A PROBABILISTIC APPROACH TO THE LEADER PROBLEM IN RANDOM GRAPHS

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ABSTRACT. Consider the classical Erdős-Rényi random graph process wherein one starts with an empty graph on \( n \) vertices at time \( t = 0 \). At each stage, an edge is chosen uniformly at random and placed in the graph. After the original fundamental work in \textsuperscript{19}, Erdős suggested that one should view the original random graph process as a “race of components”. This suggested understanding functionals such as the time for fixation of the identity of the maximal component, sometimes referred to as the “leader problem”. Using refined combinatorial techniques, Łuczak \textsuperscript{25} provided a complete analysis of this question including the close relationship to the critical scaling window of the Erdős-Rényi process. In this paper, we abstract this problem to the context of the multiplicative coalescent which by the work of Aldous in \textsuperscript{3} describes the evolution of the Erdős-Rényi random graph in the critical regime. Further, different entrance boundaries of this process have arisen in the study of heavy tailed network models in the critical regime with degree exponent \( \tau \in (3, 4) \). The leader problem in the context of the Erdős-Rényi random graph also played an important role in the study of the scaling limit of the minimal spanning tree on the complete graph \textsuperscript{2}. In this paper we provide a probabilistic analysis of the leader problem for the multiplicative coalescent in the context of entrance boundaries of relevance to critical random graphs. As a special case we recover Łuczak’s result in \textsuperscript{25} for the Erdős-Rényi random graph.

1. INTRODUCTION

Since the foundational work of Erdős-Rényi \textsuperscript{18,19}, asymptotics in the large network limit for random graph models in general, and the evolution of dynamical properties in particular have motivated an enormous amount of work in the ensuing decades. Let us briefly describe one of the motivating questions of this paper and then motivate renewed interest on this problem over the last few years. One of the main models studied in the original work of Erdős-Rényi in \textsuperscript{18} is the following “random graph process” \( \{G(n,M)\}_{M \geq 0} \) on \( [n] := \{1,2,\ldots,n\} \). At “time” \( M = 0 \), start with the empty graph \( G(n,0) \). For \( M \geq 1 \), \( G(n,M) \) is obtained from \( G(n,M-1) \) by choosing one of the \( \binom{n}{2} - M + 1 \) edges not present in \( G(n,M) \) uniformly at random and placing this in the system. Write \( |\mathcal{C}_k(M)| \) (respectively \( |\mathcal{C}_k(M)| \)) for the \( k \)-th largest component in \( G(n,M) \) (respectively the size of this component), breaking ties arbitrarily. Here we have suppressed dependence on \( n \) to simplify notation.

In \textsuperscript{19}, the following “double jump” was identified where it was shown that for \( M \ll n/2 \) \( |\mathcal{C}_{(1)}(M)| = O_P(\log n) \), if \( M = n/2 \), the so called critical regime then \( |\mathcal{C}_{(1)}(M)| = \Theta_P(n^{2/3}) \), whilst
if $M = cn/2$ with $c > 1$ then $|\mathcal{C}_1(M)| \sim f(c)n$ for a deterministic function $f$ satisfying $f(c) > 0$ for $c > 1$. This stimulated an enormous amount of work (see [12, 21, 25, 26] and the references therein) both in understanding what happens close to the critical regime and dynamic properties of the above construction wherein components merge via the addition of new edges. This resulted in the following fundamental result of Aldous [3]. Fix $\lambda \in \mathbb{R}$ and consider the process of normalized component sizes close to the critical value arranged in decreasing order:

$$\bar{C}_n(\lambda) = \left( n^{-2/3} \left| \mathcal{C}_k \left( n/2 + \lambda n^{2/3} \right) \right| \right)_{k \geq 1}$$

For any $p \geq 1$, consider the metric space,

$$l^p_1 := \left\{ x = (x_i : i \geq 1) : x_1 \geq x_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} x_i^p < \infty \right\},$$

equipped with the natural metric inherited from $l^p_1$.

**Theorem 1.1** ([3]). View the process $\{\bar{C}_n(\lambda) : -\infty < \lambda < \infty\}$ as a Markov process on $l^2_1$. Then as $n \to \infty$ the above process converges weakly to a Markov process on $l^2_1$ which is now referred to as the standard multiplicative coalescent.

We will describe this result (as well as the entrance boundary of the Markov process) in more detail in the next Section; much more extensive discussions of this process and the relationship to the evolution of the Erdős-Rényi random graph can be found in [3]. We are now in a position to state the main problem motivating this paper.

**Leader problem:** Erdős suggested that one should view the original random graph process $\{G(n,M)\}_{M \geq 0}$ as a “race of components”. One fascinating aspect of this view was studied in [25]. First we need some definitions. For any graph $G$, call the maximal component $\mathcal{C}_1 \subset G$, the leader of the graph (breaking ties arbitrarily). Now suppose we place a new edge $e$ in the graph $G$ and consider the resulting graph $G \cup \{e\}$. If $e$ does not belong to $\mathcal{C}_1$ and results in merging two components in $G$ such that the new component has size larger than $\mathcal{C}_1$ (resulting in a new maximal component), say that a change of leader has occurred and call the new maximal component of $G \cup \{e\}$ the leader of this graph. Now consider the Erdős-Rényi process $\{G(n,M)\}_{M \geq 0}$. Define,

$$L_{er}^{(n)} := \min\{s \geq 0 : \text{a change of leader does not occur in the process } \{G(n,M)\}_{M \geq s}\}.$$  

Thus this is the last time a change in leader occurs in the evolution of the above process. Then Łuczak in [25] Theorem 7 showed that

the sequence of random variables $\left\{ n^{-2/3} \left( L_{er}^{(n)} - n/2 \right) \right\}_{n \geq 1}$, is tight.  

**Aim of this paper:** The original proof in [25] is highly intricate using careful and refined combinatorial analysis of the number of components of various complexities including trees of various sizes, coupled with a “symmetry rule” relating properties of the process below and above the critical threshold. These estimates are combined with a “scanning method” to prove (1.3). This paper is motivated by the following two threads:

(i) In the last few years, a host of random graph models have been shown to belong to the Erdős-Rényi or more precisely, the multiplicative coalescent universality class [5–8, 10, 17, 20, 22, 29]. This includes the configuration model [11, 28], a large sub-class of the inhomogeneous random graph models as formulated in [13], and the so-called bounded size rules [30]. It is hard to generalize Łuczak’s result in (1.2) via the beautiful
1.1. Notation. Throughout this paper, we make use of the following standard notation. We let \( d \) denote convergence in distribution, and \( P \) convergence in probability. For a sequence of random variables \( (X_n)_{n \geq 1} \), we write \( X_n = o_p(b_n) \) when \( |X_n/b_n| \xrightarrow{P} 0 \) as \( n \to \infty \). For a non-negative function \( n \to g(n) \), we write \( f(n) = O(g(n)) \) when \( |f(n)|/g(n) \) is uniformly bounded, and \( f(n) = o(g(n)) \) when \( \lim_{n \to \infty} f(n)/g(n) = 0 \). Furthermore, we write \( f(n) = \Theta(g(n)) \) if \( f(n) = O(g(n)) \) and \( g(n) = O(f(n)) \). Throughout this paper, \( K, K' \) will denote positive constants that depend only on the sequence \( \{c_i\}_{i \geq 1} \), and their values may change from line to line. Given two functions \( f_1, f_2 : [0, \infty) \to [0, \infty) \), we write \( f_1 \leq f_2 \) on \( A \subset \mathbb{R} \) if
\[
K' f_2(x) \leq f_1(x) \leq K f_2(x) \quad \text{for all} \ x \in A.
\]
If \( A \) is not specified, then it will be understood that \( A = [M, \infty) \) for some large \( M > 0 \). Similarly, for two sequences \( \{a_n\}_{n \geq 1} \) and \( \{b_n\}_{n \geq 1} \), \( a_n \approx b_n \) or simply \( a_n = b_n \) will mean that
\[
C' b_n \leq a_n \leq C a_n \quad \text{for all} \ n \geq 1.
\]
For any \( x = (x_1, x_2, \ldots) \in [0, \infty)^N \) and \( r \geq 1 \), we will write \( \sigma_r(x) := \sum_{i \geq 1} x_i^r \in [0, \infty] \) for the \( r \)-th moment of this sequence. We say that a sequence of events \( (\mathcal{E}_n)_{n \geq 1} \) occurs with high probability (whp) when \( P(\mathcal{E}_n) \to 1 \). Throughout this note \( \alpha = 1 / (\tau - 1) \).

1.2. Organization of the paper. We start with a precise description of the multiplicative coalescent and our main results in Section 2. Our main results are stated in Section 3. Section 4 contain all the proofs.

2. Model formulation and assumptions

We start by giving a precise description of the multiplicative coalescent. We will work with \( x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}) \), \( n \geq 1 \), satisfying \( x_1^{(n)} \geq x_2^{(n)} \geq \ldots \geq x_n^{(n)} > 0 \) and one of the following two conditions:

**Condition I.** (Pure Brownian limit regime) As \( n \to \infty \),
\[
\frac{x_1^{(n)}}{\sigma_2(x^{(n)})} \to 0, \quad \frac{\sigma_3(x^{(n)})}{\sigma_2(x^{(n)})^3} \to 1, \quad \text{and} \quad \sigma_2(x^{(n)}) \to 0.
\]  \( (2.1) \)

**Condition II.** (Pure jump limit regime) There exists \( c = (c_1 \geq c_2 \geq \ldots) \) such that as \( n \to \infty \),
\[
\frac{x_j^{(n)}}{\sigma_2(x^{(n)})} \to c_j \quad \text{for each} \quad j \geq 1,
\]  \( (2.2) \)
\[
\frac{\sigma_3(x^{(n)})}{\sigma_2(x^{(n)})^3} \to \sum_{j \geq 1} c_j^3, \quad \text{and} \quad \sigma_2(x^{(n)}) \to 0.
\]

Additionally,
\[
c_i = \frac{1}{i^{1/(\tau - 1)}} = \frac{1}{i^\alpha} \quad \text{for some} \quad \tau \in (3, 4) \quad \text{(or equivalently} \quad \alpha \in (1/3, 1/2)) \quad (2.3)
\]

The scalings in the assumptions follow [3,4]. Corollary [3,4] shows how to recover the original result in [25] via a proper choice of the weight sequence.

**Remark 1.** Limiting sequences \( c \) of the form \( 2.3 \) are of particular interest to us as they describe the scaling limit of standard random graph models with degree exponent \( \tau \in (3, 4) \), see [6,16,22]. The assumption \( \tau \in (3, 4) \) (equivalently \( \alpha \in (1/3, 1/2) \)) in Condition II implies that \( c \in \ell^3_1 \setminus \ell^2_1 \), which by [4] is necessary for the existence of a scaling limit of the maximal components. The reason for the terminology “pure Brownian limit” and “pure jump” following [4] can be found in Section 4. Briefly: under Condition I, the maximal components of the multiplicative coalescent (defined below) at any fixed time converge to the excursions from zero of reflected inhomogeneous Brownian motion; whilst under condition II, the same objects are described via excursions of a so-called “Levy process without replacement”.

**Remark 2.** Many standard sequences of weights can be rescaled to satisfy one of the two above conditions. An important example is the case of uniformly elliptic and bounded weights: \( x_i^{(n)} \in [a, b] \) for all \( n \geq 1 \) and \( 1 \leq i \leq n \), where \( a > 0 \). Then the rescaled weights
\[
\tilde{x}_i^{(n)} := c x_i^{(n)}, \quad c = \frac{\sigma_3(x^{(n)})^{1/3}}{\sigma_2(x^{(n)})}
\]

satisfy Condition I.
We will start with a system of particles with weights given by $x$, and let them evolve like the multiplicative coalescent. In words, this is a Markov process such that starting at any state $z := (z_1, z_2, \ldots) \in l^2$, where we view $z_i$ as the “weight” of cluster $i$, any two “clusters” $i$ and $j$ merge at rate $z_i \cdot z_j$ into a new cluster of size $z_i + z_j$ resulting in the new state $z^{(i+j)} \in l^2$. The generator of this Markov process $\mathcal{A}$ is given by

$$\mathcal{A} g(z) = \sum_{i} \sum_{j > i} z_i z_j (g(z^{(i+j)}) - g(z)). \quad (2.4)$$

For any connected component $\mathcal{C}$, its size is given by $W(\mathcal{C}) := \sum_{v \in \mathcal{C}} x_v$. We will denote the graph at time $t$ by $\mathcal{G}(x, t)$ and the sizes of its components by $\text{MC}(x, t)$, i.e.,

$$\text{MC}(x, t) = \{W(\mathcal{C}) : \mathcal{C} \text{ component in } \mathcal{G}(x, t)\}. \quad (2.5)$$

The component containing $x_v$ in $\mathcal{G}(x, t)$ will be denoted by $\mathcal{C}(x_v; x, t)$. When the initial sequence $x^{(n)}$ is clear from the context, we will simply write $X^{(n)}(t)$ instead of $\text{MC}(x^{(n)}, t)$. Its component sizes in decreasing order will be denoted by $X_1^{(n)}(t) \geq X_2^{(n)}(t) \geq \ldots$. The following general convergence result concerning the multiplicative coalescent was shown by Aldous and Limic [4].

**Theorem 2.1** ([4]). Fix $\lambda \in \mathbb{R}$, and consider the multiplicative coalescent run with initial condition $x^{(n)}$. Then under Condition I (resp. Condition IIr), there exists a limit random vector $\mathbf{y}(\lambda)$ (resp. $\mathbf{y}_c(\lambda)$) such that $X^{(n)}([\sigma_2]^{-1} + \lambda)^{-1} \xrightarrow{d} \mathbf{y}(\lambda)$ (resp. $\mathbf{y}_c(\lambda)$) as $n \to \infty$. Here, convergence in distribution is with respect to $l^1$ topology.

Explicit description of the limits are given in Section 4.3 under Condition I and Section 4.4 under Condition IIr. The exact form of these limits are not important at this stage, rather the above result will explain the time scaling in our main result below.

**Remark 3.** As described below Proposition 4 in [3] (and used explicitly in our proof of Corollary 3.4 below), taking $x_i^{(n)} = n^{-2/3}$ for all $1 \leq i \leq n$ gives back the Erdős-Rényi random graph in the critical regime.

3. MAIN RESULTS

We can now state our main results.

**Theorem 3.1.** For each $n \geq 1$, start with a system of particles with weights $x^{(n)}$ and let it evolve like the multiplicative coalescent. Let $L^{(n)} := \inf \{t \geq 0 : \text{a change of leader does not occur after time } t\}$, and let

$$M_n := L^{(n)} - (\sigma_2 x^{(n)})^{-1}.$$

(a) If $\{x^{(n)}\}_{n \geq 1}$ satisfies either Condition I or Condition IIr for some $\tau \in (3, 4)$, then $M_n^+$ is tight.

(b) If $\{x^{(n)}\}_{n \geq 1}$ satisfies Condition I, then $M_n^{-}$ is tight.

Under Condition IIr we make the following conjecture.

**Conjecture 3.2 (Born winner/silver spoon regime).** Suppose $\{x^{(n)}\}_{n \geq 1}$ satisfies Condition IIr for some $\tau \in (3, 4)$, and let $M_n$ be as in Theorem 3.1. Then there exists $\delta > 0$ such that $\liminf_{n \to \infty} P(M_n^+ \geq K) > \delta$ for all $K > 0$.

Further, let $W_n$ denote the event that the component containing $x_1^{(n)}$ is always the leader, i.e., the identity of the leader never changes. Then, if $c_1 > c_2$, we conjecture that under Condition IIr, $\liminf_{n \to \infty} P(W_n) > 0$. 

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In contrast, under Condition I, the following result holds.

**Theorem 3.3.** Consider the setting of Theorem 3.1 and let \(N^{(n)}\) denote the number of times the leader changes in \([0, (\sigma_2(x^{(n)}))^{-1}]\). Then under Condition I, \(N^{(n)} \xrightarrow{p} \infty\).

We can recover the result for the classical Erdős-Rényi random graph from Theorem 3.1.

**Corollary 3.4.** Consider the Erdős-Rényi process \(\{G(n, M)\}_{M \geq 0}\), and let \(L^{(n)}_{\text{er}}\) be as in (1.2). Then \(n^{-2/3}(L^{(n)}_{\text{er}} - n/2)\) is tight.

4. **Proofs**

For a fixed \(t > 0\) and a weight sequence \(\mathbf{z} := (z_1, z_2, \ldots, z_n)\), recall the definition of the graph \(\mathcal{G}(\mathbf{z}, t)\) in the definition of the multiplicative coalescent in Section 2. The graph \(\mathcal{G}(\mathbf{z}, t)\) can be constructed as follows: for each \(i \neq j \in [n]\), place an edge between \(i\) and \(j\) with probability \(1 - \exp(-t z_i z_j)\), independent across edges. The following is a simple lemma that leverages this construction.

**Lemma 4.1.** Let \(\mathbf{z} = (z_1, \ldots, z_n)\), \(t > 0\), and assume that \(t \sigma_2(\mathbf{z}) < 1\). Then
\[
\mathbb{E} \left[ \mathcal{W}'(\mathcal{C}(z_j; \mathbf{z}, t)) \right] \leq \frac{z_j}{1 - t \sigma_2(\mathbf{z})}.
\]

**Proof:** Using the above description of \(\mathcal{G}(\mathbf{z}, t)\) and the simple inequality \(1 - \exp(-x) \leq x\) for \(x \geq 0\), we see that for \(i \neq j\),
\[
\mathbb{P} \left( z_i \in \mathcal{C}(z_j; \mathbf{z}, t) \right) \leq \sum_{k \geq 0} \sum_{j_1} \cdots \sum_{j_k} (t z_j z_{j_1}) \times (t z_{j_1} z_{j_2}) \times \cdots \times (t z_{j_{k-1}} z_{j_k}) \times (t z_{j_k} z_i)
\]
\[
= t z_i z_j \sum_{k \geq 0} (t \sigma_2(\mathbf{z}))^k = \frac{t z_i z_j}{1 - t \sigma_2(\mathbf{z})}.
\]

Hence
\[
\mathbb{E} \left[ \mathcal{W}'(\mathcal{C}(z_j; \mathbf{z}, t)) \right] \leq z_j + \frac{z_j t \sigma_2(\mathbf{z})}{1 - t \sigma_2(\mathbf{z})} = \frac{z_j}{1 - t \sigma_2(\mathbf{z})},
\]
as desired. \(\blacksquare\)

4.1. **Proof of Theorem 3.1(a).** The following two propositions form the heart of the proof.

**Proposition 4.2.** Consider a system of particles with weights \(\mathbf{Z}(0) := (z_0(0), z_1(0), \ldots, z_n(0))\), where the particle with size \(z_0(0)\) is “tagged,” and \(z_1(0) \geq \ldots \geq z_n(0)\). Let the system evolve like the multiplicative coalescent, and let
\[
\mathbf{Z}(t) = (z_0(t), z_1(t), \ldots)
\]
be the sizes of components at time \(t\), where \(z_0(t)\) is the size of the component containing \(z_0(0)\) and the remaining component sizes are arranged in increasing order with \(z_1(t) \geq z_2(t) \geq \ldots\). Let \(f\) be the function given by
\[
f(\mathbf{z}) := \frac{z_0^2}{\sigma_2(\mathbf{z})}
\]
acting on nonnegative vectors \(\mathbf{z} = (z_0, z_1, \ldots)\) of finite length. Then \(\{f(\mathbf{Z}(t)) : t \geq 0\}\) is a submartingale. As a consequence, if \(f(\mathbf{Z}(0)) \geq 1 - \epsilon\), then
\[
\mathbb{P} \left( z_0(t) < \max_{i \leq 1} z_i(t) \text{ for some } t \geq 0 \right) \leq 5 \epsilon.
\] (4.1)
Proposition 4.3. Assume that $x^{(n)}$ satisfies either Condition I or Condition IIr for some $\tau \in (3, 4)$. Then for all $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ depending only on $\varepsilon$ and the sequence $\{x^{(n)}\}_{n \geq 1}$ such that
\[
P \left( \frac{(X_1^{(n)}(t_{\lambda_\varepsilon}))^2}{\sigma_2(X^{(n)}(t_{\lambda_\varepsilon}))} \geq 1 - \varepsilon \right) \geq 1 - \varepsilon \quad \text{for all} \quad n \geq 1, \tag{4.2}\]
where $t_\lambda = \lambda + \left[ \sigma_2(x^{(n)}) \right]^{-1}$ for $\lambda \in \mathbb{R}$.

Completing the proof of Theorem 3.1(a): The result follows upon combining Propositions 4.2 and 4.3.

4.2. Proof of Proposition 4.2. Recall the generator $\mathcal{A}$ of the multiplicative coalescent as defined in (2.4), and $z = (z_0, z_1, \ldots)$ have finite length. Then
\[
\mathcal{A} f(z) = \sum_{i \geq 1} z_0 z_i \left( \frac{(z_0 + z_i)^2}{\sigma_2(z)} - \frac{z_0^2}{\sigma_2(z)} \right) + \sum_{i \geq 1, j > i} z_i z_j \left( \frac{z_0^2}{\sigma_2(z) + 2z_i z_j} - \frac{z_0^2}{\sigma_2(z)} \right) =: T_1 + T_2. \tag{4.3}\]

For convenience, we will write $\sigma_2$ for $\sigma_2(z)$ throughout the rest of this proof. Now
\[
T_1 = \sum_{i \geq 1} \frac{z_0 z_i}{\sigma_2(\sigma_2 + 2z_0 z_i)} \left( \frac{z_0^2 + z_i^2 + 2z_0 z_i}{\sigma_2} - \frac{z_0^2}{\sigma_2} \right) = \sum_{i \geq 1} \frac{z_0 z_i}{\sigma_2(\sigma_2 + 2z_0 z_i)} \left( \frac{z_i^2}{\sigma_2} + 2z_0 z_i \left( \frac{\sigma_2 - z_0^2}{\sigma_2} \right) \right) = z_0 \sum_{i \geq 1} \frac{z_i^3}{\sigma_2(\sigma_2 + 2z_0 z_i)} + 2z_0^2 \sum_{i \geq 1} \frac{z_i^2 (\sum_{j \geq 1} z_j^2)}{\sigma_2(\sigma_2 + 2z_0 z_i)}.\]

Using $2z_0 z_i \leq \sigma_2$, we get
\[
T_1 \geq z_0 \sum_{i \geq 1} \frac{z_i^3}{\sigma_2(\sigma_2 + 2z_0 z_i)} + \frac{z_0^2}{\sigma_2^2} \times \left( \sum_{i \geq 1} z_i^2 \right)^2. \tag{4.4}\]

Next
\[
-T_2 = z_0^2 \sum_{i \geq 1, j > i} \frac{2z_i^2 z_j^2}{\sigma_2(\sigma_2 + 2z_i z_j)} \leq z_0^2 \sum_{i \geq 1} \sum_{j > i} \frac{2z_i^2 z_j^2}{\sigma_2^2} \leq \frac{z_0^2}{\sigma_2^2} \times \left( \sum_{i \geq 1} z_i^2 \right)^2. \tag{4.5}\]

Combining (4.3), (4.4), and (4.5) yields $\mathcal{A} f \geq 0$, which shows that $\{f(Z(t)) : t \geq 0\}$ is a submartingale.

To prove (4.1), define the stopping time
\[
T = \inf \{ s \geq 0 : f(Z(s)) \leq 4/5 \},
\]
where infimum of an empty set is understood to be $+\infty$. Further, since $f(Z(t))$ is a bounded submartingale, $f(Z(\infty)) := \lim_{t \to \infty} f(Z(t))$ exists almost surely. Thus when $f(Z(0)) \geq 1 - \varepsilon$,
\[
1 - \varepsilon \leq \mathbb{E} \left[ f(Z(T)) \right] \leq \frac{4}{5} \mathbb{P} \left( f(Z(T)) \leq 4/5 \right) + \mathbb{P} \left( f(Z(T)) > 4/5 \right).
\]
This shows that $\mathbb{P}(T = \infty) \geq 1 - 5\varepsilon$. Since $f(Z(t)) > 4/5$ implies that $z_0(t) > 2 \max_{i \geq 1} z_i(t)$, (4.1) follows. \[\square\]
4.3. Proof of Proposition 4.3 when Condition I is satisfied. Fix $\lambda \in \mathbb{R}$ and let
\[ W_A(s) = B(s) + \lambda s - s^2/2, \quad s \geq 0, \] (4.6)
where $B(\cdot)$ is a standard Brownian motion. Let $\overline{W}_A(\cdot)$ denote the reflected version of the process at zero namely,
\[ \overline{W}_A(s) = W_A(s) - \min_{0 \leq u \leq s} W_A(u), \quad s \geq 0. \] (4.7)
Let $\gamma_i(\lambda), i \geq 1$, be the excursions of the reflected process from zero such that $|\gamma_1(\lambda)| > |\gamma_2(\lambda)| > \ldots$. Aldous in [3] showed that $\gamma(\lambda) := (|\gamma_i(\lambda)| : i \geq 1) \in l^2$. Further, if a weight sequence $x^{(n)}$ satisfies Condition I, then by [3, Proposition 4],
\[ MC(x^{(n)}), \lambda + [\sigma_2(x^{(n)})]^{-1} \xrightarrow{d} \gamma(\lambda) \] on $l^2$. (4.8)
Now for any $\lambda \geq 2$,
\[ \lambda t - t^2/2 \geq \lambda \quad \text{for} \quad t \in (2, 2\lambda - 2). \]
Further,
\[ \mathbb{P}\left( \sup_{0 \leq t \leq 2\lambda} |B(t)| \geq \lambda \right) \leq C \exp\left( -\lambda/4 \right), \quad \text{for all} \quad \lambda \geq 2. \]
Hence, for all $\lambda \geq 2$,
\[ \mathbb{P}(|\gamma_1(\lambda)| \geq 2\lambda - 4) \geq \mathbb{P}(W_A(t) > 0 \quad \text{for} \quad t \in (2, 2\lambda - 2)) \geq 1 - C \exp\left( -\lambda/4 \right). \] (4.9)
Fix $\varepsilon > 0$ and using (4.9), choose $\lambda'_{\varepsilon} > 8$ such that
\[ \mathbb{P}(|\gamma_1(\lambda')_{\varepsilon}| \leq 2\lambda'_{\varepsilon} - 4) \leq \varepsilon. \] (4.10)
Now to prove this result for a general initial weight sequence $x^{(n)}$ satisfying Condition I, we start by specializing to the Erdős-Rényi random graph in the critical regime so as to derive properties of the limit distributional limit $\xi(\lambda)$. Thus let $\xi_{n, i}(\lambda)$ be the $i$th largest component of the Erdős-Rényi random graph $G(n, n^{-1} + \lambda n^{-4/3})$. Using (4.3), choose $n_0(\varepsilon)$ such that
\[ \mathbb{P}\left( n/2 \geq |\xi_{n, 1}(\lambda'_{\varepsilon})| \geq (2\lambda'_{\varepsilon} - 4)n^{2/3} \right) \geq 1 - 2\varepsilon \quad \text{for all} \quad n \geq n_0(\varepsilon). \] (4.11)
Now
\[ \mathbb{P}\left( \sum_{i \geq 2} |\gamma_i(\lambda'_{\varepsilon})|^2 > 1 \right) \leq \liminf_n \mathbb{P}\left( \sum_{i \geq 2} |\xi_{n, i}(\lambda'_{\varepsilon})|^2 > n^{4/3} \right) \leq 2\varepsilon + \liminf_n \sum_x \mathbb{P}\left( \sum_{i \geq 2} |\xi_{n, i}(\lambda'_{\varepsilon})|^2 > n^{4/3} \mid \xi_{n, 1}(\lambda'_{\varepsilon}) = H \right) \mathbb{P}(\xi_{n, 1}(\lambda'_{\varepsilon}) = H), \] (4.12)
where $\sum_x$ denotes sum over all connected subgraphs $H$ of the complete graph $K_n$ such that
\[ (2\lambda'_{\varepsilon} - 4)n^{2/3} \leq |V(H)| \leq n/2. \] (4.13)
Define the event $E(n, p, m)$ as follows:
\[ E(n, p, m) := \{ \text{size of the largest component of } G(n, p) \text{ is smaller than } m \}. \]
Fix a connected subgraph $H$ of $K_n$ that satisfies (4.13) and let $m = |V(H)|$. Then conditional on $\{\xi_{n, 1}(\lambda'_{\varepsilon}) = H\}, G(n, n^{-1} + \lambda'_{\varepsilon} n^{-4/3}) \setminus \xi_{n, 1}(\lambda'_{\varepsilon})$ has the same law (after a relabeling of vertices) as $G(n - m, n^{-1} + \lambda'_{\varepsilon} n^{-4/3})$ conditional on $E(n - m, n^{-1} + \lambda'_{\varepsilon} n^{-4/3}, m)$. Letting
\[ n_1 = n - m, \quad \text{and} \quad \frac{1}{n_1} - \frac{\mu}{n_1^{4/3}} = \frac{1}{n} + \frac{\lambda'_{\varepsilon}}{n^{4/3}}, \]
and using (4.13), we see that
\[
\mu = \left(1 - \frac{1}{n_1} - \frac{\lambda'_c}{n^{1/3}}\right)n_1^{4/3} \geq \left(\frac{2\lambda'_c - 4}{n_1 n^{1/3}} - \frac{\lambda'_c}{n^{1/3}}\right)n_1^{4/3} \geq \frac{(\lambda'_c - 4)n_1^{1/3}}{n^{1/3}} \geq \beta \lambda'_c,
\]
where \(\beta := 2^{-4/3}\). Hence
\[
P\left(E\left(n_1, n^{-1} + \lambda'_c n^{-4/3}, m\right) \geq \frac{\mu}{\lambda'_c} \right) \geq P\left(\left|\mathcal{C}_{n,1}(-\beta \lambda'_c)\right| \leq m\right)
\]
and
\[
P\left(\sum_{i \geq 2} \left|\mathcal{C}_{n,i}(\lambda'_c)\right|^2 > n^{4/3} \Big| \mathcal{C}_{n,1}(\lambda'_c) = H\right) \leq \frac{P\left(\sum_{i \geq 1} \left|\mathcal{C}_{n,i}(-\beta \lambda'_c)\right|^2 > n^{4/3}\right)}{P\left(\left|\mathcal{C}_{n,1}(-\beta \lambda'_c)\right| \leq m\right)}. \quad (4.14)
\]
Now
\[
\frac{1}{n^{4/3}} \mathbb{E}\left(\sum_{i \geq 1} \left|\mathcal{C}_{n,i}(-\beta \lambda'_c)\right|^2\right) = \frac{n_1}{n^{4/3}} \mathbb{E}\left|\mathcal{C}(V_{n_1}; -\beta \lambda'_c)\right|,
\]
where \(V_{n_1} \sim \text{Uniform}[n_1]\), and \(\mathcal{C}(V_{n_1}; -\beta \lambda'_c)\) is the component of \(V_{n_1}\) in \(G(n_1, n_1^{-1} - \beta \lambda'_c n_1^{-4/3})\). Since \(\mathbb{E}(\mathcal{C}(V_{n_1}; -\beta \lambda'_c))\) is upper bounded by the total progeny of a Galton-Watson tree with a Binomial\((n_1, n_1^{-1} - \beta \lambda'_c n_1^{-4/3})\) offspring distribution,
\[
P\left(\left|\mathcal{C}_{n,1}(-\beta \lambda'_c)\right| \geq m\right) \leq P\left(\sum_{i \geq 1} \left|\mathcal{C}_{n,i}(-\beta \lambda'_c)\right|^2 > n^{4/3}\right)
\]
\[
\leq \frac{n_1}{n^{4/3}} \mathbb{E}\left|\mathcal{C}(V_{n_1}; -\beta \lambda'_c)\right| \leq \frac{n_1}{n^{4/3}} \frac{n_1^{1/3}}{\beta \lambda'_c} \leq \frac{1}{\beta \lambda'_c}.
\]
Combining this with (4.14) and (4.12), we get
\[
P\left(\sum_{i \geq 2} \left|\gamma_i(\lambda'_c)\right|^2 > 1\right) \leq 2\epsilon + C/\lambda'_c. \quad (4.15)
\]
Now let us consider the multiplicative coalescent started with a general weight sequence \(x^{(n)}\) satisfying Condition I. Combining (4.8) with (4.10) and (4.15), we see that (4.2) holds for all \(n \geq n_0''\), where \(n_0''\) depends only on \(\epsilon\) and the sequence \(\{x^{(n)}\}_{n \geq 1}\). This completes the proof.

4.4. Proof of Proposition 4.3 when Condition II is satisfied for some \(\tau \in (3, 4)\). Throughout this section \(\xi_1, \xi_2, \ldots\) will be independent random variables with \(\xi_i \sim \text{Exp}(c_i)\). Let
\[
V_c(s) := \sum_{i \geq 1} c_i \left(\mathbbm{1}_{\{|\xi_i| \leq s\}} - c_i s\right), \quad \text{and} \quad W_{\lambda, c}(s) := \lambda s + V_c(s), \quad s \geq 0.
\]
Let \(\overline{W}_{\lambda, c}(\cdot)\) be the corresponding reflected process at zero, defined as in (4.7) but now with \(W_{\lambda, c}(\cdot)\) as in (4.16). Let \(\gamma_{i, c}(\lambda), \; i \geq 1\), denote the corresponding excursions of \(\overline{W}_{\lambda, c}\) from zero with \(|\gamma_{i, c}(\lambda)| > |\gamma_{2, c}(\lambda)| > \ldots\). Under Condition II, by the general theory developed for the entrance boundary of the multiplicative coalescent [4, Proposition 7], for each fixed \(\lambda \in \mathbb{R}\), \(\gamma_c(\lambda) := (|\gamma_{i, c}(\lambda)|, \; i \geq 1) \in l_1^2\) and
\[
X^{(n)}\left(\lambda + 1/\sigma_2(x^{(n)})\right) \overset{w}{\rightarrow} \gamma_c(\lambda), \quad (4.17)
\]
with respect to \(l_1^2\) topology.
Lemma 4.4. Assume that \( c \) is as in (2.3). Then there exist positive constants \( \lambda_0, C_{4.18} \) and \( C'_{4.18} \) that depend only on the sequence \( \{c_i\}_{i \geq 1} \) such that for all \( \lambda \geq \lambda_0, \)

\[
\mathbb{P}\left( |\gamma_{1,e}(\lambda)| \leq C_{4.18} \lambda^{-\frac{1}{2} - \frac{3}{4}} \right) \leq \exp\left( -C'_{4.18} \frac{\lambda^{2(\frac{1}{2}-\frac{3}{4})} \log \log \lambda}{\log \lambda} \right). \tag{4.18}
\]

Further, there exist positive constants \( \delta \) and \( C_{4.19} \) that depend only on the sequence \( \{c_i\}_{i \geq 1} \) such that for all \( \lambda \geq \lambda_0, \)

\[
\mathbb{P}\left( \sum_{i \in \Gamma_{1,c}(\lambda)} c_i^2 \leq (1 + \delta) \lambda \right) \leq C_{4.19} \lambda^{-\frac{3}{4}}. \tag{4.19}
\]

We now finish the proof of proof of Proposition 4.3 assuming Lemma 4.4.

Completing the proof of Proposition 4.3 Write \( \sigma_2 = \sigma_2(x^n) \) for simplicity, and recall that \( t_\lambda = \lambda + [\sigma_2]^{-1} \). Let \( \mathcal{C}_{n,i}(\lambda) \) be the component of \( \mathcal{G}(x^n, t_\lambda) \) having the \( i \)th largest mass \( \lambda_i^{(n)}(t_\lambda) \). Then by [9, Lemma 5.5], for \( i \geq 1, \)

\[
\sum_{v \in \mathcal{C}_{n,i}(\lambda)} x^2_v/\sigma_2^2 \overset{w}{\to} \sum_{v \in \mathcal{C}_{n,1}(\lambda_c)} c_v^2. \tag{4.20}
\]

Fix \( \varepsilon > 0 \). Using (4.20) and Lemma 4.4 we can choose \( \lambda_\varepsilon > 0 \) and \( n_0(\varepsilon) \geq 1 \) such that with probability \( \geq 1 - \varepsilon, \)

\[
\lambda_i^{(n)}(t_\varepsilon) \geq C_{4.18} \lambda_c^{-\frac{1}{2} - \frac{3}{4}}, \quad \text{and} \quad \sum_{v \in \mathcal{C}_{n,1}(\lambda_c)} x^2_v/\sigma_2^2 \geq (1 + \delta) \lambda_c. \tag{4.21}
\]

Let \( E \) be the event in (4.21). Define

\[ x^\hat{n} := (x_v : v \notin \mathcal{C}_{n,1}(\lambda_c)), \quad \text{and} \quad \hat{\sigma}_2 := \sigma_2(x^\hat{n}) = \sigma_2 - \sum_{v \in \mathcal{C}_{n,1}(\lambda_c)} x^2_v. \]

Writing \( \lambda_c + [\sigma_2]^{-1} = -\lambda' + [\hat{\sigma}_2]^{-1}, \) it follows that on the event \( E, \)

\[
\lambda' = \frac{\sigma_2 - \hat{\sigma}_2}{\sigma_2 \hat{\sigma}_2} - \lambda_c \geq \frac{(1 + \delta) \sigma_2^2 \lambda_c}{\sigma_2 \hat{\sigma}_2} - \lambda_c \geq \delta \lambda_c. \tag{4.22}
\]

Write

\[ \hat{\lambda}_c = \lambda + [\sigma_2]^{-1}, \quad \hat{\mathbb{P}}[ \cdot ] = \mathbb{P}[ \cdot | \mathcal{C}_{n,1}(\lambda_c)], \quad \text{and} \quad \hat{\mathbb{E}}[ \cdot ] = \mathbb{E}[ \cdot | \mathcal{C}_{n,1}(\lambda_c)]. \]

Then

\[
\hat{\mathbb{E}}\left[ \sigma_2\left( \text{MC} (x^\hat{n}, \hat{\lambda}_c - \delta \hat{\lambda}_c) \right) \right] = \sum_{v \notin \mathcal{C}_{n,1}(\lambda_c)} x_v \hat{\mathbb{E}}\left[ \mathcal{W}\left( \mathcal{C}(x_v, x^\hat{n}_v, \hat{\lambda}_c - \delta \hat{\lambda}_c) \right) \right] \tag{4.23}
\]

\[
\leq \sum_{v \notin \mathcal{C}_{n,1}(\lambda_c)} x_v \frac{x_v}{1 - \hat{\sigma}_2^{-\delta \hat{\lambda}_c}} = \frac{1}{\delta \lambda_c},
\]

where the second step uses Lemma 4.4. For \( x \geq 0, \) define the event

\[ F(x) := \left\{ \mathcal{W}(\mathcal{C}) \leq x \text{ for all component } \mathcal{C} \text{ in } \mathcal{G}(x^n, t_\lambda) \right\}. \]
Note that conditional on \( \mathcal{C}_{n,1}(\lambda_c) \), the graph \( \mathcal{G}(x^{(n)}, t_{\lambda_c}) \setminus \mathcal{C}_{n,1}(\lambda_c) \) has the same distribution as \( \mathcal{G}(x^{(n)}, t_{\lambda_c}) \) conditional on \( F(X_1^{(n)}(t_{\lambda_c})) \). Hence
\[
1_E \times \hat{p}\left( \sum_{i \geq 2} X_i^{(n)}(t_{\lambda_c})^2 \geq 1 \right) = 1_E \times \hat{p}\left( \sigma_2\left( \text{MC}(x^{(n)}, t_{\lambda_c}) \right) \geq 1 \mid F(X_1^{(n)}(t_{\lambda_c})) \right)
\leq 1_E \times \frac{\hat{p}\left( \sigma_2\left( \text{MC}(x^{(n)}, t_{\lambda_c}) \right) \geq 1 \right)}{\frac{1}{1 - 1/(\delta \lambda_c)}},
\]
where the last step uses (4.22) and (4.23). Hence, for all \( n \geq n_0(\epsilon) \),
\[
\mathbb{P}\left( \sum_{i \geq 2} X_i^{(n)}(t_{\lambda_c})^2 \geq 1 \right) \leq \epsilon + \frac{1/(\delta \lambda_c)}{1 - 1/(\delta \lambda_c)}.
\]
The rest is routine. ■

The rest of this section is devoted to the proof of Lemma 4.4. We will make use of the following tail bounds.

**Lemma 4.5.** Assume that \( c \) is as in (2.3) and define
\[
Z_u^{(1)} := \sup_{s \in [0,u]} \left| \sum_{i \geq 1} c_i \left( 1 - s \right)^{\frac{1}{\lambda}} \right| \quad \text{and} \quad Z_u^{(2)} := \sup_{s \in [0,u]} \left| \sum_{i \geq 1} c_i^2 (1 - s) - \mathbb{P}(\xi \leq s) \right|.
\]
Then there exist constants \( C_1, C_2, C_3, C_4 \) that depend only on the sequence \( \{c_i\}_{i \geq 1} \) such that for every \( u \geq \max(|c_1, c_2|) \),
\[
\mathbb{P}\left( Z_u^{(1)} \geq x(u^{r-3} \log u)^{1/2} \right) \leq \exp\left( -C_1 x \log \log x \right) \quad \text{for all } x \geq C_2 \quad \text{and} \quad (4.24)
\]
\[
\mathbb{P}\left( Z_u^{(2)} \geq x(\log u)^{1/2} \right) \leq \exp\left( -C_3 x \log \log x \right) \quad \text{for all } x \geq C_4. \quad (4.25)
\]

Before starting the proof, we make note of the following asymptotics which are simple consequences of (2.3):

(i) For any \( k \geq 2 \) and \( j \geq 1, \sum_{i = j}^{\infty} c_i^k (1 - e^{-c_i s}) = s j^{k-2} e^{-c_i s} \) on \( 0 \leq s \leq 1/c_j \); \quad (4.26)

(ii) \( \sum_{i = 1}^{\infty} c_i^2 (1 - e^{-c_i s}) = s^{r-3} \) on \( s \geq 1/c_1 \); \quad (4.27)

(iii) \( \sum_{i = 1}^{\infty} c_i (c_i s - 1 - e^{-c_i s}) = s^{r-2} \) on \( s \geq 1/c_1 \). \quad (4.28)

**Proof of (4.24):** Let \( Q_u := Q \cap [0,u] \), and define
\[
Z_{u,n}^{(1)} := \sup_{q \in Q_u} \left| \sum_{i = 1}^{n} c_i \left( 1 - \mathbb{P}(\xi_i \leq q) \right) \right| \quad \text{and} \quad Z_{u,n}^{(2)} := \sup_{q \in Q_u} \left| \sum_{i = 1}^{n} c_i \left( \mathbb{P}(\xi_i \leq q) - 1(\xi_i \leq q) \right) \right|.
\]
Then note that it is enough to prove (4.24) with \( Z_{u,n}^{(1)} \) in place of \( Z_u^{(1)} \).

Next, \( Z_{u,n}^{(1)} \leq U_{u,n} + V_{u,n} \), where
\[
U_{u,n} := \sup_{q \in Q_u} \left( \sum_{i = 1}^{n} c_i \left( 1 - \mathbb{P}(\xi_i \leq q) \right) \right), \quad \text{and} \quad V_{u,n} := \sup_{q \in Q_u} \left( \sum_{i = 1}^{n} c_i \left( \mathbb{P}(\xi_i \leq q) - 1(\xi_i \leq q) \right) \right). \quad (4.29)
\]

For each \( q \in Q_u \), define
\[
R_q^j(y) := \left( u^{r-3} \log u \right)^{1/2} c_i \left( 1 - e^{-c_i q} \right) \quad i = 1, \ldots, n.
\]
Then by (4.27),
\[
\sup_{q \in Q_u} \Var \left( \sum_{i=1}^{n} R_q^i(\xi_i) \right) \leq K/\log u.
\]
Hence, by standard concentration inequalities for supremum of empirical processes (see, e.g., [23, Theorem 1.1 (b)], (4.24) holds with \( U_{u,n} \) in place of \( Z_{u,n}^{(i)} \) provided we can show that
\[
\mathbb{E} \left( U_{u,n} \right) \leq K \left( u^{r-3} \log u \right)^{1/2}.
\]
(4.30)

An identical treatment will yield a similar tail bound for \( V_{u,n} \), which combined with the tail bound for \( U_{u,n} \) will result in the desired tail bound for \( Z_{u,n}^{(i)} \). Thus, the following lemma completes the proof of (4.24).

**Lemma 4.6.** Let \( U_{u,n} \) be as in (4.29). Then there exists a constant \( K \) depending only on the sequence \( \{c_i\}_{i \geq 1} \) such that
\[
\mathbb{E} \left( U_{u,n} \right) \leq K \left( u^{r-3} \log u \right)^{1/2}, \text{ for all } u \geq \max(c_1, e).
\]

**Proof:** Let \( \varepsilon_1, \ldots, \varepsilon_n \) be i.i.d. with \( \mathbb{P}(\varepsilon_1 = 1) = 1/2 = \mathbb{P}(\varepsilon_1 = -1) \), and let \( (\xi_1', \ldots, \xi_n') \) be an independent copy of \( (\xi_1, \ldots, \xi_n) \). Then
\[
\mathbb{E} \left[ U_{u,n} \right] = \mathbb{E} \left[ \sup_{q \in Q_u} \mathbb{E}_\xi \left( \sum_{i=1}^{n} c_i (1_{\xi_i \leq q} - 1_{\xi_i' \leq q}) \right) \right]
\leq \mathbb{E} \mathbb{E}_\xi \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i (1_{\xi_i \leq q} - 1_{\xi_i' \leq q}) \right) \right] = \mathbb{E} \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i (1_{\xi_i \leq q} - 1_{\xi_i' \leq q}) \right) \right],
\]
where \( \mathbb{E}_\xi [\cdot] := \mathbb{E}[\cdot|\xi_1, \ldots, \xi_n] \). Introducing the Rademacher variables \( \varepsilon_1, \ldots, \varepsilon_n \), we see that
\[
\mathbb{E} \left[ U_{u,n} \right] \leq \mathbb{E} \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i \varepsilon_i (1_{\xi_i \leq q} - 1_{\xi_i' \leq q}) \right) \right]
\leq \mathbb{E} \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i \varepsilon_i 1_{\xi_i \leq q} \right) \right] + \mathbb{E} \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i (\varepsilon_i - \varepsilon_i') 1_{\xi_i \leq q} \right) \right]
\leq 2 \mathbb{E} \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i \varepsilon_i 1_{\xi_i \leq q} \right) \right] = 2 \mathbb{E} \mathbb{E}_\xi \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i \varepsilon_i 1_{\xi_i \leq q} \right) \right].
\]
(4.31)

Define
\[
f_q := (c_1 1_{\xi_1 \leq q}, \ldots, c_n 1_{\xi_n \leq q}) \text{ for } q \in Q_u, \text{ and let } \Delta_u := \left( \sum_{i \geq 1} c_i^2 1_{\xi_i \leq u} \right)^{1/2}.
\]
Note that \( \Delta_u \) is finite almost surely since \( c \in \ell^3 \). For \( \eta > 0 \), define
\[
N(\eta) := \max \left\{ \# \{q_1, \ldots, q_r\} : \|f_{q_i} - f_{q_j}\|_{\ell^2} > \eta \text{ for any } 1 \leq i \neq j \leq r \right\}.
\]
Then standard chaining inequalities imply (see, for example, [27, Lemma 6.1] or [24])
\[
\mathbb{E}_\xi \left[ \sup_{q \in Q_u} \left( \sum_{i=1}^{n} c_i \varepsilon_i 1_{\xi_i \leq q} \right) \right] \leq 3 \Delta_u \sum_{j=0}^{\infty} 2^{-j} \left( \log N(2^{-j-1} \Delta_u) \right)^{1/2}.
\]
Combined with (4.31), this gives
\[
\mathbb{E}[U_{u,n}] \leq 6 \sum_{j=0}^{\infty} 2^{-j} \left[ \mathbb{E}(\Delta^2_u) \right]^{1/2} \left[ \mathbb{E}\left(\log N(2^{-j-1}\Delta_u)\right)\right]^{1/2} \\
\leq K \sum_{j=0}^{\infty} 2^{-j} u^{(r-3)/2} \left[ \mathbb{E}\left(\log N(2^{-j-1}\Delta_u)\right)\right]^{1/2},
\] (4.32)
where the last step uses (4.27). Let \( k(j, u) \) be such that
\[
\sum_{i \geq k(j,u)} c_i^2 \mathbb{1}_{[\xi_i \leq u]} \leq \frac{\Delta^2_u}{2^{2j+6}} < \sum_{i \geq k(j,u)+1} c_i^2 \mathbb{1}_{[\xi_i \leq u]},
\]
and let \( \xi(1) < \ldots < \xi(k(j,u)-1) \) be the order statistics from \( \xi_i, 1 \leq i \leq k(j,u) - 1 \). Then for any \( q_1, \ldots, q_{k(j,u)} \in \mathbb{Q}_u \) satisfying
\[
q_1 < \xi(1) < q_2 < \xi(2) < \ldots < q_{k(j,u)-1} < \xi(k(j,u)-1) < q_{k(j,u)},
\]
the set \( \{f_{q_i} : 1 \leq i \leq k(j,u)\} \) is a \( 2^{-j-2}\Delta_u \)-covering of \( \{f_q : q \in \mathbb{Q}_u\} \). Hence,
\[
N(2^{-j-1}\Delta_u) \leq k(j,u).
\] (4.33)
Let \( x_{j,u} \) be the smallest number such that
\[
u \exp(x_{j,u}) \leq 1, \quad u \leq \exp\left(\frac{x_{j,u}(4-\tau)}{4(\tau-1)}\right), \quad \text{and} \quad 2^{2j+7} u \sum_{i \geq \exp(x_{j,u})} c_i^2 (1 - e^{-c_i u})^{1/2} \leq 1.
\] (4.34)
Observe that by (4.26), and (2.3),
\[
x_{j,u} \leq K(j + \log u).
\] (4.35)
Now
\[
\mathbb{E} \left[ \log k(j,u) \right] \leq x_{j,u} + \int_{x_{j,u}}^{\infty} \mathbb{P} \left( k(j,u) > e^x \right) dx.
\] (4.36)
Then,
\[
\mathbb{P} \left( k(j,u) > e^x \right) \leq \mathbb{P} \left( \sum_{i \geq [e^x]} c_i^2 \mathbb{1}_{[\xi_i \leq u]} > \frac{1}{2^{2j+6}} \sum_{i=1}^{\infty} c_i^2 \mathbb{1}_{[\xi_i \leq u]} \right)
\leq \mathbb{P} \left( \sum_{i \geq [e^x]} c_i^2 \mathbb{1}_{[\xi_i \leq u]} \geq 2 \sum_{i \geq [e^x]} c_i^2 (1 - \exp(-c_i u)) \right)
\quad + \mathbb{P} \left( 2 \sum_{i \geq [e^x]} c_i^2 (1 - \exp(-c_i u)) \geq \frac{1}{2^{2j+6}} \sum_{i=1}^{\infty} c_i^2 \mathbb{1}_{[\xi_i \leq u]} \right) =: T_1 + T_2.
\] (4.37)
To bound \( T_1 \), we write
\[
T_1 \leq \mathbb{P} \left( \sum_{i \geq [e^x]} c_i^2 \mathbb{1}_{[\xi_i \leq u]} - (1 - \exp(-c_i u)) \right) \leq \sum_{i \geq [e^x]} c_i^2 (1 - \exp(-c_i u)) \leq \frac{\text{Var} \left( \sum_{i \geq [e^x]} c_i^2 \mathbb{1}_{[\xi_i \leq u]} \right)}{\left( \sum_{i \geq [e^x]} c_i^2 (1 - \exp(-c_i u)) \right)^2} \leq K \frac{u \exp \left( -x(6-\tau)/(\tau-1) \right)}{u^2 \exp \left( -x(8-2\tau)/(\tau-1) \right)} = K \frac{u \exp \left( -x(\tau-2)/(\tau-1) \right)}{u^2 \exp \left( -x(\tau-2)/(\tau-1) \right)},
\] (4.38)
where the penultimate step uses \((4.26)\) and the fact that for \(x \geq x_{j,u}\), \(uc_{i|v|} \leq 1\) by \((4.34)\). To bound \(T_2\), note that for any \(x \geq x_{j,u}\),

\[
2^{2j+7} \left( \sum_{i \geq |v|} c_i^2 (1 - \exp(-c_i u)) \right)^{1/2} \leq u^{-1/2}.
\]

by \((4.34)\). Hence,

\[
2^{2j+7} \left( \sum_{i \geq |v|} c_i^2 (1 - \exp(-c_i u)) \right) \leq u^{-1/2} \left( \sum_{i \geq |v|} c_i^2 (1 - \exp(-c_i u)) \right)^{1/2} \leq K \exp \left( - \frac{x(4 - \tau)}{2(\tau - 1)} \right),
\]

where the last step uses \((4.26)\). Writing \(y = \exp \left( \frac{x(4 - \tau)}{2(\tau - 1)} \right)\), we see that

\[
T_2 \leq \mathbb{P} \left( \gamma \sum_{i=1}^{\infty} c_i^2 \mathbb{1}_{\{\xi_i \leq u\}} \leq K \right) \leq e^K \mathbb{E} \left[ \exp \left( - y \sum_{i=1}^{\infty} c_i^2 \mathbb{1}_{\{\xi_i \leq u\}} \right) \right]
\]

\[
= e^K \prod_{i=1}^{\infty} \left[ e^{-c_i u} + (1 - e^{-c_i u}) \exp(-yc_i^2) \right] = e^K \prod_{i=1}^{\infty} \left[ 1 - (1 - e^{-c_i u})(1 - \exp(-yc_i^2)) \right]
\]

\[
\leq e^K \exp \left( - \sum_{i=1}^{\infty} (1 - e^{-c_i u})(1 - \exp(-yc_i^2)) \right). \quad (4.39)
\]

Let \(i_0\) be the smallest integer such that \(yc_{i_0}^2 \leq 1\). This also implies that \(uc_{i_0} \leq 1\) by \((4.34)\). Note that, by \((2.3)\),

\[
i_0 = y^{(\tau - 1)/2}. \quad (4.40)
\]

Hence

\[
\sum_{i=1}^{\infty} (1 - e^{-c_i u})(1 - \exp(-yc_i^2)) \geq \sum_{i=0}^{\infty} (1 - e^{-c_i u})(1 - \exp(-yc_i^2)) \geq y \sum_{i=0}^{\infty} c_i^2 (1 - e^{-c_i u}) = yu i_0^{(r-4)/(\tau-1)} \geq yu y^{(r-4)/2} = uy^{(r-2)/2},
\]

where the third step uses \((4.26)\), and the fourth step is a consequence of \((4.40)\). Combining this with \((4.39)\), we have

\[
T_2 \leq K \exp \left( - K' u \exp \left( \frac{x(4 - \tau)(\tau - 2)}{4(\tau - 1)} \right) \right). \quad (4.41)
\]

We complete the proof of Lemma 4.6 by combining \((4.32)\), \((4.33)\), \((4.36)\), \((4.37)\), \((4.38)\), \((4.41)\), with \((4.35)\). \qed

**Proof of (4.23):** The proof is similar to the proof of \((4.24)\). Note that, in this case, the definition of \(\Delta_u\) should be modified as follows:

\[
\Delta_u := \left( \sum_{i \geq 1} c_i^4 \mathbb{1}_{\{\xi_i \leq u\}} \right)^{1/2}.
\]

Thus \(\mathbb{E}[\Delta_u^2] \leq \sum_{i \geq 1} c_i^4 < \infty\), as \(c \in \ell^3\). This explains the factor \((\log u)^{1/2}\) in \((4.25)\) (instead of \((u^{r-3} \log u)^{1/2}\) appearing in \((4.24)\)). We omit the details to avoid repetition. \qed

Define the functions \(\phi, \psi : [0, \infty) \to \mathbb{R}\) as follows:

\[
\phi(s) := \sum_{i \geq 1} c_i^2 \left( \frac{c_i s - 1 + e^{-c_i s}}{c_i s} \right), \quad \text{and} \quad \psi(s) := \sum_{i \geq 1} c_i^2 (1 - e^{-c_i s}).
\]
Observe that for each \( \lambda > 0 \), the function \( \Phi_{\lambda}(s) := \lambda s - s \phi(s) \) is concave and has a unique positive zero, which we denote by \( s_0(\lambda) \). Further, \( s_0(\lambda) \) is strictly increasing on \( (0, \infty) \), and \( s_0(\lambda) \uparrow \infty \) as \( \lambda \uparrow \infty \).

**Lemma 4.7.** Let \( \phi(\cdot), \psi(\cdot), \) and \( s_0(\cdot) \) be as above. Then the following hold.

(a) \( s_0(\lambda) = \lambda^{1/(r-3)} \) on \([1/c_1, \infty)\).

(b) There exists \( \eta_0 > 0 \) such that for all \( \eta \in (0,\eta_0) \),

\[
\sup_{s \in (2/c_1, \infty)} \frac{\phi(\eta s)}{\phi(s)} \leq \sup_{s \in (2/c_1, \infty)} \frac{\phi((1 - \eta)s)}{\phi(s)} =: 1 - \varepsilon_{\eta},
\]

where \( \varepsilon_{\eta} > 0 \). Clearly, \( \lim_{\eta \to 0} \varepsilon_{\eta} = 0 \).

(c) There exist \( \delta_0 > 0 \) and \( \lambda_0 > 0 \) such that for all \( \lambda \geq \lambda_0 \),

\[
\psi(s_0(\lambda)) \geq (1 + \delta_0) \lambda.
\]

(d) For all \( s > 0 \) and \( \eta \in (0, 1) \), \( \psi((1 - \eta)s) \geq (1 - \eta)\psi(s) \). Further, if

\[
g(\eta) := \sup_{s \geq 1/c_1} \frac{\psi(\eta s)}{\psi(s)},
\]

then \( g(\eta) \to 0 \) as \( \eta \to 0 \).

**Proof:** Note that \( \lambda = \phi(s_0(\lambda)) = (s_0(\lambda))^{r-3} \), where the final step uses (4.23). This proves (a).

Next, a direct calculation shows that the function \( u \mapsto (u - 1 + e^{-u})/u \) is increasing on \( \mathbb{R} \). This proves the claim

\[
\sup_{s \in (2/c_1, \infty)} \frac{\phi(\eta s)}{\phi(s)} \leq \sup_{s \in (2/c_1, \infty)} \frac{\phi((1 - \eta)s)}{\phi(s)} \leq 1 - \varepsilon_{\eta},
\]

for \( 0 < \eta < 1/2 \). Now

\[
\frac{\phi(s)}{\phi((1 - \eta)s)} = 1 - \frac{(1 - \eta) \sum_{i \geq 1} c_i \left( c_i s - 1 + e^{-c_i s} \right)}{(1 - \eta) s \cdot \phi((1 - \eta)s)} \leq \frac{\sum_{i \geq 1} c_i \left( \eta (1 - e^{-c_i s}) + e^{-c_i s} - e^{-c_i (1 - \eta)s} \right)}{K s^{r-2}} \leq \frac{\sum_{i \geq 1} c_i \int_{1-\eta}^1 \left( 1 - e^{-c_i s} - c_i s e^{-c_i su} \right) du}{K s^{r-2}},
\]

where the second step makes use of (4.23) and is valid whenever \( c_1 (1 - \eta)s \geq 1 \). Choose \( \eta_0 > 0 \) small so that for all \( u \in [1 - \eta_0, 1] \), \( 1 - e^{-x} - xe^{-xu} > 0 \) whenever \( x > 0 \). Now it is easy to see that,

\[
K s^{r-2} \leq \sum_{i \geq 1} c_i \left( 1 - e^{-c_i s} - c_i s e^{-c_i su} \right) \leq K' s^{r-2}
\]

uniformly for \( u \in [1 - \eta_0, 1] \) and \( s \geq 2/c_1 \). This last observation combined with (4.42) shows that

\[
\sup_{s \in (2/c_1, \infty)} \left( \frac{\psi((1 - \eta)s)}{\psi(s)} \right) < 1,
\]

which completes the proof of (b).

Since \( 1 - e^{-u} \geq (u - 1 + e^{-u})/u \) for \( u \in (0, \infty) \), it immediately follows that \( \psi(s) \geq \phi(s) \) for \( s \geq 0 \). Now

\[
\frac{\psi(s)}{\phi(s)} - 1 = \frac{1}{\phi(s)} \sum_{i \geq 1} c_i^2 \left( \frac{1 - e^{-c_i s}}{c_i s} - e^{-c_i s} \right) \geq \frac{K s^{r-3}}{\phi(s)}.
\]
when $s \geq 1/c_1$. Since $\phi(s) = s^{r-3}$ on $[1/c_1, \infty)$, it follows that there exist $\delta_0 > 0$ such that $\psi(s) \geq (1 + \delta_0)\phi(s)$ for all $s \geq s_*$, which in turn implies the existence of a $\lambda_0$ such that

$$\psi(s_0(\lambda)) \geq (1 + \delta_0)\psi(s_0(\lambda)) = (1 + \delta_0)\lambda$$

for all $\lambda \geq \lambda_0$. This proves (c).

Since $1 - \exp(-(1 - \eta)s) \geq (1 - \eta)(1 - e^{-s})$, it follows that $\psi((1 - \eta)s) \geq (1 - \eta)\psi(s)$. To prove the last claim, note that

$$g(\eta) \leq \sup_{\eta c_1 \geq 1} \frac{\psi(s)}{\psi(s)} + \sup_{1 \leq \eta c_1 \leq 1/\eta} \frac{\psi(s)}{\psi(s)} \leq K\left(\eta^{r-3} + \sup_{1 \leq \eta c_1 \leq 1/\eta} \frac{\eta s}{s^{r-3}}\right) \leq K\left(\eta^{r-3} + \frac{\eta^{r-3}}{c_1^{1-r}}\right),$$

where the second step uses (4.26) and (4.27). This completes the proof of (d).

We now turn to

**Proof of Lemma 4.4.** Let $\delta_0$, $\lambda_0$, and $g(\cdot)$ be as in Lemma 4.7. Choose $\eta \in (0, 1/2)$ such that

$$(1 - \eta - g(\eta))(1 + \delta_0) > 1 + \delta_0/2. \tag{4.43}$$

Now

$$\Phi_A((1 - \eta)s_0(\lambda)) = \left((1 - \eta)s_0(\lambda) - \lambda - \phi((1 - \eta)s_0(\lambda))\right) \geq (1 - \eta)s_0(\lambda) - \lambda - (1 - \varepsilon_0)\phi(s_0(\lambda)) = (1 - \eta)s_0(\lambda) - \lambda \leq \eta s_0(\lambda) \geq K\eta\lambda^{\frac{r-2}{r}},$$

where the second step uses Lemma 4.7(b) and the last step uses Lemma 4.7(a). Similarly $\Phi_A(s_0(\lambda)) \geq K\eta\lambda^{\frac{r-2}{r}}$. Since $\Phi_A(\cdot)$ is concave, it follows that

$$\Phi_A(s) \geq K\eta\lambda^{\frac{r-2}{r}}, \quad \text{for} \quad \eta s_0(\lambda) \leq s \leq (1 - \eta)s_0(\lambda). \tag{4.44}$$

Recall the definitions of $W_{\lambda, c}(\cdot)$ and $Z_{w}^{(1)}$ from (4.16) and Lemma 4.5 respectively. Then

$$Z_{w}^{(1)} = \sup_{s \leq s_0(\lambda)} \left|W_{\lambda, c}(s) - \Phi_A(s)\right|. \tag{4.45}$$

Using (4.44), (4.45), and applying (4.24) with $u = s_0(\lambda)$ and $x = \theta \lambda^{\frac{r-1}{2(r-3)}}(\log \lambda)^{-1/2}$ where $\theta > 0$ is very small, we see that

$$P\left(W_{\lambda, c}(s) > 0 \text{ for all } \eta s_0(\lambda) \leq s \leq (1 - \eta)s_0(\lambda)\right) \geq 1 - \exp\left(-Cf_1(\lambda)\right), \tag{4.46}$$

where $f_1(\lambda) = \lambda^{\frac{r-1}{2(r-3)}} \log \lambda / \sqrt{\log \lambda}$. Let $\gamma_+ = \gamma_+(\lambda)$ be the excursion of the reflected process $W_{\lambda, c}(\cdot)$ alive at $s = s_0(\lambda)/2$. Then (4.46) shows that $\gamma_+$ is alive when $\eta s_0(\lambda) \leq s \leq (1 - \eta)s_0(\lambda)$ with probability at least $1 - \exp\left(-Cf_1(\lambda)\right)$. Combining this with Lemma 4.7(a) proves (4.18).

Now, with probability at least $1 - \exp\left(-Cf_1(\lambda)\right)$,

$$\sum_{i \in \gamma_+} c_i^2 \geq Y := \sum_{i \geq 1} c_i^2 \mathbb{1}\{\eta s_0(\lambda) \leq \xi_i \leq (1 - \eta)s_0(\lambda)\}. \tag{4.47}$$

Further, $\left|Y - \psi((1 - \eta)s_0(\lambda)) + \psi(\eta s_0(\lambda))\right| \leq 2Z_{w}^{(2)}$. Using Lemma 4.7(c–d) and (4.43), it follows that

$$\sum_{i \in \gamma_+} c_i^2 \geq (1 + \delta_0/2)\lambda - 2Z_{w}^{(2)}.$$
with probability at least $1 - \exp\left(-C f_1(\lambda)\right)$. Using the tail bound (4.25) with $x = \theta \lambda / \sqrt{\log \lambda}$ (where $\theta > 0$ is very small), we see that there exists $\lambda_0 > 0$ such that
\[
P\left( \sum_{i \in \gamma} c_i^2 \leq (1 + \delta_0/4) \lambda \right) \leq \exp\left(-K f_2(\lambda)\right)
\] for all $\lambda \geq \lambda_0$, where $f_2(\lambda) := \lambda \log \lambda / \sqrt{\log \lambda}$.

Recall the breadth-first exploration of the graph $\mathcal{G}(x^{(n)}, t)$ from [4]. Then [9, Lemma 5.4] implies that the component $\mathcal{C}_*^{(n)} (= \mathcal{C}_*^{(n)}(\lambda))$ of $\mathcal{G}(x^{(n)}, t_\lambda)$ being explored at time $s_0(\lambda)/2$ satisfies
\[
\mathcal{W}(\mathcal{C}_*^{(n)}), \sum_{v \in \mathcal{C}_*^{(n)}} \frac{(x_v^{(n)})^2}{\sigma_2(x^{(n)})^2} \leq \left(\lambda + \frac{\delta_0}{4}\right) \lambda
\]
Thus, there exists $n_0 = n_0(\lambda)$ such that for all $n \geq n_0$, $P(E^{(n)}) \leq 2 \exp(-K f_2(\lambda))$, where
\[
E^{(n)} = \left\{ \sum_{v \in \mathcal{C}_*^{(n)}} \frac{(x_v^{(n)})^2}{\sigma_2(x^{(n)})^2} \geq \lambda\left(1 + \frac{\delta_0}{4}\right), \text{ and } \mathcal{W}(\mathcal{C}_*^{(n)}) \geq (1 - 2\eta)s_0(\lambda) \right\}
\]
Now, by an argument identical to the one given around (4.23), it follows that
\[
\mathbb{1}_{E^{(n)}} \cdot P\left( \sum_{v \in \mathcal{C}_*^{(n)}} \mathcal{W}(\mathcal{C}_*) \geq x \mid \mathcal{F} \right) \leq \frac{4}{\delta_0 x^2}
\]
where $\mathcal{C}_*^{(n)}$ is the set of components explored after exploring $\mathcal{C}_*^{(n)}$, and $\mathcal{F}$ is the sigma-field generated by the exploration process up to the exploration of $\mathcal{C}_*^{(n)}$. Taking $x = \theta' \lambda / \sqrt{n}$ (where $\theta' > 0$ is very small) and using Lemma 4.7(a), we see that $\mathcal{C}_*^{(n)} = \mathcal{C}_1^{(n)}(\lambda)$ with probability at least $KL^{-\frac{4}{13}}$, where $\mathcal{C}_1^{(n)}(\lambda)$ is the component such that $\mathcal{W}(\mathcal{C}_1^{(n)}(\lambda))$ is the largest among all components. This in turn implies that
\[
P\left( \sum_{i \in \gamma_1, c(\lambda)} c_i^2 \leq \lambda\left(1 + \frac{\delta_0}{4}\right) \right) \leq \liminf_n \mathbb{E}\left( \sum_{v \in \mathcal{C}_1^{(n)}(\lambda)} \frac{(x_v^{(n)})^2}{\sigma_2(x^{(n)})^2} \leq \lambda\left(1 + \frac{\delta_0}{4}\right) \right)
\]
\[
\leq \liminf_n \left[ \mathbb{E}\left( \sum_{v \in \mathcal{C}_1^{(n)}(\lambda)} \frac{(x_v^{(n)})^2}{\sigma_2(x^{(n)})^2} \leq \lambda\left(1 + \frac{\delta_0}{4}\right) \right) + \mathbb{P}(\mathcal{C}_*^{(n)} \text{ is not } \mathcal{C}_1^{(n)}(\lambda)) \right] \leq K \lambda^{-\frac{4}{13}}.
\]
This completes the proof of (4.19). \hfill \Box

4.5. Proof of Corollary 3.4: Consider the modified Erdős-Rényi process $\overline{G}(n, M)$, where at each step, edges are sampled uniformly and with replacement, i.e., multiple edges are allowed. A simple computation shows that the number of multiple edges created up to $M \leq n$ is tight. Thus, it is enough to show that $n^{-2/3}(\overline{T}_{er} - n/2)$ is tight, where $\overline{T}_{er}$ denotes the last time the leader changes in $\overline{G}(n, M)$.

Consider the weight sequence $x^{(n)}$ where $x_i^{(n)} = n^{-2/3}$ for $1 \leq i \leq n$. Then $x^{(n)}$ satisfies Condition I and $\sigma_2(x^{(n)}) = n^{-1/3}$. To each pair $(i, j)$, $1 \leq i < j \leq n$, associate a Poisson process $\mathcal{N}_{ij}$ having rate $n^{-4/3}$. Let
\[
\mathcal{N} = \bigcup_{1 \leq i < j \leq n} \mathcal{N}_{ij} = \{\tau_1 < \tau_2 < \ldots\}.
\]
Then $\mathcal{N}$ has rate $\lambda := (n - 1)/(2n^{1/3})$. Construct the random graph process $\overline{\mathcal{G}}(x^{(n)}, t)$ as follows: For each $k \geq 1$, if $\tau_k \in \mathcal{N}_{ij}$, then connect $i$ and $j$ by an edge at time $\tau_k$. Clearly
\( \{G(n, M)\}_{M \geq 1} \) has the same distribution as \( \{G(x^{(n)}, M)\}_{M \geq 1} \). Let \( L^{(n)} \) be the last time the leader changes in \( G(x^{(n)}, \cdot) \). For any \( y > 0 \), define \( M(y) = \lfloor n/2 + yn^{2/3} \rfloor \). Then,

\[
\mathbb{P}(L^{(n)} \geq M(y)) = \mathbb{P}(L^{(n)} \geq \tau_{M(y)}).
\]

(4.49)

Since, for any fixed \( y > 0 \),

\[
\tau_{M(y)} = \lambda^{-1} M(y) + \Theta_p(\lambda^{-1} \sqrt{M(y)}) = n^{1/3} + 2y + \Theta_p(n^{-1/6}) = \left[ \sigma_2(x^{(n)}) \right]^{-1} + 2y + \Theta_p(n^{-1/6}),
\]

an application of Theorem [3.1] shows that \( n^{-2/3}(\tau_{er}^{(n)} - n/2)^+ \) is tight. An identical argument will yield tightness of \( n^{-2/3}(\tau_{er}^{(n)} - n/2)^- \). This completes the proof.

4.6. Proof of Theorem 3.1(b). Fix \( \lambda \in \mathbb{R} \) and recall the inhomogeneous Brownian motion \( W_{\lambda}(\cdot) \) in (4.6), the corresponding reflected process \( \overline{W}_{\lambda}(\cdot) \) in (4.7) and the decreasing sequence of excursions \( \xi(\lambda) \) arising of normalized limits of component sizes in (4.8). We will need some properties of these distributional limits.

By [4], Proposition 18 and Eqn (80), for any \( \epsilon > 0 \), we can choose \( \lambda_\epsilon > 0 \) such that with probability at least \( 1 - \epsilon \), the following three assertions hold simultaneously:

\[
\lambda_\epsilon \sum_{i \geq 1} |\gamma_i(-\lambda_\epsilon)|^2 \geq 1/2, \quad \frac{1}{\sum_{i \geq 1} |\gamma_i(-\lambda_\epsilon)|^2} - \lambda_\epsilon \leq 1, \quad \text{and} \quad \lambda_\epsilon \times |\gamma_1(-\lambda_\epsilon)| \leq \epsilon.
\]

Writing \( t_\lambda = \lambda + \left[ \sigma_2(x^{(n)}) \right]^{-1} \) and using (4.9), we can choose \( n_0(\epsilon) \) such that for all \( n \geq n_0(\epsilon) \),

\[
\lambda_\epsilon \sigma_2(x^{(n)}(t_{-\lambda_\epsilon})) \geq 1/2, \quad \frac{1}{\sigma_2(x^{(n)}(t_{-\lambda_\epsilon}))} - \lambda_\epsilon \leq 1, \quad \text{and} \quad \lambda_\epsilon \times x_1^{(n)}(t_{-\lambda_\epsilon}) \leq \epsilon
\]

(4.50)

with probability \( \geq 1 - 2\epsilon \).

Let us now describe the core idea in words. Using the above estimates and Lemma 4.1, we will show that the maximal component at time \( t_{-\lambda_\epsilon} \) cannot become too “large” by time \( t_{-2} \). Further by (4.8), the maximal component at time \( t_{-2} \) is reasonably large. This implies that there has had to have been a leader change whp in the interval \( [t_{-\lambda_\epsilon}, t_{-2}] \). We now make this idea precise. Let \( x^{(n)}(t_{-\lambda_\epsilon}) =: z = (z_1 \geq z_2 \geq \ldots) \) and suppose we start the multiplicative coalescent with this as the initial sequence. Run this process for \( \lambda_\epsilon - 2 \) units of time. Using Lemma 4.1 on the event (4.50),

\[
\mathbb{E}(\overline{W}(C(z_1; z, \lambda_\epsilon - 2)) \mid z) \leq \frac{z_1}{1 - (\lambda_\epsilon - 2)\sigma_2(z)} = \frac{z_1}{\sigma_2(z)} \times \frac{1}{\sigma_2(z) - \lambda_\epsilon + 2} \leq \frac{z_1}{\sigma_2(z)} \leq \frac{\epsilon}{\lambda_\epsilon \sigma_2(z)} \leq 2\epsilon.
\]

Hence, for all \( \delta > 0 \) and \( n \geq n_0(\epsilon) \),

\[
\mathbb{P}(\overline{W}(C(z_1; z, \lambda_\epsilon - 2)) \geq \delta) \leq 2\epsilon(1 + 1/\delta).
\]

(4.51)

However by (4.8) for all \( \eta > 0 \), there exists \( \delta_\eta > 0 \) and \( n_1(\eta) \geq 1 \) such that for all \( n \geq n_1(\eta) \),

\[
\mathbb{P}(X^{(n)}_{1}(t_{-2}) \geq \delta_\eta) \geq 1 - \eta.
\]

(4.52)

Combining (4.51) and (4.52), we see that for all \( n \geq n_0(\epsilon) \vee n_1(\eta) \),

\[
\mathbb{P}(M_n^{-} \geq \lambda_\epsilon) \leq \eta + 2\epsilon(1 + 1/\delta_\eta).
\]

We take \( \epsilon = \eta \delta_\eta \) to get the desired result.
4.7. **Proof of Theorem 3.3** The proof of Theorem 3.1(b) shows that for any \( \eta > 0 \), we can choose \( \lambda^{(1)}_{\eta} > 0 \) large so that \( \Pr \{ \text{a change of leader occurs between times } t = t - \lambda^{(1)}_{\eta} \text{ and } t = 0 \} \geq 1 - \eta \). By repeating the same argument, we can choose \( \lambda^{(1)}_{\eta} < \lambda^{(2)}_{\eta} < \lambda^{(3)}_{\eta} \ldots \) such that for \( j \geq 1 \),

\[
\Pr \{ \text{a change of leader occurs between times } t = t - \lambda^{(j+1)}_{\eta} \text{ and } t = t - \lambda^{(j)}_{\eta} \} \geq 1 - \eta/2^j.
\]

It thus follows that for any \( j \geq 1 \) and \( \eta > 0 \),

\[
\limsup_n \Pr( N^{(n)}(t) \leq j) \leq 2\eta,
\]

which completes the proof.

\[ \blacksquare \]

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