COMPLETELY BOUNDED PALEY PROJECTIONS ON ANISOTROPIC SOBOLEV SPACES ON TORI

YANQI QIU

ABSTRACT. We study the existence of certain completely bounded Paley projection on the anisotropic Sobolev spaces on tori. Our result should be viewed as a generalization of a similar result obtained by Pełczyński and Wojciechowski in [3]. By a transference method, we obtain similar results on the Sobolev spaces on quantum tori.

1. Introduction

Let $S \subset \mathbb{N}^d$ be a finite subset, containing the origin and satisfying some saturation conditions. The anisotropic Sobolev space $W_1^S(T^d)$ is defined via the norm

$$
\|f\|_{S,1} = \sum_{\gamma \in S} \|\partial^\gamma f\|_{L_1(T^d)}.
$$

In [3], necessary and sufficient conditions on $S$ are given under which there exist the so-called Paley projections on $W_1^S(T^d)$.

By the definition, $W_1^S(T^d)$ embeds isometrically in $\ell_1^{|S|}(L_1(T^d))$, it is well-known that on the latter space, there exists a natural operator space structure, and we will equip $W_1^S(T^d)$ with the sub-operator space structure via the above embedding.

Following the proofs in [3], we show that under the same conditions on $S$, the projections considered by Pełczyński and Wojciechowski are in fact completely bounded. The complete boundedness of these projections can be applied to obtain similar results on the Sobolev space $W_1^S(T^d_\theta)$ associated to the quantum torus $T^d_\theta$.

2. Preliminaries

Denote by $\mathbb{N}$ the set of non-negative integers. Fix a positive integer $d \geq 1$. The usual scalar product on the Euclidian space $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$. We denote by $T^d$ the group $\mathbb{R}/2\pi \mathbb{Z}^d$ equipped with its normalized Haar measure $dx$, it will be identified with the cube $[-\pi, \pi)^d$ in a standard way. The dual group of $T^d$ is $\mathbb{Z}^d$ such that to each $n \in \mathbb{Z}^d$ is
YANQI QIU

assigned the character $\chi_n : \mathbb{T}^d \to \mathbb{C}$ defined by $\chi_n(x) = e^{i(x,n)}$. Trigonometric polynomials are complex linear combinations of characters. The set of trigonometric polynomials on $\mathbb{T}^d$ is denoted by $\mathcal{P}_d$.

To each $\gamma = (\gamma(j)) \in \mathbb{N}^d$, we associate with the partial derivative

$$\partial^\gamma = \frac{\partial^{\gamma(j)}}{\partial x_j(1) \partial x_j(2) \cdots \partial x_j(d)},$$

where $|\gamma| = \gamma(1) + \gamma(2) + \cdots + \gamma(d)$.

A smoothness $S$ is a finite subset of $\mathbb{N}^d$ which contains the origin $0$, and such that: if $\alpha = (\alpha(j)) \in S$ then every $\beta = (\beta(j)) \in \mathbb{N}$ such that $\beta(j) \leq \alpha(j)$ for $j = 1, 2, \cdots, d$ belongs to $S$.

For each $\gamma \in \mathbb{N}^d$, we define the symbol $\sigma_\gamma : \mathbb{R}^d \to \mathbb{C}$ as the function:

$$\sigma_\gamma(x) = i^{\gamma} |x^\gamma = \prod_{j=1}^d (ix(j))^{\gamma(j)},$$

otherwise, $\sigma_\gamma(x) = 0$.

The fundamental polynomial of a smoothness $S$ is

$$Q_S = \sum_{\gamma \in S} |\sigma_\gamma|^2,$$

which is a non-negative function on $\mathbb{R}^d$.

The Sobolev space $W^S_p(\mathbb{T}^d)$ is defined as the completion of $\mathcal{P}_d$ with respect to the norm defined as following: if $f \in \mathcal{P}_d$, then

$$\|f\|_{S,p} := \left( \sum_{\gamma \in S} \|\partial^\gamma f\|^p_{L^p(\mathbb{T}^d)} \right)^{1/p}.$$ 

Remark 2.1. The original definition of $\|f\|_{S,p}$ in [3] is

$$\|f\|_{S,p} = \left( \int_{\mathbb{T}^d} \left( \sum_{\gamma \in S} |\partial^\gamma f(x)|^2 \right)^{p/2} dx \right)^{1/p},$$

which is equivalent to our definition since $S$ is a finite set.

Let $f \in L_1(\mathbb{T}^d)$, its spectrum $\text{spec}(f)$ is

$$\text{spec}(f) := \{n \in \mathbb{Z}^d : \hat{f}(n) = \int_{\mathbb{T}^d} f(x)e^{-i(x,n)}dx \neq 0\}.$$

Let $\Lambda \subset \mathbb{Z}^d$ be an infinite subset. The projection $P_\Lambda : \mathcal{P}_d \to \mathcal{P}_d$ is defined by $P_\Lambda f = \sum_{n \in \Lambda} \hat{f}(n)e^{i(x,n)}$. 
Definition 2.2. In the above situation, $P_{\Lambda}$ will be called a Paley projection if there is some $K > 0$, such that
\[ \| P_{\Lambda} f \|_{S,2} \leq K \| f \|_{S,1}, \quad \text{for all } f \in \mathcal{P}_d, \]
i.e. for all $f \in \mathcal{P}_d$, we have
\[ \left( \sum_{n \in \Lambda} Q_S(n) |\hat{f}(n)|^2 \right)^{1/2} \leq K \| f \|_{S,1}. \]

If $P_{\Lambda}$ is a Paley projection, then the natural mapping $W^S_2(T^d)_{\Lambda} \to W^S_1(T^d)_{\Lambda}$ is an isomorphism. $P_{\Lambda}$ can be uniquely extended to be a projection on $W^S_1(T^d)$, which is still denoted by $P_{\Lambda} : W^S_1(T^d) \to W^S_1(T^d)$.

For the operator space theory, we refer to the book [5] for a detailed study. Here we recall that the usual $L^p$-spaces are equipped with a natural operator space structure (in short o.s.s. For the detail, see e.g. [5] p.178 - p.180). Hence $W^S_1(T^d)$ is an operator space by the embedding $W^S_1(T^d) \subset \ell^{|S|}_1(L(S^1(T^d)))$.

We will use the following useful fact: Let $E \subset L_1(\Omega, \mu)$ and $F \subset L_1(M, \nu)$ be two operator subspaces. Then a linear operator $u : E \to F$ is completely bounded iff $u \otimes I_{S_1} : E(S_1) \to F(S_1)$ is bounded, where $S_1$ is the set of trace class operators and $E(S_1)$ and $F(S_1)$ are the closures of $E \otimes S_1$ and $F \otimes S_1$ in $L_1(\Omega, \mu; S_1)$ and $L_1(M, \nu; S_1)$ respectively. Moreover,
\[ \| u \|_{cb} = \| u \otimes I_{S_1} \|. \]

Recall that the operator space $C + R$ is a homogeneous Hilbertian operator space, which is determined by the following fact: if $(e_k)$ is an orthonormal basis of $C + R$ and $(x_k)$ is a finite sequence in $S_1$, then
\[ \| \sum_k x_k \otimes e_k \|_{S_1[C+R]} = \inf \{ \| (\sum_k y_k y_k^*)^{1/2} \|_{S_1} + \| (\sum_k z_k z_k^*)^{1/2} \|_{S_1} \}, \]
where the infimum runs over all possible decompositions $x_k = y_k + z_k$.
(For the definition of $S_1[E]$, see [3]). For convenience, we will denote
\[ |||(x_k)||| := \| \sum_k x_k \otimes e_k \|_{S_1[C+R]}. \]

The following theorem of Lust-Piquard and Pisier will be used in this note.

Theorem 2.3. (Lust-Piquard & Pisier) Let $(n_k)$ be any increasing sequence which is lacunary à la Hadamard, i.e. $\lim_{n_k \to \infty} \frac{n_{k+1}}{n_k} > 1$. Then there exists $K > 0$, such that for any finite sequence $(x_k)$ in $S_1$, we
have
\[ \frac{1}{K} \| |(x_k)| | \| \leq \sum_k x_k e^{i n_k t} \|_{L_1(T; S_1)} \leq K \| |(x_k)| |. \]

**Remark 2.4.** Under the same condition as in the above theorem, by the equivalence \( (2) \), it is easy to see that if \((a_k)\) is a bounded sequence in \( \mathbb{C} \), then
\[ \| \sum_k a_k x_k e^{i n_k t} \|_{L_1(T; S_1)} \lesssim \sum_k x_k e^{i n_k t} \|_{L_1(T; S_1)}. \]

If \((a_k)\) is moreover uniformly separated from 0, i.e. \( \inf_k |a_k| > 0 \), then
\[ \| \sum_k a_k x_k e^{i n_k t} \|_{L_1(T; S_1)} \approx \sum_k x_k e^{i n_k t} \|_{L_1(T; S_1)}. \]

**Definition 2.5.** A smoothness \( S \subset \mathbb{N}^d \) is said to have Property (O) if there are \( \alpha, \beta \in S \) with \( |\alpha| \not\equiv |\beta| \mod 2 \) and \( c = (c(j)) \) with \( c(j) > 0 \) such that:

(i) \( \langle \alpha, c \rangle = \langle \beta, c \rangle = 1 \)

(ii) \( \langle \gamma, c \rangle \leq 1 \) for all \( \gamma \in S \).

**Remark 2.6.** Assume that \( S \) has property (O) and let \( \alpha, \beta \in S \) be the two points in \( S \) as in the definition of property (O). Then there exists a sequence \((n_k) \subset \mathbb{N}^d \) such that
\[ \liminf_k n_k(j) = \infty \]
and
\[ \rho = \min \{ \inf_k |\sigma_{\alpha}(n_k)|_{Q_S(n_k)^{1/2}}, \inf_k |\sigma_{\beta}(n_k)|_{Q_S(n_k)^{1/2}} \} > 0. \]

For the proof, see Proposition 1.2 in [3].

We end this section by stating the following technical proposition from [3].

**Proposition 2.7.** (Pełczyński & Wojciechowski) Let \( S \subset \mathbb{N}^d \) be a smoothness. Then given \( \varepsilon \) with \( 0 < \varepsilon < 1 \) and \( D = 1, 2, \cdots \) there exists \( \rho = \rho(D, \varepsilon) > 1 \) such that, for every \( m, n \in \mathbb{Z}^d \), if \( \min_{1 \leq j \leq d} |n(j)| \geq \rho \) and if \( \sum_{j=1}^d |n(j) - m(j)| \leq D \) then
\[ |1 - Q_S(n)Q_S(m)^{-1}| < \varepsilon; \]
\[ \sum_{\alpha \in S} \left| \frac{|\sigma_{\alpha}(m)|_{Q_S(m)^{1/2}} - |\sigma_{\alpha}(n)|_{Q_S(n)^{1/2}}}{Q_S(m)^{1/2}} \right|^2 < \varepsilon^2. \]
\[
\sum_{\alpha \in S} \left| \frac{\sigma_\alpha(m)}{Q_S(m)^{1/2}} - \frac{\sigma_\alpha(n)}{Q_S(n)^{1/2}} \right|^2 < \varepsilon^2.
\]

3. Main result

**Theorem 3.1.** If the smoothness \( S \) satisfies Property (O), then there exists a completely bounded Paley projection \( P_\Lambda : W_1^S(\mathbb{T}^d) \to W_1^S(\mathbb{T}^d) \) associated to some infinite sequence \( \Lambda = (n_k) \subset \mathbb{Z}^d \). Moreover, the linear map \( \hat{P} : W_1^S(\mathbb{T}^d)_\Lambda \to C + R \) defined by

\[
\hat{P} f = \sum_{k=1}^{\infty} Q_S(n_k)^{1/2} \hat{f}(n_k)e_k
\]

is a complete isomorphism, where \((e_k)^{\infty}_{k=1}\) is an orthonormal basis of \( C + R \).

The following lemma will be used in the proof of Theorem 3.1.

**Lemma 3.2.** Assume that \( \Sigma \subset \mathbb{Z}^d \) is an infinite subset satisfies the conditions \( n(1) \geq 1, \forall n \in \Sigma \setminus \{0\} \) and the projection to the first coordinate \( \Sigma \to \mathbb{N} \) defined by \( n \mapsto n(1) \) is injective. Assume moreover that \( \Lambda = (n_k)^{\infty}_{k=1} \) is an infinite sequence in \( \Sigma \) such that

\[
\inf_k \frac{n_k(1)}{n_{k-1}(1)} > 1.
\]

Then the natural map

\( P_{\Sigma,\Lambda} : L_1(\mathbb{T}^d)_\Sigma \to L_1(\mathbb{T}^d)_\Lambda \)

is completely bounded and \( L_1(\mathbb{T}^d)_\Lambda \) is completely isomorphic to \( C + R \).

**Proof.** We shall prove that the projection \( L_1(\mathbb{T}^d; S_1)_{\Sigma} \to L_1(\mathbb{T}^d; S_1)_{\Lambda} \) is bounded. Let \( \Gamma \) be the image of the first projection \( \Sigma \to \mathbb{N} \). The injectivity of \( n \mapsto n(1) \) on \( \Sigma \) implies that there is a map \( m : \Gamma \to \mathbb{Z}^{d-1} \) such that \( n = (n(1), m(n(1))) \) for all \( n \in \Sigma \). We write \( x \in \mathbb{T}^d \) as a pair \( x = (t, y) \in \mathbb{T} \times \mathbb{T}^{d-1} \). To each \( y \in \mathbb{T}^{d-1} \) and \( g \in L_1(\mathbb{T}^d; S_1) \) we associate with a function \( g_y : \mathbb{T} \to S_1 \) defined by \( g_y(t) = g(t, y) \). If \( g \in L_1(\mathbb{T}^d; S_1)_{\Sigma} \), then

\[
g(t, y) \sim \sum_{n \in \Sigma} \hat{g}(n)e^{i(t,y).n} = \sum_{n(1) \in \Gamma} \hat{g}(n)e^{im(1)}e^{i(g(m(n(1)))}.
\]

This implies that \( \text{spec}(g_y) \subset \Gamma \subset \mathbb{N} \) and

\[
\hat{g}_y(n(1)) = \hat{g}(n)e^{i(g(m(n(1))))}.
\]
By [2], as operator space, $L_1(\mathbb{T})_\Gamma$ is completely isomorphic to $C + R$. Hence for any fixed $y \in \mathbb{T}^{d-1}$,
\begin{align*}
\left\| \sum_{n(1) \in \Gamma} \hat{g}(n) e^{itn(1)} e^{i(y,m(n(1)))} \right\|_{L_1(\mathbb{T};S_1)} & \approx \left\| \sum_{n(1) \in \Gamma} \hat{g}(n) e^{itn(1)} \right\|_{L_1(\mathbb{T};S_1)}.
\end{align*}

It follows that
\begin{align*}
\|g\|_{L_1(\mathbb{T}^d;S_1)} &= \int_{\mathbb{T}^{d-1}} \left\| \sum_{n(1) \in \Gamma} \hat{g}(n) e^{itn(1)} e^{i(y,m(n(1)))} \right\|_{L_1(\mathbb{T};S_1)} dy \\
& \approx \left\| \sum_{n(1) \in \Gamma} \hat{g}(n) e^{itn(1)} \right\|_{L_1(\mathbb{T};S_1)} \\
& = \left\| \sum_{n(1) \in \Gamma} \hat{g}(n) e^{itn(1)} \right\|_{H^1(\mathbb{T};S_1)}.
\end{align*}

Similarly, we have
\begin{align*}
\|P_{\Sigma,\Lambda} g\|_{L_1(\mathbb{T}^d;S_1)} & \approx \left\| \sum_{k=1}^{\infty} \hat{g}(n_k) e^{itn_k(1)} \right\|_{H^1(\mathbb{T};S_1)}.
\end{align*}

The sequence $(n_k(1))_{k=1}^{\infty}$ is lacunary, thus we can apply Corollary 0.4 in [2] to obtain
\begin{align*}
\left\| \sum_{k=1}^{\infty} \hat{g}(n_k) e^{itn_k(1)} \right\|_{H^1(\mathbb{T};S_1)} & \lesssim \left\| \sum_{n(1) \in \Gamma} \hat{g}(n) e^{itn(1)} \right\|_{H^1(\mathbb{T};S_1)}.
\end{align*}

Combining the above inequalities, we have
\begin{align*}
\|P_{\Sigma,\Lambda} g\|_{L_1(\mathbb{T}^d;S_1)} & \lesssim \|g\|_{L_1(\mathbb{T}^d;S_1)}.
\end{align*}

This completes the proof that $P_{\Sigma,\Lambda} : L_1(\mathbb{T}^d)_\Sigma \to L_1(\mathbb{T}^d)_\Lambda$ is completely bounded.

The fact that $L_1(\mathbb{T}^d)_\Lambda \approx C + R$ is then easy. Indeed, if $h \in L_1(\mathbb{T}^d;S_1)_\Lambda$, then
\begin{align*}
(5) \quad \|h\|_{L_1(\mathbb{T}^d;S_1)_\Lambda} & \approx \left\| \sum_{k=1}^{\infty} \hat{h}(n_k) e^{itn_k(1)} \right\|_{H^1(\mathbb{T};S_1)} \approx \left\| \|\hat{h}(n)\| \right\|.
\end{align*}

In other words, $L_1(\mathbb{T}^d)_\Lambda \approx C + R$ completely isomorphically. \hfill \Box

**Remark 3.3.** In the situation of Lemma 3.2, the map
\[ \tilde{P}_{\Sigma,\Lambda} : L_1(\mathbb{T}^d)_\Sigma \to C + R \]
defined by $\tilde{P}_{\Sigma,\Lambda} f = \sum_{k=1}^{\infty} \hat{f}(n_k) e_k$, where $e_k$ is an orthonormal basis of $C + R$, is completely bounded.
COMPLETELY BOUNDED PALEY PROJECTIONS ON ANISOTROPIC SOBOLEV SPACES ON TORI

**Proof of Theorem 3.7.** Our proof follows the proof of Proposition 2.2 in [3]. Let \( \alpha, \beta \in S \) and \((n_k) \in \mathbb{N}^d\) be as in Remark 2.6. Since \(|\alpha| \neq |\beta| \mod 2\), one can assume that for all \(k\),

\[
\text{sign}(\frac{\sigma_\alpha(n_k)}{\sigma_\beta(n_k)}) = i^{\|\alpha|-|\beta|} = \tau,
\]

\[
\text{sign}(\frac{\sigma_\alpha(-n_k)}{\sigma_\beta(-n_k)}) = (-i)^{\|\alpha|-|\beta|} = -\tau.
\]

Here \(\text{sign}(z) := \frac{z}{|z|}\) for \(z \in \mathbb{C} \setminus 0\).

Replacing, if necessary, the sequence \((n_k)\) by a rapidly increasing subsequence, we can assume without loss of generality that the sequence \((n_k)\) satisfies the conditions:

1. \(\sum_{r=1}^{k-1} \sum_{j=1}^d n_r(j) < \min_j n_k(j)\) for \(k = 2, 3, \ldots\),
2. \(\lim_k \frac{|\sigma_\alpha(-n_k)|}{|\sigma_\beta(-n_k)|} = \lim_k \frac{|\sigma_\alpha(n_k)|}{|\sigma_\beta(n_k)|} = \ell > 0\),
3. \(\sum_{k=1}^\infty |\sigma_\alpha(-n_k) + \tau \ell \sigma_\beta(-n_k)| Q_S(n_k)^{-1/2} = \sum_{k=1}^\infty \left( |\sigma_\alpha(n_k)| - \ell |\sigma_\beta(n_k)| \right) Q_S(n_k)^{-1/2} < \frac{1}{2}\),
4. \(\sum_{k=1}^\infty \sum_{m \in B_k} |\sigma_\alpha(-m) + \tau \ell \sigma_\beta(-m)| Q_S(-m)^{1/2} < 1\),

where \(B_1 = \{n_1\}\) and for \(k = 2, 3, \ldots\),

\[
B_k = \left\{ m \in \mathbb{Z}^d : \sum_{j=1}^d |m(j) - n_k(j)| \leq \sum_{r=1}^{k-1} \sum_{j=1}^d n_r(j) \right\}.
\]

Notice that item (iv) follows from (iii), Proposition 2.7 and also the assumption that \((n_k)\) increase sufficiently fast.

Define \(M : W_1^S(\mathbb{T}^d) \to L_1(\mathbb{T}^d)\) by

\[
Mf = \partial^\alpha f + \tau \ell \partial^\beta f - \sum_{k=1}^\infty \sum_{m \in B_k} (\sigma_\alpha(-m) + \tau \ell \sigma_\beta(-m)) \hat{f}(-m) e^{-i\ell \cdot m}.
\]

Then \(M\) is completely bounded. Indeed, consider the map \(M \otimes I_{S_1} : W_1^S(\mathbb{T}^d; S_1) \to L_1(\mathbb{T}^d; S_1)\). If \(g \in W_1^S(\mathbb{T}^d; S_1)\), then

\[
\|\partial^\alpha g\|_{L_1(\mathbb{T}^d; S_1)} + \|\partial^\beta g\|_{L_1(\mathbb{T}^d; S_1)} \leq \|g\|_{W_1^S(\mathbb{T}^d; S_1)}.
\]

Remember that \(\sigma_\gamma(n) \hat{g}(n) = \int_{\mathbb{T}^d} \partial^\gamma g(x) e^{-i(x,n)} dx\), hence for any \(\gamma \in S\),

\[
\|\sigma_\gamma(n) \hat{g}(n)\|_{S_1} \leq \|\partial^\gamma g\|_{L_1(\mathbb{T}^d; S_1)} \leq \|g\|_{W_1^S(\mathbb{T}^d; S_1)}.
\]
Hence $Q_s(n)^{1/2} \| \hat{g}(n) \|_{S_1} \leq |S| \cdot \|g\|_{W^S_1(\mathbb{T}^d,S_1)}$. Combining with (iv), we have

$$
\| \sum_{k=1}^{\infty} \sum_{m \in B_k} (\sigma_\alpha(-m) + \tau \ell \sigma_\beta(-m)) \hat{g}(-m)e^{-i \langle x, m \rangle} \|_{L_1(\mathbb{T}^d,S_1)} \\
\leq |S| \cdot \|g\|_{W^S_1(\mathbb{T}^d,S_1)}.
$$

Hence $M \otimes I_{S_1}$ is bounded and $M$ is completely bounded.

Next, consider the measure $\mu_R$ on $\mathbb{T}^d$ given by the Riesz product

$$
R(x) = \prod_{k=1}^{\infty} (1 + \cos(x,n_k)) = \prod_{k=1}^{\infty} (1 + \frac{1}{2}e^{i \langle x, n_k \rangle} + \frac{1}{2}e^{-i \langle x, n_k \rangle}).
$$

Then the convolution map $M_R : L_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)$ defined by $M_R f = f * \mu_R$ is obviously completely contractive. In the definition of this Riesz product, we assume that $\frac{n_k+1}{n_k} \geq 3$, for all $k = 1, 2, \cdots$. Notice that we have

$$
\text{spec}(\mu_R) = \left\{ d_1 n_1 + d_2 n_2 + \cdots + d_k n_k : k \in \mathbb{N}, d_1, d_2, \cdots d_k \in \{-1, 0, 1\} \right\}.
$$

**Claim A:** $\text{spec}(\mu_R) \subset \{0\} \cup \bigcup_{k=1}^{\infty} (B_k \cup (-B_k))$. Indeed, if $m \in \text{spec}(\mu_R) \setminus \{0\}$, then there exist $k \geq 1$ and $d_1, d_2, \cdots, d_k \in \{-1, 0, 1\}$, such that $d_k \neq 0$ and $m = d_1 n_1 + d_2 n_2 + \cdots + d_k n_k$. Replacing $m$ by $-m$, if necessary, one may assume that $d_k = 1$, then $m - n_k = \sum_{r=1}^{k-1} d_r n_r$, it follows that $\sum_{j=1}^{d} |m(j) - n_k(j)| \leq \sum_{r=1}^{k-1} \sum_{j=1}^{d} n_r(j)$, i.e. $m \in B_k$.

**Claim B:** The projection to the first coordinate $\text{spec}(\mu_R) \to \mathbb{Z}$ is injective. Indeed, if $n, m \in \text{spec}(\mu_R)$ such that $n(1) = m(1)$, suppose that $n = d_1 n_1 + d_2 n_2 + \cdots + d_k n_k$ and $m = d'_1 n_1 + d'_2 n_2 + \cdots d'_k n_k$, then by a simple computation (cf. e.g. [II]), we have $k = k'$ and $d_1 = d'_1$, $d_2 = d'_2$, \cdots, $d_k = d'_k$, hence $n = m$. In other words, the projection to the first coordinate $\text{spec}(\mu_R) \to \mathbb{Z}$ is injective.

Let $\Sigma = \{0\} \cup \bigcup_{k=1}^{\infty} B_k$. It can be easily checked that the image $\text{Im}(M_R M)$ of the composition operator $M_R M$ is contained in $L_1(\mathbb{T}^d)_\Sigma$. By the definition of $B_k$ and condition (i) on the sequence $(n_k)$, if $m \in B_k$, then

$$
m(1) \geq n_k(1) - \sum_{r=1}^{k-1} \sum_{j=1}^{d} n_r(j) > 0.
$$

We are now in the situation of Lemma 3.2 thus we obtain a completely bounded projection $P_{\Sigma, \Lambda} : L_1(\mathbb{T}^d)_\Sigma \to L_1(\mathbb{T}^d)_\Lambda$. By composition, we obtain the following completely bounded map

$$
P_{\Sigma, \Lambda} M_R M : W^S_1(\mathbb{T}^d) \to L_1(\mathbb{T}^d)_\Lambda.$$

By computation, we have
\[ P_{\Sigma, \Lambda} f = \sum_{k=1}^{\infty} \rho_k Q_S(n_k)^{1/2} \hat{g}(n_k) e^{i(n_k)}, \]
where \( \rho_k = \frac{\sigma_\alpha(n_k) + \epsilon \sigma_\beta(n_k)}{2Q_S(n_k)^{1/2}}. \) By (4),
\[ |\rho_k| = \frac{1}{2}(|\sigma_\alpha(n_k)| + \epsilon |\sigma_\beta(n_k)|)Q_S(n_k)^{1/2} \geq \frac{1}{2} \rho(1 + \epsilon). \]
On the other hand, it is obvious that \( |\rho_k| \leq \frac{1}{2}(1 + \epsilon) \). Let \( g : \mathbb{T}^d \to S_1 \), then
\[ \|P_{\Lambda} g\|_{W^s_{\ell^d}(\mathbb{T}^d, S_1)} = \sum_{\gamma \in S} \| \sum_{k=1}^{\infty} \sigma_\gamma(n_k) \hat{g}(n_k) e^{i(n_k)} \|_{L_1(\mathbb{T}^d, S_1)} \]
By (5) \( \approx \sum_{\gamma \in S} \| \sum_{k=1}^{\infty} \sigma_\gamma(n_k) \hat{g}(n_k) e^{i\tau n_k(1)} \|_{L_1(\mathbb{T}; S_1)} \]
By Remark 2 \( \lesssim \sum_{\gamma \in S} \| \sum_{k=1}^{\infty} Q_S(n_k)^{1/2} \hat{g}(n_k) e^{i\tau n_k(1)} \|_{L_1(\mathbb{T}; S_1)} \]
By Remark 2 and \( |S| < \infty \) \( \approx \| \sum_{k=1}^{\infty} \rho_k Q_S(n_k)^{1/2} \hat{g}(n_k) e^{i\tau n_k(1)} \|_{L_1(\mathbb{T}; S_1)} \]
By (5) \( \approx \| \sum_{k=1}^{\infty} \rho_k Q_S(n_k)^{1/2} \hat{g}(n_k) e^{i(n_k)} \|_{L_1(\mathbb{T}^d; S_1)} \]
\[ = \| (P_{\Sigma, \Lambda} M_R M \otimes I_{S_1}) g \|_{L_1(\mathbb{T}^d, S_1)} \]
\[ \lesssim \| g \|_{W^s_{\ell^d}(\mathbb{T}^d, S_1)}. \]

This completes the proof that \( P_{\Lambda} : W^s_{\ell^d}(\mathbb{T}^d) \to W^s_{\ell^d}(\mathbb{T}^d) \) is completely bounded. For the second assertion of the theorem, we only need to notice that by Remark 2.6 \( |\sigma_\alpha(n_k)| \geq \rho Q_S(n_k)^{1/2} \) for all \( k \), hence if \( g \in W^s_{\ell^d}(\mathbb{T}^d, S_1) \), then
\[ \| g \|_{W^s_{\ell^d}(\mathbb{T}^d, S_1)} \geq \| \partial^\alpha g \|_{L_1(\mathbb{T}^d, S_1)} = \sum_{k=1}^{\infty} \sigma_\alpha(n_k) \hat{g}(n_k) e^{i\tau n_k(1)} \|_{L_1(\mathbb{T}^d, S_1)} \]
\[ \geq \| \sum_{k=1}^{\infty} Q_S(n_k)^{1/2} \hat{g}(n_k) e^{i\tau n_k(1)} \|_{L_1(\mathbb{T}^d, S_1)} \]
\[ \approx \| \sum_{k=1}^{\infty} Q_S(n_k)^{1/2} \hat{g}(n_k) \otimes e_k \|_{S_1[C+R]}. \]
\( \square \)
Using Theorem [3.1], then by a classical transference method, we have the following corollary, for the definition of quantum torus $\mathbb{T}^d_\theta$ and harmonic analysis on it, we refer to the paper [6].

**Corollary 3.4.** Under the same condition of Theorem [3.1], there exists a completely bounded Paley projection $P_\Lambda : W^1_1(\mathbb{T}^d_\theta) \to W^1_1(\mathbb{T}^d_\theta)$ associated to some infinite sequence $\Lambda = (n_k) \subset \mathbb{Z}^d$.

**Acknowledgements**

The author would like to thank Quanhua Xu for inviting him to Université Franche-Comté and his constant encouragement.

**References**

[1] F. R. Keogh. Riesz products. *Proc. London Math. Soc. (3)*, 14a:174–182, 1965.
[2] Françoise Lust-Piquard and Gilles Pisier. Noncommutative Khintchine and Paley inequalities. *Ark. Mat.*, 29(2):241–260, 1991.
[3] A. Pełczyński and M. Wojciechowski. Paley projections on anisotropic Sobolev spaces on tori. *Proc. London Math. Soc. (3)*, 65(2):405–422, 1992.
[4] Gilles Pisier. Non-commutative vector valued $L^p$-spaces and completely $p$-summing maps. *Astérisque*, 247:vi+131, 1998.
[5] Gilles Pisier. *Introduction to operator space theory*, volume 294 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
[6] Z. Chen, Q. Xu and Z. Yin. Harmonic analysis on quantum tori. http://arxiv.org/abs/1206.3358