Duality and $H_\infty$-Optimal Control Of Coupled ODE-PDE Systems

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Abstract—In this paper, we present a convex formulation of $H_\infty$-optimal control problem for coupled linear ODE-PDE systems with one spatial dimension. First, we reformulate the coupled ODE-PDE system as a Partial Integral Equation (PIE) system and show that stability and $H_\infty$ performance of the PIE system implies that of the ODE-PDE system. We then construct a dual PIE system and show that asymptotic stability and $H_\infty$ performance of the dual system is equivalent to that of the primal PIE system. Next, we pose a convex dual formulation of the stability and $H_\infty$-performance problems using the Linear PI Inequality (LPI) framework. Next, we use our duality results to formulate the stabilization and $H_\infty$-optimal state-feedback control problems as LPIs. LPIs are a generalization of LMIs to Partial Integral (PI) operators and can be solved using PIETOOLS, a MATLAB toolbox. Finally, we illustrate the accuracy and scalability of the algorithms by constructing controllers for several numerical examples.

I. INTRODUCTION

In this paper, we consider the problem of $H_\infty$-optimal state-feedback controller synthesis for Partial Integral Equation (PIE) systems of the form

\begin{equation}
T \dot{x}(t) = A x(t) + B u(t), \quad x(0) = x_0 \in \mathbb{R}^m \times L_2^n
\end{equation}

\begin{equation}
z(t) = C x(t) + D u(t)
\end{equation}

where $T, A, B, C, D$ are Partial Integral (PI) operators and $u(t) \in \mathbb{R}^p$. The dual (or adjoint) PIE system is then defined to be

\begin{equation}
T^* \dot{x}(t) = A^* \tilde{x}(t) + C^* \tilde{u}(t), \quad \tilde{x}(0) = \tilde{x}_0 \in \mathbb{R}^m \times L_2^n
\end{equation}

\begin{equation}
\tilde{z}(t) = B^* \tilde{x}(t) + D^* \tilde{u}(t)
\end{equation}

where $^*$ denotes the adjoint with respect to $L_2$-inner product.

Recently, it has been shown that almost any PIE system in a single spatial dimension coupled with an ODE at the boundary has an equivalent PIE system representation [11] (see Sec. IV). It should be noted, however, that the formulation in Eqn. (1) does not allow for inputs directly at the boundary - rather these must enter through the ODE or into the domain of the PDE. Use of the PIE system representation, defined by the algebra of Partial Integral (PI) operators, allows us to generalize LMIs developed for ODEs to infinite-dimensional systems. These generalizations are referred to as Linear PI Inequalities (LPIs) and can be solved efficiently using the MATLAB toolbox PIETOOLS [13]. In previous work, LPIs have been proposed for stability [10], $H_\infty$-gain [12] and $H_\infty$-optimal estimation [3] of PIE systems. However, until now the stabilization and $H_\infty$-optimal controller synthesis problems have remained unresolved. In this paper, we resolve the problems of stabilizing and $H_\infty$ state-feedback controller synthesis by proving the following results.

(A) Dual Stability Theorem: We show that the PIE system (1) is stable for $u = 0$ and any initial condition $x(0) \in L_2$ if and only if the dual PIE system (2) is stable for any initial conditions $\tilde{x}(0) \in L_2$ and $\tilde{u} = 0$.

(B) Dual $L_\infty$-Gain Theorem: For $u \in L_2([0, \infty))$ and $x(0) = 0$, any solution of the PIE system (1) satisfies $\|z\|_{L_\infty} \leq \gamma \|u\|_{L_2}$ if and only if any solution to the dual PIE system Eq. (2) satisfies $\|\tilde{z}\|_{L_\infty} \leq \gamma \|\tilde{u}\|_{L_2}$ for $\tilde{x}(0) = 0$ and $\tilde{u} \in L_2([0, \infty))$.

(C) $H_\infty$-Optimal Control of PIEs: The stabilization and $H_\infty$-optimal state-feedback controller synthesis problem for PIE systems (1) may be formulated as an LPI.

Previous work on controller synthesis for coupled ODE-PDE systems includes the well-established method of backstepping (See e.g. [6]) and discretization-based methods (See e.g. [4], [5], [11]). In the former case, backstepping methods allow for inputs at the boundary and are guaranteed to find a stabilizing controller if one exists. However, the resulting controllers are not optimal in any sense. In the latter case, $H_\infty$-optimal controllers are designed for an ODE approximation of the coupled ODE-PDE system. However, these controllers do not have provable performance properties when applied to the actual ODE-PDE, i.e. the $H_\infty$-norm of the ODE-PDE system is not same as the $H_\infty$-norm of the ODE approximation and indeed, the resulting closed-loop system is often unstable. Frequency-domain approaches, for example [14], [7], design optimal control using transfer function of the system. However, the concept of transfer function is not extendable to systems with multiple inputs/outputs.

The fundamental issue in controller synthesis for both finite-dimensional and infinite-dimensional systems is one of non-convexity. In simple terms, for either a finite or infinite-dimensional system of the form

\begin{equation}
\dot{x}(t) = A x(t) + B u(t),
\end{equation}

finding a stabilizing control $u(t) = K x(t)$ and a corresponding Lyapunov functional $V(t) = \langle x(t), P x(t) \rangle_X$ with negative time-derivative leads to a bilinear problem in variables $K$ and $P$ of the form $(A + B K) \dot{P} + P A^T \leq 0$.

In case of finite-dimensional linear systems, the linear operators $P, K, A$ and $B$ are just matrices $P, K, A$ and $B$. In absence of a controller, the Lyapunov stability test (referred to as primal stability test) can be written as an LMI in positive matrix variable $P > 0$ such that $A^T P + PA \leq 0$. In finite-dimensions, the eigenvalues of $A$ and $A^*$ are the same and hence there is an equivalent dual Lyapunov inequality of the form $AP + PA^T \leq 0$. Then

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the test for existence of a stabilizing controller $K$ and a Lyapunov functional $P$ which proves the stability of the closed-loop system can now be written as: find $P > 0$ such that $(A + BK)P + P(A + BK)^T \leq 0$. The key difference, however, is the bilinearity can now be eliminated by introducing new variable $Z = KP$ which leads to the LMI constraint $AP + BZ + (AP + BZ)^T \leq 0$.

For infinite-dimensional systems, Theorem 5.1.3 of [2] is analogous to primal stability test for ODEs. The result is similar in the sense that matrices in the constraints of primal stability test for ODE are replaced by linear operators for infinite-dimensional systems, i.e. a test for existence of a positive operator $P > 0$ that satisfies the operator-valued constraint $A^*P + PA \leq 0$. However, there does not exist a dual form of the stability test for infinite-dimensional systems. In [9], a dual Lyapunov criterion for stability in infinite-dimensional systems was presented. However, the result was restricted to infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and included constraints on the image of the operator $P$ of the form $\mathcal{P}(X) = X$ where $X = D(A)$ is the domain of the infinitesimal generator $A$. Furthermore, because $A$ for PDEs is a differential operator, this approach provides no way of enforcing negativity of the dual stability condition. These difficulties in analysis and controller synthesis for PDE systems led to the development of the PIE formulation of the problem - wherein both system parameters $A, B, C, D, T$ and the Lyapunov parameter $P$ lie in the algebra of bounded linear PI operators.

In this work, we adopt the PIE formulation of the ODE-PDE system and propose dual stability and performance tests wherein all operators lie in the PI algebra and do not include additional constraints such as $\mathcal{P}(X) = X$. Specifically, the results (A) and (B) lead to LPIs which, by allowing for the variable change trick used in finite-dimensional systems, allows us to propose convex and testable formulations of the stabilization and optimal control problems - resulting in stabilizing or $H_\infty$ optimal controllers for coupled PDE-ODE systems where the inputs enter through the ODE or in the domain. More specifically, these methods apply for linear ODE-PDE systems in 1 spatial variable with a very general set of boundary conditions including Dirichlet, Neumann, Robin, Sturm-Liouville etc. The resulting LPIs are solved numerically using PIETOOLS [13], an open-source MATLAB toolbox to handle PI variables and setup PI operator-valued optimization problems. Finally, we note that this is the first result to achieve $H_\infty$-optimal control of coupled ODE-PDE systems without discretizing the PDE. Although we are currently restricted to inputs using an ODE filter or in-domain, we believe the duality results presented here can ultimately be extended to cover inputs applied directly at the boundary.

The paper is organized as follows. After introducing preliminary notations in Section II, in Section III and IV, we present the general form of PIE and ODE-PDE under consideration. In Section V, we define the conditions under which PIE and ODE-PDE as equivalent followed by equivalence in stability and $H_\infty$-gain in Section VI. Section VII discusses the properties of adjoint PIE systems. In Section VIII and IX, we derive the dual stability theorem and dual $H_\infty$-gain theorem for PIEs. Sections XI through XIV present the LPIs developed using dual stability theorem and dual $H_\infty$-gain theorem. Examples are illustrated in Section XV and followed by conclusions in Section XVI.

II. Notation

The calligraphic font, for example $\mathcal{A}$, is used to represent linear operators on Hilbert spaces and the bold font, $x$, is used to denote functions in $L^2_0[a,b]$ which is the set of all square-integrable functions on the domain $[a,b] \subset \mathbb{R}$. The Sobolev space $W^{k,p}_2[a,b]$ is defined as $W^{k,p}_2[a,b] := \{ f \in L^2_2[a,b] \mid \frac{\partial^n f}{\partial x^n} \in L^2_0[a,b] \text{ for all } n \leq k \}$, $Z^{m,n}[a,b]$ denotes the space $\mathbb{R}^m \times L^2_0[a,b]$ that is equipped with the inner-product

$$\langle [x_1, y_1], [x_2, y_2] \rangle_Z = x_1^T y_1 + \langle x_2, y_2 \rangle_{L^2_0}, \quad [x_1, y_1], [x_2, y_2] \in Z^{m,n}.$$ 

We use $x_s$ to denote partial derivative of $\frac{\partial}{\partial x_s}$ where the number of repetitions of the subscript $s$ corresponds to the order of the partial derivative and $\dot{x}$ to denote the partial derivative $\frac{\partial}{\partial t}$.

III. Partial Integral Equations

In this section, we will define a PIE system with inputs and disturbances of the form

$$\mathcal{T}\dot{x}(t) = Ax(t) + B_1w(t) + B_2u(t), \quad x(0) \in Z^{m,n}[a,b]$$

$$z(t) = Cx(t) + D_{11}w(t) + D_{12}u(t), \quad (3)$$

where the $\mathcal{T}, A : Z^{m,n}[a,b] \rightarrow Z^{m,n}[a,b], B_1 : \mathbb{R}^q \rightarrow Z^{m,n}[a,b], B_2 : \mathbb{R}^p \rightarrow Z^{m,n}[a,b], C : Z^{m,n}[a,b] \rightarrow \mathbb{R}^r, D_{11} \in \mathbb{R}^{r \times q}$ and $D_{12} \in \mathbb{R}^{r \times p}$ are Partial Integral (PI) operators, defined as follows.

**Definition 1. (PI Operators:)** A $4$-PI operator is a bounded linear operator between $Z^{m,n}[a,b]$ and $Z^{p,n}[a,b]$ of the form

$$\mathcal{P}[P, Q_1, Q_2, R_1, R_2] : \mathbb{R}^{p \times m} \rightarrow \mathbb{R}^{p \times n}, \quad \mathcal{P}[P, Q_1, Q_2, R_1, R_2] : \mathbb{R}^{p \times m} \rightarrow \mathbb{R}^{p \times n}$$

where $P \in \mathbb{R}^{p \times m}$ is a matrix, $Q_1 : [a,b] \rightarrow \mathbb{R}^{p \times n}, Q_2 : [a,b] \rightarrow \mathbb{R}^{m \times n}$ are bounded integrable functions and $\mathcal{P}[R_1, \mathcal{P}[P, Q_1, Q_2, R_1, R_2] : L^2_n[a,b] \rightarrow L^2_n[a,b]$ is a $3$-PI operator of the form

$$R_0(s)x(s) + \int_a^s R_1(s, \theta)x(\theta)d\theta + \int_0^s R_2(s, \theta)x(\theta)d\theta.$$

**Definition 2.** For given $u \in L_2^1([0, \infty); \mathbb{R}^q), w \in L_2^2([0, \infty); \mathbb{R}^q)$ and initial conditions $x_0 \in Z^{m,n}[a,b]$, we say that $\dot{x} : [0, \infty) \rightarrow Z^{m,n}[a,b]$ and $z : [0, \infty) \rightarrow \mathbb{R}^r$ satisfy the PIE (1) defined by $\{\mathcal{T}, A, B, C, D\}$ if $x$ is Fréchet differentiable almost everywhere on $[0, \infty)$, $x(0) = x_0$ and the equations (3) are satisfied for almost all $t \geq 0$.

IV. A GENERAL CLASS OF LINEAR ODE-PDE SYSTEMS

In this paper, we consider control of the following class of coupled linear ODE-PDE systems in a single spatial variable $s \in [a,b]$.
\[
\dot{x}(t) = ax(t) + dx_3(1,t), \\
\dot{x}_1(s,t) = 0 \\
\dot{x}_2(s,t) = 1 \\
\dot{x}_3(s,t) = c \\
x_1(0,t) = kx(t), x_3(1,t) = 0
\]

where \( x_1 = \dot{w} \) and \( x_3 = w \). The parameters that define the ODE-PDE (5) are

\[
A = a, A_0 = [0 \ 0], A_2 = \begin{bmatrix} c \\ 0 \end{bmatrix}, E_0 = [0 \ 0 \ d \ 0], B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_x = \begin{bmatrix} k \end{bmatrix},
\]

and the rest of the system parameters are zero.

**Definition 3.** For given \( u \in L_2([0,\infty) ; \mathbb{R}^{n_3}) \) and initial conditions \( x_0 \in D(A_d) \) as defined in (6), we say that \( x : [0,\infty) \rightarrow \mathbb{R}^{n_3}, \ x : [0,\infty) \rightarrow \prod_{i=0}^{2} W^2_{1,1}[a,b] \) and \( z : [0,\infty) \rightarrow \mathbb{R}^{n_3} \) satisfy the ODE-PDE (5) defined by \( \{A, A_1, B_1, B_2, C_{10}, C_{20}, D, E, E_{10}, E_a, E_b\} \) if \( x \) is differentiable and \( x \) is Fréchet differentiable almost everywhere on \([0,\infty), x(0) = x_0, x(t) \in D(A_d) \) and Equations (5) hold for almost all \( t \geq 0 \).

**V. PIE REPRESENTATION OF THE ODE-PDE SYSTEM**

A coupled ODE-PDE of the form Eq. (5) can be written as a PIE Eq. (3). Furthermore, the solutions of the PIE define solutions of the ODE-PDE and vice-versa.

**Theorem 4.** For given \( u \in L_2([0,\infty) ; \mathbb{R}^{n_3}) \) and initial conditions \( x_0 \in D(A_d) \) as defined in (6), suppose \( x, x \) and \( z \) satisfy the ODE-PDE defined by \( \{A, A_1, B_1, B_2, C_{10}, C_{20}, D, E, E_{10}, E_a, E_b\} \). Then \( z \) also satisfies the PIE

\[
\hat{T}v(t) = Av(t) + Bu(t), \quad v(0) = v_0
\]

with

\[
v_0 = \begin{bmatrix} x_0 \\ x_01 \\ x_{02s} \\ x_{03ss} \end{bmatrix}, \quad v(t) := \begin{bmatrix} x(t) \\ x_1(t) \\ x_2s(t) \\ x_{3ss}(t) \end{bmatrix},
\]

where the 4-PI operators \( T, A, B, C \) and \( D \) are as defined in Eqs. (19). Conversely, for given \( u \in L_2([0,\infty) ; \mathbb{R}^{n_3}) \) and initial conditions \( v_0 \in Z^{n_3,n_p}[a,b] \), suppose \( v \) and \( z \) satisfy the PIE defined by the 4-PI operators \( T, A, B, C \) and \( D \) as defined in Equations (19). Then, \( z \) also satisfies the ODE-PDE defined by \( \{A, A_1, B_1, B_2, C_{10}, C_{20}, D, E, E_{10}, E_a, E_b\} \) with

\[
x_0 = \hat{T}v_0, \quad x(t) := \hat{T}v(t).
\]

**Proof.** Refer Lemma 3.3 and 3.4 in [11] for proof.

### Illustrative Example

To illustrate how this representation is applied to a typical ODE-PDE model, we consider a wave equation coupled with an ODE as shown below.

\[
\dot{x}(t) = ax(t) + dw(1,t), \\
\dot{w}(s,t) = cwx(s,t), \quad w(0,t) = kx(t), \quad w_{11}(1,t) = 0,
\]

where \( w(s,t) \) is the PDE state and \( x \) is the ODE state. These equations may be rewritten in the form (5)
dynamics. Finally, solutions of the PIE system are defined on $Z^{n, m}_0[a, b]$, which is a Hilbert space with respect to $Z$-inner product, whereas $D(A_d)$ is not a Hilbert space.

VI. EQUIVALENCE IN STABILITY

In this section, we show that stability in $Z$ and $L_2$-gain of the ODE-PDE system is same as that of the PIE system. First, we define asymptotic stability of PIEs and of ODE-PDEs.

Definition 5. For $w = u = 0$, the PIE (3) defined by $\{T, A, B, C, D, j\}$ is said to be asymptotically stable if for any initial condition $x_0 \in Z^{m, n}_0[a, b]$, if $x$ and $z$ satisfy the PIE, we have $\lim_{t \to \infty} \|T(x(t))\|_Z = 0$.

Definition 6. For $w = u = 0$, the ODE-PDE (5) defined by $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ is said to be asymptotically stable if for initial condition $x_0 \in D(A_d)$, if $x$ and $z$ satisfy the ODE-PDE, then

$$\lim_{t \to \infty} \|T(x(t))\|_Z = 0.$$

Lemma 7. Suppose $\{T, A, B, C, D, j\}$ and $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ satisfy Eqs. (19). Then the ODE-PDE defined by $\{A, A_i, B_i, B, B_x, C_{10}, C_a, C_b, D, E, E_{10}, E_a, E_b\}$ is asymptotically stable if the PIE defined by $\{T, A, B, C, D, j\}$ is asymptotically stable.

Proof. From Theorem 4, $x$ and $z$ satisfy the ODE-PDE for the given $x_0, z_0$ if and only if $v$ satisfies the PIE for $v_0$ where

$$T(x(t)) = T(v(t)), \quad x(t) = x_0.$$

If the PIE is stable, then $\lim_{t \to \infty} \|T(v(t))\|_Z = 0$. Since $T$ is a bounded linear operator, this implies

$$\lim_{t \to \infty} \|T(v(t))\|_Z = 0.$$

Remark 9. Consider the transport equation

$$\mathbf{v}(s, t) + \mathbf{v}_s(s, t) = 0, \quad s \in [0, 1], t > 0,$$

$$\mathbf{v}(0, t) = 0, \quad \mathbf{v}(s, 0) \in L_2[0, 1].$$

The PIE form Eq. (8) is

$$(P_{(0,1)}x)(t) = (P_{(-1,0)}x)(t), \quad t > 0.$$
for any \( x(0) \in Z^{m,n}[a,b] \). We conclude that 
\[
\lim_{t \to \infty} T^*\bar{x}(t) = 0.
\]
Since the dual and primal systems are interchangeable, necessity follows from sufficiency. \( \square \)

IX. DUAL \( L_2 \)-GAIN THEOREM

We proved that stability of a PIE system and its dual are equivalent. Now, we show that for \( x_0 = 0 \), input-output performance of primal and dual PIE in the \( L_2 \)-gain metric is equivalent.

**Theorem 11.** (Duality on \( L_2 \)-gain bound of PIEs:) Suppose \( T, A, B, C \) and \( D \) are \( 4 \)-PI operators. Then the following statements are equivalent.

1. For any \( w \in L_2((0,\infty);R^q) \) and \( x(0) = 0 \) any solution \( x(t) \in Z^{m,n} \) and \( z(t) \in R^p \) of the PIE system
\[
T\dot{x}(t) = Ax(t) + Bw(t), \quad x(0) = 0
\]
\[
z(t) = C\dot{x}(t) + Dw(t)
\]
satisfies 
\[
\parallel z \parallel_{L_2} \leq \gamma \parallel w \parallel_{L_2}.
\]

2. For any \( \bar{w} \in L_2((0,\infty);R^p) \) and \( \bar{x}(0) = 0 \) any \( \bar{x}(t) \in Z^{m,n} \) and \( \bar{z}(t) \in R^q \) of the dual PIE system
\[
\bar{T}\dot{\bar{x}}(t) = A^*\bar{x}(t) + C^*\bar{w}(t), \quad \bar{x}(0) = 0
\]
satisfies 
\[
\parallel \bar{z} \parallel_{L_2} \leq \gamma \parallel \bar{w} \parallel_{L_2}.
\]

**Proof.** Suppose that for any \( w \in L_2((0,\infty);R^q) \) and \( x(0) = 0 \) any solution \( x(t) \in Z^{m,n} \) and \( z(t) \in R^p \) of the PIE system satisfies 
\[
\parallel z \parallel_{L_2} \leq \gamma \parallel w \parallel_{L_2}.
\]
For \( \bar{w} \in L_2((0,\infty);R^p) \) and \( \bar{x}(0) = 0 \), let \( \bar{x}(t) \) and \( \bar{z}(t) \) be the corresponding dual system. Then for any finite \( t \geq 0 \), since \( x(0) = 0 \), we have
\[
\int_0^t \langle \dot{x}(t), T\bar{x}(s) \rangle \, ds = \langle x(0), T\bar{x}(t) \rangle - \langle x(t), T\bar{x}(0) \rangle + \int_0^t \langle T^*\dot{x}(\theta), x(t-\theta) \rangle \, d\theta \leq \int_0^t \langle T^*\dot{x}(\theta), x(t-\theta) \rangle \, d\theta.
\]

Furthermore, by the change variable change,
\[
\int_0^t \langle \dot{x}(t), T\bar{x}(s) \rangle \, ds = \int_0^t \langle \dot{x}(t-s), T\bar{x}(s) \rangle \, ds \leq \int_0^t \parallel \dot{x}(t-s) \parallel \parallel T\bar{x}(s) \parallel \, ds \leq \int_0^t \parallel \dot{x}(t-s) \parallel \parallel \bar{x}(s) \parallel \, ds.
\]

Combining the two equalities, we obtain
\[
\int_0^t \langle T^*\dot{x}(\theta), x(t-\theta) \rangle \, d\theta = \int_0^t \langle A^*\bar{x}(\theta), x(t-\theta) \rangle \, d\theta + \int_0^t \langle B^*\bar{x}(\theta), w(t-\theta) \rangle \, d\theta.
\]

By the definition of \( \bar{x} \), we obtain
\[
\int_0^t \langle \bar{z}(\theta), w(t-\theta) \rangle \, d\theta = \int_0^t \langle A^*\bar{x}(\theta), x(t-\theta) \rangle \, d\theta - \int_0^t \langle B^*\bar{x}(\theta), w(t-\theta) \rangle \, d\theta
\]
and
\[
\int_0^t \langle T^*\dot{x}(\theta), x(t-\theta) \rangle \, d\theta
\]
\[
\int_0^t \langle A^*\bar{x}(\theta), x(t-\theta) \rangle \, d\theta - \int_0^t \langle B^*\bar{x}(\theta), w(t-\theta) \rangle \, d\theta.
\]

PROOF END.

X. LINEAR PARTIAL INTEGRAL INEQUALITIES

Optimization problems with PI operator decision variables and Linear PI Inequality constraints are called Linear PI Inequalities (LPIs) and take the form
\[
\mathcal{P} \left[ \phi_k, \phi_{q_k} \right] \left[ q_k, \phi_{q_k} \right] \geq 0,
\]
where \( x \in R^N \) is the decision variable and \( \mathcal{P} \left[ \phi_k, \phi_{q_k} \right]: Z^{m,n}[a,b] \to Z^{m,n}[a,b] \) are known self-adjoint \( 4 \)-PI operators. LPI optimization problems can be solved using the MATLAB software package PIETOOLS [13]. In the following sections, we present applications of Theorems 10 and 11 in the form of LPI tests for dual stability, dual \( L_2 \)-gain, stabilization, and \( H_\infty \)-optimal control of PIE systems, each with associated code snippets using the PIETOOLS implementation.

XI. A DUAL LPI FOR STABILITY

Using Theorem 10, we give primal and dual LPIs for stability of a PIE system.

**Theorem 12.** (Primal LPI for Stability:) Suppose there exists a self-adjoint bounded and coercive operator \( \mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b] \) such that
\[
T^*\mathcal{P}A + A^*T^* \mathcal{P}T \leq -\epsilon T^*T
\]
for some \( \epsilon > 0 \). Then any \( x \in Z^{m,n}[a,b] \) that satisfies the system
\[
T\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in Z^{m,n}[a,b]
\]

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satisfies \( \lim_{t \to \infty} \|T x(t)\|_Z = 0 \).

**Proof.** The proof can be found in the [8].

**Theorem 13.** (Dual LPI for Stability:) Suppose there exists a self-adjoint bounded and coercive operator \( \mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b] \) such that
\[
\mathcal{T} \mathcal{P} A^* + A \mathcal{P} T^* 
\leq -\epsilon \mathcal{T} T^*
\]
for some \( \epsilon > 0 \). Then any \( x \in Z^{m,n}[a,b] \) that satisfies the system
\[
T \dot{x}(t) = A x(t), \quad x(0) = x_0 \in Z^{m,n}[a,b]
\]
also satisfies \( \lim_{t \to \infty} \|T x(t)\|_Z = 0 \).

**Proof.** The proof is same as the proof for Theorem 13. Replace \( \mathcal{A} \) by \( \mathcal{A} + \mathcal{B} \mathcal{K} \) and substitute \( \mathcal{Z} = \mathcal{K} \mathcal{P} \).

**Pseudo Code 3.**
\[
\begin{align*}
\text{prog} &= \text{sosprogram}([s,t]); \\
\{\text{prog}, P\} &= \text{sos_program}(\text{prog}, \text{dim}, I, s, t) ; \\
D &= T \star (A \star P + B \star Z) + (A \star P + B \star Z)^* + \epsilon \star T \star T' ; \\
\text{prog} &= \text{sos_opineq}(\text{prog}, -D) ; \\
\text{prog} &= \text{sosolve}(\text{prog}) ;
\end{align*}
\]

**XIV. \( H_\infty \)-OPTIMAL CONTROLLER SYNTHESIS**

Now we use Theorem 11 to pose the \( H_\infty \)-optimal controller synthesis problem as an LPI. Specifically, we formulate the following LPI for finding the \( H_\infty \)-optimal controller for a PIE in the form Eq. (3) where \( T, A : Z^{m,n}[a,b] \to Z^{m,n}[a,b], B_1 : \mathbb{R}^q \to Z^{m,n}[a,b], B_2 : \mathbb{R}^p \to Z^{m,n}[a,b], C : Z^{m,n}[a,b] \to \mathbb{R}^r, D_{11} : \mathbb{R}^r \to \mathbb{R}^r \) and \( D_{12} : \mathbb{R}^p \to \mathbb{R}^r \).

**Theorem 16.** (LPI for \( H_\infty \) Optimal Controller Synthesis:) Suppose there exist \( \gamma > 0 \), bounded linear operators \( \mathcal{P} : Z^{m,n}[a,b] \to Z^{m,n}[a,b] \) and \( \mathcal{Z} : Z^{m,n}[a,b] \to \mathbb{R}^p \), such that \( \mathcal{P} \) is self-adjoint, coercive and
\[
\begin{bmatrix}
-\gamma I & \mathcal{D} & \mathcal{C} \mathcal{T}^* \\
\mathcal{T} \mathcal{C}^* & -\gamma I & \mathcal{B}^* \\
\mathcal{C}^* \mathcal{B}^* & \mathcal{T} \mathcal{A} \mathcal{P} & -\gamma I
\end{bmatrix}
\leq 0.
\]
Then, for any \( w \in L_2 \), any \( x \) and \( z \) that satisfy the PIE (1) also satisfies
\[
\|z\|_{L_2} \leq \gamma \|w\|_{L_2}.
\]

**Proof.** The proof can be found in the appendix.

**Pseudo Code 4.**
\[
\begin{align*}
\text{prog} &= \text{sosprogram}([s,t], \text{gam}) ; \\
\{\text{prog}, P\} &= \text{sos_program}(\text{prog}, \text{dim}, I, s, t) ; \\
\text{prog}, Z &= \text{sos_opvar}(\text{prog}, \text{dim}, I, s, t) ; \\
D &= [\mathcal{D} + \mathcal{P} \mathcal{D} \mathcal{Z} + (\mathcal{C} \mathcal{P} \mathcal{D} \mathcal{Z})] ; \\
\text{prog} &= \text{sos_opineq}(\text{prog}, -D) ; \\
\text{prog} &= \text{sossetobj}(\text{prog}, \text{gam}) ; \\
\text{prog} &= \text{sosolve}(\text{prog}) ;
\end{align*}
\]

**XV. NUMERICAL EXAMPLES**

In this section, use numerical examples to demonstrate the application of the LPIs presented in this paper. First, we verify the stability of PDEs, where the stability holds for certain values of the system parameters (referred to as a stability parameter). We test for the stability of the system using the dual stability criterion and change the stability parameter continuously to identify the point at which the stability of the
system changes. The second set of examples will focus on finding in-domain controllers to stabilize an unstable system. Finally, we also present a numerical example of systems with inputs and outputs to find $H_\infty$-optimal controllers.

A. Stability Tests Using Dual Stability Criterion

Example 17. Consider the scalar diffusion-reaction equation with fixed boundary conditions.

\[ u_t(s,t) = \lambda u(s,t) + u_{ss}(s,t), \quad s \in [0,1], t > 0, \]
\[ u(0,t) = u(1,t) = 0, u(s,0) = u_0. \]

This system is stable for $\lambda \leq \pi^2$. Using LPIs, we find the maximum value for which the system is stable by varying $\lambda$ continuously and testing the feasibility of dual LPI for stability. From our tests, we find that the system is stable for $\lambda \leq (1 + 10^{-5})\pi^2$.

Example 18. Using the same PDE from the previous example, with different boundary conditions, the stability parameter changes to $\lambda \leq 2.467$.

\[ u_t(s,t) = \lambda u(s,t) + u_{ss}(s,t) + d(t) \]

where $d(t) = \int_0^s K(s)u_{ss}(s,t)ds$ is the control input. Solving the LPI in Theorem 15 we find a stabilizing controller $K(s) = 0.29s^5 - 1.01s^4 + 0.95s^3 + 0.16s^2 - 0.51s + 0.98$.

B. Finding Stabilizing Controller For Unstable PDE Systems

Example 19. In the Example 17, suppose $\lambda = 10$. Then the system is unstable. To stabilize the system, we introduce an in-domain control input as

\[ u_t(s,t) = \lambda u(s,t) + u_{ss}(s,t) + d(t) \]

where $d(t) = \int_0^s K(s)u_{ss}(s,t)ds$ is the control input. Solving the LPI in Theorem 15 we find a stabilizing controller $K(s) = 0.29s^5 - 1.01s^4 + 0.95s^3 + 0.16s^2 - 0.51s + 0.98$.

C. $H_\infty$-optimal Controller Synthesis

Example 20. Consider the following cascade of diffusion-reaction equations with a dynamic controller acting at the boundary.

\[ x_i(t) = \lambda x_i(s,t) + \sum_{k=1}^N x_k x_{ss}(s,t) + w(t), \quad i \in \{1...N\} \]
\[ x_0(t) = u(t), \quad x_0(0) = 0, x_i(0) = 0, s \in [0,1] \]
\[ z(t) = x_0(t), \quad x_i(0, t) = 0, \quad x_i(1, t) = 0 \quad \forall \ i \in \{1...N-1\}, \]
\[ x_N(0, t) = 0, \quad x_N(1, t) = x_N(t), \]

where $x_0$ is the state of the dynamic boundary controller, $x_i$ are distributed states, $z$ is the output and $w$ is the input disturbances. The control input, $u(t) = K_0 x_0(t) + \int_0^1 K(s)x(s,t)ds$ where $K : [a,b] \rightarrow \mathbb{R}^{1 \times N}$, enters the system through the ODE and acts at the boundary of the PDE state $x_N$. For $\lambda = 10$, $N = 3$ the $H_\infty$-optimal controller has a norm bound of 0.5095. In Figure 1, we plot the system response for a disturbance $w(t) = \frac{\sin(5t)}{t}$ with zero initial conditions.

XVI. CONCLUSIONS

In this article, we have proven the equivalence, in stability and $H_\infty$-norm, between a PIE and its dual. Coupled ODE-PDE have equivalent PIE representations and properties of the ODE-PDE are inherited from the PIE. Our duality results allow can be used with LPIs to find stabilizing and $H_\infty$-optimal state-feedback controllers for PDEs and these controllers can then be used to regulate the associated ODE-PDEs. We have demonstrated the accuracy and scalability of the resulting algorithms by applying the results to several illustrative examples. While the scope of the paper is limited to inputs entering through the ODE or in-domain, we believe the results can be extended to inputs at the boundary.

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APPENDIX

A. Proof of Theorem 13

**Theorem 13.** Suppose there exists a self-adjoint bounded and coercive operator $P : Z^{m,n}[a, b] \rightarrow Z^{m,n}[a, b]$ such that

$$TPA^* + APT^* \preceq -\epsilon TT^*$$

for some $\epsilon > 0$. Then any $x \in Z^{m,n}[a, b]$ that satisfies the system

$$Tx(t) = Ax(t), \quad x(0) = x_0 \in Z^{m,n}[a, b]$$

satisfies

$$\lim_{\epsilon \to 0} \|Tx(t)\|_Z = 0.$$

**Proof.** Define a Lyapunov candidate as $V(y) = \langle T^*y, P T^*y \rangle_Z$. Then there exists an $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \|T^*y\|_Z \leq V(y) \leq \beta \|T^*y\|_Z.$$

The time derivative of $V(y)$ along the solutions of the PIE

$$T^*y(t) = A^*y(t), \quad y(0) \in Z^{m,n}[a, b]$$

is given by

$$V(y(t)) = \langle T^*y(t), P T^*y(t) \rangle_Z + \langle T^*y(t), P T^*y(t) \rangle_Z$$

$$= \langle y(t), TPT^*y(t) \rangle_Z + \langle y(t), A^*y(t) \rangle_Z$$

$$\leq -\epsilon \|T^*y(t)\|_Z \leq -\frac{\epsilon}{\beta} V(y(t)).$$

Then, using Gronwall-Bellman Inequality, there exists constants $M$ and $k$ such that

$$V(y(t)) \leq V(y(0))Me^{-\epsilon kt}.$$

As $t \to \infty$, $V(y(t)) \to 0$ which implies $\|T^*y(t)\|_Z \to 0$. Then, from Theorem 10, $\|Tx(t)\|_Z \to 0$. \hfill $\square$

B. Proof of Theorem 16

**Theorem 16.** Suppose there exist $\gamma > 0$, bounded linear operators $P : Z^{m,n}[a, b] \rightarrow Z^{m,n}[a, b]$ and $Z : Z^{m,n}[a, b] \rightarrow \mathbb{R}^p$, such that $P$ is self-adjoint, coercive and

$$\begin{align*} & -\gamma I + D_{II} (CP + D_{II}Z) T^* \preceq 0, \\
& (\cdot)^* T (CP + D_{II}Z) T^* \preceq 0 \end{align*}$$

Then, for any $w \in L_2$, for $u(t) = \kappa x(t)$ where $\kappa = Z P^{-1}$, any $x$ and $z$ that satisfy the PIE (3) also satisfy $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$.

**Proof.** Define a Lyapunov candidate function $V(x) = \langle T^*x, P T^*x \rangle_Z$. Since $P$ is coercive and bounded, there exists $\alpha > 0$ and $\beta > 0$ such that

$$\alpha \|T^*x\|_Z \leq V(x) \leq \beta \|T^*x\|_Z.$$

The time derivative of $V(x)$ along the solutions of

$$T^*x(t) = (A + B_{II}K)x(t) + (C + D_{II}K)^*w(t),$$

$$z(t) = B_{II}^*x(t) + D_{II}^*w(t), \quad x(0) = 0$$

is given by

$$\dot{V}(x(t)) = \langle T^*x(t), P T^*x(t) \rangle_Z + \langle T^*x(t), P T^*x(t) \rangle_Z$$

$$= \langle (A + B_{II}K)x(t) + (C + D_{II}K)^*w(t), (A + B_{II}K)^*x(t) \rangle_Z$$

$$+ \langle (A + B_{II}K)^*x(t), P T^*x(t) \rangle_Z$$

$$+ \langle T^*x(t), P(C + D_{II}K)^*w(t) \rangle_Z$$

For any $w(t) \in \mathbb{R}^p$ and $x(t) \in Z$ that satisfies Eq. (18),

$$\dot{V}(x(t)) = \left[ \begin{array}{c} \dot{v}(t) \\
\dot{w}(t) \end{array} \right] = \begin{bmatrix} -\gamma I & D_{II} \\
-CP + D_{II}Z \end{bmatrix} T^* \begin{bmatrix} v(t) \\
w(t) \end{bmatrix} + \langle (A + B_{II}K)x(t) + (C + D_{II}K)^*w(t), (A + B_{II}K)^*x(t) \rangle_Z$$

$$+ \langle (A + B_{II}K)^*x(t), P T^*x(t) \rangle_Z + \langle T^*x(t), P(C + D_{II}K)^*w(t) \rangle_Z.$$