GRADIENT SHRINKING SASAKI-RICCI SOLITONS ON SASAKIAN MANIFOLDS OF DIMENSION UP TO SEVEN

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ABSTRACT. In this paper, we show that the uniform $L^4$-bound of the transverse Ricci curvature along the Sasaki-Ricci flow on a compact quasi-regular transverse Fano Sasakian $(2n+1)$-manifold $M$. When $M$ is dimension up to seven and the space of leaves of the characteristic foliation is well-formed, we first show that any solution of the Sasaki-Ricci flow converges in the Cheeger-Gromov sense to the unique singular orbifold Sasaki-Ricci soliton on $M\infty$ which is a $S^1$-orbibundle over the unique singular Kähler-Ricci soliton on a normal projective variety with codimension two orbifold singularities. Secondly, for $n = 1$, we show that there are only two nontrivial Sasaki-Ricci solitons on a compact quasi-regular Fano Sasakian three-sphere with its leaves space $Z_{m_1}$-teardrop and $Z_{m_1, m_2}$-football, respectively. For $n = 2, 3$, we show that the Sasaki-Ricci soliton is trivial one if $M$ is transverse $K$-stable.

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1. Introduction

Sasakian geometry is very rich as the odd-dimensional analogous of Kähler geometry. A Sasaki-Einstein 5-manifold is to say that its Kähler cone is a Calabi-Yau threefold. Such manifolds provide interesting examples of the AdS/CFT correspondence. By the second structure theorem for Sasakian manifolds, there is a $S^1$-action with the Reeb vector field which generates the finite isotropy groups. Only if the isotropy subgroup of every point is trivial is the regular free action. They are regular Sasakian manifolds and its leave spaces are smooth Kähler surfaces. In general, the space of leaves has at least the codimension two fixed point set of every non-trivial isotropy subgroup or the codimension one fixed point set of some non-trivial isotropy subgroup. They are quasi-regular Sasakian manifolds and its $S^1$-fibrations will be orbifold surfaces with orbifold singularities (cf. section 2). Furthermore, there is a well-known classification of compact Fano Kähler-Einstein smooth surfaces due to Tian-Yau and then leads to a first classification of all compact regular Sasaki-Einstein 5-manifolds. There are quasi-regular Sasaki-Einstein metrics on connected sums of $S^2 \times S^3$, rational homology 5-spheres and connected sums of these. The very first examples of irregular Sasaki-Einstein metric on $S^2 \times S^3$ was constructed in [GMSW]. We refer to [BG], [Sp] and references therein.

On the other hand, the class of simply connected, closed, oriented, smooth, 5-manifolds is classifiable under diffeomorphism due to Smale-Barden ([S], [B]). Then it is our goal to focus on the existence of Sasaki-Einstein metrics or Sasaki-Ricci solitons in a compact quasi-regular Sasakian manifolds of dimension five and seven. More precisely, by the first structure theorem (cf. section 2), any Hodge orbifold gives rise to a Sasakian manifold, one expects that existence problems of Sasaki-Einstein metrics as well as Sasaki-Ricci solitons should be intimately related to existence problems on such corresponding Hodge orbifolds. However, people has seen some success in finding Kähler-Einstein by the Kähler-Ricci flow on Kähler manifolds.

Along this spirits, in this paper we will focus on the following Sasaki-Ricci flow

$$\frac{\partial}{\partial t} \omega(t) = \omega(t) - \text{Ric}^\tau_{\omega(t)}, \quad \omega(0) = \omega_0$$

which is introduced by Smoczyk–Wang–Zhang ([SWZ]) to study the existence of Sasaki $\eta$-Einstein metrics on Sasakian manifolds. They showed that the flow has the longtime solution and asymptotic converges to a Sasaki $\eta$-Einstein metric when the basic first Chern class is negative ($c^B_1(M) < 0$) or null ($c^B_1(M) = 0$). It is wild open when a compact Sasakian $(2n + 1)$-manifold is transverse Fano ($c^B_1(M) > 0$). In the paper of [CJ], Collins and Jacob proved that the Sasaki-Ricci
flow converges exponentially fast to a Sasaki-Einstein metric if one exists, provided the automorphism group of the transverse holomorphic structure is trivial. In general, by comparing the Kähler-Ricci flow on log Fano varieties as in [BBEGZ], it is hard to deal with because the space of leaves of the characteristic foliation is a polarized, normal projective variety which endowed with the orbifold structure due to [L3].

In section 3, we first start to consider the most simple case for $n = 1$, in particular, the space $Z$ of leaves will be $(S^2, g)$ with branch divisors $k$ marked points as in Theorem 2. More precisely, let $(M, \xi, \eta, g, \Phi)$ be a compact transverse Fano Sasakian 3-manifold. It follows from [Gei] that any Sasakian 3-manifold $M$ is either canonical, anticanonical or null. $M$ is up to finite quotient a regular Sasakian 3-manifold, i.e., a circle bundle over a Riemann surface of positive genus.

In the positive case, $M$ is finite covered by $S^3$ and its Sasakian structure is a deformation of a standard Sasakian structure on $S^3$.

Now first we will focus on the Sasaki-Ricci flow (3.1) on the transverse Fano three-sphere $S^3$. In fact, it is known that

**Proposition 1.** ([WZ], [He]) For any initial Sasakian structure on $(S^3, \xi, g_0)$ with the positive transverse scalar curvature, the Sasaki-Ricci flow (3.1) converges exponentially to a gradient shrinking Sasaki-Ricci soliton

$$\nabla_i \nabla_j f - \frac{1}{2}(\Delta_B f) g_{ij} = 0.$$  

Here $f$ is the basic function defined by $\Delta_B f = RT - r$. Moreover, the Sasaki-Ricci soliton metric is a simple Sasaki metric which can be deformed to the round metric on $S^3$ through a simple deformation.

Our first goal is to remove the positive assumption of the initial transverse scalar curvature, we prove that the transverse scalar curvature $RT$ becomes positive in finite time under the Sasaki-Ricci flow. Then, by applying Proposition 1 we have

**Theorem 1.** For any initial Sasakian structure on the transverse Fano three-sphere $(S^3, \xi, g_0)$, the Sasaki-Ricci flow converges exponentially to a gradient shrinking Sasaki-Ricci soliton.

As in the paper of [H1], there are no soliton solutions other than those of constant curvature on a compact surface. However, bad orbifold surfaces do not admit metrics of constant curvature ([Wu]). Then it is very interested to know whether the Sasaki-Ricci soliton is the trivial one.

It follows from Proposition 5 and Proposition 6 that we have the following classification of Sasaki-Ricci solitons in a compact quasi-regular Fano Sasakian three-sphere.

**Theorem 2.** Let $(S^3, \xi, g_0)$ be a compact quasi-regular Fano Sasakian three-sphere and an $S^1$-orbibundle $\pi : (S^3, g_0) \to (S^2, g)$ with $k$ marked points $\{p_i\}_{i=1}^k$ of the ramification index $m_i$ and a metric $g$ on $S^2$ with a conical singularity of cone angle $\frac{2\pi}{m_i}$ at $p_i$. Then

$$k \leq 3$$

so that
(1) For \( k = 1 \): We have
\[
\chi(S^2, \beta) = 1 + \frac{1}{m_1}
\]
such that the Sasaki-Ricci soliton is nontrivial and \((S^2, \beta)\) is a \(Z_{m_1}\)-teardrop.

(2) For \( k = 2 \): We have
\[
\chi(S^2, \beta) = \frac{1}{m_1} + \frac{1}{m_2}
\]
such that
(a) if \( m_1 = m_2 \), then the Sasaki-Ricci soliton is trivial with constant transverse scalar curvature.
(b) if \( m_1 \neq m_2 \), then the Sasaki-Ricci soliton is nontrivial and \((S^2, \beta)\) is the \(Z_{m_1, m_2}\)-football.

(3) For \( k = 3 \): We have
\[
\chi(S^2, \beta) = \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} - 1
\]
such that the only possible positive numbers for \((m_1, m_2, m_3)\) are
\[(m, 2, 2); (3, 3, 2); (4, 3, 2); (5, 3, 2)\]
and then (3.9) holds. Therefore the Sasaki-Ricci soliton is trivial with constant transverse scalar curvature.

As a consequence of Theorem 2, we have

**Corollary 1.** There are only two nontrivial Sasaki-Ricci solitons on a compact quasi-regular Fano Sasakian three-sphere with its leave space \(Z_{m_1}\)-teardrop and \(Z_{m_1, m_2}\)-football, respectively.

**Remark 1.**
(1) We recapture the classification of all Sasakian structures on compact quasi-regular Fano Sasakian \(S^3\) without using the uniformization of compact 2-orbifolds. We refer to Belgun’s work ([Bel]) for another proof.

(2) The second structure theorem ([Ru]) on compact Sasakian manifolds of dimension \(2n + 1\) states that any Sasakian structure \((\xi, \eta, \Phi, g)\) on \(M\) is either quasi-regular or there is a sequence of quasi-regular Sasakian structures \((M, \xi_i, \eta_i, \Phi_i, g_i)\) converging in the compact-open \(C^\infty\)-topology to \((\xi, \eta, \Phi, g)\).

For \( n \geq 2 \) in section 4, we will assume that \(M\) is a compact quasi-regular transverse Fano Sasakian manifold and the space \(Z\) of leaves is well-formed which means its orbifold singular locus and algebro-geometric singular locus coincide, equivalently \(Z\) has no branch divisors.

Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian \((2n + 1)\)-manifold and \(Z = M/F_\xi\) denote the space of leaves of the characteristic foliation which is well-formed, a normal projective variety with codimension two orbifold singularities \(\Sigma\). Then by the first structure theorem again, \(M\) is a principal \(S^1\)-orbibundle (\(V\)-bundle) over \(Z\) which is also a \(Q\)-factorial, polarized, normal projective variety such that there is an orbifold Riemannian submersion
\[
\pi : (M, g) \rightarrow (Z, \omega)
\]
and

\[(1.3) \quad K^T_M = \pi^*(K^{orb}_Z).\]

If the orbifold structure of the leave space \(Z\) is well-formed, then the orbifold canonical divisor \(K^{orb}_Z\) and canonical divisor \(K_Z\) are the same and thus

\[K^T_M = \pi^*(\varphi^*K_Z).\]

We shall work on the Sasaki-Ricci flow (1.1) in a compact quasi-regular transverse Fano Sasakian manifold \((M, \xi, \eta_0, \Phi_0, g_0, \omega_0)\) of dimension five and seven. In the paper of [He], the author proved that the Sasaki–Ricci flow converges to a gradient Sasaki–Ricci soliton in a compact transverse Fano Sasakian \((2n+1)\)-manifold with the initial Sasaki metric of nonnegative transverse bisectional curvature. By removing the curvature assumption, we obtain the existence theorem of the gradient Shrinking Sasaki-Ricci soliton metric in a compact quasi-regular transverse Fano Sasakian manifold dimension up to seven. More precisely, first it follows from Theorem 9, Theorem 10, and Corollary 2 that we have

**Theorem 3.** Let \((M, \xi, \eta_0, g_0)\) be a compact quasi-regular transverse Fano Sasakian manifold of dimension up to seven and \((Z_0 = M/\mathcal{F}_\xi, h_0, \omega_{h_0})\) denote the space of leaves of the characteristic foliation which is a normal Fano projective Kähler orbifold surface and well-formed with codimension two orbifold singularities \(\Sigma_0\). Then, under the Sasaki-Ricci flow (1.1), \((M(t), \xi, \eta(t), g(t))\) converges to a compact quasi-regular transverse Fano Sasakian orbifold manifold \((M_\infty, \xi, \eta_\infty, g_\infty)\) with the leave space of orbifold Kähler manifold \((Z_\infty = M_\infty/\mathcal{F}_\xi, h_\infty)\) which can have at worst codimension two orbifold singularities \(\Sigma_\infty\). Furthermore, \(g^T(t_i)\) converges to a gradient Sasaki-Ricci soliton orbifold metric \(g^T_\infty\) on \(M_\infty\) with \(g^T_\infty = \pi^*(h_\infty)\) such that \(h_\infty\) is the smooth Kähler-Ricci soliton metric in the Cheeger-Gromov topology on \(Z_\infty\setminus\Sigma_\infty\).

Furthermore, in the following we will show that the gradient Sasaki-Ricci soliton orbifold metric is a Sasaki-Einstein metric if \(M\) is transverse \(K\)-stable (Definition [11]). This is an old dimensional counterpart of Yau-Tian-Donaldson conjecture on a compact \(K\)-stable Kähler manifold ([CDS1], [CDS2], [CDS3], [T5]). It can be viewed as a Sasaki analogue of Tian-Zhang’s ([TZ]) and Chen-Sun-Wang’s result ([CSW]) for the Kähler-Ricci flow.

**Theorem 4.** Let \((M, \xi, \eta_0, g_0)\) be a compact quasi-regular transverse Fano Sasakian manifold of dimension up to seven and \((Z_0 = M/\mathcal{F}_\xi, h_0, \omega_{h_0})\) be the space of leaves of the characteristic foliation which is well-formed with codimension two orbifold singularities \(\Sigma_0\). If \(M\) is transverse stable, then under the Sasaki-Ricci flow, \(M(t)\) converges to a compact transverse Fano Sasakian manifold \(M_\infty\) which is isomorphic to \(M\) endowed with a smooth Sasaki–Einstein metric.

**Remark 2.** (1) Note that by continuity method, Collins and Székelyhidi ([CZ2]) showed that a polarized affine variety admits a Ricci-flat Kähler cone metric if and only if it is \(K\)-stable. In particular, the Sasakian manifold admits a Sasaki-Einstein metric if and only if its Kähler cone is \(K\)-stable.
(2) On the other hand, instead of $K$-stability on its Kähler cone, one can have the so-called transverse $K$-stability on a compact quasi-regular transverse Fano Sasakian manifold with the space of leaves of the characteristic foliation which is well-formed and also a normal Fano projective Kähler orbifold.

(3) In the upcoming paper ([CLW2]), under the conic Sasaki-Ricci flow, we will prove the conic version of Yau-Tian-Donaldson conjecture on a log transverse Fano Sasakian manifold in which its leaf space $Z_0$ is not well-formed. It means that the orbifold structure $(Z_0, \Delta)$ has the codimension one fixed point set of some non-trivial isotropy subgroup. This is served as a Sasaki analogue of the conic Kähler-Ricci flow as in [LZ], etc.

The proofs are given in this paper, primarily along the lines of the arguments in [TZ]. In section 3, following Harnack inequality as in [Ben], the transverse scalar curvature $R^T$ becomes positive in finite time under the Sasaki-Ricci flow. Then, by applying Proposition 1 we have Theorem 1. Furthermore, by results as [Wu] and [PSSW], we have the classification of gradient Shrinking Sasaki-Ricci solitons as in Theorem 2.

In section 4, the central issue is to show the $L^4$-bound of the transverse Ricci curvature under the Sasaki-Ricci flow. Then, based on Perelman’s uniform non-collapsing condition and pseudolocality theorem of Ricci flow and a regularity theory for Sasakian manifolds with integral bounded transverse Ricci curvature, the limit solution $g^T_\infty$ is a smooth a smooth Sasaki-Ricci soliton on $(M_\infty, \omega^\infty)$ which is a $S^1$-bundle over the regular set $R$ of $Z_\infty$ and the singular set $S$ of $Z_\infty$ is the codimension two orbifold singularities (Theorem 3). It is served as a Sasaki analogue of the regularity theory of Cheeger-Colding ([CC2], [CC3]) and Cheeger-Colding-Tian ([CCT]) for manifolds with bounded Ricci curvature. Finally, as a consequence of the first structure theorem for Sasakian manifolds and the partial $C^0$-estimate (Theorem 42), the Gromov–Hausdorff limit $Z_\infty$ is a variety embedded in some $\mathbb{C}P^N$ and the singular set $S$ is a normal subvariety ([LZ, Theorem 1.6]). This will implies Theorem 5

In section 5, let $(M, \xi, \eta_0, g_0)$ be a compact quasi-regular transverse Fano Sasakian manifold of dimension up to seven with the space $(Z = M/F_0, h_0, \omega_0)$ of leaves of the characteristic foliation which is well-formed. In this special case, we can following the notions as in [T3] and [T5] to define the Sasaki analogue of a $K$-stable Fano Kähler manifold to be the so-called transverse $K$-stable on a Sasakian manifold. Finally, by applying the partial $C^0$-estimate to get a lower bound of the transverse Mabuchi $K$-energy. Furthermore, $Z_\infty$ is a normal Fano projective Kähler orbifold, the Sasaki-Futaki invariant can be extended to the generalized Sasaki-Futaki invariant (5.5). Then Theorem 4 follows easily by the transverse $K$-stable condition.

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2. Preliminaries

In this section, we will recall some preliminaries for Sasakian manifolds with foliation singularities, a Type II deformation of the Sasakian structure and the Sasaki-Ricci flow. We refer to [BG], [FOW], [Sp], [CLW], and references therein for some details.

2.1. Sasakian Structures and Foliation Singularities. Let \((M, g, \nabla)\) be a Riemannian \((2n+1)\)-manifold. \((M, g)\) is called Sasaki if the cone \((C(M), J, \omega, \eta, g) := (\mathbb{R}^+ \times M, d\tau^2 + r^2 g)\) is Kähler with \(\omega = \frac{1}{2} i \partial \bar{\partial} r^2\) and \(\eta = \frac{1}{2} g(\xi, \cdot)\) and \(\xi = J(\partial/\partial r)\).

The function \(\frac{1}{2} r^2\) is hence a global Kähler potential for the cone metric. As \(r = 1\) = \(\{1\} \times M \subset C(M)\), we may define the Reeb vector field \(\xi\) on \(M\) by
\[
\xi = J(\partial/\partial r),
\]
and the contact 1-form \(\eta\) on \(TM\)
\[
\eta = g(\xi, \cdot)
\]
Then \(\xi\) is the killing vector field with unit length such that \(\eta(\xi) = 1\) and \(d\eta(\xi, X) = 0\). The tensor field of type\((1, 1)\), defined by
\[
\Phi(Y) = \nabla_Y \xi
\]
satisfies the condition
\[
(\nabla_X \Phi)(Y) = g(\xi, Y) X - g(X, Y) \xi
\]
for any pair of vector fields \(X\) and \(Y\) on \(M\). Then such a triple \((\eta, \xi, \Phi)\) is called a Sasakian structure on a Sasakian manifold \((M, g)\). Note that the Riemannian curvature satisfying the following
\[
R(X, \xi)Y = g(\xi, Y) X - g(X, Y) \xi
\]
for any pair of vector fields \(X\) and \(Y\) on \(M\). In particular, the sectional curvature of every section containing \(\xi\) equals one.

**Definition 1.** ([BG]) Let \((M, \eta, \xi, \Phi, g)\) be a compact Sasakian \((2n+1)\)-manifold. If the orbits of the Reeb vector field \(\xi\) are all circles, then integrates to give an isometric \(S^1\)-action on \((M, g)\). It is nowhere zero and the action is locally free. Furthermore, the isotropy group of every point in \(M\) is finite. If the action is free, then the Sasakian structure is said to be regular. Otherwise, it is quasi-regular. If the orbits of are not all closed, it is called irregular. In this case, the closure of the one parameter subgroup of the isometry group of \((M, g)\) is isomorphic to a torus \(T^k\). Then the irregular Sasakian manifold has at least an \(T^2\)-isometry.

The first structure theorem on Sasakian manifolds states that

**Proposition 2.** ([Ru], [Sp], [BG]) Let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian manifold of dimension \(2n+1\) and \(Z\) denote the space of leaves of the characteristic foliation \(\mathcal{F}_\xi\) (just as topological space). Then
is defined by the local uniformizing systems
\[(a \text{ sequence of quasi-regular Sasakian structures)}
\]
where
\[G\] Sasakian structure, it admits many locally free circle actions.

That is,
\[\text{Proposition 3.}\]

(1) \(Z\) carries the structure of a Hodge orbifold \(Z = (Z, \Delta)\) with an orbifold Kähler metric \(h\) and Kähler form \(\omega\) which defines an integral class in \(H^2_{\text{orb}}(Z, \mathbb{Z})\) in such a way that \(\pi: (M, g, \omega) \to (Z, h, \omega_h)\) is an orbifold Riemannian submersion, and a principal \(S^1\)-orbibundle (\(V\)-bundle) over \(Z\). Furthermore, it satisfies \(\frac{1}{2} d\eta = \pi^*(\omega_h)\). The fibers of \(\pi\) are geodesics.

(2) \(Z\) is also a \(Q\)-factorial, polarized, normal projective algebraic variety.

(3) The orbifold \(Z\) is Fano if and only if \(Ric_g > -2\): In this case \(Z\) as a topological space is simply connected; and as an algebraic variety is uniruled with Kodaira dimension \(-\infty\).

(4) \((M, \xi, g)\) is Sasaki-Einstein if and only if \((Z, h)\) is Kähler-Einstein with scalar curvature \(4n(n + 1)\).

(5) If \((M, \eta, \xi, \Phi, g)\) is regular then the orbifold structure is trivial and \(\pi\) is a principal circle bundle over a smooth projective algebraic variety.

(6) As real cohomology classes, there is a relation between the first basic Chern class and the first orbifold Chern class
\[c_1^B(M) = c_1(F_\xi) = \pi^*c_1^{orb}(Z).\]

Conversely, let \(\pi: M \to Z\) be a \(S^1\)-orbibundle over a compact Hodge orbifold \((Z, h)\) whose first Chern class is an integral class defined by \([\omega_Z]\), and \(\eta\) be a 1-form with \(\frac{1}{2} d\eta = \pi^*\omega_Z\). Then \((M, \pi^*h + \eta \otimes \eta)\) is a Sasaki manifold if all the local uniformizing groups inject into the structure group \(U(1)\).

On the other hand, the second structure theorem on Sasakian manifolds states that

**Proposition 3.** (\([Ru]\)) Let \((M, g)\) be a compact Sasakian manifold of dimension \(2n + 1\). Any Sasakian structure \((\xi, \eta, \Phi, g)\) on \(M\) is either quasi-regular or there is a sequence of quasi-regular Sasakian structures \((M, \xi_i, \eta_i, \Phi_i, g_i)\) converging in the compact-open \(C^\infty\)-topology to \((\xi, \eta, \Phi, g)\). In particular, if \(M\) admits an irregular Sasakian structure, it admits many locally free circle actions.

We recall that

**Definition 2.** (\([BG]\)) An orbifold complex manifold is a normal, compact, complex space \(Z\) locally given by charts written as quotients of smooth coordinate charts. That is, \(Z\) can be covered by open charts \(Z = \bigcup U_i\). The orbifold charts on \((Z, U_i, \varphi_i)\) is defined by the local uniformizing systems \((\tilde{U}_i, G_i, \varphi_i)\) centered at the point \(p_i\), where \(G_i\) is the local uniformizing finite group acting on a smooth complex space \(\tilde{U}_i\) such that \(\varphi_i: \tilde{U}_i \to U_i = \tilde{U}_i/G_i\) is the biholomorphic map. A point \(x\) of complex orbifold \(X\) whose isotropy subgroup \(\Gamma_x \neq Id\) is called a singular point. Those points with \(\Gamma_x = Id\) are called regular points. The set of singular points is called the orbifold singular locus or orbifold singular set, and is denoted by \(\Sigma^{orb}(Z)\).

Now let \((M, \eta, \xi, \Phi, g)\) be a compact quasi-regular Sasakian manifold of dimension \(2n + 1\). By the first structure theorem, the underlying complex space \(Z = (Z, U_i)\) is a normal, orbifold variety with the algebro-geometric singular set \(\Sigma(Z)\). Then \(\Sigma(Z) \subset \Sigma^{orb}(Z)\) and it follows that \(\Sigma(Z) = \Sigma^{orb}(Z)\) if and only if none of the local uniformizing groups of the orbifold \((Z, U_i)\) contain a reflection. If some \(G_i\) contains a reflection, then, on \((Z, U_i)\), the reflection fixes a hyperplane.
giving rise to a ramification divisor on $\tilde{\mathcal{U}}_i$ and a branch divisor $\Delta$. More precisely, the branch divisor $\Delta$ of an orbifold $Z = (Z, \Delta)$ is a $\mathbb{Q}$-divisor on $Z$ of the form

$$\Delta = \sum_\alpha (1 - \frac{1}{m_\alpha}) D_\alpha,$$

where the sum is taken over all Weil divisors $D_\alpha$ that lie in $\Sigma^{\text{orb}}(Z)$, and $m_\alpha$ is the $\gcd$ of the orders of the local uniformizing groups taken over all points of $D_\alpha$ and is called the ramification index of $D_\alpha$.

The orbifold structure $Z = (Z, \Delta)$ is called well-formed if the fixed point set of every non-trivial isotropy subgroup has codimension at least two. Then $Z$ is well-formed if and only if its orbifold singular locus and algebro-geometric singular locus coincide, equivalently $Z$ has no branch divisors.

Example 1. For an instance, the weighted projective $\mathbb{C}P(1; 4; 6)$ has a branch divisor $\frac{1}{2}D_0 = \{z_0 = 0\}$. But $\mathbb{C}P(1; 2; 3)$ is a unramified well-formed orbifold with two singular points, $(0; 1; 0)$ with local uniformizing group the cyclic group $\mathbb{Z}_2$, and $(0; 0; 1)$ with local uniformizing group $\mathbb{Z}_3$. But both are the same varieties.

Note that the orbifold canonical divisor $K^\text{orb}_Z$ and canonical divisor $K_Z$ are related by

$$K^\text{orb}_Z = \varphi^*(K_Z + [\Delta]).$$

In particular, $K^\text{orb}_Z = \varphi^*K_Z$ if and only if there are no branch divisors.

For all previous discussions with the special case for $n = 2$, we have the following result concerning its foliation cyclic quotient singularities:

**Theorem 5.** ([CLW]) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and its leave space $Z$ of the characteristic foliation be well-formed. Then $Z$ is a $\mathbb{Q}$-factorial normal projective algebraic orbifold surface with isolated singularities of a finite cyclic quotient of $\mathbb{C}^2$. Accordingly, $p \in Z$ is analytically isomorphic to $p \in Z \simeq (0 \in \mathbb{C}^2)/\mu_r$, where $Z_r$ is a cyclic group of order $r$ and its action on such open affine neighborhood is defined by

$$\mu_r : (z_1, z_2) \rightarrow (\zeta^a z_1, \zeta^b z_2),$$

where $\zeta$ is a primitive $r$-th root of unity. We denote the cyclic quotient singularity by $\frac{1}{r}(a, b)$ with $(a, r) = 1 = (b, r)$. In particular, the action can be rescaled so that every cyclic quotient singularity corresponds to a $\frac{1}{r}(1, a)$-point with $(r, a) = 1$. It is klt (Kawamata log terminal) singularities.

**Definition 3.** (1) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian 5-manifold and its leave space $(Z, \emptyset)$ of the characteristic foliation be well-formed. Then the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called foliation cyclic quotient singularities of type $\frac{1}{r}(1, a)$ at a singular fibre $S^1$ in $M$. The foliation singular set is discrete, and hence finite.
(2) Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian $5$-manifold and its leave space $(Z, \Delta)$ has the codimension one fixed point set of some non-trivial isotropy subgroup. In this case, the action
\[
\mu_{Z_\alpha} : (z_1, z_2) \rightarrow (e^{\frac{2\pi i}{r_1}} z_1, e^{\frac{2\pi i}{r_2}} z_2),
\]
for some positive integers $r_1, r_2$ whose least common multiple is $r$, and $a_i, i = 1, 2$ are integers coprime to $r_i, i = 1, 2$. Then the foliation singular set contains some $3$-dimensional Sasakian submanifolds of $M$. More precisely, the corresponding singularities in $(M, \eta, \xi, \Phi, g)$ is called the Hopf $S^1$-orbibundle over a Riemann surface $\Sigma_h$.

2.2. The Foliated Normal Coordinate. Let $(M, \eta, \xi, \Phi, g)$ be a compact Sasakian $(2n + 1)$-manifold with $g(\xi, \xi) = 1$ and the integral curves of $\xi$ are geodesics. For any point $p \in M$, we can construct local coordinates in a neighborhood of $p$ which are simultaneously foliated and Riemann normal coordinates [GKN]. That is, we can find Riemann normal coordinates $\{x, z^1, z^2, \ldots, z^n\}$ on a neighborhood $U$ of $p$, such that $\frac{\partial}{\partial x} = \xi$ on $U$. Let $\{U_\alpha\}_{\alpha \in A}$ be an open covering of the Sasakian manifold and $\pi_\alpha : U_\alpha \rightarrow V_\alpha \subset \mathbb{C}^n$ be submersions such that $\pi_\alpha \circ \pi^{-1}_\beta : \pi_\beta(U_\alpha \cap U_\beta) \rightarrow \pi_\alpha(U_\alpha \cap U_\beta)$ is biholomorphic. On each $U_\alpha$, there is a canonical isomorphism $d\pi_\alpha : D_p \rightarrow T_{\pi_\alpha(p)}V_\alpha$ for any $p \in U_\alpha$, where $D = \text{ker} \xi \subset TM$. Since $\xi$ generates isometries, the restriction of the Sasakian metric $g$ to $D$ gives a well-defined Hermitian metric $g_\alpha^T$ on $V_\alpha$. This Hermitian metric in fact is Kähler. More precisely, let $z^1, z^2, \ldots, z^n$ be the local holomorphic coordinates on $V_\alpha$. We pull back these to $U_\alpha$ and still write the same. Let $x$ be the coordinate along the leaves with $\xi = \frac{\partial}{\partial x}$. Then we have the foliation local coordinate $\{x, z^1, z^2, \ldots, z^n\}$ on $U_\alpha$ and $(D \otimes \mathbb{C})$ is spanned by the fields $Z_j = (\frac{\partial}{\partial z^j} + ih_j \frac{\partial}{\partial x})$, $j \in \{1, 2, \ldots, n\}$ with
\[
\eta = dx - ih_j dz^j + ih_{\overline{j}} d\overline{z}^j
\]
and its dual frame
\[
\{\eta, dz^j, j = 1, 2, \ldots, n\}.
\]
Here $h$ is a basic function such that $\frac{\partial h}{\partial x} = 0$ and $h_j = \frac{\partial h}{\partial z^j}, h_{\overline{j}} = \frac{\partial^2 h}{\partial z^j \partial \overline{z}^j}$ with the foliation normal coordinate
\[
h_j(p) = 0, h_{\overline{j}}(p) = \delta^j_l, dh_{\overline{j}}(p) = 0.
\]
Moreover, we have
\[
d\eta(Z_\alpha, Z_\beta) = d\eta(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}).
\]
Then the Kähler $2$-form $\omega^T_\alpha$ of the Hermitian metric $g_\alpha^T$ on $V_\alpha$, which is the same as the restriction of the Levi form $d\eta$ to $D^n_\alpha$, the slice \{x = constant\} in $U_\alpha$, is closed. The collection of Kähler metrics $\{g^T_\alpha\}$ on $\{V_\alpha\}$ is so-called a transverse Kähler metric. We often refer to $d\eta$ as the Kähler form of the transverse Kähler metric $g^T$ in the leaf space $D^n$.

The Kähler form $d\eta$ on $D$ and the Kähler metric $g^T$ is define such that $g = g^T + \eta \otimes \eta$. Now in terms of the normal coordinate, we have
\[
g^T = g^T_\beta dz^\beta d\overline{z}^\beta.
Here $g_T^{ij} = g^T(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$. The transverse Ricci curvature $Ric^T$ of the Levi-Civita connection $\nabla^T$ associated to $g^T$ is defined by $Ric^T = Ric + 2g^T$ and then $R^T = R + 2n$. The transverse Ricci form is defined to be $\rho^T = Ric^T(\Phi, \cdot) = -iR^T_{ij}dz^i \wedge d\bar{z}^j$ with

$$R^T_{ij} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g^T_{\alpha \bar{\beta}})$$

and it is a closed basic $(1, 1)$-form $\rho^T = \rho + 2d\eta$.

2.3. The Sasaki-Ricci Flow and Type II Deformations of Sasakian Structures. We recall that a $p$-form $\gamma$ on a Sasakian $(2n+1)$-manifold is called basic if

$$i(\xi)\gamma = 0 \quad \text{and} \quad \mathcal{L}_\xi \gamma = 0.$$ 

Let $\Lambda_B^p$ be the sheaf of germs of basic $p$-forms and $\Omega_B^p$ be the set of all global sections of $\Lambda_B^p$. It is easy to check that $d\gamma$ is basic if $\gamma$ is basic. Set $d_B = d|_{\Omega_B^p}$. Then

$$d_B := \partial_B + \overline{\partial}_B : \Omega_B^p \rightarrow \Omega_B^{p+1},$$

with $\partial_B : \Lambda_B^p \rightarrow \Lambda_B^{p+1}$ and $\overline{\partial}_B : \Lambda_B^p \rightarrow \Lambda_B^{p+1}$. Moreover

$$d_B d_B^* = i\partial_B \overline{\partial}_B \quad \text{and} \quad d_B^2 = (d_B^*)^2 = 0$$

for $d_B^* := \frac{i}{2}(\overline{\partial}_B - \partial_B)$. The basic Laplacian is defined by

$$\Delta_B := d_B d_B^* + d_B^* d_B.$$ 

Then we have the basic de Rham complex $(\Omega_B^*, d_B)$ and the basic Dolbeault complex $(\Omega_B^{\nu*}, \overline{\partial}_B)$ and its cohomology ring $H^*_B(\mathcal{F}_\xi) \cong H^*_B(M, R)$ of the foliation $\mathcal{F}_\xi$ ([EKA]). Then we can define the orbifold cohomology of the leaf space $Z = M/\overline{U}(1)$ to be this basic cohomology ring

$$H^*_\text{orb}(Z, R) \cong H^*_B(\mathcal{F}_\xi)$$

and the basic first Chern class $c_B^*(M)$ by $c_B^* = \frac{[T]{\rho}}{2\pi_0}$. And a transverse Kähler-Einstein metric(or a Sasaki $\eta$-Einstein metric) means that it satisfies $[\rho^T]_B = \varepsilon[dn]_B$ for $\varepsilon = -1, 0, 1$, up to a $D$-homothetic deformation.

Example 2. Let $(M, \eta, \xi, \Phi, g)$ be a compact Sasakian $(2n+1)$-manifold. If $g^T$ is a transverse Kähler metric on $M$, then $h_\alpha = \det \left( \left( (g^T_{ij})^{-1} \right) \right)$ on $U_\alpha$ defines a basic Hermitian metric on the transverse canonical bundle $K^T_M$. The inverse $(K^T_M)^{-1}$ of $K^T_M$ is sometimes called the transverse anti-canonical bundle. Its basic first Chern class $c_1^B((K^T_M)^{-1})$ is called the basic first Chern class of $M$ and often denoted by $c_1^B(M)$. Then it follows from the previous result that $c_1^B(M)$ for any transverse Kähler metric $\omega$ on a Sasakian manifold $M$.

Definition 4. Let $(L, h)$ be a basic transverse holomorphic line bundle over a Sasakian manifold $(M, \eta, \xi, \Phi, g)$ with the basic Hermitian metric $h$. We say that $L$ is very ample if for any ordered basis $\underline{s} = (s_0, ..., s_N)$ of $H^0_B(M, L)$, the map $i_{\underline{s}} : M \rightarrow \mathbb{CP}^N$ is given by

$$i_{\underline{s}}(x) = [s_0(x), ..., s_N(x)]$$
is well-defined and an embedding which is $S^1$-equivariant with respect to the weighted $C^*$ action in $\mathbb{C}^{N+1}$ as long as not all the $s_i(x)$ vanish. We say that $L$ is ample if there exists a positive integer $m_0$ such that $L^m$ is very ample for all $m \geq m_0$.

There is a Sasakian analogue of Kodaira embedding theorem on a compact quasi-regular Sasakian $(2n + 1)$-manifold due to [RT], [HLM]:

**Proposition 4.** Let $(M, \eta, \xi, \Phi, g)$ be a compact quasi-regular Sasakian $(2n + 1)$-manifold and $(L, h)$ be a basic transverse holomorphic line bundle over $M$ with the basic Hermitian metric $h$. Then $L$ is ample if and only if $L$ is positive.

Now we consider the Type II deformations of Sasakian structures $(M, \eta, \xi, \Phi, g)$ as followings:

By fixing the $\xi$ and varying $\eta$, define

$$ \tilde{\eta} = \eta + d_B^c\varphi, $$

for $\varphi \in \Omega^0_B$. Then

$$ d\tilde{\eta} = d\eta + i\partial_B\overline{\partial}_B\varphi \quad \text{and} \quad \tilde{\omega} = \omega + i\partial_B\overline{\partial}_B\varphi. $$

Hence we have the same transversal holomorphic foliation but with the new Kähler structure on the Kähler cone $C(M)$ and new contact bundle $\tilde{D}$ with

$$ \tilde{\omega} = \frac{1}{2}(\ddbar\tilde{r}^2, \tilde{r} = re^\varphi. $$

Since $r\frac{\partial}{\partial r} = \tilde{r}\frac{\partial}{\partial r}$ and $\xi + ir\frac{\partial}{\partial r} = \xi - iJ(\xi)$ is a holomorphic vector field on $C(M)$, so we have the same holomorphic structure. Finally, by the $\partial_B\overline{\partial}_B$-Lemma in the basic Hodge decomposition, there is a basic function $F : M \to \mathbb{R}$ such that

$$ \rho^T(x, t) - \kappa d\eta(x, t) = d_B^c d_B^F = i\partial_B\overline{\partial}_B F. $$

Now we focus on finding a new $\eta$-Einstein Sasakian structure $(M, \xi, \tilde{\eta}, \tilde{\Phi}, \tilde{g})$ with $\tilde{g}^T = (g^T_{\alpha\beta} + \varphi)dz^\alpha d\overline{z}^\beta$ such that

$$ \tilde{\rho}^T = \kappa d\tilde{\eta}. $$

Hence $\tilde{\rho}^T - \rho^T = \kappa d_B^c \varphi - d_B^c F$. It follows that there is a Sasakian analogue of the Monge-Ampère equation for the orbifold version of Calabi-Yau Theorem

$$ \frac{\det(g^T_{\alpha\beta} + \varphi)}{\det(g^T_{\alpha\beta})} = e^{-\kappa \varphi + F}. $$

(2.3)

Now we consider the Sasaki-Ricci flow on $M \times [0, T)$

$$ \frac{d}{dt}g^T(x, t) = -(Ric^T(x, t) - \kappa g^T(x, t)) $$

which is equivalent to

(2.4)

$$ \frac{d}{dt} \varphi = \log \det(g^T_{\alpha\beta} + \varphi) - \log \det(g^T_{\alpha\beta}) + \kappa \varphi - F. $$

Note that, for any two Sasakian structures with the fixed Reeb vector field $\xi$, we have $Vol(M, g) = Vol(M, g')$ and

$$ \tilde{\omega}^n \wedge \eta = i^n \det(g^T_{\alpha\beta} + \varphi)dz^1 \wedge d\overline{z}^1 \wedge ... \wedge dz^n \wedge d\overline{z}^n \wedge dx. $$
As before, for an orbifold Riemannian submersion \( \pi : (M, g, \omega) \to (Z, h, \omega_h) \) with \( \omega = \pi^*(\omega_h) \). We consider the projection

(2.5) \[ \Pi : (C(M), \overline{g}, J, \overline{\omega}) \to (Z, h, \omega_h) \]

such that \( \Pi|_{(M, g, \omega)} = \pi \), then we have the relation between the volume form of the Kähler cone metric on the metric cone and the volume form of the Sasaki metric on \( M \)

(2.6) \[ i_x \overline{\omega}^{n+1} = (\Pi^*\omega_h)^n \wedge \eta. \]

Here \( \overline{\omega}^{n+1} = r^{2n+1}(\Pi^*\omega_h)^n \wedge dr \wedge \eta. \)

3. The Sasaki-Ricci Flow on Transverse Fano Three-Spheres

In this section, we will consider the Sasaki-Ricci flow on the transverse Fano three-sphere \( S^3 \). For simplicity if there is no confusion we remove the superscript \( T \), since all quantities we are considering are transverse. Let \( (x, z = x^1 + ix^2) \) be the foliation normal coordinates and

\[ g_{ij} = d\eta(\frac{\partial}{\partial x^i}, \Phi \frac{\partial}{\partial x^j}). \]

Note that \( R_{ij} = \frac{1}{2} R g_{ij} \), so we consider the Sasakian-Ricci flow as

(3.1) \[ \frac{\partial}{\partial t} g_{ij} = (r - R) g_{ij}, \]

where \( r \) is the average of the transverse scalar curvature \( R \). It’s not difficult to see \( r \) is a positive constant.

The object of this section is to remove the positive assumption of the initial transverse scalar curvature. In fact we prove that the transverse scalar curvature \( R \) becomes positive in finite time under the Sasaki-Ricci flow. Then, by applying Proposition \( \square \) We can derive the Theorem \( \square \)

It is proved (\( \square \square \)) that the evolution of the transverse scalar curvature \( R \)

(3.2) \[ \frac{\partial}{\partial t} R = \triangle_B R + R(R - r). \]

Then it is natural to consider the ordinary differential equation for \( s = s(t) \)

(3.3) \[ \frac{d}{dt} s = s(s - r), \quad s(0) < \min_{x \in M} R(x, 0) \]

which is obtained from the parabolic partial differential equation for \( R \) simply by dropping the sub-Laplacian term. Note that \( s(t) = \frac{r}{1 - c e^t} < 0 \) where \( c > 1. \)

The following Harnack inequality is due to \( [\text{Ben}] \). For completeness, we sketch the proof here for the transverse scalar curvature \( R \) which is a basic function.

**Lemma 1.** Suppose the flow (3.1) have a solution for \( t < T^* (\leq \infty) \). Then for any two space-times points \((x, \tau) \) and \((y, T) \) with \( 0 < \tau < T < T^* \), we have

(3.4) \[ R(y, T) - s(T) \geq e^{-\frac{r}{c}(T-\tau)}(R(x, \tau) - s(\tau)), \]
Where
\[ D = D((x, \tau), (y, T)) = \inf_{\gamma} \int_{\tau}^{T} \| \frac{d\gamma}{dt} \|_{gt}^2 dt. \]
Here the infimum runs over all piece-wisely smooth curves \( \gamma(t), t \in [\tau, T] \) with \( \gamma(\tau) = x \) and \( \gamma(T) = y \).

Proof. Note that
\[ \frac{\partial}{\partial t} (R - s) = \Delta_B (R - s) + (R - s)(R - r + s). \]
Since \( R - s \) is positive at \( t = 0 \), then by maximum principle, it stays positive for all time. So
\[ L = \log(R - s) \]
is well defined for all times. It follows from (3.2) and (3.3) that
\[ \frac{\partial L}{\partial t} = \Delta_B L + |\nabla L|^2_{gt} + R - r + s. \]
Let
\[ Q = \frac{\partial L}{\partial t} - |\nabla L|^2_{gt} - s = \Delta_B L + R - r. \]
We have
\[ \frac{\partial Q}{\partial t} = \Delta_B Q + 2g_t(\nabla L, \nabla Q) + 2 \left( \nabla^2 L + \frac{1}{2}(R - r)g \right)_{gt}^2 + (r - s)Q + s|\nabla L|^2 - s \]
(3.6)
It is also known that \( sR \geq -C \), where \( C \) is a positive constant. Thus we have
\[ \frac{\partial Q}{\partial t} \geq \Delta_B Q + 2g_t(\nabla L, \nabla Q) + Q^2 + (r - s)Q + s|\nabla L|^2 - C. \]
For \( sL \), we have
\[ \frac{\partial (sL)}{\partial t} = \Delta_B (sL) + s|\nabla L|^2_{gt} + s(R - r + s) + s(s - r)L. \]
Noting that \( L \geq -C - Ct \), we have
\[ \frac{\partial (sL)}{\partial t} \geq \Delta_B (sL) + 2(\nabla L, \nabla (sL))_{gt} - s|\nabla L|^2 - C. \]
Set \( P = Q + sL \). Then we have
\[ \frac{\partial (P)}{\partial t} \geq \Delta_B (P) + 2g_t(\nabla L, \nabla P) + Q^2 + (r - s)Q - C. \]
Since \( sL \) is bounded, we can find a positive constant \( C > 0 \) such that for \( t \) large enough enough
\[ \frac{\partial (P)}{\partial t} \geq \Delta_B (P) + 2g_t(\nabla L, \nabla P) + \frac{1}{2}(P^2 - C^2). \]
Applying the maximum principle yields
\[ P \geq C \frac{1 + e^{Ct}}{1 - e^{Ct}}, \]
where $c > 1$. So we have for $t$ large enough, we have

$$Q \geq -3C.$$  

Then we have

$$\frac{\partial L}{\partial t} - |\nabla L|^2 \geq -3C + s \geq -3C - 1.$$  

By the fact that $R$ is a basic function, we have

$$g_t^M \left( \nabla^M L, \frac{d}{dt} \gamma \right) = g_t^M \left( \nabla L, \frac{d}{dt} \gamma \right) = g_t \left( \nabla L, \frac{d}{dt} \gamma \right).$$  

Hence

$$\frac{d}{dt} L(t, \gamma(t)) = \frac{\partial L}{\partial t} + g_t \left( \nabla L, \frac{d}{dt} \gamma(t) \right) \geq \frac{\partial L}{\partial t} - |\nabla L|^2_{g_t} - \frac{1}{4} \left| \frac{d}{dt} \gamma \right|^2_{g_t}.$$  

Taking $\gamma(t)$ to be a path achieving the minima $D$, we have

$$L(y, T) - L(x, \tau) = \int_{\tau}^{T} \frac{d}{dt} L(t, \gamma(t)) dt$$  

$$\geq -3C(T - \tau) - \frac{D}{4}.$$  

This completes the proof of this theorem. \[\square\]

Now we are ready to prove Theorem 1:

**Proof.** It follows from Lemma 6 that the transverse scalar curvature $R$ and the diameter of $M$ are bounded. Thus, as in section 8 of [H1], the Harnack inequality (3.4) show that

$$R - s > C > 0.$$  

Since $s$ approaches zero exponentially, we conclude that $R$ becomes positive in finite time. Then Theorem 1 follows from Proposition 1. \[\square\]

Adapt the notion as in the paper of [PSSW], we first define

**Definition 5.** A metric $g$ on $\Sigma$ is said to have a conical singularity at $p_i$ if it can be expressed as

$$g = e^f |z|^{\beta_i} |dz|^2$$  

near $p_i$, with $f(z)$ a bounded function. Here $z_i$ is a local holomorphic coordinate centered at $p_i$ and $\beta_i \in (0, 1)$ is a constant. The cone angle at $p_i$ is $2(1 - \beta_i)\pi$. In the content of our current paper, we associate the conical singularity to the divisor denoted by

$$\beta = \sum_i^k \beta_i[p_i]$$  

in a compact Riemann surface $\Sigma$ and refer to the data $(\Sigma, \beta)$ as a pair.
The orbifold Euler characteristic formula reads as
\[
\chi(\Sigma, \beta) = \chi(\Sigma) - \sum_i^k \beta_i.
\]

The equation of constant Ricci curvature on the orbifold surface \((\Sigma, z_1, \ldots, z_k)\) becomes
\[
(3.7) \quad Ric(g) = \frac{1}{2}\chi(\Sigma, \beta)g - \frac{1}{2}Rg
\]
on \(\Sigma \setminus \{p_1, \ldots, p_k\}\) with the volume normalization \(\int_{\Sigma} d\mu = 2\).

When \(\chi(\Sigma, \beta) \leq 0\), it has been shown ([Tro]) that it always admits a conical metric with constant Ricci curvature and such a metric is unique up to scaling.

When \(\chi(\Sigma, \beta) > 0\), it holds only when \(\Sigma = S^2\) and
\[
(3.8) \quad \sum_i^k \beta_i < 2.
\]

Note that on a compact surface, there are no Ricci soliton solutions other than those of constant curvature ([H1]). Every bad orbifolds surface do not admit metrics of constant curvature and so every soliton solution has nonconstant curvature:

**Proposition 5.** Let \((S^2, \beta)\) be a sphere with \(k\) marked points with \(\sum_{i}^k \beta_i < 2\) and \(k \leq 2\). Then

1. For \(k = 1\): It is a tear-drop. The equation (3.7) does not admit a solution. Instead, one can construct a unique rotationally symmetric compact shrinking soliton \(g\) ([Wu], [BM]).
2. For \(k = 2\):
   (a) If \(\beta_1 = \beta_2\), there exists a unique rotationally symmetric compact shrinking solution of equation (3.7) ([Ben], [BM]).
   (b) If \(\beta_1 \neq \beta_2\), the equation (3.7) does not admit a solution. Instead, one can construct a unique rotationally symmetric compact shrinking soliton ([Wu], [BM]).

In the case of \(k \geq 3\), the equation (3.7) admits a unique compact shrinking soliton ([Tro], [LT]) if and only if
\[
(3.9) \quad 2 \max \beta_j < \sum_i^k \beta_i.
\]

In the paper of [PSSW], they consider the conical Ricci flow
\[
(3.10) \quad \frac{\partial}{\partial t} g(t) = (\chi(S^2, \beta) - R)g(t), g(0) = g_0
\]
on \(S^2 \setminus \beta\), where \(g_0\) is a metric of \(g_0 = e^{u_0}(\Pi_i(1+|z-p_i|^2)^{-\beta_i}g_{FS}), u_0 \in C^\infty(S^2), \int_{S^2} d\mu = 2\).

**Proposition 6.** ([PSSW] Theorem 1.3) Let \((S^2, \beta)\) be a sphere with \(k\) marked points with \(\sum_i^k \beta_i < 2\) and \(k \geq 3\). If \((S^2, \beta)\) is stable which is (3.9), then the flow
converges in the Gromov-Hausdorff topology in \( C^\infty(S^2 \setminus \beta) \) to the unique conical constant curvature metric \( g_\infty \in c_1(S^2) \) on \((S^2, \beta)\).

The Proof of Theorem 2:

Proof. Let \((S^3, \xi, g_0)\) be a compact quasi-regular Fano Sasakian three-sphere. By the first structure theorem, there exists a principal \( S^1 \)-orbibundle \( \pi : S^3 \rightarrow S^2 \) with \( k \) marked points \( \{p_1, ..., p_k\} \) in \((S^2, z_1, ..., z_k)\), \( z_i \) is a local holomorphic coordinate centered at \( p_i \). The orbifold structure on \((S^2, z_1, ..., z_k)\) is defined by the local uniformizing systems \( (\tilde{U}_i, C_{m_i}, \varphi_i) \) centered at the point \( z_i \), where \( C_{m_i} \) is the cyclic group of order \( m_i \) and \( \varphi_i(z) = z^{m_i} \). Then we have the following orbifold first Chern class formula with codimension one canonical divisors \( [p_i] \) of the ramification index \( m_i \)

\[
c_1^{orb}(S^2) = c_1(S^2) - \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right)
\]

Note that

\[
c_1^B(S^3) = \pi^* c_1^{orb}(S^2)
\]

and

\[
\chi(S^2, \beta) = c_1^{orb}(S^2).
\]

Thus the orbifold first Chern number is satisfying

\[
\chi(S^2, \beta) = \chi(S^2) - \sum_{i=1}^{k} (1 - \frac{1}{m_i}) = 2 - k + \sum_{i=1}^{k} \frac{1}{m_i}.
\]

Now a metric \( g \) on \( S^2 \) has a conical singularity at \( p_i \) with cone angle at \( p_i \) is \( 2(1 - \beta_i)\pi \) so that \( \beta_i = 1 - \frac{1}{m_i} \).

Hence \((S^3, \xi, g_0)\) is a Fano Sasakian 3-sphere \((c_1^B(S^3) > 0)\) only if \( k \leq 3 \) and

\[
0 < \chi(S^2, \beta) = 2 - k + \sum_{i=1}^{k} \frac{1}{m_i} = 2 - \sum_{i=1}^{k} \beta_i
\]

which is the inequality \((3.8)\).

Finally, Theorem 2 follows easily from Proposition 5 and Proposition 6 \( \square \)

4. The Sasaki-Ricci Flow on Transverse Fano Sasakian Manifolds

4.1. \( L^4 \)-Bound of the Transverse Ricci Curvature. In this subsection, we show the \( L^4 \)-bound of the transverse Ricci curvature under the Sasaki-Ricci flow.

Theorem 6. Let \((M, \xi, \eta_0, \Phi_0, g_0, \omega_0)\) be a compact transverse Fano quasi-regular Sasakian \((2n + 1)\)-manifold and its the space \( Z \) of leaves of the characteristic foliation be well-formed. Then, under the Sasaki-Ricci flow \((4.1)\), there exists a positive constant \( C \) such that

\[
\int_M |\text{Ric}^T_{\omega(t)}|^{\frac{4}{n}} \omega(t)^n \wedge \eta_0 \leq C,
\]

for all \( t \in [0, \infty) \).
We follow the line in [TZ] and [CHLW] to prove this estimate. Note that the flow (1.1) can be expressed locally as a parabolic Monge-Ampère equation on a basic Kähler potential $\varphi$ as in (2.4):

\[
\frac{d}{dt} \varphi = \log \det(g_{\alpha \beta}^T + \varphi_{\alpha \beta}) - \log \det(g_{\alpha \beta}^T) + \varphi - u(0).
\]

Here $u(0)$ is the transverse Ricci potential of $\eta_0$, defined by

\[
R_{\alpha \beta}^T - g_{\alpha \beta}^T = \partial_k \partial_l u(0).
\]

Let $u(t)$ be the evolving transverse Ricci potential. Then

\[
\partial_k \partial_l \varphi = \partial_k \partial_l u = g_{\alpha \beta}^T - R_{\alpha \beta}^T = \partial_k \partial_l u.
\]

It follows from this equation that $\varphi$ evolves by

\[
\varphi(t) = u(t) + c(t),
\]

for $c(t)$ depending only on time $t$. Then by using $c(t)$ to adjust the initial value $\varphi(0)$, we always assume that

\[
\varphi(0) = c_0 := \frac{1}{V} \int_M e^{-u(0)} \omega_0^n \wedge \eta_0.
\]

Since

\[
\partial_k \partial_l (\frac{\partial u}{\partial t}) = \partial_k \partial_l u + \partial_k \partial_l \Delta_B u,
\]

then we can

\[
a(t) = \frac{1}{V} \int_M u e^{-u(t)} \omega_0^n \wedge \eta_0
\]

such that

\[
\frac{\partial u}{\partial t} = \Delta_B u + u - a.
\]

Now by Jensen’s inequality, we have $a(t) \leq 0$ and then there exist a uniform positive constant $C_1$ such that (Co1)

\[
-C_1 \leq a(t) \leq 0
\]

for all $t \geq 0$. Moreover, it follows from the Poincaré type inequality, one can show that $a(t)$ increases along the Sasaki-Ricci flow, so we may assume

\[
\lim_{t \to \infty} a(t) = a_\infty.
\]

It follows from [Co1] (also [ST]) that

**Lemma 2.** Let $(M^{2n+1}, \xi, g_0)$ be a compact Sasakian manifold and let $g^T(t)$ be the solution of the Sasaki-Ricci flow (1.1) with the initial transverse metric $g_0^T$. Then there exists $C$ depending only on the initial metric such that

\[
||u(t)||_{C^0} + ||\nabla^T u(t)||_{C^0} + ||\Delta_B u(t)||_{C^0} \leq C
\]

for all $t \geq 0$.

In order to prove the $L^4$ bound of transverse Ricci curvature (1.1) under the normalized Sasaki-Ricci flow, it suffices to show that

\[
\int_M |\nabla \nabla^T u(t)|^4 \omega_0^n \wedge \eta_0 \leq C,
\]

for all $t \geq 0$ and for some constant $C$ independent of $t$. We need the following Lemmas.
Lemma 3. There exists a positive constant $C = C(g_0^T)$ such that
\begin{equation}
\int_M [\| \nabla^T \nabla^T u \|^2 + | \nabla^T \nabla^T u |^2 + | \text{Rm}^T |^2] \omega(t)^n \wedge \eta_0 \leq C,
\end{equation}
for all $t \in [0, \infty)$.

Proof. Applying the integration by parts, we have
\begin{equation}
\int_M [\| \nabla^T \nabla^T u \|^2 \omega(t)^n \wedge \eta_0 = \int_M (\Delta_B u)^2 \omega(t)^n \wedge \eta_0,
\end{equation}
and also
\begin{equation}
\int_M [\| \nabla^T \nabla^T u \|^2 \omega(t)^n \wedge \eta_0 = \int_M [(\Delta_B u)^2 - \langle \text{Ric}^T, \partial_B \partial_B u \rangle] \omega(t)^n \wedge \eta_0
\end{equation}
\begin{equation}
\qquad = \int_M (\Delta_B u)^2 + | \nabla^T \nabla^T u |^2 + | \nabla^T u |^2] \omega(t)^n \wedge \eta_0.
\end{equation}
Moreover, the $L^2$-bound of the transverse Riemannian curvature tensor follows from uniformly bound of the transverse scalar curvature and the Sasaki analogue of the Chern-Weil theory as in [Zh2, Lemma 7.2]:
\begin{equation}
\frac{(2n)^2}{2^{n-2}(n-2)!} \int_M [2c_B - \frac{n}{2}(c_B^2)] \wedge \omega(t)^{n-2} \wedge \eta_0
\end{equation}
\begin{equation}
= \frac{1}{2^{n-1}n!} \int_M [\text{Rm}^T] - \frac{2}{(n+1)(n-2)}(R^T)^2 - \frac{(n-1)(n+2)}{n(n+1)}((R^T)^2 + (2n(n+1))^2] \omega(t)^n \wedge \eta_0.
\end{equation}

The following integral inequalities hold by using Lemma 2 and integration by parts.

Lemma 4. There exists a universal positive constant $C = C(g_0^T)$ such that
\begin{equation}
\int_M [\nabla^T \nabla^T u |^4 \omega(t)^n \wedge \eta_0
\end{equation}
\begin{equation}
\leq C \int_M [\| \nabla^T \nabla^T u \|^2 + | \nabla^T \nabla^T u |^2] \omega(t)^n \wedge \eta_0,
\end{equation}
\begin{equation}
\int_M [\nabla^T \nabla^T u |^4 \omega(t)^n \wedge \eta_0
\end{equation}
\begin{equation}
\leq C \int_M [\| \nabla^T \nabla^T u \|^2 + | \nabla^T \nabla^T u |^2 + | \nabla^T \nabla^T u |^2] \omega(t)^n \wedge \eta_0,
\end{equation}
and
\begin{equation}
\int_M [\| \nabla^T \nabla^T \Delta_T u |^2 + | \nabla^T \nabla^T \Delta_T u |^2 + | \text{Rm}^T |^2] \omega(t)^n \wedge \eta_0
\end{equation}
\begin{equation}
\leq C \int_M [\| \nabla^T \nabla^T u |^2 + | \nabla^T \nabla^T u |^2 + | \nabla^T \nabla^T u |^2 + | \text{Rm}^T |^2] \omega(t)^n \wedge \eta_0
\end{equation}
for all $t \in [0, \infty)$.

Now we can prove a uniform bound of \(\int_M [\nabla^T \nabla^T u(t)]^4 \omega(t)^n \wedge \eta_0\) under the Sasaki-Ricci flow.

Proposition 7. There exists a positive constant $C = C(g_0^T)$ such that
\begin{equation}
\int_M [\| \nabla^T \nabla^T \nabla^T u |^2 + | \nabla^T \nabla^T \nabla^T u |^2 + | \nabla^T \nabla^T \nabla^T u |^2] \omega(t)^n \wedge \eta_0
\end{equation}
\begin{equation}
+ \int_M [\| \nabla^T \nabla^T u |^4 + | \nabla^T \nabla^T u |^4] \omega(t)^n \wedge \eta_0 \leq C,
\end{equation}
for all $t \in [0, \infty)$.
By integration by parts, we obtain
\[(\frac{\partial}{\partial t} - \Delta_B)\Delta_B u = \Delta_B u - |\nabla T \nabla T u|^2,\]
thus
\[\frac{1}{2}(\frac{\partial}{\partial t} - \Delta_b)(\Delta_B u)^2 = (\Delta_B u)^2 - |\nabla T \nabla B u| \nabla T \nabla B u|^2.\]

Integrating over the manifold gives
\[\int_M |\nabla T \nabla B u|^2 \omega^n \wedge \eta_0 = \int_M [(\Delta_B u)^2 - \Delta_B |\nabla T \nabla B u|^2 - \frac{1}{2}(\Delta_B u)^2] \omega^n \wedge \eta_0\]
\[-\int_M (\Delta_B u)^2 - \Delta_B |\nabla T \nabla B u|^2 + \frac{1}{2}(\Delta_B u)^2] \omega^n \wedge \eta_0 - \frac{1}{2} \frac{d}{dt} \int_M (\Delta_B u)^2 \omega^n \wedge \eta_0.\]

Applying the uniform bound of $\Delta_B u$ from Lemma 2 and (4.9), we obtain
(4.11)\[\int_t^{t+1} \int_M |\nabla T \nabla B u|^2 \omega(s) \wedge \eta_0 \eta ds \leq C,\]
for all $t \geq 0$. Next we compute
\[\frac{d}{dt} \int_M |\nabla T \nabla B u|^2 \omega^n \wedge \eta_0\]
\[= \int_M [\nabla T \nabla T \nabla T B u]^2 - |\nabla T \nabla B u|^2 |\nabla T \nabla B u|^2 + \Delta_B |\nabla T \nabla B u|^2] \omega^n \wedge \eta_0 - \int_M \left[\left<\nabla T |\nabla T \nabla B u|^2, \nabla T \nabla B u\right> + \left<\nabla T |\nabla T \nabla B u|^2, \nabla T \nabla B u\right>\right] \omega^n \wedge \eta_0.

By integration by parts, we obtain
\[\int_M \left<\nabla T |\nabla T \nabla B u|^2, \nabla T \nabla B u\right> \omega^n \wedge \eta_0\]
\[\leq \frac{1}{2} \int_M [\nabla T \nabla B u]^2 + \nabla T \nabla B u |\nabla T \nabla B u|^2] \omega^n \wedge \eta_0 \]
\+[C \int_M [\nabla T \nabla B u]^2 + \nabla T \nabla B u |\nabla T \nabla B u|^2] \omega^n \wedge \eta_0.

and also
\[\int_M \left<\nabla T |\nabla T \nabla B u|^2, \nabla T \nabla B u\right> \omega^n \wedge \eta_0\]
\[\leq \frac{1}{2} \int_M [\nabla T \nabla B u]^2 + \nabla T \nabla B u |\nabla T \nabla B u|^2] \omega^n \wedge \eta_0 \]
\[+C \int_M [\nabla T \nabla B u]^2 + \nabla T \nabla B u |\nabla T \nabla B u|^2] \omega^n \wedge \eta_0.

Therefore, by (4.9) and Lemma 2
\[\frac{d}{dt} \int_M |\nabla T \nabla B u|^2 \omega^n \wedge \eta_0\]
\[\leq -\frac{1}{2} \int_M |\nabla T \nabla B u|^2 + \nabla T \nabla B u |\nabla T \nabla B u|^2] \omega^n \wedge \eta_0 + C(1 + \int_M |\nabla T \nabla B u|^2 \omega^n \wedge \eta_0)\]
\[\leq C(1 + \int_M |\nabla T \nabla B u|^2 \omega^n \wedge \eta_0).

The required uniform bound of $\int_M |\nabla T \nabla B u|^2 \omega^n \wedge \eta_0$ follows from this and (4.11). \qed

4.2. Longtime Behavior of Transverse Ricci potential under the Sasaki-Ricci Flow. In this subsection, we will show that the transverse Ricci potential $u(t)$ which is a basic function behaves very well as $t \to \infty$ under the Sasaki-Ricci flow. This implies that the limit of Sasaki-Ricci flow should be the Sasaki-Ricci soliton in the $L^2$-topology.

We first define
\[\mu(g^T) = \inf \{W^T(g^T, f) : \int_M e^{-f} \omega^n \wedge \eta_0 = V\},\]
where $W^T(g^T, f) = (2\pi)^{-n} \int_M (R^T + |\nabla^T f|^2 + f)e^{-f}\omega^n \wedge \eta_0$ as in (5.9) below. Note that
\[
\mu(g^T) \leq \frac{1}{V} \int_M u e^{-u}\omega^n \wedge \eta_0.
\]

Now under the Sasaki-Ricci flow, for any backward heat equation (Co1)
\[
(4.12) \quad \frac{\partial f}{\partial t} = -\Delta_B f + |\nabla^T f|^2 + \Delta_B u,
\]
we have
\[
\frac{d}{dt} W^T(g^T, f) = \frac{1}{V} \int_M (|\nabla^T \nabla^T (u - f)|^2 + |\nabla^T \nabla^T f|^2)e^{-f}\omega^n \wedge \eta_0 \geq 0
\]
and then
\[
\mu(g^T_0) \leq \mu(g^T(t)) \leq 0
\]
for all $t \geq 0$.

**Lemma 5.** Under the Sasaki-Ricci flow, for a smooth basic function $f$
\[
\int_M |\nabla^T f|^2 \omega^n \wedge \eta_0 \leq C(g^T_0) \int_M |\nabla^T \nabla^T f|^2 \omega^n \wedge \eta_0
\]

**Proof.** We may assume that $\int_M f e^{-u}\omega^n \wedge \eta_0 = 0$. It follows from [Co1] Theorem 8.1, we have the transverse weighted Poincaré inequality
\[
\frac{1}{V} \int_M f^2 e^{-u}\omega^n \wedge \eta_0 \leq \frac{1}{V} \int_M |\nabla^T f|^2 e^{-u}\omega^n \wedge \eta_0 + (\frac{1}{V} \int_M f e^{-u}\omega^n \wedge \eta_0)^2
\]
for all basic function $f \in C^\infty_M(M; R)$. Thus
\[
\int_M f^2 e^{-u}\omega^n \wedge \eta_0 \leq \int_M |\nabla^T f|^2 e^{-u}\omega^n \wedge \eta_0.
\]
It follows from Lemma 2 that
\[
\int_M f^2 \omega^n \wedge \eta_0 \leq C(g^T_0) \int_M |\nabla^T f|^2 \omega^n \wedge \eta_0.
\]
Thus
\[
\int_M |\nabla^T f|^2 \omega^n \wedge \eta_0 = -\int_M f \Delta_B f \omega^n \wedge \eta_0 \leq \frac{1}{2C} \int_M f^2 \omega^n \wedge \eta_0 + 2C \int_M (\Delta_B f)^2 \omega^n \wedge \eta_0 \leq \frac{1}{2} \int_M |\nabla^T f|^2 \omega^n \wedge \eta_0 + 2C \int_M (\Delta_B f)^2 \omega^n \wedge \eta_0
\]
and
\[
\int_M |\nabla^T f|^2 \omega^n \wedge \eta_0 \leq C(g^T_0) \int_M (\Delta_B f)^2 \omega^n \wedge \eta_0.
\]
On the other hand,
\[
\int_M |\nabla^T \nabla^T f|^2 \omega^n \wedge \eta_0 = \int_M ((\Delta_B f)^2 - \text{Ric}^T(\nabla^T f, \nabla^T f)) \omega^n \wedge \eta_0 = \int_M ((\Delta_B f)^2 - |\nabla^T f|^2) \omega^n \wedge \eta_0 + \int_M \nabla_i \nabla_j u \nabla^T_i \nabla^T_j f \omega^n \wedge \eta_0 \leq \int_M ((\Delta_B f)^2 - |\nabla^T f|^2) \omega^n \wedge \eta_0 + \int_M (\Delta_B f)^2 + \frac{1}{2} |\nabla^T \nabla^T f|^2 + |\nabla^T u|^2 |\nabla^T f|^2) \omega^n \wedge \eta_0 \leq \int_M (2(\Delta_B f)^2 + \frac{1}{2} |\nabla^T \nabla^T f|^2 + C |\nabla^T f|^2) \omega^n \wedge \eta_0
\]
and then
\[ \int_M |\nabla^T \nabla^T f|^2 \omega^n \land \eta_0 \leq \int_M (4(\Delta_B f)^2 + 2C|\nabla^T f|^2) \omega^n \land \eta_0 \]
\[ \leq C\int_M (\Delta_B f)^2 \omega^n \land \eta_0 \]
\[ \leq C\langle g_0 \rangle \int_M |\nabla^T \nabla^T f|^2 \omega^n \land \eta_0. \]

\[ \square \]

**Theorem 7.** Under the Sasaki-Ricci flow

\[ (4.13) \]
\[ \int_0^\infty \int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \land \eta_0 \land dt < \infty. \]

In particular,

\[ (4.14) \]
\[ \int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \land \eta_0 \rightarrow 0 \]
as \( t \rightarrow \infty \).

**Proof.** Let \( f_k \) be a minimizer of \( \mu(g^T(k)) \) with \( \int_M e^{-f_k} \omega^n \land \eta_0 = V \) and \( f_k(t) \) be the solution of the backward heat equation (4.12) on the time interval \([k-1,k]\).

Then
\[ \int_{k-1}^k \int_M (|\nabla^T \nabla^T f_k|^2 + |\nabla^T \nabla^T (u-f_k)|^2)e^{-f_k} \omega^n \land \eta_0 \land dt \leq \mu(g^T(k)) - \mu(g^T(k-1)). \]

Hence by using \( \mu(g^T(t)) \leq 0 \)
\[ \sum_{k=1}^\infty \int_{k-1}^k \int_M (|\nabla^T \nabla^T f_k|^2 + |\nabla^T \nabla^T (u-f_k)|^2) \omega^n \land \eta_0 \land dt \leq C(g_0^T). \]

On the other hand, by applying the above estimate and Lemma 5 to \((u-f_k)\), we have
\[ \int_0^\infty \int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \land \eta_0 \land dt \leq \sum_{k=1}^\infty \int_M (2|\nabla^T \nabla^T f_k|^2 + 2|\nabla^T \nabla^T (u-f_k)|^2) \omega^n \land \eta_0 \land dt \]
\[ \leq \sum_{k=1}^\infty \int_M (2|\nabla^T \nabla^T f_k|^2 + 2C|\nabla^T \nabla^T (u-f_k)|^2) \omega^n \land \eta_0 \land dt \]
\[ \leq C(g_0^T). \]

This is the estimate (4.13).

Next, it follows from the straight computation that
\[ \frac{\partial}{\partial t}|\nabla^T \nabla^T u|^2 = \Delta_B |\nabla^T \nabla^T u|^2 - |\nabla^T \nabla^T \nabla^T u|^2 - |\nabla^T \nabla^T \nabla^T u|^2 - 2Rm^T (\nabla^T \nabla^T u, \nabla^T \nabla^T u) \]
and
\[ \frac{d}{dt} \int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \land \eta_0 \]
\[ \leq \int_M [|\nabla u(t)|_{C^0} + |\Delta_B u(t)|_{C^0}] |\nabla^T \nabla^T u|^2 + |\nabla^T \nabla^T \nabla^T u|^2 + |\nabla u(t)|_{C^0} |\nabla^T \nabla^T u|^2 \omega(t)^n \land \eta_0. \]
Then, from Lemma 2, Lemma 3 and Proposition 7 we have
\[ \frac{d}{dt} \int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \land \eta_0 \leq C(g_0^T). \]

Hence
\[ \int_M |\nabla^T \nabla^T u|^2 \omega(t)^n \land \eta_0 \rightarrow 0 \]
as \( t \to \infty \).

Similarly, we have

**Theorem 8.** Under the Sasaki-Ricci flow,

\[
\int_{t}^{t+1} \int_{M} |\nabla^{T}(\Delta_{B}u - |\nabla^{T} u|^{2} + u)|^{2} \omega(t)^{n} \wedge \eta_{0} \wedge ds \to 0
\]

as \( t \to \infty \) and then

\[
\int_{M} (\Delta_{B}u - |\nabla^{T} u|^{2} + u - a)^{2} \omega(t)^{n} \wedge \eta_{0} \to 0
\]

as \( t \to \infty \).

**Proof.** Note that by the transverse Ricci potential relation

\[
\nabla_{i}^{T}(\Delta_{B}u - |\nabla^{T} u|^{2} + u) = \overline{\nabla}_{j}^{T} \nabla_{i}^{T} \nabla_{j}^{T} u - \nabla_{i}^{T} \nabla_{j}^{T} u \overline{\nabla}_{j}^{T} u
\]

and then

\[
|\nabla^{T}(\Delta_{B}u - |\nabla^{T} u|^{2} + u)|^{2} \leq 2(|\nabla_{i}^{T} \nabla_{j}^{T} u|^{2}|\overline{\nabla}_{j}^{T} u|^{2} + |\overline{\nabla}_{j}^{T} \nabla_{i}^{T} \nabla_{j}^{T} u|^{2}).
\]

In order to derive (4.15a), it suffices to prove that

\[
\int_{t}^{t+1} \int_{M} |\overline{\nabla}_{j}^{T} \nabla_{i}^{T} \nabla_{j}^{T} u|^{2} \omega(t)^{n} \wedge \eta_{0} \wedge ds \to 0.
\]

Since the Reeb vector field and the transverse holomorphic structure are both invariant, all the integrands are only involved with the transverse Kähler structure \( \omega(t) \) and basic functions \( u(t) \). Hence, under the Sasaki–Ricci flow, when one applies integration by parts, the expressions involved behave essentially the same as in the Kähler-Ricci flow. Hence (4.17) follows easily from subsection 4.1 and [TZ, Proposition 3.2].

Next we denote \( \Delta_{B}u - |\nabla^{T} u|^{2} + u - a = H \) with \( \int_{M} H e^{-u} \omega(t)^{n} \wedge \eta_{0} = 0 \). Then, by weighted Poincaré inequality (cf. 4.33) and the uniform bound of \( u \), we have

\[
\int_{M} H^{2} \omega(t)^{n} \wedge \eta_{0} \leq C \int_{M} |\nabla^{T} H|^{2} \omega(t)^{n} \wedge \eta_{0}
\]

and then from (4.15a)

\[
\int_{t}^{t+1} \int_{M} H^{2} \omega(t)^{n} \wedge \eta_{0} \wedge dt \to 0
\]

as \( t \to \infty \). Since

\[
\frac{\partial H}{\partial t} = \Delta_{B}H + H - \frac{da}{dt} + |\nabla^{T} \nabla^{T} u|^{2},
\]

it follows from Proposition 7 that

\[
\frac{d}{dt} \int_{M} H^{2} \omega(t)^{n} \wedge \eta_{0} = \int_{M} 2H(\Delta_{B}H + H - \frac{da}{dt} + |\nabla^{T} \nabla^{T} u|^{2} + \frac{1}{2}H \Delta_{B}u) \omega(t)^{n} \wedge \eta_{0}
\]

\[
\leq \int_{M} 2H(\Delta_{B}u - |\nabla^{T} \nabla^{T} u|^{2} + \frac{1}{2}H \Delta_{B}u) \omega(t)^{n} \wedge \eta_{0}
\]

\[
\leq (C + |\Delta_{B}u|^{2}) \int_{M} H^{2} \omega(t)^{n} \wedge \eta_{0} + \int_{M} |\nabla^{T} \nabla^{T} u|^{4} \omega(t)^{n} \wedge \eta_{0}
\]

\[
\leq C(1 + \int_{M} H^{2} \omega(t)^{n} \wedge \eta_{0}).
\]

Thus

\[
\frac{d}{dt} \int_{M} H^{2} \omega(t)^{n} \wedge \eta_{0} \leq C(1 + \int_{M} H^{2} \omega(t)^{n} \wedge \eta_{0})
\]
and

\[ \int_M (\Delta_B u - |\nabla^T u|^2 + u - a)^2 \omega(t)^n \wedge \eta_0 \to 0 \]
as \( t \to \infty \). \qed

4.3. Regularity of the Limit Space and Its Smooth Convergence. Firstly, we will apply our previous results plus Cheeger-Colding-Tian structure theory for Kähler orbifolds ([CCT], [TZ]) to study the structure of desired limit spaces. Since \((M, \eta, \xi, \Phi, g, \omega)\) is a compact quasi-regular Sasakian manifold. By the first structure theorem on Sasakian manifolds, \(M\) is a principal \(S^1\)-orbibundle \((V\text{-bundle})\) over \(Z\) which is also a \(Q\)-factorial, polarized, normal projective orbifold such that there is an orbifold Riemannian submersion \(\pi: (M, g, \omega) \to (Z, h, \omega_h)\) with

\[ g = g^T + \eta \otimes \eta \]
and

\[ g^T = \pi^*(h); \quad \frac{1}{2}d\eta = \pi^*(\omega_h). \]
The orbit \(\xi_x\) is compact for any \(x \in M\), we then define the transverse distance function as

\[ d^T(x, y) \triangleq d_g(\xi_x, \xi_y), \]
where \(d\) is the distance function defined by the Sasaki metric \(g\). Then

\[ d^T(x, y) = d_h(\pi(x), \pi(y)). \]

We define a transverse ball \(B_{\xi, g}(x, r)\) as follows:

\[ B_{\xi, g}(x, r) = \{ y : d^T(x, y) < r \} = \{ y : d_h(\pi(x), \pi(y)) < r \}. \]

Note that when \(r\) small enough, \(B_{\xi, g}(x, r)\) is a trivial \(S^1\)-bundle over the geodesic ball \(B_h(\pi(x), r)\).

Based on Perelman’s non-collapsing theorem for a transverse ball along the unnormalizing Sasaki-Ricci flow, it follows that

**Lemma 6.** ([Co1 Proposition 7.2], [He Lemma 6.2]) Let \((M^{2n+1}, \xi, g_0)\) be a compact Sasakian manifold and let \(g^T(t)\) be the solution of the unnormalizing Sasaki-Ricci flow with the initial transverse metric \(g^T_0\). Then there exists a positive constant \(C\) such that for every \(x \in M\), if \(|S^T| \leq r^{-2}\) on \(B_{\xi, g(t)}(x, r)\) for \(r \in (0, r_0]\), where \(r_0\) is a fixed sufficiently small positive number, then

\[ \text{Vol}(B_{\xi, g(t)}(x, r)) \geq Cr^{2n}. \]

Moreover, the transverse scalar curvature \(R^T\) and transverse diameters \(d^T_{g^T(t)}\) are uniformly bounded under the Sasaki-Ricci flow. As a consequence, there is a uniform constant \(C\) such that

\[ \text{diam}(M, g(t)) \leq C. \]

Now we are ready to study the structure of the limit space([CHLW]):
Theorem 9. Let \((M_i, \eta_i, \xi_i, \Phi_i, g_i)\) be a sequence of quasi-regular Sasakian \((2n+1)\)-manifolds with Sasakian metrics \(g_i = g_i^T + \eta_i \otimes \eta_i\) such that for basic potentials \(\varphi_i\)
\[
\eta_i = \eta + d_B^C \varphi_i
\]
and
\[
d\eta_i = d\eta + \sqrt{-1} \partial_B \overline{\partial} B \varphi_i.
\]
We denote that \((Z_i, h_i, J_i, \omega_{h_i})\) are a sequence of well-formed normal projective orbifolds surfaces which are the corresponding foliation leave space with respect to \((M_i, \eta_i, \xi_i, \Phi_i, g_i)\) such that
\[
\omega_{g_i^T} = \frac{1}{2} d\eta_i = \pi^*(\omega_{h_i}); \Phi_i = \pi^*(J_i)
\]
Suppose that \((M_i, \eta_i, \xi_i, \Phi_i, g_i)\) is a compact smooth transverse Fano Sasakian \((2n+1)\)-manifolds satisfying
\[
\int_M |\text{Ric}_{g_i^T}|^p \omega_i^n \wedge \eta \leq \Lambda,
\]
and
\[
\text{Vol}(B_{\xi_i g_i^T}(x_i, 1)) \geq \nu
\]
for some \(p > n, \Lambda > 0, \nu > 0\). Then passing to a subsequence if necessary, \((M_i, \Phi_i, g_i, x_i)\) converges in the Cheeger-Gromov sense to limit length spaces \((M_\infty, \Phi_\infty, d_\infty, x_\infty)\) and then \((Z_i, J_i, h_i, \pi(x_i))\) converges to \((Z_\infty, J_\infty, h_\infty, \pi(x_\infty))\) such that

1. for any \(r > 0\) and \(p_i \in M_i\) with \(p_i \to p_\infty \in M_\infty\),
\[
\text{Vol}(B_{h_i}(\pi(p_i), r)) \to \mathcal{H}^{2n}(B_{h_\infty}(\pi(p_\infty), r))
\]
and
\[
\text{Vol}(B_{\xi_i g_i^T}(p_i, r)) \to \mathcal{H}^{2n}(B_{\xi_i g_\infty^T}(p_\infty, r)).
\]
Moreover,
\[
\text{Vol}(B(p_i, r)) \to \mathcal{H}^{2n+1}(B(p_\infty, r)),
\]
where \(\mathcal{H}^m\) denotes the m-dimensional Hausdorff measure.

2. \(M_\infty\) is a \(S^1\)-oribundle over the normal projective variety \(Z_\infty = M_\infty/F_\xi\).

3. \(Z_\infty = \mathcal{R} \cup \mathcal{S}\) such that \(\mathcal{S}\) is a closed singular set of codimension two and \(\mathcal{R}\) consists of points whose tangent cones are \(\mathbb{R}^{2n}\).

4. the convergence on the regular part of \(M_\infty\) which is a \(S^1\)-principle bundle over \(\mathcal{R}\) in the \((C^\alpha \cap L^2_B)\)-topology for any \(0 < \alpha < 2 - \frac{2}{p}\).

By the convergence theorem in Theorem 9 we have the regularity of the limit space: We define a family of Sasaki-Ricci flow \(g_i^T(t)\) by
\[
(M, g_i^T(t)) = (M, g^T(t_i + t))
\]
for \(t \geq -1\) and \(t_i \to \infty\) and for \(g_i^T(t) = \pi^*(h_i(t))\)
\[
(Z, h_i(t)) = (M, h_i(t_i + t)).
\]
Now for the associated transverse Ricci potential \(u_i(t)\) as in Lemma 2, we have
\[
||u_i(t)||_{C^0} + ||\nabla^T u_i(t)||_{C^0} + ||\Delta_B u_i(t)||_{C^0} \leq C.
\]
Moreover, it follows (1.14) that
\[
\int_M |\nabla^T \nabla^T u|^2_0 \omega(t)^n \wedge \eta_0 \to 0
\]
as \( i \to \infty \). Furthermore, by Theorem 6 and a convergence theorem 9, passing to a subsequence if necessary, we have at \( t = 0 \),
\[
(M, g^T_i(0)) \to (M_\infty, g^T_\infty, d^T_\infty)
\]
such that
\[
(Z, h_i(0)) \to (Z_\infty, h_\infty, d_h_\infty)
\]
in the Cheeger-Gromov sense. Moreover,
\[
(4.18) \quad (g^T_i(0), u_i(0)) \xrightarrow{C^\alpha \cap L^p_B} (g^T_\infty, u_\infty)
\]
on \((M_\infty)_{\text{reg}}\) which is a \( S^1 \)-principle bundle over \( \mathcal{R} \). The convergence of \( u_i(0) \) follows from the elliptic regularity ([Co1], [Co2]) of
\[
\Delta_B u_i(0) = n - R^T(g^T_i(0)) \in L^p_B.
\]

**Theorem 10.** Suppose \((4.18)\) holds, then \( g^T_\infty \) is smooth and satisfies the Sasaki-Ricci soliton equation
\[
(4.19) \quad \text{Ric}^T(g^T_\infty) + \text{Hess}^T(u_\infty) = g^T_\infty
\]
on \((M_\infty)_{\text{reg}}\) which is a \( S^1 \)-bundle over \( \mathcal{R} \). Moreover, \( \Phi_\infty \) is smooth and \( g^T_\infty \) is Kähler with respect to \( \Phi_\infty = \pi^* (J_\infty) \).

**Proof.** Since all \( g^T_\infty \) and \( u_\infty \) are basic, in the local harmonic coordinate \( \{ t, x^1, x^2, \cdots, x^{2n} \} \), the Sasaki-Ricci soliton equation \((4.19)\) is equivalent to
\[
(4.20) \quad (g^T)^{\alpha \beta} \frac{\partial^2 g^T_{\gamma \delta}}{\partial x^\beta \partial x^\alpha} = \frac{\partial^2 u_\infty}{\partial x^\gamma \partial x^\delta} + Q(g^T, \partial g^T)_{\gamma \delta} + T(g^{-1}, \partial g^T, \partial u)_{\gamma \delta} - g^T_{\gamma \delta}.
\]

By \((4.14)\), \((4.20)\) holds in \( L^2_B((M_\infty)_{\text{reg}}) \). But \( g^T_\infty \) and \( u_\infty \) are \( L^2_B \), then \((4.20)\) holds in \( L^p_B((M_\infty)_{\text{reg}}) \) too. On the other hand, by \((4.16a)\) and \((4.19)\), we have that
\[
(4.21) \quad g^T_{\alpha \beta} \frac{\partial u_\infty}{\partial x^\beta \partial x^\alpha} = (g^T)^{\alpha \beta} \frac{\partial u_\infty}{\partial x^\alpha} \frac{\partial u_\infty}{\partial x^\alpha} - 2 u_\infty + 2 a_\infty
\]
in \( L^p_B((M_\infty)_{\text{reg}}) \).

Then a bootstrap argument as in ([Pe]) to the elliptic systems \((4.20)\) and \((4.21)\) shows that \( g^T_\infty \) and \( u_\infty \) are smooth on \((M_\infty)_{\text{reg}}\). The elliptic regularity shows that \( \Phi_\infty \) is smooth since \( \nabla g^T_\infty \Phi_\infty = 0 \). \( \square \)

By applying the argument as in [TZ Theorem 1.2] (also [CHLW]) to the normal orbifold variety \( Z_\infty \) which is mainly depended on the Perelman’s pseudolocality theorem ([P1]), we have the smooth convergence of the Sasaki-Ricci flow on the regular set \((M_\infty)_{\text{reg}}\) which is a \( S^1 \)-principle bundle over \( \mathcal{R} \) and \( \mathcal{R} \) is the regular set of \( Z_\infty \):

**Theorem 11.** The limit \((M_\infty, d_\infty)\) is smooth on the regular set \((M_\infty)_{\text{reg}}\) which is a \( S^1 \)-principle bundle over the regular set \( \mathcal{R} \) of \( Z_\infty \) and \( d^T_\infty \) is induced by a smooth Sasaki-Ricci soliton \( g^T_\infty \) and \( g^T(t_i) \) converge to \( g^T_\infty \) in the \( C^\infty \)-topology on \((M_\infty)_{\text{reg}}\). Moreover, the singular set \( S \) of \( Z_\infty \) is the codimension two orbifold singularities.
Proof. Note that since $g^T$ is a transverse Kähler metric and basic. It is evolved by the Kähler–Ricci flow, then it follows from the standard computations in Kähler–Ricci flow (Co3, Mo) that

$$\frac{\partial}{\partial t} Rm^T = \Delta_B Rm^T + Rm^T \ast Rm^T + Rm^T.$$ 

Now by Perelman’s pseudolocality theorem ([P1], [TZ]), there exists $\varepsilon_0, \delta_0, r_0$ which depend on $p$ as in the Theorem 9 such that for any $(x_0, t_0)$, if

$$\text{Vol}(B_{\xi, g^T_{i}(t)}(x_0, r)) \geq (1 - \varepsilon_0) \text{Vol}(B(0, r))$$

for some $r \leq r_0$, where Vol$(B(0, r))$ denotes the volume of Euclidean ball of radius $r$ in $\mathbb{R}^{2n}$, then we have the following curvature estimate

$$|Rm^T|_{g^T_{i}(t)}(x, t) \leq \frac{1}{t - t_0}$$

for all $x \in B_{\xi, g^T_{i}(t)}(x_0, \varepsilon_0 r)$ and $t_0 < t \leq t_0 + \varepsilon_0^2 r^2$ and the volume estimate

$$\text{Vol}(B_{\xi, g^T_{i}(t)}(x_0, \delta_0 \sqrt{t - t_0})) \geq (1 - \eta) \text{Vol}(B(0, \delta_0 \sqrt{t - t_0}))$$

for $t_0 < t \leq t_0 + \varepsilon_0^2 r^2$ and $\eta \leq \varepsilon_0$ is the constant such that the $C^\alpha$ harmonic radius at $x_0$ is bounded below by $\delta_0 \sqrt{t - t_0}$.

As in Theorem 9, the metric $g^T_{i}(0)$ converges to $g^T_\infty$ in the $(C^\alpha \cap L^2)$-topology on $\mathcal{R}$. Now it is our goal to show that the metric $g^T_{i}(0)$ converges smoothly to $g^T_\infty$. For $0 < r \leq r_0$ and $t \geq -1$, define

$$\Omega_{r,i,t} := \{ x \in M \mid (4.22) \text{ holds on } B_{\xi, g^T_{i}(t)}(x, t) \}.$$ 

Then (4.24) implies that

$$\Omega_{r,i,t} \subset \Omega_{\delta_0 \sqrt{s} \sqrt{i},i,t+s}$$

for $0 < s \leq \varepsilon_0^2 r^2$.

Let $r_j$ to be a decreasing sequence of radii such that $r_j \to 0$ and $t_j = -\varepsilon_0 r_j$. Then by applying [TZ] (3.42), we may assume that

$$\Omega_{r_j,i,t_j} \subset \Omega_{r_{j+1},i,t_{j+1}}.$$ 

Then by (4.23)

$$||Rm^T||_{g^T_{i}(t)}(x, t) \leq \frac{1}{t - t_j}$$

for all $(x, t)$ with

$$d^T_{g^T_{i}(t)}(x, \Omega_{r_j,i,t_j}) \leq \varepsilon_0 r_j, t_j < t \leq 0.$$ 

By Shi’s derivative estimate ([Shi]) to the curvature, there exist a sequence of constants $C_{k,i,j}$ such that

$$||(\nabla^T)^k Rm^T||_{g^T_{i}(0)}(x, t) \leq C_{k,i,j}$$

on $\Omega_{r_j,i,t_j}$. Then Passing to a subsequence if necessary, one can find a subsequence $\{i_j\}$ of $\{j\}$ such that

$$\Omega_{r_j,i_j,t_j, g^T_{i_j}(t_j)} \underbrace{\Rightarrow}_{\text{C}^\alpha} (\Omega, g^T)$$
Theorem 9, we may also have
\[ (M, g^T M, t) \overset{d^T G,H}{\rightarrow} (M, d^T \infty) \]
with \( Z_\infty = \mathbb{R} \cup S \).
Then
\[ (4.27) \quad (M, g^T M, t) \overset{d^T}{\rightarrow} (M, d^T \infty) \]
and
\[ (4.28) \quad (M, g^T M, t) \overset{d^T}{\rightarrow} (M, d^T \infty) \]
Finally (4.27), (4.28), (4.29) and (4.30) imply the metric \( g^T M, t \) converges smoothly to \( g^T M, t \) on \( \mathbb{R} \). 

For the solution \((M, \omega(t), g^T M, t)\) of the Sasaki-Ricci flow and the line bundle \((K^T M, h(t) = \omega M, t)\) with the basic Hermitian metric \( h(t)\), we work on the evolution of the basic transverse holomorphic line bundle \((K^T M, h^m, t)\) for a large integer \( m \) such that \((K^T M, h^m, t)\) is very ample. We consider the basic embedding (Proposition 4) which is \( S^1 \)-equivariant with respect to the weighted \( C^* \) action in \( C^{N_m + 1} \)
\[ \Psi : M \rightarrow (\mathbb{C}P^{N_m}, \omega_{FS}) \]
defined by the orthonormal basic transverse holomorphic section \( \{\sigma_0, \sigma_1, \ldots, \sigma_N\} \) in \( H^B(M, (K^T M)^{-m}) \) with \( N_m = \dim H^B(M, (K^T M)^{-m}) - 1 \) with
\[ \int_M (\sigma_i, \sigma_j, h^m(t)\omega M(t) \wedge \eta_0 = \delta_{ij}. \]

Define
\[ (4.31) \quad F_m(x, t) := \sum_{a=0}^{N_m} ||\sigma_a||^2_{h^m(x)}. \]
Note that under these notations, the curvature form of the Chern connection is
\[ Ric(h(t)) = m\omega(t). \]
The following result is a Sasaki analogue of the partial \( C^0 \)-estimate which was obtained in the Kähler-Ricci flow ([TZ, Theorem 5.1]) and the proof there does carry over to our Sasaki setting due to the first structure theorem again on quasi-regular Sasakian manifolds. For completeness, we give a sketch for the proof here.
Theorem 12. Suppose \( (4.18) \) holds, we have

\[
\inf_{t_i} \inf_{x \in M} F_m(x, t_i) > 0
\]

for a sequence of \( m \to \infty \).

Proof. The main idea is that our basic function theory on quasi-regular Sasaki manifolds follows immediately from the standard function theory on the orbifold quotient (Proposition 2). In fact, since the leave space of the characteristic foliation \((Z, h, \omega_h)\) is a normal projective variety with codimension two orbifold singularities with \( \pi^* \omega_h = \omega \). Then, by integration over \( M \) via \((2.6)\) and \((2.5)\) as in \((5.8)\), this will give the desired estimates back to the space \((Z, h, \omega_h)\) which is the same as in Kähler case.

Here we described briefly two main ingredients for the reader’s completeness. These are the gradient estimate to plurianti-canonical basic sections and Hörmander’s \( L^2 \)-estimate to \( \partial_B \)-operator on basic \((0,1)\)-forms. In case of the Ricci curvature bounded, these estimates are standard as in \([DS]\) and \([T1]\). In our situation, due to the lack of the Ricci curvature bounded, the arguments should be modified as follows:

(i) The uniform bound of the Sobolev constant \( C_S(g_T^0, n) \) for the basic function along the Sasaki-Ricci flow was obtained as in [Co2, Theorem 1.1] :

\[
\int_M f^{2(2n+1)} \omega(t)^n \wedge \eta_0 \leq C_S(g_T^0, n) \int_M (|| \nabla_T f ||^2 + f^2) \omega(t)^n \wedge \eta_0
\]

for every \( f \in W^{1,2}_B(M) \). This makes it possible to apply the standard iteration arguments of Nash-Moser to the proper equations such as

\[
(\Delta_B (|| \nabla_T \sigma ||^2) = || \nabla_T \nabla_T \sigma ||^2 + || \nabla_T \nabla_T \sigma ||^2 - ((n+2)m-1) \nabla_T \sigma \nabla_T \sigma - < \nabla_T \nabla_T \sigma, \nabla_T \sigma >
\]

for any basic transverse holomorphic section \( \sigma \in H^0_B(M, (K_T^M)^{-m}) \). Here \( u \) is the transverse Ricci potential as before. This is can be done by the basic Bochner formula. Then we have \([TZ, (5.8)]\)

\[
|| \sigma ||_{C^0} + \sqrt{m} || \nabla_T \sigma ||_{C^0} \leq C m^{\frac{m}{2}} \int_M || \sigma ||^2 \omega(t)^n \wedge \eta_0.
\]

(ii) The \( L^2 \)-estimate to \( \overline{\partial}_B \)-operator for basic sections on quasi-regular Sasakian manifolds follows from the standard function theory on the orbifold quotient: There exists a \( m_0 \) such that for for any basic transverse holomorphic section \( \sigma \in H^0_B(M, K_T^{M-m}) \) and \( m \geq m_0 \) with

\[
\overline{\partial}_B \sigma = 0,
\]

one can find a solution \( \vartheta \)

\[
\overline{\partial}_B \vartheta = \sigma
\]

satisfying the property \([TZ, (5.9)]\)

\[
\int_M || \vartheta ||^2 \omega(t)^n \wedge \eta_0 \leq \frac{4}{m} \int_M || \sigma ||^2 \omega(t)^n \wedge \eta_0.
\]
In fact, it suffices to show that the Hodge basic Laplacian
\[ \Delta_{\bar{\partial}} = \bar{\partial}_B \bar{\partial}^*_{\bar{\partial}} B + \bar{\partial}^*_{\bar{\partial}} \bar{\partial}_B \geq \frac{m}{4} \]
on $C^0_B(M, T^{0,1} M \otimes (K^T_M)^{-m})$ for a larger $m$.

Note that when $r$ small enough, the transverse geodesic ball $B_{\xi,g}(x,r)$ is a trivial $S^1$-bundle over the geodesic ball $B_{h}(\pi(x), r)$ in $(Z, h, \omega_h, \pi(x))$ described as in Theorem 9 outside the singular set of codimension 4. Then the $L^2$-estimate to $\bar{\partial}$-operator for sections on the Kähler case can be applied for basic sections on quasi-regular Sasakian manifolds. Indeed, all the integrands are only involved with the transverse Kähler structure $\omega(t)$ and basic sections. Hence, under the Sasaki–Ricci flow, when one applies the Weitzenböck type formulae and integration by parts, the expressions involved behave essentially the same as in the Kähler–Ricci flow.

Finally, Theorem 12 follows easily from Theorem 9, Theorem 11, (4.34) and (4.35). We refer to [TZ, Theorem 5.1] and [T2] for some details. □

As a consequence of the first structure theorem for Sasakian manifolds and Theorem 12, the Gromov–Hausdorff limit $Z_\infty$ is a variety embedded in some $\mathbb{CP}^N$ and the singular set $S$ is a subvariety ([DS], [T2, Theorem 1.6]). Then one can refine the regularity of Theorem 11 as following :

**Corollary 2.** Let $(M, \xi, \eta_0, g_0)$ be a compact quasi-regular transverse Fano Sasakian manifold of dimension up to seven and $(Z_0 = M/\mathcal{F}_\xi, h_0, \omega_{h_0})$ denote the space of leaves of the characteristic foliation which is a normal projective variety with codimension two orbifold singularities. Then under the Sasaki-Ricci flow, $M_\infty$ is a $S^1$-orbibundle over $Z_\infty$ which is a normal projective variety and the singular set $S$ of $Z_\infty$ is a codimension two orbifold singularities.

Finally, our main theorem 3 follows easily from Theorem 9, Theorem 10, and Corollary 2.

5. Transverse $K$-Stability

In this section, all transverse quantities on Sasakian manifolds such as transverse Mabuchi $K$-energy, Sasaki-Futaki invariant, transverse Perelman’s $W$-functional can be viewed as their Kähler counterparts restricted on basic functions and transverse Kähler structure. Then all the integrands are only involved with the transverse Kähler structure. Hence, under the Sasaki–Ricci flow, the Reeb vector field and the transverse holomorphic structure are both invariant, and the metrics are bundle-like. Furthermore, when one applies integration by parts, the expressions involved behave essentially the same as in the Kähler case.

5.1. Transverse Mabuchi $K$-energy and Generalized Sasaki-Futaki Invariant. We recall the transverse Mabuchi $K$-energy (C1) on a compact transverse Fano Sasakian $(2n + 1)$-manifold along any basic transverse Kähler potential $\phi_s$ with $\phi_0 = \varphi_1$ and $\phi_1 = \varphi_2$ :

\[ K_{\eta_0}(\varphi_1, \varphi_2) := -\frac{1}{V} \int_0^1 \int_M (R^T_{\phi_s} - n) \omega_{\phi_s}^n \wedge \eta_{\phi_s} \wedge ds \]
and then also
\[ K_{\eta_0}(\varphi_1, \varphi_2) := -\frac{1}{V} \int_0^1 \int_M \phi_s (R^T_{\phi_s} - n) \omega_{\phi_s}^n \wedge \eta_0 \wedge ds. \]

It follows easily from the definition that

Lemma 7. ([CJ], [FOW])

1. \( K_{\eta_0} \) is independent of the path \( \phi_t \), where \( \phi_t = \frac{d\phi}{dt} \). Furthermore it satisfies the 1-cocycle condition
\[ K_{\eta_0}(\varphi_1, \varphi_2) + K_{\eta_0}(\varphi_2, \varphi_3) = K_{\eta_0}(\varphi_1, \varphi_3) \]
and
\[ K_{\eta_0}(\varphi_1 + C_1, \varphi_2 + C_2) = K_{\eta_0}(\varphi_1, \varphi_2). \]

2. For a family of transverse Kähler potentials \( \varphi = \varphi_t \) in (4.2), we have
\[ \frac{dK_{\eta_0}(\varphi)}{dt} = -\int_M ||\nabla^T u(t)||^2 \omega(t)^n \wedge \eta_0. \]

For the Hamiltonian holomorphic vector field \( V, d\pi_\alpha(V) \) is a holomorphic vector field on \( V_\alpha \) and the complex valued Hamiltonian function \( u_V := \sqrt{-1} \eta(V) \) satisfies
\[ \overline{\partial}_B u_V = -\frac{\sqrt{-1}}{2} i_V d\eta. \]

Assume we normalize that \( c_1^B(M) = \lfloor \frac{1}{2} d\eta \rfloor_B \), there exists a basic function \( h_\omega \) such that
\[ Ric^T(x, t) - \omega(x, t) = i\partial_B \overline{\partial}_B h_\omega. \]

Lemma 8. ([BGS], [FOW]) The Sasaki-Futaki invariant
\[ f_M(V) = \int_M V(h_\omega) \omega^n \wedge \eta \]
is only depends on the basic cohomology represented by \( d\eta \), and not on the particular transverse Kähler metric. It is clear that \( f_M \) vanishes if \( M \) has a Sasaki-Einstein metric in its basic cohomology class. One also have the following reformulation:
\[ f_M(V) = -n \int_M u_V(Ric^T_\omega - \omega) \omega^{n-1} \wedge \eta = -\int_M u_V(R^T_\omega - n) \omega^n \wedge \eta. \]

If \((M, \xi, \eta, g, \omega)\) is a transverse Fano quasi-regular Sasakian manifold and its leave space \( Z \) is well-formed and has at least codimension two fixed point set of every non-trivial isotropy subgroup. Then \((Z, h_\omega)\) is a normal Fano projective variety with codimension two orbifold singularities with \( \frac{1}{2} d\eta = \pi^* \omega_\h = \omega \). Then, by integration over the \( U(1) \)-Reeb fibre with the orbifold structure of \( Z \) via (2.6) and (2.5), (5.1) and (5.4) precisely reduce to the Mabuchi \( K \)-energy and the Futaki invariant on Kähler manifold or orbifold \( Z \) up to a proportional constant, respectively. Following the notions as in [FOW] and [DT], we have
Theorem 13. Let \((M, \xi, \eta, g, \omega)\) be a compact transverse Fano quasi-regular Sasakian manifold and its leave space \(Z\) be the normal Fano projective Kähler orbifold and well-formed. The Sasaki-Futaki invariant can be extended to the generalized Sasaki-Futaki invariant which has the following reformulation involving \((Z, h, \omega_h)\):  

\begin{equation}
(5.6) \quad f_M(V) = f_Z(X)
\end{equation}

with

\begin{equation}
(5.7) \quad f_Z(X) = -n \int_Z \theta_X (\text{Ric}_{\omega_h} - \omega_h) \omega_h^{n-1} = -n \int_Z \theta_X (R_{\omega_h} - n) \omega_h^n.
\end{equation}

Here \(f_Z(X)\) is the generalized Futaki invariant on \(Z\) as in \(5.7\) and the complex valued Hamiltonian function \(u_V\) on \(M\) satisfies \((5.3)\). Moreover, \(\pi_* V = X\) is the admissible holomorphic vector field \(X\) which is the restriction of some holomorphic vector field on \(\mathbb{C}P^N\) to \(Z\) and \(\pi^* \theta_X = u_V\) satisfies

\[ \overline{\partial} \theta_X = -\sqrt{-1} i_X \omega_h. \]

**Proof.** For the Hamiltonian holomorphic vector field \(V\), by definition the complex valued Hamiltonian function \(u_V := \sqrt{-1} \eta(V)\) satisfies

\[ \overline{\partial} u_V = -\frac{\sqrt{-1}}{2} i_V d\eta \]

and \(\pi_* V = X\) is an admissible holomorphic vector field on \(Z\). Therefore one can define the generalized Futaki invariant \((\text{DTI})\) on a Fano Kähler orbifold \((Z, h, \omega_h)\) by

\begin{equation}
(5.7) \quad f_Z(X) = \int_Z X(h_{\omega_h}) \omega_h^{n-1}
\end{equation}

with

\[ \text{Ric}_{\omega_h} - \omega_h = i \overline{\partial} h_{\omega_h}. \]

Moreover, since \(\frac{1}{2} d\eta = \pi^* \omega_h = \omega\), then for a smooth function \(\theta_X\) with \(\pi^* \theta_X = u_V\), the Futaki invariant was extended to a Fano Kähler orbifold \(Z\) \((\text{DT}, \text{T5})\) as follows:

\begin{equation}
(5.7) \quad f_Z(X) = -n \int_Z \theta_X (\text{Ric}_{\omega_h} - \omega_h) \omega_h^{n-1}
\end{equation}

with

\[ \overline{\partial} \theta_X = -\sqrt{-1} i_X \omega_h. \]

Furthermore, by \((2.6)\) and notation as above, let \(G_i\) be the local uniformizing finite group acting on a smooth complex space \(\tilde{U}_i\) such that the local uniformizing group injects into \(U(1)\) and the map

\[ \varphi_i : U(1) \times \tilde{U}_i \rightarrow U_i \]
be exactly $|G_i|$-to-one on the complement of the orbifold locus as in [CZ1] and [CHLW] section 6. Then

\begin{equation}
- n \int \theta_X (\text{Ric}_\omega - \omega_h) \omega_h^{n-1} = - n \sum_i \frac{1}{|G_i|} \int_{U_i} \theta_X \varphi_i (\text{Ric}_\omega - \omega_h) \omega_h^{n-1}
\end{equation}

\begin{align*}
&= - n \sum_i \frac{1}{|G_i|} \int_{U_i \times U_i} \pi^* \theta_X \pi^* \varphi_i \pi^* [(\text{Ric}_\omega - \omega_h) \omega_h^{n-1}] \wedge \eta \\
&= - n \sum_i \int_{U_i} u \pi^* \varphi \pi^* [(\text{Ric}_\omega - \omega_h) \omega_h^{n-1}] \wedge \eta \\
&= - n \int_M u \pi^* [(\text{Ric}_\omega - \omega_h) \omega_h^{n-1}] \wedge \eta \\
&= - n \int_M u \pi^* (\text{Ric}_\omega - \omega) \omega^{n-1} \wedge \eta.
\end{align*}

Then we are done. \qed

**Remark 3.** Note that the set of all Hamiltonian holomorphic vector fields is a Lie algebra. Furthermore in the paper of [NT], they proved that if the transverse scalar curvature $R^T$ is constant, then it is reductive which extending Lichnerowicz-Matsushima theorem in the Kähler case.

### 5.2. Transverse Perelman’s $W$-functional

We recall the transverse Perelman’s $W$-functional on a compact Sasakian $(2n+1)$-manifold:

\begin{equation}
W^T(g^T, f, \tau) = (4\pi\tau)^{-n} \int_M (\tau (R^T + |\nabla^T f|^2) + f - 2n) e^{-f} \omega^n \wedge \eta_0,
\end{equation}

for $f \in C^\infty_B(M; R)$ and $\tau > 0$ and define

$$\lambda^T(g^T, \tau) = \inf \{ W^T(g^T, f, \tau) : f \in C^\infty_B(M; R); \int_M (4\pi\tau)^{-n} e^{-f} \omega^n \wedge \eta_0 = 1 \}.$$

**Lemma 9.** ([Co1])

1. $-\infty < \lambda^T(g^T, \tau) \leq C.$
2. There exists $f_\tau \in C^\infty_B(M; R)$ so that $W^T(g^T, f_\tau, \tau) = \lambda^T(g^T, \tau).$ That is,

$$\lambda^T(g^T, \tau) = \inf \{ W^T(g^T, f, \tau) : f \in W^{1,2}_B; \int_M (4\pi\tau)^{-n} e^{-f} \omega^n \wedge \eta_0 = 1 \}.$$

Note that $W$-functional can be expressed as

$$W^T(g^T, f) = W^T(g^T, f, \frac{1}{2}) + (2\pi)^{-n}(2n)V
= (2\pi)^{-n} \int_M (\frac{1}{2}(R^T + |\nabla^T f|^2) + f) e^{-f} \omega^n \wedge \eta_0,$$

where $(g^T, f)$ satisfies $\int_M e^{-f} \omega^n \wedge \eta_0 = V$ and $\tau = \frac{1}{2}$. Again

$$\mu^T(g^T) = \inf \{ W^T(g^T, f) : f \in C^\infty_B(M; R) \text{ with } \int_M e^{-f} \omega^n \wedge \eta_0 = V \}.$$
It follows from Lemma 9 (also [He]) that

**Corollary 3.**  
(1) \( \lambda^T(g^T) \) can be attained by some \( f \) which satisfies the Euler-Lagrange equation:

\[
\Delta_B f + f + \frac{1}{2}(R^T - \| \nabla^T f \|^2) = \mu^T(g^T)
\]

and

\[
\delta \mu^T(g^T) = -\int_M <\delta g^T, \text{Ric}^T - g^T + \nabla^T \nabla^T f > e^{-f} \omega^n \wedge \eta_0.
\]

(2) \( g^T \) is a critical point of \( \mu^T(g^T) \) if and only if \( g^T \) is a gradient shrinking Sasaki-Ricci soliton

\[
\text{Ric}^T + \nabla^T \nabla^T f = g^T
\]

where \( f \) is a minimizer of \( W^T(g^T, \cdot) \).

It follows from (5.11) that

**Corollary 4.** Let \((M, \xi, \eta_0, g_0)\) be a compact quasi-regular transverse Fano Sasakian \((2n+1)\)-manifold and \((Z_0 = M/\mathcal{F}_\xi, h_0, \omega_{h_0})\) denote the space of leaves of the characteristic foliation which is well-formed normal projective variety with codimension two orbifold singularities \( \Sigma_0 \). Then, under the Sasaki-Ricci flow,

\[
\frac{d}{dt} \mu^T(g^T_t) = \int_M \| (\text{Ric}^T_{g^T_t} - g^T_t + \nabla^T \nabla^T f_t) \|^2_{g^T_t} e^{-f_t} \omega^n (g^T_t) \wedge \eta_0.
\]

Here \( f_t \) are minimizing solutions of (5.10) associated to metrics \( g^T_t \) and \( \sigma_t \) is the family of transverse diffeomorphisms of \( M \) generated by the time-dependent vector field \( \frac{1}{2} \nabla^T_{g^T_t} f_t \).

Next, we state some a priori estimates for the minimizing solution \( f_t \) of (5.10) under the Sasaki-Ricci flow. It can be viewed as their Kähler counterparts restricted on basic functions and transverse Kähler structure, etc. We refer to the details estimates as in [TZhu2, Proposition 4.2] and [PSSWe].

At first, we can improve the estimates as in Lemma 2 if the transverse Mabuchi K-energy is bounded from below.

**Lemma 10.** If in addition the transverse Mabuchi K-energy is bounded from below on the space of transverse Kähler potentials as in Lemma 2, then we have

(1) \[
\lim_{t \to \infty} |u(t)|_{C^0(M)} = 0,
\]

(2) \[
\lim_{t \to \infty} \| \nabla^T u(t) \|_{C^0(M, g^T_t)} = 0,
\]

(3) \[
\lim_{t \to \infty} |\Delta_B u(t)|_{C^0(M)} = 0.
\]

As a consequence, we have
Corollary 5. If the transverse Mabuchi $K$-energy is bounded from below on the space of transverse Kähler potentials as in Lemma 2, then there exists a sequence of $t_i \in [i, i + 1]$ such that

1. \[ \lim_{t_i \to \infty} | \Delta_B f_{t_i} |_{L^2_B(M, g_{t_i}^T)} = 0, \]
2. \[ \lim_{t_i \to \infty} \left\| \nabla_T f_{t_i} \right\|_{L^2_B(M, g_{t_i}^T)} = 0, \]
3. \[ \lim_{t_i \to \infty} \int_M f_{t_i} e^{-f_{t_i}} \omega_{g_{t_i}^T}^n \wedge \eta_0 = 0. \]

Moreover, we have

\[ \lim_{t \to \infty} \mu^T(g^T_t) = (2\pi)^{-n}(2n)V = \sup \{ \mu^T(\tilde{g}^T) : \tilde{g}^T \in c_1^B(M) \}. \]

5.3. Transverse $K$-Stability. Following the notions as in [T3] and [T5], we define the Sasaki analogue of a $K$-stable Fano Kähler manifold on a compact quasi-regular transverse Fano Sasakian $(2n + 1)$-manifold $(M, \xi, \eta_0, g_0)$ with the space $(Z = M / F, h_0, \omega_{h_0})$ of leaves of the characteristic foliation which is well-formed. Then $Z$ is a normal projective variety with codimension two orbifold singularities ([CLW]). In general, there is a so-called orbifold stability in a compact quasi-regular Sasakian manifold in the sense of Ross-Thomas ([RT]). We also refer to [CZ1, section 3] for the application.

By applying first structure theory for a quasi-regular Sasakian manifold, there exists a Riemannian submersion, $S^1$-orbibundle $\pi : M \to Z$, such that

\[ K^T_M = \pi^*(K^\text{orb}_Z) = \pi^*(\varphi^*K_Z). \]

Then by the CR Kodaira embedding theorem ([RT], [HLM]), there exists an embedding

\[ \Psi : M \to (\mathbb{CP}^N, \omega_{FS}) \]

defined by the basic transverse holomorphic section \( \{ s_0, s_1, \ldots, s_N \} \) of $H^0_B(M, (K^T_M)^{-m})$ which is $S^1$-equivariant with respect to the weighted $C^*$-action in $\mathbb{C}^{N+1}$ with $N = \dim H^0_B(M, (K^T_M)^{-m}) - 1$ for a large positive integer $m$. In fact, in our situation $Z$ is also normal Fano, there is an embedding

\[ \psi_{|mK^T_Z|^{-1}} : Z \to \mathcal{P}(H^0(Z, K_Z^{-m})). \]

Define

\[ \Psi_{|m(K^T_Z)^{-1}|} = \psi_{|mK^T_Z|^{-1}} \circ \pi \]

such that

\[ \Psi_{|m(K^T_Z)^{-1}|} : M \to \mathcal{P}(H^0_B(M, (K^T_M)^{-m})). \]

We define

\[ Diff^T(M) = \{ \sigma \in Diff(M) \mid \sigma_*\xi = \xi \quad \text{and} \quad \sigma^*g^T = (\sigma^*g)^T \} \]

and

\[ SL^T(N + 1, \mathbb{C}) = SL(N + 1, \mathbb{C}) \cap Diff^T(M). \]
Any other basis of \( H^0_B(M, (K^T_M)^{-m}) \) gives an embedding of the form \( \sigma^T \circ \Psi_{|m(K^T_M)^{-1}|} \) with \( \sigma^T \in SL^T(N + 1, \mathbb{C}) \). Now for any subgroup of the weighted \( \mathbb{C}^* \)-action \( G_0 = \{ \sigma^T(t) \}_{t \in \mathbb{C}^*} \), of \( SL^T(N + 1, \mathbb{C}) \), there is a unique limiting \[ M_\infty = \lim_{t \to 0} \sigma^T(t)(M) \subset \mathbb{C}P^N. \]

As our application of Theorem 3, \( M_\infty \) has its leave space \( Z_\infty = M_\infty / \mathcal{F}_t \) which is a normal projective Kähler orbifold with at worst codimension two orbifold singularities \( \Sigma_\infty \).

Let \( V \) be a the Hamiltonian holomorphic vector field whose real part generates the action by \( \sigma^T(e^{-s}) \). As the previous discussion and Theorem 13, if \( Z_\infty \) is normal Fano, there is a generalized Futaki invariant defined by \( f_{Z_\infty}(X) \) and then a generalized Sasaki-Futaki invariant defined by \( f_{M_\infty}(V) \) as in (5.4), (5.6) and (5.7). Thus one can introduce the Sasaki analogue of the \( K \)-stable on Kähler manifolds ([13], [15], [14]):

**Definition 6.** Let \((M, \xi, \eta, g, \omega)\) be a compact transverse Fano quasi-regular Sasakian manifold and its leave space \((Z, h, \omega_0)\) be a normal Fano projective Kähler orbifold and well-formed. We say that \(M\) is transverse \( K \)-stable with respect to \((K^T_M)^{-m}\) if the generalized Sasaki-Futaki invariant

\[ \text{Re} f_{M_\infty}(V) \geq 0 \quad \text{or} \quad \text{Re} f_{Z_\infty}(X) \geq 0 \]

for any weighted \( \mathbb{C}^* \)-action \( G_0 = \{ \sigma^T(t) \}_{t \in \mathbb{C}^*} \) of \( SL^T(N + 1, \mathbb{C}) \) with a normal Fano \( Z_\infty = M_\infty / \mathcal{F}_t \) and the equality holds if and only if \( M_\infty \) is transverse biholomorphic to \( M \). We say that \(M\) is transverse \( K \)-stable if it is transverse \( K \)-stable for all large positive integer \( m \).

With Definition 6 in mind, we are ready to show that the transverse Mabuchi \( \overline{K} \)-energy is bounded from below under the Sasaki-Ricci flow. This is served as the Sasaki analogue of the Kähler-Ricci flow due to Tian-Zhang ([13]).

**Theorem 14.** Let \((M, \xi, \eta_0, g_0)\) be a compact quasi-regular transverse Fano Sasakian manifold of dimension up to seven and \((Z_0 = M / \mathcal{F}_t, h_0, \omega_0)\) denote the space of leaves of the characteristic foliation which is a normal projective Kähler orbifold with codimension two orbifold singularities \( \Sigma_0 \). If \(M\) is transverse \( K \)-stable, then the transverse Mabuchi \( \overline{K} \)-energy is bounded from below under the Sasaki-Ricci flow

\[ K_{\eta_0}(\omega_0, \omega_t) \geq -C(g_0). \]

**Proof.** First it follows from (5.2) that \( K_{\eta_0}(\omega_0, \omega_t) \) is non-increasing in \( t \). So it suffices to show a uniform lower bound of

\[ K_{\eta_0}(\omega_0, \omega_t) \geq -C. \]

Now if \(M\) is transverse \( K \)-stable, we fix an integer \( m > 0 \) sufficiently large such that \((L^T)^{-m}\) is very-ample. Then for any orthonormal basic basis \( \{ \sigma_{t, i, m, k} \}_{k = 0}^{N_m} \) in \( H^0_B(M, (K^T_M)^{-m}) \) with \( N_m = \dim H^0_B(M, (K^T_M)^{-m}) - 1 \) at \( t_i \), we can define the \( S^1 \)-equivariant embedding ([CHLW]) with respective the weighted \( \mathbb{C}^* \)-action in \( \mathbb{C}P^{N_m+1} \)

\[ \Psi_i : M \to (\mathbb{C}P^{N_m}, \omega_{FS}) \]
with the Bergman metric \( \overline{\omega}_i := \overline{\omega}_i = \frac{1}{m} \Psi_i^*(\omega_{FS}) \) so that for any \( i \geq 1 \), there exists a \( G_i \in SL^T(N_m + 1, C) \) such that

\[
\Psi_i = G_i \circ \Psi_1.
\]

At first as in the Kähler case ([Paul1], [TZ]), the Sasaki analogue of Mabuchi K-energy will have a lower bound on \( \overline{\omega}_i \) with a fixed \( \overline{\omega}_1 \)

\[
K_{\eta_0}(\overline{\omega}_1, \overline{\omega}_i) \geq -C.
\]

By the 1-cocycle condition of the transverse Mabuchi K-energy,

\[
K_{\eta_0}(\omega_0, \omega_i) + K_{\eta_0}(\omega_i, \overline{\omega}_i) = K_{\eta_0}(\omega_0, \overline{\omega}_i) = K_{\eta_0}(\omega_0, \overline{\omega}_1) + K_{\eta_0}(\overline{\omega}_1, \overline{\omega}_i)
\]

and then

\[
K_{\eta_0}(\omega_0, \omega_i) + K_{\eta_0}(\omega_i, \overline{\omega}_i) \geq -C.
\]

Hence, to show (5.16), we only need to get an upper bound for

(5.17)

\[
K_{\eta_0}(\omega_i, \overline{\omega}_i) \leq C.
\]

For a fixed \( m \), we define

\[
\rho_i(x) := \frac{1}{m} \rho_{t_i,m}(x) := \frac{1}{m} F_{m}(x, t_i).
\]

Here \( F_m(x, t_i) \) as in (4.31). Then

\[
\omega_i = \overline{\omega}_i + \sqrt{-1} \partial_B \overline{\partial_B} \overline{\rho}_i.
\]

It follows from [T4] that the transverse Mabuchi K-energy has the following explicit expression

\[
K_{\eta_0}(\omega_i, \overline{\omega}_i) = \int_M \log \frac{\overline{\omega}_i^n}{\overline{\omega}_i^n} \wedge \eta_0 + \int_M u(\overline{\omega}_i^n - \omega_i^n) \wedge \eta_0
\]

\[
- i \sum_{k=0}^{n-1} \frac{n-k}{n+1} \int_M (\partial_B \overline{\rho}_i \wedge \overline{\partial_B} \overline{\rho}_i \wedge \omega_i^k \wedge \overline{\omega}_i^{n-k-1}) \wedge \eta_0
\]

\[
\leq \int_M \log \frac{\overline{\omega}_i^n}{\overline{\omega}_i^n} \wedge \eta_0 + \int_M u(\overline{\omega}_i^n - \omega_i^n) \wedge \eta_0
\]

\[
\leq \int_M \log \frac{\overline{\omega}_i^n}{\overline{\omega}_i^n} \wedge \eta_0 + C.
\]

Here \( u \) is the transverse Ricci potential under the Sasaki-Ricci flow and we have used the Perelman estimate that

\[
|u(t_i)| \leq C.
\]

On the other hand, it follows from (4.31) and (4.32) that

\[
\overline{\omega}_i \leq C \omega_i.
\]

Therefore (5.18) implies (5.17) and then we are done. \( \square \)
Corollary 6. Let \((M, \xi, \eta_0, g_0)\) be a compact quasi-regular transverse Fano Sasakian manifold of dimension up to seven and \((Z_0 = M/F_\xi, h_0, \omega_{h_0})\) be the space of leaves of the characteristic foliation which is a normal Fano projective Kähler orbifold with codimension two orbifold singularities. If \(M\) is transverse K-stable, then under the Sasaki-Ricci flow, \(M(t)\) converges to a compact transverse Fano Sasakian manifold \(M_\infty\) which is isomorphic to \(M\) endowed with a smooth Sasaki–Einstein metric.

Proof. Firstly, it follows from Corollary 5, (5.15) and (5.13) that \(M_\infty\) must be Sasaki-Einstein. Moreover, the Lie algebra of all Hamiltonian holomorphic vector fields is reductive ([DT], [Ber], [FOW]). If \(M_\infty\) is not equal to \(M\), there is a family of the weighted \(\mathbb{C}^*\) action \(\{G(s)\}_{s \in \mathbb{C}^*} \subset SL^T(N_m + 1, \mathbb{C})\) such that

\[
\Psi_s(M) = G(s) \circ \Psi_1(M)
\]

converges to the \(S^1\)-equivariant embedding of \(M_\infty\) with respect to the weighted \(\mathbb{C}^*\) action in \((\mathbb{C}P^N, \omega_{FS})\). Then the generalized Sasaki-Futaki invariant \(f_{M_\infty}(V)\) of \(M_\infty\) vanishes as in (5.4) and (5.5).

On the other hand, by the assumption that \(M\) is transverse K-stable, if \(M_\infty\) is not equal to \(M\), then

\[
\text{Re } f_{M_\infty}(V) > 0.
\]

This is a contradiction. Hence \(M_\infty = M\). Therefore there is a Sasaki-Einstein metric on \(M\). \(\square\)

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