Height Adjustable Triangular (HAT) Window Function for Impulse Response Modification of Signal Processing Systems

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Abstract—A window function, in signal processing and statistics, is a mathematical function that has zero values outside its chosen interval or limit of sequence, normally symmetric around the middle of the interval. Usually the middle of the window is either maximum or near maximum and tappers smoothly as it moves away to the sides. When another function or sequence of data is mathematically multiplied by the window function it forces the product to assume its nature of zero-value outside the interval and tapering from middle to the sides. Windows are finite functions and their main function is to modify an infinite impulse response sequence so as to make it finite within its chosen interval in system design. Several windows are in existence and they include Hamming, rectangular, Han, Kaiser, Triangular, Blackman, Sine, Blackman-Harris, Gaussian, Doph-Chebyshev, and Lanczos, windows. Others are Parzen, Nuttall, flat top, and Turkey windows. The window to apply in the design depends on the characteristics of the signal to be processed, types of system to be implemented and quality of output desired. In this paper, a new called height adjustable triangular (HAT) window function is developed and added to the list of windows for signal processing systems designs. The new window has a parameter called amplitude adjustment parameter that can vary its amplitude or height. The effectiveness of the window is investigated by examining its frequency characteristics. Result shows that it is stable and linear.

Index Terms—HAT, Windows, Responses in Frequency Domain.

I. INTRODUCTION

In signal processing, processing of electrical signals is very common and can among other purposes include signal separation and signal restoration. Signal separation is needed when a signal is contaminated with interference, noise or other signals, while signal restoration is used when a signal has been distorted [1]. Impulse response sequence of a system is one of the parameters a system possesses that determine whether its design conforms to specifications. In some cases, the impulse response sequence may be infinite whereas what is desired may be finite impulse sequence, or the impulse response may be outside the desired interval. In this circumstance windows are used to make the infinite impulse response finite or confine an impulse response that is outside the chosen interval within the interval.

In addition, the window also ensures that the impulse response reduces gradually from the centre and terminates smoothly at the end and not suddenly. Windows terminate at zero or near zero depending on the type of window. However, it is necessary to point out that some processing systems require infinite impulse response data sequences to function and as such do not require window modification. A situation where the disturbances from unmodified impulse response cannot be tolerated, window functions are applied to stabilize the disturbances [2-7]. Various windows are in use but in this work a new window known as height adjustable triangular (HAT) window function is developed, and its effectiveness determined. In this window a triangular window changes from being a fixed window to an adjustable window.

II. EXISTING WINDOW FUNCTIONS

With M as the length of the window, the mathematical models of some windows are presented below.

A. Triangular (Bartlett) Window

This is as obtained from [2-4],

\[ w(n) = \begin{cases} 
  2n/(M - 1), & 0 \leq n \leq (M - 1)/2 \\
  2 - 2n/(M - 1), & (M - 1)/2 \leq n \leq M - 1 
\end{cases} \]

(1)

B. Rectangular Window

This is as in [4], [8],

\[ w(n) = 1.0 \leq n \leq M - 1 \]

(2)

C. Hanning Window

This is as in [4],

\[ w(n) = 0.5 - 0.5 \cos \frac{2\pi n}{M - 1} \leq n \leq M - 1 \]

(3)

D. Hamming Window

From [4],

\[ w(n) = 0.54 - 0.46 \cos \frac{2\pi n}{M - 1} \leq n \leq M - 1 \]

(4)

Where \( 0 \leq n \leq M - 1 \).

E. Blackman Window

From [4], [9]

\[ w(n) = 0.42 + 0.5 \cos \frac{2\pi n}{M - 1} + 0.08 \cos \frac{4\pi n}{M - 1} \]

(5)
Where $0 \leq n \leq M - 1$

### F. Blackman-Harris Window

From [9], [10], [16], [17],

$$w(n) = a_0 + a_1 \cos \frac{2\pi n}{M - 1} + a_2 \cos \frac{4\pi n}{M - 1} + a_3 \cos \frac{6\pi n}{M - 1}$$

(6)

Where $-\frac{M-1}{2} \leq n \leq \frac{M-1}{2}$, and $a_0=0.35875$, $a_1 = -0.48829$, $a_2 = 0.14128$, $a_3 = -0.01168$

### G. Gaussian Window

From [9]-[11],

$$w(n) = e^{-\frac{1}{2} \left(\frac{n}{N/2}\right)^2}$$

(7)

Where $-\frac{M-1}{2} \leq n \leq M - 1$ and $\alpha \geq 2$

$\alpha$ is the reciprocal of the standard deviation. The width of the window is inversely related to the value of $\alpha$; the larger the $\alpha$, the narrower the window.

### H. Kaiser Window

This is as obtained from [4], [9], [12], [13], [18]

$$J_\alpha \left[ \beta \left( 1 - \frac{2n}{M-1} \right)^2 \right]$$

$$W_\beta (\beta, n) = \frac{J_\alpha \left[ \beta \left( 1 - \frac{2n}{M-1} \right)^2 \right]}{J_\alpha \beta}$$

(8)

Where $-\frac{M-1}{2} \leq n \leq M - 1$

$$W_\beta (\beta, n) = \frac{J_\alpha \left[ \beta \left( 1 - \frac{2n}{M-1} \right)^2 \right]}{J_\alpha \beta}$$

(9)

Jo (x) is the modified Bessel function of the first kind of order zero.

### I. Sine Window

According to [19], sine window function is given as in (10)

$$w(n) = \sin \frac{180\beta}{M-1} 0 \leq n \leq M - 1$$

(10)

### J. Nuttall Window

As can be seen from [20], [21], the window is depicted as (11)

$$w(n) = a_0 - a_1 \cos \frac{2\pi n}{M - 1} + a_2 \cos \frac{4\pi n}{M - 1} - a_3 \cos \frac{6\pi n}{M - 1}$$

(11)

Where $-\frac{M-1}{2} \leq n \leq \frac{M-1}{2}$, and $a_0=0.355768$, $a_1 = 0.487396$, $a_2 = 0.144232$, $a_3 = 0.012604$

### III. DEVELOPMENT OF HAT WINDOW FUNCTION

The HAT window function can be developed by developing the triangular window first for clarity and extending it to form the HAT window. For the triangular window, Fig. 1 is used. From the figure, line OA of the triangular window represented by OAB.

$$w_1(n) = m_1 n$$

(12)

where $m_1$ is the gradient of the line $m_1 = 1 \cdot \frac{M - 1}{2} = \frac{2}{M - 1}$ and M is the length of the window including zero position. M-1=L, where L is the order of the window. Using $m_1$ in (12) yields

$$w_1(n) = 2n / (M - 1)$$

(13)

For line AB,

$$w_2(n) = -m_2 n + g$$

(14)

where g is the point of intersection with $w' (n)$ axis and $m_2$ gradient of the line. Also $g = 2$ and $m_2 = \frac{2}{M - 1}$ from Fig.1.

Using $m_2$ and g in (12) gives
\[ w_2(n) = 2 - n/(M - 1) \]  \hspace{1cm} (15)

The mathematical model of the triangular window is a combination of (13) and (15) as shown in (16).

\[ w(n) = \begin{cases} 
2n/(M - 1), & 0 \leq n \leq (M - 1)/2 \\
2 - 2n/(M - 1), & M - 1/2 \leq n \leq M - 1 
\end{cases} \]  \hspace{1cm} (16)

If the value of \( w(n) \) at \( n=0 \) and \( n=M-1 \) is pulled up to \( \alpha \), the amplitude response appears as shown in Fig. 2, representing HAT window. The parameter \( \alpha \) is referred to as amplitude or height adjustment parameter. Therefore, (16) is modified in the form that follows. For the rising side of the window of Fig. 2, the function is

\[ W_3(n) = m_3n + \alpha \quad \text{for} \quad 0 \leq n \leq M - 1 \]  \hspace{1cm} (17)

where \( m_3 \) is the gradient of the rising side.

\[ m_3 = (1 - \alpha)M - 1)/2 = (2 - 2\alpha)/(M - 1) \]  \hspace{1cm} (18)

Using (18) in (17) gives

\[ W_3(n) = \alpha + (2 - 2\alpha)n/(M - 1), \quad \text{for} \quad 0 \leq n \leq M - 1 \]  \hspace{1cm} (19)

For the falling side of the window, the function is

\[ w_4(n) = m_4n + g_1, \quad \text{for} \quad M - 1/2 \leq n \leq M - 1 \]  \hspace{1cm} (20)

Where \( m_4 \) is the gradient of the falling side and this is a negative gradient because the side is falling, while \( g_1 \) is the differential point of intersection of the function with the vertical axis.

\[ m_4 = - (1 - \alpha)M - 1)/2 = - (2 - 2\alpha)/(M - 1) \]

\[ g_1 = 2 - \alpha. \] Substituting \( m_4 \) and \( g_1 \) in (20) gives

\[ w_4(n) = -(2 - 2\alpha)n/(M - 1) + (2 - \alpha) = 2 - \alpha - (2 - 2\alpha)n/(M - 1) = 2 - [\alpha + (2 - 2\alpha)n/(M - 1)], \quad M - 1/2 \leq n \leq M - 1 \]  \hspace{1cm} (21)

The mathematical model is a combination of (19) and (21)

\[ w(n) = \begin{cases} 
\alpha + (2 - 2\alpha)n/(M - 1), & 0 \leq n \leq M - 1/2 \\
2 - [\alpha + (2 - 2\alpha)n/(M - 1)], & M - 1/2 \leq n \leq M - 1 
\end{cases} \]  \hspace{1cm} (22)

Where \( \alpha \) varies from 0 to 1. Equation (22) is the proposed new HAT window. As the amplitude adjustment parameter \( \alpha \) varies from “0” to “1,” the function varies from triangular window to rectangular window. At the value of \( \alpha=0 \) it is a perfect triangular window function whereas at the value of \( \alpha=1 \) it is a perfect rectangular window function.

IV. FREQUENCY CHARACTERISTICS OF THE HAT WINDOW FUNCTION

The suitability of the HAT window function in system design can be determined by analyzing the behavior of the function in frequency domain. Such behaviors include amplitude response in frequency domain, magnitude response in frequency domain and phase response in frequency domain. The window adjustment parameter \( \alpha \) varies the amplitude or height of the window for a fixed window length. This amplitude variation correspondingly varies the main and side lobes of the magnitude response of the function and can also affect the linearity of the phase of the function. The effects of \( \alpha \) on the responses can be examined by considering six different values of it (0.0, 0.05, 0.1, 0.2, 0.8, 1.0) when the length of the window is M=35. Practically the value of \( \alpha \) depends on the type and frequency of signal to be processed, interference level and desired output and the length of the window, while the value of M is the same as the length of the impulse response sequence of the processing system being designed. The amplitude, magnitude and phase responses of the function in frequency domain for various values of \( \alpha \) are depicted below.

A. Responses When \( \alpha=0.0 \)

![Fig. 3. Frequency Domain Amplitude Response when \( \alpha=0.0 \)](image)

![Fig. 4. Frequency Domain Magnitude Response when \( \alpha=0.0 \)](image)
B. Responses when $\alpha=0.05$

C. Responses when $\alpha=0.1$

Fig. 5. Frequency Domain Phase Response when $\alpha=0.0$

Fig. 6. Frequency Domain Amplitude Response when $\alpha=0.05$

Fig. 7. Frequency Domain Magnitude Response when $\alpha=0.05$

Fig. 8. Frequency Domain Phase Response when $\alpha=0.05$

Fig. 9. Frequency Domain Amplitude Response when $\alpha=0.1$

Fig. 10. Frequency Domain Magnitude Response when $\alpha=0.1$
Fig. 11. Frequency Domain Phase Response when $\alpha=0.1$

Fig. 14. Frequency Domain Phase Response when $\alpha=0.2$

D. Responses when $\alpha=0.2$

Fig. 12. Frequency Domain Amplitude Response when $\alpha=0.2$

Fig. 15. Frequency Domain Amplitude Response when $\alpha=0.8$

Fig. 13. Frequency Domain Magnitude Response when $\alpha=0.2$

Fig. 16. Frequency Domain Magnitude Response when $\alpha=0.8$

E. Responses when $\alpha=0.8$
The amplitude responses of the function in frequency domain at values of \( \alpha = 0.0, 0.1 \) and 0.2 exhibit stability because they gradually reduce as the frequency increases and eventually collapse to zero. But for values of \( \alpha = 0.8 \) and 1.0 the amplitude impulse responses do not collapse to exactly zero but are indicating sustained oscillations, implying some degree of instability of the window at those \( \alpha \) values. Furthermore, at value of \( \alpha = 1.0 \) the window becomes purely a rectangular window shown as Fig. 18. The magnitude responses in frequency domain for all the values of \( \alpha \) exhibit good degree of stability because the oscillation continues to die down as the frequency increases and no sustained oscillation is observed, but as the value of \( \alpha \) increases the width of the main lobes and side lobes decrease, implying that \( \alpha \) affects the degree to which the window can effect attenuation of unwanted signals. The phase responses in frequency domain at values of \( \alpha = 0.0 \) and 0.05 exhibits linearity which implies that they will not distort any multiple frequency signal applied to the system they are used to design. But for values \( \alpha = 0.1, 0.2, 0.8 \) and 1.0 the phase responses exhibit some degree of nonlinearity which increases as \( \alpha \) increases, implying that for the length of the window under consideration it is desirable to confine the \( \alpha \) values to between 0.0 and 0.05 to avoid distortions of multiple frequency signals.

It is also necessary to point out that the value of \( \alpha \)
depends on the window length. This is demonstrated from Fig. 22 to Fig. 33 where the window length is increased to 101, giving rise to nonlinearity of phase manifesting slightly at value of $\alpha=0.02$ and significant from 0.03.

G. Responses when the Window Length is 101

Here four values of the amplitude adjustment parameter $\alpha$ are considered and they include 0.0, 0.01, 0.02, and 0.03.

![Frequency Domain Amplitude Response](image1)

**Fig. 22.** Frequency Domain Amplitude Response when the window length is 101 and $\alpha=0.00$

![Frequency Domain Magnitude Response](image2)

**Fig. 23.** Frequency Domain Magnitude Response when the window length is 101 and $\alpha=0.00$

![Frequency Domain Phase Response](image3)

**Fig. 24.** Frequency Domain Phase Response when the window length is 101 and $\alpha=0.00$

![Frequency Domain Amplitude Response](image4)

**Fig. 25.** Frequency Domain Amplitude Response when the window length is 101 and $\alpha=0.01$

![Frequency Domain Magnitude Response](image5)

**Fig. 26.** Frequency Domain Magnitude Response when the window length is 101 and $\alpha=0.01$

![Frequency Domain Phase Response](image6)

**Fig. 27.** Frequency Domain Phase Response when the window length is 101 and $\alpha=0.01$
From the characteristics of HAT window function, it can be deduced that the function is very stable and linear and can be used to design signal processing systems without compromising the integrity of the signals being processed. The degree of stability and linearity depends on the
amplitude adjustment parameter with respect to the length of the window. For example, for a window length of 35 the phase response is very linear up to the amplitude adjustment parameter of 0.05, whereas for a window length of 101 the phase response is linear only up to the amplitude adjustment parameter of 0.02. The implication therefore is that HAT window can only be used on such signal processing systems that have lengths within the good performance range of lengths of the window if the integrity of the signal to be processed is to be maintained.

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