|x' - x''|)} |K(x', y) - K(x'', y)| \, d\sigma_y \quad (6.8)$$

$$\leq \|K\|_{K_{\gamma_1, \gamma_2, \gamma_3}(\partial \Omega)} \int_{\partial \Omega \setminus B_n(x', 2|x' - x''|)} \frac{|x' - x''|^{\gamma_3}}{|x' - y|^{\gamma_2}} \, d\sigma_y$$

both in case $\gamma_2 \in ]n - 1, +\infty[ \quad$ and $\gamma_2 = n - 1 \quad$ and for all $\gamma_3 \in ]0, 1]$. 

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Under the assumptions of statement (ii), Lemma 6.8 (iii) implies that
\[
\int_{\partial \Omega \setminus B_n(x', 2|x'-x''|)} \frac{|x' - x''|^\gamma_3}{|x' - y|^\gamma_2} \, d\sigma_y \leq c_{\Omega, \gamma_2} |x' - x''|^{(n-1) - \gamma_2 + \gamma_3}. \tag{6.9}
\]
Instead, under the assumptions of statement (iii), Lemma 6.8 (iv) implies that
\[
\int_{\partial \Omega \setminus B_n(x', 2|x'-x''|)} \frac{|x' - x''|^\gamma_3}{|x' - y|^\gamma_2} \, d\sigma_y \leq c_{\Omega} |x' - x''|^\gamma_3 |\ln |x' - x''||. \tag{6.10}
\]
Then inequalities (6.9)–(6.10) imply the validity of statements (ii), (iii). □

We note that Proposition 6.3 (ii) for \( n = 3, \gamma_1 = 2 - \alpha, \gamma_2 = 3 - \alpha, \gamma_3 = 1 \) and for \( K \) fixed is known (see Kirsch and Hettlich [19, § 3.1.3, Thm. 3.17 (a)].) Next we introduce two technical lemmas, which we need to define an auxiliary integral operator.

**Lemma 6.11** Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Let \( \alpha, \beta \in ]0, 1[. \) Let \( \gamma_2 \in \mathbb{R}, \gamma_3 \in [0, 1[. \)

If \( \gamma_2 - \beta > n - 1 \), we further require that \( \gamma_3 + (n-1) - (\gamma_2 - \beta) > 0 \).

Then there exists \( c > 0 \) such that the function \( u[\partial \Omega, K, \mu] \) defined by \eqref{6.2} satisfies the following inequality
\[
|u[\partial \Omega, K, \mu](x') - u[\partial \Omega, K, \mu](x'')| \leq c \|K\|_{C^{0, \beta}(\partial \Omega)} |\mu|_{C^{0, \beta}(\partial \Omega)} \omega(|x' - x''|)
+ |\mu|_{C^{0, \beta}(\partial \Omega)} |u[\partial \Omega, K, 1](x') - u[\partial \Omega, K, 1](x'')| \quad \forall x', x'' \in \partial \Omega,
\]
for all \( (K, \mu) \in \mathcal{K}_{(n-1)-\alpha, \gamma_2, \gamma_3}(\partial \Omega) \times C^{0, \beta}(\partial \Omega), \)
where
\[
\omega(r) \equiv \begin{cases} 
\min\{\alpha + \beta, \gamma_3\} & \text{if } \gamma_2 - \beta < n - 1, \\
\max\{\alpha + \beta, \omega_{\Omega}(r)\} & \text{if } \gamma_2 - \beta = n - 1, \\
\min\{\alpha + \beta, (n-1) - (\gamma_2 - \beta)\} & \text{if } \gamma_2 - \beta > n - 1,
\end{cases} \quad \forall r \in ]0, +\infty[.
\]

**Proof.** By Remark 6.3 and by Proposition 6.3 (i), it suffices to consider case \( 0 < |x' - x''| < r_\gamma_3 \). By the triangular inequality, we have
\[
|u[\partial \Omega, K, \mu](x') - u[\partial \Omega, K, \mu](x'')| \leq \int_{\partial \Omega} |K(x', y) - K(x'', y)| (|\mu(y) - \mu(x')|) \, d\sigma_y
+ |\mu(x')| \int_{\partial \Omega} |K(x', y) - K(x'', y)| \, d\sigma_y.
\]
By the inclusion \( B_n(x', 2|x'-x''|) \subseteq B_n(x'', 3|x'-x''|) \), and by the triangular
that inequality (6.12) holds with $\omega$.

At this point we distinguish three cases. If $\gamma$ and thus inequalities (6.13) and (6.14) imply that there exists

\[
\|y - x'\|^{\beta} \leq \|y - x''\|^{\beta} + \|x' - x''\|^{\beta},
\]

we have

\[
\left| \int_{\partial \Omega} [K(x', y) - K(x'', y)](\mu(y) - \mu(x')) \, d\sigma_y \right| \leq \int_{B_n(x', 2|x' - x''|) \cap \partial \Omega} |K(x', y)| \|y - x'\|^{\beta} \, d\sigma_y \|\mu\|_{C^{\alpha, \beta}(\partial \Omega)} + \int_{B_n(x'', 3|x' - x''|) \cap \partial \Omega} |K(x'', y)| \|y - x''\|^{\beta} \, d\sigma_y \|\mu\|_{C^{\alpha, \beta}(\partial \Omega)} + \int_{\partial \Omega \setminus B_n(x', 2|x' - x''|)} |K(x', y) - K(x'', y)| \|y - x'\|^{\beta} \, d\sigma_y \|\mu\|_{C^{\alpha, \beta}(\partial \Omega)} \leq \|K\|_{K_{(n-1) - \alpha, \gamma_2, \gamma_3}(\partial \Omega)} \|\mu\|_{C^{\alpha, \beta}(\partial \Omega)} \times \left\{ \int_{B_n(x', 2|x' - x''|) \cap \partial \Omega} \frac{d\sigma_y}{|x' - x''|^{(n-1)-\alpha} + |x' - x''|^{\gamma_2} \, d\sigma_y} \right\}.
\]

At this point we distinguish three cases. If $\gamma_2 - \beta < n - 1$, then Lemma 3.8 (i) implies that

\[
\int_{\partial \Omega \setminus B_n(x', 2|x' - x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2 - \beta}} \leq \int_{\partial \Omega} \frac{d\sigma_y}{|x' - y|^{\gamma_2 - \beta}} \leq c'_{\Omega, \gamma_2 - \beta},
\]

and thus inequalities (6.13) and (6.14) imply that there exists $c > 0$ such that inequality (6.12) holds with $\omega(r) = r^{\min(\alpha + \beta, \gamma_2)}$. If $\gamma_2 - \beta = n - 1$, then Lemma 3.8 (iv) implies that

\[
\int_{\partial \Omega \setminus B_n(x', 2|x' - x''|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2 - \beta}} \leq c''_{\Omega, \gamma_2 - \beta} \ln |x' - x''|,
\]

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and thus inequalities (6.13) and (6.14) imply that there exists \( c > 0 \) such that inequality (6.12) holds with \( \omega(r) = \max\{\alpha + \beta, \omega_3(r)\} \). If \( \gamma_2 - \beta > n - 1 \), then Lemma 3.8 (iii) implies that

\[
\int_{\partial \Omega \setminus B_{\alpha}(x',2|x' - x|)} \frac{d\sigma_y}{|x' - y|^{\gamma_2 - \beta}} \leq c_{\alpha,\gamma_2 - \beta} |x'| - x^*(n-1) - (\gamma_2 - \beta),
\]

and thus inequalities (6.13) and (6.14) imply that there exists \( c > 0 \) such that inequality (6.12) holds with \( \omega(r) = r \min\{\alpha + \beta, \gamma_3 + (n-1) - (\gamma_2 - \beta)\} \).

\[\blacksquare\]

We also point out the validity of the following ‘folklore’ Lemma

**Lemma 6.15** Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Let \( \gamma_1 \in [0,1] \). Let \( G \) be a subset of \( \mathbb{R}^n \). Let \( K \in C_0((G \times \partial \Omega) \setminus \Delta_{\partial \Omega}) \) be such that

\[
\kappa_{\gamma_1} \equiv \sup_{(x,y) \in (G \times \partial \Omega) \setminus \Delta_{\partial \Omega}} |x - y|^{\gamma_1} K(x,y) < +\infty.
\]

Let \( \mu \in L^\infty(\partial \Omega) \). Then the function \( K(x,\cdot)\mu(\cdot) \) is integrable in \( \partial \Omega \) for all \( x \in G \) and the function \( u^\sharp_{\partial \Omega,K,\mu} \) from \( G \) to \( \mathbb{C} \) defined by

\[
u^\sharp_{\partial \Omega,K,\mu}(x) \equiv \int_{\partial \Omega} K(x,y)\mu(y) \, d\sigma_y \quad \forall x \in G,
\]

is continuous. If \( \sup_{x \in G} \int_{\partial \Omega} \frac{d\sigma_y}{|x - y|^{\gamma_1}} < \infty \), then \( u^\sharp_{\partial \Omega,K,\mu} \) satisfies the inequality

\[
|u^\sharp_{\partial \Omega,K,\mu}(x)| \leq \sup_{x \in G} \int_{\partial \Omega} \frac{d\sigma_y}{|x - y|^{\gamma_1}} \kappa_{\gamma_1} \|\mu\|_{L^\infty(\partial \Omega)} \quad \forall x \in G. \quad (6.16)
\]

**Proof.** The integrability of \( K(x,\cdot)\mu(\cdot) \) follows by the inequality

\[
|K(x,y)\mu(y)| \leq \frac{\kappa_{\gamma_1} \|\mu\|_{L^\infty(\partial \Omega)}}{|x - y|^{\gamma_1}} \quad a.e. \quad y \in \partial \Omega.
\]

Since \( \sup_{x \in G} \int_{\partial \Omega} \frac{d\sigma_y}{|x - y|^{\gamma_1}} < \infty \), inequality (6.16) follows and the Vitali Convergence Theorem implies that \( u^\sharp_{\partial \Omega,K,\mu} \) is continuous on \( G \) (cf. e.g., Folland [13, (2.33) p. 60, p. 180].)

We now introduce an auxiliary integral operator, and we deduce some properties which we need in the sequel by applying Proposition 6.3 and Lemma 6.11.

**Lemma 6.17** Let \( \theta \in [0,1] \). Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Then the following statements hold.
(i) Let \( Z \in C^0((\text{cl}\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega}) \) satisfy inequality
\[
\kappa_{n-1}[Z] \equiv \sup_{(x,y) \in (\text{cl}\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega}} |x-y|^{n-1}|Z(x,y)| < +\infty. \tag{6.18}
\]

Let \((f,\mu)\in C^{0,\theta}(\partial\Omega) \times L^\infty(\partial\Omega)\). Let \( H(f,Z) \) be the function from 
\((\text{cl}\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega} \to \mathbb{C} \)
defined by
\[
H^1[Z,f](x,y) \equiv (f(x) - f(y))Z(x,y) \quad \forall (x,y) \in (\text{cl}\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega}.
\]

If \( x \in \text{cl}\Omega \), then the function \( H^1[Z,f](x,\cdot) \) is Lebesgue integrable in 
\( \partial\Omega \) and the function \( Q^1[Z,f,\mu] \) from \( \text{cl}\Omega \to \mathbb{C} \)
defined by
\[
Q^1[Z,f,\mu](x) \equiv \int_{\partial\Omega} H^1[Z,f](x,y)\mu(y) \, d\sigma_y \quad \forall x \in \text{cl}\Omega,
\]
is continuous.

(ii) The map \( H \) from \( K_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega) \) to \( K_{n-1-\theta,n-1,\theta}(\partial\Omega) \),
which takes \((Z,g)\) to the function from \((\partial\Omega)^2 \setminus \Delta_{\partial\Omega} \to \mathbb{C} \)
defined by
\[
H[Z,g](x,y) \equiv (g(x) - g(y))Z(x,y) \quad \forall (x,y) \in (\partial\Omega)^2 \setminus \Delta_{\partial\Omega},
\]
is bilinear and continuous.

(iii) The map \( Q \) from \( K_{n-1,n,1}(\partial\Omega) \times C^{0,\theta}(\partial\Omega) \times L^\infty(\Omega) \)
to \( C^{0,\alpha,1}(\partial\Omega) \),
which takes \((Z,g,\mu)\) to the function from \( \partial\Omega \to \mathbb{C} \)
defined by
\[
Q[Z,g,\mu](x) \equiv \int_{\partial\Omega} H[Z,g](x,y)\mu(y) \, d\sigma_y \quad \forall x \in \partial\Omega,
\]
is trilinear and continuous.

(iv) Let \( \alpha \in [0,1[, \beta \in [0,1] \). Then there exists \( q \in [0, +\infty[ \) such that
\[
|Q[Z,g,\mu](x') - Q[Z,g,\mu](x'')| \\
\leq q|Z|_{K_{n-1,n,1}(\partial\Omega)}\|g\|_{C^{0,\alpha}(\partial\Omega)}\|\mu\|_{C^{0,\beta}(\partial\Omega)}|x' - x''|^{\alpha} \\
+ \|\mu\|_{C^{0,\beta}(\partial\Omega)}|Q[Z,g,1](x') - Q[Z,g,1](x'')| \quad \forall x', x'' \in \partial\Omega,
\]
for all \((Z,g,\mu)\in K_{n-1,n,1}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega) \times C^{0,\beta}(\partial\Omega)\).

Proof. By assumption \( [6.18] \), and by the Hölder continuity of \( f \), we have
\[
|H^1[Z,f](x,y)| \leq \frac{|f|_{\theta}}{|x-y|^{n-1-\theta}} \kappa_{n-1}[Z],
\]
for all \((x,y) \in (\text{cl}\Omega \times \partial\Omega) \setminus \Delta_{\partial\Omega} \). Then Lemma \( [6.13] \)
implies the validity of statement (i).
By the Hölder continuity of \(g\), we have

\[
|H[Z, g](x, y)| \leq \frac{|g|_\theta}{|x-y|^{\alpha}} \|Z\|_{K_{n-1,n,1}(\partial \Omega)} \quad \forall (x, y) \in (\partial \Omega)^2 \setminus \Delta_{\partial \Omega}. \tag{6.19}
\]

Now let \(x', x'' \in \partial \Omega\), \(x' \neq x''\), \(y \in \partial \Omega \setminus \mathcal{B}_n(x', 2|x'-x''|)\). Then we have

\[
|H[Z, g](x', y) - H[Z, g](x'', y)| \leq \|g\|_{C^{0,\theta}(\partial \Omega)} \|Z\|_{K_{n-1,n,1}(\partial \Omega)} \left\{ \frac{|x'-y|^\theta |x'-x''|}{|x'-y|^\alpha} + \frac{|x'-x''|^\theta}{|x'-y|^{\alpha-1}} \right\}. \tag{6.20}
\]

Since \(|x'-x''| \leq |x'-y|\), we have \(|x'-x''|^\alpha \leq |x'-y|^{1-\theta}\). Moreover, Lemma \(3.2\) (i) implies that \(|x'-y| \geq \frac{1}{2}|x'-y|\) and thus the term in braces in the right hand side of (6.20) is less or equal to

\[
\frac{|x'-y|^\theta |x'-x''|^\alpha}{|x'-y|^\alpha} + 2^{n-1}|x'-x''|^\alpha |x'-x''|^\alpha \leq (1 + 2^{n-1}) \frac{|x'-x''|^\alpha}{|x'-y|^{\alpha-1}}. \tag{6.21}
\]

Hence, inequalities (6.19)-(6.21) imply that

\[
\|H[Z, g]\|_{K_{n-1,n,1,\theta}(\partial \Omega)} \leq 2^n \|Z\|_{K_{n-1,n,1}(\partial \Omega)} \|g\|_{C^{0,\theta}(\partial \Omega)}. \tag{6.22}
\]

Hence statement (ii) holds true. We now turn to prove (iii). By Proposition \(6.3\) (iii) with \(\gamma_1 = n-1-\theta, \gamma_2 = n-1, \gamma_3 = \theta\), the map \(u[\partial \Omega, \cdot, \cdot]\) is continuous from \(K_{n-1,\theta,n-1,\theta}(\partial \Omega) \times L^\infty(\partial \Omega)\) to \(C^{0,\max\{\gamma_1,\gamma_2-\gamma_3\}}(\partial \Omega)\). Then statement (ii) implies that \(u[\partial \Omega, H[\cdot, \cdot, \cdot], \cdot]\) is continuous from \(K_{n-1,n,1}(\partial \Omega) \times C^{0,\theta}(\partial \Omega) \times L^\infty(\partial \Omega)\) to \(C^{0,\omega(1)}(\partial \Omega)\). Since

\[
u[\partial \Omega, H[Z, g], \mu] = \int_{\partial \Omega} H[Z, g](x, y) \mu(y) \, d\sigma(y) \quad \forall x \in \partial \Omega, \tag{6.23}
\]

statement (iii) holds true. Since \(C^{0,\beta_1}(\partial \Omega)\) is continuously imbedded into \(C^{0,\beta_2}(\partial \Omega)\) whenever \(0 < \beta_2 \leq \beta_1 \leq 1\), then we can assume that \(\alpha < \beta_1 < \beta_2\). Then by equality \(6.23\) and by Lemma \(6.11\) with \(\gamma_2 = n-1, \gamma_3 = \alpha\) and by statement (ii) with \(\theta = \alpha\), statement (iv) holds true. \(\square\)

7 Preliminaries on layer potentials

Let \(a\) be as in \((1.1), (1.2)\). Let \(S_\alpha\) be a fundamental solution of \(P[a, D]\). Let \(\Omega\) be a bounded open Lipschitz subset of \(\mathbb{R}^n\). If \(\mu \in L^\infty(\partial \Omega)\), Lemma
Theorem 7.2 ensures the convergence of the integral
\[ v[\partial \Omega, S_\alpha, \mu](x) \equiv \int_{\partial \Omega} S_\alpha(x - y) \mu(y) \, ds_y \quad \forall x \in \mathbb{R}^n, \]
which defines the single layer potential relative to \( \mu, S_\alpha \). We collect in the following statement some known properties of the single layer potential which we exploit in the sequel (cf. Miranda [25], Wiegner [36], Dalla Riva, Morais and Musolino [3], Dalla Riva, Morais and references therein.)

**Theorem 7.1** Let \( a \) be as in (1.7), (1.2). Let \( S_\alpha \) be a fundamental solution of \( P[a, D] \). Let \( \alpha \in [0, 1] \), \( m \in \mathbb{N} \setminus \{0\} \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{m,\alpha} \). Then the following statements hold.

(i) If \( \mu \in C^{m-1,\alpha}(\partial \Omega), \) then the function \( v^+[\partial \Omega, S_\alpha, \mu] \equiv v[\partial \Omega, S_\alpha, \mu]|_{C^{m,\alpha}(\partial \Omega)} \) belongs to \( C^{m,\alpha}(\partial \Omega) \) and the function \( v^-[\partial \Omega, S_\alpha, \mu] \equiv v[\partial \Omega, S_\alpha, \mu]|_{C^{m,\alpha}(\partial \Omega)} \) belongs to \( C^{m,\alpha}(\partial \Omega^-) \). Moreover the map which takes \( \mu \) to the function \( v^+[\partial \Omega, S_\alpha, \mu] \) is continuous from \( C^{m-1,\alpha}(\partial \Omega) \) to \( C^{m,\alpha}(\partial \Omega^-) \) and the map from the space \( C^{m-1,\alpha}(\partial \Omega) \) to \( C^{m,\alpha}(\partial \Omega^-) \) which takes \( \mu \) to \( v^-[\partial \Omega, S_\alpha, \mu]|_{C^{m,\alpha}(\partial \Omega)} \) is continuous for all \( R \in ]0, +\infty[ \) such that \( \partial \Omega \subseteq B_n(0, R) \).

(ii) Let \( l \in \{1, \ldots, n\} \). If \( \mu \in C^{0,\alpha}(\partial \Omega), \) then we have the following jump relation
\[
\frac{\partial}{\partial x_l} v^+[\partial \Omega, S_\alpha, \mu](x) = \mp \frac{v_l(x)}{2 \nu(x)^2 |\nu(x)|} \mu(x) + \int_{\partial \Omega} \partial_{x_l} S_\alpha(x - y) \mu(y) \, ds_y \quad \forall x \in \partial \Omega,
\]
where the integral in the right hand side exists in the sense of the principal value.

Then we introduce the following refinement of a classical result for homogeneous second order elliptic operators (cf. Miranda [25].)

**Theorem 7.2** Let \( a \) be as in (1.7), (1.2). Let \( S_\alpha \) be a fundamental solution of \( P[a, D] \). Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Let \( \gamma \in ]0, 1[ \). Then the operator \( v[\partial \Omega, S_\alpha, \cdot]|_{\partial \Omega} \) from \( L^\infty(\partial \Omega) \) to \( C^{0,\gamma}(\partial \Omega) \) which takes \( \mu \) to \( v[\partial \Omega, S_\alpha, \mu]|_{\partial \Omega} \) is continuous.

If we further assume that \( n > 2 \), then \( v[\partial \Omega, S_\alpha, \cdot]|_{\partial \Omega} \) is continuous from \( L^\infty(\partial \Omega) \) to \( C^{0,\gamma}(\partial \Omega) \).

**Proof.** By Lemma 1.9 we have that \( S_\alpha(x - y) \in K_{(n-1)-\gamma, n-1,1}(\partial \Omega), \) and also that \( S_\alpha(x - y) \in K_{n-2,n-1,1}(\partial \Omega) \) if we further assume that \( n > 2 \). Since
\[
v[\partial \Omega, S_\alpha, \mu]|_{\partial \Omega} = u[\partial \Omega, S_\alpha(x - y), \mu],
\]
Proposition \(6.3\) (iii) implies that \(v|\partial \Omega, S_n, \cdot|\) is continuous from \(L^\infty(\partial \Omega)\) to \(C^{0, \max\{r, \omega n(r)\}}(\partial \Omega) = C^{0, \gamma}(\partial \Omega)\), and that \(v|\partial \Omega, S_n, \cdot|\) is continuous from \(L^\infty(\partial \Omega)\) to \(C^{0, \max\{r, \omega n(r)\}}(\partial \Omega) = C^{0, \omega n(r)}(\partial \Omega)\) if we further assume that \(n > 2\).

Next we turn to the double layer potential and we introduce the following technical result (cf. Miranda [24], Wiegner [36], Dalla Riva [3], Dalla Riva, Morais and Musolino [5] and references therein.)

**Theorem 7.3** Let \(a\) be as in (7.1), (7.2). Let \(S_n\) be a fundamental solution of \(P[a, D]\). Let \(\alpha \in ]0, 1[\), \(m \in \mathbb{N} \setminus \{0\}\). Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^n\) of class \(C^{m, \alpha}\). Then the following statements hold.

(i) If \(\mu \in C^{0, \alpha}(\partial \Omega)\), then the restriction \(w|\partial \Omega, a, S_n, \mu|\) can be extended uniquely to a continuous function \(w^+|\partial \Omega, a, S_n, \mu|\) from \(\text{cl} \Omega\) to \(C\), and \(w|\partial \Omega, a, S_n, \mu|\) from \(\text{cl} \Omega^-\) to \(C\) and we have the following jump relation
\[
w^+|\partial \Omega, a, S_n, \mu|(x) = \pm \frac{1}{2} \mu(x) + w|\partial \Omega, a, S_n, \mu|(x) \quad \forall x \in \partial \Omega.
\]

(ii) If \(\mu \in C^{m, \alpha}(\partial \Omega)\), then \(w^+|\partial \Omega, a, S_n, \mu|\) belongs to \(C^{m, \alpha}(\text{cl} \Omega)\) and \(w^-|\partial \Omega, a, S_n, \mu|\) belongs to \(C^{m, \alpha}(\text{cl} \Omega^-)\). Moreover, the map from the space \(C^{m, \alpha}(\partial \Omega)\) to \(C^{m, \alpha}(\text{cl} \Omega)\) which takes \(\mu\) to \(w^+|\partial \Omega, a, S_n, \mu|\) is continuous and the map from the space \(C^{m, \alpha}(\partial \Omega)\) to \(C^{m, \alpha}(\text{cl} \Omega^-)\) which takes \(\mu\) to \(w^-|\partial \Omega, a, S_n, \mu|\) is continuous for all \(R \in ]0, +\infty[\) such that \(\text{cl} \Omega \subseteq B_n(0, R)\).

(iii) Let \(r \in \{1, \ldots, n\}\). If \(\mu \in C^{m, \alpha}(\partial \Omega)\) and \(U\) is an open neighborhood of \(\partial \Omega\) in \(\mathbb{R}^n\) and \(\tilde{\mu} \in C^{m}(U)\), \(\tilde{\mu}|\partial \Omega = \mu\), then the following equality holds
\[
\frac{\partial}{\partial x_r} w|\partial \Omega, a, S_n, \mu|(x) = \sum_{j,k=1}^n a_{jk} \frac{\partial}{\partial x_j} \left( \int_{\partial \Omega} S_n(x - y) \left[ \nu_r(y) \frac{\partial \tilde{\mu}}{\partial y_j}(y) - \nu_j(y) \frac{\partial \tilde{\mu}}{\partial y_r}(y) \right] d\sigma_y \right)
\]
\[
+ \int_{\partial \Omega} DS_n(x - y) a^{(1)}(y) + aS_n(x - y) \nu_r(y) \mu(y) d\sigma_y
\]
\[
- \int_{\partial \Omega} \partial_x S_n(x - y) \nu^{(1)}(y) a^{(1)}(y) \mu(y) d\sigma_y \quad \forall x \in \mathbb{R}^n \setminus \partial \Omega.
\]

We note that formula (7.1) for the Laplace operator with \(n = 3\) can be found in Günter [13] Ch. 2, § 10, (42)]. By combining Theorems [7.1] and [7.3] we deduce that under the assumptions of Theorem [7.3] (iii), the following
equality holds

\[ \frac{\partial}{\partial x_r} w^+[\partial \Omega, a, S_a, \mu] \]

\[= \sum_{j,l=1}^n a_{lj} \frac{\partial}{\partial x_l} v^+[\partial \Omega, S_a, M_{rj}[\mu]] + Dv^+[\partial \Omega, S_a, \nu_r \mu]_{[1]}^{(1)} + av^+[\partial \Omega, S_a, \nu_r \mu] - \frac{\partial}{\partial x_r} v^+[\partial \Omega, S_a, (\nu^\prime a^{(1)}) \mu] \] on cl\Omega.

Next we introduce the following result proved by Schauder \[30\] Hilfssatz VII, p. 112 for the Laplace operator and which we extend here to second order elliptic operators by exploiting Proposition 6.3.

**Theorem 7.6** Let \( a \) be as in (1.1), (1.2), (1.3). Let \( S_a \) be a fundamental solution of \( P[a, D] \). Let \( \alpha \in (0, 1] \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{1, \alpha} \). If \( \mu \in L^\infty(\partial \Omega) \), then \( w[\partial \Omega, a, S_a, \mu]\_{|\partial \Omega} \in C_0^{0, \alpha}(\partial \Omega) \). Moreover the operator from \( L^\infty(\partial \Omega) \) to \( C_0^{0, \alpha}(\partial \Omega) \) which takes \( \mu \) to \( w[\partial \Omega, a, S_a, \mu]\_{|\partial \Omega} \) is continuous.

**Proof.** By Lemma 5.1 the function \( K_a(x, y) \equiv B_{\Omega_{\Omega y}}(S_a(x - y)) \) belongs to \( K_{(n-1)-\alpha,n,1}(\partial \Omega) \). Since

\[ w[\partial \Omega, a, S_a, \mu]\_{|\partial \Omega} = u[\partial \Omega, K_a, \mu] , \]

Proposition 6.3 (ii) implies that \( w[\partial \Omega, a, S_a, \cdot]\_{|\partial \Omega} \) is continuous from \( L^\infty(\partial \Omega) \) to \( C_0^{0, \min\{\alpha,(n-1)-(n-\alpha)+1\}}(\partial \Omega) = C_0^{0, \alpha}(\partial \Omega) \).

\[ \square \]

**8 Auxiliary integral operators**

In order to compute the tangential derivatives of the double layer potential, we introduce the following two statements which concern two auxiliary integral operators. To shorten our notation, we define the function \( \Theta \) from \( (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n} \) to \( \mathbb{R}^n \setminus \{0\} \) by setting

\[ \Theta(x, y) \equiv y - x \quad \forall (x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n} . \]  

\[ \Theta(x, y) \equiv y - x \quad \forall (x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta_{\mathbb{R}^n} . \]  

\[ (8.1) \]

**Theorem 8.2** Let \( a \) be as in (1.1), (1.2), (1.3). Let \( S_a \) be a fundamental solution of \( P[a, D] \). Let \( r \in \{1, \ldots, n\} \). Then the following statements hold.

(i) Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Let \( \theta \in [0, 1] \). If \( (f, \mu) \in C^{0, \theta}(cl\Omega) \times L^\infty(\partial \Omega) \), then the function

\[ Q^r(\partial S_a)_{\partial x_r} \Theta, f, \mu)(x) = \int_{\partial \Omega} (f(x) - f(y)) \frac{\partial S_a}{\partial x_r}(x-y) \mu(y) d\sigma_y \quad \forall x \in cl\Omega , \]

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is continuous.

(ii) Let \( \alpha \in [0, 1], \beta, \theta \in [0, 1] \). Let \( m \in \mathbb{N} \setminus \{0\} \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{m, \alpha} \). Then the map \( Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot] \) from \( C^{m-1, \beta}(\partial \Omega) \times C^{m-1, \beta}(\partial \Omega) \) to \( C^{m-1, \min\{\alpha, \beta, \theta\}}(\partial \Omega) \), which takes \( (f, \mu) \) to \( Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, f, \mu] \) is bilinear and continuous.

**Proof.** By Lemma 4.11 (ii), statement (i) is an immediate consequence of Lemma 6.17 (i). We now consider statement (ii). By statement (i) and by the continuity of the pointwise product in Schauder spaces, (ii) implies the validity of statement (i).

Let \( \alpha \in [0, 1], \beta \in [0, 1] \). Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Then the following statement holds.

\[
Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, f, \mu](x) = f(x) \frac{\partial}{\partial x_r} \left[ \partial \Omega, S_a, f, \mu \right](x) - \frac{\partial}{\partial x_r} \left[ \Theta, S_a, f, \mu \right](x),
\]

for all \( x \in \partial \Omega \). Then the statement follows by Theorem 7.1 (i) and by continuity of the pointwise product in Schauder spaces.

Then we have the following.

**Theorem 8.3** Let \( a \) be as in (1.1), (1.2), (1.3). Let \( S_a \) be a fundamental solution of \( P[a, D] \). Then the following statement holds.

(i) Let \( \Omega \) be a bounded open Lipschitz subset of \( \mathbb{R}^n \). Then the bilinear map \( Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot] \) from \( C^{0, \beta}(\partial \Omega) \times L^\infty(\partial \Omega) \) to \( C^{0, \min\{\alpha, \beta, \theta\}}(\partial \Omega) \), which takes \( (g, \mu) \) to the function

\[
Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu](x) = \int_{\partial \Omega} (g(x) - g(y)) \frac{\partial S_a}{\partial x_r}(x - y) \mu(y) d\sigma_y \quad \forall x \in \partial \Omega,
\]

is continuous.

(ii) Let \( \alpha \in [0, 1], \beta \in [0, 1] \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{1, \alpha} \). Then the bilinear map \( Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot] \) from \( C^{0, \beta}(\partial \Omega) \times C^{0, \beta}(\partial \Omega) \) to \( C^{0, \min\{\alpha, \beta, \theta\}}(\partial \Omega) \) which takes \( (g, \mu) \) to \( Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu] \) is continuous.

**Proof.** By Lemma 4.11 we have \( \frac{\partial S_a}{\partial x_r} \in K_{n-1, 1}(\partial \Omega) \). Then Lemma 6.17 (iii) implies the validity of statement (i).

We now consider statement (ii). By statement (i) and by the continuity of the inclusion of \( C^{0, \beta}(\partial \Omega) \) into \( L^\infty(\partial \Omega) \), we already know that \( Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot] \) is continuous from \( C^{0, \beta}(\partial \Omega) \times C^{0, \beta}(\partial \Omega) \) to \( C^{0, \min\{\alpha, \beta, \theta\}}(\partial \Omega) \). Then it suffices to show that \( Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot] \) is continuous from \( C^{0, \beta}(\partial \Omega) \times C^{0, \beta}(\partial \Omega) \) to the semi-normed space \( (C^{0, \min\{\alpha, \beta, \theta\}}(\partial \Omega), \cdot : \partial \Omega)_a \). By Lemma 6.17 (iv), there exists \( q \in [0, +\infty] \) such that

\[
\left| Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu](x') - Q^1[\frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu](x'') \right| \leq q \left\| \frac{\partial S_a}{\partial x_r} \circ \Theta \right\|_{K_{n-1, 1}(\partial \Omega)} \| g \|_{C^{0, \min\{\alpha, \beta, \theta\}}(\partial \Omega)} \| \mu \|_{C^{0, \min\{\alpha, \beta, \theta\}}(\partial \Omega)} |x' - x''|^\alpha
\]
Theorem 8.2 (ii) implies that $Q$ to itself, and that accordingly, there exists $q$ for all $x$

Lemma 8.7

Let $C$ be an extension operator as in Lemma 2.6, defined on $R$ of $Q$ Then $P$ solution of $C$ is continuous from $H$.

Then by combining inequalities (8.5) and (8.6), we deduce that

Theorem 8.2 (ii) implies that $Q{\frac{\partial S_a}{\partial x_r}} \circ \Theta, g, 1]$ is continuous from $C^{0,0}(\partial \Omega)$ to itself, and that accordingly, there exists $q' \in [0, +\infty[$ such that

Then by combining inequalities (8.3) and (8.6), we deduce that $Q{\frac{\partial S_a}{\partial x_r}} \circ \Theta, \cdot, 1]$ is continuous from $C^{0,0}(\partial \Omega) \times C^{0,0}(\partial \Omega)$ to $(C^{0,0}(\partial \Omega), \cdot : \partial \Omega[\alpha])$ and thus the proof is complete.

In the next lemma, we introduce a formula for the tangential derivatives of $Q{\frac{\partial S_a}{\partial x_r}} \circ \Theta, g, \mu]$.

Lemma 8.7 Let $a$ be as in (1.1), (1.2), (1.3). Let $S_a$ be a fundamental solution of $P[a, D]$. Let $\alpha \in [0, 1], \theta \in [0, 1].$ Let $\Omega$ be a bounded open subset of $R^n$ of class $C^{2,\alpha}$. Let $r \in \{1, \ldots, n\}$. Let $g \in C^{1,\theta}(\partial \Omega), \mu \in C^1(\partial \Omega)$. Then $Q{\frac{\partial S_a}{\partial x_r}} \circ \Theta, g, \mu]$ in $C^1(\partial \Omega)$ and the following formula holds.

$$M_{ij} \left[ Q \left( \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right) \right]$$

$$= \nu_l(x)Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, D_{a,j}g, \mu \right] (x) - \nu_j(x)Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, D_{a,l}g, \mu \right] (x)$$

$$+ \nu_l(x)Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \sum_{s=1}^n M_{sj} \sum_{h=1}^n a_{sh}m_{h} \mu \left( \frac{x}{\sqrt{a^2(2)^\nu}} \right) \right] (x)$$

$$- \nu_j(x)Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \sum_{s=1}^n M_{sl} \sum_{h=1}^n a_{sh}m_{h} \mu \left( \frac{x}{\sqrt{a^2(2)^\nu}} \right) \right] (x)$$

$$+ \sum_{s,h=1}^n a_{sh} \nu_l(x) \left\{ Q \left[ \frac{\partial S_a}{\partial x_s} \circ \Theta, \nu_j, M_{hr} \left( \frac{\nu_j \mu}{\sqrt{a^2(2)^\nu}} \right) \right] (x) \right\}$$

$$- \sum_{s,h=1}^n a_{sh} \nu_j(x) \left\{ Q \left[ \frac{\partial S_a}{\partial x_s} \circ \Theta, \nu_i, M_{hr} \left( \frac{\nu_i \mu}{\sqrt{a^2(2)^\nu}} \right) \right] (x) \right\}$$

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Clearly, the notation

Now to shorten our notation, we set

\[ x \]

for all \( f \) we compute \( \frac{\nu_j}{\nu_j \alpha} \). Next we fix \( \beta \) and \( \gamma \) and we first prove the formula under the assumption that \( \mu \in C^{1,\beta}(\partial \Omega) \). By Theorem \( 8.2 \), we already know that \( Q^x \left[ \frac{\nu_j}{\nu_j \alpha} \right] \) belongs to \( C^1(\partial \Omega) \). Then we find convenient to introduce the notation

\[ M_{ij}^x[f](x) = \tilde{\nu}_i(x) \frac{\partial f}{\partial x_i}(x) - \tilde{\nu}_j(x) \frac{\partial f}{\partial x_j}(x) \quad \forall x \in \partial \Omega \]

for all \( f \in C^1(\partial \Omega) \). If necessary, we write \( M_{ij,x} \) to emphasize that we are taking \( x \) as variable of the differential operator \( M_{ij}^x \). Next we fix \( x \in \Omega \) and we compute

\[ \tilde{\nu}_i(x) \frac{\partial}{\partial x_i} Q^x \left[ \frac{\partial S_\alpha}{\partial x_\alpha} \circ \Theta, \tilde{y}, \mu \right](x) \]

\[ - \tilde{\nu}_j(x) \frac{\partial}{\partial x_j} Q^x \left[ \frac{\partial S_\alpha}{\partial x_\alpha} \circ \Theta, \tilde{g}, \mu \right](x) \]

Clearly,

\[ \frac{\partial}{\partial x_\alpha} Q^x \left[ \frac{\partial S_\alpha}{\partial x_\alpha} \circ \Theta, \tilde{y}, \mu \right](x) = \int_{\partial \Omega} \frac{\partial \tilde{y}}{\partial x_\alpha}(x) \frac{\partial}{\partial x_\alpha} S_\alpha(x - y)\mu(y) \, d\sigma_y \]

\[ + \int_{\partial \Omega} (\tilde{g}(x) - \tilde{y}(y)) \frac{\partial^2}{\partial x_\alpha \partial x_\alpha} S_\alpha(x - y)\mu(y) \, d\sigma_y \]

Now to shorten our notation, we set

\[ J_1(x) = \int_{\partial \Omega} \frac{\partial \tilde{g}}{\partial x_1}(x) \frac{\partial}{\partial x_\alpha} S_\alpha(x - y)\mu(y) \, d\sigma_y \]
Then we have
\[
\frac{\partial}{\partial x_l} Q^t \frac{\partial S_a}{\partial x_r} \circ \Theta, \tilde{g}, \mu(x) \nleq J_1(x) - \int_{\partial \Omega} (\tilde{g}(x) - \tilde{g}(y)) \\
\times \sum_{s,h=1}^n \nu_s(y) a_{sh} \nu_h(y) \frac{\partial}{\partial y_s} S_a(x - y) \mu(y) dy \\
= J_1(x) - \int_{\partial \Omega} (\tilde{g}(x) - \tilde{g}(y)) \\
\times \sum_{s=1}^n \frac{\partial}{\partial y_s} S_a(x - y) \left( \frac{\partial}{\partial x_r} S_a(x - y) \right) \mu(y) dy \\
= J_1(x) - \int_{\partial \Omega} (\tilde{g}(x) - \tilde{g}(y)) \\
\times \sum_{s=1}^n \frac{\partial}{\partial y_s} S_a(x - y) \left( \frac{\partial}{\partial x_r} S_a(x - y) \right) \mu(y) dy.
\]

By Lemma 2.8, the second term in the right hand side takes the following form
\[
\int_{\partial \Omega} (\tilde{g}(x) - \tilde{g}(y)) \sum_{s=1}^n M_{s, y} \left( \frac{\partial}{\partial x_r} S_a(x - y) \right) \left( \frac{a(2)^\nu(y)}{\nu(2)^\nu(y)} \right) \mu(y) dy \\
= - \int_{\partial \Omega} \sum_{s=1}^n M_{s, y} [\tilde{g}(x) - \tilde{g}(y)] \frac{\partial}{\partial x_r} S_a(x - y) \left( \frac{a(2)^\nu(y)}{\nu(2)^\nu(y)} \right) \mu(y) dy \\
- \int_{\partial \Omega} \sum_{s=1}^n (\tilde{g}(x) - \tilde{g}(y)) \frac{\partial}{\partial x_r} S_a(x - y) M_{s, y} \left( \frac{a(2)^\nu(y)}{\nu(2)^\nu(y)} \right) \mu(y) dy.
\]

Since \( M_{s, y} [\tilde{g}(x) - \tilde{g}(y)] = -M_{s, y} [\tilde{g}(x) - \tilde{g}(y)] \), we have
\[
\frac{\partial}{\partial x_l} Q^t \frac{\partial S_a}{\partial x_r} \circ \Theta, \tilde{g}, \mu(x) \nleq J_1(x) - \int_{\partial \Omega} (\tilde{g}(x) - \tilde{g}(y)) \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) dy \\
- \int_{\partial \Omega} \sum_{s=1}^n M_{s, y} \frac{\partial}{\partial x_r} S_a(x - y) \left( \frac{a(2)^\nu(y)}{\nu(2)^\nu(y)} \right) \mu(y) dy \\
+ \int_{\partial \Omega} \sum_{s=1}^n \frac{\partial}{\partial x_r} S_a(x - y) M_{s, y} \left( \frac{a(2)^\nu(y)}{\nu(2)^\nu(y)} \right) \mu(y) dy.
\]
By the obvious identity

and by the corresponding formula for \( M_{ij} \) on \( \partial \Omega \), and by straightforward computations, we obtain

\[
M_{ij}^{(2)} \left[ Q^4 \frac{\partial}{\partial x_r} \right] S_a(x - y) \mu(y) \, d\sigma_y
\]

Accordingly, we have

\[
M_{ij}^{(2)} \left[ Q^4 \frac{\partial}{\partial x_r} \Theta, \tilde{g}, \mu \right] (x)
\]

\[
= M_{ij}^{(2)} \left[ \tilde{g} \right] (x) \int_{\partial \Omega} \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y
\]

\[
- \int_{\partial \Omega} \sum_{s=1}^{n} \left\{ \tilde{v}_i(x) M_{sj} \tilde{g}(y) - \tilde{v}_j(x) M_{si} \tilde{g}(y) \right\} \frac{\partial}{\partial x_r} S_a(x - y)
\]

\[
\times \left( \frac{(a^{(2)} \nu(y))_s}{\nu^{(a^{(2)} \nu(y))}} \right) \mu(y) \, d\sigma_y
\]

\[
+ \int_{\partial \Omega} \sum_{s=1}^{n} \left( \tilde{g}(x) - \tilde{g}(y) \right) \frac{\partial}{\partial x_r} S_a(x - y)
\]

\[
\times \left( \tilde{v}_i(x) M_{sj} \left[ \frac{(a^{(2)} \nu(y))_s}{\nu^{(a^{(2)} \nu(y))}} \right] (y) - \tilde{v}_j(x) M_{si} \left[ \frac{(a^{(2)} \nu(y))_s}{\nu^{(a^{(2)} \nu(y))}} \right] (y) \right)
\]

\[
\times (a^{(2)} \nu)(y) \tilde{v}_i(x) \nu_j(y) \mu(y) \, d\sigma_y
\]

We now consider the first two terms in the right hand side of formula (8.9).

By the obvious identity

\[
M_{ij}^{(2)} \left[ \tilde{g} \right] (x) \int_{\partial \Omega} \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y
\]

and by the corresponding formula for \( M_{ij} \) on \( \partial \Omega \), and by formula (8.10) and by straightforward computations, we obtain

\[
M_{ij}^{(2)} \left[ \tilde{g} \right] (x) \int_{\partial \Omega} \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y
\]

\[
- \int_{\partial \Omega} \sum_{s=1}^{n} \left\{ \tilde{v}_i(x) M_{sj} \tilde{g}(y) - \tilde{v}_j(x) M_{si} \tilde{g}(y) \right\}
\]

\[
\times \frac{\partial}{\partial x_r} S_a(x - y) \left( \frac{(a^{(2)} \nu(y))_s}{\nu^{(a^{(2)} \nu(y))}} \right) \mu(y) \, d\sigma_y
\]
$$\begin{align*}
\tilde{v}_i(x) & = \tilde{v}_i(x) \left[ \frac{\partial}{\partial x_j} \tilde{g}(x) - \frac{D\tilde{g}(x)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_j(x) \right] \int_{\partial\Omega} \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y \\
- \tilde{v}_j(x) & = \tilde{v}_j(x) \left[ \frac{\partial}{\partial x_l} \tilde{g}(x) - \frac{D\tilde{g}(x)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_l(x) \right] \int_{\partial\Omega} \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y \\
- \tilde{v}_n(x) & = \tilde{v}_n(x) \left[ \frac{\partial}{\partial y_j} \tilde{g}(y) - \frac{D\tilde{g}(y)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_j(y) \right] \\
& \times \left( \sum_{s,h=1}^n \tilde{v}_s(y) \frac{\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \right) \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y \\
+ \tilde{v}_l(x) & = \tilde{v}_l(x) \left[ \frac{\partial}{\partial y_l} \tilde{g}(y) - \frac{D\tilde{g}(y)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_l(y) \right] \\
& \times \left( \sum_{s,h=1}^n \tilde{v}_s(y) \frac{\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \right) \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y \\
- \tilde{v}_j(x) & = \tilde{v}_j(x) \left[ \frac{\partial}{\partial y_s} \tilde{g}(y) - \frac{D\tilde{g}(y)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_s(y) \right] \frac{\partial}{\partial x_r} S_a(x - y) \mu(y) \, d\sigma_y \\
& \times \left( \sum_{s,h=1}^n \tilde{v}_s(y) \frac{\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \right) = 0,
\end{align*}$$

Since

$$\tilde{v}(y) = \nu(y), \quad \left( \sum_{s,h=1}^n \tilde{v}_s(y) \frac{\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \right) = 1 \quad \forall y \in \partial\Omega,$$

we have

$$\left\{ \sum_{s,h=1}^n \frac{\partial}{\partial y_s} \tilde{g}(y) - \frac{D\tilde{g}(y)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_s(y) \frac{\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \right\} = 0,$$

for all $y \in \partial\Omega$ and accordingly, the right hand side of (8.10) equals

$$\tilde{v}_i(x)Q^\ast \left[ \frac{\partial}{\partial x_r} \Theta, \frac{\partial}{\partial x_j} \tilde{g} - \frac{D\tilde{g}(y)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_j, \mu \right](x)$$

$$-\tilde{v}_j(x)Q^\ast \left[ \frac{\partial}{\partial x_r} \Theta, \frac{\partial}{\partial x_l} \tilde{g} - \frac{D\tilde{g}(y)\nu^{(2)}(y)}{\nu^{(2)}(y)\nu^{(2)}(y)} \tilde{v}_l, \mu \right](x).$$

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Next we consider the third term in the right hand side of formula (8.9), and we note that

\[
\int_{\partial \Omega} \sum_{s=1}^{n} (\tilde{g}(x) - \tilde{g}(y)) \frac{\partial}{\partial x_s} S_a(x - y) \times \left\{ \tilde{v}_t(x) M_{sj} \left[ \frac{(a^{(2)}\nu)_s^{(a^{(2)}\nu)}}{\nu^{(2)}\nu} \right] (y) - \tilde{v}_j(x) M_{sl} \left[ \frac{(a^{(2)}\nu)_s^{(a^{(2)}\nu)}}{\nu^{(2)}\nu} \right] (y) \right\} d\sigma_y
\]

\[
= \tilde{v}_j(x) Q^2 \left[ \frac{\partial S_a}{\partial y_r} \tilde{\Theta}_t \tilde{\gamma} \sum_{s=1}^{n} M_{sj} \left[ \frac{(a^{(2)}\nu)_s^{(a^{(2)}\nu)}}{\nu^{(2)}\nu} \right] (x) \right.
\]

\[
- \tilde{v}_j(x) Q^2 \left[ \frac{\partial S_a}{\partial y_r} \tilde{\Theta}_t \tilde{\gamma} \sum_{s=1}^{n} M_{sl} \left[ \frac{(a^{(2)}\nu)_s^{(a^{(2)}\nu)}}{\nu^{(2)}\nu} \right] (x) \right].
\]

Next we consider the last integral in the right hand side of formula (8.9) and we note that if \( x \in \Omega \) and \( y \in \partial \Omega \), we have

\[
\sum_{s,h=1}^{n} \frac{\partial}{\partial x_h} \left[ a_{sh} \frac{\partial}{\partial x_s} S_a(x - y) \right] + \sum_{s=1}^{n} a_s \frac{\partial}{\partial x_s} S_a(x - y) + a S_a(x - y) = 0.
\]

Thus we obtain

\[
\sum_{s,h=1}^{n} a_{sh} \nu_h(y) \frac{\partial}{\partial y_s} S_a(x - y)
\]

\[
= \sum_{s,h=1}^{n} a_{sh} \left( \nu_h(y) \frac{\partial}{\partial y_r} - \nu_r(y) \frac{\partial}{\partial y_h} \right) \left[ \frac{\partial}{\partial x_s} S_a(x - y) \right]
\]

\[
+ \nu_r(y) \sum_{s=1}^{n} a_s \frac{\partial}{\partial x_s} S_a(x - y) + \nu_r(y) a S_a(x - y),
\]

and we note that the first parenthesis in the right hand side equals \( M_{hr,y} \).

Then the last integral in the right hand side of formula (8.9) equals

\[
\int_{\partial \Omega} (\tilde{g}(x) - \tilde{g}(y)) \sum_{s,h=1}^{n} a_{sh} \nu_h(y) \frac{\partial}{\partial y_s} \left[ \frac{\partial}{\partial x_r} S_a(x - y) \right] \times \left[ \tilde{\nu}_t(x) M_{sj} \left[ \frac{(a^{(2)}\nu)_s^{(a^{(2)}\nu)}}{\nu^{(2)}\nu} \right] (y) - \tilde{\nu}_j(x) M_{sl} \left[ \frac{(a^{(2)}\nu)_s^{(a^{(2)}\nu)}}{\nu^{(2)}\nu} \right] (y) \right] d\sigma_y
\]

\[
= \int_{\partial \Omega} (\tilde{g}(x) - \tilde{g}(y)) \left\{ \sum_{s,h=1}^{n} a_{sh} M_{hr,y} \left[ \frac{\partial}{\partial x_s} S_a(x - y) \right] \right.
\]

\[
+ \nu_r(y) \sum_{s=1}^{n} a_s \frac{\partial}{\partial x_s} S_a(x - y) + \nu_r(y) a S_a(x - y) \right\}
\]

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\[
\begin{align*}
&\times \frac{\tilde{\nu}_i(x)\nu_j(y) - \tilde{\nu}_j(x)\nu_i(y)}{\nu^2(y)\nu^{(2)}(y)} \mu(y) \, d\sigma_y \\
&= \sum_{s,h=1}^n a_{sh} \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y)) M_{hr,y} \left[ \frac{\partial}{\partial x_s} S_n(x - y) \right] \\
&\times \frac{\tilde{\nu}_i(x)\tilde{\nu}_j(x) - \tilde{\nu}_j(x)\tilde{\nu}_i(x)}{\nu^2(y)\nu^{(2)}(y)} \mu(y) \, d\sigma_y \\
&+ \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y)) \left[ \sum_{s=1}^n a_s \frac{\partial}{\partial x_s} S_n(x - y) + a S_n(x - y) \right] \\
&\times \frac{\tilde{\nu}_i(x)\nu_j(y) - \tilde{\nu}_j(x)\nu_i(y)}{\nu^2(y)\nu^{(2)}(y)} \nu_r(y) \mu(y) \, d\sigma_y.
\end{align*}
\]

We now consider separately each of the terms in the right hand side of formula (8.12). By Lemma 2.3 and by equality \(-M_{hr,y}[\tilde{g}(x) - \tilde{g}(y)] = M_{hr,y}[\tilde{g}(y)]\), the first integral in the right hand side of formula (8.12) equals

\[
\begin{align*}
&\int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y)) M_{hr,y} \left[ \frac{\partial}{\partial x_s} S_n(x - y) \right] \\
&\times \frac{\tilde{\nu}_i(x)\tilde{\nu}_j(x) - \tilde{\nu}_j(x)\tilde{\nu}_i(x)}{\nu^2(y)\nu^{(2)}(y)} \mu(y) \, d\sigma_y \\
&= \int_{\partial\Omega} M_{hr}[\tilde{g}] \frac{\partial}{\partial x_s} S_n(x - y) \\
&\times \left( -\tilde{\nu}_i(x) \frac{\tilde{\nu}_j(x) - \tilde{\nu}_j(y)}{\nu^2(y)\nu^{(2)}(y)} + \tilde{\nu}_j(x) \frac{\tilde{\nu}_i(x) - \tilde{\nu}_i(y)}{\nu^2(y)\nu^{(2)}(y)} \right) \mu(y) \, d\sigma_y \\
&+ \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y)) \frac{\partial}{\partial x_s} S_n(x - y) \\
&\times \left( -\tilde{\nu}_i(x) M_{hr} \left[ \frac{\nu_i \mu}{\nu^2 a^{(2)} \nu} \right] (y) + \tilde{\nu}_j(x) M_{hr} \left[ \frac{\nu_j \mu}{\nu^2 a^{(2)} \nu} \right] (y) \right) \, d\sigma_y \\
&= -\tilde{\nu}_i(x) \int_{\partial\Omega} (\tilde{\nu}_j(x) - \tilde{\nu}_j(y)) \frac{\partial}{\partial x_s} S_n(x - y) M_{hr}[\tilde{g}] \frac{\nu_i \mu}{\nu^2 a^{(2)} \nu} \mu(y) \, d\sigma_y \\
&+ \tilde{\nu}_j(x) \int_{\partial\Omega} (\tilde{\nu}_i(x) - \tilde{\nu}_i(y)) \frac{\partial}{\partial x_s} S_n(x - y) M_{hr}[\tilde{g}] \frac{\nu_j \mu}{\nu^2 a^{(2)} \nu} \mu(y) \, d\sigma_y \\
&-\tilde{\nu}_i(x) \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y)) \frac{\partial}{\partial x_s} S_n(x - y) M_{hr} \left[ \frac{\nu_i \mu}{\nu^2 a^{(2)} \nu} \right] (y) \, d\sigma_y \\
&+ \tilde{\nu}_j(x) \int_{\partial\Omega} (\tilde{g}(x) - \tilde{g}(y)) \frac{\partial}{\partial x_s} S_n(x - y) M_{hr} \left[ \frac{\nu_j \mu}{\nu^2 a^{(2)} \nu} \right] (y) \, d\sigma_y \\
&= -\tilde{\nu}_i(x) \left\{ Q^4 \left[ \frac{\partial S_n}{\partial x_s} \circ \Theta, \tilde{\nu}_j, M_{hr}[\tilde{g}] \frac{\mu}{\nu^2 a^{(2)} \nu} \right] (x) \right. \\
&\quad \left. + Q^2 \left[ \frac{\partial S_n}{\partial x_s} \circ \Theta, \tilde{g}, M_{hr} \left[ \frac{\nu_j \mu}{\nu^2 a^{(2)} \nu} \right] \right] (x) \right\}
\end{align*}
\]
Then by combining formulas (8.9)–(8.13), we obtain the following formula

Next we note that the second integral in the right hand side of formula (8.12) equals

\[ M_l = 1_n l_j \sum_{s} g_s \sum_{r} l_j \]

\[ = \tilde{v}_j(x) \left\{ Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, g \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \right\}. \]

Then by combining formulas (8.9)–(8.13), we obtain the following formula

\[ M_{ij} = x_i Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, \mu \right](x) \]

\[ = \tilde{v}_i(x) Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, g \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \]

\[ - \tilde{v}_j(x) Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, g \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \]

\[ + \tilde{v}_i(x) Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, M_{hr} \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \]

\[ - \tilde{v}_j(x) Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, M_{hr} \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \]

\[ + \sum_{s,h=1}^{n} a_{sh} \tilde{v}_i(x) \left\{ Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, g \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \right\} \]

\[ + Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, M_{hr} \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \]

\[ - \sum_{s,h=1}^{n} a_{sh} \tilde{v}_j(x) \left\{ Q^2 \left[ \frac{\partial S_a}{\partial x} \circ \Theta, \tilde{g}, M_{hr} \frac{\nu_j \nu_r}{\nu^4 a(2) \nu} \right](x) \right\} \]
Theorem 8.2 (ii) implies that the single layer potentials in the right hand side of formula (8.14) are continuous functions if \( \phi \) belongs to \( C_0^{\alpha} \). By taking the restriction to \( \partial \Omega \), we conclude that the membership of \( Q \) to \( C^\infty(\partial\Omega) \) belongs to \( C^0(\partial\Omega) \). Now under our assumptions, the first argument of the maps \( Q = \{ \partial S_\alpha \circ \Theta, \cdot, \cdot \} \) and \( \partial S_\alpha \circ \Theta, \cdot, \cdot \), which appear in the right hand side of formula (8.14) and the single layer potentials in the right hand side of formula (8.14) are continuous in \( x \in \partial\Omega \). Then Theorem 8.2(i) implies that the right hand side of formula (8.14) defines a continuous function of the variable \( x \in \partial\Omega \). Since \( \Omega \) is of class \( C^{2,\alpha} \) and \( g \in C^{1,\beta}(\partial\Omega) \), we are assuming that \( \mu \in C^{1,\beta}(\partial\Omega) \). Hence, the equation of formula (8.14) must hold for all \( x \in \partial\Omega \), and in particular for all \( x \in \partial\Omega \). Since \( Q = \{ \partial S_\alpha \circ \Theta, \cdot, \cdot \} \) and \( M_\partial^1 = M_\partial^1 \) on \( \partial\Omega \), we conclude that (8.13) holds.

Next we assume that \( \mu \in C^1(\partial\Omega) \). We denote by \( P_{ijr}[g,\mu] \) the right hand side of (8.13). By Theorem 8.3 (i), the operators \( Q = \{ \partial S_\alpha \circ \Theta, g, \cdot, \cdot \} \), \( Q = \{ \partial S_{ijr} \circ \Theta, \cdot, \cdot \} \), \( Q = \{ \partial S_{ijr} \circ \Theta, g, \mu, \cdot, \cdot \} \) are linear and continuous from the space \( C^0(\partial\Omega) \) to \( C^0(\partial\Omega) \). Then by Theorem 7.2 and by the continuity of the pointwise product in \( C^0(\partial\Omega) \), the operator \( P_{ijr}[g,\cdot,\cdot] \) is continuous from \( C^0(\partial\Omega) \) to \( C^0(\partial\Omega) \). In particular, \( Q = \{ \partial S_{ijr} \circ \Theta, g, \mu, \cdot, \cdot \} \in C^0(\partial\Omega) \).

We now show that the weak \( M_{ijr} \)-derivative of \( Q = \{ \partial S_\alpha \circ \Theta, g, \cdot, \cdot \} \) in \( \partial\Omega \) coincides with \( P_{ijr}[g,\mu] \).

By considering an extension of \( \mu \) of class \( C^1 \) with compact support in \( \mathbb{R}^n \) and by considering a sequence of mollifiers of such an extension, and by taking the restriction to \( \partial\Omega \), we conclude that there exists a sequence of functions \( \{ \mu_b \}_{b \in \mathbb{N}} \) in \( C^2(\partial\Omega) \) converging to \( \mu \) in \( C^1(\partial\Omega) \). Then we note that if \( \varphi \in C^1(\partial\Omega) \), the validity of (8.14) for \( \mu_b \in C^2(\partial\Omega) \subseteq C^{1,\beta}(\partial\Omega) \), and the membership of \( Q = \{ \partial S_\alpha \circ \Theta, g, \mu_b \} \in C^1(\partial\Omega) \) (see Theorem 8.2(ii)), and
Hence, \( P \) from to prove that the following two statements hold.

Theorem 8.15

Let \( \mu \)

We first prove statement (i). We proceed by induction on

Proof.

Lemma 2.8 imply that

\[
\int_{\partial \Omega} Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right] M_{ij} [\varphi] \, d\sigma = \lim_{b \to \infty} \int_{\partial \Omega} Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu_b \right] M_{ij} [\varphi] \, d\sigma
\]

\[
= - \lim_{b \to \infty} \int_{\partial \Omega} M_{ij} \left[ Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu_b \right] \right] \varphi \, d\sigma
\]

\[
= - \lim_{b \to \infty} \int_{\partial \Omega} P_{ijr}[g, \mu_b] \varphi \, d\sigma = - \int_{\partial \Omega} P_{ijr}[g, \mu] \varphi \, d\sigma.
\]

Hence, \( P_{ijr}[g, \mu] \) coincides with the weak \( M_{ij} \)-derivative of \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right] \)

for all \( l, j \in \{1, \ldots, n\} \). Since both \( P_{ijr}[g, \mu] \) and \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right] \)

are continuous functions, it follows that \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right] \) is continuous. Then the bilinear map

\[
Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right]
\]

is continuous from \( C^{m,\alpha}(\partial \Omega) \) to \( C^{m,1,\omega(\cdot)}(\partial \Omega) \) which takes a pair \((g, \mu)\) classically. Hence \(\text{Lemma } 8.8\) holds also for \(\mu \in C^1(\partial \Omega)\).

By exploiting formula \(8.8\), we can prove the following.

Theorem 8.15 Let \( a \) be as in \(1.1\), \(1.2\), \(1.3\). Let \( S_a \) be a fundamental solution of \( P[a, D] \). Let \( \alpha \in [0, 1] \). Let \( m \in \mathbb{N} \setminus \{0\} \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{m,\alpha} \). Let \( r \in \{1, \ldots, n\} \). Then the following statements hold.

(i) Let \( \theta \in [0, 1] \). Then the bilinear map \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot \right] \) from the space \( C^{m,1,\theta}(\partial \Omega) \times C^{m,1}(\partial \Omega) \) to \( C^{m,1,\omega(\cdot)}(\partial \Omega) \) which takes a pair \((g, \mu)\) to \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right] \) is continuous.

(ii) Let \( \beta \in [0, 1] \). Then the bilinear map \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot \right] \) from the space \( C^{m,\alpha}(\partial \Omega) \times C^{m,1,\beta}(\partial \Omega) \) to \( C^{m,1,\omega(\cdot)}(\partial \Omega) \) which takes a pair \((g, \mu)\) to \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, g, \mu \right] \) is continuous.

Proof. We first prove statement (i). We proceed by induction on \( m \). Case \( m = 1 \) holds by Theorem 8.3 (i). We now prove that if the statement holds for \( m \), then it holds for \( m + 1 \). Thus we now assume that \( \Omega \) is of class \( C^{m+1,\alpha} \) and we turn to prove that \( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot \right] \) is bilinear and continuous from \( C^{m,\theta}(\partial \Omega) \times C^{m}(\partial \Omega) \) to \( C^{m,\omega(\cdot)}(\partial \Omega) \). By Lemma 2.9 (ii), it suffices to prove that the following two statements hold.

\( Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot \right] \) is continuous from \( C^{m,\theta}(\partial \Omega) \times C^{m}(\partial \Omega) \) to \( C^{0}(\partial \Omega) \).

\( M_{ij} \left[ Q \left[ \frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot \right] \right] \) is continuous from \( C^{m,\theta}(\partial \Omega) \times C^{m}(\partial \Omega) \) to the space \( C^{m-1,\omega(\cdot)}(\partial \Omega) \) for all \( l, j \in \{1, \ldots, n\} \).
Statement (j) holds by case \(m = 1\), and by the imbedding of \(C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)\) into \(C^{0,\theta}(\partial\Omega) \times C^0(\partial\Omega)\). We now prove statement (jj). Since \(m + 1 \geq 2\), Lemma 8.10 and the inductive assumption imply that we can actually apply \(M_{ij}\) to \(Q[\frac{\partial^r S}{\partial x_r} \circ \Theta, g, h, \mu]\). We find convenient to denote by \(P_{lj}[g, h, \mu]\) the right hand side of formula (8.8). Then we have

\[
M_{ij}[Q[\frac{\partial^r S}{\partial x_r} \circ \Theta, g, h, \mu]] = P_{lj}[g, h, \mu] \quad \forall (g, h, \mu) \in C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega).
\]

By Lemma 2.10 and by the membership of \(\nu\) in \(C^{m,\alpha}(\partial\Omega, \mathbb{R}^n)\), which is contained in \(C^{m-1,1}(\partial\Omega, \mathbb{R}^n)\), and by the continuity of the pointwise product in Schauder spaces, and by the continuity of the imbedding of \(C^m(\partial\Omega)\) into \(C^{m-1}(\partial\Omega)\), and of \(C^{m,\alpha}(\partial\Omega)\) into \(C^{m-1,\theta}(\partial\Omega)\), and by the inductive assumption on the continuity of \(Q[\frac{\partial^r S}{\partial x_r} \circ \Theta, \cdot, \cdot]\), and by the continuity of \(\nu[\partial\Omega, S_a, \cdot, \cdot, \cdot]\) from \(C^{m-1,\alpha}(\partial\Omega)\) to \(C^{m,\alpha}(\partial\Omega)\) \(\subseteq C^{m-1,\theta}(\partial\Omega)\), and by the continuity of the imbedding of \(C^m(\partial\Omega)\) into \(C^{m-1,\omega(\cdot)}(\partial\Omega)\) and of \(C^{m}(\partial\Omega)\) into \(C^{m-1,\omega_1(\cdot)}(\partial\Omega)\), and by the continuity of \(D_a\) from \(C^{m,\theta}(\partial\Omega)\) into \(C^{m-1,\theta}(\partial\Omega)\), we conclude that \(P_{lj}[\cdot, \cdot]\) is bilinear and continuous from \(C^{m,\theta}(\partial\Omega) \times C^m(\partial\Omega)\) into \(C^{m-1,\omega_1(\cdot)}(\partial\Omega)\), and the proof of statement (jj) is complete. The proof of statement (ii) follows the lines of the proof of statement (i), by replacing the use of Theorem 8.3 (i) with that of Theorem 8.3 (ii).

\[
\square
\]

**Definition 8.16** Let \(a\) be as in (1.1), (1.2), (1.3). Let \(S_a\) be a fundamental solution of \(P[a, D]\). Let \(\alpha \in [0, 1]\). Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^n\) of class \(C^{1,\alpha}\). Then we set

\[
R[g, h, \mu] = \sum_{r=1}^n a_r \left\{ Q[\frac{\partial^r S}{\partial x_r} \circ \Theta, g, h, \mu] - Q[\frac{\partial^r S}{\partial x_r} \circ \Theta, h, \mu] - Q[\frac{\partial^r S}{\partial x_r} \circ \Theta, g, \mu] \right\}
\]

\[
+ a \left\{ g[\nu[\partial\Omega, S_a, h, \mu] - h[v[\partial\Omega, S_a, g, \mu]] \right\}
\]

for all \((g, h, \mu) \in (C^{0,\alpha}(\partial\Omega))^2 \times C^0(\partial\Omega)\).

Since

\[
g(x)h(y) - g(y)h(x) = [g(x)h(x) - g(y)h(y)] - [g(x)[h(x) - h(y)] - g(y)[h(x) - h(y)] \quad \forall x, y \in \partial\Omega,
\]

we have

\[
R[g, h, \mu] = \int_{\partial\Omega} \left\{ \sum_{r=1}^n a_r \frac{\partial}{\partial x_r} S_a(x - y) + a S_a(x - y) \right\}
\]

\[
	imes [g(x)h(y) - g(y)h(x)] \mu(x) d\sigma_y \quad \forall x \in \partial\Omega.
\]

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Since $R$ is a composition of the operator $Q(\frac{\partial S_a}{\partial x_r} \circ \Theta, \cdot, \cdot)$ and of a single layer potential, Theorems 7.3, 7.6 and Theorem 8.2 and the continuity of the product in Schauder spaces and of the imbedding of $C^{m-1,\alpha}(\partial \Omega)$ into $C^{m-2,\alpha}(\partial \Omega)$ for $m \geq 2$ and of the imbedding of $C^{m-1,\alpha}(\partial \Omega)$ into $C^{m-1,\omega_\alpha(\cdot)}(\partial \Omega)$ and of $C^{\alpha,\beta}(\partial \Omega)$ into $C^{m-1,\alpha}(\partial \Omega)$, imply the validity of the following.

**Theorem 8.17** Let $a$ be as in (1.1), (1.2), (1.3). Let $S_a$ be a fundamental solution of $P[a, D]$. Let $a \in [0, 1]$. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.

(i) The trilinear map $R$ from the space $(C^{m-1,\alpha}(\partial \Omega))^2 \times C^{m-1,\omega_\alpha(\cdot)}(\partial \Omega)$ which takes a pair $(g, h, \mu)$ to $R[g, h, \mu]$ is continuous.

(ii) Let $\beta \in [0, 1]$. Then the trilinear map $R$ from the space $(C^{m-1,\alpha}(\partial \Omega))^2 \times C^{m-1,\beta}(\partial \Omega)$ to $C^{m-1,\omega_\alpha(\cdot)}(\partial \Omega)$ which takes a pair $(g, h, \mu)$ to $R[g, h, \mu]$ is continuous.

9 Tangential derivatives and regularizing properties of the double layer potential

We now exploit Theorems 6.3, 7.4 and Lemma 8.4 and Theorems 8.11, 8.14 in order to prove a formula for the tangential derivatives of the double layer potential, which generalizes the corresponding formula of Hofmann, Mitrea and Taylor (6.2.6) for homogeneous operators. We do so by means of the following.

**Theorem 9.1** Let $a$ be as in (1.1), (1.2), (1.3). Let $S_a$ be a fundamental solution of $P[a, D]$. Let $a \in [0, 1]$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $\mu \in C^1(\partial \Omega)$, then $w[\partial \Omega, a, S_a, \mu]|_{\partial \Omega} \in C^1(\partial \Omega)$ and

$$M_{lj}[w[\partial \Omega, a, S_a, \mu]|_{\partial \Omega}] = w[\partial \Omega, a, S_a, M_{lj}[\mu]|_{\partial \Omega}]$$  \hspace{1cm} (9.2)

$$+ \sum_{b,r=1}^{n} a_{br} \left\{ Q \left[ \frac{\partial S_a}{\partial x_b} \circ \Theta, \nu_j, M_{lj}[\mu] \right] - Q \left[ \frac{\partial S_a}{\partial x_b} \circ \Theta, \nu_j, M_{lj}[\mu] \right] \right\}$$

$$+ \nu Q \left[ \frac{\partial S_a}{\partial x_j} \circ \Theta, \nu \cdot a^{(1)}, \mu \right] - \nu_j Q \left[ \frac{\partial S_a}{\partial x_l} \circ \Theta, \nu_j, a^{(1)}, \mu \right]$$

$$+ \nu \cdot a^{(1)} \left\{ Q \left[ \frac{\partial S_a}{\partial x_l} \circ \Theta, \nu_j, \mu \right] - Q \left[ \frac{\partial S_a}{\partial x_l} \circ \Theta, \nu_j, \mu \right] \right\}$$

$$- \nu \cdot a^{(1)}[v[\partial \Omega, S_a, M_{lj}[\mu]] + v[\partial \Omega, S_a, \nu \cdot a^{(1)}M_{lj}[\mu]]$$

+ $R[\nu_j, \nu_j, \mu]$ on $\partial \Omega$,

for all $l, j \in \{1, \ldots, n\}$. (For $Q$ see (8.4).)
Proof. We fix $\beta \in [0, \alpha]$ and we first consider the specific case in which
$\mu \in C^{1,\beta}(\partial \Omega)$. Let $R \in [0, +\infty]$ be such that $\text{cl}\Omega \subseteq B_{R}(0, R)$. Let $\cdot \ r$ be an
extension operator of $C^{1,\beta}(\partial \Omega)$ to $C^{3,\beta}(\text{cl}\Omega)$ as in Lemma 2.3. By Theorem 7.3 (i), (ii), we have
$w^{+}[\partial \Omega, \mathbf{a}, S_{a}, \mu] \in C^{3,\beta}(\text{cl}\Omega)$ and

$$M_{ij}[w^{+}[\partial \Omega, \mathbf{a}, S_{a}, \mu]|_{\partial \Omega}] = \frac{1}{2} M_{ij}[\mu] + M_{ij}[w^{+}[\partial \Omega, \mathbf{a}, S_{a}, \mu]|_{\partial \Omega}]. \quad (9.3)$$

By the definition of $M_{ij}$ and by equality (7.5), we have

$$M_{ij}[w^{+}[\partial \Omega, \mathbf{a}, S_{a}, \mu]|_{\partial \Omega}] = \left\{ \begin{array}{l}
\frac{\partial}{\partial x_{j}} v^{+}[\partial \Omega, \mathbf{a}, S_{a}, \mu] - \frac{\partial}{\partial x_{j}} v^{+}[\partial \Omega, \mathbf{a}, S_{a}, \mu] \\
\nu_{j} \left[ \sum_{b, r=1}^{n} a_{br} \frac{\partial}{\partial x_{b}} v^{+}[\partial \Omega, \mathbf{a}, M_{jr}[\mu]] + \sum_{b=1}^{n} a_{b} \frac{\partial}{\partial x_{b}} v^{+}[\partial \Omega, \mathbf{a}, \nu_{j} \mu] \\
- \frac{\partial}{\partial x_{j}} v^{+}[\partial \Omega, \mathbf{a}, (\nu' \cdot a^{(1)}) \mu] + av^{+}[\partial \Omega, \mathbf{a}, \nu_{j} \mu] \\
- \nu_{j} \left[ \sum_{b, r=1}^{n} a_{br} \frac{\partial}{\partial x_{b}} v^{+}[\partial \Omega, \mathbf{a}, M_{ir}[\mu]] + \sum_{b=1}^{n} a_{b} \frac{\partial}{\partial x_{b}} v^{+}[\partial \Omega, \mathbf{a}, \nu_{j} \mu] \\
- \frac{\partial}{\partial x_{x}} v^{+}[\partial \Omega, \mathbf{a}, (\nu' \cdot a^{(1)}) \mu] + av^{+}[\partial \Omega, \mathbf{a}, \nu_{j} \mu] \right]
\end{array} \right. \quad (9.4)$$

We now consider the first term in braces in the right hand side of (9.4), and we note that

$$\left\{ \nu_{i}(x) \frac{\partial}{\partial x_{b}} v^{+}[\partial \Omega, \mathbf{a}, M_{jr}[\mu]](x) - \nu_{j} \frac{\partial}{\partial x_{b}} v^{+}[\partial \Omega, \mathbf{a}, M_{ir}[\mu]](x) \right\} \quad (9.5)$$

$$= - \frac{\nu_{i}(x) v_{b}(x)}{2\nu'(x) a^{(2)} \nu(x)} M_{jr}[\mu](x) + \nu_{i}(x) \int_{\partial \Omega} \frac{\partial}{\partial x_{b}} S_{a}(x - y) M_{jr}[\mu](y) \, d\sigma_{y}$$
$$+ \frac{\nu_{j}(x) v_{b}(x)}{2\nu'(x) a^{(2)} \nu(x)} M_{ir}[\mu](x) - \nu_{j}(x) \int_{\partial \Omega} \frac{\partial}{\partial x_{b}} S_{a}(x - y) M_{ir}[\mu](y) \, d\sigma_{y}$$
$$= \nu_{b}(x) - \nu_{i}(x) M_{jr}[\mu](x) + \nu_{j}(x) M_{ir}[\mu](x)$$

\[ \frac{\nu_{b}(x) - \nu_{i}(x) M_{jr}[\mu](x) + \nu_{j}(x) M_{ir}[\mu](x)}{2\nu'(x) a^{(2)} \nu(x)} \]

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Then we obtain
\[
\nu = \text{equality (9.4)}
\]
and we note that
\[
\nu = \text{equality (9.4)}
\]
Next we consider the term in the third braces in the right hand side of
\[
\sum
\]
Then we obtain
\[
\sum_{b,r=1}^{n} a_{br} v_{b} \frac{-\nu M_{jr}[\mu] + \nu_{j} M_{tr}[\mu]}{2 \nu^{2} a(2) \nu}
\]
Next we consider the term in braces in the argument of the integral in the right hand side of (9.5), and we note that equality (9.6) implies that
\[
\nu(x) M_{jr}[\mu](y) - \nu_{j}(y) M_{tr}[\mu](y) = [\nu(x) - \nu(y)] M_{jr}[\mu](y) + [\nu(y) M_{jr}[\mu](y) - \nu_{j}(y) M_{tr}[\mu](y)]
\]
\[
= [\nu(x) - \nu(y)] M_{jr}[\mu](y) - \nu_{j}(y) M_{tr}[\mu](y)
\]
\[
\forall x, y \in \partial \Omega.
\]
Next we consider the term in the second braces in the right hand side of (9.6) and we note that
\[
\nu(x) \frac{\partial}{\partial x} v^{+}[\partial \Omega, S_{a}, \nu_{j} \mu](x) - \nu_{j}(y) \frac{\partial}{\partial x} v^{+}[\partial \Omega, S_{a}, \nu_{j} \mu](x)
\]
\[
= -\nu(x) \frac{\nu_{j}(x)}{2 \nu^{2}(x) a(2) \nu}(y) \int_{\partial \Omega} \frac{\partial}{\partial x} S_{a}(x-y) \nu_{j}(y) \mu(y) d\sigma_{y}
\]
\[
+ \nu_{j}(x) \frac{\nu_{j}(x)}{2 \nu^{2}(x) a(2) \nu}(y) \int_{\partial \Omega} \frac{\partial}{\partial x} S_{a}(x-y) \nu_{j}(y) \mu(y) d\sigma_{y}
\]
\[
= \int_{\partial \Omega} \frac{\partial}{\partial x} S_{a}(x-y) [\nu(x) \nu_{j}(y) - \nu_{j}(x) \nu_{j}(y)] \mu(y) d\sigma_{y} \quad \forall x \in \partial \Omega.
\]
\( = -\nu(x) \frac{\nu_j(x)}{2\nu^2(x) \sigma^{(2)}(x) \muan}\nu'(x) \cdot \nu'^{(1)} \mu(x) \) 
\( + \nu(x) \int_{\partial \Omega} \frac{\partial \muan(x - y) \nu'(y) \cdot \nu'^{(1)} \mu(y)}{\partial x_j} \, d\sigma_y \)
\( + \nu_j(x) \int_{\partial \Omega} \frac{\nu(x)}{2\nu^2(x) \sigma^{(2)}(x) \muan}\nu'(x) \cdot \nu'^{(1)} \mu(x) \)
\( - \nu_j(x) \int_{\partial \Omega} \frac{\partial \muan(x - y) \nu'(y) \cdot \nu'^{(1)} \mu(y)}{\partial x_l} \, d\sigma_y \)

\( = -\nu(x) \int_{\partial \Omega} \left[ (\nu'(x) \cdot \nu'^{(1)}) - (\nu'(y) \cdot \nu'^{(1)}) \right] \frac{\partial \muan(x - y) \mu(y)}{\partial x_j} \, d\sigma_y \)
\( + \nu(x) \int_{\partial \Omega} \left[ (\nu'(x) \cdot \nu'^{(1)}) \frac{\partial \muan(x - y) \mu(y)}{\partial x_j} \right] \, d\sigma_y \)
\( + \nu_j(x) \int_{\partial \Omega} \left[ (\nu'(x) \cdot \nu'^{(1)}) - (\nu'(y) \cdot \nu'^{(1)}) \right] \frac{\partial \muan(x - y) \mu(y)}{\partial x_l} \, d\sigma_y \)
\( - \nu_j(x) \int_{\partial \Omega} (\nu'(x) \cdot \nu'^{(1)}) \frac{\partial \muan(x - y) \mu(y)}{\partial x_l} \, d\sigma_y \)

for all \( x \in \partial \Omega \). By Lemma 2.8, the last integral in the right hand side of (9.10) equals

\( - \int_{\partial \Omega} M_{ij,y} [\muan(x - y)] \mu(y) \, d\sigma_y = \int_{\partial \Omega} \muan(x - y) M_{ij} \mu(y) \, d\sigma_y \quad \forall x \in \partial \Omega \).

(9.11)
Thus the last term in the right hand side of (9.10) equals

\[
(\nu'(x) \cdot a^{(1)}) \int_{\partial \Omega} S_a(x - y) M_{ij}[\mu](y) \, d\sigma_y 
\]

(9.12)

\[
= \int_{\partial \Omega} \left[ (\nu'(x) \cdot a^{(1)}) - (\nu'(y) \cdot a^{(1)}) \right] S_a(x - y) M_{ij}[\mu](y) \, d\sigma_y 
\]

+ \int_{\partial \Omega} (\nu'(y) \cdot a^{(1)}) S_a(x - y) M_{ij}[\mu](y) \, d\sigma_y \quad \forall x \in \partial \Omega.
\]

In the fourth and last term in braces of equation (9.4), we have

\[
\int_{\partial \Omega} S_a(x - y) [\nu_l(x) \nu_j(y) - \nu_j(x) \nu_l(y)] \mu(y) \, d\sigma_y 
\]

(9.13)

Then by combining (9.3)–(9.5), (9.7)–(9.13), we obtain

\[
M_{ij}[w[\partial \Omega, a, S_a, \mu]](x) 
\]

\[
= \sum_{b, r=1}^n a_{br} \left\{ \int_{\partial \Omega} (\nu_l(x) - \nu_l(y)) \frac{\partial}{\partial x_b} S_a(x - y) M_{jr}[\mu](y) \, d\sigma_y 
\]

- \int_{\partial \Omega} (\nu_j(x) - \nu_j(y)) \frac{\partial}{\partial x_b} S_a(x - y) M_{lr}[\mu](y) \, d\sigma_y 
\]

- \int_{\partial \Omega} \nu_r(y) \frac{\partial}{\partial x_b} S_a(x - y) M_{ij}[\mu](y) \, d\sigma_y 
\]

+ \sum_{b=1}^n a_{br} \int_{\partial \Omega} \frac{\partial}{\partial x_b} S_a(x - y) [\nu_l(x) \nu_j(y) - \nu_j(x) \nu_l(y)] \mu(y) \, d\sigma_y 
\]

+ \nu_l(x) \int_{\partial \Omega} \left[ (\nu'(x) \cdot a^{(1)}) - (\nu'(y) \cdot a^{(1)}) \right] \frac{\partial}{\partial x_j} S_a(x - y) \mu(y) \, d\sigma_y 
\]

- \nu_j(x) \int_{\partial \Omega} \left[ (\nu'(x) \cdot a^{(1)}) - (\nu'(y) \cdot a^{(1)}) \right] \frac{\partial}{\partial x_i} S_a(x - y) \mu(y) \, d\sigma_y 
\]

- (\nu'(x) \cdot a^{(1)}) \left\{ \int_{\partial \Omega} (\nu_l(x) - \nu_l(y)) \frac{\partial}{\partial x_j} S_a(x - y) \mu(y) \, d\sigma_y 
\]

- \int_{\partial \Omega} (\nu_j(x) - \nu_j(y)) \frac{\partial}{\partial x_i} S_a(x - y) \mu(y) \, d\sigma_y 
\]

- \int_{\partial \Omega} \left[ (\nu'(x) \cdot a^{(1)}) - (\nu'(y) \cdot a^{(1)}) \right] S_a(x - y) M_{ij}[\mu](y) \, d\sigma_y 
\]

- \int_{\partial \Omega} (\nu'(y) \cdot a^{(1)}) S_a(x - y) M_{ij}[\mu](y) \, d\sigma_y 
\]

+ a \int_{\partial \Omega} S_a(x - y) [\nu_l(x) \nu_j(y) - \nu_j(x) \nu_l(y)] \mu(y) \, d\sigma_y \quad \forall x \in \partial \Omega,
\]

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Hence, \( C \) is the membership of \( t \) if and only if \( C \) is the membership of \( t \) respectively. In particular, the continuity of \( \mu \) in \( (9.2) \). By the continuity of \( \frac{\partial S}{\partial x_j} \circ \Theta, \nu, \mu \) in \( \Omega \), and from \( \Omega \), and of \( \frac{\partial S}{\partial x_j} \circ \Theta, \nu, \mu \) in \( \Omega \), we rewrite as
\[
M_{lj}[w[\partial \Omega, a, S_n, \mu]](x)
= \sum_{b, r=1}^{n} a_{b} \left\{ Q \left[ \frac{\partial S}{\partial x_b} \circ \Theta, \nu, M_{lj}[\mu] \right] (x) - Q \left[ \frac{\partial S}{\partial x_b} \circ \Theta, \nu, \mu \right] (x) \right\} \\
+ v_l(x) Q \left[ \frac{\partial S}{\partial x_j} \circ \Theta, \nu^l \cdot a^{(1)}, \mu \right] (x) - v_j(x) Q \left[ \frac{\partial S}{\partial x_j} \circ \Theta, \nu^j \cdot a^{(1)}, \mu \right] (x) \\
+ w[\partial \Omega, a, S_n, M_{lj}[\mu]](x) \\
+ (v^l(x) \cdot a^{(1)}) \left\{ Q \left[ \frac{\partial S}{\partial x_l} \circ \Theta, \nu, \mu \right] (x) - Q \left[ \frac{\partial S}{\partial x_l} \circ \Theta, \nu, \mu \right] (x) \right\} \\
- (v^j(x) \cdot a^{(1)}) v[\partial \Omega, a, S_n, M_{lj}[\mu]](x) + v[\partial \Omega, a, (w^2 \cdot a^{(1)}) M_{lj}[\mu]](x) + R[v_l, v_j, \mu](x) \quad \forall x \in \partial \Omega.
\]
Thus we have proved formula (\ref{eq:9.2}) for \( \mu \in C^{1, \beta}(\partial \Omega) \).

Next we assume that \( \mu \in C^{1}(\partial \Omega) \). We denote by \( T_{lj}[\mu] \) the right hand side of (\ref{eq:9.2}). By the continuity of \( M_{lj} \) from \( C^{1}(\partial \Omega) \) to \( C^{0}(\partial \Omega) \), and by the continuity of \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \) and of \( v[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \) from \( C^{0}(\partial \Omega) \) to \( C^{0, \alpha}(\partial \Omega) \), and by the continuity of \( Q \left[ \frac{\partial S}{\partial x_b} \circ \Theta, \nu, \mu \right] \) from \( C^{0, \alpha}(\partial \Omega) \times C^{0}(\partial \Omega) \) to \( C^{0, \omega_{\alpha}}(\partial \Omega) \), and by the continuity of \( R \) from \( C^{0, \alpha}(\partial \Omega) \times C^{0}(\partial \Omega) \) to \( C^{0, \omega_{\alpha}}(\partial \Omega) \), and by the continuity of the pointwise product in Schauder spaces, we conclude that the operators \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \) and \( T_{lj}[\cdot] \) are continuous from \( C^{1}(\partial \Omega) \) to \( C^{0, \omega_{\alpha}}(\partial \Omega) \) and from \( C^{1}(\partial \Omega) \) to \( C^{0, \omega_{\alpha}}(\partial \Omega) \), respectively. In particular, \( T_{lj}[\mu] \) and \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \) belong to \( C^{0}(\partial \Omega) \). We now show that the weak \( M_{lj} \)-derivative of \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \) coincides with \( T_{lj}[\mu] \).

By arguing so as at the end of the proof of Lemma \ref{lem:5.7}, there exists a sequence of functions \( \{ \mu_b \}_{b \in \mathbb{N}} \) in \( C^{1, \alpha}(\partial \Omega) \), which converges to \( \mu \) in \( C^{1}(\partial \Omega) \). Then we note that if \( \varphi \in C^{1}(\partial \Omega) \) the validity of (\ref{eq:7.2}) for \( \mu_b \in C^{1, \alpha}(\partial \Omega) \), and the membership of \( w[\partial \Omega, a, S_n, \mu_b] \|_{\partial \Omega} \) in \( C^{1, \alpha}(\partial \Omega) \), and the above mentioned continuity of \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \), and Lemma \ref{lem:2.3} imply that
\[
\int_{\partial \Omega} w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} M_{lj}[\varphi] \, d\sigma = \lim_{b \to \infty} \int_{\partial \Omega} w[\partial \Omega, a, S_n, \mu_b] \|_{\partial \Omega} M_{lj}[\varphi] \, d\sigma \\
= - \lim_{b \to \infty} \int_{\partial \Omega} M_{lj}[w[\partial \Omega, a, S_n, \mu_b] \|_{\partial \Omega}] \varphi \, d\sigma \\
= - \lim_{b \to \infty} \int_{\partial \Omega} T_{lj}[\mu_b] \varphi \, d\sigma = - \int_{\partial \Omega} T_{lj}[\mu] \varphi \, d\sigma.
\]
Hence, \( T_{lj}[\mu] \) coincides with the weak \( M_{lj} \)-derivative of \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \) for all \( l, j \) in \( \{1, \ldots, n\} \).

Since both \( T_{lj}[\mu] \) and \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \) are continuous functions, it follows that \( w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega} \in C^{1}(\partial \Omega) \) and that \( M_{lj}[w[\partial \Omega, a, S_n, \mu] \|_{\partial \Omega}] = T_{lj}[\mu] \) classically. Hence (\ref{eq:9.2}) holds also for \( \mu \in
By exploiting formula (9.2), we now prove the following result, which says that the double layer potential on \( \partial \Omega \) has a regularizing effect.

**Theorem 9.14** Let \( a \) be as in (1.1), (1.2), (1.3). Let \( S_a \) be a fundamental solution of \( P[a, D] \). Let \( \alpha \in [0, 1] \). Let \( m \in \mathbb{N} \setminus \{0\} \). Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) of class \( C^{m, \alpha} \). Then the following statements hold.

(i) The operator \( w[\partial \Omega, a, S_a, \cdot]\partial \Omega \) is linear and continuous from \( C^m(\partial \Omega) \) to \( C^{m, \omega_\alpha(\cdot)}(\partial \Omega) \).

(ii) Let \( \beta \in [0, \alpha] \). Then the operator \( w[\partial \Omega, a, S_a, \cdot]\partial \Omega \) is linear and continuous from \( C^{m, \beta}(\partial \Omega) \) to \( C^{m, \alpha}(\partial \Omega) \).

**Proof.** We prove statement (i) by induction on \( m \). As in the previous proof, we denote by \( T_{ij}[\mu] \) the right hand side of formula (9.2). We first consider case \( m = 1 \). By Lemma 2.9 (ii) and formula (9.2), it suffices to prove that the following two statements hold.

(j) \( w[\partial \Omega, a, S_a, \cdot]\partial \Omega \) is continuous from \( C^1(\partial \Omega) \) to \( C^0(\partial \Omega) \).

(jj) \( T_{ij}[, \cdot] \) is continuous from \( C^1(\partial \Omega) \) to \( C^{0, \omega_\alpha(\cdot)}(\partial \Omega) \) for all \( l, j \in \{1, \ldots, n\} \).

Theorem 7.6 implies the validity of (j). Statement (jj) follows by the continuity of the pointwise product in Schauder spaces and by the continuity of \( M_{ij} \) from \( C^1(\partial \Omega) \) to \( C^0(\partial \Omega) \), and by the continuity of \( w[\partial \Omega, S_a, \cdot]\partial \Omega \) from \( C^0(\partial \Omega) \) to \( C^{0, \alpha}(\partial \Omega) \) (cf. Theorems 7.3, 6.6), and by the continuity of \( Q[\frac{\partial S_a}{\partial x} \circ \Theta, \nu, \cdot, \cdot] \) from \( C^{0, \alpha}(\partial \Omega) \times C^0(\partial \Omega) \) to \( C^{0, \omega_\alpha(\cdot)}(\partial \Omega) \) (cf. Theorem 8.3 (i)) and by the continuity of \( R \) from \( (C^{0, \alpha}(\partial \Omega))^2 \times C^0(\partial \Omega) \) to \( C^{0, \omega_\alpha(\cdot)}(\partial \Omega) \) (cf. Theorem 8.17 (i)).

Next we assume that \( \Omega \) is of class \( C^{m+1, \alpha} \) and we turn to prove that \( w[\partial \Omega, a, S_a, \cdot]\partial \Omega \) is continuous from \( C^{m+1}(\partial \Omega) \) to \( C^{m+1, \omega_\alpha(\cdot)}(\partial \Omega) \). By Lemma 2.9 (ii) and formula (9.2), it suffices to prove that the following two statements hold.

(a) \( w[\partial \Omega, a, S_a, \cdot]\partial \Omega \) is continuous from \( C^{m+1}(\partial \Omega) \) to \( C^0(\partial \Omega) \).

(b) \( T_{ij}[\cdot, \cdot] \) is continuous from \( C^{m+1}(\partial \Omega) \) to \( C^{m, \omega_\alpha(\cdot)}(\partial \Omega) \). for all \( l, j \in \{1, \ldots, n\} \).

Statement (a) holds by the inductive assumption. We now prove statement (b). Since \( \Omega \) is of class \( C^{m+1, \alpha} \), then \( \nu \) is of class \( C^{m, \alpha}(\partial \Omega) \). Then Theorem 8.15 (i) ensures that \( Q[\frac{\partial S_a}{\partial x} \circ \Theta, \nu, \cdot, \cdot, \cdot] \) and \( Q[\frac{\partial S_a}{\partial x} \circ \Theta, \nu, j, \cdot, \cdot] \) are continuous from \( C^m(\partial \Omega) \) to \( C^{m, \omega_\alpha(\cdot)}(\partial \Omega) \) for all \( l, j, r \in \{1, \ldots, n\} \). Since \( M_{ij} \) is continuous from \( C^{m+1}(\partial \Omega) \) to \( C^m(\partial \Omega) \), the inductive assumption implies that \( w[\partial \Omega, a, S_a, M_{ij}[\cdot]_{|\partial \Omega} \) is continuous from \( C^{m+1}(\partial \Omega) \) to \( C^{m, \omega_\alpha(\cdot)}(\partial \Omega) \) for all \( l, j \in \{1, \ldots, n\} \).
Since $M_{ij}$ is continuous from $C^{m+1}(\partial \Omega)$ to $C^{m-\alpha}(\partial \Omega)$ and $v[\partial \Omega, S_{a \cdot \cdot} \cdot]_{\partial \Omega}$ is continuous from $C^{m-1,\alpha}(\partial \Omega)$ to $C^{\alpha}(\partial \Omega)$ and $v \in (C^{m,\alpha}(\partial \Omega))^n$ and $C^{m,\alpha}(\partial \Omega)$ is continuously imbedded into $C^{m,\omega_\alpha(\cdot)}(\partial \Omega)$, we conclude that $v[\partial \Omega, S_{a \cdot \cdot} \cdot]_{\partial \Omega}$ and $v[\partial \Omega, S_{a \cdot \cdot} \cdot \cdot]_{\partial \Omega}$ are continuous from the space $C^{m+1}(\partial \Omega)$ to $C^{m,\omega_\alpha(\cdot)}(\partial \Omega)$ for all $l$, $j$ in $\{1, \ldots, n\}$. Moreover, $R$ is continuous from $C^{m,\alpha}(\partial \Omega)^2 \times C^{m}(\partial \Omega)$ to $C^{m_{\omega_\alpha(\cdot)}(\partial \Omega)}$ (cf. Theorem 8.17 (i.).) Then statement (b) holds true.

Statement (iii) can be proved by the same argument of the proof of statement (i) by exploiting Theorem 8.15 (ii) instead of Theorem 8.15 (i) and Theorem 8.17 (ii) instead of Theorem 8.17 (i).

Since $C^{m,\omega_\alpha(\cdot)}(\partial \Omega)$ is compactly imbedded into $C^{m}(\partial \Omega)$, and $C^{m,\alpha}(\partial \Omega)$ is compactly imbedded into $C^{m,\beta}(\partial \Omega)$ for all $\beta \in [0, \alpha[$, we have the following immediate consequence of Theorem 9.13.

**Corollary 9.15** Under the assumptions of Theorem 9.14, the linear operator $w[\partial \Omega, a, S_{a \cdot \cdot} \cdot]_{\partial \Omega}$ is compact from $C^{m}(\partial \Omega)$ to itself, and from $C^{m,\omega_\alpha(\cdot)}(\partial \Omega)$ to itself, and from $C^{m,\alpha}(\partial \Omega)$ to itself.

### 10 Other layer potentials associated to $P[a, D]$

Another relevant layer potential operator associated to the analysis of boundary value problems for the operator $P[a, D]$ is the following

$$w_*[\partial \Omega, a, S_{a \cdot}, \nu](x) \equiv \int_{\partial \Omega} \mu(y) D S_a(x - y) a(y) \nu(x) \, d\sigma_y \quad \forall x \in \partial \Omega,$$

which we now turn to consider.

**Theorem 10.1** Let $a$ be as in (1.1), (1.2), (1.3). Let $S_a$ be a fundamental solution of $P[a, D]$. Let $\alpha \in [0, 1]$. Let $m \in \mathbb{N} \setminus \{0\}$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Then the following statements hold.

(i) The operator $w_*[\partial \Omega, a, S_{a \cdot} \cdot]_{\partial \Omega}$ is linear and continuous from $C^{m-1}(\partial \Omega)$ to $C^{m_{\omega_\alpha(\cdot)}(\cdot)}(\partial \Omega)$.

(ii) Let $\beta \in [0, \alpha[$. Then the operator $w_*[\partial \Omega, a, S_{a \cdot} \cdot]_{\partial \Omega}$ is linear and continuous from $C^{m-1,\beta}(\partial \Omega)$ to $C^{m-1,\alpha(\cdot)}(\partial \Omega)$.

**Proof.** We first note that

$$w_*[\partial \Omega, a, S_{a \cdot}, \nu](x) = \sum_{b, r=1}^n a_{b r} \int_{\partial \Omega} \nu_r(\partial S_a(x - y) \mu(y) \, d\sigma_y \quad (10.2)$$

$$= \sum_{b, r=1}^n a_{b r} Q \frac{\partial S_a}{\partial x_r} \cdot \Theta, \nu_r, \mu(x).$$
for all \( x \in \partial \Omega \) and for all \( \mu \in C^0(\partial \Omega) \).

If \( m = 1 \), then Theorem 7.2 implies that \( v[\partial \Omega, S_a, \mu]_{|\partial \Omega} \) is linear and continuous from \( C^{m-1}(\partial \Omega) \) to \( C^{m-1,\alpha}(\partial \Omega) \) for all \( \alpha \in (0,1) \).

If \( m > 1 \), then \( C^{m-1}(\partial \Omega) \) is continuously imbedded into \( C^{m-2,\alpha}(\partial \Omega) \), and Theorem 7.1 implies that \( v[\partial \Omega, S_a, \cdot]_{|\partial \Omega} \) is linear and continuous from \( C^{m-2,\alpha}(\partial \Omega) \) to \( C^{m-1,\alpha}(\partial \Omega) \). Hence, \( v[\partial \Omega, S_a, \cdot]_{|\partial \Omega} \) is continuous from \( C^{m-1}(\partial \Omega) \) to \( C^{m-1,\alpha}(\partial \Omega) \) for all \( m > 1 \). Then formula (10.2), and the continuity of the imbedding of \( C^{m-1,\alpha}(\partial \Omega) \) into \( C^{m-1,\omega_1(\partial \Omega)} \), and Theorems 8.15 (i), 9.14 (i) imply the validity of statement (i).

We now consider statement (ii). Since \( v[\partial \Omega, S_a, \cdot]_{|\partial \Omega} \) is continuous from \( C^{m-1,\beta}(\partial \Omega) \) to \( C^{m,\beta}(\partial \Omega) \) and \( C^{m,\beta}(\partial \Omega) \) is continuously imbedded into \( C^{m-1,\alpha}(\partial \Omega) \), then the operator \( v[\partial \Omega, S_a, \cdot]_{|\partial \Omega} \) is continuous from \( C^{m-1,\beta}(\partial \Omega) \) into \( C^{m-1,\alpha}(\partial \Omega) \). Then formula (10.2), and Theorems 8.15 (ii), 9.14 (ii) imply the validity of statement (ii).

Since the space \( C^{m-1,\omega_1(\cdot)}(\partial \Omega) \) is compactly imbedded into \( C^{m-1}(\partial \Omega) \), and \( C^{m-1,\alpha}(\partial \Omega) \) is compactly imbedded into \( C^{m-1,\beta}(\partial \Omega) \) for all \( \beta \in [0,\alpha] \), we have the following immediate consequence of Theorem 10.3 (ii).

**Corollary 10.3** Under the assumptions of Theorem 10.3, \( w[\partial \Omega, S_a, \cdot]_{|\partial \Omega} \) is compact from \( C^{m-1}(\partial \Omega) \) to itself, and from \( C^{m-1,\omega_1(\cdot)}(\partial \Omega) \) to itself, and from \( C^{m-1,\alpha}(\partial \Omega) \) to itself.

**Acknowledgement** This paper represents an extension of the work performed by F. Dondi in his ‘Laurea Magistrale’ dissertation under the guidance of M. Lanza de Cristoforis, and contains the results of [8], [9], [21]. The authors are indebted to M. Dalla Riva for a help in the formulation of Lemma 4.9 and of Theorem 4.1, and of Corollary 4.7 on the fundamental solution and of Theorems 7.1, 7.3 on layer potentials. The authors are indebted to P. Luzzini for a comment which has improved the statement of
Lemma 6.11. The authors also wish to thank D. Natroshvili for pointing out a number of references.

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