Asymptotics of Young tableaux in the strip, the $d$-sums

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September 1, 2010

Abstract. The asymptotics of the ”strip” sums $S^{(\alpha)}(n)$ and of their $d$-sums generalizations $T^{(\alpha)}_{d,ds}(dm)$ (see Definition 1.1) were calculated in [5]. It was recently noticed that when $d > 1$ there is a certain confusion about the relevant notations in [5], and the constant in the asymptotics of these $d$-sums $T^{(\alpha)}_{d,ds}(dm)$ seems to be off by a certain factor. Based on the techniques of [5] we again calculate the asymptotics of the $d$-sums $T^{(\alpha)}_{d,ds}(dm)$. We do it here carefully and with complete details. This leads to Theorem 1.2 below, which replaces Corollary 4.4 of [5] in the cases $d > 1$.

Mathematics Subject Classification: 05A16, 34M30.

1 Introduction

Let $\lambda$ be a partition and $\ell(\lambda)$ the number of non-zero parts of $\lambda$. Let $f^{\lambda}$ denote the number of standard tableaux of shape $\lambda$. For the Young-Frobenius formula for $f^{\lambda}$ see for example [2, 2.3.22], and for the ”hook” formula see for example [8, corollary 7.21.5].

The asymptotics of the sums $S^{(\alpha)}_{\ell}(n)$ and of the $d$-sums $T^{(\alpha)}_{d,ds}(dm)$ (see Definition 1.1) were studied in [4], see [5, Corollary 4.4] (there we used the notation $d_{\lambda}$ instead of $f^{\lambda}$). We recently noticed that when $d > 1$ there is a certain confusion about the notations in [5], and the constant in the asymptotics of the $d$-sums $T^{(\alpha)}_{d,ds}(dm)$ seems to be off by a certain factor.

Based on the techniques of [5] we calculate, with complete details, the asymptotics of the $d$-sums $T^{(\alpha)}_{d,ds}(dm)$. While the asymptotic formula for the sums $S^{(\alpha)}_{\ell}(n)$ remain unchanged as in [4], this leads to a new asymptotic formula for the $d$-sums $T^{(\alpha)}_{d,ds}(dm)$, given in Theorem 1.2 below.

The validity of Theorem 1.2 can be tested as follows. In few cases the $d$-sums $T^{(\alpha)}_{d,ds}(dm)$ are given by a closed formula, which yield the corresponding asymptotics directly— independent of Theorem 1.2. In all these cases, the direct asymptotics and the asymptotics deduced from Theorem 1.2— agree, see Section 3.1. Also, for small values of $d$ and $s$ it is possible to write an explicit formula for, say, $T^{(1)}_{d,ds}(dm)$. By Theorem 1.2 $T^{(1)}_{d,ds}(dm) \approx A(d, s, dm)$. 

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Now form the ratio $T^{(1)}_{d,ds}(dm)/A(d, s, dm)$. Using, say, "Mathematica", calculate that ratio for increasing values of $m$, verifying that these values become closer and closer to 1 as $m$ increases. This again tests and indicates the validity of Theorem 1.2.

1.1 The main theorem

The following definition recalls the $d$-sums from [5].

Definition 1.1. Let $m, s, d \geq 1$, then define

1. $B_d(dm) = \{ \lambda \vdash dm \mid d \text{ divides all } \lambda' \}$.

Note that $\lambda \in B_d(dm)$ if and only if $\lambda = (\mu_1^d, \mu_2^d, \ldots)$ with $(\mu_1, \mu_2, \ldots) \vdash m$, and then $d$ divides $\ell(\lambda)$.

2. $B_{d,ds}(dm) = \{ \lambda \in B_d(dm) \mid \ell(\lambda) \leq ds \}$ and

3. $T^{(\alpha)}_{d,ds}(dm) = \sum_{\lambda \in B_{d,ds}(dm)} (f^\lambda)^\alpha$.

4. When $d = 1$ we denote $T^{(\alpha)}_{1,s}(m) = S^{(\alpha)}_s(m)$. Thus

$$S^{(\alpha)}_s(m) = \sum_{\lambda \vdash m, \ell(\lambda) \leq s} (f^\lambda)^\alpha.$$ 

We correct [5, Corollary 4.4] in the case $d > 1$ by proving the following theorem (see Theorem 3.3 below). Here the variable $N$ is replaced by $s$.

Theorem 1.2. Let $1 \leq d, s \in \mathbb{Z}$ and let $0 < \alpha \in \mathbb{R}$. As $m \to \infty$,

$$T^{(\alpha)}_{d,ds}(dm) \sim$$

$$\left[ \left( \frac{1}{\sqrt{2\pi}} \right)^{ds-1} \cdot \sqrt{d} \cdot s^{d^2s/2} \cdot (2! \cdots (d - 1)!)^s \cdot \left( \frac{1}{\sqrt{m}} \right)^{(d^2s^2 + d^2s - 2)/2} \cdot (ds)^{dm} \right]^\alpha \cdot$$

$$\cdot \left( \frac{d}{s} \right)^{(s-1)(\alpha s + 2)/4} \cdot \frac{d}{\sqrt{s}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{s!} \cdot$$

$$\cdot (2\pi)^{s/2} \cdot (d^2\alpha)^{-s/2 - d^2\alpha s(s-1)/4} \cdot (\Gamma(1 + d^2\alpha/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1 + d^2\alpha j/2).$$
2 Asymptotics for a single $f^\lambda$

The following proposition corrects (and replaces) [5 (F.1.3)], and is the key for proving Theorem 1.2. Recall the notation

$$D_s(x_1, \ldots, x_s) = \prod_{1 \leq i < j \leq s} (x_i - x_j).$$

**Proposition 2.1.** Let $\lambda = (\lambda_1^d, \ldots, \lambda_s^d) \vdash dm = n$. For $1 \leq i \leq s$ write $\lambda_i = m/s + b_i \sqrt{m}$ and assume the $b_i$ are bounded, so $\lambda_i \simeq m/s$. Then, as $m$ goes to infinity,

$$f^\lambda \simeq \left( \frac{1}{\sqrt{2\pi}} \right)^{ds-1} \cdot \sqrt{d} \cdot \frac{1}{\sqrt{m}} \cdot (2! \cdots (d-1)!)^s \cdot \frac{(1)}{(d^2 s^2 + d^2 s - 2)!} \cdot (ds)^{dm} \cdot D_s(b_1, \ldots, b_s)^d \cdot e^{-(ds/2)(b_1^2 + \cdots + b_s^2)} =$$

$$= \left( \frac{1}{\sqrt{2\pi}} \right)^{ds-1} \cdot d^{(d^2 s^2 + d^2 s)/4} \cdot \frac{1}{\sqrt{m}} \cdot (2! \cdots (d-1)!)^s \cdot \frac{(1)}{(d^2 s^2 + d^2 s - 2)!} \cdot (ds)^{dm} \cdot D_s(b_1, \ldots, b_s)^d \cdot e^{-(ds/2)(b_1^2 + \cdots + b_s^2)}.$$

**Proof.** Apply, for example, the Young-Frobenius formula for $f^\lambda$: First, all $\lambda_i \simeq m/s$, hence we can write

$$f^\lambda \simeq \left( \frac{s}{m} \right)^{ds(ds-1)/2} \cdot \frac{(dm)!}{(\lambda_1)! \cdots (\lambda_s)!} \cdot H(\lambda_1, \ldots, \lambda_s) \quad (1)$$

where $H(\lambda_1, \ldots, \lambda_s)$ is the product of factors of the form $\lambda_i - \lambda_j + k$, with various $0 \leq k \leq ds$, and which we now analyze.

For $1 \leq i < j \leq s$ there are $d^2$ factors of $f^\lambda$ of the form $\lambda_i - \lambda_j + k$, with various $k$’s, all of them satisfying $\lambda_i - \lambda_j + k \simeq (b_i - b_j) \sqrt{m}$. The number of pairs $(i, j)$ where $1 \leq i < j \leq s$ is $s(s-1)/2$, and each such pair contributes $d^2$ times the factor $(b_i - b_j) \sqrt{m}$, hence the factor $D_s(b_1, \ldots, b_s)^d \cdot (\sqrt{m})^{d^2 s(s-1)/2}$ in (2) below.

In the cases $i = j$ each of the $s$ blocks $(\lambda_i^d)$ contributes $D_d(d, d-1, \ldots, 1) = 1! \cdot 2! \cdots (d-1)!$, hence the factor $(1! \cdot 2! \cdots (d-1)!)^s$ in (2) below. It follows that

$$f^\lambda \simeq \left( \frac{s}{m} \right)^{ds(ds-1)/2} \cdot (2! \cdots (d-1)!)^s \cdot D_s(b_1, \ldots, b_s)^d \cdot (\sqrt{m})^{d^2 s(s-1)/2} \cdot \frac{(dm)!}{(\lambda_1)! \cdots (\lambda_s)!} \quad (2)$$

Again since $\lambda_i \simeq m/s$,

$$\frac{m!}{(\lambda_1 + s - 1)! \cdots (\lambda_s)!} \simeq \left( \frac{s}{m} \right)^{s(s-1)/2} \cdot \frac{m!}{(\lambda_1)! \cdots (\lambda_s)!}.$$

$$\frac{m!}{(\lambda_1 + s - 1)! \cdots (\lambda_s)!} \simeq \left( \frac{s}{m} \right)^{s(s-1)/2} \cdot \frac{m!}{(\lambda_1)! \cdots (\lambda_s)!}.$$
By [5] Step 3, page 118, with $\sqrt{2\pi}$ replacing and correcting $2\pi$]

\[
\frac{m!}{(\lambda_1 + s - 1)! \cdots (\lambda_s)!} \simeq \left( \frac{1}{\sqrt{2\pi}} \right)^{s-1} \cdot \left( \frac{1}{m} \right)^{s^2/2} \cdot \left( \frac{1}{m} \right)^{(s-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2 + \cdots + b_s^2)},
\]

(4)
hence by (3) and (1)

\[
\frac{m!}{(\lambda_1!) \cdots (\lambda_s)!} \simeq \left( \frac{m}{s} \right)^{s(s-1)/2} \cdot \left( \frac{1}{\sqrt{2\pi}} \right)^{s-1} \cdot s^{s^2/2} \cdot \left( \frac{1}{m} \right)^{(s-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2 + \cdots + b_s^2)}
\]

\[= \left( \frac{1}{\sqrt{2\pi}} \right)^{s-1} \cdot s^{s/2} \cdot \left( \frac{1}{m} \right)^{(s-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2 + \cdots + b_s^2)}.\]

(5)
Now

\[
\frac{(dm)!}{(\lambda_1)! \cdots (\lambda_s)!} \simeq \left( \frac{(dm)!}{(m)!^d} \right) \left( \frac{m!}{\lambda_1! \cdots \lambda_s!} \right)^d
\]

(6)
and by Stirling’s formula

\[
\frac{(dm)!}{(m)!^d} \simeq \left( \frac{1}{\sqrt{2\pi}} \right)^{d-1} \cdot \sqrt{d} \cdot \left( \frac{1}{\sqrt{m}} \right)^{d-1} \cdot d^{dm}.
\]

(7)
It follows from (5), (6) and (7) that

\[
\frac{(dm)!}{(\lambda_1)! \cdots (\lambda_s)!^d} \simeq
\]

\[\simeq \left[ \left( \frac{1}{\sqrt{2\pi}} \right)^{d-1} \cdot \sqrt{d} \cdot \left( \frac{1}{\sqrt{m}} \right)^{d-1} \cdot d^{dm} \right] \left[ \left( \frac{1}{\sqrt{2\pi}} \right)^{s-1} \cdot s^{s/2} \cdot \left( \frac{1}{m} \right)^{(s-1)/2} \cdot s^m \cdot e^{-(s/2)(b_1^2 + \cdots + b_s^2)} \right]^d
\]

\[= \left( \frac{1}{\sqrt{2\pi}} \right)^{ds-1} \cdot \sqrt{d} \cdot s^{ds/2} \cdot \left( \frac{1}{\sqrt{m}} \right)^{ds-1} \cdot (ds)^{dm} \cdot e^{-(ds/2)(b_1^2 + \cdots + b_s^2)}.\]

(8)
Together with (2) this yields

\[
f^{\lambda} \simeq \left[ \left( \frac{s}{m} \right)^{ds(ds-1)/2} \cdot D_s(b_1, \ldots, b_s)^{d^2} \cdot (2! \cdots (d-1)!)^s \cdot (\sqrt{m})^{d^2(s-1)/2} \right] \cdot \left[ \left( \frac{1}{\sqrt{2\pi}} \right)^{ds-1} \cdot \sqrt{d} \cdot s^{ds/2} \cdot \left( \frac{1}{\sqrt{m}} \right)^{ds-1} \cdot (ds)^{dm} \cdot e^{-(ds/2)(b_1^2 + \cdots + b_s^2)} \right] =
\]

\[= \left( \frac{1}{\sqrt{2\pi}} \right)^{ds-1} \cdot \sqrt{d} \cdot s^{ds^2/2} \cdot (2! \cdots (d-1)!)^s \cdot \left( \frac{1}{\sqrt{m}} \right)^{(d^2s^2+d^2s-2)/2} \cdot (ds)^{dm} \cdot D_s(b_1, \ldots, b_s)^{d^2} \cdot e^{-(ds/2)(b_1^2 + \cdots + b_s^2)}.\]

This completes the proof of Proposition [2.1]
2.1 Some examples

Example 2.2. Using "Mathematica", Proposition 2.1 was tested and confirmed in the case 
\( d = 3, \ s = 2, \ b_1 = 1 \) and \( b_2 = -1 \), and with \( n = 3m \) getting larger and larger.

Example 2.3. The case \( s = 1 \), any \( d \), so \( \lambda = (m, \ldots, m) = (m^d) \). In this case
\[
 f^\lambda = \frac{(dm)! \cdot 2! \cdots (d-1)!}{m! \cdot (m+1)! \cdots (m+d-1)!}.
\]

By applying Stirling's formula directly we get that as \( m \to \infty \),
\[
f^\lambda \simeq \left( \frac{1}{\sqrt{2\pi}} \right)^{d-1} \cdot 2! \cdots (d-1)! \cdot \sqrt{d} \cdot \left( \frac{1}{\sqrt{m}} \right)^{d^2-1},\]

This agrees with Proposition 2.1 since the factor \( D_s(b_1, \ldots, b_s)^{d^2} \cdot e^{-(ds/2)(b_1^2 + \cdots + b_s^2)} \) in that
proposition equals 1 in this case.

Example 2.4. Here we repeat the proof of Proposition 2.1 - in the case \( d = s = 2 \), showing
more explicitly the various steps of the calculations. Let
\( \lambda = (\lambda_1, \lambda_2, \lambda_1, \lambda_2) \vdash 2m, \) so
\( (\lambda_1, \lambda_2) \vdash m. \) Let \( \lambda_j = \frac{m}{2} + b_j \sqrt{m} \simeq \frac{m}{2}. \) In that case we verify directly that
\[
f^\lambda \simeq \left( \frac{1}{\sqrt{2\pi}} \right)^3 \cdot 2^{14} \cdot \left( \frac{1}{\sqrt{2m}} \right)^{11} 4^{2m} \cdot (b_1 - b_2)^4 \cdot e^{-2(b_1^2 + b_2^2)}. \quad (9)
\]

Proof. By either the hook formula or by the Young-Frobenius formula
\[
f^\lambda = \frac{(2m)! \cdot (\lambda_1 - \lambda_2 + 1) \cdot (\lambda_1 - \lambda_2 + 2) \cdot (\lambda_1 - \lambda_2 + 3)}{(\lambda_1 + 3)! \cdot (\lambda_1 + 2)! \cdot (\lambda_2 + 1)! \cdot \lambda_2!}.
\]

Also \( \lambda_1 + j \simeq m/2 \) while \( \lambda_1 - \lambda_2 + j \simeq (b_1 - b_2) \sqrt{m}, \) hence
\[
f^\lambda \simeq \left( \frac{2}{m} \right)^6 \cdot (b_1 - b_2)^4 \cdot m^2 \cdot \frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2}. \quad (10)
\]

By [5, "Step 3" with \( \sqrt{2\pi} \) replacing \( 2\pi \) (page 118)]
\[
\frac{m!}{(\lambda_1 + 1)! \cdot \lambda_2} \simeq \frac{1}{\sqrt{2\pi}} \cdot 4 \cdot 2^m \cdot \left( \frac{1}{m} \right)^{3/2} \cdot e^{-(b_1^2 + b_2^2)}.
\]

Since \( \lambda_1 + 1 \simeq m/2, \)
\[
\frac{m!}{\lambda_1! \cdot \lambda_2!} \simeq \frac{m}{2} \cdot \frac{1}{\sqrt{2\pi}} \cdot 4 \cdot 2^m \cdot \left( \frac{1}{m} \right)^{3/2} \cdot e^{-(b_1^2 + b_2^2)} = \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot 2^m \cdot \left( \frac{1}{m} \right)^{1/2} \cdot e^{-(b_1^2 + b_2^2)}.
\]
Also
\[
\frac{(2m)!}{(m!)^2} \simeq \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{m}} \cdot 2^{2m}.
\]

Thus
\[
\frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2} = \left( \frac{m!}{\lambda_1! \cdot \lambda_2!} \right)^2 \cdot \frac{(2m)!}{(m!)^2} \simeq
\]
\[
\left[ \frac{1}{\sqrt{2\pi}} \cdot 2 \cdot 2^m \cdot \left( \frac{1}{m} \right)^{1/2} \cdot e^{-(b_1^2 + b_2^2)} \right]^2 \cdot \left[ \frac{\sqrt{2}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{m}} \cdot 2^{2m} \right]
\]

namely
\[
\frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2} \simeq \left( \frac{1}{\sqrt{2\pi}} \right)^3 \cdot 4 \cdot \sqrt{2} \cdot 4^{2m} \cdot \left( \frac{1}{m} \right)^{3/2} \cdot e^{-2(b_1^2 + b_2^2)}.
\]

Finally
\[
f^\lambda \simeq \left( \frac{2}{m} \right)^6 \cdot m^2 \cdot (b_1 - b_2)^4 \cdot \frac{(2m)!}{(\lambda_1!)^2 \cdot (\lambda_2!)^2} \simeq
\]
\[
\left( \frac{2}{m} \right)^6 \cdot m^2 \cdot (b_1 - b_2)^4 \cdot \left( \frac{1}{\sqrt{2\pi}} \right)^3 \cdot 4 \cdot \sqrt{2} \cdot 4^{2m} \cdot \left( \frac{1}{m} \right)^{3/2} \cdot e^{-2(b_1^2 + b_2^2)} =
\]
\[
= \left( \frac{1}{\sqrt{2\pi}} \right)^3 \cdot 2^{14} \cdot \left( \frac{1}{\sqrt{2m}} \right)^{11} 4^{2m} \cdot (b_1 - b_2)^4 \cdot e^{-2(b_1^2 + b_2^2)},
\]

which verifies (9). □

3 Asymptotics for the general sums

As in [5, Theorem 3.2], Proposition 2.1 implies

**Theorem 3.1.** [5, Corollary 4.4 corrected] Let \( \Omega(s) \subset \mathbb{R}^s \) denote the following domain:
\[
\Omega(s) = \{ (x_1, \ldots, x_s) \in \mathbb{R}^s \mid x_1 \geq \cdots \geq x_s \text{ and } x_1 + \cdots + x_s = 0 \}.
\]

Also recall Definition 1.1. Then, as \( m \to \infty \),
\[
T_{d,d_s}^{(\alpha)}(dm) \simeq
\]
\[
\simeq \left[ \left( \frac{1}{\sqrt{2\pi}} \right)^{d-1} \cdot \sqrt{d} \cdot s^{d^2/2} \cdot (2! \cdots (d-1)!)^s \cdot \left( \frac{1}{\sqrt{m}} \right)^{(d^2s^2+d^2s-2)/2} \cdot (ds)^{dm} \right]^\alpha \cdot \left( \sqrt{m} \right)^{s-1} \cdot I(d^2, s, \alpha)
\]

where
\[
I(d^2, s, \alpha) = \int_{\Omega(s)} \left[ D_s(x_1, \ldots, x_s)^{d^2} \cdot e^{-(ds/2)(x_1^2 + \cdots + x_s^2)} \right]^\alpha \cdot dx_1 \cdots dx_{s-1}.
\]
Remark 3.2. Note that by [5, Section 4] and by the Selberg integral [1], [3], [6]

\[ I(d^2, s, \alpha) = \left(\frac{d}{s}\right)^{(s-1)(\alpha s+2)/4} \cdot \frac{d}{\sqrt{s}} \cdot \sqrt{\frac{\alpha}{2\pi}} \cdot \frac{1}{s!} \cdot (2\pi)^{s/2} \cdot (d^2\alpha)^{-s/2-d\alpha s(s-1)/4} \cdot (\Gamma(1+d^2\alpha/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1+d^2\alpha j/2). \]

Thus Theorem 3.1 can be rewritten as follows.

**Theorem 3.3.** Let \( 1 \leq s, d \in \mathbb{Z} \) and \( 0 < \alpha \in \mathbb{R} \). Then, as \( m \to \infty \),

\[ T_{d,ds}(dm) \simeq \left[ \left(\frac{1}{\sqrt{2\pi}}\right)^{ds-1} \cdot \sqrt{d} \cdot d^{s^2/2} \cdot (2! \cdots (d-1)!)^s \cdot \left(\frac{1}{\sqrt{m}}\right)^{(d^2s^2+d^2s-2)/2} \cdot (ds)^{dm} \right]^\alpha. \]

\[ \cdot (\sqrt{m})^{s-1} \cdot \left(\frac{d}{s}\right)^{(s-1)(\alpha s+2)/4} \cdot \frac{d}{\sqrt{s}} \cdot \sqrt{\frac{\alpha}{2\pi}} \cdot \frac{1}{s!} \cdot (2\pi)^{s/2} \cdot (d^2\alpha)^{-s/2-d\alpha s(s-1)/4} \cdot (\Gamma(1+d^2\alpha/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1+d^2\alpha j/2). \]

### 3.1 Some special cases

#### 3.1.1 The case \( s = 1 \)

Let \( s = 1 \). In that case \( B_{d,d}(dm) = \{\lambda\} \) where \( \lambda = (m, \ldots, m) = (m^d) \). Thus, for Theorem 3.1 to hold, the product of the factors after the factor \([\ldots]^\alpha\) should equal 1, which is easy to verify.

#### 3.1.2 The sums \( S_{d}^{(s)}(m) \)

In the case \( d = 1 \), in the notations of [5], \( T_{d,s}^{(s)}(m) = S_{d}^{(s)}(m) \), and Theorem 3.3 becomes

**Theorem 3.4.** [5, Corollary 4.4]. Let \( d = 1, 1 \leq s \in \mathbb{Z}, 0 \leq \alpha \in \mathbb{R} \). Then, as \( m \to \infty \),

\[ T_{1,s}^{(s)}(m) = S_{d}^{(s)}(m) \simeq \left[ \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot s^{s^2/2} \cdot \left(\frac{1}{\sqrt{m}}\right)^{(s^2+s-2)/2} \cdot s^m \right]^\alpha \cdot (\sqrt{m})^{s-1} \cdot \left(\frac{1}{s}\right)^{(s-1)(\alpha s+2)/4} \cdot \frac{1}{\sqrt{s}} \cdot \sqrt{\frac{\alpha}{2\pi}} \cdot \frac{1}{s!} \cdot (2\pi)^{s/2} \cdot \alpha^{-s/2-\alpha s(s-1)/4} \cdot (\Gamma(1+\alpha/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1+\alpha j/2). \]

This agrees with the asymptotic value of \( S_{d}^{(s)}(m) \) as given by [5, Corollary 4.4] in the case \( d = 1 \).
3.1.3 The case \( d = 1 \) and \( \alpha = 1 \)

**Theorem 3.5.** Let \( d = \alpha = 1 \), then as \( m \to \infty \),

\[
T_{1,s}^{(1)}(m) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot \frac{1}{s!} \cdot \left(\frac{1}{\sqrt{m}}\right)^{s^2/2} \cdot \left(\frac{1}{\sqrt{m}}\right)^{(s^2-s-2)/2} \cdot \left(\frac{1}{\sqrt{m}}\right)^{s^3} \cdot \frac{1}{s^s} \cdot \left(\frac{1}{\sqrt{m}}\right)^{s^4} \cdot \frac{1}{s^s} \cdot \sqrt{2\pi} \cdot \frac{1}{s!}.
\]

\[
\cdot (2\pi)^{s^2/2} \cdot (\Gamma(1+1/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1+j/2) =
\]

\[
= (\sqrt{s})^{s(s-1)/2} \cdot \frac{1}{s!} \cdot \left(\frac{1}{\sqrt{m}}\right)^{s(s-1)/2} \cdot \frac{1}{s^s} \cdot (\Gamma(1+1/2))^{-s} \cdot \prod_{j=1}^{s} \Gamma(1+j/2),
\]

which agrees with [5, (E.4.5.1)].

3.1.4 The case \( d = 1 \), \( \alpha = 2 \)

Consider the case \( d = 1 \) and \( \alpha = 2 \) (any \( s \)), then

\[
T_{1,s}^{(2)}(n) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{s-1} \cdot \left(\frac{1}{\sqrt{m}}\right)^{s^2/2} \cdot (\sqrt{s})^{s/2} \cdot 2! \cdot (s-1)! \cdot \left(\frac{1}{\sqrt{n}}\right)^{s-1} \cdot s^2 \cdot n.
\]

For example, when \( s = 2 \) we have

\[
T_{1,2}^{(2)}(n) \simeq \frac{1}{\sqrt{\pi}} \cdot \frac{1}{n!} \cdot 4^n.
\]

In this case we know [4] page 64 that \( T_{1,2}^{(2)}(n) = (2n)!/(n! \cdot (n+1)) = C_n \), the \( n \)-th Catalan number, and by applying Stirling’s formula directly, we obtain the same asymptotic value.

3.1.5 The case \( s = d = 2 \) and \( \alpha = 1 \)

The case \( s = d = 2 \) and \( \alpha = 1 \). By Theorem 3.3

\[
T_{2,4}^{(1)}(2m) \simeq \left[ \left(\frac{1}{\sqrt{2\pi}}\right)^{3} \cdot \frac{1}{\sqrt{2}} \cdot 2^8 \cdot \left(\frac{1}{\sqrt{m}}\right)^{11} \cdot (4)^{2m} \right] \cdot (\sqrt{m}) \cdot \frac{2}{\sqrt{2}} \cdot \sqrt{\frac{1}{2\pi}} \cdot \frac{1}{2} \cdot 2\pi \cdot 4^{-3} \cdot \frac{2! \cdot 4!}{2! \cdot 2!} =
\]

\[
= \frac{1}{\pi} \cdot 24 \cdot \left(\frac{1}{m}\right)^{5} \cdot 4^{2m}.
\]

Note that sequence A005700 of [7] gives the following remarkable identity:

\[
T_{2,4}^{(1)}(2m) = \frac{6 \cdot (2m)! \cdot (2m + 2)!}{m! \cdot (m + 1)! \cdot (m + 2)! \cdot (m + 3)!}.
\]

Applying Stirling’s formula to the right-hand-side of (12) we obtain the same asymptotic value:

\[
\frac{6 \cdot (2m)! \cdot (2m + 2)!}{m! \cdot (m + 1)! \cdot (m + 2)! \cdot (m + 3)!} \simeq \frac{1}{\pi} \cdot 24 \cdot \left(\frac{1}{m}\right)^{5} \cdot 4^{2m},
\]

thus verifying Theorem 3.3 in this case.
References

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