A $C^0$-W EAK GALERKIN F INITE E LEMENT M ETHOD FOR T HE BIHARMONIC EQUATION

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Abstract. A $C^0$-weak Galerkin (WG) method is introduced and analyzed for solving the biharmonic equation in 2D and 3D. A weak Laplacian is defined for $C^0$ functions in the new weak formulation. This WG finite element formulation is symmetric, positive definite and parameter free. Optimal order error estimates are established in both a discrete $H^2$ norm and the $L^2$ norm, for the weak Galerkin finite element solution. Numerical results are presented to confirm the theory. As a technical tool, a refined Scott-Zhang interpolation operator is constructed to assist the corresponding error estimate. This refined interpolation preserves the volume mass of order $(k+1-d)$ and the surface mass of order $(k+2-d)$ for the $P_{k+2}$ finite element functions in $d$-dimensional space.

Key words. weak Galerkin, finite element methods, weak Laplacian, biharmonic equation, triangular mesh, tetrahedral mesh, Scott-Zhang interpolation

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1. Introduction. We consider the biharmonic equation of the form

\begin{align*}
\Delta^2 u &= f, \quad \text{in } \Omega, \\
u &= g, \quad \text{on } \partial\Omega, \\
\frac{\partial u}{\partial n} &= \phi, \quad \text{on } \partial\Omega,
\end{align*}

where $\Omega$ is a bounded polygonal or polyhedral domain in $\mathbb{R}^d$ for $d = 2, 3$. For the biharmonic problem (1.1) with Dirichlet and Neumann boundary conditions (1.2) and (1.3), the corresponding variational form is given by seeking $u \in H^2(\Omega)$ satisfying $u|_{\partial\Omega} = g$ and $\frac{\partial u}{\partial n}|_{\partial\Omega} = \phi$ such that

\begin{align*}
(\Delta u, \Delta v) = (f, v) \quad \forall v \in H^2_0(\Omega),
\end{align*}

where $H^2_0(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial\Omega$.

The conforming finite element methods for the forth order problem (1.4) require the finite element space be a subspace of $H^2(\Omega)$. It is well known that constructing $H^2$ conforming finite elements is generally quite challenging, specially in three and

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higher dimensional spaces. Weak Galerkin finite element method, first introduce in [23] (see also [22] and [16] for extensions), by design is to use nonconforming elements to relax the difficulty in the construction of conforming elements. Unlike the classical nonconforming finite element method where standard derivatives are taken on each element, the weak Galerkin finite element method relies on weak derives taken as approximate distributions for the functions in nonconforming finite element spaces. In general, weak Galerkin method refers to finite element techniques for partial differential equations in which differential operators (e.g., gradient, divergence, curl, Laplacian) are approximated by weak forms as distributions.

A weak Galerkin method for the biharmonic equation has been derived in [18] by using totally discontinuous functions of piecewise polynomials on general partitions of arbitrary shape of polygons/polyhedra. The key of the method lies in the use of a discrete weak Laplacian plus a stabilization that is parameter-free. In this paper, we will develop a new weak Galerkin method for the biharmonic equation (1.1)-(1.3) by redefining a weak Laplacian, denoted by $\Delta_w$, for $C^0$ finite element functions. Comparing with the WG method developed in [18], the $C^0$-weak Galerkin finite element formulation has less number of unknowns due to the continuity requirement. On the other hand, due to the same continuity requirement, the $C^0$-WG method allows only traditional finite element partitions (such as triangles/quadrilaterals in 2D), instead of arbitrary polygonal/polyhedral grids as allowed in [18].

A suitably-designed interpolation operator is needed for the convergence analysis of the $C^0$-weak Galerkin formulation. The Scott-Zhang operator [21] turns out to serve the purpose well with a refinement. This paper shall introduce a refined version of the Scott-Zhang operator so that it preserves the volume mass up to order $(k+1-d)$, and the surface mass up to order $(k+2-d)$, when interpolating $H^1$ functions to the $P_{k+2} C^0$-finite element space:

$$Q_0 : H^1(\Omega) \rightarrow C^0 P_{k+2},$$

$$\int_T (v - Q_0 v) p dT = 0 \quad \forall p \in P_{k+1-d}(T),$$

$$\int_E (v - Q_0 v) p dE = 0 \quad \forall p \in P_{k+2-d}(T),$$

where $T$ is any triangle ($d = 2$) or tetrahedron ($d = 3$) in the finite element, and $E$ is an edge or a face-triangle of $T$. With the operator $Q_0$, we can show an optimal order of approximation property of the $C^0$-finite element space, under the constraints of weak Galerkin formulation. Consequently, we show optimal order of convergence in both a discrete $H^2$ norm and the $L^2$ norm, for the $C^0$ weak Galerkin finite element solution.

The biharmonic equation models a plate bending problem, which is one of the first applicable problems of the finite element method, cf. [9, 2, 10, 28]. The standard finite element method, i.e., the conforming element, requires a $C^1$ function space of piecewise polynomials. This would lead to a high polynomial degree [2, 26, 27, 24], or a macro-element [10, 6, 9, 12, 20, 25], or a constraint element (where the polynomial degree is reduced at inter-element boundary) [3, 19, 28]. Mixed methods for the biharmonic equation avoid using $C^1$ element by reducing the fourth order equation to a system of two second order equations, [1, 8, 11, 14, 17]. Many other different nonconforming and discontinuous finite element methods have been developed for
solving the biharmonic equation. Morley element [13] is a well known nonconforming element for its simplicity. \( C^0 \) interior penalty methods are studied in [5, 7, 15], which are similar to our \( \frac{1}{2} \text{weak} \) Galerkin method except there is no penalty parameter here.

2. Weak Laplacian and discrete weak Laplacian. Let \( D \) be a bounded polyhedral domain in \( \mathbb{R}^d, d = 2, 3 \). We use the standard definition for the Sobolev space \( H^s(D) \) and their associated inner products \((\cdot, \cdot)_{s,D}, \| \cdot \|_{s,D} \), and semi-norms \( | \cdot |_{s,D} \) for any \( s \geq 0 \). When \( D = \Omega \), we shall drop the subscript \( D \) in the norm and in the inner product.

Let \( T \) be a triangle or a tetrahedron with boundary \( \partial T \). A weak function on the region \( T \) refers to a vector function \( v = \{v_0, v_n\} \) such that \( v_0 \in L^2(T) \) and \( v_n \cdot n \in H^{-\frac{1}{2}}(\partial T) \), where \( n \) is the outward normal direction of \( T \) on its boundary.

The first component \( v_0 \) can be understood as the value of \( v \) on \( T \) and the second component \( v_n \) represents the value \( \nabla v \) on the boundary of \( T \). Note that \( v_n \) may not be necessarily related to the trace of \( \nabla v_0 \) on \( \partial T \). Denote by \( \mathcal{W}(T) \) the space of all weak functions on \( T \); i.e.,

\[
\mathcal{W}(T) = \left\{ v = \{v_0, v_n\} : v_0 \in L^2(T), v_n \cdot n \in H^{-\frac{1}{2}}(\partial T) \right\}.
\]

Let \((\cdot, \cdot)_T\) stand for the \( L^2 \)-inner product in \( L^2(T) \), \((\cdot, \cdot)_{\partial T}\) be the inner product in \( L^2(\partial T) \). For convenience, define \( G^2(T) \) as follows

\[
G^2(T) = \{ \varphi : \varphi \in H^1(T), \Delta \varphi \in L^2(T) \}.
\]

It is clear that, for any \( \varphi \in G^2(T) \), we have \( \nabla \varphi \in H(\text{div}, T) \). It follows that \( \nabla \varphi \cdot n \in H^{-\frac{1}{2}}(\partial T) \) for any \( \varphi \in G^2(T) \).

Definition 2.1. The dual of \( L^2(T) \) can be identified with itself by using the standard \( L^2 \)-inner product as the action of linear functionals. With a similar interpretation, for any \( v \in \mathcal{W}(T) \), the weak Laplacian of \( v = \{v_0, v_n\} \) is defined as a linear functional \( \Delta_w v \) in the dual space of \( G^2(T) \) whose action on each \( \varphi \in G^2(T) \) is given by

\[
(\Delta_w v, \varphi)_T = (v_0, \Delta \varphi)_T - (v_0, \nabla \varphi \cdot n)_{\partial T} + (v_n \cdot n, \varphi)_{\partial T},
\]

where \( n \) is the outward normal direction to \( \partial T \).

The Sobolev space \( H^2(T) \) can be embedded into the space \( \mathcal{W}(T) \) by an inclusion map \( i_\mathcal{W} : H^2(T) \to \mathcal{W}(T) \) defined as follows

\[
i_\mathcal{W}(\phi) = \{ \phi|_T, \nabla \phi|_{\partial T} \}, \quad \phi \in H^2(T).
\]

With the help of the inclusion map \( i_\mathcal{W} \), the Sobolev space \( H^2(T) \) can be viewed as a subspace of \( \mathcal{W}(T) \) by identifying each \( \phi \in H^2(T) \) with \( i_\mathcal{W}(\phi) \). Analogously, a weak function \( v = \{v_0, v_n\} \in \mathcal{W}(T) \) is said to be in \( H^2(T) \) if it can be identified with a function \( \phi \in H^2(T) \) through the above inclusion map. It is not hard to see that the weak Laplacian is identical with the strong Laplacian, i.e.,

\[
\Delta_w i_\mathcal{W}(v) = \Delta v
\]

for smooth functions \( v \in H^2(T) \).
Next, we introduce a discrete weak Laplacian operator by approximating $\Delta_w$ in a polynomial subspace of the dual of $G^2(T)$. To this end, for any non-negative integer $r \geq 0$, denote by $P_r(T)$ the set of polynomials on $T$ with degree no more than $r$. A discrete weak Laplacian operator, denoted by $\Delta_{w,r,T}$, is defined as the unique polynomial $\Delta_{w,r,T}v \in P_r(T)$ that satisfies the following equation

$$ (\Delta_{w,r,T}v, \varphi)_T = (v_0, \Delta \varphi)_T - \langle v_0, \nabla \varphi \cdot n_T \rangle_{\partial T} + \langle v_n \cdot n, \varphi \rangle_{\partial T} \quad \forall \varphi \in P_r(T). $$

Recall that $v_n$ represent the $\nabla v$ on $e \in \partial T$. Define $\bar{v}_n = (\nabla v \cdot n) n = v_n n$. Obviously, $\bar{v}_2 \cdot n = v_n \cdot n$. Since the quantity of interest is not $v_n$ but $v_n \cdot n$, we can replace $v_n$ by $\bar{v}_n = v_n n$ from now on to reduce the number of unknowns. Scalar $\bar{v}_n$ represents $\nabla v \cdot n$.

### 3. Weak Galerkin Finite Element Methods

Let $T_h$ be a triangular $(d = 2)$ or a tetrahedral $(d = 3)$ partition of the domain $\Omega$ with mesh size $h$. Denote by $E_h$ the set of all edges or faces in $T_h$, and $E_h^\circ = E_h \setminus \partial \Omega$ be the set of all interior edges or faces.

Since $v_n = v_n n$ with $v_n$ representing $\nabla v \cdot n$, obviously, $v_n$ is dependent on $n$. To ensure $v_n$ a single values function on $e \in E_h$, we introduce a set of normal directions on $E_h$ as follows

$$ (\nabla v \cdot n) e \quad \text{is unit and normal to } e, \quad e \in E_h. $$

Then, we can define a weak Galerkin finite element space $V_h$ for $k \geq 0$ as follows

$$ V_h = \{ v = \{ v_0, v_n n_e \} : v_0 \in V_0, v_n|_e \in P_{k+1}(e), e \subset \partial T \}, $$

where $v_n$ can be viewed as an approximation of $\nabla v \cdot n_e$ and

$$ V_0 = \{ v \in H^1(\Omega) ; v|_T \in P_{k+2}(T) \}. $$

Denote by $V_h$ a subspace of $V_h$ with vanishing traces; i.e.,

$$ V_h^0 = \{ v = \{ v_0, v_n n_e \} \in V_h, v_0|_e = 0, v_n|_e = 0, e \subset \partial T \cap \partial \Omega \}. $$

Denote by $\Lambda_h$ the trace of $V_h$ on $\partial \Omega$ from the component $v_0$. It is easy to see that $\Lambda_h$ consists of piecewise polynomials of degree $k + 2$. Similarly, denote by $\Upsilon_h$ the trace of $V_h$ from the component $v_n$ as piecewise polynomials of degree $k + 1$. Let $\Delta_{w,k}$ be the discrete weak Laplacian operator on the finite element space $V_h$ computed by using (2.3) on each element $T$ for $k \geq 0$; i.e.,

$$ (\Delta_{w,k}v)|_T = \Delta_{w,k,T}(v|_T) \quad \forall v \in V_h. $$

For simplicity of notation, from now on we shall drop the subscript $k$ in the notation $\Delta_{w,k}$ for the discrete weak Laplacian. We also introduce the following notation

$$ (\Delta_w v, \Delta_w w)_h = \sum_{T \in T_h} (\Delta_w v, \Delta_w w)_T. $$

For any $u_h = \{ u_0, u_n n_e \}$ and $v = \{ v_0, v_n n_e \}$ in $V_h$, we introduce a bilinear form as follows

$$ s(u_h, v) = \sum_{T \in T_h} h_T^{-1} \langle \nabla u_0 \cdot n_e - u_n, \nabla v_0 \cdot n_e - v_n \rangle_{\partial T}. $$
The stabilizer $s(u_h, v)$ defined above is to enforce a connection between the normal derivative of $u_0$ along $n_e$ and its approximation $u_n$.

Weak Galerkin Algorithm 1. A numerical approximation for (1.1)-(1.3) can be obtained by seeking $u_h = \{u_0, u_n, n_e\} \in V_h$ satisfying $u_0 = Q_h g$ and $u_n = (n \cdot n_e)Q_n \phi$ on $\partial \Omega$ and the following equation:

\[ (\Delta w, u_h, v) + s(u_h, v) = (f, v_0) \quad \forall \ v = \{v_0, v_n, n_e\} \in V_h^0, \]

where $Q_h g$ and $Q_n \phi$ are the standard $L^2$ projections onto the trace spaces $\Lambda_h$ and $\Upsilon_h$, respectively.

Lemma 3.1. The weak Galerkin finite element scheme (3.6) has a unique solution.

Proof. It suffices to show that the solution of (3.6) is trivial if $f = g = \phi = 0$. To this end, assume $f = g = \phi = 0$ and take $v = u_h$ in (3.6). It follows that

\[ (\Delta w u_h, \Delta u_h) + s(u_h, u_h) = 0, \]

which implies that $\Delta w u_h = 0$ on each element $T$ and $\nabla u_0 \cdot n_e = u_n$ on $\partial T$. We claim that $\Delta u_h = 0$ holds true locally on each element $T$. To this end, it follows from $\Delta w u_h = 0$ and (2.3) that for any $\varphi \in P_k(T)$ we have

\[ 0 = (\Delta w u_h, \varphi)_T = (u_0, \Delta \varphi)_T - (u_0, \nabla \varphi \cdot n)_{\partial T} + (u_n n_e \cdot n, \varphi)_{\partial T} \]

\[ = (\Delta u_0, \varphi)_T + (u_n n_e \cdot n - \nabla u_0 \cdot n, \varphi)_{\partial T} \]

\[ = (\Delta u_0, \varphi)_T, \]

where we have used

\[ u_n n_e \cdot n - \nabla u_0 \cdot n = \pm (u_n - \nabla u_0 \cdot n_e) = 0 \]

in the last equality. The identity (3.7) implies that $\Delta u_0 = 0$ holds true locally on each element $T$. This, together with $\nabla u_0 \cdot n_e = u_n$ on $\partial T$, shows that $u_h$ is a smooth harmonic function globally on $\Omega$. The boundary condition of $u_0 = 0$ and $u_n = 0$ then implies that $u_h \equiv 0$ on $\Omega$, which completes the proof. \[\Box\]

4. Projections: Definition and Approximation Properties. In this section, we will introduce some locally defined projection operators corresponding to the finite element space $V_h$ with optimal convergent rates.

Let $Q_h : H^1(\Omega) \to V_h$ be a special Scott-Zhang interpolation operator, to be defined in (A.9) in Appendix, such that for given $v \in H^1(\Omega)$, $Q_h v \in V_h$ and for any $T \in T_h$,

\[ (Q_h v, \Delta \varphi)_T - (Q_h v, \nabla \varphi \cdot n)_{\partial T} = (v, \Delta \varphi)_T - (v, \nabla \varphi \cdot n)_{\partial T}, \quad \forall \varphi \in P_k(T), \]

and for $0 \leq s \leq 2$

\[ \sum_{T \in T_h} h^{2s} \| u - Q_h u \|^2_{s, T} \leq C h^{k+3} \| u \|_{k+3}. \]

Now we can define an interpolation operator $Q_h$ from $H^2(\Omega)$ to the finite element space $V_h$ such that on the element $T$, we have

\[ Q_h u = \{Q_h u, (Q_h (\nabla u \cdot n_e)) n_e\}, \]
where $Q_0$ is defined in (A.9) and $Q_h$ is the $L^2$ projection onto $P_{k+1}(e)$, for each $e \subset \partial T$. In addition, let $\tilde{Q}_h$ be the local $L^2$ projection onto $P_k(T)$. For any $\varphi \in P_k(T)$ we have
\[
(\Delta w, \varphi)_T = (Q_0 u, \Delta \varphi)_T - (Q_0 u, \nabla \varphi \cdot n)_{\partial T} + (Q_0 (\nabla u \cdot n_e) n_e \cdot n, \varphi)_{\partial T} \\
= (u, \Delta \varphi)_T - (u, \nabla \varphi \cdot n)_{\partial T} + (\nabla u \cdot n, \varphi)_{\partial T} \\
= (\Delta u, \varphi)_T = (Q_h \Delta u, \varphi)_T,
\]
which implies
\[
(4.4) \quad \Delta w Q_h u = Q_h (\Delta u).
\]

The above identity indicates that the discrete weak Laplacian of a projection of $u$ is a good approximation of the Laplacian of $u$.

Let $T \in T_h$ be an element with $e$ as an edge or a face triangle. It is well known that there exists a constant $C$ such that for any function $g \in H^2(T)$
\[
(4.5) \quad \|g\|^2 \leq C \left( h_T^{-1} \|g\|^2_T + h_T \|\nabla g\|^2_T \right).
\]

Define a mesh-dependent semi-norm $\| \cdot \|$ in the finite element space $V_h$ as follows
\[
(4.6) \quad \|v\|^2 = (\Delta w, \Delta w)_h + s(v, v), \quad v \in V_h.
\]

Using (4.5), we can derive the following estimates which are useful in the convergence analysis for the WG-FEM (3.6).

**Lemma 4.1.** Let $w \in H^{k+3}(\Omega)$ and $v \in V_h$. Then there exists a constant $C$ such that the following estimates hold true.

\[
(4.7) \quad \sum_{T \in T_h} |(\Delta w - Q_h \Delta w, (\nabla v_0 - v_n n_e) \cdot n)_{\partial T}| \leq C h^{k+1} \|w\|_{k+3} \|v\|,
\]
\[
(4.8) \quad \sum_{T \in T_h} h_T^{-1} |((\nabla Q_0 w \cdot n_e - Q_n (\nabla w \cdot n_e), \nabla v_0 \cdot n_e - v_n))_{\partial T}| \leq C h^{k+1} \|w\|_{k+3} \|v\|.
\]

**Proof.** To derive (4.7), we can use the Cauchy-Schwarz inequality, (3.8), the trace inequality (4.5), and the definition of $Q_h$ to obtain
\[
\sum_{T \in T_h} |(\Delta w - Q_h \Delta w, (\nabla v_0 - v_n n_e) \cdot n)_{\partial T}| \\
\leq \left( \sum_{T \in T_h} h_T \|\Delta w - Q_h \Delta w\|^2_{\partial T} \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \|\nabla v_0 \cdot n_e - v_n\|^2_{\partial T} \right)^{\frac{1}{2}} \\
\leq C \left( \sum_{T \in T_h} (\|\Delta w - Q_h \Delta w\|^2_T + h_T^2 \|\nabla (\Delta w - Q_h \Delta w)\|^2_T) \right)^{\frac{1}{2}} \|v\| \\
\leq C h^{k+1} \|w\|_{k+3} \|v\|.
\]
As to (4.8), we have from the definition of $Q_n$, the Cauchy-Schwarz inequality, the trace inequality (4.5), and (4.2) that

$$\sum_{T \in \mathcal{T}_h} h_T^{-1} |(\nabla Q_0 w) \cdot n_e - Q_n (\nabla w \cdot n_e, \nabla v_0 \cdot n_e - v_n)_{\partial T}|$$

$$= \sum_{T \in \mathcal{T}_h} h_T^{-1} |(\nabla Q_0 w) \cdot n_e - \nabla w \cdot n_e, \nabla v_0 \cdot n_e - v_n)_{\partial T}|$$

$$\leq \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} |(\nabla Q_0 w - \nabla w) \cdot n_e\|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in \mathcal{T}_h} h_T^{-1} \|\nabla v_0 \cdot n_e - v_n\|_{\partial T}^2 \right)^{\frac{1}{2}}$$

$$\leq C \left( \sum_{T \in \mathcal{T}_h} (h_T^{-2} |\nabla Q_0 w - \nabla w|_{T}^2 + \|\nabla Q_0 w - \nabla w|_{1,T}^2 \right)^{\frac{1}{2}} \|v\|$$

$$\leq C h^{k+1} \|w\|_{k+3} \|v\|.$$

This completes the proof. $\square$

5. An Error Equation. We first derive an equation that the projection of the exact solution, $Q_h u$, shall satisfy. Using (2.3), the integration by parts, and (4.4), we obtain

$$(\Delta_w Q_h u, \Delta_w v)_T = (v_0, \Delta(\Delta_w Q_h u))_T + ((v_n n_e) \cdot n, \Delta_w Q_h u)_{\partial T} - (v_0, \nabla(\Delta_w Q_h u) \cdot n)_{\partial T}$$

$$= (\Delta v_0, \Delta_w Q_h u)_T + (v_0, \nabla(\Delta_w Q_h u) \cdot n)_{\partial T} - (v_0, \Delta_w Q_h u)_{\partial T}$$

$$+ (v_n n_e) \cdot n, \Delta_w Q_h u)_{\partial T} - (v_0, \nabla(\Delta_w Q_h u) \cdot n)_{\partial T}$$

$$= (\Delta v_0, \Delta_w Q_h u)_T - ((\nabla v_0 - v_n n_e) \cdot n, \Delta_w Q_h u)_{\partial T}$$

$$= (\Delta u, \Delta v_0)_T - ((\nabla v_0 - v_n n_e) \cdot n, \Delta_h u)_{\partial T},$$

which implies that

$$(\Delta u, \Delta v_0)_T = (\Delta_w Q_h u, \Delta_w v)_T + ((\nabla v_0 - v_n n_e) \cdot n, \Delta_h u)_{\partial T}. \tag{5.1}$$

Next, it follows from the integration by parts that

$$(\Delta u, \Delta v_0)_T = (\Delta^2 u, v_0)_T + (\Delta u, \nabla v_0 \cdot n)_{\partial T} - (\nabla(\Delta u) \cdot n, v_0)_{\partial T}.$$  

Summing over all $T$ and then using the identity $\Delta^2 u, v_0) = (f, v_0)$ we arrive at

$$\sum_{T \in \mathcal{T}_h} (\Delta u, \Delta v_0)_T = (f, v_0) + \sum_{T \in \mathcal{T}_h} (\Delta u, \nabla v_0 \cdot n)_{\partial T}$$

$$= (f, v_0) + \sum_{T \in \mathcal{T}_h} (\Delta u, (\nabla v_0 - v_n n_e) \cdot n)_{\partial T}.$$ 

Combining the above equation with (5.1) leads to

$$(\Delta_w Q_h u, \Delta_w v)_h = (f, v_0) + \sum_{T \in \mathcal{T}_h} (\Delta u - \Delta_h u, (\nabla v_0 - v_n n_e) \cdot n)_{\partial T}. \tag{5.2}$$
Define the error between the finite element approximation \( u_h \) and the projection of the exact solution \( u \) as
\[
e_h := \left\{ e_0, e_n n_e \right\} = \left\{ Q_0 u - u_0, (Q_n (\nabla u \cdot n_e) - u_n)n_e \right\}.
\]

Taking the difference of (5.2) and (3.6) gives the following error equation
\[
(\Delta w_h, \Delta w)_h + s(e_h, v) = \sum_{T \in T_h} \langle \Delta u - Q_h \Delta u, (\nabla e_0 - e_n n_e) \cdot n \rangle_{\partial T}
+ s(Q_h u, v) \quad \forall v \in V_h^0.
\]

Observe that the definition of the stabilization term \( s(\cdot, \cdot) \) indicates that
\[
s(Q_h u, v) = \sum_{T \in T_h} h_T^{-1} \langle (\nabla Q_0 u) \cdot n_e - Q_n (\nabla u \cdot n_e), \nabla (e_0 - e_n) \rangle_{\partial T}.
\]

6. Error Estimates. First, we derive an estimate for the error function \( e_h \) in the natural triple-bar norm, which can be viewed as a discrete \( H^2 \)-norm.

**Theorem 6.1.** Let \( u_h \in V_h \) be the weak Galerkin finite element solution arising from (3.6) with finite element functions of order \( k + 2 \geq 2 \). Assume that the exact solution of (1.1)-(1.3) is regular such that \( u \in H^{k+3}(\Omega) \). Then, there exists a constant \( C \) such that
\[
\| u_h - Q_h u \| \leq C h^{k+1} \| u \|_{k+3}.
\]

**Proof.** By letting \( v = e_h \) in the error equation (5.3), we obtain the following identity
\[
\| e_h \|^2 = \sum_{T \in T_h} \langle \Delta u - Q_h \Delta u, (\nabla e_0 - e_n n_e) \cdot n \rangle_{\partial T}
+ \sum_{T \in T_h} h_T^{-1} \langle (\nabla Q_0 u) \cdot n_e - Q_n (\nabla u \cdot n_e), \nabla (e_0 - e_n) \rangle_{\partial T}.
\]

Using the estimates of Lemma 4.1, we arrive at
\[
\| e_h \|^2 \leq C h^{k+1} \| u \|_{k+3} \| e_h \|,
\]
which implies (6.1). This completes the proof of the theorem. \( \square \)

Next, we would like to provide an estimate for the standard \( L^2 \) norm of the first component of the error function \( e_h \). Let us consider the following dual problem
\[
\Delta^2 w = e_0 \quad \text{in } \Omega,
\]
\[
\nabla w = 0, \quad \text{on } \partial \Omega,
\]
\[
\nabla w \cdot n = 0 \quad \text{on } \partial \Omega.
\]

The \( H^4 \) regularity assumption of the dual problem implies the existence of a constant \( C \) such that
\[
\| w \|_4 \leq C \| e_0 \|.
\]
THEOREM 6.2. Let \( u_h \in V_h \) be the weak Galerkin finite element solution arising from (3.6) with finite element functions of order \( k + 2 \geq 3 \). Assume that the exact solution of (1.1)-(1.3) is regular such that \( u \in H^{k+3}(\Omega) \) and the dual problem (6.2)-(6.4) has the \( H^3 \) regularity. Then, there exists a constant \( C \) such that

\[
\|Q_0u - u_0\| \leq Ch^{k+3}\|u\|_{k+3}.
\]

Proof. Testing (6.2) by error function \( e_0 \) and then using the integration by parts gives

\[
\|e_0\|^2 = (\Delta^2 w, e_0) = \sum_{T \in T_h} (\Delta w, \Delta e_0)_T - \sum_{T \in T_h} (\Delta w, \nabla e_0 \cdot n)_{\partial T} = \sum_{T \in T_h} (\Delta w, \Delta e_0)_T - \sum_{T \in T_h} (\Delta w, (\nabla e_0 - e_n n_e) \cdot n)_{\partial T}.
\]

Using (5.1) with \( w \) in the place of \( u \), we can rewrite the above equation as follows

\[
\|e_0\|^2 = (\Delta w Q_h w, \Delta w e_h)_h - \sum_{T \in T_h} (\Delta w - Q_h \Delta w, (\nabla e_0 - e_n n_e) \cdot n)_{\partial T}.
\]

It now follows from the error equation (5.3) that

\[
(\Delta w Q_h w, \Delta w e_h)_h = \sum_{T \in T_h} (\Delta u - Q_h \Delta u, (\nabla Q_0 w - Q_n (\nabla w \cdot n_e) n_e) \cdot n)_{\partial T} - s(e_h, Q_h w) + s(Q_h u, Q_h w).
\]

Combining the two equations above gives

\[
\|e_0\|^2 = -\sum_{T \in T_h} (\Delta w - Q_h \Delta w, (\nabla e_0 - e_n n_e) \cdot n)_{\partial T}
\]

\[
+ \sum_{T \in T_h} (\Delta u - Q_h \Delta u, (\nabla Q_0 w - Q_n (\nabla w \cdot n_e) n_e) \cdot n)_{\partial T} - s(e_h, Q_h w) + s(Q_h u, Q_h w).
\]

Using the estimates of Lemma 4.1, we can bound two terms on the right-hand side of the equation above as follows

\[
\sum_{T \in T_h} |(\Delta w - Q_h \Delta w, (\nabla e_0 - e_n n_e) \cdot n))_{\partial T}| \leq Ch^2 \|w\|_4 \|e_h\|,
\]

\[
|s(e_h, Q_h w)\| \leq Ch^2 \|w\|_4 \|e_h\|.
\]

It follows from (3.8) and the definition of \( Q_n \) and \( Q_0 \) that

\[
\|\nabla Q_0 w - Q_n (\nabla w \cdot n_e) n_e\|_{\partial T} = \|(\nabla Q_0 w - Q_n (\nabla w \cdot n_e) n_e) \cdot n\|_{\partial T}
\]

\[
= \|\nabla Q_0 w \cdot n - Q_n (\nabla w \cdot n)\|_{\partial T} \leq \|\nabla Q_0 w \cdot n - \nabla w \cdot n\|_{\partial T}
\]

\[
+ \|\nabla w \cdot n - Q_n (\nabla w \cdot n)\|_{\partial T} \leq C \|\nabla Q_0 w \cdot n - \nabla w \cdot n\|_{\partial T}.
\]
Using (6.8) and (4.5), we have

\[
\sum_{T \in T_h} \langle \Delta u - Q_h \Delta u, (\nabla Q_0 w - Q_n (\nabla w \cdot n_e)) \cdot n \rangle_{\partial T} \\
\leq C \left( \sum_{T \in T_h} h \| \Delta u - Q_h \Delta u \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h^{-1} \| (\nabla Q_0 w - \nabla w) \cdot n \|_{\partial T}^2 \right)^{1/2} \\
\leq C \left( \sum_{T \in T_h} \left( h^{-2} \| (\nabla Q_0 w - \nabla w) \cdot n \|_{\partial T}^2 + \| (\nabla Q_0 w - \nabla w) \cdot n \|_{\partial T}^2 \right) \right)^{1/2} \\
\leq C h^{k+3} \| u \|_{k+3} \| w \|_4.
\]

Using (6.8) and (4.5), we have

\[
s(Q_h u, Q_h w) = \sum_{T \in T_h} h^{-1} \langle (\nabla Q_0 u) \cdot n_e - Q_n (\nabla u \cdot n_e), (\nabla Q_0 w) \cdot n_e - Q_n (\nabla w \cdot n_e) \rangle_{\partial T} \\
\leq \left( \sum_{T \in T_h} h^{-1} \| (\nabla Q_0 u - \nabla u) \cdot n \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h^{-1} \| (\nabla Q_0 w - \nabla w) \cdot n \|_{\partial T}^2 \right)^{1/2} \\
\leq C h^{k+3} \| u \|_{k+3} \| w \|_4.
\]

Substituting all above estimates into (6.7) and using (6.1) give

\[
\| e_0 \|_2^2 \leq C h^{k+3} \| u \|_{k+3} \| w \|_4.
\]

Combining the above estimate with (6.5), we obtain the desired \(L^2\) error estimate (6.6). \(\square\)

7. **Numerical Experiments.** This section shall report some numerical results for the \(C^0\) weak Galerkin finite element methods for the following biharmonic equation:

\[
\begin{align*}
\Delta^2 u &= f & \text{in } \Omega, \\
u &= g & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial n} &= \psi & \text{on } \partial \Omega.
\end{align*}
\]

For simplicity, all the numerical experiments are conducted by using \(k = 0\) or \(k = 1\) in the finite element space \(V_h\) in (3.2).

If \(\phi \in P_0(T)\) (i.e. \(k = 0\)), the above equation can be simplified as

\[
(\Delta_w v, \phi)_T = \langle v_n n_e \cdot n, \phi \rangle_{\partial T}.
\]

The error for the \(C^0\)-WG solution will be measured in four norms defined as follows:
\[ H^1 \text{ semi-norm:} \]
\[ \| v - v_0 \|^2_1 = \sum_{T \in T_h} \int_T |\nabla v - \nabla v_0|^2 dx. \]

Discrete \( H^2 \) norm:
\[ \| v \|^2 = \sum_{T \in T_h} \| \Delta w \|^2_T + \sum_{T \in T_h} h^{-1} \| \nabla v_0 \cdot n_e - v_n \|^2_{\partial T}, \]

Element-based \( L^2 \) norm:
\[ \| Q_v - v_0 \|^2 = \sum_{T \in T_h} \int_T |Q_v - v_0|^2 dx, \]

Edge-based \( L^2 \) norm:
\[ \| Q_n(\nabla v \cdot n_e) - v_n \|^2 = \sum_{e \in E_h} h \int_e |Q_n(\nabla v \cdot n_e) - v_n|^2 ds. \]

**7.1. Example 1.** Consider the biharmonic problem (7.1)-(7.3) in the square domain \( \Omega = (0,1)^2 \). Set the exact solution by
\[ u = x^2(1-x)^2 y^2(1-y)^2. \]

**Table 7.1**

| \( h \) | \( ||u - u_0||_1 \) | \( ||u - Q_h u|| \) | \( ||u_0 - Q_0 u|| \) | \( ||Q_n(\nabla u \cdot n_e) - u_n||_b \) |
|---|---|---|---|---|
| 1/4 | 6.8858e-03 | 6.0250e-02 | 1.4563e-03 | 4.3364e-03 |
| 1/8 | 1.7465e-03 | 3.0867e-02 | 3.8153e-04 | 1.4617e-03 |
| 1/16 | 4.3885e-04 | 1.5555e-02 | 9.6991e-05 | 4.0941e-04 |
| 1/32 | 1.0982e-04 | 7.7916e-03 | 2.4330e-05 | 1.0558e-04 |
| 1/64 | 2.7458e-05 | 3.8972e-03 | 6.0931e-06 | 2.6601e-05 |
| 1/128 | 6.8645e-06 | 1.9487e-03 | 1.5236e-06 | 6.6629e-06 |
| Conv.Rate | 1.9949 | 9.9160e-01 | 1.9829 | 1.8865 |

It is easy to check that
\[ u|_{\partial \Omega} = 0, \ \frac{\partial u}{\partial n} = 0. \]

The function \( f \) is given according to the equation (7.1).

The test is performed by using uniform triangular mesh. The mesh is constructed as follows: 1) partition the domain into \( n \times n \) sub-rectangles; 2) divide each square element into two triangles by the diagonal line with a negative slope. The mesh size is denoted by \( h = 1/n \). Table 7.1 shows the convergence rate for \( C^0 \)-WG solutions based on \( k = 0 \) in four norms respectively. The numerical results indicate that the WG
solution is convergent with rate $O(h^2)$ in $H^1$, $O(h^1)$ in $H^2$, and $O(h^2)$ in $L^2$ norms. The convergence rate for $\|Q_n(\nabla u \cdot n_e) - u_n\|_b$ is $O(h^2)$. Also, the same problem is tested for $k = 1$. The results are reported in Table 7.2. It indicates that the WG solution is convergent with rate $O(h^3)$ in $H^1$, $O(h^2)$ in $H^2$, and $O(h^4)$ in $L^2$ norms.

We note that the $L^2$ error is convergent at order 4, two orders higher than that of $k = 0$, confirming the sharpness of Theorem 6.6. Moreover the convergence rate for $\|Q_n(\nabla u \cdot n_e) - u_n\|_b$ is $O(h^3)$, for $k = 1$.

| $h$   | $\|u - u_0\|_1$ | $\|u_h - Q_h u\|$ | $\|u_0 - Q_0 u\|$ | $\|Q_n(\nabla u \cdot n_e) - u_n\|_b$ |
|-------|----------------|------------------|------------------|------------------|
| 1/4   | 1.5888e-03     | 1.5888e-02       | 1.5751e-04       | 1.7898e-03       |
| 1/8   | 2.6787e-04     | 4.7921e-03       | 1.3887e-05       | 2.6200e-04       |
| 1/16  | 3.8354e-05     | 1.2963e-03       | 1.0006e-06       | 3.4742e-05       |
| 1/32  | 5.0893e-06     | 3.3568e-04       | 6.6590e-08       | 4.4563e-06       |
| 1/64  | 6.5373e-07     | 8.5314e-05       | 4.2842e-09       | 5.6344e-07       |
| 1/128 | 8.2783e-08     | 2.1499e-05       | 2.7341e-08       | 7.0798e-08       |
| Conv.Rate | 2.8597       | 1.9152            | 3.8450            | 2.9336            |

7.2. Example 2. In this problem, we set $\Omega = (0, 1)^2$ and the exact solution:

$$u = \sin(\pi x) \sin(\pi y),$$

with

$$u|_{\partial \Omega} = 0, \quad \frac{\partial u}{\partial n} \neq 0.$$ 

Boundary conditions and $f$ are given according to the equation (7.1)-(7.3).

Again, the uniform triangular mesh is used in the experiment. Table 7.3 shows that the convergence rate for $C^0$-WG solutions in $H^1$, $H^2$ and $L^2$ norms is $O(h^2)$, $O(h)$ and $O(h^2)$, respectively.

| $h$   | $\|u - u_0\|_1$ | $\|u_h - Q_h u\|$ | $\|u_0 - Q_0 u\|$ | $\|Q_n(\nabla u \cdot n_e) - u_n\|_b$ |
|-------|----------------|------------------|------------------|------------------|
| 1/4   | 6.1653e-01     | 5.5381           | 1.2978e-01       | 2.7515e-01       |
| 1/8   | 1.4737e-01     | 2.7431           | 3.2219e-02       | 6.8563e-02       |
| 1/16  | 3.6122e-02     | 1.3640           | 7.9854e-03       | 1.6489e-02       |
| 1/32  | 8.9758e-03     | 6.8082e-01       | 1.9899e-03       | 4.0589e-03       |
| 1/64  | 2.2403e-03     | 3.4024e-01       | 4.9703e-04       | 1.0102e-03       |
| 1/128 | 5.5983e-04     | 1.7010e-01       | 1.2423e-04       | 2.5224e-04       |
| Conv.Rate | 2.0186       | 1.0046            | 2.0058            | 2.0209            |

7.3. Example 3. The exact solution is chosen as

$$u = \sin(\pi x) \cos(\pi y),$$
with nonhomogeneous boundary conditions.

Table 7.4 shows that the convergence rate for \( C^0 \)-WG solutions in \( H^1 \), \( H^2 \) and \( L^2 \) norms is \( O(h^2) \), \( O(h) \), and \( O(h^2) \), respectively.

| \( h \)     | \( \| u - u_0 \|_1 \) | \( \| u_h - Q_h u \| \) | \( \| u_0 - Q_0 u \| \) | \( \| Q_n(\nabla u \cdot n_e) - u_n \|_b \) |
|------------|------------------|-----------------|-----------------|------------------|
| 1/4        | 2.7134e-01       | 4.3389          | 5.9389e-01      | 2.0608           |
| 1/8        | 5.6175e-02       | 2.4888          | 2.0490e-01      | 9.4191e-01       |
| 1/16       | 1.3236e-02       | 1.3196          | 5.9347e-02      | 7.4.18609         |
| 1/32       | 3.2856e-03       | 6.737e-01       | 1.5585e-02      | 3.9554e-03       |
| 1/64       | 8.2159e-04       | 7.9441e-05      | 9.9329e-04      | 1/64              |
| 1/128      | 2.0553e-04       | 1.9812e-05      | 3.9554e-03      | 9.9329e-04       |
| Conv.Rate  | 2.0608           | 1.9812          | 9.9329e-04      | 1.8609           |

**Example 3.** Convergence rate for element \( P_2(T) - P_1(e) \) \((k = 0)\). \( L^2 \) norm is \( O(h^2) \).

7.4. Example 4. In the final example, we test a case where the exact solution has a low regularity in the domain \( \Omega = (0, 1)^2 \). The exact solution is given by

\[
    u = r^{3/2} \left( \sin \frac{3\theta}{2} - 3 \sin \frac{\theta}{2} \right),
\]

where \((r, \theta)\) are the polar coordinates. It is known that \( u \in H^{2.5}(\Omega) \). The performance for \( C^0 \) weak Galerkin finite element approximations for element \( P_2(T) - P_1(e) \) \((k = 0)\) is reported in Table 7.5. The convergence rates in \( H^1 \)-norm, \( H^2 \)-norm, and edge-based \( L^2 \)-norm are seen as \( O(h^{1.4}) \), \( O(h^{0.47}) \), and \( O(h^{1.4}) \). The corresponding theoretical prediction has the order of \( O(h^{1.5}) \), \( O(h^{0.5}) \), and \( O(h^{1.5}) \). We believe that the numerical results are in consistency with the theory. Table 7.5 indicates that the numerical convergence rate in the standard \( L^2 \) is of order \( O(h^{1.88}) \), which exceeds the theoretical prediction of \( O(h^{1.5}) \).

Table 7.6 contains some numerical results for the weak Galerkin element \( P_k(T) - P_2(e) \) \((k = 1)\). The convergence rates in \( H^1 \)-norm, \( H^2 \)-norm, and edge-based \( L^2 \)-norm are seen as \( O(h^{1.5}) \), \( O(h^{0.5}) \), and \( O(h^{1.5}) \), which are completely in consistency with the theory. For the element-based \( L^2 \) error, Table 7.6 indicates a numerical convergence rate of order \( O(h^{2.49}) \), which is also consistent with the theoretical prediction of \( O(h^{2.5}) \).

Appendix A. A mass-preserving Scott-Zhang operator. We will prove the existence of an interpolation \( Q_0 \) used in (4.1) and in the previous section, which is a special Scott-Zhang operator[21]. The new Scott-Zhang operator preserves the mass on each element and on each face, of four orders and three orders less, respectively, when interpolating \( H^1(\Omega) \) functions to the finite element \( V_h \) functions. We shall derive the optimal-order approximation properties for the interpolation in the section, which leads to a quasi-optimal convergence of the weak Galerkin finite element method (3.6).

The original Scott-Zhang operator maps \( u \in H^1(\Omega) \) functions to \( C^0 \)-Lagrange finite element functions, preserving the zero boundary condition if \( u \in H^1(\Omega) \). It is an Lagrange interpolation. All the Lagrange nodes (\(|\delta|\)) on one element are classified...
### Table 7.5

**Example 4. Convergence rate for element $P_2(T) - P_1(e)$ ($k = 0$).**

| $h$   | $\|u - u_0\|_1$ | $\|u_h - Q_h u\|$ | $\|u_0 - Q_0 u\|$ | $\|Q_n(\nabla u \cdot n_e) - u_n\|$ |
|-------|-----------------|-------------------|-------------------|-----------------|
| $1/4$ | 3.196e-02       | 9.0667e-01        | 3.3386e-03        | 1.5615e-01      |
| $1/8$ | 1.3596e-02      | 6.8589e-01        | 1.1209e-03        | 6.2562e-02      |
| $1/16$| 5.1368e-03      | 4.9952e-01        | 3.1392e-04        | 6.2562e-02      |
| $1/32$| 1.8697e-03      | 3.5808e-01        | 8.2158e-05        | 8.4733e-03      |
| $1/64$| 5.1368e-03      | 2.5488e-01        | 2.0925e-05        | 3.0321e-03      |
| $1/128$| 1.8697e-03    | 1.8081e-01        | 5.2718e-06        | 1.0784e-03      |
| Conv. Rate | 1.4233 | 4.6844e-01 | 1.8767 | 1.4415 |

### Table 7.6

**Example 4. Convergence rate for element $P_3(T) - P_2(e)$ ($k = 1$).**

| $h$   | $\|u - u_0\|_1$ | $\|u_h - Q_h u\|$ | $\|u_0 - Q_0 u\|$ | $\|Q_n(\nabla u \cdot n_e) - u_n\|$ |
|-------|-----------------|-------------------|-------------------|-----------------|
| $1/4$ | 2.5197e-02       | 5.0303e-01        | 1.3671e-03        | 4.7712e-02      |
| $1/8$ | 8.9650e-03       | 3.5619e-01        | 2.4629e-04        | 1.6900e-02      |
| $1/16$| 3.1718e-03       | 2.5190e-01        | 4.3679e-05        | 5.9764e-03      |
| $1/32$| 1.1215e-03       | 1.7812e-01        | 7.7825e-06        | 2.1130e-03      |
| $1/64$| 3.9652e-04       | 1.2595e-01        | 1.3812e-06        | 7.4708e-04      |
| $1/128$| 1.4019e-04    | 8.9063e-02        | 2.4431e-07        | 2.6413e-04      |
| Conv. Rate | 1.4984 | 4.9966e-01 | 2.4907 | 1.4995 |

into three types:

- **corner nodes** $c_j$: 3 vertex nodes in 2D, or all edge nodes in 3D,
- **middle nodes** $m_j$: all mid-edge nodes in 2D, or mid-triangle nodes in 3D,
- **internal nodes** $i_j$: all internal nodes in the triangle/tetrahedra.

The three types of nodes are illustrated in Figures A.1 and A.2. In simple words, $\{c_j\}$ are nodes shared by possibly more than two elements, $\{m_j\}$ are nodes shared by no more than two elements, and $\{i_j\}$ are nodes internal to one element.

A Lagrange nodal basis function $\phi_j$ is a $P_{k+2}$ polynomial which assumes value 1 at one node $c_j$, but vanishes at all other dim $P_{k+2} - 1$ nodes. For example, a $P^2$ nodal basis function on the reference triangle $\{0 \leq x, y, 1 - x - y \leq 1\}$, at node $(1/4, 0)$, c.f. Figure A.3, is

$$\phi_2(x, y) = \frac{x(1 - x - y)(3/4 - x - y)(2/4 - x - y)}{(1/4)(1 - 1/4 - 0)(3/4 - 1/4 - 0)(2/4 - 1/4 - 0)}.$$  
(A.1)

The restriction of a nodal basis $\phi_j$ on a lower dimensional simplex, a triangle or an edge or a vertex, is also a nodal basis function on that lower dimensional finite element. For example, this node basis function (A.1) is the restriction of the following 3D nodal basis function (at node $(1/4, 0, 0)$ on tetrahedron $\{0 \leq x, y, z, 1 - x - y - z \leq 1\}$) on the reference triangle,

$$\phi_j(x, y, z) = \frac{x(1 - x - y - z)(3/4 - x - y - z)(2/4 - x - y - z)}{(1/4)(1 - 1/4 - 0 - 0)(3/4 - 1/4 - 0 - 0)(2/4 - 1/4 - 0 - 0)}.$$  
(A.2)
The restriction of 2D basis function \( \phi_2 \) in (A.1) in 1D is, c.f. Figure A.3,
\[
\phi_j'(x) = \frac{x(1-x)(3/4-x)(2/4-x)}{(1/4)(1-1/4)(3/4-1/4)(2/4-1/4)}
\]

On each element \( T \) (an edge, a triangle, or a tetrahedron), the \( P_k \) Lagrange basis \( \{\phi_j\} \) has a dual basis \( \{\psi_j \in P_k^d\} \), satisfying
\[
\int_T \phi_j \psi_{j'} \, dx = \delta_{jj'} = \begin{cases} 
1 & \text{if } j = j', \\
0 & \text{if } j \neq j'. 
\end{cases}
\]
In other words, if writing \( \{\psi_j\} \) as linear combinations of Lagrange basis \( \{\phi_j\} \), the coefficients are simply the inverse matrix of the mass matrix, the \( L^2 \)-matrix of \( \{\phi_j\} \).
For example, the dual basis function $\psi_2$ for the nodal basis function $\phi_2$ in (A.1) (2D) is

$$\psi_2^{[2D]}(x, y) = \frac{2835}{4} x - \frac{12285}{4} x^2 + \frac{8505}{2} x^3 + 8505 x^2 y + \frac{8505}{2} xy^2 - 1890 x^4 - 5670 x^3 y - 5670 x^2 y^2 - 1890 xy^3.$$ 

We can compute the dual of $\psi_{j'}$ in (A.3) in 1D to get

$$\psi_{j'}^{[1D]}(x) = -\frac{485}{128} + \frac{2865}{32} x - \frac{21105}{64} x^2 + \frac{13615}{32} x^3 - \frac{11655}{64} x^4$$

$$= -\frac{485}{128} \phi_0 + \frac{64225}{16384} \phi_1 + \frac{345}{1024} \phi_2 - \frac{4255}{16384} \phi_3 - \frac{85}{128} \phi_4,$$

where $\phi_i$ is the nodal basis on $[0, 1]$ at $x_i = i/4$, $i = 0, 1, 2, 3, 4$. The dual function $\psi_{j'}^{[1D]}(x)$ in (A.5) is plotted in Figure A.4. Similarly we can compute the dual basis function for (A.2) in 3D. We note that both Lagrange nodal basis and its dual basis are affine invariant. That is, the Lagrange basis on the reference triangle is also the Lagrange basis on a general triangle after an affine mapping. For simplicity, we use the same notations $\phi_j$ and $\psi_j^{[2D]}$ for the nodal basis and the dual basis functions on the reference triangle and on a general triangle.

![Fig. A.4. A 1D nodal basis $\phi_j$ (A.3) and its dual basis in 1D, $\psi_{j'}^{[1D]}$ (A.5), $\int_0^1 \phi_j \psi_{j'}^{[1D]} = 1$.](image-url)

We now define the Scott-Zhang interpolation operator:

$$Q_0 : H^1(\Omega) \rightarrow V_h,$$

where $V_h$ is the $C^0$-$P_{k+2}$ finite element space defined in (3.2). $Q_0 v$ is defined by the nodal values at three types nodes.

1. For each corner node $c_j$ (shared by possibly more than two elements), we select one boundary $(d - 1)$-dimensional simplex $C_j$ if $c_j$ is a boundary node, or any one $(d - 1)$-dimensional face simplex $C_j$ on which the node is, as $c_j$’s averaging patch. C.f. Figure A.1, the boundary node $c_j$ has a boundary edge $C_j$, while the corner node $c_{j''}$ can choose any one of four edges passing it, as its averaging patch. In Figure A.2, a corner node $c_j$ has a triangle $C_j$ as its averaging patch. In both 2D and 3D, we use a same definition

$$Q_0 v(c_j) = \int_{C_j} \psi_j^{[(d-1)D]} v(x) dx.$$

2. For each middle node $m_j$, the averaging patch is the unique $(d-1)$-dimensional simplex $C_j$ containing $m_j$, see Figures A.1 and A.2. The interpolated nodal
value is then determined by the unique solution of linear equations:

\[(A.7) \int_{C_j} \left( \sum_{m,j' \in C_j} Q_0 v(m,j') \phi_j(x) + \sum_{c_j \in C_j} Q_0 v(c_j) \phi_j(x) - v(x) \right) p_i(x) dx = 0 \]

for all degree \((k + 2 - d)\) polynomials \(p_i\) on \((d-1)\)-simplex \(C_j\), where \(Q_0 v(c_j)\) is defined in \((A.6)\). In 2D, c.f. Figure A.1, after we determine the nodal values at the two end points \((c_j)\), we determine the middle-edge points’ nodal value by \((A.7)\).

3. After determine all nodal values on the surface of each element, we define the interpolation inside the element: The nodal values at internal nodes \((Q_0 v(i_{j''}))\) are determined by the unique solution of the following linear equations

\[(A.8) \int_{C_j} \left( \sum_{i,j'' \in C_j} Q_0 v(i_{j''}) \phi_{j''}(x) \right) p_i dx \]

\[= \int_{C_j} \left( v - \sum_{m,j' \in C_j} Q_0 v(m,j') \phi_{j'} - \sum_{c_j \in C_j} Q_0 v(c_j) \phi_j \right) p_i dx \]

for all degree \((k + 1 - d)\) polynomials \(p_i\) on \(d\)-simplex \(C_j\).

By \((A.6)-(A.8)\), the (refined) Scott-Zhang interpolation is

\[(A.9) Q_0 v = \sum_{x_j \in \mathcal{N}_h} Q_0 v(x_j) \phi_j(x), \]

where \(\mathcal{N}_h\) is the set of all \(C^{0}-P_{k+2}\) Lagrange nodes of triangulation \(T_h\).

**Remark A.1.** If all corner nodes have selected a same averaging patch \(C_j\) as the unique patch for the middle nodes on the patch, then the solution of \((A.7)\) is the \(L^2\)-projection, i.e.,

\[(A.10) Q_0 v(m,j') = \int_{C_j} \psi^{[(d-1)D]}_{j'} v(x) dx. \]

In fact, \((A.10)\) is the definition of the original Scott-Zhang operator in \cite{21}. In the same fashion, if all patches of the face nodes are face \((d-1)\)-simplexes of \(C_j\), then the internal nodal values are exactly that of the \(L^2\)-projection on \(C_j\), i.e., the solution of \((A.8)\) satisfies

\[(A.11) Q_0 v(i_{j''}) = \int_{C_j} \psi^{[D]}_{j''} v(x) dx. \]

But if there are more than one triangle or tetrahedron in \(T_h\), some \(C_j\) must be from neighboring elements. So \((A.10)\) and \((A.11)\) can not be satisfied in general. Otherwise the Scott-Zhang operator would preserve mass of order \((k + 2)\) both on an element and on its faces.

**Lemma A.1.** The Scott-Zhang interpolation operator \((A.9)\) is well-defined, i.e., the linear systems of equations \((A.7)\) and \((A.8)\) both have unique solutions.

**Proof.** For the linear system of equations \((A.7)\), we change the \(P_{k+2-d}^{d-1}\) basis functions \(\{p_i = 1, x, ..., x^k\}\) when \(d = 2\), or \(\{p_i = 1, x, y, x^2, xy, ..., y^{k-1}\}\) when \(d = 3\)
uniquely as linear combinations of the Lagrange basis functions on the subinterval $C_j$ ($d = 2$) or the subtriangle $C_j$ ($d = 3$), c.f. Figure A.5.

(A.12) $$p_i = \sum_j c_{i,j} \phi_j^s, \quad i = 1, 2, \ldots, \dim(P_{k+2-d}^{d-1}).$$

The nodal basis functions on a simplex $C_j$ and its subsimplex $C_s^j$ differ by a bubble functions:

(A.13) $$\phi_j = \phi_j^s \frac{b(x)}{b(x_j)},$$

where $b(x)$ is the bubble functions assuming 0 on the boundary of $C_j$. For example, c.f. Figure A.5, when $d = 2$ and $C_j = [0, 1]$,

$$b(x) = x(1 - x),$$

$$\phi_1(x) = \frac{(x - 2/4)(x - 3/4)}{(1/4 - 2/4)(1/4 - 3/4)},$$

$$\phi_2(x) = \phi_1(x) \frac{b(x)}{b(1/4)}.$$ 

d = 2:

\begin{align*}
\phi_1^s(x) &= \frac{(x - 2/4)(x - 3/4)}{(1/4 - 2/4)(1/4 - 3/4)} \\
C_s^j &\quad \bullet \quad \circ \\
C_j &\quad \bullet \\
\phi_2 &= \phi_1^s \frac{b(x)}{b(1/4)}
\end{align*}

d = 3:

\begin{align*}
\phi_1^s(x) &= \frac{(x - 2/4)(x - 3/4)}{(1/4 - 2/4)(1/4 - 3/4)} \\
C_s^j &\quad \bullet \quad \circ \\
C_j &\quad \bullet \\
\phi_2 &= \phi_1^s \frac{b(x)}{b(1/4)}
\end{align*}

Fig. A.5. Lagrange nodal basis $\phi_j^s$ on subinterval ($d = 2$) or subtriangle ($d = 3$), c.f., (A.13).

By the change of basis, (A.12) and (A.13), the linear system (A.7) is equivalent to the following weighted-mass linear systems:

(A.14) $$\sum_{m,j \in C_j} Q_{0v}(m,j') \int_{C_j} \phi_j \phi_j^s dx = \int_{C_j} v \phi_j^s dx - \sum_{j \in C_j} Q_{0v}(c_j) \int_{C_j} \phi_j \phi_j^s dx, \quad i = 1, 2, \ldots, \dim(P_{k+2-d}^{d-1}).$$

The coefficient matrix in (A.14) is the mass matrix on the subsimplex $C_s^j$ (c.f. Figure A.5) with a positive weight:

$$a_{i,j'} = \int_{C_j} \phi_i^s \phi_j dx = \int_{C_j} \phi_i^s \phi_j^s w(x) dx,$$

where

$$w(x) = \frac{b(x)}{b(x_j)} > 0 \quad \text{in interior}(C_j).$$

As the Lagrange basis $\{\phi_i^s\}$ (on the subsimplex) are linearly independent, the mass (with weight) matrix in (A.14) is invertible, and the equivalent linear system (A.7)
has a unique solution too. For example, when \( d = 2 \) and \( k = 2 \), the coefficient matrix in (A.14) and its inverse are

\[
\begin{pmatrix}
152/315 & -16/63 & 8/63 \\
-4/21 & 18/35 & -4/21 \\
8/63 & -16/63 & 152/315
\end{pmatrix}^{-1} = \begin{pmatrix}
2655/1024 & 75/64 & -225/1024 \\
225/256 & 45/16 & 225/256 \\
-225/1024 & 75/64 & 2655/1024
\end{pmatrix}.
\]

By the same argument, lifting the space dimension by 1, we can show that (A.8) has a unique solution too. In fact, the system (A.8) when \( d = 2 \) is the same system (A.7) with \( d = 3 \) there.

**Lemma A.2.** If \( d = 2 \), the Scott-Zhang interpolation operator (A.9) preserves the volume mass of order \( k - 1 \) and the face mass of order \( k \), i.e.,

\[(A.15) \quad \int_T (v - Q_0v) p_i \, dx = 0 \quad \forall T \in T_h, \ p_i \in P^2_{k-1}(T),\]

\[(A.16) \quad \int_E (v - Q_0v) p_i \, dx = 0 \quad \forall E \in E_h, \ p_i \in P_k(E),\]

where \( E_h \) is the set of edges in triangulation \( T_h \).

**Proof.** By the construction (A.7), we have

\[
\int_E (v - Q_0v) p_i \, dx = \int_E \left( v - \sum_{j' \in E} Q_0 v(m_{j'}) \phi_{j'} - \sum_{c_j \in E} Q_0 v(c_j) \phi_j \right) p_i \, dx = 0.
\]

That is (A.16). Here, because we have two missing dof (degrees of freedom) at the two end points of each edge, the polynomial degree in mass preservation is reduced by two, from \((k + 2)\) to \(k\). Similarly, (A.15) follows (A.8). Here, the polynomial degree reduction is three as each triangle has three edges where the interpolation values are not free (not determined by (A.8)).

**Lemma A.3.** If \( d = 3 \), the Scott-Zhang interpolation operator (A.9) preserves the volume mass of order \( k - 2 \) and the face mass of order \( k - 1 \), i.e.,

\[(A.17) \quad \int_T (v - Q_0v) p_i \, dx = 0 \quad \forall T \in T_h, \ p_i \in P^3_{k-2}(T),\]

\[(A.18) \quad \int_E (v - Q_0v) p_i \, dx = 0 \quad \forall E \in E_h, \ p_i \in P^2_{k-1}(E),\]

where \( E_h \) is the set of face triangles in the tetrahedral grid \( T_h \).

**Proof.** As each triangle \( E \) has three edges, where the interpolation is not determined possibly by function value on neighboring triangles, we lose dof’s on the three edges in the interpolation. That is, we lose three orders in face-mass conservation in (A.7). (A.18) is simply another expression of (A.7), as in the proof of Lemma A.2. By (A.8), (A.17) follows. Here the polynomial-degree deduction in mass conservation is 4, due to 4 face-triangles each tetrahedron.

**Remark A.2.** By Lemmas A.2 and A.3, the mass preservation on element and on faces is one order higher than the requirement (4.1), when \( d = 2 \). When \( d = 3 \), (A.17) and (A.18) imply (4.1).
**Theorem A.4.** The Scott-Zhang interpolation operator (A.9) is of optimal order in approximation, i.e., when \( k \geq -1 \),

\[
\| v - Q_0v \| + h \| v - Q_0v \|_1 \leq C h^{k+3} \| v \|_{k+3} \quad \forall v \in H^{k+3}(\Omega).
\]

Further, when \( k \geq 0 \),

\[
\left( \sum_{T \in \mathcal{T}_h} h^2 \| v - Q_0v \|^2_{H^2(T)} \right)^{1/2} \leq C h^{k+3} \| v \|_{k+3} \quad \forall v \in \bigcap H^{k+3}(\Omega).
\]

**Proof.** The Scott-Zhang operator preserves a degree \((k+2)\) polynomial on the star union of an element \( T \),

\[
\mathcal{S}_T = \bigcup_{T' \in \mathcal{T}_h : T' \neq T} T', \quad T, T' \in \mathcal{T}_h.
\]

That is,

\[
Q_0 v(c_j) = v(c_j),
\]

where \( v \in P_{k+2}(\mathcal{S}_T) \), when \( k \geq -1 \). (A.21) is shown in three steps. First, by (A.6), when \( v \in P_{k+2} \), the dual basis defines

\[
Q_0 v(c_j) = v(c_j),
\]

at all corner nodes. In the second step, by (A.22) and (A.14), (A.7) holds for \( v \in \mathcal{P}_{k+2} \) too where \( m_i \) are all middle nodes on \( C_j \). That is,

\[
\int_{C_j} \left( v - \sum_{j \in C_j} Q_0(c_j) \phi_j - \sum_{m_{j'}} Q_0 v(m_{j'}) \phi_{j'} \right) \phi_i^s dx = 0.
\]

By (A.22), as \( v \in P_{k+2} \),

\[
v - \sum_{j \in C_j} Q_0(c_j) \phi_j = v_c b(x) \quad \text{for some } v_c \in P_{k+2-d}.
\]

where \( b(x) \) is a bubble function, cf. (A.13). For the middle node basis functions, we have also, c.f. (A.13),

\[
\phi_{j'} = \phi_{j'}^s b(x) / b(x_{j'}) \quad \forall m_{j'} \in C_j.
\]

Thus (A.23) implies, where \( \sum_{m_{j'}} Q_0 v(m_{j'}) \phi_{j'} = v_m b(x) \) for some \( v_m \in P_{k+2-d} \),

\[
\int_{C_j} (v_b - v_m) \phi_i^s b(x) dx = 0,
\]

\[
\Rightarrow v_b - v_m = 0 \quad \text{and } v = Q_0v \quad \text{on } C_j.
\]

Therefore, at all middle nodes,

\[
Q_0 v(m_{j'}) = v(m_{j'}).
\]

In the third step, by (A.8) and the same argument in the second step,

\[
Q_0(i_{j''}) = v(i_{j''}),
\]
at all internal nodes. Thus $Q_0v = v \in P_{k+2}$.

We then use the standard scaling argument (on the dual basis functions) and the Sobolev inequality, as in Theorem 3.1 of [21], it follows that

$$|Q_0v|_{H^1(T)} \leq C\|v\|_{H^1(S_T)} \, \forall v \in H^1(\Omega).$$

The above stability result leads directly to the optimal-order approximation (A.19), following the standard argument (i.e., by (A.21) and the existence of local Taylor polynomials, c.f. for example, [4]), as shown in Theorem 4.1 of [21]. We note again that the Scott-Zhang operator here is a refined version of the Scott-Zhang operator in [21]. After showing the local preservation of $P_{k+2}$ polynomials above, the proof of the theorem is the same as that in [21]. (A.20) is (4.4) in [21], with $p = q = 2$, $m = 2$, and $l = k + 3$ there.

REFERENCES

[1] D. N. Arnold and F. Brezzi, Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates, RAIRO Modl. Math. Anal. Numr., 19(1), (1985), pp. 7-32.

[2] J. H. Argyris, I. Fried, D. W. Scharpf, The TUBA family of plate elements for the matrix displacement method, The Aeronautical Journal of the Royal Aeronautical Society 72 (1968), pp. 514-517.

[3] K. Bell, A refined triangular plate bending element, Internal. J. Numer. methods Engrg., 1 (1969), pp. 101-122.

[4] S. C. Brenner and L. R. Scott, The mathematical theory of finite element methods. Third edition. Texts in Applied Mathematics, 15. Springer, New York, 2008.

[5] S. Brenner and L. Sung, $C^0$ interior penalty methods for fourth order elliptic boundary value problems on polygonal domains, J. Sci. Comput., 22/23 (2005), pp. 83-118.

[6] J. Douglas Jr., T. Dupont, P. Percell and R. Scott, A family of $C^1$ finite elements with optimal approximation properties for various Galerkin methods for 2nd and 4th order problems, RAIRO Anal. Numer. 13 (1979), no. 3, pp. 227-255.

[7] G. Engel, K. Garikipati, T. Hughes, M.G. Larson, L. Mazzei, and R. Taylor, Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity, Comput. Meth. Appl. Mech. Eng., 191 (2002), pp. 3669-3750.

[8] R. Falk, Approximation of the biharmonic equation by a mixed finite element method, SIAM J. Numer. Anal. 15 (1978), pp. 556-577. Numer. Anal. 15 (1978), pp. 556-577.

[9] B. Fraelis de Veubeke, A conforming finite element for plate bending, in: O.C. Zienkiewicz and G.S. Holister (Eds.), Stress Analysis, Wiley, New York, 1965, pp. 145-197.

[10] R.W. Clough and J.L. Tocher, Finite element stiffness matrices for analysis of plates in bending, in: Proceedings of the Conference on Matrix Methods in Structural Mechanics, Wright Patterson A.F.B. Ohio, 1965.

[11] T. Gudi, N. Nataraj, A. K. Pani, Mixed Discontinuous Galerkin Finite Element Method for the Biharmonic Equation, J Sci Comput, 37 (2008), no. 2, pp. 139-161.

[12] J. Hu, Y. Huang and S. Zhang, The lowest order differentiable finite element on rectangular grids, SIAM Num. Anal. 49 (2011), no. 4, pp. 1350-1368.

[13] L.S.D. Morley, The triangular equilibrium element in the solution of plate bending problems, Aero. Quart., 19 (1968), pp. 149-169.

[14] P. Monk, A mixed finite element methods for the biharmonic equation, SIAM J. Numer. Anal. 24 (1987), pp. 737-749.

[15] I. Mozolevski and E. Süli, hp-Version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation, J. Sci. Comput. 30 (2007), no. 3, pp. 465-491.

[16] L. Mu, J. Wang, and X. Ye, Weak Galerkin finite element methods on polytopal meshes, arXiv:1204.3655v2.

[17] L. Mu, J. Wang, Y. Wang and X. Ye, Weak Galerkin mixed finite element method for the biharmonic equation, arXiv:1210.3818.
[18] L. Mu, J. Wang and X. Ye, Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes, preprint.
[19] P. Percell, On cubic and quartic Clough-Tocher finite elements, SIAM J. Numer. Anal. 13 (1976), pp. 100-103.
[20] M.J.D. Powell, M.A. Sabin, Piecewise quadratic approximations on triangles, ACM Transactions on Mathematical Software, 3-4 (1977), pp. 316-325.
[21] L. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions, Math. Comp., 54 (1990), pp. 483-493.
[22] J. Wang and X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, arXiv:1202.3655v1.
[23] J. Wang and X. Ye, A weak Galerkin finite element method for second-order elliptic problems, arXiv:1104.2897v1, 2011, Journal of Computational and Applied Mathematics, 241 (2013), pp. 103-115.
[24] A. Ženíšek, Alexander Polynomial approximation on tetrahedrons in the finite element method, J. Approximation Theory 7 (1973), pp. 334-351.
[25] S. Zhang, A C1-P2 finite element without nodal basis, M2AN 42 (2008), pp. 175-192.
[26] S. Zhang, A family of 3D continuously differentiable finite elements on tetrahedral grids, Applied Numerical Mathematics, 59 (2009), no. 1, pp. 219-233.
[27] S. Zhang, On the full C1-Qk finite element spaces on rectangles and cuboids, Adv. Appl. Math. Mech., 2 (2010), pp. 701-721.
[28] M. Zlámal, On the finite element method, Numer. math. 12 (1968), pp. 394-409.