The Chaplygin gas as a model for dark energy

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Abstract

We review the essential features of the Chaplygin gas cosmological models and provide some examples of appearance of the Chaplygin gas equation of state in modern physics. A possible theoretical basis for the Chaplygin gas in cosmology is discussed. The relation with scalar field and tachyon cosmological models is also considered.

1 Introduction

Recent years observations of the luminosity of type Ia distant supernovae [1, 2, 3] point towards an accelerated expansion of the universe, which implies that the pressure \( p \) and the energy density \( \rho \) of the universe should violate the strong energy condition, i.e. \( \rho + 3p < 0 \).

The matter responsible for this condition to be satisfied at some stage of the cosmological evolution is referred to as “dark energy” (for a review see [4, 5, 6]). There are different candidates for the role of dark energy.

The most traditional candidate is a nonvanishing cosmological constant, which can also be thought of as a perfect fluid satisfying the equation of state \( p = -\rho \). However, it remains to understand why the observed value of the cosmological constant is so small in comparison with the Planck mass scale.

Moreover, there also arises in this connection the so called ”cosmic coincidence comundrum”. It amounts to the following question: why are the energy densities of dark energy and of dust-like matter at the present epoch of the same order of magnitude? This seems to be a problem, because it would imply that at the time of recombination these two densities were different by many orders of magnitude (see for instance [7]).

A less featureless candidate to provide dark energy is represented by the so called quintessence scalar field [8]. Scalar fields are traditionally
used in inflationary models to describe the transition from the quasi-exponential expansion of the early universe to a power law expansion. It has been a natural choice to try to understand the present acceleration of the universe by also using scalar fields [9–10]. However, we now deal with the opposite task, i.e. we would like to describe the transition from a universe filled with dust-like matter to an exponentially expanding one.

Scalar fields are not the only possibility but there are (of course) alternatives. Among these one can point out the so called k-essence models, where one deals again with a scalar field, but with a non-standard kinetic term [11]. The tachyonic models of dark energy have a similar structure, where the kinetic term of the tachyon field has a form suggested by string theory (see the review [12] and references therein). One can also mention models where the role of dark energy is played by quantum corrections to the effective action of a scalar field [13].

Here we consider a recently proposed class of simple cosmological models based on the use of peculiar perfect fluids [14]. In the simplest case, we study the model of a universe filled with the so called Chaplygin gas, which is a perfect fluid characterized by the following equation of state:

\[ p = -\frac{A}{\rho}, \]

where \( A \) is a positive constant.

Chaplygin introduced this equation of state [15] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. The same model was rediscovered later in the same context [16, 17].

The convenience of the Chaplygin gas is connected with the fact that the corresponding Euler equations have a very large group of symmetry, which implies their integrability. The relevant symmetry group has been recently described in modern terms [18].

The negative pressure following from the Chaplygin equation of state could also be used for the description of certain effects in deformable solids [19], of stripe states in the context of the quantum Hall effect and of other phenomena.

It is worth mentioning a remarkable feature of the Chaplygin gas, namely that it has positive and bounded squared sound velocity

\[ v_s^2 = \frac{\partial p}{\partial \rho} = \frac{A}{\rho^2}, \]

which is a non-trivial fact for fluids with negative pressure (this follows from \( \rho^2 \geq A \), see formula (22) below).

Beyond cosmology, the Chaplygin gas equation of state has recently raised a growing attention [20] because it displays some interesting and, in some sense, intriguingly unique features.

Indeed, Eq. (1) has a nice connection with string theory and it can be obtained from the Nambu-Goto action for d-branes moving in a \((d + 2)\)-dimensional spacetime in the light-cone parametrization [21]. Also, the Chaplygin gas is the only fluid which, so far, admits a supersymmetric
generalization \[22\]. We ourselves came across this fluid \[23\] when studying the stabilization of branes \[24\] in black hole bulks \[25\]. An “anti-Chaplygin” state equation, i.e. Eq. \(1\) with negative constant \(A\), arises in the description of wiggly strings \[26, 27\].

Inspired by the fact that the Chaplygin gas possesses a negative pressure we have undertaken the simple task of studying a FRW cosmology of a universe filled with this type of fluid \[14\]. Further theoretical developments of the model were given in \[28, 29, 30, 31\]. One of its most remarkable properties is that it describes a transition from a decelerated cosmological expansion to a stage of cosmic acceleration. The inhomogeneous Chaplygin gas can do more: it is able to combine the roles of dark energy and dark matter \[20\].

Another model that has been discussed in some detail \[30\] is the generalized Chaplygin gas that has two free parameters:

\[
p = -\frac{A}{\rho^\alpha}, \quad 0 < \alpha \leq 1
\]

(2)

A further possibility is to use \[31\] a more realistic two-fluid cosmological model including both the Chaplygin gas and the usual dust-like matter; this was also studied by using the statefinder parameters \[32, 33\]. While the model looks less economical than pure Chaplygin, it is more flexible from the point of view of the comparison with observational data. Moreover, the two-fluid model can suggest a solution of the cosmic coincidence conundrum \[31\].

The cosmological models of the Chaplygin class have at least three significant features: they describe a smooth transition from a decelerated expansion of the universe to the present epoch of cosmic acceleration; they attempt to give a unified macroscopic phenomenological description of dark energy and dark matter; and, finally, they represent, perhaps, the simplest deformation of traditional ΛCDM models.

Taking into account these attractive features, it is important to try to explain what could be the microscopic origin of the presence of the Chaplygin gas in our universe. An interesting attempt \[34\] makes use of a field theory approach to the description of a \((3+1)\)-dimensional brane immersed in a \((4+1)\)-dimensional bulk \[35\]. Phenomenologically, the Chaplygin gas manifests itself as the effect of the immersion of our four-dimensional world into some multidimensional bulk. The appearance of the Chaplygin gas in such a context does not depend on the details of the theory.

Another interesting feature of the Chaplygin gas is that it can be considered as the simplest model within the family of tachyon cosmological models. Namely, the Chaplygin gas cosmological model can be identified with a tachyon field theoretical model with a constant value of the field potential \[36\]. Using this fact as a starting point, we have developed a technique of construction of tachyon models which are in general correspondence with the present observational data, describing the contemporary epoch of cosmic acceleration \[37\]. On the other hand, these models have a rather rich dynamics, opening various scenarios for the future of the universe. In particular, it is possible that the present cosmological acceleration be followed by a catastrophically decelerated expansion culminating
in a *Big Brake* cosmological singularity. This type of cosmological singularity is characterised by an infinite value of cosmic deceleration which is achieved in a finite span of time.

A significant amount of work has been devoted to the comparison of the Chaplygin cosmological predictions with observational data. In this context, one can mention the following directions of investigation:
1. Supernova of type Ia observations.
2. Cosmic microwave background radiation.
3. Growth of inhomogeneities and the large scale structure of the universe.
4. Statistics of gravitational lensing.
5. X-ray luminosity of galaxy clusters.
6. Age restrictions from high-redshift objects.
7. New methods of diagnostics of the dark energy equation of state for future supernovae observations.

The restrictions coming from the studies of the large-scale structure of the universe are crucial for the viability of the Chaplygin gas models. One can safely say that a careful investigation of the non-linear regime of the growth of inhomogeneities is necessary for the purpose of coming to definite conclusions concerning the compatibility of the Chaplygin cosmologies with the observable large-scale structure of the universe.

Finally, we remark that until recently the standard ΛCDM model was considered as the most natural candidate for the role of dark energy from the observational point of view. Thus, the usual discussions of such theoretical problems as the relation between the cosmological constant and the Planck mass scale and the cosmic coincidence conundrum coexisted with a tacit agreement that there is nothing that can fit the data better than this simplest model. One can now detect first signs of a possible change of the situation. An example is paper, where some essential arguments in favour of a non-constant ratio between the pressure and the energy density of dark energy were put forward. Moreover, the study of conditions under which future SNAP (Supernova / Acceleration Probe) and cosmic microwave background radiation observations would be able to rule out the simplest ΛCDM model is also attracting attention. In this context, further study of the Chaplygin gas cosmological model and its relatives looks promising.

The structure of this paper is the following: in sections 2 and 3 we present some theoretical foundations of the Chaplygin gas in modern physics; section 4 contains the discussion of the Chaplygin cosmological model; in section 5 we consider a two-fluid cosmological model in terms of the statefinder parameters; section 6 presents the relationships between the Chaplygin cosmological model and some corresponding scalar field and tachyon field models.
2 Branes and the Chaplygin’s equation of state

The rather special status of the Chaplygin fluid may be appreciated by reviewing how its equation of state reappears in a modern theoretical physics context.

To this end one is led to consider the Nambu-Goto action for a \(d\)-brane moving in a \((d+2)\)-dimensional spacetime in the light-cone parametrization [21]. However, to keep the discussion elementary, we restrict our attention to the 3-dimensional case [23] i.e. the string case. When written in the light-cone gauge, the Hamiltonian for such a string has the following structure:

\[
H = \frac{1}{2} \int [\Pi^2 + (\partial_\sigma x)^2] d\sigma,
\]

where \(\sigma\) is a spatial world-sheet coordinate, \(x\) is a transversal spatial coordinate and \(\Pi\) is its conjugate momentum. The Hamilton equations following from (3) are very simple:

\[
\partial_\tau x = \Pi, \tag{4}
\]

\[
\partial_\tau^2 x - \partial_\sigma^2 x = 0. \tag{5}
\]

At this point one wants to interpret the functions

\[
\rho(x) = (\partial_\sigma x)^{-1}, \tag{6}
\]

\[
v = \Pi, \tag{7}
\]

as the density and the velocity fields of a certain fluid associated with the string.

This is substantiated by a special instance of the hodograph transformation [21], that makes one move from the independent variables \(\tau\) and \(\sigma\) to the variables \(t = \tau\) and \(x\). There hold the following relations:

\[
\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \Pi \frac{\partial}{\partial x} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x}, \tag{8}
\]

\[
\frac{\partial}{\partial \sigma} = (\partial_\sigma x) \frac{\partial}{\partial x} = \frac{1}{\rho} \frac{\partial}{\partial x}. \tag{9}
\]

At this point one can easily see that the density and velocity that we have defined always satisfy the continuity equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0. \tag{10}
\]

Furthermore Eqs. (8) and (9) can be used to show that Eq. (5) for the string is equivalent to the Euler equation

\[
\rho \left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) v + \frac{\partial p}{\partial x} = 0. \tag{11}
\]

for the fluid, provided that the pressure field satisfies the Chaplygin equation of state \(\Pi\) with \(A = 1\).
The Chaplygin equation of state also arises in connection with the Randall-Sundrum model \[24\]. In this model one thinks of our four-dimensional Minkowski spacetime to be a brane in a higher dimensional manifold. The difference of the Randall-Sundrum model w.r.t. the standard Kaluza-Klein models is that the higher dimensional manifold has now a (non-factorizable) warped structure as given by its metric

\[
d s^2 = e^{-2|y|/l}(dt^2 - dx_1^2 - dx_2^2 - dx_3^2) - dy^2, \tag{12}
\]

where \(y\) is an additional fifth coordinate and \(e^{-2|y|/l}\) is the warping factor. Eq. (12) is the metric of a portion of a five-dimensional anti-de Sitter spacetime of radius \(l\).

At \(y = 0\) one has the so called orbifold boundary conditions. Here the Christoffel symbols have finite jumps while the components of the curvature tensor contain \(\delta\)-like terms. To compensate them, one should introduce a brane located at \(y = 0\) whose tension is \(\lambda = 6/l\). This tension can be interpreted as a nonvanishing cosmological constant on the 4-dimensional brane spacetime, or equivalently, as a fluid living on the brane whose state equation is \(p = -\rho\). The 5-dimensional anti-de Sitter curvature makes the graviton essentially trapped on the 4-brane.

It is possible to consider other geometries for the brane, and this in general requires other kinds of matter on it for its stabilization. We have considered \[23\] a foliation of the \((n + 2)\)-dimensional anti-de Sitter spacetime by static universes with topology \(R \times S^n\), always imposing orbifold boundary conditions. In this case \[23\] the matter on the brane is a fluid satisfying the following state equation:

\[
p = -\frac{(n - 1)\rho}{n} - \frac{4n}{\rho l^2}. \tag{13}
\]

In the three-dimensional case \((n = 1)\) this reduces again to the Chaplygin gas state equation.

### 3 A possible theoretical basis for the Chaplygin gas in cosmology

As we shall see in the next section, the Chaplygin gas cosmological model has interesting features. Before discussing the latter in detail, we would like to address the question whether there is any fundamental mechanism to produce a Chaplygin gas source term at the RHS of the Einstein equations.

An interesting attempt in this direction \[34\] makes use of a \((3 + 1)\)-brane immersed in a \((4 + 1)\)-bulk, following a recent stream of ideas \[35\]. Consider the embedding of a \((3 + 1)\)-dimensional brane in a \((4 + 1)\)-dimensional bulk described by coordinates \(x^{\mu} = (x^4, x^\lambda)\), where the index \(\mu\) runs over 0, 1, 2, 3. Denote the bulk metric by \(g_{MN}\). Then, the induced metric on the brane is given by

\[
\tilde{g}_{\mu\nu} = g_{\mu\nu} - \theta_{\mu\nu}, \tag{14}
\]
where $\theta(x^\mu)$ is a scalar field describing the embedding of the brane into the bulk.

The action on the brane has the following structure:

$$S_{brane} = \int d^4x \sqrt{-g}(-f + \cdots) = \int d^4x \sqrt{-g\sqrt{1 - g^{\mu\nu}\theta,\mu\theta,\nu}}(-f + \cdots),$$

where the constant $f$ gives a brane tension and the dots $\cdots$ stay for other possible contributions. Equation (15) follows from the identity

$$\det(a_{ij} - b_i b_j) = \det(a_{ij}) (1 - b_m(a^{-1})_{mn} b_n),$$

whose proof is straightforward.

The energy-momentum tensor following from the tension-like action (15) is then

$$T_{\mu\nu} = f \left( \frac{\theta,\mu\theta,\nu}{\sqrt{1 - g^{\mu\nu}\theta,\mu\theta,\nu}} + g_{\mu\nu}\sqrt{1 - g^{\mu\nu}\theta,\mu\theta,\nu} \right).$$

This expression corresponds to a perfect fluid energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu},$$

provided one makes the following identifications: the four-velocity

$$u_\mu = \frac{\theta,\mu}{\sqrt{g^{\mu\nu}\theta,\mu\theta,\nu}};$$

the pressure and the energy density:

$$p = -f \sqrt{1 - g^{\mu\nu}\theta,\mu\theta,\nu};$$

$$\rho = f \frac{1}{\sqrt{1 - g^{\mu\nu}\theta,\mu\theta,\nu}}.$$ (17) (18)

It follows that the pressure and energy density exactly satisfy the Chaplygin’s equation of state (11) with $A = f^2$.

4 FRW cosmology with the Chaplygin gas

We consider a homogeneous and isotropic universe with the metric

$$ds^2 = dt^2 - a^2(t)dl^2,$$

where $dl^2$ is the metric of a 3-manifold of constant curvature ($K = 0, \pm 1$), and the expansion factor $a(t)$ evolves according to the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \rho - \frac{K}{a^2}.$$ (20)

Energy conservation

$$d(\rho a^3) = -p \, d(a^3)$$ (21)
together with the equation of state \( \rho \) give the following relation:

\[
\rho = \sqrt{A + \frac{B}{a^6}},
\]

where \( B \) is an integration constant. By choosing a positive value for \( B \) we see that for small \( a \) (i.e. \( a^6 \ll B/A \)) the expression (22) is approximated by

\[
\rho \sim \frac{\sqrt{B}}{a^3}
\]

which corresponds to a universe dominated by dust-like matter. For large values of the cosmological radius \( a \) it follows that

\[
\rho \sim \sqrt{A}, \quad p \sim -\sqrt{A},
\]

which, in turn, corresponds to an empty universe with a cosmological constant \( \sqrt{A} \) (i.e. a de Sitter universe). In the flat case it is also possible to find the exact solution as follows:

\[
t = \frac{1}{6\sqrt{A}} \left( \ln \frac{\sqrt{A + \frac{B}{a^6}} + \sqrt{A}}{\sqrt{A + \frac{B}{a^6}} - \sqrt{A}} - 2 \arctan \left( \sqrt{1 + \frac{B}{Aa^6}} + \pi \right) \right).
\]

Note that \( \rho = \sqrt{A} \) solves the equation

\[
\rho + p = \rho - \frac{A}{\rho} = 0.
\]

The circumstance that this equation has a nonzero solution lies at the heart of the possibility of interpreting the model as a “quintessential” one. Let us estimate the constant \( A \) by comparing our expressions for pressure and energy with observational data. An indirect and naive way to do it is to consider the nowadays accepted values for the contributions of matter and cosmological constant to the energy density of the universe. To use these data we decompose pressure and energy density as follows:

\[
p = p_\Lambda + p_M = -\Lambda, \quad \rho = \rho_\Lambda + \rho_M = \Lambda + \rho_M.
\]

An application of Eq. \( \rho \) gives

\[
A = \Lambda (\Lambda + \rho_M).
\]

If the cosmological constant contributes seventy percent to the total energy we get \( \sqrt{A} \approx 1.2 \Lambda \). We now observe that, in the context of a Chaplygin cosmology, once an expanding universe starts accelerating it cannot decelerate any more. Indeed Eqs. (20) and (21) imply that

\[
\frac{\ddot{a}}{a} = -\frac{1}{2}(\rho + 3p).
\]

Condition \( \ddot{a} > 0 \) is equivalent to

\[
a^6 > \frac{B}{2A}.
\]
which is obviously preserved by time evolution in an expanding universe. It thus follows that the observed value of the (effective) cosmological constant will increase up to $1.2\Lambda$.

There is a relation of our Chaplygin cosmology with cosmologies based on fluids admitting a bulk viscosity proportional to a power of the density $\rho$. In the flat $K = 0$ case the FRW equations for Chaplygin fit in this scheme as a special case and indeed a transition from power law to exponential expansion was already noticed. However, since the state equation for the corresponding fluid is different from our Eq. (11) this coincidence of the solutions is destroyed by any small perturbation, for instance by a small spatial curvature or by adding another matter source.

Considering now the subleading terms in Eq. (22) at large values of $a$ (i.e. $a^6 \gg B/A$), one obtains the following expressions for the energy and pressure:

$$\rho \approx \sqrt{A} + \sqrt{\frac{B}{4A}} a^{-6},$$  \hspace{1cm} (32)  $$

$$p \approx -\sqrt{A} + \sqrt{\frac{B}{4A}} a^{-6}.\hspace{1cm} (33)$$

Eqs. (32) and (33) describe the mixture of a cosmological constant $\sqrt{A}$ with a type of matter known as “stiff” matter, described by the following equation of state:

$$p = \rho.\hspace{1cm} (34)$$

Note that a massless scalar field is a particular instance of stiff matter. Therefore, in a generic situation, a Chaplygin cosmology can be looked upon as interpolating between different phases of the universe: from a dust dominated universe to a de Sitter one passing through an intermediate phase which is the mixture just mentioned above. The interesting point, however, is that such an evolution is accounted for by using one fluid only.

For open or flat Chaplygin cosmologies ($K = -1, 0$), the universe always evolves from a decelerating to an accelerating epoch. For the closed Chaplygin cosmological models ($K = 1$), the Friedmann equations (20) and (30) tell us that it is possible to have a static Einstein universe solution $a_0 = (3A)^{-\frac{1}{4}}$ provided the following condition holds:

$$B = \frac{2}{3\sqrt{3A}}.\hspace{1cm} (35)$$

When $B > \frac{2}{3\sqrt{3A}}$ the cosmological radius $a(t)$ can take any value, while if $B < \frac{2}{3\sqrt{3A}}$ there are two possibilities: either

$$a < a_1 = \frac{1}{\sqrt{3A}} \left( \sqrt{3} \sin \frac{\varphi}{3} - \cos \frac{\varphi}{3} \right)\hspace{1cm} (36)$$

or

$$a > a_2 = \frac{2}{\sqrt{3A}} \cos \frac{\varphi}{3}.\hspace{1cm} (37)$$
where $\varphi = \pi - \arccos \frac{3\sqrt{3}}{2} AB$. The region $a_1 < a < a_2$ is not accessible. Further information on the dynamics of the Chaplygin gas cosmological model can be found in the literature [14, 73].

For the generalized Chaplygin gas [14, 30] the dependence of the energy density on the cosmological radius is

$$\rho = \left( A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}}.$$  \hfill (38)

This type of matter at the beginning of the cosmological evolution behaves like dust and at the end of the evolution like a cosmological constant, while during the intermediate stage it could be treated as a mixture of two-fluids: the cosmological constant and a perfect fluid with equation of state $p = \alpha \rho$. The generalized Chaplygin gas cosmological models have an additional free parameter $\alpha$ to play with and are convenient for the comparison with observational data. However, the prospects of the construction of a physical theory explaining the origin of these models seem even less evident than those for the true Chaplygin gas with $\alpha = 1$.

5 The statefinder parameters and a two-fluid cosmological model

Since models trying to provide a description (if not an explanation) of the cosmic acceleration are proliferating, there exists the problem of discriminating between the various contenders. To this aim a new proposal introduced in [32] may turn out useful, which exploits a pair of parameters \{r, s\}, called “statefinder”. The relevant definition is as follows:

$$r \equiv \frac{a}{aH^3}, \quad s \equiv \frac{r - 1}{3(q - 1/2)},$$  \hfill (39)

where $H \equiv \dot{a}/a$ is the Hubble constant and $q \equiv -\dddot{a}/\dot{a}a$ is the deceleration parameter. The new feature of the statefinder is that it involves the third derivative of the cosmological radius.

Trajectories in the \{s, r\}-plane corresponding to different cosmological models exhibit qualitatively different behaviours. $\Lambda$CDM model diagrams correspond to the fixed point $s = 0, r = 1$. The so-called “quiescence” models [32] are described by vertical segments with $r$ decreasing from $r = 1$ down to some definite value. Tracker models [74, 75] have typical trajectories similar to arcs of parabola lying in the positive quadrant with positive second derivative.

The current location of the parameters $s$ and $r$ in these diagrams can be calculated in models (given the deceleration parameter); it may also be extracted from data coming from future SNAP (SuperNovae Acceleration Probe)-type experiments [32]. Therefore, the statefinder diagnostic combined with forthcoming SNAP observations may possibly be used to discriminate among different dark energy models.

Here, we consider the one-fluid pure Chaplygin gas model and a two-fluid model where dust is also present [31]. We show that these models are different from those considered in [32].

10
To begin with, let us rewrite the formulae for the statefinder parameters in a form convenient for our purposes. Since

\[ \dot{p} = \frac{\partial p}{\partial \rho} \dot{\rho} = -3\sqrt{\rho}(\rho + p)\frac{\partial p}{\partial \rho}, \]  

we easily get:

\[ r = 1 + \frac{9}{2} \left( 1 + \frac{p}{\rho} \right) \frac{\partial p}{\partial \rho}, \quad s = \left( 1 + \frac{\rho}{p} \right) \frac{\partial p}{\partial \rho}. \]  

(40)

For the Chaplygin gas one has simply that

\[ v_s^2 = \frac{\partial p}{\partial \rho} = \frac{A}{\rho^2} = -\frac{p}{\rho} = 1 + s \]  

(42)

and therefore

\[ r = 1 - \frac{9}{2} s(1 + s). \]  

(43)

Thus, the curve \( r(s) \) is an arc of parabola. It is easy to see that

\[ v_s^2 = \frac{A}{A + \frac{p}{\rho^2}}. \]  

(44)

When the cosmological scale factor \( a \) varies from 0 to \( \infty \) the velocity of sound varies from 0 to 1 and \( s \) varies from \(-1\) to 0. Thus in our model the statefinder \( s \) takes negative values; this feature is not shared by quiescence and tracker models [32].

As \( s \) varies in the interval \([-1, 0]\), \( r \) first increases from \( r = 1 \) to its maximum value and then decreases to the \( \Lambda \)CDM fixed point \( s = 0, r = 1 \) (see Fig. 1).

If \( q \approx -0.5 \) the current values of the statefinder (within our model) are \( s \approx -0.3, \ r \approx 1.9. \) In [32] an interesting numerical experiment based on 1000 realizations of a SNAP-type experiment, probing a fiducial \( \Lambda \)CDM model is reported. Our values of the statefinder lie outside the three-sigma confidence region displayed in [32]. Based on this fact it can be expected that future SNAP experiments should be able to discriminate between the pure Chaplygin gas model and the standard \( \Lambda \)CDM model.

Now consider a more "realistic" cosmological model which, besides a Chaplygin’s component, contains also a dust component. For a two-component fluid Eqs. (41) take the following form:

\[ r = 1 + \frac{9}{2(p + p_1)} \left[ \frac{\partial p}{\partial \rho}(\rho + p) + \frac{\partial p_1}{\partial \rho_1}(p_1 + p_1) \right], \]  

(45)

\[ s = \frac{1}{p + p_1} \left[ \frac{\partial p}{\partial \rho}(\rho + p) + \frac{\partial p_1}{\partial \rho_1}(p_1 + p_1) \right]. \]  

(46)

If one of the fluids is dust, i.e. \( p_1 = p_d = 0 \), the above formulae become

\[ r = 1 + \frac{9(\rho + p)}{2(\rho + p_d)} \frac{\partial p}{\partial \rho}, \quad s = \frac{\rho + p}{p} \frac{\partial p}{\partial \rho}. \]  

(47)

If the second fluid is the Chaplygin gas, proceeding exactly as before we obtain the following relation:

\[ r = 1 - \frac{9}{2} \frac{s(s + 1)}{1 + \frac{s}{\rho}}. \]  

(48)
To find the term $\rho_d/\rho$ we write down the dependence of the dust density on the cosmological scale factor:

$$\rho_d = \frac{C}{a^3},$$

(49)

where $C$ is a positive constant. Eq. (44) gives $Aa^6 + B = -\frac{D}{2}$ and therefore

$$\frac{\rho_d}{\rho} = \frac{C}{\sqrt{Aa^6 + B}} = \kappa \sqrt{-s},$$

(50)

where the constant $\kappa = C/\sqrt{B}$ is the ratio between the energy densities of dust and of the Chaplygin gas at the beginning of the cosmological evolution. Thus

$$r = 1 - \frac{9}{2} \frac{s(s + 1)}{1 + \kappa \sqrt{-s}}$$

(51)

Graphs of the function (50) for different choices of $\kappa$ are plotted in Fig. 2.

In this case there are choices of the parameters so that the current values of the statefinder are close to the $\Lambda$CDM fixed point. For $\kappa = 1$ we have $s = -0.09$ and $r = 1.2835$; by increasing $\kappa$ we get closer and closer to the point $(0,1)$. Already for $\kappa = 2$ we get $s = 0.035$, $r = 1.11$ while for $\kappa \gtrsim 5$ the statefinder essentially coincides with the $\Lambda$CDM fixed point (see Fig. 2).

Thus our two-fluid cosmological models (with $\kappa$ say bigger than 5) cannot be discriminated from the $\Lambda$CDM model on the basis of the statefinder analysis.

However, even if the Chaplygin component closely mimics today the cosmological constant, this neither spoils the interest of the two-fluid
model nor makes it equivalent to ΛCDM; for instance, one advantage of the model is that it may suggest a solution to the cosmic coincidence conundrum: here the initial values of the energies of dust and of the Chaplygin gas can be of the same order of magnitude. In particular the value $\kappa = 1$ is not excluded by current observations. This may be seen by using the results of [40] and taking into account the relation

$$\kappa = \frac{\Omega_m}{(1 - \Omega_m)\sqrt{1 - \upsilon_s^2}},$$

(52)

where $\Omega_m = \frac{\rho_d}{\rho_d + \rho}$ and where $\rho, \rho_d$ and $\upsilon_s$ are evaluated at the present epoch.

Further details concerning application of the statefinder diagnostic to the study of Chaplygin gas models can be found in [31, 33].

Figure 2: s-r evolution diagram for the Chaplygin gas mixed with dust. Dots locate the current value of the statefinder.
6 The Chaplygin gas, scalar fields, tachyons and the future of the universe

It is well-known that for isotropic cosmological models, given the dependence of the cosmological radius on time it is always possible to construct a potential for a minimally coupled scalar field model, which would reproduce this cosmological evolution (see e.g. [9]), provided rather general reasonable conditions are satisfied. Sometimes, it is possible to construct an explicit scalar field potential, which, provided some special initial conditions are chosen, can reproduce the evolution arising in some perfect fluid cosmological model [99, 37].

Consider the Lagrangian

\[ L(\phi) = \frac{1}{2} \dot{\phi}^2 - V(\phi) \]  

(53)

and set the energy density of the field equal to that of the Chaplygin gas:

\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) = \sqrt{A + \frac{B}{a^6}}. \]  

(54)

The corresponding ”pressure” coincides with the Lagrangian density:

\[ p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) = -\frac{A}{\sqrt{A + \frac{B}{a^6}}}, \]  

(55)

It immediately follows that

\[ \dot{\phi}^2 = \frac{B}{a^6 \sqrt{A + \frac{B}{a^6}}} \]  

(56)

and

\[ V(\phi) = \frac{2 a^6 (A + \frac{B}{a^6}) - B}{2 a^6 \sqrt{A + \frac{B}{a^6}}}. \]  

(57)

We restrict ourselves to the flat case \( K = 0 \). Then Eq. (56) also implies that

\[ \phi' = \frac{\sqrt{B}}{a (Aa^6 + B)^{1/2}}, \]  

(58)

where prime means differentiation w.r.t. \( a \). This equation can be integrated and it follows that

\[ a^6 = \frac{4B \exp(6\phi)}{A(1 - \exp(6\phi))^2}. \]  

(59)

Finally, by substituting the latter expression for the cosmological radius in Eq. (57) one obtains the following potential, which has a surprisingly simple form:

\[ V(\phi) = \frac{1}{2} \sqrt{A} \left( \cosh 3\phi + \frac{1}{\cosh 3\phi} \right). \]  

(60)

Note that the potential does not depend on the integration constant \( B \) and therefore it reflects only the state equation \( \mathbb{H} \) as it should.
The cosmological evolution of the model with a scalar field with potential \( V(T) \) coincides with that of the Chaplygin gas model provided the initial values \( \phi(t_0) \) and \( \dot{\phi}(t_0) \) satisfy the relation
\[
\dot{\phi}^4(t_0) = 4(\dot{V}^2(\phi(t_0)) - A).
\]

We have mentioned in the Introduction, that one of the most popular candidates for the role of dark energy is a tachyon field described by the effective action\(^{[76, 77]}\)
\[
S = -\int d^4x\sqrt{-g}V(T)^2(1 - g^{\mu\nu}T_{\mu}\Tdot_{\nu}),
\]
where \( V(T) \) is a tachyon potential. For a spatially homogeneous model
\[
L = -V(T)^2(1 - T^2).
\]
The energy density of the tachyon field is
\[
\rho = \frac{V(T)^2}{1 - T^2}
\]
while the pressure is
\[
p = -\frac{V(T)^2}{1 - T^2}.
\]
It is easy to see\(^{[36]}\) that for the constant tachyon potential \( V(T) = V_0 \) the pressure\(^{[68]}\) and the energy density\(^{[62]}\) are connected by the Chaplygin state equation\(^{[1]}\) with \( A = V_0^2 \).

Thus, we see that the Chaplygin gas model coincides with the simplest tachyon cosmological model. It is interesting to construct also other tachyon potentials reproducing the dynamics of some perfect fluid models. For example\(^{[78, 79]}\), a tachyon model with a potential
\[
V(T) = \frac{4\sqrt{-k}}{g(1 + k)T^2}, \quad -1 < k < 0
\]
has a solution
\[
T = \sqrt{1 + kt},
\]
corresponding to the cosmological evolution
\[
a = a_0t^{\frac{1}{1 + k}},
\]
which, in turn, could be obtained in the model with a perfect fluid obeying the state equation
\[
p = k\rho.
\]
It is not difficult to show\(^{[37]}\), that for \( k > 0 \) it is impossible to reproduce the power-law cosmological evolution\(^{[67]}\) using a tachyon action. Instead, one can use a “pseudo” - tachyon action with the Lagrangian
\[
L = V(T)^2(1 - T^2) - 1
\]
with the potential
\[
V(T) = \frac{4\sqrt{k}}{g(1 + k)T^2}.
\]
In this case the corresponding solution (65) preserves its form.

We have considered [37] a more complicated toy tachyon model. Studying a two-fluid cosmological model, where one of the fluids is the cosmological constant and the other fluid obeys the state equation $p = k \rho$, $-1 < k < 1$, one gets the following expression for the cosmological evolution

$$a(t) = a_0 \left( \sinh \frac{3\sqrt{\Lambda}(1 + k)t}{2} \right)^{\frac{1}{3(1+k)}}. \tag{69}$$

The same evolution can be reproduced in the tachyon model with a potential

$$V(T) = \frac{\Lambda}{\sin^2 \left( \frac{3\sqrt{\Lambda}(1+k)t}{2} \right)} \sqrt{1 - (1+k) \cos^2 \left( \frac{3\sqrt{\Lambda}(1+k)t}{2} \right)}. \tag{70}$$

The solution of the tachyon equation of motion corresponding to the evolution (69) has the form

$$T(t) = \frac{2}{3\sqrt{\Lambda}(1+k)} \arctan \sinh \frac{3\sqrt{\Lambda}(1+k)t}{2} \tag{71}$$

and could be obtained provided some special initial conditions are chosen.

Considering all possible initial conditions we get a rich family of cosmological evolutions, which are rather different from (60), representing a simple two-fluid model [37]. Here we encounter two “surprises”. First, when $k > 0$, for the description of the dynamics of the model it is necessary to consider regions of the phase plane $(T, \dot{T})$, where the tachyon action (61) with the potential (70) is not well-defined and should be substituted by a pseudo-tachyon action. Second, (again for the case $k > 0$) there are two types of trajectories:

a) infinitely expanding universes;
b) universes, hitting a cosmological singularity of a special type which we call Big Brake, and which is characterised by the following behaviour of the cosmological radius

$$\ddot{a}(t_B) = -\infty, \ \dot{a}(t_B) = 0, \ 0 < a(t_B) < \infty. \tag{72}$$

Here $t_B$ means the final moment of time when the Big Brake is achieved.

In the current literature devoted to the future of the universe the following scenarios are usually considered:

a) an infinite asymptotically de Sitter expansion,
b) present accelerated expansion followed by contraction and the achieving of a Big Crunch cosmological singularity (see, e.g. [80, 81]),
c) an accelerated expansion culminating in a Big Rip cosmological singularity arising in the phantom dark energy cosmological models (see, e.g. [82, 83, 84]). At the Big Rip singularity the cosmological radius and the Hubble parameter achieve an infinite value in a finite interval of time.

On the basis of the above considerations, it seems reasonable to envisage an additional scenario, in which the present cosmic acceleration is followed by a decelerated expansion culminating in the hitting of a Big Brake cosmological singularity. The latter, as we have seen, can be described in terms of a rather simple tachyon model [37].
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