TELEPARALLEL GRAVITY: AN OVERVIEW

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The fundamentals of the teleparallel equivalent of general relativity are presented, and its main properties described. In particular, the field equations, the definition of an energy–momentum density for the gravitational field, the teleparallel version of the equivalence principle, and the dynamical role played by torsion as compared to the corresponding role played by curvature in general relativity, are discussed in some details.

1 Introduction

Teleparallel gravity corresponds to a gauge theory for the translation group. Due to the peculiar character of translations, any gauge theory including these transformations will differ from the usual internal gauge models in many ways, the most significant being the presence of a tetrad field. On the other hand, a tetrad field can naturally be used to define a linear Weitzenböck connection, which is a connection presenting torsion, but no curvature. A tetrad field can also naturally be used to define a riemannian metric, in terms of which a Levi–Civita connection can be constructed. As is well known, it is a connection presenting curvature, but no torsion. Now, torsion and curvature are properties of a connection, and many different connections can be defined on the same space. Therefore one can say that the presence of a nontrivial tetrad field in a gauge theory induces both a teleparallel and a riemannian structures in spacetime. The first is related to the Weitzenböck, and the second to the Levi–Civita connection. Owing to the universality of the gravitational interaction, it turns out to be possible to link these geometrical structures to gravitation.

In the context of teleparallel gravity, curvature and torsion are able to provide each one equivalent descriptions of the gravitational interaction. Conceptual differences, however, show up. According to general relativity, curvature is used to geometrize spacetime, and in this way successfully describe the gravitational interaction. Teleparallelism, on the other hand, attributes gravitation to torsion, but in this case torsion accounts for gravitation not by geometrizing the interaction, but by acting as a force. This means that, in the teleparallel equivalent of general relativity, there are no geodesics, but force equations quite analogous to the Lorentz force equation of electrodynamics. Thus, we can say that the gravitational interaction can be described alternatively in terms of curvature, as is usually done in general relativity, or in terms of torsion, in which case we have the so called teleparallel gravity. Whether gravitation requires a curved or a torsioned spacetime, therefore,
turns out to be a matter of convention.

In this paper, we are going to review the main features of teleparallel gravity. We start in Sec. 2 where we introduce the fundamentals of the teleparallel equivalent of general relativity, and discuss some of its main features. A detailed discussion of the energy-momentum current for the gravitational field is made in Sec. 3. In Sec. 4 the second Bianchi identity of a gauge theory for the translation group is obtained, from which we get the conservation law of the energy-momentum tensor for a general matter (source) field. In Sec. 5 we study the motion of a spinless particle in a gravitational field, and the gravitational analog of the Lorentz force equation is obtained. Then, a discussion on the riemannian and teleparallel versions of the equivalence principle is made. Finally in Sec. 6 we draw the main conclusions of the paper.

2 Teleparallel Equivalent of General Relativity

We use the Greek alphabet \((\mu, \nu, \rho, \ldots = 0, 1, 2, 3)\) to denote indices related to spacetime (base space), and the Latin alphabet \((a, b, c, \ldots = 0, 1, 2, 3)\) to denote indices related to the tangent space (fiber), assumed to be a Minkowski space with the metric 
\[ \eta_{ab} = \text{diag}(+1, -1, -1, -1). \]
A gauge transformation is defined as a local translation of the tangent-space coordinates,
\[ \delta x^a = \delta \alpha^b P_b x^a, \]
where \(P_a = \partial / \partial x^a\) the translation generators, and \(\delta \alpha^a\) the corresponding infinitesimal parameters. Denoting the gauge potentials by \(A^a_\mu\), the gauge covariant derivative of a general matter field \(\Psi\) is
\[ D_\mu \Psi = h^a_\mu \partial_a \Psi, \]
where
\[ h^a_\mu = \partial_\mu x^a + c^{-2} A^a_\mu \]
is a nontrivial tetrad field, with \(c\) the speed of light. From the covariance of \(D_\mu \Psi\), we obtain the transformation of the gauge potentials:
\[ A'^{a'}_\mu = A^a_\mu - c^2 \partial_\mu \delta \alpha^a. \]
As usual in abelian gauge theories, the field strength is given by
\[ F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu, \]
which satisfies the relation
\[ [D_\mu, D_\nu] \Psi = c^{-2} F^a_{\mu \nu} P_a \Psi. \]
It is important to remark that, whereas the tangent space indices are raised and lowered with the metric \(\eta_{ab}\), the spacetime indices are raised and lowered with the riemannian metric
\[ g_{\mu \nu} = \eta_{ab} h^a_\mu h^b_\nu. \]
A nontrivial tetrad field induces on spacetime a teleparallel structure which is directly related to the presence of the gravitational field. In fact, given a nontrivial tetrad, it is possible to define the so-called Weitzenböck connection
\[ \Gamma_{\rho \mu \nu}^a = h_a^\rho \partial_\nu h_a^{\mu}, \] (8)
which is a connection presenting torsion, but no curvature.\(^\text{12}\) As a natural consequence of this definition, the Weitzenböck covariant derivative of the tetrad field vanishes identically:
\[ \nabla_\nu h_a^\mu \equiv \partial_\nu h_a^\mu - \Gamma^a_{\theta \mu \nu} h_a^\theta = 0. \] (9)
This is the absolute parallelism condition. The torsion of the Weitzenböck connection is
\[ T_{\rho \mu \nu}^a = \Gamma_{\rho \nu \mu}^a - \Gamma_{\rho \mu \nu}^a, \] (10)
from which we see that the gravitational field strength is nothing but torsion written in the tetrad basis:
\[ F_{\mu \nu}^a = c^2 h_\rho T_{\rho \mu \nu}^a. \] (11)

A nontrivial tetrad field can also be used to define a torsionless linear connection, the Levi-Civita connection of the metric (7):
\[ \overset{\circ}{\Gamma}_{\sigma \mu \nu}^a = \frac{1}{2} g_{\sigma \rho} \left( \partial_\mu g_{\rho \nu} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu} \right). \] (12)

The Weitzenböck and the Levi–Civita connections are related by
\[ \Gamma_{\rho \mu \nu}^a = \overset{\circ}{\Gamma}_{\rho \mu \nu}^a + K_{\rho \mu \nu}^a, \] (13)
where
\[ K_{\rho \mu \nu}^a = \frac{1}{2} (T_{\mu \nu}^a + T_{\nu \mu}^a - T_{\rho \mu \nu}^a) \] (14)
is the contorsion tensor.

As already remarked, the curvature of the Weitzenböck connection vanishes identically:
\[ R^a_{\theta \mu \nu} = \partial_\mu \Gamma^a_{\theta \nu} + \Gamma^a_{\sigma \mu} \Gamma^\sigma_{\theta \nu} - (\mu \leftrightarrow \nu) \equiv 0. \] (15)
Substituting \(\Gamma_{\rho \mu \nu}^a\) as given by Eq.(13), we get
\[ R_{\rho \theta \mu \nu} = \overset{\circ}{R}^a_{\rho \theta \mu \nu} + Q_{\rho \theta \mu \nu} \equiv 0, \] (16)
where \(\overset{\circ}{R}^a_{\rho \mu \nu}\) is the curvature of the Levi–Civita connection, and
\[ Q_{\rho \theta \mu \nu} = D_{\mu} K_{\rho \theta \nu} - D_{\nu} K_{\rho \theta \mu} + K_{\sigma \theta \nu} K_{\rho \sigma \mu} - K_{\sigma \theta \mu} K_{\rho \sigma \nu} \] (17)
is a tensor written in terms of the Weitzenböck connection only. Here, \(D_{\mu}\) is the teleparallel covariant derivative, which is nothing but the Levi-Civita covariant derivative of general relativity rephrased in terms of the Weitzenböck connection.\(^\text{13}\) Acting on a spacetime vector \(V^\mu\), for example, its explicit form is
\[ D_{\rho} V^\mu \equiv \partial_\rho V^\mu + (\Gamma^\mu_{\lambda \rho} - K^\mu_{\lambda \rho}) V^\lambda. \] (18)

\(^3\)
Equation (16) has an interesting interpretation: the contribution \( \tilde{R}^\rho_{\theta\mu\nu} \) coming from the Levi–Civita connection compensates exactly the contribution \( Q^\rho_{\theta\mu\nu} \) coming from the Weitzenböck connection, yielding an identically zero curvature tensor \( R^\rho_{\theta\mu\nu} \). This is a constraint satisfied by the Levi–Civita and Weitzenböck connections, and is the fulcrum of the equivalence between the riemannian and the teleparallel descriptions of gravitation.

The gauge gravitational field Lagrangian is given by

\[
\mathcal{L}_G = \frac{hc^4}{16\pi G} S^\rho_{\mu\nu} T^\mu_{\rho\nu},
\]

where \( h = \det(h^\alpha_\mu) \), and

\[
S^\rho_{\mu\nu} = -S^\mu_{\rho\nu} \equiv \frac{1}{2} \left[ K^\mu_{\nu\rho} - g^\rho_{\nu\theta} T^{\theta\mu}_\theta + g^\rho_{\mu\nu} T^{\theta\nu}_\theta \right]
\]
is a tensor written in terms of the Weitzenböck connection only. As usual in gauge theories, it is quadratic in the field strength. By using relation (13), this lagrangian can be rewritten in terms of the Levi-Civita connection only. Up to a total divergence, the result is the Hilbert–Einstein Lagrangian of general relativity

\[
\mathcal{L} = -\frac{c^4}{16\pi G} \sqrt{-g} \tilde{R},
\]

where the identification \( h = \sqrt{-g} \) has been made.

By performing variations in relation to the gauge field \( A^\rho_a \), we obtain from the gauge lagrangian \( \mathcal{L}_G \) the teleparallel version of the gravitational field equation,

\[
\partial_\sigma (hS^\sigma_{a\rho}) - \frac{4\pi G}{c^4} (hj^a_\rho) = 0,
\]

where \( S^\sigma_{a\rho} \equiv h^\lambda_a S^\lambda_{\sigma\rho} \). Analogously to the Yang-Mills theories,

\[
hj^a_\rho \equiv \frac{\partial \mathcal{L}_G}{\partial h^a_\rho} = -\frac{c^4}{4\pi G} hh^\lambda_a S^\mu_{\nu\lambda} T^\nu_{\mu\lambda} + h^\rho_a \mathcal{L}_G
\]

stands for the gauge current, which in this case represents the energy and momentum of the gravitational field. The term \( (hS^\sigma_{a\rho}) \) is called superpotential in the sense that its derivative yields the gauge current \( (hj^a_\rho) \). Due to the anti-symmetry of \( S^\sigma_{a\rho} \) in the last two indices, \( (hj^a_\rho) \) is conserved as a consequence of the field equation:

\[
\partial_\rho (hj^a_\rho) = 0.
\]

Making use of the identity

\[
\partial_\rho h \equiv h\Gamma^\nu_{\rho\nu} = h \left( \Gamma^\nu_{\rho\nu} - K^\nu_{\rho\nu} \right),
\]

this conservation law can alternatively be written in the form

\[
D_\rho j^a_\rho \equiv \partial_\rho j^a_\rho + (\Gamma^\rho_{\lambda\rho} - K^\rho_{\lambda\rho}) j^a_\lambda = 0,
\]

with \( D_\rho \) denoting the teleparallel version of the covariant derivative, which is nothing but the Levi-Civita covariant derivative of general relativity rephrased in terms of the Weitzenböck connection.
3 Gravitational Energy-Momentum Current

Now comes an important point. As can be easily checked, the current $j_a^\rho$ transforms covariantly under a general spacetime coordinate transformation, is invariant under local (gauge) translation of the tangent-space coordinates, and transforms covariantly under a global tangent–space Lorentz transformation. This means that $j_a^\rho$, despite not covariant under a local Lorentz transformation, is a true spacetime and gauge tensor.

Let us now proceed further and find out the relation between the above gauge approach and general relativity. By using Eq. (8) to express $\partial_\rho h_a^\lambda$, the field equation (21) can be rewritten in a purely spacetime form,

$$\partial_\sigma (h S^\sigma_\lambda^\rho) - \frac{4\pi G}{c^4} (ht_\lambda^\rho) = 0 ,$$

where now

$$ht_\lambda^\rho = \frac{c^4 h}{4\pi G} \Gamma^\mu_{\nu\lambda} S_{\mu}^{\nu\rho} + \delta_{\lambda}^{\rho} L_G$$

stands for the canonical energy-momentum pseudotensor of the gravitational field. Despite not apparent, Eq. (26) is symmetric in $(\lambda\rho)$. Furthermore, by using Eq. (13), it can be rewritten in terms of the Levi-Civita connection only. As expected, due to the equivalence between the corresponding Lagrangians, it is the same as Einstein’s equation:

$$\frac{h}{2} \left[ \frac{\circ}{\circ} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \frac{\circ}{\circ} R \right] = 0 .$$

It is important to notice that the canonical energy-momentum pseudotensor $t_\lambda^\rho$ is not simply the gauge current $j_a^\rho$ with the algebraic index “$a$” changed to the spacetime index “$\lambda$”. It incorporates also an extra term coming from the derivative term of Eq. (21):

$$t_\lambda^\rho = h_a^\lambda j_a^\rho + \frac{c^4}{4\pi G} \Gamma^\mu_{\lambda\nu} S_{\mu}^{\nu\rho} .$$

We see thus clearly the origin of the connection-term which transforms the gauge current $j_a^\rho$ into the energy-momentum pseudotensor $t_\lambda^\rho$. Through the same mechanism, it is possible to appropriately exchange further terms between the derivative and the current terms of the field equation (26), giving rise to different definitions for the energy-momentum pseudotensor, each one connected to a different superpotential $(h S^\sigma_\lambda^\rho)$.

Like the gauge current $(h j_a^\rho)$, the pseudotensor $(ht_\lambda^\rho)$ is conserved as a consequence of the field equation:

$$\partial_\rho (ht_\lambda^\rho) = 0 .$$

However, in contrast to what occurs with $j_a^\rho$, due to the pseudotensor character of $t_\lambda^\rho$, this conservation law can not be rewritten with a covariant derivative.

Because of its simplicity and transparency, the teleparallel approach to gravitation seems to be much more appropriate than general relativity to deal with the energy problem of the gravitational field. In fact, Møller already noticed a long
time ago that a satisfactory solution to the problem of the energy distribution in a gravitational field could be obtained in the framework of a tetrad theory. In our notation, his expression for the gravitational energy-momentum density is

\[ h_t^\lambda{}^\rho = \frac{\partial \mathcal{L}}{\partial \partial_{\rho} h^\lambda_{\mu}} \partial_{\lambda} h^\rho_{\mu} + \delta^\lambda{}^\rho \mathcal{L}, \] (31)

which is nothing but the usual canonical energy-momentum density yielded by Noether’s theorem. Using for \( \mathcal{L} \) the gauge Lagrangian \( 11 \), it is an easy task to verify that Møller’s expression coincides exactly with the teleparallel energy-momentum density appearing in the field equation \( 24-27 \). Since \( j_\mu{}^\rho \) is a true spacetime tensor, whereas \( t^\lambda{}^\rho \) is not, we can say that the gauge current \( j_\mu{}^\rho \) is an improved version of the Møller’s energy-momentum density \( t^\lambda{}^\rho \). Mathematically, they can be obtained from each other by using the relation \( 29 \). It should be remarked, however, that both of them transform covariantly only under global tangent-space Lorentz transformations. The lack of a local Lorentz covariance can be considered as the teleparallel manifestation of the pseudotensor character of the gravitational energy-momentum density in general relativity.\[ 16 \]

### 4 Bianchi Identity and Matter Energy-Momentum Conservation

As is well known, the second Bianchi identity of general relativity is

\[ \tilde{\nabla}_\sigma R^\lambda{}_{\rho\mu\nu} + \tilde{\nabla}_\nu R^\lambda{}_{\rho\sigma\mu} + \tilde{\nabla}_\mu R^\lambda{}_{\rho\nu\sigma} = 0, \] (32)

where \( \tilde{\nabla}_\mu \) is the usual Levi-Civita covariant derivative. Its contracted form is

\[ \tilde{\nabla}_\mu \left[ R^\mu{}_{\nu} - \frac{1}{2} g^\mu{}_{\nu} R \right] = 0. \] (33)

By using Eq.(16), after a tedious but straightforward calculation, it is possible to rewrite it in terms of the Weitzenböck connection only. The result is

\[ D_\rho \left[ \partial_\sigma (h S^\lambda{}_{\sigma\rho}) - \frac{4\pi G}{c^4} (h t^\lambda{}^\rho) \right] = 0, \] (34)

where \( D_\rho \) is the teleparallel covariant derivative, defined in Eq.(25). This is the second Bianchi identity of the teleparallel equivalent of general relativity. It says that the teleparallel covariant derivative of the sourceless field equation \( 26 \) vanishes identically.

In the presence of a general matter field, the teleparallel field equation \( 26 \) becomes

\[ \partial_\sigma (h S^\lambda{}_{\sigma\rho}) - \frac{4\pi G}{c^4} (h t^\lambda{}^\rho) = \frac{4\pi G}{c^4} (h T^\lambda{}^\rho), \] (35)

with \( T^\lambda{}^\rho \) the matter energy-momentum tensor. As a consequence of the Bianchi identity \( 34 \), and using \( 24 \), we obtain

\[ D_\mu T^\lambda{}^\rho = 0. \] (36)

This is the conservation law of matter energy-momentum tensor. Therefore, we see that in teleparallel gravity, it is not the Weitzenböck covariant derivative \( \nabla_\mu \), but...
the teleparallel covariant derivative (25) that yields the correct conservation law for the energy-momentum tensors of matter fields. It should be remarked that (36) is the unique law compatible with the corresponding conservation law of general relativity,

\[ \nabla^\nu T_{\mu \rho} = \partial_\mu T^\nu_\rho + \Gamma^\nu_{\lambda \mu} T^\lambda_\rho - \tilde{\Gamma}^\lambda_{\rho \mu} T^\mu_\lambda = 0. \]  

(37)
as can easily be verified by using Eq.(13).

5 Geodesics Versus Force Equation

In the framework of the teleparallel description of gravitation, the action describing a particle of mass \( m \) submitted to a gravitational \( (A^a_\mu) \) field is

\[ S = \int^b_a \left[ -mc \, d\sigma - \frac{m}{c} A^a_\mu u_a \, dx^\mu \right], \]

(38)

where \( d\sigma = (\eta_{ab} dx^a dx^b)^{1/2} \) is the invariant tangent-space interval, and \( u_a = dx_a / d\sigma \) is the tangent-space four-velocity. The corresponding equation of motion is

\[ c^2 h^a_\mu \frac{du_a}{ds} = F^a_\rho \mu u^\mu, \]

(39)

where \( ds = (g_{\mu \nu} dx^\mu dx^\nu)^{1/2} \) is the invariant spacetime interval, and \( u^\mu = dx^\mu / ds \) is the spacetime four velocity. It is important to remark that, by using the relations

\[ h^a_\mu u^a u_\mu = 1, \]

and

\[ \frac{\partial x^\mu}{\partial x^a} u^a u_\mu = \frac{ds}{d\sigma}, \]

the action (38) reduces to its general relativity version

\[ S = -\int^b_a mc \, ds. \]

In this case, the interaction of the particle with the gravitational field is generated by the presence of the metric tensor \( g_{\mu \nu} \) in \( ds \), or alternatively in the constraint \( u^2 = g_{\mu \nu} u^\mu u^\nu = 1 \) if one opts for using a lagrangian formalism.

Now, by transforming algebra into spacetime indices, the equation of motion (37) can be rewritten alternatively in terms of magnitudes related to the Weitzenböck or to the Riemann spacetime, giving rise respectively to the teleparallel and the metric equations of motion. In fact, by using the relation

\[ h^a_\mu \frac{du_a}{ds} = \omega_\mu \equiv \frac{du_\mu}{ds} - \Gamma^\theta_{\mu \nu} u^\theta u^\nu, \]

(40)

where \( \omega_\mu \) is the spacetime particle four–acceleration, it reduces to

\[ \frac{du_\mu}{ds} - \Gamma^\theta_{\mu \nu} u^\theta u^\nu = T^\theta_{\mu \nu} u^\theta u^\nu. \]

(41)
The left–hand side of this equation is the Weitzenböck covariant derivative of $u_\mu$ along the world line of the particle. The presence of the torsion tensor on its right–
hand side shows that it plays the role of an external force. Substituting Eq.(10), it becomes
\[ \frac{du_\mu}{ds} - \Gamma_{\theta\nu\mu} u^\theta u^\nu = 0. \] (42)

It is important to remark that, as $\Gamma_{\theta\nu\mu}$ is not symmetric in the last two indices, this is not a geodesic equation. This means that the trajectories followed by spinless particles are not geodesics of the induced Weitzenböck spacetime. In a locally inertial coordinate system, $\partial_\mu g_{\theta\nu} = 0$, and the Weitzenböck connection $\Gamma_{\theta\nu\mu}$ becomes skew–symmetric in the first two indices. In this coordinate system, therefore, owing to the symmetry of $u^\theta u^\nu$, the force equation (42) becomes the equation of motion of a free particle. This is the teleparallel version of the (weak) equivalence principle.

We transform again algebra into spacetime indices, but now in such a way to get the force equation (39) written in terms of the Levi–Civita connection only. Following the same steps used earlier, we get
\[ \frac{du_\mu}{ds} - \Gamma_{\theta\mu\nu} u^\theta u^\nu = T_{\theta\mu\nu} u^\theta u^\nu. \] (43)

Then, by taking into account the symmetry of $u^\theta u^\nu$ under the exchange $(\theta \leftrightarrow \nu)$, we can rewrite it as
\[ \frac{du_\mu}{ds} - \Gamma_{\theta\mu\nu} u^\theta u^\nu = K_{\mu\theta\nu} u^\theta u^\nu. \] (44)

Noticing that $K_{\mu\theta\nu}$ is skew–symmetric in the first two indices, and using Eq.(13) to express $(K_{\theta\mu\nu} - \Gamma_{\theta\mu\nu})$, Eq.(44) becomes
\[ \frac{du_\mu}{ds} - \Gamma_{\theta\mu\nu} u^\theta u^\nu = 0. \] (45)

This is precisely the geodesic equation of general relativity, which means that the trajectories followed by spinless particles are geodesics of the induced Riemann spacetime. According to this description, therefore, the only effect of the gravitational field is to induce a curvature in spacetime, which will then be the responsible for determining the trajectory of the particle. In a locally inertial coordinate system, the first derivative of the metric tensor vanishes, the Levi–Civita connection vanishes as well, and the geodesic equation (45) becomes the equation of motion of a free particle. This is the usual version of the (weak) equivalence principle as formulated in general relativity.

Notice the difference in the index contractions between the connections and the four–velocities in equations (42) and (45). This difference is the responsible for the different characters of these equations: the first is a force equation written in the underlying Weitzenböck spacetime, and the second is a true geodesic equation written in the induced Riemann spacetime. Now, as both equations are deduced from the same force equation (39), they must be equivalent ways of describing the same physical trajectory. In fact, it is easy to see that any one of them can be obtained from the other by using the relation (13).
6 Conclusions

In general relativity, the presence of a gravitational field is expressed by the torsionless Levi–Civita metric–connection, whose curvature determines the intensity of the gravitational field. On the other hand, in the teleparallel description of gravitation, the presence of a gravitational field is expressed by the flat Weitzenböck connection, whose torsion is now the entity responsible for determining the intensity of the gravitational field. The gravitational interaction, therefore, can be described either, in terms of curvature, or in terms of torsion. Whether gravitation requires a curved or a torsioned spacetime, therefore, is a matter of convention.

An important point of the teleparallel equivalent of general relativity is that it allows for the definition of an energy-momentum gauge current \( j_a^\rho \) for the gravitational field which is covariant under a spacetime general coordinate transformation, and transforms covariantly under a global tangent-space Lorentz transformation. This means essentially that \( j_a^\rho \) is a true spacetime tensor, but not a tangent–space tensor. Then, by rewriting the gauge field equation in a purely spacetime form, it becomes Einstein’s equation, and the gauge current \( j_a^\rho \) reduces to the canonical energy-momentum pseudotensor of the gravitational field. Teleparallel gravity, therefore, seems to provide a more appropriate environment to deal with the energy problem since in the ordinary context of general relativity, the energy-momentum density for the gravitational field will always be represented by a pseudotensor.

Like in general relativity, the conservation law of the energy-momentum tensor of matter (source) fields can be obtained from the Bianchi identities. In the case of teleparallel gravity, the energy-momentum tensor turns out to be conserved with the teleparallel covariant derivative, which is the Levi-Civita covariant derivative of general relativity rephrased in terms of the Weitzenböck connection.\(^{15}\)

Finally, we have succeeded in obtaining a gravitational analog of the Lorentz force equation, which is an equation written in the underlying Weitzenböck spacetime. According to this approach, the trajectory of the particle is described in the very same way the Lorentz force describes the trajectory of a charged particle in the presence of an electromagnetic field, with torsion playing the role of force. When rewritten in terms of magnitudes related to the metric structure, it becomes the geodesic equation of general relativity, which is an equation written in the underlying Riemann spacetime. As both equations are deduced from the same force equation, they must be equivalent ways of describing the same physical trajectory.

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