Unit Consistency, Generalized Inverses, and Effective System Design Methods

Jeffrey Uhlmann
University of Missouri-Columbia
201 EBW, Columbia, MO 65211
Email: uhlmannj@missouri.edu

Abstract

This paper examines the potential role of unit consistency as a system design principle. Unit-consistent generalized matrix inverses and unit-invariant matrix decompositions are derived in support of this principle. Applications of the methods described are illustrated with examples relating to nonlinear system identification and robustness to multiplicative noise for image database retrieval.

Keywords: Drazin Inverse, Generalized Matrix Inverse, Image Databases, Inverse Problems, Linear Estimation, Linear Systems, Nonlinear Systems, Machine Learning, Matrix Analysis, Modular Systems, Moore-Penrose Pseudoinverse, Multiplicative Noise, Scale Invariance, Singular Value Decomposition, SVD, System Design, System Identification, Unit Consistency.

I. INTRODUCTION

Many of the benefits of modular system design derive from an assumption that each module has been separately tested and verified so that the composite system can be analyzed, tuned, and evaluated at a more manageable level of abstraction. However, seemingly benign decisions made as part of the design and implementation of a given module can have significant unanticipated effects on the behavior of a system in which it is used. Consider a notional representation of a module that processes a parameter vector and a dataset to produce an output:

\[ f(p, X) = Xp \rightarrow q, \]

where \( X \) is interpreted to be a matrix and \( p \) is a column vector that is to be linearly transformed by \( X \) to produce an output vector \( q \). Such a “linear transformation module” can be easily implemented and verified for correctness. For example, if the inputs \( p, X, \) and \( q \) are all defined in the same coordinate frame then the module can be tested to assess whether it is consistent with respect to an orthogonal/unitary rotation of the coordinate frame, i.e., that a unitary transformation applied to the input as \( p' = Up \) and \( X' = UXU^* \) gives \( f(p', X') \rightarrow Uq \). If consistency with respect to a rotation of the coordinate frame is assumed to hold at the system level then the integrity of the system as a whole can be tested with a change of coordinates using a random rotation matrix \( R \). More specifically, if an arbitrary rotation of the coordinate frame is applied to the inputs then the system should produce the same output but in the new rotated coordinates. If that does not occur then a fault has been detected.

Note that this kind of consistency testing does not just detect errors that are specifically related to the choice of coordinate frame; rather, it detects any error that leads to a violation of the consistency assumption. In other words, a consistency test is analogous to the use of parity bits and checksums for detecting whether a bit string has been corrupted: an error is unlikely to preserve the assumed properties and thus will be detected with high probability.

Most complex real-world systems perform transformations from inputs to outputs that are defined in an application-specific state space for which there is only an expectation of consistency with respect to the choice of units associated with state variables (e.g., kilometers-per-hour versus meters-per-second) rather than a unitary mixing of those state variables. The appropriate consistency test should therefore assess whether the system is unit consistent (UC). In the case of the linear transformation

\[ f(p, X) = Xp \rightarrow q, \]

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module, for example, the test for unit consistency might verify that a diagonal change-of-unit transformation \( D \) applied as \( p' = D p \) and \( X' = DXD^{-1} \) gives \( f(p', X') \rightarrow Dq \). In other words, a change of units for the input to the module should produce the same output but in the new units.

Unit-consistency testing represents a potentially valuable means for assessing a fundamental aspect of system integrity that does not rely in any way on a qualitative interpretation of a battery of empirical tests or comparison against a limited set of ground-truth benchmarks. Unfortunately, unit-consistency testing is not presently viable for the majority of complex systems because many modules that should be expected to exhibit unit consistency actually do not. This commonly occurs for modules that implement functions which have a multiplicity of possible solutions, e.g., due to insufficient constraints or due to the need to produce an approximate solution when not all constraints can be satisfied. If, because of habit or convenience, the solution is made unique by the arbitrary application of an ancillary criterion (e.g., least-squares) which does not ensure unit consistency then any system that uses that module will be prevented from applying unit consistency as a test of system integrity.

A goal of this paper is to promote consistency analysis, and in particular unit consistency, as a consideration during all levels of component and system development whenever applicable. Unit consistency has been suggested in the past as a consideration in specific applications (e.g., robotics [11], [9] and data fusion [32]), but to advocate it as a general design consideration requires development of UC alternatives to some of the most widely-used mathematical tools in engineering. For example, determining a “best” solution to an underdetermined or overdetermined set of equations can often be formulated – either implicitly or explicitly – in terms of a generalized matrix inverse. However, the most commonly applied inverse (Moore-Penrose [21], [23], [4]) is not unit consistent. In fact, the most commonly applied tools in linear systems analysis, the eigen and singular-value decompositions, are inherently not unit consistent and therefore require UC alternatives.

The structure of the paper is as follows: Section II motivates the importance of preserving salient system properties as a means for determining unique solutions to ill-posed inverse problems and discusses the requirements for a unit-consistent generalized inverse. Section III develops left and right unit-consistent generalized inverses. Section IV develops a unit-consistent generalized inverse for elemental nonzero matrices, and Section V develops a fully general unit-consistent generalized matrix inverse. Section VI provides an example of the use of consistency analysis to the problem of obtaining a linearized approximation to an unknown nonlinear transformation based on a set of input and output vectors. Section VII applies the techniques used to achieve unit consistency for the generalized inverse problem to develop unit-consistent and unit-invariant alternatives to the singular value decomposition (SVD) and other tools from linear algebra. Section VIII provides an example of the use of unit-invariant singular values to provide retrieval robustness to multiplicative noise in an image database application. Finally, Section IX summarizes and discusses the contributions of the paper.

II. GENERALIZED MATRIX INVERSES

For a nonsingular \( n \times n \) matrix\(^{\ref{footnote:matrix}} \) \( A \) there exists a unique matrix inverse, \( A^{-1} \), for which certain properties of scalar inverses are preserved, e.g., commutativity:

\[
AA^{-1} = A^{-1}A = I
\]

while others have direct analogs, e.g., matrix inversion distributes over nonsingular multiplicands as:

\[
(XAY)^{-1} = Y^{-1}A^{-1}X^{-1}
\]

When attempting to generalize the notion of a matrix inverse for singular \( A \) it is only possible to define an approximate inverse, \( A^{-\dagger} \), which retains a subset of the algebraic properties of a true matrix inverse. For example, a generalized inverse definition might simply require the product \( AA^{-\dagger} \) to be idempotent in analogy to the identity matrix. Alternative definitions might further require:

\[
AA^{-\dagger} = A
\]

or

\[
A^{-\dagger}A = AA^{-\dagger}
\]

and/or other properties that may be of analytic or application-specific utility.

The vast literature\(^{\ref{footnote:literature}} \) on generalized inverse theory spans more than a century and can inform the decision about which of the many possible generalized inverses is best suited to the needs of a particular application. For example, the Drazin inverse,

\footnote{Concern about unit inconsistencies resulting from naive applications of the Moore-Penrose inverse (i.e., least-squares solutions) was raised most vocally by Doty [8], [9] in the early 1990s in the context of hybrid-control systems. He referred to the problem as having “only been discussed in back hallways at conferences or in private meetings” prior to the work of Duffy [8], [11].}

\footnote{It is assumed throughout that matrices are defined over an associative normed division algebra (real, complex, and quaternion). The inverse of a unitary matrix \( U \) can therefore be expressed using the conjugate-transpose operator as \( U^* \).}

\footnote{Much of this literature is covered in the comprehensive book by Ben-Israel and Greville [4].}
A^0$, satisfies the following for any square matrix $A$ and nonsingular matrix $X$ [10], [6], [4]:

\[ A^0 A A^0 = A^0 \]  
\[ A A^0 = A^0 A \]  
\[ (X A X^{-1})^0 = X A^0 X^1 \]  

Thus it is applicable when there is need for commutativity (Eq.(6)) and/or consistency with respect to similarity transformations (Eq.(7)). On the other hand, the Drazin inverse is only defined for square matrices and does not guarantee the rank of $A^0$ to be the same as $A$. Because the rank of $A^0$ may be less than that of $A$ (and in fact is zero for all nilpotent matrices), it is not appropriate for recursive control and estimation problems (and many other applications) that cannot accommodate progressive rank reduction.

The Moore-Penrose pseudoinverse, $A^+$, is defined for any $m \times n$ matrix $A$ and satisfies conditions which include the following for any conformant nonsingular matrix $X$ and unitary matrices $U$ and $V$:

\[ \text{rank}[A^+] = \text{rank}[A] \]  
\[ A A^+ A = A \]  
\[ A^+ A A^+ = A^+ \]  
\[ (U A V)^+ = V^* A^+ U^* \]  

Its use is therefore appropriate when there is need for unitary consistency, i.e., as guaranteed by Eq.(11). Despite its near-universal use throughout many areas of science and engineering ranging from tomography [5] to genomics analysis [2], the Moore-Penrose inverse is not appropriate for many problems to which it is commonly applied, e.g., state-space applications that require consistency with respect to the choice of units for state variables. In the case of a square singular transformation matrix $A$, for example, a simple change of units applied to a set of state variables may require an inverse $A^{-1}$ to be preserved under diagonal similarity

\[ (D A D^{-1})^{-1} = D A^{-1} D^{-1} \]  

where the diagonal matrix $D$ defines an arbitrary change of units. The Moore-Penrose inverse does not satisfy this requirement because $(D A D^{-1})^+$ does not generally equal $D A^+ D^{-1}$. As a concrete example, given

\[ D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \]  

it can be verified that

\[ A^+ = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{bmatrix} \]  

and that

\[ D A^+ D^{-1} = \begin{bmatrix} 1/2 & 1/4 \\ -1 & -1/2 \end{bmatrix} \]  

which does not equal

\[ (D A D^{-1})^+ = \begin{bmatrix} 0.32 & 0.64 \\ -0.16 & -0.32 \end{bmatrix} \]  

To appreciate the significance of unit consistency, consider the standard linear model

\[ \hat{y} = A \cdot \hat{\theta} \]  

where the objective is to identify a vector $\hat{\theta}$ of parameter values satisfying the above equation for a data matrix $A$ and a known/desired state vector $\hat{y}$. If $A$ is nonsingular then there exists a unique $A^\dagger$ which gives the solution

\[ \hat{\theta} = A^\dagger \cdot \hat{y} \]  

If, however, $A$ is singular then the Moore-Penrose inverse could be applied as

\[ \hat{\theta} = A^+ \cdot \hat{y} \]  

to obtain a solution. Now suppose that $\hat{y}$ and $\hat{\theta}$ are expressed in different units as

\[ \hat{y}' = D \hat{y} \]  
\[ \hat{\theta}' = E \hat{\theta} \]  

where the diagonal matrices $D$ and $E$ represent changes of units, e.g., from imperial to metric, or rate-of-increase in liters-per-hour to a rate-of-decrease in liters-per-minute, or any other multiplicative change of units. Then Eq.(17) can be rewritten...
in the new units as
\[ \hat{y}' = D\hat{y} = (DAE^†) \cdot E\hat{\theta} = (DAE^†) \cdot \hat{\theta}' \]  
(22)

but for which
\[ E\hat{\theta} \neq (DAE^†)^* \cdot \hat{y}' \]  
(23)

In other words, the change of units applied to the input does not generally produce the same output in the new units. This is because the Moore-Penrose inverse only guarantees consistency with respect to unitary transformations and not with respect to nonsingular diagonal transformations. To ensure unit consistency in this example a generalized matrix inverse \( \mathbf{A}^- \) would have to satisfy
\[ (DAE^†)^{\sim} = E\mathbf{A}^- D^{-1} \]  
(24)

Stated more generally, if \( \mathbf{A} \) represents a mapping \( V \rightarrow W \) from a vector space \( V \) to a vector space \( W \) then the inverse transformation \( \mathbf{A}^- \) must preserve consistency with respect to the application of arbitrary changes of units to the coordinates (state variables) associated with \( V \) and \( W \).

Given a singular \( \mathbf{A} \) it is not possible to say that a solution obtained using one definition of a generalized inverse is generally “better” or “worse” than that from any other, but there are certainly application-specific considerations that can govern the choice. As has been discussed, maintenance of unit consistency in all components of a system permits the higher-level integrity of that system to be sanity-checked for unit consistency. An equally valuable practical benefit is that a fully unit-consistent system that has been extensively tuned and validated is guaranteed to produce entirely predictable results if a change of units is applied. By contrast, if multiple modules violate unit consistency in various ways, e.g., through use of the Moore-Penrose inverse, then the effect of a subsequent change of units on system performance may be highly unpredictable. This motivates the derivation of unit-consistent generalized matrix inverses and, more importantly, a general methodology for achieving unit consistency.

III. LEFT AND RIGHT UC GENERALIZED INVERSES

Inverse consistency with respect to a nonsingular left diagonal transformation, \( (DA)^L = A^+D^† \), or a right nonsingular diagonal transformation, \( (AD)^R = D^†A^+ \), is straightforward to obtain. The solution has likely been exploited implicitly in one form or another in many applications over the years; however, its formal derivation and analysis is a useful exercise to establish concepts and notation that will be used later to derive the fully-general UC solution.

**Definition III.1.** Given an \( m \times n \) matrix \( \mathbf{A} \), a left diagonal scale function, \( \mathcal{D}_L[A] \in \mathbb{R}_+^{m \times m} \), is defined as giving a positive diagonal matrix satisfying the following for all conformant positive diagonal matrices \( \mathcal{D}_L \), unitary diagonals \( \mathcal{D}_u \), permutations \( \mathcal{P} \), and unitaries \( \mathcal{U} \):

\[
\mathcal{D}_L[\mathcal{D}_L[A] \cdot (\mathcal{D}_u[A]) = \mathcal{D}_L[A] \cdot \mathcal{D}_L[A],
\]

\[
\mathcal{D}_L[\mathcal{D}_u[A] = \mathcal{D}_L[A],
\]

\[
\mathcal{D}_L[\mathcal{P}A] = \mathcal{P} \cdot \mathcal{D}_L[A],
\]

\[
\mathcal{D}_L[\mathcal{U}A] = \mathcal{D}_L[A]
\]

In other words, the product \( \mathcal{D}_L[A] \cdot \mathbf{A} \) is invariant with respect to any positive left-diagonal scaling of \( \mathbf{A} \), and \( \mathcal{D}_L[A] \) is consistent with respect to any left-permutation of \( \mathbf{A} \) and is invariant with respect to left-multiplication by any diagonal unitary and/or right-multiplication by any general unitary.

**Lemma III.2.** Existence of a left-diagonal scale function according to Definition III.1 is established by instantiating \( \mathcal{D}_L[A] = \mathbf{D} \) with

\[
\mathbf{D}(i,i) = \begin{cases} 
1/\|\mathbf{A}(i,:)\| & \|\mathbf{A}(i,:)\| > 0 \\
1 & \text{otherwise}
\end{cases}
\]

where \( \mathbf{A}(i,:) \) is row \( i \) of \( \mathbf{A} \) and \( \|\cdot\| \) is a fixed unitary-invariant vector norm\(^5\).

**Proof:** \( \mathcal{D}_L[\cdot] \) as defined by Lemma III.2 is a strictly positive diagonal as required, and the left scale-invariance condition of Eq. (25) holds trivially for any row of \( \mathbf{A} \) with all elements equal to zero and holds for every nonzero row \( i \) by homogeneity for any choice of vector norm as

\[
\mathbf{D}_L(i,i)\mathbf{A}(i,:) \parallel \|\mathbf{D}_L(i,i)\mathbf{A}(i,:)\| = \mathbf{D}_L(i,i)\mathbf{A}(i,:) \parallel \|\mathbf{A}(i,:)\|
\]

\[
= \mathbf{A}(i,:) \parallel \|\mathbf{A}(i,:)\|.
\]

\(^5\)The unitary-invariant norm used here is necessary only because of the imposed right-invariant condition of Eq. (28).
The left diagonal-unitary-invariance condition of Eq. (26) is satisfied as $|D_{i,i}| = 1$ implies $|(D_i A)(i,j)| = |A(i,j)|$ for every element $j$ of row $i$ of $D_i A$. The left permutation-invariance of Eq. (27) holds as element $D(i,i)$ is indexed with respect to the rows of $A$, and the right unitary-invariance condition of Eq. (28) is satisfied by the assumed unitary invariance of the vector norm applied to the rows of $A$.

If $A$ has full support, i.e., no row or column with all elements equal to zero, then $D_i [D_i A] = D_i^{-1} \cdot D_i [A]$. If, however, there exists a row $i$ of $A$ with all elements equal to zero then the $i$th diagonal element of $D_i [D_i A]$ is 1 according to Lemma III.2, so the corresponding element of $D_i^{-1} \cdot D_i [A]$ will be different unless $D_i^{-1}(i,i) = 1$. Eq. (25) holds because such elements are only applied to scale rows of $A$ with all elements equal to zero. The following similarly holds in general

$$D_i [D_i A] \cdot A = D_i^{-1} \cdot D_i [A] \cdot A,$$

and because any row $i$ of zeros in $A$ implies that column $i$ of $A^*$ will be zeros, the following also holds in general

$$A^* \cdot D_i [D_i A] = A^* \cdot D_i^{-1} \cdot D_i [A].$$

At this point it is possible to derive a left generalized inverse of an arbitrary $m \times n$ matrix $A$, denoted $A^\perp$, that is consistent with respect to multiplication on the left by an arbitrary nonsingular diagonal matrix.

**Theorem III.3.** For $m \times n$ matrix $A$, the operator

$$A^\perp = (D_i [A] \cdot A)^* \cdot D_i [A]$$

satisfies for any nonsingular diagonal matrix $D$:

$$AA^\perp A = A,$$

$$A^\perp AA^\perp = A^\perp,$$

$$\langle DA \rangle^\perp = A^\perp D^*,$$

$$\text{rank}[A^\perp] = \text{rank}[A]$$

and is therefore a left unit-consistent generalized inverse.

**Proof:** The first two generalized inverse properties can be established from the corresponding properties of the Moore-Penrose inverse as:

$$AA^\perp = A \cdot \{(D_i [A] \cdot A)^* \cdot D_i [A]\} \cdot A$$

$$= A \cdot \{(D_i [A] \cdot A)^* \cdot (D_i [A] \cdot A)\}$$

$$= (D_i [A]^* \cdot D_i [A]) \cdot A \cdot \{(D_i [A] \cdot A)^* \cdot (D_i [A] \cdot A)\}$$

$$= D_i [A]^* \cdot \{(D_i [A] \cdot A)^* \cdot (D_i [A] \cdot A)\}$$

$$= D_i [A]^* \cdot (D_i [A] \cdot A)$$

$$= A$$

and

$$A^\perp AA^\perp = \{(D_i [A] \cdot A)^* \cdot D_i [A]\} \cdot A \cdot \{(D_i [A]^* A)^* \cdot D_i [A]\}$$

$$= \{(D_i [A] \cdot A)^* \cdot (D_i [A] \cdot A) \cdot (D_i [A] \cdot A)^*\} \cdot D_i [A]$$

$$= (D_i [A] \cdot A)^* \cdot D_i [A]$$

$$= A^\perp$$

The left unit-consistency condition $\langle DA \rangle^\perp = A^\perp D^*$, for any nonsingular diagonal matrix $D$, can be established using a polar decomposition $D = D_u D_i$:

$$D_u = \text{Abs}[D]$$

$$D_i = DD_i^{-1}$$

and exploiting unitary-consistency of the Moore-Penrose inverse, i.e., $(UA)^* = A^* U^*$, and commutativity of $D_i [\cdot]$ with other diagonal matrices.
(DA)⁺ = (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (51)
= (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (52)
= (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (53)
= (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (54)
= (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (55)
= (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (56)
= (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (57)
= (D⁺D⁻¹[DA]D⁻¹[DA])⁺ (58)
= \{ (D⁺D⁻¹[DA]D⁻¹[DA])⁺ \} (59)
= A⁺D⁻¹ (60)

Lastly, the rank-consistency condition, rank[A⁺] = rank[A], is satisfied as every operation performed according to Lemma III.2 preserves the rank of the original matrix. In particular, the rank consistency of A⁺ derives from the fact that rank[A⁺] = rank[A].

A right unit-consistent generalized inverse clearly can be derived analogously or in terms of the already-defined left operator as

A⁺ = ((A⁺)⁺)ᵀ (61)

In terms of the linear model of Eq.(17) for determining values for parameters ̂θ,

\[ \hat{y} = A \cdot \hat{\theta} \]

\[ \downarrow \]

\[ \hat{\theta} = A⁻¹ \cdot \hat{y} \]

the inverse A⁻¹ could be instantiated with either A⁺ or A⁻¹ to provide, respectively, consistency with respect to the application of a change of units to ̂y or a change of units to ̂θ – but not both.

**IV. UC Generalized Inverse for Elemental-Nonzero Matrices**

The derivations of separate left and right UC inverses from the previous section cannot be applied to achieve general unit consistency, i.e., to obtain a UC generalized inverse A⁻¹ which satisfies

\[ (DAE)⁻¹ = E⁻¹A⁻¹D⁻¹ \] (62)

for arbitrary nonsingular diagonals D and E. However, a joint characterization of the left and right diagonal transformations can provide a basis for doing so.

**Lemma IV.1.** The transformation of an m×m matrix A as DAE, with m×m diagonal D and n×n diagonal E, is equivalent to a Hadamard (elementwise) matrix product X ⊙ A for some rank-1 matrix X.

**Proof:** Letting dₘ = Diag[D] and eₙ = Diag[E], the matrix product DAE can be expressed as

\[ DAE = (dₘIₙ) ⊙ A ⊙ (Iₘeₙᵀ) \] (63)

\[ = \{(dₘIₙ) ⊙ (Iₘeₙᵀ)\} ⊙ A \] (64)

\[ = (dₘeₙᵀ) ⊙ A \] (65)

where Iₙ is a row vector of n ones and Iₘ is a column vector of m ones. Letting X = dₘeₙᵀ completes the proof. ■

**Definition IV.2.** For an m×n matrix A, left and right general-diagonal scale functions Dₚₜ[D] ∈ Rᵐˣᵐ and Dₚₜ[A] ∈ Rⁿˣⁿ are defined as jointly satisfying the following for all conformant positive diagonal matrices D, and E, unitary diagonals D⁺ and D⁻, and permutations P and Q:

\[ Dₚₜ'[D,AEₚₜ] ⊙ (Dₚₜ'[DA,Eₚₜ]) = Dₚₜ'[A] ⊙ Dₚₜ'[A] \] (66)

\[ Dₚₜ'[PAQ] ⊙ (Dₚₜ'[PAQ]) = P ⊙ (Dₚₜ'[A] ⊙ Dₚₜ'[A]) ⊙ Q \] (67)

\[ Dₚₜ'[D,ADₚₜ] = Dₚₜ'[A] \] (68)

\[ Dₚₜ'[D,ADₚₜ] = Dₚₜ'[A] \] (69)
The function $S_u[A]$ is defined to be the rank-1 matrix guaranteed by Lemma IV.1

$$S_u[A] \circ A = D_u^+ [A] \cdot A \cdot D_u^0 [A]$$ (70)

i.e.,

$$S_u[A] = \text{Diag}[D_u^0 [A]] \cdot \text{Diag}[D_u^0 [A]]^T$$ (71)

**Definition IV.3.** A matrix $A$ is defined to be an *elemental-nonzero matrix* if and only if it does not have any element equal to zero.

The following lemma uses the elementwise matrix functions LogAbs$[\cdot]$ and Exp$[\cdot]$, where LogAbs$[A]$ represents the result of taking the logarithm of the magnitude of each element of $A$ and Exp$[A]$ represents the taking of the exponential of every element of $A$.

**Lemma IV.4.** Existence of a general diagonal scale function according to Definition IV.2 for arguments without zero elements is established by instantiating $D_u^0 [A]$ and $D_u^+ [A]$ as

$$D_u^0 [A] = \text{Diag}[x_m]$$ (72)

$$D_u^+ [A] = \text{Diag}[y_n]$$ (73)

for

$$x_m \cdot y_n^T = S_u[A] = \text{Exp} [J_m L J_n - L J_n - J_m L]$$ (74)

where $L = \text{LogAbs}[A]$ and $J_m$ has all elements equal to $1/m$ and $J_n$ has all elements equal to $1/n$.

**Proof:** First it must be shown that $\text{Exp} [J_m L J_n - L J_n - J_m L]$ is a rank-1 matrix. This can be achieved by expanding as

$$J_m L J_n - L J_n - J_m L = \left(\frac{1}{2} J_m L J_n - L J_n\right) + \left(\frac{1}{2} J_m L J_n - J_m L\right)$$ (75)

$$= \left(\frac{1}{2} J_m L - L\right)J_n + J_m\left(\frac{1}{2} L J_n - L\right)$$ (76)

$$= u_m 1_n^T + 1_m v_n^T$$ (77)

where

$$u_m = \frac{1}{n} \left(\frac{1}{2} J_m - I_m\right) L \cdot 1_n$$ (78)

$$v_n^T = 1_m^T \cdot L \left(\frac{1}{2} J_n - I_n\right) / m$$ (79)

and then noting that the elementwise exponential of $u_m 1_n^T + 1_m v_n^T$ is the strictly positive rank-1 matrix $\text{Exp}[u_m] \cdot \text{Exp}[v_n^T]$, i.e., $D_u[A] = \text{Diag}[\text{Exp}[u_m]]$ and $D_u^+ [A] = \text{Diag}[\text{Exp}[v_n]]$, which confirms existence and strict positivity as required. Eq. (66) can be established by observing that

$$D_u^0 [D A E] \cdot (D A E) \cdot D_u^+ [D A E] = S_u[D A E] \circ (D A E)$$ (80)

and letting $d_m = \text{Diag}[D]$, $e_n = \text{Diag}[E]$, $u_m = \text{LogAbs}[d_m]$, $v_n = \text{LogAbs}[e_n]$, and $L = \text{LogAbs}[A]$:

$$S_u[D A E] = S_u[\left(\text{d}_m 1_n^T\right) \circ A \circ (1_m e_n^T)]$$ (81)

$$= S_u[\text{Exp}[u_m 1_n^T] \circ A \circ \text{Exp}[1_m v_n^T]]$$ (82)

$$= \text{Exp} \left(J_m (u_m 1_n^T + L + 1_m v_n^T) J_n\right)$$ (83)

$$- (u_m 1_n^T + L + 1_m v_n^T) J_n$$ (84)

$$= \text{Exp} \left(\left(J_m \cdot u_m 1_n^T + J_m L J_n + 1_m v_n^T \cdot J_n\right)ight)$$ (86)

$$= \text{Exp} \left(-u_m 1_n^T + J_m L J_n + 1_m v_n^T \cdot J_n\right)$$ (87)

$$= \text{Exp} \left(-u_m 1_n^T \circ S_u[A] \circ \text{Exp}[-1_m v_n^T]\right)$$ (89)

$$= \text{D}_+^{-1} \cdot S_u[A] \cdot \text{E}_+^{-1}$$ (91)

where the last step recognizes that $-u_m = \text{LogAbs}[\text{Diag}[D_+^{-1}]]$ and $-v_n = \text{LogAbs}[\text{Diag}[E_+^{-1}]].$ The identity of Eq. (66) can
then be shown as:

\[ D^e[A, A E_4] \cdot (D, A E_4) \cdot D^e[A, A E_4] = S_0[D, A E_4] \cdot (D, A E_4) \]

(92)

\[ = (D_+^{-1} \cdot S_0[A] \cdot E_+^{-1}) \cdot (D, A E_4) \]

(93)

\[ = S_0[A] \cdot A \]

(94)

\[ = D^e[A] \cdot A \cdot D^e[A] \]

(95)

Eq. (67) holds as the indexing of the rows and columns of \( D^e[A] \) and \( D^e[A] \) (and \( S_0[A] \)) is the same as that of \( A \). Eqs. (68) and (69) hold directly because Lemma IV.4 only involves functions of the absolute values of the elements of the argument matrix \( A \).

Theorem IV.5. For an elemental-nonzero \( m \times n \) matrix \( A \), the operator

\[ A^u = D^e[A] \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot D^e[A] \]

(96)

satisfies for any nonsingular diagonal matrices \( D \) and \( E \):

\[ AA^u A = A, \]

(97)

\[ A^u A A^u = A^u, \]

(98)

\[ (D A E)^u = E^1 A^u D^4, \]

(99)

\[ \text{rank}[A^u] = \text{rank}[A] \]

(100)

and is therefore a general unit-consistent generalized inverse.

**Proof:** The first two generalized inverse properties can be established from the corresponding properties of the MP-inverse as:

\[ AA^u A = A \cdot \{ D^e[A] \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot D^e[A] \} \cdot A \]

(101)

\[ = (D^e[A])^\delta \cdot D^e[A] \cdot A \cdot \{ D^e[A] \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot D^e[A] \} \cdot A \]

(102)

\[ = (D^e[A])^\delta \cdot D^e[A] \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot (D^e[A]) \cdot A \cdot D^e[A] \]

(103)

\[ = (D^e[A])^\delta \cdot D^e[A] \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot D^e[A] \]

(104)

\[ = A \]

(105)

and

\[ A^u A A^u = \{ D^e[A] \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot D^e[A] \} \cdot A \]

(110)

\[ = D^e[A] \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot D^e[A] \]

(111)

\[ = (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot (D^e[A] \cdot A \cdot D^e[A]) \cdot (D^e[A] \cdot A \cdot D^e[A])^\delta \cdot D^e[A] \]

(112)

\[ = A^u \]

(113)

The general UC condition \((D A E)^u = E^1 A^u D^4\), for any nonsingular diagonal matrix \( D \), can be established using a polar...
decompositions $D = D_u D_a$ and $E = E_u E_a$:

\[
(DAE)^{\omega} = D_u^\omega [DAE] \cdot (D_u^\omega [DAE] \cdot (DAE) \cdot D_u^\omega [DAE])^s \cdot D_u^\omega [DAE]
\]

(117)

\[
= D_u^\omega [DAE] \cdot (D_u^{\omega-1} [DAE] \cdot (DAE) \cdot E_u^{-1} \cdot D_u^\omega [DAE])^s \cdot D_u^\omega [DAE]
\]

(118)

\[
= E_u^{-1} \cdot D_u^\omega [DAE] \cdot (D_u^{\omega-1} [DAE] \cdot (DAE) \cdot E_u^{-1} \cdot D_u^\omega [DAE])^s \cdot D_u^\omega [DAE]
\]

(119)

\[
= E_u^{-1} \cdot D_u^\omega [DAE] \cdot (D_u^{\omega-1} [DAE] \cdot (DAE) \cdot E_u^{-1} \cdot D_u^\omega [DAE])^s \cdot D_u^\omega [DAE]
\]

(120)

\[
= (E_u^{-1} \cdot D_u^\omega [DAE] \cdot (D_u^{\omega-1} [DAE] \cdot (DAE) \cdot E_u^{-1} \cdot D_u^\omega [DAE])^s \cdot D_u^\omega [DAE] \cdot D_u^{\omega-1}
\]

(121)

\[
= E_u^{-1} \cdot D_u^\omega [DAE] \cdot (D_u^{\omega-1} [DAE] \cdot (DAE) \cdot E_u^{-1} \cdot D_u^\omega [DAE] \cdot D_u^{\omega-1}
\]

(122)

\[
= E_u^{-1} \cdot D_u^\omega [DAE] \cdot (D_u^{\omega-1} [DAE] \cdot (DAE) \cdot E_u^{-1} \cdot D_u^\omega [DAE] \cdot D_u^{\omega-1}
\]

(123)

\[
= E_u^{-1} \cdot D_u^\omega [DAE] \cdot (D_u^{\omega-1} [DAE] \cdot (DAE) \cdot E_u^{-1} \cdot D_u^\omega [DAE] \cdot D_u^{\omega-1}
\]

(124)

The rank-consistency condition of the theorem holds exactly as for the proof of Theorem III.3.

The elemental-nonzero condition of Lemma [IV.4] is required to ensure the existence of the elemental logarithms for $L = \log \text{Abs}[A]$, so the closed-form solution for the general unit-consistent matrix inverse of Theorem [IV.5] is applicable only to matrices without zero elements. In many contexts involving general matrices there is no reason to expect any elements to be identically zero, but in some applications, e.g., compressive sensing, zeros are structurally enforced. Unfortunately, Lemma [IV.4] cannot be extended to accommodate zeros by a simple limiting strategy; however, results from matrix scaling theory can be applied to derive an unrestricted solution.

V. THE FULLY-GENERAL UNIT-CONSISTENT GENERALIZED INVERSE

Given a nonnegative matrix $A \in \mathbb{R}^{m \times n}$ with full support, $m$ positive numbers $S_1 \ldots S_m$, and $n$ positive numbers $T_1 \ldots T_n$, Rothblum & Zenios \cite{24} investigated the problem of identifying positive diagonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that the product of the nonzero elements of each row $i$ of $A' = UAV$ is $S_i$ and the product of the nonzero elements of each column $j$ of $A'$ is $T_j$. They provided an efficient solution, referred to in their paper as Program II, and analyzed its properties. Specifically, for vectors $\mu \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$, defined in their paper, they proved the following:

**Theorem V.1.** (Rothblum & Zenios) \cite{24} The following are equivalent:

1. Program II is feasible.
2. Program II has an optimal solution.
3. $\prod_{i=1}^m (S_i)^{\mu_i} = \prod_{j=1}^n (T_j)^{\eta_j}$.

If a solution exists then the matrix $A' = UAV$ is the unique positive diagonal scaling of $A$ for which the product of the nonzero elements of each row $i$ is $S_i$ and the product of the nonzero elements of each column $j$ is $T_j$.

This implies the following theorem.

**Theorem V.2.** For any nonnegative matrix $X \in \mathbb{R}^{m \times n}$ with full support there exist positive diagonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that the product of the nonzero elements of each row $i$ and column $j$ of $X' = U XV$ is 1, and $X'$ is the unique positive diagonal scaling of $X$ which has this property.

**Proof:** The existence (and uniqueness) of a solution for Program II according to Theorem [V.1] is equivalent to

\[
\prod_{i=1}^m (S_i)^{\mu_i} = \prod_{j=1}^n (T_j)^{\eta_j}
\]

(125)

which holds unconditionally, i.e., independent of $\mu$ and $\eta$, for the case in which every $S_i$ and $T_j$ is 1.

**Lemma V.3.** Given an $m \times n$ matrix $A$, let $X$ be the matrix formed by removing every row and column of $\text{Abs}[A]$ for which all elements are equal to zero, and define $r[i]$ to be the row of $X$ corresponding to row $i$ of $A$ and $c[j]$ to be the column of $X$ corresponding to column $j$ of $A$. Let $U$ and $V$ be the diagonal matrices guaranteed to exist from the application of Program II to $X$ according to Theorem [V.2]. Existence of a general-diagonal scale function according to Definition [IV.2] for $A^6$.

\* As will be seen, the precise definitions of $\mu$ and $\eta$ will prove irrelevant for purposes of this paper.

\* This theorem combines results from theorems 4.2 and 4.3 of \cite{24}. The variables $U$ and $V$ are used here for positive diagonal matrices purely for consistency with that paper despite their exclusive use elsewhere in this paper to refer to unitary matrices. Although not stated explicitly by Rothblum and Zenios, Program II is easily verified to be permutation consistent.
is provided by instantiating $D_u^r[A] = D$ and $D_v^r[A] = E$ where

$$D(i, i) = \begin{cases} U(r[i], r[i]) & \text{row } i \text{ of } A \text{ is not zero} \\ 1 & \text{otherwise} \end{cases} \quad (126)$$

$$E(j, j) = \begin{cases} V(c[j], c[j]) & \text{column } j \text{ of } A \text{ is not zero} \\ 1 & \text{otherwise} \end{cases} \quad (127)$$

**Proof:** In the case that $A$ has full support so that $X = \text{Abs}[A]$ then Theorem [V.1] guarantees that $D_v^u[X] \cdot X \cdot D_v^u[X]$ is the unique diagonal scaling of $X$ such that the product of the nonzero elements of each row and column is 1. Therefore, the scale-invariance condition of Eq. (66)

$$D_v^u[D, XE] \cdot (D, XE) \cdot D_v^u[D, XE] = D_v^u[X] \cdot X \cdot D_v^u[X] \quad (128)$$

holds for any positive diagonals $D$, and $E$, as required. For the case of general $A$ the construction defined by Lemma [V.3] preserves uniqueness with respect to nonzero rows and columns of $D_v^u[A] \cdot \text{Abs}[A] \cdot D_v^u[A]$, i.e., those which correspond to the rows and columns of $UXV$, by the guarantee of Theorem [V.1] and any row or column with all elements equal to zero is inherently scale-invariant, so Eq. (66) holds unconditionally for the construction defined by Lemma [V.3] The remaining conditions (permutation consistency and invariance with respect to unitary diagonals) hold equivalently to the proof of Lemma [V.4].

At this point it is possible to establish the existence of a fully-general, unit-consistent, generalized matrix inverse.

**Theorem V.4.** For an $m \times n$ matrix $A$ there exists an operator

$$A^u = D_v^u[A] \cdot (D_v^u[A] \cdot A \cdot D_v^u[A])^* \cdot D_v^u[A] \quad (129)$$

which satisfies for any nonsingular diagonal matrices $D$ and $E$:

$$AA^u A = A, \quad (130)$$

$$A^u AA^u = A^u, \quad (131)$$

$$(DAE)^u = E^t A^u D^t, \quad (132)$$

$$\text{rank}[A^u] = \text{rank}[A]. \quad (133)$$

**Proof:** The proof of Theorem [IV.5] applies unchanged to Theorem [V.4] except that the elemental-nonzero condition imposed by Lemma [IV.4] is removed by use of Lemma [V.3].

For completeness, the example of Eqs. (131) with

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \quad (134)$$

can be revisited to verify that

$$(DAD^*)^u = DA^u D^t = \begin{bmatrix} 1/2 & 1/4 \\ -1/2 & -1/2 \end{bmatrix} \quad (135)$$

where $(DAD^*)^u = DA^u D^t$ as expected. Extending the example with

$$E = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix} \quad (136)$$

it can be verified that

$$(DAE)^u = E^t A^u D^t = \begin{bmatrix} 1/10 & 1/20 \\ 1/6 & 1/12 \end{bmatrix} \quad (137)$$

with equality as expected.

The generalized inverse of Theorem [V.4] is unique when instantiated using the construction defined by Lemma [V.3] by virtue of the uniqueness of both the Moore-Penrose inverse and the scaling of Theorem [V.1]. However, other unit-consistent inverses may be defined using alternative inverse and scaling operators.\footnote{Additional details are discussed in Appendix A} What is most important for present purposes is that the approach for obtaining unit-consistent generalized inverses can be efficiently applied to achieve similar results for a wide variety of other matrix operators and decompositions. Some specific examples are briefly considered in the following section.

**VI. UC Example: Estimating Linearized Approximations to Nonlinear Transformations**

A fundamental problem in applications ranging from system identification to machine learning is to ascertain an estimate of an unknown nonlinear transformation based on a given sample set of input and output vectors \[18, 19, 22\]. More specifically,
given an \( m \times n \) matrix \( X \) with columns representing a domain set of \( n \) \( m \)-dimensional input vectors, and a \( q \times n \) matrix \( Y \) with columns representing the range set of \( p \)-dimensional output vectors, i.e., each column \( Y_i \) gives \( f(X_i) \) for the unknown transformation \( f(\cdot) \), determine a \( q \times m \) matrix \( F \) that linearly approximates the unknown nonlinear mapping as:

\[
FX \approx Y.
\] (138)

If \( X \) and \( Y \) are both square matrices of full rank then the unique solution \( F = YX^{-1} \) is entirely structurally determined, i.e., there are insufficient samples to reveal any information beyond a simple linear mapping. As the number of samples increases, i.e., as \( n \) becomes increasingly larger than \( m \), information about the unknown nonlinear mapping from \( X \) to \( Y \) becomes available but a unique \( X^* \) no longer exists.

Use of the Moore-Penrose inverse provides a solution \( F_r = YX^* \), and its estimation/prediction properties can be analyzed in terms of the consistency conditions that it preserves. Specifically, \( F_r \) is invariant with respect to the application of a unitary transformation \( U \) to the rows of \( X \) and \( Y \) as \( XU \) and \( YU \):

\[
F_r = (YU)(XU)^+ = YUU^*X^* = YX^* \quad \text{(139)}
\]

and therefore preserves right-unitary consistency:

\[
F_r(XU) = (F_rX)U. \quad \text{(142)}
\]

This consistency with respect to arbitrary unitary linear combinations of the input vectors (i.e., the columns of \( X \)) implies that \( F_r \) is insensitive to functional dependencies which exist purely between corresponding columns of \( X \) and \( Y \). By contrast, the solution \( F_s = YX^s \) obtained using a right unit-consistent inverse is invariant with respect to an arbitrary nonzero scaling of individual columns of \( X \) in the form of a diagonal matrix \( D \):

\[
F_s = (YD)(XD)^* = YDD^*X^* = YX^s. \quad \text{(146)}
\]

and therefore preserves right-diagonal unit consistency:

\[
F_s(XD) = (F_sX)D. \quad \text{(146)}
\]

Diagonal consistency does not enforce consistency with respect to a linear mixing of the input vectors and is therefore strongly sensitive to the functional relationship defined by the \( X_i \) and \( Y_i \) pairs. This sensitivity can be demonstrated empirically by applying a nonlinear transformation to the columns of a randomly generated \( m \times n \) matrix \( X \) to produce a matrix \( Y \) and determining the percentage of predictions \( F_sX_i \) that satisfy

\[
\|F_sX_i - Y_i\| < \|F_rX_i - Y_i\| \quad \text{(147)}
\]

for a given vector norm \( \|\cdot\| \). Figure 1 shows averaged results for a series of such tests. Each test consists of a randomly-generated nonlinear transformation applied to the columns of an \( m \times n \) matrix \( X \) of univariate-normal i.i.d-sampled elements. The transformation for each test is a degree-\( m \) polynomial function of the elements of \( X_i \) with univariate-normal i.i.d-sampled term coefficients. The percentage of vectors \( X_i \) satisfying \( \|F_sX_i - Y_i\| < \|F_rX_i - Y_i\| \) is then determined. The results show that \( F_s \) provides more accurate estimates than \( F_r \), as the dimensionality \( m \) increases, i.e., as more information about the nonlinear function becomes available.

![3D Nonlinear Mapping](image) ![5D Nonlinear Mapping](image) ![7D Nonlinear Mapping](image)

Fig. 1. Relative predictive accuracy (\( \ell_2 \) norm) of \( F_s \) versus \( F_r \) for random nonlinear polynomial transformations of \( m \)-dimensional vectors for \( m = 3 \), 5, and 7. The percentage (vertical axis) of \( F_s \) estimates with error smaller than their corresponding \( F_r \) estimates increases with the number of samples \( n \) (horizontal axis).
Figure 2 shows that for sufficiently large $n$ the predictive superiority of $F_R$ over $F_P$ as shown in Figure 1 does not depend strongly on the particular choice of vector norm.

Although the Moore-Penrose inverse is commonly associated with least-squares error minimization, Figure 2 shows that $F_R$ approaches uniform superiority even according to the $\ell_2$ vector norm. This is true because superiority is assessed here based on per-vector error rather than an average per-element error over all predictions. It must be emphasized, however, that the motivation for this example is not to provide a particular solution to a particular optimization problem but rather to show how consistency analysis can illuminate salient properties of a given problem.

VII. Unit-Consistent/Invariant Matrix Decompositions

The motivation to investigate unit consistency in the context of generalized matrix inverses extends also to other areas of matrix analysis. This clearly includes transformations $T[A]$ which can be redefined in UC form as

$$\mathcal{D}_c[A]^\dagger \cdot T[A \cdot \mathcal{D}_c[A] \cdot \mathcal{D}_c[A]^\dagger]$$

and functions $f[A]$ which can be redefined in unit scale-invariant form as $f[\mathcal{D}_c[A] \cdot A \cdot \mathcal{D}_c[A]]$, but it also extends to matrix decompositions.

The Singular Value Decomposition (SVD) is among the most powerful and versatile tools in linear algebra and data analytics [31], [20], [15], [1]. The Moore-Penrose generalized inverse of $A$ can be obtained from the SVD of $A$

$$A = USV^*$$

as

$$A^* = VS^{-1}U^*$$

where $U$ and $V$ are unitary, $S$ is the diagonal matrix of singular values of $A$, and $S^{-1}$ is the matrix obtained from inverting the nonzero elements of $S$. This motivates the following definition.

Definition VII.1. The Unit-Invariant Singular-Value Decomposition (UI-SVD) is defined as

$$A = D \cdot USV^* \cdot E$$

with $D = \mathcal{D}_c[A]^\dagger$, $E = \mathcal{D}_c[A]$, and $USV^*$ is the SVD of $X = \mathcal{D}_c[A] \cdot A \cdot \mathcal{D}_c[A]$. The diagonal elements of $S$ are referred to as the unit-invariant (UI) singular values of $A$.

Given the UI-SVD of a matrix $A$

$$A = D \cdot USV^* \cdot E$$

the UC generalized inverse of $A$ can be expressed as

$$A^g = E^* \cdot VS^{-1}U^* \cdot D^*.$$  

Unlike the singular values of $A$, which are invariant with respect to arbitrary left and right unitary transformations of $A$, the UI singular values are invariant with respect to arbitrary left and right nonsingular diagonal transformations. Thus, functions of the unit-invariant singular values are unit-invariant with respect to $A$.

The largest $k$ singular values of a matrix (e.g., representing a photograph, video sequence, or other object of interest) can be used to define a unitary-invariant signature [7], [17], [12], [16], [14], [13] which supports computationally efficient similarity

A left UI-SVD can be similarly defined as $A = D \cdot USV^*$ with $D = \mathcal{D}_c[A]^\dagger$ and $USV^*$ being the SVD of $X = \mathcal{D}_c[A] \cdot A$. The resulting left unit-invariant singular values are invariant with respect to nonsingular left diagonal transformations and right unitary transformations of $A$ (the latter property is what motivated the right unitary-invariance requirement of Definition VII.1 and the consequent use of a unitary-invariant norm in the construction of Lemma III.2). A right UI-SVD can be defined analogously to interchange the left and right invariants.
testing. However, many sources of error in practical applications are not unitary. As a concrete example, consider a system in which a passport or driving license is scanned to produce a rectilinearly-aligned and scaled image that is to be used as a key to search an existing image database. The signature formed from the largest $k$ unit-invariant singular values can be used for this purpose to provide robustness to amplitude variations among the rows and/or columns of the image due to the scanning process (details are provided in the example of Section VIII).

The UI-SVD may also offer advantages as an alternative to the conventional SVD, or truncated SVD, used by existing methods for image and signal processing, cryptography, digital watermarking, tomography, and other applications in order to provide state-space or coordinate-aligned robustness to noise.

More generally, the approach used to define unit scale-invariant singular values can be applied to other matrix decompositions, though the invariance properties may be different. In the case of scale-invariant eigenvalues for square $A$, i.e., $\text{eig}[D_{\infty}^5[A] \cdot A \cdot D_{\infty}^5[A]]$, the invariance is limited to diagonal transformations $\text{DAE}$ such that $\text{DE}$ is nonnegative real, e.g., $D_5 A E_5$, $D_5 A D_5$ (or $D_5 A D_5$ for complex $D$), and $D_5 A D_5$. In applications in which changes of units can be assumed to take the form of positive diagonal transformations, the scale-invariant (SI) eigenvalues can therefore be taken as a complementary or alternative signature to that of provided by the UI singular values.

VIII. UI EXAMPLE: SCANNED-KEY IMAGE RETRIEVAL

Most flatbed image scanners produce a digital copy of an image by mechanically moving a linear light source, e.g., a flourescent tube, horizontally across a given document/image while continuously recording the amplitude of the reflected light using a CDD. The fidelity of such a scanner can degrade over time as oil residue and dust accumulate along the surface of the light source and as mechanical components begin to wear. In particular, variations in illumination along the length of the light source produce horizontal amplitude artifacts in the resulting image while variations in the distance of the light source to the document due to non-smooth motion during the scanning process produces vertical amplitude artifacts.

Robustness to row and column amplitude perturbations i.e., multiplicative noise, may be needed for reliable image retrieval when scanned images are to be used as keys for searching an image database. One mechanism for mitigating such artifacts is to associate a signature with every image in the form of a vector of its largest $k$ normalized sorted singular values (NSVs). If $k = 5$, for example, the NSV signature would be a vector of length 5 consisting of the first five singular values. The vector is normalized to have unit magnitude so that the signature is invariant with respect to a uniform scaling of pixel intensities (amplitudes), which is necessary to accommodate global amplitude differences among different scanners. This normalization also provides robustness with respect to image decimation and super-resolution, i.e., the NSV of a given image will tend to be similar to the NSV of the same image scaled to a lower or higher resolution.

Because NSV keys are nonnegative vectors, they can be compared using the angular distance metric:

$$\text{adist } [p, q] = \frac{1}{\pi} \cos^{-1} \left( \frac{p \cdot q}{\|p\| \|q\|} \right)$$

which simplifies in the present context to

$$\text{adist } [p, q] = \frac{1}{\pi} \cos^{-1} |p \cdot q|$$

because NSV vectors are defined to have unit magnitude. The use of a metric comparison function is needed to permit efficient retrieval when the image database is implemented using a metric search structure.

Figure 3 shows source images: LENA, RUSSELL, and JARRY, followed by their respective query images: $\sim$LENA, $\sim$RUSSELL, and $\sim$JARRY, which have been corrupted with the same set of horizontal and vertical variations in amplitude. Letting their respective 5-element NSV signatures be denoted as $L_{NSV}$, $R_{NSV}$, $J_{NSV}$, $\bar{L}_{NSV}$, $\bar{R}_{NSV}$, and $\bar{J}_{NSV}$, the effect of the scanning artifacts can be quantified as:

$$\text{adist } [L_{NSV}, \bar{L}_{NSV}] = 0.009$$
$$\text{adist } [R_{NSV}, \bar{R}_{NSV}] = 0.005$$
$$\text{adist } [J_{NSV}, \bar{J}_{NSV}] = 0.003$$

where a distance of zero would imply that the keys are identical and a distance of 1 would imply that they are orthogonal. Unit-invariant singular values are invariant with respect to the scaling of rows and columns of a matrix and therefore should be invariant with respect to the scanning artifacts depicted in Figure 3. This motivates an alternative to NSVs in which singular values are replaced with UI singular values. As should be expected, the resulting UNSV keys are significantly less sensitive

\[\text{In applications in which the signature provided by the set of UI singular values may be too concise, e.g., because it is permutation invariant, a decimation of the matrix } X = D_5^5[A] \cdot A \cdot D_5^5[A] \text{ can be used as a more discriminating signature.}\]
Fig. 3. The first three images (left to right) represent high-quality original source images, e.g., as might be stored in an image database, followed by versions of each that have been corrupted with horizontal and vertical scaling artifacts. The three corrupted images were subjected to the same horizontal and vertical scaling process, e.g., as might be expected if obtained from the same poor-quality flatbed scanner.

to amplitude artifacts and are not identically zero only because of noise due to 8-bit discretization of pixel values:

\[
\text{adist}[\hat{L}_{\text{UNSV}}, \tilde{L}_{\text{UNSV}}] = 1.48 \times 10^{-5} \tag{159}
\]

\[
\text{adist}[\hat{R}_{\text{UNSV}}, \tilde{R}_{\text{UNSV}}] = 3.41 \times 10^{-5} \tag{160}
\]

\[
\text{adist}[\hat{J}_{\text{UNSV}}, \tilde{J}_{\text{UNSV}}] = 2.73 \times 10^{-4}. \tag{161}
\]

However, the effectiveness of a signature must be assessed in terms of both specificity and discrimination. In this example the latter can be seen by examining the extent to which each source image can be distinguished from signatures derived from a different image. The following shows the distance of the NSV signature for each source image to the NSV signatures obtained from scans of different (i.e., non-matching) images:

\[
\text{adist}[L_{\text{NSV}}, \tilde{R}_{\text{NSV}}] = 0.021 \tag{162}
\]

\[
\text{adist}[L_{\text{NSV}}, \tilde{J}_{\text{NSV}}] = 0.048 \tag{163}
\]

\[
\text{adist}[R_{\text{NSV}}, \tilde{L}_{\text{NSV}}] = 0.020 \tag{164}
\]

\[
\text{adist}[R_{\text{NSV}}, \tilde{J}_{\text{NSV}}] = 0.038 \tag{165}
\]

\[
\text{adist}[J_{\text{NSV}}, \tilde{L}_{\text{NSV}}] = 0.048 \tag{166}
\]

\[
\text{adist}[J_{\text{NSV}}, \tilde{R}_{\text{NSV}}] = 0.036 \tag{167}
\]

as compared to those obtained from UNSV signatures:

\[
\text{adist}[L_{\text{UNSV}}, \tilde{R}_{\text{UNSV}}] = 0.080 \tag{168}
\]

\[
\text{adist}[L_{\text{UNSV}}, \tilde{J}_{\text{UNSV}}] = 0.171 \tag{169}
\]

\[
\text{adist}[R_{\text{UNSV}}, \tilde{L}_{\text{UNSV}}] = 0.080 \tag{170}
\]

\[
\text{adist}[R_{\text{UNSV}}, \tilde{J}_{\text{UNSV}}] = 0.108 \tag{171}
\]

\[
\text{adist}[J_{\text{UNSV}}, \tilde{L}_{\text{UNSV}}] = 0.171 \tag{172}
\]

\[
\text{adist}[J_{\text{UNSV}}, \tilde{R}_{\text{UNSV}}] = 0.109. \tag{173}
\]

The UNSV signatures provide much larger differences in distance between correct and incorrect pairings than NSV signatures. For example, the UNSV signatures are much more effective at distinguishing the image of British mathematician Bertrand Russell from that of French absurdist Alfred Jarry. These differences are critical because the choice of tolerance \(\epsilon\) for searching the database must also accommodate image perturbations other than amplitude variations. Thus, if images are not sufficiently distinguished the retrieval process will return a large fraction of the images in the database as being potential matches.

The artifacts in this example impact the effectiveness of NSV signatures in terms of both specificity and discrimination. In the case of specificity, the distance between the source image LENA and its scanned counterpart, \(\sim\)LENA, is relatively large:

\[
\text{adist}[L_{\text{NSV}}, \tilde{L}_{\text{NSV}}] = 0.009 \tag{174}
\]

and this is due entirely to the presence amplitude deviations. Because these same artifacts are present in every scan, the NSV signatures for LENA and \(\sim\)RUSSELL interpret the artifacts as being features that the two images have in common, hence the distance between them – i.e., the ability to discriminate them – is reduced:

\[
\text{adist}[L_{\text{NSV}}, \tilde{R}_{\text{NSV}}] = 0.021; \tag{175}
\]

More specifically, the NSV distance of LENA to \(\sim\)LENA and the NSV distance of LENA to \(\sim\)RUSSELL differ by only 0.012, whereas the corresponding UNSV difference is almost a factor of 7 larger.

SVD-based keys can of course only be used for coarse-grain similarity and discrimination testing, e.g., as query keys for searching a database, but they provide a good example of how consistency and invariance considerations can be applied to
mitigate the effects of a known source error such as multiplicative noise\textsuperscript{11}

IX. DISCUSSION

A stated purpose of this paper is to promote unit consistency as a system design principle so that the functional integrity of complex systems can be sanity-checked in a manner that is relatively general and application independent. For this principle to be applied in practice it is necessary to establish that unit-consistent and/or unit-invariant methods exist to address a broad range of real-world engineering problems. It is of course impossible to enumerate and consider every kind of problem, but it has been argued that the techniques applied to develop unit-consistent generalized matrix inverses, unit-invariant SVD, etc., are applicable to a wide variety of problems for which unit consistency is commonly sacrificed through the reflexive use of least-squares and other non-UC optimization criteria.

At a high level, consistency testing of a system can be summarized most generally as follows:

1) A valid though otherwise arbitrary input $x$ is provided to the system to produce an output $y$:

$$x \rightarrow \text{SYSTEM} \rightarrow y. \quad (176)$$

2) If the system is assumed to be consistent with respect to some transformation $T(x)$ then:

$$T(x) \rightarrow \text{SYSTEM} \rightarrow y' \neq T(y) \implies \text{Fault}. \quad (177)$$

Of course the practical application of unit-consistency testing to large-scale systems will almost certainly have to accommodate modules – or even subsystems – that functionally should maintain unit consistency but do not. For example, a module for calculating an orbital trajectory may require inputs to be provided in specific units because internally-used gravitational and other constants are defined in those units. To test a system that includes such a module it is necessary to implement a wrapper to convert from the testing coordinates to the module-required coordinates and back again. This expenditure of effort can be justified in that it allows the non-UC character of the module to be explicitly recognized before testing rather than being identified later following a failed unit-consistency test of the system. In other words, the UC testing process facilitates the explicit identification of all sources of unit inconsistency, whether before testing or as a result of testing.

It must be emphasized that the motivation for enforcing unit consistency is not limited to simply avoiding coordinate-mismatch faults, e.g., like that which felled the Mars Climate Orbiter \textsuperscript{29}. Rather, it is to provide a means for identifying a broad range of design and implementation flaws for which a violation of unit consistency is just a side-effect.

More generally, it is hoped that a greater focus on the consistency properties of engineering solutions will yield additional performance and reliability benefits across a diverse spectrum of applications.

APPENDIX A

ALTERNATIVE CONSTRUCTIONS

The proofs of Theorems \textsuperscript{III.3} and \textsuperscript{V.5} (and consequently Theorem \textsuperscript{V.4}) do not actually require the general unitary consistency property of the Moore-Penrose inverse and instead only require diagonal unitary consistency, e.g., in Eqs.(55)-(56) as

$$(D,A)^p = A^D_u^* \quad (178)$$

and in Eqs.(120)-(121) as

$$(D,AE)_u = E_u A^D_u^* \quad (179)$$

for unitary diagonal matrices $D_u$ and $E_u$. Thus, the Moore-Penrose inverse could be replaced with an alternative which maintains the other required properties but satisfies this weaker condition in place of general unitary consistency.

Similarly, the scalings defined by Lemmas \textsuperscript{IV.4} and \textsuperscript{V.3} are not necessarily the only ones that may be used to satisfy the conditions of Definition \textsuperscript{V.2} More specifically, Lemmas \textsuperscript{IV.4} and \textsuperscript{V.3} define left and right nonnegative diagonal scaling functions $D_u[A]$ and $D_u[A]$ satisfying

$$D_u[A] \cdot A \cdot D_u[A] = D_u[D,AE_u] \cdot D_u[A] \cdot D_u[D,AE_u] \quad (180)$$

for all positive diagonals $D_u$ and $E_u$. Because the unitary factors of the elements of $A$ are unaffected by the nonnegative scaling, the scalings can be constructed without loss of generality from $\text{Abs}[A]$. If nonnegative $A$ is square, irreducible, and has full support then such a scaling can be obtained by alternately normalizing the rows and columns to have unit sum using the Sinkhorn iteration \textsuperscript{26}, \textsuperscript{27}. The requirement for irreducibility stems from the fact that the process cannot always converge to a finite left and right scaling. For example, the matrix

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad (181)$$

\textsuperscript{11}It should be noted that invariance with respect to additive noise can be homomorphically \textsuperscript{28} obtained from the elemental logarithms of the diagonal scaling values used to compute the UI singular values.
cannot be scaled so that the rows and columns sum to unity unless the off-diagonal element \( b \) is driven to zero, which is not possible for any finite scaling. In other words, the Sinkhorn unit-sum condition cannot be jointly satisfied with respect to both the set of row vectors and the set of column vectors. What is needed, therefore, is a measure of vector “size” that can be applied within a Sinkhorn-type iteration but is guaranteed to converge to a finite scaling.\(^{12}\)

**Definition A.1.** For all vectors \( \mathbf{u} \) with elements from a normed division algebra, a nonnegative composable size function \( s[\mathbf{u}] \) is defined as satisfying the following conditions for all \( \alpha \):

\[
\begin{align*}
s[\mathbf{u}] &= 0 \iff \mathbf{u} = \mathbf{0} \\
\alpha s[\mathbf{u}] &= \alpha s[\mathbf{u}] \\
s[\mathbf{b}] &= 1 \quad \forall \mathbf{b} \in \{0,1\}^n \\
s[\mathbf{u} \otimes \mathbf{b}] &= s[\mathbf{u} \otimes \mathbf{b}] = s[\mathbf{b} \otimes \mathbf{u}] \quad \forall \mathbf{b} \in \{0,1\}^n - \mathbf{0}_n
\end{align*}
\]

The defined size function provides a measure of scale that is homogeneous, permutation-invariant, and invariant with respect to tensor expansions involving identity and zero elements. More intuitively, however, \( s[\mathbf{u}] \) can be thought of as a “mean-like” measure taken over the magnitudes of the nonzero elements of \( \mathbf{u} \). With the imposed condition \( s[\mathbf{0}] = 0 \) the following instantiations can also be verified to satisfy the definition:

\[
\begin{align*}
s_{Ax}[\mathbf{u}] &= \left( \prod_{i \in S} |\mathbf{u}_i|^{1/|S|} \right) \quad j \in S \iff \mathbf{u}(j) \neq 0 \\
s_p[\mathbf{u}] &= \|\mathbf{u}\|_p / |S|^{1/p} \quad j \in S \iff \mathbf{u}(j) \neq 0 \\
s_{a,b}[\mathbf{u}] &= \left( \sum_i |\mathbf{u}_i|^{a+b} \right)^{1/b} \quad a > 0, \quad b > 0
\end{align*}
\]

The first case, \( s_{Ax}(\mathbf{u}) \), is more easily interpreted as the geometric mean of the nonzero elements of \( \mathbf{u} \). Its application in a Sinkhorn-type iteration converges to a unique scaling in which the product of the nonzero elements in each row and column has unit magnitude. If \( a, b, \) and \( c \) are positive for the matrix of Eq.(181) then the scaled result using \( s_p[\mathbf{u}] \) is

\[
\begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix}
\]

where the product of the nonzero elements in each row and column is unity and the particular left and right diagonal scalings are determined by the values of \( a, b, \) and \( c \). It can be shown that for all elemental nonzero matrices that the scaling produced using \( s_{Ax}(\mathbf{u}) \) is equivalent to that produced by the constructions defined by Lemmas IV.4 and V.3 and that the iteration is fast-converging.

The row/column conditions imposed by \( s_p[\mathbf{u}] \) can most easily be understood in the case of \( p = 1 \), for which it is equivalent to the mean of the absolute values of the nonzero elements of \( \mathbf{u} \). In the case of \( p = 2 \), if a vector \( \mathbf{v} \) is formed from the \( m \) nonzero elements of \( \mathbf{u} \) then

\[
s_2[\mathbf{u}] = \|\mathbf{v}\|_2 / m^{1/2}
\]

In the example of the \( 2 \times 2 \) matrix of Eq.(181) the scaled result produced using \( s_p[\mathbf{u}] \) for any \( p > 0 \) happens to be the same as that produced using \( s_{Ax}(\mathbf{u}) \). For nontrivial matrices, however, the results for different \( p \) are not generally (nor typically) equivalent to each other or to that produced by \( s_{Ax}(\mathbf{u}) \).

The third size function, \( s_{a,b}[\mathbf{u}] \), satisfies the required conditions without imposing special treatment of zero elements. In other words, it is a continuous function of the elements of \( \mathbf{u} \) and would therefore appear to be a more natural choice for instantiating \( D^t_v[\mathbf{A}] \) and \( D^o_v[\mathbf{A}] \) for analysis purposes.

The diagonal scalings \( D^t_v[\mathbf{A}] \) and \( D^o_v[\mathbf{A}] \) can alternatively be defined to provide specific properties for the resulting UC inverse. For example, they could be defined as the positive diagonal solutions for the following matrix to have equal row sums and equal column sums:

\[
(D^t_v[\mathbf{A}] \cdot \mathbf{A} \cdot D^o_v[\mathbf{A}])^T \circ (D^t_v[\mathbf{A}] \cdot \mathbf{A} \cdot D^o_v[\mathbf{A}])^o.
\]

Unfortunately, solutions to such problems are not typically efficient to evaluate.

\(^{12}\)This definition and the subsequently-defined instance, \( s_{a,b}[\mathbf{u}] \), may be of independent interest for analyzing properties of low-rank subspace embeddings in high-dimensional vector spaces, e.g., infinite-dimensional spaces.
**APPENDIX B
IMPLEMENTATIONS**

Below are basic Octave/Matlab implementations of some of the methods developed in the paper. Although not coded for maximum efficiency or numerical robustness, they should be sufficient for experimental corroboration of theoretically-established properties.

The following function computes $A^d$ for $m \times n$ real or complex matrix $A$. It has complexity dominated by the Moore-Penrose inverse calculation, which is $O(mn \cdot \min(m,n))$.

```octave
function Ai = uinv(A)
    [S dl dr] = dscale(A);
    Ai = pinv(S) .* (dl * dr)';
end
```

The following function evaluates the UC/UI singular values of the real or complex matrix $A$.

```octave
function s = usvd(A)
    s = svd(dscale(A));
end
```

The following function evaluates the UC/UI singular-value decomposition of the $m \times n$ real or complex matrix $A$.

```octave
function [D U S V E] = usv_decomp(A)
    [S dl dr] = dscale(A);
    D = diag(1./dl); E = diag(1./dr);
    [U S V] = svd(S);
end
```

The following function computes the unique (up to unitary factors) scaled matrix $S = D_L^b[A] \cdot A \cdot D_R^b[A]$ with diagonal left and right scaling matrices $D_L^b[A] = \text{diag}[dl]$ and $D_R^b[A] = \text{diag}[dr]$. It has $O(mn)$ complexity for $m \times n$ real or complex matrix $A$.

```octave
function [S dl dr] = dscale(A)
    tol = 1e-15;
    [m, n] = size(A);
    L = zeros(m, n); M = ones(m, n);
    S = sign(A); A = abs(A);
    idx = find(A > 0.0);
    L(idx) = log(A(idx));
    idx = setdiff(1 : numel(A), idx);
    L(idx) = 0; A(idx) = 0; M(idx) = 0;
    r = sum(M, 2); c = sum(M, 1);
    u = zeros(m, 1); v = zeros(1, n);
    dx = 2*tol;
    while (dx > tol)
        idx = c > 0;
        p = sum(L(:, idx), 1) ./ c(idx);
        L(:, idx) = L(:, idx) - repmat(p, m, 1) .* M(:, idx);
        v(idx) = v(idx) - p; dx = mean(abs(p));
        idx = r > 0;
        p = sum(L(idx, :), 2) ./ r(idx);
        L(idx, :) = L(idx, :) - repmat(p, 1, n) .* M(idx, :);
        u(idx) = u(idx) - p; dx += mean(abs(p));
    end
    dl = exp(u); dr = exp(v);
    S .*= exp(L);
end
```

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