BIELLIPTIC INTERMEDIATE MODULAR CURVES

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Abstract. We determine which of the modular curves $X_\Delta(N)$, that is, curves lying between $X_0(N)$ and $X_1(N)$, are bielliptic. Somewhat surprisingly, we find that one of these curves has exceptional automorphisms. Finally we find all $X_\Delta(N)$ that have infinitely many quadratic points over $\mathbb{Q}$.

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0. Introduction

For any positive integer $N$, let $\Gamma_1(N)$, $\Gamma_0(N)$ be the congruence subgroups of $\Gamma(1) = \text{SL}_2(\mathbb{Z})$ consisting of the matrices $(a\ b\ c\ d)$ congruent modulo $N$ to $(1\ 1\ 0\ 1)$, $(a\ b\ c\ d)$ respectively. We let $X_1(N)$, $X_0(N)$ be the modular curves associated to $\Gamma_1(N)$, $\Gamma_0(N)$ respectively. Let $\Delta$ be a subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$, and let $X_\Delta(N)$ be the modular curve defined over $\mathbb{Q}$ associated to the modular group $\Gamma_\Delta(N)$:

$$\Gamma_\Delta(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | c \equiv 0 \mod N, (a \mod N) \in \Delta \right\}.$$ 

Without loss of generality we always assume that $-1 \in \Delta$. If $\Delta = \{\pm 1\}$ (resp. $\Delta = (\mathbb{Z}/N\mathbb{Z})^*$) then $X_\Delta(N)$ is equal to $X_1(N)$ (resp. $X_0(N)$). If $\Gamma_{\Delta_1}(N)$ is a subgroup of $\Gamma_{\Delta_2}(N)$, the inclusions $\pm\Gamma_1(N) \subseteq \Gamma_{\Delta_1}(N) \subseteq \Gamma_{\Delta_2}(N) \subseteq \Gamma_0(N)$ induce natural Galois covers $X_1(N) \to X_{\Delta_1}(N) \to X_{\Delta_2}(N) \to X_0(N)$. Denote the genus of $X_{\Delta}(N)$ by $g_{\Delta}(N)$.

We point out that each such curve always has a $\mathbb{Q}$-rational point, namely the cusp $0$. This simplifies the discussion of properties like being a rational curve, being elliptic, or being hyperelliptic.

The next interesting property after being hyperelliptic is being bielliptic. A smooth, projective curve $X$ of genus $g(X) \geq 2$ is called bielliptic if it admits a map $\phi : X \to E$ of degree 2 onto an elliptic curve $E$. Equivalently, $X$ has an involution $v$ (called bielliptic involution) such that $X/v \cong E$.

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Harris and Silverman [H-S, Corollary 3] showed that if a curve $X$ with $g(X) \geq 2$ defined over a number field $F$ is neither hyperelliptic nor bielliptic, then the set of quadratic points on $X$,

$$\{ P \in X(F) : [F(P) : F] \leq 2 \}$$

is finite. See [B2, Theorem 2.14] (cited as Theorem 6.1 below) for a precise “if-and-only-if” statement.

Bars [B1] determined all the bielliptic modular curves of type $X_0(N)$ and also found all the curves $X_0(N)$ which have infinitely many quadratic points over $\mathbb{Q}$. The first two authors [J-K1] did the same work for the curves $X_1(N)$. Also, [Go] contains bielliptic equations for the bielliptic $X_0(N)$ that are not hyperelliptic.

In this paper we shall determine all the bielliptic intermediate modular curves $X_\Delta(N)$ and find all the curves $X_\Delta(N)$ which have infinitely many quadratic points over $\mathbb{Q}$. This type of curves is in some sense more difficult to handle than the other ones, because there are various subgroups $\Delta$ and the automorphism groups of the curves $X_\Delta(N)$ are not determined yet for most cases.

Recently [Ze] determined the normalizer of $\Gamma_\Delta(N)$ in $SL_2(\mathbb{R})$, which furnishes already quite a range of automorphisms of $X_\Delta(N)$. Any automorphism that is not of this form is called exceptional.

The unpublished preprint [Mo] states that if $N$ is square-free then $X_\Delta(N)$ has no exceptional automorphisms, except for the well-known curve $X_0(37)$. However, deciding the biellipticity of $X_\Delta(37)$ where $\Delta_3 = \{ \pm 1, \pm 6, \pm 8, \pm 10, \pm 11, \pm 14 \}$ revealed that this curve also has exceptional automorphisms (see Lemma 5.2 and Theorem 5.4 below).

As exactly the curve $X_\Delta(37)$ requires lengthy extra treatment in [Mo], it is quite likely that the mistake occurs there and that the rest of [Mo] is probably correct. But checking this would require more algebraic geometry than we are comfortable with.

It might seem unfair that we elaborate on this mistake. After all, [Mo] is an unpublished preprint. But some of the proofs in the paper [I-M] by Ishii and Momose use results from [Mo]; so their correctness is an issue. And on the other hand, after eliminating the mistake in [Mo] one would have a result that is much more general than the currently established ones. The paper [Ka] determines the automorphism group of $X_\Delta(N)$ for $N$ a prime bigger than 311 (which is too big to be of any help to us). Both, [Ka] itself and its review in MathSciNet suggest that it should be possible to generalize to the case of square-free $N$.

This paper is organized as follows. We begin with necessary basic notions in Section 1.

In Section 2 we have a more detailed look at two types of automorphisms of $X_\Delta(N)$, namely those from the Galois group of $X_\Delta(N)$ over $X_0(N)$ and, more interestingly, the possible lifts of the Atkin-Lehner involutions to $X_\Delta(N)$. These can behave quite differently from the Atkin-Lehner involutions on $X_0(N)$.

Section 3 presents a systematic method to find all the fixed points on $X_0(N)$ and $X_\Delta(N)$ by Atkin-Lehner involutions. Formulas for the number of fixed points on $X_0(N)$ were already known to Newman and Ogg. Also Delaunay [De] suggested a method to find all the
fixed points. In fact, he gave an algorithm to give all the candidates for the fixed points, but didn’t explain how to choose the exact fixed points among them explicitly.

In Section 4 we exclude all the non-bielliptic curves $X_\Delta(N)$ by using various criteria, and in Section 5 we show that the remaining $X_\Delta(N)$ are bielliptic curves.

Lastly, in Section 6 we find all the curves $X_\Delta(N)$ which have infinitely many quadratic points over $\mathbb{Q}$ and give some examples in terms of the moduli problem described by $X_\Delta(N)$.

1. Preliminaries

We begin with a proposition which describes the image of a bielliptic curve.

**Proposition 1.1.** [H-S, Proposition 1] If $X$ is a bielliptic curve, and if $X \to Y$ is a finite map, then $Y$ is subhyperelliptic (i.e. rational, elliptic or hyperelliptic) or bielliptic.

To determine all the bielliptic $X_\Delta(N)$ it therefore suffices to consider only those $N$ for which $X_0(N)$ is either subhyperelliptic or bielliptic. We summarize the classification of $X_0(N)$ accomplished by Ogg [O2] for the hyperelliptic curves and by Bars [B1] for the bielliptic curves as follows:

**Theorem 1.2.**

1. $X_0(N)$ is rational if and only if $N$ is one of the following 15 values:
   
   $1, \ldots, 10, 12, 13, 16, 18, 25$.

2. $X_0(N)$ is elliptic if and only if $N$ is one of the following 12 values:
   
   $11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49$.

3. $X_0(N)$ is hyperelliptic if and only if $N$ is one of the following 19 values:
   
   $22, 23, 26, 28, 29, 30, 31, 33, 35, 37, 39, 40, 41, 46, 47, 48, 50, 59, 71$.

4. $X_0(N)$ is bielliptic exactly for the following 41 values of $N$ (eleven of which are also hyperelliptic):
   
   $22, 26, 28, 30, 33, 34, 35, 37, 38, 39, 40, 42, 43, 44, 45, 48, 50, 51, 53, 54, 55, 56, 60, 61, 62, 63, 64, 65, 69, 72, 75, 79, 81, 83, 89, 92, 94, 95, 101, 119, 131$.

When dealing with an individual curve, the following facts are very useful.

**Theorem 1.3.** (Castelnuovo’s Inequality) Let $F$ be a function field with constant field $k$ of characteristic zero. Suppose there are two subfields $F_1$ and $F_2$ with constant field $k$ satisfying

1. $F = F_1F_2$ is the compositum of $F_1$ and $F_2$.
2. $[F : F_i] = n_i$, and $F_i$ has genus $g_i$ ($i = 1, 2$).

Then the genus $g$ of $F$ is bounded by

$$g \leq n_1g_1 + n_2g_2 + (n_1 - 1)(n_2 - 1).$$

A proof can be found in [St3, Theorem III.10.3]. Or see [Ac2, Theorem 3.5] for a proof in the language of Riemann surfaces.

The following useful fact is just a special case of the Hurwitz formula.
Proposition 1.4. Let $v$ be any involution on a smooth projective curve $X$ over an algebraically closed field $k$ of characteristic different from 2 and let $n_v$ denote the number of fixed points of $v$. Then we have the following genus formula:

$$g(X/v) = \frac{1}{4}(2g(X) + 2 - n_v).$$

In particular, $v$ is a bielliptic involution if and only if it has exactly $2g - 2$ fixed points.

Proposition 1.5. Let $K$ be a field of characteristic 0 and $X$ a curve defined over $K$ of genus $g \geq 6$. If $X$ is bielliptic, then the bielliptic involution is unique, defined over $K$ and lies in the center of $\text{Aut}(X)$.

Proof. Castelnuovo’s Inequality shows uniqueness, and the uniqueness implies the rest. \hfill \square

Here, as in the rest of the paper, we use the notation $\text{Aut}(X) = \text{Aut}_K(X)$.

2. Automorphisms

In this section we describe some automorphisms of $X_\Delta(N)$. We mainly need two types of non-exceptional automorphisms, that are easy to handle.

Note that $X_\Delta(N) \to X_0(N)$ is a Galois covering with Galois group $\Gamma_0(N)/\Gamma_\Delta(N)$ which is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^* / \Delta$. For an integer $a$ prime to $N$, let $[a]$ denote the automorphism of $X_\Delta(N)$ represented by $\gamma \in \Gamma_0(N)$ such that $\gamma \equiv (a, *) \mod N$. Sometimes we regard $[a]$ as a matrix.

For each divisor $d|N$ with $(d, N/d) = 1$, consider the matrices of the form $\begin{pmatrix} dx & y \\ Nz & dw \end{pmatrix}$ with $x, y, z, w \in \mathbb{Z}$ and determinant $d$. Then these matrices define a unique involution of $X_0(N)$, which is called the Atkin-Lehner involution and denoted by $W_d$. In particular, if $d = N$, then $W_N$ is called the full Atkin-Lehner involution. We also denote by $W_d$ a matrix of the above form.

If we fix a matrix $W_d$ then $W_d$ may not belong to the normalizer of $\Gamma_\Delta(N)$ in $\text{PSL}_2(\mathbb{R})$ and might therefore not define an automorphism of $X_\Delta(N)$. Now we will find a criterion for which $W_d$ defines an automorphism on $X_\Delta(N)$. Each $\gamma \in \Gamma_\Delta(N)$ is of the form $\begin{pmatrix} a & b \\ c & \overline{a} \end{pmatrix}$ where $(a \mod N) \in \Delta$ and $\overline{a}$ is an integer with $a\overline{a} \equiv 1 \mod N$. For $W_d = \begin{pmatrix} dx & y \\ Nz & dw \end{pmatrix}$ and $\gamma = \begin{pmatrix} a & b \\ c & \overline{a} \end{pmatrix} \in \Gamma_\Delta(N)$, one can easily compute that $W_d \gamma W_d^{-1} \in \Gamma_\Delta(N)$ if and only if the following condition holds:

$$dxwa - \frac{N}{4}yz\overline{a} \in \Delta. \quad (1)$$
Since \( d^2xw - Nyz = d \), we have \( dxw - \frac{N}{d}yz = 1 \), and hence the following holds:

\[
dxwa - \frac{N}{d}yza \equiv \begin{cases} a \pmod{\frac{N}{d}}, \\ a \pmod{d}. \end{cases}
\]

Note that \( \overline{a} \) is the multiplicative inverse of \( a \) modulo \( d \). Now we define an isomorphism \( t_d : (\mathbb{Z}/N\mathbb{Z})^* \to (\mathbb{Z}/N\mathbb{Z})^* \) by

\[
t_d(a) \equiv \begin{cases} a \pmod{\frac{N}{d}}, \\ a \pmod{d}. \end{cases}
\]

Since \((\mathbb{Z}/N\mathbb{Z})^*\) is isomorphic to the direct product \((\mathbb{Z}/d\mathbb{Z})^* \times (\mathbb{Z}/N\mathbb{Z})^*\), one can show that the condition (1) holds if and only if \( t_d(a) \in \Delta \). Therefore we have the following result:

**Theorem 2.1.** A matrix \( W_d \) defines an automorphism of \( X_\Delta(N) \) if and only if \( t_d(\Delta) = \Delta \).

**Remark 2.2.** If \( W_d \) defines an automorphism of \( X_\Delta(N) \) then so does \( W_{dN} \) because \( t_{dN} = t_d^{-1} \).

Moreover, if \( a \) is running through \((\mathbb{Z}/N\mathbb{Z})^*/\Delta \), then \([a]W_d\) gives the different automorphisms of \( X_\Delta(1) \) that induce the same Atkin-Lehner involution \( W_d \) on \( X_0(1) \).

**Example 2.3.** Let \( N = 65 \). Then \( t_5 \) interchanges \( \Delta_1 = \{\pm 1, \pm 8\} \) and \( \Delta_2 = \{\pm 1, \pm 18\} \). Correspondingly, \( W_5 \) does not give automorphisms of \( X_{\Delta_1}(65) \); rather it gives isomorphisms from \( X_{\Delta_1}(65) \) to \( X_{\Delta_2}(65) \).

**Example 2.4.** Even for square-free \( N \) it can happen that \( W_d \) gives automorphisms of \( X_\Delta(N) \), but none of these automorphisms is an involution.

Take for example \( N = 65 \) and \( \Delta = \{\pm 1, \pm 14\} \). Then any \( W_5 = \begin{pmatrix} 5x \\ 65z \\ y \end{pmatrix} \) with determinant 5 is an automorphism of \( X_\Delta(1) \) by Theorem 2.1. However, \( \frac{1}{5}W_5^2 \) is a matrix in \( \Gamma_0(65) \) with upper left entry \( 5x^2 + 13yz \), which cannot be congruent to \( \pm 1 \) or \( \pm 14 \) modulo 65, as \( \pm 5 \) is not a square modulo 13. So \( W_5^2 \) is a non-trivial automorphism of \( X_\Delta(65) \).

By the way, the same then holds as well for the automorphisms \( W_5 \) of \( X_1(65) \).

Now we derive some conditions under which an automorphism \( W_d \) of \( X_\Delta(N) \) necessarily is an involution.

Assume that \( W_d \) has a non-cuspidal fixed point on \( X_\Delta(N) \). Then we can multiply \( W_d \) on the left by a matrix from \( \Gamma_\Delta(N) \) without changing the automorphism such that the new matrix actually has a fixed point on the upper half plane \( \mathbb{H} \).

Recall that an elliptic element of \( \text{GL}_2^+(\mathbb{R}) \) is a matrix \( A \) which has a fixed point on \( \mathbb{H} \). This is equivalent to \( |\text{tr}(A)|^2 < 4 \text{det}(A) \).

If \( d > 3 \) and \( W_d \) has a fixed point on \( \mathbb{H} \), this forces \( \text{tr}(W_d) = 0 \). In particular, then \( W_d \) is an involution. Moreover, if \( \text{tr}(W_d) = 0 \), from \( \text{det}(W_d) = d \) we get 

\[-dx^2 - \frac{N}{d}yz = 1; \]

so \( -d \) necessarily is congruent to a square modulo \( \frac{N}{d} \).

Conversely, suppose the following conditions hold:

for \( d \neq 2, 3, \)

(i) \( t_d(\Delta) = \Delta \).
(ii) \(-d\) is congruent to a square modulo \(N/d\).

for \(d = 2\) or \(3\), (i) and (ii)’ \(dx(t - x) \equiv 1 \pmod{N/d}\) has a solution for some \(t \in \{0, 1, -1\}\).

First consider the case \(d \neq 2, 3\). Then there exist \(x, y\) satisfying \(-dx^2 - y^2 N/d = 1\), and for uniqueness if we set \(x_0\) to be the smallest nonnegative such integer and \(y_0 = -dy_0^2 - 1\) then the matrix

\[
\hat{W}_d = \begin{pmatrix} dx_0 & y_0 \\
N & -dx_0 \end{pmatrix}
\]

defines an involution on \(X_\Delta(N)\), and it has a non-cuspidal fixed point on \(X_\Delta(N)\). Note that the condition (ii) is equivalent to the existence of an elliptic element \(W_d\).

For the case \(d = 2\) or \(3\), we encounter some different situation. One can check easily that the condition (ii)’ is equivalent to the existence of an elliptic element \(W_d\). But in this case, such \(W_d\) may not define an involution on \(X_\Delta(N)\). We can choose \(\hat{W}_d\) uniquely of the form

\[
\hat{W}_d = \begin{pmatrix} dx_0 & y_0 \\
N & dx_0(t - x_0) \end{pmatrix}
\]

with the smallest nonnegative integer \(x_0\) satisfying (ii)’.

In the above, we use the notation \(\hat{W}_d\) to distinguish it from the Atkin-Lehner involution on \(X_0(N)\). Moreover \(\hat{W}_d\) means the matrix of the above form \((2)\) or \((3)\), or the automorphism on \(X_\Delta(N)\) defined by such a matrix.

If \(\hat{W}_d\) exists, all automorphisms of \(X_\Delta(N)\) coming from the matrices \(W_d\) are of the form \(\hat{W}_d\) for some suitable \(a\) relatively prime to \(N\).

In particular, we always have \(\hat{W}_N = \begin{pmatrix} 0 & -1 \\
N & 0 \end{pmatrix}\). One easily checks that \(\hat{W}_N\) normalizes the Galois group of \(X_\Delta(N)\) over \(X_0(N)\) with \(\hat{W}_N[a]\hat{W}_N^{-1} = [a]\). This also shows that all automorphisms \(W_N\) are involutions on \(X_\Delta(N)\).

**Example 2.5.** However, if \(3 < d \neq N\), the existence of \(\hat{W}_d\) does not imply that all \(W_d\) are involutions. Let for example \(N = 55\) and \(\Delta_3 = \{\pm1, \pm16, \pm19, \pm24, \pm26\}\). Then \(t_{11}(\Delta_3) = \Delta_3\) and \(\hat{W}_{11}\) exists because \(-11\) is a square modulo 5. But \(W_{11} = \begin{pmatrix} 11 & 2 \\
55 & 11 \end{pmatrix}\) has order 4 because its square is \([21]\).

**Example 2.6.** Even if \(W_{d_1}\) and \(W_{d_2}\) are Atkin-Lehner involutions of \(X_\Delta(N)\), their product is not necessarily an involution. In particular, Atkin-Lehner involutions on \(X_\Delta(N)\) do not always commute.

For example, \(\hat{W}_5 = \begin{pmatrix} 1 & -3 \\
35 & -10 \end{pmatrix}\) and \(\hat{W}_{35} = \begin{pmatrix} 0 & -1 \\
35 & 0 \end{pmatrix}\) are involutions on \(X_{\Delta_2}(35)\) where \(\Delta_2 = \{\pm1, \pm11, \pm16\}\). But \((\hat{W}_5\hat{W}_{35})^2 = [13]\) has order 4; so \(\hat{W}_5\hat{W}_{35}\) has order 8.

**Remark 2.7.** In contrast to \(X_0(N)\), the involutions \(\hat{W}_d\) of \(X_\Delta(N)\) are in general not defined over \(\mathbb{Q}\), not even \(\hat{W}_N\) and not even if \(N\) is prime.
For example the involution $\hat{W}_{37}$ of $X_{\Delta 4}(37)$ is only defined over $\mathbb{Q}(\sqrt{37})$. This will turn out to be useful in the proof of Proposition 4.20. By the way, this also explains that the genus 2 curve $X_{\Delta 4}(37)/\hat{W}_{37}$ does not show up in the tables of [B-G-G-P].

3. Fixed points of Atkin-Lehner involutions

In this section we explain a method to determine whether a matrix $W_d$ on $X_{\Delta}(N)$ gives a bielliptic involution or not. Thanks to Proposition 1.4, it suffices to know how to calculate the number of the fixed points of $W_d$.

First we point out that if $d \neq 4$, then $W_d$ does not fix any cusps by [O2, Proposition 3]. For our purposes it will suffice to find the number of non-cuspidal fixed points of $W_d$ on $X_{\Delta}(N)$.

Delaunay [De] suggested a method to find all the fixed points of $W_d$ on $X_0(N)$. In fact, he gave an algorithm to give all the candidates for the fixed points of $W_d$, but didn’t explain how to choose the exact fixed points among them explicitly.

We briefly explain the algorithm of Delaunay. Suppose $d \neq 2, 3$. If $W_d$ has a non-cuspidal fixed point on $X_{\Delta}(N)$ at all (and only these $W_d$ are of interest to us), then $W_d$ is given by an elliptic element (see Section 2). So $x = -w$ and

$$W_d = \begin{pmatrix} dx & y \\ Nz & -dx \end{pmatrix}.$$ 

Furthermore, $d = \det(W_d) = -d^2x^2 - yNz$ so we have $(2dx)^2 = -4d - 4Nyz$ and the point

$$\tau = \frac{2dx + \sqrt{-4d}}{2Nz}$$

(4) is a fixed point of $W_d$. Conversely, one can check that every fixed point has the form (4).

In order to find all the fixed points:

(i) We search $\beta \pmod{2N}$ such that

$$\beta^2 \equiv -4d \pmod{4N} \text{ with } \beta = 2dx, \ x \in \mathbb{Z}.$$ 

(ii) For each divisor $z$ of $(\beta^2 + 4d)/4N$, we get the fixed point

$$\tau = \frac{2dx + \sqrt{-4d}}{2Nz}.$$ 

(5)

Moreover, as Delaunay pointed out, the fixed points in (5) can be considered as Heegner points.

For $d = 2$ or 3, we slightly modify his method. If $W_d$ has a non-cuspidal fixed point at all, then, as explained in the last section, we have $|x + w| = 0$ or $|x + w| = 1$. The first case is discussed above, for the second, putting $w = 1 - x$ the point

$$\tau = \frac{-d(1 - 2x) + \sqrt{d^2 - 4d}}{2Nz}$$

(7)
is fixed by
\[ W_d = \left( \frac{dx}{Nz}, \frac{y}{d(1-x)} \right). \]

In order to find all the fixed points in the second case:

(i) We search \( \beta \pmod{2N} \) such that
\[ \beta^2 \equiv d^2 - 4d \pmod{4N} \]
with \( \beta = d(1-2x), \ x \in \mathbb{Z}. \)

(ii) For each divisor \( z \) of \( (\beta^2 - d^2 + 4d)/4N \), we get the fixed point
\[ \tau = -d(1-2x) + \sqrt{d^2 - 4d} \]
\[ \frac{2Nz}{2} \] \tag{6}

Now we will propose a systematic way to find inequivalent points modulo \( \Gamma_0(N) \) among the fixed points in (5) and (6). For this purpose we need to introduce quadratic forms. For a negative integer \( D \) congruent to 0 or 1 modulo 4, we denote by \( Q_D \) the set of positive definite integral binary quadratic forms
\[ Q(x, y) = ax^2 + bxy + cy^2 \]
with discriminant \( D = b^2 - 4ac \). Then \( \Gamma(1) \) acts on \( Q_D \) by
\[ Q \circ \gamma(x, y) = Q(px + qy, rx + sy) \]
where \( \gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \). A primitive positive definite form \([a, b, c]\) is said to be reduced form if
\[ |b| \leq a \leq c, \ b \geq 0 \] if either \(|b| = a\) or \(a = c\).

Let \( Q_D^0 \subset Q_D \) be the subset of primitive forms. Then \( \Gamma(1) \) also acts on \( Q_D^0 \). As is well known, there is a 1-1 correspondence between the set of classes \( \Gamma(1) \backslash Q_D^0 \) and the set of reduced primitive definite forms.

We define
\[ Q_{D,N,\beta}^0 = \{ [aN, b, c] \in Q_D | \gcd(a, b, c) = 1 \} \]
where \( \beta \equiv b \pmod{2N} \). Set \( m = \gcd \left( N, \beta, \frac{\beta^2 - D}{4N} \right) \). For \( Q = [aN, b, c] \in Q_{D,N,\beta}^0 \) we have \( \gcd(N, b, ac) = m \) and \( \gcd(a, b, c) = 1 \). Then \( \gcd(N, b, a) = m_1 \) and \( \gcd(N, b, c) = m_2 \) are coprime and \( m = m_1 m_2 \). Conversely we have the following:

**Proposition 3.1.** \([G-K-Z]\) Define \( m = \gcd \left( N, \beta, \frac{\beta^2 - D}{4N} \right) \) and fix a decomposition \( m = m_1 m_2 \) with \( m_1, m_2 > 0 \) and \( \gcd(m_1, m_2) = 1 \). Let
\[ Q_{D,N,\beta,m_1,m_2}^0 = \{ [aN, b, c] \in Q_{D,N,\beta}^0 | \gcd(N, b, a) = m_1, \gcd(N, b, c) = m_2 \} \].

Then \( \Gamma_0(N) \) acts on \( Q_{D,N,\beta,m_1,m_2}^0 \) and there is an 1-1 correspondence between
\[ Q_{D,N,\beta,m_1,m_2}^0/\Gamma_0(N) \rightarrow Q_D^0/\Gamma(1) \]
\[ [aN, b, c] \mapsto [aN_1, b, cN_2] \]
where \( N_1 N_2 \) is any decomposition of \( N \) into coprime factors such that \( \gcd(m_1, N_2) = \gcd(m_2, N_1) = 1 \).
The inverse image $[aN_2,b,c/N_2]$ of any primitive form $[a,b,c]$ of discriminant $D$ under the 1-1 correspondence in the Theorem 3.1 is obtained by solving the following equations:

\[
\begin{align*}
a &= \bar{a}p^2 + bpr + \bar{c}r^2 \\
b &= 2\bar{a}pq + \bar{b}(ps + qr) + 2\bar{c}rs \\
c &= \bar{a}q^2 + \bar{b}qs + \bar{c}s^2
\end{align*}
\]

satisfying $a \equiv 0 \pmod{N_1}$, $b \equiv \beta \pmod{2N}$, $c \equiv 0 \pmod{N_2}$ and \( \begin{pmatrix} p & q \\ r & s \end{pmatrix} \) $\in \Gamma(1)$.

In this way we have the following $\Gamma_0(N)$-invariant decomposition:

\[
\mathcal{Q}_{D,N,\beta} = \bigcup_{\ell \geq 0, \ell \mid N} \bigcup_{\substack{\lambda(2N) \\ \Delta^2 D/\ell^2 (2N)}} \ell \mathcal{Q}_{D/\ell^2, N, \lambda} \tag{7}
\]

Note that each fixed point in (5) (resp. (6)) can be considered as the Heegner point of a quadratic form $[Nz, -2\Delta x, -y]$ (resp. $[Nz, d(1-2x), -y]$). Our strategy is to find inequivalent quadratic forms $[Nz, -2\Delta x, -y]$ mod $\Gamma_0(N)$ by using Proposition 3.1 and the decomposition (7). One can show that for $d = 3$ the Heegner points in (6) are the same as the Heegner points in (5) of the quadratic forms in $2\mathcal{Q}_{D,N,\lambda}$. Thus it suffices to handle the case $d = 2$ separately.

Finally we explain a method of determining the number of fixed points of an elliptic element $W_d$ on $X_\Delta(N)$. Note that the fixed points of $W_d$ on $X_\Delta(N)$ are lying above the fixed points of $W_d$ on $X_0(N)$. Let $z_1, z_2, \ldots, z_n \in \mathbb{H}$ be the inequivalent elliptic points which represent all the fixed points of $W_d$ on $X_0(N)$, and let $W_{d,1}, W_{d,2}, \ldots, W_{d,n}$ be corresponding elliptic elements. Note that $G = \Gamma_0(N)/\Delta(N)$ is the Galois group of the covering $X_\Delta(N) \rightarrow X_0(N)$, and the elements of $G$ are the automorphisms of the form $[a]$ with $a \in (\mathbb{Z}/N\mathbb{Z})^*$. Thus for each $j$ the points on $X_\Delta(N)$ lying above $z_j$ are represented by $[a]z_j$ with $[a] \in G$. Then one can easily show that $W_{d,i}$ fixes $[a]z_j$ if and only if $W_{d,j}[a]W_{d,j}^{-1}[a]^{-1} \in \Delta(N)$. Thus one can calculate the number of fixed points of $W_{d,i}$ lying above $z_j$, and hence determine whether $W_{d,i}$ defines a bielliptic involution or not on $X_\Delta(N)$.

**Example 3.2.** Consider $X_{\Delta}(28)$ of genus 4. Since $W_7$ is the hyperelliptic involution on $X_0(28)$, $W_7$ has 6 fixed points on $X_0(28)$. The method described above allows us to determine 6 elliptic elements which give 6 fixed points of $W_7$ on $X_0(28)$ as follows:

\[
\begin{align*}
E_1 &= \begin{pmatrix} 7 & -2 \\ 28 & -7 \end{pmatrix}, E_2 = \begin{pmatrix} 7 & -1 \\ 56 & -7 \end{pmatrix}, E_3 = \begin{pmatrix} -7 & -2 \\ 28 & 7 \end{pmatrix}, \\
E_4 &= \begin{pmatrix} -7 & -1 \\ 56 & 7 \end{pmatrix}, E_5 = \begin{pmatrix} 21 & -8 \\ 56 & -21 \end{pmatrix}, E_6 = \begin{pmatrix} -21 & -8 \\ 56 & 21 \end{pmatrix}.
\end{align*}
\]

For each $i$, denote the fixed point of $E_i$ by $z_i$. Then $E_1$, $E_2$ and $E_3$ give the same involution $\tilde{W}_7$ on $X_{\Delta}(28)$. Also $E_4$, $E_5$ and $E_6$ give the same involution $[11]\tilde{W}_7$ on $X_{\Delta}(28)$. Note
that the map $X_{D\Delta}(28) \to X_0(28)$ is of degree 2 and each $E_i$ fixes two points lying above $z_i$. Thus $\hat{\omega}_7$ and $[11] W_7$ each have 6 fixed points on $X_{\Delta\Delta}(28)$ and they are bielliptic involutions.

4. NON-BIELLIPTIC CURVES

In this section, we exclude all the non-bielliptic curves $X_{\Delta}(N)$. For the extremal cases $X_0(N)$ and $X_1(N)$ it is known which are bielliptic ([BL], [JK1]). So $\Delta$ is always meant to be strictly between $\{\pm 1\}$ and $(\mathbb{Z}/N\mathbb{Z})^*$.

Note that among the list of the in total 76 values of $N$ in Theorem 1.2 there are 20 values for which there exist no intermediate curves $X_{\Delta}(N)$, namely for:

$$N = 1, 2, \ldots, 12, 14, 18, 22, 23, 46, 47, 59, 94.$$ 

Applying Proposition 1.1 to Theorem 1.2 we therefore have

**Lemma 4.1.** There are at most 56 possible values of $N$ for which there might exist an intermediate modular curve $X_{\Delta}(N)$ that is bielliptic, namely:

$$13, 15 - 17, 19 - 21, 24 - 45, 48 - 51, 53 - 56, 60 - 65, 69, 71, 72, 75, 79, 81, 83, 89, 92, 95, 101, 119, 131.$$ 

We recall that if $X$ is a bielliptic curve, there exists an involution $v$, called a bielliptic involution, such that $X/v$ is an elliptic curve. If the genus $g(X) \geq 6$, by Proposition 1.5 then $v$ is unique, defined over $\mathbb{Q}$, and lies in the center of the automorphism group $\text{Aut}(X)$ of $X$. Consider a Galois covering $X \to Y$ with Galois group $G$. Then either $v \in G$ or it induces an involution $\hat{v}$ on $Y$ such that $Y/\hat{v}$ is a rational or elliptic curve. Thus $\hat{v}$ is a hyperelliptic or bielliptic involution if the genus $g(Y) \geq 2$.

4.1. CUSPS. Let $X(N)$ be the modular curve defined over $\mathbb{Q}$ associated to the modular group $\Gamma(N)$:

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid a \equiv d \equiv 1 \text{ mod } N, b \equiv c \equiv 0 \text{ mod } N \right\}.$$ 

By virtue of [O1] the cusps of $X(N)$ can be regarded as pairs $\pm \left( \frac{x}{y} \right)$, where $x, y \in \mathbb{Z}/N\mathbb{Z}$, and are relatively prime, and $\left( \frac{x}{y} \right), \left( -\frac{x}{y} \right)$ are identified; $\Gamma_{\Delta}/\Gamma(N)$ operates naturally on the left, and so a cusp of $X_{\Delta}(N)$ can be regarded as an orbit of $\Gamma_{\Delta}(N)/\Gamma(N)$.

**Lemma 4.2.** The curves $X_{\Delta}(N)$ are not bielliptic curves for the following $N$:

$$31, 43, 53, 61, 65, 71, 75, 79, 83, 89, 95, 101, 119, 131.$$ 

**Proof.** Let $N$ be one of the numbers of the above list. Then the curve $X_{0}(N)$ is either hyperelliptic or bielliptic, but not both. From the tables in [BL, O2], we know that the hyperelliptic or bielliptic involution is the full Atkin-Lehner involution. Note that $g_{\Delta}(N) \geq 6$ for all possible $\Delta$. Suppose that $X_{\Delta}(N)$ is bielliptic and let $v$ be the unique bielliptic involution. Since $g_{0}(N) \geq 2$, the involution $v$ induces an involution $\hat{v}$ on $X_{0}(N)$ which is the full Atkin-Lehner involution. Then $\hat{v}$ maps the cusp $0 = \left( \frac{0}{1} \right)$ to $\infty = \left( \frac{1}{0} \right)$. Thus $v$ maps...
the cusps lying above 0 to the cusps lying above $\infty$. Note that the cusps over 0 are rational but the cusps over $\infty$ are non-rational. This is a contradiction. □

For the reader’s convenience, by using the genus formula in [J-K3, Theorem 1.1] we tabulate in Table 4 at the end of the paper all the curves $X_\Delta(N)$ and their genera for the 42 currently remaining values of $N$ which are listed in Lemma 4.1 and not excluded by Lemma 4.2.

First of all, we make a list of all $X_\Delta(N)$ that are rational, elliptic or hyperelliptic:

**Theorem 4.3.** (1) All the rational $X_\Delta(N)$ are as follows:

- $X_{\Delta_1}(13)$, $X_{\Delta_2}(13)$, $X_{\Delta_1}(16)$, $X_{\Delta_2}(25)$.

(2) All the elliptic $X_\Delta(N)$ are as follows:

- $X_{\Delta_1}(15)$, $X_{\Delta_1}(17)$, $X_{\Delta_2}(17)$, $X_{\Delta_1}(19)$, $X_{\Delta_1}(20)$, $X_{\Delta_2}(21)$, $X_{\Delta_3}(24)$, $X_{\Delta_1}(27)$, $X_{\Delta_2}(32)$.

(3) $X_{\Delta_1}(21)$ is the only hyperelliptic curve.

**Proof.** This is essentially shown in [I-M] and [J-K3]. However, the hyperellipticity of $X_{\Delta_1}(21)$ had been overlooked in [I-M]. Moreover, the proof that $X_{\Delta_3}(37)$ is not hyperelliptic in [I-M] is based on the claim from [Mo] that this curve has no exceptional automorphisms. But we shall see in Lemma 5.2 and Theorem 5.4 that $X_{\Delta_3}(37)$ actually has exceptional automorphisms.

So, concerning non-hyperellipticity, we want to provide a proof at least for $N = 37$. Obviously it suffices to treat the two maximal cases $\Delta_3$ and $\Delta_4$.

The curve $X_{\Delta_3}(37)$ is of genus 4 and it is a degree 3 Galois cover of the genus 2 curve $X_0(37)$. By the Hurwitz formula this implies that the covering map is unramified. If $X_{\Delta_3}(37)$ were hyperelliptic, $Gal(X_{\Delta_3}(37)/X_0(37))$, a group of order 3, would act without fixed points on the 10 fixed points of the hyperelliptic involution.

The automorphism group of $X_{\Delta_3}(37)$ has the subgroup $\langle [2], \hat{W}_{37}\rangle$, of order 4, whose quotient is the elliptic curve $X_0^+(37)$. If $X_{\Delta_3}(37)$ were hyperelliptic, this group would act on the 10 fixed points of the hyperelliptic involution. So at least one of these points would be fixed by the hyperelliptic involution and another involution, contradicting the fact that the stabilizer of a point is always cyclic. □

By the cusp argument used in the proof of Lemma 4.2, we have the following result:

**Lemma 4.4.** The following $X_\Delta(N)$ are not bielliptic:

- $X_{\Delta_1}(29)$, $X_{\Delta_1}(33)$, $X_{\Delta_1}(35)$, $X_{\Delta_2}(35)$, $X_{\Delta_1}(38)$, $X_{\Delta_2}(39)$, $X_{\Delta_1}(41)$, $X_{\Delta_2}(41)$, $X_{\Delta_6}(41)$, $X_{\Delta_1}(42)$, $X_{\Delta_2}(42)$, $X_{\Delta_1}(51)$, $X_{\Delta_2}(55)$, $X_{\Delta_3}(55)$, $X_{\Delta_4}(60)$, $X_{\Delta_5}(60)$, $X_{\Delta_6}(60)$, $X_{\Delta_1}(62)$, $X_{\Delta_2}(62)$, $X_{\Delta_1}(69)$, $X_{\Delta_1}(92)$, $X_{\Delta_2}(92)$.

**Proof.** Suppose that a curve $X_\Delta(N)$ from the above list is bielliptic. Since $g_\Delta(N) \geq 6$, $X_\Delta(N)$ has a unique bielliptic involution $v$ which induces an involution $\tilde{v}$ on $X_0(N)$. Note that any hyperelliptic or bielliptic involution on the corresponding $X_0(N)$ is equal to one of
the Atkin-Lehner involutions $W_d$ with $d \neq 2$. Thus $\tilde{v}$ should be $W_d$ with $d \neq 2$. Note that $W_d$ is represented by a matrix $\begin{pmatrix} dx & y \\ Nz & dw \end{pmatrix}$ where $x, y, z, w \in \mathbb{Z}$ and $\det W_d = d$. We can choose $w = 1$ and $(y, d) = 1$. Then $\tilde{v}$ maps the cusp 0 to $(\frac{y}{d})$. By using the Lemma 1.2 of [I-M], one can check that the cusps lying above $(\frac{y}{d})$ are non-rational. Thus $v$ maps rational cusps to non-rational cusps. This gives rise to a contradiction. □

For each $X(\Delta(N))$ in the list of Lemma 4.4 and $\Delta' \subseteq \Delta$ the curve $X(\Delta)(N)$ cannot be bielliptic by Proposition 1.1. Thus we get the following result:

**Corollary 4.5.** The following $X(\Delta(N))$ are not bielliptic:

$X(51), X(51), X(55), X(60), X(60), X(60)$.

**Lemma 4.6.** The modular curve $X(39)$ is not bielliptic.

*Proof.* Suppose $X(39)$ is bielliptic. Since $X(39)$ is of genus 9, the bielliptic involution $v$ is unique and must have 16 fixed points. The induced involution $\tilde{v}$ on $X_0(39)$ must be equal to $W_3$ or $W_{39}$ [B1, O1]. By the same reason as in the proof of Lemma 4.2, $\tilde{v}$ is not $W_{39}$. Suppose $\tilde{v}$ is equal to $W_3$. Note that $W_3$ is the only bielliptic involution on $X_0(39)$ and it has 4 fixed points [B1]. Since the covering $X(39) \rightarrow X_0(39)$ is of degree 3, $v$ has at most 12 fixed points on $X_3(39)$, which is a contradiction. □

From Proposition 1.1 we have the following result:

**Corollary 4.7.** The modular curve $X(39)$ is not bielliptic.

**Lemma 4.8.** The modular curves $X(40)$ and $X(40)$ are not bielliptic.

*Proof.* Suppose $v_i$ are the unique bielliptic involutions on $X(40)$ with $i = 4, 5$. Then on $X_0(40)$ both of the $v_i$ with $i = 4, 5$ must induce the hyperelliptic involution $w = \begin{pmatrix} -10 & 1 \\ -120 & 10 \end{pmatrix}$ which is not of Atkin-Lehner type (cf. [O]). Note that $w\infty = (\frac{1}{4})$ and $w0 = (\frac{1}{10})$. By using the Lemma 1.2 of [L-M], we know that the cusps on $X(40)$ (resp. $X(40)$) lying over $(\frac{1}{4})$ (resp. 0) are all rational, but the cusps on $X(40)$ (resp. $X(40)$) lying over $\infty$ (resp. $(\frac{1}{10})$) are all non-rational. Therefore, we have contradictions. □

From Proposition 1.1 we have the following result:

**Corollary 4.9.** The modular curve $X(40)$ is not bielliptic.

### 4.2. Unramified coverings

We give a criterion to determine whether $X(\Delta(N))$ is bielliptic or not.

**Lemma 4.10.** Suppose $X$ is a bielliptic curve of genus $g(X) \geq 6$ and $\phi : X \rightarrow Y$ is a finite Galois covering. If $Y$ is not subhyperelliptic, then $\phi$ is totally unramified. In particular, we then have

$$g(X) - 1 = \deg(\phi)(g(Y) - 1).$$
Proof. As \( g(X) \geq 6 \), by Proposition 1.5, the bielliptic involution \( v \) is unique and commutes with all automorphisms. As \( Y \) is not subhyperelliptic, it induces a bielliptic involution \( \tilde{v} \) on \( Y \), and \( \text{Gal}(X/Y) \) induces a group of automorphisms on \( X/v \) with quotient \( Y/\tilde{v} \).

Any ramification point of \( \phi \) would induce a ramification point of the covering of elliptic curves \( X/v \to Y/\tilde{v} \).

Alternatively in characteristic different from 2: Each of the \( 2g(X) - 2 \) fixed points of \( v \) on \( X \) must lie above one of the \( 2g(Y) - 2 \) fixed points of \( \tilde{v} \) on \( Y \). So \( 2g(X) - 2 \leq \deg(\phi)(2g(Y) - 2) \). By the Hurwitz formula this implies equality and no ramification. \( \square \)

By applying the above lemma, we have the following result:

**Lemma 4.11.** The following \( X_\Delta(N) \) are not bielliptic:

\[
X_{\Delta_1}(37), X_{\Delta_2}(37), X_{\Delta_2}(40), X_{\Delta_1}(40), X_{\Delta_2}(44), X_{\Delta_3}(45), X_{\Delta_4}(48), X_{\Delta_5}(48), X_{\Delta_6}(56), X_{\Delta_7}(56), X_{\Delta_6}(63), X_{\Delta_7}(63), X_{\Delta_8}(63), X_{\Delta_9}(64), X_{\Delta_6}(72), X_{\Delta_7}(72).
\]

For each \( X = X_\Delta(N) \) in the list of Lemma 4.11 we suggest a finite Galois covering \( \phi : X \to Y \) and its degree in Table 1 which enables us to conclude that \( X \) is not bielliptic by Lemma 4.10.

| \( \phi : X \to Y \) | degree |
|----------------------|--------|
| \( X_{\Delta_1}(37) \to X_{\Delta_3}(37) \) | 3 |
| \( X_{\Delta_2}(37) \to X_{\Delta_1}(37) \) | 2 |
| \( X_{\Delta_2}(40) \to X_{\Delta_1}(40) \) | 2 |
| \( X_{\Delta_3}(40) \to X_1(20) \) | 2 |
| \( X_{\Delta_4}(44) \to X_1(22) \) | 2 |
| \( X_{\Delta_4}(44) \to X_0(44) \) | 2 |
| \( X_{\Delta_5}(45) \to X_0(45) \) | 3 |
| \( X_{\Delta_6}(48) \to X_{\Delta_4}(24) \) | 2 |
| \( X_{\Delta_6}(48) \to X_{\Delta_6}(24) \) | 2 |
| \( X_{\Delta_6}(56) \to X_0(56) \) | 2 |
| \( X_{\Delta_7}(56) \to X_0(56) \) | 2 |
| \( X_{\Delta_6}(63) \to X_0(63) \) | 3 |
| \( X_{\Delta_8}(63) \to X_0(63) \) | 3 |
| \( X_{\Delta_6}(63) \to X_0(63) \) | 3 |
| \( X_{\Delta_7}(64) \to X_{\Delta_6}(64) \) | 2 |
| \( X_{\Delta_6}(72) \to X_0(72) \) | 2 |
| \( X_{\Delta_7}(72) \to X_0(72) \) | 2 |
From Proposition 1.1 we have the following result:

**Corollary 4.12.** The following $X_{\Delta}(N)$ are not bielliptic:

- $X_{\Delta_1}(45)$, $X_{\Delta_1}(48)$, $X_{\Delta_2}(48)$, $X_{\Delta_3}(48)$, $X_{\Delta_1}(56)$, $X_{\Delta_2}(56)$, $X_{\Delta_4}(56)$, $X_{\Delta_1}(63)$
- $X_{\Delta_2}(63)$, $X_{\Delta_3}(63)$, $X_{\Delta_1}(64)$, $X_{\Delta_1}(72)$, $X_{\Delta_4}(72)$, $X_{\Delta_1}(72)$.

4.3. **Castelnuovo’s inequality.** Consider $X_{\Delta_1}(34)$ of genus 9. Note that there is a natural map $\phi : X_{\Delta_1}(34) \to X_{\Delta_1}(17)$ of degree 3, and $X_{\Delta_1}(17)$ is of genus 1. Suppose $X_{\Delta_1}(34)$ is bielliptic. Then there is a map of degree 2 from $X_{\Delta_1}(34)$ to an elliptic curve $E$. Let $F$ (resp. $F_1, F_2$) be the function field of $X_{\Delta_1}(34)$ (resp. $X_{\Delta_1}(17)$, $E$). Applying Castelnuovo’s inequality, we get a contradiction. Thus $X_{\Delta_1}(34)$ is not bielliptic. By using the same argument, we have the following result:

**Lemma 4.13.** The following $X_{\Delta}(N)$ are not bielliptic:

- $X_{\Delta_1}(34)$, $X_{\Delta_2}(45)$, $X_{\Delta_1}(49)$, $X_{\Delta_1}(50)$, $X_{\Delta_4}(54)$, $X_{\Delta_5}(56)$, $X_{\Delta_1}(63)$, $X_{\Delta_5}(72)$, $X_{\Delta_8}(72)$, $X_{\Delta_3}(81)$.

For each $X = X_{\Delta}(N)$ in the list of Lemma 4.13, we suggest a finite morphism $\phi : X \to Y$ and its degree in Table 2 which enables us to conclude that $X$ is not bielliptic by applying Castelnuovo’s inequality.

**Table 2:** List of maps $\phi : X \to Y$ and their degrees

| $\phi : X \to Y$ | degree |
|------------------|--------|
| $X_{\Delta_1}(34) \to X_{\Delta_1}(17)$ | 3 |
| $X_{\Delta_1}(45) \to X_{\Delta_1}(15)$ | 3 |
| $X_{\Delta_1}(49) \to X_{\Delta_1}(49)$ | 7 |
| $X_{\Delta_1}(50) \to X_{\Delta_1}(25)$ | 3 |
| $X_{\Delta_4}(54) \to X_{\Delta_1}(27)$ | 3 |
| $X_{\Delta_5}(56) \to X_{\Delta_1}(28)$ | 2 |
| $X_{\Delta_1}(63) \to X_{\Delta_1}(21)$ | 3 |
| $X_{\Delta_2}(72) \to X_{\Delta_1}(24)$ | 9 |
| $X_{\Delta_5}(72) \to X_{\Delta_1}(24)$ | 3 |
| $X_{\Delta_8}(81) \to X_{\Delta_1}(27)$ | 3 |

From Proposition 1.1 we have the following result:

**Corollary 4.14.** The following $X_{\Delta}(N)$ are not bielliptic:

- $X_{\Delta_3}(56)$, $X_{\Delta_5}(63)$, $X_{\Delta_3}(72)$, $X_{\Delta_1}(81)$.

4.4. **Elliptic elements.** We can actually refine the unramified covering method a little bit.
Lemma 4.15. Suppose that $X_0(N)$ has a bielliptic involution $\tilde{v}$ that is given by an elliptic element $w \in GL_2^+(\mathbb{R})$ such that $w$ also normalizes $\Gamma_\Delta(N)$. Suppose further that there exists $a \in (\mathbb{Z}/N\mathbb{Z})^* - \Delta$ such that $[a]w$ is an elliptic element.

If $X_\Delta(N)$ is bielliptic and $q_\Delta(N) \geq 6$, then the (unique) bielliptic involution of $X_\Delta(N)$ cannot induce the bielliptic involution $\tilde{v}$ on $X_0(N)$.

Proof. Suppose the (unique) bielliptic involution $v$ of $X_\Delta(N)$ induces $\tilde{v}$ on $X_0(N)$. As the automorphisms $v$ and $w$ of $X_\Delta(N)$ both induce $\tilde{v}$ on $X_0(N)$, we must have $v = [b]w$ for some $b \in (\mathbb{Z}/N\mathbb{Z})^*$. As in the proof of Lemma 4.10 the covering $X_\Delta(N) \to X_0(N)$ must be totally unramified and $v = [b]w$ must fix all the points lying above the fixed points of $\tilde{v}$.

In particular, let $z, u \in \mathbb{H}$ be the fixed point of $w$ resp. of $[a]w$ and denote by $\overline{v}$ and $\overline{w}$ their images on $X_\Delta(N)$. Then we have $\overline{z} = \overline{v}(z) = [b]w\overline{z} = [b]\overline{w}$, which implies $[b] \in \Delta$, and $[a]w\overline{w} = \overline{w} = v\overline{w} = [b]w\overline{w}$, which would imply that $[a]$ must also be in $\Delta$.

By applying the above lemma, we have the following result:

Lemma 4.16. The following $X_\Delta(N)$ are not bielliptic:

- $X_{\Delta_8}(56)$, $X_{\Delta_0}(63)$, $X_{\Delta_2}(69)$.

Proof. We only treat the case $X_{\Delta_8}(56)$ in detail. Suppose $v$ is a unique bielliptic involution on $X_{\Delta_8}(56)$ and $\tilde{v}$ is the induced involution on $X_0(56)$. Note that $X_0(56)$ is not hyperelliptic but bielliptic, and all the bielliptic involutions of $X_0(56)$ are $W_{56}$, $W_7$ and $W_7S_2W_8S_2$ where $S_2 = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$ (see [B1] Theorem 3.15). Thus $\tilde{v}$ is equal to one of the three bielliptic involutions.

One easily checks that $S_2$ normalizes $\Gamma_{\Delta_8}(56)$, as do $W_7$ and $W_8$. So $W_7S_2W_8S_2$ also normalizes $\Gamma_{\Delta_8}(56)$. By the same argument in the proof of Lemma 4.10 $\tilde{v}$ cannot be $W_{56}$.

Take $W_7 = \begin{pmatrix} 7 & -1 \\ 56 & -7 \end{pmatrix}$ and $[5] = \begin{pmatrix} 5 & -1 \\ 56 & -11 \end{pmatrix}$. Then $[5]W_7 = \begin{pmatrix} -21 & 2 \\ 224 & -21 \end{pmatrix}$. By Lemma 4.10 $\tilde{v}$ cannot be $W_7$. Finally take $W_7 = \begin{pmatrix} 7 & -6 \\ 56 & 49 \end{pmatrix}$, $W_8 = \begin{pmatrix} -64 & -3 \\ -168 & -8 \end{pmatrix}$ and $[5] = \begin{pmatrix} 61 & 6 \\ -112 & -11 \end{pmatrix}$. Then $W_7S_2W_8S_2 = \begin{pmatrix} -28 & -15 \\ 56 & 28 \end{pmatrix}$ and $[5]W_7S_2W_8S_2 = \begin{pmatrix} -1372 & -747 \\ 2520 & 1372 \end{pmatrix}$.

Thus $\tilde{v}$ cannot be $W_7S_2W_8S_2$, and hence $X_{\Delta_8}(56)$ is not bielliptic.

The other cases are dealt with in the same manner. Note that the curve $X_0(63)$ has exceptional automorphisms. But by [B1] Theorem 3.15 all its bielliptic involutions come from $PSL_2(\mathbb{R})$. 

4.5. Bring’s curve. Next we will treat the modular curve $X_{\Delta_1}(25)$. Kubert denoted this curve by $B$ in [Ku]. We first thought that Kubert used such a notation because it would be isomorphic to Bring’s curve. Bring’s curve is the unique curve of genus 4 with full automorphism group $S_5$. Also it is isomorphic to the modular curve $X_1(5,10)$ and it is a bielliptic curve (cf. [C-D], [Hu]). But it turns out that $X_{\Delta_1}(25)$ is not Bring’s curve and not even a bielliptic curve.
Theorem 4.17. Bring’s curve is the unique bielliptic curve of genus 4 over \( \mathbb{C} \) that has an automorphism of order 5.

Proof. The automorphism \( \sigma \) of order 5 acts by conjugation on the bielliptic involutions. If it fixes one of them, this induces an automorphism of order 5 with fixed points on a curve of genus 1, which is impossible. So the number of bielliptic involutions must be divisible by 5. By [C-D, Corollary 6.9] this number cannot be 5, and 10 means Bring’s curve. \( \square \)

Lemma 4.18. \( X_{\Delta_1}(25) \) is not a bielliptic curve.

Proof. Obviously \([6]\) is an automorphism of \( X_{\Delta_1}(25) \) of order 5. So if the curve were bielliptic, it would be isomorphic to Bring’s curve \( X_1(5,10) \). Both curves are defined over some number field, so an isomorphism would be defined over the algebraic closure of \( \mathbb{Q} \) and hence over a suitable number field. By conjugating with the diagonal matrix \( \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \), one gets an isomorphism between \( X_1(5,10) \) and \( X_{\Delta_2}(50) \). The latter curve covers \( X_0(50) \).

By Cremona’s table [C2] there are elliptic curves over \( \mathbb{Q} \) with conductor 50. These have multiplicative reduction at 2 and hence also multiplicative reduction over any number field at any prime above 2. They are isogeny factors of the Jacobian of the modular curve isomorphic to \( X_1(5,10) \), which therefore must have bad reduction above 2. On the other hand, \( X_{\Delta_1}(25) \) is covered by \( X_1(25) \) and therefore has good reduction outside the primes above 5. \( \square \)

4.6. The case \( N=37 \).

Lemma 4.19. The group \( \langle [2], \hat{W}_{37} \rangle \cong C_2 \times C_2 \) is a Sylow 2-subgroup of \( \text{Aut}(X_{\Delta_4}(37)) \). In particular, \( X_{\Delta_4}(37) \) is not bielliptic.

Proof. To ease notation we write \( X \) for \( X_{\Delta_4}(37) \) and \( A \) for its full automorphism group. Also, \( \langle [2], \hat{W}_{37} \rangle \) will be denoted by \( M \).

We first show that \( X_{\Delta_4}(37) \) is bielliptic if and only if \( |A| \) is divisible by 8.

Each of the three involutions in \( M \) has exactly 2 fixed points. So if \( X \) has a bielliptic involution \( v \), the 2-Sylow subgroup containing \( M \) must also contain a conjugate of \( v \), and hence its order must be divisible by 8.

Conversely, if 8 divides \( |A| \), then, by the theory of p-groups, there exists a group \( D \subseteq A \) of order 8 that contains \( M \). Since \( D \) has at least 3 involutions, it cannot be cyclic or a quaternion group. On the other hand, the groups \( C_4 \times C_2 \) and \( C_2 \times C_2 \times C_2 \) cannot act as automorphisms on a genus 4 Riemann surface [K1 Proposition 2]. So \( D \) must be a dihedral group \( D_4 \). Applying the Hurwitz formula to the covering \( X \to X/D \) shows that the two involutions \( v_1, v_2 \) outside \( M \) are bielliptic.

For the same reason as in the proof of Lemma 4.18 the curve \( X \) is not Bring’s curve. So by [C-D Corollary 6.9] it has at most 6 bielliptic involutions. Thus any possible bielliptic involution must be defined over an extension of degree at most 6 of \( \mathbb{Q} \). On the other hand, \( [2] \) is defined over \( \mathbb{Q} \), whereas \( \hat{W}_{37} \) and \( [2]\hat{W}_{37} \) are defined over \( \mathbb{Q}(\sqrt{37}) \).
By using the computer algebra system SAGE, we can get the \( q \)-expansions of a basis of the cusps forms of weight 2 for \( \Gamma_{\Delta_4}(37) \). Using these \( q \)-expansions and following the method described in Section 2 in \([\text{Ha-Sh}]\), one obtains a model of the curve \( X_{\Delta_4}(37) \) over \( \mathbb{C} \) as the common zero-set of the following two homogeneous polynomials:

\[
Q_1: \quad -x_2^2 + x_3x_1 - 2x_4x_3,
\]

\[
Q_2: \quad 9x_2^2x_1 - 20x_2^3 - 9x_3x_1^2 + 12x_3x_2x_1 - 8x_3x_2^2 + 16x_3^2x_1 - 8x_3^2x_2
\]

\[
-12x_3^3 + 8x_4x_1^2 + 18x_4x_2^2 - 20x_4^2x_1 - 24x_4^2x_2 + 24x_4^3.
\]

The curve \( C \) defined by \( Q_1 \) and \( Q_2 \) is already defined over \( \mathbb{Q} \) and smooth at \( p = 5 \). Let \( \overline{C} \) denote its reduction modulo 5.

We emphasize that the question whether over \( \mathbb{Q} \) the curve \( C \) is already a model of \( X_{\Delta_4}(37) \) or rather of one of its twists is not important for us. We only need the connection over \( \mathbb{C} \), which implies \( \text{Aut}_C(X_{\Delta_4}(37)) \cong \text{Aut}_C(C) \). Moreover, \( \text{Aut}_C(C) \) embeds into \( \text{Aut}_{\mathbb{F}_5}(C) \).

So if \( |A| \) is divisible by 8, or equivalently \( X \) is bielliptic, then by the preceding discussion \( \text{Aut}_{\mathbb{F}_5}(C) \) contains a pair of conjugate, non-commuting bielliptic involutions that generate a group \( D_4 \). By the preceding argumentation, these two bielliptic involutions must both be defined over \( \mathbb{F}_{5^k} \) with \( k \leq 6 \).

But using the computer algebra system MAGMA one can easily calculate for \( k \leq 12 \) that \( |\text{Aut}_{\mathbb{F}_{5^k}}(C)| \) equals 2 if \( k \) is odd and 4 if \( k \) is even.

This finally proves the lemma. \( \square \)

Having come so far, we cannot resist working out the full automorphism group of \( X_{\Delta_4}(37) \).

**Proposition 4.20.** The modular curve \( X_{\Delta_4}(37) \) has no exceptional automorphisms. In other words, \( \text{Aut}(X_{\Delta_4}(37)) = \langle [2], \hat{W}_{37} \rangle \).

**Proof.** We keep the notation from the proof of the previous lemma, where we have shown that \( M = \langle [2], \hat{W}_{37} \rangle \) is a 2-Sylow subgroup of \( A = \text{Aut}(X_{\Delta_4}(37)) \). As a consequence the only possibilities for \( |A| \) are 4, 12, 20, 36, and 60. (See for example \([\text{Ki}]\) or \([\text{K-K}]\) for the possible orders of automorphism groups for genus 4.)

Also note that by \([\text{Ac1}]\) Corollary 1, p. 346] an unramified degree 3 Galois cover of a genus 2 curve is bielliptic. So every element of order 3 in \( A \) must have 3 or 6 fixed points.

If the normalizer of \( M \) in \( A \) is bigger than \( M \), then there is an element of order 3 or 5 in \( A \) inducing an automorphism with fixed points on the genus 1 curve \( X/M = X_{\Delta_4}^+(37) \). But this is not possible. (Note that the \( j \)-invariant of \( X_{\Delta_4}^+(37) \) is not 0.)

So we can assume from now on that \( M \) is its own normalizer. Then by Burnside’s Transfer Theorem (see for example \([\text{Ha}]\) Theorem 14.3.1]) \( A \) contains a normal subgroup \( N \) of index 4. If for example \( |A| = 60 \), then \( N \), being of order 15, is cyclic, and hence \( A \) has a unique 5-Sylow group \( F \). An involution can only act as identity or as inversion on \( F \). Applying the Galois automorphism of \( \mathbb{Q}(\sqrt{37})/\mathbb{Q} \) to \( A \) maps \( F \) into itself and \( \hat{W}_{37} \) to \([2] \hat{W}_{37} \) (compare Remark 2.7). This shows that \( \hat{W}_{37} \) and \([2] \hat{W}_{37} \) must act on \( F \) in the
same manner. Consequently \([2]\) commutes with the elements in \(F\), and we end up with an automorphism of order 5 on the curve \(X_0(37)\). The same arguments apply if \(N\) has order 3, or 5, or is a cyclic group of order 9.

So we are left with the case \(N \cong C_3 \times C_3\). Then by the Hurwitz formula each nontrivial element in \(N\) has 3 fixed points. The action of \(N\) cuts up the Jacobian \(J\) of \(X\) into a product (up to isogeny) of 4 elliptic curves \(E_i\), each \(E_i\) being the quotient of \(X\) by an element \(t_i\) of order 3, and \(N/\langle t_i \rangle\) inducing an automorphism of order 3 with fixed points on \(E_i\). This is only possible if \(j(E_i) = 0\). So \(J\) must have potentially good reduction everywhere. This contradicts \(J\) having the isogeny factor \(X_0^+(37)\) with multiplicative reduction at places above 37. □

Finally, the first two authors [J-K2, Lemma 2.6] already proved that \(X_{\Delta_1}(36)\) is not bielliptic by using properties of the normalizer of \(\Gamma_{\Delta_1}(36)\).

**Lemma 4.21.** The modular curve \(X_{\Delta_1}(36)\) is not bielliptic.

**Remark 4.22.** The reader might wonder why we didn’t systematically use [J-K2], which decides for every curve \(X_1(M, N)\) whether it is bielliptic or not. After all, each \(X_1(M, N)\) is isomorphic to some \(X_{\Delta}(MN)\). The answer is that this would mainly apply to curves which we have settled by Proposition 1.1 as corollaries to other cases that are not isomorphic to any \(X_1(M, N)\), and that hence had to be dealt with anyway.

5. **Bielliptic curves**

In this section, we will show that the remaining curves \(X_{\Delta}(N)\) are bielliptic. By [J-K2] the curves \(X_1(2, 14), X_1(2, 16), X_1(3, 12), X_1(4, 12), X_1(5, 10), X_1(7, 7),\) and \(X_1(8, 8)\) are bielliptic. The first four are isomorphic (over \(\mathbb{C}\)) to \(X_{\Delta_1}(28), X_{\Delta_1}(32), X_{\Delta_2}(36), X_{\Delta_6}(48)\), respectively. The other three are already known under different names.

**Remark 5.1.** The following curves are famous extremal examples.

- \(X_1(5, 10)\), which over \(\mathbb{C}\) is isomorphic to \(X_{\Delta_2}(50)\), is Bring’s curve, which we already discussed in the previous section. It has 10 bielliptic involutions, and is the unique curve of genus 4 with more than 6 bielliptic involutions. (See [C-D] Corollary 6.9.)
- \(X(7)\), which over \(\mathbb{C}\) is isomorphic to \(X_{\Delta_2}(49)\), is the famous Klein quartic. Its automorphism group is the simple group \(PSL_2(\mathbb{F}_7)\) of order 168, and it has 21 bielliptic involutions, the maximum possible for a curve of genus 3.
- \(X(8)\), which over \(\mathbb{C}\) is isomorphic to \(X_{\Delta_3}(64)\), is the Wiman curve, the unique curve of genus 5 with the maximum possible of 192 automorphisms. It has exactly 3 bielliptic involutions. (See [B-K-X], Lemma 4.5 and [K-M-V].)

Some of the remaining curves \(X_{\Delta}(N)\) are easily seen to be bielliptic, as they are double covers of an elliptic curve \(X_0(N)\), to wit, \(X_{\Delta_1}(24), X_{\Delta_2}(24)\), and \(X_{\Delta_2}(36)\). Correspondingly, there is a bielliptic involution of the form \([a]\).

Similarly, there is a degree 2 map from \(X_{\Delta_1}(32)\) to the elliptic curve \(X_{\Delta_2}(32)\) and a degree 2 map from \(X_{\Delta_1}(28)\) to the elliptic curve \(X_1(14)\).
More generally, there can be a degree 2 map from $X_\Delta(N)$ to an elliptic curve $X_\Delta'(N/2)$. This happens for $X_{\Delta_6}(40), X_{\Delta_6}(48),$ and $X_{\Delta_3}(64),$ mapping to $X_{\Delta_1}(20), X_{\Delta_3}(24), X_{\Delta_2}(32),$ respectively.

The curves $X_{\Delta_1}(30), X_{\Delta_2}(33), X_{\Delta_4}(35), X_{\Delta_4}(39), X_{\Delta_6}(40), X_{\Delta_4}(41), X_{\Delta_6}(48),$ are of genus 5 and double covers of the hyperelliptic genus 3 curve $X_0(N)$. By [Ac2, p. 50] they must be hyperelliptic (which is excluded by Theorem 4.3) or bielliptic. Consequently, in these cases the two possible lifts of the hyperelliptic involution of $X_0(N)$ to involutions of $X_{\Delta_i}(N)$ both are bielliptic involutions.

Another criterion by Accola [Ac1, Corollary 1] says that a curve $X$ of genus 4 that is an unramified degree 3 Galois cover of a genus 2 curve $Y$ will be bielliptic. More precisely, by [Ac1, Lemma 2] then $X$ has at least 3 bielliptic involutions, namely the 3 lifts of the hyperelliptic involution of $Y$. This applies to $X_{\Delta_1}(26), X_{\Delta_1}(28),$ and $X_{\Delta_3}(37),$ all three being unramified degree 3 covers of the corresponding $X_0(N)$. The latter curve is highly interesting, and we investigate it in detail.

Lemma 5.2. $X_{\Delta_3}(37)$ is a bielliptic curve, but any bielliptic involution must be an exceptional automorphism.

Proof. $X_{\Delta_3}(37)$ has genus 4 and is an unramified degree 3 Galois cover of the genus 2 curve $X_0(37)$. So by [Ac1, Corollary 1, p. 346] it is bielliptic.

The non-exceptional automorphisms of $X_{\Delta_3}(37)$ form a group $S_3$ generated by $[2]$ (of order 3) and the involution $\hat{W}_{37}$. The quotient by this group is the elliptic curve $X_0(37)/\hat{W}_{37}$. Applying the Hurwitz formula to this $S_3$-covering, one sees that each of the 3 (conjugate) involutions has 2 fixed points. So any bielliptic involution must be exceptional.

Remark 5.3. This gives us another proof that $X_{\Delta_3}(37)$ is not hyperelliptic, as by the Castelnuovo inequality a bielliptic curve of genus $g > 3$ cannot be hyperelliptic.

But the more important point is that this contradicts the claim in [Mo] that $X_0(37)$ is the only curve for square-free $N$ with exceptional automorphisms. Given the situation, of course we want to determine the complete automorphism group of $X_{\Delta_3}(37)$. For that it will be convenient to prove the preceding lemma with a slightly more constructive approach.

Alternative Proof of Lemma 5.2. Let $u (= [2])$ be the automorphism of $X_{\Delta_3}(37)$ of order 3 whose quotient is $X_0(37)$. Then $\text{Aut}(X_{\Delta_3}(37))$ contains the group of non-exceptional automorphisms $B = \langle u, \hat{W}_{37} \rangle \cong S_3$.

$X_{\Delta_3}(37)$ has genus 4 and is an unramified Galois cover of the genus 2 curve $X_0(37)$. So by [Ac2, Corollary 4.13] the hyperelliptic involution of $X_0(37)$ lifts to an involution $w$ on $X_{\Delta_3}(37)$. Moreover, $w$ normalizes $B$, and hence they generate a non-abelian group of order 12 that has a normal 3-Sylow subgroup whose quotient group is non-cyclic. So the group of order 12 is a dihedral group $D_6$.

The center of this dihedral group is an involution $v$. So $u$, which commutes with $v$, permutes the fixed points of $v$. If there are two fixed points, $u$ must fix them. But we know that $u$ has no fixed points. So $v$ has 6 fixed points and it is a bielliptic involution.
Theorem 5.4. The full automorphism group of the curve $X_{\Delta_3}(37)$ is isomorphic to a dihedral group $D_6$ of order 12. The exceptional automorphisms are two automorphisms of order 6 and the 4 bielliptic involutions, of which 3 are conjugate in $\text{Aut}(X_{\Delta_3}(37))$ and one is central. The quotient by the central involution is one of the two curves that are 3-isogenous to the strong Weil curve $X_-(37)$.

Proof. In the previous elaborations we have seen that $A := \text{Aut}(X_{\Delta_3}(37))$ contains a subgroup $D$, isomorphic to $D_6$, that contains $u (= [2])$, an automorphism of order 3 with $X_{\Delta_3}(37)/\langle u \rangle = X_0(37)$. Now we show equality.

Let $p$ be a prime dividing $|A|$. Then $p \leq 2g + 1 = 9$ by the Hurwitz formula. But $p = 7$ is not possible, as then the automorphism of order 7 would have exactly one fixed point, which is known to be impossible (see for example [Ac2, 4.15.3, p. 41]). Also, $p = 5$ is not possible, for by the same arguments as in the proof of Lemma 4.18, $X_{\Delta_3}(37)$ is not isomorphic to Bring’s curve. Finally, $|A|$ is not divisible by 9. Otherwise the nontrivial center of the 3-Sylow subgroup would lead to an automorphism of order 3 on $X_0(37)$.

As $\text{Aut}(X_0(37))$ has order 4, $D$ is the normalizer of $u$ in $A$. So the index of $D$ in $A$ equals the number of 3-Sylow subgroups, and hence must be congruent to 1 modulo 3. Since for genus 4 the order of the automorphism group is bounded by 120, this leaves only 12 and 48 as possible orders for $|A|$. If $|A| = 48$, the action on the four 3-Sylow subgroups induces a homomorphism from $A$ into $S_4$. The kernel of this homomorphism must be a normal subgroup of $D$. But it cannot contain $u$; otherwise $A$ would have a subgroup $C_3 \times C_3$. So this kernel can only be the center of $D$, which is the bielliptic involution $v$. Thus, if $|A| = 48$, then $A$ would induce a group of automorphisms $A/\langle v \rangle \cong S_4$ on $X_{\Delta_3}(37)/\langle v \rangle$, which is known to be impossible on a genus 1 curve.

So we have shown $A \cong D_6$ and the rest of the theorem follows easily from the structure of that group. □

Example 5.5. The genus 3 curve $X_{\Delta_1}(21)$ was already shown in [J-K3] to be hyperelliptic. Using its equation

$$y^2 = (x^2 - x + 1)(x^6 + x^5 - 6x^4 - 3x^3 + 14x^2 - 7x + 1)$$

from [B-G-G-P, Table 11], MAGMA determines its automorphism group to be $D_6$; so there are no exceptional automorphisms. The center is the hyperelliptic involution $\hat{W}_3 = (9_{21} - 4)$. There are exactly 3 (conjugate) bielliptic involutions, namely $\hat{W}_{21}$, $[2] \hat{W}_{21}$, and $[4] \hat{W}_{21}$.

So far our elaborations account for 18 of the remaining 25 curves. In most cases, the bielliptic involutions on $X_\Delta(N)$ are coming from $W_d$. By applying the same method used in Example 3.2 in Section 3, we have the following result:

Theorem 5.6. The full list of bielliptic modular curves $X_\Delta(N)$ is given by the following 25 curves. Some of these curves have more bielliptic involutions than the ones we listed.
Table 3: List of all bielliptic \(X_\Delta(N)\) and some of their bielliptic involutions

| \(X_\Delta(N)\) | genus | some bielliptic involutions |
|------------------|-------|----------------------------|
| \(X_{\Delta_1}(21)\) | 3     | \(\tilde{W}_{21}, [2]\tilde{W}_{21}, [4]\tilde{W}_{21}\) |
| \(X_{\Delta_1}(24)\) | 3     | \(\tilde{W}_{24}, [7]\tilde{W}_{24}\) |
| \(X_{\Delta_2}(24)\) | 3     | \(\tilde{W}_{24}, [5]\tilde{W}_{24}\) |
| \(X_{\Delta_1}(26)\) | 4     | \(\tilde{W}_{26}, [3]\tilde{W}_{26}, [9]\tilde{W}_{26}\) |
| \(X_{\Delta_2}(26)\) | 4     | \(\tilde{W}_{26}\) |
| \(X_{\Delta_1}(28)\) | 4     | \(\left(\begin{smallmatrix} 1 & 0 \\ 14 & 0 \end{smallmatrix}\right), W_7, [3]W_7, [9]W_7\) |
| \(X_{\Delta_2}(28)\) | 4     | \(W_7, [11]W_7\) |
| \(X_{\Delta_1}(29)\) | 4     | \(\tilde{W}_{29}\) |
| \(X_{\Delta_1}(30)\) | 5     | \(\tilde{W}_{15}, [7]\tilde{W}_{15}\) |
| \(X_{\Delta_1}(32)\) | 5     | \([7]\) |
| \(X_{\Delta_2}(33)\) | 5     | \(\tilde{W}_{11}, [5]\tilde{W}_{11}\) |
| \(X_{\Delta_2}(34)\) | 5     | \(\tilde{W}_2\) |
| \(X_{\Delta_4}(35)\) | 7     | \(W_5\) |
| \(X_{\Delta_2}(35)\) | 5     | \(\tilde{W}_{35}, [2]\tilde{W}_{35}\) |
| \(X_{\Delta_2}(36)\) | 3     | \([5], \tilde{W}_{36}, [5]\tilde{W}_{36}\) |
| \(X_{\Delta_3}(37)\) | 4     | see Theorem 5.4 |
| \(X_{\Delta_4}(39)\) | 5     | \(\tilde{W}_{39}, [2]\tilde{W}_{39}\) |
| \(X_{\Delta_2}(40)\) | 5     | \(\left(\begin{smallmatrix} 1 & 0 \\ 20 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -10 & 1 \\ -120 & 10 \end{smallmatrix}\right), [3] \left(\begin{smallmatrix} -10 & 1 \\ -120 & 10 \end{smallmatrix}\right)\) |
| \(X_{\Delta_4}(41)\) | 5     | \(\tilde{W}_{41}, [3]\tilde{W}_{41}\) |
| \(X_{\Delta_4}(45)\) | 5     | \(W_9\) |
| \(X_{\Delta_4}(48)\) | 5     | \(\left(\begin{smallmatrix} 1 & 0 \\ 24 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -6 & 1 \\ -48 & 6 \end{smallmatrix}\right), [5] \left(\begin{smallmatrix} -6 & 1 \\ -48 & 6 \end{smallmatrix}\right)\) |
| \(X_{\Delta_2}(49)\) | 3     | \(\tilde{W}_{49}\) |
| \(X_{\Delta_2}(50)\) | 4     | \(\tilde{W}_{50}\) |
| \(X_{\Delta_4}(55)\) | 9     | \(\tilde{W}_{11}\) |
| \(X_{\Delta_4}(64)\) | 5     | \(\left(\begin{smallmatrix} 1 & 0 \\ 52 & 1 \end{smallmatrix}\right)\) |

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6. Quadratic points

In this section, we determine all the $X_\Delta(N)$ which admit infinitely many quadratic points over $\mathbb{Q}$. If $X_\Delta(N)$ is subhyperelliptic, then it has infinitely many quadratic points because there exists $\mathbb{Q}$-rational map of degree 2 to the projective line.

Now suppose that $X_\Delta(N)$ has infinitely many quadratic points but is not subhyperelliptic. Then it must be bielliptic by [H-S, Corollary 3]. This reduces the remaining candidates to those listed in Theorem 5.6.

More precisely, we have the following “if and only if”-statement over a fixed based field $k$.

**Theorem 6.1.** [B2, Theorem 2.14] Let $k$ be a number field and $X$ a non-hyperelliptic curve of genus $g \geq 3$ over $k$ with a $k$-rational point. Then $X$ has infinitely many quadratic points over $k$ if and only if it has a bielliptic involution $v$ defined over $k$ such that the elliptic curve $X/v$ has positive rank over $k$.

Actually, except for the case $N = 37$ we only need the weaker statement that if a non-subhyperelliptic $X$ has infinitely many quadratic points over $\mathbb{Q}$, then its Jacobian variety must contain an elliptic curve $E$ with positive rank over $\mathbb{Q}$.

Now, if the Jacobian of $X_\Delta(N)$ or even of $X_1(N)$ contains an elliptic curve $E$ over $\mathbb{Q}$, then the conductor of $E$ must divide $N$. From Cremona’s tables [Cr], for the numbers $N$ from the Table in Theorem 5.6 there exists no elliptic curve of positive rank over $\mathbb{Q}$ whose conductor divides $N$, except for $N = 37$.

**Lemma 6.2.** $X_\Delta(37)$ has only finitely many quadratic points.

**Proof.** Suppose $X_\Delta(37)$ has infinitely many quadratic points over $\mathbb{Q}$. Then its Jacobian variety must contain an elliptic curve $E$ of positive rank over $\mathbb{Q}$ and there must be a map of degree 2 from $X_\Delta(37)$ to $E$. Now according to Stein’s table [St], the Jacobian variety of $X_1(37)$ contains only one elliptic curve $E$ with positive rank over $\mathbb{Q}$, namely the quotient curve of $X_0(37)$ by $W_{37}$. There also is the natural map of degree 6 from $X_\Delta(37)$ via $X_0(37)$ to $E$. Since the $\mathbb{Q}$-isogeny class of $E$ contains only one curve, a map of degree 2 would imply the existence of an endomorphism of $E$ of degree 3. But all endomorphisms of $E$ have degree $n^2$, since the endomorphism ring of $E$ is $\mathbb{Z}$. Note that CM-curves have integral $j$-invariant, but from Cremona’s table [Cr] the $j$-invariant of $E$ has a pole at 37. □

**Remark 6.3.** With practically the same proof one can see that the curve $X_\Delta(37)$ has only finitely many quadratic points. So to obtain that fact one doesn’t need the lengthy proof that $X_\Delta(37)$ is not bielliptic.

Summarizing, we have the following:

**Theorem 6.4.** Let $\{\pm 1\} \subseteq \Delta \subseteq (\mathbb{Z}/N\mathbb{Z})^\times$. Then $X_\Delta(N)$ has infinitely many quadratic points over $\mathbb{Q}$ if and only if it is a subhyperelliptic curve. Explicitly these are:

- rational: $X_\Delta_1(13), X_\Delta_2(13), X_\Delta_1(16), X_\Delta_2(25)$;
- elliptic: $X_{\Delta_1}(15), X_{\Delta_1}(17), X_{\Delta_2}(17), X_{\Delta_1}(19), X_{\Delta_1}(20), X_{\Delta_2}(21), X_{\Delta_3}(24), X_{\Delta_1}(27), X_{\Delta_2}(32)$;

- hyperelliptic: $X_{\Delta_1}(21)$ (genus 3).

Finally, we say a few words about the moduli problem described by the curves $X_\Delta(N)$. Let $\{\pm 1\} \subseteq \Delta \subseteq (\mathbb{Z}/N\mathbb{Z})^*$. A non-cuspidal $K$ rational point of $X_\Delta(N)$ corresponds to an isomorphism class of an elliptic curve $E$ over $K$ (in short Weierstrass form $y^2 = x^3 + ax + b$) and a primitive $N$-torsion point $P$ of $E$ such that the set $\{aP : a \in \Delta\}$ is Galois stable under $\text{Gal}(\overline{K}/K)$.

If $\Delta = \{\pm 1\}$, this means that the $x$-coordinate $x(P)$ of $P$ is in $K$. Consequently $(y(P))^2 \in K$. We can take a quadratic twist of $E$ over $K$ that multiplies $x$ with $(y(P))^2$ and $y$ with $(y(P))^3$ and obtain an elliptic curve $E'$ over $K$ with a $K$-rational $N$-torsion point.

For general $\Delta$ the $x$-coordinate of $P$ can have degree up to $\frac{1}{2} |\Delta|$ over $K$. Let $L = K(x(P))$. As above, we can twist $E$ over $L$ and get an elliptic curve $E'$ over $L$ with $L$-rational $N$-torsion point.

But if $\Delta$ is bigger than $\{\pm 1\}$, in general we cannot get an elliptic curve $E''$ over $K$ with an $N$-torsion point of degree $\leq \frac{1}{2} |\Delta|$ over $K$.

**Example 6.5.** The curve $X_{\Delta_1}(17)$ where $\Delta_1 = \{\pm 1, \pm 4\}$ has genus 1 and hence infinitely many quadratic points. A non-cuspidal one corresponds to an elliptic curve $E$ over a quadratic number field $K$ with a 17-torsion point $P$ whose $x$-coordinate has degree 2 over $K$.

Suppose we could construct from this an elliptic curve $E'$ over a quadratic number field $L$ with a 17-torsion point $Q$ that is quadratic over $L$. As the automorphism group of $(\mathbb{Z}/17\mathbb{Z})^*$ is cyclic, the non-trivial Galois automorphism of $L(Q)/L$ can map $Q$ only to $-Q$. But then $(E', Q)$ would correspond to a non-cuspidal $L$-rational point on $X_1(17)$, which is known not to exist.

Or interpreted differently, then we could twist $E'$ and get an elliptic curve $E''$ over the quadratic number field $L$ with an $L$-rational 17-torsion point, which is known not to exist.

**Example 6.6.** The curve $X_{\Delta_1}(21)$ has infinitely many quadratic points over $\mathbb{Q}$. So there are infinitely many elliptic curves $E$ over quadratic number fields $K$ (depending on $E$) with a $K$-rational 21-isogeny containing a $K$-rational 7-torsion point. Equivalently, we can say that there are infinitely many elliptic curves $E$ over quadratic number fields $K$ that have a $K$-rational 7-torsion point and a $K$-rational 3-isogeny.

As the curve $X_1(21)$ is known not to have any quadratic points outside the cusps, the underlying 3-torsion point cannot be $K$-rational. Of course, taking a suitable twist of $E$ will make the 3-torsion point $K$-rational, but then the (y-coordinate of the) 7-torsion point will no longer be $K$-rational.

Group-theoretically the underlying feature is that the intersection of $\pm \Gamma_1(7)$ and $\pm \Gamma_1(3)$ is $\Gamma_{\Delta_1}(21)$, and not $\pm \Gamma_1(21)$.

**Example 6.7.** By [B11, Theorem 4.3] the curve $X_0(61)$ has infinitely many quadratic points over $\mathbb{Q}$. This means that there are infinitely many elliptic curves $E$ over quadratic number
fields \( K \) (depending on \( E \)) with a \( K \)-rational 61-isogeny. (Infinitely many here means with infinitely many different \( j \)-invariants.)

However, none of the six intermediate curves \( X_\Delta(61) \) has infinitely many quadratic points. So for almost all of these \( j \)-invariants the \( x \)-coordinate of the underlying 61-torsion point of \( E \) will generate an extension of \( K \) of degree 30.

**Example 6.8.** Fix a number field \( F \). As the curve \( X_0(41) \) is hyperelliptic, it has infinitely many quadratic points over \( F \). So as \( K \) varies over all quadratic extensions of \( F \), there will be in total infinitely many elliptic curves \( E \) over \( K \) with a \( K \)-rational 41-isogeny. In general, the \( x \)-coordinate of the underlying 41-torsion point will generate an extension of degree 20 of \( K \).

If the elliptic curve \( X_\Delta_4(41)/\hat{W}_{41} \) has positive rank over \( F \), however, then \( X_\Delta_4(41) \) has infinitely many quadratic points over \( F \), and for these there are elliptic curves with the \( x \)-coordinate of the 41-torsion point lying already in a degree 10 extension of \( K \).

But no matter what \( F \) is, there will always only be finitely many cases for which the \( x \)-coordinate of the 41-torsion point generates an extension of \( K \) of degree less than 10. This is because none of the other intermediate curves \( X_\Delta(41) \) is subhyperelliptic or bielliptic. So over any number field \( F \) they will always have only finitely many quadratic points.

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| $N$ | $\{\pm 1\} \subsetneq \Delta \subsetneq (\mathbb{Z}/N\mathbb{Z})^*$ | $g_\Delta(N)$ | treated in |
|-----|--------------------------------------------------|--------------|-----------|
| 13  | $\Delta_1 = \{\pm 1, \pm 5\}$                  | 0            | Theorem 4.3 |
|     | $\Delta_2 = \{\pm 1, \pm 3, \pm 4\}$          | 0            | Theorem 4.3 |
| 15  | $\Delta_1 = \{\pm 1, \pm 4\}$                  | 1            | Theorem 4.3 |
| 16  | $\Delta_1 = \{\pm 1, \pm 7\}$                  | 0            | Theorem 4.3 |
| 17  | $\Delta_1 = \{\pm 1, \pm 4\}$                  | 1            | Theorem 4.3 |
|     | $\Delta_2 = \{\pm 1, \pm 2, \pm 4, \pm 8\}$   | 1            | Theorem 4.3 |
| 19  | $\Delta_1 = \{\pm 1, \pm 7, \pm 8\}$          | 1            | Theorem 4.3 |
| 20  | $\Delta_1 = \{\pm 1, \pm 9\}$                  | 1            | Theorem 4.3 |
| 21  | $\Delta_1 = \{\pm 1, \pm 8\}$                  | 3            | Example 5.5 |
|     | $\Delta_2 = \{\pm 1, \pm 4, \pm 5\}$          | 1            | Theorem 4.3 |
| 24  | $\Delta_1 = \{\pm 1, \pm 5\}$                  | 3            | Theorem 5.6 |
|     | $\Delta_2 = \{\pm 1, \pm 7\}$                  | 3            | Theorem 5.6 |
|     | $\Delta_3 = \{\pm 1, \pm 11\}$                 | 1            | Theorem 4.3 |
| 25  | $\Delta_1 = \{\pm 1, \pm 7\}$                  | 4            | Lemma 4.18 |
|     | $\Delta_2 = \{\pm 1, \pm 4, \pm 6, \pm 9, \pm 11\}$ | 0            | Theorem 4.3 |
|   |   |   |
|---|---|---|
| 26 | $\Delta_1 = \{\pm 1, \pm 5\}$ | 4 | Theorem 5.6 |
|   | $\Delta_2 = \{\pm 1, \pm 3, \pm 9\}$ | 4 | Theorem 5.6 |
| 27 | $\Delta_1 = \{\pm 1, \pm 8, \pm 10\}$ | 1 | Theorem 4.3 |
| 28 | $\Delta_1 = \{\pm 1, \pm 13\}$ | 4 | Theorem 5.6 |
|   | $\Delta_2 = \{\pm 1, \pm 3, \pm 9\}$ | 4 | Example 3.2 |
| 29 | $\Delta_1 = \{\pm 1, \pm 12\}$ | 8 | Lemma 4.4 |
|   | $\Delta_2 = \{\pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 13\}$ | 4 | Theorem 5.6 |
| 30 | $\Delta_1 = \{\pm 1, \pm 11\}$ | 5 | Theorem 5.6 |
| 32 | $\Delta_1 = \{\pm 1, \pm 15\}$ | 5 | Theorem 5.6 |
|   | $\Delta_2 = \{\pm 1, \pm 7, \pm 9, \pm 15\}$ | 1 | Theorem 4.3 |
| 33 | $\Delta_1 = \{\pm 1, \pm 10\}$ | 11 | Lemma 4.3 |
|   | $\Delta_2 = \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$ | 5 | Theorem 5.6 |
| 34 | $\Delta_1 = \{\pm 1, \pm 13\}$ | 9 | Lemma 4.3 |
|   | $\Delta_2 = \{\pm 1, \pm 9, \pm 13, \pm 15\}$ | 5 | Theorem 5.6 |
| 35 | $\Delta_1 = \{\pm 1, \pm 6\}$ | 13 | Lemma 4.4 |
|   | $\Delta_2 = \{\pm 1, \pm 11, \pm 16\}$ | 9 | Lemma 4.3 |
|   | $\Delta_3 = \{\pm 1, \pm 6, \pm 8, \pm 13\}$ | 7 | Theorem 5.6 |
|   | $\Delta_4 = \{\pm 1, \pm 4, \pm 6, \pm 9, \pm 11, \pm 16\}$ | 5 | Theorem 5.6 |
| 36 | $\Delta_1 = \{\pm 1, \pm 17\}$ | 7 | Lemma 4.1 |
|   | $\Delta_2 = \{\pm 1, \pm 11, \pm 13\}$ | 3 | Theorem 5.6 |
| 37 | $\Delta_1 = \{\pm 1, \pm 6\}$ | 16 | Lemma 4.1 |
|   | $\Delta_2 = \{\pm 1, \pm 10, \pm 11\}$ | 10 | Lemma 4.1 |
|   | $\Delta_3 = \{\pm 1, \pm 6, \pm 8, \pm 10, \pm 11, \pm 14\}$ | 4 | Theorem 5.4 |
|   | $\Delta_4 = \{\pm 1, \pm 3, \pm 4, \pm 7, \pm 9, \pm 10, \pm 11, \pm 12, \pm 16\}$ | 4 | Lemma 4.1 |
| 38 | $\Delta_1 = \{\pm 1, \pm 7, \pm 11\}$ | 10 | Lemma 4.3 |
| 39 | $\Delta_1 = \{\pm 1, \pm 14\}$ | 17 | Corollary 4.7 |
|   | $\Delta_2 = \{\pm 1, \pm 16, \pm 17\}$ | 9 | Lemma 4.3 |
|   | $\Delta_3 = \{\pm 1, \pm 5, \pm 8, \pm 14\}$ | 9 | Lemma 4.3 |
|   | $\Delta_4 = \{\pm 1, \pm 4, \pm 10, \pm 14, \pm 16, \pm 17\}$ | 5 | Theorem 5.6 |
| 40 | $\Delta_1 = \{\pm 1, \pm 9\}$ | 13 | Corollary 4.9 |
|   | $\Delta_2 = \{\pm 1, \pm 11\}$ | 13 | Lemma 4.11 |
|   | $\Delta_3 = \{\pm 1, \pm 19\}$ | 9 | Lemma 4.11 |
|   | $\Delta_4 = \{\pm 1, \pm 3, \pm 9, \pm 13\}$ | 7 | Lemma 4.8 |
|   | $\Delta_5 = \{\pm 1, \pm 7, \pm 9, \pm 17\}$ | 7 | Lemma 4.8 |
|   | $\Delta_6 = \{\pm 1, \pm 9, \pm 11, \pm 19\}$ | 5 | Theorem 5.6 |
| 41 | $\Delta_1 = \{\pm 1, \pm 9\}$ | 21 | Lemma 4.4 |
|   | $\Delta_2 = \{\pm 1, \pm 3, \pm 9, \pm 14\}$ | 11 | Lemma 4.3 |
|   | $\Delta_3 = \{\pm 1, \pm 4, \pm 10, \pm 16, \pm 18\}$ | 11 | Lemma 4.3 |
|   | $\Delta_4 = \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 9, \pm 10, \pm 16, \pm 18, \pm 20\}$ | 5 | Theorem 5.6 |
| 42 | $\Delta_1 = \{\pm 1, \pm 13\}$ | 13 | Lemma 4.4 |
| \( \Delta_1 \) | \( \Delta_2 \) | \( \Delta_3 \) | \( \Delta_4 \) | \( \Delta_5 \) | \( \Delta_6 \) |
|----------------|----------------|----------------|----------------|----------------|----------------|
| \{±1, ±19\}   | \{±1, ±5, ±17\} | \{±1, ±14, ±16\} | \{±1, ±5, ±19, ±23\} | \{±1, ±7, ±17, ±23\} | \{±1, ±11, ±13, ±23\} |
| 21             | 19             | 19             | 7              | 7              | 5              |
| Corollary 4.12 | Corollary 4.12 | Corollary 4.12 | Lemma 4.11     | Lemma 4.11     | Theorem 5.6     |
| 44             | 45             | 48             | 49             | 50             | 51             |
| \( \Delta_1 \) | \( \Delta_2 \) | \( \Delta_3 \) | \( \Delta_4 \) | \( \Delta_5 \) | \( \Delta_6 \) |
| \{±1, ±1, ±17\} | \{±1, ±17\}   | \{±1, ±4, ±13, ±16\} | \{±1, ±2, ±4, ±8, ±13, ±16, ±19, ±25\} | \{±1, ±17, ±19\} | \{±1, ±17, ±19\} |
| 44             | 48             | 49             | 50             | 54             | 51             |
| \( \Delta_1 \) | \( \Delta_2 \) | \( \Delta_3 \) | \( \Delta_4 \) | \( \Delta_5 \) | \( \Delta_6 \) |
| \{±1, ±18, ±19\} | \{±1, ±6, ±8, ±13, ±15, ±20, ±22\} | \{±1, ±9, ±11, ±19, ±21\} | \{±1, ±4, ±6, ±9, ±14, ±16, ±19, ±21, ±24, ±26\} | \{±1, ±13\} | \{±1, ±13\} |
| 44             | 49             | 50             | 54             | 55             | 56             |
| \( \Delta_1 \) | \( \Delta_2 \) | \( \Delta_3 \) | \( \Delta_4 \) | \( \Delta_5 \) | \( \Delta_6 \) |
| \{±1, ±1, ±21\} | \{±1, ±12, ±21, ±23\} | \{±1, ±16, ±19, ±24, ±26\} | \{±1, ±4, ±6, ±9, ±14, ±16, ±19, ±21, ±24, ±26\} | \{±1, ±13\} | \{±1, ±13\} |
| 44             | 48             | 49             | 50             | 55             | 56             |
| \( \Delta_1 \) | \( \Delta_2 \) | \( \Delta_3 \) | \( \Delta_4 \) | \( \Delta_5 \) | \( \Delta_6 \) |
| \{±1, ±11\}   | \{±1, ±19\}   | \{±1, ±27\}   | \{±1, ±9, ±25\} | \{±1, ±13, ±15, ±27\} | \{±1, ±9, ±11, ±13, ±25\} |
| 60             | 61             | 57             | 58             | 59             | 60             |
| \( \Delta_1 \) | \( \Delta_2 \) | \( \Delta_3 \) | \( \Delta_4 \) | \( \Delta_5 \) | \( \Delta_6 \) |
| \{±1, ±5, ±25\} | \{±1, ±7, ±11, ±17\} | \{±1, ±11, ±13, ±23\} | \{±1, ±11, ±19, ±29\} | \{±1, ±5, ±25\} | \{±1, ±5, ±25\} |
| 62             | 62             | 62             | 62             | 62             | 62             |
| $\Delta_2 = \{\pm 1, \pm 15, \pm 23, \pm 27, \pm 29\}$ | 19 | Lemma 4.4 |
| $\Delta_1 = \{\pm 1, \pm 8\}$ | 49 | Corollary 4.12 |
| $\Delta_2 = \{\pm 1, \pm 4, \pm 16\}$ | 33 | Corollary 4.12 |
| $\Delta_3 = \{\pm 1, \pm 5, \pm 25\}$ | 33 | Corollary 4.12 |
| $\Delta_4 = \{\pm 1, \pm 17, \pm 26\}$ | 25 | Corollary 4.14 |
| $\Delta_5 = \{\pm 1, \pm 20, \pm 22\}$ | 25 | Corollary 4.14 |
| $\Delta_6 = \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 31\}$ | 17 | Lemma 4.11 |
| $\Delta_7 = \{\pm 1, \pm 5, \pm 8, \pm 11, \pm 23, \pm 25\}$ | 17 | Lemma 4.11 |
| $\Delta_8 = \{\pm 1, \pm 8, \pm 10, \pm 17, \pm 19, \pm 26\}$ | 17 | Lemma 4.11 |
| $\Delta_9 = \{\pm 1, \pm 8, \pm 13, \pm 20, \pm 22, \pm 29\}$ | 13 | Lemma 4.16 |
| $\Delta_{10} = \{\pm 1, \pm 4, \pm 5, \pm 16, \pm 17, \pm 20, \pm 22, \pm 25, \pm 26\}$ | 9 | Lemma 4.13 |
| $\Delta_1 = \{\pm 1, \pm 31\}$ | 37 | Corollary 4.12 |
| $\Delta_2 = \{\pm 1, \pm 15, \pm 17, \pm 31\}$ | 13 | Lemma 4.11 |
| $\Delta_3 = \{\pm 1, \pm 7, \pm 9, \pm 15, \pm 17, \pm 23, \pm 25, \pm 31\}$ | 5 | Remark 5.1 |
| $\Delta_1 = \{\pm 1, \pm 22\}$ | 67 | Lemma 4.4 |
| $\Delta_2 = \{\pm 1, \pm 4, \pm 5, \pm 11, \pm 13, \pm 14, \pm 16, \pm 17, \pm 20, \pm 25, \pm 31\}$ | 13 | Lemma 4.16 |
| $\Delta_1 = \{\pm 1, \pm 17\}$ | 49 | Corollary 4.12 |
| $\Delta_2 = \{\pm 1, \pm 19\}$ | 49 | Corollary 4.12 |
| $\Delta_3 = \{\pm 1, \pm 35\}$ | 41 | Corollary 4.14 |
| $\Delta_4 = \{\pm 1, \pm 23, \pm 25\}$ | 25 | Corollary 4.12 |
| $\Delta_5 = \{\pm 1, \pm 17, \pm 19, \pm 35\}$ | 21 | Lemma 4.18 |
| $\Delta_6 = \{\pm 1, \pm 5, \pm 19, \pm 23, \pm 25, \pm 29\}$ | 13 | Lemma 4.11 |
| $\Delta_7 = \{\pm 1, \pm 7, \pm 17, \pm 23, \pm 25, \pm 31\}$ | 13 | Lemma 4.11 |
| $\Delta_8 = \{\pm 1, \pm 11, \pm 13, \pm 23, \pm 25, \pm 35\}$ | 9 | Lemma 4.13 |
| $\Delta_1 = \{\pm 1, \pm 26, \pm 28\}$ | 46 | Corollary 4.14 |
| $\Delta_2 = \{\pm 1, \pm 8, \pm 10, \pm 17, \pm 19, \pm 26, \pm 28, \pm 35, \pm 37\}$ | 10 | Lemma 4.13 |
| $\Delta_1 = \{\pm 1, \pm 45\}$ | 100 | Lemma 4.3 |
| $\Delta_2 = \{\pm 1, \pm 7, \pm 9, \pm 11, \pm 13, \pm 15, \pm 19, \pm 25, \pm 29, \pm 41, \pm 43\}$ | 20 | Lemma 4.4 |

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