Non-isothermal filtration of a viscous compressible fluid in a viscoelastic porous medium

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Abstract. The system of equations of one-dimensional unsteady fluid motion in a viscous heat-conducting medium is considered. The mathematical model is based on the equations of conservation of mass for liquid and solid phases, Darcy’s law, rheological relation, the law of conservation of balance of forces and the equation for the temperature of the medium. The transition to Lagrange variables in the case of an incompressible fluid allows us to reduce the initial system of governing equations to a third-order equation for porosity and a second-order equation for temperature, respectively. A calculation algorithm is proposed and a numerical study of the obtained initial-boundary value problem is carried out.

1. Introduction
The following quasilinear system of equations of composite type is studied in the work:

\[
\frac{\partial \phi \rho_f}{\partial t} + \text{div}(\phi \vec{v}_f \rho_f) = 0, \quad \frac{\partial \rho_s(1 - \phi)}{\partial t} + \text{div}((1 - \phi)\vec{v}_s \rho_s) = 0,
\]

\[
\phi(\vec{v}_f - \vec{v}_s) = -k(\phi)(\nabla p_f - \rho_f \vec{g}),
\]

\[
\text{div}\vec{v}_s = -a_1(\phi)\xi_1(\theta) p_c, \quad \nabla p_{tot} - \rho_{tot} \vec{g} = 0,
\]

\[
(\rho_f c_f \phi + \rho_s c_s (1 - \phi)) \frac{\partial \theta}{\partial t} + (\rho_f c_f \phi \vec{v}_f + \rho_s c_s (1 - \phi) \vec{v}_s) \nabla \theta = \text{div}(K \nabla \theta).
\]

This initial-boundary-value problem describes the one-dimensional motion between the impermeable heat-insulated walls of a two-phase mixture consisting of a viscous fluid and a movable solid skeleton [1]. Here \(\rho_i, \vec{v}_i, i = s, f\) are the true densities and velocities of the solid and liquid phases, respectively; \(\phi\) is the porosity; \(p_{tot} = \phi p_f + (1 - \phi) p_s\) is the total pressure; \(\rho_{tot} = \phi \rho_f + (1 - \phi) \rho_s\) is the density of the two-phase medium; \(p_f, p_s\) are the pressure of the liquid and solid phases, respectively; \(p_c = p_{tot} - p_f\) is the effective pressure; \(\theta\) is the temperature of the medium; \(K = k_f \phi + (1 - \phi) k_s\) is the thermal conductivity, where \(k_f\) is the thermal conductivity of the liquid phase, \(k_s\) is the thermal conductivity of the porous skeleton; \(c_i = \text{const} > 0\) is the specific heat of the \(i\)-th phase with a constant volume; \(\vec{g} = (0, 0, -g)\) is the vector of gravity; \(k(\phi) = k\phi^n / \mu\) is the filtration coefficient, where \(k\) is the permeability of the porous medium, \(\mu\) is the viscosity of the liquid; \(a_1(\phi\theta) = \phi^m\) is the coefficient of bulk viscosity; \(\xi_1(\theta) = 1/\eta(\theta)\).
Densities of solid and liquid phases are assumed constant. The problem is written in the Euler coordinates \((\vec{x}, t) \in Q_T\). Structurally similar models were considered in [1–10].

A numerical analysis of the one-dimensional motion of magma in the absence of mass forces and at a constant temperature was carried out in [2]. The numerical solution of the one-dimensional non-stationary isothermal fluid filtration problem in a viscous porous medium was carried out in [3]. The substantiation of isothermal filtration models in individual model cases was dealt with in [4,5,7–9].

2. The Lagrange Variables

In the one-dimensional case, in mass Lagrangian variables, instead of (1) – (4) we obtain the system of equations [4]

\[
\frac{\partial (1 - \phi)}{\partial t} + (1 - \phi)\frac{\partial v_s}{\partial x} = 0,
\]

\[
\frac{\partial}{\partial t} \left( \frac{\phi}{1 - \phi} \right) + \frac{\partial}{\partial x} (\phi(v_f - v_s)) = 0,
\]

\[
\phi(v_f - v_s) = -k(\phi) \left( (1 - \phi) \frac{\partial p_f}{\partial x} + \rho_f g \right),
\]

\[
(1 - \phi) \frac{\partial p_{tot}}{\partial x} = -\rho_{tot} g,
\]

\[
(1 - \phi) \frac{\partial v_s}{\partial x} = -a_1(\phi)\xi_1(\theta) p_c,
\]

\[
\left( \rho_c (\frac{\phi}{1 - \phi} + \rho_s c_s) \right) \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left( K(1 - \phi) \frac{\partial \theta}{\partial x} \right) - c_f \rho_f \phi(v_f - v_s) \frac{\partial \theta}{\partial x}.
\]

System (5)–(10) is solved in the domain \((x, t) \in Q_T = (0,1) \times (0,T)\), with boundary and initial conditions

\[
v_s(0, t) = v_s(1, t) = v_f(0, t) = v_f(1, t) = 0, \quad \frac{\partial \theta}{\partial x} |_{x=0,x=1} = 0,
\]

\[
\theta(x, 0) = \theta^0(x), \quad \phi(x, 0) = \phi^0(x).
\]

The equation of mass conservation for the liquid phase, taking into account Darcy’s law, will take the form

\[
\frac{\partial}{\partial t} \left( \frac{\phi}{1 - \phi} \right) - \frac{\partial}{\partial x} (k(\phi)((1 - \phi) \frac{\partial p_f}{\partial x} + \rho_f g)) = 0.
\]

For the viscosity of the solid phase, we use the dependence [10]:

\[
\eta(\theta) = \eta_r \exp \left( \frac{Q(1 - \frac{\theta}{\theta_R})}{\theta R} \right),
\]

where \(\eta_r\) is the viscosity at the temperature \(\theta_R\), \(Q\) is the creep activation energy, \(R\) is the universal gas constant.

From equations (5) and (9) it follows

\[
\frac{1}{1 - \phi} \frac{\partial \phi}{\partial t} = -a_1(\phi)\xi_1(\theta)(p_{tot} - p_f).
\]
The last equation can be represented as

\[ p_f - p_{\text{tot}} = \frac{1}{\xi_1(\theta)} \frac{\partial G(\phi)}{\partial t}, \]  

where the function \( G(\phi) \) is determined by the equality

\[ \frac{\partial G(\phi)}{\partial t} = \frac{1}{a_1(\phi)(1 - \phi)}. \]

The equation (11) taking into account (8) and (12) is rewritten in the form

\[ \frac{\partial}{\partial t} \left( \frac{\phi}{1 - \phi} \right) = \frac{\partial}{\partial x} \left( k(\phi)(1 - \phi) \frac{\partial G(\phi)}{\partial x} \right) + k(\phi)g(\rho_f - \rho_{\text{tot}}). \]  

(13)

The boundary condition for (13) follows from (7) and, taking into account the boundary conditions for phase velocities, has the form

\[ \left( k(\phi)(1 - \phi) \frac{\partial}{\partial x} \left( \frac{1}{\xi_1(\theta)} \frac{\partial G(\phi)}{\partial t} \right) + k(\phi)g(\rho_f - \rho_{\text{tot}}) \right) |_{x=0,x=1} = 0. \]  

(14)

The equation for temperature, taking into account (7) will take the following form

\[ \left( \rho_f c_f \frac{\phi}{1 - \phi} + \rho_s c_s \right) \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x} \left( K(1 - \phi) \frac{\partial \theta}{\partial x} \right) + c_f \rho_f k(\phi) \left( (1 - \phi) \frac{\partial p_f}{\partial x} + \rho_f g \right) \frac{\partial \theta}{\partial x}. \]  

(15)

In dimensionless variables

\[ x' = \frac{x}{L}, \quad t' = \frac{t}{T}, \quad \theta' = \frac{\theta}{\tilde{\theta}}, \]

the system (13)–(15) for finding the functions \( \phi \) and \( \theta \) takes the following form

\[ zd(\phi, \theta) = \lambda \frac{\partial}{\partial x} \left( a(\phi) \frac{\partial z}{\partial x} + \varepsilon b(\phi) \right), \quad \left( a(\phi) \frac{\partial z}{\partial x} + \varepsilon b(\phi) \right) |_{x=0,x=1} = 0, \]  

\[ z = \frac{1}{a_1(\phi)(1 - \phi)\xi_1(\theta)} \frac{\partial \phi}{\partial t}, \]  

\[ Q \frac{\partial \theta}{\partial x} = \omega \frac{\partial}{\partial x} \left( K_1 \frac{\partial \theta}{\partial x} \right) + \lambda \frac{\partial}{\partial x} \left( a(\phi) \frac{\partial z}{\partial x} + \varepsilon b(\phi) \right), \]

where

\[ \lambda = \frac{k_{\eta r}}{\mu L^2}, \quad \varepsilon = \frac{r TL \rho_s}{\eta r}, \quad \omega = \frac{k_s T}{\rho_f c_f L^2}, \]

are dimensionless parameters, \( a(\phi) = \phi^\alpha (1 - \phi), b(\phi) = \phi^\alpha (\rho_f / \rho_s - (1 - \phi) - \phi \rho_f / \rho_s), d(\phi) = \xi_1(\theta) \phi^\alpha / (1 - \phi), \xi_1(\theta) = \exp(Q(1 - \theta / \theta_r)/R \theta), Q = \phi/(1 - \phi) + \rho_s c_s / \rho_f c_f, K = (k_f / k_s) \phi (1 - \phi) + (1 - \phi)^2 \) are dimensionless coefficients.
3. Finite-difference approximation of the problem
In this section, the implicit difference discretization of equations (16)–(18) is proposed. In the
space - time domain \([0,1] \times [0,T]\) we define a uniform mesh \(\bar{\omega}_h = \omega_h \times \bar{\omega}_\tau : \omega_h = \{x_i = ih, \ i = 0,1,\ldots,N, \ Nh = 1\}, \ \bar{\omega}_\tau = \{t_n = n\tau, \ n = 0,1,\ldots,M, \ M\tau = T\}, h \) is the spatial
coordinate step, \(\tau\) is the time step. Numerical solutions at grid nodes \((x_i,t_n)\) are denoted by \(\phi_i^n = \phi(x_i,t_n), \ \theta_i^n = \theta(x_i,t_n), \ z_i^n = z(x_i,t_n)\). Following [11,12], equations (16)–(18) are
approximated by the following difference schemes:

\[
z_i^{n+1}d_i^n = \frac{1}{h} \left[ \frac{a_i^n + a_{i+1}^n}{2} \frac{z_i^{n+1} - z_i^{n+1}}{h} - \frac{a_i^n + a_{i-1}^n}{2} \frac{z_i^{n+1} - z_i^{n+1}}{h} \right] + \varepsilon \left[ \frac{b_i^n - b_{i-1}^n}{2h} \right], \quad (19)
\]

\[
\frac{z_i^{n+1} - z_i^n}{h} = -\frac{\varepsilon b_i^n}{a_i^n}, \quad i = 1,\ldots,N - 1,
\]

\[
\frac{\phi_i^{n+1} - \phi_i^n}{\tau} = \frac{1}{\rho_i} \left[ K_i^n + K_{i+1}^n \frac{\phi_i^{n+1} - \phi_i^{n+1}}{h} - K_i^n + K_{i-1}^n \frac{\phi_i^{n+1} - \phi_i^{n+1}}{h} \right] + \\
+ \lambda \left[ \frac{\phi_{i+1}^{n} - \phi_i^{n}}{2h} \right] \left[ a_i^n \frac{z_i^{n+1} - z_i^{n}}{2h} + \varepsilon b_i^n \right], \quad (21)
\]

where

\[
k_1 = z_i^{n+1}a_i(\phi_i^n)(1 - \phi_i^n)\bar{\xi}_i(\theta_i^n) = f(z,\phi,\theta), \quad k_2 = f(z,\phi + \tau k_1/2,\theta),
\]

\[
k_3 = f(z,\phi + \tau k_2/2,\theta), \quad k_4 = f(z,\phi + \tau k_3,\theta),
\]

\[
a_i^n = a(\phi_i^n), \quad b_i^n = b(\phi_i^n), \quad d_i^n = d(\phi_i^n,\theta_i^n), \quad \bar{K}_i^n = \bar{K}(\phi_i^n), \quad Q_i^n = Q(\phi_i^n).
\]

The calculation algorithm is as follows: using the initial value of porosity and temperature, we
find \(z_i^n\) from equation (19), then we find \(\phi_i^n\) from equation (20), the next step is finding \(\theta_i^n\)
from equation (21). Repeat the algorithm \(M\) times.

Testing of the numerical algorithm for the considered implicit difference schemes was carried
out on sequences of spatial grids. The calculation was performed on three grids in steps of
\(h_1 = 0.01, h_2 = 0.005, h_3 = 0.0025, \tau_i = \lambda h_i, \lambda = const, i = 1,2,3\). Numerical experiments
showed a slight difference in the obtained values of the sought quantities.

In numerical calculations, the following parameters were used [1,10]:

\[
L = 1m, \quad T = 1day, \quad \rho_f = 1000kg/m^3, \mu = 2.6 \cdot 10^{-4}Pa \cdot s,
\]

\[
\theta_r = 293K, \quad \tilde{\theta} = 323K, \quad R = 8.31 J/mol \cdot K, \quad Q_e = 40J,
\]

\[
c_f = 4183 J/kg \cdot K, \quad k_f = 0.6 W/m \cdot K, \quad g = 10 m/s^2.
\]

The initial condition for temperature is taken in the following form: \(\theta^0(x) = ((300(x)L)^2(L -
xL)^2 + 278))/\theta\). As an example, consider a soil with the following characteristics: density \(\rho_s =
1800kg/m^3\), specific heat \(c_s = 800J/kg \cdot K\), thermal conductivity coefficient \(k_s = 0.5W/m \cdot K\),
permeability \(k = 2.6 \cdot 10^{-11}m^2\), viscosity \(\eta_p = 10^8 Pa \cdot s\) and porosity \(\phi^0(x) = 0.4\).
In the zone of high porosity (Figure 1) the liquid phase predominates, the specific heat of which is much larger than the heat capacity of the soil, so the temperature here changes slightly over time (Figure 2).

Consider the effect of dynamic soil viscosity on porosity (Figures 3, 4). The variable nature of porosity in the regions $x = 0$ and $x = 1$ is due to the influence of gravity. The porosity is minimal at $x = 0$, since, probably, the maximum concentration of soil particles in this region is. The porosity is close to unity at $x = 1$, since soil particles are transported by gravity. Soil with a viscosity equal to $\eta_r = 10^8 Pa \cdot s$ is soft plastic, therefore, it is more deformed and its porosity at the boundary of the region under consideration varies significantly. A hard-plastic soil with a viscosity of $\eta_r = 10^{10} Pa \cdot s$ is practically not deformed, therefore its porosity is constant.

Figure 1. Porosity distribution.

Figure 2. Temperature distribution.

Figure 3. Soil porosity at $x = 0$, $(1 - \eta_r = 10^8 Pa \cdot s, 2 - \eta_r = 10^9 Pa \cdot s, 3 - \eta_r = 10^{10} Pa \cdot s)$.

Figure 4. Soil porosity at $x = 1$, $(1 - \eta_r = 10^8 Pa \cdot s, 2 - \eta_r = 10^9 Pa \cdot s, 3 - \eta_r = 10^{10} Pa \cdot s)$.

The velocities of moving solid particles are shown in Figure 5. In motionless soil, the porosity does not change. We can conclude that the porosity of the medium changes due to its motion and deformation. Calculations were carried out in the absence of gravity (Figure 6). It was found that the porosity of the soil does not change and is equal to a given initial value in the entire region. Thus, taking gravity into account is an important factor in solving the problems under consideration.
Figure 5. The speed of the solid phase at $x = 0.5$, (1 - $\eta_r = 10^8 \text{Pa} \cdot \text{s}$, 2 - $\eta_r = 10^9 \text{Pa} \cdot \text{s}$, 3 - $\eta_r = 10^{10} \text{Pa} \cdot \text{s}$).

Figure 6. The effect of gravity on soil porosity, $x = 1/2$, $\eta_r = 10^8 \text{Pa} \cdot \text{s}$.

4. Conclusion
The model of non-isothermal filtration of an incompressible fluid in a deformable porous medium is studied. A numerical study was carried out. The model can be used to study the processes occurring in soils and other environments where it is important to take into account the deformation of the medium and changes in its porosity.

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