Formal groups over Hopf algebras

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1 The formal group, connected with FBSP

The aim of this section is to define some generalization of the notion of formal group. More precisely, we consider the analog of formal groups with coefficients belonging to a Hopf algebra. We also study some example of a formal group over a Hopf algebra, which generalizes the formal group of geometric cobordisms.

Recently some important connections between the Landweber-Novikov algebra and the formal group of geometric cobordisms were established (1).

Let \((H, \mu, \eta, \Delta, \varepsilon, S)\) be a (topological) Hopf algebra over ring \(R\) (where \(\mu: H \otimes H \to H\) is the multiplication, \(\eta: R \to H\) is the unit, \(\Delta: H \to H \otimes H\) is the diagonal (comultiplication), \(\varepsilon: H \to R\) is the counit, and \(S: H \to H\) is the antipode).

Definition 1. A formal series \(F(x \otimes 1, 1 \otimes x) \in H \otimes H[[x \otimes 1, 1 \otimes x]]\) is called a formal group over the Hopf algebra \((H, \mu, \eta, \Delta, \varepsilon, S)\) if the following conditions hold:

1) (associativity)

\[
((\text{id}_H \otimes \Delta)F)(x \otimes 1 \otimes 1, 1 \otimes F(x \otimes 1, 1 \otimes x)) =
((\Delta \otimes \text{id}_H)F)(F(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x);
\]

2) (unit)

\[
((\text{id}_H \otimes \varepsilon)F)(x \otimes 1, 0) = x \otimes 1,
((\varepsilon \otimes \text{id}_H)F)(0, 1 \otimes x) = 1 \otimes x;
\]
3) (inverse element) there exists the series $\Theta(x) \in H[[x]]$ such that

$$((\mu \circ (\text{id}_H \otimes S))\mathcal{F})(x, \Theta(x)) = 0 = ((\mu \circ (S \otimes \text{id}_H))\mathcal{F})(\Theta(x), x).$$

If for a formal group $\mathcal{F}(x \otimes 1, 1 \otimes x)$ over a commutative and cocommutative Hopf algebra $H$ the equality $\mathcal{F}(x \otimes 1, 1 \otimes x) = \mathcal{F}(1 \otimes x, x \otimes 1)$ holds, then it is called commutative. Below we shall deal only with the commutative case.

**Remark 2.** Note that a formal group $\mathcal{F}(x \otimes 1, 1 \otimes x)$ over Hopf algebra $H$ over ring $R$ defines the formal group (in the usual sense) $\mathcal{F}(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]]$ such that $\mathcal{F}(\Theta(x), x) = 0 = (\Theta(x), x)$.

**Remark 3.** By definition, put $\tilde{\Delta}(x) = \mathcal{F}(x \otimes 1, 1 \otimes x)$, $\tilde{\varepsilon}(x) = 0$, $\tilde{S}(x) = \Theta(x)$ and $\tilde{\Delta} \mid H = \Delta$, $\tilde{\varepsilon} \mid H = \varepsilon$, $\tilde{S} \mid H = S$. We claim that $(H[[x]], \tilde{\mu}, \tilde{\eta}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{S})$ is the Hopf algebra (here $\tilde{\mu}, \tilde{\eta}$ are evidently extensions of $\mu$, $\eta$). Indeed, the
The commutativity of the diagram

\[
\begin{array}{ccc}
H[[x]] & \xrightarrow{\tilde{\Delta}} & H[[x]] \hat{\otimes} H[[x]] \\
\tilde{\Delta} \downarrow & & \downarrow \text{id}_{H[[x]]} \otimes \tilde{\Delta} \\
H[[x]] \hat{\otimes} H[[x]] & \xrightarrow{\tilde{\Delta} \otimes \text{id}_{H[[x]]}} & H[[x]] \hat{\otimes} H[[x]] \hat{\otimes} H[[x]]
\end{array}
\]  

(2)

follows from the equations

\[
((\text{id}_{H[[x]]} \otimes \tilde{\Delta})(\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x)) = ((\text{id}_{H} \otimes \Delta)(\tilde{\mathcal{F}})(x \otimes 1 \otimes 1, 1 \otimes \tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x)) =
\]

\[
((\Delta \otimes \text{id}_{H})\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x) = (\tilde{\Delta} \otimes \text{id}_{H[[x]]})(\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x)).
\]

The commutativity of the diagram

\[
\begin{array}{ccc}
R \otimes H[[x]] & \xleftarrow{\tilde{\mathcal{F}} \otimes \text{id}_{H[[x]]}} & H[[x]] \hat{\otimes} H[[x]] \\
\cong & \Delta & \cong \\
H[[x]] & \xrightarrow{\Delta} & H[[x]] \hat{\otimes} R
\end{array}
\]  

follows from the equations

\[
((\text{id}_{H[[x]]} \otimes \tilde{\varepsilon}) \circ \tilde{\Delta})(x) = ((\text{id}_{H} \otimes \varepsilon)(\tilde{\mathcal{F}})(x \otimes 1, 1 \otimes \tilde{\varepsilon}(x)) = x \otimes 1,
\]

\[
((\tilde{\varepsilon} \otimes \text{id}_{H[[x]]}) \circ \tilde{\Delta})(x) = ((\varepsilon \otimes \text{id}_{H})(\tilde{\mathcal{F}})(\tilde{\varepsilon}(x) \otimes 1, 1 \otimes x) = 1 \otimes x.
\]

The axiom of antipode

\[
(\tilde{\mu} \circ (\text{id}_{H[[x]]} \otimes \tilde{\mathcal{S}}) \circ \tilde{\Delta})(x) = (\tilde{\mu} \circ (\tilde{\mathcal{S}} \otimes \text{id}_{H[[x]]}) \circ \tilde{\Delta})(x) = (\tilde{\eta} \circ \tilde{\varepsilon})(x) = 0
\]

follows from the condition 3) of Definition [1].

**Remark 4.** We may rewrite the conditions 1), 2), 3) of Definition [1] in terms of series \(\tilde{\mathcal{F}}(x \otimes 1, 1 \otimes x)\) in the next way. Let

\[
\sum_{i,j \geq 0} A_{i,j} x^i \otimes x^j =
\]

\[
\sum_{i,j \geq 0} (\sum_{k} a_{i,j}^k \otimes b_{i,j}^k) x^i \otimes x^j \in H \hat{\otimes} H[[x \otimes 1, 1 \otimes x]]
\]

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be the series $\tilde{F}(x \otimes 1, 1 \otimes x)$. Then the condition 1) is equivalent to the following equality:

$$
\sum_{i,j \geq 0} \left( \sum_k a_{i,j}^k \otimes \Delta(b_{i,j}^k) \right) x^i \otimes \tilde{F}(x \otimes 1, 1 \otimes x)^j = \\
\sum_{i,j \geq 0} \left( \sum_k \Delta(a_{i,j}^k) \otimes b_{i,j}^k \right) \tilde{F}(x \otimes 1, 1 \otimes x)^i \otimes x^j
$$

The condition 2) is equivalent to

$$
\sum_k a_{i,0}^k \varepsilon(b_{i,0}^k) = 0, \quad \text{if} \quad i \neq 1, \quad \sum_k a_{1,0}^k \varepsilon(b_{1,0}^k) = 1,
$$

$$
\sum_k \varepsilon(a_{0,j}^k) b_{0,j}^k = 0, \quad \text{if} \quad j \neq 1, \quad \sum_k \varepsilon(a_{0,1}^k) b_{0,1}^k = 1.
$$

The condition 3) also may be rewritten in terms of series.

Let us consider some examples of defined objects.

**Example 5.** (Trivial extension) Let $F(x \otimes 1, 1 \otimes x)$ be a formal group (in the usual sense) over a ring $R$, and $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra over the same ring $R$. Then $\tilde{F}(x \otimes 1, 1 \otimes x) = ((\eta \otimes \eta)F)(x \otimes 1, 1 \otimes x) \in H \hat{} \otimes H[[x \otimes 1, 1 \otimes x]]$ is the formal group over the Hopf algebra $H$ (recall that we identify $R \otimes R$ and $R$).

**Example 6.** Now we construct a nontrivial extension $\tilde{F}(x \otimes 1, 1 \otimes x)$ of the formal group of geometric cobordisms $F(x \otimes 1, 1 \otimes x) \in \Omega^*_U(pt)[[x \otimes 1, 1 \otimes x]]$ by the Hopf algebra $\Omega^*_U(Gr)$. For this let us consider the map $\hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} \xrightarrow{\hat{\phi}_{kl,mn}} \hat{Gr}_{km,klmn}$, $(km, ln) = 1$. By $x |_{km,ln}$ denote the cobordism’s class in $\Omega^2_U(\hat{Gr}_{km,klmn})$ such that its restriction to every fiber of the bundle

$\mathbb{C}P^{km-1} \hookrightarrow \hat{Gr}_{km,klmn} \xrightarrow{\pi} Gr_{km,klmn}$

is the standard generator in $\Omega^2_U(\mathbb{C}P^{km-1})$. Let $x |_{k,l}$ and $x |_{m,n}$ be analogously elements in $\Omega^2_U(\hat{Gr}_{k,l})$ and $\Omega^2_U(\hat{Gr}_{m,mn})$ respectively. Then we obtain that

$$
\hat{\phi}_{kl,mn}^*(x |_{km,ln}) = \sum_{0 \leq i \leq k-1 \atop 0 \leq j \leq m-1} A_{i,j} |_{kl,mn} (x |_{k,l})^i \otimes (x |_{m,n})^j,
$$

where
where $A_{i,j} |_{kl, mn} \in \Omega^2_{U} (Gr_{k,kl} \times Gr_{m,mn})$. Applying the functor of unitary cobordisms to the following injective system of the spaces and their maps

$$\begin{align*}
\hat{Gr}_{p,pq} \times \hat{Gr}_{t,tu} &\xrightarrow{\phi_{pq,tu}} \hat{Gr}_{pt,pqtu} \\
\uparrow & \uparrow \\
\hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} &\xrightarrow{\phi_{kl,mn}} \hat{Gr}_{km,klmn}.
\end{align*}$$

(4)

(under the conditions $k \mid p$, $l \mid q$, $m \mid t$, $n \mid u$, and $(pt, qu) = 1$), we obtain the formal series

$$\mathfrak{F}(x \otimes 1, 1 \otimes x) = \sum_{i,j \geq 0} A_{i,j} x^i \otimes x^j \in \Omega_{U}^* (Gr) \otimes_{\Omega^*_{U}(pt)} \Omega_{U}^* (Gr) [[x \otimes 1, 1 \otimes x]]$$

such that $i_{kl}^* A_{i,j} = A_{i,j} |_{kl}$ for injection $i_{kl}: Gr_{k,kl} \hookrightarrow Gr$ (for every pair $\{k, l\}$ such that $(k, l) = 1$).

By $R$ and $H$ denote the ring $\Omega^*_{U}(pt)$ and the Hopf algebra $\Omega^*_{U}(Gr)$ (over the ring $\Omega^*_{U}(pt)$) respectively (recall that we consider the space $Gr$ with the $H$-group structure, induced by the multiplication of FBSP).

**Proposition 7.** The series $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ is the formal group over the Hopf algebra $H$.

**Proof.** To prove $((id_H \otimes \Delta) \mathfrak{F})(x \otimes 1 \otimes 1, 1 \otimes \mathfrak{F}(x \otimes 1, 1 \otimes x)) = ((\Delta \otimes id_H) \mathfrak{F})(x \otimes 1, 1 \otimes x \otimes 1, 1 \otimes 1 \otimes x)$, we need the following commutative diagram $((kmt, lnu) = 1)$:

$$\begin{align*}
\hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} \times \hat{Gr}_{t,tu} &\xrightarrow{\phi_{kl,mn}} \hat{Gr}_{km,klmn} \\
\downarrow & \downarrow \\
\hat{Gr}_{km,klmn} \times \hat{Gr}_{t,tu} &\xrightarrow{\phi_{kl,mn}} \hat{Gr}_{km,klmn}.
\end{align*}$$

(5)

To prove $((id_H \otimes \varepsilon) \mathfrak{F})(x \otimes 1, 0) = x \otimes 1$, we need the following commutative diagram $((km, ln) = 1)$:

$$\begin{align*}
\hat{Gr}_{k,kl} \times \hat{Gr}_{m,mn} &\xrightarrow{} \hat{Gr}_{km,klmn} \\
\uparrow & \uparrow \\
\hat{Gr}_{k,kl} \times CP^{m-1} &\xleftarrow{} \hat{Gr}_{k,kl} \times \{pt\}.
\end{align*}$$

(6)

where right-hand vertical arrow is the standard inclusion.

To prove $((\mu \circ (id_H \otimes S)) \mathfrak{F})(x, \Theta(x)) = 0$, let us construct the fiber map $\hat{\nu}: \hat{Gr} \rightarrow \hat{Gr}$ such that the following two conditions are satisfied:
1) the restriction of $\hat{\nu}$ to any fiber ($\cong \mathbb{C}P^\infty$) is the inversion in the $H$-group $\mathbb{C}P^\infty$;

2) $\hat{\nu}$ covers the $\nu$: $Gr \to Gr$ (where $\nu$ is the inversion in the $H$-group $Gr$).

Let us remember that $\mathcal{P}^{k-1} \times Q^{l-1}$ is the canonical FBSP over $Gr_{k,kl}$ and we have denoted by $\hat{Gr}_{k,kl}$ the bundle space $\mathcal{P}^{k-1}$. Let $\hat{Gr}_{k,kl}$ (or $\hat{Gr}'$) be the bundle space of the “second half” $Q^{l-1}$ of the canonical FBSP over $Gr_{k,kl}$ (lim $(k,l) = 1 \hat{Gr}_{k,kl}$ respectively).

First note that there exists the fiber isomorphism $\hat{\nu}'_{k,l}: \hat{Gr}_{k,kl} \to \hat{Gr}'_{l,lk}$ that covers the inverse map $\nu_{k,l}: Gr_{k,kl} \to Gr'_{l,lk}$ (in other words, the map $\nu_{k,l}$ takes each subalgebra $A_k \cong M_k(\mathbb{C})$ in the $M_{kl}(\mathbb{C})$ to its centralizer $Z_{M_{kl}(\mathbb{C})}(A_k) \cong M_l(\mathbb{C})$ in the $M_{kl}(\mathbb{C})$). Let $c_{l,k}: \hat{Gr}'_{l,lk} \to \hat{Gr}_{l,lk}$ be the fiber map such that the following two conditions are satisfied:

1) $c_{l,k}$ covers the identity mapping of the base $Gr_{l,lk}$;

2) the restriction of $c_{l,k}$ to any fiber $\cong \mathbb{C}P^{k-1}$ is the complex conjugation.

Let $\hat{\nu}_{k,l}: \hat{Gr}_{k,kl} \to \hat{Gr}'_{l,lk}$ be the composition $c_{l,k} \circ \hat{\nu}'_{k,l}$. It is easy to prove that the map $\hat{\nu} = \lim_{(k,l) = 1} \hat{\nu}_{k,l}: \lim_{(k,l) = 1} \hat{Gr}_{k,kl} \to \lim_{(l,k) = 1} \hat{Gr}'_{l,lk}$ is required. In particular, there exists the fiber isomorphism between $\hat{Gr}$ and $\hat{Gr}'$.

The map $\hat{\nu}$ defines (by the same way, as $\hat{\phi}$ in the beginning of the example) the formal series $\Theta(x) \in H[[x]]$ (note that $\epsilon(\Theta)(x) = \theta(x)$, where $\theta(x) \in R[[x]]$ is the inverse element in the group of geometric cobordisms).

Now we claim that $((\mu \circ (id_H \otimes S)) 3)(x, \Theta(x)) = 0$. Indeed, this follows from the next commutative diagram:

$$
\begin{align*}
\begin{array}{c}
\hat{Gr} \xrightarrow{\text{diag}} \hat{Gr} \times \hat{Gr} \\
\downarrow \\
Gr \xrightarrow{\text{diag}} Gr \times Gr
\end{array} \\
\begin{array}{c}
\xrightarrow{id \times \hat{\nu}} \hat{Gr} \times \hat{Gr} \\
\xrightarrow{\hat{\phi}} \hat{Gr}
\end{array} \\
\begin{array}{c}
\downarrow \\
\xrightarrow{id \times \nu} Gr \times Gr \\
\xrightarrow{\phi} Gr
\end{array} \\
\end{align*}
(7)

(we see that the composition $\hat{\phi} \circ (id \times \hat{\nu}) \circ \text{diag}$ is homotopic (in class of fiber homotopies) to the map $\hat{Gr} \to \text{pt} \in \hat{Gr}$). □
Let $\kappa: \widetilde{Gr} \to \mathbb{C}P^\infty$ be the direct limit of the fiber maps

$$\widetilde{Gr}_{k,kl} \xrightarrow{\kappa_{k,l}} \mathbb{C}P^{kl-1} \quad \text{(8)}$$

It defines (in the same way, as $\hat{\phi}$ and $\hat{\nu}$ above) the formal series

$$\mathfrak{G}(x, y) = \sum_{i,j \geq 0} B_{i,j} x^i y^j \in H[[x, y]].$$

**Proposition 8.** $\mathfrak{G}(x, y) = ((\mu \circ (\text{id}_H \otimes S)) \hat{\mathfrak{g}})(x, y)$, i.e.

$$B_{i,j} = \sum_k a_{i,j}^k S(b_{i,j}^k).$$

**Proof.** Recall that in the proof of Proposition 7 the fiber maps $\hat{\nu}'_{k,l}: \hat{\mathcal{G}}_{k,kl} \to \hat{\mathcal{G}}_{l,kl}$ were defined. By $\hat{\nu}'$ denote the direct limit $\lim_{(k,l)=1} \hat{\nu}'_{k,l}: \hat{\mathcal{G}} \to \hat{\mathcal{G}} \to \hat{\mathcal{G}}$. Note that $\hat{\nu}'$ covers the inversion $\nu: \mathcal{G} \to \mathcal{G}$ in the $H$-group $\mathcal{G}$.

Now the proof follows from the next composition of the bundle maps:

$$\begin{align*}
\widetilde{Gr} &\xrightarrow{\phi} \widetilde{Gr} \\
\hat{\mathcal{G}}_{k,kl} &\xrightarrow{\kappa_{k,l}} \mathbb{C}P^{kl-1} \\
\mathcal{G} &\xrightarrow{\nu} \mathcal{G} \xrightarrow{\phi} \mathcal{G}.
\end{align*}$$

We see that the upper composition in fact is the map $\widetilde{Gr} \to \mathbb{C}P^\infty$ and it coincides with the map $\kappa$. Let $y$ be $\hat{\nu}'^*(x)$. The upper composition gives $x \mapsto \mathfrak{F}(x \otimes 1, 1 \otimes y) \mapsto ((\mu \circ (\text{id}_H \otimes S)) \mathfrak{g}')(x \otimes 1, 1 \otimes y)$. Without loss of sense we may write $x$ and $y$ instead of $x \otimes 1$ and $1 \otimes y$ respectively.

The series $\mathfrak{G}(x, y)$ has the following interesting property.

**Proposition 9.**

$$(\Delta \mathfrak{G})(\mathfrak{F}(x \otimes 1, 1 \otimes x), ((S \otimes S) \mathfrak{g}')(y \otimes 1, 1 \otimes y)) =$$

$$F(\mathfrak{G}(x, y) \otimes 1, 1 \otimes \mathfrak{G}(x, y)),$$

where $F(x, y) \in R[[x, y]]$ is the formal group of geometric cobordisms.
Proof. We give two variants of the proof.

1). "Topological proof" follows from the commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}P^{kl-1} \times \mathbb{C}P^{mn-1} & \to & \mathbb{C}P^{klmn-1} \\
\uparrow & & \uparrow \\
\tilde{Gr}_{k,kl} \times \tilde{Gr}_{m,mn} & \to & \tilde{Gr}_{km,klmn}
\end{array}
\]

(10)

[((km, ln) = 1) combining with the decomposition of the map \(\kappa\), which was obtained in previous proof.

2). By \(\tilde{S}'\) denote the homomorphism \(\tilde{\nu}^*: H[[x]] \to H[[y]]\) (recall that \(\tilde{\nu}^*|_H = S: H \to H\), where \(S\) is the antipode). Let us consider the following composition of homomorphisms of Hopf algebras:

\[
H[[x]] \xrightarrow{\tilde{\Delta}} H[[x]] \hat{\otimes} H[[x]] \xrightarrow{id \otimes \tilde{S}'} H[[x]] \hat{\otimes} H[[y]] \xrightarrow{(\mu)} H[[x, y]],
\]

where \((\mu)\) is the homomorphism, induced by multiplication

\[
\mu: H \hat{\otimes} H \to H.
\]

It follows from the axiom of antipode

\[
\mu \circ (id_H \otimes S) \circ \Delta = \eta \circ \varepsilon
\]

that

\[
(\mu) \circ (id_{H[[x]]} \otimes \tilde{S}') \circ \tilde{\Delta} |_H = \eta \circ \varepsilon.
\]

Hence there exists the homomorphism of Hopf algebras

\[
(\eta): R[[x]] \to H[[x, y]]
\]

such that the following diagram

\[
\begin{array}{ccc}
H[[x]] & \xrightarrow{\tilde{\Delta}} & H[[x]] \hat{\otimes} H[[x]] \xrightarrow{id \otimes \tilde{S}'} H[[x]] \hat{\otimes} H[[y]] \\
\downarrow \varepsilon & & \downarrow (\mu) \\
R[[x]] & \xrightarrow{(\eta)} & H[[x, y]]
\end{array}
\]

is commutative (here \((\varepsilon)\) is the homomorphism, induced by \(\varepsilon\)). Note that

\[
(\eta)(x) = \mathcal{E}(x, y).
\]
This completes the proof that
\[ \Delta_{R[[x]]}(x) = F(x \otimes 1, 1 \otimes x), \]
where \( F(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]] \) is the formal group of geometric cobordisms. \( \square \)

It is very important that we consider the maps \( \hat{\phi}, \hat{\nu}, \) and \( \kappa \) as fiber maps in this example. Otherwise instead of \( \mathfrak{F}(x \otimes 1, 1 \otimes x) \) we obtain the usual formal group of geometric cobordisms because the \( H \)-space \( \widehat{Gr} \) is isomorphic to the \( H \)-space \( BSU_\otimes \times \mathbb{CP}^\infty \).

It is well known ([2]), that the formal group of geometric cobordisms is the universal formal group.

**Conjecture 10.** The formal group \( \mathfrak{F}(x \otimes 1, 1 \otimes x) \) is the universal object in the category of formal groups over a (topological) Hopf algebras.

Let \( R' \) be a ring and \( F'(x \otimes 1, 1 \otimes x) \) be a formal group over \( R' \). Note that we may consider the \( R' \) as the Hopf algebra over \( R' \) with respect to the \( \Delta_{R'}: R' \cong R' \otimes R', \eta_{R'} = \varepsilon_{R'} = S_{R'} = \text{id}_{R'}: R' \to R' \). If \( \chi: H \to R' \) is a homomorphism of the Hopf algebras from \((H, \mu, \eta, \Delta, \varepsilon, S)\) to \((R', \mu_{R'}, \eta_{R'}, \Delta_{R'}, \varepsilon_{R'}, S_{R'})\), then \( \chi = (\chi \circ \eta) \circ \varepsilon = \chi |_{R} \circ \varepsilon \). Hence there exists the natural bijection \( \text{Hom}_{\text{Hopf alg.}}(H, R') \leftrightarrow \text{Hom}_{\text{Ring}}(R, R') \). Therefore the Conjecture implies the universal property of the formal group of geometric cobordisms.

## 2 Extensions of the formal group of geometric cobordisms, generated by \( \mathfrak{F}(x \otimes 1, 1 \otimes x) \)

Now we construct the denumerable set of extensions of the formal group of geometric cobordisms \( F(x \otimes 1, 1 \otimes x) \) by the Hopf algebra \( H = \Omega_\ast (Gr) \).

Let \( F_i(x \otimes 1, 1 \otimes x), i = 1, 2 \) be formal groups over ring \( R \). Recall the following definition.

**Definition 11.** A homomorphism of formal groups \( \varphi: F_1 \to F_2 \) is a formal series \( \varphi(x) \in R[[x]] \) such that \( \varphi(F_1(x \otimes 1, 1 \otimes x)) = F_2(\varphi(x) \otimes 1, 1 \otimes \varphi(x)) \).
Let $H$ be a Hopf algebra over ring $R$ with diagonal $\Delta$; let $\mathfrak{F}_i(x \otimes 1, 1 \otimes x)$, $i = 1, 2$ be formal groups over $H$.

**Definition 12.** A homomorphism of formal groups over Hopf algebra $H$ $\Phi: \mathfrak{F}_1 \rightarrow \mathfrak{F}_2$ is a formal series $\Phi(x) \in H[[x]]$ such that $(\Delta \Phi)(\mathfrak{F}_1(x \otimes 1, 1 \otimes x)) = \mathfrak{F}_2(\Phi(x) \otimes 1, 1 \otimes \Phi(x))$.

Note that $\varepsilon(\Phi): (\varepsilon \otimes \varepsilon)(\mathfrak{F}_1) \rightarrow (\varepsilon \otimes \varepsilon)(\mathfrak{F}_2)$ is the homomorphism of the formal groups over the ring $R$ (where $\varepsilon$ is the counit of the Hopf algebra $H$). We say that the homomorphism $\Phi$ covers the homomorphism $\varepsilon(\Phi)$.

Let $R$ be the ring $\Omega^*_U(pt)$; let $\mathfrak{F}(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]]$ be the formal group of geometric cobordisms. Let $H$ be the Hopf algebra $\Omega^*_U(Gr)$. By definition, put $\varphi^{(1)}(x) = x$, $\varphi^{(-1)}(x) = \theta(x)$ and $\varphi^{(n)}(x) = F(x, \varphi^{(n-1)}(x))$, where $\theta(x) \in R[[x]]$ is the inverse element in $F$. Clearly, that $\varphi^{(n)}: F \rightarrow F$ is the homomorphism for every $n \in \mathbb{Z}$. Power systems were considered by S. P. Novikov and V. M. Buchstaber in [3].

Below for any $n \in \mathbb{Z}$ we construct the extension $\mathfrak{F}^{(n)}(x \otimes 1, 1 \otimes x)$ of $\mathfrak{F}(x \otimes 1, 1 \otimes x)$ by $H$ and the homomorphism $\Phi^{(n)}: \mathfrak{F} \rightarrow \mathfrak{F}^{(n)}$ such that

(i) $\mathfrak{F}^{(1)} = \mathfrak{F}$;

(ii) $\varepsilon(\Phi^{(n)}) = \varphi^{(n)}$.

Let $n$ be a positive integer. Let us take the product of the FBSP $\widehat{Gr}_{k,kl}$ (over $Gr_{k,kl}$) with itself $n$ times. It is the FBSP over $Gr_{k,kl}$ with a fiber $\mathbb{C}P^{kn-1} \times \mathbb{C}P^{ln-1}$. By $\widehat{Gr}^{(n)}_{k,kl}$ denote the obtained FBSP. Let $\widehat{Gr}^{(n)}_{k,kl}$ be the corresponding bundle over $Gr_{k,kl}$ with fiber $\mathbb{C}P^{kn-1}$. Let $\overline{Gr}^{(n)}_{k,kl} = \lim_{(k,l) \rightarrow 1} \widehat{Gr}^{(n)}_{k,kl}$. We have the evident fiber maps $\widehat{Gr}_{k,kl} \rightarrow \widehat{Gr}^{(n)}_{k,kl}$, $\lambda^{(n)}: \widehat{Gr} \rightarrow \widehat{Gr}^{(n)}$ and the following commutative diagrams ($(km, ln) = 1$):

$$
\begin{array}{ccc}
\widehat{Gr}_{km,klmn} & \rightarrow & \widehat{Gr}^{(n)}_{km,klmn} \\
\uparrow & & \uparrow \\
\widehat{Gr}_{k,kl} \times \widehat{Gr}_{m,mn} & \rightarrow & \widehat{Gr}^{(n)}_{k,kl} \times \widehat{Gr}^{(n)}_{m,mn}, \\
\end{array}
$$

(11)

$$
\begin{array}{ccc}
\widehat{Gr} & \xrightarrow{\lambda^{(n)}} & \widehat{Gr}^{(n)} \\
\uparrow \varphi \uparrow & & \uparrow \varphi^{(n)} \\
\widehat{Gr} \times \widehat{Gr} & \xrightarrow{\lambda^{(n)} \times \lambda^{(n)}} & \widehat{Gr}^{(n)} \times \widehat{Gr}^{(n)}. \\
\end{array}
$$

(12)
By $x$ denote the class of cobordisms in $\Omega^2 \hat{U}(\hat{Gr}^{(n)})$ such that its restriction to any fiber $\cong \mathbb{C}P^\infty$ is the generator $x|_{\mathbb{C}P^\infty} \in \Omega^2 \hat{U}(\mathbb{C}P^\infty)$. Let $\Phi^{(n)}(x) \in H[[x]]$ be the series, defined by the fiber map $\lambda^{(n)}$. Let

$$\hat{\Phi}^{(n)}(x \otimes 1, 1 \otimes x) \in H \hat{\otimes} H[[x \otimes 1, 1 \otimes x]]$$

be the series, corresponds to the fiber map $\hat{Gr}^{(n)} \times \hat{Gr}^{(n)} \to \hat{Gr}^{(n)}$; note that $Gr^{(n)}$ is the $H$-group with the multiplication $\hat{\phi}^{(n)}$. Clearly, that $\hat{\Phi}^{(n)}(x \otimes 1, 1 \otimes x)$ is an extension of $F(x \otimes 1, 1 \otimes x)$ by $H$ (in particular, it is the formal group over Hopf algebra $H$). Note that $\lambda^{(n)}$ covers the identity map of the base $Gr$. It follows from diagram (15) that

$$(\Delta \Phi^{(n)})(\hat{\Phi}(x \otimes 1, 1 \otimes x)) = \hat{\Phi}^{(n)}(\Phi^{(n)}(x) \otimes 1, 1 \otimes \Phi^{(n)}(x)).$$

It is clear that $\varepsilon(\Phi^{(n)}(x)) = \varphi^{(n)}(x)$.

For $n = 0$ let $\hat{Gr}^{(0)} = Gr \times \mathbb{C}P^\infty$ and let $\lambda^{(0)}$ be the composition

$$\hat{Gr} \to pt \to \hat{Gr}^{(0)}.$$

It defines the series $\hat{\Phi}^{(0)} = F$ and $\Phi^{(0)} = 0$.

Let $\lambda^{(-1)}$ be the fiber map $\hat{Gr} \to \hat{Gr}^{(-1)} = \hat{Gr}$ such that the following conditions hold:

(i) the restriction of $\lambda^{(-1)}$ to any fiber is the inversion in the $H$-group $\mathbb{C}P^\infty$ (i.e. the complex conjugation);

(ii) $\lambda^{(-1)}$ covers the map $\nu: Gr \to Gr$, where $\nu$ is the inversion in the $H$-group $Gr$.

Let $\Phi^{(-1)}(x) \in H[[x]]$ be the series, defined by $\lambda^{(-1)}$. Trivially, that $\varepsilon(\Phi^{(-1)}(x)) = \theta(x)$. Note that the $\lambda^{(-1)}$ coincides with $\hat{\nu}$. Consequently, $\Phi^{(-1)} = \Theta(x)$. Now we can define $\hat{\Phi}^{(n)}$ and $\Phi^{(n)}$ for negative integer $n$ by the obvious way.

By $S$ denote the antipode of the Hopf algebra $H$. Let $\mu$ be the multiplication in the Hopf algebra $H$. By definition, put $(1) = \text{id}_H$, $(-1) = S: H \to H$ and $(n) = \mu \circ ((n - 1) \otimes (1)) \circ \Delta: H \to H$ (in particular, $(0) = \eta \circ \varepsilon: H \to H$, where $\eta$ is the unit in $H$).
Proposition 13. \( \mathfrak{F}(n)(x \otimes 1, 1 \otimes x) = ((n) \otimes (n)) \mathfrak{F}(x \otimes 1, 1 \otimes x) \) for any \( n \in \mathbb{Z} \).

Proof. By \( \phi: Gr \times Gr \to Gr \) denote the multiplication in the \( H \)-space \( Gr \). Suppose \( n \) a positive integer. By definition, put \( \phi(1) = \text{id}_{Gr}, \phi(n) = \phi \circ (\phi(n-1) \times \text{id}_{Gr}), \text{and diag}(n) = (\text{diag}(n-1) \times \text{id}_{Gr}) \circ \text{diag}, \) where \( \text{diag}(1) = \text{id}_{Gr}, \text{diag} = \text{diag}(2): Gr \to Gr \times Gr. \) Note that the composition \( \phi(n) \circ \text{diag}(n): Gr \to Gr \) induces the homomorphism \( (n): H \to H. \)

Let us consider the classifying map \( \alpha(n): Gr \to Gr \) for the bundle \( \hat{Gr}^{(n)} \) over \( Gr \). We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C}P^\infty & \xrightarrow{=} & \mathbb{C}P^\infty \\
\downarrow & & \downarrow \\
\hat{Gr}^{(n)} & \xrightarrow{\hat{\alpha}(n)} & \hat{Gr} \\
\downarrow & & \downarrow \\
Gr & \xrightarrow{\alpha(n)} & Gr \\
\end{array}
\]

It is easy to prove that \( \alpha(n) = \phi(n) \circ \text{diag}(n) \). Hence \( \alpha(n)^* = (n): H \to H. \)

Note that the following diagram

\[
\begin{array}{ccc}
\hat{Gr}^{(n)} \times \hat{Gr}^{(n)} & \xrightarrow{\hat{\alpha}(n) \times \hat{\alpha}(n)} & \hat{Gr} \times \hat{Gr} \\
\downarrow \hat{\phi}^{(n)} & & \downarrow \hat{\phi} \\
\hat{Gr}^{(n)} & \xrightarrow{\hat{\alpha}(n)} & \hat{Gr} \\
\end{array}
\]

(13)

is commutative. This completes the proof for positive \( n \). For negative \( n \) proof is similar. \( \Box \)

We can define the structure of group on the set \( \{\mathfrak{F}(n); n \in \mathbb{Z}\} \) in the following way. Recall that for any Hopf algebra \( H \) the triple \( (\text{Hom}_{\text{Alg}, H_{opf}}(H, H), *, \eta \circ \varepsilon) \) is the algebra with respect to the convolution \( f \ast g = \mu \circ (f \otimes g) \circ \Delta: H \to H. \) It follows from the previous Proposition that the formal group \( \mathfrak{F}(n) \) corresponds to the homomorphism \( (n): H \to H \) (see Conjecture [10]). Clearly, that \( (m) \ast (n) = (m + n) \) for any \( m, n \in \mathbb{Z}. \)
3 Logarithms of formal groups over Hopf algebras

In this section by \((H, \mu, \eta, \Delta, \varepsilon, S)\) denote a commutative Hopf algebra over ring \(R\) without torsion and by \(\mathfrak{F}(x \otimes 1, 1 \otimes x)\) denote a formal group over Hopf algebra \(H\). By \(H_\mathbb{Q}\) denote the Hopf algebra \(H \otimes \mathbb{Q}\) over ring \(R_\mathbb{Q} = R \otimes \mathbb{Q}\).

We shall write \(\mu, \eta, \ldots\) instead of \(\mu_\mathbb{Q}, \eta_\mathbb{Q}, \ldots\).

The aim of this section is to prove the following result.

**Proposition 14.** For any commutative formal group \(\mathfrak{F}(x \otimes 1, 1 \otimes x)\), which is considered as a formal group over \(H_\mathbb{Q}\), there exists a homomorphism to a formal group of the form \(c + x \otimes 1 + 1 \otimes x\), where \(c \in H_\mathbb{Q} \hat{\otimes} H_\mathbb{Q}\) such that 
\[(\text{id} \otimes \varepsilon)c = 0 = (\varepsilon \otimes \text{id})c.\]

We recall that the notion of a homomorphism of formal groups over Hopf algebra was given in previous Section. Below we shall use notations of Section 1.

To prove the Proposition, we need the following Lemma.

**Lemma 15.** A symmetric series of the form \(c + x \otimes 1 + 1 \otimes x \in H_\mathbb{Q} \hat{\otimes} H_\mathbb{Q}[x \otimes 1, 1 \otimes x]\) is a formal group over the Hopf algebra \(H_\mathbb{Q}\) if and only if the following two conditions hold:

\[(i) \quad (\text{id} \otimes \Delta)c + 1 \otimes c - (\Delta \otimes \text{id})c - c \otimes 1 = 0;\]

\[(ii) \quad (\text{id} \otimes \varepsilon)c = 0 = (\varepsilon \otimes \text{id})c.\]

(Note that the condition (i) means, that \(c\) is a 2-cocycle in the cobar complex of the Hopf algebra \(H_\mathbb{Q}\).)

**Proof of the Lemma.** The conditions (i) and (ii) are equivalent to the associativity axiom and to the unit axiom for formal groups respectively. Let us show that the series \(\Theta(x) = -(\mu \circ (\text{id} \otimes S))c - x\) is the inverse element. Indeed,

\[(\mu \circ (\text{id} \otimes S))c + x + \Theta(x) = (\mu \circ (\text{id} \otimes S))c + x - (\mu \circ (\text{id} \otimes S))c - x = 0.\]

The symmetric condition follows from the equality \((\mu \circ (\text{id} \otimes S))c = (\mu \circ (S \otimes \text{id}))c\). □
Proof of the Proposition. By definition, put
\[ \tilde{\omega}(x) = (\text{id} \otimes \tilde{\varepsilon}) \frac{\partial \tilde{\mathcal{F}}(x, z)}{\partial z} \in H[[x]] \]
(here \( \tilde{\varepsilon} : H[[z]] \to R \) is the map such that \( \tilde{\varepsilon}|_H = \varepsilon : H \to R, \) \( \tilde{\varepsilon}(z) = 0 \)). Recall that \( \tilde{\Delta} : H[[x]] \to H[[x]] \otimes H[[z]] = H \otimes_R H[[x \otimes 1, 1 \otimes x]] \) is the map such that \( \tilde{\Delta}|_H = \Delta, \) \( \tilde{\Delta}(x) = \mathcal{F}(x \otimes 1, 1 \otimes x) \). We have
\[
(\Delta \tilde{\omega})(\mathcal{F}(x \otimes 1, 1 \otimes x)) = \tilde{\Delta}(\tilde{\omega}(x)) = (\Delta \circ (\text{id} \otimes \tilde{\varepsilon}) \circ \frac{\partial}{\partial z})(\mathcal{F}(x, z)) =
\]
\[
((\text{id} \otimes \text{id} \otimes \tilde{\varepsilon}) \circ (\tilde{\Delta} \times \text{id}) \circ \frac{\partial}{\partial z})(\mathcal{F}(x, z)) =
\]
\[
\left( (\text{id} \otimes \text{id} \otimes \tilde{\varepsilon}) \circ \frac{\partial}{\partial z} \right) ((\text{id} \otimes \text{id})\tilde{\mathcal{F}})(x, 1, 1 \otimes x, z) =
\]
\[
\left( (\text{id} \otimes \text{id} \otimes \tilde{\varepsilon}) \circ \frac{\partial}{\partial z} \right) ((\Delta \times \text{id})\tilde{\mathcal{F}})(x, 1, 1 \otimes x, z) =
\]
\[
(\text{id} \otimes \text{id} \otimes \tilde{\varepsilon}) \left( \frac{\partial ((\Delta \times \text{id})\tilde{\mathcal{F}})(x, 1, 1 \otimes x, z)}{\partial \tilde{\mathcal{F}}(1 \otimes x, z)} \right) \times (1 \otimes \tilde{\omega})(1 \otimes x).
\]
Therefore, we have
\[
(\Delta \tilde{\omega})(\mathcal{F}(x \otimes 1, 1 \otimes x)) = \frac{\partial \mathcal{F}(x \otimes 1, 1 \otimes x)}{\partial (1 \otimes x)} \cdot (1 \otimes \tilde{\omega})(1 \otimes x). \tag{14}
\]
If
\[ \mathcal{F}(x, z) = \sum_{i,j \geq 0} A_{i,j} x^i z^j \quad (A_{i,j} \in H \otimes_R H), \]
then
\[ \tilde{\omega}(x) = (\text{id} \otimes \tilde{\varepsilon}) \sum_{i,j} A_{i,j} x^i z^{i+1} = (\text{id} \otimes \tilde{\varepsilon}) A_{0,1} + \sum_{i \geq 1} ((\text{id} \otimes \varepsilon) A_{i,1}) x^i, \]
where \( (\varepsilon \circ (\text{id} \otimes \varepsilon)) A_{0,1} = 1 \neq 0 \). Therefore
\[ \frac{1}{\tilde{\omega}(x)} \in H[[x]] \quad \text{and} \]

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\[ \tilde{\Delta} \left( \frac{1}{\tilde{\omega}(x)} \right) = \frac{1}{\Delta(\tilde{\omega}(x))} = \frac{1}{(\Delta \tilde{\omega})(\tilde{\omega}(x) \otimes 1, 1 \otimes x)} \in H \otimes H[[x \otimes 1, 1 \otimes x]]. \]

Therefore (14) may be rewritten in the form

\[ \frac{d(1 \otimes x)}{(1 \otimes \tilde{\omega})(1 \otimes x)} = \frac{d(\tilde{\omega}(x) \otimes 1, 1 \otimes x)}{(\Delta \tilde{\omega})(\tilde{\omega}(x) \otimes 1, 1 \otimes x)). \tag{15} \]

It is clear that

\[ \frac{1}{\tilde{\omega}(x)} = b_0 + b_1 x + \ldots, \]

where \( b_i \in H, \varepsilon(b_0) = 1 \). By \( g(x) \) denote the series

\[ \int_o^x \frac{dt}{\tilde{\omega}(t)} \in H_Q[[x]]. \]

Equality (15) implies

\[ c' + (1 \otimes g)(1 \otimes x) = (\Delta g)(\tilde{\omega}(x) \otimes 1, 1 \otimes x)), \tag{16} \]

where \( c' \) is independent of \( 1 \otimes x \). The application of \( id \otimes \tilde{\varepsilon} \) to relation (16) yields

\[ (id \otimes \tilde{\varepsilon})c' = ((id \otimes \varepsilon) \circ \Delta)g)(x \otimes 1) = (g \otimes 1)(x \otimes 1), \]

and the application of \( \tilde{\varepsilon} \otimes id \) to relation (16) yields

\[ (\tilde{\varepsilon} \otimes id)c' + (1 \otimes g)(1 \otimes x) = (1 \otimes g)(1 \otimes x). \]

Hence

\[ (\Delta g)(\tilde{\omega}(x) \otimes 1, 1 \otimes x)) = c + (g \otimes 1)(x \otimes 1) + (1 \otimes g)(1 \otimes x), \tag{17} \]

where \( c' = (g \otimes 1)(x \otimes 1) + c, c \in H_Q \otimes H_Q \) and \( (id \otimes \varepsilon)c = 0 = (\varepsilon \otimes id)c \).

To complete the proof we must check the condition (i) of the previous Lemma. For this purpose we apply \( id \otimes \tilde{\Delta} \) and \( \tilde{\Delta} \otimes id \) to equation (17). We have

\[ (((id \otimes \Delta) \circ \Delta)g)((id \otimes \Delta)\tilde{\omega}(x \otimes 1 \otimes 1, 1 \otimes \tilde{\omega}(x \otimes 1, 1 \otimes x))) = \]

\[ (id \otimes \Delta)c + g(x) \otimes 1 \otimes 1 + 1 \otimes (\Delta g)(\tilde{\omega}(x \otimes 1, 1 \otimes x)) = \]

\[ (((\Delta \otimes id) \circ \Delta)g)((\Delta \otimes id)\tilde{\omega}(\tilde{\omega}(x \otimes 1, 1 \otimes x) \otimes 1, 1 \otimes 1 \otimes x)) = \]
\[(\Delta \otimes \text{id})c + (\Delta g)(\mathfrak{g}(x \otimes 1, 1 \otimes x)) \otimes 1 + 1 \otimes 1 \otimes g(x),\]
i. e.

\[(\text{id} \otimes \Delta)c + g(x) \otimes 1 \otimes 1 + 1 \otimes c + 1 \otimes g(x) \otimes 1 + 1 \otimes 1 \otimes g(x) =\]

\[(\Delta \otimes \text{id})c + c \otimes 1 + g(x) \otimes 1 \otimes 1 + 1 \otimes g(x) \otimes 1 + 1 \otimes 1 \otimes g(x).\]

This completes the proof. □

**Lemma 16.** “Linear” formal groups \(c_i + x \otimes 1 + 1 \otimes x, \ i = 1, 2\) over \(H_\mathbb{Q}\) are isomorphic if and only if cohomology classes \([c_1]\) and \([c_2]\) are equal.

**Proof.** Suppose \([c_1] = [c_2]\); then there exists \(\lambda \in H_\mathbb{Q}\) such that \(\varepsilon(\lambda) = 0\) and \(c_2 - c_1 = \Delta \lambda - \lambda \otimes 1 - 1 \otimes \lambda\). Hence \(\Delta \lambda + c_1 + x \otimes 1 + 1 \otimes x = c_2 + (\lambda + x) \otimes 1 + 1 \otimes (\lambda + x)\). This shows that \(g(x) = \lambda + x\) is an isomorphism from \(c_1 + x \otimes 1 + 1 \otimes x\) to \(c_2 + x \otimes 1 + 1 \otimes x\). The proof of the converse statement is clear. □

We may obtain more precise result than in the previous Proposition.

**Corollary 17.** A formal group \(\mathfrak{g}(x \otimes 1, 1 \otimes x)\) over \(H_\mathbb{Q}\) is isomorphic to the “trivial“ group \(x \otimes 1 + 1 \otimes x\) if and only if the 2-cocycle \(c\) is a coboundary.

**Proof.** Let

\[(\Delta g)(\mathfrak{g}(x \otimes 1, 1 \otimes x)) = c + g(x) \otimes 1 + 1 \otimes g(x).\]

Let \(c = \lambda \otimes 1 - \Delta \lambda + 1 \otimes \lambda\). Let us consider the isomorphism \(h(x) = \lambda + g(x)\).

We have

\[(\Delta h)(\mathfrak{g}(x \otimes 1, 1 \otimes x)) = \Delta \lambda + (\Delta g)(\mathfrak{g}(x \otimes 1, 1 \otimes x)) =\]

\[\Delta \lambda + c + g(x) \otimes 1 + 1 \otimes g(x) = \Delta \lambda - \lambda \otimes 1 - 1 \otimes \lambda + c + h(x) \otimes 1 + 1 \otimes h(x) =\]

\[h(x) \otimes 1 + 1 \otimes h(x).\] □

**Remark 18.** Note that this proof generalizes the standard proof of the analogous result for formal groups over rings (see [4]).
Remark 19. Note that in the proof we assign for any formal group $F(x \otimes 1, 1 \otimes x)$ over $H$ some 2-cocycle $c$ in the cobar complex of the coalgebra $H_Q$.

Remark 20. Note that $(\varepsilon g)(x) \in R_Q[[x]]$ is the logarithm of the formal group $((\varepsilon \otimes \varepsilon)F)(x \otimes 1, 1 \otimes x) = F(x \otimes 1, 1 \otimes x) \in R[[x \otimes 1, 1 \otimes x]]$ over ring $R$.

Remark 21. Since $g(x) = b_0 x + b_1 x^2 + \ldots$ and $\varepsilon(b_0) = 1$, there exists the series $(\Delta g)^{-1}(x) = (\Delta(g^{-1}))(x) \in H_Q \otimes H_Q[[x]]$. Using (17), we get

$$F(x \otimes 1, 1 \otimes x) = (\Delta g)^{-1}(c + g(x) \otimes 1 + 1 \otimes g(x)).$$

4 The tensor category, connected with cobordisms rings of FBSP

Below we study some properties of category, connected with cobordism rings of FBSP. In particular, we shall show that it is the tensor category.

Let us consider the series $\mathfrak{G}(x, y) \in H[[x, y]] = \Omega^*_U(\widetilde{Gr})$, where $H = \Omega^*_U(Gr)$. Recall that it corresponds to the direct limit $\kappa$ of the maps $\kappa_{k,l}: \widetilde{Gr}_{k,kl} \rightarrow \mathbb{C}P^{kl-1}$, where $\widetilde{Gr}_{k,kl}$ is the canonical FBSP over $Gr_{k,kl}$ ($(k, l) = 1$). Previously some properties of $\mathfrak{G}(x, y)$ were studied. In particular, it was shown that

$$(\varepsilon \mathfrak{G})(x, y) = F(x, y),$$

where $\varepsilon: H \rightarrow R = \Omega^*_U(\text{pt})$ is the counit of the Hopf algebra $H$ and $F(x, y) \in R[[x, y]]$ is the formal group of geometric cobordisms.

Let $\varphi_{k,l}$ be the map

$$\kappa_{k,l} \times \text{id}_{\widetilde{Gr}_{k,kl}}: \widetilde{Gr}_{k,kl} \rightarrow \mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl}.$$ 

The commutativity of the following diagram

$$
\begin{array}{ccc}
\mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl} & \xrightarrow{\varphi_{k,l}} & \mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl} \\
\downarrow \text{id}_{\mathbb{C}P^{kl-1}} \times \varphi_{k,l} & & \downarrow \text{id}_{\mathbb{C}P^{kl-1} \times \mathbb{C}P^{kl-1}} \\
\mathbb{C}P^{kl-1} \times \mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl} & \xrightarrow{\text{diag}} & \mathbb{C}P^{kl-1} \times \mathbb{C}P^{kl-1} \times \widetilde{Gr}_{k,kl}
\end{array}
$$

(18)
allows us to define on the algebra $H[[x, y]]$ the structure of $R[[z]] = \Omega_U^*(\mathbb{CP}^\infty)$-module such that $z$ acts as the multiplication by $\mathfrak{G}(x, y)$. Let us denote this $R[[z]]$-module by $(H[[x, y]]; \mathfrak{G}(x, y))$.

Let us consider $R[[z]] = \Omega_U^*(\mathbb{CP}^\infty)$ as a Hopf algebra. Recall that $\Delta_{R[[z]]}(z) = F(z \otimes 1, 1 \otimes z)$.

**Proposition 22.** $H[[x, y]]$ is the module coalgebra over $R[[z]]$, i.e. $R[[z]] \hat{\otimes} R H[[x, y]] \rightarrow H[[x, y]]$ is the homomorphism of coalgebras.

**Proof.** The proof follows from the following commutative diagram ( $(km, ln) = 1$):

\[
\begin{array}{ccccccccc}
\mathbb{CP}^{kl-1} \times \widetilde{Gr}_{k,kl} \times \mathbb{CP}^{mn-1} \times \widetilde{Gr}_{m,mn} & \xrightarrow{\varphi_{kl, mn}} & \mathbb{CP}^{km, klmn} \\
\downarrow & & \downarrow \\
\mathbb{CP}^{klmn-1} \times \widetilde{Gr}_{km,klmn} & \xrightarrow{\varphi_{km, ln}} & \widetilde{Gr}_{km,klmn}.
\end{array}
\]

Let us consider the next commutative diagram ( $(km, ln) = 1$):

\[
\begin{array}{ccccccccc}
\widetilde{Gr}_{k,kl} \times \widetilde{Gr}_{m,mn} & \xrightarrow{\psi_{kl, mn}} & \widetilde{Gr}_{km,klmn} \\
\downarrow & & \downarrow \\
\widetilde{Gr}_{k,kl} \times \widetilde{Gr}_{m,mn} & \xrightarrow{\phi_{kl, mn}} & \widetilde{Gr}_{km,klmn},
\end{array}
\]

(19)

where $\widetilde{Gr}_{k,kl} \times \widetilde{Gr}_{m,mn}$ is the FBSP over $Gr_{k,kl} \times Gr_{m,mn}$, induced by the map $\phi_{kl, mn}$. Clearly that the bundle $\widetilde{Gr}_{k,kl} \times \widetilde{Gr}_{m,mn}$ (with fiber $\mathbb{CP}^{km-1} \times \mathbb{CP}^{ln-1}$) is ("external") Segre’s product of the canonical FBSP over $Gr_{k,kl}$ and $Gr_{m,mn}$.

By definition, put

\[
\widetilde{Gr} \times Gr = \lim_{(km, ln) = 1} \widetilde{Gr}_{k,kl} \times \widetilde{Gr}_{m,mn},
\]

\[
\psi = \lim_{(km, ln) = 1} \psi_{km, ln} : \widetilde{Gr} \times Gr \rightarrow \widetilde{Gr}.
\]

We have the homomorphism of $R[[z]]$-modules

\[
\Psi : (H[[x, y]]; \mathfrak{G}(x, y)) \rightarrow (\widetilde{H} \hat{\otimes} H[[x, y]]; (\Delta \mathfrak{G})(x, y)),
\]

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defined by the fiber map \( \psi \) (recall that \( \Delta \) is the comultiplication in the Hopf algebra \( H = \Omega^*_U(Gr) \)). Clearly that the restriction \( \Psi|_H \) coincides with \( \Delta \).

Let \( \mathcal{P}^{k-1 \times l-1}_X \) be a FBSP over a finite CW-complex \( X \) with fiber \( \mathbb{C}P^{k-1} \times \mathbb{C}P^{l-1} \). Recall that if \( k \) and \( l \) are sufficiently large then there exist a classifying map \( f_{k,l} \) and the corresponding fiber map

\[
\mathcal{P}^{k-1 \times l-1}_X \rightarrow \tilde{Gr}_{k,kl}
\]

\[
\downarrow \hspace{1cm} \downarrow \\
X \rightarrow \tilde{Gr}_{k,kl}
\]

which are unique up to homotopy and up to fiber homotopy respectively. Let \( \mathcal{P}^{km-1 \times ln-1}_X \), \( (km, ln) = 1 \) be Segre's product of \( \mathcal{P}^{k-1 \times l-1}_X \) with the trivial FBSP \( X \times \mathbb{C}P^{m-1} \times \mathbb{C}P^{n-1} \). Let us pass to the direct limit

\[
\mathcal{P}_X \times \mathcal{Q} = \lim_{\to} (\mathcal{P}^{km_i-1 \times ln_i-1}_X),
\]

where \( (km_i, ln_i) = 1 \), \( m_i \mid m_{i+1}, n_i \mid n_{i+1}, m_i, n_i \to \infty \), as \( i \to \infty \). The stable equivalence class of FBSP \( \mathcal{P}^{k-1 \times l-1}_X \) may be unique restored by the direct limit \( \mathcal{P}_X \times \mathcal{Q} \). We have also a classifying map \( f = \lim_{(k,l) \to 1} f_{k,l} \) and the corresponding fiber map

\[
\mathcal{P}_X \times \mathcal{Q} \rightarrow \tilde{Gr}
\]

\[
\downarrow \hspace{1cm} \downarrow \\
X \rightarrow \tilde{Gr}
\]

Let us define the category \( FBSP_{finite} \) by the following way.

(i) \( Ob(FBSP_{finite}) \) is the class of direct limits \( \mathcal{P}_X \times \mathcal{Q} \) of FBSP over finite CW-complexes \( X \) (in other words, the class of stable equivalence classes of FBSP);

(ii) \( Mor_{\mathcal{C}}(\mathcal{P}_X \times \mathcal{Q}, \mathcal{P}'_Y \times \mathcal{Q}') \) is the set of fiber maps

\[
\mathcal{P}_X \times \mathcal{Q} \rightarrow \mathcal{P}'_Y \times \mathcal{Q}'
\]

\[
\downarrow \hspace{1cm} \downarrow \\
X \rightarrow Y
\]
such that its restrictions to any fiber \((\cong \mathbb{C}P^{\infty} \times \mathbb{C}P^{\infty})\) are isomorphisms.

Applying the functor of unitary cobordisms \(\Omega^*_U\) to an object \(\mathcal{P} \times \mathcal{Q} \in \text{Ob}(\mathfrak{FBS}_P)\), we get the \(R[z]\)-module \((A[[x,y]]; (f^* \mathfrak{G})(x,y)) \in \text{Ob}(\Omega^*_U(\mathfrak{FBS}_P))\), where \(A = \Omega^*_U(X)\) and \(f: X \to Gr\) is a classifying map for \(\mathcal{P} \times \mathcal{Q}\). It is clear that \(((\varepsilon_A \circ f^*) \mathfrak{G})(x,y) = F(x,y)\), where \(\varepsilon_A: A \to R\) is the homomorphism, induced by an embedding of a point \(pt \to X\). In other words, for any object in the category \(\Omega^*_U(\mathfrak{FBS}_P)\), there exists the canonical morphism \((A[[x,y]]; (f^* \mathfrak{G})(x,y)) \to (R[[x,y]]; F(x,y))\).

Hence there exist the initial object \((H[[x,y]]; \mathfrak{G}(x,y))\) and the final object \((R[[x,y]]; F(x,y))\) in the category \(\Omega^*_U(\mathfrak{FBS}_P)\).

Let’s consider a pair \((A[[x,y]]; (f^* \mathfrak{G})(x,y)), (B[[x,y]]; (g^* \mathfrak{G})(x,y)) \in \text{Ob}(\Omega^*_U(\mathfrak{FBS}_P))\), where \((B[[x,y]]; (g^* \mathfrak{G})(x,y)) = \Omega^*_U(\mathcal{P} \times \mathcal{Q}').\) Let’s define their “tensor product” as the object \(((A \otimes B) [[x,y]]; ((f^* \otimes g^*) \circ \Delta) \mathfrak{G})(x,y)) = \text{Ob}(\Omega^*_U(\mathfrak{FBS}_P))\) (recall that \(\Delta: H \to H \otimes H\) is the co-multiplication in the Hopf algebra \(H\)).

**Proposition 23.** The category \(\Omega^*_U(\mathfrak{FBS}_P)\) is the tensor category with the just defined tensor product and the unit \(1 = (R[[x,y]]; F(x,y))\).

**Proof.** The proof is trivial. For example, the associativity axiom follows from the identity \(((\Delta \otimes \text{id}_H) \circ \Delta) \mathfrak{G})(x,y) = (((\text{id}_H \otimes \Delta) \circ \Delta) \mathfrak{G})(x,y)\) which follows from the next commutative diagram \(((kmt,lnu) = 1):\)

\[
\begin{array}{ccc}
Gr_{km,klmn} \times Gr_{t,stu} & \rightarrow & \widetilde{Gr}_{kmt,klmntu} \\
\uparrow & & \uparrow \\
Gr_{k,kl} \times Gr_{m,mn} \times Gr_{t,stu} & \rightarrow & Gr_{k,kl} \times Gr_{mt,mntu},
\end{array}
\]  

(23)

where \(Gr_{k,kl} \times Gr_{m,mm} \times Gr_{t,stu}\) is external Segre’s product of the canonical FBSP over \(Gr_{k,kl}, Gr_{m,mm}\) and \(Gr_{t,stu}\) (it is the bundle over \(Gr_{k,kl} \times Gr_{m,mm} \times Gr_{t,stu}\) with fiber \(\mathbb{C}P^{kmt-1} \times \mathbb{C}P^{lnu-1}\)). □

Note that there exist the canonical homomorphisms \(p_1, p_2:\)

\[
\begin{array}{c}
(A[[x,y]]; ((f^* \otimes g^*) \circ \Delta) \mathfrak{G})(x,y) \\
\longrightarrow \quad \longrightarrow \\
(A[[x,y]]; (f^* \mathfrak{G})(x,y)) \quad (B[[x,y]]; (g^* \mathfrak{G})(x,y)),
\end{array}
\]  

\(p_1\) \quad \(p_2\)
such that $p_1 |_{A \otimes B}^R = \text{id}_A \otimes \varepsilon_B$, $p_2 |_{A \otimes B} = \varepsilon_A \otimes \text{id}_B$.

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References

[1] B. I. Botvinnik, V. M. Buchstaber, S. P. Novikov, S. A. Yuzvinsky Algebraic aspects of the theory of multiplications in the complex cobordisms theory. — UMN., 55:4 (2000), 5–24. (in Russian)

[2] Quillen D. On the formal group low of unoriented and complex cobordism theory.— Bull. Amer. Math. Soc., 75:6 (1969), 1293–1298.

[3] V. M. Buchstaber, S. P. Novikov Formal groups, power systems and operators of Adams.— Matematichesky sbornik (new series), 84(126):1 (1971), 81–118. (in Russian)

[4] Honda T. Formal groups and zeta-functions.— Osaka Journal of Math., 5:2 (1968), 199–213.