MINKOWSKI BASES ON ALGEBRAIC SURFACES WITH RATIONAL POLYHEDRAL PSEUDO-EFFECTIVE Cone

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Abstract. The purpose of this note is to study the number of elements in Minkowski bases on algebraic surfaces with rational polyhedral pseudo-effective cone.

1. Introduction

Lazarsfeld and Mustaţă in [6] and Kaveh and Khovanskii in [3] initiated a systematic study of Okounkov bodies. These are convex bodies $\Delta(D)$ in $\mathbb{R}^n$ attached to big divisors $D$ on a smooth projective variety $X$ of dimension $n$. They depend on the choice of a flag of subvarieties $(Y_n, Y_{n-1}, \ldots, Y_1)$ of codimensions $n, n-1, \ldots, 1$ in $X$ respectively, such that $Y_n$ is a non-singular point of each of the $Y_i$’s. We refer to [6] for details of the construction and a very enjoyable introduction to this circle of ideas.

Okounkov bodies are a subject of intensive ongoing research. Luszcz-Swidecka observed in [7] that for a del Pezzo surface there are finitely many basic bodies, called the Minkowski basis, such that all other bodies are obtained as their Minkowski sums (hence the name “basis”). Building upon these ideas, Luszcz-Swidecka and Schmitz introduced in [8] an effective algorithmic construction of Minkowski bases for algebraic surfaces with rational polyhedral pseudo-effective cone $\text{Eff}(X)$.

In the present note we consider a natural question of how many elements there are in a Minkowski basis in the set-up of [8]. The answer is closely related to the partition, governed by Zariski decompositions, of the big cone of arbitrary smooth projective surfaces introduced in [2].
2. Preliminaries

In this section we introduce the notation and collect some basic ideas underlying the present note. By a curve we mean here an irreducible and reduced complete subscheme of dimension 1. For a divisor \( D \) on a smooth projective surface \( X \) we denote by \( D^\perp \) the set of curves intersecting \( D \) with multiplicity zero, that is,

\[ D^\perp := \{ C \subset X : D.C = 0 \}. \]

We begin with a tool fundamental for understanding linear series on algebraic surfaces.

**Definition 2.1** (Zariski decomposition). Let \( X \) be a smooth projective surface and let \( D \) be a pseudo-effective \( \mathbb{Q} \)-divisor on \( X \). Then there exist \( \mathbb{Q} \)-divisors \( P_D \) and \( N_D \) such that

a) \( D = P_D + N_D \);
b) \( P_D \) is a nef divisor and \( N_D \) is either zero or it is supported on a union of curves \( N_1, \ldots, N_r \) with negative definite intersection matrix;
c) \( N_i \in (P_D)^\perp \) for each \( i = 1, \ldots, r \).

Let \((x,C)\) be a flag on a surface \( X \). Let \( D \) be a big divisor on \( X \) with Zariski decomposition \( D = P_D + N_D \). Lazarsfeld and Mustaţă give in [6] the description of \( \Delta(D) \) as the area enclosed between the graphs of functions \( \alpha(t) \) and \( \beta(t) \) defined for real numbers \( t \) between 0 and \( \sup \{ s \in \mathbb{R} : D-sC \text{ is effective} \} \) as follows.

\[ \alpha(t) = \text{ord}_x(N_{D-tC}) \quad \text{and} \quad \beta(t) = \alpha(t) + \text{vol}_X|C(P_{D-tC}) = \alpha(t) + P_{D-tC} \cdot C. \]

The description of Okounkov bodies on surfaces was later on rendered by the following result [4, Theorem B].

**Theorem 2.2** (Küronya, Lozovanu, Maclean). Let \( X \) be a smooth projective surface and let \( D \) be a big \( \mathbb{R} \)-divisor. Then the Okounkov body \( \Delta(D) \) is a polygon.

Recently, the authors of [8] presented a different approach to describing Okounkov bodies for a certain class of smooth complex projective surfaces. Note that from this description one also gets as a corollary that Okounkov bodies (on surfaces where this approach applies), being Minkowski sums of segments and triangles, are polygons.

**Theorem 2.3** (Luszcz-Świdecka, Schmitz). Let \( X \) be a smooth complex projective surface with \( \text{Eff}(X) \) rational polyhedral. Given a flag \((x,C)\), where \( x \) is a general point and \( C \) is a big and nef curve on \( X \), there exists a finite set of nef divisors \( \text{MB}(x,C) = \{ P_1, \ldots, P_r \} \) such that for a big and nef \( \mathbb{R} \)-divisor \( D \) there exist uniquely determined non-negative real numbers \( a_i \geq 0 \) with

\[ D = \sum_{i=1}^{r} a_i P_i \quad \text{and} \quad \Delta(D) = \sum_{i=1}^{r} a_i \Delta(P_i), \]

where the first sum indicates the numerical equivalence of divisors and the second sum is the Minkowski sum of convex bodies.

The Theorem above justifies the following definition.

**Definition 2.4** (Minkowski basis). The set \( \text{MB}(x,C) \) in Theorem 2.3 is called the Minkowski basis of \( X \) with respect to the flag \((x,C)\).

**Remark 2.5.** Note that in general \( \text{MB}(x,C) \) is not a basis of the Neron-Severi space \( N^1(X)_{\mathbb{R}} \) (treated as an \( \mathbb{R} \)-vector space).
The proof of Theorem 2.3 in [8] gives in particular a simple way to construct Minkowski basis elements based on the Bauer-Küronya-Szemberg decomposition of the big cone \( \text{Big}(X) \) [2].

**Theorem 2.6 (BKS decomposition).** Let \( X \) be a smooth complex projective surface. Then there is a locally finite decomposition of the big cone of \( X \) into rational locally polyhedral subcones \( \Sigma \) such that in the interior of each subcone \( \Sigma \) the support \( \text{Neg}(\Sigma) \) of the negative part of the Zariski decomposition of the divisors in the subcone is constant.

The idea of Luszcz-Šwidecka and Schmitz is to assign to a chamber \( \Sigma \) an element in the Minkowski basis \( M_\Sigma \). Specifically, let \( C \) be a big and nef curve in the interior of \( \Sigma \). Then

\[
M_\Sigma = dC + \sum_{i=1}^{r} a_i N_i, \quad \text{where } N_1, \ldots, N_r \in \text{Neg}(\Sigma) \text{ and } a_i \text{ are coefficients that are the solution of the following system of equations}
\]

\[
S(a_1, \ldots, a_r)^T = -d(C.N_1, \ldots, N_r)^T, \tag{1}
\]

where \( S \) is the \( r \times r \) intersection matrix of negative curves \( N_1, \ldots, N_r \). Since \( S \) is negatively defined, by an auxiliary result in [2] the inverse matrix \( S^{-1} \) has only negative entries and thus all numbers \( a_i \) are non-negative.

It is convenient to work in the sequel with a compact slice \( \text{Nef}_H(X) \) of the nef cone \( \text{Nef}(X) \) defined as

\[
\text{Nef}_H(X) = \{ D \in \text{Nef}(X) : D.H = 1 \}
\]

for a fixed ample divisor \( H \) on \( X \). We denote by \( f_i \) the number of \( i \)-dimensional faces of \( \text{Nef}_H(X) \) for \( i = 0, \ldots, \rho(X) - 1 \). Moreover we write \( f_0 = (f_0)_b + (f_0)_{nb} \), where \( (f_0)_b \) is the number of big vertices in \( \text{Nef}_H(X) \) and \( (f_0)_{nb} \) is the number of non-big vertices, see also Lemma 3.1.

Finally, we write \( \text{NnB}(X) \) for the number of numerical equivalence classes of nef and non-big integral divisors in \( \text{Nef}_H(X) \), and we write \( \text{Zar}(X) \) for the number of Zariski chambers in the BKS-decomposition of \( \text{Big}(X) \).

### 3. The cardinality of Minkowski bases

In the view of Remark 2.5 it is natural to ask how many elements there are in the Minkowski basis. We will show here that the answer depends on the choice of the flag and that the number

\[
1 + \text{NnB}(X) + \text{Zar}(X) \tag{2}
\]

is a sharp upper bound for the number of elements in the Minkowski basis. The number of negative curves on surfaces with \( \text{Eff}(X) \) rational polyhedral is finite, hence the number of Zariski chambers on such surfaces is finite as well. This number can be large. For example, for del Pezzo surfaces \( X_i \) obtained as the blow ups of \( \mathbb{P}^2 \) in \( i \in \{1, \ldots, 8\} \) general points we have

\[
\begin{array}{c|cccccccc}
  i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \text{Zar}(X_i) & 2 & 5 & 18 & 76 & 393 & 2764 & 33645 & 1501681 \\
\end{array}
\tag{3}
\]

see [1].

Now, we explain that the second summand in (2) is also finite.

**Lemma 3.1 (Nef, non-big divisors).** Let \( X \) be a surface with \( \text{Eff}(X) \) rational polyhedral. Then there is only a finite number of nef and non-big divisors in \( \text{Nef}_H(X) \).
Proof. Assume to the contrary that there are two divisors $N_1, N_2$, which are nef and not big, such that for all $t \in [0,1]$ the divisors $tN_1 + (1-t)N_2$ lie on the common face (here the rational polyhedrality assumption comes into the play). Thus $(tN_1 + (1-t)N_2)^2 = 0$ for every $t \in [0,1]$, which implies that $N_1.N_2 = 0$. This means that the intersection matrix of $N_1, N_2$ is the zero matrix of size $2 \times 2$, which contradicts the index theorem. □

Now we relate the number in (2) to the geometry of the solid NefH(X).

Proposition 3.2. Let $X$ be a smooth complex projective surface with $\text{Eff}(X)$ rational polyhedral. Then
\[
\sum_{i=0}^{\rho-1} f_i = 1 + \text{NnB}(X) + \text{Zar}(X).
\] (4)

Proof. Let $G$ be a face of NefH(X). If $G = \text{NefH}(X)$, then this corresponds to $f_{\rho-1} = 1$ and is accounted for by the right side in the formula (4). Otherwise we distinguish two cases: either $G$ is a vertex of NefH(X) which is not big, hence $G^2 = 0$, or $G$ is a big vertex or a face of dimension $\geq 1$.

The first case occurs $(f_0)_{nb}$ times and is accounted for by the second summand on the right in (4).

The second case corresponds to the third summand in (4). Indeed, given a nef and big divisor $D$ there exists a Zariski chamber $\Sigma_D$ with Neg($\Sigma_D$) = $D^\perp$. This follows from Nakamaye’s result [9, Theorem 1.1]. Thus the inequality $\leq$ in (4) is established.

For the reverse inequality, it suffices to show that distinct Zariski chambers determine distinct faces of NefH(X). To this end, let $\Sigma$ be a Zariski chamber. By [2] there is a face of NefH(X) orthogonal to the support of Neg($\Sigma$). The injectivity of this assignment $\Sigma \to \text{Neg}(\Sigma)^\perp$ follows again from the aforementioned result of Nakamaye. □

Now we are in a position to prove our main result.

Theorem 3.3. Let $X$ be a smooth complex projective surface with $\overline{\text{Eff}(X)}$ rational polyhedral. Given a flag $(x,A)$, where $A$ is an ample curve and $x$ is a smooth point on $A$, there is
\[
\# \text{MB}(x, A) = 1 + \text{NnB}(X) + \text{Zar}(X).
\]

Proof. Given Zariski chambers $\Sigma_1, \Sigma_2$ with Neg($\Sigma_1$) = $\{N_1, \ldots, N_n\}$ and Neg($\Sigma_2$) = $\{N_{n+1}, \ldots, N_m\}$, one associates to them the Minkowski basis elements
\[
M_{\Sigma_1} = b_1A + \sum_{j=1}^{n} a_j N_j \quad \text{and} \quad M_{\Sigma_2} = b_2A + \sum_{j=n+1}^{m} a_j N_j.
\]
Suppose that Neg($\Sigma_1$) $\neq$ Neg($\Sigma_2$) and assume to the contrary that $M_{\Sigma_1} = M_{\Sigma_2}$. Furthermore we may assume after reordering that the symmetric difference between these negative supports is $\{N_k, \ldots, N_m\}$ for a certain $k \in \{1, \ldots, m\}$. By the construction of Minkowski basis elements [8] we know that
\[
M_{\Sigma_1} = M_{\Sigma_2} \in N_i^\perp \quad \text{for all} \quad i \in \{1, \ldots, m\}.
\]
This implies that for every $N_i$ we have $N_i.M_{\Sigma_1} = N_i.M_{\Sigma_2} = 0$. Let us take one of the elements from $\{N_k, \ldots, N_m\}$. This implies in particular that $N_i.A = 0$, a
contradiction. Proceeding in the same spirit, one can show that the symmetric difference is empty and \( \text{Neg}(\Sigma_1) = \text{Neg}(\Sigma_2) \), which ends the proof. □

**Example 3.4** (Del Pezzo surfaces). Using the above theorem we can compute the cardinality of Minkowski basis for del Pezzo surfaces \( X_i \) with respect to a fixed ample flag \((x, A)\). To this end we need to compute the number of nef non-big curves on \( X_i \). Let \( C = aH - \sum b_jE_j \) be such a curve, where as usual \( \pi_i : X_i \to \mathbb{P}^2 \) is the blow up of \( \mathbb{P}^2 \) at \( i \) general points with exceptional divisors \( E_1, \ldots, E_i \) and \( H = \pi_i^*(\mathcal{O}_{\mathbb{P}^2}(1)) \). First we observe that \( C \) is a rational curve. This follows from the adjunction since

\[
2(p_a(C) - 1) = K_{X_i} \cdot C + C^2 = K_{X_i} \cdot C < 0
\]

implies \( p_a(C) = 0 \). Hence

\[
2 = -K_{X_i} \cdot C = 3a - \sum b_j. \tag{5}
\]

On the other hand

\[
0 = C^2 = a^2 - \sum b_j^2. \tag{6}
\]

It is elementary to check that (5) and (6) have only finitely many integral solutions, listed (up to permutation) in the following table

|   | a  | b_1 | b_2 | b_3 | b_4 | b_5 | b_6 | b_7 | b_8 |
|---|----|-----|-----|-----|-----|-----|-----|-----|-----|
| C(1)| 1  | 1   | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| C(2)| 2  | 1   | 1   | 1   | 1   | 0   | 0   | 0   | 0   |
| C(3)| 3  | 2   | 1   | 1   | 1   | 1   | 1   | 0   | 0   |
| C(4)| 4  | 2   | 2   | 2   | 1   | 1   | 1   | 0   | 0   |
| C(5)| 5  | 3   | 2   | 2   | 1   | 1   | 1   | 1   | 1   |
| C(6)| 6  | 3   | 3   | 2   | 2   | 2   | 1   | 1   | 1   |
| C(7)| 7  | 3   | 3   | 3   | 3   | 2   | 2   | 2   | 1   |
| C(8a)| 8  | 4   | 3   | 3   | 3   | 3   | 2   | 2   | 2   |
| C(8b)| 8  | 3   | 3   | 3   | 3   | 3   | 3   | 3   | 3   |
| C(9)| 9  | 4   | 4   | 3   | 3   | 3   | 3   | 3   | 3   |
| C(10)| 10 | 4   | 4   | 4   | 3   | 3   | 3   | 3   | 3   |
| C(11)| 11 | 4   | 4   | 4   | 4   | 4   | 4   | 4   | 3   |

Note that all solutions can be obtained from \( C(1) \) applying standard Cremona transformations. This verifies again that an irreducible nef non-big curve on a del Pezzo surface is rational.

Counting all curves \( C(j) \) on the appropriate surface \( X_i \) and taking (3) into account we have

\[
\# \text{MB}(x, A) = 3 \quad 7 \quad 21 \quad 81 \quad 403 \quad 2797 \quad 33764 \quad 1503721.
\]

**Remark 3.5.** For \( r = 0, \ldots, 8 \) let \( X_r \) be a del Pezzo surface arising as the blow up of the projective plane \( \mathbb{P}^2 \) in \( r \) general points. Let \( C \) be a curve in the anti-canonical system \(-K_{X_r}\). There is a Weyl group action on \( \text{Eff}(X_r) \), which fixes the anti-canonical class, see [5]. In this situation, there is a Weyl invariant Minkowski basis \( \text{MB}(x, C) \). Indeed, it can be constructed taking for each \( j = 0, \ldots, \rho(X_r) - 1 \) an element \( M_j \) corresponding to a \( j \)-dimensional face of \( \text{Nef}_H(X_r) \) and then all of its images under the action of the Weyl group, see also [2, Section 3.1].
Now we show that for a special choice of a flag \((x, C)\), it might happen that the number of divisors in the Minkowski basis is strictly smaller than the number in (2). In fact we get Minkowski bases with any number of elements between 3 and 7 on the del Pezzo surface \(X_2\).

**Example 3.6.** For the del Pezzo surface \(X_2\) we have the following possibilities:

- Fix a toric flag for \(X_2\), i.e., \((x, L_1)\) with \(L_1 \in |H - E_1|\) and \(x = L_1 \cap L_2\) for a fixed line \(L_2 \in |H - E_2|\). Then by [10]

\[
\text{MB}(x, L_1) = \{H, H - E_1, H - E_2\}.
\]

From now on \(x\) denotes a general point on the flag curve \(C\).

- For the flag \((x, C)\), where \(C \in |H|\), we have

\[
\text{MB}(x, C) = \{2H - E_1, H - E_1, H - E_2, 2H - E_1 - E_2\}.
\]

- For a curve \(C \in |2H - E_1|\), we get

\[
\text{MB}(x, C) = \{2H - E_1, H - E_1, H - E_2, H, 3H - 2E_1 - E_2\}.
\]

- For a curve \(C \in |2H - E_1 - E_2|\) we have

\[
\text{MB}(x, C) = \{2H - E_1 - E_2, H - E_1, H - E_2, 2H - E_1 - E_2, 2H - E_2, H\}.
\]

- For the anticanonical flag \((x, C)\) with a curve \(C \in |-K_{X_2}|\) we have

\[
\text{MB}(x, C) = \{-K_{X_2}, H, H - E_1, H - E_2, 2H - E_1 - E_2, 3H - E_1, 3H - E_2\}.
\]

**Remark 3.7.** It would be interesting to know effective lower bounds on the number of elements in the Minkowski basis.

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