Unitary theories in the work of Mira Fernandes (beyond general relativity and differential geometry)

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Abstract
An analysis of the work of Mira Fernandes on unitary theories is presented. First it is briefly mentioned the Portuguese scientific context of the 1920s. A short analysis of the extension of Riemann geometries to new generalized geometries with new affine connections, such as those of Weyl and Cartan, is given. Based on these new geometries, the unitary theories of the gravitational and electromagnetic fields, proposed by Weyl, Eddington, Einstein, and others are then explained. Finally, the book and one paper on connections and two papers on unitary theories, all written by Mira Fernandes, are analyzed and put in context.

1 Introduction

1.1 Mira Fernandes background and the Portuguese context
Aureliano de Mira Fernandes, born in 1884 in Portugal, was professor of differential and integral calculus, rational mechanics, and other lecture courses in mathematics, at Instituto Superior Técnico (IST), from its foundation in 1911, onwards, until his retirement. IST was situated provisionally at Rua do Instituto Industrial (by Rua Conde Barão) near the river Tejo. Provisionally means until the end of the 1930s, when IST moved to the place where

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it is now, in the middle of the town. He was also professor of mathematical analysis at the Instituto Superior de Ciências Económicas e Financeiras (ISCEF), what is now the Instituto Superior de Economia e Gestão (ISEG).

Mira Fernandes has a vast work in theoretical physics and mathematics, his complete works have now been published [1, 2, 3]. Mira Fernandes, by formation, was a mathematician not a physicist. His Doctoral dissertation in 1911, supervised by Sidónio Pais, on “Galois theory”, was submitted when he was 27 years old [4] (see also [5]). From his dissertation to 1924, when he was 40 years old, there are no publications. From 1924 onwards there are many publications on several subjects, namely, group theory, differential geometry, unitary theories and rational mechanics. There is no direct explanation for this 13-year gap in publications, the only reasonable one is that during those 13 years he was busy in preparing the lectures he had to deliver as well as acquainting himself with the new subjects he was interested. The most important papers were published in Rendiconti della Accademia dei Lincei, due to his friendship with Levi-Civita, the great Italian mathematical physicist. After Levi-Civita’s compulsory retirement, Mira Fernandes published mainly in Portuguese journals.

He corresponded heavily with Levi-Civita (see [6]) and also corresponded with Élie Cartan. Cartan in his work “Les espaces de Finsler” [7] writes (my translation) “It is after an exchange of letters with M. Aurelio (sic) de Mira-Fernandes, that I have perceived of the possibility of this simplification”. This means he had relations of value to him and to his country. He also corresponded with Portuguese mathematicians [8].

At the time, in mathematics in Portugal, there was the towering figure of Gomes Teixeira in Porto, a worldwide recognized mathematician with works in the theory of curves and surfaces. Also in Porto, there was Leonardo Coimbra, a philosopher who occasionally wrote on physics. In Lisbon, Mira Fernandes had no peer. He was a member of the Lisbon Academy of Sciences from 1928 onwards. In 1932 he proposed Levi-Civita and Einstein to be foreign members of the Academy, a proposal immediately accepted by the President of the Academy, Egas Moniz, the future Nobel prize in medicine. These proposals by him were apt, since these two figures were pioneers in differential and Riemann geometry, general relativity and unitary theories, areas for which Mira Fernandes devoted a great part of his scientific life. In these matters there was some interest by some community in Portugal, although mostly dilettante, the exceptions being António Santos Lucas, a
professor in the Faculdade de Ciências of Lisbon, who delivered lectures on general relativity there, and perhaps some other instances, although it seems that Eddington’s expedition to Príncipe in 1919 to observe the light shift due to the gravitational field of the sun, was not greeted with enthusiasm and interest by the Portuguese scientific community. For the details of the scientific context in Portugal in Mira Fernandes’ time, see the excellent studies in [9, 10] (see also [11]).

1.2 Aim and plan of work

In this article we will study Mira Fernandes works related to unitary theories, what are now called theories of unification. These theories tried to unify the gravitational and the electromagnetic fields, the two known fields at the time, into a single field. Since these theories are related to the theory of connections in differential geometry, a theme that was dear to Mira, we also review his works on the theory of connections.

The plan of the work is as follows. Above we outlined the scientific context of the epoch in which Mira Fernandes was immersed. In section 2 we will outline the scientific context of the unitary theories of the time, a time that spans from about 1916, the year of the creation of general relativity, to about 1934, the year of the last work of Mira Fernandes on the subject. Some generalities related to general relativity and Riemann geometry, the geometry on which the theory is based will be laid out. The extension of Riemann geometry to Weyl geometry, the first unification scheme in this context proposed by Weyl himself, and its potential development performed by Eddington, will be revised. The extension of Riemann geometry to include torsion given by Cartan, will also be mentioned. We will display the spectrum of unitary theories based on the different geometries and connections of the time, and mention the various unsuccessful attempts made by Einstein to find the true unitary theory. We will also refer to a field, the $C$-field, which makes a bridge between the contravariant and the covariant vectors and tensors. Then in section 3 we will delve into Mira Fernandes’ works. First we will analyze his works on connections, namely, the book and the 1931 paper in Rendiconti dei Lincei, and we will comment on them. Finally we will study his two works of 1932 and 1933, also published in the same journal, which apply the theory of connections to some unitary theories of gravitation and electromagnetism. In these works Mira Fernandes finds
an application for the $C$-field as the physical field of electromagnetism in the unitary theory of Straneo. This is unique. In section 4 we conclude, commenting on other interesting works of Mira in this conjunction, and on the fate of the unitary theories. Finally, the main sources to write this article on history of unitary theories and Mira Fernandes will be discussed, and the motivations for writing it plus the acknowledgments will be given. The text is thus divided as,

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2 The scientific context of unitary theories

2.1 Generalities and general relativity (1916)

The idea of unification in physics is an old one. One of the first attempts to unify fields and particles in the same scheme was provided by Mie in 1912 [12]. In [13] an English translation of the original paper, as well as of other important papers with a complete set of comments is given. An important follow up of this idea came later through the work of Born and Infeld in 1934 [14], who implemented this type of unification by modifying the Maxwell Lagrangian, providing a non-linear extension with particle solutions of the Maxwell equations. Nordström in 1914 [15] tried a different type of unification, not of fields and particles that generate the fields themselves, but a unification of the different fields. At the time there were two known fields, the gravitational and the electromagnetic. In his attempt to unify his theory of gravitation, a scalar one, with the Maxwell electromagnetic theory, Nordström used a fifth spatial dimension, being thus the precursor of the Kaluza-Klein theories.

The appearance of general relativity in 1916 [16], inspired new forms of unification. For instance Hilbert [17] tried to use Mie’s ideas [12] in conjunction with general relativity to produce a theory of particles and fields. Soon, from its beautiful structure based on Riemann geometry, general relativity would further lead to many other unification schemes. To start with, general relativity put the gravitational field in a special relativity framework. However, it left electromagnetism out. Defining $G_{\alpha\beta}$, $F_{\alpha\beta}$, $\tau_{\alpha\beta}^{em}$, and $j_{\alpha}$, as the Einstein tensor, the Maxwell tensor, the electromagnetic energy-momentum tensor, and the electric current, respectively, one may still argue that since the Einstein-Maxwell equations lead to $G_{\alpha\beta} = 8\pi \tau_{\alpha\beta}^{em}$ and $F_{\alpha\beta} = j_{\alpha}$ (Newton’s constant $G = 1$, and the velocity of light $c = 1$), there is a sense of unification. This was put forward by Rainich in 1924 [18] and continued by Misner and Wheeler in 1954 [19] in what they called an already unified theory. But those in pursue of unification wanted more.

The argument for the unification went as follows, see Figure 1. The electric and magnetic fields had been unified into the electromagnetic field, later shown that the whole unified scheme was consistent only using special relativity and the corresponding spacetime arena. Thus, one might have argued, gravity (and so general relativity) and electromagnetism, the two
known fields of the time, should be unifiable in a unitary theory using some special world background as the correct arena. This was advocated by many, in particular by Eddington, see [20]. What this special world background could be was left imprecise. This rationale works if one considers general relativity as a field theory, on the same footing of electromagnetic theory. But even this is controversial. Is general relativity a field or is it an arena as special relativity?

![Figure 1](image)

Here, we simply note that from 1916 onwards unification schemes have always been forefront problems.

### 2.2 Weyl geometry and Weyl theory of gravitation and electromagnetism (1918)

The first attempt to unify gravitation and electromagnetism was proposed by Weyl in 1918 [21, 22]. In this theory the electromagnetic potential is introduced as a geometrical quantity which determines the transport law of a length scale. The idea can be decomposed into two parts. First, one has to develop a new geometry, which in turn embodies the Riemann geometry, second one has to set up a physical theory of gravitation and electromagnetism
which in a particular instance yields the Einstein-Maxwell equations. Let us analyze first the Weyl geometry, then the Weyl theory.

In Weyl geometry the transport of a vector $\xi$ with components $\xi^\alpha$, is given, as in Riemann geometry, by the equation

$$\delta \xi^\alpha = \Gamma^\alpha_{\beta\gamma} dx^\beta \xi^\gamma,$$

where $\delta \xi^\alpha$ is the change of the vector under transport, $\Gamma^\alpha_{\beta\gamma}$ is the connection, and $dx^\beta$ is a local displacement. The difference to the Riemann geometry is that the connection $\Gamma^\alpha_{\beta\gamma}$ is not given by the Christoffel symbols $\{^\alpha_{\beta\gamma}\}$, composed of the metric alone, but is more general. The idea of a general transport, independent of the metric, had been developed at about this time by Levi-Civita and others [23] (see also [24] for the ideas of Weyl in relation to differential geometry and transport laws). Now, comes the new geometrical requirement. At a point the length of a vector $\xi^\alpha$ is given by $l^2 = g_{\alpha\beta} \xi^\alpha \xi^\beta$, where $g_{\alpha\beta}$ is a symmetric metric. For Weyl the length can change under transport as

$$\delta l = \phi_\beta dx^\beta l$$

in analogy with equation (1), and where $\phi_\beta$ is a new field. With the two requirements given by Eqs. (1)-(2) one can deduce after some algebra, (see, e.g., [25]) two things. One, that the Weyl connection $\Gamma^\alpha_{\beta\gamma}$ should be given in terms of the metric $g_{\alpha\beta}$ and the field $\phi_\beta$ as

$$\Gamma^\alpha_{\beta\gamma} = \{^\alpha_{\beta\gamma}\} + g^{\sigma\rho} (g_{\sigma\beta} \phi_\gamma + g_{\sigma\gamma} \phi_\beta - g_{\beta\gamma} \phi_\sigma).$$

The other is that the covariant derivative of the metric $g_{\mu\nu;\alpha}$ is given by

$$g_{\mu\nu;\alpha} = \phi_\alpha g_{\mu\nu},$$

where a semicolon denotes a covariant derivative. Now, in this more general geometry, the Riemann tensor is decomposed into two parts $R_{\alpha\beta\gamma\delta} = K_{\alpha\beta\gamma\delta} + T_{\alpha\beta\gamma\delta}$, where $K_{\alpha\beta\gamma\delta}$ is the Riemann-Christoffel curvature made of Christoffel symbols only, and $T_{\alpha\beta\gamma\delta}$ is the $\phi$-dependent curvature. Several other important conclusions can be drawn from this new geometry. Perform the following transformations, $g_{\alpha\beta}$ goes into $\hat{g}_{\alpha\beta} = f(x^\lambda) g_{\alpha\beta}$ and $\phi_\alpha$ into $\hat{\phi}_\alpha = \phi_\alpha + \frac{1}{2} (\log f)_\alpha$, for some function $f(x^\lambda)$, a coma denoting simple derivative. Then with the help of Eq. (3) one can work out that $\Gamma^\alpha_{\beta\gamma} = \hat{\Gamma}^\alpha_{\beta\gamma}$. 

This set of transformations forms the Weyl group. Since the gauge of the length can be changed under these transformations, but not the transport law (given by the $\Gamma^\alpha_{\beta\gamma}$), one says that the geometry is invariant under gauge transformations. Other points worth remarking is that angles between vectors and ratios of lengths are preserved under the Weyl transport, and the light cone structure too. On the other hand local lengths change as $l^2 = f(x^\lambda)l^2$. Note also the interesting result that if $\phi_\alpha = 0$ then the geometry is Riemannian.

The condition to be Riemannian is that $\phi_{\alpha;\beta} - \phi_{\beta;\alpha} = 0$ (indeed, under closed transport the length changes by $\oint_C \frac{dl}{l} = \oint_C \phi_\alpha dx^\alpha$, and this is zero if and only if $\phi_{\alpha;\beta} - \phi_{\beta;\alpha} = 0$). Recall that when $R_{\alpha\beta\gamma\delta} = 0$ there is no change in direction of the transported vector along a closed path. Thus, the quantity $F_{\alpha\beta}$ defined by $F_{\alpha\beta} = \phi_{\alpha;\beta} - \phi_{\beta;\alpha} = 0$ is, in this context, analogous to the Riemann tensor $R_{\alpha\beta\gamma\delta}$, in that when $F_{\alpha\beta} = 0$ there is no change in length of the transported vector along a closed path. Moreover, there are further analogies between both tensors. For instance, the tensor $F_{\alpha\beta}$ possesses symmetries, with some affinities to the Riemann tensor symmetries. They are, $F_{\alpha\beta} = -F_{\beta\alpha}$, and $F_{\{\alpha\beta;\gamma\}} = 0$.

Having established a geometry in which directions and lengths have similar behaviors in relation to transport, Weyl made the first attempt to unify gravitation in the form of general relativity, and electromagnetism in the form of Maxwell theory. His idea was, given that the curvature tensor and its contractions provide a basis for a physical picture of tidal forces and gravitation as in general relativity, an extended geometry with its new connection and field $\phi_\alpha$ can provide a basis for a gravitoelectromagnetic unitary theory. Similarly to having a Riemann geometry and proposing a theory based on it as Einstein did for general relativity, Weyl proposed a theory based on his own geometry. He looked for an action, invariant under coordinate and gauge transformations, and found that an Einstein-Hilbert term, proportional to the Ricci scalar $R$, was not gauge-invariant and would not do. Thus, he had to resort to an $R^2$ term, the full action being,

$$ S = \int \left( R^2 + a F_{\alpha\beta} F^{\alpha\beta} \right) \sqrt{-g} \, d^4 x, \quad (5) $$

where $a$ is a coupling constant, and $g$ is the determinant of the metric. Applying a careful variational procedure one finds that the equations governing the Weyl theory are (see, e.g., [25])

$$ G_{\alpha\beta} = 8\pi \tau_{\alpha\beta}, \quad (6) $$

8
and

\[ F^{\alpha\beta} = j^\alpha, \quad (7) \]

where \( G_{\alpha\beta} \) is the Einstein tensor related to the Christoffel connection alone, \( F^{\alpha\beta} \) is the Maxwell tensor as above, and \( \tau_{\alpha\beta} \) and \( j^\alpha \) are the corresponding energy-momentum tensor and charge current, respectively, constructed from within the theory alone. Thus, Weyl was able to reproduce Einstein’s and Maxwell’s equation within a single scheme. However, when confronted with observations the theory does not hold, it must be rejected on fundamental physical grounds, as pointed out first by Einstein (see, e.g., [25]). Indeed, since the length of objects as well as intervals of time of particle trajectories depend on the paths taken and thus on their past history, one should observe that atoms arriving at the earth from different distances in the cosmos would have different physical properties, which we do not observe. In spite of this demolishing problem, Weyl’s idea of gauging was one of the most fruitful ideas in the history of physics. London [26] tried first to apply the gauge ideas of Weyl to quantum mechanics. Then Weyl himself [27] understood that instead of gauging the metric tensor, he could gauge the quantum mechanical wave function \( \psi \) by a phase \( \psi \rightarrow \lambda \psi \) with \( \lambda = e^{ie \int A_\mu dx^\mu} \) and couple it to electromagnetism by changing the normal derivative to a covariant derivative \( \partial_\mu \rightarrow \nabla_\mu = \partial_\mu - ieA_\mu \), where \( e \) is the electric charge. These transformations should have been called phase transformations, but due to the similarity with Weyl’s previous work the name of 1918 stuck, see [28] for this wonderful story.

Notwithstanding its problems in relation to unification of gravitation and electromagnetism the door to unification schemes was open. There is a Brazilian saying that says “Onde passa um boi, passa uma boiada” (Where one ox passes a herd of oxen passes). It applies neatly here.

### 2.3 Eddington theory (1921)

In this conjunction, Eddington’s theory was the next [29] (see also [20]). Eddington set forward the idea that, perhaps, the connection \( \Gamma^\lambda_{\mu\nu} \) is the primary quantity, rather than the metric \( g_{\mu\nu} \) itself. Assuming a symmetric connection, which he did, the Ricci tensor can be decomposed as

\[ R_{\mu\nu} = R^{\text{symm}}_{\mu\nu} + R^{\text{antisymm}}_{\mu\nu}, \quad (8) \]
where $R_{\mu^\nu}$ is the usual part of the Ricci tensor, and

$$R_{\mu^\nu}^{\text{antisym}} = \frac{1}{2} \left( \frac{\partial \Gamma^\lambda_{\mu \lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^\lambda_{\nu \lambda}}{\partial x^{\mu}} \right), \quad (9)$$

which is nonzero in general, it is zero for a metric Christoffel connection. Then, one can identify first, the electromagnetic tensor with $R_{\mu^\nu}^{\text{antisym}}$, namely, $F_{\mu^\nu} \equiv R_{\mu^\nu}^{\text{antisym}}$, and second, the new potential of the theory with $\Gamma^\lambda_{\mu \lambda}$, namely, $\phi_\mu \equiv \Gamma^\lambda_{\mu \lambda}$, $\phi_\mu$ being thus the electromagnetic potential. The metric tensor $g_{\mu^\nu}$, not being fundamental anymore, has nonetheless to be recovered. One postulates then $g_{\mu^\nu} \equiv \frac{1}{\Lambda} R_{\mu^\nu}^{\text{symm}}$, with $\Lambda$ being a new fundamental constant. The line element squared, $ds^2 = g_{\mu^\nu} dx^{\mu} dx^{\nu}$ is now written as, $ds^2 = \frac{1}{\Lambda} R_{\mu^\nu}^{\text{symm}} dx^{\mu} dx^{\nu}$. Given the essentials of the geometry, Eddington goes on and proposes an action of the type, $S = \int \sqrt{|R_{\mu^\nu}|} d^4 x$ (this type of action was taken up later in the Born-Infeld theory of electromagnetism [14]). It is a theory based on an affine connection, indeed it is the first affine theory. It is also the only one, probably due to its awkwardness, despite its ingenuity. Einstein in between the years of 1923 and 1925 fiddled with the theory, trying to find out field equations, but could not progress (see [30], see also [31] for a recent perspective on the action and equations of Eddington’s affine theory).

### 2.4 Cartan’s torsion, differential geometry, Einstein’s attempts and the spectrum of unitary theories, and the $C$-field

After the appearance of general relativity, differential geometry and manifold theory started to be considered an important branch of mathematics. Indeed, the ideas on connections and parallel transport of Hessenberg (1917), Levi-Civita (1917), and Schouten (1917) sprang from the establishment of the beauty and power of general relativity (see [34] for the display of the new connections). These ideas were then used by Weyl (1918) [21, 22] and Eddington (1921) [29] (see also [20]) to propose new geometries and new physical theories of gravitation and electromagnetism. In turn these theories inspired new ways to explore theories of general connections and their properties. For instance, Cartan in 1922 discovered the notion of torsion, which is given by the antisymmetric part of the connection [32, 33] and from which follows the Riemann-Cartan geometry, see Schouten’s book [34] (for a
textbook see [35]). Finally, with the help of this paraphernalia of connections
new unitary theories were invented and proposed. For all these theories see
the thorough book of Mme. Tonnelat (1965) [36], and the excellent review
by Goenner (2004) [37].

A general connection $\Gamma$ (dropping the indices, which we will do whenever
we think it is appropriate and facilitates the reading) has a metric part as in
Riemann geometry, a homothetic part as in Weyl geometry, and a torsion
as in Cartan geometry. Thus, besides the Riemann-Christoffel curvature,
one gets a homothetic curvature, and a torsion curvature. Unitary theories,
that tried to unify the gravitational and electromagnetic fields used one or
all these new connections and curvatures. Let us enumerate some of these,
see [36, 37] for precise citations: (i) Theories with Riemann-Christoffel and
homothetic curvatures, without torsion, i.e., $\Gamma$ is symmetric, were based on
the original one, constructed by Weyl (1918). (ii) Theories with Riemann-
Christoffel and torsion curvatures, without homothetic curvature, have an
asymmetric $\Gamma$ which can be written as $\Gamma = \Gamma_{\text{sym}} + \Gamma_{\text{antisym}}$. In their full
generality this type of theories was started by Cartan (1923), and the original
theory, along with developments, is now called Einstein-Cartan theory.
In a particular case, namely, in the case one could use the notion of distant
parallelism, these teleparallel versions were explored by Weitzenböck (1925),
Einstein (1925), Infeld (1928), and others. (iii) Theories with all three curva-
tures, where also $\Gamma = \Gamma_{\text{sym}} + \Gamma_{\text{antisym}}$, were tried by Schouten (1924), Eyraud
(1926), Infeld (1928), and Straneo (1931). The original theory of Eddington
(1921), explored by Einstein (1923), starts from a manifold with a connection
only, the metric being a derived entity, and follows outside this scheme, perhaps.
Einstein (1942) and Schrödinger (1943) even tried theories where the
fundamental tensor $g_{\alpha\beta}$ has an antisymmetric part, $g_{\alpha\beta} = g_{\alpha\beta \text{ sym}} + g_{\alpha\beta \text{ antisym}}$. See [36, 37].

Another idea on connections that sprang from all these differential geo-
metries, and is seldom mentioned, is that the manifold can see a connec-
tion $\Gamma$ for contravariant vectors $v$ and a different connection $\Gamma'$ for covari-
ant vectors $u$. Thus, for each connection, $\Gamma$ and $\Gamma'$, one gets the usual
Riemann-Christoffel curvature, a torsion curvature, and a homothetic curva-
ture. These two distinct connections give rise to a new three-index tensor
field $C$ which in turn makes the bridge between the connections themselves,
and so between the contravariant and the covariant vectors and tensors.
The field $C$ is defined as the covariant derivative of the identity $I$, namely,
$C_{\alpha\beta}^\gamma \equiv \Gamma^\gamma_{\alpha\beta} = \Gamma_{\alpha\beta}^\gamma + \Gamma'_{\alpha\beta}^\gamma$. For most physicists and in most theories, this $C$ field was put to zero, probably because of its apparent lack of physical meaning. As we will see, not for Mira Fernandes. In the years between 1926 and 1933 he explored some of the proposed theories by adding to them the $C$ field, while trying to physically interpret it.

At the time there were ways, other than modifying the connection structure of spacetime, to try unification between the gravitational and electromagnetic fields. An important scheme is still under study. In this scheme one sticks to Riemann geometry and Einstein’s equations (or some modifications of these) but in spacetime dimensions higher than four $d > 4$, so that the gravitational field in the extra dimensions is seen in four dimensions instead as an electromagnetic or some other field. Such an idea was pursued by Kaluza (1921), Klein (1926), Einstein and Mayer (1931), Einstein, Bargmann and Bergmann (1941), Jordan (1945) and Thiry (1945), and Podolanski (1950), and others. These theories are generically called Kaluza-Klein theories.

With these new ideas and connections many different schemes were constructed [36, 37].

3 The works of Mira Fernandes on connections and on unitary theories of gravitation and electromagnetism

Having put forward the ideas on unitary theories of gravitation and electromagnetism in the context of the 1920s and beginnings of 1930s we are now ready to understand the works of Mira Fernandes, first on connections, then on unitary theories themselves. The works on connections [38, 39] are based on the books and papers of the mathematicians and mathematical physicists previously referred to. The works on unitary theories are based on ideas developed by the Italian mathematical physicist Paolo Straneo, which in turn are based on the theories of Weyl, Eddington, Cartan, Einstein and others already mentioned. Indeed, using Straneo’s ideas on gravitational and electromagnetic fields and their relations to connections [40, 41, 42, 43, 44], Mira Fernandes wrote a paper in 1932 [45]. Then Mira Fernandes became interested in another Straneo’s idea related to teleparallel theories [46] upon which he wrote a paper in 1933 [47]. Let us see all these works in detail.
3.1 The work on connections: (i) The book 1926 (ii) Rendiconti 1931

(i) The book 1926 “Fundamentos da geometria diferencial dos espaços lineares” (Foundations of differential geometry of the linear spaces) (in Portuguese) [38].

This book was published by the press of Museu Comercial in 1926 and has been reprinted [38]. In its foreword Mira acknowledges the Dutch mathematicians Schouten and Struik, the German mathematicians Blaschke and Weyl, and the English physicist and astrophysicist Eddington. In his book Mira Fernandes follows Schouten’s book of 1924 “Ricci-Kalkül” [34].

After some preliminary definitions on tensors and their properties, which take about 70 pages, the book goes on to define the linear transport for vectors (throughout a consistent mixture of the notation and conventions adopted by Mira Fernandes in the book and the papers will be followed). For a contravariant vector $v^\alpha$ the linear transport is defined as

$$Dv^\alpha = dv^\alpha + \Gamma^\alpha_{\mu\beta} v^\mu dx^\beta,$$

where $D$ means covariant derivative, $d$ simple derivative, $dx^\beta$ is the displacement vector along which $v^\alpha$ is transported, and $\Gamma^\alpha_{\mu\beta}$ is the connection for contravariant vectors. The linear transport for generic contravariant tensors of any number of indices $v^{\alpha\beta\gamma\cdots}$ can be generalized in the usual way. In addition, the linear transport for a covariant vector $u_\alpha$ is defined as

$$Du_\alpha = du_\alpha + \Gamma'_{\alpha\beta} ^\mu u_\mu dx^\beta,$$

where $\Gamma'_{\alpha\beta} ^\mu$ is the connection for covariant vectors, in general different from $\Gamma^\alpha_{\mu\beta}$. The linear transport for generic covariant tensors of any number of indices $u_{\alpha\beta\gamma\cdots}$ can also be generalized in the usual way. A prime as a superscript will indicate from now on quantities related to $\Gamma'$. The identity tensor $I_\alpha^\beta$ has then covariant derivative given by $I_\alpha^\beta;\mu = I_\alpha^\beta,\mu + \Gamma^\gamma_{\nu\mu} I_\gamma^\alpha + \Gamma'^\gamma_{\nu\mu} I_\gamma^\beta$, where, of course, $I_\alpha^\beta;\mu = 0$, a comma denoting simple derivative. Thus, one can define a $C$-field, a three index tensor, as $C_{\mu\alpha}^\beta \equiv I_{\alpha;\mu}^\beta$. One then has

$$C_{\mu\alpha}^\beta \equiv I_{\alpha;\mu}^\beta = \Gamma^\beta_{\nu\mu} I_\nu^\alpha + \Gamma'^\beta_{\nu\mu} I_\nu^\beta.$$

The tensor field $C$ links the connection for the contravariant vectors and tensors with the connection for covariant vectors and tensors. Due to the
nonvanishing of the covariant derivative of the identity tensor in general, one has that
\[(u_\alpha v^\alpha)_;\beta = u_{\alpha;\beta} v^\alpha + u_\alpha v_{\alpha;\beta} - C^\mu_{\beta\alpha} u_\mu v^\alpha.\] (13)
When \(C^\mu_{\beta\alpha} = 0\), the case we are used to, then the Leibniz rule for the differentiation of a product, here a contraction, holds, and one says, with Mira Fernandes, that the transport is invariant by contraction.

Now, the tensor \(C^\mu_{\beta\alpha}\) is quite general, and unwieldy to handle, so it is of interest to simplify it, as Schouten first suggested (see [34]). One puts,
\[C^\mu_{\beta\alpha} = C^I_{\beta\mu},\] (14)
i.e., the three index tensor field \(C^\mu_{\beta\alpha}\) turns essentially into a simple vector \(C^I_{\beta}\). The derivative of a vector contraction becomes now
\[(u_\alpha v^\alpha)_;\beta = u_{\alpha;\beta} v^\alpha + u_\alpha v_{\alpha;\beta} - C^I_{\beta} (u_\alpha v^\alpha).\] (15)
Mira Fernandes in the book, as well as in some of his papers, works in \(n\)-dimensions, usually to be considered spacetime dimensions. He then states that when \(u_\alpha v^\alpha = 0\), i.e., \(v^\alpha\) belongs to the \((n-1)\)-hyperplane defined by the covector \(u_\alpha\), the Leibniz rule for the differentiation of the product is verified. When \(u_\alpha v^\alpha = 0\) the vectors \(u\) and \(v\) are said to be incident vectors. In this case the transport is said invariant by incidence. Thus, there are transports invariant by contraction and transports invariant by incidence.

Within each connection, \(\Gamma\) or \(\Gamma'\), there is an important quantity related to the antisymmetric part of it, called torsion. The torsion \(S^\gamma_{\alpha\beta}\) is defined by
\[S^\gamma_{\alpha\beta} = \frac{1}{2} \left( \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} \right),\] (16)
and there is an analogous definition for \(S'_{\alpha\beta}^\gamma\),
\[S'_{\alpha\beta}^\gamma = \frac{1}{2} \left( \Gamma'^\gamma_{\alpha\beta} - \Gamma'^\gamma_{\beta\alpha} \right).\] (17)
When the torsion is nonzero, the transport of a vector along a closed path of the manifold is mapped into a transport in a nonclosed path in the associated tangent space. There are two particular cases of relevance. When \(S^\gamma_{\alpha\beta} = 0\) the transport is said, in the book, contravariant symmetric. When \(S^\gamma_{\alpha\beta} = S[\beta I^\gamma_{\alpha}]\) the transport is said contravariant hemisymmetric, a nomenclature
which followed Schouten [34]. The same applies to \( S'_{\alpha \beta \gamma} \), the torsion for the transport of covariant vectors.

The fundamental tensor \( g_{\alpha \beta} \) is the generalization of the metric tensor, used in Riemann geometry, to more general geometries. In general, \( g_{\alpha \beta} \) can have no symmetries, the symmetric part of it defines the lengths of vectors at a point. In the book, the fundamental tensor \( g_{\alpha \beta} \) is always considered symmetric. Weyl geometry gives the simplest example of such a fundamental tensor. As in Weyl geometry there is now a quantity \( Q'_{\alpha \beta \gamma} \), useful for contravariant vectors \( v^\alpha \), see Eq. (4), defined by

\[
Q'_{\alpha \beta \gamma} = g_{\beta \gamma; \alpha},
\]

where \( Q' \) is called the nonmetricity tensor. It tells how the fundamental tensor \( g_{\alpha \beta} \) deviates from being a pure metric tensor. Again, there are two particular cases of relevance. When \( Q'_{\alpha \beta \gamma} = 0 \) the transport is said contravariant metric, since in this case \( g_{\alpha \beta} \) is indeed a metric tensor for contravariant vectors \( v^\alpha \). When \( Q'_{\alpha \beta \gamma} = Q'_{\alpha \beta \gamma} \), the transport is contravariant conform, which is the case in Weyl theory, see Eq. (4). An analogous quantity \( Q^{'\beta \gamma} \) holds for covariant vectors. \( Q^{'\beta \gamma} \) is defined as

\[
Q^{'\beta \gamma} = g^{'\beta \gamma; \alpha},
\]

It tells how the raised fundamental tensor \( g^{\alpha \beta} \) deviates from being a pure metric tensor. Again, there are two particular cases of relevance. When \( Q^{'\alpha \beta \gamma} = 0 \) the transport is said covariant metric, since in this case \( g^{\alpha \beta} \) is indeed a metric tensor for covariant vectors \( u_\alpha \). When \( Q^{'\alpha \beta \gamma} = Q^{'\alpha \beta \gamma} \), the transport is covariant conform.

With these definitions one can now express the connections \( \Gamma \) and \( \Gamma' \) in the Christoffel symbols and in the fields \( C, g, S, Q, S', \) and \( Q' \) (see Eq. (3) for the particular case of the Weyl geometry). Indeed, it is shown in the book, that

\[
\Gamma^\lambda_{\alpha \gamma} = \{_{\beta \gamma}\}^\alpha_+ T^\lambda_{\alpha \gamma},
\]

\[
\Gamma'^\lambda_{\alpha \gamma} = -\{_{\beta \gamma}\}^\alpha_+ T'^\lambda_{\alpha \gamma},
\]

where

\[
T^\lambda_{\alpha \gamma} = C^\lambda_{\gamma \alpha} - T'^\lambda_{\alpha \gamma},
\]
and

\[
T'_{\alpha\gamma}^\lambda = \frac{1}{2} (Q_{\gamma\alpha\beta} + Q_{\alpha\gamma\beta} - Q_{\beta\alpha\gamma}) g^{\beta\gamma} - S'_{\beta\gamma}^{\nu} g_{\alpha\nu} g^{\beta\lambda} - S'_{\mu\alpha}^{\nu} g_{\gamma\nu} g_{\gamma\mu} g^{\beta\lambda} + S'_{\alpha\nu}^{\lambda},
\]

(23)

with the last three terms involving a linear combination of the torsion being sometimes called the contorsion.

Having properly defined the connections of a manifold one can go on to define the associated curvature. The curvature of a manifold has the same expression in terms of the connection as the Riemann-Christoffel curvature has in terms of the Christoffel symbols connection (a connection defined solely in terms of the metric). The expression for the curvature for contravariant vectors is

\[
R_{\nu\beta\lambda}^\alpha = \Gamma_{\lambda\nu;\beta}^\alpha - \Gamma_{\lambda\beta;\nu}^\alpha + \Gamma_{\mu\beta}^\alpha \Gamma_{\lambda\nu}^\mu - \Gamma_{\mu\nu}^\alpha \Gamma_{\lambda\beta}^\mu,
\]

(24)

whereas the expression for the curvature for covariant vectors is

\[
R'_{\nu\beta\lambda}^\alpha = \Gamma'_{\lambda\nu;\beta}^\alpha - \Gamma'_{\lambda\beta;\nu}^\alpha + \Gamma'_{\mu\beta}^\alpha \Gamma'_{\lambda\nu}^\mu - \Gamma'_{\mu\nu}^\alpha \Gamma'_{\lambda\beta}^\mu.
\]

(25)

When the curvature \( R_{\nu\beta\lambda}^\alpha = 0 \) the manifold is flat for the transport of a contravariant vector, the transport is called contravariant parallel. When the curvature \( R'_{\nu\beta\lambda}^\alpha = 0 \) the manifold is flat for the transport of covariant vectors, the transport is called covariant parallel. The curvature is also used to define a transport for a bivector that is called contravariant equivalent in the book. A bivector \( v^{\alpha\beta} \) is a tensor such that \( v^{\beta\alpha} = -v^{\alpha\beta} \), i.e., it is an antisymmetric two-indice tensor. Define the quantity \( V_{\nu\mu} \), as Mira Fernandes does in the book, as

\[
V_{\nu\mu} = R_{\nu\mu\alpha}^\alpha.
\]

Then, if the transport of \( v^{\alpha\beta} \) along a closed path is zero it is called contravariant equivalent or equiaffine. The same rationale holds for a covariant bivector \( u_{\beta\alpha} = -u_{\alpha\beta} \).

There are particular important cases, all of them analyzed towards the end of the book. Riemann transport is the one for which \( C = 0, S = 0 \), and \( Q = 0 \), and leads to general relativity. Weyl transport is such that \( C = 0, S = 0 \), and \( Q_{\gamma\alpha\beta} = Q_{\gamma} g_{\alpha\beta} \), and leads to Weyl’s theory. The so called affine transport is such that \( C = 0, S = 0 \), and \( Q \) is any arbitrary quantity, like in Eddington’s theory.

Most geometries studied throughout the years have \( C = 0 \). This is mathematically a relief, since the field \( C \) complicates the expressions tremendously. However, for some reason, Mira Fernandes used manifolds in which the connections are linked by a nonzero \( C \)-field, and tried to give a physical meaning
to $C$, such as an electromagnetic field in the unitary schemes he and others developed, as we will see.

(ii) Rendiconti 1931 “Proprietà di alcune connessioni lineari” (Properties of some linear connections) (in Italian) [39].

This paper [39] shows some seven properties of connections which are neither in Schouten’s book [34], nor in Mira Fernandes’ book [38]. Let us see a typical one. Mira assumes that the connection is invariant by incidence, i.e., $C_{\alpha\beta\gamma} = C_\alpha \Gamma^\gamma_\beta$, is covariant symmetric, i.e., $S = 0$, and metric conform, i.e., $Q_{\alpha\beta\gamma} = Q_\alpha g_{\beta\gamma}$. Then, using a result of Schouten he writes that the Riemann curvature for the contravariant vectors $R_{\rho\beta\lambda}^\alpha$ and the Riemann curvature for the covariant vectors $R'_{\rho\beta\lambda}^\alpha$ are linked through

$$R_{\rho\beta\lambda}^\alpha = R'_{\rho\beta\lambda}^\alpha + 2C_{[\rho;\beta]} I^\alpha_\lambda.$$  \hspace{1cm} (26)

Now, contract in $\alpha$ and $\rho$ to get the link between the Ricci tensors,

$$R_{\beta\lambda} = R'_{\beta\lambda} + 2C_{[\lambda;\beta]}.$$  \hspace{1cm} (27)

Assume further now $Q'_{\alpha} = 0$, so that the connection is metric. Then, in this case, the connection $\Gamma'_{\alpha\beta}^\gamma$ is given by the Christoffel symbols, and $R_{\rho\beta\lambda}^\alpha = K'_{\rho\beta\lambda}^\alpha$, where $K'_{\rho\beta\lambda}^\alpha$ is the Riemann-Christoffel curvature. Upon antisymmetrization, he finds

$$R_{[\beta\lambda]} = 2C_{[\lambda;\beta]},$$  \hspace{1cm} (28)

since $K'_{\beta\lambda}$ is symmetric in $\beta\lambda$. He now gladly proclaims, “this formula resembles the formula of Eddington”, namely,

$$R_{[\beta\lambda]} = R'_{[\beta\lambda]} = [\beta; T_{\lambda}]^\alpha_\alpha,$$  \hspace{1cm} (29)

where $T$ is part of the connection that is not Christoffel, see Eqs. (20)-(23). But there are differences. Eddington’s theory is an affine theory with $C = 0$, $S' = 0$ and arbitrary $Q'$.

In summary, for Mira Fernandes, $C_{\alpha\beta\gamma} = C_\alpha \Gamma^\gamma_\beta$, $S' = 0$, $Q' = 0$, and one gets $R_{[\beta\lambda]} = 2C_{[\lambda;\beta]}$ and $R'_{[\beta\lambda]} = 0$. For Eddington, $C = 0$, $S' = 0$, arbitrary $Q'$, and one gets $R_{[\beta\lambda]} = [\beta; T_{\lambda}]^\alpha_\alpha$ and $R'_{[\beta\lambda]} = R_{[\beta\lambda]}$. Can Mira Fernandes improve on these similarities? Yes. Since $\Gamma + \Gamma' = C$, one also has $T + T' = C$.
(the Christoffel symbols disappear when summed). Moreover, since here $T' = 0$, one gets $T = \mathcal{C}$, i.e., restoring indices, $T_{\beta\alpha} = C_\alpha I_\beta$. Contracting in $\gamma\alpha$ he obtains, $T_{\lambda\alpha} = \mathcal{C}_\lambda$. Thus, taking the covariant derivative yields, $\mathcal{C}_{[\lambda;\beta]} = [\beta;T_{\lambda}]^\alpha_\alpha$. Finally, using Eq. (28), he finds
\[ R_{[\beta\lambda]} = 2[\beta;T_{\lambda}]^\alpha_\alpha, \] (30)
indeed of Eddington’s form (see Eq. (29)), apart a factor 2! This is the first instance where Mira Fernandes tries to give a theoretical application to the $C$-field, the field that connects the connections. The next two papers develop this idea.

3.2 Application to unitary theories of gravitation and electromagnetism: (i) Rendiconti 1932 (ii) Rendiconti 1933

(i) Rendiconti 1932 “Sulla teoria unitaria dello spazio fisico” (About the unitary theory of the physical space) (in Italian) [45].

In this paper of 1932 [45] Mira Fernandes ventures into unitary theories. He has already given a hint that he likes this type of theories and speculations in the previous paper when he mentions Eddington. But now he embraces it in full. Mira Fernandes analyzes Paolo Straneo’s papers, an Italian mathematical physicist who belonged to the group of Levi-Civita. Straneo published papers on a certain type of unitary theories on Rendiconti dei Lincei in the years 1931-1932 [40, 41, 42, 43] and a review of his ideas is given in La Rivista del Nuovo Cimento in 1931 [44].

In his paper [45], Mira Fernandes states (I translate freely) “In a number of Notes published in these Proceedings prof. Paolo Straneo establishes a unitary theory of gravitation and electromagnetism, which, constituting a geometrical synthesis of the physical phenomena, reduces itself to the theory of Einstein in the absence of electrical phenomena”. Straneo’s connection that most interested Mira Fernandes is
\[ \Gamma^\alpha_{\beta\gamma} = \{\alpha\} + (I^\alpha_\mu \psi_\nu - I^\alpha_\nu \psi_\mu), \] (31)
where $\psi_\nu$ is an additional vector field of the theory, to be equated physically to the electromagnetic potential, and $I^\beta_\beta$ is the unit tensor. Equation (31) is based, in a sense, on Weyl’s connection, and modifies it. However, it does not have the same mathematical substratum, neither the same physical
background. Nonetheless it was of interest at the time. For the connection
(31) one can show that the curvature tensor is given by
\[ R^\alpha_{\rho\mu\nu} = K^\alpha_{\rho\mu\nu} + 2I^\alpha_{\rho} (\psi_{\rho,\nu} - \psi_{\nu,\rho}) , \]  
where \( K^\alpha_{\rho\mu\nu} \) is the Riemann-Christoffel curvature. Contracting in \( \alpha\rho \) yields
\[ R_{\mu\nu} = K_{\mu\nu} + 2 (\psi_{\mu,\nu} - \psi_{\nu,\mu}) . \]  
Contracting again gives
\[ R = K . \]  
Mira Fernandes then writes Straneo’s gravitational field equation, namely,
\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 2\psi_{\mu\nu} = \chi \Theta_{\mu\nu} , \]  
where \( \chi \) is some coupling constant, \( \psi_{\mu\nu} \equiv \psi_{\mu,\nu} - \psi_{\nu,\mu} \), and \( \Theta_{\mu\nu} \) is the energy-
momentum tensor. Mira Fernandes does not write the field equation for \( \psi_{\nu} \),
presumably it is \( \psi_{\mu} ^{\nu} = J_{\mu} \), where \( J_{\mu} \) is some current. There is a similar
theory proposed by Infeld in 1928 which is previous to Straneo’s and Mira
Fernandes mentions it in passing, see also [36]. Note that in a clear sense
these equations do not fulfill a scheme for full unification, as envisaged by
some at the time, since an energy-momentum tensor appears.

Mira Fernandes then states: “The aim of this Note is to formulate some
considerations about the connection of Straneo, and about other connections
that lead to the same field equations and of which the author has occupied
himself in a previous paper”. This previous paper is the Rendiconti 1931 [39]
commented above. Then Mira points out several things.

To start he points out that Straneo’s connection in not contravariant
metric, i.e., \( Q'_{a\mu\nu} \neq 0 \). Further, he takes some time to show that assuming
\( Q'_{a\mu\nu} = 0 \) and \( Q_{a} ^{\mu\nu} = 0 \) there is no way one can find the curvature tensor
of Straneo, a result one could have guessed beforehand given the experience
with Weyl’s connection.

He wants to go further and derive the field \( \psi_{\nu} \) from the connection itself!
So he supposes that the connection is invariant by incidence, covariant
symmetric, and contravariant metric, in brief: \( C_{a\beta} ^{\gamma} = C_{a} ^{\alpha} I_{\beta} ^{\gamma} \), \( S'_{a\beta} ^{\gamma} = 0 \), and
\( Q'_{a\beta\gamma} = 0 \). Then he finds
\[ \Gamma_{\beta\gamma} ^{\alpha} = \{ a \} _{\beta\gamma} ^{\alpha} + T_{\beta\gamma} ^{\alpha} , \]  
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\[ \Gamma^\alpha_{\beta\gamma} = -\{^\alpha_{\beta\gamma}\} + T'_{\beta\gamma} \alpha, \]  \tag{37} 

with

\[ T_{\beta\gamma} \alpha = C_{\gamma} I_\alpha^\beta, \]  \tag{38} 

\[ T'_{\beta\gamma} \alpha = 0. \]  \tag{39} 

Now, in his previous Rendiconti [39] he displayed

\[ R_{\alpha\beta\gamma} \delta = R'_{\alpha\beta\gamma} \delta + 2C_{[\alpha;\beta]} I_\gamma^\delta. \]  \tag{40} 

For the connection under study one has, \( R'_{\alpha\beta\gamma} \delta = K'_{\alpha\beta\gamma} \delta \), the Riemann-Christoffel tensor. One can show without difficulty that \( K'_{\alpha\beta\gamma} \delta = K_{\alpha\beta\gamma} \delta \), i.e., Riemann-Christoffel tensor for covariant vectors is the same as the Riemann-Christoffel tensor for contravariant vectors. Contracting in \( \alpha\delta \), and noting that since \( S' = 0 \) one has \( C_{[\alpha;\beta]} = C_{[\alpha,\beta]} \), he finds

\[ R_{\beta\gamma} = K_{\beta\gamma} + (C_{\gamma,\beta} - C_{\beta,\gamma}). \]  \tag{41} 

Contracting again gives

\[ R = K. \]  \tag{42} 

Comparing Straneo’s equation, Eq. (33), with Mira Fernandes’ equation, Eq. (41), it is clear that Straneo’s Ricci tensor is recovered if one puts

\[ C_\mu = -2\psi_\mu. \]  \tag{43} 

Thus, the field \( C \), that links the distinct connections for contravariant and covariant vector fields, provides the electromagnetic field potential \( \psi \). It is perhaps the first instance that the field \( C \) receives a physical interpretation.

Moreover, the field \( C \), and thus the electromagnetic field \( \psi \), is also related to both the torsion and the nonmetricity tensors. The link with the torsion goes as follows: since quite generally \( C_{[\alpha\beta]} \gamma = S_{\beta\alpha} \gamma + S'_{\beta\alpha} \gamma \), and here \( S' = 0 \), one finds \( S_{\beta\alpha} \gamma = S_{[\alpha} I_{\beta]} \gamma \), with \( S_{\alpha} \equiv C_{\alpha} \), which means, in the nomenclature of Schouten [34] and Mira Fernandes [38], that the connection is contravariant hemisymmetric. The link with the nonmetricity tensor can be easily seen since for \( C_{[\alpha\beta]} \gamma = C_{\alpha} I_{\beta} \gamma \) one finds that \( Q'_{\alpha\beta\gamma} = -g_{\beta\mu} g_{\gamma\nu} Q_{\alpha}^{\mu\nu} + 2C_{\alpha} g_{\beta\gamma} \). Since here \( Q' = 0 \) one has \( Q_{\alpha\beta\gamma} = Q_{\alpha} g_{\beta\gamma} \) with \( Q_{\alpha} = 2C_{\alpha} \), providing the link. The connection is contravariant conform, of Weyl type. A final property
referred in the paper comes from the fact that a contraction in $\gamma\delta$ implies
that $V'_{\alpha\beta} = 0$, so the connection is covariant equivalent or equiaffine.

For a four-dimensional spacetime, $n = 4$, the connection, in Mira Fernandes’ words, satisfies “all the conditions attributed by Straneo to the structure of the physical space”. The vector $\psi_\mu$ representing the electromagnetic potential is such that

$$\psi_\mu = -\frac{1}{2} C_\mu,$$

and the torsion and nonmetricity quantities are given by

$$S_\mu = \frac{1}{2} Q_\mu = C_\mu.$$  \hspace{1cm} (45)

The field $C$ provides much of the physical and geometrical content of the analysis.

Summarizing, the linear connection proposed by Mira Fernandes is metric contravariant but not metric covariant. It is contravariant and covariant conform, i.e., conserves the angles and the ratios of lengths in both transports and preserves lengths only in the contravariant transport. It also conserves covariant bivectors and covariant p-vectors (totally antisymmetric tensors with any number of indices) in the transport. Mira Fernandes concludes the section stating that the connection “is distinct from Weyl since $C_{\alpha\beta\gamma} \neq 0$”. The connection is invariant by incidence, not by contraction. When $C_\alpha = C_{\alpha,\alpha}$, i.e., $C_\alpha$ is the gradient of some function $C$, then one recovers Weyl. In the rest of the paper he does a similar analysis for a connection that is invariant by incidence, contravariant symmetric, and covariant metric, i.e., $C_{\alpha\beta\gamma} = C_\alpha I^\gamma_\beta$, $S_{\alpha\beta\gamma} = 0$, and $Q_{\alpha\beta\gamma} = 0$. It is of no great use to repeat this analysis here since it is much the same as the previous one.

(ii) Rendiconti 1933 “Sulla teoria unitaria dello spazio fisico” (About the unitary theory of the physical space) (in Italian) [47].

This paper of Mira Fernandes [47] has the same title as the previous one [45], which was perhaps a common practice in the Rendiconti when the author wrote on the same subject. In this paper, as usual, he states the definitions of the fields $C$, $S$, $S'$, $Q$, $Q'$, $\Gamma(g,T)$, $\Gamma'(g,T')$, $T(Q,S,g)$, $T'(Q',S',g)$, where again we have dropped the indices. Some properties related to these quantities are $T_{[\alpha\gamma]} = S_{\alpha\gamma}$, $C_{[\alpha\gamma]} = S_{\gamma\alpha} + S'_{\gamma\alpha}$, and $Q' = Q'(S,Q,g)$ a complicated function which we do not need to show here explicitly. He takes
from his book [38] the relation between the Riemann $R'$ and the Riemann $R$,

$$R'_{\alpha\beta\gamma}{}^{\delta} = R_{\alpha\beta\gamma}{}^{\delta} + 2S'_{\alpha\beta}{}^{\mu}C_{\mu\gamma}{}^{\delta} + C_{\beta\gamma}{}^{\delta,\alpha} - C_{\alpha\gamma}{}^{\delta,\beta}, \quad (46)$$

and the relation between the Riemann $R$ and the Riemann-Christoffel $K$,

$$R_{\alpha\beta\gamma}{}^{\delta} = K_{\alpha\beta\gamma}{}^{\delta} + T_{\gamma\alpha}{}^{\delta,\beta} - T_{\gamma\beta}{}^{\delta,\alpha} + T_{\gamma\delta}{}^{\mu}T_{\mu\alpha}{}^{\delta} - T_{\gamma\alpha}{}^{\mu}T_{\mu\delta}{}^{\beta}. \quad (47)$$

A similar relation between $R'_{\alpha\beta\gamma}{}^{\delta}$ and the other prime quantities also holds. Suppose now that the connection is metric covariant $Q_{\alpha\beta\gamma} = 0$, and that $T$ is antisymmetric in the lower indices $\alpha\gamma$ so that $T_{\alpha\gamma}{}^{\lambda} = S_{\alpha\gamma}{}^{\lambda}$. Then $T_{\alpha\gamma}{}^{\gamma} = 0$, with no sum in the repeated indices. Suppose further that spacetime is flat for contravariant vectors, i.e., $R_{\alpha\beta\gamma}{}^{\delta} = 0$. Then, from the symmetries of $R_{\alpha\beta\gamma}{}^{\delta}$ and the properties mentioned above, one finds upon using (47) that

$$K_{\alpha\beta\gamma}{}^{\delta} - T_{\alpha\beta}{}^{\delta,\gamma} + T_{\alpha\beta}{}^{\mu}T_{\mu\gamma}{}^{\delta} = 0. \quad (48)$$

Mira Fernandes states that these equations “are the fundamental equations of the unitary theory of prof. Straneo”. Unfortunately, he does not quote the paper in which Straneo writes these equations. They are not in Rendiconti [40, 41, 42, 43] neither in Nuovo Cimento [44]. It is most certain that Mira Fernandes is referring to a paper of Straneo (in German) “Einheitlich Feldtheorie der Gravitation und Elektrizität” published in Zeitschrift für Physik in 1932 [46]. The equations (48) define an absolute transport for covariant vectors. Straneo recovers distant parallelism of Cartan, Weitzenböck, Einstein, and others [36, 37]. With a somewhat pompous stance, Mira Fernandes then states (my translation): “The above equation translates a remarkable structure of physical space characterizing a chronotope of contravariant curvature zero and metric covariant.” Also, since here $\Gamma_{\mu\nu}{}^{\alpha} = -\Gamma'_{\mu\nu}{}^{\alpha}$, one has that $C_{\mu\nu}{}^{\alpha} = 0$ and $Q'_{\alpha\beta\gamma} = 0$, so that $R'_{\alpha\beta\gamma}{}^{\delta} = 0$. There is also distant, or absolute, parallelism for covariant transport.

Mira Fernandes then turns again to the tensor $C_{\alpha\beta}{}^{\gamma}$ and shows that if this is nonzero then the equations of Straneo still hold for contravariant vectors, but now $R'_{\alpha\beta\gamma}{}^{\delta}$ can be nonzero, i.e., there is absolute transport for contravariant vectors but not for covariant vectors. Mira’s final remark is: “E non sarà privo d’interesse, per future utilizzazioni della teoria unitaria l’aver constatato che le equazioni del prof. Straneo sono compatibili con connessioni lineari (in numero infinito) in cui il tensore $(C_{\alpha\beta}{}^{\gamma})$ non è nullo;
And it will be not without interest, to a future use of the unitary theory to have ascertained that the equations of prof. Straneo are compatible with linear connections (in infinite number) in which the tensor \((C_{\alpha\beta\gamma})\) is nonzero; i.e., that they are not invariant by contraction”. In this way Mira Fernandes tries to push forward once again the field \(C\) that links the connections. However, this time, there is no attempt, not even indirectly, to relate it to the electromagnetic potential.

4 Conclusions

4.1 What else?

There are other related publications which I did not delved into and certainly deserve a closer scrutiny.

In 1924 Mira Fernandes published his first book, “Elements of the theory of the quadratic forms” (in Portuguese) [48]. It is divided into two parts: Algebraic forms and Differential forms. It is self-contained, written at a somewhat advanced level but of easy reading. It would be of interest to know which are the sources Mira Fernandes used to write this book.

In 1934 there is another publication in Rendiconti on unitary theories with the title “The unitary theory of physical space and the relativistic equations of atomic mechanics” (in Italian) [49]. It is a paper on Dirac’s equation and tries to unify general relativity and wave mechanics. As remarked in [9] it is a work that certainly deserves interpretation in a historical context.

In 1945 and in 1950 Mira Fernandes published two papers in Portugaliae Mathematica in order to develop fresh Einstein work on bivectors [50, 51], where Einstein tried to find fundamental equations without the use of differential equations. The first paper of the set has the the title “Finite connections” (in Italian) [52], and the second “Finite transports” (in Italian) [53].

In 1950, in Revista da Faculdade de Ciências, Mira Fernandes published a paper with the title “The geodesics of the unitary space” (in Italian) [54]. The paper is on complex manifolds, generalizing results of Coburn, an American mathematician, and it has nothing to do with the papers on the unitary theories of the physical space. This paper of Mira Fernandes is quoted in the book “Ricci Calculus” [55], the 1954 second edition, now in English, of
Schouten, a citation that must have given him much pleasure. The paper is also quoted in the book of 1955 of Mme. Tonnelat on the unitary theories of Einstein and Schrödinger [56].

There are other papers of Mira Fernandes on differential geometry of much interest but they are outside the main theme of our work on the extension of general relativity into unitary theories of gravitation and electromagnetism using the full possibilities offered by the connections of differential geometry. One that perhaps is worth quoting, published in 1932, is about the brachitochronous problem of Zermelo [57], which, in turn, can be put in a Finsler geometric context, as has been shown recently, see [58].

4.2 The fate of unitary theories

In the 1920s and beginnings of the 1930s the only two fields known were the gravitational and electromagnetic fields, assumed to be classical in the proposed unitary theories. Since then two more fields have been discovered, the weak and strong fields, and these, together with the electromagnetic field, have proved to be quantum fields. The mere existence of these two additional fields already puts in jeopardy the program of unifying the gravitational and electromagnetic fields alone. The fact that the fields are quantum in character dismisses definitely the whole program, based on a classical setup. Nonetheless, there are important ramifications taken out from the unification idea.

First, although the theories which change Riemann geometry, as those used by Mira Fernandes, are not in fashion nowadays as theories of unification of gravitation and electromagnetism, some of them were reverted to theories that embody gravitation, torsion, energy-momentum and elementary spin, and are called Einstein-Cartan theories, or Einstein-Cartan-Kibble-Sciama theories, the latter two names appearing because they showed first that the Einstein-Cartan theory can be formulated as a gauge theory with local Poincaré invariance in flat spacetime, see, e.g., [59] (see also [35]).

Second, the idea of unification still persists but on a different basis. The electromagnetic field, and its associated massless quantum particle, the photon, has been already joined with the weak field to produce the electroweak field. There remains the possibility that all three fields, electromagnetic, weak and strong, can be unified in a grand unified theory with their associated massless quantum particles. One can then hope that the gravitational
field with its associated massless graviton, and the grand unified field and particles, can be united in an ultimate theory of unification. The most current celebrated theories make use of the gravitational field in extra dimensions in order to try to obtain, in four dimensions, the gravitational field itself and the grand unified field (which itself contains and generalizes the electromagnetic field sought for in the early attempts). These theories are reminiscent of the ideas of Kaluza and Klein back in the 1920s, that were then used in their naïve form by Einstein and others [36, 37], but not touched or mentioned by Mira Fernandes. Nowadays these theories are generically called Kaluza-Klein theories. They were incorporated into supergravity [60], and then reappeared in string theory in a prominent form, see, e.g., [61].

The name of such theories has been changing, unitary theories at first, then unified field theories, and nowadays theories of everything. Will their fate be the same as Mie’s theory?

Sources

I have benefited from several sources which helped me to put the works of Mira Fernandes in context.

Direct sources:

· Schouten 1924 [34]. Jan Arnoldus Schouten was Dutch and, at the time, to write in German gave a much wider audience. The book Ricci-Kalkil [34] is written in German. I have used this book to connect the formulas in Mira Fernandes’ book [38] with the formulas in Schouten’s book [34].

· Mira Fernandes’ works 1926-1933 [38, 39, 45, 47]. Mira Fernandes works on differential geometry along with other works have been now reprinted by Gulbenkian Foundation in three volumes [1, 2, 3]. Prior to the publication of volumes 2 and 3, several papers, including the ones published in Rendiconti in the year 1931 to 1933 commented above [39, 45, 47], were facilitated to me by the staff in CEMAPRE (Centre for Applied Mathematics and Economics) of ISEG (Instituto Superior de Economia e Gestão), where copies of all the works and papers of Mira Fernandes are kept. ISEG is one of the places where Mira Fernandes taught.

· Straneo’s papers 1931-1932 [40, 41, 42, 43, 44, 46]. These papers were essential for our analysis, since Mira Fernandes bases his works on unitary theories on them. Paolo Straneo was a mathematical physicist from Genoa. Levi-Civita presented to the Academy of Lincei papers on unitary theories from Paolo Straneo, Attilio Palatini, Pia Nalli, Mira Fernandes and others. Initially, I have had access to
his review paper published in 1931 in La Rivista del Nuovo Cimento [44] and to a paper published in 1932 in Zeitschrift für Physik [46] only. Now, many of his papers are on the internet.

Indirect sources:
- Gagean and Leite 1990 [9]. Gagean and Leite wrote a remarkable article [9]. In this article there are sections devoted to Mira Fernandes where a historical description in context of Mira Fernandes' work and stance is given. It was the first article on Mira Fernandes I came across.
- Adler, Bazin, Schiffer 1965 [25]. This book is a master piece. It is perhaps the first text book in general relativity written from a physical point of view, and superbly so. The other previous good text books were written by mathematicians, even Eddington’s book [20], its title says it all. Adler’s book contains a chapter in which Weyl’s theory is masterly explained, and it should not be missed by the interested reader. In the second edition of 1974 the chapter is maintained.
- Pais 1982 [30]. This is a tour de force biography, all scientific work of Einstein is reviewed. Although there are some flaws, and understandably so given the huge scope and commitment of the book, the part concerned with unified field theories is fantastically clear.
- Tonnelat 1965 [36]. This book [36] is very important in our context. It makes a thorough review up to 1965 of the whole sets and ramifications of unified theories in vogue and out of fashion. It does not quote Mira Fernandes works on unitary theories. However, it quotes works by Straneo and Infeld (Straneo’s in a footnote), helping tremendously to connect Mira Fernandes to the main stream of the time. Without Mme. Tonnelat’s book it would have been much more difficult to put Mira Fernandes’ work on unitary theories in context.
- Goenner 2004 [37]. This is a review paper on unified field theories, the first part of it up to the beginning of the 1930s [37] and the second part yet to be published. This work connects smoothly to Tonnelat’s book of 1965 [36], although it is clear that the author has done thoroughly independent work. It also quotes Straneo’s work, and so also helps in connecting Mira Fernandes’ works to the main stream.

For a shorter version of this article see [63].

Acknowledgments

It is hard to write on others’ work, specially from a historical standpoint, because first one has to understand the overall context in which the works have been written, and second one has to understand the works themselves as well as their
motivations. This task of trying to understand Mira Fernandes’ works on unitary theories, differential geometry and general relativity, took me more than a year. Of course, on one hand I had to be motivated to do it, on the other hand several people have helped.

My motivations to understand Mira Fernandes’ work started while I was in Cambridge University during my Ph.D. times. In 1983, in Galloway and Porter bookshop in Sidney Street, a second hand bookshop, I came across the book “Relativity: the general theory” by Synge [62]. I turned the pages and at the end in the bibliography I hit upon the name of Mira Fernandes, his paper of 1932 published in Rendiconti [45] was cited. I was surprised and glad, seeing a Portuguese scientist being quoted, posthumously, in such an important book. The book was expensive but I bought it immediately with no more qualms or regrets. A further motivation came from Manuel Sande Lemos, my grandfather, who lived 102 years and managed to cross three centuries. He got the degree of chemical engineer at Instituto Superior Técnico (IST) in the beginning of the 1920s, when IST was at Rua do Instituto Industrial, Conde Barão. He attended Mira Fernandes courses on Differential and Integral Calculus and on Rational Mechanics, having a great admiration for him. He was always telling me about him, and advising me that I should study his works, specially the equations named Mira Fernandes. I still don’t know which equations are these, perhaps he also did not know. When the opportunity came I had no doubts I should embrace this project of studying his works on unitary theories.

I was helped by several people to whom I grateful thank. Nuno Crato has pushed me into giving the talk, and João Teixeira Pinto and Luís Saraiva have invited me to give it in the “Mira Fernandes e a sua época, Historical Conference in honor of Aureliano de Mira Fernandes (1884-1958)” realized in 2009 at IST, this article being in the Proceedings of it. Vera Lameiras, secretary of Cemapre-ISEG, has helped to find Mira’s papers in their archives. Although I do not know them personally, those who have collected Mira Fernandes papers and made his complete works possible, namely, Jaime Campos Ferreira, Luis Canto Loura, Joaquim Moura Ramos, Dulce Cabrita, Vicente Gonçalves, and others certainly, have made a significant contribution to put in real perspective Mira Fernandes’ work. Manuel Fiolhais has sent me from the Library of the University of Coimbra two papers of Straneo [44, 46] which helped in the analysis of the whole context.

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