Two-colour QCD at non-zero quark-number density.

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Abstract

We have simulated two-colour four-flavour QCD at non-zero chemical potential $\mu$ for quark number. Simulations were performed on $8^4$ and $12^3 \times 24$ lattices. Clear evidence was seen for the formation of a colourless diquark condensate which breaks quark number spontaneously, for $\mu > \mu_c \sim m_\pi/2$. The transition appears to be second order. We have measured the spectrum of scalar and pseudoscalar bosons which shows clear evidence for the expected Goldstone boson. Our results are in qualitative agreement with those from effective Lagrangians for the potential Goldstone excitations of this theory.
I. INTRODUCTION

QCD at finite quark/baryon-number density at zero and at finite temperature describes nuclear matter. Nuclear matter at high temperatures (and possibly densities) was certainly present in the early universe. Neutron stars consist of dense cold nuclear matter. RHIC and the CERN heavy-ion program promise to produce hot nuclear matter in the laboratory. Calculating the properties of high density nuclear matter could predict if and where strange matter could be produced. Any method which can be used to determine the properties of nuclear matter could be adapted to nuclear physics calculations.

Finite quark-number density is best achieved by introducing a chemical potential $\mu$ for quark-number, and using the grand-canonical partition function. Unfortunately, this renders the Euclidean-time fermion determinant complex, with a real part which can change sign. Standard lattice simulations, which rely on importance sampling, fail in this case. Attempting to circumvent these problems by using canonical (fixed quark number) ensembles fail except at high temperatures because of sign problems.

Until a simulation method is found which avoids these difficulties, it is useful to study models which exhibit some of the properties of QCD at high $\mu$. Now it is expected that, at zero temperature, nuclear matter undergoes a phase transition at $\mu$ of order one third the mass of the nucleon. It has been proposed that at still higher $\mu$ the ground state is characterised by a diquark condensate. Such a condensate would not only cause spontaneous breaking of baryon number, but would also spontaneously break colour. Since colour is a gauge symmetry, such breaking is realized in the Higgs mode. Thus nuclear matter would become a colour superconductor at high $\mu$.

For this reason we have simulated 2-colour QCD, i.e. SU(2) Yang-Mills theory with fermion matter fields ('quarks') in the fundamental representation of $SU(2)_{\text{colour}}$, and finite $\mu$. As well as having colour confinement this theory does exhibit diquark condensation as we shall demonstrate in this paper, but for $\mu > \mu_c \sim m_\pi/2$, since the diquark 'baryons' in the same multiplet as the pions, also have mass $m_\pi$. For $\mu \lesssim m_\pi$ the phenomenon is describable.
as a rotation of the condensate from the chiral to the diquark direction as predicted by
effective Lagrangian analyses [7]. Unlike in true (3-colour) QCD, the diquark condensates
are colourless, and the broken symmetry is realized in the Goldstone mode, and there is no
colour superconductivity, but rather superfluidity, as in liquid $^3$He. Despite this, we shall
later argue that this theory is more similar to 2-flavour QCD than one might think (see
section 4).

Since 2-colour QCD has a non-negative determinant and pfaffian, even at non-zero $\mu$,
standard simulation methods can be used. We use the hybrid molecular-dynamics method
and simulate the theory with 4 flavours of staggered quarks [8]. Pfaffian simulations of a
4-fermion model including a diquark source term have been reported in [8]. We have run
simulations at a moderately large quark mass and an intermediate gauge coupling on $8^4$
and $12^3 \times 24$ lattices, i.e. at zero temperature. We measured order parameters including
the chiral and diquark condensates, the quark-number and energy densities, and the Wil-
son Line (Polyakov loop). The larger lattice allowed us to observe finite size effects and,
more importantly to measure the scalar and pseudoscalar meson and diquark masses, which
include all the potential Goldstone bosons in the theory. Preliminary results from these sim-
ulations were reported at Lattice’2000, Bangalore [10]. The extension of these calculations
to finite temperature was reported in a recent letter [11]. This work builds on early work
with 8 quark flavours which presented far less conclusive results [12]. Previous studies of
diquark condensation in this model with various numbers of flavours have either used the
approximation where $\lambda = 0$ in the updating algorithm [13] (as does [12]), or been in the
strong gauge-coupling regime [14].

Section 2 introduces the staggered-fermion lattice port of 2-colour QCD. The results of
our simulation are presented in section 3. Section 4 gives our conclusions and indicates
future avenues of research.
II. LATTICE 2-COLOUR QCD

Because in 2-colour QCD fundamental quarks and antiquarks lie in the same representation of $SU(2)_{\text{colour}}$ the flavour symmetry group for $N_f$ flavours is enlarged from $SU_L(N_f) \times SU_R(N_f) \times U_V(1)$ to $SU(2N_f)$. The pattern of chiral symmetry breaking is $SU(2N_f) \to Sp(2N_f)$ rather than the usual $SU_L(N_f) \times SU_R(N_f) \to SU_V(N_f)$ \cite{15,7}. 3-colour QCD with 1 staggered quark (4 continuum flavours) has a $U_V(1) \times U(1)_{\varepsilon}$ flavour symmetry. For 2 colours this is enhanced to $U(2)$ \cite{12}. Chiral or quark-number symmetry breaking (the chiral and diquark condensates lie in the same $U(2)$ multiplet) occurs according to the pattern $U(2) \to U(1)$.

The quark action for 2-colour QCD with one staggered quark is

$$S_f = \sum_{\text{sites}} \left\{ \bar{\chi} [\mathcal{D}(\mu) + m] \chi + \frac{1}{2} \lambda [\chi^T \tau_2 \chi + \bar{\chi} \tau_2 \bar{\chi}^T] \right\}$$

(1)

where $\mathcal{D}(\mu)$ is the normal staggered covariant finite difference operator with $\mu$ introduced by multiplying the links in the $+t$ direction by $e^\mu$ and those in the $-t$ direction by $e^{-\mu}$. The superscript $T$ stands for transposition. Note that we have introduced a gauge-invariant Majorana mass $\lambda$ which explicitly breaks quark-number symmetry. Such an explicit symmetry breaking term is needed to observe spontaneous symmetry breaking on a finite lattice. We shall be interested in the limit $\lambda \to 0$. Integrating out these fermion fields yields

$$\text{pfaffian} \begin{pmatrix} \lambda \tau_2 & \mathcal{A} \\ -\mathcal{A}^T & \lambda \tau_2 \end{pmatrix} = \sqrt{\det(\mathcal{A}^\dagger \mathcal{A} + \lambda^2)}$$

(2)

where

$$\mathcal{A} \equiv \mathcal{D}(\mu) + m$$

(3)

We note that $\mathcal{A}^\dagger \mathcal{A} + \lambda^2$ is positive definite for finite $\lambda$. Hence the pfaffian never vanishes and thus, by continuity arguments never changes sign and can be chosen to be positive. Note that $\det(\mathcal{A})$ has been shown to be positive in \cite{14}. Denoting the $2 \times 2$ matrix in equation \cite{2} by $\mathcal{M}$ we have seen that its determinant is the determinant of a positive definite matrix and thus can be used directly in our simulations. To do this we define $\tilde{\mathcal{M}}$ by
\[ \tilde{M} = \begin{bmatrix} 1 & 0 \\ 0 & \tau_2 \end{bmatrix} \tilde{M} \begin{bmatrix} \tau_2 & 0 \\ 0 & 1 \end{bmatrix} \] (4)

so that

\[ \tilde{M} = \begin{bmatrix} \lambda & A \\ -A^\dagger & \lambda \end{bmatrix} \] (5)

and

\[ \tilde{M}^\dagger \tilde{M} \rightarrow \begin{bmatrix} A^\dagger A + \lambda^2 & 0 \\ 0 & A A^\dagger + \lambda^2 \end{bmatrix} \] (6)

We note that \( \det[A A^\dagger + \lambda^2] = \det[A^\dagger A + \lambda^2] \). Because we are now dealing with the matrix \( \tilde{M}^\dagger \tilde{M} \) we can use the hybrid molecular dynamics method with ‘noisy’ fermions [8] to simulate this theory. Here, although we generate gaussian noise with both upper and lower components, we keep only the upper components of the pseudo-fermion field to calculate \( \det[A A^\dagger + \lambda^2] \), which means that we only need to invert \( A^\dagger A + \lambda^2 \) at each update. Keeping only half the components of the pseudo-fermion field is entirely analogous to keeping only the fermion fields on even sites in normal QCD simulations. The square root of equation 2 is obtained by inserting a factor of \( \frac{1}{2} \) in front of the fermion term in the stochastic action in the standard manner.

We now give a brief discussion of symmetry breaking in this model. This is covered in more detail in [12]. At \( \mu = m = \lambda = 0 \), the \( U(2) \) symmetry will break spontaneously. Two directions in which it will choose to break are of particular interest. The first is where it breaks to give a non-zero chiral condensate \( \langle \bar{\chi} \chi \rangle \). There will be 3 Goldstone bosons corresponding to the 3 broken generators of \( U(2) \). These states and their corresponding \( U(2) \) generators are

\[ 1 \implies \bar{\chi} \epsilon \chi \]

\[ \sigma_1 \implies \chi^T \tau_2 \chi - \bar{\chi} \tau_2 \bar{\chi}^T \] (7)

\[ \sigma_2 \implies \chi^T \tau_2 \chi + \bar{\chi} \tau_2 \bar{\chi}^T \]
\(\sigma_3\) remains unbroken. The second is where \(U(2)\) breaks to give a non-zero diquark condensate 
\(\frac{1}{2}\langle \chi^T \tau_2 \chi + \bar{\chi}^T \tau_2 \bar{\chi} \rangle\). This time the 3 Goldstone bosons and corresponding generators are
\[
\begin{align*}
1 & \mapsto \chi^T \tau_2 \epsilon \chi + \bar{\chi}^T \tau_2 \epsilon \bar{\chi}^T \\
\sigma_2 & \mapsto \bar{\chi} \chi \\
\sigma_3 & \mapsto \chi^T \tau_2 \chi - \bar{\chi}^T \tau_2 \bar{\chi}^T
\end{align*}
\]
\(\sigma_1\) remains unbroken. When \(\mu \neq 0\) only 2 Goldstone bosons remain, \(\chi^T \tau_2 \epsilon \chi + \bar{\chi}^T \tau_2 \epsilon \bar{\chi}^T\) and 
\(\chi^T \tau_2 \chi - \bar{\chi}^T \tau_2 \bar{\chi}^T\). When in addition \(m \neq 0\), only the latter state remains a Goldstone boson.

A more detailed study of the patterns of symmetry breaking can be performed in terms of an effective Lagrangian for the Goldstone modes as in [7]. The only difference is that the effective field, denoted \(\Sigma\) in that work, here belongs to a symmetric \(2 \times 2\) tensor representation of \(U(2)\) rather than to the antisymmetric \(2N_f \times 2N_f\) representation of \(SU(2N_f)\) of the continuum case.

One can see the remnant \(U(1)\) symmetry when \(\lambda \rightarrow 0\) by allowing \(\lambda\) to become complex. The Majorana mass term in equation [1] then becomes \(\frac{1}{2}[\lambda \chi^T \tau_2 \chi + \lambda^* \bar{\chi}^T \tau_2 \bar{\chi}^T]\). \(\lambda^2\) is replaced by \(|\lambda|^2\) in the pfaffian.

Although the 2-flavour theory would be of more interest in the continuum, we have chosen to simulate the 4-flavour theory because this represents a single staggered quark species and thus has well defined symmetries and a well defined spectrum at all lattice spacings. Unlike the 8-flavour case simulated in [12] it probably does have a sensible continuum limit.

### III. SIMULATIONS OF 2-COLOUR, 4-FLAVOUR LATTICE QCD

We have simulated 2-colour QCD with 1 staggered quark species (4 continuum flavours) on \(8^4\) and \(12^3 \times 24\) lattices. The simulations reported here are all at \(\beta = 4/g^2 = 1.5\) which is roughly the \(\beta_c\) for the finite temperature transition on an \(N_t = 4\) lattice [17]. This first set of simulations has been performed with quark mass \(m = 0.1\) in lattice units. These simulations are currently being repeated at \(m = 0.025\), where the smaller pion mass will
give a richer spectrum of Goldstone and pseudo-Goldstone bosons and where a larger portion of the relevant phase diagram should be described by effective Lagrangians. (We have also performed some zero temperature simulations at $\beta = 1.0$ and $m = 0.05$. This has been reported in our finite temperature/finite $\mu$ letter [11].) Since we wished to take the limit $\lambda \to 0$, we needed $\lambda << m$. The values we chose were $\lambda = 0.01$ and $\lambda = 0.02$ for $m = 0.1$. (At low $\mu$'s we also ran at $\lambda = 0$.)

The smaller lattice was used to map out the interesting range of $\mu$ values, measuring order parameters including the diquark condensate $\langle \chi^T \tau_2 \chi \rangle$, the chiral condensate $\langle \bar{\chi} \chi \rangle = \langle \bar{\psi} \psi \rangle$, and the number density $j_0 = \frac{1}{V} \frac{\partial \ln S_f}{\partial \mu}$. In addition to measuring these quantities on the larger lattice, we also calculated the spectrum of potential Goldstone bosons. The length of each ‘run’ was 2000 molecular dynamics time units. $dt$ had to be chosen as low as 0.005 for $\lambda = 0.01$ and $0.4 < \mu \leq 0.975$. In figure 1 we have plotted the diquark condensate as a function of $\mu$ at each $\lambda$ for both lattice sizes.

Since we are interested in the limit where the symmetry breaking parameter $\lambda \to 0$, we have performed a linear extrapolation of the diquark condensate to $\lambda = 0$. Note that the effective Lagrangian calculations [7] suggest that linear extrapolations are the correct approach for $\lambda$ sufficiently small, except at $\mu_c$. The results of these extrapolations are plotted in figure 2. What we first notice is that for $\mu \leq 0.2$, the extrapolated diquark condensate is small enough that we can believe that it should be zero. For $\mu \geq 0.35$ it is clearly non-zero. The points at $\mu = 0.3$ would appear to show finite size rounding were it not for the fact that the $8^4$ and $12^3 \times 24$ points are so close together. We think it more likely that $\mu = 0.3$ is so close to the transition that the linear extrapolation has broken down. We therefore conclude that the system undergoes a phase transition at $\mu = \mu_c \approx 0.3 < m_\pi/2 = 0.37622(5)$. If this holds true, the fact that $\mu_c$ is less than $m_\pi/2$ would indicate that the diquark ‘baryons’ do not exist as free particles but bind into ‘nuclear’ matter. Over the range $0.35 \leq \mu \leq 0.6$, the condensate increases with a curvature consistent with a critical index $\beta_{qq} < 1$ (the tree level effective Lagrangian analysis predicts the mean field result $\beta_{qq} = \frac{1}{2}$). Since the condensate starts to decrease soon after $\mu = 0.6$, the scaling region is narrow and it would require
FIG. 1. $\langle \chi^T \tau_2 \chi \rangle$ as a function of $\mu$: a) on an $8^4$ lattice and b) on a $12^3 \times 24$ lattice. The arrow is at $\mu = m_\pi$. 
more points to even try to extract this critical index. For $\mu \gtrsim m_\pi$, the condensate starts to decrease, approaching zero for large $\mu$, which would appear to be a saturation effect. In fact figure 2 suggests that the condensate vanishes for $\mu > \mu_s \approx 1$, indicating a saturation phase transition. We shall have more to say about this later. Note that this decrease in the condensate with $\mu$ is not predicted by the effective Lagrangian analysis, which is only...
expected to be valid for small $\mu$ and $m_\pi$. This is not surprising if it is indeed a saturation effect. Saturation is a result of the fermi statistics of the quarks, and should not be seen in a model which only considers the system’s bosonic excitations.

We now turn to a consideration of the chiral condensate, $\langle \bar{\chi} \chi \rangle = \langle \bar{\psi} \psi \rangle$. This is plotted in figure 3 for our 2 different lattice sizes. In the $\lambda \to 0$ limit, this is expected to be constant at its $\mu = 0$ value for $\mu < \mu_c$. These plots are consistent with this expectation. Above $\mu_c$, effective Lagrangian studies predict that the condensate merely rotates from the chiral direction to the diquark direction, so that the magnitude of the condensate, i.e. $\sqrt{\langle \bar{\chi} \chi \rangle^2 + \langle \chi^T \tau_2 \chi \rangle^2}$, should remain constant and independent of $\lambda$. Since the diquark condensate increases up to $\mu \sim m_\pi$, this means that the chiral condensate should fall, as it does. It, however, continues to fall past this point, because of saturation effects, appearing to vanish for $\mu > \mu_s$. To test the prediction that the principal effect for $\mu < m_\pi$ is merely a rotation, we have tabulated $\sqrt{\langle \bar{\chi} \chi \rangle^2 + \langle \chi^T \tau_2 \chi \rangle^2}$ in table 1. We see that, for $\mu \leq 0.6$, the magnitude of the condensate is approximately constant and independent of $\lambda$, so that the main effect is a rotation of the condensate from the direction of chiral symmetry breaking to that of quark-number breaking. For $\mu \geq 0.8$, it decreases with increasing $\mu$ approaching zero for large $\mu$.

In figure 3 we plot the quark-number density $j_0 = \frac{1}{V} \frac{\partial \ln S_f}{\partial \mu}$ as a function of $\mu$ for our 2 lattices. $j_0$ is consistent with zero as $\lambda \to 0$, for $\mu < \mu_c$. (Note effective Lagrangians predict that it vanishes quadratically with $\lambda$ in this region.) Above $\mu_c$ it increases with increasing $\mu$ approaching the saturation value of 2 (1 staggered quark field of each colour/site, the maximum value allowed by fermi statistics) for large $\mu$. Note that figure 4 suggests that $j_0 = 2$ for $\mu > \mu_s$ with $\mu_s$ consistent with that predicted from the condensates. This indicates that it is this saturation which causes the condensates to vanish for $\mu > \mu_s$. It also tells us that this transition is an artifact of the finite lattice spacing and would recede to infinite $\mu$ in the continuum limit.

On the $12^3 \times 24$ lattice we have measured the propagators for all the potential Goldstone bosons, both scalar and pseudoscalar using a single noisy point source for the connected
FIG. 3. $\langle \bar{\psi} \psi \rangle$ as a function of $\mu$: a) on an $8^4$ lattice and b) on a $12^3 \times 24$ lattice.
TABLE I. Magnitude of the condensate $\sqrt{\langle \bar{\chi}\chi \rangle^2 + \langle \chi^T \tau_2 \chi \rangle^2}$ as functions of $\mu$ and $\lambda$ on a $12^2 \times 24$ lattice. Errors are not quoted but they are in the next or a subsequent digit after the least significant quoted digit.

| $\mu$  | $\lambda = 0.01$  | $\lambda = 0.02$  |
|--------|------------------|------------------|
| 0.000  | 0.835            | 0.836            |
| 0.200  | 0.834            | 0.836            |
| 0.300  | 0.835            | 0.839            |
| 0.350  | 0.841            | 0.840            |
| 0.400  | 0.845            | 0.846            |
| 0.500  | 0.850            | 0.856            |
| 0.600  | 0.856            | 0.860            |
| 0.800  | 0.792            | 0.799            |
| 0.900  | 0.657            | 0.667            |
| 0.950  | 0.503            | 0.534            |
| 0.975  | 0.379            | 0.433            |
| 1.000  | 0.212            | 0.311            |
| 1.200  | 0.019            | 0.038            |

propagators and multiple (5) noisy point sources for the disconnected propagators. We have measured both diagonal and off-diagonal zero momentum propagators.

In figure 5 we show the masses from fitting the propagator for the diquark state produced by applying the operator $\chi^T \tau_2 \chi - \bar{\chi} \tau_2 \bar{\chi}^T$ to the vacuum, to the form

$$P_G(T) = A \left\{ \exp[-m_G T] + \exp[-m_G (N_t - T)] \right\},$$

valid for large temporal separations $T$. As argued in section 2, this is the expected Goldstone boson for $\mu > \mu_c$ and $\lambda = 0$. The effective Lagrangian approach indicates that, in the low $\mu$ phase, this mass should be quadratic in $\lambda$, while in the high $\mu$ phase, it should vanish as $\sqrt{\lambda}$. 

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FIG. 4. Quark-number density as a function of $\mu$: a) on an $8^4$ lattice and b) on a $12^3 \times 24$ lattice.

SU(2) $N_t=4$ $\beta=1.5$ $m=0.1$ $8^4$ lattice

(a)

SU(2) $N_t=4$ $\beta=1.5$ $m=0.1$ $12^3 \times 24$ lattice

(b)
FIG. 5. Mass of the diquark state expected to become a Goldstone boson in the diquark condensed phase, as a function of $\mu$ for $\lambda = 0.1$ and $\lambda = 0.2$. The line is the expected linear behaviour expected for the symmetric phase. Also included are the point at $\lambda = 0, \mu = 0$ and $\lambda \to 0$ extrapolations.

We have performed extrapolations based on this and plotted them in the figure. For the 3 points closest to the transition $\mu = 0.3, 0.35, 0.4$, we have performed both extrapolations.
For $\mu < m_\pi/2$ and $\lambda = 0$, we expect

$$m_G = m_\pi - 2\mu$$

with $m_\pi = m_\pi(\mu = 0)$. We see that for $\mu \leq 0.3$, the points obtained from quadratic extrapolation lie on this straight line, the point at $\mu = 0.35$ is close to the line and that for $\mu = 0.4$ lies above this line. All points for $0.35 \leq \mu \leq 1.0$ obtained by square root extrapolation are negative. The points at $\mu = 0.35, 0.4$ show the most significant negative values (although both are greater than $-0.05$), and should be considered as transitional. The other points are so close to zero that one can easily attribute the difference as being due to a combination of the systematic errors in extracting masses from point source propagators and higher order terms in the extrapolation. Thus our results are consistent with a transition to a phase with a massless (Goldstone) scalar diquark at $\mu \approx m_\pi/2 = 0.37622(5)$, as expected.

Finally the large value of this diquark mass at $\mu = 1.2$ is further indication that the system has passed through the saturation phase transition.

We now turn to consideration of the other potential Goldstone bosons of the theory. Since at this quark mass, $m_\pi^2$ is relatively large, we concentrate on those states which are expected to have masses of order $m_\pi$ or less over the range of interest. For this reason, we consider the pion itself. For $\mu < \mu_c$, its mass should remain constant at $m_\pi$, and it is created by the operator $\bar{\chi}\epsilon \chi$. For $\mu > \mu_c$, it is expected to mix with the pseudoscalar diquark state created from the vacuum by the operator $\chi^T \tau_2 \epsilon \chi + \bar{\chi} \tau_2 \epsilon \bar{\chi}^T$. Rather than trying to diagonalize the propagator over these 2 states, we have instead calculated the diagonal propagators for each of these states separately. These we fit to the form

$$P_\pi(T) = A \{ \exp[-m_\pi T] + \exp[-m_\pi (N_t - T)] \} + B (-1)^T \{ \exp[-m_{\psi 1} T] + \exp[-m_{\psi 1} (N_t - T)] \}$$

or the form with $B = 0$, and a similar form for the diquark state. These masses are plotted in figure 3. Note that these 2 graphs are almost identical. For $\mu > \mu_c$, this is expected, since the 2 states mix. For $\mu < \mu_c$ this is not, a priori, expected since the diquark state should
FIG. 6. Pion (a) and pseudoscalar diquark (b) masses as functions of $\mu$. 

$\text{SU}(2) \ N_f=4 \ \beta=1.5 \ m=0.1$
exhibit a linear decrease with $\mu$ for $\lambda = 0$. However, since $\lambda \neq 0$ this gives a small mixing with the pion state, which is then the lowest mass particle contributing to this pseudoscalar diquark propagator in this region. It is precisely because this mixing is so small that the errors are so large. The final indicator that this is the correct interpretation was that the $\mu = 0, \lambda = 0$ propagator yielded no sensible mass fit, a sure indicator that the mass was large where there can be no mixing. The pion mass fits in this low $\mu$ domain are consistent with the expectation that the $\lambda = 0$ pion mass is independent of $\mu$ in this region. For $\mu > \mu_c$ these masses fall faster than the expected $m_\pi^2/2\mu$. At least part of the reason for this more rapid falloff is the saturation effect. Note that the reason that this mass tends to zero at large $\mu$ (before the saturation transition) is because the pseudoscalar diquark would be a Goldstone boson if $m = 0$, and as $\mu$ increases the relative importance of $m$ diminishes. The fact that the $\pi$ mass becomes more poorly determined for large $\mu$ is because here, the lowest mass state in this channel is predominantly the pseudoscalar diquark.

It is interesting to contrast our findings with those of a study of adjoint quarks in 2-colour QCD \cite{16} with $\lambda$ set to zero, implying strictly no mixing; in this case $m_\pi$ was found to rise as $m_\pi \approx 2\mu$ for $\mu > \mu_c$, with the signal becoming appreciably noisier in the high density phase. Both behaviours are in accord with the predictions of chiral perturbation theory for a meson formed from quarks with a symmetric combination of quantum numbers under the residual global symmetry, the difference arising due to the distinct Dyson indices of each model \cite{7}.

IV. CONCLUSIONS

Two-colour lattice QCD with one staggered quark species (4 continuum flavours) has been studied at finite chemical potential $\mu$ for quark number. We have shown conclusive evidence that this theory undergoes a phase transition to a phase characterized by a diquark condensate which spontaneously breaks quark-number, at $\mu = \mu_c \sim m_\pi/2$. This transition appears to be second order. The simulations were performed at an intermediate value of the
coupling (close to $\beta_c$ for the finite temperature transition at $\mu = 0$ on a lattice with $N_t = 4$). Because of this, the relevant symmetry was the lattice flavour symmetry $U(2)$ rather than the $SU(8)$ of the continuum 4-flavour theory. We have presented convincing evidence that the scalar diquark is the Goldstone boson associated with the spontaneous breaking of quark number for $\mu > \mu_c$. In addition we have measured the lightest mass in the pion channel as a function of $\mu$. For $\mu \lesssim m_\pi$ the chiral condensate and diquark condensate are well described as a single condensate of constant magnitude which rotates from the chiral direction for $\mu < \mu_c$ towards the diquark direction as $\mu$ is increased above $\mu_c$. Despite the fact that the quark mass $m = 0.1$ was large enough that $m_\pi$ was not small relative to the 2-colour QCD scale, we saw good qualitative agreement with previous calculations in terms of effective Lagrangians (of the chiral perturbation theory form) for $\mu \lesssim m_\pi$. This includes the fact that the quark number density increases from zero at $\mu = \mu_c$.

As $\mu$ becomes large, this relationship with effective Lagrangians is no longer valid. Both chiral and diquark condensates decrease towards zero as $\mu$ becomes large. However, because at least part of this is due to the fact that $j_0$ saturates at 2 fermions/site, a finite lattice spacing artifact, and because we see large finite size effects in the condensates for these $\mu$ values, studies at smaller lattice spacings as well as on larger lattices would be needed to determine how much of this observed high $\mu$ behaviour is real.

We are currently extending these calculations to lower quark mass ($m = 0.025$) where we should be able to measure the complete spectrum of Goldstone and pseudo-Goldstone bosons. For this $m$, the pion mass should be half that at $m = 0.1$ and the assumptions of the effective Lagrangian approach should have more validity, allowing a more quantitative analysis. In addition, there is a larger range of $\mu/m_\pi$ between $\mu_c$ and the turnover point. This means we can hope to measure the critical index $\beta_{qq}$ for this transition. (Preliminary attempts to extract this index from short runs at relatively strong coupling were reported in our letter on finite $T$ and $\mu$.)

Let us now compare 2-colour QCD with 3-colour QCD, at finite $\mu$. Most of this discussion is condensed from the published literature [4–6]. Since for 2-colour QCD, the diquark
condensate is a colour singlet, the spontaneous breaking of quark number is realized in the Goldstone mode. This contrasts with true (3-colour) QCD where the condensate is, of necessity, coloured and the symmetry breaking is realized in the Higgs mode. To see similarities, we consider the mode of symmetry-breaking for normal QCD with 2 light quark flavours. Here the expected condensate is a flavour singlet and a colour anti-triplet, i.e. antisymmetric in both flavour and colour. The pattern of colour breaking is \( SU(3) \times U(1) \rightarrow SU(2) \times U(1) \), where \( q \) and \( Q \) refer to quark number before and after the breaking. The 5 gluons corresponding to symmetries broken by the condensate, gain masses via the Higgs mechanism by combining with the 5 would-be Goldstone bosons associated with colour/quark-number breaking. With respect to the remnant \( SU(2) \) colour symmetry, the condensate is a flavour and colour singlet. This is precisely the condensate that would be formed from the 2 quarks which are in a colour doublet with respect to this unbroken \( SU(2) \) (the third quark is a singlet with respect to this group) if we ignored the interactions of the gluons which gain masses due to the Higgs mechanism. Since the gluon masses produced from the Goldstone bosons produced in this manner are of order \( \alpha_s^{1/2}(\Lambda_{QCD}) f_\pi \), ignoring such light particle interactions is presumably a rather drastic approximation. However, one might hope that it has at least qualitative validity. (Note that \( f_\pi \) is of the order of 100 MeV, \( \alpha_s(\Lambda_{QCD}) \sim 1 \), giving gluon masses of the same order of magnitude as previous estimates.) Since this remnant \( SU(2) \) theory is also at finite \( \mu \) this condensate spontaneously breaks only 1 symmetry, quark number. As seen above, the Goldstone boson associated with this breaking gives mass to one of the gluons via the Higgs mechanism.

Finally, we note that with an appropriate reinterpretation of the fields, 2-colour QCD with a finite chemical potential for quark number can be reinterpreted as 2-colour QCD with a finite chemical potential for isospin. It is clear that one can simulate true 3-colour QCD for 2 flavours with a finite chemical potential for isospin. Such a program is underway. The 2-colour theory can be used as a guide to the pattern of symmetry breaking and what to expect.
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