Abstract
In this paper we study the $L^p$-$L^q$ boundedness of the Fourier multipliers in the setting where the underlying Fourier analysis is introduced with respect to the eigenfunctions of an anharmonic oscillator $A$. Using the notion of a global symbol that arises from this analysis, we extend a version of the Hausdorff–Young–Paley inequality that guarantees the $L^p$-$L^q$ boundedness of these operators for the range $1 < p \leq 2 \leq q < \infty$. The boundedness results for spectral multipliers acquired, yield as particular cases Sobolev embedding theorems and time asymptotics for the $L^p$-$L^q$ norms of the heat kernel associated with the anharmonic oscillator. Additionally, we consider functions $f(A)$ of the anharmonic oscillator on modulation spaces and prove that Linskii’s trace formula holds true even when $f(A)$ is simply a nuclear operator.

Keywords Anharmonic oscillator · Fourier multipliers · Spectral multipliers · Paley inequality · Hausdorff–Young–Paley inequality · $L^p$-$L^q$ boundedness · Heat kernels

Mathematics Subject Classification 42B15 · 58J40. · Secondary 47B10 · 47G30 · 35P10

1 Introduction

A short exposition of the results we present here can be found in [19]. The current work contains the proofs of the statements presented in the aforementioned work, and
provides the reader with a detailed discussion of the involved ideas. The aim of [19] and of the current paper, is to establish sufficient conditions for the $L^p-L^q$ boundedness of the Fourier multipliers associated with a family of anharmonic oscillators on $\mathbb{R}^n$ as considered in [18]. In particular, the boundedness of Fourier multipliers is a central theme of harmonic analysis with far reaching applications in different areas. Such analysis is traced back to Hörmander’s seminal paper [29] in 1960. Later, the $L^p$-boundedness of Fourier multipliers and spectral multipliers was thoroughly investigated by several researchers in many different settings, we cite here [4–6, 11, 12, 40] to mention a very few of them.

Here we deal with $L^p-L^q$ multipliers as opposed to the $L^p$-multipliers, for which theorems of Mihlin–Hörmander or Marcinkiewicz type, provide results in different settings based on the regularity of the symbol. The $L^p-L^q$ boundedness of Fourier multipliers and spectral multipliers on locally compact unimodular groups and on compact homogeneous spaces is recently settled by Akylzhanov et al. [1–3]. Also, see [31, 32] for some recent results in different settings.

Anharmonic oscillators on $\mathbb{R}^n$ are important operators in analysis and mathematical physics, but also in number theory. The analysis of the energy levels of the Schrödinger operator $i\partial_t \psi = -\Delta \psi + V(x)\psi$ can be reduced to the corresponding eigenvalue problem of an operator of the form $A = -\Delta + V(x)$; the so-called anharmonic oscillator.

Unlike the case of the harmonic oscillator (case $V(x) = |x|^2$) where the eigenfunctions are the Hermite functions and are well-understood (cf. [37, 42]), the exact solution of the eigenvalue problem associated with the anharmonic oscillator is still unknown. Despite the intensive research on this topic over the last 40 years (cf. [45]), the exact solution of it, even in the quartic case, is still unknown. The literature on the subject is vast and overviews can be found, for instance in [18].

Here we consider operators which in [18] has been regarded as “prototype” in the class under consideration. For such operators the potential is given as $V(x) = |x|^{2k}$, where $k \geq 1$ is an integer, and we generalise $A$ by considering higher order derivatives. In particular, for $l \geq 1$ integer, we define $A$ to be

$$A = A_{k,l} = (-\Delta)^l + |x|^{2k} + 1. \quad (1.1)$$

The Weyl–Hörmander calculus associated with the anharmonic oscillator $A_{k,l}$ that has been developed in [18], will be used in order to prove the continuous inclusion of the Sobolev spaces associated to $A_{k,l}$ (in the sense of Weyl–Hörmander calculus) to some suitable space, and finally get an estimate for the operator norm of Fourier multipliers.

To prove the above, one needs to follow classical techniques developed by Hörmander [29]. The Paley-type inequality [34] describes the growth of the Fourier transform of a function in terms of its $L^p$-norm. Interpolating the latter with the Hausdorff–Young inequality, one can obtain the following Hörmander’s version of the Hausdorff–Young–Paley inequality,
\begin{equation}
\left( \int_{\mathbb{R}^n} \left| \mathcal{F} f(\xi) \phi(\xi) \right|^r \right)^{\frac{1}{r}} \leq \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p \leq r \leq p' < \infty, \quad 1 < p < 2,
\end{equation}

where $\mathcal{F} f$ stands for the Fourier transform of $f$ on $\mathbb{R}^n$ and $\phi$ is a positive function defined on $\mathbb{R}^n$. As a consequence of the Hausdorff–Young–Paley inequality, Hörmander [29, p. 106] proves that the condition

\begin{equation}
\sup_{t>0} t^b \{ \xi \in \mathbb{R}^n : m(\xi) \geq t \} < \infty, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{b},
\end{equation}

where $1 < p \leq 2 \leq q < \infty$, implies the existence of a bounded extension of a Fourier multiplier $T_m$ with symbol $m$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Recently, Akylzhanov and Ruzhansky studied Hörmander’s classical results for unimodular locally compact groups [3]. In their paper the key idea is the extension of Hörmander’s theorem to the unimodular locally compact groups is the reformulation of this classical theorem in terms of noncommutative Lorentz spaces and group von-Neumann algebras of unimodular groups.

The following reads as the Hausdorff–Young–Paley inequality in our setting where the appearing Fourier transform $\mathcal{F}_{A_{k,l}}$ is taken with respect to the eigenfunctions of $A_{k,l}$ precisely defined in Sect. 3.

**Theorem 1.1** Let $1 < p \leq 2$, and let $1 < p \leq b \leq p' < \infty$, where $p' = \frac{p}{p-1}$. If $\varphi(j)$ is a positive sequence in $\mathbb{N}$ such that

\begin{equation}
M_{\varphi} := \sup_{t>0} \sum_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \text{ is finite, then for every } f \in L^p(\mathbb{R}^n) \text{ there exists a constant } C := C(p) > 0 \text{ such that}
\end{equation}

\begin{equation}
\left( \sum_{j \in \mathbb{N}} \left| \mathcal{F}_{A_{k,l}} f(j) \varphi(j)^{\frac{1}{b} - \frac{1}{p}} \right|^b \|u_j\|_{L^\infty(\mathbb{R}^n)}^{2-b} \right)^{\frac{1}{b}} \leq CM_{\varphi}^{\frac{1}{p} - \frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.4}
\end{equation}

Here, $(u_j)_{j \in \mathbb{N}}$ are the eigenfunctions of the anharmonic oscillator $A_{k,l}$ corresponding to the eigenvalues $(\lambda_j)_{j \in \mathbb{N}}$ in decreasing order. Although, the form of the eigenfunctions $u_j$ is unknown, the Sobolev estimates in this setting as in [18], show that the $L^\infty$ norm of them is finite, and as a result (1.4) is well-defined; see Lemma 2.2. For a complete exposition of the necessary notation and ideas involved in (1.4) we refer to Sects. 2 and 3.
The analysis on the asymptotic behaviour of the eigenvalues of the anharmonic oscillator $A_{k,l}$ as appeared in [10], or in more generality in [18], together with Theorem 1.1 and the analysis prior to this, leads to the following result on the $L^p - L^q$ boundedness of the Fourier multipliers associated with the anharmonic oscillator.

We note that the following result is new even in the setting of the simpler case of the harmonic oscillator in any dimension. Precisely, a simplification of the arguments used here, would lead to an analogous to Theorem 1.1 result in the setting of the harmonic oscillator where, of course, the Fourier analysis, should be regarded as the one associated with it.

**Theorem 1.2** Let $1 < p \leq 2 \leq q < \infty$. Suppose that $m$ is a $A_{k,l}$-Fourier multiplier with $A_{k,l}$-symbol $\sigma_m$ on $\mathbb{R}^n$. Then we have

$$\|m\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \lesssim \sup_{s > 0} s \left( \sum_{j \in \mathbb{N}} \|u_j\|^2_{L^\infty(\mathbb{R}^n)} \right)^{\frac{1}{p} - \frac{1}{q}}.$$

As an application of Theorem 1.2 we get the following theorem on the boundedness of spectral multipliers of the anharmonic oscillator.

**Corollary 1.3** Let $1 < p \leq 2 \leq q < \infty$, and let $\varphi$ be any decreasing function on $[1, \infty)$ such that $\lim_{u \to \infty} \varphi(u) = 0$. Then,

$$\|\varphi(A_{k,l})\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \lesssim \sup_{u > 1} \varphi(u) \left( u^{1 + \frac{(k+l)n}{2kl}} \right)^{\frac{1}{p} - \frac{1}{q}}. \quad (1.5)$$

Corollary 1.3 yields time asymptotic for the $L^p-L^q$ norms of heat kernel associated with the anharmonic oscillator $A_{k,l}$; that is the fundamental solution of the heat equation when the Laplace operator is replaced by the differential operator $A_{k,l}$ as in Remark 4.10, and Sobolev-type estimates; see Remark 4.9. Finally, it allows to find the range of $m > 0$ for which $A_{k,l}^{-m}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, see Corollary 4.8.

In the last part of the paper, we study the $r$-nuclearity of functions of the anharmonic oscillator when acting on the modulation spaces $M_{w}^{p,q}$ on $\mathbb{R}^n$. In this concept, we are investigating the conditions on the decomposition of the operator that allow for the operator to be $r$-nuclear and also for Lindskii’s trace formula (also called the nuclear trace) to hold in cases where $r \in (0, 1]$ is not in the standard range $(0, 2/3)$. These concepts have been studied by many authors in different settings; for instance, we refer to [7, 13–17, 21–23, 25, 30, 43, 44]. The following is our main theorem in this context.

**Theorem 1.4** Let $0 < r \leq 1, 1 \leq p, q < \infty$ and $w$ be a submultiplicative polynomially moderate weight. The operator $f(A_{k,l})$ is $r$-nuclear on $M_{w}^{p,q}(\mathbb{R}^n)$, provided that

$$\sum_{j=1}^{\infty} |f(\lambda_j)| r \|u_j\|_{M_{w}^{p,q}} \|u_j\|_{M_{w}^{p^{'},q^{'}}}^{-r} < \infty. \quad (1.6)$$
If, in particular, (5.6) holds for \( r = 1 \), then we have the trace formula

\[
\text{Tr}(f(A_k,l)) = \sum_{j=1}^{\infty} f(\lambda_j),
\]

where the series \( \sum_{j=1}^{\infty} f(\lambda_j) \) converges absolutely.

2 Basics of the Anharmonic Oscillator

Let us begin by recalling some basic facts about the anharmonic oscillator as in [18], and performing some preliminary analysis on it. For the rest of this section, and the subsequent ones, we assume that the integers \( k, l \geq 1 \) in (1.1) are fixed, i.e., we can write

\[
A = (-\Delta)^l + |x|^{2k} + 1, \quad x \in \mathbb{R}^n,
\]

when referring to the anharmonic oscillator, where \( \Delta \) stands for the Laplace–Belrami operator, and \( |\cdot| \) for the Euclidean norm on \( \mathbb{R}^n \). Let us note that the constant in (2.1) allows for the invertibility of \( A \).

If \( A \) is viewed as a pseudo-differential operator, then it arises as the Weyl-quantization of the symbol \( a(x, \xi) = |\xi|^{2l} + |x|^{2k} + 1 \), where \( x, \xi \in \mathbb{R}^n \), and we write \( A = a^w \) to denote this relation. In [18] the authors proved that, in the setting of Weyl–Hörmander calculus, we have

\[
a \in S(M, G),
\]

where the Hörmander metric \( G \) is given by

\[
G(dx, d\xi) = \frac{dx^2}{(|\xi|^{2l} + |x|^{2k} + 1)^{\frac{1}{k}}} + \frac{d\xi^2}{(|\xi|^{2l} + |x|^{2k} + 1)^{\frac{1}{l}}},
\]

and the \( G \)-weight \( M \) is given by

\[
M(x, \xi) = |\xi|^{2l} + |x|^{2k} + 1.
\]

It is well-known from the general theory of pseudo-differential operators (c.f. [35, Sect. 1.7]) that if \( \sigma^w \) is an injective pseudo-differential operator on the Schwartz space \( S(\mathbb{R}^n) \), whose symbol \( \sigma \in S(m, g) \) is real and elliptic, then \( \sigma^w \) is an isomorphism on \( S(\mathbb{R}^n) \) and on \( S'(\mathbb{R}^n) \); the Schwartz space and its dual, and moreover, its inverse is a pseudo-differential operator with real and elliptic Weyl-symbol in the class \( S(m^{-1}, g) \). We recall here that a symbol \( \sigma \in S(m, g) \) is elliptic if for some \( R > 0 \)

\[
\sigma(x, \xi) \geq Cm(x, \xi), \quad \text{for} \quad x, \xi > R, C > 0.
\]
Simple calculations show that $A$ is an injective operator on $S(R^n)$, while one can check that its symbol is elliptic in the class $S(M, G)$ defined above. Hence, its inverse $A^{-1}$ is a positive, and also a compact operator since $M^{-1}(x, \xi) \to \infty$ as $x, \xi \to 0$; cf. [35, Theorem 1.4.2].

The operator $A$ admits a unique self-adjoint extension on $L^2(R^n)$, and its spectrum is purely discrete and lies in $(0, \infty)$. By the spectral theorem there exists an orthonormal basis, say $(u_j)_{j \in \mathbb{N}}$, of $L^2(R^n)$ made of eigenfunctions of it. Let $(\lambda_j)_{j \in \mathbb{N}}$ be the corresponding eigenvalues arranged in increasing order.

The functional calculus allows for the definition of the operator $A^m$, for each $m \in \mathbb{R}$, with domain

$$\text{Dom} \left( A^m \right) = \left\{ u \in L^2(R^n) : \sum_{j=1}^{\infty} \lambda_j^{2m} |\langle u_j, u \rangle_{L^2}|^2 < \infty \right\}, \quad (2.4)$$

so that if $u \in \text{Dom} \left( A^m \right)$, then we can write

$$A^m u = \sum_{j=1}^{\infty} \lambda_j^m \langle u_j, u \rangle_{L^2} u_j.$$ 

In the context of the anharmonic oscillator $A$ the below Sobolev spaces are introduced in [18, Definition 4.7]: for fixed $k, l \geq 1$ integers, and for $m \in \mathbb{R}$, we denote by $\mathcal{H}^m_{k, l}$, or simply by $\mathcal{H}^m$, the set of tempered distributions $u$ such that

$$A^m u \in L^2(R^n),$$

and for $u \in \mathcal{H}^m$ we define

$$\| u \|_{\mathcal{H}^m} := \| A^m u \|_{L^2(R^n)}. \quad (2.5)$$

The Sobolev spaces $\mathcal{H}^m$ reads as

$$\mathcal{H}^m \equiv H \left( M^m \frac{\pi}{\tau}, G \right),$$

with regards to the notion of Sobolev spaces adjusted to the Weyl–Hörmander calculus, where $M, G$ are as in (2.3) and (2.2), respectively.

Recall that for some Hörmander metric $g$ and some $g$-weight $m$, the Sobolev space denoted by $H(m, g)$ is the set of tempered distributions $u$ such that

$$a^m u \in L^2(R^n), \quad \forall a \in S(m, g),$$

where $a^m$ denotes the Weyl-quantization of the symbol $a$.

For the Sobolev embeddings in the scale of Weyl–Hörmander calculus, we have the following result due to Chemin and Xu [20, Theorem 1.9].
Theorem 2.1 Let \( m_1, m_2 \) be two g-weights such that
\[
\lim_{x, \xi \to \infty} \frac{m_1(x, \xi)}{m_2(x, \xi)} = \infty,
\]
then \( H(m_1, g) \) is compactly included in \( H(m_2, g) \).

The next technical lemma will be useful for our purposes.

Lemma 2.2 Let \((u_j)_{j \in \mathbb{N}}\) be the set of eigenvalues of \( A \) with corresponding eigenvalues \((\lambda_j)_{j \in \mathbb{N}}\). There exists \( C > 0 \) such that
\[
\|u_j\|_{L^\infty(\mathbb{R}^n)} \leq C \sqrt{\lambda_j} \quad \text{for every} \quad j \in \mathbb{N}.
\]

Proof It is known, cf. [9, Theorem 8.8], that the usual Sobolev space of order \( p \); that is the space \( H^p(\mathbb{R}^n) = H(\langle \xi \rangle^p, g^{1,0}) \), where \( g^{1,0} \) is defined by
\[
g^{1,0}(dx, d\xi) = |dx|^2 + |d\xi|^2 / \langle \xi \rangle^{-2},
\]
is continuously included in \( L^\infty(\mathbb{R}^n) \) for any \( p \in [1, \infty) \). On the other hand, for \( m \in \mathbb{R} \), we have \( H \left( M^m, g^{1,0} \right) \), where \( M \) as in (2.3), is compactly included in \( H^p(\mathbb{R}^n) \) for \( p \leq lm \), where \( k, l \geq 1 \) are the fixed integers associated to \( A \). Indeed the last is true by Theorem 2.1, since
\[
\lim_{x, \xi \to \infty} \frac{(1 + |x|^{2k} + |\xi|^{2l})^m}{\langle \xi \rangle^p} = \infty, \quad \text{for} \quad lm \geq p.
\]
For \( g \) as in (2.2) we have the inclusion \( H \left( M^m, g \right) \subset H \left( M^m, g^{k,l} \right) \). Summarising the above, and choosing \( p = 1 \) we conclude that the space \( H \left( M^m, g^{k,l} \right) \) is continuously embedded in the space \( L^\infty(\mathbb{R}^n) \) under the condition \( lm \geq 1 \). The latter implies that
\[
\|u_j\|_{L^\infty(\mathbb{R}^n)} \leq C \left\| A^m u_j \right\|_{L^2(\mathbb{R}^n)} \leq C \lambda_j^m \|u\|_{L^2(\mathbb{R}^n)} = C \lambda_j^m, \quad C > 0,
\]
for every \( m \in \mathbb{R} \) such that \( lm \geq 1 \), and so also for \( m = 1 \) and this completes the proof of Lemma 2.2. \( \square \)

Finally, let us briefly recall some spectral properties of the anharmonic oscillator \( A \). In [18, Corollary 5.6] that authors prove that \( A^{-1} \in S_r(L^2(\mathbb{R}^n)) \) for \( r > \frac{(k+l)n}{2kl} \), where by \( S_r(L^2(\mathbb{R}^n)) \) we have denoted the \( r \)th Schatten–von Neumann class of operators on the Hilbert space \( L^2(\mathbb{R}^n) \). For an operator \( T \) in the class \( S_r \), with singular values \( (s_j(T))_j \), \(^1\) it is known that
\[
s_j(T) = o \left( j^{-\frac{1}{r}} \right), \quad \text{as} \quad j \to \infty,
\]
\(^1\) The singular values of a compact operator \( T \) between Hilbert spaces are the square roots of non-negative eigenvalues of the self-adjoint operator \( T^*T \), where \( T^* \) denotes the adjoint of \( T \).
where the singular values appear in decreasing order. Hence, in the particular case of the compact, self-adjoint $A^{-1}$, in which we have $s_j = \lambda_j$, the above property reads as follows

$$\lambda_j^{-1} = o\left(j^{\frac{1}{r}}\right), \quad \text{as} \quad j \to \infty,$$

(2.6)

for $r > \frac{(k+l)n}{2kl}$ where the eigenvalues $(\lambda_j^{-1})_j$ appear in decreasing order. Moreover, if $N(\Lambda)$ is the eigenvalue counting function associated with the anharmonic oscillator $A$; that is

$$N(\Lambda) := \{ j : \lambda_j \leq \Lambda \},$$

then we have the following estimate

$$N(\Lambda) \lesssim \Lambda^{n\left(\frac{k+l}{k+l} + \frac{1}{2}\right)}, \quad \text{as} \quad \Lambda \to \infty,$$

(2.7)

cf. [10, Theorem 3.2].

3 Fourier Analysis Associated with the Anharmonic Oscillator

In this section we present the Fourier analysis associated with the eigenfunction expansion of the anharmonic oscillator $A$ as in (2.1). The theory presented here is a natural analogue of Fourier analysis developed in [41] in the setting of nonharmonic analysis associated with a model operator. Therefore we will not present any proofs for any of our statements which can be derived verbatim as in [41].

As before, we denote by $(u_j)_{j \in \mathbb{N}}$ the eigenvalues of the operator $A$, and by $(\lambda_j)_{j \in \mathbb{N}}$ the corresponding eigenvalues in increasing order.

Let us now define the space of functions

$$C^\infty_A(\mathbb{R}^n) := \cap_{m=1}^\infty \text{Dom}(A^m),$$

(3.1)

where the domain of $A^m$ is as in (2.4), whose Fréchet topology is given by the family of norms

$$\|f\|_{C^m_A} := \max_{n \leq m} \|A^n f\|_{L^2(\mathbb{R}^n)}, \quad m \in \mathbb{N}_0, \quad f \in C^\infty_A(\mathbb{R}^n).$$

Since $(u_j)_{j \in \mathbb{N}}$ is dense in $L^2(\mathbb{R}^n)$ we have that $C^\infty_A(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.

A-Fourier transform: Let $S(\mathbb{N})$ be the space of rapidly decreasing functions $\phi : \mathbb{N} \to \mathbb{C}$, i.e., for any $N < \infty$, there exists a constant $C_{\phi,N}$ such that $|\phi(j)| \leq C_{\phi,N} j^{-N}$ for all $j \in \mathbb{N}$. The space $S(\mathbb{N})$ forms a Fréchet space if endowed with the family $\{p_r\}_{r \in \mathbb{N}}$ of semi-norms

$$p_r(\phi) := \sup_{j \in \mathbb{N}} (j^r |\phi(j)|).$$

© Birkhäuser
We define the \textit{A-Fourier transform} to be that bijective homeomorphism

\[ \mathcal{F}_A : C^\infty_A(\mathbb{R}^n) \to \mathcal{S}(\mathbb{N}) \]

defined by

\[ (\mathcal{F}_A f)(j) := \hat{f}(j) := \int_{\mathbb{R}^n} f(x) u_j(x) \, dx. \quad (3.2) \]

The inverse operator \( \mathcal{F}_A^{-1} : \mathcal{S}(\mathbb{N}) \to C^\infty_A(\mathbb{R}^n) \) is given by

\[ (\mathcal{F}_A^{-1} h)(x) := \sum_{j \in \mathbb{N}} h(j) u_j(x), \]

so that the Fourier inversion formula becomes

\[ f(x) = \sum_{j \in \mathbb{N}} \hat{f}(j) u_j(x), \quad f \in C^\infty_A(\mathbb{R}^n). \quad (3.3) \]

The Plancherel identity holds true for the \( A \)-Fourier transform of functions in the space \( l^2_A \) that we define as the (linear) space of complex-valued sequences \( a \) on \( \mathbb{N} \) such that \( \mathcal{F}_A^{-1} a \in L^2(\mathbb{R}^n) \). The latter means that there exists \( f \in L^2(\mathbb{R}^n) \) such that \( \mathcal{F}_A f = a \). The space of sequences \( l^2_A \) is a Hilbert space under the norm

\[ \|a\| = \left( \sum_{j \in \mathbb{N}} |a(j)|^2 \right)^{\frac{1}{2}}, \]

since by formal calculations

\[ (a, b)_{l^2_A} = (\mathcal{F}_A^{-1} a, \mathcal{F}_A^{-1} b)_{L^2}, \quad a, b \in l^2_A, \]

and the inner product is then given by

\[ (a, b)_{l^2_A} := \sum_{j \in \mathbb{N}} a(j) \overline{b(j)}, \]

for arbitrary \( a, b \in l^2_A \). For any \( f \in L^2(\mathbb{R}^n) \) we have \( \hat{f} \in l^2_A \), and the Plancherel identity is satisfied

\[ \|f\|_{L^2} = \|\hat{f}\|_{l^2_A}. \quad (3.4) \]
Let us now define the generalisation of the space $l^2_A$ to be the space of functionals $a$ on $S(\mathbb{N})$ denoted by $l^p_A$, for $1 \leq p < \infty$, such that
\[
\|a\|_{l^p(A)} := \left( \sum_{j \in \mathbb{N}} |a(j)|^p \|u_j\|_{L^\infty(\mathbb{R}^n)}^{2-p} \right)^{1/p} < \infty. \tag{3.5}
\]
The space $l^\infty_A$ shall be defined as the set of $a \in S'(\mathbb{N})$ such that
\[
\|a\|_{l^\infty(A)} := \sup_{j \in \mathbb{N}} \left( |a(j)| \cdot \|u_j\|_{L^\infty(\mathbb{R}^n)}^{-1} \right) < \infty.
\]

The following theorem reads as the Hausdorff–Young inequality in our setting, and can be proved using standard interpolation properties as in [41, Theorem 7.6].

**Theorem 3.1** (Hausdorff–Young inequality) Let $1 \leq p \leq 2$ and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then, there is a constant $C_p \geq 1$ such that, for all $f \in L^p(\mathbb{R}^n)$ and for all $a \in l^p(A)$, we have
\[
\|\hat{f}\|_{l^{p'}(A)} \leq C_p \|f\|_{L^p}, \tag{3.6}
\]
and
\[
\|\mathcal{F}_A^{-1}a\|_{L^{p'}} \leq C_p \|a\|_{l^p(A)}.
\]

**4 $L^p$-$L^q$ Boundedness of Fourier Multipliers for $1 < p \leq 2 \leq q < \infty$**

In this section we present our boundedness results, first in a general form, and later on we specialise by considering particular examples of operators. Our analysis begins with proving our version of two well-known inequalities; namely, the Paley-type inequality, and the Hausdorff–Young–Paley-type inequality.

**4.1 Hausdorff–Young–Paley Inequality**

The Hausdorff–Young–Paley inequality is a generalisation of the Paley-type inequality that follows. For the proof of it interpolation techniques as in Corollary 4.3 are necessary.

**Theorem 4.1** (Paley-type inequality) Let $1 < p \leq 2$. If $\varphi(j)$ is a positive sequence in $\mathbb{N}$ such that
\[
M_\varphi := \sup_{t > 0} t \sum_{j \in \mathbb{N}} \frac{\|u_j\|^2_{L^\infty(\mathbb{R}^n)}}{t^{\varphi(j)}} < \infty,
\]
then for every \( f \in L^p(\mathbb{R}^n) \) we have

\[
\left( \sum_{j \in \mathbb{N}} |\mathcal{F}_A(f)(j)|^p \|u_j\|_{L^\infty(\mathbb{R}^n)}^{2-p} \varphi(j)^{2-p} \right)^{\frac{1}{p}} \lesssim M_{\varphi} \|f\|_{L^p(\mathbb{R}^n)}. \quad (4.1)
\]

**Proof** Let \( \nu \) be the measure on \( \mathbb{N} \) defined by \( \nu(\{j\}) := \varphi^2(j)\|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \) for \( j \in \mathbb{N} \). Now, we define weighted spaces \( L^p(\mathbb{N}, \nu) \), \( 1 \leq p \leq 2 \), as the spaces of complex (or real) sequences \( a = \{a_j\}_{j \in \mathbb{N}} \) such that

\[
\|a\|_{L^p(\mathbb{N}, \nu)} := \left( \sum_{j \in \mathbb{N}} |a_j|^p \varphi^2(j) \|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \right)^{\frac{1}{p}} < \infty. \quad (4.2)
\]

We show that the subadditive operator \( T : L^p(\mathbb{R}^n) \to L^p(\mathbb{N}, \nu) \) defined by

\[
Tf := \left\{ \frac{|\mathcal{F}_A(f)(j)|}{\|u_j\|_{L^\infty(\mathbb{R}^n)} \varphi(j)} \right\}_{j \in \mathbb{N}}
\]

is well-defined and bounded from \( L^p(\mathbb{R}^n) \) to \( L^p(\mathbb{N}, \nu) \) for \( 1 < p \leq 2 \). In other words, we claim that we have the estimate

\[
\|Tf\|_{L^p(\mathbb{N}, \nu)} = \left( \sum_{j \in \mathbb{N}} \left( \frac{|\mathcal{F}_A(f)(j)|}{\|u_j\|_{L^\infty(\mathbb{R}^n)} \varphi(j)} \right)^p \varphi^2(j) \|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \right)^{\frac{1}{p}} \lesssim M_{\varphi} \|f\|_{L^p(\mathbb{R}^n)}, \quad (4.3)
\]

which would give us (4.1) and where we set

\[
M_{\varphi} := \sup_{t > 0} \sum_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\mathbb{R}^n)}^2 |\varphi(j) \geq t|.
\]

To prove this we will show that \( T \) is of weak type \((2, 2)\) and of weak type \((1, 1)\). More precisely, for the distribution function,

\[
v_N(y; Tf) = \sum_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \varphi^2(j), \quad \text{where} \quad y \in \mathbb{N}
\]

we show that

\[
v_N(y; Tf) \leq \left( \frac{M_2 \|f\|_{L^2(\mathbb{R}^n)}}{y} \right)^2 \quad \text{with norm} \quad M_2 = 1, \quad (4.4)
\]
and
\[ v_N(y; T_f) \leq \frac{M_1 \|f\|_{L^1(\mathbb{R}^n)}}{y} \] with norm \( M_1 = M \phi \). \hfill (4.5)

Then (4.3) will follow by the Marcinkiewicz interpolation theorem, cf. [8, Theorem 1.1.3]. Now, to show (4.4), using the Plancherel identity as in Proposition 3.4 we get
\[
y^2 v_N(y; T_f) \leq \|T_f\|_{L^2(N, v)}^2 = \sum_{j \in \mathbb{N}} \left( \frac{|\mathcal{F}_A(f)(j)|}{\varphi(j)\|u_j\|_{L^\infty(\mathbb{R}^n)}} \right)^2 \varphi^2(j)\|u_j\|_{L^\infty(\mathbb{R}^n)}^2
\]
\[ = \sum_{j \in \mathbb{N}} |\mathcal{F}_A(f)(j)|^2 = \mathcal{F}_A(f)\|_{l^2(L^1)} = \|f\|_{L^2(\mathbb{R}^n)}. \hfill (4.6)\]

Thus, \( T \) is of weak type \((2, 2)\) with norm \( M_2 \leq 1 \). Further, we show that \( T \) is of weak type \((1, 1)\) with norm \( M_1 = M \phi \); more precisely, we show that
\[
y \in \{ j \in \mathbb{N} : |\mathcal{F}_A(f)(j)|/\varphi(j)\|u_j\|_{L^\infty(\mathbb{R}^n)} > y \} \leq M \phi \|f\|_{L^1(\mathbb{R}^n)} / y. \hfill (4.7)\]

Here, the left hand-side is the weighted sum \( \sum_{j \in \mathbb{N}} \varphi^2(j)\|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \) taken over those \( j \in \mathbb{N} \) such that \( |\mathcal{F}_A(f)(j)|/\varphi(j)\|u_j\|_{L^\infty(\mathbb{R}^n)} > y \). From the definition of the Fourier transform and Hölder’s inequality it follows that
\[ |\mathcal{F}_A(f)(j)| \leq \|u_j\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}. \]

Therefore, we get
\[
y < \frac{|\mathcal{F}_A(f)(j)|}{\varphi(j)\|u_j\|_{L^\infty(\mathbb{R}^n)}} \leq \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\varphi(j)}. \]

Using this, we get
\[
\{ j \in \mathbb{N} : |\mathcal{F}_A(f)(j)|/\varphi(j)\|u_j\|_{L^\infty(\mathbb{R}^n)} > y \} \subset \{ j \in \mathbb{N} : \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\varphi(j)} > y \}
\]
for any \( y > 0 \). Consequently,
\[
v \left\{ j \in \mathbb{N} : |\mathcal{F}_A(f)(j)|/\varphi(j)\|u_j\|_{L^\infty(\mathbb{R}^n)} > y \right\} \leq v \left\{ j \in \mathbb{N} : \frac{\|f\|_{L^1(\mathbb{R}^n)}}{\varphi(j)} > y \right\}. \]

By setting \( w := \frac{\|f\|_{L^1(\mathbb{R}^n)}}{y} \), we get
\[
v \left\{ j \in \mathbb{N} : |\mathcal{F}_A(f)(j)|/\varphi(j) > y \right\} \leq \sum_{j \in \mathbb{N}} \frac{\varphi^2(j)\|u_j\|_{L^\infty(\mathbb{R}^n)}^2}{\varphi(j) \leq w}.\]
We claim that
\[ \sum_{j \in \mathbb{N}} \varphi(j) \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \lesssim M_\varphi w. \] (4.8)

In fact, we have
\[ \sum_{j \in \mathbb{N}} \varphi(j) \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 = \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \int_0^{\varphi(j)} d\tau. \]

We can interchange sum and integration to get
\[ \sum_{j \in \mathbb{N}} \varphi(j) \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \leq \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \int_0^{\varphi(j)} d\tau. \]

Further, we make a substitution \( \tau = t^2 \), yielding
\[ \int_0^{w} t \sum_{\tau \leq \varphi(j) \leq w} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \leq 2 \int_0^{w} t \sum_{\tau \leq \varphi(j) \leq w} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \leq 2 \int_0^{w} t \sum_{t \leq \varphi(j) \leq w} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2. \]

Now, since
\[ t \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \leq \sup_{t > 0} t \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 = M_\varphi \]
is finite by assumption, we have
\[ 2 \int_0^{w} t \sum_{t \leq \varphi(j) \leq w} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \lesssim M_\varphi w = \frac{M_\varphi \| f \|_{L^1(\mathbb{R}^n)}}{y}. \]

This proves (4.8). Therefore, we have proved inequality (4.4) and (4.5). Then by using the Marcinkiewicz interpolation theorem we obtain that for \( 1 < p < 2 \)
\[ \left( \sum_{j \in \mathbb{N}} \left( \frac{|\mathcal{F}A(f)(j)|}{\| u_j \|_{L^\infty(\mathbb{R}^n)} \varphi(j)} \right)^p \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \varphi(j)^2 \right)^{\frac{1}{p}} = \| Tf \|_{L^p(\mathbb{N}, \nu)} \lesssim_p M_\varphi \frac{2-p}{p} \| f \|_{L^p(\mathbb{R}^n)}. \]
At the same time (4.6) suggest that the last inequality holds true for \( p = 2 \) as an equality. The proof is now complete. \( \square \)

The following theorem, see e.g. [8], is an interpolation theorem of \( L^p \) spaces with change of measure.

**Theorem 4.2** Let \( d\mu_0(x) = \omega_0(x)d\mu(x) \), \( d\mu_1(x) = \omega_1(x)d\mu(x) \). Let us also write \( L^p(\omega) = L^p(\omega d\mu) \) for the weight \( \omega \). Suppose that \( 0 < p_0, p_1 < \infty \). Then

\[
(L^{p_0}(\omega_0), L^{p_1}(\omega_1))_{\theta,p} = L^p(\omega), \quad 0 < \theta < 1,
\]

where \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \) and \( \omega = \omega_0^{p/p_0} \omega_1^{p/p_1} \).

As a corollary of Theorem 4.2 we get the following extension (cf. [8, Corollary 5.5.4]) of the interpolation theorem of Stein-Weiss.

**Corollary 4.3** (Stein–Weiss) Assume that \( 1 \leq p_0, p_1, q_0, q_1 < \infty \), and that

\[
T : L^{p_0}(\omega_0 d\mu) \to L^{q_0}(\tilde{\omega}_0 d\nu),
T : L^{p_1}(\omega_1 d\mu) \to L^{q_1}(\tilde{\omega}_1 d\nu),
\]

with norms \( M_0 \) and \( M_1 \), respectively. Then

\[
T : L^p(\omega d\mu) \to L^q(\tilde{\omega} d\nu),
\]

with norm \( M \) such that

\[
M \leq M_0^{1-\theta} M_1^{\theta},
\]

where

\[
\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},
\]

\[
\omega = \omega_0^{p(1-\theta)/p_1} \omega_1^{p\theta/p_1}, \quad \tilde{\omega} = \omega_0^{q(1-\theta)/q_0} \tilde{\omega}_1^{q\theta/q_1}.
\]

Using the above, we can now prove a version of the Hausdorff–Young–Paley inequality adapted to our setting.

**Theorem 4.4** (Hausdorff–Young–Paley inequality) Let \( 1 < p \leq 2 \), and let \( 1 < p \leq b \leq p' \leq \infty \), where \( p' = \frac{p}{p-1} \). If \( \varphi(j) \) is a positive sequence in \( \mathbb{N} \) such that

\[
M_{\varphi} := \sup_{t > 0} \sum_{j \in \mathbb{N}} \|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \sum_{j \leq \varphi(j)} \|u_j\|_{L^\infty(\mathbb{R}^n)}
\]
is finite, then for every $f \in L^p(\mathbb{R}^n)$ we have

$$\left(\sum_{j \in \mathbb{N}} \left| \mathcal{F}_A f(j) \varphi(j) \frac{1}{b^p} \right|^b \|u_j\|^{2-b}_{L^\infty(\mathbb{R}^n)} \right)^{\frac{1}{b}} \lesssim M_{\varphi}^{\frac{1}{b} - \frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n)}. \quad (4.9)$$

**Proof** By the Paley-type inequality as in Theorem 4.1, the operator $T$ defined by

$$Tf(j) := \frac{\mathcal{F}_A f(j)}{\|u_j\|_{L^\infty(\mathbb{R}^n)}},$$

is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{N}, \omega_0)$, where $\omega_0(j) = \|u_j\|_{L^\infty(\mathbb{R}^n)} \varphi(j)^{2-p}$, and the space $L^p(\mathbb{N}, \omega)$ has been endowed with the norm

$$\| \cdot \|_{L^p(\mathbb{N}, \omega)} := \left(\sum_{j \in \mathbb{N}} |\cdot(j)|^p w(j) \right)^{\frac{1}{p}},$$

From Hausdorff–Young inequality as in Theorem 3.1, we deduce that $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{N}, \omega_1)$ with $\omega_1(j) = \|u_j\|_{L^\infty(\mathbb{R}^n)}^2 \varphi(j)^{2-p}$, admits a bounded extension, while, by Corollary 4.3, the operator $T : L^p(\mathbb{R}^n) \to L^b(\mathbb{N}, \omega)$, is continuous for $p \leq b \leq p'$ such that $\frac{1}{b} = \frac{1-\theta}{p} + \frac{\theta}{p'}$, where $\theta \in (0, 1)$. By fixing $\theta = \frac{p-b}{b(p-2)}$, we get

$$\omega = \omega_0^{\frac{p(1-\theta)}{p(1-\theta)}} \omega_1^{\frac{p\theta}{p(1-\theta)}} = \varphi(j)^{1-\frac{b}{p'}} \|u_j\|^{2-b}_{L^\infty(\mathbb{R}^n)},$$

so that for the operator norm we have $\|T\|_{L^p \to L^b} \leq M_{\varphi}^{\frac{1}{b} - \frac{1}{p'}}$. The proof is now complete. \hfill \Box

Naturally, the Hausdorff–Young–Paley inequality (4.9) reduces to the Hausdorff–Young inequality (3.6) when $b = p'$ and to the Paley-type inequality (4.1) when $b = p$.

### 4.2 $L^p$-$L^q$ Boundedness

In this paragraph we prove the $L^p$-$L^q$ boundedness of Fourier multipliers in our setting. Analogous results have been proved in the case of the torus in [36] using a different method. We recall here that $m$ is an $A$-Fourier multiplier on $\mathbb{R}^n$ if it satisfies

$$\mathcal{F}_A(mf)(j) = \sigma_m(j) \mathcal{F}_A f(j), \quad j \in \mathbb{N},$$

where $\sigma_m : \mathbb{N} \to \mathbb{C}$ is a function that shall be called the $A$-symbol, or simply symbol, of the multiplier $m$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Example of a Fourier multiplier.}
\end{figure}
Theorem 4.5 Let \( 1 < p \leq 2 \leq q < \infty \). Suppose that \( m \) is an \( A \)-Fourier multiplier with symbol \( \sigma_m \) on \( \mathbb{R}^n \). Then we have

\[
\| m \|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \lesssim \sup_{s > 0} \left( \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \right)^{-\frac{1}{p}}. 
\]

\textbf{Proof} Let us first assume that \( p \leq q' \), where \( \frac{1}{q} + \frac{1}{q'} = 1 \). Since \( q' \leq 2 \), the Hausdorff–Young inequality Theorem 3.1 gives that

\[
\| mf \|_{L^q(\mathbb{R}^n)} \leq \| \mathcal{F}_A(mf) \|_{\ell^{q'}(A)} = \| \sigma_m \mathcal{F}_A(f) \|_{\ell^{q'}(A)} 
= \left( \sum_{j \in \mathbb{N}} |\sigma_m(j)|^{q'} |\mathcal{F}_A(f)(j)|^{q'} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \right)^{-\frac{1}{q'}}. 
\]

(4.10)

The case \( q' \leq (p')' = p \) can be reduced to the case \( p \leq q' \) as follows. Using the duality of \( L^p \)-spaces we have \( \| m \|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} = \| m^* \|_{L'^{q'}(\mathbb{R}^n) \to L'^{p'}(\mathbb{R}^n)} \). The symbol \( \sigma_{m^*}(j) \) of the adjoint operator \( m^* \), which is also an \( A \)-Fourier multiplier, is equal to \( \sigma_m(j) \) (cf. [41, Proposition 9.7]), and obviously we have \( |\sigma_m(j)| = |\sigma_{m^*}(j)| \). Therefore (4.10) holds true for \( 1 < p \leq 2 \leq q < \infty \). Now, we are in a position to apply Hausdorff–Young–Paley inequality Theorem 4.4. Set \( \frac{1}{p} - \frac{1}{q} = \frac{1}{r} \). Then, applying Theorem 4.4 with \( \varphi(j) = |\sigma_m(j)|^r \) with \( b = q' \) we get

\[
\| \sigma_m \mathcal{F}_A(f) \|_{\ell^{q'}(A)} \lesssim \left( \sup_{s > 0} \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \right)^{-\frac{1}{r}} \| f \|_{L^p(\mathbb{R}^n)}
\]

for all \( f \in L^p(\mathbb{R}^n) \), in view of \( \frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'} = \frac{1}{r} \). Thus, for \( 1 < p \leq 2 \leq q < \infty \), we obtain

\[
\| mf \|_{L^q(\mathbb{R}^n)} \lesssim \left( \sup_{s > 0} \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \right)^{-\frac{1}{r}} \| f \|_{L^p(\mathbb{R}^n)},
\]

and this completes the proof Theorem 4.5 under the observation that

\[
\left( \sup_{s > 0} \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \right)^{-\frac{1}{r}} \leq \left( \sup_{s > 0} \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \right)^{-\frac{1}{q'}} \leq \left( \sup_{s > 0} \sum_{j \in \mathbb{N}} \| u_j \|_{L^\infty(\mathbb{R}^n)}^2 \right)^{-\frac{1}{q'}}.
\]
Corollary 4.6  Let $1 < p \leq 2 \leq q < \infty$, and let $\varphi$ be any decreasing function on $[1, \infty)$ such that $\lim_{u \to \infty} \varphi(u) = 0$. Then,

$$\|\varphi(A)\|_{op} \lesssim \sup_{u > 1} \varphi(u) \left( u^{1 + \frac{(k+l)m}{2d}} \right) \frac{1}{p - \frac{1}{q}},$$

(4.11)

where $\| \cdot \|_{op}$ denotes the operator norm from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

**Proof** Observe that the operator $\varphi(A)$ is a Fourier multiplier with symbol $(\varphi(\lambda_j))_{j \in \mathbb{N}}$. Therefore an application of Theorem 4.5 yields

$$\|\varphi(A)\|_{op} \lesssim \sup_{s > 0} s \left( \sum_{j \in \mathbb{N}} \|u_j\|^2_{L^\infty(\mathbb{R}^n)} \right)^{\frac{1}{p} - \frac{1}{q}} = \sup_{0 < s \leq \varphi(1)} s \left( \sum_{j \in \mathbb{N}} \|u_j\|^2_{L^\infty(\mathbb{R}^n)} \right)^{\frac{1}{p} - \frac{1}{q}},$$

(4.12)

since $\varphi \leq \varphi(1)$, so that $s \in (0, \varphi(1)]$ can be written as $s = \varphi(u)$. Now, by Lemma 2.2 together with (4.12) and by using the monotonicity of $\varphi$ we get

$$\|\varphi(A)\|_{op} \lesssim \sup_{u > 1} \varphi(u) \left( \sum_{j \in \mathbb{N}} \|u_j\|^2_{L^\infty(\mathbb{R}^n)} \right)^{\frac{1}{p} - \frac{1}{q}} \lesssim \sup_{u > 1} \varphi(u) \left( \sum_{j \in \mathbb{N}} \lambda_j \right)^{\frac{1}{p} - \frac{1}{q}}.$$  

(4.13)

Now, notice that as $u \to \infty$, there exists some $j_0 \in \mathbb{N}$ big enough such that $u \sim \lambda_{j_0}$. Then one can estimate

$$\sum_{j \in \mathbb{N}} \lambda_j \lesssim u \sum_{j \in \mathbb{N}} 1 \lesssim u N(\lambda_{j_0}) \lesssim u \lambda_{j_0}^{\frac{(k+l)m}{2d}} \sim u^{1 + \frac{(k+l)m}{2d}},$$

(4.14)
where the eigenvalue counting function $N(\lambda_{j_0})$ associated to $A$ is as in (2.7). Therefore, by combining (4.13) with (4.14) one gets

$$
\|\varphi(A)\|_{op} \lesssim \sup_{u>1} \varphi(u) \left( u^{1+\frac{(k+l)n}{2kl}} \right)^{\frac{1}{p} - \frac{1}{q}},
$$

and the last completes the proof of Corollary 4.6. \(\square\)

**Remark 4.7** Notice that one can estimate the operator norm $\|\varphi(A)\|_{op}$ with $\varphi$ as in Corollary 4.6 just by using the fact that $A^{-1} \in S_r(L^2(\mathbb{R}^n))$ for any $r > \frac{(k+l)n}{2kl}$. Indeed, by using that $u \sim \lambda_{j_0}$ for some $j_0$ big enough, while also by the eigenvalue asymptotics (2.6) we have that $j^2 \lesssim \lambda_j$ for any $r > \frac{(k+l)n}{2kl}$, one can alternatively get the following estimate

$$
\sum_{\lambda_j < u} \lambda_j^2 \lesssim \sum_{\lambda_j < \lambda_{j_0}} \lambda_j^2 \sum_{j=1}^{j_0} 1 = j_0 \lambda_{j_0}^2 \lesssim \lambda_{j_0}^2 \lambda_{j_0}^r \sim u^{2+r}.
$$

A combination of (4.13) together with (4.15) yields

$$
\|\varphi(A)\|_{op} \lesssim \sup_{u>1} \varphi(u) \left( u^{2+r} \right)^{\frac{1}{p} - \frac{1}{q}}.
$$

Observe that the estimate (4.11) is sharper than the one given by (4.16).

**Corollary 4.8** The operator $A^{-m}$ for $m > \left( 1 + \frac{(k+l)n}{2kl} \right) \left( \frac{1}{p} - \frac{1}{q} \right)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. In particular, the operator $A^{-m}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for any $m \geq 0$.

**Proof** Let $\varphi = \varphi(u) := u^{-m}$, for $u \geq 1$ and $m > 0$. Then, $\varphi$ satisfies the hypothesis of Corollary 4.6 and we have

$$
\|A^{-m}\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} \lesssim \sup_{u>1} u^{-m} u^{C_n,k,l} \left( \frac{1}{p} - \frac{1}{q} \right) < \infty,
$$

for $C_{n,k,l} \left( \frac{1}{p} - \frac{1}{q} \right) \leq m$, where $C_{n,k,l} = 1 + \frac{(k+l)n}{2kl}$. \(\square\)

**Remark 4.9** (Sobolev-type estimates) If we choose

$$
\varphi = \varphi(u) := \frac{1}{u^{a-b}}, \quad u \geq 1,
$$

then $\varphi$ satisfies the hypothesis of Corollary 4.6, and so for $f \in S(\mathbb{R}^n)$ we have

$$
\|A^{-a} A^b f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)},
$$

\(\copyright\) Birkhäuser
for \( a, b \) being such that \( a - b \geq C_{n,l,k} \cdot \left( \frac{1}{p} - \frac{1}{q} \right) \), where \( C_{n,l,k} = 1 + \frac{(k+l)n}{2kl} \), and for \( 1 < p \leq 2 \leq q < \infty \), or equivalently
\[
\| A^b f \|_{L^q(\mathbb{R}^n)} \lesssim \| A^a f \|_{L^p(\mathbb{R}^n)}. \tag{4.17}
\]

As particular cases of (4.17) we get:

1. For \( p = q = 2 \) we recover the inclusion
\[
\| f \|_{H^b} \lesssim \| f \|_{H^a}, \quad a \geq b,
\]
where the Sobolev norms \( \| \cdot \|_{H^m} \) have been defined in (2.5).

2. For \( a \geq C_{n,l,k} \cdot \left( \frac{1}{p} - \frac{1}{q} \right) \), \( b = 0 \), we get the Sobolev-type estimate
\[
\| f \|_{L^q(\mathbb{R}^n)} \leq C \| A^a f \|_{L^p(\mathbb{R}^n)}.
\]

**Remark 4.10** (The \( A \)-heat equation) for the operator \( A \) we consider the heat equation
\[
\partial_t v + Av = 0, \quad v(0) = v_0. \tag{4.18}
\]

One can check that for each \( t > 0 \) the function \( v(t) = v(t, x) := e^{-tA}v_0 \) is a solution of the initial value problem (4.18). For \( t > 0 \), consider the function \( \varphi = \varphi(u) := e^{-tu} \). Now, \( \varphi \) satisfies the assumptions of Corollary 4.6 and we get
\[
\| e^{-tA} \|_{L^p(\mathbb{R}^n)} \rightarrow L^q(\mathbb{R}^n)} \lesssim \sup_{u > 0} e^{-tu} \cdot u^{-C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right)}, \tag{4.19}
\]
for \( 1 < p \leq 2 \leq q < \infty \) and \( C_{n,k,l} = 1 + \frac{(k+l)n}{2kl} \). Using techniques of standard mathematical analysis we see that
\[
\sup_{u > 0} e^{-tu} \cdot u^{C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right)} = \left( \frac{C_{n,k,l}}{t} \left( \frac{1}{p} - \frac{1}{q} \right) \right)^{C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right)} \cdot e^{-C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right)} = C_{p,q} t^{-C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right) \cdot e^{-C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right) \cdot e^{-C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right)}}. \tag{4.20}
\]

Indeed, let us consider the function \( g(u) = e^{-tu} u^{C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right)} \), and compute its derivative
\[
g'(u) = u^{C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right) - 1} e^{-tu} \left( C_{n,k,l}\left( \frac{1}{p} - \frac{1}{q} \right) - ut \right).
\]
The only zero of the derivative \( g' \) is taken for \( u_0 = \frac{C_{n,k,l}}{t} \left( \frac{1}{p} - \frac{1}{q} \right) \) and \( g' \) changes sign from positive to negative at \( u_0 \). Thus, \( g \) attends a maximum at \( u_0 \) and we have proved (4.20). Finally, combining (4.19) with (4.20) we obtain
\[ \| v(t, \cdot) \|_{L^q(\mathbb{R}^n)} \lesssim C_{p,q,t}^{-1} \left( \frac{1}{p} - \frac{1}{q} \right) \| v_0 \|_{L^q(\mathbb{R}^n)}, \]

where \( C_{n,k,l} = 1 + \frac{(k+l)n}{2kl} \).

5 Nuclearity and Traces of Spectral Multipliers of Anharmonic Oscillator on Modulation Spaces

This section is devoted to the study of \( r \)-nuclearity, where \( 0 < r \leq 1 \), and traces of spectral multipliers of the anharmonic oscillator \( A \).

First we give a short exposition of nuclear operators on Banach spaces with the metric approximation property. Later on, we apply these ideas to the study of our operators when acting on modulation spaces \( \mathcal{M}^{p,q}_{w} \), where \( 1 \leq p, q < \infty \), on \( \mathbb{R}^n \).

Let \( T \in \mathcal{L}(\mathcal{B}, \mathcal{B}) \) be a linear operator from some Banach space \( \mathcal{B} \) to \( \mathcal{B} \). We say that \( T \) is a nuclear operator on \( \mathcal{B} \), and we write \( T \in \mathcal{N}(\mathcal{B}) \), if \( T \) admits a decomposition of the form

\[ T = \sum_{j=1}^{\infty} \varphi_j \otimes \psi_j', \quad \text{where } (\varphi_j)_{j=1}^{\infty} \subset \mathcal{B} \quad \text{and} \quad (\psi_j')_{j=1}^{\infty} \subset \mathcal{B}', \quad (5.1) \]

and we have \( \sum_{j=1}^{\infty} \| \psi_j' \|_{\mathcal{B}'} \| \varphi_j \|_{\mathcal{B}} < \infty \).

Grothendieck in [28] proved that if \( \mathcal{B} \) has the metric approximation property, and \( T \in \mathcal{N}(\mathcal{B}) \), then the trace of \( T \), denoted by \( \text{Tr}(T) \), is well defined, i.e., we have that

\[ \text{Tr}(T) = \sum_{j=1}^{\infty} \langle \varphi_j, \psi_j' \rangle_{\mathcal{B},\mathcal{B}'} < \infty, \quad (5.2) \]

Recall that the Banach space \( \mathcal{B} \) has the metric approximation property if for every compact set \( K \subset \mathcal{B} \) and \( \epsilon > 0 \), there exists an operator \( F \) of finite rank such that \( \| x - Fx \|_{\mathcal{B}} < \epsilon \) for all \( x \in K \), see [38, Lemma 10.2.20] for this characterisation of Banach spaces with the metric approximation property.

We note that nuclear operators agree with trace class operators in the setting of Hilbert spaces. For such operators Lidskiĭ [33] proved that

\[ \text{Tr}(T) = \sum_{j=1}^{\infty} \lambda_j, \quad (5.3) \]

where each eigenvalue \( \lambda_j \) is repeated accordingly to multiplicity. Formula (5.3) is known as Lidskiĭ’s formula. However, as we will see later, Lidskiĭ’s formula can still hold true for operators on some Banach spaces, under certain assumptions. To become more precise, let us first recall the following definition.
Let $\mathcal{B}$ be a Banach space and let $0 < r \leq 1$. A linear operator $T \in \mathcal{L}(\mathcal{B}, \mathcal{B})$ is called $r$-nuclear if $T$ has a representation as in (5.1) that satisfies

$$\sum_{j=1}^{\infty} \|\psi_j^r\|_{\mathcal{B}'}\|\varphi_j\|_{\mathcal{B}} < \infty.$$  

In [28], Grothendieck proved that if $T \in \mathcal{L}(\mathcal{B}, \mathcal{B})$ is a $\frac{2}{3}$-nuclear operator (and so also $r'$-nuclear, with $r' \leq 2/3$), and $\mathcal{B}$ has the metric approximation property, then the trace of $T$ can be calculated using Lidskiǐ’s formula (5.3).

The operators we consider here, act on the modulation spaces $\mathcal{M}_w^{p,q}$ on $\mathbb{R}^n$ introduced by Feichtinger [26]. These spaces are, under some conditions on the weight function $w$, Banach spaces with the metric approximation property. Roughly speaking a modulation space is defined by imposing a quasi-norm estimate on the short-time Fourier transform (STFT) $V_g f$ of the involved distributions $f$. Formally, we give the following definition

**Definition 5.1** Let $1 \leq p, q < \infty$. For a suitable weight $w$ on $\mathbb{R}^{2n}$, and a window function $g \in \mathcal{S}(\mathbb{R}^n)$, we define the modulation space $\mathcal{M}_w^{p,q}(\mathbb{R}^n)$ to be the set of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{M}_w^{p,q}} := \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |V_g f(x, \xi)|^p w(x, \xi)^p \, dx \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{q}} < \infty,$$

where $V_g f$ denotes the short-time Fourier transform of $f$ with respect to $g$ at the point $(x, \xi)$, i.e., we can write

$$V_g f(x, \xi) = \int_{\mathbb{R}^n} f(y) \overline{g(y - x)} e^{-iy \cdot \xi} \, dy.$$  

The modulation space $\mathcal{M}_w^{p,q}(\mathbb{R}^n)$ endowed with the above norm becomes a Banach space.

We recall that a weight function is a non-negative, locally integrable function on $\mathbb{R}^{2n}$. A weight function $u$ is called submultiplicative if

$$u(x_1 + x_2) \leq u(x_1)u(x_2), \quad \text{for all } x_1, x_2 \in \mathbb{R}^{2n};$$

a weight function $v$ is called $u$-moderate if

$$v(x_1 + x_2) \leq u(x_1)v(x_2), \quad \text{for all } x_1, x_2 \in \mathbb{R}^{2n}.$$  

Weights $v_s$, $s \in \mathbb{R}$, of polynomial type; that is weights of the form

$$v_s(x, \xi) = (1 + |x|^2 + |\xi|^2)^{\frac{s}{2}},$$

are submultiplicative and $u$-moderate.
play an important role. Any $v_s$-moderated weight function for some $s$ will be called \textit{polynomially moderated}.

Polynomially moderate weights give rise to modulation spaces with the metric approximation property. In particular, we have the following Corollary as in [24, Corollary 3.1].

\textbf{Corollary 5.2} Let $1 \leq p, q < \infty$, and let $w$ be a submultiplicative polynomially moderate weight. Then $\mathcal{M}_{w}^{p,q}(\mathbb{R}^n)$ has the metric approximation property.

Thus, for $T \in \mathcal{N}(\mathcal{M}_{w}^{p,w})$, where $T = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j'$, and $(\phi_j)_j \subset \mathcal{M}_{w}^{p,q}$, $(\psi_j')_j \subset \mathcal{M}_{w^{-1}}^{p',q'}$ we can write

$$\text{Tr}(T) = \sum_{j=1}^{\infty} \langle \phi_j, \psi_j' \rangle_{\mathcal{M}_{w}^{p,q}, \mathcal{M}_{w^{-1}}^{p',q'}},$$

where the duality has been defined via

$$(f, h)_{\mathcal{M}_{w}^{p,q}, \mathcal{M}_{w}^{p',q'}} = \int_{\mathbb{R}^{2n}} Vg f(x, \xi) \overline{Vg h(x, \xi)} \, dx \, d\xi, \quad (5.4)$$

for $f \in \mathcal{M}_{w}^{p,q}$, $h \in \mathcal{M}_{w^{-1}}^{p',q'} = (\mathcal{M}_{w}^{p,q})'$.

Note that the duality (5.4) is different, with respect to taking the complex conjugate, from the one in [27] to make it compatible with the Grothendieck’s theory.

The main result of this chapter is an application of the next corollary as in [24, Corollary 5.1].

\textbf{Corollary 5.3} Let $0 < r \leq 1$, $1 \leq p, q < \infty$ and $w$ be a submultiplicative polynomially moderate weight. An operator $T \in \mathcal{L}(\mathcal{M}_{w}^{p,q}, \mathcal{M}_{w}^{p,q})$ is $r$-nuclear if and only if its kernel $k(x, y)$ can be written in the form

$$k(x, y) = \sum_{j=1}^{\infty} \phi_j \otimes \psi_j',$$

with $\phi_j \in \mathcal{M}_{w}^{p,q}$, $\psi_j' \in \mathcal{M}_{w^{-1}}^{p',q'}$ and

$$\sum_{j=1}^{\infty} \| \phi_j \|_{\mathcal{M}_{w}^{p,q}}^r \| \psi_j' \|_{\mathcal{M}_{w^{-1}}^{p',q'}}^r < \infty.$$ 

Moreover, if $T$ is $r$-nuclear with $r \leq \frac{2}{3}$, then

$$\text{Tr}(T) = \sum_{j=1}^{\infty} \lambda_j, \quad (5.5)$$

where $\lambda_j$, $j = 1, 2, \ldots$, are the eigenvalues of $T$ repeated according to multiplicity.
We note that in the context of a general Banach space with the approximation property, the range of \( r, r \leq \frac{2}{3} \), in the \( r \)-nuclearity is sharp for the trace formula (5.5). However, the above condition on \( r \) can be relaxed if one considers for instance traces in the \( L^p \) spaces (cf. [22, 39]). In addition, as the next theorem shows, the trace formula (5.5) can still holds true in the context of an operator \( f(A) \in \mathcal{L}(\mathcal{M}^{p,q}_w, \mathcal{M}^{p,q}_w) \) for even larger values of \( r \), and in particular even for simply nuclear operators (case \( r = 1 \)).

We can write with convergence in \( L^2(\mathbb{R}^n) \)

\[
g = \sum_{j=1}^{\infty} (g, u_j) L^2 u_j,
\]

where \( (u_j) \subset L^2(\mathbb{R}^n) \) is the set of eigenfunctions of \( A \).

Hence, the kernel of \( A \) can be written as

\[
k(x, y) = \sum_{j=1}^{\infty} Au_j(x)\overline{u}_j(y) = \sum_{j=1}^{\infty} \lambda_j u_j(x)\overline{u}_j(y).
\]

This can be justified by taking \( N > 0 \) large enough so that \( \sum_{j=1}^{\infty} \lambda_j^{-N} < \infty \) and the kernel of \( A^{-N} \) can be decomposed as

\[
k_{A^{-N}}(x, y) = \sum_{j=1}^{\infty} \lambda_j^{-N} u_j(x)\overline{u}_j(y).
\]

For functions \( f \) of \( A \), the kernel can be expressed in the form

\[
k_{f(A)} = \sum_{j=1}^{\infty} f(\lambda_j) u_j(x)\overline{u}_j(y).
\]

**Theorem 5.4** Let \( 0 < r \leq 1, 1 \leq p, q < \infty \) and \( w \) be a submultiplicative polynomially moderate weight. The operator \( f(A) \) is \( r \)-nuclear on \( \mathcal{M}^{p,q}_w(\mathbb{R}^n) \), provided that

\[
\sum_{j=1}^{\infty} |f(\lambda_j)|^r \| u_j \|_{\mathcal{M}^{p,q}_w}^{r} \| u_j \|_{\mathcal{M}^{p',q'}_w}^{r-1} < \infty.
\]

(5.6)

If, in particular, (5.6) holds for \( r = 1 \), then we have the trace formula

\[
\text{Tr}(f(A)) = \sum_{j=1}^{\infty} f(\lambda_j),
\]

(5.7)

where the series \( \sum_{j=1}^{\infty} f(\lambda_j) \) converges absolutely.
We note that the modulation space $\mathcal{M}^{p,q}_{w}$ corresponds to a weight of the form $w(x, \xi) = (1 + |x|^2 + |\xi|^2)^{\frac{r}{2}}$, while the STFT of $f$ with respect to $g$, $V_g f$ is taken with respect to the standard Gaussian window $g(x) = 2^{\frac{n}{2}} e^{-\pi x^2}$.

**Proof** The first part follows from Corollary 5.3. Formula (5.7) is expected from the general Grothendieck’s theory in the case where $r \leq \frac{2}{3}$. For $r = 1$, by (5.2) we obtain

$$\text{Tr}(f(A)) = \sum_{j=1}^{\infty} |f(\lambda_j)| \langle u_j, \overline{u}_j \rangle_{\mathcal{M}^{p,q}_{w}, \mathcal{M}^{p',q'}_{w-1}} = \sum_{j=1}^{\infty} f(\lambda_j),$$

since by the duality (5.4) we have

$$\langle u_j, \overline{u}_j \rangle_{\mathcal{M}^{p,q}_{w}, \mathcal{M}^{p',q'}_{w-1}} = (V_g u_j, V_g u_j)_{L^2} = (u_j, u_j)_{L^2} \| g \|_{L^2} = 1,$$

where $(u_j, u_j)_{L^2} = \| g \|_{L^2} = 1$, where we have used the orthogonality relations for the STFT, i.e., that

$$(V_{g_1} f_1, V_{g_2} f_2)_{L^2} = (f_1, f_2)_{L^2} \langle g_1, g_2 \rangle_{L^2}, \quad \text{for } f_1, f_2, g_1, g_2 \in L^2.$$

The series in (5.7) converges absolutely in view of

$$\sum_{j=1}^{\infty} |f(\lambda_j)| \| u_j \|_{\mathcal{M}^{p,q}_{w}} \| \overline{u}_j \|_{\mathcal{M}^{p',q'}_{w-1}} \leq \sum_{j=1}^{\infty} f(\lambda_j) \| u_j \|_{\mathcal{M}^{p,q}_{w}} \| \overline{u}_j \|_{\mathcal{M}^{p',q'}_{w-1}} < \infty,$$

which is finite by assumption. This completes the proof. □

Observe that the proof of Theorem 5.4 relies exclusively on the fact that the operator $A^{-1}$ is compact and self-adjoint on $L^2(\mathbb{R}^n)$. The following corollary is then immediate.

**Corollary 5.5** Let $0 < r \leq 1$, $1 \leq p, q < \infty$ and $w$ be a submultiplicative polynomially weight. Let also $T$ be an operator on $L^2(\mathbb{R}^n)$ with discrete spectrum, and whose eigenvalues form an orthonormal basis in $L^2(\mathbb{R}^n)$. Then the operator $f(T)$ is $r$-nuclear on $\mathcal{M}^{p,q}_{w}(\mathbb{R}^n)$ provided that

$$\sum_{j=1}^{\infty} |f(\lambda_j)|^r \| u_j \|_{\mathcal{M}^{p,q}_{w}}^r \| \overline{u}_j \|_{\mathcal{M}^{p',q'}_{w-1}}^r < \infty, \quad (5.8)$$

where $\{u_j\}_{j \in \mathbb{N}}$, $\{\lambda_j\}_{j \in \mathbb{N}}$ denote the sets of the eigenfunctions and eigenvalues of $T$, respectively. If in particular, (5.8) holds for $r = 1$, then we have the trace formula

$$\text{Tr}((f(T))) = \sum_{j=1}^{\infty} f(\lambda_j),$$

where the above series converges absolutely.

© Birkhäuser
Acknowledgements  The authors would like to thank the reviewers for their valuable comments and suggestions which helped to improve the manuscript. The authors are also grateful to Professor Michael Ruzhansky for his valuable suggestions. The authors are supported by the FWO Odysseus 1 Grant G.0F94.18N: Analysis and Partial Differential Equations, the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant Number 01M01021). Marianna Chatzakou is a Postdoctoral Fellow of the Research Foundation—Flanders (FWO) under the Postdoctoral Grant No. 12B1223N. Vishvesh Kumar is also supported FWO Senior Research Grant G011522N.

References

1. Akylzhanov, R., Nursultanov, E., Ruzhansky, M.: Hardy–Littlewood–Paley inequalities and Fourier multipliers on $SU(2)$. Stud. Math. 234(1), 1–29 (2016)
2. Akylzhanov, R., Nursultanov, E., Ruzhansky, M.: Hardy–Littlewood, Hausdorff–Young–Paley inequalities, and $L^p$–$L^q$ Fourier multipliers on compact homogeneous manifolds. J. Math. Anal. Appl. 479(2), 1519–1548 (2019)
3. Akylzhanov, R., Ruzhansky, M.: $L^p$–$L^q$ multipliers on locally compact groups. J. Funct. Anal. (2019). https://doi.org/10.1016/j.jfa.2019.108324
4. Anker, J.-P.: Fourier multipliers on Riemannian symmetric space of the noncompact type. Ann. Math. 132(3), 597–628 (1990)
5. Anker, J.-P.: Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces. Duke Math. J. 65(2), 257–297 (1992)
6. Bagchi, S., Thangavelu, S.: On Hermite pseudo-multipliers. J. Funct. Anal. 268(1), 140–170 (2015)
7. Barraza, E.S., Cardona, D.: On nuclear $L^p$ multipliers associated to the harmonic oscillator. In: Ruzhansky, M., Delgado, J. (eds) Analysis and Partial Differential Equations: Perspectives from Developing Countries, Springer Proceedings in Mathematics and Statistics, Imperial College, London, 2016. Springer (2019)
8. Bergh, J., Lofström, J.: Interpolation Spaces, Grundlehren der mathematischen Wissenschaften (1976)
9. Brezis, H.: Functional Analysis. Sobolev Spaces and Partial Differential Equations, Springer, New York (2002)
10. Buzano, E., Boggiatto, P., Rodino, L.: Global Hypoellipticity and Spectral Theory. Mathematical Research, vol. 92. Akademie Verlag, Berlin (1996)
11. Cardona, D., Ruzhansky, M.: Littlewood–Paley theorem, Nikolskii inequality, Besov spaces, Fourier and spectral multipliers on graded Lie groups (2016). arXiv:1610.04701
12. Cardona, D., Ruzhansky, M.: Hörmander condition for pseudo-multipliers associated to the harmonic oscillator (2018). arXiv:1810.01260
13. Cardona, D.: On the nuclear trace of Fourier Integral Operators. Rev. Integr. temas Mat. 37(2), 219–249 (2019)
14. Cardona, D.: Nuclear pseudo-differential operators in Besov spaces on compact Lie groups. J. Fourier Anal. Appl. 23(5), 1238–1262 (2017)
15. Cardona, D., Kumar, V.: $L^p$-boundedness and $L^p$-nuclearity of multilinear pseudo-differential operators on $\mathbb{R}^n$ and the torus $\mathbb{T}^n$. J. Fourier Anal. Appl. 25(6), 2973–3017 (2019)
16. Cardona, D., Kumar, V.: Multilinear analysis for discrete and periodic pseudo-differential operators in $L^p$ spaces. Rev. Integr. temas Mat. 36(2), 151–164 (2018)
17. Cardona, D., Corral, C., Kumar, V.: Dixmier traces for discrete pseudo-differential operators. J. Pseudo-Differ. Oper. Appl. 11(2), 647–656 (2020)
18. Chatzakou, M., Delgado, M., Ruzhansky, M.: On a class of anharmonic oscillators. J. Math. Pures Appl. 9(153), 1–29 (2021)
19. Chatzakou, M., Kumar, V.: $L^p$–$L^q$ boundedness of spectral multipliers of the anharmonic oscillator. C. R. Math. Acad. Sci. Paris 360, 343–347 (2022)
20. Chemin, J.Y., Xu, C.J.: Sobolev embedding in Weyl–Hörmander calculus. In: Colombini, F., Lerner, N. (eds.) Geometrical Optics and Related Topics. Progress in Nonlinear Differential Equations and Their Applications, vol. 32. Birkhäuser, Boston (1997)
21. Delgado, J., Ruzhansky, M.: Fourier multipliers, symbols, and nuclearity on compact manifolds. J. Anal. Math. 135(2), 757–800 (2018)
22. Delgado, J., Ruzhansky, M.: $L^p$-nuclearity, traces, and Grothendieck–Lidski˘ı formula on compact Lie groups. J. Math. Pures Appl. 102(1), 153–172 (2014)
23. Delgado, J., Wong, M.W.: $L^p$-nuclear pseudo-differential operators on $\mathbb{Z}$ and $S^1$. Proc. Am. Math. Soc. 141(11), 3935–3942 (2013)
24. Delgado, J., Ruzhansky, M., Wang, B.: Approximation property and nuclearity on mixed-norm $L^p$, modulation and Wiener amalgam spaces. J. Lond. Math. Soc. 94(2), 391–408 (2016)
25. Delgado, J., Ruzhansky, M., Tokmagambetov, M.: Schatten classes, nuclearity and nonharmonic analysis on compact manifolds with boundary. J. Math. Pures Appl. 107, 758–783 (2017)
26. Feichtinger, H.G.: Modulation spaces: looking back and ahead. Sampl. Theory Signal Image Process. 5(2), 109–140 (2006)
27. Gröchenig, K.: Foundations of Time–Frequency Analysis. Applied and Numerical Harmonic Analysis, Birkhäuser Boston Inc, Boston (2001)
28. Grothendieck, A.: Produits Tensoriels Topologiques et Espaces Nucléaires. Mem. Am. Math. Soc. 69, 193–200 (1955)
29. Hörmander, L.: Estimates for translation invariant operators in $L^p$ spaces. Acta Math. 104(1–2), 93–140 (1960)
30. Kumar, V., Mondal, S.S.: Schatten Class and nuclear pseudo-differential operators on homogeneous spaces of compact groups. Monatsh. Math. 197(1), 149–176 (2022)
31. Kumar, V., Ruzhansky, M.: $L^p$–$L^q$ boundedness of $(k, a)$-Fourier multipliers with applications to nonlinear equations. Int. Math. Res. Not. (2021). https://doi.org/10.1093/imrn/rnab256
32. Kumar, V., Ruzhansky, M.: Hardy–Littlewood inequality and $L^p$–$L^q$ Fourier multipliers on compact hypergroups. J. Lie Theory 32(2), 475–498 (2022)
33. Lidkski˘ı, V.B.: Non-self adjoint operators with a trace. Dokl. Ak. Nauk. SSSR 125, 485–487 (1959)
34. Littlewood, J.E., Paley, R.E.A.: Theorems on Fourier series and power series (II). Proc. Lond. Mater. Soc. s2–42, 52–89 (1937)
35. Nicola, F., Rodino, L.: Global Pseudo-differential Calculus on Euclidean Spaces. Pseudo-Differential Operators. Theory and Applications, vol. 4. Birkhäuser Verlag, Basel (2010)
36. Nursultanov, E.D., Tleukhanova, N.T.: Lower and upper bounds for the norm of multipliers of multiple trigonometric Fourier series in Lebesgue space. Funkt. Anal. i Prilozhen. 34(2), 151–153 (2000)
37. Parmegianni, A.: Spectral Theory of Non-commutative Harmonic Oscillators: An Introduction. Lecture Notes in Mathematics, vol. 1992. Springer, Berlin (2010)
38. Pietsch, A.: Operator Ideals. North-Holland Mathematical Library, vol. 20. North-Holland Publishing Co., Amsterdam (1980). (Translated from German by the author)
39. Reinov, O.I., Laif, Q.: Grothendieck–Lindski˘ı theorem for subspaces of $L^p$ spaces. Math. Nachr. 286(2–3), 282–297 (2013)
40. Ruzhansky, M., Wirth, J.: $L^p$ Fourier multipliers on compact Lie groups. Math. Z. 280, 621–642 (2015)
41. Ruzhansky, M., Tokmagambetov, N.: Nonharmonic analysis of boundary value problems. Int. Math. Res. Not. 12, 3548–3615 (2016)
42. Thangavelu, S.: Lectures on Hermite and Laguerre Expansions (MN-42), Vol. 42. Princeton University Press, Princeton (1993)
43. Toft, J.: Schatten–von Neumann properties in the Weyl calculus and calculus of metrics on symplectic vector space. Ann. Glob. Anal. Geom. 30(2), 169–209 (2006)
44. Toft, J.: Schatten properties for pseudo-differential on modulation spaces. In: Pseudo-differential Operators. Lecture Notes in Mathematics, vol. 1949, pp. 175–202. Springer, Berlin (2008)
45. Voros, A.: Oscillator quartic and méthodes semi-classiques, Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi "Séminaire Goulaouic-Schwart" (1979–1980), Exposé no. 6, 6 p
46. Weyl, H.: Inequalities between the two kinds of eigenvalues of linear transformation. Proc. Natl Acad. Sci. USA 35, 408–411 (1949)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.