A deterministic small-world network created by edge iterations

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Abstract

Small-world networks are ubiquitous in real-life systems. Most previous models of small-world networks are stochastic. The randomness makes it more difficult to gain a visual understanding on how do different nodes of networks interact with each other and is not appropriate for communication networks that have fixed interconnections. Here we present a model that generates a small-world network in a simple deterministic way. Our model has a discrete exponential degree distribution. We solve the main characteristics of the model.

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1 Introduction

Small-world networks describe many real-life networks, such as the World Wide Web, communication networks, the neuronal network of the worm C. elegans, the electric power grid of southern California, or social networks that achieve both a strong local clustering and a small average path length [1,2,3,4,5].

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Generally, small-world networks are characterized by three main properties. First, their average path length (APL) doesn’t increase linearly with the system size, but grows logarithmically with the number of nodes or slower. Second, average node degree of the network is small. Third, the network has a high average clustering [6] compared to an Erdős-Rényi (ER) random network [7,8] of equal size and average node degree. Average path length can be used to estimate the average transmission delay. One can always reduce the average path length by adding more edges. But for some economical and technical reasons, the average edge number of nodes must be controlled within an acceptable range. Moreover, the cliquishness tendency of network nodes leads to a large clustering coefficient.

The first successful attempt to generate networks with high clustering coefficients and small APL is the model introduced by Watts and Strogatz (WS model) [6]. This pioneering work of Watts and Strogatz started an avalanche of research on the properties of small-world networks and the Watts-Strogatz (WS) model. A much-studied variant of the WS model was proposed by Newman and Watts [9,10], in which edges are added between randomly chosen pairs of nodes, but no edges are removed from the regular lattice. In 1999, Kasturirangan proposed an alternative model to WS small-world network [11]. The model starts also with one ring lattice, then a number of extra nodes are added in the middle of the lattice which are connected to a large number of sites chosen randomly on the main lattice. In fact, even in the case where only one extra node is added, the model shows the small-world effect if that node is sufficiently highly connected, which has been solved exactly by Dorogovtsev and Mendes [12]. To investigate the small-world effect further, Kleinberg has presented a generalization of the WS model which is based on a two-dimensional square lattice [13]. Besides, in order to study other mechanisms for forming small-world networks, Ozik, Hunt and Ott introduced a simple evolution model of growing small-world networks, in which all connections are made locally to geographically nearby sites [14].

The above models are all random. It is known to us all, stochasticity is a common feature of complex network models that generate small-world and scale-free topologies. That is, new nodes connect using a probabilistic rule to the nodes already present in the system. But as mentioned by Barabási et al., the randomness, while in line with the major features of real-life networks, makes it harder to gain a visual understanding of how networks are shaped, and how do different nodes relate to each other [15]. In addition, the probabilistic analysis techniques and random placement or addition of edges used in most previous studies are not appropriate for communication networks that have fixed interconnections, such as neural networks, computer networks, electronic circuits, and so on. Therefore, it would be not only of major theoretical interest but also of great practical significance to construct models that lead to scale-free networks [15,16,17,18,19,20,21,22,23,24,25] and small-world net-
works [26,27] in deterministic fashions. A strong advantage of deterministic networks is that it is often possible to compute analytically their properties, for example, degree distribution, clustering coefficient, average path length, diameter and adjacency matrix whose eigenvalue spectrum characterizes the topology.

In this paper, we focus on the small-world network topology generated in a deterministic way. In 2000, using graph-theoretic methods Comellas, Ozón and Peters introduced deterministic small-world communication networks [26]. Two years later, Comellas and Sampels presented other two deterministic techniques for small-world networks with constant and variable degree distributions, respectively [27]. Here we propose a simple construction technique generating a deterministic small-world network by attaching to edges which was used by Dorogovtsev et al. to generate pseudofractal scale-free web [17]. This exact approach of our model enables one to obtain the analytic solution for relevant network parameters: degree distribution, clustering coefficient and diameter. As a result, our network has strong clustering and small diameter.

2 The iterative algorithm for the deterministic small-world network

It is well known that the number of nodes in most of networks in real world increases exponentially with time. The World Wide Web, for example, has been increasing in size exponentially from a few thousand nodes in the early 1990s to hundreds of millions today. Therefore, our deterministic model is constructed in this evolutionary way. That is to say, our model is a growing network, whose size (the number of nodes) increases exponentially with time.

We denote our network after $t$ step evolution by $N(t)$. The network is created by an iterative method. Its construction algorithm is the following: For $t = 0$, $N(0)$ is a triangle whose three nodes connect one another. For $t \geq 1$, $N(t)$ is obtained from $N(t - 1)$ by adding for each edge created at step $t - 1$ a new node and attaching it to both end nodes of the edge (see Fig. 1). Notice that in [17] Dorogovtsev et al. introduced a pseudofractal scale-free web which is constructed as follows: At each time step, for every edge of the web (not only the ones created at the last time step as in our network), a new node is added, which is attached to both end nodes of the edge. So our model is a variant of the pseudofractal scale-free web. Now we compute the size and order of $N(t)$. In the evolution process of the model, for each new node added, two new edges are created in the network. And for each of the newly-created edges, a node will be created and connected to both the ends of the edge in the next step.
Therefore,
\[ \Delta n_v(t) = n_v(t) - n_v(t-1) = 2 \Delta n_v(t-1), \quad t > 1 \]  
(1)

where \( n_v \) is the total number of nodes in the network. Since \( n_v(0) = 3 \), and \( \Delta n_v(1) = 3 \), it follows that
\[ \Delta n_v(t) = 3 \cdot 2^{t-1}, n_v(t) = 3 \cdot 2^t \]
(2)

The addition of each new node leads to two new edges. Therefore,
\[ \Delta n_e(t) = n_e(t) - n_e(t-1) = 2 \Delta n_v(t) = 3 \cdot 2^t \]
(3)

where \( n_e \) is the total number of edges in the network. As \( n_e(0) = 3 \)
\[ n_e(t) = 3 \cdot 2^{t+1} - 3 \]
(4)

The average node degree is then
\[ \langle k \rangle = \frac{2n_e}{v_v} = \frac{2(3 \cdot 2^{t+1} - 3)}{3 \cdot 2^t} = 4 \cdot \left(1 - \frac{1}{2^{t+1}}\right) \]
(5)
For large $t$ it is small and approximately equal to 4. We can see when $t$ is large enough the resulting network is a sparse graph whose nodes have many fewer connections than is possible.

3 Relevant characteristics of the deterministic small-world Network

Thanks to its deterministic and discrete nature, the model described above can be solved exactly. In the following we concentrate on the degree, clustering coefficient and diameter.

3.1 Degree distribution

The degree distribution is one of the most important statistical characteristics of a network. By definition, the degree of a node $i$ is the number of edges incident from $i$. We denote the degree of node $i$ at step $t$ by $k_i(t)$. Degree distribution $P(k)$ is the probability that a randomly selected node has exactly $k$ edges. By construction, we have

$$k_i(t + 1) = k_i(t) + 2$$

if $t_{c,i}$ is the step at which a node $i$ is created, then $k_i(t_{c,i}) = 2$ and hence

$$k_i(t) = 2(t - t_{c,i} + 1)$$

Therefore, the degree spectrum of the present network is series of discrete values: at time $t$, the number of nodes of degree $k = 2 \cdot 1, 2 \cdot 2, 2 \cdot 3, \cdots, 2 \cdot (t-1), 2 \cdot t, 2 \cdot (t+1)$, equals to $\Delta n_v(t), \Delta n_v(t-1), \Delta n_v(t-2), \cdots, \Delta n_v(2), \Delta n_v(1), n_v(0)$, respectively. Other values of degree are absent in the spectrum. Due to the discreteness of this degree spectrum, it is convenient to obtain the degree via its cumulative distribution [18], i.e.

$$P(k) = P(k' > k - 1) - P(k' > k).$$

Using the fact, $P(k' > k) = P \left( t_c < \tau = t - (\frac{k}{2} - 1) \right)$, where $t_c$ is the birth time of nodes, we obtain that

$$P(k' > k) = \frac{n_v(0)}{n_v(t)} + \sum_{t_c=1}^{\tau-1} \frac{\Delta n_v(t_c)}{n_v(t)} = 2^{-\frac{k}{2}}$$

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Using Eq. (8), we have

\[ P(k) = P(k' > k - 1) - P(k' > k) = (\sqrt{2} - 1)2^{-\frac{k}{2}} \]  

(10)

Obviously, when the size of the network is large, the degree distribution \( P(k) \) is an exponential of a power of degree \( k \), so this deterministic small-world model can be called an exponential network. Note that most small-world networks including WS network belong to this type [28].

### 3.2 Clustering coefficient

Clustering is another important property of a network, which provides a measure of the local structure within the network. The most immediate measure of clustering is the clustering coefficient \( C_i \) for every node \( i \). By definition, clustering coefficient [6] \( C_i \) of a node \( i \) is the ratio of the total number \( E_i \) of edges that actually exist between all \( k_i \) its nearest neighbors and the number \( k_i(k_i - 1)/2 \) of all possible edges between them, i.e. \( C_i = 2E_i/[k_i(k_i - 1)] \). The clustering coefficient \( \langle C \rangle \) of the whole network is the average of all individual \( C_i \)'s. Next we will compute the clustering coefficient of every node and their average value.

When a new node \( i \) joins the network, its degree \( k_i \) and \( E_i \) is 2 and 1, respectively. Each subsequent addition of a link to that node increases both \( k_i \) and \( E_i \) by one. Thus, \( E_i \) equals to \( k_i - 1 \) for all nodes at all steps. So one can see that, in this network there is a one-to-one correspondence between the degree of a node and its clustering. For a node \( v \) of degree \( k \), the exact expression for its clustering coefficient is \( 2/k \). This expression indicates that the local clustering scales as \( C(k) \sim k^{-1} \). It is interesting to notice that a similar scaling has been observed empirically in several real-life networks [19].

Clearly, the number of nodes with clustering coefficient \( C = 1, 1/2, 1/3, \ldots, 1/(t-1), 1/t, 1/(t+1) \), is equal to \( \Delta n_v(t), \Delta n_v(t-1), \Delta n_v(t-2), \ldots, \Delta n_v(2), \Delta n_v(1), n_v(0) \), respectively. The average clustering coefficient \( \langle C \rangle \) can be easily obtained for arbitrary \( t \),

\[
\langle C \rangle = 2\langle \frac{1}{k} \rangle = \frac{1}{n_v(0)} \left[ \sum_{i=1}^{t} \frac{1}{i} \cdot \Delta n_v(t - 1 - i) + \frac{1}{i+1} \cdot n_v(0) \right]
\]

\[
= 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{5} + \cdots + \frac{1}{t-1} \cdot \frac{1}{t} + \frac{1}{t} \cdot \frac{1}{2} + \frac{1}{t+1} \cdot \frac{1}{2}
\]

\[
= \sum_{m=1}^{t} \frac{1}{m} \left( \frac{1}{2} \right)^m + \frac{1}{t+1} \cdot \frac{1}{2}
\]

For infinite \( t \), \( \langle C \rangle = -ln(1 - \frac{1}{2}) = ln2 \), which approaches to a constant value 0.6931, and so the clustering is high.
3.3 Diameter

The small-world concept describes the fact that there is a relatively short distance between most pairs of nodes in most real-life networks. The distance between two nodes is the least number of edges to get from one node to the other. The average path length is the smallest number of links connecting a pair of nodes, averaged over all pairs of nodes. The longest shortest path between all pairs of nodes is called diameter, which is one of the most important evaluation indexes because it characterizes the maximum communication delay in the network. Usually those graphs of smaller node degree and with smaller diameter or average path length are preferred as interconnection networks. Small diameter is consistent with the concept of small-world network and it is easier to compute. So we will study the diameter instead of average path length. Below we give the precise analytical computation of diameter of $N(t)$ denoted by $Diam(N(t))$.

Clearly, at step $t = 0$ (resp. $t = 1$), the diameter is equal to 1 (resp. 2). At each step $t \geq 2$, one can easily see that the diameter always lies between a pair of nodes that have just been created at this step. We will call such newly-created nodes outer nodes. At any step $t \geq 2$, we note that an outer node cannot be connected with two or more nodes that were created during the same step $t' \geq t - 1$. Indeed, we know that from step 2, no outer node is connected to a node of the initial triangle $N(0)$. Thus, for any step $t \geq 2$, any outer node is connected with nodes that appeared at pairwise different steps. Now consider two outer nodes created at step $t \geq 2$, say $v_t$ and $w_t$. Then $v_t$ is connected to two nodes, and one of them must have been created before or during step $t - 2$. We repeat this argument, and we end up with two cases: (1) $t = 2m$ is even. Then, if we make $m$ ”jumps”, from $v_t$ we arrive in the initial triangle $N(0)$, in which we can reach any $w_t$ by using an edge of $N(0)$ and making $m$ jumps to $w_t$ in a similar way. Thus $Diam(N(2m)) \leq 2m + 1$. (2) $t = 2m + 1$ is odd. In this case we can stop after $m$ jumps at $N(1)$, for which we know that the diameter is 2, and make $m$ jumps in a similar way to reach $w_t$. Thus $Diam(N(2m+1)) \leq 2(m+1)$. It is easily seen that the bound can be reached by pairs of outer nodes created at step $t$. More precisely, those two nodes $v_t$ and $w_t$ share the property that they are connected to two nodes that appeared respectively at steps $t - 1$, $t - 2$.

Hence, formally, $Diam(N(t)) = t + 1$ for any $t \geq 0$. Note that the logarithm value of total number of nodes $N_t$ is approximately equals to $(t + 1)ln2$ for large $t$. Thus the diameter grows logarithmically with the number of nodes and the average path length increases more slowly than $ln(N_t)$.

To see why the diameter and average path length grow so slowly, one can pull the nodes of the networks represented in Fig. 1 to the circumference of a
circle. The older nodes that had once been nearest neighbors along the circle are pushed apart as new nodes are inserted into the interval between them. From Fig. 1, we can see when new nodes are introduced into the system, the original nodes are not adjacent but, rather, have a great number of new nodes between them. Thus, growth leads to long-range edges between old nodes, and these long-range edges similar to the shortcuts in the WS model [6] are responsible for short diameter and average path length.

Based on the above discussions, our model is a deterministic small-world network, because it is a sparse one with high clustering and short diameter and average path length, which satisfy the three main necessary properties for small-world network.

4 Conclusion and discussion

In conclusion, we have presented a simple model that allows one to construct deterministic small-world networks. We have obtained the analytic solution for relevant network parameters of the deterministic model, which are close to those for usual random small-world networks [6,9,10,11,12,13,14]. We believe that our model may help engineers in network topology-designing and performance-analyzing, it may also help to demystify the small-world phenomenon. In addition, the model under consideration is a planar graph which can be drawn on a plane without edges crossing. Many real-life networks are planar graphs for technical or natural requirements, such as layout of printed circuits and vein networks clinging to cutis. We believe our network may help to understand some properties of real-world planar networks.

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References

[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74 (2002) 47.
[2] S.N. Dorogovtsev and J.F.F. Mendes, Adv. Phys. 51 (2002) 1079.
[3] M. E. J. Newman, SIAM Review 45 (2003) 167.
[4] S. H. Strogatz, Nature 410 (2001) 268.
[5] M. E. J. Newman, J. Stat. Phys. 101 (2000) 819.
[6] D.J. Watts and S. H. Strogatz, Nature 393 (1998) 440.
[7] P. Erdős and A. Rényi, Publ. Math. 6 (1959) 290.
[8] P. Erdős, and A. Rényi, Publ. Math. Ins. Hung. Acad. Sci. 5 (1960) 17.
[9] M. E. J. Newman and D. J. Watts, Phys. Lett. A 263 (1999) 341.
[10] M. E. J. Newman and D. J. Watts, Phys. Rev. E 60 (1999) 7332.
[11] R. Kasturirangan cond-mat/9904055 (1999).
[12] S.N. Dorogovtsev and J.F.F. Mendes, Europhys. Lett. 50 (2000) 1.
[13] J. Kleinberg, Nature 406 (2000) 845.
[14] J. Ozik, B.-R. Hunt, and E. Ott, Phys. Rev. E 69 (2004) 026108.
[15] A.-L. Barabási, E. Ravasz, and T. Vicsek, Physica A 299 (2001) 559.
[16] K. Iguchi and H. Yamada, Phys. Rev. E 71 (2005) 036144.
[17] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. E 65 (2002) 066122.
[18] S. Jung, S. Kim, and B. Kahng, Phys. Rev. E 65 (2002) 056101.
[19] E. Ravasz and A.-L. Barabási, Phys. Rev. E 67 (2003) 026112.
[20] J.D. Noh, Phys. Rev. E 67 (2003) 045103.
[21] F. Comellas, G. Fertin, and A. Raspaud, Phys. Rev. E 69 (2004) 037104.
[22] J. S. Andrade Jr., H. J. Herrmann, R. F. S. Andrade and L. R. da Silva, Phys. Rev. Lett. 94 (2005) 018702.
[23] J. P. K. Doye and C. P. Massen, Phys. Rev. E 71 (2005) 016128.
[24] Z.Z. Zhang, F. Comellas, G. Fertin and L.L Rong, arXiv: cond-mat/0503316
[25] T. Zhou, B. H. Wang, P. M. Hui and K. P. Chan, arXiv: cond-mat/0405258
[26] F. Comellas, J. Ozón, J.G. Peters, Inf. Process. Lett. 76 (2000) 83.
[27] F. Comellas and M. Sampels, Physica A 309 (2002) 231.
[28] A. Barrat, and M. Weigt, Eur. Phys. J. B 13 (2000) 547.