Binary dynamics on star networks under external perturbations

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Abstract

We study a binary dynamical process that is a representation of the voter model with opinion makers. The process models an election with two candidates but can also describe the frequencies of a biallelic gene in a population or atoms with two spin orientations in a magnetic material. The system is represented by a network whose nodes have internal states labeled 0 or 1, and nodes that are connected can influence each other. The network is perturbed by a set of external nodes whose states are fixed in 0 or 1 and that can influence all nodes of the network. The fixed nodes play the role of opinion makers in the voter model, mutation rates in population genetics or temperature in a magnetic material. The quantity of interest is the probability \( \rho(m,t) \) that \( m \) nodes are in state 1 at time \( t \). Here we study this process on star networks and compare the results with those obtained for networks that are fully connected. In both cases a transition from disordered to ordered equilibrium states is observed as the number of external fixed nodes becomes small, but it differs significantly between the two network topologies. For fully connected networks the probability distribution becomes uniform at the critical point, which is independent of the network size. For star networks, on the other hand, the equilibrium distribution \( \rho(m) \) splits in two peaks, reflecting the two possible states of the central node. We obtain approximate analytical solutions that hold near the transition and that clarify the role of the central node in the process. We show that the different character of the transition also manifest itself in the magnetization of system, obtained in the limit of large \( N \). Finally, extending the analysis to two star networks we compare our results with simulations in scalefree networks, detecting the presence of hubs.

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I. INTRODUCTION

Network science has provided a large body of theoretical tools to investigate complex systems, from physics to social sciences and biology [1–3]. Much work has been devoted to the study of networks topological properties [1, 3, 4, 7–10] and dynamical processes on networks have been shown to depend sensitively on the network structure [11–20]. However, the response of networks to external perturbations has only recently been investigated [7, 21–25]. Here we consider a dynamical process which is a version of the voter model with opinion makers as a general binary dynamical process subjected to an external environment.

The voter model provides one of the simplest dynamical systems that can be studied on networks [26, 27]. It consists of a set of individuals trying to decide in which of two candidates to vote for. Their opinion can be influenced by their friends and by opinion makers, such as journalists or politicians, whose power of persuasion toward one of the candidates extents over a large portion of the population. Moreover, among the friends, there can be stubborn individuals, whose opinions are fixed. Real social networks are usually complex and can be modeled by specific network structures. The opinion makers, on the other hand, can be modeled by additional sets of external 'fixed' nodes whose state never change and that reach all voters equally, acting as external perturbations to the intrinsic network dynamics (Fig. 1).

In the model the intention of a voter is quantified by its state being 0 or 1 and the number of fixed nodes (opinion makers) for candidates 0 and 1 are $N_0$ and $N_1$ respectively. In the absence of opinion makers and stubborn individuals, the population eventually reaches a consensus and the network stabilizes with all nodes 0 or all nodes 1, which are the only absorbing states. As long as stubborn agents or opinion makers are present for both candidates, the network never stabilizes, but it does reach a statistical equilibrium where the probability that candidate 1 has so many votes becomes independent of the time.

This dynamical process can model other interesting systems besides as an election with two candidates [28, 29], as the Glauber dynamics of the Ising model [25, 32] where $N_0 + N_1$ plays the role of temperature and $N_0 - N_1$ of a magnetic field and a population of sexually reproducing (haploid) organisms where the internal state represents one of two alleles of a gene and the fixed nodes map into mutation rates [33, 34].

While it is not possible to solve the voter model for arbitrary network topologies, progress
has been made by considering simple networks and specific distribution of fixed nodes. In particular, the voter model without opinion makers was studied with a single stubborn agent in regular lattices [30] and with arbitrary number of them in fully connected networks, where analytic solutions were obtained in the limit where the number of voters go to infinity [31]. In this case each stubborn individual is also an opinion maker, since it is connected to all undecided voters. The problem was finally solved for fully connected networks of arbitrary size in [25] where it was shown that the solution was also a good approximation for networks of different topologies, as long as number of opinion makers $N_0$ and $N_1$ were rescaled according to the average degree of the network. The numbers $N_0$ and $N_1$ were also analytically extended to real numbers smaller than 1, to represent weak coupling between the voters and the opinion makers. It was shown (see also [31]) that a phase transition exists between ordered states, where most voters have the same opinion, to a disordered state, where approximately half the votes go to each candidate, as $N_0$ and $N_1$ go from very small to very large numbers. The transition occurs exactly at $N_0 = N_1 = 1$ for fully connected networks of any size.

Here we study the voter model without stubborn agents but with arbitrary number of opinion makers in the star network. Being, in a way, the opposite of fully connected networks, and also a model for hubs in scale free networks, it provides a test for the robustness of the results derived in [25]. We show that, in the regimes of strong coupling $N_0, N_1 >> 1$ and weak coupling $N_0, N_1 << 1$, the approximation provided by the fully connected theory works surprisingly well even for star networks. However, near the phase transition a completely new behavior is obtained and clearly reveals the influence of the central node. We derive approximate analytical solutions for the star network and extend the results for multiple stars, which can be used as a simplified model for a scalefree network.

II. MODEL

Consider a network with $N$ nodes specified by the adjacency matrix $A$, defined by $A_{ij} = 1$ if nodes $i$ and $j$ are connected and $A_{ij} = 0$ if $i$ and $j$ are not connected. We also define $A_{ii} = 0$ and assume there are no disconnected nodes. Each node has an internal state which can take only the values 0 or 1. The nodes are also connected to $N_0$ external nodes whose states are fixed at 0 and to $N_1$ nodes whose states are fixed at 1, as illustrated in Fig.1.
FIG. 1. (color online) Representation of the voter model on a network. The different colors indicate the internal states of the voters, which can be undecided (circles), stubborns (squares labeled $a$ and $b$) and opinion makers (external squares). Opinion makers can affect all voters of the network but undecided voters and stubborn individuals can only influence their connected neighbors.

In order to distinguish between the two kinds of nodes, we call the $N_0 + N_1$ external nodes fixed and the $N$ nodes of the network, whose states are variable, free. The free nodes can change their internal state according to the following dynamical rule: at each time step a free node is selected at random and, with probability $p$ its state remains the same; with probability $1-p$ the node copies the state of one of its connected neighbors, free or fixed, also chosen at random.

Let

$$x = \{x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_N\}$$  \hspace{1cm} (1)$$

denote a microscopic state of the network with $x_i = 0$ or $x_i = 1$ denoting the state of node $i$. There is a total of $2^N$ possible microscopic states and we call $P_t(x)$ the probability of finding the network in state $x$ at time $t$. Since a single free node can change state per time step, it is useful to define the auxiliary state $x^k$ which is identical to $x$ at every node except
at node \( k \), whose state is the opposite of \( x_k \), i.e., \( x_k^k = 1 - x_k \). Explicitly,

\[
x^k = \{ x_1, x_2, \ldots, x_{k-1}, 1 - x_k, x_{k+1}, \ldots, x_N \}.
\] (2)

With these definitions, the evolution equation for the probabilities can be written as

\[
P_{t+1}(x) = pP_t(x) + (1 - p) \frac{1}{N} P_t(x) \sum_{i=1}^{N} T(x \rightarrow x_i) + (1 - p) \sum_{i=1}^{N} \frac{1}{N} P_t(x^i) T(x^i \rightarrow x_i).
\] (3)

The first two terms take into account the probability that the network was already in state \( x \) and the selected node \( i \) did not change its state or (ii) copied the state of a neighbor which was identical to its own state. The last term is the probability that the network was in a state differing from \( x \) by a single node, which was selected and copied the state of neighbor opposite to its own.

According to the dynamical rules, the transition probabilities can be written as

\[
T(x_i \rightarrow x_i) = \frac{1}{k_i + N_0 + N_1} \left[ \sum_{j=1}^{N} A_{ij} |1 - x_i - x_j| + x_i N_1 + (1 - x_i) N_0 \right]
\] (4)

and

\[
T(x^i \rightarrow x_i) = \frac{1}{k_i + N_0 + N_1} \left[ \sum_{j=1}^{N} A_{ij} |x^i_i - x_j| + (1 - x^i_i) N_1 + x^i_i N_0 \right]
\] (5)

where \( k_i \) is the degree of node \( i \). Using \( x^i_i = 1 - x_i \) we find that the two transition probabilities are identical and obtain

\[
P_{t+1}(x) = pP_t(x) + \frac{(1 - p)}{N} \sum_{i=1}^{N} \left[ \frac{P_t(x) + P_t(x^i)}{k_i + N_0 + N_1} \left[ \sum_{j=1}^{N} A_{ij} |1 - x_i - x_j| + x_i N_1 + (1 - x_i) N_0 \right] \right].\] (6)

III. FULLY CONNECTED NETWORKS

For networks that are fully connected the nodes are indistinguishable and the state of the network is fully specified by the number \( m \) of nodes with internal state 1 [18, 25]. Each of these macroscopic states corresponds to a set of \( N! / [(N - m)! m!] \) degenerated microscopic network states. Because there are only \( N + 1 \) macroscopic states equations (6) are greatly simplified. Using Eq. (6) we determine that the probability of finding the network with \( m \)
nodes 1 at time \( t \) satisfies the equation

\[
P_{t+1}(m) = P_t(m) \left\{ p + \frac{(1 - p)}{N(N + N_0 + N_1 - 1)} \left[ m(m + N_1 - 1) + (N - m)(N + N_0 - m - 1) \right] \right\} + \\
P_t(m - 1) \frac{(1 - p)}{N(N + N_0 + N_1 - 1)} (m + N_1 - 1)(N - m + 1) + \\
P_t(m + 1) \frac{(1 - p)}{N(N + N_0 + N_1 - 1)} (m + 1)(N + N_0 - m - 1).
\]

(7)

and the equilibrium solution \( \rho_{FC}(m) \) is given by \[25\]

\[
\rho_{FC}(m) = A(N, N_0, N_1) \frac{\Gamma(N_1 + m) \Gamma(N + N_0 - m)}{\Gamma(N - m + 1) \Gamma(m + 1)},
\]

(8)

where

\[
A(N, N_0, N_1) = \frac{\Gamma(N + 1) \Gamma(N_0 + N_1)}{\Gamma(N + N_0 + N_1) \Gamma(N_1) \Gamma(N_0)}.
\]

(9)

Because of the fixed nodes, the dynamics never stabilizes, but continuously changes from one state to another with mean occupation \( m_0 = \sum_m m \rho_{FC}(m) \) and variance \( \sigma^2 = \sum_m m^2 \rho_{FC}(m) - m_0^2 \) given by \[33\]

\[
m_0 = N \frac{N_1}{N_0 + N_1},
\]

(10)

and

\[
\sigma^2 = \frac{NN_1N_0(N_1 + N_0 + N)}{(N_1 + N_0)^2(1 + N_1 + N_0)}.
\]

(11)

The interesting feature of this solution is that for \( N_0 = N_1 = 1 \) it gives \( \rho_{FC}(m) = 1/(N + 1) \), meaning that all states are equally likely. This is illustrated in Fig.2.

For networks of different topologies the effect of the fixed nodes is amplified. The probability that a free node copies a fixed node is \( P_i = (N_0 + N_1)/(N_0 + N_1 + k_i) \) where \( k_i \) is the degree of the node. For fully connected networks \( k_i = N - 1 \) and \( P_{FC} \equiv (N_0 + N_1)/(N_0 + N_1 + N - 1) \).

For general networks an average value \( P_{av} \) can be calculated by replacing \( k_i \) by the average degree. Effective numbers of fixed nodes \( N_{0ef} \) and \( N_{1ef} \) can be then defined as the values of \( N_0 \) and \( N_1 \) in \( P_{FC} \) for which \( P_{av} \equiv P_{FC} \). This leads to

\[
N_{0ef} = fN_0, \quad N_{1ef} = fN_1,
\]

(12)

where \( f = (N - 1)/k_{av} \). It was shown that Eq.(8) with the above rescaling of fixed nodes fits very well the probability distribution for a variety of topologies. The formula was tested
FIG. 2. (color online) Equilibrium probability distribution for a fully connected network with $N = 100$ nodes and several values of $N_0$ and $N_1$. The transition between ordered and disordered states occurs at $N_0 = N_1 = 1$.

for relatively small networks of the types random, 2-D regular lattice, Barabasi-Albert scale free and small world.

IV. STAR NETWORKS

A. Master equation

For a star network it is convenient to set the total number of nodes to $N + 1$. Node 1 is the central node and is connected to all peripheral $N$ nodes. The peripheral nodes, on the other hand, are only connected to the central node. The peripheral nodes are indistinguishable from each other and, similar to the fully connected network, there are only $2(N + 1)$ macroscopic states, characterized by having $m$ peripheral nodes in state 1 ($N + 1$
possibilities) and the central node in state 1 or 0.

The evolution equation for the macroscopic states can be obtained from equations (6) if we define \( r_1(m, t) \) and \( r_0(m, t) \) as the probabilities of having \( m \) peripheral nodes in state 1 at time \( t \) with the central node in state 1 and 0 respectively. We obtain

\[
r_1(m, t + 1) = r_1(m, t) \left\{ p + \frac{(1 - p)}{(N + 1)} \left[ \frac{m(N_1 + 1) + (N - m)N_0}{1 + N_1 + N_0} + \frac{(m + N_1)}{(N + N_0 + N_1)} \right] \right. \\
+ r_1(m + 1, t) \frac{(m + 1)N_0}{(N + 1)(1 + N_0 + N_1)} \\
+ r_1(m - 1, t) \frac{(N - m + 1)(N_1 + 1)}{(N + 1)(1 + N_0 + N_1)} \\
+ r_0(m, t) \frac{(m + N_1)}{(N + 1)(N + N_0 + N_1)} \right. \\
\]

and

\[
r_0(m, t + 1) = r_0(m, t) \left\{ p + \frac{(1 - p)}{(N + 1)} \left[ \frac{mN_1 + (N - m)(N_0 + 1)}{1 + N_1 + N_0} + \frac{(N - m + N_0)}{(N + N_0 + N_1)} \right] \right. \\
+ r_0(m + 1, t) \frac{(m + 1)(N_0 + 1)}{(N + 1)(1 + N_0 + N_1)} \\
+ r_0(m - 1, t) \frac{(N - m + 1)N_1}{(N + 1)(1 + N_0 + N_1)} \\
+ r_1(m, t) \frac{(N - m + N_0)}{(N + 1)(N + N_0 + N_1)} \right. \\
\]

The first term in the first line of each equation refers to the probability \( p \) that the selected node does not change its state. The other terms in the first line indicate the probability that the selected node (peripheral or central) copies the state of a neighbor whose state is identical to its own. The other terms account for the probability that states differing at a single node change back to \( m \), considering that the node in the different state can be peripheral (second and third lines) or central (forth line).

The probability of having \( m \) nodes in state 1 in the star network is, therefore,

\[
\rho_S(m, t) = r_1(m - 1, t) + r_0(m, t). \tag{15}
\]

A comparison between the results provided by equations (13) and (14) and numerical simulations is shown in Fig.2. The main feature of these results is the different way in which the transition between ordered and disordered states occurs: instead of the meltdown of the
FIG. 3. (color online) Comparison between equations (13) and (14) (solid lines) and numerical simulations (dots). On the vertical axis is the probability of having $m$ nodes in the state 1 as given by eq. (15) for $T = 2.3 \times 10^4$ time steps, $N = 100$ peripheral nodes and several values of $N_0$ and $N_1$. For the simulations $10^6$ realizations of the dynamics were performed. Line colors: green: $N_0 = N_1 = 10$; red: $N_0 = N_1 = 1$; black: $N_0 = N_1 = 0.5$; blue: $N_0 = N_1 = 0.05$.

Gaussian distribution observed for fully connected networks displayed in Fig. 2, the Gaussian state splits in two peaks that move toward the boundaries $m = 0$ and $m = N$ as $N_0$ and $N_1$ are decreased.

B. Approximate solutions

The main difficulty in solving equations (13) and (14) is that they are coupled through the central node. Although we have not found exact solutions, a simple enough approximation
can be readily obtained if the central node is momentarily considered to be fixed. If the central node is fixed in state 1, any peripheral node sees \( N_0 \) fixed nodes in state 0 and \( N_1 + 1 \) nodes fixed in state 1. The problem reduces to that of a totally connected network with a single node, which we solve using the results of section III. The asymptotic probability that a peripheral node is in state 1 is

\[
\nu_1 = \frac{1 + N_1}{1 + N_0 + N_1}.
\]  

(16)

Therefore, the probability that \( m \) nodes are in state 1 (the central node plus \( m - 1 \) peripheral nodes) becomes

\[
p_1(m) = \binom{N}{m-1} \nu_1^{m-1}(1 - \nu_1)^{N-m+1}.
\]  

(17)

Similarly, fixing the central node in the state 0, the asymptotic probability that a peripheral node is in state 1 is

\[
\nu_0 = \frac{N_1}{1 + N_0 + N_1},
\]  

(18)

and the probability that \( m \) nodes are in state 1 is

\[
p_0(m) = \binom{N}{m} \nu_0^m(1 - \nu_0)^{N-m}.
\]  

(19)

Adding these results we obtain the approximate expression

\[
\rho_S(m) \approx \frac{N_1}{N_0 + N_1} p_1(m) + \frac{N_0}{N_0 + N_1} p_0(m),
\]  

(20)

where we have introduced the weights \( N_1/(N_0 + N_1) \) and \( N_0/(N_0 + N_1) \) of the central node to be in the state 1 or 0 respectively.

Fig. 4 shows a comparison between simulations and the approximate formula (20). The two peaks are clearly related to the two states of the central node and are well described by the approximation. The region within peaks is not well represented, since it has important contributions from flips of the central node that have been discarded.

The critical point for star networks can be defined as the value of \( N_0 = N_1 = N_c \) for which the Gaussian-like distribution breaks in two peaks. Using the approximate solution (20) we can estimate \( N_c \) as follows: for large \( N \), the contributions \( p_0 \) and \( p_1 \) for \( \rho_S \) become Gaussians centered at \( N \nu_0 \) and \( N \nu_1 \) with variances \( \sigma_0^2 = 4N \nu_0(1 - \nu_0) \) and \( \sigma_1^2 = 4N \nu_1(1 - \nu_1) \) respectively. The two-peak structure becomes visible when the distance between the two centers is of the order of the variance. This gives \( N_c \sim \sqrt{N} \) and numerical calculations
FIG. 4. (color online) Comparison between numerical simulations (thick lines) and the approximate equilibrium distribution Eq.(20) (thin lines). For $N_0 = N_1 = 10$ we also show the result for a fully connected network with the rescaling Eq.(12) corresponding to $f = 99/2$ (red). The parameters are $T = 2 \times 10^4$, $p = 0.5$, $N = 100$. For the simulations $10^5$ realizations were performed.

indicate that $N_c \approx \sqrt{N}/4$. For the parameters in Fig.4 $N_c \approx 2.6$ but the peaks will only be clearly separated for $N_0 = N_1 \sim 1$. This behavior persists for larger network sizes: for $N = 10^4$ we obtain $N_c \approx 27$ but once again the peaks only clearly separate for $N_0 = N_1 \sim 1$, displaying a slow and smooth transition. Fig.4(a) also shows the approximation (12) obtained via rescaling of the expression for fully connected networks, which works well for $N_0, N_1 \gg 1$. 
C. Magnetization

The different character of the transition between disordered and ordered states that takes place in fully connected and star networks has consequences for the magnetization of the system. In analogy with the Ising model we run a single simulation for each network and save the number of nodes in state 1, \( n_1 \), as a function of the time. We plot

\[
M = \frac{2n_1}{N} - 1
\]  

so that \(-1 \leq M \leq +1\). Figure 5 shows the results for \( N_0 = N_1 = 10, 1 \) and 0.05 (see Figs. 2 and 3). In these plots one unit of time corresponds to \( N \) steps of the dynamics, so that all nodes are updated, on the average, at each time step.

For the fully connected network with \( N = 20000 \), \( M \) fluctuates around zero for \( N_0 = N_1 = 10 \) (green). The fluctuations increase as the critical value is approached and for \( N_0 = N_1 = 1 \) (red) they take the entire range of \( M \). For \( N_0 = N_1 = 0.05 \) the system stays a substantial amount of time magnetized at \( M = +1 \) or \( M = -1 \), alternating from one extreme to the other. The lower the values of \( N_0 \) and \( N_1 \) the longer the times the system stays in each state for a fixed value of \( N \), and similarly for increasing \( N \) for fixed \( N_0 \) and \( N_1 \).

For the star network the results are rather different. Although the size of the oscillations
also increase as $N_0$ and $N_1$ decreases, the system never stays too long in a state of extreme values of $M$. These fast oscillations are clearly driven by flips of the central node, that is responsible for the two peaks in the probability distribution and that pushes the majority of the nodes with it. In this case the *average magnetization* is always zero for $N_0, N_1 > 0$.

D. Generalizations

Star networks where the center is composed not by a single node, but by a group of totally connected nodes can also be studied within this approximation. If the center has $N_c$ nodes a stationary solution can be constructed by freezing the state of the center into $m$ ones and $N_c - m$ zeros and assigning a weight to this state according to the fully connected distribution $\rho_{FC}(m)$, given by Eq.(8). Equation (20) readily generalizes to

$$\rho(m) \approx \sum_{k=0}^{N_c} \rho_{FC}(k) \binom{N}{m-k} \nu_k^{m-k}(1 - \nu_k)^{N-m+k},$$

where

$$\nu_k = \frac{N_1 + k}{N_c + N_0 + N_1}$$

and $\rho(k)_{FC}$ is given by equation [8] with $N$ replaced by $N_c$. Fig. 6 shows and example with $N_c = 2$ where a three peak structure is clearly visible close to the phase transition $N_0 = N_1 = 1$. The approximation (22) captures well the position of the peaks, but overshoots their height to compensate for the lost interference between the peaks.

As a second application we consider the joint effect of two hubs in a complex network. First we approximate the hubs as independent star networks, each with a single central node. The probability of finding $m$ nodes in the state 1 is simply given by

$$\rho(m) = \sum_{j=0}^{m} \rho_{S,P_1}(j)\rho_{S,P_2}(m-j),$$

where we have indicated explicitly the number of peripheral nodes in the distribution of the star networks.

Figure 7 shows the stationary distribution for two values of $P_1$ and $P_2$. For $P_1 = P_2 = 100$ (top left) the individual distributions split at the same value $N_0 = N_1 \approx 2.6$ and the resulting total distribution shows three symmetric peaks after the splitting. For $P_1 = 100$ and $P_2 = 10$ (top right) the splitting occurs at $N_0 = N_1 \approx 2.6$ and $N_0 = N_1 \approx 1.0$ respectively. However,
FIG. 6. (color online) Equilibrium probability distribution $\rho(m)$ for a star networks with $N_c = 2$ center nodes and $N = 200$ peripheral nodes for $N_0 = N_1 = 1$ (black) and $N_0 = N_1 = 0.1$ (red). Thick curves show the result of simulations and thin curves the approximation eq.(22).

because the separation between the two peaks of the small star is small, its effect is felt only at much smaller values of the perturbation, when the two peaks of the large hub approach the borders and the distribution becomes thin. The distribution displays four peaks instead of three. Finally we show the equilibrium distribution for a more complex network with 100 nodes constructed with preferential attachment (Fig. 7, bottom left). The network has two main hubs (shown in red in the inset) with 16 and 13 peripheral nodes. The peaks in the distribution show a clear signature of hubs. For comparison we also show the distribution corresponding to two stars of the same sizes coupled by their centers, by two peripheral nodes or by one center and one peripheral node.
FIG. 7. (color online) Top: equilibrium distribution for a pair of independent star networks with $P_1 = P_2 = 100$ (left) and $P_1 = 100$ and $P_2 = 10$ (right). Bottom left: distribution for a scale-free network with 100 nodes and $N_0 = N_1 = 0.01$. The two largest hubs (shown in red in the inset) have 16 and 13 peripheral nodes and are connected by their centers. On the left is the result for two star networks with $P_1 = 16$, $P_2 = 13$ and $N_0 = N_1 = 0.05$ connected by their centers (black), by two peripheral nodes (blue), by one center and one peripheral node (red) and disconnected (green).

V. CONCLUSIONS

The voter model with opinion makers is one of the simplest dynamical systems that can be represented on a network. It models an election between candidates where undecided voters are influenced by their social contacts and by external factors such as journalists and politicians. The system can also be used to model the spreading of alleles in a population.
subjected to mutations or a magnetic material in contact with a thermal bath. If the number of opinion makers is zero the population is certain to reach a consensus toward one of the candidates independently of the structure of the network. This corresponds to the fixation of one allele or to a magnetized state. The power of the opinion makers, however, depends strongly on the average degree of the network. For a completely connected network the transition between nearly consensus (ordered state) and a tie (disordered state) happens exactly at $N_0 = N_1 = 1$ independently of the population size $N$. For networks with average degree $k_{av}$ the effect of the fixed nodes is amplified by a factor $f = (N - 1)/k_{av}$, which can be very large for natural populations. Much above or much below the transition from disordered to ordered states the influence of the network structure is negligible and only shows up in the rescaling of the opinion makers influence, that is large when the network is weakly connected. This is in contrast with processes describing the spreading of epidemics or synchronization of oscillators, where the topology plays a crucial role [11–20]. However, close to the critical point, the network structure leaves important signatures in the probability distribution $\rho(m)$.

For the particular case of star networks with a single central node, the Gaussian-like distribution displayed by $\rho_S(m)$ for large values of $N_0$ and $N_1$ splits into two peaks centered at $N(N_1 + 1)/(N_1 + N_0 + 1)$ and $NN_1/(N_1 + N_0 + 1)$ according to the state of the central node being 1 or 0. The central node controls the entire system and the distribution behaves approximately as a single giant node with two collective states only. For $N_0 = N_1 = 1$ the peaks are centered at $2N/3$ and $N/3$ respectively, which is rather different from the distribution of fully connected networks where $\rho_{FC}(m) = 1/(N + 1)$ is constant. In the former case the election will be own by one of the candidates with approximately 67% of the votes, whereas in the latter, the winner can have any number of votes with equal probability. When more than one central node exists, or when the system is controlled by more than one hub, more complex behaviors can be obtained. However, signatures of the network topology are only clearly visible near the transition between ordered and disordered states. Away from the critical point Eqs. (8) and (12) provide good the approximations for the equilibrium probability.

Since hubs and clusters are ubiquitous in complex networks, the superimposition of these structures give clues as to the behavior of more realistic systems. Here we showed examples where two stars form a simple network mimicking the scalefree topology. The features
displayed by such simple structures can indeed be recognized in more complex networks. As a final remark we note that fully connected and stars with arbitrary number of central nodes seem to be the only network topologies where a simple treatment via master equations similar to (7) and (13)-(14) is possible. Even the highly symmetric ring structure (a 1D lattice with periodic boundary conditions) will not behave as if all nodes were identical, since the order of zeros and ones along the ring matters.

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