Does the Big Rip survive quantization?

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It is known that certain quantum cosmological models present quantum behavior for large scale factors. Since quantization can suppress past singularities, it is natural to inquire whether quantum effects can prevent future singularities. To this end, a Friedmann-Robertson-Walker quantum cosmological model dominated by a phantom energy fluid is investigated. The classical model displays accelerated expansion ending in a Big Rip. The quantization is performed in three different ways, which turn out to lead to the same result, namely there is a possibility that quantum gravitational effects could not remove the Big Rip.

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I. INTRODUCTION

Ever since there appeared compelling evidence that the Universe is expanding at an increasing rate [1] many ideas have been put forward to make sense of this unexpected discovery. One of the most popular is the assumption that the Universe is homogeneously permeated with a mysterious dark energy which, although unseen, dominates the present energy content of the Universe (for an up-to-date review see [2]). Accelerated expansion driven by negative pressure is characteristic of a wide variety of proposals such as cosmological constant [3], quintessence [4], k-essence [5], braneworlds [6], Chaplygin gas [7], quintom [8], and holography [9]. The generalized Chaplygin gas, described in terms of a complex scalar field, has been first proposed to unify dark matter and dark energy in [10] and [11]. Quartessence [12], which also relies on exotic equations of state, and a dark fluid implemented by a complex scalar field with an exponential potential [13] are other attempts at unifying dark matter and dark energy. A striking candidate to dark energy is a so-called phantom field [14] whose equation of state is $p = w\rho$ with $w < -1$. A recent analysis

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shows that the possibility that our Universe contains phantom energy cannot be ruled out by the present observational data. Instead of decreasing as the Universe expands, the phantom energy density grows without bound and eventually dominates any other form of energy. The scale factor, as well as the phantom energy density, becomes infinite a finite time from now. This hypothetical catastrophic end of the Universe, with everything torn apart by the infinite phantom energy density, has been christened a “Big Rip”.

Several suggestions have been made either to shun phantom energy or to retain it while avoiding the Big Rip. The most radical of such proposals do away with dark energy altogether either by ascribing the observed acceleration of the Universe to back-reaction of cosmological perturbations or to a postulated modification of general relativity through the addition of inverse curvature terms to the Einstein-Hilbert action. Other models without dark energy are based on such disparate schemes as taking the Universe to be an expanding spherical 3-brane or imputing the origin of the cosmic acceleration to the short range interaction of fundamental particles. Cosmological evolution in the presence of phantom energy without a Big Rip seems possible, as in the so-called “hessence” models, in which a non-canonical complex scalar field plays the role of dark energy and the equation-of-state parameter can cross the phantom divide . It has also been claimed that avoidance of the Big Rip in phantom cosmology can be accomplished by gravitational back-reaction.

Phantom-like behavior can arise in a dilatonic brane-world scenario with induced gravity or be caused by a non-Hermitian but CPT symmetric Hamiltonian. At the phenomenological level, in which phantom energy is described as a perfect fluid, thermodynamical considerations suggest that phantom energy has negative entropy, which appears to defy a sensible physical interpretation.

There have been studies not only of classical cosmological scenarios motivated by quantum effects and containing phantom energy but also of the influence of quantized fields on the evolution of classical cosmological models containing a Big Rip, in addition to attempts at reformulating semiclassical effects in terms of an effective phantom fluid in loop quantum cosmology. There are some examples of quantum cosmological models that present quantum behavior for large scale factors. Since, in many cases, quantization has the power to suppress past singularities, it is reasonable to inquire whether quantum gravitational effects might prevent the Big Rip, which is a future singularity.

A series of investigations of quantum effects in phantom cosmology based on one-loop effective action corrections or back-reaction of quantum fields indicate that the future singularity may be
softened or even suppressed \[33\]. It has recently been found that modifications to Friedmann’s 
equation induced by loop quantum gravity lead to avoidance of certain future singularities, in-
cluding the Big Rip \[34\]. Similarly, a quantum phantom cosmology based on the Wheeler-DeWitt 
equation and the representation of phantom energy by a scalar field has indicated that wave packets 
tend to disperse in the region that corresponds to the Big Rip, thus removing the future singularity 
\[35\].

Since there are several independent ways of doing quantum cosmology, and it is not at all clear 
how they are related, one should try to find out what these different formulations have to tell about 
future singularities. One widely explored approach to quantum cosmology consists in describing 
the matter content by means of fluids, making use of some action principle to describe gravity plus 
matter and then performing a canonical quantization. To our knowledge, within this formalism no 
analysis of quantum cosmological models containing phantom energy, with emphasis on the possible 
avoidance of the Big Rip by quantum gravitational effects, has appeared so far. As a preliminary 
step in this direction, here we investigate a quantum cosmological model dominated by phantom 
energy, the latter mimicked by a perfect fluid. As will be seen, this quantization scheme is worth 
pursuing since it is exactly soluble and produces interesting results. Of course it would be more 
realistic not to assume that the phantom fluid dominates the dynamics of the Universe throughout 
its history, as we do to make the model tractable. Nevertheless, this may be considered not too 
serious a weakness since we are interested in the possible influence of quantum effects on the very 
late time dynamics.

The quantization is performed in three different ways, in order to reach a more satisfactory level 
of generality and gain confidence that the results are not a mere artifact of a specious method of 
quantization. We find that for a certain class of quantum states a phantom fluid brings about a 
Big Rip, just as it does classically.

Here is a summary of the paper. In Section 2 the classical model is formulated on the basis 
of the canonical formalism of Schutz and the classical equations of motion are solved. In the case 
of a phantom fluid the model displays a Big Rip. In Section 3 a quantization is performed with 
a particular order of noncommuting operators in the Wheeler-DeWitt equation, which requires a 
special inner product. The expectation value of the scale factor grows without bound and becomes 
infinite at a finite value of cosmic time. In Section 4 the order of noncommuting operators is left 
arbitrary so that the quantization can be carried out with the standard inner product. In Section 
5, after reduction of the classical Hamiltonian to that of a free particle with the help of a canonical 
transformation, a new quantization is achieved with the standard inner product. The results of
Sections 4 and 5 confirm those of Section 3: accelerated expansion and a Big Rip. Section 6 is devoted to our conclusions.

II. THE CLASSICAL MODEL

Our starting point is the canonical formalism developed by Schutz \[31\] to describe gravity plus a perfect fluid. The action is

\[
S = \int_M d^4x \sqrt{-g} \left( R + 2 \sqrt{h} \, h_{ab} K^{ab} + \int_M d^4x \sqrt{-g} p \right), \tag{1}
\]

where \( K^{ab} \) is the extrinsic curvature and \( h_{ab} \) is the metric of the boundary \( \partial M \) of the four-dimensional domain \( M \). Units are chosen such that \( 16\pi G = 1, c = 1, \) and \( \hbar = 1 \). The pressure \( p \) is related to the energy density \( \rho \) by the equation of state \( p = w \rho \) where \( w \) is constant. The four-velocity of the fluid is expressed in terms of five velocity potentials \( \epsilon, \zeta, \beta, \theta, S \) as

\[
U_\nu = \frac{1}{\mu} \left( \epsilon_\nu + \zeta \beta_\nu + \theta S_\nu \right), \tag{2}
\]

where \( \mu \) is the specific enthalpy, and the comma denotes partial derivative. In the case of Friedmann-Robertson-Walker (FRW) models the only nonvanishing potentials are \( \epsilon, \theta, \) and the specific entropy \( S \). The normalization condition \( U_\nu U^\nu = -1 \) determines \( \mu \) in terms of the potentials.

The FRW metric is

\[
ds^2 = -N^2(t)dt^2 + a^2(t) \sigma_{ij} dx^i dx^j \tag{3}
\]

where \( N \) is the lapse function, \( a \) is the scale factor, and \( \sigma_{ij} \) is the metric of the constant-curvature spatial sections. With the help of thermodynamical considerations, and taking the constraints into account, insertion of (3) into (1) yields the reduced action

\[
S = \int (\dot{\alpha} p_a + \hat{T} p_T - \mathcal{H}) dt, \tag{4}
\]
with the super-Hamiltonian

\[ \mathcal{H} = -\frac{p_a^2}{24a} - 6ka + \frac{p_T}{a^{3w}} \].

(5)

Here \( T \) is a new canonical variable \( [32] \) that describes the only remaining degree of freedom of the fluid, while \( k = 0, 1 \) or \(-1\) depending on whether the curvature of the spatial sections is zero, positive or negative. From now on we shall discuss only the flat case \((k = 0)\).

With \( k = 0 \) the equations of motion are

\[ \dot{a} = \frac{\partial}{\partial p_a} (N \mathcal{H}) = -\frac{Np_a}{12a} , \quad \dot{p}_a = -\frac{\partial}{\partial a} (N \mathcal{H}) = -\frac{Np_a^2}{24a^2} + 3w \frac{Np_T}{a^{3w+1}} , \]

(6)

\[ \dot{T} = \frac{\partial}{\partial p_T} (N \mathcal{H}) = \frac{N}{a^{3w}} , \quad \dot{p}_T = -\frac{\partial}{\partial T} (N \mathcal{H}) = 0 , \]

(7)

supplemented by the super-Hamiltonian constraint

\[ \mathcal{H} = -\frac{p_a^2}{24a} + \frac{p_T}{a^{3w}} = 0 \].

(8)

Solving this constraint for \( p_a \) we get

\[ p_a = -\sqrt{24p_T} a^{-(3w-1)/2} \],

(9)

where \( p_T \) is a positive constant, and the negative sign has been selected in order to produce an expanding universe. In the cosmic-time gauge \( N = 1 \) the first of equations \( [6] \) becomes

\[ \dot{a} = \frac{\sqrt{24p_T}}{12} a^{-(3w+1)/2} , \]

(10)

which is solved by

\[ a^{3(w+1)/2} = a_0^{3(w+1)/2} + \sqrt{\frac{p_T}{24}} 3(w + 1) (t - t_0) , \]

(11)

where \( a_0 \) is the present value of the scale factor. If \( w < -1 \) then \( w + 1 = -|w + 1| < 0 \) and
\[ a(t) = \left[ a_0^{3(w+1)/2} - \sqrt{\frac{p_T}{24}} 3|w + 1|(t - t_0) \right]^{-2/3|w+1|}, \tag{12} \]

which becomes infinite at the finite future time

\[ t = t_0 + \sqrt{\frac{24}{p_T}} \frac{a_0^{3(w+1)/2}}{3|w+1|}. \tag{13} \]

This is the famous Big Rip [16].

### III. QUANTIZATION WITH A SPECIAL INNER PRODUCT

The Wheeler-DeWitt equation results from making the correspondence \( p_a \rightarrow \hat{p}_a = -i\partial/\partial a, \) \( p_T \rightarrow \hat{p}_T = i\partial/\partial \tau \) and requiring \( \hat{H} \Psi = 0, \) where \( \hat{H} \) is the operator corresponding to the super-Hamiltonian \( \hat{H}. \) This means that the time has been chosen as \( \tau = -T. \) With a particular choice of operator ordering the Wheeler-DeWitt equation becomes

\[ i a^{1-3w} \frac{\partial \Psi}{\partial \tau} = -\frac{1}{24} \frac{\partial^2 \Psi}{\partial a^2}. \tag{14} \]

As discussed in [32], this equation takes the form of a Schrödinger equation \( i\partial \Psi/\partial \tau = \hat{H} \Psi \) with a Hermitian Hamiltonian operator \( \hat{H} \) if the inner product is

\[ (\Phi, \Psi) = \int_0^\infty a^{1-3w} \Phi(a, \tau)^* \Psi(a, \tau) da. \tag{15} \]

By superposing stationary states a solution to (14) can be found [32] in the form (\( \gamma \) is a positive constant)

\[ \Psi(a, \tau) = a \exp \left\{ -\frac{a^{3(1-w)}}{4[\gamma - i3(1-w)^2 \tau/32]} \right\}. \tag{16} \]

The expectation value of the scale factor is [32]

\[ \langle a \rangle(\tau) \propto (b^2 + \tau^2)^{1/3(1-w)}, \quad b = \frac{3(1-w)^2}{32\gamma}. \tag{17} \]
Note, however, that $\tau$ is not cosmic time. Recall that $\tau = -T$, where $T$ is the only remaining degree of freedom of the fluid. It follows from the first of equations (7) that $|N| = a^{3w}$. Since $\langle a \rangle$ plays the role of the scale factor in the quantum theory, we write $dt = \langle a \rangle^{3w} d\tau$. Therefore, choosing a common origin for both times,

$$ t \propto \int_0^\tau (b^2 + \lambda^2)^{w/(1-w)} d\lambda . \quad (18) $$

Near infinity the above integral behaves as

$$ \int_0^\tau \lambda^{2w/(1-w)} d\lambda \propto \tau^{(1+w)/(1-w)} . \quad (19) $$

If $w < -1$ the integral is convergent and $t \to t_{\text{rip}}$ for $\tau \to \infty$, where $t_{\text{rip}}$ is finite. Thus the expectation value of the scale factor becomes infinite at the finite cosmic time $t_{\text{rip}}$ which corresponds to $\tau = \infty$. The quantum model displays a Big Rip just like the classical model.

One might suspect that this result lacks generality in view of: (i) the particular order of noncommuting operators employed in the construction of the Wheeler-DeWitt equation (14); (ii) the unusual inner product (15) that, oddly enough, depends on the equation of state through $w$. We proceed to address these issues.

IV. OPERATOR ORDERING AND QUANTIZATION WITH THE STANDARD INNER PRODUCT

The classical Hamiltonian corresponding to the Wheeler-DeWitt equation (14) is

$$ H = \frac{a^{3w-1}}{24} p_a^2 , \quad (20) $$

to which we associate the Hamiltonian operator

$$ \hat{H}_{\lambda\mu\nu} = \frac{1}{24} \frac{1}{2} \left( a^\lambda \hat{p}_a a^{\mu} \hat{p}_a a^{\nu} + a^{\nu} \hat{p}_a a^{\mu} \hat{p}_a a^{\lambda} \right) , \quad \lambda + \mu + \nu = 3w - 1 . \quad (21) $$

This operator is formally self-adjoint for any choice of the ordering parameters $\lambda, \mu, \nu$ with the standard inner product

$$ (\Phi, \Psi) = \int_0^\infty \Phi(a, \tau)^* \Psi(a, \tau) da . \quad (22) $$
Now the Wheeler-DeWitt equation takes the form

\[ i \frac{\partial \Psi}{\partial \tau} = \hat{H}_{\lambda \mu \nu} \Psi, \]  

(23)

whose stationary solutions \( \Psi(a, \tau) = \psi(a)e^{-iE\tau} \) satisfy

\[ \hat{H}_{\lambda \mu \nu}\psi - E\psi = 0. \]  

(24)

After some algebraic labor this equation can be put in the form

\[ a^2 \frac{d^2 \psi}{da^2} + 2ra \frac{d\psi}{da} + [\epsilon + 24Ea^{2(1-r)}] \psi = 0 \]  

(25)

where \( \epsilon = [\nu(2r - \lambda - 1) + \lambda(2r - \nu - 1)] \) and \( r = (3w - 1)/2 \). The solutions of this equation are

\[ \psi(a) = a^{(1-2r)/2}Z_p\left(\frac{\sqrt{24E}}{1-r}a^{1-r}\right), \quad r \neq 1 \]  

(26)

where

\[ p = \frac{1}{1-r} \sqrt{\left(\frac{1-2r}{2}\right)^2 - \epsilon} \]  

(27)

and \( Z_p \) is a Bessel function either of the first or of the second kind of order \( p \). We shall consider only \( Z_p = J_p \), the Bessel function of the first kind, in which case a complete solution to equation (23) is given by

\[ \Psi_E(a, \tau) = A e^{-iE\tau} a^{(1-2r)/2} J_p\left(\frac{\sqrt{24E}}{1-r}a^{1-r}\right). \]  

(28)

Our next step is to generate a normalizable wave packet by superposition of the solutions (28) in the form

\[ \Psi(a, \tau) = \int_0^\infty dE A(E) e^{-iE\tau} a^{(1-2r)/2} J_p\left(\frac{\sqrt{24E}}{1-r}a^{1-r}\right). \]  

(29)

Setting \( \lambda = \sqrt{24E}/(1-r) \) this takes the more convenient form
\[ \Psi(a, \tau) = \int_0^\infty d\lambda C(\lambda) e^{-i(1-r)^2\lambda^2\tau/24} a^{(1-r)} J_p(\lambda a^{1-r}) . \] (30)

The choice

\[ C(\lambda) = \lambda^{p+1} e^{-\gamma \lambda^2} , \quad \gamma > 0 \] (31)

allows the integration to be performed in closed form and we obtain

\[ \Psi(a, \tau) = a^{-1/2} \left[ \frac{a^{1-r}}{2\gamma + \frac{(1-r)^2\tau}{12}} \right]^{1+p} \exp \left[ - \frac{a^{2(1-r)}}{4\gamma + \frac{(1-r)^2\tau}{6}} \right] . \] (32)

The expectation value of the scale factor is defined by

\[ \langle a \rangle(\tau) = \frac{\int_0^\infty a |\Psi(a, \tau)|^2 da}{\int_0^\infty |\Psi(a, \tau)|^2 da} \] (33)

and a straightforward computation furnishes

\[ \langle a \rangle(\tau) = \frac{\Gamma \left[ \frac{2(1-r)(1+p)+1}{2(1-r)} \right]}{\Gamma(p+1)} \left\{ \frac{1}{2\gamma} \left[ \gamma^2 + \frac{(1-r)^4}{576\tau^2} \right] \right\}^{1/2(1-r)} . \] (34)

Recalling that \( r = (3w - 1)/2 \), equation (34) yields

\[ \langle a \rangle(\tau) \propto (b^2 + \tau^2)^{1/3(1-w)} , \] (35)

the same as (17). Thus, an arbitrary operator order with the standard inner product leads to the same result as the one obtained when a special operator order is adopted which requires the strange inner product (15). This lends support to our previous claim that there exist evolving quantum states such that the phantom energy gives rise to accelerated expansion and a Big Rip.

It should be stressed that we are trying to find out whether quantum effects necessarily remove the future singularity. If this were the case, for any quantum state the expectation value of the scale factor would have to remain finite for any value of the cosmic time. We have just presented an example of a quantum state such that the expectation value of the scale factor becomes infinite in a finite cosmic time. This shows that, at least in this model, and for the above quantization
procedure, it is not generally the case that quantum effects prevent the Big Rip. This reasoning also justifies our particular choices (28) and (31), which allow the integration to be performed and the wave packet to be expressed in terms of elementary functions, which, in turn, makes easy the computation of the expectation value of the scale factor.

V. CANONICAL TRANSFORMATION AND ANOTHER QUANTIZATION WITH THE STANDARD INNER PRODUCT

The phase-space transformation \((a, p_a) \rightarrow (x, p)\) defined by

\[
x = \frac{2\sqrt{12}}{3(1 - w)} a^{3(1-w)/2}, \quad p = \frac{a^{3w-1/2}}{\sqrt{12}} p_a
\]

is canonical inasmuch as the Poisson bracket \(\{x, p\}_{(a, p_a)} = 1\). In terms of these new canonical variables the classical Hamiltonian (20) reduces to the free-particle Hamiltonian

\[
H = \frac{p^2}{2}
\]

and the corresponding Wheeler-DeWitt equation takes the form

\[
i\frac{\partial \Psi}{\partial \tau} = -\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2}.
\]

This is the Schrödinger wave equation for a free particle on the half-line \([0, \infty)\). Self-adjointness of the Hamiltonian operator requires \([38]\) that the domain of \(\hat{H}\) be restricted to those wave functions that satisfy the boundary condition

\[
\Psi(0, \tau) = \xi \Psi'(0, \tau)
\]

where \(\Psi' = \partial \Psi / \partial x\) and \(\xi \in [0, \infty]\). For the sake of simplicity we shall consider only the cases \(\xi = 0\) and \(\xi = \infty\), that is, either the wave function or its derivative vanishes at \(x = 0\). The propagator for arbitrary \(\xi\) is known \([39]\), but it is sufficiently complicated to prevent us from constructing manageable wave packets. However, inasmuch as we are searching for examples of quantum states that lead to a future singularity, if we are successful our particular choices of \(\xi\) will still allow us to conclude that it is not true that the Big Rip is suppressed \textit{in general}, that is, for arbitrary \(\xi\).
In short, our special choices for $\xi$ are technically convenient and do not impair the logic of our reasoning.

The propagators for the cases $\xi = 0$ and $\xi = \infty$ are given by

$$
G_0(x, x', \tau) = G(x, x', \tau) - G(x, -x', \tau) ,
G_\infty(x, x', \tau) = G(x, x', \tau) + G(x, -x', \tau) ,
$$

(40)

where $G(x, x', \tau)$ is the usual free-particle propagator, which in our case ($m = 1$, $\hbar = 1$) takes the form

$$
G(x, x', \tau) = \left( \frac{1}{2\pi i\tau} \right)^{1/2} \exp \left[ i \frac{(x - x')^2}{2\tau} \right] .
$$

(41)

Let us first consider the initial normalized wave function

$$
\Psi_e(x, 0) = \left( \frac{8\sigma}{\pi} \right)^{1/4} e^{-\beta x^2} , \quad \beta = \sigma + i\gamma , \quad \sigma > 0 .
$$

(42)

Since its derivative vanishes at $x = 0$, it satisfies the boundary condition (39) with $\xi = \infty$. Therefore

$$
\Psi_e(x, \tau) = \int_0^\infty G_\infty(x, y, \tau) \Psi_e(y, 0) dy = \int_{-\infty}^\infty G(x, y, \tau) \Psi(y, 0) dy ,
$$

(43)

where we have taken advantage of the fact that $\Psi_e(x, 0)$ is an even function to extend the integration to the whole real line. A simple computation yields

$$
\Psi_e(x, \tau) = \left( \frac{8\sigma}{\pi} \right)^{1/4} (1 + 2i\beta\tau)^{-1/2} \exp \left[ - \frac{\beta x^2}{1 + 2i\beta\tau} \right] .
$$

(44)

In accordance with (46) the expectation value of the scale factor is defined by

$$
\langle a \rangle_e(\tau) = \left[ \frac{3(1 - w)}{2\sqrt{12}} \right]^{2/3(1-w)} \int_0^\infty x^{2/3(1-w)} |\Psi_e(x, \tau)|^2 dx ,
$$

(45)

and a simple calculation gives

$$
\langle a \rangle_e(\tau) = \frac{1}{\sqrt{\pi}} \left[ \frac{3(1 - w)}{2\sqrt{12}} \right]^{2/3(1-w)} \Gamma \left[ \frac{5 - 3w}{3(1 - w)} \right] \left[ \frac{4\sigma^2\tau^2 + (1 - 2\gamma\tau)^2}{2\sigma} \right]^{1/3(1-w)} ,
$$

(46)
whose asymptotic behavior coincides with that of (17).

Finally, consider the initial normalized wave function

$$
\Psi_o(x,0) = \left( \frac{128\sigma^3}{\pi} \right)^{1/4} x e^{-\beta x^2}, \quad \beta = \sigma + i\gamma, \quad \sigma > 0,
$$

which satisfies the boundary condition (39) with $\xi = 0$. As in the previous case, the fact that $\Psi_o(x,0)$ is an odd function allows us to write

$$
\Psi_o(x,\tau) = \int_0^\infty G_0(x,y,\tau)\Psi_o(y,0)\,dy = \int_{-\infty}^\infty G(x,y,\tau)\Psi_o(y,0)\,dy,
$$

which gives

$$
\Psi_o(x,\tau) = \left( \frac{128\sigma^3}{\pi} \right)^{1/4} (1 + 2i\beta\tau)^{-3/2} x \exp \left[ -\frac{\beta x^2}{1 + 2i\beta\tau} \right].
$$

The expectation value of the scale factor is now given by

$$
\langle a \rangle_o(\tau) = \frac{5 - 3w}{3(1 - w)\sqrt{\pi}} \left[ \frac{3(1 - w)}{2\sqrt{12}} \right]^{2/(3(1-w))} \Gamma \left[ \frac{5 - 3w}{3(1 - w)} \right] \left[ \frac{4\sigma^2\tau^2 + (1 - 2\gamma\tau)^2}{2\sigma} \right]^{1/(3(1-w))},
$$

or, equivalently,

$$
\langle a \rangle_o(\tau) = \frac{5 - 3w}{3(1 - w)} \langle a \rangle_e(\tau).
$$

Thus, we recover the same asymptotic behavior of the previous case, which entails accelerated expansion and a Big Rip.

**VI. CONCLUSIONS**

We have investigated the quantum features of a FRW cosmological model dominated by a phantom energy fluid. The classical model was formulated on the basis of the canonical formalism of Schutz, which takes into account the degrees of freedom of the fluid. In the case of a phantom fluid (equation of state $p = w\rho$ with $w < -1$) the classical equations of motion predict an accelerated expansion for the Universe ending in a catastrophic Big Rip a finite time from now. Then the model was quantized in three different ways. Firstly, the Wheeler-DeWitt equation was set up with a
special choice of the order of noncommuting operators, which requires an unusual inner product that depends on the equation of state for the fluid. It was found that the expectation value of the scale factor becomes infinite at a finite cosmic time, and the Universe winds up in a Big Rip. Next, an arbitrary order of noncommuting operators was allowed for the purpose of quantizing the model with the standard inner product. A particular wave packet was constructed, and it was found that the expectation value of the scale factor reproduces the previous behavior. Finally, a third method of quantization was explored based on the previous reduction of the classical Hamiltonian to that of a free particle by means of a canonical transformation. Since the propagator of the Schrödinger equation for a free particle on the half-line is known, it is possible, in principle, to find explicitly the wave function at any time once its form is given at an initial time. It turns out that two initial wave functions obeying different boundary conditions give rise to essentially the same expectation value of the scale factor with exactly the same behavior predicted by the two other quantization schemes. This indicates that our results are not merely an artifact of a spurious quantization procedure. The quantum behavior of the scale factor appears to be robust, and supports our conclusion that, at least in our model, quantum effects do not, in general, prevent a Big Rip.

We do not claim that our result is conclusive. In order to check its generality, the matter content should be enriched to describe the early universe more realistically, which we intend to do in a future work. Since the phantom energy dominates the late time dynamics, we suspect that the enrichment of the matter content will not change our result, but this conjecture must be checked. We anticipate that this will not be an easy task because it will surely be much more difficult to obtain exact normalizable solutions to the Wheeler-DeWitt equation. We do not expect that approximate solutions, such as those furnished by the WKB method, are suitable to discuss the existence of quantum singularities, that is, we believe that only a fully quantum treatment can give us a solid clue as to the persistence of future singularities in the quantum regime of phantom cosmological models. We also feel that a study of the Bohmian trajectories of the scale factor is likely to be helpful in clarifying whether quantum gravitational effects can suppress the Big Rip. This is an investigation we expect to undertake in the near future.

Our result differs from those obtained by means of other formalisms [33, 34, 35], namely that the Big Rip is softened or avoided by quantum effects. On the one hand, since it is not known how these different formulations of quantum cosmology are related, we confess to our ignorance about the reason for the discrepancy. On the other hand, it is hard to make clear-cut statements on presence or absence of cosmological singularities in the quantum regime because there is no general agreement on what constitutes a quantum singularity. These issues certainly deserve a
deeper investigation.

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