Jacobian Elliptic Wave Solutions for the Wadati-Segur-Ablowitz Equation

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Abstract

Jacobian elliptic travelling wave solutions for a new Hamiltonian amplitude equation determining some instabilities of modulated wave train are obtained. By mere variation of the Jacobian elliptic parameter $k^2$ from zero to one, these solutions are transformed from a trivial one to the known solitary wave solutions [1], [2].

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1 Introduction

A new Hamiltonian amplitude equation governing modulated wave instabilities was reported by Wadati et. al [1] in 1992. This new Hamiltonian amplitude equation that is introduced,

\[ i\psi_x + \psi_{tt} + 2\sigma|\psi|^2\psi - \epsilon\psi_{xt} = 0, \quad \sigma = \pm 1, \quad \epsilon \ll 1, \tag{1} \]

governs certain instabilities of modulated wave trains. The subscripts here denote partial derivatives. The addition of the term \(-\epsilon\psi_{xt}\) overcomes the ill-posedness of the unstable nonlinear Schrodinger equation. Wadati et. al [1] found that this new equation is apparently not integrable because it does not satisfy the Painleve property but it is a Hamiltonian analogue of the Kuramoto-Sivashinsky equation which arises in dissipative systems.

By assuming a boundary condition for \(\psi(x, t)\) such that it is either rapidly decreasing or periodic in \(x\), Eq. (1) has at least three conserved quantities given by [1]

\[ I_1 = \int dx (\psi_t^*\psi_x - \frac{i}{2}\psi_x^*\psi - \frac{1}{2}i\epsilon\psi^*\psi_x), \tag{2} \]
\[ I_2 = \int dx (\frac{i}{2}\psi_x^*\psi - \frac{i}{2}\psi_t^*\psi + \sigma|\psi|^4 + \psi_t^*\psi_t), \tag{3} \]

which is the Hamiltonian and

\[ I_3 = \int dx (\psi_x^*\psi_t + \psi_t^*\psi_x - \epsilon\psi_x^*\psi_x). \tag{4} \]

The two solitary wave solutions that have been reported are [1]

\[ \psi(x, t) = \pm \sqrt{-\frac{2m}{n}} \tanh \left( \sqrt{-\frac{m}{l}}(x - vt + \xi_0) \right) e^{i(Kx - \Omega t)}, \tag{5} \]

where \(\sigma = 1, \quad l = v^2 + \epsilon v > 0, \quad -m = K + \Omega^2 + \epsilon K\Omega > 0, \quad n = 2\sigma, \quad \xi_0 = \text{constant}, \quad \) and [2]

\[ \psi(x, t) = \pm \sqrt{-\frac{m}{n}} \frac{1}{tanh} \left( \sqrt{-\frac{m}{2l}}(x - vt + \xi_0) \right) e^{i(Kx - \Omega t)}, \tag{6} \]

where \(\sigma = -1, \quad l = v^2 + \epsilon v > 0, \quad -m = K + \Omega^2 + \epsilon K\Omega < 0.\)

In this paper, we show that Eq. (1) does not only support the solitary wave solutions (5) and (6) but it also supports Jacobian elliptic travelling wave solutions as well. The solitary wave solutions (5) and (6) can be obtained from the Jacobian elliptic wave solutions when the Jacobian elliptic parameter \(k^2\) takes the value one for the three basic Jacobian elliptic functions, \(sn(u), cn(u), \) and \(dn(u).\) These elliptic wave solutions will be discussed in the next section and we will end with some remarks in the last section.
2 The Jacobian Elliptic Wave Solutions

We start off with the travelling wave ansatz of Wadati et. al [1],
\[ \psi(x,t) = e^{i\eta} \phi(\xi), \quad \xi = x - vt, \quad \eta = Kx - \Omega t, \] (7)
and find that
\[ \psi_x = e^{i\eta} (\phi'(\xi) + iK\phi(\xi)), \]
\[ \psi_{xt} = e^{i\eta} (\phi''(\xi) - i(Kv + \Omega)\phi'(\xi) + K\Omega \phi(\xi)), \]
\[ \psi_{tt} = e^{i\eta} (v^2 \phi'''(\xi) + 2i\Omega v\phi'(\xi) - \Omega^2 \phi(\xi)). \] (8)

Upon substitution of Eqs. (8) into Eq. (1), the Hamiltonian amplitude equation then reads,
\[ (v^2 + \epsilon v)\phi''(\xi) + i[1 + 2\Omega v + \epsilon(Kv + \Omega)]\phi'(\xi) \]
\[ - (K + \Omega^2 + \epsilon K\Omega)\phi(\xi) + 2\sigma \phi^3(\xi) = 0. \] (9)

As exact, explicit solutions cannot be found for Eq. (9) without making,
\[ 1 + 2\Omega v + \epsilon(Kv + \Omega) = 0, \]
as had been done by Wadati et. al [1], we choose to do the same and reduce Eq. (9) into the solvable form,
\[ l\phi''(\xi) + m\phi(\xi) + n\phi^3(\xi) = 0, \]
\[ l = v^2 + \epsilon v, \quad \Omega = -\left(\frac{1 + \epsilon vK}{2v + \epsilon}\right), \]
\[ m = -(K + \Omega^2 + \epsilon K\Omega) \]
\[ = -\left(\frac{(1 + 4Kv^2) + \epsilon(4Kv - \epsilon^2(K^2v^2) - \epsilon^3(K^2v))}{(2v + \epsilon)^2}\right), \]
\[ n = 2\sigma. \] (10)

By noticing that Eq. (10) is just the Jacobian elliptic functions differential equation [3] - [5], the solution to Eq. (10) is therefore given by
\[ \phi(\xi) = fE(u); \quad u = g(\xi + \xi_0), \] (11)
where \( f, \ g, \) and \( \xi_0 \) are constants, and \( E(u) \) is any of the twelve Jacobian elliptic functions. Upon substitution, Eq. (10) reduces to the Jacobian elliptic functions differential equation,
\[ E''(u) + aE(u) + bE^3(u) = 0, \]
\[ (E'(u))^2 + aE^2(u) + \frac{1}{2}bE^4(u) = c, \] (12)
where \( a = a(k) = \frac{m}{2\kappa} \), \( b = b(k) = \frac{n\kappa^2}{2\theta} \), \( c = \) integration constant, and \( 0 \leq k^2 \leq 1 \) is the Jacobian elliptic parameter. The constants of solution (11) can also be written as

\[
g = \sqrt{-\frac{m}{l a}} = \sqrt{\frac{-(1 + 4Kv^2) - \epsilon(4Kv + \epsilon^2(K^2v^2) + \epsilon^3(K^2v))}{a(2v + \epsilon)^2(v + \epsilonv)}},
\]

\[
f = \sqrt{\frac{mb}{na}} = \sqrt{\frac{b[-(1 + 4Kv^2) - \epsilon(4Kv + \epsilon^2(K^2v^2) + \epsilon^3(K^2v^2))]}{2\sigma a(2v + \epsilon)^2}},
\]

and \( \xi_0 \) is an arbitrary constant. The constants \( a, b, \) and \( c \) for the twelve Jacobian elliptic functions are, \( ^3 \)

| Function | \( a \) | \( b \) | \( c \) |
|----------|---------|---------|---------|
| \( E(u) \) | \( 1 + k^2 \) | \(-2k^2\) | \( c \) |
| \( sn(u) \) | \( 1 - 2k^2 \) | \( 2k^2 \) | \( 1 - k^2 \) |
| \( cn(u) \) | \(-2 - k^2\) | \( 2 \) | \(-1 - k^2\) |
| \( dn(u) \) | \( 1 + k^2 \) | \(-2\) | \( k^2 \) |
| \( nc(u) \) | \( 1 - 2k^2 \) | \(-2(1 - k^2)\) | \(-k^2 \) |
| \( nd(u) \) | \(-2 - k^2\) | \( 2(1 - k^2) \) | \(-1 \) |
| \( sc(u) \) | \(-2 - k^2\) | \(-2(1 - k^2)\) | \( 1 \) |
| \( sd(u) \) | \( 1 - 2k^2 \) | \( 2k^2(1 - k^2) \) | \( 1 \) |
| \( cs(u) \) | \(-2 - k^2\) | \(-2\) | \( 1 - k^2 \) |
| \( cd(u) \) | \( 1 + k^2 \) | \(-2k^2\) | \( 1 \) |
| \( ds(u) \) | \( 1 - 2k^2 \) | \(-2\) | \(-k^2(1 - k^2) \) |
| \( dc(u) \) | \( 1 + k^2 \) | \(-2\) | \( k^2 \) |

Since the wave solution, \( \phi(\xi) \), is singular when \( a = 0 \), the parameter \( k^2 \) must not take on the value \( \frac{1}{2} \) when \( E(u) \) is \( cn(u), nc(u), sd(u), \) or \( ds(u) \). However for the other eight Jacobian elliptic solutions, \( a \neq 0 \) when \( k^2 \) runs from zero to one.

We note that the Jacobian elliptic solution,

\[
\psi(x,t) = \pm \sqrt{-\frac{2mk^2}{n(1 + k^2)}} \frac{sn}{l(1 + k^2)}(\xi + \xi_0) e^{i(Kx - \Omega t)},
\]

(14)

where \( \frac{m}{n} < 0 \), \( \frac{m}{\kappa} > 0 \); or equivalently \( \sigma = -1 \), \( l = v^2 + \epsilonv > 0 \), \( -m = K + \Omega^2 + \epsilon K \Omega < 0 \), reduces to the solitary wave solution of Kong and Zhang \( ^2 \) when the parameter \( k^2 = 1 \), that is,

\[
\phi(\xi) = \pm \sqrt{-\frac{m}{n}} \tanh \left( \sqrt{-\frac{m}{2l}}(\xi + \xi_0) \right).
\]

However \( \phi(\xi) \) tends to zero as \( k^2 \to 0 \). Hence the solution (14) evolves from the trival zero solution to the solitary wave solution of Eq. (10), as \( k^2 \) runs from zero to one.

Similarly the Jacobian elliptic wave solutions,

\[
\psi(x,t) = \pm \sqrt{-\frac{2m}{n(2 - k^2)}} \frac{dn}{l(2 - k^2)}(\xi + \xi_0) e^{i(Kx - \Omega t)},
\]

(15)

with \( 0 \leq k^2 \leq 1 \) and
\[ \psi(x, t) = \pm \sqrt{\frac{2mk^2}{n(1-2k^2)}} cn \left( \sqrt{\frac{m}{l(1-2k^2)}}(\xi + \xi_0) \right) e^{i(Kx-\Omega t)}, \quad (16) \]

with \( \frac{1}{2} < k^2 \leq 1 \), reduce to the solitary wave solution of Wadati et al. \[ \text{[1]} \] when \( k^2 = 1 \), that is,

\[ \phi(\xi) = \pm \sqrt{-\frac{2m}{n}} \text{sech} \left( \sqrt{-\frac{m}{l}}(\xi + \xi_0) \right). \]

Here we have \( \frac{m}{n} < 0 \), \( \frac{m}{l} < 0 \); or equivalently \( \sigma = +1 \), \( l = v^2 + \epsilon v > 0 \), \( -m = K + \Omega^2 + \epsilon K \Omega > 0 \), for both the solutions \[ \text{(15)} \] and \[ \text{(16)} \]. Solution \[ \text{(15)} \] unlike solution \[ \text{(16)} \], approaches the trivial solution,

\[ \psi(x, t) = \pm \sqrt{-\frac{m}{n}} e^{i(Kx-\Omega t)}. \quad (17) \]

when \( k^2 \) approaches zero, whereas solution \[ \text{(16)} \] blows up when \( k^2 \) approaches \( \frac{1}{2} \), as it is singular there at \( k^2 = \frac{1}{2} \).

Hence we see that the solitary wave solution \[ \text{(5)} \], unlike the solitary wave solution \[ \text{(6)} \], actually evolves from two solutions \[ \text{(15)} \] and \[ \text{(16)} \], instead of one. This shows that bifurcation in the conserved quantities, Eq. \[ \text{(2)} \] to Eq. \[ \text{(4)} \], may occur here. Further work in this area is still in progress.

### 3 Remarks

1) The nonlinear Schrodinger equation,

\[ i\psi_x + \psi_t + 2\sigma|\psi|^2\psi = 0, \quad \sigma = \pm 1, \quad (18) \]

which has been proposed as a model for nonlinear modulation of stable plane waves in unstable media \[ \text{[3]} - \text{[4]} \] is a special case of the new Hamiltonian amplitude equation \[ \text{[1]} \], when \( \epsilon = 0 \) \[ \text{[1]} \]. Hence beside the two solitary wave solutions of Ref. \[ \text{[1]} \] and \[ \text{[2]} \], the nonlinear Schrodinger equation \[ \text{[18]} \] also possesses Jacobian elliptic travelling waves solutions,

\[ \psi(x, t) = \pm \frac{1}{2\sqrt{2v}} \sqrt{-\frac{b(1+4Kv^2)}{2a\sigma}} E \left( \frac{1}{2v^2} \sqrt{-\frac{(1+4Kv^2)}{a}}(x - vt) \right) e^{i(Kx-\Omega t)} \quad (19) \]

when \( E(u) \) is any of the twelve Jacobian elliptic functions.

2) The conserved quantities of Eq. \[ \text{(2)} \] to \[ \text{(4)} \] when integrated over all space is finite only for the solitary wave solution \[ \text{(3)} \], and not the solitary wave solution \[ \text{(5)} \]. However when these conserved quantities are integrated over half a period, that is from \(-K(k)\) to \(K(k)\) \[ \text{[3]} - \text{[5]} \], then they are finite only for the solutions when \( E(u) \) is \( \text{cnu, snu, dnu, ndu, sdu, and cdu} \) as these solutions are regular over all space. More will be discussed about these conserved quantities in a longer report.

3) Beside the solutions when \( E(u) \) is \( \text{nsu, ncu, scu, csu, dcu, and dsu} \) which are singular, all the other Jacobian elliptic wave solutions are regular over all space with boundary condition that is either rapidly decreasing or periodic in \( x \). We believe that
these Jacobian elliptic wave solutions had been overlooked by Wadati et. al \cite{[1]} and Kong and Zhang \cite{[2]} as these authors were not looking for periodic solutions but rather solutions with boundary condition that is rapidly decreasing in $x$.

4) These Jacobian elliptic wave solutions are the most general solutions of the Wadati-Segur-Ablowitz equation; all the other solitary wave solutions arise from changing the parameter $k^2$, or correspondingly in an experimental setup the experimental condition, which we can control sometimes.

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