It is shown that $SL(n, R)$ KdV hierarchy can be expressed as definite nonpolynomials in Kac Moody currents and their derivatives by the action of Borel subgroup of $SL(n, R)$ on the phase space of centrally extended $sl(n, R)$ Kac Moody currents. Construction of Lax pair is shown, confirming Drinfeld Sokolov type Hamiltonian reduction. This suggests an example of a moduli space with simplectic structure corresponding to extended conformal
symmetries.
Drinfeld Sokolov Hamiltonian reduction procedure [1] in the theory of nonlinear integrable systems has opened up the most interesting aspect viz. the connection between conformal field theory and intergable systems [2,3,4]. This came out via the correspondence between Poisson bracket algebra among the KdV type fields and the quantum Dirac bracket algebra among normal ordered quantities. The direction was from quantum algebra to Poisson bracket algebra in \( \hbar \to 0 \) limit [2]. This was the case due to normal ordering ambiguity in the modes of periodic nonlinear fields. From these lines there were hints that there are hidden quantum symmetries or symmetries reflected through extended conformal algebras or some current algebra whose classical counterparts are the Gelfand Dikki Poisson bracket algebras of KdV type nonlinear integrable systems. Apart from the pioneering works of Zamolodchikov and others [5,6], the quantum symmetries having hidden symmetries were shown to exist by Polyakov [7] and Bershadsky and Ooguri [3]. Polyakov constructed diffeomorphisms from restricted \( SL(2, R) \) and \( SL(3, R) \) transformations leading to Virasoro algebra and \( W_3 \) algebra. He demonstrated the connection between the Wess Zumino Novikov Witten (WZNW) action and its gravitational analogue implementing the above observation in the \( SL(2, R) \) case. The classical underplay in the restricted gauge fixing procedure is essentially Drinfeld Sokolov Hamiltonian reduction for \( SL(n, R) \) which essentially rests on the important physical principle of gauge equivalence between Lax operators. Bradshadsky and Ooguri [3] observed that the physical Hilbert space of the right moving sector of constrained WZNW model can give rise to irreducible representations of \( W_n \) algebra [6]. This symmetry is again the quantum analogue of Drinfeld Sokolov Hamiltonian reduction [1].

At this point it is worthwhile to enquire whether there are new systems admitting Drinfeld Sokolov type Hamiltonian reduction and what kind of simplectic structure the reduced phase space can have and to what quantum symmetry this structure may correspond. In this letter we want to look into a moduli space (explained below) which is inequivalent to the usual Drinfeld Sokolov moduli space [1] and to show that it is, in fact,
classical analogue of the same quantum symmetries, shown in the references 2 and 3.

Drinfeld Sokolov Hamiltonian reduction has been studied for the action of the subgroup of $SL(n, R)$ of upper triangular matrices with 1’s in the main diagonal (group of such matrices are denoted by N) on the space of centrally extended $sl(n, R)$ Kac Moody currents and the possible quantum generalization of the procedure [2,3]. The emerging $sl(n, R)$ classical KdV fields in this case are polynomials in currents and their derivatives.

In this letter we enlarge the action of the above subgroup by taking Borel subgroup $\tilde{N}$ of $SL(n, R)$ and show that $sl(n, R)$ KdV fields are definite rational functions (not polynomials) of the Kac Moody currents and their derivatives. Subsequently our construction of canonical Lax equation, which is gauge equivalence preserving [1], ensures Hamiltonian reduction. We also discuss the corresponding quantum symmetry of the classical KdV system. We explicitly demonstrate our formulation for $SL(2, R)$ and $SL(3, R)$.

Consider the space of first order differential operators

$$\mathcal{L} = k \frac{d}{dx} + v(x),$$

(1)

$v(x)$ taking values in $sl(n, R)$. $\mathcal{L}$ is said to be equivalent to $S^{-1}\mathcal{L}S$; $S \in C^\infty(S^1, SL(n, R))$, which implies

$$v(x) \sim S^{-1}v(x)S + kS^{-1}\partial_x S.$$ \hspace{1cm} (2)

In particular, if $S \in C^\infty(S^1, \tilde{N})$, $\tilde{N}$ being the Borel subgroup of $SL(n, R)$, there is a unique $S$ such that

---

$\dagger$ Borel subgroup $SL(n, R)$ is the group generated by $\{H_i, J^+_l; \ i = 1, 2, .., n - 1; l = 1, 2, .., \frac{1}{2} n(n - 1)\}$ where $\{H_i, J^\pm_l\}$ is the standard Cartan Weyl basis of $sl(n, R)$ algebra. If we denote this group by $\tilde{N}$, $N$, the group of upper triangular matrices of $SL(n, R)$ with 1’s in the main diagonal and generated by $\{J^+_l\}$ is a subgroup of $\tilde{N}$: $N \subset \tilde{N} \subset SL(n, R)$. Sometimes $N$ is also termed as Borel subgroup in the literature.
\[
S^{-1}v(x)S + kS^{-1} \partial_x S = \sum_{i=1}^{n} w_{n+1-i} e_{in} + \Lambda,
\] (3)
where
\[
\Lambda = \sum_{i=1}^{n-1} e_{i+1,i} \quad ; \quad w_1 = 0 \quad (4)
\]
and \(e_{ij}\) denotes the \(n \times n\) matrix with 1 in \((i, j)\) th. position and zero elsewhere. \(w\)'s in (3) are rational functions in the elements of \(v(x)\) and their derivatives. \(w_n, w_{n-1}, \ldots, w_2\) are gauge invariant quantities i.e. \(\delta_S w_i = 0, \quad i = 2, 3, \ldots, n\) for any \(S \in C^\infty(S^1, \tilde{N})\).

Relation (2) defines an action of \(\tilde{N}\) on \(\tilde{M}\) where \(\tilde{M}\) is the phase space with coordinates \(\{h_i(x), j^\pm_i(x), k\}\). The phase space coordinates satisfy the centrally extended Kac Moody algebra with central extension \(k \in \mathbb{R}\).

\(v(x)\) in (1) has the form
\[
v(x) = \sum_{i=1}^{n-1} h_i(x) H_i + \sum_{i=1}^{\frac{1}{2} n(n-1)} j^+_i(x) J^+_i + \sum_{i=1}^{\frac{1}{2} n(n-1)} j^-_i(x) J^-_i. \quad (5a)
\]

Here \(\{H_i, J^\pm_i\}\) is the Cartan Weyl basis of \(sl(n, R)\).

(5a) together with (3) and (4) gives
\[
j^-_i(x) = 0; \quad n \leq i \leq \frac{1}{2} n(n - 1) \quad (5b)
\]

It is easy to verify from (2) and (5) that \(w_i\)'s in (3) together with central extension \(k\) are the coordinates of the moduli space \(\frac{\tilde{M}}{\tilde{N}}\). Moreover, any gauge invariant quantity can be expressed in terms of \(w_i\)'s and their derivatives only.

In order to ensure now that our procedure corresponds to Hamiltonian reduction \(a la'\) Drinfeld Sokolov, we have to formulate the construction of gauge equivalence preserving canonical Lax equation \([1]\). For this we first assert that if
\[
\mathcal{L}(x, \lambda) = k \frac{d}{dx} + \bar{\Lambda}(x) + \sum_{i=1}^{n-1} h_i(x) H_i + \sum_{i=1}^{\frac{1}{2} n(n-1)} j^+_i(x) J^+_i \quad (6a)
\]
\[ \tilde{\Lambda}(x) = \frac{\lambda}{j_1 j_2 \cdots j_{n-1}} e_{1n} + \sum_{i=1}^{n-1} j_i^- (x) e_{i+1,i}, \quad (6b) \]

\( \lambda \) being the spectral parameter, there is a unique

\[ T = \sum_{i=0}^{\infty} T_i(x) \lambda^{-i} \quad (6c) \]

with the first column of \( T_0 = (1, 0, \ldots, 0)^T \) and the first column of \( T_i = (0, 0, \ldots, 0)^T \) \((i \neq 0)\) such that

\[ \tilde{\mathcal{L}}_0(x, \lambda) = T \tilde{\mathcal{L}}(x, \lambda) T^{-1} \quad (7a) \]

is of the form

\[ \tilde{\mathcal{L}}_0(x, \lambda) = k \frac{d}{dx} \tilde{\Lambda}(x) + \sum_{i=0}^{\infty} f_i(x) \tilde{\Lambda}(x) + \frac{k}{2} \sum_{i=1}^{n-1} A_i(x) H_i \quad (7b) \]

where \( f_i(x) \) are functions of \( x \) and \( A_i(x) = (j_i^-(x))^{-1} \partial j_i^-(x) \) \((i = 1, 2, \ldots, n-1 \) and so sum over \( i \)). \( f_i(x) \) and \( T \) can be uniquely determined from the recurrence relations obtained from (6) and (7). \( \tilde{\Lambda}(x) \) in (6b) has the property that for \( SL(n, R) \) group

\[ (\tilde{\Lambda}(x))^n = \mathbf{E}, \quad (7c) \]

\( n \times n \) identity matrix.

Notice that \( \tilde{\mathcal{L}} \) in (6) reduces to the initial linear differential operator \( \mathcal{L} \) (1,5) for \( \lambda = 0 \). Whereas, in order to get \( SL(n, R) \) KdV hierarchy one has to suitably modify \( \mathcal{L} \) by injecting the spectral parameter \( \lambda \) as in (6).

The Lax equation, given by

\[ \frac{d\tilde{\mathcal{L}}}{dt} = [\tilde{\Lambda}, \tilde{\mathcal{L}}] \quad (8) \]
will be gauge equivalence preserving if $\tilde{A}$ is chosen in such a way that both sides of (8) are independent of the spectral parameter $\lambda$ and the time evolution of $w_i$’s from (8) is expressed as polynomials in $w_i$’s and their derivatives only. Following a procedure similar to that of Drinfeld Sokolov we can choose $\tilde{A}$ as

$$\tilde{A} = \sum_{i=1}^{m} C_i \Phi(\tilde{\Lambda}^i(x)),$$  \hspace{1cm} (9a)

$m$ being positive integer and $C_i = 0$ modulo $n$ because of (7c). $\Phi(\tilde{\Lambda}^i(x))$ is the polynomial part of $\Phi(\tilde{\Lambda}^i(x))$, where $\Phi(\tilde{\Lambda}^i(x))$ can be defined by

$$\Phi(\tilde{\Lambda}^i(x)) = T^{-1}(\tilde{\Lambda}^i(x))T = i \sum_{l=-\infty}^{r} \phi_l(x)\tilde{\Lambda}^l(x),$$  \hspace{1cm} (9b)

$r$ being a positive integer and $\phi_l(x)$ diagonal matrices. $T$ in (9b) is defined in (6c). Using (6b,c) one can determine $\phi_l(x)$ and thus can also obtain $\tilde{A}$.

It is interesting to note that our procedure reduces to that of Drinfeld Sokolov if we further impose the constraints

$$j_1^+ = j_2^+ = \ldots = j_{n-1}^+ = 1$$

on the phase space coordinates.

We now demonstrate our construction with the examples of $SL(2, R)$ and $SL(3, R)$.

$SL(2, R)$ case :

The first order differential operator in this case is given by

$$\mathcal{L} = k\frac{d}{dx} + \begin{pmatrix} h(x) & j^+(x) \\ j^-(x) & -h(x) \end{pmatrix}$$  \hspace{1cm} (10)

and the Borel subgroup $\tilde{N}$ is the group of matrices $\begin{pmatrix} a & \alpha \\ 0 & a^{-1} \end{pmatrix}$ with

$$S = \begin{pmatrix} a(x) & \alpha(x) \\ 0 & a^{-1}(x) \end{pmatrix}$$  \hspace{1cm} (11)
Substituting (10) and (11) in (3) we can solve for \( a(x) \) and \( \alpha(x) \) from the relation

\[
S^{-1} \begin{pmatrix} h(x) & j^+(x) \\ j^-(x) & -h(x) \end{pmatrix} S + S^{-1} \partial_x S = \begin{pmatrix} 0 & w_2(x) \\ 1 & 0 \end{pmatrix}
\]

whence

\[
w_2(x) = h^2(x) + j^+(x)j^-(x) + k \partial h(x) - kh(x)A(x) + \frac{k^2}{4} A^2(x) - \frac{k^2}{2} \partial A(x)
\]  

(12)

with \( A(x) = (j^-(x))^{-1} \partial j^-(x) \).

Notice that \( w_2(x) \) is not polynomial in \( h(x), j^\pm(x) \) unlike the previous cases [2,4]. This suggests that upon quantization our system would correspond to some constrained gauged WZNW model which will be different from that considered in [3]. One can easily check that \( \delta_S w_2(x) = 0 \) for arbitrary infinitesimal \( a(x) \) and \( \alpha(x) \) confirming gauge invariance of \( w_2(x) \). The coordinates, \( \{h(x), j^\pm(x), k\} \) of \( \tilde{M} \) satisfy \( \mathfrak{sl}(2, R) \) Kac Moody algebra which induces the Poisson bracket algebra of the coordinates \( \{w_2(x), k\} \) of the reduced phase space \( \tilde{\mathcal{M}} / N \). Writing

\[
U(x) = -\frac{1}{k} w_2(x)
\]

(13)

we have

\[
\{U(x), U(y)\} = 2U(x) \partial_x \delta(x-y) + \partial_x U(x) \delta(x-y) + \frac{k}{2} \partial_x^3 \delta(x-y)
\]

(14)

which looks like Gelfand Dikki Poisson bracket of second kind for KdV fields. We will, however, show shortly that \( U(x) \), indeed, satisfies KdV equation. Notice that \( \tilde{\mathcal{M}} / N \) is a subspace of \( \tilde{\mathcal{M}} / N \) since \( w_2(x) \) in (12) reduces to the same expression of the gauge invariant quantity obtained in [2,4] only when \( j^-(x) = 1 \). From (8) and (9) we have

(i) When \( C_1 \neq 0 \) and other \( C_i \)'s are zero
\[
\partial_t h(x) = -k\partial_x (h - \frac{k}{2}A) - j^+ J^- \\
\partial_t j^+(x) = 2j^+ (h - \frac{k}{2}A) \\
\partial_t j^-(x) = 0
\]

These equations lead to

\[
\partial_t U(x) = \partial_x U(x)
\]

(15a)

\(\partial_t U(x)\) being defined in (11) and (12).

(ii) When \(C_3 \neq 0\) and other \(C_i\)'s are zero

\[
\partial_t h(x) = \frac{k^3}{4} \partial_x^2 U - \frac{k^2}{2} \partial_x U (h - \frac{k}{2}A) - \frac{k^2}{2} U \partial_x (h - \frac{k}{2}A) - \frac{k}{2} U j^+ j^- \\
\partial_t j^+(x) = -\frac{k}{2} j^+ \partial_x U + k j^+ U (h - \frac{k}{2}A) \\
\partial_t j^- = 0
\]

These equations lead to

\[
\partial_t U(x) = \partial_x^3 U(x) + 6U(x) \partial_x U(x)
\]

(15b)

after proper rescaling of \(U(x)\) and \(x\). (15) is the wellknown KdV equation. Similarly for other nonzero \(C_i\)'s one can obtain KdV hierarchy. It is rather important to observe that for each \(C_i\), as in (14) and (15) for \(C_1\) and \(C_3\), the time evolution of the coordinate \(U(x)\) (or \(w_2(x)\)) of \(\frac{\mathcal{N}}{\mathcal{N}}\) is gauge invariant. This clearly shows a consistent formulation of Hamiltonian reduction in our case.

\(SL(3, R)\) case:

In this case the linear differential operator has the form

\[
\mathcal{L} = k \frac{d}{dx} + \begin{pmatrix}
  h_1(x) & j_1^+(x) & j_3^+(x) \\
  j_1^-(x) & -h_1(x) + h_2(x) & j_2^+(x) \\
  j_3^-(x) & j_2^-(x) & -h_2(x)
\end{pmatrix}
\]

(16)
where \( h_1(x), h_2(x), j^+_1(x), j^+_2(x) \) and \( j^+_3(x) \) satisfy \( sl(3, R) \) Kac Moody algebra. The Borel subgroup \( \tilde{N} \) of \( SL(3, R) \) is the group of matrices

\[
S = \begin{pmatrix} a(x) & p(x) & q(x) \\ 0 & a^{-1}(x)b(x) & n(x) \\ 0 & 0 & b^{-1}(x) \end{pmatrix}
\]

(17)

Substituting (16) and (17) in (3) we can solve for \( a(x), p(x), q(x), n(x) \) and \( b(x) \) from the relation

\[
S^{-1} \begin{pmatrix} h_1(x) & j^+_1(x) & j^+_2(x) \\ j^-_1(x) & -h_1(x) + h_2(x) & j^+_3(x) \\ j^-_3(x) & j^-_2(x) & -h_2(x) \end{pmatrix} S + S^{-1} \partial_x S = \begin{pmatrix} 0 & 0 & w_3(x) \\ 1 & 0 & w_2(x) \\ 0 & 1 & 0 \end{pmatrix}
\]

where

\[
w_2(x) = j^-_1(x)j^-_2(x) + j^+_2(x)j^-_2(x) + \tilde{h}^2_1(x) + \tilde{h}^2_2(x) - \tilde{h}_1(x)\tilde{h}_2(x) + k\partial_x \tilde{h}_1(x) + k\partial_x \tilde{h}_2(x),
\]

(18a)

\[
w_3(x) = j^-_1(x)j^-_2(x)j^+_3(x) + j^-_1(x)j^+_1(x)\tilde{h}_2(x) - j^-_2(x)j^+_2(x)\tilde{h}_1(x) + \tilde{h}_1^2(x)\tilde{h}_2(x)
\]

\[- \tilde{h}_1(x)\tilde{h}_2^2(x) + k\partial_x (j^-_1(x)j^+_1(x)) + 2k\tilde{h}_1(x)\partial_x \tilde{h}_1(x) - k\tilde{h}_1(x)\partial_x \tilde{h}_2(x) + k^2\partial_x^2 \tilde{h}_1(x)
\]

(18b)

and

\[
j^-_3(x) = 0
\]

(18c)

with

\[
\tilde{h}_1(x) = h_1(x) - \frac{2}{3}(j^-_1(x))^{-1}\partial_x j^-_1(x) - \frac{1}{3}(j^-_2(x))^{-1}\partial_x j^-_2(x)
\]

and
\[ h_2(x) = h_2(x) - \frac{1}{3} (j_1^-(x))^{-1} \partial_x j_1^-(x) - \frac{2}{3} (j_2^-(x))^{-1} \partial_x j_2^-(x) \]

Unlike the previous cases [2,4] again \( w_2(x) \) and \( w_3(x) \) are non polynomials in Kac Moody currents. One can easily check that

\[
\delta_S w_2(x) = 0 \\
\delta_S w_3(x) = 0
\]

for arbitrary infinitesimal \( a(x), b(x), p(x), q(x) \) and \( n(x) \), confirming the gauge invariance of \( w_2(x) \) and \( w_3(x) \). It can also be shown that \( sl(3,R) \) Kac Moody algebra among the coordinates, \( h_1(x), h_2(x), j_1^\pm(x), j_2^\pm(x), j_3^\pm(x) \) and \( k \) induces the Poisson brackets of the coordinates, \( w_2(x), w_3(x) \) and \( k \) of the phase space \( \tilde{M}/N \).

Writing

\[
W_2(x) = \frac{1}{k} w_2(x) \\
W_3(x) = \frac{1}{k} w_3(x)
\]

we have

\[
\{ W_2(x), W_2(y) \} = [W_2(x) + W_2(y)] \partial_x \delta(x - y) + 2k \partial_x^3 \delta(x - y) \]  \hspace{1cm} (21a)

\[
\{ W_2(x), W_3(y) \} = [W_3(x) + 2W_3(y)] \partial_x \delta(x - y) \]  \hspace{1cm} (21b)

\[
\{ W_3(x), W_3(y) \} = - \frac{1}{6} k^3 \delta(x - y) - \frac{1}{3} [W_2^2(x) + W_2^2(y)] \partial_x \delta(x - y) + \frac{1}{4} [\partial_x^2 W_2(x) + \partial_x^2 (y)] \partial_x \delta(x - y) - \frac{5}{12} k [W_2(x) + W_2(y)] \partial_x^3 \delta(x - y) \]  \hspace{1cm} (21c)

These look like the Gelfand Dikki Poisson brackets of second kind for KdV and W fields.

\( SL(3,R) \) KdV hierarchy can be obtained now from the spectral parameterful Lax operator (6a)
\[ \tilde{\mathcal{L}}(x, \lambda) = k \frac{d}{dx} + \tilde{\Lambda}(x) + j_1^+(x)J_1^+ + j_2^+(x)J_2^+ + j_3^+(x)J_3^+ + h_1(x)H_1 + h_2(x)H_2 \quad (22a) \]

with

\[ \tilde{\Lambda}(x) = \begin{pmatrix} 0 & 0 & \lambda \\ j_1^- & 0 & 0 \\ 0 & j_2^- & 0 \end{pmatrix} \quad (22b) \]

and from the operator in \((7b)\),

\[ \tilde{\mathcal{L}}_0(x, \lambda) = k \frac{d}{dx} + \tilde{\Lambda}(x) + \sum_{i=0}^{\infty} f_i(x)\tilde{\Lambda}(x) + \frac{k}{2} A_1 H_1 + \frac{k}{2} A_2 H_2 \quad (23) \]

satisfying the relation \((7a)\) where, \(A_1 = (j_1^-(x))^{-1}\partial_x j_1^-(x)\) and \(A_2 = (j_2^-(x))^{-1}\partial_x j_2^-(x)\). \(J_1^\pm, J_2^\pm, J_3^\pm, H_1\) and \(H_2\) in the above equations are the generators of \(SL(3, R)\) group.

The coefficients \(C_i\) of the series \((9a)\) in this case is zero modulo 3, \(i.e.\ C_0 = C_3 = C_6 = ... = 0\). It can be shown by a lengthy but straightforward calculation that \(w_2(x)\) and \(w_3(x)\) satisfy the equations of motions when \(C_4, C_5 \neq 0\) and other \(C_i\)'s are zero, which can be recast into the Boussinesq equations. Boussinesq hierarchy can be obtained in a similar way by choosing other higher \(C_i\)'s to be non zero. This, in fact, confirms the Hamiltonian reduction in \(SL(3, R)\) case. We should also mention that the expressions for \(w_2(x)\) and \(w_3(x)\) reduce to the similar expressions in \([4]\) only when \(j_1^-(x) = j_2^-(x) = 1\), implying once again \(\frac{M}{N} \subset \frac{M}{N}\).

We now come to the question of quantum algebra reflecting the symmetry of a quantum system so that the classical limit will be the Poisson bracket algebra in the reduced phase space \(\tilde{M}/N\). Since \(\omega_i\)'s \((i = 2, 3, ..., n)\) in our case are rational functions of currents in \((5)\) we have to choose a suitable representation to make the rational functions into polynomial in terms of the new fields. This can be done first by considering the Wakimoto representation \([3, 8]\) of Kac Moody currents and then by taking the exponential form of
(β, γ) representation, introduced by Gerasimov et. al. [9]. For convenience, however, we develop the quantum formalism first with the example of sl(2, R) case.

The classical Wakimoto representation for sl(2, R) phase space is given by

\[ h(x) = \beta(x)\gamma(x) + \sqrt{\frac{k}{2}}\partial\phi(x) \]  \hspace{1cm} (24a)

\[ j^+(x) = -\beta(x)(\gamma(x))^2 - \sqrt{2}k\gamma(x)\partial\phi(x) - k\partial\gamma(x) \] \hspace{1cm} (24b)

and

\[ j^-(x) = \beta(x) \] \hspace{1cm} (24c)

If we write \( \beta(x) \) and \( \gamma(x) \) as

\[ \beta(x) = \exp(-u(x) - iv(x)) \] \hspace{1cm} (25a)

\[ \gamma(x) = \sqrt{2}k\gamma(x)\partial\phi(x) - k\partial\gamma(x) \] \hspace{1cm} (25b)

(24) becomes

\[ h(x) = i\partial v(x) + \sqrt{\frac{k}{2}}\partial\phi(x) \] \hspace{1cm} (26a)

\[ j^+(x) = [(k+1)(\partial v(x))^2 - ik\partial^2 v(x) - i\sqrt{2}k\partial v(x)\partial\phi(x) - ik\partial u(x)v(x)]\exp(u(x) + iv(x)) \] \hspace{1cm} (26b)

and

\[ j^-(x) = \exp(-u(x) - iv(x)) \] \hspace{1cm} (26c)
and the phase space structure is now given by
\[
\{ \phi(x), \partial \phi(y) \} = \delta(x - y)
\]
\[
\{ \beta(x), \gamma(y) \} = \delta(x - y) \tag{27}
\]
\[
\{ u(x), \partial v(y) \} = i \delta(x - y)
\]

(24), (26) together with (27) satisfy the classical \( sl(2, R) \) Kac Moody algebra with central extension \( k \). This is, in fact, \( \hbar \to 0 \) limit of the \( sl(2, R) \) quantum Kac Moody algebra. Following this limiting procedure to come from \( sl(2, R) \) quantum Kac Moody algebra to \( sl(2, R) \) classical Kac Moody algebra we have definite transition from classical to quantum fields properly scaled by \( \hbar \). Together with the limits taken in the reference 2 we have the following the transitions in (26).

\[
\begin{align*}
  k_{\text{quantum}} &\to -\hbar^{-1} k_{\text{cl}} \\
  \phi_{\text{quantum}} &\to \sqrt{\hbar} \hbar^{-1} \phi_{\text{cl}} \\
  u_{\text{quantum}} &\to u_{\text{cl}} \\
  v_{\text{quantum}} &\to v_{\text{cl}}
\end{align*} \tag{28}
\]

as \( \hbar \to 0 \). Using (26) the expression for \( U(x) \) (12, 13) in terms of \( u(x), v(x), \phi(x) \) takes the form

\[
U(x) = -\frac{1}{2} [\partial \phi(x) + \sqrt{\frac{k}{2}} \partial (u(x) + iv(x))]^2 - \sqrt{\frac{k}{2}} \partial^2 [\phi(x) + \sqrt{\frac{2}{k}} (u(x) + iv(x))] \tag{29}
\]

This is obviously the classical limit of free field representation of the stress tensor, \( T(z) \) with back ground. Using the limits in (28) and \( T(z) \to \hbar U(x) \) as \( \hbar \to 0 \), we have the following form for \( T(z) \) in the quantum case.

\[
T(z) = -\frac{1}{2} [\partial \phi(x) - i \sqrt{\frac{k + 2}{2}} \partial (u(x) + iv(x))]^2 + \frac{i k + 1}{\sqrt{2} \sqrt{k + 2}} \partial^2 [\phi(x) - i \sqrt{\frac{k + 2}{2}} (u(x) + iv(x))] \tag{30}
\]
giving the central charge of the quantum theory as

\[ c = \frac{3k}{k+2} - 6k - 2 \]  \hspace{1cm} (31)

Thus the bracket algebra of the reduced phase space, \( \tilde{M} \tilde{N} \), in this case, corresponds to the Virasoro algebra for a theory which is unitary. Notice that the exact form of \( T(z) \) in (30) is unique subject to the condition that its classical limit is (12) and \( T(z) \) corresponds to unitary, irreducible representation of Virasoro algebra.

Next we point out the procedure for \( \text{sl}(3, R) \) case. In this case one can follow exactly the same procedure as we did in \( \text{sl}(2, R) \) case by taking the explicit \( (\beta, \gamma) \) representation for quantum \( \text{sl}(3, R) \) Kac Moody algebra [3] and then substituting the exponential forms for ghosts as in [9]. It is worth mentioning here that the advantage of choosing exponential representations for ghosts becomes much more transparent in \( \text{sl}(3, R) \) case. In this representation the constraint \( j_3^- (x) = 0 \) in (18c) gets linearized. We omit here the calculation since it is lengthy but straightforward. It is, however, important to mention that the expression for \( w_2(x) \) and \( w_3(x) \) in (18) are respectively the classical limit of \( T(z) \) and \( W_3(z) \) which correspond to a unitary theory with a central charge for \( W_3 \) algebra [5]. Clearly one can proceed in the same way for \( \text{sl}(4, R) \) and onwards. In each case we have a larger moduli space \( \tilde{M} \tilde{N} \) leading to same definite quantum symmetry as in [6]. So we conclude that for Drinfeld Sokolov procedure the maximum action is that of Borel subgroup on the Kac Moody phase space giving rise to a moduli space whose symmetry correspond to that of \( Z_n \) [6] symmetry in the quantum theory.

Authors are thankful to P. Majumdar for discussions. One of the authors (SKP) is thankful to P. Mitra and D. Gangopadhyay for discussions and also acknowledges his discussions with Ryu Sasaki. SKP would like to thank Institute of Mathematical Sciences, Madras for providing him with a visiting position while the work was initiated and SG would like to thank S.N. Bose National Centre for Basic Sciences, Calcutta for providing him with a visiting position to continue the work.
References:

1. V.G. Drinfeld and V.V. Sokolov; Jour. Sov. Math. 30 (1984) 1975.

2. A. Belavin; KdV type equations and W- algebras, Hand written manuscript, 1988.

3. M. Bershadsky and H. Ooguri, Comm. Math. Phys. 126 (1989) 49.

4. I. Bakas, Phys. Lett. B219 (1989) 283.

5. A.B. Zamolodchikov and V.A. Fateev, Nucl. Phys. B280 [FS 18] (1987) 644.

6. V.A. Fateev and S.K. Lykyanov, Int. Jour. Mod. Phys. A3 (1988) 507.

7. A.M. Polyakov, Int. Jour. Mod Phys A5 (1990) 833.

8. M. Wakimoto, Comm. Math. Phys. 104 (1986) 604; V.I.S. Dotsenko, Nucl. Phys. B338 (1990) 747.

9. A. Gerasimov, A. Marshakov and A. Morozov; preprint ITEP 89-139.